Neap

HSC Trial Examination 2020

Mathematics Extension 2

Solutions and marking guidelines

Section I

Sample answer	Syllabus content, outcomes and targeted performance bands
Question 1 **A** Let P be $n^2 + 4n + 1$ is even and let Q be n is odd. The contrapositive of $P \Rightarrow Q$ is $(\sim Q) \Rightarrow (\sim P)$. So the contrapositive is "If n is even, then $n^2 + 4n + 1$ is odd".	MEX-P1 The Nature of Proof MEX12–8 Bands E2–E3
Question 2 **A** Integration by parts takes the form $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Let $u = x^2$ and $\frac{dv}{dx} = e^{-x}$. So $\frac{du}{dx} = 2x$ and $v = -e^{-x}$. Substituting into $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ gives: $\int x^2 e^{-x} dx = -x^2 e^{-x} - \int (-e^{-x})(2x)dx$ So $\int x^2 e^{-x} dx = -x^2 e^{-x} + \int 2xe^{-x} dx$.	MEX-C1 Further Integration MEX12–5 Bands E2–E3
Question 3 **D** In general, if the factor in the denominator is $(ax+b)^n$, then the corresponding term(s) in the partial fraction decomposition is $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$. Here, $(x-3)^2$ corresponds to $\frac{A}{x-3} + \frac{B}{(x-3)^2}$. If the factor in the denominator is $ax^2 + bx + c$ (with no linear factors), then the corresponding term in the partial fraction decomposition is $\frac{Cx+D}{ax^2+bx+c}$. So the partial fraction form is $\frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+2}$.	MEX-C1 Further Integration MEX12–5 Bands E2–E3
Question 4 **B** Geometrically, to form OC from OA, we rotate OA anticlockwise through an angle of $\frac{\pi}{2}$ and then double the length of OA. Anticlockwise rotation through $\frac{\pi}{2}$ is equivalent to multiplying by i, and doubling the length is achieved by multiplying by 2. So the complex number that corresponds to vertex C is $2iw$.	MEX-N1 Introduction to Complex Numbers MEX12–4 Bands E2–E3

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Question 5 **C** To disprove a statement of the form $P \Rightarrow Q$, we require a counterexample for which *P* is true and *Q* is not true. Here *P* is $x > y$. For **A**, *Q* is $x^2 > y^2$. For example, with $x = -2$ and $y = -3$, $-2 > -3$ is true but $4 > 9$ is not true, and so *Q* is not true. **A** is incorrect. For **B**, *Q* is $\frac{1}{x} < \frac{1}{y}$. For example, with $x = 3$ and $y = -2$, $3 > -2$ is true but $\frac{1}{3} < -\frac{1}{2}$ is not true and so *Q* is not true. **B** is incorrect. For **C**, *Q* is $x + z > y + z$. An inequality is unchanged when the same amount is added to both sides, so *Q* is true and hence **C** is correct. For **D**, *Q* is $xz > yz$. For example, with $x = 3$, $y = 2$ and $z = -2$, $3 > 2$ is true $-6 > -4$ is not true and so *Q* is not true. **D** is incorrect.	MEX-P1 The Nature of Proof MEX12–2 Bands E2–E3
Question 6 **A** The period is 3 seconds. $3 = \frac{2\pi}{n} \Rightarrow n = \frac{2\pi}{3}$ When $t = 0$, $x = 12$. So the equation of motion is $x = 12\cos\frac{2\pi t}{3}$.	MEX-M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4
Question 7 **D** The radius of the sphere is given by $r^2 = \lvert\underset{\sim}{a} - \underset{\sim}{c}\rvert^2$. $\underset{\sim}{a} - \underset{\sim}{c} = (3\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}) - (-3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$ $= 6\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ $r^2 = 6^2 + 2^2 + 3^2$ $= 49$ The equation of the sphere is $\lvert\underset{\sim}{v} - \underset{\sim}{c}\rvert^2 = r^2$ where $\underset{\sim}{v} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$. In Cartesian form, the equation of the sphere is $(x+3)^2 + (y-1)^2 + (z+2)^2 = 49$.	MEX-V1 Further Work with Vectors MEX12–3 Bands E3–E4
Question 8 **B** If $z = x + yi$, then $\bar{z} = x - yi$. Substituting these into $i\bar{z} - iz = 2$ gives: $i(x - yi) - i(x + yi) = 2$ $-2i^2y = 2$ $2y = 2$ $y = 1$	MEX-N1 Introduction to Complex Numbers MEX12–4 Bands E3–E4

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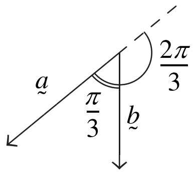
Question 9 **C** $\frac{dv}{dx} = \frac{1}{2v}$ and so $v\frac{dv}{dx} = a = v\left(\frac{1}{2v}\right) = \frac{1}{2}$. Therefore, the acceleration is constant. Since the acceleration is also positive, the velocity is increasing.	MEX-M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4
Question 10 **C** The angle between the two vectors is defined as the angle between their positive directions – the angle formed when the two vectors are placed 'tail-to-tail'.  Hence the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$. $\underset{\sim}{a} \cdot \underset{\sim}{b} = \lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta$ $= a^2\cos\frac{\pi}{3}$ since $\lvert\underset{\sim}{a}\rvert = \lvert\underset{\sim}{b}\rvert$ and $\lvert\underset{\sim}{a}\rvert = a$ $= \frac{1}{2}a^2$ since $\cos\frac{\pi}{3} = \frac{1}{2}$	MEX-V1 Further Work with Vectors MEX12–3 Bands E3–E4

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Section II

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 11	
(a) Let the other root be β. $-i\beta = 1 - i$ (product of roots) $\beta = \frac{1-i}{-i} \times \frac{i}{i}$ $= 1 + i$ $-i + 1 + i = -b$ (sum of roots) $b = -1$	MEX-N1 Introduction to Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution . . . . .2 • Uses product of roots to attempt to find the other root 1
(b) (i) $\underset{\sim}{a} + \underset{\sim}{c} = (2+m)\underset{\sim}{i} + (2+n)\underset{\sim}{j} + \underset{\sim}{k}$ As $\underset{\sim}{a} + \underset{\sim}{c}$ is parallel to $\underset{\sim}{b}$, $\underset{\sim}{a} + \underset{\sim}{c} = \lambda\underset{\sim}{b}$. $\lambda\underset{\sim}{b} = 2\lambda\underset{\sim}{j} + 2\lambda\underset{\sim}{k}$ $(2+m)\underset{\sim}{i} + (2+n)\underset{\sim}{j} + \underset{\sim}{k} = 2\lambda\underset{\sim}{j} + 2\lambda\underset{\sim}{k}$ Equating components of $\underset{\sim}{a} + \underset{\sim}{c}$ and $\lambda\underset{\sim}{b}$ and solving for λ gives $\lambda = \frac{1}{2}$. So $2 + m = 0$ and $2 + n = 1$. Hence $m = -2$ and $n = -1$.	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . .2 • Obtains correct expressions for $\underset{\sim}{a} + \underset{\sim}{c}$ and $\lambda\underset{\sim}{b}$ 1
(ii) If $\underset{\sim}{c}$ is a unit vector, then $\lvert\underset{\sim}{c}\rvert = 1$, and so $\sqrt{m^2 + n^2} = 1$. Squaring both sides gives $m^2 + n^2 = 1$. (1) Applying the condition $\underset{\sim}{c} \cdot \underset{\sim}{a} = 0$ gives: $(m\underset{\sim}{i} + n\underset{\sim}{j}) \cdot (2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) = 0$ $2m + 2n = 0$ $m = -n$ (2) Substituting (2) into (1) and solving for n gives: $2n^2 = 1 \Rightarrow n = \pm\frac{1}{\sqrt{2}}$ Substituting into (2) gives: $m = -\frac{1}{\sqrt{2}}, n = \frac{1}{\sqrt{2}}$ or $m = \frac{1}{\sqrt{2}}, n = -\frac{1}{\sqrt{2}}$	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . .2 • Obtains two valid equations in m and n. . .1

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(c) $u = 1 - \sin 2x$ $du = -2\cos 2x\,dx$ $= -2(2\cos^2 x - 1)dx$ $= 2(1 - 2\cos^2 x)dx$ When $x = \frac{\pi}{4}$, $u = 0$ and when $x = \frac{\pi}{2}$, $u = 1$. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 - \sin 2x}(1 - 2\cos^2 x)dx = \int_0^1 u^{\frac{1}{2}}\left(\frac{1}{2}du\right)$ $= \frac{1}{2} \times \frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^1$ $= \frac{1}{3}(1 - 0)$ $= \frac{1}{3}$	MEX-C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution . . . . 4 • Obtains appropriate integral OR equivalent merit . . . . 3 • Correctly rewrites integrand and limits OR equivalent merit . . . . 2 • Makes suggested substitution and rewrites integrand 1
(d) (i) $\lvert z \rvert = \sqrt{(\sqrt{3})^2 + 1^2}$ $= 2$ $\text{Arg}\, z = \tan^{-1}\frac{1}{\sqrt{3}}$ $= \frac{\pi}{6}$ So $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$.	MEX-N1 Introduction to Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution . . . . 2 • Gives exact argument in radians OR equivalent merit 1

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(ii) $\bar{z} = \sqrt{3} - i$ $= 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ Applying de Moivre's theorem gives $z^n = 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right)$ and $\bar{z}^n = 2^n\left(\cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)\right)$. Using $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ to form $z^n - \bar{z}^n$ gives: $z^n - \bar{z}^n = 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right) -$ $2^n\left(\cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)\right)$ $= 2^n\cos\frac{n\pi}{6} - 2^n\cos\frac{n\pi}{6} + 2^n i\sin\frac{n\pi}{6} + 2^n i\sin\frac{n\pi}{6}$ $= 2^{n+1} i\sin\frac{n\pi}{6}$ $z^n - \bar{z}^n = 0 \Rightarrow \sin\frac{n\pi}{6} = 0$ $\sin\frac{n\pi}{6} = \sin k\pi$ $k = 1 \Rightarrow n = 6$ Hence the smallest possible integer is 6.	MEX-N1 Introduction to Complex Numbers MEX12–4 Bands E2–E4 • Gives the correct solution . . . . .3 • Obtains $z^n - \bar{z}^n = 2^{n+1} i\sin\frac{n\pi}{6}$ OR equivalent merit. . . . . .2 • Applies de Moivre's theorem on z and $\bar{z}$ to form expressions for z^n and $\bar{z}^n$ 1
Question 12	
(a) Let $u = u_1 + 2i$ and $v = -1 + v_2 i$. $u + v = (u_1 - 1) + (2 + v_2)i$ (1) $-uv = (u_1 + 2v_2) + (2 - u_1 v_2)i$ (2) Equating the real components of (1) and (2) gives $u_1 - 1 = u_1 + 2v_2 \Rightarrow v_2 = -\frac{1}{2}$. Equating the imaginary components of (1) and (2) with $v_2 = -\frac{1}{2}$ gives $-\frac{1}{2} = \frac{1}{2}u_1 \Rightarrow u_1 = -1$. So the two complex numbers are $u = -1 + 2i$ and $v = -1 - \frac{1}{2}i$.	MEX-N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution . . . . .2 • Uses $u + v$ and $-uv$ to form two valid equations. 1

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(b)	$\left\|e^{2i\theta}+e^{-2i\theta}\right\|=1$ Use of Euler's formula and $\cos(-2\theta)=\cos 2\theta$ and $\sin(-2\theta)=-\sin 2\theta$ gives: $\|\cos 2\theta+i\sin 2\theta+\cos(-2\theta)+i\sin(-2\theta)\|=1$ $\|\cos 2\theta+\cos 2\theta+i\sin 2\theta-i\sin 2\theta\|=1$ $2\|\cos 2\theta\|=1$ So $\cos 2\theta=\pm\frac{1}{2}$ and $-2\pi<2\theta\le 2\pi$. $\cos 2\theta=\frac{1}{2}\Rightarrow 2\theta=\pm\frac{\pi}{3},\pm\left(2\pi-\frac{\pi}{3}\right)$ and $\cos 2\theta=-\frac{1}{2}\Rightarrow 2\theta=\pm\frac{2\pi}{3},\pm\left(2\pi-\frac{2\pi}{3}\right)$ $\theta=\pm\frac{\pi}{6},\pm\frac{5\pi}{6},\pm\frac{\pi}{3},\pm\frac{2\pi}{3}$	MEX-N2 Using Complex Numbers MEX12–7, 12–8 Bands E2–E4 • Finds all EIGHT correct values for θ. . . . 3 • Obtains $\cos 2\theta=\pm\frac{1}{2}$ . . . . 2 • Uses Euler's formula OR equivalent merit 1
(c)	Consider $n=1$. $\frac{dy}{dx}=-\frac{1}{(1+x)^2}$ and $\frac{(-1)^1 1!}{(1+x)^{1+1}}=-\frac{1}{1+x^2}=\frac{dy}{dx}$ True when $n=1$. Suppose true for $n=k$. So $\frac{d^k y}{dx^k}=\frac{(-1)^k k!}{(1+x)^{k+1}}$. Required to show it is true for $n=k+1$. That is, $\frac{d^{k+1}y}{dx^{k+1}}=\frac{(-1)^{k+1}(k+1)!}{(1+x)^{k+2}}$. $\text{LHS}=\frac{d^{k+1}y}{dx^{k+1}}$ $=\frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$ $=\frac{d}{dx}\left(\frac{(-1)^k k!}{(1+x)^{k+1}}\right)$ $=(-1)^k k!(-(k+1)(1+x)^{-(k+2)})$ $=\frac{(-1)^{k+1}(k+1)!}{(1+x)^{k+2}}$ $=\text{RHS}$ If true for $n=k$, then true for $n=k+1$. Hence, by mathematical induction, true for $n\ge 1$.	MEX-P2 Further Proof by Mathematical Induction MEX12–2 Bands E2–E3 • Gives the correct proof . . . . 3 • Establishes an inductive step OR equivalent merit . . . . 2 • Establishes the $n=1$ case OR equivalent merit 1

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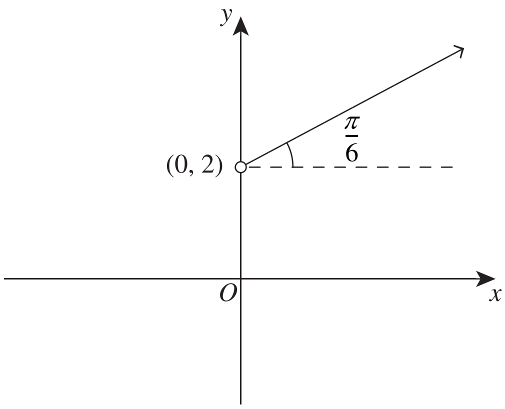
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(d) (i) Substituting $z = x + yi$ into $\text{Arg}(z - 2i) = \frac{\pi}{6}$ gives: $\text{Arg}(x + (y-2)i) = \frac{\pi}{6}$ $\frac{y-2}{x} = \tan\frac{\pi}{6}$ $\frac{y-2}{x} = \frac{1}{\sqrt{3}}$ $y - 2 = \frac{1}{\sqrt{3}}x$ $y = \frac{1}{\sqrt{3}}x + 2,\ x > 0$	MEX-N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution . . . . .2 • Substitutes $z = x + yi$ and attempts to express y in terms of x OR equivalent merit 1
(ii) 	MEX-N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Correctly sketches the relation 1

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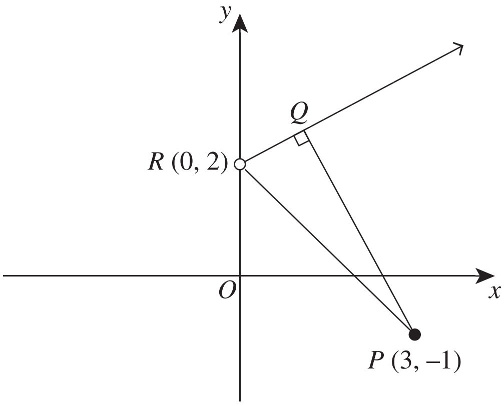
(iii) 	MEX-N2 Using Complex Numbers MEX12–4 Bands E3–E4 • Gives the correct solution . . . . 4 • Obtains the exact value of $\sin\frac{5\pi}{12}$ . . . . . . 3 • Obtains RP and $\angle QRP$ OR equivalent merit . . . . 2 • Recognises that the minimum distance from P to Q occurs when $\angle RQP = \frac{\pi}{2}$ . . . 1
The complex number z lies on the ray $\text{Arg}(z - 2i) = \frac{\pi}{6}$. $\lvert z - 3 + i\rvert$ is the distance from the point $P(3, -1)$ to the ray. The minimum distance from point P to a point Q on the ray occurs when $\angle RQP = \frac{\pi}{2}$. This is the condition for the least possible value of $\lvert z - 3 + i\rvert$. $RP = \sqrt{3^2 + 3^2} = 3\sqrt{2}$ $\angle QRP = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$ $\frac{QP}{3\sqrt{2}} = \sin\frac{5\pi}{12}$ The compound angle formula $\sin(A + B) = \sin A\cos B + \cos A\sin B$ can be used to find the exact value of $\sin\frac{5\pi}{12}$. $\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$ $QP = \frac{3\sqrt{2}(\sqrt{3} + 1)}{2\sqrt{2}} = \frac{3(\sqrt{3} + 1)}{2}$ So the least possible exact value of $\lvert z - 3 + i\rvert$ is $\frac{3(\sqrt{3} + 1)}{2}$.	

 TENME2_SS_20.FM

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Question 13	
(a) Assume that there exists $p, q \in N$ such that $\log_2 5 = \frac{p}{q}$ where $q \geq 1$ and the highest common factor of p and q is 1. $\log_2 5 = \frac{p}{q} \Rightarrow 5 = 2^{\frac{p}{q}}$ Taking the qth power of both sides gives $5^q = 2^p$. **EITHER:** Therefore, 5 is a factor of 5^q but not a factor of 2^p. **OR** 2 is a factor of 2^p but not a factor of 5^q. **OR** 5^q is odd and 2^p is even. Then: No p and q such that $p, q \in N$ satisfies the equation $5^q = 2^p$ and this equation must be a contradiction. So $\log_2 5$ is an irrational number.	MEX-P1 The Nature of Proof MEX12–2 Bands E2–E3 • Gives the correct proof . . . . .3 • Determines $\log_2 5 = \frac{p}{q} \Rightarrow 5 = 2^{\frac{p}{q}} \Rightarrow 5^q = 2^p$ . . . . .2 • Attempts proof by contradiction 1
(b) (i) $x = a\sin(nt + \alpha)$ When $t = 0$, $x = \frac{a}{2}$ and so $\frac{a}{2} = a\sin\alpha \Rightarrow \alpha = \frac{\pi}{6}$. So $x = a\sin\left(nt + \frac{\pi}{6}\right)$.	MEX-M1 Applications of Calculus to Mechanics MEX12–6, MEX12–7 Bands E2–E4 • Gives the correct solution. 1
(ii) Find the value of t when $x = a$. $a = a\sin\left(nt + \frac{\pi}{6}\right) \Rightarrow \sin\left(nt + \frac{\pi}{6}\right) = 1$ $nt + \frac{\pi}{6} = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{3n}$	MEX-M1 Applications of Calculus to Mechanics MEX12–6, MEX12–7 Bands E2–E4 • Gives the correct solution. 1

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(iii) $v^2 = n^2(a^2 - x^2)$ and when $x = \frac{2a}{3}$, $\lvert v \rvert = V$. *Note: $v^2 = n^2(a^2 - x^2)$ can be derived from the differential equation $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x - c)$, where $c = 0$.* $V^2 = \frac{5n^2a^2}{9}$ $a^2 = \frac{9V^2}{5n^2}$ $a = \frac{3V}{\sqrt{5}n}$ Using $v_{\max} = na$ gives: $v_{\max} = n\left(\frac{3V}{\sqrt{5}n}\right)$ $= \frac{3V}{\sqrt{5}}$ (m s^{-1})	MEX-M1 Applications of Calculus to Mechanics MEX12–6, MEX12–7 Bands E2–E4 • Gives the correct solution . . . . 2 • Obtains a valid expression for V^2 in terms of *a* and *n* 1
(c) (i) The position vector of the point $(-1, 2, -3)$ is $\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$. So the vector equation of l_1 is $\underset{\sim}{r}_1 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution 1
(ii) The vector equation of l_2 can be written as $\underset{\sim}{r}_2 = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. So $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ is a vector parallel to l_2.	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution 1

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(iii) At the point of intersection, $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$. Equating the three components: $s - 1 = -t + 1 \Rightarrow s + t = 2 \quad (1)$ $-2s + 2 = 2t - 2 \Rightarrow -2s - 2t = -4 \quad (2)$ $2s - 3 = 3t + 6 \Rightarrow 2s - 3t = 9 \quad (3)$ (2) + (3) gives: $-5t = 5 \Rightarrow t = -1$ Substituting $t = -1$ into (3) and solving gives: $2s + 3 = 9 \Rightarrow s = 3$ Substituting $s = 3$ and $t = -1$ into (1) gives: $3 - 1 = 2$ Hence the lines intersect. Thus from $\underset{\sim}{r}_2$: $\underset{\sim}{r}_2 = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} - 1\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$ So the point of intersection is $(2, -4, 3)$. Note that substituting $s = 3$ into $\underset{\sim}{r}_1$ gives: $\underset{\sim}{r}_1 = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$ This gives the same point.	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . .3 • Attempts to solve two of the equations AND checks that the lines intersect. . . . . .2 • Equates three components AND forms three valid equations 1
(iv) The angle between l_1 and l_2 is the angle between the two direction vectors of the lines. l_1 is parallel to $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and l_2 is parallel to $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. Let θ be the required angle. $\cos\theta = \dfrac{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\right\|\left\|\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}\right\|}$ $= \dfrac{-1 - 4 + 6}{\sqrt{1^2 + (-2)^2 + 2^2} \times \sqrt{(-1)^2 + 2^2 + 3^2}}$ $= \dfrac{1}{3\sqrt{14}}$ So $\theta = \cos^{-1}\left(\dfrac{1}{3\sqrt{14}}\right)$. Converting to degrees, $\theta = 84.9°$ (correct to one decimal place).	MEX-V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . .3 • Obtains the correct value for $\cos\theta$ . . . . . .2 • Attempts to find the angle between the two direction vectors 1

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Question 14	
(a) (i) The forces acting on the parachutist are the force due to weight (acting in the same direction as the motion) and the force due to air resistance (acting in the opposite direction to the motion). From Newton's second law, the equation of motion is $m\ddot{x} = mg - mkv^2$. Hence $\ddot{x} = g - kv^2$.	MEX-M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution 1
(ii) $\ddot{x} = v\dfrac{dv}{dx} = g - kv^2$ Separating variables gives: $\int dx = \int \dfrac{v}{g - kv^2} dv$ Using integration by substitution (or recognition) gives: $x + c_1 = -\dfrac{1}{2k}\ln\lvert g - kv^2\rvert$ $Ae^{-2kx} = g - kv^2$ where $A = e^{-2kc_1}$ When $x = 0$, $v = 0$ and so $A = g$. $ge^{-2kx} = g - kv^2$ $kv^2 = g(1 - e^{-2kx})$ $v^2 = \dfrac{g}{k}(1 - e^{-2kx})$	MEX-C1 Further Integration MEX12–5, 12–7 Bands E2–E4 • Gives the correct solution . . . . 2 • Obtains $x + c_1 = -\dfrac{1}{2k}\ln\lvert g - kv^2\rvert$ OR equivalent merit 1
(iii) Terminal velocity is achieved when $\ddot{x} = 0$. $g - kv^2 = 0 \Rightarrow k = \dfrac{g}{v^2}$ Substituting $v = 6g$ gives: $k = \dfrac{g}{36g^2}$ $= \dfrac{1}{36g}$ **OR** $v^2 = \dfrac{g}{k}(1 - e^{-2kx})$ As $x \to \infty$, $1 - e^{-2kx} \to 1$. $v^2 = \dfrac{g}{k} \Rightarrow k = \dfrac{g}{v^2}$ Substituting $v = 6g$ gives: $k = \dfrac{g}{36g^2}$ $= \dfrac{1}{36g}$	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iv) From Newton's second law, the equation of motion is $m\ddot{x} = mg - \frac{1}{10}mgv$. $\ddot{x} = \frac{dv}{dt} = g\left(1 - \frac{v}{10}\right)$ Separating variables gives: $\int g\,dt = \int \frac{1}{1 - \frac{v}{10}}dv$ $gt + c_2 = -10\ln\left\|1 - \frac{v}{10}\right\|$ $Be^{-\frac{gt}{10}} = 1 - \frac{v}{10}$ where $B = e^{-\frac{c_2}{10}}$ When $t = 0$, $v = 5g$ and so $B = 1 - \frac{g}{2}$. $\frac{v}{10} = 1 - \left(1 - \frac{g}{2}\right)e^{-\frac{gt}{10}}$ $v = 10 + 5(g - 2)e^{-\frac{gt}{10}}$	MEX-C1 Further Integration MEX12–5, 12–6, 12–7 Bands E2–E4 • Gives the correct solution . . . . .3 • Obtains $gt + c_2 = -10\ln\left\|1 - \frac{v}{10}\right\|$ OR equivalent merit. . . . . .2 • Obtains $\frac{dv}{dt} = g\left(1 - \frac{v}{10}\right)$ OR equivalent merit. 1
(v) Substituting $v = 2g$ and $t = T$ into $v = 10 + 5(g - 2)e^{-\frac{gt}{10}}$ and solving for T gives: $2(g - 5) = 5(g - 2)e^{-\frac{gT}{10}}$ $e^{-\frac{gT}{10}} = \frac{2(g-5)}{5(g-2)}$ $e^{\frac{gT}{10}} = \frac{5(g-2)}{2(g-5)}$ $\frac{gT}{10} = \ln\left[\frac{5(g-2)}{2(g-5)}\right]$ $T = \frac{10}{g}\ln\left[\frac{5(g-2)}{2(g-5)}\right]$	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution 1

	Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) (i)	The z^5 terms cancel, leaving a quartic equation that will have only four roots. Hence the equation does not have five roots. **OR** As $e^{2k\pi i} = 1$, the equation can be written as $e^{2k\pi i}z^5 = (z+1)^5$. Taking the fifth root of each side of the equation gives: $e^{\frac{2k\pi i}{5}}z = z + 1, k = 0, 1, 2, 3, 4$ However, $k = 0$ must be excluded because this gives $z = z + 1$. So $k = 1, 2, 3, 4$ corresponds to four roots. Hence the equation does not have five roots.	MEX-N2 Using Complex Numbers MEX12–4, 12–7, 12–8 Bands E2–E4 • Gives a correct explanation 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Using $e^{2k\pi i} = 1$ gives $e^{2k\pi i}z^5 = (z+1)^5$. Hence $e^{\frac{2k\pi i}{5}}z = z + 1$, $k = 1, 2, 3, 4$. $k = 0$ is excluded because this gives $z = z + 1$. *Note: The above work may be seen in part (b) (i).* Solving for z where $k = 1, 2, 3, 4$ gives: $z\left(e^{\frac{2k\pi i}{5}} - 1\right) = 1$ $z = \dfrac{1}{e^{\frac{2k\pi i}{5}} - 1}$ Multiplying the numerator and denominator of the RHS by $e^{-\frac{k\pi i}{5}}$ gives: $z = \dfrac{e^{-\frac{k\pi i}{5}}}{\left(e^{\frac{2k\pi i}{5}}e^{-\frac{k\pi i}{5}}\right) - e^{-\frac{k\pi i}{5}}}$ $= \dfrac{e^{-\frac{k\pi i}{5}}}{e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}}}$ Using $\sin\theta = \dfrac{1}{2i}(e^{i\theta} - e^{-i\theta})$ gives $e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}} = 2i\sin\dfrac{k\pi}{5}$. $z = \dfrac{\cos\left(-\frac{k\pi}{5}\right) + i\sin\left(-\frac{k\pi}{5}\right)}{2i\sin\frac{k\pi}{5}}$ $= \dfrac{\cos\frac{k\pi}{5} - i\sin\frac{k\pi}{5}}{2i\sin\frac{k\pi}{5}}$ $= \dfrac{1}{2i}\cot\dfrac{k\pi}{5} - \dfrac{1}{2}$ $= -\dfrac{1}{2} - \dfrac{1}{2}i\cot\dfrac{k\pi}{5}$, $k = 1, 2, 3, 4$	MEX-N2 Using Complex Numbers MEX12–4, 12–7, 12–8 Bands E2–E4 • Gives the correct solutions in the correct form . . . . .4 • Determines $e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}} = 2i\sin\frac{k\pi}{5}$ and finds $z = \dfrac{\cos\left(-\frac{k\pi}{5}\right) + i\sin\left(-\frac{k\pi}{5}\right)}{2i\sin\frac{k\pi}{5}}$ OR equivalent merit. . . . . .3 • Obtains $z = \dfrac{e^{-\frac{k\pi i}{5}}}{e^{\frac{k\pi i}{5}} - e^{-\frac{k\pi i}{5}}}$ OR equivalent merit. . . . . .2 • Obtains $z = \dfrac{1}{e^{\frac{2k\pi i}{5}} - 1}$ OR equivalent merit. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) The roots of $z^5 = (z+1)^5$ are $z = -\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5}, k = 1, 2, 3, 4.$ The roots of $iz^5 = (iz+1)^5$ are $iz = -\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5}, k = 1, 2, 3, 4.$ $z = -i\left(-\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5}\right), k = 1, 2, 3, 4$ Multiplication by $-i$ is equivalent to a clockwise rotation through $\frac{\pi}{2}$. Hence the roots of $iz^5 = (iz+1)^5$ can be obtained by rotating the points $z = -\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5}, k = 1, 2, 3, 4$ through $\frac{\pi}{2}$ clockwise about the origin.	MEX-N2 Using Complex Numbers MEX12–4, 12–7, 12–8 Bands E2–E4 • Gives a correct geometric description . . . 2 • Obtains $z = -i\left(-\frac{1}{2} - \frac{1}{2}i\cot\frac{k\pi}{5}\right), k = 1, 2, 3, 4$ 1
Question 15	
(a) (i) Using $\overrightarrow{OM} \cdot \overrightarrow{MC} = 0$ gives: $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$ $= \underset{\sim}{a} + k\underset{\sim}{b}$ $\overrightarrow{MC} = \overrightarrow{MA} + \overrightarrow{AO} + \overrightarrow{OC}$ $= -k\underset{\sim}{b} - \underset{\sim}{a} + \underset{\sim}{b}$ $= (1-k)\underset{\sim}{b} - \underset{\sim}{a}$ $\overrightarrow{OM} \cdot \overrightarrow{MC} = (\underset{\sim}{a} + k\underset{\sim}{b}) \cdot ((1-k)\underset{\sim}{b} - \underset{\sim}{a})$ $= (1-k)(\underset{\sim}{a} \cdot \underset{\sim}{b}) - (\underset{\sim}{a} \cdot \underset{\sim}{a}) + k(1-k)(\underset{\sim}{b} \cdot \underset{\sim}{b}) - k(\underset{\sim}{a} \cdot \underset{\sim}{b})$ $= (1-k)(\underset{\sim}{a} \cdot \underset{\sim}{b}) - \|\underset{\sim}{a}\|^2 + k(1-k)\|\underset{\sim}{b}\|^2 - k(\underset{\sim}{a} \cdot \underset{\sim}{b})$ $= (1-2k)(\underset{\sim}{a} \cdot \underset{\sim}{b}) - \|\underset{\sim}{a}\|^2 + k(1-k)\|\underset{\sim}{b}\|^2$ $= 2(1-2k)\cos\alpha\|\underset{\sim}{a}\|^2 - \|\underset{\sim}{a}\|^2 + 4k(1-k)\|\underset{\sim}{a}\|^2$ (as $\|\underset{\sim}{b}\| = 2\|\underset{\sim}{a}\|$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2\|\underset{\sim}{a}\|^2\cos\alpha$) $= 2(1-2k)\cos\alpha\|\underset{\sim}{a}\|^2 - (1-4k+4k^2)\|\underset{\sim}{a}\|^2$ $= 2(1-2k)\cos\alpha\|\underset{\sim}{a}\|^2 - (1-2k)^2\|\underset{\sim}{a}\|^2$ $= (1-2k)\|\underset{\sim}{a}\|^2(2\cos\alpha - (1-2k))$ Given $\overrightarrow{OM} \cdot \overrightarrow{MC} = 0$, so $(1-2k)\|\underset{\sim}{a}\|^2(2\cos\alpha - (1-2k)) = 0.$	MEX-V1 Further Work with Vectors MEX12–3, 12–7 Bands E2–E4 • Gives the correct solution. . . . . 4 • Uses $\|\underset{\sim}{b}\| = 2\|\underset{\sim}{a}\|$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2\|\underset{\sim}{a}\|^2\cos\alpha$ AND attempts to simplify . . . . 3 • Obtains $(1-2k)(\underset{\sim}{a} \cdot \underset{\sim}{b}) - \|\underset{\sim}{a}\|^2 + k(1-k)\|\underset{\sim}{b}\|^2$ OR equivalent merit . . . . 2 • Obtains ONE of $\overrightarrow{OM}$ OR $\overrightarrow{MC}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and k 1

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Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $(1-2k)\lvert \underset{\sim}{a}\rvert^2(2\cos\alpha-(1-2k))=0$ $1-2k=0$ or $2\cos\alpha-(1-2k)=0$ $(\lvert \underset{\sim}{a}\rvert^2\neq 0)$ $k=\frac{1}{2}$ or $k=\frac{1}{2}-\cos\alpha$ As $0\le k\le 1$, $0\le\frac{1}{2}-\cos\alpha\le 1$. $-\frac{1}{2}\le\cos\alpha\le\frac{1}{2}$ So $\frac{\pi}{3}\le\alpha\le\frac{2\pi}{3}$, $\alpha\neq\frac{\pi}{2}$. *Note: If $\alpha=\frac{\pi}{2}$, there is only one possible position for M.*	MEX-V1 Further Work with Vectors MEX12–3, 12–7 Bands E2–E4 • Gives the correct solution . . . . .2 • Determines $k=\frac{1}{2}$ OR $k=\frac{1}{2}-\cos\alpha$ 1
(b) (i) Let $t=\tan\frac{x}{2}$ and so $\frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}$. Using the identity $\sec^2\frac{x}{2}=1+\tan^2\frac{x}{2}$ and knowing $t=\tan\frac{x}{2}$ gives: $\frac{dt}{dx}=\frac{1}{2}(1+t^2)$ $dx=\frac{2dt}{1+t^2}$ From t-formulae, $\cos x=\frac{1-t^2}{1+t^2}$. When $x=0$, $t=0$ and $x=\frac{\pi}{2}$, $t=1$. Substituting into $I=\int_0^{\frac{\pi}{2}}\frac{2}{3+5\cos x}dx$ gives: $I=\int_0^1\frac{2}{3+5\left(\frac{1-t^2}{1+t^2}\right)}\times\frac{2dt}{1+t^2}$ $=\int_0^1\frac{4}{8-2t^2}dt$ $=\int_0^1\frac{2}{4-t^2}dt$	MEX-C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution . . . . .2 • Makes suggested substitution and rewrites integrand and limits OR equivalent merit. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $\dfrac{2}{4-t^2}=\dfrac{2}{(2+t)(2-t)}=\dfrac{A}{2+t}+\dfrac{B}{2-t}$	MEX-C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution 2
$2=A(2-t)+B(2+t)$ Using the cover up method: Substituting $t=2$ and solving for B and substituting $t=-2$ and solving for A. $2=4B\Rightarrow B=\dfrac{1}{2}$; $2=4A\Rightarrow A=\dfrac{1}{2}$ Using the equating coefficients method: Forming $2=2A+2B$ and $0=-A+B$ and solving for A and B. $A=\dfrac{1}{2}$ and $B=\dfrac{1}{2}$ Then: $I=\dfrac{1}{2}\int_0^1\dfrac{1}{2+t}+\dfrac{1}{2-t}dt$ $=\dfrac{1}{2}\Big[\ln\lvert 2+t\rvert-\ln\lvert 2-t\rvert\Big]_0^1$ $=\dfrac{1}{2}(\ln 3-0-(\ln 2-\ln 2))$ $=\ln\sqrt{3}$	• Obtains $A=B=\dfrac{1}{2}$ 1

		Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c)	(i)	We are required to find T such that $\frac{g}{4} + gT\sin\theta - \frac{g}{2}T^2 = 0.$ $T = \dfrac{-g\sin\theta \pm \sqrt{g^2\sin^2\theta - 4\left(-\frac{g}{2}\right)\left(\frac{g}{4}\right)}}{-g}$ $= \sin\theta \pm \frac{1}{\sqrt{2}}\sqrt{2\sin^2\theta + 1}$ $\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow 2\sin^2\theta = 1 - \cos 2\theta$ $\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}\sqrt{1 - \cos 2\theta}\ (\sin\theta > 0)$ Substituting for $2\sin^2\theta$ and $\sin\theta$ into $T = \sin\theta \pm \frac{1}{\sqrt{2}}\sqrt{2\sin^2\theta + 1}$ gives: $T = \frac{1}{\sqrt{2}}\sqrt{1 - \cos 2\theta} \pm \frac{1}{\sqrt{2}}\sqrt{2 - \cos 2\theta}$ $\sqrt{2 - \cos 2\theta} > \sqrt{1 - \cos 2\theta}.$ We require $T > 0$ and so $T = \frac{1}{\sqrt{2}}\sqrt{1 - \cos 2\theta} + \frac{1}{\sqrt{2}}\sqrt{2 - \cos 2\theta}.$ Hence $T = \frac{1}{\sqrt{2}}(\sqrt{1 - \cos 2\theta} + \sqrt{2 - \cos 2\theta}).$	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution . . . . .3 • Substitutes for $2\sin^2\theta$ and $\sin\theta$ into $T = \sin\theta \pm \frac{1}{\sqrt{2}}\sqrt{2\sin^2\theta + 1}$ OR equivalent merit . . . . .2 • Attempts to solve $\frac{g}{4} + gT\sin\theta - \frac{g}{2}T^2 = 0$ for T 1
	(ii)	$R = (g\cos\theta)T$ where $T = \frac{1}{\sqrt{2}}(\sqrt{1 - \cos 2\theta} + \sqrt{2 - \cos 2\theta}).$ $\cos 2\theta = 2\cos^2\theta - 1$ $\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}\sqrt{1 + \cos 2\theta}\ (\cos\theta > 0)$ Substituting for T and $\cos\theta$ into $R = (g\cos\theta)T$ gives: $R = \frac{g}{\sqrt{2}}\sqrt{1 + \cos 2\theta}\left(\frac{1}{\sqrt{2}}(\sqrt{1 - \cos 2\theta} + \sqrt{2 - \cos 2\theta})\right)$ So $R = \frac{g}{2}(\sqrt{1 - \cos^2 2\theta} + \sqrt{2 + \cos 2\theta - \cos^2 2\theta}).$	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) When $\theta = 45°$, $R = \frac{g}{2}(1 + \sqrt{2})$. When $\cos 2\theta = \frac{1}{5}$, $R = \frac{g}{2}\left(\sqrt{\frac{24}{25}} + \sqrt{\frac{54}{25}}\right)$. Let d represent the extra distance attained. $\sqrt{24} = 2\sqrt{6}$ and $\sqrt{54} = 3\sqrt{6}$. $d = \frac{g}{2}\left(\frac{2\sqrt{6}}{5} + \frac{3\sqrt{6}}{5}\right) - \frac{g}{2}(1 + \sqrt{2})$ $= \frac{g}{2}(\sqrt{6} - \sqrt{2} - 1)$ So the extra distance attained is $\frac{g}{2}(\sqrt{6} - \sqrt{2} - 1)$ metres.	MEX-M1 Applications of Calculus to Mechanics MEX12–6, 12–7 Bands E2–E4 • Gives the correct solution 1
Question 16	
(a) (i) Integration by parts takes the form $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Use $u = \tan^{-1}x$ and $\frac{dv}{dx} = x^n$. So $\frac{du}{dx} = \frac{1}{1 + x^2}$ and $v = \frac{x^{n+1}}{n+1}$. Substituting into $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ gives: $I_n = \int_0^1 x^n \tan^{-1}x\,dx$ $= \left[(\tan^{-1}x)\left(\frac{x^{n+1}}{n+1}\right)\right]_0^1 - \int_0^1\left(\frac{x^{n+1}}{n+1}\right)\left(\frac{1}{1+x^2}\right)dx$ $= \left(\frac{\pi}{4}\right)\left(\frac{1}{n+1}\right) - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}dx$ Multiply both sides by $(n + 1)$. So $(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx$.	MEX-C1 Further Integration MEX12–5, 12–8 Bands E2–E4 • Gives the correct solution . . . . 2 • Uses integration by parts with correct substitutions 1

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Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Using part (a) (i): Setting $n = 0$ gives: $I_0 = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx$ $= \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$ So $I_0 = \frac{\pi}{4} - \frac{1}{2}\ln 2$. **OR** Integration by parts takes the form $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$. Put $u = \tan^{-1}x$ and $\frac{dv}{dx} = 1$. So $\frac{du}{dx} = \frac{1}{1+x^2}$ and $v = x$. Substituting into $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ gives: $I_0 = \int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}dx$ $I_0 = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx$ $= \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$ $= \frac{\pi}{4} - \frac{1}{2}\ln 2$	MEX-C1 Further Integration MEX12–5 Bands E2–E3 • Gives the correct solution 1
(iii) $(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx$ Replacing n with $n + 2$ gives: $(n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}dx$ Attempting to form $(n+3)I_{n+2} + (n+1)I_n$: $= \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}dx + \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}dx$ $= \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}(1+x^2)}{1+x^2}dx$ $= \frac{\pi}{2} - \int_0^1 x^{n+1}dx$ $= \frac{\pi}{2} - \frac{1}{n+2}$	MEX-C1 Further Integration MEX12–5, 12–8 Bands E2–E4 • Gives the correct solution . . . . .2 • Replaces n with $n + 2$ AND attempts to form a valid expression for $(n+3)I_{n+2} + (n+1)I_n$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iv) $(n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \frac{1}{n+2}$ Substituting $n = 0$ gives: $3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2} \quad (1)$ Substituting $n = 2$ gives: $5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4} \quad (2)$ $(2) - (1)$ gives $5I_4 - I_0 = \frac{1}{4}$. $I_4 = \frac{1}{5}\left(I_0 + \frac{1}{4}\right)$ Substituting $I_0 = \frac{\pi}{4} - \frac{1}{2}\ln 2$ into $I_4 = \frac{1}{5}\left(I_0 + \frac{1}{4}\right)$ gives: $I_4 = \frac{1}{5}\left(\frac{\pi}{4} - \frac{1}{2}\ln 2 + \frac{1}{4}\right)$ $= \frac{1}{20}(1 + \pi - 2\ln 2)$	MEX-C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution . . . . 2 • Obtains the two equations AND attempts to solve for I_4 1
(b) (i) Since squares cannot be negative, $(\sqrt{a_1} - \sqrt{a_2})^2 \ge 0$. Expanding the LHS gives: $a_1 - 2\sqrt{a_1 a_2} + a_2 \ge 0$ $a_1 + a_2 \ge 2\sqrt{a_1 a_2}$ So $\frac{a_1 + a_2}{2} \ge \sqrt{a_1 a_2}$. **OR** $\frac{a_1 + a_2}{2} - \sqrt{a_1 a_2} = \frac{1}{2}(a_1 + a_2 - 2\sqrt{a_1 a_2})$ $= \frac{1}{2}((\sqrt{a_1})^2 + (\sqrt{a_2})^2 - 2\sqrt{a_1 a_2})$ $= \frac{1}{2}(\sqrt{a_1} - \sqrt{a_2})^2$ ≥ 0 So $\frac{a_1 + a_2}{2} \ge \sqrt{a_1 a_2}$.	MEX-P1 The Nature of Proof MEX12–2 Bands E2–E3 • Gives the correct proof 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Let $x = (a_1 a_2 \ldots a_n)^{\frac{1}{n}} > 0$. Taking the *n*th power of both sides of the equality gives: $x^n = a_1 a_2 \ldots a_n \Rightarrow \frac{a_1 a_2 \ldots a_n}{x^n} = 1$ $\left(\frac{a_1}{x}\right)\left(\frac{a_2}{x}\right)\ldots\left(\frac{a_n}{x}\right) = 1$ Note that $\frac{a_1}{x} > 0, \frac{a_2}{x} > 0, \ldots, \frac{a_n}{x} > 0$. If $a_1 a_2 \ldots a_n = 1$, then $a_1 + a_2 + \ldots + a_n \geq n$. $\frac{a_1}{x} + \frac{a_2}{x} + \ldots + \frac{a_n}{x} \geq n$ $\frac{a_1 + a_2 + \ldots + a_n}{n} \geq x$ $\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$	MEX-P1 The Nature of Proof MEX12–2, 12–7 Bands E3–E4 • Gives the correct proof . . . . .3 • Obtains $\left(\frac{a_1}{x}\right)\left(\frac{a_2}{x}\right)\ldots\left(\frac{a_n}{x}\right) = 1$ AND attempts to use the given result . . . . .2 • Obtains $\frac{a_1 a_2 \ldots a_n}{x^n} = 1$ OR equivalent merit. 1
(iii) $2^n - 1 = 2^{n-1} + 2^{n-2} + \ldots + 2 + 1$ $2^{n-1}, 2^{n-2}, \ldots, 2, 1$ are unequal positive numbers. Hence using the strong inequality $\frac{a_1 + a_2 + \ldots + a_n}{n} > (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$ gives: $\frac{2^n - 1}{n} > (2^{n-1} \times 2^{n-2} \times \ldots 2 \times 1)^{\frac{1}{n}}$ $= (2^{(n-1)+(n-2)+\ldots+2+1})^{\frac{1}{n}}$ $= \left(2^{\frac{n(n-1)}{2}}\right)^{\frac{1}{n}}$ $= 2^{\frac{n-1}{2}}$ $\frac{2^n - 1}{n} > 2^{\frac{n-1}{2}} \Rightarrow 2^n - 1 > n\left(2^{\frac{n-1}{2}}\right)$ So $2^n - 1 > n\sqrt{2^{n-1}}$ for integers $n \geq 1$.	MEX-P1 The Nature of Proof MEX12–2, 12–7 Bands E3–E4 • Gives the correct proof . . . . .4 • Obtains $(2^{n-1} \times 2^{n-2} \times \ldots 2 \times 1)^{\frac{1}{n}} = 2^{\frac{n-1}{2}}$ . . . . .3 • Establishes $\frac{2^n - 1}{n} > (2^{n-1} \times 2^{n-2} \times \ldots 2 \times 1)^{\frac{1}{n}}$ . . . . .2 • Obtains $2^n - 1 = 2^{n-1} + 2^{n-2} + \ldots + 2 + 1$ OR equivalent merit. 1