YEAR **11**

MATHEMAT
ADVAN

CambridgeMATHS
STAGE 6

BILL PENDER

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INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Rationale

CambridgeMATHS Advanced Year 11 covers all syllabus dotpoints for Year 11 of the Mathematics Advanced course being implemented in 2019. This rationale serves as a guide to how the book covers the dotpoints of the syllabus. Further documents are available in the teacher resources.

The Exercises

No-one should try to do all the questions! We have written long exercises with a great variety of questions so that everyone will find enough questions of a suitable standard — they cater for differentiated teaching to a wide range of students. The division of all exercises into Foundation, Development and Challenge sections helps with this. Each student will need to tackle a selection of questions, and there should be plenty left for revision.

The **Foundation** section in each exercise provides a gentle start to each exercise with many straightforward questions on each new skill and idea. Students need encouragement to assimilate comfortably the new ideas and methods presented in the text so that they are prepared and confident before tackling later problems.

The **Development** section is usually the longest, and is graded from reasonably straightforward questions to harder problems. The later questions may require the new content to be applied, they may require proof or explanation, or they may require content from previous sections to put the new ideas into a wider context.

The **Challenge** section is intended to match some of the more demanding questions that HSC Advanced examination papers often contain — we assume that this will continue with the HSC examinations on the new syllabuses. They may be algebraically or logically challenging, they may establish more difficult connections between topics, or they may require less obvious explanations or proofs. The section may be inappropriate at first reading.

Syllabus Coverage of the Chapters

Chapter 1: Methods in algebra

References: F1.1 dotpoints 2–3

Chapter 1 revises algebraic techniques. The Advanced course relies heavily on fluency in algebra, and the chapter is intended for students who need the revision. If students are confident with this material already from Years 9–10, they should not spend time on it, but simply know that it is there if they find later that some techniques need revision.

F1.1 dotpoint 1 index laws & Sections 7A–7B: The index laws from Years 9–10 are not reviewed here. They need a clearer and fuller presentation in Chapter 7 in the context of exponential and logarithmic functions and their graphs.

F1.1 dotpoint 1 surds & Sections 2C–2D: Surds are dealt with thoroughly in Chapter 2.

F1.1 dotpoint 2 & Sections 3D–3F: When quadratics graphs are developed in Chapter 3, the quadratic techniques of this dotpoint are covered again, but at a higher level.

Chapter 2: Numbers and surds

References: F1.1 dotpoint 1 surds; F1.2 dotpoint 2 intervals

Chapter 2 reviews the four standard sets of numbers: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ — students should be clear about these before tackling functions. In particular, they must know how to manipulate surds. They must also

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understand the geometric basis of the real numbers as the points on the number line, and be able to approximate them in various ways. The syllabus mostly takes all this as assumed knowledge from Years 9–10, but it is mentioned specifically in the two dotpoints above.

This chapter may be mostly review, and time should not be spent on it unnecessarily.

Types of intervals and their line graphs are introduced in Section 2B. 'Interval notation' in the form $3 < x < 6$ is used throughout the Year 11 book. The alternative 'interval notation' (3, 6) is introduced in Appendix C (Bracket interval notation) and can follow section 3H.

Chapter 3: Functions and graphs

References: F1.2 dotpoints 1–4, 6, 8; F1.3; F1.4 dotpoints 1–4

Section 3A, 3B and 3I introduce the more sophisticated notions of functions, relations and graphs required in the course and mentioned in F1.2 dotpoints 1–4, 6, 8.

Functions are introduced in Section 3A using a function machine, $f(x)$ notation, and tables of values. This leads in Section 3B to the graph, and to the use of the vertical line test to distinguish between a function and the more general concept of a relation.

The final Section 3I returns to the general theory and covers several related matters — reading a graph backwards using horizontal lines, graphical solutions of $f(x) = k$ (F1.3 dotpoint 3, sub-dotpoint), the horizontal line test, the classification of relations as one-to-one, many-to-one, one-to-many and many-to-many (F1.2 dotpoint 4), and 'mappings between sets'. Dotpoint 1 of F1.2 presumably implies sets others than numbers, as they occur in databases and spreadsheets. This section may be too challenging so early, and could be left until later in the year.

F1.2 dotpoints 4 & Section 7C: A full treatment of inverse functions is not part of the Advanced course, but some ideas about them are needed for the square root function $y = \sqrt{x}$, and in Chapter 7 where the idea is briefly introduced and formalised in Sections 7C, 7D and 7F so that the laws of logarithms can be established and the graphs discussed. The discussion in Section 3I of reading a graph backwards, and of one-to-one and many-to-one functions, is good preparation for logarithms.

F1.2 dotpoint 7 & Section 4E: Composite functions are delayed until Chapter 4, after translations, reflections and absolute values have already provided specific examples.

F1.2 dotpoints 6–7 & Chapter 8: The functions of algebra almost all involve sums, differences, quotients, products and composites of functions. Apart from composites, it would not be helpful at this stage to formalise rules for domains and ranges because the rules quickly become very complicated, and things are best done case-by-case throughout Chapter 3 and beyond. Each of these five ways to build functions is considered separately as the rules of differentiation are developed in Chapter 8: Differentiation.

Sections 3C–3H present, in the unified manner introduced in Sections 3A–3B, a variety of graphs that are partly known from Years 9–10, particularly linear graphs quadratics, square roots, and also higher powers of x, polynomials, circles, and graphs with asymptotes. Section 3G sketches cubics and general polynomials factored into linear factors, using a table of values to test the sign.

F1.3 dotpoint 1 & Chapter 6: Lines are only briefly mentioned in Section 3C, and are discussed in far more detail in Chapter 6: The coordinate plane. Gradient is the key idea with linear graphs, and gradient in turn requires $\tan\theta$ and the ability to solve, for example, the trigonometric equation $\tan\theta = -1$, where $0° \le \theta < 180°$ (see also the mention at C1.1 dotpoint 3).

Chapter 4: Transformations and symmetry

References: F1.2 dotpoints 5, 7; F1.4 dotpoints 5–9

Chapter 4 begins with translations of graphs, then reflections of them in the x-axis and y-axis, stressing the equivalence of the geometric and algebraic formulations. It then introduces even and odd functions and explains them in terms of the symmetry involving these two reflections.

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With these two sets of symmetries, the graphing of simple functions involving absolute value can mostly be done using transformations and symmetry rather than the confusing algebra with cases.

Composites fit well into this chapter because translations horizontally and vertically, and reflections in either axis, are special cases of composites.

Translations & F2 from Year 12: Translations have been brought forward from Year 12 Functions because translations are able to unify many things about graphs and their equations, particularly with the graphs of circles, parabolas and absolute value, and make them much easier to deal with. Dilations are also important in such unification, but they cause more difficulty and have been left in Year 12 where the syllabus has them.

F1.4 dotpoint 9 & Section 3G and 4A: Circles with centre the origin are in Section 3G, and circles with other origins are in Section 4A.

Chapter 5: Trigonometry

References: T1 omitting radian measure; T2

Chapter 5 begins to turn the attention of trigonometry towards the trigonometric functions and their graphs. This requires their definition and calculation for angles of any magnitude, the solving of simple trigonometric equations, and the examination of their graphs. The wave properties of the sine and cosine graphs are the most important goal here, because so much of science is concerned with cycles and periodic events.

The chapter is thus partly a review of Years 9–10 triangle trigonometry, extended to three dimensions, and partly a change of attention to the functions and their graphs, with all the required calculations and identities.

T1.1 dotpoint 3 & Section 5J–5K: The use of the cosine rule in the ambiguous case is not mentioned in the syllabus, but is briefly touched on in Exercise 5J and Section 5K.

T1.2 (radians) & Chapter 9: Radians are not introduced in Chapter 5. First, there is too much trigonometry to review, including general angles and the six graphs, ideas which students already find unfamiliar and difficult. Secondly, radian measure is unmotivated at this stage, because what motivates it is the derivative $\frac{d}{dx}(\sin x) = \cos x$. We have therefore placed radian measure later in the book, in the second half of Chapter 9: Extending calculus, which begins the extension of calculus to the exponential and trigonometric functions.

Given one trigonometric function, find another & Section 5F: This material is not in the syllabus, but is in the support material, so is covered in Section 5F.

Chapter 6: The coordinate plane

References: F1.3 dotpoint 1; C1.1 dotpoint 3

Chapter 6 is mostly, if not all, a review of Years 9–10 material. Its main purpose is the analysis of gradient and the equation of lines — revising gradient is an immediate preparation for calculus, which takes place initially in the coordinate plane, where the derivative is a gradient.

As further preparation, length and midpoint are also revised. The final section, which may be regarded as Challenge, asks for coordinate plane calculations involving pronumeral constants — this is required in many calculus problems — and explores the geometric implications.

Chapter 7: Exponential and logarithmic functions

References: F1.1 dotpoint 1 (index laws); E1.1–E1.2 E1.4

Chapter 7 begins with a thorough treatment of indices, then introduces and develops logarithms, before turning attention to the exponential and logarithmic functions, their graphs, and their applications.

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Indices and logarithms base e are not covered in this chapter, but are presented in the first part of Chapter 9: Extending calculus, which follows Chapter 8: Differentiation and deals with e^x, then with radian measure of angles. It is important that students first encounter exponential and logarithmic functions with familiar bases such as 2, 3 and 10 so that they are well prepared for Euler's number and its surprising definition, and for base e calculations.

This chapter has been placed before differentiation because indices, including negative and fractional indices, are essential for the introductory chapter on differentiation that follows.

Exponential equations reducible to quadratics & Year 12 Functions: These equations are not in the syllabus, but are in the support material. They will be discussed in Year 12, along with algebraic equations reducible to quadratics, when students are more confident with indices.

Chapter 8: Differentiation

References: F1.2 dotpoint 6; C1.1–1.4

This long chapter follows very closely the development given in C1, except that continuity and differentiability have been placed at the end of the chapter — experience with tangents is necessary to understand the significance of these ideas.

Sections 8A–8I present the standard theory of the derivative, first geometrically using tangents, then using the definition of the derivative as a limit. The geometric implications are stressed throughout, particularly as they apply to quadratics and higher-degree polynomials. As negative and fractional indices are introduced, the geometry of hyperbolas and other more exotic curves is investigated.

The proofs of the derivative of x^n, and of the chain, product and quotient rule, have been placed in an appendix at the end of the chapter so as not to disturb the exposition of the theory with difficult proofs. They are part of the course.

As remarked in the notes to Chapter 3, extending differentiation to the sums, differences, products, quotients and composites of functions are significant advances in the theory in Sections 8C and 8G–8I, and draw attention to these five ways of combining functions.

Families of curves with a common derivative are covered lightly here in Section 8D. The intention is to break the ground for their later use in integration and boundary value problems.

C1.1 dotpoint 3 & Chapter 6: The angle of inclination of a line is part of the theory of gradient, and was covered in Chapter 6, apart from the mention of a tangent.

C1.2 dotpoint 1 and C1.3 dotpoint 7 & Year 12 Curve-sketching-using-calculus and integration: The terms 'increasing', 'decreasing', 'stationary' and so forth are used informally in this chapter and earlier, and are then defined formally in terms of tangents in Section 8J: Rates of change. Their proper treatment, however, occurs in the Year 12 syllabus, where turning points, concavity, inflections, and increasing over an interval, are also essential parts of the story. In many problems here, however, one is required to find the points where the derivative is zero, and other questions deal lightly with the other concepts.

Section 8J interprets the derivative as a rate of change, as in C1.4 dotpoints 4–7.

C1.4 dotpoints 6–7 & Year 12 Motion chapter: Motion receives only a brief mention in Section 8J because the necessary theory for motion has not yet been developed. In particular, curve-sketching-using calculus is needed even to discuss stationary points, derivatives of the sine the cosine functions are needed for many interesting situations, and without integration not even the motion of a body falling under gravity can be derived. Students find motion difficult, and need an extended treatment to understand it.

Sections 8K–8L present continuity at a point, and then differentiability at a point, informally, without limits, as in C1.1 dotpoint 1. 'Differentiable' and 'doubly differentiable' are mentioned several times in the syllabus glossary.

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Chapter 9: Extending calculus

References: E1.3–E1.4; T1.2

After the derivative has been developed for algebraic functions, it then needs to be developed for the two groups of special functions: the exponential and logarithmic functions, and the trigonometric functions. This two-part chapter first develops the derivative of e^x, then develops radian measure in preparation for the derivatives of the trigonometric functions.

The intention of this two-part chapter is to allow some of the parallels between the two sets of functions to emerge naturally, particularly between the new base e required for exponential functions and the new units of radians, based on, required for the trigonometric functions.

Sections 9A–9F explain carefully the definition of e, taking care to emphasise the importance in calculus of $y = e^x$ having gradient 1 at its y-intercept. They then establish the derivatives of e^x and e^{ax+b}, introduce $\log_e x$ without its derivative, and show how calculus can be applied to find the rate of change of a quantity that is a function of time expressed in terms of the base e.

Sections 9G–9J explain radian measure, use it in problems involving arcs, sectors and segments, and develop the theory of the general angle and the graphs of the trigonometric functions in radians, which is their true form from the point of view of calculus. The derivative of $\sin x$, however, is not developed, although it is reasonably clear from the graph that $y = \sin x$ has gradient 1 at the origin and that $\frac{d}{dx}(\sin x) = \cos x$ looks like $y = \cos x$.

The final Exercise 9J has been constructed as an investigation on the large variety of symmetries that these six graphs display.

Chapter 10: Probability

References: S1.1

This chapter follows the usual exposition of probability, with Section 10C on sets and Venn diagrams necessary because sets will often not have been taught in earlier years.

Uniform sample spaces — sample spaces consisting of equally likely possible outcomes — are in this course the basis of the theory and of the whole idea of probability. As mentioned at the end of the chapter, various probability formulae are valid in much more general situations. Probability in such general situations, however, is not easy to define, and the sort of axiomatisation required for rigorous proofs is beyond school mathematics.

Experimental probability is introduced briefly in terms of relative frequency, in preparation for the more extended account in Chapter 11.

The final section on the new topic of conditional probability still begins with a definition that uses equally likely possible outcomes, and is based on the reduced sample space. From this, formulae with more general applicability can be derived.

Chapter 11: Discrete probability distributions

References: S1.2

Discrete probability distributions are new to NSW mathematics courses, and there are thus no accepted conventions about definitions, notations, and layouts, nor have any been specified in the syllabus. The choices in this chapter were carefully made on the basis of logical coherence, and of being the most straightforward way to present the material in the senior classroom and perform the necessary calculations with the minimum of fuss.

Bill Pender, November 2018

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Overview

As part of the *CambridgeMATHS* series, this resource is part of a continuum from Year 7 through to 12. The four components of *Mathematics Advanced Year 11* — the print book, downloadable PDF textbook, online Interactive Textbook and Online Teaching Resource — contain a range of resources available to schools in a single package at a convenient low price. There are no extra subscriptions or per-student charges to pay.

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4 Plenty of numbered worked examples are provided, with video versions for most of them.
5 Important concepts are formatted in numbered boxes for easy reference.
6 Investigation exercises and suggestions for projects are included.
7 Proofs for important results are provided as appendices to certain chapters.
8 Chapter review exercises assess learning in the chapter.

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The Interactive Textbook is an online HTML version of the print textbook powered by the HOTmaths platform, completely designed and reformatted for on-screen use, with easy navigation. It is included with the print book, or available as digital-only purchase. Its features include:

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11 All exercises including chapter reviews have the option of being done interactively on line, using **workspaces** and **self-assessment tools**. Students rate their level of confidence in answering the question and can flag the ones that gave them difficulty. Working and answers, whether typed or handwritten, and drawings, can be saved and submitted to the teacher electronically. Answers displayed on screen if selected and worked solutions (if enabled by the teacher) open in pop-up windows.
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17 Chapter Quizzes of automatically marked multiple-choice questions are provided for students to test their progress.
18 Definitions pop up for key terms in the text, and are also provided in a dictionary.
19 Spreadsheet files are provided to support questions and examples based on the use of such technology.
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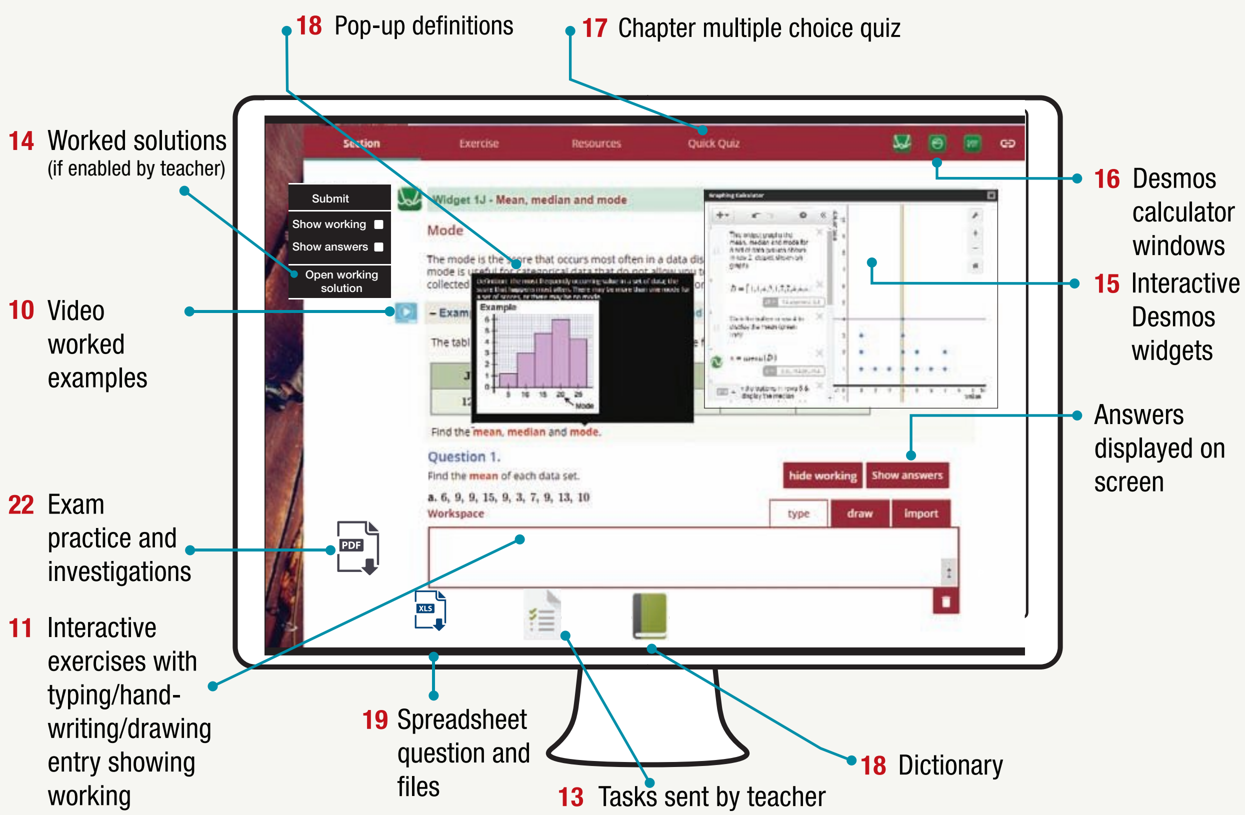

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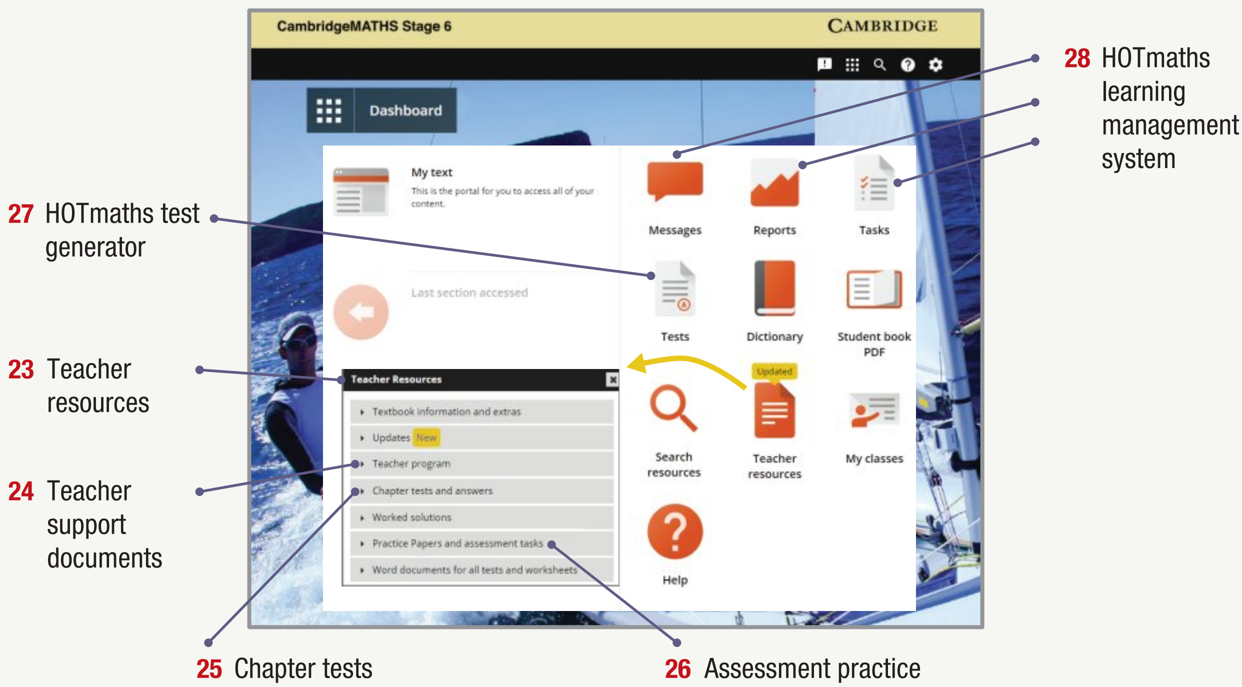

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The mathematician's patterns, like the painter's or the poet's,
must be beautiful. The ideas, like the colours or the words,
must fit together in a harmonious way. Beauty is the first test.

— The English mathematician G. H. Hardy (1877–1947)

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1 Methods in algebra

Fluency in algebra, particularly in factoring, is absolutely vital for everything in this course. This chapter is intended as a review of earlier algebraic techniques, and readers should do as much or as little of it as is necessary.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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1A Arithmetic with pronumerals

A *pronumeral* is a symbol that stands for a number. The pronumeral may stand for a known number or for an unknown number, or it may be a *variable* and stand for any one of a whole set of possible numbers. Pronumerals, being numbers, can take part in all the operations that are possible with numbers, such as addition, subtraction, multiplication, and division (except by zero).

Like and unlike terms

An *algebraic expression* consists of pronumerals, numbers and the operations of arithmetic. Here is an example:

$$x^2 + 2x + 3x^2 - 4x - 3 = 4x^2 - 2x - 3$$

This particular algebraic expression can be *simplified* by combining *like terms*.

- The two like terms x^2 and $3x^2$ can be combined to give $4x^2$.
- Another pair of like terms $2x$ and $-4x$ can be combined to give $-2x$.
- This yields three *unlike terms*, $4x^2$, $-2x$ and -3, which cannot be combined.

Example 1 1A

Simplify each expression by combining like terms.

a $7a + 15 - 2a - 20$ **b** $x^2 + 2x + 3x^2 - 4x - 3$

SOLUTION

a $7a + 15 - 2a - 20 = 5a - 5$ **b** $x^2 + 2x + 3x^2 - 4x - 3 = 4x^2 - 2x - 3$

Multiplying and dividing

To simplify a product such as $3y \times (-6y)$, or a quotient such as $10x^2y \div 5y$, work systematically through the signs, then the numerals, and then each pronumeral in turn.

Example 2 1A

Simplify these products and quotients.

a $3y \times (-6y)$ **b** $4ab \times 7bc$ **c** $10x^2y \div 5y$

SOLUTION

a $3y \times (-6y) = -18y^2$ **b** $4ab \times 7bc = 28ab^2c$ **c** $10x^2y \div 5y = 2x^2$

Index laws

Here are the standard laws for dealing with indices. They will be covered in more detail in Chapter 7.

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1 THE INDEX LAWS

- To multiply powers of the same base, add the indices: $a^x a^y = a^{x+y}$
- To divide powers of the same base, subtract the indices: $\dfrac{a^x}{a^y} = a^{x-y}$
- To raise a power to a power, multiply the indices: $(a^x)^n = a^{xn}$
- The power of a product is the product of the powers: $(ab)^x = a^x b^x$
- The power of a quotient is the quotient of the powers: $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

In expressions with several factors, work systematically through the signs, then the numerals, and then each pronumeral in turn.

Example 3 1A

Use the index laws above to simplify each expression.

a $3x^4 \times 4x^3$ **b** $(20x^7y^3) \div (4x^5y^3)$ **c** $(3a^4)^3$

d $(-x^2)^3 \times (2xy)^4$ **e** $\left(\dfrac{2x}{3y}\right)^4$

SOLUTION

a $3x^4 \times 4x^3 = 12x^7$ (multiplying powers of the same base)

b $(20x^7y^3) \div (4x^5y^3) = 5x^2$ (dividing powers of the same base)

c $(3a^4)^3 = 27a^{12}$ (raising a power to a power)

d $(-x^2)^3 \times (2xy)^4 = -x^6 \times 16x^4y^4$ (two powers of products)
$= -16x^{10}y^4$ (multiplying powers of the same base)

e $\left(\dfrac{2x}{3y}\right)^4 = \dfrac{16x^4}{81y^4}$ (a power of a quotient)

Exercise 1A FOUNDATION

1 Simplify:
a $3x + x$ **b** $3x - x$ **c** $-3x + x$ **d** $-3x - x$

2 Simplify:
a $-2a + 3a + 4a$ **b** $-2a - 3a + 4a$ **c** $-2a - 3a - 4a$ **d** $-2a + 3a - 4a$

3 Simplify:
a $-x + x$ **b** $2y - 3y$ **c** $-3a - 7a$ **d** $-8b + 5b$
e $4x - (-3x)$ **f** $-2ab - ba$ **g** $-3pq + 7pq$ **h** $-5abc - (-2abc)$

4 Simplify:
a $-3a \times 2$ **b** $-4a \times (-3a)$ **c** $a^2 \times a^3$ **d** $(a^2)^3$

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5 Simplify:

a $-10a \div 5$ b $-24a \div (-8a)$ c $a^9 \div a^3$ d $7a^2 \div 7a$

6 Simplify:

a $t^2 + t^2$ b $t^2 - t^2$ c $t^2 \times t^2$ d $t^2 \div t^2$

7 Simplify:

a $-6x + 3x$ b $-6x - 3x$ c $-6x \times 3x$ d $-6x \div 3x$

8 If $a = -2$, find the value of:

a $3a + 2$ b $a^3 - a^2$ c $3a^2 - a + 4$ d $a^4 + 3a^3 + 2a^2 - a$

DEVELOPMENT

9 Simplify:

a $6x + 3 - 5x$ b $-2 + 2y - 1$

c $3a - 7 - a + 4$ d $3x - 2y + 5x + 6y$

e $-8t + 12 - 2t - 17$ f $2a^2 + 7a - 5a^2 - 3a$

g $9x^2 - 7x + 4 - 14x^2 - 5x - 7$ h $3a - 4b - 2c + 4a + 2b - c + 2a - b - 2c$

10 Simplify:

a $\dfrac{5x}{x}$ b $\dfrac{-7m^3}{-m}$ c $\dfrac{-12a^2b}{ab}$ d $\dfrac{-27p^6q^7r^2}{9p^3q^3r}$

11 Subtract:

a x from $3x$ b $-x$ from $3x$ c $2a$ from $-4a$ d $-b$ from $-5b$

12 Multiply:

a $5a$ by 2 b $6x$ by -3 c $-3a$ by a

d $-2a^2$ by $-3ab$ e $4x^2$ by $-2x^3$ f $-3p^2q$ by $2pq^3$

13 Divide:

a $-2x$ by x b $3x^3$ by x^2 c x^3y^2 by x^2y

d a^6x^3 by $-a^2x^3$ e $14a^5b^4$ by $-2a^4b$ f $-50a^2b^5c^8$ by $-10ab^3c^2$

14 Simplify:

a $2a^2b^4 \times 3a^3b^2$ b $-6ab^5 \times 4a^3b^3$ c $(-3a^3)^2$ d $(-2a^4b)^3$

15 If $x = 2$ and $y = -3$, find the value of:

a $3x + 2y$ b $y^2 - 5x$ c $8x^2 - y^3$ d $x^2 - 3xy + 2y^2$

CHALLENGE

16 Simplify:

a $\dfrac{3a \times 3a \times 3a}{3a + 3a + 3a}$ b $\dfrac{3c \times 4c^2 \times 5c^3}{3c^2 + 4c^2 + 5c^2}$ c $\dfrac{ab^2 \times 2b^2c^3 \times 3c^3a^4}{a^3b^3 + 2a^3b^3 + 3a^3b^3}$

17 Simplify:

a $\dfrac{(-2x^2)^3}{-4x}$ b $\dfrac{(3xy^3)^3}{3x^2y^4}$ c $\dfrac{(-ab)^3 \times (-ab^2)^2}{-a^5b^3}$ d $\dfrac{(-2a^3b^2)^2 \times 16a^7b}{(2a^2b)^5}$

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1B Expanding brackets

Expanding brackets is routine in arithmetic. For example, we can calculate 7×61 as

$$7 \times (60 + 1) = 7 \times 60 + 7 \times 1$$

which quickly gives the result $7 \times 61 = 420 + 7 = 427$. The algebraic version of this procedure can be written as:

2 EXPANDING BRACKETS IN ALGEBRA

$$a(x + y) = ax + ay \qquad \text{and} \qquad (x + y)a = xa + ya$$

There may then be like terms to collect.

Example 4 1B

Expand and simplify each expression.

a $3x(4x - 7)$ **b** $5a(3 - b) - 3b(1 - 5a)$

SOLUTION

a $3x(4x - 7) = 12x^2 - 21x$

b $5a(3 - b) - 3b(1 - 5a) = 15a - 5ab - 3b + 15ab$
$= 15a + 10ab - 3b$ (collecting like terms)

Expanding the product of two bracketed terms

Expand one pair of brackets, then expand the other pair of brackets. Then collect any like terms.

Example 5 1B

Expand and simplify each expression.

a $(x + 3)(x - 5)$ **b** $(3 + x)(9 + 3x + x^2)$

SOLUTION

a $(x + 3)(x - 5)$
$= x(x - 5) + 3(x - 5)$
$= x^2 - 5x + 3x - 15$
$= x^2 - 2x - 15$

b $(3 + x)(9 + 3x + x^2)$
$= 3(9 + 3x + x^2) + x(9 + 3x + x^2)$
$= 27 + 9x + 3x^2 + 9x + 3x^2 + x^3$
$= 27 + 18x + 6x^2 + x^3$

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Special expansions

These three identities are important and must be memorised. Examples of these expansions occur constantly, and knowing the formulae greatly simplifies the working. They are proven in the exercises.

3 SPECIAL EXPANSIONS

- Square of a sum: $(A + B)^2 = A^2 + 2AB + B^2$
- Square of a difference: $(A - B)^2 = A^2 - 2AB + B^2$
- Difference of squares: $(A + B)(A - B) = A^2 - B^2$

Example 6 1B

Use the three special expansions above to simplify:

a $(x + 4)^2$ **b** $(s - 3t)^2$ **c** $(x + 3y)(x - 3y)$

SOLUTION

a $(x + 4)^2 = x^2 + 8x + 16$ (the square of a sum)

b $(s - 3t)^2 = s^2 - 6st + 9t^2$ (the square of a difference)

c $(x + 3y)(x - 3y) = x^2 - 9y^2$ (the difference of squares)

Exercise 1B FOUNDATION

1 Expand:

a $3(x - 2)$ **b** $2(x - 3)$ **c** $-3(x - 2)$
d $-2(x - 3)$ **e** $-3(x + 2)$ **f** $-2(x + 3)$
g $-(x - 2)$ **h** $-(2 - x)$ **i** $-(x + 3)$

2 Expand:

a $3(x + y)$ **b** $-2(p - q)$ **c** $4(a + 2b)$
d $x(x - 7)$ **e** $-x(x - 3)$ **f** $-a(a + 4)$
g $5(a + 3b - 2c)$ **h** $-3(2x - 3y + 5z)$ **i** $xy(2x - 3y)$

3 Expand and simplify:

a $2(x + 1) - x$ **b** $3a + 5 + 4(a - 2)$ **c** $2 + 2(x - 3)$
d $-3(a + 2) + 10$ **e** $3 - (x + 1)$ **f** $b + c - (b - c)$
g $(2x - 3y) - (3x - 2y)$ **h** $3(x - 2) - 2(x - 5)$ **i** $4(2a - 3b) - 3(a + 2b)$
j $4(s - t) - 6(s + t)$ **k** $2x(x + 6y) - x(x - 5y)$ **l** $5(2a - 5b) - 6(-a - 4b)$

4 Expand and simplify:

a $(x + 2)(x + 3)$ **b** $(y + 4)(y + 7)$ **c** $(t + 6)(t - 3)$
d $(x - 4)(x + 2)$ **e** $(t - 1)(t - 3)$ **f** $(2a + 3)(a + 5)$
g $(u - 4)(3u + 2)$ **h** $(4p + 5)(2p - 3)$ **i** $(2b - 7)(b - 3)$
j $(5a - 2)(3a + 1)$ **k** $(6 - c)(c - 3)$ **l** $(2d - 3)(4 + d)$

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DEVELOPMENT

5 **a** By expanding $(A+B)(A+B)$, prove the special expansion $(A+B)^2 = A^2 + 2AB + B^2$.

b Similarly, prove the special expansions:

i $(A-B)^2 = A^2 - 2AB + B^2$ **ii** $(A-B)(A+B) = A^2 - B^2$

6 Use the special expansions to expand:

a $(x+y)^2$ **b** $(x-y)^2$ **c** $(x-y)(x+y)$ **d** $(a+3)^2$

e $(b-4)^2$ **f** $(c+5)^2$ **g** $(d-6)(d+6)$ **h** $(7+e)(7-e)$

i $(8+f)^2$ **j** $(9-g)^2$ **k** $(h+10)(h-10)$ **l** $(i+11)^2$

m $(2a+1)^2$ **n** $(2b-3)^2$ **o** $(3c+2)^2$ **p** $(2d+3e)^2$

q $(2f+3g)(2f-3g)$ **r** $(3h-2i)(3h+2i)$ **s** $(5j+4)^2$ **t** $(4k-5l)^2$

u $(4+5m)(4-5m)$ **v** $(5-3n)^2$ **w** $(7p+4q)^2$ **x** $(8-3r)^2$

CHALLENGE

7 Expand and simplify:

a $\left(t+\frac{1}{t}\right)^2$ **b** $\left(t-\frac{1}{t}\right)^2$ **c** $\left(t+\frac{1}{t}\right)\left(t-\frac{1}{t}\right)$

8 By writing 102 as $(100+2)$, and adopting a similar approach for parts **b** and **c**, use the special expansions to find (without using a calculator) the value of:

a 102^2 **b** 999^2 **c** 203×197

9 Expand and simplify:

a $(a-b)(a^2+ab+b^2)$ **b** $(x+2)^2 - (x+1)^2$

c $(a-3)^2 - (a-3)(a+3)$ **d** $(2x+3)(x-1) - (x-2)(x+1)$

e $(x-2)^3$ **f** $(p+q+r)^2 - 2(pq+qr+rp)$

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1C Factoring

Factoring is the reverse process of expanding brackets, and is needed routinely throughout the course. There are four basic methods, but in every situation common factors should always be taken out first.

4 THE FOUR BASIC METHODS OF FACTORING

- **Highest common factor**: *Always try this first*.
- **Difference of squares**: This involves two terms.
- **Quadratics**: This involves three terms.
- **Grouping**: This involves four or more terms.

Factoring should continue until each factor is *irreducible*, meaning that it cannot be factored further.

Factoring by taking out the highest common factor

Always look first for any common factors of all the terms, and then take out the highest common factor.

Example 7 — 1C

Factor each expression by taking out the highest common factor.

a $4x^3 + 3x^2$ **b** $9a^2b^3 - 15b^3$

SOLUTION

a The highest common factor of $4x^3$ and $3x^2$ is x^2,
so $4x^3 + 3x^2 = x^2(4x + 3)$.

b The highest common factor of $9a^2b^3$ and $15b^3$ is $3b^3$,
so $9a^2b^3 - 15b^3 = 3b^3(3a^2 - 5)$.

Factoring by difference of squares

The expression must have two terms, both of which are squares. Sometimes a common factor must be taken out first.

Example 8 — 1C

Use the difference of squares to factor each expression.

a $a^2 - 36$ **b** $80x^2 - 5y^2$

SOLUTION

a $a^2 - 36 = (a + 6)(a - 6)$

b $80x^2 - 5y^2 = 5(16x^2 - y^2)$ (take out the highest common factor)
$ = 5(4x - y)(4x + y)$ (use the difference of squares)

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Factoring monic quadratics

A quadratic is called *monic* if the coefficient of x^2 is 1. Suppose that we want to factor the monic quadratic expression $x^2 - 13x + 36$. Look for two numbers:

- whose sum is -13 (the coefficient of x), and
- whose product is $+36$ (the constant term).

Example 9 1C

Factor these monic quadratics.

a $x^2 - 13x + 36$ **b** $a^2 + 12a - 28$

SOLUTION

a The numbers with sum -13 and product $+36$ are -9 and -4,
so $x^2 - 13x + 36 = (x - 9)(x - 4)$.

b The numbers with sum $+12$ and product -28 are $+14$ and -2,
so $a^2 + 12a - 28 = (a + 14)(a - 2)$.

Factoring non-monic quadratics

In a *non-monic* quadratic such as $2x^2 + 11x + 12$, where the coefficient of x^2 is not 1, look for two numbers:

- whose sum is 11 (the coefficient of x), and
- whose product is $12 \times 2 = 24$ (the constant times the coefficient of x^2).

Then split the middle term into two terms.

Example 10 1C

Factor these non-monic quadratics.

a $2x^2 + 11x + 12$ **b** $6s^2 - 11s - 10$

SOLUTION

a The numbers with sum 11 and product $12 \times 2 = 24$ are 8 and 3,

$$\begin{aligned} \text{so } 2x^2 + 11x + 12 &= (2x^2 + 8x) + (3x + 12) && \text{(split } 11x \text{ into } 8x + 3x) \\ &= 2x(x + 4) + 3(x + 4) && \text{(take out the HCF of each group)} \\ &= (2x + 3)(x + 4) && (x + 4 \text{ is a common factor}) \end{aligned}$$

b The numbers with sum -11 and product $-10 \times 6 = -60$ are -15 and 4,

$$\begin{aligned} \text{so } 6s^2 - 11s - 10 &= (6s^2 - 15s) + (4s - 10) && \text{(split } -11s \text{ into } -15s + 4s) \\ &= 3s(2s - 5) + 2(2s - 5) && \text{(take out the HCF of each group)} \\ &= (3s + 2)(2s - 5) && (2s - 5 \text{ is a common factor}) \end{aligned}$$

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Factoring by grouping

When there are four or more terms, it is sometimes possible to factor the expression by grouping.

- Split the expression into groups.
- Then factor each group in turn.
- Then factor the whole expression by taking out a common factor or by some other method.

Example 11 1C

Factor each expression by grouping.

a $12xy - 9x - 16y + 12$ **b** $s^2 - t^2 + s - t$

SOLUTION

a $12xy - 9x - 16y + 12 = 3x(4y - 3) - 4(4y - 3)$ (take out the HCF of each pair)
$= (3x - 4)(4y - 3)$ ($4y - 3$ is a common factor)

b $s^2 - t^2 + s - t = (s + t)(s - t) + (s - t)$ (factor $s^2 - t^2$ using the difference of squares)
$= (s - t)(s + t + 1)$ ($s - t$ is a common factor)

Exercise 1C

FOUNDATION

1 Factor each expression by taking out any common factors:

a $2x + 8$ **b** $6a - 15$ **c** $ax - ay$ **d** $20ab - 15ac$
e $x^2 + 3x$ **f** $p^2 + 2pq$ **g** $3a^2 - 6ab$ **h** $12x^2 + 18x$
i $20cd - 32c$ **j** $a^2b + b^2a$ **k** $6a^2 + 2a^3$ **l** $7x^3y - 14x^2y^2$

2 Factor each expression by grouping in pairs:

a $mp + mq + np + nq$ **b** $ax - ay + bx - by$ **c** $ax + 3a + 2x + 6$
d $a^2 + ab + ac + bc$ **e** $z^3 - z^2 + z - 1$ **f** $ac + bc - ad - bd$
g $pu - qu - pv + qv$ **h** $x^2 - 3x - xy + 3y$ **i** $5p - 5q - px + qx$
j $2ax - bx - 2ay + by$ **k** $ab + ac - b - c$ **l** $x^3 + 4x^2 - 3x - 12$
m $a^3 - 3a^2 - 2a + 6$ **n** $2t^3 + 5t^2 - 10t - 25$ **o** $2x^3 - 6x^2 - ax + 3a$

3 Factor each expression by using the difference of squares:

a $a^2 - 1$ **b** $b^2 - 4$ **c** $c^2 - 9$ **d** $d^2 - 100$
e $25 - y^2$ **f** $1 - n^2$ **g** $49 - x^2$ **h** $144 - p^2$
i $4c^2 - 9$ **j** $9u^2 - 1$ **k** $25x^2 - 16$ **l** $1 - 49k^2$
m $x^2 - 4y^2$ **n** $9a^2 - b^2$ **o** $25m^2 - 36n^2$ **p** $81a^2b^2 - 64$

4 Factor each quadratic expression. They are all monic quadratics.

a $a^2 + 3a + 2$ **b** $k^2 + 5k + 6$ **c** $m^2 + 7m + 6$ **d** $x^2 + 8x + 15$
e $y^2 + 9y + 20$ **f** $t^2 + 12t + 20$ **g** $x^2 - 4x + 3$ **h** $c^2 - 7c + 10$
i $a^2 - 7a + 12$ **j** $b^2 - 8b + 12$ **k** $t^2 + t - 2$ **l** $u^2 - u - 2$
m $w^2 - 2w - 8$ **n** $a^2 + 2a - 8$ **o** $p^2 - 2p - 15$ **p** $y^2 + 3y - 28$
q $c^2 - 12c + 27$ **r** $u^2 - 13u + 42$ **s** $x^2 - x - 90$ **t** $x^2 + 3x - 40$
u $t^2 - 4t - 32$ **v** $p^2 + 9p - 36$ **w** $u^2 - 16u - 80$ **x** $t^2 + 23t - 50$

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DEVELOPMENT

5 Factor each quadratic expression. They are all non-monic quadratics.

a $3x^2 + 4x + 1$	b $2x^2 + 5x + 2$	c $3x^2 + 16x + 5$	d $3x^2 + 8x + 4$
e $2x^2 - 3x + 1$	f $5x^2 - 13x + 6$	g $5x^2 - 11x + 6$	h $6x^2 - 11x + 3$
i $2x^2 - x - 3$	j $2x^2 + 3x - 5$	k $3x^2 + 2x - 5$	l $3x^2 + 14x - 5$
m $2x^2 - 7x - 15$	n $2x^2 + x - 15$	o $6x^2 + 17x - 3$	p $6x^2 - 7x - 3$
q $6x^2 + 5x - 6$	r $5x^2 + 23x + 12$	s $5x^2 + 4x - 12$	t $5x^2 - 19x + 12$
u $5x^2 - 11x - 12$	v $5x^2 + 28x - 12$	w $9x^2 - 6x - 8$	x $3x^2 + 13x - 30$

6 Use the techniques of the previous questions to factor each expression.

a $a^2 - 25$	b $b^2 - 25b$	c $c^2 - 25c + 100$
d $2d^2 + 25d + 50$	e $e^3 + 5e^2 + 5e + 25$	f $16 - f^2$
g $16g^2 - g^3$	h $h^2 + 16h + 64$	i $i^2 - 16i - 36$
j $5j^2 + 16j - 16$	k $4k^2 - 16k - 9$	l $2k^3 - 16k^2 - 3k + 24$
m $2a^2 + ab - 4a - 2b$	n $6m^3n^4 + 9m^2n^5$	o $49p^2 - 121q^2$
p $t^2 - 14t + 40$	q $3t^2 + 2t - 40$	r $5t^2 + 54t + 40$
s $5t^2 + 33t + 40$	t $5t^3 + 10t^2 + 15t$	u $u^2 + 15u - 54$
v $3x^3 - 2x^2y - 15x + 10y$	w $(p + q)^2 - r^2$	x $4a^2 - 12a + 9$

CHALLENGE

7 Factor each expression as fully as possible. (Take out any common factors first.)

a $3a^2 - 12$	b $x^4 - y^4$	c $x^3 - x$	d $5x^2 - 5x - 30$
e $25y - y^3$	f $16 - a^4$	g $4x^2 + 14x - 30$	h $a^4 + a^3 + a^2 + a$
i $c^3 + 9c^2 - c - 9$	j $x^3 - 8x^2 + 7x$	k $x^4 - 3x^2 - 4$	l $ax^2 - a - 2x^2 + 2$

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1D Algebraic fractions

An *algebraic fraction* is a fraction that contains pronumerals. Algebraic fractions are manipulated in the same way as arithmetic fractions, and factoring may play a major role.

Adding and subtracting algebraic fractions

A common denominator is needed. Finding the lowest common denominator may involve factoring each denominator.

5 ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

- First factor each denominator.
- Then work with the lowest common denominator.

Example 12 1D

Use a common denominator to simplify each algebraic fraction.

a $\frac{x}{2} - \frac{x}{3}$ **b** $\frac{5x}{6} + \frac{11x}{4}$ **c** $\frac{2}{3x} - \frac{3}{5x}$ **d** $\frac{1}{x-4} - \frac{1}{x}$

SOLUTION

a $$\begin{aligned}\frac{x}{2} - \frac{x}{3} &= \frac{3x}{6} - \frac{2x}{6}\\ &= \frac{x}{6}\end{aligned}$$

b $$\begin{aligned}\frac{5x}{6} + \frac{11x}{4} &= \frac{10x}{12} + \frac{33x}{12}\\ &= \frac{43x}{12}\end{aligned}$$

c $$\begin{aligned}\frac{2}{3x} - \frac{3}{5x} &= \frac{10}{15x} - \frac{9}{15x}\\ &= \frac{1}{15x}\end{aligned}$$

d $$\begin{aligned}\frac{1}{x-4} - \frac{1}{x} &= \frac{x-(x-4)}{x(x-4)}\\ &= \frac{4}{x(x-4)}\end{aligned}$$

Example 13 1D

Factor the denominators of $\frac{2+x}{x^2-x} - \frac{5}{x-1}$, then simplify the expression.

SOLUTION

$$\begin{aligned}\frac{2+x}{x^2-x} - \frac{5}{x-1} &= \frac{2+x}{x(x-1)} - \frac{5}{x-1}\\ &= \frac{2+x-5x}{x(x-1)}\\ &= \frac{2-4x}{x(x-1)}\end{aligned}$$

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Cancelling algebraic fractions

The key step here is to factor the numerator and denominator completely before cancelling factors.

6 CANCELLING ALGEBRAIC FRACTIONS

- First factor the numerator and the denominator.
- Then cancel out all common factors.

Example 14 1D

Simplify each algebraic fraction.

a $\dfrac{6x+8}{6}$ **b** $\dfrac{x^2-x}{x^2-1}$

SOLUTION

a $$\frac{6x+8}{6} = \frac{2(3x+4)}{6} = \frac{3x+4}{3} \quad \left(\text{which could be written as } x+\tfrac{4}{3}\right)$$

b $$\frac{x^2-x}{x^2-1} = \frac{x(x-1)}{(x+1)(x-1)} = \frac{x}{x+1}$$

Multiplying and dividing algebraic fractions

These processes are done exactly as for arithmetic fractions.

7 MULTIPLYING ALGEBRAIC FRACTIONS AND DIVIDING BY AN ALGEBRAIC FRACTION

Multiplying algebraic fractions

- First factor all numerators and denominators completely.
- Then cancel common factors.

Dividing by an algebraic fraction

- To divide by an algebraic fraction, multiply by its reciprocal. For example: $\dfrac{3}{x} \div \dfrac{4}{y} = \dfrac{3}{x} \times \dfrac{y}{4}$.
- The reciprocal of the fraction $\dfrac{4}{y}$ is $\dfrac{y}{4}$.

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Example 15 1D

Simplify these products and quotients of algebraic fractions.

a $\dfrac{2a}{a^2-9} \times \dfrac{a-3}{5a}$ **b** $\dfrac{12x}{x+1} \div \dfrac{6x}{x^2+2x+1}$

SOLUTION

a
$$\begin{aligned}\frac{2a}{a^2-9} \times \frac{a-3}{5a} &= \frac{2a}{(a-3)(a+3)} \times \frac{a-3}{5a} && \text{(factor } a^2-9\text{)}\\ &= \frac{2}{5(a+3)} && \text{(cancel } a-3 \text{ and } a\text{)}\end{aligned}$$

b
$$\begin{aligned}\frac{12x}{x+1} \div \frac{6x}{x^2+2x+1} &= \frac{12x}{x+1} \times \frac{x^2+2x+1}{6x} && \text{(multiply by the reciprocal)}\\ &= \frac{12x}{x+1} \times \frac{(x+1)^2}{6x} && \text{(factor } x^2+2x+1\text{)}\\ &= 2(x+1) && \text{(cancel } x+1 \text{ and } 6x\text{)}\end{aligned}$$

Simplifying compound fractions

A *compound fraction* is a fraction in which either the numerator or the denominator is itself a fraction.

8 SIMPLIFYING COMPOUND FRACTIONS

- Find the lowest common multiple of the denominators on the top and the bottom.
- Multiply top and bottom by this lowest common multiple.

This will clear all the fractions from the top and bottom together.

Example 16 1D

Simplify each compound fraction.

a $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{4}+\frac{1}{6}}$ **b** $\dfrac{\frac{1}{t}+1}{\frac{1}{t}-1}$

SOLUTION

a
$$\begin{aligned}\frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{4}+\frac{1}{6}} &= \frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{4}+\frac{1}{6}} \times \frac{12}{12}\\ &= \frac{6-4}{3+2}\\ &= \frac{2}{5}\end{aligned}$$

b
$$\begin{aligned}\frac{\frac{1}{t}+1}{\frac{1}{t}-1} &= \frac{\frac{1}{t}+1}{\frac{1}{t}-1} \times \frac{t}{t}\\ &= \frac{1+t}{1-t}\end{aligned}$$

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Exercise 1D

FOUNDATION

1 Simplify:

a $\frac{x}{x}$ **b** $\frac{2x}{x}$ **c** $\frac{x}{2x}$ **d** $\frac{a}{a^2}$ **e** $\frac{3x^2}{9xy}$ **f** $\frac{12ab}{4a^2b}$

2 Simplify:

a $\frac{x}{3} \times \frac{3}{x}$ **b** $\frac{a}{4} \div \frac{a}{2}$ **c** $x^2 \times \frac{3}{x}$ **d** $\frac{1}{2b} \times b^2$

e $\frac{3x}{4} \times \frac{2}{x^2}$ **f** $\frac{5}{a} \div 10$ **g** $\frac{2ab}{3} \times \frac{6}{ab^2}$ **h** $\frac{8ab}{5} \div \frac{4ab}{15}$

3 Write as a single fraction:

a $x + \frac{x}{2}$ **b** $\frac{y}{4} + \frac{y}{2}$ **c** $\frac{m}{3} - \frac{m}{9}$ **d** $\frac{n}{2} + \frac{n}{5}$

e $\frac{x}{8} - \frac{y}{12}$ **f** $\frac{2a}{3} + \frac{3a}{2}$ **g** $\frac{7b}{10} - \frac{19b}{30}$ **h** $\frac{xy}{30} - \frac{xy}{12}$

4 Write as a single fraction:

a $\frac{1}{a} + \frac{1}{a}$ **b** $\frac{1}{x} - \frac{2}{x}$ **c** $\frac{1}{a} + \frac{1}{2a}$ **d** $\frac{1}{2x} - \frac{1}{3x}$ **e** $\frac{3}{4a} + \frac{4}{3a}$ **f** $\frac{5}{6x} - \frac{1}{3x}$

DEVELOPMENT

5 Simplify:

a $\frac{x+1}{2} + \frac{x+2}{3}$ **b** $\frac{2x-1}{5} + \frac{2x+3}{4}$ **c** $\frac{x+3}{6} + \frac{x-3}{12}$

d $\frac{x+2}{2} - \frac{x+3}{3}$ **e** $\frac{2x+1}{4} - \frac{2x-3}{5}$ **f** $\frac{2x-1}{3} - \frac{2x+1}{6}$

6 Factor where possible and then simplify:

a $\frac{2p+2q}{p+q}$ **b** $\frac{3t-12}{2t-8}$ **c** $\frac{x^2+3x}{3x+9}$

d $\frac{a}{ax+ay}$ **e** $\frac{3a^2-6ab}{2a^2b-4ab^2}$ **f** $\frac{x^2+2x}{x^2-4}$

g $\frac{a^2-9}{a^2+a-12}$ **h** $\frac{x^2+2x+1}{x^2-1}$ **i** $\frac{x^2+10x+25}{x^2+9x+20}$

7 Simplify:

a $\frac{1}{x} + \frac{1}{x+1}$ **b** $\frac{1}{x} - \frac{1}{x+1}$ **c** $\frac{1}{x+1} + \frac{1}{x-1}$

d $\frac{2}{x-3} + \frac{3}{x-2}$ **e** $\frac{3}{x+1} - \frac{2}{x-1}$ **f** $\frac{2}{x-2} - \frac{2}{x+3}$

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8 Simplify:

a $\dfrac{3x+3}{2x} \times \dfrac{x^2}{x^2-1}$

b $\dfrac{a^2+a-2}{a+2} \times \dfrac{a^2-3a}{a^2-4a+3}$

c $\dfrac{c^2+5c+6}{c^2-16} \div \dfrac{c+3}{c-4}$

d $\dfrac{x^2-x-20}{x^2-25} \times \dfrac{x^2-x-2}{x^2+2x-8} \div \dfrac{x+1}{x^2+5x}$

CHALLENGE

9 Simplify:

a $\dfrac{b-a}{a-b}$

b $\dfrac{1}{a-b} - \dfrac{1}{b-a}$

c $\dfrac{m}{m-n} + \dfrac{n}{n-m}$

d $\dfrac{x^2-5x+6}{2-x}$

10 Simplify:

a $\dfrac{1}{x^2+x} + \dfrac{1}{x^2-x}$

b $\dfrac{1}{x-y} + \dfrac{2x-y}{x^2-y^2}$

c $\dfrac{3}{x^2+2x-8} - \dfrac{2}{x^2+x-6}$

d $\dfrac{1}{x^2-4x+3} + \dfrac{1}{x^2-5x+6} - \dfrac{1}{x^2-3x+2}$

11 Study the worked example on compound fractions and then simplify:

a $\dfrac{1-\frac{1}{2}}{1+\frac{1}{2}}$

b $\dfrac{2+\frac{1}{3}}{5-\frac{2}{3}}$

c $\dfrac{\frac{1}{2}-\frac{1}{5}}{1+\frac{1}{10}}$

d $\dfrac{\frac{17}{20}-\frac{3}{4}}{\frac{4}{5}-\frac{3}{10}}$

e $\dfrac{\frac{1}{x}}{1+\frac{2}{x}}$

f $\dfrac{t-\frac{1}{t}}{t+\frac{1}{t}}$

g $\dfrac{1}{\frac{1}{b}+\frac{1}{a}}$

h $\dfrac{\frac{x}{y}+\frac{y}{x}}{\frac{x}{y}-\frac{y}{x}}$

i $\dfrac{1-\frac{1}{x+1}}{\frac{1}{x}+\frac{1}{x+1}}$

j $\dfrac{\frac{3}{x+2}-\frac{2}{x+1}}{\frac{5}{x+2}-\frac{4}{x+1}}$

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1E Solving linear equations

The first principle in solving any equation is to simplify it by doing the same things to both sides. Linear equations can be solved completely this way.

9 SOLVING LINEAR EQUATIONS

- Any number can be added to or subtracted from both sides.
- Both sides can be multiplied or divided by any non-zero number.

An equation involving algebraic fractions can often be reduced to a linear equation by following these steps.

Example 17 1E

Solve each equation.

a $6x + 5 = 4x - 9$

b $\dfrac{4 - 7x}{4x - 7} = 1$

SOLUTION

a
$$\begin{aligned} 6x + 5 &= 4x - 9 \\ \boxed{-4x}\quad 2x + 5 &= -9 \\ \boxed{-5}\quad 2x &= -14 \\ \boxed{\div 2}\quad x &= -7 \end{aligned}$$

b
$$\begin{aligned} \frac{4 - 7x}{4x - 7} &= 1 \\ \boxed{\times (4x - 7)}\quad 4 - 7x &= 4x - 7 \\ \boxed{+7x}\quad 4 &= 11x - 7 \\ \boxed{+7}\quad 11 &= 11x \\ \boxed{\div 11}\quad x &= 1 \end{aligned}$$

Changing the subject of a formula

Similar sequences of operations allow the subject of a formula to be changed from one pronumeral to another.

Example 18 1E

Change the subject of each formula to x.

a $y = 4x - 3$

b $y = \dfrac{x + 1}{x + 2}$

SOLUTION

a
$$\begin{aligned} y &= 4x - 3 \\ \boxed{+3}\quad y + 3 &= 4x \\ \boxed{\div 4}\quad \frac{y + 3}{4} &= x \\ x &= \frac{y + 3}{4} \end{aligned}$$

b
$$\begin{aligned} y &= \frac{x + 1}{x + 2} \\ \boxed{\times (x + 2)}\quad xy + 2y &= x + 1 \\ \text{Rearranging,}\quad xy - x &= 1 - 2y \\ \text{Factoring,}\quad x(y - 1) &= 1 - 2y \\ \boxed{\div (y - 1)}\quad x &= \frac{1 - 2y}{y - 1} \end{aligned}$$

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Exercise 1E

FOUNDATION

1 Solve:

a $2x + 1 = 7$
b $5p - 2 = -2$
c $\frac{a}{2} - 1 = 3$
d $3 - w = 4$
e $3x - 5 = 22$
f $4x + 7 = -13$
g $1 - \frac{x}{2} = 9$
h $11 - 6x = 23$

2 Solve:

a $3n - 1 = 2n + 3$
b $4b + 3 = 2b + 1$
c $5x - 2 = 2x + 10$
d $5 - x = 27 + x$
e $16 + 9a = 10 - 3a$
f $13y - 21 = 20y - 35$
g $13 - 12x = 6 - 3x$
h $3x + 21 = 18 - 2x$

DEVELOPMENT

3 Solve:

a $\frac{a}{12} = \frac{2}{3}$
b $\frac{y}{20} = \frac{4}{5}$
c $\frac{1}{x} = 3$
d $\frac{2}{a} = 5$
e $1 = \frac{3}{2y}$
f $\frac{2x + 1}{5} = -3$
g $\frac{3a - 5}{4} = 4$
h $\frac{7 - 4x}{6} = \frac{3}{2}$
i $\frac{20 + a}{a} = -3$
j $\frac{9 - 2t}{t} = 13$
k $\frac{3}{x - 1} = -1$
l $\frac{3x}{2x - 1} = \frac{5}{3}$

4 **a** If $v = u + at$, find a when $t = 4$, $v = 20$ and $u = 8$.

b Given that $v^2 = u^2 + 2as$, find the value of s when $u = 6$, $v = 10$ and $a = 2$.

c Suppose that $\frac{1}{u} + \frac{1}{v} = \frac{1}{t}$. Find v, given that $u = -1$ and $t = 2$.

d If $S = -15$, $n = 10$ and $a = -24$, find ℓ, given that $S = \frac{n}{2}(a + \ell)$.

e The formula $F = \frac{9}{5}C + 32$ relates temperatures in degrees Fahrenheit and Celsius. Find the value of C that corresponds to $F = 95$.

f Suppose that c and d are related by the formula $\frac{1}{c + 1} = \frac{5}{d - 1}$. Find c when $d = 4$.

5 Solve each problem by forming, and then solving, a linear equation.

a Three less than four times a certain number is equal to 21. Find the number.

b Five more than twice a certain number is one more than the number itself. What is the number?

c Bill and Derek collect Batman cards. Bill has three times as many cards as Derek, and altogether they have 68 cards. How many cards does Derek have?

d If I paid \$1.45 for an apple and an orange, and the apple cost 15 cents more than the orange, how much did the orange cost?

6 Solve:

a $y + \frac{y}{2} = 1$
b $\frac{x}{3} - \frac{x}{5} = 2$
c $\frac{a}{10} - \frac{a}{6} = 1$
d $\frac{x}{6} + \frac{2}{3} = \frac{x}{2} - \frac{5}{6}$
e $\frac{x}{3} - 2 = \frac{x}{2} - 3$
f $\frac{1}{x} - 3 = \frac{1}{2x}$
g $\frac{1}{2x} - \frac{2}{3} = 1 - \frac{1}{3x}$
h $\frac{x - 2}{3} = \frac{x + 4}{4}$
i $\frac{3}{x - 2} = \frac{4}{2x + 5}$
j $\frac{x + 1}{x + 2} = \frac{x - 3}{x + 1}$

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7 Rearrange each formula so that the pronumeral written in square brackets is the subject.

a $a = bc - d \quad [b]$

b $t = a + (n - 1)d \quad [n]$

c $\dfrac{p}{q + r} = t \quad [r]$

d $u = 1 + \dfrac{3}{v} \quad [v]$

8 Expand the brackets on both sides of each equation, then solve it.

a $(x - 3)(x + 6) = (x - 4)(x - 5)$

b $(1 + 2x)(4 + 3x) = (2 - x)(5 - 6x)$

c $(x + 3)^2 = (x - 1)^2$

d $(2x - 5)(2x + 5) = (2x - 3)^2$

CHALLENGE

9 Solve:

a $\dfrac{a + 5}{2} - \dfrac{a - 1}{3} = 1$

b $\dfrac{3}{4} - \dfrac{x + 1}{12} = \dfrac{2}{3} - \dfrac{x - 1}{6}$

c $\dfrac{3}{4}(x - 1) - \dfrac{1}{2}(3x + 2) = 0$

d $\dfrac{4x + 1}{6} - \dfrac{2x - 1}{15} = \dfrac{3x - 5}{5} - \dfrac{6x + 1}{10}$

10 Rearrange each formula so that the pronumeral written in square brackets is the subject.

a $\dfrac{a}{2} - \dfrac{b}{3} = a \quad [a]$

b $\dfrac{1}{f} + \dfrac{2}{g} = \dfrac{5}{h} \quad [g]$

c $x = \dfrac{y}{y + 2} \quad [y]$

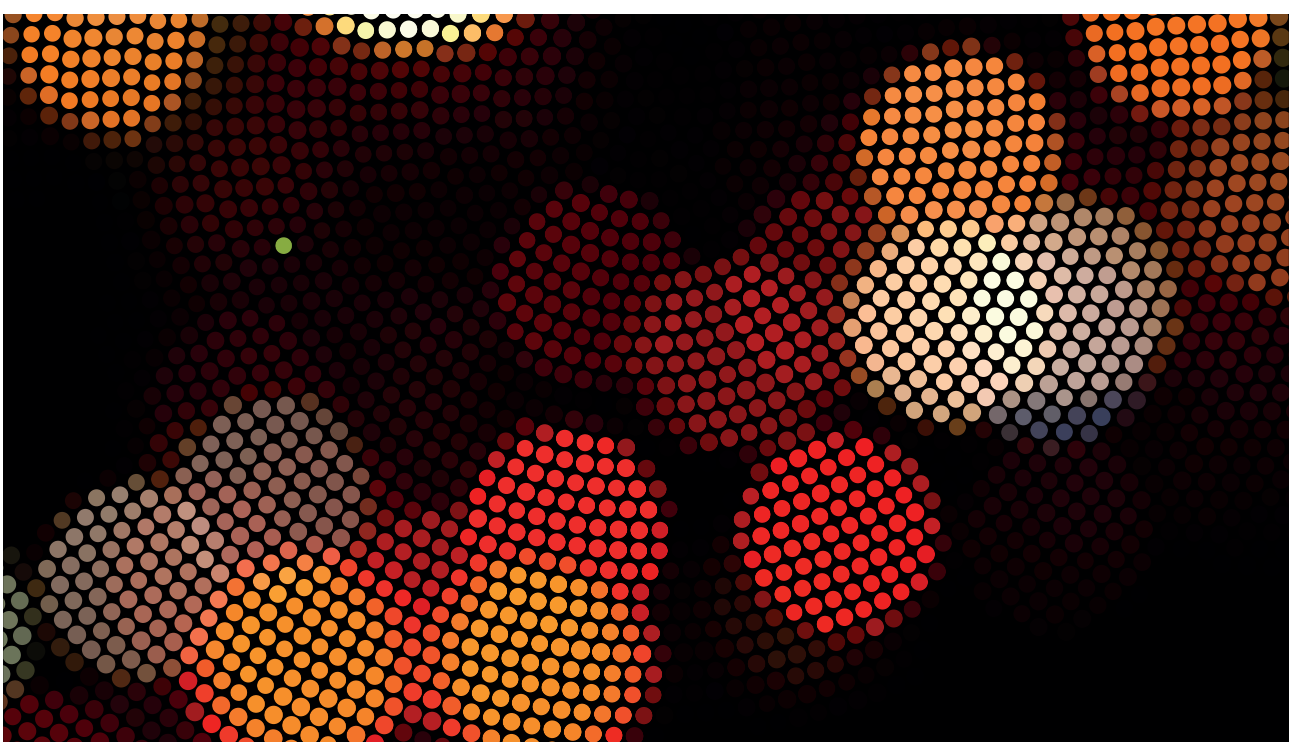
d $a = \dfrac{b + 5}{b - 4} \quad [b]$

11 Solve each problem by forming, and then solving, a linear equation.

a My father is 40 years older than me and he is three times my age. How old am I?

b A basketballer has scored 312 points in 15 games. How many points must he average per game in his next 3 games to take his overall average to 20 points per game?

c A cyclist rides for 5 hours at a certain speed and then for 4 hours at a speed 6 km/h greater than his original speed. If he rides 294 km altogether, what was his first speed?



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1F Solving quadratic equations

There are three approaches to solving a quadratic equation:

- factoring
- completing the square
- the quadratic formula.

This section reviews factoring and the quadratic formula. Completing the square is reviewed in Section 1H.

Solving a quadratic by factoring

This method is the simplest, but it only works in special cases.

10 SOLVING A QUADRATIC BY FACTORING

- Get all the terms on the left, then factor the left-hand side.
- Then use the principle that if $A \times B = 0$, then $A = 0$ or $B = 0$.

Example 19 1F

Solve the quadratic equation $5x^2 + 34x - 7 = 0$ by factoring.

SOLUTION

$$
\begin{aligned}
5x^2 + 34x - 7 &= 0 \\
5x^2 + 35x - x - 7 &= 0 && \text{(35 and -1 have sum 34 and product $-7 \times 5 = -35$)} \\
5x(x + 7) - (x + 7) &= 0 \\
(5x - 1)(x + 7) &= 0 && \text{(the LHS is now factored)} \\
5x - 1 = 0 \quad &\text{or} \quad x + 7 = 0 && \text{(one of the factors must be zero)} \\
x = \tfrac{1}{5} \quad &\text{or} \quad x = -7 && \text{(there are two solutions)}
\end{aligned}
$$

Solving a quadratic by the formula

This method works whether the solutions are rational numbers or involve surds. It will be proven in the last challenge question of Exercise 3E.

11 THE QUADRATIC FORMULA

- The solutions of $ax^2 + bx + c = 0$ are:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

- Always calculate $b^2 - 4ac$ first.
(Later, this quantity will be called the *discriminant* and given the symbol Δ.)

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Example 20 1F

Solve each quadratic equation using the quadratic formula.

a $5x^2 + 2x - 7 = 0$

b $3x^2 + 4x - 1 = 0$

SOLUTION

a For $5x^2 + 2x - 7 = 0$,
$a = 5$, $b = 2$ and $c = -7$.

Hence $b^2 - 4ac = 2^2 + 140$
$= 144$
$= 12^2$,

so $x = \dfrac{-2 + 12}{10}$ or $\dfrac{-2 - 12}{10}$
$= 1$ or $-1\frac{2}{5}$.

b For $3x^2 + 4x - 1 = 0$,
$a = 3$, $b = 4$ and $c = -1$.

Hence $b^2 - 4ac = 4^2 + 12$
$= 28$
$= 4 \times 7$,

so $x = \dfrac{-4 + 2\sqrt{7}}{6}$ or $\dfrac{-4 - 2\sqrt{7}}{6}$
$= \dfrac{-2 + \sqrt{7}}{3}$ or $\dfrac{-2 - \sqrt{7}}{3}$.

Exercise 1F

FOUNDATION

1 Solve:

a $x^2 = 9$ **b** $y^2 = 25$ **c** $a^2 - 4 = 0$
d $c^2 - 36 = 0$ **e** $1 - t^2 = 0$ **f** $x^2 = \frac{9}{4}$
g $4x^2 - 1 = 0$ **h** $9a^2 - 64 = 0$ **i** $25y^2 = 16$

2 Solve by factoring:

a $x^2 - 5x = 0$ **b** $y^2 + y = 0$ **c** $c^2 + 2c = 0$ **d** $k^2 - 7k = 0$
e $t^2 = t$ **f** $3a = a^2$ **g** $2b^2 - b = 0$ **h** $3u^2 + u = 0$
i $4x^2 + 3x = 0$ **j** $2a^2 = 5a$ **k** $3y^2 = 2y$ **l** $3n + 5n^2 = 0$

3 Solve by factoring:

a $x^2 + 4x + 3 = 0$ **b** $x^2 - 3x + 2 = 0$ **c** $x^2 + 6x + 8 = 0$
d $a^2 - 7a + 10 = 0$ **e** $t^2 - 4t - 12 = 0$ **f** $c^2 - 10c + 25 = 0$
g $n^2 - 9n + 8 = 0$ **h** $p^2 + 2p - 15 = 0$ **i** $a^2 - 10a - 24 = 0$
j $y^2 + 4y = 5$ **k** $p^2 = p + 6$ **l** $a^2 = a + 132$
m $c^2 + 18 = 9c$ **n** $8t + 20 = t^2$ **o** $u^2 + u = 56$
p $k^2 = 24 + 2k$ **q** $50 + 27h + h^2 = 0$ **r** $a^2 + 20a = 44$

DEVELOPMENT

4 Solve by factoring:

a $2x^2 + 3x + 1 = 0$ **b** $3a^2 - 7a + 2 = 0$ **c** $4y^2 - 5y + 1 = 0$
d $2x^2 + 11x + 5 = 0$ **e** $2x^2 + x - 3 = 0$ **f** $3n^2 - 2n - 5 = 0$
g $3b^2 - 4b - 4 = 0$ **h** $2a^2 + 7a - 15 = 0$ **i** $2y^2 - y - 15 = 0$
j $3y^2 + 10y = 8$ **k** $5x^2 - 26x + 5 = 0$ **l** $4t^2 + 9 = 15t$
m $13t + 6 = 5t^2$ **n** $10u^2 + 3u - 4 = 0$ **o** $25x^2 + 1 = 10x$
p $6x^2 + 13x + 6 = 0$ **q** $12b^2 + 3 + 20b = 0$ **r** $6k^2 + 13k = 8$

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5 Solve each equation using the quadratic formula. Give exact answers, followed by approximations correct to four significant figures where appropriate.

a $x^2 - x - 1 = 0$ **b** $y^2 + y = 3$ **c** $a^2 + 12 = 7a$
d $u^2 + 2u - 2 = 0$ **e** $c^2 - 6c + 2 = 0$ **f** $4x^2 + 4x + 1 = 0$
g $2a^2 + 1 = 4a$ **h** $5x^2 + 13x - 6 = 0$ **i** $2b^2 + 3b = 1$
j $3c^2 = 4c + 3$ **k** $4t^2 = 2t + 1$ **l** $x^2 + x + 1 = 0$

6 Solve by factoring:

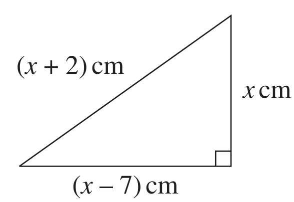
a $x = \dfrac{x+2}{x}$ **b** $a + \dfrac{10}{a} = 7$
c $y + \dfrac{2}{y} = \dfrac{9}{2}$ **d** $(5b - 3)(3b + 1) = 1$

7 Find the exact solutions of:

a $x = \dfrac{1}{x} + 2$ **b** $\dfrac{4x - 1}{x} = x$ **c** $a = \dfrac{a+4}{a-1}$ **d** $\dfrac{5m}{2} = 2 + \dfrac{1}{m}$

8 a If $y = px - ap^2$, find p, given that $a = 2$, $x = 3$ and $y = 1$.
b Given that $(x - a)(x - b) = c$, find x when $a = -2$, $b = 4$ and $c = 7$.
c Suppose that $S = \dfrac{n}{2}\big(2a + (n - 1)d\big)$. Find the positive value of n that gives $S = 80$ when $a = 4$ and $d = 6$.

9 a Find a positive integer that, when increased by 30, is 12 less than its square.
b Two positive numbers differ by 3 and the sum of their squares is 117. Find the numbers.
c Find the value of x in the diagram opposite.



CHALLENGE

10 Solve each equation.

a $\dfrac{5k + 7}{k - 1} = 3k + 2$ **b** $\dfrac{u + 3}{2u - 7} = \dfrac{2u - 1}{u - 3}$ **c** $\dfrac{y + 1}{y + 2} = \dfrac{3 - y}{y - 4}$
d $2(k - 1) = \dfrac{4 - 5k}{k + 1}$ **e** $\dfrac{2}{a + 3} + \dfrac{a + 3}{2} = \dfrac{10}{3}$ **f** $\dfrac{k + 10}{k - 5} - \dfrac{10}{k} = \dfrac{11}{6}$
g $\dfrac{3t}{t^2 - 6} = \sqrt{3}$ **h** $\dfrac{3m + 1}{3m - 1} - \dfrac{3m - 1}{3m + 1} = 2$

11 Solve each problem by constructing and then solving a quadratic equation.
a A rectangular area can be completely tiled with 200 square tiles. If the side length of each tile was increased by 1 cm, it would take only 128 tiles to tile the area. Find the side length of each tile.
b A photograph is 18 cm by 12 cm. It is to be surrounded by a frame of uniform width whose area is equal to the area of the photograph. Find the width of the frame.
c Two trains each make a journey of 330 km. One of the trains travels 5 km/h faster than the other and takes 30 minutes less time. Find the speeds of the trains.

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1G Solving simultaneous equations

There are two algebraic approaches to solving simultaneous equations — substitution and elimination. They can be applied to both linear and non-linear simultaneous equations.

Solution by substitution

This method can be applied whenever one of the equations can be solved for one of the variables.

12 SOLVING SIMULTANEOUS EQUATIONS BY SUBSTITUTION

- Solve one of the equations for one of the variables.
- Then substitute it into the other equation.

Example 21 1G

Solve each pair of simultaneous equations by substitution.

a $3x - 2y = 29$ (1)
$4x + y = 24$ (2)

b $y = x^2$ (1)
$y = x + 2$ (2)

SOLUTION

a Solving (2) for y, $y = 24 - 4x$. (2A)

Substituting (2A) into (1), $3x - 2(24 - 4x) = 29$
$x = 7.$

Substituting $x = 7$ into (1), $21 - 2y = 29$
$y = -4.$

Hence $x = 7$ and $y = -4$. (This should be checked in the original equations.)

b Substituting (1) into (2),

$$\begin{aligned} x^2 &= x + 2 \\ x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \\ x &= 2 \text{ or } -1. \end{aligned}$$

From (1), when $x = 2$, $y = 4$, and when $x = -1$, $y = 1$.
Hence $x = 2$ and $y = 4$, or $x = -1$ and $y = 1$. (Check in the original equations.)

Solution by elimination

This method, when it can be used, is more elegant, and usually involves less algebraic manipulation.

13 SOLVING SIMULTANEOUS EQUATIONS BY ELIMINATION

- Take suitable multiples of the equations so that one variable is eliminated when the equations are added or subtracted.

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Example 22 1G

Solve each pair of simultaneous equations by elimination.

a $3x - 2y = 29 \quad (1)$
$4x + 5y = 8 \quad (2)$

b $x^2 + y^2 = 53 \quad (1)$
$x^2 - y^2 = 45 \quad (2)$

SOLUTION

a Taking $4 \times (1)$ and $3 \times (2)$,

$$12x - 8y = 116 \qquad (1A)$$
$$12x + 15y = 24. \qquad (2A)$$

Subtracting (1A) from (2A),

$$23y = -92$$
$$\boxed{\div 23} \qquad y = -4.$$

Substituting into (1).

$$3x + 8 = 29$$
$$x = 7.$$

Hence $x = 7$ and $y = -4$.

b Adding (1) and (2),

$$2x^2 = 98$$
$$x^2 = 49.$$

Subtracting (2) from (1),

$$2y^2 = 8$$
$$y^2 = 4.$$

Hence $x = 7$ and $y = 2$,
or $x = 7$ and $y = -2$,
or $x = -7$ and $y = 2$,
or $x = -7$ and $y = -2$.

Exercise 1G

FOUNDATION

1 Solve by substituting the first equation into the second:

a $y = x$ and $2x + y = 9$
b $y = 2x$ and $3x - y = 2$
c $y = x - 1$ and $2x + y = 5$
d $a = 2b + 1$ and $a - 3b = 3$
e $p = 2 - q$ and $p - q = 4$
f $v = 1 - 3u$ and $2u + v = 0$

2 Solve by either adding or subtracting the two equations:

a $x + y = 5$ and $x - y = 1$
b $3x - 2y = 7$ and $x + 2y = -3$
c $2x + y = 9$ and $x + y = 5$
d $a + 3b = 8$ and $a + 2b = 5$
e $4c - d = 6$ and $2c - d = 2$
f $p - 2q = 4$ and $3p - 2q = 0$

DEVELOPMENT

3 Solve by substitution:

a $y = 2x$ and $3x + 2y = 14$
b $y = -3x$ and $2x + 5y = 13$
c $y = 4 - x$ and $x + 3y = 8$
d $x = 5y + 4$ and $3x - y = 26$
e $2x + y = 10$ and $7x + 8y = 53$
f $2x - y = 9$ and $3x - 7y = 19$
g $4x - 5y = 2$ and $x + 10y = 41$
h $2x + 3y = 47$ and $4x - y = 45$

4 Solve by elimination:

a $2x + y = 1$ and $x - y = -4$
b $2x + 3y = 16$ and $2x + 7y = 24$
c $3x + 2y = -6$ and $x - 2y = -10$
d $5x - 3y = 28$ and $2x - 3y = 22$
e $3x + 2y = 7$ and $5x + y = 7$
f $3x + 2y = 0$ and $2x - y = 56$
g $15x + 2y = 27$ and $3x + 7y = 45$
h $7x - 3y = 41$ and $3x - y = 17$
i $2x + 3y = 28$ and $3x + 2y = 27$
j $3x - 2y = 11$ and $4x + 3y = 43$

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5 Solve by substitution:

a $y = 2 - x$ and $y = x^2$

b $y = 2x - 3$ and $y = x^2 - 4x + 5$

c $y = 3x^2$ and $y = 4x - x^2$

d $x - y = 5$ and $y = x^2 - 11$

e $x - y = 2$ and $xy = 15$

f $3x + y = 9$ and $xy = 6$

6 Solve each problem by forming and then solving a pair of simultaneous equations.

a Find two numbers that differ by 16 and have a sum of 90.

b I paid 75 cents for a pen and a pencil. If the pen cost four times as much as the pencil, find the cost of each item.

c If 7 apples and 2 oranges cost \$4, and 5 apples and 4 oranges cost \$4.40, find the cost of each apple and orange.

d Twice as many adults as children attended a certain concert. If adult tickets cost \$8 each, child tickets cost \$3 each, and the total takings were \$418, find the numbers of adults and children who attended.

e A man is 3 times as old as his son. In 12 years' time he will be twice as old as his son. How old is each of them now?

f At a meeting of the members of a certain club, a proposal was voted on. If 357 members voted and the proposal was carried by a majority of 21, how many voted for and how many voted against?

CHALLENGE

7 Solve simultaneously:

a $x + y = 15$ and $x^2 + y^2 = 125$

b $x - y = 3$ and $x^2 + y^2 = 185$

c $2x + y = 5$ and $4x^2 + y^2 = 17$

d $x + y = 9$ and $x^2 + xy + y^2 = 61$

e $x + 2y = 5$ and $2xy - x^2 = 3$

f $3x + 2y = 16$ and $xy = 10$

8 Set up a pair of simultaneous equations to solve each problem.

a Kathy paid \$320 in cash for a CD player. If she paid in \$20 and \$10 notes, and there were 23 notes altogether, how many of each type were there?

b Two people are 16 km apart on a straight road. They start walking at the same time. If they walk towards each other, they will meet in 2 hours, but if they walk in the same direction (so that the distance between them is decreasing), they will meet in 8 hours. Find their walking speeds.

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1H Completing the square

Completing the square can be done with all quadratic equations, whereas factoring is only possible in special cases.

The review in this section is mostly restricted to monic quadratics, in which the coefficient of x^2 is 1. Chapter 3 will deal with non-monic quadratics. Chapter 3 will also require completing the square in a quadratic *function*, which is only slightly different from completing the square in a quadratic *equation*.

Perfect squares

The expansion of the quadratic $(x+5)^2$ is

$$(x+5)^2 = x^2 + 10x + 25.$$

Notice that the coefficient of x is twice 5, and the constant is the square of 5.

Reversing the process, the constant term in a perfect square can be found by taking half the coefficient of x and squaring the result.

14 COMPLETING THE SQUARE IN AN EXPRESSION $x^2 + bx + \cdots$

Halve the coefficient b of x, and square the result.

Example 23 **1H**

Complete the square for each expression.

a $x^2 + 16x + \cdots$ **b** $x^2 - 3x + \cdots$

SOLUTION

a The coefficient of x is 16, half of 16 is 8, and $8^2 = 64$,
so $x^2 + 16x + 64 = (x+8)^2$.

b The coefficient of x is -3, half of -3 is $-1\frac{1}{2}$, and $\left(-1\frac{1}{2}\right)^2 = 2\frac{1}{4}$,
so $x^2 - 3x + 2\frac{1}{4} = \left(x - 1\frac{1}{2}\right)^2$.

Solving a quadratic equation by completing the square

This process always works.

15 SOLVING A QUADRATIC EQUATION BY COMPLETING THE SQUARE

Complete the square in the quadratic by adding the same to both sides.

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Example 24 1H

Solve each quadratic equation by completing the square.

a $t^2 + 8t = 20$ **b** $x^2 - x - 1 = 0$

SOLUTION

a

$$\begin{aligned} t^2 + 8t &= 20 \\ \boxed{+16} \quad t^2 + 8t + 16 &= 36 \\ (t+4)^2 &= 36 \end{aligned}$$

$$t + 4 = 6 \text{ or } t + 4 = -6$$

$$t = 2 \text{ or } -10$$

b

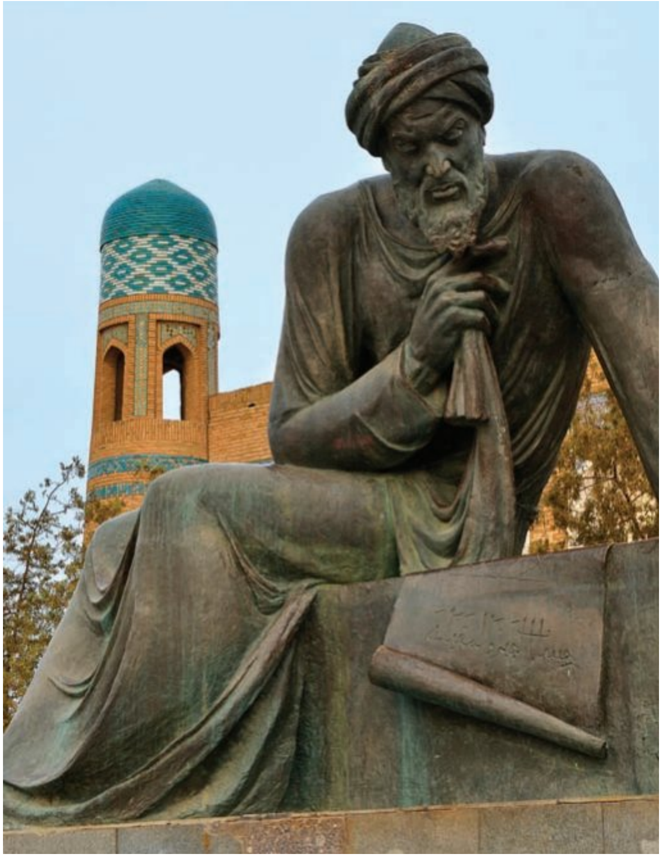
$$\begin{aligned} x^2 - x - 1 &= 0 \\ \boxed{+1} \quad x^2 - x &= 1 \\ \boxed{+\tfrac{1}{4}} \quad x^2 - x + \tfrac{1}{4} &= 1\tfrac{1}{4} \\ \left(x - \tfrac{1}{2}\right)^2 &= \tfrac{5}{4} \end{aligned}$$

$$x - \tfrac{1}{2} = \tfrac{1}{2}\sqrt{5} \text{ or } x - \tfrac{1}{2} = -\tfrac{1}{2}\sqrt{5}$$

$$x = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} \text{ or } \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}$$

The word 'Algebra'

Al-Khwarizmi was a famous and influential Persian mathematician who worked in Baghdad during the early ninth century when the Baghdad Caliphate excelled in science and mathematics. The Arabic word 'algebra' comes from *al-jabr*, a word from the title of his most important work, and means 'the restoration of broken parts' — a reference to the balancing of terms on both sides of an equation. Al-Khwarizmi's own name came into modern European languages as 'algorithm'.



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Exercise 1H

FOUNDATION

1 What constant must be added to each expression in order to create a perfect square?
a $x^2 + 2x$ b $y^2 - 6y$ c $a^2 + 10a$ d $m^2 - 18m$
e $c^2 + 3c$ f $x^2 - x$ g $b^2 + 5b$ h $t^2 - 9t$

2 Factor:
a $x^2 + 4x + 4$ b $y^2 + 2y + 1$ c $p^2 + 14p + 49$ d $m^2 - 12m + 36$
e $t^2 - 16t + 64$ f $x^2 + 20x + 100$ g $u^2 - 40u + 400$ h $a^2 - 24a + 144$

3 Copy and complete:
a $x^2 + 6x + \ldots = (x + \ldots)^2$ b $y^2 + 8y + \ldots = (y + \ldots)^2$
c $a^2 - 20a + \ldots = (a - \ldots)^2$ d $b^2 - 100b + \ldots = (b - \ldots)^2$
e $u^2 + u + \ldots = (u + \ldots)^2$ f $t^2 - 7t + \ldots = (t - \ldots)^2$
g $m^2 + 50m + \ldots = (m + \ldots)^2$ h $c^2 - 13c + \ldots = (c - \ldots)^2$

DEVELOPMENT

4 Solve each quadratic equation by completing the square.
a $x^2 - 2x = 3$ b $x^2 - 6x = 0$ c $a^2 + 6a + 8 = 0$
d $x^2 + 4x + 1 = 0$ e $x^2 - 10x + 20 = 0$ f $y^2 + 3y = 10$
g $b^2 - 5b - 14 = 0$ h $y^2 - y + 2 = 0$ i $a^2 + 7a + 7 = 0$

CHALLENGE

5 Solve, by dividing both sides by the coefficient of x^2 and then completing the square.
a $2x^2 - 4x - 1 = 0$ b $2x^2 + 8x + 3 = 0$ c $3x^2 + 6x + 5 = 0$
d $4x^2 + 4x - 3 = 0$ e $4x^2 - 2x - 1 = 0$ f $2x^2 - 10x + 7 = 0$

6 a If $x^2 + y^2 + 4x - 2y + 1 = 0$, show that $(x + 2)^2 + (y - 1)^2 = 4$.
b Show that the equation $x^2 + y^2 - 6x - 8y = 0$ can be written in the form
$$(x - a)^2 + (y - b)^2 = c,$$
where a, b and c are constants. Hence find a, b and c.
c If $x^2 + 1 = 10x + 12y$, show that $(x - 5)^2 = 12(y + 2)$.
d Find values for A, B and C if $y^2 - 6x + 16y + 94 = (y + C)^2 - B(x + A)$.

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Chapter 1 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 1 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Simplify:

a $-8y + 2y$ **b** $-8y - 2y$ **c** $-8y \times 2y$ **d** $-8y \div 2y$

2 Simplify:

a $-2a^2 - a^2$ **b** $-2a^2 - (-a^2)$ **c** $-2a^2 \times (-a^2)$ **d** $-2a^2 \div (-a^2)$

3 Simplify:

a $3t - 1 - t$ **b** $-6p + 3q + 10p$
c $7x - 4y - 6x + 2y$ **d** $2a^2 + 8a - 13 + 3a^2 - 11a - 5$

4 Simplify:

a $-6k^6 \times 3k^3$ **b** $-6k^6 \div 3k^3$ **c** $(-6k^6)^2$ **d** $(3k^3)^3$

5 Expand and simplify:

a $4(x + 3) + 5(2x - 3)$ **b** $8(a - 2b) - 6(2a - 3b)$ **c** $-(a - b) - (a + b)$
d $-4x^2(x + 3) - 2x^2(x - 1)$ **e** $(n + 7)(2n - 3)$ **f** $(r + 3)^2$
g $(y - 5)(y + 5)$ **h** $(3x - 5)(2x - 3)$ **i** $(t - 8)^2$
j $(2c + 7)(2c - 7)$ **k** $(4p + 1)^2$ **l** $(3u - 2)^2$

6 Factor:

a $18a + 36$ **b** $20b - 36$ **c** $9c^2 + 36c$
d $d^2 - 36$ **e** $e^2 + 13e + 36$ **f** $f^2 - 12f + 36$
g $36 - 25g^2$ **h** $h^2 - 9h - 36$ **i** $i^2 + 5i - 36$
j $2j^2 + 11j + 12$ **k** $3k^2 - 7k - 6$ **l** $5l^2 - 14l + 8$
m $4m^2 + 4m - 15$ **n** $mn + m + pn + p$ **o** $p^3 + 9p^2 + 4p + 36$
p $qt - rt - 5q + 5r$ **q** $u^2w + vw - u^2x - vx$ **r** $x^2 - y^2 + 2x - 2y$

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7 Simplify:

a $\frac{x}{2} + \frac{x}{4}$ **b** $\frac{x}{2} - \frac{x}{4}$ **c** $\frac{x}{2} \times \frac{x}{4}$ **d** $\frac{x}{2} \div \frac{x}{4}$

e $\frac{3a}{2b} + \frac{2a}{3b}$ **f** $\frac{3a}{2b} - \frac{2a}{3b}$ **g** $\frac{3a}{2b} \times \frac{2a}{3b}$ **h** $\frac{3a}{2b} \div \frac{2a}{3b}$

i $\frac{x}{y} + \frac{y}{x}$ **j** $\frac{x}{y} - \frac{y}{x}$ **k** $\frac{x}{y} \times \frac{y}{x}$ **l** $\frac{x}{y} \div \frac{y}{x}$

8 Simplify:

a $\frac{x+4}{5} + \frac{x-5}{3}$ **b** $\frac{5}{x+4} + \frac{3}{x-5}$ **c** $\frac{x+1}{2} - \frac{x-4}{5}$

d $\frac{2}{x+1} - \frac{5}{x-4}$ **e** $\frac{x}{2} - \frac{x+3}{4}$ **f** $\frac{2}{x} - \frac{4}{x+3}$

9 Factor each expression where possible, then simplify it.

a $\frac{6a+3b}{10a+5b}$ **b** $\frac{2x-2y}{x^2-y^2}$ **c** $\frac{x^2+2x-3}{x^2-5x+4}$

d $\frac{2x^2+3x+1}{2x^3+x^2+2x+1}$ **e** $\frac{a+b}{a^2+2ab+b^2}$ **f** $\frac{3x^2-19x-14}{9x^2-4}$

10 Solve each linear equation.

a $3x+5=17$ **b** $3(x+5)=17$ **c** $\frac{x+5}{3}=17$

d $\frac{x}{3}+5=17$ **e** $7a-4=2a+11$ **f** $7(a-4)=2(a+11)$

g $\frac{a-4}{7}=\frac{a+11}{2}$ **h** $\frac{a}{7}-4=\frac{a}{2}+11$

11 Solve each quadratic equation by factoring the left-hand side.

a $a^2-49=0$ **b** $b^2+7b=0$ **c** $c^2+7c+6=0$

d $d^2+6d-7=0$ **e** $e^2-5e+6=0$ **f** $2f^2-f-6=0$

g $2g^2-13g+6=0$ **h** $3h^2+2h-8=0$

12 Solve using the quadratic formula. Write the solutions in simplest exact form.

a $x^2-4x+1=0$ **b** $y^2+3y-3=0$ **c** $t^2+6t+4=0$

d $3x^2-2x-2=0$ **e** $2a^2+5a-5=0$ **f** $4k^2-6k-1=0$

13 Solve each quadratic by completing the square on the left-hand side.

a $x^2+4x=6$ **b** $y^2-6y+3=0$ **c** $x^2-2x=12$ **d** $y^2+10y+7=0$

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2 Numbers and surds

Arithmetic is the study of numbers and operations on them. This short chapter reviews whole numbers, integers, rational numbers and real numbers, with particular attention to the arithmetic of surds and their approximations. Most of this material will be familiar from earlier years.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

2A Whole numbers, integers and rationals

Our ideas about numbers arise from the two quite distinct sources:

- The *whole numbers*, the *integers* and the *rational numbers* are developed from counting.
- The *real numbers* are developed from geometry and the number line.

This section very briefly reviews whole numbers, integers and rational numbers, with particular attention to percentages and recurring decimals.

The whole numbers

Counting is the first operation in arithmetic. Counting things such as people in a room requires *zero* (if the room is empty) and then the successive numbers 1, 2, 3, …, generating all the *whole numbers*:

$$0, 1, 2, 3, 4, 5, 6, \dots$$

The number zero is the first number on this list, but there is no last number, because every number is followed by another number, distinct from all previous numbers. The list is therefore called *infinite*, which means that it never 'finishes'. The symbol $\mathbb{N}$ is generally used for the set of whole numbers.

A non-zero whole number can be factored, in one and only one way, into the product of prime numbers, where a *prime number* is a whole number greater than 1 whose only divisors are itself and 1. The primes form a sequence whose distinctive pattern has confused every mathematician since Greek times:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, \dots$$

The whole numbers greater than 1 and not prime are called *composite numbers*:

$$4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, 32, \dots$$

The whole numbers 0 and 1 are special cases, being neither prime nor composite.

1 THE SET $\mathbb{N}$ OF WHOLE NUMBERS

- The *whole numbers* $\mathbb{N}$ are 0, 1, 2, 3, 4, 5, 6, …
- Every whole number except 0 and 1 is either *prime* or *composite*, and every composite number can be factored into primes in one and only one way.
- When whole numbers are added or multiplied, the result is a whole number.

The integers

Any two whole numbers can be *added* or *multiplied*, and the result is another whole number. *Subtraction*, however, requires the *negative integers* as well:

$$\dots, -6, -5, -4, -3, -2, -1$$

so that calculations such as $5 - 7 = -2$ can be completed. The symbol $\mathbb{Z}$ (from German 'Zahlen' meaning numbers) is conventionally used for the set of integers.

2 THE SET $\mathbb{Z}$ OF INTEGERS

- The *integers* $\mathbb{Z}$ are $\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots$
- When integers are added, subtracted or multiplied, the result is an integer.

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The rational numbers

A problem such as, 'Divide 7 cakes into 3 equal parts', leads naturally to *fractions*, where the whole is 'fractured' or 'broken' into pieces. Thus we have the system of *rational numbers*, which are numbers that can be written as the 'ratio' of two integers. Here are some examples of rational numbers written as single fractions:

$$2\tfrac{1}{3} = \frac{7}{3} \qquad -\frac{1}{3} = \frac{-1}{3} \qquad 30 \div 24 = \frac{5}{4} \qquad 3.72 = \frac{372}{100} \qquad 4 = \frac{4}{1}$$

The symbol $\mathbb{Q}$ for 'quotient' is conventionally used for the set of rational numbers.

Operations on the rational numbers

Addition, multiplication, subtraction and division (except by 0) can all be carried out within the rational numbers.

- Rational numbers are simplified by dividing top and bottom by their HCF (highest common factor). For example, 21 and 35 have HCF 7, so

$$\begin{aligned}\frac{21}{35} &= \frac{21 \div 7}{35 \div 7}\\ &= \frac{3}{5}\end{aligned}$$

- Rational numbers are added and subtracted using the LCM (lowest common multiple) of their denominators. For example, 6 and 8 have LCM 24, so

$$\begin{aligned}\frac{1}{6} + \frac{5}{8} &= \frac{1 \times 4}{24} + \frac{5 \times 3}{24}\\ &= \frac{19}{24}\end{aligned} \qquad \begin{aligned}\frac{1}{6} - \frac{5}{8} &= \frac{1 \times 4}{24} - \frac{5 \times 3}{24}\\ &= -\frac{11}{24}\end{aligned}$$

- Fractions are multiplied by multiplying the numerators and multiplying the denominators, after first cancelling out any common factors. To divide by a fraction, multiply by its reciprocal.

$$\begin{aligned}\frac{10}{21} \times \frac{9}{25} &= \frac{2}{7} \times \frac{3}{5}\\ &= \frac{6}{35}\end{aligned} \qquad \begin{aligned}\frac{8}{21} \div \frac{3}{4} &= \frac{8}{21} \times \frac{4}{3}\\ &= \frac{32}{63}\end{aligned}$$

The symbol $\mathbb{Q}$ for 'quotient' is conventionally used for the set of rational numbers.

3 THE SET $\mathbb{Q}$ OF RATIONAL NUMBERS

- The *rational numbers* $\mathbb{Q}$ are the numbers that can be written as *fractions* $\frac{a}{b}$, where a and b are integers and $b \neq 0$.
- Every integer a can be written as a fraction $\frac{a}{1}$, and so is a rational number.
- When rational numbers are added, subtracted, multiplied and divided (but not by zero), the result is a rational number.

Decimal notation — terminating or recurring decimals

Decimal notation extends place value to negative powers of 10. For example:

$$123.456 = 1 \times 10^2 + 2 \times 10^1 + 3 \times 10^0 + 4 \times 10^{-1} + 5 \times 10^{-2} + 6 \times 10^{-3}$$

Such a number can be written as a fraction $\frac{123\,456}{1000}$, and so is a rational number.

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If a rational number can be written as a fraction whose denominator is a power of 10, then it can easily be written as a *terminating decimal*:

$$\tfrac{3}{25} = \tfrac{12}{100} = 0.12 \qquad \text{and} \qquad 578\tfrac{3}{50} = 578 + \tfrac{6}{100} = 578.06$$

If a rational number cannot be written with a power of 10 as its denominator, then repeated division will yield an infinite string of digits in its decimal representation. This string will cycle once the same remainder recurs, giving a *recurring decimal*.

$$\begin{aligned} \tfrac{2}{3} &= 0.666666666666\ldots = 0.\dot{6} && \text{(which has cycle length 1)} \\ 6\tfrac{3}{7} &= 6.428571428571\ldots = 6.\dot{4}2857\dot{1} && \text{(which has cycle length 6)} \\ 4\tfrac{7}{22} &= 4.31818181818\ldots = 4.3\dot{1}\dot{8} && \text{(which has cycle length 2)} \end{aligned}$$

Conversely, every recurring decimal can be written as a fraction — such calculations are discussed in Year 12 in the context of geometric series.

Percentages

Many practical situations involving fractions, decimals and ratios are commonly expressed in terms of percentages. The symbol % evolved from the handwritten 'per centum', meaning 'per hundred' — interpret the symbol as '/100', that is, 'over 100'.

4 PERCENTAGES

- To convert a fraction to a percentage, multiply by $\tfrac{100}{1}\%$:
$$\frac{3}{20} = \frac{3}{20} \times \frac{100}{1}\% = 15\%$$
- To convert a percentage to a fraction, replace % by $\times \tfrac{1}{100}$:
$$15\% = 15 \times \frac{1}{100} = \frac{3}{20}$$

Many problems are best solved by the *unitary method*, illustrated below.

Example 1 2A

a A table marked \$1400 has been discounted by 30%. How much does it now cost?

b A table discounted by 30% now costs \$1400. What was the original price?

SOLUTION

a
$$\begin{array}{lrll} & 100\% & \text{is} & \$1400 \\ \boxed{\div 10} & 10\% & \text{is} & \$140 \\ \boxed{\times 7} & 70\% & \text{is} & \$980 \end{array}$$
so the discounted price is \$980.

b
$$\begin{array}{lrll} & 70\% & \text{is} & \$1400 \\ \boxed{\div 7} & 10\% & \text{is} & \$200 \\ \boxed{\times 10} & 100\% & \text{is} & \$2000 \end{array}$$
so the original price was \$2000.

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Exercise 2A

FOUNDATION

Note: Questions **1–11** are non-calculator questions.

1 Write as a fraction in lowest terms:

a 30% b 80% c 75% d 5%

2 Write as a decimal:

a 60% b 27% c 9% d 16.5%

3 Write as a percentage:

a $\frac{1}{4}$ b $\frac{2}{5}$ c $\frac{6}{25}$ d $\frac{13}{20}$

4 Write as a percentage:

a 0.32 b 0.09 c 0.225 d 1.5

5 Factor into primes:

a 35 b 18 c 90 d 220

6 Cancel each fraction down to lowest terms.

a $\frac{4}{12}$ b $\frac{8}{10}$ c $\frac{10}{15}$ d $\frac{21}{28}$ e $\frac{16}{40}$

f $\frac{21}{45}$ g $\frac{24}{42}$ h $\frac{45}{54}$ i $\frac{36}{60}$ j $\frac{54}{72}$

7 Express each fraction as a decimal by rewriting it with denominator 10, 100 or 1000.

a $\frac{1}{2}$ b $\frac{1}{5}$ c $\frac{3}{5}$ d $\frac{3}{4}$

e $\frac{1}{25}$ f $\frac{7}{20}$ g $\frac{1}{8}$ h $\frac{5}{8}$

8 Express each terminating decimal as a fraction in lowest terms.

a 0.4 b 0.25 c 0.15 d 0.16

e 0.78 f 0.005 g 0.375 h 0.264

9 Express each fraction as a recurring decimal by dividing the numerator by the denominator.

a $\frac{1}{3}$ b $\frac{2}{3}$ c $\frac{1}{9}$ d $\frac{5}{9}$

e $\frac{3}{11}$ f $\frac{1}{11}$ g $\frac{1}{6}$ h $\frac{5}{6}$

10 Find the lowest common denominator, then simplify:

a $\frac{1}{2}+\frac{1}{4}$ b $\frac{3}{10}+\frac{2}{5}$ c $\frac{1}{2}+\frac{1}{3}$ d $\frac{2}{3}-\frac{2}{5}$

e $\frac{1}{6}+\frac{1}{9}$ f $\frac{5}{12}-\frac{3}{8}$ g $\frac{7}{10}+\frac{2}{15}$ h $\frac{2}{25}-\frac{1}{15}$

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11 Find the value of:

a $\frac{1}{4} \times 20$ **b** $\frac{2}{3} \times 12$ **c** $\frac{1}{2} \times \frac{1}{5}$ **d** $\frac{1}{3} \times \frac{3}{7}$ **e** $\frac{2}{5} \times \frac{5}{8}$

f $2 \div \frac{1}{3}$ **g** $\frac{3}{4} \div 3$ **h** $\frac{1}{3} \div \frac{1}{2}$ **i** $1\frac{1}{2} \div \frac{3}{8}$ **j** $\frac{5}{12} \div 1\frac{2}{3}$

DEVELOPMENT

12 a Find 12% of \$5.

b Find 7.5% of 200 kg.

c Increase \$6000 by 30%.

d Decrease $1\frac{1}{2}$ hours by 20%.

13 Express each fraction as a decimal.

a $\frac{33}{250}$ **b** $\frac{1}{40}$ **c** $\frac{5}{16}$ **d** $\frac{27}{80}$

e $\frac{7}{12}$ **f** $1\frac{9}{11}$ **g** $\frac{2}{15}$ **h** $\frac{13}{55}$

14 a Steve's council rates increased by 5% this year to \$840. What were his council rates last year?

b Joanne received a 10% discount on a pair of shoes. If she paid \$144, what was the original price?

c Marko spent \$135 this year at the Easter Show, a 12.5% increase on last year. How much did he spend last year?

CHALLENGE

15 Express each fraction in lowest terms, without using a calculator.

a $\frac{588}{630}$ **b** $\frac{455}{1001}$ **c** $\frac{500}{1000000}$

16 a Use your calculator to find the the recurring decimals for $\frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \frac{4}{11}, \ldots, \frac{10}{11}$. Is there a pattern?

b Use your calculator to find the recurring decimals for $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$ and $\frac{6}{7}$. Is there a pattern?

17 The numbers you obtain in this question may vary depending on the calculator used.

a Use your calculator to express $\frac{1}{3}$ as a decimal by entering $1 \div 3$.

b Subtract 0.33333333 from this, multiply the result by 10^8, and then take the reciprocal.

c Show arithmetically that the final answer in part **b** is 3. Is the answer on your calculator also equal to 3? What does this tell you about the way fractions are stored on a calculator?

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2B Real numbers and approximations

This section introduces the set $\mathbb{R}$ of real numbers, which are based not on counting, but on geometry — they are the points on the number line. They certainly include all rational numbers, but as we shall see, they also include many more numbers that cannot be written as fractions.

Dealing with real numbers that are not rational requires special symbols, such as $\sqrt{}$ and π, but when a real number needs to be approximated, a decimal is usually the best approach, written to as many decimal places as is necessary.

Decimals are used routinely in mathematics and science for two good reasons:

- Any two decimals can easily be compared with each other.
- Any quantity can be approximated 'as closely as we like' by a decimal.

Every measurement is only approximate, no matter how good the instrument, and rounding using decimals is a useful way of showing how accurate it is.

Rounding to a certain number of decimal places

The rules for rounding a decimal are:

5 RULES FOR ROUNDING A DECIMAL

To round a decimal to, say, *two decimal places*, look at the *third digit*.

- If the *third* digit is 0, 1, 2, 3 or 4, leave the *second* digit alone.
- If the *third* digit is 5, 6, 7, 8 or 9, increase the *second* digit by 1.

Always use $\doteqdot$ rather than $=$ when a quantity has been rounded or approximated.

For example,

$3.8472 \doteqdot 3.85$, correct to two decimal places. (look at 7, the third digit)
$3.8472 \doteqdot 3.8$, correct to one decimal place. (look at 4, the second digit)

Scientific notation and rounding to a certain number of significant figures

The very large and very small numbers common in astronomy and atomic physics are easier to comprehend when they are written in *scientific notation*:

$1\,234\,000 = 1.234 \times 10^6$ (there are four significant figures)
$0.000065432 = 6.5432 \times 10^{-5}$ (there are five significant figures)

The digits in the first factor are called the *significant figures* of the number. It is often more sensible to round a quantity correct to a given number of significant figures rather than to a given number of decimal places.

To round, say to *three significant figures*, look at the *fourth* digit. If it is 5, 6, 7, 8 or 9, increase the *third* digit by 1. Otherwise, leave the *third* digit alone.

$3.0848 \times 10^9 \doteqdot 3.08 \times 10^9$, correct to three significant figures.
$2.789654 \times 10^{-29} \doteqdot 2.790 \times 10^{-29}$, correct to four significant figures.

The number can be in normal notation and still be rounded this way:

$31.203 \doteqdot 31.20$, correct to four significant figures.

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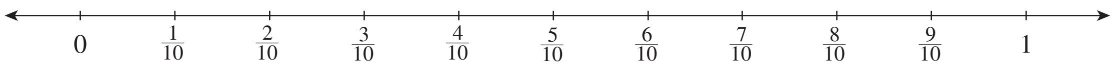
Unfortunately, this may be ambiguous. For example, when we see a number such as 3200, we do not know whether it has been rounded to 2, 3 or 4 significant figures. That can only be conveyed by changing to scientific notation and writing,

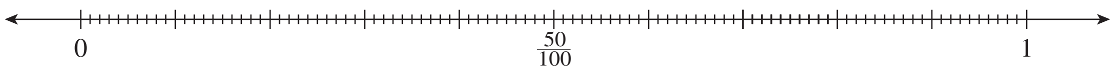
$$3.2 \times 10^3 \quad \text{or} \quad 3.20 \times 10^3 \quad \text{or} \quad 3.200 \times 10^3.$$

There are numbers that are not rational

At first glance, it would seem reasonable to believe that all the numbers on the number line are rational, because the rational numbers are clearly spread 'as finely as we like' along the whole number line. Between 0 and 1 there are 9 rational numbers with denominator 10:



Between 0 and 1 there are 99 rational numbers with denominator 100:



Most points on the number line, however, represent numbers that cannot be written as fractions, and are called *irrational numbers*. Some of the most important numbers in this course are irrational, such as $\sqrt{2}$ and π, and the number e that will be introduced in Chapter 9.

The square root of 2 is irrational

The number $\sqrt{2}$ is particularly important, because by Pythagoras' theorem, $\sqrt{2}$ is the diagonal of a unit square. Here is a proof by contradiction that $\sqrt{2}$ is an irrational number — regard this proof as extension.

Suppose that $\sqrt{2}$ were a rational number.

Then $\sqrt{2}$ could be written as a fraction $\dfrac{a}{b}$ in lowest terms.

That is, $\sqrt{2} = \dfrac{a}{b}$, where a and b have no common factor.

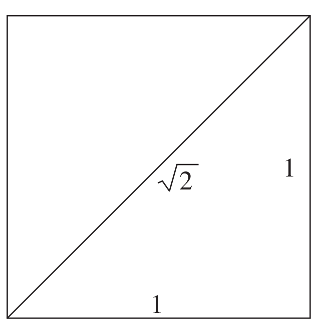
We know that $b > 1$ because $\sqrt{2}$ is not a whole number.

Squaring, $2 = \dfrac{a^2}{b^2}$, where $b^2 > 1$ because $b > 1$.

Because $\dfrac{a}{b}$ is in lowest terms, $\dfrac{a^2}{b^2}$ is also in lowest terms,

which is impossible, because $\dfrac{a^2}{b^2} = 2$, but $b^2 > 1$.

This is a contradiction, so $\sqrt{2}$ cannot be a rational number.



The Greek mathematicians were greatly troubled by the existence of irrational numbers. Their concerns can still be seen in modern English, where the word 'irrational' means both 'not a fraction' and 'not reasonable'.

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The real numbers and the number line

The whole numbers, the integers, and the rational numbers are based on *counting*. The existence of irrational numbers, however, means that this approach to arithmetic is inadequate, and a more general idea of number is needed. We have to turn away from counting and make use of *geometry*.

6 DEFINITION OF THE SET $\mathbb{R}$ OF REAL NUMBERS

- The *real numbers* $\mathbb{R}$ are defined to be all the points on the number line.
- All rational numbers are real, but real numbers such as $\sqrt{2}$ and π are irrational.

At this point, geometry replaces counting as the basis of arithmetic.

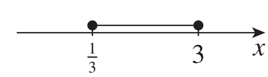
An irrational real number cannot be written as a fraction, or as a terminating or recurring decimal. In this course, such as a number is usually specified in exact form, such as $x = \sqrt{2}$ or $x = \pi$, or as a decimal approximation correct to a certain number of significant figures, such as $x \doteqdot 1.4142$ or $x \doteqdot 3.1416$. Very occasionally, a fractional approximation is useful or traditional, such as $\pi \doteqdot 3\frac{1}{7}$.

The real numbers are often referred to as the *continuum*, because the rationals, despite being dense, are scattered along the number line like specks of dust, but do not 'join up'. For example, the rational multiples of $\sqrt{2}$, which are all irrational, are just as dense on the number line as the rational numbers. It is only the real line itself that is completely joined up, to be the continuous line of geometry rather than falling apart into an infinitude of discrete points.

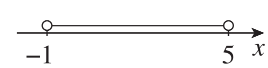
Open and closed intervals

Any connected part of the real number line is called an *interval*.

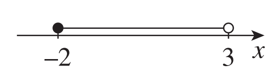
- An interval such as $\frac{1}{3} \le x \le 3$ is called a *closed interval* because it contains all its endpoints.



- An interval such as $-1 < x < 5$ is called an *open interval* because it does not contain any of its endpoints.



- An interval such as $-2 \le x < 3$ is neither open nor closed (the word *half-closed* is often used).

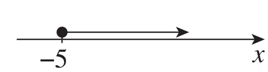


In diagrams, an endpoint is represented by a *closed circle* • if it is contained in the interval, and by an *open circle* ∘ if it is not contained in the interval.

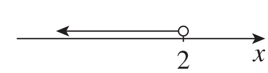
Bounded and unbounded intervals

The three intervals above are *bounded* because they have two endpoints, which bound the interval. An *unbounded interval* in contrast is either open or closed, and the direction that continues towards ∞ or $-\infty$ is represented by an arrow.

- The unbounded interval $x \ge -5$ is a *closed interval* because it contains all its endpoints (it only has one).



- The unbounded interval $x < 2$ is an *open interval* because it does not contain any of its endpoints (it only has one).



- The real line itself is an unbounded interval without any endpoints.

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7 INTERVALS

- An *interval* is a connected part of the number line.
- A *closed interval* such as $\frac{1}{3} \le x \le 3$ contains its endpoints.
- An *open interval* such as $-1 < x < 5$ does not contain its endpoints.
- An interval such as $-2 \le x < 3$ is neither open nor closed.
- A *bounded interval* has two endpoints, which bound the interval.
- An *unbounded interval* such as $x \ge -5$ continues to ∞ or to $-\infty$ (or both).

A single point is regarded as a *degenerate interval*, and is closed. The empty set is sometimes also regarded as a degenerate interval.

An alternative notation using round and square brackets will be introduced in Year 12.

Exercise 2B

FOUNDATION

1 Classify these real numbers as rational or irrational. Express those that are rational in the form $\frac{a}{b}$ in lowest terms, where a and b are integers.

a -3 **b** $1\frac{1}{2}$ **c** $\sqrt{3}$ **d** $\sqrt{4}$ **e** $\sqrt[3]{27}$

f $\sqrt[4]{8}$ **g** $\sqrt{\frac{4}{9}}$ **h** 0.45 **i** 12% **j** 0.333

k $0.\dot{3}$ **l** $3\frac{1}{7}$ **m** π **n** 3.14 **o** 0

2 Write each number correct to one decimal place.

a 0.32 **b** 5.68 **c** 12.75

d 0.05 **e** 3.03 **f** 9.96

3 Write each number correct to two significant figures.

a 0.429 **b** 5.429 **c** 5.029

d 0.0429 **e** 429 **f** 4290

4 Use a calculator to find each number correct to three decimal places.

a $\sqrt{10}$ **b** $\sqrt{47}$ **c** $\frac{9}{16}$

d $\frac{37}{48}$ **e** π **f** π^2

5 Use a calculator to find each number correct to three significant figures.

a $\sqrt{58}$ **b** $\sqrt[3]{133}$ **c** 62^2

d 14^5 **e** $\sqrt[4]{0.3}$ **f** 124^{-1}

6 To how many significant figures is each of these numbers accurate?

a 0.04 **b** 0.40 **c** 0.404

d 0.044 **e** 4.004 **f** 400

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DEVELOPMENT

7 **a** Classify each interval as open or closed or neither (that is, half-closed).

i $0 \le x \le 7$ **ii** $x > 5$ **iii** $x \le 7$

iv $5 < x \le 15$ **v** $x < -1$ **vi** $-4 < x < 10$

vii $x \ge 6$ **viii** $-4 \le x < -3$

b Classify each interval in part **a** as bounded or unbounded.

8 Write each interval in symbols, then sketch it on a separate number line.

a The real numbers greater than -2 and less than 5.

b The real numbers greater than or equal to -3 and less than or equal to 0.

c The real numbers less than 7.

d The real numbers less than or equal to -6.

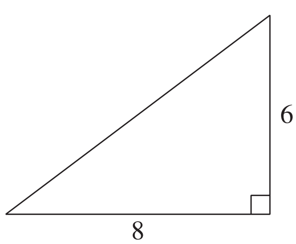
9 Use a calculator to evaluate each expression correct to three decimal places.

a $\dfrac{67 \times 29}{43}$ **b** $\dfrac{67 + 29}{43}$ **c** $\dfrac{67}{43 \times 29}$

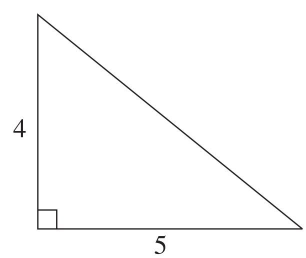
d $\dfrac{67}{43 + 29}$ **e** $\dfrac{67 + 29}{43 + 71}$ **f** $\dfrac{67 + 71}{43 \times 29}$

10 Use Pythagoras' theorem to find the length of the unknown side in each triangle, and state whether it is rational or irrational.

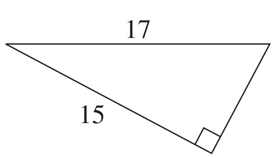
a



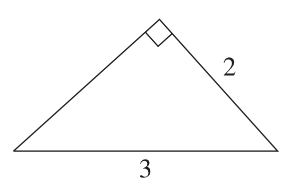
b



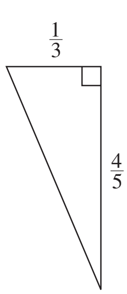
c



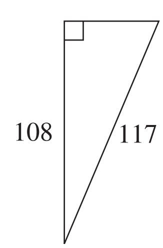
d



e



f



11 Calculate, correct to four significant figures:

a $10^{-0.4}$

b $\dfrac{1}{240 - 13 \times 17}$

c $\dfrac{\sqrt{6.5 + 8.3}}{2.7}$

d $\sqrt[3]{10.57 \times 12.83}$

e $\dfrac{3.5 \times 10^4}{2.3 \times 10^5}$

f $20000 \times (1.01)^{25}$

g $\dfrac{11.3}{\sqrt{19.5 - 14.7}}$

h $\dfrac{3\frac{2}{3} + 5\frac{1}{4}}{4\frac{1}{2} + 6\frac{4}{5}}$

i $(87.3 \times 10^4) \div (0.629 \times 10^{-8})$

j $\dfrac{\sqrt{3} + \sqrt[3]{4}}{\sqrt[4]{5} + \sqrt[5]{6}}$

k $\dfrac{\left(\frac{2}{5}\right)^4 \times \left(\frac{3}{4}\right)^5}{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{3}\right)^3}$

l $\sqrt{\dfrac{36.41 - 19.57}{23.62 - 11.39}}$

CHALLENGE

12 a Identify the approximation of π that seems to be used in 1 Kings 7:23, '*He made the Sea of cast metal, circular in shape, measuring ten cubits from rim to rim and five cubits high. It took a line of thirty cubits to measure around it.*'

b Many centuries later, in the 3rd century BC, Archimedes used regular polygons with 96 sides to prove that $223/71 < \pi < 22/7$. To how many significant figures is this correct?

c What is the current record for the computation of π?

d It is well known that a python has a length of about 3.14159 yards. How many pythons can be lined up between the wickets of a cricket pitch (22 yards)?

Use a calculator to answer questions **13** and **14**. Write each answer in scientific notation.

13 The speed of light is approximately 2.997925×10^8 m/s.

a How many metres are there in a light-year (which is the distance that light travels in a year)? Assume that there are $365\frac{1}{4}$ days in a year and write your answer in metres, correct to three significant figures.

b The nearest large galaxy is Andromeda, which is estimated to be 2560000 light-years away. How far is that in metres, correct to two significant figures?

c The time since the Big Bang is estimated to be 13.8 billion years. How long is that in seconds, correct to three significant figures?

d How far has light travelled since the Big Bang? Give your answer in metres, correct to two significant figures.

14 The mass of a proton is 1.6726×10^{-27} kg and the mass of an electron is 9.1095×10^{-31} kg.

a Calculate, correct to four significant figures, the ratio of the mass of a proton to the mass of an electron.

b How many protons, correct to one significant figure, are there in 1 kg?

15 Prove that $\sqrt{3}$ is irrational. (Adapt the given proof that $\sqrt{2}$ is irrational.)

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2C Surds and their arithmetic

Numbers such as $\sqrt{2}$ and $\sqrt{3}$ occur constantly in this course because they occur in the solutions of quadratic equations and when using Pythagoras' theorem. The last three sections of this chapter review various methods of dealing with them.

Square roots and positive square roots

The square of any real number is positive, except that $0^2 = 0$. Hence a negative number cannot have a square root, and the only square root of 0 is 0 itself.

A positive number, however, has two square roots, which are the opposites of each other. For example, the square roots of 9 are 3 and -3.

Note that the well-known symbol $\sqrt{x}$ does not mean 'the square root of x'. It is defined to mean the positive square root of x (or zero, if $x = 0$).

8 DEFINITION OF THE SYMBOL $\sqrt{x}$

- For $x > 0$, $\sqrt{x}$ means the *positive* square root of x.
- For $x = 0$, $\sqrt{0} = 0$.
- For $x < 0$, $\sqrt{x}$ is undefined.

For example, $\sqrt{25} = 5$, even though 25 has two square roots, -5 and 5. The symbol for the negative square root of 25 is $-\sqrt{25}$.

Cube roots

Cube roots are less complicated. Every number has exactly one cube root, so the symbol $\sqrt[3]{x}$ simply means 'the cube root of x'. For example:

$$\sqrt[3]{8} = 2 \quad \text{and} \quad \sqrt[3]{-8} = -2 \quad \text{and} \quad \sqrt[3]{0} = 0.$$

What is a surd?

The word *surd* is often used to refer to any expression involving a square or higher root. More precisely, however, surds do not include expressions such as $\sqrt{\frac{4}{9}}$ and $\sqrt[3]{8}$, which can be simplified to rational numbers.

9 SURDS

An expression $\sqrt[n]{x}$, where x is a rational number and $n \geq 2$ is an integer, is called a *surd* if it is not itself a rational number.

The word 'surd' is related to 'absurd' — surds are irrational.

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Simplifying expressions involving surds

Here are some laws from earlier years for simplifying expressions involving square roots. The first pair restate the definition of the square root, and the second pair are easily proven by squaring.

10 LAWS CONCERNING SURDS

Let a and b be positive real numbers. Then:

$$\sqrt{a^2} = a \qquad\qquad \sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

and

$$\left(\sqrt{a}\right)^2 = a \qquad\qquad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

Taking out square divisors

A surd such as $\sqrt{500}$ is not regarded as being simplified, because 500 is divisible by the square number 100, so $\sqrt{500}$ can be written as $10\sqrt{5}$:

$$\sqrt{500} = \sqrt{100 \times 5} = \sqrt{100} \times \sqrt{5} = 10\sqrt{5}.$$

11 SIMPLIFYING A SURD

- Check the number inside the square root for divisibility by one of the squares

  $4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, \ldots$

- Continue until the number inside the square root sign has no more square divisors (apart from 1).

Example 2 — 2C

Simplify these expressions involving surds.

a $\sqrt{108}$ **b** $5\sqrt{27}$ **c** $\sqrt{216}$

SOLUTION

a $\sqrt{108} = \sqrt{36 \times 3}$
$= \sqrt{36} \times \sqrt{3}$
$= 6\sqrt{3}$

b $5\sqrt{27} = 5\sqrt{9 \times 3}$
$= 5 \times \sqrt{9} \times \sqrt{3}$
$= 15\sqrt{3}$

c $\sqrt{216} = \sqrt{4} \times \sqrt{54}$
$= \sqrt{4} \times \sqrt{9} \times \sqrt{6}$
$= 6\sqrt{6}$

Example 3 — 2C

Simplify the surds in these expressions, then collect like terms.

a $\sqrt{44} + \sqrt{99}$ **b** $\sqrt{72} - \sqrt{50} + \sqrt{12}$

SOLUTION

a $\sqrt{44} + \sqrt{99} = 2\sqrt{11} + 3\sqrt{11}$
$= 5\sqrt{11}$

b $\sqrt{72} - \sqrt{50} + \sqrt{12}$
$= 6\sqrt{2} - 5\sqrt{2} + 2\sqrt{3}$
$= \sqrt{2} + 2\sqrt{3}$

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Exercise 2C

FOUNDATION

1 Write down the value of:

a $\sqrt{16}$ b $\sqrt{36}$ c $\sqrt{81}$ d $\sqrt{121}$
e $\sqrt{144}$ f $\sqrt{400}$ g $\sqrt{2500}$ h $\sqrt{10000}$

2 Simplify:

a $\sqrt{12}$ b $\sqrt{18}$ c $\sqrt{20}$ d $\sqrt{27}$
e $\sqrt{28}$ f $\sqrt{40}$ g $\sqrt{32}$ h $\sqrt{99}$
i $\sqrt{54}$ j $\sqrt{200}$ k $\sqrt{60}$ l $\sqrt{75}$
m $\sqrt{80}$ n $\sqrt{98}$ o $\sqrt{800}$ p $\sqrt{1000}$

3 Simplify:

a $\sqrt{3} + \sqrt{3}$ b $5\sqrt{7} - 3\sqrt{7}$ c $2\sqrt{5} - \sqrt{5}$
d $-3\sqrt{2} + \sqrt{2}$ e $4\sqrt{3} + 3\sqrt{2} - 2\sqrt{3}$ f $-5\sqrt{5} - 2\sqrt{7} + 6\sqrt{5}$
g $7\sqrt{6} + 5\sqrt{3} - 4\sqrt{6} - 7\sqrt{3}$ h $-6\sqrt{2} - 4\sqrt{5} + 3\sqrt{2} - 2\sqrt{5}$ i $3\sqrt{10} - 8\sqrt{5} - 7\sqrt{10} + 10\sqrt{5}$

DEVELOPMENT

4 Simplify:

a $3\sqrt{8}$ b $5\sqrt{12}$ c $2\sqrt{24}$ d $4\sqrt{44}$
e $3\sqrt{45}$ f $6\sqrt{52}$ g $2\sqrt{300}$ h $2\sqrt{96}$

5 Write each expression as a single square root. For example, $3\sqrt{2} = \sqrt{9} \times \sqrt{2} = \sqrt{18}$.

a $2\sqrt{5}$ b $5\sqrt{2}$ c $8\sqrt{2}$ d $6\sqrt{3}$
e $5\sqrt{5}$ f $4\sqrt{7}$ g $2\sqrt{17}$ h $7\sqrt{10}$

6 Simplify fully:

a $\sqrt{8} + \sqrt{2}$ b $\sqrt{12} - \sqrt{3}$ c $\sqrt{50} - \sqrt{18}$
d $\sqrt{54} + \sqrt{24}$ e $\sqrt{45} - \sqrt{20}$ f $\sqrt{90} - \sqrt{40} + \sqrt{10}$
g $\sqrt{27} + \sqrt{75} - \sqrt{48}$ h $\sqrt{45} + \sqrt{80} - \sqrt{125}$ i $\sqrt{2} + \sqrt{32} + \sqrt{72}$

CHALLENGE

7 Simplify fully:

a $\sqrt{600} + \sqrt{300} - \sqrt{216}$ b $4\sqrt{18} + 3\sqrt{12} - 2\sqrt{50}$ c $2\sqrt{175} - 5\sqrt{140} - 3\sqrt{28}$

8 Find the value of x if:

a $\sqrt{63} - \sqrt{28} = \sqrt{x}$ b $\sqrt{80} - \sqrt{20} = \sqrt{x}$ c $2\sqrt{150} - 3\sqrt{24} = \sqrt{x}$

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2D Further simplification of surds

This section deals with the simplification of more complicated surdic expressions. The usual rules of algebra, together with the methods of simplifying surds given in the last section, are all that is needed.

Simplifying products of surds

The product of two surds is found using the identity

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}.$$

It is important to check whether the answer needs further simplification.

Example 4 — 2D

Simplify each product.

a $\sqrt{15} \times \sqrt{5}$

b $5\sqrt{6} \times 7\sqrt{10}$

SOLUTION

a
$$\begin{aligned}\sqrt{15} \times \sqrt{5} &= \sqrt{75}\\ &= \sqrt{25 \times 3}\\ &= 5\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}5\sqrt{6} \times 7\sqrt{10} &= 35\sqrt{60}\\ &= 35\sqrt{4 \times 15}\\ &= 35 \times 2\sqrt{15}\\ &= 70\sqrt{15}\end{aligned}$$

Using binomial expansions

All the usual algebraic methods of expanding binomial products can be applied to surdic expressions.

Example 5 — 2D

Expand these products and then simplify them.

a $\left(\sqrt{15} + 2\right)\left(\sqrt{3} - 3\right)$

b $\left(\sqrt{15} - \sqrt{6}\right)^2$

SOLUTION

a
$$\begin{aligned}\left(\sqrt{15} + 2\right)\left(\sqrt{3} - 3\right) &= \sqrt{15}\left(\sqrt{3} - 3\right) + 2\left(\sqrt{3} - 3\right)\\ &= \sqrt{45} - 3\sqrt{15} + 2\sqrt{3} - 6\\ &= 3\sqrt{5} - 3\sqrt{15} + 2\sqrt{3} - 6\end{aligned}$$

b
$$\begin{aligned}\left(\sqrt{15} - \sqrt{6}\right)^2 &= 15 - 2\sqrt{90} + 6 \qquad \left(\text{using the identity } (A - B)^2 = A^2 - 2AB + B^2\right)\\ &= 21 - 2 \times 3\sqrt{10}\\ &= 21 - 6\sqrt{10}\end{aligned}$$

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Exercise 2D

FOUNDATION

1 Simplify:

a $\left(\sqrt{3}\right)^2$ **b** $\sqrt{2}\times\sqrt{3}$ **c** $\sqrt{7}\times\sqrt{7}$ **d** $\sqrt{6}\times\sqrt{5}$

e $2\times 3\sqrt{2}$ **f** $2\sqrt{5}\times 5$ **g** $2\sqrt{3}\times 3\sqrt{5}$ **h** $6\sqrt{2}\times 5\sqrt{7}$

i $\left(2\sqrt{3}\right)^2$ **j** $\left(3\sqrt{7}\right)^2$ **k** $5\sqrt{2}\times 3\sqrt{2}$ **l** $6\sqrt{10}\times 4\sqrt{10}$

2 Simplify:

a $\sqrt{15}\div\sqrt{3}$ **b** $\sqrt{42}\div\sqrt{6}$ **c** $3\sqrt{5}\div 3$ **d** $2\sqrt{7}\div\sqrt{7}$

e $3\sqrt{10}\div\sqrt{5}$ **f** $6\sqrt{33}\div 6\sqrt{11}$ **g** $10\sqrt{14}\div 5\sqrt{2}$ **h** $15\sqrt{35}\div 3\sqrt{7}$

3 Expand:

a $\sqrt{5}\left(\sqrt{5}+1\right)$ **b** $\sqrt{2}\left(\sqrt{3}-1\right)$ **c** $\sqrt{3}\left(2-\sqrt{3}\right)$

d $2\sqrt{2}\left(\sqrt{5}-\sqrt{2}\right)$ **e** $\sqrt{7}\left(7-2\sqrt{7}\right)$ **f** $\sqrt{6}\left(3\sqrt{6}-2\sqrt{5}\right)$

DEVELOPMENT

4 Simplify fully:

a $\sqrt{6}\times\sqrt{2}$ **b** $\sqrt{5}\times\sqrt{10}$ **c** $\sqrt{3}\times\sqrt{15}$

d $\sqrt{2}\times 2\sqrt{22}$ **e** $4\sqrt{12}\times\sqrt{3}$ **f** $3\sqrt{8}\times 2\sqrt{5}$

5 Expand and simplify:

a $\sqrt{2}\left(\sqrt{10}-\sqrt{2}\right)$ **b** $\sqrt{6}\left(3+\sqrt{3}\right)$ **c** $\sqrt{5}\left(\sqrt{15}+4\right)$

d $\sqrt{6}\left(\sqrt{8}-2\right)$ **e** $3\sqrt{3}\left(9-\sqrt{21}\right)$ **f** $3\sqrt{7}\left(\sqrt{14}-2\sqrt{7}\right)$

6 Expand and simplify:

a $\left(\sqrt{3}+1\right)\left(\sqrt{2}-1\right)$ **b** $\left(\sqrt{5}-2\right)\left(\sqrt{7}+3\right)$ **c** $\left(\sqrt{5}+\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)$

d $\left(\sqrt{6}-1\right)\left(\sqrt{6}-2\right)$ **e** $\left(\sqrt{7}-2\right)\left(2\sqrt{7}+5\right)$ **f** $\left(3\sqrt{2}-1\right)\left(\sqrt{6}-\sqrt{3}\right)$

7 Use the special expansion $(a+b)(a-b)=a^2-b^2$ to expand and simplify:

a $\left(\sqrt{5}+1\right)\left(\sqrt{5}-1\right)$ **b** $\left(3-\sqrt{7}\right)\left(3+\sqrt{7}\right)$ **c** $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$

d $\left(3\sqrt{2}-\sqrt{11}\right)\left(3\sqrt{2}+\sqrt{11}\right)$ **e** $\left(2\sqrt{6}+3\right)\left(2\sqrt{6}-3\right)$ **f** $\left(7-2\sqrt{5}\right)\left(7+2\sqrt{5}\right)$

8 Expand and simplify the following, using the special expansions

$(a+b)^2=a^2+2ab+b^2$ and $(a-b)^2=a^2-2ab+b^2$.

a $\left(\sqrt{3}+1\right)^2$ **b** $\left(\sqrt{5}-1\right)^2$ **c** $\left(\sqrt{3}+\sqrt{2}\right)^2$

d $\left(\sqrt{7}-\sqrt{5}\right)^2$ **e** $\left(2\sqrt{3}-1\right)^2$ **f** $\left(2\sqrt{5}+3\right)^2$

g $\left(2\sqrt{7}+\sqrt{5}\right)^2$ **h** $\left(3\sqrt{2}-2\sqrt{3}\right)^2$ **i** $\left(3\sqrt{5}+\sqrt{10}\right)^2$

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CHALLENGE

9 Simplify fully:

a $\dfrac{\sqrt{40}}{\sqrt{10}}$

b $\dfrac{\sqrt{18}}{\sqrt{50}}$

c $\dfrac{2\sqrt{6} \times \sqrt{5}}{\sqrt{10}}$

d $\dfrac{5\sqrt{7} \times \sqrt{3}}{\sqrt{28}}$

e $\dfrac{\sqrt{15} \times \sqrt{20}}{\sqrt{12}}$

f $\dfrac{6\sqrt{3} \times 8\sqrt{2}}{\sqrt{32} \times \sqrt{27}}$

10 Use Pythagoras' theorem to find the hypotenuse of the right-angled triangle in which the lengths of the other two sides are:

a $\sqrt{2}$ and $\sqrt{7}$

b $\sqrt{5}$ and $2\sqrt{5}$

c $\sqrt{7} + 1$ and $\sqrt{7} - 1$

d $2\sqrt{3} - \sqrt{6}$ and $2\sqrt{3} + \sqrt{6}$

11 Simplify by forming the lowest common denominator:

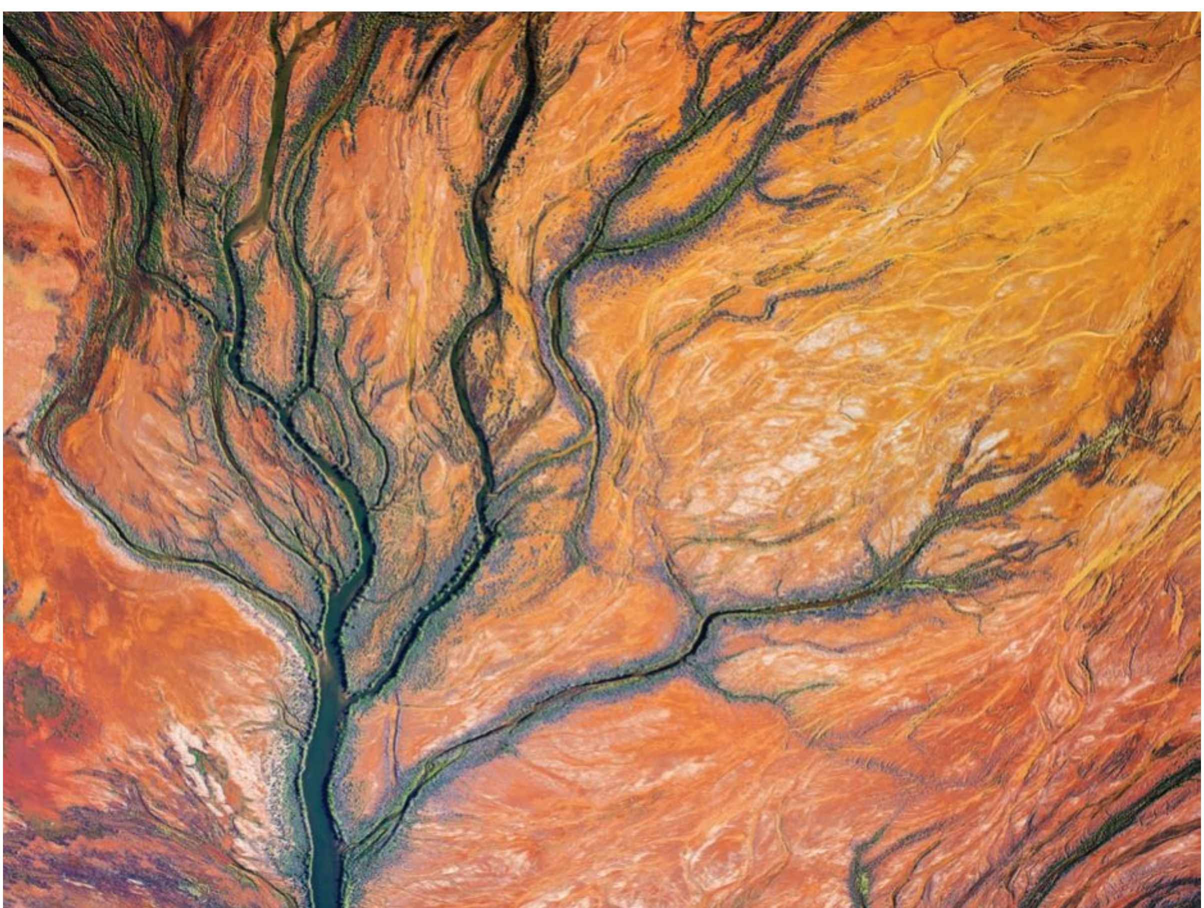
a $\dfrac{1}{\sqrt{3} + 1} + \dfrac{1}{\sqrt{3} - 1}$

b $\dfrac{3}{2\sqrt{5} - \sqrt{7}} - \dfrac{3}{2\sqrt{5} + \sqrt{7}}$

12 a Write down the expansion of $(a + b)^2$.

b Use the expansion in part **a** to square $\sqrt{6 + \sqrt{11}} - \sqrt{6 - \sqrt{11}}$.

c Hence simplify $\sqrt{6 + \sqrt{11}} - \sqrt{6 - \sqrt{11}}$.



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2E Rationalising the denominator

When dealing with surdic expressions, it is usual to remove any surds from the denominator, a process called *rationalising the denominator*. There are two cases.

The denominator has a single term

In the first case, the denominator is a surd or a multiple of a surd.

12 RATIONALISING A SINGLE-TERM DENOMINATOR

In an expression such as $\dfrac{\sqrt{7}}{2\sqrt{3}}$, multiply top and bottom by $\sqrt{3}$.

Example 6 2E

Simplify each expression by rationalising the denominator.

a $\dfrac{\sqrt{7}}{2\sqrt{3}}$

b $\dfrac{55}{\sqrt{11}}$

SOLUTION

a
$$\begin{aligned}\frac{\sqrt{7}}{2\sqrt{3}} &= \frac{\sqrt{7}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{\sqrt{21}}{2 \times 3}\\ &= \frac{\sqrt{21}}{6}\end{aligned}$$

b
$$\begin{aligned}\frac{55}{\sqrt{11}} &= \frac{55}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}}\\ &= \frac{55\sqrt{11}}{11}\\ &= 5\sqrt{11}\end{aligned}$$

The denominator has two terms

The second case involves a denominator with two terms, one or both of which contain a surd. The method uses the difference of squares identity

$$(A + B)(A - B) = A^2 - B^2$$

to square the unwanted surds and convert them to integers.

13 RATIONALISING A BINOMIAL DENOMINATOR

- In an expression such as $\dfrac{3}{5 + \sqrt{3}}$, multiply top and bottom by $5 - \sqrt{3}$.
- Then use the difference of squares.

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Example 7 2E

Rationalise the denominator in each expression.

a $\dfrac{3}{5+\sqrt{3}}$

b $\dfrac{1}{2\sqrt{3}-3\sqrt{2}}$

SOLUTION

a
$$\begin{aligned}\frac{3}{5+\sqrt{3}} &= \frac{3}{5+\sqrt{3}} \times \frac{5-\sqrt{3}}{5-\sqrt{3}}\\ &= \frac{15-3\sqrt{3}}{25-3}\\ &= \frac{15-3\sqrt{3}}{22}\end{aligned}$$

Using the difference of squares:
$$\begin{aligned}\left(5+\sqrt{3}\right)\left(5-\sqrt{3}\right) &= 5^2-\left(\sqrt{3}\right)^2\\ &= 25-3\end{aligned}$$

b
$$\begin{aligned}\frac{1}{2\sqrt{3}-3\sqrt{2}} &= \frac{1}{2\sqrt{3}-3\sqrt{2}} \times \frac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}\\ &= \frac{2\sqrt{3}+3\sqrt{2}}{4\times 3-9\times 2}\\ &= -\frac{2\sqrt{3}+3\sqrt{2}}{6}\end{aligned}$$

Using the difference of squares:
$$\begin{aligned}\left(2\sqrt{3}-3\sqrt{2}\right)\left(2\sqrt{3}+3\sqrt{2}\right) &= \left(2\sqrt{3}\right)^2-\left(3\sqrt{2}\right)^2\\ &= 4\times 3-9\times 2\end{aligned}$$

Exercise 2E

FOUNDATION

1 Rewrite each fraction with a rational denominator.

a $\dfrac{1}{\sqrt{3}}$ **b** $\dfrac{1}{\sqrt{7}}$ **c** $\dfrac{3}{\sqrt{5}}$ **d** $\dfrac{5}{\sqrt{2}}$

e $\dfrac{\sqrt{2}}{\sqrt{3}}$ **f** $\dfrac{\sqrt{5}}{\sqrt{7}}$ **g** $\dfrac{2\sqrt{11}}{\sqrt{5}}$ **h** $\dfrac{3\sqrt{7}}{\sqrt{2}}$

2 Rewrite each fraction with a rational denominator.

a $\dfrac{1}{\sqrt{3}-1}$ **b** $\dfrac{1}{\sqrt{7}+2}$ **c** $\dfrac{1}{3+\sqrt{5}}$ **d** $\dfrac{1}{4-\sqrt{7}}$

e $\dfrac{1}{\sqrt{5}-\sqrt{2}}$ **f** $\dfrac{1}{\sqrt{10}+\sqrt{6}}$ **g** $\dfrac{1}{2\sqrt{3}+1}$ **h** $\dfrac{1}{5-3\sqrt{2}}$

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DEVELOPMENT

3 Simplify each expression by rationalising the denominator.

a $\dfrac{2}{\sqrt{2}}$ **b** $\dfrac{5}{\sqrt{5}}$ **c** $\dfrac{6}{\sqrt{3}}$ **d** $\dfrac{21}{\sqrt{7}}$

e $\dfrac{3}{\sqrt{6}}$ **f** $\dfrac{5}{\sqrt{15}}$ **g** $\dfrac{8}{\sqrt{6}}$ **h** $\dfrac{14}{\sqrt{10}}$

4 Rewrite each fraction with a rational denominator.

a $\dfrac{1}{2\sqrt{5}}$ **b** $\dfrac{1}{3\sqrt{7}}$ **c** $\dfrac{3}{5\sqrt{2}}$ **d** $\dfrac{2}{7\sqrt{3}}$

e $\dfrac{10}{3\sqrt{2}}$ **f** $\dfrac{9}{4\sqrt{3}}$ **g** $\dfrac{\sqrt{3}}{2\sqrt{10}}$ **h** $\dfrac{2\sqrt{11}}{5\sqrt{7}}$

5 Rewrite each fraction with a rational denominator.

a $\dfrac{3}{\sqrt{5}+1}$ **b** $\dfrac{4}{2\sqrt{2}-\sqrt{3}}$ **c** $\dfrac{\sqrt{7}}{5-\sqrt{7}}$ **d** $\dfrac{3\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

e $\dfrac{2\sqrt{7}}{2\sqrt{7}-5}$ **f** $\dfrac{\sqrt{5}}{\sqrt{10}-\sqrt{5}}$ **g** $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ **h** $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

i $\dfrac{3-\sqrt{7}}{3+\sqrt{7}}$ **j** $\dfrac{3\sqrt{2}+\sqrt{5}}{3\sqrt{2}-\sqrt{5}}$ **k** $\dfrac{\sqrt{10}-\sqrt{6}}{\sqrt{10}+\sqrt{6}}$ **l** $\dfrac{7+2\sqrt{11}}{7-2\sqrt{11}}$

CHALLENGE

6 Simplify each expression by rationalising the denominator.

a $\dfrac{\sqrt{3}-1}{2-\sqrt{3}}$ **b** $\dfrac{2\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$

7 Show that each expression is rational by first rationalising the denominators.

a $\dfrac{3}{\sqrt{2}}+\dfrac{3}{2+\sqrt{2}}$ **b** $\dfrac{1}{3+\sqrt{6}}+\dfrac{2}{\sqrt{6}}$

c $\dfrac{4}{2+\sqrt{2}}+\dfrac{1}{3-2\sqrt{2}}$ **d** $\dfrac{8}{3-\sqrt{7}}-\dfrac{6}{2\sqrt{7}-5}$

8 If $x=\dfrac{\sqrt{5}+1}{2}$, show that $1+\dfrac{1}{x}=x$.

9 The expression $\dfrac{\sqrt{6}+1}{\sqrt{3}+\sqrt{2}}$ can be written in the form $a\sqrt{3}+b\sqrt{2}$. Find a and b.

10 **a** Expand $\left(x+\dfrac{1}{x}\right)^2$.

b Suppose that $x=\sqrt{7}+\sqrt{6}$.

i Show that $x+\dfrac{1}{x}=2\sqrt{7}$.

ii Use the result in part **a** to find the value of $x^2+\dfrac{1}{x^2}$.

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Chapter 2 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 2 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

Review

1 Classify each of these real numbers as rational or irrational. Express those that are rational in the form $\frac{a}{b}$, where a and b are integers.

a 7 **b** $-2\frac{1}{4}$ **c** $\sqrt{9}$ **d** $\sqrt{10}$

e $\sqrt[3]{15}$ **f** $\sqrt[4]{16}$ **g** -0.16 **h** π

2 Use a calculator to write each number correct to:

i two decimal places **ii** two significant figures.

a $\sqrt{17}$ **b** $\sqrt[3]{102}$ **c** 1.16^7

d $\frac{49}{64}$ **e** 7.3^{-2} **f** $\pi^{5.5}$

3 Evaluate, correct to three significant figures:

a $\frac{7.93}{8.22 - 3.48}$ **b** $-4.9 \times (-5.8 - 8.5)$ **c** $\sqrt[4]{1.6 \times 2.6}$

d $\frac{13^5}{11^6 + 17^4}$ **e** $\frac{\frac{4}{9} - \frac{2}{7}}{\frac{5}{8} - \frac{3}{10}}$ **f** $\sqrt{2.4^{-1.6}}$

g $\sqrt{\frac{1.347}{2.518 - 1.679}}$ **h** $\frac{2.7 \times 10^{-2}}{1.7 \times 10^{-5}}$ **i** $\frac{\sqrt{\frac{1}{2}} + \sqrt[3]{\frac{1}{3}}}{\sqrt[4]{\frac{1}{4}} + \sqrt[5]{\frac{1}{5}}}$

4 Simplify:

a $\sqrt{24}$ **b** $\sqrt{45}$ **c** $\sqrt{50}$

d $\sqrt{500}$ **e** $3\sqrt{18}$ **f** $2\sqrt{40}$

5 Simplify:

a $\sqrt{5} + \sqrt{5}$ **b** $\sqrt{5} \times \sqrt{5}$ **c** $\left(2\sqrt{7}\right)^2$

d $2\sqrt{5} + \sqrt{7} - 3\sqrt{5}$ **e** $\sqrt{35} \div \sqrt{5}$ **f** $6\sqrt{55} \div 2\sqrt{11}$

g $\sqrt{8} \times \sqrt{2}$ **h** $\sqrt{10} \times \sqrt{2}$ **i** $2\sqrt{6} \times 4\sqrt{15}$

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Review

6 Simplify:

a $\sqrt{27} - \sqrt{12}$

b $\sqrt{18} + \sqrt{32}$

c $3\sqrt{2} + 3\sqrt{8} - \sqrt{50}$

d $\sqrt{54} - \sqrt{20} + \sqrt{150} - \sqrt{80}$

7 Expand:

a $\sqrt{7}\left(3 - \sqrt{7}\right)$

b $\sqrt{5}\left(2\sqrt{6} + 3\sqrt{2}\right)$

c $\sqrt{15}\left(\sqrt{3} - 5\right)$

d $\sqrt{3}\left(\sqrt{6} + 2\sqrt{3}\right)$

8 Expand and simplify:

a $\left(\sqrt{5} + 2\right)\left(3 - \sqrt{5}\right)$

b $\left(2\sqrt{3} - 1\right)\left(3\sqrt{3} + 5\right)$

c $\left(\sqrt{7} - 3\right)\left(2\sqrt{5} + 4\right)$

d $\left(\sqrt{10} - 3\right)\left(\sqrt{10} + 3\right)$

e $\left(2\sqrt{6} + \sqrt{11}\right)\left(2\sqrt{6} - \sqrt{11}\right)$

f $\left(\sqrt{7} - 2\right)^2$

g $\left(\sqrt{5} + \sqrt{2}\right)^2$

h $\left(4 - 3\sqrt{2}\right)^2$

9 Write with a rational denominator:

a $\dfrac{1}{\sqrt{5}}$

b $\dfrac{3}{\sqrt{2}}$

c $\dfrac{\sqrt{3}}{\sqrt{11}}$

d $\dfrac{1}{5\sqrt{3}}$

e $\dfrac{5}{2\sqrt{7}}$

f $\dfrac{\sqrt{2}}{3\sqrt{10}}$

10 Write with a rational denominator:

a $\dfrac{1}{\sqrt{5} + \sqrt{2}}$

b $\dfrac{1}{3 - \sqrt{7}}$

c $\dfrac{1}{2\sqrt{6} - \sqrt{3}}$

d $\dfrac{\sqrt{3}}{\sqrt{3} + 1}$

e $\dfrac{3}{\sqrt{11} + \sqrt{5}}$

f $\dfrac{3\sqrt{7}}{2\sqrt{5} - \sqrt{7}}$

11 Rationalise the denominator of each fraction.

a $\dfrac{\sqrt{7} - \sqrt{2}}{\sqrt{7} + \sqrt{2}}$

b $\dfrac{3\sqrt{3} + 5}{3\sqrt{3} - 5}$

12 Find the value of x if $\sqrt{18} + \sqrt{8} = \sqrt{x}$.

13 Simplify $\dfrac{3}{\sqrt{5} - 2} + \dfrac{2}{\sqrt{5} + 2}$ by forming the lowest common denominator.

14 Find the values of p and q such that $\dfrac{\sqrt{5}}{\sqrt{5} - 2} = p + q\sqrt{5}$.

15 Show that $\dfrac{2}{6 - 3\sqrt{3}} - \dfrac{1}{2\sqrt{3} + 3}$ is rational by first rationalising each denominator.

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3 Functions and graphs

The principal purpose of this course is the study of functions. Now that the real numbers have been reviewed, this chapter develops the idea of functions and relations and their graphs. A variety of known graphs are then discussed, with particular emphasis on quadratics and their parabolic graphs, on factored cubics and polynomials, and on graphs with asymptotes.

The final section uses the vertical and horizontal line tests to classify four types of relations. It also gives examples of relations as they occur in databases and spreadsheets, where the elements of the two sets are not restricted to numbers. This section is demanding, and could be delayed until later in the year.

Curve-sketching software is useful in emphasising the basic idea that a function has a graph. It is, of course, also useful in sketching quickly a large number of graphs and recognising their relationships. Nevertheless, readers must eventually be able to construct a graph from its equation on their own.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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3A Functions and function notation

Many of the graphs studied in previous years are examples of functions. This section will make the idea more precise and introduce some new notation.

Functions

Here is a situation that naturally leads to a function. An electrician charges \$100 to visit a home, and then charges \$40 for each power point that he installs.

> Let x be the number of power points he installs.
> Let y be the total cost, in dollars.
> Then $y = 100 + 40x$.

This is an example of a function. Informally we say that *y is a function of x* because the value of y is determined by the value of x, and we call x and y *variables* because they take many different values. The variable x is called the *independent variable* of the function, and the variable y is called the *dependent variable* because its value depends on x.

Thus a function is a rule. We input a value of x, and the rule produces an output y. In this example, x must be a whole number 0, 1, 2, …, and we can add this restriction to the rule, describing the function as:

> 'The function $y = 100 + 40x$, where x is a whole number.'

A *table of values* is a useful tool — a few values of the function are selected and arranged in a table. Here is a representative table of values showing the total cost y dollars of installing x power points:

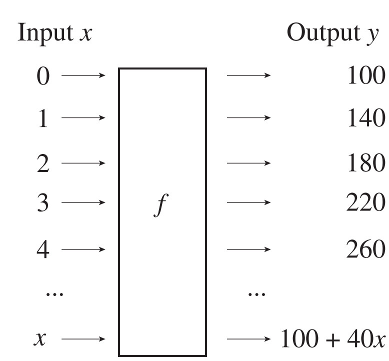
x	0	1	2	3	4	5	6
y	100	140	180	220	260	300	340

1 INFORMALLY, A FUNCTION IS A RULE

- *A variable y is a function of a variable x* when y is completely determined by x as a result of some rule.
- The variable x is called the *independent variable* of the function, and the variable y is called the *dependent variable* because its value depends on x.
- The function rule is almost always an equation, possibly with a restriction. For example:
  'The function $y = 100 + 40x$, where x is a whole number'.

The function machine

The function and its rule can be regarded as a 'machine' with inputs and outputs. For example, the numbers in the right-hand column are the outputs from the function $y = 100 + 40x$, when the numbers 0, 1, 2, 3 and 4 are the inputs.



This model of a function has become far more intuitive in the last few decades because computers and calculators routinely produce output from a given input.

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Function notation

In the function machine diagram above, we gave the name f to our function. We can now write the results of the input–output routines as follows:

$$f(0) = 100,\ \ f(1) = 140,\ \ f(2) = 180,\ \ f(3) = 220,\ \ \ldots$$

This is read aloud as, 'f of zero is equal to one hundred', and so on. When x is the input, the output is $100 + 40x$. Thus, using the well-known notation introduced by Euler in 1735, we can write the *function rule* as

$$f(x) = 100 + 40x, \text{ where } x \text{ is a whole number.}$$

This $f(x)$ notation will be used throughout the course alongside the $y = \cdots$ notation. The previous table of values can therefore also be written as

x	0	1	2	3	4	5	6
$f(x)$	100	140	180	220	260	300	340

Example 1 — 3A

A function is defined by the rule $f(x) = x^2 + 5x$. Find $f(3)$, $f(0)$ and $f(-3)$.

SOLUTION

$f(3) = 3^2 + 5 \times 3$
$= 9 + 15$
$= 24$

$f(0) = 0^2 + 0 \times 3$
$= 0 + 0$
$= 0$

$f(-3) = (-3)^2 + 5 \times (-3)$
$= 9 - 15$
$= -6$

Example 2 — 3A

[This is an example of a function that is described verbally.]

A function $g(x)$ is defined by the rule, 'Cube the number and subtract 7'. Write down its function rule as an equation, then draw up a table of values for -2, -1, 0, 1 and 2.

SOLUTION

The function rule is $g(x) = x^3 - 7$.

x	-2	-1	0	1	2
$g(x)$	-15	-8	-7	-6	1

Example 3 — 3A

[Other pronumerals, and even expressions, can be substituted into a function.]

If $f(x) = x^2 + 5$, find and simplify $f(a)$, $f(a + 1)$ and $f(a + h)$.

SOLUTION

$f(a) = a^2 + 5$

$f(a + 1) = (a + 1)^2 + 5$
$= a^2 + 2a + 1 + 5$
$= a^2 + 2a + 6$

$f(a + h) = (a + h)^2 + 5$
$= a^2 + 2ah + h^2 + 5$

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Example 4 3A

[Many functions have natural restrictions on the variables. These restrictions are part of the function.]

Sadie the snail is crawling at a steady 10 cm per minute vertically up a wall 3 metres high, starting at the bottom. Write down the height y metres as a function of the time x minutes of climbing, adding the restriction on x.

SOLUTION

The snail's height after x minutes is $\frac{1}{10}x$ metres, and the snail will take 30 minutes to get to the top, so the function is

$$y = \frac{1}{10}x, \text{ where } 0 \le x \le 30.$$

Exercise 3A

FOUNDATION

1 Let $f(x) = 2x + 3$. Calculate the following function values.

a $f(1)$ **b** $f(0)$ **c** $f(-2)$ **d** $f(4)$

2 Let $p(x) = x^2 - 2x - 3$. Determine:

a $p(0)$ **b** $p(4)$ **c** $p(3)$ **d** $p(-2)$

3 Find the value of the function $y = 5 + 2x - x^2$ at each value of x.

a 0 **b** 5 **c** -2 **d** -1

4 Find $f(2)$, $f(0)$ and $f(-2)$ for each function.

a $f(x) = 3x - 1$ **b** $f(x) = 4 - x^2$ **c** $f(x) = x^3 + 8$ **d** $f(x) = 2^x$

5 Find $h(-3)$, $h(1)$ and $h(5)$ for each function.

a $h(x) = 2x + 2$ **b** $h(x) = \dfrac{1}{x}$ **c** $h(x) = 3x - x^2$ **d** $h(x) = \sqrt{x + 4}$

DEVELOPMENT

6 Copy and complete the table of values for each function.

a $y = 2x + 1$

x	-1	0	1
y			

b $y = x^2 - 2x$

x	-1	0	1	2	3
y					

c $f(x) = 1 - x^2$

x	-2	-1	0	1	2
$f(x)$					

d $f(x) = x^3 - 4x$

x	-3	-2	-1	0	1	2	3
$f(x)$							

7 For the function $L(x) = 3x + 1$, determine:

a $L(2) + 1$ **b** $L(1) - 2$ **c** $3L(-1)$ **d** $L(5) \div 4$

e $L(1) + L(2)$ **f** $L(1) - L(-1)$ **g** $L(-2) \times L(2)$ **h** $L(9) \div L(2)$

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8 Given that $f(x) = x^2 - 3x + 5$, find the value of:

a $\frac{1}{2}\Big(f(2) + f(3)\Big)$

b $\frac{1}{4}\Big(f(-1) + 2f(0) + f(1)\Big)$

c $\frac{1}{6}\Big(f(-1) + 4f(0) + f(1)\Big)$

d $\frac{1}{6}\Big(f(0) + 4f(2) + f(4)\Big)$

9 Given that $P(x) = x^2 - 2x - 4$, find the value of:

a $P\big(\sqrt{2}\big)$

b $P\big(\sqrt{7}\big)$

10 a Write the equation $3x + 4y + 5 = 0$ as a function with independent variable x.

b Write the equation $3x + 4y + 5 = 0$ as a function with independent variable y.

c Write the equation $4 + xy = 0$ as a function with dependent variable y.

d The volume of a cube with side length s is $V = s^3$, and its surface area is $A = 6s^2$. Write each formula as a function with dependent variable s.

e If a rectangle has area $100\,\text{m}^2$ and sides ℓ and b, then $\ell b = 100$. Write this formula as a function with:

i ℓ as the dependent variable **ii** ℓ as the independent variable.

11 A restaurant offers a special deal to groups by charging a cover fee of \$50, then \$20 per person. Write down C, the total cost of the meal in dollars, as a function of x, the number of people in the group.

12 In each case explain why the function value cannot be found.

a $F(0)$, where $F(x) = \sqrt{x - 4}$.

b $H(3)$, where $H(x) = \sqrt{1 - x^2}$.

c $g(-2)$, where $g(x) = \dfrac{1}{2 + x}$.

d $f(0)$, where $f(x) = \dfrac{1}{x}$.

CHALLENGE

13 Again using $P(x) = x^2 - 2x - 4$, find the value of:

a $P\big(1 + \sqrt{5}\big)$

b $P\big(\sqrt{3} - 1\big)$

14 Find $g(a)$, $g(-a)$ and $g(a + 1)$ for each function.

a $g(x) = 2x - 4$ **b** $g(x) = 2 - x$ **c** $g(x) = x^2$ **d** $g(x) = \dfrac{1}{x - 1}$

15 Find $F(t) - 2$ and $F(t - 2)$ for each function.

a $F(x) = 5x + 2$ **b** $F(x) = \sqrt{x}$ **c** $F(x) = x^2 + 2x$ **d** $F(x) = 2 - x^2$

16 If $f(x) = x^2 + 5x$, find in simplest form:

a $\dfrac{f(1 + h) - f(1)}{h}$ **b** $\dfrac{f(p) - f(q)}{p - q}$ **c** $\dfrac{f(x + h) - f(x)}{h}$

17 a If $f(x) = x^4 + 2x^2 + 3$, show that $f(-x) = f(x)$ for all values of x.

b If $g(x) = x^3 + \dfrac{4}{x}$, show that $g(-x) = -g(x)$ whenever $x \neq 0$.

c If $h(x) = \dfrac{x}{x^2 + 1}$, show that $h\Big(\frac{1}{x}\Big) = h(x)$ whenever $x \neq 0$.

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3B Functions, relations and graphs

A graph is the most important and helpful way to represent a function. Most things that we will discover about a function can be seen on its graph, and the sketching of graphs is consequently central to this course.

A function and its graph

A ball is thrown vertically upwards. While the ball is in the air, its height y metres, x seconds after it is thrown, is

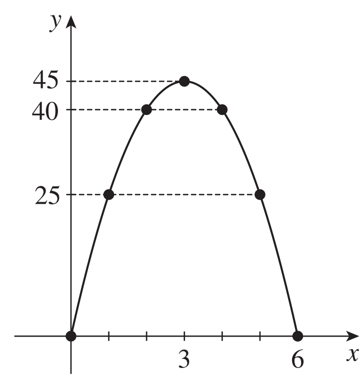
$$y = 5x(6 - x).$$

Here is a representative table of values of the function.

x (time)	0	1	2	3	4	5	6
y (height)	0	25	40	45	40	25	0

Each x-value and its corresponding y-value can be put into an ordered pair ready to plot on a graph of the function. The seven ordered pairs calculated here are:

$$(0, 0), (1, 25), (2, 40), (3, 45), (4, 40), (5, 25), (6, 0)$$



and the graph is sketched opposite. The seven representative points have been plotted, but there are infinitely many such ordered pairs, and they all join up to make the nice smooth curve shown to the right.

Like all graphs of functions, this graph has a crucial property — no two points have the same x-coordinate. This is because at any one time, the ball can only be in one position. In function-machine language, no input can have two outputs.

2 THE GRAPH OF A FUNCTION

- The *graph of a function* consists of all the ordered pairs (x, y) plotted on a pair of axes, where x and y are values of the input and output variables.
- No two points on the graph ever have the same x-coordinate.
- In most graphs in this course, the points join up to make a smooth curve.

A function as a set of ordered pairs

These ideas of the graph and its ordered pairs allow a more formal definition of a function — a function can be defined simply as a set of ordered pairs satisfying the crucial property mentioned above.

3 MORE FORMALLY, A FUNCTION IS A SET OF ORDERED PAIRS SATISFYING A CONDITION

A *function* is a set of ordered pairs (x, y) in which:

- no two ordered pairs have the same x-coordinate.

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Domain and range

There are two restrictions on the time x in our example above:

- The time variable x cannot be negative, because the ball had not been thrown then.
- The time variable cannot be greater than 6, because the ball hits the ground after 6 seconds.

The *domain* of the function is the set of possible x-values, so the domain is the closed interval $0 \le x \le 6$. Thus the function is more correctly written as

$$y = 5x(6 - x), \text{ where } 0 \le x \le 6.$$

The endpoints of the graph are marked with *closed (filled-in) circles* • to indicate that these endpoints are included in the graph. If they were not included, they would be marked with *open circles* ∘. These are the same conventions that were used with intervals in Section 2B.

From the graph, we can see that the height of the ball ranges from 0 on the ground to 45 metres. The *range* is the set of possible y-values, so the range is the interval $0 \le y \le 45$, which includes the two endpoints $y = 0$ and $y = 45$.

4 THE DOMAIN AND RANGE OF A FUNCTION

- The *domain* of a function is the set of all possible x-coordinates.
- The *range* of a function is the set of all possible y-coordinates.

It is usually easier to find the range after the graph has been drawn.

Reading the domain and range from the graph

Sometimes we have the graph of a function, but not its equation or rule. We may be able to read the domain and range from such a graph.

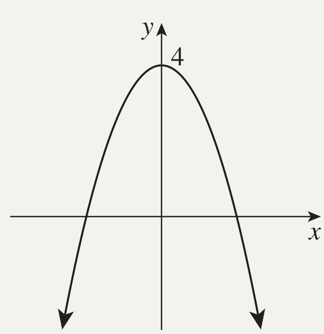
5 READING THE DOMAIN AND RANGE FROM THE GRAPH

- **Domain:** Take all the values on the x-axis that have graph points above or below them.
- **Range:** Take all the values on the y-axis that have graph points to the left or right of them.

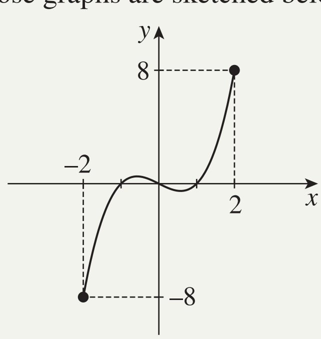
Example 5 3B

Write down the domain and the range of the functions whose graphs are sketched below.

a



b



SOLUTION

a Domain: all real x, range: $y \le 4$.

b Domain: $-2 \le x \le 2$, range: $-8 \le y \le 8$.

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The natural domain

When the equation of a function is given with no restriction, we assume *as a convention* that the domain is all the x-values that can validly be substituted into the equation. This is called the *natural domain*.

6 THE NATURAL DOMAIN

If no restriction is given, the domain is all x-values that can validly be substituted into the equation. This is called the *natural domain*.

There are many reasons why a number cannot be substituted into an equation. So far, the two most common reasons are:

- We cannot divide by zero.
- We cannot take square roots of negative numbers.

Example 6 3B

Find the natural domain of each function:

a $y = \dfrac{1}{x-2}$

b $y = \sqrt{x-2}$

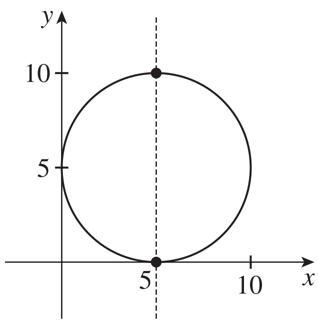
SOLUTION

a One cannot divide by zero.
Hence the domain is $x - 2 \neq 0$
that is, $x \neq 2$.

b One cannot take square roots of negative numbers.
Hence the domain is $x - 2 \geq 0$
that is, $x \geq 2$.

Relations

We shall often be dealing with graphs such as the circle sketched to the right. This graph is a set of ordered pairs. But it is not a function, because for example, the points $(5, 0)$ and $(5, 10)$ have the same x-coordinate,



so that the input $x = 5$ has the two outputs $y = 0$ and $y = 10$. The graph thus fails the crucial property that no input can have more than one output. The more general word 'relation' is used for any graph in the plane, whether it is a function or not.

7 RELATIONS

A *relation* is any set of ordered pairs.

- Like a function, a relation has a graph.
- Like a function, a relation has a *domain* and a *range*.
- Unlike a function, a relation may have two or more points with the same x-coordinate.

A function is thus a special type of relation, just as a square is a special type of rectangle.

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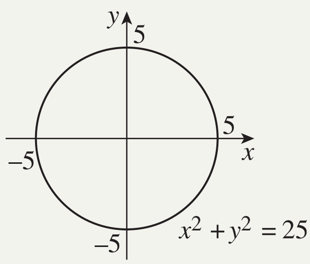
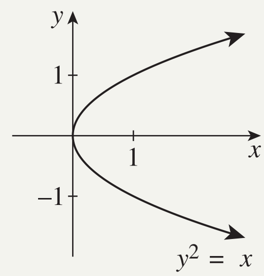
Example 7 3B

In each part, show that the relation is not a function by writing down two ordered pairs on the graph with the same x-coordinate. Illustrate this by connecting the two points by a vertical line. Then write down the domain and range.

a

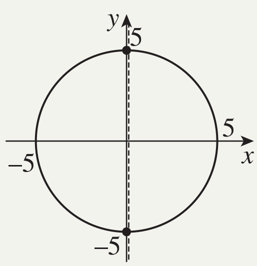


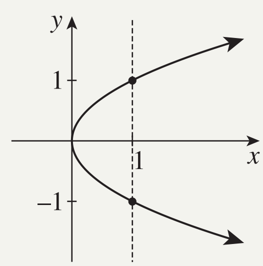
b



SOLUTION

a The points $(0, 5)$ and $(0, -5)$ on the graph have the same x-coordinate $x = 0$.
Thus when $x = 0$ is the input, there are two outputs: $y = 5$ and $y = -5$.
The vertical line $x = 0$ meets the graph at $(0, 5)$ and at $(0, -5)$.
Domain: $-5 \le x \le 5$, range: $-5 \le y \le 5$





b The points $(1, 1)$ and $(1, -1)$ on the graph have the same x-coordinate $x = 1$.
Thus when $x = 1$ is the input, there are two outputs: $y = -1$ and $y = 1$.
The vertical line $x = 1$ meets the graph at $(1, -1)$ and $(1, 1)$.
Domain: $x \ge 0$, range: all real y.

The vertical line test

Example 7 above shows that we can easily use vertical lines on a graph to see whether or not it is a function.

To show that the graph is not a function, we need to identify just two points with the same x-coordinate. That means, we need to draw just one vertical line that crosses the graph twice.

8 THE VERTICAL LINE TEST

If at least one vertical line crosses the graph of a relation more than once, then the relation is not a function.

The word ‘map’

Functions (but not relations in general) are also called *maps* or *mappings*. The word *map* may also be used as a verb. Thus in the function $f(x) = 2^x$, we may say that ‘3 is mapped to 8’, and that ‘8 is mapped to from 3’.

A map of NSW is a one-to-one correspondence from points on the surface of the Earth to points on a piece of paper or on a screen.

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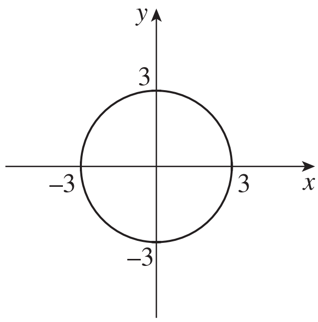
Exercise 3B

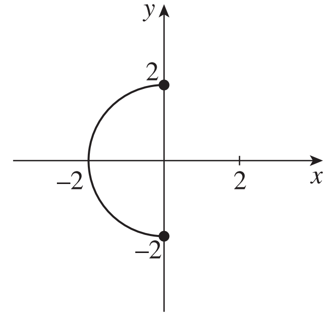
FOUNDATION

1 In each case, copy the graph, then draw a vertical line to show that the curve does not represent a function.

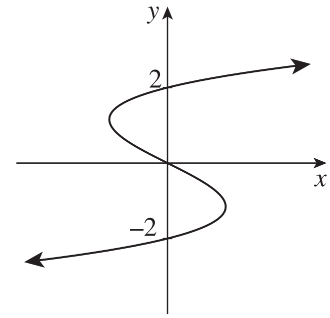
a



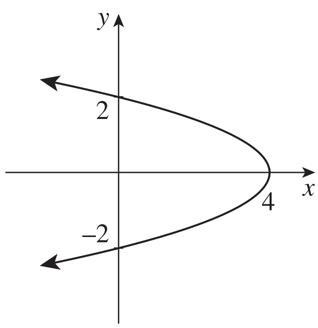
b



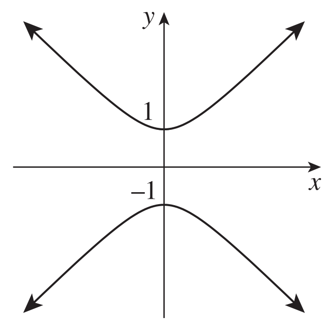
c



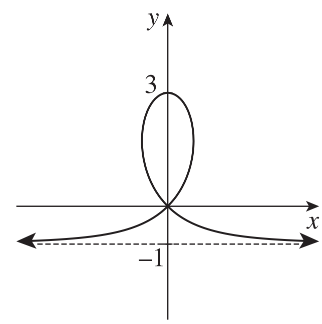
d



e

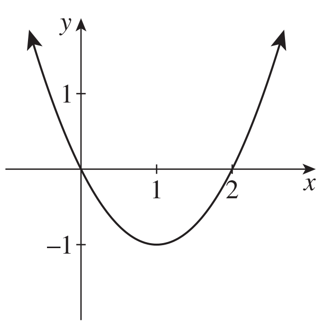


f

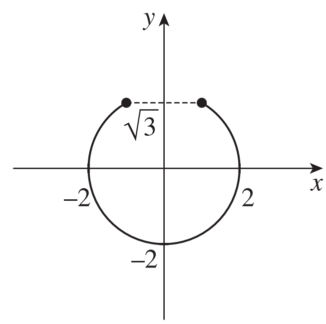


2 Use the vertical line test to find which of the following graphs represent functions.

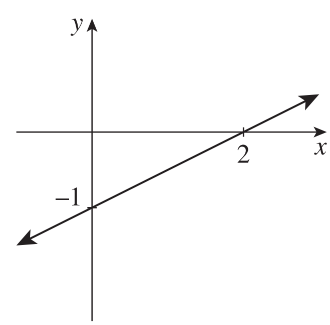
a



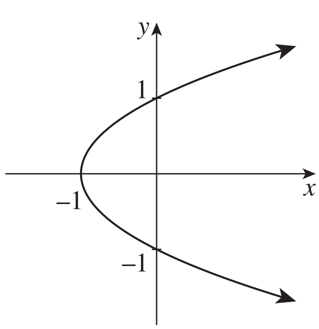
b



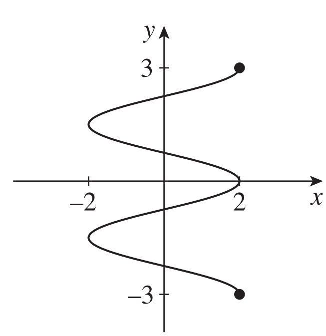
c



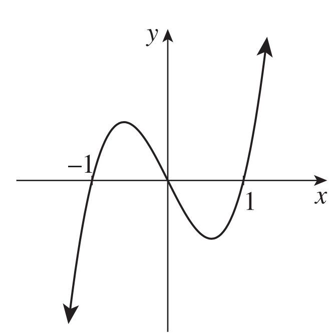
d



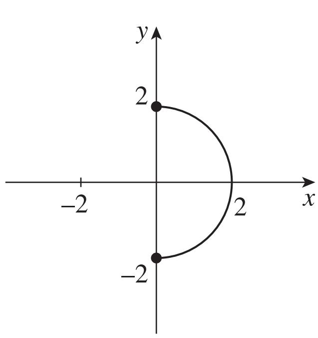
e



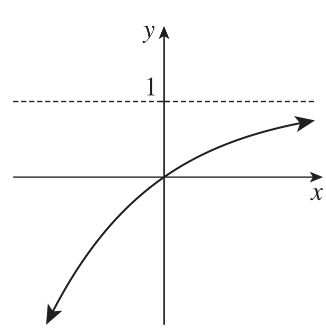
f



g



h



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3 What are the domain and range of each relation in Question **2**?

4 For each of the following functions:

i copy and complete the table of values

ii plot the points in the table and hence sketch the function

iii then write down the domain and range.

a $y = -2x + 3$

x	0	1	2
y			

b $y = x^2 + 2x + 1$

x	-3	-2	-1	0	1
y					

5 Use the fact that division by zero is undefined to find the natural domain of each function.

a $f(x) = \dfrac{1}{x}$ **b** $f(x) = \dfrac{1}{x - 3}$ **c** $f(x) = \dfrac{1}{x + 1}$ **d** $f(x) = \dfrac{1}{2 + x}$

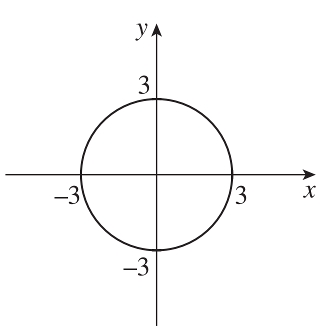
6 Use the fact that a negative number does not have a square root to find the natural domain of each function.

a $f(x) = \sqrt{x}$ **b** $f(x) = \sqrt{x - 2}$ **c** $f(x) = \sqrt{x + 3}$ **d** $f(x) = \sqrt{5 + x}$

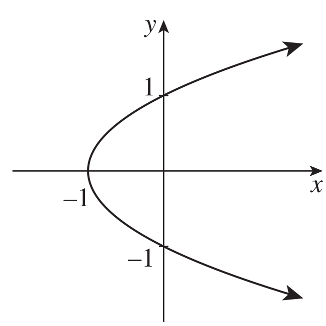
DEVELOPMENT

7 The following relations are not functions. Write down the coordinates of two points on each graph that have the same x-coordinate.

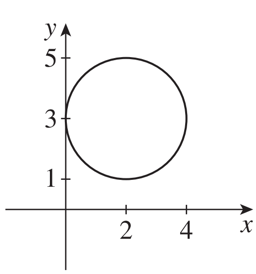
a



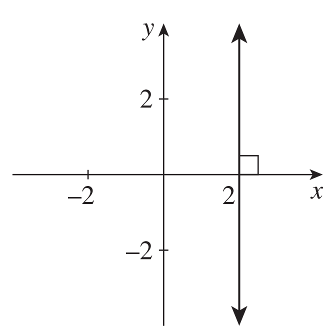
b



c



d



8 Find the natural domain of each function.

a $f(x) = 4x$ **b** $f(x) = 7 - 3x$ **c** $f(x) = \dfrac{1}{4 - x}$ **d** $f(x) = \dfrac{3}{2x - 1}$

e $f(x) = \sqrt{x + 4}$ **f** $f(x) = \sqrt{2x + 1}$ **g** $f(x) = \sqrt{5 - x}$ **h** $f(x) = \sqrt{4 - 2x}$

i $f(x) = \dfrac{1}{\sqrt{x}}$ **j** $f(x) = \dfrac{1}{\sqrt{x + 1}}$ **k** $f(x) = \dfrac{2}{\sqrt{1 - x}}$ **l** $f(x) = \dfrac{1}{\sqrt{2x - 3}}$

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9 For each of the following functions:

- **i** copy and complete the table of values
- **ii** plot the points in the table and hence sketch the function
- **iii** then write down the domain and range.

a $y = 2^x$

x	-2	-1	0	1	2
y					

b $y = \frac{1}{3}(4x - x^3)$

x	-3	-2	-1	0	1	2	3
y							

10 Let $R(x) = \sqrt{x}$.

a What is the natural domain of $R(x)$?

b Copy and complete the table of values. Use a calculator to approximate values correct to one decimal place where necessary.

x	0	$\frac{1}{2}$	1	2	3	4	5
$R(x)$							

c Plot these points and join them with a smooth curve starting at the origin. This curve may look similar to a curve you know. Describe it.

11 Let $h(x) = \dfrac{2}{x}$.

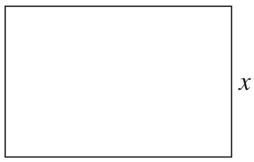
a What is the natural domain of $h(x)$?

b Copy and complete the table of values. Why is there a star for the value where $x = 0$?

x	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4
$h(x)$					*				

c Plot these points and join them with a smooth curve in two parts. This curve is called a *rectangular hyperbola.*

12 Jordan is playing with a 20 cm piece of copper wire, which he bends into the shape of a rectangle, as shown. Let x be the length of a side of the rectangle.



a What is the length of the opposite side?

b What is the length of each of the remaining sides?

c Write down the area A of the rectangle as a function of x.

d Use the fact that lengths must be positive to find the domain of A.

e Use a table of values to graph A for the domain you found in part **d**.

13 Solve each equation for y and hence show that it represents a function.

a $2x - y + 3 = 0$ **b** $xy = 4$ **c** $xy - 2y = 3$ **d** $y + 2 = \sqrt{9 - x^2}$

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CHALLENGE

14 State the natural domain of each function.

a $f(x) = \dfrac{x}{\sqrt{x+2}}$

b $f(x) = \dfrac{2}{x^2 - 4}$

c $f(x) = \dfrac{1}{x^2 + x}$

d $f(x) = \dfrac{1}{x^2 - 5x + 6}$

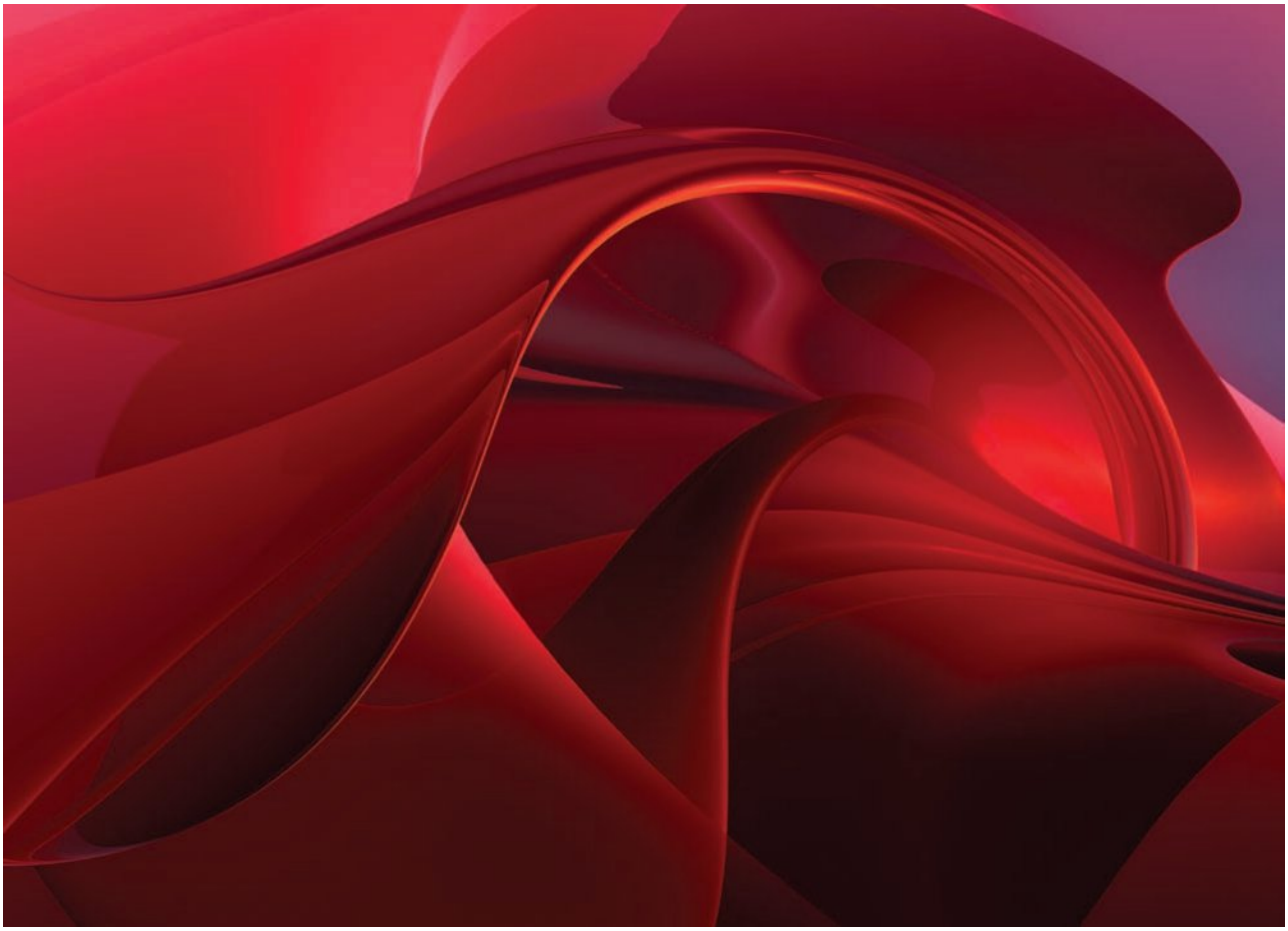
e $f(x) = \sqrt{x^2 - 4}$

f $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$

15 Let $f(x) = \begin{cases} 2 + x, & \text{for } x \le 0, \\ 2 - x, & \text{for } x > 0. \end{cases}$

a Create a table of values for $-3 \le x \le 3$.

b Hence sketch this function. (The graph is not smooth.)



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3C Review of linear graphs

The next few sections will review some functions and relations that have been introduced in earlier years, and the sketching of their graphs. Linear graphs are briefly reviewed in this section and are the main subject of Chapter 6.

Linear functions

A function is called *linear* if its graph is a straight line. Its equation is then something like

$$y = 2x - 3$$

with a term in x and a constant term. We often write a linear function with all its terms on the left — the equation of the function above then becomes

$$2x - y - 3 = 0$$

where the coefficient of y cannot be zero, because we must be able to solve for y.

9 LINEAR FUNCTIONS

- A *linear function* has a graph that is a straight line.
- The equation of a linear function can be written in *gradient–intercept form*,

  $$y = mx + b.$$
- Alternatively, the equation of a linear function can be written in *general form*,

  $ax + by + c = 0$, where the coefficient of y is non-zero.

Sketching linear functions

When all three terms of the equation are non-zero, the easiest way to sketch a linear function is to find the two intercepts with the axes.

10 SKETCHING A LINEAR FUNCTION WHOSE EQUATION HAS THREE NON-ZERO TERMS

- Find the x-intercept by putting $y = 0$.
- Find the y-intercept by putting $x = 0$.

Example 8 3C

a Sketch each linear function by finding its x-intercept and y-intercept.

i $x + 2y - 6 = 0$ **ii** $y = x - 3$

b Estimate from the graph where the two lines intersect.

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SOLUTION

a i Consider $x + 2y - 6 = 0$

When $y = 0$, $x - 6 = 0$

$x = 6$

When $x = 0$, $2y - 6 = 0$

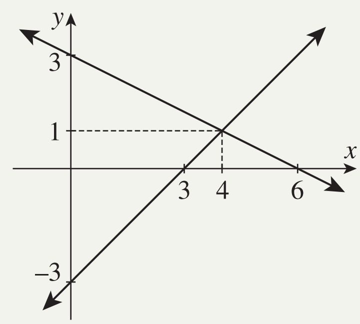
$y = 3$

ii Consider $y = x - 3$

When $y = 0$, $x - 3 = 0$

$x = 3$

When $x = 0$, $y = -3$



b From the diagram, the lines appear to meet at (4, 1).

Using simultaneous equations

Solve the simultaneous equations in Example 8, and check that the solution agrees with the estimate from the graph.

SOLUTION

The equations are $x + 2y - 6 = 0$ (1)

$x - y - 3 = 0$ (2)

Subtracting (2) from (1), $3y - 3 = 0$

$y = 1$

Substituting into (1), $x + 2 - 6 = 0$

$x = 4$ Check this by substitution into (2).

Thus the lines meet at (4, 1), as seen in the graph of Example 8.

Linear relations

A *linear relation* is a relation whose graph is a straight line. By the vertical line test, every linear relation is a function except for vertical lines, which fail the vertical line test dramatically. The equation of such a relation has no term in y, such as the relation:

$$x = 3$$

Its graph consists of all points whose x-coordinate is 3, giving a vertical line, which is sketched in Example 9.

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Three special cases

The two-intercept method for sketching graphs won't work for sketching the graph of $ax + by + c = 0$ if any of the constants a, b or c is zero.

11 SKETCHING SPECIAL CASES OF LINEAR GRAPHS $ax + by + c = 0$

Horizontal lines: If $a = 0$, the equation can be put into to the form $y = k$.

- Its graph is a horizontal line with y-intercept k.

Vertical lines: If $b = 0$, the equation can be put into the form $x = \ell$.

- Its graph is a vertical line with x-intercept ℓ.
- This is the only type of linear graph that is a relation, but not a function.

Lines through the origin: If $c = 0$, and the line is neither horizontal nor vertical, then the equation can be put into the form $y = mx$, where $m \neq 0$.

- Both intercepts are zero, so the graph passes through the origin.
- Find one more point on it, by substituting a value such as $x = 1$.

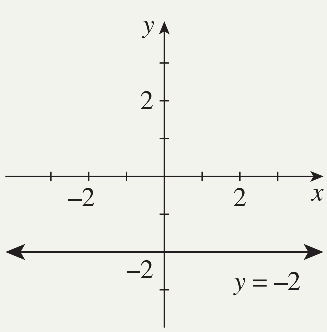
Example 9 3C

Sketch the following three lines. Which of the three is not a function?

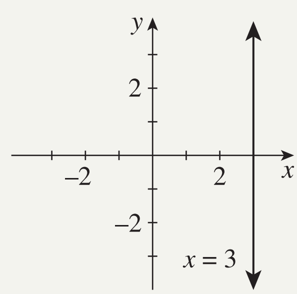
a $y + 2 = 0$ **b** $x - 3 = 0$ **c** $x + 2y = 0$

SOLUTION

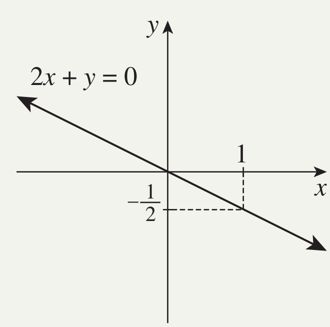
a



b



c



a The line $y + 2 = 0$, or $y = -2$, is horizontal with y-intercept -2.

b The line $x - 3 = 0$, or $x = 3$, is vertical with x-intercept 3.

c The line $x + 2y = 0$ passes through the origin, and when $x = 1$, $y = -\frac{1}{2}$.

The vertical line is a relation, but not a function — it fails the vertical line test.

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Exercise 3C

FOUNDATION

1 Follow these steps for the linear function $y = 2x - 2$.
  a Find the y-intercept by putting $x = 0$.
  b Find the x-intercept by putting $y = 0$.
  c Plot these intercepts and hence sketch the line.

2 Carry out the following steps for the straight line $x + 2y - 4 = 0$.
  a Find the y-intercept by putting $x = 0$.
  b Find the x-intercept by putting $y = 0$.
  c Plot these intercepts and hence sketch the line.

3 Repeat the three steps in the previous two questions for each line.
  a $y = x + 1$ **b** $y = 4 - 2x$ **c** $y = \frac{1}{2}x - 3$ **d** $y = -3x - 6$
  e $x + y - 1 = 0$ **f** $2x - y + 2 = 0$ **g** $x - 3y - 3 = 0$ **h** $x - 2y - 4 = 0$
  i $2x - 3y - 12 = 0$ **j** $x + 4y + 6 = 0$ **k** $5x + 2y - 10 = 0$ **l** $-5x + 2y + 15 = 0$

4 A linear function has equation $y = -2x$.
  a Show that the x-intercept and the y-intercept are both zero.
  b Substitute $x = 1$ to find a second point on the line, then sketch the line.

5 Repeat the steps in Question **4** for each line.
  a $y = x$ **b** $y = 3x$ **c** $y = -4x$
  d $x + y = 0$ **e** $x - 2y = 0$ **f** $3x + 2y = 0$

6 Sketch the following vertical and horizontal lines.
  a $x = 1$ **b** $y = 2$ **c** $x = -2$
  d $y = 0$ **e** $2y = -3$ **f** $3x = 5$

7 **a** State which lines in Question **6** are not functions.
  b For each line that is not a function, write down the coordinates of two points on it with the same x-coordinate.

DEVELOPMENT

8 For parts **e**–**l** in Question **3**, solve the equation for y to show that it is a function.

9 Determine, by substitution, whether or not the given point lies on the line.
  a $(3, 1)$ $y = x - 2$ **b** $(7, 4)$ $y = 20 - 2x$
  c $(1, -2)$ $y = -3x + 1$ **d** $(-5, 3)$ $2x + 3y + 1 = 0$
  e $(-1, -4)$ $3x - 2y - 5 = 0$ **f** $(-6, -4)$ $4x - 5y - 4 = 0$

10 Consider the lines $x + y = 5$ and $x - y = 1$.
  a Graph the lines on a number plane, using a scale of 1 cm to 1 unit on each axis.
  b Read off the point of intersection of the two lines.
  c Confirm your answer to part **b** by solving the two equations simultaneously.

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11 Repeat the previous question for the following pairs of lines.

a $x + y = 2$
$x - y = -4$

b $x - y = 3$
$2x + y = 0$

c $x + 2y = -4$
$2x - y = -3$

12 Looksmart Shirts charges \$60 for one shirt and \$50 for each shirt after this.

a Find, as a function of n, the cost C in dollars of n shirts.

b Delivery costs \$10 for one shirt and \$2 for each subsequent shirt.

i Find, as a function of n, the cost D in dollars of delivering n shirts.

ii Find, as a function of n, the total cost T in dollars of buying n shirts and having them delivered.

CHALLENGE

13 Consider the linear equation $y = \frac{1}{2}x + c$.

a Sketch on one number plane the four lines corresponding to the following values of c:

i $c = -2$

ii $c = -1$

iii $c = 1$

iv $c = 2$

b What do you notice about all these lines?

14 Consider the linear equation $y - 2 = m(x - 1)$.

a Sketch on one number plane the four lines corresponding to the following values of m:

i $m = 1$

ii $m = 2$

iii $m = -\frac{1}{2}$

iv $m = 0$

b Which point in the number plane do all these lines pass through?

c Now prove that the line $y - 2 = m(x - 1)$ passes through the point found in the previous part, regardless of the value of m.

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3D Quadratic functions — factoring and the graph

A *quadratic function* is a function that can be written in the form

$f(x) = ax^2 + bx + c$, where a, b and c are constants, and $a \neq 0$.

A *quadratic equation* is an equation that can be written in the form

$ax^2 + bx + c = 0$, where a, b and c are constants, and $a \neq 0$.

The requirement that $a \neq 0$ means that the term in x^2 cannot vanish. Thus linear functions and equations are not regarded as special cases of quadratics.

The word 'quadratic' comes from the Latin word *quadratus*, meaning 'square', and reminds us that quadratics tend to arise as the areas of regions in the plane.

Monic quadratics

A quadratic function $f(x) = ax^2 + bx + c$ is called *monic* if $a = 1$. Calculations are usually easier in monic quadratics than in non-monic quadratics.

12 MONIC QUADRATICS

A quadratic is called *monic* if the coefficient of x^2 is 1. For example:

$y = x^2 - 8x + 15$ is monic

$y = -x^2 + 8x - 15$ is non-monic.

Zeroes and roots

The solutions of a quadratic equation are called the *roots* of the equation, and the x-intercepts of a quadratic function are called the *zeroes* of the function. This distinction is often not strictly observed, however, because questions about quadratic functions and their graphs are so closely related to questions about quadratic equations.

Five questions about the graph of a quadratic

The graph of any quadratic function $y = ax^2 + bx + c$ is a *parabola*, as seen in earlier years. Before sketching the parabola, five questions need to be answered.

13 FIVE QUESTIONS ABOUT THE PARABOLA $y = ax^2 + bx + c$

1. Which way up is the parabola? **Answer:** Look at the sign of a.
2. What is the y-intercept? **Answer:** Put $x = 0$, and then $y = c$.
3. What are the x-intercepts, or zeroes, if there are any?
4. What is the axis of symmetry?
5. What is the vertex? **Method:** Substitute the axis back into the quadratic.

The first two questions are easy to answer:

- If a is positive, the curve is concave up. If a is negative, it is concave down.
- To find the y-intercept, put $x = 0$, then $y = c$.

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And once the axis of symmetry has been found, the y-coordinate of the vertex can be found by substituting back into the quadratic.

But finding the x-intercepts, and finding the axis of symmetry, need careful working — there are three standard approaches:

- factoring (this section)
- completing the square (Section 3E)
- formulae (Section 3F).

Factoring and the zeroes

Factoring of monic and non-monic quadratics was reviewed in Chapter 1. Most quadratics cannot easily be factored, but when straightforward factoring is possible, this is usually the quickest approach. After factoring, the zeroes can be found by putting $y = 0$ and using the principle:

$$\text{If } A \times B = 0 \quad \text{then} \quad A = 0 \text{ or } B = 0.$$

For example, $y = x^2 - 2x - 3$ is a quadratic function.

1 Its graph is concave up because $a = 1$ is positive.

2 Substituting $x = 0$ gives $\quad y = 0 + 0 - 3$
so the y-intercept is $\quad y = -3$.

3 Factoring, $\quad y = (x + 1)(x - 3)$.
Substituting $y = 0$ gives

$$\begin{aligned} 0 &= (x + 1)(x - 3) \\ x + 1 = 0 &\text{ or } x - 3 = 0 \\ x &= -1 \text{ or } x = 3, \end{aligned}$$

so the x-intercepts are $\quad x = -1$ and $x = 3$.

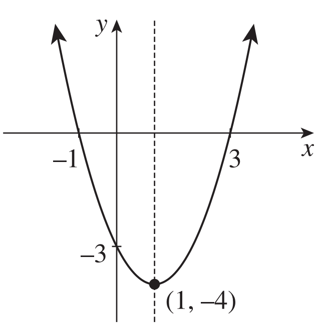
Finding the axis of symmetry and the vertex from the zeroes

The axis of symmetry is always the vertical line midway between the x-intercepts. Thus, its x-intercept is the average of the zeroes.

Continuing with our example of $y = x^2 - 2x - 3$, which factors as $y = (x + 1)(x - 3)$, and so has zeroes -1 and 3:

4 Taking the average of the zeroes, the axis of symmetry is

$$\begin{aligned} x &= \tfrac{1}{2}(-1 + 3) \\ x &= 1. \end{aligned}$$



5 Substituting $x = 1$ into the factored quadratic,

$$\begin{aligned} y &= (1 + 1)(1 - 3) \\ &= -4, \end{aligned}$$

so the vertex is $(1, -4)$.

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14 THE ZEROES AND INTERCEPTS OF A FACTORED QUADRATIC

Suppose that we have managed to factor a quadratic as $y = a(x - \alpha)(x - \beta)$.

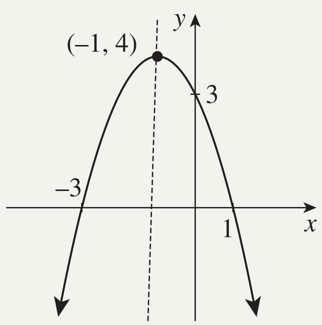
- Its x-intercepts (zeroes) are $x = \alpha$ and $x = \beta$.
- Its axis is the line $x = \frac{1}{2}(\alpha + \beta)$. Take the average of the zeroes.
- Substitute the axis into the factored form of the quadratic to find the y-coordinate of the vertex.

Example 10 3D

[This is an example of a non-monic quadratic.]

Sketch the curve $y = -x^2 - 2x + 3$.

SOLUTION



1 Because $a < 0$, the curve is concave down.

2 When $x = 0$, $y = 3$.

3 Factoring, $y = -(x^2 + 2x - 3)$
$= -(x + 3)(x - 1)$.

When $y = 0$, $x + 3 = 0$ and $x - 1 = 0$,
so the zeroes are $x = -3$ and $x = 1$.

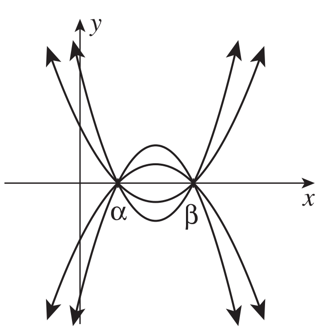
4 Take the average of the zeroes, the axis of symmetry is

$$x = \tfrac{1}{2}(-3 + 1)$$
$$x = -1.$$

5 When $x = -1$, $y = -(-1 + 3) \times (-1 - 1)$
$= 4$,

so the vertex is $(-1, 4)$.

Quadratics with given zeroes



If a quadratic $f(x)$ has zeroes at $x = \alpha$ and $x = \beta$, then its equation must have the form

$$f(x) = a(x - \alpha)(x - \beta)$$

where a is the coefficient of x^2. By taking different values of the coefficient a, this equation forms a *family* of quadratics, all with the same x-intercepts. The sketch to the right shows four of the curves in the family, two concave up with positive values of a, and two concave down with negative values of a.

15 THE FAMILY OF QUADRATICS WITH GIVEN ZEROES

The quadratics with zeroes at $x = \alpha$ and $x = \beta$ form a *family* of parabolas with equation
$y = a(x - \alpha)(x - \beta)$, for some non-zero value of a.

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Example 11 3D

a Write down the family of quadratics with zeroes $x = -2$ and $x = 4$.
b Then find the equation of such a quadratic if:
  i the y-intercept is 16 **ii** the curve passes through $(5, 1)$.

SOLUTION

a The family of quadratics with zeroes -2 and 4 is $y = a(x + 2)(x - 4)$.

b **i** Substituting the point $(0, 16)$ gives $16 = a \times 2 \times (-4)$
$a = -2$,
so the quadratic is $y = -2(x + 2)(x - 4)$.

ii Substituting the point $(5, 1)$ gives $1 = a \times 7 \times 1$
$a = \frac{1}{7}$
so the quadratic is $y = \frac{1}{7}(x + 2)(x - 4)$.

Exercise 3D

FOUNDATION

1 **a** The parabola with equation $y = (x - 1)(x - 3)$ is concave up.
  i Write down its y-intercept.
  ii Put $y = 0$ to find the x-intercepts.
  iii Hence determine the equation of the axis of symmetry.
  iv Use the axis of symmetry to find the coordinates of the vertex.
  v Sketch the parabola, showing these features.
b Follow the steps of part **a** to sketch these concave-up parabolas.
  i $y = (x - 1)(x + 3)$ **ii** $y = (x - 1)(x + 1)$

2 **a** The parabola $y = -x(x - 2)$ is concave down.
  i Write down its y-intercept.
  ii Put $y = 0$ to find the x-intercepts.
  iii Hence determine the equation of the axis of symmetry.
  iv Use the axis of symmetry to find the coordinates of the vertex.
  v Sketch the parabola, showing these features.
b Follow the steps of part **a** to sketch these concave-down parabolas.
  i $y = (2 + x)(2 - x)$ **ii** $y = (x + 2)(4 - x)$

3 **a** The parabola $y = (x - 1)^2$ is a perfect square.
  i Write down its y-intercept.
  ii Put $y = 0$ to find its single x-intercept at the vertex.
  iii Sketch the parabola showing the y-intercept and the vertex.
  iv Use symmetry and the y-intercept to find another point on the parabola.
b Follow similar steps to part **a** to sketch the parabolas of these perfect squares.
  i $y = (x + 1)^2$ **ii** $y = -(x - 2)^2$

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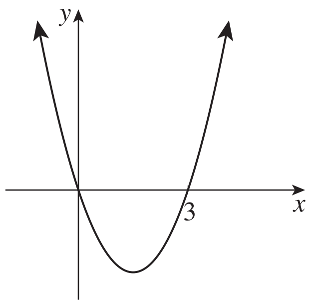
4 Write down, in factored form, the equation of the monic quadratic function with zeroes:

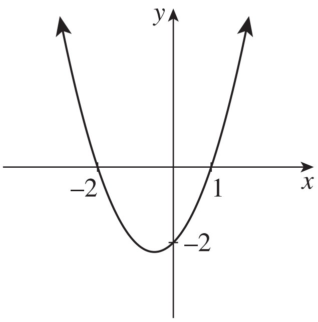
a 4 and 6 **b** 0 and 3 **c** −3 and 5 **d** −6 and −1

5 Write down, in factored form, the equation of each quadratic function sketched below, given that the coefficient of x^2 is either 1 or −1.

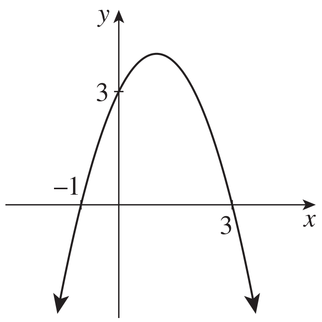
a



b



c



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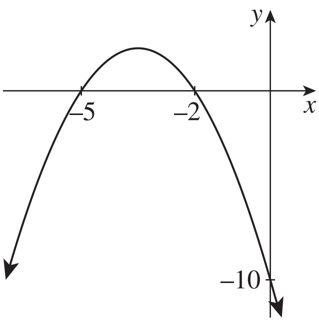

DEVELOPMENT

6 Use factoring to find the zeroes of each quadratic function. Hence sketch the graph of $y = f(x)$, showing all intercepts and the coordinates of the vertex.

a $f(x) = x^2 - 9$ **b** $f(x) = x^2 + 4x - 5$ **c** $f(x) = x^2 + 4x - 12$

7 Use factoring to find the zeroes of each quadratic function. Hence sketch the graph of $y = f(x)$, showing all intercepts and the coordinates of the vertex.

a $f(x) = 4x - x^2$ **b** $f(x) = -x^2 + 2x + 3$ **c** $f(x) = 8 - 2x - x^2$

8 Sketch these parabolas involving perfect squares. Begin by factoring the quadratic. In each case, symmetry will be needed to find a third point on the parabola.

a $y = x^2 - 6x + 9$ **b** $y = -x^2 + 2x - 1$

9 Use factoring to sketch the graphs of the following non-monic quadratic functions, clearly indicating the vertex and the intercepts with the axes.

a $y = 2x^2 + 7x + 5$ **b** $y = 2x^2 + 5x - 3$ **c** $y = 3x^2 + 2x - 8$

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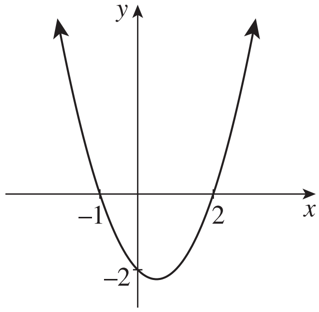
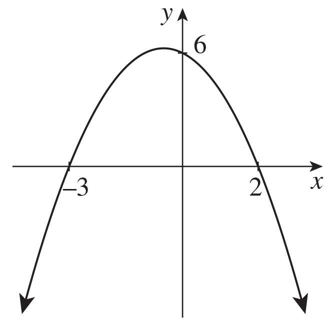
10 Find the equations of the quadratic functions sketched below.

a



b



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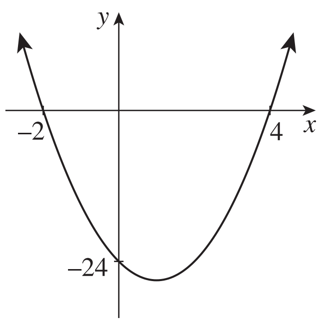
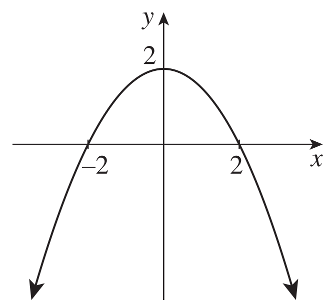


d



11 Find, in factored form, the equations of the parabolas with the given intercepts.

a $x = 1, 3$
 $y = 6$

b $x = -2, 1$
 $y = 4$

c $x = -1, 5$
 $y = 15$

d $x = -2, -4$
 $y = 2$

12 [Technology]

a Use computer graphing software to plot accurately on the one number plane the parabola $y = a(x - 1)(x - 3)$ for the following values of a.

i $a = 2$ **ii** $a = 1$ **iii** $a = -1$ **iv** $a = -2$

b Which two points do all these parabolas pass through?

CHALLENGE

13 Use factoring to find the zeroes of each quadratic function. Sketch a graph of the function, clearly indicating all intercepts and the coordinates of the vertex.

a $y = 2x^2 - 18$ **b** $y = 3x^2 + x - 4$ **c** $y = 7x - 3 - 4x^2$

14 The general form of a quadratic with zeroes $x = 2$ and $x = 8$ is $y = a(x - 2)(x - 8)$. Find the equation of such a quadratic for which:

a the coefficient of x^2 is 3

b the y-intercept is -16

c the vertex is $(5, -12)$

d the curve passes through $(1, -20)$.

15 Consider the quadratic function $f(x) = x^2 - 2x - 8$.

a Factor the quadratic and hence find the equation of the axis of symmetry.

b **i** Expand and simplify $f(1 + h)$ and $f(1 - h)$. What do you notice?

ii What geometric feature of the parabola does this result demonstrate?

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3E Completing the square and the graph

Completing the square is the most general method of dealing with quadratics. It works in every case, whereas factoring really only works in exceptional cases.

We first review the algebra of completing the square, and extend it to non-monic quadratics. Once this has been done, sketching the graph follows easily.

Completing the square in a monic quadratic

The quadratic $y = x^2 + 6x + 5$ is not a perfect square, but it can be made into the sum of a perfect square and a constant. As explained in Chapter 1 the procedure is:

- Look just at the two terms in x, that is, $x^2 + 6x$.
- Halve the coefficient 6 of x to get 3, then square to get 9.
- Add and subtract 9 on the RHS to produce a perfect square plus a constant.

$$\begin{aligned} y &= x^2 + 6x + 5 \\ &= (x^2 + 6x + 9) - 9 + 5 \text{ (add and subtract 9)} \\ &= (x + 3)^2 - 4 \end{aligned}$$

16 COMPLETING THE SQUARE IN A MONIC QUADRATIC

- Take the coefficient of x, halve it, then square the result.
- Add and subtract this number to produce a perfect square plus a constant.

Example 12 **3E**

Complete the square in each quadratic.

a $y = x^2 - 4x - 5$ **b** $y = x^2 + x + 1$

SOLUTION

a Here $y = x^2 - 4x - 5$.

The coefficient of x is -4. Halve it to get -2, then square to get $(-2)^2 = 4$.

Hence $y = (x^2 - 4x + 4) - 4 - 5$ (add and subtract 4)
$= (x - 2)^2 - 9$.

b Here $y = x^2 + x + 1$.

The coefficient of x is 1. Halve it to get $\frac{1}{2}$, then square to get $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

Hence $y = (x^2 + x + \frac{1}{4}) - \frac{1}{4} + 1$ $\left(\text{add and subtract } \frac{1}{4}\right)$
$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$.

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Completing the square in a non-monic quadratic

For a *non-monic quadratic* such as $y = 2x^2 - 12x + 16$, where the coefficient of x^2 is not 1, divide through by the coefficient of x^2 before completing the square. This slightly more difficult procedure was not covered in Chapter 1.

17 COMPLETING THE SQUARE IN A NON-MONIC QUADRATIC

- Divide through by the coefficient of x^2 so that the coefficient of x^2 is 1.
- Complete the square in the resulting monic quadratic.

Example 13 3E

Complete the square in each quadratic.

a $y = 2x^2 - 12x + 16$ **b** $y = -x^2 + 8x - 15$

SOLUTION

a $y = 2x^2 - 12x + 16$

$\frac{y}{2} = x^2 - 6x + 8$ (divide through by the coefficient 2 of x^2)

$\frac{y}{2} = (x^2 - 6x + 9) - 9 + 8$ (complete the square on the RHS)

$\frac{y}{2} = (x - 3)^2 - 1$

$y = 2(x - 3)^2 - 2$ (multiply by 2 to make y the subject again)

b $y = -x^2 + 8x - 15$

$-y = x^2 - 8x + 15$ (divide through by the coefficient -1 of x^2)

$-y = (x^2 - 8x + 16) - 16 + 15$ (complete the square on the RHS)

$-y = (x - 4)^2 - 1$

$y = -(x - 4)^2 + 1$ (multiply by -1 to make y the subject again)

Finding the vertex from the completed square

The completed square allows the vertex to be found using one fundamental fact about squares:

18 A SQUARE CAN NEVER BE NEGATIVE

- $x^2 = 0$, when $x = 0$,
- $x^2 > 0$, when $x \neq 0$.

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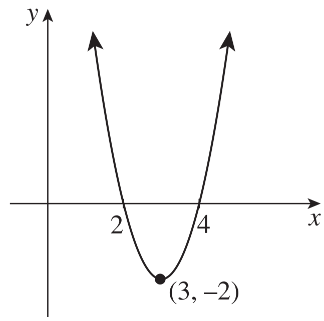
Thus in part **a** above, where

$$y = 2(x - 3)^2 - 2$$

the term $2(x - 3)^2$ can never be negative.



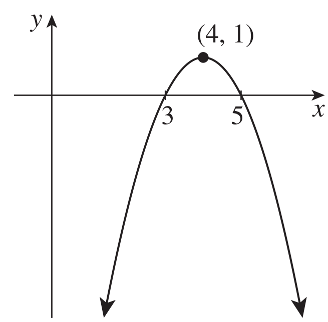
Substituting $x = 3$ gives $y = 0 - 2$
$= -2,$
but if $x \neq 3$, then $y > -2.$

Hence the graph passes through the point $V(3, -2)$, but never goes below it. This means that $V(3, -2)$ is the vertex of the parabola, and $x = 3$ is its axis of symmetry.

But in part **b** above, where

$$y = -(x - 4)^2 + 1,$$

the term $-(x - 4)^2$ can never be positive.



Substituting $x = 4$ gives $y = 0 + 1,$
$= 1,$
but if $x \neq 4$, then $y < 1.$

Hence the graph passes through the point $V(4, 1)$, but never goes above it. This means that $V(4, 1)$ is the vertex of the parabola, and $x = 4$ is its axis of symmetry.

There is no need to repeat this argument every time. The result is simple:

19 FINDING THE AXIS AND VERTEX FROM THE COMPLETED SQUARE

For the quadratic $y = a(x - h)^2 + k$,
the axis is $x = h$ and the vertex is $V(h, k)$.

In Chapter 4, we will interpret this result as a translation — you may have done this already in earlier years.

Finding the zeroes from the completed square

The completed square also allows the zeroes to be found in the usual way.

20 FINDING THE ZEROES FROM THE COMPLETED SQUARE

To find the x-intercepts from the completed square, put $y = 0$.
- There may be two zeroes, in which case they may or may not involve surds.
- There may be no zeroes.
- There may be exactly one zero, in which case the quadratic is a *perfect square*.
  The x-axis is a tangent to the graph, and the zero is called a *double zero* of the quadratic.

These methods are illustrated in the following four examples.

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Example 14 3E

[Examples with and without surds]

Use completing the square to sketch the graphs of these quadratics.

a $y = x^2 - 4x - 5$ **b** $y = x^2 - 4x - 1$

SOLUTION

a $y = x^2 - 4x - 5$ is concave up, with y-intercept -5.

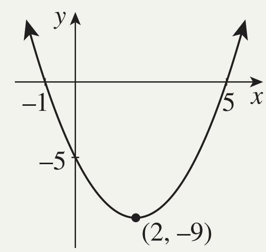
The square was completed earlier in this section,

$$y = (x - 2)^2 - 9,$$

so the axis is $x = 2$ and the vertex is $(2, -9)$.

Put $y = 0$, then $(x - 2)^2 = 9$

$$x - 2 = 3 \text{ or } x - 2 = -3$$
$$x = 5 \text{ or } x = -1.$$



b $y = x^2 - 4x - 1$ is concave up, with y-intercept 1.

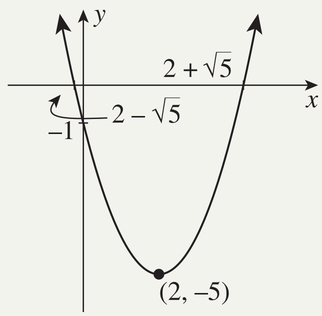
Completing the square, $y = (x^2 - 4x + 4) - 4 - 1$

$$= (x - 2)^2 - 5,$$

so the axis is $x = 2$ and the vertex is $(2, -5)$.

Put $y = 0$, then $(x - 2)^2 = 5$

$$x - 2 = \sqrt{5} \text{ or } x - 2 = -\sqrt{5}$$
$$x = 2 + \sqrt{5} \text{ or } x = 2 - \sqrt{5}.$$



Example 15 3E

[An example with no zeroes, and an example with one zero.]

Use completing the square to sketch the graphs of these quadratics.

a $y = x^2 + x + 1$ **b** $y = x^2 + 6x + 9$

SOLUTION

a $y = x^2 + x + 1$ is concave up, with y-intercept 1.

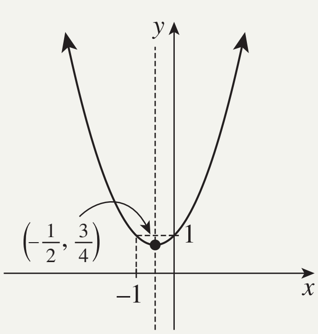
The square was completed earlier in this section.

$$y = \left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4},$$

so the axis is $x = -\frac{1}{2}$, and the vertex is $\left(-\frac{1}{2}, \frac{3}{4}\right)$.

Put $y = 0$, then $\left(x + \frac{1}{2}\right)^2 = -\frac{3}{4}$.

Because negative numbers do not have square roots,
this equation has no solutions, so there are no x-intercepts.



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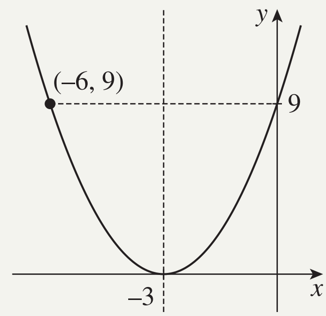
b $y = x^2 + 6x + 9$ is concave up, with y-intercept 9.
This quadratic is already a perfect square,

$$y = (x + 3)^2 + 0,$$

so the axis is $x = -3$ and the vertex is $(-3, 0)$.

Put $y = 0$, then
$$\begin{aligned}(x + 3)^2 &= 0\\ x + 3 &= 0\\ x &= -3 \text{ (a double zero).}\end{aligned}$$



Notice that the symmetric point $(-6, 9)$ has been plotted so that the parabolic graph has at least three points on it.

Note: With this method, the axis of symmetry and the vertex are read directly off the completed-square form — the zeroes are then calculated afterwards. Compare this with factoring, where the zeroes are found first — and the axis and vertex can then be calculated from them.

Example 16 3E

Use the completed squares from Example 13 to sketch the graphs of these quadratics.

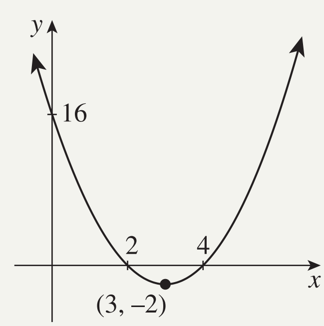
a $y = 2x^2 - 12x + 16$

b $y = -x^2 + 8x - 15$

SOLUTION

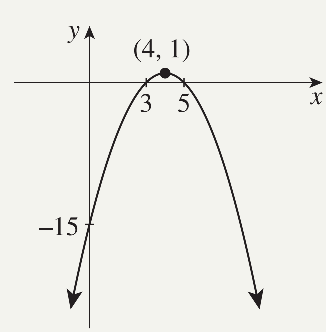
a $y = 2x^2 - 12x + 16$ is concave up, with y-intercept 16.
Completing the square, $y = 2(x - 3)^2 - 2$,
so the axis is $x = 3$ and the vertex is $(3, -2)$.

Put $y = 0$, then
$$\begin{aligned}2(x - 3)^2 &= 2\\ (x - 3)^2 &= 1\\ x - 3 = 1 &\text{ or } x - 3 = -1\\ x = 4 &\text{ or } x = 2.\end{aligned}$$



b $y = -x^2 + 8x - 15$ is concave down, with y-intercept -15.
Completing the square, $y = -(x - 4)^2 + 1$,
so the axis is $x = 4$ and the vertex is $(4, 1)$.

Put $y = 0$, then
$$\begin{aligned}(x - 4)^2 &= 1\\ x - 4 = 1 &\text{ or } x - 4 = -1\\ x = 5 &\text{ or } x = 3.\end{aligned}$$



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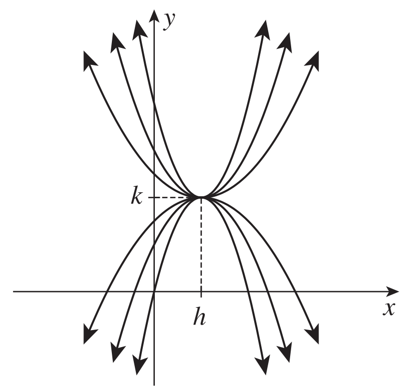
The family of quadratics with a common vertex

A quadratic with vertex (h, k) must have an equation of the form

$$y = a(x - h)^2 + k$$

where a is the coefficient of x^2. This equation gives a *family* of parabolas all with vertex (h, k), as different values of a are taken. The sketch to the right shows six curves in the family, three with a positive, and three with a negative.



21 THE FAMILY OF QUADRATICS WITH A COMMON VERTEX

The quadratics with vertex (h, k) form a family of parabolas, all with equation $y = a(x - h)^2 + k$, for some value of a.

Example 17 — 3E

Write down the family of quadratics with vertex $(-3, 2)$. Then find the equation of such a quadratic:

a if $x = 5$ is one of its zeroes

b if the coefficient of x is equal to 1.

SOLUTION

The family of quadratics with vertex $(-3, 2)$ is

$$y = a(x + 3)^2 + 2. \qquad (*)$$

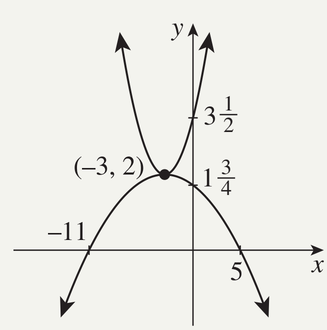
a Substituting $(5, 0)$ into the equation $(*)$ gives $0 = a \times 64 + 2$,

so $a = -\frac{1}{32}$, and the quadratic is $y = -\frac{1}{32}(x + 3)^2 + 2$.

b Expanding the equation $(*)$, $y = ax^2 + 6ax + (9a + 2)$,

so $6a = 1$.

Hence $a = \frac{1}{6}$, and the quadratic is $y = \frac{1}{6}(x + 3)^2 + 2$.



Exercise 3E

FOUNDATION

1 a The equation of a parabola in completed square form is $y = (x - 2)^2 - 1$.

- **i** What is the concavity of this parabola?
- **ii** Substitute $x = 0$ to find the y-intercept.
- **iii** Put $y = 0$ to find the x-intercepts.
- **iv** Use the results of Box 19 to write down the equation of the axis of symmetry and the coordinates of the vertex.
- **v** Hence sketch this parabola, showing this information.

b Repeat the steps of part **a** in order to sketch the graphs of these quadratic functions.

- **i** $y = (x + 1)^2 - 4$
- **ii** $y = (x - 1)^2 - 9$

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2 **a** A concave-down parabola has the completed square form $y = -(x - 1)^2 + 1$.

i Why is the parabola concave-down?

ii Substitute $x = 0$ to find the y-intercept.

iii Put $y = 0$ to find the x-intercepts.

iv Use the results of Box 19 to write down the equation of the axis of symmetry and the coordinates of the vertex.

v Hence sketch this parabola, showing this information.

b Repeat the steps of part **a** in order to sketch the graphs of these parabolas.

i $y = -(x + 2)^2 + 4$ **ii** $y = -(x - 2)^2 + 9$

3 Complete the square for each monic quadratic function.

a $f(x) = x^2 - 4x + 5$ **b** $f(x) = x^2 + 6x + 11$ **c** $f(x) = x^2 - 2x + 8$

d $f(x) = x^2 - 10x + 1$ **e** $f(x) = x^2 + 2x - 5$ **f** $f(x) = x^2 + 4x - 1$

4 Follow the steps in Question **1** to sketch these quadratics. The x-intercepts involve surds.

a $y = (x + 1)^2 - 3$ **b** $y = (x - 4)^2 - 7$ **c** $y = (x - 3)^2 - 2$

DEVELOPMENT

5 Complete the square in each quadratic. Then sketch the graph of each function, showing the vertex and the intercepts with the axes.

a $y = x^2 - 2x$ **b** $y = x^2 - 4x + 3$ **c** $y = x^2 - 2x - 5$

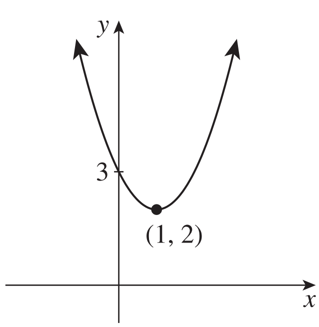
d $y = x^2 + 2x - 1$ **e** $y = x^2 + 2x + 2$ **f** $y = x^2 - 3x + 4$

6 Find the zeroes of each quadratic function by first completing the square. Then show that the same answer is obtained by factoring.

a $f(x) = x^2 - 4x + 3$ **b** $f(x) = x^2 + 2x - 3$ **c** $f(x) = x^2 - x - 2$

7 Use the formula in Box 21 to write down the equation of each of the quadratic functions sketched below. In each case, the coefficient of x^2 is either 1 or -1.

a



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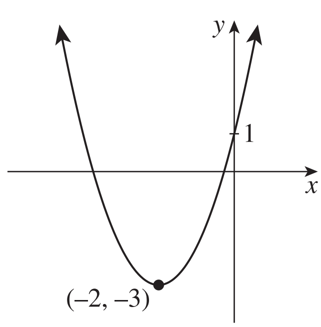

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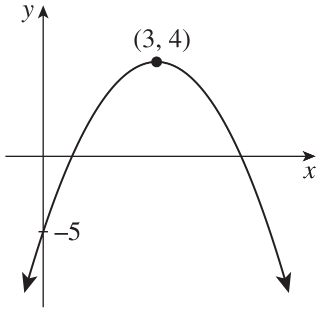

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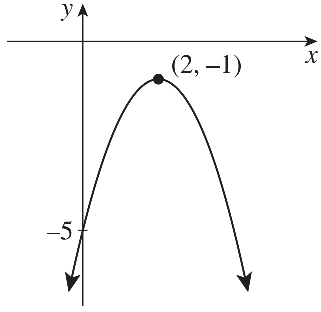

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8 Write down the equation of the monic quadratic with vertex:

a $(2, 5)$ **b** $(0, -3)$ **c** $(-1, 7)$ **d** $(3, -11)$

9 [Technology]

a Use computer graphing software to plot accurately on the one number plane the parabola $y = a(x - 1)^2 - 2$ for the following values of a.

i $a = 2$ **ii** $a = 1$ **iii** $a = -1$ **iv** $a = -2$

b Which point do all these parabolas pass through?

c For which values of a does the parabola have x-intercepts?

d Explain your answer to part **c** geometrically.

CHALLENGE

10 Write down the coordinates of the vertex and the concavity for each parabola. Hence determine the number of x-intercepts.

a $y = 2(x - 3)^2 - 5$ **b** $y = 3 - (x + 1)^2$ **c** $y = -3(x + 2)^2 - 1$

d $y = 2(x - 4)^2 + 3$ **e** $y = 4(x + 1)^2$ **f** $y = -(x - 3)^2$

11 Complete the square for these non-monic quadratics. (In each case, notice that the coefficient of x^2 is not 1.) Then sketch each curve, showing the vertex and any intercepts.

a $y = -x^2 - 2x$ **b** $y = -x^2 + 4x + 1$ **c** $y = 2x^2 - 4x + 3$

d $y = 4x^2 - 8x + 1$ **e** $y = 2x^2 + 6x + 2$ **f** $y = -2x^2 - 8x - 11$

12 Complete the square for each quadratic function. Hence write each quadratic in factored form. Your answers will involve surds.

a $f(x) = x^2 + 2x - 1$ **b** $f(x) = x^2 - 4x + 1$ **c** $f(x) = -x^2 - 2x + 4$

13 Explain why $y = a(x + 4)^2 + 2$ is the general form of a quadratic with vertex $(-4, 2)$. Then find the equation of such a quadratic for which:

a the quadratic is monic

b the coefficient of x^2 is 3

c the y-intercept is 16

d the curve passes through the origin.

14 Consider the general quadratic function $y = ax^2 + bx + c$.

a Divide through by a and then complete the square in x in order to show that

$$\frac{y}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}.$$

b Now make y the subject, and hence write down the coordinates of the vertex and the equation of the axis of symmetry.

c Put $y = 0$ in order to find the x-intercepts.

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3F The quadratic formulae and the graph

Completing the square in the general quadratic yields formulae for its axis of symmetry and for its zeroes. These formulae are extremely useful, and like factoring and completing the square, allow the graph to be sketched.

The algebra of completing the square in a general quadratic was made into a structured question at the end of the Exercise 3E.

The formula for the axis of symmetry

The structured question at the end of Exercise 3E yields the formula for the axis of symmetry:

22 THE AXIS OF SYMMETRY OF $y = ax^2 + bx + c$

- The axis of symmetry is the line $x = -\dfrac{b}{2a}$.
- Substitute back into the quadratic to find the y-coordinate of the vertex.

Remember just the formula for the axis of symmetry, and find the y-coordinate of the vertex by substituting back into the quadratic.

The formula for the zeroes

Further working in the structured question of Exercise 3E shows that putting $y = 0$ into $y = ax^2 + bx + c$ gives

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The quantity $b^2 - 4ac$ is very important in the theory of quadratics. It is called the *discriminant* because it discriminates. It has the symbol Δ (Greek uppercase delta, corresponding to 'D'). Using Δ in this formula makes it much easier to deal with and to remember.

23 THE ZEROES (x-INTERCEPTS) OF THE QUADRATIC $y = ax^2 + bx + c$

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}, \text{ where } \Delta = b^2 - 4ac.$$

- Always calculate the discriminant first when finding the zeroes of a quadratic.
- If $\Delta > 0$, there are two zeroes, because positives have two square roots.
- If $\Delta = 0$, there is only one zero because 0 is the only square root of 0, and in this situation, the x-axis is tangent to the graph.
- If $\Delta < 0$, there are no zeroes, because negatives don't have square roots.

You will soon see from solving some quadratic equations that if $\Delta > 0$ and all three coefficients are rational numbers, then we can 'discriminate' further:

- If Δ is a perfect square, then the zeroes are rational.
- If Δ is not a perfect square, then the zeroes involve surds.

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Example 18 3F

Use the quadratic formulae to sketch each quadratic. Give any irrational zeroes first in simplified surd form, then approximated correct to three decimal places.

a $y = -x^2 + 6x + 1$ **b** $y = 3x^2 - 6x + 4$

SOLUTION

a The curve $y = -x^2 + 6x + 1$ is concave down, with y-intercept 1.

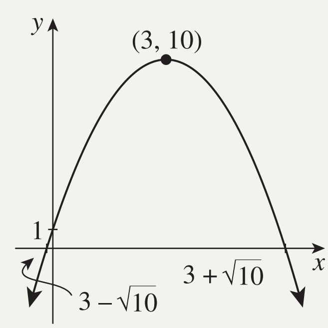
The formulae are now applied with $a = -1$, $b = 6$ and $c = 1$.

First, the axis of symmetry is
$$x = -\frac{b}{2a}$$
$$x = -\left(\frac{6}{-2}\right)$$
$$x = 3.$$

When $x = 3$, $y = -9 + 18 + 1$
$= 10$,

so the vertex is $(3, 10)$.

Secondly, $\Delta = b^2 - 4ac$
$= 40$
$= 4 \times 10$ (take out square factors),



So $y = 0$ when $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ or $\dfrac{-b - \sqrt{\Delta}}{2a}$

$= \dfrac{-6 + 2\sqrt{10}}{-2}$ or $\dfrac{-6 - 2\sqrt{10}}{-2}$

$= 3 - \sqrt{10}$ or $3 + \sqrt{10}$

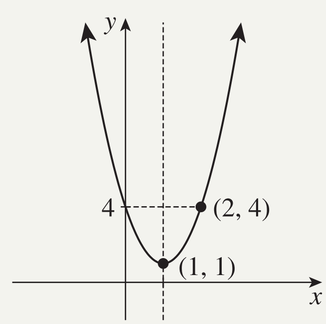
$\fallingdotseq -0.162$ or 6.162.

b The curve $y = 3x^2 - 6x + 4$ is concave up, with y-intercept 4.

Apply the formulae with $a = 3$, $b = -6$ and $c = 4$.

First, the axis of symmetry is
$$x = -\frac{b}{2a}$$
$$x = -\left(\frac{-6}{6}\right)$$
$$x = 1.$$

and substituting $x = 1$, the vertex is $(1, 1)$.



Secondly, $\Delta = 36 - 48$
$= -12$

which is negative, so there are no zeroes, because negatives do not have square roots.

Notice that the symmetric point $(2, 4)$ has been plotted so that the parabolic graph has at least three points on it.

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Example 19 3F

a Use the discriminant to find the number of zeroes of $y = 5x^2 - 20x + 20$.

b What would be a better approach to this question?

SOLUTION

a $y = 5x^2 - 20x + 20$

$\Delta = 20^2 - 4 \times 5 \times 20$

$= 0,$

so there is exactly one zero.

b Better, take out the common factor:

$y = 5x^2 - 20x + 20$

$= 5(x^2 - 4x + 4)$

$= 5(x - 2)^2.$

Exercise 3F

FOUNDATION

1 Answer the following questions for the parabola $y = x^2 - 2x - 1$.

a **i** Use the value of a to determine the concavity of the parabola.

ii Write down the value of the y-intercept.

iii Use the formula $x = \frac{-b}{2a}$ to find the axis of symmetry.

iv Use the axis of symmetry to find the y-coordinate of the vertex.

v Calculate the discriminant $\Delta = b^2 - 4ac$.

vi Explain why this parabola must have x-intercepts.

vii Find the x-intercepts by the formula $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ or $\dfrac{-b - \sqrt{\Delta}}{2a}$.

b Sketch the parabola, showing these features.

2 Follow the steps of Question **1** to sketch these parabolas.

a $y = x^2 + 4x + 1$

b $y = -x^2 + 2x + 5$

3 Answer the following questions for the parabola $y = x^2 + 2x + 3$.

a **i** Use the value of a to determine the concavity of the parabola.

ii Write down the value of the y-intercept.

iii Use the formula $x = \frac{-b}{2a}$ to find the axis of symmetry.

iv Use the axis of symmetry to find the y-coordinate of the vertex.

v Calculate the discriminant $\Delta = b^2 - 4ac$.

vi Explain why this parabola has no x-intercepts.

b Sketch the parabola, showing the y-intercept and vertex. Then use symmetry to locate another point on the parabola and add this to your sketch.

4 Follow the steps of Question **3** to sketch these parabolas.

a $y = x^2 - 2x + 2$

b $y = -x^2 - 2x - 4$

5 Find the discriminant $\Delta = b^2 - 4ac$ of each quadratic function, then find the zeroes. Give the zeroes first in surd form, then correct to two decimal places.

a $f(x) = x^2 + 2x - 2$

b $f(x) = x^2 - 4x + 1$

c $f(x) = x^2 + 3x - 2$

d $f(x) = -x^2 - 2x + 4$

e $f(x) = 3x^2 - 2x - 2$

f $f(x) = 2x^2 + 4x - 1$

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DEVELOPMENT

6 Sketch a graph of each parabola by carefully following the steps outlined in Question **1** or Question **3** as appropriate.

a $y = x^2 + 6x + 4$ **b** $y = x^2 - 4x + 5$ **c** $y = -x^2 + 2x + 2$
d $y = -2x^2 + 4x - 3$ **e** $y = 3x^2 + 6x - 1$ **f** $y = 2x^2 + 2x - 1$

7 In each case, find the zeroes of the quadratic function first by factoring, then by completing the square, and finally by using the quadratic formula. Observe that the answers to all three methods are the same for each function.

a $f(x) = x^2 - 3x - 4$ **b** $f(x) = x^2 - 5x + 6$ **c** $f(x) = -x^2 + 4x + 12$

8 **a** Consider the parabola $y = x^2 + 2$.
- **i** Calculate Δ and explain why this parabola has no x-intercept.
- **ii** What do you notice about the y-intercept and the vertex?
- **iii** Sketch the parabola showing the y-intercept and vertex. Then add to your sketch the point where $x = 1$.
- **iv** Complete the sketch with another point found by symmetry.

b Follow similar working to sketch these parabolas.
- **i** $y = -x^2 - 1$ **ii** $y = \frac{1}{2}x^2 + 1$

9 Find the discriminant $\Delta = b^2 - 4ac$ of each quadratic. Use this and the concavity to state how many zeroes the function has, without drawing its graph.

a $f(x) = x^2 + 3x - 2$ **b** $f(x) = 9x^2 - 6x + 1$ **c** $f(x) = -2x^2 + 5x - 4$

10 Consider the parabola with equation $y = -x^2 + 2x + 3$.

a Use algebra to find the x-coordinates of any intersection points of this parabola with each of the following lines.
- **i** $y = 2$ **ii** $y = 4$ **iii** $y = 6$

b Graph the situation.

c For what values of k does the parabola intersect the line $y = k$ twice?

CHALLENGE

11 Use the quadratic formula to find the roots α and β of each quadratic equation. Hence show in each case that $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

a $x^2 - 6x + 1 = 0$ **b** $x^2 - 2x - 4 = 0$ **c** $-3x^2 + 10x - 5 = 0$

12 Use the quadratic formula to find the zeroes α and β of each quadratic function. Hence write the function in factored form, $f(x) = (x - \alpha)(x - \beta)$.

a $f(x) = x^2 - 6x + 4$ **b** $f(x) = x^2 + 2x - 1$ **c** $f(x) = x^2 - 3x + 1$

13 The parabola $y = ax^2 + bx + c$ has axis of symmetry $x = -\frac{b}{2a}$.

a Use this value to show that the vertex of the parabola is $\left(-\dfrac{b}{2a}, -\dfrac{\Delta}{4a}\right)$.

b Use this formula to check the coordinates of the vertices in Question **6**.

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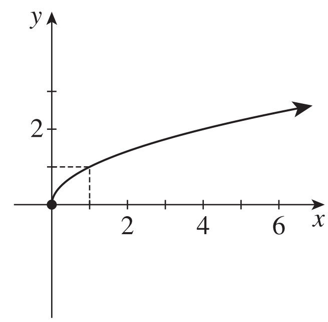
3G Powers, polynomials and circles

This section deals with the graphs of cubes, fourth powers, and higher powers of x, and of the square root of x, with particular attention to sketching cubics and polynomials that have been factored into linear factors. Then it reviews circles and semicircles.

The function $y = \sqrt{x}$

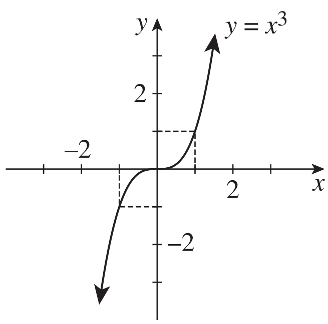


The graph of $y = \sqrt{x}$ is the upper half of a parabola on its side, as can be seen by squaring both sides to give $y^2 = x$. Remember that the symbol $\sqrt{x}$ means the *positive* square root of x, so the lower half of the parabola $y^2 = x$ is excluded.

$y = \sqrt{x}$

x	0	$\frac{1}{4}$	1	2	4
y	0	$\frac{1}{2}$	1	$\sqrt{2}$	2

The cube of x



Graphed to the right is the cubic function $y = x^3$. It has a zero at $x = 0$, it is positive when x is positive, and negative when x is negative.

$y = x^3$

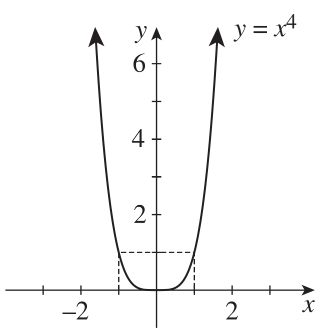
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	-8	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8

The curve becomes flat at the origin. You can see this by substituting a small number such as $x = 0.1$. The corresponding value of y is $y = 0.001$, which is far smaller.

The origin is called a *horizontal inflexion* of the curve — 'inflexion' means that the curve 'flexes' from concave down on the left to concave up on the right, and 'horizontal' means that the curve is momentarily horizontal. Inflexions in general will be studied in Year 12.

The graphs of odd powers $y = x^3$, $y = x^5$, $y = x^7$, … look similar. The origin is always a horizontal inflexion. As the index increases, they become flatter near the origin, and steeper further away.

The fourth power of x



The graph to the right shows $y = x^4$. It has a zero at $x = 0$, and is positive elsewhere.

$y = x^4$

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	16	1	$\frac{1}{16}$	0	$\frac{1}{16}$	1	16

All even powers $y = x^6$, $y = x^8$, … look similar. Like odd powers, as the index increases they become flatter near the origin and steeper further away.

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Polynomials

Polynomials are expressions such as

$$3x^4 + 4x^2 - 2x \quad \text{and} \quad x^3 - x^2 \quad \text{and} \quad 7 - 2x^3 + \tfrac{3}{4}x^6$$

that can be written as the sum of multiples of x, x^2, x^3, … and a constant.

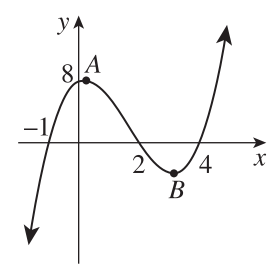
A polynomial is usually written with the powers in descending or ascending order, as above. The index of the power with highest index is called the *degree* of the polynomial, thus the first polynomial above has degree 4, the second has degree 3, and the third has degree 6.

We have already been studying three types of polynomials.

- Quadratic expressions such as $3x^2 + 4x + 5$ are polynomials of degree 2.
- Linear expressions such as $-4x + 2$, where the coefficient of x is non-zero, are polynomials of degree 1.
- Numbers such as 7 and 12 are polynomials of degree 0, except that the *zero polynomial* 0 is regarded as having no terms, and thus no degree.

Polynomials of degree 3 are called *cubic polynomials* — the volume of a cube is x^3. Polynomials of degree 4 are called *quartic polynomials*, and further such names can be used for polynomials of higher degrees. The word 'polynomial' means 'many terms'.

Sketching a cubic factored into linear factors



At this stage, little systematic graphing can be done of polynomial functions of degree 3 and higher. If, however, the polynomial has already been factored into linear factors, its basic shape can be established. This is done by drawing up a table of values to test its sign. For example, consider the cubic

$$y = (x + 1)(x - 2)(x - 4).$$

The function has zeroes at $x = -1$, $x = 2$ and $x = 4$. Its domain is all real x, and it is continuous for all x, so the zeroes are the only places where it can change sign. We can draw up a table of values dodging these zeroes,

x	-2	-1	0	2	3	4	5
y	-24	0	8	0	-4	0	18
sign	$-$	0	$+$	0	$-$	0	$+$

We also know that, as with any cubic function, y becomes very large positive or negative for large positive or negative values of x, and combining this with the table above allows us to draw a sketch.

Be careful! We cannot yet find the two points A and B on the curve because that requires calculus. Notice that if we draw tangents there, the tangents are horizontal, which means that the curve is neither increasing nor decreasing there. Don't ever think that those points A and B are midway between the zeroes — this curve is not a parabola. Mark the zeroes and the y-intercept, and nothing else.

Cubics with square factors

The cubic in the following worked example has a square factor, so that there are only two zeroes.

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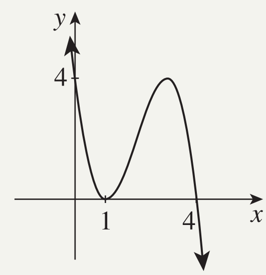
Example 20 3G

Sketch $y = -(x-1)^2(x-4)$. What is happening at $x = 1$?



SOLUTION

The cubic has zeroes at $x = 1$ and $x = 4$.

Here is a table of test values dodging around these zeroes,

x	0	1	2	4	5
y	4	0	2	0	-16
sign	$+$	0	$+$	0	$-$

At $x = 1$, the cubic has a zero, but does not cross the x-axis.

Instead, the x-axis is a tangent to the curve at $(1, 0)$.

24 SKETCHING A CUBIC FACTORED INTO LINEAR FACTORS:

To sketch a cubic that has been factored into linear factors as

$$y = a(x-\alpha)(x-\beta)(x-\gamma).$$

- Draw up a table of values dodging around the zeroes to test the sign.
- If the curve meets the x-axis at $x = \alpha$ without crossing it, then the x-axis is tangent to the curve at $x = \alpha$.

Cubics such as $y = 3(x-2)^3$ will be sketched in question 14 of Exercise 4A. Their basic shape has already been discussed at the start of this section.

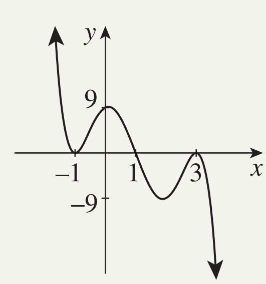
Sketching a polynomial factored into linear factors

This same method can be applied to polynomials of higher degree that have been factored into linear factors.

The factoring may now involve a cubic or higher degree linear factor such as $(x-5)^3$, but we will not consider these situations yet because the behaviour at the corresponding x-intercept requires calculus to analyse.

Example 21 3G

Sketch $y = -(x+1)^2(x-1)(x-3)^2$. What is happening at $x = -1$ and at $x = 3$?



SOLUTION

The polynomial has zeroes at $x = -1$, $x = 1$ and $x = 3$.

Here is a table of test values dodging around these zeroes,

x	-2	-1	0	1	2	3	4
y	75	0	9	0	-9	0	-75
sign	$+$	0	$+$	0	$-$	0	$-$

At $x = -1$ and $x = 3$, the curve meets the x-axis, without crossing it, so the x-axis is a tangent to the curve at those two places.

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25 SKETCHING A POLYNOMIAL FACTORED INTO LINEAR FACTORS:

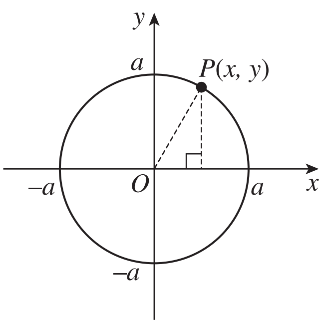
To sketch a polynomial that has been factored into linear factors as

$$y = a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) \ldots$$

- Draw up a table of values dodging around the zeroes to test the sign.
- If the curve meets the x-axis at $x = \alpha$ without crossing it, then the x-axis is tangent to the curve at $x = \alpha$.

Circles and semicircles

The equation of the circle with centre the origin and radius a can be found using Pythagoras' theorem, in the form of the distance formula.



A point $P(x, y)$ in the plane will lie on the circle if its distance from the centre is the radius a.

That is, if

$$\begin{aligned} OP &= a \\ OP^2 &= a^2 \\ (x - 0)^2 + (y - 0)^2 &= a^2 \\ x^2 + y^2 &= a^2. \end{aligned}$$

To put it very briefly, the equation of a circle is Pythagoras' theorem.

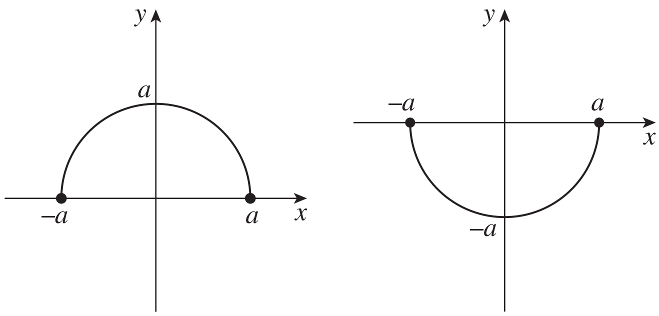
This graph fails the vertical line test, so is not a function. This can also be seen algebraically — solving the equation for y yields

$$y = \sqrt{a^2 - x^2} \qquad \text{or} \qquad y = -\sqrt{a^2 - x^2}$$

giving two values of y for some values of x.

The *positive square root* $y = \sqrt{a^2 - x^2}$, however, is a function, whose graph is the *upper semicircle* on the left below.

Similarly, the *negative square root* $y = -\sqrt{a^2 - x^2}$ is also a function, whose graph is the *lower semicircle* on the right below.



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Exercise 3G

FOUNDATION

1 Write down the coordinates of the centre and the radius of each circle.

a $x^2 + y^2 = 16$ b $x^2 + y^2 = 49$ c $x^2 + y^2 = \frac{1}{9}$ d $x^2 + y^2 = 1.44$

2 Sketch graphs of these circles, marking all intercepts with the axes, then write down the domain and range of each.

a $x^2 + y^2 = 1$ b $x^2 + y^2 = 9$ c $x^2 + y^2 = \frac{1}{4}$ d $x^2 + y^2 = \frac{9}{4}$

3 Consider the curve $y = x^3$.

a Copy and complete the following table of values:

x	-1.5	-1	-0.5	0	0.5	1	1.5
y							

b Plot the points in the table, using a scale of 2 cm to 1 unit on each axis, and then join the points with a smooth curve.

4 Repeat Question **3** for the curve $y = x^4$.

5 [Technology]

a Use computer graphing software to plot accurately on the one number plane the graphs for $y = x$, $y = x^3$ and $y = x^5$.

b Which three points do all these graphs pass through?

c Which curve is nearest the x-axis for:

i $0 < x < 1$? ii $x > 1$?

d Which curve is nearest the x-axis for:

i $-1 < x < 0$? ii $x < -1$?

e Rotate each curve by 180° about the origin. What do you notice?

f Try finding other powers of x that have the same feature found in part **e**. What do you notice about the index of these functions?

6 [Technology]

a Use computer graphing software to plot accurately on the one number plane $y = x^2$, $y = x^4$ and $y = x^6$.

b Which three points do all these graphs pass through?

c Which curve is nearest the x-axis for:

i $0 < x < 1$? ii $x > 1$?

d Which curve is nearest the x-axis for:

i $-1 < x < 0$? ii $x < -1$?

e Reflect each curve in the y-axis. What do you notice?

f Try finding other powers of x that have the feature found in part **e**. What do you notice about the index of these functions?

7 In each case, state whether or not the function is a polynomial. If it is a polynomial, write down its degree and the coefficient of x.

a $a(x) = 2x + 3$ b $b(x) = x^3 - 4x^2 + 5$ c $c(x) = 3x^2 - \dfrac{1}{x}$

d $d(x) = \sqrt{x} + x - 1$ e $e(x) = -\dfrac{x^3}{6} + \dfrac{x^2}{2} - x + 1$ f $f(x) = \sqrt{x^2 - 9} + 1$

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8 Consider the curve $y = \sqrt{x}$.

a Copy and complete the following table of values:

x	0	0.25	1	2.25	4	6.25
y						

b Plot the points in the table, using a scale of 2 cm to 1 unit on each axis, and then join the points with a smooth curve.

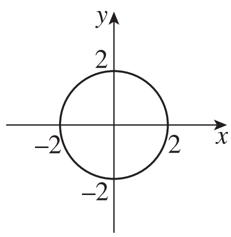
9 Write down the zeroes of each cubic, use a table of values to test its sign, then sketch it, showing the y-intercept.

a $y = (x - 1)(x - 3)(x - 5)$ **b** $y = -3(x + 4)x(x - 2)$ **c** $y = 2x^2(3 - x)$

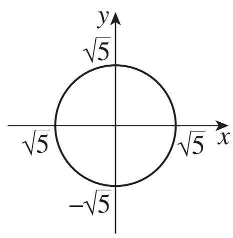
DEVELOPMENT

10 Write down the equation of each circle.

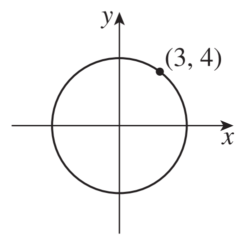
a



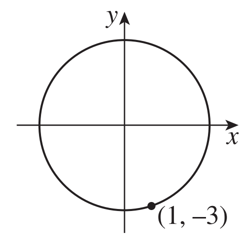
b



c



d



11 Consider the circle $x^2 + y^2 = 25$.

a Copy and complete the following table of values, correct to one decimal place where necessary. (Remember that a positive number has *two* square roots.)

x	0	1	2	3	4	5
$y \geq 0$						
$y \leq 0$						

b Plot the points in the table, using a scale of 1 cm to 1 unit on each axis.

c Reflect the points plotted in part **b** in the y-axis, and so sketch the entire circle.

12 Sketch each semicircle, and state the domain and range.

a $y = \sqrt{4 - x^2}$ **b** $y = -\sqrt{4 - x^2}$ **c** $y = -\sqrt{1 - x^2}$

d $y = \sqrt{\frac{25}{4} - x^2}$ **e** $y = -\sqrt{\frac{9}{4} - x^2}$ **f** $y = \sqrt{0.64 - x^2}$

13 Write down the zeroes of each polynomial, use a table of values to test its sign, then sketch it, showing the y-intercept.

a $y = (x + 2)(x + 1)x(x - 1)(x - 2)$

b $y = -(x - 3)^2(x + 2)^2$

c $y = 2x^2(x - 2)^4(x - 4)$

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14 Consider the curve $y = -\sqrt{x}$.

a Copy and complete the following table of values:

x	0	0.25	1	2.25	4	6.25
y						

b Plot the points in the table, using a scale of 2 cm to 1 unit on each axis, and then join the points with a smooth curve.

15 [Technology]

a Graph these cubic polynomials accurately using computer graphing software.

i $y = \frac{1}{4}x^3 + 2$ **ii** $y = \frac{1}{2}(x^3 - 6x^2 + 9x)$ **iii** $y = \frac{1}{2}(x^3 - 2x^2 - 5x + 6)$

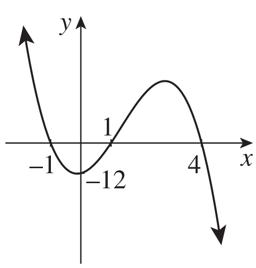
b In each case, check that the y-intercept is equal to the constant term.

c Read the x-intercepts from the screen and then check these values by substituting them into the corresponding polynomial.

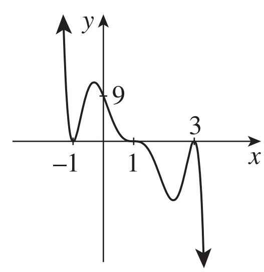
CHALLENGE

16 These graphs are known to be polynomials, and the second is known to have degree 7. Write down their equations factored into linear factors.

a



b



17 a Use the results of Question **8** to sketch the graph of $y = \sqrt{x}$.

b On the same number plane, sketch the graph of $y = -\sqrt{x}$, by using Question **14**.

c What shape has been formed?

d Explain why this has happened.

18 [Technology]

As you discovered in the previous sections, the graphs of quadratic functions fall into one of two categories: concave up and concave down. In contrast, cubic functions have six basic types.

a Use computer graphing software to plot accurately the cubic $y = c(x)$ for:

i $c(x) = x^3 + x$ **ii** $c(x) = x^3$ **iii** $c(x) = x^3 - x$

b These three curves have some similarities.

i For large positive and negative values of x, which two quadrants does the graph of $y = c(x)$ lie in?

ii Rotate each curve by 180° about the origin. What do you notice?

iii By considering your investigation in Question **5**, explain why that might be.

c Look carefully at each curve as it passes through the origin. What distinguishes each graph there?

d Conclude this investigation by accurately graphing $y = -c(x)$ for each cubic function in part **a**. Then look for any similarities and differences.

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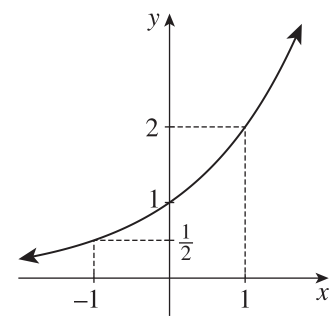
3H Two graphs that have asymptotes

This section reviews exponential graphs and rectangular hyperbolas. They are grouped together because both types of graphs have asymptotes, which need further discussion. Then direct and inverse variation are briefly reviewed.

Exponential functions

Functions of the form $y = a^x$, where the base a is positive and $a \neq 1$, are called *exponential functions*, because the variable x is in the *exponent* or *index*.

Here is a sketch of the function $y = 2^x$.



$y = 2^x$

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

Three key features should be shown when sketching this graph:

- The y-intercept is $y = 1$, because $2^0 = 1$.
- When $x = 1$, $y = 2$, which is the base 2, because $2^1 = 2$.
- The x-axis is a horizontal asymptote, as discussed below.

Limits and asymptotes of exponential functions

On the far left, as x becomes a very large negative number, $y = 2^x$ becomes very small. Indeed, we can make y 'as small as we like' by choosing sufficiently large negative values of x. We say that 'as x approaches negative infinity, y approaches the limit zero', and write:

$$\text{as } x \to -\infty,\ y \to 0 \quad \text{or} \quad \lim_{x \to -\infty} y = 0.$$

The x-axis is called an *asymptote* of the curve (from the Greek word *asymptotos*, meaning 'apt to fall together'), because the curve gets 'as close as we like' to the x-axis for sufficiently large negative values of x.

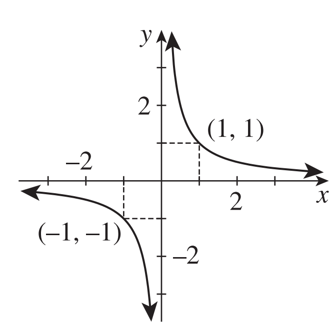
Rectangular hyperbolas

The *reciprocal function* $y = \dfrac{1}{x}$ can also be written as $xy = 1$, and has a graph that is called a *rectangular hyperbola*. This graph has two disconnected parts called *branches*.

$y = \dfrac{1}{x}$

x	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1	2	5	10
y	$*$	10	5	2	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$

x	-10	-5	-2	-1	$-\frac{1}{2}$	$-\frac{1}{5}$	$-\frac{1}{10}$
y	$-\frac{1}{10}$	$-\frac{1}{5}$	$-\frac{1}{2}$	-1	-2	-5	-10



The asterisk ($*$) at $x = 0$ means that the function is not defined there.

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Limits and asymptotes of rectangular hyperbolas

The x-axis is an asymptote to this curve on both sides of the graph. We can make y 'as small we like' by choosing sufficiently large positive or negative values of x. We say that 'as x approaches $-\infty$, y approaches the limit zero', and write:

As $x \to \infty$, $y \to 0$ or $\lim\limits_{x \to \infty} y = 0$ and As $x \to -\infty$, $y \to 0$ or $\lim\limits_{x \to -\infty} y = 0$

The y-axis is a second asymptote to the graph. On the right-hand side of the origin, when x is a very small positive number, y becomes very large. We can make y 'as large as we like' by taking sufficiently small but still positive values of x. We say that 'as x approaches zero from the right, y approaches ∞', and write:

As $x \to 0^+$, $y \to \infty$.

On the left-hand side of the origin, y is negative and can be made 'as large negative as we like' by taking sufficiently small negative values of x. We say that 'as x approaches zero from the left, y approaches $-\infty$', and write:

As $x \to 0^-$, $y \to -\infty$.

These notations, with their superscript + or −, will not often be used in this course — regard them as extension.

Direct and inverse variation

Direct variation and inverse variation were introduced in previous years, and are easily summarised.

26 DIRECT AND INVERSE VARIATION

Direct variation: A variable y *varies directly* with a variable x if

$y = kx$, for some non-zero *constant k of proportionality*.

- The graph of y as a function of x is thus a line through the origin.

Inverse variation: A variable y *varies inversely* with a variable x if

$y = \dfrac{k}{x}$, for some non-zero *constant k of proportionality*.

- The graph of y as a function of x is thus a rectangular hyperbola whose asymptotes are the x-axis and the y-axis.

Inverse variation is also called 'indirect variation', and the word 'proportion' is often used instead of 'variation'. Most applications begin by finding the constant of proportionality, as in the next two worked examples.

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Example 22 [Direct variation] 3H

a Garden mulch is sold in bulk, with the cost C proportional to the volume V in cubic metres. Write this algebraically.

b The shop quotes \$270 for 7.5 m^3. Find the constant of proportionality, and graph the function.

c How much does 12 m^3 cost?

d How much can I buy for \$600?

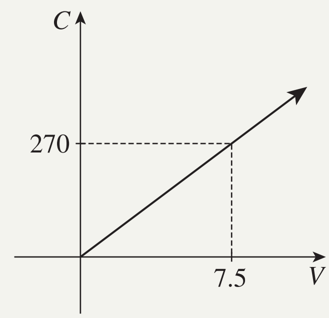
SOLUTION

a $C = kV$, for some constant k.

b Substituting the known values, $270 = k \times 7.5$

$$k = 36.$$

(More precisely, $k = \$36/\text{m}^3$.)



c $C = 36 \times 12$

$= \$432$

d $600 = 36V$

$V = 16\tfrac{2}{3}\ \text{m}^3$

Example 23 [Inverse variation] 3H

a The wavelength λ in metres of a musical tone is inversely proportional to its frequency f in vibrations per second. Write this algebraically.

b The frequency of middle C is about 260 s^{-1} ('260 vibrations per second'), and its wavelength is about 1.319 m. Find the constant of proportionality.

c Find the wavelength of a sound wave with frequency 440 s^{-1}.

d Find the frequency of a sound wave with wave length 1 m.

e What is the approximate speed of sound in air, and why?

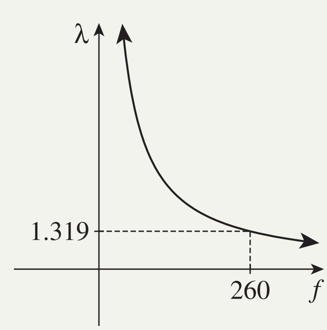
SOLUTION

a $\lambda = \dfrac{k}{f}$, for some constant k.

b Substituting the known values, $1.319 = \dfrac{k}{260}$

$$k \doteqdot 343.$$

(More precisely, $k = 343\,\text{m}\,\text{s}^{-1}$.)



c $\lambda = \dfrac{k}{440}$

$\doteqdot 0.779\,\text{m}$

d $\lambda = \dfrac{k}{f}$

$f \doteqdot 343\,\text{s}^{-1}$

e About $343\,\text{m}\,\text{s}^{-1}$, because 343 waves, each 1 metre long, go past in 1 second.

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Direct and inverse variation are closely related

Direct and inverse variation are very closely related, despite their contrasting graphs. For example, if a rectangle has area A and adjacent sides x and y, then:

$$A = xy \quad \text{and} \quad y = \frac{A}{x}.$$

- If the side y is constant, the area A is directly proportional to the other side x.
- If the area A is constant, the side y is inversely proportional to the side x.

Exercise 3H

FOUNDATION

Note: If computer graphing software is not available, the two technology questions can be completed using tables of values.

1 **a** Copy and complete the following table of values for the hyperbola $y = \frac{2}{x}$.

x	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4
y					*				

b Plot the points, using a scale of 1 cm to 1 unit on each axis, then sketch the hyperbola.

c Which two quadrants do the branches of the curve lie in?

d Write down the equations of the two asymptotes of the hyperbola.

e Write down the domain and range of the function.

2 Construct a table of values for each hyperbola, then sketch it. State the domain and range of each hyperbola and the equations of the two asymptotes. Also state which quadrants the branches lie in.

a $y = \frac{4}{x}$ **b** $y = \frac{3}{x}$

3 [Technology]

Use computer graphing software (if unavailable, tables of value will also do) to plot accurately these graphs on the one number plane.

$$y = \frac{1}{x} \quad \text{and} \quad y = \frac{4}{x} \quad \text{and} \quad y = \frac{9}{x}$$

a Which two quadrants do the the branches of each hyperbola lie in?

b Write down the equations of the two asymptotes of each hyperbola.

c Write down the domain and range of each hyperbola.

d Write down the coordinates of the points on each hyperbola closest to the origin. What do you notice?

4 Consider the exponential curve $y = 3^x$.

a Copy and complete the following table of values of the exponential function $y = 3^x$. Give answers correct to one decimal place where necessary.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y									

b Plot these points, using scales of 1 cm to 1 unit on both axes, then sketch the curve.

c What is the y-intercept of the function?

d What is the y-coordinate when $x = 1$?

e Write down the equation of the asymptote.

f Write down the domain and range of the function.

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5 Construct a table of values for each exponential function, then sketch its graph. Write down the domain and range in each part and the equation of the asymptote. Also state the y-intercept and the y-coordinate at $x = 1$.

a $y = 4^x$ **b** $y = 1.5^x$

6 [Technology]
Use computer graphing software to plot accurately on the one number plane: $y = 2^x$, $y = 3^x$ and $y = 4^x$.

a Which point is common to all three graphs?
b Write down the equation of the asymptote of each exponential curve.
c Write down the domain and range of each exponential function.
d Confirm by observation that at $x = 1$, the y-coordinate equals the base.
e Which curve increases more rapidly to the right of the y-axis, and why?
f Which curve approaches the asymptote more quickly to the left of the y-axis? Why?

DEVELOPMENT

7 **a** Construct a table of values for each hyperbola then sketch it.

i $y = -\dfrac{2}{x}$ **ii** $y = -\dfrac{4}{x}$ **iii** $y = -\dfrac{3}{x}$

b Compare these equations and graphs with those in Questions **1** and **2**.
i In which quadrants do the graphs of part **a** lie?
ii What has changed in the equation to cause this difference?

8 Rewrite each equation with x as the independent variable and then sketch its graph.

a $xy = \frac{1}{2}$ **b** $xy = -6$

9 **a** Construct a table of values for each exponential function then sketch it.

i $y = 3^{-2x}$ **ii** $y = 4^{-2x}$ **iii** $y = 1.5^{-2x}$

b Compare these equations and graphs with those in Questions **4** and **5**.
i Has the y-intercept changed?
ii Has the asymptote changed?
iii For the graphs in part **a**, at what value of x is the y-coordinate equal to the base?
iv Heading to the right along each of these curves, describe how the y-coordinate changes.
v What has changed in the equation to cause these differences?

10 Sketch these exponential graphs without resorting to a table of values. Ensure the key features are shown.

a $y = 5^x$ **b** $y = 2^{-x}$

11 [Direct Variation]

a The amount of paint P, in litres, used to paint a building is directly proportional to the area A, in square metres, to be covered. Write this algebraically.
b A certain building requires 48 L to cover an area of $576\,\text{m}^2$. Find the constant of proportionality, and graph the function.
c A larger building has an area of $668\,\text{m}^2$ to be painted. How many litres of paint will this require?
d A paint supplier sells 40 L buckets and 4 L tins of paint. How many buckets and tins must be bought in order to paint the building completely?

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12 [Inverse Variation]
The owners of the Fizgig Manufacturing Company use a uniformly elastic demand curve to model their sales. Thus if the price of a Fizgig is p and the quantity sold per year is q, then the turnover from sales is constant. That is, $pq = T$, for some constant T.

a Last year, the price of a Fizgig was $p = \$6$ and the quantity sold was $q = 400\,000$. Find T.
b The company's board of directors want to raise the price to $p = \$8$ next year. How many Fizgigs can the company expect to sell if this happens?
c Under this model, what will happen to the sales if the price is doubled instead?
d Sketch the graph of the demand curve with q on the horizontal axis and p on the vertical axis.

CHALLENGE

13 This question requires the language of limits from the first two pages of this section.
a In Question **4** above, the line $y = 0$ is an asymptote to $y = 3^x$. Write a statement using limits to justify this.
b In Question **4b**, the line $y = 0$ is an asymptote to the exponential curve $y = 2^{-x}$. Write a statement using limits to justify this.
c In Question **1**, the lines $y = 0$ and $x = 0$ are asymptotes to the hyperbola $y = \dfrac{2}{x}$. Write four statements using limits to justify this.

14 Does the equation $xy = 0$ represent a hyperbola? Explain your answer.

15 a Construct a table of values, and hence sketch the graph of the function $y = \left(\frac{1}{2}\right)^x$.
b Use the index laws to explain why $y = \left(\frac{1}{2}\right)^x$ has the same graph as $y = 2^{-x}$.

16 a Where does the hyperbola $xy = c^2$ intersect the line $y = x$?
b Confirm your answer by plotting the situation when $c = 2$.

17 [Inverse Variation]
An architect is designing a building. The client has insisted that one of the rooms in the building must have an area of $48\,\text{m}^2$. For ease of design, the length ℓ and the breadth b must each be a whole number of metres. Because of the furniture that must go into the room, no wall may be less than 4 m. What are the possible dimensions of the room?

18 Some curves can cross their asymptote.
a Complete the following table of values for $y = \dfrac{2x}{x^2 + 1}$.

x	-8	-4	-2	-1	-0.5	0	0.5	1	2	4	8
y											

b Plot the points, using a scale of 1 cm to 1 unit on each axis, and join them with a smooth curve.
c What is the horizontal asymptote of this curve?
d Where does the curve cross its asymptote?

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3I Four types of relations

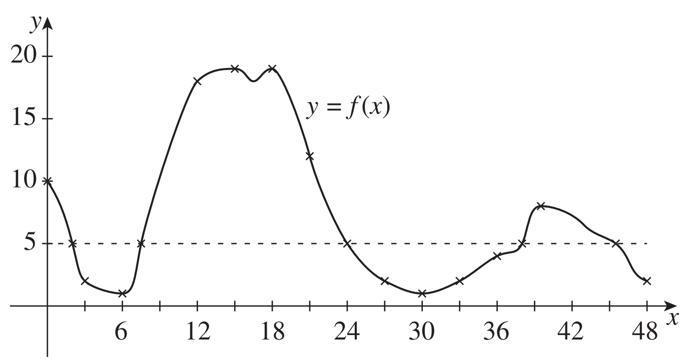
In Section 3B, functions were generalised to relations, and a relation was defined as any set of ordered pairs. The *vertical line test* was used to establish from the graph whether or not a relation is a function.

This section introduces the corresponding *horizontal line test*, and classifies relations into four types depending on whether their graphs pass or fail one or the other of these two tests. Reading the graph backwards is the key idea because it makes it clear why horizontal lines are so important on a graph.

There are also a few examples of relations and functions whose domain and range do not consist of numbers. Such relations do not appear very often in this course, but they are vitally important in the databases and spreadsheets that we use every day on the internet and in our record-keeping.

Using horizontal lines to read a graph backwards

Here is a graph of the temperature $y°$C measured x hours after midnight over two days. Call the function $y = f(x)$.



The straightforward way to read this graph is to take a time, say $x = 2$, on the horizontal axis, draw the vertical line $x = 2$ to the graph, and read the temperature off the y-axis. This graph is a function, there are never two answers.

'After 2 hours, the temperature was 5°C'.

Algebraically, substituting $x = 2$ gives $f(2) = 5$.

We can also read the graph backwards. Take a temperature, say 5°C, on the vertical axis, draw a horizontal line to the graph, and read off the times when the temperature was 5°C,

'The temperature was 5°C after 2, $7\frac{1}{2}$, 24, 38 and $45\frac{1}{2}$ hours'.

Algebraically, the solutions of $f(x) = 5$ are $x = 2, 7\frac{1}{2}, 24, 38$ and $45\frac{1}{2}$.

27 USING HORIZONTAL LINES TO READ A GRAPH BACKWARDS

- To solve $f(x) = b$ from the graph, draw the horizontal line $y = b$ and read off the corresponding x-values.
- Thus we are *reading the graph backwards* — starting with the y-values and looking for the corresponding x-values.

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The horizontal line test

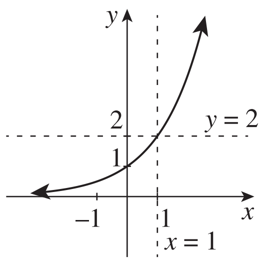
The graph above is called a *many-to-one* function, because many x-values all map to the one y-value. To formalise this, we introduce the *horizontal line test*. This test is the companion of the vertical line test — the two definitions simply exchange the words 'vertical' and 'horizontal'.

28 THE VERTICAL AND HORIZONTAL LINE TESTS

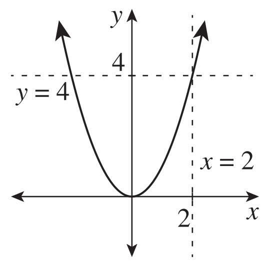
- **Vertical line test:**
  No vertical line meets the graph more than once.
- **Horizontal line test:**
  No horizontal line meets the graph more than once.

Using the horizontal line test with functions

The two graphs below are both functions, because they pass the vertical line test.



$y = 2^x$ is one-to-one



$y = x^2$ is many-to-one

The first graph of $y = 2^x$ also passes the horizontal line test. This means that the situation is symmetric in the following way:

- Every number in the domain corresponds to exactly one number in the range.
- Every number in the range corresponds to exactly one number in the domain.

This graph is therefore called *one-to-one*, and the function is a *one-to-one correspondence* between the domain, all real x, and the range, $y > 0$. When the graph is read backwards, there is never more than one answer.

The second graph of $y = x^2$ fails the horizontal line test. For example, the horizontal line $y = 4$ meets the curve twice. This is why 4 has two square roots, 2 and -2. The situation is no longer symmetric vertically and horizontally.

- Every number in the domain corresponds to exactly one number in the range.
- At least one number in the range (say 4) corresponds to more than one number in the domain (to 2 and -2).

The graph is therefore called *many-to-one*, because, for example, many x-values map to the one y-value 4. When we read the graph backwards, at least one value of y will give more than one answer.

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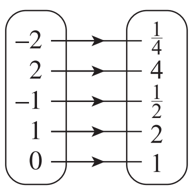
29 ONE-TO-ONE AND MANY-TO-ONE

Suppose that a graph passes the vertical line test, that is, it is a function.

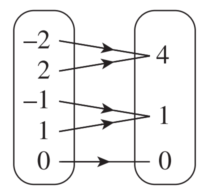
- If it also passes the horizontal line test, it is called *one-to-one*.
  - Whether the graph is read forwards or backwards, there will be no more than one answer.
  - It is a one-to-one correspondence between the domain and the range.
- If it fails the horizontal line test, it is called *many-to-one*.
  - When the graph is read forwards, there is never more than one answer.
  - When the graph is read backwards, at least one value of y gives more than one answer. Many values of x map to this one value of y.

Two diagrams of a different type may help to clarify the situation. Both functions are something-to-one because every number in the domain maps to exactly one number in the range.



$y = 2^x$ is one-to-one

For every input there is at most one output, and for every output there is at most one input.

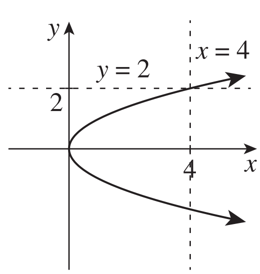


$y = x^2$ is many-to-one

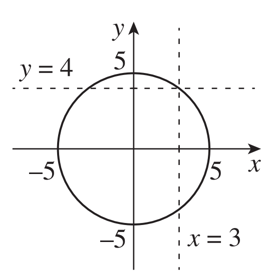
For every input there is at most one output, but for at least one output there is more than one input.

Using the horizontal line test with relations that are not functions

The two graphs below are relations, but not functions, because they fail the vertical line test.



$y^2 = x$ is one-to-many



$x^2 + y^2 = 25$ is many-to-many

The left-hand graph of $y^2 = x$ fails the vertical line test, but passes the horizontal line test. For example, the vertical line $x = 4$ meets the curve where $y = 2$ and where $y = -2$, but no horizontal line meets the graph more than once. The situation here is not symmetric vertically and horizontally.

- At least one number (say 4) in the domain corresponds to more than one number (2 and -2) in the range.
- Every number in the range corresponds to exactly one number in the domain.

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This graph is therefore called *one-to-many*, because, for example, the one number 4 in the domain corresponds to the two numbers 2 and -2 in the range.

The right-hand graph of $x^2 + y^2 = 25$ fails both horizontal and vertical line tests. For example, the vertical line $x = 3$ meets the curve twice, and the horizontal line $y = 4$ meets the curve twice. The situation is again symmetric.

- At least one number in the domain corresponds to more than one number in the range.
- At least one number in the range corresponds to more than one number in the domain.

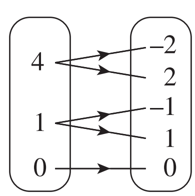
The graph is therefore called *many-to-many*, because, for example, the one numbers 3 in the domain corresponds to the two numbers 4 and -4 in the range, and the one number 4 in the range corresponds to the two numbers 3 and -3 in the domain.

30 ONE-TO-MANY AND MANY-TO-MANY

Suppose that a graph fails the vertical line test, that is, it is not a function.

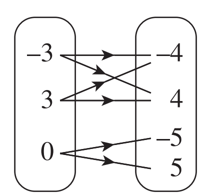
- If it passes the horizontal line test, it is called *one-to-many*.
  - When the graph is read forwards, at least one value of x gives more than one answer. Many values of y correspond to this one value of x.
  - When the graph is read backwards, there is never more than one answer.
- If it also fails the horizontal line test, it is called *many-to-many*.
  - Whether the graph is read forwards or backwards, there will be values of x or of y that give more than one answer.

Again, two further diagrams help to clarify the situation. Both relations are something-to-many because at least one element of the domain maps to at least two elements of the range



$y^2 = x$ is one-to-many

For at least one number in the domain, there are more than one corresponding numbers in the range, but for every number in the range, there is only one number in the domain.



$x^2 + y^2 = 25$ is many-to-many

For at least one number in the domain, there are more than one corresponding numbers in the range, and for at least one number in the range, there are more than one corresponding numbers in the domain.

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The four types of relations

Here is a table of the four types of relations:

Type	Vertical line test	Horizontal line test
One-to-one	Passes (so a function)	Passes
Many-to-one	Passes (so a function)	Fails
One-to-many	Fails (so not a function)	Passes
Many-to-many	Fails (so not a function)	Fails

Relations may involve objects other than numbers

Function and relations do not have to act on numbers. They can act on anything at all, and the four types of relations defined in this section are as common in database logic as in mathematics.

Example 24 3I

In a firm of 10000 employees, a database records the postcode (y-value) that each person (x-value) has nominated as a home address.

a Explain why this is a function, and whether it is one-to-one or many-to-one.

b What change may there be for a firm of 10 employees?

SOLUTION

a It is a function because every person's nominated home has a postcode. It will not pass the horizontal line test because there are fewer than 10000 postcodes in NSW.

b It is quite possible that the 10 employees all live in different postcodes, in which case the function would be one-to-one.

Example 25 3I

Many people (x-values) in the suburb of Blue Hills own a pet (y-value).

a What sort of relation is this?

b Council registers each pet (y-value) with only one owner (x-value). What sort of relation is this?

SOLUTION

a It fails the vertical line test because many residents have several pets. It fails the horizontal line test because many pets are owned by everyone in the family. Hence the relation is many-to-many.

b This relation passes the horizontal line test because each pet has no more than one person registering it. Hence the relation is one-to-many.

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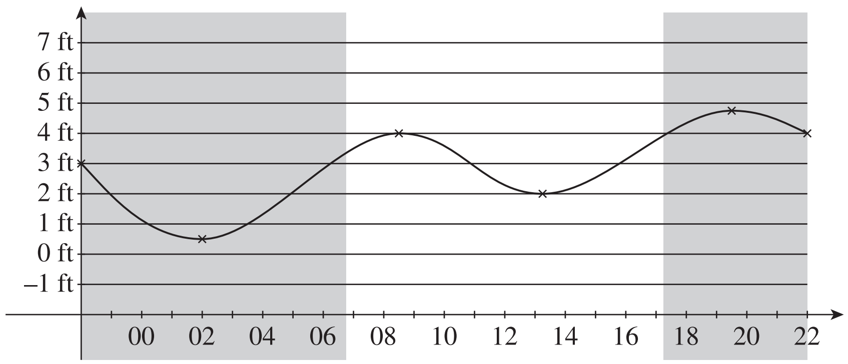
Exercise 3I

FOUNDATION

1 Printed below is a graph of the tides at Benicia, Carquinez Strait, California, on Sunday, September 16th 2016. It is in feet because the US uses imperial measure, and there are no vertical lines, which is inconvenient. Your numerical answers will only be approximate.



a Does the graph pass or fail the vertical line test? Is it a function?

b Does the graph pass or fail the horizontal line test? Is the graph one-to-one, many-to-one, one-to-many or many-to-many?

c The graph runs for 24 hours. What are the starting and finishing times?

d What were the tide heights at 6:00 am and 5:00 pm?

e When was the tide height:

i 3 ft? **ii** 2 ft? **iii** 6 ft?

iv What are the possible numbers of solutions of the equation $f(x) = k$, as k varies, where $f(x)$ is the function?

2 Go back to the temperature graph printed at the start of this Section 3I.

a Why is the graph a function, and why is it classified as many-to-one?

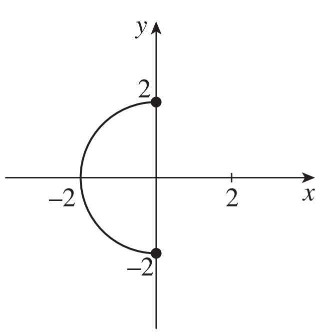
b What was the temperature at 6:00 am on the second day?

c When was the temperature 20°C, and when was it 8°C?

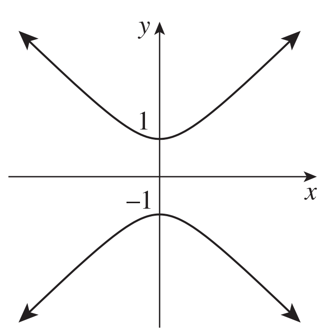
d What are the possible numbers of solutions of the equation $f(x) = k$, as k varies?

3 a Say whether each relation sketched below passes the vertical line test, and whether it passes the horizontal line test.

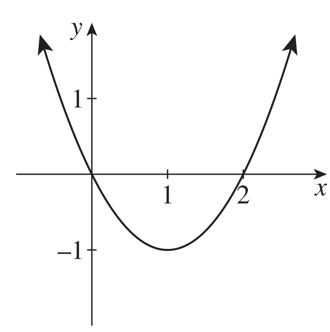
i



ii



iii



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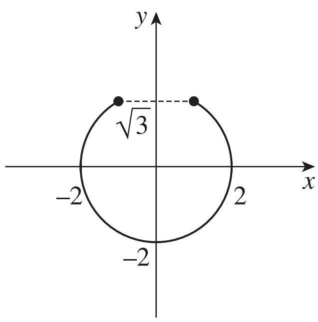
iv


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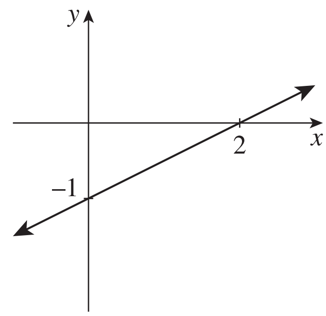

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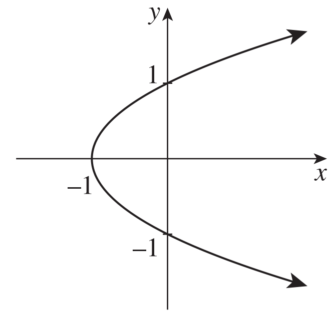


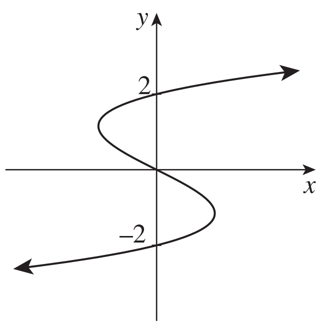
b Which graphs above give no more than one answer when read forwards from x-value to y-value(s)?

c Which graphs above give no more than one answer when read backwards from y-value to x-value(s)?

d Which graphs above are a one-to-one correspondence between domain and range?

e Classify each graph as one-to-one, many-to-one, one-to-many or many-to-many.

4 Classify each graph as one-to-one, many-to-one, one-to-many or many-to-many.

a


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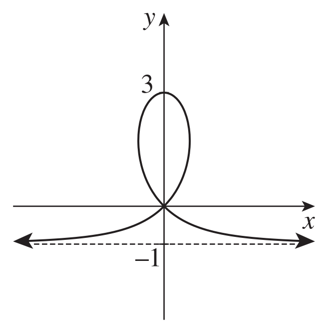

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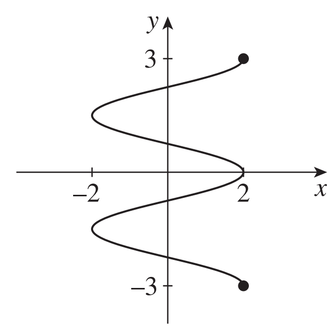

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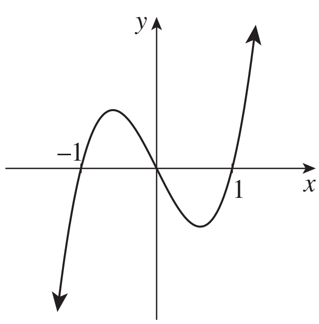

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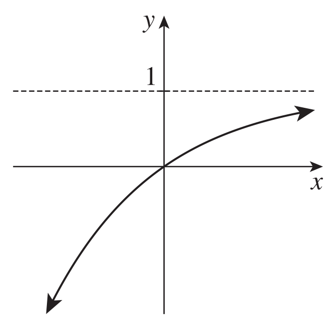

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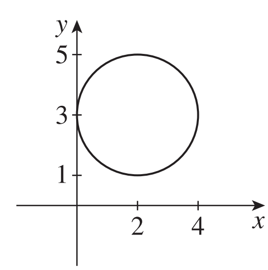

DEVELOPMENT

5 **a** Explain, with an example using a y-value, why each function is many-to-one.

i $y = x^2 - 4$ **ii** $y = (x - 1)x(x + 1)$ **iii** $y = x^4 + 1$

b Hence classify each relation below.

i $x = y^2 - 4$ **ii** $x = (y - 1)y(y + 1)$ **iii** $x = y^4 + 1$

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6 **a** By solving for x, show that each function is one-to-one. (This method works only if x can be made the subject. Then the relation is one-to-one if there is never more than one answer, and many-to-one otherwise.)

i $y = 3x - 1$ **ii** $y = 5 - 2x$ **iii** $y = 8x^3$ **iv** $y = \dfrac{5}{x}$

b Hence classify each relation below.

i $x = 3y - 1$ **ii** $x = 5 - 2y$ **iii** $x = 8y^3$ **iv** $x = \dfrac{5}{y}$

7 By giving an example using an x-value, and an example using a y-value, show how each relation fails the horizontal and vertical line tests, and hence is many-to-many.

a $(x - 3)^2 + (y + 1)^2 = 25$ **b** $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ **c** $x^2 - y^2 = 1$

8 **a** A database records all the doctors that a person has visited. Does this relation pass the horizontal or vertical line tests, and how should it be classified? (The x-values are all people in Australia, and the y-values are the doctors.)

b The database is queried to report the last doctor (y-value) that a person (x-value) has visited. Does this change the answers in part **a**?

9 Every student (x-value) in a class has a preferred name (y-value).

a What types of relation could this be?

b Explain what extra condition could make the classification unambiguous?

CHALLENGE

10 **a** A dancer points north, closes his eyes, and spins, ending up pointing east. Through how many degrees has the dancer turned (take clockwise as positive)?

b Regard the final position as the x-value, and the degrees the dancer turns as the y-value. Classify this relation.

c Now regard the degrees the dancer turns as the x-value, and the final position as the y-value. Classify this relation.

11 A person living in a block of flats is called a co-habitant of another person if they both live in the same flat.

a Classify this relation (the x-values and y-values are both any person in the block).

b Under what condition would this relation be one-to-one, and what would then be particularly special about the relation?

c What happens if the block is being renovated and no one is living there?

12 Classify each relation as one-to-one, many-to-one, one-to-many or many-to-many.

a $y = 4$ **b** $x = -3$ **c** $x + y = 0$ **d** $x^2 + y^2 = 0$

e $x^2 - y^2 = 0$ **f** $x = y^2 - 5y + 6$ **g** $y = x^3 - 7x^2 + 12x$ **h** $y = x^3 + 8$

i $y = (x + 8)^3$ **j** $xy = 36$ **k** $x^2y^2 = 36$ **l** $x^3y^3 = 1000$

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Chapter 3 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 3 Multiple-choice quiz

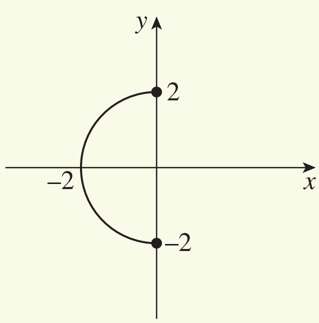
- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Determine which of the following relations are functions.

a



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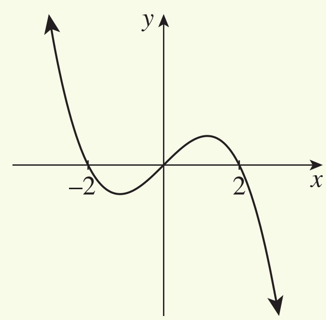

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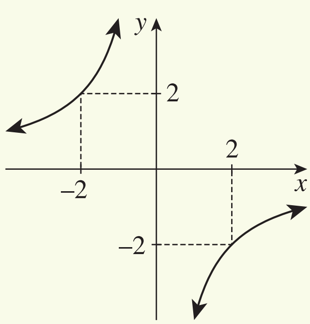

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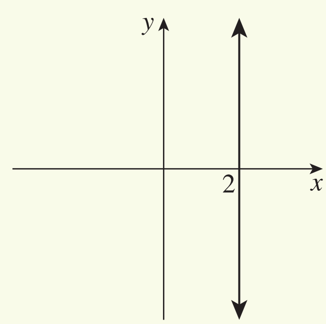

2 Write down the domain and range of each relation in Question **1**.

3 Find $f(3)$ and $f(-2)$ for each function.

a $f(x) = x^2 + 4x$

b $f(x) = x^3 - 3x^2 + 5$

4 Find the natural domain of each function.

a $f(x) = \dfrac{1}{x-2}$

b $f(x) = \sqrt{x-1}$

c $f(x) = \sqrt{3x+2}$

d $f(x) = \dfrac{1}{\sqrt{2-x}}$

5 Find $F(a) - 1$ and $F(a-1)$ for each function.

a $F(x) = 2x + 3$

b $F(x) = x^2 - 3x - 7$

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Review

6 Suppose that $f(x) = \begin{cases} x + 1, & \text{for } x < 0, \\ 1 - x, & \text{for } x \ge 0. \end{cases}$

Complete the table of values for $f(x)$, then sketch its graph.

x	-3	-2	-1	0	1	2	3
y							

7 Find the x-intercept and the y-intercept of each line, then sketch it.

a $y = 2x + 2$ **b** $x - 3y + 6 = 0$

8 Sketch each of these lines through the origin.

a $y = 2x$ **b** $x + 2y = 0$

9 Sketch each vertical or horizontal line.

a $y = -1$ **b** $x - 3 = 0$

10 Use factoring where necessary to find the zeroes of each quadratic function. Hence sketch the graph of $y = f(x)$, showing all intercepts and the coordinates of the vertex. Also state the domain and range.

a $f(x) = 16 - x^2$ **b** $f(x) = x(x + 2)$ **c** $f(x) = (x - 2)(x - 6)$
d $f(x) = -(x + 5)(x - 1)$ **e** $f(x) = x^2 + x - 6$ **f** $f(x) = -x^2 + 2x + 8$

11 Complete the square in each quadratic. Then sketch the graph of each function, showing the vertex and the intercepts with the axes.

a $y = x^2 + 2x - 5$ **b** $y = -x^2 + 6x - 6$
c $y = -x^2 + 2x - 3$ **d** $y = x^2 + 6x + 10$

12 Sketch the graph of each parabola. Use the discriminant to determine whether or not there are any x-intercepts. Show the vertex and the intercepts with the axes.

a $y = -x^2 - 2x + 1$ **b** $y = x^2 - 4x + 2$
c $y = x^2 - 4x + 8$ **d** $y = -x^2 + 6x - 15$

13 Use a table of values to test the sign, and then sketch each polynomial.

a $y = (x - 1)(x - 3)(x - 6)$ **b** $y = -x^2(x + 2)(x - 2)^2$

14 Sketch each circle.

a $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 100$

15 Sketch each semicircle, and state the domain and range in each case.

a $y = \sqrt{16 - x^2}$ **b** $y = -\sqrt{25 - x^2}$

16 Construct a table of values for each hyperbola, then sketch it. State the domain and range in each case.

a $y = \dfrac{8}{x}$ **b** $y = -\dfrac{4}{x}$

17 Construct a table of values for each exponential function, then sketch it. State the domain and range in each case.

a $y = 2^x$ **b** $y = 3^{-x}$

18 Construct a table of values for each function, then sketch it.

a $y = x^3 - 3x^2$ **b** $y = x^4 - 4x^2$ **c** $y = \sqrt{x + 1}$

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Review

19 [A revision medley of curve sketches]

Sketch each set of graphs on a single pair of axes, showing all significant points. A table of values may help in certain cases.

a $y = 2x$, $y = 2x + 3$, $y = 2x - 1$

b $y = -\frac{1}{2}x$, $y = -\frac{1}{2}x + 1$, $y = -\frac{1}{2}x - 2$

c $y = x^2$, $y = (x + 2)^2$, $y = (x - 1)^2$

d $x + y = 0$, $x + y = 2$, $x + y = -3$

e $y = x^2$, $y = 2x^2$, $y = \frac{1}{2}x^2$

f $x - y = 0$, $x - y = 1$, $x - y = -2$

g $x^2 + y^2 = 4$, $x^2 = 1 - y^2$, $y^2 = 25 - x^2$

h $y = 3x$, $x = 3y$, $y = 3x + 1$, $x = 3y + 1$

i $y = 2^x$, $y = 3^x$, $y = 4^x$

j $y = -x$, $y = 4 - x$, $y = x - 4$, $x = -4 - y$

k $y = x^2 - x$, $y = x^2 - 4x$, $y = x^2 + 3x$

l $y = x^2 - 1$, $y = 1 - x^2$, $y = 4 - x^2$, $y = -1 - x^2$

m $y = (x + 2)^2$, $y = (x + 2)^2 - 4$, $y = (x + 2)^2 + 1$

n $y = x^2 - 1$, $y = x^2 - 4x + 3$, $y = x^2 - 8x + 15$

o $y = \sqrt{9 - x^2}$, $y = -\sqrt{4 - x^2}$, $y = \sqrt{1 - x}$

p $y = \dfrac{1}{x}$, $y = \dfrac{2}{x}$, $y = -\dfrac{3}{x}$

q $y = \sqrt{x}$, $y = 2 - \sqrt{x}$, $y = \sqrt{1 - x}$

r $y = x^3$, $y = x^3 + 1$, $y = (x + 1)^3$

s $y = x^4$, $y = (x - 1)^4$, $y = x^4 - 1$

t $y = 2^{-x}$, $y = \left(\frac{1}{2}\right)^x$, $y = \dfrac{1}{2^x}$

20 Classify each relation below as one-to-one, many-to-one, one-to-many or many-to-many. It's probably best to work from a sketch of the curves, but you may want to approach the question algebraically.

a $y = 5x - 7$

b $(x - 2)^2 + (y - 3)^2 = 4$

c $y^2 = x - 2$

d $y = x^4 + 1$

21 **a** Twenty people (x-values) in an office have their country of birth (y-values) recorded in the office manager's spreadsheet. Is this relation a function? Classify it as one-to-one, many-to-one, one-to-many or many-to-many.

b What condition would make this relation one-to-one?

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4 Transformations and symmetry

In the previous chapter, various graphs of functions and relations were reviewed or introduced. This chapter deals with transformations of these graphs under vertical and horizontal translations, under reflections in the x-axis and y-axis, and under rotations of 180° about the origin (dilations are introduced in Year 12). These procedures allow a wide variety of new graphs to be obtained, and relationships amongst different graphs to be discovered.

Many graphs are unchanged under one or more of these transformations, which means that they are symmetric in some way. This chapter deals with line symmetry under reflection in the y-axis, and point symmetry under rotation of 180° about the origin. These transformations and symmetries are described geometrically and algebraically, and the theme remains the interrelationship between the algebra and the graphs.

The absolute value function is then introduced, together with its own transformations and reflection symmetry. The final section generalises the transformations of this chapter to the far more general idea of composite functions.

As always, computer sketching of curves is very useful in demonstrating how the features of a graph are related to the algebraic properties of its equation, and to gain familiarity with the variety of graphs and their interrelationships.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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4A Translations of known graphs

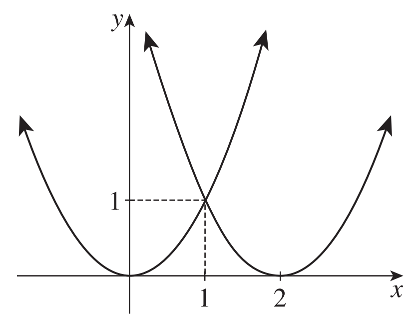
Once a graph has been drawn, it can be *shifted* (or *translated*) vertically or horizontally to produce further graphs. These procedures work generally on all functions and relations, and greatly extend the range of functions and relations whose graphs can be quickly recognised and drawn.

In particular, translations are very helpful when dealing with parabolas and circles, where they are closely related to completing the square.

Shifting right and left

The graphs of $y = x^2$ and $y = (x - 2)^2$ are sketched from their tables of values.

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$(x-2)^2$	16	9	4	1	0	1	4



- The values for $(x - 2)^2$ in the third row are the values of x^2 in second row shifted 2 steps to the right.
- Hence the graph of $y = (x - 2)^2$ is obtained by shifting the graph of $y = x^2$ to the right by 2 units.

1 SHIFTING (OR TRANSLATING) RIGHT AND LEFT

- To shift a graph h units to the *right*, replace x by $x - h$.
- Alternatively, if the graph is a function, the new function rule is $y = f(x - h)$.

Shifting a graph h units to the left means shifting it $-h$ units to the right, so x is replaced by $x - (-h) = x + h$.

Example 1 — 4A

a Draw up tables of values for $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x+1}$.

b Sketch the two graphs, and state the asymptotes of each graph.

c What transformation maps $y = \dfrac{1}{x}$ to $y = \dfrac{1}{x+1}$?

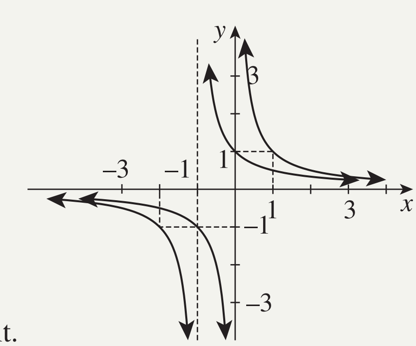
SOLUTION

a

x	-3	-2	-1	0	1	2	3
$\frac{1}{x}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$	$\frac{1}{3}$
$\frac{1}{x+1}$	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

b $y = \dfrac{1}{x}$ has asymptotes $x = 0$ and $y = 0$.

$y = \dfrac{1}{x+1}$ has asymptotes $x = -1$ and $y = 0$.



c Because x is replaced by $x + 1 = x - (-1)$, it is a shift left of 1 unit.

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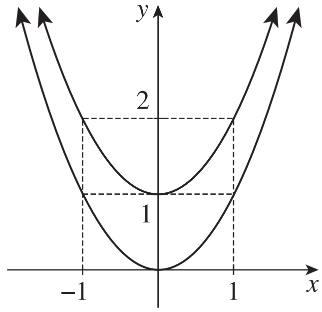
Shifting up and down

The graphs of $y = x^2$ and $y = x^2 + 1$ are sketched on the right from their tables of values.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 1$	10	5	2	1	2	5	10



- The values for $x^2 + 1$ in the third row are each 1 more than the corresponding values of x^2 in second row.
- Hence the graph of $y = x^2 + 1$ is produced by shifting the graph of $y = x^2$ upwards 1 unit.

Rewriting the transformed graph as $y - 1 = x^2$ makes it clear that the shifting has been obtained by replacing y by $y - 1$, giving a rule that is completely analogous to that for horizontal shifting.

2 SHIFTING (OR TRANSLATING) UP AND DOWN

- To shift a graph k units *upwards*, replace y by $y - k$.
- Alternatively, if the graph is a function, the new function rule is $y = f(x) + k$.

Shifting a graph k units down means shifting it $-k$ units up, so y is replaced by $y - (-k) = y + k$.

Example 2 — 4A

The graph of $y = 2^x$ is shifted down 2 units.

a Write down the equation of the shifted graph.

b Construct tables of values, and sketch the two graphs.

c State the asymptotes of the two graphs.

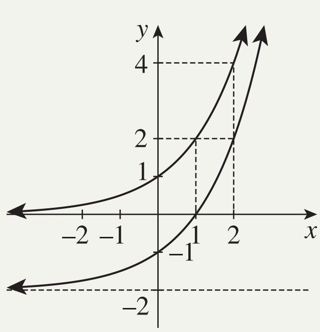
SOLUTION

a Replace y by $y - (-2) = y + 2$, so the new function is

$$y + 2 = 2^x, \qquad \text{that is,} \qquad y = 2^x - 2.$$

b

x	-2	-1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$2^x - 2$	$-1\frac{3}{4}$	$-1\frac{1}{2}$	-1	0	2



c $y = 2^x$ has asymptote $y = 0$.
$y = 2^x - 2$ has asymptote $y = -2$.

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Combining horizontal and vertical translations

When a graph is shifted horizontally and vertically, the order in which the translations are applied makes no difference. The following example shows the effect of two translations on a cubic graph.

Example 3 4A

a How is the graph of $y = (x + 2)^3 - 4$ obtained from the graph of $y = x^3$ by a horizontal translation followed by a vertical translation?

b Draw up a table of values for the two functions and the intermediate function.

c Sketch the two curves, together with the intermediate graph.

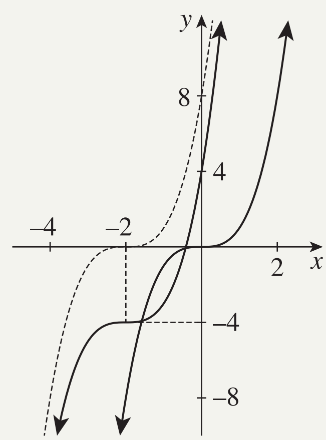
SOLUTION

a Shifting $y = x^3$ left 2 gives $y = (x + 2)^3$.
Shifting $y = (x + 2)^3$ down 4 gives $y + 4 = (x + 2)^3$,
which can be written as $y = (x + 2)^3 - 4$.

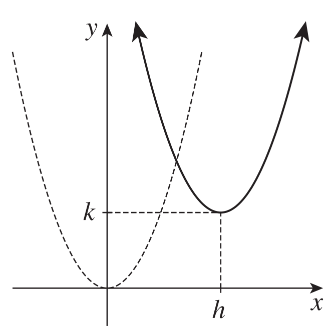
b

x	-4	-3	-2	-1	0	1	2
x^3	-64	-27	-8	-1	0	1	8
$(x + 2)^3$	-8	-1	0	1	8	27	64
$(x + 2)^3 - 4$	-12	-5	-4	-3	4	23	60

c



Translations and the vertex of a parabola



When we complete the square in a quadratic, it has the form

$$y = a(x - h)^2 + k, \qquad \text{that is} \qquad y - k = a(x - h)^2.$$

This is a translation of the quadratic $y = ax^2$. The parabola has been shifted h units right and k units up.

This gives a clear and straightforward motivation for completing the square.

3 THE COMPLETED SQUARE AND THE VERTEX OF A PARABOLA

The completed square form of a quadratic

$$y = a(x - h)^2 + k \qquad \text{or} \qquad y - k = a(x - h)^2$$

displays its graph as the parabola $y = ax^2$ shifted right h units and up k units.

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Example 4 4A

In each part, complete the square in the quadratic. Then identify its graph as a translation of a parabola with vertex at the origin, and sketch the two graphs.

a $y = x^2 - 4x + 5$ **b** $y = -2x^2 - 4x$

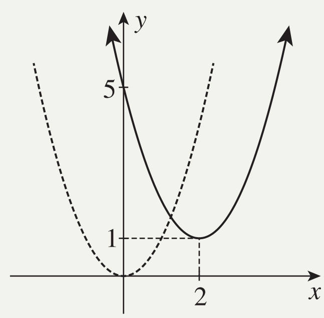
SOLUTION

a $y = x^2 - 4x + 5$

$y = (x^2 - 4x + 4) - 4 + 5$

$y = (x - 2)^2 + 1$ or $y - 1 = (x - 2)^2$

This is $y = x^2$ shifted right 2 and up 1.



b $y = -2x^2 - 4x$

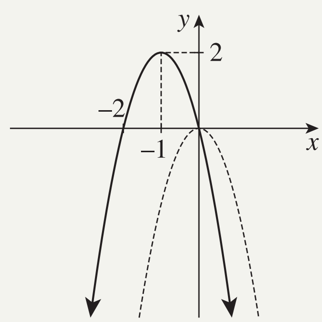
$-\frac{y}{2} = x^2 + 2x$

$-\frac{y}{2} = (x^2 + 2x + 1) - 1$

$-\frac{y}{2} = (x + 1)^2 - 1$

$y = -2(x + 1)^2 + 2$ or $y - 2 = -2(x + 1)^2$

This is $y = -2x^2$ shifted left 1 and up 2.



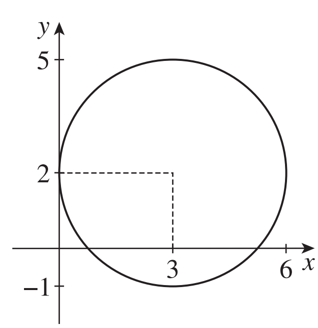
Translations and the centre of a circle

The circle drawn to the right has centre (3, 2) and radius 3. To find its equation, we start with the circle with centre the origin and radius 3,

$$x^2 + y^2 = 9,$$

then translate it 3 to the right and 2 up,

$$(x - 3)^2 + (y - 2)^2 = 9.$$



This formula can also be established directly by Pythagoras' theorem in the form of the distance formula, but as we saw with parabolas, translations make things clearer and more straightforward.

When the squares in the equation of a circle have both been expanded, the centre and radius can be found by completing the squares in x and in y.

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Example 5 4A

a Complete the squares in x and in y of the relation $x^2 + y^2 - 6x + 8y = 0$.
b Identify the circle with centre the origin that can be translated to it, and state the translations.
c Sketch both circles on the same diagram, and explain why each circle passes through the centre of the other circle.

SOLUTION

a Completing the squares in x and in y,

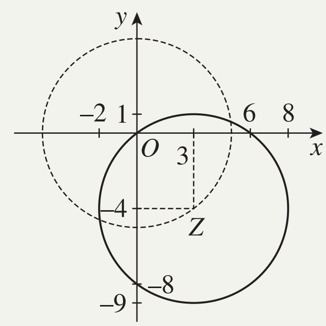
$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 9 + 16$$
$$(x^2 - 3)^2 + (y + 4)^2 = 25.$$

b It is the circle $x^2 + y^2 = 5^2$ shifted right 3 and down 4, so its centre is $Z(3, -4)$ and its radius is 5.

c Using the distance formula,

$$OZ^2 = 3^2 + 4^2$$
$$OZ = 5,$$

which is the radius of each circle.



Exercise 4A

FOUNDATION

1 a Copy and complete the table of values for $y = x^2$ and $y = (x - 1)^2$.

x	-2	-1	0	1	2	3
x^2						
$(x-1)^2$						

b Sketch the two graphs and state the vertex of each.
c What transformation maps $y = x^2$ to $y = (x - 1)^2$?

2 a Copy and complete the table of values for $y = \frac{1}{4}x^3$ and $y = \frac{1}{4}x^3 + 2$.

x	-3	-2	-1	0	1	2	3
$\frac{1}{4}x^3$							
$\frac{1}{4}x^3 + 2$							

b Sketch the two graphs and state the y-intercept of each.
c What transformation maps $y = \frac{1}{4}x^3$ to $y = \frac{1}{4}x^3 + 2$?

3 How far and in which direction has the parabola $y = x^2$ been shifted to produce each of these parabolas?
a $y = x^2 + 2$ **b** $y = x^2 - 5$ **c** $y = (x + 4)^2$ **d** $y = (x - 3)^2$

4 How far and in which direction has the hyperbola $y = \dfrac{1}{x}$ been shifted to produce each of these hyperbolas?
a $y = \dfrac{1}{x - 2}$ **b** $y = \dfrac{1}{x + 3}$ **c** $y = \dfrac{1}{x} - 4$ **d** $y = \dfrac{1}{x} + 5$

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5 Sketch each parabola by shifting $y = x^2$ either horizontally or vertically. Mark all intercepts with the axes.

a $y = x^2 + 1$ b $y = x^2 - 1$ c $y = (x - 1)^2$ d $y = (x + 1)^2$

6 Sketch each hyperbola by shifting $y = \dfrac{1}{x}$ either horizontally or vertically. Mark any intercepts with the axes.

a $y = \dfrac{1}{x + 1}$ b $y = \dfrac{1}{x - 1}$ c $y = \dfrac{1}{x} + 1$ d $y = \dfrac{1}{x} - 1$

7 Sketch each circle by shifting $x^2 + y^2 = 1$ either horizontally or vertically. Mark all intercepts with the axes.

a $(x - 1)^2 + y^2 = 1$ b $x^2 + (y - 1)^2 = 1$ c $x^2 + (y + 1)^2 = 1$ d $(x + 1)^2 + y^2 = 1$

DEVELOPMENT

8 Write down the new equation for each function or relation after the given translation has been applied. Then sketch the graph of the new curve.

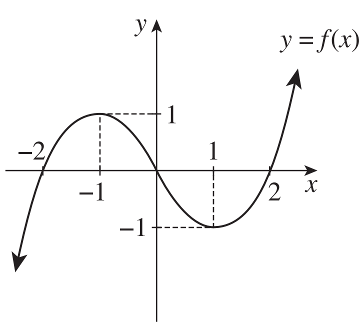
a $y = x^2$: right 1 unit b $y = 2^x$: down 3 units c $y = x^3$: left 1 unit

d $y = \dfrac{1}{x}$: right 3 units e $x^2 + y^2 = 4$: up 1 unit f $y = x^2 - 4$: left 1 unit

g $xy = 1$: down 1 unit h $y = \sqrt{x}$: up 2 units

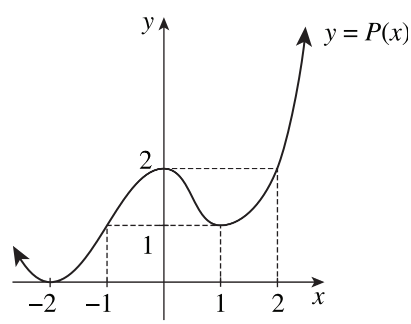
9 In each case an unknown function has been drawn. Draw the functions specified below it.

a



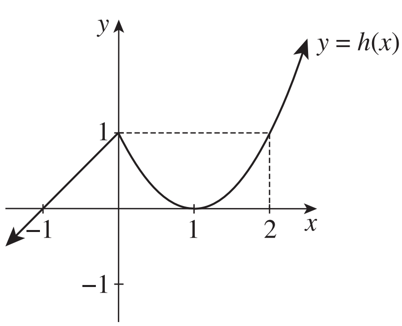
i $y = f(x - 2)$
ii $y = f(x + 1)$

b



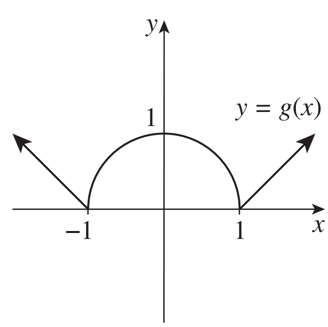
i $y = P(x + 2)$
ii $y = P(x + 1)$

c



i $y - 1 = h(x)$
ii $y = h(x) - 1$

d



i $y - 1 = g(x)$
ii $y = g(x - 1)$

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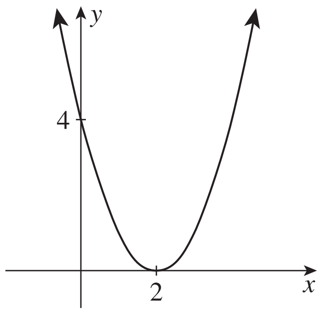
10 In each part, complete the square in the quadratic. Then identify its graph as a translation of a parabola with vertex at the origin. Finally, sketch its graph.

a $y = x^2 + 2x + 3$ **b** $y = x^2 - 2x - 2$ **c** $y = -x^2 + 4x + 1$

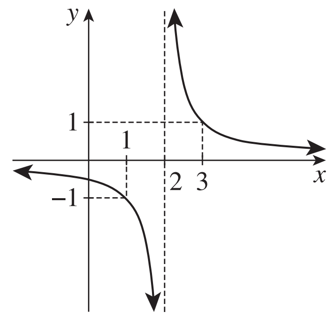
d $y = -x^2 - 4x - 5$ **e** $y = 2x^2 - 4x - 2$ **f** $y = \frac{1}{2}x^2 - x - 2$

11 Describe each graph below as the parabola $y = x^2$ or the hyperbola $xy = 1$ transformed by shifts, and hence write down its equation.

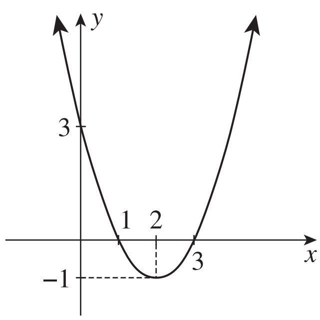
a



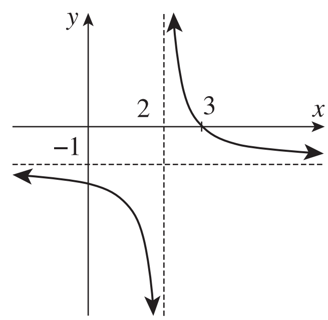
b



c



d



12 Use shifting, and completing the squares where necessary, to determine the centre and radius of each circle.

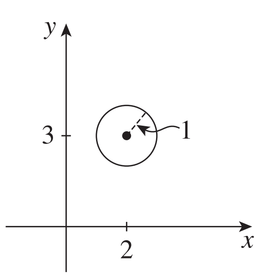
a $(x + 1)^2 + y^2 = 4$ **b** $(x - 1)^2 + (y - 2)^2 = 1$

c $x^2 - 2x + y^2 - 4y - 4 = 0$ **d** $x^2 + 6x + y^2 - 8y = 0$

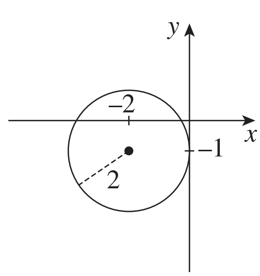
e $x^2 - 10x + y^2 + 8y + 32 = 0$ **f** $x^2 + 14x + 14 + y^2 - 2y = 0$

13 Describe each graph below as the circle $x^2 + y^2 = r^2$ transformed by shifts, and hence write down its equation.

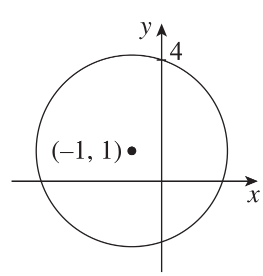
a



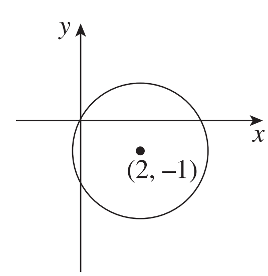
b



c



d



14 a Use a table of values to sketch $y = \frac{1}{2}x^3$. Then use translations to sketch:

i $y = \frac{1}{2}x^3 - 2$ **ii** $y = \frac{1}{2}(x - 2)^3$ **iii** $y = \frac{1}{2}(x + 3)^3 + 1$

b Use a table of values to sketch $y = -2x^3$. Then use translations to sketch:

i $y = 3 - 2x^3$ **ii** $y = -2(x + 3)^3$ **iii** $y = -2(x - 1)^3 - 2$

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CHALLENGE

15 Consider the straight line equation $x + 2y - 4 = 0$.

a The line is translated 2 units left. Find the equation of the new line.

b The original line is translated 1 unit down. Find the equation of this third line.

c Comment on your answers, and draw the lines on the same number plane.

16 Sketch $y = \frac{1}{x}$, then use shifting to sketch the following graphs. Find any x-intercepts and y-intercepts, and mark them on your graphs.

a $y = \dfrac{1}{x-2}$ b $y = 1 + \dfrac{1}{x-2}$ c $y = \dfrac{1}{x-2} - 2$

d $y = \dfrac{1}{x+1} - 1$ e $y = 3 + \dfrac{1}{x+2}$ f $y = \dfrac{1}{x-3} + 4$

17 In each part, explain how the graph of each subsequent equation is a transformation of the first graph (there may be more than one answer), then sketch each function.

a From $y = 2x$:

i $y = 2x + 4$ ii $y = 2x - 4$

b From $y = x^2$:

i $y = x^2 + 9$ ii $y = x^2 - 9$ iii $y = (x-3)^2$

c From $y = -x^2$:

i $y = 1 - x^2$ ii $y = -(x+1)^2$ iii $y = -(x+1)^2 + 2$

d From $y = \sqrt{x}$:

i $y = \sqrt{x+4}$ ii $y = \sqrt{x} + 4$ iii $y = \sqrt{x+4} - 2$

e From $y = \dfrac{2}{x}$:

i $y = \dfrac{2}{x} + 1$ ii $y = \dfrac{2}{x+2}$ iii $y = \dfrac{2}{x+2} + 1$

18 a The circle $x^2 + y^2 = r^2$ has centre the origin and radius r. This circle is shifted so that its centre is at $C(h, k)$. Write down its equation.

b The point $P(x, y)$ lies on the circle with centre $C(h, k)$ and radius r. That is, P lies on the shifted circle in part **a**. This time use the distance formula for the radius PC to obtain the equation of the circle.

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4B Reflections in the x-axis and y-axis

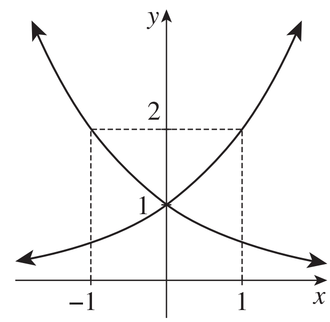
Reflecting only in the x-axis and the y-axis may seem an unnecessary restriction, but in fact these two transformations are the key to understanding many significant properties of functions, particularly the symmetry of graphs.

When these two reflections are combined, they produce a rotation of 180° about the origin, which again is the key to the symmetry of many graphs.

Reflection in the y-axis

The graphs of $y = 2^x$ and $y = 2^{-x}$ have been sketched to the right from their tables of values.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



- The second and third rows are the reverse of each other.
- Hence the graphs are reflections of each other in the y-axis.

4 REFLECTION IN THE Y-AXIS

- To reflect a graph in the y-axis, replace x by $-x$.
- Alternatively, if the graph is a function, the new function rule is $y = f(-x)$.

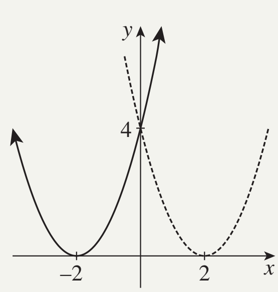
Reflection is mutual — it maps each graph to the other graph.

Example 6 — 4B

a Sketch the parabola $y = (x - 2)^2$, and on the same set of axes, sketch its reflection in the y-axis.
b Use the rule in the box above to write down the equation of the reflected graph.
c Why can this equation be written as $y = (x + 2)^2$?
d What are the vertices of the two parabolas?
e What other transformation would move the first parabola to the second?

SOLUTION

a



b Replacing x by $-x$ gives the equation
$y = (-x - 2)^2$

c Taking out the factor -1 from the brackets,
$$y = (-x - 2)^2$$
$$y = (-1)^2 \times (x + 2)^2$$
$$y = (x + 2)^2$$

d The vertices are $(2, 0)$ and $(-2, 0)$.

e The second parabola is also the first parabola shifted left 4 units.
This replaces x by $x + 4$, so the new equation is the same as before,
$$y = \big((x + 4) - 2\big)^2, \quad \text{or} \quad y = (x + 2)^2.$$
The reason why there are two possible transformations is that the parabola has line symmetry in its axis of symmetry.

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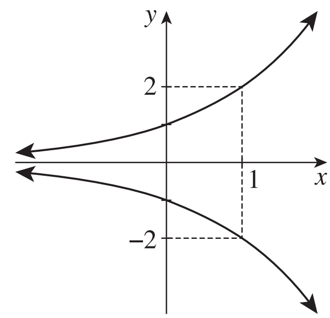
Reflection in the x-axis

The graphs of $y = 2^x$ and $y = -2^x$ have been sketched to the right from the table of values.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
-2^x	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-8



- The values in the second and third rows are the opposites of each other.
- Hence the graphs are reflections of each other in the x-axis.

Rewriting the transformed graph as $-y = 2^x$ makes it clear that the reflection has been obtained by replacing y by $-y$, giving a rule that is completely analogous to that for reflection in the y-axis.

5 REFLECTION IN THE X-AXIS

- To reflect a graph in the x-axis, replace y by $-y$.
- Alternatively, if the graph is a function, the new function rule is $y = -f(x)$.

Again, reflection is mutual — it maps each graph to the other graph.

Example 7 4B

The graph of $y = (x - 2)^2$ is reflected in the x-axis.

a Write down the equation of the reflected graph.
b Construct tables of values, and sketch the two graphs.
c What are the vertices of the two parabolas?

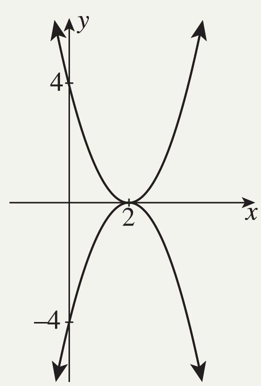
SOLUTION

a Replace y by $-y$, so the new function is

$$-y = (x - 2)^2, \qquad \text{that is,} \qquad y = -(x - 2)^2.$$

b

x	0	1	2	3	4
$(x-2)^2$	4	1	0	1	4
$-(x-2)^2$	-4	-1	0	-1	-4



c They both have vertex $(2, 0)$.

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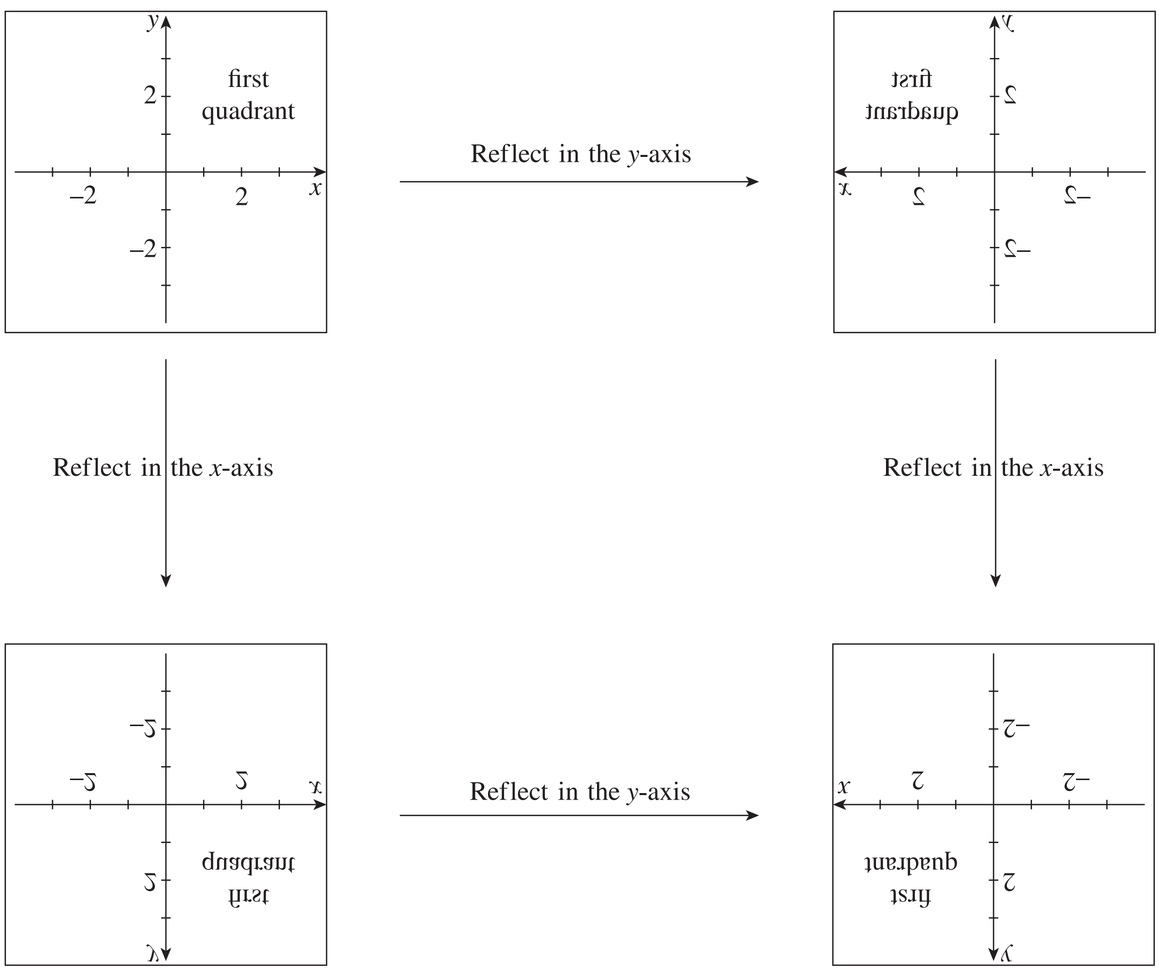
Combining the two reflections — rotating 180° about the origin

- Draw an x-axis and y-axis on a thin, semi-transparent sheet of paper.
- Hold the paper out flat, and regard it as a two-dimensional object.
- Reflect it in the x-axis — do this by holding the sheet steady at the two ends of the x-axis and rotating it 180° so that you are now looking at the back of the sheet. Then reflect it in the y-axis — do this by holding the sheet at the two ends of the y-axis and rotating it 180° so that you are looking at the front of the sheet again. What has happened?
- Reflect it in the y-axis, then in the x-axis. What happens?



This little experiment should convince you of two things:

- Performing successive reflections in the x-axis and in the y-axis results in a rotation of 180° about the origin.
- The order of these two reflections does not matter.

This rotation of 180° about the origin is sometimes called *reflection in the origin* — every point in the plane is moved along a line through the origin to a point the same distance from the origin on the opposite side.

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6 ROTATION OF 180° ABOUT THE ORIGIN

- To rotate a graph 180° about the origin, replace x by $-x$ and y by $-y$.
- Successive reflections in the x-axis and the y-axis are the same as a rotation of 180° about the origin.
- The order in which these two successive reflections are done does not matter.
- Rotation of 180° about the origin is also called *reflection in the origin*, because every point is moved through the origin to a point the same distance from the origin on the opposite side.

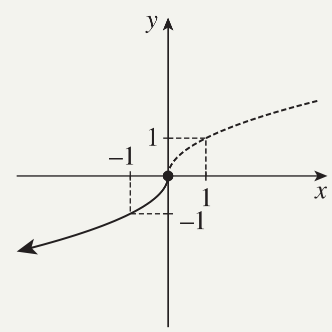
Rotation of 180° about the origin is also mutual — it maps each graph to the other graph.

Example 8 4B

a From the graph of $y = \sqrt{x}$, deduce the graph of $y = -\sqrt{-x}$ using reflections.

b What single transformation maps each graph to the other?

SOLUTION



a The equation $y = -\sqrt{-x}$ can be rewritten as

$$-y = \sqrt{-x}$$

so the graph is obtained from the graph of $y = \sqrt{x}$ by successive reflections in the x-axis and the y-axis, where the reflections may be done in either order.

b This is the same as rotation of 180° about the origin.

Exercise 4B FOUNDATION

1 Consider the parabola $y = x^2 - 2x$.

a Show that when y is replaced by $-y$, the equation becomes $y = 2x - x^2$.

b Copy and complete the table of values for $y = x^2 - 2x$ and $y = 2x - x^2$.

x	-2	-1	0	1	2	3	4
$x^2 - 2x$							
$2x - x^2$							

c Sketch the two parabolas and state the vertex of each.

d What transformation maps $y = x^2 - 2x$ to $y = 2x - x^2$?

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2 Consider the hyperbola $y = \dfrac{2}{x-2}$.

a Show that when x is replaced by $-x$ the equation becomes $y = -\dfrac{2}{x+2}$.

b Copy and complete the table of values for $y = \dfrac{2}{x-2}$ and $y = -\dfrac{2}{x+2}$.

x	-4	-3	-2	-1	0	1	2	3	4
$\dfrac{2}{x-2}$							*		
$-\dfrac{2}{x+2}$			*						

c Sketch the two hyperbolas and state the vertical asymptote of each.

d What transformation maps $y = \dfrac{2}{x-2}$ to $y = -\dfrac{2}{x+2}$?

3 a Sketch the graph of the quadratic function $y = x^2 - 2x - 3$, showing the intercepts and vertex.

b In each case, determine the equation of the result when this parabola is reflected as indicated. Then draw a sketch of the new function.

i In the y-axis **ii** In the x-axis **iii** In both axes

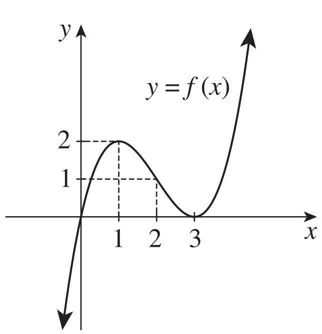
4 a Sketch the graph of the exponential function $y = 2^{-x}$, showing the y-intercept and the coordinates at $x = -1$, and clearly indicating the asymptote.

b In each case, find the equation of the curve when this exponential graph is reflected as indicated. Then draw a sketch of the new function.

i In the y-axis **ii** In the x-axis **iii** In both axes

5 In each case, an unknown function has been drawn. Draw the reflections of the function specified below it.

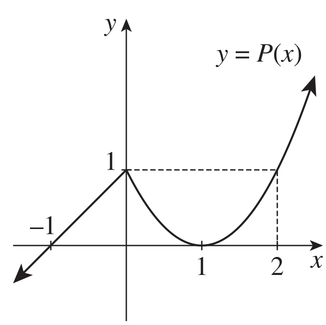
a



i $y = f(-x)$

ii $y = -f(x)$

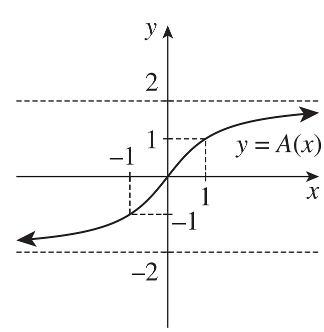
b



i $y = -P(x)$

ii $y = -P(-x)$

c



i $y = A(-x)$

ii $y = -A(-x)$

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DEVELOPMENT

6 Write down the new equation for each function or relation after the given transformation has been applied. Then sketch the graph of the new curve.

a $y = x^2$: reflect in the x-axis

b $y = x^3$: reflect in the y-axis

c $y = 2^x$: rotate by 180°

d $y = 2x - x^2$: rotate by 180°

e $x^2 + y^2 = 9$: reflect in the y-axis

f $y = \dfrac{1}{x}$: reflect in the x-axis

7 Consider the hyperbola $y = \dfrac{1}{x+2} - 1$.

a Sketch this hyperbola.

b In each case, determine the reflection or rotation required to achieve the specified result. Then write down the equation of the new hyperbola and sketch it.

i The vertical asymptote is unchanged, but the horizontal asymptote changes sign.

ii The intercepts with the axes are positive.

8 **a** Sketch the circles $(x - 3)^2 + y^2 = 4$ and $(x + 3)^2 + y^2 = 4$.

b What transformation maps each circle onto the other?

c Confirm your answer by making an appropriate substitution into the first equation.

9 Consider $x^2 + y^2 = r^2$, the circle with centre the origin and radius r.

a Show that this equation is unchanged when reflected in either the x-axis or the y-axis.

b Explain this result geometrically.

CHALLENGE

10 In each part, explain how the graph of each subsequent equation is a reflection of the first graph or a rotation of 180°, then sketch each one.

a From $y = \frac{1}{2}x + 1$:

i $y = -\frac{1}{2}x + 1$ **ii** $y = -\frac{1}{2}x - 1$ **iii** $y = \frac{1}{2}x - 1$

b From $y = 4 - x$:

i $y = x - 4$ **ii** $y = x + 4$ **iii** $y = -x - 4$

c From $y = (x - 1)^2$:

i $y = -(x + 1)^2$ **ii** $y = (x + 1)^2$ **iii** $y = -(x - 1)^2$

d From $y = \sqrt{x}$:

i $y = -\sqrt{-x}$ **ii** $y = -\sqrt{x}$ **iii** $y = \sqrt{-x}$

e From $y = 3^x$:

i $y = -3^x$ **ii** $y = -3^{-x}$ **iii** $y = 3^{-x}$

f From $y = 1 + \dfrac{1}{x-1}$:

i $y = 1 - \dfrac{1}{x+1}$ **ii** $y = -1 + \dfrac{1}{x+1}$ **iii** $y = -1 + \dfrac{1}{1-x}$

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11 Consider the two parabolas $y = x^2 - 4x + 3$ and $y = x^2 + 4x + 3$.

a Sketch both quadratic functions on the same set of axes.

b What reflection maps each parabola onto the other?

c How can the second parabola be obtained by shifting the first?

d Confirm your answer to part **c** algebraically.

e Investigate which parts of Question **10** could also have been achieved by shifting instead.

12 a Let $c(x) = \dfrac{2^x + 2^{-x}}{2}$. Show that $c(-x) = c(x)$, and explain this geometrically.

b Let $t(x) = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$. Show that $-t(-x) = t(x)$, and explain this geometrically.

c [Technology]
Confirm your observations in parts **a** and **b** by plotting each function using graphing software.

13 Consider the parabola $y = (x - 1)^2$. Sketches or plots done on graphing software may help answer the following questions.

a i The parabola is shifted right 1 unit. What is the new equation?

ii This new parabola is then reflected in the y-axis. Write down the equation of the new function.

b i The original parabola is reflected in the y-axis. What is the new equation?

ii This fourth parabola is then shifted right 1 unit. What is the final equation?

c Parts **a** and **b** both used a reflection in the y-axis and a shift right 1 unit. Did the order of these affect the answer?

d Investigate other combinations of shifts and reflections. In particular, what do you notice if the shift is parallel with the axis of reflection?

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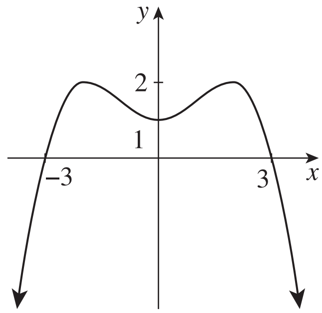
4C Even and odd symmetry

It has been said that all mathematics is the study of symmetry. Two simple types of symmetry occur so often in the functions of this course that every function should be tested routinely for them.

Even functions and line symmetry in the y-axis

A relation or function is called *even* if its graph has *line symmetry in the y-axis*. This means that the graph is unchanged by reflection in the y-axis, as with the graph to the right.



As explained in Section 4B, when the graph of $y = f(x)$ is reflected in the y-axis, the new curve has equation $y = f(-x)$. Hence for a function to be *even*, the graphs of $y = f(x)$ and $y = f(-x)$ must coincide, that is,

$$f(-x) = f(x), \text{ for all } x \text{ in the domain.}$$

7 EVEN FUNCTIONS

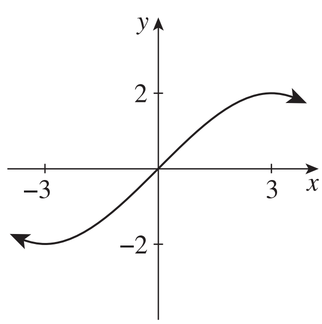
- A relation or function is called *even* if its graph has *line symmetry in the y-axis*.
- Algebraically, a function $f(x)$ is even if

  $$f(-x) = f(x), \text{ for all } x \text{ in the domain.}$$

  More generally, a relation is even if its equation is unchanged when x is replaced by $-x$.

Odd functions and point symmetry in the origin

A relation or function is called *odd* if its graph has *point symmetry in the origin*. This means that the graph is unchanged by a rotation of 180° about the origin, or equivalently, by successive reflections in the x-axis and the y-axis.



When the graph of $y = f(x)$ is reflected in the x-axis and then in the y-axis, the new curve has equation $y = -f(-x)$. Hence for a function to be *odd*, the graph of $-f(-x)$ must coincide with the graph of $f(x)$, that is,

$$f(-x) = -f(x), \text{ for all } x \text{ in the domain.}$$

8 ODD FUNCTIONS AND POINT SYMMETRY IN THE ORIGIN

- A relation or function is called *odd* if its graph has *point symmetry in the origin*.
- Algebraically, a function $f(x)$ is odd if

  $$f(-x) = -f(x), \text{ for all } x \text{ in the domain.}$$

  More generally, a relation is odd if its equation is unchanged when x is replaced by $-x$ and y is replaced by $-y$.
- *Point symmetry in the origin* means that the graph is mapped onto itself by a rotation of 180° about the origin.
- Equivalently, it means that the graph is mapped onto itself by successive reflections in the x-axis and the y-axis. The order of these two reflections does not matter.

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Testing functions algebraically for evenness and oddness

A single test will pick up both these types of symmetry in functions.

9 TESTING FOR EVENNESS AND ODDNESS (OR NEITHER)

- Simplify $f(-x)$ and note whether it is $f(x)$, $-f(x)$ or neither.

Most functions are neither even nor odd.

Example 9 4C

Test each function for evenness or oddness, then sketch it.

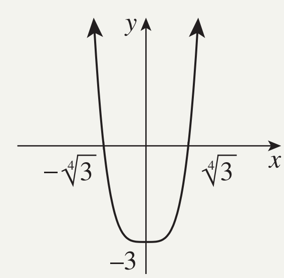
a $f(x) = x^4 - 3$ **b** $f(x) = x^3$ **c** $f(x) = x^2 - 2x$

SOLUTION

a Here $f(x) = x^4 - 3.$

Substituting $-x$ for x,
$$\begin{aligned} f(-x) &= (-x)^4 - 3 \\ &= x^4 - 3 \\ &= f(x). \end{aligned}$$

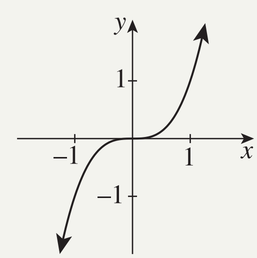
Hence $f(x)$ is an even function.



b Here $f(x) = x^3.$

Substituting $-x$ for x,
$$\begin{aligned} f(-x) &= (-x)^3 \\ &= -x^3 \\ &= -f(x) \end{aligned}$$

Hence $f(x)$ is an odd function.

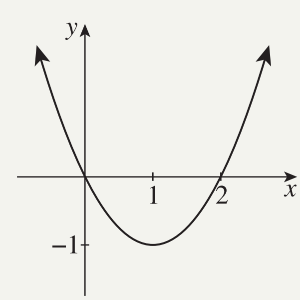


c Here $f(x) = x^2 - 2x.$

Substituting $-x$ for x,
$$\begin{aligned} f(-x) &= (-x)^2 - 2(-x) \\ &= x^2 + 2x. \end{aligned}$$

Because $f(-x)$ is equal neither to $f(x)$ nor to $-f(x)$, the function is neither even nor odd.

(The parabola does, however, have line symmetry, not in the y-axis, but in its axis of symmetry $x = 1$.)



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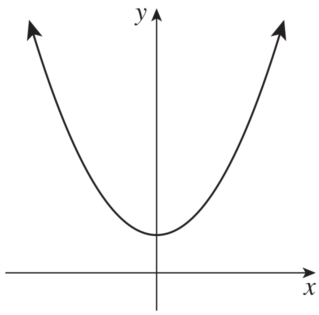
Exercise 4C

FOUNDATION

1 Classify each function $y = f(x)$ as even, odd or neither.

a


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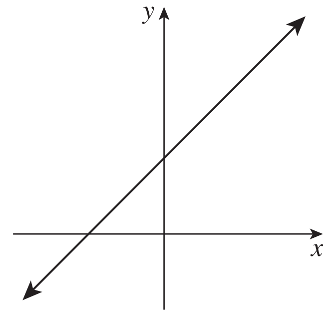

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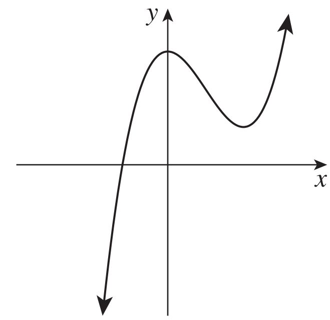

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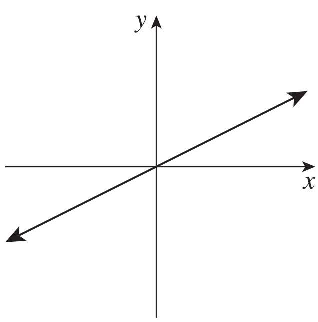

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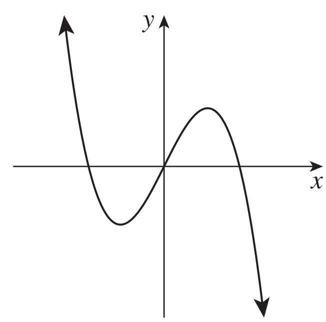

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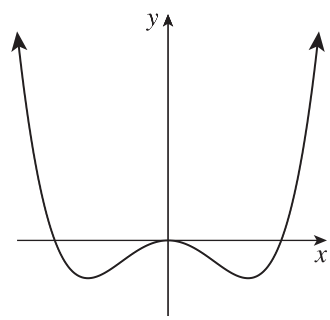
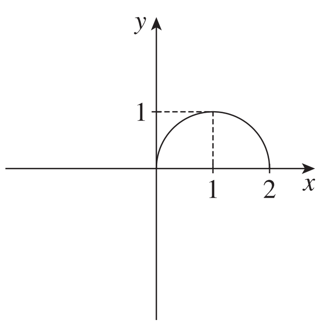


2 In each diagram below, complete the graph so that:

i $f(x)$ is even **ii** $f(x)$ is odd.

a


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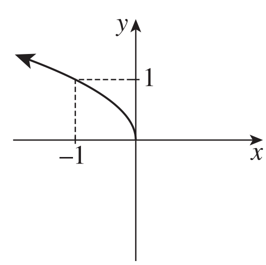

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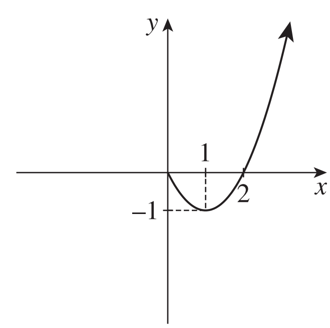

3 Consider the function $f(x) = x^4 - 2x^2 + 1$.

a Simplify $f(-x)$. **b** Hence show that $f(x)$ is an even function.

4 Consider the function $g(x) = x^3 - 3x$.

a Simplify $g(-x)$. **b** Hence show that $g(x)$ is an odd function.

5 Consider the function $h(x) = x^3 + 3x^2 - 2$.

a Simplify $h(-x)$. **b** Hence show that $h(x)$ is neither even nor odd.

6 Simplify $f(-x)$ for each function, and hence determine whether it is even, odd or neither.

a $f(x) = x^2 - 9$ **b** $f(x) = x^2 - 6x + 5$ **c** $f(x) = x^3 - 25x$

d $f(x) = x^4 - 4x^2$ **e** $f(x) = x^3 + 5x^2$ **f** $f(x) = x^5 - 16x$

g $f(x) = x^5 - 8x^3 + 16x$ **h** $f(x) = x^4 + 3x^3 - 9x^2 - 27x$

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7 On the basis of the previous questions, copy and complete these sentences:

a 'A polynomial function is odd if …'.

b 'A polynomial function is even if …'.

DEVELOPMENT

8 Factor each polynomial in parts **a**–**f** of Question **6** above and write down its zeroes (that is, its x-intercepts). Then use a table of values to sketch its graph. Confirm that the graph exhibits the symmetry established above.

9 [Algebra and Technology]

In Questions **3**–**8**, the odd and even functions were all polynomials. Other functions can also be classified as odd or even. In each case following, simplify $f(-x)$ and compare it with $f(x)$ and $-f(x)$ to determine whether the function is odd or even. Then confirm your answer by plotting the function on appropriate graphing software.

a $f(x) = \dfrac{2^x + 2^{-x}}{2}$ **b** $f(x) = \dfrac{2^x - 2^{-x}}{2}$ **c** $f(x) = \sqrt[3]{x}$ **d** $f(x) = \left(\sqrt[3]{x}\right)^2$

e $f(x) = \dfrac{x}{x^2 - 4}$ **f** $f(x) = \dfrac{2}{x^2 - 4}$ **g** $f(x) = \sqrt{9 - x^2}$ **h** $f(x) = x\sqrt{9 - x^2}$

10 Determine whether each function is even, odd or neither.

a $f(x) = 2^x$ **b** $f(x) = 2^{-x}$ **c** $f(x) = \sqrt{3 - x^2}$

d $f(x) = \dfrac{1}{x^2 + 1}$ **e** $f(x) = \dfrac{4x}{x^2 + 4}$ **f** $f(x) = 3^x + 3^{-x}$

g $f(x) = 3^x - 3^{-x}$ **h** $f(x) = 3^x + x^3$

CHALLENGE

11 a Explain why the relation $x^2 + (y - 5)^2 = 49$ is even.

b Explain why the relation $x^2 + y^2 = 49$ is both even and odd.

12 a Prove that if $f(x)$ is an odd function defined at $x = 0$, then $y = f(x)$ passes through the origin. Hint: Apply the condition for a function to be odd to $f(0)$.

b If $f(x)$ is an even function defined at $x = 0$, does the graph of $y = f(x)$ have to pass through the origin? Either prove the statement or give a counter-example.

13 a Suppose that $h(x) = f(x) \times g(x)$.

i Show that if f and g are both even or both odd, then $h(x)$ is even.

ii Show that if one of $f(x)$ and $g(x)$ is even and the other odd, then $h(x)$ is odd.

b Suppose that $h(x) = f(x) + g(x)$.

i Show that if $f(x)$ and $g(x)$ are both even, then $h(x)$ is even.

ii Show that if $f(x)$ and $g(x)$ are both odd, then $h(x)$ is odd.

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4D The absolute value function

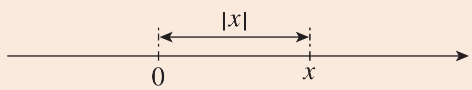
Often it is the size or magnitude of a number that is significant, rather than whether it is positive or negative. *Absolute value* is the mathematical name for this concept.

Absolute value as distance

Distance is the clearest way to define absolute value.

10 ABSOLUTE VALUE AS DISTANCE

- The absolute value $|x|$ of a number x is the distance from x to the origin on the number line.



For example, $|-5| = 5$ and $|0| = 0$ and $|5| = 5$.
- Distance is always positive or zero, so
  $|x| \ge 0$, for all real numbers x.
- The numbers x and $-x$ are equally distant from the origin, so
  $|-x| = |x|$, for all real numbers x.

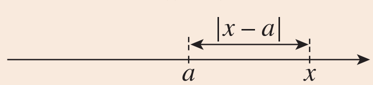
Thus absolute value is a measure of the *size* or *magnitude* of a number. In the examples above, the numbers -5 and $+5$ both have the same magnitude 5, and differ only in their signs.

Distance between numbers

Replacing x by $x - a$ in the previous definition gives a measure of the distance from x to a on the number line.

11 DISTANCE BETWEEN NUMBERS

- The distance from x to a on the number line is $|x - a|$.



For example, the distance between 5 and -2 is $|5 - (-2)| = 7$.
- If follows that $|x - a| = |a - x|$, for all real numbers x and a.

An expression for absolute value involving cases

If x is a negative number, then the absolute value of x is $-x$, the opposite of x. This gives an alternative definition:

12 ABSOLUTE VALUE INVOLVING CASES

For any real number x, define $|x| = \begin{cases} x, & \text{for } x \ge 0, \\ -x, & \text{for } x < 0. \end{cases}$

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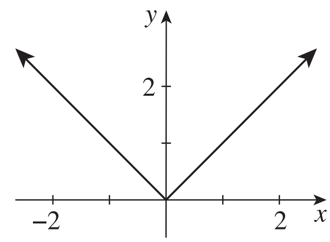
The two cases lead directly to the graph of $y = |x|$.
A table of values confirms the sharp point at the origin where the two branches meet at right angles.



x	-2	-1	0	1	2
$\|x\|$	2	1	0	1	2

- The domain is the set of all real numbers, and the range is $y \geq 0$.
- The function is even, because the graph has line symmetry in the y-axis.
- The function has a zero at $x = 0$, and is positive for all other values of x.

Graphing functions with absolute value

Transformations can now be applied to the graph of $y = |x|$ to sketch many functions involving absolute value. More complicated functions, however, require the approach involving cases.

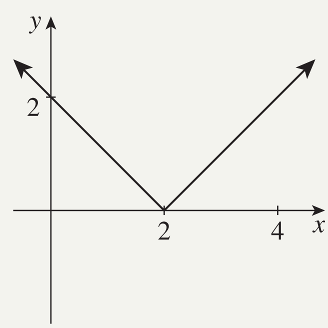
A short table of values is always an excellent safety check.

Example 10 — 4D

a Sketch $y = |x - 2|$ using shifting.
b Check the graph using a table of values.
c Write down the equations of the two branches.

SOLUTION

a This is $y = |x|$ shifted 2 units to the right.



b

x	0	1	2	3	4
y	2	1	0	1	2

c From the expression using cases, or from the graph:

$$y = \begin{cases} x - 2, & \text{for } x \geq 2, \\ -x + 2, & \text{for } x < 2. \end{cases}$$

Example 11 — 4D

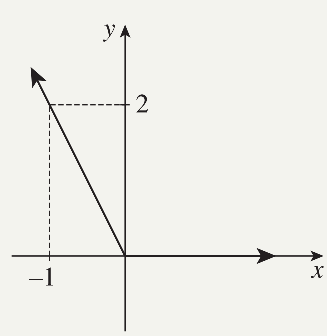
a Use cases to sketch $y = |x| - x$.
b Check using a table of values.

SOLUTION

a Considering separately the cases $x \geq 0$ and $x < 0$,

$$y = \begin{cases} x - x, & \text{for } x \geq 0, \\ -x - x, & \text{for } x < 0, \end{cases}$$

that is, $y = \begin{cases} 0, & \text{for } x \geq 0, \\ -2x, & \text{for } x < 0. \end{cases}$



b Checking using a table of values,

x	-2	-1	0	1	2
y	4	2	0	0	0

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Solving absolute value equations:

Three observations should make everything clear:

- An equation such as $|3x + 6| = -21$ has no solutions, because an absolute value can never be negative.
- An equation such as $|3x + 6| = 0$ is true when $3x + 6 = 0$.
- An equation such as $|3x + 6| = 21$ can be solved by realising that:

$$|3x + 6| = 21 \quad \text{is true when} \quad 3x + 6 = 21 \quad \text{or} \quad 3x + 6 = -21.$$

13 TO SOLVE THE EQUATION $|ax + b| = k$

- If $k < 0$, the equation has no solutions.
- If $k = 0$, the equation has one solution, found by solving
  $ax + b = 0$.
- If $k > 0$, the equation has two solutions, found by solving
  $ax + b = k \qquad \text{or} \qquad ax + b = -k$.

Example 12 — 4D

Solve each absolute value equation.

a $|3x + 6| = -21$ **b** $|3x + 6| = 0$ **c** $|3x + 6| = 21$

SOLUTION

a $|3x + 6| = -21$ has no solutions, because an absolute value cannot be negative.

b
$$\begin{aligned} |3x + 6| &= 0 \\ 3x + 6 &= 0 \\ \boxed{-6} \qquad 3x &= -6 \\ \boxed{\div 3} \qquad x &= -2 \end{aligned}$$

c
$$\begin{aligned} |3x + 6| &= 21 \\ 3x + 6 &= 21 \text{ or } 3x + 6 = -21 \\ \boxed{-6} \qquad 3x &= 15 \text{ or } 3x = -27 \\ \boxed{\div 3} \qquad x &= 5 \text{ or } x = -9 \end{aligned}$$

Example 13 — 4D

Solve each absolute value equation.

a $|x - 2| = 3$ **b** $|7 - \frac{1}{4}x| = 3$

SOLUTION

a
$$\begin{aligned} |x - 2| &= 3 \\ x - 2 &= 3 \text{ or } x - 2 = -3 \\ \boxed{+2} \qquad x &= 5 \text{ or } x = -1 \end{aligned}$$

b
$$\begin{aligned} |7 - \tfrac{1}{4}x| &= 3 \\ 7 - \tfrac{1}{4}x &= 3 \text{ or } 7 - \tfrac{1}{4}x = -3 \\ \boxed{-7} \qquad -\tfrac{1}{4}x &= -4 \text{ or } -\tfrac{1}{4}x = -10 \\ \boxed{\times (-4)} \qquad x &= 16 \text{ or } x = 40 \end{aligned}$$

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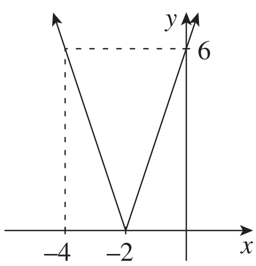
Sketching $y = |ax + b|$:

To sketch a function such as $y = |3x + 6|$, the first step is always to find the x-intercept and y-intercept. As in the example above,

Put $y = 0$, then $|3x + 6| = 0$
$$3x + 6 = 0$$
$$x = -2.$$

Put $x = 0$
Then $y = |0 + 6|$
$$= 6.$$

Plot those two points $(-2, 0)$ and $(0, 6)$.



The graph is symmetric about the vertical line $x = -2$, so the point $(-4, 6)$ also lies on the curve.
Now join the points up in the characteristic V shape.
Alternatively, draw up a small table of values,

x	-4	-3	-2	-1	0
y	6	3	0	3	6

A good final check: The two branches of the curve should have gradients 3 and –3.

Absolute value as the square root of the square

Taking the absolute value of a number means stripping any negative sign from the number. We already have algebraic functions capable of doing this job — we can square the number, then apply the function $\sqrt{\ }$ that says 'take the positive square root (or zero)'.

14 ABSOLUTE VALUE AS THE POSITIVE SQUARE ROOT OF THE SQUARE

- For all real numbers x, $\quad |x|^2 = x^2 \quad$ and $\quad |x| = \sqrt{x^2}$.

For example, $|-3|^2 = 9 = (-3)^2 \quad$ and $\quad |-3| = \sqrt{9} = \sqrt{(-3)^2}$.

Identities involving absolute value

Here are some standard identities.

15 IDENTITIES INVOLVING ABSOLUTE VALUE

- $|-x| = |x|$, for all x.
- $|x - y| = |y - x|$, for all x and y.
- $|xy| = |x||y|$, for all x and y.
- $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, for all x, and for all $y \neq 0$.

Substitution of some positive and negative values for x and y should be sufficient to demonstrate these results.

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Exercise 4D

FOUNDATION

1 Evaluate:

a $|3|$ b $|-3|$ c $|4-7|$ d $|7-4|$
e $|14-9-12|$ f $|-7+8|$ g $|3^2-5^2|$ h $|11-16|-8$

2 Solve each absolute value equation, then graph the solution on a number line.

a $|x|=1$ b $|x|=3$ c $|4x|=8$
d $|2x|=10$ e $|2x|=6$ f $|3x|=12$

3 Solve each equation and graph the solution on a number line.

a $|x-4|=1$ b $|x-3|=7$ c $|x-3|=3$ d $|x-7|=2$
e $|x+5|=2$ f $|x+2|=2$ g $|x+1|=6$ h $|x+3|=1$

4 a Copy and complete the tables of values for the functions $y=|x-1|$ and $y=|x|-1$.

x	-2	-1	0	1	2	3
$\|x-1\|$						

x	-2	-1	0	1	2	3
$\|x\|-1$						

b Draw the graphs of the two functions on separate number planes, and observe the similarities and differences between them.

c Explain how each graph is obtained by shifting $y=|x|$.

5 Show that each statement is true when $x=-3$.

a $|5x|=5|x|$ b $|-x|=|x|$ c $|x|^2=x^2$
d $|x-7|=|7-x|$ e $x\le|x|$ f $-|x|\le x$

6 Show that each statement is false when $x=-2$.

a $|x|=x$ b $|-x|=x$ c $|x+2|=|x|+2$
d $|x+1|=x+1$ e $|x-1|<|x|-1$ f $|x|^3=x^3$

DEVELOPMENT

7 In each case, use the rules of Box 13 to solve the equation for x.

a $|7x|=35$ b $|2x+1|=3$ c $|2x-1|=11$
d $|7x-3|=-11$ e $|3x+2|=-8$ f $|5x+2|=0$
g $|3x-5|=0$ h $|6x-7|=5$ i $|5x+4|=6$

8 a Consider the equation $|1-2x|=3$.

i Explain why this equation has the same solution as $|2x-1|=3$. (Refer to Box 11.)

ii Hence, or otherwise, solve the equation.

b Likewise, solve these equations.

i $|3-2x|=1$ ii $|1-3x|=2$

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9 **a** Use cases to help sketch the branches of $y = |x|$.

b In each part, identify the shift or shifts of $y = |x|$ and hence sketch the graph. Then write down the equations of the two branches.

i $y = |x - 3|$ **ii** $y = |x + 2|$ **iii** $y = |x| - 2$

iv $y = |x| + 3$ **v** $y = |x - 2| - 1$ **vi** $y = |x + 1| - 1$

10 Sketch each function using cases. Check the graph with a table of values.

a $y = |2x|$ **b** $y = \left|\frac{1}{2}x\right|$

11 Find the x-intercept and y-intercept of each function. Then sketch the graph using symmetry, and confirm with a small table of values.

a $y = |2x - 6|$ **b** $y = |9 - 3x|$ **c** $y = |5x|$

d $y = |4x + 10|$ **e** $y = -|3x + 7|$ **f** $y = -|7x|$

12 [Technology]

Use suitable graphing software to help solve these problems.

a **i** Sketch $y = |x - 4|$ and $y = 1$ on the same set of axes, clearly showing the points of intersection.

ii Hence write down the solution of $|x - 4| = 1$.

b Now use similar graphical methods to solve each of the following.

i $|x + 3| = 1$ **ii** $|2x + 1| = 3$ **iii** $|3x - 3| = -2$ **iv** $|2x - 5| = 0$

CHALLENGE

13 Consider the absolute value function $f(x) = |x|$.

a Use the result $f(x) = \sqrt{x^2}$ given in Box 14 to help prove that the absolute value function is even.

b Why was this result obvious from the graph of $y = |x|$?

14 **a** For what values of x is $y = \dfrac{|x|}{x}$ undefined?

b Use a table of values of x from -3 to 3 to sketch the graph.

c Hence write down the equations of the two branches of $y = \dfrac{|x|}{x}$.

15 Sketch each graph by drawing up a table of values for $-3 \le x \le 3$. Then use cases to determine the equation of each branch of the function.

a $y = |x| + x$ **b** $y = |x| - x$

c $y = 2(x + 1) - |x + 1|$ **d** $y = x^2 - |2x|$

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4E Composite functions

Shifting, reflecting, and taking absolute value, are all examples of a far more general procedure of creating a composite function from two given functions. The example below shows how a translation left 3 and a translation up 3 are obtained using composites. In this section, however, attention is on the algebra rather than on the final graph.

Composition of functions

Suppose that we are given the two functions $f(x) = x + 3$ and $g(x) = x^2$. We can put them together by placing their function machines so that the output of the first function is the input of the second function

x				$f(x)$				$g(f(x))$
-2	$\longrightarrow$	Add 3	$\longrightarrow$	1	$\longrightarrow$	Square	$\longrightarrow$	1
-1	$\longrightarrow$		$\longrightarrow$	2	$\longrightarrow$		$\longrightarrow$	4
0	$\longrightarrow$		$\longrightarrow$	3	$\longrightarrow$		$\longrightarrow$	9
1	$\longrightarrow$		$\longrightarrow$	4	$\longrightarrow$		$\longrightarrow$	16
2	$\longrightarrow$		$\longrightarrow$	5	$\longrightarrow$		$\longrightarrow$	25
x	$\longrightarrow$		$\longrightarrow$	$x + 3$	$\longrightarrow$		$\longrightarrow$	$(x + 3)^2$

The middle column is the output of the first function 'Add 3'. This output is then the input of the second function 'Square'. The result is the *composite function*

$$g\big(f(x)\big) = (x + 3)^2 \qquad \text{'Add 3, then square.'}$$

If the functions are composed the other way around, the result is different,

$$f\big(g(x)\big) = x^2 + 3 \qquad \text{'Square, then add 3.'}$$

Notice how in this example, both ways around are examples of translations.

- The composite graph $y = g\big(f(x)\big)$ is $y = g(x)$ shifted left 3.
- The composite graph $y = f\big(g(x)\big)$ is $y = g(x)$ shifted up 3.

Example 14 **4E**

Find and simplify $k\big(h(x)\big)$ and $h\big(k(x)\big)$ when $h(x) = 2x + 3$ and $k(x) = 1 - 5x$.

SOLUTION

$$\begin{aligned} k\big(h(x)\big) &= k(2x + 3) \\ &= 1 - 5(2x + 3) \\ &= -10x - 14 \end{aligned}$$

$$\begin{aligned} h\big(k(x)\big) &= h(1 - 5x) \\ &= 2(1 - 5x) + 3 \\ &= -10x + 5 \end{aligned}$$

Domain and range of the composite function

In Example 14 above, $h(x)$ and $k(x)$ have domain and range all real numbers, so there are no problems, and the domains and ranges of both composites are all real numbers.

In the original example with the function machines, $f(x)$ and $g(x)$ again both have domain all real numbers, and so do $g\big(f(x)\big)$ and $f\big(g(x)\big)$. But while $f(x)$ has range all real numbers, $g(x)$ has range $y \ge 0$. Thus the range of $g\big(f(x)\big) = (x + 3)^2$ is $y \ge 0$, and the range of $f\big(g(x)\big) = x^2 + 3$ is $y \ge 3$.

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In general, neither the domain nor the range of $g\big(f(x)\big)$ are all real numbers.

- For a real number a to be in the domain of $g\big(f(x)\big)$, a must be in the domain of $f(x)$, and $f(a)$ must be in the domain of $g(x)$.
- The range of $g\big(f(x)\big)$ is the range of $g(x)$ when it is restricted just to the range of $f(x)$.

Example 15 4E

Find the domain and range of $g\big(f(x)\big)$ if $f(x) = \sqrt{x-4}$ and $g(x) = \frac{1}{x}$.

SOLUTION

The domain of $f(x)$ is $x \ge 4$, and the domain of g(x) is $x \ne 0$.
When $x = 4$, $f(4) = 0$, and when $x > 4$, $f(x) > 0$, so $g\big(f(x)\big)$ is not defined at $x = 4$, but is defined for $x > 4$.
Thus the domain of $g\big(f(x)\big)$ is $x > 4$.
When restricted to $x > 4$, the range of $f(x)$ is $y > 0$, and when g(x) is restricted to $x > 0$, its range is $y > 0$ (note the change of variable as output becomes input). Thus the range of $g\big(f(x)\big)$ is $y > 0$.

This approach, however, is rather elaborate. It is almost always enough to find the equation of the composite and look at it as a single function. In this example,

$$g\big(f(x)\big) = \frac{1}{\sqrt{x-4}},$$

from which it is easily seen that the domain is $x > 4$ and the range is $y > 0$.

16 COMPOSITE FUNCTIONS:

- The *composites* of two functions $f(x)$ and $g(x)$ are $g\big(f(x)\big)$ and $f\big(g(x)\big)$.
- For a real number a to be in the domain of $g\big(f(x)\big)$, a must be in the domain of $f(x)$, and then $f(a)$ must be in the domain of $g(x)$.
- The range of $g\big(f(x)\big)$ is the range of $g(x)$ when it is restricted just to the range of $f(x)$.

It is almost always enough to read the domain and range from the equation of the composite function.

The notations $(g \circ f)(x)$ for the composite $g\big(f(x)\big)$, and $(f \circ g)(x)$ for $f\big(g(x)\big)$, are also widely used.

The empty function

Let $f(x) = -x^2 - 1$ and $g(x) = \sqrt{x}$. Then

$$g\big(f(x)\big) = \sqrt{-x^2 - 1}.$$

This is a problem, because $\sqrt{-x^2 - 1}$ is undefined, whatever the value of x. The range of $f(x)$ is all real numbers less than or equal to -1, and g(x) is undefined on all of these because you can't take the square root of a negative. Thus $g\big(f(x)\big)$ has domain the empty set, and its range is therefore also the empty set. It is *the empty function*.

The empty function has domain the empty set, and its range is therefore also the empty set. For those interested in trivialities, the empty function is one-to-one.

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Exercise 4E

FOUNDATION

1 Consider the function $f(x) = x + 2$.

a Find the values of:

i $f\big(f(0)\big)$ **ii** $f\big(f(3)\big)$ **iii** $f\big(f(-1)\big)$ **iv** $f\big(f(-8)\big)$

b Find expressions for:

i $f\big(f(x)\big)$ **ii** $f\Big(f\big(f(x)\big)\Big)$

c Find the value of x for which $f\big(f(x)\big) = 0$.

2 Consider the function $F(x) = 2x$.

a Find the values of $F\big(F(0)\big)$, $F\big(F(7)\big)$, $F\big(F(-3)\big)$ and $F\big(F(-11)\big)$.

b Find expressions for $F\big(F(x)\big)$ and for $F\Big(F\big(F(x)\big)\Big)$

c Find the value of x for which $F\big(F(x)\big) = 32$.

3 Consider the function $g(x) = 2 - x$.

a Find the values of $g\big(g(0)\big)$, $g\big(g(4)\big)$, $g\big(g(-2)\big)$ and $g\big(g(-9)\big)$.

b Show that $g\big(g(x)\big) = x$.

c Show that $g\Big(g\big(g(x)\big)\Big) = g(x)$.

4 Consider the function $h(x) = 3x - 5$.

a Find the values of $h\big(h(0)\big)$, $h\big(h(5)\big)$, $h\big(h(-1)\big)$ and $h\big(h(-5)\big)$.

b Find expressions for $h\big(h(x)\big)$ and for $h\Big(h\big(h(x)\big)\Big)$.

5 Two linear functions are defined by $f(x) = x + 1$ and $g(x) = 2x - 3$.

a Find the values of $f\big(g(7)\big)$, $g\big(f(7)\big)$, $f\big(f(7)\big)$ and $g\big(g(7)\big)$.

b Find expressions for:

i $f\big(g(x)\big)$ **ii** $g\big(f(x)\big)$ **iii** $f\big(f(x)\big)$ **iv** $g\big(g(x)\big)$

c What transformation maps the graph of $y = g(x)$ to the graph of $y = g\big(f(x)\big)$?

d What transformation maps the graph of $y = g(x)$ to the graph of $y = f\big(g(x)\big)$?

6 Two functions $\ell(x)$ and $q(x)$ are defined by $\ell(x) = x - 3$ and $q(x) = x^2$.

a Find the values of $\ell\big(q(-1)\big)$, $q\big(\ell(-1)\big)$, $\ell\big(\ell(-1)\big)$ and $q\big(q(-1)\big)$.

b Find:

i $\ell\big(q(x)\big)$ **ii** $q\big(\ell(x)\big)$ **iii** $\ell\big(\ell(x)\big)$ **iv** $q\big(q(x)\big)$

c Determine the domains and ranges of:

i $\ell\big(q(x)\big)$ **ii** $q\big(\ell(x)\big)$

d What transformation maps the graph of $y = q(x)$ to the graph of $y = q\big(\ell(x)\big)$?

e What transformation maps the graph of $y = q(x)$ to the graph of $y = \ell\big(q(x)\big)$?

DEVELOPMENT

7 Suppose that $F(x) = 4x$ and $G(x) = \sqrt{x}$.

a Find the values of $F\big(G(25)\big)$, $G\big(F(25)\big)$, $F\big(F(25)\big)$ and $G\big(G(25)\big)$.

b Find $F\big(G(x)\big)$.

c Find $G\big(F(x)\big)$.

d Hence show that $F\big(G(x)\big) = 2G\big(F(x)\big)$.

e State the domain and range of $F\big(G(x)\big)$.

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8 Two functions f and h are defined by $f(x) = -x$ and $h(x) = \frac{1}{x}$.

a Find the values of $f\big(h\big(-\tfrac{1}{4}\big)\big)$, $h\big(f\big(-\tfrac{1}{4}\big)\big)$, $f\big(f\big(-\tfrac{1}{4}\big)\big)$ and $h\big(h\big(-\tfrac{1}{4}\big)\big)$.

b Show that for all $x \neq 0$:

i $f\big(h(x)\big) = h\big(f(x)\big)$ **ii** $f\big(f(x)\big) = h\big(h(x)\big)$

c Write down the domain and range of $f\big(h(x)\big)$.

d Describe how the graph of $h(x)$ is transformed to obtain the graph of $h\big(f(x)\big)$.

9 Suppose that $f(x) = -5 - |x|$ and $g(x) = \sqrt{x}$.

a Find $f\big(g(x)\big)$, state its domain and range, and sketch its graph.

b Explain why $g\big(f(x)\big)$ is the empty function.

10 a Show that if $f(x)$ and $g(x)$ are odd functions, then $g\big(f(x)\big)$ is odd.

b Show that if $f(x)$ is an odd function and $g(x)$ is even, then $g\big(f(x)\big)$ is even.

c Show that if $f(x)$ is an even function, then $g\big(f(x)\big)$ is even.

CHALLENGE

11 Find the composite functions $g\big(f(x)\big)$ and $g\big(f(x)\big)$.

a $f(x) = 4$, for all x, and $g(x) = 7$, for all x.

b $f(x) = x$, $g(x)$ any function.

12 Let $f(x) = 2x + 3$ and $g(x) = 5x + b$, where b is a constant.

a Find expressions for $g\big(f(x)\big)$ and $f\big(g(x)\big)$.

b Hence find the value of b so that $g\big(f(x)\big) = f\big(g(x)\big)$, for all x.

13 a Let $f(x)$ be any function, and let $g(x) = x - a$, where a is a constant. Describe each composite graph as a transformation of the graph of $y = f(x)$.

i $y = g\big(f(x)\big)$ **ii** $y = f\big(g(x)\big)$

b Let $f(x)$ be any function, and let $g(x) = -x$. Describe each composite graph as a transformation of the graph of $y = f(x)$.

i $y = g\big(f(x)\big)$ **ii** $y = f\big(g(x)\big)$

14 Let $f(x) = 2x + 3$ and $g(x) = ax + b$, where b is a constant.

a Find expressions for $g\big(f(x)\big)$ and $f\big(g(x)\big)$.

b Hence find the values of a and b so that $g\big(f(x)\big) = x$, for all x.

c Show that if a and b have these values, then $f\big(g(x)\big) = x$, for all x.

15 Let $f(x) = x^2 + x - 3$ and $g(x) = |x|$.

a Find $f\big(g(0)\big)$, $g\big(f(0)\big)$, $f\big(g(-2)\big)$ and $g\big(f(-2)\big)$.

b Write an expression for $f\big(g(x)\big)$ without any use of absolute value, given that:

i $x \geq 0$ **ii** $x < 0$

16 Let $L(x) = x + 1$ and $Q(x) = x^2 + 2x$.

a State the ranges of $L(x)$ and $Q(x)$.

b Find $L\big(Q(x)\big)$ and determine its range.

c Find $Q\big(L(x)\big)$ and determine its range.

d Find the zeroes of $Q\big(L(x)\big)$.

e Show that $Q\big(L(\tfrac{1}{x+1})\big) = \dfrac{(x+2)(3x+4)}{(x+1)^2}$, provided that $x \neq -1$.

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Chapter 4 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 4 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 **a** Copy and complete the table of values for $y = x^2$ and $y = (x - 2)^2$.

x	-2	-1	0	1	2	3	4
x^2							
$(x - 2)^2$							

b Sketch the two graphs and state the vertex of each.

c What transformation maps $y = x^2$ to $y = (x - 2)^2$?

2 Consider the parabola $y = x^2 - 2x$.

a Show that when this is reflected in the y-axis the equation becomes $y = x^2 + 2x$.

b Copy and complete the table of values for $y = x^2 - 2x$ and $y = x^2 + 2x$.

x	-3	-2	-1	0	1	2	3
$x^2 - 2x$							
$x^2 + 2x$							

c Sketch the two parabolas and state the vertex of each.

3 Evaluate:

a $|-7|$ **b** $|4|$ **c** $|3 - 8|$
d $|-2 - (-5)|$ **e** $|-2| - |-5|$ **f** $|13 - 9 - 16|$

4 Solve for x:

a $|x| = 5$ **b** $|3x| = 18$ **c** $|x - 2| = 4$
d $|x + 3| = 2$ **e** $|2x - 3| = 5$ **f** $|3x - 4| = 7$

5 Explain how to shift the graph of $y = x^2$ to obtain each function.

a $y = x^2 + 5$ **b** $y = x^2 - 1$ **c** $y = (x - 3)^2$ **d** $y = (x + 4)^2 + 7$

6 Write down the equation of the monic quadratic with vertex:

a $(1, 0)$ **b** $(0, -2)$ **c** $(-1, 5)$ **d** $(4, -9)$

7 Write down the centre and radius of each circle. Shifting may help locate the centre.

a $x^2 + y^2 = 1$ **b** $(x + 1)^2 + y^2 = 4$
c $(x - 2)^2 + (y + 3)^2 = 5$ **d** $x^2 + (y - 4)^2 = 64$

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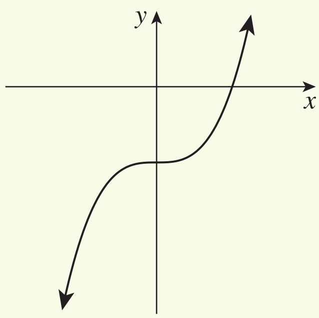
8 In each case, find the function obtained by the given reflection or rotation.

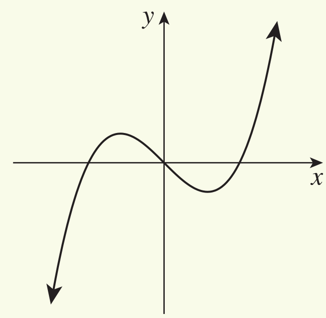
a $y = x^3 - 2x + 1$: reflect in the y-axis
b $y = x^2 - 3x - 4$: reflect in the x-axis
c $y = 2^x - x$: rotate 180° about the origin
d $y = \sqrt{9 - x^2}$: reflect in the y-axis

9 Classify each function $y = f(x)$ as odd, even or neither.

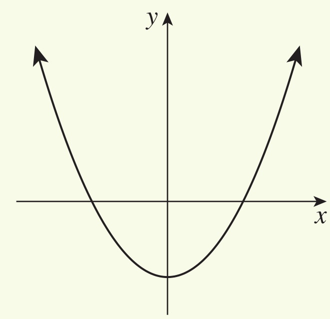
a



b



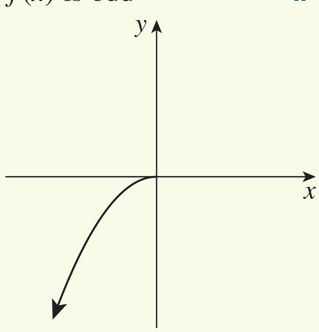
c



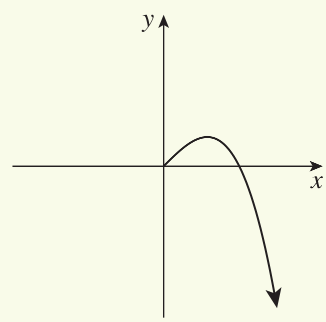
10 In each diagram below, complete the graph so that:

i $f(x)$ is odd **ii** $f(x)$ is even.

a



b



11 Sketch each graph by shifting $y = |x|$, or by using a table of values. Mark all x- and y-intercepts.

a $y = |x| - 2$ **b** $y = |x - 2|$ **c** $y = |x + 2|$ **d** $y = |x| + 2$

12 Sketch each graph by finding the x-intercept and y-intercept and then using symmetry. Perhaps also use a table of values to confirm the graph.

a $y = |3x + 9|$ **b** $y = -|2x - 8|$ **c** $y = |4x + 13|$

13 Solve these absolute value equations:

a $|3x| = 15$ **b** $|x + 4| = 5$ **c** $|x + 4| = -5$ **d** $|5 - x| = 7$
e $|2x + 7| = 9$ **f** $|3x - 8| = 4$ **g** $|7x + 2| = 0$ **h** $|x^2 - 25| = 0$

14 Find $f(-x)$ for each function, and then decide whether the function is odd, even or neither.

a $f(x) = x + 3$ **b** $f(x) = 2x^2 - 5$ **c** $f(x) = \dfrac{1}{x}$ **d** $f(x) = \dfrac{x}{x^2 + 1}$

15 For each parabola, complete the square to find the coordinates of the vertex.

a $y = x^2 - 2x + 5$ **b** $y = x^2 + 4x - 3$ **c** $y = 2x^2 + 8x + 11$ **d** $y = -x^2 + 6x + 1$

16 Use completion of the square to help sketch the graph of each quadratic function. Indicate the vertex and all intercepts with the axes.

a $y = x^2 + 2x + 3$ **b** $y = x^2 - 4x + 1$ **c** $y = 2 + 2x - x^2$ **d** $y = x^2 - x - 1$

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17 Complete the squares to find the centre and radius of each circle.

a $x^2 + y^2 - 2y = 3$ **b** $x^2 + 6x + y^2 + 8 = 0$

c $x^2 - 4x + y^2 + 6y - 3 = 0$ **d** $x^2 + y^2 - 8x + 14y = 35$

18 Consider the cubic with equation $y = x^3 - x$.

a Use an appropriate substitution to show that when the graph of this function is shifted right 1 unit the result is $y = x^3 - 3x^2 + 2x$.

b [Technology]

Plot both cubics using graphing software to confirm the outcome.

19 Given that $f(x) = 5x - 2$ and $g(x) = x^2 + 3$, find:

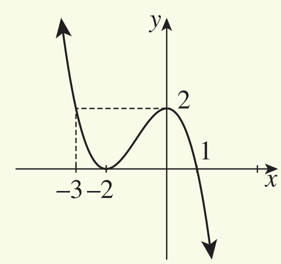
a $f\big(g(0)\big)$ **b** $g\big(f(0)\big)$ **c** $f\big(g(4)\big)$

d $g\big(f(4)\big)$ **e** $f\big(g(a)\big)$ **f** $g\big(f(a)\big)$

20 Find the domain and range of the composite functions $f\big(g(x)\big)$ and $f\big(g(x)\big)$ given that:

a $f(x) = x - 1$ and $g(x) = \sqrt{x}$ **b** $f(x) = \dfrac{1}{x}$ and $g(x) = x^2 + 1$

21 The graph drawn to the right shows the curve $y = f(x)$. Use this graph to sketch the following.



a $y = f(x + 1)$ **b** $y = f(x) + 1$

c $y = f(x - 1)$ **d** $y = f(x) - 1$

e $y = f(-x)$ **f** $y = -f(x)$

g $y = -f(-x)$

22 [A revision medley of curve sketches]

Sketch each set of graphs on a single pair of axes, showing all significant points. Use transformations, tables of values, or any other convenient method.

a $y = 2x$, $y = 2x + 3$, $y = 2x - 2$

b $y = -\frac{1}{2}x$, $y = -\frac{1}{2}x + 1$, $y = -\frac{1}{2}x - 2$

c $y = x + 3$, $y = 3 - x$, $y = -x - 3$

d $y = (x - 2)^2 - 1$, $y = (x + 2)^2 - 1$, $y = -(x + 2)^2 + 1$

e $y = x^2$, $y = (x + 2)^2$, $y = (x - 1)^2$

f $(x - 1)^2 + y^2 = 1$, $(x + 1)^2 + y^2 = 1$, $x^2 + (y - 1)^2 = 1$

g $y = x^2 - 1$, $y = 1 - x^2$, $y = 4 - x^2$

h $y = (x + 2)^2$, $y = (x + 2)^2 - 4$, $y = (x + 2)^2 + 1$

i $y = -|x|$, $y = -|x| + 1$, $y = |x - 2|$

j $y = \sqrt{x}$, $y = \sqrt{x} + 1$, $y = \sqrt{x + 1}$

k $y = 2^x$, $y = 2^x - 1$, $y = 2^{x-1}$

l $y = \dfrac{1}{x}$, $y = \dfrac{1}{x - 2}$, $y = \dfrac{1}{x + 1}$

m $y = x^3$, $y = x^3 - 1$, $y = (x - 1)^3$

n $y = x^4$, $y = (x - 1)^4$, $y = x^4 + 1$

o $y = \sqrt{x}$, $y = -\sqrt{x}$, $y = 2 - \sqrt{x}$

p $y = 2^{-x}$, $y = 2^{-x} - 2$, $y = 2 - 2^x$

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5 Trigonometry

Trigonometry is important in science principally because the graphs of the sine and cosine functions are waves. Waves appear everywhere in the natural world, for example as water waves, as sound waves, or as the electromagnetic waves that are responsible for radio, heat, light, ultraviolet radiation, X-rays and gamma rays. In quantum mechanics, a wave is associated with every particle.

Trigonometry began in classical times, however, in practical situations such as building, surveying and navigation. It used the relationships between the angles and the side lengths in a triangle, and its name comes from the Greek words *trigonon*, ‘triangle’, and *metron*, ‘measure’. This chapter develops the trigonometric functions and their graphs from the geometry of triangles and circles, and applies the trigonometric functions in practical problems.

Some of this chapter will be new to most readers, in particular the extension of the trigonometric functions to angles of any magnitude, the graphs of these functions, trigonometric identities and equations, and three-dimensional problems.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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5A Trigonometry with right-angled triangles

This section and Section 5B will review the definitions of the trigonometric functions for acute angles, and apply them to problems involving right-angled triangles.

Pythagoras' theorem and similarity

You will know from your previous study that the trigonometry of triangles begins with these two fundamental ideas.

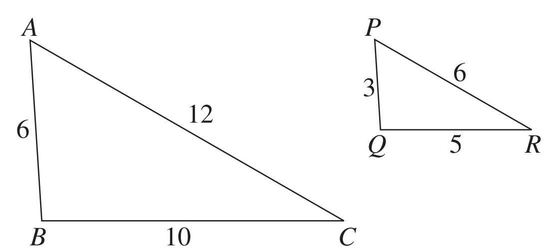
First, Pythagoras' theorem tells us how to find the third side of a right-angled triangle. The theorem is the best-known of all theorems in mathematics, and has been mentioned several times already in earlier chapters. Here is what it says:

The square on the hypotenuse of a right-angled triangle is the sum of the squares on the other two sides.

The diagrams below provide a very simple proof. Can you work out how the four shaded triangles have been pushed around inside the square to prove the theorem?

Secondly, similarity is required even to define the trigonometric functions, because each function is defined as the ratio of two sides of a triangle.

- Two figures are called *congruent* if one can be obtained from the other by translations, rotations and reflections.
- They are called *similar* if enlargements are allowed as well.

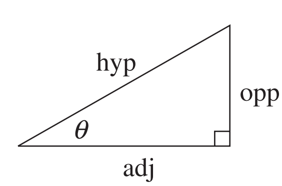


In two similar figures:

matching angles are equal, and matching sides are in ratio.

The trigonometric functions for acute angles

Let θ be any acute angle, $0° < \theta < 90°$. Construct a right-angled triangle with an acute angle θ, and label the sides:



hyp — the *hypotenuse*, the side opposite the right angle,
opp — the side *opposite* the angle θ,
adj — the third side, *adjacent* to θ but is not the hypotenuse.

1 THE TRIGONOMETRIC FUNCTIONS FOR AN ACUTE ANGLE θ

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\operatorname{cosec}\theta = \frac{\text{hyp}}{\text{opp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

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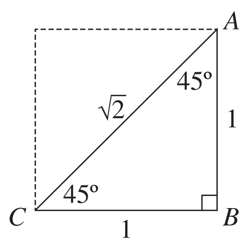
Any two triangles with angles of 90° and θ are similar, by the AA similarity test. Hence the values of the six trigonometric functions are the same, whatever the size of the triangle. The full names of the six trigonometric functions are:

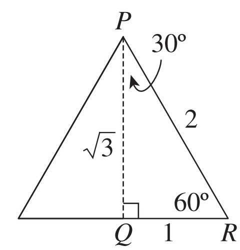
sine, cosine, tangent, cosecant, secant, cotangent.

Question **14** in Exercise 5A gives some clues about these names, and the way in which the functions were originally defined.

Special angles

The values of the trigonometric functions for the three acute angles 30°, 45° and 60° can be calculated exactly using half a square and half an equilateral triangle, and applying Pythagoras' theorem.





Take half a square with side length 1.
The resulting right-angled triangle ABC has two angles of 45°.
By Pythagoras' theorem, the hypotenuse AC has length $\sqrt{2}$

Take half an equilateral triangle with side length 2 by dropping an altitude.
The resulting right-angled ΔPQR has angles of 60° and 30°.
By Pythagoras' theorem, $PQ = \sqrt{3}$.

Applying the definitions in Box 1 gives the values in the table below.

2 A TABLE OF EXACT VALUES

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$

Trigonometric functions of other angles

A calculator is usually used to approximate trigonometric functions of other angles. Alway check first that the calculator is in degrees mode — there is usually a key labelled $\boxed{\text{mode}}$ or $\boxed{\text{DRG}}$ or something similar. Later, you will be swapping between degrees mode and radian mode (ignore the 'grads' unit).

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Make sure also that you can enter angles in degrees, minutes and seconds, and that you can convert decimal output to degrees, minutes and seconds — there is usually a key labelled $\boxed{\circ\,'\,''}$ or $\boxed{\text{DMS}}$ or something similar. Check that you can perform these two procedures:

$$\sin 53°47' \doteqdot 0.8068 \qquad \text{and} \qquad \sin\theta = \tfrac{5}{8}, \qquad \text{so } \theta \doteqdot 38°41'.$$

The reciprocal trigonometric functions

The functions cosecant, secant and cotangent can mostly be avoided by using the sine, cosine and tangent functions.

3 AVOIDING THE RECIPROCAL TRIGONOMETRIC FUNCTIONS

The three reciprocal trigonometric functions can mostly be avoided because

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

Finding an unknown side of a right triangle

Calculators only have the sine, cosine and tangent functions, so it is best to use only these three functions in problems.

4 TO FIND AN UNKNOWN SIDE OF A RIGHT-ANGLED TRIANGLE

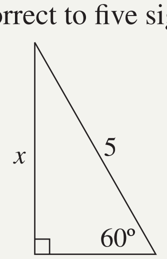
1 Start by writing $\dfrac{\text{unknown side}}{\text{known side}} = \ldots$ (place the unknown at top left).

2 Complete the RHS with sin, cos or tan, or the reciprocal of one of these.

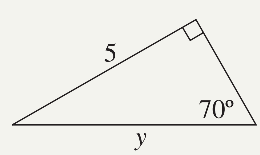
Example 1 — 5A

Find the side marked with a pronumeral in each triangle. Give the answer in exact form if possible, or else correct to five significant figures.

a



b



SOLUTION

a

$$\frac{x}{5} = \sin 60° \qquad \left(\frac{\text{opposite}}{\text{hypotenuse}}\right)$$

$$\boxed{\times 5} \qquad x = 5\sin 60°$$

$$= \frac{5\sqrt{3}}{2}$$

b

$$\frac{y}{5} = \frac{1}{\sin 70°} \qquad \left(\frac{\text{hypotenuse}}{\text{opposite}}\right)$$

$$\boxed{\times 5} \qquad y = \frac{5}{\sin 70°}$$

$$\doteqdot 5.3209$$

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Finding an unknown angle of a right triangle

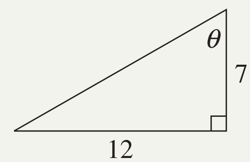
As before, use only sin, cos and tan.

5 FINDING AN UNKNOWN ANGLE, GIVEN TWO SIDES OF A RIGHT-ANGLED TRIANGLE

Work out from the known sides which one of $\cos\theta$, $\sin\theta$ or $\tan\theta$ to use.

Example 2 5A

Find θ in the triangle drawn to the right.



SOLUTION

The given sides are the opposite and the adjacent sides, so $\tan\theta$ is known.

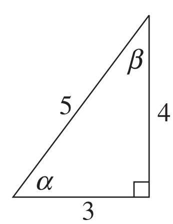
$$\tan\theta = \frac{12}{7} \quad \left(\frac{\text{opposite}}{\text{adjacent}}\right)$$

$$\theta \doteqdot 59°45'.$$

Exercise 5A

FOUNDATION

1 Find the following values for the triangle on the right:



a $\cos\alpha$ **b** $\tan\beta$
c $\sin\alpha$ **d** $\cos\beta$
e $\sin\beta$ **f** $\tan\alpha$

2 Use your calculator to find, correct to four decimal places:

a $\sin 24°$ **b** $\cos 61°$ **c** $\tan 35°$ **d** $\sin 87°$
e $\tan 2°$ **f** $\cos 33°$ **g** $\sin 1°$ **h** $\cos 3°$

3 Use your calculator to find, correct to four decimal places:

a $\tan 57°30'$ **b** $\cos 32°24'$ **c** $\tan 78°40'$ **d** $\cos 16°51'$
e $\sin 43°6'$ **f** $\sin 5°50'$ **g** $\sin 8'$ **h** $\tan 57'$

4 Use your calculator to find the acute angle θ, correct to the nearest degree, if:

a $\tan\theta = 4$ **b** $\cos\theta = 0.7$ **c** $\sin\theta = \frac{1}{5}$
d $\sin\theta = 0.456$ **e** $\cos\theta = 2$ **f** $\cos\theta = \frac{7}{9}$
g $\tan\theta = 1\frac{3}{4}$ **h** $\sin\theta = 1.1$

5 Use your calculator to find the acute angle α, correct to the nearest minute, if:

a $\cos\alpha = \frac{3}{4}$ **b** $\tan\alpha = 2$ **c** $\sin\alpha = 0.1$
d $\tan\alpha = 0.3$ **e** $\sin\alpha = 0.7251$ **f** $\cos\alpha = \frac{7}{13}$

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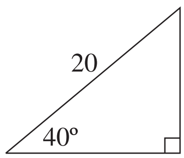
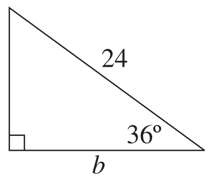
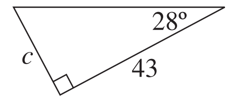
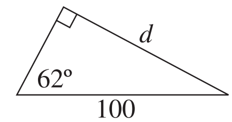
6 Find the value of each pronumeral, correct to the nearest whole number.

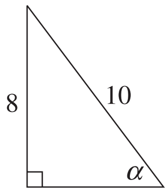
a 

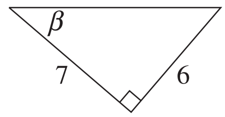
b 

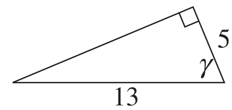
c 

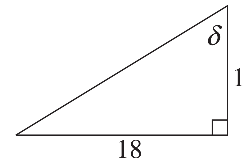
d 

7 Find, correct to the nearest degree, the size of each angle marked with a pronumeral.

a 

b 

c 

d 

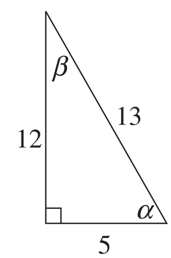
DEVELOPMENT

8 From the diagram opposite, write down the values of:

a $\sin\alpha$ **b** $\tan\beta$

c $\sec\beta$ **d** $\cot\alpha$

e $\operatorname{cosec}\alpha$ **f** $\sec\alpha$

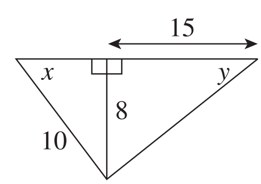


9 **a** Use Pythagoras' theorem to find the unknown side in each of the two right-angled triangles in the diagram opposite.

b Write down the values of:

i $\cos y$ **ii** $\sin x$ **iii** $\cot x$

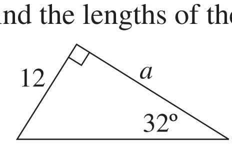
iv $\operatorname{cosec} y$ **v** $\sec x$ **vi** $\cot y$

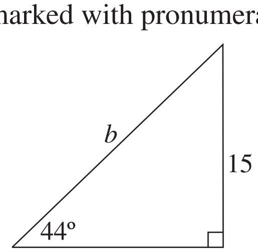


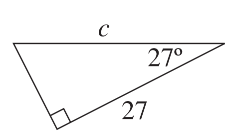
10 Draw the two special triangles containing the acute angles 30°, 60° and 45°. Hence write down these exact values:

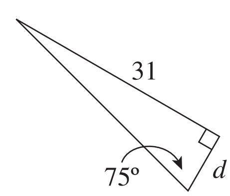
a $\sin 60°$ **b** $\tan 30°$ **c** $\cos 45°$ **d** $\sec 60°$ **e** $\operatorname{cosec} 45°$ **f** $\cot 30°$

11 Find the lengths of the sides marked with pronumerals, correct to one decimal place.

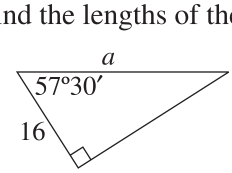
a 

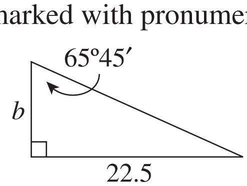
b 

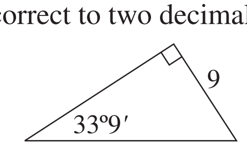
c 

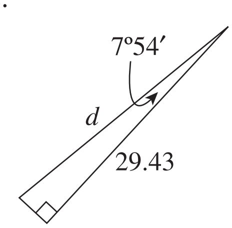
d 

12 Find the lengths of the sides marked with pronumerals, correct to two decimal places.

a 

b 

c 

d 

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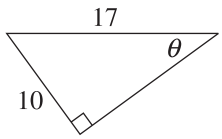
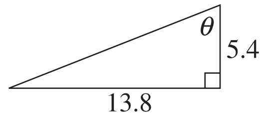
13 Find θ, correct to the nearest minute, in each diagram below.

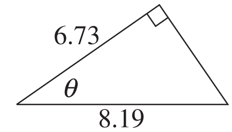
a



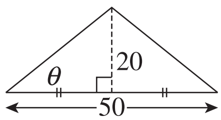
b



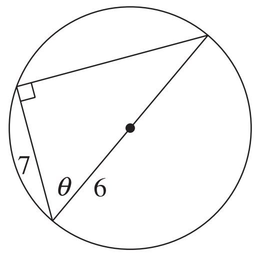
c



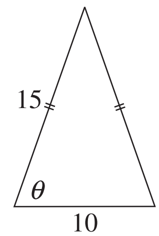
d



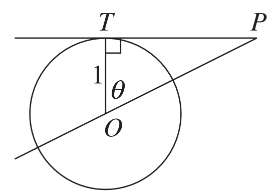
e



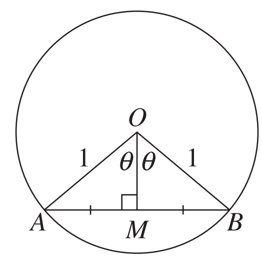
f



14 a The names *tangent* and *secant* refer to the lengths of the tangent TP and secant OP in the diagram to the right, where the tangent TP subtends an angle θ at the centre O of the circle of radius 1. Show that $TP = \tan\theta$ and $OP = \sec\theta$.



b In classical times, the sine function was usually defined to be the length of the semi-chord AM in the diagram to the right, where the semi-chord AM subtends an angle θ at the centre O of a circle of radius 1. Show that $AM = \sin\theta$.



CHALLENGE

15 It is given that α is an acute angle and that $\tan\alpha = \frac{\sqrt{5}}{2}$.

a Draw a right-angled triangle (of any size), one of whose angles is α, and show this information.

b Use Pythagoras' theorem to find the length of the unknown side.

c Hence write down the exact values of $\sin\alpha$ and $\cos\alpha$.

d Show that $\sin^2\alpha + \cos^2\alpha = 1$.

16 Suppose that β is an acute angle and $\sec\beta = \frac{\sqrt{11}}{3}$.

a Find the exact values of:

i $\operatorname{cosec}\beta$ **ii** $\cot\beta$

b Show that $\operatorname{cosec}^2\beta - \cot^2\beta = 1$.

17 Find without using a calculator, the value of:

a $\sin 45^\circ \cos 45^\circ + \sin 30^\circ$ **b** $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$

c $1 + \tan^2 60^\circ$ **d** $\operatorname{cosec}^2 30^\circ - \cot^2 30^\circ$

18 Without using a calculator, show that:

a $1 + \tan^2 45^\circ = \sec^2 45^\circ$ **b** $2\sin 30^\circ \cos 30^\circ = \sin 60^\circ$

c $\cos^2 60^\circ - \cos^2 30^\circ = -\frac{1}{2}$ **d** $\dfrac{2\tan 30^\circ}{1 - \tan^2 30^\circ} = \tan 60^\circ$

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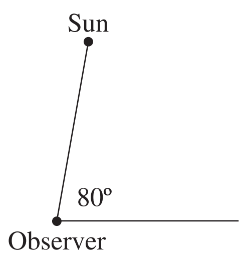
5B Problems involving right-angled triangles

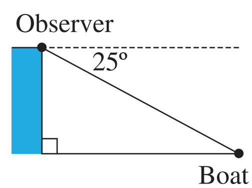
The trigonometry developed so far can be used to solve practical problems involving right-angled triangles. The examples below are typical of problems involving compass bearings and angle of elevation or depression.

When the figure has two or more triangles, always name the triangle that you are working in.

Angles of elevation and depression

Angles of elevation and depression are always measured from the horizontal, and are always acute angles.





The *angle of elevation* of the sun in the diagram above is 80°, because the angle at the observer between the sun and the horizontal is 80°.

For an observer on top of the cliff, the *angle of depression* of the boat is 25°, because the angle at the observer between boat and horizontal is 25°.

Example 3 5B

From a plane flying at 9000 metres above level ground, I can see a church at an angle of depression of 35° from the cabin of the plane. Find how far the church is from the plane, correct to the nearest 100 metres:

a measured along the ground,

b measured along the line of sight.

SOLUTION

The situation is illustrated in the diagram by ΔPGC.

a $\dfrac{GC}{9000} = \tan 55°$

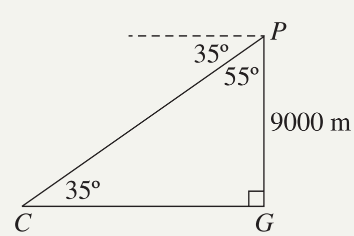
$GC = 9000 \tan 55°$

$\fallingdotseq 12\,900$ metres

b $\dfrac{PC}{9000} = \dfrac{1}{\cos 55°}$

$PC = \dfrac{9000}{\cos 55°}$

$\fallingdotseq 15\,700$ metres



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Example 4 5B

A walker on level ground is 1 kilometre from the base of a 300-metre vertical cliff.

a Find the angle of elevation of the top of the cliff, correct to the nearest minute.

b Find the line-of-sight distance to the top of the cliff, correct to the nearest metre.

SOLUTION

The situation is illustrated by ΔCWB in the diagram to the right.

a $\tan\theta = \dfrac{CB}{WB}$

$= \dfrac{300}{1000}$

$= \frac{3}{10}$

$\theta \doteqdot 16°42'$

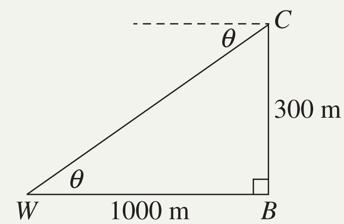
b Using Pythagoras' theorem,

$CW^2 = CB^2 + WB^2$

$= 1000^2 + 300^2$

$= 1\,090\,000$

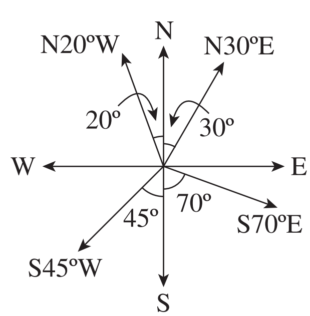
$CW \doteqdot 1044$ metres.



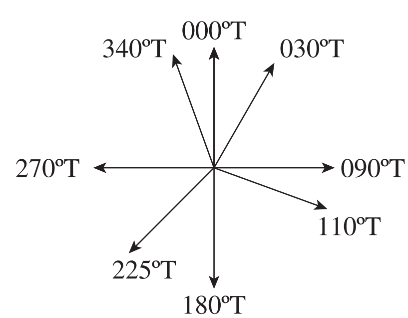
Compass bearings and true bearings

Compass bearings are based on north, south, east and west. Any other direction is specified by the deviation from north or south towards the east or west. The diagram to the left below gives four examples. Note that S45°W can also be written simply as SW (that is, south west).

True bearings are measured clockwise from north (not anticlockwise as in the coordinate plane). The diagram on the right below gives the same four directions expressed as true bearings. Three digits are used, even for angles less than 100°.



Compass bearing



True bearing

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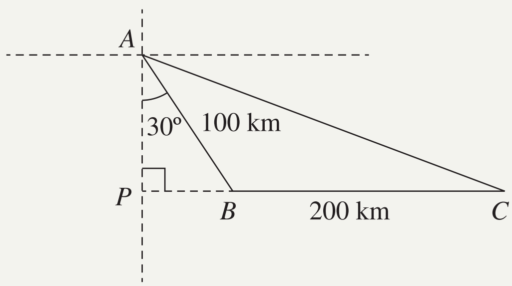
Example 5 5B

[Compass bearings and true bearings]

A plane flies at 400 km per hour, and flies from A to B in the direction S30°E for 15 minutes. The plane then turns sharply to fly due east for 30 minutes to C.

a Find how far south and east of A the point B is.

b Find the true bearing of C from A, correct to the nearest degree.



SOLUTION

a The distances AB and BC are 100 km and 200 km respectively.

Working in ΔPAB,

$$\frac{PB}{100} = \sin 30°$$

$$PB = 100 \sin 30°$$
$$= 50\text{ km},$$

and $AP = 100 \cos 30°$
$$= 50\sqrt{3}\text{ km}.$$

Hence B is $50\sqrt{3}$ km south of A and 50 km east.

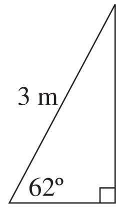
b Using opposite over adjacent in ΔPAC,

$$\tan \angle PAC = \frac{PC}{AP}$$
$$= \frac{50 + 200}{50\sqrt{3}}$$
$$= \frac{5}{\sqrt{3}}$$
$$\angle PAC \doteqdot 71°.$$

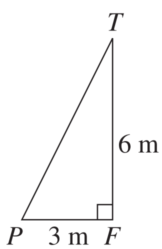
Hence the bearing of C from A is about 109°T.

Exercise 5B FOUNDATION

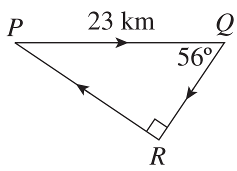
1 A ladder of length 3 metres is leaning against a wall and is inclined at 62° to the ground. How far does it reach up the wall? (Answer in metres correct to two decimal places.)



2 Determine, correct to the nearest degree, the angle of elevation of the top T of a 6-metre flagpole FT from a point P on level ground 3 metres from F.



3 Ben cycles from P to Q to R and then back to P in a road race. Find the distance he has ridden, correct to the nearest kilometre.



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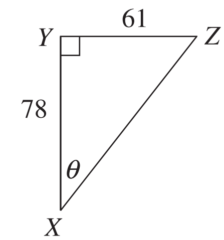
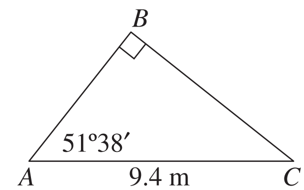
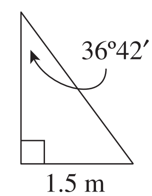
4 A ship sails 78 nautical miles due north from X to Y, then 61 nautical miles due east from Y to Z. Find θ, the bearing of Z from X, correct to the nearest degree.



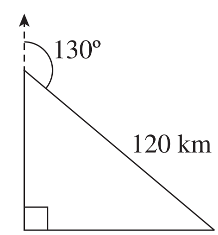
5 A tree snapped into two sections AB and BC in high winds and then fell. The section BA is inclined at $51°38'$ to the horizontal and AC is 9.4 metres long. Find the height of the original tree, in metres correct to one decimal place.



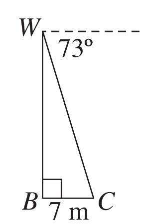
6 A ladder makes an angle of $36°42'$ with a wall, and its foot is 1.5 metres out from the base of the wall. Find the length of the ladder, in metres correct to one decimal place.



7 Eleni drives 120 km on a bearing of 130°T. She then drives due west until she is due south of her starting point. How far is she from her starting point, correct to the nearest kilometre?



8 John is looking out of the window W at a car C parked on the street below. If the angle of depression of C from W is 73° and the car is 7 metres from the base B of the building, find the height WB of the window, correct to the nearest metre.



DEVELOPMENT

9 A ladder of length 5 metres is placed on level ground against a vertical wall. If the foot of the ladder is 1.5 metres from the base of the wall, find the angle at which the ladder is inclined to the ground, correct to the nearest degree.

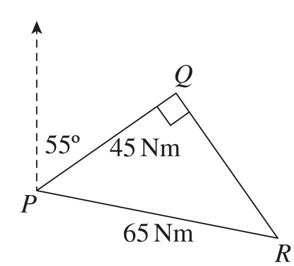
10 Find, correct to the nearest tenth of a metre, the height of a tower, if the angle of elevation of the top of a tower is $64°48'$ from a point on horizontal ground 10 metres from the base of a tower.

11 A boat is 200 metres out to sea from a vertical cliff of height 40 metres. Find the angle of depression of the boat from the top of the cliff, correct to the nearest degree.

12 Port Q is 45 nautical miles from port P on a bearing of 055°T. Port R is 65 nautical miles from port P, and $\angle PQR = 90°$.

a Find $\angle QPR$ to the nearest degree.

b Hence find the bearing of R from P, correct to the nearest degree.



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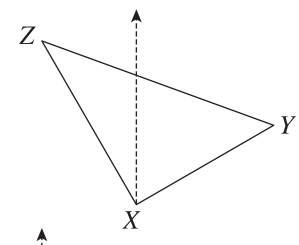
13 The bearings of towns Y and Z from town X are 060°T and 330°T respectively.

a Show that $\angle ZXY = 90°$.

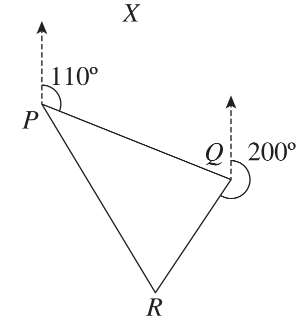
b Given that town Z is 80 km from town X and that $\angle XYZ = 50°$, find, correct to the nearest kilometre, how far town Y is from town X.



14 A ship leaves port P and travels 150 nautical miles to port Q on a bearing of 110°T. It then travels 120 nautical miles to port R on a bearing of 200°T.

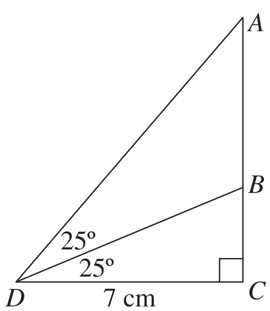
a Explain why $\angle PQR = 90°$.

b Find the bearing of port R from port P, correct to the nearest degree.



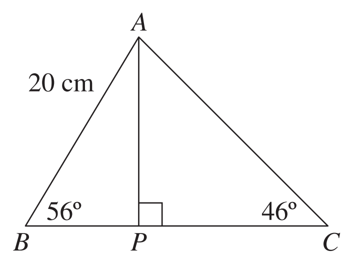
CHALLENGE

15 a



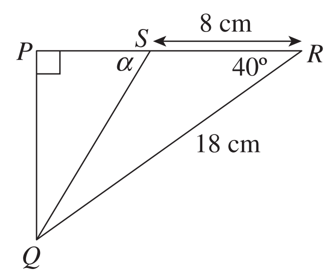
Show that $AC = 7\tan 50°$ and $BC = 7\tan 25°$, and hence find the length AB, correct to 1 mm.

b



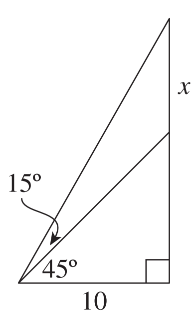
Show that $AP = 20\sin 56°$, and hence find the length of PC, giving your answer correct to 1 cm.

c



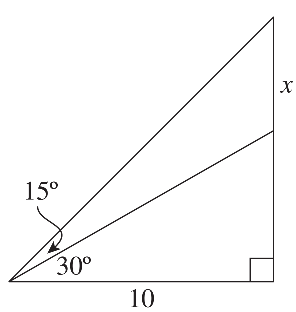
Show that $PR = 18\cos 40°$, find an expression for PQ, and hence find the angle α, correct to the nearest minute.

16 a



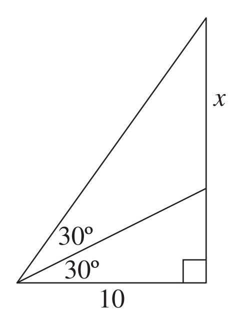
Show that $x = 10(\sqrt{3} - 1)$.

b



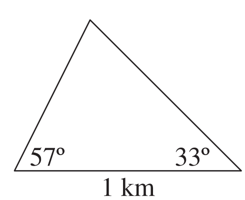
Show that $x = \frac{10}{3}(3 - \sqrt{3})$.

c



Show that $x = \frac{20}{3}\sqrt{3}$.

17 From the ends of a straight horizontal road 1 km long, a balloon directly above the road is observed to have angles of elevation of 57° and 33° respectively. Find, correct to the nearest metre, the height of the balloon above the road.



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5C Three-dimensional trigonometry

Trigonometry is based on triangles, which are two-dimensional objects. When trigonometry is applied to a three-dimensional problem, the diagram must be broken up into a collection of triangles in space, and trigonometry applied for each triangle in turn. A carefully drawn diagram is always essential.

Two new ideas about angles are needed — the angle between a line and a plane, and the angle between two planes. Pythagoras' theorem remains fundamental.

Trigonometry and Pythagoras' theorem in three dimensions

Here are the steps in a successful approach to a three-dimensional problem.

6 TRIGONOMETRY AND PYTHAGORAS' THEOREM IN THREE DIMENSIONS

1. Draw a careful sketch of the situation.
2. Note carefully all the triangles in the figure.
3. Mark, or note, all right angles in these triangles.
4. Always name the triangle you are working with.

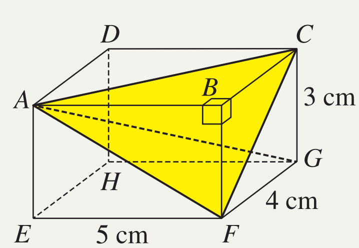
Example 6 5C

The rectangular prism sketched below has the following dimensions:

$EF = 5\,\text{cm}$ and $FG = 4\,\text{cm}$ and $CG = 3\,\text{cm}$

a Use Pythagoras' theorem in ΔCFG to find the length of the diagonal FC.
b Similarly find the lengths of the diagonals AC and AF.
c Use Pythagoras' theorem in ΔACG to find the length of the space diagonal AG.
d Use trigonometry in ΔBAF to find $\angle BAF$ (nearest minute).
e Use trigonometry in ΔGAF to find $\angle GAF$ (nearest minute).

SOLUTION



a In $\Delta\, CFG$, $FC^2 = 3^2 + 4^2$, using Pythagoras,
$FC = 5\,\text{cm}$.

b In $\Delta\, ABC$, $AC^2 = 5^2 + 4^2$, using Pythagoras,
$AC = \sqrt{41}\,\text{cm}$.
In $\Delta\, ABF$, $AF^2 = 5^2 + 3^2$, using Pythagoras,
$AF = \sqrt{34}\,\text{cm}$.

c In ΔACG, the angle $\angle ACG$ is a right angle, and $AC = \sqrt{41}$ and $CG = 3$.

Hence $AG^2 = AC^2 + CG^2$, using Pythagoras,
$= 41 + 3^2$
$AG = \sqrt{50}$
$= 5\sqrt{2}\,\text{cm}$.

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d In ΔBAF, the angle $\angle ABF$ is a right angle, and $AB = 5$ and $BF = 3$.

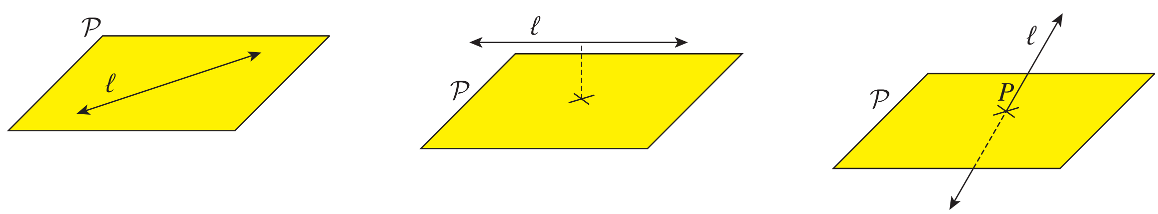
$$\text{Hence } \tan \angle BAF = \frac{BF}{AB}$$
$$= \tfrac{3}{5}$$
$$\angle BAF \doteqdot 30°58'$$

e In ΔGAF, the angle $\angle AFG$ is a right angle, and $AF = \sqrt{34}$ and $FG = 4$.

$$\text{Hence } \tan \angle GAF = \frac{FG}{AF}$$
$$= \frac{4}{\sqrt{34}}$$
$$\angle GAF \doteqdot 34°27'$$

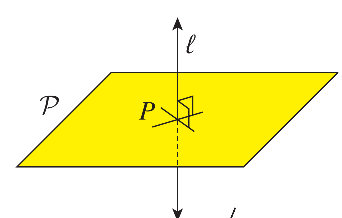
The angle between a line and a plane

In three-dimensional space, a plane $\mathcal{P}$ and a line ℓ can be related in three different ways:



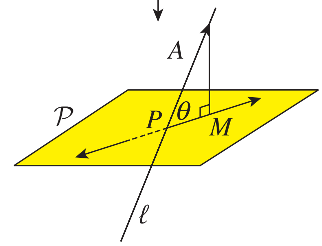
- In the first diagram above, the line lies wholly within the plane.
- In the second diagram, the line never meets the plane. We say that the line and the plane are *parallel*.
- In the third diagram, the line intersects the plane in a single point P.

When the line ℓ meets the plane $\mathcal{P}$ in the single point P, it can do so in two distinct ways.
In the upper diagram, the line ℓ is perpendicular to every line in the plane through P. We say that the line is *perpendicular* to the plane.



In the lower diagram, the line ℓ is not perpendicular to $\mathcal{P}$. To construct the angle θ between the line and the plane:

- choose another point A on the line ℓ
- construct the point M in the plane $\mathcal{P}$ so that $AM \perp \mathcal{P}$.



Then $\angle APM$ is the angle between the plane and the line.

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Example 7 5C

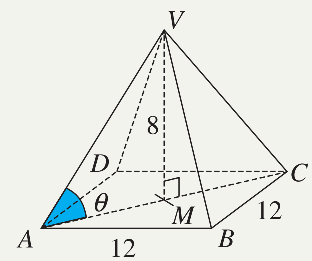
Find the angle between a slant edge and the base in a square pyramid of height 8 metres whose base has side length 12 metres.

SOLUTION

Using Pythagoras' theorem in the base $ABCD$,

$$AC^2 = 12^2 + 12^2$$
$$AC = 12\sqrt{2} \text{ metres.}$$

The perpendicular from the vertex V to the base meets the base at the midpoint M of the diagonal AC.



$$\text{In } \Delta MAV, \tan \angle MAV = \frac{MV}{MA}$$
$$= \frac{8}{6\sqrt{2}}$$
$$\angle MAV \doteqdot 43°19',$$

and this is the angle between the slant edge AV and the base.

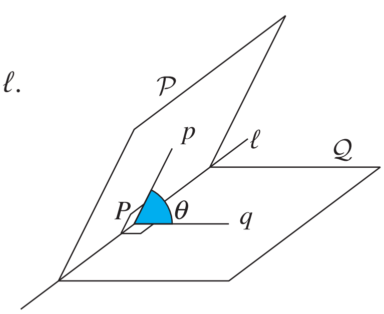
The angle between two planes

In three-dimensional space, two planes that are not parallel intersect in a line ℓ.

To construct the angle between the planes:

- Take any point P on this line of intersection.
- Construct the line p through P perpendicular to ℓ lying in the plane $\mathcal{P}$.
- Construct the line q through P perpendicular to ℓ lying in the plane $\mathcal{Q}$.

The angle between the planes $\mathcal{P}$ and $\mathcal{Q}$ is the angle between the two lines p and q.



Example 8 5C

In the pyramid of Example 7, find the angle between an oblique face of the pyramid and the base.

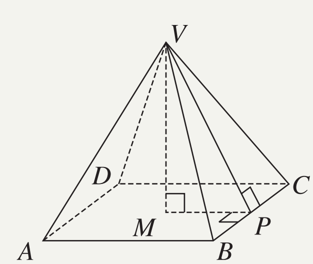
SOLUTION

Let P be the midpoint of the edge BC.

Then $VP \perp BC$ and $MP \perp BC$,

so $\angle VPM$ is the angle between the oblique face and the base.

$$\text{In } \Delta\, VPM, \tan \angle\, VPM = \frac{VM}{PM}$$
$$= \tfrac{8}{6}$$
$$\angle\, VPM \doteqdot 53°8'.$$



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Challenge — three-dimensional problems in which no triangle can be solved

In the following classic problem, there are four triangles forming a tetrahedron, but no triangle can be solved, because no more than two measurements are known in any one of these triangles. The method is to introduce a pronumeral for the height, then work around the figure until *four* measurements are known in terms of h in the base triangle — at this point an equation in h can be formed and solved.

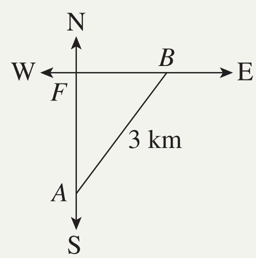
Example 9 5C

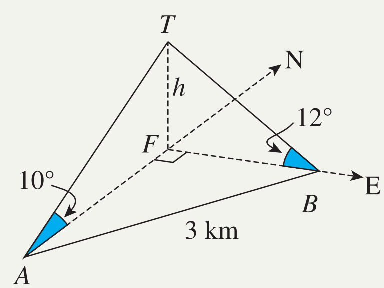
A motorist driving on level ground sees, due north of her, a tower whose angle of elevation is 10°. After driving 3 km further in a straight line, the tower is in the direction due west, with angle of elevation 12°.

a How high is the tower?

b In what direction is she driving?

SOLUTION





Let the tower be TF, and let the motorist be driving from A to B.

a There are four triangles, none of which can be solved.
Let h be the height of the tower.
In ΔTAF, $AF = h\cot 10°$.
In ΔTBF, $BF = h\cot 12°$.
We now have expressions for four measurements in ΔABF, so we can use Pythagoras' theorem to form an equation in h.

In ΔABF,
$$\begin{aligned} AF^2 + BF^2 &= AB^2 \\ h^2\cot^2 10° + h^2\cot^2 12° &= 3^2 \\ h^2(\cot^2 10° + \cot^2 12°) &= 9 \\ h^2 &= \frac{9}{\cot^2 10° + \cot^2 12°} \\ h &\doteqdot 0.407 \text{ km,} \end{aligned}$$

so the tower is about 407 metres high.

b Let $\theta = \angle FAB$, then in ΔAFB,
$$\begin{aligned} \sin\theta &= \frac{FB}{AB} \\ &= \frac{h\cot 12°}{3} \\ \theta &\doteqdot 40°, \end{aligned}$$

so her direction is about N40°E.

The general method of approach

Here is a summary of what has been said about three-dimensional problems (apart from the ideas of angles between lines and planes and between planes and planes).

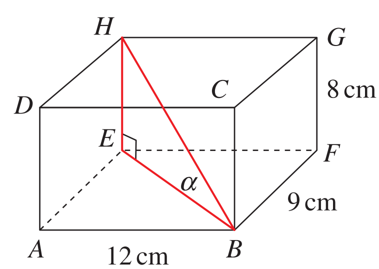
7 THREE-DIMENSIONAL TRIGONOMETRY

1 Draw a careful diagram of the situation, marking all right angles.
2 A plan diagram, looking down, is usually a great help.
3 Identify every triangle in the diagram, to see whether it can be solved.
4 If one triangle can be solved, then work from it around the diagram until the problem is solved.
5 If no triangle can be solved, assign a pronumeral to what is to be found, then work around the diagram until an equation in that pronumeral can be formed and solved.

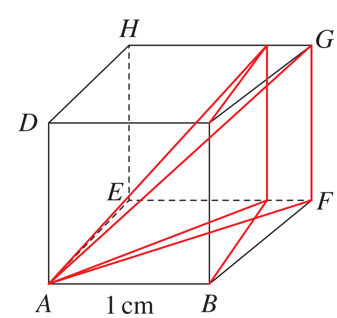
Exercise 5C

FOUNDATION

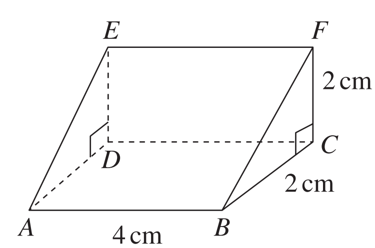
1 The diagram opposite shows a rectangular prism.
  a Use Pythagoras' theorem to find the length of the base diagonal BE.
  b Hence find the length of the prism diagonal BH.
  c Find, correct to the nearest degree, the angle α that BH makes with the base of the prism.



2 The diagram opposite shows a cube.
  a Write down the size of:
    i $\angle ABF$ **ii** $\angle AFG$ **iii** $\angle ABG$
  b Use Pythagoras' theorem to find the exact length of:
    i AF **ii** AG
  c Hence find, correct to the nearest degree:
    i $\angle GAF$ **ii** $\angle AGB$



3 The diagram opposite shows a triangular prism.
  a Find the exact length of:
    i AC **ii** AF
  b What is the size of $\angle ACF$?
  c Find $\angle AFC$, correct to the nearest degree.



DEVELOPMENT

4 The diagram to the right shows a square pyramid. The point C is the centre of the base, and TC is perpendicular to the base.

a Write down the size of:

i $\angle CMQ$ **ii** $\angle TCM$ **iii** $\angle TCQ$

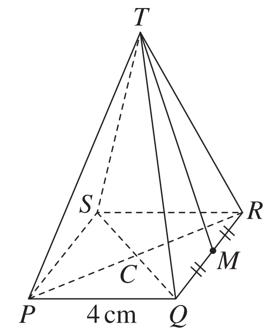
b Find the length of:

i CM **ii** CQ

c Find, correct to the nearest degree:

i the angle between a side face and the base,

ii the angle between a slant edge and the base.

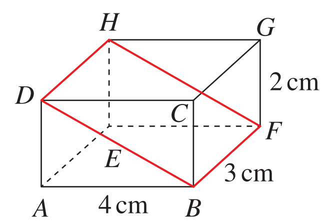


5 The diagram opposite shows a rectangular prism.

a Write down the size of:

i $\angle ABF$ **ii** $\angle DBF$

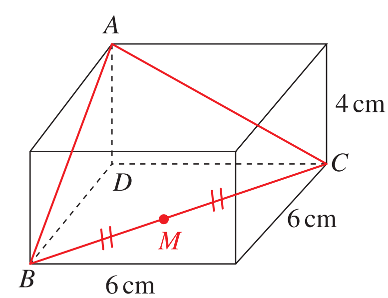
b Find, correct to the nearest degree, the angle that the diagonal plane $DBFH$ makes with the base of the prism.



6 The diagram opposite shows a square prism. The plane ABC is inside the prism, and M is the midpoint of the base diagonal BC.

a Find the exact length of MD.

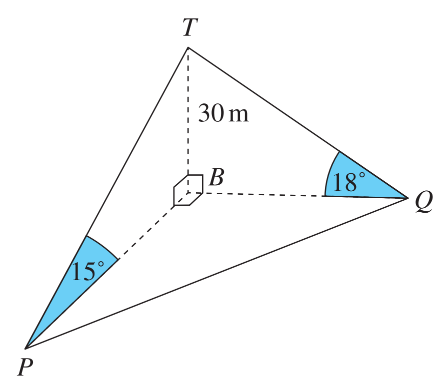
b Hence find, correct to the nearest degree, the angle that the plane ABC makes with the base of the prism.



7 Two landmarks P and Q on level ground are observed from the top T of a vertical tower BT of height 30 m. Landmark P is due south of the tower, while landmark Q is due east of the tower. The angles of elevation of T from P and Q are 15° and 18° respectively.

a Show that $BP = 30 \tan 75°$, and find a similar expression for BQ.

b Find, correct to the nearest metre, the distance between the two landmarks.

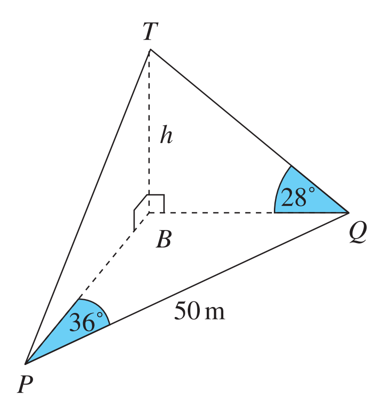


8 A tree BT is due north of an observer at P and due west of an observer at Q. The two observers are 50 m apart and the bearing of Q from P is 36°. The angle of elevation of T from Q is 28°.

a Show that $BQ = 50 \sin 36°$.

b Hence find the height h of the tree, correct to the nearest metre.

c Find the angle of elevation of T from P, correct to the nearest degree.



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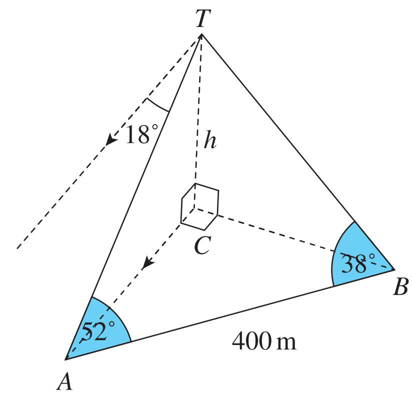
9 Two monuments A and B are 400 m apart on a horizontal plane. The angle of depression of A from the top T of a tall building is $18°$. Also $\angle TAB = 52°$ and $\angle TBA = 38°$.

a Show that $TA = 400\cos 52°$.

b Find the height h of the building, correct to the nearest metre.

c Find the angle of depression of B from T, correct to the nearest degree.



CHALLENGE

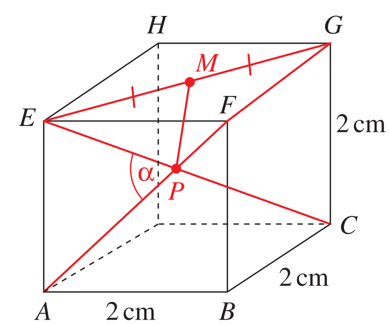
10 The diagram to the right shows a cube of side 2 cm, with diagonals AG and CE intersecting at P. The point M is the midpoint of the face diagonal EG. Let α be the acute angle between the diagonals AG and CE.

a What is the length of PM?

b Find the exact length of EM.

c Find the exact value of $\tan \angle EPM$.

d Hence find α, correct to the nearest minute.



11 A balloon B is due north of an observer P and its angle of elevation is $62°$. From another observer Q 100 metres from P, the balloon is due west and its angle of elevation is $55°$. Let the height of the balloon be h metres and let C be the point on the level ground vertically below B.

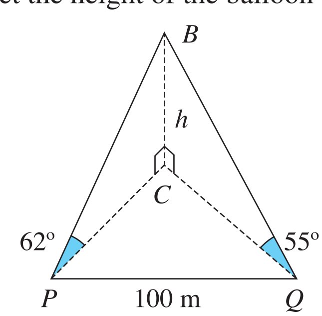
a Show that $PC = h\cot 62°$, and write down a similar expression for QC.

b Explain why $\angle PCQ = 90°$.

c Use Pythagoras' theorem in ΔCPQ to show that

$$h^2 = \frac{100^2}{\cot^2 62° + \cot^2 55°}.$$

d Hence find h, correct to the nearest metre.



12 From a point P due south of a vertical tower, the angle of elevation of the top of the tower is $20°$. From a point Q situated 40 metres from P and due east of the tower, the angle of elevation is $35°$. Let h metres be the height of the tower.

a Draw a diagram to represent the situation.

b Show that $h = \dfrac{40}{\sqrt{\tan^2 70° + \tan^2 55°}}$, and evaluate h, correct to the nearest metre.

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5D Trigonometric functions of a general angle

The definitions of the trigonometric functions given in Section 5A only apply to acute angles, because in a right-angled triangle, both other angles are acute angles.

This section introduces more general definitions based on circles in the coordinate plane (whose equations are Pythagoras' theorem, as we saw in Section 3G). The new definitions will apply to any angle, but will, of course, give the same values as the previous definitions for acute angles.

Putting a general angle on the coordinate plane

Let θ be any angle — possibly negative, possibly obtuse or reflex, possibly greater than 360°. We shall associate with θ a ray with vertex at the origin.

8 THE RAY CORRESPONDING TO θ

- The positive direction of the x-axis is the ray representing the angle 0°.
- For all other angles, rotate this ray anticlockwise through an angle θ.

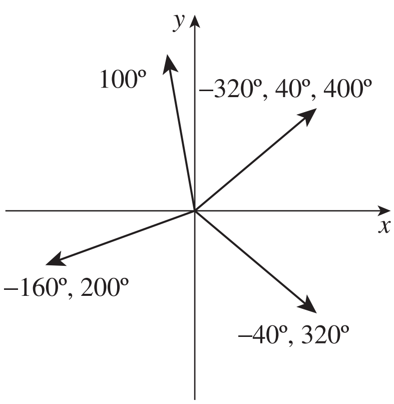
If the angle is negative, the ray is rotated backwards, which means clockwise.

Here are some examples of angles and the corresponding rays. The angles have been written at the ends of the arrows representing the rays.

Notice that one ray can correspond to many angles. For example, all the following angles have the same ray as 40°:

$$\ldots, -680°, -320°, 40°, 400°, 760°, \ldots$$

A given ray thus corresponds to infinitely many angles, all differing by multiples of 360°. The relation from rays to angles is thus one-to-many.



9 CORRESPONDING ANGLES AND RAYS

- To each angle, there corresponds exactly one ray.
- To each ray, there correspond infinitely many angles, all differing from each other by multiples of 360°.

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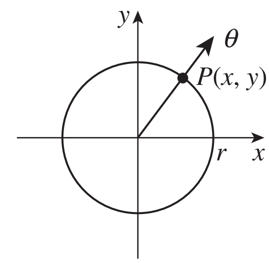
Defining the trigonometric functions for general angles



Let θ be any angle, positive or negative.

Construct a circle with centre the origin and any positive radius r. Let the ray corresponding to θ intersect the circle at the point $P(x, y)$.

The six trigonometric functions are now defined in terms of x, y and r as follows:

10 DEFINITIONS OF THE SIX TRIGONOMETRIC FUNCTIONS

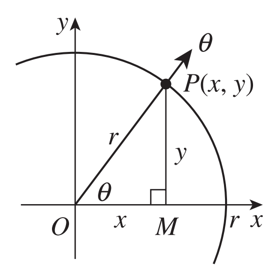
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\operatorname{cosec}\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

Note: We chose r to be 'any positive radius'. If a different radius were chosen, the two figures would be similar, so the lengths x, y and r would stay in the same ratio. Because the definitions depend only on the ratios of the lengths, the values of the trigonometric functions would not change.

In particular, we may use a circle of radius 1 in the definitions. When this is done, however, we would then lose the intuition that a trigonometric function is not a length, but is the ratio of two lengths.

Agreement with the earlier definition



Let θ be an acute angle.
Construct the ray corresponding to θ.
Let the perpendicular from P meet the x-axis at M.
Then $\theta = \angle POM$, so relating the sides to the angle θ,

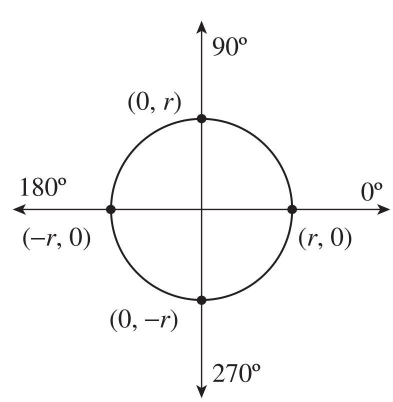
$$\text{hypotenuse} = OP = r, \qquad \text{opposite} = PM = y, \qquad \text{adjacent} = OM = x.$$

Hence the old and the new definitions are in agreement.

Note: Most people find that the diagram above is the easiest way to learn the new definitions of the trigonometric functions. Take the old definitions in terms of hypotenuse, opposite and adjacent sides, and make the replacements,

$$\text{hypotenuse} \longleftrightarrow r, \qquad \text{opposite} \longleftrightarrow y, \qquad \text{adjacent} \longleftrightarrow x.$$

Boundary angles



Integer multiples of 90°, that is

$$\ldots, -90°, 0°, 90°, 180°, 270°, 360°, 450°, \ldots$$

are called *boundary angles* because they lie on the boundaries between quadrants.

The values of the trigonometric functions at these boundary angles are not always defined, and are 0, 1 or −1 when they are defined. The diagram to the right can be used to calculate them, and the results are shown in the table on next page (where the star * indicates that the value is undefined).

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11 THE BOUNDARY ANGLES

θ	$0°$	$90°$	$180°$	$270°$
x	r	0	$-r$	0
y	0	r	0	$-r$
r	r	r	r	r
$\sin\theta$	0	1	0	-1
$\cos\theta$	1	0	-1	0
$\tan\theta$	0	*	0	*
$\operatorname{cosec}\theta$	*	1	*	-1
$\sec\theta$	1	*	-1	*
$\cot\theta$	*	0	*	0

In practice, the answer to any question about the values of the trigonometric functions at these boundary angles should be read off the graphs of the functions. These graphs need to be known very well indeed.

The domains of the trigonometric functions

The trigonometric functions are defined everywhere except where the denominator is zero.

12 DOMAINS OF THE TRIGONOMETRIC FUNCTIONS

- $\sin\theta$ and $\cos\theta$ are defined for all angles θ.
- $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$, that is, when
  $$\theta = \ldots, -90°, 90°, 270°, 450°, \ldots$$
- $\cot\theta$ and $\operatorname{cosec}\theta$ are undefined when $y = 0$, that is, when
  $$\theta = \ldots, -180°, 0°, 180°, 360°, \ldots$$

Exercise 5D

FOUNDATION

1 On a number plane, draw rays representing the following angles.

a $40°$ **b** $110°$ **c** $190°$
d $290°$ **e** $420°$ **f** $500°$

2 On another number plane, draw rays representing the following angles.

a $-50°$ **b** $-130°$ **c** $-250°$
d $-350°$ **e** $-440°$ **f** $-550°$

3 For each of the angles in Question **1**, write down the size of the negative angle between $-360°$ and $0°$ that is represented by the same ray.

4 For each of the angles in Question **2**, write down the size of the positive angle between $0°$ and $360°$ that is represented by the same ray.

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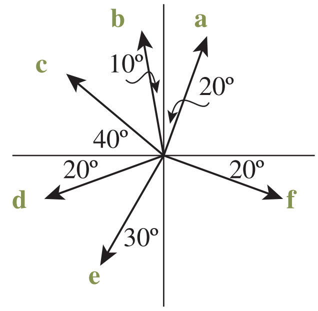
5 Write down two positive angles between 0° and 720° and two negative angles between −720° and 0° that are represented by each of the rays in the diagram to the right.

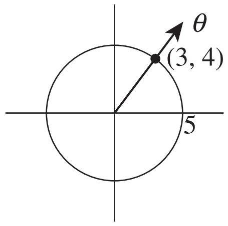


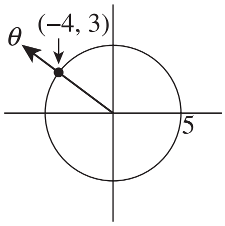
DEVELOPMENT

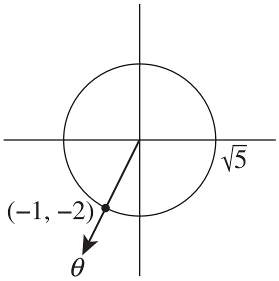
6 Use the definitions

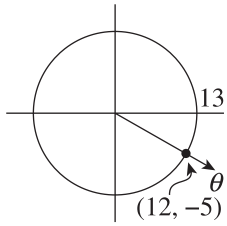
$$\sin\theta = \frac{y}{r} \quad \text{and} \quad \cos\theta = \frac{x}{r} \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

to write down the values of the six trigonometric ratios of the angle θ in each diagram.

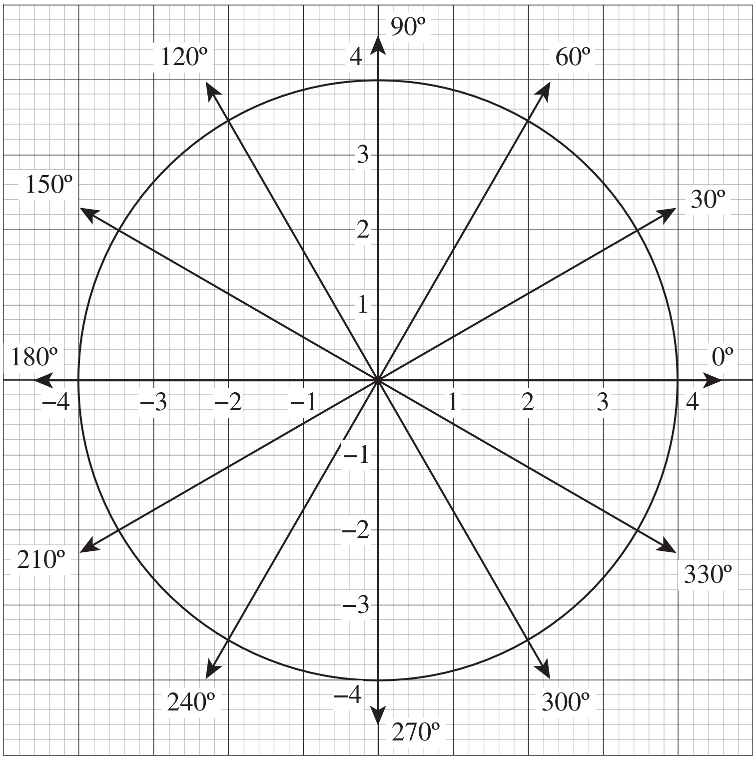
a 

b 

c 

d 

7 [The graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$]
The diagram shows angles from 0° to 360° at 30° intervals. The circle has radius 4 units.



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a Use the diagram and the definitions of the three trigonometric ratios to complete the following table. Measure the values of x and y correct to two decimal places, and use your calculator only to perform the necessary divisions.

θ	$-30°$	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$	$390°$
x															
y															
r															
$\sin\theta$															
$\cos\theta$															
$\tan\theta$															

b Use your calculator to check the accuracy of the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ that you obtained in part **a**.

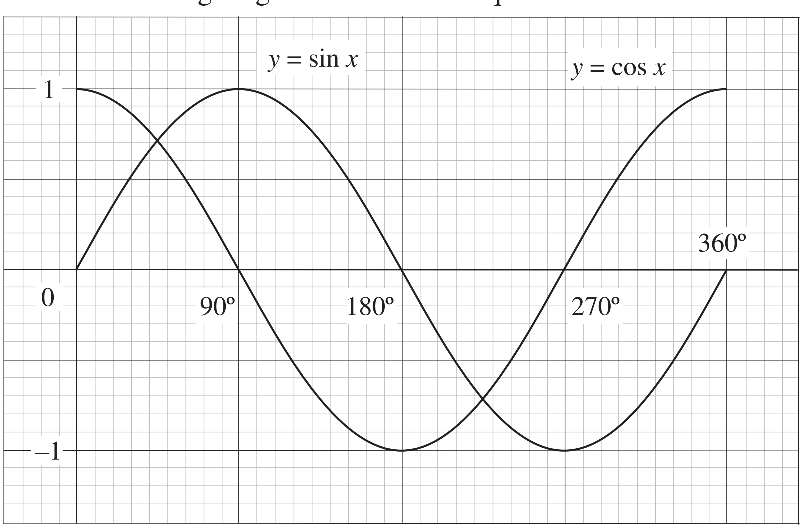
c Using the table of values in part **a**, graph the curves $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ as accurately as possible on graph paper. Use the following scales:

On the horizontal axis, let 2 mm represent 10°.

On the vertical axis, let 2 cm represent 1 unit.

CHALLENGE

8 Use the following diagram to answer the questions.



a Read off the diagram, correct to two decimal places where necessary, the values of:

i $\cos 60°$ **ii** $\sin 210°$ **iii** $\sin 72°$ **iv** $\cos 18°$
v $\sin 144°$ **vi** $\cos 36°$ **vii** $\cos 153°$ **viii** $\sin 27°$
ix $\sin 234°$ **x** $\cos 306°$

b Find from the graphs two values of x between 0° and 360° for which:

i $\sin x = 0.5$ **ii** $\cos x = -0.5$ **iii** $\sin x = 0.9$ **iv** $\cos x = 0.6$
v $\sin x = 0.8$ **vi** $\cos x = -0.8$ **vii** $\sin x = -0.4$ **viii** $\cos x = -0.3$

c Find two values of x between 0° and 360° for which $\sin x = \cos x$.

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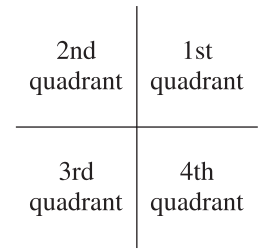
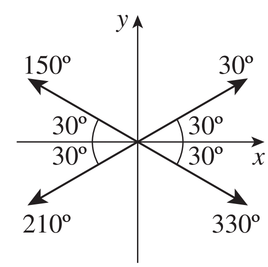
5E Quadrant, sign, and related acute angle

Symmetry is an essential aspect of trigonometric functions. In this section we will use symmetry to express the values of the trigonometric functions of any angle in terms of trigonometric functions of acute angles.



The diagram shows the conventional anticlockwise numbering of the four quadrants of the coordinate plane. Acute angles are in the first quadrant, obtuse angles are in the second, and reflex angles are in the third a fourth.

The quadrant and the related acute angle



The diagram to the right shows the four rays corresponding to the four angles 30°, 150°, 210° and 330°.
These four rays lie in each of the four quadrants of the plane, and they all make the same acute angle of 30° with the x-axis. The four rays are thus the reflections of each other in the two axes.

13 QUADRANT AND RELATED ACUTE ANGLE

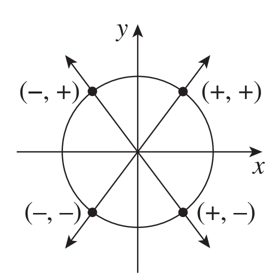
Let θ be any angle.
- The *quadrant* of θ is the quadrant (1, 2, 3 or 4) in which the ray lies.
- The *related acute angle* of θ is the acute angle between the ray and the x-axis.

Each of the four angles above has the same related acute angle 30°. Notice that θ and its related angle are only the same when θ is an acute angle.

The signs of the trigonometric functions

The signs of the trigonometric functions depend only on the signs of x and y. (The radius r is always positive.) The signs of x and y depend in turn only on the quadrant in which the ray lies. Thus we can easily compute the signs of the trigonometric functions from the accompanying diagram and the definitions.

Quadrant	1	2	3	4	
x	+	−	−	+	
y	+	+	−	−	
r	+	+	+	+	
$\sin\theta$	+	+	−	−	
$\cos\theta$	+	−	−	+	
$\tan\theta$	+	−	+	−	
$\operatorname{cosec}\theta$	+	+	−	−	(same as $\sin\theta$)
$\sec\theta$	+	−	−	+	(same as $\cos\theta$)
$\cot\theta$	+	−	+	−	(same as $\tan\theta$)



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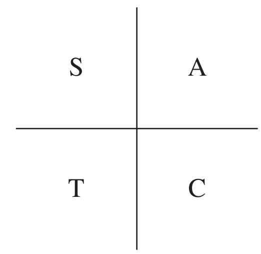
In NSW, these results are usually remembered by the phrase:

14 SIGNS OF THE TRIGONOMETRIC FUNCTIONS

‘All Stations To Central’

indicating that the four letters A, S, T and C are placed successively in the four quadrants as shown. The significance of the letters is:

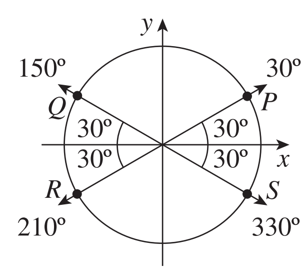
A means *all* six functions are positive.
S means only *sine* (and cosecant) are positive.
T means only *tangent* (and cotangent) are positive.
C means only *cosine* (and secant) are positive.



Three graphs of the trigonometric functions were constructed in Exercise 5D, and all six are drawn together at the end of this section. Study each of them to see how the table of signs above, and the ASTC rule, agree with your observations about when the graph is above the x-axis and when it is below.

The angle and the related acute angle

In the diagram to the right, a circle of radius r has been added to the earlier diagram showing the four angles 30°, 150°, 210° and 330°.



The four points P, Q, R and S where the four rays meet the circle are all reflections of each other in the x-axis and y-axis. Because of this symmetry, the coordinates of these four points are identical, apart from their signs.

Hence the trigonometric functions of these angles will all be the same too, except that the signs may be different.

15 THE ANGLE AND THE RELATED ACUTE ANGLE

- The trigonometric functions of any angle θ are the same as the trigonometric functions of its related acute angle, apart from a possible change of sign.
- The sign is best found using the ASTC diagram.

Evaluating the trigonometric functions of any angle

This gives a straightforward way of evaluating the trigonometric functions of any angle.

16 TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

Draw a quadrant diagram then:

- Place the ray in the correct quadrant, and use the ASTC rule to work out the sign of the answer.
- Find the related acute angle, and work out the value of the trigonometric function at this related angle.

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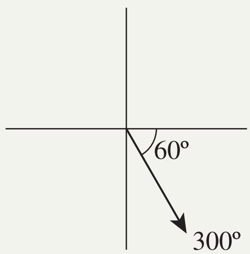
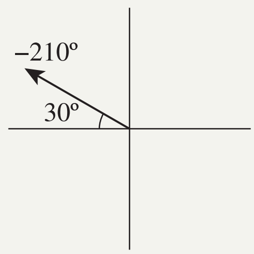
Example 10 5E

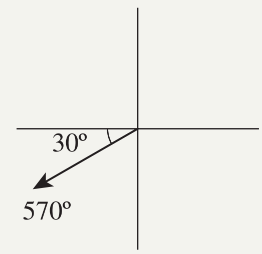
Find the exact value of:

a $\tan 300°$ **b** $\sin(-210°)$ **c** $\cos 570°$

SOLUTION







a $300°$ is in quadrant 4,
the related angle is $60°$,
so $\tan 300° = -\tan 60°$
$= -\sqrt{3}$.

b $-210°$ is in quadrant 2,
the related angle is $30°$,
so $\sin(-210°) = +\sin 30°$
$= \frac{1}{2}$.

c $570°$ is in quadrant 3,
the related angle is $30°$,
so $\cos 570° = -\cos 30°$
$= -\dfrac{\sqrt{3}}{2}$.

Note: A calculator will give approximate values of the trigonometric functions without any need to find the related acute angle. It will *not* give exact values, however, when these values involve surds.

General angles with pronumerals

This quadrant-diagram method can be used to generate formulae for expressions such as $\sin(180° + A)$ or $\cot(360° - A)$. The trick is to place A on the quadrant diagram *as if it were acute*.

17 SOME FORMULAE WITH GENERAL ANGLES

$\sin(180° - A) = \sin A$	$\sin(180° + A) = -\sin A$	$\sin(360° - A) = -\sin A$
$\cos(180° - A) = -\cos A$	$\cos(180° + A) = -\cos A$	$\cos(360° - A) = \cos A$
$\tan(180° - A) = -\tan A$	$\tan(180° + A) = \tan A$	$\tan(360° - A) = -\tan A$

Some people prefer to learn this list of identities to evaluate trigonometric functions, but this seems unnecessary when the quadrant-diagram method is so clear.

The graphs of the six trigonometric functions

The diagrams on the next page show the graphs of the six trigonometric functions over a domain extending beyond $-360° \le x \le 360°$. With this extended domain, it becomes clear how the graphs are built up by infinite repetition of a simple element.

The sine and cosine graphs are waves, and they are the basis of all the mathematics that deals with waves. The later trigonometry in this course will mostly deal with these wave properties. These two graphs each repeat every $360°$, and the graphs are therefore said to have a *period* of $360°$.

The piece of the sine wave from $\theta = 0°$ to $\theta = 90°$ is enough to construct the whole sine wave and the whole cosine wave — use reflections, rotations and translations.

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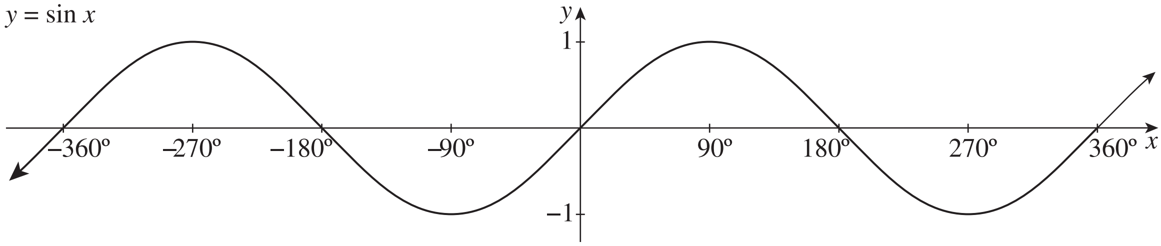
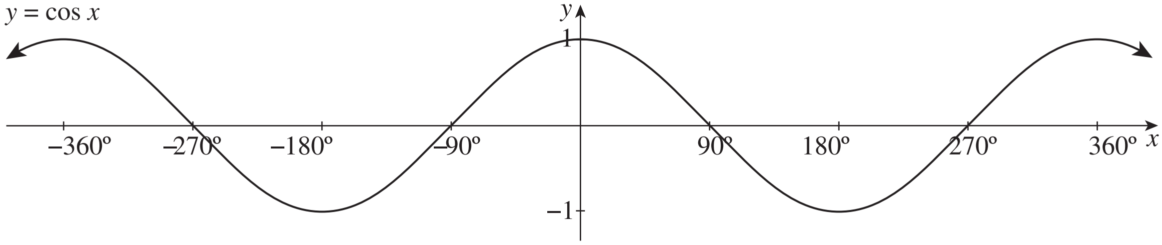
The other four graphs also repeat themselves periodically. The graphs of $\operatorname{cosec} x$ and $\sec x$ each have period 360° because they are reciprocals of $\sin x$ and $\cos x$. The graphs of $\tan x$ and $\cot x$, on the other hand; each have period 180°. This will all be discussed in more detail in Chapter 9.

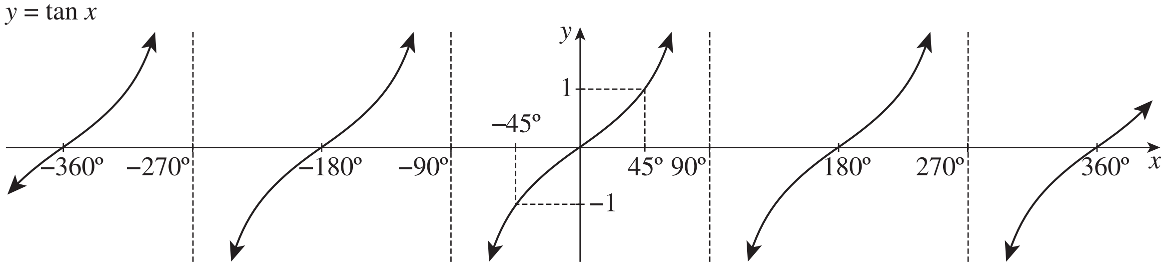
$y = \sin x$



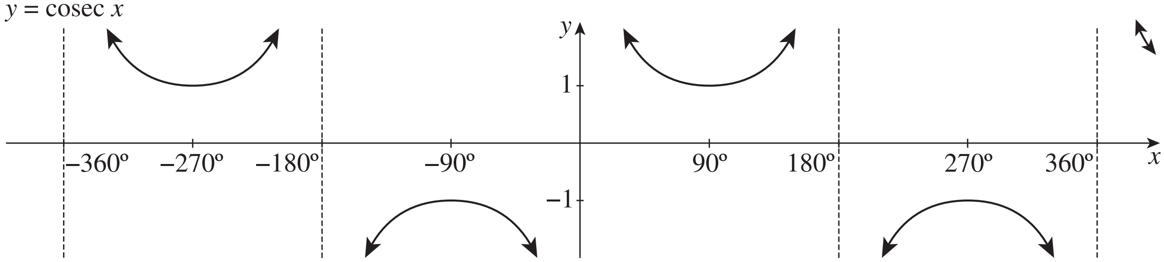
$y = \cos x$



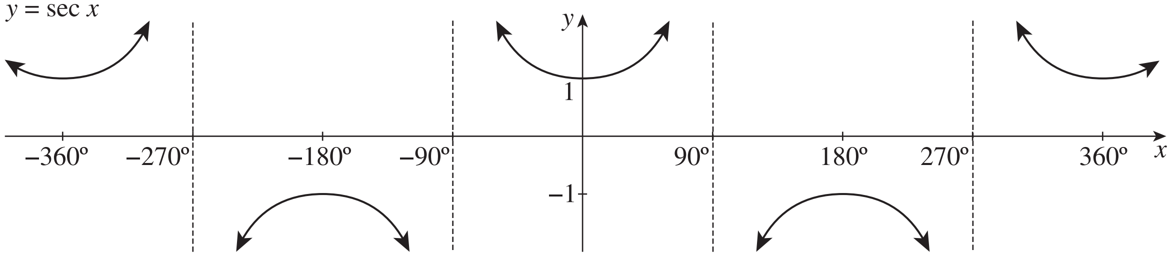
$y = \tan x$



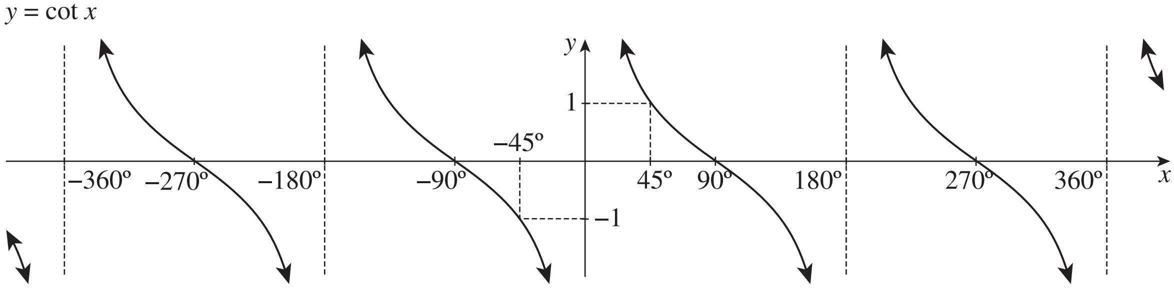
$y = \operatorname{cosec} x$



$y = \sec x$



$y = \cot x$



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Exercise 5E

FOUNDATION

1 Use the ASTC rule to determine the sign (+ or −) of each trigonometric ratio.

a $\sin 20°$	b $\cos 50°$	c $\cos 100°$	d $\tan 140°$
e $\tan 250°$	f $\sin 310°$	g $\sin 200°$	h $\cos 280°$
i $\sin 340°$	j $\cos 350°$	k $\tan 290°$	l $\cos 190°$
m $\tan 170°$	n $\sin 110°$	o $\tan 80°$	p $\cos 170°$

2 Find the related acute angle of each angle.

a $10°$	b $150°$	c $310°$	d $200°$	e $80°$
f $250°$	g $290°$	h $100°$	i $350°$	j $160°$

3 Write each trigonometric ratio as the ratio of an acute angle with the correct sign attached.

a $\tan 130°$	b $\cos 310°$	c $\sin 220°$	d $\tan 260°$
e $\cos 170°$	f $\sin 320°$	g $\cos 185°$	h $\sin 125°$
i $\tan 325°$	j $\sin 85°$	k $\cos 95°$	l $\tan 205°$

4 Use the trigonometric graphs to find the values (if they exist) of these trigonometric ratios of boundary angles.

a $\sin 0°$	b $\cos 180°$	c $\cos 90°$	d $\tan 0°$
e $\sin 90°$	f $\cos 0°$	g $\sin 270°$	h $\tan 270°$
i $\sin 180°$	j $\cos 270°$	k $\tan 90°$	l $\tan 180°$

5 Find the exact value of:

a $\sin 60°$	b $\sin 120°$	c $\sin 240°$	d $\sin 300°$
e $\cos 45°$	f $\cos 135°$	g $\cos 225°$	h $\cos 315°$
i $\tan 30°$	j $\tan 150°$	k $\tan 210°$	l $\tan 330°$

DEVELOPMENT

6 Find the exact value of:

a $\cos 120°$	b $\tan 225°$	c $\sin 330°$	d $\sin 135°$
e $\tan 240°$	f $\cos 210°$	g $\tan 315°$	h $\cos 300°$
i $\sin 225°$	j $\cos 150°$	k $\sin 210°$	l $\tan 300°$

7 Find the exact value of:

a $\operatorname{cosec} 150°$	b $\sec 225°$	c $\cot 120°$
d $\cot 210°$	e $\sec 330°$	f $\operatorname{cosec} 300°$

8 Use the trigonometric graphs to find (if they exist):

a $\sec 0°$	b $\operatorname{cosec} 270°$	c $\sec 90°$
d $\operatorname{cosec} 180°$	e $\cot 90°$	f $\cot 180°$

9 Find the related acute angle of each angle.

a $-60°$	b $-200°$	c $-150°$	d $-300°$
e $430°$	f $530°$	g $590°$	h $680°$

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10 Find the exact value of:

a $\cos(-60°)$ **b** $\sin(-120°)$ **c** $\tan(-120°)$
d $\sin(-315°)$ **e** $\tan(-210°)$ **f** $\cos(-225°)$
g $\tan 420°$ **h** $\cos 510°$ **i** $\sin 495°$
j $\sin 690°$ **k** $\cos 600°$ **l** $\tan 585°$

11 Classify the six trigonometric graphs as one-to-one, many-to-one, one-to-many or many-to-many.

CHALLENGE

12 Given that $\sin 25° \doteqdot 0.42$ and $\cos 25° \doteqdot 0.91$, write down approximate values, without using a calculator, for:

a $\sin 155°$ **b** $\cos 205°$
c $\cos 335°$ **d** $\sin 335°$
e $\sin 205° - \cos 155°$ **f** $\cos 385° - \sin 515°$

13 Given that $\tan 35° \doteqdot 0.70$ and $\sec 35° \doteqdot 1.22$, write down approximate values, without using a calculator, for:

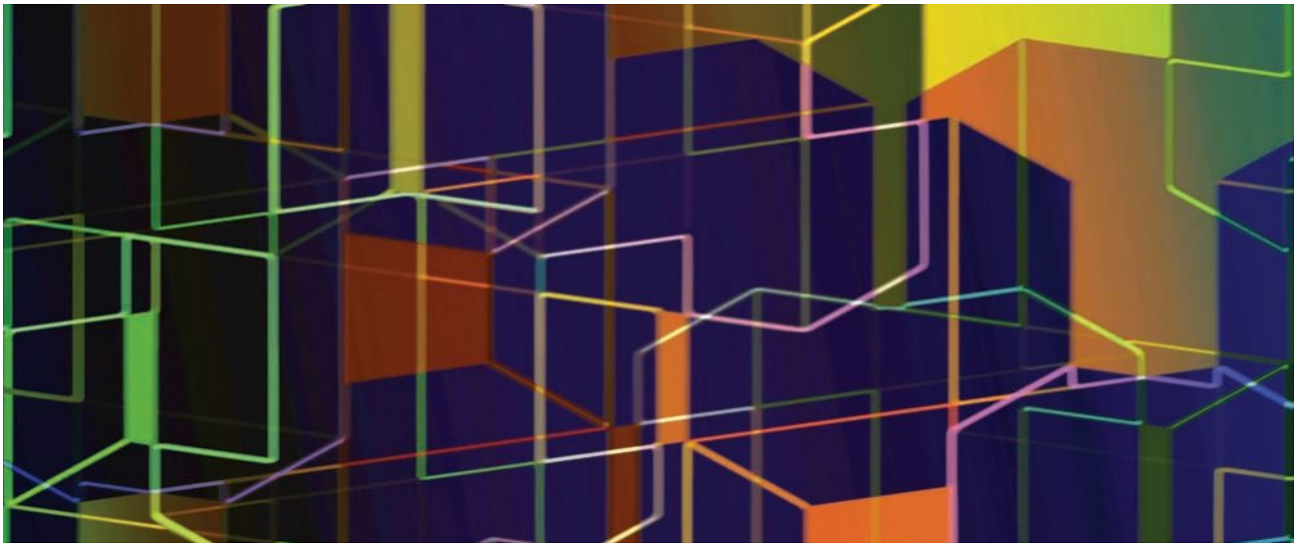
a $\tan 145°$ **b** $\sec 215°$
c $\tan 325°$ **d** $\tan 215° + \sec 145°$
e $\sec 325° + \tan 395°$ **f** $\sec(-145)° - \tan(-215)°$

14 Show by substitution into LHS and RHS that each trigonometric identity is satisfied by the given values of the angles.

a Show that $\sin 2\theta = 2\sin\theta\cos\theta$, when $\theta = 150°$.
b Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, when $\theta = 225°$.
c Show that $\sin(A + B) = \sin A\cos B + \cos A\sin B$, when $A = 300°$ and $B = 240°$.

15 Write as a trigonometric ratio of θ, with the correct sign attached:

a $\sin(-\theta)$ **b** $\cos(-\theta)$ **c** $\tan(-\theta)$
d $\sec(-\theta)$ **e** $\sin(180° - \theta)$ **f** $\sin(360° - \theta)$
g $\cos(180° - \theta)$ **h** $\tan(180° + \theta)$



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5F Given one trigonometric function, find another

When the exact value of one trigonometric function is known for an angle, the exact values of the other trigonometric functions can easily be found using the circle diagram and Pythagoras' theorem.

18 GIVEN ONE TRIGONOMETRIC FUNCTION, FIND ANOTHER

- Place a ray or rays on a circle diagram in the quadrants allowed in the question.
- Complete the triangle and use Pythagoras' theorem to find whichever of x, y and r is missing.

Example 11 5F

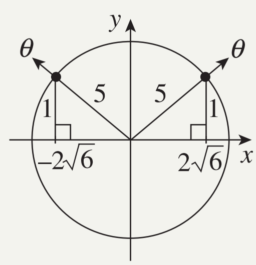
It is known that $\sin\theta = \frac{1}{5}$.

a Find the possible values of $\cos\theta$.

b Find $\cos\theta$ if it is also known that $\tan\theta$ is negative.

SOLUTION

a The angle must be in quadrant 1 or 2, because $\sin\theta$ is positive.



Because $\sin\theta = \frac{y}{r} = \frac{1}{5}$, we can take $y = 1$ and $r = 5$,

so by Pythagoras' theorem, $x = \sqrt{24}$ or $-\sqrt{24}$,

so $\cos\theta = \frac{2\sqrt{6}}{5}$ or $-\frac{2\sqrt{6}}{5}$.

b Because $\tan\theta$ is negative, θ can only be in quadrant 2,

so $\cos\theta = -\frac{2\sqrt{6}}{5}$.

Note: In the diagrams of this section, some of the 'side lengths' on the horizontal and vertical sides of triangles are marked as negative because they are really displacements from the origin rather than lengths. Always trust the quadrants to give you the correct sign.

Exercise 5F FOUNDATION

Note: Diagrams have been drawn for questions **1–4**, and similar diagrams should be drawn for the subsequent questions. Many answers will involve surds, but it is not important to rationalise denominators.

Do not use the calculator at all in this exercise, because you are looking for exact values, not approximations.

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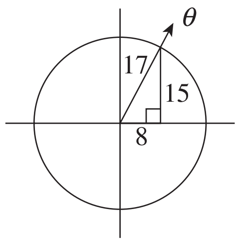
1 Write down the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in each part.

a



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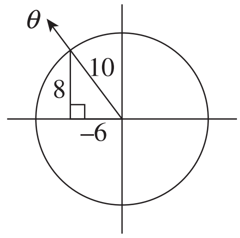

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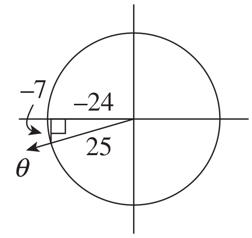
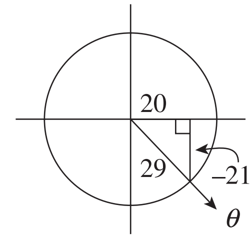
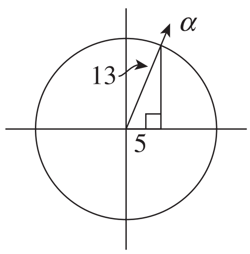


d

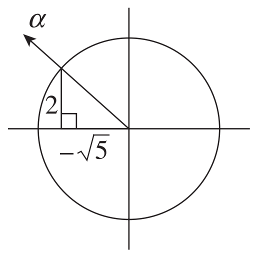


2 In each part use Pythagoras' theorem to find whichever of x, y or r is unknown. Then write down the values of $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$.

a



b



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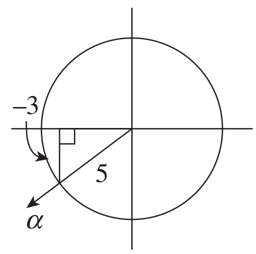

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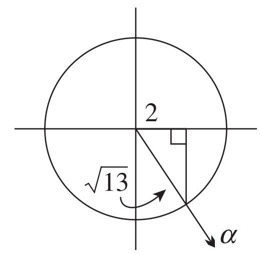
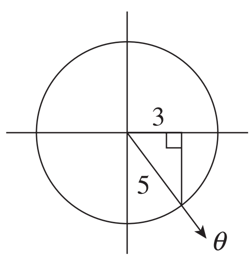


3 **a**

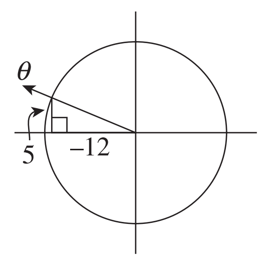


Let $\cos\theta = \frac{3}{5}$, where $270° < \theta < 360°$.

i Find $\sin\theta$.

ii Find $\tan\theta$.

b

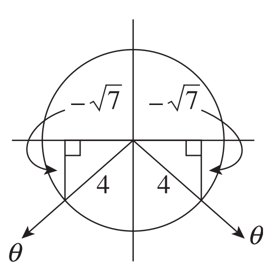


Let $\tan\theta = -\frac{5}{12}$, where θ is obtuse.

i Find $\sin\theta$.

ii Find $\cos\theta$.

4 **a**

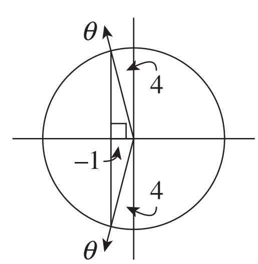


Suppose that $\sin\theta = -\frac{\sqrt{7}}{4}$.

i Find the possible values of $\cos\theta$.

ii Find the possible values of $\tan\theta$.

b



Suppose that $\cos\theta = -\frac{1}{4}$.

i Find the possible values of $\sin\theta$.

ii Find the possible values of $\tan\theta$.

5 Draw similar diagrams for this question and subsequent questions.

a If $\cos\theta = \frac{1}{3}$ and θ is acute, find $\tan\theta$.

b If $\cos\theta = -\frac{4}{5}$ and θ is obtuse, find $\tan\theta$.

c If $\cos\theta = \frac{1}{2}$ and θ is reflex, find $\sin\theta$.

d Find $\cos\theta$ if $\tan\theta = -\frac{2}{3}$ and θ is reflex.

e Find $\sin\theta$, given that $\cos\theta = -\frac{40}{41}$ and $0° \le \theta \le 180°$.

f Find $\tan\theta$, given that $\sin\theta = \frac{1}{\sqrt{5}}$ and $-90° \le \theta \le 90°$.

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DEVELOPMENT

6 In this question, each part has two possible answers.

a If $\tan\alpha = \frac{1}{3}$, find $\sin\alpha$.

b If $\cos\theta = \frac{2}{\sqrt{5}}$, find $\sin\theta$.

c If $\sin\theta = \frac{3}{5}$, find $\cos\theta$.

d Find $\tan\theta$, given that $\cos\theta = -\frac{2}{3}$.

e Find $\tan\theta$, given that $\sin\theta = -\frac{12}{13}$.

f Find $\cos\theta$, given that $\tan\theta = -\frac{2}{\sqrt{3}}$.

7 a If $\cos\alpha = \frac{4}{5}$ and $\sin\alpha < 0$, find $\tan\alpha$.

b If $\tan\theta = -\frac{8}{15}$ and $\sin\theta > 0$, find $\cos\theta$.

c Find $\cos\theta$, given that $\sin\theta = \frac{1}{4}$ and $\tan\theta < 0$.

d Find $\sin\theta$, given that $\tan\theta = \frac{35}{12}$ and $\cos\theta > 0$.

e Find $\tan\theta$, given that $\sin\theta = -\frac{21}{29}$ and $\cos\theta > 0$.

f Find $\sin\theta$, given that $\cos\theta = -\frac{5}{6}$ and $\tan\theta < 0$.

8 a Find $\sec\theta$, given that $\sin\theta = \frac{1}{\sqrt{2}}$.

b Find $\tan\theta$, given that $\sec\theta = -\frac{17}{8}$.

c If $\sec C = -\frac{\sqrt{7}}{\sqrt{3}}$, find $\cot C$.

d If $\cot D = \frac{\sqrt{11}}{5}$, find $\operatorname{cosec} D$.

9 a Find $\sec\theta$, given that $\operatorname{cosec}\theta = \frac{3}{2}$ and θ is obtuse.

b Find $\sec\theta$, given that $\cot\theta = \frac{9}{40}$ and θ is reflex.

c Find $\tan\theta$, given that $\sec\theta = -\frac{17}{8}$ and $0° \le \theta \le 180°$.

d Find $\operatorname{cosec}\theta$, given that $\cot\theta = \frac{2}{\sqrt{3}}$ and $-90° \le \theta \le 90°$.

10 a If $\sin A = -\frac{1}{3}$ and $\tan A < 0$, find $\sec A$.

b If $\operatorname{cosec} B = \frac{7}{3}$ and $\cos B < 0$, find $\tan B$.

c Find $\cot\theta$, given that $\sec\theta = -\sqrt{2}$ and $\operatorname{cosec}\theta < 0$.

d Find $\cos\theta$, given that $\operatorname{cosec}\theta = -\frac{13}{5}$ and $\cot\theta < 0$.

CHALLENGE

11 Given that $\sin\theta = \dfrac{p}{q}$, with θ obtuse and p and q both positive, find $\cos\theta$ and $\tan\theta$.

12 If $\tan\alpha = k$, where $k > 0$, find the possible values of $\sin\alpha$ and $\sec\alpha$.

13 a Prove the algebraic identity $(1 - t^2)^2 + (2t)^2 = (1 + t^2)^2$.

b If $\cos x = \dfrac{1 - t^2}{1 + t^2}$ where x is acute and t is positive, find expressions for $\sin x$ and $\tan x$.

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5G Trigonometric identities

Working with the trigonometric functions requires knowledge of a number of formulae called *trigonometric identities*, which relate trigonometric functions to each other. This section introduces eleven trigonometric identities in four groups:

- the three *reciprocal identities*
- the two *ratio identities*
- the three *Pythagorean identities*
- the three *identities concerning complementary angles.*

The three reciprocal identities

It follows immediately from the definitions of the trigonometric functions in terms of x, y and r that:

19 THE RECIPROCAL IDENTITIES

For any angle θ:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad (\text{provided that } \sin\theta \neq 0)$$

$$\sec\theta = \frac{1}{\cos\theta} \quad (\text{provided that } \cos\theta \neq 0)$$

$$\cot\theta = \frac{1}{\tan\theta} \quad (\text{provided that } \tan\theta \neq 0 \text{ and } \cot\theta \neq 0)$$

Note: We cannot use a calculator to find $\cot 90^\circ$ or $\cot 270^\circ$ by first finding $\tan 90^\circ$ or $\tan 270^\circ$, because both are undefined. We do already know, however, that

$$\cot 90^\circ = \cot 270^\circ = 0.$$

The two ratio identities

Again using the definitions of the trigonometric functions:

20 THE RATIO IDENTITIES

For any angle θ:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad (\text{provided that } \cos\theta \neq 0)$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (\text{provided that } \sin\theta \neq 0)$$

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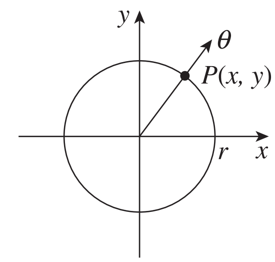
The three Pythagorean identities

The point $P(x, y)$ lies on the circle with centre O and radius r, so its coordinates satisfy

$$x^2 + y^2 = r^2.$$



Dividing through by r^2, $\dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1,$

that is, $\sin^2\theta + \cos^2\theta = 1.$

Dividing through by $\cos^2\theta$ and using the ratio and reciprocal identities,

$$\tan^2\theta + 1 = \sec^2\theta, \quad \text{provided that } \cos\theta \neq 0.$$

Dividing instead by $\sin^2\theta$, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$, provided that $\sin\theta \neq 0$.

These identities are called the *Pythagorean identities* because they rely on the circle equation $x^2 + y^2 = r^2$, which is a restatement of Pythagoras' theorem.

21 THE PYTHAGOREAN IDENTITIES

For any angle θ:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta \qquad (\text{provided that } \cos\theta \neq 0)$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta \qquad (\text{provided that } \sin\theta \neq 0)$$

The three identities for complementary angles

The angles θ and $90° - \theta$ are called *complementary angles* because they add to a right angle. Three trigonometric identities relate the values of the trigonometric functions at an angle θ and the complementary angle $90° - \theta$.

22 THE COMPLEMENTARY ANGLE IDENTITIES

For any angle θ:

$$\cos(90° - \theta) = \sin\theta$$

$$\cot(90° - \theta) = \tan\theta \qquad (\text{provided that } \tan\theta \text{ is defined})$$

$$\operatorname{cosec}(90° - \theta) = \sec\theta \qquad (\text{provided that } \sec\theta \text{ is defined})$$

For example, $\cos 20° = \sin 70°$,

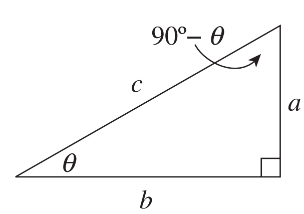
$\cot 20° = \tan 70°$,

$\operatorname{cosec} 20° = \sec 70°$.

Proof

A [Acute angles]

The triangle to the right shows that when a right-angled triangle is viewed from $90° - \theta$ instead of from θ, then the opposite side and the adjacent side are exchanged. Hence



$$\cos(90° - \theta) = \frac{a}{c} = \sin\theta,$$

$$\cot(90° - \theta) = \frac{a}{b} = \tan\theta,$$

$$\operatorname{cosec}(90° - \theta) = \frac{c}{b} = \sec\theta.$$

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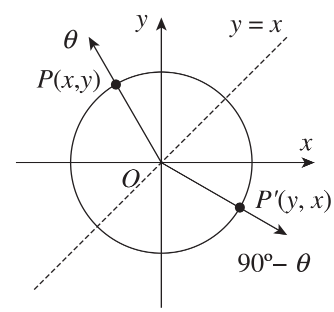
B [General angles]

For general angles, we take the full circle diagram, and reflect it in the diagonal line $y = x$. Let P' be the image of P under this reflection.

1 The image OP' of the ray OP corresponds to the angle $90° - \theta$.

2 The image P' of $P(x, y)$ has coordinates $P'(y, x)$, because reflection in the line $y = x$ reverses the coordinates of each point.



Applying the definitions of the trigonometric functions to the angle $90° - \theta$:

$$\cos(90° - \theta) = \frac{y}{r} = \sin\theta,$$

$$\cot(90° - \theta) = \frac{y}{x} = \tan\theta, \text{ (provided that } x \neq 0).$$

$$\operatorname{cosec}(90° - \theta) = \frac{r}{x} = \sec\theta, \text{ (provided that } x \neq 0).$$

Cosine, cosecant and cotangent

The complementary identities are the origin of the names ‘cosine’, ‘cosecant’ and ‘cotangent’: the prefix ‘co-’ has the same meaning as the prefix ‘com-’ of ‘complementary’ angle.

23 COSINE, COSECANT AND COTANGENT

$\underline{co}\text{sine}\,\theta = \text{sine}\,(\underline{com}\text{plement of}\,\theta)$
$\underline{co}\text{tangent}\,\theta = \text{tangent}\,(\underline{com}\text{plement of}\,\theta)$
$\underline{co}\text{secant}\,\theta = \text{secant}\,(\underline{com}\text{plement of}\,\theta)$

Proving identities

An *identity* is a statement that is true for all values of θ for which both sides are defined, and an identity needs to be proven. It is quite different from an *equation*, which needs to be solved and to have its solutions listed.

24 PROVING TRIGONOMETRIC IDENTITIES

- Work separately on the LHS and the RHS of the identity until they are the same.
- Use the four sets of identities in boxes 19–22 above.

Mostly it is only necessary to work on one of the two sides. The important thing is never to treat it as an equation, moving terms from one side to the other.

Example 12 5G

Prove that $(1 - \cos\theta)(1 + \cos\theta) = \sin^2\theta$.

SOLUTION

$$\begin{aligned} \text{LHS} &= 1 - \cos^2\theta && \text{(use the difference of squares identity, from algebra)} \\ &= \sin^2\theta && \text{(use the Pythagorean identities in Box 21)} \\ &= \text{RHS}. \end{aligned}$$

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Example 13 5G

Prove that $\sin A \sec A = \tan A$.

SOLUTION

$$\begin{aligned} \text{LHS} &= \sin A \times \frac{1}{\cos A} && \text{(use the reciprocal identities in Box 19)} \\ &= \tan A && \text{(use the ratio identities in Box 20)} \\ &= \text{RHS.} \end{aligned}$$

Example 14 5G

Prove that $\dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta} = \sec^2\theta\,\operatorname{cosec}^2\theta$.

SOLUTION

$$\begin{aligned} \text{LHS} &= \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} && \text{(use a common denominator)} \\ &= \frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta} && \text{(use the Pythagorean identities in Box 21)} \\ &= \frac{1}{\sin^2\theta\cos^2\theta} && \text{(use the reciprocal identities in Box 19)} \\ &= \sec^2\theta\,\operatorname{cosec}^2\theta \\ &= \text{RHS.} \end{aligned}$$

Exercise 5G

FOUNDATION

1 Use your calculator to verify that:

a $\sin 16° = \cos 74°$
b $\tan 63° = \cot 27°$
c $\sec 7° = \operatorname{cosec} 83°$
d $\sin^2 23° + \cos^2 23° = 1$
e $1 + \tan^2 55° = \sec^2 55°$
f $\operatorname{cosec}^2 32° - 1 = \cot^2 32°$

2 Simplify:

a $\dfrac{1}{\sin\theta}$
b $\dfrac{1}{\tan\alpha}$
c $\dfrac{\sin\beta}{\cos\beta}$
d $\dfrac{\cos\phi}{\sin\phi}$

3 Simplify:

a $\sin\alpha\operatorname{cosec}\alpha$
b $\cot\beta\tan\beta$
c $\cos\theta\sec\theta$

4 Prove:

a $\tan\theta\cos\theta = \sin\theta$
b $\cot\alpha\sin\alpha = \cos\alpha$
c $\sin\beta\sec\beta = \tan\beta$

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5 Use the complementary identities to simplify:

a $\sin(90° - \theta)$ **b** $\sec(90° - \alpha)$ **c** $\dfrac{1}{\cot(90° - \beta)}$ **d** $\dfrac{\cos(90° - \phi)}{\sin(90° - \phi)}$

6 Use the Pythagorean identities to simplify:

a $\sin^2\alpha + \cos^2\alpha$ **b** $1 - \cos^2\beta$ **c** $1 + \tan^2\phi$ **d** $\sec^2 x - \tan^2 x$

7 Use the Pythagorean identities to simplify:

a $1 - \sin^2\beta$ **b** $1 + \cot^2\phi$ **c** $\operatorname{cosec}^2 A - 1$ **d** $\cot^2\theta - \operatorname{cosec}^2\theta$

DEVELOPMENT

8 Use the reciprocal and ratio identities to simplify:

a $\dfrac{1}{\sec^2\theta}$ **b** $\dfrac{\sin^2\beta}{\cos^2\beta}$ **c** $\dfrac{\cos^2 A}{\sin^2 A}$ **d** $\sin^2\alpha\operatorname{cosec}^2\alpha$

9 Use the reciprocal and ratio identities to prove:

a $\cos A\operatorname{cosec} A = \cot A$ **b** $\operatorname{cosec} x\cos x\tan x = 1$ **c** $\sin y\cot y\sec y = 1$

10 Use the reciprocal and ratio identities to simplify:

a $\dfrac{\cos\alpha}{\sec\alpha}$ **b** $\dfrac{\sin\alpha}{\operatorname{cosec}\alpha}$ **c** $\dfrac{\tan A}{\sec A}$ **d** $\dfrac{\cot A}{\operatorname{cosec} A}$

11 Prove the identities:

a $(1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta$
b $(1 + \tan^2\alpha)\cos^2\alpha = 1$
c $(\sin A + \cos A)^2 = 1 + 2\sin A\cos A$
d $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$
e $\tan^2\phi\cos^2\phi + \cot^2\phi\sin^2\phi = 1$
f $3\cos^2\theta - 2 = 1 - 3\sin^2\theta$
g $2\tan^2 A - 1 = 2\sec^2 A - 3$
h $1 - \tan^2\alpha + \sec^2\alpha = 2$
i $\cos^4 x + \cos^2 x\sin^2 x = \cos^2 x$
j $\cot\theta(\sec^2\theta - 1) = \tan\theta$

12 Prove the identities:

a $\tan\alpha\operatorname{cosec}\alpha = \sec\alpha$
b $\cot\beta\sec\beta = \operatorname{cosec}\beta$
c $\operatorname{cosec}^2\gamma + \sec^2\gamma = \operatorname{cosec}^2\gamma\sec^2\gamma$
d $\tan\delta + \cot\delta = \operatorname{cosec}\delta\sec\delta$
e $\operatorname{cosec}\phi - \sin\phi = \cos\phi\cot\phi$
f $\sec\theta - \cos\theta = \tan\theta\sin\theta$

CHALLENGE

13 Prove the identities:

a $\sin\theta\cos\theta\operatorname{cosec}^2\theta = \cot\theta$
b $(\cos\phi + \cot\phi)\sec\phi = 1 + \operatorname{cosec}\phi$
c $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$
d $\sin\beta + \cot\beta\cos\beta = \operatorname{cosec}\beta$
e $\dfrac{1 + \tan^2 x}{1 + \cot^2 x} = \tan^2 x$
f $\dfrac{1 + \cot x}{1 + \tan x} = \cot x$
g $\dfrac{1}{1 + \sin\theta} + \dfrac{1}{1 - \sin\theta} = 2\sec^2\theta$
h $\dfrac{1}{\sec\phi - \tan\phi} - \dfrac{1}{\sec\phi + \tan\phi} = 2\tan\phi$

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5H Trigonometric equations

This piece of work is absolutely vital, because so many problems in later work end up with a trigonometric equation that has to be solved.

There are many small details and qualifications in the methods, and the subject needs a great deal of careful study.

Pay attention to the domain

To begin with a warning, before any other details:

25 THE DOMAIN

Always pay careful attention to the domain in which the angle can lie.

Equations involving boundary angles

Boundary angles are a special case because they do not lie in any quadrant.

26 THE BOUNDARY ANGLES

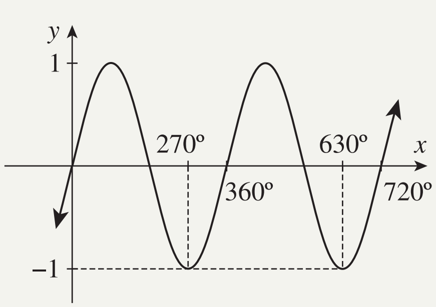
If the solutions are boundary angles, read the solutions off a sketch of the graph.

Example 15 5H

a Solve $\sin x = -1$, for $0° \le x \le 720°$.
b Solve $\sin x = 0$, for $0° \le x \le 720°$.

SOLUTION

a The graph of $y = \sin x$ is shown to the right.
Examine where the curve touches the line $y = -1$, and read off the x-coordinates of these points.
The solution is $x = 270°$ or $630°$.



b Examine where the graph crosses the x-axis.
The solution is $x = 0°, 180°, 360°, 540°$ or $720°$.

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The standard method — quadrants and the related acute angle

Most trigonometric equations eventually come down to one or more equations such as

$$\sin x = -\tfrac{1}{2}, \text{ where } -180° \le x \le 180°.$$

Provided that the angle is not a boundary angle, the method is:

27 THE QUADRANTS-AND-RELATED-ANGLE METHOD

1. Draw a quadrant diagram, then draw a ray in each quadrant that the angle could be in.
2. Find the related acute angle (only work with positive numbers here):
   - **a** using special angles, or
   - **b** using the calculator to find an approximation.

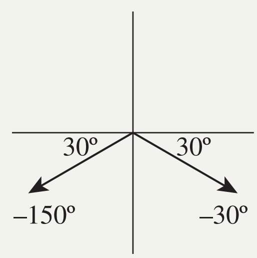
   Never enter a negative number into the calculator at this point.
3. Mark the angles on the ends of the rays, taking account of any restrictions on x, and write a conclusion.

Example 16 — 5H

Solve the equation $\sin x = -\frac{1}{2}$, for $-180° \le x \le 180°$.

SOLUTION

Here $\quad \sin x = -\frac{1}{2}$, where $-180° \le x \le 180°$.
Because $\sin x$ is negative, x is in quadrant 3 or 4.
The sine of the related acute angle is $+\frac{1}{2}$,
so the related angle is 30°.
Hence $\quad x = -150°$ or $-30°$.

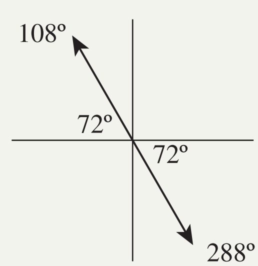


Example 17 — 5H

Solve the equation $\tan x = -3$, for $0° \le x \le 360°$, correct to the nearest degree.

SOLUTION

Here $\quad \tan x = -3$, where $0° \le x \le 360°$.
Because $\tan x$ is negative, x is in quadrant 2 or 4.
The tangent of the related acute angle is $+3$,
so the related angle is about 72°.
Hence $\quad x \doteqdot 108°$ or $288°$.



Note: When using a calculator, *never enter a negative number and take an inverse trigonometric function of it*.

In the example above, the calculator was used to find the *related acute angle* whose tan was 3, which is $71°34'$, correct to the nearest minute. The positive number 3 was entered, not -3.

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The three reciprocal functions

The calculator doesn't have specific keys for secant, cosecant and cotangent. These functions should be converted to sine, cosine and tangent as quickly as possible.

28 THE RECIPROCAL FUNCTIONS

Take reciprocals to convert the three reciprocal functions secant, cosecant and cotangent to the three more common functions.

Example 18 5H

a Solve $\operatorname{cosec} x = -2$, for $-180° \le x \le 180°$.
b Solve $\sec x = 0.7$, for $-180° \le x \le 180°$.

SOLUTION

a Taking the reciprocals of both sides gives

$$\sin x = -\tfrac{1}{2},$$

which was solved in Example 16,

so $x = -150°$ or $-30°$.

b Taking the reciprocals of both sides gives

$$\cos x = \tfrac{10}{7},$$

which has no solutions, because $\cos\theta$ can never be greater than 1.

Equations with compound angles

In some equations, the angle is a function of x rather than simply x itself. For example,

$$\tan 2x = \sqrt{3}, \quad \text{where } 0° \le x \le 360°, \quad \text{or}$$

$$\sin(x - 250°) = \frac{\sqrt{3}}{2}, \quad \text{where } 0° \le x \le 360°.$$

These equations are really trigonometric equations in the *compound angles* $2x$ and $(x - 250°)$ respectively. The secret lies in solving for the *compound angle*, and in *first calculating the domain for that compound angle*.

29 EQUATIONS WITH COMPOUND ANGLES

1. Let u be the compound angle.
2. Find the restrictions on u from the given restrictions on x.
3. Solve the trigonometric equation for u.
4. Hence solve for x.

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Example 19 5H

Solve $\tan 2x = \sqrt{3}$, where $0° \le x \le 360°$.

SOLUTION

Let $u = 2x$.

Then $\tan u = \sqrt{3}$.

The restriction on x is $0° \le x \le 360°$

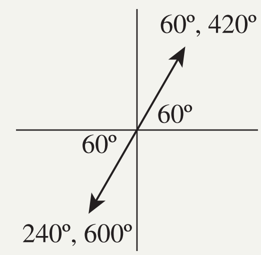
$\boxed{\times 2}$ $0° \le 2x \le 720°$

and replacing $2x$ by u, $0° \le u \le 720°$.

(The restriction on u is the key step here.)

Hence from the diagram, $u = 60°, 240°, 420°$ or $600°$.

Because $x = \frac{1}{2}u$, $x = 30°, 120°, 210°$ or $300°$.



Example 20 5H

Solve $\sin(x - 250°) = \dfrac{\sqrt{3}}{2}$, where $0° \le x \le 360°$.

SOLUTION

Let $u = x - 250°$.

Then $\sin u = \dfrac{\sqrt{3}}{2}$.

The restriction on x is $0° \le x \le 360°$

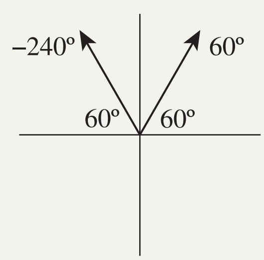
$\boxed{-250°}$ $-250° \le x - 250° \le 110°$

and replacing $x - 250°$ by u, $-250° \le u \le 110°$.

(Again, the restriction on u is the key step here.)

Hence from the diagram, $u = -240°$ or $60°$.

Because $x = u + 250°$, $x = 10°$ or $310°$.



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Equations with more than one trigonometric function

Some trigonometric equations involve more than one trigonometric function. For example,

$$\sin x + \sqrt{3}\cos x = 0$$

The general approach is to use trigonometric identities to produce an equation in only one trigonometric function.

Example 21 5H

Solve $\sin x + \sqrt{3}\cos x = 0$, where $0° \le x \le 360°$.

SOLUTION

$$\sin x + \sqrt{3}\cos x = 0$$

$\boxed{\div \cos x}$ $$\tan x + \sqrt{3} = 0$$

$$\tan x = -\sqrt{3}, \text{ where } 0° \le x \le 360°.$$

Because $\tan x$ is negative, x is in quadrants 2 or 4.

The tan of the related acute angle is $\sqrt{3}$, so the related angle is $60°$.

Hence $x = 120°$ or $300°$.

Exercise 5H

FOUNDATION

1 Solve each equation for $0° \le \theta \le 360°$. (Each related acute angle is $30°$, $45°$ or $60°$.)

a $\sin\theta = \frac{\sqrt{3}}{2}$
b $\sin\theta = \frac{1}{2}$
c $\tan\theta = 1$
d $\tan\theta = \sqrt{3}$
e $\cos\theta = -\frac{1}{\sqrt{2}}$
f $\tan\theta = -\sqrt{3}$
g $\sin\theta = -\frac{1}{2}$
h $\cos\theta = -\frac{\sqrt{3}}{2}$

2 Solve each equation for $0° \le \theta \le 360°$. (The trigonometric graphs are helpful here.)

a $\sin\theta = 1$
b $\cos\theta = 1$
c $\cos\theta = 0$
d $\cos\theta = -1$
e $\tan\theta = 0$
f $\sin\theta = -1$

3 Solve each equation for $0° \le x \le 360°$. Use your calculator to find the related acute angle in each case, and give solutions correct to the nearest degree.

a $\cos x = \frac{3}{7}$
b $\sin x = 0.1234$
c $\tan x = 7$
d $\sin x = -\frac{2}{3}$
e $\tan x = -\frac{20}{9}$
f $\cos x = -0.77$

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DEVELOPMENT

4 Solve for $0° \le \alpha \le 360°$. Give solutions correct to the nearest minute where necessary.

a $\sin\alpha = 0.1$ **b** $\cos\alpha = -0.1$ **c** $\tan\alpha = -1$ **d** $\operatorname{cosec}\alpha = -1$

e $\sin\alpha = 3$ **f** $\sec\alpha = -2$ **g** $\sqrt{3}\tan\alpha + 1 = 0$ **h** $\cot\alpha = 3$

5 Solve for $-180° \le x \le 180°$. Give solutions correct to the nearest minute where necessary.

a $\tan x = -0.3$ **b** $\cos x = 0$ **c** $\sec x = \sqrt{2}$ **d** $\sin x = -0.7$

6 Solve each equation for $0° \le \theta \le 720°$.

a $2\cos\theta - 1 = 0$ **b** $\cot\theta = 0$ **c** $\operatorname{cosec}\theta + 2 = 0$ **d** $\tan\theta = \sqrt{2} - 1$

CHALLENGE

7 Solve each equation for $0° \le x \le 360°$. (Let $u = 2x$.)

a $\sin 2x = \frac{1}{2}$ **b** $\tan 2x = \sqrt{3}$

c $\cos 2x = -\frac{1}{\sqrt{2}}$ **d** $\sin 2x = -1$

8 Solve each equation for $0° \le \alpha \le 360°$. (Let u be the compound angle.)

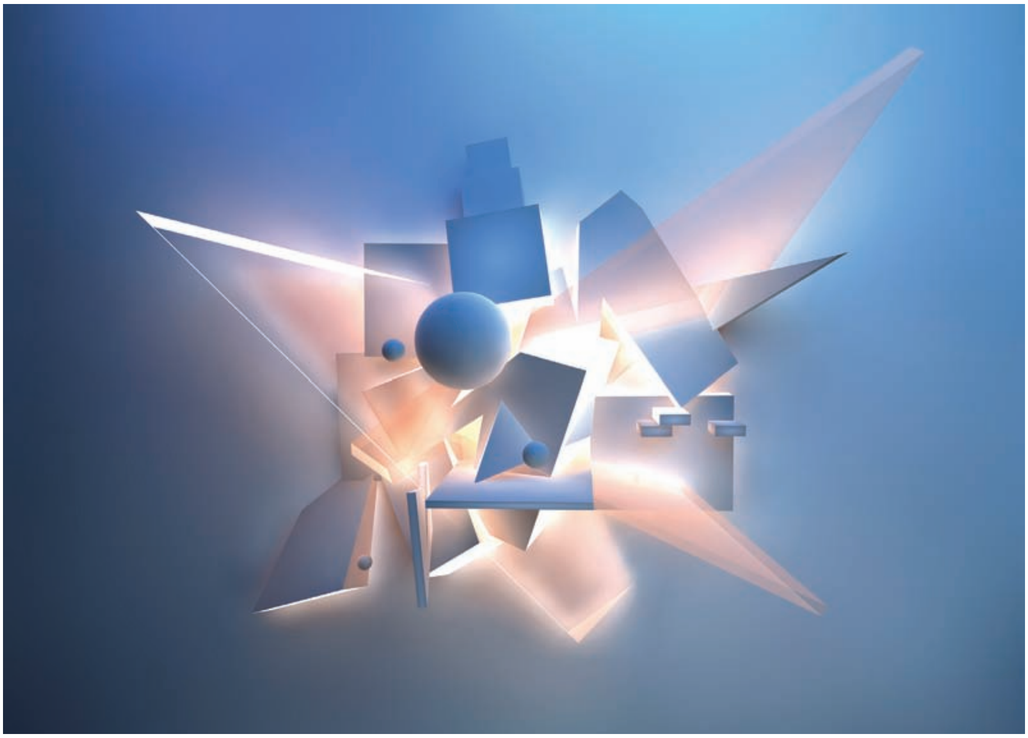
a $\tan(\alpha - 45°) = \frac{1}{\sqrt{3}}$ **b** $\sin(\alpha + 30°) = -\frac{\sqrt{3}}{2}$

c $\cos(\alpha + 60°) = 1$ **d** $\cos(\alpha - 75°) = -\frac{1}{\sqrt{2}}$

9 Solve each equation for $0° \le \theta \le 360°$.

a $\sin\theta = \cos\theta$ **b** $\sin\theta + \cos\theta = 0$

c $\sin\theta = \sqrt{3}\cos\theta$ **d** $\sqrt{3}\sin\theta + \cos\theta = 0$



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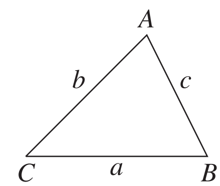
5I The sine rule and the area formula

The last three sections of this chapter review the sine rule, the area formula and the cosine rule. These three rules extend trigonometry to non-right-angled triangles, and are closely connected to the standard congruence tests of Euclidean geometry.

The usual statement of all three rules uses the convention shown in the diagram to the right. The vertices are named with upper-case letters, then each side takes its name from the lower-case letter of the opposite vertex.



Statement of the sine rule

The sine rule states that the ratio of each side of a triangle to the sine of its opposite angle is constant for the triangle.

30 THE SINE RULE

In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

'The ratio of each side to the sine of the opposite angle is constant.'

The sine rule is easily proven by dropping an altitude from one of the vertices. The details of the proof are given in the appendix to this chapter.

Using the sine rule to find a side — the AAS congruence situation

When using the sine rule to find a side, one side and two angles must be known. This is the situation described by the AAS congruence test from geometry, so we know that there will only be one solution.

31 USING THE SINE RULE TO FIND A SIDE

In the AAS congruence situation:

$$\frac{\text{unknown side}}{\text{sine of its opposite angle}} = \frac{\text{known side}}{\text{sine of its opposite angle}}.$$

Always place the unknown side at the top left of the equation.

If two angles of a triangle are known, so is the third, because the angles add to 180°.

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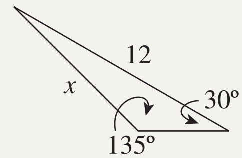
Example 22 5I

Find the side x in the triangle shown to the right.



SOLUTION

Using the sine rule, and placing the unknown at the top left,

$$\frac{x}{\sin 30^\circ} = \frac{12}{\sin 135^\circ}$$

$\boxed{\times \sin 30^\circ}$ $$x = \frac{12\sin 30^\circ}{\sin 135^\circ}.$$

Using special angles, $\sin 30^\circ = \frac{1}{2}$,

and $\sin 135^\circ = +\sin 45^\circ$ (sine is positive for obtuse angles)

$= \frac{1}{\sqrt{2}}$. (the related acute angle is 45°)

Hence $$x = 12 \times \frac{1}{2} \times \frac{\sqrt{2}}{1}$$ $$= 6\sqrt{2}.$$

The area formula

The well-known area formula, area $= \frac{1}{2} \times$ base $\times$ height, can be generalised to a formula involving two sides and the included angle.

32 THE AREA FORMULA

In any triangle ABC,

$$\text{area } \Delta ABC = \tfrac{1}{2}bc\sin A.$$

'The area of a triangle is half the product of any two sides times the sine of the included angle'.

The proof of the area formula uses the same methods as the proof of the sine rule, and is given in the appendix to this chapter.

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Using the area formula — the SAS congruence situation

The area formula requires the SAS congruence situation in which two sides and the included angle are known.

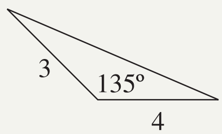
33 USING THE AREA FORMULA

In the SAS congruence situation:

area = (half the product of two sides) × (sine of the included angle).

Example 23 5I

Find the area of the triangle shown to the right.



SOLUTION

Using the formula, area $= \frac{1}{2} \times 3 \times 4 \times \sin 135°$.

$$\begin{aligned} \text{Because} \quad \sin 135° &= \frac{1}{\sqrt{2}}, && \text{(as in Example 22)} \\ \text{area} &= 6 \times \frac{1}{\sqrt{2}} \\ &= 6 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} && \text{(rationalise the denominator)} \\ &= 3\sqrt{2} \text{ square units.} \end{aligned}$$

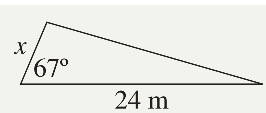
Using the area formula to find a side or an angle

Substituting into the area formula when the area is known may allow an unknown side or angle to be found.

When finding an angle, the formula will always give a single answer for $\sin\theta$. There will be two solutions for θ, however, one acute and one obtuse.

Example 24 5I

Find x, correct to four significant figures, given that the triangle to the right has area $72\,\text{m}^2$.



SOLUTION

Substituting into the area formula,

$$72 = \tfrac{1}{2} \times 24 \times x \times \sin 67°$$
$$72 = 12 \times x \times \sin 67°$$
$$\boxed{\div 12\sin 67°} \quad x = \frac{6}{\sin 67°}$$
$$\doteqdot 6.518 \text{ metres.}$$

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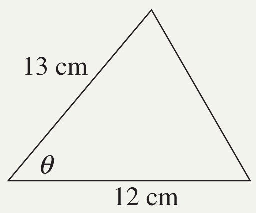
Example 25 5I

Find θ, correct to the nearest minute, given that the triangle below has area $60\,\text{cm}^2$.

SOLUTION

Substituting into the area formula,



$$60 = \tfrac{1}{2} \times 13 \times 12 \times \sin\theta$$

$$60 = 6 \times 13 \sin\theta$$

$$\boxed{\div (6 \times 13)} \quad \sin\theta = \tfrac{10}{13}.$$

Hence $\theta \doteqdot 50°17'$ or $129°43'$.

Notice that the second angle is the supplement of the first.

Using the sine rule to find an angle — the ambiguous ASS situation

The SAS congruence test requires that the angle be included between the two sides. When two sides and a non-included angle are known, the resulting triangle may not be determined up to congruence, and two triangles may be possible. This situation may be referred to as 'the ambiguous ASS situation'.

When the sine rule is applied in the ambiguous ASS situation, there is only one answer for the sine of an angle. There may be two possible solutions for the angle itself, however, one acute and one obtuse.

34 USING THE SINE RULE TO FIND AN ANGLE

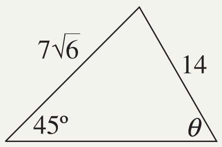
In the ambiguous ASS situation, in which two sides and a non-included angle of the triangle are known,

$$\frac{\text{sine of unknown angle}}{\text{its opposite side}} = \frac{\text{sine of known angle}}{\text{its opposite side}}.$$

Always check the angle sum to see whether both answers are possible.

Example 26 5I

Find the angle θ in the triangle drawn to the right.



SOLUTION

$$\frac{\sin\theta}{7\sqrt{6}} = \frac{\sin 45°}{14} \quad \text{(always place the unknown at the top left)}$$

$$\sin\theta = 7 \times \sqrt{6} \times \frac{1}{14} \times \frac{1}{\sqrt{2}}, \text{ since } \sin 45° = \frac{1}{\sqrt{2}},$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60° \text{ or } 120°.$$

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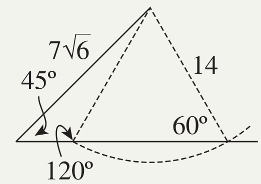
Note: There are *two* angles whose sine is $\dfrac{\sqrt{3}}{2}$, one of them acute and the other obtuse. Moreover,

$$120° + 45° = 165°$$

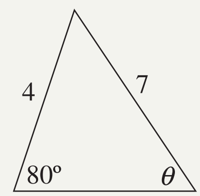
leaving just 15° for the third angle in the obtuse case, so it all seems to work. Opposite is the ruler-and-compasses construction of the triangle, showing how two different triangles can be produced from the same given ASS measurements.



In many examples, however, the obtuse angle solution can be excluded using the fact that the angle sum of a triangle cannot exceed 180°.

Example 27 5I

Find the angle θ in the triangle drawn to the right, and show that there is only one solution.



SOLUTION

$$\frac{\sin\theta}{4} = \frac{\sin 80°}{7} \quad \text{(always place the unknown at the top left)}$$

$$\sin\theta = \frac{4\sin 80°}{7}$$

$$\theta \doteqdot 34°15' \text{ or } 145°45'.$$

But $\theta \doteqdot 145°45'$ is impossible, because the angle sum would then exceed 180°,

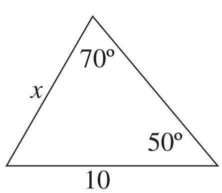
so $\theta \doteqdot 34°15'$ is the only solution.

Exercise 5I

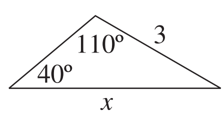
FOUNDATION

1 Find x in each triangle, correct to one decimal place.

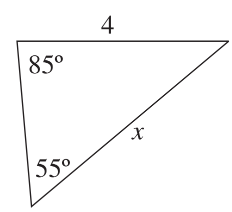
a



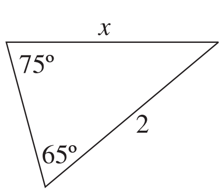
b



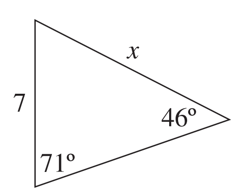
c



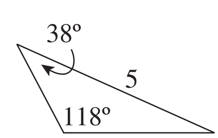
d



e

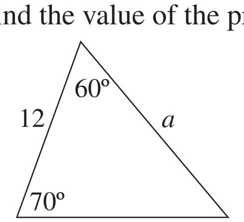


f



2 Find the value of the pronumeral in each triangle, correct to two decimal places.

a



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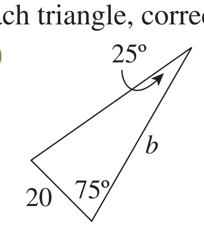

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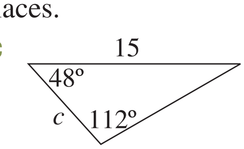

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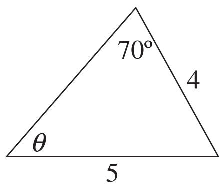
3 Find θ in each triangle, correct to the nearest degree.

a


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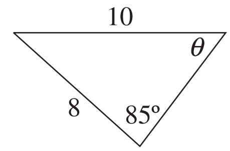

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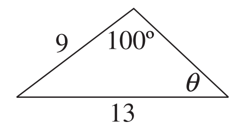

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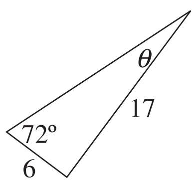
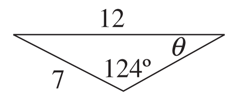
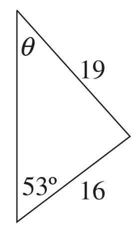
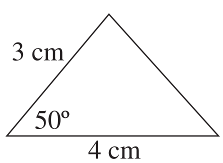


e


f


4 Find the area of each triangle, correct to the nearest square centimetre.

a


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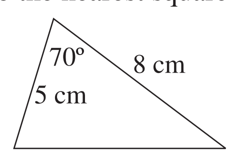

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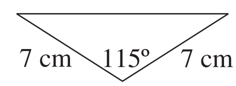


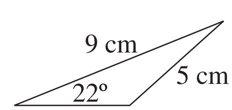
DEVELOPMENT

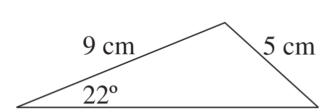
5 **a** Sketch ΔABC in which $A = 43°$, $B = 101°$ and $a = 7.5$ cm.
b Find b and c, in cm correct to two decimal places.

6 **a** Sketch ΔXYZ in which $y = 32$ cm, $Y = 58°$ and $Z = 52°$.
b Find the perimeter of ΔXYZ, correct to the nearest centimetre.

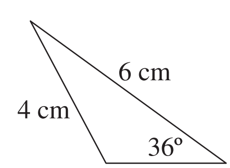
7 Sketch ΔABC in which $a = 2.8$ cm, $b = 2.7$ cm and $A = 52°21'$.
a Find B, correct to the nearest minute.
b Hence find C, correct to the nearest minute.
c Hence find the area of ΔABC in cm^2, correct to two decimal places.

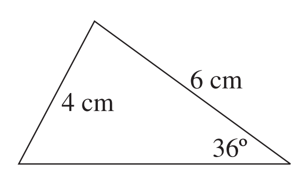
8 There are two triangles that have sides 9 cm and 5 cm, and in which the angle opposite the 5 cm side is 22°. Find, in each case, the size of angle opposite the 9 cm side, correct to the nearest degree.





9 Two triangles are shown, with sides 6 cm and 4 cm, in which the angle opposite the 4 cm is 36°. Find, in each case, the angle opposite the 6 cm side, correct to the nearest degree.





10 Sketch ΔPQR in which $p = 7$ cm, $q = 15$ cm and $\angle P = 25°50'$.
a Find the two possible sizes of $\angle Q$, correct to the nearest minute.
b For each possible size of $\angle Q$, find r in cm, correct to one decimal place.

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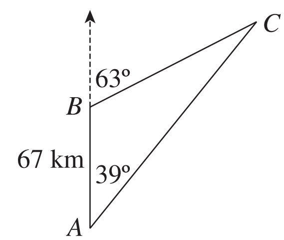
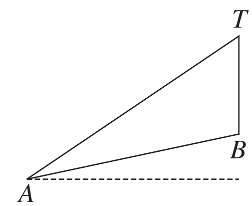
11 A travelling salesman drove from town A to town B, then to town C, and finally directly home to town A.
Town B is 67 km north of town A, and the bearings of town C from towns A and B are 039°T and 063°T respectively.
Find how far the salesman drove, correct to the nearest kilometre.



12 Melissa is standing at A on a path that leads to the base B of a vertical flagpole. The path is inclined at 12° to the horizontal, and the angle of elevation of the top T of the flagpole from A is 34°.

a Explain why $\angle TAB = 22°$ and $\angle ABT = 102°$.

b Given that $AB = 20$ m, find the height of the flagpole, correct to the nearest metre.

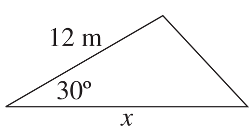


13 a In ΔABC, $\sin A = \frac{1}{4}$, $\sin B = \frac{2}{3}$ and $a = 12$. Find the value of b.

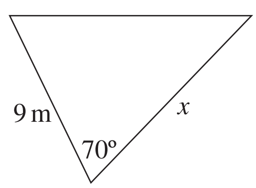
b In ΔPQR, $p = 25$, $q = 21$ and $\sin Q = \frac{3}{5}$. Find the value of $\sin P$.

14 Substitute into the area formula to find the side length x, given that each triangle has area 48 m^2. Give your answers in exact form, or correct to the nearest centimetre.

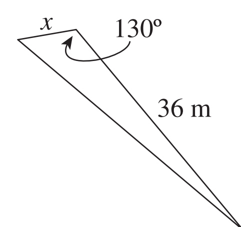
a



b

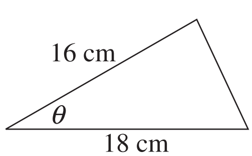


c



15 Substitute into the area formula to find the angle θ, given that each triangle has area 72 cm^2. Give answers correct to the nearest minute, where appropriate.

a



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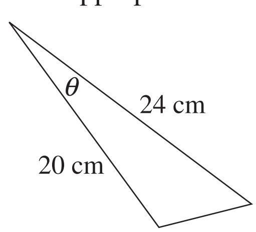

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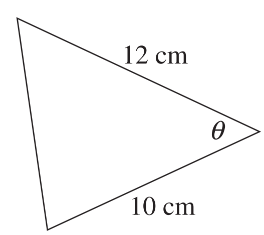
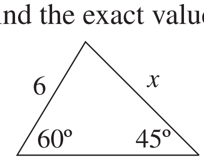


CHALLENGE

16 Find the exact value of x in each diagram.

a



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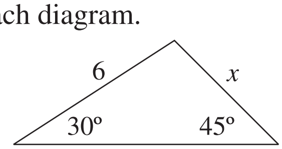

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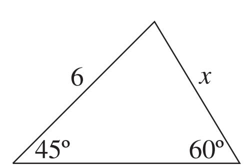

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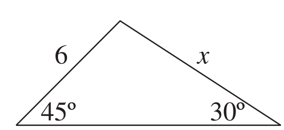
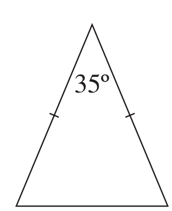


17 The diagram to the right shows an isosceles triangle in which the apex angle is 35°.
Its area is 35 cm^2.
Find the length of the equal sides, correct to the nearest millimetre.



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5J The cosine rule

The cosine rule is a generalisation of Pythagoras' theorem to non-right-angled triangles. It gives a formula for the square of any side in terms of the squares of the other two sides and the cosine of the opposite angle.

35 THE COSINE RULE

In any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

'The square of any side of a triangle equals:
the sum of the squares of the other two sides, minus
twice the product of those sides and the cosine of their included angle.'

The proof is based on Pythagoras' theorem, and again begins with the construction of an altitude. The details are in the appendix to this chapter, but the following points need to be understood when solving problems using the cosine rule.

36 THE COSINE RULE AND PYTHAGORAS' THEOREM

- When $\angle A = 90°$, then $\cos A = 0$ and the cosine rule is Pythagoras' theorem.
- The last term is thus a correction to Pythagoras' theorem when $\angle A \neq 90°$.
- When $\angle A < 90°$, then $\cos A$ is positive, so $a^2 < b^2 + c^2$.
  When $\angle A > 90°$, then $\cos A$ is negative, so $a^2 > b^2 + c^2$.

Using the cosine rule to find a side — the SAS situation

For the cosine rule to be applied to find a side, the other two sides and their included angle must be known. This is the SAS congruence situation.

37 USING THE COSINE RULE TO FIND A SIDE

In the SAS congruence situation:
(square of any side) = (sum of squares of other two sides)
− (twice the product of those sides) × (cosine of their included angle).

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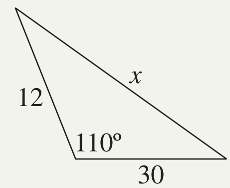
Example 28 5J

Find x in the triangle drawn to the right.



SOLUTION

Applying the cosine rule to the triangle,

$$\begin{aligned} x^2 &= 12^2 + 30^2 - 2 \times 12 \times 30 \times \cos 110° \\ &= 144 + 900 - 720 \cos 110° \\ &= 1044 - 720 \cos 110°, \end{aligned}$$

and since $\cos 110° = -\cos 70°$, (cosine is negative in the second quadrant)

$x^2 = 1044 + 720 \cos 70°$ (until this point, all calculations have been exact).

Using the calculator to approximate x^2, and then to take the square root,

$x \doteqdot 35.92.$

Using the cosine rule to find an angle — the SSS situation

To use the cosine rule to find an angle, all three sides need to be known, which is the SSS congruence test. Finding the angle is done most straight forwardly by substituting into the usual form of the cosine rule:

38 USING THE COSINE RULE TO FIND AN ANGLE

In the SSS congruence situation:

- Substitute into the cosine rule and solve for $\cos\theta$.

There is an alternative approach. Solving the cosine rule for $\cos A$ gives a formula for $\cos A$. Some readers may prefer to remember and apply this second form of the cosine rule — but the triangle may then need to be relabelled.

39 THE COSINE RULE WITH $\cos A$ AS SUBJECT

In any triangle ABC,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Notice that $\cos\theta$ is positive when θ is acute, and is negative when θ is obtuse. Hence, there is only ever one solution for the unknown angle, unlike the situation for the sine rule, in which there are often two possible angles.

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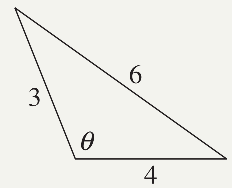
Example 29 5J

Find θ in the triangle drawn to the right.



SOLUTION

Substituting into the cosine rule,

$$\begin{aligned} 6^2 &= 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos\theta \\ 24\cos\theta &= -11 \\ \cos\theta &= -\frac{11}{24} \\ \theta &\doteqdot 117^\circ 17'. \end{aligned}$$

OR

Using the box 39 formula,

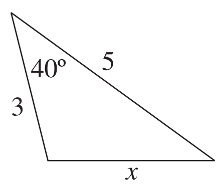
$$\begin{aligned} \cos\theta &= \frac{3^2 + 4^2 - 6^2}{2 \times 3 \times 4} \\ \cos\theta &= \frac{-11}{24} \\ \theta &\doteqdot 117^\circ 17'. \end{aligned}$$

Exercise 5J

FOUNDATION

1 Find x in each triangle, correct to one decimal place.

a



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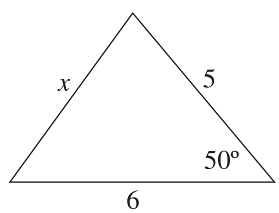

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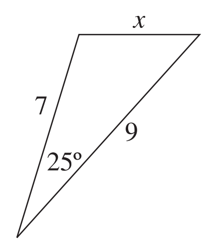

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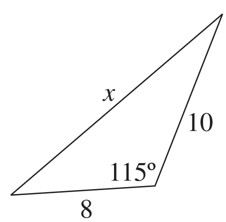

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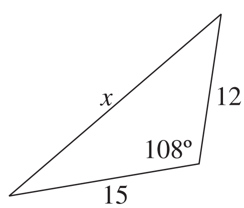

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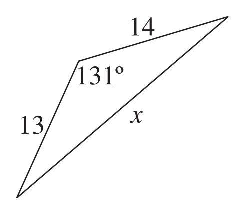
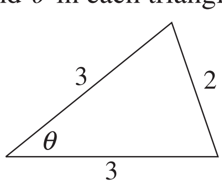


2 Find θ in each triangle, correct to the nearest degree.

a



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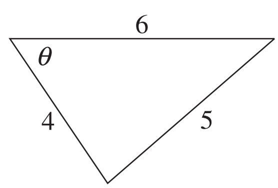

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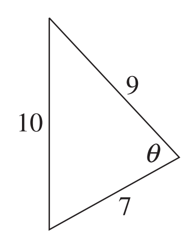

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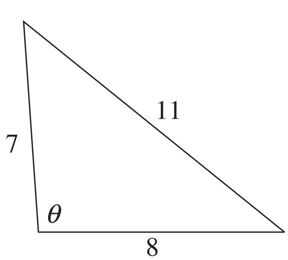

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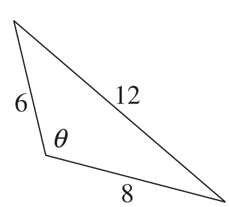

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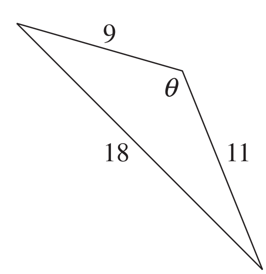

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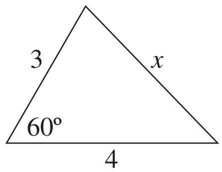
DEVELOPMENT

3 Using the fact that $\cos 60° = \frac{1}{2}$ and $\cos 120° = -\frac{1}{2}$, find x as a surd in each triangle.

a



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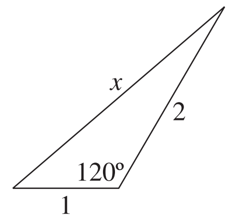
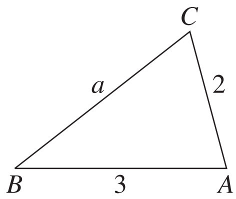
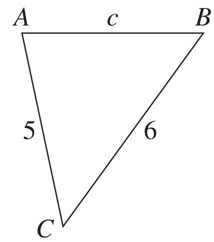


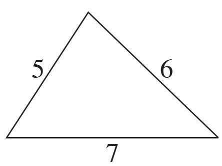
4 a If $\cos A = \frac{1}{4}$, find the exact value of a.



b If $\cos C = \frac{2}{3}$, find the exact value of c.

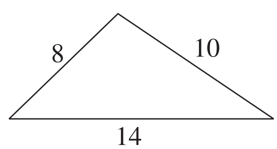


5 a



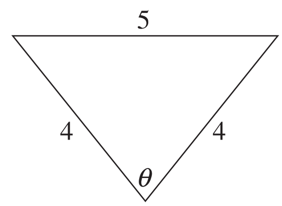
Find the smallest angle of the triangle, correct to the nearest minute.

b



Find the largest angle of the triangle, correct to the nearest minute.

c

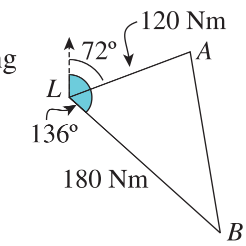


Find the value of $\cos\theta$.

6 There are three landmarks, P, Q and R. It is known that R is 8.7 km from P and 9.3 km from Q, and that $\angle PRQ = 79°32'$. Draw a diagram and find the distance between P and Q, in kilometres correct to one decimal place.

7 In the diagram to the right, ship A is 120 nautical miles from lighthouse L on a bearing of 072°T, while ship B is 180 nautical miles from L on a bearing of 136°T.

Calculate the distance between the two ships, correct to the nearest nautical mile.



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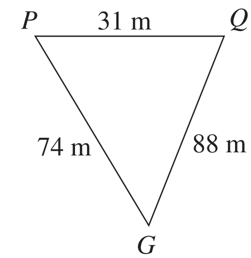
8 A golfer at G wishes to hit a shot between two trees P and Q, as shown in the diagram opposite. The trees are 31 metres apart, and the golfer is 74 metres from P and 88 metres from Q. Find the angle within which the golfer must play the shot, correct to the nearest degree.

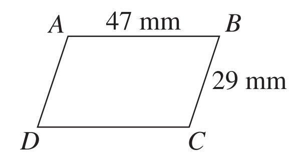


9 A parallelogram $ABCD$ has sides $AB = DC = 47$ mm and $AD = BC = 29$ mm. The longer diagonal BD is 60 mm.

a Use the cosine rule to find the size of $\angle BCD$.

b Use co-interior angles to find the size of $\angle ABC$.

Give your answers correct to the nearest minute.



10 The sides of a triangle are in the ratio $5:16:19$. Find the smallest and largest angles of the triangle, correct to the nearest minute where necessary.

11 In ΔABC, $a = 31$ units, $b = 24$ units and $\cos C = \frac{59}{62}$.

a Show that $c = 11$ units.

b Show that $A = 120°$.

12 In ΔPQR, $p = 5\sqrt{3}$ cm, $q = 11$ cm and $R = 150°$.

a Find r.

b Find $\cos P$.

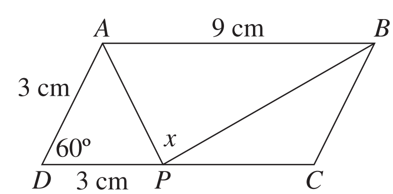
CHALLENGE

13 In a parallelogram $ABCD$, $\angle ADC = 60°$, $AB = 9$ cm and $AD = 3$ cm. The point P lies on DC such that $DP = 3$ cm.

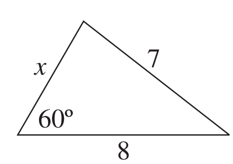
a Explain why ΔADP is equilateral, and hence find AP.

b Use the cosine rule in ΔBCP to find BP.

c Let $\angle APB = x$. Show that $\cos x = -\dfrac{\sqrt{7}}{14}$.



14 Use the cosine rule to find the two possible values of x in the diagram to the right.

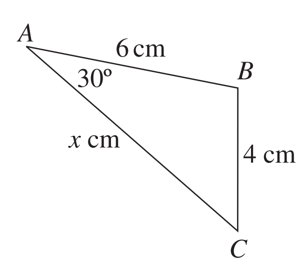


15 The diagram shows ΔABC in which $\angle A = 30°$, $AB = 6$ cm and $BC = 4$ cm. Let $AC = x$ cm.

a Use the cosine rule to show that $x^2 - 6\sqrt{3}x + 20 = 0$.

b Use the quadratic formula to show that AC has length $3\sqrt{3} + \sqrt{7}$ cm or $3\sqrt{3} - \sqrt{7}$ cm.

c Copy the diagram and indicate on it (approximately) the other possible position of the point C.



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5K Problems involving general triangles

A triangle has three lengths and three angles, and most triangle problems involve using three of these six measurements to calculate some of the others. The key to deciding which formula to use is to see which congruence situation applies.

Trigonometry and the congruence tests

There are four standard congruence tests — RHS, AAS, SAS and SSS. These tests can also be regarded as theorems about constructing triangles from given data.

If you know three measurements including one length, then apart from the ambiguous ASS situation, any two triangles with these three measurements will be congruent.

40 THE SINE, COSINE AND AREA RULES AND THE STANDARD CONGRUENCE TESTS

In a right-angled triangle, use simple trigonometry and Pythagoras. Otherwise:
AAS: Use the sine rule to find each of the other two sides.
ASS: [The ambiguous situation] Use the sine rule to find the unknown angle opposite a known side.
There may or may not be two possible solutions.
SAS: Use the cosine rule to find the third side.
Use the area formula to find the area.
SSS: Use the cosine rule to find any angle.

In the ambiguous ASS situation, it is also possible to use the cosine rule to find the third side. See Questions **14** and **15** of the previous exercise.

Problems requiring two steps

Various situations with non-right-angled triangles require two steps for their solution, for example, finding the other two angles in an SAS situation, or finding the area given AAS, ASS or SSS situations.

Example 30 **5K**

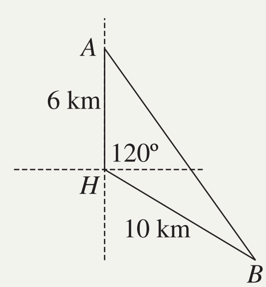
A boat sails 6 km due north from the harbour H to A, and a second boat sails 10 km from H to B on a bearing of 120°T.

a What is the distance AB?

b What is the bearing of B from A, correct to the nearest minute?

SOLUTION

a This is an SAS situation,
so we use the cosine rule to find AB:



$$\begin{aligned} AB^2 &= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 120° \\ &= 36 + 100 - 120 \times \left(-\tfrac{1}{2}\right) \\ &= 196 \\ AB &= 14\,\text{km}. \end{aligned}$$

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b Because AB is now known, this is an SSS situation, so we use the cosine rule in reverse to find $\angle A$:

$$10^2 = 14^2 + 6^2 - 2 \times 14 \times 6 \times \cos A$$
$$12 \times 14 \times \cos A = 196 + 36 - 100$$
$$\cos A = \frac{132}{12 \times 14}$$
$$= \frac{11}{14}$$
$$A \doteqdot 38°13', \text{ and the bearing of } B \text{ from } A \text{ is about } 141°47'\text{T}.$$

Challenge — using the cosine rule in three-dimensional problem:

The three-dimensional problem at the end of the Section 5C involved four triangles, none of which could be solved. We assigned the pronumerals h to the height, then worked around the diagram until we knew four things in terms of h in the base triangle, and could therefore from an equations in h.

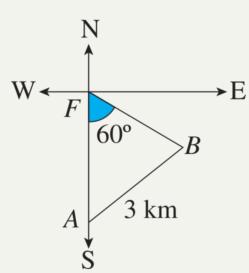
That final triangle was right-angled. The following problem has only one small change from the previous problem, but as a consequence, we need to apply the cosine rule instead of Pythogoras' theorem.

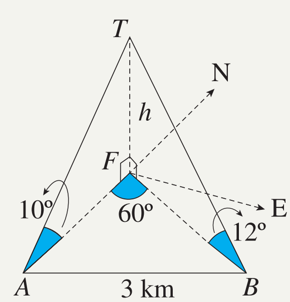
Example 31 5C

A motorist driving on level ground sees, due north of her, a tower whose angle of elevation is 10°. After driving 3 km further in a straight line, the tower is in the direction N60°W, with angle of elevation 12°.

a How high is the tower? **b** In what direction is she driving?

SOLUTION





Let the tower be TF, and let the motorist be driving from A to B.
There are four triangles, none of which can be solved.

a Let h be the height of the tower.
In ΔTAF, $AF = h\cot 10°$.
In ΔTBF, $BF = h\cot 12°$.
We now have expressions for four measurements in ΔABF, so we can use the cosine rule to form an equation in h.

In ΔABF, $3^2 = h^2\cot^2 10° + h^2\cot^2 12° - 2h^2\cot 10°\cot 12° \times \cos 60°$

$$9 = h^2(\cot^2 10° + \cot^2 12° - \cot 10°\cot 12°)$$
$$h^2 = \frac{9}{\cot^2 10° + \cot^2 12° - \cot 10°\cot 12°},$$
$$h \doteqdot 0.571\,\text{km}$$

so the tower is about 571 metres high.

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b Let $\theta = \angle FAB.$

In ΔAFB, $\dfrac{\sin\theta}{h\cot 12^\circ} = \dfrac{\sin 60^\circ}{3}$

$$\sin\theta = h\cot 12^\circ \times \frac{\sqrt{3}}{6}$$

$$\theta \doteqdot 51^\circ,$$

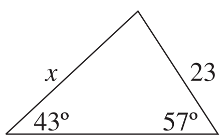
so her direction is about N51°E.

Exercise 5K

FOUNDATION

1 Use right-angled triangle trigonometry, the sine rule or the cosine rule to find x in each triangle, correct to one decimal place.

a



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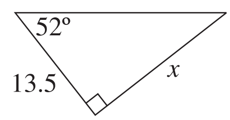

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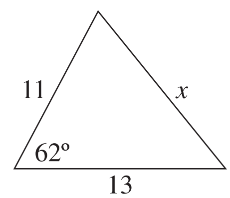

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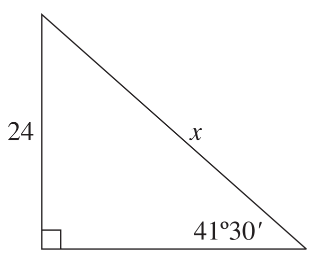

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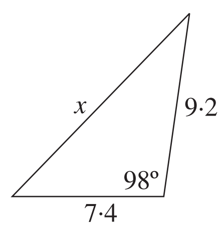

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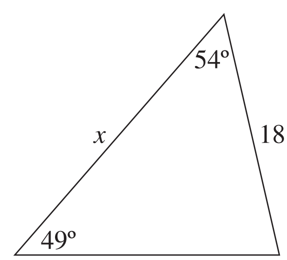
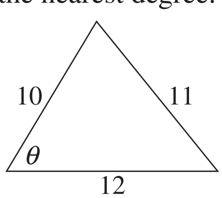


2 Use right-angled triangle trigonometry, the sine rule or the cosine rule to find θ in each triangle, correct to the nearest degree.

a



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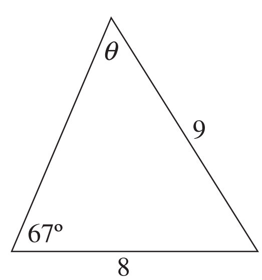

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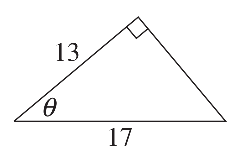

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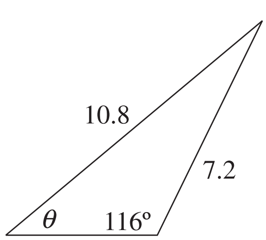

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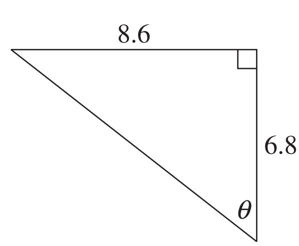

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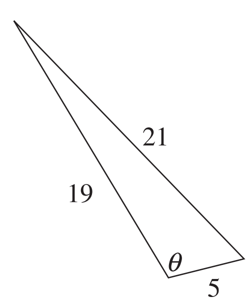

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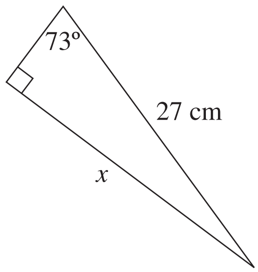
DEVELOPMENT

3 [This question is designed to show that the sine and cosine rules work in right-angled triangles, but are *not* the most efficient methods.]
In each part find the pronumeral (correct to the nearest cm or to the nearest degree), using either the sine rule or the cosine rule. Then check your answer using right-angled triangle trigonometry.

a



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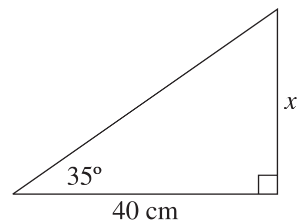

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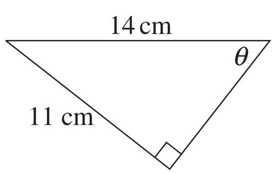

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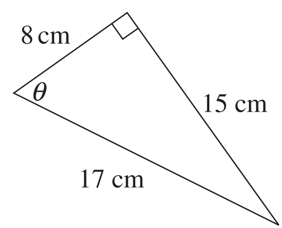
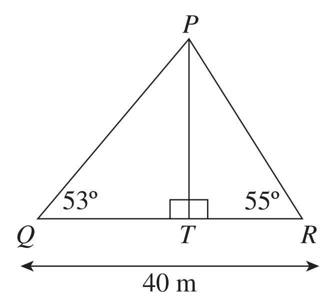


4 In ΔPQR, $\angle Q = 53°$, $\angle R = 55°$ and $QR = 40\,\text{m}$. The point T lies on QR such that $PT \perp QR$.

a Use the sine rule in ΔPQR to show that $PQ = \dfrac{40\sin 55°}{\sin 72°}$.

b Use ΔPQT to find PT, correct to the nearest metre.

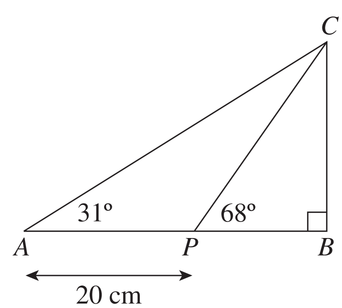


5 In ΔABC, $\angle B = 90°$ and $\angle A = 31°$. The point P lies on AB such that $AP = 20\,\text{cm}$ and $\angle CPB = 68°$.

a Explain why $\angle ACP = 37°$.

b Use the sine rule to show that $PC = \dfrac{20\sin 31°}{\sin 37°}$.

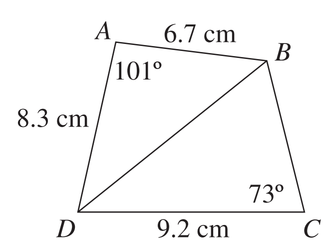
c Hence find PB, correct to the nearest centimetre.



6 In the diagram to the right, $AB = 6.7$ cm, $AD = 8.3\,\text{cm}$ and $DC = 9.2$ cm. Also, $\angle A = 101°$ and $\angle C = 73°$.

a Use the cosine rule to find the diagonal BD, correct to the nearest millimetre.

b Hence use the sine rule to find $\angle CBD$, correct to the nearest degree.



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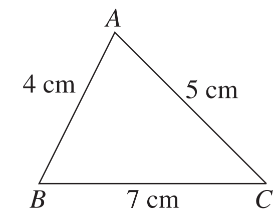
7 In ΔABC, $AB = 4\,\text{cm}$, $BC = 7\,\text{cm}$ and $CA = 5\,\text{cm}$.

a Use the cosine rule to find $\angle ABC$, correct to the nearest minute.

b Hence calculate the area of ΔABC, correct to the nearest square centimetre.



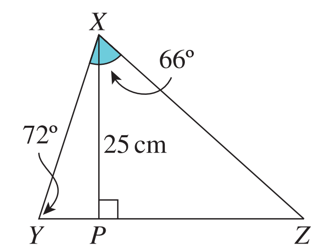
8 A triangle has sides 13 cm, 14 cm and 15 cm. Use the cosine rule to find one of its angles, and hence show that its area is $84\,\text{cm}^2$.

9 In triangle XYZ, $\angle Y = 72°$ and $\angle YXZ = 66°$, $XP \perp YZ$ and $XP = 25\,\text{cm}$.

a Use the sine ratio in ΔPXY to show that $XY \doteqdot 26.3\,\text{cm}$.

b Hence use the sine rule in ΔXYZ to find YZ, correct to the nearest centimetre.

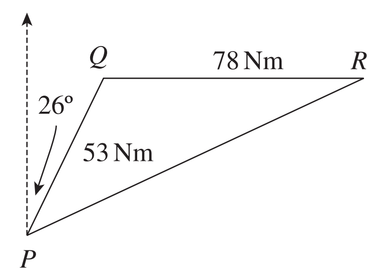
c Check your answer to part **b** by using the tangent ratio in triangles PXY andPXZ to find PY and PZ.



10 A ship sails 53 nautical miles from P to Q on a bearing of 026°T. It then sails 78 nautical miles due east from Q to R.

a Explain why $\angle PQR = 116°$.

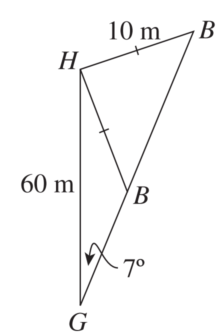
b How far apart are P and R, correct to the nearest nautical mile?



11 A golfer at G, 60 metres from the hole H, played a shot that landed at B, 10 metres from the hole. The direction of the shot was 7° away from the direct line between G and H.

a Find, correct to the nearest minute, the two possible sizes of $\angle GBH$.

b Hence find the two possible distances the ball has travelled. (Answer in metres correct to one decimal place.)



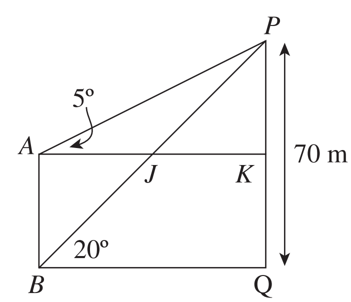
12 Two towers AB and PQ stand on level ground. The angles of elevation of the top of the taller tower from the top and bottom of the shorter tower are 5° and 20° respectively. The height of the taller tower is 70 metres.

a Explain why $\angle APJ = 15°$.

b Show that $AB = \dfrac{BP\sin 15°}{\sin 95°}$.

c Show that $BP = \dfrac{70}{\sin 20°}$.

d Hence find the height of the shorter tower, correct to the nearest metre.



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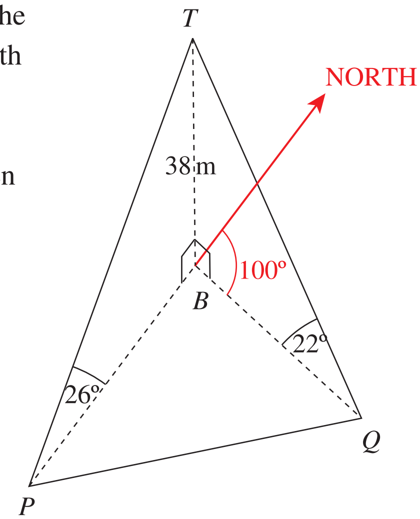
13 From two points P and Q on level ground, the angles of elevation of the top T of a 38 m tower are 26° and 22° respectively. Point P is due south of the tower, and the bearing of Q from the tower is 100°T.

a Show that $PB = 38\tan 64°$, and find a similar expression for QB.

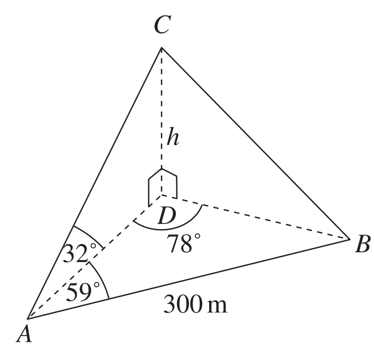
b Hence determine, correct to the nearest metre, the distance between P and Q.



14 Two observers at A and B on horizontal ground are 300 m apart. From A, the angle of elevation of the top C of a tall building DC is 32°. It is also known that $\angle DAB = 59°$ and $\angle ADB = 78°$.

a Show that $AD = \dfrac{300\sin 43°}{\sin 78°}$.

b Hence find the height of the building, correct to the nearest metre.



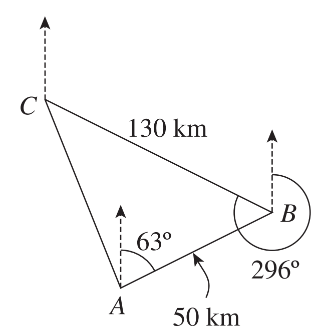
CHALLENGE

15 A ship sails 50 km from port A to port B on a bearing of 063°T, then sails 130 km from port B to port C on a bearing of 296°T.

a Show that $\angle ABC = 53°$.

b Find, correct to the nearest km the distance of port A from port C.

c Use the cosine rule to find $\angle ACB$, and hence find the bearing of port A from port C, correct to the nearest degree.



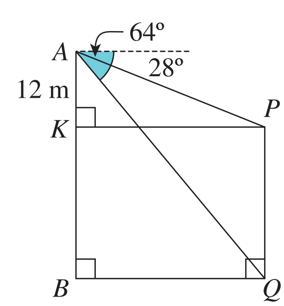
16 Two towers AB and PQ stand on level ground. Tower AB is 12 metres taller than tower PQ. From A, the angles of depression of P and Q are 28° and 64° respectively.

a Use ΔAKP to show that $KP = BQ = 12\tan 62°$.

b Use ΔABQ to show that $AB = 12\tan 62° \tan 64°$.

c Hence find the height of the shorter tower, correct to the nearest metre.

d Solve the problem again by using ΔAKP to find AP, and then using the sine rule in ΔAPQ.



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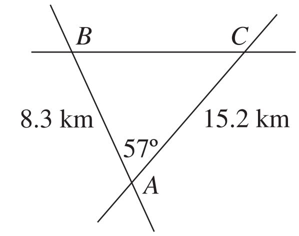
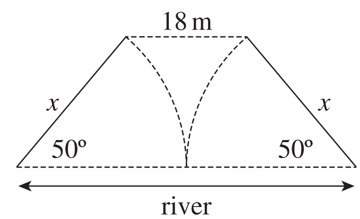
17 The diagram shows three straight roads, AB, BC and CA, where $AB = 8.3$ km, $AC = 15.2$ km, and the roads AB and AC intersect at 57°.
Two cars, P_1 and P_2, leave A at the same instant. Car P_1 travels along AB and then BC at 80 km/h while P_2 travels along AC at 50 km/h. Which car reaches C first, and by how many minutes? (Answer correct to one decimal place.)



18 A bridge spans a river, and the two identical sections of the bridge, each of length x metres, can be raised to allow tall boats to pass. When the two sections are fully raised, they are each inclined at 50° to the horizontal, and there is an 18-metre gap between them, as shown in the diagram. Calculate the width of the river in metres, correct to one decimal place.



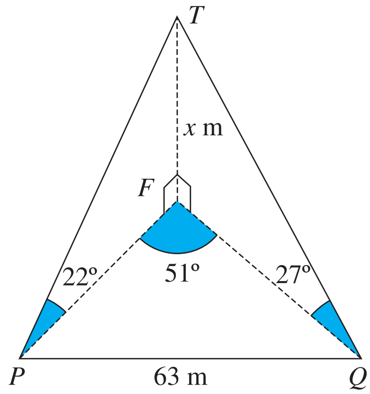
19 In the diagram, TF represents a vertical tower of height x metres standing on level ground. From P and Q at ground level, the angles of elevation of T are 22° and 27° respectively. $PQ = 63$ metres and $\angle PFQ = 51°$.

a Show that $PF = x \cot 22°$ and write down a similar expression for QF.

b Use the cosine rule to show that

$$x^2 = \frac{63^2}{\cot^2 22° + \cot^2 27° - 2\cot 22° \cot 27° \cos 51°}$$

c Use a calculator to show that $x \doteqdot 32$.



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Chapter 5 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 5 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Find, correct to four decimal places:

a $\cos 73°$

b $\tan 42°$

c $\sin 38°24'$

d $\cos 7°56'$

2 Find the acute angle θ, correct to the nearest minute, given that:

a $\sin\theta = 0.3$

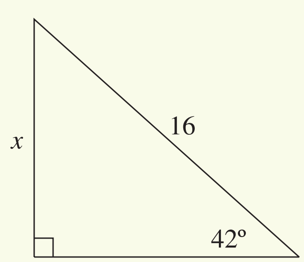
b $\tan\theta = 2.36$

c $\cos\theta = \frac{1}{4}$

d $\tan\theta = 1\frac{1}{3}$

3 Find, correct to two decimal places, the side marked x in each triangle below.

a



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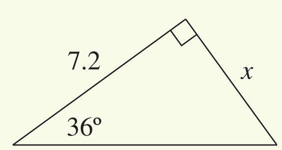
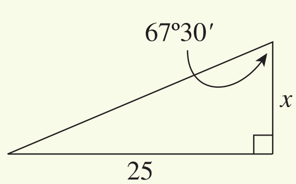


c



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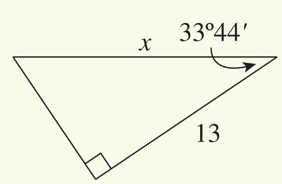

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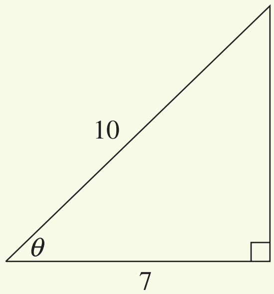
4 Find, correct to the nearest minute, the angle θ in each triangle below.

a



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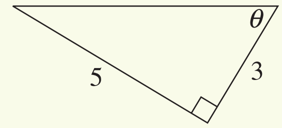

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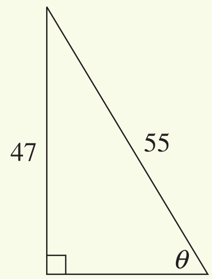

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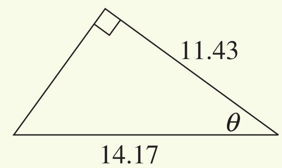


5 Use the special triangles to find the exact values of:

a $\tan 60°$ **b** $\sin 45°$ **c** $\cos 30°$
d $\cot 45°$ **e** $\sec 60°$ **f** $\operatorname{cosec} 60°$

6 A vertical pole stands on level ground. From a point on the ground 8 metres from its base, the angle of elevation of the top of the pole is 38°. Find the height of the pole, correct to the nearest centimetre.

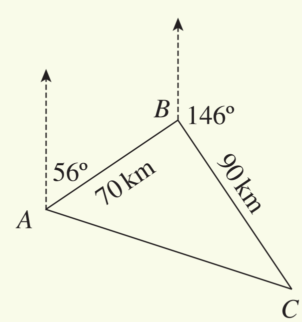
7 At what angle, correct to the nearest degree, is a 6-metre ladder inclined to the ground if its foot is 2.5 metres out from the wall?

8 A motorist drove 70 km from town A to town B on a bearing of 056°T, and then drove 90 km from town B to town C on a bearing of 146°T.

a Explain why $\angle ABC = 90°$.

b How far apart are the towns A and C, correct to the nearest kilometre?

c Find $\angle BAC$, and hence find the bearing of town C from town A, correct to the nearest degree.



9 Sketch each graph for $0° \le x \le 360°$.

a $y = \sin x$ **b** $y = \cos x$ **c** $y = \tan x$

10 Write each trigonometric ratio as the ratio of its related acute angle, with the correct sign attached.

a $\cos 125°$ **b** $\sin 312°$ **c** $\tan 244°$ **d** $\sin 173°$

11 Find the exact value of:

a $\tan 240°$ **b** $\sin 315°$ **c** $\cos 330°$ **d** $\tan 150°$

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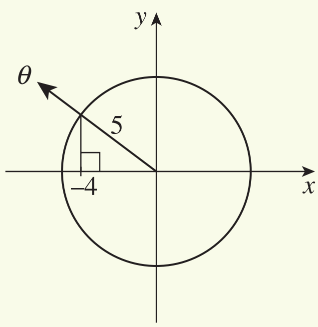
12 Use the graphs of the trigonometric functions to find these values, if they exist.

a $\sin 180°$ **b** $\cos 180°$ **c** $\tan 90°$ **d** $\sin 270°$

13 Use Pythagoras' theorem to find whichever of x, y or r is unknown. Then write down the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

a



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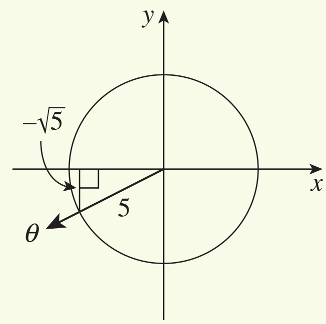

14 **a** If $\tan\alpha = \frac{12}{5}$ and α is acute, find the values of $\sin\alpha$ and $\cos\alpha$.

b If $\sin\beta = \frac{2\sqrt{6}}{7}$ and β is acute, find the values of $\cos\beta$ and $\tan\beta$.

c If $\tan\alpha = -\frac{9}{40}$ and $270° < \alpha < 360°$, find the values of $\sin\alpha$ and $\cos\alpha$.

d If $\sin\beta = \frac{2\sqrt{6}}{7}$ and $90° < \beta < 180°$, find the values of $\cos\beta$ and $\tan\beta$.

15 Simplify:

a $\dfrac{1}{\cos\theta}$ **b** $\dfrac{1}{\cot\theta}$ **c** $\dfrac{\sin\theta}{\cos\theta}$

d $1 - \sin^2\theta$ **e** $\sec^2\theta - \tan^2\theta$ **f** $\operatorname{cosec}^2\theta - 1$

16 Prove the following trigonometric identities.

a $\cos\theta\sec\theta = 1$ **b** $\tan\theta\operatorname{cosec}\theta = \sec\theta$

c $\dfrac{\cot\theta}{\cos\theta} = \operatorname{cosec}\theta$ **d** $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

e $4\sec^2\theta - 3 = 1 + 4\tan^2\theta$ **f** $\cos\theta + \tan\theta\sin\theta = \sec\theta$

17 Solve each trigonometric equation for $0° \le x \le 360°$.

a $\cos x = \frac{1}{2}$ **b** $\sin x = 1$ **c** $\tan x = -1$ **d** $\cos x = 0$

e $\sqrt{3}\tan x = 1$ **f** $\tan x = 0$ **g** $\sqrt{2}\sin x + 1 = 0$ **h** $2\cos x + \sqrt{3} = 0$

i $\cos^2 x = \frac{1}{2}$ **j** $\cos 2x = \frac{1}{2}$ **k** $\cos(x - 75°) = \frac{1}{2}$ **l** $\sin x = -\sqrt{3}\cos x$

18 Solve each equation for $0° \le \theta \le 360°$ by reducing it to a quadratic equation in u. Give your solution correct to the nearest minute where necessary.

a $2\sin^2\theta + \sin\theta = 0$ **b** $\cos^2\theta - \cos\theta - 2 = 0$ **c** $2\tan^2\theta + 5\tan\theta - 3 = 0$

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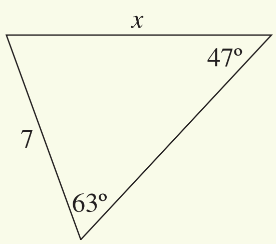
Review

19 Use the sine rule or the cosine rule in each triangle to find x, correct to one decimal place.

a



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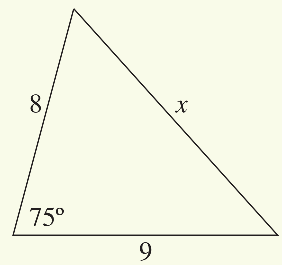

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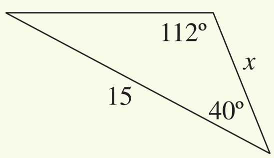

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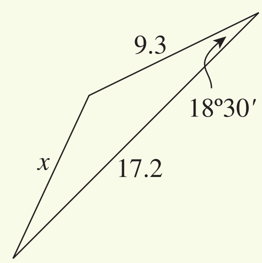
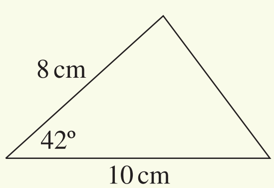


20 Calculate the area of each triangle, correct to the nearest cm^2.

a



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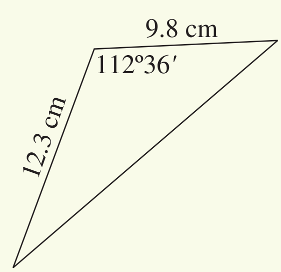
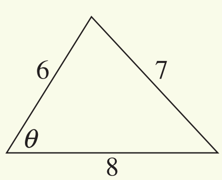


21 Use the sine rule or the cosine rule in each triangle to find θ, correct to the nearest minute.

a



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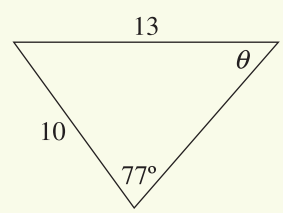

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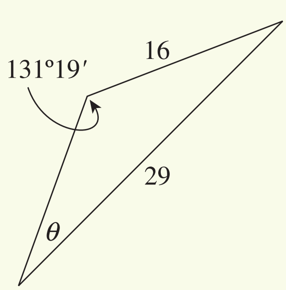

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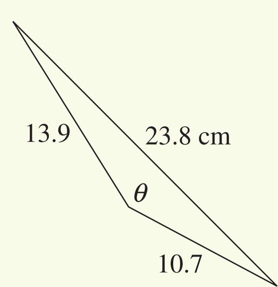

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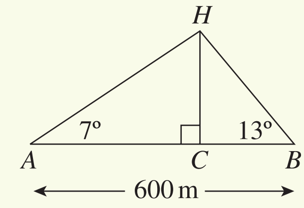
22 A triangle has sides 7 cm, 8 cm and 10 cm. Use the cosine rule to find one of its angles, and hence find the area of the triangle, correct to the nearest cm^2.

23 a Find the side a in ΔABC, where $\angle C = 60°$, $b = 24$ cm and the area is 30 cm^2.

b Find the size of $\angle B$ in ΔABC, where $a = 9$ cm, $c = 8$ cm and the area is 18 cm^2.

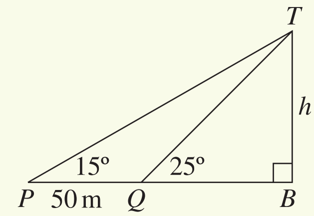
24 A helicopter H is hovering above a straight, horizontal road AB of length 600 m. The angles of elevation of H from A and B are 7° and 13° respectively. The point C lies on the road directly below H.



a Use the sine rule to show that $HB = \dfrac{600\sin 7°}{\sin 160°}$.

b Hence find the height CH of the helicopter above the road, correct to the nearest metre.

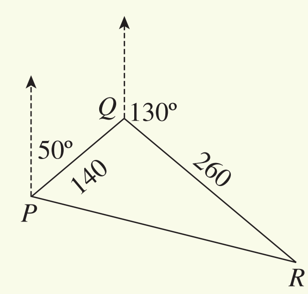
25 A man is sitting in a boat at P, where the angle of elevation of the top T of a vertical cliff BT is 15°. He then rows 50 metres directly towards the cliff to Q, where the angle of elevation of T is 25°.



a Show that $TQ = \dfrac{50\sin 15°}{\sin 10°}$.

b Hence find the height h of the cliff, correct to the nearest tenth of a metre.

26 A ship sailed 140 nautical miles from port P to port Q on a bearing of 050°T. It then sailed 260 nautical miles from port Q to port R on a bearing of 130°T.



a Explain why $\angle PQR = 100°$.

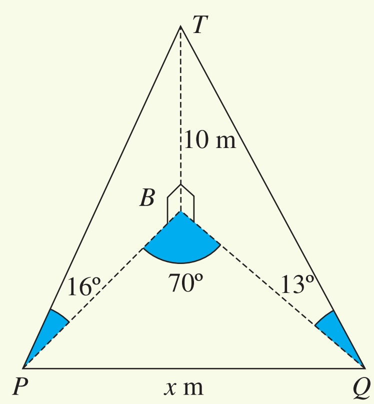
b Find the distance between ports R and P, correct to the nearest nautical mile.

c Find the bearing of port R from port P, correct to the nearest degree.

27 From two points P and Q on horizontal ground, the angles of elevation of the top T of a 10 m monument are 16° and 13° respectively. It is known that $\angle PBQ = 70°$, where B is the base of the monument.

a Show that $PB = 10\tan 74°$, and find a similar expression for QB.

b Hence determine the distance between P and Q, correct to the nearest metre.



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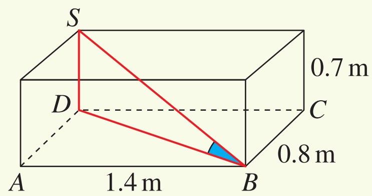
Review

28 The diagram below shows an open wooden crate in the shape of a rectangular prism. The base is 1.4 metres by 0.8 metres, and the height is 0.7 metres.

a Find, correct to the nearest millimetre the length of the base diagonal BD.



b Find, correct to the nearest millimetre the length of the longest metal rod BS that will fit in the box.

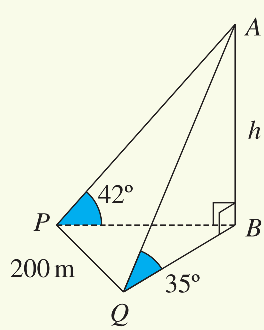
c Find, correct to the nearest minute the angle that the rod BS makes with the base.

29 The points P, Q, and B lie in a horizontal plane. From P, which is due west of B, the angle of elevation of the top of a tower AB of height h meters is 42°. From Q, which is on a bearing of 196° from the tower, the angle of elevation of the top of the tower is 35°. The distance PQ is 200 metres.

a Explain why $\angle PBQ = 74°$.

b Show that $h^2 = \dfrac{200^2}{\cot^2 42° + \cot^2 35° - 2\cot 35° \cot 42° \cos 74°}$.

c Hence find the height of the tower, correct to the nearest metre.



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Appendix: Proofs of the sine, cosine and area rules

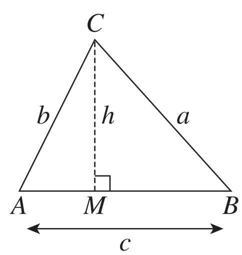
Proof of the sine rule

The sine rule says that in any triangle ABC,

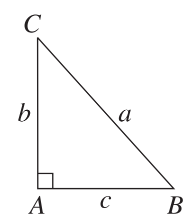
$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

The proof below begins by constructing an altitude. This breaks the triangle into two right-angled triangles, for which previous methods can be used.

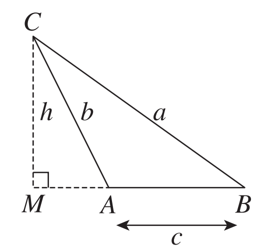
Given: Let ABC be any triangle. There are three cases, depending on whether $\angle A$ is an acute angle, a right angle, or an obtuse angle.



Case 1: $\angle A$ is acute



Case 2: $\angle A = 90°$



Case 3: $\angle A$ is obtuse

Aim: To prove that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

In case 2, $\sin A = \sin 90° = 1$, and $\sin B = \dfrac{b}{a}$, so the result is clear.

Construction: In the remaining cases 1 and 3, construct the altitude from C, meeting AB, produced if necessary, at M. Let h be the length of CM.

Proof:

Case 1: Suppose that $\angle A$ is acute.

In ΔACM, $\dfrac{h}{b} = \sin A$

$\boxed{\times b}$ $h = b\sin A$.

In ΔBCM, $\dfrac{h}{a} = \sin B$

$\boxed{\times a}$ $h = a\sin B$.

Equating these, $b\sin A = a\sin B$

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

Case 3: Suppose that $\angle A$ is obtuse.

In ΔACM, $\dfrac{h}{b} = \sin(180° - A)$,

and because $\sin(180° - A) = \sin A$,

$\boxed{\times b}$ $h = b\sin A$.

In ΔBCM, $\dfrac{h}{a} = \sin B$

$\boxed{\times a}$ $h = a\sin B$.

Equating these, $b\sin A = a\sin B$

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

Proof of the area formula

The area formula says that in any triangle ABC,

$$\text{area } \Delta ABC = \tfrac{1}{2}bc\sin A.$$

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Proof:
Use the same diagrams as for the proof of the sine rule.

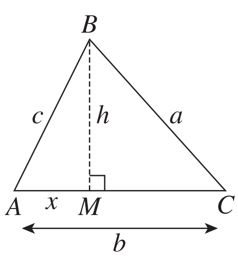
In case 2, $\angle A = 90°$ and $\sin A = 1$, so area $= \frac{1}{2}bc = \frac{1}{2}bc\sin A$, as required.

Otherwise:
$$\begin{aligned}\text{area} &= \tfrac{1}{2} \times \text{base} \times \text{height}\\ &= \tfrac{1}{2} \times AB \times h\\ &= \tfrac{1}{2} \times c \times b\sin A,\end{aligned}$$
because we proved before that $h = b\sin A$, in both cases.

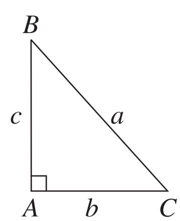
Proof of the cosine rule

The cosine rule says that in any triangle ABC,
$$a^2 = b^2 + c^2 - 2bc\cos A.$$

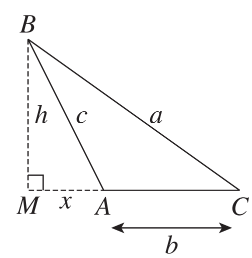
Given: Let ABC be any triangle. Again, there are three cases, depending on whether $\angle A$ is acute, a right angle or obtuse.



Case 1: $\angle A$ is acute



Case 2: $\angle A = 90°$



Case 3: $\angle A$ is obtuse

Aim: To prove that $a^2 = b^2 + c^2 - 2bc\cos A$.

In case 2, $\cos A = 0$, and this is just Pythagoras' theorem.

Construction: In the remaining cases 1 and 3, construct the altitude from B, meeting AC, produced if necessary, at M. Let $BM = h$ and $AM = x$.

Proof:

Case 1: Suppose that $\angle A$ is acute.
By Pythagoras' theorem in ΔBMC,
$$a^2 = h^2 + (b - x)^2.$$
By Pythagoras' theorem in ΔBMA,
$$\begin{aligned} h^2 &= c^2 - x^2,\\ \text{so } a^2 &= c^2 - x^2 + (b - x)^2\\ &= c^2 - x^2 + b^2 - 2bx + x^2\\ &= b^2 + c^2 - 2bx. \qquad (*)\end{aligned}$$
Using trigonometry in ΔABM,
$$\begin{aligned} x &= c\cos A,\\ \text{so } a^2 &= b^2 + c^2 - 2bc\cos A.\end{aligned}$$

Case 3: Suppose that $\angle A$ is obtuse.
By Pythagoras' theorem in ΔBMC,
$$a^2 = h^2 + (b + x)^2.$$
By Pythagoras' theorem in ΔBMA,
$$\begin{aligned} h^2 &= c^2 - x^2,\\ \text{so } a^2 &= c^2 - x^2 + (b + x)^2\\ &= c^2 - x^2 + b^2 + 2bx + x^2\\ &= b^2 + c^2 + 2bx. \qquad (*)\end{aligned}$$
Using trigonometry in ΔABM,
$$\begin{aligned} x &= c\cos(180° - A)\\ &= -c\cos A,\\ \text{so } a^2 &= b^2 + c^2 - 2bc\cos A.\end{aligned}$$

Note: The identity $\cos(180° - A) = -\cos A$ is the key step in Case 3 of the proof. The cosine rule appears in Euclid's geometry book, but without any mention of the cosine ratio — the form given there is approximately the two statements marked with (*) in the proof above.

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6 The coordinate plane

The earlier chapters have used graphs to turn functions into geometric objects within the coordinate plane, thus allowing them to be visualised and studied by geometric as well as algebraic methods. In the coordinate plane:

- Points are represented by pairs of numbers.
- Lines are represented by linear equations.
- Circles, parabolas and other curves are represented by non-linear equations.

Points, lines and intervals are the main concern of this chapter. Its purpose is to prepare for chapter 8, where the new topic of calculus will be introduced in the coordinate plane, heavily based on geometric ideas of tangents and areas.

This material may all have been covered in earlier years. Readers should do as much or as little of it as they need to in order to master the skills.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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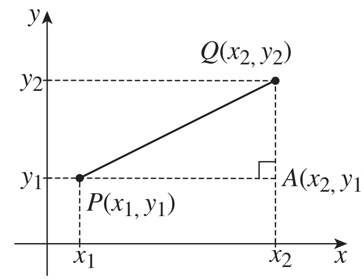
6A Lengths and midpoints of intervals

An *interval* is completely determined by its two endpoints. *In this chapter, unlike in Section 2B, the word 'interval' will always mean 'bounded interval'*. There are simple formulae for the length of an interval and for the midpoint of an interval.

The distance formula



The formula for the length of an interval PQ is just Pythagoras' theorem in different notation.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane. Construct the right-angled triangle ΔPQA, where $A(x_2, y_1)$ lies level with P and vertically above or below Q. Then $PA = |x_2 - x_1|$ and $QA = |y_2 - y_1|$, so by Pythagoras' theorem in ΔPQA,

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

1 DISTANCE FORMULA

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane. Then:

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

- First find the square PQ^2 of the distance.
- Then take the square root to find the distance PQ.

Example 1 6A

Find the lengths of the sides AB and AC of the triangle with vertices $A(1, -2)$, $B(-4, 2)$ and $C(5, -7)$, and say why ΔABC is isosceles.

SOLUTION

First,
$$\begin{aligned} AB^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (-4 - 1)^2 + \big(2 - (-2)\big)^2 \\ &= (-5)^2 + 4^2 \\ &= 41, \end{aligned}$$
so $AB = \sqrt{41}$.

Second,
$$\begin{aligned} AC^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (5 - 1)^2 + \big(-7 - (-2)\big)^2 \\ &= 4^2 + (-5)^2 \\ &= 41, \end{aligned}$$
so $AC = \sqrt{41}$.

Because the two sides AB and AC are equal, the triangle is isosceles.

The midpoint formula

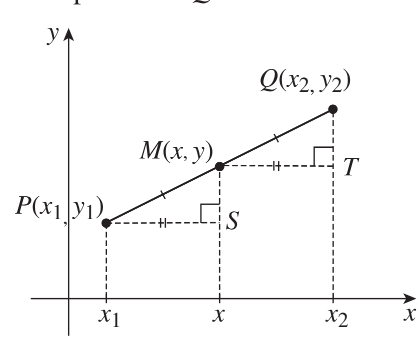
The midpoint of an interval is found by taking the averages of the coordinates of the two points. Congruence is the basis of the proof below.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane, and let $M(x, y)$ be the midpoint of PQ.
Construct $S(x, y_1)$ and $T(x_2, y)$, as shown.
Then $\Delta PMS \equiv \Delta MQT$ (AAS).

Hence $PS = MT$ (matching sides of congruent triangles).

$$x - x_1 = x_2 - x$$
$$2x = x_1 + x_2$$
$$x = \frac{x_1 + x_2}{2}, \quad \text{which is the average of } x_1 \text{ and } x_2.$$

The calculation of the y-coordinate of M is similar.



2 MIDPOINT FORMULA

Let $M(x, y)$ be the midpoint of the interval joining $P(x_1, y_1)$ and $Q(x_2, y_2)$. Then:

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2} \quad \text{(take the averages of the coordinates)}$$

Example 2 6A

The interval joining the points $A(3, -1)$ and $B(-7, 5)$ is a diameter of a circle. Find, for this circle:

a the centre M, **b** the radius, **c** the equation.

SOLUTION

a The centre of the circle is the midpoint $M(x, y)$ of the interval AB.

Using the midpoint formula,

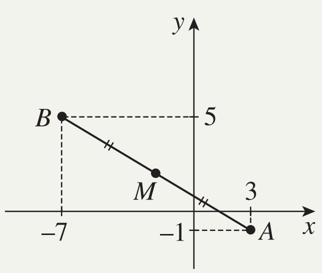
$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$
$$= \frac{3 - 7}{2} \qquad\qquad = \frac{-1 + 5}{2}$$
$$= -2, \qquad\qquad = 2,$$

so the centre is $M(-2, 2)$.

b Using the distance formula,

$$\begin{aligned} AM^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (-2 - 3)^2 + \big((2 - (-1)\big)^2 \\ &= 34, \\ AM &= \sqrt{34}. \end{aligned}$$

Hence the circle has radius $\sqrt{34}$.



c The equation of the circle is therefore $(x + 2)^2 + (y - 2)^2 = 34$.

Testing for special quadrilaterals

Many questions in this chapter ask for a proof that a quadrilateral is of a particular type. The most obvious way is to test the definition itself.

3 DEFINITIONS OF THE SPECIAL QUADRILATERALS

- A *trapezium* is a quadrilateral with one pair of opposite sides parallel.
- A *parallelogram* is a quadrilateral with both pairs of opposite sides parallel.
- A *rhombus* is a parallelogram with a pair of adjacent sides equal.
- A *rectangle* is a parallelogram with one angle a right angle.
- A *square* is both a rectangle and a rhombus.

There are, however, several further standard tests that the exercises assume, and that were developed in earlier years. (Tests involving angles are omitted here, as they are irrelevant in this chapter.)

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4 FURTHER STANDARD TESTS FOR SPECIAL QUADRILATERALS

A quadrilateral is a parallelogram:

- if the opposite sides are equal, or
- if one pair of opposite sides are equal and parallel, or
- if the diagonals bisect each other.

A quadrilateral is a rhombus:

- if all sides are equal, or
- if the diagonals bisect each other at right angles.

A quadrilateral is a rectangle:

- if the diagonals are equal and bisect each other.

Exercise 6A

FOUNDATION

Note: Diagrams should be drawn wherever possible.

1 Find the midpoint of each interval AB. Use $x = \dfrac{x_1 + x_2}{2}$ and $y = \dfrac{y_1 + y_2}{2}$.

a $A(3, 5)$ and $B(1, 9)$
b $A(4, 8)$ and $B(6, 4)$
c $A(-4, 7)$ and $B(8, -11)$
d $A(-3, 6)$ and $B(3, 1)$
e $A(0, -8)$ and $B(-11, -12)$
f $A(4, -7)$ and $B(4, 7)$

2 Find the length of each interval. Use $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, then find AB.

a $A(1, 4)$, $B(5, 1)$
b $A(-2, 7)$, $B(3, -5)$
c $A(-5, -2)$, $B(3, 4)$
d $A(3, 6)$, $B(5, 4)$
e $A(-4, -1)$, $B(4, 3)$
f $A(5, -12)$, $B(0, 0)$

3 a Find the midpoint M of the interval joining $P(-2, 1)$ and $Q(4, 9)$.
b Find the lengths PM and MQ, and verify that $PM = MQ$.

4 a Find the length of each side of the triangle formed by $P(0, 3)$, $Q(1, 7)$ and $R(5, 8)$.
b Hence show that ΔPQR is isosceles.

5 The vertices of the ΔABC are given by $A(0, 0)$, $B(9, 12)$ and $C(25, 0)$.
a Find the lengths of the three sides AB, BC, AC.
b By checking that $AB^2 + BC^2 = AC^2$, show that ΔABC is right-angled.

6 a Find the length of each side of ΔABC, where $A = (0, 5)$, $B = (3, -2)$ and $C = (-3, 4)$.
b Find the midpoint of each side of this triangle ABC.

DEVELOPMENT

7 a A circle with centre $O(0, 0)$ passes through $A(5, 12)$. What is its radius?
b A circle with centre $B(4, 5)$ passes through the origin. What is its radius?
c Find the centre of the circle with diameter CD, where $C = (2, 1)$ and $D = (8, -7)$.
d Show that $E(-12, -5)$ lies on the circle with centre the origin and radius 13.

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8 **a** Find the midpoint of the interval joining $A(4, 9)$ and $C(-2, 3)$.
b Find the midpoint of the interval joining $B(0, 4)$ and $D(2, 8)$.
c What can you conclude about the diagonals of the quadrilateral $ABCD$?
d What sort of quadrilateral is $ABCD$? (Hint: See Box 4 above.)

9 The points $A(3, 1)$, $B(10, 2)$, $C(5, 7)$ and $D(-2, 6)$ are the vertices of a quadrilateral.
a Find the lengths of all four sides.
b What sort of quadrilateral is $ABCD$? (Hint: See Box 4 above.)

10 **a** Find the side lengths of the triangle with vertices $X(0, -4)$, $Y(4, 2)$ and $Z(-2, 6)$.
b Show that ΔXYZ is a right-angled isosceles triangle by showing that its side lengths satisfy Pythagoras' theorem.
c Hence find the area of ΔXYZ.

11 **a** Find the distance of each point $A(1, 4)$, $B(2, \sqrt{13})$, $C(3, 2\sqrt{2})$ and $D(4, 1)$ from the origin O. Hence explain why the four points lie on a circle with centre the origin.
b What are the radius, diameter, circumference and area of this circle?

12 The point $M(3, 7)$ is the midpoint of the interval joining $A(1, 12)$ and $B(x_2, y_2)$. Find the coordinates x_2 and y_2 of B by substituting into the formulae:

$$x = \frac{x_1 + x_2}{2} \qquad \text{and} \qquad y = \frac{y_1 + y_2}{2}.$$

13 Solve each problem using the same method as in the previous question.
a If $A(-1, 2)$ is the midpoint of $S(x, y)$ and $T(3, 6)$, find the coordinates of S.
b The midpoint of the interval PQ is $M(2, -7)$. Find the coordinates of P if:
i $Q = (0, 0)$ **ii** $Q = (5, 3)$ **iii** $Q = (-3, -7)$
c Find B, if AB is a diameter of a circle with centre $Q(4, 5)$, and $A = (8, 3)$.
d Given that $P(4, 7)$ is one vertex of the square $PQRS$, and that the centre of the square is $M(8, -1)$, find the coordinates of the opposite vertex R.

14 **a** Write down any two points A and B whose midpoint is $M(4, 6)$.
b Write down any two points C and D that are 10 units apart.

CHALLENGE

15 Each set of three points given below forms a triangle of one of these types:
A isosceles, **B** equilateral, **C** right-angled, **D** none of these.
Find the side lengths of each triangle below and hence determine its type.
a $A(-1, 0)$, $B(1, 0)$, $C(0, \sqrt{3})$ **b** $P(-1, 1)$, $Q(0, -1)$, $R(3, 3)$
c $D(1, 1)$, $E(2, -2)$, $F(-3, 0)$ **d** $X(-3, -1)$, $Y(0, 0)$, $Z(-2, 2)$

16 As was discussed in Section 4A, the circle with centre (h, k) and radius r has equation $(x - h)^2 + (y - k)^2 = r^2$. By identifying the centre and radius, find the equations of:
a the circle with centre $(5, -2)$ and passing through $(-1, 1)$,
b the circle with $K(5, 7)$ and $L(-9, -3)$ as endpoints of a diameter.

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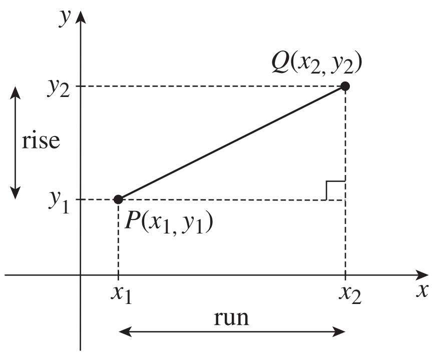
6B Gradients of intervals and lines

Gradient is the key idea that will be used in the next section to bring lines and their equations into the coordinate plane.

The gradient of an interval and the gradient of a line

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane.
The *gradient* of the interval PQ is a measure of its steepness, as someone walks along the interval from P to Q.
The *rise* is the vertical difference $y_2 - y_1$,
and the *run* is the horizontal difference $x_2 - x_1$.
The *gradient* of the interval PQ is defined to be the ratio of the rise and the run:

$$\text{gradient of } PQ = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$



5 THE GRADIENT OF AN INTERVAL AND THE GRADIENT OF A LINE

The gradient of an interval

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points. Then

$$\text{gradient of } PQ = \frac{\text{rise}}{\text{run}} \qquad \text{that is,} \qquad \text{gradient of } PQ = \frac{y_2 - y_1}{x_2 - x_1}.$$

- Horizontal intervals have gradient zero, because the rise is always zero.
- Vertical intervals don't have a gradient — the run is always zero, so the fraction is undefined.

The gradient of a line:

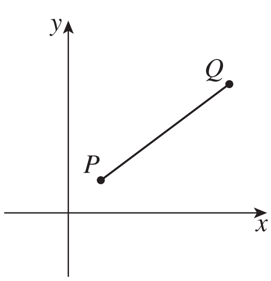
- The *gradient of a line* is found by taking any two distinct points P and Q on the line and finding the gradient of the interval PQ.
- It doesn't matter which two points on the line are taken, because the ratio of rise over run will be the same, as is easily seen using similar triangles.

Positive and negative gradients

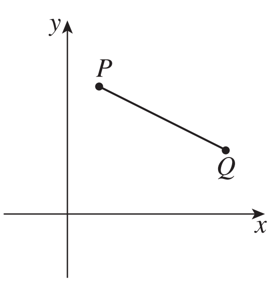
If the rise and the run have the same sign, then the gradient is *positive*, as in the first diagram below. In this case the interval slopes *upwards* as we move from left to right.

If the rise and run have opposite signs, then the gradient is *negative*, as in the second diagram. The interval slopes *downwards* as we move from left to right.

If the points P and Q are interchanged, in either diagram, then the rise and the run both change sign, but the gradient remains the same.



Positive gradient



Negative gradient

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Example 3 6B

Find the gradients of the sides of ΔXYZ, where $X = (2, 5)$, $Y = (5, -2)$ and $Z = (-3, 4)$.

SOLUTION

$$\begin{aligned}\text{gradient of } XY &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{-2-5}{5-2}\\ &= \frac{-7}{3}\\ &= -\tfrac{7}{3}\end{aligned}$$

$$\begin{aligned}\text{gradient of } YZ &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{4-(-2)}{-3-5}\\ &= \frac{4+2}{-8}\\ &= -\tfrac{3}{4}\end{aligned}$$

$$\begin{aligned}\text{gradient of } ZX &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{5-4}{2-(-3)}\\ &= \frac{1}{2+3}\\ &= \tfrac{1}{5}\end{aligned}$$

A condition for two lines to be parallel

Any two vertical lines are parallel. Otherwise, lines are parallel when their gradients are equal.

6 PARALLEL LINES

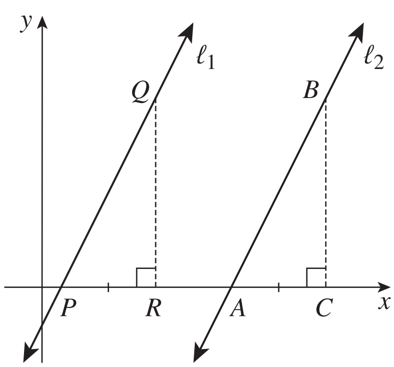
Two lines are parallel if and only if:

- they have the same gradient OR they are both vertical.

The phrase 'if and only if' means that:

- if the condition holds, then the lines are parallel, and
- if the lines are parallel then the condition holds.

The two statements are called *converses* of each other. 'If and only if' is a very convenient abbreviation, but its logic needs attention.



Proof (Assuming neither line is vertical)
The two lines ℓ_1 and ℓ_2 meet the x-axis at P and A.
Construct $PR = AC$, then construct QR and BC perpendicular to the x-axis.

A If the lines ℓ_1 and ℓ_2 are parallel, then

$\angle QPR = \angle BAC$ (corresponding angles on parallel lines)

Hence $\Delta PQR \equiv \Delta ABC$ (AAS)

so $QR = BC$ (matching sides of congruent triangles)

$\dfrac{QR}{PR} = \dfrac{BC}{AC}$ (because $PR = AC$).

gradient ℓ_1 = gradient ℓ_2.

B Conversely, if the two gradients are equal, then

$\dfrac{QR}{PR} = \dfrac{BC}{AC}$

$QR = BC$ (because $PR = AC$).

Hence $\Delta PQR \equiv \Delta ABC$ (SAS)

so $\ell_1 \parallel \ell_2$ (corresponding angles are equal).

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Example 4 6B

Given the four points $A(3, 6)$, $B(7, -2)$, $C(4, -5)$ and $D(-1, 5)$, show that the quadrilateral $ABCD$ is a trapezium with $AB \parallel CD$.

SOLUTION

$$\begin{aligned}\text{gradient of } AB &= \frac{-2-6}{7-3}\\ &= \frac{-8}{4}\\ &= -2,\end{aligned} \qquad \begin{aligned}\text{gradient of } CD &= \frac{5-(-5)}{-1-4}\\ &= \frac{10}{-5}\\ &= -2.\end{aligned}$$

Hence $AB \parallel CD$ because their gradients are equal, so $ABCD$ is a trapezium.

Testing for collinear points

Three points are called *collinear* if they all lie on one line.

7 TESTING FOR COLLINEAR POINTS

- To test whether three points A, B and C are collinear, test whether AB and BC are parallel. (Are they both vertical, or do they have the same gradient?)
- If AB and BC are parallel, then the three points are collinear, because then AB and BC are parallel lines passing through a common point B.

Example 5 6B

Test whether the three points $A(-2, 5)$, $B(1, 3)$ and $C(7, -1)$ are collinear.

SOLUTION

$$\begin{aligned}\text{gradient of } AB &= \frac{3-5}{1+2}\\ &= -\tfrac{2}{3},\end{aligned} \qquad \begin{aligned}\text{gradient of } BC &= \frac{-1-3}{7-1}\\ &= -\tfrac{2}{3}.\end{aligned}$$

Because the gradients are equal, the points A, B and C are collinear.

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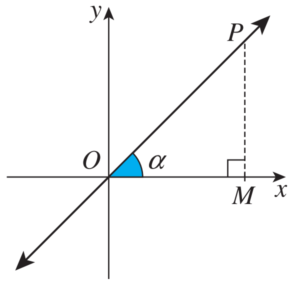
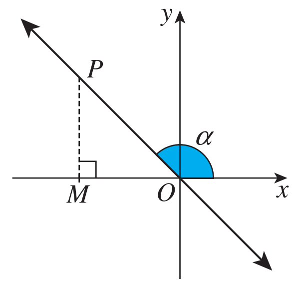
Gradient and the angle of inclination

The *angle of inclination* of a line is the angle between the upward direction of the line and the positive direction of the x-axis.





The two diagrams above show that lines with positive gradients have acute angles of inclination, and lines with negative gradients have obtuse angles of inclination. They also illustrate the trigonometric relationship between the gradient and the angle of inclination α.

8 ANGLE OF INCLINATION

Suppose that a line has angle of inclination α. Then:

- gradient of line $= \tan\alpha$ OR the line is vertical and $\alpha = 90°$.

Proof

A When α is acute, as in the first diagram above, then the rise MP and the run OM are the opposite and adjacent sides of the triangle POM, so

$$\tan\alpha = \frac{MP}{OM} = \text{gradient of } OP.$$

B When α is obtuse, as in the second diagram, then $\angle POM = 180° - \alpha$, so

$$\tan\alpha = -\tan\angle POM = -\frac{MP}{OM} = \text{gradient of } OP.$$

Example 6 6B

a Given the points $A(-3, 5)$, $B(-6, 0)$ and $O(0, 0)$, find the angles of inclination of the intervals AB and AO.

b What sort of triangle is ΔABO?

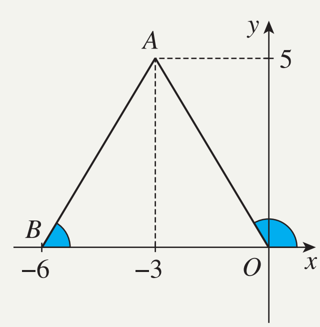
SOLUTION

a First,

$$\text{gradient of } AB = \frac{0-5}{-6+3} = \tfrac{5}{3},$$

and using a calculator to solve $\tan\alpha = \frac{5}{3}$,

angle of inclination of $AB \doteqdot 59°$.



Secondly,

$$\text{gradient of } AO = \frac{0-5}{0+3} = -\tfrac{5}{3},$$

and using a calculator to solve $\tan\alpha = -\frac{5}{3}$,

angle of inclination of $AO \doteqdot 121°$.

b Hence $\angle AOB = 59°$ (straight angle).

Thus the base angles of ΔAOB are equal, so the triangle is isosceles.

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A condition for lines to be perpendicular

A vertical and a horizontal line are perpendicular. Otherwise, the condition is that the product of their gradients is -1.

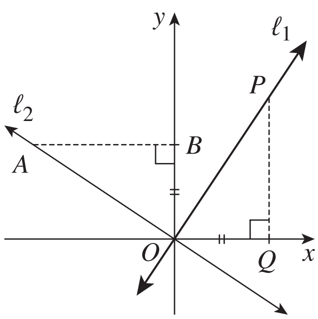
9 PERPENDICULAR LINES

Two lines are perpendicular if and only if:

- $m_1 m_2 = -1$, where m_1 and m_2 are the gradients of the two lines.
  OR
- one is vertical and the other is horizontal.

$m_1 m_2 = -1$ can also be written as $m_2 = -\dfrac{1}{m_1}$.

Proof (Assuming that neither is vertical)



Shift each line sideways, without rotating it, so that the lines intersect at the origin. One line must have positive gradient and the other negative gradient, otherwise one of the angles between them would be acute.
So let ℓ_1 be a line with positive gradient through the origin,
and let ℓ_2 be a line with negative gradient through the origin.
Construct the two triangles POQ and AOB as shown in the diagram, with the run OQ of ℓ_1 equal to the rise OB of ℓ_2.

Then $m_1 \times m_2 = \dfrac{QP}{OQ} \times \left(-\dfrac{OB}{AB}\right) = -\dfrac{QP}{AB}$, because $OQ = OB$.

A If the lines are perpendicular, then $\angle AOB = \angle POQ$. (adjacent angles at O).
Hence $\Delta AOB \equiv \Delta POQ$ (AAS)
so $QP = AB$ (matching sides of congruent triangles)
so $m_1 \times m_2 = -1$.

B Conversely, if $m_1 \times m_2 = -1$, then $QP = AB$.
Hence $\Delta AOB \equiv \Delta POQ$ (SAS)
so $\angle AOB = \angle POQ$, (matching angles of congruent triangles)
and so ℓ_1 and ℓ_2 are perpendicular.

Example 7 6B

What is the gradient of a line perpendicular to a line with gradient $\frac{2}{3}$?

SOLUTION

Perpendicular gradient $= -\frac{3}{2}$ $\left(\text{take the opposite of the reciprocal of } \frac{2}{3}\right)$

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Example 8 — 6B

Show that the diagonals of the quadrilateral $ABCD$ are perpendicular, where the vertices are $A(3, 7)$, $B(-1, 6)$, $C(-2, -3)$ and $D(11, 0)$.

SOLUTION

$$\text{Gradient of } AC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 7}{-2 - 3} = 2,$$

$$\text{Gradient of } BD = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 6}{11 + 1} = -\tfrac{1}{2}.$$

Hence $AC \perp BD$, because the product of the gradients of AC and BD is -1.

Example 9 — 6B

The interval joining the points $C(-6, 0)$ and $D(-1, a)$ is perpendicular to a line with gradient 10. Find the value of a.

SOLUTION

$$\text{The interval } CD \text{ has gradient} = \frac{a - 0}{-1 + 6} = \frac{a}{5}.$$

Because the interval CD is perpendicular to a line with gradient 10,

$$\frac{a}{5} \times \frac{10}{1} = -1 \quad \text{(the product of the gradients is } -1\text{)}$$

$$a \times 2 = -1$$

$$\boxed{\div 2} \qquad a = -\tfrac{1}{2}.$$

Exercise 6B — FOUNDATION

Note: Diagrams should be drawn wherever possible.

1 **a** Write down the gradient of a line parallel to a line with gradient:

i 2 **ii** $\frac{3}{4}$ **iii** $-1\frac{1}{2}$

b Find the gradient of a line perpendicular to a line with gradient:

i 2 **ii** $\frac{3}{4}$ **iii** $-1\frac{1}{2}$

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2 Use the formula gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$ to find the gradient of each interval AB below. Then find the gradient of a line perpendicular to it.

a $A(1, 4), B(5, 0)$ b $A(-2, -7), B(3, 3)$ c $A(-5, -2), B(3, 2)$
d $A(3, 6), B(5, 5)$ e $A(-1, -2), B(1, 4)$ f $A(-5, 7), B(15, -7)$

3 Each of the lines AB in this question are horizontal (with gradient zero), vertical (with gradient undefined) or neither. State which in each case.

a $A(1, 2), B(1, 3)$ b $A(4, 3), B(7, 3)$ c $A(5, 1), B(4, 5)$
d $A(6, -3), B(1, -3)$ e $A(-2, 2), B(2, -2)$ f $A(0, 4), B(0, -4)$

4 The points $A(2, 5)$, $B(4, 11)$, $C(12, 15)$ and $D(10, 9)$ form a quadrilateral.

a Find the gradients of AB and DC, and hence show that $AB \parallel DC$.
b Find the gradients of BC and AD, and hence show that $BC \parallel AD$.
c What type of quadrilateral is $ABCD$? (Hint: Look at the definitions in Box 3.)

5 a Show that $A(-2, -6)$, $B(0, -5)$, $C(10, -7)$ and $D(8, -8)$ form a parallelogram.
b Show that $A(2, 5)$, $B(3, 7)$, $C(-4, -1)$ and $D(-5, 2)$ do not form a parallelogram.

6 Use the formula gradient $= \tan\alpha$ to find the gradient, correct to two decimal places where necessary, of a line with angle of inclination:

a $15°$ b $135°$ c $22\frac{1}{2}°$ d $72°$

7 Use the formula gradient $= \tan\alpha$ to find the angle of inclination, correct to the nearest degree where necessary, of a line with gradient:

a 1 b $-\sqrt{3}$ c 4 d $\dfrac{1}{\sqrt{3}}$

8 The gradient measures the steepness of a line. Each of the following lines AB has positive gradient. By comparing the gradient to 1, determine if the angle of inclination is less than, equal to, or more than 45°.

a $A(1, 3), B(6, 5)$ b $A(4, 1), B(7, 4)$
c $A(-3, 2), B(1, 5)$ d $A(-3, 4), B(-7, -2)$

DEVELOPMENT

9 The quadrilateral $ABCD$ has vertices $A(-1, 1)$, $B(3, -1)$, $C(5, 3)$ and $D(1, 5)$. Use the definitions of the special quadrilaterals in Box 3 to answer these questions.

a Show that the opposite sides are parallel, and hence that $ABCD$ is a parallelogram.
b Show that $AB \perp BC$, and hence that $ABCD$ is a rectangle.
c Show that $AB = BC$, and hence that $ABCD$ is a square.

10 Use gradients to show that each quadrilateral $ABCD$ is a parallelogram. Then use the definitions of the special quadrilaterals in Box 3 to show that it is:

a a rhombus, for the vertices $A(2, 1)$, $B(-1, 3)$, $C(1, 0)$ and $D(4, -2)$,
b a rectangle, for the vertices $A(4, 0)$, $B(-2, 3)$, $C(-3, 1)$ and $D(3, -2)$,
c a square, for the vertices $A(3, 3)$, $B(-1, 2)$, $C(0, -2)$ and $D(4, -1)$.

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11 Find the gradients of PQ and QR, and hence determine whether P, Q and R are collinear.

a $P(-2, 7)$, $Q(1, 1)$, $R(4, -6)$ **b** $P(-5, -4)$, $Q(-2, -2)$, $R(1, 0)$

12 Show that the four points $A(2, 5)$, $B(5, 6)$, $C(11, 8)$ and $D(-16, -1)$ are collinear.

13 The triangle ABC has vertices $A(-1, 0)$, $B(3, 2)$ and $C(4, 0)$. Calculate the gradient of each side, and hence show that ΔABC is a right-angled triangle.

14 Similarly, show that each triangle below is right-angled. Then find the lengths of the sides enclosing the right angle, and calculate the area of each triangle.

a $P(2, -1)$, $Q(3, 3)$, $R(-1, 4)$ **b** $X(-1, -3)$, $Y(2, 4)$, $Z(-3, 2)$

15 a Write down two points A and B for which the interval AB has gradient 3.

b Write down two points A and B for which the interval AB is vertical.

16 The interval PQ has gradient -3. A second line passes through $A(-2, 4)$ and $B(1, k)$.

a Find k if AB is parallel to PQ.

b Find k if AB is perpendicular to PQ.

17 Find the points A and B where each line below meets the x-axis and y-axis respectively. Hence find the gradient of AB and its angle of inclination α (correct to the nearest degree).

a $y = 3x + 6$ **b** $y = -\frac{1}{2}x + 1$ **c** $3x + 4y + 12 = 0$ **d** $\frac{x}{3} - \frac{y}{2} = 1$

18 The quadrilateral $ABCD$ has vertices $A(1, -4)$, $B(3, 2)$, $C(-5, 6)$ and $D(-1, -2)$.

a Find the midpoints P of AB, Q of BC, R of CD, and S of DA.

b Prove that $PQRS$ is a parallelogram by showing that $PQ \parallel RS$ and $PS \parallel QR$.

19 a Show that the points $A(-5, 0)$, $B(5, 0)$ and $C(3, 4)$ all lie on the circle $x^2 + y^2 = 25$.

b Explain why AB is a diameter of the circle.

c Show that $AC \perp BC$.

CHALLENGE

20 Find the gradient, correct to two decimal places where appropriate, of a line sloping upwards, if its acute angle with the y-axis is:

a $15°$ **b** $45°$ **c** $22\frac{1}{2}°$ **d** $72°$

21 Given the points $X(-1, 0)$, $Y(1, a)$ and $Z(a, 2)$, find a if $\angle YXZ = 90°$.

22 For the four points $P(k, 1)$, $Q(-2, -3)$, $R(2, 3)$ and $S(1, k)$, it is known that PQ is parallel to RS. Find the possible values of k.

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6C Equations of lines

In the coordinate plane, a line is represented by a linear equation in x and y. This section and the next develop various useful forms for the equation of a line.

Horizontal and vertical lines

All the points on a vertical line must have the same x-coordinate, but the y-coordinate can take any value.

10 VERTICAL LINES

The vertical line through the point $P(a, b)$ has equation

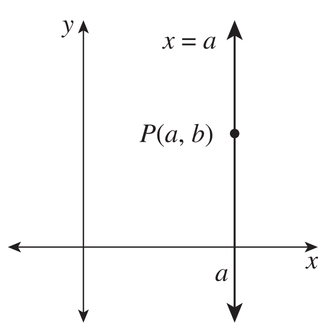
$$x = a.$$

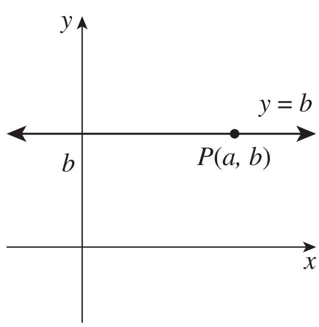
All the points on a horizontal line must have the same y-coordinate, but the x-coordinate can take any value.

11 HORIZONTAL LINES

The horizontal line through the point $P(a, b)$ has equation

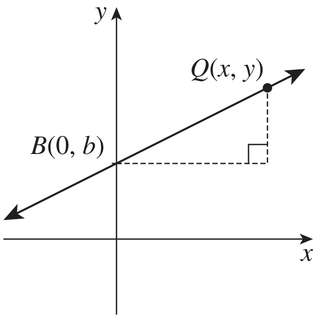
$$y = b.$$





Gradient–intercept form

There is a simple form of the equation of a line whose gradient and y-intercept are known.
Let ℓ have gradient m and y-intercept b, and pass through the point $B(0, b)$.
Let $Q(x, y)$ be any other point in the plane.
Then the condition for Q to lie on the line ℓ is



$$\text{gradient of } BQ = m,$$

that is,

$$\frac{y-b}{x-0} = m \quad \text{(this is the formula for gradient)}$$

$$y - b = mx$$

$$y = mx + b.$$

12 GRADIENT–INTERCEPT FORM

The line with gradient m and y-intercept b is

$$y = mx + b.$$

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Example 10 6C

a Write down the gradient and the y-intercept of the line ℓ: $y = 3x - 2$.
b Find the equation of the line through $B(0, 5)$ parallel to ℓ.
c Find the equation of the line through $B(0, 5)$ perpendicular to ℓ.

SOLUTION

a The line ℓ: $y = 3x - 2$ has gradient 3 and y-intercept -2.

b The line through $B(0, 5)$ parallel to ℓ has gradient 3 and y-intercept 5, so its equation is $y = 3x + 5$.

c The line through $B(0, 5)$ perpendicular to ℓ has gradient $-\frac{1}{3}$ and y-intercept 5, so its equation is $y = -\frac{1}{3}x + 5$.

General form

It is often useful to write the equation of a line so that all the terms are on the LHS and only zero is on the RHS. This is called *general form*.

13 GENERAL FORM

The equation of a line is said to be in *general form* if it is

$ax + by + c = 0$, where a, b and c are constants.

When an equation is given in general form, it should usually be *simplified* by multiplying out all fractions and dividing through by all common factors.

Example 11 6C

a Put the equation of the line $3x + 4y + 5 = 0$ into gradient–intercept form.
b Hence find the gradient and y-intercept of the line $3x + 4y + 5 = 0$.

SOLUTION

a Solving the equation for y, $4y = -3x - 5$

$\boxed{\div 4}$ $y = -\frac{3}{4}x - \frac{5}{4}$, which is gradient–intercept form.

b Hence the line has gradient $-\frac{3}{4}$ and y-intercept $-1\frac{1}{4}$.

Example 12 6C

Find, in general form, the equation of the line passing through $B(0, -2)$ and:

a perpendicular to ℓ: $3y = 2x$,

b having angle of inclination 60°.

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SOLUTION

a The line ℓ is $y = \frac{2}{3}x$, so its gradient is $\frac{2}{3}$. The required line thus has perpendicular gradient $-\frac{3}{2}$ and its y-intercept is -2,

so its equation is $\quad y = -\frac{3}{2}x - 2 \quad$ (this is gradient–intercept form)

$\boxed{\times 2} \quad 2y = -3x - 4$

$3x + 2y + 4 = 0. \quad$ (this is general form)

b The required line has gradient $\tan 60^\circ = \sqrt{3}$, and its y-intercept is -2,

so its equation is $\quad y = x\sqrt{3} - 2 \quad$ (this is gradient–intercept form)

$x\sqrt{3} - y - 2 = 0. \quad$ (this is general form)

Exercise 6C

FOUNDATION

1 Determine, by substitution, whether the point $A(3, -2)$ lies on the line:
a $y = 4x - 10$ **b** $8x + 10y - 4 = 0$ **c** $x = 3$

2 Find the x-intercept and y-intercept of each line.
a $3x + 4y = 12$ **b** $y = 4x - 6$ **c** $x - 2y = 8$

3 Write down the coordinates of any three points on the line $x + 3y = 24$.

4 Write down the equations of the vertical and horizontal lines through:
a $(1, 2)$ **b** $(0, -4)$ **c** $(5, 0)$

5 Write down the gradient and y-intercept of each line.
a $y = 4x - 2$ **b** $y = \frac{1}{5}x - 3$ **c** $y = 2 - x$ **d** $y = -\frac{5}{7}x$

6 Use the formula $y = mx + b$ to write down the equation of the line with gradient -3 and:
a y-intercept 5 **b** y-intercept $-\frac{2}{3}$ **c** y-intercept 0

7 Use the formula $y = mx + b$ to write down the equation of the line with y-intercept -4 and:
a gradient 5 **b** gradient $-\frac{2}{3}$ **c** gradient 0

8 Use the formula $y = mx + b$ to write down the equation of the line:
a with gradient 1 and y-intercept 3
b with gradient -2 and y-intercept 5
c with gradient $\frac{1}{5}$ and y-intercept -1
d with gradient $-\frac{1}{2}$ and y-intercept 3.

9 Solve each equation for y and hence write down its gradient m and y-intercept b.
a $x - y + 3 = 0$ **b** $y + x - 2 = 0$ **c** $x - 3y = 0$ **d** $3x + 4y = 5$

10 Write down the gradient m of each line. Then use the formula gradient $= \tan\alpha$ to find its angle of inclination α, correct to the nearest minute where appropriate.
a $y = x + 3$ **b** $y = -x - 16$ **c** $y = 2x$ **d** $y = -\frac{3}{4}x$

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DEVELOPMENT

11 Substitute $y = 0$ and $x = 0$ into the equation of each line below to find the points A and B where the line crosses the x-axis and y-axis respectively. Hence sketch the line.

a $5x + 3y - 15 = 0$ **b** $2x - y + 6 = 0$ **c** $3x - 5y + 12 = 0$

12 Find the gradient of the line through each pair of given points. Then find its equation, using gradient–intercept form. Give your final answer in general form.

a $(0, 4), (2, 8)$ **b** $(0, 0), (1, -1)$ **c** $(-9, -1), (0, -4)$

13 Find the gradient of each line below. Hence find, in gradient–intercept form, the equation of a line passing through $A(0, 3)$ and:

i parallel to it

ii perpendicular to it.

a $2x + y + 3 = 0$ **b** $5x - 2y - 1 = 0$ **c** $3x + 4y - 5 = 0$

14 In each part, find the gradients of the four lines, and hence state the sort of special quadrilateral they enclose.

a $3x + y + 7 = 0$, $x - 2y - 1 = 0$, $3x + y + 11 = 0$, $x - 2y + 12 = 0$

b $4x - 3y + 10 = 0$, $3x + 4y + 7 = 0$, $4x - 3y - 7 = 0$, $3x + 4y + 1 = 0$

15 Find the gradients of the three lines $5x - 7y + 5 = 0$, $2x - 5y + 7 = 0$ and $7x + 5y + 2 = 0$. Hence show that they enclose a right-angled triangle.

16 Draw a sketch, then find the equations of the sides of:

a the rectangle with vertices $P(3, -7)$, $Q(0, -7)$, $R(0, -2)$ and $S(3, -2)$

b the triangle with vertices $F(3, 0)$, $G(-6, 0)$ and $H(0, 12)$.

17 In each part below, the angle of inclination α and the y-intercept A of a line are given. Use the formula gradient $= \tan\alpha$ to find the gradient of each line, then find its equation in general form.

a $\alpha = 45°$, $A = (0, 3)$ **b** $\alpha = 60°$, $A = (0, -1)$

c $\alpha = 30°$, $A = (0, -2)$ **d** $\alpha = 135°$, $A = (0, 1)$

CHALLENGE

18 A triangle is formed by the x-axis and the lines $5y = 9x$ and $5y + 9x = 45$.

a Find (correct to the nearest degree) the angles of inclination of the two lines.

b What sort of triangle has been formed?

19 Consider the lines ℓ_1: $3x - y + 4 = 0$ and ℓ_2: $x + ky + \ell = 0$. Find the value of k if:

a ℓ_1 is parallel to ℓ_2

b ℓ_1 is perpendicular to ℓ_2.

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6D Further equations of lines

This section introduces another standard form of the equation of a line, called the *point–gradient form*. It also deals with lines through two given points, and the point of intersection of two lines.

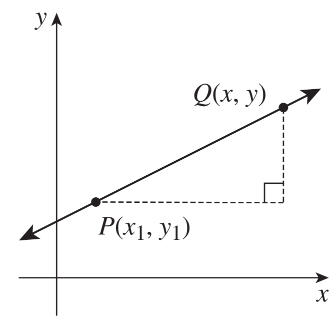
People often ask what form the equation of a line should be put in when the question specifies no form. There are two forms that are acceptable in this situation:

gradient–intercept form and general form.

If general form is used, then in most circumstances it should be simplified by multiplying out any fractions, and dividing through by any common factor.

Point–gradient form

Point–gradient form gives the equation of a line with gradient m passing through a particular point $P(x_1, y_1)$.



Let $Q(x, y)$ be any other point in the plane. Then the condition for Q to lie on the line is

gradient of $PQ = m$

that is, $\dfrac{y - y_1}{x - x_1} = m$ (this is the formula for gradient)

$y - y_1 = m(x - x_1)$.

14 POINT–GRADIENT FORM

The line with gradient m through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

Example 13 **6D**

a Find the equation of the line through $(-2, -5)$ and parallel to $y = 3x + 2$.

b Express the answer in gradient–intercept form, and hence find its y-intercept.

SOLUTION

a The line $y = 3x + 2$ has gradient 3.

Hence the required line is $y - y_1 = m(x - x_1)$ (this is point–gradient form)

$y + 5 = 3(x + 2)$

$y + 5 = 3x + 6$

$y = 3x + 1$. (this is gradient–intercept form)

b Hence the new line has y-intercept 1.

The line through two given points

Given two distinct points, there is just one line passing through them both. Its equation is best found by a two-step approach.

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15 THE LINE THROUGH TWO GIVEN POINTS

- First find the gradient of the line, using the gradient formula.
- Then find the equation of the line, using point–gradient form.

Example 14 6D

Find the equation of the line passing through $A(1, 5)$ and $B(4, -1)$.

SOLUTION

First, using the gradient formula,

$$\begin{aligned} \text{gradient of } AB &= \frac{-1-5}{4-1} \\ &= -2 \end{aligned}$$

Then, using point–gradient form for a line with gradient -2 through $A(1, 5)$, the line AB is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= -2(x - 1) \\ y - 5 &= -2x + 2 \\ y &= -2x + 7. \end{aligned}$$

Note: Using the coordinates of $B(4, -1)$ rather than $A(1, 5)$ would give the same equation.

Example 15 6D

Given the points $A(6, 0)$ and $B(0, 9)$, find, in general form, the equation of the perpendicular bisector of AB.

SOLUTION

First,

$$\begin{aligned} \text{gradient of } AB &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{9-0}{0-6} \\ &= -\tfrac{3}{2}, \end{aligned}$$

so any line perpendicular to AB has gradient $\frac{2}{3}$.

Secondly,

$$\begin{aligned} \text{midpoint of } AB &= \left(\frac{6+0}{2}, \frac{0+9}{2}\right) \\ &= \left(3, 4\tfrac{1}{2}\right). \end{aligned}$$

Hence, using point–gradient form, the perpendicular bisector of AB is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4\tfrac{1}{2} &= \tfrac{2}{3}(x - 3) \\ \boxed{\times 6} \qquad 6y - 27 &= 4x - 12 \\ 4x - 6y + 15 &= 0. \end{aligned}$$

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Intersection of lines — concurrent lines

The point where two distinct lines intersect can be found using simultaneous equations, as discussed in Section 1G.

Three distinct lines are called *concurrent* if they all pass through the same point.

16 TESTING FOR CONCURRENT LINES

To test whether three lines are concurrent:

- First find the point of intersection of two of the lines.
- Then test, by substitution, whether this point lies on the third line.

Example 16 6D

Test whether the following three lines are concurrent.

ℓ_1: $5x - y - 10 = 0$, ℓ_2: $x + y - 8 = 0$, ℓ_3: $2x - 3y + 9 = 0$.

SOLUTION

A Solve ℓ_1 and ℓ_2 simultaneously.

Adding ℓ_1 and ℓ_2, $6x - 18 = 0$

$x = 3$,

and substituting into ℓ_2, $3 + y - 8 = 0$

$y = 5$

so the lines ℓ_1 and ℓ_2 intersect at $(3, 5)$.

B Substituting the point $(3, 5)$ into the third line ℓ_3,

$$\begin{aligned}\text{LHS} &= 6 - 15 + 9\\ &= 0\\ &= \text{RHS},\end{aligned}$$

so the three lines are concurrent, meeting at $(3, 5)$.

Exercise 6D

FOUNDATION

1 Use point–gradient form $y - y_1 = m(x - x_1)$ to find the equation of the line through the point $P(3, 5)$ with gradient 3. Rearrange your answer into general form.

2 Use point–gradient form $y - y_1 = m(x - x_1)$ to find the equation of the line through the point $P(-2, 7)$ with each given gradient. Rearrange your answer into general form.

a gradient 6 **b** gradient -2 **c** gradient $\frac{2}{3}$ **d** gradient $-\frac{7}{2}$

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3 Use point–gradient form $y - y_1 = m(x - x_1)$ to find the equation of the line through each given point with gradient $-\frac{3}{5}$. Rearrange your answer into general form.

a through $(1, 2)$ b through $(6, 0)$ c through $(-5, 3)$ d through $(0, -4)$

4 Find, in general form, the equation of the line:

a through $(1, 1)$ with gradient 2,

b with gradient -1 through $(3, 1)$,

c through $(0, 0)$ with gradient -5,

d through $(-1, 3)$ with gradient $-\frac{1}{3}$,

e with gradient $-\frac{4}{5}$ through $(3, -4)$.

5 Find, in gradient–intercept form, the equation of:

a the line through $(2, 5)$ and parallel to $y = 2x + 5$,

b the line through $(2, 5)$ and perpendicular to $y = 2x + 5$,

c the line through $(5, -7)$ and perpendicular to $y = -5x$,

d the line through $(-7, 6)$ and parallel to $y = \frac{3}{7}x - 8$,

e the line through $(-4, 0)$ and perpendicular to $y = -\frac{2}{5}x$.

6 a Find the gradient of the line through the points $A(4, 7)$ and $B(6, 13)$.

b Hence use point–gradient form to find, in general form, the equation of the line AB.

7 Find the gradient of the line through each pair of points, and hence find its equation.

a $(3, 4)$, $(5, 8)$ b $(-1, 3)$, $(1, -1)$ c $(5, 6)$, $(-1, 4)$

d $(-1, 0)$, $(0, 2)$ e $(0, -1)$, $(-4, 0)$ f $(0, -3)$, $(3, 0)$

8 a Find the gradient of the line through $A(1, -2)$ and $B(-3, 4)$.

b Hence find, in general form, the equation of:

i the line AB,

ii the line through A and perpendicular to AB.

9 Find the equation of the line parallel to $2x - 3y + 1 = 0$ and:

a passing through $(2, 2)$,

b passing through $(3, -1)$.

10 Find the equation of the line perpendicular to $3x + 4y - 3 = 0$ and:

a passing through $(-1, -4)$,

b passing through $(-2, 1)$.

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DEVELOPMENT

11 a Find the point M of intersection of the lines ℓ_1: $x + y = 2$ and ℓ_2: $4x - y = 13$.

b Show that M lies on ℓ_3: $2x - 5y = 11$, and hence that ℓ_1, ℓ_2 and ℓ_3 are concurrent.

c Use the same method to test whether each set of lines is concurrent.

i $2x + y = -1$, $x - 2y = -18$ and $x + 3y = 15$

ii $6x - y = 26$, $5x - 4y = 9$ and $x + y = 9$

12 Put the equation of each line into gradient–intercept form and hence write down the gradient. Then find, in gradient–intercept form, the equation of the line that is:

i parallel to it through $A(3, -1)$,

ii perpendicular to it through $B(-2, 5)$.

a $2x + y + 3 = 0$ **b** $5x - 2y - 1 = 0$ **c** $4x + 3y - 5 = 0$

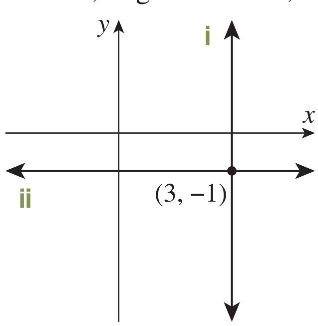
13 The angle of inclination α and a point A on a line are given below. Use the formula gradient $= \tan\alpha$ to find the gradient of each line, then find its equation in general form.

a $\alpha = 45°$, $A = (1, 0)$ **b** $\alpha = 120°$, $A = (-1, 0)$

c $\alpha = 30°$, $A = (4, -3)$ **d** $\alpha = 150°$, $A = (-2, -5)$

14 Determine, in general form, the equation of each straight line sketched below.

a



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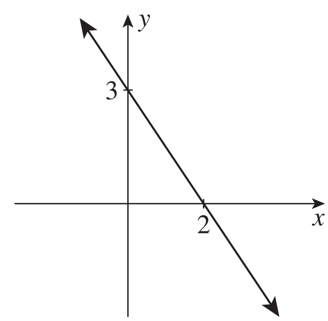

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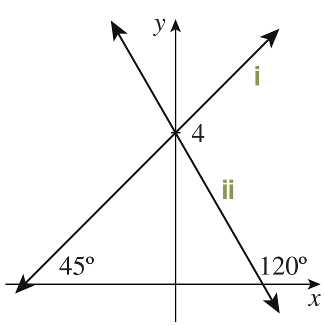
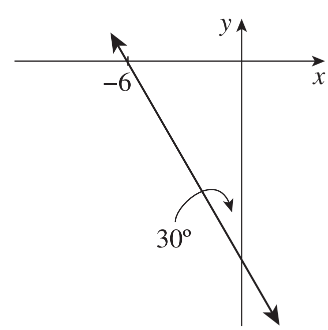


d



15 Explain why the four lines ℓ_1: $y = x + 1$, ℓ_2: $y = x - 3$, ℓ_3: $y = 3x + 5$ and ℓ_4: $y = 3x - 5$ enclose a parallelogram. Then find the vertices of this parallelogram.

16 Show that the triangle with vertices $A(1, 0)$, $B(6, 5)$ and $C(0, 2)$ is right-angled. Then find the equation of each side.

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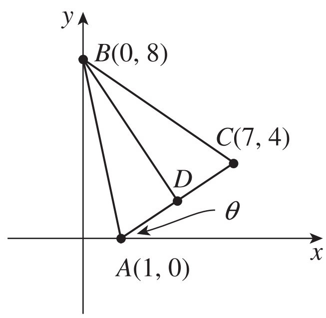
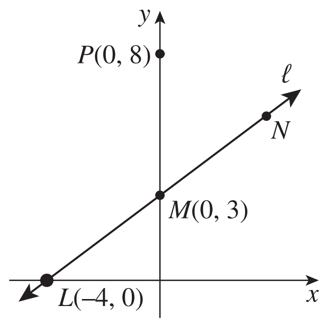
17 The three points $A(1, 0)$, $B(0, 8)$ and $C(7, 4)$ form a triangle. Let θ be the angle between AC and the x-axis.



- **a** Find the gradient of the line AC and hence determine θ, correct to the nearest degree.
- **b** Derive the equation of AC.
- **c** Find the coordinates of the midpoint D of AC.
- **d** Show that AC is perpendicular to BD.
- **e** What type of triangle is ABC?
- **f** Find the area of this triangle.
- **g** Write down the coordinates of a point E such that the parallelogram $ABCE$ is a rhombus.

CHALLENGE

18 The line ℓ crosses the x- and y-axes at $L(-4, 0)$ and $M(0, 3)$. The point N lies on ℓ, and P is the point $P(0, 8)$.



- **a** Copy the sketch and find the equation of ℓ.
- **b** Find the lengths of ML and MP and hence show that LMP is an isosceles triangle.
- **c** If M is the midpoint of LN, find the coordinates of N.
- **d** Show that $\angle NPL = 90°$.
- **e** Write down the equation of the circle through N, P and L.

19 Find k if the lines ℓ_1: $x + 3y + 13 = 0$, ℓ_2: $4x + y - 3 = 0$ and ℓ_3: $kx - y - 10 = 0$ are concurrent. (Hint: Find the point of intersection of ℓ_1 and ℓ_2 and substitute into ℓ_3.)

20 Consider the two lines ℓ_1: $3x + 2y + 4 = 0$ and ℓ_2: $6x + \mu y + \lambda = 0$. Write down the value of μ if:

- **a** ℓ_1 is parallel to ℓ_2,
- **b** ℓ_1 is perpendicular to ℓ_2.

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6E Using pronumerals in place of numbers

Most problems of this chapter so far have used numbers for the coordinates of points, and for the coefficients of x and y. When constant pronumerals are used, however, far more general results can be obtained. For example, Chapter 3 used pronumeral constants systematically in the study of quadratics and their parabolic graphs. Accordingly, this final section consolidates the methods of the last four sections, but uses pronumerals, rather than numbers, for points and coefficients. Because of the generality, most of the questions have an interpretation as a geometric result.

When the French mathematician and philosopher René Descartes introduced the coordinate plane in the 17th century, he intended it as an alternative approach to geometry, in which all the ancient theorems of Euclid would be proven again using algebraic rather than geometric arguments. In our course, however, Descartes' methods are used with the reverse purpose — we are turning functions that have been defined algebraically into geometric objects that can be visualised and interpreted using the methods of geometry. Questions here should be interpreted both algebraically and geometrically.

The procedures in this section are more demanding and sustained than in the previous sections, and the whole section may be regarded as extension.

Example 17 6E

(This example has an introductory part **a** that uses numbers. Part **b** uses pronumerals, and is the important part.)

a A triangle has vertices at $A(1, -3)$, $B(3, 3)$ and $C(-3, 1)$.

  i Find the coordinates of the midpoint P of AB and the midpoint Q of BC.

  ii Show that $PQ \parallel AC$ and that $PQ = \frac{1}{2}AC$.

b Repeat this procedure for a triangle with vertices $A(2a, 0)$, $B(2b, 2c)$ and $C(0, 0)$, where $a > 0$.

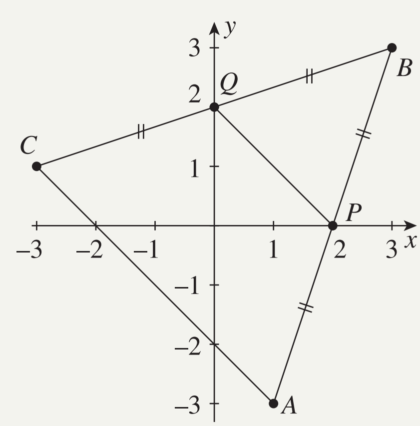
SOLUTION

a i Using the midpoint formula:

For P, $x = \dfrac{1+3}{2} = 2$ $\qquad y = \dfrac{-3+3}{2} = 0.$

For Q, $x = \dfrac{3-3}{2} = 0$ $\qquad y = \dfrac{3+1}{2} = 2.$

Hence $P = (2, 0)$ and $Q = (0, 2)$.



ii The lines PQ and AC are parallel because they both have gradient -1.

gradient $PQ = \dfrac{2-0}{0-2} = -1$ $\qquad$ gradient $AC = \dfrac{1+3}{-3-1} = -1.$

Using the distance formula:

$PQ^2 = (0-2)^2 + (2-0)^2 = 8$ $\qquad AC^2 = (-3-1)^2 + (1+3)^2 = 32$

so $PQ = 2\sqrt{2}$ and $AC = 4\sqrt{2} = 2PQ$

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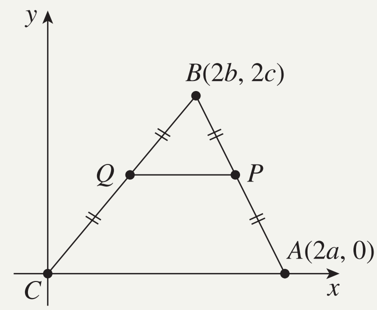
b i Using the midpoint formula:

$$\text{For } P,\ x = \frac{2a+2b}{2} = a+b \qquad y = \frac{0+2c}{2} = c.$$

$$\text{For } Q,\ x = \frac{2b+0}{2} = b \qquad y = \frac{2c+0}{2} = c.$$

Hence $P = (a+b, c)$ and $Q = (b, c)$.



ii The lines PQ and AC are parallel because they are both horizontal.

Using the distance formula:

$$PQ^2 = (b-a-b)^2 + (c-c)^2 = a^2 \qquad AC^2 = (0-2a)^2 + (0-0)^2 = 4a^2$$

So $PQ = a$ and $AC = 2a = 2PQ$.

Any triangle can be translated and rotated into the position of the triangle in part **b**, so this example demonstrates that the interval joining the midpoints of two sides of a triangle is parallel to the base and half its length.

Exercise 6E

FOUNDATION

Note: Diagrams should be drawn wherever possible.

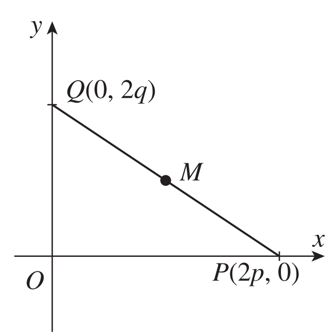
Many of these questions involve a particular triangle with numerical coordinates, before applying your understanding to a triangle for which the coordinates include pronumerals.

1 **a** On a number plane, plot the points $O(0, 0)$, $A(6, 0)$, $B(6, 6)$ and $C(0, 6)$, which form a square.
- **i** Find the gradients of the diagonals OB and AC.
- **ii** Hence show that the diagonals OB and AC are perpendicular.

b Let the vertices of a square be $O(0, 0)$, $A(a, 0)$, $B(a, a)$ and $C(0, a)$.
- **i** Find the gradients of the diagonals OB and AC.
- **ii** Hence show that the diagonals OB and AC are perpendicular.

In this question you have shown that *the diagonals of a square are perpendicular*.

2 **a** The points $O(0, 0)$, $P(8, 0)$ and $Q(0, 10)$ form a right-angled triangle. Let M be the midpoint of PQ.
- **i** Find the coordinates of M.
- **ii** Find the distances OM, PM and QM, and show that M is equidistant from each of the vertices.
- **iii** Explain why a circle with centre M can be drawn through the three vertices O, P and Q.



b Consider a right-angled triangle with vertices at $O(0, 0)$, $P(2p, 0)$ and $Q(0, 2q)$ and repeat the steps of part **a**.

In this question you have shown that *the midpoint of the hypotenuse of a right-angled triangle is the centre of a circle through all three vertices*.

3 **a** A triangle has vertices at $A(3, 1)$, $B(7, 3)$ and $C(1, -1)$.

i Find the coordinates of the midpoint P of AB and the midpoint Q of BC.

ii Show that the equation of the line ℓ through P parallel to AC is $y = x - 3$.

iii Show that Q lies on the line found in **ii**. That is, show that ℓ bisects BC.

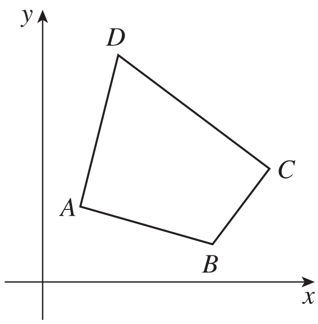
iv Show that $PQ = \frac{1}{2}AC$.

b Consider a triangle by placing its vertices at $A(2a, 0)$, $B(2b, 2c)$ and $C(0, 0)$, where $a > 0$, and repeat the steps in part **a**.

In this question you have shown that *the interval through the midpoint of a side and parallel to the base bisects the third side and is half the length of the base.*

DEVELOPMENT

4 Let the vertices of the quadrilateral be $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$ and $D(d_1, d_2)$, as shown in the diagram.



a Find the midpoints P, Q, R and S of the sides AB, BC, CD and DA respectively. (The figure $PQRS$ is also a quadrilateral.)

b Find the midpoints of the diagonals PR and QS.

c Explain why this proves that $PQRS$ is a parallelogram.

In this question you have showed that *the midpoints of the sides of a quadrilateral form a parallelogram.*

5 The points $A(a, 0)$ and $Q(q, 0)$ are points on the positive x-axis, and the points $B(0, b)$ and $P(0, p)$ lie on the positive y-axis. Show that $AB^2 - AP^2 = QB^2 - QP^2$.

6 The triangle OBA has its vertices at the origin $O(0, 0)$, at $A(3, 0)$ and at $B(0, 4)$. The point C lies on AB, and OC is perpendicular to AB. Draw a diagram showing this information.

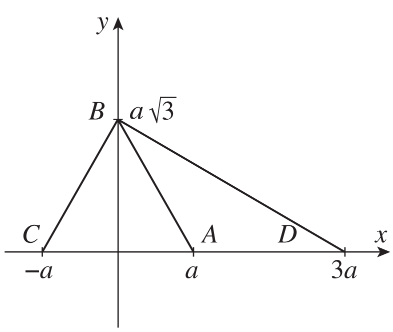
a Find the equations of AB and OC and hence find the coordinates of C.

b Find the lengths OA, AB, OC, BC and AC.

c Thus confirm these important results for a right-angled triangle:

i $OC^2 = AC \times BC$ **ii** $OA^2 = AC \times AB$

7 The diagram opposite shows the points A, B, C and D on the number plane.



a Show that ΔABC is equilateral.

b Show that ΔABD is isosceles, with $AB = AD$.

c Show that $AB^2 = \frac{1}{3}BD^2$.

8 Place three vertices of a parallelogram at $A(0, 0)$, $B(2a, 2b)$ and $D(2c, 2d)$.

a Use gradients to show that with $C = (2a + 2c, 2b + 2d)$, the quadrilateral $ABCD$ is a parallelogram.

b Find the midpoints of the diagonals AC and BD.

c Explain why this proves that *the diagonals of a parallelogram bisect each other.*

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CHALLENGE

9 **a** The points $A(1, -2)$, $B(5, 6)$ and $C(-3, 2)$ are the vertices of a triangle, and P, Q and R are the midpoints of BC, CA and AB respectively.

i Find the equations of the three medians BQ, CR and AP.

ii Find the intersection of BQ and CR, and show that it lies on the third median AP.

b Now consider the more general situation of a triangle with vertices at $A(6a, 6b)$, $B(-6a, -6b)$ and $C(0, 6c)$, and follow these steps.

i Find the midpoints P, Q and R of BC, CA and AB respectively.

ii Show that the median through C is $x = 0$ and find the equations of the other two medians.

iii Find the point where the median through C meets the median through A, and show that this point lies on the median through B.

In this question you have shown that *the three medians of a triangle are concurrent.* (Their point of intersection is called the *centroid*.)

10 Place a triangle in the plane with vertices $A(2a, 0)$, $B(-2a, 0)$ and $C(2b, 2c)$.

a Find the gradients of AB, BC and CA.

b Hence find the equations of the three perpendicular bisectors.

c Find the intersection M of any two bisectors, and show that it lies on the third bisector.

d Explain why the point of intersection of the three perpendicular bisectors is equidistant from each vertex.

In this question you have shown that *the perpendicular bisectors of the sides of a triangle are concurrent*. Their point of intersection (called the *circumcentre*) is thus the centre of a circle through all three vertices (called the *circumcircle*).

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Chapter 6 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 6 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Let $X = (2, 9)$ and $Y = (14, 4)$. Use the standard formulae to find:

a the midpoint of XY, **b** the gradient of XY, **c** the length of XY.

2 A triangle has vertices $A(1, 4)$, $B(-3, 1)$ and $C(-2, 0)$.

a Find the lengths of all three sides of ΔABC.

b What sort of triangle is ΔABC?

3 A quadrilateral has vertices $A(2, 5)$, $B(4, 9)$, $C(8, 1)$ and $D(-2, -7)$.

a Find the midpoints P of AB, Q of BC, R of CD and S of DA.

b Find the gradients of PQ, QR, RS and SP.

c What sort of quadrilateral is $PQRS$?

4 A circle has diameter AB, where $A = (2, -5)$ and $B = (-4, 7)$.

a Find the centre C and radius r of the circle.

b Use the distance formula to test whether $P(6, -1)$ lies on the circle.

5 a Find the gradients of the sides of ΔLMN, given $L(3, 9)$, $M(8, -1)$ and $N(-1, 7)$.

b Explain why ΔLMN is a right-angled triangle.

6 a Find the gradient of the interval AB, where $A = (3, 0)$ and $B = (5, -2)$.

b Find a if $AP \perp AB$ and $P = (a, 5)$.

c Find the point $Q(b, c)$ if B is the midpoint of AQ.

d Find d if the interval AD has length 5, where $D = (6, d)$.

7 Find, in general form, the equation of the line:

a with gradient -2 and y-intercept 5,

b with gradient $\frac{2}{3}$ through the point $A(3, 5)$,

c through the origin and perpendicular to $y = 7x - 5$,

d through $B(-5, 7)$ and parallel to $y = 4 - 3x$,

e with y-intercept -2 and angle of inclination $60°$.

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8 Put the equation of each line into gradient–intercept form, and hence find its y-intercept b, its gradient m and its angle of inclination α (correct to the nearest minute when necessary).

a $5x - 6y - 7 = 0$

b $4x + 4y - 3 = 0$

9 Find the gradient of each line AB, then find its equation in general form.

a $A(3, 0)$ and $B(4, 8)$

b $A(5, -2)$ and $B(7, -7)$

10 a Are the points $L(7, 4)$, $M(13, 2)$ and $N(25, -3)$ collinear?

b Are the lines $2x + 5y - 29 = 0$, $4x - 2y + 2 = 0$ and $7x - 3y + 1 = 0$ concurrent?

11 a Determine whether the lines $8x + 7y + 6 = 0$, $6x - 4y + 3 = 0$ and $2x + 3y + 9 = 0$ enclose a right-angled triangle.

b Determine what sort of figure the lines $4x + 8y + 3 = 0$, $5x - 2y + 7 = 0$, $x + 2y - 6 = 0$ and $9x - 3y = 0$ enclose.

12 a Find the points where the line $5x + 4y - 30 = 0$ meets the x-axis and y-axis.

b Hence find the area of the triangle formed by the line, the x-axis and the y-axis.

13 A sketch is essential in this question.

a Find the gradient, length and midpoint M of the interval joining $A(10, 2)$ and $B(2, 8)$.

b Show that the perpendicular bisector of the interval AB has equation $4x - 3y - 9 = 0$.

c Find the point C where the perpendicular bisector meets the line $x - y + 2 = 0$.

d Use the distance formula to show that C is equidistant from A and B.

e Show that $CM = 15$ and hence find the area of ΔABC.

f Let $\theta = \angle ACB$. Use the area formula area $= \frac{1}{2} \times AC \times BC \times \sin\theta$ to find θ, correct to the nearest minute.

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7 Exponential and logarithmic functions

Suppose that the amount of mould on a piece of cheese is doubling every day.

- In one day, the mould increases by a factor of 2.
- In two days, the mould increases by a factor of $2^2 = 4$.
- In three days, the mould increases by a factor of $2^3 = 8$.
- In four days, the mould increases by a factor of $2^4 = 16$.

And so on, until the cheese is thrown out. Applying mathematics to situations such as this requires indices and logarithms, and their graphs. Logarithms may be quite unfamiliar to many readers.

The application questions throughout, and particularly the final exercise, indicate some of the extraordinary variety of situations where exponential and logarithmic functions are needed.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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7A Indices

This section introduces powers such as 2^3 and 5^{-2}, whose indices are integers. Then the various index laws for working with them are developed.

Power, base and index

Each of these three words means a different thing.

1 POWER, BASE AND INDEX

- An expression a^n is called a *power*.
- The number a is called the *base*.
- The number n is called the *index* or *exponent*.

Thus 2^3 is a *power* with *base 2* and *index 3*.

The words *exponent* and *index* (plural *indices*) mean exactly the same thing.

Powers whose indices are positive integers

Powers are defined differently depending on what sort of number the index is. When the index is a positive integer, the definition is quite straightforward.

2 POWERS WHOSE INDICES ARE POSITIVE INTEGERS

For any number a,

$$a^1 = a, \quad a^2 = a \times a, \quad a^3 = a \times a \times a, \quad \ldots$$

In general, $a^n = \overbrace{a \times a \times a \times \cdots \times a}^{n \text{ factors}}$, for all positive integers n.

For example: $7^1 = 7$ $\qquad 3^4 = 3 \times 3 \times 3 \times 3 = 81$ $\qquad 9^5 = 9 \times 9 \times 9 \times 9 \times 9 = 59\,049$

With large numbers such as 9^5, the index form may be more convenient.

Scientific calculators can help to evaluate or approximate powers. The button for powers is labelled $\boxed{x^y}$ or $\boxed{\wedge}$, depending on the calculator. For example:

$$2^{26} = 67\,108\,864 \qquad 3^{50} \doteqdot 7.179 \times 10^{23} \qquad \pi^4 \doteqdot 97.41$$

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Index laws — combining powers with the same base

The first three index laws show how to combine powers when the base is fixed.

3 INDEX LAWS — PRODUCTS, QUOTIENTS AND POWERS OF POWERS

- To multiply powers with the same base, add the indices:
  $a^m \times a^n = a^{m+n}$
- To divide powers with the same base, subtract the indices:
  $a^m \div a^n = a^{m-n} \quad \left(\text{also written as } \dfrac{a^m}{a^n} = a^{m-n}\right)$
- To raise a power to a power, multiply the indices:
  $(a^m)^n = a^{mn}$

Demonstrating the results when $m = 5$ and $n = 3$ should make these laws clear.

- $$\begin{aligned} a^5 \times a^3 &= (a \times a \times a \times a \times a) \times (a \times a \times a) \\ &= a^8 \\ &= a^{5+3} \end{aligned}$$
- $$\begin{aligned} a^5 \div a^3 &= \frac{a \times a \times a \times a \times a}{a \times a \times a} \\ &= a^2 \\ &= a^{5-3} \end{aligned}$$
- $$\begin{aligned} (a^5)^3 &= a^5 \times a^5 \times a^5 \\ &= a^{5+5+5} \quad \text{(to multiply powers of the same base, add the indices)} \\ &= a^{5\times 3} \end{aligned}$$

Example 1 7A

Use the index laws above to simplify each expression.

a $x^3 \times x^7$ **b** $3^x \times 3^{5x}$ **c** $w^{12} \div w^3$
d $10^{a+b} \div 10^b$ **e** $(y^6)^9$ **f** $(2^{3x})^{2y}$

SOLUTION

a $x^3 \times x^7 = x^{10}$ **b** $3^x \times 3^{5x} = 3^{6x}$ **c** $w^{12} \div w^3 = w^9$
d $10^{a+b} \div 10^b = 10^a$ **e** $(y^6)^9 = y^{54}$ **f** $(2^{3x})^{2y} = 2^{6xy}$

Index laws — powers of products and quotients

The next two index laws show how to work with powers of products and powers of quotients.

4 INDEX LAWS — POWERS OF PRODUCTS AND QUOTIENTS

- The power of a product is the product of the powers:
  $(ab)^n = a^n \times b^n$
- The power of a quotient is the quotient of the powers:
  $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

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Demonstrating the results when $n = 3$ should make the general results obvious.

- $(ab)^3 = ab \times ab \times ab$
  $= a \times a \times a \times b \times b \times b$
  $= a^3b^3$
- $\left(\frac{a}{b}\right)^3 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$
  $= \frac{a^3}{b^3}$

Example 2 7A

Expand the brackets in each expression.

a $(10a)^3$ **b** $(3x^2y)^3$ **c** $\left(\frac{x}{3}\right)^4$ **d** $\left(\frac{a^3}{3b}\right)^2$

SOLUTION

a $(10a)^3 = 10^3 \times a^3$
$= 1000a^3$

b $(3x^2y)^3 = 3^3 \times (x^2)^3 \times y^3$
$= 27x^6y^3$

c $\left(\frac{x}{3}\right)^4 = \frac{x^4}{3^4}$
$= \frac{x^4}{81}$

d $\left(\frac{a^3}{3b}\right)^2 = \frac{(a^3)^2}{3^2 \times b^2}$
$= \frac{a^6}{9b^2}$

Zero and negative indices

The index laws were demonstrated only when the indices that were all positive integers. Powers with negative indices are defined in such a way that these laws are valid for negative indices as well.

Zero and negative indices involve division by the base a, so a cannot be zero.

1. We know that $a^3 \div a^3 = 1$.
   If we use the index laws, however, $a^3 \div a^3 = a^{3-3} = a^0$.
   Hence it is convenient to define $a^0 = 1$.
2. We know that $a^2 \div a^3 = \frac{1}{a}$.
   If we used the index laws, however, $a^2 \div a^3 = a^{2-3} = a^{-1}$.
   Hence it is convenient to define $a^{-1} = \frac{1}{a}$.
3. Similarly, we shall define $a^{-2} = \frac{1}{a^2}$, and $a^{-3} = \frac{1}{a^3}$, and so on.

5 ZERO AND NEGATIVE INDICES

Let a be any non-zero number.

- Define $a^0 = 1$.
- Define $a^{-1} = \frac{1}{a}$, define $a^{-2} = \frac{1}{a^2}$, define $a^{-3} = \frac{1}{a^3}$, …
- In general, define $a^{-n} = \frac{1}{a^n}$, for all positive integers n.

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Thus the negative sign in the index says, 'Take the reciprocal'. Always do this first, and always write the reciprocal of $\frac{2}{3}$ as $\frac{3}{2}$, never as $\dfrac{1}{\frac{2}{3}}$. For example:

$$\left(\tfrac{2}{3}\right)^{-3} = \left(\tfrac{3}{2}\right)^{3} = \tfrac{27}{8}.$$

Example 3 7A

Write each expression using fractions instead of negative indices, then simplify it.

a 12^{-1} **b** 7^{-2} **c** $\left(\tfrac{2}{3}\right)^{-1}$ **d** $\left(\tfrac{2}{3}\right)^{-4}$

SOLUTION

a $12^{-1} = \frac{1}{12}$

b $7^{-2} = \dfrac{1}{7^2}$
$= \frac{1}{49}$

c $\left(\tfrac{2}{3}\right)^{-1} = \frac{3}{2}$ (the negative index says, 'take the reciprocal')

d $\left(\tfrac{2}{3}\right)^{-4} = \left(\tfrac{3}{2}\right)^{4}$ (take the reciprocal first)
$= \frac{81}{16}$ (then take the fourth power)

Example 4 7A

Write each expression using negative indices instead of fractions.

a $\dfrac{1}{2^{20}}$ **b** $\dfrac{1}{x^3}$ **c** $\dfrac{5}{a^7}$ **d** $\dfrac{w^2}{v^4}$

SOLUTION

a $\dfrac{1}{2^{20}} = 2^{-20}$

b $\dfrac{1}{x^3} = x^{-3}$

c $\dfrac{5}{a^7} = 5 \times \dfrac{1}{a^7}$
$= 5a^{-7}$

d $\dfrac{w^2}{v^4} = w^2 \times \dfrac{1}{v^4}$
$= w^2v^{-4}$

Example 5 7A

Use the index laws, extended to negative indices, to simplify these expressions, giving answers in fraction form without negative indices.

a $x^4 \times x^{-10}$ **b** $12a^2b^4 \div 6a^3b^4$ **c** $(2^n)^{-3}$

SOLUTION

a $x^4 \times x^{-10} = x^{4-10}$
$= x^{-6}$
$= \dfrac{1}{x^6}$

b $12a^2b^4 \div 6a^3b^4 = \frac{12}{6}a^{2-3}b^{4-4}$
$= 2a^{-1}b^0$
$= 2a^{-1}$
$= \dfrac{2}{a}$

c $(2^n)^{-3} = 2^{n\times(-3)}$
$= 2^{-3n}$
$= \dfrac{1}{2^{3n}}$

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Exercise 7A

FOUNDATION

Note: Do not use a calculator in this exercise at all, except for the very last question.

1 The town of Goldhope had a small population in 1970, but the population has been increasing by a factor of 3 every decade since then.

a By what factor had the population increased from its 1970 value in 1980, 1990, 2000, 2010 and 2020?

b If the town's population was 10 000 in 1970, during which decade did the population pass 1 000 000?

2 **Preparation**: The answers to this question will be needed for all the exercises in this chapter. Keep the results in a place where you can refer to them easily.

a Write down all the powers of 2 from $2^0 = 1$ to $2^{12} = 4096$.

b In the same way, write down:

i the powers of 3 to $3^6 = 729$

ii the powers of 5 to $5^5 = 3125$

iii the powers of 6 to $6^3 = 216$

iv the powers of 7 to $7^3 = 343$

v the powers of 10 to $10^6 = 1\,000\,000$

vi the powers of 20 to $20^6 = 64\,000\,000$.

c From your list of powers of 2, read off:

i powers of 4

ii powers of 8.

d From your list of powers of 3, read off powers of 9.

e From your list of powers of 5, read off powers of 25.

3 Simplify:

a 2^3 b 2^6 c 3^4 d 9^3 e $\left(\frac{2}{3}\right)^2$

f $\left(\frac{2}{3}\right)^3$ g $\left(\frac{3}{10}\right)^4$ h $\left(\frac{4}{7}\right)^2$ i $\left(\frac{5}{9}\right)^1$ j 1^1

4 Simplify:

a 3^0 b 7^0 c 5^{-1} d 11^{-1} e 6^{-2}

f 10^{-2} g 3^{-3} h 5^{-3} i 2^{-5} j 10^{-6}

5 Simplify. With fractions, a negative index says, 'Take the reciprocal'.

a $\left(\frac{1}{5}\right)^{-1}$ b $\left(\frac{1}{11}\right)^{-1}$ c $\left(\frac{2}{7}\right)^{-1}$ d $\left(\frac{7}{2}\right)^{-1}$ e $\left(\frac{3}{4}\right)^{-1}$

f $\left(\frac{10}{23}\right)^{-1}$ g $(10)^{-1}$ h $(0.1)^{-1}$ i $(0.01)^{-1}$ j $(0.02)^{-1}$

6 Simplify. First take the reciprocal, then take the power.

a 5^{-2} b $\left(\frac{1}{5}\right)^{-2}$ c $\left(\frac{1}{5}\right)^{-3}$ d $\left(\frac{1}{2}\right)^{-4}$ e $\left(\frac{1}{10}\right)^{-6}$

f $\left(\frac{2}{3}\right)^{-2}$ g $\left(\frac{2}{3}\right)^{-4}$ h $\left(\frac{3}{2}\right)^{-4}$ i $\left(\frac{2}{5}\right)^{-2}$ j $\left(\frac{3}{7}\right)^{0}$

7 Simplify, leaving your answers in index form.

a $2^9 \times 2^5$ b $a^8 \times a^7$ c $7^2 \times 7^{-10}$ d $x^7 \times x^{-5}$

e $9^6 \times 9^{-6}$ f $a^5 \times a^{-5}$ g 5×5^{-4} h $8^4 \times 8 \times 8^{-4}$

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8 Simplify, leaving your answers in index form.

a $7^8 \div 7^3$ **b** $a^5 \div a^7$ **c** $x^{10} \div x^{-2}$ **d** $x^{-10} \div x^2$
e $2^8 \div 2^{-8}$ **f** $2^8 \div 2^8$ **g** $y^{12} \div y$ **h** $y \div y^{12}$

9 Simplify, leaving your answers in index form.

a $x^5 \times x^5 \times x^5$ **b** $(x^5)^3$ **c** $(z^2)^7$ **d** $a^{-2} \times a^{-2} \times a^{-2}$
e $(a^{-2})^3$ **f** $(5^4)^{-7}$ **g** $(y^{-2})^{-5}$ **h** $(2^{-4})^{-4}$

10 Write down the solutions of these index equations.

a $3^x = 9$ **b** $2^x = 16$ **c** $5^x = 125$ **d** $2^x = 64$
e $7^x = \frac{1}{7}$ **f** $5^x = \frac{1}{5}$ **g** $9^x = \frac{1}{81}$ **h** $2^x = \frac{1}{8}$
i $\left(\frac{1}{3}\right)^x = 3$ **j** $\left(\frac{5}{6}\right)^x = \frac{6}{5}$ **k** $9^x = 1$ **l** $12^x = 1$

11 Expand the brackets in each expression.

a $(3x)^2$ **b** $(5a)^3$ **c** $(2c)^6$ **d** $(3st)^4$ **e** $(7xyz)^2$
f $\left(\frac{1}{x}\right)^5$ **g** $\left(\frac{3}{x}\right)^2$ **h** $\left(\frac{y}{5}\right)^2$ **i** $\left(\frac{7a}{5}\right)^2$ **j** $\left(\frac{3x}{2y}\right)^3$

DEVELOPMENT

12 A huge desert rock is entirely made up of coarse grains of sandstone, each about 1 mm^3 in size, and is very roughly 3 km long, 2 km wide, and 500 m high.

a Find the approximate volume of the rock in cubic kilometres.
b How many cubic millimetres are in a cubic kilometre? Answer in index notation.
c Find the approximate number of grains of sandstone in the rock.

13 Write each expression as a fraction without negative indices or brackets.

a 9^{-1} **b** x^{-1} **c** b^{-2} **d** $-a^{-4}$ **e** $(7x)^{-1}$
f $7x^{-1}$ **g** $-9x^{-1}$ **h** $(3a)^{-2}$ **i** $3a^{-2}$ **j** $4x^{-3}$

14 Write each fraction in index form.

a $\frac{1}{x}$ **b** $-\frac{1}{x^2}$ **c** $-\frac{12}{x}$ **d** $\frac{9}{x^2}$ **e** $-\frac{1}{x^3}$
f $\frac{12}{x^5}$ **g** $\frac{7}{x^3}$ **h** $-\frac{6}{x}$ **i** $\frac{1}{6x}$ **j** $-\frac{1}{4x^2}$

15 First, change each mixed numeral or decimal to a fraction, then simplify the expression.

a $\left(1\frac{1}{2}\right)^{-1}$ **b** $\left(2\frac{1}{3}\right)^{-1}$ **c** $\left(2\frac{2}{3}\right)^{-1}$ **d** $\left(2\frac{1}{2}\right)^{-2}$
e $\left(3\frac{1}{3}\right)^{-3}$ **f** $\left(6\frac{2}{3}\right)^{-2}$ **g** 0.2^{-1} **h** 2.4^{-1}
i 2.25^{-1} **j** 2.5^{-2} **k** 2.5^{-3} **l** 0.05^{-2}

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16 Write down the solutions of these index equations.

a $\left(\frac{5}{8}\right)^x = \frac{25}{64}$ **b** $\left(\frac{5}{8}\right)^x = \frac{8}{5}$ **c** $\left(\frac{3}{5}\right)^x = \frac{25}{9}$ **d** $\left(\frac{3}{2}\right)^x = \frac{8}{27}$

e $x^2 = \frac{100}{169}$ **f** $x^{-2} = \frac{1}{4}$ **g** $x^{-3} = 27$ **h** $x^{-2} = \frac{64}{81}$

17 Simplify each expression, giving the answer without negative indices.

a $2^x \times 2^3$ **b** $3^x \times 3$ **c** $7^{4x} \times 7^{-5x}$ **d** $5^{2x} \div 5^3$

e $(10^{2x})^3$ **f** $(5^{4x})^{-2}$ **g** $(6^x)^4 \times (6^{2x})^5$ **h** $(2^x)^3 \div 2^4$

18 Simplify each expression, giving the answer without negative indices.

a $x^2y \times x^4y^3$ **b** $x^{-5}y^3 \times x^3y^{-2}$ **c** $3ax^{-2} \times 7a^2x$

d $\frac{1}{15}s^{-2}t^3 \times 5st^{-5}$ **e** $56x^2y^6 \div 8xy^8$ **f** $35a^{-2}b^4 \div 28a^4b^{-6}$

g $(s^2y^{-3})^3$ **h** $(5c^{-2}d^3)^{-1}$ **i** $(3x^2y^3)^3 \times (xy^4)^2$

j $(2a^{-2}y^3)^{-2} \times (2ay^{-3})^3$ **k** $(5st^2)^3 \div (5s^{-1}t^3)^2$ **l** $(10x^2y^{-2})^3 \div (2x^{-1}y^3)^2$

CHALLENGE

19 Expand and simplify, answering without using negative indices.

a $(x + x^{-1})^2$ **b** $(x - x^{-1})^2$ **c** $(x^2 - x^{-2})^2$

20 Write each expression as a single power.

a 2×2^x **b** $2^x + 2^x$ **c** 3×3^x **d** $3^x + 3^x + 3^x$ **e** 4×2^x

f 32×2^x **g** 125×5^x **h** 81×3^x **i** $\frac{1}{2} \times 2^x$ **j** $\frac{1}{9} \times 3^x$

21 Explain why if $3^{3x-1} = 9$, then $3x - 1 = 2$, and so $x = 1$. Similarly, solve:

a $5^{x+3} = 25$ **b** $3^{2x-8} = 81$ **c** $4^{7-x} = \frac{1}{4}$

d $4^{5x+2} = \frac{1}{64}$. **e** $\left(\frac{1}{3}\right)^{x+1} = 27$ **f** $\left(\frac{1}{5}\right)^{x+1} = \left(\frac{1}{125}\right)^{x-1}$

22 A calculator is required in this question. Use scientific notation throughout.

a The mass of a neutron is about 1.675×10^{-27} kg. About how many neutrons are there in 1 kg of neutrons?

b The radius of a neutron is about 1.11×10^{-15} m. Use the formula for the volume of a sphere $V = \frac{4}{3}\pi r^3$ to find its approximate volume.

c Use the formula density $= \dfrac{\text{mass}}{\text{volume}}$ to find its approximate density in kg/m^3.

Note: This question makes several extremely naive assumptions about what a neutron actually is, but the extraordinarily high density that this calculation gives is close to the density of neutron stars, which are the densest objects in the universe.

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7B Fractional indices

In this section, the definition of powers is extended to powers such as $4^{\frac{1}{2}}$ and $27^{-\frac{2}{3}}$, where the index is a positive or negative fraction. Again, the definitions are given so that the index laws work for all indices.

Fractional indices involve taking square roots and other roots of the base a, so a cannot be negative.

Fractional indices with numerator 1

Let the base a be any real number $a \geq 0$.

1 We know that $\left(\sqrt{a}\right)^2 = a$

If we use the index laws, however, $\left(a^{\frac{1}{2}}\right)^2 = a$.

Hence it is convenient to define $a^{\frac{1}{2}} = \sqrt{a}$.

(Remember that $\sqrt{a}$ means the non-negative square root of a.)

2 We know that $\left(\sqrt[3]{a}\right)^3 = a$.

If we use the index laws, however, $\left(a^{\frac{1}{3}}\right)^3 = a$.

Hence it is convenient to define $a^{\frac{1}{3}} = \sqrt[3]{a}$.

6 FRACTIONAL INDICES WITH NUMERATOR 1

Let a be any real number $a \geq 0$.

- Define $a^{\frac{1}{2}} = \sqrt{a}$, define $a^{\frac{1}{3}} = \sqrt[3]{a}$, define $a^{\frac{1}{4}} = \sqrt[4]{a}$, …
- In general, define $a^{\frac{1}{n}} = \sqrt[n]{a}$, for all positive integers n, where in every case, the *non-negative root* of a is to be taken.

Example 6 7B

Write each expression using surd notation, then simplify it.

a $64^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $10000^{\frac{1}{4}}$ **d** $32^{\frac{1}{5}}$

SOLUTION

a $64^{\frac{1}{2}} = \sqrt{64} = 8$

b $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

c $10000^{\frac{1}{4}} = \sqrt[4]{10000} = 10$

d $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$

Example 7 7B

Write each expression using a fractional index.

a $\sqrt{2}$ **b** $\sqrt[3]{15}$ **c** $\sqrt{x}$ **d** $\sqrt[5]{x}$

SOLUTION

a $\sqrt{2} = 2^{\frac{1}{2}}$ **b** $\sqrt[3]{15} = 15^{\frac{1}{3}}$ **c** $\sqrt{x} = x^{\frac{1}{2}}$ **d** $\sqrt[5]{x} = x^{\frac{1}{5}}$

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General fractional indices

Let the base a be any positive real number. If the index laws are to apply to a power such as $a^{\frac{2}{3}}$, then we must be able to write

$$\left(a^{\frac{1}{3}}\right)^2 = a^{\frac{2}{3}} \quad \text{and} \quad (a^2)^{\frac{1}{3}} = a^{\frac{2}{3}}$$

because to raise a power to a power, we multiply the indices.

Hence we shall define $a^{\frac{2}{3}}$ as $\left(a^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{a}\right)^2$, which is the same as $(a^2)^{\frac{1}{3}} = \sqrt[3]{a^2}$.

7 GENERAL FRACTIONAL INDICES

Let the base a be any positive real number, and m and n positive integers.

- Define $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$, which is the same as $\sqrt[n]{a^m}$.
- Define $a^{-\frac{m}{n}} = \dfrac{1}{a^{\frac{m}{n}}}$.

When there are negative numbers and fractions in the index, first deal with the negative sign, then deal with the fractional index.

8 DEAL WITH COMPLICATED INDICES IN THIS ORDER

1 If the index is negative, take the reciprocal.
2 If the index has a denominator, take the corresponding root.
3 Finally, take the power indicated by the numerator of the index.

For example: $\left(\frac{4}{9}\right)^{-\frac{3}{2}} = \left(\frac{9}{4}\right)^{\frac{3}{2}} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$.

Example 8 — 7B

Simplify each power. First take the root indicated by the denominator.

a $125^{\frac{2}{3}}$ **b** $100^{\frac{3}{2}}$ **c** $\left(\frac{16}{81}\right)^{\frac{3}{4}}$

SOLUTION

a $125^{\frac{2}{3}} = 5^2$
$= 25$

b $100^{\frac{3}{2}} = 10^3$
$= 1000$

c $\left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{2}{3}\right)^3$
$= \frac{8}{27}$

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Example 9 7B

Simplify each power. First take the reciprocal as indicated by the negative index.

a $25^{-\frac{3}{2}}$ **b** $1000^{-\frac{2}{3}}$ **c** $\left(\frac{8}{27}\right)^{-\frac{4}{3}}$

SOLUTION

a $25^{-\frac{3}{2}} = \left(\frac{1}{25}\right)^{\frac{3}{2}} = \left(\frac{1}{5}\right)^{3} = \frac{1}{125}$

b $1000^{-\frac{2}{3}} = \left(\frac{1}{1000}\right)^{\frac{2}{3}} = \left(\frac{1}{10}\right)^{2} = \frac{1}{100}$

c $\left(\frac{8}{27}\right)^{-\frac{4}{3}} = \left(\frac{27}{8}\right)^{\frac{4}{3}} = \left(\frac{3}{2}\right)^{4} = \frac{81}{16}$

Example 10 7B

Write each expression using a fractional index.

a $\sqrt{x^3}$ **b** $\frac{1}{\sqrt{x^3}}$ **c** $\sqrt[3]{x^2}$ **d** $\frac{1}{\sqrt[3]{x^2}}$

SOLUTION

a $\sqrt{x^3} = x^{\frac{3}{2}}$ **b** $\frac{1}{\sqrt{x^3}} = x^{-\frac{3}{2}}$ **c** $\sqrt[3]{x^2} = x^{\frac{2}{3}}$ **d** $\frac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}}$

Irrational indices: Irrational indices need only be defined informally — the calculator knows how to approximate them. To define for example 2^{π}, take the sequence 3, 3.1, 3.14, 3.141, 3.1415, … of truncated decimals for π, then form the sequence of rational powers $2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, \ldots$. Everything then works as expected.

Exercise 7B FOUNDATION

Note: Do not use a calculator in this exercise at all, unless the question asks for it. Make sure that you can refer easily to the the list of powers of 2, 3, 5, … from the previous exercise.

1 Simplify these powers, giving each answer as a whole number.

a $25^{\frac{1}{2}}$ **b** $36^{\frac{1}{2}}$ **c** $100^{\frac{1}{2}}$ **d** $27^{\frac{1}{3}}$ **e** $64^{\frac{1}{3}}$

f $1000^{\frac{1}{3}}$ **g** $81^{\frac{1}{4}}$ **h** $32^{\frac{1}{5}}$ **i** $1\,000\,000^{\frac{1}{6}}$ **j** $1\,000\,000^{\frac{1}{2}}$

2 Simplify these powers. First take the root indicated by the denominator.

a $25^{\frac{3}{2}}$ **b** $9^{\frac{3}{2}}$ **c** $27^{\frac{2}{3}}$ **d** $8^{\frac{2}{3}}$ **e** $16^{\frac{3}{4}}$

f $81^{\frac{3}{4}}$ **g** $27^{\frac{4}{3}}$ **h** $4^{\frac{5}{2}}$ **i** $32^{\frac{3}{5}}$ **j** $64^{\frac{2}{3}}$

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3 Simplify these powers, giving each answer as a fraction.

a $\left(\frac{1}{49}\right)^{\frac{1}{2}}$ **b** $\left(\frac{1}{8}\right)^{\frac{1}{3}}$ **c** $\left(\frac{25}{49}\right)^{\frac{1}{2}}$ **d** $\left(\frac{27}{8}\right)^{\frac{1}{3}}$

e $\left(\frac{1}{4}\right)^{\frac{3}{2}}$ **f** $\left(\frac{1}{25}\right)^{\frac{3}{2}}$ **g** $\left(\frac{4}{9}\right)^{\frac{3}{2}}$ **h** $\left(\frac{9}{100}\right)^{\frac{3}{2}}$

4 Use the calculator button labelled $\boxed{x^y}$ or $\boxed{\wedge}$ to find these powers. Give each answer exactly if possible, or else correct to four significant figures.

a 13^4 **b** 3.235^4 **c** 2^{50} **d** $759\,375^{\frac{1}{5}}$

e $10^{\frac{1}{3}}$ **f** $17^{\frac{1}{4}}$ **g** 7^{-12} **h** $15^{-\frac{3}{5}}$

5 Use the index laws to simplify these expressions, leaving your answers in index form.

a $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$ **b** $x^{2\frac{1}{2}} \times x^{3\frac{1}{2}}$ **c** $x^4 \times x^{-\frac{1}{2}}$

d $x^{4\frac{1}{2}} \div x^{3\frac{1}{2}}$ **e** $x \div x^{\frac{1}{2}}$ **f** $x^{-7} \div x^{-2\frac{1}{2}}$

g $\left(x^{\frac{1}{3}}\right)^6$ **h** $\left(x^{-\frac{2}{3}}\right)^6$ **i** $(x^9)^{\frac{2}{3}}$

6 Use the index laws to simplify these expressions, giving answers as integers or fractions.

a $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ **b** $2^{\frac{1}{2}} \times 2^{-\frac{1}{2}}$ **c** $2^{\frac{1}{2}} \times 2^{2\frac{1}{2}}$

d $3^{4\frac{1}{2}} \div 3^{5\frac{1}{2}}$ **e** $25 \div 25^{\frac{1}{2}}$ **f** $7^{\frac{2}{3}} \div 7^{\frac{2}{3}}$

g $3^3 \times 3^4 \div 3^{10}$ **h** $\left(3^{-\frac{1}{3}}\right)^6$ **i** $(9^8)^{\frac{1}{4}}$

7 Write down the solutions of these index equations.

a $9^x = 3$ **b** $121^x = 11$ **c** $81^x = 3$

d $64^x = 2$ **e** $\left(\frac{1}{25}\right)^x = \frac{1}{5}$ **f** $\left(\frac{1}{8}\right)^x = \frac{1}{2}$

8 Rewrite each expression using surds instead of fractional indices.

a $x^{\frac{1}{2}}$ **b** $x^{\frac{1}{3}}$ **c** $7x^{\frac{1}{2}}$ **d** $(7x)^{\frac{1}{2}}$

e $15x^{\frac{1}{4}}$ **f** $x^{\frac{3}{2}}$ **g** $6x^{\frac{5}{2}}$ **h** $x^{\frac{4}{3}}$

9 Rewrite each expression using fractional indices instead of surds.

a $\sqrt{x}$ **b** $3\sqrt{x}$ **c** $\sqrt{3x}$ **d** $12\sqrt[3]{x}$

e $9\sqrt[6]{x}$ **f** $\sqrt{x^3}$ **g** $\sqrt{x^9}$ **h** $25\sqrt[5]{x^6}$

DEVELOPMENT

10 Simplify these powers by first taking the reciprocal as indicated by the negative index.

a $25^{-\frac{1}{2}}$ **b** $100^{-\frac{1}{2}}$ **c** $125^{-\frac{1}{3}}$ **d** $16^{-\frac{1}{4}}$

e $16^{-\frac{3}{4}}$ **f** $27^{-\frac{2}{3}}$ **g** $81^{-\frac{3}{4}}$ **h** $49^{-\frac{3}{2}}$

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11 Simplify these powers.

a $\left(\frac{1}{16}\right)^{-\frac{1}{4}}$ **b** $\left(\frac{1}{125}\right)^{-\frac{1}{3}}$ **c** $\left(\frac{1}{49}\right)^{-\frac{1}{2}}$ **d** $\left(\frac{1}{27}\right)^{-\frac{1}{3}}$

e $\left(\frac{1}{16}\right)^{-\frac{3}{4}}$ **f** $\left(\frac{1}{81}\right)^{-\frac{3}{4}}$ **g** $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$ **h** $\left(\frac{125}{8}\right)^{-\frac{2}{3}}$

12 Simplify each expression, giving the answer in index form.

a $3x^{\frac{1}{2}}y \times 3x^{\frac{1}{2}}y^2$ **b** $5a^{\frac{1}{3}}b^{\frac{2}{3}} \times 7a^{-\frac{1}{3}}b^{\frac{1}{3}}$ **c** $\frac{1}{8}s^{2\frac{1}{2}} \times 24s^{-2}$

d $x^2y^3 \div x^{\frac{1}{2}}y^{\frac{1}{2}}$ **e** $a^{\frac{1}{2}}b^{\frac{1}{2}} \div a^{-\frac{1}{2}}b^{\frac{1}{2}}$ **f** $(a^{-2}b^4)^{\frac{1}{2}}$

g $(8x^3y^{-6})^{\frac{1}{3}}$ **h** $\left(p^{\frac{1}{5}}q^{-\frac{3}{5}}\right)^{10}$ **i** $\left(x^{\frac{3}{4}}\right)^4 \times \left(x^{\frac{4}{3}}\right)^3$

13 Rewrite these expressions, using fractional and negative indices.

a $\frac{1}{\sqrt{x}}$ **b** $\frac{12}{\sqrt{x}}$ **c** $-\frac{5}{\sqrt{x}}$ **d** $\frac{15}{\sqrt[3]{x}}$

e $-\frac{4}{\sqrt[3]{x^2}}$ **f** $x\sqrt{x}$ **g** $\frac{5}{x\sqrt{x}}$ **h** $8x^2\sqrt{x}$

14 Given that $x = 16$ and $y = 25$, evaluate:

a $x^{\frac{1}{2}} + y^{\frac{1}{2}}$ **b** $x^{\frac{1}{4}} - y^{\frac{1}{2}}$ **c** $x^{-\frac{1}{2}} - y^{-\frac{1}{2}}$ **d** $(y - x)^{\frac{1}{2}} \times (4y)^{-\frac{1}{2}}$

15 The cost of installing a standard swimming pool is rising exponentially according to the formula $C = \$6000 \times (1.03)^n$, where n is the number of years after 1 January 2015.

a What was the original cost on 1 January 2015?

b What was the cost on 1 January 2016?

c Use your calculator to find, correct to the nearest \$10, the cost of installing a pool on these dates.

i 1 January 2020 **ii** 1 July 2015 **iii** 1 July 2018

16 Use the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$ on your calculator to approximate these powers. Express your answers in scientific notation correct to four significant figures.

a 7^8 **b** $565^{\frac{2}{5}}$ **c** $5^{-0.12}$ **d** $(0.001)^{0.7}$

e 2^{π} **f** $10^{\sqrt{2}}$ **g** $\left(\sqrt{7}\right)^{-\sqrt{7}}$ **h** $0.001^{-\frac{\pi}{2}}$

CHALLENGE

17 Expand and simplify, giving the answers in index form:

a $\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2$ **b** $\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)^2$ **c** $\left(x^{\frac{5}{2}} - x^{-\frac{5}{2}}\right)^2$

18 Write down the solutions of these index equations.

a $49^x = \frac{1}{7}$ **b** $81^x = \frac{1}{3}$ **c** $8^x = 4$ **d** $8^x = \frac{1}{4}$ **e** $4^x = 8$

f $4^x = \frac{1}{8}$ **g** $81^x = 27$ **h** $27^x = \frac{1}{81}$ **i** $\left(\frac{1}{25}\right)^x = 5$ **j** $\left(\frac{8}{125}\right)^x = \frac{25}{4}$

19 By taking 6th powers of both sides, show that $11^{\frac{1}{3}} < 5^{\frac{1}{2}}$. Using similar methods (followed perhaps by a check on the calculator), compare:

a $3^{\frac{1}{3}}$ and $2^{\frac{1}{2}}$ **b** $2^{\frac{1}{2}}$ and $5^{\frac{1}{5}}$ **c** $7^{\frac{3}{2}}$ and 20 **d** $5^{\frac{1}{5}}$ and $3^{\frac{1}{3}}$

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7C Logarithms

The introduction to this chapter mentioned that the mould growing on cheese was doubling every day. Thus in 2 days the amount of mould will increase by a factor of $2^2 = 4$, in 3 days by $2^3 = 8$, in 4 days by $2^4 = 16$, and so on.

Now suppose that someone asks the question backwards, asking how many days it takes for the mould to increase by a factor of 8. The answer is 3 days, because $8 = 2^3$.

This index 3 is the *logarithm base 2 of 8*, and is written as $\log_2 8 = 3$:

$$\log_2 8 = 3 \qquad \text{means that} \qquad 8 = 2^3.$$

Read this as, 'The log base 2 of 8', mentioning the base first, and then the number.

Logarithms

The logarithm is the index. More precisely:

9 THE LOGARITHM IS THE INDEX

The *logarithm base a* of a positive number x is the index, when the number x is expressed as a power of the base a:

$$y = \log_a x \qquad \text{means that} \qquad x = a^y.$$

The base a must always be positive, and not equal to 1.

The basic skill with logarithms is converting between statements about indices and statements about logarithms. The next box should be learnt off by heart.

10 A SENTENCE TO COMMIT TO MEMORY

This one sentence should fix most problems:

$$\text{‘} \log_2 8 = 3 \qquad \text{means that} \qquad 8 = 2^3.\text{’}$$

Examine the sentence and notice that:

- The base of the log is the base of the power (in this case 2).
- The log is the index (in this case 3).

Example 11 7C

Rewrite each index statement in logarithmic form.

a $10^3 = 1000$ **b** $3^4 = 81$

SOLUTION

a $10^3 = 1000$
$\log_{10} 1000 = 3$

b $3^4 = 81$
$\log_3 81 = 4$

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Example 12 7C

Rewrite each logarithmic statement in index form. Then state whether it is true or false.

a $\log_2 16 = 4$

b $\log_3 27 = 4$

SOLUTION

a $\log_2 16 = 4$
$16 = 2^4$, which is true.

b $\log_3 27 = 4$
$27 = 3^4$, which is false.

Finding logarithms

Change questions about logarithms to questions about indices.

Example 13 7C

a Find $\log_2 32$.

b Find $\log_{10} 1\,000\,000$.

SOLUTION

a Let $x = \log_2 32$.
Then $2^x = 32$.
Hence $x = 5$.

b Let $x = \log_{10} 1\,000\,000$.
Then $10^x = 1\,000\,000$.
Hence $x = 6$.

Negative and fractional indices

A logarithmic equation can involve a negative or a fractional index.

Example 14 7C

Solve each logarithmic equation by changing it to an index equation.

a $x = \log_7 \frac{1}{7}$

b $\frac{1}{3} = \log_8 x$

c $\log_x 9 = -1$

SOLUTION

a $x = \log_7 \frac{1}{7}$
$\frac{1}{7} = 7^x$
$x = -1$

b $\frac{1}{3} = \log_8 x$
$x = 8^{\frac{1}{3}}$
$= 2$

c $\log_x 9 = -1$
$x^{-1} = 9$
$x = \frac{1}{9}$

Logarithms on the calculator

Most scientific calculators only allow direct calculations of logarithms base 10, using the button labelled $\boxed{\text{log}}$. (They also allow calculations of logarithms using the important mathematical constant *e* as the base — this is the purpose of the button marked $\boxed{\text{ln}}$, as explained later in chapter 9.)

The function $\boxed{10^x}$ is usually on the same key as $\boxed{\text{log}}$, and is accessed by pressing $\boxed{\text{SHIFT}}$ followed by $\boxed{\text{log}}$. These two functions $\boxed{\text{log}}$ and $\boxed{10^x}$ are the inverse of each other — when used one after the other, the original number returns.

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Example 15 7C

Write each statement below in logarithmic form, then use the function labelled $\boxed{\text{log}}$ on your calculator to approximate x, correct to four significant figures. Then check your calculation using the button labelled $\boxed{10^x}$.

a $10^x = 750$

b $10^x = 0.003$

SOLUTION

a
$$\begin{aligned} 10^x &= 750 \\ x &= \log_{10} 750 \\ &\doteqdot 2.875, \text{ using } \boxed{\text{log}}. \end{aligned}$$

Check this using the $\boxed{10^x}$ button:
$10^{2.875} \doteqdot 750$.

b
$$\begin{aligned} 10^x &= 0.003 \\ x &= \log_{10} 0.003 \\ &\doteqdot -2.523, \text{ using } \boxed{\text{log}}. \end{aligned}$$

Check this using the $\boxed{10^x}$ button:
$10^{-2.523} \doteqdot 0.003$.

Locating a logarithm between two integers

Consider again the cheese with a mould that doubles in amount every day. Suppose that someone asks how many days it takes for the amount of mould to increase by a factor of 10.

The answer is $\log_2 10$ days, but this is an awkward question, because the answer is not an integer.

- In three days the mould increases 8-fold.
- In four days it increases 16-fold.

Thus we can at least say that the answer lies between 3 days and 4 days.

Note: The change-of-base formula in the next section will allow $\log_2 10$ to be approximated on a calculator.

Example 16 7C

Use a list of powers of 2 to explain why $\log_2 10$ is between 3 and 4.

SOLUTION

We know that $2^3 = 8$ and $2^4 = 16$, so $2^3 < 10 < 2^4$.
Hence taking logarithms base 2, $\quad 3 < \log_2 10 < 4$.

Exercise 7C FOUNDATION

Note: Do not use a calculator in this exercise unless the question asks for it. Make sure that you can refer easily to the the list of powers of 2, 3, 5, … from Exercise 7A.

1 Copy and complete each sentence.

a $\log_2 8 = 3$ because …

b $\log_5 25 = 2$ because …

c $\log_{10} 1000 = 3$ because …

d $7^2 = 49$, so $\log_7 49 =$ …

e $3^4 = 81$, so …

f $10^5 = 100\,000$, so …

2 Copy and complete the following statements of the meaning of logarithms.

a '$y = \log_a x$ means that …'

b '$y = a^x$ means that …'

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3 Rewrite each equation in index form, then solve it for x.

a $x = \log_{10} 10000$ **b** $x = \log_{10} 1000$ **c** $x = \log_{10} 100$ **d** $x = \log_{10} 10$

e $x = \log_{10} 1$ **f** $x = \log_{10} \frac{1}{10}$ **g** $x = \log_{10} \frac{1}{100}$ **h** $x = \log_{10} \frac{1}{1000}$

4 Rewrite each equation in index form, then solve it for x.

a $x = \log_3 9$ **b** $x = \log_5 125$ **c** $x = \log_7 49$ **d** $x = \log_2 64$

e $x = \log_4 64$ **f** $x = \log_8 64$ **g** $x = \log_8 8$ **h** $x = \log_8 1$

i $x = \log_7 \frac{1}{7}$ **j** $x = \log_{12} \frac{1}{12}$ **k** $x = \log_{11} \frac{1}{121}$ **l** $x = \log_6 \frac{1}{36}$

m $x = \log_4 \frac{1}{64}$ **n** $x = \log_8 \frac{1}{64}$ **o** $x = \log_2 \frac{1}{64}$ **p** $x = \log_5 \frac{1}{125}$

5 Rewrite each equation in index form, then solve it for x.

a $\log_7 x = 2$ **b** $\log_9 x = 2$ **c** $\log_5 x = 3$ **d** $\log_2 x = 5$

e $\log_4 x = 3$ **f** $\log_{100} x = 3$ **g** $\log_7 x = 1$ **h** $\log_{11} x = 0$

i $\log_{13} x = -1$ **j** $\log_7 x = -1$ **k** $\log_{10} x = -2$ **l** $\log_{12} x = -2$

m $\log_5 x = -3$ **n** $\log_7 x = -3$ **o** $\log_2 x = -5$ **p** $\log_3 x = -4$

6 Rewrite each equation in index form, then solve it for x. Remember that a base must be positive and not equal to 1.

a $\log_x 49 = 2$ **b** $\log_x 8 = 3$ **c** $\log_x 27 = 3$ **d** $\log_x 10000 = 4$

e $\log_x 10000 = 2$ **f** $\log_x 64 = 6$ **g** $\log_x 64 = 2$ **h** $\log_x 125 = 1$

i $\log_x 11 = 1$ **j** $\log_x \frac{1}{17} = -1$ **k** $\log_x \frac{1}{6} = -1$ **l** $\log_x \frac{1}{7} = -1$

m $\log_x \frac{1}{9} = -2$ **n** $\log_x \frac{1}{49} = -2$ **o** $\log_x \frac{1}{8} = -3$ **p** $\log_x \frac{1}{81} = -2$

DEVELOPMENT

7 Rewrite each equation in index form and then solve it for x (where a is a constant).

a $x = \log_a a$ **b** $\log_a x = 1$ **c** $\log_x a = 1$

d $x = \log_a \frac{1}{a}$ **e** $\log_a x = -1$ **f** $\log_x \frac{1}{a} = -1$

g $x = \log_a 1$ **h** $\log_a x = 0$ **i** $\log_x 1 = 0$

8 Given that a is a positive real number not equal to 1, evaluate:

a $\log_a a$ **b** $\log_a \frac{1}{a}$ **c** $\log_a a^3$ **d** $\log_a \frac{1}{a^2}$

e $\log_a \frac{1}{a^5}$ **f** $\log_a \sqrt{a}$ **g** $\log_a \frac{1}{\sqrt{a}}$ **h** $\log_a 1$

9 Use the list of powers of 2 prepared in Exercise 7A to find which two integers each expression lies between.

a $\log_2 3$ **b** $\log_2 7$ **c** $\log_2 1.8$ **d** $\log_2 13$

e $\log_2 50$ **f** $\log_2 1000$ **g** $\log_2 \frac{4}{5}$ **h** $\log_2 \frac{1}{3}$

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10 Use the list of powers of 2, 3, 5, … prepared in Exercise 7A to find which two integers each expression lies between.

a $\log_{10} 35$ **b** $\log_{10} 6$ **c** $\log_{10} 2000$ **d** $\log_3 2$
e $\log_3 50$ **f** $\log_3 100$ **g** $\log_5 100$ **h** $\log_7 20$
i $\log_{10} 0.4$ **j** $\log_{10} 0.05$

11 Use the calculator buttons $\boxed{\text{log}}$ and $\boxed{10^x}$ to approximate each expression. Give answers correct to three significant figures, then check each answer using the other button.

a $\log_{10} 2$ **b** $\log_{10} 20$ **c** $10^{0.301}$ **d** $10^{1.301}$
e $10^{0.5}$ **f** $10^{1.5}$ **g** $\log_{10} 3.16$ **h** $\log_{10} 31.6$
i $\log_{10} 1000$ **j** $\log_{10} 1\,000\,000$ **k** 10^3 **l** 10^6
m $\log_{10} 0.7$ **n** $\log_{10} 0.007$ **o** $10^{-0.155}$ **p** $10^{-2.15}$

12 Problems arise when a logarithm is written down with too few significant figures.

a Find $\log_{10} 45$ correct to two significant figures.
b Then find 10 to that index, again correct to two significant figures.

CHALLENGE

13 Solve each equation for x. These questions involve fractional indices.

a $x = \log_7 \sqrt{7}$ **b** $x = \log_{11} \sqrt{11}$ **c** $\log_9 x = \frac{1}{2}$ **d** $\log_{144} x = \frac{1}{2}$
e $\log_x 3 = \frac{1}{2}$ **f** $\log_x 13 = \frac{1}{2}$ **g** $x = \log_6 \sqrt[3]{6}$ **h** $x = \log_9 3$
i $\log_{64} x = \frac{1}{3}$ **j** $\log_{16} x = \frac{1}{4}$ **k** $\log_x 2 = \frac{1}{3}$ **l** $\log_x 2 = \frac{1}{6}$
m $x = \log_8 2$ **n** $x = \log_{125} 5$ **o** $\log_7 x = \frac{1}{2}$ **p** $\log_7 x = -\frac{1}{2}$
q $\log_x \frac{1}{7} = -\frac{1}{2}$ **r** $\log_x \frac{1}{20} = -\frac{1}{2}$ **s** $x = \log_4 \frac{1}{2}$ **t** $x = \log_{27} \frac{1}{3}$
u $\log_{121} x = -\frac{1}{2}$ **v** $\log_{81} x = -\frac{1}{4}$ **w** $\log_x \frac{1}{2} = -\frac{1}{4}$ **x** $\log_x 2 = -\frac{1}{4}$

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7D The laws for logarithms

Logarithms are indices, so the laws for manipulating indices can be rewritten as laws for manipulating logarithms.

Logarithmic and exponential functions are inverse functions

We have already seen that the functions $\boxed{\log}$ and $\boxed{10^x}$ are *inverse functions*. That is, when the two functions are applied to a number one after the other, in any order, the original number returns. Check, for example, using the $\boxed{\log}$ and $\boxed{10^x}$ buttons on your calculator, that

$$\log_{10} 10^3 = 3 \quad \text{and} \quad 10^{\log_{10} 3} = 3.$$

These relationships are true whatever the base of the logarithm, simply because of the definition of logarithms. For example:

$\log_2 2^6 = 6$, because 6 is the index, when 2^6 is written as a power of 2.

$2^{\log_2 64} = 64$, because $\log_2 64$ is the index, when 64 is written as a power of 2.

The exponential and logarithmic functions to any base a are thus called *inverse functions*.

11 THE FUNCTIONS $y = a^x$ AND $y = \log_a x$ ARE INVERSE FUNCTIONS

- Let the base a be any positive real number not equal to 1. Then
  $$\log_a a^x = x, \quad \text{for all real } x \text{, and}$$
  $$a^{\log_a x} = x, \quad \text{for all real } x > 0.$$
- In general, two functions $f(x)$ and $g(x)$ are called *inverse functions* if
  $$f\big(g(x)\big) = x, \quad \text{for all } x \text{ where the calculation is possible, and}$$
  $$g\big(f(x)\big) = x, \quad \text{for all } x \text{ where the calculation is possible.}$$

Example 17 — 7D

a Simplify $\log_7 7^{12}$.

b Simplify $5^{\log_5 11}$.

SOLUTION

a Using the first identity above,
$$\log_7 7^{12} = 12.$$

b Using the second identity above,
$$5^{\log_5 11} = 11.$$

Example 18 — 7D

a Write 3 as a power of 10.

b Write 7 as a power of 2.

SOLUTION

Using the second identity above:

a $3 = 10^{\log_{10} 3}$

b $7 = 2^{\log_2 7}$

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Three laws for logarithms

Three laws are particularly useful when working with logarithms:

12 THREE LAWS FOR LOGARITHMS

1 The log of a product is the sum of the logs:

$$\log_a xy = \log_a x + \log_a y$$

2 The log of a quotient is the difference of the logs:

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

3 The log of a power is the multiple of the log:

$$\log_a x^n = n \log_a x$$

The simple reason for these three laws is that logarithms are indices, and these laws therefore mirror the index laws.

- When multiplying powers, you add the indices.
- When dividing powers, you subtract the indices.
- And when raising a power to a power, you multiply the indices.

Here is a more formal proof.

Proof: To prove each law, show that $a^{\text{LHS}} = a^{\text{RHS}}$, by using the second identity $a^{\log_a x} = x$ and the index laws.

$$\begin{aligned} a^{\log_a x + \log_a y} &= a^{\log_a x} \times a^{\log_a y} && \text{(to multiply powers, add the indices)} \\ &= xy \\ &= a^{\log_a (xy)} \end{aligned}$$

$$\begin{aligned} a^{\log_a x - \log_a y} &= a^{\log_a x} \div a^{\log_a y} && \text{(to divide powers, subtract the indices)} \\ &= x \div y \\ &= a^{\log_a \left(\frac{x}{y}\right)} \end{aligned}$$

$$\begin{aligned} a^{n(\log_a x)} &= (a^{\log_a x})^n && \text{(to raise a power to a power, multiply the indices)} \\ &= x^n \\ &= a^{\log_a (x^n)} \end{aligned}$$

Example 19 7D

Suppose that it has been found that $\log_3 2 \doteqdot 0.63$ and $\log_3 5 \doteqdot 1.46$. Use the log laws to find approximations for:

a $\log_3 10$ **b** $\log_3 \frac{2}{5}$ **c** $\log_3 32$ **d** $\log_3 18$

SOLUTION

a Because $10 = 2 \times 5$,

$$\begin{aligned} \log_3 10 &= \log_3 2 + \log_3 5 \\ &\doteqdot 0.63 + 1.46 \\ &\doteqdot 2.09. \end{aligned}$$

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b $\log_3 \frac{2}{5} = \log_3 2 - \log_3 5$
$\doteqdot 0.63 - 1.46$
$\doteqdot -0.83.$

c Because $32 = 2^5$,
$\log_3 32 = 5\log_3 2$
$\doteqdot 3.15.$

d Because $18 = 2 \times 3^2$,
$\log_3 18 = \log_3 2 + 2\log_3 3$
$= \log_3 2 + 2$, because $\log_3 3 = 1$
$\doteqdot 2.63.$

Some particular values of logarithmic functions

Some particular values of logarithmic functions occur very often and are worth committing to memory.

13 SOME PARTICULAR VALUES AND IDENTITIES OF LOGARITHMIC FUNCTIONS

$\log_a 1 = 0$, because $1 = a^0$.

$\log_a a = 1$, because $a = a^1$.

$\log_a \sqrt{a} = \frac{1}{2}$, because $\sqrt{a} = a^{\frac{1}{2}}$.

$\log_a \dfrac{1}{a} = -1$, because $\dfrac{1}{a} = a^{-1}$.

$\log_a \dfrac{1}{x} = -\log_a x$, because $\log_a \dfrac{1}{x} = \log_a x^{-1} = -\log_a x$.

Example 20 7D

Use the log laws to expand:

a $\log_3 7x^3$

b $\log_5 \dfrac{5}{x}$

SOLUTION

a $\log_3 7x^3 = \log_3 7 + \log_3 x^3$ (the log of a product is the sum of the logs)
$= \log_3 7 + 3\log_3 x$ (the log of a power is the multiple of the log)

b $\log_5 \dfrac{5}{x} = \log_5 5 - \log_5 x$ (the log of a quotient is the difference of the logs)
$= 1 - \log_5 x$ ($\log_5 5 = 1$, because $5 = 5^1$)

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Exercise 7D

FOUNDATION

Note: Do not use a calculator in this exercise. Make sure that you can refer easily to the list of powers of 2, 3, 5, … from Exercise 7A.

1 **a** Use the first log law in Box 12 to find $\log_6 9 + \log_6 4$.
b Use the second log law in Box 12 to find $\log_5 75 - \log_5 3$.
c Use the first and second log laws in Box 12 to find $\log_2 12 + \log_2 6 - \log_2 9$.

2 Use the log law $\log_a x + \log_a y = \log_a xy$ to simplify:
a $\log_6 2 + \log_6 3$ **b** $\log_{15} 3 + \log_{15} 5$ **c** $\log_{10} 4 + \log_{10} 25$
d $\log_{12} 72 + \log_{12} 2$ **e** $\log_{10} 50 + \log_{10} 20$ **f** $\log_6 18 + \log_6 2$

3 Use the log law $\log_a x - \log_a y = \log_a \dfrac{x}{y}$ to simplify:
a $\log_3 15 - \log_3 5$ **b** $\log_4 20 - \log_4 5$ **c** $\log_2 24 - \log_2 3$
d $\log_5 50 - \log_5 2$ **e** $\log_3 810 - \log_3 10$ **f** $\log_2 96 - \log_2 3$

4 Use the log laws to simplify:
a $\log_{30} 2 + \log_{30} 3 + \log_{30} 5$ **b** $\log_{12} 9 + \log_{12} 8 + \log_{12} 2$ **c** $\log_2 12 + \log_2 6 - \log_2 9$
d $\log_3 6 + \log_3 12 - \log_3 8$ **e** $\log_5 12 + \log_5 2 - \log_5 24$ **f** $\log_5 2 - \log_5 50$
g $\log_2 6 - \log_2 48$ **h** $\log_2 12 + \log_2 \frac{1}{3}$ **i** $\log_7 \frac{1}{9} + \log_7 9$

5 Use the log law $\log_a x^n = n\log_a x$ to write each expression in terms of $\log_a 2$.
a $\log_a 8$ **b** $\log_a 16$ **c** $\log_a 64$ **d** $\log_a \frac{1}{2}$
e $\log_a \frac{1}{8}$ **f** $\log_a \frac{1}{32}$ **g** $\log_a \sqrt{2}$ **h** $\log_a \dfrac{1}{\sqrt{2}}$

6 Express each logarithm in terms of $\log_2 3$ and $\log_2 5$. Remember that $\log_2 2 = 1$.
a $\log_2 9$ **b** $\log_2 25$ **c** $\log_2 6$ **d** $\log_2 10$
e $\log_2 18$ **f** $\log_2 20$ **g** $\log_2 \frac{2}{3}$ **h** $\log_2 2\frac{1}{2}$

7 Given that $\log_2 3 \doteqdot 1.58$ and $\log_2 5 \doteqdot 2.32$, use the log laws to find approximations for:
a $\log_2 15$ **b** $\log_2 9$ **c** $\log_2 10$ **d** $\log_2 50$
e $\log_2 \frac{3}{2}$ **f** $\log_2 \frac{3}{5}$ **g** $\log_2 \frac{2}{3}$ **h** $\log_2 75$

DEVELOPMENT

8 Use the identity $\log_a a^x = x$ to simplify:
a $\log_{10} 10^3$ **b** $\log_7 7^5$ **c** $\log_{12} 12^{1.3}$ **d** $\log_8 8^n$

9 Use the identity $a^{\log_a x} = x$ to simplify:
a $10^{\log_{10} 100}$ **b** $3^{\log_3 7}$ **c** $4^{\log_4 3.6}$ **d** $6^{\log_6 y}$

10 Use the log law $\log_a x^n = n\log_a x$ and the identity $\log_a a = 1$ to simplify:
a $\log_a a^2$ **b** $5\log_a a^3$ **c** $\log_a \frac{1}{a}$ **d** $12\log_a \sqrt{a}$

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11 Use the log law $\log_a x^n = n\log_a x$ to write each expression in terms of $\log_a x$.

a $\log_a x^3$ **b** $\log_a \frac{1}{x}$ **c** $\log_a \sqrt{x}$

d $\log_a \frac{1}{x^2}$ **e** $\log_a x^3 - \log_a x^5$ **f** $\log_a x^4 + \log_a \frac{1}{x^2}$

g $2\log_a a^4 - \log_a x^8$ **h** $\log_a \frac{1}{\sqrt{x}} + 3\log_a \sqrt{x}$

12 Write these expressions in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

a $\log_a yz$ **b** $\log_a \frac{z}{y}$ **c** $\log_a y^4$ **d** $\log_a \frac{1}{x^2}$

e $\log_a xy^3$ **f** $\log_a \frac{x^2y}{z^3}$ **g** $\log_a \sqrt{y}$ **h** $\log_a \sqrt{xz}$

13 Given that $\log_{10} 2 \doteqdot 0.30$ and $\log_{10} 3 \doteqdot 0.48$, use the fact that $\log_{10} 5 = \log_{10} 10 - \log_{10} 2$ to find an approximation for $\log_{10} 5$. Then use the log laws to find approximations for:

a $\log_{10} 20$ **b** $\log_{10} 0.2$ **c** $\log_{10} 360$ **d** $\log_{10} \sqrt{2}$

e $\log_{10} \sqrt{8}$ **f** $\log_{10} \frac{1}{\sqrt{10}}$ **g** $\log_{10} \sqrt{12}$ **h** $\log_{10} \frac{1}{\sqrt{5}}$

14 If $x = \log_a 2$, $y = \log_a 3$ and $z = \log_a 5$, simplify each expression.

a $\log_a 64$ **b** $\log_a \frac{1}{30}$ **c** $\log_a 27a^5$ **d** $\log_a \frac{100}{a}$

e $\log_a 1.5$ **f** $\log_a \frac{18}{25a}$ **g** $\log_a 0.04$ **h** $\log_a \frac{8}{15a^2}$

CHALLENGE

15 Using the identity $x = a^{\log_a x}$, express:

a 10 as a power of 3, **b** 3 as a power of 10, **c** 0.1 as a power of 2.

7E Equations involving logarithms and indices

Calculators allow approximation of logarithms base 10. This section explains how to obtain approximations to any other base. Index equations can then be solved approximately, whatever the base.

The change-of-base formula

Finding approximations of logarithms to other bases requires a formula that converts logarithms from one base to another.

14 THE CHANGE-OF-BASE FORMULA

To write log base a in terms of log base b, use the formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Remember this as 'the log of the number over the log of the base'.

The calculator button $\boxed{\text{log}}$ gives approximations for logs base 10, so to find logs to another base a, put $b = 10$ and use the formula

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}.$$

Proof

To prove this formula, let $y = \log_a x$.

Then by the definition of logs, $x = a^y$

and taking log base b of both sides, $\log_b x = \log_b a^y$.

Using the log laws, $\log_b x = y \log_b a$

and rearranging, $y = \dfrac{\log_b x}{\log_b a}$, as required.

Example 21 7E

Find, correct to four significant figures:

a $\log_2 5$

b $\log_3 0.02$

Then check your approximations using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

SOLUTION

In each part, use the change-of-base formula to change to log base 10.

a $$\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} \doteqdot 2.322$$

Checking, $2^{2.322} \doteqdot 5$.

b $$\log_3 0.02 = \frac{\log_{10} 0.02}{\log_{10} 3} \doteqdot -3.561$$

Checking, $3^{-3.561} \doteqdot 0.02$.

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Solving index equations

An index equation such as $5^x = 18$ is solved, in exact form, just by rewriting it in terms of logarithms,

$$5^x = 18$$
$$x = \log_5 18.$$

To find an approximate solution, however, first use the change-of-base formula to convert to log base 10,

$$x = \frac{\log_{10} 18}{\log_{10} 5}$$
$$\doteqdot 1.796.$$

Example 22 7E

Solve, correct to four significant figures, then check your approximations using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

a $2^x = 7$

b $3^x = 0.05$

SOLUTION

a $2^x = 7$

Solving for x, $x = \log_2 7$

Changing to base 10, $x = \dfrac{\log_{10} 7}{\log_{10} 2}$

$\doteqdot 2.807$

Checking, $2^{2.807} \doteqdot 7$.

b $3^x = 0.05$

Solving for x, $x = \log_3 0.05$

Changing to base 10, $x = \dfrac{\log_{10} 0.05}{\log_{10} 3}$

$\doteqdot -2.727$

Checking, $3^{-2.727} \doteqdot 0.05$.

Example 23 7E

The rabbit population on Kanin Island is doubling every year. A few years ago, a study estimated that there were 500 rabbits there. How many years later will the rabbit population be 10 000? Answer in exact form and correct to the nearest tenth of a year.

SOLUTION

Let P be the rabbit population n years after the study.

Then $P = 500 \times 2^n$

Put $P = 10000$, then $10000 = 500 \times 2^n$

$$2^n = 20$$

$$n = \log_2 20 \quad \text{(the exact answer)}$$

$$= \frac{\log_{10} 20}{\log_{10} 2}$$

$$\doteqdot 4.3 \text{ years} \quad \text{(correct to 0.1 years)}$$

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Solving exponential and logarithmic inequations

Provided that the base a is greater than 1, the exponential function $y = a^x$ and the logarithmic function $y = \log_a x$ both increase as x increases, so that there are no difficulties solving inequations such as $2^x \le 64$ and $\log_{10} x > 4$.

Example 24 7E

Solve:

a $2^x \le 64$

b $\log_{10} x \ge 4$

SOLUTION

a
$$2^x \le 64$$
$$2^x \le 2^6$$
$$x \le 6$$

b
$$\log_{10} x \ge 4$$
$$x \ge 10^4$$
$$x \ge 10\,000$$

Exercise 7E

FOUNDATION

1 Use the change-of-base formula, $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$, and your calculator, to verify that:

a $\log_2 8 = 3$ **b** $\log_3 81 = 4$ **c** $\log_{100} 10\,000 = 2$

2 Use the change-of-base formula $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$ to approximate these logarithms, correct to four significant figures. Check each answer using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

a $\log_2 7$ **b** $\log_2 26$ **c** $\log_2 0.07$ **d** $\log_3 4690$
e $\log_5 2$ **f** $\log_7 31$ **g** $\log_6 3$ **h** $\log_{12} 2$
i $\log_3 0.1$ **j** $\log_3 0.0004$ **k** $\log_{11} 1000$ **l** $\log_{0.8} 0.2$
m $\log_{0.03} 0.89$ **n** $\log_{0.99} 0.003$ **o** $\log_{0.99} 1000$

3 Rewrite each equation with x as the subject, using logarithms. Then use the change-of-base formula to solve it, giving your answers correct to four significant figures. Check each answer using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

a $2^x = 15$ **b** $2^x = 5$ **c** $2^x = 1.45$ **d** $2^x = 0.1$
e $2^x = 0.0007$ **f** $3^x = 10$ **g** $3^x = 0.01$ **h** $5^x = 10$
i $12^x = 150$ **j** $8^x = \dfrac{7}{9}$ **k** $6^x = 1.4$ **l** $30^x = 2$
m $0.7^x = 0.1$ **n** $0.98^x = 0.03$ **o** $0.99^x = 0.01$

4 Give exact solutions to these inequations. Do not use a calculator.

a $2^x > 32$ **b** $2^x \le 32$ **c** $2^x < 64$ **d** $3^x \ge 81$
e $5^x > 5$ **f** $4^x \le 1$ **g** $2^x < \dfrac{1}{2}$ **h** $10^x \le 0.001$

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DEVELOPMENT

5 Give exact solutions to these inequations. Do not use a calculator.

a $\log_2 x < 3$ **b** $\log_2 x \ge 3$ **c** $\log_{10} x > 3$ **d** $\log_{10} x \ge 1$
e $\log_5 x > 0$ **f** $\log_6 x < 1$ **g** $\log_5 x \le 3$ **h** $\log_6 x > 2$

6 Rewrite each inequation in terms of logarithms, with x as the subject. Then use the change-of-base formula to solve it, giving your answer correct to three significant figures.

a $2^x > 12$ **b** $2^x < 100$ **c** $2^x < 0.02$ **d** $2^x > 0.1$
e $5^x < 100$ **f** $3^x < 0.007$ **g** $1.2^x > 10$ **h** $1.001^x > 100$

7 A few years ago, Rahul and Fiona were interested in building a new garage, whose price is rising with inflation at 5% per annum. Its price then was \$12 000.

a Explain why the cost C in dollars is $C = 12\,000 \times 1.05^n$, where n is the years since.

b Substitute into the formula to find, correct to the nearest 0.1 years, how many years later the cost will be \$18 000.

8 Use the change-of-base formula, $\log_a x = \dfrac{\log_b x}{\log_b a}$, to prove that:

a $\log_8 x = \dfrac{1}{3}\log_2 x$ **b** $\log_{a^n} x = \dfrac{1}{n}\log_a x$

9 Use the log laws, including possibly the change-of-base formula, to prove that:

a $\log_a \dfrac{b}{c} = -\log_a \dfrac{c}{b}$ **b** $\log_{\frac{1}{a}} x = -\log_a x$ **c** $\log_{\frac{1}{a}} x^{-1} = \log_a x$

CHALLENGE

10 Solve these equations and inequations, correct to three significant figures if necessary.

a $2^{x+1} = 16$ **b** $3^{2x} = 81$ **c** $5^{x-1} < 1$ **d** $10^{\frac{1}{3}x} \le 1000$
e $\left(\frac{1}{2}\right)^{x-3} = 8$ **f** $\left(\frac{1}{10}\right)^{5x} = \frac{1}{10}$ **g** $2^{x-2} < 7$ **h** $3^{x+5} > 10$

11 a Solve $2^x < 10^{10}$. How many positive integer powers of 2 are less than 10^{10}?

b Solve $3^x < 10^{50}$. How many positive integer powers of 3 are less than 10^{50}?

12 a Explain why $\log_{10} 300$ lies between 2 and 3.

b If x is a two-digit number, what two integers does $\log_{10} x$ lie between?

c If $\log_{10} x = 4.7$, how many digits does x have to the left of the decimal point?

d Find $\log_{10} 25^{20}$, and hence find the number of digits in 25^{20}.

e Find $\log_{10} 2^{1000}$, and hence find the number of digits in 2^{1000}.

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7F Exponential and logarithmic graphs

The function $y = a^x$ is an *exponential function*, because the variable x is in the *exponent* or *index*.
The function $y = \log_a x$ is a *logarithmic function*.

The graphs of $y = 2^x$ and $y = \log_2 x$

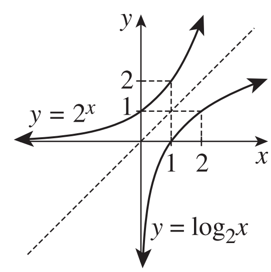
These two graphs will demonstrate the characteristic features of all exponential and logarithmic graphs.
Here are their tables of values:

$y = 2^x$

x	-2	-1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

$y = \log_2 x$

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	-2	-1	0	1	2



- The two graphs are reflections of each other in the diagonal line $y = x$. This is because they are *inverse functions of each other*, so their tables of values are the same, except that the x-values and y-values have been swapped.
- For $y = 2^x$, the domain is all real x and the range is $y > 0$.
  For $y = \log_2 x$, the domain is $x > 0$ and the range is all real y.
- For $y = 2^x$, the x-axis is a horizontal asymptote — as $x \to -\infty$, $y \to 0$.
  For $y = \log_2 x$, the y-axis is a vertical asymptote — as $x \to 0^+$, $y \to -\infty$.
- As x increases, $y = 2^x$ also increases, getting steeper all the time.
  As x increases, $y = \log_2 x$ also increases, but gets flatter all the time.
- The graph of $y = 2^x$ is concave up, but $y = \log_2 x$ is concave down.

15 EXPONENTIAL AND LOGARITHMIC FUNCTIONS ARE INVERSE FUNCTIONS

Let the base a be any positive number not equal to 1.

- The functions $y = a^x$ and $y = \log_a x$ are inverse functions, meaning that
  $$\log_a (a^x) = x \quad \text{and} \quad a^{\log_a x} = x$$
- The graphs of $y = a^x$ and $y = \log_a x$ are reflections of each other in the diagonal line $y = x$.
- The domains, the asymptotes, the steepness, and the concavity of the two graphs are clearly seen from their graphs and this reflection property.

In the language of Section 3I, $y = 2^x$ is a *one-to-one function*, satisfying both the *vertical line test* and the *horizontal line test*. Taking logarithms base 2 is *reading the graph of $y = 2^x$ backwards*, and the inverse function $y = \log_2 x$ is also a *one-to-one function*.

Transformations of exponential and logarithmic functions

The usual transformations of shifting and reflecting apply to exponential and logarithmic functions. Transformations of some exponential functions occurred earlier in Chapter 4, but logarithmic functions were only introduced in this chapter.

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Example 25 [Reflections] 7F

For each pair of functions:

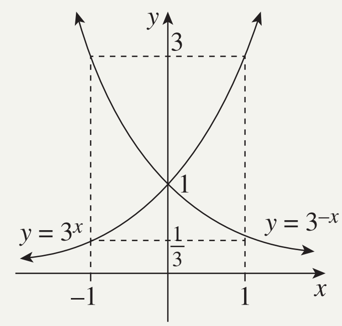
- Draw up convenient tables of values for them.
- Sketch both functions on one set of axes.
- Explain what transformation transforms one graph onto the other, and why.
- State their domain and range.

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = -3^x$

SOLUTION

a

x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
3^{-x}	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$

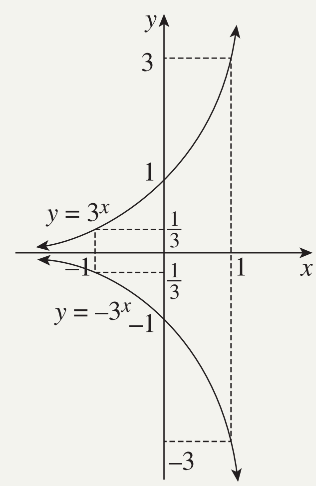


The graphs are reflections of each other in the y-axis, because x has been replaced by $-x$.

For both functions, the domain is all real x, the range is $y > 0$.

b

x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
-3^x	$-\frac{1}{9}$	$-\frac{1}{3}$	-1	-3	-9



The graphs are reflections of each other in the x-axis, because y has been replaced by $-y$. Both functions have domain all real x.

For $y = 3^x$, the range is $y > 0$.

For $y = -3^x$, the range is $y < 0$.

Example 26 [Translations] 7F

Repeat Example 25 for:

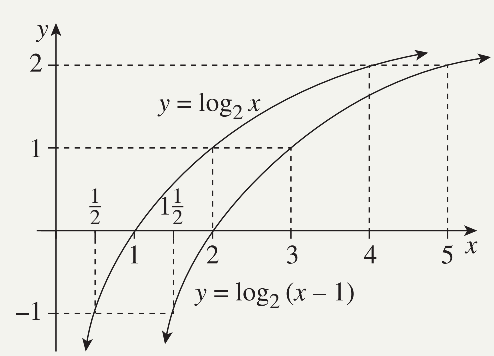
a $y = \log_2 x$ and $y = \log_2 (x - 1)$ **b** $y = \log_2 x$ and $y = 3 + \log_2 x$

SOLUTION

a

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$	-2	-1	0	1	2

x	$1\frac{1}{4}$	$1\frac{1}{2}$	2	3	5
$\log_2 (x - 1)$	-2	-1	0	1	2



The second graph is the first shifted right by 1 unit, because x has been replaced by $x - 1$.

Both functions have range all real y.

$y = \log_2 x$ has domain $x > 0$, and $y = \log_2 (x - 1)$ has domain $x > 1$.

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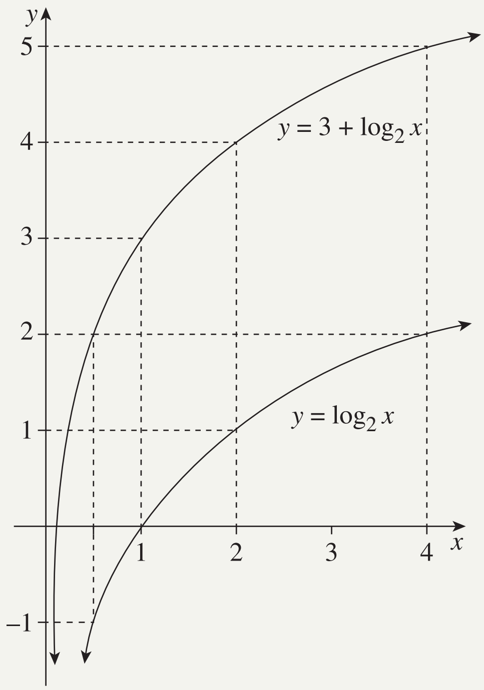
b

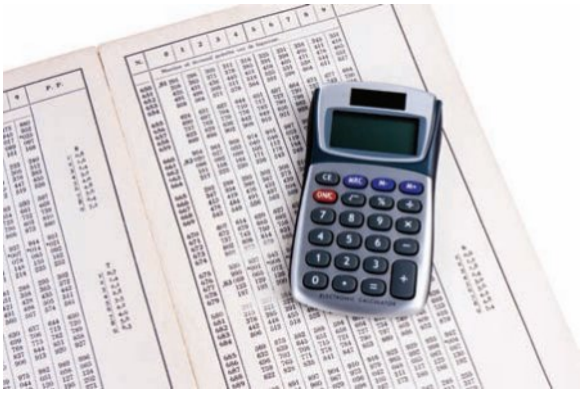
x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$	-2	-1	0	1	2
$3 + \log_2 x$	1	2	3	4	5

The second graph is the first shifted up by 3 units, because y has been replaced by $y - 3$ (or because 3 has been added to each value of y).
Both functions have domain $x > 0$ and range all real y.



The word 'logarithm'

The Scottish mathematician John Napier (1550–1617) formed his new word 'log|arithm' from the two Greek words *logos*, meaning 'ratio' or 'calculation', and *arithmos*, meaning 'number'. Until the invention of calculators, the routine method for performing difficult calculations in arithmetic was to use tables of logarithms, invented by Napier, to convert products to sums, quotients to differences, and powers to multiples.



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Exercise 7F

FOUNDATION

1 **a** Use the calculator button log to complete the following table of values, giving each entry correct to two significant figures where appropriate.

x	0.1	0.25	0.5	0.75	1	2	3	4	5	6	7	8	9	10
$\log_{10} x$														

b Hence sketch the graph of $y = \log_{10} x$. Use a large scale, with the same scale on both axes. Ideally use graph paper so that you can see the shape of the curve.

2 **a** Copy and complete these two tables of values.

i

x	-3	-2	-1	0	1	2	3
2^x							

ii

x	-3	-2	-1	0	1	2	3
2^{-x}							

b Hence sketch the graphs of $y = 2^x$ and $y = 2^{-x}$ on one set of axes.

c How are the two tables of values related to each other?

d What symmetry does the diagram of the two graphs display, and why?

e Write down the domain and range of:

i $y = 2^x$, **ii** $y = 2^{-x}$.

f Write down the equation of the asymptote of:

i $y = 2^x$, **ii** $y = 2^{-x}$.

g Copy and complete:

i 'As $x \to -\infty$, $2^x \to \ldots$' **ii** 'As $x \to \infty$, $2^x \to \ldots$'

h Copy and complete:

i 'As $x \to -\infty$, $2^{-x} \to \ldots$' **ii** 'As $x \to \infty$, $2^{-x} \to \ldots$'

3 **a** Copy and complete these two tables of values.

i

x	-2	-1	0	1	2
3^x					

ii

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\log_3 x$					

b Hence sketch the graphs of $y = 3^x$ and $y = \log_3 x$ on one set of axes.

c How are the two tables of values related to each other?

d What symmetry does the diagram of the two graphs display, and why?

e Write down the domain and range of:

i $y = 3^x$ **ii** $y = \log_3 x$

f Write down the equations of the asymptotes of:

i $y = 3^x$, **ii** $y = \log_3 x$.

g Copy and complete:

i 'As $x \to -\infty$, $3^x \to \ldots$' **ii** 'As $x \to 0^+$, $\log_3 x \to \ldots$'

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DEVELOPMENT

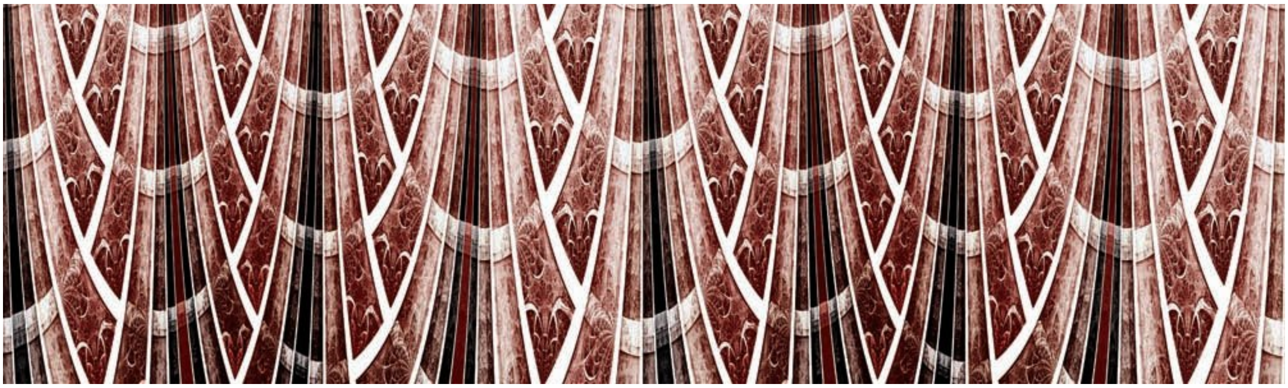
4 **a** Sketch the graphs of $y = 3^x$ and $y = 3^{-x}$ on one set of axes.
b Sketch the graphs of $y = 10^x$ and $y = 10^{-x}$ on one set of axes.

5 Sketch these four graphs on one set of axes.
a $y = 2^x$
b $y = -2^x$
c $y = 2^{-x}$
d $y = -2^{-x}$

6 Sketch each set of three graphs on one set of axes, clearly indicating the asymptote, the y-intercept, and the x-intercept if it exists. Use shifting of the graphs in the previous questions, but also use a table of values to confirm your diagram.
a $y = 2^x$
$y = 2^x + 3$
$y = 2^x - 1$
b $y = -2^x$
$y = 2 - 2^x$
$y = -2 - 2^x$
c $y = 2^{-x}$
$y = 2^{-x} + 1$
$y = 2^{-x} - 2$

7 Use reflections in the x-axis and y-axis to sketch the four graphs below on one set of axes.
a $y = \log_2 x$
b $y = -\log_2 x$
c $y = \log_2(-x)$
d $y = -\log_2(-x)$

8 Use shifting to sketch each set of graphs on one set of axes, clearly indicating the asymptote and the intercepts with the axes.
a $y = \log_2 x$
$y = \log_2 x + 1$
$y = \log_2 x - 1$
b $y = -\log_2 x$
$y = 2 - \log_2 x$
$y = -2 - \log_2 x$



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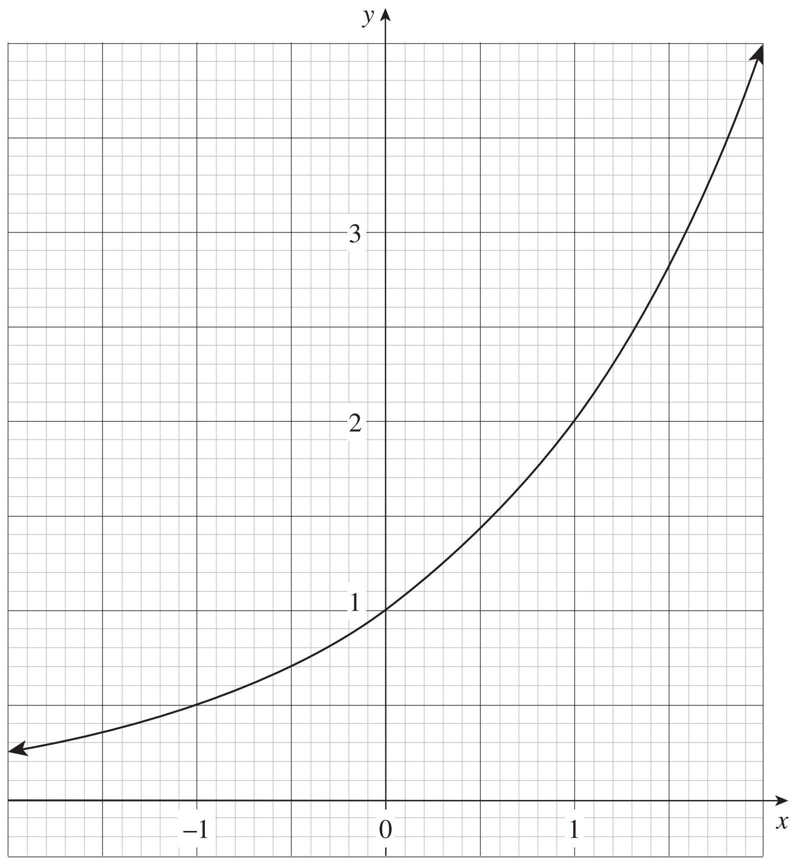

The diagram above shows the graph of $y = 2^x$. Use the graph to answer the questions below, giving your answers correct to no more than two decimal places.

a Read these values from the graph.

i 2^2 **ii** 2^{-2} **iii** $2^{1.5}$ **iv** $2^{0.4}$ **v** $2^{-0.6}$

b Find x from the graph.

i $2^x = 2$ **ii** $2^x = 3$ **iii** $2^x = 1.2$ **iv** $2^x = 0.4$

c Find the values of x for which:

i $1 \le 2^x \le 4$ **ii** $1 \le 2^x \le 2$ **iii** $1.5 \le 2^x \le 3$ **iv** $0.5 \le 2^x \le 2$

d Read the graph backwards to find:

i $\log_2 4$ **ii** $\log_2 3$ **iii** $\log_2 1.4$ **iv** $\log_2 0.8$

CHALLENGE

10 Sketch each graph, on separate axes.

a $y = 2^{x-2}$ **b** $y = 2^{x+1}$ **c** $y = \log_2(x + 1)$

d $y = \log_2(x - 1)$ **e** $y = \frac{1}{2}(2^x + 2^{-x})$ **f** $y = \frac{1}{2}(2^x - 2^{-x})$

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7G Applications of these functions

Some applications of these functions have been scattered through the preceding exercises — the applications in this exercise are longer and more sustained. The intention of all the questions is to indicate how diverse the applications of exponential and logarithmic functions are in various sciences and technologies.

The questions require conversion between a statement in exponential form and a statement in logarithmic form. The pattern for this is

$$2^3 = 8 \qquad \text{means that} \qquad 3 = \log_2 8.$$

The change-of-base formula is also needed to convert logs to base 10,

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}.$$

The later questions in this exercise are more difficult than in other exercises, but they are included because of their vital interest to readers studying computing, physics, geology and chemistry. Many of them could easily be adapted to projects that would go into the subjects in more detail. The exercise concludes with an online project that would investigate web data to see whether it is exponential.

Exercise 7G

FOUNDATION

1 A quantity Q is varying over time t according to the formula $Q = 5 \times 10^{\frac{t}{2}}$. Give answers correct to four significant figures.

a Find Q when $t = 6$, and when $t = 5.43$.

b Rewrite the formula with t as the subject.

c Find t when $Q = 500$, and when $Q = 256$.

2 A quantity Q is varying over time t according to the formula $t = 20\log_2 2Q$.

a Find t when $Q = 4$, and when $Q = 6$.

b Rewrite the formula with Q as the subject.

c Find Q when $t = 40$, and when $t = 45$.

DEVELOPMENT

3 The population of a country is doubling every 30 years, and was 3 000 000 at the last census.

a Explain why the population P after another n years is $P = 3\,000\,000 \times 2^{\frac{n}{30}}$.

b Find the population (to the nearest million) after:

i 90 years,

ii 100 years.

c By substituting into the equation, find when the population will be:

i 48 000 000,

ii 60 000 000 (to the nearest year).

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4 The price of a particular metal has been rising with inflation at 5% per annum, from a base price of \$100 per kilogram in 1900. Let \$$P$ be the price n years since 1900, so that $P = 100 \times (1.05)^n$.

a Copy and complete the following table of values, giving values correct to the nearest whole number:

n	0	20	40	60	80	100
P	100					

b Sketch the graph of the function.

c Now copy and complete this table for $\log_{10} P$, correct to two decimal places.

n	0	20	40	60	80	100
$\log_{10} P$	2					

d Draw a graph with n on the horizontal axis and $\log P$ on the vertical axis.

e Use the log laws to prove that $\log_{10} P = 2 + n\log_{10}(1.05)$, and hence explain the shape of the second graph.

5 [Moore's law]

Gordon Moore predicted in 1965 that over the next few decades, the number of transistors within a computer chip would very roughly double every two years: this is called *Moore's law*. Let D_0 be the density in 1975.

a Explain why the density D after n more years is predicted to be $D = D_0\, 2^{\frac{n}{2}}$.

b What was the density predicted for 2015?

c Substitute into the formula to find, correct to the nearest year, the prediction of the year when the density increases by a factor of 10 000 000.

6 [Newton's law of cooling]

A container of water, originally just below boiling point at 96°C, is placed in a refrigerator whose temperature is 0°C. The container is known to cool in such a way that its temperature halves every 20 minutes.

a Explain why the temperature T°C after n hours in the refrigerator is given by the function $T = 96 \times \left(\frac{1}{2}\right)^{3n}$.

b Draw up a table of values of the function and sketch its graph.

c What is the temperature after 2 hours?

d Write the equation with the time n in hours as the subject.

e How long does it take, correct to the nearest minute, for the temperature to fall to 1°C?

7 [Decay of uranium-235]

Uranium-235 is a naturally-occurring isotope of uranium. It is particularly important because it can be made to fission, when it releases vast amounts of energy in a nuclear power station or a nuclear bomb. It has a half-life of about 700 million years, which means that if you store a mass of uranium-235 for 700 million years, half of it will then have decayed into other elements.

a Explain why the mass of uranium-235 in the Earth n years after the present is given by $M = M_0 \times \left(\frac{1}{2}\right)^{\frac{n}{700\,000\,000}}$, where M_0 is the mass in the Earth now.

b The Andromeda Galaxy will collide with our Milky Way galaxy in about 4 billion years. What percentage of the present uranium-235 will still be present in the Earth then?

c The Earth itself is about 4.5 billion years old. How many times more uranium-235 was present in the Earth when it was formed?

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CHALLENGE

8 [The Richter scale]

Earthquake strengths are usually reported on the Richter scale, which has several minor variants. They are all based on the log base 10 of the 'shaking amplitude' of the earthquake wave at some standardised distance from the epicentre.

(An earthquake occurs kilometres underground — the *epicentre* is the point on the surface above the place where the earthquake occurs.)

a An earthquake of strength 4.0–4.9 is classified as 'light', and will cause only minimal damage. An earthquake of strength 7.0 or above is classified as 'major', and will cause damage or total collapse to most buildings. What is the ratio of the shaking amplitude of the smallest major quake to the smallest light quake?

b The energy released by the quake is proportional to the $\frac{3}{2}$th power of the shaking amplitude. What is the ratio of the energies released by the quakes in part **a**?

c An earthquake of magnitude 9.0 will cause total destruction. What are the ratios of the shaking amplitudes and the energies released of such an earthquake and the smallest light quake?

9 [pH of a solution]

The pH of a liquid is traditionally defined on a logarithmic scale as $\text{pH} = -\log_{10}[\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen ion concentration in units of moles per litre (mol/L). It is not necessary to understand the units, except to know that the greater the concentration of hydrogen ions, the greater the acidity.

a Rewrite the formula with $[\text{H}^+]$ as the subject.

b Pure water has a pH of about 7. What is its hydrogen ion concentration?

c Lemon juice typically has a pH of about 2.0. What is its hydrogen ion concentration? How many times more acidic is it than pure water?

d Sea water typically has a pH of about 8.1. What is its hydrogen ion concentration? How many times more alkaline (meaning 'less acidic') is it than pure water?

Exponential data investigation

See the Interactive Textbook for this printable sheet of an investigation using logarithms to test whether data is exponential.

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Chapter 7 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 7 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Write each expression as an integer or fraction.

a 5^3 **b** 2^8 **c** 10^9 **d** 17^{-1}

e 9^{-2} **f** 2^{-3} **g** 3^{-4} **h** 27^0

i $\left(\frac{2}{3}\right)^3$ **j** $\left(\frac{7}{12}\right)^{-1}$ **k** $\left(\frac{5}{6}\right)^{-2}$ **l** $36^{\frac{1}{2}}$

m $27^{\frac{1}{3}}$ **n** $8^{\frac{2}{3}}$ **o** $9^{\frac{5}{2}}$ **p** $\left(\frac{4}{49}\right)^{\frac{1}{2}}$

q $\left(\frac{14}{59}\right)^0$ **r** $\left(\frac{9}{25}\right)^{-\frac{1}{2}}$ **s** $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ **t** $\left(\frac{9}{100}\right)^{-\frac{3}{2}}$

2 Write each expression in index form.

a $\frac{1}{x}$ **b** $\frac{7}{x^2}$ **c** $-\frac{1}{2x}$ **d** $\sqrt{x}$

e $5\sqrt{36x}$ **f** $\frac{4}{\sqrt{x}}$ **g** $\frac{y}{x}$ **h** $2y\sqrt{x}$

3 Simplify:

a $(x^4)^5$ **b** $\left(\frac{3}{a^3}\right)^4$ **c** $(25x^6)^{\frac{1}{2}}$ **d** $\left(\frac{8r^3}{t^6}\right)^{\frac{1}{3}}$

4 Simplify each expression, leaving the answer in index form.

a $x^2y \times y^2x$ **b** $15xyz \times 4y^2z^4$ **c** $3x^{-2}y \times 6xy^{-3}$

d $4a^{-2}bc \times a^5b^2c^{-2}$ **e** $x^3y \div xy^3$ **f** $14x^{-2}y^{-1} \div 7xy^{-2}$

g $m^{\frac{1}{2}}n^{-\frac{1}{2}} \times m^{1\frac{1}{2}}n^{-\frac{1}{2}}$ **h** $(2st)^3 \times (3s^3)^2$ **i** $(4x^2y^{-2})^3 \div (2xy^{-1})^3$

5 Solve each equation for x.

a $3^x = 81$ **b** $5^x = 25$ **c** $7^x = \frac{1}{7}$ **d** $2^x = \frac{1}{32}$

e $\left(\frac{1}{3}\right)^x = \frac{1}{9}$ **f** $\left(\frac{2}{3}\right)^x = \frac{8}{27}$ **g** $25^x = 5$ **h** $8^x = 2$

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6 Rewrite each logarithmic equation as an index equation and solve it for x.

a $x = \log_2 8$ **b** $x = \log_3 9$ **c** $x = \log_{10} 10000$ **d** $x = \log_5 \frac{1}{5}$

e $x = \log_7 \frac{1}{49}$ **f** $x = \log_{13} 1$ **g** $x = \log_9 3$ **h** $x = \log_2 \sqrt{2}$

i $2 = \log_7 x$ **j** $-1 = \log_{11} x$ **k** $\frac{1}{2} = \log_{16} x$ **l** $\frac{1}{3} = \log_{27} x$

m $2 = \log_x 36$ **n** $3 = \log_x 1000$ **o** $-1 = \log_x \frac{1}{7}$ **p** $\frac{1}{2} = \log_x 4$

7 Use the log laws and identities to simplify:

a $\log_{22} 2 + \log_{22} 11$ **b** $\log_{10} 25 + \log_{10} 4$ **c** $\log_7 98 - \log_7 2$

d $\log_5 6 - \log_5 150$ **e** $\log_3 54 + \log_3 \frac{1}{6}$ **f** $\log_{12} \frac{2}{7} + \log_{12} \frac{7}{2}$

8 Write each expression in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

a $\log_a xyz$ **b** $\log_a \dfrac{x}{y}$ **c** $\log_a x^3$ **d** $\log_a \dfrac{1}{z^2}$

e $\log_a x^2y^5$ **f** $\log_a \dfrac{y^2}{xz^2}$ **g** $\log_a \sqrt{x}$ **h** $\log_a \sqrt{xyz}$

9 Between which two consecutive integers do the following logarithms lie?

a $\log_{10} 34$ **b** $\log_7 100$ **c** $\log_3 90$ **d** $\log_2 35$

e $\log_{10} 0.4$ **f** $\log_{10} 0.007$ **g** $\log_2 0.1$ **h** $\log_7 0.1$

10 Use a calculator, and the change-of-base formula if necessary, to approximate these logarithms, correct to four significant figures.

a $\log_{10} 215$ **b** $\log_{10} 0.0045$ **c** $\log_7 50$ **d** $\log_2 1000$

e $\log_5 0.215$ **f** $\log_{0.01} 2$ **g** $\log_{0.5} 8$ **h** $\log_{0.99} 0.001$

11 Use logarithms to solve these index equations, correct to four significant figures.

a $2^x = 11$ **b** $2^x = 0.04$ **c** $7^x = 350$ **d** $3^x = 0.67$

e $1.01^x = 5$ **f** $1.01^x = 0.2$ **g** $0.8^x = \frac{1}{10}$ **h** $0.99^x = \frac{1}{100}$

12 a Sketch on one set of axes $y = 3^x$, $y = 3^{-x}$, $y = -3^x$ and $y = -3^{-x}$.

b Sketch on one set of axes $y = 2^x$ and $y = \log_2 x$.

c Sketch on one set of axes $y = 3^x$, $y = 3^x - 1$ and $y = 3^x + 2$

13 The number of bacteria growing in a Petri dish is doubling every four hours. Initially there were 100 bacteria.

a Explain why the formula for the population P after n hours is $P = 100 \times 2^{\frac{n}{4}}$.

b How many bacteria are there after:

i 12 hours? **ii** 13 hours?

c Write the formula with n as the subject.

d How long, correct to the nearest hour, before there are 10000000 bacteria?

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8 Differentiation

The next step in studying functions and their graphs is called *calculus*. Calculus begins with two processes called *differentiation* and *integration*.

- *Differentiation* looks at the changing steepness of a curve.
- *Integration* looks at the areas of regions bounded by curves.

Both processes involve taking limits. They were well known to the Greeks, but it was not until the late 17th century that Gottfried Leibniz in Germany and Sir Isaac Newton in England independently gave systematic accounts of them.

This chapter deals with differentiation, and introduces the derivative as the gradient of the tangent to a curve. Section 8J on rates of change begins to show how essential differentiation is for science, for economics, and for solving problems.

The proofs of some of the basic results here are difficult, and have been given only in the appendix at the end of this chapter. They could be regarded as Challenge material during a second reading.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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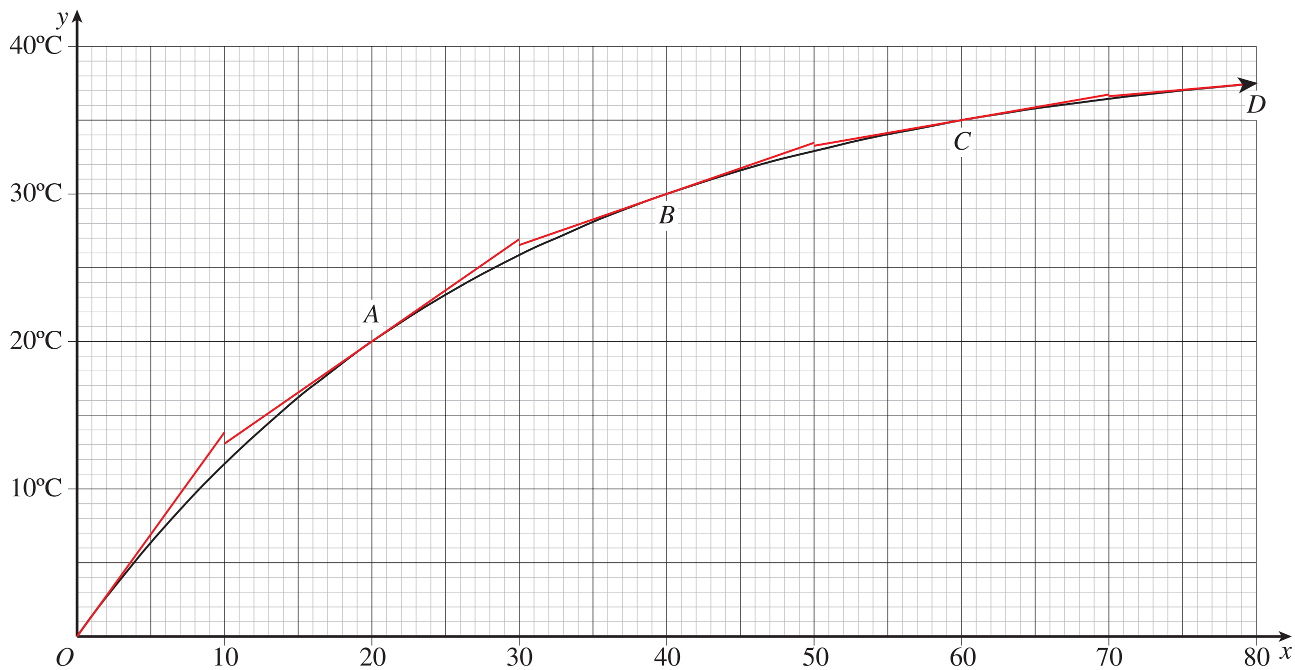
8A Tangents and the derivative



A bottle of water was taken out of a refrigerator on a hot day when the air temperature was 40°C. The graph $y = f(x)$ above shows how the temperature increased over the next 80 minutes. The horizontal axis gives the time x in minutes, and the vertical axis gives the temperature y°C.

The water temperature was originally 0°C, and 20 minutes later it was 20°C. Thus during the first 20 minutes, the temperature was rising at an *average rate* of 1° per minute. This rate is the gradient of the chord OA.

Measuring the *instantaneous rate of temperature increase*, however, requires a tangent to be drawn. The gradient of the tangent is the instantaneous rate of increase at the time x — such gradients are easy to measure on the graph paper by counting little divisions and using the formula gradient $= \dfrac{\text{rise}}{\text{run}}$.

The gradient of the tangent to the curve $y = f(x)$ at any point is called the *derivative*, which is written as $f'(x)$. Measuring the gradients at the marked points O, A, B and C gives a table of values of the derivative:

x	0	20	40	60
$f'(x)$	1.39	0.69	0.35	0.17

Notice that the derivative $f'(x)$ is a new function. It has a table of values, and it can be sketched, just like the original function. You will be sketching the derivative in questions 1 and 2 of the first investigation exercise below.

Geometric definition of the derivative

Here is the essential definition of the derivative.

1 THE DERIVATIVE $f'(x)$ DEFINED GEOMETRICALLY

$f'(x)$ is the gradient of the tangent to $y = f(x)$ at each point on the curve.

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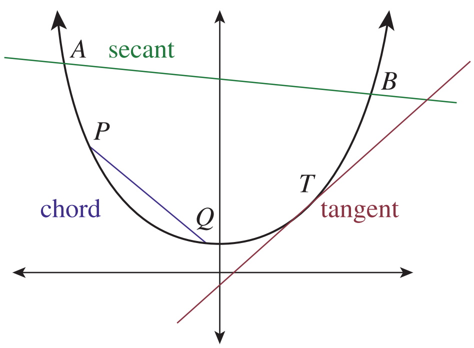
Chords, secants and tangents

Some new terminology has been used in this section. The words 'chord', 'secant' and 'tangent' were probably introduced in earlier years only for circles, but are now being used in the context of graphs.



- A *chord* (meaning 'cord', such as a bowstring) is the interval joining two distinct points P and Q on the graph.
- A *secant* (meaning 'cutting') is the line through two distinct points A and B on the graph — it therefore contains the chord PQ.
- Informally, a *tangent* (meaning 'touching') at a point T on the graph is the line that continues in the direction the curve is going at the point T — think of the ray of light from the headlights of a car going around a bend. The definition used with circles, 'meets the curve only at T', doesn't work for graphs, because if the curve twists around, the tangent may meet the curve again at one or more other points. Section 8B has a more formal definition.

Some preliminary investigations

Investigations Exercise 8A applies the definition of the derivative directly to several functions.
The investigations may be done on graph paper by photocopying the graphs, or by using graphing software, such as the Desmos graphing calculator in the Interactive Textbook.

Section 8B will then develop these ideas into a more rigorous treatment of the derivative.

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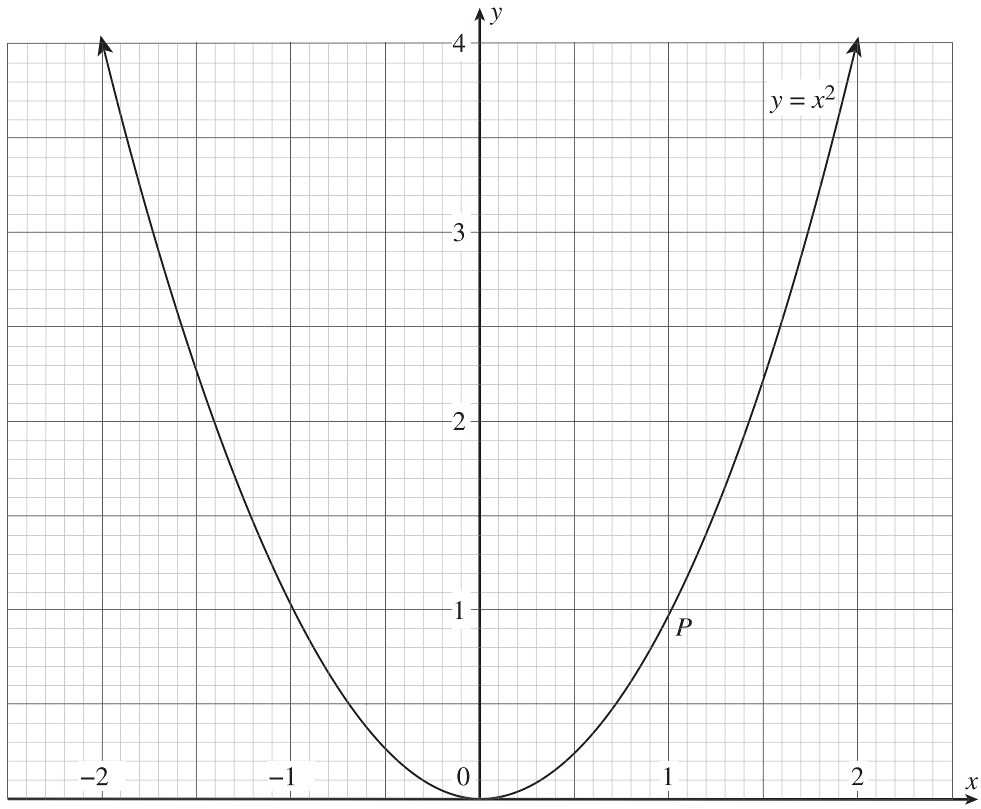
Exercise 8A

INVESTIGATIONS

1 [Graph paper]



a Photocopy the sketch above of $f(x) = x^2$.

b At the point $P(1,\ 1)$, construct the tangent. Place your pencil point on P, bring your ruler to the pencil, then rotate the ruler about P until it seems reasonably like a tangent.

c Use the definition gradient $= \dfrac{\text{rise}}{\text{run}}$ to measure the gradient of this tangent correct to at most two decimal places. Choose the run to be 10 little divisions, and count how many vertical divisions the tangent rises as it runs across the 10 horizontal divisions.

d Copy and complete the following table of values of the derivative $f'(x)$ by constructing a tangent at each of the nine points on the curve and measuring its gradient.

x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
$f'(x)$									

e On a separate set of axes, use your table of values to sketch the curve $y = f'(x)$.

f Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

2 [The same investigation using technology]

a Use graphing software to sketch the graph of $y = x^2$. Use the same scale on both axes so that gradients are represented correctly. The values on the y-axis should run from at least $y = 0$ to $y = 4$, and on the x-axis from at least $x = -2$ to $x = 2$.

b Construct the tangent at the point $P(1,\ 1)$.

c Find the gradient of this tangent, either by asking the software, or by producing a graph-paper background and counting little squares as in Question **1**.

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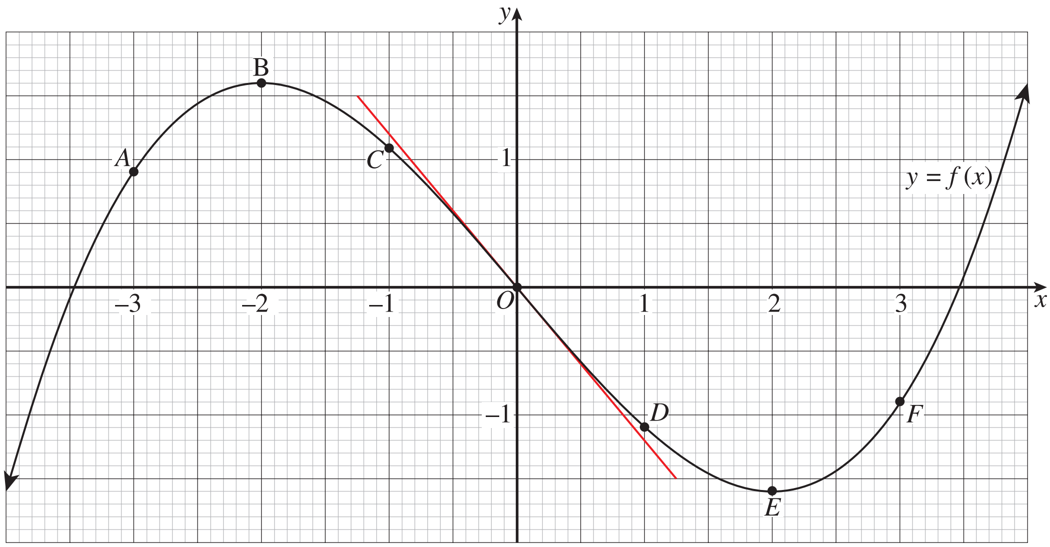
d Copy and complete the table of values of $f'(x)$ as in Question **1d**.

e On a new set of axes, plot these values of $f'(x)$, and join them up to form the graph of $y = f'(x)$.

f Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

3 [Graph paper, but easily adapted to technology]



a Photocopy the cubic graph above.

b The tangent at the origin $O(0, 0)$ has been drawn — notice that it crosses the curve at the origin. Use the definition gradient $= \dfrac{\text{rise}}{\text{run}}$ to measure the gradient of this tangent.

c Copy and complete the following table of values of the derivative $f'(x)$ by constructing a tangent at each of the seven points on the curve and measuring its gradient.

x	-3	-2	-1	0	1	2	3
$f'(x)$							

d On a separate set of axes, use your table of values to sketch the curve $y = f'(x)$.

4 [Technology, but easily done on graph paper]

a Use graphing software to sketch the graph of $y = x^3$, using the same scale on both axes. The values on the y-axis should run from just $y = -1$ to $y = 1$, and on the x-axis from $x = -1$ to $x = 1$. (Larger values of x will send the graph off the screen.)

b Using the software to construct tangents and calculate their gradients, copy and complete the following table of values of the derivative $f'(x)$.

x	-1	-0.9	-0.8	-0.5	0	0.5	0.8	0.9	1
$f'(x)$									

c On a new set of axes, plot the graph of $y = f'(x)$.

d Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

CONCLUSIONS

- Either of the first two investigations should demonstrate reasonably clearly that the derivative of the quadratic function $f(x) = x^2$ is the linear function $y = 2x$.
- Either of the last two investigations should demonstrate at least that the derivative of a cubic curve looks very much like a quadratic curve — the last investigation may have indicated that $f(x) = x^3$ has derivative $f'(x) = 3x^2$.

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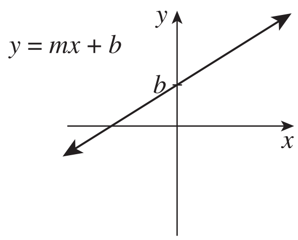
8B The derivative as a limit

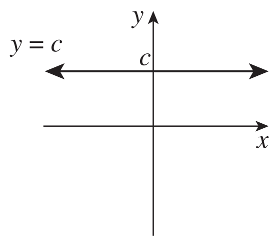
This section introduces a limiting process to find the gradient of a tangent at a point P on a curve. This will be the formal definition of the derivative.

Linear and constant functions

Before any limiting process, however, we can deal very quickly with the derivative of linear and constant functions.

When a graph is a straight line, as in the diagram on the left below, the tangent at every point on the graph is just the line itself. Thus if $f(x) = mx + b$ is a line with gradient m, the derivative at every point is m. Hence the derivative is the constant function $f'(x) = m$.





In particular, a horizontal straight line has gradient zero, as in the diagram on the right above. Hence the tangent to the graph of a constant function $f(x) = c$ at any point is horizontal, so the derivative is the zero function $f'(x) = 0$.

2 THE DERIVATIVE OF LINEAR AND CONSTANT FUNCTIONS

- If $f(x) = mx + b$ is a linear function, then
  $f'(x) = m$ is a constant function.
- In particular, if $f(x) = c$ is a constant function, then
  $f'(x) = 0$ is the zero function.

Example 1 8B

Write down the derivative $f'(x)$ of each linear function.

a $f(x) = 3x + 2$ **b** $f(x) = 5 - 2x$ **c** $f(x) = -4$

SOLUTION

In each case the derivative is just the gradient of the line:

a $f'(x) = 3$ **b** $f'(x) = -2$ **c** $f'(x) = 0$

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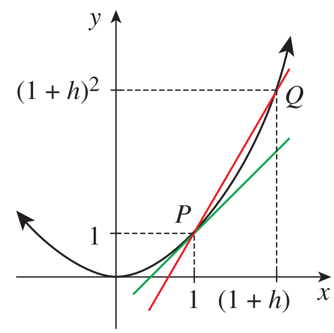
The tangent as the limit of secants

When the function is not linear, we need to look at secants through P that cross the curve again at another point Q near P, and then take the limit as Q moves towards P.

The diagram below shows the graph of $f(x) = x^2$ and the tangent at the point $P(1, 1)$ on the curve.
Let Q be another point on the curve, and join the secant PQ.
Let the x-coordinate of Q be $1 + h$, where $h \neq 0$.
Then the y-coordinate of Q is $(1 + h)^2$.



$$
\begin{aligned}
\text{Hence gradient } PQ &= \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(this is rise over run)}\\
&= \frac{(1+h)^2 - 1}{(1+h) - 1}\\
&= \frac{2h + h^2}{h}\\
&= 2 + h, \qquad \text{because } h \neq 0.
\end{aligned}
$$

As Q moves along the curve, to the right or left of P, the secant PQ changes.
The closer Q is to the point P, the closer the secant PQ is to the tangent at P.
The gradient of the secant PQ becomes 'as close as we like' to the gradient of the tangent as Q moves sufficiently close to P.
This is called 'taking the limit as Q approaches P', with notation $\lim\limits_{Q \to P}$.

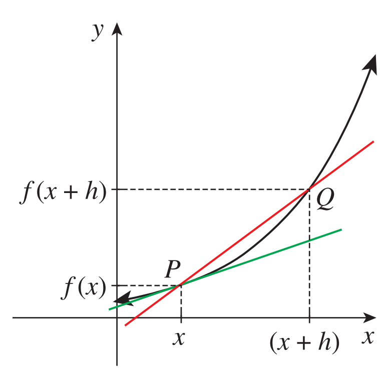
$$
\begin{aligned}
\text{gradient (tangent at } P) &= \lim_{Q \to P} (\text{gradient } PQ)\\
&= \lim_{h \to 0} (2 + h), \qquad \text{because } h \to 0 \text{ as } Q \to P\\
&= 2 \qquad \text{because } 2 + h \to 2 \text{ as } h \to 0.
\end{aligned}
$$

Thus the tangent at P has gradient 2, which means that $f'(1) = 2$.

Notice that the point Q cannot actually coincide with P, that is, h cannot be zero. Otherwise both rise and run would be zero, and the calculation would be invalid. The point Q may, however, be on the left or the right of P.

The derivative as a limit

This same process can be applied to any function $f(x)$.
Let $P\big(x, f(x)\big)$ be any point on the curve.
Let Q be any other point on the curve, to the left or right of P, and let Q have x-coordinate $x + h$, where $h \neq 0$, and y-coordinate $f(x + h)$.



Then gradient of the secant $PQ = \dfrac{f(x+h) - f(x)}{h}$ (rise over run).

As $h \to 0$, the point Q moves 'as close as we like' to P, and the gradient of the secant PQ becomes 'as close as we like' to the gradient of the tangent at P.

Hence gradient of the tangent at $P = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

This last expression is the *limit formula for the derivative* and it in turn, allows a more formal definition of a tangent:

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3 THE DERIVATIVE $f'(x)$ AS A LIMIT, AND THE FORMAL DEFINITION OF A TANGENT

- For each value of x, the derivative is defined as
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(if the limit exists)}$$
- The tangent at a point $P\big(a, f(a)\big)$ on the graph of $y = f(x)$ is the line through P with gradient $f'(a)$.

That is, tangents are now formally defined in terms of the derivative.

The expression $\dfrac{f(x+h) - f(x)}{h}$ is sometimes called a *difference quotient*, because the numerator $f(x+h) - f(x)$ is the difference between the heights at P and Q, and the denominator h is the difference between the x-coordinates of P and Q. Geometrically, it is the gradient of the secant PQ.

Using the definition of the derivative — first-principles differentiation

Using the limit formula above to find the derivative is called *first-principles differentiation*.

Example 2 8B

If $f(x) = x^2$, use first-principles differentiation to show that $f'(5) = 10$.

SOLUTION

$$\begin{aligned}\frac{f(5+h) - f(5)}{h} &= \frac{(5+h)^2 - 5^2}{h} \\ &= \frac{25 + 10h + h^2 - 25}{h} \\ &= \frac{10h + h^2}{h} \\ &= 10 + h, \qquad \text{provided that } h \ne 0.\end{aligned}$$

Taking the limit as $h \to 0$, $f'(5) = 10$.

Example 3 8B

a Use first-principles differentiation to find the derivative $f'(x)$ of $f(x) = x^2$.

b Substitute $x = 5$ to confirm that $f'(5) = 10$, as in the previous exercise.

SOLUTION

a
$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= 2x + h, \qquad \text{provided that } h \ne 0.\end{aligned}$$

Taking the limit as $h \to 0$, $f'(x) = 2x$.

b Substituting $x = 5$, $\quad f'(5) = 10$, as established previously.

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Example 4 8B

a Find the derivative of $f(x) = x^2 + 4x$ by first-principles differentiation.

b Find the gradient of the tangent at the point $(1, 5)$ on the curve $y = x^2 + 4x$.

SOLUTION

a For all $h \neq 0$,
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 + 4(x+h) - (x^2 + 4x)}{h}$$
$$= \frac{x^2 + 2xh + h^2 + 4x + 4h - x^2 - 4x}{h}$$
$$= \frac{2xh + h^2 + 4h}{h}$$
$$= 2x + h + 4, \quad \text{because } h \neq 0.$$

Taking the limit as $h \to 0$, $f'(x) = 2x + 4$.

b At the point $(1, 5)$, gradient of tangent $= f'(1)$ (this is what $f'(1)$ means)
$= 6$.

Exercise 8B

FOUNDATION

Note: Almost all questions in this exercise use the formula for the derivative as a limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1 Use the fact that the derivative of $f(x) = mx + b$ is $f'(x) = m$ to write down $f'(x)$ for these functions.

a $f(x) = 3x$ **b** $f(x) = -7x$ **c** $f(x) = 5x + 10$

d $f(x) = 5 - 3x$ **e** $f(x) = \frac{1}{2}x - 7$ **f** $f(x) = 12$

2 Consider the function $f(x) = 5x^2$.

a Show that $f(1) = 5$ and $f(1 + h) = 5 + 10h + 5h^2$.

b Hence find $f(1 + h) - f(1)$.

c Show that $\dfrac{f(1+h) - f(1)}{h} = 10 + 5h$.

d Take the limit as $h \to 0$ to show that $f'(1) = 10$.

3 Consider again the function $f(x) = 5x^2$.

a Show that $f(x + h) = 5x^2 + 10xh + 5h^2$.

b Hence find $f(x + h) - f(x)$.

c Show that $\dfrac{f(x+h) - f(x)}{h} = 10x + 5h$.

d Take the limit as $h \to 0$ to show that $f'(x) = 10x$.

e Substitute $x = 1$ into $f'(x)$ to confirm that $f'(1) = 10$, as found in Question **2**.

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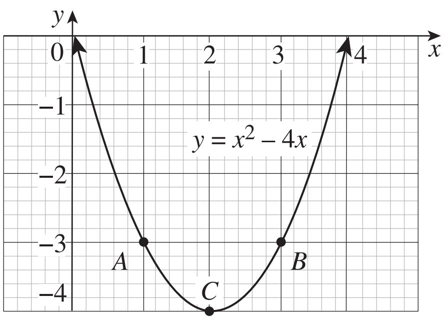
4 Consider the function $f(x) = x^2 - 4x$.



a Show that $f(x + h) = x^2 + 2xh + h^2 - 4x - 4h$.

b Show that $\dfrac{f(x + h) - f(x)}{h} = 2x + h - 4$.

c Show that $f'(x) = 2x - 4$ by taking the limit as $h \to 0$.

d Evaluate $f'(1)$ to find the gradient of the tangent at $A(1, -3)$.

e Similarly, find the gradients of the tangents at $B(3, -3)$ and $C(2, -4)$.

f The function $f(x) = x^2 - 4x$ is graphed on the right. Place your ruler on the curve at A, B and C to check the reasonableness of the results obtained above.

5 In Question **1c**, you found that the linear function $f(x) = 5x + 10$ has derivative $f'(x) = 5$. This question confirms that the limit formula for $f'(x)$ gives this same answer.

a Show that $f(x + h) = 5x + 5h + 10$.

b Hence find $f(x + h) - f(x)$.

c Find $\dfrac{f(x + h) - f(x)}{h}$.

d Take the limit as $h \to 0$ to find $f'(x)$.

DEVELOPMENT

6 Write each of these linear functions in the form $f(x) = mx + b$. Then write down $f'(x)$.

a $f(x) = 5(2x + 4)$ **b** $f(x) = \frac{2}{3}(x + 4)$ **c** $f(x) = \frac{1}{4}(3 - 4x)$

7 Consider the function $f(x) = x^2 + 2$.

a Find $f(x + h) - f(x)$ and hence show that $\dfrac{f(x + h) - f(x)}{h} = 2x + h$.

b Take the limit as $h \to 0$ to find $f'(x)$.

c Evaluate $f'(0)$ to find the gradient of the tangent at the point where $x = 0$.

d Evaluate $f'(3)$ to find the gradient of the tangent at the point where $x = 3$.

8 Consider the function $f(x) = x^2 + 4x$.

a Simplify $\dfrac{f(x + h) - f(x)}{h}$, then take the limit as $h \to 0$ to find $f'(x)$.

b Find the gradients of the tangents to $y = x^2 + 4x$ at the points where $x = 0$ and $x = -2$.

9 a Find $\dfrac{f(x + h) - f(x)}{h}$ for the function $f(x) = x^2 - 2x$ and hence find its derivative.

b Hence find the gradients of the tangents at the points where $x = 0$ and $x = 2$.

10 a Find $\dfrac{f(x + h) - f(x)}{h}$ for the function $f(x) = x^2 + 6x + 8$ and hence find its derivative.

b Hence find the gradients of the tangents at the points where $x = 0$ and $x = -3$.

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11 a Find $\dfrac{f(2+h)-f(2)}{h}$ for the function $f(x) = 1 + x^2$ and hence find $f'(2)$.

b Find $\dfrac{f(h)-f(0)}{h}$ for the function $f(x) = 2x^2 + 3x$ and hence find $f'(0)$.

c Find $\dfrac{f(-1+h)-f(-1)}{h}$ for the function $f(x) = x^2 - 4x$ and hence find $f'(-1)$.

CHALLENGE

12 a If $f(x) = x^3$, show that $\dfrac{f(x+h)-f(x)}{h} = 3x^2 + 3xh + h^2$ and hence find $f'(x)$.

You will need to use the expansion $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$.

b Prove the necessary expansion of $(x+h)^3$ given in part **a**. Begin with:

$$\begin{aligned}(x+h)^3 &= (x+h)(x+h)^2\\ &= (x+h)(x^2+2xh+h^2)\end{aligned}$$

13 a If $f(x) = x^4$, show that $\dfrac{f(x+h)-f(x)}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3$, then find $f'(x)$.

You will need to use the expansion $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

b Prove the necessary expansion of $(x+h)^4$ given in part **a**. Use the previous question, and begin:

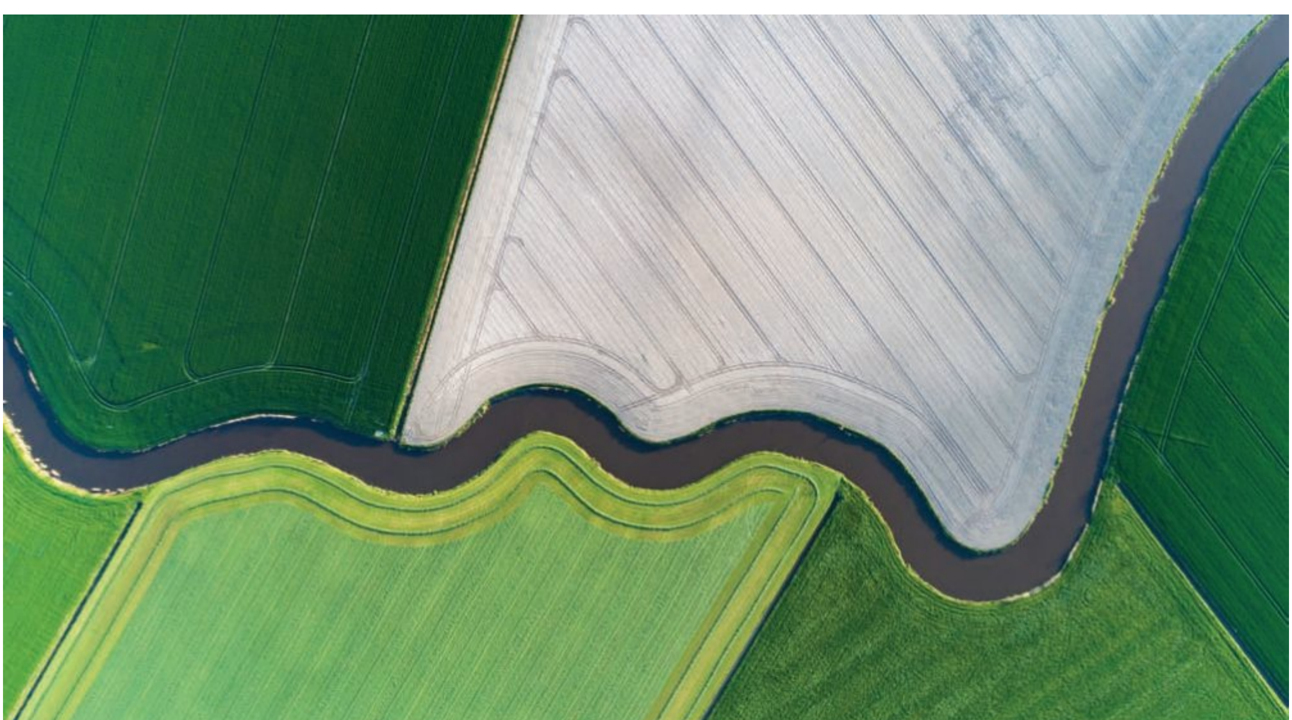
$$\begin{aligned}(x+h)^4 &= (x+h)(x+h)^3\\ &= (x+h)(x^3+3x^2h+3xh^2+h^3)\end{aligned}$$

14 a Use first-principles differentiation to show that if $f(x) = 5x + 7$, then $f'(x) = 5$.

b Show similarly that if $f(x) = mx + b$, where m and b are constants, then $f'(x) = m$.

c Show similarly that if $f(x) = c$, where c is a constant, then $f'(x) = 0$.

d Show similarly that if $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$.



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8C A rule for differentiating powers of x

The long calculations of $f'(x)$ in the previous exercise had quite simple answers, as you will probably have noticed. Here is the pattern:

$f(x)$	1	x	x^2	x^3	x^4	x^5	$\cdots$
$f'(x)$	0	1	$2x$	$3x^2$	$4x^3$	$5x^4$	$\cdots$

These results are examples of a simple rule for differentiating any power of x.

4 THE DERIVATIVE OF ANY POWER OF x

Let $f(x) = x^n$, where n is any real number. Then the derivative is

$$f'(x) = nx^{n-1}.$$

This rule is usually memorised as:

'Take the index as a factor, and reduce the index by 1.'

The proof is complicated, and is given in the appendix at the end of this chapter only for positive integers n. Questions in the Challenge sections of Exercises 8E and 8F develop the further proofs for $n = -1$ and $n = \frac{1}{2}$ respectively.

This course, however, will assume that the result is true for all real numbers n. Here are some examples of the formula applied to powers with positive integers.

Example 5 8C

Differentiate each function.

a $f(x) = x^5$ **b** $f(x) = x^8$ **c** $f(x) = x^{37}$

SOLUTION

a $f(x) = x^5$
$f'(x) = 5x^4$

b $f(x) = x^8$
$f'(x) = 8x^7$

c $f(x) = x^{37}$
$f'(x) = 37x^{36}$

Linear combinations of functions

Functions formed by taking sums and multiples of simpler functions can be differentiated in the obvious way, one term at a time. The proofs are not difficult and are given in the appendix for this chapter.

5 LINEAR COMBINATIONS OF FUNCTIONS

- If $f(x) = g(x) + k(x)$, then $f'(x) = g'(x) + k'(x)$.
- If $f(x) = ag(x)$, then $f'(x) = ag'(x)$.

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Example 6 — 8C

Differentiate each function.

a $f(x) = x^2 + 3x + 5$

b $f(x) = 4x^2 - 3x + 2$

c $f(x) = 3x^{10} + 4x^9$

d $f(x) = \frac{1}{2}x^6 - \frac{1}{6}x^3$

SOLUTION

a
$$\begin{aligned} f(x) &= x^2 + 3x + 5 \\ f'(x) &= 2x + 3 + 0 \\ &= 2x + 3 \end{aligned}$$

b
$$\begin{aligned} f(x) &= 4x^2 - 3x + 2 \\ f'(x) &= 4 \times 2x - 3 + 0 \\ &= 8x - 3 \end{aligned}$$

c
$$\begin{aligned} f(x) &= 3x^{10} + 4x^9 \\ f'(x) &= 3 \times 10x^9 + 4 \times 9x^8 \\ &= 30x^9 + 36x^8 \end{aligned}$$

d
$$\begin{aligned} f(x) &= \tfrac{1}{2}x^6 - \tfrac{1}{6}x^3 \\ f'(x) &= \tfrac{1}{2} \times 6x^5 - \tfrac{1}{6} \times 3x^2 \\ &= 3x^5 - \tfrac{1}{2}x^2 \end{aligned}$$

Expanding products

Sometimes a product needs to be expanded before the function can be differentiated.

Example 7 — 8C

Differentiate each function after first expanding the brackets.

a $f(x) = x^3(x - 10)$

b $f(x) = (x + 2)(2x + 3)$

SOLUTION

a
$$\begin{aligned} f(x) &= x^3(x - 10) \\ &= x^4 - 10x^3 \\ f'(x) &= 4x^3 - 30x^2 \end{aligned}$$

b
$$\begin{aligned} f(x) &= (x + 2)(2x + 3) \\ &= 2x^2 + 7x + 6 \\ f'(x) &= 4x + 7 \end{aligned}$$

The angle of inclination of a tangent

The *gradient of a curve* at a particular point P on it is defined to be the gradient of the tangent at P — the gradient of the curve at P is thus the value of the derivative at P.

The steepness of a tangent can be expressed either by giving its gradient, or by giving its angle of inclination. As we saw in Chapter 6, negative gradients correspond to obtuse angles of inclination.

6 GRADIENT OF A CURVE, AND ANGLE OF INCLINATION OF A TANGENT

- The *gradient of a curve* at a point P on the curve is the gradient of the tangent at P.
- The steepness of a tangent can also be expressed by its angle of inclination.

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Example 8 8C

a Differentiate $f(x) = x^2 + 2x$.

b Find the gradient and angle of inclination of the curve at the origin.

c Find the gradient and angle of inclination of the curve at the point $A(-2, 0)$.

d Sketch the curve and the tangents, marking their angles of inclination.

SOLUTION

a Differentiating $f(x) = x^2 + 2x$ gives $f'(x) = 2x + 2$.

b At the origin $O(0, 0)$, the gradient of the tangent $= f'(0)$
$= 2$.

Let α be the angle of inclination of the tangent.

Then $\tan\alpha = 2$, where $0° \le \alpha < 180°$,

and $\alpha \doteqdot 63°26'$.

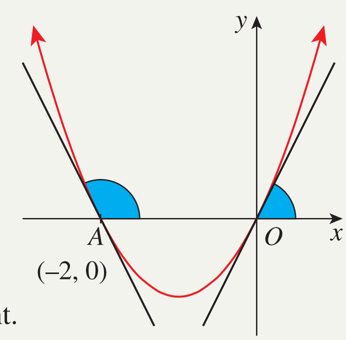
c At $A(-2, 0)$, gradient of tangent $= f'(-2)$
$= -4 + 2$
$= -2$.

Let β be the angle of inclination of the tangent.

Then $\tan\beta = -2$, where $0° \le \beta < 180°$,

and $\beta \doteqdot 180° - 63°26'$
$\doteqdot 116°34'$.

Method: Enter 2 into the calculator, take $\tan^{-1} 2$, then take its supplement.



Finding points on a curve with a given gradient

The derivative can be used to find the points on a curve where the tangent has a particular gradient.

7 FINDING POINTS ON A CURVE WITH A GIVEN GRADIENT

- To find the points where the tangent has a given gradient m, solve the equation $f'(x) = m$.
- To find the the y-coordinates of the points, substitute back into $f(x)$.

The points on the curve where the tangent is horizontal are particularly important.

Example 9 8C

Find the point on $f(x) = 6x - x^2$ where the tangent is horizontal, then sketch the curve.

SOLUTION

Here $f(x) = 6x - x^2$,

So $f'(x) = 6 - 2x$.

Put $f'(x) = 0$ (because the tangent is horizontal)

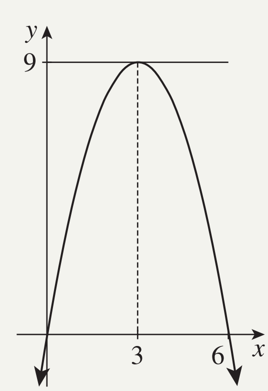
Then $6 - 2x = 0$

$x = 3$.

Substituting, $f'(3) = 18 - 9$
$= 9$,

so the tangent is horizontal at $(3, 9)$.

(This is, of course, the vertex of the parabola.)



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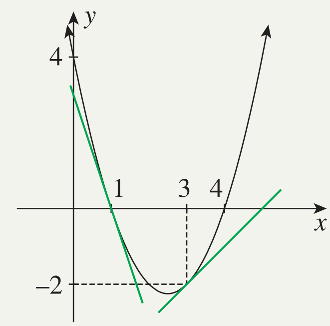
Example 10 8C

Find the points on the graph of $f(x) = x^2 - 5x + 4$ where:

a the tangent has gradient -3,

b the tangent has angle of inclination $45°$.



SOLUTION

Here $f(x) = x^2 - 5x + 4$,
so $f'(x) = 2x - 5$.

a Put $f'(x) = -3$.

Then
$$\begin{aligned} 2x - 5 &= -3 \\ 2x &= 2 \\ x &= 1. \end{aligned}$$

Substituting $x = 1$ into the function,
$$\begin{aligned} f(1) &= 1 - 5 + 4 \\ &= 0, \end{aligned}$$

so the tangent has gradient -3 at the point $(1, 0)$.

b First, $\tan 45° = 1$,
so put $f'(x) = 1$.

Then
$$\begin{aligned} 2x - 5 &= 1 \\ 2x &= 6 \\ x &= 3. \end{aligned}$$

Substituting $x = 3$ into the function,
$$\begin{aligned} f(3) &= 9 - 15 + 4 \\ &= -2, \end{aligned}$$

so the tangent has angle of inclination $45°$ at $(3, -2)$.

Exercise 8C

FOUNDATION

1 Use the rule that the derivative of x^n is nx^{n-1} to write down the derivative $f'(x)$ of:

a $f(x) = x^7$ **b** $f(x) = x^5$ **c** $f(x) = -x^{24}$

d $f(x) = 9x^5$ **e** $f(x) = 3x^2$ **f** $f(x) = -5x^{12}$

g $f(x) = \frac{1}{3}x^6$ **h** $f(x) = \frac{1}{2}x^8$ **i** $f(x) = -\frac{2}{3}x^9$

2 **a** Write $f(x) = 9x$ as $f(x) = 9x^1$, and then use the rule to show that $f'(x) = 9$.

b Write $g(x) = 4$ as $g(x) = 4x^0$, and then use the rule to show that $g'(x) = 0$.

3 For each function, find the derivative $f'(x)$.

a $f(x) = 5x + 7$ **b** $f(x) = 8 - x$ **c** $f(x) = x^2 + 5x + 7$

d $f(x) = 3x^2 - 5x$ **e** $f(x) = x^4 - 5x^2 + 5$ **f** $f(x) = 2 - 3x - 5x^3$

g $f(x) = x^4 + x^3 + x^2 + x + 1$ **h** $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2$ **i** $f(x) = \frac{1}{3}x^6 - \frac{1}{2}x^4 + x^2 - 2$

4 **a** Find the derivative of $f(x) = x^2 + 7x - 2$.

b Evaluate $f'(0)$ to find the gradient of the tangent at the point $(0, -2)$.

5 Differentiate each function, then find the gradient of the tangent at the point where $x = 2$.

a $f(x) = 7 - x^2$ **b** $f(x) = x^3 + 6x$ **c** $f(x) = 10x^2 - x^4$

6 Differentiate $f(x) = x^3$, and hence find the gradient of the tangent to $y = x^3$ at:

a $(2, 8)$ **b** $(1, 1)$ **c** $(0, 0)$ **d** $(-1, -1)$ **e** $(-2, -8)$

7 **a** Differentiate $f(x) = x^2 - 4$.

b Solve $f'(x) = 0$ to find the point on $y = x^2 - 4$ where the tangent is horizontal.

c Solve $f'(x) = 6$ to find the point on $y = x^2 - 4$ where the tangent has gradient 6.

d Solve $f'(x) = -6$ to find the point on $y = x^2 - 4$ where the tangent has gradient -6.

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DEVELOPMENT

8 Expand each product, then find the derivative.

a $f(x) = x(4 - x)$ **b** $f(x) = x(x^2 + 1)$ **c** $f(x) = x^2(3 - 4x^2)$
d $f(x) = (x + 4)(x - 2)$ **e** $f(x) = (2x + 1)(2x - 1)$ **f** $f(x) = (x^2 + 3)^2$
g $f(x) = (7 - x)^2$ **h** $f(x) = (x^2 + 3)(x - 5)$ **i** $f(x) = (3x - 5)^2$

9 a Find the derivative of $f(x) = x^2 + x + 1$.
b Evaluate $f'(0)$ to show that the tangent at the point $(0, 1)$ has gradient 1.
c Solve $\tan\alpha = 1$ to find the angle of inclination of the tangent at $(0, 1)$.

10 a Find the derivative of $f(x) = 9 - x + x^2$.
b Evaluate $f'(2)$ to show that the tangent at the point $(2, 11)$ has gradient 3.
c Find the angle of inclination of the tangent at $(2, 11)$, correct to the nearest minute.

11 Differentiate $f(x) = x^2 - 3x - 6$. Hence find the gradient and the angle of inclination (correct to the nearest minute, where appropriate) of the tangent at the point where:

a $x = 3$ **b** $x = 2$ **c** $x = 1\frac{1}{2}$ **d** $x = 1$ **e** $x = 0$

12 a Find the derivative of $f(x) = 8x - x^2$.
b Use the factoring $f(x) = x(8 - x)$ to find the x-intercepts of $y = f(x)$, then sketch it.
c Find the gradients of the tangents at these two x-intercepts.
d Show that the tangent is horizontal at the point where $x = 4$.

13 a Find the derivative of $f(x) = x^2 + 8x + 7$.
b Solve $f'(x) = 0$. Hence find the point on $y = f(x)$ where the tangent is horizontal.
c Solve $f'(x) = 12$. Hence find the point on $y = f(x)$ where the tangent has gradient 12.

14 a Find the derivative of $f(x) = 3 - 2x^2$.
b Solve $f'(x) = 0$. Hence find the point on $y = f(x)$ where the tangent is horizontal.
c Solve $f'(x) = -20$. Hence find the point where the tangent has gradient -20.

15 Differentiate, then find the points on each curve where the tangent is horizontal.
a $f(x) = x^2 - 2x + 7$
b $f(x) = x^2 + 4x - 10$
c $f(x) = x^2 - 10x + 15$

16 a Find the derivative of $f(x) = x^2 - 6x$.
b Use the factoring $f(x) = x(x - 6)$ to find the x-intercepts of $y = f(x)$, then sketch it.
c Find the gradients of the tangents at these two x-intercepts.
d Find the point on the curve where the tangent is horizontal.

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17 Differentiate $f(x) = x^2 - 5x + 1$ and hence find the points on the curve where:

a the gradient is 3,

b the gradient is -5,

c the angle of inclination is 45°,

d the angle of inclination is 135°.

CHALLENGE

18 a Differentiate $f(x) = x^3 - 3x + 2$. Hence find the two points on $y = x^3 - 3x + 2$ where the tangent is horizontal.

b Differentiate $f(x) = x^4 - 18x^2$. Hence find the three points on $y = x^4 - 18x^2$ where the tangent is horizontal.

c Differentiate $f(x) = x^3 + 6$. Hence find the two points on $y = x^3 + 6$ where the tangent has gradient 75.

19 Sketch the graph of $f(x) = x^2 - 6x$ and find the gradient of the tangent at the point $A(a, a^2 - 6a)$ on the curve.

a Hence find the value of a if the tangent has gradient:

i 0

ii 2

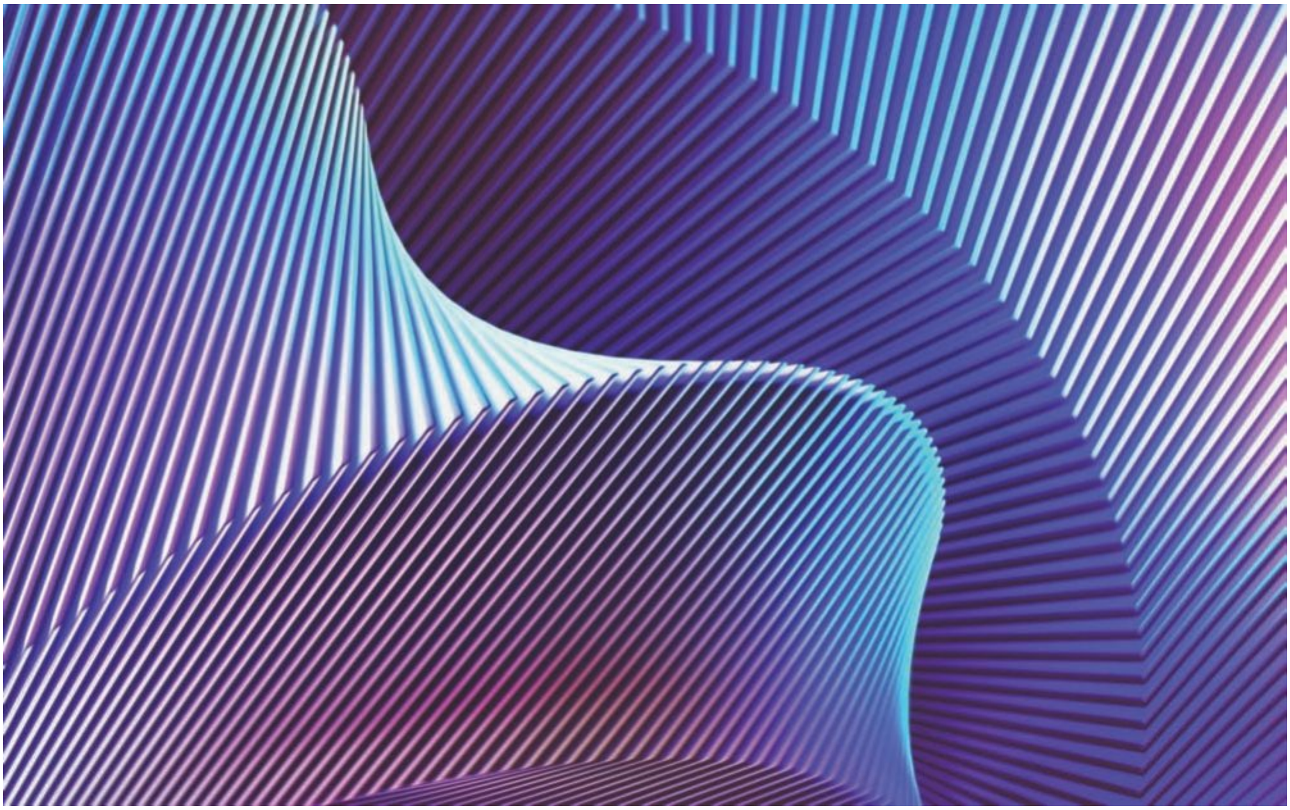
iii $\frac{1}{2}$

b Find the value of a if the tangent has angle of inclination 135°,

c Find the value of a if the tangent is,

i parallel to $2x - 3y + 4 = 0$,

ii perpendicular to $2x - 3y + 4 = 0$.



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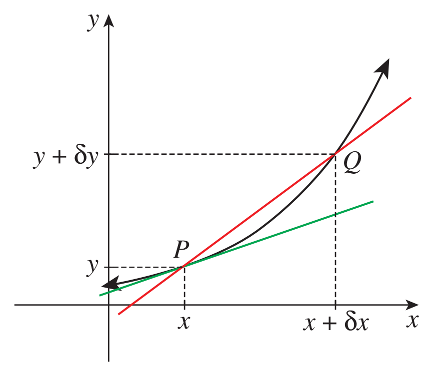
8D Tangents and normals — $\dfrac{dy}{dx}$ notation

Leibniz's original notation for the derivative remains the most widely used and best-known notation, particularly in science. It is even said that Dee Why Beach was named after the derivative $\dfrac{dy}{dx}$. The notation is extremely flexible, and clearly expresses the fact that the derivative behaves very much like a fraction.

Small changes in *x* and in *y*

The diagram used to explain $\dfrac{dy}{dx}$ is exactly the same as for $f'(x)$, but the notation is a little different.

Let $P(x, y)$ be any point on the graph of a function.



Let x change by a small amount δx to $x + \delta x$, and let y change by a corresponding amount δy to $y + \delta y$.

Thus $Q(x + \delta x, y + \delta y)$ is another point on the curve. Then

$$\text{gradient } PQ = \frac{\delta y}{\delta x} \quad \text{(rise over run)}.$$

When δx is small, the secant PQ is almost the same as the tangent at P, and the derivative is the limit of $\dfrac{\delta y}{\delta x}$ as Q gets 'as close as we like' to P, that is, as $\delta x \to 0$. This is the basis for Leibniz's notation.

8 AN ALTERNATIVE NOTATION FOR THE DERIVATIVE

Let δy be the small change in y resulting from a small change δx in x. Then

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

(The letter δ is the lower case Greek letter delta, corresponding to d.)

The object dx is intuitively understood as an 'infinitesimal change' in x, and dy is understood as the corresponding 'infinitesimal change' in y. The derivative $\dfrac{dy}{dx}$ is then understood as the ratio of these two infinitesimal changes. 'Infinitesimal changes', however, are for the intuition only — the logic of the situation is that:

9 FRACTION NOTATION AND THE DERIVATIVE $\dfrac{dy}{dx}$

The derivative $\dfrac{dy}{dx}$ is not a fraction, but is the limit of the fraction $\dfrac{\delta y}{\delta x}$ as $\delta x \to 0$.

The notation is very clever because the derivative is a gradient, the gradient is a fraction $\dfrac{\text{rise}}{\text{run}}$, and the notation $\dfrac{dy}{dx}$ preserves the intuition of fractions.

The words 'differentiation' and 'calculus'

The *small differences* δx and δy, and the *infinitesimal differences* dx and dy, are the origins of the word 'differentiation'.

Calculus is a Latin word meaning 'stone'. An *abacus* consists of *stones* sliding on bars and was once used routinely to help with arithmetic — this is the origin of the modern word 'calculate'. The word 'calculus' in English can refer to any systematic method of calculation, but is most often used for the twin theories of differentiation and integration studied in this course.

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Second and higher derivatives

The derivative of a function is another function, so it can in turn be differentiated to give the *second derivative* of the original function. This is written as $f''(x)$ in $f(x)$ notation, and as $\dfrac{d^2y}{dx^2}$ in the new $\dfrac{dy}{dx}$ notation.

Thus for a function such as $x^5 + 5x^3$, the two notations are:

$$y = x^5 + 5x^3 \qquad f(x) = x^5 + 5x^3$$
$$\frac{dy}{dx} = 5x^4 + 15x^2 \qquad f'(x) = 5x^4 + 15x^2$$
$$\frac{d^2y}{dx^2} = 20x^3 + 30x \qquad f''(x) = 20x^3 + 30x$$

This process can be continued indefinitely to *third* and *higher derivatives*:

$$\frac{d^3y}{dx^3} = 60x^2 + 30 \qquad f'''(x) = 60x^2 + 30$$
$$\frac{d^4y}{dx^4} = 120x \qquad f^{(4)}(x) = 120x$$
$$\frac{d^5y}{dx^5} = 120 \qquad f^{(5)}(x) = 120$$

After that, the sixth and higher derivatives are all zero. Notice that to avoid counting multiple dashes, we usually write $f^{(4)}(x)$ rather than $f''''(x)$.

The second derivative $f''(x)$ is the *gradient function of the gradient function*. Next year, we will interpret the second derivative in terms of curvature, but there are no obvious geometric interpretations of higher derivatives.

Setting out using $\dfrac{dy}{dx}$ notation

The remaining examples in this section show how the new notation is used in calculations on the geometry of a curve.

Example 11 — 8D

a Differentiate $y = 4 - x^2$.
b Find the gradient of the tangent at the point $P(-1, 3)$ on the curve.
c Find the point on the curve where the gradient of the tangent is 6.

SOLUTION

a $y = 4 - x^2$

$\dfrac{dy}{dx} = -2x$ (applying the usual rule for differentiating)

b At the point $P(-1, 3)$, the gradient of the tangent is

$\dfrac{dy}{dx} = (-2) \times (-1)$ (substituting $x = -1$ into the formula for $\dfrac{dy}{dx}$)

$= 2.$

c Put $\dfrac{dy}{dx} = 6$ (to find where the gradient is 6)

then $-2x = 6$

$x = -3.$

When $x = -3$, $y = 4 - 9$,

$= -5$,

so the gradient is 6 at the point $(-3, -5)$.

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Tangents and normals to a curve

Let P be any point on a curve $y = f(x)$. Using the derivative, it is a simple procedure to find the equations of the tangent and the normal to a curve at P.

10 TANGENTS AND NORMALS TO A CURVE

- The *tangent* at P is the line through P whose gradient is the derivative at P.
- The *normal* at P is the line through P perpendicular to the tangent at P.

Equations of tangents and normals are easily found using gradient–intercept form,

$$y - y_1 = m(x - x_1)$$

Two gradients m_1 and m_2 are perpendicular when their product is -1,

$$m_1 m_2 = -1, \qquad \text{that is,} \qquad m_2 = -\frac{1}{m_1}.$$

Example 12 8D

a Find the gradient and angle of inclination of the tangent to $y = (x + 1)^2$ at $A(0, 1)$.

b Find the gradient and angle of inclination of the normal at A, and draw a sketch.

SOLUTION

a Expanding: $y = x^2 + 2x + 1$,

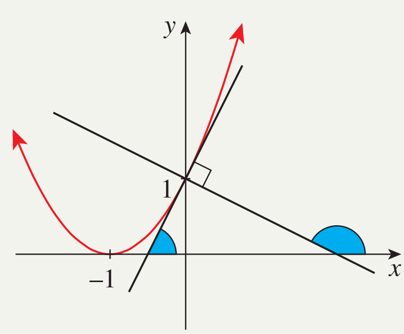
so $\dfrac{dy}{dx} = 2x + 2$.

When $x = 0$, $\dfrac{dy}{dx} = 0 + 2$,

so the tangent at $A(0, 1)$ has gradient 2.

Solving $\tan\alpha = 2$, where $0° \le \alpha < 180°$,

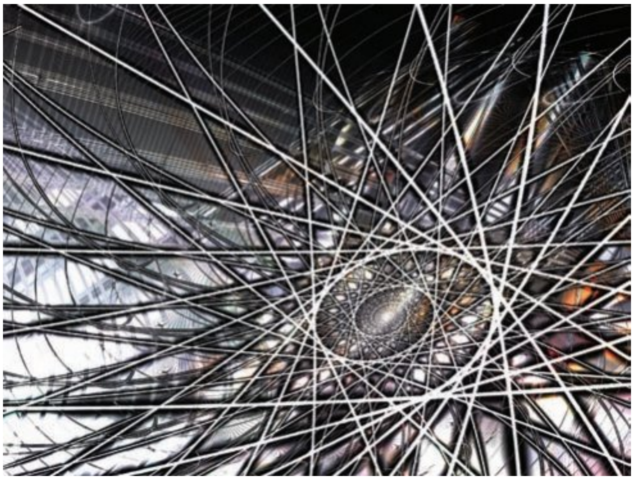
its angle of inclination is about $63°26'$.



b The normal has gradient $-\frac{1}{2}$ (opposite of the reciprocal).

Solving $\tan\beta = -\frac{1}{2}$, where $0° \le \beta < 180°$,

its angle of inclination is about $153°26'$.



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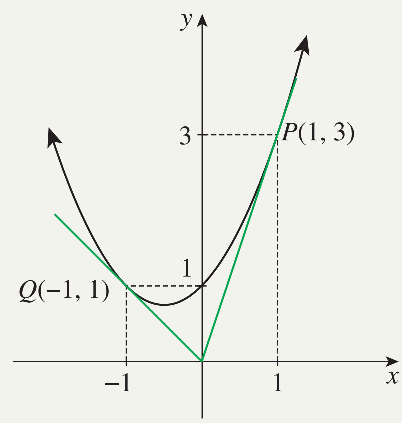
Example 13 8D

a Find the equations of the tangents to $y = x^2 + x + 1$ at $P(1, 3)$ and $Q(-1, 1)$.

b At what point do these tangents intersect?

SOLUTION



a Differentiating, $\dfrac{dy}{dx} = 2x + 1.$

When $x = 1$, $\dfrac{dy}{dx} = 3,$

so, using point–gradient form, the tangent at $P(1, 3)$ is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= 3(x - 1) \\ y &= 3x. \end{aligned}$$

When $x = -1$, $\dfrac{dy}{dx} = -1,$

so the tangent at $Q(-1, 1)$ is $y - y_1 = m(x - x_1)$

$$\begin{aligned} y - 1 &= -1(x + 1) \\ y &= -x. \end{aligned}$$

b Both $y = 3x$ and $y = -x$ pass through the origin,

so the origin is the point of intersection of the two tangents.

Geometric figures created by tangents and normals

Tangents and normals to a graph, together with the axes, often produce geometric figures that can be studied further. The following example finds the area of a triangle formed by the tangent and normal at a point, and the y-axis.

Example 14 8D

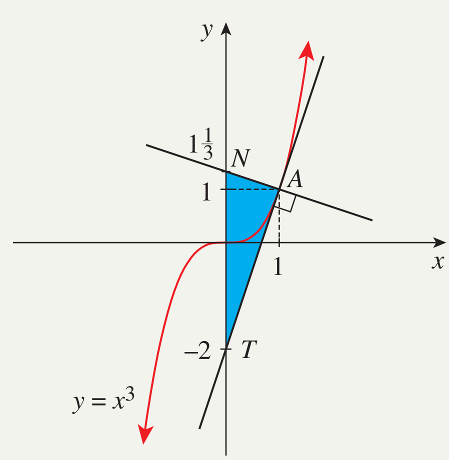
a Find the equation of the tangent to $y = x^3$ at $A(1, 1)$.

b Find the equation of the normal to the curve at $A(1, 1)$.

c Find the coordinates of the y-intercepts T and N of the tangent and normal.

d Sketch the situation and find the area of ΔANT.

SOLUTION



a Differentiating, $\dfrac{dy}{dx} = 3x^2.$

When $x = 1$, $\dfrac{dy}{dx} = 3,$

so the tangent at A has gradient 3.

Using point–gradient form, the tangent is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 3(x - 1) \\ y &= 3x - 2. \end{aligned}$$

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b The normal has gradient $-\frac{1}{3}$.

Using point–gradient form, the normal is

$$y = y_1 = m(x - x_1)$$
$$y - 1 = -\tfrac{1}{3}(x - 1)$$
$$y = -\tfrac{1}{3}x + 1\tfrac{1}{3}.$$

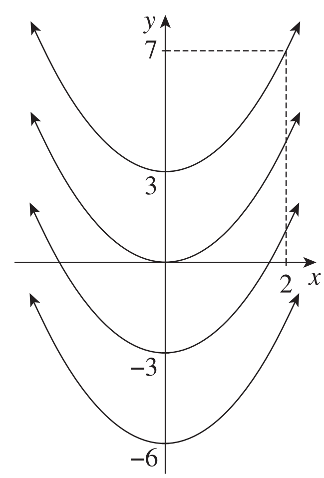
c Both lines are already in gradient–intercept form, and their y-intercepts are therefore $T(0, -2)$ and $N(0, 1\frac{1}{3})$.

d The triangle ANT has base NT and perpendicular height 1,

where $\quad NT = 2 + 1\frac{1}{3}$
$\quad = 3\frac{1}{3}$,

so $\quad$ area $\Delta ANT = \frac{1}{2} \times 3\frac{1}{3} \times 1$
$\quad = 1\frac{2}{3}$ square units.

Families of curves with the same derivative



The four functions $y = x^2$, $y = x^2 + 3$, $y = x^2 - 3$ and $y = x^2 - 6$ are all seen to be related when they are differentiated. In all four cases,

$$\frac{dy}{dx} = 2x$$

because the derivative of any constant term is zero. More generally, we can prove that the curves with derivative $2x$ form an infinite family of curves

$$y = x^2 + C, \text{ where } C \text{ is a constant.}$$

First, all the curves in this family have derivative $2x$.

Conversely, if some function y has derivative $\frac{dy}{dx} = 2x$, then

$$\frac{d}{dx}(y - x^2) = 2x - 2x = 0,$$

so $y - x^2$ is a constant function C, because its gradient is always zero.

That is, $\quad y - x^2 = C,$
$\quad y = x^2 + C.$

The diagram above shows four members of this family. Notice that no two curves in the family ever intersect. Notice also that we can pick out the curve that goes through the particular point $(2, 7)$, because a simple substitution shows that this curve is $y = x^2 + 3$.

11 FAMILIES OF CURVES WITH THE SAME DERIVATIVE:

The functions in a *family of curves with the same derivative* all differ by a constant.

For example, the family of curves with derivative $\frac{dy}{dx} = 2x$ is

$$y = x^2 + C, \qquad \text{for some constant } C.$$

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These remarks are the beginning of the very large subject of *integration*, which is the reverse process of *differentiation*, and will be a major concern in Year 12. This is not the place to formalise any rules — the table below contains all the forms that will be used in this volume. Just check that each entry in the second line below is the family of curves whose derivative is the entry above it.

$\dfrac{dy}{dx} = 1$	$\dfrac{dy}{dx} = x$	$\dfrac{dy}{dx} = x^2$	$\dfrac{dy}{dx} = x^3$	$\dfrac{dy}{dx} = x^4$
$y = x + C$	$y = \dfrac{x^2}{2} + C$	$y = \dfrac{x^3}{3} + C$	$y = \dfrac{x^4}{4} + C$	$y = \dfrac{x^5}{5} + C$

Example 15

Use the table above to find the family of curves whose derivative is:

a $\dfrac{dy}{dx} = 9x^2 - 5$

b $\dfrac{dy}{dx} = x^3 + 6x$

SOLUTION

a Given that $\dfrac{dy}{dx} = 9x^2 - 5$

it follows that $y = \dfrac{9x^3}{3} - 5x + C$, for some constant C

$y = 3x^3 - 5x + C$. (Check by differentiating.)

b Given that $\dfrac{dy}{dx} = x^3 + 6x$

it follows that $y = \dfrac{x^4}{4} + \dfrac{6x^2}{2} + C$, for some constant C

$y = \dfrac{x^4}{4} + 3x^2 + C$. (Check by differentiating.)

Exercise 8D

FOUNDATION

1 Find the derivative $\dfrac{dy}{dx}$ of each function.

a $y = x^2$
b $y = x^2 + 7x - 10$
c $y = x^3 + 3x^2 + 6x + 8$
d $y = x^4 + x^2 + 8x$
e $y = 4x - 7$
f $y = 7$

2 Using $\dfrac{dy}{dx}$ notation, find the first, second and third derivatives of each function.

a $y = x^6 + 2x$
b $y = 5x^2 - x^5$
c $y = 4x$

3 Using $f(x)$ notation, find the 1st, 2nd, 3rd and 4th derivatives of each function.

a $f(x) = 10x^3 + x$
b $f(x) = 2x^4 + 2$
c $f(x) = 5$

4 Differentiate $y = x^2 + x$. Then solve $\dfrac{dy}{dx} = 7$ and hence find the coordinates of any points on $y = x^2 + x$ where the tangent has gradient 7.

5 Differentiate $y = x^3 - 1$. Then solve $\dfrac{dy}{dx} = 12$ and hence find the coordinates of any points on $y = x^3 - 1$ where the tangent has gradient 12.

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6 **a** Differentiate the function $y = x^2 - 3x$.

b Substitute into $\frac{dy}{dx}$ to find the gradient of the tangent at the point P where $x = 2$.

c Substitute into the function to find the y-coordinate of the point P.

d Use point–gradient form to find the equation of the tangent to the curve at P.

e Write down the gradient of the normal at P.

f Use point–gradient form to find the equation of the normal to the curve at P.

7 **a** Differentiate $y = x^5 + x^3 + 2x$.

b Find the values of y and $\frac{dy}{dx}$ at the point P where $x = 0$.

c Use point–gradient form to find the equation of the tangent to the curve at P.

d Write down the gradient of the normal at P.

e Use point–gradient form to find the equation of the normal to the curve at P.

8 **a** Show that each function below has the same derivative:

$$y = x^3 + 7x + 4, \quad y = x^3 + 7x - 6, \quad y = x^3 + 7x - 3\tfrac{1}{2}, \quad y = x^3 + 7x + 100$$

What transformations maps the first curve to each of the other three?

b Which function below is not in the same family as the other three?

$y = \frac{1}{2}x^4 + 3x^2 + 6$, $y = \frac{1}{2}x^4 - 7 + 3x^2$, $y = 3x^2 - \frac{1}{2}x^4 + 1$, $y = 3x^2 + \frac{1}{2}x^4 - 1$

DEVELOPMENT

9 Find the derivative of each function. You will need to expand the brackets first.

a $y = (x + 1)(x - 1)$ **b** $y = x^2(3 - x^3)$ **c** $y = (x - 1)(x - 2)$

10 **a** Differentiate $y = x^3 - 2x$ and find the values of $\frac{dy}{dx}$ when $x = 1$ and when $x = 2$.

b Find, correct to the nearest degree where appropriate, the angles of inclination of the tangents at the points where $x = 1$ and $x = 2$. (Hint: You will need the formula gradient $= \tan\alpha$, where α is the angle of inclination of the tangent.)

11 Differentiate each function. Then solve $\frac{dy}{dx} = 0$ and hence find the coordinates of any points on each curve where the tangent is horizontal.

a $y = 3 - 2x + x^2$ **b** $y = x^4 + 18x^2$

12 Differentiate $y = x^2 - 8x + 15$. Hence find the equations of the tangent and the normal at:

a $A(1, 8)$ **b** $B(6, 3)$ **c** $C(0, 15)$ **d** $D(4, -1)$

13 In each part, find the derivative $\frac{dy}{dx}$ of the function. Then find the equations of the tangent and the normal to the curve at the point indicated.

a $y = x^2 - 6x$ at $O(0, 0)$ **b** $y = x^3 - 4x$ at $P(2, 0)$

c $y = x^2 - x^4$ at $Q(-1, 0)$ **d** $y = x^3 - 3x + 2$ at $R(1, 0)$

14 **a** Find the equations of the tangent and normal to $y = x^2$ at the point $H(2, 4)$.

b Find the points A and B where the tangent and the normal respectively meet the y-axis.

c Sketch the situation, then find the length AB and the area of ΔABH.

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15 a Find the equations of the tangent and the normal to $y = 9 - x^2$ at the point $K(1, 8)$.

b Find the points A and B where the tangent and the normal respectively meet the x-axis.

c Sketch the situation, then find the length AB and the area of ΔABK.

16 a Show that the line $y = -3$ meets the parabola $y = x^2 - 4$ at $D(1, -3)$ and $E(-1, -3)$.

b Find the equations of the tangents to $y = x^2 - 4$ at D and E.

c Find the point where these tangents intersect. Sketch the situation.

17 a Differentiate $y = 4 - x^2$, and hence find the coordinates of the points A and B on the curve where the tangent has gradient 2 and -2 respectively. Sketch the situation.

b Find the equations of the tangents at A and B, and find their point of intersection.

c Find the equations of the normals at A and B, and find their point of intersection.

18 The tangent and the normal to $f(x) = x^3$ at $U(1, 1)$ meet the y-axis at P and Q respectively. Find their equations, the coordinates of P and Q, and the area of ΔQUP.

19 a Differentiate $y = x^5$ successively until the result is zero. How many successive derivatives of $y = x^5$ are non-zero?

b How many successive derivatives of $y = x^6$ are non-zero?

c How many successive derivatives of $y = x^n$ are non-zero, where n is a whole number?

20 Find the families of curves with derivative:

a $\dfrac{dy}{dx} = x + x^2$

b $\dfrac{dy}{dx} = 6x^2 - 7$

c $\dfrac{dy}{dx} = 5x^3 + 3x^2 - 4$

d $\dfrac{dy}{dx} = 10x^4 - 12x^2 - 24$

CHALLENGE

21 a Differentiate the quadratic function $y = ax^2 + bx + c$, where a, b and c are constants.

b Find the x-coordinate of the point V on the curve where the tangent is horizontal.

c In Chapter 3, you used completing the square to show that this parabola has axis of symmetry $x = -\dfrac{b}{2a}$. Explain why parts **a** and **b** are another proof of this formula.

22 Let $P(a, a^2)$ be any point on the parabolic graph of $y = x^2$. Let $A(a - h, (a - h)^2)$ and $B(a + h, (a + h)^2)$ be two other points on the curve, so that the x-coordinates of A, P and B are equally spaced along the x-axis.

a Find the gradient of the tangent at P.

b Find the gradient of the chord AB, and show that the chord is parallel to the tangent.

c What property of the parabola has been proven here?

23 Differentiate $y = x^2 + bx + c$, and hence find b and c, given that:

a the parabola passes through the origin, and the tangent there has gradient 7,

b the parabola has y-intercept -3 and gradient -2 there,

c the parabola is tangent to the x-axis at the point $(5, 0)$,

d when $x = 3$ the gradient is 5, and $x = 2$ is a zero

e the parabola is tangent to $3x + y - 8 = 0$ at the point $T(3, -1)$,

f the line $3x + y - 8 = 0$ is a normal at the point $T(3, -1)$.

8E Differentiating powers with negative indices

The formula for the derivative of x^n applies unchanged when n is negative (first-principles differentiation of $y = x^{-1}$ is in the Challenge section of Exercise 8E).

12 DIFFERENTIATING POWERS WITH NEGATIVE INDICES

'Take the index as a factor and reduce the index by 1'.

For example, if $y = x^{-4}$, then $\dfrac{dy}{dx} = -4x^{-5}$.

Example 16 — 8E

Differentiate each function.

a $y = x^{-20}$ **b** $y = 6x^{-1}$ **c** $y = -5x^{-4}$

SOLUTION

a
$$y = x^{-20}$$
$$\frac{dy}{dx} = -20x^{-21}$$

b
$$y = 6x^{-1}$$
$$\frac{dy}{dx} = -6x^{-2}$$

c
$$y = -5x^{-4}$$
$$\frac{dy}{dx} = 20x^{-5}$$

Converting reciprocals of powers to powers with negative indices

Using the formula for differentiation often requires quick conversion between a power with a negative index and the reciprocal of a power. For example:

$$\frac{1}{x} = x^{-1} \quad \text{and} \quad 5x^{-2} = \frac{5}{x^2} \quad \text{and} \quad \frac{1}{3x^5} = \tfrac{1}{3}x^{-5}$$

Example 17 — 8E

Differentiate each function, giving your answer without negative indices.

a $f(x) = \dfrac{1}{x}$ **b** $f(x) = \dfrac{5}{x^2}$ **c** $f(x) = \dfrac{1}{3x^5}$

SOLUTION

In each example, first convert the expression to negative indices, then apply the rule, then convert back to fraction notation.

a
$$\begin{aligned} f(x) &= \frac{1}{x} \\ &= x^{-1} \\ f'(x) &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

b
$$\begin{aligned} f(x) &= \frac{5}{x^2} \\ &= 5x^{-2} \\ f'(x) &= -10x^{-3} \\ &= -\frac{10}{x^3} \end{aligned}$$

c
$$\begin{aligned} f(x) &= \frac{1}{3x^5} \\ &= \tfrac{1}{3}x^{-5} \\ f'(x) &= -\tfrac{5}{3}x^{-6} \\ &= -\frac{5}{3x^6} \end{aligned}$$

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Example 18 8E

a Differentiate $f(x) = \dfrac{16}{x^3} - \dfrac{24}{x^2}$.

b Use a common denominator to write your answer as a single fraction.

c Hence find the point on the curve $y = f(x)$ where the tangent is horizontal.

SOLUTION

a $f(x) = \dfrac{16}{x^3} - \dfrac{24}{x^2}$

$= 16x^{-3} - 24x^{-2}$ (rewrite the function using negative indices)

$f'(x) = -48x^{-4} + 48x^{-3}$ (apply the usual rule for differentiation)

b $f'(x) = -\dfrac{48}{x^4} + \dfrac{48}{x^3}$ (go back to fraction notation)

$= \dfrac{-48 + 48x}{x^4}$ (combine the fractions, using a common denominator)

c $f'(x) = \dfrac{48(x-1)}{x^4}$ (factor the numerator)

Hence $f'(x) = 0$ when $\dfrac{48(x-1)}{x^4} = 0$

that is, when $x = 1$. (the numerator must be zero)

Since $f(1) = 16 - 24 = -8$, the tangent is horizontal at the point $(1, -8)$.

Dividing through by the denominator

If the denominator of a function is a single term, divide through by it first, and then differentiate.

Example 19 8E

In each function, first divide through by the denominator. Then differentiate.

a $y = \dfrac{x^3 + x^2 + x}{x}$

b $y = \dfrac{5 - x^2 + 5x^4}{x^2}$

SOLUTION

a $y = \dfrac{x^3 + x^2 + x}{x}$

$= x^2 + x + 1$

$\dfrac{dy}{dx} = 2x + 1$

b $y = \dfrac{5 - x^2 + 5x^4}{x^2}$

$= 5x^{-2} - 1 + 5x^2$

$\dfrac{dy}{dx} = -10x^{-3} + 10x$

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Exercise 8E

1 **a** Change the fraction $\dfrac{1}{x^3}$ to a term with a negative index.

b Hence use the rule that the derivative of x^n is nx^{n-1} to write down the derivative of $f(x) = \dfrac{1}{x^3}$.

c Convert the answer to part **b** back to a fraction.

2 Write down the derivative $f'(x)$ of each function. Leave your answers with negative indices.

a $f(x) = x^{-1}$ **b** $f(x) = x^{-5}$ **c** $f(x) = 3x^{-1}$

d $f(x) = 5x^{-2}$ **e** $f(x) = -\frac{4}{3}x^{-3}$ **f** $f(x) = 2x^{-2} + \frac{1}{2}x^{-8}$

3 Rewrite each function using a negative power of x, then differentiate it, then convert the answer back to a fraction.

a $f(x) = \dfrac{1}{x}$ **b** $f(x) = \dfrac{1}{x^2}$ **c** $f(x) = \dfrac{1}{x^4}$ **d** $f(x) = \dfrac{3}{x}$

4 Find the derivative $\dfrac{dy}{dx}$ of each function. Then find the gradient of the tangent to the curve at the point where $x = 1$.

a $y = 3x^{-2}$ **b** $y = 15x^{-4}$ **c** $y = 2x - 2x^{-1}$

d $y = x + 5 + 5x^{-6}$ **e** $y = \frac{1}{2}x^{-2} + \frac{1}{3}x^{-3}$ **f** $y = x^6 + x^{-6}$

5 For each function, first divide through by the denominator. Then find the derivative. Leave your answer with negative indices where appropriate.

a $y = \dfrac{x^3 + x^2}{x}$ **b** $y = \dfrac{x^3 + x^2}{x^5}$ **c** $y = \dfrac{4x - 5x^4}{x^2}$ **d** $y = \dfrac{6x^2 + 6x^6}{2x^6}$

DEVELOPMENT

6 **a** Differentiate $f(x) = \dfrac{1}{x}$. Hence find $f'(3)$ and $f'\left(\frac{1}{3}\right)$.

b Find the two points on $y = \dfrac{1}{x}$ where the tangent has gradient -1.

c Find the two points on $y = \dfrac{1}{x}$ where the tangent has gradient -4.

d Are there any points on $y = \dfrac{1}{x}$ where the tangent has zero gradient?

e Are there any points on $y = \dfrac{1}{x}$ where the tangent has negative gradient?

7 **a** Differentiate $f(x) = -\dfrac{3}{x}$. Hence find $f'(2)$ and $f'(6)$.

b Find the two points on the curve where the tangent has gradient 3.

8 **a** Differentiate $f(x) = \dfrac{12}{x}$. Hence find $f'(2)$ and $f'(6)$.

b Hence find the tangent and normal to the curve at $M(2, 6)$ and at $N(6, 2)$.

c Find the points on the curve where the tangent has gradient -12. Sketch the situation.

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9 Rewrite each function using negative powers of x, then differentiate it. Give each final answer in fractional form without negative indices.

a $f(x) = \dfrac{1}{x^6} - \dfrac{1}{x^8}$ **b** $f(x) = \dfrac{1}{3x}$ **c** $f(x) = \dfrac{5}{x^3}$ **d** $f(x) = \dfrac{1}{5x^4}$

e $f(x) = -\dfrac{7}{x}$ **f** $f(x) = \dfrac{7}{2x}$ **g** $f(x) = -\dfrac{7}{3x}$ **h** $f(x) = -\dfrac{3}{5x^5}$

10 Find the derivative $\dfrac{dy}{dx}$ of each function. Then find the value of $\dfrac{dy}{dx}$ when $x = 1$.

a $y = x^4 - x^2 + 1$ **b** $y = \frac{1}{3}x^6 - \frac{1}{2}x^4 + x^2$ **c** $y = \frac{1}{10}x^5 - \frac{1}{6}x^3 + \frac{1}{2}x$

d $y = (x+1)(x-1)$ **e** $y = x^2(3 - x^3)$ **f** $y = (2x-1)(x-2)$

g $y = 4x^{-1}$ **h** $y = \frac{2}{3}x^{-3}$ **i** $y = x^{-1} - x^{-2}$

11 For each function, first divide through by the denominator, then find the derivative. Leave your answer with negative indices where appropriate.

a $y = \dfrac{3x^4 - 5x^2}{x}$ **b** $y = \dfrac{x^4 - 4x^2}{x^2}$ **c** $y = \dfrac{5x^6 + 4x^5}{3x^3}$

d $y = \dfrac{3x^2 - 1}{x}$ **e** $y = \dfrac{1 + 7x}{x^3}$ **f** $y = \dfrac{3x^5 - 5x^3 + x}{x^2}$

12 Differentiate $y = \dfrac{a}{x} - \dfrac{b}{cx^2}$, where a, b and c are constants. Give the answer in fractional form.

CHALLENGE

13 Consider the function $f(x) = x^{-1}$.

a Find $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$ and $f^{(5)}(x)$.

b Find $f'(1)$, $f''(1)$, $f'''(1)$, $f^{(4)}(1)$ and $f^{(5)}(1)$.

c Describe the pattern in the results of part **b**.

d Describe the pattern if -1 is substituted rather than 1.

14 This question shows how to differentiate the function $f(x) = \dfrac{1}{x}$ by first principles.

a Show that $\dfrac{1}{x+h} - \dfrac{1}{x} = -\dfrac{h}{(x+h)x}$.

b Hence show that $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-1}{x(x+h)}$.

c Take the limit as $h \to 0$ to show that $f'(x) = -\dfrac{1}{x^2}$.

15 **a** Show that the tangent to $y = x^2 + 15x + 36$, at the point P where $x = a$, has equation $y = (2a + 15)x - a^2 + 36$.

b By substituting $O(0, 0)$ into the equation and solving it for a, find the equations of any tangents passing through the origin.

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8F Differentiating powers with fractional indices

The formula for the derivative of x^n applies also when n is a fraction, or indeed any real number (first-principles differentiation of $y = x^{\frac{1}{2}}$ is in the Challenge of Exercise 8F).

13 DIFFERENTIATING POWERS WITH FRACTIONAL INDICES

'Take the index as a factor and reduce the index by 1'.

For example, if $y = x^{\frac{5}{2}}$, then $\dfrac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}}$.

This formula also applies to negative fractional indices, as in part **c** and **d** of the example below.

Example 20 8F

Differentiate each function.

a $f(x) = x^{\frac{4}{3}}$ **b** $f(x) = x^{\frac{1}{4}}$ **c** $f(x) = x^{-\frac{1}{5}}$ **d** $f(x) = x^{-2\frac{1}{2}}$

SOLUTION

a $f(x) = x^{\frac{4}{3}}$

$f'(x) = \frac{4}{3}x^{\frac{1}{3}}$

b $f(x) = x^{\frac{1}{4}}$

$f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$

c $f(x) = x^{-\frac{1}{5}}$

$f'(x) = -\frac{1}{5}x^{-\frac{6}{5}}$

d $f(x) = x^{-2\frac{1}{2}}$

$f'(x) = -\frac{5}{2}x^{-3\frac{1}{2}}$

Converting between surd notation and powers with fractional indices

When using the formula for differentiation, we often need to convert between fractional index notation and surd notation:

$$\sqrt{x} = x^{\frac{1}{2}} \quad \text{and} \quad \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \quad \text{and} \quad x^{-1\frac{1}{2}} = \frac{1}{x\sqrt{x}}$$

Remember that $\sqrt{x}$ and $x^{\frac{1}{2}}$ mean exactly the same thing — the *positive square root* of x — and that they are not defined when x is a negative number. Thus

$$\sqrt{9} = 9^{\frac{1}{2}} = 3 \text{ (not } -3) \qquad \text{and} \qquad \sqrt{-9} \text{ and } (-9)^{\frac{1}{2}} \text{ are undefined.}$$

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Example 21 8F

Write each function using a fractional index, then differentiate it, then convert the answer back to surd form.

a $y = 12\sqrt{x}$ **b** $y = \sqrt{16x}$ **c** $y = \dfrac{3}{\sqrt{x}}$

SOLUTION

a
$$\begin{aligned} y &= 12\sqrt{x} \\ &= 12x^{\frac{1}{2}} \\ f'(x) &= 12 \times \tfrac{1}{2} \times x^{-\frac{1}{2}} \\ &= \frac{6}{\sqrt{x}} \end{aligned}$$

b
$$\begin{aligned} f(x) &= \sqrt{16x} \\ &= 4x^{\frac{1}{2}} \\ f'(x) &= 4 \times \tfrac{1}{2} \times x^{-\frac{1}{2}} \\ &= \frac{2}{\sqrt{x}} \end{aligned}$$

c
$$\begin{aligned} f(x) &= \frac{3}{\sqrt{x}} \\ &= 3x^{-\frac{1}{2}} \\ f'(x) &= -\tfrac{1}{2} \times 3 \times x^{-1\frac{1}{2}} \\ &= -\frac{3}{2x\sqrt{x}} \end{aligned}$$

Sometimes surds and whole-number powers need to be combined, and the index laws applied, as in the following two more difficult examples.

Example 22 8F

Write each function in terms of a power of x, then differentiate it.

a $f(x) = x^3\sqrt{x}$ **b** $f(x) = \dfrac{1}{x^3\sqrt{x}}$

SOLUTION

a
$$\begin{aligned} f(x) &= x^3\sqrt{x} \\ &= x^3 \times x^{\frac{1}{2}} \\ &= x^{3\frac{1}{2}} \\ f'(x) &= \tfrac{7}{2}x^{3\frac{1}{2}} \end{aligned}$$

b
$$\begin{aligned} f(x) &= \frac{1}{x^3\sqrt{x}} \\ &= \frac{1}{x^3 \times x^{\frac{1}{2}}} \\ &= x^{-3\frac{1}{2}} \\ f'(x) &= -\tfrac{7}{2}x^{-4\frac{1}{2}} \end{aligned}$$

Example 23 8F

Divide through by the denominator, then differentiate.

a $y = \dfrac{10x - 6}{\sqrt{x}}$ **b** $y = \dfrac{3x + 4\sqrt{x}}{x}$

SOLUTION

a
$$\begin{aligned} y &= \frac{10x^1}{x^{\frac{1}{2}}} - \frac{6}{x^{\frac{1}{2}}} \\ &= 10x^{\frac{1}{2}} - 6x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= 5x^{-\frac{1}{2}} + 3x^{-1\frac{1}{2}} \end{aligned}$$

b
$$\begin{aligned} y &= \frac{3x}{x} + \frac{4x^{\frac{1}{2}}}{x^1} \\ &= 3 + 4x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= -2x^{-1\frac{1}{2}} \end{aligned}$$

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Exercise 8F

FOUNDATION

1 a Rewrite the function $y = 20\sqrt{x}$ using index notation.
  b Use the rule that the derivative of x^n is nx^{n-1} to write down the derivative.
  c Convert your answer back to surd and fraction notation.

2 Use the rule that the derivative of x^n is nx^{n-1} to write down the derivative of each function. Leave answers in index notation.

a $y = x^{\frac{1}{2}}$ b $y = x^{-\frac{1}{2}}$ c $y = x^{1\frac{1}{2}}$

d $y = 6x^{\frac{2}{3}}$ e $y = 12x^{-\frac{1}{3}}$ f $y = 4x^{\frac{1}{4}} + 8x^{-\frac{1}{4}}$

g $y = 7x^{2\frac{1}{2}}$ h $y = 5x^{-\frac{2}{3}}$ i $y = -10x^{-0.6}$

3 Write each function as a power with a fractional index and then differentiate it.

a $y = \sqrt{x}$ b $y = \sqrt[3]{x}$ c $y = \sqrt[4]{x}$ d $y = 10\sqrt{x}$

4 Find the first and second derivatives of $y = x^{\frac{1}{2}}$.

5 a Explain why $x\sqrt{x} = x^{\frac{3}{2}}$, and hence differentiate $y = x\sqrt{x}$.
  b Similarly, write $y = x^2\sqrt{x}$ as a single power of x, and hence differentiate it.
  c Write $y = \dfrac{1}{\sqrt{x}}$ as a single power of x, and hence differentiate it.
  d Write $y = \dfrac{1}{x\sqrt{x}}$ as a single power of x, and hence differentiate it.

DEVELOPMENT

6 a Show that the derivative of $y = \sqrt{x}$ is $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$.
  b Find the gradients of the tangents to $y = \sqrt{x}$ at the points $A(1, 1)$ and $B(4, 2)$.
  c Find the equations of the tangents to $y = \sqrt{x}$ at A and B.
  d Find the gradients of the normals to $y = \sqrt{x}$ at A and B.
  e Find the equations of the normals to $y = \sqrt{x}$ at A and B.

7 a Show that the derivative of $y = \dfrac{4}{\sqrt{x}}$ is $\dfrac{dy}{dx} = -\dfrac{2}{x\sqrt{x}}$.
  b Find the equations of the tangent and the normal to the curve at the point where $x = 4$.
  c Explain why the domain is $x > 0$, and why every tangent has negative gradient.

8 a Differentiate $y = 2\sqrt{x}$ and find the gradients of the tangent and the normal at $P(4, 4)$.
  b Find the equations of the tangent and the normal at P.
  c Find the points A and B where the tangent and the normal respectively meet the x-axis.
  d Find the length AB and hence find the area of ΔPAB.

9 Find any points on these curves where the tangent has gradient -1.

a $y = \dfrac{1}{x}$ b $y = \frac{1}{2}x^{-2}$ c $y = -\sqrt{x}$ d $y = \frac{1}{4}x^4 - \frac{1}{2}x^2 - x$

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10 Divide each function through by the denominator, using index notation, then differentiate.

a $y = \dfrac{x^2 + 6x\sqrt{x} + x}{x}$ **b** $y = \dfrac{3x - 3x\sqrt{x} - 8x^2}{x}$ **c** $y = \dfrac{3x - 2\sqrt{x}}{\sqrt{x}}$

11 Write each function as a power with a fractional index and then differentiate it.

a $f(x) = 24\sqrt{x}$ **b** $f(x) = \sqrt{64x}$ **c** $f(x) = \sqrt{25x}$ **d** $f(x) = 2x\sqrt{x}$

e $f(x) = 12x\sqrt{x}$ **f** $f(x) = 4x^2\sqrt{x}$ **g** $f(x) = 24\sqrt[3]{x}$ **h** $f(x) = \sqrt[3]{x^2}$

i $f(x) = 30\sqrt[3]{x^2}$ **j** $f(x) = \dfrac{1}{\sqrt{x}}$ **k** $f(x) = \dfrac{6}{\sqrt{x}}$ **l** $f(x) = \dfrac{5}{x\sqrt{x}}$

CHALLENGE

12 This question shows how to differentiate the function $f(x) = \sqrt{x}$ by first principles.

a Multiply top and bottom of the fraction $\dfrac{\sqrt{x+h} - \sqrt{x}}{h}$ by $\sqrt{x+h} + \sqrt{x}$ (use the difference of squares identity), and show that it equals $\dfrac{1}{\sqrt{x+h} + \sqrt{x}}$.

b Write down $\dfrac{f(x+h) - f(x)}{h}$ for the function $f(x) = \sqrt{x}$, then use part **a** to show that

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

c Take the limit as $h \to 0$ to prove that $f'(x) = \dfrac{1}{2\sqrt{x}}$.

13 a Find the gradient of the tangent to $y = x^2 - 10x + 9$ at the point P where $x = a$.

b Find the value of y at $x = a$, and hence show that the tangent at P has equation $y = (2a - 10)x - a^2 + 9$.

c By substituting $(0, 0)$ into the tangent, find the value of a when the tangent passes through the origin, and hence find the equations of the tangents through the origin.

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8G The chain rule

In Section 8C we showed that the derivative of a *sum* or *difference* of functions is the sum or difference of the derivatives, and that the derivative of a *multiple* of a function is the multiple of the derivative. These were easy rules. Differentiating the *composite* or *product* or *quotient* of two functions, however, is more complicated. Sections 8G, 8H and 8I develop rules for differentiating such compound functions.

Composition of functions — a chain of functions

We begin by reviewing composite functions from Section 4E. They are formed by putting two functions into a *chain* so that the output of the first function becomes the input of the second.

We now want to apply the process in reverse. For example, the semicircle function $y = \sqrt{25 - x^2}$ can be decomposed into a chain of two functions — 'square and subtract from 25', followed by 'take the positive square root'.

x				u				y
0	$\longrightarrow$	Square and subtract from 25	$\longrightarrow$	25	$\longrightarrow$	Take the positive square root	$\longrightarrow$	5
3	$\longrightarrow$		$\longrightarrow$	16	$\longrightarrow$		$\longrightarrow$	4
-4	$\longrightarrow$		$\longrightarrow$	9	$\longrightarrow$		$\longrightarrow$	3
x	$\longrightarrow$		$\longrightarrow$	$25 - x^2$	$\longrightarrow$		$\longrightarrow$	$\sqrt{25 - x^2}$

The middle column is the output of the first function 'Square and subtract from 25'. This output is then the input of the second function 'Take the positive square root'. The resulting decomposition of the original function $y = \sqrt{25 - x^2}$ into the chain of two functions may be expressed as follows:

'Let $u = 25 - x^2$. Then $y = \sqrt{u}$.'

The resulting function is called the *composite* of the two functions. We met such composite functions first in Section 4E, but this is the first time that we have reversed the process and decomposed a function into two parts.

The chain rule

Suppose then that y is a function of u, where u is a function of x.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\
&= \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x} \right) && \text{(multiplying top and bottom by } \delta u\text{)} \\
&= \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} && \text{(because } \delta u \to 0 \text{ as } \delta x \to 0\text{)} \\
&= \frac{dy}{du} \times \frac{du}{dx}.
\end{aligned}$$

Although the proof uses limits, the usual attitude to this rule is that 'the du's cancel out'.

The chain rule should be remembered in this form:

14 THE CHAIN RULE — DIFFERENTIATING A COMPOSITE FUNCTION

Suppose that y is a function of u, where u is a function of x. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

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Example 24 8G

Use the chain rule to differentiate each function.

a $(x^2 + 1)^6$

b $7(3x + 4)^5$

Note: The working in the right-hand column is the recommended setting out of the calculation. The calculation should begin with that working, because the first step is the decomposition of the function into a chain of two functions.

SOLUTION

a Let $y = (x^2 + 1)^6$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 6(x^2 + 1)^5 \times 2x$

$= 12x(x^2 + 1)^5$.

Let $u = x^2 + 1$.

Then $y = u^6$.

Hence $\dfrac{du}{dx} = 2x$

and $\dfrac{dy}{du} = 6u^5$.

b Let $y = 7(3x + 4)^5$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 35(3x + 4)^4 \times 3$

$= 105(3x + 4)^4$.

Let $u = 3x + 4$.

Then $y = 7u^5$.

Hence $\dfrac{du}{dx} = 3$

and $\dfrac{dy}{du} = 35u^4$.

Powers of a linear function

Here is the derivative of $(ax + b)^n$.

Let $y = (ax + b)^n$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= n(ax + b)^{n-1} \times a$

$= an(ax + b)^{n-1}$.

Let $u = ax + b$.

Then $y = u^n$.

Hence $\dfrac{du}{dx} = a$

and $\dfrac{dy}{du} = nu^{n-1}$.

This result occurs so often that it should be remembered as a formula for differentiating any linear function of x raised to a power.

15 POWERS OF A LINEAR FUNCTION

$$\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$$

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Example 25 8G

Use the formula $\dfrac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$ to differentiate each function.

Convert the answers to parts **b** and **c** back to fraction or surd form.

a $y = (4x-1)^7$ **b** $y = \dfrac{1}{3x-8}$ **c** $y = \sqrt{5x+3}$

SOLUTION

a $y = (4x-1)^7$

$\dfrac{dy}{dx} = 4 \times 7 \times (4x-1)^6$ (here $a = 4$, $b = -1$ and $n = 7$)

$= 28(4x-1)^6$

b $y = (3x-8)^{-1}$ (convert to index form)

$\dfrac{dy}{dx} = 3 \times (-1) \times (3x-8)^{-2}$ (here $a = 3$, $b = -8$ and $n = -1$)

$= -\dfrac{3}{(3x-8)^2}$

c $y = (5x+3)^{\frac{1}{2}}$ (convert to index form)

$\dfrac{dy}{dx} = 5 \times \frac{1}{2} \times (5x+3)^{-\frac{1}{2}}$ (here $a = 5$, $b = 3$ and $n = \frac{1}{2}$)

$= \dfrac{5}{2\sqrt{5x+3}}$

A shorter setting-out

It is important to practise the full setting-out, first, because that is the best way to handle many tricky calculations, and secondly, because the process will be reversed in Year 12 and must be clearly understood.

Many people like to shorten the setting-out — if so, it is safer to write down at least the function u on the right, and take at least one middle step in the working.

Example 26 8G

Differentiate $y = (3-5x^3)^7$ with a shorter setting-out.

SOLUTION

$y = (3-5x^3)^7$ Let $u = 3 - 5x^3$.

$\dfrac{dy}{dx} = 7(3-5x^3)^6 \times (-15x^2)$ (further steps if it gets tricky)

$= -105x^2(3-5x^3)^6$

Another approach to shortening the setting-out is to remember a new standard form for differentiating a power of a function:

$$\frac{d}{dx}\big(f(x)\big)^n = n\big(f(x)\big)^{n-1}f'(x)$$

Again, it is probably safer to write down the function $f(x)$ on the right.

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Example 27 8G

Differentiate $y = (3 - 5x^3)^7$ using the standard form above.

SOLUTION

$$y = (3 - 5x^3)^7$$
$$\frac{dy}{dx} = 7(3 - 5x^3)^6 \times (-15x^2)$$
$$= -105x^2(3 - 5x^3)^6$$

Let $f(x) = 3 - 5x^3$.
(further steps if it gets tricky)

The chain rule and tangents

The usual methods of dealing with tangents apply when the chain rule is used to find the derivative.

Example 28 8G

Differentiate $y = (1 - x^4)^4$, and hence find the points on the curve where the tangent is horizontal.

SOLUTION

Let $y = (1 - x^4)^4$,

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$= 4(1 - x^4)^3 \times (-4x^3)$
$= 16x^3(1 - x^4)^3$,

Let $u = 1 - x^4$.
Then $y = u^4$.
Hence $\dfrac{du}{dx} = -4x^3$
and $\dfrac{dy}{du} = 4u^3$.

which is zero when $x = -1$, 0 or 1,
so the points are $(-1, 0)$, $(0, 1)$ and $(1, 0)$.

Exercise 8G

FOUNDATION

Note: Use the standard form $\dfrac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ when the function is a power of a linear function. Otherwise use the full setting-out of the chain rule, or use a shorter form if you are quite confident to do so.

1 Copy and complete the setting out below to differentiate $(x^2 + 9)^5$ by the chain rule.

Let $y = (x^2 + 9)^5$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$= \cdots \times \cdots$
$= \cdots$.

Let $u = x^2 + 9$
Then $y = u^5$.
Hence $\dfrac{du}{dx} = \cdots$
and $\dfrac{dy}{du} = \cdots$.

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2 Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ to differentiate each function. Use the full setting-out above, first identifying u as a function of x, and y as a function of u.

a $y = (3x + 7)^4$ b $y = (5x - 9)^6$ c $y = (5 - 4x)^7$
d $y = (1 - x)^4$ e $y = (x^2 + 1)^{12}$ f $y = (x^2 - 2)^7$

3 Use the full setting-out of the chain rule to differentiate these functions.

a $y = (7x + 2)^{-1}$ b $y = 3(x - 1)^{-2}$ c $y = (x^3 - 12)^{-4}$
d $y = (5x^2 - 2)^{-3}$ e $y = 8(7 - x^2)^4$ f $y = -3(x^3 + x + 1)^6$

4 Use the standard form $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ to differentiate these functions.

a $y = (5x - 7)^5$ b $y = (7x + 3)^7$ c $y = 9(5x + 3)^4$
d $y = (4 - 3x)^7$ e $y = 11(3 - x)^2$ f $y = (4x - 5)^{-7}$
g $y = 2(3x + 7)^{-5}$ h $y = (10 - 3x)^{-4}$ i $y = 3(5 - 7x)^{-4}$

5 a Differentiate $y = (x - 3)^2$ by expanding the RHS and differentiating each term.
b Differentiate $y = (x - 3)^2$, using the chain rule.

6 a Differentiate $y = 3(1 - 2x)^2$ by expanding the RHS and differentiating each term.
b Differentiate $y = 3(1 - 2x)^2$, using the chain rule.

7 Find the first and second derivatives of $f(x) = (2x + 3)^5$.

DEVELOPMENT

8 Differentiate these functions.

a $y = (5 - x^2)^3$ b $y = (3x^2 + 7)^7$ c $y = (x^4 + 1)^4$
d $y = (3x^3 - 7)^5$ e $y = (x^3 - x^2)^{-5}$ f $y = (x^2 + 3x + 1)^{-9}$

9 Rewrite each function with negative index notation. Then differentiate it using the formula $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$. Give each answer as a fraction without negative indices.

a $y = \frac{1}{2x + 7}$ b $y = \frac{1}{2 - x}$ c $y = \frac{1}{3 + 5x}$
d $y = \frac{7}{4 - 3x}$ e $y = \frac{4}{(3x - 1)^5}$ f $y = -\frac{5}{(x + 1)^3}$

10 a Find the derivative of $y = (5x - 4)^4$.
b Find the values of y and $\frac{dy}{dx}$ at the point P where $x = 1$.
c Hence find the equations of the tangent and the normal at P.

11 Find the equation of the tangent to each curve at the point P where $x = 1$.

a $y = (x^2 + 1)^3$ b $y = (3x - 2)^{-1}$ c $y = (x^2 + 1)^{-1}$

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12 Differentiate each function using the chain rule. Then solve $\frac{dy}{dx} = 0$ to find the x-coordinate and y-coordinate of any points where the tangent is horizontal.

a $y = (x - 5)^4$ **b** $y = (x^2 - 1)^3$ **c** $y = (2x + x^2)^5$

d $y = \dfrac{1}{5x + 2}$ **e** $y = 4 + (x - 5)^6$ **f** $y = \dfrac{1}{1 + x^2}$

13 Find the x-coordinates of any points on $y = (4x - 7)^3$ where the tangent is:

a parallel to $y = 108x + 7$, **b** perpendicular to $y = -\frac{1}{12}x + 2$.

14 Differentiate each function, then find the gradient of the tangent at the point where $x = 2$.

a $y = (6x + 4)^{\frac{1}{2}}$ **b** $y = \sqrt{2x + 5}$ **c** $y = \sqrt{x^2 - 3}$

15 Differentiate each function, then find the x-coordinate and y-coordinate of any points where the tangent is horizontal.

a $y = \sqrt{3 - 2x}$ **b** $y = \sqrt{x^2 - 2x + 5}$ **c** $y = \sqrt{x^2 - 2x}$

16 a Find the derivative of the semicircle $y = \sqrt{25 - x^2}$, and sketch the semicircle.

b Show that the tangent at $P(3, 4)$ is $3x + 4y = 25$.

c Verify that the tangent at P is perpendicular to the radius OP. Add the tangent and radius to your sketch.

CHALLENGE

17 Differentiate:

a $y = (\sqrt{x} - 3)^{11}$ **b** $y = 3\sqrt{4 - \frac{1}{2}x}$

c $y = \dfrac{3}{1 - x\sqrt{2}}$ **d** $y = (5 - x)^{-\frac{1}{2}}$

e $y = \dfrac{-a}{\sqrt{1 + ax}}$ **f** $y = \dfrac{b}{\sqrt{c - x}}$

g $y = -4\left(x + \dfrac{1}{x}\right)^4$ **h** $y = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^6$

18 a Differentiate $y = (x - a)^3$, then find the value of a if $\frac{dy}{dx} = 12$ when $x = 6$.

b Differentiate $y = \dfrac{1}{x + a}$, then find the value of a if $\frac{dy}{dx} = -1$ when $x = 6$.

19 Differentiate $y = a(x + b)^2 - 8$, then find a and b if the parabola:

a passes through the origin with gradient 16,

b has tangent $y = 2x$ at the point $P(4, 8)$.

20 a Find the equation of the tangent to $y = \dfrac{1}{x - 4}$ at the point L where $x = b$.

b Hence find the equation of the tangent to the curve passing through:

i the origin,

ii $W(6, 0)$.

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8H The product rule

The product rule extends the methods for differentiation to functions that are products of two simpler functions. For example,

$$y = x(x-10)^4 \quad \text{is the product} \quad y = x \times (x-10)^4,$$

and we shall decompose y as

$$y = uv, \qquad \text{where} \qquad u = x \qquad \text{and} \qquad v = (x-10)^4.$$

Statement of the product rule

The proof of the product rule is given in the appendix at the end of this chapter.

16 THE PRODUCT RULE

Suppose that the function

$$y = u \times v$$

is the *product* of two functions u and v, each of which is a function of x. Then

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} \qquad \text{or} \qquad y' = vu' + uv'.$$

The second form uses the convention of the dash $'$ to represent differentiation with respect to x. That is,

$$y' = \frac{dy}{dx} \qquad \text{and} \qquad u' = \frac{du}{dx} \qquad \text{and} \qquad v' = \frac{dv}{dx}.$$

Using the product rule

The working in the right-hand column is the recommended setting out. The first step is to decompose the function into the product of two simpler functions.

Example 29 **8H**

Differentiate each function, writing the result in fully factored form. Then state for what value(s) of x the derivative is zero.

a $x(x-10)^4$ **b** $x^2(3x+2)^3$

SOLUTION

a Let $y = x(x-10)^4$.

Then
$$\begin{aligned}\frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= (x-10)^4 \times 1 + x \times 4(x-10)^3\\ &= (x-10)^3(x-10+4x)\\ &= (x-10)^3(5x-10)\\ &= 5(x-10)^3(x-2),\end{aligned}$$

so the derivative is zero for $x = 10$ and for $x = 2$.

Let $u = x$
and $v = (x-10)^4$.

Then $\dfrac{du}{dx} = 1$

and $\dfrac{dv}{dx} = 4(x-10)^3$ (using the chain rule).

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b Let $y = x^2(3x + 2)^3$.

Then $y' = vu' + uv'$

$= (3x + 2)^3 \times 2x + x^2 \times 9(3x + 2)^2$

$= x(3x + 2)^2(6x + 4 + 9x)$

$= x(3x + 2)^2(15x + 4)$,

Let $u = x^2$

and $v = (3x + 2)^3$.

Then $u' = 2x$

and $v' = 9(3x + 2)^2$.

so the derivative is zero for $x = 0$, $x = -\frac{2}{3}$ and for $x = -\frac{4}{15}$.

Note: The product rule can be difficult to use with the algebraic functions under consideration at present, because the calculations can easily become quite involved. The rule will seem far more straightforward later in the context of exponential and trigonometric functions.

Exercise 8H

FOUNDATION

1 Copy and complete the setting out below to differentiate $5x(x - 2)^4$ by the product rule.

Let $y = 5x(x - 2)^4$.

Then $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

$= \cdots \times \cdots + \cdots \times \cdots$

$= 5(x - 2)^4 + 20x(x - 2)^3$

$= \cdots$ (take out the common factor)

$= 5(x - 2)^3(5x - 2)$.

Let $u = 5x$

and $v = (x - 2)^4$.

Then $\dfrac{du}{dx} = \cdots$

and $\dfrac{dv}{dx} = \cdots$

2 **a** Differentiate $y = x^3(x - 2)$ by expanding the product and differentiating each term.

b Differentiate $y = x^3(x - 2)$ using the product rule with $u = x^3$ and $v = x - 2$. Use the setting out shown in the worked examples above.

3 **a** Differentiate $y = (2x + 1)(x - 5)$ by expanding and then differentiating each term.

b Differentiate $y = (2x + 1)(x - 5)$ using the product rule with $u = 2x + 1$ and $v = x - 5$.

4 **a** Differentiate $y = (x^2 - 3)(x^2 + 3)$ by expanding and then differentiating each term.

b Differentiate $y = (x^2 - 3)(x^2 + 3)$ using the product rule with $u = x^2 - 3$ and $v = x^2 + 3$.

5 **a** Given that $u = x^4$ and $v = (2x - 1)^5$, find $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.

b Use the product rule and the results of part **a** to show that if $y = x^4(2x - 1)^5$, then $\dfrac{dy}{dx} = 4x^3(2x - 1)^5 + 10x^4(2x - 1)^4$.

c By taking out the common factor $2x^3(2x - 1)^4$, show that $\dfrac{dy}{dx} = 2x^3(2x - 1)^4(9x - 2)$.

d Hence find the x-coordinates of the points on the curve where the tangent is horizontal.

6 Differentiate these functions by the product rule, giving your answers in factor form.

a $y = x(3x + 5)^3$ **b** $y = x^2(x - 1)^3$ **c** $y = x^4(1 - 5x)^6$

7 Show that the derivative of $y = x(1 - x)^6$ is $\dfrac{dy}{dx} = (7x - 1)(x - 1)^5$, then find the tangent and normal to the curve at the origin.

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DEVELOPMENT

8 Differentiate these functions using the product rule, identifying the factors u and v in each example. Express the answers in fully factored form, and state the values of x for which the derivative is zero.

a $y = x(x-1)^4$ **b** $y = x(x+5)^5$ **c** $y = x(4-3x)^5$
d $y = x(3-2x)^5$ **e** $y = x^3(x+1)^4$ **f** $y = x^3(3x-2)^4$
g $y = x^5(1-x)^7$ **h** $y = (x-1)(x-2)^3$ **i** $y = (x+2)(x+5)^6$

9 **a** Show that the derivative of $y = x^3(1-x)^5$ is $y' = x^2(1-x)^4(3-8x)$.
b Hence find the points on the curve where the tangents are horizontal.

10 **a** Show that the derivative of $y = (2x-1)^3(x-2)^4$ is $\dfrac{dy}{dx} = 2(2x-1)^2(x-2)^3(7x-8)$.
b Hence find the tangent and normal to the curve at the point $A(1, 1)$.

11 Show that the derivative of $y = (x+1)(x+3)^3$ is $\dfrac{dy}{dx} = 2(2x+3)(x+3)^2$, then find the tangent and normal to the curve at $A(-1, 0)$.

12 **a** Differentiate $y = (x^2+1)^5$ using the chain rule. Then use the product rule to differentiate $y = x(x^2+1)^5$.
b Differentiate $y = (1-x^2)^4$. Hence differentiate $y = 2x^3(1-x^2)^4$.
c Differentiate $y = (x^2+x+1)^3$, then differentiate $y = -2(x^2+x+1)^3x$.
d Similarly, differentiate $y = x(4-9x^4)^4$.

13 **a** Differentiate $y = (x^2-10)^3x^4$, using the chain rule to differentiate the first factor.
b Hence find the points on the curve where the tangent is horizontal.

CHALLENGE

14 Differentiate each function using the product rule. Then combine terms using a common denominator, and factor the numerator completely. State the values of x for which the derivative is zero.

a $y = 6x\sqrt{x+1}$ **b** $y = -4x\sqrt{1-2x}$ **c** $y = 10x^2\sqrt{2x-1}$

15 Find the derivative of each function below, giving the answer in fully factored form. Then state the values of x for which the derivative is zero.

a $y = (x+1)^3(x+2)^4$ **b** $y = (2x-3)^4(2x+3)^5$ **c** $y = x\sqrt{1-x^2}$

16 **a** Differentiate $y = a(x-1)(x-5)$, using the product rule, then sketch the curve.
b Show that the tangents at the x-intercepts $(1, 0)$ and $(5, 0)$ have opposite gradients.
c Find the equations of these tangents and their point M of intersection.
d Find the point V where the tangent is horizontal.
e Show that M is vertically below V and twice as far from the x-axis.

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8I The quotient rule

The quotient rule extends the formulae for differentiation to functions that are quotients of two simpler functions. For example,

$$y = \frac{2x+1}{2x-1} \text{ is the quotient of the two functions } 2x+1 \text{ and } 2x-1.$$

Statement of the quotient rule

The proof of the quotient rule is also given in the appendix at the end of this chapter.

17 THE QUOTIENT RULE

Suppose that the function

$$y = \frac{u}{v}$$

is the *quotient* of two functions u and v, each of which is a function of x. Then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \quad \text{or} \quad y' = \frac{vu' - uv'}{v^2}.$$

Example 30 — 8I

Differentiate each function, stating any values of x where the derivative is zero.

a $y = \dfrac{2x+1}{2x-1}$

b $y = \dfrac{x}{x^2+1}$

SOLUTION

a Let $y = \dfrac{2x+1}{2x-1}$.

Let $u = 2x + 1$
and $v = 2x - 1$.

Then $\dfrac{du}{dx} = 2$
and $\dfrac{dv}{dx} = 2$.

Then
$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$
$$= \frac{2(2x-1) - 2(2x+1)}{(2x-1)^2}$$
$$= \frac{-4}{(2x-1)^2}, \quad \text{which is never zero.}$$

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b Let $y = \dfrac{x}{x^2+1}$.

Let $u = x$
and $v = x^2 + 1$.
Then $u' = 1$
and $v' = 2x$.

Then
$$y' = \frac{vu' - uv'}{v^2}$$
$$= \frac{(x^2+1) - 2x^2}{(x^2+1)^2}$$
$$= \frac{1-x^2}{(x^2+1)^2}$$
$$= \frac{(1-x)(1+x)}{(x^2+1)^2},$$
which is zero when $x = 1$ or $x = -1$

Note: Both these functions could have been differentiated using the product rule after writing them as $(2x+1)(2x-1)^{-1}$ and $x(x^2+1)^{-1}$. The quotient rule, however, makes the calculations much easier.

Exercise 8I

FOUNDATION

1 Copy and complete the setting out below to differentiate $\dfrac{2x+3}{3x+2}$ by the quotient rule.

Let $y = \dfrac{2x+3}{3x+2}$.

Let $u = 2x + 3$
and $v = 3x + 2$.
Then $u' = \cdots$
and $v' = \cdots$.

Then
$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$
$$= \frac{\cdots \times \cdots - \cdots \times \cdots}{\cdots}$$
$$= \frac{-5}{(3x+2)^2}.$$

2 Differentiate each function using the quotient rule. Identify u and v first, and use the setting out shown in Example 30 and in Question **1**.

a $y = \dfrac{x}{x+1}$ **b** $y = \dfrac{2x}{x+2}$ **c** $y = \dfrac{x}{1-3x}$ **d** $y = \dfrac{x+1}{x-1}$

e $y = \dfrac{x+2}{x-2}$ **f** $y = \dfrac{x-2}{x+2}$ **g** $y = \dfrac{3x-2}{2x-3}$ **h** $y = \dfrac{5-4x}{5+4x}$

3 Differentiate each function using the quotient rule. Express your answer in fully factored form, then state any values of x for which the tangent is horizontal.

a $y = \dfrac{x^2}{x+1}$ **b** $y = \dfrac{x}{3-x^2}$ **c** $y = \dfrac{x^2}{1-x}$

d $y = \dfrac{x}{1-x^2}$ **e** $y = \dfrac{x^2-1}{x^2+1}$ **f** $y = \dfrac{x^2-9}{x^2-4}$

4 Differentiate $y = \dfrac{1}{3x-2}$ in two different ways:

a Use the chain rule with $u = 3x - 2$ and $y = \dfrac{1}{u}$. This is the better method.

b Use the quotient rule with $u = 1$ and $v = 3x - 2$. This method is longer.

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DEVELOPMENT

5 **a** Show that the derivative of $y = \dfrac{x}{5 - 3x}$ is $\dfrac{dy}{dx} = \dfrac{5}{(5 - 3x)^2}$.

b Hence find the gradient of the tangent at $K(2, -2)$ and its angle of inclination.

c Find the equations of the tangent and the normal at K.

6 **a** Show that the derivative of $y = \dfrac{x^2 - 4}{x - 1}$ is $\dfrac{dy}{dx} = \dfrac{x^2 - 2x + 4}{(x - 1)^2}$.

b Hence find the gradient of the tangent at $L(4, 4)$ and its angle of inclination.

c Find the equations of the tangent and the normal at L.

7 **a** Find the equation of the tangent to $y = \dfrac{x}{x + 1}$ at the origin $O(0, 0)$.

b Show that the tangent to $y = \dfrac{x}{x + 1}$ at the point $P\left(1, \frac{1}{2}\right)$ is $y = \frac{1}{4}x + \frac{1}{4}$.

c Find the points A and B where the tangent in part **b** crosses the x-axis and the y-axis respectively.

d Find the area of the triangle ABO, where O is the origin.

e Use simultaneous equations to find where the tangents at O and P intersect.

CHALLENGE

8 **a** Differentiate $y = \dfrac{x^2}{x + 1}$, and hence find the value of c if $y' = 0$ at $x = c$.

b Differentiate $y = \dfrac{x^2 + k}{x^2 - k}$, and hence find the value of k if $y' = 1$ at $x = -3$.

9 Differentiate, using the most appropriate method. Factor each answer completely.

a $(3x - 7)^4$ **b** $\dfrac{x^2 + 3x - 2}{x}$ **c** $(2x + 3)(2x - 3)$ **d** $\dfrac{1}{x^2 - 9}$

e $x(4 - x)^3$ **f** $\dfrac{3 - x}{3 + x}$ **g** $(x^4 - 1)^5$ **h** $\dfrac{1}{\sqrt{2 - x}}$

i $(x^3 + 5)^2$ **j** $\dfrac{x^2 + x + 1}{2\sqrt{x}}$ **k** $\frac{2}{3}x^2(x^3 - 1)$ **l** $\dfrac{x}{x + 5}$

m $x\sqrt{x} + x^2\sqrt{x}$ **n** $\left(x - \dfrac{1}{x}\right)^2$ **o** $x^3(x - 1)^8$ **p** $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$

10 **a** Show that the derivative of $y = \dfrac{x - a}{x - b}$ is $y' = \dfrac{a - b}{(x - b)^2}$ (where a and b are constants).

b Explain why all tangents have positive gradient when $a > b$, and all tangents have negative gradient when $a < b$.

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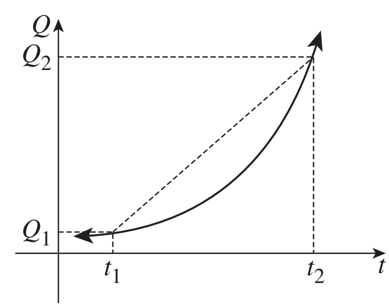
8J Rates of change

So far, differentiation has been developed geometrically in terms of tangents to curves, and many more geometric ideas will be involved as calculus is developed. Calculus has many interpretations, however. It has always been a very practical subject, and is an essential part of engineering, economics and all the sciences.

This section will begin to interpret the derivative as a *rate of change*. In various practical situations, some variable quantity Q is a function of time t. Differentiation with respect to time will give the *rate* at which the quantity Q is changing over time.

Two important type of rates are velocity and acceleration, but we have left the discussion of motion until Year 12 because motion requires a great deal more theory. A couple of questions in Exercise 8J, however, indicate how velocity and acceleration are particular types of rates.

Average rates and instantaneous rates



Suppose that a quantity Q is given as a function of time t, as in the diagram to the right. There are two types of rates:

- An *average rate of change* corresponds to a chord. In the diagram, the quantity Q takes the value Q_1 at time t_1, and the value Q_2 at time t_2. The average rate of change from time t_1 to time t_2 is the usual gradient formula:

$$\text{average rate} = \frac{Q_2 - Q_1}{t_2 - t_1}$$

- An *instantaneous rate of change* corresponds to a tangent.

  The instantaneous rate of change at time t_1 is the value of the derivative $\dfrac{dQ}{dt}$ at time $t = t_1$:

$$\text{instantaneous rate} = \frac{dQ}{dt}, \quad \text{evaluated at } t = t_1.$$

One can see from the diagram that as the time t_2 gets closer to t_1 the average rate from $t = t_1$ to $t = t_2$ gets closer to the instantaneous rate at t_1 (provided that the curve is continuous and smooth at $t = t_1$). This is exactly the way in which first-principles differentiation was defined using a limit in Section 8B.

18 AVERAGE AND INSTANTANEOUS RATES OF CHANGE

Suppose that a quantity Q is a function of time t.

- The *average rate of change* from the time $t = t_1$, when the value is Q_1, to the time $t = t_2$, when the value is Q_2, is the gradient of the chord,

$$\text{average rate} = \frac{Q_2 - Q_1}{t_2 - t_1}.$$

- The *instantaneous rate of change* at time $t = t_1$ is the gradient of the tangent, that is, the value of the derivative $\dfrac{dQ}{dt}$ evaluated at $t = t_1$,

$$\text{instantaneous rate} = \frac{dQ}{dt}, \quad \text{evaluated at } t = t_1.$$

- In this course, the unqualified phrase 'rate of change' will always mean the 'instantaneous rate of change'.

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In many problems, it is useful to sketch two graphs, one of the quantity Q as a function of time t, the other of the rate of change $\dfrac{dQ}{dt}$ also as a function of time t.

The quantity Q is usually replaced by a more convenient pronumeral, such as P for population, V for volume, or M for mass. Take particular care to use the correct units for the quantity, the time and the rate, in final answers.

Example 31 8J

A cockroach plague hit the suburb of Berrawong last year, but was gradually brought under control. The council estimated that the cockroach population P in millions, t months after 1 January, was given by

$$P = 7 + 6t - t^2.$$

a Differentiate to find the rate of change $\dfrac{dP}{dt}$ of the cockroach population.

b Find the cockroach population on 1 January, and the rate at which the population was increasing at that time. Be careful with the units.

c Find the cockroach population on 1 April, and the rate at which the population was increasing at that time.

d When did the council manage to stop the cockroach population increasing any further, and what was the population then?

e When were the cockroaches finally eliminated?

f What was the average rate of increase from 1 January to 1 April?

g Draw the graphs of P and $\dfrac{dP}{dt}$ as functions of time t, and state their domains and ranges. Add to your graph of P the tangents and chords corresponding to parts **b**, **d** and **f**.

SOLUTION

a The population function was $\quad P = 7 + 6t - t^2.$

Differentiating, the rate of change was $\quad \dfrac{dP}{dt} = 6 - 2t.$

b When $t = 0$, $\quad P = 7,$

and, when $t = 0$, $\quad \dfrac{dP}{dt} = 6.$

Thus on 1 January, the population was 7 million,
and the population was increasing at 6 million per month.
(The time t is in months, so the rate is 'per month'.)

c On 1 April, $t = 3$, so $\quad P = 16$ million,

and $\quad \dfrac{dP}{dt} = 0$ cockroaches/month.

d We found in part **b** that the population stopped increasing on 1 April, and that the population then was 16 million.

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e To find the time when the population was zero,

put $\qquad P = 0,$

that is, $\qquad 7 + 6t - t^2 = 0$

$\boxed{\times(-1)}$ $\qquad t^2 - 6t - 7 = 0$

$$(t-7)(t+1) = 0$$

$$t = 7 \text{ or } -1.$$

Hence $t = 7$, and the cockroaches were finally eliminated on 1 August.

(The negative solution is irrelevant because council action started at $t = 0$.)

f From part **b**, the population was 7 million on 1 January,

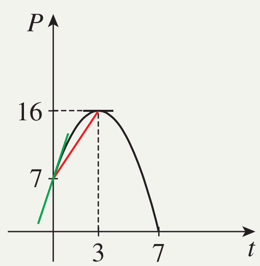
and from part **c**, the population was 16 million on 1 April.

Hence average rate of increase $= \dfrac{P_2 - P_1}{t_2 - t_1}$ (gradient of chord formula)

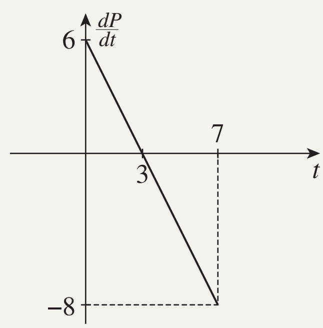
$$= \frac{16 - 7}{3}$$

$$= 3 \text{ million per month.}$$

g



Domain: $0 \le t \le 7$,
range: $0 \le P \le 16$



Domain: $0 \le t \le 7$,
range: $-8 \le \dfrac{dP}{dt} \le 6$

Increasing, decreasing and stationary

We can see from the left-hand graph above that the cockroach population was *increasing* when $0 < t < 3$ — every tangent to the left of $t = 3$ has positive gradient. For $3 < t < 7$, the cockroach population was *decreasing* — every tangent to the right of $t = 3$ has negative gradient.

At $t = 3$, the cockroach population was *stationary*, that is, it was neither increasing nor decreasing and the tangent is horizontal. These three words are used throughout calculus to describe functions, whether or not a rate is involved.

19 INCREASING, DECREASING, STATIONARY

Suppose that $y = f(x)$ is defined at $x = a$, with $f'(a)$ also defined.

- If $f'(a)$ is positive (that is, if the tangent slopes upwards), then the curve is called *increasing* at $x = a$.
- If $f'(a)$ is negative (that is, if the tangent slopes downwards), then the curve is called *decreasing* at $x = a$.
- If $f'(a)$ is zero (that is, if the tangent is horizontal), then the curve is called *stationary* at $x = a$.

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The diagram on the right above is a graph of the rate at which the population is changing. It is a straight line with negative gradient — this means that *the rate of change of the cockroach population was decreasing at a constant rate*.

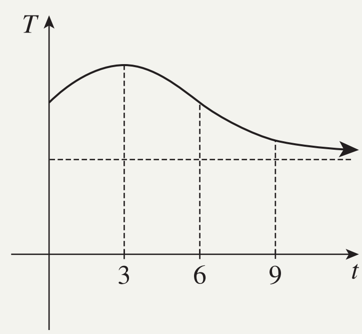
Questions with a diagram or a graph instead of an equation

In some problems about rates, the graph of some quantity Q over time is known, but its equation is unknown. Such problems require careful reading of the graph — pay attention to zeroes of the quantity and of the rate, and to where the quantity and rates are increasing, decreasing and stationary.

As always, it is often useful to draw an approximate sketch of the rate of change. The gradient of this second rate-of-change graph is the derivative of the rate of change $\dfrac{dQ}{dt}$, and is thus the second derivative $\dfrac{d^2Q}{dt^2}$ of the quantity.

Example 32 8J

The graph to the right shows the temperature T of a patient with Symond's Syndrome at time t hours after her admission to hospital at midnight.



a When did her temperature reach its maximum?

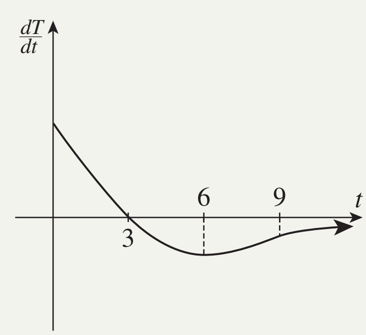
b When was her temperature increasing most rapidly, and when was it decreasing most rapidly and when was it stationary?

c What happened to her temperature eventually?

d Sketch the graph of the rate $\dfrac{dT}{dt}$ at which the temperature is changing.

e When was the rate of change decreasing, and when was it increasing?

SOLUTION



a The maximum temperature occurred when $t = 3$, that is, at 3:00 am.

b Her temperature was increasing most rapidly at the start when $t = 0$, that is, at midnight. It was decreasing most rapidly at 6:00 am. It was stationary at 3:00 am.

c The patient's temperature eventually stabilised.

d The graph of $\dfrac{dT}{dt}$ is zero at $t = 3$, and minimum at $t = 6$.

As $t \to \infty$, $\dfrac{dT}{dt} \to 0$, so the t-axis is a horizontal asymptote.

e The rate of change of temperature was decreasing from midnight to 6.00 am, and increasing from 6.00 am onwards.

Exercise 8J FOUNDATION

Note: Except in questions **1**–**3**, be careful to give the correct units in all final answers.

1 **a** If $Q = t^3 - 10t^2$, find the function $\dfrac{dQ}{dt}$ for the rate of change of Q.

b Hence find the quantity Q and the rate $\dfrac{dQ}{dt}$ when $t = 2$.

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2 **a** If $Q = t^2 + 6t$, find the function $\frac{dQ}{dt}$ for the rate of change of Q.

b Hence find the quantity Q and the rate $\frac{dQ}{dt}$ when $t = 2$.

c When is the quantity Q:

i stationary, **ii** increasing, **iii** decreasing.

3 **a** If $Q = 8t - t^2$, find the values of Q when $t = 1$ and $t = 3$.

b Hence find the average rate of change from $t = 1$ to $t = 3$.

c Similarly find the average rate of change from $t = 5$ to $t = 7$.

4 Orange juice is being poured at a constant rate into a glass. After t seconds there are V mL of juice in the glass, where $V = 60t$.

a How much juice is in the glass after 3 seconds?

b Show that the glass was empty to begin with.

c If the glass takes 5 seconds to fill, what is its capacity?

d Differentiate to find the rate at which the glass is being filled.

e What sort of function is the derivative in part **d**?

5 The volume V litres of fuel in a tanker, t minutes after it has started to empty, is given by $V = 200(400 - t^2)$. Initially the tanker was full.

a Find the initial volume of fuel in the tanker.

b Find the volume of fuel in the tanker after 15 minutes.

c Find the time taken for the tanker to empty.

d Show that $\frac{dV}{dt} = -400t$, and hence find the rate at which the tanker is emptying after 5 minutes.

DEVELOPMENT

6 Water surges into a rock pool, and then out again. The mass M in kilograms of water in the pool t seconds after time zero is given by the function $M = 10t - t^2$.

a Find the rate $\frac{dM}{dt}$ as a function of time t.

b Find (with units) the values of M and $\frac{dM}{dt}$ when $t = 4$.

c Find the value of M when $t = 2$, and hence find the average rate of change over the time interval from $t = 2$ to $t = 4$.

d At what time is M zero?

e For how long was there water in the pool?

f How long after time zero is $\frac{dM}{dt}$ zero?

g For how long was the water level in the pool rising, and for how long was it falling?

h Sketch the graphs of M and $\frac{dM}{dt}$ as functions of t, showing these results. Add the chord corresponding to the result in part **c** and the tangent corresponding to the result in part **f**.

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7 The share price $\$P$ of the Eastcom Bank t years after it opened on 1 January 1970 was $P = -0.4t^2 + 4t + 2$.

a What was the initial share price?

b What was the share price after one year?

c At what rate was the share price increasing after two years?

d Find when the share price was stationary.

e Explain why the maximum share price was \$12, at the start of 1975.

f Explain why the rate of change of the price decreased at a constant rate.

g The directors decided to close the bank when the share price fell back to its initial value. When did this happen?

8 For a certain medicine, the amount M present in the blood after t hours is given by $M = 3t^2 - t^3$, for $0 \le t \le 3$.

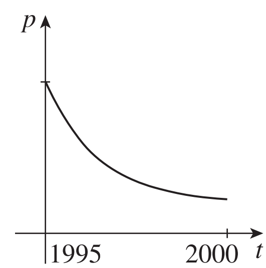
a Sketch a graph of M as a function of t.

b When is the amount of medicine in the blood a maximum?

c Differentiate to find the rate $\dfrac{dM}{dt}$, and find the axis of symmetry of the resulting parabola.

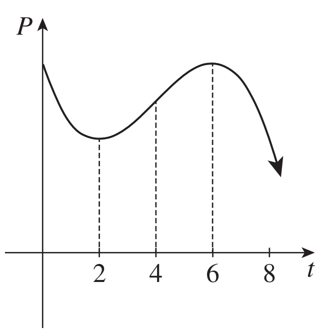
d When is the amount of medicine increasing most rapidly?

9 The graph shows the level of pollution in a river between 1995 and 2000. In 1995, the local council implemented a scheme to reduce the level of pollution in the river.



Comment briefly on whether this scheme worked and how the level of pollution changed. Include mention of the rate of change.

10 The graph to the right shows the share price P in Penn & Penn Stationery Suppliers t months after 1 January.



a When is the price stationary, neither increasing nor decreasing?

b When is the price maximum and when is it minimum?

c When is the price increasing and when is it decreasing?

d When is the share price increasing most rapidly?

e When is the share price increasing at an increasing rate?

f Sketch a possible graph of $\dfrac{dP}{dt}$ as a function of time t.

11 A lightning bolt hits the ground with a tremendous noise. The noise spreads out across the suburbs in a circle of area A, startling people.

a The speed of sound in air is about $\frac{1}{3}$ km/s. Show that the area A in which people are startled as a function of the time t in seconds after the strike is $A = \frac{\pi}{9}t^2$.

b Find the rate at which the area of startled people is increasing.

c Find, correct to four significant figures, the time when the area is 5 km^2, and the rate at which the area is increasing at that time.

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12 [A question about motion]

A stone is dropped from the top of a tall building. The stone's height h metres above the ground t seconds later is $h = 80 - 5t^2$.

a How high is the building?

b How many seconds does the stone take to reach the ground?

c The velocity v is the rate of change $\frac{dh}{dt}$ of the height. Find the velocity function.

d How fast is the stone travelling when it hits the ground?

e Acceleration is the rate of change $\frac{dv}{dt}$ of the velocity. Find the acceleration of the stone (the units are 'metres per second, per second', usually written as m/s^2).

13 [A question about motion]

A truck driver was travelling on a straight, flat road. He later stated in court that at a certain 'time zero', he applied the brakes in such a way that his velocity decreased at a constant rate. His speed-monitoring equipment record showed that t seconds after time zero, his velocity was $v = -\frac{1}{2}t + 50$, in units of m/s.

a Acceleration is the rate of change of velocity — differentiate to find his acceleration as a function of time. Was the truck driver's statement in court correct, and why?

b His GPS recorded that his displacement x, relative to where the truck was at 'time zero', was $x = -\frac{1}{4}t^2 + 50t$. Velocity is the rate of change of displacement — differentiate this function, and say whether his GPS record agreed with his speed-monitoring equipment.

c The police car behind him tracked, by laser and GPS, his displacement y relative to where the police car was at 'time zero'. They found that $y = -\frac{1}{4}t^2 + 50t + 450$. Differentiate this function, and say whether the police car's record agreed with the truck's speed-monitoring equipment.

d How far behind the truck was the police car at 'time zero'?

e How long did the truck take to stop, and how far did it go in that time?

f What was the truck's speed at 'time zero' in km/h, and why was he later in court?

CHALLENGE

14 A 3000 cm long water trough has the shape of a triangular prism. Its cross-section is an isosceles triangle with apex 90°, the trough is built with the apex at the bottom.

a **i** Find the area of an isosceles triangle with apex angle 90°, and height h centimetres.

ii Find the volume of water in the water trough when the water height is h cm.

b The water trough was initially empty, and then was filled with water at a constant rate of 27 litres per minute.

i Find the height h cm of the water after t minutes, and the rate at which the water was rising.

ii Find the water height, and the rate at which the water was rising after 25 minutes.

iii Find when the water height was 30 cm, and the rate at which the water was then rising.

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8K Continuity

A tangent can only be drawn at a point on a curve if:

- the curve has no break at the point (it is continuous at the point), and
- there is no sharp corner there (it is smooth at the point).

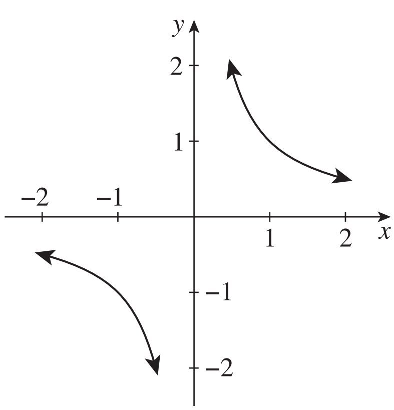
Sections 8K and 8L will make these two ideas a little more precise.

Continuity at a point

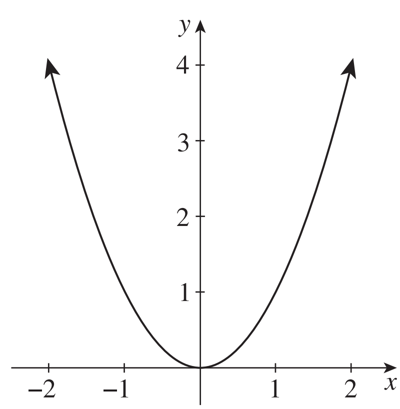
Continuity at a point means that there is no break in the curve at that point. This is an informal definition, but it will be enough for our purposes.

20 CONTINUITY AT A POINT — INFORMAL DEFINITION

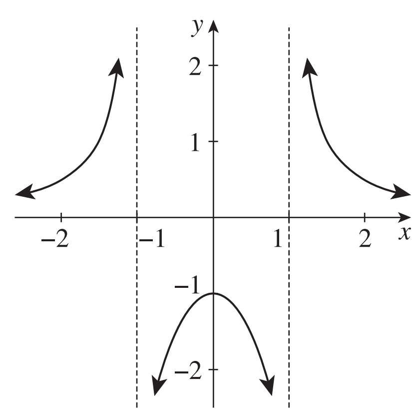
- A function $f(x)$ is called *continuous at* $x = a$ if the graph of $y = f(x)$ can be drawn *through* the point where $x = a$ without lifting the pencil off the paper.
- If there is a break in the curve, we say that there is a *discontinuity* at $x = a$.



$y = \dfrac{1}{x}$ has a discontinuity at $x = 0$, and is continuous everywhere else.



$y = x^2$ is continuous for all values of x.



$y = \dfrac{1}{x^2 - 1}$ has discontinuities at $x = 1$ and at $x = -1$, and is continuous everywhere else.

An assumption of continuity

It is intuitively obvious that a function such as $y = x^2$ is continuous for every value of x. Without further formality, we will make a general assumption of continuity, loosely stated as follows:

21 ASSUMPTION

The functions in this course are continuous for every value of x in their domain, except where there is an obvious problem.

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Piecewise-defined functions

A *piecewise-defined function* has different definitions in different parts of its domain, for example:

$$f(x) = \begin{cases} 4 - x^2, & \text{for } x \le 0, \\ 4 + x, & \text{for } x > 0. \end{cases}$$

The two pieces of this graph are a parabola and line, but we need to find out whether these two parts join up or leave a gap. The most obvious way is to construct two tables of values, one for each line of the function's definition.

Example 33 — 8K

Let $f(x) = \begin{cases} 4 - x^2, & \text{for } x \le 0, \\ 4 + x, & \text{for } x > 0. \end{cases}$

a Find $f(0)$. Then construct tables of values of $4 - x^2$ and $4 + x$ to establish whether the graph is continuous at $x = 0$.

b Sketch the graph.

c Hence write down its domain and range.

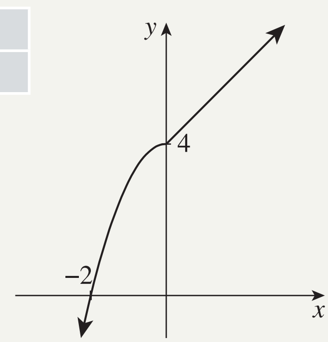
SOLUTION

a First, $f(0) = 4 - 0 = 4$ (using the top line of the definition).

Secondly, the values of $4 - x^2$ and $4 - x$ are:

x	-2	-1	0
$4 - x^2$	0	3	4

x	0	1	2
$4 + x$	4	5	6



Hence the curve is *continuous* at $x = 0$.
(Reason: When $x = 0$, the values of $4 - x^2$ and $4 + x$ are 4, which is the same as $f(0)$, so the two pieces of the graph join up.)

b The graph is made up of the parabolic piece on the left, and the linear piece on the right.

c The domain is all real x, and the range is all real y.

Testing whether piecewise-defined functions join up

In the previous example, all we really needed to do to test whether the pieces joined up at $x = 0$ was:

- Find $f(0)$.
- Substitute $x = 0$ into the formulae on the left.
- Substitute $x = 0$ into the formulae on the right.

This is the approach taken in the next example.

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Example 34 8K

Let $f(x) = \begin{cases} 2^x, & \text{for } x < 1, \\ 1, & \text{for } x = 1, \\ x^2, & \text{for } x > 1. \end{cases}$

a Establish whether the graph is continuous at $x = 1$.

b Sketch the graph.

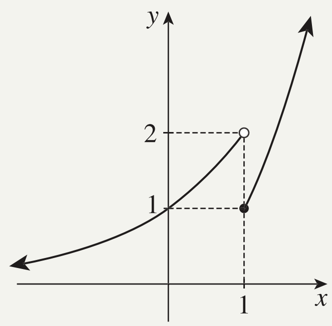
c Hence write down its domain and range.

SOLUTION

a First, $f(x) = 1$.
Secondly, $2^x = 2$, when $x = 1$.
Thirdly, $x^2 = 1$, when $x = 1$.

Hence the function is *not continuous* at $x = 1$.

b



c The domain is all real x, and the range is $y > 0$.

Note: When $x = 1$, the value of $f(x)$ is 1, so the endpoint $(1, 1)$ of the right-hand piece is included in the graph, as indicated by the closed circle. But the endpoint $(1, 2)$ of the left-hand piece is not included in the graph, as indicated by the open circle there.

22 OPEN AND CLOSED CIRCLES, AND ARROWS

- A *closed circle* marks an endpoint that is included in the graph.
- An *open circle* marks an endpoint that is not included in the graph.
- An *arrow* marks a curve that continues forever.

We used these same conventions in Section 2B when drawing intervals.

Exercise 8K

FOUNDATION

1 State the discontinuities of each function.

a $f(x) = \dfrac{5}{6 - x}$ **b** $f(x) = \dfrac{3x}{(x-1)(x-3)(x-5)}$ **c** $f(x) = \dfrac{x(x+1)}{(x+2)(x+3)}$

2 Let $f(x) = \begin{cases} 1 - x, & \text{for } x < 0, \\ 1 + x^2, & \text{for } x \ge 0. \end{cases}$

a Find $f(0)$. Then copy and complete these tables of values:

x	-2	-1	0
$1 - x$			

x	0	1	2
$1 + x^2$			

b Is $f(x)$ continuous at $x = 0$?

c Sketch the graph.

d Hence write down the domain and range of $f(x)$.

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3 Let $y = \begin{cases} 2 - x, & \text{for } x \le 1, \\ x - 1, & \text{for } x > 1. \end{cases}$

a Find $f(1)$. Then find the values of $2 - x$ and $x - 1$ when $x = 1$.
b Is the function continuous at $x = 1$?
c Sketch the graph.
d Hence write down the domain and range.

DEVELOPMENT

4 Factor the denominator of each function, and hence write down its discontinuities.

a $f(x) = \dfrac{1}{x^2 - 5x}$ **b** $f(x) = \dfrac{x}{x^2 - 5x + 6}$ **c** $f(x) = \dfrac{x^2 - 16}{x^2 - 9}$

5 For each function, find whether the function is continuous at $x = 2$. Then sketch the curve and state the domain and the range.

a $f(x) = \begin{cases} x^3, & \text{for } x \le 2, \\ 10 - x, & \text{for } x > 2. \end{cases}$

b $f(x) = \begin{cases} 3^x, & \text{for } x < 2, \\ 13 - x^2, & \text{for } x > 2, \\ 4, & \text{for } x = 2. \end{cases}$

c $f(x) = \begin{cases} \dfrac{1}{x}, & \text{for } 0 < x < 2, \\ 1 - \dfrac{1}{4}x, & \text{for } x > 2, \\ \dfrac{1}{2}, & \text{for } x = 2. \end{cases}$

d $f(x) = \begin{cases} x, & \text{for } x < 2, \\ 2 - x, & \text{for } x > 2, \\ 2, & \text{for } x = 2. \end{cases}$

CHALLENGE

6 **a** Draw up a table of values for $y = \dfrac{|x|}{x}$.
b Is the function continuous at $x = 0$? Give a reason.
c Sketch the curve, and write down its domain and range.

7 Let $f(x) = \dfrac{3x + 3}{x + 1}$.
a Is the graph continuous at $x = -1$?
b Sketch the graph, and state its domain and range.
c Explain why the graph looks like a straight line.

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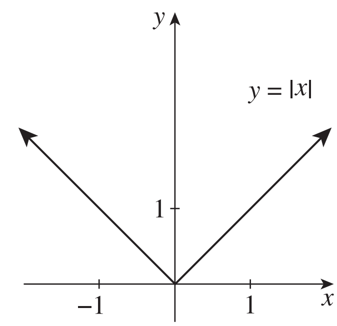
8L Differentiability



Look at the origin on the graph of $y = |x|$ to the right. The graph is continuous at $x = 0$ because the two pieces join up.

But this chapter is about differentiation, and there is no way that a tangent can be drawn at the origin, because there is a sharp corner where the gradient changes abruptly — from -1 on the left of the origin, to 1 on the right of the origin. Thus there is no tangent at the origin, and the derivative $f'(0)$ does not exist. We say that *the function is not differentiable at* $x = 0$.

A tangent can only be drawn at a point P on a curve if the curve is *smooth* at that point, meaning that the curve can be drawn through P without lifting the pencil off the paper, and without any sharp change of direction. The words *smooth* and *differentiable* are used interchangeably, and we have already used the word *smooth* on several occasions because it is intuitively obvious what it means.

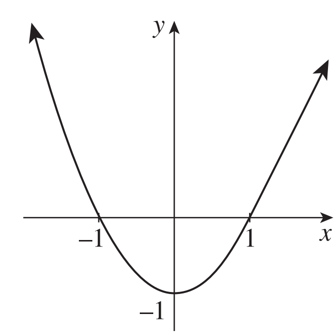
23 DIFFERENTIABLE AT A POINT

A function $f(x)$ is called *differentiable* (or *smooth*) at $x = a$ if it satisfies two conditions there.
1. The graph is continuous at $x = a$.
2. The graph passes through the point where $x = a$ without any sharp corner, and the tangent is not vertical there.

In this situation, $f'(a)$ is the gradient of the tangent at $x = a$.

If a tangent is vertical, its gradient, and therefore the derivative, is undefined.

Piecewise-defined functions



If the pieces of a piecewise-defined function don't join up at $x = a$, then the function not continuous at $x = a$, so it is certainly not differentiable there. If it is continuous there, take the derivative in the two pieces, and see whether the gradients join up smoothly at $x = a$.

For example, the sketch opposite shows $f(x) = \begin{cases} x^2 - 1, & \text{for } x < 1, \\ 2x - 2, & \text{for } x \ge 1. \end{cases}$

A The graph is continuous, because the two pieces join at $P(1, 0)$:

$$f(1) = 0 \quad \text{and} \quad x^2 - 1 = 0 \text{ when } x = 1 \quad \text{and} \quad 2x - 2 = 0 \text{ when } x = 1.$$

B Now differentiate to find the gradient functions on the left and right of $x = 1$:

$$f'(x) = \begin{cases} 2x, & \text{for } x < 1, \\ 2, & \text{for } x > 1. \end{cases}$$

We can see from this that when the two pieces join, they do so with the same gradient 2, because

$$2x = 2 \text{ when } x = 1 \qquad \text{and} \qquad 2 = 2 \text{ when } x = 1.$$

The combined curve is therefore *smooth* or *differentiable* at the point $P(1, 0)$, with derivative $f'(1) = 2$, and a well-defined tangent there of gradient 2.

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24 DIFFERENTIABILITY OF A PIECEWISE-DEFINED FUNCTION

To test a function for differentiability at a join between pieces at $x = a$:

1. Test whether the function is continuous at $x = a$.
2. If it is, look at $f'(x)$ on the left and right to see whether the join is smooth.

Example 35 — 8L

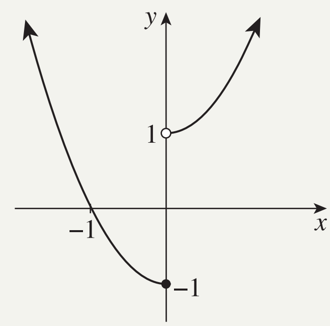
Test each function for continuity, and then for differentiability, at $x = 0$. Then sketch the curve, and write down its domain and range.

a $f(x) = \begin{cases} x^2 - 1, & \text{for } x \le 0, \\ x^2 + 1, & \text{for } x > 0. \end{cases}$

b $f(x) = \begin{cases} x, & \text{for } x \le 0, \\ x - x^2, & \text{for } x > 0. \end{cases}$

SOLUTION

a First, $x^2 - 1 = -1$ when $x = 0$, and $x^2 + 1 = 1$ when $x = 0$,
so the function is not continuous at $x = 0$,
let alone differentiable there.
Domain: all real x, range: $y \ge -1$

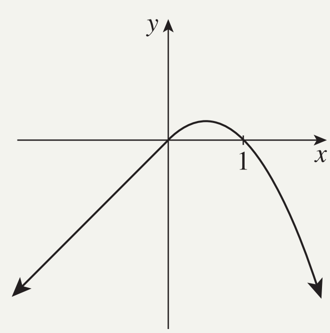


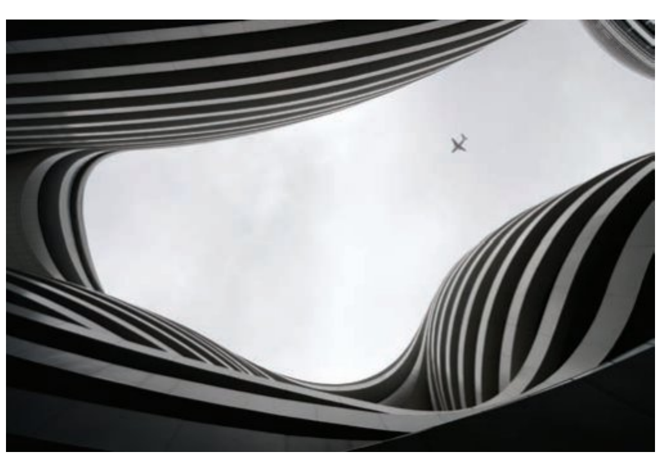
b First, $f(x) = f(0) = 0$, and $x = 0$ when $x = 0$,
and $x - x^2 = 0$ when $x = 0$,
so the function is continuous at $x = 0$.

Secondly, $f'(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - 2x, & \text{for } x > 0, \end{cases}$

and $1 = 1$ when $x = 0$, and $1 - 2x = 1$ when $x = 0$,
so the function is differentiable at $x = 0$, with $f'(0) = 1$.
Domain: all real x, range: $y \le \frac{1}{4}$

The quadratic piece has vertex $\left(\frac{1}{2}, \frac{1}{4}\right)$ and a zero at $x = 1$.





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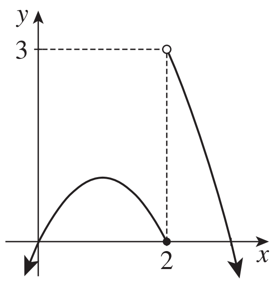
Exercise 8L

FOUNDATION

1 State whether each function $f(x)$ is continuous at $x = 0$ and at $x = 2$, and whether it is differentiable (smooth) there.

a


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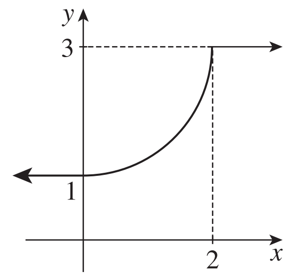

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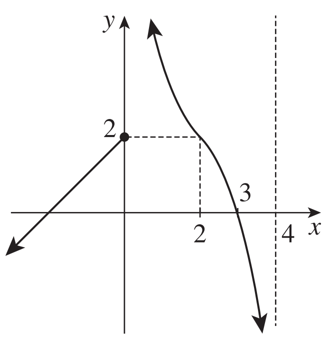

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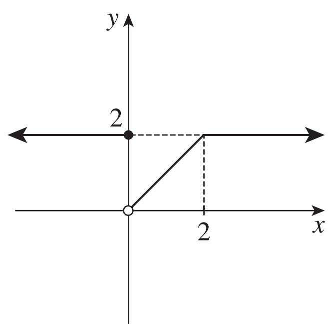

2 a Sketch the graph of the function $f(x) = \begin{cases} x^2, & \text{for } x \le 1, \\ 2x - 1, & \text{for } x > 1. \end{cases}$

b Show that the function is continuous at $x = 1$.

c Explain why $f'(x) = \begin{cases} 2x, & \text{for } x < 1, \\ 2, & \text{for } x > 1. \end{cases}$

d Hence explain why there is a tangent at $x = 1$, state its gradient, and state $f'(1)$.

DEVELOPMENT

3 Sketch each function. State any values of x where it is not continuous or not differentiable.

a $y = |x + 2|$

b $y = |x| + 2$

4 Test each function for continuity at $x = 1$. If the function is continuous there, check for differentiability at $x = 1$. Then sketch the graph.

a $f(x) = \begin{cases} (x + 1)^2, & \text{for } x \le 1, \\ 4x - 2, & \text{for } x > 1. \end{cases}$

b $f(x) = \begin{cases} 3 - 2x, & \text{for } x < 1, \\ \dfrac{1}{x}, & \text{for } x \ge 1. \end{cases}$

c $f(x) = \begin{cases} 2 - x^2, & \text{for } x \le 1, \\ (x - 2)^2, & \text{for } x > 1. \end{cases}$

d $f(x) = \begin{cases} (x - 1)^3, & \text{for } x \le 1, \\ (x - 1)^2, & \text{for } x > 1. \end{cases}$

CHALLENGE

5 Sketch $y = \begin{cases} \sqrt{1 - (x + 1)^2}, & \text{for } -2 \le x \le 0, \\ \sqrt{1 - (x - 1)^2}, & \text{for } 0 < x \le 2. \end{cases}$

(Hint: Each part is a semicircle, which can easily be seen by squaring it.)

6 Find whether each function is differentiable at $x = 0$:

a $y = |x^2|$

b $y = |x^3|$

c $y = \sqrt{|x|}$

d $y = \sqrt[3]{x}$

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Chapter 8 Review

Review activity

Create your own summary of this chapter on paper or in a digital document.

Chapter 8 Multiple-choice quiz

This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Use the definition of the derivative $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find the derivative of each function by first principles.

a $f(x) = x^2 + 5x$ **b** $f(x) = 6 - x^2$ **c** $f(x) = 3x^2 - 2x + 7$

2 Write down the derivative of each function. You will need to expand any brackets.

a $y = x^3 - 2x^2 + 3x - 4$ **b** $y = x^6 - 4x^4$ **c** $y = 3x^2(x - 2x^3)$

d $y = (x + 3)(x - 2)$ **e** $y = (2x - 1)(2 - 3x)$ **f** $y = 3x^{-2} - 2x^{-1}$

g $y = 4x^3 - 4x^{-3}$ **h** $y = 3x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$ **i** $y = x^{-2}(x^2 - x + 1)$

3 Find the first and second derivatives of each function:

a $f(x) = x^4 + x^3 + x^2 + x + 1$ **b** $f(x) = 5x^{-2}$ **c** $f(x) = 8x^{-\frac{1}{2}}$

4 Find the family of curves whose derivative is:

a $\dfrac{dy}{dx} = 3x^2 + 4$ **b** $\dfrac{dy}{dx} = 7 - 12x - 12x^2$ **c** $\dfrac{dy}{dx} = 20x^4 - 12x^2 + 4$

5 Rewrite each function in index notation and then differentiate it. Write each final answer without negative or fractional indices.

a $y = \dfrac{3}{x}$ **b** $y = \dfrac{1}{6x^2}$ **c** $y = 7\sqrt{x}$

d $y = \sqrt{144x}$ **e** $y = -3x\sqrt{x}$ **f** $y = \dfrac{6}{\sqrt{x}}$

6 Divide each function through by the numerator and then differentiate it. Give each final answer without negative or fractional indices.

a $y = \dfrac{3x^4 - 2x^3}{x^2}$ **b** $y = \dfrac{x^3 - x^2 + 7x}{2x}$ **c** $y = \dfrac{5x^3 - 7}{x}$

d $y = \dfrac{x^2 + 2x + 1}{x^2}$ **e** $y = \dfrac{4x + 5\sqrt{x}}{\sqrt{x}}$ **f** $y = \dfrac{2x^2\sqrt{x} + 3x\sqrt{x}}{x}$

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Review

7 Use the formula $\dfrac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$ to differentiate these functions.

a $y = (3x+7)^3$ **b** $y = (5-2x)^2$ **c** $y = \dfrac{1}{5x-1}$

d $y = \dfrac{1}{(2-7x)^2}$ **e** $y = \sqrt{5x+1}$ **f** $y = \dfrac{1}{\sqrt{1-x}}$

8 Use the chain rule to differentiate these functions.

a $y = (7x^2-1)^3$ **b** $y = (1+x^3)^{-5}$ **c** $y = (1+x-x^2)^8$

d $y = \dfrac{1}{(x^2-1)^3}$ **e** $y = \sqrt{9-x^2}$ **f** $y = \dfrac{1}{\sqrt{9-x^2}}$

9 Differentiate each function, using the product or quotient rule. Factor each answer completely.

a $y = x^9(x+1)^7$ **b** $y = \dfrac{x^2}{1-x}$ **c** $y = x^2(4x^2+1)^4$

d $y = \dfrac{2x-3}{2x+3}$ **e** $y = (x+1)^5(x-1)^4$ **f** $y = \dfrac{x^2+5}{x-2}$

10 Differentiate $y = x^2 + 3x + 2$. Hence find the gradient and the angle of inclination, correct to the nearest minute when appropriate, of the tangents at the points where:

a $x = 0$ **b** $x = -1$ **c** $x = -2$

11 a Find the equations of the tangent and the normal to $y = x^3 - 3x$ at the origin.

b Find the equation of the tangent and the normal to the curve at the point $P(1, -2)$.

c By solving $f'(x) = 0$, find the points on the curve where the tangent is horizontal.

d Find the points on the curve where the tangent has gradient 9.

12 a Find the equations of the tangent and normal to $y = x^2 - 5x$ at the point $P(2, -6)$.

b The tangent and the normal at P meet the x-axis at A and B respectively. Find the coordinates of A and B.

c Find the length of the interval AB and the area of the triangle PAB.

13 Find the tangents to the curve $y = x^4 - 4x^3 + 4x^2 + x$ at the origin and at the point where $x = 2$, and show that these two lines are the same.

14 Find any points on each curve where the tangent has the given angle of inclination.

a $y = \frac{1}{3}x^3 - 7$, $45°$ **b** $y = x^2 + \frac{1}{3}x^3$, $135°$

15 a Find the points on $y = x^3 - 4x$ where the tangents are parallel to $\ell: x + y + 2 = 0$.

b Find the equations of these two tangents, and show that ℓ is one of them.

16 A huge fireball high in the air is exploding at the speed of sound, which is about $\frac{1}{3}$ km/s.

a Using the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere, show that the volume V of the fireball after t seconds is $V = \frac{4\pi}{81}t^3$.

b Differentiate to find the rate at which the volume is expanding.

c Find, correct to two significant figures, the volume and the rate after 0.1 seconds.

d Find, correct to two significant figures, when the volume is increasing by 1 km^3/s.

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17 Let $f(x) = \begin{cases} x^2, & \text{for } x \le 0, \\ x^2 + 1, & \text{for } x > 0. \end{cases}$

a Is the curve continuous at $x = 0$?

b Write down the domain and range of the function.

18 Let $f(x) = \begin{cases} x^2 - 4, & \text{for } x < 2, \\ 4x - 8, & \text{for } x \ge 2. \end{cases}$

a Is the curve continuous at $x = 2$?

b Is the curve differentiable at $x = 2$?

c What are the domain and range of the function?

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Appendix: Proving differentiation rules

Differentiating powers of x

First, the rule for differentiating powers of x is very simple:

Theorem: Let $y = x^n$, where n is any real number. Then $\dfrac{dy}{dx} = nx^{n-1}$.

Unfortunately, the proof is complicated. The proof is only developed below in the case where n is a positive integer. (Challenge questions in Exercise 8E and 8F develop the proofs for $n = -1$ and $n = -\frac{1}{2}$ respectively). Here are two of many different approaches.

1 An approach using the expansion of $(x + h)^n$

Look carefully at the expansion of $(x + h)^4$. We can easily work out what the first two terms are:

$$\begin{aligned}(x+h)^4 &= (x+h)(x+h)(x+h)(x+h)\\ &= x^4 + (xxxh + xxhx + xhxx + hxxx) + \text{(terms with at least two } h\text{s)}\\ &= x^4 + 4x^3h + \text{(terms in } h^2 \text{ and above)}.\end{aligned}$$

The general case of the expansion of $(x + h)^n$ is similar:

$$\begin{aligned}(x+h)^n &= (x+h)(x+h)\cdots(x+h)(x+h) \quad \text{(there are } n \text{ factors)}\\ &= x^n + (x^{n-1}h + x^{n-2}hx + x^{n-3}hx^2 + \cdots + hx^{n-1}) + \text{(terms with at least two } h\text{s)}\\ &= x^n + nx^{n-1}h + \text{(terms in } h^2 \text{ and above)}.\end{aligned}$$

Hence
$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\ &= \lim_{h\to 0}\frac{nx^{n-1}h + \text{(terms in } h^2 \text{ and above)}}{h}\\ &= \lim_{h\to 0}(nx^{n-1} + \text{(terms in } h \text{ and above))}\\ &= nx^{n-1}.\end{aligned}$$

2 An approach using the product rule

The method of proof used in this approach is formalised in the Extension 1 course as *mathematical induction*. We already know that the result is true for $y = x$ and $y = x^2$.

For $y = x^3$, we can write y as a *product*,

$$y = x \times x^2.$$

Then
$$\begin{aligned}\frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= x^2 \times 1 + x \times 2x\\ &= 3x^2, \quad \text{as required.}\end{aligned}$$

Let $u = x$
and $v = x^2$
Hence $\dfrac{du}{dx} = 1$
and $\dfrac{dv}{dx} = 2x.$

For $y = x^4$, we can also write y as a *product*,

$$y = x \times x^3.$$

Then
$$\begin{aligned}\frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= x^3 \times 1 + x \times 3x^2\\ &= 4x^3, \quad \text{as required.}\end{aligned}$$

Let $u = x$
and $v = x^3$
Hence $\dfrac{du}{dx} = 1$
and $\dfrac{dv}{dx} = 3x^2.$

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Continue in this way through the positive integers. In general, suppose that we have already proven that the derivative of x^n is nx^{n-1} for some index n.

Then for the next index $n + 1$, we can write:

$$\begin{aligned} y &= x^{n+1} \\ &= x \times x^n \end{aligned}$$

Hence
$$\begin{aligned} \frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\ &= x^n \times 1 + x \times nx^{n-1} \\ &= (n+1)x^n. \end{aligned}$$

Let $u = x$
and $v = x^n$
Then $\dfrac{du}{dx} = 1$
and $\dfrac{dv}{dx} = nx^{n-1}$.

Thus the result is also true for the next index $n + 1$.

Linear combinations of functions

Two results need to be proven:

Derivative of a sum: If $f(x) = g(x) + k(x)$, then $f'(x) = g'(x) + k'(x)$.

Derivative of a multiple: If $f(x) = ag(x)$, then $f'(x) = ag'(x)$.

Proof

For the first:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{g(x+h) + k(x+h) - g(x) - k(x)}{h} \\ &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} \\ &= g'(x) + k'(x). \end{aligned}$$

For the second:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{ag(x+h) - ag(x)}{h} \\ &= a\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= ag'(x). \end{aligned}$$

Proof of the product rule

The product rule states that:

Theorem: Let $y = uv$, where u and v are functions of x. Then:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Proof

Suppose that x changes to $x + \delta x$.

Suppose that as a result, u changes to $u + \delta u$ and v changes to $v + \delta v$, and that finally y changes to $y + \delta y$.

Then $\quad y = uv,$

and $\quad y + \delta y = (u + \delta u)(v + \delta v)$
$\quad = uv + v\delta u + u\delta v + \delta u \delta v,$

so $\quad \delta y = v\delta u + u\delta v + \delta u\delta v.$

Hence, dividing by δx, $\quad \dfrac{\delta y}{\delta x} = v\dfrac{\delta u}{\delta x} + u\dfrac{\delta v}{\delta x} + \dfrac{\delta u}{\delta x} \times \dfrac{\delta v}{\delta x} \times \delta x,$

and taking limits as $\delta x \to 0$, $\quad \dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx} + 0$, as required.

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Proof of quotient rule

The quotient rule states that:

Theorem

Let $y = \dfrac{u}{v}$, where u and v are functions of x. Then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Proof

Differentiate uv^{-1} using the product rule.

Let $\quad y = uv^{-1}.$

Then $\quad \dfrac{dy}{dx} = V\dfrac{dU}{dx} + U\dfrac{dV}{dx}$

$$= v^{-1}\,\frac{du}{dx} - uv^{-2}\,\frac{dv}{dx}$$

$$= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$\left(\text{after multiplying by } \dfrac{v^2}{v^2}\right).$

Let $\quad U = u$

and $\quad V = v^{-1}.$

Then $\quad \dfrac{dU}{dx} = \dfrac{du}{dx}$

and by the chain rule,

$$\frac{dV}{dx} = \frac{dV}{dv} \times \frac{dv}{dx}$$

$$= -v^{-2}\,\frac{dv}{dx}.$$

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9 Extending calculus

So far, calculus has been developed for algebraic functions such as

$$f(x) = x^3 + 8x \quad \text{and} \quad f(x) = \sqrt{x} \quad \text{and} \quad f(x) = x^2 - \frac{1}{x^2}$$

that involve only powers and the four operations of arithmetic. Some of the most important functions in science, however, cannot be written in this way. This chapter begins to extend calculus to two important groups of non-algebraic functions — exponential and logarithmic functions, and trigonometric functions.

Exponential functions are functions such as $y = 2^x$ and $y = 10^x$, where the variable is in the exponent (index). They were introduced in Chapter 7. These functions model many very common natural phenomena where something is dying away, such as radioactive decay or the noise of a plucked guitar string, or where something is growing, such as populations or inflation.

Trigonometric functions include functions such as $\sin x$ and $\cos x$, whose graphs are waves. They were introduced in Chapter 5. These functions model all the many wave-like phenomena in science, such as sound waves, light and radio waves, vibrating strings, tides and economic cycles.

Differentiating trigonometric functions requires a new unit for measuring angles, based on $\pi \doteqdot 3.1416$ — this is hardly surprising, because trigonometric functions are based on circles. Differentiating exponential functions requires a new base $e \doteqdot 2.7183$ — this new number e is just as important in mathematics as π, but only makes its first appearance in this chapter, in the exponential function $y = e^x$.

Both numbers π and e are real numbers. But neither number is a fraction, although the proofs of this are beyond the present course.

This chapter provides only an introduction, in preparation for much further development next year. First, the differentiation of exponential functions is introduced (Sections 9A–9F). Secondly, radian measure of angles is introduced (Sections 9G–9J) in preparation for the calculus of the trigonometric functions in Year 12.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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9A The exponential function base e

As indicated in the introduction, differentiating exponential functions requires a new base $e \doteqdot 2.7183$. The fundamental result of this section is surprisingly simple — the function $y = e^x$ is its own derivative:

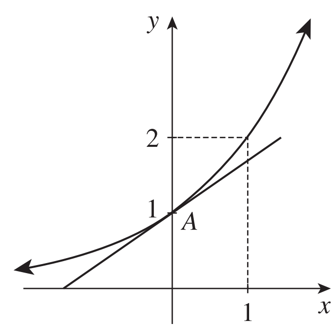
$$\frac{d}{dx}e^x = e^x$$

An investigation of the graph of $y = 2^x$

We begin by examining the tangents to the familiar exponential function $y = 2^x$. Before reading the following argument, carry out the investigation in Question **1** of Exercise 9A. The investigation looks at the tangent to $y = 2^x$ at each of several points on the curve, and relates its gradient to the height of the curve at the point. The procedures demonstrate graphically what the following argument proves.

Differentiating $y = 2^x$

Below is a sketch of $y = 2^x$, with the tangent drawn at its y-intercept $A(0, 1)$. Differentiating $y = 2^x$ requires first-principles differentiation, because the theory so far hasn't provided any rule for differentiating 2^x.



The formula for first-principles differentiation is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Applying this formula to the function $f(x) = 2^x$,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} \\ &= \lim_{h \to 0} \frac{2^x \times 2^h - 2^x}{h}, \qquad \text{because } 2^x \times 2^h = 2^{x+h}. \end{aligned}$$

and taking out the common factor 2^x in the numerator,

$$f'(x) = 2^x \times \lim_{h \to 0} \frac{2^h - 1}{h}$$

$$f'(x) = 2^x \times m, \qquad \text{where } m = \lim_{h \to 0} \frac{2^h - 1}{h}.$$

This limit $m = \lim_{h \to 0} \frac{2^h - 1}{h}$ cannot be found by algebraic methods.

But substituting $x = 0$ gives a very simple geometric interpretation of the limit:

$$\begin{aligned} f'(0) &= 2^0 \times m \\ &= m, \qquad \text{because } 2^0 = 1, \end{aligned}$$

so m is just the gradient of the tangent to $y = 2^x$ at its y-intercept $(0, 1)$.

The conclusion of all this is that

$$\frac{d}{dx}2^x = 2^x \times m, \text{ where } m \text{ is the gradient of } y = 2^x \text{ at its } y\text{-intercept.}$$

The investigation in Question **1** of Exercise 9A shows that $m \doteqdot 0.7$.

This argument applied to the base 2. But exactly the same argument can be applied to any exponential function $y = a^x$, whatever the base a, simply by replacing 2 by a in the argument above.

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1 DIFFERENTIATING $y = a^x$

For all positive real numbers a,

$$\frac{d}{dx}a^x = a^x \times m, \text{ where } m \text{ is the gradient of } y = a^x \text{ at its } y\text{-intercept.}$$

That is, the derivative of an exponential function is a multiple of itself.

The gradient m of the curve at its y-intercept will of course change as the base a changes.

The definition of e

It now makes sense to choose the base that will make the gradient exactly 1 at the y-intercept, because the value of m will then be exactly 1. This base is given the symbol e, and has the value $e \doteqdot 2.7183$.

2 THE DEFINITION OF e

Define e to be the number such that the exponential function $y = e^x$ has gradient exactly 1 at its y-intercept. Then

$$e \doteqdot 2.7183.$$

The function $y = e^x$ is called ***the*** *exponential function* to distinguish it from all other exponential functions $y = a^x$.

Investigations to approximate e

Question **4** of Exercise 9A uses a simple argument with no technology to prove that e is between 2 and 4. The investigation in Question **5** of Exercise 9A uses technology to find ever closer approximations to e. Both questions are based on the definition of e given above.

The derivative of e^x

The fundamental result of this section then follows immediately from the previous two boxed results.

$$\begin{aligned}\frac{d}{dx}e^x &= e^x \times m, \text{ where } m \text{ is the gradient of } y = e^x \text{ at its } y\text{-intercept,}\\ &= e^x \times 1, \text{ by the definition of } e,\\ &= e^x.\end{aligned}$$

3 THE EXPONENTIAL FUNCTION $y = e^x$ IS ITS OWN DERIVATIVE

$$\frac{d}{dx}e^x = e^x$$

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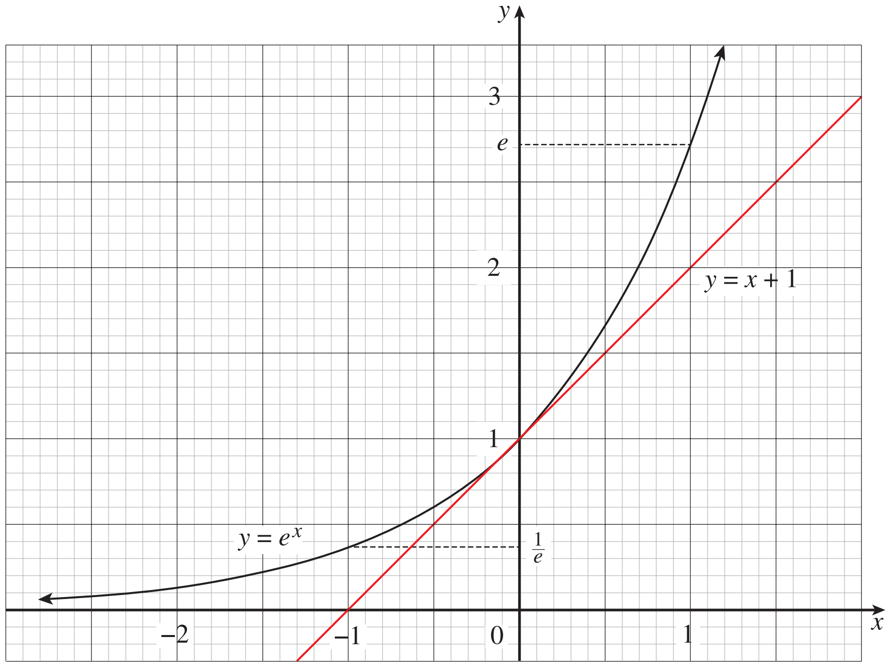
The graph of e^x

The graph of $y = e^x$ has been drawn below on graph paper. The tangent has been drawn at the y-intercept (0, 1) — it has gradient exactly 1.

This graph of $y = e^x$ is one of the most important graphs in the whole course, and its shape and properties need to be memorised.



- The domain is all real x.
  The range is $y > 0$.
- There are no zeroes.
  The curve is always above the x-axis.
- The x-axis $y = 0$ is a horizontal asymptote to the curve on the left.
  That is, as $x \to -\infty$, $y \to 0$.
- On the right-hand side, the curve rises steeply.
  That is, as $x \to \infty$, $y \to \infty$ and $\dfrac{dy}{dx} \to \infty$.
- The curve has gradient 1 at its y-intercept (0, 1).
- The curve is always increasing — and increasing at an increasing rate — and is always concave up.

Gradient equals height

The fact that the derivative of the exponential function e^x is the same function has a striking geometrical interpretation in terms of its graph. If $y = e^x$, then $\dfrac{dy}{dx} = e^x$, which means that for this function

$$\frac{dy}{dx} = y, \qquad \text{that is,} \qquad \text{gradient} = \text{height}.$$

Thus at each point on the curve $y = e^x$, the gradient $\dfrac{dy}{dx}$ of the curve is equal to the height y of the curve above the x-axis. We have already seen this happening at the y-intercept (0, 1), where the gradient is 1 and the height is also 1.

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4 GRADIENT EQUALS HEIGHT

At each point on the graph of the exponential function $y = e^x$,

$$\frac{dy}{dx} = y.$$

That is, the gradient of the curve is always equal to its height above the x-axis.

This property of the exponential function is the reason why the function is so important in calculus.

An investigation of the graph of $y = e^x$

The investigation in Question **2** of Exercise 9A confirms these properties of the graph of $y = e^x$, particularly the fact that the gradient and the height are equal at every point on the curve.

Exercise 9A

FOUNDATION

Technology: The first two investigations are written as graph-paper exercises, so that they are independent of any device. The instructions are identical if graphing software is used. When using the graph-paper method, the diagrams should first be photocopied.

1 [Graph paper, but easily transferred to graphing software]
The graph is the function $y = 2^x$.

a Photocopy the graph of $y = 2^x$, and on it draw the tangent at the point (0, 1). Extend the tangent across the diagram.

b Use 'rise over run' to measure the gradient $\dfrac{dy}{dx}$ of this tangent. The run should be chosen as 10 or 20 little divisions, then count how many little divisions the rise is.

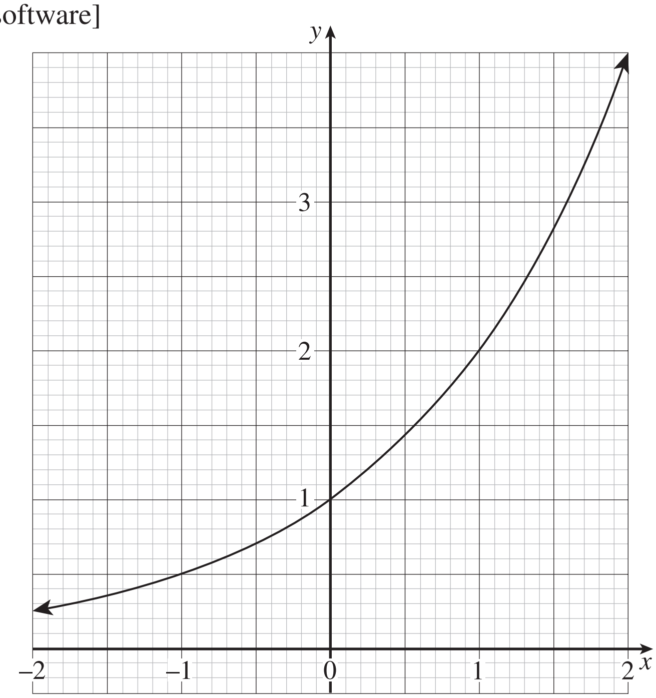
c Similarly, draw tangents at four more points on the curve where $x = -2, -1, 1$ and 2.
Measure the gradient $\dfrac{dy}{dx}$ of each tangent.

d Copy and complete the table of values to the right (the values will only be rough).

e What do you notice about the ratios of gradient to height?

f Hence copy and complete:

$$\frac{dy}{dx} = \cdots y$$



x	-2	-1	0	1	2
height y					
gradient $\dfrac{dy}{dx}$					
$\dfrac{\text{gradient}}{\text{height}}$					

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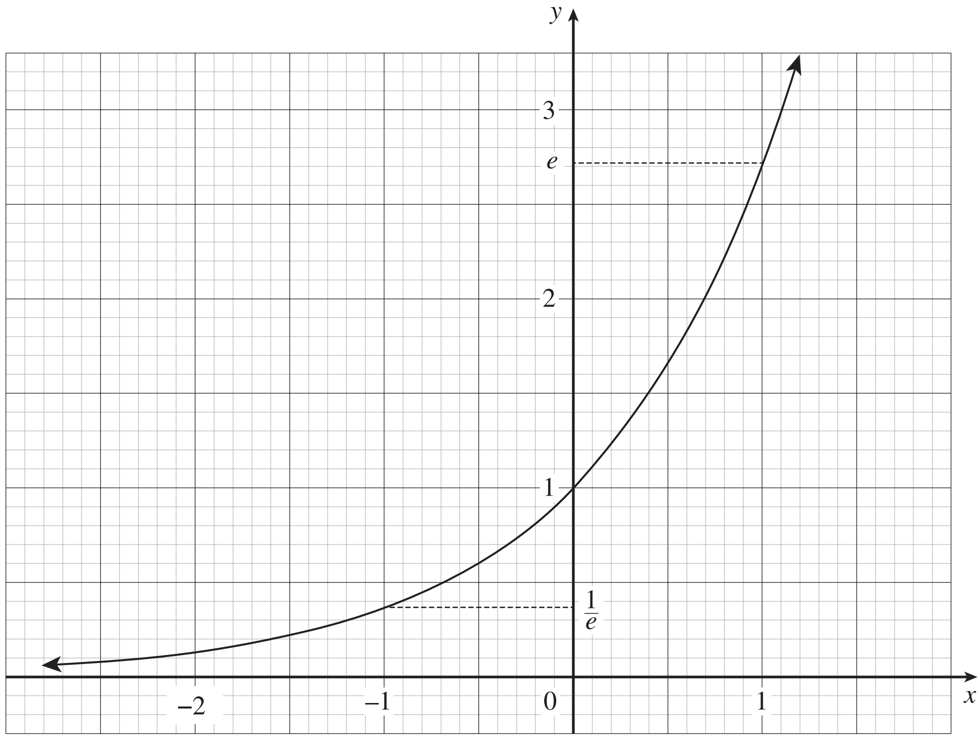
2 [Graph paper, but easily transferred to graphing software]



These questions refer to the graph of $y = e^x$ drawn above.

a Photocopy the graph of $y = e^x$ above and on it draw the tangent at the point $(0, 1)$ where the height is 1. Extend the tangent across the diagram.

b Measure the gradient of this tangent and confirm that it is equal to the height of the exponential graph at the point of contact.

c Copy and complete the table of values to the right by measuring the gradient $\frac{dy}{dx}$ of the tangent at the points where the height y is $\frac{1}{2}$, 1, 2 and 3.

height y	$\frac{1}{2}$	1	2	3
gradient $\frac{dy}{dx}$				
$\frac{\text{gradient}}{\text{height}}$				

d What do you notice about the ratios of gradient to height?

e What does this tell you about the derivative of $y = e^x$?

3 **a** Photocopy again the graph of $y = e^x$ in Question **2** (or use graphing software).

b Draw the tangents at the points where $x = -2, -1, 0$ and 1, extending each tangent down to the x-axis.

c Measure the gradient of each tangent and confirm that it is equal to the height of the graph at the point.

d What do you notice about the x-intercepts of all the tangents?

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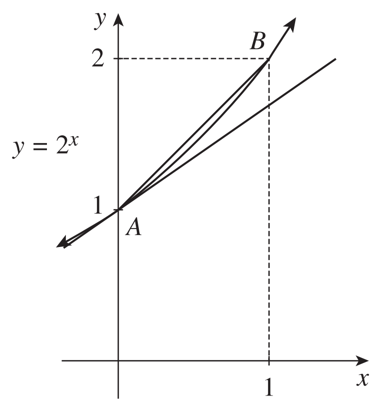
DEVELOPMENT

4 Approximating e to many significant figures is beyond this course. The argument below, however, at least shows very quickly that the number e lies between 2 and 4.

Here are tables of values and sketches of $y = 2^x$ and $y = 4^x$. On each graph, the tangent at the y-intercept $A(0, 1)$ has been drawn.

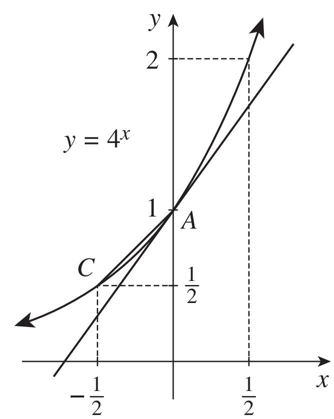
For $y = 2^x$

x	0	1
2^x	1	2



For $y = 4^x$

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
2^x	$\frac{1}{2}$	1	2



a **i** In the first diagram, find the gradient of the chord AB.
ii Hence explain why the tangent at A has gradient less than 1.

b **i** In the second diagram, find the gradient of the chord CA.
ii Hence explain why the tangent at A has gradient greater than 1.

c Use these two results to explain why $2 < e < 4$.

5 [Technology]
This question requires graphing software.

a Use graphing software to graph $y = 2^x$ and $y = x + 1$ on the same number plane, and hence observe that the gradient of $y = 2^x$ at $(0, 1)$ is less than 1.

b Similarly, graph $y = 3^x$ and $y = x + 1$ on the same number plane and hence observe that the gradient of $y = 3^x$ at $(0, 1)$ is greater than 1.

c Choose a sequence of bases between 2 and 3 that make the line $y = x + 1$ more and more like a tangent to the curve at $(0, 1)$ — that is, the gradient at $(0, 1)$ becomes closer and closer to 1. In this way a reasonable approximation for e can be obtained.

CHALLENGE

6 [Technology]
At the start of this section it was shown by first-principles differentiation that the gradient of $y = 2^x$ at its y-intercept is given by $\lim\limits_{h \to 0} \dfrac{2^h - 1}{h}$. The investigation in Question **1** showed that the limit is about 0.7. Use a calculator or a spreadsheet to evaluate $\dfrac{2^h - 1}{h}$, correct to five decimal places, at least for the following values of h:

a 1 **b** 0.1 **c** 0.01 **d** 0.001 **e** 0.0001 **f** 0.00001

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9B Transformations of exponential functions

Transformations of exponential functions with bases other than e were discussed in Chapter 7. This short exercise first gives you practice using the calculator to evaluate powers of e, and is then concerned with transformations of $y = e^x$.

Using the calculator to approximate powers of e

A calculator will provide approximate values of e^x correct to about ten significant figures. Most calculators label the function $\boxed{e^x}$ and locate it above the button labelled $\boxed{\ln}$ where pressing $\boxed{\text{SHIFT}}$ first may be required.

Example 1 — 9B

Use your calculator to find, correct to four significant figures:

a e^2 **b** e^{-4} **c** $e^{\frac{1}{3}}$

SOLUTION

Using the function labelled $\boxed{e^x}$ on the calculator:

a $e^2 \doteqdot 7.389$ **b** $e^{-4} \doteqdot 0.01832$ **c** $e^{\frac{1}{3}} \doteqdot 1.396$

The next three examples must first be put into index form before using the calculator. In particular, e itself must be written as e^1, so that an approximation for the number e can be found using the function $\boxed{e^x}$ with input $x = 1$.

Example 2 — 9B

Write each as a power of e, then approximate it correct to five decimal places.

a e **b** $\dfrac{1}{e}$ **c** $\sqrt{e}$

SOLUTION

a $e = e^1$
$\doteqdot 2.71828$

b $\dfrac{1}{e} = e^{-1}$
$\doteqdot 0.36788$

c $\sqrt{e} = e^{\frac{1}{2}}$
$\doteqdot 1.64872$

Transformations of $y = e^x$

The usual methods of shifting and reflecting graphs can be applied to $y = e^x$. When the graph is shifted vertically, the horizontal asymptote at $y = 0$ will be shifted also. A small table of approximate values can be a very useful check.

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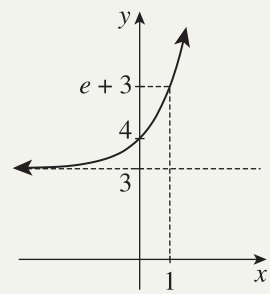
Example 3 9B

Use transformations of the graph of $y = e^x$, confirmed by a table of values, to generate a sketch of each function. Show and state the y-intercept and the horizontal asymptote, and state the range.

a $y = e^x + 3$ **b** $y = e^{-x}$ **c** $y = e^{x-2}$

SOLUTION

a Graph $y = e^x + 3$ by shifting $y = e^x$ up 3 units.



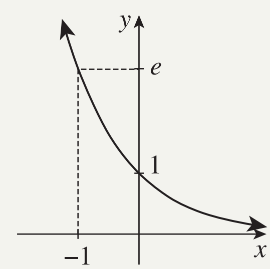
x	-1	0	1
y	$e^{-1} + 3$	4	$e + 3$
approximation	3.37	4	5.72

y-intercept: $(0, 4)$

asymptote: $y = 3$

range: $y > 3$

b Graph $y = e^{-x}$ by reflecting $y = e^x$ in the y-axis.



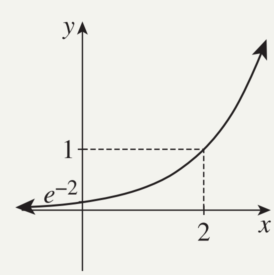
x	-1	0	1
y	e	1	e^{-1}
approximation	2.72	1	0.37

y-intercept: $(0, 1)$

asymptote: $y = 0$ (the x-axis)

range: $y > 0$

c Graph $y = e^{x-2}$ by shifting $y = e^x$ to the right by 2 units.



x	0	1	2	3
y	e^{-2}	e^{-1}	1	e
approximation	0.14	0.37	1	2.72

y-intercept: $(0, e^{-2})$

asymptote: $y = 0$ (the x-axis)

range: $y > 0$

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Exercise 9B

FOUNDATION

Technology: The transformations of the graph of the exponential function $y = e^x$ in Questions **4–7** and **9**, and the procedure in Question **8**, can be confirmed using graphing software. Experimentation with further graphs can then be done.

1 Use the button labelled $\boxed{e^x}$ on your calculator to approximate the following correct to four decimal places.

a e^2 **b** e^3 **c** e^{10}
d e^0 **e** e^1 **f** e^{-1}
g e^{-2} **h** $e^{\frac{1}{2}}$ **i** $e^{-\frac{1}{2}}$

2 Write each expression as a power of e. Then approximate it correct to four significant figures.

a e **b** $\dfrac{1}{e}$ **c** $\dfrac{1}{e^4}$
d $\sqrt{e}$ **e** $\sqrt[3]{e}$ **f** $\dfrac{1}{\sqrt{e}}$

3 Write using powers of e. Then approximate correct to four significant figures.

a $5e^2$ **b** $\frac{1}{64}e^6$ **c** $7\sqrt{e}$
d $\frac{3}{5}\sqrt{e}$ **e** $\dfrac{4}{e}$ **f** $\dfrac{5}{7e^4}$

4 Sketch the graph of $y = e^x$, then use your knowledge of transformations to graph the following functions, showing the horizontal asymptote. For each function, state the transformation, give the equation of the asymptote, and state the range.

a $y = e^x + 1$ **b** $y = e^x + 2$ **c** $y = e^x - 1$ **d** $y = e^x - 2$

5 a Copy and complete the following tables of values for the functions $y = e^x$ and $y = e^{-x}$, giving your answers correct to two decimal places.

x	-2	-1	0	1	2
e^x					

x	-2	-1	0	1	2
e^{-x}					

b Sketch both graphs on one number plane, and draw the tangents at each y-intercept.

c What transformation exchanges these graphs?

d We saw that the tangent to $y = e^x$ at its y-intercept has gradient 1. What is the gradient of $y = e^{-x}$ at its y-intercept? Explain why the two tangents are perpendicular.

6 a Sketch the graph of $y = e^{-x}$. State the horizontal asymptote, and the range.

b Then use your knowledge of transformations to graph each function below. State the transformation of $y = e^{-x}$, give the equation of the asymptote, and state the range.

i $y = e^{-x} + 1$ **ii** $y = e^{-x} + 2$ **iii** $y = e^{-x} - 1$ **iv** $y = e^{-x} - 2$

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DEVELOPMENT

7 **a** What transformation maps $y = e^x$ to $y = e^{x-1}$?
b Sketch $y = e^{x-1}$.
c Similarly sketch these functions:
i $y = e^{x-3}$ **ii** $y = e^{x+1}$ **iii** $y = e^{x+2}$

8 Let A, B, C and D be the points on $y = e^x$ with x-coordinates 0, 1, 2 and 3.
a Find the y-coordinates of A, B, C and D, and sketch the situation.
b **i** Find the gradient of the chord AB.
ii Hence use $y - y_1 = m(x - x_1)$ to write down the equation of the chord AB.
iii Put $y = 0$ to show that the x-intercept of the chord AB is $-\dfrac{1}{e-1}$.
c Repeat the procedure with the chord BC, and show that its x-intercept is $1 - \dfrac{1}{e-1}$.
d Repeat the procedure with the chord CD, and show that its x-intercept is $2 - \dfrac{1}{e-1}$.

CHALLENGE

9 Use the graph of $y = e^x$ and your knowledge of transformations to graph the following functions. Show the horizontal asymptote and state the range in each case.
a $y = 1 - e^x$ **b** $y = 3 - e^x$ **c** $y = -e^{-x}$ **d** $y = 1 - e^{-x}$

10 [A calculator or spreadsheet investigation]
The function e^x may be approximated by adding up a few terms of the infinite power series

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \times 2} + \frac{x^3}{1 \times 2 \times 3} + \frac{x^4}{1 \times 2 \times 3 \times 4} + \cdots.$$

Use this power series to approximate each of the following, correct to two decimal places. Then compare your answers with those obtained on your calculator.
a e **b** e^{-1} **c** e^2 **d** e^{-2}

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9C Differentiation of exponential functions

Now that the new standard form $\frac{d}{dx}e^{x} = e^{x}$ has been established, the familiar chain, product and quotient rules can now be applied to functions involving e^{x}. Those procedures, however, will be developed in Year 12. At this stage, all that is necessary is a simple chain-rule extension to the standard form

$$\frac{d}{dx}e^{ax+b} = a\,e^{ax+b}.$$

Standard forms for differentiating e^{ax+b}

The chain rule can be applied to differentiating functions such as $y = e^{3x+4}$, where the index $3x + 4$ is a linear function.

Example 4 — 9C

Use the chain rule to differentiate these functions.

a $y = e^{3x+4}$ **b** $y = e^{ax+b}$

SOLUTION

a Let $y = e^{3x+4}$.

Then using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= 3\,e^{3x+4}.\end{aligned}$$

Let $u = 3x + 4$.
Then $y = e^{u}$.
Hence $\frac{du}{dx} = 3$
and $\frac{dy}{du} = e^{u}$.

b Let $y = e^{ax+b}$.

Then using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx}\\ &= a\,e^{ax+b}.\end{aligned}$$

Let $u = ax + b$.
Then $y = e^{u}$.
Hence $\frac{du}{dx} = a$
and $\frac{dy}{du} = e^{u}$.

This situation occurs so often that the result should be learnt as a standard form.

5 STANDARD FORMS FOR DIFFERENTIATING EXPONENTIAL FUNCTIONS

$$\frac{d}{dx}e^{x} = e^{x}$$

$$\frac{d}{dx}e^{ax+b} = a\,e^{ax+b}$$

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Example 5 9C

Use the standard forms above to find the derivatives of these functions.

a $y = e^{4x-7}$ **b** $y = 2e^{-9x}$ **c** $y = e^{\frac{1}{2}(6-x)}$

SOLUTION

Each function needs to be written in the form $y = e^{ax+b}$.

a $y = e^{4x-7}$

$\frac{dy}{dx} = 4e^{4x-7}$ (here $a = 4$ and $b = -7$)

b $y = 2e^{-9x}$

$\frac{dy}{dx} = -18e^{-9x}$ (here $a = -9$ and $b = 0$)

c $y = e^{\frac{1}{2}(6-x)}$

$y = e^{3-\frac{1}{2}x}$ (expand the brackets in the index)

$\frac{dy}{dx} = -\frac{1}{2}e^{3-\frac{1}{2}x}$ (here $a = -\frac{1}{2}$ and $b = 3$)

EXERCISE 9C FOUNDATION

Technology: Programs that perform algebraic differentiation can be used to confirm the answers to many of these exercises.

1 Use the standard form $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = e^{2x}$ **b** $y = e^{7x}$ **c** $y = e^{-x}$ **d** $y = -e^{5x}$

e $y = e^{\frac{1}{2}x}$ **f** $y = 6e^{\frac{1}{3}x}$ **g** $y = e^{-\frac{1}{3}x}$ **h** $y = 5e^{\frac{1}{5}x}$

2 Using the standard form $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $f(x) = e^{x+2}$ **b** $f(x) = e^{x-3}$ **c** $f(x) = e^{5x+1}$ **d** $f(x) = e^{2x-1}$

e $f(x) = e^{-4x+1}$ **f** $f(x) = e^{-3x+4}$ **g** $f(x) = e^{-3x-6}$ **h** $f(x) = 2e^{\frac{1}{2}x+4}$

3 Differentiate these functions.

a $f(x) = e^{x} + e^{-x}$ **b** $f(x) = e^{2x} - e^{-3x}$ **c** $f(x) = \frac{e^{2x}}{2} + \frac{e^{3x}}{3}$

d $f(x) = \frac{e^{4x}}{4} + \frac{e^{5x}}{5}$ **e** $f(x) = \frac{e^{x} - e^{-x}}{2}$ **f** $f(x) = \frac{e^{x} + e^{-x}}{3}$

4 **a** Differentiate $y = e^{2x}$.

b Hence find $\frac{dy}{dx}$ in exact form when $x = 0$ and when $x = 4$.

5 **a** Differentiate $f(x) = e^{-x+3}$.

b Hence find $f'(x)$ in exact form when $x = 0$ and when $x = 4$.

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DEVELOPMENT

6 For each function, first find $\frac{dy}{dx}$. Then evaluate $\frac{dy}{dx}$ at $x = 2$, giving your answer first in exact form and then correct to two decimal places.

a $y = e^{3x}$ **b** $y = e^{-2x}$ **c** $y = e^{\frac{3}{2}x}$

7 **a** For the function $f(x) = e^{-x}$, write down $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.
b What is the pattern in these derivatives?

8 **a** For the function $f(x) = e^{2x}$, write down $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.
b What is the pattern in these derivatives?

9 **a** For the function $f(x) = e^{x}$, write down $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.
b What is the pattern in these derivatives?

10 What happens when $y = e^x + x^2 + x + 1$ is differentiated successively?

11 Differentiate:

a $f(x) = e^{5x} + e^{7x}$ **b** $f(x) = e^{4x+2} + e^{5+8x}$
c $f(x) = 4e^{-x} + 4e^{-3x}$ **d** $f(x) = 6e^{-2x-3} - 7e^{5-6x}$
e $f(x) = 5x^2 - 4x - 3e^{-x}$ **f** $f(x) = e^{\frac{1}{2}x} + x^{\frac{1}{2}}$

12 Use the standard form $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = e^{ax}$ **b** $y = e^{-kx}$ **c** $y = Ae^{kx}$ **d** $y = Be^{-\ell x}$

CHALLENGE

13 Write each expression as a simple power of e, and then differentiate it.

a $y = \sqrt{e^x}$ **b** $y = \sqrt[3]{e^x}$ **c** $y = \frac{1}{\sqrt{e^x}}$ **d** $y = \frac{1}{\sqrt[3]{e^x}}$

14 Use the standard form $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = Ce^{px+q}$ **b** $\frac{e^{ax}}{a} + \frac{e^{-px}}{p}$

15 **a** Show by substitution that the function $y = e^{5x}$ satisfies the equation $\frac{dy}{dx} = 5y$.

b Show by substitution that the function $y = 3e^{2x}$ satisfies the equation $\frac{dy}{dx} = 2y$.

c Show by substitution that the function $y = 5e^{-4x}$ satisfies the equation $\frac{dy}{dx} = -4y$.

16 Define two new functions, $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$.

a Show that $\frac{d}{dx}\cosh x = \sinh x$ and $\frac{d}{dx}\sinh x = \cosh x$.
b Find the second derivative of each function, and show that they both satisfy $y'' = y$.
c Show that $\cosh^2 x - \sinh^2 x = 1$.

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9D Differentiation and the graph

Differentiation can now be applied as usual to the graphs of functions whose equations involve e^x. This section uses the derivative to deal with the tangents and normals to such graphs.

The graphs of e^x and e^{-x}

The graphs of $y = e^x$ and $y = e^{-x}$ are fundamental to this whole course. Before dealing with further graphs, it is worth reviewing the most basic results obtained so far in this chapter.

6 REVIEW OF THE MOST BASIC FACTS ABOUT $y = e^x$

- The real number $e \doteqdot 2.7183$ is defined to be the base so that the exponential function $y = e^x$ has gradient 1 at its y-intercept.
- The function $y = e^x$ is it own derivative. That is,
$$\frac{d}{dx} e^x = e^x.$$
- We have obtained the further standard form,
$$\frac{d}{dx} e^{ax+b} = a e^{ax+b}.$$

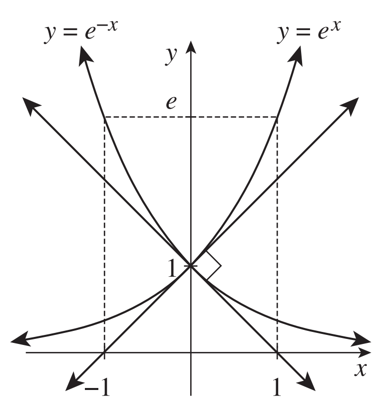
Because x has been replaced by $-x$ in the second equation, the two graphs $y = e^x$ and $y = e^{-x}$ are reflections of each other in the y-axis.

For $y = e^x$:

x	-2	-1	0	1	2
y	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2

For $y = e^{-x}$:

x	-2	-1	0	1	2
y	e^2	e	1	$\frac{1}{e}$	$\frac{1}{e^2}$



The two curves cross at $(0, 1)$.

By the definition of e, the gradient of $y = e^x$ at its y-intercept $(0, 1)$ is 1.

Hence by reflection, the gradient of $y = e^{-x}$ at its y-intercept $(0, 1)$ is -1.

Thus the curves are perpendicular at their point of intersection.

Note: The function $y = e^{-x}$ is as important as $y = e^x$ in applications, or even more important. It describes a great many physical situations where a quantity 'dies away exponentially', such as the dying away of the sound of a plucked string.

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Example 6 9D

a Differentiate $y = e^x - x$.
b For what value of x is the curve stationary?

SOLUTION

a Differentiating, $\dfrac{dy}{dx} = e^x - 1$ (the derivative of e^x is e^x).

b Put $\dfrac{dy}{dx} = 0$ to find where the curve is stationary,

$$\begin{aligned} e^x - 1 &= 0 \\ e^x &= 1 \\ x &= 0. \end{aligned}$$

The geometry of tangents and normals

Example 7 shows how to work with tangents and normals of exponential functions.

Remember that the word ‘normal’ means ‘perpendicular’. The *normal* to a curve at a point P on it is the line through P perpendicular to the tangent at P.

Example 7 9D

Let A be the point on the curve $y = 2e^x$ where $x = 1$.
a Find the equation of the tangent to the curve at the point A.
b Show that the tangent at A passes through the origin.
c Find the equation of the normal at the point A.
d Find the point B where this normal crosses the x-axis.
e Find the area of ΔAOB.

SOLUTION

a The given function is $y = 2e^x$
and differentiating, $y' = 2e^x$.
When $x = 1$, $y = 2e^1$
$= 2e$ (this is the y-coordinate of A)
and also when $x = 1$, $y' = 2e^1$
$= 2e$ (this is the gradient of the tangent at A)

so A has coordinates $A(1, 2e)$ and the tangent at A has gradient $2e$.

Hence, using point–gradient form, the tangent at A is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2e &= 2e(x - 1) \\ y &= 2ex. \end{aligned}$$

b Because its y-intercept is zero, the tangent passes through the origin.

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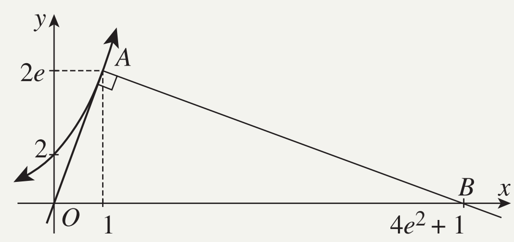
c From part **a**, the tangent at A has gradient $2e$,

so the normal has gradient $-\dfrac{1}{2e}$ (it is perpendicular to the tangent).

Thus the normal at A is $y - 2e = -\dfrac{1}{2e}(x - 1)$

$$2ey - 4e^2 = -x + 1$$
$$x + 2ey = 4e^2 + 1.$$



d To find the x-intercept, put $y = 0$,

thus $x = 4e^2 + 1$,

so B has coordinates $(4e^2 + 1, 0)$.

e Hence area $\Delta AOB = \dfrac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times (4e^2 + 1) \times 2e$$
$$= e(4e^2 + 1) \text{ square units.}$$

Exercise 9D

FOUNDATION

Point–gradient form: Most questions in this exercise require the point–gradient form of the equation of a line. A line with gradient m passing through (x_1, y_1) has equation

$$y - y_1 = m(x - x_1).$$

1 **a** Use calculus to find the gradient of the tangent to $y = e^x$ at $Q(0, 1)$.
b Hence find the equation of the tangent at Q, and prove that it passes through $A(-1, 0)$.

2 **a** Use calculus to find the gradient of the tangent to $y = e^x$ at $P(1, e)$.
b Hence find the equation of the tangent at P, and prove that it passes through O.

3 **a** Use calculus to find the gradient of the tangent to $y = e^x$ at $R\left(-1, \frac{1}{e}\right)$.
b Hence find the equation of the tangent at R, and prove that it passes through $B(-2, 0)$.

4 **a** Find the y-coordinate of the point A on the curve $y = e^{2x-1}$ where $x = \frac{1}{2}$.
b Find the derivative of $y = e^{2x-1}$, and show that the gradient of the tangent at A is 2.
c Hence find the equation of the tangent at A, and prove that it passes through O.

5 **a** Explain why $y = e^x$ is always increasing.
b Explain why $y = e^{-x}$ is always decreasing.

6 **a** What is the y-coordinate of the point P on the curve $y = e^x - 1$ where $x = 1$?
b Find $\dfrac{dy}{dx}$ for this curve, and the value of $\dfrac{dy}{dx}$ when $x = 1$.
c Hence find the equation of the tangent at P.
d For what values of x is the curve:
i stationary, **ii** increasing, **iii** decreasing.

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DEVELOPMENT

7 **a** Write down the coordinates of the point R on the curve $y = e^{3x+1}$ where $x = -\frac{1}{3}$.

b Find $\dfrac{dy}{dx}$ and hence show that the gradient of the tangent at R is 3.

c What is the gradient of the normal at R?

d Hence find the equation of the normal at R in general form.

8 **a** Find the gradient of the tangent to $y = e^{-x}$ at the point $P(-1, e)$.

b Thus write down the gradient of the normal at this point.

c Hence determine the equation of this normal.

d Find the x- and y-intercepts of the normal.

e Find the area of the triangle whose vertices lie at the intercepts and the origin.

9 **a** Use the derivative to find the gradient of the tangent to $y = e^x$ at $B(0, 1)$.

b Hence find the equation of this tangent and show that it meets the x-axis at $F(-1, 0)$.

c Use the derivative to find the gradient of the tangent to $y = e^{-x}$ at $B(0, 1)$.

d Hence find the equation of this tangent and show that it meets the x-axis at $G(1, 0)$.

e Sketch $y = e^x$ and $y = e^{-x}$ on the same set of axes, showing the two tangents.

f What sort of triangle is ΔBFG, and what is its area?

10 Find the gradient, and the angle of inclination correct to the nearest minute, of the tangent to $y = e^x$ at the points where:

a $x = 0$ **b** $x = 1$ **c** $x = -2$ **d** $x = 5$

11 A curve is defined by $y = e^{2x-4}$. The points A and B on it have x-coordinates 1 and 2. Find the coordinates of A and B. Then find the gradients of:

a the tangent at A, **b** the tangent at B, **c** the chord AB.

CHALLENGE

12 **a** Find the equation of the tangent to $y = e^x$ at $x = t$.

b Hence show that the x-intercept of this tangent is $t - 1$.

c Compare this result with those of Questions **2** and **3**.

13 **a** Show that the equation of the tangent to $y = 1 - e^{-x}$ at the origin is $y = x$.

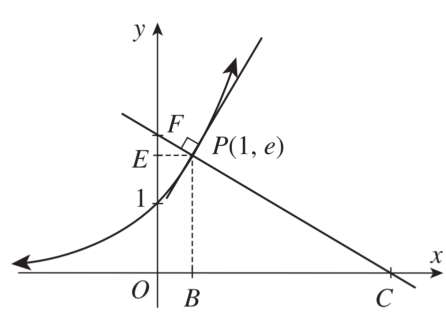
b Deduce the equation of the normal at the origin without further use of calculus.

c What is the equation of the asymptote of this curve?

d Sketch the curve, showing the points T and N where the tangent and the normal respectively cut the asymptote.

e Find the area of ΔOTN.

14 On the graph of $y = e^x$ are drawn the tangent and the normal to the curve at the point $P(1, e)$.



a Show that the tangent at P passes through the origin.

b Find the coordinates of each of the points B, C, E and F.

c Hence show that:

i $OB = 1$ **ii** $BC = e^2$ **iii** $OE = e$ **iv** $EF = \dfrac{1}{e}$

d **i** What is the area of ΔOCP? **ii** What is the area of ΔOFP?

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9E The logarithmic function base e

We have seen that $e \doteqdot 2.7183$ is the most natural base to use for exponential functions in calculus. Work with the exponential function $y = e^x$ necessarily involves work with the logarithmic function $\log_e x$, because it is the inverse function of $y = e^x$. Thus e is also the most natural base to use for logarithms in calculus.

This section introduces $y = \log_e x$ and its graph, describing its relationship with the graph of $y = e^x$, and the transformations of its graph.

The logarithmic function

Because e is the natural base for logarithms, logarithms base e are called *natural logarithms*, and the function $y = \log_e x$, is called ***the*** *logarithmic function* to distinguish it from other logarithmic functions such as $\log_2 x$ and $\log_{10} x$ that have other bases.

7 THE LOGARITHMIC FUNCTION

- Logarithms base e are called *natural logarithms*.
- The logarithmic function with base e,

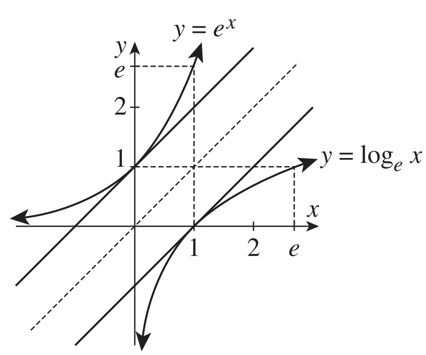
  $$y = \log_e x$$

  is called ***the*** *logarithmic function* to distinguish it from all other logarithmic functions $y = \log_a x$.

Below are tables of values and the graphs of the inverse functions $y = e^x$ and $y = \log_e x$. You can see that they are inverse functions because the rows are swapped. These are two of the most important graphs in the course.

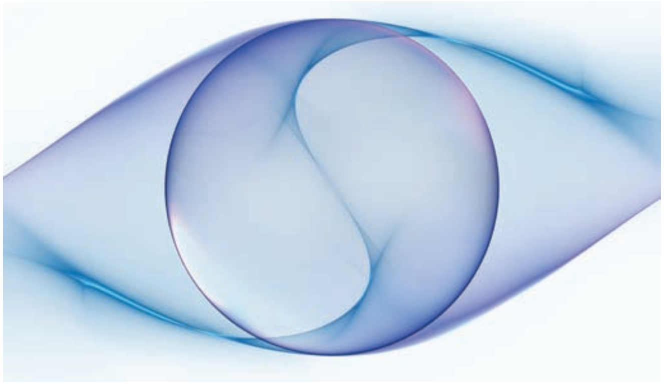
x	-2	-1	0	1	2
e^x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2

x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2
$\log_e x$	-2	-1	0	1	2



We know already that the tangent to $y = e^x$ at its y-intercept $(0, 1)$ has gradient 1.

When this graph is reflected in the line $y = x$, this tangent is reflected to a tangent to $y = \log_e x$ at its x-intercept $(1, 0)$. Both these tangents therefore have gradient 1.



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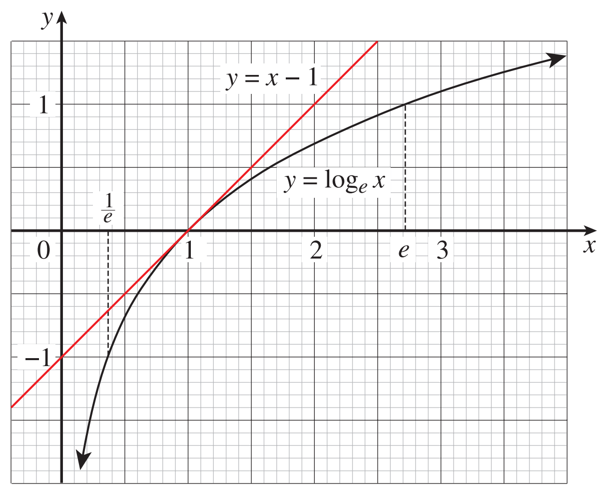
Properties of the graph of $y = \log_e x$

The graph of $y = \log_e x$ alone is drawn below on graph paper. The tangent has been drawn at the x-intercept $(1, 0)$ to show that the gradient of the curve there is exactly 1.



The graph of $y = \log_e x$ and its properties must be thoroughly known. Its properties correspond to the properties listed earlier of its inverse function $y = e^x$.

- The domain is $x > 0$.
  The range is all real y.
- The y-axis $x = 0$ is a vertical asymptote to the curve.
- As $x \to 0^+$, $y \to -\infty$.
  As $x \to \infty$, $y \to \infty$.
- The curve has gradient 1 at its x-intercept $(1, 0)$.
- The curve is always concave down.

The notations for the logarithmic function — $\log_e x$, $\log x$ and $\ln x$

In calculus, and in mathematics generally, $\log_e x$ is the only logarithmic function that matters, and is more often written simply as $\log x$. The problem is that many calculators, and even some mathematical software, contradict this mathematics convention, and use an engineering convention, where '$\log x$' means $\log_{10} x$.

The function $\log_e x$ is also written as $\ln x$, where the 'n' stands for 'natural logarithms'. The 'n' also stands for 'Napierian logarithms', in honour of the Scottish mathematician John Napier (1550–1617), who first developed tables of logarithms for calculations (first published in 1614).

The most sensible way to deal with this confusion is:

- Never write '$\log x$' without a base — always write either $\log_{10} x$ or $\log_e x$.
- Use either $\log_e x$ or $\ln x$ for logarithms base e.

Note: Be careful of the different convention on calculators, where $\boxed{\text{log}}$ means $\log_{10} x$, and the function labelled $\boxed{\text{ln}}$ is used to find logarithms base e. The function $\boxed{e^x}$ is usually located on the same button as $\boxed{\text{ln}}$ because the two functions $y = e^x$ and $y = \log_e x$ are inverses of each other.

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8 THE NOTATION FOR THE LOGARITHMIC FUNCTION

- In this text, logarithms base e are written as $\log_e x$, and less often as $\ln x$.
- On calculators, however:
  - ln is used to approximate $\log_e x$. It is the inverse function of e^x.
  - log is used to approximate $\log_{10} x$. It is the inverse function of 10^x.

Example 8 9E

a Use your calculator to find, correct to four significant figures:

i $\log_e 10$ **ii** $\log_e \frac{1}{10}$ **iii** $\log_e 100$

b How are the answers to parts **ii** and **iii** related to the answer to part **i**?

SOLUTION

a Using the function labelled ln on the calculator,

i $\log_e 10 \doteqdot 2.303$ **ii** $\log_e \frac{1}{10} \doteqdot -2.303$ **iii** $\log_e 100 \doteqdot 4.605.$

b Using the log laws:

$$\log_e \frac{1}{10} = \log_e 10^{-1} = -\log_e 10 \qquad \text{and} \qquad \log_e 10^2 = 2\log_e 10,$$

and these relationships are clear from the approximations above.

Combining the logarithmic and exponential functions

As with any base, when the logarithmic and exponential functions base e are applied successively to a number, the result is the original number.

9 THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS ARE INVERSE FUNCTIONS

$$\log_e e^x = x \qquad \text{and} \qquad e^{\log_e x} = x$$

These identities follow immediately from the definition of the logarithmic function as the inverse function of the exponential function.

Example 9 9E

Use the functions labelled ln and w on your calculator to demonstrate that:

a $\log_e e^{10} = 10$ **b** $e^{\log_e 10} = 10$

SOLUTION

a $\log_e e^{10} = \log_e 22\,026.46\ldots$
$\doteqdot 10$

b $e^{\log_e 10} = e^{2.302\,585\ldots}$
$\doteqdot 10$

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Transformations of the logarithmic graph

The usual methods of transforming graphs can be applied to $y = \log_e x$. When the graph is shifted sideways, the vertical asymptote at $x = 0$ will also be shifted.

A small table of approximate values can be a very useful check, particulary when a sequence of transformations is involved. Remember that $y = \log_e x$ has domain $x > 0$ and that the vertical asymptote is at $x = 0$.

Example 10 9E

Use transformations of the graph of $y = \log_e x$, confirmed by a table of values, to generate a sketch of each function. State the transformation used, and write down the domain, the x-intercept, and the vertical asymptote.

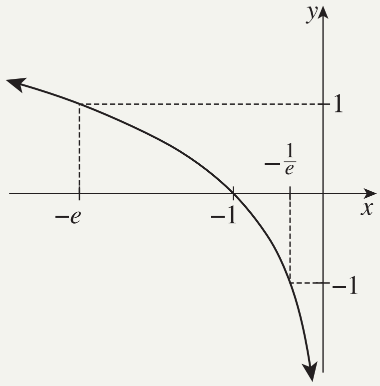
a $y = \log_e(-x)$ **b** $y = \log_e x - 2$ **c** $y = \log_e(x + 3)$

SOLUTION

a Graph $y = \log_e(-x)$ by reflecting $y = \log_e x$ in the y-axis.

x	$-e$	-1	$-\frac{1}{e}$
y	1	0	-1

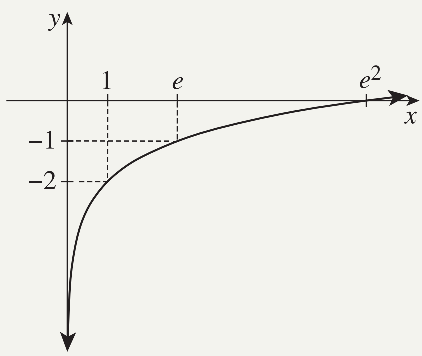
domain: $x < 0$
x-intercept: $(-1, 0)$
asymptote: $x = 0$ (the y-axis)



b Graph $y = \log_e x - 2$, which is $y + 2 = \log_e x$, by shifting $y = \log_e x$ down 2 units.

x	$\frac{1}{e}$	1	e	e^2
y	-3	-2	-1	0

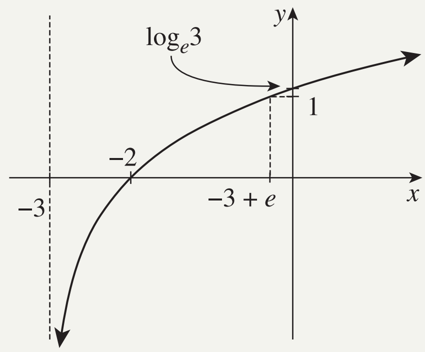
domain: $x > 0$
x-intercept: $(e^2, 0)$
asymptote: $x = 0$ (the y-axis)



c Graph $y = \log_e(x + 3)$ by shifting $y = \log_e x$ left 3 units.

x	$\frac{1}{e} - 3$	-2	$e - 3$
y	-1	0	1

domain: $x > -3$
x-intercept: $(-2, 0)$
asymptote: $x = -3$



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Exercise 9E

FOUNDATION

Note: Remember that $\ln x$ means $\log_e x$, the natural logarithm. On the calculator, the key $\boxed{\ln}$ means $\log_e x$, and the key $\boxed{\log}$ unfortunately means $\log_{10} x$.

1 Use your calculator to approximate the following, correct to four decimal places where necessary. Read the note above and remember to use the $\boxed{\ln}$ key on the calculator.

a $\log_e 1$ **b** $\log_e 2$ **c** $\ln 3$ **d** $\ln 8$

e $\log_e \frac{1}{2}$ **f** $\log_e \frac{1}{3}$ **g** $\ln \frac{1}{8}$ **h** $\ln \frac{1}{10}$

2 Use the functions labelled $\boxed{\ln}$ and $\boxed{e^x}$ on your calculator to demonstrate that:

a $\log_e e^2 = 2$ **b** $\log_e e^3 = 3$ **c** $\log_e e^1 = 1$

d $\log_e e^{-2} = -2$ **e** $\log_e e^{-3} = -3$ **f** $\log_e e^{-1} = -1$

3 Use the functions labelled $\boxed{\ln}$ and $\boxed{e^x}$ on your calculator to demonstrate that:

a $e^{\log_e 2} = 2$ **b** $e^{\log_e 3} = 3$ **c** $e^{\log_e 1} = 1$

d $e^{\log_e 10} = 10$ **e** $e^{\log_e \frac{1}{2}} = \frac{1}{2}$ **f** $e^{\log_e \frac{1}{10}} = \frac{1}{10}$

4 Solve each equation of x by first rewriting it as an index statement using the definition:

$$x = \log_e a \quad \text{means} \quad e^x = a.$$

a $x = \log_e 1$ **b** $x = \log_e e$ **c** $x = \log_e e^2$

d $x = \log_e \frac{1}{e}$ **e** $x = \log_e \frac{1}{e^2}$ **f** $x = \log_e \sqrt{e}$

5 Simplify each expression using the logarithm law $\log_e a^n = n \log_e a$, then evaluate it using the fact that $\log_e e = 1$.

a $\log_e e^2$ **b** $\log_e e^5$ **c** $\log_e e^{200}$ **d** $\log_e e^{-6}$ **e** $\log_e \frac{1}{e^6}$

f $\log_e e^{-1}$ **g** $\log_e \frac{1}{e}$ **h** $\log_e e^{\frac{1}{2}}$ **i** $\log_e \sqrt{e}$ **j** $\log_e \frac{1}{\sqrt{e}}$

DEVELOPMENT

6 Sketch the graph of $y = \log_e x$, then use your knowledge of transformations to graph the following functions. Note that in each case the y-axis is a vertical asymptote and the domain is $x > 0$.

a $y = \log_e x + 1$ **b** $y = \log_e x + 2$ **c** $y = \ln x - 1$ **d** $y = \ln x - 2$

7 a Copy and complete the following tables of values for the functions $y = \log_e x$ and $y = -\log_e x$, giving your answers correct to two decimal places.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_e x$					

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$-\log_e x$					

b Sketch both graphs on the same number plane, and draw the tangent to each at the x-intercept.

c It was shown in the text that the tangent to $y = \log_e x$ at its y-intercept has gradient 1. Use this fact to find the gradient of $y = -\log_e x$ at its y-intercept, and hence explain why the two curves meet at right angles.

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8 Use the log laws to simplify:

a $e \log_e e$ **b** $\frac{1}{e}\log_e \frac{1}{e}$ **c** $3\log_e e^2$

d $\log_e \sqrt{e}$ **e** $e\log_e e^3 - e\log_e e$ **f** $\log_e e + \log_e \frac{1}{e}$

9 Express as a single logarithm:

a $\log_e 3 + \log_e 2$ **b** $\log_e 100 - \log_e 25$

c $\log_e 2 - \log_e 3 + \log_e 6$ **d** $\log_e 54 - \log_e 10 + \log_e 5$

10 Use the graph of $y = \log_e x$ and your knowledge of transformations to graph the following functions. Show the vertical asymptote and state the domain in each case.

a $y = \log_e(x-1)$ **b** $y = \log_e(x-3)$ **c** $y = \log_e(x+1)$

d $y = \log_e(x+2)$ **e** $y = -\log_e x$ **f** $y = \log_e(-x)$

11 Use the log laws and the change-of-base formula to prove:

a $\log_e \frac{a}{b} = -\log_e \frac{b}{a}$ **b** $\log_{\frac{1}{e}} x = -\log_e x$ **c** $\log_{\frac{1}{e}} x^{-1} = \log_e x$

CHALLENGE

12 Sketch the graph of $y = -\log_e x$ and use your knowledge of transformations to graph the following functions. Note that in each case the y-axis is a vertical asymptote and the domain is $x > 0$.

a $y = -\log_e x + 1$ **b** $y = -\log_e x + 2$

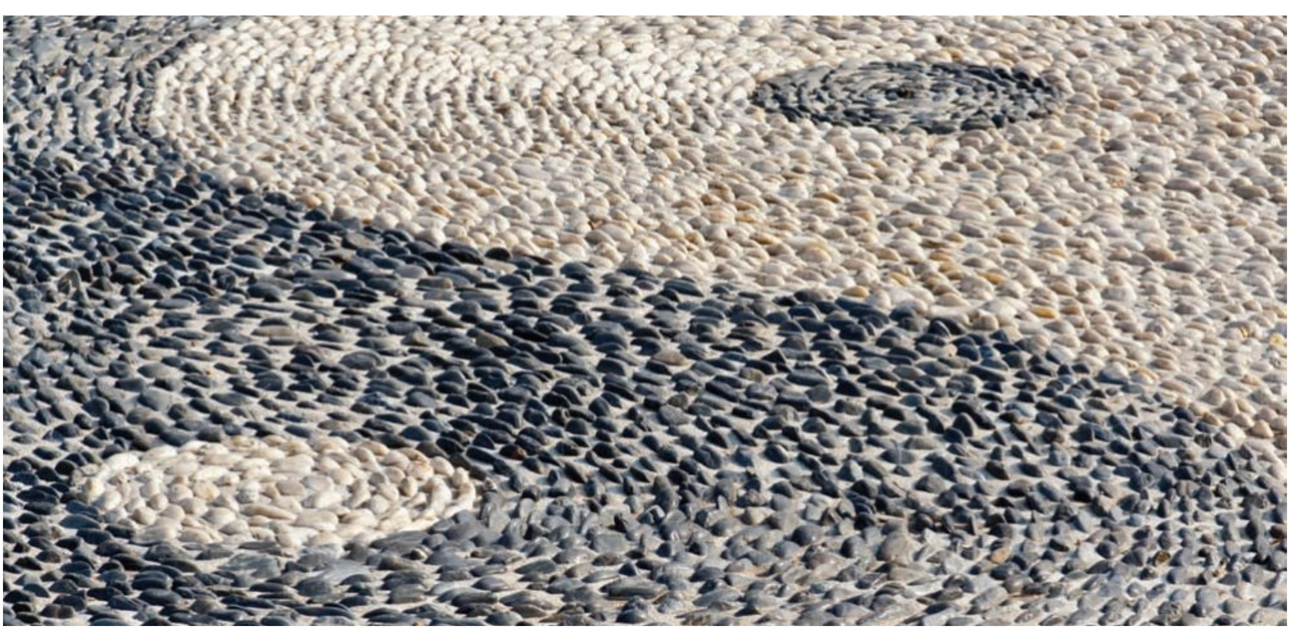
c $y = -\log_e x - 1$ **d** $y = -\log_e x - 2$

13 For $-1 < x \le 1$, the function $\log_e(1+x)$ may be approximated using the *power series*

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Use this power series to approximate each of the following, correct to two decimal places. Then compare your answers with those obtained on your calculator.

a $\log_e 1\frac{1}{2}$ **b** $\log_e \frac{5}{4}$ **c** $\log_e \frac{1}{2}$ **d** $\log_e \frac{1}{3}$



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9F Exponential rates using the base *e*

Now that we have e^x, we can differentiate an exponential function and find the rate at which a quantity is changing at a particular time.

This short exercise is only concerned with rates involving e^{kx}, where k is a given constant. The topic will be developed further next year.

Example 11 9F

The number B of bacteria in a laboratory culture is growing according to the formula $B = 4000\,e^{0.1t}$, where t is the time in hours after the experiment was started. Answer these questions correct to three significant figures:

a How many bacteria are there after 5 hours?

b Find the rate $\dfrac{dB}{dt}$ at which the number of bacteria is increasing.

c At what rate are the bacteria increasing after 5 hours?

d What is the average rate of increase over the first 5 hours?

e When are there 10000 bacteria?

f When are the bacteria increasing by 10000 per hour?

SOLUTION

a When $t = 5$,
$$\begin{aligned} B &= 4000 \times e^{0.5} \\ &\doteqdot 6590 \text{ bacteria.} \end{aligned}$$

b Differentiating, $\dfrac{dB}{dt} = 400\,e^{0.1t}$.

c When $t = 5$,
$$\begin{aligned} \frac{dB}{dt} &= 400 \times e^{0.5} \\ &\doteqdot 659 \text{ bacteria per hour} \end{aligned}$$

d When $t = 0$, $B = 4000 \times e^{0} = 4000$, and when $t = 5$, $B \doteqdot 6590$.

$$\begin{aligned} \text{Hence average rate} &= \frac{B_2 - B_1}{t_2 - t_1} \\ &\doteqdot \tfrac{2590}{5} \\ &\doteqdot 518 \text{ bacteria per hour.} \end{aligned}$$

e Put $B = 10000$.

$$\begin{aligned} \text{Then} \quad 4000\,e^{0.1t} &= 10000 \\ e^{\frac{1}{10}t} &= \tfrac{5}{2} \\ \tfrac{1}{10}t &= \log_e 2.5 \\ t &= 10 \log_e 2.5 \\ &\doteqdot 9.16 \text{ hours.} \end{aligned}$$

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f Put $\dfrac{dB}{dt} = 10\,000.$

Then
$$\begin{aligned} 400\,e^{0.1t} &= 10\,000 \\ e^{\frac{1}{10}t} &= 25 \\ \tfrac{1}{10}t &= \log_e 25 \\ t &= 10 \log_e 25 \\ &\doteqdot 32.2 \text{ hours.} \end{aligned}$$

Exercise 9F

FOUNDATION

Note: Give approximations correct to four significant figures, unless otherwise stated.

1 **a** A quantity Q is given by $Q = 300\,e^{3t}$. Find the rate of change $\dfrac{dQ}{dt}$ of Q at time t.

b Evaluate Q and $\dfrac{dQ}{dt}$ when $t = 2$.

c Find the average rate of change from $t = 0$ to $t = 2$.

2 **a** A quantity Q is given by $Q = 10\,000\,e^{-2t}$. Find the rate of change $\dfrac{dQ}{dt}$ of Q at time t.

b Evaluate Q and $\dfrac{dQ}{dt}$ when $t = 4$.

c Find the average rate of change from $t = 0$ to $t = 4$.

3 **a** A quantity Q is given by $Q = 5\,e^{2t}$. Find the rate of change $\dfrac{dQ}{dt}$ of Q at time t.

b Find when the quantity Q is 1000.

c Find when the rate $\dfrac{dQ}{dt}$ is 1000.

DEVELOPMENT

4 A population P is varying with time t in hours according to the formula $P = 2000\,e^{0.3t}$.

a Find the population when $t = 5$.

b Find, as a function of t, the rate $\dfrac{dP}{dt}$ at which the population is changing.

c Find the rate at which the population is changing when $t = 5$.

d Find the average rate of change over the first 5 hours.

5 The concentration $C = 2000\,e^{-2t}$ of a chemical is varying with time t in years.

a Find the concentration when $t = 2$.

b Find, as a function of t, the rate $\dfrac{dC}{dt}$ at which the concentration is changing.

c Find the rate at which the concentration is changing when $t = 2$.

d Find the average rate of change over the first 2 years.

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6 The price $\$P$ of an item is rising with inflation according to the formula $P = 150e^{0.04t}$, where t is the time in years.

a When will the price be \$300?

b Find the rate $\dfrac{dP}{dt}$ at which the price is rising.

c When will the price be rising at \$300 per year?

7 The population P of cats on Snake Island is $P = 1000e^{0.4t}$, where t is the time in years after time zero when the population was first estimated.

a Find the rate at which the population is increasing.

b Find the population, and the rate of increase, when $t = 5$. Give both answers correct to two significant figures.

c When will the cat population reach 20000 cats, correct to one decimal place?

d When will the rate of increase reach 20000 cats per year, correct to one decimal place?

8 **a** Explain why $Q = e^t$ is always increasing at an increasing rate.

b Explain why $Q = e^{-t}$ is always decreasing at a decreasing rate.

CHALLENGE

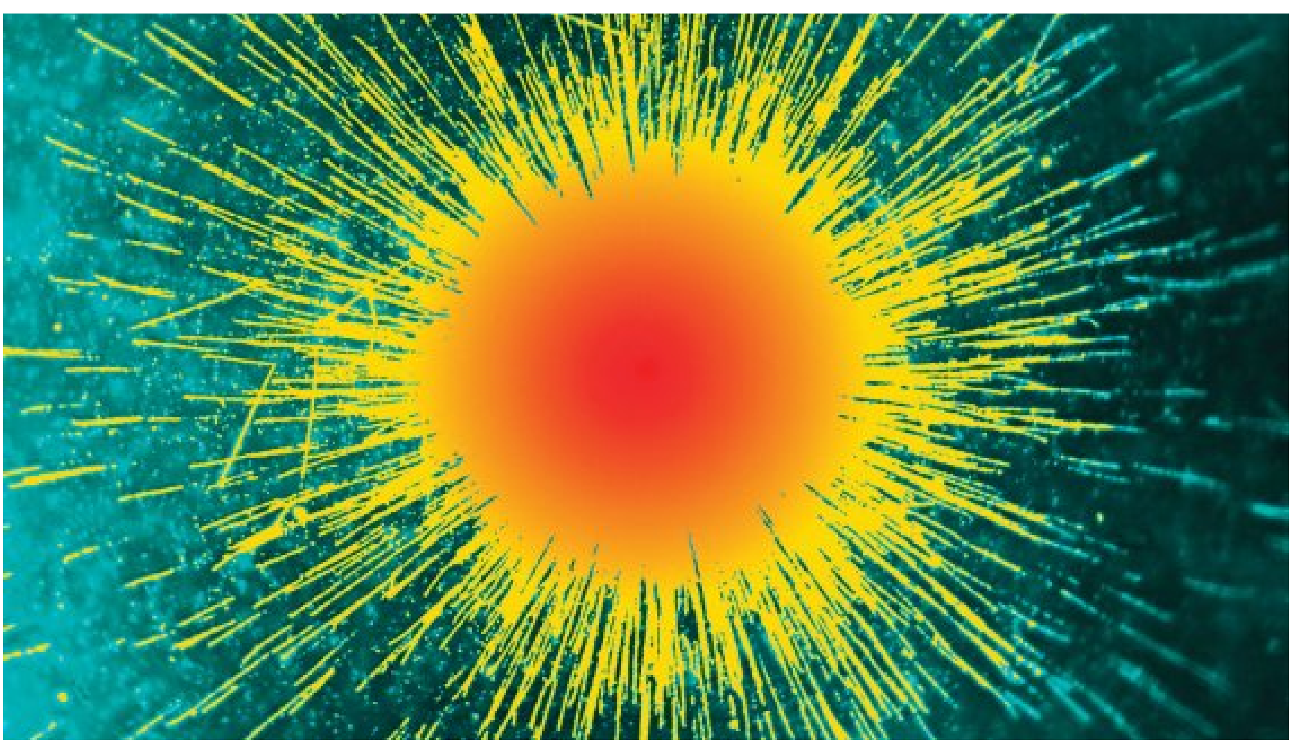
9 A radioactive substance decays in such a way that the mass M of a sample t years after it has been measured and stored is $M = M_0e^{-0.1t}$, where M_0 is its original mass.

a How long until half the sample is gone?

b Find the rate $\dfrac{dM}{dt}$ at which the sample is decaying.

c What percentage of the remaining sample decays each year?

d When will the rate of decay be 1% of the original mass M_0 per year?



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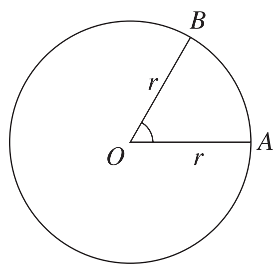
9G Radian measure of angle size

This section begins the second part of the chapter. It introduces a new way of measuring angle size in *radians*, based on the number π. Radian measure of angles will be needed next year for the calculus of the trigonometric functions.

The use of degrees to measure angle size is based on astronomy, not on mathematics. There are 360 days in the year, correct to the nearest convenient whole number, so $1°$ is the angle through which the Sun moves against the fixed stars each day. (After the work of Copernicus and Galileo, $1°$ is the angle swept out by the Earth each day in its orbit around the Sun.) Mathematics is far too general a discipline to be tied to the particularities of our solar system, so it is quite natural to develop a new system for measuring angles based on mathematics alone.

Radian measure of angle size



The size of an angle in radians is defined as the ratio of two lengths in a circle.

Given an angle with vertex O, construct a circle with centre O meeting the two arms of the angle at A and B.

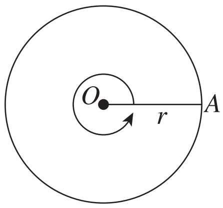
The size of $\angle AOB$ in radians is the ratio of the arc length AB and the radius OA.

10 RADIAN MEASURE

$$\text{Size of } \angle AOB = \frac{\text{arc length } AB}{\text{radius } OA}$$

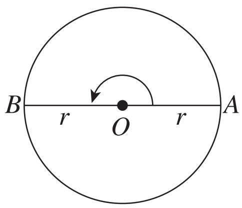
This definition gives the same angle size, whatever the radius of the circle, because all circles are similar to one another.

The arc subtended by a revolution is the whole circumference of the circle,



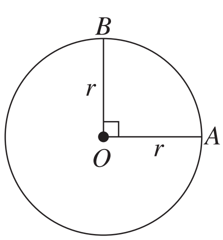
$$\begin{aligned}\text{so 1 revolution} &= \frac{\text{arc length}}{\text{radius}}\\ &= \frac{\text{circumference}}{\text{radius}}\\ &= \frac{2\pi r}{r} = 2\pi.\end{aligned}$$

Similarly, a straight angle subtends a semicircle,



$$\begin{aligned}\text{so 1 straight angle} &= \frac{\text{arc length of semicircle}}{\text{radius}}\\ &= \frac{\pi r}{r} = \pi.\end{aligned}$$

A right angle subtends a quarter-circle,



$$\begin{aligned}\text{so 1 right angle} &= \frac{\text{arc length of quarter-circle}}{\text{radius}}\\ &= \frac{\frac{1}{2}\pi r}{r} = \frac{\pi}{2}.\end{aligned}$$

These three basic conversions should be memorised very securely.

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11 CONVERSIONS BETWEEN DEGREES AND RADIAN MEASURE

$$360^\circ = 2\pi \qquad 180^\circ = \pi \qquad 90^\circ = \frac{\pi}{2}$$

Because $180^\circ = \pi$ radians, an angle size in radians can be converted to an angle size in degrees by multiplying by $\dfrac{180^\circ}{\pi}$.

Conversely, degrees are converted to radians by multiplying by $\dfrac{\pi}{180}$.

12 CONVERTING BETWEEN DEGREES AND RADIANS

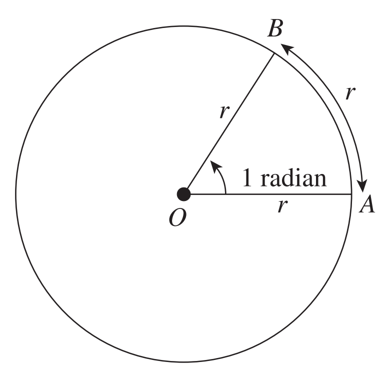
- To convert from radians to degrees,

  multiply by $\dfrac{180^\circ}{\pi}$.
- To convert from degrees to radians,

  multiply by $\dfrac{\pi}{180}$.
- One radian and one degree:

$$1 \text{ radian} = \frac{180^\circ}{\pi} \doteqdot 57^\circ 18' \qquad \text{and} \qquad 1 \text{ degree} = \frac{\pi}{180} \doteqdot 0.0175.$$

One radian is the angle subtended at the centre of a circle by an arc of length equal to the radius. Notice that the sector OAB in the diagram to the right is almost an equilateral triangle, so 1 radian is about 60°. This makes sense of the value given above, that 1 radian is about 57°.



Note: The size of an angle in radians is a ratio of lengths, so it is a dimensionless real number. It is therefore unnecessary to mention radians. For example, 'an angle of size 1.3' means an angle of 1.3 radians.

This definition of angle size is very similar to the definitions of the six trigonometric functions, which are also ratios of lengths and so are also pure numbers.

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Example 12 9G

Express these angle sizes in radians.

a $60°$ **b** $270°$ **c** $495°$ **d** $37°$

SOLUTION

a $60° = 60 \times \dfrac{\pi}{180}$

$= \dfrac{\pi}{3}$

b $270° = 270 \times \dfrac{\pi}{180}$

$= \dfrac{3\pi}{2}$

c $495° = 495 \times \dfrac{\pi}{180}$

$= \dfrac{11\pi}{4}$

d $37° = 37 \times \dfrac{\pi}{180}$

$= \dfrac{37\pi}{180}$

Example 13 9G

Express these angle sizes in degrees. Give exact answers, and then where appropriate give answers correct to the nearest degree.

a $\dfrac{\pi}{6}$ **b** 0.3 **c** $\frac{3\pi}{4}$ **d** 20

SOLUTION

a $\dfrac{\pi}{6} = \dfrac{\pi}{6} \times \dfrac{180°}{\pi}$

$= 30°$

b $0.3 = \dfrac{3}{10} \times \dfrac{180°}{\pi}$

$= \dfrac{54°}{\pi}$

$\doteqdot 17°$

c $\dfrac{3\pi}{4} = \dfrac{3\pi}{4} \times \dfrac{180°}{\pi}$

$= 135°$

d $20 = 20 \times \dfrac{180°}{\pi}$

$= \dfrac{3600°}{\pi}$

$\doteqdot 1146°$

Evaluating trigonometric functions of special angles

The trigonometric function of an angle is the same whether the angle size is given in degrees or radians.

With angles whose related angle is one of the three special angles, it is a matter of recognising the special angles

$$\frac{\pi}{6} = 30° \qquad \text{and} \qquad \frac{\pi}{4} = 45° \qquad \text{and} \qquad \frac{\pi}{3} = 60°.$$

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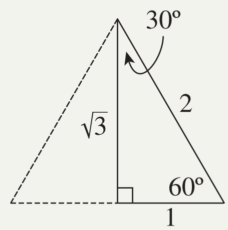
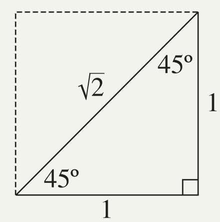
Example 14 9G

Evaluate these trigonometric functions, using the special triangles shown:

a $\sin \frac{\pi}{6}$

b $\operatorname{cosec} \frac{\pi}{4}$





SOLUTION

a $\sin \frac{\pi}{6} = \sin 30°$
$= \frac{1}{2}$

b $\operatorname{cosec} \frac{\pi}{4} = \dfrac{1}{\sin \frac{\pi}{4}}$
$= \dfrac{1}{\sin 45°}$
$= \sqrt{2}$

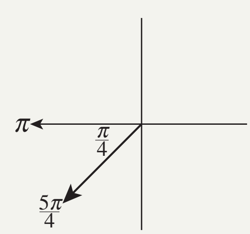
Example 15 [Angles whose related angle is a special angle] 9G

Use special angles to evaluate these trigonometric functions.

a $\sin \frac{5\pi}{4}$ **b** $\sec \frac{11\pi}{6}$

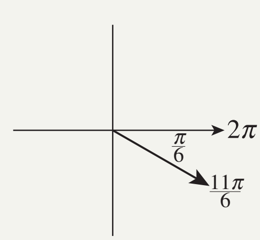
SOLUTION

a Because $\frac{5\pi}{4}$ is in the third quadrant, with related angle $\frac{\pi}{4}$,



$\sin \frac{5\pi}{4} = -\sin \frac{\pi}{4}$
$= -\sin 45°$
$= -\frac{1}{\sqrt{2}}$.

b Because $\frac{11\pi}{6}$ is in the fourth quadrant, with related angle $\frac{\pi}{6}$,



$\sec \frac{11\pi}{6} = +\sec \frac{\pi}{6}$
$= \dfrac{1}{\cos \frac{\pi}{6}}$
$= \dfrac{1}{\cos 30°}$
$= \dfrac{2}{\sqrt{3}}$.

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Approximating trigonometric functions of other angles

For other angles, exact values of the trigonometric functions cannot normally be found. When calculator approximations are required, *it is vital to set the calculator to radians mode first*.

Your calculator has a key labelled mode or something similar to make the change — calculators set to the wrong mode routinely cause havoc at this point!

13 SETTING THE CALCULATOR TO RADIANS MODE OR DEGREES MODE

From now on, always decide whether the calculator should be in radians mode or degrees mode before using any of the trigonometric functions.

Example 16 9G

Evaluate correct to four decimal places:

a $\cos 1$ **b** $\cot 1.3$

SOLUTION

Here the calculator must be set to radians mode.

a $\cos 1 \doteqdot 0.5403$

b $\cot 1.3 = \dfrac{1}{\tan 1.3}$
$\doteqdot 0.2776$

Exercise 9G FOUNDATION

Note: Be very careful throughout the remainder of this chapter whether your calculator is set in radians or degrees. The button used to make the change is usually labelled mode.

1 Express the following angles in radians.

a $90°$ **b** $45°$ **c** $30°$ **d** $60°$
e $120°$ **f** $150°$ **g** $135°$ **h** $225°$
i $360°$ **j** $300°$ **k** $270°$ **l** $210°$

2 Express the following angles in degrees.

a π **b** 2π **c** 4π **d** $\dfrac{\pi}{2}$
e $\dfrac{\pi}{3}$ **f** $\dfrac{\pi}{4}$ **g** $\dfrac{2\pi}{3}$ **h** $\dfrac{5\pi}{6}$
i $\dfrac{3\pi}{4}$ **j** $\dfrac{3\pi}{2}$ **k** $\dfrac{4\pi}{3}$ **l** $\dfrac{7\pi}{4}$
m $\dfrac{11\pi}{6}$

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3 Use your calculator in radian mode to evaluate, correct to two decimal places:
a sin 1 **b** cos 2 **c** tan 3
d sin 0.7 **e** cos 1.23 **f** tan 5.678

4 Use your calculator to express these angles in radians, correct to three decimal places:
a $73°$ **b** $14°$ **c** $168°$ **d** $21°36'$ **e** $95°17'$ **f** $211°12'$

5 Use your calculator to express these angles in degrees and minutes, correct to the nearest minute:
a 2 radians **b** 0.3 radians **c** 1.44 radians
d 0.123 radians **e** 3.1985 radians **f** 5.64792 radians

DEVELOPMENT

6 Using the two special triangles, find the exact value of:
a $\sin\frac{\pi}{6}$ **b** $\sin\frac{\pi}{4}$ **c** $\cos\frac{\pi}{6}$ **d** $\tan\frac{\pi}{3}$
e $\tan\frac{\pi}{4}$ **f** $\cos\frac{\pi}{3}$ **g** $\sec\frac{\pi}{4}$ **h** $\cot\frac{\pi}{3}$

7 Express these angles in radians in terms of π:
a $20°$ **b** $22.5°$ **c** $36°$ **d** $100°$ **e** $112.5°$ **f** $252°$

8 Express these angles in degrees:
a $\dfrac{\pi}{12}$ **b** $\dfrac{2\pi}{5}$ **c** $\dfrac{20\pi}{9}$ **d** $\dfrac{11\pi}{8}$ **e** $\dfrac{17\pi}{10}$ **f** $\dfrac{23\pi}{15}$

9 **a** Find the complement of $\frac{\pi}{6}$.
b Find the supplement of $\frac{\pi}{6}$.

10 Two angles of a triangle are $\frac{\pi}{3}$ and $\frac{2\pi}{9}$. Find, in radians, the third angle.

11 Using the two special triangles and your knowledge of angles of any magnitude, find the exact value of:
a $\sin\frac{2\pi}{3}$ **b** $\cos\frac{2\pi}{3}$ **c** $\cos\frac{5\pi}{6}$ **d** $\tan\frac{4\pi}{3}$
e $\tan\frac{3\pi}{4}$ **f** $\cos\frac{5\pi}{3}$ **g** $\sin\frac{5\pi}{4}$ **h** $\tan\frac{7\pi}{6}$

12 Give each answer in degrees and in radians.
a What angle do the hour and minute hands of a clock move through in 1 hour?
b Find the non-reflex angle between the hour and minute hands of a clock at:
i 2:00 pm **ii** 2:15 pm **iii** 2:30 pm **iv** 2:45 pm

CHALLENGE

13 If $f(x) = \sin x$, $g(x) = \cos 2x$ and $h(x) = \tan 3x$, find, correct to three significant figures:
a $f(1) + g(1) + h(1)$ **b** $f\Big(g\big(h(1)\big)\Big)$

14 Find, correct to three decimal places, the angle in radians through which:
a the second hand of a clock turns in 7 seconds,
b the hour hand of a clock turns between 6 am and 6:40 am.

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9H Solving trigonometric equations

Solving trigonometric equations in radians

Solving a trigonometric equation is done in exactly the same way whether the solution is to be given in radians or degrees.

14 SOLVING A TRIGONOMETRIC EQUATION IN RADIANS

- First establish on a diagram which quadrants the angle can lie in.
- Then find the related angle — but use radian measure, not degrees.
- Use the diagram and the related angle to find all the answers, taking account of any restrictions on the angle.

Example 17 [Acute angles] 9H

Solve each trigonometric equation in radians, where the angle x is an acute angle. Give each answer in exact form if possible, otherwise give an approximation correct to five significant figures.

a $\sin x = \frac{1}{2}$ **b** $\tan x = 3$

SOLUTION

With acute angles, a quadrants diagram is not necessary.

a $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ (the special angle 30°, which is $\frac{\pi}{6}$ radians)

b $\tan x = 3$

$x \doteqdot 1.2490$ (use the calculator here)

Example 18 [General angles] 9H

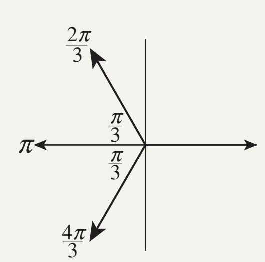
Solve these trigonometric equations. Give answers in exact form if possible, otherwise give the answer correct to five significant figures.

a $\cos x = -\frac{1}{2}$, where $0 \le x \le 2\pi$ **b** $\sin x = -\frac{1}{3}$, where $0 \le x \le 2\pi$

Note: When using the calculator's inverse trigonometric functions, *do not work with a negative number*. Always enter the absolute value of the number in order to find the related angle.

SOLUTION

a $\cos x = -\frac{1}{2}$, where $0 \le x \le 2\pi$.



Because $\cos x$ is negative, x is in quadrant 2 or 3.

The acute angle whose cosine is $+\frac{1}{2}$ is the special angle 60°, which in radian measure is $\frac{\pi}{3}$.

Hence, from the diagram to the right,

$$x = \pi - \frac{\pi}{3} \quad \text{or} \quad \pi + \frac{\pi}{3}$$
$$= \frac{2\pi}{3} \quad \text{or} \quad \frac{4\pi}{3}.$$

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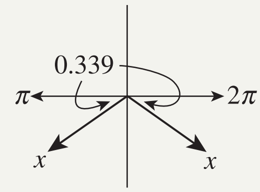
b $\sin x = -\frac{1}{3}$, where $0 \le x \le 2\pi$.

Because $\sin x$ is negative, x is in quadrant 3 or 4.

With the calculator in radians mode, enter $+\frac{1}{3}$,

then the related angle is 0.339 836 … (store this in memory).

Hence, from the diagram to the right,



$$x = \pi + 0.339836\ldots \quad \text{or} \quad 2\pi - 0.339836\ldots$$
$$\doteqdot 3.4814 \quad \text{or} \quad 5.9433$$

Solving trigonometric equations reducible to quadratics

Some trigonometric equations are quadratic equations in a trigonometric function. Some such equations can be solved directly, but it is often useful to make an algebraic substitution. Both methods are used in the following examples.

Example 19 9H

Solve these trigonometric equations in radians, for $0 \le x \le 2\pi$.

a $\sin^2 x = \frac{3}{4}$ **b** $2\cos^2 x - \cos x - 1 = 0$

SOLUTION

a This is easily done without any substitution.

$$\sin^2 x = \tfrac{3}{4}$$
$$\sin x = \tfrac{\sqrt{3}}{2} \quad \text{or} \quad \sin x = -\tfrac{\sqrt{3}}{2}$$

The related angle is $\frac{\pi}{3}$, and the angle can be in any of the four quadrants,

so, $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

b Because of the factorising required, substitution is a better approach here.

$$2\cos^2 x - \cos x - 1 = 0$$

Let $u = \cos x$.

Then $2u^2 - u - 1 = 0$, thus 'reducing' it to a quadratic equation.

$$2u^2 - 2u + u - 1 = 0$$
$$2u(u - 1) + (u - 1) = 0$$
$$(2u + 1)(u - 1) = 0$$
$$u = -\tfrac{1}{2} \quad \text{or} \quad u = 1$$
$$\cos x = -\tfrac{1}{2} \quad \text{or} \quad \cos x = 1.$$

When $\cos x = -\frac{1}{2}$, the related angle is $\frac{\pi}{3}$, and the angle is in quadrants 2 or 3,

so $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

When $\cos x = 1$, the solutions are boundary angles, so we use the graph.

From the graphs in Chapter 5 page 171, the answers are 0° and 360°.

Converting to radians, or using the graphs in radians in the later Section 9J (page 397),

$$x = 0 \quad \text{or} \quad 2\pi.$$

Hence the solutions are $x = 0$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or 2π.

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Exercise 9H

FOUNDATION

1 Find, in radians, the acute angle θ that satisfies each equation.

a $\tan\theta = 1$ b $\sin\theta = \frac{1}{2}$ c $\cos\theta = \frac{1}{\sqrt{2}}$

d $\tan\theta = \frac{1}{\sqrt{3}}$ e $\sin\theta = \frac{\sqrt{3}}{2}$ f $\cos\theta = \frac{1}{2}$

2 Find, correct to three decimal places, the value of x between 0 and $\frac{\pi}{2}$ that satisfies each equation. (Your calculator needs to be in radians mode.)

a $\tan x = 3$ b $\sin x = 0.8$ c $\cos x = 0.4$

d $\sin x = 0.234$ e $\cos x = 0.987$ f $\tan x = 100$

3 Solve these equations for x over the domain $0 \le x \le 2\pi$:

a $\sin x = \frac{1}{2}$ b $\cos x = -\frac{1}{2}$ c $\tan x = -1$ d $\sin x = 1$

e $2\cos x = \sqrt{3}$ f $\sqrt{3}\tan x = 1$ g $\cos x + 1 = 0$ h $\sqrt{2}\sin x + 1 = 0$

DEVELOPMENT

4 Solve each equation for $0 \le \theta \le 2\pi$. Remember that a positive number has two square roots.

a $\sin^2\theta = 1$ b $\tan^2\theta = 1$ c $\cos^2\theta = \frac{1}{4}$ d $\cos^2\theta = \frac{3}{4}$

5 Consider the equation $\cos^2\theta - \cos\theta = 0$, for $0 \le \theta \le 2\pi$.

a Write the equation as a quadratic equation in u by letting $u = \cos\theta$.

b Solve the quadratic equation for u.

c Hence find the values of θ that satisfy the original equation.

6 Consider the equation $\tan^2\theta - \tan\theta - 2 = 0$, for $0 \le \theta \le 2\pi$.

a Write the equation as a quadratic equation in u by letting $u = \tan\theta$.

b Solve the quadratic equation for u.

c Hence find the values of θ that satisfy the original equation. Give the solutions correct to two decimal places where necessary.

7 Solve these equations for $0 \le \theta \le 2\pi$, by transforming each equation into a quadratic equation in u. Give your solutions correct to two decimal places where necessary.

a $\tan^2\theta + \tan\theta = 0$ (Let $u = \tan\theta$.) b $2\sin^2\theta - \sin\theta = 0$ (Let $u = \sin\theta$.)

c $\sin^2\theta + \sin\theta - 2 = 0$ (Let $u = \sin\theta$.) d $\tan^2\theta + \tan\theta - 6 = 0$ (Let $u = \tan\theta$.)

e $2\cos^2\theta + \cos\theta - 1 = 0$ f $2\sin^2\theta - \sin\theta - 1 = 0$

g $3\sin^2\theta + 8\sin\theta - 3 = 0$ h $3\cos^2\theta - 8\cos\theta - 3 = 0$

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CHALLENGE

8 [Technology]
Graphing programs provide an excellent way to see what is happening when an equation has many solutions. For example, the equations in Question **2** are quite simple to graph, because $y = \text{LHS}$ is a single trigonometric function and $y = \text{RHS}$ is a horizontal line. Every point of intersection corresponds to a solution. Graph the LHS of each equation in question 2, and verify the solutions you obtained.

9 Use the trigonometric identities from Chapter 5 to transform each equation so that it only involves one trigonometric function. Then solve it for $0 \le x \le 2\pi$. Give solutions correct to two decimal places where necessary.

a $2\sin^2 x + \cos x = 2$ **b** $\sec^2 x - 2\tan x - 4 = 0$

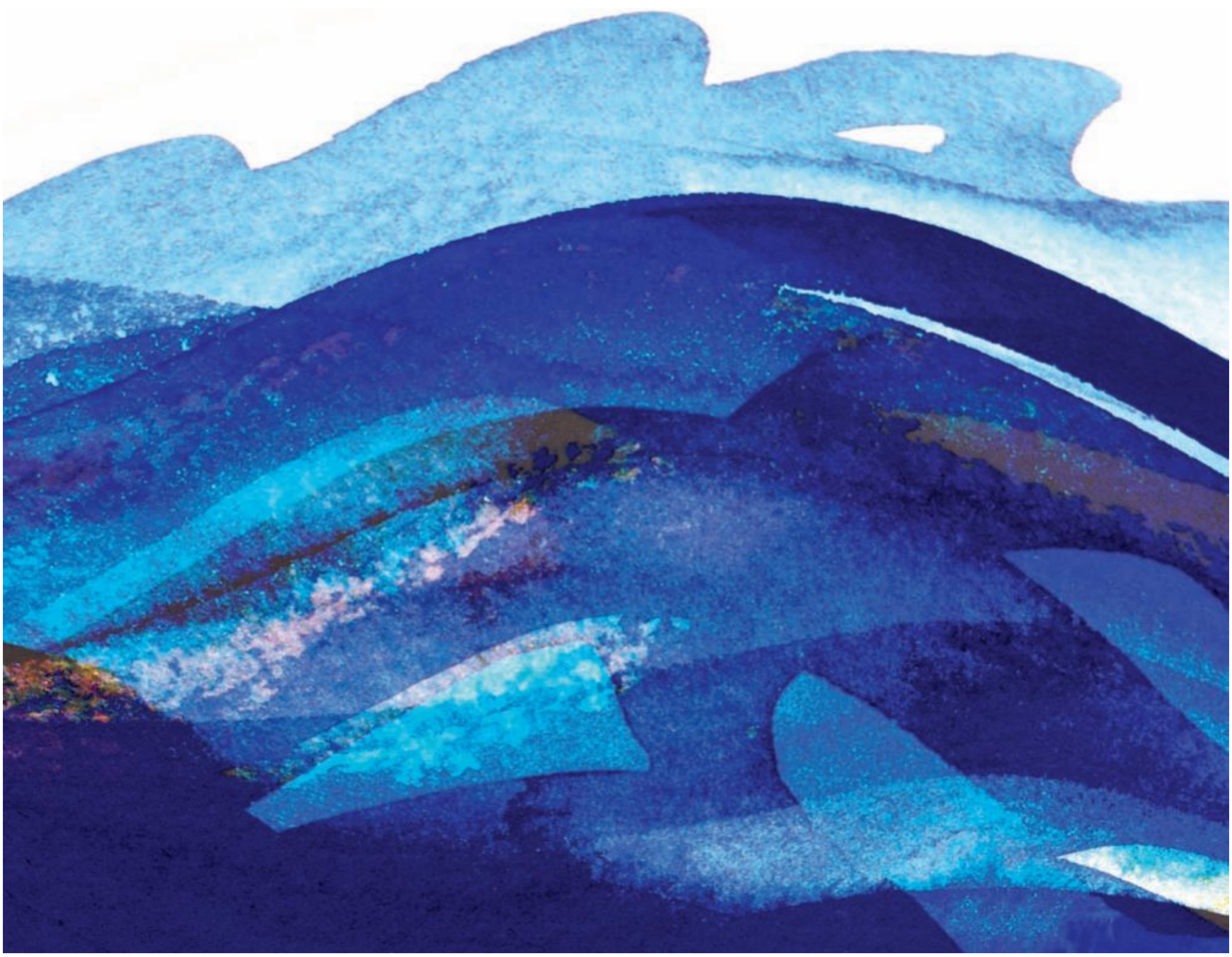
c $8\cos^2 x = 2\sin x + 7$ **d** $6\tan^2 x = 5\sec x$

10 Solve each equation for $0 \le \alpha \le 2\pi$, giving solutions correct to two decimal places where necessary. Again, you will need trigonometric identities.

a $3\sin\alpha = \operatorname{cosec}\alpha + 2$ **b** $3\tan\alpha - 2\cot\alpha = 5$

11 Solve each equation for $0 \le x \le 2\pi$ by first dividing through by $\cos^2 x$. Give solutions correct to two decimal places where necessary.

a $\sin^2 x + \sin x \cos x = 0$ **b** $\sin^2 x - 5\sin x\cos x + 6\cos^2 x = 0$



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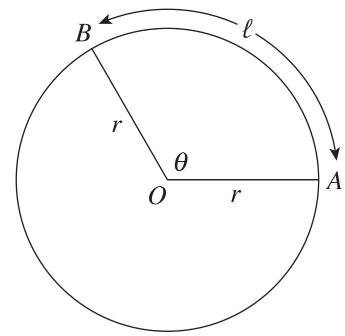
9I Arcs and sectors of circles

The lengths of arcs and the areas of sectors and segments can already be calculated using fractions of circles, but radian measure allows the formulae to be expressed in more elegant forms.

Arc length



In the diagram to the right, the *arc AB* has length ℓ and subtends an angle θ at the centre O of a circle with radius r.

The definition of angle size in radians is $\theta = \dfrac{\ell}{r}$.

Multiplying through by r, $\qquad \ell = r\theta$.

This is the standard formula for arc length.

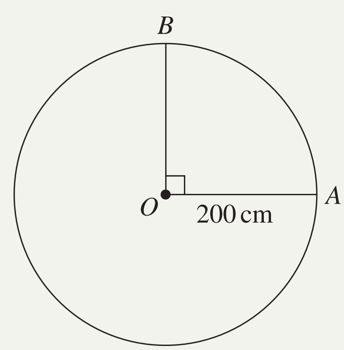
15 ARC LENGTH

An arc subtending an angle θ at the centre of a circle of radius r has length

$$\ell = r\theta$$

Example 20 9I

What is the length of an arc subtending a right angle at the centre of a circle of radius 200 cm?

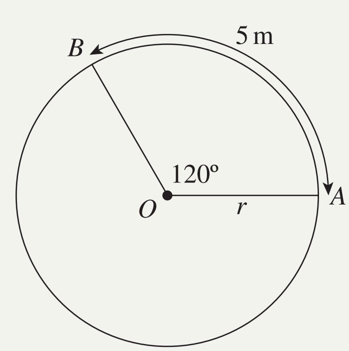


SOLUTION

$$\begin{aligned}\text{Arc length} &= r\theta \\ &= 200 \times \frac{\pi}{2} \quad \left(\text{a right angle has size } \tfrac{\pi}{2}\right) \\ &= 100\pi \text{ cm.}\end{aligned}$$

Example 21 9I

What is the radius of a circle in which an arc of length 5 m subtends an angle of 120° at the centre? Answer correct to the nearest centimetre.



SOLUTION

Substituting into $\ell = r\theta$,

$$5 = r \times \frac{2\pi}{3} \quad \left(120^\circ \text{ in radians is } \tfrac{2\pi}{3}\right)$$

$$\boxed{\times \frac{3}{2\pi}} \qquad r = \frac{15}{2\pi}$$

$$\doteqdot 2.39 \text{ m}$$

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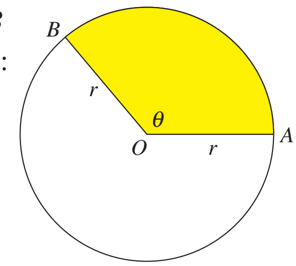
Area of a sector

In the diagram to the right, the *sector AOB* is the shaded area bounded by the arc AB and the two radii OA and OB. Its area can be calculated as a fraction of the total area:



$$\begin{aligned}\text{area of sector} &= \frac{\theta}{2\pi} \times \text{area of circle}\\ &= \frac{\theta}{2\pi} \times \pi r^2\\ &= \tfrac{1}{2} r^2 \theta.\end{aligned}$$

16 AREA OF A SECTOR

An arc subtending an angle θ at the centre of a circle of radius r has area

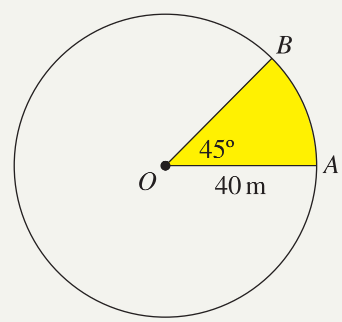
$$\text{Area} = \tfrac{1}{2} r^2 \theta$$

Example 22 9I

What are the area and the perimeter of a sector subtending an angle of 45° at the centre of a circle of radius 40 m?

SOLUTION

$$\begin{aligned}\text{Area of sector} &= \tfrac{1}{2} r^2 \theta\\ &= \frac{1}{2} \times 1600 \times \frac{\pi}{4} \quad \left(45^\circ \text{ in radians is } \tfrac{\pi}{4}\right)\\ &= 200\pi \text{ m}^2.\\ \text{Perimeter} &= \text{arc length} + 2r\\ &= \tfrac{\pi}{4} \times 40 + 2 \times 40\\ &= 10\pi + 80 \text{ m}.\end{aligned}$$

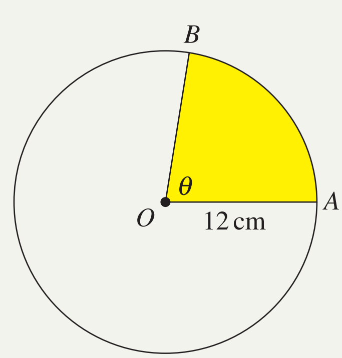


Example 23 9I

A circular cake has radius 12 cm. What is the angle subtended at the centre by a sector of area 100 cm^2? Answer correct to the nearest degree.

SOLUTION

$$\begin{aligned}\text{Area of sector} &= \tfrac{1}{2} r^2 \theta.\\ 100 &= \frac{1}{2} \times 144 \times \theta\\ \boxed{\div 72} \qquad \theta &= \frac{100}{72} \qquad \text{(This answer is in radians.)}\\ &= \frac{25}{18} \times \frac{180^\circ}{\pi} \qquad \text{(Converting to degrees.)}\\ &\doteqdot 80^\circ.\end{aligned}$$



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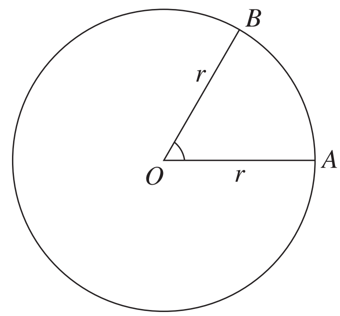
Major and minor arcs and sectors



In the diagram to the right, the phrase 'the arc AB' is ambiguous, because there are two arcs AB.

- The *minor arc AB* is the arc subtending the marked angle at the centre. This angle is less than a straight angle, and the minor arc is less than half the circumference.
- The *major arc AB* is the *opposite arc* subtending the unmarked reflex angle at the centre. This angle is more than a straight angle, and the major arc is more than half the circumference.

The words 'major' and 'minor' are originally Latin words that simply mean 'greater' and 'lesser'. They apply also to sectors in the obvious way — there are two *opposite sectors AOB*: the *minor sector* containing the marked angle, and the *major sector* containing the unmarked reflex angle.

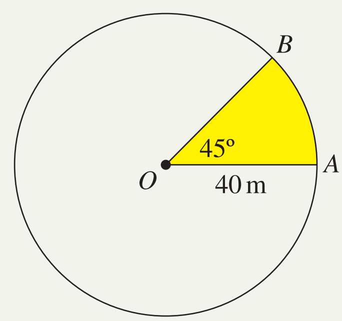
The length of a major arc and the area of a major sector can be calculated with the usual arc length and sector formulae, but using the reflex angle that they subtend, as in the example below. Alternatively, they can be calculated by subtraction from the circumference or area of the whole circle.

Example 24 [The opposite arc and sector of an earlier example] 9I

Find the area and the perimeter of the major sector AOB in the diagram below.

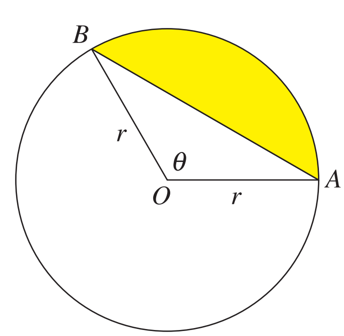
SOLUTION

The major arc and sector each subtend $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ at the centre, and are unshaded in the diagram.



$$\begin{aligned}
\text{Area of sector} &= \tfrac{1}{2}r^2\theta \\
&= \frac{1}{2} \times 1600 \times \frac{7\pi}{4} \qquad \left(315^\circ \text{ in radians is } \tfrac{7\pi}{4}\right) \\
&= 1400\pi \text{ m}^2. \\
\text{Perimeter} &= \text{arc length} + 2r \\
&= \tfrac{7\pi}{4} \times 40 + 2 \times 40 \\
&= 70\pi + 80 \text{ m}.
\end{aligned}$$

Area of a segment



In the diagram to the right, the chord AB divides the circle into two *segments*. The *minor segment* has been shaded, and the rest of the circle is the *major segment*.

Drawing the two radii OA and OB produces an isosceles triangle AOB, together with two opposite sectors. The areas of the two segments can now be found by adding or subtracting appropriate areas.

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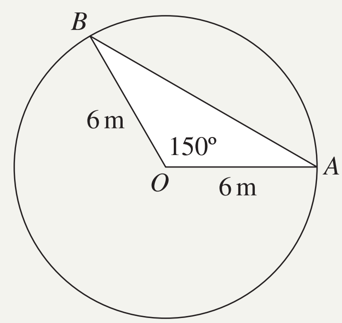
17 AREA OF A SEGMENT

To find the area of a segment:
- Construct the radii from the ends of the chord, to form an isosceles triangle and two opposite sectors.
- Add or subtract the appropriate areas.

Example 25 9I



a Find the lengths of the minor and major arcs formed by two radii of a circle of radius 6 m meeting at 150°.
b Find the areas of the minor and major sectors.
c Find the area of ΔAOB.
d Find the areas of the major and minor segments.

SOLUTION

The minor arc subtends 150° at the centre, which in radians is $\frac{5\pi}{6}$, and the major arc subtends 210° at the centre, which in radians is $\frac{7\pi}{6}$.

a Minor arc $= r\theta$
$= 6 \times \frac{5\pi}{6}$
$= 5\pi$ m.

Major arc $= r\theta$
$= 6 \times \frac{7\pi}{6}$
$= 7\pi$ m.

b Minor sector $= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 6^2 \times \frac{5\pi}{6}$
$= 15\pi$ m^2.

Major sector $= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 6^2 \times \frac{7\pi}{6}$
$= 21\pi$ m^2.

c Area of $\Delta AOB = \frac{1}{2}r^2 \sin\theta$
$= \frac{1}{2} \times 6^2 \times \sin 150°$ $\left(\text{alternatively } \sin\frac{5\pi}{6}\right)$
$= \frac{1}{2} \times 36 \times \frac{1}{2}$
$= 9$ m^2.

d The minor segment area is obtained by subtraction.

Minor segment = minor sector − ΔAOB
$= 15\pi - 9$ m^2.

The major segment area is obtained by addition.

Major segment = major sector + ΔAOB
$= 21\pi + 9$ m^2.

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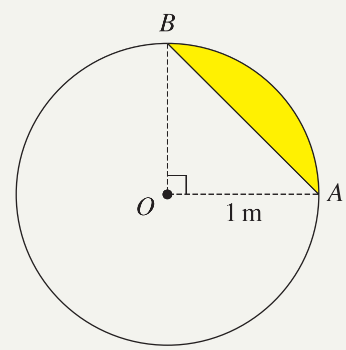
Example 26 9I

In a circle of radius 1 m, what is the area of a segment subtending a right angle at the centre? Give your answer correct to the nearest cm^2.

SOLUTION



First, area of sector $AOB = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 10000 \times \frac{\pi}{2}$

$= 2500\pi$.

Secondly, area of $\Delta AOB = \frac{1}{2}r^2 \sin\theta$

$= \frac{1}{2} \times 10000 \times \sin\frac{\pi}{2}$

$= 5000$ because $\sin\frac{\pi}{2} = 1$.

Hence area of segment $= 2500\pi - 5000$

$\doteqdot 2854\ \text{cm}^2$.

Example 27 9I

Find, correct to the nearest mm, the radius of a circle in which:

a a sector, **b** a segment,

of area 1 square metre subtends an angle of 90° at the centre of the circle.

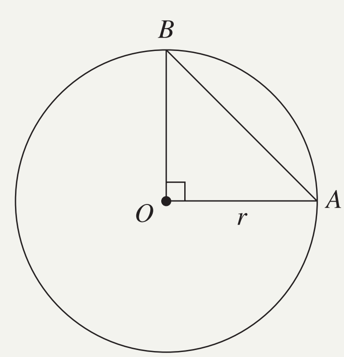
SOLUTION

Let the radius of the circle be r metres.

Then area of sector $AOB = \frac{1}{2}r^2 \times \frac{\pi}{2}$

$= \frac{\pi}{4}r^2$.

and area of $\Delta AOB = \frac{1}{2}r^2$ (it is half a square).



a Substituting into the formula for the area of a sector,

$\frac{\pi}{4} \times r^2 = 1$ (the area is $1\ \text{m}^2$)

$\boxed{\times \frac{4}{\pi}}$ $r^2 = \frac{4}{\pi}$

$r \doteqdot 1.128\ \text{m}$.

b Subtracting, area of segment $= \frac{\pi}{4}r^2 - \frac{1}{2}r^2$

$= \frac{1}{4}r^2(\pi - 2)$.

Hence $\frac{1}{4}r^2(\pi - 2) = 1$ (the area is $1\ \text{m}^2$)

$\boxed{\times \frac{4}{\pi - 2}}$ $r^2 = \frac{4}{\pi - 2}$

$r \doteqdot 1.872\ \text{m}$.

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Exercise 9I

FOUNDATION

Note: Are you working with radians or degrees? Remember the button labelled $\boxed{\text{mode}}$.

1 In the formula $\ell = r\theta$:

a Find ℓ, if $r = 18$ and $\theta = \frac{\pi}{6}$.

b Find ℓ, if $r = 10$ and $\theta = \frac{\pi}{4}$.

c Find r, if $\ell = 15$ and $\theta = 2$.

d Find r, if $\ell = 6\pi$ and $\theta = \frac{\pi}{4}$.

e Find θ, if $\ell = 2\pi$ and $r = 8$.

f Find θ, if $\ell = 3\pi$ and $r = 1.5$.

2 In the formula $A = \frac{1}{2}r^2\theta$:

a Find A, if $r = 4$ and $\theta = \frac{\pi}{4}$.

b Find A, if $r = 2$ and $\theta = \frac{2\pi}{3}$.

c Find θ, if $A = 16$ and $r = 4$.

d Find θ, if $A = 12\pi$ and $r = 6$.

e Find r, if $A = 54$ and $\theta = 3$.

f Find r, if $A = 40\pi$ and $\theta = \frac{\pi}{5}$.

3 A circle has radius 6 cm. Find the length of an arc of this circle that subtends an angle at the centre of:

a 2 radians
b 0.5 radians
c $\frac{\pi}{3}$ radians
d $\frac{\pi}{4}$ radians

4 A circle has radius 8 cm. Find the area of a sector of this circle that subtends an angle at the centre of:

a 1 radian
b 3 radians
c $\frac{\pi}{4}$ radians
d $\frac{3\pi}{8}$ radians

DEVELOPMENT

5 What is the radius of the circle in which an arc of length 10 cm subtends an angle of 2.5 radians at the centre?

6 If a sector of a circle of radius 4 cm has area 12 cm^2, find the angle at the centre in radians.

7 A circle has radius 3.4 cm. Find, correct to the nearest millimetre, the length of an arc of this circle that subtends an angle at the centre of:

a $40°$
b $73°38'$

(Hint: Remember that θ must be in radians.)

8 Find, correct to the nearest square metre, the area of a sector of a circle of radius 100 metres if the angle at the centre is $100°$.

9 A circle has radius 12 cm. Find, in exact form:

a the length of an arc that subtends an angle of $120°$ at the centre,

b the area of a sector in which the angle at the centre is $40°$.

10 An arc of a circle of radius 7.2 cm is 10.6 cm in length. Find the angle subtended at the centre by this arc, correct to the nearest degree.

11 A sector of a circle has area 52 cm^2 and subtends an angle of $44°16'$. Find the radius in cm, correct to one decimal place.

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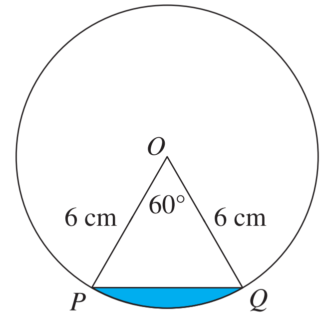
12 Consider the diagram to the right.

a Find the exact area of sector OPQ.

b Find the exact area of ΔOPQ.

c Hence find the exact area of the shaded minor segment.



13 A chord of a circle of radius 4 cm subtends an angle of 150° at the centre.

a Use the same method as the previous question to show that the area of the minor segment cut off by the chord is $\frac{4}{3}(5\pi - 3)\,\text{cm}^2$.

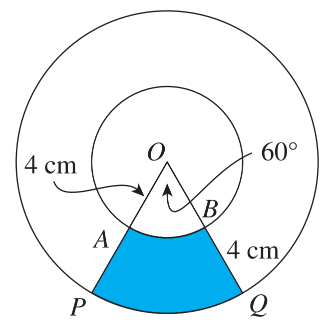
b By subtracting the area of the minor segment from the area of the circle, show that the area of the major segment cut off by the chord is $\frac{4}{3}(7\pi + 3)\,\text{cm}^2$.

14 A circle has centre C and radius 5 cm, and an arc AB of this circle has length 6 cm. Find the area of the sector ACB.

15 The diagram to the right shows two concentric circles with common centre O.

a Find the exact perimeter of the region $APQB$.

b Find the exact area of the region $APQB$.



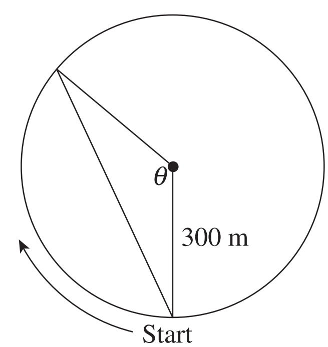
16 An athlete runs at a steady 4 m/s around a circular track of radius 300 metres. She runs clockwise, starting at the southernmost point.

a How far has she run after 3 minutes?

b What angle does this distance subtend at the centre?

c How far, in a direct line across the field, is she from her start, correct to the nearest 0.01 m?

d What is her true bearing from the centre then, correct to the nearest minute?



CHALLENGE

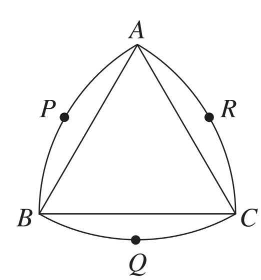
17 Triangle ABC is equilateral with side length 2 cm. Circular arcs AB, BC and CA have centres C, A and B respectively. Answer each part in exact form.

a Find the length of the arc AB.

b Find the area of the sector $CAPBC$.

c Find the length of the perimeter $APBQCRA$.

d Find the area of ΔABC and hence find the area enclosed by the perimeter $APBQCRA$.



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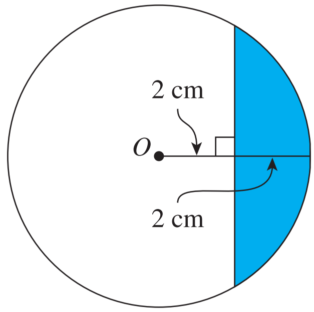
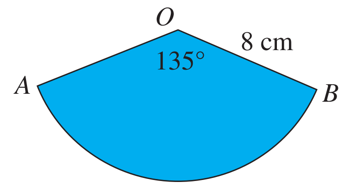
18 Find the exact area of the shaded region of the circle shown in the diagram to the right.



19 A piece of paper is cut in the shape of a sector of a circle. The radius is 8 cm and the angle at the centre is 135°. The straight edges of the sector are placed together so that a cone is formed.

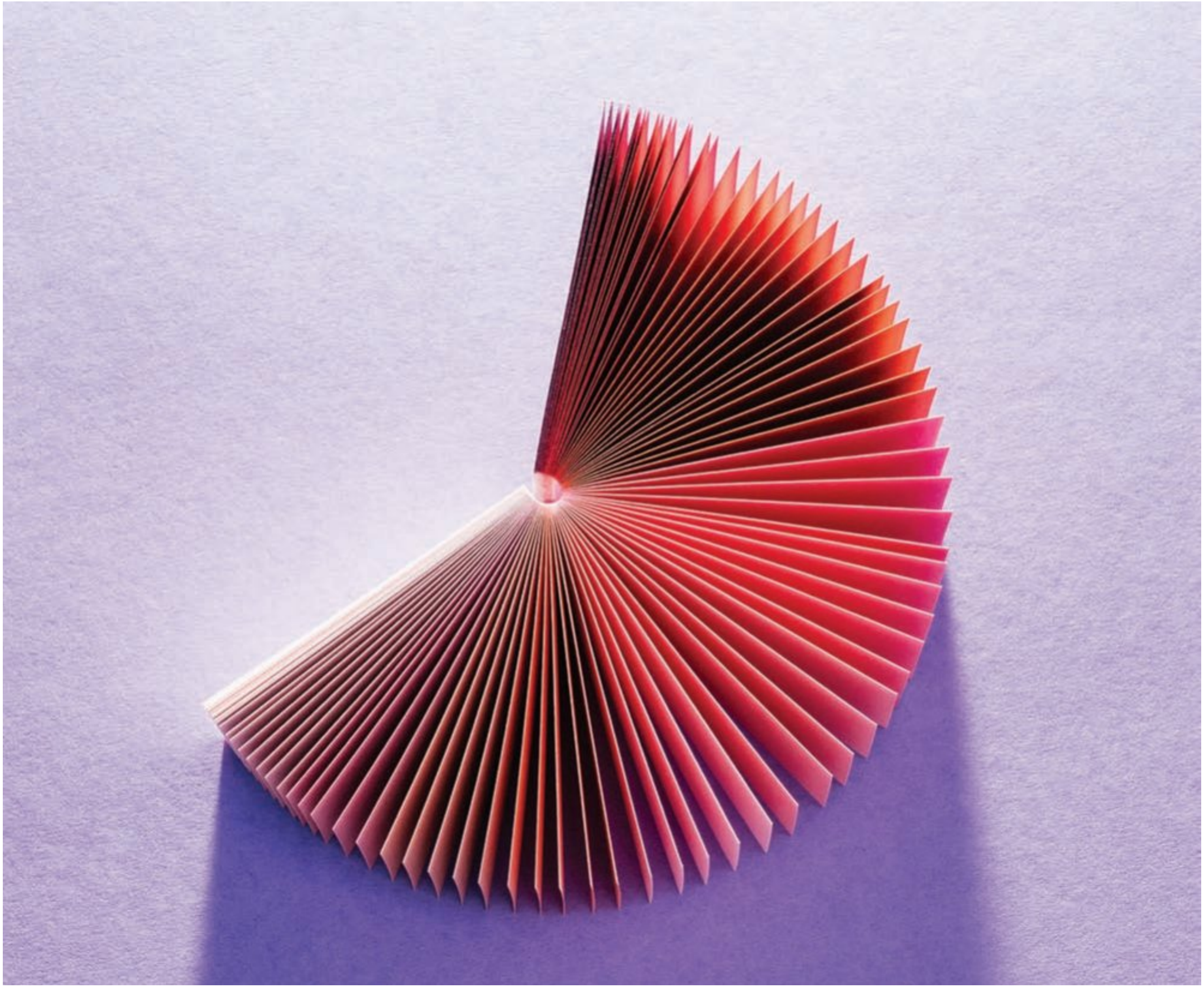


a Show that the base of the cone has radius 3 cm.

b Show that the cone has perpendicular height $\sqrt{55}$ cm.

c Hence find, in exact form, the volume of the cone.

d Find the curved surface area of the cone.



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9J Trigonometric graphs in radians

Now that angle size has been defined as a ratio, that is, as a pure number, the trigonometric functions can be drawn in their true shapes. On the next page, the graphs of the six functions have all been drawn using the same scale on the x-axis and y-axis. This means that the gradient of the tangent at each point now equals the true value of the derivative there.

For example, place a ruler on the graph of $y = \sin x$ so that it makes a tangent to the curve at the origin. The ruler should lie along the line $y = x$, indicating that the tangent at the origin has gradient 1. In the language of calculus, this means that the derivative of $\sin x$ has value 1 when $x = 0$. This is where we will begin when the topic is taken up again in Year 12.

Amplitude of the sine and cosine functions

The *amplitude* of a wave is the maximum height of the wave above the mean position. Both $y = \sin x$ and $y = \cos x$ have a maximum value of 1, a minimum value of -1 and a mean value of 0. Thus both have amplitude 1.

18 AMPLITUDE OF THE SINE AND COSINE FUNCTIONS

- $y = \sin x$ and $y = \cos x$ both have amplitude 1.

The other four trigonometric functions increase without bound near their asymptotes, so the idea of amplitude makes no sense. We can conveniently tie down the vertical scale of $y = \tan x$, however, by using the fact that $\tan \frac{\pi}{4} = 1$.

The periods of the trigonometric functions

The trigonometric functions are called *periodic functions* because each graph repeats itself exactly over and over again. The *period* of such a function is the length of the smallest repeating unit.

The graphs of $y = \sin x$ and $y = \cos x$ on the next page are waves, with a pattern that repeats every revolution. Thus they both have period 2π.

The graph of $y = \tan x$, on the other hand, has a pattern that repeats every half-revolution. Thus it has period π.

19 THE PERIODS OF THE SINE, COSINE AND TANGENT FUNCTIONS

- $y = \sin x$ and $y = \cos x$ have period 2π (that is, a full revolution).
- $y = \tan x$ has period π (that is, half a revolution).

The secant and cosecant functions are reciprocals of the cosine and sine functions and so have the same period 2π as they do. Similarly, the cotangent function has the same period π as the tangent function.

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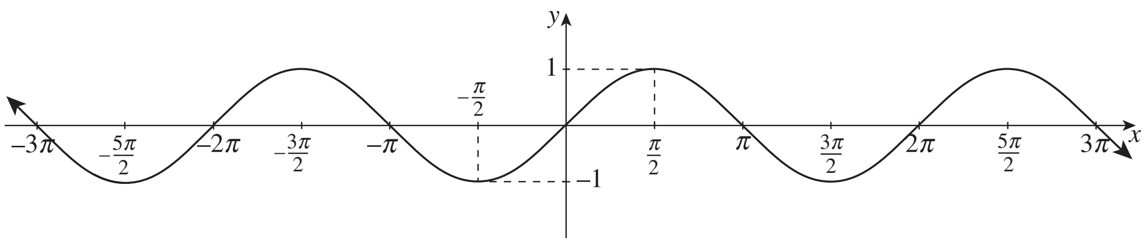
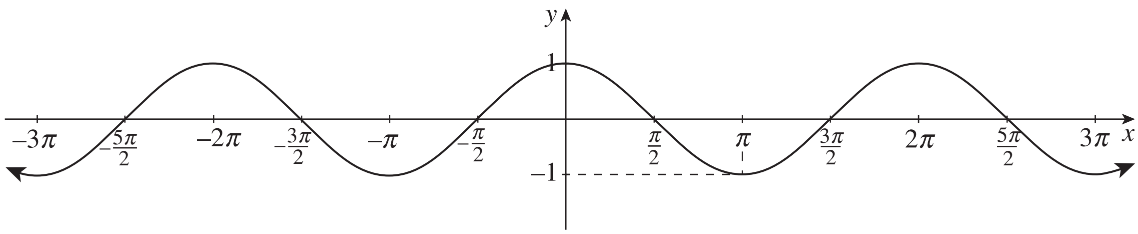
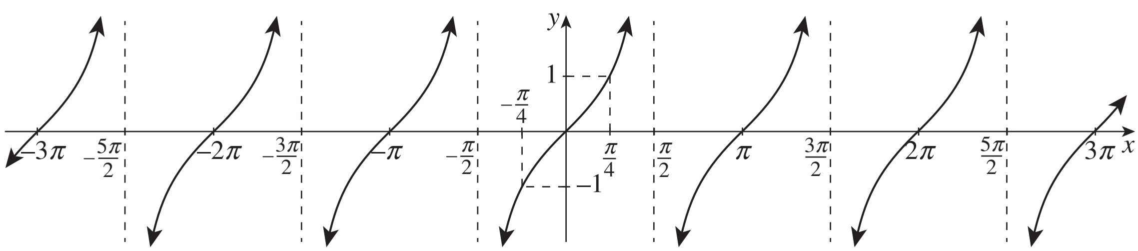
$y = \sin x$



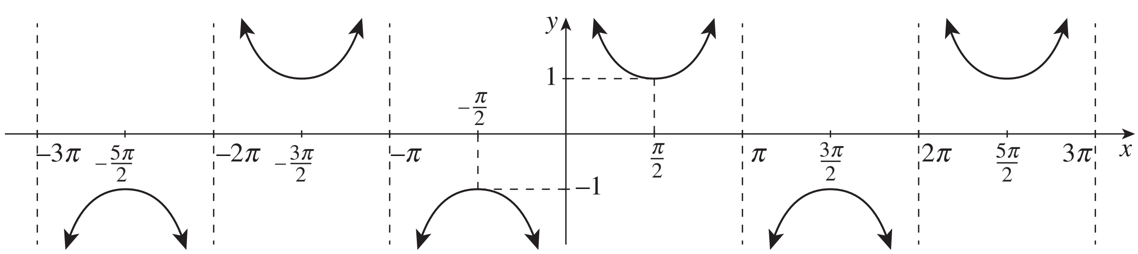
$y = \cos x$



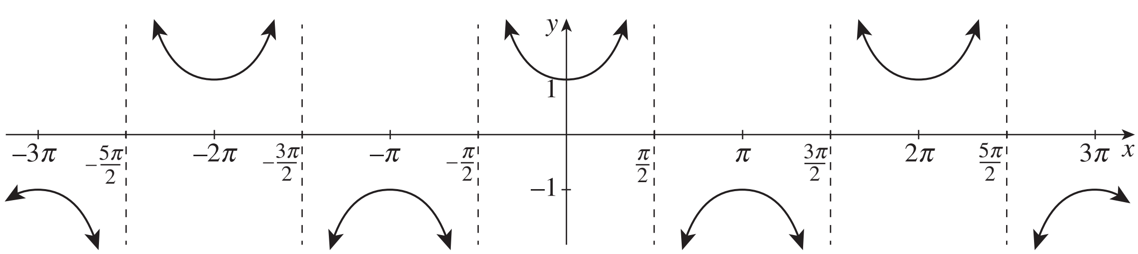
$y = \tan x$



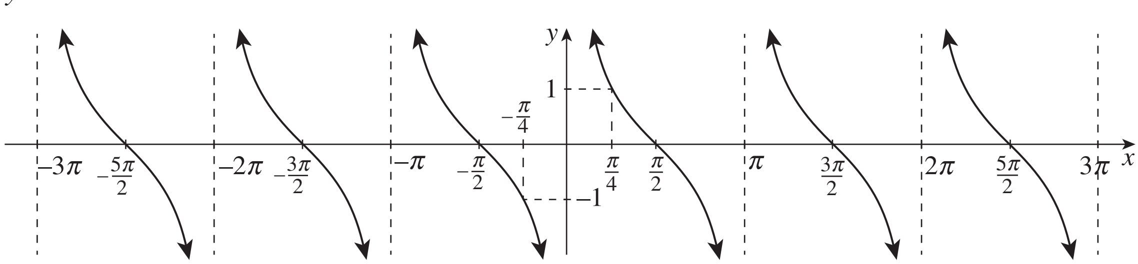
$y = \operatorname{cosec} x$



$y = \sec x$



$y = \cot x$



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Oddness and evenness of the trigonometric functions

The graphs of $y = \sin x$ and $y = \tan x$ have point symmetry in the origin, as can easily be seen from their graphs on page 397. This means that the functions $\sin x$ and $\tan x$ are odd functions. Algebraically, $\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$.

The graph of $y = \cos x$, however, has line symmetry in the y-axis. This means that the function $\cos x$ is an even function. Algebraically, $\cos(-x) = \cos x$.

20 ODDNESS AND EVENNESS OF THE TRIGONOMETRIC FUNCTIONS:

- The functions $\sin x$ and $\tan x$ are odd functions. Thus
  $$\sin(-x) = -\sin x \quad \text{and} \quad \tan(-x) = -\tan x.$$
- The function $\cos x$ is an even function. Thus
  $$\cos(-x) = \cos x.$$

The functions $\operatorname{cosec} x$ and $\cot x$ are odd because their reciprocal functions $\sin x$ and $\tan x$ are odd. The function $\sec x$ is even because its reciprocal function $\cos x$ is even.

A preview of what is to come

Now that radian measure has been introduced, the way is clear next year to differentiate the trigonometric functions $\sin x$ and $\cos x$. The results are surprisingly simple:

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

These results are also reasonably obvious from the graphs.

- Look at the graph of $y = \sin x$ on page 397. Look at the places where the tangent to the graph is horizontal, where the tangent has its greatest positive slope, and where the tangent has its greatest negative slope. Then on a separate set of axes, begin a sketch of $y = \dfrac{d}{dx}(\sin x)$. This second sketch looks very much like $y = \cos x$.
- Do the same with the graph of $y = \cos x$. This time, the second sketch looks very much like $y = -\sin x$.

So differentiating a wave gives another wave shifted backwards by $\frac{\pi}{2}$, which is a quarter-revolution. But all that is next year's story.

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Exercise 9J

INVESTIGATION

This exercise is an investigation. Its purpose is familiarity with the six trigonometric functions, and particularly familiarity with their symmetries.

Except for Question **1**, refer all the time to the page at the start of this section where all six trigonometric graphs have been drawn in radians with the same scale on both axes.

An accurate drawing and an accurate gradient:

1 **a** On graph paper, using a scale on both axes of 1 unit = 2 cm, draw an accurate graph of $y = \sin x$ for $-\pi \le x \le \pi$. Use your calculator to obtain a table of values, and in the interval from $x = -0.5$ to $x = 0.5$, plot points every 0.1 units. Also plot the important points at $x = \frac{\pi}{6}$, $x = \frac{\pi}{2}$, $x = \frac{5\pi}{6}$, $x = \pi$, and the opposites of these values.

b Place your ruler on the graph to confirm that the gradient of the tangent at the origin appears to be 1.

Reading the graphs backwards: Examine the graphs of the six trigonometric functions.

2 **a** Classify each graph as one-to-one, many-to-one, one-to-many or many-to-many.

b Use horizontal lines on the graph of $y = \sin x$ to find, if possible, at least six positive solutions of:

i $\sin x = 0$ **ii** $\sin x = \frac{1}{2}$ **iii** $\sin x = 1$ **iv** $\sin x = 2$

Line symmetries: Examine the graphs of the six trigonometric functions.

3 **a** List all the vertical lines in which the graph of $y = \sin x$ is symmetric.

b Which other trigonometric graphs are symmetric in the same vertical lines, and why?

c In which vertical lines are the graphs of $y = \tan x$ and $y = \cot x$ symmetric?

4 **a** List all the vertical lines in which the graph of $y = \cos x$ is symmetric.

b Which of these lines is related to the fact that $\cos x$ is even, that is, $\cos(-x) = \cos x$?

c Which other trigonometric graphs are symmetric in the same vertical lines, and why?

Point symmetries: Examine the graphs of the six trigonometric functions.

5 **a** List all the points in which the graph of $y = \sin x$ has point symmetry.

b Which of these points is related to the fact that $\sin x$ is odd, that is, $\sin(-x) = -\sin x$?

c Which other graph has the same point symmetries as $y = \sin x$, and why?

6 **a** List all the points in which the graph of $y = \cos x$ has point symmetry.

b Which other graph has the same point symmetries as $y = \cos x$, and why?

7 **a** About which points do $y = \tan x$ and $y = \cot x$ have point symmetry?

b Relate these symmetries to the oddness or evenness of $y = \tan x$ and $y = \cot x$.

Translation symmetries: Examine the graphs of the six trigonometric functions.

8 **a** What translation symmetry does $y = \sin x$ have? That is, what translations map the graph of $y = \sin x$ onto itself?

b What other trigonometric graphs have exactly these same translation symmetries?

c What translation symmetries do the other trigonometric graphs have?

d What periods do the six trigonometric functions have?

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Transformations: Examine the graphs of the six trigonometric functions.

9 a Reflection in some vertical lines transforms the graph of $y = \sin x$ onto the graph of $y = \cos x$. Identify all such vertical lines.

b For what other pair of trigonometric functions is the answer to the question in part **a** the same?

c Identify all vertical lines that reflect $y = \tan x$ onto $y = \cot x$.

10 a Identify all translations that shift $y = \sin x$ onto the graph of $y = \cos x$.

b For what values of θ is it true that $\sin(x - \theta) = \cos x$?

c Identify all translations that shift $y = \tan x$ onto the graph of $y = \cot x$.

11 Identify any rotations about the origin that rotate the graph of one trigonometric function onto another.

Intersections: Examine the graphs of the six trigonometric functions.

12 a Sketch $y = \sin x$ and $y = \cos x$ on one set of axes. Where do they intersect?

b Write down the corresponding trigonometric equation and solve it algebraically.

13 a Sketch $y = \sin x$ and $y = \tan x$ on one set of axes. Where do they intersect?

b Write down the corresponding trigonometric equation and solve it algebraically.

14 Sketch $y = \cos x$ and $y = \tan x$ together. Very approximately, what is the smallest positive value of x where they intersect?

15 a Describe the situation when $y = \sin x$ and $y = \operatorname{cosec} x$ are drawn together.

b What other pair of graphs have the same relationship?

c Name all the pairs of trigonometric functions whose graphs do not intersect when drawn together.

A harder question:

16 a Write down the trigonometric equation corresponding to the intersections of $y = \cos x$ and $y = \tan x$, and solve it algebraically.

b Write down the trigonometric equation corresponding to the intersections of $y = \tan x$ and $y = \sec x$, and show algebraically that it has no solutions.

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Chapter 9 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 9 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Use the index laws to simplify the following. Leave your answers in index form.

a $3^4 \times 3^5$ **b** $(3^6)^2$ **c** $\dfrac{3^{10}}{3^5}$ **d** $2^5 \times 3^5$

2 Write as fractions:

a 5^{-1} **b** 10^{-2} **c** x^{-3} **d** 3^{-x}

3 Simplify:

a $9^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $8^{\frac{2}{3}}$

d $16^{-\frac{1}{2}}$ **e** $27^{-\frac{2}{3}}$ **f** $100^{-\frac{3}{2}}$

4 Simplify:

a $2^x \times 2^{2x}$ **b** $\dfrac{2^{6x}}{2^{2x}}$ **c** $(2^{3x})^2$

d $2^x \times 5^x$ **e** $2^{x+1} \times 2^{x+2}$ **f** $\dfrac{2^{3x+2}}{2^{x+3}}$

5 Sketch the graphs of $y = 2^x$ and $y = 2^{-x}$ on the same number plane. Then write down the equation of the line that reflects each graph onto the other graph.

6 Use your calculator to approximate the following, correct to four significant figures:

a e **b** e^4 **c** e^{-2} **d** $e^{\frac{3}{2}}$

7 Simplify:

a $e^{2x} \times e^{3x}$ **b** $e^{7x} \div e^x$ **c** $\dfrac{e^{2x}}{e^{6x}}$ **d** $(e^{3x})^3$

8 Sketch the graph of each function on a separate number plane, and state its range.

a $y = e^x$ **b** $y = e^{-x}$ **c** $y = e^x + 1$ **d** $y = e^{-x} - 1$

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Review

9 Differentiate:

a $y = e^x$ **b** $y = e^{3x}$ **c** $y = e^{x+3}$ **d** $y = e^{2x+3}$ **e** $y = e^{-x}$

f $y = e^{-3x}$ **g** $y = e^{3-2x}$ **h** $y = 3e^{2x+5}$ **i** $y = 4e^{\frac{1}{2}x}$ **j** $y = \dfrac{2e^{6x-5}}{3}$

10 Write each function as a single power of e, and then differentiate it.

a $y = e^{3x} \times e^{2x}$ **b** $y = \dfrac{e^{7x}}{e^{3x}}$ **c** $y = \dfrac{e^x}{e^{4x}}$ **d** $y = (e^{-2x})^3$

11 Find the gradient of the tangent to the curve $y = e^{2x}$ at the point $(0, 1)$.

12 Find the equation of the tangent to the curve $y = e^x$ at the point where $x = 2$, and find the x-intercept and y-intercept of this tangent.

13 Use the appropriate button on your calculator to approximate the following, correct to four decimal places.

a $\log_{10} 27$ **b** $\log_{10} \frac{1}{2}$ **c** $\log_e 2$ **d** $\log_e 14$

14 Write each equation below in logarithmic form. Then use the appropriate button on your calculator to approximate x correct to four decimal places.

a $10^x = 15$ **b** $10^x = 3$ **c** $e^x = 7$ **d** $e^x = \frac{1}{3}$

15 Use the fact that $y = e^x$ and $y = \log_e x$ are inverse functions to simplify:

a $\log_e e^5$ **b** $\log_e e^{-\frac{1}{4}}$ **c** $e^{\log_e 3}$ **d** $e^{\log_e \frac{1}{5}}$

16 Use the log laws to simplify:

a $e \log_e e$ **b** $\log_e e^3$ **c** $\log_e \dfrac{1}{e}$ **d** $2e \log_e \sqrt{e}$

17 Sketch graphs of the following, clearly indicating the vertical asymptote in each case.

a $y = \log_2 x$ **b** $y = -\log_2 x$ **c** $y = \log_2 (x - 1)$ **d** $y = \log_2 (x + 3)$

18 Sketch graphs of the following, clearly indicating the vertical asymptote in each case.

a $y = \log_e x$ **b** $y = \log_e (-x)$ **c** $y = \log_e (x - 2)$ **d** $y = \log_e x + 1$

19 The lizards on Goanna Island are gradually dying out. The predicted population P of lizards, t years after the first observations were made, is $P = P_0 e^{-0.01t}$, where P_0 is the initial population.

a At what rate is the population changing?

b At what rate is the population changing 45 years later? Answer correct to two significant figures, in terms of P_0.

c What percentage of the original population remains 45 years later, correct to the nearest 1%?

d When will the population decline to 10% of the original population, correct to the nearest year?

20 Express in radians in terms of π:

a 180° **b** 20° **c** 240° **d** 315°

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21 Express in degrees:

a $\frac{\pi}{6}$ **b** $\frac{3\pi}{5}$ **c** 3π **d** $\frac{5\pi}{3}$

22 Find the exact value of:

a $\sin\frac{\pi}{3}$ **b** $\tan\frac{5\pi}{6}$

23 Solve for $0 \le x \le 2\pi$:

a $\cos x = \frac{1}{\sqrt{2}}$ **b** $\tan x = -\sqrt{3}$

24 Solve each equation for $0 \le \theta \le 2\pi$ by reducing it to a quadratic equation in u. Give your solutions in terms of π, or approximated correct to two decimal places, as appropriate.

a $2\sin^2\theta + \sin\theta = 0$

b $\cos^2\theta - \cos\theta - 2 = 0$

c $2\tan^2\theta + 5\tan\theta - 3 = 0$

25 A circle has radius 12 cm. Find, in exact form:

a the length of an arc that subtends an angle at the centre of 45°,

b the area of a sector in which the angle at the centre is 60°.

26 A chord of a circle of radius 8 cm subtends an angle at the centre of 90°. Find, correct to three significant figures, the area of the minor segment cut off by the chord.

27 Find, correct to the nearest minute, the angle subtended at the centre of a circle of radius 5 cm by an arc of length 13 cm.

28 a Which of the six trigonometric graphs have amplitudes, and what are they?

b Which of the six trigonometric graphs are periodic, and what are their periods?

c Which of the six trigonometric functions are odd, and which are even?

29 a Give the smallest positive value of θ for which $\sin(x - \theta) = \cos x$.

b Give the smallest positive value of θ for which $\cos(x - \theta) = \sin x$.

c What is the smallest positive value of x for which $\sin x = \cos x$?

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10 Probability

Probability arises when we perform an experiment that has various possible outcomes, but there is insufficient information to predict which of these outcomes will occur. The classic examples of this are tossing a coin, throwing a die, and drawing a card from a pack. Probability, however, is involved in almost every experiment done in science, and is fundamental to understanding statistics.

This chapter reviews some basic ideas of probability and develops a more systematic approach to solving probability problems. It concludes with the new topic of conditional probability.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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10A Probability and sample spaces

The first task is to develop a workable formula for probability that can serve as the foundation for the topic, we will do this when the possible results of an experiment can be divided into a finite number of equally likely possible outcomes.

Equally likely possible outcomes

The idea of equally likely possible outcomes is well illustrated by the experiment of throwing a die and recording the number shown. (A *die*, plural *dice*, is a cube with its corners and edges rounded so that it rolls easily, and with the numbers 1–6 printed on its six sides.) There are six *possible outcomes*: 1, 2, 3, 4, 5, 6. This is a complete list of the possible outcomes, because each time the die is rolled, one and only one of these outcomes will occur.

Provided that we believe the die to be completely symmetric and not *biased* in any way, there is no reason for us to expect that any one outcome is more likely to occur than any of the other five. We may thus regard these six possible outcomes as *equally likely possible outcomes*.

With the results of the experiment now divided into six equally likely possible outcomes, the probability $\frac{1}{6}$ is assigned to each of these six outcomes. Notice that these six probabilities are equal, and that they all add up to 1.

1 EQUALLY LIKELY POSSIBLE OUTCOMES

Suppose that the possible results of an experiment can be divided into a finite number n of *equally likely possible outcomes* — meaning that:

- one and only one of these n outcomes will occur, and
- we have no reason to expect one outcome to be more likely than another.

Then the probability $\dfrac{1}{n}$ is assigned to each of these equally likely possible outcomes.

Randomness

The explanations above assumed that the terms ‘equally likely’ and ‘more likely’ already have a meaning in the mind of the reader. There are many ways of interpreting these words. In the case of a thrown die, we could interpret the phrase ‘equally likely’ as a statement about the natural world, in this case that the die is perfectly symmetric. Alternatively, we could interpret it as saying that we lack entirely the knowledge to make any statement of preference for one outcome over another.

The word *random* can be used here. In the context of equally likely possible outcomes, saying that a die is thrown ‘randomly’ means that we are justified in assigning the same probability to each of the six possible outcomes. In a similar way, a card can be drawn ‘at random’ from a pack, or a queue of people can be formed in a ‘random order’.

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The basic formula for probability

Suppose that a throw of at least 3 on a die is needed to win a game. Getting at least 3 on a die is called the particular *event* under discussion. The outcomes 3, 4, 5 and 6 are called *favourable* for this event, and the other two possible outcomes 1 and 2 are called *unfavourable*. The probability assigned to getting a score of at least 3 is then

$$\begin{aligned} P\,(\text{scoring at least 3}) &= \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} \\ &= \frac{4}{6} \\ &= \frac{2}{3}. \end{aligned}$$

Again, this is easily generalised.

2 THE BASIC FORMULA FOR PROBABILITY

If the results of an experiment can be divided into a finite number of equally likely possible outcomes, some of which are favourable for a particular event and the others unfavourable, then:

$$P\,(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

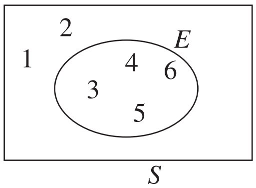
An *experiment* always includes what is recorded. For example, ‘tossing a coin and recording heads or tails’ is a different experiment from ‘tossing a coin and recording its distance from the wall’.

The sample space and the event space

Venn diagrams and the language of sets are a great help when explaining probability or solving probability problems. Section 10C will present a short account of sets and Venn diagrams, but at this stage the diagrams should be self-explanatory.

The Venn diagram to the right shows the six equally likely possible outcomes when a die is thrown. The set of all these possible outcomes is

$$S = \{1, 2, 3, 4, 5, 6\}.$$



This set S is called a *uniform sample space*, and is represented by the outer rectangular box. The word ‘uniform’ indicates that the six possible outcomes are all equally likely, and the word ‘sample’ refers to the fact that running the experiment can be regarded as ‘sampling’. The event ‘scoring at least 3’ is identified with the set

$$E = \{3, 4, 5, 6\}$$

which is called the *event space*. It is represented by the ellipse, which is inside the rectangle because E is a subset of S.

Sample spaces and uniform sample spaces

In general, a sample space need not consist of equally likely possible outcomes. The only condition is that one and only one of the possible outcomes must occur when the experiment is performed.

A finite sample space is called *uniform* if it does consist of equally likely possible outcomes, as when a die is thrown.

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3 SAMPLE SPACE

A set of possible outcomes of an experiment is called a *sample space* if:

- one and only one of these outcomes will occur.

Uniform sample space

A finite sample space is called *uniform* if:

- all its possible outcomes are equally likely.

Event space

The set of all favourable outcomes is called the *event space*.

The assumptions of the chapter

First, the sample spaces in this chapter are assumed to be finite — this qualification applies without further mention throughout the chapter. Infinite sample spaces appear in the next chapter and in Year 12.

Secondly, the discussions are restricted to situations where the results of experiment can be reduced to a set of equally likely possible outcomes, that is, to a uniform sample space (assumed finite, as discussed above).

Thus, the basic probability formula can now be restated in set language as:

4 THE BASIC FORMULA FOR PROBABILITY

Suppose that an event E is a subset of a uniform sample space S. Then

$$P(E) = \frac{|E|}{|S|},$$

where the symbols $|E|$ and $|S|$ mean the number of members of E and S.

Probabilities involving playing cards

So many questions in probability involve a pack of playing cards that anyone studying probability needs to be familiar with them. You are encouraged to acquire some cards and play some simple games with them. A pack of cards consists of 52 cards organised into four *suits*, each containing 13 cards. The four suits are:

two black suits:	♣ clubs	♠ spades
two red suits:	♦ diamonds	♥ hearts

Each of the four suits contains 13 cards:

A (ace), 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king)

An ace is also regarded as a 1. Three cards in each suit are called *court cards* (or *picture cards* or *face cards*) because they depict people in the royal court:

J (Jack), Q (Queen), K (King)

It is assumed that when a pack of cards is shuffled, the order is totally *random*, meaning that there is no reason to expect any one ordering of the cards to be more likely to occur than any other.

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Example 1 10A

A card is drawn at random from a standard pack of 52 playing cards. Find the probability that the card is:

a the seven of hearts
b any heart
c any seven
d any red card
e any court card (that is, a jack, a queen, or a king)
f any green card
g any red or black card.

SOLUTION

In each case, there are 52 equally likely possible outcomes.

a There is 1 seven of hearts, so $P(7\heartsuit) = \frac{1}{52}$.

b There are 13 hearts, so
$$\begin{aligned} P(\text{heart}) &= \tfrac{13}{52} \\ &= \tfrac{1}{4}. \end{aligned}$$

c There are 4 sevens, so
$$\begin{aligned} P(\text{seven}) &= \tfrac{4}{52} \\ &= \tfrac{1}{13}. \end{aligned}$$

d There are 26 red cards, so
$$\begin{aligned} P(\text{red card}) &= \tfrac{26}{52} \\ &= \tfrac{1}{2}. \end{aligned}$$

e There are 12 court cards, so
$$\begin{aligned} P(\text{court card}) &= \tfrac{12}{52} \\ &= \tfrac{3}{13}. \end{aligned}$$

f No card is green, so
$$\begin{aligned} P(\text{green card}) &= \tfrac{0}{52} \\ &= 0. \end{aligned}$$

g All 52 cards are red or black, so
$$\begin{aligned} P(\text{red or black card}) &= \tfrac{52}{52} \\ &= 1. \end{aligned}$$

Impossible and certain events

Parts **f** and **g** of Example 1 were intended to illustrate the probabilities of events that are impossible or certain.

- Getting a green card is impossible because there are no green cards. Hence there are no favourable outcomes, and the probability is 0.
- Getting a red or black card is certain because all the cards are either red or black. Hence all possible outcomes are favourable, and the probability is 1.
- For the other five events, the probability lies between 0 and 1.

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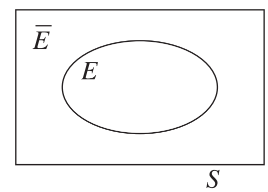
5 IMPOSSIBLE AND CERTAIN EVENTS

- An impossible event has probability zero.
- A certain event has probability 1.
- The probability of any other event E lies in the interval $0 < P(E) < 1$.

Complementary events and the word 'not'

It is often easier to find the probability that an event does ***not*** occur than the probability that it does occur. The *complement* of an event E is the event 'E does ***not*** occur'. It is written as $\overline{E}$, or sometimes as E' or E^c.

Let S be a uniform sample space of the experiment. The complementary event $\overline{E}$ is represented by the region outside the circle in the Venn diagram to the right.



Because $|\overline{E}| = |S| - |E|$, it follows that

$$P\left(\overline{E}\right) = \frac{|\overline{E}|}{|S|} = \frac{|S| - |E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|} = 1 - P(E).$$

6 COMPLEMENTARY EVENTS AND THE WORD 'NOT'

- The *complement* $\overline{E}$ of an event E is the event 'E does ***not*** occur', normally read as 'not E'?
- $P\left(\overline{E}\right) = 1 - P(E)$.
- The symbols E' and E^c are also used for the complementary event.
- Always consider using complementary events in problems, particularly when the word 'not' occurs.

In Section 10C, the *complement* $\overline{E}$ of a set E will be defined to be the set of things in S but ***not*** in E. The notation $\overline{E}$ for complementary event is quite deliberately the same notation as that for the complement of a set.

Example 2 10A

A card is drawn at random from a pack of playing cards. Find the probability that it is

a not a spade
b not a court card (jack, queen or king)
c neither a red 2 nor a black 6.

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SOLUTION

a Thirteen cards are spades, so

$$\begin{aligned} P\,(\text{spade}) &= \tfrac{13}{52} \\ &= \tfrac{1}{4}. \end{aligned}$$

Hence, using complementary events,

$$\begin{aligned} P\,(\text{not a spade}) &= 1 - \tfrac{1}{4} \\ &= \tfrac{3}{4}. \end{aligned}$$

b Twelve cards are court cards, so

$$\begin{aligned} P\,(\text{court card}) &= \tfrac{12}{52} \\ &= \tfrac{3}{13}. \end{aligned}$$

Hence, using complementary events,

$$\begin{aligned} P\,(\text{court card}) &= 1 - \tfrac{3}{13} \\ &= \tfrac{10}{13}. \end{aligned}$$

c There are two red 2s and two black 6s, so

$$\begin{aligned} P\,(\text{red 2 or black 6}) &= \tfrac{4}{52} \\ &= \tfrac{1}{13}. \end{aligned}$$

Hence, using complementary events,

$$\begin{aligned} P\,(\text{neither}) &= 1 - \tfrac{1}{13} \\ &= \tfrac{12}{13}. \end{aligned}$$

Invalid arguments

Arguments offered in probability theory can be invalid for all sorts of subtle reasons, and it is common for a question to ask for comment on a given argument. It is most important in such a situation that any fallacy in the given argument be explained — it is not sufficient to offer only an alternative argument with a different conclusion.

Example 3 10A

Comment on the validity of this argument.

'Brisbane is one of 16 League teams, so the probability that Brisbane wins the premiership is $\frac{1}{16}$.'

SOLUTION

[Identifying the fallacy]

The division into 16 possible outcomes is correct (assuming that a tie for first place is impossible), but no reason has been offered as to why each team is equally likely to win, so the argument is invalid.

[Offering a replacement argument]

If a team is selected at random from the 16 teams, then the probability that it is the premiership-winning team is $\frac{1}{16}$. Also, if someone knows nothing whatsoever about League, then for them the statement is correct. But these are different experiments.

Note: It is difficult to give a satisfactory account of this situation, indeed the idea of exact probabilities seems to have no meaning. Those with knowledge of the game would have some idea of ranking the 16 teams in order from most likely to win to least likely to win. If there is an organised system of betting, one may, or may not, agree to take this as an indication of the community's collective wisdom on Brisbane's chance of winning, and nominate it as 'the probability'.

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Experimental probability — relative frequency

When a drawing pin is thrown, there are two possible outcomes, point-up and point-down. But these two outcomes are not equally likely, and there seems to be no way to analyse the results of the experiment into equally likely possible outcomes.

In the absence of a fancy argument from physics about rotating pins falling on a smooth surface, however, some estimate of probability can be gained by performing the experiment many times. We can then use the *relative frequencies* of the drawing pin landing pin-up and pin-down as estimates of the probabilities, as in the following example. (Relative frequencies may have been introduced in earlier years.) But an exact value is unobtainable by these methods; indeed, the very idea of an exact value may well have no meaning.

The questions in the following example could raise difficult issues beyond this course, but the intention here is only that the questions be answered briefly in a common-sense manner. Chapter 11 pursues these issues in more detail, particularly in Section 11D.

Example 4 — 10A

A drawing pin was thrown 400 times and fell point-up 362 times.

a What were the relative frequencies of the drawing pin falling point-up, and of falling point-down?

b What probability does this experiment suggest for the result 'point-up'?

c A machine later repeated the experiment 1 000 000 times, and the pin fell point-up 916 203 times. Does this change your estimate in part **b**?

SOLUTION

a
$$\begin{aligned}\text{Relative frequency of point-up} &= \tfrac{362}{400}\\ &= 0.905\end{aligned}$$
$$\begin{aligned}\text{Relative frequency of point-down} &= \tfrac{38}{400}\\ &= 0.095\end{aligned}$$

b These results suggest that $P(\text{point-up}) \doteqdot 0.905$, but with only 400 trials, there would be little confidence in this result past the second decimal place, or even the first decimal place, because different *runs* of the same experiment would be expected to differ by small numbers. The safest conclusion would be that $P(\text{point-up}) \doteqdot 0.9$.

c The new results suggest that the estimate of the probability can now be refined to $P(\text{point-up}) \doteqdot 0.916$ — we can now be reasonably sure that the rounding to 0.9 in part **a** gave a value that was too low. (Did the machine throw the pin in a random manner, whatever that may mean?)

Exercise 10A — FOUNDATION

1 A pupil has 3 tickets in the class raffle. If there were 60 tickets in the raffle and one ticket is drawn, find the probability that the pupil:

a wins

b does not win.

2 A coin is tossed. Write down the probability that it shows:

a a head

b a tail

c either a head or a tail

d neither a head nor a tail.

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3 If a die is rolled, find the probability that the uppermost face is:

a a three
b an even number
c a number greater than four
d a multiple of three.

4 A bag contains five red and seven blue marbles. If one marble is drawn from the bag at random, find the probability that it is:

a red
b blue
c green.

5 A bag contains eight red balls, seven yellow balls and three green balls. A ball is selected at random. Find the probability that it is:

a red
b yellow or green
c not yellow.

6 In a bag there are four red capsicums, three green capsicums, six red apples and five green apples. One item is chosen at random. Find the probability that it is:

a green
b red
c an apple
d a capsicum
e a red apple
f a green capsicum.

7 A letter is chosen at random from the word TASMANIA. Find the probability that it is:

a an A
b a vowel
c a letter of the word HOBART.

8 A letter is randomly selected from the 26 letters in the English alphabet. Find the probability that the letter is:

a the letter S
b a vowel
c a consonant
d the letter γ
e a C, D or E
f one of the letters of the word MATHS.

Note: The letter Y is normally classified as a consonant.

9 A student has a 22% chance of being chosen as a prefect. What is the chance that they will not be chosen as a prefect?

10 When breeding labradors, the probability of breeding a black dog is $\frac{3}{7}$.

a What is the probability of breeding a dog that is not black?
b If you bred 56 dogs, how many would you expect not to be black?

11 A box containing a light bulb has a chance of $\frac{1}{15}$ of holding a defective bulb.

a If 120 boxes were checked, how many would you expect to hold defective bulbs?
b What is the probability that the box holds a bulb that works?

12 When a die is rolled, the theoretical probability of throwing a six is $\frac{1}{6}$, which is about 17%.

a How many sixes would you expect to get if a die is thrown 60 times?
b In an experiment, a certain die was thrown 60 times and a six turned up 18 times.
  i What was the relative frequency of throwing a six?
  ii What does this experiment suggest is the experimental probability of throwing a six?
  iii Comment on whether the die is *biased* or *fair*.

13 Every day the principal goes early to the tuckshop to buy a sandwich for his lunch. He knows that the canteen makes 400 sandwiches a day — equal numbers of four types: bacon-lettuce-tomato; chicken; ham and cheese; and Vegemite. He always chooses a sandwich at random.

a What is his theoretical probability of obtaining a chicken sandwich?
b Over 20 days, the principal found he had bought 8 chicken sandwiches. What was his relative frequency, and therefore his experimental probability, of obtaining a chicken sandwich?
c Comment on why your answers for the theoretical and experimental probability might differ.

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DEVELOPMENT

14 A number is selected at random from the integers 1, 2, 3, …, 19, 20. Find the probability of choosing:

a the number 4
b a number greater than 15
c an even number
d an odd number
e a prime number
f a square number
g a multiple of 4
h the number e
i a rational number.

15 From a regular pack of 52 cards, one card is drawn at random. Find the probability that:

a it is black
b it is red
c it is a king
d it is the jack of hearts
e it is a club
f it is a picture card
g it is a heart or a spade
h it is a red 5 or a black 7
i it is less than a 4.

16 A book has 150 pages. The book is randomly opened at a page. Find the probability that the page number is:

a greater than 140
b a multiple of 20
c an odd number
d a number less than 25
e either 72 or 111
f a three-digit number.

17 An integer x, where $1 \le x \le 200$, is chosen at random. Determine the probability that it:

a is divisible by 5
b is a multiple of 13
c has two digits
d is a square number
e is greater than 172
f has three equal digits.

18 A bag contains three times as many yellow marbles as blue marbles, and no other colours. If a marble is chosen at random, find the probability that it is:

a yellow
b blue.

19 Fifty tagged fish were released into a dam known to contain fish. Later, a sample of 30 fish was netted from this dam, of which eight were found to be tagged. Estimate the total number of fish in the dam just prior to the sample of 30 being removed.

CHALLENGE

20 Comment on the following arguments. Identify precisely any fallacies in the arguments, and, if possible, give some indication of how to correct them.

a ‘On every day of the year it either rains or it doesn’t. Therefore the chance that it will rain tomorrow is $\frac{1}{2}$.’

b ‘When the Sydney Swans play Hawthorn, Hawthorn wins, the Swans win or the game is a draw. Therefore the probability that the next game between these two teams results in a draw is $\frac{1}{3}$.’

c ‘When answering a multiple-choice question for which there are four possible answers to each question, the chance that Peter answers the question correctly is $\frac{1}{4}$.’

d ‘A bag contains a number of red, white and black beads. If you choose one bead at random from the bag, the probability that it is black is $\frac{1}{3}$.’

e ‘Four players play in a knockout tennis tournament resulting in a single winner. A man with no knowledge of the game or of the players declares that one particular player will win his semi-final, but lose the final. The probability that he is correct is $\frac{1}{4}$.’

21 A rectangular field is 60 metres long and 30 metres wide. A cow wanders randomly around the field. Find the probability that the cow is:

a more than 10 metres from the edge of the field,
b not more than 10 metres from a corner of the field.

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10B Sample space graphs and tree diagrams

Many experiments consist of several *stages*. For example, when a die is thrown twice, the two throws can be regarded as two separate stages of the one experiment. This section develops two approaches — graphing and tree diagrams — to investigating the sample space of a multi-stage experiment.

Graphing the sample space

The word 'sample space' is used in probability, rather than 'sample set', because the sample space of a multi-stage experiment takes on some of the characteristics of a space. In particular, the sample space of a two-stage experiment can be displayed on a two-dimensional graph, and the sample space of a three-stage experiment can be displayed in a three-dimensional graph.

The following example shows how a two-dimensional dot diagram can be used for calculations with the sample space of a die thrown twice.

Example 5 10B

A die is thrown twice. Find the probability that:

a the pair is a double

b at least one number is a four

c both numbers are greater than four

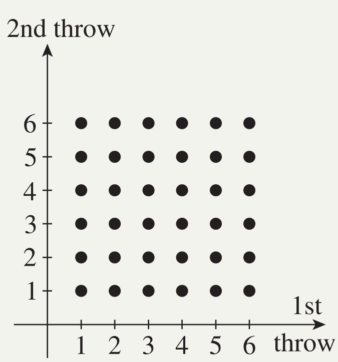
d both numbers are even

e the sum of the two numbers is six

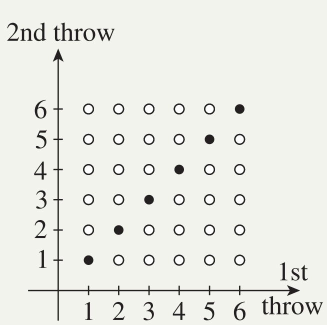
f the sum is at most four.

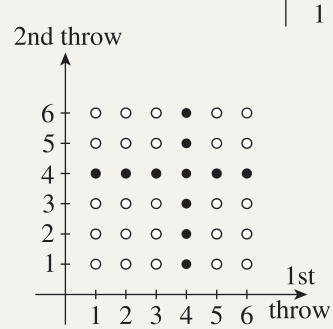
SOLUTION

The horizontal axis in the diagram to the right represents the six possible outcomes of the first throw, and the vertical axis represents the six possible outcomes of the second throw. Each dot is an ordered pair, such as (3, 5), representing the first and second outcomes. The 36 dots therefore represent all 36 different possible outcomes of the two-stage experiment, all equally likely. This is the full sample space.



Parts **a**–**f** can now be answered by counting the dots representing the various event spaces.





a There are 6 doubles, so

$$P(\text{double}) = \frac{6}{36} \quad \text{(count the dots)}$$
$$= \frac{1}{6}.$$

b 11 pairs contain a 4, so

$$P(\text{at least one is a 4}) = \frac{11}{36}.$$

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c 4 pairs consist only of 5 or 6, so $P\,(\text{both greater than 4}) = \frac{4}{36} = \frac{1}{9}.$

d 9 pairs have two even members, so $P\,(\text{both even}) = \frac{9}{36} = \frac{1}{4}.$

e 5 pairs have sum 6, so $P\,(\text{sum is 6}) = \frac{5}{36}.$

f 6 pairs have sum 2, 3 or 4, so $P\,(\text{sum at most 4}) = \frac{6}{36} = \frac{1}{6}.$

7 GRAPHING THE SAMPLE SPACE OF A TWO-STAGE EXPERIMENT

- The horizontal axis represents the first stage of the experiment.
- The vertical axis represents the second stage.

Complementary events in multi-stage experiments

The following example continues the two-stage experiment in the previous example, but this time it involves working with complementary events.

Example 6 10B

A die is thrown twice.

a Find the probability of failing to throw a double six.

b Find the probability that the sum is not seven.

SOLUTION

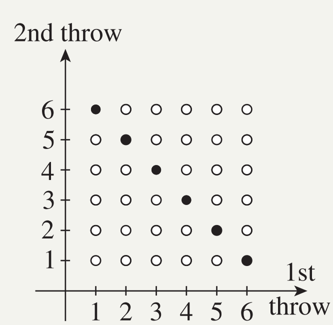
The same sample space can be used as in the previous example.

a The double six is one outcome out of 36,

so $P\,(\text{double six}) = \frac{1}{36}.$

Hence
$$P\,(\text{not throwing a double six}) = 1 - P\,(\text{double six}) = \frac{35}{36}.$$

b The graph shows that there are six outcomes giving a sum of seven,



so $P\,(\text{sum is seven}) = \frac{6}{36} = \frac{1}{6}.$

Hence
$$P\,(\text{sum is not seven}) = 1 - P\,(\text{sum is seven}) = 1 - \frac{1}{6} = \frac{5}{6}.$$

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Tree diagrams

Listing the sample space of a multi-stage experiment can be difficult, and the dot diagrams of the previous paragraph are hard to draw in more than two dimensions. Tree diagrams provide a very useful alternative way to display the sample space. Such diagrams have a column for each stage, plus an initial column labelled 'Start' and a final column listing the possible outcomes.

Example 7 10B

A three-letter word is chosen in the following way. The first and last letters are chosen from the three vowels 'A', 'O' and 'U', with repetition not allowed, and the middle letter is chosen from 'L' and 'M'. List the sample space, then find the probability that:

a the word is 'ALO'

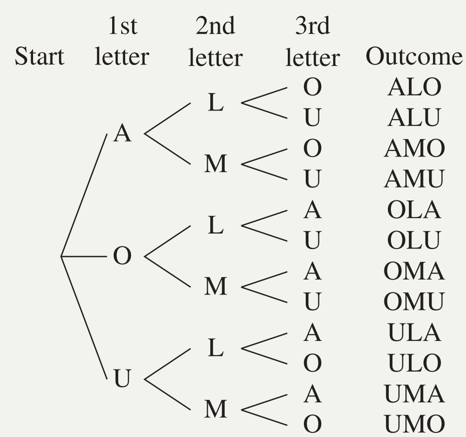
b the letter 'O' does not occur

c 'M' and 'U' do not both occur

d the letters are in alphabetical order.

SOLUTION

The tree diagram to the right lists all 12 equally likely possible outcomes. The two vowels must be different, because repetition was not allowed.



a $P(\text{'ALO'}) = \frac{1}{12}$

b $P(\text{no'O'}) = \frac{4}{12}$

$= \frac{1}{3}$

c $P(\text{not both 'M' and 'U'}) = \frac{8}{12}$

$= \frac{2}{3}$

d $P(\text{alphabetical order}) = \frac{4}{12}$

$= \frac{1}{3}$

The meaning of 'word'

In the example above and throughout this chapter, 'word' simply means an arrangement of letters — the arrangement doesn't have to have any meaning or be a word in the dictionary. Thus 'word' simply becomes a convenient device for discussing arrangements of things in particular orders.

8 A TREE DIAGRAM OF A MULTI-STAGE EXPERIMENT

- The first column is headed 'Start'.
- The last column is headed 'Output', and lists all the paths through the tree.
- Each other column represents all the equally likely possible outcomes in the particular stage of the experiment.

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Invalid arguments

The following example illustrates another invalid argument in probability. As always, the solution first offers an explanation of the fallacy, before then offering an alternative argument with a different conclusion.

Example 8 10B

Comment on the validity of this argument.

'When two coins are tossed together, there are three outcomes: two heads, two tails and one of each. Hence the probability of getting one of each is $\frac{1}{3}$.'

SOLUTION

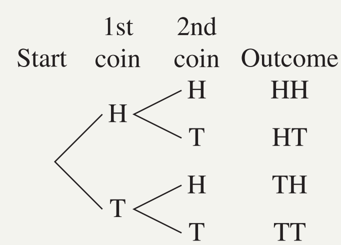
[Identifying the fallacy]

The division of the results into the three given outcomes is correct, but no reason is offered as to why these outcomes are equally likely.

[Supplying the correct argument]

On the right is a tree diagram of the sample space. It divides the results of the experiment into four *equally likely* possible outcomes. Two of these outcomes, HT and TH, are favourable to the event 'one of each', so

$$P\,(\text{one of each}) = \frac{2}{4} = \frac{1}{2}.$$



Exercise 10B

FOUNDATION

1 A fair coin is tossed twice.

a Use a tree diagram to list the four possible outcomes.

b Hence find the probability that the two tosses result in:

i two heads

ii a head and a tail

iii a head on the first toss and a tail on the second.

2 A coin is tossed and a die is thrown.

a Use a tree diagram to list all 12 possible outcomes.

b Hence find the probability of obtaining:

i a head and an even number

ii a tail and a number greater than 4

iii a tail and a number less than 4

iv a head and a prime number.

3 Two tiles are chosen at random, one after the other, from three Scrabble tiles T, O and E, and placed in order on the table.

a List the six possible outcomes in the sample space.

b Find the probability of choosing:

i two vowels

ii a consonant then a vowel

iii a T.

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4 From a group of four students, Anna, Bill, Charlie and David, two are chosen at random, one after the other, to be on the Student Representative Council.

a List the 12 possible outcomes in the sample space.

b Hence find the probability that:

i Bill and David are chosen
ii Anna is chosen
iii Charlie is chosen but Bill is not
iv neither Anna nor David is selected
v Bill is chosen first
vi Charles is not chosen second.

5 From the digits 2, 3, 8 and 9, a two-digit number is formed in which no digit is repeated.

a Use a tree diagram to list the possible outcomes.

b If the number was formed at random, find the probability that it is:

i the number 82
ii a number greater than 39
iii an even number
iv a multiple of 3
v a number ending in 2
vi a perfect square.

6 A captain and vice-captain of a cricket team are to be chosen from Amanda, Belinda, Carol, Dianne and Emma.

a Use a tree diagram to list the possible pairings, noting that order is important.

b If the choices were made at random, find the probability that:

i Carol is captain and Emma is vice-captain
ii Belinda is either captain or vice-captain
iii Amanda is not selected for either position
iv Emma is vice-captain.

DEVELOPMENT

7 A coin is tossed three times. Draw a tree diagram to illustrate the possible outcomes. Then find the probability of obtaining:

a three heads
b a head and two tails
c at least two tails
d at most one head
e more heads than tails
f a head on the second toss.

8 A green die and a red die are thrown simultaneously. List the set of 36 possible outcomes on a two-dimensional graph and hence find the probability of:

a obtaining a 3 on the green die
b obtaining a 4 on the red die
c a double 5
d a total score of 7
e a total score greater than 9
f an even number on both dice
g at least one 2
h neither a 1 nor a 4 appearing
i a 5 and a number greater than 3
j the same number on both dice.

9 Suppose that the births of boys and girls are equally likely.

a In a family of two children, determine the probability that there are:

i two girls
ii no girls
iii one boy and one girl.

b In a family of three children, determine the probability that there are:

i three boys
ii two girls and one boy
iii more boys than girls.

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10 A coin is tossed four times. Find the probability of obtaining:

a four heads

b exactly three tails

c at least two heads

d at most one head

e two heads and two tails

f more tails than heads.

11 A hand of five cards contains a 10, jack, queen, king and ace. From the hand, two cards are drawn in succession, the first card not being replaced before the second card is drawn. Find the probability that:

a the ace is drawn

b the king is not drawn

c the queen is the second card drawn.

CHALLENGE

12 Three-digit numbers are formed from the digits 2, 5, 7 and 8, without repetition.

a Use a tree diagram to list all the possible outcomes. How many outcomes are there?

b Hence find the probability that the number is:

i greater than 528

ii divisible by 3

iii divisible by 13

iv prime.

13 If a coin is tossed n times, where $n > 1$, find the probability of obtaining:

a n heads

b at least one head and at least one tail.

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10C Sets and Venn diagrams

This section is a brief account of sets and Venn diagrams for those who have not met these ideas already. The three key ideas needed in probability are the intersection of sets, the union of sets, and the complement of a set.

Logic is very close to the surface when we talk about sets and Venn diagrams. The three ideas of intersection, union and complement mentioned above correspond very precisely to the words 'and', 'or' and 'not'.

Listing sets and describing sets

A *set* is a collection of things called *elements* or *members*. When a set is specified, it needs to be made absolutely clear what things are its members. This can be done by *listing* the members inside curly brackets. For example,

$$S = \{1, 3, 5, 7, 9\},$$

which is read as 'S is the set whose members are 1, 3, 5, 7 and 9'.

It can also be done by *writing a description* of the members inside curly brackets. For example,

$$T = \{\text{odd integers from 0 to 10}\},$$

which is read as 'T is the set of odd integers from 0 to 10'.

Equal sets

Two sets are called *equal* if they have exactly the same members. Hence the sets S and T in the previous paragraph are equal, which is written as $S = T$. The order in which the members are written doesn't matter at all, and neither does repetition. For example,

$$\{1, 3, 5, 7, 9\} = \{3, 9, 7, 5, 1\} = \{5, 9, 1, 3, 7\} = \{1, 3, 1, 5, 1, 7, 9\}.$$

The size of a set

A set may be *finite*, like the set above of positive odd numbers less than 10, or *infinite*, like the set of all integers. Only finite sets are needed here.

If a set S is finite, then the symbol $|S|$ is used to mean the number of members of S. For example:

If $A = \{5, 6, 7, 8, 9, 10\}$, then $|A| = 6$.
If $B = \{\text{letters in the alphabet}\}$, then $|B| = 26$.
If $C = \{12\}$, then $|C| = 1$.
If $D = \{\text{odd numbers between 1.5 and 2.5}\}$, then $|D| = 0$.

The empty set

The last set D above is called the *empty set*, because it has no members at all. The usual symbol for the empty set is $\varnothing$. There is only one empty set, because any two empty sets have exactly the same members (that is, none at all) and so are equal.

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9 SET TERMINOLOGY AND NOTATION

- A *set* is a collection of *elements*. For example,

  $S = \{1, 3, 5, 7, 9\}$ and $T = \{\text{odd integers from 0 to 10}\}$.
- Two sets are called *equal* if they have exactly the same members.
- The number of elements in a finite set S is written as $|S|$.
- The *empty set* is written as $\varnothing$ — there is only one empty set.

Intersection and union

There are two obvious ways of combining two sets A and B. The *intersection* $A \cap B$ of A and B is the set of everything belonging to A ***and*** B. The *union* $A \cup B$ of A and B is the set of everything belonging to A ***or*** B. For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 3, 6\}$, then

$$A \cap B = \{1, 3\}$$
$$A \cup B = \{0, 1, 2, 3, 6\}.$$

Two sets A and B are called *disjoint* if they have no elements in common, that is, if $A \cap B = \varnothing$. For example, if $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7\}$, then

$$A \cap B = \varnothing,$$

so A and B are disjoint.

‘And’ means intersection, ‘or’ means union

The important mathematical words ‘and’ and ‘or’ can be interpreted in terms of union and intersection:

$$A \cap B = \{\text{elements that are in } A \text{ \underline{\textbf{and}} in } B\}$$
$$A \cup B = \{\text{elements that are in } A \text{ \underline{\textbf{or}} in } B\}$$

Note: The word ‘or’ in mathematics always means ‘and/or’. Correspondingly, all the elements of $A \cap B$ are members of $A \cup B$.

10 THE INTERSECTION AND UNION OF SETS

Intersection of sets

- The *intersection* of two sets A and B is the set of elements in A *and* in B:

  $$A \cap B = \{\text{elements in } A \text{ \underline{\textbf{and}} in } B\}.$$
- The sets A and B are called *disjoint* when $A \cap B = \varnothing$.

Union of sets

- The *union* of A and B is the set of elements in A *or* in B:

  $$A \cup B = \{\text{elements in } A \text{ \underline{\textbf{or}} in } B\}.$$
- The word ‘or’ in mathematics always means ‘and/or’.

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Subsets of sets

A set A is called a *subset* of a set B if every member of A is a member of B. This relation is written as $A \subset B$. For example,

$$\begin{aligned} \{\text{teenagers in Australia}\} &\subset \{\text{people in Australia}\} \\ \{2, 3, 4\} &\subset \{1, 2, 3, 4, 5\} \\ \{\text{vowels}\} &\subset \{\text{letters in the alphabet}\}. \end{aligned}$$

Because of the way subsets have been defined, every set is a subset of itself. Also, the empty set is a subset of every set. For example,

$$\begin{aligned} \{1, 3, 5\} &\subset \{1, 3, 5\}, \qquad \text{and} \\ \varnothing &\subset \{1, 3, 5\}. \end{aligned}$$

The universal set and the complement of a set

A *universal set* is the set of everything under discussion in a particular situation. For example, if $A = \{1, 3, 5, 7, 9\}$, then possible universal sets could be the set of all positive integers less than 11, or the set of all integers, or even the set of all real numbers.

Once a universal set E is fixed, then the *complement* $\overline{A}$ of any set A is the set of all members of that universal set which are ***not*** in A. For example,

If $A = \{1, 3, 5, 7, 9\}$ and $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
then $\overline{A} = \{2, 4, 6, 8, 10\}$.

Every member of the universal set is either in A or in $\overline{A}$, but never in both A and $\overline{A}$. This means that

$A \cap \overline{A} = \varnothing$, the empty set, and
$A \cup \overline{A} = E$, the universal set.

'Not' means complement

As mentioned in Section 10A, the important mathematical word 'not' can be interpreted in terms of the complementary set:

$\overline{A} = \{$members of E that are ***not*** members of $A\}$

11 SUBSETS, THE UNIVERSAL SET, AND COMPLEMENTS

- A set A is a *subset* of a set S, written as $A \subset S$, if every element of A is also an element of S.
- If A and B are sets, then $A \cap B \subset A \cup B$.
- A *universal set* is a conveniently chosen set that contains all the elements relevant to the situation.
- Let A be a subset of a universal set E. The *complement* of A is the set of all elements of E that are *not* in A,

  $\overline{A} = \{$elements of E that are ***not*** in $A\}$.

The notations A' and A^c are also used for the complement of A.

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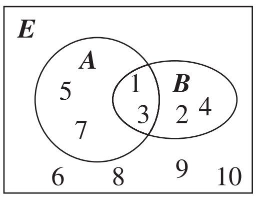
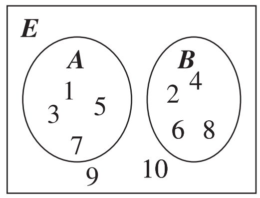
Venn diagrams

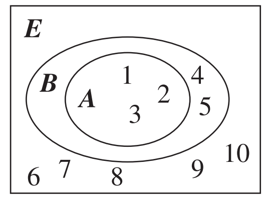
A *Venn diagram* is a diagram used to represent the relationship between sets. For example, the four diagrams below represent four different possible relationships between two sets A and B. In each case, the universal set is again $E = \{1, 2, 3, \ldots, 10\}$.



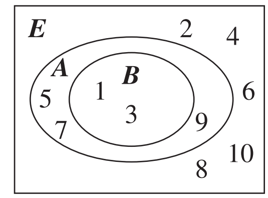
$A = \{1, 3, 5, 7\}$
$B = \{1, 2, 3, 4\}$



$A = \{1, 3, 5, 7\}$
$B = \{2, 4, 6, 8\}$

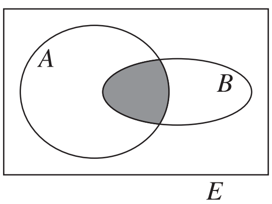


$A = \{1, 2, 3\}$
$B = \{1, 2, 3, 4, 5\}$

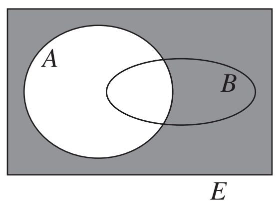


$A = \{1, 3, 5, 7, 9\}$
$B = \{1, 3\}$

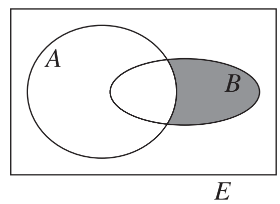
Sets can also be visualised by shading regions of the Venn diagram, as in the following examples:



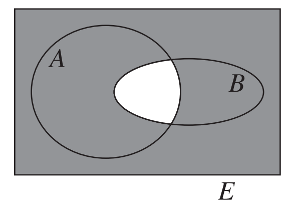
$A \cap B$



$\overline{A}$



$\overline{A} \cap B$



$\overline{A \cap B}$

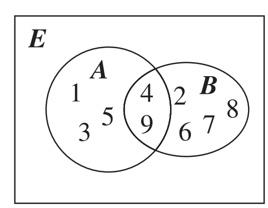
The counting rule for finite sets

To calculate the size of the union $A \cup B$ of two finite sets, adding the sizes of A and of B will not do, because the members of the intersection $A \cap B$ would be counted twice. Hence $|A \cap B|$ needs to be subtracted again, and the rule is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For example, the Venn diagram to the right shows the sets

$$A = \{1, 3, 4, 5, 9\} \qquad \text{and} \qquad B = \{2, 4, 6, 7, 8, 9\}.$$



From the diagram, $|A \cup B| = 9$, $|A| = 5$, $|B| = 6$ and $|A \cap B| = 2$, and the formula works because

$$9 = 5 + 6 - 2.$$

When two sets are disjoint, there is no overlap between A and B to cause any double counting. With $|A \cap B| = 0$, the counting rule becomes

$$|A \cup B| = |A| + |B|.$$

12 THE COUNTING RULE FOR FINITE SETS

- Let A and B be finite sets. Then

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

- Let A and B be disjoint finite sets. Then $|A \cap B| = 0$, so:

  $$|A \cup B| = |A| + |B|$$

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Problem solving using Venn diagrams

A Venn diagram is a very convenient way to sort out problems involving overlapping sets. But once there are more than a handful of elements, listing the elements is most inconvenient. Instead, we work only with the number of elements in each region, and show that number, in brackets, in the corresponding region.

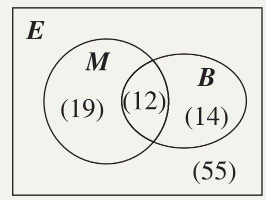
Example 9 10B

100 Sydneysiders were surveyed to find out how many of them had visited the cities of Melbourne and Brisbane. The survey showed that 31 people had visited Melbourne, 26 people had visited Brisbane and 12 people had visited both cities. Find how many people had visited:

a Melbourne or Brisbane
b Brisbane but not Melbourne
c only one of the two cities
d neither city.

SOLUTION

Let M be the set of people who have visited Melbourne, let B be the set of people who have visited Brisbane, and let E be the universal set of all people surveyed. Calculations should begin with the 12 people in the intersection of the two regions. Then the numbers shown in the other three regions of the Venn diagram can easily be found.



a Number visiting Melbourne or Brisbane $= 19 + 14 + 12$
$= 45$

b Number visiting Brisbane but not Melbourne $= 14$

c Number visiting only one city $= 19 + 14$
$= 33$

d Number visiting neither city $= 100 - 45$
$= 55$

Exercise 10C

FOUNDATION

1 Write down the following sets by listing their members.
  a Odd positive integers less than 10.
  b The first six positive multiples of 6.
  c The numbers on a die.
  d The factors of 20.

2 Find $A \cup B$ and $A \cap B$ for each pair of sets.
  a $A = \{1, 3, 5\}$, $B = \{3, 5, 7\}$
  b $A = \{1, 3, 4, 8, 9\}$, $B = \{2, 4, 5, 6, 9, 10\}$
  c $A = \{\text{h, o, b, a, r, t}\}$, $B = \{\text{b, i, c, h, e, n, o}\}$
  d $A = \{\text{j, a, c, k}\}$, $B = \{\text{e, m, m, a}\}$
  e $A = \{\text{prime numbers less than } 10\}$, $B = \{\text{odd numbers less than } 10\}$

3 If $A = \{1, 4, 7, 8\}$ and $B = \{1, 2, 4, 5, 7\}$, state whether the following statements are true or false.
  a $A = B$
  b $|A| = 4$
  c $|B| = 6$
  d $A \subset B$
  e $A \cup B = \{1, 2, 4, 5, 7, 8\}$
  f $A \cap B = \{1, 4, 7\}$

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4 If $A = \{1, 3, 5\}$, $B = \{3, 4\}$ and $E = \{1, 2, 3, 4, 5\}$, find:

a $|A|$ **b** $|B|$ **c** $A \cup B$ **d** $|A \cup B|$

e $A \cap B$ **f** $|A \cap B|$ **g** $\overline{A}$ **h** $\overline{B}$

5 If $A = \{$students who study Japanese$\}$ and $B = \{$students who study History$\}$, carefully describe each of the following sets.

a $A \cap B$ **b** $A \cup B$

6 If $A = \{$students with blue eyes$\}$ and $B = \{$students with blond hair$\}$, with universal set $E = \{$students at Clarence High School$\}$, carefully describe each of the following:

a $\overline{A}$ **b** $\overline{B}$ **c** $A \cup B$ **d** $A \cap B$

7 List all the subsets of each of these sets.

a $\{a\}$ **b** $\{a, b\}$ **c** $\{a, b, c\}$ **d** $\varnothing$

8 State in each case whether or not $A \subset B$ (i.e. whether A is a subset of B).

a $A = \{2, 3, 5\}$, $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

b $A = \{3, 6, 9, 12\}$, $B = \{3, 5, 9, 11\}$

c $A = \{d, a, n, c, e\}$, $B = \{e, d, u, c, a, t, i, o, n\}$

d $A = \{a, m, y\}$, $B = \{s, a, r, a, h\}$

e $A = \varnothing$, $B = \{51, 52, 53, 54\}$

DEVELOPMENT

9 Let $A = \{1, 3, 7, 10\}$ and $B = \{4, 6, 7, 9\}$, and take the universal set to be the set $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. List the members of:

a $\overline{A}$ **b** $\overline{B}$ **c** $A \cap B$

d $\overline{A \cap B}$ **e** $A \cup B$ **f** $\overline{A \cup B}$

10 Let $A = \{1, 3, 6, 8\}$ and $B = \{3, 4, 6, 7, 10\}$, and take the universal set to be the set $E = \{1, 2, 3, \ldots, 10\}$. List the members of:

a $\overline{A}$ **b** $\overline{B}$ **c** $\overline{A} \cup \overline{B}$ **d** $\overline{A} \cap \overline{B}$

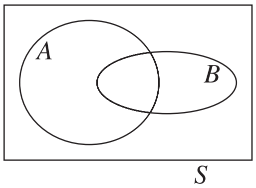
e $A \cup B$ **f** $\overline{A \cup B}$ **g** $A \cap B$ **h** $\overline{A \cap B}$

11 In each part, draw a copy of the diagram to the right and shade the corresponding region.

a $A \cap B$ **b** $\overline{A} \cap B$

c $A \cap \overline{B}$ **d** $A \cup B$

e $\overline{A \cup B}$ **f** $\overline{\varnothing}$ ($\varnothing$ is the empty set)



12 Answer true or false:

a If $A \subset B$ and $B \subset A$, then $A = B$.

b If $A \subset B$ and $B \subset C$, then $A \subset C$.

13 Copy and complete:

a If $P \subset Q$, then $P \cup Q = \ldots$

b If $P \subset Q$, then $P \cap Q = \ldots$

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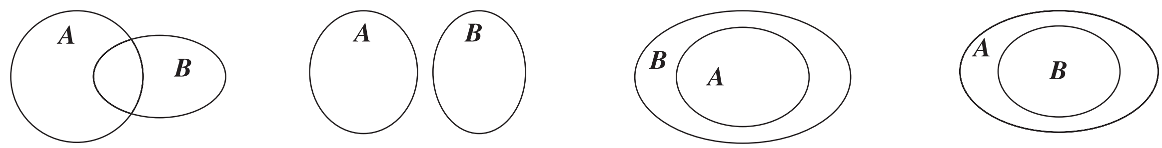
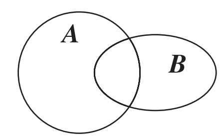
14 Select the Venn diagram that best shows the relationship between each pair of sets A and B.



I II III IV

- **a** $A = \{\text{Tasmania}\},\ B = \{\text{states of Australia}\}$
- **b** $A = \{7, 1, 4, 8, 3, 5\},\ B = \{2, 9, 0, 7\}$
- **c** $A = \{\text{l, e, a, r, n}\},\ B = \{\text{s, t, u, d, y}\}$
- **d** $A = \{\text{politicians in Australia}\},\ B = \{\text{politicians in NSW}\}$

15 **a** Explain the counting rule $|A \cup B| = |A| + |B| - |A \cap B|$ by making reference to the Venn diagram opposite.



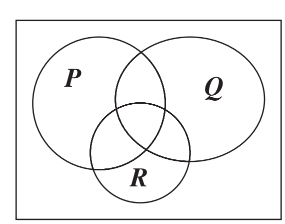
- **b** If $|A \cup B| = 17$, $|A| = 12$ and $|B| = 10$, find $|A \cap B|$.
- **c** Show that the relationship in part **a** is satisfied when $A = \{3, 5, 6, 8, 9\}$ and $B = \{2, 3, 5, 6, 7, 8\}$.

16 Use a Venn diagram to solve each problem.

- **a** In a group of 20 people, there are 8 who play the piano, 5 who play the violin and 3 who play both. How many people play neither instrument?
- **b** Each person in a group of 30 plays either tennis or golf. 17 play tennis, while 9 play both. How many play golf?
- **c** In a class of 28 students, there are 19 who like geometry and 16 who like trigonometry. How many students like both if there are 5 students who don't like either?

CHALLENGE

17 Shade each of the following regions on the given diagram (use a separate copy of the diagram for each part).



- **a** $P \cap Q \cap R$
- **b** $(P \cap R) \cup (Q \cap R)$
- **c** $\overline{P} \cup \overline{Q} \cup \overline{R}$

18 A group of 80 people was surveyed about their approaches to keeping fit.
It was found that 20 jog, 22 swim and 18 go to the gym on a regular basis.
Further questioning found that 10 people both jog and swim, 11 people both jog and go to the gym, and 6 people both swim and go to the gym.
Finally 43 people do none of these activities.
How many of the people do all three?

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10D Venn diagrams and the addition theorem

In probability situations where an event is described using the logical words 'and', 'or' and 'not', Venn diagrams and the language of sets are a useful way to visualise the sample space and the event space.

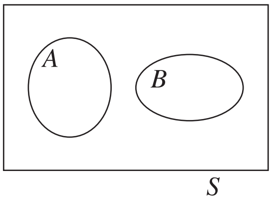
A uniform sample space S is taken as the universal set because it includes all the equally likely possible outcomes. The event spaces are then subsets of S.

Mutually exclusive events and disjoint sets

Two events A and B are called *mutually exclusive* if they cannot both occur. For example:

If a die is thrown, the events 'throwing a number less than three' and 'throwing a number greater than four' cannot both occur and so are mutually exclusive.

If a card is drawn at random from a pack, the events 'drawing a red card' and 'drawing a spade' cannot both occur and so are mutually exclusive.



In the Venn diagram of such a situation, the two events A and B are represented as disjoint sets (*disjoint* means that their intersection is empty). The event 'A and B' is impossible, and therefore has probability zero.
On the other hand, the event 'A or B' is represented on the Venn diagram by the union $A \cup B$ of the two sets. Because $|A \cup B| = |A| + |B|$ for disjoint sets, it follows that

$$\begin{aligned} P(A \text{ or } B) &= \frac{|A \cup B|}{|S|} \\ &= \frac{|A| + |B|}{|S|} \qquad \text{(because } A \text{ and } B \text{ are disjoint)} \\ &= P(A) + P(B). \end{aligned}$$

13 MUTUALLY EXCLUSIVE EVENTS

Suppose that two mutually exclusive events A and B are subsets of the same uniform sample space S. Then the event 'A or B' is represented by $A \cup B$, and

$$P(A \text{ or } B) = P(A) + P(B).$$

The event 'A and B' cannot occur, and has probability zero.

Example 10 10D

If a die is thrown, find the probability that it is less than three or greater than four.

SOLUTION

The events 'throwing a number less than three' and 'throwing a number greater than four' are mutually exclusive,

$$\begin{aligned} \text{so } P(\text{less than three or greater than four}) &= P(\text{less than three}) + P(\text{greater than four}) \\ &= \tfrac{1}{3} + \tfrac{1}{3} \\ &= \tfrac{2}{3}. \end{aligned}$$

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Example 11 10D

If a card is drawn at random from a standard pack, find the probability that it is a red card or a spade.

SOLUTION

'Drawing a red card' and 'drawing a spade' are mutually exclusive,

$$\begin{aligned}\text{so } P(\text{a red card or a spade}) &= P(\text{a red card}) + P(\text{a spade})\\ &= \tfrac{1}{2} + \tfrac{1}{4}\\ &= \tfrac{3}{4}.\end{aligned}$$

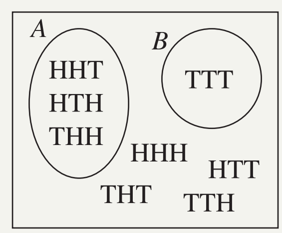
Example 12 [Mutually exclusive events in multi-stage experiments] 10D

If three coins are tossed, find the probability of throwing an odd number of tails.

SOLUTION

Let A be the event 'one tail' and B the event 'three tails'. Then A and B are mutually exclusive, with

$$A = \{\text{HHT, HTH, THH}\} \qquad \text{and} \qquad B = \{\text{TTT}\}.$$

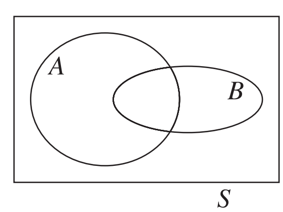


The full sample space has eight members altogether (Question **7** in the Exercise 10B lists them all), so:

$$P(A \text{ or } B) = \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{1}{2}.$$

The events 'A and B' and 'A or B' — the addition rule

More generally, suppose that two events A and B are subsets of the same uniform sample space S, not necessarily mutually exclusive. The Venn diagram of the situation now represents the two events A and B as overlapping sets within the same universal set S.



The event 'A and B' is represented by the intersection $A \cap B$ of the two sets, and the event 'A or B' is represented by the union $A \cup B$.

The general counting rule for sets is $|A \cup B| = |A| + |B| - |A \cap B|$, because the members of the intersection $A \cap B$ are counted in A, and are counted again in B, and so have to be subtracted. It follows then that

$$\begin{aligned}P(A \text{ or } B) &= \frac{|A \cup B|}{|S|}\\ &= \frac{|A| + |B| - |A \cap B|}{|S|}\\ &= P(A) + P(B) - P(A \text{ and } B).\end{aligned}$$

This rule is the *addition rule* of probability.

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14 THE EVENTS '*A* OR *B*' AND '*A* AND *B*'

Suppose that two events A and B are subsets of the same uniform sample space S.

- The event 'A and B' is represented by the intersection $A \cap B$.
- The event 'A or B' is represented by the union $A \cup B$.

The addition rule

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

The symbols ∩ and ∪, and the words 'and', 'or' and 'but'

As explained in the previous section, the word 'and' is closely linked with the intersection of event spaces. For this reason, the event A and B is also written as $A \cap B$, and the following two notations are used interchangeably:

$$P(A \text{ and } B) \qquad \text{means the same as} \qquad P(A \cap B).$$

Similarly, the word 'or' is closely linked with the union of event spaces — 'or' always means 'and/or' in logic and mathematics. Thus the event A or B is also written as $A \cup B$, and the following two notations are used interchangeably:

$$P(A \text{ or } B) \qquad \text{means the same as} \qquad P(A \cup B).$$

The word 'but' has the same logical meaning as 'and' — the difference in meaning between 'and' and 'but' is rhetorical, not logical. Using this notation, the addition rule now looks much closer to the counting rule for finite sets:

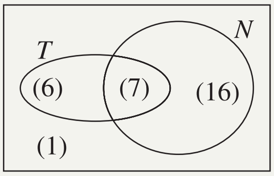
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Example 13 10D

In a class of 30 girls, 13 play tennis and 23 play netball. If 7 girls play both sports, what is the probability that a girl chosen at random plays neither sport?

SOLUTION

Let T be the event 'she plays tennis', and let N be the event 'she plays netball'.



Then $\quad P(T) = \frac{13}{30}$

$\quad P(N) = \frac{23}{30}$

and $\quad P(N \cap T) = \frac{7}{30} \quad \big(\text{meaning } P(N \text{ and } T)\big).$

Hence $\quad P(N \cup T) = \frac{13}{30} + \frac{23}{30} - \frac{7}{30} \quad \big(\text{meaning } P(N \text{ or } T)\big)$

$\quad = \frac{29}{30},$

and $\quad P(\text{neither sport}) = 1 - P(N \text{ or } T)$

$\quad = \frac{1}{30}.$

Note: An alternative approach is shown in the diagram above. Starting with the 7 girls in the intersection, the numbers 6 and 16 can then be written into the respective regions 'tennis but not netball' and 'netball but not tennis'. These numbers add to 29, leaving only one girl playing neither tennis nor netball.

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Complementary events and the addition rule

In the following example, the addition rule has to be applied in combination with the idea of complementary events. Some careful thinking is required when the words 'and' and 'or' are combined with 'not'.

Example 14 10D

A card is drawn at random from a pack.

a Find the probability that it is not an ace, but also not a two.

b Find the probability that it is an even number, or a court card, or a red card (the court cards are jack, queen and king).

Note: The word 'or' always means 'and/or' in logic and mathematics. Thus in part **b**, there is no need to add 'or any two of these, or all three of these'.

SOLUTION

a The complementary event $\overline{E}$ is drawing a card that is an ace or a two.

There are eight such cards, so $P(\text{ace or two}) = \frac{8}{52}$

$= \frac{2}{13}$.

Hence $P(\text{not an ace, but also not a two}) = 1 - \frac{2}{13}$

$= \frac{11}{13}$.

(The words 'and' and 'but' have different emphasis, but they have the same logical meaning.)

b The complementary event $\overline{E}$ is drawing a card that is a black odd number less than 10. This complementary event has 10 members:

$$\overline{E} = \{\,A\clubsuit, 3\clubsuit, 5\clubsuit, 7\clubsuit, 9\clubsuit, A\spadesuit, 3\spadesuit, 5\spadesuit, 7\spadesuit, 9\spadesuit\,\}.$$

There are 52 possible cards to choose, so:

$$P\left(\overline{E}\right) = \frac{10}{52} = \frac{5}{26}.$$

Hence, using the complementary event formula:

$$P(E) = 1 - P\left(\overline{E}\right) = \frac{21}{26}.$$

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Exercise 10D

FOUNDATION

1 A die is rolled. If n denotes the number on the uppermost face, find:

a $P(n = 5)$

b $P(n \neq 5)$

c $P(n = 4 \text{ or } n = 5)$

d $P(n = 4 \text{ and } n = 5)$

e $P(n \text{ is even or odd})$

f $P(n \text{ is neither even nor odd})$

g $P(n \text{ is even and divisible by three})$

h $P(n \text{ is even or divisible by three})$

2 A card is selected from a regular pack of 52 cards. Find the probability that the card:

a is a jack

b is a 10

c is a jack or a 10

d is a jack and a 10

e is neither a jack nor a 10

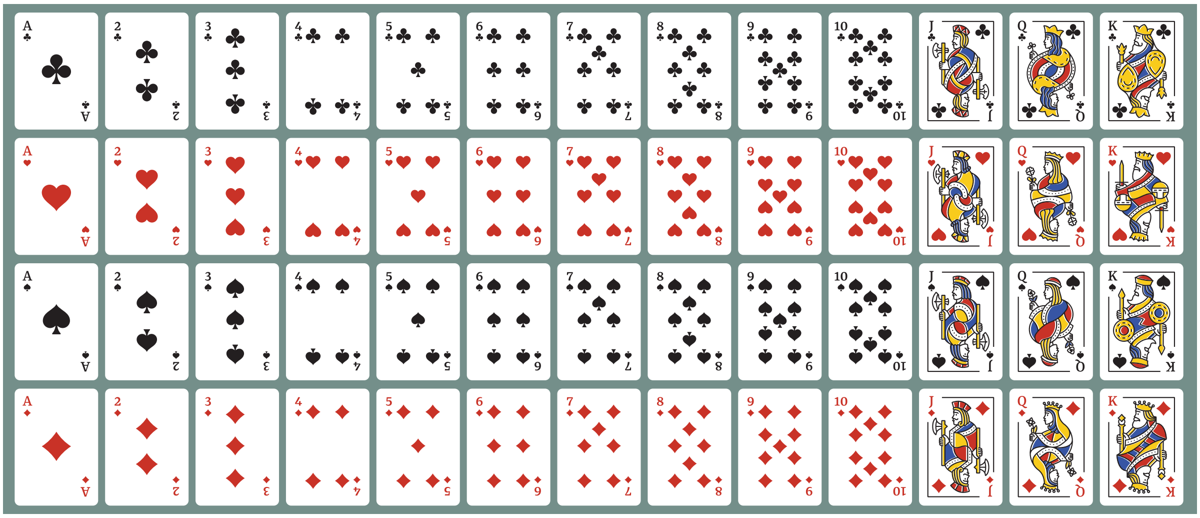
f is black

g is a picture card

h is a black picture card

i is black or a picture card

j is neither black nor a picture card.



3 Show that the following events A and B are disjoint by showing that $A \cap B = \varnothing$, and check that $P(A \text{ or } B) = P(A) + P(B)$.

a Two coins are thrown and $A = \{\text{two heads}\}$, $B = \{\text{one tail}\}$.

b A committee of two is formed by choosing at random from two men, Ricardo and Steve, and one woman, Tania. Suppose $A = \{\text{committee of two men}\}$, $B = \{\text{committee includes Tania}\}$.

4 A die is thrown. Let A be the event that an even number appears. Let B be the event that a number greater than 2 appears.

a Are A and B mutually exclusive?

b Find:

i $P(A)$

ii $P(B)$

iii $P(A \cap B)$

iv $P(A \cup B)$

5 Two dice are thrown. Let a and b denote the numbers rolled. Find:

a $P(a \text{ is odd})$

b $P(b \text{ is odd})$

c $P(a \text{ and } b \text{ are odd})$

d $P(a \text{ or } b \text{ is odd})$

e $P(\text{neither } a \text{ nor } b \text{ is odd})$

f $P(a = 1)$

g $P(b = a)$

h $P(a = 1 \text{ and } b = a)$

i $P(a = 1 \text{ or } b = a)$

j $P(a \neq 1 \text{ and } a \neq b)$

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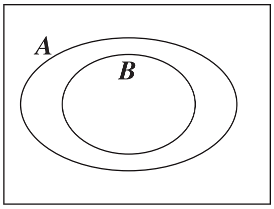
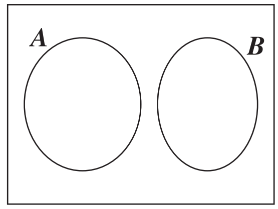
DEVELOPMENT

6 a 

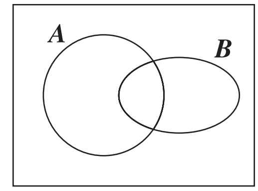
$P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$.

Find:
i $P(\overline{A})$
ii $P(\overline{B})$
iii $P(A \text{ and } B)$
iv $P(A \text{ or } B)$
v $P(\text{neither } A \text{ nor } B)$

b 

$P(A) = \frac{2}{5}$ and $P(B) = \frac{1}{5}$.

Find:
i $P(\overline{A})$
ii $P(\overline{B})$
iii $P(A \text{ or } B)$
iv $P(A \text{ and } B)$
v $P(\text{not both } A \text{ and } B)$

c 

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \text{ and } B) = \frac{1}{6}$.

Find:
i $P(\overline{A})$
ii $P(\overline{B})$
iii $P(A \text{ or } B)$
iv $P(\text{neither } A \text{ nor } B)$
v $P(\text{not both } A \text{ and } B)$

7 Use the addition rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to answer the following questions.

a If $P(A) = \frac{1}{5}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{15}$, find $P(A \cup B)$.

b If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{5}{6}$, find $P(A \cap B)$.

c If $P(A \cup B) = \frac{9}{10}$, $P(A \cap B) = \frac{1}{5}$ and $P(A) = \frac{1}{2}$, find $P(B)$.

d If A and B are mutually exclusive and $P(A) = \frac{1}{7}$ and $P(B) = \frac{4}{7}$, find $P(A \cup B)$.

8 An integer n is picked at random, where $1 \le n \le 20$. The events A, B, C and D are:

A: an even number is chosen,
B: a number greater than 15 is chosen,
C: a multiple of 3 is chosen,
D: a one-digit number is chosen.

a i Are the events A and B mutually exclusive?
ii Find $P(A)$, $P(B)$ and $P(A \text{ and } B)$ and hence evaluate $P(A \text{ or } B)$.

b i Are the events A and C mutually exclusive?
ii Find $P(A)$, $P(C)$ and $P(A \text{ and } C)$ and hence evaluate $P(A \text{ or } C)$.

c i Are the events B and D mutually exclusive?
ii Find $P(B)$, $P(D)$ and $P(B \text{ and } D)$, and hence evaluate $P(B \text{ or } D)$.

9 In a group of 50 students, there are 26 who study Latin and 15 who study Greek and 8 who study both languages. Draw a Venn diagram and find the probability that a student chosen at random:

a studies only Latin
b studies only Greek
c does not study either language.

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10 During a game, all 21 members of an Australian Rules football team consume liquid. Some players drink only water, some players drink only Gatorade^{TM} and some players drink both. There are 14 players who drink water and 17 players who drink Gatorade^{TM}.

a How many drink both water and Gatorade^{TM}?

b If one team member is selected at random, find the probability that:

  i he drinks water but not Gatorade^{TM},

  ii he drinks Gatorade^{TM} but not water.

11 Each student in a music class of 28 studies either the piano or the violin or both. It is known that 20 study the piano and 15 study the violin. Find the probability that a student selected at random studies both instruments.

CHALLENGE

12 List the 25 primes less than 100. A number is drawn at random from the integers from 1 to 100. Find the probability that:

a it is prime

b it has remainder 1 after division by 4

c it is prime and it has remainder 1 after division by 4

d it is either prime or it has remainder 1 after division by 4.

13 A group of 60 students was invited to try out for three sports: rugby, soccer and cross country. Of these, 32 tried out for rugby, 29 tried out for soccer, 15 tried out for cross country, 11 tried out for rugby and soccer, 9 tried out for soccer and cross country, 8 tried out for rugby and cross country, and 5 tried out for all three sports. Draw a Venn diagram and find the probability that a student chosen at random:

a tried out for only one sport

b tried out for exactly two sports

c tried out for at least two sports

d did not try out for a sport.

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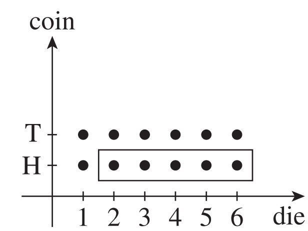
10E Multi-stage experiments and the product rule

This section deals with experiments that have a number of stages. The full sample space of such an experiment can quickly become too large to be conveniently listed, but instead a rule can be developed for multiplying together the probabilities associated with each stage.

Two-stage experiments — the product rule

Here is a simple question about a two-stage experiment:

> 'Throw a die, then toss a coin. What is the probability of obtaining at least 2 on the die, followed by a head?'



Graphed to the right are the twelve possible outcomes of the experiment, all equally likely, with a box drawn around the five favourable outcomes. Thus

$$P(\text{at least 2 and a head}) = \tfrac{5}{12}.$$

Now consider the two stages separately. The first stage is throwing a die, and the desired outcome is $A =$ 'getting at least 2' — here there are six possible outcomes and five favourable outcomes, giving probability $\frac{5}{6}$. The second stage is tossing a coin, and the desired outcome is $B =$ 'tossing a head' — here there are two possible outcomes and one favourable outcome, giving probability $\frac{1}{2}$.

The full experiment then has $6 \times 2 = 12$ possible outcomes and there are $5 \times 1 = 5$ favourable outcomes. Hence

$$P(AB) = \frac{5 \times 1}{6 \times 2} = \frac{5}{6} \times \frac{1}{2} = P(A) \times P(B).$$

Thus the probability of the compound event 'getting at least 2 and a head' can be found by multiplying together the probabilities of the two stages. The argument here can easily be generalised to any two-stage experiment.

15 TWO-STAGE EXPERIMENTS

If A and B are events in successive independent stages of a two-stage experiment, then

$$P(AB) = P(A) \times P(B)$$

where the word 'independent' means that the outcome of one stage does not affect the probabilities of the other stage.

Independent events

The word 'independent' needs further discussion. In the example above, the throwing of the die clearly does not affect the tossing of the coin, so the two events are independent.

Here is a very common and important type of experiment where the two stages are not independent:

> 'Choose an elector at random from the NSW population. First note the elector's gender. Then ask the elector if he or she voted Labor or non-Labor in the last State election.'

In this example, one might suspect that the gender and the political opinion of a person may not be independent and that there is *correlation* between them. This is in fact the case, as almost every opinion poll has shown over the years. More on this in Year 12.

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Example 15 10E

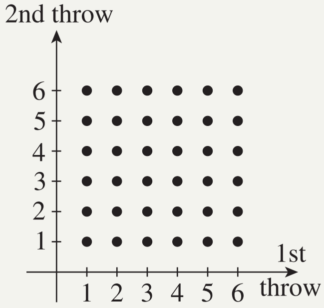
A pair of dice is thrown twice. What is the probability that the first throw is a double, and the second throw gives a sum of at least 4?

SOLUTION

We saw in Section 10B that when two dice are thrown, there are 36 possible outcomes, graphed in the diagram to the right.

There are six doubles among the 36 possible outcomes,

so $$P(\text{double}) = \frac{6}{36} = \frac{1}{6}.$$



All but the pairs (1, 1), (2, 1) and (1, 2) give a sum at least 4,

so $$P(\text{sum is at least } 4) = \frac{33}{36} = \frac{11}{12}.$$

Because the two stages are independent,

$$P(\text{double, sum at least } 4) = \frac{1}{6} \times \frac{11}{12} = \frac{11}{72}.$$

Multi-stage experiments — the product rule

The same arguments clearly apply to an experiment with any number of stages.

15 MULTI-STAGE EXPERIMENTS

If $A_1, A_2, \ldots, A_n$ are events in successive independent stages of a multi-stage experiment, then

$$P(A_1 A_2 \ldots A_n) = P(A_1) \times P(A_2) \times \ldots \times P(A_n).$$

Example 16 10E

A coin is tossed five times. Find the probability that:

a every toss is a head

b no toss is a head

c there is at least one head.

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SOLUTION

The five tosses are independent events.

a $P(\text{HHHHH}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{32}$

b $P(\text{no heads}) = P(\text{TTTTT})$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{32}$

c $P(\text{at least one head}) = 1 - P(\text{no heads})$

$= 1 - \frac{1}{32}$

$= \frac{31}{32}$

Sampling without replacement — an extension of the product rule

The product rule can be extended to the following question, where the two stages of the experiment are not independent.

Example 17 10E

A box contains five discs numbered 1, 2, 3, 4 and 5.
Two numbers are drawn in succession, without replacement.
What is the probability that both are even?

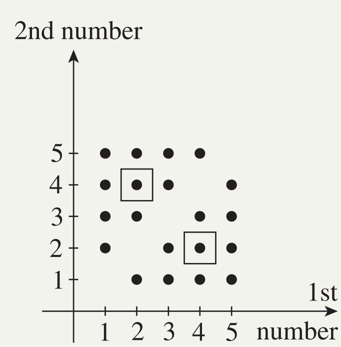
SOLUTION

The probability that the first number is even is $\frac{2}{5}$.

When this even number is removed, one even and three odd numbers remain, so the probability that the second number is also even is $\frac{1}{4}$.

Hence $P(\text{both even}) = \frac{2}{5} \times \frac{1}{4}$

$= \frac{1}{10}$.



Note: The graph to the right allows the calculation to be checked by examining its full sample space.

Because doubles are not allowed (i.e. there is no replacement), there are only 20 possible outcomes.

The two boxed outcomes are the only outcomes that consist of two even numbers, giving the same probability of $\frac{2}{20} = \frac{1}{10}$.

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Listing the favourable outcomes

The product rule is often combined with a listing of the favourable outcomes. A tree diagram may help in producing that listing, although this is hardly necessary in the straightforward example below, which continues an earlier example.

Example 18 10E

A coin is tossed four times. Find the probability that:

a the first three coins are heads

b the middle two coins are tails.

c Copy and complete this table:

Number of heads	0	1	2	3	4
Probability					

SOLUTION

a $P\text{(the first three coins are heads)} = P(\text{HHHH}) + P(\text{HHHT})$

(notice that the two events HHHH and HHHT are mutually exclusive)

$$= \frac{1}{16} + \frac{1}{16}$$

$\left(\text{because each of these two probabilities is } \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$

$$= \frac{1}{8}.$$

b $P\text{(middle two are tails)} = P(\text{HTTH}) + P(\text{HTTT}) + P(\text{TTTH}) + P(\text{TTTT})$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$= \frac{1}{4}.$$

c $P\text{(no heads)} = P(\text{TTTT})$

$$= \frac{1}{16}$$

$P\text{(one head)} = P(\text{HTTT}) + P(\text{THTT}) + P(\text{TTHT}) + P(\text{TTTH})$

$$= \frac{4}{16}$$

$P\text{(two heads)} = P(\text{HHTT}) + P(\text{HTHT}) + P(\text{THHT}) + P(\text{HTTH}) + P(\text{THTH}) + P(\text{TTHH})$

(these are all six possible orderings of H, H, T and T)

$= \frac{6}{16}$, and the last two results follow by symmetry.

Number of heads	0	1	2	3	4
Probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

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Describing the experiment in a different way

Sometimes, the manner in which an experiment is described makes calculation difficult, but the experiment can be described in a different way so that the probabilities are the same but the calculations are much simpler.

Example 19 10E

Wes is sending Christmas cards to ten friends. He has two cards with angels, two with snow, two with reindeer, two with trumpets and two with Santa Claus.
What is the probability that Harry and Helmut get matching cards?

SOLUTION

Describe the process in a different way as follows:

> 'Wes decides that he will choose Harry's card first and Helmut's card second. Then he will choose the cards for his remaining eight friends.'

All that matters now is whether the card that Wes chooses for Helmut is the same as the card that he has already chosen for Harry. After he chooses Harry's card, there are nine cards remaining, of which only one matches Harry's card. Thus the probability that Helmut's card matches is $\frac{1}{9}$.

Exercise 10E

FOUNDATION

1 Suppose that A, B, C and D are independent events, with $P(A) = \frac{1}{8}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$ and $P(D) = \frac{2}{7}$. Use the product rule to find:
a $P(AB)$ b $P(AD)$ c $P(BC)$ d $P(ABC)$ e $P(BCD)$ f $P(ABCD)$

2 A coin and a die are tossed. Use the product rule to find the probability of obtaining:
a a three and a head
b a six and a tail
c an even number and a tail
d a number less than five and a head

3 One set of cards contains the numbers 1, 2, 3, 4 and 5, and another set contains the letters A, B, C, D and E. One card is drawn at random from each set. Use the product rule to find the probability of drawing:
a 4 and B
b 2 or 5, then D
c 1, then A or B or C
d an odd number and C
e an even number and a vowel
f a number less than 3, and E
g the number 4, followed by a letter from the word MATHS.

4 Two marbles are picked at random, one from a bag containing three red and four blue marbles, and the other from a bag containing five red and two blue marbles. Find the probability of drawing:
a two red marbles
b two blue marbles
c a red marble from the first bag and a blue marble from the second.

5 A box contains five light bulbs, two of which are faulty. Two bulbs are selected, one at a time without replacement. Find the probability that:
a both bulbs are faulty
b neither bulb is faulty
c the first bulb is faulty and the second one is not
d the second bulb is faulty and the first one is not.

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DEVELOPMENT

6 A die is rolled twice. Using the product rule, find the probability of throwing:

a a six and then a five
b a one and then an odd number
c a double six
d two numbers greater than four
e a number greater than four and then a number less than four.

7 A box contains 12 red and 10 green discs. Three discs are selected, one at a time without replacement.

a What is the probability that the discs selected are red, green, red in that order?
b What is the probability of this event if the disc is replaced after each draw?

8 a From a standard pack of 52 cards, two cards are drawn at random without replacement. Find the probability of drawing:

i a spade and then a heart
ii two clubs
iii a jack and then a queen
iv the king of diamonds and then the ace of clubs.

b Repeat the question if the first card is replaced before the second card is drawn.

9 A coin is weighted so that it is twice as likely to fall heads as it is tails.

a Write down the probabilities that the coin falls:

i heads
ii tails.

b If you toss the coin three times, find the probability of:

i three heads
ii three tails
iii head, tail, head in that order.

10 If a coin is tossed repeatedly, find the probability of obtaining at least one head in:

a two tosses
b five tosses
c ten tosses.

11 [Valid and invalid arguments]
Identify any fallacies in the following arguments. If possible, give some indication of how to correct them.

a ‘The probability that a Year 12 student chosen at random likes classical music is 50%, and the probability that a student plays a classical instrument is 20%. Therefore the probability that a student chosen at random likes classical music and plays a classical instrument is 10%.’

b ‘The probability of a die showing a prime is $\frac{1}{2}$, and the probability that it shows an odd number is $\frac{1}{2}$. Hence the probability that it shows an odd prime number is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.’

c ‘I choose a team at random from an eight-team competition. The probability that the team wins any game is $\frac{1}{2}$, so the probability that it defeats all of the other seven teams is $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$.’

d ‘A normal coin is tossed and shows heads eight times. Nevertheless, the probability that it shows heads the next time is still $\frac{1}{2}$.’

12 A die is rolled twice. Using the product rule, find the probability of rolling:

a a double two
b any double
c a number greater than three, then an odd number
d a one and then a four
e a four and then a one
f a one and a four in any order
g an even number, then a five
h a five and then an even number
i an even number and a five in any order.

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13 An archer fires three shots at a bullseye. He has a 90% chance of hitting the bullseye. Using H for hit and M for miss, list all eight possible outcomes. Then, assuming that successive shots are independent, use the product rule to find the probability that he will:

a hit the bullseye three times
b miss the bullseye three times
c hit the bullseye on the first shot only
d hit the bullseye exactly once
e miss the bullseye on the first shot only
f miss the bullseye exactly once.

(Hint: Part **d** requires adding the probabilities of HMM, MHM and MMH, and part **f** requires a similar calculation.)

CHALLENGE

14 One layer of tinting material on a window cuts out $\frac{1}{5}$ of the sun's UV light coming through it.

a What fraction would be cut out by using two layers?
b How many layers would be required to cut out at least $\frac{9}{10}$ of the UV light?
c What relation does this question have to questions about probability?

15 In a lottery, the probability of the jackpot prize being won in any draw is $\frac{1}{60}$.

a What is the probability that the jackpot prize will be won in each of four consecutive draws?
b How many consecutive draws need to be made for there to be a greater than 98% chance that at least one jackpot prize will have been won?

16 (This question is best done by retelling the story of the experiment, as explained in the notes above.) Nick has five different pairs of socks to last the working week, and they are scattered loose in his drawer. Each morning, he gets up before light and chooses two socks at random. Find the probability that he wears a matching pair:

a on the first morning
b on the last morning
c on the third morning
d the first two mornings
e every morning
f every morning but one.

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10F Probability tree diagrams

In more complicated problems, and particularly in unsymmetric situations, a probability tree diagram can be very useful in organising the various cases, in preparation for the application of the product rule.

Constructing a probability tree diagram

A *probability tree diagram* differs from the tree diagrams used in Section 10B for counting possible outcomes, in that the relevant probabilities are written on the branches and then multiplied together in accordance with the product rule. An example will demonstrate the method.

As before, these diagrams have one column labelled 'Start', a column for each stage, and a column listing the outcomes, but there is now an extra column labelled 'Probability' at the end.

Example 20 10F

One card is drawn from each of two packs. Use a probability tree diagram to find the probability that:

a both cards are court cards (jack, queen or king)

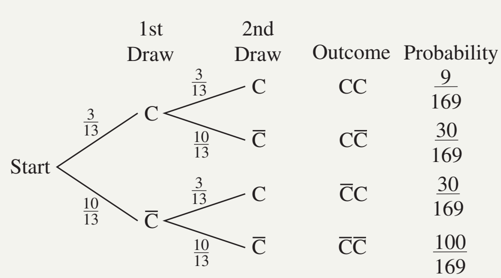
b neither card is a court card

c one card is a court card and the other is not.

SOLUTION

In each pack of cards, there are 12 court cards out of 52 cards,

so $P(\text{court card}) = \frac{3}{13}$ and $P(\text{not a court card}) = \frac{10}{13}$.



Multiply the probabilities along each arm because they are successive stages.
Add the probabilities in the final column because they are mutually exclusive.

a $P(\text{two court cards}) = \frac{3}{13} \times \frac{3}{13}$
$= \frac{9}{169}$

b $P(\text{no court cards}) = \frac{10}{13} \times \frac{10}{13}$
$= \frac{100}{169}$

c $P(\text{one court card}) = \frac{3}{13} \times \frac{10}{13} + \frac{10}{13} \times \frac{3}{13}$
$= \frac{30}{169} + \frac{30}{169}$
$= \frac{60}{169}$

Note: The four probabilities in the last column of the tree diagram add exactly to 1, which is a useful check on the working. The three answers also add to 1.

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17 PROBABILITY TREE DIAGRAMS

In a *probability tree diagram*:

- Probabilities are written on the branches.
- There is a final column giving the probability of each outcome.

Example 21 [A more complicated experiment] 10F

A bag contains six white marbles and four blue marbles. Three marbles are drawn in succession. At each draw, if the marble is white it is replaced, and if it is blue it is not replaced.

Find the probabilities of drawing:

a no blue marbles

b one blue marble

c two blue marbles

d three blue marbles.

SOLUTION

With the ten marbles all in the bag:

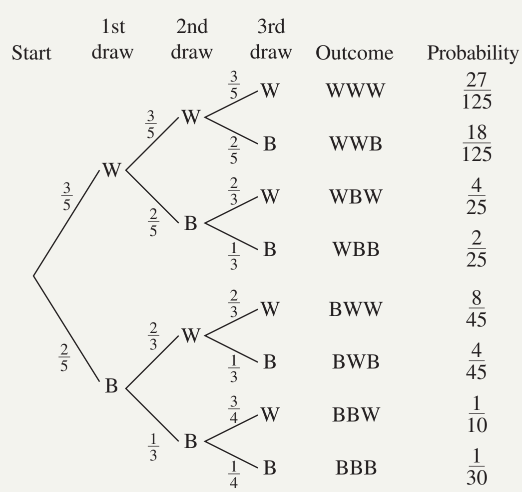
$$P(W) = \tfrac{6}{10} = \tfrac{3}{5} \qquad \text{and} \qquad P(B) = \tfrac{4}{10} = \tfrac{2}{5}.$$

If one blue marble is removed, there are six white and three blue marbles, so

$$P(W) = \tfrac{6}{9} = \tfrac{2}{3} \qquad \text{and} \qquad P(B) = \tfrac{3}{9} = \tfrac{1}{3}.$$

If two blue marbles are removed, there are six white and two blue marbles, so

$$P(W) = \tfrac{6}{8} = \tfrac{3}{4} \qquad \text{and} \qquad P(B) = \tfrac{2}{8} = \tfrac{1}{4}.$$



In each part, multiply the probabilities along each arm and then add the cases.

a $P(\text{no blue marbles}) = \frac{27}{125}$

b $P(\text{one blue marble}) = \frac{18}{125} + \frac{4}{25} + \frac{8}{45}$
$= \frac{542}{1125}$

c $P(\text{two blue marbles}) = \frac{2}{25} + \frac{4}{45} + \frac{1}{10}$
$= \frac{121}{450}$

d $P(\text{three blue marbles}) = \frac{1}{30}$

Note: Again, as a check on the working, your calculator will show that the eight probabilities in the last column of the diagram add exactly to 1 and that the four answers above also add to 1.

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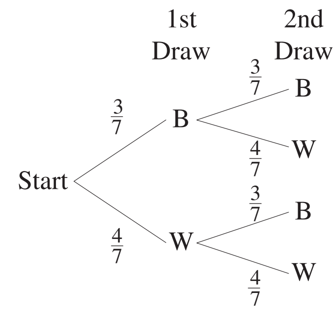
Exercise 10F

FOUNDATION

1 A bag contains three black and four white discs. A disc is selected from the bag, its colour is noted, and the disc is then returned to the bag before a second disc is drawn.



a By multiplying along the branches of the tree, find:

i $P(\text{BB})$ **ii** $P(\text{BW})$

iii $P(\text{WB})$ **iv** $P(\text{WW})$

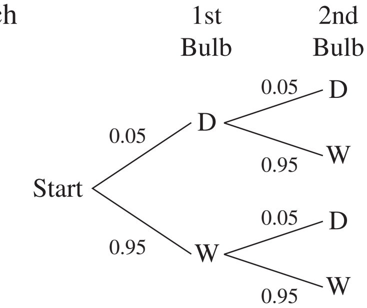
b Hence, by adding, find the probability that:

i both discs drawn have the same colour

ii the discs drawn have different colours.

c Draw your own tree diagram, and repeat part **b** if the first disc is not replaced before the second one is drawn.

2 Two light bulbs are selected at random from a large batch of bulbs in which 5% are defective.



a By multiplying along the branches of the tree, find:

i $P(\text{both bulbs work})$

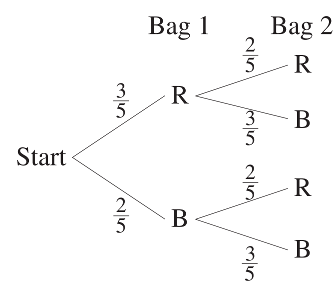
ii $P(\text{the first works, but the second is defective})$

iii $P(\text{the first is defective, but the second works})$

iv $P(\text{both bulbs are defective})$.

b Hence find the probability that at least one bulb works.

3 One bag contains three red and two blue balls and another bag contains two red and three blue balls. A ball is drawn at random from each bag.



a By multiplying along the branches of the tree, find:

i $P(\text{RR})$ **ii** $P(\text{RB})$

iii $P(\text{BR})$ **iv** $P(\text{BB})$

b Hence, by adding, find the probability that:

i the balls are the same colour

ii the balls are different colours.

4 In group A there are three girls and seven boys, and in group B there are six girls and four boys. One person is chosen at random from each group. Draw a probability tree diagram.

a By multiplying along the branches of the tree, find:

i $P(\text{GG})$ **ii** $P(\text{GB})$ **iii** $P(\text{BG})$ **iv** $P(\text{BB})$

b Hence, by adding, find the probability that:

i two boys or two girls are chosen **ii** one boy and one girl are chosen.

5 There is an 80% chance that Garry will pass his driving test and a 90% chance that Emma will pass hers. Draw a probability tree diagram, and find the chance that:

a Garry passes and Emma fails

b Garry fails and Emma passes

c only one of Garry and Emma passes

d at least one fails.

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DEVELOPMENT

6 The probability that a set of traffic lights will be green when you arrive at them is $\frac{3}{5}$. A motorist drives through two sets of lights. Assuming that the two sets of traffic lights are not synchronised, find the probability that:

a both sets of lights will be green

b at least one set of lights will be green.

7 A factory assembles calculators. Each calculator requires a chip and a battery. It is known that 1% of chips and 4% of batteries are defective. Find the probability that a calculator selected at random will have at least one defective component.

8 The probability of a woman being alive at 80 years of age is 0.2, and the probability of her husband being alive at 80 years of age is 0.05. Find the probability that:

a they will both live to be 80 years of age

b only one of them will live to be 80.

9 Alex and Julia are playing in a tennis tournament. They will play each other twice, and each has an equal chance of winning the first game. If Alex wins the first game, his probability of winning the second game is increased to 0.55. If he loses the first game, his probability of winning the second game is reduced to 0.25. Find the probability that Alex wins exactly one game.

10 One bag contains four red and three blue discs, and another bag contains two red and five blue discs. A bag is chosen at random and then a disc is drawn from it. Find the probability that the disc is blue.

11 In a raffle in which there are 200 tickets, the first prize is drawn and then the second prize is drawn without replacing the winning ticket. If you buy 15 tickets, find the probability that you win:

a both prizes

b at least one prize.

12 A box contains 10 chocolates, all of identical appearance. Three of the chocolates have caramel centres and the other seven have mint centres. Hugo randomly selects and eats three chocolates from the box. Find the probability that:

a the first chocolate Hugo eats is caramel

b Hugo eats three mint chocolates

c Hugo eats exactly one caramel chocolate.

13 In an aviary there are four canaries, five cockatoos and three budgerigars. If two birds are selected at random, find the probability that:

a both are canaries

b neither is a canary

c one is a canary and one is a cockatoo

d at least one is a canary.

14 Max and Jack each throw a die. Find the probability that:

a they do not throw the same number

b the number thrown by Max is greater than the number thrown by Jack

c the numbers they throw differ by three.

15 In a large co-educational school, the population is 47% female and 53% male. Two students are selected from the school population at random. Find, correct to two decimal places, the probability that:

a both are male

b a girl and a boy are selected.

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16 The numbers 1, 2, 3, 4 and 5 are each written on a card. The cards are shuffled and one card is drawn at random. The number is noted and the card is then returned to the pack. A second card is selected, and in this way a two-digit number is recorded. For example, a 2 on the first draw and a 3 on the second results in the number 23.

- **a** What is the probability of the number 35 being recorded?
- **b** What is the probability of an odd number being recorded?

17 A twenty-sided die has the numbers from 1 to 20 on its faces.

- **a** If it is rolled twice, what is the probability that the same number appears on the uppermost face each time?
- **b** If it is rolled three times, what is the probability that the number 15 appears on the uppermost face exactly twice?

18 An interviewer conducts a poll in Sydney and Melbourne on the popularity of the prime minister. In Sydney, 52% of the population approve of the prime minister, and in Melbourne her approval rating is 60%. If one of the two capital cities is selected at random and two electors are surveyed, find the probability that:

- **a** both electors approve of the prime minister
- **b** at least one elector approves of the prime minister.

19 In a bag there are four green, three blue and five red discs.

- **a** Two discs are drawn at random, and the first disc is not replaced before the second disc is drawn. Find the probability of drawing:
  - **i** two red discs
  - **ii** one red and one blue disc
  - **iii** at least one green disc
  - **iv** a blue disc on the first draw
  - **v** two discs of the same colour
  - **vi** two differently coloured discs.
- **b** Repeat part **a** if the first disc is replaced before the second disc is drawn.

CHALLENGE

20 In a game, two dice are rolled and the score given is the higher of the two numbers on the uppermost faces. For example, if the dice show a three and a five, the score is a five.

- **a** Find the probability that you score a one in a single throw of the two dice.
- **b** What is the probability of scoring three consecutive ones in three rolls of the dice?
- **c** Find the probability that you score a six in a single roll of the dice.

21 In each game of Sic Bo, three regular six-sided dice are thrown once.

- **a** In a single game, what is the probability that all three dice show six?
- **b** What is the probability that exactly two of the dice show six?
- **c** What is the probability that exactly two of the dice show the same number?
- **d** What is the probability of rolling three different numbers on the dice?

22 A bag contains two green and two blue marbles. Marbles are drawn at random, one by one without replacement, until two green marbles have been drawn. What is the probability that exactly three draws will be required?

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10G Conditional probability

Sometimes our knowledge of the results of an experiment change over time, and as we gain more information, our probabilities change. Conditional probability allows us to calculate these changing probabilities.

Reduced sample space and reduced event space

Drago will win a certain game if an odd number is thrown on a die. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$, and the event space is $A = \{1, 3, 5\}$, so

$$\begin{aligned} P\,(\text{he wins}) &= \tfrac{3}{6} \\ &= \tfrac{1}{2}. \end{aligned}$$

Drago nervously turns away as the die is thrown, and someone calls out, 'It's a prime number!' What, for him, is now the probability that he wins the game?

- First, the sample space has now been reduced to the set $B = \{2, 3, 5\}$ of prime numbers on the die. This is called the *reduced sample space*.
- Secondly, the event space has now been reduced to the set $\{3, 5\}$ of odd prime numbers on the die. This *reduced event space* is the intersection $A \cap B$ of the event space A and the reduced event space B. Hence

$$\begin{aligned} P\,(\text{he wins, given that the number is prime}) &= \frac{|\text{reduced event space}|}{|\text{reduced sample space}|} \\ &= \tfrac{2}{3}. \end{aligned}$$

Alternatively, with B as the reduced sample space and $A \cap B$ as the reduced event space, we can write the working using a concise formula,

$$\begin{aligned} P\,(A\,|\,B) &= \frac{|A \cap B|}{|B|} \\ &= \tfrac{2}{3}. \end{aligned}$$

where $P\,(A\,|\,B)$ is the symbol for conditional probability, and means 'the probability of A, given that B has occurred'.

(Of course we are assuming that the person calling out was telling the truth.)

Conditional probability

It is only a matter of convenience whether we work with the reduced sample space and reduced event space, or with formulae.

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18 CONDITIONAL PROBABILITY

Suppose that two events A and B are subsets of same uniform sample space S. Then the *conditional probability* $P(A|B)$ of A, given that B has occurred, is obtained by removing all the elements not in B from the sample space S, and from the event space A. Thus there are two equivalent formulae,

$$P(A|B) = \frac{|\text{reduced event space}|}{|\text{reduced sample space}|} \quad \text{and} \quad P(A|B) = \frac{|A \cap B|}{|B|}.$$

Read $P(A|B)$ as 'the probability of A, given that B has occurred'.

Example 22 10G

A card is drawn at random from a normal pack of 52 cards. Find the probability that it is a 9, 10, jack or queen if:

a Nothing more is known.

b It is known to be a court card (jack, queen or king).

c It is known to be an 8 or a 10.

d It is known not to be a 2, 3, 4 or 5.

SOLUTION

Let the sample space S be the set of all 52 cards, and let

$A = \{9♣, 9♦, 9♥, 9♠, 10♣, 10♦, 10♥, 10♠, J♣, J♦, J♥, J♠, Q♣, Q♦, Q♥, Q♠\}$.

a There is no reduction of the sample space or the event space,

$$\text{so } P(A) = \frac{|A|}{|S|} = \tfrac{16}{52} = \tfrac{4}{13}.$$

b The reduced sample space is

$$B = \{\text{court cards}\},$$

and the reduced event space is

$$A \cap B = \{\text{jacks and queens}\},$$

$$\text{so } P(A|B) = \frac{|A \cap B|}{|B|} = \tfrac{8}{12} = \tfrac{2}{3}.$$

c The reduced sample space is

$$B = \{8\text{s and }10\text{s}\}$$

and the reduced event space is

$$A \cap B = \{10s\},$$

$$\text{so } P(A|B) = \frac{|A \cap B|}{|B|} = \tfrac{4}{8} = \tfrac{1}{2}.$$

d The reduced sample space is

$$B = \{\text{cards not } 2, 3, 4 \text{ or } 5\}$$

and the reduced event space is

$$A \cap B = A,$$

$$\text{so } P(A|B) = \frac{|A \cap B|}{|B|} = \tfrac{16}{36} = \tfrac{4}{9}.$$

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Another formula for conditional probability

Suppose as before that two events A and B are subsets of the same uniform sample space S. The previous formula gave $P(A|B)$ in terms of the sizes of the sets $A \cap B$ and B. By dividing through by $|S|$, we obtain a formula for $P(A|B)$ in terms of the probabilities of $A \cap B$ and B:

$$\begin{aligned} P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \frac{|A \cap B|}{|S|} \div \frac{|B|}{|S|} \\ &= \frac{P(A \cap B)}{P(B)} \end{aligned}$$

19 ANOTHER FORMULA FOR CONDITIONAL PROBABILITY

For two events A and B,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

The great advantage of this formula is that it can be used when nothing is known about the actual sizes of the sample space and the event space, as in the example below.

Example 23 10G

In a certain population, 35% of people have blue eyes, 15% have blond hair, and 10% have blue eyes and blond hair. A person is chosen from this population at random.

a Find the probability that they have blond hair, given that they have blue eyes.

b Find the probability that they have blue eyes, given that they have blond hair.

SOLUTION

From the given percentages, we know that

$$P(\text{blue eyes}) = 0.35, \quad P(\text{blond hair}) = 0.15, \quad P(\text{blue eyes and blond hair}) = 0.1\,.$$

a
$$\begin{aligned} P(\text{blond hair, given blue eyes}) &= \frac{P(\text{blond hair and blue eyes})}{P(\text{blue eyes})} \\ &= \frac{0.1}{0.35} \\ &\doteqdot 0.29 \end{aligned}$$

b
$$\begin{aligned} P(\text{blue eyes, given blond hair}) &= \frac{P(\text{blond hair and blue eyes})}{P(\text{blond hair})} \\ &= \frac{0.1}{0.15} \\ &\doteqdot 0.67 \end{aligned}$$

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Independent events

We say that an event A is *independent* of an event B if knowing whether or not B has occurred does not affect our probability of A. That is,

$$A \text{ and } B \text{ are independent} \qquad \text{means that} \qquad P(A|B) = P(A).$$

20 INDEPENDENT EVENTS

- Two events A and B are called *independent* if the probability of A is not affected by knowing whether or not B has occurred:
  $$P(A|B) = P(A)$$
- If A is independent of B, then B is independent of A.

The second dot point may seem trivial, but it needs proving.
Suppose that A is independent of B, meaning that $P(A|B) = P(A)$.
Then to prove that B is independent of A,

$$\begin{aligned}
P(B|A) &= \frac{P(B \cap A)}{P(A)} && \big(\text{using the formula for } P(B|A)\big)\\
&= \frac{P(A \cap B)}{P(B)} \times \frac{P(B)}{P(A)} && \big(\text{multiply top and bottom by } P(B)\big)\\
&= P(A|B) \times \frac{P(B)}{P(A)} && \big(\text{using the formula for } P(A|B)\big)\\
&= P(A) \times \frac{P(B)}{P(A)} && \big(\text{we are assuming that } P(A|B) = P(A)\big)\\
&= P(B), \quad \text{as required.}
\end{aligned}$$

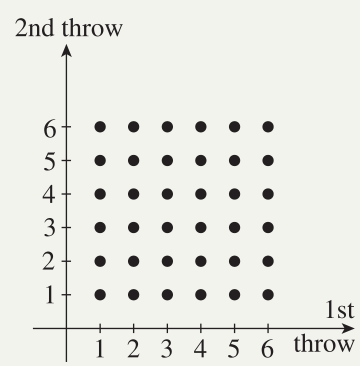
The consequence of this is that we can just say that ‘the two events A and B are independent’, which is what we would expect.

Example 24 10G

Two dice are thrown one after the other.

- Let A be the event ‘the first die is odd’.
- Let B be the event ‘the second die is 1, 2 or 3’.
- Let C be the event ‘the sum is five’.

Which of the three pairs of events are independent?



SOLUTION

The sample space S has size 36, and

$$P(A) = \tfrac{18}{36} = \tfrac{1}{2}, \quad P(B) = \tfrac{18}{36} = \tfrac{1}{2}, \quad P(C) = \tfrac{4}{36} = \tfrac{1}{9}.$$

The last result follows because $5 = 1 + 4 = 2 + 3 = 3 + 2 = 4 + 1$ are all the ways that a sum of five can be obtained. Taking intersections,

$$P(A \cap B) = \tfrac{9}{36} = \tfrac{1}{4}, \quad P(B \cap C) = \tfrac{3}{36} = \tfrac{1}{12}, \quad P(C \cap A) = \tfrac{2}{36} = \tfrac{1}{18}.$$

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Testing the definition of independence on the three pairs of events:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{4} \div \frac{1}{2} = \frac{1}{2} = P(A)$$

$$P(B|C) = \frac{P(B \cap C)}{P(C)} = \frac{1}{12} \div \frac{1}{9} = \frac{3}{4} \neq P(B)$$

$$P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{1}{18} \div \frac{1}{2} = \frac{1}{9} = P(C)$$

Hence A and B are independent, and C and A are independent, but B and C are not independent.

Note: The word 'independent' here means 'mathematically independent', and carries no scientific implication. A scientist who makes a statement about the natural world, on the basis of such mathematical arguments, is doing science, not mathematics.

Note: The word 'independent' was introduced in Section 10E with multi-stage experiments. Its use in this section is a generalisation of its use in the multi-stage situation. For example, the events A and B above are independent in both senses, but the independence of C and A is a new idea.

The product rule

The formula $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ can be rearranged as

$$P(A \cap B) = P(A|B) \times P(B).$$

If the events are independent, then $P(A|B) = P(A)$, so

$$P(A \cap B) = P(A) \times P(B).$$

Conversely, if $P = (A \cap B) = P(A) \times P(B)$, then $P(A|B) = P(A)$, meaning that the events are independent. Thus we have both a formula that applies to independent events, and another test for independence:

21 INDEPENDENT EVENTS AND THE PRODUCT RULE

- For any events A and B,
  $$P(A \text{ and } B) = P(A|B) \times P(B).$$
- If A and B are independent events, then
  $$P(A \text{ and } B) = P(A) \times P(B).$$
- Conversely, if $P(A \text{ and } B) = P(A) \times P(B)$, then the events are independent.

Remember that $P(A \text{ and } B)$ and $P(A \cap B)$ mean exactly the same thing.

All this is a generalisation of the product rule introduced with the multi-stage experiments and tree diagrams in Section 10E.

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Example 25 10G

Calculate $P(A) \times P(B)$ and $P(B) \times P(C)$ and $P(C) \times P(A)$ in the previous example. Then use the third dotpoint of Box 21 above to confirm that these calculations give the same results for the independence of A and B, of B and C, and of C and A.

SOLUTION

$$\begin{aligned} P(A) \times P(B) &= \tfrac{1}{2} \times \tfrac{1}{2} \\ &= \tfrac{1}{4} \\ &= P(A \cap B) \end{aligned} \qquad \begin{aligned} P(B) \times P(C) &= \tfrac{1}{2} \times \tfrac{1}{9} \\ &= \tfrac{1}{18} \\ &\neq P(B \cap C) \end{aligned} \qquad \begin{aligned} P(C) \times P(A) &= \tfrac{1}{9} \times \tfrac{1}{2} \\ &= \tfrac{1}{18} \\ &= P(C \cap A) \end{aligned}$$

again showing that A and B are independent, that C and A are independent, but that B and C are not independent.

Example 26 10G

Example 23 described the incidence of blue eyes and blond hair in a population:

a Why does the product rule formula show immediately that 'blue eyes' and 'blond hair' are not independent?

b What would be the probability of having blue eyes and blond hair if the two characteristics were independent?

SOLUTION

a Box 21 says that 'blue eyes' and blond hair are independent if and only if

$$P(\text{blue eyes and blond hair}) = P(\text{blue eyes}) \times P(\text{blond hair}).$$

But according to the data given in Example 23, LHS $= 0.1$ and RHS $= 0.35 \times 0.15 = 0.0525$, so the events are not independent.

b If the events were independent, then the product rule formula would hold, so

$$\begin{aligned} P(\text{blue eyes and blond hair}) &= P(\text{blue eyes}) \times P(\text{blond hair}) \\ &= 0.35 \times 0.15 \\ &= 0.0525. \end{aligned}$$

Using the sample space without any formulae

Sometimes, however, it is easier not to use any of the machinery in this section, but to go back to basics and deal directly with the sample space, with little or no notation.

Example 27 10G

Yanick and Oskar can't agree on which movie to watch, so they play a game throwing two dice.

- If the sum is 7 or 8, Yanick chooses the movie.
- If the sum is 9, 10, 11 or 12, Oskar chooses the movie.
- If the sum is less than 7, they throw the dice again.

Who has the higher probability of choosing the movie?

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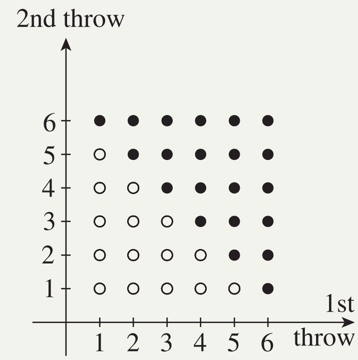
SOLUTION

There are 36 possible throws in the original sample space.
But we can ignore the throws whose sum is less than 7, represented by the open circles, because these 15 throws are discarded if they occur.
This leaves a sample space with only the 21 closed circles.
There are $6 + 5 = 11$ throws with sum 7 or 8, so

$$P(\text{Yanick chooses}) = \tfrac{11}{21}$$

and $$P(\text{Oskar chooses}) = \tfrac{10}{21}.$$

Hence Yanick has the higher probability of choosing.



A final note about general sample spaces

The theory developed in this chapter can be summarised in four results:

- $P(\text{not } A) = 1 - P(A)$
- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$
- $P(A \text{ and } B) = P(A) \times P(B)$ if and only if A and B are independent events.

These four results were all developed using equally likely possible outcomes, which restricts them to experiments in which there is a finite sample space that is uniform. The four results, however, hold in all theories of probability, either as axioms and definitions, or as results proven by different methods.

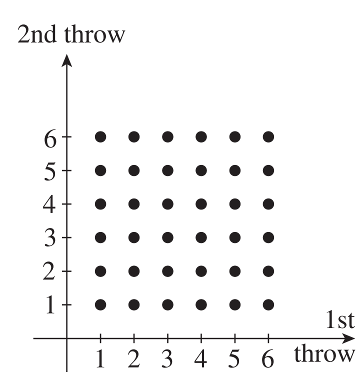
Exercise 10G

FOUNDATION

Note: Several questions in this exercise, and elsewhere, require the assumptions that a baby is equally likely to be a boy or a girl, that these are the only two possibilities, and that the sex of a child is independent of the sex of any previous children. These assumptions are simplifications of complex scientific and social considerations, and they do not describe the real situation.

1 Two dice are thrown. The sample space of this experiment is shown in the dot diagram alongside. Following the throw, it is revealed that the first die shows an even number.

- **a** Copy the diagram and circle the reduced sample space.
- **b** Find the conditional probability of getting two sixes.
- **c** Find the conditional probability of getting at least one six.
- **d** Find the conditional probability that the sum of the two numbers is five.



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2 A poll is taken amongst 1000 people to determine their voting patterns in the last election.

	Coalition	Labor	Other	Total
Male	130	340	110	580
Female	210	190	20	420
Total	340	530	130	1000

a Determine the probability that a particular person in the group voted Coalition.
b What is the probability that a particular person voted Labor, given that they were female?
c What is the probability that a particular member of the group was male, if it is known that they voted Coalition?
d Robin voted neither Coalition nor Labor. What is the probability that Robin was female?

3 A student is investigating if there is any relationship between those who choose Mathematics Extension 1 (M) and those who choose English Extension 1 (E) at his school.

	M	$\overline{M}$	Total
E	29	27	
$\overline{E}$	95	42	
Total			

a Copy and complete the table by filling in the totals.
b Find the probability that a particular student chose:
i neither Maths Extension 1 nor English Extension 1?
ii English Extension 1, given that they chose Maths Extension 1?
iii Maths Extension 1, if it is known that they chose English Extension 1?
iv Maths Extension 1, given that they did not choose English Extension 1?

4 Two cards are drawn from a standard pack — the first card is replaced and the pack shuffled before the second card is drawn — and the suit of each card (S, H, D or C) is noted by the game master. A player wants to know the probability that both cards are hearts.
a What is the probability if nothing else is known?
b The first card is known to be a heart. List the reduced sample space. What is the conditional probability of two hearts?
c List the reduced sample space if at least one of the cards is known to be a heart. What is the probability of two hearts?
d List the reduced sample space if the first card is known to be red. What is the probability of two hearts?

5 In a certain game, the player tosses two coins and then throws a die. The sample space is shown in the table.

The rules of the game assign one point for each head, and zero points for each tail. This score is then added to the score on the die.

	1	2	3	4	5	6
HH						
HT						
TH						
TT						

a Copy the table and fill in the point score for each outcome.
b Find the probability that a player gets more than 7 points.
c Suppose it is known that he has thrown two heads. Find the probability that a player gets more than 7 points.
d Find the probability that a player got an odd number of heads, given that their score was odd.

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6 Use the formula $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ to answer the following questions.

a Find $P(A|B)$ if $P(A \cap B) = 0.5$ and $P(B) = 0.7$.

b Find $P(A|B)$ if $P(A \cap B) = 0.15$ and $P(B) = 0.4$.

c Find $P(E|F)$ if $P(E \cap F) = 0.8$ and $P(F) = 0.95$.

7 The two events A and B in the following experiments are known to be independent.

a $P(A) = 0.4$, and $P(B) = 0.6$. Find $P(A \cap B)$.

b $P(A) = 0.3$, and $P(B) = 0.5$. Find $P(A \cap B)$.

c $P(A) = 0.4$, and $P(B) = 0.6$. Find $P(A|B)$.

d $P(A) = 0.7$, and $P(B) = 0.2$. Find $P(A|B)$.

8 Each of the following experiments involves two events, A and B. State in each case whether they are dependent or independent.

a $P(A|B) = 0.5$ and $P(A) = 0.4$ and $P(B) = 0.5$

b $P(A|B) = 0.3$ and $P(A) = 0.3$ and $P(B) = 0.6$

c $P(A|B) = \frac{3}{4}$ and $P(A) = \frac{2}{5}$ and $P(B) = \frac{3}{10}$

d $P(A) = 0.3$ and $P(B) = 0.7$ and $P(A \cap B) = 0.21$

e $P(A) = 0.2$ and $P(B) = 0.4$ and $P(A \cap B) = 0.8$

f $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{3}$

DEVELOPMENT

9 a Draw a table showing the sample space if two dice are thrown in turn and their sum is recorded.

b Highlight the reduced sample space if the sum of the two dice is 5. Given that the sum of the two dice is 5, find the probability that:

i the first die shows a 1,

ii at least one dice shows a 1,

iii at least one of the dice shows an odd number.

10 a For two events A and B it is known that $P(A \cup B) = 0.6$ and $P(A) = 0.4$ and $P(B) = 0.3$.

i Use the addition formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to find $P(A \cap B)$.

ii Use the formula for conditional probability to find $P(A|B)$.

iii Find $P(B|A)$.

b Suppose that $P(V \cup W) = 0.7$ and $P(V) = 0.5$ and $P(W) = 0.35$. Find $P(V|W)$.

c Find $P(X|Y)$ if $P(X \cap Y) = 0.2$ and $P(X) = 0.3$ and $P(Y) = 0.4$.

d Find $P(A|B)$ if $P(A \cup B) = \frac{1}{3}$ and $P(A) = \frac{1}{5}$ and $P(B) = \frac{3}{10}$.

11 Two dice are rolled. A three appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than seven.

12 The members of a cricket team know that in half of the games they played last season, they won the game and their star player Arnav was playing. Arnav consistently plays in 80% of the games. What is the probability that they will win this Saturday if Arnav is playing?

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13 **a** A couple has two children, the older of which is a boy. What is the probability that they have two boys?
b A couple has two children and at least one is a boy. What is the probability that they have two boys?

14 A couple has three children, each being either a boy or girl.
a List the sample space.
b Given that at least one is a boy, what is the probability that the oldest is male?
c Given that at least one of the first two children is a boy, what is the probability that the oldest is male?

15 A card is drawn from a standard pack. The dealer tells the players that it is a court card (jack, queen or king).
a What is the probability that it is a jack?
b What is the probability that it is either a jack or a red card?
c What is the probability that the next card drawn is a jack? Assume that the first card was not replaced.

16 Two dice are tossed in turn and the outcomes recorded. Let A be the probability that the first die is odd. Let S be the probability that the sum is odd. Let M be the probability that the product is odd.
a Use the definition of independence in Box 20 to find which of the three pairs of events are independent.
b Confirm your conclusions using the product-rule test for independence in Box 21.

17 The two events A and B in the following experiments are known to be independent.
a $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A \cup B)$.
b The probability of event A occurring is 0.6 and the probability of event B occuring is 0.3. What is the probability that either A or B occurs?

18 A set of four cards contains two jacks, a queen and a king. Bob selects one card and then, without replacing it, selects another. Find the probability that:
a both Bob's cards are jacks,
b at least one of Bob's cards is a jack
c given that one of Bob's cards is a jack, the other is also a jack.

19 A small committee of two is formed from a group of 4 boys and 8 girls. If at least one of the members of the committee is a girl, what is the probability that both members are girls?

20 Jack and Ben have been tracking their success rate of converting goals in their rugby games. Jack converts 70% of his goals and Ben converts 60%. At a recent home game, both get a kick, but only one converts his goal. What is the probability that it was Ben?

21 Susan picks two notes at random from four \$5, three \$10 and two \$20 notes. Given that at least one of the notes was \$10, what is the probability that Susan has picked up a total of \$20 or more?

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CHALLENGE

22 Researchers are investigating a potential new test for a disease, because although the usual test is totally reliable, it is painful and expensive.

The new test is intended to show a positive result when the disease is present. Unfortunately the test may show a *false positive*, meaning that the test result is positive, even though the disease is not present. The test may also show a *false negative*, meaning that the test result is negative even though the disease is in fact present.

The researchers tested a large sample of people who had symptoms that vaguely suggested that it was worth testing for the disease. They used the new test, and checked afterwards for the disease with the old reliable test, and came up with the following results:

- Of this sample group, 1% had the disease.
- Of those who had the disease, 80% tested positive and 20% tested negative.
- Of those who did not have the disease, 5% tested positive and 95% tested negative.

a What percentage of this group would test positive to this new test?

b Use the conditional probability formula $P(A \cap B) = P(A|B)P(B)$ to find the probability that a person tests positive, but does not have the disease.

c Find the probability that the new test gives a false positive. That is, find the probability the patient does not have the disease, given that the patient has tested positive.

d Find the probability that the new test gives a false negative. That is, find the probability the patient has the disease, given that the patient has tested negative.

e Comment on the usefulness of the new test.

23 a Prove that for events A and B,

$$P(A|B) = \frac{P(B|A)}{P(B)} \times P(A).$$

b Use this formula to produce an alternative solution to the probability of a false positive or false negative in the previous question.

24 Prove the *symmetry of independence*, that is, if B is independent of A then A is independent of B.

25 [A notoriously confusing question]
In a television game show, the host shows the contestant three doors, only one of which conceals the prize, and the game proceeds as follows. First, the contestant chooses a door. Secondly, the host then opens one of the other two doors, showing the contestant that it is not the prize door. Thirdly, the host invites the contestant to change their choice, if they wish. Analyse the game, and advise the contestant what to do.

26 A family has two children. Given that at least one of the children is a girl who was born on a Monday, what is the probability that both children are girls?

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Chapter 10 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 10 multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 If a die is rolled, find the probability that the uppermost face is:

a a two
b an odd number
c a number less than two
d a prime number.

2 A number is selected at random from the integers $1, 2, \ldots, 10$. Find the probability of choosing:

a the number three
b an even number
c a square number
d a negative number
e a number less than 20
f a multiple of three.

3 From a regular pack of 52 cards, one card is drawn at random. Find the probability that the chosen card is:

a black
b red
c a queen
d the ace of spades
e a club or a diamond
f not a seven.

4 A student has a 63% chance of passing his driving test. What is the chance that he does not pass?

5 A fair coin is tossed twice. Find the probability that the two tosses result in:

a two tails
b a head followed by a tail
c a head and a tail.

6 Two dice are thrown. Find the probability of:

a a double three
b a total score of five
c a total greater than nine
d at least one five
e neither a four nor a five appearing
f a two and a number greater than four
g the same number on both dice
h a two on at least one die.

7 In a group of 60 tourists to Tasmania, 31 visited Queenstown, 33 visited Strahan and 14 visited both places. Draw a Venn diagram and find the probability that a tourist:

a visited Queenstown only
b visited Strahan only
c did not visit either place.

8 A die is thrown. Let A be the event that an odd number appears. Let B be the event that a number less than five appears.

a Are A and B mutually exclusive?
b Find:

i $P(A)$
ii $P(B)$
iii $P(A \text{ and } B)$
iv $P(A \text{ or } B)$

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Review

9 The events A, B and C are independent, with $P(A) = \frac{1}{4}$, and $P(B) = \frac{1}{3}$ and $P(C) = \frac{3}{5}$.
Use the product rule to find:

a $P(AB)$ **b** $P(BC)$ **c** $P(AC)$ **d** $P(ABC)$

10 **a** From a standard pack of 52 cards, two cards are drawn at random without replacement. Find the probability of drawing:

i a club then a diamond, **ii** two hearts,
iii a seven then an ace, **iv** the queen of hearts then the eight of diamonds.

b Repeat part **a** if the first card is replaced before the second card is drawn.

11 There is a 70% chance that Harold will be chosen for the boys' debating team, and an 80% chance that Grace will be chosen for the girls' team. Draw a probability tree diagram and find the chance that:

a Harold is chosen, but Grace is not **b** Grace is chosen, but Harold is not
c only one of Harold and Grace is chosen **d** neither Harold nor Grace is chosen.

12 In a park there are four Labradors, six German shepherds and five beagles. If two dogs are selected at random, find the probability that:

a both are beagles **b** neither is a Labrador
c at least one is a Labrador **d** a beagle and German shepherd are chosen.

13 There are 500 tickets sold in a raffle. The winning ticket is drawn and then the ticket for second prize is drawn, without replacing the winning ticket. If you buy 20 tickets, find the probability that you win:

a both prizes **b** at least one prize.

14 In a certain experiment, the probability of events A and B are $P(A) = 0.6$ and $P(B) = 0.3$ respectively. In which of these cases are the events independent?

a $P(A \cap B) = 0.18$ **b** $P(A|B) = 0.3$ **c** $P(A \cup B) = 0.72$

15 Two dice are rolled. A five appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than nine.

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11 Probability distributions

In the previous chapter, the probabilities of individual outcomes of an experiment were calculated. In this chapter, we look at the *probability distribution* of the experiment, which consists of all the probabilities of all the possible outcomes. This provides an overview of the whole experiment.

Having established the probability distribution, we can calculate the *expected value* or *mean* of the experiment, which is a measure of central tendency. We can also calculate the *variance* and the *standard deviation*, which are measures of spread. The standard deviation is the square root of the variance.

The last section on sampling invites readers to perform some experiments of their own. This should give them some feeling for how the theory of probability distributions, expected value, and variance is reflected in practical experiments that they carry out themselves. It also has suggestions about analysing data from the internet. The relationship between theoretical probabilities and practical experiments is at the centre of the much-discussed subject of *statistics*.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *Overview* at the front of the textbook for details.

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11A The language of probability distributions

This section explains what a probability distribution is and introduces some necessary language. After a distinction is made between discrete and continuous probability distributions, the rest of the chapter deals only with discrete distributions, and continuous probability distributions are left until Year 12.

Discrete probability distributions

Four coins are tossed and the number of heads is recorded. There are five possible outcomes, 0, 1, 2, 3 and 4, with different probabilities. Using the methods of the previous chapter, we can calculate all five probabilities and arrange them in a table — this particular example was done in Example 18 of Section 10E:

Number of heads	0	1	2	3	4
Probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

The set $\{0, 1, 2, 3, 4\}$ is the sample space of the experiment, because each time the experiment is performed, the result will be one, and only one, of these five outcomes. It is not a uniform sample space, however, because these possible outcomes are not equally likely.

The set of these possible outcomes, together with the corresponding probabilities given in the table, is a *probability distribution* — in the context of probability distributions, we usually use the word *values* for the possible outcomes in the sample space. This particular probability distribution is a *numeric distribution* because all its values are numbers, and it is a *discrete distribution* because it is numeric and its values can be *listed*, which means they can be written down in some order (in this case we listed them as 0, 1, 2, 3, 4).

1 DISCRETE PROBABILITY DISTRIBUTIONS

Suppose that the sample space of an experiment can be *listed* — meaning that it can be written down in some order. Suppose also that the possible outcomes, usually called *values*, are numeric.

Then these possible outcomes, together with their probabilities, are called a *discrete probability distribution*.

The five probabilities in this example were originally calculated by dividing the results of tossing four coins into 16 equally likely possible outcomes. Many of our examples, however, will have probabilities that are obtained without any appeal to such a uniform finite sample space — some probabilities may be obtained by other forms of calculation, and some may be estimates obtained empirically.

Random experiments and random variables

Throwing four coins and recording the number of heads is a *random experiment* because there is more than one possible outcome. Throwing four coins into an empty bucket and counting the number of coins in the bucket is a *deterministic experiment* because 4 is the only possible outcome.

Denote by X the number of heads when four coins are thrown. This variable X is called a *random variable* because it is the result of a random experiment, and it can take the values 0, 1, 2, 3 and 4. Using the standard probability notation from the previous chapter, we can write

$$P(X = 3) = \tfrac{4}{16}.$$

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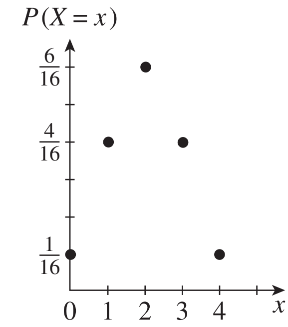
This is read as 'the probability that X takes the value 3', and means that the probability that there are three heads is $\frac{4}{16}$. Similarly,

$$P(X > 2) = \tfrac{5}{16} \qquad \text{and} \qquad P(X \text{ is odd}) = \tfrac{8}{16} \qquad \text{and} \qquad P(X \text{ is prime}) = \tfrac{10}{16}.$$

Using x (lower case) for the number of heads, the table of probabilities can now be written as:

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$



The distribution is graphed to the right, with the values on the horizontal axis and the probabilities on the vertical axis.

Categorical data

The values of a probability distribution don't have to be numbers. They may be names, or objects, or any verbal description. The general word for this is 'category', and such data is called *categorical*.

For example, the table to the right was claimed on the web to be the world's population by continent in 2016.

Continent	Population	Percentage
Asia	4436224000	59.7%
Africa	1216130000	16.4%
Europe	738849000	9.9%
North America	579024000	7.8%
South America	422535000	5.7%
Oceania	39901000	0.5%
Totals	7432663000	100%

Now consider the following experiment. A person is chosen at random from the world's population, and the continent he or she lives in recorded as the value of a random variable X. This experiment would be very hard indeed to carry out without bias, but we can imagine it. The percentages now become probabilities, which we can arrange in a probability distribution whose values are the six continents:

x	Asia	Africa	Europe	NA	SA	Oceania
$P(X = x)$	0.597	0.164	0.099	0.078	0.057	0.005

and, for example, $P(X = \text{Europe or North America}) = 0.099 + 0.078 = 0.177$.

2 RANDOM VARIABLE

- A *deterministic experiment* is an experiment with one possible outcome.
- A *random experiment* is an experiment with more than one possible outcome.
- A *random variable*, usually denoted by an upper-case letter such as X, is the result of running a random experiment. It may be *numeric* or *categorical*.

The calculations in the next two sections require the values to be numeric, so distributions with categorical values will not greatly concern us. A word of warning, however. A probability distribution whose values are categorical and can be listed is usually classified as discrete — but in this course, such a distribution is definitely not discrete because it is not numeric.

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Uniform discrete probability distributions

A discrete or categorical probability distribution is called *uniform* if all its values have the same probability. This means that the values are *equally likely possible outcomes*, and its sample space is a *uniform sample space*. This is what we were dealing with throughout the previous chapter.

3 UNIFORM DISCRETE PROBABILITY DISTRIBUTION

A discrete or categorical probability distribution is called *uniform* if the probabilities of all its values are the same. That is, the values are equally likely possible outcomes, and the sample space is a uniform sample space.

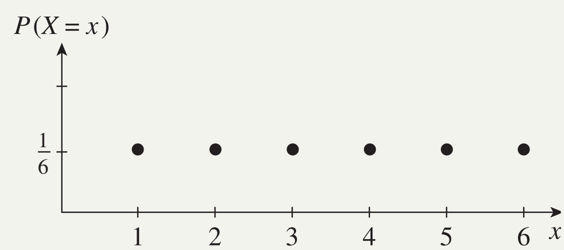
Example 1 **11A**

A die is thrown. Write out the probability distribution for the number shown on the die, and graph the distribution.

SOLUTION

Each value has probability $\frac{1}{6}$, so the table of the distribution is:

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

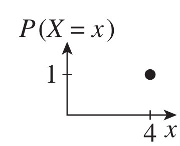


The probability distribution of a deterministic experiment

x	4
$P(X = x)$	1

To the right is the trivial probability distribution of the deterministic experiment, 'Throw four coins into an empty bucket and record the number of coins in the bucket'.

The sample space has one possible outcome 4, which is certain to occur, and so has probability 1. The graph is drawn underneath. Every deterministic experiment has a similar trivial distribution, and trivially it is a uniform distribution.



What is a list?

Our definition of discrete probability distribution requires that the values can be listed. Obviously any finite set can be listed, but some infinite sets can also be listed. The best example is the set of whole numbers, which can be listed as 0, 1, 2, 3, 4, 5, … The list will not terminate, but every whole number will eventually appear once and once only. For this reason, an infinite set that can be listed is called *countably infinite*.

The integers can be listed, for example as 0, 1, −1, 2, −2, 3, −3, … But the set of real numbers in an interval such as $0 \le x \le 1$ cannot be listed, as the German mathematician Georg Cantor proved in the late 19th century.

4 LISTING THE VALUES OF THE SAMPLE SPACE

- Any finite set can be listed.
- The whole numbers, and the integers, do not form finite sets, but can be listed:

  0, 1, 2, 3, 4, … and 0, 1, −1, 2, −2, 3, −3, …
- The real numbers in an interval such as $0 \le x \le 1$ cannot be listed.

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Example 2 [An infinite sample space that can be listed] 11A

Wulf is a determined person. He has decided to keep tossing a coin until it shows heads. Adolfa is counting how many times he tosses the coin. What is the probability distribution for this experiment?

SOLUTION

The result of this experiment is how many times Wulf tosses the coin to get a head. If he is very, very unlucky, he may have to toss the coin many, many times.

$P(\text{Wulf requires 1 toss}) = P(H) = \frac{1}{2}$

$P(\text{Wulf requires 2 tosses}) = P(TH) = \frac{1}{4}$

$P(\text{Wulf requires 3 tosses}) = P(TTH) = \frac{1}{8}$

$P(\text{Wulf requires 4 tosses}) = P(TTTH) = \frac{1}{16}$, and so on, giving the table:

x	1	2	3	4	5	$\cdots$	n
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\cdots$	$\frac{1}{2^n}$

Continuous probability distributions

In a continuous probability distribution, the sample space is typically a closed interval such as $0 \le x \le 100$ on the real number line. Examples are the heights or weights of people, and speeds of cars on an expressway.

Suppose that the police have set up a speed camera to measure the speeds of oncoming cars. If we regard the speed as a real number, then we are dealing with a continuous probability distribution. If, however, we take into account that the speed camera only records speeds correct to the nearest 0.01 km/h, then we are dealing with a discrete probability distribution. Such distinctions arise all the time in statistics, because any measurement, no matter how accurate, will only be correct to some number of decimal places.

In a continuous probability distribution, the probability of any one particular value occurring is zero. For example, a speed of 56.0123456789 km/h has almost no chance of ever being recorded, even if it could be measured. But worse, the decimal expansion of most real numbers never terminates or repeats. The probabilities involved in a continuous distribution must therefore be recorded as the probabilities that the random variable lies within an interval, for example as $P(55 \le X \le 60)$. The required machinery for this is *integration*, which is the other process in calculus and is not covered until Year 12, so continuous probability distributions will not concern us in this chapter.

5 CONTINUOUS PROBABILITY DISTRIBUTIONS

In a continuous probability distribution, the sample space is typically a closed interval such as $0 \le x \le 100$ on the real number line.

The sample space is therefore infinite and cannot even be listed.

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Exercise 11A

FOUNDATION

1 State whether each probability distribution is numeric or categorical. If it is numeric, state whether it is *discrete* or *continuous*.

- **a** The number showing when a die is thrown.
- **b** The weight of a randomly chosen adult male in Australia.
- **c** Whether it rains or not on a spring day in Sydney.
- **d** The daily rainfall in Sydney on a September day.
- **e** The colour of a ball drawn from a bag containing four red and three green balls.
- **f** The colours of two balls drawn together from a bag containing four red and three green balls.
- **g** The shoe size of a randomly chosen adult female in Australia.
- **h** ATAR results for a particular year.

2 **a** Complete the table for the probability distribution obtained when two coins are thrown one after the other and the successive results are recorded.

Outcome	HH	HT	TH	TT
Probability				

Look at the probabilities you have obtained. What sort of distribution is this?

b Complete the table for the probability distribution obtained when two coins are thrown and the numbers of heads and tails recorded.

Outcome	2 heads	1 head and 1 tail	2 tails
Probability			

3 Construct tables for the following probability distributions.

- **a** A ball is drawn from a bag containing four red and three green balls, and its colour is noted.
- **b** One hundred tickets are sold in the class lottery. Three friends Jack, Kylie and Lochlan buy six, eight and four tickets respectively. The winner is the person whose ticket is drawn first. Construct the distribution table of the probability that the winner is Jack, Kylie, Lochlan, or Other.
- **c** A letter is chosen at random from the word ‘Parramatta’.
- **d** A whole number is chosen at random between 1 and 1000 inclusive, and the number of digits is recorded.
- **e** A whole number between 10 and 19 inclusive is selected at random. A record is made whether it is even, prime, or neither.

4 Each experiment below returns a numerical result. Define a random variable X for each experiment and record its distribution in a table.

- **a** The word lengths in the sentence, ‘The ginger cat ran off with the meat.’
- **b** Two coins are thrown and the number of heads recorded.
- **c** A digit is chosen at random from the 12-digit number 1.41421356237.
- **d** Jeff has a collection of marbles with digits on them. He puts marbles numbered 1, 1, 2, 4 into one bag, then puts marbles numbered 2, 3, 3, 5 into a second bag. He randomly selects one of the bags, then randomly selects a marble from that bag and records its number.

5 Amy scoops up coins at random from a pile containing one 10 c coin and two 5 c coins.

- **a** Name the 10 c coin T and the two 5 c coins F1 and F2, and write down as sets the seven possible non-empty scoops.
- **b** Let the random variable X be the value of the money that she picks up. Assuming that she is equally likely to pick up any of these seven non-empty scoops, draw up a probability distribution table for X.

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6 Which of the following are probability distributions? Remember that the probabilities must be all non-negative and add to 1.

a

x	1	2	3	4
$P(X = x)$	0.1	0.6	0.2	0.1

b

x	1	2	3	4
$P(X = x)$	0.5	0.3	−0.2	0.4

c

x	1	2	3	4
$P(X = x)$	0.25	0.25	0.25	0.25

d

x	1	2	3	4
$P(X = x)$	30%	20%	40%	10%

e

x	1	2	3	4
$P(X = x)$	0.7	0.2	0.4	0.2

f

x	1	2	3	4
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$

7 A discrete probability distribution is tabulated below.

x	0	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.3	0.25	0.05	0.1

Find:

a $P(X = 1)$
b $P(2 \le X \le 4)$
c $P(1 \le X < 4)$
d $P(X = 6)$
e $P(X \le 2)$
f $P(X < 4)$
g $P(X \ge 1)$
h $P(X > 1)$
i $P(X \text{ is even})$

DEVELOPMENT

8 These questions all require probability tree diagrams with identical stages.

a A pack of cards has 52 cards, of which 12 are court cards (jack, queen or king). A card is drawn at random, and it is recorded whether or not it is a court card. The card is replaced, then a second card is drawn and again it is recorded whether or not it is a court card.

i Draw up a probability tree diagram with two identical stages.

ii Let X be the number of court cards drawn. Use the results of part **i** to draw up a probability distribution table for X.

b Three dice are thrown, and the number of even numbers is recorded.

i Draw up a probability tree diagram with three identical stages, each stage being, ‘Throw a die and record whether the result is even or odd’.

ii Let X be the number of even numbers when the three dice are thrown. Use the results of part **i** to draw up a probability distribution table for X.

c A class has 10 girls and 15 boys. Three times the teacher chooses a student at random to answer a question, not caring whether he has called that name before. Use a similar method to draw up a probability distribution table for the number of boys called.

d The Spring Hill Zoo has 20 friendly wallabies. Six are from Snake Gully, five are from Dingo Ridge, and nine are from Acacia Flat. On three days last week, the zookeepers selected a wallaby at random and took it to breakfast with them, then returned it to the enclosure. Draw up a probability distribution table for the number X of Snake Gully wallabies taken to breakfast.

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9 Find the unknown constant a in the following probability distributions. Use the facts that $0 \le P(X = x) \le 1$ for each value x, and that the sum of the probabilities is 1.

a

x	1	2	3	4	5
$P(X = x)$	$4a$	$2a$	$9a$	$3a$	$7a$

b

x	1	2	3	4	5
$P(X = x)$	$3a$	a	$4a$	a	$5a$

c

x	-2	-1	0	1	2
$P(X = x)$	a	$3a$	$5a$	$7a$	$11a$

d

x	10	20	30	40	50
$P(X = x)$	$1 - 3a$	a	$1 - 9a$	$1 - 10a$	a

e

x	1	2	3	4	5
$P(X = x)$	$0.2a$	$0.1a$	$0.5a$	$0.1a$	$0.1a$

10 These questions all involve probability tree diagrams in which the stages are different because the sampling has been done ‘without replacement’.

a A bag contains six marbles numbered 1, 2, 3, 4, 5, 6. A marble is drawn, and it is recorded whether the number is odd or even. Without replacing the marble, a second marble is drawn, and it is recorded again whether the number is odd or even.

i Draw up a probability tree diagram for the four outcomes.

ii Draw up a probability distribution table for the number X of even numbers chosen.

b Two students are chosen from a group of four boys and two girls.
Use a similar method to draw up a probability distribution table for the number X of girls chosen.

c Five tiles marked E, E, E, R, T are turned upside down, and two are selected at random one after the other.

i Draw a probability tree diagram for the seven possible selections of two tiles.

ii Draw up a probability distribution table for the number X of Es selected.

11 A small pack of cards consists of three 4s, two 2s and one 5 (the suit is not important for this experiment). A card is selected at random and the value recorded. The card is then returned to the pack. This is repeated. Complete the probabilities for the outcomes of the categorical random variable X in the following table.

x	22	44	55	24 or 42	25 or 52	45 or 54
$P(X = x)$						

(This is an example of a *multinomial distribution* — a multi-stage experiment with identical stages, each with multiple outcomes. Here there are two stages, because we select a card twice, and the stages are identical because the card is replaced.)

12 Construct tables for the following distributions. In each case the outcome is categorical, namely a pair of colours or a pair of suits.

a A ball is drawn from a bag containing four red and three green balls. Its colour is noted and the ball is returned. A second ball is drawn and its colour also noted.

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b A ball is drawn from a bag containing four red and three green balls and the colour is recorded. The ball is returned, a second ball is drawn without replacement and the colour is also recorded.

c Two cards are drawn from a pack. Their suits are noted (but not their order).

13 [Simulation experiment] Cut five identical pieces of paper or cardboard. Label three of them ‘Red’ and the other two ‘Green’. Place all the pieces in a hat or bag.

In this experiment, two pieces of paper are drawn out, the number of pieces labelled ‘Green’ is recorded, and the pieces are then returned. Let X be the random variable for the number of ‘Greens’.

a Copy the table below for recording the results of your experiment.

x	0	1	2
Tally			

b Repeat the experiment 40 times, recording each outcome in the tally column.

c Add two rows to your table, as below. Complete the table to calculate the experimental probabilities of each outcome.

x	0	1	2
Tally			
Frequency f			
Relative frequency $f_r = f/40$			

d Compare your results with other members of the class. Does this suggest that the results are accurate? Can you explain any differences in the results obtained?

e Calculate the theoretical probabilities for this experiment and add this as a further row to your table. How do your results compare?

f Comment on any aspects of your experimental design that assist the accuracy of the results. Is there any way you could have improved the reliability and randomness of your experiment?

CHALLENGE

14 A bag has six marbles marked 1, 2, 3, 4, 5, 6. Three marbles are drawn in succession, without being replaced, and the number of marbles with even numbers is recorded. Let X be the number of even-numbered marbles chosen. Draw up a probability distribution table for X.

15 Find the unknown constant a in the following probability distributions.

a

x	1	2	3	4	5
$P(X = x)$	$a(a+1)$	$3a^2$	$1-3a$	$1-4a$	a

b

x	1	2	3	4
$P(X = x)$	$\frac{1}{6}(a+1)$	$\frac{1}{4}a^2$	$\frac{1}{8}(5-3a)$	$\frac{1}{6}(3-2a)$

16 a Kylie has a hand of four cards: 7 of hearts, 7 of diamonds, 6 of clubs and 8 of spades. She takes three of the cards at random, places them on the table, and adds the cards’ values. Construct a table showing the probability distribution of the sum.

b Repeat this experiment if Kylie has a hand of five cards, including three 7s, a 6 and an 8, and still chooses three cards at random.

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11B Expected value

The first experiment in the previous section was ‘Toss four coins and count the number of heads’.

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

The notation $P(X = x)$ will be too unwieldy for what follows, so we will write it instead using standard function notation as $p(x)$. For each value x in the distribution, $p(x)$ means ‘the probability that $X = x$’.

$$p(x) = P(X = x), \text{ for all values } x \text{ in the distribution.}$$

With this more concise notation, the table above becomes:

x	0	1	2	3	4
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Where is the centre of the distribution?

Someone looking at this table might want to say,

‘I would expect to get the result two heads.’

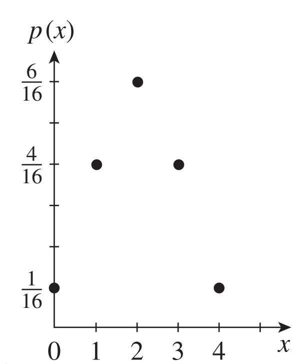
When asked, they might try to explain what they mean by saying,

‘When we do the experiment a large number of times, the average number of heads will be about 2, and the more times we do it, the closer the average will be to two’.

This is a good explanation, because it is based on the interpretation of probability as running the experiment a large number of times and averaging things out.

They might also try to explain it by symmetry:

‘If I plot the five probabilities, the pattern is symmetric about the value 2.’



This is also a good explanation, but is completely different. It ignores the idea of running trials, and is based solely on the symmetry of the distribution’s graph. It observes that each smaller number of heads on the left of $x = 2$ is balanced out by a larger number of heads on the right of $x = 2$ with the same probability, so $x = 2$ can be regarded as the centre of the distribution.

This section introduces a clearer and more general way of dealing with this ‘expected value’, which is better described as the ‘mean’ of the distribution. From now on, values must be numeric and not categorical.

Expected value

Everyone is familiar with the idea of the mean or average of a set of n numbers — add them up and divide by n,

$$\text{mean} = \frac{0 + 1 + 2 + 3 + 4}{5} = \frac{10}{5} = 2.$$

Alternatively, multiply each number by $\frac{1}{n}$ and add the results:

$$\text{mean} = 0 \times \tfrac{1}{5} + 1 \times \tfrac{1}{5} + 2 \times \tfrac{1}{5} + 3 \times \tfrac{1}{5} + 4 \times \tfrac{1}{5} = \tfrac{10}{5} = 2.$$

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But to calculate the mean of a discrete probability distribution, we need a *weighted mean* of the values, giving more *weight* to more likely values, which will occur more often, and less weight to the less likely values, which will occur less often. So instead of multiplying each value by $\frac{1}{n}$, we *weight* each value by its probability,

$$\begin{aligned}\text{weighted mean} &= 0 \times \tfrac{1}{16} + 1 \times \tfrac{4}{16} + 2 \times \tfrac{6}{16} + 3 \times \tfrac{4}{16} + 4 \times \tfrac{1}{16} \\ &= 0 + \tfrac{4}{16} + \tfrac{12}{16} + \tfrac{12}{16} + \tfrac{4}{16} \\ &= \tfrac{32}{16} \\ &= 2, \qquad \text{which is called the } \textit{expected value } \mathrm{E}(X).\end{aligned}$$

This process is called *weighting the values by their probabilities*, and is the way the expected value $\mathrm{E}(X) = 2$ of this distribution is calculated. Here the result 2 happens to be the same as the ordinary mean of the five values, but this is only because of the mirror symmetry of the probabilities about $x = 2$.

The formula for expected value

The usual notation for this procedure will be completely unfamiliar. In any distribution, the weighting of a value x by its probability $p(x)$ means that we take the product of the value x and its probability $p(x)$, written as the product

$$xp(x).$$

Now we need to add up all these products, for all the values in the distribution. The symbol in mathematics for 'take the sum' is $\sum$ which is the Greek upper-case sigma corresponding to the Latin letter 'S' and standing for 'sum'. Thus the formula for the expected value $\mathrm{E}(X)$ is written as

$$\mathrm{E}(X) = \sum xp(x), \quad \text{summing over all values of the distribution.}$$

The phrase, 'summing over all values of the distribution' means that we take all the products $xp(x)$, for all the values of the distribution, and add them up.

Setting out the calculation of expected value

The actual calculation is best set out not as done above, nor by substituting into the formula, but in tabular form, extending the table of values and probabilities of the distribution with an extra row and an extra column:

x	0	1	2	3	4	Sum
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1
$xp(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2

as we obtained before

The probabilities must add to 1 — summing the middle row is only a check.

Question 3 of Exercise 11B is an investigation that involves tossing four coins a large number of times and seeing how close the average value is to the expected value of 2 calculated above.

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An alternative setting out using columns rather than rows

Many people prefer to use columns rather than rows for these calculations. There is a very good reason for this — it is easier to add up a column of numbers than to add up a row of numbers.

If you use columns rather than rows, then your table will be *transposed*, meaning that the rows becomes columns and the columns become rows.

This setting out is shown to the right.

x	$p(x)$	$xp(x)$
0	$\frac{1}{16}$	0
1	$\frac{4}{16}$	$\frac{4}{16}$
2	$\frac{6}{16}$	$\frac{12}{16}$
3	$\frac{4}{16}$	$\frac{12}{16}$
4	$\frac{1}{16}$	$\frac{4}{16}$
Sum	1	2

Expectation is a measure of central tendency

As mentioned earlier, the expected value $\text{E}(X)$ of a distribution is also called the *mean* of the distribution, because it is obtained as the weighted mean of the values. It is a *measure of central tendency*, meaning that it is a measure of where the middle of the distribution lies.

For this reason, the expected value $\text{E}(X)$ is often assigned the pronumeral μ, a Greek lower-case letter corresponding to Latin 'm', and standing for 'mean'.

6 EXPECTED VALUE OF A DISCRETE PROBABILITY DISTRIBUTION

The *expected value* of a discrete probability distribution with numeric data is the weighted mean of the values x weighted by the probabilities $p(x)$,

$$\text{E}(X) = \sum xp(x), \qquad \text{summing over all values of the distribution.}$$

- The symbol $\sum$ is the Greek upper-case sigma, and stands for 'sum'.
- Each term in the sum is the product $xp(x)$ of a value and its probability.
- Take all these products $xp(x)$, for all the values of the distribution, and add them up.
- The expected value is also called the *mean* of the distribution. It is usually assigned the pronumeral μ — the Greek letter 'mu' stands for 'mean'.
- The expected value is a *measure of central tendency*.

The example above is symmetric, which is why the mean of the values turned out to be the same as the expected value. The following example shows how well the idea of expected value is able to pick out the central tendency in quite unsymmetric data.

Example 3 11B

Twenty friends go out to dinner at a restaurant. One has \$560 cash, four have \$350 cash, two have \$180 cash, three have \$80 cash, four have \$50 cash, and one has \$40 cash. Five have no cash, and are expecting to borrow from the others. An armed robber bursts in and seizes one of the friends at random.
She threatens him, grabs all his cash, and runs away. Graph the distribution, and find her expected criminal gain.

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SOLUTION

Let $\$x$ be the amount the criminal seizes. The probability distribution with the added row and column is

x	0	40	50	80	180	350	560	Sum
$p(x)$	$\frac{5}{20}$	$\frac{1}{20}$	$\frac{4}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{1}{20}$	1
$xp(x)$	0	2	10	12	18	70	28	140

Hence her expected criminal gain is \$140.

Expected value of a uniform distribution

In a uniform distribution, all the values have the same probability. We could add the values and then divide by the number of values, but it is better to continue with the same setting out. There are two things to notice here:

- The sample space of a uniform distribution consists of equally likely possible outcomes. These were the sorts of sample spaces that most of the calculations in Chapter 10 were based on.
- In a uniform distribution with n values, each value has probability $\frac{1}{n}$. Thus to weight each value by its probability, we multiply it by $\frac{1}{n}$, so that calculating the expected value is the same as calculating the mean of the values.

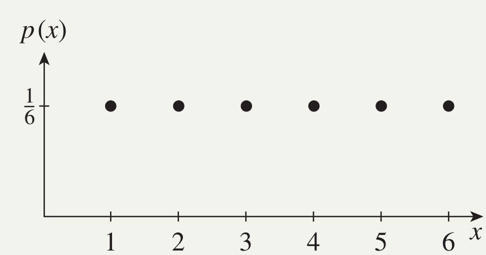
Example 4 — 11B

A die is thrown. Graph the distribution and find its expected value.

SOLUTION

Each value has probability $\frac{1}{6}$, so the table is

x	1	2	3	4	5	6	Sum
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1
$xp(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{21}{6}$



Hence the expected value on the die is $\mu = 3\frac{1}{2}$.

Do you find this answer just a little unsettling? How can we 'expect' an answer that we will never see on the die?

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The same thing happened in the restaurant robbery above — the expected value was \$140, but none of the friends had \$140 cash. There is actually no problem here, because the expected value is just a fancy name, and perhaps a slightly misleading name, for the mean of the distribution, and we know from countless examples that the mean of a set of numbers is usually not one of the numbers.

Exercise 11B

FOUNDATION

1 Complete each table and find the expected value $E(X)$ for the distribution, where the function $p(x)$ is defined as usual by $p(x) = P(X = x)$, for all values x.

a

x	0	1	2	3	Sum
$p(x)$	0.4	0.1	0.2	0.3	
$xp(x)$					

b

x	2	4	6	8	Sum
$p(x)$	0.1	0.4	0.4	0.1	
$xp(x)$					

c

x	−50	−20	0	30	100	Sum
$p(x)$	0.1	0.35	0.4	0.1	0.05	
$xp(x)$						

2 A simple gambling game involves the throw of a die. Players are charged 40 cents if the die shows a 1, 2 or 3, they are not charged for a 4, and they receive 30 cents or 60 cents respectively for a 5 or 6.
Let the random variable X be the payout to the player.

x	$p(x)$	$xp(x)$
−40		
0		
30		
60		
Sum	1	

a Copy and complete the table to the right, written this time in columns instead of rows.
b Calculate the expected value by summing the third column.
c What does this expected value represent?
d How much profit would the casino expect to make on 100 games?

3 Four coins are tossed and the number of heads listed. In the theory for this section, we constructed the table reproduced below. The expected value was calculated to be 2.

x	0	1	2	3	4	Sum
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1
$xp(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2

In this question we shall simulate the experiment and see if we have experimental agreement with these results.

a Toss four coins and record the number of heads, using a tally row and filling in a copy of the table below. Repeat this experiment 32 times.

x	0	1	2	3	4	Sum
Tally						—
Frequency f						
Relative frequency $f_r = \dfrac{f}{32}$						

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b Check whether your frequencies agree reasonably with the results 2, 8, 12, 8, 2 that the theoretical distribution would predict. Check also that the relative frequencies agree reasonably with the corresponding probabilities 0.0625, 0.25, 0.375, 0.25, 0.0625. If your results do not agree closely, you might like to repeat the experiment a further 16 or 32 times (and divide by 48 or 64 to calculate the relative frequencies.)

c Using your calculator, or by hand using the table below, calculate the mean for this data using the values x and their frequencies.

x	0	1	2	3	4	Sum
Frequency f						
$x \times f$						

$$\text{Mean} = \frac{\text{sum of } xf}{\text{sum of frequencies}}$$
$$= \dots$$

d Does your answer for the mean approximate the theoretical expected value $E(X) = 2$?

4 [An alternative notation] Sometimes the values of a discrete probability experiment are indexed $x_1, x_2, x_3, \dots$, with corresponding probabilities $p_1, p_2, p_3, \dots$. These values and probabilities can be abbreviated to x_i and p_i, where i is called the *index* variable because it *indexes* the values and probabilities. The calculations then proceed as before.

Find the expected value for these distributions.

a

x_i	2	4	6	8	10	Sum
p_i	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	
$x_i p_i$						

b

x_i	-3	1	2	5	6	Sum
p_i	0.1	0.3	0.2	0.3	0.1	
$x_i p_i$						

DEVELOPMENT

5 Fiona's mother tells her that she may buy a pencil, eraser or pen from the stationery shop. A pencil costs \$1.50, an eraser costs \$2.10 and a pen costs \$2.40. There are five boxes of pencils, four boxes of erasers and three boxes of pens.

a If Fiona chooses a box at random and takes one item from the box, what is the expected cost?

b If Fiona and her 99 friends each choose at random, what would be the expected total cost?

6 In this question we investigate what happens to the expected value if we transform the random variable, such as by doubling all the values, or increasing them all by 1. Because we are dealing with more than one random variable, we have retained the notation $P(X = x)$, $P(Y = y)$, and so on from Section 11A.

A random variable X records the outcomes of a spinner with sectors labelled 1, 2, 3, 4. The spinner is biased because it has been weighted.

x	1	2	3	4
$P(X = x)$	0.1	0.1	0.5	0.3
$x \times P(X = x)$				

a Copy and complete the table to calculate $E(X)$.

b A second random variable defined by $Y = 2X$ records twice the outcome of the weighted spinner.

y	2	4	6	8
$P(Y = y)$	0.1	0.1	0.5	0.3
$y \times P(Y = y)$				

i Calculate $E(Y)$ from this table.

ii Does your result agree with the result $E(aX) = aE(X)$?

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c A third random variable defined by $Z = X + 1$ records the outcome of the weighted spinner plus one.

i Calculate $\mathrm{E}(Z)$ from this table.

ii Does your result agree with the result $\mathrm{E}(X + b) = \mathrm{E}(X) + b$?

z	2	3	4	5
$P(Z = z)$	0.1	0.1	0.5	0.3
$z \times P(Z = z)$				

The general result that these examples illustrate is

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b, \text{ for all constants } a \text{ and } b.$$

7 A random variable X is known to have the property that $\mathrm{E}(X) = 5$. Use the formula $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ to calculate:

a $\mathrm{E}(3X)$ **b** $\mathrm{E}(X + 5)$ **c** $\mathrm{E}\left(\frac{1}{2}X\right)$ **d** $\mathrm{E}(X - 2)$ **e** $\mathrm{E}(10 - 2X)$ **f** $\mathrm{E}(4X - 2)$

8 A coin is tossed three times and the number of heads is recorded. Construct a table showing the probability distribution, and calculate the expected value.

(This is an example of a *binomial distribution* — a multi-stage experiment with identical stages, where at each stage there are two possible outcomes. Here there are three stages, because we toss the coin three times and there are two possible outcomes, heads or tails, for each throw.)

9 Two cards are selected at random from a standard pack, and the number of hearts is recorded. Note that this is equivalent to selecting two cards without replacement. Construct a table showing the probability distribution and calculate the expected value.

(This is an example of a *hypergeometric distribution* — a multi-stage experiment involving two possible outcomes, where at each stage the object is selected without replacement. Here there are two stages, because we select two cards.)

10 [Simulation experiment] Two dice are thrown. Let X be the difference between the two numbers thrown, so that the sample space consists of the integers 0, 1, 2, 3, 4, 5.

a Conduct an experiment to determine the experimental probability of the six outcomes. You should conduct the experiment 36 times and record your results in a copy of the table below.

x	0	1	2	3	4	5	Sum
Tally							—
Frequency f							
Relative frequency $f_r = \dfrac{f}{36}$							
$x \times f_r$							

b Use the relative frequencies as estimates for the probabilities of each outcome. Calculate the experimental expected value (the symbol for this experimental expected value is $\bar{x}$ to distinguish it from the theoretical expected value μ).

c Compare your results with those of other members of the class.

d **i** Calculate the theoretical probabilities using the normal 6×6 array of dots.

ii Also copy and complete the table to the right to calculate the theoretical expected value.

x	0	1	2	3	4	5	Sum
$p(x)$							
$xp(x)$							

e How do your results compare?

f If you think your results are inaccurate, consider any design faults in your experiment.

g Do your results agree more closely with the theoretical if you increase the number of trials?

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CHALLENGE

11 The backers of the game in Question **2** have weighted the die so that it is 50% more likely to turn up each of the results 1, 2 or 3 than to turn up 4, 5 or 6. (This is illegal.)

a What are the probabilities now of each outcome?

b Find the expected value now.

12 A company is designing a new gambling slot machine for a casino. A player pulls a lever and the machine randomly rolls three outcomes from Orange (O), Strawberry (S) and Apple (A), for example AOS or SSO. The probabilities of the various fruits turning up are in the ratio 1 : 2 : 3, so that Apple is three times as likely to occur in a given position as Orange. The machine pays out in the ratio 11 : 2 : 1 if it turns up three Oranges, three Strawberries or three Apples, respectively. No other combination pays the player. A player initiates a roll by feeding a \$1 coin into the machine.

a Find the probabilities that Orange, Strawberry or Apple turn up in a given position.

b Suppose the triple AAA pays $\$k$. Construct a probability distribution table showing the outcomes OOO, SSS, AAA and Other for the random variable X representing the payout.

c Determine how much a player would get for the highest paying outcome OOO if the machine is designed so that it will break even (this is unlikely).

13 [Expected value when the sample space is infinite, but can be listed]

In Example 2 of Section 11A, Wulf tosses a coin repeatedly until it shows a head. The probability table for the number X of tosses is infinite:

x	1	2	3	4	5	$\cdots$	n	$\cdots$
$p(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$

Thus to calculate the expected value we need to calculate the infinite sum:

$$\mu = 1 \times \tfrac{1}{2} + 2 \times \tfrac{1}{4} + 3 \times \tfrac{1}{8} + 4 \times \tfrac{1}{16} + 5 \times \tfrac{1}{32} + 6 \times \tfrac{1}{64} + \cdots$$

a Write down the sum for 2μ and by carefully subtracting like fractions, show that

$$2\mu - \mu = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \cdots$$

and hence write down a second infinite sum for μ.

b Repeat the process of doubling and subtracting with this new infinite sum for μ, and hence calculate μ.

c Explain what this expected value means practically and design an experiment to test your theoretical result.

Note: The operations on infinite series used here are valid for these particular series because they are convergent (this concept is explained in Year 12, and even then the proofs are difficult). The operations are certainly not true for all infinite series.

14 [St Petersburg Paradox] Patrons at a casino may play a game involving a single coin, which is tossed until a head turns up. A jackpot is set aside. It initially contains \$2, but its value is doubled on each toss of the coin turning up a tail. The winner receives the contents of the pot when the first head is tossed. Thus the player would win \$2 if the initial toss is a head, \$4 if the second toss is the first head, \$8 if the third toss is the first head, and so on.

A manager at the casino suggests that a player should be charged \$40 to play the game. Calculate the player's expected return, and comment on the manager's advice.

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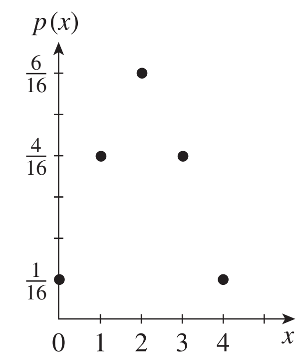
11C Variance and standard deviation

After discussing E(X), which is a measure of the central tendency of a discrete probability distribution, we now turn to variance and standard deviation, which are two measures of spread. Variance is easier to define and calculate, so we will work first with the variance. But standard deviation, which is just the square root of the variance, is important because it has the same units as the values of the distribution, and increases in proportion when all the values are increased by some factor.

The variance — first formula

Return again to the number of heads when four coins are tossed:

x	0	1	2	3	4
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$



In the last section, the expected value of this distribution, that is its mean, was found to be $\mu = 2$. We now want a measure of how spread out the values are from the mean. The usual measure of this is the *variance* Var(X).

To find the variance, take the difference $x - \mu$ of each value from the mean μ. This difference is called the *deviation* of the value x from the mean μ. Then we square the deviation to give $(x - \mu)^2$. This squared deviation is a good measure of how far x is from the mean for two reasons:

- The square $(x - \mu)^2$ is always a positive number or zero, whether the value x is on the left or the right of μ, because of the squaring.
- The square $(x - \mu)^2$ gets larger as x moves away from the mean, and the square means it gets larger very quickly. For example, doubling the distance of x from the mean has four times the effect on the square.

Then, as with expected value, take the weighted mean of these squared deviations $(x - \mu)^2$, weighted as before by the probabilities. Using sigma notation,

$$\mathrm{Var}(X) = \sum (x - \mu)^2 p(x), \quad \text{summed over the distribution.}$$

As with expected value, the calculation is best done by adding more rows to the probability distribution table. Here are the calculations for the number of heads on four coins. The table calculates the expected value and variance together:

x	0	1	2	3	4	Sum	
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1	(a check)
$xp(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2	(the mean μ)
$(x - \mu)^2$	4	1	0	1	4	—	
$(x - \mu)^2 p(x)$	$\frac{4}{16}$	$\frac{4}{16}$	0	$\frac{4}{16}$	$\frac{4}{16}$	1	(the variance)

Thus the mean is 2 and the variance is 1.

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Using columns instead of rows

To the right is the alternative setting out that uses columns instead of rows.

Take your pick — use the layout that you find more convenient.

x	$p(x)$	$xp(x)$	$(x-\mu)^2$	$(x-\mu)^2 p(x)$
0	$\frac{1}{16}$	0	4	$\frac{4}{16}$
1	$\frac{4}{16}$	$\frac{4}{16}$	1	$\frac{4}{16}$
2	$\frac{6}{16}$	$\frac{12}{16}$	0	0
3	$\frac{4}{16}$	$\frac{12}{16}$	1	$\frac{4}{16}$
4	$\frac{1}{16}$	$\frac{4}{16}$	4	$\frac{4}{16}$
Sum	1	2	—	1

The variance as an expected value

The variance is the weighted mean of $(x-\mu)^2$, weighted by the probabilities. Thus the variance is the expected value of $(x-\mu)^2$,

$$\mathrm{Var}(X) = \mathrm{E}\big((X-\mu)^2\big).$$

This is a very useful form of the variance formula, because it will apply also to the continuous probability distributions that will be introduced in Year 12.

Example 5 — 11C

A die is thrown, producing a uniform distribution. Find the mean and variance of the distribution using the formula $\mathrm{Var}(X) = \mathrm{E}\big((X-\mu)^2\big)$.

SOLUTION

x	1	2	3	4	5	6	Sum	
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1	(a check)
$xp(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$3\frac{1}{2}$	(the mean μ)
$(x-\mu)^2$	$\frac{25}{4}$	$\frac{9}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{9}{4}$	$\frac{25}{4}$	—	
$(x-\mu)^2 p(x)$	$\frac{25}{24}$	$\frac{9}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{9}{24}$	$\frac{25}{24}$	$\frac{70}{24}$	(the variance)

Hence the mean is $3\frac{1}{2}$ and the variance is $\frac{70}{24} = 2\frac{11}{12}$.

The variance — alternative formula

The method of calculation in the table for the four tossed coins above worked well because the mean was a whole number. Things became a little turgid in Example 5 in which a die is thrown because the mean, $3\frac{1}{2}$, was a fraction. If the denominator had been a larger number such as 287 rather than 2, things would have looked a great deal messier.

Fortunately, there is another formula for variance, again using weighted means, and this alternative formula makes calculations much easier:

$$\mathrm{Var}(X) = \sum x^2 p(x) - \mu^2$$

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This formula uses the squares of the values, and takes the weighted mean of these squares, weighted as always by the probabilities. Thus, like the first formula, it can be written in terms of expected values, this time in terms of the expected value of X^2,

$$\text{Var}(X) = \text{E}(X^2) - \mu^2.$$

The alternative formula makes the calculations more straightforward, whether or not μ is a whole number. The following table uses this alternative formula to recalculate the variance for the distribution obtained by throwing four coins and counting the heads.

x	0	1	2	3	4	Sum	
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1	(a check)
$xp(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2	(the mean μ)
$x^2p(x)$	0	$\frac{4}{16}$	$\frac{24}{16}$	$\frac{36}{16}$	$\frac{16}{16}$	5	$\big($this is $\text{E}(X^2)\big)$

From the third row: $\text{E}(X) = 2$, which is μ

From the last row: $\text{Var}(X) = \text{E}(X^2) - \mu^2$
$= 5 - 2^2$
$= 1$, as we obtained before.

Example 6 11C

A die is thrown, producing a uniform distribution. Find the mean and variance of the distribution using the alternative formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$.

SOLUTION

x	1	2	3	4	5	6	Sum	
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1	(a check)
$xp(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$3\frac{1}{2}$	(the mean μ)
$x^2p(x)$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{9}{6}$	$\frac{16}{6}$	$\frac{25}{6}$	$\frac{36}{6}$	$\frac{91}{6}$	$\big($this is $\text{E}(X^2)\big)$

From the third row, $\text{E}(X) = 3\frac{1}{2}$

From the last row, $\text{Var}(X) = \text{E}(X^2) - \mu^2$
$= \frac{91}{6} - \frac{49}{4}$
$= 15\frac{1}{6} - 12\frac{1}{4}$
$= 2\frac{11}{12}$, as we obtained before.

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7 VARIANCE OF A DISCRETE PROBABILITY DISTRIBUTION:

- The *variance* of a discrete probability distribution with numeric data is
$$\mathrm{Var}(X) = \mathrm{E}\big((X-\mu)^2\big) \quad \text{or equivalently} \quad \mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2.$$
- Written in terms of weighted means,
$$\mathrm{Var}(X) = \sum (x-\mu)^2 p(x) \quad \text{or} \quad \sum x^2 p(x) - \mu^2$$
where the sums are taken over all the values in the distribution.
- The first of each pair gives the best intuitive understanding of the variance.
- The second of each pair is usually easier for calculation.
- Each difference $x - \mu$ is called the *deviation* of x from the mean μ.

Extension — proof of the alternative formula

The proof looks complicated only because of sigma notation. To prove the alternative formula for $\mathrm{Var}(X)$, start with the first formula, and move to the second.

$$\begin{aligned}\mathrm{Var}(X) &= \sum (x-\mu)^2 p(x)\\ &= \sum (x^2 - 2\mu x + \mu^2) p(x) \quad \text{(expand the square)}\\ &= \sum x^2 p(x) - \sum 2\mu x p(x) + \sum \mu^2 p(x) \quad \text{(expand the brackets)}\\ &= \sum x^2 p(x) - 2\mu \sum x p(x) + \mu^2 \sum p(x) \quad \text{(take out common factors)}\end{aligned}$$

Using the fact that $\sum p(x) = 1$ and $\sum xp(x) = \mu$,

$$\begin{aligned}\mathrm{Var}(X) &= \sum x^2 p(x) - 2\mu^2 + \mu^2\\ &= \sum x^2 p(x) - \mu^2.\end{aligned}$$

Standard deviation

The units of variance are the square of whatever units the values have. The *standard deviation* is the square root of the variance, and therefore has the same units as the values. Its usual pronumeral is σ, the lower-case Greek letter sigma (Latin 's'), standing for 'standard',

$$\sigma = \sqrt{\mathrm{Var}(X)} \qquad \text{or equivalently} \qquad \sigma^2 = \mathrm{Var}(X)$$

In the four-coins example above, the units of the values and of the standard deviation are heads, and in Example 7 which follows, their units are dollars.

The term *standard deviation* implies that the deviations from the mean have been taken into account across all values of the distribution.

The other nice thing about the standard deviation is its proportionality. Standard deviation, like variance, is a measure of spread. If we stretch out all the values by doubling them, then we correspondingly double the standard deviation. If we spread out all the values by a factor of k, the standard deviation is multiplied by k. This will be proven in the Challenge section of Exercise 11C.

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Example 7 11C

Find the standard deviation of the distribution in Example 3 about the robbery in the restaurant in Section 11B.

SOLUTION

We use the alternative formula for variance, then take the square root.

x	0	40	50	80	180	350	560	Sum
$p(x)$	$\frac{5}{20}$	$\frac{1}{20}$	$\frac{4}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{1}{20}$	1
$xp(x)$	0	2	10	12	18	70	28	140
$x^2p(x)$	0	80	500	960	3240	24 500	15 680	44 960

From the last row,
$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - \mu^2 \\ &= 44\,960 - 140^2 \\ &= 25\,360.\end{aligned}$$

Taking the square root,
$$\begin{aligned}\sigma &= \sqrt{\text{Var}(X)} \\ &\doteqdot \$159.25,\end{aligned}$$

giving a spread of \$159.25 about the mean of \$140.

8 STANDARD DEVIATION OF A DISCRETE PROBABILITY DISTRIBUTION

- The *standard deviation* is the square root of the variance. It is usually assigned the pronumeral σ, the Greek letter 'sigma' which stands for 'standard'.

$$\sigma = \sqrt{\text{Var}(X)} \qquad \text{or equivalently} \qquad \sigma^2 = \text{Var}(X)$$

- The pronumeral for variance is thus σ^2.
- The standard deviation has the same units as the values.
- The variance and the standard deviation are *measures of spread*.

Exercise 11C

FOUNDATION

1 Consider a random variable X whose probability distribution is given in the table to the right.

x	1	2	3	4	Sum
$p(x)$	0.3	0.5	0.1	0.1	
$xp(x)$					
$(x-\mu)^2$					—
$(x-\mu)^2p(x)$					

a Copy and complete the table to calculate the mean $\text{E}(X) = \mu$ and the variance $\text{Var}(X)$ using the definition
$$\text{Var}(X) = \text{E}\big((X-\mu)^2\big).$$

b Calculate the standard deviation
$$\sigma = \sqrt{\text{Var}(X)}.$$

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2 This question uses the alternative formula for Var(X), rather than the definition, to calculate the variance for the random variable in the previous question.

a Copy and complete the table.

b Now calculate the variance using the alternative formula

$$\text{Var}(X) = \text{E}(X^2) - \mu^2.$$

x	1	2	3	4	Sum
$p(x)$	0.3	0.5	0.1	0.1	
$xp(x)$					
x^2					—
$x^2p(x)$					

3 For each random variable, calculate $\mu = \text{E}(X)$. Then calculate the variance Var(X) twice, first using the definition $\text{Var}(X) = \text{E}\big((X-\mu)^2\big)$, then using the alternative formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$. Use columns instead of rows in parts **c** and **d**.

a

x	0	1	2	3	4
$p(x)$	0.2	0.2	0.2	0.2	0.2

b

x	0	1	2	3	4
$p(x)$	0.0	0.1	0.2	0.3	0.4

c

x	$p(x)$
−2	0.3
−1	0.1
0	0.2
1	0.1
2	0.3

d

x	$p(x)$
1	0.1
2	0.4
3	0.2
4	0.2
5	0.1

DEVELOPMENT

4 a Calculate the expected value, variance and standard deviation for each random variable. For the variance, choose whether to use the definition $\text{Var}(X) = \text{E}\big((X-\mu)^2\big)$ or the alternative formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$.

i

y	0	1	2	3	4
$P(Y=y)$	0	0.5	0	0.5	0

ii

z	0	1	2	3	4
$P(Z=z)$	0.5	0	0	0	0.5

iii

v	$P(V=v)$
0	0.3
1	0.5
2	0.1
3	0.1
4	0

iv

w	$P(W=w)$
0	0
1	0.1
2	0.1
3	0.5
4	0.3

b Use the idea that the expected value measures the centre of the data, and the variance and standard deviation measure its spread, to comment on:

i how E(Y) and Var(Y) compare with E(Z) and Var(Z),

ii how E(V) and Var(V) compare with E(W) and Var(W).

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5 A distribution that takes a single value is called *deterministic*, because it is no longer random. Our formulae for expected value and variance may still be calculated and the results are not a surprise, as this question demonstrates.

Calculate $\mathrm{E}(X)$ and $\mathrm{Var}(X) = \mathrm{E}\big((X - \mu)^2\big)$ for the distribution:

x	1	2	3	4	5
$P(X = x)$	0	1	0	0	0

6 John and Liam are keen basketballers and keep track of the number of baskets they score in games. Using the data from a large number of games, they have estimated the probability of scoring in any one game. Let the random variables J and L be the number of baskets scored by John and Liam respectively in a game. Their probability data is recorded in the tables below.

j	0	1	2	3	4
$P(J = j)$	0.35	0.2	0.1	0.25	0.1

ℓ	0	1	2	3	4
$P(L = \ell)$	0.2	0.3	0.4	0.1	0

a Calculate the expected value and variance for J and L using the alternative form $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$.
b With reference to expected value, comment on who is the better player.
c With reference to variance, comment on who is the more consistent player.

7 The random variable X records the number on a spinner with sectors of equal size marked $1,\ 2,\ 3$.
a What is the probability of each outcome, and what sort of distribution is this?
b Calculate the expected value $\mathrm{E}(X)$.
c Calculate the variance $\mathrm{Var}(X)$.

8 The *deviation of a score x from the mean* is often expressed in terms of how many standard deviations x lies from the mean μ. The formula for this is:

$$\text{number of standard deviations from the mean} = \frac{x - \mu}{\sigma},$$

where a negative sign means that the score x is below the mean.
a Englebert's score in his English test was 55. The test mean was $\mu = 65$ and the standard deviation was $\sigma = 5$. How many standard deviations was his score below the mean?
b Matthew's score in his Mathematics test was 54. The test mean was $\mu = 72$ and the standard deviation was $\sigma = 12$. How many standard deviations was his score below the mean?
c Comment on which score was more impressive, by noting which score was furthest from the mean in terms of the number of standard deviations.

9 Using the method of the previous question, that is, how many standard deviations a score is from the mean, decide which of each pair of test scores below is better.
a A score of 45 for Visual Arts (mean 60, standard deviation 15) or a score of 46 for Music (mean of 67, standard deviation of 12).
b A score of 88 for Earth Science (mean 70, standard deviation 9) or a score of 90 for Biology (mean of 75, standard deviation of 10).
c A score of 62 for Chinese (mean 50, standard deviation 6) or a score of 63 for Sanskrit (mean of 55, standard deviation of 4).

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10 Jasmine is practising her accuracy with bow and arrow over 10 shots. The random variable X is the number of bullseyes obtained. She repeats this experiment 20 times and records the relative frequency as an estimate to the probability of each outcome. Her results are tabulated below.

x	0	1	2	3	4	5	6	7	8
$p(x)$	0	0.1	0.15	0.3	0.4	0	0	0	0.05

a Calculate the expected value and standard deviation for the experiment.
b An *outlier* is informally a value that is a long way away from the mean, and from the rest of the data, of the distribution. What value(s) would you think are outliers in this distribution?
c There are various quantitative ways of defining outliers. One such definition is *any value three or more standard deviations from the mean*. Are there any outliers, using this definition?
d Jasmine realises that due to poor handwriting, her results have been wrongly interpreted. The corrected table is below. Recalculate the expected value and standard deviation for this new table.

x	0	1	2	3	4	5
$p(x)$	0	0.1	0.15	0.3	0.4	0.05

e Because of their distance from the mean, outliers can have a big influence on the value of the mean and standard deviation. Comment on the differences in the expected value and variance between the two tables above.
f Would it be valid in general to discard or 'correct' any outliers?

11 For some value of k, a random variable X with values 1, 2, 3, 4 is defined by

$$P(X = x) = kx, \text{ for } x = 1, 2, 3, 4.$$

Find k, then find the expected value and the standard deviation.

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CHALLENGE

12 A distribution is said to be *uniform* if every outcome has the same probability. Consider the random variable X of a uniform distribution with values $1, 2, \ldots, n$.

To complete this question you will find these formulae useful:

$$1 + 2 + 3 + \ldots + n = \tfrac{1}{2}n(n+1)$$

$$1 + 4 + 9 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

a What is the probability of $P(X = k)$ for integers $1 \le k \le n$?

b Calculate the expected value $\mathrm{E}(X)$.

c Calculate the variance $\mathrm{Var}(X)$.

d Compare your answers for expected value and variance with those for Question 7 involving a three-valued spinner, and with the example in the theory about throwing a standard six-sided die.

13 The expected value $\mathrm{E}(X)$ of a discrete probability distribution is μ.

a A constant a is added to all the values in the distribution. Let the random variable of this new distribution be Z, so that $Z = X + a$. Show that the expected value $\mathrm{E}(Z)$ of the new distribution is $\mu + a$.

b Each value in the distribution is multiplied by a constant k. Let the random variable of this new distribution be Z, so that $Z = kX$. Show that the expected value $\mathrm{E}(Z)$ of the new distribution is $k\mu$.

You have now proven that for all constants k and a,

$$\mathrm{E}(kX + a) = k\mathrm{E}(X) + a.$$

14 The mean and standard deviation of a discrete probability distribution are μ and σ. Use the formula $\mathrm{Var}(X) = \mathrm{E}\big((X - \mu)^2\big)$ to solve these two problems.

a A constant a is added to all the values in the distribution. Let the random variable of this new distribution be Z, so that $Z = X + a$. Show that the standard deviation $\sqrt{\mathrm{Var}(Z)}$ of the new distribution is also σ.

b Each value in the distribution is multiplied by a constant $k > 0$. Let the random variable of this new distribution be Z, so that $Z = kX$. Show that the standard deviation $\sqrt{\mathrm{Var}(Z)}$ of the new distribution is $k\sigma$.

You have now proven that for all constants $k > 0$ and a,

$$(\text{standard deviation of } kX + a) = k\sigma$$

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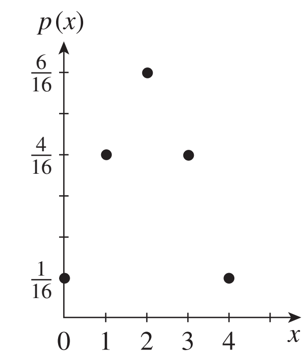
11D Sampling

The word *sampling* is used when we perform an experiment a number of times and record the results. The set of results is a *sample*. One example of sampling is the large numbers of surveys from researchers, the Government, and marketers. These surveys are constantly asking what we think about some product or service, how many people live in our house, whether we agree with the latest Government policy, how many holidays we intend to take in the next five years, and so forth. The purpose of most of these surveys is to gain some idea of probabilities, usually so that future behaviours can be predicted.

Now that discrete probability distributions, expected value and variance have been introduced, this section discusses the relationship between sampling and probabilities. The subject is fundamental to statistics, and extremely complicated. Only a few very basic ideas are accessible here.

Sampling of a theoretical probability distribution

We return once again to the experiment of tossing four coins and recording the number of heads.



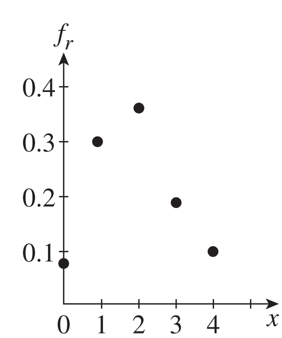
x	0	1	2	3	4	Sum
$p(x)$	0.06	0.25	0.38	0.25	0.06	1
$xp(x)$	0	0.25	0.75	0.75	0.25	2
$x^2p(x)$	0	0.25	1.50	2.25	1.00	5

We concluded that $\mathrm{E}(X) = 2$ and $\mathrm{Var}(X) = 5 - 2^2 = 1$.

Each fraction in the earlier table has been rounded correct to two decimal places for comparison with the sample below — the last two rows were not recalculated. Notice that because of rounding errors, the first row may no longer add to 1, and the entry 1.50 in the middle of the bottom row is not $2^2 \times 0.38$. In this distribution, the errors could have been removed by rounding to four decimal places, but in general, rounding errors are a constant issue in statistics.

Can we confirm these probabilities and the subsequent calculations empirically? We could do a number of *simulations* or *trials*, which are independent *runs* of the experiment. It would be most unlikely that the resulting frequencies would be in the exact proportions of the theoretical table above, but it may be interesting to perform say 100 simulations and see what happens. The authors have done this using random numbers in an Excel spreadsheet rather than coins, because they only had three coins between them. Here are the results:

x	0	1	2	3	4	Sum
f	7	29	34	21	9	100
f_r	0.07	0.29	0.34	0.21	0.09	1
xf_r	0	0.29	0.68	0.63	0.36	$1.96 = \bar{x}$
x^2f_r	0	0.29	1.36	1.89	1.44	4.98



The resulting *sample distribution* or *frequency table* has been structured as closely as possible to the theoretical table above. Its graph is drawn to the right.

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In the table:

- The second line is the *frequency* f. There is no corresponding line in the probability distribution table.
- The third line is the *relative frequency* f_r, obtained by dividing each frequency by 100, which is the number of simulations performed. These relative frequencies correspond to the probabilities in the theoretical distribution. The graph of the sample distribution has been drawn, and should be compared with the theoretical graph above it.

Did you expect the relative frequencies to be closer to the probabilities, and the sample graph to be closer to the theoretical graph? That is a very difficult question, and beyond this course, but at least we can begin to see what may happen in practice.

- The fourth line gives us the *sample mean*, which is written as $\bar{x} = 1.96$ — placing a bar over the pronumeral x is the usual notation for the sample mean. It is the mean of the values weighted by their relative frequencies, just as the *theoretical mean* $\mu = 2$ (the expected value) is the mean of the values weighted by their probabilities.
- The last line gives us the mean of the squares of the values, again weighted by the relative frequency, corresponding to $E(X^2)$ in the probability distribution table. It allows us to calculate the *sample variance* s^2 and the *sample standard deviation* s in exactly the same way that the *theoretical variance* σ^2 and the *theoretical standard deviation* σ were calculated:

$$\begin{aligned} s^2 &= \sum x^2 f_r - \bar{x}^2 \\ &= 4.98 - (1.96)^2 \\ &\doteqdot 1.14 \quad \text{(compare with 1)} \end{aligned} \qquad \begin{aligned} s &= \sqrt{s^2} \\ &\doteqdot 1.07 \text{ heads} \quad \text{(compare with 1)} \end{aligned}$$

Although this course does not have the machinery to characterise whether the sample distribution differs significantly from the theoretical distribution, at least we have the basic data to begin the comparison.

9 SAMPLE DISTRIBUTION

Suppose that a number of independent *runs* (or *simulations* or *trials*) are performed of an experiment whose possible outcomes are discrete.

- The resulting *sample* then yields a *sample distribution* or *frequency table*.
- The relative frequencies f_r in the sample distribution correspond to the probabilities $p(x)$ in a probability distribution, and can be graphed in the same way.

Now suppose also that the possible outcomes are numeric.

- The *mean* of the sample is denoted by $\bar{x}$, and corresponds to the expected value of the theoretical probability distribution:

$$\bar{x} = \sum x f_r$$

- The *sample variance* s^2 and the *sample standard deviation* s are given by:

$$s^2 = \sum (x - \bar{x})^2 f_r \qquad \text{or} \qquad s^2 = \sum x^2 f_r - \bar{x}^2$$

- The calculation of the mean and sample variance can be done in tabular form in the same way as the expected value and variance of a probability distribution.

The sample mean $\bar{x}$ will seldom be the same as the theoretical mean μ, which is why different symbols must be used. Similarly, the sample standard deviation s will seldom be the same as the theoretical standard deviation σ.

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Populations — census and survey

A school asks all its 1000 students to report how many brothers and sisters the student has — this is a *census*, because everyone is asked. Another similar-sized school chooses 100 students at random and asks them the same question — this is a *survey* or *sample*, because not everyone is asked. When a marketer or the Government wants to know something about the Australian population, almost always a survey is used because it is quicker and much cheaper than a census. Every five years, however, the Federal Government performs a compulsory census of the whole Australian population, and the results are eagerly awaited and discussed.

In any one census question, the relative frequencies are exact probabilities of a particular result when a person or thing is chosen at random. A *population distribution* is therefore akin to a probability distribution that has been calculated exactly using the laws of probability, so the pronumerals μ and σ are assigned to its mean and standard deviation.

When designing a survey, it is vitally important to know whether the sample is truly random. For example, many websites offer internet surveys, but such a sample is in no way random, because it includes only people visiting that website.

Also, the people chosen in the survey may be a significant proportion of the population — in the school survey above, 100 students were chosen out of only 1000. That should be taken into account, but it is beyond the present course.

Everyone wants to use surveys to approximate the statistical parameters of a population, but there are many problems, as the following example shows.

Example 8 11D

In 1992, astronomers confirmed the first discovery of an *exoplanet*, which is a planet orbiting a star other than our Sun. They soon found examples of multiple exoplanets orbiting the same star. Up to 28th November 2017, the website http://www.openexoplanetcatalogue.com reported that 3653 planets had been discovered, in 2657 nearby star systems in our galaxy, including our Sun. Here is the sample distribution for the number of stars that have x known planets orbiting them — multiple-star systems are excluded because they are hard to classify, and only stars with at least one discovered planet are included. The frequency table below thus involves 3457 known planets orbiting 2527 single stars.

x	1	2	3	4	5	6	7	8	9	Sum
f	1921	401	133	47	14	5	3	1	2	2527 suns
xf	1921	802	399	188	70	30	21	8	18	3457 planets
f_r	0.7602	0.1587	0.0526	0.0186	0.0055	0.0020	0.0012	0.0004	0.0008	1
$x f_r$	0.7602	0.3174	0.1579	0.0744	0.0277	0.0119	0.0083	0.0032	0.0071	1.3680
$x^2 f_r$	0.7602	0.6347	0.4737	0.2976	0.1385	0.0712	0.0582	0.0253	0.0641	2.5235

$$\bar{x} = \sum x f_r = 1.37 \text{ planets}$$

$$s^2 = \sum x^2 f_r - \bar{x}^2 = 2.5235 - 1.3680^2 \doteqdot 0.65$$

$$s = \sqrt{s^2} \doteqdot 0.81 \text{ planets.}$$

This is a sample of the population of all the single-star-system stars in our Milky Way Galaxy. Why would scientists never use this table alone to infer the distribution of exoplanets across all the single-star systems in the galaxy?

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SOLUTION

[Some of the more obvious reasons]

- Only larger planets are counted because smaller planets are hard to find. The number of planets observed to orbit each star in the table will probably be less than the actual number of planets orbiting that star.
- The value 0 is missing because we can't yet show that a star has no planets.
- These stars are near our solar system, so they may not represent the average situation in the galaxy. In particular, in denser regions, stars are closer, and planets may perhaps be more easily stripped away by near collisions with other stars.
- This is an extremely small sample of the 100–400 billion stars in the galaxy.

10 POPULATION — CENSUS AND SURVEY

- A *census* performs an experiment (perhaps by asking a question) on everyone or everything in a population. A *survey* samples only some of the population.
- The relative frequencies in a population distribution are the exact probabilities of a particular result when a person or thing is chosen at random from the population.
- The population mean is therefore assigned the pronumeral μ, and the population standard deviation is assigned the pronumeral σ.

Sampling when there are no exact probabilities

Until now, almost everything has been based on equally likely possible outcomes. The four-coins probabilities were based on 16 equally likely outcomes when four coins are tossed. The discussion of the number of exoplanets was based on the unknown population of stars in the galaxy, with the equally likely possible outcomes being choosing a single-star-system star in the galaxy at random and counting its planets (not that we have any idea how such an experiment could be performed).

In 1903, a German researcher recorded the number x of baby mice in each litter of mice in his laboratory (Yule & Kendall, *An introduction to the theory of statistics*, Charles Griffin, page 121).

x	1	2	3	4	5	6	7	8	9	Sum
f	7	11	16	17	26	31	11	1	1	121 litters
xf	7	22	48	68	130	186	77	8	9	555 babies
f_r	$\frac{7}{121}$	$\frac{11}{121}$	$\frac{16}{121}$	$\frac{17}{121}$	$\frac{26}{121}$	$\frac{31}{121}$	$\frac{11}{121}$	$\frac{1}{121}$	$\frac{1}{121}$	1
xf_r	$\frac{7}{121}$	$\frac{22}{121}$	$\frac{48}{121}$	$\frac{68}{121}$	$\frac{130}{121}$	$\frac{186}{121}$	$\frac{77}{121}$	$\frac{8}{121}$	$\frac{9}{121}$	$\frac{555}{121} = \bar{x}$
$x^2 f_r$	$\frac{7}{121}$	$\frac{44}{121}$	$\frac{144}{121}$	$\frac{272}{121}$	$\frac{650}{121}$	$\frac{1116}{121}$	$\frac{539}{121}$	$\frac{64}{121}$	$\frac{81}{121}$	$\frac{2917}{121}$

$$\begin{aligned}\bar{x} &= \sum x f_r \\ &= \frac{555}{121} \\ &\doteqdot 4.59 \text{ baby mice}\end{aligned} \qquad \begin{aligned} s^2 &= \sum x^2 f_r - \bar{x}^2 \\ &= \frac{2917}{121} - \frac{308\,025}{14\,641} \\ &\doteqdot 3.0689\end{aligned} \qquad \begin{aligned} s &= \sqrt{s^2} \\ &\doteqdot 1.75 \text{ baby mice}\end{aligned}$$

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The sample statistics were easily calculated above. The obvious intention, however, is to talk about all mice, and this time there are no theoretical probabilities based on equally likely possible outcomes, and no population. We would normally interpret the relative frequencies as probabilities that a mouse litter, chosen at random, has this many baby mice, but what would such a probability mean?

- Is such a probability a statement about the physical world, in this case mice?
- Or is it a statement about our lack of knowledge about the size of a litter?

In the second case, the probability is never fixed, but changes all the time as further information comes in.

Then there are also the usual questions about the experiment. For example:

- Are these laboratory mice better fed and housed than wild mice?
- Is there some subtle genetic difference between these mice and other mice?
- Does weather affect litter size, and were recent weather patterns normal, whatever that means?

Predicting the future

The most important, and the most tricky, use of sampling is to predict the future. From the website http://www.bom.gov.au, here are the numbers of years from 1913 to 2012 when there were x rainy days in April, recorded at Observatory Hill, Sydney (where 'rainy day' is a day when the rainfall was greater than zero.)

x	3	4	5	6	7	8	9	10	11	12	13
f	1	1	3	2	7	9	3	9	10	9	4
f_r	0.01	0.01	0.03	0.02	0.07	0.09	0.03	0.09	0.10	0.09	0.04
$x f_r$	0.03	0.04	0.15	0.12	0.49	0.72	0.27	0.90	1.10	1.08	0.52
$x^2 f_r$	0.09	0.16	0.75	0.72	3.43	5.76	2.43	9.00	12.10	12.96	6.76

x	14	15	16	17	18	19	20	21	22	23	Sum
f	9	2	4	9	7	5	2	1	2	1	100
f_r	0.09	0.02	0.04	0.09	0.07	0.05	0.02	0.01	0.02	0.01	1
$x f_r$	1.26	0.30	0.64	1.53	1.26	0.95	0.40	0.21	0.44	0.23	12.64
$x^2 f_r$	17.64	4.50	10.24	26.01	22.68	18.05	8.00	4.41	9.68	5.29	180.66

$$\bar{x} = \sum x f_r = 12.64 \text{ rainy days}$$

$$s^2 = \sum x^2 f_r - \bar{x}^2 = 180.66 - 12.64^2 \doteqdot 20.89$$

$$s = \sqrt{s^2} \doteqdot 4.57 \text{ rainy days}$$

In the five subsequent years from 2013 to 2017, the successive numbers of rainy days in April were 16, 15, 23, 13 and 10. Here are some questions that you may raise, but the course cannot answer.

- Are these subsequent results consistent with the probabilities suggested by the relative frequencies above?
- Droughts last for many years, so perhaps rainy Aprils come in groups? If there are cycles in weather with periods of say 15 years, or 1100 years, or 125 000 years, then how could we use this table to predict future weather? Are the last five years' results consistent with a permanent and unprecedented shift in climate?
- What does the probability of 10 rainy days in April actually mean? Is it a statement about the physical world, in this case the climate, or is it statement about our lack of knowledge? If such probabilities exist, do they vary over time, in which case how do we define them? There are certainly no equally likely possible outcomes in sight here on which to base a theoretical approach.

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Clearly, a single sample distribution such as this does nothing to answer the urgent questions people are asking. Far more data on all sorts of different phenomena, together with sophisticated statistical tests that can quantify exceptional results and take into account possible cyclical data, are required before any reliable conclusions can be made. Climate science is extremely complicated.

Extension — a correction factor for the sample variance

When we know the theoretical mean μ and we are sampling to find the variance, there is no problem with the formulae for the sample variance. However, when we are sampling to both find the mean and to find the variance, then the sample mean will drift very slightly towards the sample results, with the effect that the sample variance will tend to be slightly smaller than it should be.

This phenomenon is much discussed, and the standard answer is that the sample variance should be multiplied by a correction factor $\dfrac{n}{n-1}$ where n is the size of the sample.

Thus, in the example above about rainy days in April, there were 100 results, and we were using a sample mean rather than a theoretical mean, so the correction factor is $\frac{100}{99}$. Using the correction factor would yield the following values:

$$\begin{array}{lll} \bar{x} = 12.64 \text{ rainy days} & s^2 = 20.89 \times \frac{100}{99} & s = \sqrt{s^2} \\ \text{(as before)} & \quad \doteqdot 21.10 & \quad \doteqdot 4.59 \text{ rainy days} \end{array}$$

The larger the size n of the sample, the less difference the correction makes.

Your calculator may have a button $\boxed{\sigma_n}$ or something equivalent. This gives the standard deviation without the correction factor, and is all that is required in this course. It may also have a button labelled $\boxed{\sigma_{n-1}}$ or equivalent, which applies the correction factor — this button is not required in this course.

Approaching the following exercise

The following exercise contains many suggestions for experiments that will generate data. In each situation you should be asking a number of important questions:

- How can you ensure that the data are randomly generated?
- Are there any biases due to the population used for your experiment? For example, measuring the heights of students in the class will give results different from measuring the heights of students in the school.
- How can you complete the experiment efficiently and accurately, including recording the data? Tally tables, and accurate record-keeping, are two important skills in experiments.
- What special equipment is needed, such as stopwatches, tape measures, metre rulers?
- How big does the sample size need to be to get a reasonable estimate? This might become clearer as the experiment proceeds, depending on whether the data appear stable.
- Is your data from a population, with mean μ and standard deviation σ, or from a sample, with sample mean $\bar{x}$ and sample standard deviation s?
- What are the mean and standard deviation measuring — what is their interpretation in the context of the experiment? Are there theoretical or population means and standard deviations to compare the data with? If so, what conclusions can you draw?

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Using a spreadsheet

The probabilities of a distribution need to be calculated, and the frequencies of a sample need to be tallied. Once this is done, however, the calculations of mean, variance and standard deviation can be automated on a spreadsheet, provided that you know how to do two things:

1 Write formulae into cells.
2 Fill down and fill right.

You are invited to use a spreadsheet in this exercise to save time in the calculations. Once a calculation has been set up, it can be copied and adapted.

Many simulations can be done using *random numbers* — see question **5** below, for example, and see the table of random numbers on page 497. Excel has two very useful random number functions:

`RANDBETWEEN(a, b)` generates a random integer in the interval $a \le x \le b$. For example, `RANDBETWEEN(11, 20)` may generate 14, or 20, or 11, or 15.

`RAND()` generates a random number in the interval $0 < x < 1$. For example, it may generate 0.865348256572775, or 0.44037476507047, or 0.0174096864227915.

Once you have a random number in a cell, the commands fill down (`Ctrl + D`) and fill right (`Ctrl + R`) will give you as many random numbers as you need.

Exercise 11D

FOUNDATION INVESTIGATIONS

This is not a normal exercise with a normal set of questions. Apart from Question **1**, it is a set of suggestions for activities and investigations, some of which could easily be turned into projects. Most questions ask for a number of simulations (or trials or runs) of an experiment in order to create a sample, followed by an analysis of the results.

First do Question **1**. After that, there is no expectation that anyone should simply work through the exercise, because the subsequent questions all take some time, however they are done. They are intended to be adapted to the needs of the class or the individual.

These simulations could be done by individual students, but they could also be done in groups, or by the class as a whole. Most could also be done on a spreadsheet using random numbers — this requires setting up a spreadsheet, which also takes time.

1 Three dice are thrown. Let X be the number of dice showing a 5 or 6.

a The probability distribution for this experiment is shown here (it can easily be calculated using a probability tree diagram). Graph the distribution. Then calculate the theoretical mean μ, the theoretical variance σ^2, and the theoretical standard deviation σ.

x	0	1	2	3
$p(x)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

b The frequency table shown gives the results when the experiment was done 100 times. Calculate the relative frequencies, and graph them. Then calculate the sample mean $\bar{x}$, the sample variance s^2 and the sample standard deviation s.

x	0	1	2	3	Sum
f	33	47	16	4	100

c Do the sample results appear to be consistent with the theoretical results?

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2 Two dice are thrown. Let X be the sum of the two numbers on the dice.

a Here is the theoretical probability distribution of the experiment, together with the calculations for obtaining the mean and the variance. From the table, write down the mean μ, the variance σ^2, and the standard deviation σ.

x	2	3	4	5	6	7	8	9	10	11	12	Sum
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1
$xp(x)$	$\frac{2}{36}$	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{20}{36}$	$\frac{30}{36}$	$\frac{42}{36}$	$\frac{40}{36}$	$\frac{36}{36}$	$\frac{30}{36}$	$\frac{22}{36}$	$\frac{12}{36}$	7
$(x-\mu)^2$	25	16	9	4	1	0	1	4	9	16	25	—
$(x-\mu)^2p(x)$	$\frac{25}{36}$	$\frac{32}{36}$	$\frac{27}{36}$	$\frac{16}{36}$	$\frac{5}{36}$	0	$\frac{5}{36}$	$\frac{16}{36}$	$\frac{27}{36}$	$\frac{32}{36}$	$\frac{25}{36}$	$\frac{210}{36}$

b Throw a pair of dice 72 times and record your sample distribution in a relative frequency table.

c How do your probabilities in part **a** compare with the relative frequencies obtained in your experiment? Draw the two graphs and compare them.

d Use the sample distribution table to calculate the sample mean $\bar{x}$, the sample variance s^2 and the sample standard deviation s. The sample mean will not be a whole number, so use the alternative formula for variance.

e How do the sample statistics $\bar{x}$ and s compare with the theoretical statistics μ and σ?

f To improve your estimation, combine your results with those from other members of your class.

3 Design some censuses in your class. Draw up a population distribution table for each, and calculate the population mean μ, the population variance σ^2, and the population standard deviation σ. Here are some suggestions.

a The month number of each student's birthday.

b The number of brothers and sisters of each student.

c Each student's writing speed — count the number of times in one minute that a student can fully write the four words, 'The quick brown fox.'

d The number of sports they do, or the number of musical instruments they play.

e The number of pets at home — if they have goldfish, decide whether that result is an outlier or should be rejected.

f How many days in the last week did each student cook a meal?

Why is, or is not, each class census an unbiased sample of the school population?

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4 Many experiments in this chapter have involved throwing a die. It is important to know *how random* the results will be that are obtained from throwing a die.

a Throw a die 40 times and write down the results one after the other.

b Are the probabilities approximately *uniform*, that is, the same for each outcome?

c We now investigate whether each outcome on the die is *independent* of the previous roll. Perhaps the way you roll the die affects things here?

i Write down the 39 *differences* between successive rolls, discarding any minus sign.

ii Draw up a sample distribution table, and calculate the sample mean $\bar{x}$ and sample standard deviation s.

iii Question **10** of Exercise 11B asked for the probability distribution table of an experiment equivalent to this, namely 'Throw two dice and record their difference'. Copy and complete the table to find the standard deviation σ.

x	0	1	2	3	4	5	Sum
$p(x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	1
$xp(x)$	0	$\frac{10}{36}$	$\frac{16}{36}$	$\frac{18}{36}$	$\frac{16}{36}$	$\frac{10}{36}$	$\frac{70}{36}$

iv Check whether your experimental probabilities agree with the theoretical results. Draw the two graphs and compare them.

5 Many calculators have a random-number generator. One calculator has a button labelled $\boxed{\text{Ran\#}}$ (returning a three-digit number between 0 and 1) and a button labelled $\boxed{\text{ranint()}}$ (for example $\boxed{\text{ranint}}$ (1, 13) returns a random integer between 1 and 13 inclusive). Use your manual, or ask others, to find what functions are available on your calculator.

Excel's RAND function is a random-number generator. Many websites provide random numbers. There is also a list of 10 000 random numbers at the end of this exercise. If your method generates a three-digit number and only one digit is required, simply discard the unwanted digits.

a Generate a single-digit random number 50 times, keeping the numbers. Calculate the sample mean $\bar{x}$ and the sample standard deviation s. Compare them with the uniform distribution for the numbers $0, 1, \ldots, 9$, where $\mu = 4\frac{1}{2}$ and $\sigma^2 = 8\frac{1}{4}$ (these results were calculated in general in Question **12** of Exercise 11C).

b Discard the results of part **a** if the digit is 0 or 9. Calculate the new sample mean $\bar{x}$ and sample standard deviation s, and compare them with the corresponding uniform distribution for the numbers $1, 2, \ldots, 8$, where $\mu = 4\frac{1}{2}$ and $\sigma^2 = 5\frac{1}{4}$. This models an eight-sided die used in some games.

DEVELOPMENT INVESTIGATIONS

6 This question models a simple lottery. You will be using random-number simulations to estimate the probabilities and payouts — theoretical probabilities are not required.

Write down any four distinct single-digit numbers between 0 and 9 inclusive. These will be your winning numbers for the next four parts.

a Generate, by any method, four distinct random single-digit numbers — if you get a number that has already occurred, just discard it and generate a new number.

b Record the number of matches you have between your four random numbers and your four winning numbers. The number of matches is your random variable X.

c Repeat this experiment say 30 times and tabulate your results as a frequency table for X. These are your experimental estimates of the probability of each outcome.

d Calculate your expected payout if 4 matches wins you \$100 and 3 matches wins you \$10. Then calculate your expected profit or loss if entering the game costs \$2.

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[A longer investigation] You may change the range of numbers, the number of numbers, the payouts, and the cost of entering the game, to generate other results. How much of the procedures in parts **a–d** can you automate if you write your simulations in Excel?

7 [Benford's law] This question models *Benford's law*, which is used, among other things, in forensic analysis of large data sets, for example to spot tax fraud. It models the frequency distribution of the leading digit in each score of the data by the following table:

x	1	2	3	4	5	6	7	8	9
$p(x)$	30.1%	17.6%	12.5%	9.7%	7.9%	6.7%	5.8%	5.1%	4.6%

a Copy and complete the table to find the mean μ and standard deviation σ.

b We can test Benford's law using the first digit of the powers of 2 from 2^0 to 2^{99}. Generate a random two-digit number d (whose leading digit may be 0). Define a random variable X to be the leading digit of 2^d. For example,

If $d = 36$, then $2^d \doteqdot 6.8719 \times 10^{10}$, so $X = 6$.

Repeat this experiment 40 times, record your results in a frequency table, and calculate the sample mean $\bar{x}$ and the sample standard deviation s. Do your results agree reasonably with Benford's law?

Applying Benford's law to internet data: Obtain a large set of numerical data, such as stock prices, house prices, population numbers, death rates, lengths of rivers. Choose data that varies widely in its range across several orders of magnitude — a distribution with a range from say 10 to 100 would not be a good set of data for this experiment. See if the distribution of the leading digits in the data agrees reasonably with Benford's law.

A possible project: What formula generated the nine percentages in the table above? How can it be proven, and what assumptions are being made in that proof? What is the history of Benford's law? What are some typical situations, in all sorts of contrasting fields, where Benford's law is useful? In what situations does Benford's law begin to break down, or break down completely?

8 This experiment estimates throwing accuracy. First, you need an object that you can throw but will not bounce much, such as a small bag filled with sand, or tightly rolled paper. Secondly, find a clear wall that you can stand some distance away from. On the wall at ground-level, mark a bullseye that a thrower is to aim for. Use tape to place successive marks at 10 cm intervals from the bullseye, horizontally to the left and to the right, at the base of the wall. Label these marks …, −2, −1, 0, 1, 2, … Later, if people are missing the bullseye badly, you may have to increase the number of marks. Thirdly, mark a position quite some distance away to throw from.

Throw the projectile 50 times and record the integer nearest to where the projectile lands — the successive throws can be done to test the accuracy of an individual or of a group. Calculate the sample mean and sample standard deviation of the experiment.

This experiment can also be done as a continuous probability experiment. If the results are graphed, the result might be expected to be the bell-shaped curve of the *normal distribution* that you will study in Year 12. A discrete version of this curve may be evident if you graph your results.

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9 **a** Open a novel on a random page, and choose a word at random. Let X be the random variable recording the number of letters in the word.

b Compare results obtained from a 'light' novel and a more serious text or classic, such as you might study in English.

c Will your results be different if you use a textbook from history or science? Perhaps different groups in the class could use different types of books.

10 The distribution of letters in English is well documented and may be used in code cracking, as well as having application to games such as Scrabble.

a Use the internet to find the expected distribution of letters in an English document. (A search for 'distribution of letters in English' should return a suitable result.)

b Again using a random page in a novel, count the number of occurrences of each letter within a block of 200 letters, and hence make your own estimate of the frequencies of each letter.

c Do the results depend on the type of document selected?

11 Many irrational mathematical constants are called *normal*, meaning that the distribution of the digits in their decimal expansion is uniform, with every digit being equally likely. Use the internet to find the decimal expansions of constants such as π and $\sqrt{2}$, and find the frequency of each digit in their first 50 digits.

12 [Experiments involving estimation] These experiments involve certain difficulties with randomisation that you will need to minimise in your experimental design.

a First read the list of experiments below and discuss what issues may occur.

b Complete the experiment in groups or with the whole class.

i Draw a line on the board. Each member of the group is to estimate the length of the line, correct to the nearest centimetre.

ii Each member is asked to draw a 10 cm line on the board without a ruler. Then measure the line correct to the nearest centimetre.

iii Each member is to draw a circle of radius 20 cm on the board. Then estimate its roundness by measuring the difference between the longest and shortest diagonal, correct to the nearest 1 cm.

Again, these experiments could also be done as continuous probability experiments.

13 **Applying the theory to internet data**: The internet is a vast source of data. Here are some suggestions of how some internet data can be investigated — they should prompt many more similar investigations.

a Most countries in the world have land borders with other countries. Let X be the number of adjacent countries of a country, and draw up a frequency table for X. Then comment on your results, with the help of people with a knowledge of geography.

You could do the same analysis of the states of one country, such as USA, Germany or Australia (decide first whether to regard the Northern Territory and ACT as states).

b There are fewer than 100 naturally occurring elements, but more than 300 naturally occurring isotopes of these elements. Let X be the number of isotopes of an element, and draw up a frequency table for X, with mean and standard deviation. Then comment on your results, with the help of people with a knowledge of physics and chemistry.

You will soon run into huge problems of classification. What shall we class as 'naturally occurring'? Do we include radioactive isotopes that have extremely short half-lives? Do we include isotopes not found on Earth? Do we include isotopes and elements produced in nuclear reactors?

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c Natural disasters are rich sources of data. Let X be the number of major earthquakes worldwide in a particular year. Draw up a frequency table for X, and comment on your results. You will need to define the term 'major earthquake', probably after looking at some data, and also decide whether several large earthquakes close together in time and space count as one earthquake or many. You may also like to consider smaller earthquakes and restrict the study to some region of the world.

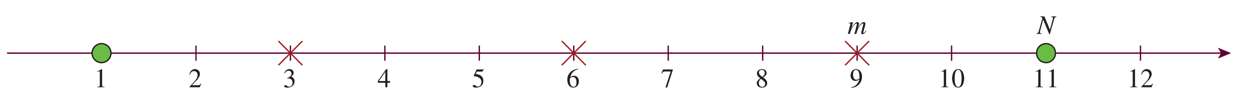
Major hurricanes could be the subject of a similar study.

CHALLENGE INVESTIGATION

14 [The German tank problem] The enemy has N tanks, each labelled with a unique serial number from 1 to N. Unfortunately the number of tanks is unknown, but that would be useful information to the war effort. Tanks are occasionally captured and their serial numbers read. Counter-intelligence operatives want to use these serial numbers to estimate the total number N of tanks.

They begin with the assumption that these serial numbers lie randomly within the interval from 1 to N. For the purposes of modelling this problem, we will work as if the serial numbers are equally spaced.

Suppose that k serial numbers have been discovered, with m being the largest of them. The case $k = 3$ and $m = 9$ is shown in the diagram.



a Explain why the average number of undiscovered serial numbers between the k samples on the interval from 1 to m is $\dfrac{m-k}{k}$.

b If this same gap extends past the largest discovered serial number, explain why

$$N = m + \frac{m}{k} - 1.$$

c Test this estimate by modelling the tank problem.

- **i** Working in pairs, the enemy partner chooses a two-digit number N, representing the number of tanks they have. They do not share this.
- **ii** The enemy tank owner generates a random number between 1 and N and allows the counter-intelligence partner to 'discover' this serial number. The counter-intelligence operative tabulates the number k of discovered serials, the maximum serial number m discovered, and their estimate of N using the previous formula.
- **iii** How quickly and accurately does the scheme estimate the correct value of N? How many tanks must be discovered before the estimate is within 10% of the correct value?

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A table of random numbers

Some cautions about using these or any table of random numbers:

- Any printed set of numbers such as these is only a simulation of random numbers. Once they are written down they are no longer random.
- If you are repeating an experiment, never take your random numbers from a place on this table where you have taken numbers before.
- It may be safer to search for 'table of random numbers' on the internet than to use these.
- Excel generates random numbers with the functions `RAND` and `RANDBETWEEN`, and most calculators generate them. How do they generate them?

37743	65520	55183	10968	38039	67295	66872	49849	24546	23324	77131	09038	02082	50408
41801	02262	85716	34590	09006	66392	21083	70211	01972	13007	29700	23977	26832	04955
79271	24018	05443	57916	79007	73955	33197	66964	99935	91957	61833	58130	20211	74807
07252	64966	95430	86966	82324	30105	53945	74630	54709	31824	17172	04797	29660	35607
44907	29330	57843	70710	05639	74016	68147	31718	71858	52629	25494	05112	40372	05998
13655	20830	77618	66061	43060	28829	89975	44934	83615	00648	19074	48971	22506	55971
62113	17825	47021	14189	17171	88516	52356	70962	53516	84486	74430	99783	06920	29668
47698	78539	99974	87544	87961	41074	87394	25210	00183	53751	26340	02438	11297	17016
71726	85738	28023	91959	96532	87421	19680	51877	37115	29660	25994	58836	63527	33556
89917	65236	74820	33196	15434	40370	19366	14720	50378	21987	40163	80380	94243	92858
45152	37604	30995	82334	02529	13058	86885	92913	61053	91709	13261	51796	95365	80887
94383	87457	76296	99349	80233	31202	66181	91339	44328	43728	39172	65651	66531	66694
26550	59696	40868	99275	87036	28556	08335	99728	02106	59054	14716	11957	96302	08814
93391	17072	82546	70322	73149	09950	32449	59990	11004	00805	71630	32220	50003	41740
00019	18462	02044	78320	37180	91282	55680	26126	92615	53419	08942	72322	61082	51852
27142	02218	42903	16862	25693	50829	89282	26090	77915	79441	72611	93606	47853	77171
81790	83966	48086	87531	42531	51278	22542	59990	62679	03263	54306	26536	02516	90318
98440	03241	70082	19577	39788	16819	41482	93977	75793	56193	06199	76316	24307	49053
37407	78320	56407	63424	87287	74701	54201	34224	07333	42224	93650	14828	93670	23765
52371	53632	57363	17378	42455	95610	19732	90354	68279	08886	32663	18284	09316	36288
23921	49125	27951	71912	67462	92257	02289	62690	78746	32550	83974	34946	35050	69424
80994	45961	20780	80912	65499	60536	97782	12724	06612	52555	00657	92307	28239	31544
57072	67909	91887	94239	33074	72642	04470	01144	03400	13543	51126	78094	46792	85639
53028	42821	66307	11004	50198	56513	34633	02830	09316	00376	64396	54481	56535	77200
03481	53562	35294	37918	55334	20976	89131	33172	24283	74992	69490	89872	85848	20303
68542	12761	82050	62092	76990	10975	41729	45090	66887	49608	25854	17644	99446	26275
97832	20775	57850	72669	18829	02244	05069	81133	68073	55611	66672	56212	88173	84300
38636	91223	55429	90662	20162	69815	75145	90034	71484	92532	01942	49805	96687	28373
85068	53478	51514	91358	38715	89730	76181	42089	37326	66477	53189	14313	35914	49938
90136	66483	15368	28691	27533	58321	52769	02744	71611	52382	16702	97590	52136	40617
87569	03275	05223	95931	02900	38457	81146	48751	27668	39470	77154	88585	49599	74388
95278	90584	99483	82082	25149	67110	96494	36290	51806	44290	18492	34675	88514	63587
97345	12886	34569	08356	39236	83625	58118	55409	69976	00707	03967	51579	05318	48122
26393	15280	46270	54100	46962	63071	99405	30169	37334	61219	22175	35222	27080	02257
93225	64579	36496	06554	07701	64046	34410	48872	97568	68980	20999	09930	76920	68352
35601	52966	12289	93810	89525	25301	17966	33739	23866	84899	76304	72323	92317	51934
83037	53612	68980	85346	17591	70582	81182	84410	38393	79195	67283	15293	03191	37464
59490	62620	14356	69518	91673	06038	89207	54660	32172	21772	55311	41429	41614	09193
70806	86669	55723	42381	69705	19586	93131	46682	32216	30809	88198	00277	80048	86458

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48813	60741	96728	67204	39217	16605	72911	32330	87809	63919	76696	08795	19962	07818
48931	05297	15832	89078	23535	51333	93176	35684	72804	61267	15158	39390	77725	23384
23167	57283	23289	13961	94625	38343	18805	36323	25804	75008	43210	84912	39633	79735
15381	34341	18518	86564	89450	53466	62145	77678	75117	44440	70303	34968	82958	06781
50686	16595	94742	26433	51716	58551	58742	89828	20345	01223	10048	22948	84876	40955
15876	55389	62771	06611	45887	09424	93465	61135	67494	14062	96178	28119	66176	44113
52727	43359	90793	84253	15340	31422	23893	18080	39469	07207	77032	42184	41597	02507
11080	05202	82059	17571	81536	71887	72847	97671	08840	12168	10429	34657	73687	15209
33555	93526	37338	28479	06074	63982	86933	21808	24745	36996	05804	76451	52050	06376
68784	99154	74006	95318	62528	61211	82977	07981	30813	62531	85487	06040	44863	20697
35035	10254	98654	85034	89902	27024	29227	75663	95969	99961	62004	47571	94526	86443
81097	79671	55086	59182	24623	88864	33197	32783	05015	69100	53255	15310	54334	55445
16316	84845	94584	13138	32907	91051	33021	51660	80844	81737	82469	63789	88037	54421
06631	89667	40308	32886	90400	05865	14201	68751	35174	39438	34796	68027	79168	06994
58519	42683	42723	78754	80957	87208	98625	67616	69671	80173	67412	43213	22006	10792
37689	70982	47903	30488	11438	76221	77716	29452	74337	33157	95020	38207	91062	79001
94427	54174	25610	53018	64393	93225	86605	60584	95038	66257	97613	62254	10952	80745
92453	87817	29012	67312	35511	75367	41192	31286	67052	35359	22375	29264	73532	39672
63169	70677	82912	35678	27715	44349	84968	85296	36502	70061	76606	29060	74181	75115
32172	75178	22374	95893	98768	13599	17488	04423	16560	55070	33528	59941	64095	95204
89910	85449	28611	03537	84271	69284	01348	37595	50285	91677	97866	81955	40472	76316
03349	12306	34013	66907	42927	84514	69563	28741	76982	48682	03059	75250	21462	88216
90537	19577	93195	83882	95946	51130	80199	02875	23952	37154	36997	12876	77762	49284
04928	21118	41927	15114	57461	38447	90159	94759	70504	68385	18143	82414	17721	21192
01349	32918	90842	52928	77845	13319	44975	29518	56107	79434	68921	15811	47838	77411
80482	89646	25772	70294	46781	80911	06493	89582	29566	35609	58243	11796	55527	04918
33740	60037	11609	59254	27984	30552	65324	27333	61075	44293	10061	60237	86020	23485
35401	20472	21553	93181	22430	96031	09939	07832	64049	39460	99793	02844	10289	31917
53891	71142	38621	72237	81979	64735	49272	90992	98589	21276	85534	89556	05852	15624
79077	04164	01309	36408	20265	94980	10721	87589	53464	60154	45888	15024	97357	74888
87455	58024	90255	55495	92727	25923	60412	48467	28484	99634	21334	44533	28789	58216
92096	17721	77803	30436	42649	89425	17599	71328	64138	17463	80969	63760	64210	10142
49244	44994	87330	65321	35101	58249								

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Chapter 11 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 11 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 State whether each random variable is numeric or categorical. If it is numeric, state whether it is discrete or continuous.

a The maximum temperature in Sydney on a given day.
b The number of test cricket games in Australia in a given year.
c A die is thrown until it shows a six, and the number of throws required is recorded.
d The state of origin of a Rugby League player.

2 Which of the following are probability distributions?

a

x	1	2	3	4
$P(X = x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

b

x	1	2	3	4
$P(X = x)$	0.4	0.1	0.5	0.2

c

x	1	2	3	4
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{6}$	$\frac{1}{8}$

3 Give three reasons why the following table is not a valid probability distribution.

x	1	2	3	4
$P(X = x)$	0.2	−0.1	0.5	1.3

4 Copy and complete each table to find the expected value $E(X)$ of the distribution.

a

x	0	1	2	3	Sum
$p(x)$	0.2	0.3	0.4	0.1	
$xp(x)$					

b

x	−2	−1	0	1	Sum
$p(x)$	0.5	0.1	0.1	0.3	
$xp(x)$					

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Review

5 When Jack first visited the Thai Pin Restaurant, he read the menu and assigned each meal a probability indicating how likely he was to order it in the future — this was determined by how much the meal interested him. The fish cost \$27 and he rated it $\frac{4}{9}$, the steak cost \$32 and he rated it $\frac{2}{9}$, the vegetarian option cost \$23 and he rated it $\frac{1}{9}$, the chicken cost \$25 and he rated it $\frac{2}{9}$.

a What was Jack's expected cost in buying a meal at the restaurant?

b This is now Jack's favourite restaurant, and he visits it once a week (52 times a year). What is his expected cost over the next year, assuming his interest ratings do not change and the prices remain constant?

6 Copy and complete the probability distribution tables below to calculate $\mathrm{Var}(X)$ using the definition $\mathrm{Var}(X) = \mathrm{E}\big((X - \mu)^2\big)$. Also write down σ .

a

x	1	2	3	4	Sum
$p(x)$	0.4	0.3	0.2	0.1	
$xp(x)$					
$(x-\mu)^2$					—
$(x-\mu)^2 p(x)$					

b

x	4	5	6	7	Sum
$p(x)$	0.2	0.6	0.1	0.1	
$xp(x)$					
$(x-\mu)^2$					—
$(x-\mu)^2 p(x)$					

7 Now use the alternative formula $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$ for the variance for the distributions of the previous question. Copy and complete the tables, then calculate the variance.

a

0	1	2	3	4	Sum
$p(x)$	0.4	0.3	0.2	0.1	
$xp(x)$					
$x^2 p(x)$					

b

x	4	5	6	7	Sum
$p(x)$	0.2	0.6	0.1	0.1	
$xp(x)$					
$x^2 p(x)$					

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8 Calculate the mean, the variance and the standard deviation of each probability distribution.

a

x	0	1	2	3	4
$p(x)$	0.05	0.1	0.8	0	0.05

b

x	0	1	2	3	4
$p(x)$	0.3	0.1	0.2	0.1	0.3

9 Explain briefly the meaning and significance of the expected value of a probability distribution.

10 Explain briefly the meaning and significance of the variance and standard deviation of a probability distribution.

11 For the random variable X, it is known that $E(X) = 6$ and $\mathrm{Var}(X) = 2$.

a Write down $E(2X)$, $\mathrm{Var}(2X)$ and σ for the new distribution $2X$.

b Write down $E(X + 5)$, $\mathrm{Var}(X + 5)$ and σ for the new distribution $X + 5$.

c Write down $E(3X - 1)$, $\mathrm{Var}(3X - 1)$ and σ for the new distribution $3X - 1$.

12 a The sample space for a certain random variable X is $\{5, 6, 7, 8, 9\}$. The probability distribution is known to be uniform. Write down the table for this probability distribution and find the expected value μ and the standard deviation σ.

b Use random numbers to simulate this experiment. First decide how you will obtain the random numbers, how you will use them, and how many simulations your sample will have. Then draw up a relative frequency table and calculate the sample mean $\bar{x}$ and the sample standard deviation s.

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Answers

Chapter 1

Exercise 1A

1a $4x$ **b** $2x$ **c** $-2x$ **d** $-4x$
2a $5a$ **b** $-a$ **c** $-9a$ **d** $-3a$
3a 0 **b** $-y$ **c** $-10a$ **d** $-3b$ **e** $7x$ **f** $-3ab$
g $4pq$ **h** $-3abc$
4a $-6a$ **b** $12a^2$ **c** a^5 **d** a^6
5a $-2a$ **b** 3 **c** a^6 **d** a
6a $2t^2$ **b** 0 **c** t^4 **d** 1
7a $-3x$ **b** $-9x$ **c** $-18x^2$ **d** -2
8a -4 **b** -12 **c** 18 **d** 2
9a $x + 3$ **b** $2y - 3$ **c** $2a - 3$ **d** $8x + 4y$
e $-10t - 5$ **f** $4a - 3a^2$ **g** $-5x^2 - 12x - 3$
h $9a - 3b - 5c$
10a 5 **b** $7m^2$ **c** $-12a$ **d** $-3p^3q^4r$
11a $2x$ **b** $4x$ **c** $-6a$ **d** $-4b$
12a $10a$ **b** $-18x$ **c** $-3a^2$ **d** $6a^3b$
e $-8x^5$ **f** $-6p^3q^4$
13a -2 **b** $3x$ **c** xy **d** $-a^4$ **e** $-7ab^3$ **f** $5ab^2c^6$
14a $6a^5b^6$ **b** $-24a^4b^8$ **c** $9a^6$ **d** $-8a^{12}b^3$
15a 0 **b** -1 **c** 59 **d** 40
16a $3a^2$ **b** $5c^4$ **c** a^2bc^6
17a $2x^5$ **b** $9xy^5$ **c** b^4 **d** $2a^3$

Exercise 1B

1a $3x - 6$ **b** $2x - 6$ **c** $-3x + 6$ **d** $-2x + 6$
e $-3x - 6$ **f** $-2x - 6$ **g** $-x + 2$ **h** $-2 + x$ **i** $-x - 3$
2a $3x + 3y$ **b** $-2p + 2q$ **c** $4a + 8b$ **d** $x^2 - 7x$
e $-x^2 + 3x$ **f** $-a^2 - 4a$ **g** $5a + 15b - 10c$
h $-6x + 9y - 15z$ **i** $2x^2y - 3xy^2$
3a $x + 2$ **b** $7a - 3$ **c** $2x - 4$ **d** $4 - 3a$
e $2 - x$ **f** $2c$ **g** $-x - y$ **h** $x + 4$ **i** $5a - 18b$
j $-2s - 10t$ **k** $x^2 + 17xy$ **l** $16a - b$
4a $x^2 + 5x + 6$ **b** $y^2 + 11y + 28$ **c** $t^2 + 3t - 18$
d $x^2 - 2x - 8$ **e** $t^2 - 4t + 3$ **f** $2a^2 + 13a + 15$
g $3u^2 - 10u - 8$ **h** $8p^2 - 2p - 15$
i $2b^2 - 13b + 21$ **j** $15a^2 - a - 2$
k $-c^2 + 9c - 18$ **l** $2d^2 + 5d - 12$
5a Answers will vary
b **i** Answers will vary
ii Answers will vary
6a $x^2 + 2xy + y^2$ **b** $x^2 - 2xy + y^2$ **c** $x^2 - y^2$
d $a^2 + 6a + 9$ **e** $b^2 - 8b + 16$ **f** $c^2 + 10c + 25$
g $d^2 - 36$ **h** $49 - e^2$ **i** $64 + 16f + f^2$
j $81 - 18g + g^2$ **k** $h^2 - 100$ **l** $i^2 + 22i + 121$
m $4a^2 + 4a + 1$ **n** $4b^2 - 12b + 9$ **o** $9c^2 + 12c + 4$
p $4d^2 + 12de + 9e^2$ **q** $4f^2 - 9g^2$ **r** $9h^2 - 4i^2$
s $25j^2 + 40j + 16$ **t** $16k^2 - 40k\ell + 25\ell^2$
u $16 - 25m^2$ **v** $25 - 30n + 9n^2$
w $49p^2 + 56pq + 16q^2$ **x** $64 - 48r + 9r^2$
7a $t^2 + 2 + \frac{1}{t^2}$ **b** $t^2 - 2 + \frac{1}{t^2}$ **c** $t^2 - \frac{1}{t^2}$
8a $10\,404$ **b** $998\,001$ **c** $39\,991$
9a $a^3 - b^3$ **b** $2x + 3$ **c** $18 - 6a$ **d** $x^2 + 2x - 1$
e $x^3 - 6x^2 + 12x - 8$ **f** $p^2 + q^2 + r^2$

Exercise 1C

1a $2(x + 4)$ **b** $3(2a - 5)$ **c** $a(x - y)$
d $5a(4b - 3c)$ **e** $x(x + 3)$ **f** $p(p + 2q)$
g $3a(a - 2b)$ **h** $6x(2x + 3)$ **i** $4c(5d - 8)$
j $ab(a + b)$ **k** $2a^2(3 + a)$ **l** $7x^2y(x - 2y)$
2a $(p + q)(m + n)$ **b** $(x - y)(a + b)$
c $(x + 3)(a + 2)$ **d** $(a + b)(a + c)$
e $(z - 1)(z^2 + 1)$ **f** $(a + b)(c - d)$
g $(p - q)(u - v)$ **h** $(x - 3)(x - y)$
i $(p - q)(5 - x)$ **j** $(2a - b)(x - y)$
k $(b + c)(a - 1)$ **l** $(x + 4)(x^2 - 3)$
m $(a - 3)(a^2 - 2)$ **n** $(2t + 5)(t^2 - 5)$
o $(x - 3)(2x^2 - a)$
3a $(a - 1)(a + 1)$ **b** $(b - 2)(b + 2)$ **c** $(c - 3)(c + 3)$
d $(d - 10)(d + 10)$ **e** $(5 - y)(5 + y)$
f $(1 - n)(1 + n)$ **g** $(7 - x)(7 + x)$
h $(12 - p)(12 + p)$ **i** $(2c - 3)(2c + 3)$
j $(3u - 1)(3u + 1)$ **k** $(5x - 4)(5x + 4)$
l $(1 - 7k)(1 + 7k)$ **m** $(x - 2y)(x + 2y)$
n $(3a - b)(3a + b)$ **o** $(5m - 6n)(5m + 6n)$
p $(9ab - 8)(9ab + 8)$
4a $(a + 1)(a + 2)$ **b** $(k + 2)(k + 3)$ **c** $(m + 1)(m + 6)$
d $(x + 3)(x + 5)$ **e** $(y + 4)(y + 5)$ **f** $(t + 2)(t + 10)$
g $(x - 1)(x - 3)$ **h** $(c - 2)(c - 5)$ **i** $(a - 3)(a - 4)$
j $(b - 2)(b - 6)$ **k** $(t + 2)(t - 1)$ **l** $(u - 2)(u + 1)$
m $(w - 4)(w + 2)$ **n** $(a + 4)(a - 2)$
o $(p - 5)(p + 3)$ **p** $(y + 7)(y - 4)$ **q** $(c - 3)(c - 9)$
r $(u - 6)(u - 7)$ **s** $(x - 10)(x + 9)$ **t** $(x + 8)(x - 5)$
u $(t - 8)(t + 4)$ **v** $(p + 12)(p - 3)$
w $(u - 20)(u + 4)$ **x** $(t + 25)(t - 2)$

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5a $(3x+1)(x+1)$ **b** $(2x+1)(x+2)$
c $(3x+1)(x+5)$ **d** $(3x+2)(x+2)$
e $(2x-1)(x-1)$ **f** $(5x-3)(x-2)$
g $(5x-6)(x-1)$ **h** $(3x-1)(2x-3)$
i $(2x-3)(x+1)$ **j** $(2x+5)(x-1)$
k $(3x+5)(x-1)$ **l** $(3x-1)(x+5)$
m $(2x+3)(x-5)$ **n** $(2x-5)(x+3)$
o $(6x-1)(x+3)$ **p** $(2x-3)(3x+1)$
q $(3x-2)(2x+3)$ **r** $(5x+3)(x+4)$
s $(5x-6)(x+2)$ **t** $(5x-4)(x-3)$
u $(5x+4)(x-3)$ **v** $(5x-2)(x+6)$
w $(3x-4)(3x+2)$ **x** $(3x-5)(x+6)$

6a $(a-5)(a+5)$ **b** $b(b-25)$ **c** $(c-5)(c-20)$
d $(2d+5)(d+10)$ **e** $(e+5)(e^2+5)$
f $(4-f)(4+f)$ **g** $g^2(16-g)$ **h** $(h+8)^2$
i $(i-18)(i+2)$ **j** $(j+4)(5j-4)$
k $(2k+1)(2k-9)$ **l** $(k-8)(2k^2-3)$
m $(2a+b)(a-2)$ **n** $3m^2n^4(2m+3n)$
o $(7p-11q)(7p+11q)$ **p** $(t-4)(t-10)$
q $(3t-10)(t+4)$ **r** $(5t+4)(t+10)$
s $(5t+8)(t+5)$ **t** $5t(t^2+2t+3)$
u $(u+18)(u-3)$ **v** $(3x-2y)(x^2-5)$
w $(p+q-r)(p+q+r)$ **x** $(2a-3)^2$

7a $3(a-2)(a+2)$ **b** $(x-y)(x+y)(x^2+y^2)$
c $x(x-1)(x+1)$ **d** $5(x+2)(x-3)$
e $y(5-y)(5+y)$ **f** $(2-a)(2+a)(4+a^2)$
g $2(2x-3)(x+5)$ **h** $a(a+1)(a^2+1)$
i $(c+1)(c-1)(c+9)$ **j** $x(x-1)(x-7)$
k $(x-2)(x+2)(x^2+1)$ **l** $(x-1)(x+1)(a-2)$

Exercise 1D

1a 1 **b** 2 **c** $\frac{1}{2}$ **d** $\frac{1}{a}$ **e** $\frac{x}{3y}$ **f** $\frac{3}{a}$

2a 1 **b** $\frac{1}{2}$ **c** $3x$ **d** $\frac{b}{2}$ **e** $\frac{3}{2x}$ **f** $\frac{1}{2a}$ **g** $\frac{4}{b}$ **h** 6

3a $\frac{3x}{2}$ **b** $\frac{3y}{4}$ **c** $\frac{2m}{9}$ **d** $\frac{7n}{10}$ **e** $\frac{3x-2y}{24}$ **f** $\frac{13a}{6}$
g $\frac{b}{15}$ **h** $-\frac{xy}{20}$

4a $\frac{2}{a}$ **b** $-\frac{1}{x}$ **c** $\frac{3}{2a}$ **d** $\frac{1}{6x}$ **e** $\frac{25}{12a}$ **f** $\frac{1}{2x}$

5a $\frac{5x+7}{6}$ **b** $\frac{18x+11}{20}$ **c** $\frac{x+1}{4}$
d $\frac{x}{6}$ **e** $\frac{2x+17}{20}$ **f** $\frac{2x-3}{6}$

6a 2 **b** $\frac{3}{2}$ **c** $\frac{x}{3}$ **d** $\frac{1}{x+y}$ **e** $\frac{3}{2b}$ **f** $\frac{x}{x-2}$ **g** $\frac{a+3}{a+4}$
h $\frac{x+1}{x-1}$ **i** $\frac{x+5}{x+4}$

7a $\frac{2x+1}{x(x+1)}$ **b** $\frac{1}{x(x+1)}$ **c** $\frac{2x}{(x+1)(x-1)}$
d $\frac{5x-13}{(x-2)(x-3)}$ **e** $\frac{x-5}{(x+1)(x-1)}$ **f** $\frac{10}{(x+3)(x-2)}$

8a $\frac{3x}{2(x-1)}$ **b** a **c** $\frac{c+2}{c+4}$ **d** x

9a -1 **b** $\frac{2}{a-b}$ **c** 1 **d** $3-x$

10a $\frac{2}{x^2-1}$ **b** $\frac{3x}{x^2-y^2}$ **c** $\frac{x+1}{(x-2)(x+3)(x+4)}$
d $\frac{x}{(x-1)(x-2)(x-3)}$

11a $\frac{1}{3}$ **b** $\frac{7}{13}$ **c** $\frac{3}{11}$ **d** $\frac{1}{5}$ **e** $\frac{1}{x+2}$ **f** $\frac{t^2-1}{t^2+1}$
g $\frac{ab}{a+b}$ **h** $\frac{x^2+y^2}{x^2-y^2}$ **i** $\frac{x^2}{2x+1}$ **j** $\frac{x-1}{x-3}$

Exercise 1E

1a $x=3$ **b** $p=0$ **c** $a=8$ **d** $w=-1$
e $x=9$ **f** $x=-5$ **g** $x=-16$ **h** $x=-2$

2a $n=4$ **b** $b=-1$ **c** $x=4$ **d** $x=-11$ **e** $a=-\frac{1}{2}$
f $y=2$ **g** $x=\frac{7}{9}$ **h** $x=-\frac{3}{5}$

3a $a=8$ **b** $y=16$ **c** $x=\frac{1}{3}$ **d** $a=\frac{2}{5}$ **e** $y=\frac{3}{2}$
f $x=-8$ **g** $a=7$ **h** $x=-\frac{1}{2}$ **i** $a=-5$ **j** $t=\frac{3}{5}$
k $x=-2$ **l** $x=5$

4a $a=3$ **b** $s=16$ **c** $v=\frac{2}{3}$ **d** $\ell=21$
e $C=35$ **f** $c=-\frac{2}{5}$

5a 6 **b** −4 **c** 17 **d** 65 cents

6a $y=\frac{2}{3}$ **b** $x=15$ **c** $a=-15$ **d** $x=\frac{9}{2}$ **e** $x=6$
f $x=\frac{1}{6}$ **g** $x=\frac{1}{2}$ **h** $x=20$ **i** $x=-\frac{23}{2}$ **j** $x=-\frac{7}{3}$

7a $b=\frac{a+d}{c}$ **b** $n=\frac{t-a+d}{d}$ **c** $r=\frac{p-qt}{t}$
d $v=\frac{3}{u-1}$

8a $x=\frac{19}{6}$ **b** $x=\frac{3}{14}$ **c** $x=-1$ **d** $x=\frac{17}{6}$

9a $a=-11$ **b** $x=2$ **c** $x=-\frac{7}{3}$ **d** $x=-\frac{5}{2}$

10a $a=-\frac{2b}{3}$ **b** $g=\frac{2fh}{5f-h}$
c $y=\frac{2x}{1-x}$ **d** $b=\frac{4a+5}{a-1}$

11a 20 **b** 16 **c** 30 km/h

Exercise 1F

1a $x=3$ or -3 **b** $y=5$ or -5 **c** $a=2$ or -2
d $c=6$ or -6 **e** $t=1$ or -1 **f** $x=\frac{3}{2}$ or $-\frac{3}{2}$
g $x=\frac{1}{2}$ or $-\frac{1}{2}$ **h** $a=2\frac{2}{3}$ or $-2\frac{2}{3}$ **i** $y=\frac{4}{5}$ or $-\frac{4}{5}$

2a $x=0$ or 5 **b** $y=0$ or -1 **c** $c=0$ or -2
d $k=0$ or 7 **e** $t=0$ or 1 **f** $a=0$ or 3 **g** $b=0$ or $\frac{1}{2}$
h $u=0$ or $-\frac{1}{3}$ **i** $x=-\frac{3}{4}$ or 0 **j** $a=0$ or $\frac{5}{2}$
k $y=0$ or $\frac{2}{3}$ **l** $n=0$ or $-\frac{3}{5}$

3a $x=-3$ or -1 **b** $x=1$ or 2 **c** $x=-4$ or -2
d $a=2$ or 5 **e** $t=-2$ or 6 **f** $c=5$ **g** $n=1$ or 8
h $p=-5$ or 3 **i** $a=-2$ or 12 **j** $y=-5$ or 1
k $p=-2$ or 3 **l** $a=-11$ or 12 **m** $c=3$ or 6
n $t=-2$ or 10 **o** $u=-8$ or 7 **p** $k=-4$ or 6
q $h=-25$ or -2 **r** $a=-22$ or 2

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4a $x = -\frac{1}{2}$ or -1 **b** $a = \frac{1}{3}$ or 2 **c** $y = \frac{1}{4}$ or 1
d $x = -5$ or $-\frac{1}{2}$ **e** $x = -1\frac{1}{2}$ or 1 **f** $n = -1$ or $1\frac{2}{3}$
g $b = -\frac{2}{3}$ or 2 **h** $a = -5$ or $1\frac{1}{2}$ **i** $y = -2\frac{1}{2}$ or 3
j $y = -4$ or $\frac{2}{3}$ **k** $x = \frac{1}{5}$ or 5 **l** $t = \frac{3}{4}$ or 3 **m** $t = -\frac{2}{5}$ or 3
n $u = -\frac{4}{5}$ or $\frac{1}{2}$ **o** $x = \frac{1}{5}$ **p** $x = -\frac{2}{3}$ or $\frac{3}{2}$
q $b = -\frac{3}{2}$ or $-\frac{1}{6}$ **r** $k = -\frac{8}{3}$ or $\frac{1}{2}$
5a $x = \frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$, $x \doteqdot 1.618$ or -0.6180
b $x = \frac{-1+\sqrt{13}}{2}$ or $\frac{-1-\sqrt{3}}{2}$, $x \doteqdot 1.303$ or -2.303
c $a = 3$ or 4
d $u = -1 + \sqrt{3}$ or $-1 - \sqrt{3}$, $u \doteqdot 0.7321$ or -2.732
e $c = 3 + \sqrt{7}$ or $3 - \sqrt{7}$, $c \doteqdot 5.646$ or 0.3542
f $x = -\frac{1}{2}$
g $a = \frac{2+\sqrt{2}}{2}$ or $\frac{2-\sqrt{2}}{2} a \doteqdot 1.707$ or 0.2929
h $x = -3$ or $\frac{2}{5}$
i $b = \frac{-3+\sqrt{17}}{4}$ or $\frac{-3-\sqrt{17}}{4}$, $b \doteqdot 0.2808$ or -1.781
j $c = \frac{2+\sqrt{13}}{3}$ or $\frac{2-\sqrt{13}}{3}$, $c \doteqdot 1.869$ or -0.5352
k $t = \frac{1+\sqrt{5}}{4}$ or $\frac{1-\sqrt{5}}{4}$, $t \doteqdot 0.8090$ or -0.3090
l no solutions
6a $x = -1$ or 2 **b** $a = 2$ or 5 **c** $y = \frac{1}{2}$ or 4 **d** $b = -\frac{2}{5}$ or $\frac{2}{3}$
7a $x = 1 + \sqrt{2}$ or $1 - \sqrt{2}$ **b** $x = 2 + \sqrt{3}$ or $2 - \sqrt{3}$
c $a = 1 + \sqrt{5}$ or $1 - \sqrt{5}$ **d** $m = \frac{2+\sqrt{14}}{5}$ or $\frac{2-\sqrt{14}}{5}$
8a $p = \frac{1}{2}$ or 1 **b** $x = -3$ or 5 **c** $n = 5$
9a 7 **b** 6 and 9 **c** $x = 15$
10a $k = -1$ or 3 **b** $u = \frac{4}{3}$ or 4 **c** $y = 1 + \sqrt{6}$ or $1 - \sqrt{6}$
d $k = \frac{-5+\sqrt{73}}{4}$ or $\frac{-5-\sqrt{73}}{4}$ **e** $a = -\frac{7}{3}$ or 3
f $k = -4$ or 15 **g** $t = 2\sqrt{3}$ or $-\sqrt{3}$
h $m = \frac{1+\sqrt{2}}{3}$ or $\frac{1-\sqrt{2}}{3}$
11a 4 cm **b** 3 cm **c** 55 km/h and 60 km/h

Exercise 1G

1a $x = 3, y = 3$ **b** $x = 2, y = 4$ **c** $x = 2, y = 1$
d $a = -3, b = -2$ **e** $p = 3, q = -1$ **f** $u = 1, v = -2$
2a $x = 3, y = 2$ **b** $x = 1, y = -2$ **c** $x = 4, y = 1$
d $a = -1, b = 3$ **e** $c = 2, d = 2$ **f** $p = -2, q = -3$
3a $x = 2, y = 4$ **b** $x = -1, y = 3$ **c** $x = 2, y = 2$
d $x = 9, y = 1$ **e** $x = 3, y = 4$ **f** $x = 4, y = -1$
g $x = 5, y = 3\frac{3}{5}$ **h** $x = 13, y = 7$
4a $x = -1, y = 3$ **b** $x = 5, y = 2$ **c** $x = -4, y = 3$
d $x = 2, y = -6$ **e** $x = 1, y = 2$ **f** $x = 16, y = -24$
g $x = 1, y = 6$ **h** $x = 5, y = -2$ **i** $x = 5, y = 6$
j $x = 7, y = 5$
5a $x = 1$ & $y = 1$ or $x = -2$ & $y = 4$
b $x = 2$ & $y = 1$ or $x = 4$ & $y = 5$
c $x = 0$ & $y = 0$ or $x = 1$ & $y = 3$
d $x = -2$ & $y = -7$ or $x = 3$ & $y = -2$
e $x = -3$ & $y = -5$ or $x = 5$ & $y = 3$
f $x = 1$ & $y = 6$ or $x = 2$ & $y = 3$
6a 53 and 37
b The pen cost 60 c, the pencil cost 15 c.
c Each apple cost 40 c, each orange cost 60 c.
d 44 adults, 22 children
e The man is 36, the son is 12.
f 189 for, 168 against
7a $x = 5$ & $y = 10$ or $x = 10$ & $y = 5$
b $x = -8$ & $y = -11$ or $x = 11$ & $y = 8$
c $x = \frac{1}{2}$ & $y = 4$ or $x = 2$ & $y = 1$
d $x = 4$ & $y = 5$ or $x = 5$ & $y = 4$
e $x = 1$ & $y = 2$ or $x = \frac{3}{2}$ & $y = \frac{7}{4}$
f $x = 2$ & $y = 5$ or $x = \frac{10}{3}$ & $y = 3$
8a 9 \$20 notes, 14 \$10 notes **b** 5 km/h, 3 km/h

Exercise 1H

1a 1 **b** 9 **c** 25 **d** 81 **e** $\frac{9}{4}$ **f** $\frac{1}{4}$ **g** $\frac{25}{4}$ **h** $\frac{81}{4}$
2a $(x + 2)^2$ **b** $(y + 1)^2$ **c** $(p + 7)^2$ **d** $(m - 6)^2$
e $(t - 8)^2$ **f** $(x + 10)^2$ **g** $(u - 20)^2$ **h** $(a - 12)^2$
3a $x^2 + 6x + 9 = (x + 3)^2$ **b** $y^2 + 8y + 16 = (y + 4)^2$
c $a^2 - 20a + 100 = (a - 10)^2$
d $b^2 - 100b + 2500 = (b - 50)^2$
e $u^2 + u + \frac{1}{4} = = \left(u + \frac{1}{2}\right)^2$ **f** $t^2 - 7t + \frac{49}{4} = \left(t - \frac{7}{2}\right)^2$
g $m^2 + 50m + 625 = (m + 25)^2$
h $c^2 - 13c + \frac{169}{4} = \left(c - \frac{13}{2}\right)^2$
4a $x = -1$ or 3 **b** $x = 0$ or 6 **c** $a = -4$ or -2
d $x = -2 + \sqrt{3}$ or $-2 - \sqrt{3}$ **e** $x = 5 + \sqrt{5}$ or $5 - \sqrt{5}$
f $y = -5$ or 2 **g** $b = -2$ or 7 **h** no solution for y
i $a = \frac{-7+\sqrt{21}}{2}$ or $\frac{-7-\sqrt{21}}{2}$
5a $x = \frac{2+\sqrt{6}}{2}$ or $\frac{2-\sqrt{6}}{2}$
b $x = \frac{-4+\sqrt{10}}{2}$ or $\frac{-4-\sqrt{10}}{2}$
c no solution for x **d** $x = -\frac{3}{2}$ or $\frac{1}{2}$
e $x = \frac{1+\sqrt{5}}{4}$ or $\frac{1-\sqrt{5}}{4}$ **f** $x = \frac{5+\sqrt{11}}{2}$ or $\frac{5-\sqrt{11}}{2}$
6a Answers will vary **b** $a = 3, b = 4$ and $c = 25$
c Answers will vary **d** $A = -5, B = 6$ and $C = 8$

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Chapter 1 review exercise

1a $-6y$ **b** $-10y$ **c** $-16y^2$ **d** -4
2a $-3a^2$ **b** $-a^2$ **c** $2a^4$ **d** 2
3a $2t - 1$ **b** $4p + 3q$ **c** $x - 2y$ **d** $5a^2 - 3a - 18$
4a $-18k^9$ **b** $-2k^3$ **c** $36k^{12}$ **d** $27k^9$
5a $14x - 3$ **b** $-4a + 2b$ **c** $-2a$ **d** $-6x^3 - 10x^2$
e $2n^2 + 11n - 21$ **f** $r^2 + 6r + 9$ **g** $y^2 - 25$
h $6x^2 - 19x + 15$ **i** $t^2 - 16t + 64$ **j** $4c^2 - 49$
k $16p^2 + 8p + 1$ **l** $9u^2 - 12u + 4$
6a $18(a + 2)$ **b** $4(5b - 9)$ **c** $9c(c + 4)$
d $(d - 6)(d + 6)$ **e** $(e + 4)(e + 9)$ **f** $(f - 6)^2$
g $(6 - 5g)(6 + 5g)$ **h** $(h - 12)(h + 3)$
i $(i + 9)(i - 4)$ **j** $(2j + 3)(j + 4)$ **k** $(3k + 2)(k - 3)$
l $(5\ell - 4)(\ell - 2)$ **m** $(2m - 3)(2m + 5)$
n $(n + 1)(m + p)$ **o** $(p + 9)(p^2 + 4)$ **p** $(q - r)(t - 5)$
q $(u^2 + v)(w - x)$ **r** $(x - y)(x + y + 2)$
7a $\frac{3x}{4}$ **b** $\frac{x}{4}$ **c** $\frac{x^2}{8}$ **d** 2 **e** $\frac{13a}{6b}$ **f** $\frac{5a}{6b}$ **g** $\frac{a^2}{b^2}$ **h** $\frac{9}{4}$
i $\frac{x^2 + y^2}{xy}$ **j** $\frac{x^2 - y^2}{xy}$ **k** 1 **l** $\frac{x^2}{y^2}$
8a $\frac{8x - 13}{15}$ **b** $\frac{8x - 13}{(x + 4)(x - 5)}$ **c** $\frac{3x + 13}{10}$
d $\frac{-3x - 13}{(x + 1)(x - 4)}$ **e** $\frac{x - 3}{4}$ **f** $\frac{-2x + 6}{x(x + 3)}$
9a $\frac{3}{5}$ **b** $\frac{2}{x + y}$ **c** $\frac{x + 3}{x - 4}$ **d** $\frac{x + 1}{x^2 + 1}$ **e** $\frac{1}{a + b}$ **f** $\frac{x - 7}{3x - 2}$
10a $x = 4$ **b** $x = \frac{2}{3}$ **c** $x = 46$ **d** $x = 36$ **e** $a = 3$
f $a = 10$ **g** $a = -17$ **h** $a = -42$
11a $a = -7$ or 7 **b** $b = -7$ or 0 **c** $c = -6$ or -1
d $d = -7$ or 1 **e** $e = 2$ or 3 **f** $f = -\frac{3}{2}$ or 2
g $g = \frac{1}{2}$ or 6 **h** $h = -2$ or $\frac{4}{3}$
12a $x = 2 + \sqrt{3}$ or $2 - \sqrt{3}$
b $y = \frac{-3 + \sqrt{21}}{2}$ or $\frac{-3 - \sqrt{21}}{2}$
c $y = -3 + \sqrt{5}$ or $-3 - \sqrt{5}$ **d** $y = \frac{1 + \sqrt{7}}{3}$ or $\frac{1 - \sqrt{7}}{3}$
e $y = \frac{-5 + \sqrt{65}}{4}$ or $\frac{-5 - \sqrt{65}}{4}$
f $y = \frac{3 + \sqrt{13}}{4}$ or $\frac{3 - \sqrt{13}}{4}$
13a $x = -2 + \sqrt{10}$ or $-2 - \sqrt{10}$
b $x = 3 + \sqrt{6}$ or $3 - \sqrt{6}$
c $x = 1 + \sqrt{13}$ or $1 - \sqrt{13}$
d $x = -5 + 3\sqrt{2}$ or $-5 - 3\sqrt{2}$

Chapter 2

Exercise 2A

1a $\frac{3}{10}$ **b** $\frac{4}{5}$ **c** $\frac{3}{4}$ **d** $\frac{1}{20}$
2a 0.6 **b** 0.27 **c** 0.09 **d** 0.165
3a 25% **b** 40% **c** 24% **d** 65%
4a 32% **b** 9% **c** 22.5% **d** 150%
5a 5×7 **b** 2×3^2 **c** $2 \times 3^2 \times 5$ **d** $2^2 \times 5 \times 11$
6a $\frac{1}{3}$ **b** $\frac{4}{5}$ **c** $\frac{2}{3}$ **d** $\frac{3}{4}$ **e** $\frac{2}{5}$ **f** $\frac{7}{15}$ **g** $\frac{4}{7}$ **h** $\frac{5}{6}$ **i** $\frac{3}{5}$ **j** $\frac{3}{4}$
7a 0.5 **b** 0.2 **c** 0.6 **d** 0.75 **e** 0.04 **f** 0.35
g 0.125 **h** 0.625
8a $\frac{2}{5}$ **b** $\frac{1}{4}$ **c** $\frac{3}{20}$ **d** $\frac{4}{25}$ **e** $\frac{39}{50}$ **f** $\frac{1}{200}$ **g** $\frac{3}{8}$ **h** $\frac{33}{125}$
9a $0.\dot{3}$ **b** $0.\dot{6}$ **c** $0.\dot{1}$ **d** $0.\dot{5}$ **e** $0.\dot{2}\dot{7}$ **f** $0.\dot{0}\dot{9}$
g $0.1\dot{6}$ **h** $0.8\dot{3}$
10a $\frac{3}{4}$ **b** $\frac{7}{10}$ **c** $\frac{5}{6}$ **d** $\frac{4}{15}$ **e** $\frac{5}{18}$ **f** $\frac{1}{24}$ **g** $\frac{5}{6}$ **h** $\frac{1}{75}$
11a 5 **b** 8 **c** $\frac{1}{10}$ **d** $\frac{1}{7}$ **e** $\frac{1}{4}$ **f** 6 **g** $\frac{1}{4}$ **h** $\frac{2}{3}$ **i** 4 **j** $\frac{1}{4}$
12a 60 c **b** 15 kg **c** \$7800 **d** 72 min or $1\frac{1}{5}$ h
13a 0.132 **b** 0.025 **c** 0.3125 **d** 0.3375 **e** $0.58\dot{3}$
f $1.\dot{8}\dot{1}$ **g** $0.1\dot{3}$ **h** $0.2\dot{3}\dot{6}$
14a \$800 **b** \$160 **c** \$120
15a $\frac{14}{15}$ **b** $\frac{5}{11}$ **c** $\frac{1}{2000}$
16a $\frac{1}{11} = 0.\dot{0}\dot{9}$, $\frac{2}{11} = 0.\dot{1}\dot{8}$, …, $\frac{5}{11} = 0.\dot{4}\dot{5}$, $\frac{6}{11} = 0.\dot{5}\dot{4}$, …, $\frac{10}{11} = 0.\dot{9}\dot{0}$. The first digit runs from 0 to 9, the second digit runs from 9 to 0.
b $\frac{1}{7} = 0.\dot{1}4285\dot{7}$, $\frac{2}{7} = 0.\dot{2}8571\dot{4}$, etc. The digits of each cycle are in the same order but start at a different place in the cycle.
17c $3.000\,000\,3 \neq 3$, showing that some fractions are not stored exactly.

Exercise 2B

1a rational, $\frac{-3}{1}$ **b** rational, $\frac{3}{2}$ **c** irrational **d** rational, $\frac{2}{1}$
e rational, $\frac{3}{1}$ **f** irrational **g** rational, $\frac{2}{3}$ **h** rational, $\frac{9}{20}$
i rational, $\frac{3}{25}$ **j** rational, $\frac{333}{1000}$
k rational, $\frac{1}{3}$ **l** rational, $\frac{22}{7}$ **m** irrational
n rational, $3\frac{7}{50}$ **o** rational, $\frac{0}{1}$
2a 0.3 **b** 5.7 **c** 12.8 **d** 0.1 **e** 3.0 **f** 10.0
3a 0.43 **b** 5.4 **c** 5.0 **d** 0.043 **e** 430 **f** 4300
4a 3.162 **b** 6.856 **c** 0.563 **d** 0.771
e 3.142 **f** 9.870
5a 7.62 **b** 5.10 **c** 3840 **d** 538 000
e 0.740 **f** 0.00806
6a 1 **b** 2 **c** 3 **d** 2 **e** 4 **f** either 1, 2 or 3
7a i closed **ii** open **iii** closed **iv** neither open nor closed
v open **vi** open **vii** closed **viii** neither open nor closed
b i bounded **ii** unbounded **iii** unbounded
iv bounded **v** unbounded **vi** bounded
vii unbounded **viii** bounded

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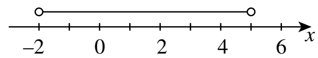
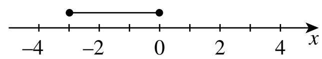
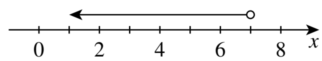
8a $-2 < x < 5$



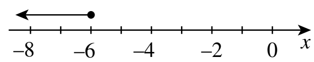
b $-3 \le x \le 0$



c $x < 7$



d $x \le -6$



9a 45.186 **b** 2.233 **c** 0.054 **d** 0.931
e 0.842 **f** 0.111

10a 10, rational **b** $\sqrt{41}$, irrational **c** 8, rational
d $\sqrt{5}$, irrational **e** $\frac{13}{15}$, rational **f** 45, rational

11a 0.3981 **b** 0.05263 **c** 1.425 **d** 5.138
e 0.1522 **f** 25 650 **g** 5.158 **h** 0.7891
i 1.388×10^{14} **j** 1.134 **k** 0.005892 **l** 1.173

12a The passage seems to take $\pi \doteqdot 3$.
b 3 significant figures. **c** Search the internet.
d 7, with a gap of about 0.3 inches.

13a 9.46×10^{15} m **b** 2.4×10^{22} m
c 4.35×10^{17} seconds **d** 1.3×10^{26} m

14a 1.836×10^{3} **b** 6×10^{26}

15 Answers will vary

Exercise 2C

1a 4 **b** 6 **c** 9 **d** 11 **e** 12 **f** 20 **g** 50 **h** 100

2a $2\sqrt{3}$ **b** $3\sqrt{2}$ **c** $2\sqrt{5}$ **d** $3\sqrt{3}$ **e** $2\sqrt{7}$ **f** $2\sqrt{10}$
g $4\sqrt{2}$ **h** $3\sqrt{11}$ **i** $3\sqrt{6}$ **j** $10\sqrt{2}$ **k** $2\sqrt{15}$ **l** $5\sqrt{3}$
m $4\sqrt{5}$ **n** $7\sqrt{2}$ **o** $20\sqrt{2}$ **p** $10\sqrt{10}$

3a $2\sqrt{3}$ **b** $2\sqrt{7}$ **c** $\sqrt{5}$ **d** $-2\sqrt{2}$ **e** $2\sqrt{3} + 3\sqrt{2}$
f $\sqrt{5} - 2\sqrt{7}$ **g** $3\sqrt{6} - 2\sqrt{3}$ **h** $-3\sqrt{2} - 6\sqrt{5}$
i $-4\sqrt{10} + 2\sqrt{5}$

4a $6\sqrt{2}$ **b** $10\sqrt{3}$ **c** $4\sqrt{6}$ **d** $8\sqrt{11}$ **e** $9\sqrt{5}$ **f** $12\sqrt{13}$
g $20\sqrt{3}$ **h** $8\sqrt{6}$

5a $\sqrt{20}$ **b** $\sqrt{50}$ **c** $\sqrt{128}$ **d** $\sqrt{108}$ **e** $\sqrt{125}$ **f** $\sqrt{112}$
g $\sqrt{68}$ **h** $\sqrt{490}$

6a $3\sqrt{2}$ **b** $\sqrt{3}$ **c** $2\sqrt{2}$ **d** $5\sqrt{6}$ **e** $\sqrt{5}$ **f** $2\sqrt{10}$ **g** $4\sqrt{3}$
h $2\sqrt{5}$ **i** $11\sqrt{2}$

7a $4\sqrt{6} + 10\sqrt{3}$ **b** $2\sqrt{2} + 6\sqrt{3}$ **c** $4\sqrt{7} - 10\sqrt{35}$

8a 7 **b** 20 **c** 96

Exercise 2D

1a 3 **b** $\sqrt{6}$ **c** 7 **d** $\sqrt{30}$ **e** $6\sqrt{2}$ **f** $10\sqrt{5}$ **g** $6\sqrt{15}$
h $30\sqrt{14}$ **i** 12 **j** 63 **k** 30 **l** 240

2a $\sqrt{5}$ **b** $\sqrt{7}$ **c** $\sqrt{5}$ **d** 2 **e** $3\sqrt{2}$ **f** $\sqrt{3}$ **g** $2\sqrt{7}$ **h** $5\sqrt{5}$

3a $5 + \sqrt{5}$ **b** $\sqrt{6} - \sqrt{2}$ **c** $2\sqrt{3} - 3$ **d** $2\sqrt{10} - 4$
e $7\sqrt{7} - 14$ **f** $18 - 2\sqrt{30}$

4a $2\sqrt{3}$ **b** $5\sqrt{2}$ **c** $3\sqrt{5}$ **d** $4\sqrt{11}$ **e** 24 **f** $12\sqrt{10}$

5a $2\sqrt{5} - 2$ **b** $3\sqrt{6} + 3\sqrt{2}$ **c** $5\sqrt{3} + 4\sqrt{5}$
d $4\sqrt{3} - 2\sqrt{6}$ **e** $27\sqrt{3} - 9\sqrt{7}$ **f** $21\sqrt{2} - 42$

6a $\sqrt{6} - \sqrt{3} + \sqrt{2} - 1$ **b** $\sqrt{35} + 3\sqrt{5} - 2\sqrt{7} - 6$
c $\sqrt{15} + \sqrt{10} + \sqrt{6} + 2$ **d** $8 - 3\sqrt{6}$
e $4 + \sqrt{7}$ **f** $7\sqrt{3} - 4\sqrt{6}$

7a 4 **b** 2 **c** 1 **d** 7 **e** 15 **f** 29

8a $4 + 2\sqrt{3}$ **b** $6 - 2\sqrt{5}$ **c** $5 + 2\sqrt{6}$ **d** $12 - 2\sqrt{35}$
e $13 - 4\sqrt{3}$ **f** $29 + 12\sqrt{5}$ **g** $33 + 4\sqrt{35}$
h $30 - 12\sqrt{6}$ **i** $55 + 30\sqrt{2}$

9a 2 **b** $\frac{3}{5}$ **c** $2\sqrt{3}$ **d** $\frac{5\sqrt{3}}{2}$ **e** 5 **f** 4

10a 3 **b** 5 **c** 4 **d** 6

11a $\sqrt{3}$ **b** $\frac{6\sqrt{7}}{13}$

12a $a^2 + 2ab + b^2$
b $6 + \sqrt{11} - 2\sqrt{36 - 11} + 6 - \sqrt{11} = 2$ **c** $\sqrt{2}$

Exercise 2E

1a $\frac{\sqrt{3}}{3}$ **b** $\frac{\sqrt{7}}{7}$ **c** $\frac{3\sqrt{5}}{5}$ **d** $\frac{5\sqrt{2}}{2}$ **e** $\frac{\sqrt{6}}{3}$ **f** $\frac{\sqrt{35}}{7}$ **g** $\frac{2\sqrt{55}}{5}$ **h** $\frac{3\sqrt{14}}{2}$

2a $\frac{\sqrt{3} + 1}{2}$ **b** $\frac{\sqrt{7} - 2}{3}$ **c** $\frac{3 - \sqrt{5}}{4}$ **d** $\frac{4 + \sqrt{7}}{9}$
e $\frac{\sqrt{5} + \sqrt{2}}{3}$ **f** $\frac{\sqrt{10} - \sqrt{6}}{4}$ **g** $\frac{2\sqrt{3} - 1}{11}$ **h** $\frac{5 + 3\sqrt{2}}{7}$

3a $\sqrt{2}$ **b** $\sqrt{5}$ **c** $2\sqrt{3}$ **d** $3\sqrt{7}$ **e** $\frac{\sqrt{6}}{2}$ **f** $\frac{\sqrt{15}}{3}$
g $\frac{4\sqrt{6}}{3}$ **h** $\frac{7\sqrt{10}}{5}$

4a $\frac{\sqrt{5}}{10}$ **b** $\frac{\sqrt{7}}{21}$ **c** $\frac{3\sqrt{2}}{10}$ **d** $\frac{2\sqrt{3}}{21}$ **e** $\frac{5\sqrt{2}}{3}$ **f** $\frac{3\sqrt{3}}{4}$
g $\frac{\sqrt{30}}{20}$ **h** $\frac{2\sqrt{77}}{35}$

5a $\frac{3\sqrt{5} - 3}{4}$ **b** $\frac{8\sqrt{2} + 4\sqrt{3}}{5}$ **c** $\frac{5\sqrt{7} + 7}{18}$ **d** $\frac{3\sqrt{15} - 9}{2}$
e $\frac{28 + 10\sqrt{7}}{3}$ **f** $\sqrt{2} + 1$ **g** $2 - \sqrt{3}$ **h** $\frac{7 + 2\sqrt{10}}{3}$
i $8 - 3\sqrt{7}$ **j** $\frac{23 + 6\sqrt{10}}{13}$ **k** $4 - \sqrt{15}$ **l** $\frac{93 + 28\sqrt{11}}{5}$

6a $\sqrt{3} + 1$ **b** $4 - \sqrt{10}$

7a 3 **b** 1 **c** 7 **d** 2

8 Answers will vary

9 $a = -1, b = 2$

10a $x^2 + 2 + \frac{1}{x^2}$ **b i** Answers will vary
ii $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = 28 - 2 = 26$

Chapter 2 review exercise

1a rational, $\frac{7}{1}$ **b** rational, $\frac{-9}{4}$ **c** rational, $\frac{3}{1}$ **d** irrational
e irrational **f** rational, $\frac{2}{1}$ **g** rational, $\frac{-4}{25}$ **h** irrational

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2a i 4.12 **ii** 4.1 **b i** 4.67 **ii** 4.7 **c i** 2.83 **ii** 2.8
d i 0.77 **ii** 0.77 **e i** 0.02 **ii** 0.019 **f i** 542.41 **ii** 540
3a 1.67 **b** 70.1 **c** 1.43 **d** 0.200 **e** 0.488
f 0.496 **g** 1.27 **h** 1590 **i** 0.978
4a $2\sqrt{6}$ **b** $3\sqrt{5}$ **c** $5\sqrt{2}$ **d** $10\sqrt{5}$ **e** $9\sqrt{2}$ **f** $4\sqrt{10}$
5a $2\sqrt{5}$ **b** 5 **c** 28 **d** $\sqrt{7}-\sqrt{5}$ **e** $\sqrt{7}$ **f** $3\sqrt{5}$ **g** 4
h $2\sqrt{5}$ **i** $24\sqrt{10}$
6a $\sqrt{3}$ **b** $7\sqrt{2}$ **c** $4\sqrt{2}$ **d** $8\sqrt{6}-6\sqrt{5}$
7a $3\sqrt{7}-7$ **b** $2\sqrt{30}+3\sqrt{10}$ **c** $3\sqrt{5}-5\sqrt{15}$
d $3\sqrt{2}+6$
8a $\sqrt{5}+1$ **b** $13+7\sqrt{3}$ **c** $2\sqrt{35}+4\sqrt{7}-6\sqrt{5}-12$
d 1 **e** 13 **f** $11-4\sqrt{7}$ **g** $7+2\sqrt{10}$ **h** $34-24\sqrt{2}$
9a $\frac{\sqrt{5}}{5}$ **b** $\frac{3\sqrt{2}}{2}$ **c** $\frac{\sqrt{33}}{11}$ **d** $\frac{\sqrt{3}}{15}$ **e** $\frac{5\sqrt{7}}{14}$ **f** $\frac{\sqrt{5}}{15}$
10a $\frac{\sqrt{5}-\sqrt{2}}{3}$ **b** $\frac{3+\sqrt{7}}{2}$ **c** $\frac{2\sqrt{6}+\sqrt{3}}{21}$ **d** $\frac{3-\sqrt{3}}{2}$
e $\frac{\sqrt{11}-\sqrt{5}}{2}$ **f** $\frac{6\sqrt{35}+21}{13}$
11a $\frac{9-2\sqrt{14}}{5}$ **b** $26+15\sqrt{3}$
12 $x=50$
13 $5\sqrt{5}+2$
14 $p=5, q=2$
15 $\frac{7}{3}$

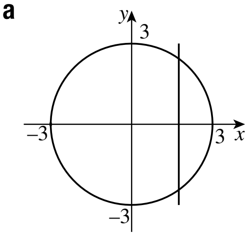
Chapter 3

Exercise 3A

1a 5 **b** 3 **c** -1 **d** 11
2a -3 **b** 5 **c** 0 **d** 5
3a 5 **b** -10 **c** -3 **d** 2
4a $5, -1, -7$ **b** $0, 4, 0$ **c** $16, 8, 0$ **d** $4, 1, \frac{1}{4}$
5a $-4, 4, 12$ **b** $-\frac{1}{3}, 1, \frac{1}{5}$ **c** $-18, 2, -10$ **d** $1, \sqrt{5}, 3$
6a y: $-1, 1, 3$ **b** y: $3, 0, -1, 0, 3$
c $f(x)$: $-3, 0, 1, 0, -3$ **d** $f(x)$: $-15, 0, 3, 0, -3, 0, 15$
7a 8 **b** 2 **c** -6 **d** 4 **e** 11 **f** 6 **g** -35 **h** 4
8a 4 **b** $5\frac{1}{2}$ **c** $5\frac{1}{3}$ **d** $4\frac{1}{3}$
9a $-2-2\sqrt{2}$ **b** $3-2\sqrt{7}$
10a $y=-\dfrac{3}{4}x-\dfrac{5}{4}$ **b** $x=-\dfrac{4}{3}y-\dfrac{5}{3}$ **c** $y=-\dfrac{4}{x}$
d $s=\sqrt[3]{V}, s=\sqrt{\dfrac{A}{6}}$ **e i** $\ell=\dfrac{100}{b}$ **ii** $b=\dfrac{100}{\ell}$
11 $C=50+20x$
12a The square root of a negative is undefined.
b The square root of a negative is undefined.
c Division by zero is undefined.
d Division by zero is undefined.
13a 0 **b** $2-4\sqrt{3}$
14a $2a-4, -2a-4, 2a-2$ **b** $2-x, 2+a, 1-a$
c a^2, a^2, a^2+2a+1 **d** $\frac{1}{a-1}, \frac{1}{-a-1}=-\frac{1}{a+1}, \frac{1}{a}$
15a $5t, 5t-8$ **b** $\sqrt{t}-2, \sqrt{t-2}$
c t^2+2t-2, t^2-2t **d** $-t^2, -t^2+4t-2$
16a $7+h$ **b** $p+q+5$ **c** $2x+h+5$
17a, b, c Answers will vary

Exercise 3B

1 Notice that the y-axis is such a line in every case. Shown below are some other vertical lines that intersect at least twice.

a


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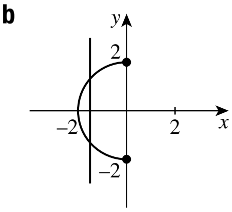

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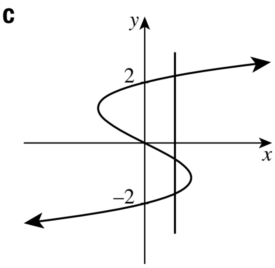

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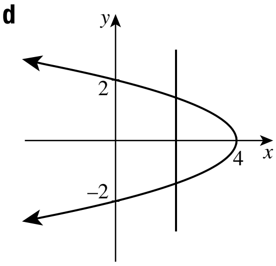
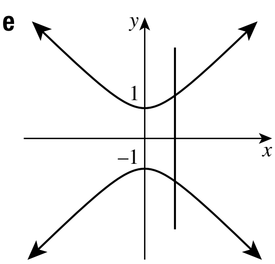
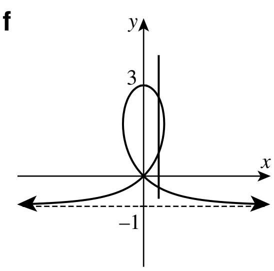
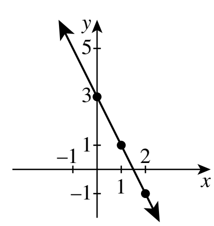
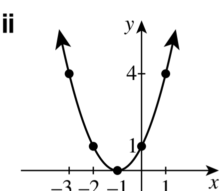


e


f


2 a, c, f, h
3a domain: all real x, range: $y \geq -1$
b domain: $-2 \leq x \leq 2$, range: $-2 \leq y \leq \sqrt{3}$
c domain: all real x, range: all real y
d domain: $-1 \leq x$, range: all real y
e domain: $-2 \leq x \leq 2$, range: $-3 \leq y \leq 3$
f domain: all real x, range: all real y
g domain: $0 \leq x \leq 2$, range: $-2 \leq y \leq 2$
h domain: all real x, range: $y < 1$
4a i $3, 1, -1$
ii


iii domain: all real x,
range: all real y

b i $4, 1, 0, 1, 4$
ii


iii domain: all real x,
range: $y \geq 0$.

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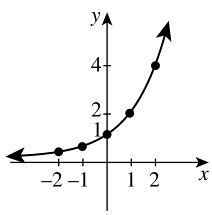
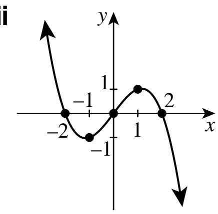
5a $x \neq 0$ **b** $x \neq 3$ **c** $x \neq -1$ **d** $x \neq -2$
6a $x \geq 0$ **b** $x \geq 2$ **c** $x \geq -3$ **d** $x \geq -5$
7a $(0, 3)$ and $(0, -3)$ **b** $(0, 1)$ and $(0, -1)$
c $(2, 1)$ and $(2, 5)$ **d** $(2, 2)$ and $(2, -2)$
8a all real x **b** all real x **c** $x \neq 4$ **d** $x \neq \frac{1}{2}$
e $x \geq -4$ **f** $x \geq -\frac{1}{2}$ **g** $x \leq 5$ **h** $x \leq 2$
i $x > 0$ **j** $x > -1$ **k** $x < 1$ **l** $x > 1\frac{1}{2}$
9a i $\frac{1}{4}, \frac{1}{2}, 1, 2, 4$ **b i** $5, 0, -1, 0, 1, 0, -5$

ii



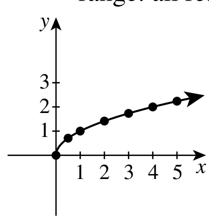
ii



iii domain: all real x, range: $y > 0$

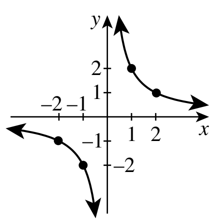
iii domain: all real x, range: all real y

10a $x \geq 0$
b 0, 0.7, 1, 1.4, 1.7, 2, 2.2
c It is the top half of a concave right parabola.



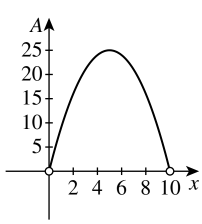
11a $x \neq 0$
b $-\frac{1}{2}, -1, -2, -4,$
$*, 4, 2, 1, \frac{1}{2}$
Division by zero is undefined.

c



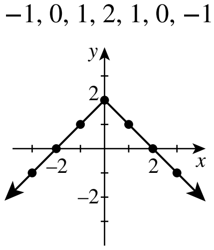
12a x **b** $(10 - x)$
c $A = x(10 - x)$
d Both $10 - x > 0$ and $x > 0$. Thus $10 > x$ and $x > 0$. Hence the domain is $0 < x < 10$.

e



13a $y = 2x + 3$ **b** $y = \dfrac{4}{x}$ **c** $y = \frac{3}{x - 2}$
d $y = -2 + \sqrt{9 - x^2}$
14a $x > -2$ **b** $x \neq 2$ and $x \neq -2$
c $x \neq -1$ and $x \neq 0$ **d** $x \neq 2$ and $x \neq 3$
e $x \leq -2$ or $x \geq 2$ **f** $-1 < x < 1$
15a $-1, 0, 1, 2, 1, 0, -1$

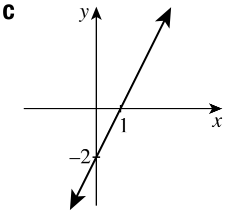
b



Exercise 3C

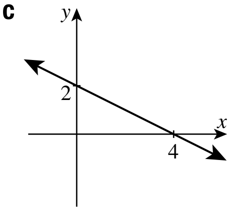
1a $y = -2$ **b** $x = 1$

c

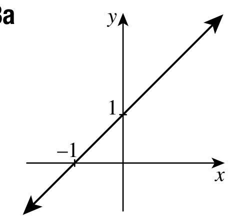


2a $y = 2$ **b** $x = 4$

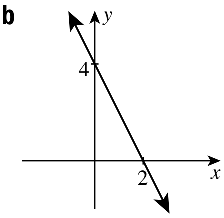
c



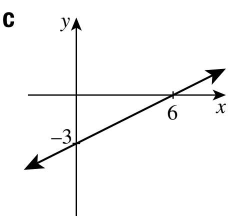
3a



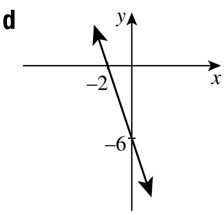
b



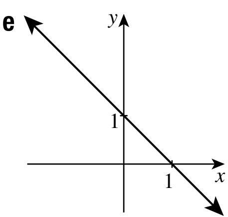
c



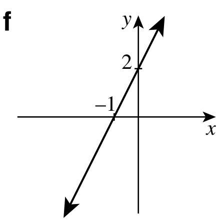
d



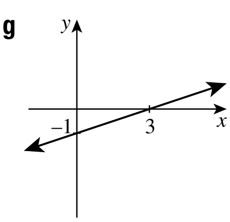
e



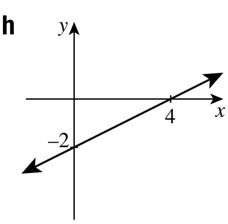
f



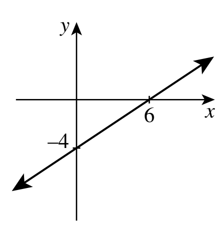
g



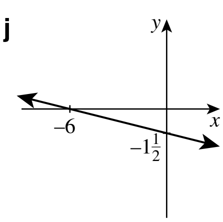
h



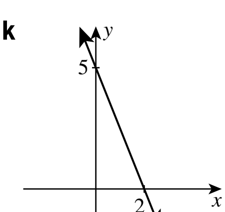
i



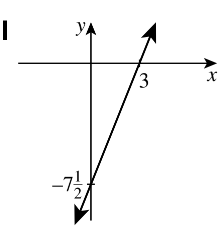
j



k



l



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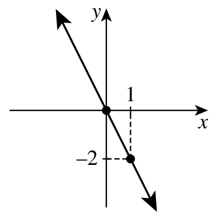
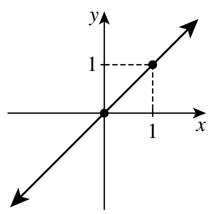
4a When $x = 0$, $y = 0$.

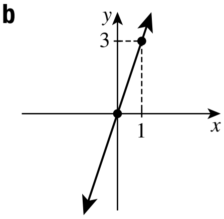
b $(1, -2)$



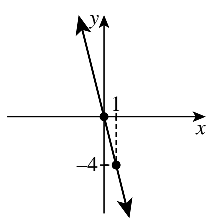
5a



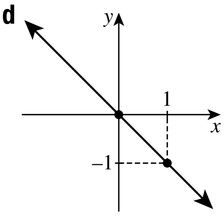
b



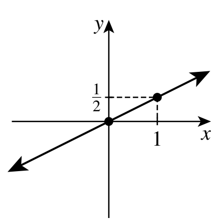
c



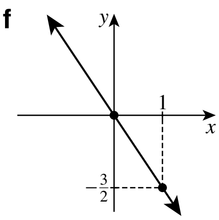
d



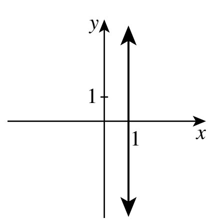
e



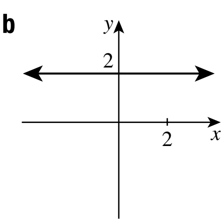
f



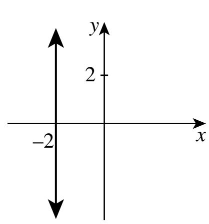
6a



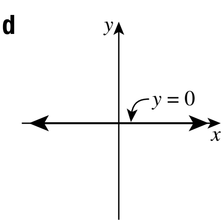
b



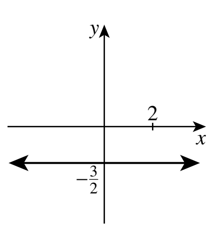
c



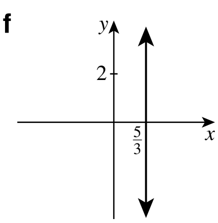
d



e



f



7a a, c, f

b $(1, 0)$ and $(1, 1)$ are on $x = 1$.

$(-2, 0)$ and $(-2, 1)$ are on $x = -2$.

$\left(\frac{5}{3}, 0\right)$ and $\left(\frac{5}{3}, 1\right)$ are on $3x = 5$.

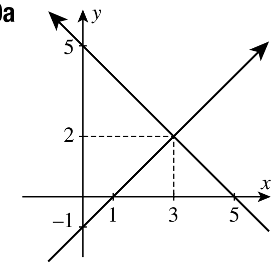
8e $y = 1 - x$ **f** $y = 2x + 2$ **g** $y = \frac{1}{3}x - 1$

h $y = \frac{1}{2}x - 2$ **i** $y = \frac{2}{3}x - 4$ **j** $y = -\frac{1}{4}x - \frac{3}{2}$

k $y = -\frac{5}{2}x + 5$ **l** $y = \frac{5}{2}x - \frac{15}{2}$

9a yes **b** no **c** yes **d** yes **e** yes **f** no

10a



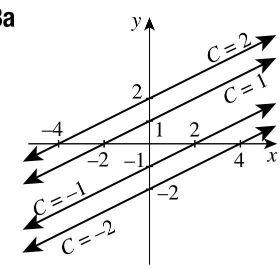
b $(3, 2)$ **c** See 10b

11a $(-1, 3)$ **b** $(1, -2)$ **c** $(-2, -1)$

12a $C(n) = 60 + 50n$

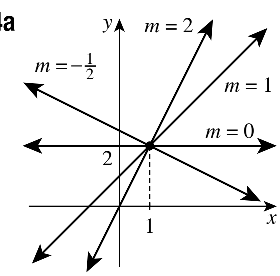
b i $D(n) = 10 + 2n$ **ii** $T = C + D$ so $T(n) = 70 + 52n$

13a



b They are parallel.
The value of c gives the y-intercept.

14a



b $(1, 2)$

c Answers will vary

Exercise 3D

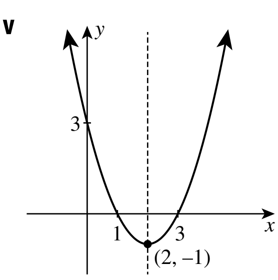
1a i $y = 3$

ii $x = 1, 3$

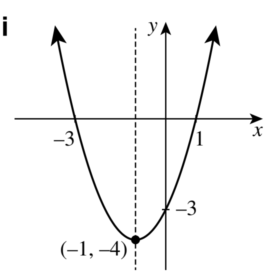
iii $x = 2$

iv $(2, -1)$

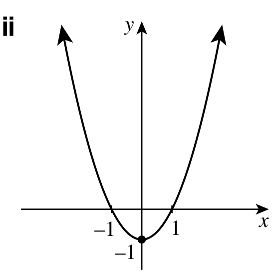
v



b i



ii



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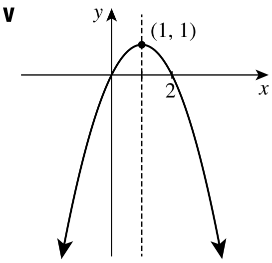
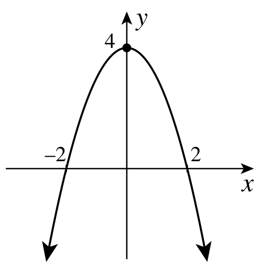
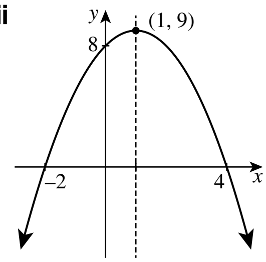
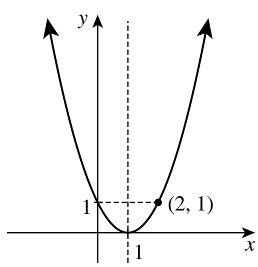
2a i $y = 0$
ii $x = 0, 2$
iii $x = 1$
iv $(1, 1)$

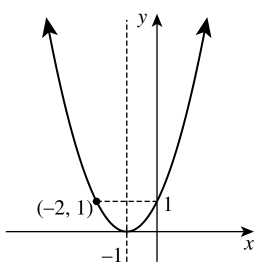
v


b i


ii


3a i $(0, 1)$
ii $(1, 0)$
iv $(2, 1)$

iii


b i


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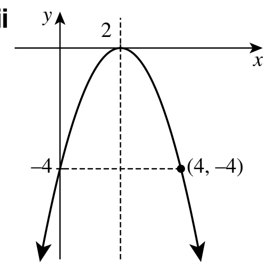
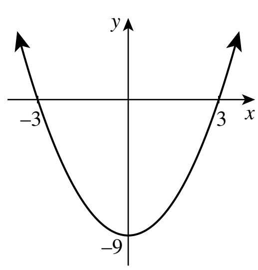
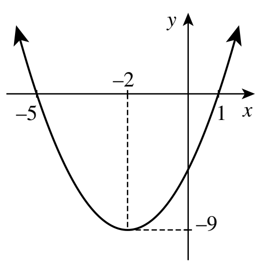
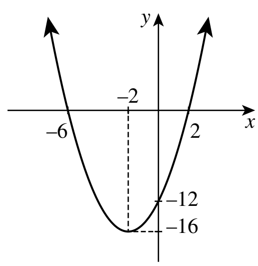
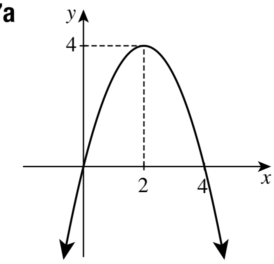


4a $y = (x - 4)(x - 6)$ **b** $y = x(x - 3)$
c $y = (x + 3)(x - 5)$ **d** $y = (x + 6)(x + 1)$
5a $y = x(x - 3)$ **b** $y = (x + 2)(x - 1)$
c $y = -(x + 1)(x - 3)$ **d** $y = -(x + 2)(x + 5)$

6a


b


c


7a


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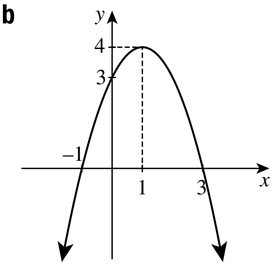
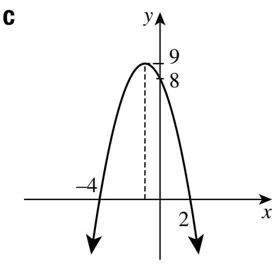
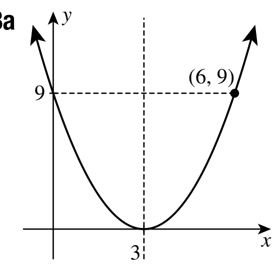
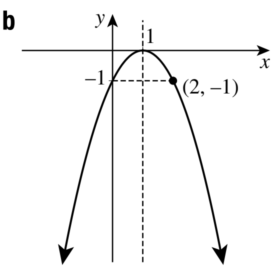


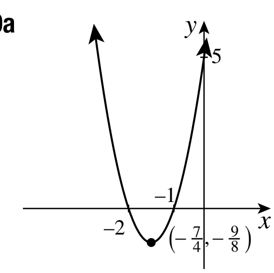
c


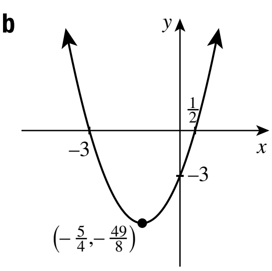
8a


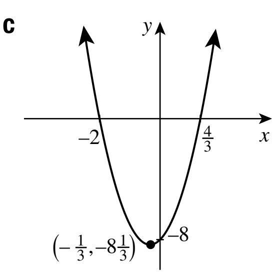
$y = (x - 3)^2$

b


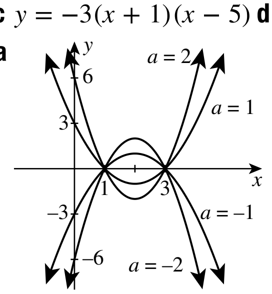
$y = -(x - 1)^2$

9a


b


c


10a $y = (x + 1)(x - 2)$ **b** $y = -(x + 3)(x - 2)$
c $y = 3(x + 2)(x - 4)$ **d** $y = -\frac{1}{2}(x - 2)(x + 2)$
11a $y = 2(x - 1)(x - 3)$ **b** $y = -2(x + 2)(x - 1)$
c $y = -3(x + 1)(x - 5)$ **d** $y = \frac{1}{4}(x + 2)(x + 4)$

12a


b $(1, 0)$ and $(3, 0)$

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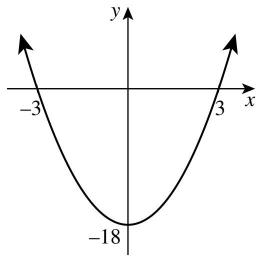
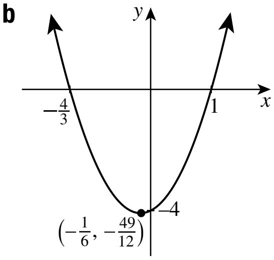
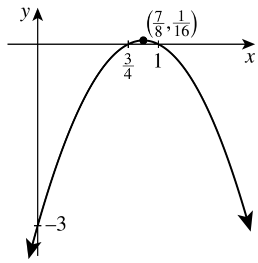
13a



b



c

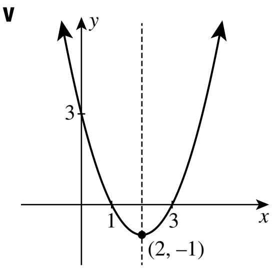


14a $y = 3(x - 2)(x - 8)$ **b** $y = -(x - 2)(x - 8)$
c $y = \frac{4}{3}(x - 2)(x - 8)$ **d** $y = -\frac{20}{7}(x - 2)(x - 8)$

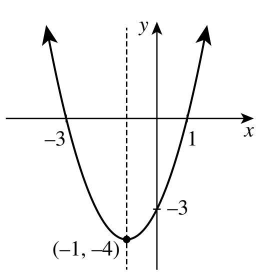
15a $f(x) = (x - 4)(x + 2)$, so the axis is $x = 1$.
b i Both $f(1 + h) = h^2 - 9$ and $f(1 - h) = h^2 - 9$.
ii The parabola is symmetric in the line $x = 1$.

Exercise 3E

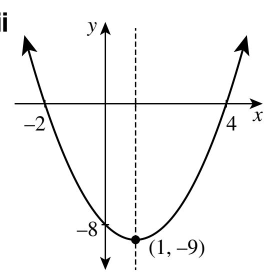
1a i $a = 1$, concave up
ii $y = 3$
iii $x = 1, 3$
iv $x = 2$, $V(2, -1)$
v



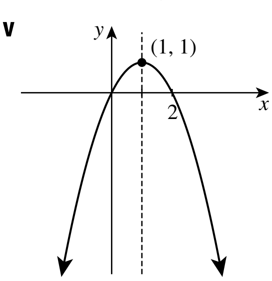
b i



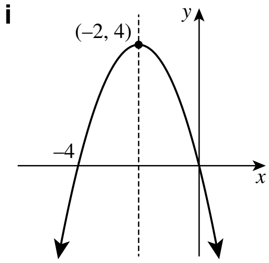
ii



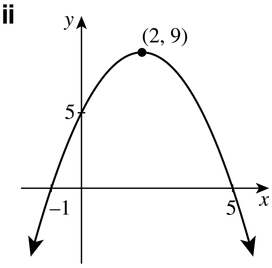
2a i $a = -1$
ii $y = 0$
iii $x = 0, 2$
iv $x = 1$, $V(1, 1)$
v



b i

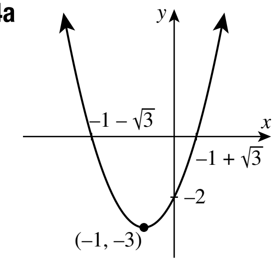


ii

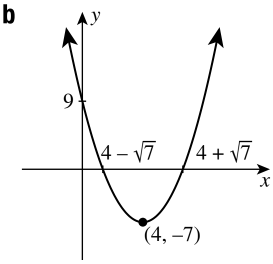


3a $f(x) = (x - 2)^2 + 1$ **b** $f(x) = (x + 3)^2 + 2$
c $f(x) = (x - 1)^2 + 7$ **d** $f(x) = (x - 5)^2 - 24$
e $f(x) = (x + 1)^2 - 6$ **f** $f(x) = (x + 2)^2 - 5$

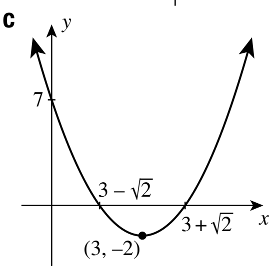
4a



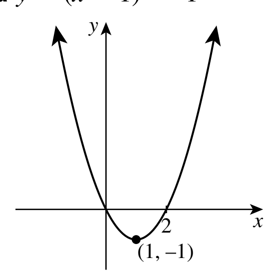
b



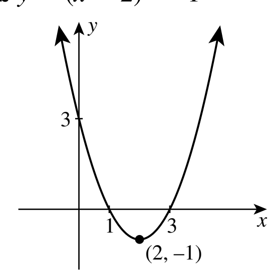
c



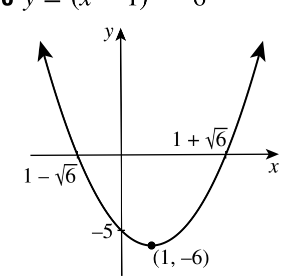
5a $y = (x - 1)^2 - 1$



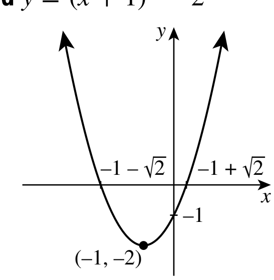
b $y = (x - 2)^2 - 1$



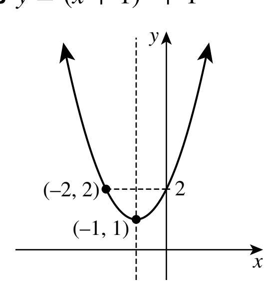
c $y = (x - 1)^2 - 6$



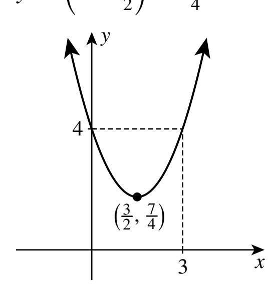
d $y = (x + 1)^2 - 2$



e $y = (x + 1)^2 + 1$



f $y = \left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$



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6a $x = 1, 3$ **b** $x = -3, 1$ **c** $x = -1, 2$

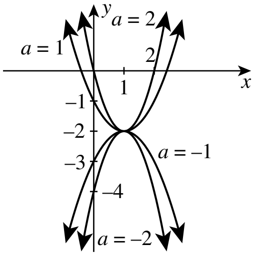
7a $y = (x - 1)^2 + 2$ **b** $y = (x + 2)^2 - 3$

c $y = -(x - 3)^2 + 5$ **d** $y = -(x - 2)^2 - 1$

8a $y = (x - 2)^2 + 5$ **b** $y = x^2 - 3$

c $y = (x + 1)^2 + 7$ **d** $y = (x - 3)^2 - 11$

9a



b $(1, -2)$ **c** $a > 0$

d The vertex is below the x-axis.
Thus the parabola will only intersect the x-axis if it is concave up.

10a $V = (3, -5)$, concave up, two x-intercepts.

b $V = (-1, 3)$, concave down, two x-intercepts.

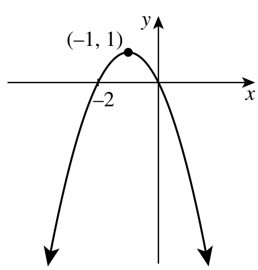
c $V = (-2, -1)$, concave down, no x-intercepts.

d $V = (4, 3)$, concave up, no x-intercepts.

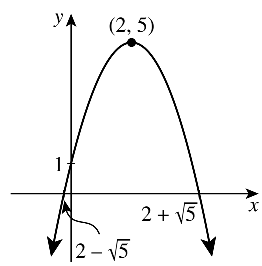
e $V = (-1, 0)$, concave up, one x-intercept.

f $V = (3, 0)$, concave down, one x-intercept.

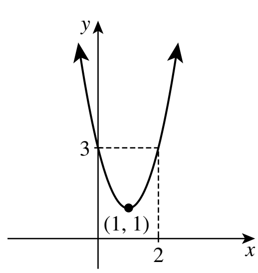
11a $y = -(x + 1)^2 + 1$



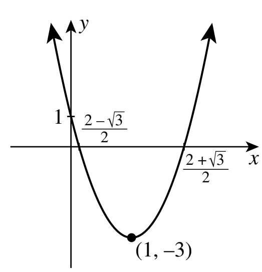
b $y = -(x - 2)^2 + 5$



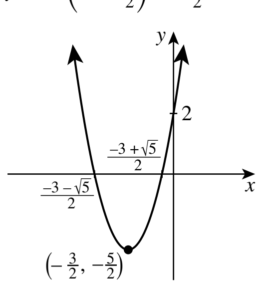
c $y = 2(x - 1)^2 + 1$



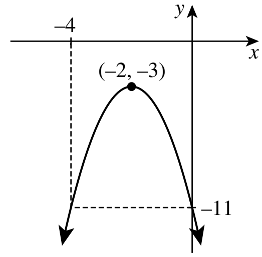
d $y = 4(x - 1)^2 - 3$



e $y = 2\left(x + \frac{3}{2}\right)^2 - \frac{5}{2}$



f $y = -2(x + 2)^2 - 3$



12a $f(x) = \left(x + 1 + \sqrt{2}\right)\left(x + 1 - \sqrt{2}\right)$

b $f(x) = \left(x - 2 + \sqrt{3}\right)\left(x - 2 - \sqrt{3}\right)$

c $f(x) = -\left(x + 1 + \sqrt{5}\right)\left(x + 1 - \sqrt{5}\right)$

13 Put $h = -4$ and $k = 2$ into the formula $y = a(x - h)^2 + k$.

a $y = (x + 4)^2 + 2$ **b** $y = 3(x + 4)^2 + 2$

c $y = \frac{7}{8}(x + 4)^2 + 2$ **d** $y = -\frac{1}{8}(x + 4)^2 + 2$

14a Answer in question

b $y = a\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2 - 4ac}{4a}$ with vertex $\left(-\dfrac{b}{2a}, -\dfrac{b^2 - 4ac}{4a}\right)$ and axis of symmetry $x = -\dfrac{b}{2a}$.

c $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

Exercise 3F

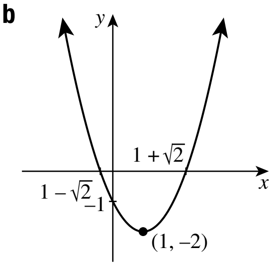
1a i concave up **ii** $y = -1$

iii $x = 1$ **iv** $= (1, -2)$

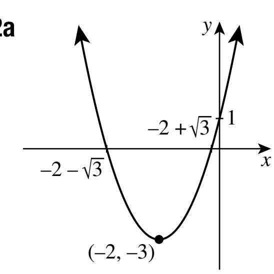
v $\Delta = 8$ **vi** $\Delta > 0$

vii $x = 1 - \sqrt{2}, 1 + \sqrt{2} \doteqdot -0.41, 2.41$.

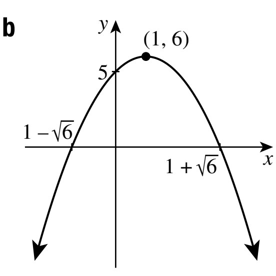
b



2a



b



3a i concave up

ii $y = 3$

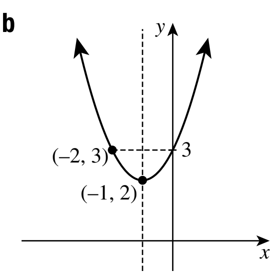
iii $x = -1$

iv $= (-1, 2)$

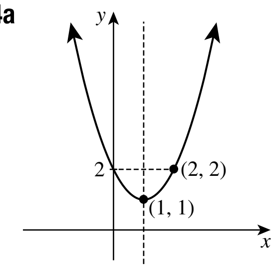
v $\Delta = -8$

vi $\Delta < 0$

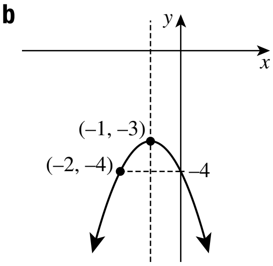
b



4a



b



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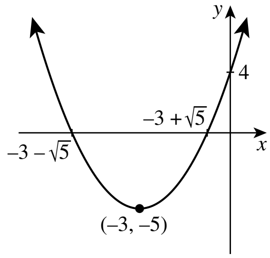
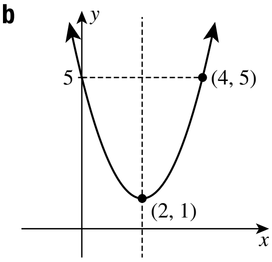
5a $-1 + \sqrt{3}$ or $-1 - \sqrt{3}$, 0.73 or -2.73
b $2 + \sqrt{3}$ or $2 - \sqrt{3}$, 3.73 or 0.27
c $\frac{1}{2}\left(-3 + \sqrt{17}\right)$ or $\frac{1}{2}\left(-3 - \sqrt{17}\right)$, 0.56 or -3.56
d $-1 + \sqrt{5}$ or $-1 - \sqrt{5}$, 1.23 or -3.23
e $\frac{1}{3}\left(1 + \sqrt{7}\right)$ or $\frac{1}{3}\left(1 - \sqrt{7}\right)$, 1.22 or -0.55
f $\frac{1}{2}\left(-2 + \sqrt{6}\right)$ or $\frac{1}{2}\left(-2 - \sqrt{6}\right)$, 0.22 or -2.22

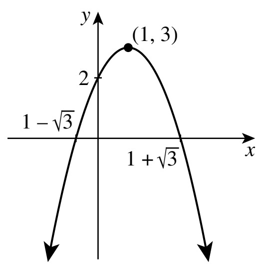
6a



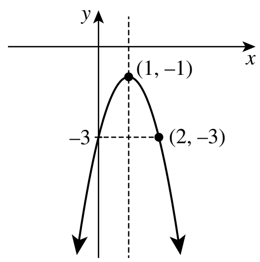
b



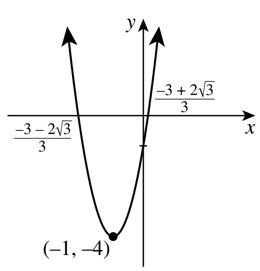
c



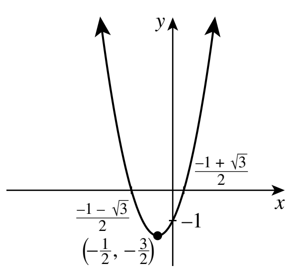
d



e

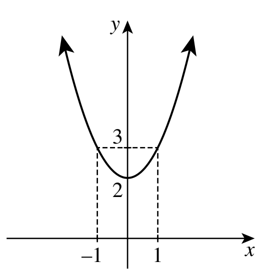


f

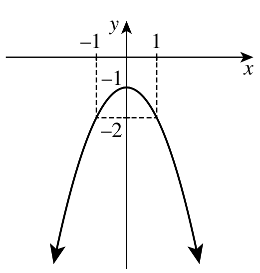


7a $x = -1, 4$ **b** $x = 2, 3$ **c** $x = -2, 6$

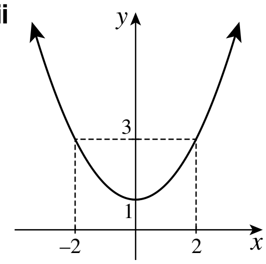
8a i $\Delta = -8 < 0$
ii Both equal $(0, 2)$.
iii $(1, 3)$
iv $(-1, 3)$



b i



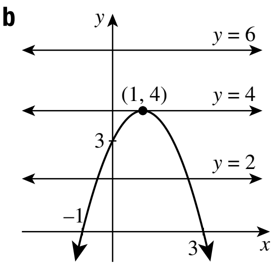
ii



9a $\Delta = 17$, two zeroes **b** $\Delta = 0$, one zero
c $\Delta = -7$, no zeroes

10a i $x = 1 - \sqrt{2}$ and $x = -1 + \sqrt{2}$
ii $x = 1$ **iii** There are none.

b



c $k < 4$

11a $3 + 2\sqrt{2}, 3 - 2\sqrt{2}$ **b** $1 + \sqrt{5}, 1 - \sqrt{5}$
c $\frac{5 + \sqrt{10}}{3}, \frac{5 - \sqrt{10}}{3}$

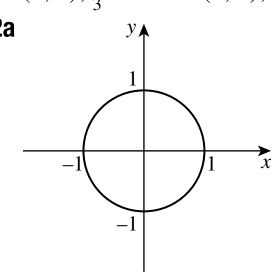
12a $\left(x - 3 + \sqrt{5}\right)\left(x - 3 - \sqrt{5}\right)$
b $\left(x + 1 + \sqrt{2}\right)\left(x + 1 - \sqrt{2}\right)$
c $\left(x - \frac{3 + \sqrt{5}}{2}\right)\left(x - \frac{3 - \sqrt{5}}{2}\right)$

13a Answer in question
b Compare to answer to question 6

Exercise 3G

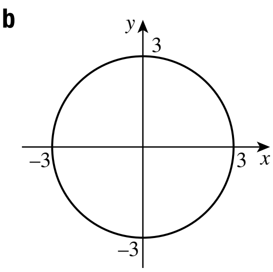
1a (0, 0), 4 units **b** (0, 0), 7 units
c (0, 0), $\frac{1}{3}$ units **d** (0, 0), 1.2 units

2a



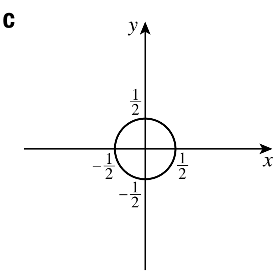
$-1 \le x \le 1$
$-1 \le y \le 1$

b



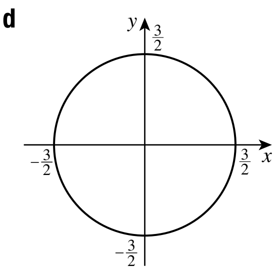
$-3 \le x \le 3$
$-3 \le y \le 3$

c



$-\frac{1}{2} \le x \le \frac{1}{2}$
$-\frac{1}{2} \le y \le \frac{1}{2}$

d



$-\frac{3}{2} \le x \le \frac{3}{2}$
$-\frac{3}{2} \le y \le \frac{3}{2}$

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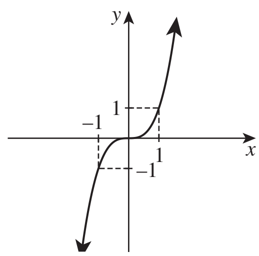
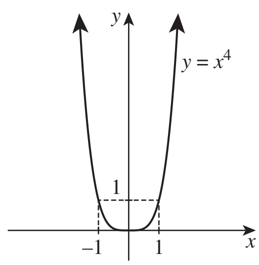
3a $-3.375, -1, -0.125, 0, 0.125, 1, 3.375$

b

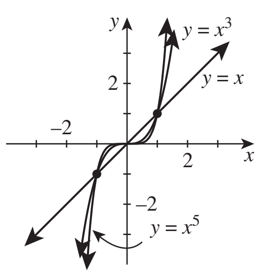


4a $5.0625, 1, 0.0625, 0, 0.0625, 1, 5.0625$

b



5a



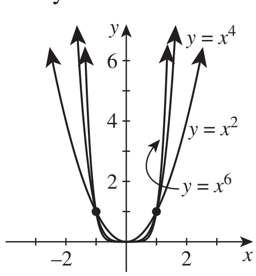
b $(-1, -1)$, $(0, 0)$ and $(1, 1)$

c i $y = x^5$ **ii** $y = x$ **d i** $y = x^5$ **ii** $y = x$

e In each case, the result is the same curve.

f Every index is odd.

6a



b $(-1, 1)$, $(0, 0)$ and $(1, 1)$

c i $y = x^6$ **ii** $y = x^2$ **d i** $y = x^6$ **ii** $y = x^2$

e In each case, the result is the same curve.

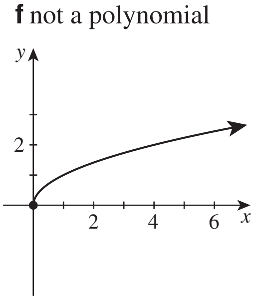
f Every index is even.

7a degree 1, coefficient 2 **b** degree 3, coefficient 0

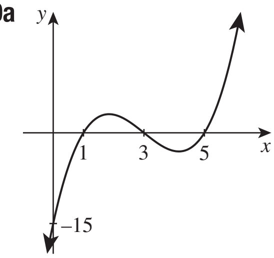
c not a polynomial **d** not a polynomial

e degree 3, coefficient -1 **f** not a polynomial

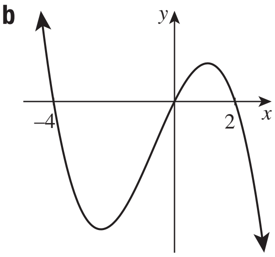
8a $0, 0.5, 1, 1.5, 2, 2.5$ **b**



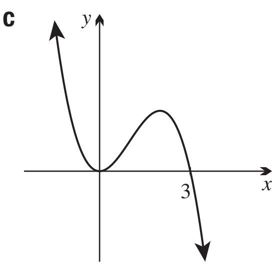
9a



b



c

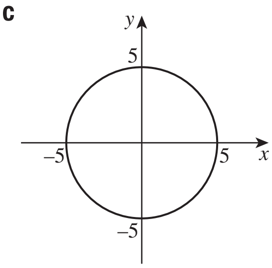


10a $x^2 + y^2 = 4$ **b** $x^2 + y^2 = 5$ **c** $x^2 + y^2 = 25$

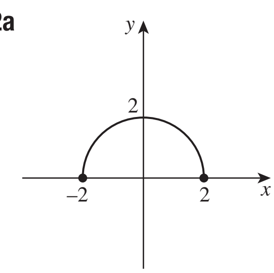
d $x^2 + y^2 = 10$

11a 5 or -5, 4.9 or -4.9, 4.6 or -4.6, 4 or -4, 3 or -3, 0

c

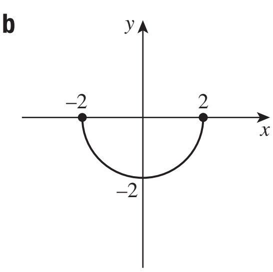


12a



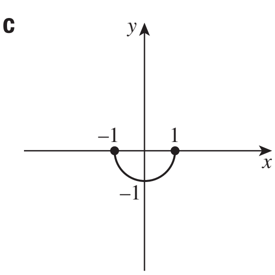
domain: $-2 \le x \le 2$,
range: $0 \le y \le 2$

b



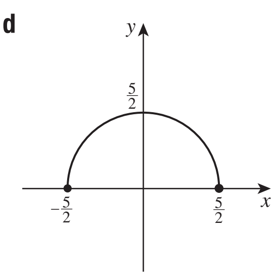
domain: $-2 \le x \le 2$,
range: $-2 \le y \le 0$

c



domain: $-1 \le x \le 1$,
range: $-1 \le y \le 0$

d



domain: $-\frac{5}{2} \le x \le \frac{5}{2}$,
range: $0 \le y \le \frac{5}{2}$

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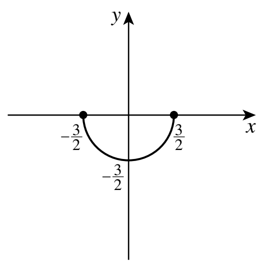
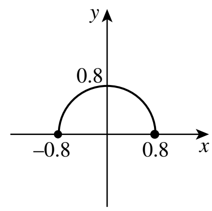
e



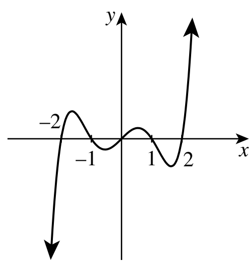
domain: $-\frac{3}{2} \le x \le \frac{3}{2}$,
range: $-\frac{3}{2} \le y \le 0$

f

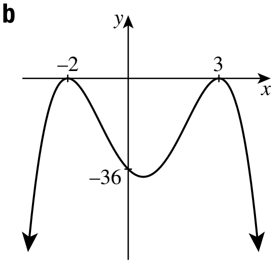


domain: $-0.8 \le x \le 0.8$,
range: $0 \le y \le 0.8$

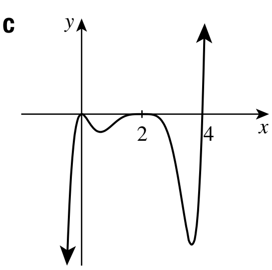
13a



b

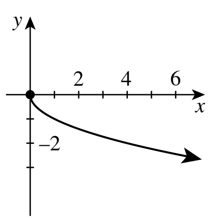


c

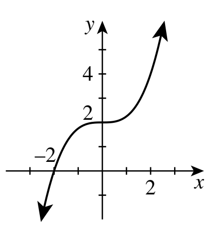


14a 0, −0.5, −1, −1.5, −2, −2.5

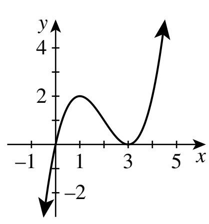
b



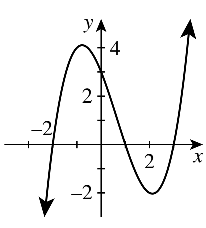
15a i



ii



iii

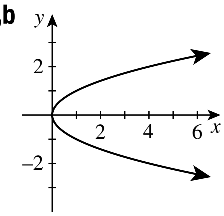


b Answer in question **c** Answer in question

16a $y = -3(x + 1)(x - 1)(x - 4)$

b $y = -(x + 1)^2(x - 1)^3(x - 3)^2$

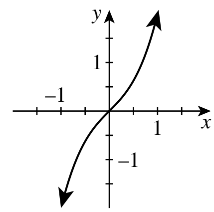
17a,b



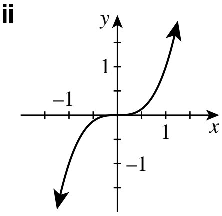
c It is a concave right parabola.

d In both cases, squaring gives $x = y^2$. This is the result of swapping x and y in $y = x^2$.

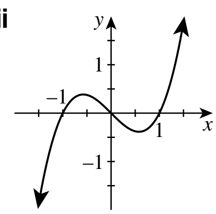
18a i



ii



iii



b i 1st and 3rd

ii In each case, the result is the same curve.

iii Every index is odd.

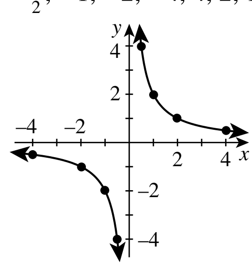
c The slope: $x^3 + x$ is upwards, x^3 is horizontal, $x^3 - x$ is downwards.

d Answers will vary

Exercise 3H

1a $-\frac{1}{2}$, −1, −2, −4, 4, 2, 1, $\frac{1}{2}$

b



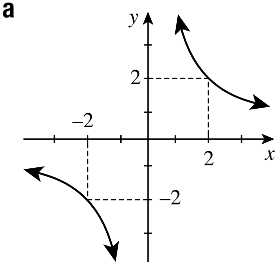
c 1st and 3rd

d the x-axis ($y = 0$) and the y-axis ($x = 0$)

e domain: $x \ne 0$, range: $y \ne 0$

2 In each case, the domain is $x \ne 0$, the range is $y \ne 0$. The asymptotes are $y = 0$ and $x = 0$. The branches are in quadrants 1 and 3.

a



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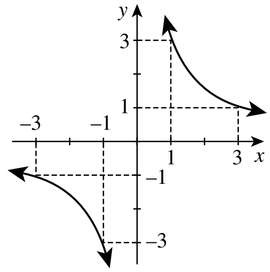
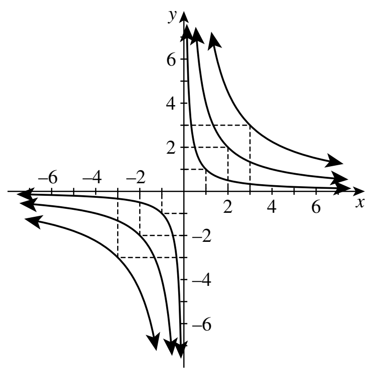
b



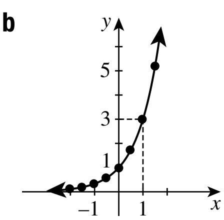
3



a 1st and 3rd
b the x-axis ($y = 0$) and the y-axis ($x = 0$)
c $x \neq 0, y \neq 0$
d $(1, 1)$ and $(-1, -1)$ on $y = \frac{1}{x}$
$(2, 2)$ and $(-2, -2)$ on $y = \frac{4}{x}$
$(3, 3)$ and $(-3, -3)$ on $y = \frac{9}{x}$
The values are the square roots of the numerator.

4a 0.1, 0.2, 0.3, 0.6, 1, 1.7, 3, 5.2, 9

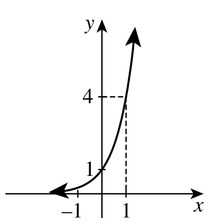
b



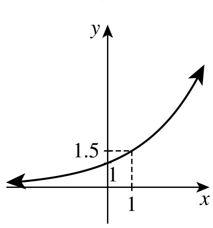
c $(0, 1)$ **d** 3, the base **e** the x-axis ($y = 0$)
f domain: all real x, range: $y > 0$

5 In each case, the domain is all real x, the range is $y > 0$. The asymptote is $y = 0$. The y-intercept is $(0, 1)$. At $x = 1$, $y =$ the base.

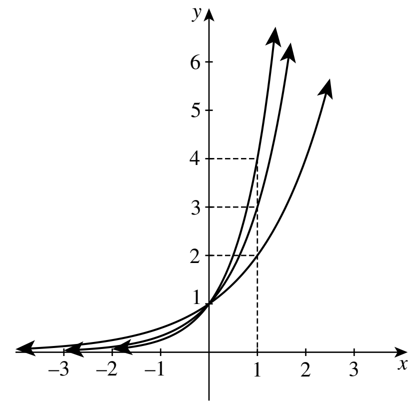
a



b

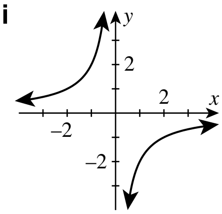


6

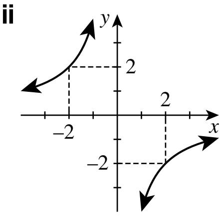


a $(0, 1)$ **b** the x-axis ($y = 0$) **c** all real x, $y > 0$
d No answer required.
e $y = 4^x$, it has the greater base.
f $y = 4^x$ again, it has the greater base.

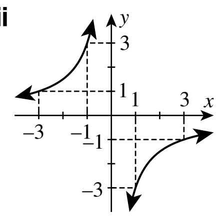
7a i



ii

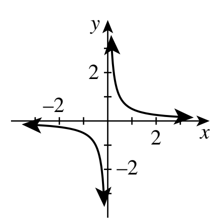


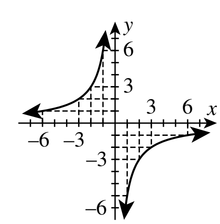
iii



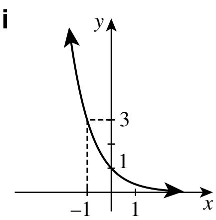
b i quadrants 2 and 4
ii The minus sign has caused the quadrants to change.

8a $y = \dfrac{1}{2x}$ **b** $y = \dfrac{6}{x}$

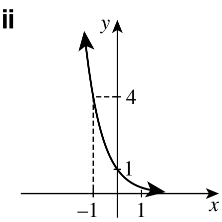




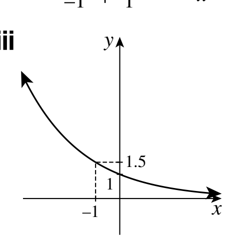
9a i



ii



iii



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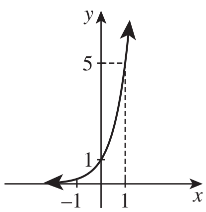
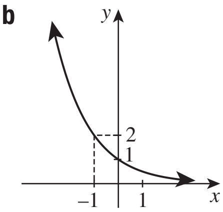
b i No: it is $(0, 1)$. **ii** No: it is the x-axis. **iii** $x = -1$
iv In Questions 4 and 5, the y-values grow. In these questions they decay away.
v The minus sign has caused the changes.

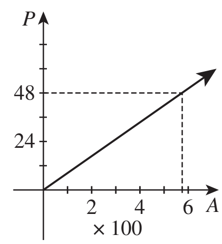
10a



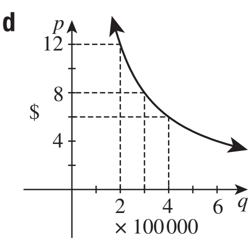
b



11a $P = kA$
b $k = \frac{1}{12}$
c $55\frac{2}{3}$L
d 1 bucket, 4 tins

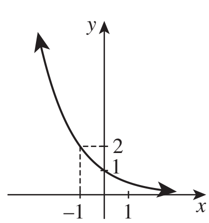


12a $T = 2\,400\,000$ **b** 300 000 **c** Sales will halve.
d



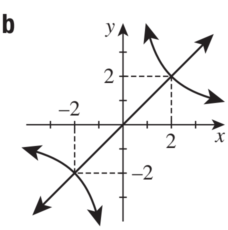
13a $y \to 0$ as $x \to -\infty$.
b $y \to 0$ as $x \to \infty$.
c $y \to 0$ as $x \to \infty$ and as $x \to -\infty$, $y \to \infty$ as $x \to 0^+$, $y \to -\infty$ as $x \to 0^-$.
14 No, because the only points that satisfy the equation lie on the x- and y-axes.

15a



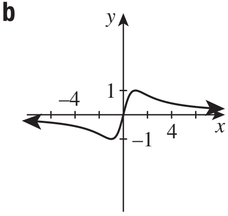
b $\left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x$
so $\left(\frac{1}{2}\right)^x = 2^{-x}$

16a (c, c) and $(-c, -c)$
b



17 4 m × 12 m or 6 m × 8 m
18a $-\frac{16}{65}, -\frac{8}{17}, -\frac{4}{5}, -1, -\frac{4}{5}, 0, \frac{4}{5}, 1, \frac{4}{5}, \frac{8}{17}, \frac{16}{65}$

b



c x-axis $(y = 0)$
d $(0, 0)$

Exercise 3I

1a Vertical line test: Yes. It is a function.
b Horizontal line test: No. Many-to-one
c 10:00 pm on Saturday to 10:00 pm on Sunday
d 3 ft and 4 ft
e i 10:00 pm, 6:00 am, 10:30 am and 3:30 pm
ii 11:00 pm, 4:45 am and 1:00 pm
iii Never
f 0, 1, 2, 3 and 4
2a It passes the vertical line test, so it is a function. Also, it fails the horizontal line test, so it is many-to-one.
b 1°C
c It was never 20°C. It was 8°C at 01:00 am, 8:00 am and 10:30 pm on the first day, and at about 3:30 pm on the second day.
d 0, 2, 3, 4, 5 (Whether 1 is omitted depends on how accurately you are supposed to read the graph.)
3a i Vertical line test: No. Horizontal line test: Yes.
ii Vertical line test: No. Horizontal line test: No.
iii Vertical line test: Yes. Horizontal line test: No.
iv Vertical line test: No. Horizontal line test: No.
v Vertical line test: Yes. Horizontal line test: Yes.
vi Vertical line test: No. Horizontal line test: Yes.
b iii, v
c i, v, vi
d v
e i One-to-many
ii Many-to-many
iii Many-to-one
iv Many-to-many
v One-to-one
vi One-to-many
4a one-to-many **b** many-to-many
c one-to-many **d** many-to-one
e one-to-one **f** many-to-many
5a i When $y = 0$, $x = 2$ or -2
ii When $y = 0$, $x = 1$ or 0 or -1
iii When $y = 2$, $x = 1$ or -1
b They are all one-to-many, because x and y are reversed.

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6a i $x = \frac{1}{3}y + \frac{1}{3}$ **ii** $x = -\frac{1}{2}y + \frac{5}{2}$

iii $x = \frac{1}{2}\sqrt[3]{y}$ **iv** $x = \frac{5}{y}$

b i–iv They are all one-to-one also, because x and y are reversed.

7a When $x = 3$, $y = 4$ or -6. When $y = -1$, $x = 8$ or -2

b When $x = 0$, $y = 3$ or -3. When $y = 0$, $x = 2$ or -2

c When $x = 2$, $y = \sqrt{3}$ or $-\sqrt{3}$. When $y = 0$, $x = 1$ or -1

8a It passes neither test, and is thus many-to-many.

b Vertical line test: Yes, Horizontal line test: No. It is many-to-one, and therefore a function.

9a It is a function, but it may be one-to-one or many-to-one.

b If there are two of more students with the same preferred name, it is many-to-one. Otherwise it is one-to-one.

10a …, −270°, 90°, 450°, …

b one-to-many

c many-to-one

11a Probably many-to-many, but just possibly one-to-one.

b The condition to be one-to-one is that every flat has no more than one occupant, and in this case, every inhabitant is mapped to himself, that is, $f(x) = x$, for every inhabitant x. Otherwise the relation is many-to-many.

c The relation is then the *empty relation*, which is discussed later in Section 4E. This empty relation is a one-to-one function, because it trivially passes the vertical and horizontal lines tests.

12a many-to-one

b one-to-many

c one-to-one

d one-to-one (trivially because the graph has only one point)

e many-to-many

f one-to-many (factor as $x = (y - 2)(y - 3)$)

g many-to-one (factor as $y = x(x - 3)(x - 4)$)

h one-to-one

i one-to-one

j one-to-one

k many-to-many

l one-to-one

Chapter 3 review exercise

1a not a function **b** function **c** function **d** not a function

2a $-2 \le x \le 0$, $-2 \le y \le 2$ **b** all real x, all real y

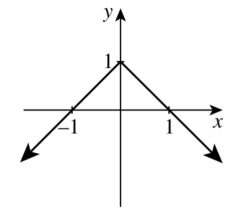
c $x \ne 0$, $y \ne 0$ **d** $x = 2$, all real y

3a 21, −4 **b** 5, −15

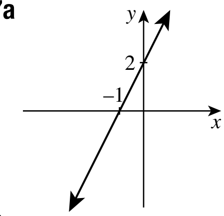
4a $x \ne 2$ **b** $x \ge 1$ **c** $x \ge -\frac{2}{3}$ **d** $x < 2$

5a $2a + 2$, $2a + 1$ **b** $a^2 - 3a - 8$, $a^2 - 5a - 3$

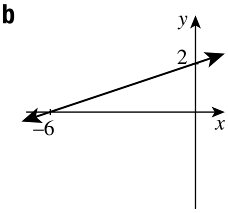
6 −2, −1, 0, 1, 0, −1, −2



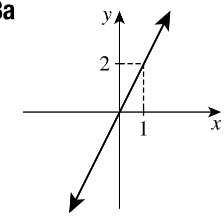
7a



b



8a



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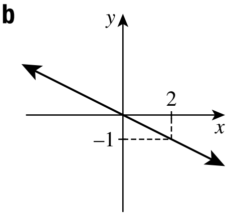
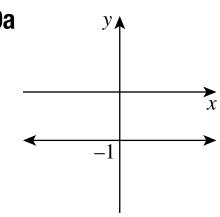


9a



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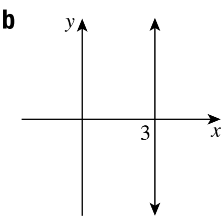
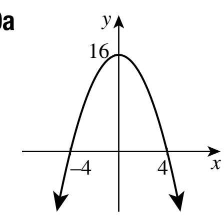
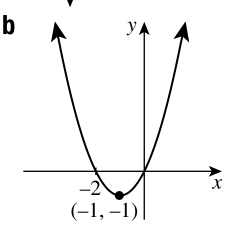


10a



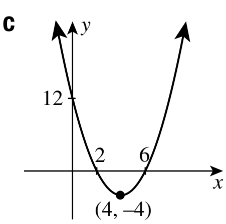
domain: all real x, range: $y \le 16$

b



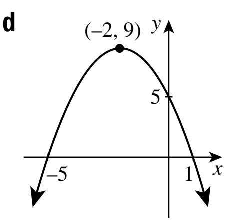
domain: all real x, range: $y \ge -1$

c



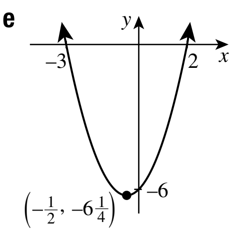
domain: all real x, range: $y \ge -4$

d



domain: all real x, range: $y \le 9$

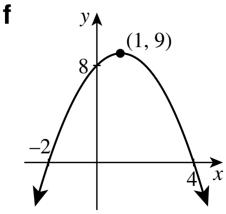
e



$\left(-\frac{1}{2}, -6\frac{1}{4}\right)$

domain: all real x, range: $y \ge -6\frac{1}{4}$

f



domain: all real x, range: $y \le 9$

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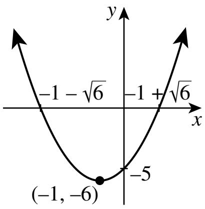
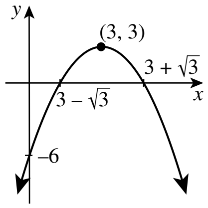
11a



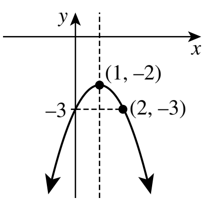
$y = (x + 1)^2 - 6$

b



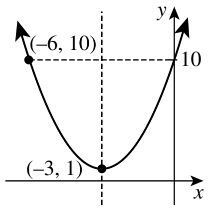
$y = -(x - 3)^2 + 3$

c



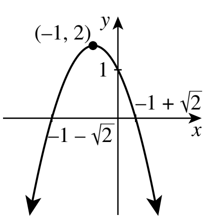
$y = -(x - 1)^2 - 2$

d

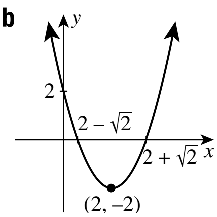


$y = (x + 3)^2 + 1$

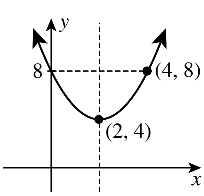
12a



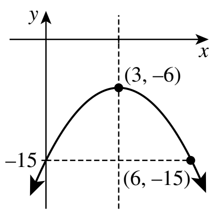
b



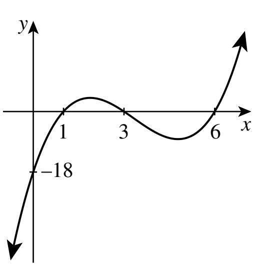
c



d



13a



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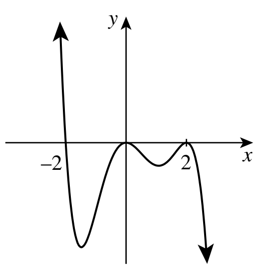
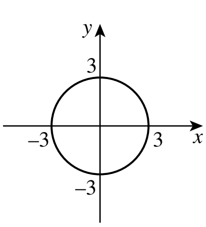
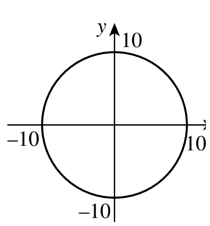


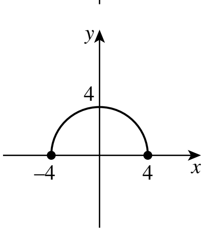
14a



b

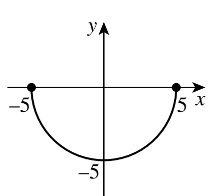


15a



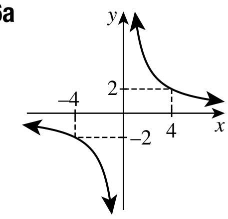
domain: $-4 \le x \le 4$,
range: $0 \le y \le 4$

b



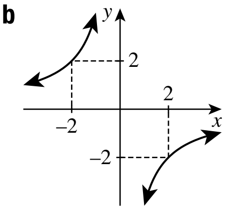
domain: $-5 \le x \le 5$,
range: $-5 \le y \le 0$

16a



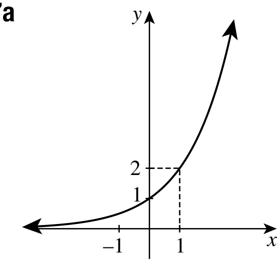
domain: $x \neq 0$,
range: $y \neq 0$

b



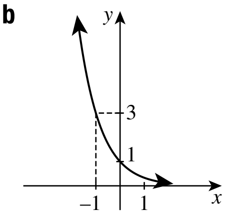
domain: $x \neq 0$,
range: $y \neq 0$

17a



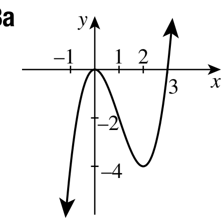
domain: all real x,
range: $y > 0$

b



domain: all real x,
range: $y > 0$

18a



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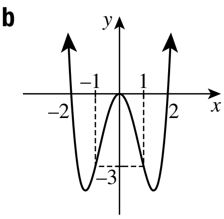

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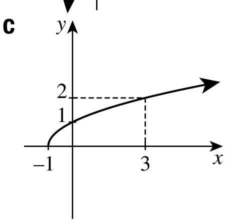
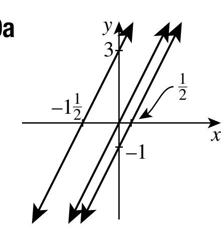


19a



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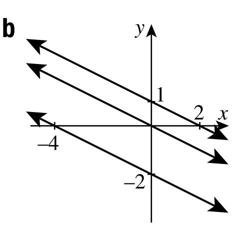

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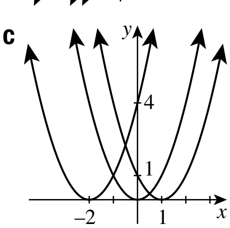

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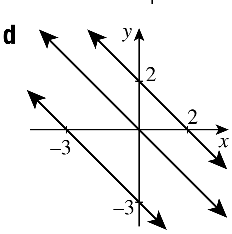

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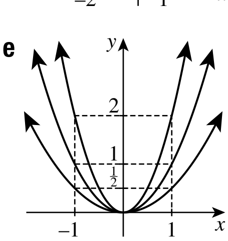

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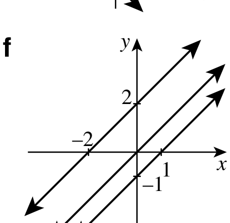

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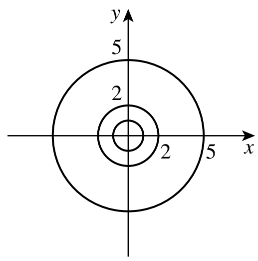
g



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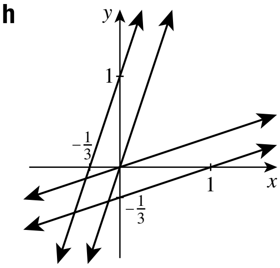

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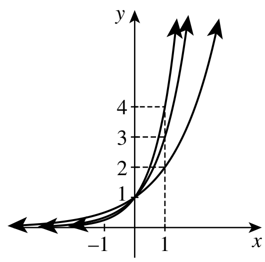

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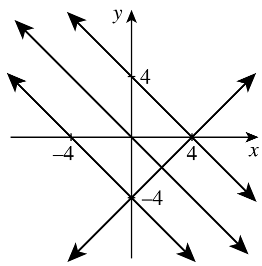

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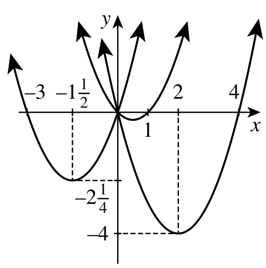

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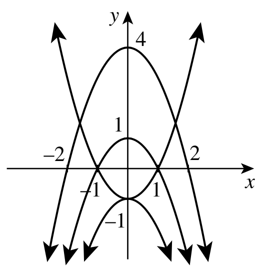

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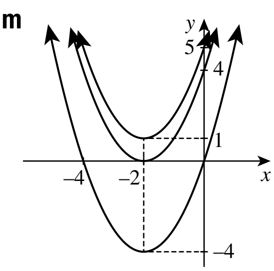

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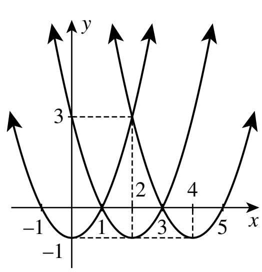

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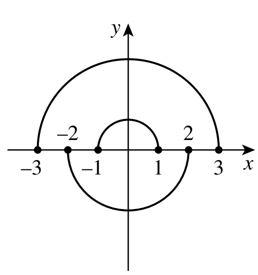

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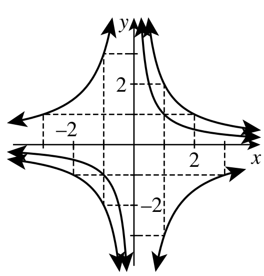

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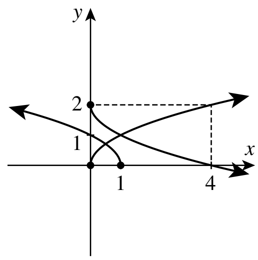

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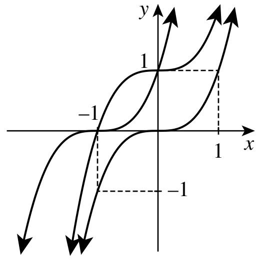

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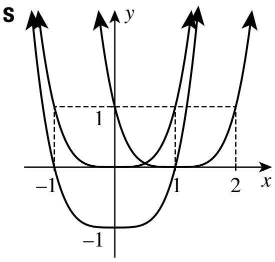

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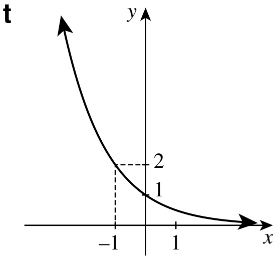


All three are the same.

20a one-to-one **b** many-to-many

c one-to-many **d** many-to-one

21a It is probably a many-to-one function, but it is possibly a one-to-one function

b If every person was born in a different country, the function is one-to-one. Otherwise it is many-to-one.

Chapter 4

Exercise 4A

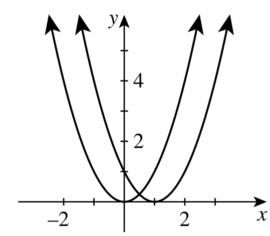
1a x^2: 4, 1, 0, 1, 4, 9

$(x-1)^2$: 9, 4, 1, 0, 1, 4

b $y = x^2$, $V = (0, 0)$

$y = (x-1)^2$, $V = (1, 0)$

c Here x is replaced by $(x-1)$, so it is a shift right by 1 unit.

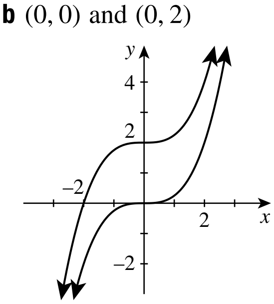


2a $\frac{1}{4}x^3$: $-6\frac{3}{4}$, -2, $-\frac{1}{4}$, 0, $\frac{1}{4}$, 2, $6\frac{3}{4}$

$\left(\frac{1}{4}x^3 + 2\right)$: $-4\frac{3}{4}$, 0, $1\frac{3}{4}$, 2, $2\frac{1}{4}$, 4, $8\frac{3}{4}$

b (0, 0) and (0, 2)

c The second equation is also $y - 2 = \frac{1}{4}x^3$. Here y is replaced by $(y - 2)$, so it is a shift up by 2 units.



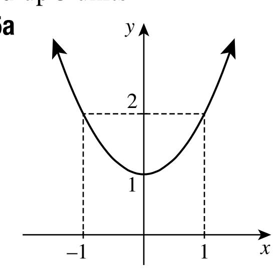
3a up 2 units **b** down 5 units **c** left 4 units

d right 3 units

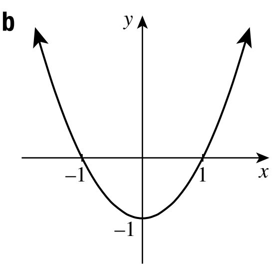
4a right 2 units **b** left 3 units **c** down 4 units

d up 5 units

5a



b



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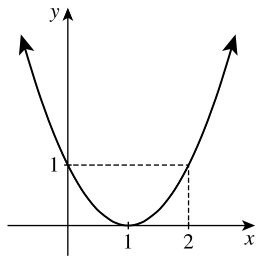
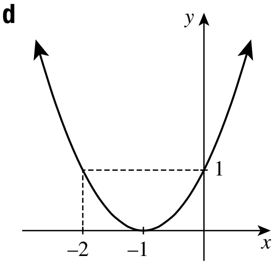
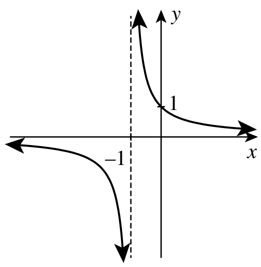
c



d



6a



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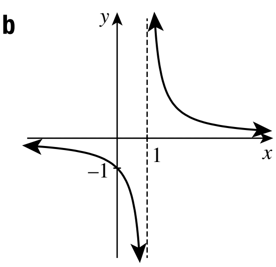

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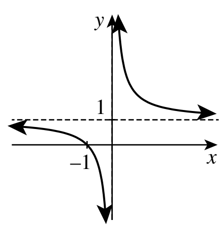

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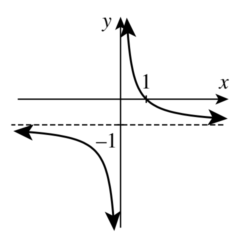
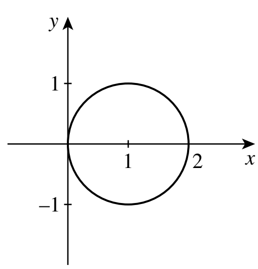


7a



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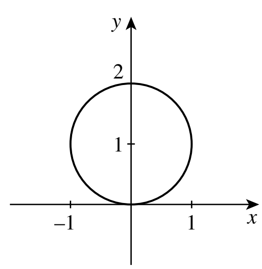
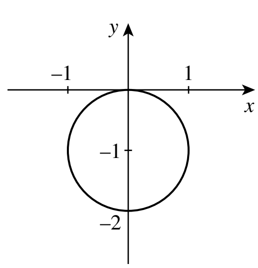
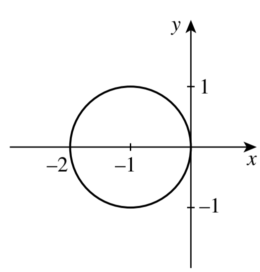


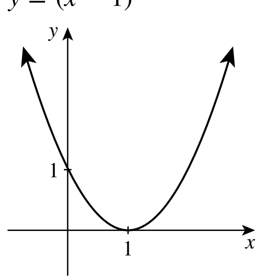
c



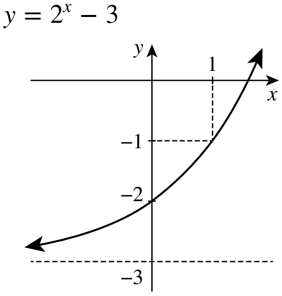
d



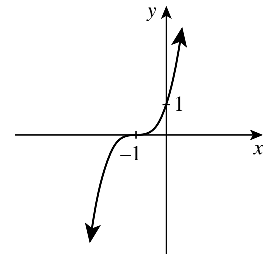
8a $y = (x - 1)^2$



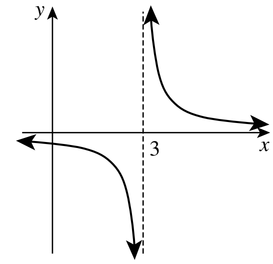
b $y = 2^x - 3$



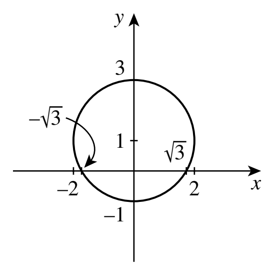
c $y = (x + 1)^3$



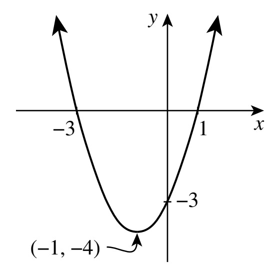
d $y = \dfrac{1}{x - 3}$



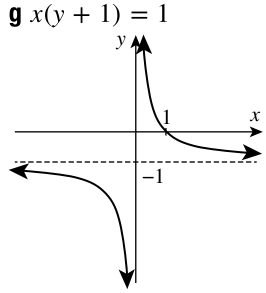
e $x^2 + (y - 1)^2 = 4$



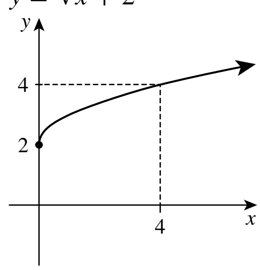
f $y = (x + 1)^2 - 4$



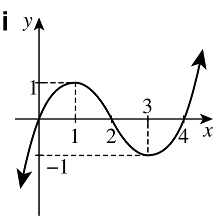
g $x(y + 1) = 1$



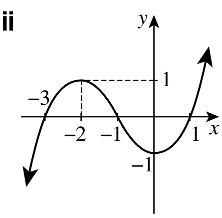
h $y = \sqrt{x} + 2$



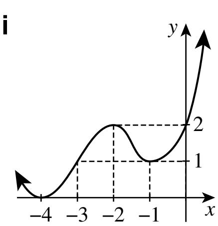
9a i



ii



b i



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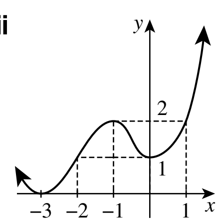
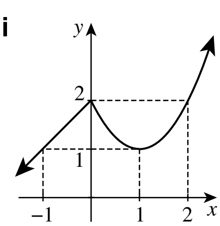


c i



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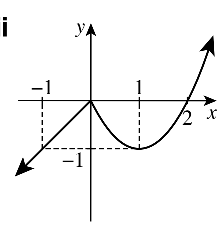
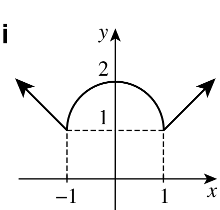


d i



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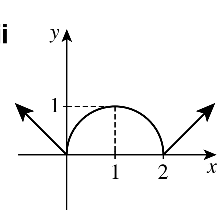

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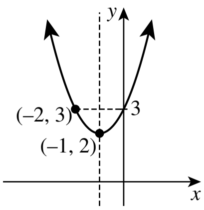
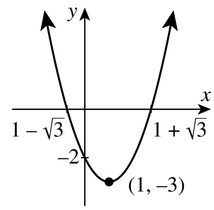
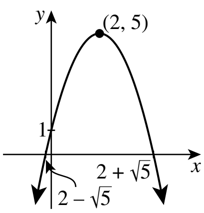
10a $y = (x + 1)^2 + 2$
This is $y = x^2$ shifted left 1 and up 2.



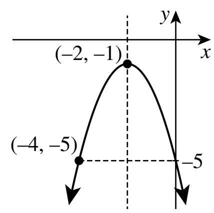
b $y = (x - 1)^2 - 3$
This is $y = x^2$ shifted right 1 and down 3.



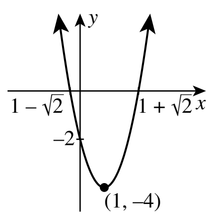
c $y = -(x - 2)^2 + 5$
This is $y = -x^2$ shifted right 2 and up 5.



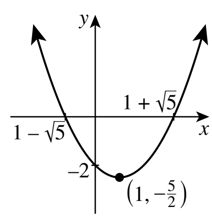
d $y = -(x + 2)^2 - 1$
This is $y = -x^2$ shifted left 2 and down 1.



e $y = 2(x - 1)^2 - 4$
This is $y = 2x^2$ shifted right 1 and down 4.



f $y = \frac{1}{2}(x - 1)^2 - \frac{5}{2}$
This is $y = \frac{1}{2}x^2$ shifted right 1 and down $\frac{5}{2}$.

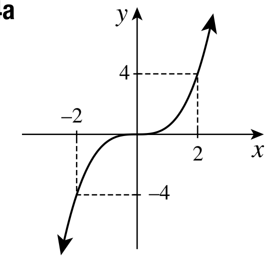


11a the parabola $y = x^2$ translated right 2, $y = (x - 2)^2$
b the hyperbola $xy = 1$ translated right 2, $(x - 2)y = 1$ or $y = \frac{1}{x - 2}$
c the parabola $y = x^2$ translated right 2, down 1, $y + 1 = (x - 2)^2$
d the hyperbola $xy = 1$ translated right 2, down 1, $(x - 2)(y + 1) = 1$ or $y + 1 = \frac{1}{x - 2}$

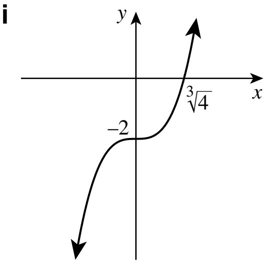
12a $r = 2$, $(-1, 0)$ **b** $r = 1$, $(1, 2)$
c $r = 3$, $(1, 2)$ **d** $r = 5$, $(-3, 4)$
e $r = 3$, $(5, -4)$ **f** $r = 6$, $(-7, 1)$

13a the circle $x^2 + y^2 = 1$ translated right 2, up 3, $(x - 2)^2 + (y - 3)^2 = 1$
b the circle $x^2 + y^2 = 4$ translated left 2, down 1, $(x + 2)^2 + (y + 1)^2 = 4$
c the circle $x^2 + y^2 = 10$ translated left 1, up 1, $(x + 1)^2 + (y - 1)^2 = 10$
d the circle $x^2 + y^2 = 5$ translated right 2, down 1, $(x - 2)^2 + (y + 1)^2 = 5$

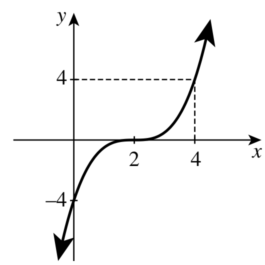
14a



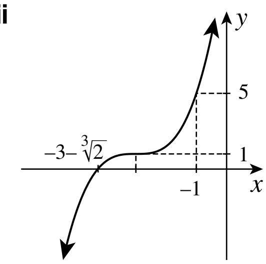
i



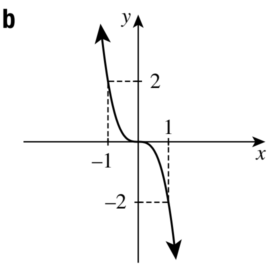
ii



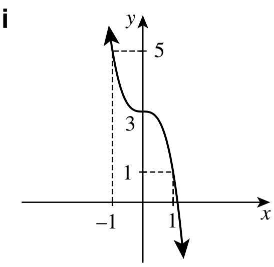
iii



b



i



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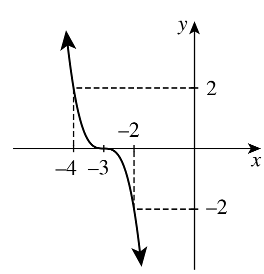
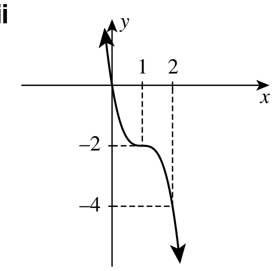
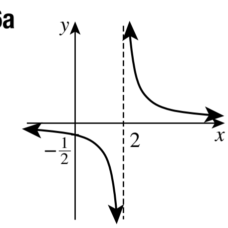


iii

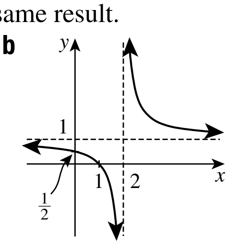


15a $x + 2y - 2 = 0$ **b** $x + 2y - 2 = 0$
c Both translations yield the same result.

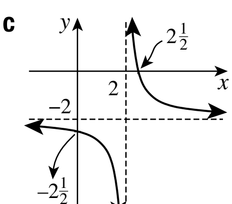
16a



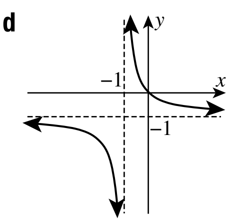
b



c



d



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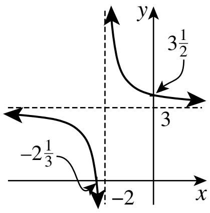
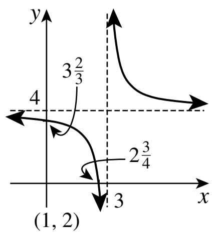
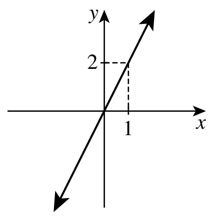
e



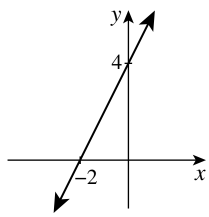
f



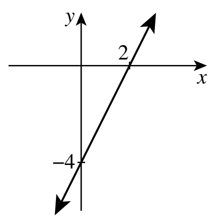
17a From $y = 2x$:



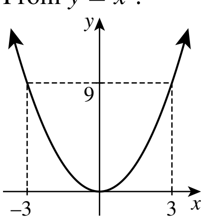
i shift up 4 (or left 2)



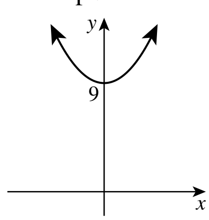
ii shift down 4 (or right 2)



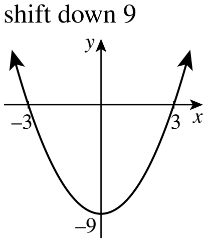
b From $y = x^2$:



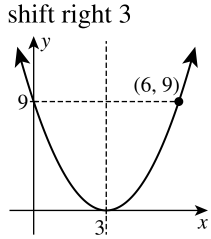
i shift up 9



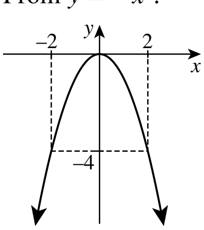
ii shift down 9



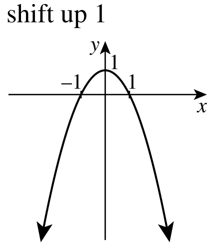
iii shift right 3



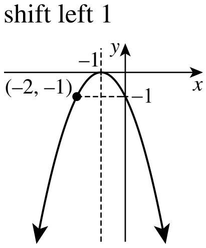
c From $y = -x^2$:



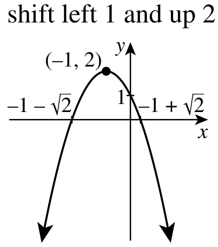
i shift up 1



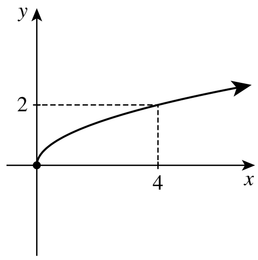
ii shift left 1



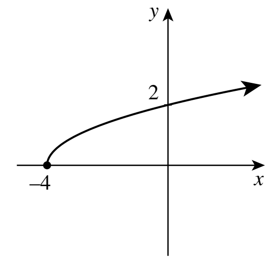
iii shift left 1 and up 2



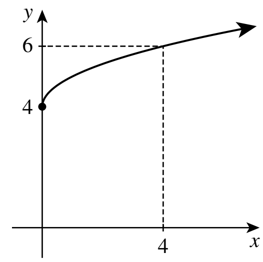
d From $y = \sqrt{x}$:



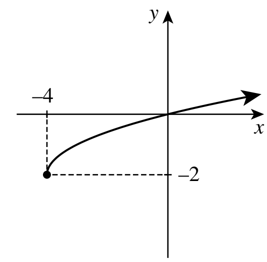
i shift left 4



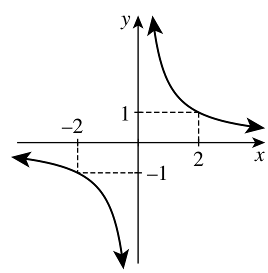
ii shift up 4



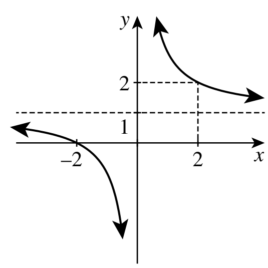
iii shift left 4 and down 2



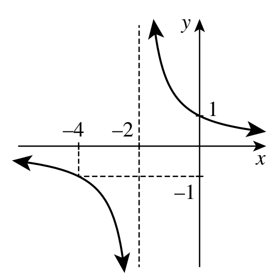
e From $y = \dfrac{2}{x}$:



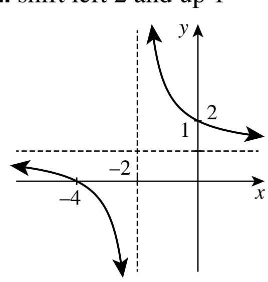
i shift up 1



ii shift left 2



iii shift left 2 and up 1



18a $(x - h)^2 + (y - k)^2 = r^2$

b Answer same as 18a

Exercise 4B

1a Answers will vary

b $y = x^2 - 2x$: 8, 3, 0, −1, 0, 3, 8

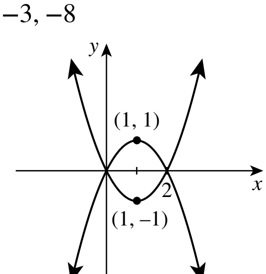
$y = 2x - x^2$: −8, −3, 0, 1, 0, −3, −8

c $y = x^2 - 2x$:

$V = (1, -1)$

$y = 2x - x^2$: $V = (1, 1)$

d Here y is replaced with $-y$, so it is a reflection in the x-axis.



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2a Answers will vary

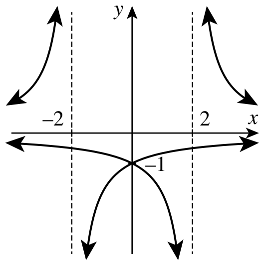
b $y = \frac{2}{x-2}$: $-\frac{1}{3}, -\frac{2}{5}, -\frac{1}{2}, -\frac{2}{3}, -1, -2, *, 2, 1$

$y = -\frac{2}{x+2}$: $1, 2, *, -2, -1, -\frac{2}{3}, -\frac{1}{2}, -\frac{2}{5}, -\frac{1}{3}$

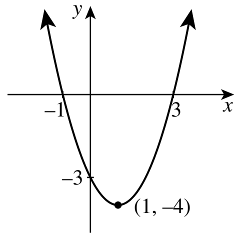
c $y = \frac{2}{x-2}$: $x = 2$

$y = -\frac{2}{x+2}$: $x = -2$

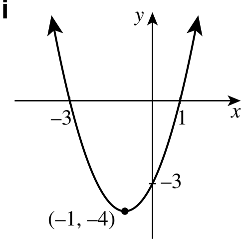
d Here x is replaced with $-x$, so it is a reflection in the y-axis.



3a

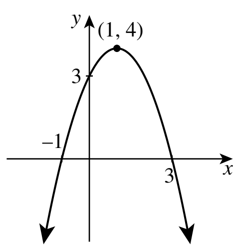


b i



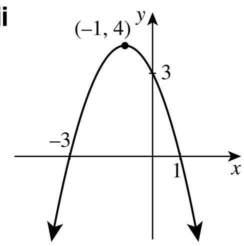
$y = x^2 + 2x - 3$

ii



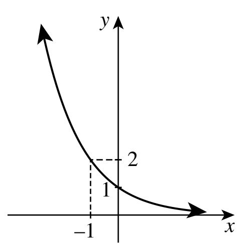
$y = -x^2 + 2x + 3$

iii

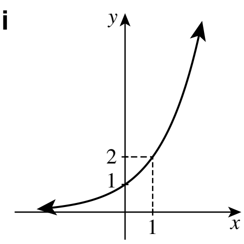


$y = -x^2 - 2x + 3$

4a

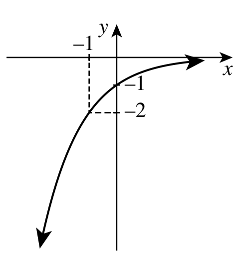


b i



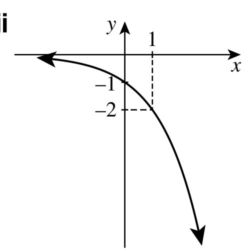
$y = 2^x$

ii



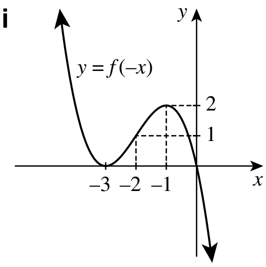
$y = -2^{-x}$

iii

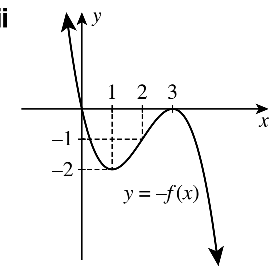


$y = -2^x$

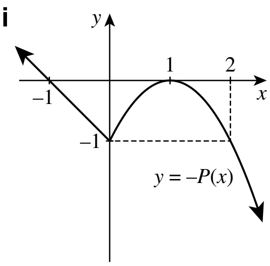
5a i



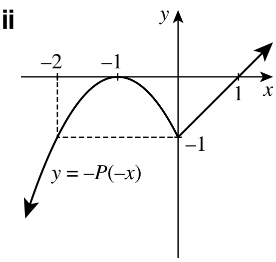
ii



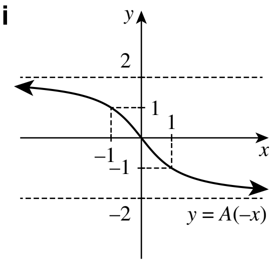
b i



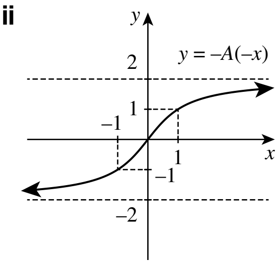
ii



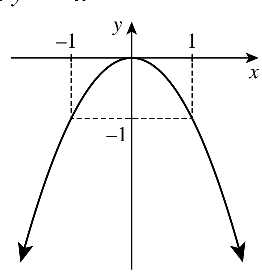
c i



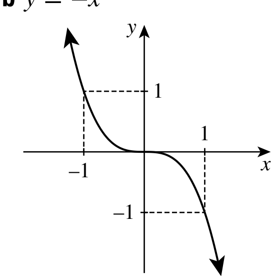
ii



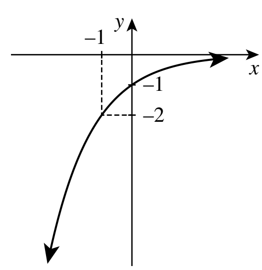
6a $y = -x^2$



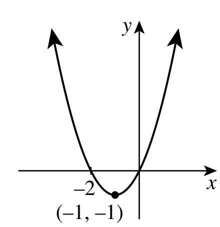
b $y = -x^3$



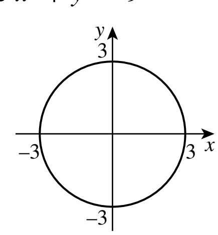
c $y = -2^{-x}$



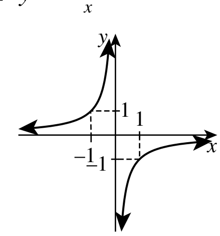
d $y = x^2 + 2x$



e $x^2 + y^2 = 9$



f $y = -\frac{1}{x}$



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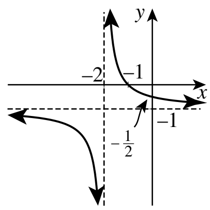
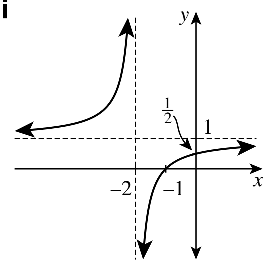
7a

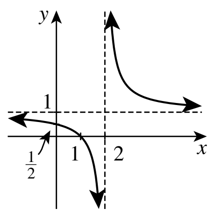


b i



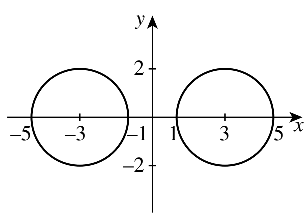
Reflect in the x-axis:
$y = 1 - \frac{1}{x + 2}$

ii



Rotate by180°: $y = 1 - \frac{1}{2 - x}$

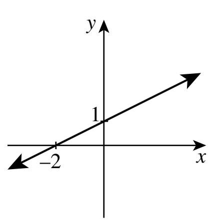
8a



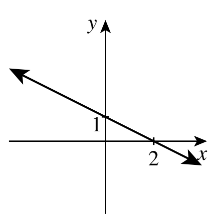
b Reflect in the y-axis.
c You will need to use $(-x - 3)^2 = (x + 3)^2$.

9a Answers will vary
b The circle is symmetric in both axes.

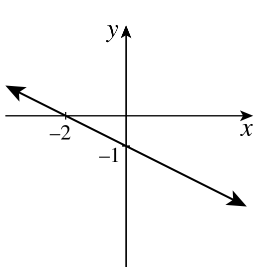
10a From $y = \frac{1}{2}x + 1$:



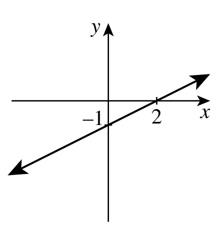
i reflect in the y-axis



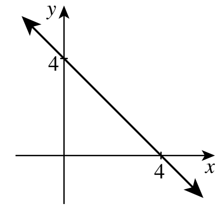
ii reflect in the x-axis



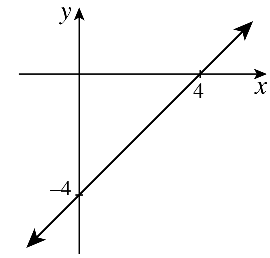
iii rotate by 180°



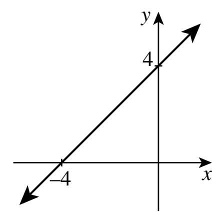
b From $y = 4 - x$:



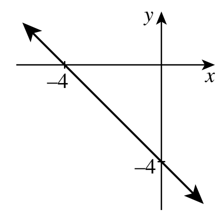
i reflect in the x-axis



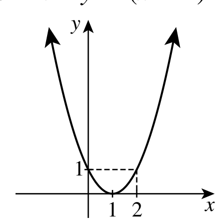
ii reflect in the y-axis



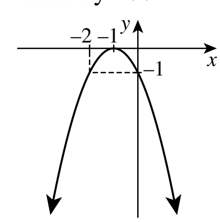
iii rotate by 180°



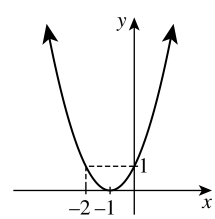
c From $y = (x - 1)^2$:



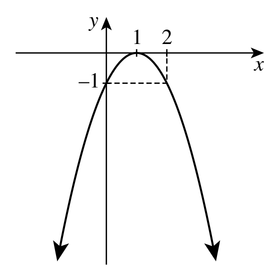
i rotate by 180°



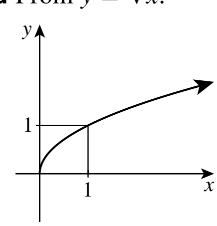
ii reflect in the y-axis



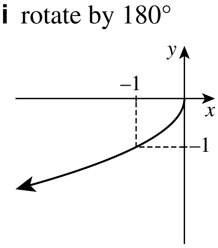
iii reflect in the x-axis



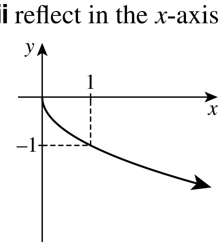
d From $y = \sqrt{x}$:



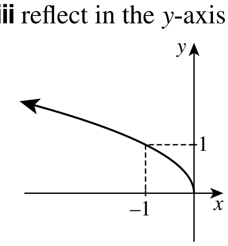
i rotate by 180°



ii reflect in the x-axis



iii reflect in the y-axis



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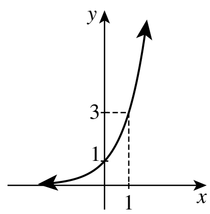
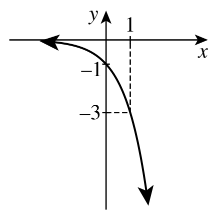
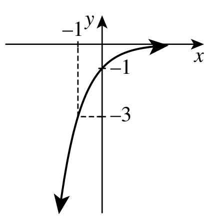
e From $y = 3^x$:



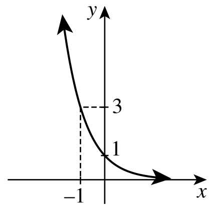
i reflect in the x-axis



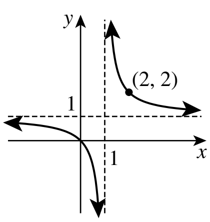
ii rotate by 180°



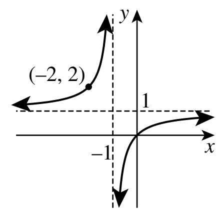
iii reflect in the y-axis



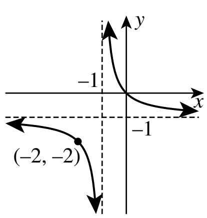
f From $y = 1 + \frac{1}{x-1}$:



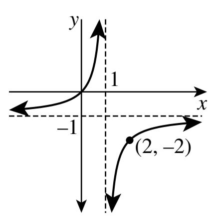
i reflect in the y-axis



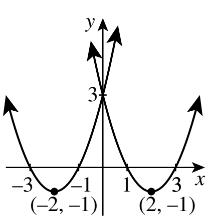
ii rotate by 180°



iii reflect in the x-axis



11a



b Reflect in the y-axis. **c** Shift left 4 units.

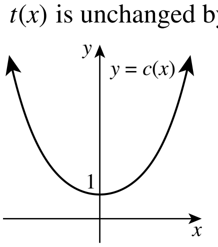
d $(x + 4)^2 - 4(x + 4) + 3 = x^2 + 4x + 3$

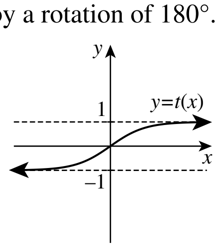
e a iii, b iii, c ii, f ii

12a $c(x)$ is the same when reflected in the y-axis.

b $t(x)$ is unchanged by a rotation of 180°.

c





13a i $y = (x - 2)^2$ **ii** $y = (x + 2)^2$

b i $y = (x + 1)^2$ **ii** $y = x^2$

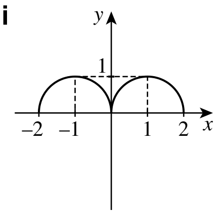
c Yes: the answer depends on the order.

d The order is irrelevant when the shift is parallel with the axis of reflection.

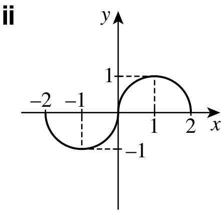
Exercise 4C

1a even **b** neither **c** neither **d** odd **e** odd **f** even

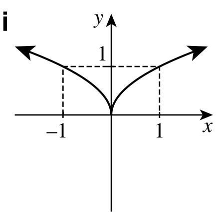
2a i



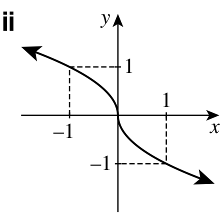
ii



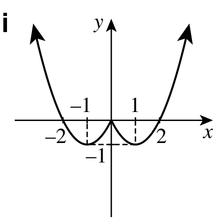
b i



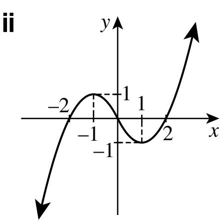
ii



c i



ii



3a $f(-x) = x^4 - 2x^2 + 1$

b $f(-x) = f(x)$, so it is even.

4a $g(-x) = -x^3 + 3x$

b $-g(x) = -(x^3 - 3x) = g(-x)$, so it is odd.

5a $h(-x) = -x^3 + 3x^2 - 2$

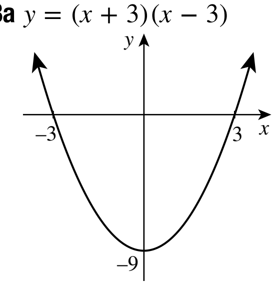
b $-h(x) = -x^3 - 3x^2 + 2$. Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, it is neither.

6a even **b** neither **c** odd **d** even **e** neither **f** odd **g** odd **h** neither

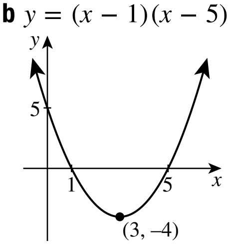
7a … if all powers of x are odd.

b … if all powers of x are even.

8a $y = (x + 3)(x - 3)$



b $y = (x - 1)(x - 5)$



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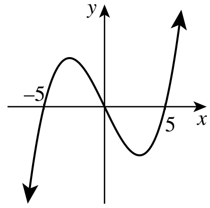
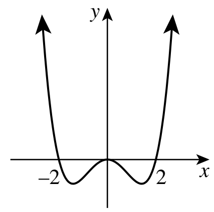
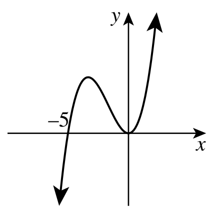
c $y = x(x - 5)(x + 5)$



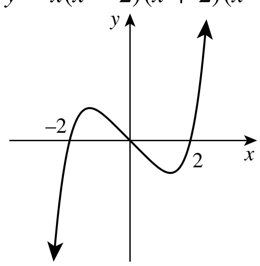
d $y = x^2(x - 2)(x + 2)$



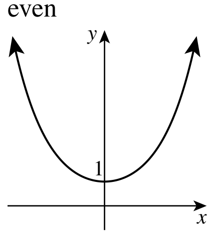
e $y = x^2(x + 5)$



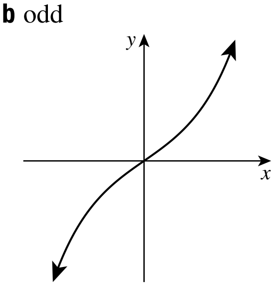
f $y = x(x - 2)(x + 2)(x^2 + 4)$



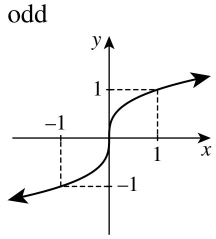
9a even



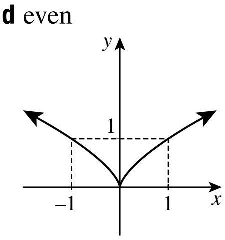
b odd



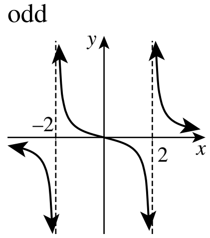
c odd



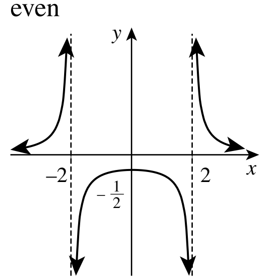
d even



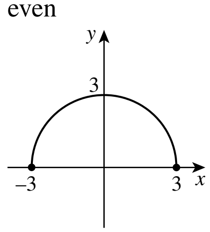
e odd



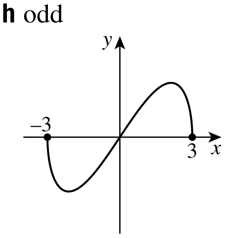
f even



g even



h odd



10a neither **b** neither **c** even **d** even **e** odd
f even **g** odd **h** neither

11a Either show that the equation is unchanged when x is replaced by $-x$. Or use the fact that the circle graph has line symmetry in the y-axis.

b Either show that the equation is unchanged when x is replaced by $-x$, and when x and y are replaced by $-x$ and $-y$. Or use the fact that that the circle graph has line symmetry in the x-axis and in the y-axis.

12a Suppose $f(0) = c$. Then since $f(x)$ is odd, $f(0) = -f(0) = -c$. So $c = -c$, and hence $c = 0$.

b No. A counter-example is $y = x^2 + 1$.

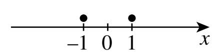
13a i–ii Answers will vary

b i–ii Answers will vary

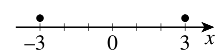
Exercise 4D

1a 3 **b** 3 **c** 3 **d** 3 **e** 7 **f** 1 **g** 16 **h** –3

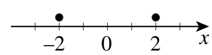
2a $x = 1$ or -1



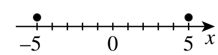
b $x = 3$ or -3



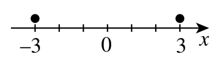
c $x = 2$ or -2



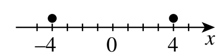
d $x = 5$ or -5



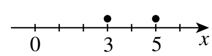
e $x = -3$ or 3



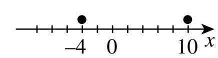
f $x = -4$ or 4



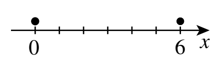
3a $x = 3$ or 5



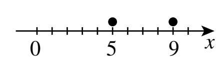
b $x = -4$ or 10



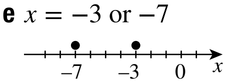
c $x = 0$ or 6



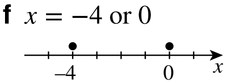
d $x = 5$ or 9



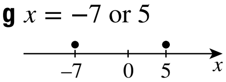
e $x = -3$ or -7



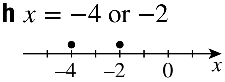
f $x = -4$ or 0



g $x = -7$ or 5

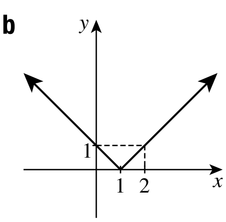


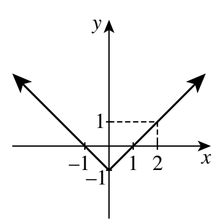
h $x = -4$ or -2



4a For $|x - 1|$: 3, 2, 1, 0, 1, 2
For $|x| - 1$: 1, 0, –1, 0, 1, 2

b





The two graphs overlap for $x > 1$.

c The first is $y = |x|$ shifted right 1 unit, the second is $y = |x|$ shifted down 1 unit.

5a LHS = RHS = 15 **b** LHS = RHS = 3
c LHS = RHS = 9 **d** LHS = RHS = 10
e $-3 < 3$ **f** $-3 \le -3$

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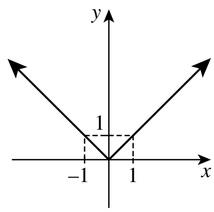
6a LHS = 2, RHS = −2 **b** LHS = 2, RHS = −2
c LHS = 0, RHS = 4 **d** LHS = 1, RHS = −1
e LHS = 3, RHS = 1 **f** LHS = 8, RHS = −8

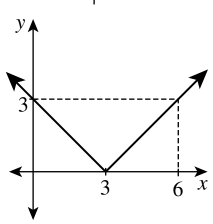
7a $x = 5$ or -5 **b** $x = -2$ or 1 **c** $x = 6$ or -5
d no solution **e** no solution **f** $x = -\frac{2}{5}$ **g** $x = \frac{5}{3}$
h $x = \frac{1}{3}$ or 2 **i** $x = -2$ or $\frac{2}{5}$

8a i $|1 - 2x| = |2x - 1|$ **ii** $x = -1$ or 2
b i $x = 1$ or 2 **ii** $x = -\frac{1}{3}$ or 1

9a



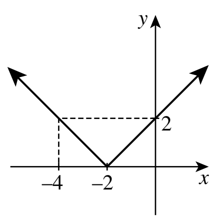
b i



shift right 3,

$$y = \begin{cases} x - 3, & \text{for } x \ge 3, \\ 3 - x, & \text{for } x < 3. \end{cases}$$

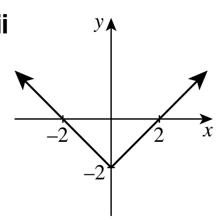
ii



shift left 2,

$$y = \begin{cases} x + 2, & \text{for } x \ge -2, \\ -x - 2, & \text{for } x < -2. \end{cases}$$

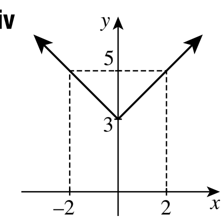
iii



shift down 2,

$$y = \begin{cases} x - 2, & \text{for } x \ge 0, \\ -x - 2, & \text{for } x < 0. \end{cases}$$

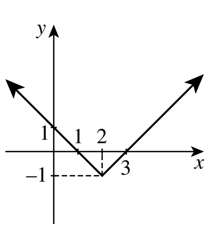
iv



shift up 3,

$$y = \begin{cases} x + 3, & \text{for } x \ge 0, \\ 3 - x, & \text{for } x < 0. \end{cases}$$

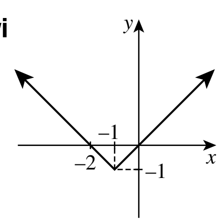
v



shift right 2, down 1,

$$y = \begin{cases} x - 3, & \text{for } x \ge 2, \\ 1 - x, & \text{for } x < 2. \end{cases}$$

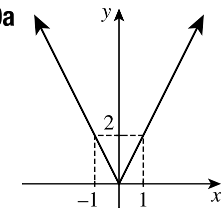
vi



shift left 1, down 1,

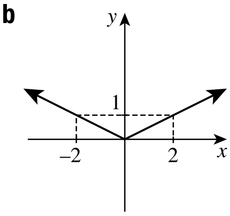
$$y = \begin{cases} x, & \text{for } x \ge -1, \\ -x - 2, & \text{for } x < -1. \end{cases}$$

10a



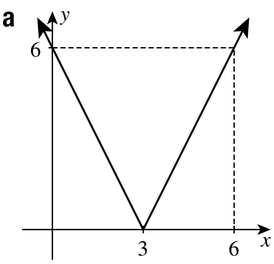
$$y = \begin{cases} 2x, & \text{for } x \ge 0, \\ -2x, & \text{for } x < 0. \end{cases}$$

b

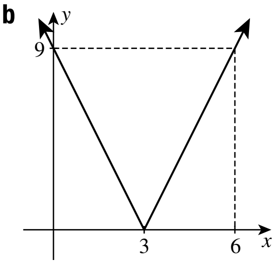


$$y = \begin{cases} \frac{1}{2}x, & \text{for } x \ge 0, \\ -\frac{1}{2}x, & \text{for } x < 0. \end{cases}$$

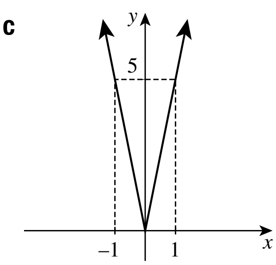
11a



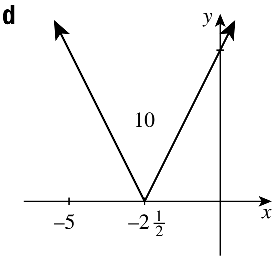
b



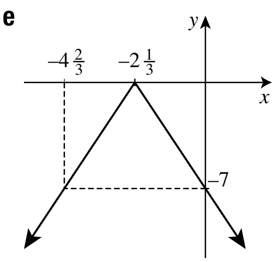
c



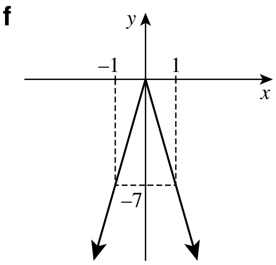
d



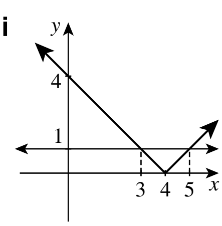
e



f

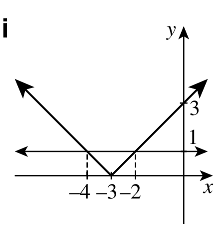


12a i



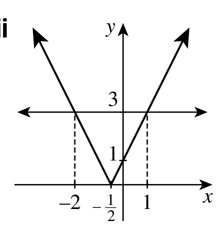
ii The x-coordinates of the points of intersection give: $x = 3$ or 5

b i



$x = -4$ or -2

ii



$x = -2$ or 1

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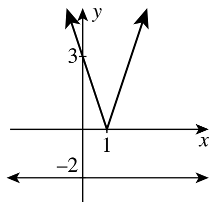
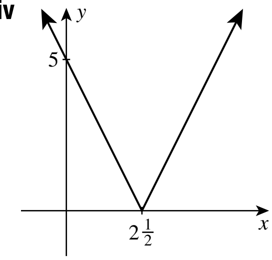
iii



no solution

iv



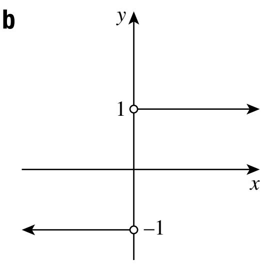
$x = 2\frac{1}{2}$

13a Answers will vary

b The graph is symmetric in the y-axis.

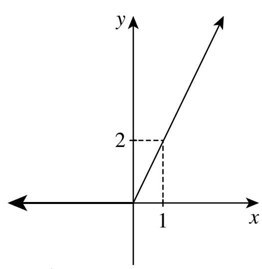
14a $x = 0$

b



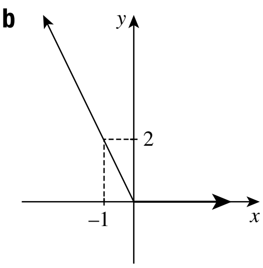
c $y = \begin{cases} 1, & \text{for } x \ge 0, \\ -1, & \text{for } x < 0. \end{cases}$

15a



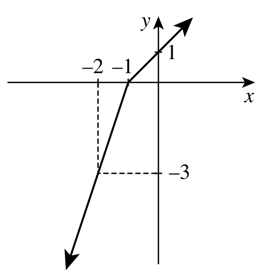
$y = \begin{cases} 2x, & \text{for } x \ge 0, \\ 0, & \text{for } x < 0. \end{cases}$

b



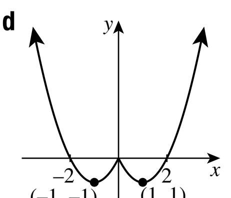
$y = \begin{cases} 0, & \text{for } x \ge 0, \\ -2x, & \text{for } x < 0. \end{cases}$

c



$y = \begin{cases} x + 1, & \text{for } x \ge -1, \\ 3x + 3, & \text{for } x < -1. \end{cases}$

d



$y = \begin{cases} x^2 - 2x, & \text{for } x \ge 0, \\ x^2 + 2x, & \text{for } x < 0. \end{cases}$

Exercise 4E

1a i 4 **ii** 7 **iii** 3 **iv** −4 **b i** $x + 4$ **ii** $x + 6$ **c** $x = -4$

2a $F(F(0)) = 0$, $F(F(7)) = 28$, $F(F(-3)) = -12$, $F(F(-11)) = -44$

b $F(F(x)) = 4x$, $F(F(F(x))) = 8x$ **c** $x = 8$

3a $g(g(0)) = 0$, $g(g(4)) = 4$, $g(g(-2)) = -2$, $g(g(-9)) = -9$

b $g(g(x)) = 2 - (2 - x) = x$

c $g(g(g(x))) = g(x)$

4a $h(h(0)) = -20$, $h(h(5)) = 25$, $h(h(-1)) = -29$, $h(h(-5)) = -65$

b $h(h(x)) = 9x - 20$, $h(h(h(x))) = 27x - 65$

5a $f(g(7)) = 12$, $g(f(7)) = 13$, $f(f(7)) = 9$, $g(g(7)) = 19$

b i $2x - 2$ **ii** $2x - 1$ **iii** $x + 2$ **iv** $4x - 9$

c Shift 1 unit to the left (or shift two units up).

d Shift 1 unit up (or shift $\frac{1}{2}$ left).

6a $\ell(q(-1)) = -2$, $q(\ell(-1)) = 16$, $\ell(\ell(-1)) = -7$, $q(q(-1)) = 1$

b i $x^2 - 3$ **ii** $(x - 3)^2$ **iii** $x - 6$ **iv** x^4

c i Domain: all real x, range: $y \ge -3$

ii Domain: all real x, range: $y \ge 0$

d It is shifted 3 units to the right.

e It is shifted 3 units down.

7a $F(G(25)) = 20$, $G(F(25)) = 10$, $F(F(25)) = 400$, $G(G(25)) = \sqrt{5}$

b $4\sqrt{x}$ **c** $\sqrt{4x} = 2\sqrt{x}$

d Answers will vary **e** Domain: $x \ge 0$, range: $y \ge 0$

8a $f\left(h\left(-\frac{1}{4}\right)\right) = 4$, $h\left(f\left(-\frac{1}{4}\right)\right) = 4$, $f\left(f\left(-\frac{1}{4}\right)\right) = -\frac{1}{4}$, $h\left(h\left(-\frac{1}{4}\right)\right) = -\frac{1}{4}$

b i Both sides equal $-\frac{1}{x}$, for all $x \ne 0$.

ii Both sides equal x, for all $x \ne 0$.

c Domain: $x \ne 0$, range $y \ne 0$.

d It is reflected in the y-axis (or in the x-axis).

9a $f(g(x)) = -5 - \sqrt{x}$.
Domain: $x \ge 0$, range: $y \le -5$. Take the graph of $y = \sqrt{x}$, reflect it in the y-axis, then shift down 5.

b $f(x) = -5 - |x|$, which is negative for all x, so $g(f(x)) = \sqrt{-5 - |x|}$ is never defined.

10a $g(f(-x)) = g(-f(x)) = -g(f(x))$

b $g(f(-x)) = g(-f(x)) = g(f(x))$

c $g(f(-x)) = g(f(x))$

11a $g(f(x)) = 7$ for all x, $f(g(x)) = 4$ for all x

b $g(f(x)) = g(x)$, $f(g(x)) = g(x)$

12a $g(f(x)) = 10x + 15 + b$, $f(g(x)) = 10x + 2b + 3$

b $b = 12$

13a i Translation down a **ii** Translation right a

b i Reflection in the x-axis **ii** Reflection in the y-axis

14a $g(f(x)) = 2ax + 3a + b$, $f(g(x)) = 2ax + 2b + 3$

b First, $2a = 1$, so $a = \frac{1}{2}$. Secondly, $3a + b = 0$, so $b = -1\frac{1}{2}$.

c Answers will vary

15a $f(g(0)) = -3$, $g(f(0)) = 3$, $f(g(-2)) = 3$, $g(f(-2)) = 1$

b i $x^2 + x - 3$ **ii** $x^2 - x - 3$

16a All real y and $y \ge -1$.

b $x^2 + 2x + 1 = (x + 1)^2$, Range: $y \ge 0$

c $x^2 + 4x + 3 = (x + 1)(x + 3)$, Range: $y \ge -1$

d −1 and −3.

e Answers will vary

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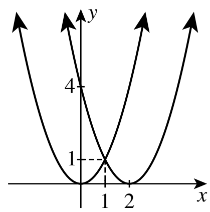
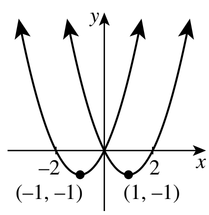
Chapter 4 review exercise

1a x^2: 4, 1, 0, 1, 4, 9, 16
$(x-2)^2$: 16, 9, 4, 1, 0, 1, 4
b $y = x^2$, $V = (0, 0)$
$y = (x-2)^2$, $V = (2, 0)$
c Here x is replaced by $(x - 2)$, so it is a shift right by 2 units.

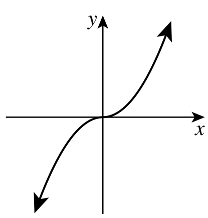


2a Replace x with $-x$.
b $y = x^2 - 2x$:
15, 8, 3, 0, −1, 0, 3
$y = x^2 + 2x$:
3, 0, −1, 0, 3, 8, 15
c $y = x^2 - 2x$: $(1, -1)$
$y = x^2 + 2x$: $(-1, -1)$

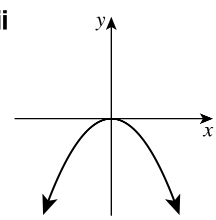


3a 7 **b** 4 **c** 5 **d** 3 **e** −3 **f** 12
4a $x = -5$ or 5 **b** $x = -6$ or 6 **c** $x = -2$ or 6
d $x = -5$ or -1 **e** $x = -1$ or 4 **f** $x = -1$ or $3\frac{2}{3}$
5a Shift $y = x^2$ up by 5 units.
b Shift $y = x^2$ down by 1 unit.
c Shift $y = x^2$ right by 3 units.
d Shift $y = x^2$ left by 4 units and up by 7 units.
6a $y = (x-1)^2$ **b** $y = x^2 - 2$
c $y = (x+1)^2 + 5$ **d** $y = (x-4)^2 - 9$
7a $C(0, 0)$, $r = 1$ **b** $C(-1, 0)$, $r = 2$
c $C(2, -3)$, $r = \sqrt{5}$ **d** $C(0, 4)$, $r = 8$
8a $y = -x^3 + 2x + 1$ **b** $y = -x^2 + 3x + 4$
c $y = -2^{-x} - x$ **d** $y = \sqrt{9 - x^2}$
9a neither **b** odd **c** even

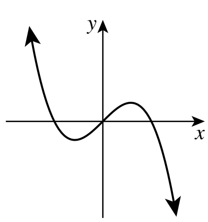
10a i



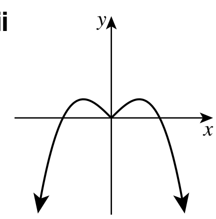
ii



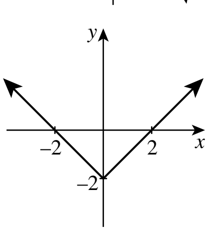
b i



ii



11a



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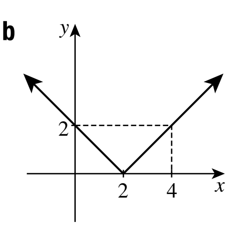
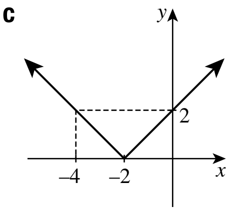
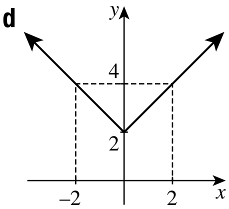


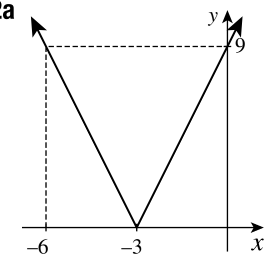
c



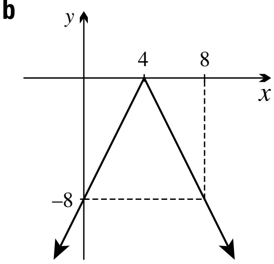
d



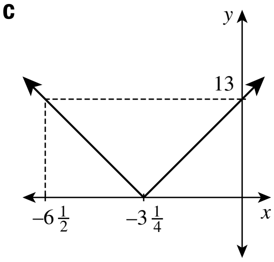
12a



b



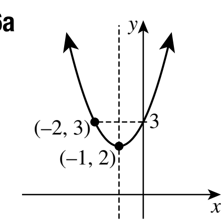
c



13a 5 or −5 **b** 1 or −9 **c** no solutions **d** 12 or −2
e 1 or −8 **f** 4 or $\frac{4}{3}$ **g** $\frac{-2}{7}$ **h** 5 or −5

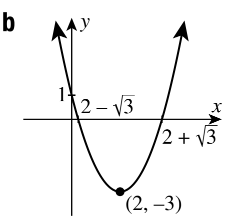
14a neither **b** even **c** odd **d** odd
15a $y = (x-1)^2 + 4$, $V = (1, 4)$
b $y = (x+2)^2 - 7$, $V = (-2, -7)$
c $y = 2(x+2)^2 + 3$, $V = (-2, 3)$
d $y = -(x-3)^2 + 10$, $V = (3, 10)$

16a



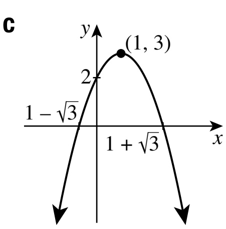
$y = (x+1)^2 + 2$

b



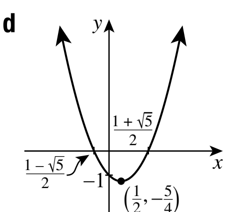
$y = (x-2)^2 - 3$

c



$y = 3 - (x-1)^2$

d



$y = \left(x - \frac{1}{2}\right)^2 - \frac{5}{4}$

17a $C(0, 1)$, $r = 2$ **b** $C(-3, 0)$, $r = 1$
c $C(2, -3)$, $r = 4$ **d** $C(4, -7)$, $r = 10$

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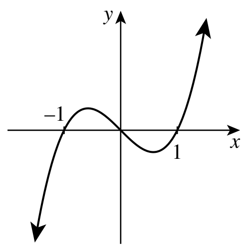
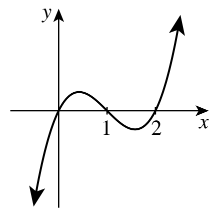
18a Answers will vary

b



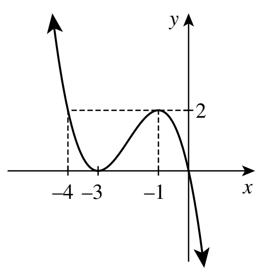


19a 13 **b** 7 **c** 93 **d** 327 **e** $5a^2 + 13$ **f** $25a^2 - 20a + 7$

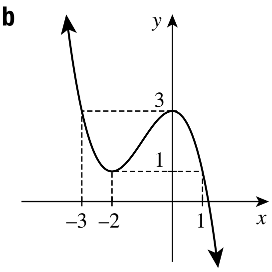
20a $f(g(x))$ has domain $x \geq 0$ and range $y \geq -1$, $g(f(x))$ has domain $x \geq 1$ and range $y \geq 0$.

b $g((x))$ has domain all real x and range $0 < y \leq 1$, $g(f(x))$ has domain all real x, $x \neq 0$ and range $y > 1$.

21a



b



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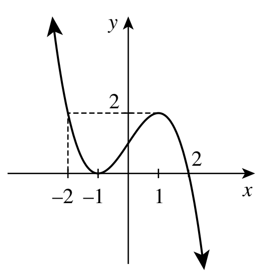

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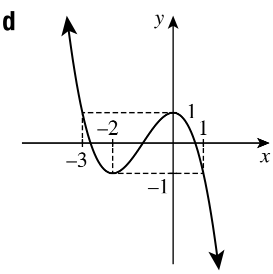

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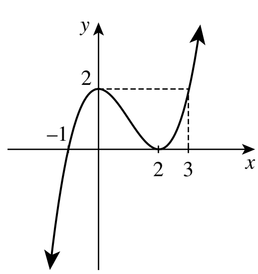

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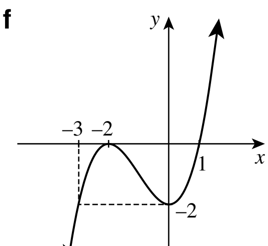

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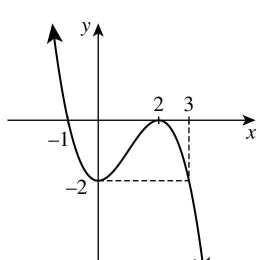

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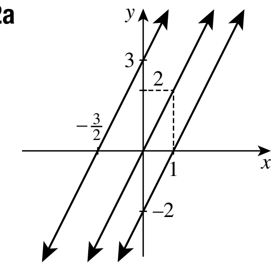

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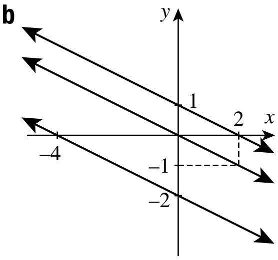

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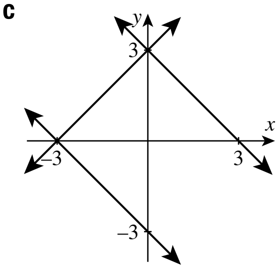

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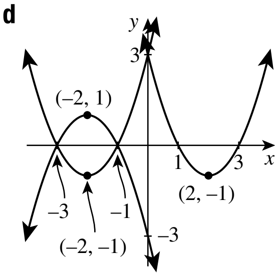

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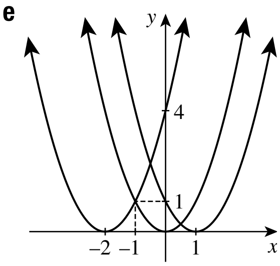

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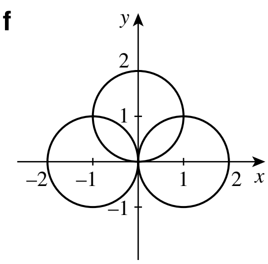

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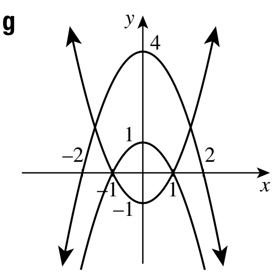

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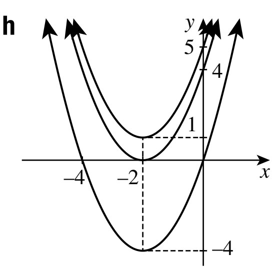

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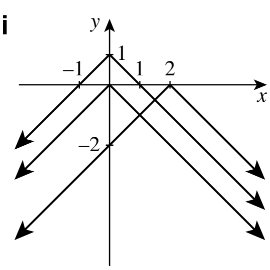

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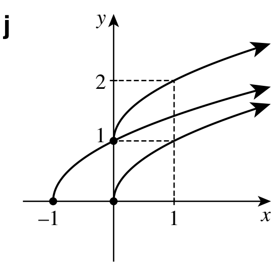

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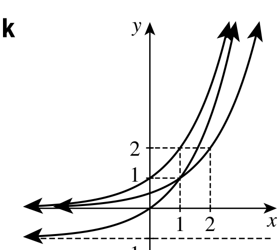

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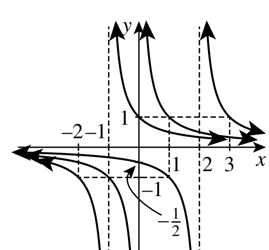

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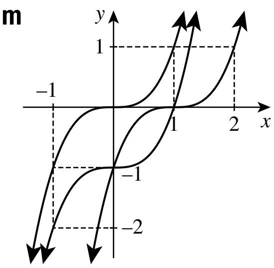
m



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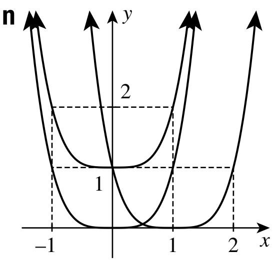

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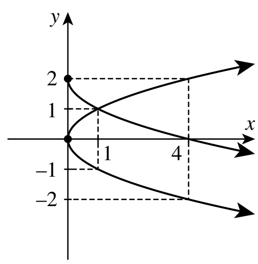

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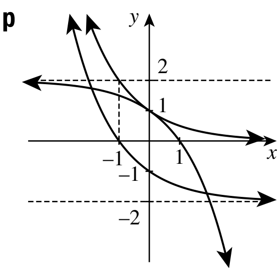


Chapter 5

Exercise 5A

1a $\frac{3}{5}$ **b** $\frac{3}{4}$ **c** $\frac{4}{5}$ **d** $\frac{4}{5}$ **e** $\frac{3}{5}$ **f** $\frac{4}{3}$

2a 0.4067 **b** 0.4848 **c** 0.7002 **d** 0.9986
e 0.0349 **f** 0.8387 **g** 0.0175 **h** 0.9986

3a 1.5697 **b** 0.8443 **c** 4.9894 **d** 0.9571
e 0.6833 **f** 0.1016 **g** 0.0023 **h** 0.0166

4a 76° **b** 46° **c** 12° **d** 27°
e No such angle — cos θ cannot exceed 1. **f** 39° **g** 60°
h No such angle — sin θ cannot exceed 1.

5a 41°25′ **b** 63°26′ **c** 5°44′ **d** 16°42′ **e** 46°29′ **f** 57°25′

6a 13 **b** 19 **c** 23 **d** 88

7a 53° **b** 41° **c** 67° **d** 59°

8a $\frac{12}{13}$ **b** $\frac{5}{12}$ **c** $\frac{13}{12}$ **d** $\frac{5}{12}$ **e** $\frac{13}{12}$ **f** $\frac{13}{5}$

9a 6 and 17 **b i** $\frac{15}{17}$ **ii** $\frac{4}{5}$ **iii** $\frac{3}{4}$ **iv** $\frac{17}{8}$ **v** $\frac{5}{3}$ **vi** $\frac{15}{8}$

10a $\frac{\sqrt{3}}{2}$ **b** $\frac{1}{\sqrt{3}}$ **c** $\frac{1}{\sqrt{2}}$ **d** 2 **e** $\sqrt{2}$ **f** $\sqrt{3}$

11a 19.2 **b** 21.6 **c** 30.3 **d** 8.3

12a 29.78 **b** 10.14 **c** 16.46 **d** 29.71

13a 36°2′ **b** 68°38′ **c** 34°44′ **d** 38°40′ **e** 54°19′ **f** 70°32′

14a Answers will vary **b** Answers will vary

15a Answers will vary **b** 3 **c** $\frac{1}{3}\sqrt{5}$, $\frac{2}{3}$ **d** Answers will vary

16a i $\frac{1}{2}\sqrt{22}$ **ii** $\frac{3}{2}\sqrt{2}$ **b** Answers will vary

17a 1 **b** $\frac{1}{2}$ **c** 4 **d** 1

18a–d Answers will vary

Exercise 5B

1 2.65 m
2 63°
3 55 km
4 038° T
5 13.2 m
6 2.5 m
7 77 km
8 23 m
9 73°
10 21.3 m
11 11°
12a 46° **b** 101° T
13a Answers will vary **b** 67 km
14a $\angle PQR = 360° - (200° + 70°) = 90°$
(using co-interior angles on parallel lines and the fact that a revolution is 360°)
b $110° + 39° = 149°$ T
15a 5.1 cm **b** 16 cm **c** $PQ = 18 \sin 40°$, 63°25′
16a–c Answers will vary
17 457 m

Exercise 5C

1a 15 cm **b** 17 cm **c** 28°

2a i 90° **ii** 90° **iii** 90°
b i $\sqrt{2}$ **ii** $\sqrt{3}$
c i 35° **ii** 35°

3a i $2\sqrt{5}$ cm **ii** $2\sqrt{6}$ cm **b** 90° **c** 66°

4a i 90° **ii** 90° **iii** 90°
b i 2 cm **ii** $2\sqrt{2}$ cm
c i 72° **ii** 65°

5a i 90° **ii** 90° **b** 27°

6a $3\sqrt{2}$ cm **b** 43°

7a $BQ = 30 \tan 72°$ **b** 145 m

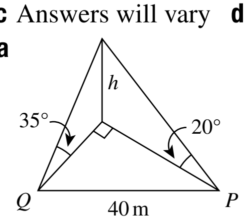
8a Answers will vary **b** 16 m **c** 21°

9a Answers will vary **b** 76 m **c** 14°

10a 1 cm **b** $\sqrt{2}$ cm **c** $\sqrt{2}$ **d** 70°32′

11a $h \cot 55°$ **b** It is the angle between south and east.
c Answers will vary **d** 114 m

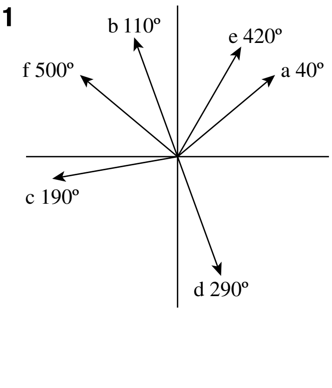
12a



b 13 metres

Exercise 5D

1



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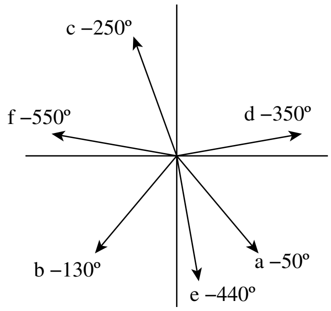
2



3a $-320°$ **b** $-250°$ **c** $-170°$ **d** $-70°$
e $-300°$ **f** $-220°$

4a $310°$ **b** $230°$ **c** $110°$ **d** $10°$ **e** $280°$ **f** $170°$

5a $70°, 430°, -290°, -650°$
b $100°, 460°, -260°, -620°$
c $140°, 500°, -220°, -580°$
d $200°, 560°, -160°, -520°$
e $240°, 600°, -120°, -480°$
f $340°, 700°, -20°, -380°$

6a $\sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}, \tan\theta = \frac{4}{3},$
$\operatorname{cosec}\theta = \frac{5}{4}, \sec\theta = \frac{5}{3}, \cot\theta = \frac{3}{4}$
b $\sin\theta = \frac{3}{5}, \cos\theta = -\frac{4}{5}, \tan\theta = -\frac{3}{4},$
$\operatorname{cosec}\theta = \frac{5}{3}, \sec\theta = -\frac{5}{4}, \cot\theta = -\frac{4}{3}$
c $\sin\theta = -\frac{2}{\sqrt{5}}, \cos\theta = -\frac{1}{\sqrt{5}}, \tan\theta = 2,$
$\operatorname{cosec}\theta = -\frac{\sqrt{5}}{2}, \sec\theta = -\sqrt{5}, \cot\theta = \frac{1}{2}$
d $\sin\theta = -\frac{5}{13}, \cos\theta = \frac{12}{13}, \tan\theta = -\frac{5}{12},$
$\operatorname{cosec}\theta = -\frac{13}{5}, \sec\theta = \frac{13}{12}, \cot\theta = -\frac{12}{5}$

7a–c All six trigonometric functions are sketched in Section 5E.

8a i 0.5 **ii** −0.5 **iii** 0.95 **iv** 0.95 **v** 0.59 **vi** 0.81
vii −0.89 **viii** 0.45 **ix** −0.81 **x** 0.59
b i $30°, 150°$ **ii** $120°, 240°$ **iii** $64°, 116°$
iv $53°, 307°$ **v** $53°, 127°$ **vi** $143°, 217°$
vii $204°, 336°$ **viii** $107°, 253°$
c $45°, 225°$

Exercise 5E

1a + **b** + **c** − **d** − **e** + **f** − **g** − **h** + **i** − **j** +
k − **l** − **m** − **n** + **o** + **p** −

2a $10°$ **b** $30°$ **c** $50°$ **d** $20°$ **e** $80°$ **f** $70°$
g $70°$ **h** $80°$ **i** $10°$ **j** $20°$

3a $-\tan 50°$ **b** $\cos 50°$ **c** $-\sin 40°$ **d** $\tan 80°$
e $-\cos 10°$ **f** $-\sin 40°$ **g** $-\cos 5°$ **h** $\sin 55°$
i $-\tan 35°$ **j** $\sin 85°$ **k** $-\cos 85°$ **l** $\tan 25°$

4a 0 **b** −1 **c** 0 **d** 0 **e** 1 **f** 1 **g** −1 **h** undefined
i 0 **j** 0 **k** undefined **l** 0

5a $\frac{\sqrt{3}}{2}$ **b** $\frac{\sqrt{3}}{2}$ **c** $-\frac{\sqrt{3}}{2}$ **d** $-\frac{\sqrt{3}}{2}$ **e** $\frac{1}{\sqrt{2}}$ **f** $-\frac{1}{\sqrt{2}}$
g $-\frac{1}{\sqrt{2}}$ **h** $\frac{1}{\sqrt{2}}$ **i** $\frac{1}{\sqrt{3}}$ **j** $-\frac{1}{\sqrt{3}}$ **k** $\frac{1}{\sqrt{3}}$ **l** $-\frac{1}{\sqrt{3}}$

6a $-\frac{1}{2}$ **b** 1 **c** $-\frac{1}{2}$ **d** $\frac{1}{\sqrt{2}}$ **e** $\sqrt{3}$ **f** $-\frac{\sqrt{3}}{2}$ **g** −1 **h** $\frac{1}{2}$
i $-\frac{1}{\sqrt{2}}$ **j** $-\frac{\sqrt{3}}{2}$ **k** $-\frac{1}{2}$ **l** $-\sqrt{3}$

7a 2 **b** $-\sqrt{2}$ **c** $-\frac{1}{\sqrt{3}}$ **d** $\sqrt{3}$ **e** $\frac{2}{\sqrt{3}}$ **f** $-\frac{2}{\sqrt{3}}$

8a 1 **b** −1 **c** undefined **d** undefined **e** 0 **f** undefined

9a $60°$ **b** $20°$ **c** $30°$ **d** $60°$ **e** $70°$ **f** $10°$
g $50°$ **h** $40°$

10a $\frac{1}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $\sqrt{3}$ **d** $\frac{1}{\sqrt{2}}$ **e** $-\frac{1}{\sqrt{3}}$ **f** $-\frac{1}{\sqrt{2}}$ **g** $\sqrt{3}$
h $-\frac{\sqrt{3}}{2}$ **i** $\frac{1}{\sqrt{2}}$ **j** $-\frac{1}{2}$ **k** $-\frac{1}{2}$ **l** 1

11 All six graphs are many-to-one.

12a 0.42 **b** −0.91 **c** 0.91 **d** −0.42
e 0.49 **f** 0.49

13a −0.70 **b** −1.22 **c** −0.70 **d** −0.52
e 1.92 **f** −0.52

14a–c Answers will vary

15a $-\sin\theta$ **b** $\cos\theta$ **c** $-\tan\theta$ **d** $\sec\theta$ **e** $\sin\theta$
f $-\sin\theta$ **g** $-\cos\theta$ **h** $\tan\theta$

Exercise 5F

1a $\sin\theta = \frac{15}{17}, \cos\theta = \frac{8}{17}, \tan\theta = \frac{15}{8}$
b $\sin\theta = \frac{4}{5}, \cos\theta = -\frac{3}{5}, \tan\theta = -\frac{4}{3}$
c $\sin\theta = -\frac{7}{25}, \cos\theta = -\frac{24}{25}, \tan\theta = \frac{7}{24}$
d $\sin\theta = -\frac{21}{29}, \cos\theta = \frac{20}{29}, \tan\theta = -\frac{21}{20}$

2a $y = 12, \sin\alpha = \frac{12}{13}, \cos\alpha = \frac{5}{13}, \tan\alpha = \frac{12}{5}$
b $r = 3, \sin\alpha = \frac{2}{3}, \cos\alpha = -\frac{\sqrt{5}}{3}, \tan\alpha = -\frac{2}{\sqrt{5}}$
c $x = -4, \sin\alpha = -\frac{3}{5}, \cos\alpha = -\frac{4}{5}, \tan\alpha = \frac{3}{4}$
d $y = -3, \sin\alpha = -\frac{3}{\sqrt{13}}, \cos\alpha = \frac{2}{\sqrt{13}}, \tan\alpha = -\frac{3}{2}$

3a i $\sin\theta = -\frac{4}{5}$ **ii** $\tan\theta = -\frac{4}{3}$
b i $\sin\theta = \frac{5}{13}$ **ii** $\cos\theta = -\frac{12}{13}$

4a i $\cos\theta = -\frac{3}{4}$ **ii** $\tan\theta = \frac{\sqrt{7}}{3}$,
or $\cos\theta = \frac{3}{4}$ or $\tan\theta = -\frac{\sqrt{7}}{3}$
b i $\sin\theta = \frac{\sqrt{15}}{4}$ **ii** $\tan\theta = -\sqrt{15}$,
or $\sin\theta = -\frac{\sqrt{15}}{4}$ or $\tan\theta = \sqrt{15}$

5a $2\sqrt{2}$ **b** $-\frac{3}{4}$ **c** $-\frac{\sqrt{3}}{2}$ **d** $\frac{3}{\sqrt{13}}$ **e** $\frac{9}{41}$ **f** $\frac{1}{2}$

6a $\frac{1}{\sqrt{10}}$ or $-\frac{1}{\sqrt{10}}$ **b** $\frac{1}{\sqrt{5}}$ or $-\frac{1}{\sqrt{5}}$ **c** $\frac{4}{5}$ or $-\frac{4}{5}$
d $\frac{\sqrt{5}}{2}$ or $-\frac{\sqrt{5}}{2}$ **e** $\frac{12}{5}$ or $-\frac{12}{5}$ **f** $\frac{\sqrt{3}}{\sqrt{7}}$ or $-\frac{\sqrt{3}}{\sqrt{7}}$

7a $-\frac{3}{4}$ **b** $-\frac{15}{17}$ **c** $-\frac{\sqrt{15}}{4}$ **d** $\frac{35}{37}$ **e** $-\frac{21}{20}$ **f** $\frac{\sqrt{11}}{6}$

8a $\sqrt{2}$ or $-\sqrt{2}$ **b** $\frac{15}{8}$ or $-\frac{15}{8}$ **c** $\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$ **d** $\frac{6}{5}$ or $-\frac{6}{5}$

9a $-\frac{3}{\sqrt{5}}$ **b** $-\frac{41}{9}$ **c** $-\frac{15}{8}$ **d** $\frac{\sqrt{7}}{\sqrt{3}}$

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10a $\frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$ **b** $-\frac{3}{2\sqrt{10}} = -\frac{3\sqrt{10}}{20}$ **c** 1 **d** $\frac{12}{13}$

11 $\cos\theta = -\frac{\sqrt{q^2 - p^2}}{q}$, $\tan\theta = -\frac{p}{\sqrt{q^2 - p^2}}$

12 $\sin\alpha = \frac{k}{\sqrt{1 + k^2}}$ or $-\frac{k}{\sqrt{1 + k^2}}$,
$\sec\alpha = \sqrt{1 + k^2}$ or $-\sqrt{1 + k^2}$

13a Answers will vary **b** $\sin x = \frac{2t}{1 + t^2}$, $\tan x = \frac{2t}{1 - t^2}$

Exercise 5G

1a–f Answer is in question
2a $\operatorname{cosec}\theta$ **b** $\cot\alpha$ **c** $\tan\beta$ **d** $\cot\phi$
3a 1 **b** 1 **c** 1
4a–c Answers will vary
5a $\cos\theta$ **b** $\operatorname{cosec}\alpha$ **c** $\cot\beta$ **d** $\tan\phi$
6a 1 **b** $\sin^2\beta$ **c** $\sec^2\phi$ **d** 1
7a $\cos^2\beta$ **b** $\operatorname{cosec}^2\phi$ **c** $\cot^2 A$ **d** -1
8a $\cos^2\theta$ **b** $\tan^2\beta$ **c** $\cot^2 A$ **d** 1
9a–c Answers will vary
10a $\cos^2\alpha$ **b** $\sin^2\alpha$ **c** $\sin A$ **d** $\cos A$
11a–j Answers will vary
12a–f Answers will vary
13a–h Answers will vary

Exercise 5H

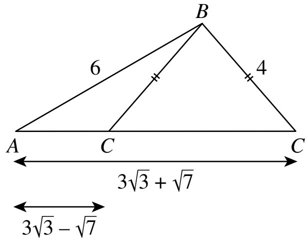
1a $\theta = 60°$ or $120°$ **b** $\theta = 30°$ or $150°$
c $\theta = 45°$ or $225°$ **d** $\theta = 60°$ or $240°$
e $\theta = 135°$ or $225°$ **f** $\theta = 120°$ or $300°$
g $\theta = 210°$ or $330°$ **h** $\theta = 150°$ or $210°$
2a $\theta = 90°$ **b** $\theta = 0°$ or $360°$ **c** $\theta = 90°$ or $270°$
d $\theta = 180°$ **e** $\theta = 0°$ or $180°$ or $360°$ **f** $\theta = 270°$
3a $x \doteqdot 65°$ or $295°$ **b** $x \doteqdot 7°$ or $173°$ **c** $x \doteqdot 82°$ or $262°$
d $x \doteqdot 222°$ or $318°$ **e** $x \doteqdot 114°$ or $294°$
f $x \doteqdot 140°$ or $220°$
4a $\alpha \doteqdot 5°44'$ or $174°16'$ **b** $\alpha \doteqdot 95°44'$ or $264°16'$
c $\alpha = 135°$ or $315°$ **d** $\alpha = 270°$ **e** no solutions
f $\alpha = 120°$ or $240°$ **g** $\alpha = 150°$ or $330°$
h $\alpha \doteqdot 18°26'$ or $198°26'$
5a $x \doteqdot -16°42'$ or $163°18'$ **b** $x = 90°$ or $-90°$
c $x = 45°$ or $-45°$ **d** $x \doteqdot -135°34'$ or $-44°26'$
6a $\theta = 60°, 300°, 420°$ or $660°$
b $\theta = 90°, 270°, 450°$ or $630°$
c $\theta = 210°, 330°, 570°$ or $690°$
d $\theta = 22°30', 202°30', 382°30'$ or $562°30'$
7a $x = 15°, 75°, 195°$ or $255°$
b $x = 30°, 120°, 210°$ or $300°$
c $x = 67°30', 112°30', 247°30'$ or $292°30'$
d $x = 135°$ or $315°$
8a $\alpha = 75°$ or $255°$ **b** $\alpha = 210°$ or $270°$
c $\alpha = 300°$ **d** $\alpha = 210°$ or $300°$
9a $\theta = 45°$ or $225°$ **b** $\theta = 135°$ or $315°$
c $\theta = 60°$ or $240°$ **d** $\theta = 150°$ or $330°$

Exercise 5I

1a 8.2 **b** 4.4 **c** 4.9 **d** 1.9 **e** 9.2 **f** 3.5
2a 14.72 **b** 46.61 **c** 5.53
3a 49° **b** 53° **c** 43° **d** 20° **e** 29° **f** 42°
4a 5 cm² **b** 19 cm² **c** 22 cm²
5b $b \doteqdot 10.80$ cm, $c \doteqdot 6.46$ cm
6b 97 cm
7a $49°46'$ **b** $77°53'$ **c** 3.70 cm²
8 42°, 138°
9 62°, 118°
10a $69°2'$ or $110°58'$ **b** 16.0 cm or 11.0 cm
11 317 km
12a Answers will vary **b** 9 m
13a 32 **b** $\frac{5}{7}$
14a 16 m **b** 11.35 m **c** 3.48 m
15a 30° or 150° **b** $17°27'$ or $162°33'$
c No solutions, because $\sin\theta = 1.2$ is impossible.
16a $3\sqrt{6}$ **b** $3\sqrt{2}$ **c** $2\sqrt{6}$ **d** $6\sqrt{2}$
17 11.0 cm

Exercise 5J

1a 3.3 **b** 4.7 **c** 4.0 **d** 15.2 **e** 21.9 **f** 24.6
2a 39° **b** 56° **c** 76° **d** 94° **e** 117° **f** 128°
3a $\sqrt{13}$ **b** $\sqrt{7}$
4a $\sqrt{10}$ **b** $\sqrt{21}$
5a $44°25'$ **b** $101°32'$ **c** $\frac{7}{32}$
6 11.5 km
7 167 nautical miles
8 20°
9a $101°38'$ **b** $78°22'$
10 $13°10'$, 120°
11a Answers will vary **b** Answers will vary
12a 19 cm **b** $\frac{37}{38}$
13a $\angle DAP = \angle DPA = 60°$ (angle sum of isosceles triangle), so ΔADP is equilateral. Hence $AP = 3$ cm.
b $3\sqrt{7}$ cm **c** Answers will vary
14 3 or 5
15a Answers will vary **b** Answers will vary
c



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Exercise 5K

1a 28.3 **b** 17.3 **c** 12.5 **d** 36.2 **e** 12.6 **f** 23.2
2a 59° **b** 55° **c** 40° **d** 37° **e** 52° **f** 107°
3a 26 cm **b** 28 cm **c** 52° **d** 62°
4a Answers will vary **b** 28 m
5a $\angle ACP + 31° = 68°$ (exterior angle of ΔACP)
b Answers will vary **c** 6 cm
6a 11.6 cm **b** 49°
7a 44°25′ **b** 10 cm^2
8 Answer is in question
9a Answers will vary **b** 36 cm **c** Answers will vary
10a PQ is inclined at 26° to a north–south line through Q, because of alternate angles on parallel lines. Then $\angle PQR = 26° + 90°$.
b 112 nautical miles
11a 46°59′ or 133°1′ **b** 66.4 m or 52.7 m
12a $\angle PJK = \angle PBQ = 20°$ (corresponding angles on parallel lines), but $\angle PJK = \angle PAJ + \angle APJ$ (exterior angle of triangle), so $\angle APJ = 20° - 5° = 15°$.
b Answers will vary **c** Answers will vary **d** 53 m
13a 38 tan 68° **b** 111 m
14a Answers will vary **b** 131 m
15a Answers will vary **b** 108 km
c $\angle ACB \doteqdot 22°$, bearing $\doteqdot$ 138° T
16a Answers will vary **b** Answers will vary **c** 34 m
d Answers will vary
17 P_1 by 2.5 min
18 50.4 m
19a $x \cot 27°$ **b** Answers will vary **c** Answers will vary

Chapter 5 review exercise

1a 0.2924 **b** 0.9004 **c** 0.6211 **d** 0.9904
2a 17°27′ **b** 67°2′ **c** 75°31′ **d** 53°8′
3a 10.71 **b** 5.23 **c** 10.36 **d** 15.63
4a 45°34′ **b** 59°2′ **c** 58°43′ **d** 36°14′
5a $\sqrt{3}$ **b** $\frac{1}{\sqrt{2}}$ **c** $\frac{\sqrt{3}}{2}$ **d** 1 **e** 2 **f** $\frac{2}{\sqrt{3}}$
6 6.25 metres
7 65°
8a Answers will vary **b** 114 km **c** 108° T
9 All six trigonometric graphs are drawn on page 175.
10a −cos 55° **b** −sin 48° **c** tan 64° **d** sin 7°
11a $\sqrt{3}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $\frac{\sqrt{3}}{2}$ **d** $-\frac{1}{\sqrt{3}}$
12a 0 **b** −1 **c** undefined **d** −1
13a $y = 3$, $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$
b $x = -2\sqrt{5}$, $\sin\theta = -\frac{\sqrt{5}}{5}$, $\cos\theta = -\frac{2\sqrt{5}}{5}$, $\tan\theta = \frac{1}{2}$
14a $\sin\alpha = \frac{12}{13}$, $\cos\alpha = \frac{5}{13}$ **b** $\cos\beta = \frac{5}{7}$, $\tan\beta = \frac{2\sqrt{6}}{5}$
c $\sin\alpha = -\frac{9}{41}$, $\cos\alpha = \frac{40}{41}$ **d** $\cos\beta = -\frac{5}{7}$, $\tan\beta = -\frac{2\sqrt{6}}{5}$
15a $\sec\theta$ **b** $\tan\theta$ **c** $\tan\theta$ **d** $\cos^2\theta$ **e** 1 **f** $\cot^2\theta$
16a–f Answers will vary
17a $x = 60°$ or $300°$ **b** $x = 90°$ **c** $x = 135°$ or $315°$
d $x = 90°$ or $270°$ **e** $x = 30°$ or $210°$
f $x = 0°$, $180°$ or $360°$ **g** $x = 225°$ or $315°$
h $x = 150°$ or $210°$ **i** $x = 45°$, $135°$, $225°$ or $315°$
j $x = 30°$, $150°$, $210°$ or $330°$ **k** $x = 15°$ or $135°$
l $\tan x = -\sqrt{3}$, $x = 120°$ or $300°$
18a $\sin\theta = 0$ or $-\frac{1}{2}$, $\theta = 0°, 180°, 210°, 330°$ or $360°$
b $\cos\theta = -1$ or 2, $\theta = 180°$ **c** $\tan\theta = \frac{1}{2}$ or -3,
$\theta = 26°34′, 108°26′, 206°34′$ or $288°26′$
19a 8.5 **b** 10.4 **c** 7.6 **d** 8.9
20a 27 cm^2 **b** 56 cm^2
21a 57°55′ **b** 48°33′ **c** 24°29′ **d** 150°26′
22 28 cm^2
23a $\frac{5\sqrt{3}}{3}$ cm **b** 30° or 150°
24a Answers will vary **b** 48 m
25a Answers will vary **b** 31.5 m
26a Answers will vary **b** 316 nautical miles **c** 104° T
27a 10 tan 77° **b** 45 m
28a 1.612 m **b** 1.758 m **c** 23°28′
29a Answers will vary **b** Answers will vary **c** 129 m

Chapter 6

Exercise 6A

1a (2, 7) **b** (5, 6) **c** (2, −2) **d** $\left(0, 3\frac{1}{2}\right)$
e $\left(-5\frac{1}{2}, -10\right)$ **f** (4, 0)
2a 5 **b** 13 **c** 10 **d** $\sqrt{8} = 2\sqrt{2}$ **e** $\sqrt{80} = 4\sqrt{5}$ **f** 13
3a $M(1, 5)$ **b** $PM = MQ = 5$
4a $PQ = QR = \sqrt{17}$, $PR = \sqrt{50} = 5\sqrt{2}$
b Answers will vary
5a $AB = 15$, $BC = 20$ and $AC = 25$
b LHS $= AB^2 + BC^2 = 15^2 + 20^2 = 625 =$ RHS
6a $AB = \sqrt{58}$, $BC = \sqrt{72} = 6\sqrt{2}$, $CA = \sqrt{10}$
b AB: $\left(1\frac{1}{2}, 1\frac{1}{2}\right)$, BC: (0, 1), CA: $\left(-1\frac{1}{2}, 4\frac{1}{2}\right)$
7a 13 **b** $\sqrt{41}$ **c** (5, −3) **d** Answers will vary
8a (1, 6) **b** (1, 6) **c** The diagonals bisect each other.
d parallelogram
9a All sides are $5\sqrt{2}$. **b** rhombus
10a $XY = YZ = \sqrt{52} = 2\sqrt{13}$, $ZX = \sqrt{104} = 2\sqrt{26}$
b $XY^2 + YZ^2 = 104 = ZX^2$ **c** 26 square units
11a Each point is $\sqrt{17}$ from the origin.
b $\sqrt{17}, 2\sqrt{17}, 2\pi\sqrt{17}, 17\pi$
12 (5, 2)
13a $S(-5, -2)$
b i $P = (4, -14)$ **ii** $P = (-1, -17)$ **iii** $P = (7, -7)$
c $B = (0, 7)$ **d** $R = (12, -9)$
14a $A(3, 5)$ and $B(5, 7)$ will do.
b $C(0, 0)$ and $D(6, 8)$ will do.

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15a ABC is an equilateral triangle.
b PQR is a right triangle.
c DEF is none of these.
d XYZ is an isosceles triangle.
16a $(x-5)^2+(y+2)^2=45$
b $(x+2)^2+(y-2)^2=74$

Exercise 6B

1a i 2 **ii** $\frac{3}{4}$ **iii** $-1\frac{1}{2}$ **b i** $-\frac{1}{2}$ **ii** $-\frac{4}{3}$ **iii** $\frac{2}{3}$
2a -1, 1 **b** 2, $-\frac{1}{2}$ **c** $\frac{1}{2}$, -2 **d** $-\frac{1}{2}$, 2
e 3, $-\frac{1}{3}$ **f** $-\frac{7}{10}$, $\frac{10}{7}$
3a Vertical **b** Horizontal **c** Neither
d Horizontal **e** Neither **f** Vertical
4a 3 **b** $\frac{1}{2}$ **c** parallelogram
5a $m_{AB}=m_{CD}=\frac{1}{2}, m_{BC}=m_{DA}=-\frac{1}{5}$.
b $m_{AB}=2, m_{CD}=-3$
6a 0.27 **b** -1.00 **c** 0.41 **d** 3.08
7a 45° **b** 120° **c** 76° **d** 30°
8a less **b** equal **c** less **d** more
9a $m_{AB}=m_{CD}=-\frac{1}{2}, m_{BC}=m_{DA}=2$
b $m_{AB}=m_{BC}=-1$ **c** $AB=BC=2\sqrt{5}$
10 In each case, show that each pair of opposite sides is parallel.
a Show also that two adjacent sides are equal.
b Show also that two adjacent sides are perpendicular.
c Show that it is both a rhombus and a rectangle.
11a $-2, -\frac{7}{3}$, non-collinear **b** $\frac{2}{3}, \frac{2}{3}$, collinear
12 The gradients of AB, BC and CD are all $\frac{1}{3}$.
13 $m_{AB}=\frac{1}{2}$, $m_{BC}=-2$ and $m_{AC}=0$, so $AB\perp BC$.
14a $m_{PQ}=4$, $m_{QR}=-\frac{1}{4}$ and $m_{PR}=-\frac{5}{3}$, so $PQ\perp QR$. Area $=8\frac{1}{2}$ square units
b $m_{XY}=\frac{7}{3}$, $m_{YZ}=\frac{2}{5}$ and $m_{XZ}=-\frac{5}{2}$, so $XZ\perp YZ$. Area $=14\frac{1}{2}$ square units
15a $A(0,0)$ and $B(1,3)$ will do.
b $A(1,1)$ and $B(1,4)$ will do.
16a -5 **b** 5
17a $A(-2,0)$, $B(0,6)$, $m=3$, $\alpha \doteqdot 72°$
b $A(2,0), B(0,1), m=-\frac{1}{2}, \alpha \doteqdot 153°$
c $A(-4,0), B(0,-3), m=-\frac{3}{4}, \alpha \doteqdot 143°$
d $A(3,0), B(0,-2), m=\frac{2}{3}, \alpha \doteqdot 34°$
18a $P=(2,-1)$, $Q=(-1,4)$, $R=(-3,2)$, $S=(0,-3)$
b $m_{PQ}=m_{RS}=-\frac{5}{3}$ and $m_{PS}=m_{QR}=1$
19a They all satisfy the equation, or they all lie 5 units from O.
b The centre $O(0,0)$ lies on AB.
c $m_{AC}=\frac{1}{2}$, $m_{BC}=-2$
20a 3.73 **b** 1 **c** 2.41 **d** 0.32
21 $a=-\frac{1}{2}$
22 $k=2$ or -1

Exercise 6C

1a not on the line **b** on the line **c** on the line
2a (4, 0) and (0, 3) **b** (1.5, 0) and (0, −6)
c (8, 0) and (0, −4)
3 Check the points in your answer by substitution. (0, 8), (3, 7) and (6, 6) will do.
4a $x=1,\ y=2$ **b** $x=0,\ y=-4$ **c** $x=5,\ y=0$
5a $m=4,\ b=-2$ **b** $m=\frac{1}{5},\ b=-3$
c $m=-1,\ b=2,$ **d** $m=-\frac{5}{7},\ b=0$
6a $y=-3x+5$ **b** $y=-3x-\frac{2}{3}$ **c** $y=-3x$
7a $y=5x-4$ **b** $y=-\frac{2}{3}x-4$ **c** $y=-4$
8a $x-y+3=0$ **b** $2x+y-5=0$
c $x-5y-5=0$ **d** $x+2y-6=0$
9a $m=1,\ b=3$ **b** $m=-1,\ b=2$ **c** $m=\frac{1}{3},\ b=0$
d $m=-\frac{3}{4},\ b=\frac{5}{4}$
10a $m=1,\ \alpha=45°$ **b** $m=-1,\ \alpha=135°$
c $m=2,\ \alpha \doteqdot 63°26'$ **d** $m=-\frac{3}{4},\ \alpha \doteqdot 143°8'$
11 The sketches required are clear from the intercepts.
a $A(3,0), B(0,5)$ **b** $A(-3,0), B(0,6)$
c $A(-4,0), B\left(0, 2\frac{2}{5}\right)$
12a $y=2x+4,\ 2x-y+4=0$
b $y=-x,\ x+y=0$
c $y=-\frac{1}{3}x-4,\ x+3y+12=0$
13a i $y=-2x+3$ **ii** $y=\frac{1}{2}x+3$
b i $y=\frac{5}{2}x+3$ **ii** $y=-\frac{2}{5}x+3$
c i $y=-\frac{3}{4}x+3$ **ii** $y=\frac{4}{3}x+3$
14a $-3,\ \frac{1}{2},\ -3,\ \frac{1}{2}$, parallelogram
b $\frac{4}{3}, -\frac{3}{4}, \frac{4}{3}, -\frac{3}{4}$, rectangle
15 The gradients are $\frac{5}{7}$, $\frac{2}{5}$ and $-\frac{7}{5}$, so the first and last are perpendicular.
16a $x=3,\ x=0,\ y=-7,\ y=-2$
b $y=0,\ y=-4x+12,\ y=2x+12$
17a $x-y+3=0$ **b** $-\sqrt{3}x+y+1=0$
c $x-\sqrt{3}y-2\sqrt{3}=0$ **d** $x+y-1=0$
18a They are about 61° and 119°.
b It is isosceles. (The two interior angles with the x-axis are equal.)
19a $k=-\frac{1}{3}$ **b** $k=3$

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Exercise 6D

1 $3x - y - 4 = 0$

2a $6x - y + 19 = 0$ **b** $2x + y - 3 = 0$
c $2x - 3y + 25 = 0$ **d** $7x + 2y = 0$

3a $3x + 5y - 13 = 0$ **b** $3x + 5y - 18 = 0$
c $3x + 5y = 0$ **d** $3x + 5y + 20 = 0$

4a $2x - y - 1 = 0$ **b** $x + y - 4 = 0$
c $5x + y = 0$ **d** $x + 3y - 8 = 0$ **e** $4x + 5y + 8 = 0$

5a $y = 2x + 1$ **b** $y = -\frac{1}{2}x + 6$ **c** $y = \frac{1}{5}x - 8$
d $y = \frac{3}{7}x + 9$ **e** $y = \frac{5}{2}x + 10$

6a 3 **b** $3x - y - 5 = 0$

7a $2,\ 2x - y - 2 = 0$ **b** $-2,\ 2x + y - 1 = 0$
c $\frac{1}{3},\ x - 3y + 13 = 0$ **d** $2,\ 2x - y + 2 = 0$
e $-\frac{1}{4},\ x + 4y + 4 = 0$ **f** $1,\ x - y - 3 = 0$

8a $-\frac{3}{2}$ **b i** $3x + 2y + 1 = 0$ **ii** $2x - 3y - 8 = 0$

9a $2x - 3y + 2 = 0$ **b** $2x - 3y - 9 = 0$

10a $4x - 3y - 8 = 0$ **b** $4x - 3y + 11 = 0$

11a $M(3, -1)$ **b** Answers vary.
c i No, the first two intersect at $(-4, 7)$, which does not lie on the third.
ii They all meet at $(5, 4)$.

12a i $y = -2x + 5$ **ii** $y = \frac{1}{2}x + 6$
b i $y = 2\frac{1}{2}x - 8\frac{1}{2}$ **ii** $y = -\frac{2}{5}x + 4\frac{1}{5}$
c i $y = -1\frac{1}{3}x + 3$ **ii** $y = \frac{3}{4}x + 6\frac{1}{2}$

13a $x - y - 1 = 0$ **b** $\sqrt{3}x + y + \sqrt{3} = 0$
c $x - y\sqrt{3} - 4 - 3\sqrt{3} = 0$
d $x + \sqrt{3}y + 2 + 5\sqrt{3} = 0$

14a i $x - 3 = 0$ **ii** $y + 1 = 0$ **b** $3x + 2y - 6 = 0$
c i $x - y + 4 = 0$ **ii** $\sqrt{3}x + y - 4 = 0$
d $x\sqrt{3} + y + 6\sqrt{3} = 0$

15 $\ell_1 \parallel \ell_2$, and $\ell_3 \parallel \ell_4$ so there are two pairs of parallel sides. The vertices are $(-2, -1),\ (-4, -7),\ (1, -2),\ (3, 4)$.

16 $m_{BC} \times m_{AC} = -1$ so $BC \perp AC$.
AB: $y = x - 1$, BC: $y = \frac{1}{2}x + 2$,
AC: $y = 2 - 2x$

17a $m_{AC} = \frac{2}{3}, \theta \doteqdot 34°$ **b** $2x - 3y - 2 = 0$ **c** $D(4, 2)$
d $m_{AC} \times m_{BD} = \frac{2}{3} \times -\frac{3}{2} = -1$, hence they are perpendicular. **e** isosceles
f area $= \frac{1}{2} \times AC \times BD = \frac{1}{2} \times \sqrt{52} \times \sqrt{52} = 26$
g $E(8, -4)$

18a $4y = 3x + 12$ **b** $ML = MP = 5$ **c** $N(4, 6)$
d Answers will vary **e** $x^2 + (y - 3)^2 = 25$

19 $k = 2\frac{1}{2}$

20a $\mu = 4$ **b** $\mu = -9$

Exercise 6E

1a i $1, -1$ **ii** The product of their gradients is -1.
b i $1, -1$ **ii** The product of their gradients is -1.

2a i $M = (4, 5)$ **ii** $OM = PM = QM = \sqrt{41}$
iii OM, PM and QM are three radii of the circle.
b $M = (p, q)$, $OM = PM = QM = \sqrt{p^2 + q^2}$

3a i $P(5, 2)$ and $Q(4, 1)$ **ii, iii** Answers will vary
iv $AC = 2\sqrt{2}$ and $PQ = \sqrt{2}$
b $P(a + b, c)$, $Q(b, c)$, $y = c$ and so $Q(b, c)$ lies on $y = c$. Also, $AC = 2a$ and $PQ = a$ so $PQ = \frac{1}{2}AC$.

4a $P = \left(\frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2)\right)$,
$Q = \left(\frac{1}{2}(b_1 + c_1), \frac{1}{2}(b_2 + c_2)\right)$,
$R = \left(\frac{1}{2}(c_1 + d_1), \frac{1}{2}(c_2 + d_2)\right)$,
$S = \left(\frac{1}{2}(d_1 + a_1), \frac{1}{2}(d_2 + a_2)\right)$.
b Both midpoints are,
$\left(\frac{1}{4}(a_1 + b_1 + c_1 + d_1), \frac{1}{4}(a_2 + b_2 + c_2 + d_2)\right)$.
c Part **b** shows that its diagonals bisect each other, so it is a parallelogram.

5 Answers will vary

6a $\frac{x}{3} + \frac{y}{4} = 1$ and $4y = 3x$, thus $C = \left(\frac{48}{25}, \frac{36}{25}\right)$.
b $OA = 3, AB = 5, OC = \frac{12}{5}, BC = \frac{16}{5}, AC = \frac{9}{5}$
c i Answers will vary **ii** Answers will vary

7a $AB = BC = CA = 2a$ **b** $AB = AD = 2a$
c $BD = 2a\sqrt{3}$

8a AB and DC have gradient $\dfrac{b}{a}$; AD and BC have gradient $\dfrac{d}{c}$.
b Both the midpoints are $(a + c,\ b + d)$.
c The midpoints coincide.

9a i $P = (1, 4)$, $Q = (-1, 0)$ and $R = (3, 2)$,
BQ: $x - y + 1 = 0$, CR: $y - 2 = 0$, AP: $x = 1$
ii The medians intersect at $(1, 2)$.
b i $P(-3a, 3c - 3b), Q(3a, 3c + 3b), R(0, 0)$
ii The median passing through B is
$3a(y + 6b) = (c + 3b)(x + 6a)$.
The median passing through A is
$-3a(y - 6b) = (c - 3b)(x - 6a)$.
iii The medians intersect at $(0, 2c)$.

10a gradient $AB = 0$, gradient $BC = \dfrac{c}{b + a}$, gradient $CA = \dfrac{c}{b - a}$
b perpendicular bisector of AB: $x = 0$,
of BC: $c(c - y) = (b + a)(x - b + a)$,
of AC: $c(c - y) = (b - a)(x - b - a)$
c They all meet at $(0, \frac{c^2 + b^2 - a^2}{c})$.
d Any point on the perpendicular bisector of an interval is equidistant from the endpoints of that interval.

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Chapter 6 review exercise

1a $\left(8, 6\frac{1}{2}\right)$ **b** $-\frac{5}{12}$ **c** 13

2a $AB = 5, BC = \sqrt{2}, CA = 5$ **b** isosceles

3a $P(3, 7), Q(6, 5), R(3, -3), S(0, -1)$

b PQ and RS have gradient $-\frac{2}{3}$, QR and SP have gradient $\frac{8}{3}$. **c** parallelogram

4a $C(-1, 1), r = \sqrt{45} = 3\sqrt{5}$ **b** $PC = \sqrt{53}$, no

5a $m_{LM} = -2, m_{MN} = -\frac{8}{9}, m_{NL} = \frac{1}{2}$

b $m_{LM} \times m_{NL} = -1$

6a -1 **b** $a = 8$ **c** $Q(7, -4)$

d $d^2 = 16$, so $d = 4$ or -4.

7a $2x + y - 5 = 0$ **b** $2x - 3y + 9 = 0$

c $x + 7y = 0$ **d** $3x + y + 8 = 0$ **e** $x\sqrt{3} - y - 2 = 0$

8a $b = -\frac{7}{6},\ m = \frac{5}{6},\ \alpha \doteqdot 39°48'$

b $b = \frac{3}{4}, m = -1, \alpha = 135°$

9a $8x - y - 24 = 0$ **b** $5x + 2y - 21 = 0$

10a No; $m_{LM} = -\frac{1}{3}$ and $m_{MN} = -\frac{5}{12}$.

b Yes; they all pass through $(2, 5)$.

11a Yes; the 2nd and 3rd lines have gradients $\frac{3}{2}$ and $-\frac{2}{3}$ and are perpendicular.

b Trapezium; the 1st and 3rd lines are parallel.

12a $A = (6, 0), B = (0, 7\frac{1}{2})$ **b** $22\frac{1}{2}$ square units

13a $m_{AB} = -\frac{3}{4}, AB = 10, M(6, 5)$ **b** Answers will vary

c $C(15, 17)$ **d** $AC = BC = 5\sqrt{10}$ **e** 75 units^2

f $\sin\theta = \frac{3}{5},\ \theta \doteqdot 36°52'$

Chapter 7

Exercise 7A

1a The factors are $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243$.

b Population in 2010 = 810000, population in 2020 = 2430000, so the decade was 2010–2020.

2a 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096

b i 1, 3, 9, 27, 81, 243, 729

ii 1, 5, 25, 125, 625, 3125

iii 1, 6, 36, 216 **iv** 1, 7, 49, 343

v 1, 10, 100, 1000, 10000, 100000, 1000000

vi 1, 20, 400, 8000, 160000, 3200000, 64000000

c i 1, 4, 16, 64, 256, 1024, 4096

ii 1, 8, 64, 512, 4096

d 1, 9, 81, 729 **e** 1, 25, 625

3a 8 **b** 64 **c** 81 **d** 729 **e** $\frac{4}{9}$ **f** $\frac{8}{27}$ **g** $\frac{81}{10000}$ **h** $\frac{16}{49}$

i $\frac{5}{9}$ **j** 1

4a 1 **b** 1 **c** $\frac{1}{5}$ **d** $\frac{1}{11}$ **e** $\frac{1}{36}$ **f** $\frac{1}{100}$ **g** $\frac{1}{27}$ **h** $\frac{1}{125}$

i $\frac{1}{32}$ **j** $\frac{1}{1000000}$

5a 5 **b** 11 **c** $\frac{7}{2}$ or $3\frac{1}{2}$ **d** $\frac{2}{7}$ **e** $\frac{4}{3}$ or $1\frac{1}{3}$ **f** $\frac{23}{10}$ or $2\frac{3}{10}$

g $\frac{1}{10}$ or 0.1 **h** 10 **i** 100 **j** 50

6a $\frac{1}{25}$ **b** 25 **c** 125 **d** 16 **e** 1000000 **f** $\frac{9}{4}$ **g** $\frac{81}{16}$

h $\frac{16}{81}$ **i** $\frac{25}{4}$ **j** 1

7a 2^{14} **b** a^{15} **c** 7^{-8} **d** x^2 **e** $9^0 = 1$ **f** $a^0 = 1$

g 5^{-3} **h** 8

8a 7^5 **b** a^{-2} **c** x^{12} **d** x^{-12} **e** 2^{16} **f** 1 **g** y^{11} **h** y^{-11}

9a x^{15} **b** x^{15} **c** z^{14} **d** a^{-6} **e** a^{-6} **f** 5^{-28} **g** y^{10} **h** 2^{16}

10a $x = 2$ **b** $x = 4$ **c** $x = 3$ **d** $x = 6$ **e** $x = -1$

f $x = -1$ **g** $x = -2$ **h** $x = -3$ **i** $x = -1$

j $x = -1$ **k** $x = 0$ **l** $x = 0$

11a $9x^2$ **b** $125a^3$ **c** $64c^6$ **d** $81s^4t^4$

e $49x^2y^2z^2$ **f** $\frac{1}{x^5}$ **g** $\frac{9}{x^2}$ **h** $\frac{y^2}{25}$ **i** $\frac{49a^2}{25}$ **j** $\frac{27x^3}{8y^3}$

12a $3\,\text{km}^3$ **b** $(10^3 \times 10^3)^3 = 10^{18}$ **c** 3×10^{18}

13a $\frac{1}{9}$ **b** $\frac{1}{x}$ **c** $\frac{1}{b^2}$ **d** $-\frac{1}{a^4}$ **e** $\frac{1}{7x}$ **f** $\frac{7}{x}$ **g** $-\frac{9}{x}$ **h** $\frac{1}{9a^2}$

i $\frac{3}{a^2}$ **j** $\frac{4}{x^3}$

14a x^{-1} **b** $-x^{-2}$ **c** $-12x^{-1}$ **d** $9x^{-2}$ **e** $-x^{-3}$ **f** $12x^{-5}$

g $7x^{-3}$ **h** $-6x^{-1}$ **i** $\frac{1}{6}x^{-1}$ **j** $-\frac{1}{4}x^{-2}$

15a $\frac{2}{3}$ **b** $\frac{3}{7}$ **c** $\frac{3}{8}$ **d** $\frac{4}{25}$ **e** $\frac{27}{1000}$ **f** $\frac{9}{400}$ **g** 5

h $\frac{5}{12}$ **i** $\frac{4}{9}$ **j** $\frac{4}{25}$ **k** $\frac{8}{125}$ **l** 400

16a $x = 2$ **b** $x = -1$ **c** $x = -2$ **d** $x = -3$

e $x = \frac{10}{13}$ **f** $x = 2$ **g** $x = \frac{1}{3}$ **h** $x = \frac{9}{8}$

17a 2^{x+3} **b** 3^{x+1} **c** $\frac{1}{7^x}$ **d** $\frac{5^{2x}}{5^3}$ **e** 10^{6x}

f $\frac{1}{5^{8x}}$ **g** 6^{14x} **h** $\frac{2^{3x}}{2^4}$

18a x^6y^4 **b** $\frac{y}{x^2}$ **c** $\frac{21a^3}{x}$ **d** $\frac{1}{3st^2}$ **e** $\frac{7x}{y^2}$ **f** $\frac{5b^{10}}{4a^6}$ **g** $\frac{s^6}{y^9}$

h $\frac{c^2}{5d^3}$ **i** $27x^8y^{17}$ **j** $\frac{2a^7}{y^{15}}$ **k** $5s^5$ **l** $\frac{250x^8}{y^{12}}$

19a $x^2 + 2 + \frac{1}{x^2}$ **b** $x^2 - 2 + \frac{1}{x^2}$ **c** $x^4 - 2 + \frac{1}{x^4}$

20a 2^{x+1} **b** 2^{x+1} **c** 3^{x+1} **d** 3^{x+1} **e** 2^{x+2} **f** 2^{x+5}

g 5^{x+3} **h** 3^{x+4} **i** 2^{x-1} **j** 3^{x-2}

21a $x = -1$ **b** $x = 6$ **c** $x = 8$ **d** $x = -1$

e $x = -4$ **f** $x = 2$

22a Take the reciprocal: 5.97×10^{26}

b $5.73 \times 10^{-45}\,\text{m}^3$ **c** $2.9 \times 10^{17}\,\text{kg/m}^3$

Exercise 7B

1a 5 **b** 6 **c** 10 **d** 3 **e** 4 **f** 10 **g** 3 **h** 2 **i** 10 **j** 1000

2a 125 **b** 27 **c** 9 **d** 4 **e** 8 **f** 27 **g** 81 **h** 32 **i** 8 **j** 16

3a $\frac{1}{7}$ **b** $\frac{1}{2}$ **c** $\frac{5}{7}$ **d** $\frac{3}{2}$ **e** $\frac{1}{8}$ **f** $\frac{1}{125}$ **g** $\frac{8}{27}$ **h** $\frac{27}{1000}$

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4a 28 561 **b** 109.5 **c** 1.126×10^{15} **d** 15 **e** 2.154
f 2.031 **g** 7.225×10^{-11} **h** 0.1969

5a x **b** x^6 **c** $x^{3\frac{1}{2}}$ **d** x **e** $x^{\frac{1}{2}}$ **f** $x^{-4\frac{1}{2}}$ **g** x^2 **h** x^{-4} **i** x^6

6a $2^1 = 2$ **b** $2^0 = 1$ **c** $2^3 = 8$ **d** $3^{-1} = \frac{1}{3}$ **e** $25^{\frac{1}{2}} = 5$
f $7^0 = 1$ **g** $3^{-3} = \frac{1}{27}$ **h** $3^{-2} = \frac{1}{9}$ **i** $9^2 = 81$

7a $x = \frac{1}{2}$ **b** $x = \frac{1}{2}$ **c** $x = \frac{1}{4}$ **d** $x = \frac{1}{6}$ **e** $x = \frac{1}{2}$ **f** $x = \frac{1}{3}$

8a $\sqrt{x}$ **b** $\sqrt[3]{x}$ **c** $7\sqrt{x}$ **d** $\sqrt{7x}$ **e** $15\sqrt[4]{x}$ **f** $\sqrt{x^3}$ or $\left(\sqrt{x}\right)^3$
g $6\sqrt{x^5}$ or $6\left(\sqrt{x}\right)^5$ **h** $\sqrt[3]{x^4}$ or $\left(\sqrt[3]{x}\right)^4$

9a $x^{\frac{1}{2}}$ **b** $3x^{\frac{1}{2}}$ **c** $(3x)^{\frac{1}{2}}$ **d** $12x^{\frac{1}{3}}$ **e** $9x^{\frac{1}{6}}$ **f** $x^{\frac{3}{2}}$ **g** $x^{\frac{9}{2}}$ **h** $25x^{\frac{6}{5}}$

10a $\frac{1}{5}$ **b** $\frac{1}{10}$ **c** $\frac{1}{5}$ **d** $\frac{1}{2}$ **e** $\frac{1}{8}$ **f** $\frac{1}{9}$ **g** $\frac{1}{27}$ **h** $\frac{1}{343}$

11a 2 **b** 5 **c** 7 **d** 3 **e** 8 **f** 27 **g** $\frac{27}{8}$ **h** $\frac{4}{25}$

12a $9xy^3$ **b** $35b$ **c** $3s^{\frac{1}{2}}$ **d** $x^{1\frac{1}{2}}y^{2\frac{1}{2}}$ **e** a **f** $a^{-1}b^2$
g $2xy^{-2}$ **h** p^2q^{-6} **i** x^7

13a $x^{-\frac{1}{2}}$ **b** $12x^{-\frac{1}{2}}$ **c** $-5x^{-\frac{1}{2}}$ **d** $15x^{-\frac{1}{3}}$ **e** $-4x^{-\frac{2}{3}}$
f $x^{1\frac{1}{2}}$ **g** $5x^{-1\frac{1}{2}}$ **h** $8x^{2\frac{1}{2}}$

14a 9 **b** -3 **c** $\frac{1}{20}$ **d** $\frac{3}{10}$

15a $\$6000 \times (1.03)^0 = \$6000$
b $\$6000 \times (1.03)^1 = \$6180$
c i $\$6000 \times (1.03)^5 \doteqdot \$6960$
ii $\$6000 \times (1.03)^{\frac{1}{2}} \doteqdot \$6090$
iii $\$6000 \times (1.03)^{\frac{7}{2}} \doteqdot \$6650$

16a 5.765×10^6 **b** 1.261×10^1 **c** 8.244×10^{-1}
d 7.943×10^{-3} **e** 8.825×10^0 **f** 2.595×10^1
g 7.621×10^{-2} **h** 5.157×10^4

17a $x + 2 + x^{-1}$ **b** $x - 2 + x^{-1}$ **c** $x^5 - 2 + x^{-5}$

18a $x = -\frac{1}{2}$ **b** $x = -\frac{1}{4}$ **c** $x = \frac{2}{3}$ **d** $x = -\frac{2}{3}$ **e** $x = \frac{3}{2}$
f $x = -\frac{3}{2}$ **g** $x = \frac{3}{4}$ **h** $x = -\frac{4}{3}$ **i** $x = -\frac{1}{2}$ **j** $x = -\frac{2}{3}$

19a $3^{\frac{1}{3}} > 2^{\frac{1}{2}}$ **b** $2^{\frac{1}{2}} > 5^{\frac{1}{5}}$ **c** $7^{\frac{3}{2}} < 20$ **d** $5^{\frac{1}{5}} < 3^{\frac{1}{3}}$

Exercise 7C

1a because $2^3 = 8$. **b** because $5^2 = 25$.
c because $10^3 = 1000$. **d** so $\log_7 49 = 2$.
e so $\log_3 81 = 4$. **f** so $\log_{10} 100\,000 = 5$.

2a $x = a^y$ **b** $x = \log_a y$

3a $10^x = 10\,000, x = 4$ **b** $10^x = 1000, x = 3$
c $10^x = 100, x = 2$ **d** $10^x = 10, x = 1$
e $10^x = 1, x = 0$ **f** $10^x = \frac{1}{10}, x = -1$
g $10^x = \frac{1}{100}, x = -2$ **h** $10^x = \frac{1}{1000}, x = -3$

4a $3^x = 9, x = 2$ **b** $5^x = 125, x = 3$ **c** $7^x = 49, x = 2$
d $2^x = 64, x = 6$ **e** $4^x = 64, x = 3$ **f** $8^x = 64, x = 2$
g $8^x = 8, x = 1$ **h** $8^x = 1, x = 0$ **i** $7^x = \frac{1}{7}, x = -1$
j $12^x = \frac{1}{12}, x = -1$ **k** $11^x = \frac{1}{121}, x = -2$
l $6^x = \frac{1}{36}, x = -2$ **m** $4^x = \frac{1}{64}, x = -3$
n $8^x = \frac{1}{64}, x = -2$ **o** $2^x = 64, x = -6$
p $5^x = \frac{1}{125}, x = -3$

5a $x = 7^2 = 49$ **b** $x = 9^2 = 81$ **c** $x = 5^3 = 125$
d $x = 2^5 = 32$ **e** $x = 4^3 = 64$ **f** $x = 100^3 = 1\,000\,000$
g $x = 7^1 = 7$ **h** $x = 11^0 = 1$ **i** $x = 13^{-1} = \frac{1}{13}$
j $x = 7^{-1} = \frac{1}{7}$ **k** $x = 10^{-2} = \frac{1}{100}$ **l** $x = 12^{-2} = \frac{1}{144}$
m $x = 5^{-3} = \frac{1}{125}$ **n** $x = 7^{-3} = \frac{1}{343}$ **o** $x = 2^{-5} = \frac{1}{32}$
p $x = 3^{-4} = \frac{1}{81}$

6a $x^2 = 49, x = 7$ **b** $x^3 = 8, x = 2$ **c** $x^3 = 27, x = 3$
d $x^4 = 10\,000, x = 10$ **e** $x^2 = 10\,000, x = 100$
f $x^6 = 64, x = 2$ **g** $x^2 = 64, x = 8$ **h** $x^1 = 125, x = 125$
i $x^1 = 11, x = 11$ **j** $x^{-1} = \frac{1}{17}, x = 17$
k $x^{-1} = \frac{1}{6}, x = 6$ **l** $x^{-1} = \frac{1}{7}, x = 7$ **m** $x^{-2} = \frac{1}{9}, x = 3$
n $x^{-2} = \frac{1}{49}, x = 7$ **o** $x^{-3} = \frac{1}{8}, x = 2$
p $x^{-2} = \frac{1}{81}, x = 9$

7a $a^x = a, x = 1$ **b** $x = a^1 = a$ **c** $x^1 = a, x = a$
d $a^x = \frac{1}{a}, x = -1$ **e** $x = a^{-1} = \frac{1}{a}$ **f** $x^{-1} = \frac{1}{a}, x = a$
g $a^x = 1, x = 0$ **h** $x = a^0 = 1$
i $x^0 = 1$ where x can be any positive number.

8a 1 **b** -1 **c** 3 **d** -2 **e** -5 **f** $\frac{1}{2}$ **g** $-\frac{1}{2}$ **h** 0

9a 1 and 2 **b** 2 and 3 **c** 0 and 1 **d** 3 and 4
e 5 and 6 **f** 9 and 10 **g** -1 and 0 **h** -2 and -1

10a 1 and 2 **b** 0 and 1 **c** 3 and 4 **d** 0 and 1 **e** 3 and 4
f 4 and 5 **g** 2 and 3 **h** 1 and 2 **i** -1 and 0
j -2 and -1

11a 0.301 **b** 1.30 **c** 2.00 **d** 20.0 **e** 3.16 **f** 31.6
g 0.500 **h** 1.50 **i** 3 **j** 6 **k** 1000 **l** 1 000 000
m -0.155 **n** -2.15 **o** 0.700 **p** 0.00708

12a $\log_{10}45 \doteqdot 1.7$ **b** $10^{1.7} \doteqdot 50$

13a $7^x = \sqrt{7}, x = \frac{1}{2}$ **b** $11^x = \sqrt{11}, x = \frac{1}{2}$
c $x = 9^{\frac{1}{2}} = 3$ **d** $x = 144^{\frac{1}{2}} = 12$ **e** $x^{\frac{1}{2}} = 3, x = 9$
f $x^{\frac{1}{2}} = 13, x = 169$ **g** $6^x = \sqrt[3]{6}, x = \frac{1}{3}$
h $9^x = 3, x = \frac{1}{2}$ **i** $x = 64^{\frac{1}{3}} = 4$ **j** $x = 16^{\frac{1}{4}} = 2$
k $x^{\frac{1}{3}} = 2, x = 8$ **l** $x^{\frac{1}{6}} = 2, x = 64$ **m** $8^x = 2, x = \frac{1}{3}$

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n $125^x = 5,\ x = \frac{1}{3}$ **o** $x = 7^{\frac{1}{2}}$ or $\sqrt{7}$ **p** $x = 7^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{7}}$
q $x^{-\frac{1}{2}} = \frac{1}{7},\ x = 49$ **r** $x^{-\frac{1}{2}} = \frac{1}{20},\ x = 400$
s $4^x = \frac{1}{2},\ x = -\frac{1}{2}$ **t** $27^x = \frac{1}{3},\ x = -\frac{1}{3}$
u $x = 121^{-\frac{1}{2}} = \frac{1}{11}$ **v** $x = 81^{-\frac{1}{4}} = \frac{1}{3}$ **w** $x^{-\frac{1}{4}} = \frac{1}{2},\ x = 16$
x $x^{-\frac{1}{4}} = 2,\ x = \frac{1}{16}$

Exercise 7D

1a $\log_6 36 = 2$ **b** $\log_5 25 = 2$ **c** $\log_2 8 = 3$
2a $\log_6 6 = 1$ **b** $\log_{15} 15 = 1$ **c** $\log_{10} 100 = 2$
d $\log_{12} 144 = 2$ **e** $\log_{10} 1000 = 3$ **f** $\log_6 36 = 2$
3a $\log_3 3 = 1$ **b** $\log_4 4 = 1$ **c** $\log_2 8 = 3$
d $\log_5 25 = 2$ **e** $\log_3 81 = 4$ **f** $\log_2 32 = 5$
4a 1 **b** 2 **c** 3 **d** 2 **e** 0 **f** −2 **g** −3 **h** 2 **i** 0
5a $3 \log_a 2$ **b** $4 \log_a 2$ **c** $6 \log_a 2$ **d** $-\log_a 2$
e $-3 \log_a 2$ **f** $-5 \log_a 2$ **g** $\frac{1}{2} \log_a 2$ **h** $-\frac{1}{2} \log_a 2$
6a $2 \log_2 3$ **b** $2 \log_2 5$ **c** $1 + \log_2 3$ **d** $1 + \log_2 5$
e $1 + 2 \log_2 3$ **f** $2 + \log_2 5$ **g** $1 - \log_2 3$
h $-1 + \log_2 5$
7a 3.90 **b** 3.16 **c** 3.32 **d** 5.64 **e** 0.58
f −0.74 **g** −0.58 **h** 6.22
8a 3 **b** 5 **c** 1.3 **d** n
9a 100 **b** 7 **c** 3.6 **d** y
10a 2 **b** 15 **c** −1 **d** 6
11a $3 \log_a x$ **b** $-\log_a x$ **c** $\frac{1}{2} \log_a x$ **d** $-2 \log_a x$
e $-2 \log_a x$ **f** $2 \log_a x$ **g** $8 - 8 \log_a x$ **h** $\log_a x$
12a $\log_a y + \log_a z$ **b** $\log_a z - \log_a y$ **c** $4 \log_a y$
d $-2 \log_a x$ **e** $\log_a x + 3 \log_a y$
f $2 \log_a x + \log_a y - 3 \log_a z$ **g** $\frac{1}{2} \log_a y$
h $\frac{1}{2} \log_a x + \frac{1}{2} \log_a z$
13a 1.30 **b** −0.70 **c** 2.56 **d** 0.15 **e** 0.45
f −0.50 **g** 0.54 **h** −0.35
14a $6x$ **b** $-x - y - z$ **c** $3y + 5$ **d** $2x + 2z - 1$
e $y - x$ **f** $x + 2y - 2z - 1$ **g** $-2z$ **h** $3x - y - z - 2$
15a $10 = 3^{\log_3 10}$ **b** $3 = 10^{\log_{10} 3}$ **c** $0.1 = 2^{\log_2 0.1}$

Exercise 7E

1a–c Answer is in question
2a 2.807 **b** 4.700 **c** −3.837 **d** 7.694
e 0.4307 **f** 1.765 **g** 0.6131 **h** 0.2789
i −2.096 **j** −7.122 **k** 2.881 **l** 7.213
m 0.03323 **n** 578.0 **o** −687.3
3a $x = \log_2 15 \doteqdot 3.907$
b $x = \log_2 5 \doteqdot 2.322$
c $x = \log_2 1.45 \doteqdot 0.5361$
d $x = \log_2 0.1 \doteqdot -3.322$
e $x = \log_2 0.0007 \doteqdot -10.48$
f $x = \log_3 10 \doteqdot 2.096$
g $x = \log_3 0.01 \doteqdot -4.192$
h $x = \log_5 10 \doteqdot 1.431$ **i** $x = \log_{12} 150 \doteqdot 2.016$
j $x = \log_8 \frac{7}{9} \doteqdot -0.1209$ **k** $x = \log_6 1.4 \doteqdot 0.1878$
l $x = \log_{30} 2 \doteqdot 0.2038$ **m** $x = \log_{0.7} 0.1 \doteqdot 6.456$
n $x = \log_{0.98} 0.03 \doteqdot 173.6$
o $x = \log_{0.99} 0.01 \doteqdot 458.2$
4a $x > 5$ **b** $x \le 5$ **c** $x < 6$ **d** $x \ge 4$ **e** $x > 1$
f $x \le 0$ **g** $x < -1$ **h** $x \le -3$
5a $0 < x < 8$ **b** $x \ge 8$ **c** $x > 1000$ **d** $x \ge 10$
e $x > 1$ **f** $0 < x < 6$ **g** $0 < x \le 125$ **h** $x > 36$
6a $x > \log_2 12 \doteqdot 3.58$
b $x < \log_2 100 \doteqdot 6.64$
c $x < \log_2 0.02 \doteqdot -5.64$
d $x > \log_2 0.1 \doteqdot -3.32$
e $x < \log_5 100 \doteqdot 2.86$
f $x < \log_3 0.007 \doteqdot -4.52$
g $x > \log_{1.2} 10 \doteqdot 12.6$
h $x > \log_{1.001} 100 \doteqdot 4610$
7a After 1 year, the price is 1.05 times greater, after 2 years, it is $(1.05)^2$ times greater, and so on.
b $\log_{1.05} 1.5 \doteqdot 8.3$ years
8a $\log_8 x = \frac{\log_2 x}{\log_2 8} = \frac{1}{3} \log_2 x$
b $\log_{a^n} x = \frac{\log_a x}{\log_a a^n} = \frac{1}{n} \log_a x$
9a–c Answers will vary
10a $x = 3$ **b** $x = 2$ **c** $x < 1$ **d** $x \le 9$ **e** $x = 0$ **f** $x = \frac{1}{5}$
g $x < 4.81$ **h** $x > -2.90$
11a $x < 33.2$, 33 powers **b** $x < 104.8$, 104 powers
12a $10^2 < 300 < 10^3$ **b** $1 \le \log_{10} x < 2$ **c** 5 digits
d 27.96, 28 digits **e** $1000 \log_{10} 2 = 301.03$, 302 digits

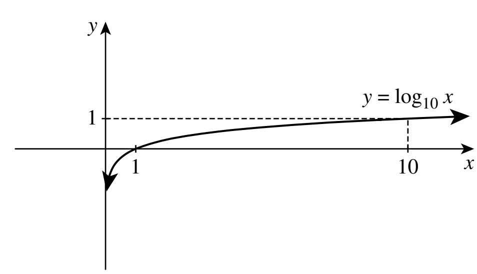
Exercise 7F

1a

x	0.1	0.25	0.5	0.75	1	2
$\log_{10} x$	−1	−0.60	−0.30	−0.12	0	0.30

x	3	4	5	6	7	8	9	10
$\log_{10} x$	0.48	0.60	0.70	0.78	0.85	0.90	0.95	1

b



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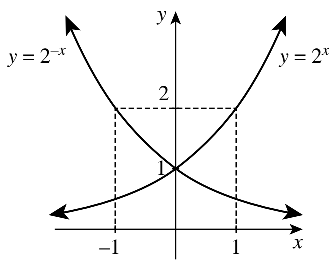
2a i

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

ii

x	-3	-2	-1	0	1	2	3
2^x	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

b



c The values of $y = 2^{-x}$ are the values of $y = 2^x$ in reverse order.

d The two graphs are reflections of each other in the y-axis, because x has been replaced with $-x$.

e i and **ii** For both, domain: all real x, range: $y > 0$

f i and **ii** For both, the asymptote is $y = 0$ (the x-axis).

g i 'As $x \to -\infty$, $2^x \to 0$.'

ii 'As $x \to \infty$, $2^x \to \infty$.'

h i 'As $x \to -\infty$, $2^{-x} \to \infty$.'

ii 'As $x \to \infty$, $2^{-x} \to 0$.'

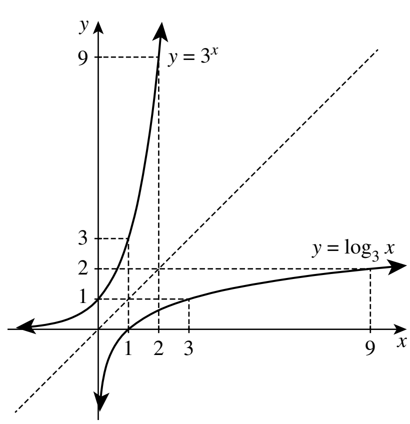
3a i

x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

ii

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\log_3 x$	-2	-1	0	1	2

b



c The two rows have been exchanged.

d The two graphs are reflections of each other in the diagonal line $y = x$, because the two functions are inverses of each other.

e i domain: all real x, range: $y > 0$

ii domain $x > 0$, range: all real y

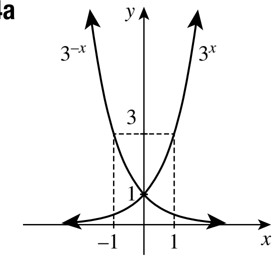
f i $y = 0$ (the x-axis)

ii $x = 0$ (the y-axis)

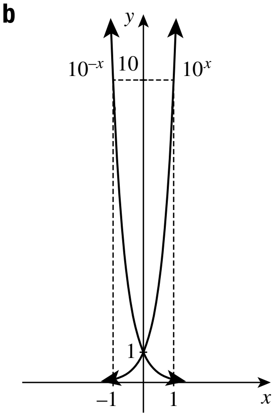
g i 'As $x \to -\infty$, $3^x \to 0$.'

ii 'As $x \to 0^+$, $\log_3 x \to -\infty$.'

4a



b



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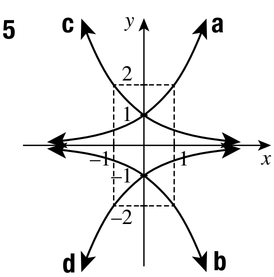

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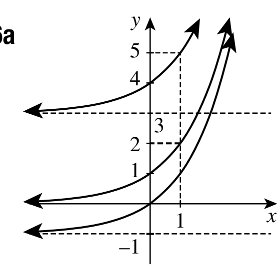

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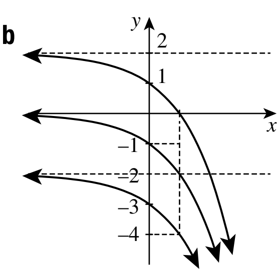

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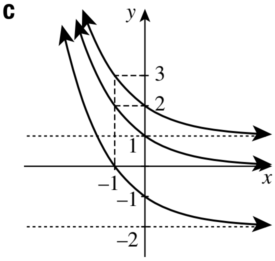

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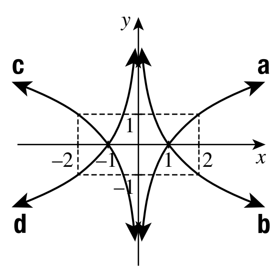

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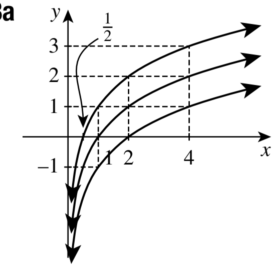
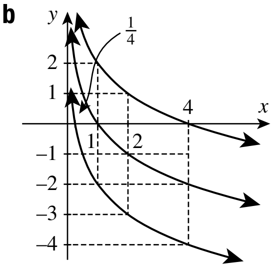
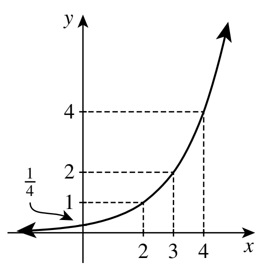
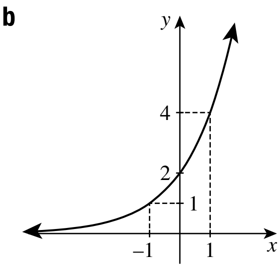
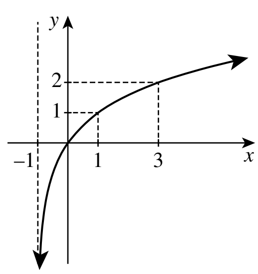
8a 

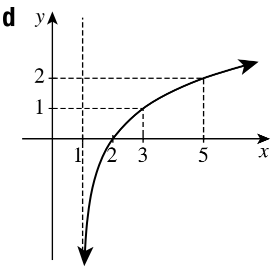
b 

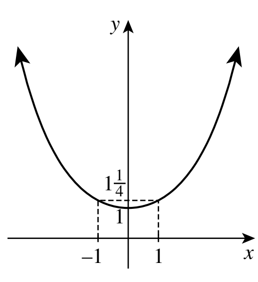
9a **i** 4 **ii** $\frac{1}{4}$ **iii** 2.83 **iv** 1.32 **v** 0.66
b **i** 1 **ii** 1.58 **iii** 0.26 **iv** −1.32
c **i** $0 \le x \le 2$ **ii** $0 \le x \le 1$ **iii** $0.58 \le x \le 1.58$
iv $-1 \le x \le 1$
d **i** 2 **ii** 1.58 **iii** 0.49 **iv** −0.32

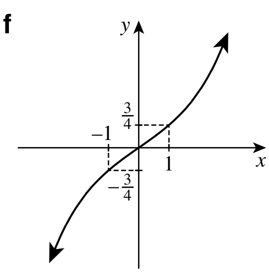
10a 

b 

c 

d 

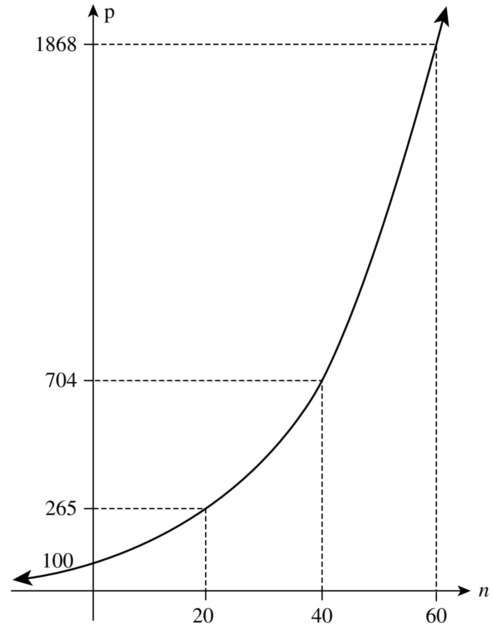
e 

f 

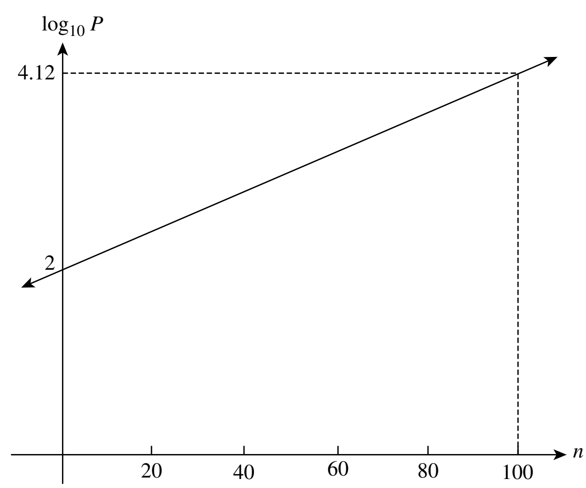
Exercise 7G

1a 5000, 2594 **b** $\frac{t}{2} = \log_{10}\frac{Q}{5}$, so $t = 2\log_{10}\frac{Q}{5}$
c 4, 3.419
2a 60, $20\log_2 12 = \frac{20\log_{10}12}{\log_{10}2} \doteqdot 71.70$
b $\frac{t}{20} = \log_2 2Q$, so $2Q = 2^{\frac{t}{20}}$, so $Q = \frac{1}{2} \times 2^{\frac{t}{20}}$
c 2, 2.378
3a There are $\frac{n}{30}$ thirty–year intervals in n years.
b **i** 24 000 000 **ii** 30 000 000
c **i** 120 years **ii** $30\log_2 20 \doteqdot 130$ years

4a P: 100, 265, 704, 1868, 4956, 13 150
b $P = 100 \times (1.05)^n$



c The values are about 2, 2.42, 2.85, 3.27, 3.70, 4.12.
d $\log_{10} P$



e The new graph is a straight line, and $\log_{10} P$ is a linear function of n.
5a $\frac{n}{2}$ is the number of 2–year periods
b $D = 2^{20}D_0 \doteqdot 1\,050\,000\,D_0$
c $2^{\frac{n}{2}} = 10^7$, so $\frac{n}{2} = \log_2 10^7$, so $n = 2\log_2 10^7 \doteqdot 47$ years, that is, in 2022.

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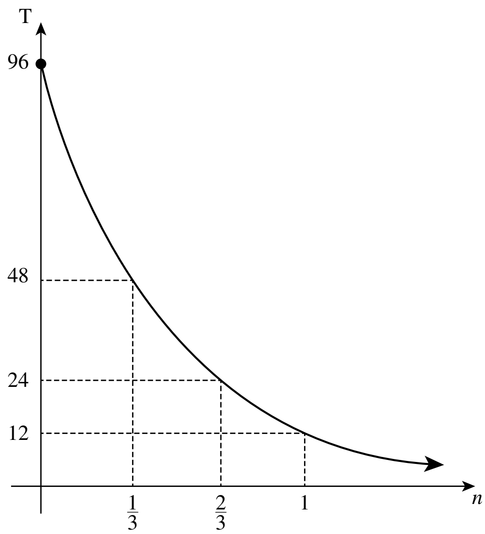
6a $3n$ is the number of 20-minutes periods in n hours.

b $T = 96 \times \left(\frac{1}{2}\right)^{3n}$



c $96 \times \left(\frac{1}{2}\right)^{6} = 1\frac{1}{2}°C$

d $3n = \log_{\frac{1}{2}} \frac{T}{96}$ $\left(\text{Alternatively } n = -\frac{1}{3}\log_2 \frac{T}{96}.\right)$

e $n = \frac{1}{3}\log_{\frac{1}{2}} \frac{1}{96} = 2.1949\ldots \text{ h} \doteqdot 2\text{ h }12\text{ min}$

7a The mass halves every 700 000 000 years.

b When $n = 4$ billion, $\frac{n}{700000000} = \frac{40}{7}$,

so $M = M_0 \times \left(\frac{1}{2}\right)^{\frac{40}{7}} \doteqdot 1.9\%$ of M_0

c When $n = -4.5$ billion, $\frac{n}{700000000} = -\frac{45}{7}$,

so $M = M_0 \times \left(\frac{1}{2}\right)^{-\frac{45}{7}} \doteqdot 86 M_0$

8a 1000 **b** $1000^{\frac{3}{2}} \doteqdot 32000$

c Ratio of shaking amplitudes is 10^5, ratio of energies released is about 3.2×10^7.

9a $[H^+] = 10^{-pH}$ **b** About 10^{-7} mol/L

c About 10^{-2} mol/L, about 100 000 times more acidic than water

d About 7.94×10^{-9} mol/L, about 12.6 times more alkaline than water.

Chapter 7 review exercise

1a 125 **b** 256 **c** 1 000 000 000 **d** $\frac{1}{17}$ **e** $\frac{1}{81}$ **f** $\frac{1}{8}$ **g** $\frac{1}{81}$

h 1 **i** $\frac{8}{27}$ **j** $\frac{12}{7}$ **k** $\frac{36}{25}$ **l** 6 **m** 3 **n** 4 **o** 243 **p** $\frac{2}{7}$ **q** 1

r $\frac{5}{3}$ **s** $\frac{4}{9}$ **t** $\frac{1000}{27}$

2a x^{-1} **b** $7x^{-2}$ **c** $-\frac{1}{2}x^{-1}$ **d** $x^{\frac{1}{2}}$

e $30x^{\frac{1}{2}}$ **f** $4x^{-\frac{1}{2}}$ **g** yx^{-1} **h** $2yx^{\frac{1}{2}}$

3a x^{20} **b** $\frac{81}{a^{12}}$ **c** $5x^3$ **d** $\frac{2r}{t^2}$

4a x^3y^3 **b** $60xy^3z^5$ **c** $18x^{-1}y^{-2}$ **d** $4a^3b^3c^{-1}$ **e** x^2y^{-2}

f $2x^{-3}y$ **g** m^2n^{-1} **h** $72s^9t^3$ **i** $8x^3y^{-3}$

5a 4 **b** 2 **c** -1 **d** -5 **e** 2 **f** 3 **g** $\frac{1}{2}$ **h** $\frac{1}{3}$

6a $2^x = 8, x = 3$ **b** $3^x = 9, x = 2$ **c** $10^x = 10000, x = 4$

d $5^x = \frac{1}{5}, x = -1$ **e** $7^x = \frac{1}{49}, x = -2$ **f** $13^x = 1, x = 0$

g $9^x = 3, x = \frac{1}{2}$ **h** $2^x = \sqrt{2}, x = \frac{1}{2}$ **i** $7^2 = x, x = 49$

j $11^{-1} = x, x = \frac{1}{11}$ **k** $16^{\frac{1}{2}} = x, x = 4$ **l** $27^{\frac{1}{3}} = x, x = 3$

m $x^2 = 36, x = 6$ **n** $x^3 = 1000, x = 10$

o $x^{-1} = \frac{1}{7}, x = 7$ **p** $x^{\frac{1}{2}} = 4, x = 16$

7a 1 **b** 2 **c** 2 **d** -2 **e** 2 **f** 0

8a $\log_a x + \log_a y + \log_a z$ **b** $\log_a x - \log_a y$

c $3\log_a x$ **d** $-2\log_a z$ **e** $2\log_a x + 5\log_a y$

f $2\log_a y - \log_a x - 2\log_a z$ **g** $\frac{1}{2}\log_a x$

h $\frac{1}{2}\log_a x + \frac{1}{2}\log_a y + \frac{1}{2}\log_a z$

9a 1 and 2 **b** 2 and 3 **c** 4 and 5 **d** 5 and 6 **e** -1 and 0

f -3 and -2 **g** -4 and -3 **h** -2 and -1

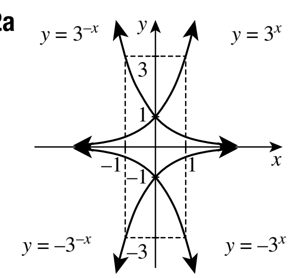
10a 2.332 **b** -2.347 **c** 2.010 **d** 9.966 **e** -0.9551

f 69.66 **g** -3 **h** 687.3

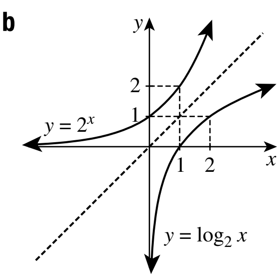
11a 3.459 **b** -4.644 **c** 3.010 **d** -0.3645 **e** 161.7

f -161.7 **g** 10.32 **h** 458.2

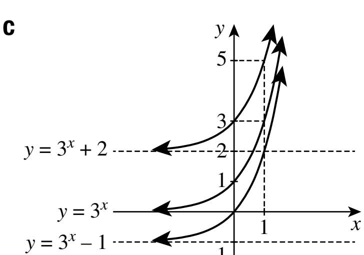
12a



b



c



13a There are $\frac{n}{4}$ four-hour periods in n hours.

b i 800 **ii** $100 \times 2^{3.25} \doteqdot 950$

c $\frac{n}{4} = \log_2 \frac{P}{100}$, so $n = 4\log_2 \frac{P}{100}$

d $4\log_2 100000 \doteqdot 66$ hours

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Chapter 8

Exercise 8A

1 The values of $f'(x)$ should be about $-4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4$. The graph of $y = f'(x)$ should approximate a line of gradient 2 through the origin; its exact equation is $f'(x) = 2x$.
2 Answers are the same as for question 1.
3 The values of $f'(x)$ should be about $1\frac{1}{2},\ 0,\ -0.9,\ -1.2,\ -0.9,\ 0,\ 1\frac{1}{2}$. The graph of $f'(x)$ is a parabola crossing the x-axis at $x = -2$ and $x = 2$.
4 The eventual graph of $f'(x)$ is a parabola with its vertex at the origin. Depending on the software, you may be able to see that it is $y = 3x^2$.

Exercise 8B

1a 3 **b** -7 **c** 5 **d** -3 **e** $\frac{1}{2}$ **f** 0
2a Answers will vary **b** $10h + 5h^2$
c Answers will vary **d** Answers will vary
3a Answers will vary **b** $10xh + 5h^2$
c, d and e Answers will vary
4a, b and c Answers will vary **d** At A, $f'(1) = -2$
e At B, $f'(3) = 2$; at C, $f'(2) = 0$. **f** Answers will vary
5a Answers will vary **b** $5h$ **c** 5
d The value of $\dfrac{f(x+h) - f(x)}{h}$ is a constant 5, so trivially its limit is 5 as $h \to 0$.
6a 10 **b** $\frac{2}{3}$ **c** -1
7a $2xh + h^2$ **b** $f'(x) = 2x$ **c** $f'(0) = 0$ **d** $f'(3) = 6$
8a $2x + h + 4,\ 2x + 4$ **b** $f'(0) = 4,\ f'(-2) = 0$
9a $2x + h - 2,\ f'(x) = 2x - 2$ **b** $f'(0) = -2,\ f'(2) = 2$
10a $2x + h + 6,\ f'(x) = 2x + 6$ **b** $f'(0) = 6,\ f'(-3) = 0$
11a $4 + h,\ f'(2) = 4$ **b** $2h + 3,\ f'(0) = 3$
c $-6 + h,\ f'(-1) = -6$
12a $3x^2$ **b** Answers will vary
13a $4x^3$ **b** Answers will vary
14a–d Answers will vary

Exercise 8C

1a $7x^6$ **b** $5x^4$ **c** $-24x^{23}$ **d** $45x^4$ **e** $6x$ **f** $-60x^{11}$
g $2x^5$ **h** $4x^7$ **i** $-6x^8$
2a Answers will vary **b** Answers will vary
3a 5 **b** -1 **c** $2x + 5$ **d** $6x - 5$ **e** $4x^3 - 10x$
f $-3 - 15x^2$ **g** $4x^3 + 3x^2 + 2x + 1$ **h** $x^3 + x^2 + x$
i $2x^5 - 2x^3 + 2x$
4a $2x + 7$ **b** $f'(0) = 7$
5a $f'(x) = -2x,\ f'(2) = -4$
b $f'(x) = 3x^2 + 6,\ f'(2) = 18$
c $f'(x) = 20x - 4x^3,\ f'(2) = 8$
6a 12 **b** 3 **c** 0 **d** 3 **e** 12
7a $f'(x) = 2x$ **b** $(0, -4)$ **c** $(3, 5)$ **d** $(-3, 5)$
8a $4 - 2x$ **b** $3x^2 + 1$ **c** $6x - 16x^3$ **d** $2x + 2$ **e** $8x$
f $4x^3 + 12x$ **g** $2x - 14$ **h** $3x^2 - 10x + 3$ **i** $18x - 30$
9a $2x + 1$ **b** $f'(0) = 1$ **c** $45°$
10a $-1 + 2x$ **b** Answers will vary **c** $71°34'$
11 $f'(x) = 2x - 3$ **a** $3,\ 71°34'$ **b** $1,\ 45°$ **c** $0,\ 0°$
d $-1,\ 135°$ **e** $-3,\ 108°26'$
12a $f'(x) = 8 - 2x$
b It is a concave-down parabola with x-intercepts $x = 0$ and $x = 8$. **c** $f'(0) = 8,\ f'(8) = -8$ **d** $f'(4) = 0$
13a $2x + 8$ **b** $x = -4,\ (-4, -9)$ **c** $x = 2,\ (2, 27)$
14a $-4x$ **b** $x = 0,\ (0, 3)$ **c** $x = 5,\ (5, -47)$
15a $2x - 2,\ (1, 6)$ **b** $2x + 4,\ (-2, -14)$
c $2x - 10,\ (5, -10)$
16a $f'(x) = 2x - 6$ **b** It is a concave-up parabola with x-intercepts $x = 0$ and $x = 6$.
c $f'(0) = -6,\ f'(6) = 6$ **d** $(3, -9)$
17 $f'(x) = 2x - 5$ **a** $(4, -3)$ **b** $(0, 1)$ **c** $(3, -5)$
d $(2, -5)$
18a $f'(x) = 3x^2 - 3,\ (1, 0),\ (-1, 4)$
b $f'(x) = 4x^3 - 36x,\ (0, 0),\ (3, -81),\ (-3, -81)$
c $f'(x) = 3x^2,\ (5, 131),\ (-5, -119)$
19 The tangent has gradient $2a - 6$.
a i 3 **ii** 4 **iii** $3\frac{1}{4}$ **b** $2\frac{1}{2}$ **c i** $3\frac{1}{3}$ **ii** $2\frac{1}{4}$

Exercise 8D

1a $2x$ **b** $2x + 7$ **c** $3x^2 + 6x + 6$ **d** $4x^3 + 2x + 8$
e 4 **f** 0
2a $\dfrac{dy}{dx} = 6x^5 + 2,\ \dfrac{d^2y}{dx^2} = 30x^4,\ \dfrac{d^3y}{dx^3} = 120x^3$
b $\dfrac{dy}{dx} = 10x - 5x^4,\ \dfrac{d^2y}{dx^2} = 10 - 20x^3,\ \dfrac{d^3y}{dx^3} = -60x^2$
c $\dfrac{dy}{dx} = 4,\ \dfrac{d^2y}{dx^2} = 0,\ \dfrac{d^3y}{dx^3} = 0$
3a $f'(x) = 30x^2 + 1,\ f''(x) = 60x,\ f'''(x) = 60,\ f^{(4)}(x) = 0$
b $f'(x) = 8x^3,\ f''(x) = 24x^2,\ f'''(x) = 48x,\ f^{(4)}(x) = 48$
c $f'(x) = 0,\ f''(x) = 0,\ f'''(x) = 0,\ f^{(4)}(x) = 0$
4 $\dfrac{dy}{dx} = 2x + 1,\ x = 3,\ (3, 12)$
5 $\dfrac{dy}{dx} = 3x^2,\ x = 2$ or $-2,\ (2, 7),\ (-2, -9)$
6a $2x - 3$ **b** 1 **c** $(2, -2)$ **d** $y = x - 4$ **e** -1 **f** $y = -x$
7a $5x^4 + 3x^2 + 2$ **b** $y = 0,\ \dfrac{dy}{dx} = 2$ **c** $y = 2x$ **d** $-\frac{1}{2}$
e $y = -\frac{1}{2}x$
8a They all have derivative $3x^2 + 7$. First to second, shift down 10. First to third, shift down $7\frac{1}{2}$. First to fourth, shift up 96.
b The third has derivative $-2x^3 + 6x$. The other three have derivative $2x^3 + 6x$.
9a $2x$ **b** $6x - 5x^4$ **c** $2x - 3$

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10a $3x^2 - 2,\ 1,\ 10$ **b** $45°,\ 84°17'$
11a $-2 + 2x,\ (1,\ 2)$ **b** $4x^3 + 36x,\ (0,\ 0)$
12 $\dfrac{dy}{dx} = 2x - 8$ **a** $y = -6x + 14,\ x - 6y + 47 = 0$
b $y = 4x - 21,\ x + 4y - 18 = 0$
c $y = -8x + 15,\ x - 8y + 120 = 0$
d $y = -1, x = 4$
13a $2x - 6, y = -6x, y = \frac{1}{6}x$
b $3x^2 - 4,\ y = 8x - 16,\ x + 8y - 2 = 0$
c $2x - 4x^3, y = 2x + 2, x + 2y + 1 = 0$
d $3x^2 - 3, y = 0, x = 1$
14a $y = 4x - 4, y = -\frac{1}{4}x + 4\frac{1}{2}$
b $A = (0, -4),\ B = \left(0, 4\frac{1}{2}\right)$
c $AB = 8\frac{1}{2}$, area $= 8\frac{1}{2}$ square units
15a $y = -2x + 10, x - 2y + 15 = 0$
b $A = (5, 0), B = (-15, 0)$
c $AB = 20$, area $= 80$ square units
16a Answers will vary
b At D, $y = 2x - 5$, at E, $y = -2x - 5$ **c** $(0, -5)$
17a $\dfrac{dy}{dx} = -2x, A = (-1, 3)$ and $B = (1, 3)$
b tangent at A: $y = 2x + 5$, tangent at B: $y = -2x + 5$. They meet at $(0, 5)$.
c normal at A: $y = -\frac{1}{2}x + 2\frac{1}{2}$, normal at B: $y = \frac{1}{2}x + 2\frac{1}{2}$. They meet at $\left(0, 2\frac{1}{2}\right)$.
18 $y = 3x - 2,\ x + 3y = 4,\ P = (0, -2)$,
$Q = \left(0, 1\frac{1}{3}\right)$, $|\Delta QUP| = 1\frac{2}{3}$ square units
19a $5x^4, 20x^3, 60x^2, 120x, 120$. Five successive derivatives are non-zero. **b** 6 **c** n
20a $y = \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$ **b** $y = 2x^3 - 7x + C$
c $y = \frac{5}{4}x^4 + x^3 - 4x + C$ **d** $y = 2x^5 - 4x^3 - 24x + C$
21a $\dfrac{dy}{dx} = 2ax + b$ **b** $x = -\frac{b}{2a}$
c The vertex is the only point on the parabola where the tangent is horizontal.
22a $2a$ **b** $2a$
c Here is one way to express the result: 'Let M be the midpoint of a chord AB of a parabola, and let the line through M parallel to the axis of symmetry meet the parabola at P. Then the chord AB is parallel to the tangent at P.'
23a $b = 7, c = 0$ **b** $b = -2,\ c = -3$ **c** $b = -10,\ c = 25$
d $b = -1,\ c = -2$ **e** $b = -9,\ c = 17$ **f** $b = -5\frac{2}{3},\ c = 7$

Exercise 8E

1a $1x^{-3}$ **b** $-3x^{-4}$ **c** $-\frac{3}{x^4}$
2a $-x^{-2}$ **b** $-5x^{-6}$ **c** $-3x^{-2}$ **d** $-10x^{-3}$ **e** $4x^{-4}$
f $-4x^{-3} - 4x^{-9}$
3a $f(x) = x^{-1},\ f'(x) = -x^{-2} = -\frac{1}{x^2}$
b $f(x) = x^{-2},\ f'(x) = -2x^{-3} = -\frac{2}{x^3}$
c $f(x) = x^{-4},\ f'(x) = -4x^{-5} = -\frac{4}{x^5}$
d $f(x) = 3x^{-1},\ f'(x) = -3x^{-2} = -\frac{3}{x^2}$
4a $\dfrac{dy}{dx} = -6x^{-3}, -6$ **b** $\dfrac{dy}{dx} = -60x^{-5}, -60$
c $\dfrac{dy}{dx} = 2 + 2x^{-2}, 4$ **d** $\dfrac{dy}{dx} = 1 - 30x^{-7}, -29$
e $\dfrac{dy}{dx} = -x^{-3} - x^{-4}, -2$ **f** $\dfrac{dy}{dx} = 6x^5 - 6x^{-7}, 0$
5a $y = x^2 + x, \dfrac{dy}{dx} = 2x + 1$
b $y = x^{-2} + x^{-3}, \dfrac{dy}{dx} = -2x^{-3} - 3x^{-4}$
c $y = 4x^{-1} - 5x^2, \dfrac{dy}{dx} = -4x^{-2} - 10x$
d $y = 3x^{-4} + 3, \dfrac{dy}{dx} = -12x^{-5}$
6a $f'(x) = -\frac{1}{x^2}, f'(3) = -\frac{1}{9}, f'\left(\frac{1}{3}\right) = -9$
b $(1, 1),\ (-1, -1)$ **c** $\left(\frac{1}{2}, 2\right),\ \left(-\frac{1}{2}, -2\right)$
d No; the derivative $-\frac{1}{x^2}$ can never be zero.
e Yes, all of them; the derivative $-\frac{1}{x^2}$ is negative for all points on the curve.
7a $f'(x) = \frac{3}{x^2}, f'(2) = \frac{3}{4}, f'(6) = \frac{1}{12}$
b $(1, -3)$ and $(-1, 3)$
8a $f'(x) = -\frac{12}{x^2}, f'(2) = -3, f'(6) = -\frac{1}{3}$
b At $M(2, 6)$, tangent: $y = -3x + 12$, normal: $x - 3y + 16 = 0$. At $N(6, 2)$, tangent: $y = -\frac{1}{3}x + 4$, normal: $y = 3x - 16$.
c $(1,\ 12)$ and $(-1,\ -12)$
9a $-\frac{6}{x^7} + \frac{8}{x^9}$ **b** $-\frac{1}{3x^2}$ **c** $-\frac{15}{x^4}$ **d** $-\frac{4}{5x^5}$ **e** $\frac{7}{x^2}$ **f** $-\frac{7}{2x^2}$
g $\frac{7}{3x^2}$ **h** $\frac{3}{x^6}$
10a $4x^3 - 2x,\ 2$ **b** $2x^5 - 2x^3 + 2x,\ 2$
c $\frac{1}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{2},\ \frac{1}{2}$ **d** $2x,\ 2$ **e** $6x - 5x^4,\ 1$
f $4x - 5, -1$ **g** $-4x^{-2}, -4$ **h** $-2x^{-4}, -2$
i $-x^{-2} + 2x^{-3},\ 1$
11a $y = 3x^3 - 5x,\ \dfrac{dy}{dx} = 9x^2 - 5$ **b** $y = x^2 - 4,\ \dfrac{dy}{dx} = 2x$
c $y = \frac{5}{3}x^3 + \frac{4}{3}x^2,\ \dfrac{dy}{dx} = 5x^2 + \frac{8}{3}x$
d $y = 3x - x^{-1},\ \dfrac{dy}{dx} = 3 + x^{-2}$
e $y = x^{-3} + 7x^{-2}, \dfrac{dy}{dx} = -3x^{-4} - 14x^{-3}$
f $y = 3x^3 - 5x + x^{-1}, \dfrac{dy}{dx} = 9x^2 - 5 - x^{-2}$
12 $-\dfrac{a}{x^2} + \dfrac{2b}{cx^3}$

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13a $f'(x) = -x^{-2}, f'''(x) = 2x^{-3}, f'''(x) = -6x^{-4},$ $f^{(4)}(x) = 24x^{-5}, f^{(5)}(x) = -120x^{-6}$

b $f'(1) = -1, f''(1) = 2, f'''(1) = -6, f^{(4)}(1) = 24,$ $f^{(5)}(1) = -120$

c Start with -1, then multiply by $-n$ to get each next term.

d Same as before, except that all the terms are negative.

15a The tangent has gradient $2a + 15$, and it passes through $(a, a^2 + 15a + 36)$. Now use point–gradient form.

b Solving $0 = 0 - a^2 + 36$ gives $a = 6$ or $a = -6$. Substituting these values into the equation of the tangent gives $y = 27x$ or $y = 3x$.

Exercise 8F

1a $y = 20x^{\frac{1}{2}}$ **b** $\frac{dy}{dx} = 10x^{-\frac{1}{2}}$ **c** $\frac{dy}{dx} = \frac{10}{x^{\frac{1}{2}}} = \frac{10}{\sqrt{x}}$

2a $\frac{1}{2}x^{-\frac{1}{2}}$ **b** $-\frac{1}{2}x^{-1\frac{1}{2}}$ **c** $\frac{3}{2}x^{\frac{1}{2}}$ **d** $4x^{-\frac{1}{3}}$ **e** $-4x^{-1\frac{1}{3}}$

f $x^{-\frac{3}{4}} - 2x^{-\frac{5}{4}}$ **g** $\frac{49}{3}x^{1\frac{1}{3}}$ **h** $-\frac{10}{3}x^{-1\frac{2}{3}}$ **i** $6x^{-1.6}$

3a $\frac{1}{2}x^{-\frac{1}{2}}$ **b** $\frac{1}{3}x^{-\frac{2}{3}}$ **c** $\frac{1}{4}x^{-\frac{3}{4}}$ **d** $5x^{-\frac{1}{2}}$

4 $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}, \frac{d^2y}{dx} = -\frac{1}{4}x^{-1\frac{1}{2}}$

5a $y = x\sqrt{x} = x^1 \times x^{\frac{1}{2}} = x^{1\frac{1}{2}} = x^{\frac{3}{2}}, \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

b $y = x^2\sqrt{x} = x^2 \times x^{\frac{1}{2}} = x^{2\frac{1}{2}} = x^{\frac{5}{2}}, \frac{dy}{dx} = \frac{5}{2}x^{1\frac{1}{2}}$

c $y = x^{-\frac{1}{2}}, \frac{dy}{dx} = -\frac{1}{2}x^{-1\frac{1}{2}}$

d $y = \frac{1}{x^1 \times x^{\frac{1}{2}}} = x^{-1\frac{1}{2}} = x^{-\frac{3}{2}}, \frac{dy}{dx} = -\frac{3}{2}x^{-2\frac{1}{2}}$

6a $y = x^{\frac{1}{2}}, \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ **b** $\frac{1}{2}$ and $\frac{1}{4}$

c $y = \frac{1}{2}x + \frac{1}{2}, y = \frac{1}{4}x + 1$ **d** $-2, -4$

e $y = -2x + 3, y = -4x + 18$

7a $y = 4x^{-\frac{1}{2}}, \frac{dy}{dx} = -2x^{-1\frac{1}{2}} = -\frac{2}{x\sqrt{x}}$

b $y = -\frac{1}{4}x + 3, y = 4x - 14$

c $\frac{1}{\sqrt{x}}$ is undefined when $x \le 0$; $-\frac{2}{x\sqrt{x}} < 0$, for all $x > 0$

8a $\frac{1}{\sqrt{x}}, \frac{1}{2}, -2$ **b** $y = \frac{1}{2}x + 2, y = -2x + 12$

c $A(-4, 0),\ B(6, 0)$ **d** $AB = 10$, area $= 20$ square units

9a $(1, 1)$ and $(-1, -1)$ **b** $\left(1, \frac{1}{2}\right)$ **c** $\left(\frac{1}{4}, -\frac{1}{2}\right)$

d $(0, 0),\ \left(1, -1\frac{1}{4}\right), \left(-1, \frac{3}{4}\right)$

10a $y = x + 6x^{\frac{1}{2}} + 1, \frac{dy}{dx} = 1 + 3x^{-\frac{1}{2}}$

b $y = 3 - 3x^{\frac{1}{2}} - 8x, \frac{dy}{dx} = -\frac{3}{2}x^{-\frac{1}{2}} - 8$

c $y = 3x^{\frac{1}{2}} - 2, \frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}$

11a $f(x) = 24x^{\frac{1}{2}}, f'(x) = 12x^{-\frac{1}{2}}$

b $f(x) = 8x^{\frac{1}{2}}, f'(x) = 4x^{-\frac{1}{2}}$

c $f(x) = 5x^{\frac{1}{2}}, f'(x) = \frac{5}{2}x^{-\frac{1}{2}}$ **d** $f(x) = 2x^{1\frac{1}{2}}, f'(x) = 3\sqrt{x}$

e $f(x) = 12x^{1\frac{1}{2}}, f'(x) = 18\sqrt{x}$

f $f(x) = 4x^{2\frac{1}{2}}, f'(x) = 10x^{\frac{3}{2}}$

g $f(x) = 24x^{\frac{1}{3}}, f'(x) = 8x^{-\frac{2}{3}}$ **h** $f(x) = x^{\frac{2}{3}}, f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

i $f(x) = 30x^{\frac{2}{3}}, f'(x) = 20x^{-\frac{1}{3}}$

j $f(x) = x^{-\frac{1}{2}}, f'(x) = -\frac{1}{2}x^{-1\frac{1}{2}}$

k $f(x) = 6x^{-\frac{1}{2}}, f'(x) = -3x^{-\frac{3}{2}}$

l $f(x) = 5x^{-1\frac{1}{2}}, f'(x) = -\frac{15}{2}x^{-2\frac{1}{2}}$

12a–c Answers will vary

13a At P, $\frac{dy}{dx} = 2a - 10$. **b** At P, $y = a^2 - 10a + 9$.

c $a = 3$ and $y = -4x$, or $a = -3$ and $y = -16x$

Exercise 8G

1 $\frac{du}{dx} = 2x, \frac{dy}{du} = 5u^4,$

$\frac{dy}{dx} = 5(x^2 + 9)^4 \times 2x = 10x(x^2 + 9)^4$

2a $12(3x + 7)^3$ **b** $30(5x - 9)^5$ **c** $-28(5 - 4x)^6$

d $-4(1 - x)^3$ **e** $24x(x^2 + 1)^{11}$ **f** $14x(x^2 - 2)^6$

3a $-7(7x + 2)^{-2}$ **b** $-6(x - 1)^{-3}$ **c** $-12x^2(x^3 - 12)^{-5}$

d $-30x(5x^2 - 2)^{-4}$ **e** $-64x(7 - x^2)^3$

f $-18(3x^2 + 1)(x^3 + x + 1)^5$

4a $25(5x - 7)^4$ **b** $49(7x + 3)^6$ **c** $180(5x + 3)^3$

d $-21(4 - 3x)^6$ **e** $-22(3 - x)$ **f** $-28(4x - 5)^{-8}$

g $-30(3x + 7)^{-6}$ **h** $12(10 - 3x)^{-5}$ **i** $84(5 - 7x)^{-5}$

5a and **b** $2x - 6$

6a and **b** $24x - 12$

7 $f'(x) = 10(2x + 3)^4, f''(x) = 80(2x + 3)^3$

8a $-6x(5 - x^2)^2$ **b** $42x(3x^2 + 7)^6$ **c** $16x^3(x^4 + 1)^3$

d $45x^2(3x^3 - 7)^4$ **e** $-5(3x^2 - 2x)(x^3 - x^2)^{-6}$

f $-9(2x + 3)(x^2 + 3x + 1)^{-10}$

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9a $y = (2x+7)^{-1}, \dfrac{dy}{dx} = \dfrac{-2}{(2x+7)^2}$

b $y = (2-x)^{-1}, \dfrac{dy}{dx} = \dfrac{1}{(2-x)^2}$

c $y = (3+5x)^{-1}, \dfrac{dy}{dx} = \dfrac{-5}{(3+5x)^2}$

d $y = 7(4-3x)^{-1}, \dfrac{dy}{dx} = \dfrac{21}{(4-3x)^2}$

e $y = 4(3x-1)^{-5}, \dfrac{dy}{dx} = \dfrac{-60}{(3x-1)^6}$

f $y = -5(x+1)^{-3}, \dfrac{dy}{dx} = \dfrac{15}{(x+1)^4}$

10a $20(5x-4)^3$ **b** $y = 1, \dfrac{dy}{dx} = 20$

c $y = 20x - 19,\ x + 20y = 21$

11a $y = 24x - 16$ **b** $3x + y = 4$ **c** $x + 2y = 2$

12a $4(x-5)^3, (5, 0)$

b $6x(x^2-1)^2, (0,-1), (1,0), (-1,0)$

c $10(x+1)(2x+x^2)^4, (0,0), (-2,0), (-1,-1)$

d $\dfrac{-5}{(5x+2)^2}$, none **e** $6(x-5)^5, (5,4)$

f $\dfrac{-2x}{(1+x^2)^2}, (0,1)$

13a $2\frac{1}{2}$ and 1 **b** 2 and $1\frac{1}{2}$

14a $\dfrac{dy}{dx} = 3(6x+4)^{-\frac{1}{2}}, \dfrac{3}{4}$ **b** $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2x+5}}, \dfrac{1}{3}$

c $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-3}}, 2$

15a $-\dfrac{1}{\sqrt{3-2x}}$, none **b** $\dfrac{x-1}{\sqrt{x^2-2x+5}}, (1,2)$

c $\dfrac{x-1}{\sqrt{x^2-2x}}$, none ($x = 1$ is outside the domain.)

16a $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{25-x^2}}$ **b** Answers will vary **c** The tangent at P has gradient $-\frac{3}{4}$, the radius OP has gradient $\frac{4}{3}$.

17a $\dfrac{11(\sqrt{x}-3)^{10}}{2\sqrt{x}}$ **b** $\dfrac{-3}{4\sqrt{4-\frac{1}{2}x}}$ **c** $\dfrac{3\sqrt{2}}{\left(1-x\sqrt{2}\right)^2}$

d $\frac{1}{2}(5-x)^{-1\frac{1}{2}}$ **e** $\frac{1}{2}a^2(1+ax)^{-1\frac{1}{2}}$ **f** $\frac{1}{2}b(c-x)^{-1\frac{1}{2}}$

g $-16\left(1-\dfrac{1}{x^2}\right)\left(x+\dfrac{1}{x}\right)^3$

h $3\left(\dfrac{1}{\sqrt{x}} - \dfrac{1}{x\sqrt{x}}\right)\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^5$

18a $\dfrac{dy}{dx} = 3(x-a)^2, a = 4$, or $a = 8$

b $\dfrac{dy}{dx} = \dfrac{-1}{(x+a)^2}$, $a = -5$ or $a = -7$

19a $a = 8,\ b = 1$ **b** $a = \dfrac{1}{16},\ b = 12$

20a $x + y(b-4)^2 = 2b - 4$ **b i** $x + 4y = 0$ **ii** $x + y = 6$

Exercise 8H

1 Let $u = 5x$
and $v = (x-2)^4$
Then $\dfrac{du}{dx} = 5$
and $\dfrac{dv}{dx} = 4(x-2)^3$
Let $y = 5x(x-2)^4$

$$\begin{aligned} \text{Then } \frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\ &= (x-2)^4 \times 5 + 5x \times 4(x-2)^3 \\ &= 5(x-2)^4 + 20x(x-2)^3 \\ &= 5(x-2)^3((x-2)+4x) \\ &= 5(x-2)^3(5x-2) \end{aligned}$$

2a and **b** $2x^2(2x-3) = 4x^3 - 6x^2$

3a and **b** $4x - 9$

4a and **b** $4x^3$

5a $u' = 4x^3, v' = 10(2x-1)^4$ **b** Answers will vary

c Answers will vary **d** $x = 0,\ x = \frac{1}{2},\ x = \frac{2}{9}$

6a $(3x+5)^2(12x+5)$ **b** $x(x-1)^2(5x-2)$

c $2x^3(1-5x)^5(2-25x)$

7 $y = x, y = -x$

8a $(x-1)^3(5x-1), 1, \frac{1}{5}$ **b** $(x+5)^4(6x+5), -5, -\frac{5}{6}$

c $2(4-3x)^4(2-9x), \frac{4}{3}, \frac{2}{9}$

d $3(3-2x)^4(1-4x), 1\frac{1}{2}, \frac{1}{4}$

e $x^2(x+1)^3(7x+3), 0, -1, -\frac{3}{7}$

f $3x^2(3x-2)^3(7x-2), 0, \frac{2}{3}, \frac{2}{7}$

g $x^4(1-x)^6(5-12x), 0, 1, \frac{5}{12}$

h $(x-2)^2(4x-5), 2, \frac{5}{4}$

i $(x+5)^5(7x+17), -5, -\frac{17}{7}$

9a Answers will vary **b** $(0, 0)$, $(1, 0)$, and $\left(\frac{3}{8}, \left(\frac{3}{8}\right)^3 \times \left(\frac{5}{8}\right)^5\right)$

10a Answers will vary **b** $y = 2x - 1,\ y = -\frac{1}{2}x + 1\frac{1}{2}$

11 $y = 8x + 8,\ x + 8y + 1 = 0$

12a $10x(x^2+1)^4, (x^2+1)^4(11x^2+1)$,

b $-8x(1-x^2)^3, 2x^2(1-x^2)^3(3-11x^2)$

c $3(2x+1)(x^2+x+1)^2$,
$-2(x^2+x+1)^2(7x^2+4x+1)$

d $(4-9x^4)^3(4-153x^4)$

13a $10x^3(x^2-10)^2(x^2-4)$

b $(0,0), \left(\sqrt{10}, 0\right), \left(-\sqrt{10}, 0\right), (2, -3456), (-2, -3456)$

14a $\dfrac{3(3x+2)}{\sqrt{x+1}}, -\frac{2}{3}$ **b** $\dfrac{4(3x-1)}{\sqrt{1-2x}}, \frac{1}{3}$ **c** $\dfrac{10x(5x-2)}{\sqrt{2x-1}}$, none
(0 and $\frac{2}{5}$ are outside the domain.)

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15a $(x + 1)^2(x + 2)^3(7x + 10), -1, -2, -\frac{10}{7}$

b $6(2x - 3)^3(2x + 3)^4(6x - 1),\ 1\frac{1}{2}, -1\frac{1}{2}, \frac{1}{6}$

c $\dfrac{1 - 2x^2}{\sqrt{1 - x^2}}, \sqrt{\frac{1}{2}}$ and $-\sqrt{\frac{1}{2}}$

16a $y' = 2a(x - 3)$ **b** $y'(1) = -4a, y'(5) = 4a$

c $y = -4ax + 4a, y = 4ax - 20a, M = (3, -8a)$

d $V = (3, -4a)$ **e** Answers will vary

Exercise 8I

1 Let $u = 2x + 3$

and $v = 3x + 2$

Then $u' = 2$

and $v' = 3$

Let $y = \dfrac{2x + 3}{3x + 2}$

$$\begin{aligned}\text{Then } \frac{dy}{dx} &= \frac{vu' - uv'}{v^2}\\ &= \frac{(3x + 2) \times 2 - (2x + 3) \times 3}{(3x + 2)^2}\\ &= \frac{6x + 4 - 6x - 9}{(3x + 2)^2}\\ &= \frac{-5}{(3x + 2)^2}\end{aligned}$$

2a $\dfrac{1}{(x + 1)^2}$ **b** $\dfrac{4}{(x + 2)^2}$ **c** $\dfrac{1}{(1 - 3x)^2}$ **d** $\dfrac{-2}{(x - 1)^2}$

e $\dfrac{-4}{(x - 2)^2}$ **f** $\dfrac{4}{(x + 2)^2}$ **g** $\dfrac{-5}{(2x - 3)^2}$ **h** $\dfrac{-40}{(5 + 4x)^2}$

3a $\dfrac{x(x + 2)}{(x + 1)^2}, x = 0, x = -2$ **b** $\dfrac{3 + x^2}{(3 - x^2)^2}$, none

c $\dfrac{x(2 - x)}{(1 - x)^2}, x = 0, x = 2$ **d** $\dfrac{1 + x^2}{(1 - x^2)^2}$, none

e $\dfrac{4x}{(x^2 + 1)^2}, x = 0$ **f** $\dfrac{10x}{(x^2 - 4)^2}, x = 0$

4a and **b** $\dfrac{-3}{(3x - 2)^2}$

5a Answers will vary **b** $5, 78°41'$

c $y = 5x - 12, x + 5y + 8 = 0$

6a Answers will vary **b** $\frac{4}{3}, 53°8'$

c $4x - 3y = 4,\ 3x + 4y = 28$

7a $y = x$ **b** Answers will vary **c** $A(-1, 0), B\left(0, \frac{1}{4}\right)$

d area $= \frac{1}{8}$ square units

e $\left(\frac{1}{3}, \frac{1}{3}\right)$

8a $y' = \dfrac{x^2 + 2x}{(x + 1)^2}, c = 0$ or -2

b $y' = \dfrac{-4kx}{(x^2 - k)^2}, \dfrac{12k}{(9 - k)^2} = 1, k = 3$ or 27

9a $12(3x - 7)^3$ **b** $\dfrac{x^2 + 2}{x^2}$ **c** $8x$ **d** $\dfrac{-2x}{(x^2 - 9)^2}$

e $4(1 - x)(4 - x)^2$ **f** $\dfrac{-6}{(3 + x)^2}$ **g** $20x^3(x^4 - 1)^4$

h $\dfrac{1}{2(2 - x)^{\frac{3}{2}}}$ **i** $6x^2(x^3 + 5)$ **j** $\dfrac{3x^2 + x - 1}{4x\sqrt{x}}$

k $\frac{2}{3}x(5x^3 - 2)$ **l** $\dfrac{5}{(x + 5)^2}$ **m** $\frac{1}{2}\sqrt{x}(3 + 5x)$

n $\dfrac{2(x - 1)(x + 1)(x^2 + 1)}{x^3}$ **o** $x^2(x - 1)^7(11x - 3)$

p $\dfrac{(x + 1)(x - 1)}{x^2}$

10a Answers will vary

b The denominator is positive, being a square, so the sign of y' is the sign of $a - b$.

Exercise 8J

1a $\dfrac{dQ}{dt} = 3t^2 - 20t$ **b** When $t = 2, Q = -32, \dfrac{dQ}{dt} = -28$.

2a $\dfrac{dQ}{dt} = 2t + 6$ **b** When $t = 2, Q = 16, \dfrac{dQ}{dt} = 10$

c i $t = -3$ **ii** $t > -3$ **iii** $t < -3$

3a 7 and 15 **b** $\frac{15 - 7}{3 - 1} = 4$ **c** $\frac{7 - 15}{7 - 5} = -4$

4a 180 mL **b** When $t = 0,\ V = 0$. **c** 300 mL **d** 60 mL/s

e The derivative is a constant function.

5a 80 000 litres **b** 35 000 litres **c** 20 min

d 2 000 litres/min

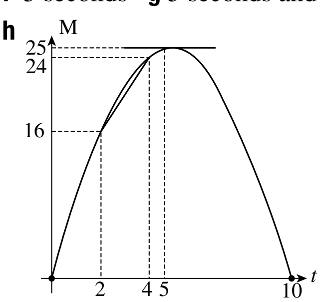
6a $\dfrac{dM}{dt} = 10 - 2t$ **b** $M = 24\,\text{kg}, \dfrac{dM}{dt} = 2\,\text{kg/s}$

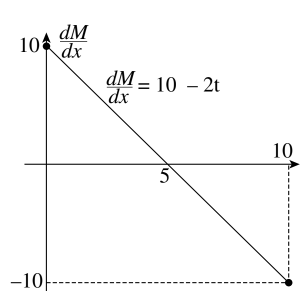
c $M = 16\,\text{kg}$, average rate $= \dfrac{24 - 16}{4 - 2} = 4\,\text{kg/s}$

d 0 seconds and 10 seconds **e** 10 seconds

f 5 seconds **g** 5 seconds and 5 seconds

h





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7a \$2 **b** \$5.60

c $\dfrac{dP}{dt} = -0.8t + 4$, \$2.40 per annum

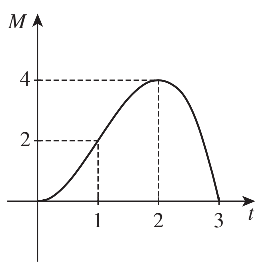
d $t = 5$, the start of 1975

e The price was increasing before then, and decreasing afterwards.

f $\dfrac{dP}{dt}$ is linear with negative gradient -0.8.

g At the start of 1980.

8a



b $t = 2$

c $\dfrac{dM}{dt} = 6t - 3t^2$, $t = 1$

d $t = 1$

9 The scheme appears to have worked initially and the level of pollution decreased, but the rate at which the pollution decreased gradually slowed down and was almost zero in 2000. A new scheme would have been required to remove the remaining pollution.

10a The graph is stationary on 1 July and 1 March.

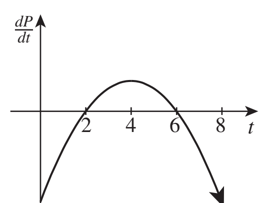
b It is maximum on 1 July and on 1 January. The price is locally minimum on 1 March, but globally the graph has no minimum.

c It is increasing from 1 March to 1 July. It is decreasing from 1 January to 1 March and after 1 July.

d on 1 May

e from 1 March to 1 May

f



11a $A = \pi r^2 = \pi\left(\frac{t}{3}\right)^2 = \frac{\pi}{9}t^2$ **b** $\dfrac{dA}{dt} = \frac{2\pi}{9}t$

c When $A = 5$, $t = \sqrt{\frac{45}{\pi}} \doteqdot 3.785$ s and

$\dfrac{dA}{dt} = \frac{2\pi}{9}\sqrt{\frac{45}{\pi}} \doteqdot 2.642$ km^2/s

12a When $t = 0$, $h = 80$, so the building is 80 m tall.

b When $h = 0$, $t = 4$, so it takes 4 seconds.

c $v = -10t$

d When $t = 4$, $v = -40$, so the stone hits the ground at 40 m/s.

e 10 m/s^2 downwards

13a Yes. $\dfrac{dv}{dt} = -\frac{1}{2}$, meaning his velocity decreased at a constant rate of $\frac{1}{2}$ m/s every second, just as he said.

b Yes. $\dfrac{dx}{dt} = -\frac{1}{2}t + 50$, which is what the truck's speed monitor recorded.

c Yes. $\dfrac{dy}{dt} = -\frac{1}{2}t + 50$, which also agrees with the truck's speed monitor.

d When $t = 0$, $x = 0$ and $y = 450$, so the truck was 450 m ahead.

e Solving $-\frac{1}{2}t + 50 = 0$ gives 100 seconds. When $t = 100$, $x = 2500$ m or 2.5 km.

f When $t = 0$, $v = 50$ m/s, which is 180 km/h. He was in court for speeding.

14a i Area $= h^2$ cm^2 **ii** Volume $= 3000h^2$ cm^3

b i $h = 3\sqrt{t}$, $\dfrac{dh}{dt} = \frac{3}{2\sqrt{t}}$

ii $h = 15$ cm, $\dfrac{dh}{dt} = \frac{3}{10}$ cm/min

iii 100 min, $\frac{3}{20}$ cm/min

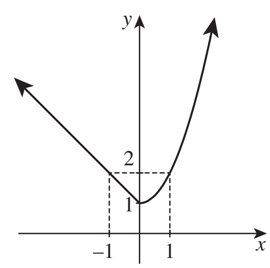
Exercise 8K

1a $x = 6$ **b** $x = 1, x = 3, x = 5$. **c** $x = -2, x = -3$

2a $f(0) = 1$. First table: 3, 2, 1. Second table: 1, 2, 5

b Yes

c

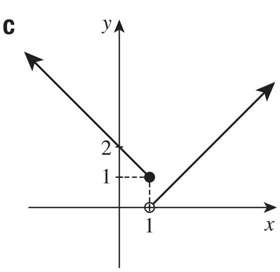


d domain: all real x, range: $y \geq 1$

3a $f(1) = 1$. When $x = 1$, $2 - x = 1$ and $x - 1 = 0$.

b No

c



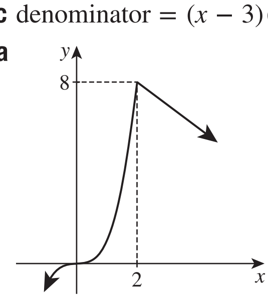
d domain: all real x, range: $y > 0$

4a denominator $= x(x - 5)$, $x = 0, x = 5$

b denominator $= (x - 2)(x - 3)$, $x = 2, x = 3$

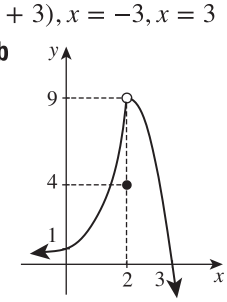
c denominator $= (x - 3)(x + 3)$, $x = -3, x = 3$

5a



$f(2) = 8$. When $x = 2$, $x^3 = 8$ and $10 - x = 8$. Thus $f(x)$ is continuous at $x = 2$.
domain: all real x,
range: $y \leq 8$

b



$f(2) = 4$. When $x = 2$, $3^x = 9$ and $13 - x^2 = 9$. Thus $f(x)$ is not continuous at $x = 2$.
domain: all real x,
range: $y < 9$

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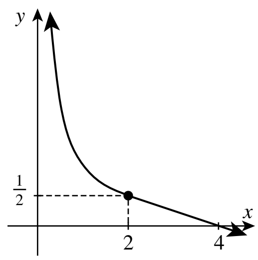
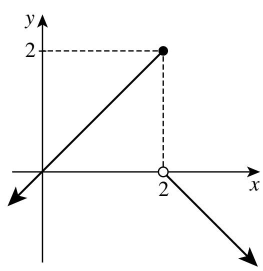
c



$f(2) = \frac{1}{2}$. When $x = 2, \frac{1}{x} = \frac{1}{2}$ and $1 - \frac{1}{4}x = \frac{1}{2}$.
Thus $f(x)$ is continuous at $x = 2$.
domain: $x > 0$, range: all real y

d



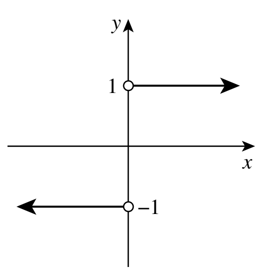
$f(2) = 2$. When $x = 2$, $x = 2$, but $2 - x = 0$.
Thus $f(x)$ is not continuous at $x = 2$.
domain: all real x, range $y \le 2$

6a The table of values should make it clear that

$$y = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0, \\ \text{undefined}, & \text{for } x = 0. \end{cases}$$

b The curve is not continuous at $x = 0$ — it is not even defined there.

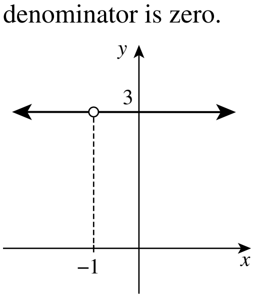
c



domain: $x \neq 0$, range: $y = 1$ or -1

7a The graph is not continuous at $x = -1$ because the denominator is zero.

b



domain: $x \neq -1$, range: $y = 3$

c $y = \dfrac{3(x+1)}{x+1} = \begin{cases} 3, & \text{when } x \neq -1 \\ \text{undefined}, & \text{when } x = -1 \end{cases}$

Hence the graph is the line $y = 3$, except that the point $(-1,\ 3)$ is removed.

Exercise 8L

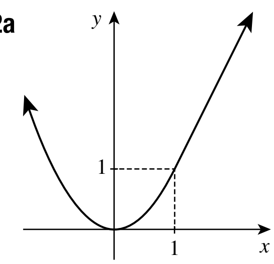
1a continuous and differentiable at $x = 0$, neither at $x = 2$

b continuous and differentiable at $x = 0$, continuous but not differentiable at $x = 2$

c neither at $x = 0$, continuous and differentiable at $x = 2$

d neither at $x = 0$, continuous but not differentiable at $x = 2$

2a

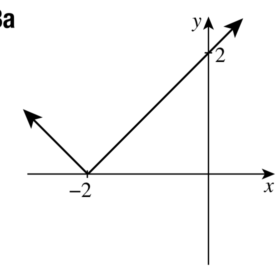


b $f(1) = 1$,
$x^2 = 1$ when $x = 1$,
$2x - 1 = 1$ when $x = 1$

c Answers will vary

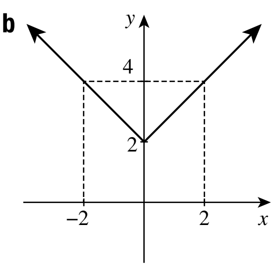
d $2x = 2$ when $x = 1$, and $2 = 2$ when $x = 1$. The tangent at $x = 1$ has gradient 2, so $f'(1) = 2$.

3a



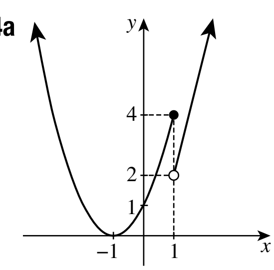
continuous but not differentiable at $x = -2$

b



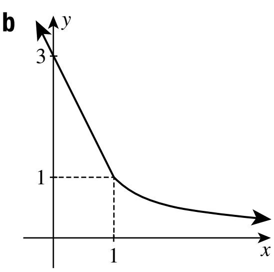
continuous but not differentiable at $x = 0$

4a



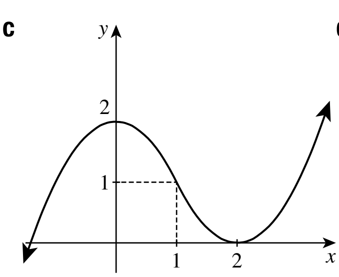
not even continuous at $x = 1$

b



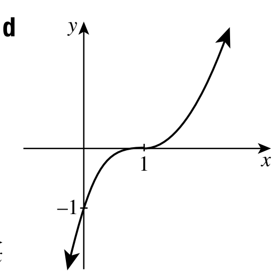
continuous but not differentiable at $x = 1$

c



differentiable at $x = 1$, $f'(1) = -2$

d



differentiable at $x = 1$, $f'(1) = 0$

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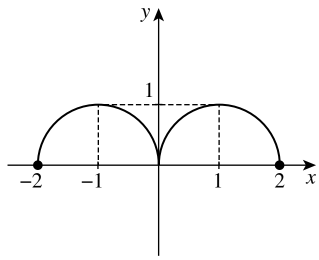
5



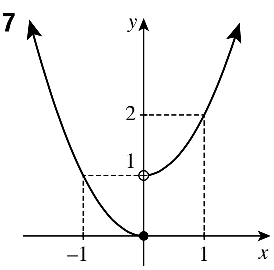
6a Differentiable at $x = 0$. x^2 is never negative, so $|x^2| = x^2$ for all x.
b Differentiable at $x = 0$. x^3 is flat at $x = 0$, so $|x^3|$ is also flat at $x = 0$.
c Continuous, but not differentiable, at $x = 0$. The graph of $y = \sqrt{x}$ becomes vertical near $x = 0$.
d Continuous, but not differentiable, at $x = 0$. The graph has a vertical tangent at the origin.

Chapter 8 review exercise

1a $2x + 5$ **b** $-2x$ **c** $6x - 2$
2a $3x^2 - 4x + 3$ **b** $6x^5 - 16x^3$ **c** $9x^2 - 30x^4$ **d** $2x + 1$
e $-12x + 7$ **f** $-6x^{-3} + 2x^{-2}$ **g** $12x^2 + 12x^{-4}$
h $\frac{3}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-1\frac{1}{2}}$ **i** $x^{-2} - 2x^{-3}$
3a $f'(x) = 4x^3 + 3x^2 + 2x + 1, f''(x) = 12x^2 + 6x + 2$
b $f'(x) = -10x^{-3}, f''(x) = 30x^{-4}$
c $f'(x) = -4x^{-\frac{3}{2}}, f''(x) = 6x^{-\frac{5}{2}}$
4a $y = x^3 + 4x + C$
b $y = 7x - 6x^2 - 4x^3 + C$
c $y = 4x^5 - 4x^3 + 4x + C$
5a $-\dfrac{3}{x^2}$ **b** $-\dfrac{1}{3x^3}$ **c** $\dfrac{7}{2\sqrt{x}}$ **d** $\dfrac{6}{\sqrt{x}}$ **e** $-\dfrac{9}{2}\sqrt{x}$ **f** $-\dfrac{3}{x\sqrt{x}}$
6a $6x - 2$ **b** $x - \frac{1}{2}$ **c** $10x + \dfrac{7}{x^2}$ **d** $-\dfrac{2}{x^2} - \dfrac{2}{x^3}$ **e** $\dfrac{2}{\sqrt{x}}$
f $3\sqrt{x} + \dfrac{3}{2\sqrt{x}}$
7a $9(3x + 7)^2$ **b** $-4(5 - 2x)$ **c** $-\dfrac{5}{(5x - 1)^2}$
d $\dfrac{14}{(2 - 7x)^3}$ **e** $\dfrac{5}{2\sqrt{5x + 1}}$ **f** $\dfrac{1}{2(1 - x)^{\frac{3}{2}}}$
8a $42x(7x^2 - 1)^2$ **b** $-15x^2(1 + x^3)^{-6}$
c $8(1 - 2x)(1 + x - x^2)^7$ **d** $-6x(x^2 - 1)^{-4}$
e $-\dfrac{x}{\sqrt{9 - x^2}}$ **f** $\dfrac{x}{(9 - x^2)^{\frac{3}{2}}}$

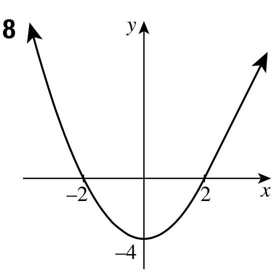
9a $x^8(x + 1)^6(16x + 9)$ **b** $\dfrac{x(2 - x)}{(1 - x)^2}$
c $2x(4x^2 + 1)^3(20x^2 + 1)$ **d** $\dfrac{12}{(2x + 3)^2}$
e $(9x - 1)(x + 1)^4(x - 1)^3$ **f** $\dfrac{(x - 5)(x + 1)}{(x - 2)^2}$
10 $\dfrac{dy}{dx} = 2x + 3$ **a** $3, 71°34'$ **b** $1, 45°$ **c** $-1, 135°$
11a tangent: $y = -3x$, normal: $3y = x$
b tangent: $y = -2$, normal: $x = 1$
c $(1, -2)$ and $(-1, 2)$
d $(2, 2)$ and $(-2, -2)$
12a $y = -x - 4, y = x - 8$ **b** $A(-4, 0), B(8, 0)$
c $AB = 12$, area $= 36$ square units
13 The tangent is $y = x$.
14a $\left(1, -6\frac{2}{3}\right), \left(-1, -7\frac{1}{3}\right)$ **b** $\left(-1, \frac{2}{3}\right)$
15 At $(1, -3)$ the tangent is l: $x + y + 2 = 0$, at $(-1, 3)$ the tangent is $x + y - 2 = 0$.
16a $V = \frac{4}{3}\pi \times \left(\frac{t}{3}\right)^3 = \frac{4\pi}{81}t^3$ **b** $\dfrac{dV}{dt} = \frac{4\pi}{27}t^2$
c $V \doteqdot \frac{4\pi}{81} \times 0.001 \doteqdot 0.00016\,\text{km}^3$,
$\dfrac{dV}{dt} \doteqdot \frac{4\pi}{27} \times 0.01 \doteqdot 0.0047\,\text{km}^3/\text{s}$ **d** $t^2 = \frac{27}{4\pi}, t \doteqdot 1.5\,\text{s}$

17



a $f(0) = 0$, $x^2 = 0$ when $x = 0$, $x^2 + 1 = 1$ when $x = 0$, so it is not continuous at $x = 0$.
b domain: all real x, range: $y \ge 0$

18



a $f(0) = 2$, $x^2 - 4 = 0$ when $x = 2$, $4x - 8 = 0$ when $x = 2$, so it is continuous at $x = 2$.
b $f'(2) = 4$ when $x < 2$ (substitute into $2x$), $f'(2) = 4$ when $x > 2$ (substitute into 4), so it is differentiable at $x = 2$, with $f'(2) = 4$
c domain: all real x, range: $y \ge 4$

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Chapter 9

Exercise 9A

1d

x	-2	-1	0	1	2
height y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
gradient $\frac{dy}{dx}$	0.17	0.35	0.69	1.39	2.77
$\frac{\text{gradient}}{\text{height}}$	0.69	0.69	0.69	0.69	0.69

e All the ratios are about 0.7. **f** $\frac{dy}{dx} \doteqdot 0.7y$

2b Both are equal to 1.

c

height y	$\frac{1}{2}$	1	2	3
gradient $\frac{y}{x}$	$\frac{1}{2}$	1	2	3
$\frac{\text{gradient}}{\text{height}}$	1	1	1	1

d They are all equal to 1. **e** $\frac{dy}{dx} = y$

3c The values are: 0.14, 0.37, 1, 2.72.

d The x-intercept is always 1 unit to the left of the point of contact.

4a i AB has gradient 1.

ii The curve is concave up, so the chord is steeper than the tangent.

b i CA has gradient 1.

ii The curve is concave up, so the chord is not as steep as the tangent.

c As the base increases, the gradient at the y-intercept increases. With $y = 2^x$, the gradient at the y-intercept is less than 1, and with $y = 4^x$, the gradient at the y-intercept is greater than 1. Hence the base e for which the y-intercept is exactly 1 is between 2 and 4.

6 a–f The values get closer and closer to the limit: $\log_e 2 \doteqdot 0.69315$

Exercise 9B

1a 7.3891 **b** 20.0855 **c** 22 026.4658

d 1.0000 **e** 2.7183 **f** 0.3679

g 0.1353 **h** 1.6487 **i** 0.6065

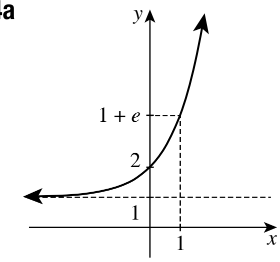
2a $e^1 \doteqdot 2.718$ **b** $e^{-1} \doteqdot 0.3679$ **c** $e^{-4} \doteqdot 0.01832$

d $e^{\frac{1}{2}} \doteqdot 1.649$ **e** $e^{\frac{1}{3}} \doteqdot 1.396$ **f** $e^{-\frac{1}{2}} \doteqdot 0.6065$

3a $5e^2 \doteqdot 36.95$ **b** $\frac{1}{64}e^6 \doteqdot 6.304$ **c** $7e^{\frac{1}{2}} \doteqdot 11.54$

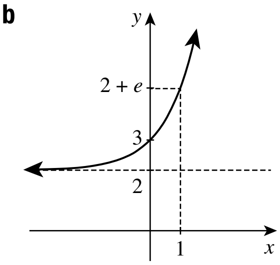
d $\frac{3}{5}e^{\frac{1}{2}} \doteqdot 0.9892$ **e** $4e^{-1} \doteqdot 1.472$ **f** $\frac{5}{7}e^{-4} \doteqdot 0.01308$

4a



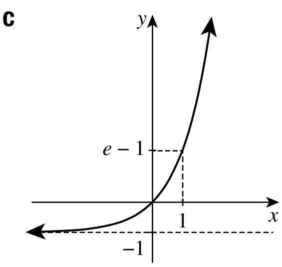
Shift up 1 unit, asymptote: $y = 1$, range: $y > 1$

b



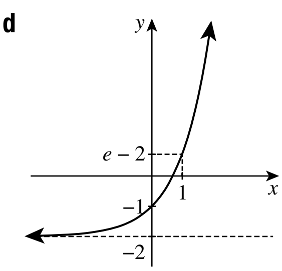
Shift up 2 units, asymptote: $y = 2$, range: $y > 2$

c



Shift down 1 unit, asymptote: $y = -1$, range: $y > -1$

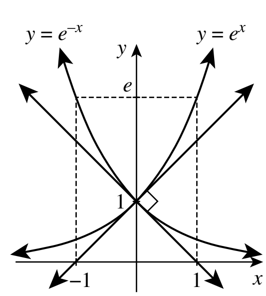
d



Shift down 2 units, asymptote: $y = -2$, range: $y > -2$

5a e^x: 0.14, 0.37, 1.00, 2.72, 7.39.
e^{-x}: 7.39, 2.72, 1.00, 0.37, 0.14

b



c Reflection in the y-axis.

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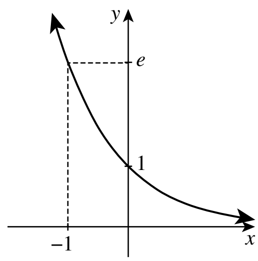
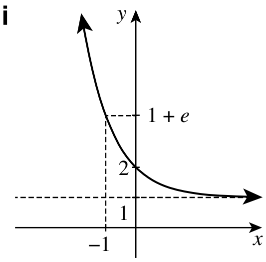
d The graph of $y = e^{-x}$ is the reflection of $y = e^x$ in the y-axis, so its gradient at the y-intercept is -1. Hence the two tangents are perpendicular because the product of their gradients is -1 (or because $45° + 45° = 90°$).

6a



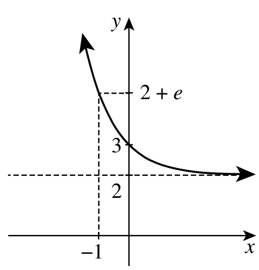
Asymptote: $y = 0$, range: $y > 0$

b i



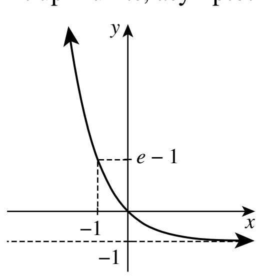
Shift up 1 unit, asymptote: $y = 1$, range: $y > 1$

ii



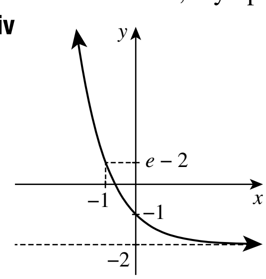
Shift up 2 units, asymptote: $y = 2$, range: $y > 2$

iii



Shift down 1 unit, asymptote: $y = -1$, range: $y > -1$

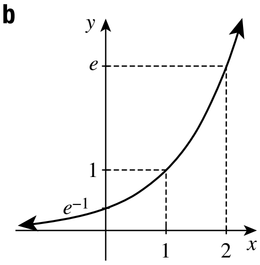
iv



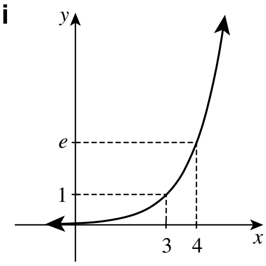
Shift down 2 units, asymptote: $y = -2$, range: $y > -2$

7a Shift right 1 unit

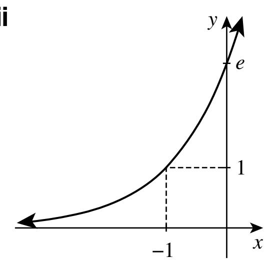
b



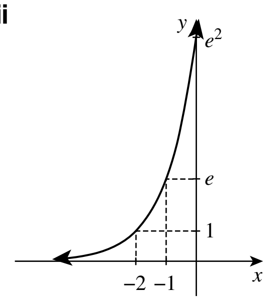
c i



ii



iii



8a $1, e, e^2, e^3$

b i gradient of $AB = e - 1$

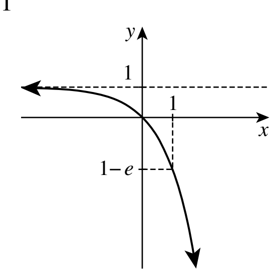
ii AB: $y - 1 = (e - 1)x$

iii Answers will vary

c gradient of $BC = e^2 - e$, BC: $y - e = (e^2 - e)(x - 1)$

d gradient of $CD = e^3 - e^2$, BC: $y - e^2 = (e^3 - e^2)(x - 2)$

9a $y < 1$



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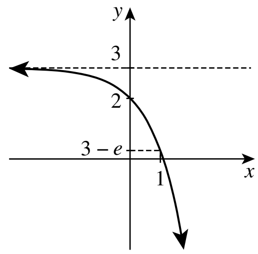
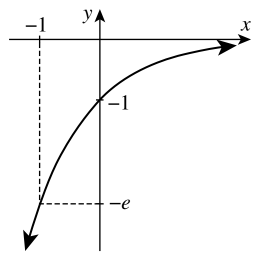
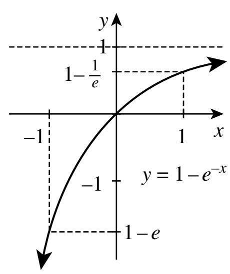
b $y < 3$



c $y < 0$



d $y < 1$



Exercise 9C

1a $2e^{2x}$ **b** $7e^{7x}$ **c** $-e^{-x}$ **d** $-5e^{5x}$

e $\frac{1}{2}e^{\frac{1}{2}x}$ **f** $2e^{\frac{1}{3}x}$ **g** $-\frac{1}{3}e^{-\frac{1}{3}x}$ **h** $e^{\frac{1}{5}x}$

2a $f'(x) = e^{x+2}$ **b** $f'(x) = e^{x-3}$

c $f'(x) = 5e^{5x+1}$ **d** $f'(x) = 2e^{2x-1}$

e $f'(x) = -4e^{-4x+1}$ **f** $f'(x) = -3e^{-3x+4}$

g $f'(x) = -3e^{-3x-6}$ **h** $f'(x) = e^{\frac{1}{2}x+4}$

3a $e^x - e^{-x}$ **b** $2e^{2x} + 3e^{-3x}$

c $e^{2x} + e^{3x}$ **d** $e^{4x} + e^{5x}$

e $\frac{e^x + e^{-x}}{2}$ **f** $\frac{e^x - e^{-x}}{3}$

4a $y' = 2e^{2x}$

b When $x = 0$, $f'(y') = 2$. When $x = 4$, $f'(y') = 2e^8$.

5a $f'(x) = -e^{-x+3}$

b When $x = 0$, $f'(x) = -e^3$. When $x = 4$, $f'(x) = -e^{-1}$.

6a $y' = 3e^{3x}, y'(2) = 3e^6 \doteqdot 1210.29$

b $y' = -2e^{-2x}, y'(2) = -2e^{-4} \doteqdot -0.04$

c $y' = \frac{3}{2}e^{\frac{3}{2}x}, y'(2) = \frac{3}{2}e^3 \doteqdot 30.13$

7a $-e^{-x}, e^{-x}, -e^{-x}, e^{-x}$

b Successive derivatives alternate in sign.

More precisely, $f^{(n)}(x) = \begin{cases} e^{-x} & \text{if } n \text{ is even,} \\ -e^{-x} & \text{if } n \text{ is odd.} \end{cases}$

8a $2e^{2x}, 4e^{2x}, 8e^{2x}, 16e^{2x}$

b Each derivative is twice the previous one. More precisely, $f^{(n)}(x) = 2^n e^{2x}$.

9a e^x, e^x, e^x, e^x

b All derivatives are the same, and are equal to the original function.

10 $y' = e^x + 2x + 1, y'' = e^x + 2, y'''$ and all subsequent derivatives are e^x.

11a $5e^{5x} + 7e^{7x}$ **b** $4e^{4x+2} + 8e^{5+8x}$

c $-4e^{-x} - 12e^{-3x}$ **d** $-12e^{-2x-3} + 42e^{5-6x}$

e $10x - 4 + 3e^{-x}$ **f** $\frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{2}x^{-\frac{1}{2}}$

12a $y' = ae^{ax}$ **b** $y' = -ke^{-kx}$

c $y' = Ake^{kx}$ **d** $y' = -B\ell e^{-\ell x}$

13a $y' = \frac{1}{2}\sqrt{e^x}$ **b** $y' = \frac{1}{3}\sqrt[3]{x}$

c $y' = -\dfrac{1}{2\sqrt{e^x}}$ **d** $y' = -\dfrac{1}{3\sqrt[3]{e^x}}$

14a $y' = pCe^{px+q}$ **b** $e^{ax} - e^{-px}$

15a a–c Answer is in question

16a a–c Answer is in question

Exercise 9D

1a 1 **b** $y = x + 1$

2a e **b** $y = ex$

3a $\frac{1}{e}$ **b** $y = \frac{1}{e}(x + 2)$

4a $A = \left(\frac{1}{2}, 1\right)$ **b** $y' = 2e^{2x-1}$ **c** $y = 2x$

5a $y' = e^x$, which is always positive.

b $y' = -e^{-x}$, which is always negative.

6a $e - 1$ **b** $\frac{dy}{dx} = e^x$. When $x = 1$, $\frac{dy}{dx} = e$.

c $y = ex - 1$

d i never **ii** all real x **iii** never

7a $R = \left(-\frac{1}{3}, 1\right)$ **b** $y' = 3e^{3x+1}$

c $-\frac{1}{3}$ **d** $3x + 9y - 8 = 0$.

8a $-e$ **b** $\frac{1}{e}$

c $x - ey + e^2 + 1 = 0$

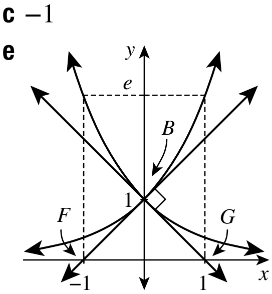
d x-intercept $-e^2 - 1$, y-intercept $e + e^{-1}$

e $\frac{1}{2}(e^3 + 2e + e^{-1})$

9a 1 **b** $y = x + 1$

c -1 **d** $y = -x + 1$

e



f isosceles right triangle, 1 square unit

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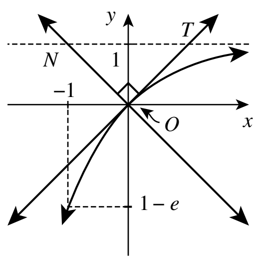
10a 1, 45° **b** e, 69°48′
c e^{-2}, 7°42′ **d** e^5, 89°37′

11 $A = (1, e^{-2}), B = (2, 1), y' = 2e^{2x-4}$
a $y' = 2e^{-2}$ **b** $y' = 2$ **c** $1 - e^{-2}$

12a $y = e^t(x - t + 1)$ **b and c** Answers will vary

13a Answers will vary **b** $y = -x$ **c** $y = 1$

d



e 1 square unit

14a Answers will vary
b $B(1, 0), C(1 + e^2, 0), E(0, e), F\left(0, e + \frac{1}{e}\right)$
c i–iv Answers will vary
d i $A = \frac{e}{2}(e^2 + 1)$ **ii** $A = \frac{1}{2e}(e^2 + 1)$

Exercise 9E

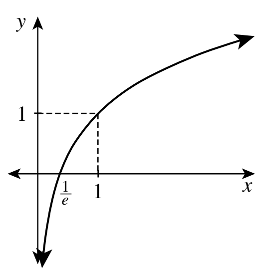
1a 0 **b** 0.6931 **c** 1.0986 **d** 2.0794 **e** −0.6931
f −1.0986 **g** −2.0794 **h** −2.3026

2, 3 a–f Answer is in question

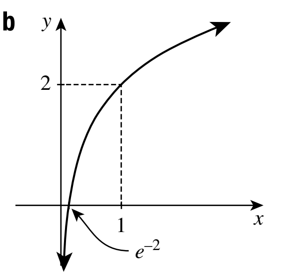
4a $e^x = 1, x = 0$ **b** $e^x = e, x = 1$ **c** $e^x = e^2, x = 2$
d $e^x = \frac{1}{e}, x = -1$ **e** $e^x = \frac{1}{e^2}, x = -2$ **f** $e^x = \sqrt{e}, x = \frac{1}{2}$

5a $2\log_e e = 2$ **b** $5\log_e e = 5$ **c** $200\log_e e = 200$
d $-6\log_e e = -6$ **e** $\log_e e^{-6} = -6\log_e e = -6$
f $-\log_e e = -1$ **g** $\log_e e^{-1} = -\log_e e = -1$
h $\frac{1}{2}\log_e e = \frac{1}{2}$ **i** $\frac{1}{2}\log_e e = \frac{1}{2}$ **j** $\log_e e^{-\frac{1}{2}} = -\frac{1}{2}\log_e e = -\frac{1}{2}$

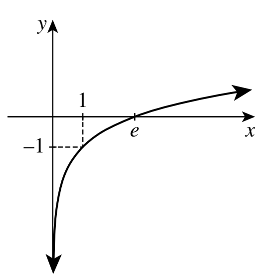
6a



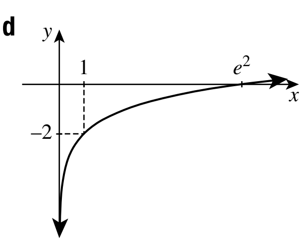
b



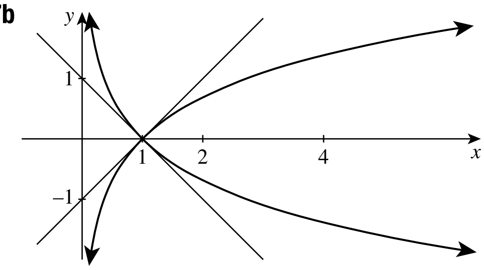
c



d



7b

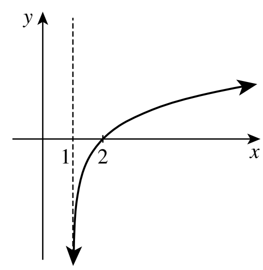


c The graph of $y = -\log_e x$ is obtained by reflecting the first in the x-axis. Hence its tangent has gradient −1, and the two are perpendicular.

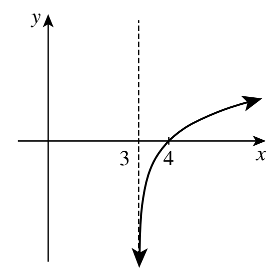
8a e **b** $-\frac{1}{e}$ **c** 6 **d** $\frac{1}{2}$ **e** $2e$ **f** 0

9a $\log_e 6$ **b** $\log_e 4$ **c** $\log_e 4$ **d** $\log_e 27$

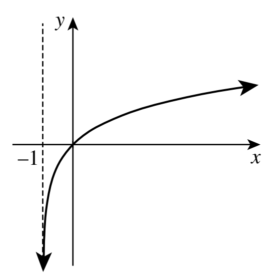
10a $x > 1$



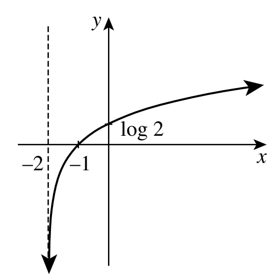
b $x > 3$



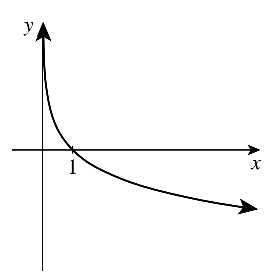
c $x > -1$



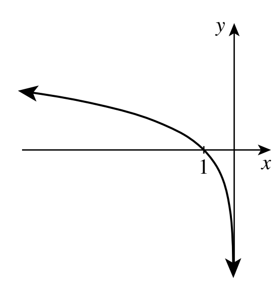
d $x > -2$



e $x > 0$



f $x < 0$



11a $\log_e \frac{a}{b} = \log_e a - \log b$ and $-\log_e \frac{b}{a} = -\log_e b + \log_e a$

b $\log_{\frac{1}{e}} x = \dfrac{\log_e x}{\log_e \frac{1}{e}} = \dfrac{\log_e x}{-1} = -\log_e x$

c Using part **b**, $\log_{\frac{1}{e}} x^{-1} = -\log_e x^{-1} = +\log_e x$

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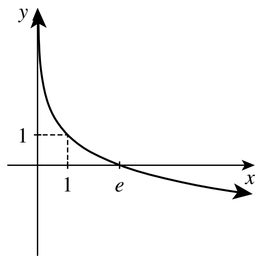
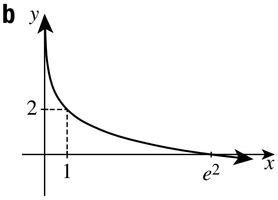
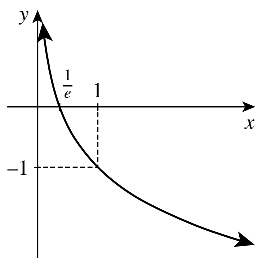
12a



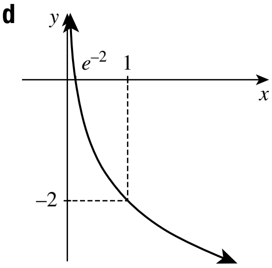
b



c



d



Exercise 9F

1a $\dfrac{dQ}{dt} = 900e^{3t}$

b $Q = 300e^6 \doteqdot 121\,000$, $\dfrac{dQ}{dt} = 900e^6 \doteqdot 363\,100$

c 60 360

2a $\dfrac{dQ}{dt} = -20\,000e^{-2t}$

b $Q = 10\,000e^{-8} \doteqdot 3.355$, $\dfrac{dQ}{dt} = -20\,000e^{-8} \doteqdot -6.709$

c −2499

3a $\dfrac{dQ}{dt} = 10e^{2t}$ **b** Put $1000 = 5e^{2t}$, $t = \frac{1}{2}\log 200 \doteqdot 2.649$

c Put $1000 = 10e^{2t}$, $t = \frac{1}{2}\log 100 \doteqdot 2.303$

4a $P = 2000e^{1.5} \doteqdot 8963$ individuals **b** $\dfrac{dP}{dt} = 600e^{0.3t}$

c $\dfrac{dP}{dt} = 600e^{1.5} \doteqdot 2689$ individuals/hour

d 1393 individuals per hour

5a $C = 2000e^{-4} \doteqdot 36.63$ **b** $\dfrac{dC}{dt} = -4000e^{-2t}$

c $\dfrac{dC}{dt} = -4000e^{-4} \doteqdot -73.26$ per year

d −981.7 per year

6a $t = 25\log_e 2 \doteqdot 17.33$ years **b** $\dfrac{dP}{dt} = 6e^{0.04t}$

c $t = 25\log_e 50 \doteqdot 97.80$ years

7a $\dfrac{dP}{dt} = 400e^{0.4t}$

b $P = 1000e^2 \doteqdot 7400$ cats, $\dfrac{dP}{dt} = 400e^2 \doteqdot 3000$ cats/year

c $t = \frac{5}{2}\log_e 20 \doteqdot 7.5$ years **d** $t = \frac{5}{2}\log_e 50 \doteqdot 9.8$ years

8a $\dfrac{dQ}{dt} = e^t$, which is always positive, so Q is increasing. Also $\dfrac{dQ}{dt}$ is increasing, so Q is increasing at an increasing rate.

b $\dfrac{dQ}{dt} = -e^{-t}$, which is always negative, so Q is decreasing. Also $\dfrac{dQ}{dt}$ is increasing, so the rate of change of Q is increasing, thus Q is decreasing at a decreasing rate. (The language here is not entirely satisfactory — more on this in year 12.)

9a $t = -10\log_e\left(\frac{1}{2}\right) = 10\log_e 2 \doteqdot 6.931$ years

b $\dfrac{dM}{dt} = -\frac{1}{10}M_0e^{-0.1t}$

c $(1 - e^{-0.1}) \times 100\% \doteqdot 9.516\%$

d When $\dfrac{dM}{dt} = -\frac{1}{100}M_0$,

$t = -10\log_e\left(\frac{1}{10}\right) = 10\log_e 10 \doteqdot 23.03$ years

Exercise 9G

1a $\frac{\pi}{2}$ **b** $\frac{\pi}{4}$ **c** $\frac{\pi}{6}$ **d** $\frac{\pi}{3}$ **e** $\frac{2\pi}{3}$ **f** $\frac{5\pi}{6}$ **g** $\frac{3\pi}{4}$ **h** $\frac{5\pi}{4}$ **i** 2π

j $\frac{5\pi}{3}$ **k** $\frac{3\pi}{2}$ **l** $\frac{7\pi}{6}$

2a 180° **b** 360° **c** 720° **d** 90° **e** 60° **f** 45° **g** 120°

h 150° **i** 135° **j** 270° **k** 240° **l** 315° **m** 330°

3a 0.84 **b** −0.42 **c** −0.14 **d** 0.64 **e** 0.33 **f** −0.69

4a 1.274 **b** 0.244 **c** 2.932 **d** 0.377

e 1.663 **f** 3.686

5a 114°35′ **b** 17°11′ **c** 82°30′ **d** 7°3′

e 183°16′ **f** 323°36′

6a $\frac{1}{2}$ **b** $\frac{1}{\sqrt{2}}$ **c** $\frac{\sqrt{3}}{2}$ **d** $\sqrt{3}$ **e** 1 **f** $\frac{1}{2}$ **g** $\sqrt{2}$ **h** $\frac{1}{\sqrt{3}}$

7a $\frac{\pi}{9}$ **b** $\frac{\pi}{8}$ **c** $\frac{\pi}{5}$ **d** $\frac{5\pi}{9}$ **e** $\frac{5\pi}{8}$ **f** $\frac{7\pi}{5}$

8a 15° **b** 72° **c** 400° **d** 247.5° **e** 306° **f** 276°

9a $\frac{\pi}{3}$ **b** $\frac{5\pi}{6}$

10 $\frac{4\pi}{9}$

11a $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{2}$ **c** $-\frac{\sqrt{3}}{2}$ **d** $\sqrt{3}$ **e** −1 **f** $\frac{1}{2}$ **g** $-\frac{1}{\sqrt{2}}$ **h** $\frac{1}{\sqrt{3}}$

12a Hour hand: 30° or $\frac{\pi}{6}$ radians, minute hand: 360° or 2π radians.

b i 60° or $\frac{\pi}{3}$ radians **ii** $22\frac{1}{2}$° or $\frac{\pi}{8}$ radians

iii 105° or $\frac{7\pi}{12}$ radians **iv** $172\frac{1}{2}$° or $\frac{23\pi}{24}$ radians

13a 0.283 **b** 0.819

14a 0.733 **b** 0.349

Exercise 9H

1a $\frac{\pi}{4}$ **b** $\frac{\pi}{6}$ **c** $\frac{\pi}{4}$ **d** $\frac{\pi}{6}$ **e** $\frac{\pi}{3}$ **f** $\frac{\pi}{3}$

2a $x \doteqdot 1.249$ **b** $x \doteqdot 0.927$ **c** $x \doteqdot 1.159$ **d** $x \doteqdot 0.236$

e $x \doteqdot 0.161$ **f** $x \doteqdot 1.561$

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3a $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ **b** $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ **c** $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ **d** $x = \frac{\pi}{2}$

e $x = \frac{\pi}{6}$ or $\frac{11\pi}{6}$ **f** $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$ **g** $x = \pi$ **h** $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

4a $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ **b** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

c $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ or $\frac{5\pi}{3}$ **d** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

5a $u^2 - u = 0$ **b** $u = 0$ or $u = 1$ **c** $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$ or 2π

6a $u^2 - u - 2 = 0$ **b** $u = -1$ or $u = 2$

c $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ or $\theta \doteqdot 1.11$ or 4.25

7a $\theta = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$ or 2π **b** $\theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$ or 2π

c $\theta = \frac{\pi}{2}$ **d** $\theta \doteqdot 1.11, 1.89, 4.25$ or 5.03

e $\theta = \frac{\pi}{3}, \pi$ or $\frac{5\pi}{3}$ **f** $\theta = \frac{\pi}{2}, \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

g $\theta \doteqdot 0.34$ or 2.80 **h** $\theta \doteqdot 1.91$ or 4.37

8 Compare to the answer to question **2**

9a $x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}$ or $\frac{5\pi}{3}$ **b** $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$, or $x \doteqdot 1.25$ or 4.39

c $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$, or $x \doteqdot 0.25$ or 2.89 **d** $x \doteqdot 0.84$ or 5.44

10a $\alpha = \frac{\pi}{2}$, or $\alpha \doteqdot 3.48$ or 5.94

b $\alpha \doteqdot 1.11, 2.82, 4.25$ or 5.96

11a $x = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$ or 2π

b $x \doteqdot 1.11, 1.25, 4.25$ or 4.39

Exercise 9I

1a 3π **b** $\frac{5\pi}{2}$ **c** 7.5 **d** 24 **e** $\frac{\pi}{4}$ **f** 2π

2a 2π **b** $\frac{4\pi}{3}$ **c** 2 **d** $\frac{2\pi}{3}$ **e** 6 **f** 20

3a 12 cm **b** 3 cm **c** 2π cm **d** $\frac{3\pi}{2}$ cm

4a $32\,\text{cm}^2$ **b** $96\,\text{cm}^2$ **c** $8\pi\,\text{cm}^2$ **d** $12\pi\,\text{cm}^2$

5 4 cm

6 1.5 radians

7a 2.4 cm **b** 4.4 cm

8 $8727\,\text{m}^2$

9a 8π cm **b** $16\pi\,\text{cm}^2$

10 84°

11 11.6 cm

12a $6\pi\,\text{cm}^2$ **b** $9\sqrt{3}\,\text{cm}^2$ **c** $3\left(2\pi - 3\sqrt{3}\right)\text{cm}^2$

14 $15\,\text{cm}^2$

15a $4(\pi + 2)$ cm **b** $8\pi\,\text{cm}^2$

16a 720 metres

b 2.4 radians (about 137°31′)

c 559.22 metres

d 317°31′T

17a $\frac{2\pi}{3}$ cm **b** $\frac{2\pi}{3}\text{cm}^2$ **c** 2π cm **d** $\sqrt{3}\,\text{cm}^2, 2\left(\pi - \sqrt{3}\right)\text{cm}^2$

18 $\frac{4}{3}\left(4\pi - 3\sqrt{3}\right)\text{cm}^2$

19a, b Answers will vary **c** $3\sqrt{55}\pi\,\text{cm}^3$ **d** $24\pi\,\text{cm}^2$

Exercise 9J

1a and **b** Refer to teacher.

2a All six graphs are many-to-one

b i $\pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$ **ii** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$

iii $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}, \frac{21\pi}{2}$ **iv** There are no solutions.

3a $x = \frac{\pi}{2}$, w$x = -\frac{\pi}{2}, x = \frac{3\pi}{2}, x = -\frac{3\pi}{2}, x = \frac{5\pi}{2}, x = -\frac{5\pi}{2}, \dots$

b $y = \operatorname{cosec} x$, the reciprocal of $y = \sin x$.

c Neither graph has any line symmetries.

4a $x = 0, x = \pi, x = -\pi, x = 2\pi, x = -2\pi, \dots$

b Line symmetry in the y-axis $x = 0$

c $y = \sec x$, the reciprocal of $y = \cos x$.

5a $(0, 0), (\pi, 0), (-\pi, 0), (2\pi, 0), (-2\pi, 0), \dots$

b Point symmetry in the origin $(0, 0)$

c $y = \operatorname{cosec} x$, the reciprocal of $y = \sin x$.

6a $\left(\frac{\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(-\frac{3\pi}{2}, 0\right), \dots$

b $y = \sec x$, the reciprocal of $y = \cos x$.

7a $(0, 0), \left(\frac{\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), (\pi, 0), (-\pi, 0), \left(\frac{3\pi}{2}, 0\right), \left(-\frac{3\pi}{2}, 0\right), \dots$

b Both functions are odd, because both have point symmetry in the origin. Neither is even, because neither have line symmetry in the y-axis.

8a Translations left or right by multiples of 2π.

b $y = \cos x$, $y = \operatorname{cosec} x$ and $y = \sec x$.

c $y = \tan x$ and $y = \cot x$ can each be mapped onto themselves by translations left or right by multiples of π.

d $y = \sin x$, $y = \cos x$,
$y = \operatorname{cosec} x$, $y = \sec x$ each has period 2π.
$y = \tan x$, $y = \cot x$ each has period π.

9a $x = \frac{\pi}{4}, x = -\frac{3\pi}{4}, x = \frac{5\pi}{4}, x = -\frac{7\pi}{4}, \dots$

b $y = \operatorname{cosec} x$ and $y = \sec x$

c $x = \frac{\pi}{4}, x = -\frac{\pi}{4}, x = \frac{3\pi}{4}, x = -\frac{3\pi}{4}, x = \frac{5\pi}{4}, x = -\frac{5\pi}{4}, \dots$

10a Translations left $\frac{\pi}{2}, \frac{5\pi}{2}, \dots$, and right $\frac{3\pi}{2}, \frac{7\pi}{2}, \dots$

b $y = \sin(x - \theta)$ is $y = \sin x$ shifted right by θ, so $\sin(x - \theta) = \cos x$ for $\theta = \frac{3\pi}{2}, -\frac{\pi}{2}, \frac{7\pi}{2}, -\frac{5\pi}{2}, \frac{11\pi}{2}, -\frac{9\pi}{2}, \dots$

c There are none.

11 There are none.

12a $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(-\frac{3\pi}{4}, -\frac{1}{\sqrt{2}}\right), \left(\frac{5\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(-\frac{7\pi}{4}, -\frac{1}{\sqrt{2}}\right), \dots$

b $\sin x = \cos x$, so $\tan x = 1$.

13a $(0, 0), (\pi, 0), (-\pi, 0), (2\pi, 0), (-2\pi, 0), \dots$

b $\sin x = \dfrac{\sin x}{\cos x}$, so $\sin x \cos x = \sin x$, so $\sin x(\cos x - 1) = 0$, so $\sin x = 0$ or $\cos x = 1$.

14 Roughly 0.7 (radians).

15a They touch each other at their maxima and minima.

b $y = \cos x$ and $y = \sec x$.

c $y = \sin x$ & $y = \sec x$, $y = \cos x$ & $y = \operatorname{cosec} x$,
$y = \tan x$ & $y = \sec x$, $y = \cot x$ & $y = \operatorname{cosec} x$

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16a
$$\begin{aligned} \cos x &= \frac{\sin x}{\cos x} \\ \boxed{\times \cos x}\ \cos^2 x &= \sin x \text{ and } \cos x \neq 0 \\ 1 - \sin^2 x &= \sin x \\ \sin^2 x + \sin x - 1 &= 0 \\ \Delta &= 1 + 4 = 5 \\ \sin x &= \frac{-1 + \sqrt{5}}{2} \end{aligned}$$

giving solutions in the first and second quadrants.
$\left(\dfrac{-1 - \sqrt{5}}{2} < -1\text{, so } \sin x = \dfrac{-1 - \sqrt{5}}{2} \text{ has no solutions.}\right)$

b
$$\begin{aligned} \frac{\sin x}{\cos x} &= \frac{1}{\cos x} \\ \times \cos x \quad \sin x &= 1 \text{ and } \cos x \neq 0 \end{aligned}$$

There are no solutions,
because if $\sin x = 1$, then $\cos x = 0$.

Chapter 9 review exercise

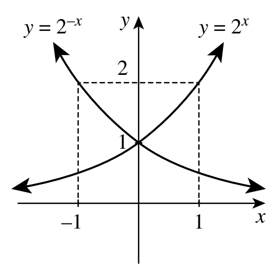
1a 3^9 **b** 3^{12} **c** 3^5 **d** 6^5

2a $\frac{1}{5}$ **b** $\frac{1}{100}$ **c** $\frac{1}{x^3}$ **d** $\frac{1}{3^x}$

3a 3 **b** 3 **c** 4 **d** $\frac{1}{4}$ **e** $\frac{1}{9}$ **f** $\frac{1}{1000}$

4a 2^{3x} **b** 2^{4x} **c** 2^{6x} **d** 10^x **e** 2^{2x+3} **f** 2^{2x-1}

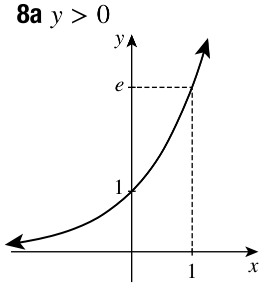
5 Each graph is reflected onto the other graph in the line $x = 0$.



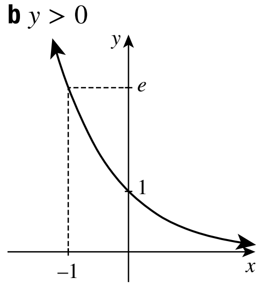
6a 2.718 **b** 54.60 **c** 0.1353 **d** 4.482

7a e^{5x} **b** e^{6x} **c** e^{-4x} **d** e^{9x}

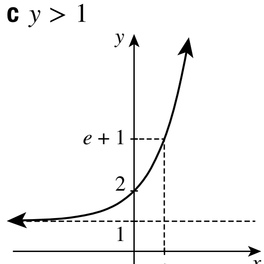
8a $y > 0$



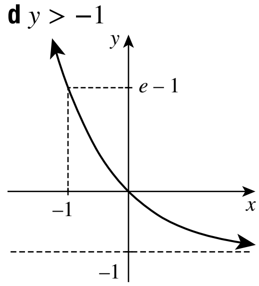
b $y > 0$



c $y > 1$



d $y > -1$



9a e^x **b** $3e^{3x}$ **c** e^{x+3} **d** $2e^{2x+3}$ **e** $-e^{-x}$ **f** $-3e^{-3x}$
g $-2e^{3-2x}$ **h** $6e^{2x+5}$ **i** $2e^{\frac{1}{2}x}$ **j** $4e^{6x-5}$

10a $5e^{5x}$ **b** $4e^{4x}$ **c** $-3e^{-3x}$ **d** $-6e^{-6x}$

11 2

12 $y = e^2x - e^2$, x-intercept 1, y-intercept $-e^2$.

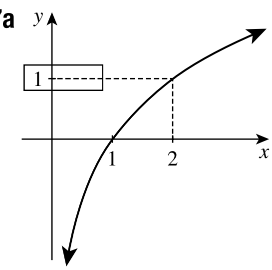
13a 1.4314 **b** −0.3010 **c** 0.6931 **d** 2.6391

14a 1.1761 **b** 0.4771 **c** 1.9459 **d** −1.0986

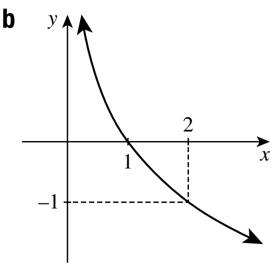
15a 5 **b** $-\frac{1}{4}$ **c** 3 **d** $\frac{1}{5}$

16a e **b** 3 **c** −1 **d** e

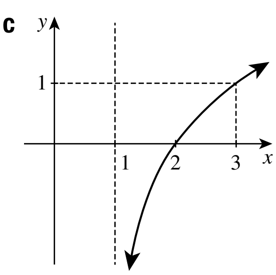
17a



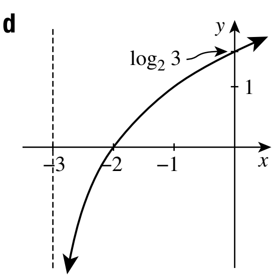
b



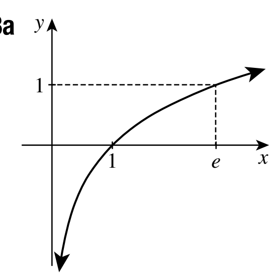
c



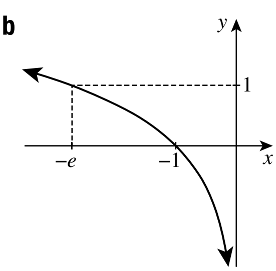
d



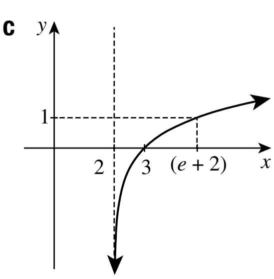
18a



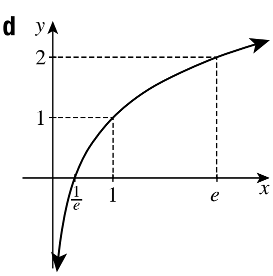
b



c



d



19a $\dfrac{dP}{dt} = -\frac{1}{100}P_0e^{-0.01t}$

b $\dfrac{dP}{dt} = -\frac{1}{100}P_0e^{-0.45} = -0.0064P_0$ lizards per year.

c $P = P_0e^{-0.45} \doteqdot 64\%$ of the original population.

d $e^{-0.01t} = \frac{1}{10}$, so $t = 100 \log_e 10 \doteqdot 230$ years

20a π **b** $\frac{\pi}{9}$ **c** $\frac{4\pi}{3}$ **d** $\frac{7\pi}{4}$

21a 30° **b** 108° **c** 540° **d** 300°

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22a $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{3}}$

23a $x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$ **b** $x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$

24a $\sin\theta = 0$ or $-\frac{1}{2}$, $\theta = 0,\ \pi,\ \frac{7\pi}{6},\ \frac{11\pi}{6}$ or 2π

b $\cos\theta = -1$ or 2, $\theta = \pi$ ($\cos\theta = 2$ has no solutions.)

c $\tan\theta = \frac{1}{2}$ and $\theta \doteqdot 0.46$ or 3.61, or $\tan\theta = -3$ and $\theta \doteqdot 1.89$ or 5.03

25a 3πcm **b** $12\pi\text{cm}^2$

26 $16(\pi - 2) \doteqdot 18.3\text{cm}^2$

27 148°58′

28a $y = \sin x$ and $y = \cos x$ both have amplitude 1.

b $y = \sin x$, $y = \cos x$, $y = \operatorname{cosec} x$ and $y = \sec x$ all have period 2π, $y = \tan x$ and $y = \cot x$ both have period π.

c $y = \sin x$, $y = \tan x$, $y = \operatorname{cosec} x$ and $y = \cot x$ are all odd, $y = \cos x$, and $y = \sec x$ are both even.

29 a $\theta = \frac{3\pi}{2}$ **b** $\theta = \frac{\pi}{2}$ **c** $x = \frac{\pi}{4}$

Chapter 10

Exercise 10A

1a $\frac{1}{20}$ **b** $\frac{19}{20}$

2a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** 1 **d** 0

3a $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** $\frac{1}{3}$ **d** $\frac{1}{3}$

4a $\frac{5}{12}$ **b** $\frac{7}{12}$ **c** 0

5a $\frac{4}{9}$ **b** $\frac{5}{9}$ **c** $\frac{11}{18}$

6a $\frac{4}{9}$ **b** $\frac{5}{9}$ **c** $\frac{11}{18}$ **d** $\frac{7}{18}$ **e** $\frac{1}{3}$ **f** $\frac{1}{6}$

7a $\frac{3}{8}$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$

8a $\frac{1}{26}$ **b** $\frac{5}{26}$ **c** $\frac{21}{26}$ **d** 0 **e** $\frac{3}{26}$ **f** $\frac{5}{26}$

9 78%

10a $\frac{4}{7}$ **b** 32

11a 8 **b** $\frac{14}{15}$

12a 10 sixes

b i $\frac{18}{60} = 30\%$

ii The experiment suggest a probability of about 30%

iii The theoretical probability suggests that for an unbiased die, we would expect to get a six on one-sixth of the throws, that is, 10 times. The large number of sixes turning up suggests that this die is biased.

13a $\frac{100}{400} = \frac{1}{4} = 25\%$ **b** $\frac{8}{20} = \frac{2}{5} = 40\%$

c We would expect him to get chicken one-quarter of the time, that is on 5 occasions. He may have got more chicken sandwiches because of the way the canteen makes or sells the sandwiches, for example making the chicken sandwiches early and placing them at the front of the display, or making more Vegemite sandwiches as they sell out. Possibly also the sample is too small and the result would approach $\frac{1}{4}$ if the experiment were continued over a longer time. The experimental probability is only an estimate, and in fact it is possible he may have got no chicken sandwiches over the 20 days.

14a $\frac{1}{20}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$ **d** $\frac{1}{2}$ **e** $\frac{2}{5}$ **f** $\frac{1}{5}$ **g** $\frac{1}{4}$ **h** 0 **i** 1

15a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{13}$ **d** $\frac{1}{52}$ **e** $\frac{1}{4}$ **f** $\frac{3}{13}$ **g** $\frac{1}{2}$

h $\frac{1}{13}$ **i** $\frac{3}{13}$ (counting an ace as a one)

16a $\frac{1}{15}$ **b** $\frac{7}{150}$ **c** $\frac{1}{2}$ **d** $\frac{4}{25}$ **e** $\frac{1}{75}$ **f** $\frac{17}{50}$

17a $\frac{1}{5}$ **b** $\frac{3}{40}$ **c** $\frac{9}{20}$ **d** $\frac{7}{100}$ **e** $\frac{7}{50}$ **f** $\frac{1}{200}$

18a $\frac{3}{4}$ **b** $\frac{1}{4}$

19 187 or 188

20a The argument is invalid, because on any one day the two outcomes are not equally likely. The argument really can’t be corrected.

b The argument is invalid. One team may be significantly better than the other, the game may be played in conditions that suit one particular team, and so on. Even when the teams are evenly matched, the high-scoring nature of the game makes a draw an unlikely event. The three outcomes are not equally likely. The argument really can’t be corrected.

c The argument is invalid, because we would presume that Peter has some knowledge of the subject, and is therefore more likely to choose one answer than another. The argument would be valid if the questions were answered at random.

d The argument is only valid if there are equal numbers of red, white and black beads, otherwise the three outcomes are not equally likely.

e The argument is missing, but the conclusion is correct. Exactly one of the four players will win his semi-final and then lose the final. Our man is as likely to pick this player as he is to pick any of the other three players.

21a $\frac{2}{9}$ **b** $\frac{\pi}{18}$

Exercise 10B

1a HH, HT, TH, TT **b i** $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{4}$

2a H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6

b i $\frac{1}{4}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{4}$ **iv** $\frac{1}{4}$

3a TO, OT, OE, EO, ET, TE

b i $\frac{1}{3}$ **ii** $\frac{1}{3}$ **iii** $\frac{2}{3}$

4a AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC

b i $\frac{1}{6}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{3}$ **iv** $\frac{1}{6}$ **v** $\frac{1}{4}$ **vi** $\frac{3}{4}$

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5a 23, 32, 28, 82, 29, 92, 38, 83, 39, 93, 89, 98

b i $\frac{1}{12}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$ **iv** $\frac{1}{6}$ **v** $\frac{1}{4}$ **vi** 0

6a The captain is listed first and the vice-captain second: AB, AC, AD, AE, BC, BD, BE, CD, CE, DE, BA, CA, DA, EA, CB, DB, EB, DC, EC, ED

b i $\frac{1}{20}$ **ii** $\frac{2}{5}$ **iii** $\frac{3}{5}$ **iv** $\frac{1}{5}$

7 HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

a $\frac{1}{8}$ **b** $\frac{3}{8}$ **c** $\frac{1}{2}$ **d** $\frac{1}{2}$ **e** $\frac{1}{2}$ **f** $\frac{1}{2}$

8 11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66

a $\frac{1}{6}$ **b** $\frac{1}{6}$ **c** $\frac{1}{36}$ **d** $\frac{1}{6}$ **e** $\frac{1}{6}$ **f** $\frac{1}{4}$ **g** $\frac{11}{36}$ **h** $\frac{4}{9}$ **i** $\frac{5}{36}$ **j** $\frac{1}{6}$

9a i $\frac{1}{4}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{2}$ **b i** $\frac{1}{8}$ **ii** $\frac{3}{8}$ **iii** $\frac{1}{2}$

10a $\frac{1}{16}$ **b** $\frac{1}{4}$ **c** $\frac{11}{16}$ **d** $\frac{5}{16}$ **e** $\frac{3}{8}$ **f** $\frac{5}{16}$

11a $\frac{2}{5}$ **b** $\frac{3}{5}$ **c** $\frac{1}{5}$

12a 24 **b i** $\frac{2}{3}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{12}$ **iv** $\frac{1}{6}$

13a $\dfrac{1}{2^n}$ **b** $1 - 2^{1-n}$

Exercise 10C

1a $\{1, 3, 5, 7, 9\}$ **b** $\{6, 12, 18, 24, 30, 36\}$

c $\{1, 2, 3, 4, 5, 6\}$ **d** $\{1, 2, 4, 5, 10, 20\}$

2a $A \cup B = \{1, 3, 5, 7\}$, $A \cap B = \{3, 5\}$

b $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A \cap B = \{4, 9\}$

c $A \cup B = \{\text{h, o, b, a, r, t, i, c, e, n}\}$, $A \cap B = \{\text{h, o, b}\}$

d $A \cup B = \{\text{j, a, c, k, e, m}\}$, $A \cap B = \{\text{a}\}$

e $A \cup B = \{1, 2, 3, 5, 7, 9\}$, $A \cap B = \{3, 5, 7\}$

3a false **b** true **c** false **d** false **e** true **f** true

4a 3 **b** 2 **c** $\{1, 3, 4, 5\}$ **d** 4 **e** $\{3\}$ **f** 1

g $\{2, 4\}$ **h** $\{1, 2, 5\}$

5a students who study both Japanese and History

b students who study either Japanese or History or both

6a students at Clarence High School who do not have blue eyes

b students at Clarence High School who do not have blond hair

c students at Clarence High School who have blue eyes or blond hair or both

d students at Clarence High School who have blue eyes and blond hair

7a $\varnothing$, $\{a\}$ **b** $\varnothing$, $\{a\}$, $\{b\}$, $\{a, b\}$

c $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{a, b, c\}$

d $\varnothing$

8a true **b** false **c** true **d** false **e** true

9a $\{2, 4, 5, 6, 8, 9\}$ **b** $\{1, 2, 3, 5, 8, 10\}$

c $\{7\}$ **d** $\{1, 2, 3, 4, 5, 6, 8, 9, 10\}$

e $\{1, 3, 4, 6, 7, 9, 10\}$ **f** $\{2, 5, 8\}$

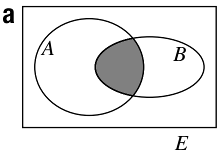
10a $\{2, 4, 5, 7, 9, 10\}$ **b** $\{1, 2, 5, 8, 9\}$

c $\{1, 2, 4, 5, 7, 8, 9, 10\}$ **d** $\{2, 5, 9\}$

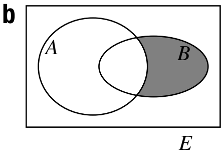
e $\{1, 3, 4, 6, 7, 8, 10\}$ **f** $\{2, 5, 9\}$

g $\{3, 6\}$ **h** $\{1, 2, 4, 5, 7, 8, 9, 10\}$

11a



b



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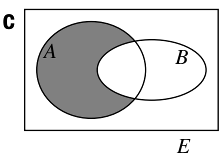
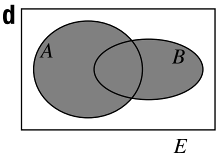


d



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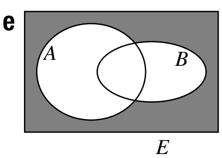

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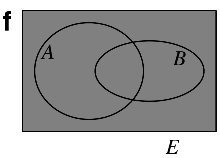


12a true **b** true

13a Q **b** P

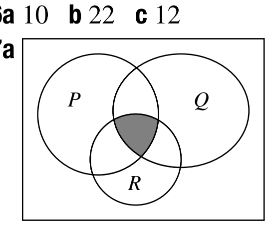
14a III **b** I **c** II **d** IV

15a $|A \cap B|$ is subtracted so that it is not counted twice.

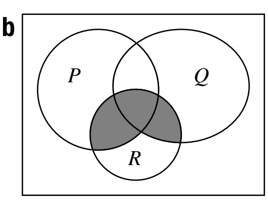
b 5 **c** LHS = 7, RHS = 5 + 6 – 4 = 7

16a 10 **b** 22 **c** 12

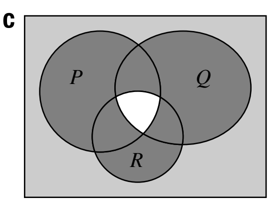
17a



b



c



18 4

Exercise 10D

1a $\frac{1}{6}$ **b** $\frac{5}{6}$ **c** $\frac{1}{3}$ **d** 0 **e** 1 **f** 0 **g** $\frac{1}{6}$ **h** $\frac{2}{3}$

2a $\frac{1}{13}$ **b** $\frac{1}{13}$ **c** $\frac{2}{13}$ **d** 0 **e** $\frac{11}{13}$ **f** $\frac{1}{2}$

g $\frac{3}{13}$ **h** $\frac{3}{26}$ **i** $\frac{8}{13}$ **j** $\frac{5}{13}$

3a $A = \{HH\}$, $B = \{HT, TH\}$, $P(A \text{ or } B) = \frac{3}{4}$, $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{4}$

b $A = \{RS\}$, $B = \{RS, ST\}$, $P(A \text{ or } B) = \frac{3}{3}$, $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{3}$

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4a no **b i** $\frac{1}{2}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{3}$ **iv** $\frac{5}{6}$

5a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{4}$ **d** $\frac{3}{4}$ **e** $\frac{1}{4}$ **f** $\frac{1}{6}$

g $\frac{1}{6}$ **h** $\frac{1}{36}$ **i** $\frac{11}{36}$ **j** $\frac{25}{36}$

6a i $\frac{1}{2}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{3}$ **iv** $\frac{1}{2}$ **v** $\frac{1}{2}$

b i $\frac{3}{5}$ **ii** $\frac{4}{5}$ **iii** $\frac{3}{5}$ **iv** 0 **v** 1

c i $\frac{1}{2}$ **ii** $\frac{2}{3}$ **iii** $\frac{2}{3}$ **iv** $\frac{1}{3}$ **v** $\frac{5}{6}$

7a $\frac{7}{15}$ **b** 0 **c** $\frac{3}{5}$ **d** $\frac{5}{7}$

8a i no **ii** $\frac{1}{2}, \frac{1}{4}, \frac{3}{20}, \frac{3}{5}$

b i no **ii** $\frac{1}{2}, \frac{3}{10}, \frac{3}{20}, \frac{13}{20}$

c i yes **ii** $\frac{1}{4}, \frac{9}{20}, 0, \frac{7}{10}$

9a $\frac{9}{25}$ **b** $\frac{7}{50}$ **c** $\frac{17}{50}$

10a 10 **b i** $\frac{4}{21}$ **ii** $\frac{1}{3}$

11 $\frac{1}{4}$

12 2, 3, 5, 7, 11,13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

a $\frac{1}{4}$ **b** $\frac{1}{4}$ **c** $\frac{11}{100}$ **d** $\frac{39}{100}$

13a $\frac{7}{12}$ **b** $\frac{13}{60}$ **c** $\frac{3}{10}$ **d** $\frac{7}{60}$

Exercise 10E

1a $\frac{1}{24}$ **b** $\frac{1}{28}$ **c** $\frac{1}{12}$ **d** $\frac{1}{96}$ **e** $\frac{1}{42}$ **f** $\frac{1}{336}$

2a $\frac{1}{12}$ **b** $\frac{1}{12}$ **c** $\frac{1}{4}$ **d** $\frac{1}{3}$

3a $\frac{1}{25}$ **b** $\frac{2}{25}$ **c** $\frac{3}{25}$ **d** $\frac{3}{25}$ **e** $\frac{4}{25}$ **f** $\frac{2}{25}$ **g** $\frac{1}{25}$

4a $\frac{15}{49}$ **b** $\frac{8}{49}$ **c** $\frac{6}{49}$

5a $\frac{1}{10}$ **b** $\frac{3}{10}$ **c** $\frac{3}{10}$ **d** $\frac{3}{10}$

6a $\frac{1}{36}$ **b** $\frac{1}{12}$ **c** $\frac{1}{36}$ **d** $\frac{1}{9}$ **e** $\frac{1}{6}$

7a $\frac{1}{7}$ **b** $\frac{180}{1331}$

8a i $\frac{13}{204}$ **ii** $\frac{1}{17}$ **iii** $\frac{4}{663}$ **iv** $\frac{1}{2652}$

b $\frac{1}{16}, \frac{1}{16}, \frac{1}{169}, \frac{1}{2704}$

9a i $\frac{2}{3}$ **ii** $\frac{1}{3}$ **b i** $\frac{8}{27}$ **ii** $\frac{1}{27}$ **iii** $\frac{4}{27}$

10a $\frac{3}{4}$ **b** $\frac{31}{32}$ **c** $\frac{1023}{1024}$

11a The argument is invalid, because the events ‘liking classical music’ and ‘playing a classical instrument’ are not independent. One would expect that most of those playing a classical instrument would like classical music, whereas a smaller proportion of those not playing a classical instrument would like classical music. The probability that a student does both cannot be discovered from the given data — one would have to go back and do another survey.

b The argument is invalid, because the events ‘being prime’ and ‘being odd’ are not independent — two out of the three odd numbers less than 7 are prime, but only one out of the three such even numbers is prime. The correct argument is that the odd prime numbers amongst the numbers 1, 2, 3, 4, 5 and 6 are 3 and 5, hence the probability that the die shows an odd prime number is $\frac{2}{6} = \frac{1}{3}$.

c The teams in the competition may not be of equal ability, and factors such as home-ground advantage may also affect the outcome of a game, hence assigning a probability of $\frac{1}{2}$ to winning each of the seven games is unjustified. Also, the outcomes of successive games are not independent — the confidence gained after winning a game may improve a team’s chances in the next one, a loss may adversely affect their chances, or a team may receive injuries in one game leading to a depleted team in the next. The argument really can’t be corrected.

d This argument is valid. The coin is normal, not biased, and tossed coins do not remember their previous history, so the next toss is completely unaffected by the previous string of heads.

12a $\frac{1}{36}$ **b** $\frac{1}{6}$ **c** $\frac{1}{4}$ **d** $\frac{1}{36}$ **e** $\frac{1}{36}$ **f** $\frac{1}{18}$ **g** $\frac{1}{12}$ **h** $\frac{1}{12}$ **i** $\frac{1}{6}$

13 HHH, HHM, HMH, MHH, HMM, MHM, MMH, MMM

a $P(\text{HHH}) = 0.9^3 = 0.729$ **b** $P(\text{MMM}) = 0.1^3 = 0.001$

c $P(\text{HMM}) = 0.9 \times 0.1^2 = 0.009$

d $P(\text{HMM}) + P(\text{MHM}) + P(\text{MMH}) = 3 \times 0.009 = 0.027$

e 0.081 **f** 0.243

14a $\frac{9}{25}$ **b** 11

c Compare it with question 13 above, replacing 90% there with 80% here.

15a $\frac{1}{12960000}$ **b** 233

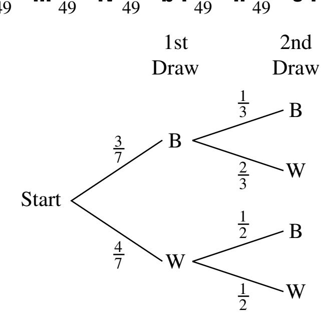
16a $\frac{1}{9}$ **b** $\frac{1}{9}$. Retell as ‘Nick begins by picking out two socks for the last morning and setting them aside’.

c $\frac{1}{9}$. Retell as ‘Nick begins by picking out two socks for the third morning and setting them aside’.

d $\frac{1}{63}$ **e** $\frac{1}{9 \times 7 \times 5 \times 3}$ **f** zero

Exercise 10F

1a i $\frac{9}{49}$ **ii** $\frac{12}{49}$ **iii** $\frac{12}{49}$ **iv** $\frac{16}{49}$ **b i** $\frac{25}{49}$ **ii** $\frac{24}{49}$ **c i** $\frac{3}{7}$ **ii** $\frac{4}{7}$



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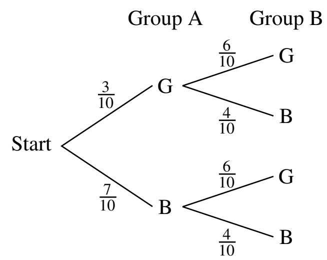
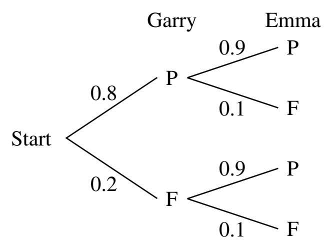
2a i 90.25% **ii** 4.75% **iii** 4.75% **iv** 0.25%
b 99.75%

3a i $\frac{6}{25}$ **ii** $\frac{9}{25}$ **iii** $\frac{4}{25}$ **iv** $\frac{6}{25}$ **b i** $\frac{12}{25}$ **ii** $\frac{13}{25}$

4a i $\frac{9}{50}$ **ii** $\frac{3}{25}$ **iii** $\frac{21}{50}$ **iv** $\frac{7}{25}$ **b i** $\frac{23}{50}$ **ii** $\frac{27}{50}$



5a 8% **b** 18% **c** 26% **d** 28%



6a $\frac{9}{25}$ **b** $\frac{21}{25}$

7 4.96%

8a 0.01 **b** 0.23

9 0.35

10 $\frac{4}{7}$

11a $\frac{21}{3980}$ **b** $\frac{144}{995}$

12a $\frac{3}{10}$ **b** $\frac{7}{24}$ **c** $\frac{21}{40}$

13a $\frac{1}{11}$ **b** $\frac{14}{33}$ **c** $\frac{10}{33}$ **d** $\frac{19}{33}$

14a $\frac{5}{6}$ **b** $\frac{5}{12}$ **c** $\frac{1}{6}$

15 The term 'large school' is code for saying that the probabilities do not change for the second choice because the sample space hardly changes.
a 0.28 **b** 0.50

16a $\frac{1}{25}$ **b** $\frac{3}{5}$

17a $\frac{1}{20}$ **b** $\frac{57}{8000}$

18a 31.52% **b** 80.48%

19a i $\frac{5}{33}$ **ii** $\frac{5}{22}$ **iii** $\frac{19}{33}$ **iv** $\frac{1}{4}$ **v** $\frac{19}{66}$ **vi** $\frac{47}{66}$
b i $\frac{25}{144}$ **ii** $\frac{5}{24}$ **iii** $\frac{5}{9}$ **iv** $\frac{1}{4}$ **v** $\frac{25}{72}$ **vi** $\frac{47}{72}$

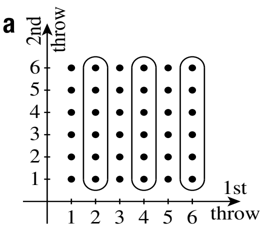
20a $\frac{1}{36}$ **b** $\frac{1}{46656}$ **c** $\frac{11}{36}$

21a $\frac{1}{216}$ **b** $\frac{5}{72}$ **c** $\frac{5}{12}$ **d** $\frac{5}{9}$

22 $\frac{1}{3}$

Exercise 10G

1a



b $\frac{1}{18}$ **c** $\frac{4}{9}$ **d** $\frac{1}{9}$

2a $\frac{340}{1000} = \frac{17}{50} = 0.34$ **b** $\frac{190}{420} = \frac{19}{42} \doteqdot 0.45$
c $\frac{130}{340} = \frac{13}{34} \doteqdot 0.38$ **d** $\frac{20}{130} = \frac{2}{13} \doteqdot 0.15$

3a Totals in last column: 56, 137, 193
Totals in last row: 124, 69, 193
b i $\frac{42}{193} \doteqdot 0.22$ **ii** $\frac{29}{124} \doteqdot 0.23$
iii $\frac{29}{56} \doteqdot 0.52$ **iv** $\frac{95}{137} \doteqdot 0.69$

4a $\frac{1}{16}$ **b** HH, HD, HC, HS; $\frac{1}{4}$
c HH, HD, HC, HS, DH, CH, SH; $\frac{1}{7}$
d HH, HD, HC, HS, DH, DD, DC, DS; $\frac{1}{8}$

5a

	1	2	3	4	5	6
HH	3	4	5	6	7	8
HT	2	3	4	5	6	7
TH	2	3	4	5	6	7
TT	1	2	3	4	5	6

b $\frac{1}{24}$ **c** $\frac{1}{6}$ **d** $\frac{1}{2}$

6a $\frac{5}{7}$ **b** $\frac{3}{8}$ **c** $\frac{16}{19}$

7a $P(A \cap B) = 0.24$ **b** $P(A \cap B) = 0.15$
c $P(A|B) = 0.4$ **d** $P(A|B) = 0.7$

8a dependent **b** independent **c** dependent
d independent
e impossible; $P(A \cap B)$ cannot be bigger than $P(A)$ or $P(B)$
f independent

9a

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

b The cases 1 + 4, 2 + 3, 3 + 2 and 4 + 1 make up the reduced sample space.
i $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** 1

10a i 0.1 **ii** $\frac{1}{3}$ **iii** $\frac{1}{4}$ **b** $\frac{3}{7}$ **c** $\frac{1}{2}$ **d** $\frac{5}{9}$

11 $\frac{4}{11}$

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12 $\frac{5}{8}$ or 62.5%

13a $\frac{1}{2}$ **b** $\frac{1}{3}$

14a BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG
b $\frac{4}{7}$ **c** $\frac{2}{3}$

15a $\frac{1}{3}$ **b** $\frac{2}{3}$ **c** $\frac{11}{153}$

16a and **b** $P(A|B) = P(A \cap B)/P(B) = \frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$, while $P(A) = \frac{1}{2}$. Hence the events are independent.

17a $P(A \cup B) = 0.76$ **b** $P(A \cup B) = 0.72$

18a $\frac{1}{6}$ **b** $\frac{5}{6}$ **c** $\frac{1}{5}$

19 $\frac{7}{15}$

20 $\frac{9}{23}$

21 $\frac{3}{7}$

22a 5.75% **b** 4.95% **c** 86% **d** 0.21%
e It is most important that the number of false negatives is low — that almost all patients with the disease are picked up. False positives are scary for the patient, but further tests should determine that they do not have the disease.

23a and **b**
$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(B \cap A)}{P(B)} \\ &= \frac{P(B|A)P(A)}{P(B)} \\ &= \frac{P(B|A)}{P(B)} \times P(A) \end{aligned}$$

24 If B is independent of A then,
$$\begin{aligned} P(A|B) &= \frac{P(B|A)}{P(B)} \times P(A) \\ &= \frac{P(B)}{P(B)} \times P(A) \\ &= P(A) \end{aligned}$$
which states that A is independent of B.

25 Suppose first that the contestant changes her choice. If her original choice was correct, she loses, otherwise she wins, so her chance of winning is $\frac{2}{3}$. Suppose now that the contestant does not change her choice. If her original choice was correct, she wins, otherwise she loses, so her chance of winning is $\frac{1}{3}$. Thus the strategy of changing will double her chance of winning.

26 Let $G1$ be, 'A girl is born on a Sunday', let $B1$ be, 'A boy is born on a Sunday', let $G2$ be, 'A girl is born on a Monday', …, giving 14 equally likely events at the birth of every child. In this particular family, there are two children, giving $14^2 = 196$ equally likely possible outcomes for the two successive births in this family. Draw up the 2×2 sample space, showing at least all the entries in the row indexed by $G2$ and the column indexed by $G2$.
Let A be, 'At least one child is a girl born on a Monday.' There are 27 favourable outcomes for A.
Let B be, 'Both children are girls.' There are 13 favourable outcomes for the event $A \cap B$.
Hence $P(B|A) = |A \cap B| / |A| = \frac{13}{27}$

Chapter 10 review exercise

1a $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** $\frac{1}{6}$ **d** $\frac{1}{2}$

2a $\frac{1}{10}$ **b** $\frac{1}{2}$ **c** $\frac{3}{10}$ **d** 0 **e** 1 **f** $\frac{3}{10}$

3a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{13}$ **d** $\frac{1}{52}$ **e** $\frac{1}{2}$ **f** $\frac{12}{13}$

4 37%

5a $\frac{1}{4}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$

6a $\frac{1}{36}$ **b** $\frac{1}{9}$ **c** $\frac{1}{6}$ **d** $\frac{11}{36}$ **e** $\frac{4}{9}$ **f** $\frac{1}{9}$ **g** $\frac{1}{6}$ **h** $\frac{11}{36}$

7a $\frac{17}{60}$ **b** $\frac{19}{60}$ **c** $\frac{1}{6}$

8a No **b i** $\frac{1}{2}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{3}$ **iv** $\frac{5}{6}$

9a $\frac{1}{12}$ **b** $\frac{1}{5}$ **c** $\frac{3}{20}$ **d** $\frac{1}{20}$

10a i $\frac{13}{204}$ **ii** $\frac{1}{17}$ **iii** $\frac{4}{663}$ **iv** $\frac{1}{2652}$
b i $\frac{1}{16}$ **ii** $\frac{1}{16}$ **iii** $\frac{1}{169}$ **iv** $\frac{1}{2704}$

11a 14% **b** 24% **c** 38% **d** 6%

12a $\frac{2}{21}$ **b** $\frac{11}{21}$ **c** $\frac{10}{21}$ **d** $\frac{2}{7}$

13a $\frac{19}{12\,475}$ **b** $\frac{979}{12\,475}$

14a independent **b** dependent
c independent, with $P(A \cap B) = 0.18$

15 $\frac{3}{11}$

Chapter 11

Exercise 11A

1a numeric, discrete **b** numeric, continuous
c categorical **d** numeric, continuous
e categorical **f** categorical
g On a standard scale of shoes sizes, this is numeric and discrete. The length of a person's foot would be a continuous distribution.
h Numeric, discrete. Reported ATAR scores are between 30 and 99.95 in steps of 0.05. There are about 1400 different scores awarded.

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2a

Outcome	HH	HT	TH	TT
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Uniform distribution (and categorical).

b

Outcome	2 heads	1 head and 1 tail	2 tails
Probability	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

3a

Outcome	red	green
Probability	$\frac{4}{7}$	$\frac{3}{7}$

b

Outcome	J	K	L	O
Probability	0.06	0.08	0.04	0.82

c

Outcome	P	A	R	M	T
Probability	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$

d

Outcome	1	2	3	4
Probability	$\frac{9}{1000}$	$\frac{90}{1000}$	$\frac{900}{1000}$	$\frac{1}{1000}$

e

Outcome	even	prime	neither
Probability	$\frac{5}{10}$	$\frac{4}{10}$	$\frac{1}{10}$

4a Let X be the number of letters in a randomly chosen word.

Outcome x	3	4	6
Probability $P(X = x)$	$\frac{5}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

b Let X be the number of heads recorded when two coins are thrown.

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

c Let X be the digits recorded from the first 12 digits of $\sqrt{2}$.

x	1	2	3	4	5	6	7
$P(X = x)$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

d Let X be the number selected.

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

(Note that the answer is the same if the sets are amalgamated. Why?)

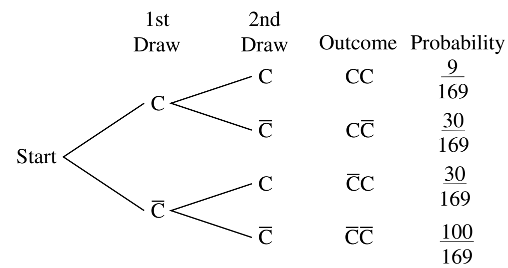
5a { T }, { F1 }, { F2 }, { T, F1 }, { T, F2 }, { F1, F2 }, { T, F1, F2 }

b

x	5	10	15	20
$P(X = x)$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{1}{7}$

6a yes **b** no **c** yes **d** yes **e** no **f** yes
7a 0.2 **b** 0.6 **c** 0.75 **d** 0 **e** 0.6 **f** 0.85 **g** 0.9 **h** 0.7 **i** 0.45

8a i Let C be the event, ‘A court card is drawn.’



ii

x	0	1	2
$P(X = x)$	$\frac{100}{169}$	$\frac{60}{169}$	$\frac{9}{169}$

b i The eight outcomes EEE, EEO, EOE, EOO, OEE, OEO, OOE, OOO each have probability $\frac{1}{8}$.

ii

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

c GGG has probability $\frac{8}{125}$, GGB, GBG, BGG each have probability $\frac{12}{125}$, GBB, BGB, BBG each have probability $\frac{18}{125}$, BBB has probability $\frac{27}{125}$.

x	0	1	2	3
$P(X = x)$	$\frac{8}{125}$	$\frac{36}{125}$	$\frac{54}{125}$	$\frac{27}{125}$

d Let S be the event, ‘A wallaby from Snake Ridge was selected’. SSS has probability 0.027,
$\overline{S}SS$, $S\overline{S}S$, $SS\overline{S}$ each have probability 0.063,
$\overline{S}\,\overline{S}S$, $\overline{S}S\overline{S}$, $S\overline{S}\,\overline{S}$ each have probability 0.147,
$\overline{S}\,\overline{S}\,\overline{S}$ has probability 0.343.

x	0	1	2	3
$P(X = x)$	0.343	0.441	0.189	0.027

9a $a = \frac{1}{25}$ **b** $a = \frac{1}{14}$ **c** $a = \frac{1}{27}$ **d** $a = \frac{1}{10}$ **e** $a = 1$

10a i EE and OO each have probability $\frac{1}{5}$,
EO and OE each have probability $\frac{3}{10}$.

ii

x	0	1	2
$P(X = x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

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b i BB has probability $\frac{2}{5}$, BG and GB each have probability $\frac{4}{15}$, GG has probability $\frac{1}{15}$.

ii

x	0	1	2
$P(X = x)$	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$

c i EE has probability $\frac{3}{10}$, ER, RE, ET, TE each have probability $\frac{3}{20}$, RT and TR each have probability $\frac{1}{20}$.

ii

x	0	1	2
$P(X = x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

11

x	22	44	55	24 or 42	25 or 52	45 or 54
$P(X = x)$	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{1}{36}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{6}$

12a

Outcome	RR	RG	GR	GG
Probability	$\frac{16}{49}$	$\frac{12}{49}$	$\frac{12}{49}$	$\frac{9}{49}$

b

Outcome	RR	RG	GR	GG
Probability	$\frac{12}{42}$	$\frac{12}{42}$	$\frac{12}{42}$	$\frac{6}{42}$

c

Outcome	HH	DD	SS	CC
Probability	$\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{17}$

HS or SH	HC or CH	HD or DH
$\frac{13}{102}$	$\frac{13}{102}$	$\frac{13}{102}$
SC or CS	SD or DS	CD or DC
$\frac{13}{102}$	$\frac{13}{102}$	$\frac{13}{102}$

13 a–c Answers will vary

d There is no guarantee that the results will be identical, though you would expect more *trials* (repeats of the experiment) would bring the results closer to each other and to the theoretical probabilities.

e Theoretical results: $P(X = 0) = 0.3$, $P(X = 1) = 0.6$, $P(X = 2) = 0.1$

f It might be easier to perform the experiment with coloured balls or tokens. Running the experiment in pairs with a nominated recorder also helps. The paper pieces need to be indistinguishable and well mixed in the bag. You could increase the number of trials or combine the class results.

14 EEE and OOO each have probability $\frac{1}{20}$, the other six possible outcomes each have probability $\frac{3}{20}$,

x	0	1	2	3
$P(X = x)$	$\frac{1}{20}$	$\frac{9}{20}$	$\frac{9}{20}$	$\frac{1}{20}$

15a The condition that the sum of the probabilities is 1 gives $a = \frac{1}{4}$ or $a = 1$. But $a = 1$ gives probabilities outside the interval $0 \le p \le 1$, so the only valid answer is $a = \frac{1}{4}$.

b $a = 1$ or $\frac{7}{6}$ (both are valid)

16a Let X be the sum of the numbers on the three cards. This question is best done by asking what card is *discarded.*

x	20	21	22
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b

x	20	21	22
$P(X = x)$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{3}{10}$

Exercise 11B

1a

x	0	1	2	3	Sum
$p(x)$	0.4	0.1	0.2	0.3	1
$xp(x)$	0	0.1	0.4	0.9	1.4

Hence $E(X) = 1.4$.

b

x	2	4	6	8	Sum
$p(x)$	0.1	0.4	0.4	0.1	1
$xp(x)$	0.2	1.6	2.4	0.8	5

Hence $E(X) = 5$.

c

x	−50	−20	0	30	100	Sum
$p(x)$	0.1	0.35	0.4	0.1	0.05	1
$xp(x)$	−5	−7	0	3	5	−4

Hence $E(X) = -4$.

2a

x	−40	0	30	60	Sum
$p(x)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1
$xp(x)$	−20	0	5	10	−5

b Expected value $= -5$

c The average cost to the player per game is 5 cents.

d $100 \times (-5) = -500$ cents. Thus the player expects to lose 500 cents and the casino expects to make 500 cents profit. This is an expected average value, not guaranteed.

3a–d Answers will vary

4a

x_i	2	4	6	8	10	Sum
p_i	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1
$x_i p_i$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{8}{5}$	$\frac{10}{5}$	6

So $E(X) = 6$.

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b

x_i	-3	1	2	5	6	Sum
p_i	0.1	0.3	0.2	0.3	0.1	1
$x_i p_i$	-0.3	0.3	0.4	1.5	0.6	2.5

So $\mathrm{E}(X) = 2.5$.

5a

x	1.50	2.10	2.40	Sum
$p(x)$	$\frac{5}{12}$	$\frac{4}{12}$	$\frac{3}{12}$	1
$xp(x)$	0.625	0.7	0.60	1.925

The expected value is \$1.925.

b If 100 purchases are made at random, the expected cost is \$192.50.

6a $\mathrm{E}(X) = 3$

b i $\mathrm{E}(Y) = 6$ **ii** Yes

c i $\mathrm{E}(Z) = 4$ **ii** Yes

7a 15 **b** 10 **c** $\frac{5}{2}$ **d** 3 **e** 0 **f** 18

8

x	0	1	2	3	Sum
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1
$xp(x)$	0	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{3}{8}$	$\frac{12}{8}$

The expected value is $1\frac{1}{2}$, as might be expected from the symmetry of the table of probabilities.

9

x	0	1	2	Sum
$p(x)$	$\frac{19}{34}$	$\frac{13}{34}$	$\frac{1}{17}$	1
$xp(x)$	0	$\frac{13}{34}$	$\frac{2}{17}$	$\frac{17}{34}$

The expected value is $\frac{1}{2}$.

10a–c Answers will vary

d

x	0	1	2	3	4	5	Sum
$p(x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	1
$xp(x)$	0	$\frac{10}{36}$	$\frac{16}{36}$	$\frac{18}{36}$	$\frac{16}{36}$	$\frac{10}{36}$	$\frac{70}{36}$

Hence $\mathrm{E}(X) = \frac{35}{18}$.

e Answers will vary

f In any dice experiment, it is important to check the randomness of your dice rolls. This can depend on your rolling technique. Try throwing a die 12 times and see if every outcome is equally likely. Does each outcome seem independent of the last?

g Answers will vary

11a $\frac{3}{15}, \frac{3}{15}, \frac{3}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}$

b -12, so the casino expects to make 12 cents each game, on average.

12a $P(\text{Orange}) = \frac{1}{6}$, $P(\text{Strawberry}) = \frac{2}{6}$, $P(\text{Apple}) = \frac{3}{6}$

b

Outcome	OOO	SSS	AAA	Other	Sum
x	$11k$	$2k$	k	0	—
$p(x)$	$\frac{1}{216}$	$\frac{8}{216}$	$\frac{27}{216}$	$\frac{180}{216}$	1

c The payout will be \$44 and their profit would be \$43, accounting for the \$1 entry fee.

13a $\mu = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 4 \times \frac{1}{16}$
$+ 5 \times \frac{1}{32} + 6 \times \frac{1}{64} + \ldots \qquad (1)$

Doubling:

$2\mu = 1 \times 1 + 2 \times \frac{1}{2} + 3 \times \frac{1}{4} + 4 \times \frac{1}{8}$
$+ 5 \times \frac{1}{16} + 6 \times \frac{1}{32} + \cdots \qquad (2)$

Subtracting (1) from (2):

$\mu = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \qquad (3)$

b Doubling:

$2\mu = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \qquad (4)$

Subtracting (3) from (4): $\mu = 2$.

c On average, we would expect to get a head on the second throw. You could test this by recording how many throws it takes over say 50 trials and averaging the results.

14 $\mathrm{E}(X) = 2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} + 16 \times \frac{1}{16} + \cdots$
$= 1 + 1 + 1 + 1 + \cdots$

The expected value ‘increases without bound’, that is, $\mathrm{E}(X) \to \infty$ as the game continues.

This suggests that there is no reasonable price the casino could put on this game and expect to break even. There are various issues with this scenario in real life. Casinos would not provide a game which had no upper limit to the payout. Patrons would also be unwilling to pay a large price for a game with such low apparent probabilities for the later stages of the game. The calculation of a simple expected value may not be the best way to analyse this game.

Exercise 11C

1a

x	1	2	3	4	Sum
$p(x)$	0.3	0.5	0.1	0.1	1
$xp(x)$	0.3	1	0.3	0.4	2
$(x-\mu)^2$	1	0	1	4	—
$(x-\mu)^2 p(x)$	0.3	0	0.1	0.4	0.8

$\mu = 2$, $\mathrm{Var}(X) = 0.8$

b $\sigma = \sqrt{0.8} \doteqdot 0.89$

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2a

x	1	2	3	4	Sum
$p(x)$	0.3	0.5	0.1	0.1	1
$xp(x)$	0.3	0.1	0.3	0.4	2
x^2	1	4	9	16	—
$x^2p(x)$	0.3	2	0.9	1.6	4.8

b $\text{Var}(X) = 4.8 - 2^2 = 0.8$

3a $\text{E}(X) = 2$, $\text{Var}(X) = 2$ **b** $\text{E}(X) = 3$, $\text{Var}(X) = 1$

c $\text{E}(X) = 0$, $\text{Var}(X) = 2.6$

d $\text{E}(X) = 2.8$, $\text{Var}(X) = 1.36$

4a i $\text{E}(Y) = 2$, $\text{Var}(Y) = 1$, $\sigma = 1$

ii $\text{E}(Z) = 2$, $\text{Var}(Z) = 4$, $\sigma = 2$

iii $\text{E}(V) = 1$, $\text{Var}(V) = 0.8$, $\sigma \doteqdot 0.89$

iv $\text{E}(W) = 3$, $\text{Var}(W) = 0.8$, $\sigma \doteqdot 0.89$

b i Both sets of data are centred around 2 and the expected value of each is, unsurprisingly, 2. The second data set is more spread out — in fact in moving from Y to Z the distances from the mean to each data point have been doubled and the standard deviation is doubled.

ii The data has been 'flipped over', but is no more spread out than before; the variance is unchanged, the expected value has changed. You may notice that $W = 4 - V$.

5 $\text{E}(X) = 2$, $\text{Var}(X) = 0$

6a $\text{E}(J) = 1.55$, $\text{Var}(J) = 2.05$, $\text{E}(L) = 1.4$, $\text{Var}(L) = 0.84$

b Over the season John might be expected to score more baskets, because his expected value is higher.

c Liam is the more consistent player, with the lower variance. Coaches may prefer a more consistent player, particularly if it is more important to score *some* goals, rather than the maximum number. This may also be a sign that John needs to work on the consistency of his game.

7a Each outcome has probability $\frac{1}{3}$. This is a uniform distribution. **b** $\text{E}(X) = 2$ **c** $\text{Var}(X) = \frac{2}{3}$

8a Two standard deviations

b It is one and a half standard deviations below the mean.

c The English score was more standard deviations below the mean than the Mathematics result, so it may be considered less impressive.

9a Visual Arts is 1 standard deviation below the mean, Music is 1.75 standard deviations below the mean, hence the Visual Arts score is better.

b Earth Science is 2 standard deviations above the mean, Biology is 1.5 standard deviations above the mean, hence the Earth Science score is more impressive.

c Chinese is 2 standard deviations above the mean, Sanskrit is also 2 standard deviations above the mean, hence the scores are equally impressive.

10a $E(X) = 3.3$, $\sigma = 1.45$

b 8 appears to be a long way from 3.3 and well removed from the rest of the data.

c 8 is 3.2 standard deviations above the mean and thus would be an outlier by this definition.

d $E(X) = 3.15$, $\sigma = 1.06$

e The mean and standard deviation have changed significantly, especially the standard deviation.

f Outliers are interesting values in any distribution and should be a flag to investigate more closely. Were results recorded correctly? Was there an error in the experiment; e.g. Jasmine used a more powerful bow with greater range, or perhaps she used a new set of arrows with better fletching? It may, however, be that Jasmine is inconsistent, occasionally getting much better results, but often getting fairly poor results — in which case the large standard deviation is warranted as a measure of this distribution. Over 20 trials, a probability of 0.05 only represents one set of 10 shots, so a larger set of results may give a better picture of her long term accuracy and reduce the impact of one strong result amongst many other weaker scores.

11 $k = \frac{1}{10}$, $E(X) = 3$, $\sigma = 1$

12a $\frac{1}{n}$ **b** $\frac{n+1}{2}$ **c** $\frac{1}{12}(n^2 - 1)$ **d** Answers will vary

13a Because $Z = X + a$:

$$\begin{aligned}
\text{E}(Z) &= \sum zP(Z = z) \\
&= \sum (x + a)P(X + a = x + a) \\
&= \sum (x + a)P(X = x) \\
&= \sum xP(X = x) + \sum aP(X = x) \\
&= \sum xP(X = x) + a\sum P(X = x) \\
&= \mu + a
\end{aligned}$$

because $\sum P(X = x) = 1$.

b Because $Z = kX$:

$$\begin{aligned}
\text{E}(Z) &= \sum zP(Z = z) \\
&= \sum (kx)P(kX = kx) \\
&= \sum (kx)P(X = x), \\
&= k \times \sum xP(X = x) \\
&= k\mu
\end{aligned}$$

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14a The mean of Z is $\mu + a$, by the previous question.

$$\begin{aligned}\text{Hence } \mathrm{Var}(Z) &= \mathrm{E}((Z - (\mu + a))^2)\\ &= \mathrm{E}((Z - a - \mu)^2)\\ &= \mathrm{E}((X - \mu)^2)\\ &= \mathrm{Var}(X)\end{aligned}$$

Hence the standard deviation of the new distribution remains σ. This is to be expected, because the distribution is no more spread out than previously.

b The mean of Z is $k\mu$, by the previous question. Hence

$$\begin{aligned}\mathrm{Var}(Z) &= \mathrm{E}((Z - k\mu)^2)\\ &= \mathrm{E}((kX - k\mu)^2)\\ &= k^2 \times \mathrm{E}((X - \mu)^2)\\ &= k^2\mathrm{Var}(X)\end{aligned}$$

The standard deviation of the new distribution is $\sqrt{k^2\sigma^2} = k\sigma$.

Exercise 11D

1a

x	0	1	2	3	Sum
$p(x)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$	1
$xp(x)$	0	$\frac{12}{27}$	$\frac{12}{27}$	$\frac{3}{27}$	1
$x^2p(x)$	0	$\frac{12}{27}$	$\frac{24}{27}$	$\frac{9}{27}$	$1\frac{2}{3}$

$\mu = 1, \sigma^2 = 1\frac{2}{3} - 1^2 = \frac{2}{3}, \sigma \doteqdot 0.82$

b

x	0	1	2	3	Sum
f	33	47	16	4	100
f_r	0.33	0.47	0.16	0.04	1
xf_r	0	0.47	0.32	0.12	0.91
x^2f_r	0	0.47	0.64	0.36	1.47

$\bar{x} = 0.91, s^2 = 1.47 - (0.91)^2 = 0.6419, s \doteqdot 0.80$

c The sample results are a little below what is predicted by the theoretical probabilities.

2a $\mu = 7, \sigma^2 = \frac{35}{6}, \sigma \doteqdot 2.42$ **b–f** Answers will vary

3a–f Answers will vary

4a Answers will vary **b** Answers will vary

c iii

x	0	1	2	3	4	5	Sum
$p(x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	1
$xp(x)$	0	$\frac{10}{36}$	$\frac{16}{36}$	$\frac{18}{36}$	$\frac{16}{36}$	$\frac{10}{36}$	$\frac{70}{36}$
$x^2p(x)$	0	$\frac{10}{36}$	$\frac{32}{36}$	$\frac{54}{36}$	$\frac{64}{36}$	$\frac{50}{36}$	$\frac{210}{36}$

$\mu \doteqdot 1.94, \sigma^2 = \frac{210}{36} - \left(\frac{70}{36}\right)^2, \sigma \doteqdot 1.43$

i, ii, iv Answers may vary

5a Answers will vary **b** Answers will vary

6a–d Answers will vary

7a $\mu = 3.441, \sigma \doteqdot 2.46$ **b** Answers will vary

8 Answers will vary

9a–c Answers will vary

10a–c Answers will vary

11 Answers will vary

12a Later people taking part in the experiment will be influenced by earlier guesses, particularly if the previous guesses have been measured for accuracy. Perhaps students could record their estimate, or draw their estimated shape, at the same time and before any measuring occurs. Perhaps students go into a separate room for the experiment.

b i–iii Answers will vary

13a–c Answers will vary

14a $m - k$ is the number of serial numbers not yet discovered in the range from 1 to m. If these serial numbers are spread between the k gaps, the average size of the gap (number of undiscovered serials) is $\dfrac{m-k}{k}$.

b The gap of $\dfrac{m-k}{k}$ integers should extend past m to $m + \dfrac{m-k}{k}$. Using this estimate the last serial will be:

$$\begin{aligned}N &= m + \frac{m-k}{k}\\ &= m + \frac{m}{k} - 1\end{aligned}$$

c i–iii Answers will vary

Chapter 11 review exercise

1a numeric, continuous **b** numeric, discrete

c numeric, discrete (and infinite)

d categorical

2a yes **b** no **c** no

3 The probabilities are not all positive, do not sum to 1, and are not all less than 1.

4a $\mathrm{E}(X) = 1.4$ **b** $\mathrm{E}(X) = -0.8$

5a $\mathrm{E}(X) = 27.22$

b His expected cost is $\$27.22 \times 52 = \$1415.56$.

6a $E(X) = 2, \mathrm{Var}(X) = 1, \sigma = 1$

b $E(X) = 5.1, \mathrm{Var}(X) = 0.69, \sigma \doteqdot 0.83$

7a $E(X) = 2, E(X^2) = 5, \mathrm{Var}(X) = 1$

b $E(X) = 5.1, E(X^2) = 26.70, \mathrm{Var}(X) = 0.69$

8a $E(X) = 1.9, \mathrm{Var}(X) = 0.49, \sigma = 0.7$

b $E(X) = 2, \mathrm{Var}(X) = 2.6, \sigma \doteqdot 1.61$

9 Expected value is a measure of central tendency — it measures the centre of the data set. It may also be thought of as a weighted mean (weighted by the probabilities of the distribution). If the experiment is carried out experimentally a large number of times we would expect that the average of the outcomes would approach the expected value.

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10 The standard deviation is the square root of the variance. Both measure the spread of the data, so that a distribution with a larger standard deviation is more spread out than a distribution with a smaller standard deviation. Both are zero if the distribution only takes one value — that is. if it is not spread out at all. If the distribution is stretched (multiplied) by a constant k the standard deviation also increases by a factor k.

11a $12, 8, 2\sqrt{2}$ **b** $11, 2, \sqrt{2}$ **c** $17, 18, 3\sqrt{2}$

12a

x	5	6	7	8	9
$p(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$\mathrm{E}(X) = 7$, $\mathrm{Var}(X) = 2$, $\sigma = \sqrt{2}$

b Answers will vary

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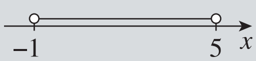
Appendix C: Bracket interval notation

Section 2B discussed intervals, using inequalities such as $\frac{1}{3} \le x \le 3$ to describe them. This extra section for Chapter 3 introduces an alternative notation for intervals that can often make the notation a little more concise. The notation encloses the endpoints of the interval in brackets, using a square bracket if the endpoint is included and a round bracket if the endpoint is not included.

Here are the five examples from Section 2B written in both notations:

Diagram	Using inequalities	Using brackets
	$\frac{1}{3} \le x \le 3$	$\left[\frac{1}{3}, 3\right]$
Read this as, 'The closed interval from $\frac{1}{3}$ to 3'.		
	$-1 < x < 5$	$(-1, 5)$
Read this as, 'The open interval from -1 to 5'.		
	$-2 \le x < 3$	$[-2, 3)$
Read this as, 'The interval from -2 to 3, including -2 but excluding 3'.		
	$x \ge -5$	$[-5, \infty)$
Read this as, 'The closed ray from -5 to the right'.		
	$x < 2$	$(-\infty, 2)$
Read this as, 'The open ray from 2 to the left'.		

The first interval $\left[\frac{1}{3}, 3\right]$ is *closed*, meaning that it contains all its endpoints.

The second interval $(-1, 5)$ is *open*, meaning that it does not contain any of its endpoints.

The third interval $[-2, 3)$ is neither open nor closed — it contains one of its endpoints, but does not contain the other endpoint.

The fourth interval $[-5, \infty)$ is *unbounded on the right*, meaning that it continues towards infinity. It only has one endpoint -5, which it contains, so it is closed.

The fifth interval $(-\infty, 2)$ is *unbounded on the left*, meaning that it continues towards negative infinity. It only has one endpoint 2, which it does not contain, so it is open.

'Infinity' and 'negative infinity', with their symbols ∞ and $-\infty$, are not numbers. They are ideas used in specific situations and phrases to make language and notation more concise. Here, they indicate that an interval is unbounded on the left or right, and the symbol $(-\infty, 2)$ means 'all real numbers less than 2'.

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Bracket interval notation has some details that need attention.

- The variable x or y or whatever is missing. This can be confusing when we are talking about domain and range, or solving an inequation for some variable. When, however, we are just thinking about 'all real numbers greater than 100', no variable is involved, so the notation $(100, \infty)$ is more satisfactory than $x > 100$.
- The notation can be dangerously ambiguous. For example, the open interval $(-1, 5)$ can easily be confused with the point $(-1, 5)$ in the coordinate plane.
- Infinity and negative infinity are not numbers, as remarked above.
- The set **R** of all real numbers can be written as $(-\infty, \infty)$.
- The notation $[4, 4]$ is the one-member set $\{4\}$, called a *degenerate interval* because it has length zero.
- Notations such as $(4, 4)$, $(4, 4]$, $[7, 3]$ and $[7, 3)$ all suggest the empty set, if they mean anything at all, and should be avoided in this course.

1 BRACKET INTERVAL NOTATION

- A square bracket means that the endpoint is included, and a round bracket means that the endpoint is not included.
- For $a < b$, we can form the four *bounded intervals* below. The first is closed, the last is open, and the other two are neither open nor closed.
$$[a, b] \quad\text{and}\quad [a, b) \quad\text{and}\quad (a, b] \quad\text{and}\quad (a, b).$$
- For any real number a, we can form the four *unbounded intervals* below. The first two are closed, and the last two are open.
$$[a, \infty) \quad\text{and}\quad (-\infty, a] \quad\text{and}\quad (a, \infty) \quad\text{and}\quad (-\infty, a).$$
- The notation $(-\infty, \infty)$ means the whole real number line **R**.
- The notation $[a, a]$ is the one-member set $\{a\}$, called a *degenerate interval*.
- An interval is called *closed* if it contains all its endpoints, and *open* if it doesn't contain any of its endpoints.

For those who enjoy precision, the unbounded interval $(-\infty, \infty)$ is both open and closed (it has no endpoints), and a degenerate interval $[a, a]$ is closed.

Using bracket interval notation for domain and range

When the domain or range of a function or relation is an interval, bracket interval notation is a straightforward alternative notation for the domain or range, as in the next worked exercise.

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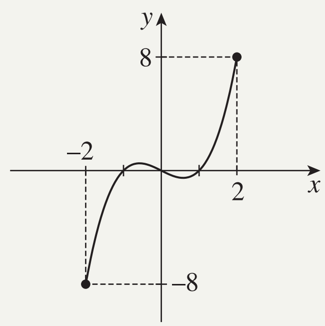
Example 1

Write down the domain and range of these relations:

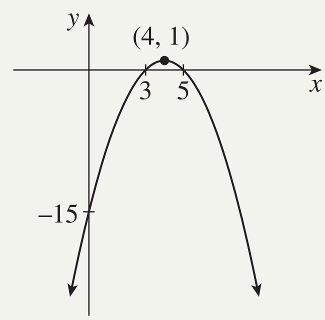
i using inequality interval notation,

ii using bracket interval notation.

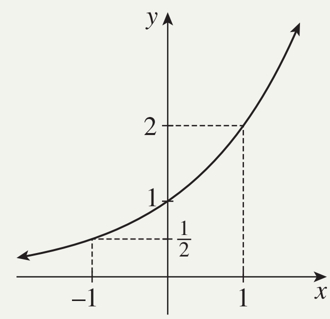
a



b



c



SOLUTION

a **i** domain: $-2 \le x \le 2$
range: $-8 \le y \le 8$

ii domain $= [-2, 2]$
range $= [-8, 8]$

b **i** domain: all real x
range: $y \le 1$

ii domain $= (-\infty, \infty)$
range $= (-\infty, 1]$

c **i** domain: all real x
range: $y > 0$

ii domain $= (-\infty, \infty)$
range $= (0, \infty)$

Using bracket interval notation for solutions of inequations

When the solutions of an inequation form an interval, bracket interval notation is a straightforward alternative notation.

Remember to reverse the inequality when multiplying or dividing by a negative.

Example 2

Write down the solution of these inequations, first using inequality interval notation, then using bracket interval notation.

a $-15 \le 5x < 30$

b $-2x < 10$

SOLUTION

a
$$-15 \le 5x < 30$$
$$\boxed{\div 5} \quad -3 \le x < 6$$

The solution is $[-3, 6)$.

b
$$-2x < 10$$
$$\boxed{\div (-2)} \quad x > -5$$

The solution is $(-5, \infty)$.

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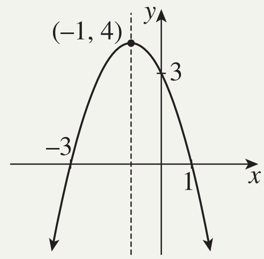
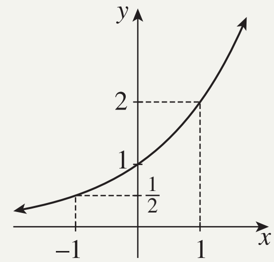
Example 3

State where $y \geq 0$ for each function, using both interval notations.

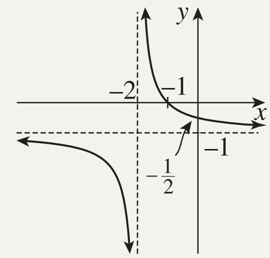
a



b



c



SOLUTION

a The solution is $-3 \leq x \leq 1$, or in the bracket notation, $[-3, 1]$.

b The solution is all real x, or in the bracket notation, $(-\infty, \infty)$.

c The solution is $-2 < x \leq -1$, or in the bracket notation, $(-2, -1]$.

Notation when the domain or range or set of solutions is not an interval

The domain of the function $f(x) = \dfrac{1}{x}$ is $x \neq 0$, which is not an interval. This domain is, however, the union of the two intervals $(-\infty, 0)$ and $(0, \infty)$. Notation in such situations is discussed in Section 3A of the Year 12 book, after the union and intersection of sets have been reviewed in in the context of probability (Section 12C in the Year 11 book).

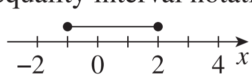
Exercise Appendix C

FOUNDATION

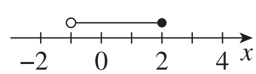
1 For each number line, write the graphed interval using:

i inequality interval notation, **ii** bracket interval notation.

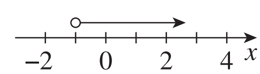
a



b



c



2 For each interval given by inequality interval notation:

i draw the interval on a number line,

ii write it using bracket interval notation.

a $-1 \leq x < 2$ **b** $x \leq 2$ **c** $x < 2$

3 For each interval given by bracket interval notation:

i draw the interval on a number line,

ii write it using inequality interval notation.

a $[-1, \infty)$ **b** $(-1, 2)$ **c** $(-\infty, \infty)$

4 Solve each inequation, writing the solution using both interval notations.

a $7x < 35$ **b** $5x \geq -45$ **c** $-3x \leq 15$

d $-6 < 2x < 8$ **e** $10 < -5x \leq 35$ **f** $-180 \leq -10x \leq 260$

5 There are fifteen intervals in Q1–Q4.

a Which are open? **b** Which are closed? **c** Which are unbounded?

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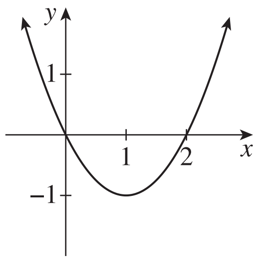
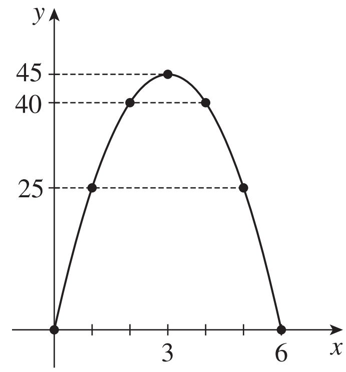
DEVELOPMENT

6 Write down the domains and ranges of each function using both interval notations.

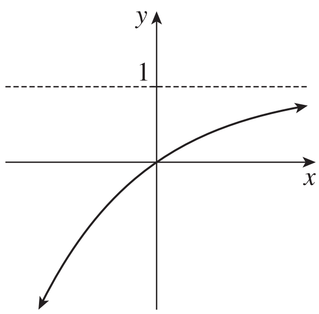
a



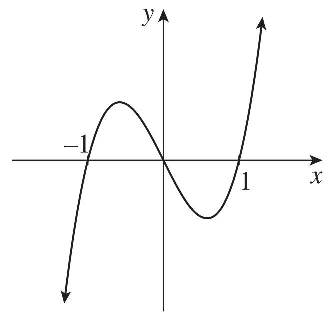
b



c

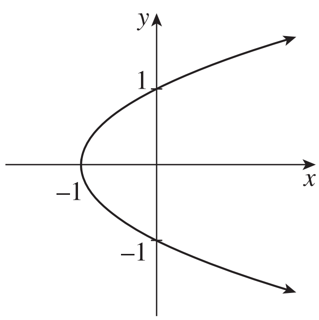


d

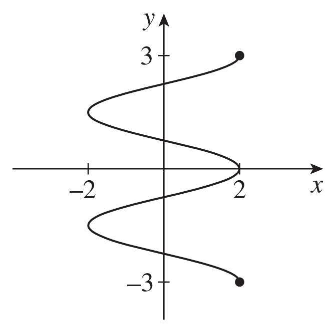


7 Write down the domains and ranges of each relation using both interval notations.

a



b



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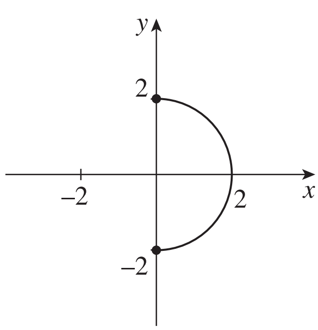
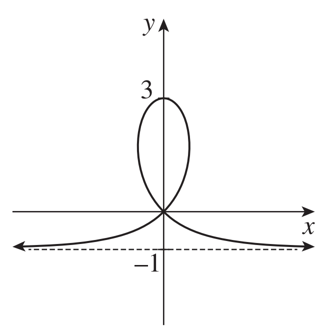


d



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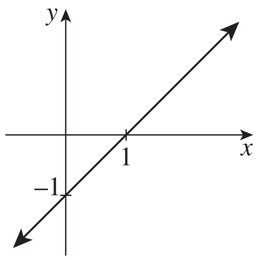
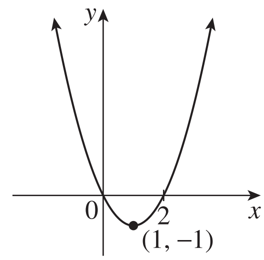
8 For each function, state the values of x for which $y < 0$, using both interval notations.

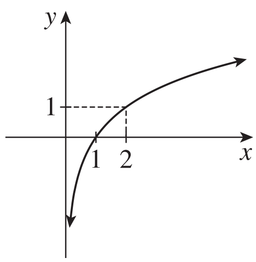
a



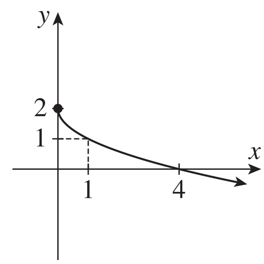
b



c



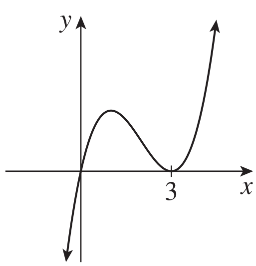
d



9 Repeat the previous question, finding the values of x for which $y \leq 0$.

10 Use the given graph of the LHS to help solve each inequation using both interval notations.

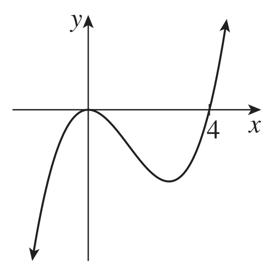
a



i $x(x - 3)^2 \geq 0$

ii $x(x - 3)^2 < 0$

b



i $x^2(x - 4) > 0$

ii $x^2(x - 4) \leq 0$

CHALLENGE

11 Use bracket interval notation in this question, where $f(x) = 2^{x-3} + 1$.

a Write down the domain and range of $f(x)$.

b Write down the solutions of:

i $f(x) \geq 1$ **ii** $f(x) \geq 2$ **iii** $f(x) < 9$ **iv** $1\frac{1}{2} \leq f(x) < 3$

12 Write the solutions of each inequation using bracket interval notation.

a $x^2 \leq 0$ **b** $x^3 \leq 0$ **c** $|x| \leq 0$ **d** $|x| + x \leq 0$

13 As defined in the notes above, an interval is called *closed* if it contains all its endpoints, and *open* if it doesn't contain any of its endpoints.

a Explain why the degenerate interval $[5, 5]$ is closed.

b Explain why the interval $(-\infty, \infty)$ is closed.

c Explain why the same interval $(-\infty, \infty)$ is also open.

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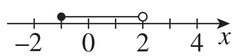
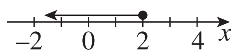
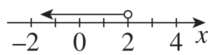
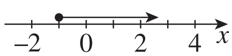
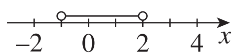
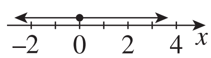
Answers

Exercise Appendix C

1 a i $-1 \le x \le 2$ **ii** $[-1, 2]$
b i $-1 < x \le 2$ **ii** $(-1, 2]$
c i $x > -1$ **ii** $(-1, \infty)$

2 a i  **ii** $[-1, 2)$
b i  **ii** $(-\infty, 2]$
c i  **ii** $(-\infty, 2)$

3 a i  **ii** $x \ge -1$
b i  **ii** $-1 < x < 2$
c i  **ii** all real x

(The interval can't be written using inequalities.)

4 a $x < 5$ OR $(-\infty, 5)$
b $x \ge -9$ OR $[-9, \infty)$
c $x \ge -5$ OR $[-5, \infty)$
d $-3 < x < 4$ OR $(-3, 4)$
e $-7 \le x < -2$ OR $[-7, -2)$
f $-26 \le x \le 18$ OR $[-26, 18]$

5 a Open: Q1c, Q2c, Q3b & c, Q4a & d
b Closed: Q1a, Q2b, Q3a & c, Q4b, c & f
c Unbounded: Q1c, Q2b & c, Q3a & c, Q4a, b & c

6 a domain: all real x OR $(-\infty, \infty)$
range: $y \ge -1$ OR $[-1, \infty)$
b domain: $0 \le x \le 6$ OR $[0, 6]$
range: $0 \le y \le 45$ OR $[0, 45]$
c domain: all real x OR $(-\infty, \infty)$
range: $y < 1$ OR $(-\infty, 1)$
d domain: all real x OR $(-\infty, \infty)$
range: all real y OR $(-\infty, \infty)$

7 a domain: $x \ge -1$ OR $[-1, \infty)$
range: all real y OR $(-\infty, \infty)$
b domain: $-2 \le x \le 2$ OR $[-2, 2]$
range: all real y OR $(-\infty, \infty)$
c domain: $0 \le x \le 2$ OR $[0, 2]$
range: $-2 \le y \le 2$ OR $[-2, 2]$
d domain: all real x OR $(-\infty, \infty)$
range: $-1 < y \le 3$ OR $(-1, 3]$

8 a $x < 1$ OR $(-\infty, 1)$
b $0 < x < 2$ OR $(0, 2)$
c $0 < x < 1$ OR $(0, 1)$
d $x > 4$ OR $(4, \infty)$

9 a $x \le 1$ OR $(-\infty, 1]$
b $0 \le x \le 2$ OR $[0, 2]$
c $0 < x \le 1$ OR $(0, 1]$
d $x \ge 4$ OR $[4, \infty)$

10 a i $x \ge 0$ OR $[0, \infty)$
ii $x < 0$ OR $(-\infty, 0)$
b i $x > 4$ OR $(4, \infty)$
ii $x \le 4$ OR $(-\infty, 4]$

11 a domain: $(-\infty, \infty)$, range: $(1, \infty)$
b i $(-\infty, \infty)$ **ii** $[3, \infty)$
iii $(-\infty, 6)$ **iv** $[2, 4)$

12 a $[0, 0]$ **b** $(-\infty, 0]$ **c** $[0, 0]$ **d** $(-\infty, 0]$

13 a It has one endpoint 5, which it contains.
b It contains all its endpoints (there are none).
c It does not contain any of its endpoints (there are none).

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