Neap

Trial Examination 2021

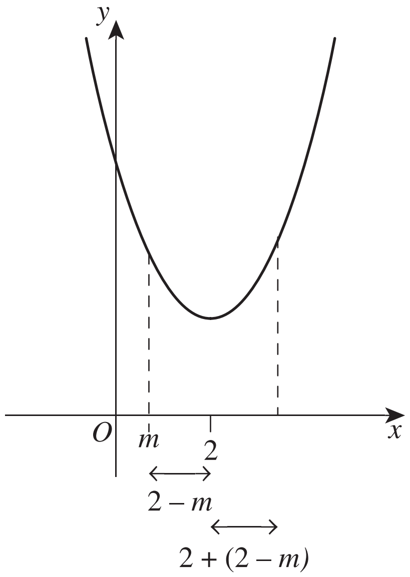
HSC Year 12 Mathematics Extension 1

Solutions and marking guidelines

SECTION I

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 1 B $P(-3)=0$ $P'(-3)\neq 0$	ME–F2 Polynomials ME11–1 Bands E2–E3
Question 2 D $\frac{dy}{dx}=\cos x.\cos^{-1}x-\frac{\sin x}{\sqrt{1-x^2}}$	ME– C2 Further Calculus Skills ME12–1 Bands E2–E3
Question 3 C Domain: $-1\leq\frac{4x}{\pi}\leq 1$ $-\pi\leq 4x\leq\pi$ $-\frac{\pi}{4}\leq x\leq\frac{\pi}{4}$ Range: $0\leq y\leq\pi$	ME–T1 Inverse Trigonometric Functions ME11–3 Bands E2–E3
Question 4 A Scenario 1: 4 algebra and 3 other calculus books ${}^8C_4\times{}^7C_3=2450$ Scenario 2: 5 algebra and 2 other calculus books ${}^8C_5\times{}^7C_2=1176$ Therefore, the total is $2450+1176=3626$.	ME–A1 Working with Combinatorics ME11–5 Bands E2–E3
Question 5 A $\overrightarrow{AC}$ is NOT perpendicular to $\overrightarrow{BC}$. $\therefore$ **A** is correct. $\overrightarrow{AB}\perp\overrightarrow{BC}$ This statement is true. $\therefore$ **B** is incorrect. $AC=DB$ This statement is true. $\therefore$ **C** is incorrect. $\overrightarrow{AD}=\overrightarrow{BC}=\underset{\sim}{c}-\underset{\sim}{b}$ This statement is true. $\therefore$ **D** is incorrect.	ME–V1 Introduction to Vectors ME12–2 Bands E2–E3

Answer and explanation	**Syllabus content, outcomes and targeted performance bands**
Question 6 B $\int \frac{\ln 2}{\sqrt{\pi - x^2}} dx = \ln 2 \times \int \frac{1}{\sqrt{\pi - x^2}} dx$ Now: $a^2 = \pi$ $a = \sqrt{\pi}$ Also: $\int \frac{1}{\sqrt{\pi - x^2}} dx = \sin^{-1}\left(\frac{x}{\sqrt{\pi}}\right) + C$ $\ln 2 \times \int \frac{1}{\sqrt{\pi - x^2}} dx = \ln 2 \times \sin^{-1}\left(\frac{x}{\sqrt{\pi}}\right) + C$	ME– C2 Further Calculus Skills ME12–1 Bands E2–E3
Question 7 C $23 = 2 + 5 + 5 \times 3 + 1$	ME–A1 Working with Combinatorics ME11–5 Bands E2–E3
Question 8 B **B** is correct. It shows $y' = x + e^x$. **A** is incorrect. It shows $y' = e^{-x} + e^x$. **C** is incorrect. It shows $y' = \frac{x}{y} + e^x$. **D** is incorrect. It shows $y' = e^{-x} + x$.	ME– C3 Applications of Calculus ME12–1, 12–4 Bands E3–E4
Question 9 B $\frac{dy}{dx} = 5 - y$ $\frac{1}{5-y} dy = 1 dx$ $-\ln(5 + y) = x + C$ $5 - y = Ae^{-x}$, where $A = e^{-C}$ $y = 5 - Ae^{-x}$ Substituting $y = 4$, $x = 3$ results in $A = e^3$. $\therefore y = 5 - e^{3-x}$	ME– C3 Applications of Calculus ME12–1, 12–4 Bands E3–E4

Answer and explanation	**Syllabus content, outcomes and targeted performance bands**
Question 10 D $f(x)$ maps m outside the domain to $2+2-m=4-m$, which is inside the domain. Now $f^{-1}(f(m))=f^{-1}\big(f(4-m)\big)=4-m$. 	ME–T1 Inverse Trigonometric Functions ME11–1 Bands E3–E4

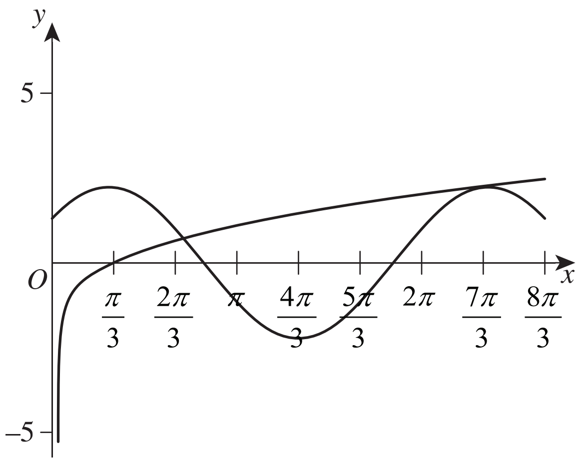
SECTION II

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 11	
(a) (i) $\cos 7x \sin 5x = \frac{1}{2}(\sin 12x - \sin 2x)$	ME–T2 Further Trigonometric Identities ME11–3 Band E2 • Gives the correct solution 1
(ii) $2\cos 7x \sin 5x = \sin 12x - \frac{\sqrt{3}}{2}$ Consider the LHS: $2\cos 7x \sin 5x = 2 \times \frac{1}{2}(\sin 12x - \sin 2x)$ (using the part (a) (i) result) $= \sin 12x - \sin 2x$ $\sin 12x - \sin 2x = \sin 12x - \frac{\sqrt{3}}{2}$ $-\sin 2x = -\frac{\sqrt{3}}{2}$ $\sin 2x = \frac{\sqrt{3}}{2}$ $2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$ $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$	ME–T3 Trigonometric Equations ME12–3 Bands E2–E3 • Gives the correct solutions . . . . 3 ——— • Correctly simplifies the equation to $\sin 2x = \frac{\sqrt{3}}{2}$. AND • Solves for at least ONE value of x . . . . 2 ——— • Correctly simplifies the equation to $\sin 2x = \frac{\sqrt{3}}{2}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) Let $S(n)$ be the statement $2^{2n}+6n-1$ is divisible by 3 for all integers $n \ge 1$. Step 1: Prove that $S(1)$ is true. $2^{2\times 1}+6\times 1-1=9$, which is divisible by 3. Therefore, $S(1)$ is true. Step 2: Assume that $S(k)$ is true. $2^{2k}+6k-1=3M$, where M is an integer. $2^{2k}=3M-6k+1$ Step 3: Prove that $S(k+1)$ is true. $2^{2(k+1)}+6(k+1)-1=3N$, where N is an integer. $2^{2(k+1)}+6(k+1)-1=2^{2k+2}+6k+6-1$ $=4\times 2^{2k}+6k+5$ $=4\times(3M-6k+1)+6k+5$ $=12M-24k+4+6k+5$ $=12M-18k+9$ $=3(4M-6k+3)$ $=3N$, where N is an integer Therefore, $S(k+1)$ is true if $S(k)$ is true. By mathematical induction, $S(n)$ is true for all integers $n \ge 1$.	ME–P1 Proof by Mathematical Induction ME12–1 Bands E2–E3 • Gives the correct proof for all steps. 3 • Gives the correct proof for step 1. AND • Makes some progress in using the assumption for step 2 . . . . 2 • Gives the correct proof for step 1 1
(c) (i) $a=1, b=0, c=-2p, d=q$ Taking sum of the roots, one at a time: $\alpha+\beta+(\alpha+\beta)=-\dfrac{b}{a}$ $2(\alpha+\beta)=0$ $\alpha+\beta=0$ $\therefore \alpha+\beta=0$ $\alpha=-\beta$	ME–F2 Polynomials ME11–1, 11–2 Bands E2–E3 • Gives the correct proof 1

		Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
	(ii)	Taking sum of the roots, three at a time OR taking product of the roots: $\alpha\beta(\alpha+\beta)=-q$ $q=0$ as $\alpha+\beta=0$ $P(x)=x^3-2px+q$ $=x^3-2px$ $=x\left(x^2-2p\right)$ $x\left(x^2-2p\right)=0$ $x=0$ or $x^2-2p=0$ $x^2-2p=0$ $x^2=2p$ $x=\pm\sqrt{2p}$ Therefore, the roots are $x=0, \pm\sqrt{2p}$.	ME–F2 Polynomials ME11–1, 11–2 Bands E2–E3 • Gives the correct solutions . . . . 3 • Correctly finds the value of q AND ONE root. . . . . 2 • Correctly finds the value of q. 1
(d)	(i)	If $\overrightarrow{PA}=\underset{\sim}{a}$ and $\overrightarrow{SB}=\underset{\sim}{b}$, then $\overrightarrow{AQ}=\underset{\sim}{a}$ and $\overrightarrow{BR}=\underset{\sim}{b}$, as A and B are the midpoints of PQ and RS respectively. $\overrightarrow{QR}=\overrightarrow{QA}+\overrightarrow{AB}+\overrightarrow{BR}$ $=-\underset{\sim}{a}+\overrightarrow{AB}+\underset{\sim}{b}$ $=\underset{\sim}{b}-\underset{\sim}{a}+\overrightarrow{AB}$	ME–V1 Introduction to Vectors ME12–2 Bands E2–E3 • Gives the correct solution . . . . 2 • Demonstrates that $\overrightarrow{AQ}=\underset{\sim}{a}$ and $\overrightarrow{BR}=\underset{\sim}{b}$ OR equivalent merit. 1
	(ii)	$\overrightarrow{PS}=\overrightarrow{PA}+\overrightarrow{AB}+\overrightarrow{BS}$ $=\underset{\sim}{a}+\overrightarrow{AB}-\underset{\sim}{b}$ $=\underset{\sim}{a}-\underset{\sim}{b}+\overrightarrow{AB}$ Adding $\overrightarrow{QR}$ to $\overrightarrow{PS}$: $\overrightarrow{QR}+\overrightarrow{PS}=\underset{\sim}{b}-\underset{\sim}{a}+\overrightarrow{AB}+\underset{\sim}{a}-\underset{\sim}{b}+\overrightarrow{AB}$ (using the part (d) (i) result) $=2\overrightarrow{AB}$ $\therefore \overrightarrow{AB}=\frac{1}{2}\left(\overrightarrow{PS}+\overrightarrow{QR}\right)$	ME–V1 Introduction to Vectors ME12–2 Bands E2–E3 • Gives the correct proof . . . . 2 • Correctly finds $\overrightarrow{PS}$ in terms of $\underset{\sim}{a}, \underset{\sim}{b}$ and $\overrightarrow{AB}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 12	
(a) (i) $\sqrt{3}\sin x + \cos x \equiv R\sin(x+\alpha)$ $= R\sin x\cos\alpha + R\cos x\sin\alpha$ Therefore: $R\cos\alpha = \sqrt{3}$ $R\sin\alpha = 1$ $R^2\left((\cos x)^2 + (\sin x)^2\right) = 4$ $R = 2$ In addition: $\dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{1}{\sqrt{3}}$ $\tan\alpha = \dfrac{1}{\sqrt{3}}$ $\alpha = \dfrac{\pi}{6}$ $\therefore \sqrt{3}\sin x + \cos x = 2\sin\left(x + \dfrac{\pi}{6}\right)$	ME–T3 Trigonometric Equations ME12–3 Bands E2–E3 • Gives the correct solution . . . . 2 • Finds the correct value for R or α 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) The graphs of $y = 2\sin\left(x + \frac{\pi}{6}\right)$ and $y = \ln x$:  Inspect number of points of interest near $x = \frac{7\pi}{3}$: $\sqrt{3}\sin x + \cos x = 2\sin\left(x + \frac{\pi}{6}\right)$ (from part (a) (i) result) $= 2\sin\left(\frac{7\pi}{3} + \frac{\pi}{6}\right)$ $= 2\sin\left(\frac{5\pi}{2}\right)$ $= 2$ $\ln x = 1.99$ Therefore, there exist two POI at $x = \frac{7\pi}{3}$. Hence, there are three solutions to the equation $\sqrt{3}\sin x + \cos x = \ln x$.	ME–T3 Trigonometric Equations ME12–3, 12–7 Bands E2–E4 • Gives the correct solution AND clearly justifies all POI at $x = \frac{7\pi}{3}$ . . . . 3 • Correctly sketches both graphs . . . . 2 • Correctly sketches the graph of $y = 2\sin\left(x + \frac{\pi}{6}\right)$. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) Consider the identity $(1+x)^m(1+x)^n=(1+x)^{m+n}$. The term x^4 of $(1+x)^{m+n}$ is: $\binom{m+n}{4}x^4 \quad (1)$ The term x^4 of $(1+x)^m(1+x)^n$ is: $\binom{m}{4}x^4\times 1+\binom{m}{3}x^3\times\binom{n}{1}x+\binom{m}{2}x^2\times\binom{n}{2}x^2+\binom{m}{1}x\times\binom{n}{3}x^3+1\times\binom{n}{4}x^4=\left[\binom{m}{4}+\binom{m}{3}\binom{n}{1}+\binom{m}{2}\binom{n}{2}+\binom{m}{1}\binom{n}{3}+\binom{n}{4}\right]x^4 \quad (2)$ Compare the coefficients of x^4 from (1) and (2): $\binom{m+n}{4}=\binom{m}{4}+\binom{m}{3}\binom{n}{1}+\binom{m}{2}\binom{n}{2}+\binom{m}{1}\binom{n}{3}+\binom{n}{4}$	ME–A1 Working with Combinatorics ME11–5 Bands E2–E3 • Gives the correct proof .... 2 • Finds the correct coefficient of x^4 of $(1+x)^{m+n}$ AND demonstrates some progress in achieving the coefficient of $(1+x)^m(1+x)^n$ 1

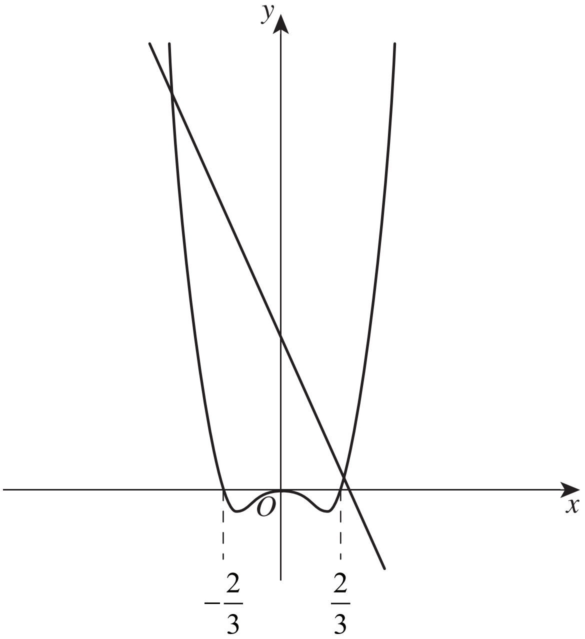
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide

Sample answer

(c) The graph of $y = 3|x|^3 - 2|x|^2$ is $y = f|x|$, which can be drawn by taking the graph for $x \geq 0$ and reflecting it over the y-axis.

$$f(x) = 3x^2 - 2x^2$$
$$= x^2(3x - 2)$$

The function $f(x) = x^2(3x - 2)$ has two x-intercepts at 0 and $\frac{2}{3}$. Therefore, when reflected, the graph of $y = 3|x|^3 - 2|x|^2$ should have three x-intercepts at 0, $\frac{2}{3}$ and $-\frac{2}{3}$. There are now two POI between the straight line and $y = 3|x|^3 - 2|x|^2$.



The solutions $x \in (-\infty, -2) \cup (1, \infty)$ give the x-coordinates of these POI to be –2 and 1.

Substituting these into $y = 3|x|^3 - 2|x|^2$ gives:

$$y = 3|-2|^3 - 2|-2|^2$$
$$= 16$$
$$y = 3|1|^3 - 2|1|^2$$
$$= 1$$

(continues on page 12)

Syllabus content, outcomes, targeted performance bands and marking guide

ME–F1 Further Work with Functions
ME11–1, 11–2, 11–7 Bands E2–E3

- Gives the correct solution 3

- Sketches the correct graph of $y = 3|x|^3 - 2|x|^2$.

AND

- Finds the POI between $f(x)$ and $g(x)$ 2

- Sketches the correct graph of $y = 3|x|^3 - 2|x|^2$.

OR

- Finds the POI between $f(x)$ and $g(x)$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (continued) Therefore, the straight line passes through the points (–2, 16) and (1, 1). $m = \frac{16-1}{-2-1}$ $= -5$ $y = -5x + b$ Substituting (1, 1): $1 = -5 \times 1 + b$ $b = 6$ $\therefore g(x) = -5x + 6.$	
(d) (i) ${}^{150}C_{135}(0.9)^{135}(0.1)^{15} = 0.107970$ $= 0.108$	ME–S1 The Binomial Distribution ME12–5 Bands E2–E3 • Gives the correct solution 1
(ii) $E(\hat{p}) = E\left(\frac{X}{n}\right)$ $= \frac{E(X)}{n}$ $= \frac{np}{n}$ $= p$ $= 0.9$ $\sigma(\hat{p}) = \sigma\left(\frac{X}{n}\right)$ $= \frac{\sigma(X)}{n}$ $= \frac{\sqrt{npq}}{n}$ $= \frac{\sqrt{150 \times 0.9 \times 0.1}}{150}$ $= 0.0245$	ME–S1 The Binomial Distribution ME12–5 Bands E2–E3 • Gives the correct proofs. . . . . 2 • Correctly shows the expected value OR standard deviation 1

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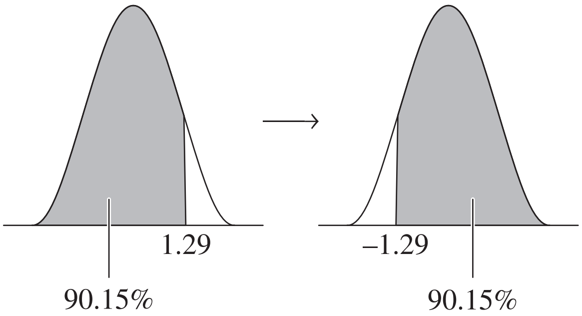
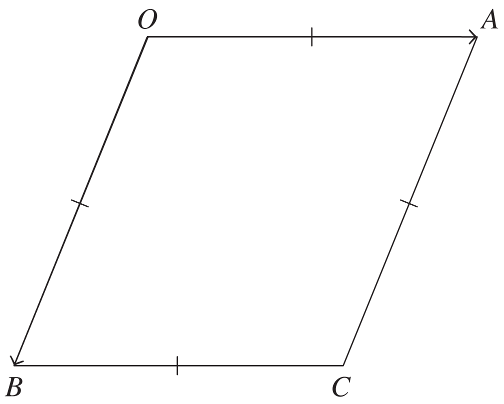
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) From the table, $0.90147 = 90.15\%$ probability. This corresponds to a z-score of 1.29. However, for the least number of furniture pieces expected to be delivered, we need to use the symmetry of the normal distribution curve for a z-score of –1.29.  We now need to find x (the sample proportion) that corresponds to this z-score. $-1.29 = \frac{x-0.9}{0.0245}$ (using part (d) (ii) result) $x = -1.29 \times 0.0245 + 0.9$ $= 0.868395$ Therefore, the minimum number of furniture expected to be delivered within 54 hours is $0.868395 \times 150 = 130.25925$, which would round to 130 pieces of furniture.	ME–S1 The Binomial Distribution ME12–5 Bands E3–E4 • Gives the correct solution . . . . 2 • Correctly finds the z-score of –1.29. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 13	

(a)



If $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$ then $\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b}$.

$$\cos\angle AOC = \frac{\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b})}{|\underset{\sim}{a}| \cdot |\underset{\sim}{a} + \underset{\sim}{b}|}$$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}| \cdot |\underset{\sim}{a} + \underset{\sim}{b}|}$$

$$= \frac{|\underset{\sim}{a}|^2 + \underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}| \cdot |\underset{\sim}{a} + \underset{\sim}{b}|} \quad (1)$$

$$\cos\angle BOC = \frac{\underset{\sim}{b} \cdot (\underset{\sim}{a} + \underset{\sim}{b})}{|\underset{\sim}{b}| \cdot |\underset{\sim}{a} + \underset{\sim}{b}|}$$

$$= \frac{\underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}| \cdot |\underset{\sim}{a} + \underset{\sim}{b}|}$$

$$= \frac{|\underset{\sim}{b}|^2 + \underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}| \cdot |\underset{\sim}{a} + \underset{\sim}{b}|} \quad (2)$$

But $|\underset{\sim}{a}| = |\underset{\sim}{b}|$ (*OA* = *OB*, adjacent sides are equal in a rhombus).

From (1) and (2), $\cos\angle AOC$ or $\cos\angle BOC$ and, as none of these angles is a reflex angle, $\angle AOC = \angle BOC$.

Therefore, the diagonal *OC* bisects $\angle AOB$.

ME–V1 Introduction to Vectors
ME12–2 Bands E2–E4

- Gives the correct proof 2

- Finds $\cos\angle AOC$ or $\cos\angle BOC$ . . 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) (i) $V = \pi\int_0^h y^2 dx$ $= \pi\int_0^h \left(\frac{3}{\sqrt{9+x^2}}\right)^2 dx$ $= \pi\int_0^h \frac{9}{9+x^2} dx$ $= 9\pi\int_0^h \frac{1}{9+x^2} dx$ $= \frac{9\pi}{3}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_0^h$ $= 3\pi\left[\tan^{-1}\left(\frac{h}{3}\right) - 0\right]$ $= 3\pi\tan^{-1}\left(\frac{h}{3}\right)$ cubic units	ME– C3 Applications of Calculus ME12– 4 Bands E2–E4 • Gives the correct proofs. . . . . 2 • Correctly integrates $\left(\frac{3}{\sqrt{9+x^2}}\right)^2$ 1
(ii) $\frac{dh}{dt} = 3$ cm/s $\frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt}$ (1) Now: $V = 3\pi\tan^{-1}\left(\frac{h}{3}\right)$ $\frac{dV}{dh} = 3\pi\times\frac{3}{9+h^2}$ $= \frac{9\pi}{9+h^2}$ Substitute $\frac{dh}{dt} = 3$ cm/s, $\frac{dV}{dh} = \frac{9\pi}{9+h^2}$ and $h = 6$ cm into (1): $\frac{dV}{dt} = \frac{9\pi}{9+6^2}\times 3$ $= \frac{3\pi}{5}$ cm^3/s	ME– C1 Rates of Changes ME11– 4 Bands E2–E3 • Gives the correct solution . . . . 2 • Correctly finds $\frac{dV}{dh}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) Refer to the original graph; as $x = h, y = \dfrac{3}{\sqrt{9+x^2}}$, this y-value is the radius of the water surface area A. Therefore: $A = \pi r^2$ $= \pi\left(\dfrac{3}{\sqrt{9+x^2}}\right)^2$ $= \dfrac{9\pi}{9+h^2}$ $= 9\pi\left(9+h^2\right)^{-1}$ $\dfrac{dA}{dh} = -9\pi\left(9+h^2\right)^{-2} \times 2h$ $= \dfrac{-18\pi h}{\left(9+h^2\right)^2}$ $\dfrac{dA}{dt} = \dfrac{dA}{dh} \times \dfrac{dh}{dt}$ Substitute and $\dfrac{dh}{dt} = 3$ cm/s, $\dfrac{dA}{dh} = \dfrac{-18\pi h}{\left(9+h^2\right)^2}$ $h = 6$ cm into $\dfrac{dA}{dt} = \dfrac{dA}{dh} \times \dfrac{dh}{dt}$: $\dfrac{dA}{dt} = \dfrac{-18\pi \times 6}{\left(9+6^2\right)^2} \times 3$ $= \dfrac{-4\pi}{25}$ Therefore the area is decreasing at $\dfrac{4\pi}{25}$ cm^2/s.	ME–C1 Rates of Changes ME11–4 Bands E3–E4 • Gives the correct solution . . . . 2 • Correctly finds $\dfrac{dA}{dh}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) $S = \dfrac{2000}{1+199e^{-0.4t}}$ $= 2000\left(1+199e^{-0.4t}\right)^{-1}$ $\dfrac{dS}{dt} = -2000\left(1+199e^{-0.4t}\right)^{-2} \times -0.4 \times 199e^{-0.4t}$ $= \dfrac{-2000 \times -0.4 \times 199e^{-0.4t}}{\left(1+199e^{-0.4t}\right)^2}$ $= \dfrac{159\,200e^{-0.4t}}{\left(1+199e^{-0.4t}\right)^2} \quad (1)$ $\dfrac{S}{5}\left(2-\dfrac{S}{1000}\right) = \dfrac{2000}{5\times\left(1+199e^{-0.4t}\right)} \times \left(2-\dfrac{2000}{1000\left(1+199e^{-0.4t}\right)}\right)$ $= \dfrac{400}{1+199e^{-0.4t}}\left(2-\dfrac{2}{1+199e^{-0.4t}}\right)$ $= \dfrac{800}{1+199e^{-0.4t}}\left(1-\dfrac{1}{1+199e^{-0.4t}}\right)$ $= \dfrac{800}{1+199e^{-0.4t}}\left(\dfrac{1+199e^{-0.4t}-1}{1+199e^{-0.4t}}\right)$ $= \dfrac{800}{1+199e^{-0.4t}}\left(\dfrac{199e^{-0.4t}}{1+199e^{-0.4t}}\right)$ $= \dfrac{159\,200e^{-0.4t}}{\left(1+199e^{-0.4t}\right)^2} \quad (2)$ From (1) and (2): $\dfrac{dS}{dt} = \dfrac{S}{5}\left(2-\dfrac{S}{1000}\right)$	ME– C3 Applications of Calculus ME12– 4 Bands E3–E4 • Gives the correct proof . . . . 3 • Finds $\dfrac{dS}{dt}$ AND demonstrates some progress . . . . 2 • Finds $\dfrac{dS}{dt}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Region B From the differential equation proven in part (c) (i), the solution curve S must be bounded by two constant solutions, 0 and 2000. These constant solutions are shown by horizontal slope line segments. Substituting $t = 0$: $S = \dfrac{2000}{1+199e^{0}}$ $= 10$ This initial S value is in the interval [0, 2000]. All these characteristics of the solution curve S occur in region B.	ME– C3 Applications of Calculus ME12– 4, 12–7 Bands E3–E4 • Gives the correct solution . . . . 2 • Correctly finds constant solutions 1
(iii) The graph $\dfrac{dS}{dt}$ versus S is a parabola with two intercepts with the t-axis: 0 and 2000. As the parabola concaves down, the maximum is the vertex that occurs at $S = \dfrac{1}{2}\left(0+2000\right) = 1000$. Substituting $S = 1000$ into $S = \dfrac{2000}{1+199e^{-0.4t}}$ gives: $1000 = \dfrac{2000}{1+199e^{-0.4t}}$ $1+199e^{-0.4t} = 2$ $199e^{-0.4t} = 1$ $e^{-0.4t} = \dfrac{1}{199}$ $-0.4t = \ln\dfrac{1}{199}$ $t = \dfrac{\ln\frac{1}{199}}{-0.4}$ $= 13.233$ ≈ 13 days	ME– C3 Applications of Calculus ME12– 4 Bands E3–E4 • Gives the correct solution . . . . 2 • Correctly finds the maximum point with correct S values. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 14	

(a) Consider $f(x)=\dfrac{x^2+1}{x}=x+x^{-1}$.

At stationary points, $f'(x)=1-x^{-2}=0$.

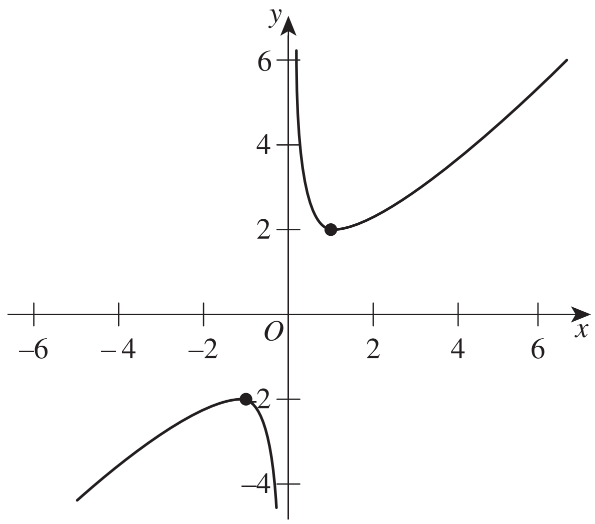
$1-\dfrac{1}{x^2}=0$

$x^2=1$

$x=\pm 1$

Substituting these values into $f(x)=\dfrac{x^2+1}{x}$ gives the stationary points (1, 2) and (–1, –2).

By inspection, the graph of $f(x)$ can be completed as shown.



Consider $g(x)=\dfrac{1}{f(x)}$.

The minimum point on $f(x)$ becomes the maximum point on $\dfrac{1}{f(x)}$.

$\therefore y=\dfrac{1}{f(x)}$ has a maximum point at

$\left(1,\dfrac{1}{f(x)}\right)=\left(1,\dfrac{1}{2}\right)$.

(continues on page 20)

ME–F1 Further Work with Functions
ME11–1, 11–2, 11–7 Bands E2–E4

- Gives the correct solution 3

- Correctly draws the graph of $g(x)=\dfrac{1}{\sqrt{f(x)}}$ without turning points.

OR

- Correctly draws the graph of $g(x)=\dfrac{1}{f(x)}$ with turning points. 2

- Correctly draws the graph of $g(x)=\dfrac{1}{f(x)}$ with turning points. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(a) (continued)	

As $x \to \infty, f(x) \to \infty, \dfrac{1}{f(x)} \to 0^+$.

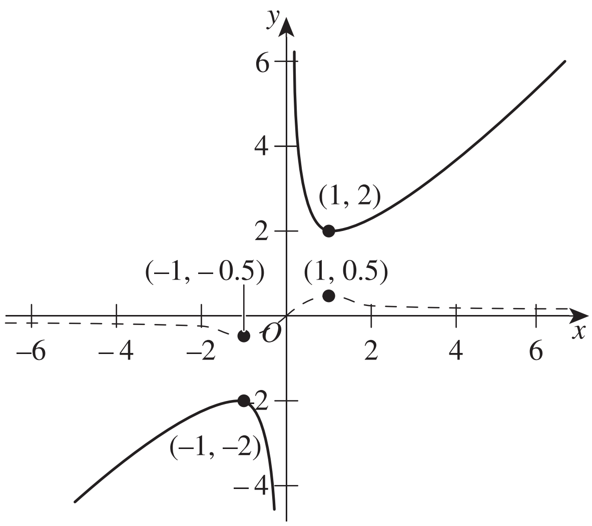
At $x = 0, \dfrac{1}{f(x)} = 0$.

Both $y = f(x)$ and $y = \dfrac{1}{f(x)}$ are odd functions.

Hence, the graphs of $y = f(x)$ (full)

and $y = g(x) = \dfrac{1}{f(x)}$ (dashed)

are as shown.



Consider:

$$h(x) = \frac{1}{\sqrt{f(x)}}$$
$$= \sqrt{\frac{1}{f(x)}}$$
$$= \sqrt{g(x)}$$

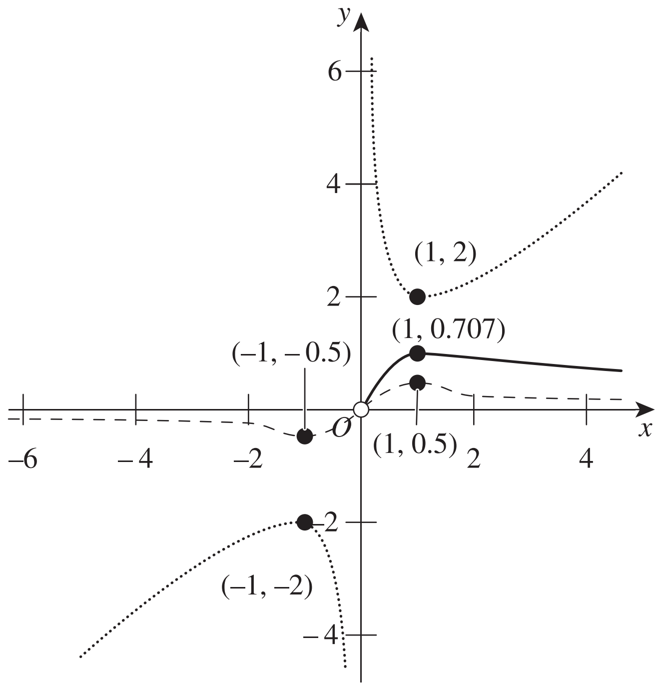
x only exists when $g(x) \ge 0$.

$g(0) = 0; h(0) = \sqrt{g(0)} = 0$

The maximum point at $x = 1$ remains a maximum point on $h(x)$.

$g(1) = 2; h(1) = \sqrt{g(2)} = \sqrt{0.5} \approx 0.7071$

(continues on page 21)

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(a) (continued) As $g(x) < 1$ for $x > 0$, the square root function $\sqrt{g(x)}$ is above $g(x)$ (the square root of a number less than 1 is more than the original number). Hence, the graph of $y = \dfrac{1}{f(x)}$ is shown (full). 	

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) (i) For $t = \tan\frac{x}{2}$, $\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$ $= \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)$ $= \frac{1}{2}\left(1+t^2\right)$ $\int \frac{1}{5+3\cos x}dx = \int \frac{1}{5+3\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}dt$ since $\frac{dx}{dt} = \frac{2}{1+t^2}, dx = \frac{2}{1+t^2}dt.$ $= \int \frac{1}{\frac{5\left(1+t^2\right)+3\left(1-t^2\right)}{1+t^2}} \times \frac{2}{1+t^2}dt$ $= \int \frac{1}{\frac{5\left(1+t^2\right)+3\left(1-t^2\right)}{1+t^2}} \times \frac{2}{1+t^2}dt$ $= \int \frac{2}{5\left(1+t^2\right)+3\left(1-t^2\right)}dt$ $= \int \frac{2}{8+2t^2}dt$ $= \int \frac{1}{4+t^2}dt$ $= \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+C$ $= \frac{1}{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right)+C$ $= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)+C$	ME–T2 Further Trigonometric Identities, ME– C2 Further Calculus Skills ME11–3, 12–3 Bands E3–E4 • Gives the correct proof . . . . 3 • Achieves the complete integrand in terms of t . . . . 2 • Correctly finds $\frac{dx}{dt}$ 1

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Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $A = \int_0^{\pi} \frac{1}{5+3\cos x}\,dx$ $= \left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)\right]_0^{\pi}$ However, $\tan\frac{\pi}{2}$ is undefined. Therefore, due to the symmetry of the curve about $x = \pi$: $A = \int_0^{\pi} \frac{1}{5+3\cos x}\,dx$ Let $t = \tan\frac{x}{2}$. When $x = 0, t = \tan 0 = 0$. When $x = \pi, t = \tan\frac{\pi}{2} = \infty$. $\int_0^{\infty} \frac{1}{4+t^2}\,dt = \left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{\infty}$ $= \frac{1}{2}\tan^{-1}(\infty) - \frac{1}{2}\tan^{-1}(0)$ $= \frac{1}{2}\cdot\frac{\pi}{2} - 0$ $= \frac{\pi}{4}$	ME– C3 Applications of Calculus ME12–1, 12–7 Bands E3–E4 • Gives the correct proof . . . . 2 • Achieves the result that $A = \frac{1}{2}\times\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2}\right)\right]_0^{2\pi}$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) At $t=0, u=90$ m/s, $\theta=30°$, $g=10$ m/s^2 and $\underset{\sim}{a}=-10\underset{\sim}{j}$. $\underset{\sim}{v}=\int \underset{\sim}{a}dt$ $=\int -10\underset{\sim}{j}dt$ $=-10t\underset{\sim}{j}+C_1$ At $t=0$, $\underset{\sim}{v}=90\cos 30°\underset{\sim}{i}+90\sin 30°\underset{\sim}{j}$ $=45\sqrt{3}\underset{\sim}{i}+45\underset{\sim}{j}$ $\therefore C_1=45\sqrt{3}\underset{\sim}{i}+45\underset{\sim}{j}$ $\therefore \underset{\sim}{v}=45\sqrt{3}\underset{\sim}{i}+(45-10t)\underset{\sim}{j}$ $\underset{\sim}{s}=\int \underset{\sim}{v}dt$ $=\int 45\sqrt{3}\underset{\sim}{i}+(45-10t)\underset{\sim}{j}dt$ $=45\sqrt{3}t\underset{\sim}{i}+(45t-5t^2)\underset{\sim}{j}+C_2$ At $t=0, \underset{\sim}{s}=0$. $\therefore C_2=0$ $\therefore \underset{\sim}{s}=45\sqrt{3}t\underset{\sim}{i}+(45t-5t^2)\underset{\sim}{j}$	ME–V1 Introduction to Vectors ME12–2 Bands E2–E4 • Gives the correct proof . . . . 2 • Correctly derives the velocity vector 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Let t, in seconds, be the time travelled by the golf ball. The time travelled by the stone is then $t^* = (t-5)$ seconds. At $t^* = 0, u = V$ m/s, $\theta = 0°$, $g = 10$ m/s^2 and $\underset{\sim}{a} = -10\underset{\sim}{j}$. $\underset{\sim}{v} = \int \underset{\sim}{a}\,dt^*$ $= \int -10\underset{\sim}{j}\,dt^*$ $= -10t^*\underset{\sim}{j} + C_1$ At $t^* = 0, v = V\cos 0\underset{\sim}{i} + V\sin 0\underset{\sim}{j} = V\underset{\sim}{i}$. $\therefore C_1 = V\underset{\sim}{i}$ $\therefore \underset{\sim}{v} = V\underset{\sim}{i} - 10t^*\underset{\sim}{j}$ $\underset{\sim}{s} = \int \underset{\sim}{v}\,dt$ $= \int V\underset{\sim}{i} - 10t^*\underset{\sim}{j}\,dt^*$ $= Vt^*\underset{\sim}{i} - 5t^{*2}\underset{\sim}{j} + C_2$ At $t^* = 0, \underset{\sim}{s} = 20\underset{\sim}{j}$. $\therefore C_2 = 20\underset{\sim}{j}$ $\therefore \underset{\sim}{s}_2 = Vt^*\underset{\sim}{i} + \left(20 - 5t^{*2}\right)\underset{\sim}{j}$ Substitute in $t^* = (t-5)$ since $\frac{dt^*}{dt} = 1$: *Note: This substitution does not affect any of the integration processes above.* $\therefore \underset{\sim}{s}_2 = V(t-5)\underset{\sim}{i} + \left(20 - 5(t-5)^2\right)\underset{\sim}{j}$, where $\underset{\sim}{s}_2$ is the position vector of the stone, and $\underset{\sim}{s}_1 = 45\sqrt{3}t\underset{\sim}{i} + \left(45t - 5t^2\right)\underset{\sim}{j}$, where $\underset{\sim}{s}_1$ is the position vector of the golf ball. At the time of collision, the two position vectors are equal: $45\sqrt{3}t\underset{\sim}{i} + \left(45t - 5t^2\right)\underset{\sim}{j} = V(t-5)\underset{\sim}{i} + \left(20 - 5(t-5)^2\right)\underset{\sim}{j}$ (continues on page 26)	ME–V1 Introduction to Vectors ME12–2 Bands E3–E4 • Gives the correct proof . . . . 3 • Finds the displacement/position vector for the stone . . . . 2 • Finds the velocity vector of the stone 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) (continued) $\therefore 45\sqrt{3}t = V(t-5) \quad (1)$ and $(45t - 5t^2) = (20 - 5(t-5)^2) \quad (2)$ Using (2): $9t - t^2 = 4 - (t-5)^2$ $9t - t^2 = 4 - t^2 + 10t - 25$ $9t = 10t - 21$ $t = 21$ Therefore, the two objects collided after the golf ball has travelled for 21 seconds.	
(iii) Substituting $t = 21$ into $45\sqrt{3}t = V(t-5)$ (from part (c) (ii)) gives: $45\sqrt{3} \times 21 = V(21-5)$ $V = \dfrac{45\sqrt{3} \times 21}{21-5}$ $= \dfrac{945\sqrt{3}}{16}$ m/s Using the velocity vector $\underset{\sim}{v} = V\underset{\sim}{i} - 10t^* \underset{\sim}{j}$, substitute $V = \dfrac{945\sqrt{3}}{16}$ and $t^* = 21 - 5 = 16$: $\underset{\sim}{v} = \dfrac{945\sqrt{3}}{16}\underset{\sim}{i} - 10 \times 16\underset{\sim}{j}$ Therefore, the speed of the stone at the time of collision is: $\lvert\underset{\sim}{v}\rvert = \sqrt{\left(\dfrac{945\sqrt{3}}{16}\right)^2 + (160)^2}$ $= 189.908$ ≈ 190 m/s	ME–V1 Introduction to Vectors ME12–2 Bands E3–E4 • Gives the correct solution . . . . 2 • Finds the initial speed of the stone. 1