CSSA

2024
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

DO NOT REMOVE PAPER FROM EXAMINATION ROOM

Centre Number

Student Number

Mathematics Extension 2

Morning Session
Monday, 12 August 2024

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- Use the Multiple-Choice Answer Sheet provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks: 100

Section I – 10 marks (pages 2–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7–14)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1–10

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1 What is the negation of the statement "Violet likes sleeping and crawling"?

A. Violet likes sleeping but does not like crawling.

B. Violet does not like sleeping but likes crawling.

C. Violet does not like sleeping or does not like crawling.

D. Violet does not like sleeping and does not like crawling.

2 A particle moves in simple harmonic motion such that the relationship between its velocity v ms^{-1} and its displacement x m is given by the equation

$$v^2 = -n^2\left((x-c)^2 - a^2\right),$$

where n, c, and a are constants and $n > 1$.

Which of the following is true at the endpoints of its oscillation?

A. $n(x-c)^2 = a$

B. $n(x-c)^2 = a^2$

C. $(x-c)^2 = a$

D. $(x-c)^2 = a^2$

3 Points A, B and C are represented by the vectors $\underset{\sim}{a} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 4 \\ m \\ -1 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} 1 \\ -7 \\ n \end{pmatrix}$ respectively.

Which values of m and n will ensure that A, B and C are collinear?

A. $m = 1$ and $n = -5$

B. $m = -1$ and $n = -5$

C. $m = 1$ and $n = 5$

D. $m = -1$ and $n = 5$

4 Which of the following expressions is equivalent to $3\sqrt{3} - 3i$?

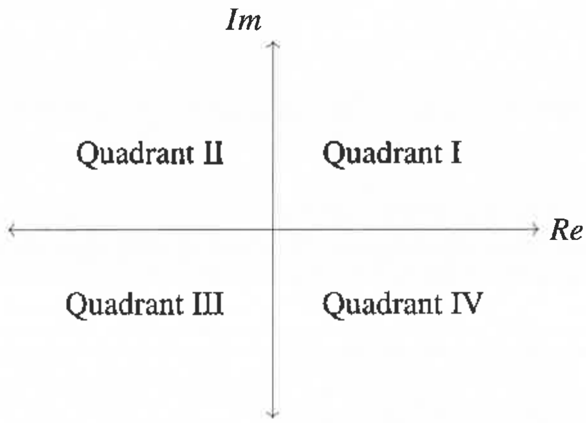
A. $6e^{\frac{5i\pi}{6}}$

B. $6e^{-\frac{i\pi}{6}}$

C. $3\sqrt{2}e^{\frac{i5\pi}{6}}$

D. $3\sqrt{2}e^{-\frac{i\pi}{6}}$

5 For this question we define the quadrants of the Argand diagram to be as follows:



In which quadrants of the Argand diagram do the square roots of $-3 - 4i$ lie?

A. Quadrants I and III

B. Quadrants I and IV

C. Quadrants II and III

D. Quadrants II and IV

6 A projectile is launched from horizontal ground in the direction $-\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$ at a speed of 10 ms^{-1} where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are horizontal and $\underset{\sim}{k}$ is vertical.

Assuming no air resistance, at what angle, to the nearest degree, does the projectile hit the ground?

A. 35 degrees

B. 45 degrees

C. 60 degrees

D. 63 degrees

7 Which of the following expressions is equivalent to $(1+i\tan\theta)^n+(1-i\tan\theta)^n$ for all integers n?

A. $\dfrac{2\cos n\theta}{(\sin\theta)^n}$

B. $\dfrac{2\cos n\theta}{(\cos\theta)^n}$

C. $\dfrac{2\sin n\theta}{(\sin\theta)^n}$

D. $\dfrac{2\sin n\theta}{(\cos\theta)^n}$

8 Given $P(z)$ is a polynomial function with real coefficients, which of the following is true?

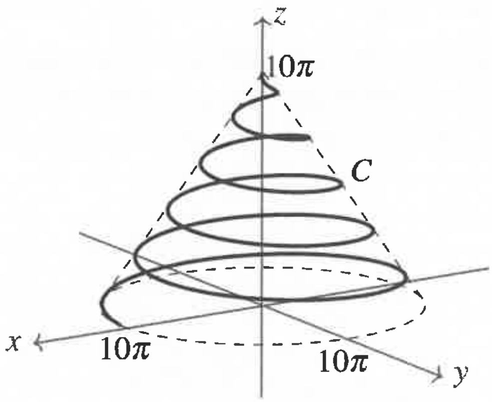
A. $\forall\alpha\in\mathbb{C}, P(\alpha)=0 \implies P(\overline{\alpha})=0$

B. $(P(\alpha)=0 \implies P(\overline{\alpha})=0)$ iff $\alpha\neq 0$

C. $(P(\alpha)=0 \implies P(\overline{\alpha})=0)$ iff $\alpha\in\mathbb{R}$

D. $(P(\alpha)=0 \implies P(\overline{\alpha})=0)$ iff $\alpha\notin\mathbb{R}$

9 A curve C spirals around a cone as shown in the diagram.



A particle moves along C, starting at the point $(0,0,10\pi)$ and ending at the point $(10\pi,0,0)$.

Which of the following vector equations best describes the path of the particle for $0 \le t \le 10\pi$?

A. $\underset{\sim}{r}_A = t\cos(t)\underset{\sim}{i} + t\sin(t)\underset{\sim}{j} + t\underset{\sim}{k}$

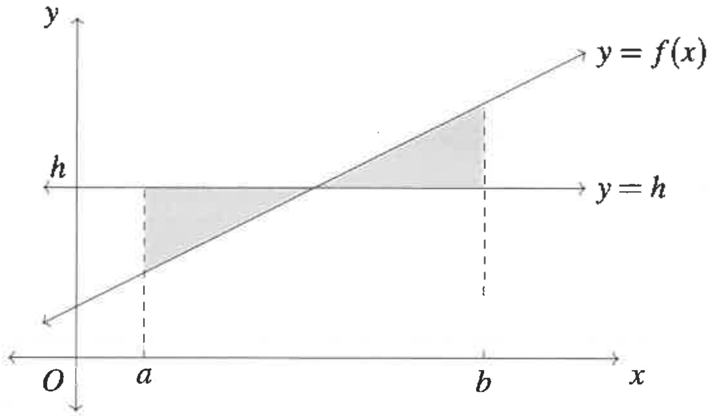
B. $\underset{\sim}{r}_B = t\sin(t)\underset{\sim}{i} + t\cos(t)\underset{\sim}{j} + t\underset{\sim}{k}$

C. $\underset{\sim}{r}_C = t\cos(t)\underset{\sim}{i} + t\sin(t)\underset{\sim}{j} + (10\pi - t)\underset{\sim}{k}$

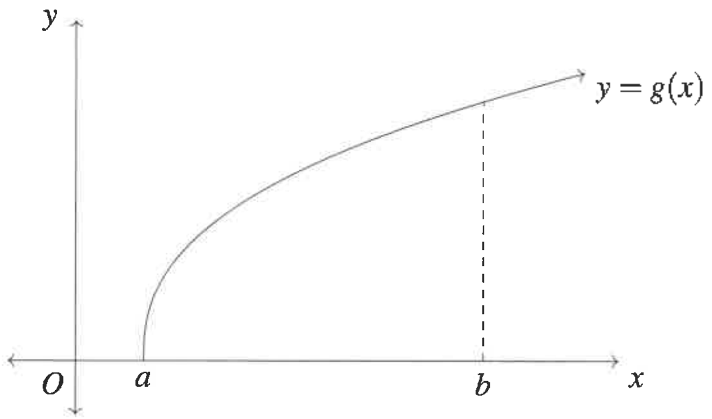
D. $\underset{\sim}{r}_D = t\sin(t)\underset{\sim}{i} + t\cos(t)\underset{\sim}{j} + (10\pi - t)\underset{\sim}{k}$

10 The average height h of a continuous function $y = f(x)$ on the interval $[a, b]$ is given by the formula $h = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Consider the areas between the function $y = f(x)$ and the line $y = h$ on the interval $[a, b]$ as shown in the diagram. Here, h is the average height of $y = f(x)$ on $[a, b]$ if the sum of areas above $y = h$ is equal to the sum of the areas below $y = h$.



Now consider the function $y = g(x)$ in the diagram below which has the property that $g'(x) > 0$ and $g''(x) < 0$ for all $x > a$.



Let h be the average height of $y = g(x)$ on the interval $[a, b]$ where $0 < a < b$. You may want to sketch $y = h$ on the diagram above. This will not be marked.

Which of the following statements is true?

A. $h < g\left(\frac{a+b}{2}\right)$ and $g^{-1}(h) < \frac{a+b}{2}$

B. $h < g\left(\frac{a+b}{2}\right)$ and $g^{-1}(h) > \frac{a+b}{2}$

C. $h > g\left(\frac{a+b}{2}\right)$ and $g^{-1}(h) < \frac{a+b}{2}$

D. $h > g\left(\frac{a+b}{2}\right)$ and $g^{-1}(h) > \frac{a+b}{2}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

Your responses for Questions 11–16 should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Consider $z = 2 - 5i$ and $w = -3 - i$.

Find simplified expressions for the following.

(i) $z - w$ **1**

(ii) $\dfrac{z}{w}$ **1**

(iii) $z\bar{z}$ **1**

(b) Prove algebraically that the difference between two consecutive square numbers is odd. **2**

(c) Find $\displaystyle\int_{-1}^{1} \frac{12}{x^2 - 9}\, dx$. **3**

(d) Given $z = 2e^{i\frac{2\pi}{3}}$, express z^4 in the form $a + ib$. **2**

(e) Find the acute angle between the vectors $\underset{\sim}{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$. Give your answer correct to the nearest degree. **2**

(f) An 80 kg solid metal ball is placed on the surface of a deep lake and released. **3**

The ball experiences acceleration due to gravity g m/s^2. It also experiences a resistive force R newtons proportional to velocity v m/s such that $R = -1000v$, and an upwards buoyancy force B of magnitude $10g$ newtons.

Assuming $g = 10$, calculate the terminal velocity of the ball in the liquid.

Question 12 (16 marks)

(a) Prove by contradiction that if a is rational and b is irrational, then $a+b$ is irrational. **2**

(b) A sequence is defined by $u_1 = 4$, and $u_n = 5u_{n-1} - 3$ for integers $n \geq 1$. Prove by mathematical induction that $u_n = \frac{1}{20}(13 \times 5^n + 15)$, for all positive integers n. **3**

(c) Consider $z = \cos\theta + i\sin\theta$.

(i) Show that $z^n + z^{-n} = 2\cos n\theta$. **1**

(ii) Hence solve $3(z^2 + z^{-2}) - (z + z^{-1}) + 2 = 0$. **3**

(d) The positions of two particles A and B in metres after t seconds are given by the vector equations below.

$$\underset{\sim}{r}_A(t) = t^2\underset{\sim}{i} + (6-t)\underset{\sim}{j} + (2t-4)\underset{\sim}{k}$$

$$\underset{\sim}{r}_B(t) = (t+2)\underset{\sim}{i} + t^2\underset{\sim}{j} + (t-2)\underset{\sim}{k}$$

(i) Find the time when the particles collide. **2**

(ii) Find the speed of particle B at the time the particles collide. **2**

(e) One tonne of sand is being carried around in a truck when the back gate falls open, spilling sand onto the street. The rate at which the mass M tonnes of sand is spilling out of the truck at time t minutes after the gate opens is given by $M'(t) = te^{-t}$ tonnes per minute. **3**

How much sand is left in the truck after 5 minutes? Give your answer correct to the nearest kilogram.

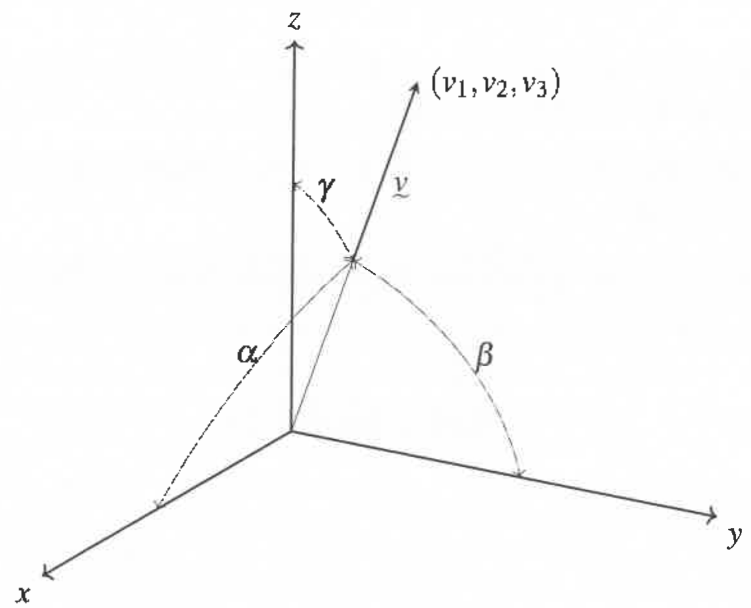
Question 13 (15 marks)

(a) Find the vector equation of the sphere $x^2+y^2+z^2+2x-14y+25=0$. **2**

(b) Given the complex number $z=x+iy$, sketch and shade the subset on the Argand diagram such that $|z| \leq \sqrt{10}$ and $\text{Im}\left(z^2\right) \geq 6$. **3**

(c) Find $\displaystyle\int \frac{x^2+4x}{x^2+4x+13}\,dx$. **3**

(d) The vector $\underset{\sim}{v}$ is represented by the point with coordinates (v_1, v_2, v_3) as shown in the diagram.



Let α, β and γ be the angles between $\underset{\sim}{v}$ and the positive x, y and z axes, respectively.

The direction cosines of a vector are the cosines of the angles between the vector and the three positive coordinate axes. That is, the direction cosines of $\underset{\sim}{v}$ are $\cos\alpha$, $\cos\beta$ and $\cos\gamma$.

(i) Show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. **2**

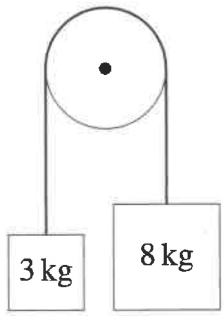
(ii) A vector $\underset{\sim}{r}$ makes an angle of 72° with the positive x-axis and 36° with the positive y-axis. **1**

Find the angle of elevation of $\underset{\sim}{r}$ from the xy plane.

Question 13 continues on page 10

Question 13 (continued)

(e) A light inextensible string passes over a smooth pulley. Particles of mass 3 kg and 8 kg are attached to each end of the string as shown in the diagram below.



The two particles are both held at rest 1.62 metres above the floor. The system is released from rest and each mass now experiences constant air resistance of 2.5 newtons.

The 8 kg particle reaches the floor before the 3 kg particle reaches the pulley. Let the acceleration due to gravity be $9.8\ \text{m/s}^2$.

(i) Show that the tension in the string before the 8 kg mass reaches the floor is 43.9 newtons. **2**

(ii) Determine the speed at which the 8 kg particle hits the floor. **2**

End of Question 13

Question 14 (14 marks)

(a) A stationary object suspended in a vacuum is moved by two forces F_A and F_B where the magnitude of F_A is twice the magnitude of F_B. **2**

If the direction of F_A is $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and the direction of F_B is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, calculate the direction the object moves. Give your answer in vector form.

(b) The complex number $z \neq -1$ lies on the unit circle in the Argand diagram and $w = \dfrac{1}{z+1}$. **2**

Show that $\text{Re}(w) = \frac{1}{2}$.

(c) Find the two square roots of $z = e^{i\frac{\pi}{3}} + 1$. Express your answer in exponential form. **3**

(d) Consider $a, b \in \mathbb{R}$.

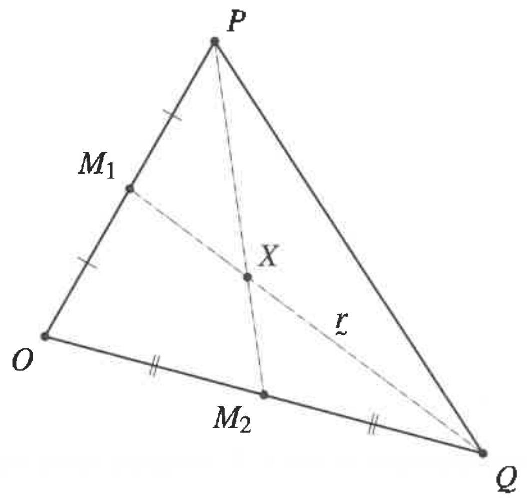
(i) Show that $a^2 + 9b^2 \geq 6ab$. **1**

(ii) Hence show that $a^2 + 5b^2 + 9c^2 \geq 3(ab + bc + ac)$. **2**

Question 14 continues on page 12

Question 14 (continued)

(e) The points P and Q have position vectors $\overrightarrow{OP} = \underset{\sim}{p}$ and $\overrightarrow{OQ} = \underset{\sim}{q}$ respectively, where $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are non-zero and non-parallel.



As shown in the diagram above, OPQ forms a triangle. The midpoints of OP and OQ are M_1 and M_2, respectively.

(i) Show that the vector equation of the line $\underset{\sim}{r}$ that passes through Q and M_1 is given by **1**

$$\underset{\sim}{r} = \underset{\sim}{q} + \lambda\left(\frac{1}{2}\underset{\sim}{p} - \underset{\sim}{q}\right), \text{ for } \lambda \in \mathbb{R}.$$

(ii) The point of intersection of QM_1 and PM_2 is X. Find $\underset{\sim}{x}$, the position vector of X, in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$. **3**

End of Question 14

Question 15 (15 marks)

(a) A particle moves along the x-axis according to the differential equation **4**

$$\frac{d^2x}{dt^2} = 2 - x.$$

Initially the particle is at rest at $x = 5$. Find the displacement of the particle as a function of time.

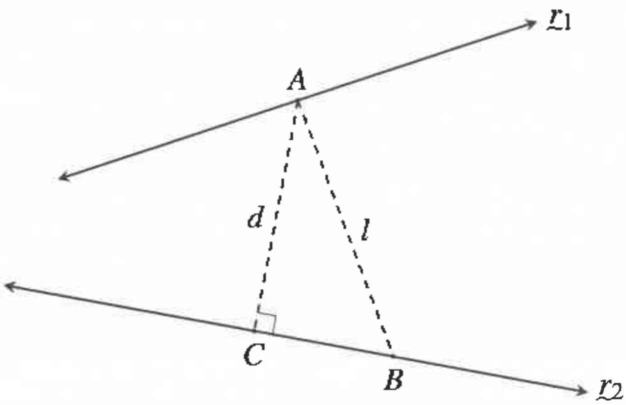
(b) Let $I_n = \int_1^e \frac{1}{x^2} (\log_e x)^n \, dx$ for all integers $n \geq 0$.

(i) Show that $I_n = nI_{n-1} - \frac{1}{e}$ for all integers $n \geq 1$. **2**

(ii) Hence show that $I_n = n! - \frac{1}{e}\left({}^nP_0 + {}^nP_1 + {}^nP_2 + \cdots + {}^nP_n\right)$. **3**

(c) Consider the lines $\underset{\sim}{r_1} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $\underset{\sim}{r_2} = \begin{pmatrix} 5 \\ 3 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda, \mu \in \mathbb{R}$.

Let l represent the distance from the general point $A(-1+\lambda, 1+\lambda, 4-\lambda)$ on $\underset{\sim}{r_1}$ to the point $B(5, 3, -3)$ on $\underset{\sim}{r_2}$ as shown in the diagram.



Let d be the perpendicular distance between the point A and the line $\underset{\sim}{r_2}$.

(i) Show $l^2 = 3\lambda^2 - 30\lambda + 89$. **2**

(ii) Using a projection, or otherwise, show $d^2 = \frac{6}{5}(\lambda^2 - 6\lambda + 14)$. **3**

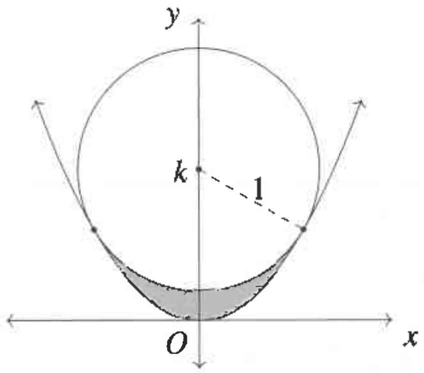
(iii) Hence find the minimum distance between the lines $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$. **1**

Question 16 (15 marks)

(a) Prove $m!n! < (m+n)!$ for positive integers m and n. **2**

(b) Find $\int \cos\sqrt{x}\,dx$. **3**

(c) The diagram shows a circle of radius 1, with its centre on the y-axis, that is tangent to the parabola $y = x^2$ at two distinct points. This circle has centre $(0, k)$ for some value $k > 0$ and therefore has equation $x^2 + (y-k)^2 = 1$. (Do NOT prove this.) **5**



By showing the x-values of the tangent points are $x = \pm\frac{\sqrt{3}}{2}$, find the exact area between the circle and the parabola.

(d) Let $f(x) = 1 + \frac{1}{x}$ and $\varphi = \frac{1+\sqrt{5}}{2}$.

Define $f \circ f(x)$ to be the composition of $f(x)$ with itself twice, that is $f \circ f(x) = f\big(f(x)\big)$.

Define $f^n(x)$ to be the function $f(x)$ composed with itself n times, for integers $n \geq 0$, that is

$$f^n(x) = \begin{cases} 1 & \text{if } n = 0 \\ \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}(x) & \text{if } n \geq 1. \end{cases}$$

(i) Show that $1 + \frac{1}{\varphi} = \varphi$ and $1 + \frac{1}{1-\varphi} = 1 - \varphi$. **1**

(ii) Show using mathematical induction that for all integers $n \geq 0$, **3**

$$f^n(1) = \frac{\varphi^{n+2} - (1-\varphi)^{n+2}}{\varphi^{n+1} - (1-\varphi)^{n+1}}.$$

(iii) Hence, or otherwise, find the value of the infinite composition $\lim_{n\to\infty} f^n(1)$. **1**

End of Examination

EXAMINERS

Geoff Carroll (Convenor)	Sydney Grammar School, Darlinghurst
Stephen Ewington	Ascham School, Edgecliff
Catheryn Gray	School of Mathematics and Statistics, UNSW
David Houghton	Oxley College, Burradoo
Rebekah Johnson	Loreto Kirribilli, Kirribilli
Svetlana Onisczenko	SCEGGS Darlinghurst, Darlinghurst