Neap

Trial Examination 2021

HSC Year 12 Mathematics Extension 2

Solutions and marking guidelines

SECTION I

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 1 **B** $2\underset{\sim}{a}-\underset{\sim}{b}=2\left(3\underset{\sim}{i}-4\underset{\sim}{j}+\underset{\sim}{k}\right)-\left(-\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}\right)$ $=6\underset{\sim}{i}-8\underset{\sim}{j}+2\underset{\sim}{k}+\underset{\sim}{i}-2\underset{\sim}{j}+3\underset{\sim}{k}$ $=7\underset{\sim}{i}-10\underset{\sim}{j}+5\underset{\sim}{k}$	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E3
Question 2 **C** $z=\dfrac{\sqrt{3}-i}{1+i}$ $\|z\|=\dfrac{\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}}{\sqrt{1^2+(-1)^2}}$ $=\dfrac{2}{\sqrt{2}}$ $=\sqrt{2}$ $\arg\left(\dfrac{\sqrt{3}-i}{1+i}\right)=\arg\left(\sqrt{3}-i\right)-\arg\left(1+i\right)$ $=-\dfrac{\pi}{6}-\dfrac{\pi}{4}$ $=-\dfrac{5\pi}{12}$	MEX–N1 Introduction to Complex Numbers MEX12–4 Bands E2–E3
Question 3 **A** The parametric equations are $x=-5+2\lambda$ and $y=6-3\lambda$. $x=-5+2\lambda \Rightarrow \lambda=\dfrac{x+5}{2}$ Substituting $\lambda=\dfrac{x+5}{2}$ into $y=6-3\lambda$ gives $y=6-3\left(\dfrac{x+5}{2}\right)$. Multiply all terms by 2: $2y=12-3\left(x+5\right)$ $2y=12-3x-15$ Rearranging gives $2y+3x=-3$.	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E3

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 4 **B** The centre of C is the midpoint of AB, which is given by $\frac{1-5i+3-i}{2}=2-3i.$ $AB=\sqrt{(3-1)^2+(-1+5)^2}$ $=\sqrt{2^2+4^2}$ $=\sqrt{20}$ $=2\sqrt{5}$ So the radius of C is $\sqrt{5}$. The equation of C is $\left\|z-(2-3i)\right\|=\sqrt{5}$, and so $\left\|z-2+3i\right\|=\sqrt{5}$.	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E3
Question 5 **B** $\frac{2\pi}{n}=\pi\Rightarrow n=2$ Using $v^2=n^2\left(a^2-x^2\right)$ with v = 2.40, n = 2 and x = 0.50: $2.40^2=4\left(a^2-0.50^2\right)$ $a^2=\frac{2.40^2}{4}+0.50^2$ $a=\sqrt{1.69}$ $=1.3\text{ (m)}$ Let the maximum speed be $v_{\max}$, which occurs at the centre of motion $\left(\text{where }\ddot{x}=0\right)$. Using $v_{\max}=na$ with $n=2$ and $a=1.30$: $v_{\max}=2\times1.30$ $=2.60\text{ (m/s)}$	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4
Question 6 **D** $\int\frac{8x+1}{x^2+9}dx=4\int\frac{2x}{x^2+9}dx+\int\frac{1}{x^2+9}dx$ $=4\ln\left(x^2+9\right)+\frac{1}{3}\tan^{-1}\frac{x}{3}+c$	MEX–C1 Further Integration MEX12–5 Bands E2–E3

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 7 **C** **C** is correct. When $x=1, y=\frac{1}{3}$, which is not a natural number, and so this counterexample demonstrates that **C** is a false statement. **A** is incorrect. For all *x* there exists a *y* such that $x-y=0$. Consider $y=x$. **B** is incorrect. For all *x* there exists a *y* such that $3x-y=0$. Consider $y=3x$. **D** is incorrect. For example, $x=2$ and $y=6$.	MEX–P1 The Nature of Proof MEX12–2 Bands E3–E4
Question 8 **D** **D** is correct. In a negation of the statement, the phrases 'for all' and 'there exists' must be interchanged, and '=' must change to '≠'. The negation of the given statement will therefore be: 'There exists odd primes $p<q$ such that for all positive non-primes $r<s, p^2+q^2 \neq r^2+s^2$'. **A**, **B** and **C** are incorrect. These options do not show an accurate negation of the statement.	MEX–V1 The Nature of Proof MEX12–2 Bands E3–E4
Question 9 **D** **D** is correct. Let the square roots of z be z_1 and z_2. $z=r(\cos\theta+i\sin\theta)$ and so $\sqrt{z}=\pm\sqrt{r}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$. Hence, $z_1=\sqrt{r}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$ and $z_2=-\sqrt{r}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$. If z_1 has coordinates (x_1, y_1), then z_2 has coordinates $(-x_1, -y_1)$, where $x_1=\sqrt{r}\cos\frac{\theta}{2}$ and $y_1=\sqrt{r}\sin\frac{\theta}{2}$. Points *C* and *E* satisfy this condition. **A**, **B** and **C** are incorrect. These options do not satisfy the above condition.	MEX–N1 Introduction to Complex Numbers MEX12–4 Bands E3–E4

Answer and explanation	Syllabus content, outcomes and targeted performance bands
Question 10 **A** **A** is correct. As $a = vx$ and $v\dfrac{dv}{dx} = a,$ we have $\dfrac{dv}{dx} = x.$ Thus, $\int \dfrac{dv}{dx}dx = \int x dx$ and so $v = \dfrac{x^2}{2} + c.$ The graph of v against x must be a parabola. **B**, **C** and **D** are incorrect. These options do not show a parabola as required.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4

SECTION II

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 11	
(a) The other root is $\sqrt{5}+i$. The sum of roots is $\left(\sqrt{5}-i\right)+\left(\sqrt{5}+i\right)=2\sqrt{5}$. The product of roots is $\left(\sqrt{5}-i\right)\left(\sqrt{5}+i\right)=6$. So, $p=-2\sqrt{5}$ and $q=6$.	MEX–N1 Introduction to Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution . . . . 2 • Attempts to find the sum and product of roots OR equivalent merit 1
(b) Integration by parts takes the form $\int u\frac{dv}{dx}dx=uv-\int v\frac{du}{dx}dx$. Let $u=\sin^{-1}x$ and $\frac{dv}{dx}=1$. So, $\frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$ and $v=x$. $\int_0^1 \sin^{-1}x\,dx=\left[x\sin^{-1}x\right]_0^1-\int_0^1\frac{x}{\sqrt{1-x^2}}dx$ $=\left(\frac{\pi}{2}-0\right)+\frac{1}{2}\int_0^1\frac{-2x}{\sqrt{1-x^2}}dx$ $=\frac{\pi}{2}+\frac{1}{2}\left[2\sqrt{1-x^2}\right]_0^1$ $=\frac{\pi}{2}+(0-1)$ $=\frac{\pi}{2}-1$	MEX–C1 Further Integration MEX12–5 Bands E2–E3 • Gives the correct solution . . . . 3 • Correctly applies integration by parts OR equivalent merit. . . . . . 2 • Correctly identifies the two functions to be used in integration by parts OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $= x + 3$ with $t = 0, x = 0$ and $v = 3$ $\frac{1}{2}v^2 = \int (x+3)dx$ $v^2 = 2\left(\frac{x^2}{2} + 3x\right) + c$ $= x^2 + 6x + c$ Applying the initial condition $x = 0$ and $v = 3$ gives $c = 9$. $v^2 = x^2 + 6x + 9$ $= (x+3)^2$ $v = \pm(x+3)$ As $v = 3$ when $t = 0$, $v = x + 3$ (taking the positive root as $v > 0$).	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E3 • Gives the correct solution . . . . 2 • Attempts to use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to find an expression for v^2 OR equivalent merit. 1
(ii) $v = \frac{dx}{dt} = x + 3$ with $t = 0, x = 0$ $\frac{dt}{dx} = \frac{1}{x+3}$ $\Rightarrow t = \int \frac{1}{x+3}dx$ $t = \ln(x+3) + d$ $e^{t-d} = x + 3$ $x = Ae^t - 3$, where $A = e^{-d}$ Applying the condition $t = 0, x = 0$ gives $A = 3$. So $x = 3e^t - 3$. *Note: The value of the constant can be determined immediately after integrating.*	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E3 • Gives the correct solution . . . . 2 • Attempts to integrate a correct expression for $\frac{dt}{dx}$ OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(d) Given $\hat{\underset{\sim}{u}} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, form three equations involving x, y and z. $\underset{\sim}{b} \cdot \hat{\underset{\sim}{u}} = 0 \Rightarrow 2x + 3y - z = 0$ (1) $\underset{\sim}{c} \cdot \hat{\underset{\sim}{u}} = 0 \Rightarrow 4x - y + 5z = 0$ (2) $\lvert\hat{\underset{\sim}{u}}\rvert = 1 \Rightarrow x^2 + y^2 + z^2 = 1$ (3) For example, $2 \times (1) - (2)$ gives $y = z$. Substituting $z = y$ into (1) and solving gives $x = -y$. Substituting $y = z$ and $x = -y$ into (3) and solving $3x^2 = 1$ for x gives $x = \frac{1}{\sqrt{3}}$ since $x > 0$. So, $y = -\frac{1}{\sqrt{3}}$ and $z = -\frac{1}{\sqrt{3}}$. Hence, $\hat{\underset{\sim}{u}} = \frac{1}{\sqrt{3}}\left(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)$. As $\underset{\sim}{a} = \sqrt{3}\,\hat{\underset{\sim}{u}}$, $\underset{\sim}{a}$ is perpendicular to both $\underset{\sim}{b}$ and $\underset{\sim}{c}$. Hence, $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are mutually perpendicular.	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . 3 • Finds $\hat{\underset{\sim}{u}} = \frac{1}{\sqrt{3}}\left(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right)$ . . . . 2 • Attempts to form three equations in x, y and z 1

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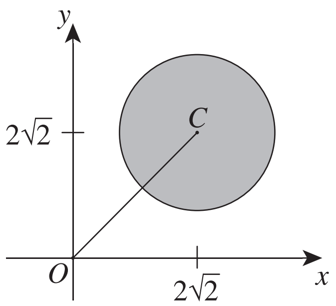
<table>
<tr><th>Sample answer</th><th>Syllabus content, outcomes, targeted performance bands and marking guide</th></tr>
<tr><td>(e) Method 1:
$1-e^{2i\theta}=(1-\cos 2\theta)-i\sin 2\theta.$
$\overline{1-e^{2i\theta}}=(1-\cos 2\theta)+i\sin 2\theta$
$\frac{2}{1-e^{2i\theta}}=\frac{2\left((1-\cos 2\theta)+i\sin 2\theta\right)}{(1-\cos 2\theta)^2+\sin^2 2\theta}$
$=\frac{2\left((1-\cos 2\theta)+i\sin 2\theta\right)}{2(1-\cos 2\theta)}$
$\left(\text{since } \sin^2 2\theta+\cos^2 2\theta=1\right)$
$=1+\frac{i\sin 2\theta}{1-\cos 2\theta}$
$=1+\frac{2i\sin\theta\cos\theta}{1-\left(1-2\sin^2\theta\right)}$
(use of double-angle formula)
$=1+\frac{i\cos\theta}{\sin\theta}$
So $z=1+i\cot\theta$.
Method 2:
$\frac{2}{1-e^{2i\theta}}=\frac{2}{e^{i\theta}\left(e^{-i\theta}-e^{i\theta}\right)}$
$=\frac{2e^{-i\theta}}{e^{-i\theta}-e^{i\theta}}$
$=\frac{2(\cos\theta-i\sin\theta)}{(\cos\theta-i\sin\theta)-(\cos\theta+i\sin\theta)}$
$=\frac{2(\cos\theta-i\sin\theta)}{-2i\sin\theta}$
$=1-\frac{1}{i}\cot\theta$
$=1+i\cot\theta$</td><td>MEX–N2 Using Complex Numbers
MEX12–4 Bands E2–E4
• Gives the correct solution 3
• Uses $\sin^2 2\theta+\cos^2 2\theta=1$ and appropriate double-angle formulae OR equivalent merit 2
• Multiplies and divides by $\overline{1-e^{2i\theta}}=(1-\cos 2\theta)+i\sin 2\theta$ OR equivalent merit. . . . . 1</td></tr>
</table>

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 12	
(a) The equation of motion has the form $x = a\cos nt$. $a = 4$ and $\frac{2\pi}{n} = \frac{\pi}{3}$, so $n = 6$. $x = 4\cos 6t$ and so $v = -24\sin 6t$. The maximum speed is 24 m/s and so half the maximum speed is 12 m/s. Find the first two values of t such that $12 = 24\sin 6t$. $\sin 6t = \frac{1}{2}$ $6t = \frac{\pi}{6}, \frac{5\pi}{6}$ So, $t = \frac{\pi}{36}, \frac{5\pi}{36}$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 3 • Attempts to formulate the required equation $24\sin 6t = 12$ . . . . 2 • Finds $a = 4$ and $n = 6$ 1
(b) (i) The equation of motion for particle *A* is: $5mg - T = \frac{5mg}{4}$ $T = 5mg - \frac{5mg}{4}$ $= \frac{15mg}{4}$ (newtons)	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 2 • Attempts to apply Newton's second law OR equivalent merit. 1
(ii) The equation of motion for particle *B* is: $T - kmg = \frac{kmg}{4}$ $\frac{15mg}{4} = kmg + \frac{kmg}{4}$ $\frac{15mg}{4} = \frac{5kmg}{4}$ Therefore, $k = 3$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 2 • Attempts to apply Newton's second law OR equivalent merit. 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) Firstly consider the descending motion of particle A: $\ddot{x} = \frac{g}{4}$ $\dot{x} = \frac{gt}{4} + c_1$ When $t = 0, \dot{x} = 0$ and so $c_1 = 0$. Hence, $\dot{x} = \frac{gt}{4}$. When $t = 1, \dot{x} = \frac{g}{4}$ and so particle A impacts the floor at $\frac{g}{4}$ m/s. $x = \frac{gt^2}{8} + c_2$ When $t = 0, x = 0$ and so $c_2 = 0$. Hence, $x = \frac{gt^2}{8}$. When $t = 1, x = \frac{g}{8}$ and so particle A descends $\frac{g}{8}$ metres before impacting the floor. Let s_1 be the initial height of particle A above the floor. So, $s_1 = \frac{g}{8}$ (m). Let h be the greatest height reached by particle B above the floor. $h = 2s_1 + s_2$, where s_2 is the distance particle B travels under gravity. (continues on page 12)	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E3–E4 • Gives the correct solution .... 3 • Attempts to find the distance particle B travels under gravity .... 2 • Attempts to find the velocity of particle A when it impacts the floor AND its initial height 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) (continued) Now consider the ascending motion of particle B under gravity: $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -g$ $\frac{1}{2}v^2 = -gx + d$ When $x = 0, v = \frac{g}{4}$ and so $d = \frac{g^2}{32}$. So, $\frac{1}{2}v^2 = -gx + \frac{g^2}{32}$. When $v = 0, x = s_2$ and so $s_2 = \frac{g}{32}$ (m). Substituting $s_1 = \frac{g}{8}$ and $s_2 = \frac{g}{32}$ into $h = 2s_1 + s_2$ gives: $h = 2\left(\frac{g}{8}\right) + \frac{g}{32}$ $= \frac{9g}{32}$ So, the greatest height reached by particle B is $\frac{9g}{32}$ m.	
(c) (i) 	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Correctly sketches the relation 1

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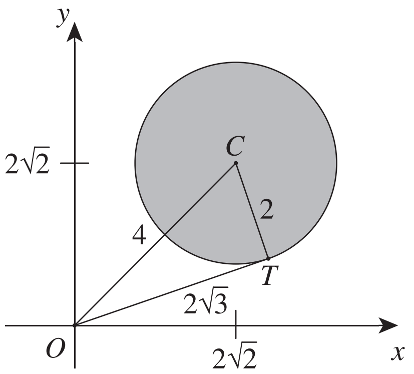
Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) Let d be the distance from O to the centre of the circle C. $d = \sqrt{\left(2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2}$ $= 4$ The radius of the circle is 2. So, the minimum value of $\lvert z \rvert$ is $4 - 2 = 2$ and the maximum value of $\lvert z \rvert$ is $4 + 2 = 6$.	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Finds the minimum AND maximum value of $\lvert z \rvert$ . . . . 2 • Finds the minimum OR maximum value of $\lvert z \rvert$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) OC makes an angle of $\frac{\pi}{4}$ with the x-axis. **Either:** $\sin\angle COT = \frac{2}{4} = \frac{1}{2}$ $\Rightarrow \angle COT = \frac{\pi}{6}$ **Or:** Form a right-angled triangle with OC, the radius of the circle, and T, the point of tangency to the circle. $OT = \sqrt{4^2 - 2^2} = 2\sqrt{3}$ $\tan\angle COT = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$ $\Rightarrow \angle COT = \frac{\pi}{6}$ Then:  So, the minimum value of $\text{Arg}\, z$ is $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$. By similar considerations, the maximum value of $\text{Arg}\, z$ is $\frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$.	MEX–N2 Using Complex Numbers MEX12–4 Bands E3–E4 • Finds the minimum AND maximum value of Arg z . . . . 2 • Finds the minimum OR maximum value of Arg z 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 13	
(a) (i) If x is even, then x^2-6x+5 is odd.	MEX–P1 The Nature of Proof MEX12–2 Bands E2–E3 • Gives the correct contrapositive . . . 1
(ii) Suppose that x is even. Let $x=2k$ for integer k. Substituting $x=2k$ into x^2-6x+5 gives $(2k)^2-6(2k)+5$. $(2k)^2-6(2k)+5=4k^2-12k+5$ $=2(2k^2-6k+2)+1$ Therefore, $x^2-6x+5=2b+1$, where b is the integer $2k^2-6k+2$. Hence, x^2-6x+5 is odd and the statement is proven by proving the contrapositive.	MEX–P1 The Nature of Proof MEX12–2 Bands E2–E3 • Proves the contrapositive. . . . . 2 • Substitutes $x=2k$ into x^2-6x+5 AND attempts to express it in the form $2b+1$ 1

	Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b)	(i) Let $\overrightarrow{OP}$ be the point of intersection of $\overrightarrow{OC}$ and $\overrightarrow{MB}$. $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$ $= \underset{\sim}{a} + \underset{\sim}{b}$ The equation of line OC is $\underset{\sim}{r} = \lambda\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$. The equation of line MB is $\underset{\sim}{r} = \overrightarrow{OM} + \mu\left(\overrightarrow{OB} - \overrightarrow{OM}\right)$. $\underset{\sim}{r} = \frac{1}{5}\underset{\sim}{a} + \mu\left(-\frac{1}{5}\underset{\sim}{a} + \underset{\sim}{b}\right)$ Equating: $\lambda\underset{\sim}{a} + \lambda\underset{\sim}{b} = \frac{1}{5}\underset{\sim}{a} - \frac{\mu}{5}\underset{\sim}{a} + \mu\underset{\sim}{b}$ $\lambda = \frac{1-\mu}{5}$ and $\lambda = \mu \Rightarrow \lambda = \frac{1}{6}$ The point of intersection is $\overrightarrow{OP} = \frac{1}{6}\left(\underset{\sim}{a} + \underset{\sim}{b}\right)$. So, $\overrightarrow{OP} = \frac{1}{6}\overrightarrow{OC}$ and P is a common point of OC and MB. Hence, P lies on OC.	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E4 • Gives the correct solution . . . . 3 • Equates equations of lines OC and MB. AND • Attempts to solve. OR • Attempts to write $\overrightarrow{OP}$ as a multiple of $\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b}$. OR • Equivalent merit . . . . 2 • Finds equation of line OC. OR • Writes $\overrightarrow{OP}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{MB}$. OR • Equivalent merit 1
	(ii) $\overrightarrow{OP} = \frac{1}{6}\overrightarrow{OC}$ $OP : PC = 1 : 5$	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E4 • Gives the correct solution 1
(c)	(i) $e^{in\theta} + e^{-in\theta} = \left(\cos n\theta + i\sin n\theta\right) + \left(\cos n\theta - i\sin n\theta\right)$ $= 2\cos n\theta$	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E3 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $\left(e^{i\theta}+e^{-i\theta}\right)^5 = e^{5i\theta}+5\left(e^{4i\theta}\right)\left(e^{-i\theta}\right)$ $+10\left(e^{3i\theta}\right)\left(e^{-2i\theta}\right)$ $+10\left(e^{2i\theta}\right)\left(e^{-3i\theta}\right)$ $+5\left(e^{i\theta}\right)\left(e^{-4i\theta}\right)+e^{-5i\theta}$ $=\left(e^{5i\theta}+e^{-5i\theta}\right)+5\left(e^{3i\theta}+e^{-3i\theta}\right)$ $+10\left(e^{i\theta}+e^{-i\theta}\right)$ $=2\cos 5\theta+10\cos 3\theta+20\cos\theta$ Also, $\left(e^{i\theta}+e^{-i\theta}\right)^5 = 2^5\cos^5\theta$. So, $\cos^5\theta=\frac{1}{16}\left(\cos 5\theta+5\cos 3\theta+10\cos\theta\right)$.	MEX–N2 Using Complex Numbers MEX12–4 Bands E2–E4 • Gives the correct solution . . . . 3 • Correctly uses binomial theorem and groups conjugate pairs OR equivalent merit . . . . 2 • Obtains $\left(e^{i\theta}+e^{-i\theta}\right)^5=32\cos^5\theta$ OR attempts to use binomial theorem OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) **Method 1:** $\int_0^{\frac{\pi}{2}} \cos^5 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)d\theta$ $= \frac{1}{16}\left[\frac{1}{5}\sin 5\theta + \frac{5}{3}\sin 3\theta + 10\sin\theta\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{16}\left(\frac{1}{5}\sin\frac{5\pi}{2} + \frac{5}{3}\sin\frac{3\pi}{2} + 10\sin\frac{\pi}{2} - (0+0+0)\right)$ $= \frac{1}{16}\left(\frac{1}{5} - \frac{5}{3} + 10\right)$ $= \frac{8}{15}$ *Note: Consequential on answer to Question 13 part (c)(ii).* **Method 2:** $\int_0^{\frac{\pi}{2}} \cos^5 \theta d\theta = \int_0^{\frac{\pi}{2}} \left(\cos^2\theta\right)^2 \cos\theta d\theta$ $= \int_0^{\frac{\pi}{2}} \left(1 - \sin^2\theta\right)^2 \cos\theta d\theta$ Let $u = \sin\theta$ and so $\frac{du}{d\theta} = \cos\theta$. When $\theta = 0, u = 0$ and when $\theta = \frac{\pi}{2}, u = 1$. $\int_0^1 \left(1-u^2\right)^2 du = \int_0^1 \left(1 - 2u^2 + u^4\right)du$ $= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1$ $= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0 - 0 + 0)$ $= \frac{8}{15}$	MEX–N2 Using Complex Numbers MEX12–7 Bands E2–E4 • Gives the correct solution . . . . 2 • Uses part (c)(ii) and attempts to integrate OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iv) $16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$ $16\cos^5\theta - \cos\theta = \cos 5\theta + 5\cos 3\theta + 9\cos\theta$ Hence, to solve the given equation we solve the equation $16\cos^5\theta - \cos\theta = 0$. $\cos\theta = 0, \pm\frac{1}{2}$ $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ and $\cos\theta = \pm\frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}$ $\theta = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$	MEX–N2 Using Complex Numbers MEX12–7 Bands E2–E4 • Gives the correct solution . . . . 2 • Determines that $16\cos^5\theta - \cos\theta = 0$ 1
Question 14	
(a) (i) **Method 1:** Since squares cannot be negative, $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$. Expanding the LHS gives: $a - 2\sqrt{ab} + b \geq 0$ $a + b \geq 2\sqrt{ab}$ So $\frac{a+b}{2} \geq \sqrt{ab}$. **Method 2:** Consider: $\frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(a + b - 2\sqrt{ab}\right)$ $= \frac{1}{2}\left(\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 - 2\sqrt{ab}\right)$ $= \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2$ ≥ 0 So $\frac{a+b}{2} \geq \sqrt{ab}$.	MEX–P1 The Nature of Proof MEX12–2 Bands E2–E3 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) From the AM–GM inequality: $\frac{a+b}{2} \geq \sqrt{ab}, \frac{b+c}{2} \geq \sqrt{bc}$ and $\frac{c+a}{2} \geq \sqrt{ca}$. Multiply together: $\left(\frac{a+b}{2}\right)\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right) \geq \left(\sqrt{ab}\right)\left(\sqrt{bc}\right)\left(\sqrt{ca}\right)$ $\frac{1}{8}(a+b)(b+c)(c+a) \geq \left(\sqrt{a^2b^2c^2}\right)$ So, $(a+b)(b+c)(c+a) \geq 8abc$.	MEX–P1 The Nature of Proof MEX12–2 Bands E2–E3 • Gives the correct solution 1
(iii) Let $a+b=z, b+c=x$ and $c+a=y$. $a=\frac{y+z-x}{2}, b=\frac{z+x-y}{2}$ and $c=\frac{x+y-z}{2}$. Using the result $(a+b)(b+c)(c+a) \geq 8abc$ with the above substitutions: $zxy \geq 8\left(\frac{y+z-x}{2}\right)\left(\frac{z+x-y}{2}\right)\left(\frac{x+y-z}{2}\right)$ So, $xyz \geq (y+z-x)(z+x-y)(x+y-z)$.	MEX–P1 The Nature of Proof MEX12–2 Bands E3–E4 • Gives the correct solution .... 2 • Correctly uses the result $(a+b)(b+c)(c+a) \geq 8abc$ 1
(b) Assume there exists an $x \in \left[0, \frac{\pi}{2}\right]$ for which $\sin\theta + \cos\theta < 1$. Since $x \in \left[0, \frac{\pi}{2}\right]$, neither $\sin\theta$ nor $\cos\theta$ is negative, so $0 \leq \sin\theta + \cos\theta < 1$. $0^2 \leq (\sin\theta + \cos\theta)^2 < 1^2$ $0^2 \leq \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta < 1^2$ As $\sin^2\theta + \cos^2\theta = 1$, this becomes $0 \leq 1 + 2\sin\theta\cos\theta < 1$. So, $1 + 2\sin\theta\cos\theta < 1$. Hence, $2\sin\theta\cos\theta < 0$. This contradicts the fact that neither $\sin\theta$ nor $\cos\theta$ is negative. So, $\sin\theta + \cos\theta \geq 1$ for $0 \leq \theta \leq \frac{\pi}{2}$.	MEX–P1 The Nature of Proof MEX12–2 Bands E3–E4 • Gives the correct solution .... 3 • Attempts to establish that the assumption leads to $2\sin\theta\cos\theta < 0$ OR equivalent merit .... 2 • Assumes there exists an $x \in \left[0, \frac{\pi}{2}\right]$ for which $\sin\theta + \cos\theta < 1$ OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(c) (i) $\frac{d\dot{x}}{dt} = -k\dot{x}$ Separating variables gives: $\int \frac{1}{\dot{x}} d\dot{x} = -k \int dt$ $\ln \dot{x} = -kt + c_1$ $\dot{x} = Ae^{-kt}$, where $A = e^{c_1}$ When $t = 0, \dot{x} = v_1$ and so $A = v_1$. $\dot{x} = v_1 e^{-kt}$ $x = \int v_1 e^{-kt} dt$ $= -\frac{v_1}{k} e^{-kt} + d_1$ When $t = 0, x = 0$ and so $d_1 = \frac{v_1}{k}$. So, $x = \frac{v_1}{k}\left(1 - e^{-kt}\right)$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 2 • Correctly separates variables, attempts to integrate to find $\dot{x}$ as a function of t and evaluates the constant OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $\dfrac{d\dot{y}}{dt} = -k\dot{y} - g$ Separating variables gives: $\displaystyle\int \frac{1}{\dot{y}+\frac{g}{k}} d\dot{y} = -k\int dt$ $\ln\left(\dot{y}+\frac{g}{k}\right) = -kt + c_2$ $\dot{y}+\frac{g}{k} = Be^{-kt}$, where $B = e^{c_2}$ When $t = 0, \dot{y} = v_2$ and so $B = v_2 + \frac{g}{k}$. $\dot{y} = \left(v_2 + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}$ $y = \displaystyle\int\left(\left(v_2 + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}\right)dt$ $= -\frac{1}{k}\left(v_2 + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}t + d_2$ When $t = 0, y = 0$ and so $d_2 = \frac{1}{k}\left(v_2 + \frac{g}{k}\right)$. $y = -\frac{1}{k}\left(v_2 + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}t + \frac{1}{k}\left(v_2 + \frac{g}{k}\right)$ $= \frac{1}{k}\left(v_2 + \frac{g}{k}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t$ So, $y = \dfrac{kv_2 + g}{k^2}\left(1 - e^{-kt}\right) - \dfrac{g}{k}t$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 3 • Attempts to integrate to find y as a function of t and evaluates the constant OR equivalent merit. . . . . 2 • Correctly separates variables attempts to integrate to find $\dot{y}$ as a function of t and evaluates the constant OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) Rearranging $x = \frac{v_1}{k}\left(1 - e^{-kt}\right)$ gives: $1 - e^{-kt} = \frac{kx}{v_1}$ Solving for t gives: $t = -\frac{1}{k}\ln\left(1 - \frac{kx}{v_1}\right)$ Substituting $t = -\frac{1}{k}\ln\left(1 - \frac{kx}{v_1}\right)$ into y gives: $y = \frac{kv_2 + g}{k^2}\left(1 - \left(1 - \frac{kx}{v_1}\right)\right) + \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_1}\right)$ So, $y = \left(\frac{kv_2 + g}{kv_1}\right)x + \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_1}\right)$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E4 • Gives the correct solution . . . . 2 • Attempts to solve $x = \frac{v_1}{k}\left(1 - e^{-kt}\right)$ for t 1
(iv) Substituting $x = 6, v_1 = v_2 = 10, k = 0.1$ and $g = 9.8$ into y gives: $y = 4.1621...$ (m) Hence, the particle will clear the wall.	MEX–M1 Applications of Calculus to Mechanics MEX12–7 Bands E2–E3 • Gives the correct solution 1
Question 15	
(a) (i) Equating components: $1 + 2\lambda_1 = 4 + \lambda_2$ (1) $2\lambda_1 = -2 + 2\lambda_2$ (2) $2 - 3\lambda_1 = 9 - 2\lambda_2$ (3) (1) – (2) gives: $1 = 6 - \lambda_2 \Rightarrow \lambda_2 = 5$ Substituting $\lambda_2 = 5$ into (1) and solving gives: $1 + 2\lambda_1 = 9 \Rightarrow \lambda_1 = 4$ Substituting $\lambda_1 = 4$ and $\lambda_2 = 5$ into (3) gives: $-10 \neq -1$ Since the equations are inconsistent, the lines l_1 and l_2 do not intersect.	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution . . . . 3 • Correctly finds λ_1 and λ_2 OR equivalent merit. . . . . 2 • Finds at least one equation linking λ_1 and λ_2 OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) $d=\sqrt{\left(3-2\lambda_1+\lambda_2\right)^2+\left(-2-2\lambda_1+2\lambda_2\right)^2+\left(7+3\lambda_1-2\lambda_2\right)^2}$ $=\sqrt{17\lambda_1^2+9\lambda_2^2-24\lambda_1\lambda_2+38\lambda_1-30\lambda_2+62}$ $=\sqrt{\left(9\lambda_2^2-24\lambda_1\lambda_2+16\lambda_1^2+40\lambda_1-30\lambda_2+25\right)+\left(\lambda_1^2-2\lambda_1+1\right)+36}$ $9\lambda_2^2-24\lambda_1\lambda_2+16\lambda_1^2+40\lambda_1-30\lambda_2+25$ $=\left(3\lambda_2\right)^2-2\left(3\lambda_2\right)\left(4\lambda_1+5\right)+\left(4\lambda_1+5\right)^2$ $=\left(3\lambda_2-4\lambda_1-5\right)^2$ So $d=\sqrt{\left(3\lambda_2-4\lambda_1-5\right)^2+\left(\lambda_1-1\right)^2+36}$. *Note: The result* $a^2+b^2+c^2=\left(a+b+c\right)^2-2\left(ab+ac+bc\right)$ *can be used where* $a=3-2\lambda_1+\lambda_2$, $b=-2-2\lambda_1+2\lambda_2$ *and* $c=7+3\lambda_1-2\lambda_2$.	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E4 • Gives the correct solution . . . . 4 • Attempts to factorise an expression for *d* into the required form . . . . 3 • Attempts to find an expression for *d* in expanded form . . . . 2 • Finds a correct expression for *d* in terms of λ_1 and λ_2 1
(iii) $\left(3\lambda_2-4\lambda_1-5\right)^2\geq 0$ and $\left(\lambda_1-1\right)^2\geq 0$ for all λ_1,λ_2. The minimum value of *d* is obtained by setting $3\lambda_2-4\lambda_1-5=0$ and $\lambda_1-1=0$. $\lambda_1-1=0$ $\lambda_1=1$ $3\lambda_2-4\times 1-5=0$ $3\lambda_2=9$ $\lambda_2=3$ So, the minimum value of *d* is 6.	MEX–V1 Further Work with Vectors MEX12–3 Bands E3–E4 • Gives the correct solution 1

		Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
	(iv)	When $\lambda_1 = 1, \underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$ and so the corresponding point is (3, 2, –1). When $\lambda_2 = 3, \underset{\sim}{r} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and so the corresponding point is (7, 4, 3). The points that are the minimum distance apart are (3, 2, –1) and (7, 4, 3).	MEX–V1 Further Work with Vectors MEX12–3 Bands E2–E3 • Gives the correct solution 1
(b)	(i)	The parametric equations are: $x = vt\cos\theta$ (1) $y = vt\sin\theta - \frac{1}{2}gt^2$ (2) From (1), $t = \frac{x}{v\cos\theta}$. Substituting $t = \frac{x}{v\cos\theta}$ into (2) gives: $y = \frac{vx\sin\theta}{v\cos\theta} - \frac{1}{2}g\left(\frac{x^2}{v^2\cos^2\theta}\right)$ So, $y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}$.	MEX–M1 Applications of Calculus to Mechanics MEX12–6 Bands E2–E3 • Gives the correct solution . . . . 2 • Attempts to eliminate t OR equivalent merit. 1

<table>
<tr><th>Sample answer</th><th>Syllabus content, outcomes, targeted performance bands and marking guide</th></tr>
<tr><td>
(ii) $y = x\tan\theta - \dfrac{gx^2}{2v^2\cos^2\theta}$ (1)

Substituting $x = d_1$ and $y = d_2$ into (1) gives:

$d_2 = d_1\tan\theta - \dfrac{gd_1^2}{2v^2\cos^2\theta}$ (2)

Substituting $x = d_2$ and $y = d_1$ into (1) gives:

$d_1 = d_2\tan\theta - \dfrac{gd_2^2}{2v^2\cos^2\theta}$ (3)

Rearranging (2) and (3) gives:

$\dfrac{gd_1^2}{2v^2\cos^2\theta} = d_1\tan\theta - d_2$ (4)

$\dfrac{gd_2^2}{2v^2\cos^2\theta} = d_2\tan\theta - d_1$ (5)

Multiplying (4) by d_2^2, multiplying (5) by d_1^2

and equating the RHSs gives:

$(d_1\tan\theta - d_2)d_2^2 = (d_2\tan\theta - d_1)d_1^2$

Expanding gives:

$d_1d_2^2\tan\theta - d_2^3 = d_1^2d_2\tan\theta - d_1^3$
$\left(d_1d_2^2 - d_1^2d_2\right)\tan\theta = d_2^3 - d_1^3$

(continues on page 27)
</td><td>
MEX–M1 Applications of Calculus to Mechanics
MEX12–6 Bands E3–E4
• Gives the correct solution 4

• Factorises $d_2^3 - d_1^3$ 3

• Attempts to form an equation involving d_1, d_2 and $\tan\theta$ only OR equivalent merit. 2

• Substitutes into the Cartesian equation to form two equations OR equivalent merit. 1
</td></tr>
</table>

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) (continued) Factorising the LHS and applying the result $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$ to the RHS gives: $d_1 d_2 (d_2 - d_1) \tan\theta = (d_2 - d_1)\left(d_2^{\,2} + d_1 d_2 + d_1^{\,2}\right)$ Dividing by $d_1 d_2 (d_2 - d_1)(\neq 0)$ gives: $\tan\theta = \dfrac{d_1^{\,2} + d_1 d_2 + d_2^{\,2}}{d_1 d_2}$ So, $\theta = \tan^{-1}\left(\dfrac{d_1^{\,2} + d_1 d_2 + d_2^{\,2}}{d_1 d_2}\right)$.	

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
Question 16	
(a) Let $P(n)$ be the given proposition. Consider $P(4)$: If $n = 4$, the polygon is a quadrilateral, which has two diagonals. Also, for $n = 4, \frac{1}{2}n(n-3) = \frac{1}{2}(4)(1) = 2$. So, $P(4)$ is true. Suppose $P(n)$ is true for $n = k$. A convex polygon with k vertices has $\frac{1}{2}k(k-3)$ diagonals for $k \geq 4$. It is required to show that $P(k+1)$ is true. That is, a convex polygon with $(k + 1)$ vertices has $\frac{1}{2}(k+1)((k+1)-3) = \frac{1}{2}(k+1)(k-2)$ diagonals. When another vertex is added, we have $\frac{1}{2}k(k-3)$ existing diagonals + $(k - 2)$ extra diagonals (formed from the added vertex to all other vertices except the two adjacent vertices) + 1 diagonal (formerly a side of the polygon that is now a diagonal). Hence, we have $\frac{1}{2}k(k-3)+(k-2)+1$ diagonals. $\frac{1}{2}k(k-3)+(k-2)+1 = \frac{1}{2}(k(k-3)+2k-2)$ $= \frac{1}{2}(k^2-k-2)$ $= \frac{1}{2}(k+1)(k-2)$ Hence, $P(k+1)$ is true. So, $P(n)$ is true for (all integers) $n \geq 4$ by induction.	MEX–P2 Further Proof by Mathematical Induction MEX12–2, 12–8 Bands E2–E4 • Gives the correct solution . . . . 4 • Determines that there are $\frac{1}{2}k(k-3)+(k-2)+1$ diagonals . . . . 3 • Assumes true for $n = k$ and attempts to prove true for $n = k+1$ . . . . 2 • Establishes the initial case $(n = 4)$ 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(b) (i) **Method 1:** $t = \tan\frac{x}{2} \Rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ The identity $\sec^2\frac{x}{2} = 1 + \tan^2\frac{x}{2}$. $t = \tan\frac{x}{2}$ gives $\frac{dt}{dx} = \frac{1}{2}\left(1+t^2\right)$. So, $\frac{dx}{dt} = \frac{2}{1+t^2}$. **Method 2:** $t = \tan\frac{x}{2} \Rightarrow x = 2\tan^{-1} t$ So, $\frac{dx}{dt} = \frac{2}{1+t^2}$.	MEX– C1 Further Integration MEX12–5 Bands E2–E3 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(ii) **Method 1 (starting on the LHS):** $\text{LHS} = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ $= \dfrac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}}$ $= \dfrac{2\sin\frac{x}{2}\cos^2\frac{x}{2}}{\cos\frac{x}{2}}$ $= 2\sin\frac{x}{2}\cos\frac{x}{2}$ $= \sin x$ $= \text{RHS}$ **Method 2 (starting on the RHS):** $\text{RHS} = \sin x$ $= 2\sin\frac{x}{2}\cos\frac{x}{2}$ $= 2\left(\dfrac{t}{\sqrt{1+t^2}}\right)\left(\dfrac{1}{\sqrt{1+t^2}}\right)$ $= \dfrac{2t}{1+t^2}$ As $t = \tan\frac{x}{2}$, $\dfrac{2t}{1+t^2} = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \text{LHS}.$	MEX– C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) When $x = \frac{\pi}{2}, t = 1$ and when $x = 0$, $t = 0$. $\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x}dx = \int_0^1 \frac{\frac{2}{1+t^2}}{1+\frac{2kt}{1+t^2}}dt$ $= \int_0^1 \frac{2}{t^2+2kt+1}dt$ $= \int_0^1 \frac{2}{\left(1-k^2\right)+\left(k+t\right)^2}dt$ $= \frac{2}{\sqrt{1-k^2}}\left[\tan^{-1}\left(\frac{k+t}{\sqrt{1-k^2}}\right)\right]_0^1$ $= \frac{2}{\sqrt{1-k^2}}\left(\tan^{-1}\left(\frac{k+t}{\sqrt{1-k^2}}\right)\right.$ $\left.-\tan^{-1}\left(\frac{k}{\sqrt{1-k^2}}\right)\right)$ $\tan\left(A-B\right) = \frac{\tan A - \tan B}{1+\tan A\tan B}$ $\Rightarrow A - B = \tan^{-1}\left(\frac{\tan A - \tan B}{1+\tan A\tan B}\right)$: (continues on page 32)	MEX– C1 Further Integration MEX12–5 Bands E3–E4 • Gives the correct solution . . . . 4 • Applies $A - B = \tan^{-1}\left(\frac{\tan A - \tan B}{1+\tan A\tan B}\right)$ . . . . 3 • Obtains $\frac{2}{\sqrt{1-k^2}}\left[\tan^{-1}\left(\frac{k+t}{\sqrt{1-k^2}}\right)\right]_0^1$ OR equivalent merit. . . . . 2 • Makes a substitution to convert to a definite integral involving t OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(iii) (continued) $\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x}dx = \frac{2}{\sqrt{1-k^2}} \times \tan^{-1}\left(\frac{\frac{k+1-k}{\sqrt{1-k^2}}}{1+\frac{k(k+1)}{1-k^2}}\right)$ $= \frac{2}{\sqrt{1-k^2}} \times \tan^{-1}\left(\frac{\sqrt{1-k^2}}{1-k^2+k+k^2}\right)$ $= \frac{2}{\sqrt{1-k^2}} \times \tan^{-1}\left(\frac{\sqrt{1-k}\sqrt{1+k}}{1+k}\right)$ So, $\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x} = \frac{2}{\sqrt{1-k^2}}\tan^{-1}\sqrt{\frac{1-k}{1+k}}$.	
(iv) $I_{n+1}+2I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1}x}{2+\sin x}dx + 2\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{2+\sin x}dx$ $= \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1}x+2\sin^n x}{2+\sin x}dx$ $= \int_0^{\frac{\pi}{2}} \frac{\sin^n x(\sin x+2)}{2+\sin x}dx$ So, $I_{n+1}+2I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$.	MEX– C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(v) Using $I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$ with $n = 0$ gives: $I_1 + 2I_0 = \int_0^{\frac{\pi}{2}} 1dx$ $= \frac{\pi}{2}$ $I_0 = \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}dx$ $= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{2}\sin x}dx$ $= \frac{1}{2}\frac{2}{\sqrt{1-\left(\frac{1}{2}\right)^2}}\tan^{-1}\sqrt{\frac{1-\frac{1}{2}}{1+\frac{1}{2}}}$ (using result from part (b)(iii) with $k = \frac{1}{2}$) $= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$ So, $I_0 = \frac{\pi}{3\sqrt{3}}$. Using $I_1 = \frac{\pi}{2} - 2I_0$ with $I_0 = \frac{\pi}{3\sqrt{3}}$ gives $I_1 = \frac{\pi}{2} - 2\left(\frac{\pi}{3\sqrt{3}}\right)$. (continues on page 34)	MEX– C1 Further Integration MEX12–5 Bands E2–E4 • Gives the correct solution . . . . 4 • Correctly uses $I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$ with $n = 1$ OR equivalent merit . . . . 3 • Correctly applies the part (b)(iii) result to find I_0 OR equivalent merit. . . . . 2 • Forms $I_1 + 2I_0 = \frac{\pi}{2}$ OR equivalent merit 1

Sample answer	Syllabus content, outcomes, targeted performance bands and marking guide
(v) (continued) Using $I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ with $n = 1$ gives: $I_2 + 2I_1 = \int_0^{\frac{\pi}{2}} \sin x dx$ $I_2 = -[\cos x]_0^{\frac{\pi}{2}} - 2I_1$ $= -(0-1) - 2\left(\frac{\pi}{2} - 2\left(\frac{\pi}{3\sqrt{3}}\right)\right)$ So, $I_2 = \pi\left(\frac{4\sqrt{3}}{9} - 1\right) + 1.$	