2024

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MARKING GUIDELINES

Mathematics Extension 2

Section I
10 marks

Multiple Choice Answer Key

Question	Answer	Outcomes Assessed	Targeted Performance Bands
1	C	MEX12-2	E1-E2
2	D	MEX12-6	E1-E2
3	D	MEX12-3	E1-E2
4	B	MEX12-4	E3
5	D	MEX12-4	E3
6	A	MEX12-6	E2-E3
7	B	MEX12-4	E3-E4
8	A	MEX12-2, MEX12-4	E2-E3
9	C	MEX12-3	E3-E4
10	A	MEX12-5	E4

Question 1 (1 mark)
***Outcomes Assessed:** MEX12-2*
***Targeted Performance Bands:** E1-E2*

Solution	Mark
X: Violet likes sleeping Y: Violet likes crawling $$\neg(X \wedge Y) \Leftrightarrow (\neg X \vee \neg Y)$$ Thus the negation would be *Violet does not like sleeping or does not like crawling.* Hence C	1

Question 2 (1 mark)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E1-E2*

Solution	Mark
At endpoints of its oscillation $v = 0$. That is $n^2\left((x-c)^2 - a^2\right) = 0$. So $(x-c)^2 = a^2$. Hence D	1

Question 3 (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E1-E2*

Solution	Mark
A, B and C are collinear iff $\underset{\sim}{c} - \underset{\sim}{a} = \lambda(\underset{\sim}{b} - \underset{\sim}{a})$ for some $\lambda \in \mathbb{R}$. $\underset{\sim}{b} - \underset{\sim}{a} = -\underset{\sim}{i} + (m-1)\underset{\sim}{j} + 2\underset{\sim}{k}$ $\underset{\sim}{c} - \underset{\sim}{a} = -4\underset{\sim}{i} - 8\underset{\sim}{j} + (n+3)\underset{\sim}{k}$ Now, to equate $\underset{\sim}{i}$ components set $\lambda = 4$: $4(\underset{\sim}{b} - \underset{\sim}{a}) = -4\underset{\sim}{i} + (4m-1)\underset{\sim}{j} + 8\underset{\sim}{k}$ Equating $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components: $4m - 4 = -8 \Rightarrow m = -1$ $n + 3 = 8 \Rightarrow n = 5$ Hence D	1

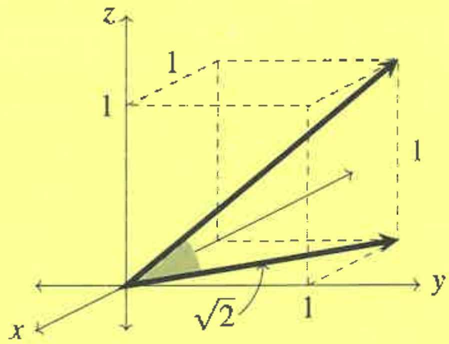
Question 4 (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E3*

Solution	Mark
Let $z = 3\sqrt{3} - 3i$. $\text{Arg}(z) = \tan^{-1}\frac{-3}{3\sqrt{3}} = -\frac{\pi}{6}$. $\lvert z\rvert = \sqrt{\left(3\sqrt{3}\right)^2 + (-3)^2} = 6$. So $3\sqrt{3} - 3i = 6e^{-\frac{i\pi}{6}}$ Hence B	1

Question 5 (1 mark)
***Outcomes Assessed:** MEX12-4*
***Targeted Performance Bands:** E3*

Solution	Mark
The complex number $-3-4i$ is in Quadrant III. By De Moivre's Theorem, halving the principal argument of $-3-4i$ indicates one of the roots of $-3-4i$ is in Quadrant IV. Hence the other root must be diagonally opposite in Quadrant II. Alternatively: $\sqrt{-3-4i} = a+ib$ $-3-4i = a^2-b^2+2abi$ Equating Re and Im gives $a^2-b^2=-3$ and $ab=-2 \Rightarrow a=\pm 1$ and $b=\mp 2$. So the square roots are $1-2i$ (Quadrant IV) and $-1+2i$ (Quadrant II). Hence D	1

Question 6 (1 mark)
***Outcomes Assessed:** MEX12-6*
***Targeted Performance Bands:** E2-E3*

Solution	Mark
Without air resistance, the path of flight is a parabola. By symmetry, the angle of impact equals the angle of projection. With respect to the direction vector, the vertical displacement is 1 unit against a horizontal displacement of $\sqrt{1^2+(-1)^2}=\sqrt{2}$ units. So the angle of impact is $\tan^{-1}\frac{1}{\sqrt{2}} \approx 35$ degrees.  Hence A	1

Question 7 (1 mark)
***Outcomes Assessed:** MEX12-4*
***Targeted Performance Bands:** E3-E4*

Solution	Mark
$(1+i\tan\theta)^n+(1-i\tan\theta)^n = \left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n+\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n$ $= \frac{\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta}{(\cos\theta)^n}$ by De Moivre's Thm $= \frac{2\cos n\theta}{(\cos\theta)^n}$ Hence B	1

Question 8 (1 mark)
***Outcomes Assessed:** MEX12-2, MEX12-4*
***Targeted Performance Bands:** E2-E3*

Solution	Mark
For polynomials with real coefficients, roots occur in complex conjugate pairs. To be clear the set of complex numbers includes the set of real numbers. Further, the complex conjugate of a real number is itself. A claims the theorem applies to all complex numbers (including reals), which is correct. B claims the theorem only applies to non-zero numbers. OR B claims that if $P(0)=0$ then $P(\overline{0})\neq 0$, which is false. $(\overline{0}=0)$ C claims the theorem only applies to real numbers, which is false. D claims the theorem only applies to non-real numbers. OR D claims that if $\alpha\in\mathbb{R}$ then if $P(\alpha)=0$ then $P(\overline{\alpha})\neq 0$, which is false. $(\overline{\alpha}=\alpha)$ Hence A	1

Question 9 (1 mark)
***Outcomes Assessed:** MEX12-3*
***Targeted Performance Bands:** E3-E4*

Solution	Mark
When $t=0$, $\underset{\sim}{r}_A=\underset{\sim}{r}_B=0\underset{\sim}{i}+0\underset{\sim}{j}+0\underset{\sim}{k}$. So only C or D works. When $t=10\pi$, $\underset{\sim}{r}_C=10\pi\cos(10\pi)\underset{\sim}{i}+10\pi\sin(10\pi)\underset{\sim}{j}+0\underset{\sim}{k}=10\pi\underset{\sim}{i}+0\underset{\sim}{j}+0\underset{\sim}{k}$. So C works. When $t=10\pi$, $\underset{\sim}{r}_D=10\pi\sin(10\pi)\underset{\sim}{i}+10\pi\cos(10\pi)\underset{\sim}{j}+0\underset{\sim}{k}=0\underset{\sim}{i}+10\pi\underset{\sim}{j}+0\underset{\sim}{k}$. So D doesn't work. Hence C	1

Question 10 (1 mark)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E4*

<table>
<tr><th>Solution</th><th>Mark</th></tr>
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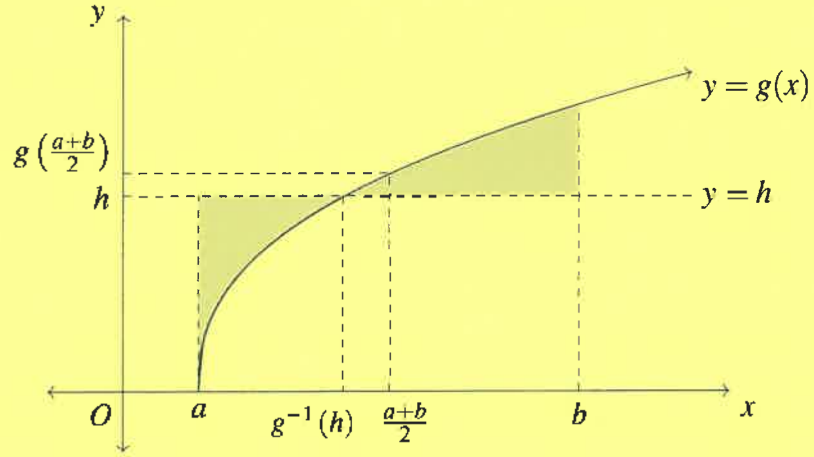
Consider the increasing concave down function $y = g(x)$ on the domain $[a, b]$ below with $\dfrac{a+b}{2}$ and $g\left(\dfrac{a+b}{2}\right)$ marked on the x- and y-axes respectively.



Let $h = \dfrac{1}{b-a}\displaystyle\int_a^b g(x)\,dx$ be the average height of the function over the interval $a \leq x < b$.

Set $y = h$ to be the line where the 'positive' and 'negative' areas between $y = g(x)$ and $y = h$ are equal.
From the graph, it is clear that for an increasing concave down function $y = h$ is below $y = g\left(\dfrac{a+b}{2}\right)$.
Dropping a line from the point of intersection of $y = g(x)$ and $y = h$ to the x-axis gives us $x = g^{-1}(h)$. From the graph, it is clear that $g^{-1}(h) < \dfrac{a+b}{2}$.

Hence A

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Section II

90 marks

Question 11 (15 marks)

Question 11(a) (i) (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1*

Criteria	Mark
• correct solution	1

Sample Answer:

$z - w = 2 - 5i - (-3 - i) = 5 - 4i$

Question 11(a) (ii) (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1-E2*

Criteria	Mark
• correct solution	1

Sample Answer:

$$\frac{z}{w} = \frac{2-5i}{-3-i} \times \frac{-3+i}{-3+i} = \frac{-6+2i+15i+5}{9+1} = -\frac{1}{10} + \frac{17}{10}i$$

Question 11(a) (iii) (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1-E2*

Criteria	Mark
• correct solution	1

Sample Answer:
$z\bar{z} = (2-5i)(2+5i) = 4 + 10i - 10i + 25 = 29$
OR $z\bar{z} = |z|^2 = 2^2 + 5^2 = 29$

Question 11(b) (2 marks)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	2
• setting up the proof w.r.t. a pair of consecutive numbers such as n and $n+1$	1

Sample Answer:

Let the two square numbers be n^2 and $(n+1)^2$, where $n \in \mathbb{Z}$.
$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$, which is odd.

Question 11(c) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E2*

Criteria	Marks
• substitutes limits to evaluate integral	3
• correctly integrates the expression	2
• finds correct partial fraction expression to integrate, or equivalent merit	1

Sample Answer:

$$\begin{aligned}\int_{-1}^{1}\frac{12}{x^2-9}\,dx &= \int_{-1}^{1}\frac{12}{(x-3)(x+3)}\,dx\\ &= \int_{-1}^{1}\left(\frac{2}{x+3}-\frac{2}{x-3}\right)dx\\ &= 2\Big[\ln|x+3|-\ln|x-3|\Big]_{-1}^{1}\\ &= 2\left[\ln\left|\frac{x-3}{x+3}\right|\right]_{-1}^{1}\\ &= 2\left(\ln\left|\frac{-2}{4}\right|-\ln\left|\frac{4}{2}\right|\right)\\ &= 2\ln\left(\frac{1}{4}\right)\\ &= -4\ln 2\end{aligned}$$

If $\dfrac{12}{(x-3)(x+3)} \equiv \dfrac{A}{x-3}+\dfrac{B}{x+3}$
Then $12 \equiv A(x+3)+B(x-3)$
Hence $A=2$, $B=-2$.

Question 11(d) (2 marks)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1*

Criteria	Marks
• provides correct solution	2
• applies De Moivre's theorem, or equivalent merit	1

Sample Answer:

$$\begin{aligned}\left(2e^{i\frac{2\pi}{3}}\right)^4 &= 16e^{i\frac{8\pi}{3}}\\ &= 16e^{i\frac{2\pi}{3}}\\ &= 16\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\\ &= -8+8\sqrt{3}i\end{aligned}$$

Question 11(e) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Criteria	Marks
• provides correct solution	2
• finds the dot product of $\underset{\sim}{a} \cdot \underset{\sim}{b}$, or equivalent merit	1

Sample Answer:

$$\begin{aligned}\cos\theta &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\\ &= \frac{(-1\times 2)+(1\times 4)+(1\times 1)}{\sqrt{1+1+1}\sqrt{4+16+1}}\\ &= \frac{3}{\sqrt{3}\times\sqrt{21}}\\ \text{So } \theta &= \cos^{-1}\left(\frac{3}{\sqrt{63}}\right)\\ &\approx 67.7923\ldots\\ &\approx 68^\circ \quad \text{(to the nearest degree)}\end{aligned}$$

Question 11(f) (3 marks)
Outcomes Assessed: *MEX12-6*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	3
• state and use $\ddot{x} = 0$, or equivalent merit	2
• sets up equation resolving forces	1

Sample Answer:

Assuming down to be positive, resolving forces gives

$$\begin{aligned}F = m\ddot{x} &= mg - 10g - 1000v\\ 80\ddot{x} &= 800 - 100 - 1000v\\ &= 700 - 1000v.\end{aligned}$$

Terminal velocity occurs when $\ddot{x} = 0$, that is when $700 - 1000v = 0$.
So $v = 0.7$ metres per second.

Question 12 (16 marks)

Question 12(a) (2 marks)
Outcomes Assessed: MEX12-2
Targeted Performance Bands: E2-E3

Criteria	Marks
• provides correct proof	2
• set up the proof by contradiction	1

Sample Answer:

Let's assume the result is false, that is a is rational, b is irrational but $a+b$ is rational.
Let $a = \dfrac{m}{n}$ and $a+b = \dfrac{p}{q}$ where m, n, p, and $q \in \mathbb{Z}$, also n and $q \neq 0$.
So $b = \dfrac{p}{q} - \dfrac{m}{n} = \dfrac{pn-qm}{qn}$.
Since p, q, m and n are integers, then $pn-qm$ and qn are integers.
Therefore b is rational, which is a contradiction.
Therefore the sum of a rational number and an irrational number must be irrational.

Question 12(b) (3 marks)
Outcomes Assessed: MEX12-2
Targeted Performance Bands: E2-E3

Criteria	Marks
• provides correct proof	3
• proves the inductive step, or equivalent merit	2
• establishes the base case, or equivalent merit	1

Sample Answer:

Step 1: By definition $u_1 = 4$, and by the formula $u_1 = \frac{1}{20}\left(13\times 5^1 + 15\right) = 4$.
So the result is true for $n = 1$.

Step 2: Suppose the statement is true for $n = k$, that is $u_k = \frac{1}{20}\left(13\times 5^k + 15\right)$.
Prove the statement is true for $n = k+1$. Now, by definition,

$$\begin{aligned} u_{k+1} &= 5u_k - 3 \\ &= 5\times\tfrac{1}{20}\left(13\times 5^k + 15\right) - 3 \\ &= \tfrac{1}{20}\left(13\times 5\times 5^k + 75\right) - \tfrac{60}{20} \\ &= \tfrac{1}{20}\left(13\times 5^{k+1} + 15\right) \end{aligned}$$

Step 3: We have shown that if the result is true for $n = k$ then it is also true for $n = k+1$.
Since the result is true for $n = 1$, using the principle of mathematical induction, the result is true for all positive integers n.

Question 12(c) (i) (1 mark)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E1-E2*

Criteria	Mark
• correct proof	1

Sample Answer:

$$\begin{aligned} z^n + z^{-n} &= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} \\ &= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta) \qquad \text{by De Moivre's Theorem} \\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta \qquad \text{since sine is odd and cosine even} \\ &= 2\cos n\theta \end{aligned}$$

Question 12(c) (ii) (3 marks)
Outcomes Assessed: *MEX12-4*
Targeted Performance Bands: *E2*

Criteria	Marks
• correct solution	3
• finds correct values of $\cos\theta$	2
• correctly substitutes result from (i), or equivalent merit	1

Sample Answer:

$$\begin{aligned} 3\left(z^2 + z^{-2}\right) - \left(z + z^{-1}\right) + 2 &= 0 \\ 3 \times 2\cos 2\theta - 2\cos\theta + 2 &= 0 \qquad \text{from part (i)} \\ 3\cos 2\theta - \cos\theta + 1 &= 0 \\ 3(2\cos^2\theta - 1) - \cos\theta + 1 &= 0 \\ 6\cos^2\theta - \cos\theta - 2 &= 0 \\ (3\cos\theta - 2)(2\cos\theta + 1) &= 0 \end{aligned}$$

Now $\cos\theta = \dfrac{2}{3} \Rightarrow \sin\theta = \pm\dfrac{\sqrt{5}}{3}$ and $\cos\theta = -\dfrac{1}{2} \Rightarrow \sin\theta = \pm\dfrac{\sqrt{3}}{2}$.

So $z = \dfrac{2 \pm i\sqrt{5}}{3}$ or $z = \dfrac{-1 \pm i\sqrt{3}}{2}$.

Question 12(d) (i) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Criteria	Marks
• explicitly shows that $\underset{\sim}{r}_A(2) = \underset{\sim}{r}_B(2)$	2
• finds $t = 2$	1

Sample Answer:

It's simplest to equate k components of the vectors, since they are linear expressions in t.
So $2t - 4 = t - 2 \Rightarrow t = 2$ seconds.
Substituting into $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components confirms $\underset{\sim}{r}_A(2) = \underset{\sim}{r}_B(2) = 4\underset{\sim}{i} + 4\underset{\sim}{j} + 0\underset{\sim}{k}$.

Question 12(d) (ii) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	2
• correctly differentiates $\underset{\sim}{r}_B(t)$	1

Sample Answer:

Differentiating $\underset{\sim}{r}_B(t)$ gives $\underset{\sim}{v}_B(t) = \underset{\sim}{i} + 2t\underset{\sim}{j} + \underset{\sim}{k}$. When $t = 2$, $\underset{\sim}{v}_B(2) = \underset{\sim}{i} + 4\underset{\sim}{j} + \underset{\sim}{k}$.

So the speed of particle B at $t = 2$ is $|\underset{\sim}{v}_B(2)| = \sqrt{1^2 + 4^2 + 1^2} = 3\sqrt{2}$ metres per second.

Question 12(e) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E2-E3*

Criteria	Marks
• correct solution	3
• correct evaluation of integral	2
• sets up integral and attempts integration by parts	1

Sample Answer:

$$M'(t) = te^{-t}$$
$$u = t \Rightarrow du = dt$$
$$dv = e^{-t}dt \Rightarrow v = -e^{-t}$$

Total mass of sand spilled is given by:

$$\begin{aligned} M(5) &= \int_0^5 te^{-t}dt \\ &= \left[-te^{-t}\right]_0^5 - \int_0^5 -e^{-t}dt \\ &= -5e^{-5} - \left[e^{-t}\right]_0^5 \\ &= -5e^{-5} - e^{-5} + 1 \\ &\approx 0.9596 \end{aligned}$$

So approximately 40 kg of sand remains.

Question 13 (15 marks)

Question 13(a) (2 marks)

***Outcomes Assessed:** MEX12-3*

***Targeted Performance Bands:** E2*

Criteria	Marks
• correct solution	2
• completes the square on x and y, or equivalent merit	1

Sample Answer:

$$x^2+y^2+z^2+2x-14y+25=0$$
$$x^2+2x+1+y^2-14y+49+z^2=-25+1+49$$
$$(x+1)^2+(y-7)^2+z^2=25$$

Therefore $\left|\underset{\sim}{r}-\begin{pmatrix}-1\\7\\0\end{pmatrix}\right|=5$

Question 13(b) (3 marks)

***Outcomes Assessed:** MEX12-4, MEX12-8*

***Targeted Performance Bands:** E3*

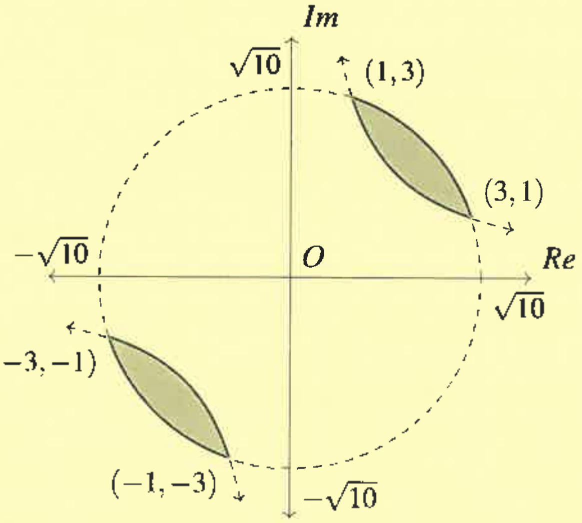
Criteria	Marks
• correct region shaded	3
• correctly sketched both shapes and at least 1 point of intersection, or correctly sketched region without points of intersection	2
• correctly sketched circle or hyperbola	1

Sample Answer:

$|z|\le\sqrt{10}\Rightarrow x^2+y^2\le 10.$
$\mathrm{Im}(z^2)\ge 6\Rightarrow \mathrm{Im}(x^2-y^2+2ixy)\ge 6$
$\Rightarrow xy\ge 3.$
Solving $x^2+y^2=10$ and $xy=3$ gives

$$x^2+\left(\tfrac{3}{x}\right)^2=10$$
$$\left(x^2\right)^2-10x^2+9=0$$
$$\left(x^2-9\right)\left(x^2-1\right)=0$$

So $x=\pm3$ or $x=\pm1$.



So the points of intersection are $3+i$, $-3-i$, $1+3i$, and $-1-3i$.
Note that $xy\ge3$ is equivalent to $y\ge\frac{3}{x}$ when $x\ge0$, but $y\le\frac{3}{x}$ when $x\le0$.

Question 13(c) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	3
• completing the square in the quadratic denominator	2
• progress using a correct algebraic manipulation or a division	1

Sample Answer:

$$\begin{aligned}\int \frac{x^2+4x}{x^2+4x+13}\,dx &= \int \frac{x^2+4x+13-13}{x^2+4x+13}\,dx \\ &= \int\left(1-\frac{13}{x^2+4x+13}\right)dx \\ &= \int\left(1-\frac{13}{(x+2)^2+3^2}\right)dx \\ &= x-\frac{13}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c\end{aligned}$$

Question 13(d) (i) (2 marks)
Outcomes Assessed: *MEX12-3, MEX12-7*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	2
• finding an expression for $\cos\alpha$ or similar.	1

Sample Answer:

Consider the basis vectors $\underset{\sim}{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\underset{\sim}{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\underset{\sim}{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$.

For $\underset{\sim}{v}\neq\underset{\sim}{0}$, $\quad \cos\alpha=\dfrac{\underset{\sim}{v}\cdot\underset{\sim}{i}}{|\underset{\sim}{v}||\underset{\sim}{i}|}=\dfrac{v_1}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}$

Similarly, $\cos\beta=\dfrac{v_2}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}$, and $\cos\gamma=\dfrac{v_3}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}$.

$$\begin{aligned}\text{So, } \cos^2\alpha+\cos^2\beta+\cos^2\gamma &= \left(\frac{v_1}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}\right)^2+\left(\frac{v_2}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}\right)^2 \\ &\quad +\left(\frac{v_3}{\sqrt{{v_1}^2+{v_2}^2+{v_3}^2}}\right)^2 \\ &= \frac{{v_1}^2}{{v_1}^2+{v_2}^2+{v_3}^2}+\frac{{v_2}^2}{{v_1}^2+{v_2}^2+{v_3}^2}+\frac{{v_3}^2}{{v_1}^2+{v_2}^2+{v_3}^2} \\ &= 1\end{aligned}$$

Question 13(d) (ii) (1 mark)
***Outcomes Assessed:** MEX12-3*
***Targeted Performance Bands:** E2-E3*

Criteria	Mark
• correct solution	1

Sample Answer:

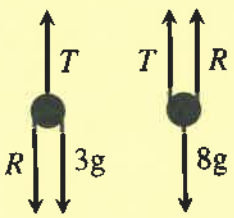
$\cos^2 72° + \cos^2 36° + \cos^2\gamma = 1$. Hence, by calculator, $\cos^2\gamma = \frac{1}{4} \Rightarrow \cos\gamma = \pm\frac{1}{2}$.
Choosing the positive acute solution gives $\gamma = 60°$.
So the angle of elevation is $90° - 60° = 30°$.

Question 13(e) (i) (2 marks)
***Outcomes Assessed:** MEX12-6, MEX12-7*
***Targeted Performance Bands:** E2-E3*

Criteria	Marks
• correct solution	2
• resolves at least one set of forces correctly	1

Sample Answer:

Consider the forces on each mass where $g = 9.8\text{ m/s}^2$ and air resistance $R = 2.5\text{ N}$.



Resolving forces where $F = ma$ gives:

$8a = 8\times 9.8 - T - 2.5$, and
$3a = -3\times 9.8 + T - 2.5$

Adding these gives:
$11a = 5\times 9.8 - 5 \Rightarrow a = 4$
Substituting back gives:
$32 = 8\times 9.8 - T - 2.5$
$\therefore T = 43.9\text{N}$

Question 13(e) (ii) (2 marks)
***Outcomes Assessed:** MEX12-6, MEX12-5*
***Targeted Performance Bands:** E2-E3*

Criteria	Marks
• provides correct solution	2
• integrates correctly to find an equation for displacement	1

Sample Answer:

From part (i), $a = 4$, and working w.r.t. time gives $v = 4t + c_1$.
When $t = 0$, $v = 0$ and hence $c_1 = 0$.
So $x = 2t^2 + c_2$
When $t = 0$, $x = 0$ and hence $c_2 = 0$.

Hence impact at $x = 1.62 \Rightarrow 2t^2 = 1.62$
So $t = \sqrt{0.81} = 0.9\text{ s}$
Substituting into v gives $v = 4\times 0.9 = 3.6$ m/s .

Question 14 (14 marks)

Question 14(a) (2 marks)

***Outcomes Assessed:** MEX12-6, MEX12-3*
***Targeted Performance Bands:** E2-E3*

Criteria	Marks
• correct solution	2
• finds the unit direction vectors of F_A and F_B	1

Sample Answer:

The unit direction vectors of F_A and F_B are $\widehat{F_A} = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix}$ and $\widehat{F_B} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$.

Given the magnitude of F_A is twice that of F_B, the direction of the forces acting together is

$$2\widehat{F_A} + \widehat{F_B} = \frac{2}{\sqrt{6}}\begin{pmatrix}-1\\2\\1\end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{\sqrt{2}}{\sqrt{3}}\begin{pmatrix}-1\\2\\1\end{pmatrix} + \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1-\sqrt{2}\\1+2\sqrt{2}\\1+\sqrt{2}\end{pmatrix}.$$

Note that any multiple of $\begin{pmatrix}1-\sqrt{2}\\1+2\sqrt{2}\\1+\sqrt{2}\end{pmatrix}$ is valid.

Question 14(b) (2 marks)

***Outcomes Assessed:** MEX12-4*
***Targeted Performance Bands:** E3*

Criteria	Marks
• correct solution	2
• correctly realises denominator or equivalent merit	1

Sample Answer:

Write z in Cartesian form as $z = x + iy$.
Since z lies on the unit circle, $x^2 + y^2 = 1$.

$$\text{Then } w = \frac{1}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x+1)-iy}{(x+1)^2 - i^2y^2}$$

$$\text{Thus } \operatorname{Re}(w) = \frac{x+1}{x^2+2x+1+y^2} = \frac{x+1}{2x+2}, \quad (\text{since } x^2+y^2=1) = \frac{1}{2}$$

Question 14(c) (3 marks)
***Outcomes Assessed:** MEX12-4, MEX12-8*
***Targeted Performance Bands:** E3*

Criteria	Marks
• correct solution	3
• finding two arguments for the square root of z	2
• finds $\text{Arg}(z)$ and $\lvert z \rvert$	1

Sample Answer:

$$z = e^{i\frac{\pi}{3}} + 1 = \cos\tfrac{\pi}{3} + 1 + i\sin\tfrac{\pi}{3}$$
$$= \tfrac{3}{2} + i\tfrac{\sqrt{3}}{2}$$

So $\text{Arg}(z) = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ and

$$|z| = \sqrt{\tfrac{9}{4} + \tfrac{3}{4}} = \sqrt{3}$$

Let $w^2 = z$ and by De Moivre's theorem, $2\,\text{Arg}(w) = \frac{\pi}{6}$ or $-\frac{11\pi}{6}$ and $|w| = 3^{1/4}$.

So the two roots are
$3^{1/4}e^{i\pi/12}$ and $3^{1/4}e^{-11i\pi/12}$.

Question 14(d) (i) (1 mark)
***Outcomes Assessed:** MEX12-2*
***Targeted Performance Bands:** E2-E3*

Criteria	Mark
• provides correct proof	1

Sample Answer:

$$(a-3b) \in \mathbb{R} \Rightarrow (a-3b)^2 \geq 0$$
$$a^2 - 6ab + 9b^2 \geq 0$$
$$a^2 + 9b^2 \geq 6ab$$

Question 14(d) (ii) (2 marks)
***Outcomes Assessed:** MEX12-2*
***Targeted Performance Bands:** E3*

Criteria	Marks
• provides correct proof	2
• some progress towards correct result	1

Sample Answer:

From (i): $a^2 + 9b^2 \geq 6ab$
$$a^2 + 9c^2 \geq 6ac$$
$$b^2 + 9c^2 \geq 6bc$$

Adding these inequalities gives

$$2a^2 + 10b^2 + 18c^2 \geq 6(ab + ac + bc)$$
$$a^2 + 5b^2 + 9c^2 \geq 3(ab + ac + bc),$$

as required.

Question 14(e) (i) (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E2*

Criteria	Mark
• correct solution	1

Sample Answer:

$\overrightarrow{OM_1} = \frac{1}{2}\underset{\sim}{p}$. The direction vector of $\underset{\sim}{r}$ is $\overrightarrow{QM_1} = \overrightarrow{OM_1} - \overrightarrow{OQ} = \frac{1}{2}\underset{\sim}{p} - \underset{\sim}{q}$.

Since $\underset{\sim}{r}$ passes through the point Q, the equation is $\underset{\sim}{r} = \underset{\sim}{q} + \lambda\left(\frac{1}{2}\underset{\sim}{p} - \underset{\sim}{q}\right)$ as required.

Question 14(e) (ii) (3 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correct solution	3
• finds μ and λ, or similar progress	2
• finds equation of line through PM_2, or similar progress	1

Sample Answer:

As in part (i), it can be shown that $\underset{\sim}{s}$, the equation for the line through PM_2, is given by

$\underset{\sim}{s} = \underset{\sim}{p} + \mu\left(\frac{1}{2}\underset{\sim}{q} - \underset{\sim}{p}\right)$ for some scalar μ.

To find X, we equate $\underset{\sim}{r}$ and $\underset{\sim}{s}$. Thus $\underset{\sim}{p} + \mu\left(\frac{1}{2}\underset{\sim}{q} - \underset{\sim}{p}\right) = \underset{\sim}{q} + \lambda\left(\frac{1}{2}\underset{\sim}{p} - \underset{\sim}{q}\right)$

Rearranging, we have $\left(1 - \mu - \frac{\lambda}{2}\right)\underset{\sim}{p} - \left(1 - \lambda - \frac{\mu}{2}\right)\underset{\sim}{q} = \underset{\sim}{0}$

However, since the vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are not parallel, this is only possible if both

$1 - \mu - \frac{\lambda}{2} = 0$, AND $1 - \lambda - \frac{\mu}{2} = 0$,
which has the solution $\lambda = \mu = \frac{2}{3}$. Hence $\underset{\sim}{x} = \underset{\sim}{p} + \frac{2}{3}\left(\frac{1}{2}\underset{\sim}{q} - \underset{\sim}{p}\right) = \frac{1}{3}\left(\underset{\sim}{p} + \underset{\sim}{q}\right)$.

Question 15 (15 marks)

Question 15(a) (4 marks)

***Outcomes Assessed:** MEX12-6, MEX12-7*

***Targeted Performance Bands:** E3-E4*

Criteria	Marks
• correct solution	4
• finds correct expression for t as an intergral dx or equivalent merit	3
• makes a justification for their choice of $v < 0$ or $v > 0$ for dx/dt	2
• finds correct expression for v^2 or equivalent merit	1

Sample Answer:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - x$$

$$\frac{1}{2}v^2 = \int (2-x)\,dx$$

$$\frac{1}{2}v^2 = 2x - \frac{x^2}{2} + c_1$$

$$v^2 = 4x - x^2 + c_2$$

When $x = 5$, $v = 0$, so $0 = 20 - 25 + c_2$.

$\therefore c_2 = 5 \Rightarrow v^2 = 4x - x^2 + 5$.

When $x = 5$, $v = 0$ and $a = 2 - x = -3$, so the particle's subsequent motion will be towards the origin, i.e. $v < 0$.

$$v = \frac{dx}{dt} = -\sqrt{4x - x^2 + 5}$$

$$\text{So } \frac{dt}{dx} = \frac{-1}{\sqrt{4x - x^2 + 5}} = \frac{-1}{\sqrt{3^2 - (x-2)^2}}$$

$$\text{So } t = \int \frac{-1}{\sqrt{3^2 - (x-2)^2}}\,dx = -\sin^{-1}\left(\frac{x-2}{3}\right) + c_3$$

When $t = 0$, $x = 5$, so $0 = -\sin^{-1} 1 + c_3$.

$\therefore c_3 = \frac{\pi}{2} \Rightarrow t = -\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{\pi}{2}$.

Rearranging gives $x = 2 - 3\sin\left(t - \frac{\pi}{2}\right)$.

Note, $x = 3\cos t + 2$, $x = 2 - 3\cos(t + \pi)$ and $x = 2 + 3\sin\left(t + \frac{\pi}{2}\right)$ are acceptable also.

Question 15(b) (i) (2 marks)

***Outcomes Assessed:** MEX12-5*

***Targeted Performance Bands:** E3*

Criteria	Marks
• correct solution	2
• progress towards integration by parts	1

Sample Answer:

Applying integration by parts to

$$I_n = \int_1^e \frac{1}{x^2}(\log_e x)^n\,dx$$

$$u = (\log_e x)^n \Rightarrow du = n(\log_e x)^{n-1} \times \frac{1}{x}\,dx$$

$$dv = x^{-2}\,dx \Rightarrow v = -x^{-1}$$

$$\text{So } I_n = \left[-\frac{1}{x}(\log_e x)^n\right]_1^e - \int_1^e -\frac{n(\log_e x)^{n-1}}{x^2}\,dx$$

$$= \left(-\frac{1}{e} + 0\right) + n\int_1^e \frac{(\log_e x)^{n-1}}{x^2}\,dx$$

$$= nI_{n-1} - \frac{1}{e} \text{ as required.}$$

Question 15(b) (ii) (3 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correct solution	3
• correctly iterates to I_0 and evaluates I_0 or equivalent merit	2
• correctly iterates to I_{n-3}	1

Sample Answer:

$$I_n = \int_1^e \frac{1}{x^2}(\log_e x)^n\,dx$$
$$= nI_{n-1} - \frac{1}{e}$$
$$= n\left[(n-1)I_{n-2} - \frac{1}{e}\right] - \frac{1}{e}$$
$$= n(n-1)I_{n-2} - \frac{1}{e}(1+n)$$
$$= n(n-1)\left[(n-2)I_{n-3} - \frac{1}{e}\right] - \frac{1}{e}(1+n)$$
$$= n(n-1)(n-2)I_{n-3} - \frac{1}{e}(1+n+n(n-1))$$

Iterating n times gives

$$I_n = n(n-1)(n-2)\times\cdots\times 1\times I_0 - \frac{1}{e}\left[1+n+n(n-1)+\cdots+\Big(n(n-1)(n-2)\times\cdots\times 2\Big)\right]$$
$$= n!I_0 - \frac{1}{e}\left({}^nP_0 + {}^nP_1 + {}^nP_2 + \cdots + {}^nP_{n-1}\right)$$

But $I_0 = \int_1^e \frac{1}{x^2}\,dx = -\left[\frac{1}{x}\right]_1^e = 1 - \frac{1}{e}$. Then substituting this in gives:

$$I_n = n!\left(1-\frac{1}{e}\right) - \frac{1}{e}\left({}^nP_0 + {}^nP_1 + {}^nP_2 + \cdots + {}^nP_{n-1}\right) \qquad \text{and since } n! = {}^nP_n$$
$$= n! - \frac{1}{e}\left({}^nP_0 + {}^nP_1 + {}^nP_2 + \cdots + {}^nP_{n-1} + {}^nP_n\right) \text{ as required.}$$

Question 15(c) (i) (2 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	2
• finds an expression for l^2 in terms of λ	1

Sample Answer:

Applying vector distance formula gives

$$l^2 = \left|\overrightarrow{AB}\right|^2 = (5-(-1+\lambda))^2 + (3-(1+\lambda))^2 + (-3-(4-\lambda))^2$$
$$= (6-\lambda)^2 + (2-\lambda)^2 + (-7+\lambda)^2$$
$$= 36 - 12\lambda + \lambda^2 + 4 - 4\lambda + \lambda^2 + 49 - 14\lambda + \lambda^2$$
$$= 3\lambda^2 - 30\lambda + 89 \text{ as required.}$$

Question 15(c) (ii) (3 marks)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3*

Criteria	Marks
• correct solution	3
• determining the magnitude of $\overrightarrow{BC}$, or equivalent merit	2
• evaluation of $\overrightarrow{BC}$ using a projection, or equivalent merit	1

Sample Answer:

Project $\overrightarrow{BA} = \begin{pmatrix} -1+\lambda \\ 1+\lambda \\ 4-\lambda \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -6+\lambda \\ -2+\lambda \\ 7-\lambda \end{pmatrix}$ onto the direction vector of $\underset{\sim}{r}_2$: $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

$$\overrightarrow{BC} = \text{proj}_{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} -6+\lambda \\ -2+\lambda \\ 7-\lambda \end{pmatrix} = \frac{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -6+\lambda \\ -2+\lambda \\ 7-\lambda \end{pmatrix}}{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{12-2\lambda+7-\lambda}{4+1} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \frac{19-3\lambda}{5} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$d^2 = l^2 - \left|\overrightarrow{BC}\right|^2 = 3\lambda^2 - 30\lambda + 89 - \frac{(19-3\lambda)^2}{25}\left((-2)^2+1^2\right)$$

$$= 3\lambda^2 - 30\lambda + 89 - \frac{361-114\lambda+9\lambda^2}{5} = \frac{6}{5}(\lambda^2 - 6\lambda + 14) \text{ as required.}$$

Question 15(c) (iii) (1 mark)
Outcomes Assessed: *MEX12-3*
Targeted Performance Bands: *E3-E4*

Criteria	Mark
• correct solution	1

Sample Answer:

Completing the square on the expression for d^2 gives
$d^2 = \frac{6}{5}(\lambda^2 - 6\lambda + 14) = \frac{6}{5}\left((\lambda-3)^2+5\right)$, which is minimum when $\lambda = 3$.

So the minimum distance between $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ is $d = \sqrt{\frac{6}{5}(5)} = \sqrt{6}$ units.

Question 16 (15 marks)

Question 16(a) (2 marks)

Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• provides correct proof	2
• deduces that $m < m+n$, or equivalent merit	1

Sample Answer:

For $n > 1$, the following inequalities are true

$$1 < n+1$$
$$2 < n+2$$
$$\vdots$$
$$m < n+m$$

Multiplying these inequalities gives

$$m! < (n+1)(n+2)(n+3) \times \cdots \times (n+m)$$

Multiplying both sides by $n!$ gives

$$m!n! < n!(n+1)(n+2)(n+3) \times \cdots \times (n+m) = (n+m)! \text{ as required.}$$

Question 16(b) (3 marks)

Outcomes Assessed: *MEX12-6, MEX12-7*
Targeted Performance Bands: *E3-E4*

Criteria	Marks
• correct solution	3
• correct reformulation of integral using parts	2
• correct substitution $\theta = \sqrt{x}$ or $\theta^2 = x$, or equivalent merit	1

Sample Answer:

Let $\theta^2 = x \Rightarrow 2\theta\, d\theta = dx$.

So $\int \cos\sqrt{x}\, dx = 2\int \theta\cos\theta\, d\theta$

Now, let $u = \theta \Rightarrow du = d\theta$

and $dv = \cos\theta d\theta \Rightarrow v = \sin\theta$

$$= 2\left(\theta\sin\theta - \int \sin\theta\, d\theta\right)$$
$$= 2(\theta\sin\theta + \cos\theta) + c$$
$$= 2\left(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right) + c$$

Question 16(c) (5 marks)
Outcomes Assessed: *MEX12-5*
Targeted Performance Bands: *E4*

Criteria	Marks
• correct solution	5
• makes correct substitution into integral, or equivalent merit	4
• finds correct expression for the area, or equivalent merit	3
• finds x-values of the tangent points	2
• sets up simultaneous equation, or equivalent merit	1

Sample Answer:

Solving simultaneously gives $y+(y-k)^2=1 \quad \Rightarrow \quad y^2+(1-2k)y+k^2-1=0$.
This is a quadratic in y with one solution so $\Delta=0$
$\Rightarrow (1-2k)^2-4(k^2-1)=0 \quad \Rightarrow \quad 1-4k+4k^2-4k^2+4=0 \quad \Rightarrow \quad k=\frac{5}{4}$
Substituting this back in gives: $y^2+\left(1-\frac{5}{2}\right)y+\frac{25}{16}-1=0 \quad \Rightarrow \quad y^2-\frac{3}{2}y+\frac{9}{16}=0$
$\Rightarrow \quad \left(y-\frac{3}{4}\right)^2=0$. So $y=\frac{3}{4}$ and $x=\pm\frac{\sqrt{3}}{2}$.
For eqn of semicircle: $x^2+(y-\frac{5}{4})^2=1 \Rightarrow y-\frac{5}{4}=-\sqrt{1-x^2} \Rightarrow y=\frac{5}{4}-\sqrt{1-x^2}$.

$$\text{So Area } = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(\tfrac{5}{4}-\sqrt{1-x^2}-x^2\right)dx$$
$$= 2\int_0^{\frac{\sqrt{3}}{2}} \left(\tfrac{5}{4}-\sqrt{1-x^2}-x^2\right)dx$$
$$= 2\left[\tfrac{5}{4}x-\tfrac{1}{3}x^3\right]_0^{\frac{\sqrt{3}}{2}} - 2\int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\,dx$$

Now, let $x=\sin\theta$, $dx=\cos\theta\, d\theta$

$$= 2\left[\tfrac{5\sqrt{3}}{8}-\tfrac{\sqrt{3}}{8}\right] - 2\int_0^{\frac{\pi}{3}} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta$$

$$\therefore \text{Area } = \sqrt{3}-\int_0^{\frac{\pi}{3}} 2\cos^2\theta\, d\theta$$
$$= \sqrt{3}-\int_0^{\frac{\pi}{3}} (\cos 2\theta+1)\, d\theta$$
$$= \sqrt{3}-\left[\tfrac{1}{2}\sin 2\theta+\theta\right]_0^{\frac{\pi}{3}}$$
$$= \sqrt{3}-\left(\tfrac{\sqrt{3}}{4}+\tfrac{\pi}{3}\right)$$
$$= \tfrac{3\sqrt{3}}{4}-\tfrac{\pi}{3} \text{ units}^2 \approx 0.25 \text{ units}^2$$

Question 16(d) (i) (1 mark)
Outcomes Assessed: *MEX12-2*
Targeted Performance Bands: *E2-E3*

Criteria	Mark
• correct solution	1

Sample Answer:

$$1+\tfrac{1}{\varphi} = 1+\tfrac{2}{1+\sqrt{5}} = \tfrac{3+\sqrt{5}}{1+\sqrt{5}}\times\tfrac{1-\sqrt{5}}{1-\sqrt{5}} = \tfrac{-2-2\sqrt{5}}{-4} = \tfrac{1+\sqrt{5}}{2} = \varphi \text{ as required.}$$

$$1+\tfrac{1}{1-\varphi} = 1+\tfrac{2}{1-\sqrt{5}} = \tfrac{3-\sqrt{5}}{1-\sqrt{5}}\times\tfrac{1+\sqrt{5}}{1+\sqrt{5}} = \tfrac{-2+2\sqrt{5}}{-4} = \tfrac{1-\sqrt{5}}{2} = 1-\varphi \text{ as required.}$$

Question 16(d) (ii) (3 marks)
Outcomes Assessed: MEX12-2
Targeted Performance Bands: E4

Criteria	Marks
• correct solution	3
• correctly using assumption	2
• correctly showing $f^0(1)=1$	1

Sample Answer:

Step 1: Show $f^0(1)=1$.

$$f^0(1)=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^2-\left(\frac{1-\sqrt{5}}{2}\right)^2}{\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}}$$
$$=\frac{\left(1+2\sqrt{5}+5\right)-\left(1-2\sqrt{5}+5\right)}{2\left(1+\sqrt{5}-1+\sqrt{5}\right)}$$
$$=\frac{4\sqrt{5}}{4\sqrt{5}}=1 \text{ as required. So true for } n=0.$$

OR $f^0(1)=\dfrac{\varphi^2-(1-\varphi)^2}{\varphi^1-(1-\varphi)^1}$
$$=\frac{\varphi^2-1+2\varphi-\varphi^2}{2\varphi-1}$$
$$=\frac{2\varphi-1}{2\varphi-1}=1, \text{ as required.}$$

So true for $n=0$.

Step 2: Suppose the statement is true for $n=k$, that is $f^k(1)=\dfrac{\varphi^{k+2}-(1-\varphi)^{k+2}}{\varphi^{k+1}-(1-\varphi)^{k+1}}$.

We are now r.t.p. the statement is true for $n=k+1$, that is $f^{k+1}(1)=\dfrac{\varphi^{k+3}-(1-\varphi)^{k+3}}{\varphi^{k+2}-(1-\varphi)^{k+2}}$.

$$f^{k+1}(1)=f\circ f^k(1)$$
$$=1+\frac{1}{f^k(1)}$$
$$=1+\frac{\varphi^{k+1}-(1-\varphi)^{k+1}}{\varphi^{k+2}-(1-\varphi)^{k+2}} \quad \text{by assumption}$$
$$=\frac{\varphi^{k+2}-(1-\varphi)^{k+2}+\varphi^{k+1}-(1-\varphi)^{k+1}}{\varphi^{k+2}-(1-\varphi)^{k+2}}$$
$$=\frac{\varphi^{k+2}-(1-\varphi)^{k+2}+\varphi^{k+2}\left(\frac{1}{\varphi}\right)-(1-\varphi)^{k+2}\left(\frac{1}{1-\varphi}\right)}{\varphi^{k+2}-(1-\varphi)^{k+2}}$$
$$=\frac{\varphi^{k+2}\left(1+\frac{1}{\varphi}\right)-(1-\varphi)^{k+2}\left(1+\frac{1}{1-\varphi}\right)}{\varphi^{k+2}-(1-\varphi)^{k+2}}$$
$$=\frac{\varphi^{k+3}-(1-\varphi)^{k+3}}{\varphi^{k+2}-(1-\varphi)^{k+2}} \text{ from part (i) as required.}$$

Step 3: We have shown that if the result is true for $n=k$ then it is also true for $n=k+1$. Since the result is true for $n=0$, using the principle of mathematical induction, the result is true for all integers $n\geq 0$.

Question 16(d) (iii) (1 mark)
***Outcomes Assessed:** MEX12-2*
***Targeted Performance Bands:** E3-E4*

Criteria	Mark
• correct solution	1

Sample Answer:

$$
\begin{aligned}
\lim_{n\to\infty} f^n(1) &= \lim_{n\to\infty} \frac{\varphi^{n+2}-(1-\varphi)^{n+2}}{\varphi^{n+1}-(1-\varphi)^{n+1}} \\
&= \lim_{n\to\infty} \frac{\varphi^{n+2}}{\varphi^{n+1}} \qquad \text{as } |1-\varphi|<1,\ \lim_{k\to\infty}(1-\varphi)^k = 0 \\
&= \lim_{n\to\infty} \frac{\varphi}{1} \qquad \text{dividing top and bottom by } \varphi^{n+1} \\
&= \varphi
\end{aligned}
$$