2024
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

CSSA

MARKING GUIDELINES

Mathematics Extension I

Section I
10 marks

Multiple Choice Answer Key

Question	Answer	Outcomes Assessed	Targeted Performance Bands
1	D	ME11-5	E1-E2
2	D	ME12-1	E1-E2
3	B	ME12-5	E1-E2
4	A	ME11-3	E2-E3
5	D	ME11-2	E2-E3
6	B	ME11-5	E2-E3
7	C	ME12-4	E3-E4
8	B	ME12-4	E3-E4
9	A	ME11-2	E3-E4
10	C	ME11-5	E4

Question 1 (1 mark)
Outcomes Assessed: *ME11-5*
Targeted Performance Bands: *E1-E2*

Solution	Mark
The term containing x^3 is $\binom{7}{4}(2x)^3(-3)^4 = 22\,680x^3$ $\therefore$ coefficient of x^3 is 22680 Hence D	1

Question 2 (1 mark)
Outcomes Assessed: *ME12-1*
Targeted Performance Bands: *E1-E2*

Solution	Mark
$$\frac{d}{dx}\left(2\sin^{-1}\frac{x}{2}\right) = 2\times\frac{1}{2}\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} = \frac{2}{\sqrt{4-x^2}}$$ Hence D	1

Question 3 (1 mark)
Outcomes Assessed: *ME12-5*
Targeted Performance Bands: *E1-E2*

Solution	Mark
$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.3\times 0.7}{50}} \approx 0.0648$$ Hence B	1

Question 4 (1 mark)
Outcomes Assessed: *ME11-3*
Targeted Performance Bands: *E2-E3*

Solution	Mark
The graphs of $y=\cos(\sin^{-1}x)$ and $y=\sin(\cos^{-1}x)$ both have their domain restricted to $-1\le x\le 1$, so this eliminates options C and D. Considering $y=\sin^{-1}(\cos x)$, we know that $y=\cos x$ is an even function so the graph in $-\pi\le x\le 0$ is the reflection of the graph in $0\le x\le\pi$; this eliminates option B. Also the range of $y=\cos^{-1}x$ is $0\leqslant y\leqslant\pi$. Hence A	1

Question 5 (1 mark)
Outcomes Assessed: *ME11-2*
Targeted Performance Bands: *E2-E3*

Solution	Mark
$P(x) = (4x^2 - 1)Q(x) + 2x + 3$ $= (2x-1)(2x+1)Q(x) + 2x + 3$ Substitute $x = 0.5$ to find remainder when divided by $(2x-1)$ $P(0.5) = (2(0.5)-1)(2(0.5)+1)Q(0.5) + 2(0.5) + 3$ $P(0.5) = 2(0.5) + 3$ $= 4$ Hence D	1

Question 6 (1 mark)
Outcomes Assessed: *ME11-5*
Targeted Performance Bands: *E2-E3*

Solution	Mark
Place an odd number in any of the eight possible positions. Then the next odd number can be placed in any of the three remaining positions in the same row, and so on for the next two odd numbers. This leaves four positions for the even numbers, giving: $\frac{8 \times 3 \times 2 \times 1 \times 4!}{8!} = \frac{1152}{40320} = \frac{1}{35}$ Hence B	1

Question 7 (1 mark)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E3-E4*

Solution	Mark
The function $f(t) = \cos t + k$ oscillates about an average value of k. If $k > 0$, then over the course of one period (2π), the solution will increase, which eliminates A and B. Furthermore, $\cos t + k > 0$ for $t < \frac{\pi}{2}$, which eliminates D. Consequently, the answer is C. Hence C	1

Question 8 (1 mark)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E3-E4*

Solution	Mark
$y = e^{-x} + ex, \ \ x \geq -1$ interchanging x and y, $\ x = e^{-y} + ey, \ \ y \geq -1$ (inverse function) $\dfrac{dy}{dx}$ of inverse function $= \dfrac{1}{dx/dy}$ $\therefore \ \dfrac{dx}{dy} = -e^{-y} + e$ when $x = 1, \ y = 0$ $\dfrac{dx}{dy} = -1 + e$ $\therefore \ \dfrac{dy}{dx}$ of inverse function at $x = 1$ is $\dfrac{1}{e-1}$ Hence B	1

Question 9 (1 mark)
Outcomes Assessed: *ME11-2*
Targeted Performance Bands: *E3-E4*

Solution	Mark
After dilations $x = 2at$ becomes $x = 4at$ and $y = at^2$ becomes $y = 3at^2$. Solving simultaneously to eliminate t gives $y = \frac{3x^2}{16a}$ $\frac{dy}{dx} = \frac{3x}{8a}$ at $x = a$, $\frac{dy}{dx} = \frac{3}{8}$ Hence A	1

Question 10 (1 mark)
Outcomes Assessed: *ME11-5*
Targeted Performance Bands: *E4*

Solution	Mark
To move from the origin to (m, n) we need $m \times \begin{pmatrix}1\\0\end{pmatrix}$ and $n \times \begin{pmatrix}0\\1\end{pmatrix}$ cards. So there is a total of $m + n$ cards, where $\begin{pmatrix}1\\0\end{pmatrix}$ is repeated m times and $\begin{pmatrix}0\\1\end{pmatrix}$ is repeated n times. Total number of unique combinations is $\frac{(m+n)!}{m!n!}$ Hence C	1

Section II

60 marks

Question 11 (15 marks)

Question 11(a) (3 marks)
Outcomes Assessed: *ME12-3*
Targeted Performance Bands: *E1-E2*

Criteria	Marks
• provides correct solution	3
• factorises the equation correctly and finds two correct solutions	2
• correctly uses the double angle formula for $\sin 2\theta$	1

Sample Answer:

$$\sin 2\theta + \cos\theta = 0$$
$$2\sin\theta\cos\theta + \cos\theta = 0$$
$$\cos\theta\,(2\sin\theta + 1) = 0$$

So either $\cos\theta = 0$ or $\sin\theta = -\frac{1}{2}$

If $\cos\theta = 0, \theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$ If $\sin\theta = -\frac{1}{2}, \theta = \frac{7\pi}{6}$ or $\theta = \frac{11\pi}{6}$

$$\therefore \quad \theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$$

Question 11(b) (3 marks)
Outcomes Assessed: *ME12-2*
Targeted Performance Bands: *E1-E2*

Criteria	Marks
• provides correct solution	3
• obtains $-1 = \sqrt{50}\cos\theta$	2
• evaluates $\underset{\sim}{a} \cdot \underset{\sim}{b} = -1$ correctly or finds correct magnitude of both $\underset{\sim}{a}$ and $\underset{\sim}{b}$	1

Sample Answer:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$
$$-3 + 2 = \sqrt{10}\sqrt{5}\cos\theta$$
$$\cos\theta = \frac{-1}{\sqrt{50}}$$
$$\therefore \quad \theta \approx 98^\circ$$

Question 11(c) (3 marks)
Outcomes Assessed: *ME11-2*
Targeted Performance Bands: *E2*

Criteria	Marks
• provides correct solution	3
• correctly finds the two critical points of the solution	2
• multiplies by the square of the denominator or recognises that $x \neq 1$	1

Sample Answer:

$$\begin{aligned} \frac{1}{x-1} &\geqslant 2, \quad x \neq 1 \\ x-1 &\geqslant 2(x-1)^2 \\ 2(x-1)^2-(x-1) &\leqslant 0 \\ (x-1)(2x-3) &\leqslant 0 \\ \therefore \quad 1 < x &\leqslant \frac{3}{2} \end{aligned}$$

Question 11(d) (3 marks)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	3
• integrates correctly or work of equivalent merit	2
• finds correct limits or integrand in terms of u or work of equivalent merit	1

Sample Answer:

$$\int_0^1 x(1-x)^{10}\,dx \qquad \text{let } u = 1-x \quad \text{when } x=0,\ u=1$$

$$du = -dx \quad \text{when } x=1,\ u=0$$

$$\begin{aligned} \int_0^1 x(1-x)^{10}\,dx &= -\int_1^0 (1-u)u^{10}\,du \\ &= \int_0^1 \left(u^{10}-u^{11}\right) du \\ &= \left[\frac{u^{11}}{11}-\frac{u^{12}}{12}\right]_0^1 \\ &= \frac{1}{11}-\frac{1}{12} \\ &= \frac{1}{132} \end{aligned}$$

Question 11(e) (3 marks)
Outcomes Assessed:* *ME12-4
Targeted Performance Bands:* *E2-E3

Criteria	Marks
• provides correct solution	3
• integrates correctly or work of equivalent merit	2
• obtains correct integral expression for the volume or work of equivalent merit	1

Sample Answer:

The volume of the hyperboloid can be found using symmetry:

$$\begin{aligned} V &= 2\pi \int_0^4 x^2\,dy \\ &= 2\pi \int_0^4 \left(9+y^2\right)\,dy \\ &= 2\pi \left[9y + \frac{y^3}{3}\right]_0^4 \\ &= 2\pi \left(9(4) + \frac{4^3}{3} - (0-0)\right) \\ \therefore \quad V &= \frac{344\pi}{3}\text{ units}^3 \end{aligned}$$

Question 12 (14 marks)

Question 12(a) (2 marks)
Outcomes Assessed:* *ME11-2
Targeted Performance Bands:* *E2

Criteria	Marks
• provides correct solution	2
• finds one of the roots or equivalent merit	1

Sample Answer:

Let the roots be α, $-\alpha$ and β.

$$\begin{aligned} \text{sum of roots}: \alpha - \alpha + \beta &= -1 \\ \beta &= -1 \\ \text{by the factor theorem} \quad P(-1) &= 0 \\ 0 &= (-1)^3 + (-1)^2 - m(-1) - 3 \\ \therefore \quad m &= 3 \end{aligned}$$

Question 12(b) (2 marks)
Outcomes Assessed: *ME12-3*
Targeted Performance Bands: *E2*

Criteria	Marks
• provides correct solution	2
• finds correct R or α or equivalent merit	1

Sample Answer:

Using the compound angle formula to expand,

$$\begin{aligned} R\sin(x+\alpha) &= R(\sin x\cos\alpha + \cos x\sin\alpha) \\ &= R\cos\alpha\sin x + R\sin\alpha\cos x \\ &= 3\sin x + \sqrt{3}\cos x \end{aligned}$$

Equating coefficients, $R\cos\alpha = 3$ and $R\sin\alpha = \sqrt{3}$

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 9+3$$

$$\therefore \quad R^2 = 12$$

$$R = 2\sqrt{3} \quad \text{and} \quad \frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{3}$$

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6} \quad (\alpha \text{ is acute})$$

$$\therefore \quad 3\sin x + \sqrt{3}\cos x = 2\sqrt{3}\sin\left(x + \frac{\pi}{6}\right)$$

Question 12(c) (i) (2 marks)
Outcomes Assessed: *ME12-2*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	2
• finds t in terms of V, or equivalent merit	1

Sample Answer:

The object clears the wall if $y > 15$, when $x = 20$.

$$\text{i.e.}\quad Vt = 20$$
$$\therefore\quad t = \frac{20}{V}$$
$$\text{So}\quad 45 - 5\left(\frac{20}{V}\right)^2 > 15$$
$$V^2 > \frac{400}{6}$$
$$V > \frac{20}{\sqrt{6}}\quad (V > 0),\quad \text{as required}$$

Question 12(c) (ii) (2 marks)
Outcomes Assessed: *ME12-2*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	2
• differentiates to find velocity components or finds $t = 3$	1

Sample Answer:

$$\underset{\sim}{v}(t) = \begin{bmatrix} V \\ -10t \end{bmatrix}$$

object hits ground when $y = 0$, i.e. $45 - 5t^2 = 0$

$$t = 3 \quad (t > 0)$$

$$\text{so}, \quad \underset{\sim}{v}(3) = \begin{bmatrix} 9 \\ -30 \end{bmatrix}$$

$$\therefore \quad \text{impact speed} = \sqrt{9^2 + (-30)^2}$$

$$= \sqrt{981} \text{ m/s}$$

Question 12(d) (3 marks)
Outcomes Assessed: *ME11-2, ME12-4*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	3
• correctly integrates	2
• correct division or work of equivalent merit	1

Sample Answer:

Using polynomial division:

$$\begin{array}{r|l} & x^2 - x + 1 \\ x+1 & x^3 \\ & \underline{x^3 + x^2} \\ & -x^2 \\ & \underline{-x^2 - x} \\ & x \\ & \underline{x + 1} \\ & -1 \end{array}$$

$$\text{so } \int_0^1 \frac{x^3}{x+1}\,dx = \int_0^1 x^2 - x + 1 - \frac{1}{x+1}\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1|\right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{2} + 1 - \ln 2\right) - (0 - \ln 1)$$

$$= \frac{5}{6} - \ln 2$$

Question 12(e) (3 marks)
Outcomes Assessed: *ME12-5, ME12-7*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	3
• correct expression to find the probability in terms of Z	2
• finds both $E(X)$ and $\text{Var}(X)$	1

Sample Answer:

$X \sim \text{Bin}(25, 0.6) \quad \therefore \quad \text{E}(X) = np = (25)(0.6) = 15 \quad$ and $\quad \text{Var}(X) = npq = (25)(0.6)(0.4) = 6$

Since this (discrete) binomial distribution can be approximated by the (continuous) normal distribution, $np = 15$ and $nq = 10$, and applying the continuity correction,

$$\begin{aligned} P(\text{more red balls}) &\approx P(X > 12.5) \\ &\approx P\left(Z > \frac{12.5 - 15}{\sqrt{6}}\right) \\ &\approx P(Z < 1.02) \\ &\approx 0.8461 \end{aligned}$$

Without continuity correction,

$$\begin{aligned} P(\text{more red balls}) &\approx P(X \geqslant 13) \\ &\approx P\left(Z > \frac{13 - 15}{\sqrt{6}}\right) \\ &\approx P(Z < 0.82) \\ &\approx 0.7393 \end{aligned}$$

Question 13 (16 marks)

Question 13(a) (2 marks)

Outcomes Assessed: *ME11-3, ME12-3, ME12-4*

Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides either correct solution	2
• uses products to sums or double angle formula to simplify the integral	1

Sample Answer:

$$\begin{aligned}\int \sin x \sin 2x\,dx &= \frac{1}{2}\int \cos(-x) - \cos 3x\,dx\\ &= \frac{1}{2}\int \cos x - \cos 3x\,dx\\ &= \frac{1}{2}\left[\sin x - \frac{\sin 3x}{3}\right] + c\end{aligned}$$

Alternate solution

$$\begin{aligned}\int \sin x \sin 2x\,dx &= \int 2\sin^2 x \cos x\,dx\\ &= \frac{2\sin^3 x}{3} + c\end{aligned}$$

Question 13(b) (3 marks)
Outcomes Assessed: *ME12-1, ME12-7*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct proof	3
• proves the inductive step, or equivalent merit	2
• establishes the base case, or equivalent merit	1

Sample Answer:

Step 1: Prove the result true for $n = 1$.

$$\text{LHS} = \frac{1}{2}$$
$$\text{RHS} = 2 - \frac{3}{2} = \frac{1}{2}$$

so the statement is true for $n = 1$

Step 2: Suppose the statement is true for some integer $n = k$.

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$$

Step 3: Prove the result is true for $n = k+1$.

i.e. Prove that $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$

$$\begin{aligned}
\text{LHS} &= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} \\
&= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \qquad \text{By Step 2} \\
&= 2 - \frac{2k+4}{2^{k+1}} + \frac{k+1}{2^{k+1}} \\
&= 2 - \frac{2k+4-k-1}{2^{k+1}} \\
&= 2 - \frac{k+3}{2^{k+1}} \\
&= \text{RHS}
\end{aligned}$$

Since the statement is true for $n = k$ and $n = k+1$, by mathematical induction, the statement is true for all integers $n \geq 1$.

Question 13(c) (3 marks)

Outcomes Assessed: *ME12-2, ME12-7*

Targeted Performance Bands: *E3*

Criteria	Marks
• provides correct solution, recognising that $p_1^2 + p_2^2 = r^2$	3
• finds both $\underset{\sim}{p} - \underset{\sim}{a}$ and $\underset{\sim}{p} - \underset{\sim}{b}$ and expands $\lvert \underset{\sim}{p} - \underset{\sim}{a} \rvert^2 + \lvert \underset{\sim}{p} - \underset{\sim}{b} \rvert^2$ or equivalent merit	2
• finds $\underset{\sim}{p} - \underset{\sim}{a}$ or $\underset{\sim}{p} - \underset{\sim}{b}$ or equivalent merit	1

Sample Answer:

$$\underset{\sim}{p} - \underset{\sim}{a} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \begin{pmatrix} -r \\ 0 \end{pmatrix} = \begin{pmatrix} p_1 + r \\ p_2 \end{pmatrix}$$

Similarly,
$$\underset{\sim}{p} - \underset{\sim}{b} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} p_1 - r \\ p_2 \end{pmatrix}$$

Thus
$$\begin{aligned} \lvert \underset{\sim}{p} - \underset{\sim}{a} \rvert^2 + \lvert \underset{\sim}{p} - \underset{\sim}{b} \rvert^2 &= \begin{pmatrix} p_1 + r \\ p_2 \end{pmatrix} \cdot \begin{pmatrix} p_1 + r \\ p_2 \end{pmatrix} + \begin{pmatrix} p_1 - r \\ p_2 \end{pmatrix} \cdot \begin{pmatrix} p_1 - r \\ p_2 \end{pmatrix} \\ &= (p_1 + r)^2 + p_2^2 + (p_1 - r)^2 + p_2^2 \\ &= p_1^2 + 2rp_1 + r^2 + p_1^2 - 2rp_1 + r^2 + p_2^2 \\ &= 2\left(p_1^2 + p_2^2\right) + 2r^2 \end{aligned}$$

Since $\underset{\sim}{p}$ lies on the circle, $p_1^2 + p_2^2 = r^2$

$$\therefore \quad \lvert \underset{\sim}{p} - \underset{\sim}{a} \rvert^2 + \lvert \underset{\sim}{p} - \underset{\sim}{b} \rvert^2 = 4r^2, \quad \text{as required.}$$

Question 13(d) (3 marks)
Outcomes Assessed: *ME11-4*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• provides correct solution	3
• correct equation for chain rule	2
• finds $\frac{dV}{dh}$ or equal merit	1

Sample Answer:

$$\frac{dV}{dt} = 0.03\,\text{m}^3/\text{min}$$

since $R = 0.5$, $\quad V = \frac{\pi h^2}{2} - \frac{\pi h^3}{3} \quad$ and $\quad \frac{dV}{dh} = \frac{2\pi h}{2} - \frac{3\pi h^2}{3} = \pi h - \pi h^2$

$$\text{so} \quad \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{\pi h - \pi h^2} \times \frac{3}{100}$$

$$\text{at} \quad h = 0.25, \quad \frac{dh}{dt} = \frac{1}{\pi(0.25) - \pi(0.25)^2} \times \frac{3}{100}$$

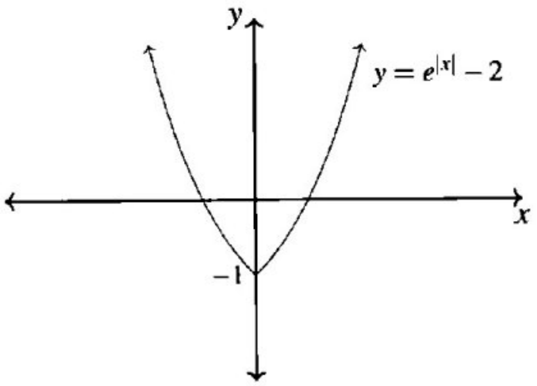
$$= \frac{4}{25\pi}$$

$$\therefore \quad \frac{dh}{dt} = \frac{4}{25\pi} \text{ m/minute.}$$

Question 13(e) (i) (1 mark)
Outcomes Assessed: *ME11-2, ME11-7*
Targeted Performance Bands: *E2*

Criteria	Mark
• provides correct sketch of $y = f(\|x\|)$ with cusp clearly indicated	1

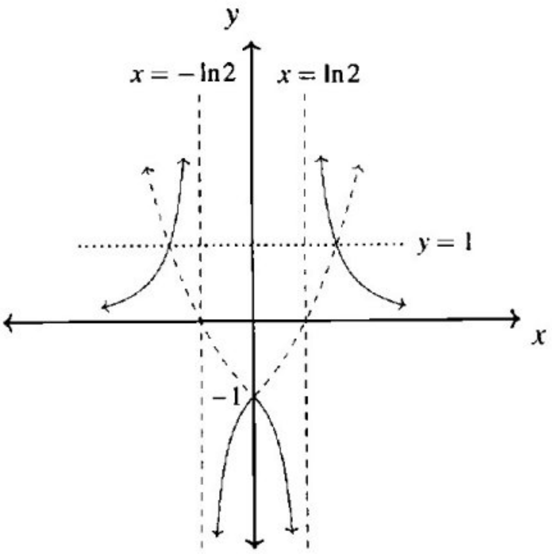
Sample Answer:



Question 13(e) (ii) (2 marks)
Outcomes Assessed: *ME11-2, ME11-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• provides a correct sketch of $y = \dfrac{1}{f(\|x\|)}$, with important features indicated	2
• indicates the asymptotes correctly, or equivalent merit	1

Sample Answer:



Question 13(e) (iii) (2 marks)
Outcomes Assessed: *ME11-2*
Targeted Performance Bands: *E4*

Criteria	Marks
• provides all solutions	2
• finds at least one solution or some progress towards an algebraic solution	1

Sample Answer:

The functions intersect at $y = \pm 1$ and since they are even functions we consider the case $x > 0$.

$$\begin{aligned} \text{Solving} \quad e^x - 2 &= \pm 1 \\ e^x &= 3,\ 1 \\ \therefore \quad x &= 0 \text{ or } \ln 3 \\ \text{so by symmetry} \quad x &= 0, \pm \ln 3 \end{aligned}$$

Alternate algebraic solution: for $x \geqslant 0$, $e^x - 2 = \dfrac{1}{e^x - 2}$

$$\begin{aligned} e^{2x} - 4e^x + 3 &= 0 \\ (e^x - 1)(e^x - 3) &= 0 \\ e^x &= 1,\ 3 \\ \therefore \quad x &= 0,\ \ln 3 \end{aligned}$$

for $x < 0$, $e^{-x} - 2 = \dfrac{1}{e^{-x} - 2}$

$$\begin{aligned} e^{-2x} - 4e^{-x} + 3 &= 0 \\ 3e^{2x} - 4e^x + 3 &= 0 \\ (3e^x - 1)(e^x - 1) &= 0 \\ 3e^x &= 1, \text{or } e^x = 1 \\ \therefore \quad x &= \ln \tfrac{1}{3} \quad (x = 0 \text{ is not in this domain}) \\ x &= -\ln 3 \end{aligned}$$

$\therefore$ solutions are $x = 0,\ \pm \ln 3$

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Question 14 (15 marks)

Question 14(a) (i) (2 marks)
Outcomes Assessed: *ME12-2*
Targeted Performance Bands: *E2*

Criteria	Marks
• provides correct solution	2
• finds $\overrightarrow{AB}$ or gives the correct formula for $\text{proj}_{\overrightarrow{AB}}\overrightarrow{AO}$	1

Sample Answer:

$$\overrightarrow{AO} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad \text{and} \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \text{proj}_{\overrightarrow{AB}}\overrightarrow{AO} &= \frac{\overrightarrow{AO} \cdot \overrightarrow{AB}}{|\overrightarrow{AB}|^2}\overrightarrow{AB} \\ &= \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix} \end{aligned}$$

Question 14(a) (ii) (2 marks)
Outcomes Assessed: *ME12-2*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	2
• correct vector description to obtain the distance (or distance squared), or equivalent merit	1

Sample Answer:

Let the perpendicular distance of $\overrightarrow{AB}$ from O be d

$$\begin{aligned} d^2 &= |\overrightarrow{OA}|^2 - |\text{proj}_{\overrightarrow{AB}}\overrightarrow{AO}|^2 \\ &= (\sqrt{5})^2 - \frac{1}{25}(\sqrt{25})^2 \\ &= 4 \\ \therefore \quad d &= 2 \end{aligned}$$

Question 14(b) (5 marks)

Outcomes Assessed: *ME11-3, ME12-2*

Targeted Performance Bands: *E4*

Criteria	Marks
• correct solution	5
• integrates correctly	4
• correctly recognises that a double angle formula is required	3
• substitutes and simplifies correctly into the formula for ℓ	2
• identifies the correct limits of integration or finds both $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$	1

Sample Answer:

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = r\sin\theta$$

$$\begin{aligned}
\therefore \quad \ell &= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta \\
&= r\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\
&= r\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\tfrac{\theta}{2}}\, d\theta \\
&= 2r\int_0^{2\pi} \sin\tfrac{\theta}{2}\, d\theta \\
&= 4r\left[-\cos\tfrac{\theta}{2}\right]_0^{2\pi} \\
&= 4r[-(-1) - (-(1))] \\
&= 8r
\end{aligned}$$

Question 14(c) (i) (2 marks)
Outcomes Assessed: *ME12-4, ME12-7*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correctly justifies the value of r	2
• identifies the maximum growth rate is when $P = 500$	1

Sample Answer:

The maximum of $\frac{dP}{dt}$ is when $P = 500$.

$$\begin{aligned}\frac{dP}{dt} &= r(500)(1000-500)\\ &= 250\,000\,r\end{aligned}$$

Under ideal conditions the population increases by 125 $\left(250 \text{ parent pairs } \times \frac{1}{2}\right)$.

$$\begin{aligned}125 &= 250\,000\,r\\ r &= \frac{1}{2000} \quad \text{(as required)}\end{aligned}$$

Question 14(c) (ii) (3 marks)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E4*

Criteria	Marks
• provides correct solution	3
• integrates both sides correctly, or equivalent merit	2
• attempts to separate the variables in the differential equation, or equivalent merit	1

Sample Answer:

$$\text{If} \quad \frac{dP}{dt} = \frac{P}{2000}(1000-P),$$

$$\int \frac{1}{P(1000-P)}\,dP = \int \frac{dt}{2000}$$

$$\int \frac{1}{1000}\left(\frac{1}{P} + \frac{1}{1000-P}\right) dP = \int \frac{1}{2000}\,dt$$

$$\int \left(\frac{1}{P} + \frac{1}{1000-P}\right) dP = \int \frac{dt}{2}$$

$$\ln|P| - \ln|1000-P| = \frac{t}{2} + C$$

$$\ln\left|\frac{P}{1000-P}\right| = \frac{t}{2} + C$$

Since $0 < P(0) < 1000$,we may drop the absolute values and thus

$$\frac{P}{1000-P} = Ae^{0.5t}, \quad \text{where } A = e^C.$$

When $t = 0,\ P = 200,$ so $A = \frac{200}{800} = \frac{1}{4}$

$$\therefore \quad \frac{P}{1000-P} = \frac{e^{0.5t}}{4}$$

$$4P = (1000-P)e^{0.5t}$$

$$P(4+e^{0.5t}) = 1000e^{0.5t}$$

$$P = \frac{1000e^{0.5t}}{4+e^{0.5t}} \times \frac{e^{-0.5t}}{e^{-0.5t}}$$

$$= \frac{1000}{1+4e^{0.5t}}$$

Question 14(c) (iii) (1 mark)
Outcomes Assessed: *ME12-4*
Targeted Performance Bands: *E3*

Criteria	Mark
• correct solution	1

Sample Answer:

$$\begin{aligned}
\text{If}\quad P &= 900, \\
900 &= \frac{1000}{1+4e^{-0.5t}} \\
1+4e^{-0.5t} &= \frac{10}{9} \\
e^{-0.5t} &= \frac{1}{36} \\
-\frac{t}{2} &= \ln\frac{1}{36} \\
t &= 2\ln 36 \\
&\approx 7.167038
\end{aligned}$$

It will take approximately 7 years.