CSSA

2021

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

DO NOT REMOVE PAPER FROM EXAM ROOM

Centre Number

Student Number

Mathematics Extension 2

Morning Session
Thursday, 29 July 2021

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 100

Section I Pages 2 - 5

10 marks

- Attempt Questions 1 - 10
- Allow 15 minutes for this section

Section II Pages 6 - 14

90 marks

- Attempt Questions 11 - 16
- Allow about 2 hours and 45 minutes for this section

6300-1

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1–10.

---

1 Consider the statement "If an animal is a bird then it can fly or swim."
What is the contrapositive statement?

(A) If an animal cannot fly or cannot swim then it is a bird
(B) If an animal cannot fly or cannot swim then it is not a bird
(C) If an animal cannot fly and cannot swim then it is a bird
(D) If an animal cannot fly and cannot swim then it is not a bird

2 What value of z satisfies $z^2 = 7 + 24i$?

(A) $4 + 3i$
(B) $-4 + 3i$
(C) $-3 + 4i$
(D) $3 + 4i$

3 What is the angle between vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, to the nearest degree?

(A) 77°
(B) 83°
(C) 84°
(D) 96°

4 $A(1,2,2)$, $B(3,-12,4)$, $C(1,2,0)$ and $D(3,-12,0)$ are four positional vectors. What is the vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{CD}$?

(A) $2\underset{\sim}{i}-14\underset{\sim}{j}+2\underset{\sim}{k}$

(B) $2\underset{\sim}{i}-14\underset{\sim}{j}+4\underset{\sim}{k}$

(C) $2\underset{\sim}{i}-14\underset{\sim}{j}$

(D) $-2\underset{\sim}{i}+14\underset{\sim}{j}$

5 For all non-zero integers x and y, if $x>y$ then $\frac{1}{x}<\frac{1}{y}$. What is a counter example to the statement above?

(A) $x=2, y=-1$

(B) $x=0, y=0$

(C) $x=4, y=3$

(D) $x=-2, y=1$

6 A particle is moving in simple harmonic motion with displacement x metres. Its acceleration, $\ddot{x}$, is given by $\ddot{x}=-4x+3$. What are the centre and period of motion?

(A) centre of motion $=3$, period $=\frac{\pi}{2}$

(B) centre of motion $=-3$, period $=\pi$

(C) centre of motion $=\frac{3}{4}$, period $=\pi$

(D) centre of motion $=\frac{3}{4}$, period $=\frac{\pi}{2}$

7 It is given that $z = 2 + i$ is a root of $z^3 + az^2 - bz + 5 = 0$, where a and b are real numbers.

What is the value of a?

(A) -5

(B) -3

(C) 3

(D) 5

8 Which integral has the smallest value?

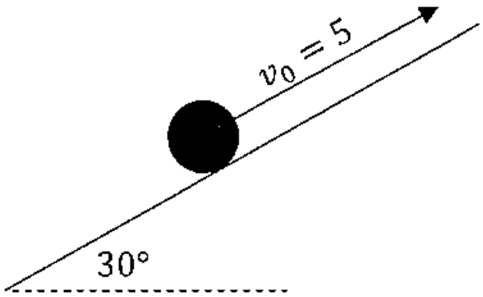
(A) $\int_0^{\frac{\pi}{4}} \sin^2 x dx$

(B) $\int_0^{\frac{\pi}{4}} \cos^2 x dx$

(C) $\int_0^{\frac{\pi}{4}} \sin x \cos x dx$

(D) $\int_0^{\frac{\pi}{4}} \sin x \tan x dx$

9 A ball is rolled up a frictionless 30° ramp, with an initial velocity of 5 m/s. Assuming $g = 10$ m/s^2, what is the net acceleration on the ball?



(A) $5\sqrt{3}$ m/s^2 directed down the ramp

(B) $5\sqrt{3}$ m/s^2 directed up the ramp

(C) 5 m/s^2 directed down the ramp

(D) 5 m/s^2 directed up the ramp

10 What value of a will minimise the integral $\int_0^1 (x^2 - a)^2 dx$?

(A) $a = \frac{1}{2}$

(B) $a = \frac{1}{\sqrt{2}}$

(C) $a = \frac{4}{45}$

(D) $a = \frac{1}{3}$

End of Section I

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Given $w = 2 + 5i$ and $z = 4 - 3i$, evaluate

(i) $|w + \bar{z}|$. **2**

(ii) $(w + \bar{z})(\bar{w} + z)$. **2**

(b) Find the square roots of $15 - 8i$. **3**

(c) Use the substitution $t = \tan\frac{\theta}{2}$, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \sin\theta + \cos\theta}$. **4**

(d) The acceleration, a, of a particle moving in a straight line is given by

$a = 6\left(1 - \frac{1}{2}x^2\right)$, where x is its displacement in metres. The particle is initially at the origin and travelling with velocity of 2 m/s.

(i) Show that the velocity of the particle is described by $v^2 = 4 + 12x - 2x^3$. **2**

(ii) Show that the particle returns to the origin. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Use integration by parts to find $\int x3^x dx$. **3**

(b) By writing $\frac{8-2x}{(1+x)(4+x^2)}$ in the form $\frac{a}{1+x}+\frac{bx+c}{4+x^2}$, **4**

evaluate $\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}dx$.

(c) On the same Argand diagram, draw a neat sketch of $|z-4-4i|=2$ and $\arg(z)=\frac{\pi}{4}$. **4**

Hence write down all the values of z which satisfy simultaneously

$|z-4-4i|=2$ and $\arg(z)=\frac{\pi}{4}$.

(d) Find the scalar projection of the vector $\underset{\sim}{u}=\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}$ onto the vector $4\underset{\sim}{i}-4\underset{\sim}{j}+7\underset{\sim}{k}$. **2**

(e) Given $\underset{\sim}{a}=\begin{pmatrix}-1\\3\\4\end{pmatrix}$ and $\underset{\sim}{b}=\begin{pmatrix}-2\\1\\-4\end{pmatrix}$, and $\underset{\sim}{a}-\underset{\sim}{b}+2\underset{\sim}{c}=0$, find $\underset{\sim}{c}$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Suppose a and b are positive integers, where a is even and b is odd. Show that the sum of a and b is odd. **1**

(ii) Let P be the proposition **1**

"For all positive integers m and n, if $m+n$ is even then m and n are both even or m and n are both odd."

Using (i) above, prove that the proposition P is true by proving that the contrapositive statement is true.

(b) Let $I_1 = \int_0^{\pi} \frac{x\sin x dx}{1+\cos^2 x}$ and $I_2 = \int_0^{\pi} \frac{(\pi - x)\sin x dx}{1+\cos^2 x}$

(i) Using the substitution, $u = \pi - x$, show that $I_1 = I_2$. **2**

(ii) Hence, or otherwise, evaluate I_1. **2**

(c) (i) If a and b are positive integers with $a > b$, prove that $6(a+b)^2 - 2(a-b)^2$ is divisible by 4. **2**

(ii) It is given that x and y are positive integers with $x > y$, $M = 6x^2 - 2y^2$ and $D = x - y$. **2**

The result in (i) proves only one of the following statements to be true.

P: M is a multiple of 4 if D is an odd integer only, or

Q: M is a multiple of 4 if D is an even integer only, or

R: M is a multiple of 4 if D is an odd integer or an even integer.

Which of these statements is true? Justify your answer.

Question 13 continues on page 9

(d) A particle is travelling in a straight line. Its displacement, x cm, from O at a given time, t seconds after the start of motion, is given by $x = 3 + \sin^2 t$.

(i) Prove that the particle is undergoing simple harmonic motion. **2**

(ii) Find the period of the motion. **1**

(iii) Find the total distance travelled by the particle in the first π seconds. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Prove that $\log_3 7$ is irrational. **2**

(b) The scalar product of $\underset{\sim}{i} - 2\lambda \underset{\sim}{j} - \underset{\sim}{k}$, and the sum of $\underset{\sim}{i} - \lambda \underset{\sim}{k}$ and $\lambda \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, is 6. **2**

Find λ.

(c) Prove by mathematical induction that $(2n)! < (n!)^2\, 4^{n-1}$ for $n \geq 5$. **3**

(d) Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = 3\underset{\sim}{a} + 2\underset{\sim}{b}$.

(i) Prove that if $\overrightarrow{OD} = \dfrac{1}{5}\overrightarrow{OC}$ then D lies on AB. **3**

(ii) Is the point D closer to point A or point B? Justify your answer. **1**

(e) (i) Given $z = \cos\theta + i\sin\theta$ prove that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$. **1**

(ii) Hence by considering the expansion $\left(z - \dfrac{1}{z}\right)^5$, show that **3**

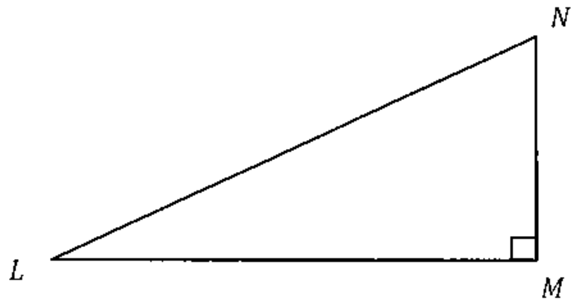
$$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta.$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove that for non-zero vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ that $(\underset{\sim}{a}+\underset{\sim}{b}).(\underset{\sim}{a}+\underset{\sim}{b}) = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$ if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular. **2**

(ii)



In ΔLMN, let $\overrightarrow{LM} = \underset{\sim}{a}$ and $\overrightarrow{MN} = \underset{\sim}{b}$.

By finding an expression for the side LN in terms of the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$,

or otherwise, prove that $\left|\overrightarrow{LN}\right|^2 = \left|\overrightarrow{LM}\right|^2 + \left|\overrightarrow{MN}\right|^2$. **2**

(b) Find $\int x^2\sqrt{1-x^2}\,dx$. **3**

(c) It is given that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ for $k \in \mathbb{N}$.

(i) Prove $\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$. **1**

(ii) If $x_1, x_2, \ldots, x_n$ are positive integers, not necessarily consecutive, such that **2**

$1 < x_1 < x_2 < \ldots < x_n$,

prove that $\frac{1}{{x_1}^2} + \frac{1}{{x_2}^2} + \frac{1}{{x_3}^2} + \ldots + \frac{1}{{x_{n-1}}^2} < 1$.

Question 15 continues on page 12

(d) A particle of unit mass is moving vertically downward in a medium which exerts a resistance force proportional to the square of the speed, v, of the particle. It is released from rest at O and its terminal velocity is U.

(i) Show that the distance it has fallen below O is given by **2**

$$x = \frac{1}{2k}\ln\left|\frac{g}{g - kv^2}\right|.$$

(ii) Prove that the time taken, T, for the particle to fall from O to when its velocity is half of its terminal velocity, U, is given by **3**

$$T = \frac{U}{2g}\ln 3.$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) State all the roots of $z^7 - 1 = 0$ in exponential form. **2**

(ii) Using $z^7 - 1 = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$, or otherwise, **3**

prove that $\frac{2\pi}{7}$, $\frac{4\pi}{7}$ and $\frac{6\pi}{7}$ are solutions to

$$2\cos 3\theta + 2\cos 2\theta + 2\cos\theta + 1 = 0.$$

(iii) Hence or otherwise, prove that **2**

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}.$$

(b) (i) Given that $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2}\, dx$ where n is a positive integer, **3**

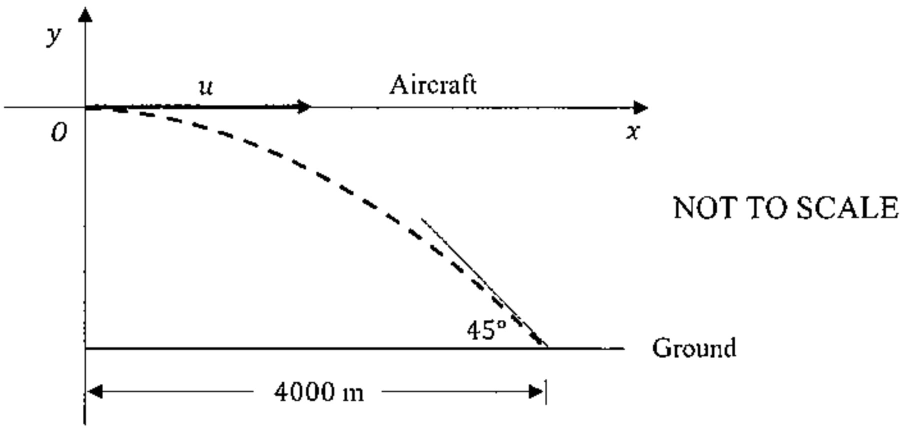
prove that $I_{2n+1} = \frac{e}{2} - nI_{2n-1}$.

(ii) Hence, or otherwise, prove that $2\int_0^1 x^{2n-1}(1 + x^2)e^{x^2}\, dx \leq e$ for $n \geq 1$. **2**

Question 16 continues on page 14

Question 16 (continued)

(c) An aircraft flying horizontally at u m/s delivers an emergency medical supply package that hits the ground 4000 m away, measured horizontally. The package experiences an air resistance of $0.1v$ where v is the velocity at time t and g is the acceleration due to gravity. The package hits the ground an angle of $45°$ to the horizontal.



Assume that t seconds after release, the position vector is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} 10u(1-e^{-0.1t}) \\ 100g(1-e^{-0.1t}) - 10gt \end{pmatrix}. \qquad \text{(Do NOT prove this.)}$$

(i) Show that the velocity vector $\underset{\sim}{v}(t)$ of the particle is given by **1**

$$\underset{\sim}{v}(t) = \begin{pmatrix} ue^{-0.1t} \\ -10g(1-e^{-0.1t}) \end{pmatrix}.$$

(ii) Find the time when the package hits the ground and the speed on impact, where $g = 10$ m/s^2. **2**

End of examination

BLANK PAGE

EXAMINERS

Piv Huot (Convenor)	Mount St Joseph Catholic College, Milperra
Eric Choy	Educational Consultant
Khiet Hoang	John Edmondson High School, Horningsea Park
Steve Howard	Cowra High School, Cowra
Praneetha Singh	The Hills Grammar School, Kenthurst
Vito Zurlo	St Scholastica's College, Glebe

2021
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

CSSA

MARKING GUIDELINES

Mathematics Extension 2

Section I
10 marks
Multiple Choice Answer Key

Question	Answer
1	D
2	A
3	C
4	C
5	A
6	C
7	B
8	A
9	C
10	D

6300-2

Question 1 (1 mark)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E2-E3

Solution	Mark
The parts of the statement $P \Rightarrow Q$ are: P: is a bird and Q: fly or swim The negations are: $\neg P$: is not a bird and $\neg Q$: cannot fly and cannot swim The contrapositive is $\neg Q \Rightarrow \neg P$, so: "If an animal cannot fly and cannot swim then it is not a bird" Hence (D)	1

Question 2 (1 mark)
Outcomes Assessed: N1.1/MEX12-4
Targeted Performance Bands: E2-E3

Solution	Mark
$(4 + 3i)^2 = 16 + 24i + 9i^2$ $= 7 + 24i$ Hence (A)	1

Question 3 (1 mark)
Outcomes Assessed: V1.2/MEX12-3
Targeted Performance Bands: E2-E3

Solution	Mark
$\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$ $\underset{\sim}{u} \cdot \underset{\sim}{v} = 2 - 3 + 2 = 1$ $\lvert\underset{\sim}{u}\rvert = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$ and $\lvert\underset{\sim}{v}\rvert = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$ Angle θ between $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is given by: $\theta = \cos^{-1}\left(\frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{\lvert\underset{\sim}{u}\rvert\,\lvert\underset{\sim}{v}\rvert}\right)$ $= \cos^{-1}\left(\frac{1}{\sqrt{14} \times \sqrt{6}}\right) = 83.73604728 = 84^0$ Hence (C)	1

Question 4 (1 mark)
Outcomes Assessed: V1.1/MEX12-3
Targeted Performance Bands: E2-E3

Solution	Mark
$A(1,2,2)$, $B(3,-12,4)$, $C(1,2,0)$ and $D(3,-12,0)$ $\overrightarrow{AB} = 2\underset{\sim}{i} - 14\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\overrightarrow{CD} = 2\underset{\sim}{i} - 14\underset{\sim}{j}$ When $\overrightarrow{AB}$ is projected onto $\overrightarrow{CD} = \frac{\overrightarrow{AB}\times\overrightarrow{CD}}{\lvert\overrightarrow{CD}\rvert^2} \times \overrightarrow{CD}$ Therefore the vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{CD}$ is $2\underset{\sim}{i} - 14\underset{\sim}{j}$ Hence (C)	1

Question 5 (1 mark)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E2-E3

Solution	Mark
A counter example is one which shows that the statement is false (or not true). For option A , $x = 2,\ y = -1 \Rightarrow\ 2 > -1$ then $\frac{1}{2} < \frac{1}{-1}$ but this is false. Hence (A)	1

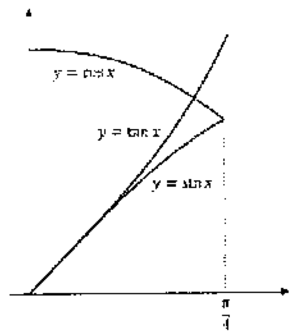
Question 6 (1 mark)
Outcomes Assessed: M1.1/MEX12-6
Targeted Performance Bands: E3-E4

Solution	Mark
$\ddot{x} = -4x + 3$ $= -2^2(x - \frac{3}{4})$ $\therefore\ n = 2$, centre of motion $= \frac{3}{4}$ period $= \frac{2\pi}{2}$ $= \pi$ Hence (C)	1

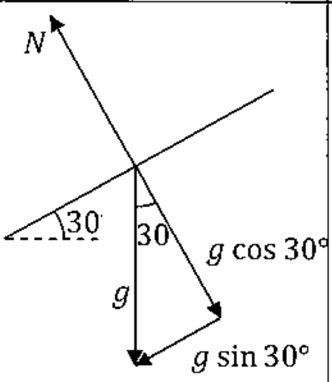
Question 7 (1 mark)
Outcomes Assessed: N2.1/MEX12-4
Targeted Performance Bands: E3-E4

Solution	Mark
Since $z_1 = 2+i$ is a root then $z_2 = 2-i$ is also a root since all the coefficients are real. Let the roots be z_1, z_2, α Using product of roots, $z_1 z_2 \alpha = -5$ $(2+i)(2-i)\,\alpha = -5$ $5\alpha = -5$ $\alpha = -1$ Using the sum of roots, $z_1 + z_2 + \alpha = -a$ $2+i+2-i+-1 = -a$ $a = -3$ Hence (B)	1

Question 8 (1 mark)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Solution	Mark
For $\left[0, \frac{\pi}{4}\right]$ $\sin x < \cos x$ and $\sin x < \tan x$ $\therefore\ \sin^2 x < \sin x \cos x < \cos^2 x$ and $\sin^2 x < \sin x \tan x$ $\therefore$ the smallest integral is $$\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$$  Hence (A)	1

Question 9(1 mark)
Outcomes Assessed: M1.2/MEX12-6
Targeted Performance Bands: E3-E4

Solution	Mark
The acceleration due to gravity can be split into two components: • the component perpendicular to the ramp, $g\cos 30°$, is balanced by the normal reactive force • the component parallel to the ramp, $g\sin 30°$, points down the ramp • The acceleration is given by $a = 10\sin 30° = 5$ m/s^2 down the ramp  Hence (C)	1

Question 10 (1 mark)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Solution	Mark
$\int_0^1 (x^2 - a)^2\,dx$ $= \int_0^1 (x^4 - 2ax^2 + a^2)\,dx$ $= \left[\frac{x^5}{5} - \frac{2a}{3}x^3 + a^2x\right]_0^1$ $= \left(\frac{1}{5} - \frac{2a}{3} + a^2\right) - (0)$ $= a^2 - \frac{2a}{3} + \frac{1}{5}$ Minimum value at the axis of symmetry when $a = -\frac{-\frac{2}{3}}{2(1)} = \frac{1}{3}$ Hence (D)	1

Section II
90 marks

Question 11 (15 marks)

11 (a) (i) (2 marks)
Outcomes Assessed: N1.1/MEX12-4
Targeted Performance: E2-E3

Criteria	Marks
• Provides correct solution	2
• Obtains the correct expression of the sum, or equivalent merit	1

Sample Answer:

$$w + \bar{z} = (2 + 5i) + \left(\overline{(4 - 3i)}\right)$$
$$= 2 + 5i + 4 + 3i$$
$$= 6 + 8i$$

$$|w + \bar{z}| = \sqrt{6^2 + 8^2} = 10$$

11 (a) (ii) (2 marks)
Outcomes Assessed: N1.1/MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Arrives at $(w+\bar{z})\left(\overline{(w+\bar{z})}\right)$ OR • Attempts to apply result from part(i), or equivalent merit	1

Sample Answer:

$$(w+\bar{z})(\bar{w}+z) = (w+\bar{z})\left(\overline{(w+\bar{z})}\right)$$
$$= |(w+\bar{z})|^2 = 100$$

11 (b) (3 marks)
Outcomes Assessed: N1.1/MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	3
• Obtains a value for either a or b	2
• Obtains the 2 simultaneous equations OR • Attempts to find a value for either a or b, or equivalent merit	1

Sample Answer:
Let a square root of $15 - 8i$ be a + bi

$15 - 8i = (1 + bi)^2$
$a^2 - b^2 = 15$
$ab = -4$
$a = \pm 4, b = \mp 1$ (by inspection)
The square roots of $15 - 8i$ are $\pm (4 - i)$

Alternatively:

$$15 - 18i = (a + bi)^2 \quad a, b \in R$$
$$a^2 - b^2 = 15$$
$$ab = -4$$
$$b = \frac{-4}{a}$$
$$a^2 - \frac{16}{a^2} = 15$$
$$a^4 - 15a^2 - 16 = 0$$
$$(a^2 + 1)(a^2 - 16) = 0$$
$$a^2 = -1, NS \qquad a = \pm 4, b = \mp 1$$
$$\therefore \sqrt{15 - 18i} = \pm(4 - i)$$

11 (c) (4 marks)
Outcomes assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	4
• Obtains correct integrand, or equivalent merit	3
• Correctly substitutes and attempts to simplify (ignoring limits), or equivalent merit	2
• Uses given substitution, or equivalent merit	1

Sample Answer:

$t = \tan\frac{\theta}{2} \Rightarrow \theta = 2\tan^{-1}t$

$d\theta = \frac{2}{1+t^2}dt$

$\theta = 0, t = 0$ and $\theta = \frac{\pi}{2}, t = 1$

$$\int_0^1 \frac{1}{(1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2})} \times \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2dt}{1+t^2+2t+1-t^2}$$

$$= \int_0^1 \frac{dt}{1+t}$$

$$= [\ln|1+t|]_0^1$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

11 (d)(i) (2 marks)
Outcomes Assessed: M1.2/MEX12-6
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Obtains integral for v^2 in terms of x, or equivalent merit	1

Sample Answer:

$a = 6\left(1 + \frac{1}{2}x^2\right)$

i.e. $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6 - 3x^2$

$\frac{1}{2}v^2 = \int (6 - 3x^2)dx$

$= 6x - x^3 + C$

At $x = 0, v = 2$

$\therefore \frac{1}{2}(4) = 6(0) - 0^3 + C$

$\therefore C = 2$

$\frac{1}{2}v^2 = 6x - x^3 + 2$

$v^2 = 12x - 2x^3 + 4$

11 (d)(ii) (2 marks)
Outcomes Assessed: M1.2/MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Arrives at $v = \pm 2$ m/s	1

Sample Answer:
$v^2 = 4 + 12x - 2x^3 = 0$ at approximately $x = -2.3, -0.3$ and 2.6 (using trial and error), so the particle oscillates between -0.3 and 2.6 given initial conditions, with the particle repeatedly passing through the origin.

Alternatively:
The particle is at rest when $x = 0$

$\therefore v^2 = 4 + 12(0) - 0$

$= 4$

$v = \pm 2$ m/s

Since the particle was initially at the origin with a velocity of 2 m/s, a velocity of -2 m/s indicates it passes through the origin in the reverse direction to that of its original motion

Question 12

12 (a) (2 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	3
• Correctly applies integration by parts, or equivalent merit	2
• Attempts to use integration by parts, or equivalent merit	1

Sample Answer:

$$\int x3^x\,dx$$

$$u = x \qquad \frac{dv}{dx} = 3^x$$
$$\frac{du}{dx} = 1 \qquad v = \frac{3^x}{\ln 3}$$

$$= \frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\int 3^x\,dx$$

$$= \frac{x3^x}{\ln 3} - \frac{1}{\ln 3}\left(\frac{3^x}{\ln 3}\right) + c$$

$$= \frac{x3^x}{\ln 3} - \frac{3^x}{\ln^2 3} + c$$

12 (b) (4 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E2-E4

Criteria	Marks
• Provides correct solution	4
• Integrates correctly in terms of ln, or equivalent merit	3
• Obtains 3 values correctly	2
• Obtains at least 2 values correctly, or equivalent merit	1

Sample Answer:

$$\frac{8-2x}{(1+x)(4+x^2)}=\frac{a}{1+x}+\frac{bx+c}{4+x^2}$$

$8-2x=a(4+x^2)+(bx+c)(1+x)$

let $x=-1$

$10=5a$

$a=2$

let $x=0$

$c=0$

Equating the coefficient of x^2 :

$0=2+b$

$b=-2$

$$\int_0^4 \left(\frac{2}{1+x}+\frac{-2x}{4+x^2}\right)dx$$

$=\left[2\ln|1+x|-\ln|4+x^2|\right]_0^4$

$=2\ln 5-\ln 20-2\ln 1+\ln 4$

$=2(\ln 5-\ln 1)-(\ln 20-\ln 4)$

$=\ln 5$

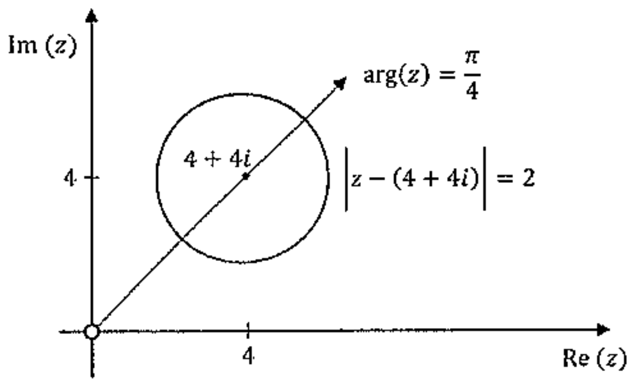
12 (c) (5 marks)
Outcomes Assessed: N2.2/MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	4
• Correct sketch for $\|z-(4+4i)\|=2$ and $\arg z=\frac{\pi}{4}$ and correctly finds one intersection point	3
• Correct sketch for $\|z-(4+4i)\|=2$ and $\arg z=\frac{\pi}{4}$	2
• Correct sketch for $\|z-(4+4i)\|=2$ or $\arg z=\frac{\pi}{4}$	1

Sample Answer:

$|z-(4+4i)|=2 \Rightarrow$ circle centre at $(4,4)$ and radius 2 units $\Rightarrow (x-4)^2+(y-4)^2=4$

$\arg z=\frac{\pi}{4}$ $\Rightarrow$ The angle that the vector from 0 to z makes with the positive direction of the x axis is always $\frac{\pi}{4}$ $\Rightarrow$ The line $y=x$ (NOT including the origin as arg (0) is undefined)



$|z-(4+4i)|=2 \quad \Rightarrow$ circle centre at $(4,4)$ and radius 2 units

$\Rightarrow (x-4)^2+(y-4)^2=4 \qquad (1)$

$\arg(z)=\frac{\pi}{4} \quad \Rightarrow \quad y=x \quad (\text{for } x>0) \quad (2)$

Solving (1) and (2) simultaneously

$$(x-4)^2+(x-4)^2=4$$

$$x=4\pm\sqrt{2}$$

$x=4+\sqrt{2}$, $y=4+\sqrt{2}$ and $x=4-\sqrt{2}$, $y=4-\sqrt{2}$

Hence the values of z which satisfy simultaneously $|z-(4+4i)|=2$ and $\arg(z)=\dfrac{\pi}{4}$ are

$z=(4-\sqrt{2})+(4-\sqrt{2})i$ and $z=(4+\sqrt{2})+(4+\sqrt{2})i$

12(d) (2 marks)

Outcomes Assessed: V1.1/MEX12-3

Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Correctly finds $\underset{\sim}{u}.\underset{\sim}{v}$, or equivalent merit	1

Sample Answer:

Let $\underset{\sim}{u} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$.

Scalar projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v} = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|}$

$$= \frac{\left(\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) \cdot \left(4\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}\right)}{\left|4\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}\right|}$$

$$= \frac{4 + 8 + 7}{\sqrt{4^4 + (-4)^2 + 7^2}}$$

$$= \frac{19}{9}$$

12(e) (2 marks)
Outcomes Assessed: V1.1/MEX12-3
Targeted Performance Bands: E2-E4

Criteria	Marks
• Provides correct solution	2
• Attempts to use $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ in a calculation	1

Sample Answer:

$$\begin{pmatrix} -1+2 \\ 3-1 \\ 4+4 \end{pmatrix} + 2\underset{\sim}{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} + 2\underset{\sim}{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2\underset{\sim}{c} = \begin{pmatrix} -1 \\ -2 \\ -8 \end{pmatrix}$$

$$\underset{\sim}{c} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ -4 \end{pmatrix}$$

Question 13 (15 marks)
13 (a) (i) (1 mark)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	1

Sample Answer:
By the definition of even and odd, $\exists r, s$ such that $a = 2r$ and $b = 2s + 1$ for $r, s \in \mathbb{N}$.

$\therefore a + b = 2r + (2s + 1)$

$\qquad = 2(r + s) + 1$

Let $k = r + s$ then $k \in \mathbb{N}$ because r and s are integers and the sum of integers are integers.

$\therefore a + b = 2k + 1$

$\Rightarrow a + b$ is odd.

13 (a) (ii) (1 mark)

Outcomes Assessed: P1/MEX12-2

Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	1

Sample Answer:

Proof by contrapositive.

Suppose m and n are positive integers such that one of m and n is even and the other is odd.

From part (i), the sum of any even integer and any odd integer is an odd.

$\Rightarrow m+n$ is odd

$\therefore P$ is true.

13 (b) (i) (2 marks)

Outcomes Assessed: C1/MEX12-5

Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Attempts to use given substitution, or equivalent merit	1

Sample Answer:

$$I_1 = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$u = \pi - x \;\Rightarrow\; du = -dx$

$x = 0, u = \pi$ and $x = \pi, u = 0$

$$I_1 = \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}(-du)$$

$$= \int_0^{\pi} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}\,du$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$$

$$= I_2$$

since u is a dummy variable and $\sin(\pi - A) = \sin A$

$\Rightarrow \sin(\pi - u) = \sin x$

13 (b) (ii) (2 marks)

Outcomes Assessed: C1/MEX12-5

Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Arrives at a correct integrated expression for $I_1 + I_2$	1

Sample Answer:

$$I_1 + I_2 = \int_0^{\pi} \frac{x\sin x + (\pi - x)\sin x}{1+\cos^2 x}\,dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= -\pi \int_0^{\pi} \frac{-\sin x}{1+\cos^2 x}\,dx$$

$$= -\pi\left[\tan^{-1}(\cos x)\right]_o^{\pi}$$

$$= -\pi\left(\tan^{-1}(-1) - \tan^{-1} 1\right)$$

$$= -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right)$$

$$= \frac{\pi^2}{2}$$

As $I_1 = I_2$

$$I_1 = \frac{1}{2} \times \frac{\pi^2}{2}$$

$$= \frac{\pi^2}{4}$$

13 (c) (i) (2 marks)

Outcomes Assessed: P1/MEX12-2

Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Correctly simplify the expression	1

Sample Answer:

$6(a+b)^2-2(a-b)^2$

$=6(a^2+2ab+b^2)-2(a^2-2ab+b^2)$

$=6a^2+12ab+6b^2-2a^2+4ab-2b^2$

$=4(a^2+b^2+4ab)$

$=4p$ for integral p since a,b integral

$\therefore$ if a and b are integers, then $6(a+b)^2-2(a-b)^2$ is divisible by 4

13 (c) (ii) (2 marks)

Outcomes Assessed: P1/MEX12-2

Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Attempts to use part (i), or equivalent merit	1

Sample Answer:

Let $x=a+b, y=a-b$

$D=x-y$

$=a+b-a+b$ from (i)

$=2b$

$\therefore$ D is even since b is integral

$\therefore$ part (i) proves that M is a multiple of 4 when D is even, so statement Q is correct.

13 (d)(i) (2 marks)
Outcomes Assessed: M1.1/MEX12-6
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Provides correct derivative in terms of t	1

Sample Answer:

$x = 3 + \sin^2 t$

$v = 2\sin t\cos t$

$$\begin{aligned}\frac{dv}{dt} &= \cos t(2\cos t) + 2\sin t(-\sin t)\\ &= 2\left[\cos^2 t - \sin^2 t\right]\\ &= 2\left[1 - \sin^2 t - \sin^2 t\right]\\ &= 2\left[1 - 2\sin^2 t\right]\\ &= 2\left[1 - 2(x-3)\right]\\ &= 2\left[1 - 2x + 6\right]\\ &= 14 - 4x\\ &= -4\left(x - \frac{7}{2}\right)\end{aligned}$$

Hence the particle is undergoing SHM.

13 (d)(ii) (1 mark)
Outcomes Assessed: M1.1/MEX12-6
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	1

Sample Answer:

$$n = 2, \ \therefore \ \text{Period} = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$$

13 (d)(iii) (2 marks)
Outcomes Assessed: M1.1/MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Correctly finds at least two values of t when $v = 0$, or equivalent merit	1

Sample Answer:

$v = 0$

$v = 2\sin t\cos t$

$2\sin t\cos t = 0$

$\therefore \sin t = 0 \qquad \cos t = 0$

$t = 0, \pi, 2\pi, 3\pi... \qquad t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}.....$

Using $x = 3 + \sin^2 t$

$t = 0 \ \rightarrow \ x = 3$

$t = \frac{\pi}{2} \ \rightarrow x = 4$

$t = \pi \ \rightarrow x = 3$

$\therefore$ total distance = 2 cm

Question 14 (15 marks)

14 (a) (2 marks)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E2-E4

Criteria	Marks
• Provides correct solution	2
• Assumes $\log_3 7$ is rational and attempts to eliminate the logarithm, or equivalent merit	1

Sample Answer:
Assume $\log_3 7$ is a rational number

Assume $\log_3 7 = \frac{a}{b}$ where a and b are integers with $HCF = 1$ and $b \neq 0$ $(*)$

$$\therefore\ 3^{\frac{a}{b}} = 7$$

$$\therefore\ 3^a = 7^b$$

Since 7 is not divisible by 3 then $3^a = 7^b$ only when $a = b = 0$ which contradicts $(*)$.

Hence $\log_3 7$ is an irrational number.

14 (b) (2 marks)
Outcomes Assessed: V1.2/MEX12-3
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Obtains a correct expression for the scalar product OR • Obtains the correct response from the incorrect order of operations	1

Sample Answer:

$$\begin{pmatrix} 1 \\ -2\lambda \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1+\lambda \\ 0+2 \\ -\lambda-1 \end{pmatrix} = 6$$

$$1 + \lambda - 4\lambda + \lambda + 1 = 6$$

$$-2\lambda = 4$$

$$\lambda = -2$$

14 (c) (3 marks)
Outcomes Assessed: P2/MEX12-2
Targeted Performance Bands: E2-E4

Criteria	Marks
• Provides correct solution	3
• Proves base case, states the induction hypothesis and uses it in simplifying $P(k+1)$	2
• Proves base case OR • States the induction hypothesis and uses it in simplifying $P(k+1)$	1

Sample Answer:
Let $P(n)$ represent the proposition

$P(5)$ is true since LHS = $(2\times 5)! = 3\ 628\ 800$, RHS = $(5!)^2 \times 4^{5-1} = 3\ 686\ 000$, $\therefore$ LHS < RHS

If $P(k)$ is true for some $k \geq 5$ then $(2k)! < (k!)^2 4^{k-1}$

RTP : $P(k+1)$ $\left(2(k+1)\right)! < \left((k+1)!\right)^2 4^k$

$$\begin{aligned}\text{LHS} &= \left(2(k+1)\right)! \\ &= (2k+2)(2k+1)(2k)! \\ &< (2k+2)(2k+1)\times(k!)^2 4^{k-1} \quad \text{from } P(k) \\ &= 2(k+1)(2k+1)\times(k!)^2 4^{k-1} \\ &< 2(k+1)2(k+1)\times(k!)^2 4^{k-1} \quad \text{since } 2k+1 < 2k+2 = 2(k+1) \text{ for } k \geq 5 \\ &= 4(k+1)^2(k!)^2 4^{k-1} \\ &= ((k+1)!)^2 4^k\end{aligned}$$

$\therefore\ P(k) \Rightarrow P(k+1)$

Hence $P(n)$ is true for $n \geq 5$ by induction.

14 (d) (i) (3 marks)
Outcomes Assessed: V1.3/MEX12-3
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Obtains an expression for $\overrightarrow{OD}$ and the equation of the line through A and B	2
• Obtains an expression for $\overrightarrow{OD}$ or the equation of the line through A and B	1

Sample Answer:

Let $\underset{\sim}{d} = \overrightarrow{OD}$

$$\underset{\sim}{d} = \frac{1}{5}(3\underset{\sim}{a} + 2\underset{\sim}{b})$$
$$= \frac{3}{5}\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}$$

Let the line AB be $\underset{\sim}{r}_{AB} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$

$$= (1-\lambda)\underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Let $\lambda = \frac{2}{5}$

$$\therefore \underset{\sim}{r}_{AB} = \left(1 - \frac{2}{5}\right)\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}$$
$$= \frac{3}{5}\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}$$
$$= \underset{\sim}{d}$$

$\therefore$ D lies on AB

14 (d) (ii) (1 mark)
Outcomes Assessed: V1.3/MEX12-3
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	1

Sample Answer:

D is $\frac{2}{5}$ of the distance from A to B, so D is closer to A since $\frac{2}{5} < \frac{1}{2}$

14 (e) (i) (1 mark)
Outcomes Assessed: N2.1/MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	1

Sample Answer:

$$z^n = \cos n\theta + i\sin n\theta$$
$$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta$$
$$z^n - z^{-n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$$
$$= 2i\sin(n\theta)$$

Alternatively:

$z^{-n} = \overline{(z^n)}$ since $|z^n| = |z|^n = 1$

$$\therefore\ z^n - z^{-n} = z^n - \overline{(z^n)}$$
$$= 2\,Im(z^n)$$
$$= 2i\,sin(n\theta)$$

14 (e) (ii) (3 marks)
Outcomes Assessed: N2.1/MEX12-4
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Rearranges the equation into conjugate pairs and uses the result from (i), or equivalent merit	2
• Expands $\left(z-\frac{1}{z}\right)^5$ correctly	1

Sample Answer:

$$\left(z-\frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$
$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$
$$(2i\sin\theta)^5 = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$
$$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$
$$\therefore\ \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$$

Question 15 (15 marks)

15 (a) (i) (2 marks)
Outcomes Assessed: V1.2/MEX12-3
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Obtains the correct expansion	1

Sample Answer:

$$(\underset{\sim}{a}+\underset{\sim}{b}).(\underset{\sim}{a}+\underset{\sim}{b}) = \underset{\sim}{a}.\underset{\sim}{a} + 2\underset{\sim}{a}.\underset{\sim}{b} + \underset{\sim}{b}.\underset{\sim}{b}$$
$$= |\underset{\sim}{a}|^2 + 2\underset{\sim}{a}.\underset{\sim}{b} + |\underset{\sim}{b}|^2$$
$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 \text{ since } \underset{\sim}{a}.\underset{\sim}{b} = 0$$

15 (a) (ii) (2 marks)
Outcomes Assessed: V1.2/MEX12-3
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Correctly finds $\overrightarrow{LN}$	1

Sample Answer:

$$\left|\overrightarrow{LN}\right|^2 = \left|\overrightarrow{LM} + \overrightarrow{MN}\right|^2$$
$$= |\underset{\sim}{a}+\underset{\sim}{b}|^2$$
$$= (\underset{\sim}{a}+\underset{\sim}{b}).(\underset{\sim}{a}+\underset{\sim}{b})$$
$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 \text{ from } (i)$$
$$= \left|\overrightarrow{LM}\right|^2 + \left|\overrightarrow{MN}\right|^2$$

15 (b) (3 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Finds $-\frac{1}{4}-\frac{\pi}{2}-\left[\frac{1}{2}\sin 2x\frac{2}{2}\right]_0^{\frac{\pi}{2}}$ OR • Finds $\frac{\pi}{4}-\frac{1}{4}\left[x-\frac{1}{2}\sin 2x\frac{2}{2}\right]_0^{\frac{\pi}{2}}$	2
• Rewrites the integrand as $x\sin 2x$ OR • Determines $u = x\sin x$ and $\frac{dv}{dx} = \cos x$ or $u = x\cos x$ and $\frac{dv}{dx} = \sin x$	1

Sample Answer:

$$\int x^2\sqrt{1-x^2}\,dx = \int \sin^2\theta\sqrt{1-\sin^2\theta}\times\cos\theta\,d\theta$$

$$= \int \sin^2\theta\cos^2\theta\,d\theta = \frac{1}{4}\int \sin^2 2\theta\,d\theta$$

$$= \frac{1}{4}\int\frac{1}{2}(1-\cos 4\theta)\,d\theta = \frac{\theta}{8}-\frac{\sin 4\theta}{32}+c$$

$$= \frac{\sin^{-1}x}{8}-\frac{2\sin 2\theta\cos 2\theta}{32}+c$$

$$= \frac{\sin^{-1}x}{8}-\frac{2\sin\theta\cos\theta\,(1-2\sin^2\theta)}{16}+c$$

$$= \frac{\sin^{-1}x}{8}-\frac{x\sqrt{1-x^2}(1-2x^2)}{8}+c$$

$$= \frac{\sin^{-1}x+(2x^3-x)\sqrt{1-x^2}}{8}+c$$

$x = \sin\theta$
$dx = \cos\theta\,d\theta$

15 (c) (i) (1 mark)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	1

Sample Answer:

$$\text{LHS} = \frac{1}{(k+1)^2}$$

$$= \frac{1}{(k+1)(k+1)}$$

$$< \frac{1}{k(k+1)}$$

$$= \frac{1}{k} - \frac{1}{k+1}$$

$$\therefore \frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$$

15 (c) (ii) (2 marks)
Outcomes Assessed: P1/MEX12-2
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Attempts to apply result from part(i), or equivalent merit	1

Sample Answer:

Since $1 < x_1 < x_2 < \ldots\ldots\ldots\ldots\ldots < x_n$,

$$\therefore \frac{1}{{x_1}^2}+\frac{1}{{x_2}^2}+\frac{1}{{x_3}^2}+\ldots+\frac{1}{{x_{n-1}}^2} < \frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{{x_n}^2}$$

Now from a(i)

$$\frac{1}{(k+1)^2} < \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\therefore \quad \frac{1}{2^2} < \frac{1}{1} - \frac{1}{2}$$

$$\frac{1}{3^2} < \frac{1}{2} - \frac{1}{3}$$

$$\ldots\ldots$$

$$\frac{1}{{x_n}^2} < \frac{1}{x_{n-1}} - \frac{1}{x_n}$$

$$\therefore \frac{1}{{x_1}^2}+\frac{1}{{x_2}^2}+\frac{1}{{x_3}^2}+\ldots+\frac{1}{{x_{n-1}}^2} < \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{x_{n-1}}-\frac{1}{x_n}\right)$$

$$< 1 - \frac{1}{x_n}$$

$$< 1$$

Alternatively:

$$\frac{1}{x_1^2} < \frac{1}{x_1 - 1} - \frac{1}{x_1} \text{ from (i)}$$

$$\leq \frac{1}{1} - \frac{1}{x_1} \quad \text{since } 1 < x_1$$

$$\therefore \frac{1}{x_1^2} < 1 - \frac{1}{x_1}$$

Similarly:

$$\frac{1}{x_2^2} < \frac{1}{x_1} - \frac{1}{x_2}, \ldots, \frac{1}{x_{n-1}^2} < \frac{1}{x_{n-2}} - \frac{1}{x_{n-1}}$$

$$\therefore \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \ldots + \frac{1}{x_{n-1}^2} < 1 - \frac{1}{x_1} + \frac{1}{x_1} - \frac{1}{x_2} + \frac{1}{x_2} - \frac{1}{x_3} + \ldots + \frac{1}{x_{n-2}} - \frac{1}{x_{n-1}}$$

$$= 1 - \frac{1}{x_{n-1}}$$

$$< 1 \qquad \text{since } x_{n-1} > 1$$

15 (d) (i) (2 marks)
Outcomes Assessed: M1.3/MEX12-6
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Correctly separates the variables, or equivalent merit	1

Sample Answer:
Choose point 0 as origin and ↓ as positive direction.

Equation of motion: $\dot{v} = g - kv^2$.
Initial conditions: $t = 0,\ x = 0,\ v = 0$.
Terminal velocity U hence $g = kU^2 \Rightarrow k = g/U^2$.
Relation between v and x:

$$\dot{v} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$-2k\,dx = \frac{-2kv\,dv}{g - kv^2}$$

$$-2kx + C = \ln\left|g - kv^2\right|$$

$$x = 0,\ v = 0 \Rightarrow C = \ln g,$$

$$x = \frac{1}{2k}\ln\left|\frac{g}{g - kv^2}\right|.$$

Alternatively:

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$x = \int_0^v \frac{v}{g - kv^2}\,dv$$

$$= -\frac{1}{2k}\int_0^v -\frac{2kv}{g - kv^2}\,dv$$

$$= \frac{1}{2k}\left[\ln\left|g - kv^2\right|\right]_v^0$$

$$= \frac{1}{2k}\left(\ln|g| - \ln\left|g - kv^2\right|\right)$$

$$= \frac{1}{2k}\ln\left|\frac{g}{g - kv^2}\right|$$

15 (d) (ii) (3 marks)

Outcomes Assessed: M1.3/MEX12-6

Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Finds $t = \dfrac{1}{2\sqrt{kg}} \ln\left\lvert \dfrac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v} \right\rvert$	2
• Correctly separates the variables OR • Finds $\dfrac{dt}{dv} = \dfrac{1}{g-kv^2}$	1

Sample Answer:

Relation between v and t :

$$\frac{dv}{dt} = g - kv^2$$

$$\sqrt{k}dt = \frac{\sqrt{k}dv}{g-\left(\sqrt{k}v\right)^2}$$

$$\sqrt{k}dt = \frac{\sqrt{k}dv}{g-kv^2}$$

$$\sqrt{k}dt = \left\{\frac{1}{\sqrt{g}-\sqrt{k}v} + \frac{1}{\sqrt{g}+\sqrt{k}v}\right\}\frac{\sqrt{k}dv}{2\sqrt{g}}$$

$$2\sqrt{kg}t + C = \ln\left\lvert\frac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v}\right\rvert,$$

$$t = 0, v = 0 \Rightarrow C = 0,$$

$$t = \frac{1}{2\sqrt{kg}}\ln\left\lvert\frac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v}\right\rvert \qquad (1)$$

If $v = \dfrac{U}{2}$ and $k = \dfrac{g}{U^2}$, then from (1) $t = \dfrac{U}{2g}\ln 3$.

$$\frac{1}{g-kv^2} = \frac{1}{\left(\sqrt{g}-\sqrt{k}v\right)\left(\sqrt{g}+\sqrt{k}v\right)}$$

$$\frac{1}{\left(\sqrt{g}-\sqrt{k}v\right)\left(\sqrt{g}+\sqrt{k}v\right)} = \frac{A}{\left(\sqrt{g}-\sqrt{k}v\right)} + \frac{B}{\left(\sqrt{g}+\sqrt{k}v\right)}$$

$$v = A\left(\sqrt{g}+\sqrt{k}v\right) + B\left(\sqrt{g}-\sqrt{k}v\right)$$

$$v = \frac{\sqrt{g}}{\sqrt{k}}, \quad 1 = 2A\sqrt{g} \quad \Rightarrow A = \frac{1}{2\sqrt{g}}$$

$$v = -\frac{\sqrt{g}}{\sqrt{k}}, \quad 1 = 2B\sqrt{g} \quad \Rightarrow B = \frac{1}{2\sqrt{g}}$$

$$\frac{v}{g-kv^2} = \frac{1}{2\sqrt{g}\left(\sqrt{g}-\sqrt{k}v\right)} + \frac{1}{2\sqrt{g}\left(\sqrt{g}+\sqrt{k}v\right)}$$

Alternatively:

$$\frac{dv}{dt} = g - kv^2$$

$$\frac{dt}{dv} = \frac{1}{g - kv^2}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$t = \frac{1}{2\sqrt{g}}\int_0^{\frac{U}{2}}\left(\frac{1}{\sqrt{g}-\sqrt{k}v} + \frac{1}{\sqrt{g}+\sqrt{k}v}\right)dv$$

$$= \frac{1}{2\sqrt{kg}}\left[-\ln\left|\sqrt{g}-\sqrt{k}v\right| + \ln\left|\sqrt{g}+\sqrt{k}v\right|\right.$$

$$= \frac{1}{2\sqrt{kg}}\left[\ln\left|\frac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v}\right|\right]_v^{\frac{U}{2}}$$

$$= \frac{1}{2\sqrt{kg}}\left(\ln\left|\frac{\sqrt{g}+\sqrt{k}\times\frac{U}{2}}{\sqrt{g}-\sqrt{k}\times\frac{U}{2}}\right| - \ln\left|\frac{\sqrt{g}}{\sqrt{g}}\right|\right)$$

$$= \frac{1}{2\sqrt{kg}}\ln\left|\frac{2\sqrt{g}+U\sqrt{k}}{2\sqrt{g}-U\sqrt{k}}\right|$$

$$\frac{1}{g-kv^2} = \frac{1}{\left(\sqrt{g}-\sqrt{k}v\right)\left(\sqrt{g}+\sqrt{k}v\right)}$$

$$\frac{1}{\left(\sqrt{g}-\sqrt{k}v\right)\left(\sqrt{g}+\sqrt{k}v\right)} = \frac{A}{\left(\sqrt{g}-\sqrt{k}v\right)} + \frac{B}{\left(\sqrt{g}+\sqrt{k}v\right)}$$

$$v = A\left(\sqrt{g}+\sqrt{k}v\right) + B\left(\sqrt{g}-\sqrt{k}v\right)$$

$$v = \frac{\sqrt{g}}{\sqrt{k}},\quad 1 = 2A\sqrt{g} \quad\Rightarrow A = \frac{1}{2\sqrt{g}}$$

$$v = -\frac{\sqrt{g}}{\sqrt{k}},\quad 1 = 2B\sqrt{g} \quad\Rightarrow B = \frac{1}{2\sqrt{g}}$$

$$\frac{v}{g-kv^2} = \frac{1}{2\sqrt{g}\left(\sqrt{g}-\sqrt{k}v\right)} + \frac{1}{2\sqrt{g}\left(\sqrt{g}+\sqrt{k}v\right)}$$

At terminal velocity:

$$0 = g - kU^2 \Rightarrow \sqrt{k} = \frac{\sqrt{g}}{U}$$

$$\therefore t = \frac{1}{2\sqrt{g}\left(\frac{\sqrt{g}}{U}\right)}\ln\left|\frac{2\sqrt{g}+U\frac{\sqrt{g}}{U}}{2\sqrt{g}-U\frac{\sqrt{g}}{U}}\right|$$

$$= \frac{U}{2g}\ln\frac{3\sqrt{g}}{\sqrt{g}}$$

$$= \frac{U}{2g}\ln 3.$$

Question 16 (15 marks)

16 (a) (i) (2 marks)
Outcomes Assessed: N2.2/MEX12-4
Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	2
• Demonstrates significant progress towards solution	1

Sample Answer:

$z^7 - 1 = 0 \Rightarrow z^7 = 1$

$z_1 = 1 = \cos 0 + i\sin 0 = e^{0i}$

$z_2 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} = e^{\frac{2\pi}{7}i}$

$z_3 = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} = e^{\frac{4\pi}{7}i}$

$z_4 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7} = e^{\frac{6\pi}{7}i}$

$z_5 = \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7} = \cos\left(-\frac{6\pi}{7}\right) + i\sin\left(-\frac{6\pi}{7}\right) = \overline{z_4} = e^{-\frac{6\pi}{7}i}$

$z_6 = \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7} = \cos\left(-\frac{4\pi}{7}\right) + i\sin\left(-\frac{4\pi}{7}\right) = \overline{z_3} = e^{-\frac{4\pi}{7}i}$

$z_7 = \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7} = \cos\left(-\frac{2\pi}{7}\right) + i\sin\left(-\frac{2\pi}{7}\right) = \overline{z_2} = e^{-\frac{2\pi}{7}i}$

16 (a) (ii) (3 marks)
Outcomes Assessed: N2.2/MEX12-4
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Shows some of the roots by substitution, or equivalent merit	2
• Shows one of the roots by substitution, or equivalent merit	1

Sample Answer:

$z^7 - 1 = 0$

$(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$

$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0 \quad (1)$

$\dfrac{z^6 + z^5 + z^4 + z^3 + z^2 + z + 1}{z^3} = \dfrac{0}{z^3}$

$z^3 + z^2 + z + 1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \dfrac{1}{z^3} = 0$

$\left(z^3 + \dfrac{1}{z^3}\right) + \left(z^2 + \dfrac{1}{z^2}\right) + \left(z + \dfrac{1}{z}\right) + 1 = 0$

since $z^3 + \dfrac{1}{z^3} = 2\cos 3\theta$, $z^2 + \dfrac{1}{z^2} = 2\cos 2\theta$ and $z + \dfrac{1}{z} = 2\cos\theta$

$\Rightarrow \; 2\cos 3\theta + 2\cos 2\theta + 2\cos\theta + 1 = 0 \quad (2)$

The solutions of (1) are the non-real seventh roots of unity, so $\theta = \pm\dfrac{2\pi}{7}, \; \pm\dfrac{4\pi}{7}, \; \pm\dfrac{6\pi}{7}$

$\therefore$ Since (1) and (2) are equivalent then $\dfrac{2\pi}{7}$, $\dfrac{4\pi}{7}$ and $\dfrac{6\pi}{7}$ are also solutions to (2).

16 (a) (iii) (2 marks)
Outcomes Assessed: N2.2/MEX12-4
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Substitute $\theta = \frac{2\pi}{7}$ in (ii) OR • Use roots and coefficients to show sum of the roots is $-\frac{1}{2}$ only	1

Sample Answer:

Let $\theta = \frac{2\pi}{7}$ in (ii):

$$\therefore\ 2\cos 3\left(\frac{2\pi}{7}\right) + 2\cos 2\left(\frac{2\pi}{7}\right) + 2\cos\left(\frac{2\pi}{7}\right) + 1 = 0$$

$$2\left(\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right) = -1$$

$$\cos\left(\frac{6\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$$

16 (b)(i) (3 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	3
• Obtains $I_{2n+1} = \frac{1}{2}\left[x^{2n}e^{x^2}\frac{2}{2}\right]_0^1 - n\int_0^1 x^{2n-1}e^{x^2}\,dx$	2
• Obtains correct expressions for u and $\frac{dv}{dx}$	1

Sample Answer:

$$I_{2n+1} = \int_0^1 x^{2n+1}e^{x^2}\,dx$$
$$= \frac{1}{2}\left[x^{2n}e^{x^2}\frac{2}{2}\right]_0^1 - n\int_0^1 x^{2n-1}e^{x^2}\,dx$$
$$= \frac{1}{2}(e-0) - nI_{2n-1}$$
$$= \frac{e}{2} - nI_{2n-1}$$

$u = \frac{1}{2}x^{2n}$	$\frac{dv}{dx} = 2xe^{x^2}$
$\frac{du}{dx} = nx^{2n-1}$	$v = e^{x^2}$

16 (b)(ii) (2 marks)
Outcomes Assessed: C1/MEX12-5
Targeted Performance Bands: E3-E4

Criteria	Marks
Provides the correct solution	2
Finds the integral in terms of $I_{2n-1}$and I_{2n+1}, or equivalent merit	1

Sample Answer:

$$\text{LHS} = 2\int_0^1 x^{2n-1}(1+x^2)e^{x^2}\,dx$$
$$= 2\left[\int_0^1 x^{2n-1}e^{x^2}\,dx + \int_0^1 x^{2n+1}e^{x^2}\,dx\right]$$
$$= 2\left[I_{2n-1} + I_{2n+1}\frac{2}{2}\right]$$
$$\le 2\left[I_{2n+1}\frac{2}{2} + nI_{2n-1}\frac{2}{2}\right] \qquad I_{2n-1} > 0 \text{ since } x^{2n-1} \ge 0, e^{x^2} > 0 \text{ for } [0,1]$$
$$\le 2\left[\frac{e}{2} - nI_{2n-1} + nI_{2n-1}\frac{2}{2}\right] \quad \text{from (i)}$$
$$\le e$$

16 (c) (i) (1 mark)

Outcomes Assessed: M1.4/MEX12-6

Targeted Performance Bands: E2-E3

Criteria	Marks
• Provides correct solution	1

Sample Answer:

$$x = 10u(1 - e^{-0.1t}) \Rightarrow \dot{x} = 0.1 \times 10ue^{-0.1t}$$
$$= ue^{-0.1t}$$

$$y = 100g(1 - e^{-0.1t}) - 10gt \Rightarrow \dot{y} = 10ge^{-0.1t} - 10g$$
$$= -10g\left(1 - e^{-0.1t}\right)$$

$$\therefore \underset{\sim}{v}(t) = \begin{pmatrix} ue^{-0.1t} \\ -10g(1 - e^{-0.1t}) \end{pmatrix}$$

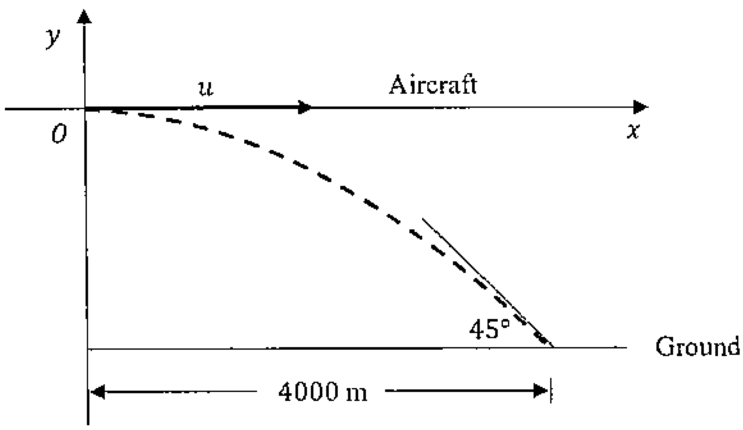
16 (c) (ii) (2 marks)

Outcomes Assessed: M1.4/MEX12-6

Targeted Performance Bands: E3-E4

Criteria	Marks
• Provides correct solution	2
• Finds the correct time on impact or the velocity on impact	1

Sample Answer:



$x = 10u(1 - e^{-0.1t})$

When $x = 4000$, $4000 = 10u(1 - e^{-0.1t})$

$u = \dfrac{400}{1 - e^{-0.1t}}$ (1)

$\tan\theta = \dfrac{\dot{y}}{\dot{x}}$

$\Rightarrow \tan(-45°) = \dfrac{-100(1 - e^{-0.1t})}{ue^{-0.1t}}$

$-1 = \dfrac{-100(1 - e^{-0.1t})}{ue^{-0.1t}}$ (2)

Sub (1) into (2),

$$-1=\frac{-100(1-e^{-0.1t})}{\left(\frac{400}{1-e^{-0.1t}}\right)e^{-0.1t}}$$

$$4=\frac{(1-e^{-0.1t})^2}{e^{-0.1t}}$$

Let $m=e^{-0.1t}\Rightarrow 4m=(1-m)^2$

$$m^2-6m+1=0$$

$$m=\frac{6\pm\sqrt{36-4}}{2}$$

$$=\frac{6\pm4\sqrt{2}}{2}$$

$$=3\pm2\sqrt{2}$$

$$e^{-0.1t}=3\pm2\sqrt{2}$$

when $e^{-0.1t}=3+2\sqrt{2}$, $\quad t=-\dfrac{\ln(3+2\sqrt{2})}{0.1}=-17.6$ sec

when $e^{-0.1t}=3-2\sqrt{2}$, $\quad t=-\dfrac{\ln(3-2\sqrt{2})}{0.1}=17.6$ sec (1 dec)

$$t=17.6,\ u=\frac{400}{1-e^{-0.1(17.6)}}$$

$$=483$$

$$\dot{x}=ue^{-0.1t}\Rightarrow\dot{x}=483e^{-0.1(17.6)}$$

$$=83.1$$

$$\dot{y}=-100(1-e^{-0.1t})\Rightarrow\dot{y}=-100(1-e^{-0.1(17.6)})$$

$$=-82.8$$

$$|\underset{\sim}{v}|=\sqrt{|\dot{x}|^2+|\dot{y}|^2}\Rightarrow|\underset{\sim}{v}|=\sqrt{83.1^2+(-82.8)^2}$$

$$=117.2 \text{ m/s (1 dp).}$$

Alternatively:

On impact $x = 4000$ (given) and $\dot{x} = -\dot{y}$ since the angle of impact is 45°.

$$\therefore 10u(1 - e^{-0.1t}) = 4000$$

$$1 - e^{-0.1t} = \frac{400}{u}$$

$$e^{-0.1t} = \frac{400}{1-u} \qquad (1)$$

Also $\quad ue^{-0.1t} = 100(1 - e^{-0.1t}) \qquad (2)$

Sub (1) in (2):

$$u\left(\frac{u-400}{u}\right) = 100\left(1 - \frac{u-400}{u}\right)$$

$$u - 400 = \frac{100(u - u + 400)}{u}$$

$$u - 400 = \frac{40000}{u}$$

$$u^2 - 400u = 40000$$

$$u^2 - 400u - 40000 = 0$$

$$u = \frac{400 \pm \sqrt{400^2 - 4(1)(-40000)}}{2}$$

$$= \frac{400 \pm 400\sqrt{2}}{2}$$

$$= 200\left(\sqrt{2} + 1\right) \text{ since } u > 0$$

Sub in (1):

$$e^{-0.1t} = \frac{200\left(\sqrt{2}+1\right) - 400}{2\left(\sqrt{2}+1\right)}$$

$$e^{0.1t} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$t = 10\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\approx 17.6 \text{ sec (1 dp)}$$

On impact $v = \sqrt{2}\dot{x}$ since angle impact is 45°

$$\therefore\ v = \sqrt{2}ue^{-0.1t}$$

$$= \frac{\sqrt{2}u(u-400)}{u}$$

$$= \sqrt{2}\left(200\left(\sqrt{2}+1\right)-400\right)$$

$$= \sqrt{2}\left(200\left(\sqrt{2}-1\right)\right)$$

$$= 200(2-\sqrt{2})$$

$$\approx 117.2 \text{ m/s (1 dp)}.$$

BLANK PAGE

BLANK PAGE