YEAR 11

MATHEMATICS EXTENSION 1

CambridgeMATHS

STAGE 6

BILL PENDER

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Extension 1

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Online appendices

See the interactive textbook for guides to spreadsheets, the Desmos graphing calculator, and links to scientific calculator guides.

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Rationale

CambridgeMATHS Extension 1 Year 11 covers all syllabus dotpoints for Year 11 of the Mathematics Advanced and Extension 1 courses being implemented in 2019. This rationale serves as a guide to how the book covers the dotpoints of the syllabus. Further documents are available in the teacher resources.

The Exercises

No-one should try to do all the questions! We have written long exercises with a great variety of questions so that everyone will find enough questions of a suitable standard — they cater for differentiated teaching to a wide range of students. The division of all exercises into Foundation, Development and Enrichment sections helps with this. Each student will need to tackle a selection of questions, and there should be plenty left for revision.

Compared to our previous Extension 1 textbooks, the **Foundation** section in each exercise provides a gentler start. There are more questions, and they are straightforward. Students need encouragement to assimilate comfortably the new ideas and methods presented in the text so that they are prepared and confident before tackling later problems.

The **Development** sections are graded from reasonably straightforward questions to harder problems. They are intended to bring students up to Extension 1 standard. The examinations have traditionally included questions that students find difficult, and we assume that this will continue with the HSC examinations on the new syllabuses — there are questions at the end of the Development sections intended to prepare students for these demands.

The **Enrichment** sections contain 'interesting' questions that are probably beyond the Extension 1 examination standard (although one can never be sure of this) and often beyond the Extension 1 syllabus. One of their purposes is to highlight issues significant in the Extension 2 course. Students aiming to do Extension 2 in Year 12 should 'have a go' at some of these questions, but should keep in mind that some of this material is demanding even at Extension 2 level.

The Extension 1 material

This book presents the Year 11 Advanced course and Year 11 Extension 1 course within the one volume. Despite the tight integration of the two courses, there are various practical reasons why Advanced material and Extension 1 material need to be easily identified and separated. In this volume:

The eleven chapters from the Advanced textbook, and their sections, have been retained. Some of the text and exercises in those chapters has been made slightly more demanding so that it is suitable for the higher standards customary in Extension 1.

The Extension 1 material has been placed into six separate chapters distributed appropriately through the volume.

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Chapter 1: Methods in algebra

1A Arithmetic with pronumerals
1B Expanding brackets
1C Factoring
1D Algebraic fractions
1E Solving linear equations
1F Solving quadratic equations
1G Solving simultaneous equations
1H Completing the square
Chapter Review Exercise

Chapter 1 revises algebraic techniques. The calculus courses rely heavily on fluency in algebra, and the chapter is intended for students who need the revision. If students are confident with this material already from Years 9–10, they should not spend time on it, but simply know that it is there if they find later that some techniques need revision.

MA-F1.1 dotpoints 2–3

F1.1 dotpoint 1 index laws & Sections 8A–8B: The index laws from Years 9–10 are not reviewed here. They need a clearer and fuller presentation in Chapter 8 in the context of exponential and logarithmic functions and their graphs.

F1.1 dotpoint 1 surds & Sections 2C–2D: Surds are dealt with thoroughly in Chapter 2.

F1.1 dotpoint 2 & Sections 3D–3F: When quadratics graphs are developed in Chapter 3, the quadratic techniques of this dotpoint are covered again, but at a higher level.

Chapter 2: Numbers and surds

2A Whole numbers, integers and rationals
2B Real numbers and approximations
2C Surds and their arithmetic
2D Further simplification of surds
2E Rationalising the denominator
Chapter Review Exercise

Chapter 2 reviews the four standard sets of numbers: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ — students should be clear about these before tackling functions. In particular, they must know how to manipulate surds. They must also understand the geometric basis of the real numbers as the points on the number line, and be able to approximate them in various ways. The syllabus mostly takes all this as assumed knowledge from Years 9–10, but it is mentioned specifically in.

F1.1, dotpoint 1 (surds), F1.2, dotpoint 2 (intervals)

This chapter is mostly intended as a quick review, and time should not be spent on it unnecessarily.

Types of intervals and their line graphs are introduced in Section 2B. 'Interval notation' in the form $3 < x < 6$ is used throughout the Year 11 book. The alternative 'interval notation' (3, 6) is introduced in Appendix C (Bracket interval notation) and can follow section 3H.

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Chapter 3: Functions and graphs

Section 3A, 3B and 3I introduce the more sophisticated notions of functions, relations and graphs required in the course and mentioned in

MA-F1.2, apart from odd and even functions and composite functions

Functions are introduced first in Section 3A using a function machine, $f(x)$ notation, and tables of values. This leads in Section 3B to the graph, and to the use of the vertical line test to distinguish between a function and the more general concept of a relation.

The final Section 3I returns to the general theory and covers several related matters — reading a graph backwards using horizontal lines, graphical solutions of $f(x) = k$ (F1.3 dotpoint 3, sub-dotpoint), the horizontal line test, the classification of relations as one-to-one, many-to-one, one-to-many and many-to-many (F1.2 dotpoint 4), and mappings between sets (F1.2, dotpoint 1). This prepares for the Extension 1 topic of inverse functions in Chapter 5.

MA-F1.2 dotpoint 7 & section 4E Composite functions: Composite functions are covered in Chapter 4, after translations, reflections and absolute values have already provided specific examples.

MA-F1.2 dotpoint 6 & section 5D Graphing sums and products: The graphs of sums, differences and products are treated in some detail in section 5D.

MA-F1.2 dotpoints 6–7 & Chapter 9 Differentiation: The functions of algebra almost all involve sums, differences, quotients, products and composites of functions. Apart from composites, it would not be helpful at this stage to formalise rules for domains and ranges because the rules quickly become very complicated, and things are best done case-by-case. Each of these five ways to build functions is considered separately as the rules of differentiation are developed in Chapter 9.

Sections 3C–3H present, in the unified manner introduced in Sections 3A–3B, a variety of graphs that are partly known from Years 9–10, particularly quadratics, and also higher powers of x, polynomials, circles, and graphs with asymptotes. Section 3G sketches cubics and general polynomials factored into linear factors, using a table of values to test the sign, but restricted to square linear factors because the situation at multiple will be addressed later in Chapter 10 using calculus.

MA-F1.3 all dotpoints, and MA-F1.4 dotpoints 1–4

MA-F1.3 dotpoint 1 & Chapter 7 The coordinate plane: Lines are only briefly covered in Section 3C, and covered in far more detail in Chapter 7. Gradient is the key idea with linear graphs, and gradient in turn requires $\tan\theta$ and the ability to solve, for example, $\tan\theta = -1$, where $0° \le \theta < 180°$ (see also the mention at MA-C1.1 dotpoint 3).

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Chapter 4: Transformations and symmetry

4A Translations of known graphs
4B Reflections in the x-axis and y-axis
4C Even and odd symmetry
4D The absolute value function
4E Composite functions
Chapter Review Exercise

Chapter 4 presents translations of graphs, then reflections of them in the x-axis and y-axis, stressing the equivalence of the geometric and algebraic formulations. It then presents odd and even functions, which are functions whose symmetry is defined using these two reflections.

With these two sets of symmetries, the graphing of simple functions involving absolute value can then mostly be done using transformations rather than the confusing algebra with cases.

Composites fit well into this chapter because translations horizontally and vertically, and reflections in either axis, are special cases of composites.

MA-F1.2 dotpoint 5, and MA-F1.4 dotpoints 5–9

Translations have been brought forward from MA-F2 in Year 12 Functions because translations are able to unify many things about graphs and their equations, particularly with the graphs of circles, parabolas and absolute value, and make them much easier to deal with. Dilations are also important in such unification, but they cause more difficulty and have been left to Year 12.

MA-F1.4 dotpoint 9 & Section 3G–4A: Circles with centre the origin are in Section 3G, and circles with other origins are in Section 4A.

Chapter 5: Extension 1—Further graphs

5A Solving inequations
5B The sign of a function
5C Reciprocals and asymptotes
5D Graphing sums and products
5E Absolute value and square roots
5F Inverse relations and functions
5G Inverse function notation
5H Defining functions and relations parametrically
Chapter Review Exercise

ME-F1.1–1.4

The material in Chapter 5 comes from four interrelated syllabus sections:

ME-F1.1 Transformations involving reciprocals, sums, products, absolute value and squares
ME-F1.2 Inequations (also called inequalities), with algebraic and graphical approaches.
ME-F1.3 Inverse functions.
ME-F1.4 Parametric equations of functions and relations.

The placement of this chapter may need to be varied. The final section on parameters requires trigonometric functions of a general angle and Pythagorean identities, and could be delayed until after trigonometry if students need trigonometry reviewed. In fact, the whole chapter is rather demanding and could be delayed, but it should be done before Chapter 9: Differentiation.

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Sections 5A–5C are a unit. They begin with the solution of various types of inequations using combinations of algebraic and graphical techniques. Section 5B introduces the *table of signs*, which is a table of values dodging zeroes and discontinuities that tests the sign of the function over its whole domain, and uses it to solve various other inequalities and to establish the behaviour of a curve near a vertical asymptote. This leads naturally to the reciprocal transformation, where the graph of $y = \left(f(x)\right)^{-1}$ is sketched from the known graph of $y = f(x)$.

All the elements of a general systematic pre-calculus curve-sketching routine are now all in place, but they are not put together until Year 12.

Sections 5D–5E develop procedures for sketching further transformations of given graphs. Section 5D sketches the sum (and difference) and product of two functions (and the square of one function). Section 5E sketches various absolute value and square root transformations.

Sections 5F–5G develop the theory of inverse relations in terms of swapping the coordinates of all the ordered pairs, so that the theory applies also to relations such as circles and is easily related to reflection in $y = x$, which is yet another transformation of the graph. Using the horizontal line test and the distinction between one-to-one and many-to-one functions, inverse function notation is introduced, and a very brief account is given of how to restrict the domain of a many-to-one function so that the restriction has an inverse.

ME-F1.3 dotpoint 5 & Section 17A: Restricting the domain before taking the inverse seems a step too far at this early stage, apart from $y = \sqrt{x}$, which is covered here. It becomes crucial, however, for the inverse trigonometric functions. More general restrictions of the domain have therefore been left until the first section of Chapter 17 before introducing inverse trigonometric functions. Section 17A could be covered here if appropriate for the class.

Section 5H on parametric functions requires the trigonometry of the general angle and the three Pythagorean identities, as remarked above.

Chapter 6: Trigonometry

6A Trigonometry with right-angled triangles
6B Problems involving right-angled triangles
6C Three-dimensional trigonometry
6D Trigonometric functions of a general angle
6E Quadrant, sign, and related acute angle
6F Given one trigonometric function, find another
6G Trigonometric identities
6H Trigonometric equations
6I The sine rule and the area formula
6J The cosine rule
6K Problems involving general triangles
Chapter Review Exercise

Chapter 6 begins to turn the attention of trigonometry towards the trigonometric functions and their graphs. This requires their definition and calculation for angles of any magnitude, the solving of simple trigonometric equations, and the examination of their graphs. The wave properties of the sine and cosine graphs are the most important goal here, because so much of science is concerned with cycles and periodic events.

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The chapter is thus partly a review of Years 9–10 triangle trigonometry, extended to three dimensions, and partly a change of attention to the functions and their graphs, with all the required calculations and identities. The corresponding syllabus items are:

MA-T1 — all dotpoints except for radians

MA-T2 — all dotpoints

MA-T1.1 dotpoint 3 & Section 6J–6K: The use of the cosine rule in the ambiguous case is not mentioned in the syllabus, but is briefly touched on in Exercise 6J and Section 6K.

MA-T1.2 (radians) & Chapter 11 Extending calculus: Radians are not introduced in Chapter 5. First, there is too much trigonometry to review, including general angles and the six graphs, ideas which students already find unfamiliar and difficult. Secondly, radian measure is unmotivated at this stage, because what motivates it is the derivative $\dfrac{d}{dx}(\sin x) = \cos x$. We have therefore placed radian measure later in the book, in the second half of Chapter 9: Extending calculus, which begins the extension of calculus to the exponential and trigonometric functions.

Given one trigonometric function, find another & Section 6F: This material is not explicitly in the syllabus dotpoints, but is in the support material, so is covered in Section 6F.

Chapter 7: The coordinate plane

7A Lengths and midpoints of intervals
7B Gradients of intervals and lines
7C Equations of lines
7D Further equations of lines
7E Using pronumerals in place of numbers
Chapter Review Exercise

Chapter 7 is mostly, if not all, a review of Years 9–10 material. Its main purpose is the analysis of gradient and the equation of lines — revising gradient is an immediate preparation for calculus, which takes place initially in the coordinate plane, where the derivative is a gradient.

MA-F1.3 dotpoint 1, MA-C1.1 dotpoint 3

As further preparation, length and midpoint are also revised in Section 7A. The final section, which may be regarded as enrichment, asks for coordinate plane calculations involving pronumeral constants — this is required in many calculus problems.

Chapter 8: Exponential and logarithmic functions

8A Indices
8B Fractional indices
8C Logarithms
8D The laws for logarithms
8E Equations involving logarithms and indices
8F Exponential and logarithmic graphs
8G Applications of these functions
Chapter Review Exercise

MA-F1.1 first dotpoint, MA-E1.1, MA-E1.2, MA-E1.4

Chapter 8 begins with a thorough treatment of indices, then introduces and develops logarithms, before turning attention to the exponential and logarithmic functions, their graphs, and their applications.

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Indices and logarithms base e are not covered in this chapter, but are presented in the first part of Chapter 11: Extending calculus, which follows Chapter 9: Differentiation and deals with e^x, then with radian measure of angles. It is important that students first encounter exponential and logarithmic functions with familiar bases such as 2, 3 and 10, so that they are well prepared for Euler's number and its surprising definition, and for base e calculations.

This chapter has been placed before differentiation because indices, including negative and fractional indices, are essential for the introductory chapter on differentiation that follows.

Exponential equations reducible to quadratics & Year 12 Functions: These equations are not in the syllabus, but are in the support material. They are therefore covered briefly in Section 8E and will be reviewed in Year 12.

Chapter 9: Differentiation

9A Tangents and the derivative
9B The derivative as a limit
9C A rule for differentiating powers of x
9D The notation $\dfrac{dy}{dx}$ for the derivative
9E The chain rule
9F Differentiating powers with negative indices
9G Differentiating powers with fractional indices
9H The product rule
9I The quotient rule
9J Rates of change
9K Continuity
9L Differentiability
Chapter Review Exercise

MA-F1.2 dotpoint 6, MA-C1.1–1.4 all dotpoints

This long chapter follows very closely the development given MA-C1, except that continuity and differentiability have been placed at the end of the chapter — experience with tangents is necessary to understand the significance of these ideas.

MA-C1.1 dotpoint 3 & Chapter 7 The coordinate plane: This dotpoint is part of the theory of gradient, and was covered (apart from the mention of a tangent) in Chapter 7.

Sections 9A–9I present the standard theory of the derivative, first geometrically using tangents, then using the definition of the derivative as a limit. The geometric implications are stressed throughout, particularly as they apply to quadratics and higher-degree polynomials, but as negative and fractional indices are introduced, the geometry of hyperbolas and other more exotic curves is investigated.

As remarked in the notes to Chapter 3, the extension of differentiation to the sums, differences, products, quotients and composites of functions are significant advances in the theory in Sections 9C, 9E and 9H–9I, and draw attention to these five ways of combining functions.

C1.2 dotpoint 1 & C1.3 dotpoint 7 & Year 12 Curve-sketching-using-calculus and integration: The terms 'increasing', 'decreasing', 'stationary' and so forth are used informally in this chapter and earlier, and are then defined formally in terms of tangents in Section 9J: Rates of change. Their proper treatment, however, occurs in the Year 12 syllabus, where turning points, concavity and inflections are also essential parts of the story. In many problems here, however, one is required to find the points where the derivative is zero, and other questions deal lightly with the other concepts.

Families of curves with a common derivative are covered lightly here in Section 9D. The intention is to break the ground for their later use in integration, boundary value problems, and DEs.

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Section 9J interprets the derivative as a rate of change, as in C1.4 dotpoints 4–7.

MA-C1.4 dotpoints 6–7 & The Year 12 Motion chapter: Motion is only briefly mentioned in Section 9J, because the necessary theory for motion has not yet been developed. In particular, curve-sketching-using calculus is needed even to discuss stationary points, derivatives of the sine the cosine functions are needed for many interesting situations, and without integration not even the motion of a body falling under gravity can be derived. Students find motion difficult, and need an extended treatment to understand it.

Section 9K–9L present continuity at a point, and then differentiability at a point, informally, without limits, as in C1.1 dotpoint 1 (differentiability is mentioned several times in the Glossary).

Chapter 10: Extension 1—Polynomials

10A The language of polynomials
10B Graphs of polynomial functions
10C Division of polynomials
10D The remainder and factor theorems
10E Consequences of the factor theorem
10F Sums and products of zeroes
10G Multiple zeroes
10H Geometry using polynomial techniques
Chapter Review Exercise

Besides basic sketching of polynomial curves, the goal of the chapter is to introduce some standard algebraic approaches to polynomials — long division, the factor theorem, the relationship between the coefficients and sums of products of roots, and the analysis of multiple zeroes. The final exercise relates these things back to the graph.

ME-F2 F2.1–F2.2

Some of this material will be known to some students from Years 9–10, some terminology and sketching was introduced in Section 3G, and sketching was also done in Section 5A, but the use of calculus in the sketching is new.

The ground has been prepared in several ways. Section 3G makes some initial definitions, and without calculus introduces the concepts of stationary inflexions and turning points on the x-axis, which is reviewed using calculus in this chapter. Section 9J on rates talks about functions that are increasing, decreasing and stationary at a point. Even after this, the turning points and inflexions away from the x-axis are not accessible until Year 12 curve-sketching-with-calculus.

Chapter 11: Extending calculus

11A The exponential function base e
11B Transformations of exponential functions
11C Differentiation of exponential functions
11D Differentiation and the graph
11E The logarithmic function base e
11F Exponential rates using the base e
11G Radian measure of angle size
11H Solving trigonometric equations
11I Arcs and sectors of circles
11J Trigonometric graphs in radians
Chapter Review Exercise

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After the derivative has been developed for algebraic functions, it then needs to be developed for the two groups of special functions: the exponential and logarithmic functions, and the trigonometric functions. This two-part chapter first develops the derivative of e^x, then develops radian measure in preparation for the derivative of the trigonometric functions.

The intention is to allow some of the parallels between the two sets of functions to emerge naturally, particularly between the new base e required for exponential functions and the new units of radians, based on π, required for trigonometric functions

Sections 11A–11F explain carefully the definition of e, taking care to emphasise the importance in calculus of $y = e^x$ having gradient 1 at its y-intercept. They then establish the derivatives of e^x and e^{ax+b}, introduce $\log_e x$ without its derivative, and show how calculus can be applied to find the rate of change of a quantity that is a function of time expressed in terms of the base e.

MA-E1.3 and MA-E1.4 dotpoints 2–4 and 6–7

Sections 11G–11J explain radian measure, use it in problems involving arcs and sections, and develop the theory of the general angle and the graphs of the trigonometric functions in radians, which is their true form from the point of view of calculus. The derivative, of $\sin x$, however, is not developed, although it is reasonably clear from the graph that $\dfrac{d}{dx}(\sin x) = \cos x$ looks like $y = \cos x$ and has gradient 1 at the origin. The final section is an investigation exercise on the six graphs and their symmetries. The four sections cover:

MA-T1.2 all dotpoints

Chapter 12: Probability

12A Probability and sample spaces
12B Sample space graphs and tree diagrams
12C Sets and Venn diagrams
12D Venn diagrams and the addition theorem
12E Multi-stage experiments and the product rule
12F Probability tree diagrams
12G Conditional probability
Chapter Review Exercise

This chapter follows the usual exposition of probability, with Section 12C on sets and Venn diagrams necessary because the topic will often not have been taught in earlier years.

MA-S1.1 all dotpoints

Uniform sample spaces — sample spaces consisting of equally likely possible outcomes — are in this course the basis of the theory and of the whole idea of probability. As mentioned at the end of the chapter, various probability formulae are valid in much more general situations. Probability in such general situations, however, is not easy to define, and the sort of axiomatisation required for rigorous proofs is beyond school mathematics.

Experimental probability is introduced briefly in terms of relative frequency, in preparation for the more extended account in Chapter 13.

The final section on conditional probability still begins with a definition that uses equally likely possible outcomes, and is based on the reduced sample space. From this, formulae with more general applicability can be derived.

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Chapter 13: Discrete probability distributions

13A The language of probability distributions
13B Expected value
13C Variance and standard deviation
13D Sampling
Chapter Review Exercise

Discrete probability distributions are new to NSW mathematics courses, and there are thus no accepted conventions about definitions, notations, and layouts, nor have any been specified in the syllabus. The choices in this chapter were made very carefully on the basis of logical coherence, and of being the most straightforward way to present the material in the senior classroom and perform the necessary calculations with the minimum of fuss. The extent and depth of treatment are also unclear at this stage.

MA-S1.2 all dotpoints

Chapter 14: Extension 1—Combinatorics

14A Factorial notation
14B Counting ordered selections
14C Ordered selections and grouping
14D Ordered selections with identical elements
14E Counting unordered selections
14F Using counting in probability
14G Arrangements in a circle
14H The pigeonhole principle
Chapter Review Exercise

ME-A1.1 all dotpoints

Chapter 14 presents standard methods of counting that are essential in all branches of mathematics, and in particular for the probability and the algebraic expansions in this course. The chapter mostly follows the standard material on the old syllabus, but begins with a more explicit treatment of the multiplication principle. It ends with the new topic of the pigeonhole principle, which needs care because it can become very difficult very quickly.

Chapter 15: Extension 1—The binomial expansion and Pascal's triangle

15A Binomial expansions and Pascal's triangle
15B Further binomial expansions
15C The binomial theorem
15D Identities in Pascal's triangle
15E Enrichment — Using the general term
Chapter Review Exercise

ME-A1 A1.2 all dotpoints

In the new syllabus, the coefficients of the binomial expansion are obtained by counting arguments, so that this chapter is now closely linked with the previous chapter on counting, and is no longer burdened by a difficult induction proof. At the time of writing, the depth of treatment required is unclear, so the final Section 15E is marked Enrichment.

Combinatorial methods of obtaining Pascal's triangle identities have been explicitly added in this syllabus. It therefore seemed reasonable to consign identities obtained by equating like terms to the Enrichment section of Exercise 15E, and restrict algebraic methods to substitution and differentiation, but at the time of writing the status of algebraic methods throughout this chapter is unclear.

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ME-S1 S1.1, dotpoint 4 & Section 15D: The identities obtained by differentiating the binomial expansion once, and then twice, and substituting, are required for the mean and variance of the binomial distribution in Year 12. It seemed best to obtain them here in the course of discussing binomial expansions and Pascal triangle identities, then use them later in Year 12.

More generally, the material in this and the previous probability chapters is required for the treatment of the binomial distribution in Year 12.

Chapter 16: Extension 1—Further rates

16A Related rates
16B Exponential growth and decay
16C Modified exponential growth and decay
Chapter Review Exercise

ME-C1 C1.1–C1.3

This chapter extends the Advanced treatment of rates with two further techniques.

ME-C1 C1.3 & Section 16A: When the one phenomenon involves two or more different rates, the chain rule may allow these different rates to be related.

ME-C1 C1.2 & Section 16B–16C: When a rate of change of a quantity Q is exponential, the rate of change is often related to the quantity Q by a differential equation. This section continues the discussion of exponential rates begun in Section 11F, but places more attention on the differential equations involved.

ME-C1 C1.1 & The rest of the books: Section 1.1 does not need separate attention because it is part of the continuing discussion of rates covered in various places throughout the two books. This is a convenient place to summarise these scattered accounts:

Section 8G was a pre-calculus treatment of rates and average rates.

Section 9J introduced rates as a derivative and contrasted them with average rates. Section 11F and Sections 16A–16B deal specifically with exponential rates.

In Year 12, APs and GPs model average rates, amongst many other things.

In Year 12, curve-sketching with calculus models rates, and is the appropriate time for some of the language in C1.1 to be discussed.

In Year 12, calculating the quantity from the rate will follow integration, particularly of exponential functions.

In Year 12, motion is a wonderful example of rates, but its language causes difficulties, so only scattered examples of motion occur before this chapter.

In Year 12, projectile motion extends the idea of motion as a rate.

In Year 12, finance involves average and instantaneous rates.

Throughout the books, trigonometric functions describe periodic rates.

Chapter 17: Extension 1—Further trigonometry

17A Restricting the domain
17B Defining the inverse trigonometric functions
17C Graphs involving inverse trigonometric functions
17D Trigonometric functions of compound angles
17E The double-angle formulae
17F The *t*-formulae
17G Products to sums
Chapter Review Exercise

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This chapter consists of important material needed for the calculus of the trigonometric functions in Year 12. It may appear rather disconnected in Year 11, and the future purposes of the material needs to be explained as much as is possible at this stage.

ME-T1 all dotpoints, ME-F1 F1.3 dotpoint 5

Sections 17A–17C introduce the inverse trigonometric functions in preparation for their appearance in Year 12 in the integrals of $\dfrac{1}{\sqrt{1 - x^2}}$ and $\dfrac{1}{1 + x^2}$. These functions require that the domain of the original functions be restricted so that the inverse relation is a function, and this theory is done in general in Section 17A (see ME-F1 F1.3 dotpoint 5).

Section 17D–17G move systematically through various trigonometric identities that will be important later, particularly for the calculus of the trigonometric functions.

Bill Pender, October 2018

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Overview

As part of the *CambridgeMATHS* series, this resource is part of a continuum from Year 7 through to 12. The four components of *Mathematics Extension 1 Year 11* — the print book, downloadable PDF textbook, online Interactive Textbook and Online Teaching Resource — contain a range of resources available to schools in a single package at a convenient low price. There are no extra subscriptions or per-student charges to pay.

Features of the print textbook

1 Refer to the *Rationale* for details of question categories in the exercises and syllabus coverage.
2 Chapters 1 & 2 in particular provide revision of required knowledge.
3 Each section begins at the top of the page to make them easy to find and access.
4 Plenty of numbered worked examples are provided, with video versions for most of them.
5 Important concepts are formatted in numbered boxes for easy reference.
6 Investigation exercises and suggestions for projects are included.
7 Proofs for important results are provided in certain chapters.
8 Chapter review exercises assess learning in the chapter.

Downloadable PDF textbook

9 The convenience of a downloadable PDF textbook has been retained for times when users cannot go online. PDF search and commenting tools are enabled.

Digital resources in the Interactive Textbook powered by the HOTmaths platform *(shown on the page opposite)*

The Interactive Textbook is an online HTML version of the print textbook powered by the HOTmaths platform, completely designed and reformatted for on-screen use, with easy navigation. It is included with the print book, or available as digital-only purchase. Its features include:

10 Video versions of the examples to encourage independent learning.
11 All exercises including chapter reviews have the option of being done interactively on line, using **workspaces** and **self-assessment tools**. Students rate their level of confidence in answering the question and can flag the ones that gave them difficulty. Working and answers, whether typed or handwritten, and drawings, can be saved and submitted to the teacher electronically. Answers displayed on screen if selected and worked solutions (if enabled by the teacher) open in pop-up windows.
12 Teachers can give feedback to students on their self-assessment and flagged answers.
13 The full suite of the HOTmaths learning management system and communication tools are included in the platform, with similar interfaces and controls.
14 Worked solutions are included and can be enabled or disabled in the student accounts by the teacher.
15 Interactive widgets and activities based on embedded Desmos windows demonstrate key concepts and enable students to visualise the mathematics.
16 Desmos scientific and graphics calculator windows are also included.

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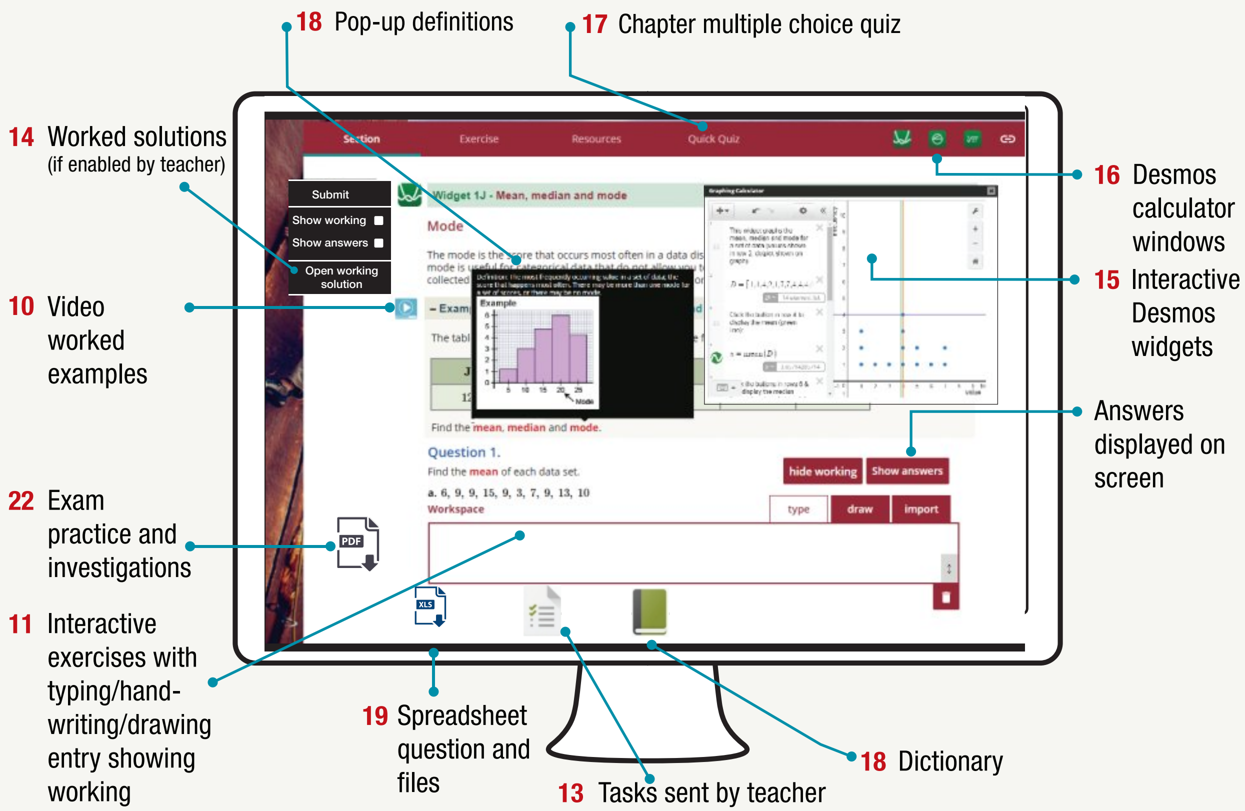
17 Chapter Quizzes of automatically marked multiple-choice questions are provided for students to test their progress.
18 Definitions pop up for key terms in the text, and are also provided in a dictionary.
19 Spreadsheet files are provided to support questions and examples based on the use of such technology.
20 Online guides are provided to spreadsheets and the Desmos graphing calculator, while links to scientific calculator guides on the internet are provided.
21 Links to selected HOTmaths lessons are provided for revision of prior knowledge.
22 Examination and assessment practice items are available

INTERACTIVE TEXTBOOK POWERED BY THE *HOTmaths* PLATFORM

Numbers refer to the descriptions on the opposite page. HOTmaths platform features are updated regularly. Content shown is from Mathematics Standard.



Online Teaching Suite powered by the HOTmaths platform *(shown on the next page)*

23 The Online Teaching Suite is automatically enabled with a teacher account and appears in the teacher's copy of the Interactive Textbook. All the assets and resources are in Teacher Resources panel for easy access.
24 Teacher support documents include editable teaching programs with a scope and sequence document and curriculum grid.
25 Chapter tests are provided as printable PDFs or editable Word documents.
26 Assessment practice items (unseen by students) are included in the teacher resources.
27 The HOTmaths test generator is included.
28 The HOTmaths learning management system with class and student reports and communication tools is included.

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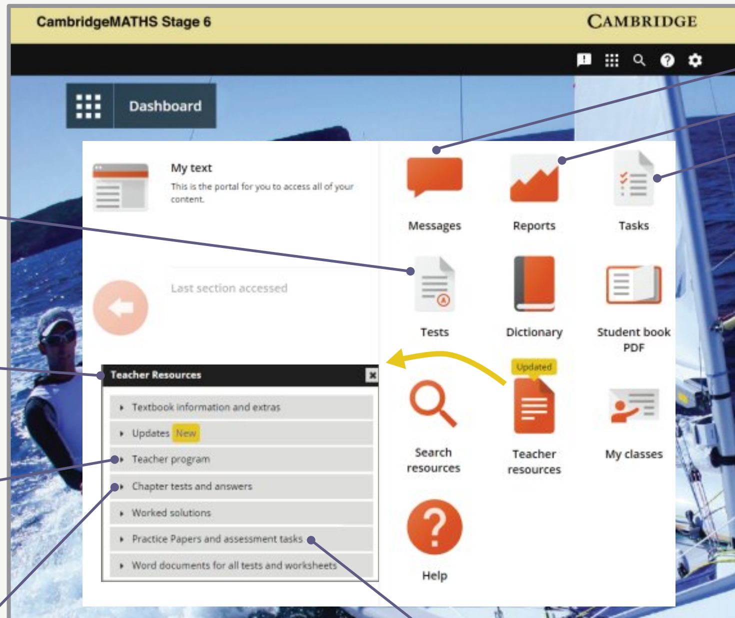

28 HOTmaths learning management system

27 HOTmaths test generator

23 Teacher resources

24 Teacher support documents

25 Chapter tests

26 Assessment practice

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The mathematician's patterns, like the painter's or the poet's,
must be beautiful. The ideas, like the colours or the words,
must fit together in a harmonious way. Beauty is the first test.

— The English mathematician G. H. Hardy (1877–1947)

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1 Methods in algebra

Fluency in algebra, particularly in factoring, is absolutely vital for everything in this course. This chapter is intended as a review of earlier algebraic techniques, and readers should do as much or as little of it as is necessary.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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1A Arithmetic with pronumerals

A *pronumeral* is a symbol that stands for a number. The pronumeral may stand for a known number or for an unknown number, or it may be a *variable* and stand for any one of a whole set of possible numbers. Pronumerals, being numbers, can take part in all the operations that are possible with numbers, such as addition, subtraction, multiplication, and division (except by zero).

Like and unlike terms

An *algebraic expression* consists of pronumerals, numbers and the operations of arithmetic. Here is an example:

$$x^2 + 2x + 3x^2 - 4x - 3 = 4x^2 - 2x - 3$$

This particular algebraic expression can be *simplified* by combining *like terms*.

- The two like terms x^2 and $3x^2$ can be combined to give $4x^2$.
- Another pair of like terms $2x$ and $-4x$ can be combined to give $-2x$.
- This yields three *unlike terms*, $4x^2$, $-2x$ and -3, which cannot be combined.

Example 1 — 1A

Simplify each expression by combining like terms.

a $7a + 15 - 2a - 20$ **b** $x^2 + 2x + 3x^2 - 4x - 3$

SOLUTION

a $7a + 15 - 2a - 20 = 5a - 5$

b $x^2 + 2x + 3x^2 - 4x - 3 = 4x^2 - 2x - 3$

Multiplying and dividing

To simplify a product such as $3y \times (-6y)$, or a quotient such as $10x^2y \div 5y$, work systematically through the signs, then the numerals, and then each pronumeral in turn.

Example 2 — 1A

Simplify these products and quotients.

a $3y \times (-6y)$ **b** $4ab \times 7bc$ **c** $10x^2y \div 5y$

SOLUTION

a $3y \times (-6y) = -18y^2$

b $4ab \times 7bc = 28ab^2c$

c $10x^2y \div 5y = 2x^2$

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Index laws

Here are the standard laws for dealing with indices. They will be covered in more detail in Chapter 8.

1 THE INDEX LAWS

- To multiply powers of the same base, add the indices: $a^x a^y = a^{x+y}$
- To divide powers of the same base, subtract the indices: $\dfrac{a^x}{a^y} = a^{x-y}$
- To raise a power to a power, multiply the indices: $(a^x)^n = a^{xn}$
- The power of a product is the product of the powers: $(ab)^x = a^x b^x$
- The power of a quotient is the quotient of the powers: $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

In expressions with several factors, work systematically through the signs, then the numerals, and then each pronumeral in turn.

Example 3 — 1A

Use the index laws above to simplify each expression.

a $3x^4 \times 4x^3$ **b** $(20x^7y^3) \div (4x^5y^3)$ **c** $(3a^4)^3$

d $(-x^2)^3 \times (2xy)^4$ **e** $\left(\dfrac{2x}{3y}\right)^4$

SOLUTION

a $3x^4 \times 4x^3 = 12x^7$ (multiplying powers of the same base)

b $(20x^7y^3) \div (4x^5y^3) = 5x^2$ (dividing powers of the same base)

c $(3a^4)^3 = 3^3 \times a^{4\times 3}$ (raising a power to a power)

$= 27a^{12}$

d $(-x^2)^3 \times (2xy)^4 = -x^6 \times 16x^4y^4$ (two powers of products)

$= -16x^{10}y^4$ (multiplying powers of the same base)

e $\left(\dfrac{2x}{3y}\right)^4 = \dfrac{16x^4}{81y^4}$ (a power of a quotient)

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Exercise 1A

FOUNDATION

1 Simplify:

a $3x + x$ b $3x - x$ c $-3x + x$ d $-3x - x$

2 Simplify:

a $-2a + 3a + 4a$ b $-2a - 3a + 4a$ c $-2a - 3a - 4a$ d $-2a + 3a - 4a$

3 Simplify:

a $-x + x$ b $2y - 3y$ c $-3a - 7a$ d $-8b + 5b$
e $4x - (-3x)$ f $-2ab - ba$ g $-3pq + 7pq$ h $-5abc - (-2abc)$

4 Simplify:

a $-3a \times 2$ b $-4a \times (-3a)$ c $a^2 \times a^3$ d $(a^2)^3$

5 Simplify:

a $-10a \div 5$ b $-24a \div (-8a)$ c $a^9 \div a^3$ d $7a^2 \div 7a$

6 Simplify:

a $t^2 + t^2$ b $t^2 - t^2$ c $t^2 \times t^2$ d $t^2 \div t^2$

7 Simplify:

a $-6x + 3x$ b $-6x - 3x$ c $-6x \times 3x$ d $-6x \div 3x$

8 If $a = -2$, find the value of:

a $3a + 2$ b $a^3 - a^2$
c $3a^2 - a + 4$ d $a^4 + 3a^3 + 2a^2 - a$

DEVELOPMENT

9 Simplify:

a $6x + 3 - 5x$ b $-2 + 2y - 1$
c $3a - 7 - a + 4$ d $3x - 2y + 5x + 6y$
e $-8t + 12 - 2t - 17$ f $2a^2 + 7a - 5a^2 - 3a$
g $9x^2 - 7x + 4 - 14x^2 - 5x - 7$ h $3a - 4b - 2c + 4a + 2b - c + 2a - b - 2c$

10 Simplify:

a $\dfrac{5x}{x}$ b $\dfrac{-7m^3}{-m}$ c $\dfrac{-12a^2b}{ab}$ d $\dfrac{-27p^6q^7r^2}{9p^3q^3r}$

11 Subtract:

a x from $3x$ b $-x$ from $3x$ c $2a$ from $-4a$ d $-b$ from $-5b$

12 Multiply:

a $5a$ by 2 b $6x$ by -3 c $-3a$ by a
d $-2a^2$ by $-3ab$ e $4x^2$ by $-2x^3$ f $-3p^2q$ by $2pq^3$

13 Divide:

a $-2x$ by x b $3x^3$ by x^2 c x^3y^2 by x^2y
d a^6x^3 by $-a^2x^3$ e $14a^5b^4$ by $-2a^4b$ f $-50a^2b^5c^8$ by $-10ab^3c^2$

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14 Simplify:

a $2a^2b^4 \times 3a^3b^2$ **b** $-6ab^5 \times 4a^3b^3$ **c** $(-3a^3)^2$ **d** $(-2a^4b)^3$

15 If $x = 2$ and $y = -3$, find the value of:

a $3x + 2y$ **b** $y^2 - 5x$ **c** $8x^2 - y^3$ **d** $x^2 - 3xy + 2y^2$

16 Simplify:

a $\dfrac{3a \times 3a \times 3a}{3a + 3a + 3a}$ **b** $\dfrac{3c \times 4c^2 \times 5c^3}{3c^2 + 4c^2 + 5c^2}$ **c** $\dfrac{ab^2 \times 2b^2c^3 \times 3c^3a^4}{a^3b^3 + 2a^3b^3 + 3a^3b^3}$

17 Simplify:

a $\dfrac{(-2x^2)^3}{-4x}$ **b** $\dfrac{(3xy^3)^3}{3x^2y^4}$ **c** $\dfrac{(-ab)^3 \times (-ab^2)^2}{-a^5b^3}$ **d** $\dfrac{(-2a^3b^2)^2 \times 16a^7b}{(2a^2b)^5}$

18 **a** What must be added to $4x^3 - 3x^2 + 2$ to give $3x^3 + 7x - 6$?

b Take the sum of $2a - 3b - 4c$ and $-4a + 7b - 5c$ from the sum of $4c - 2b$ and $5b - 2a - 2c$.

c If $X = 2b + 3c - 5d$ and $Y = 4d - 7c - b$, take $X - Y$ from $X + Y$.

d Divide the product of $(-3x^7y^5)^4$ and $(-2xy^6)^3$ by $(-6x^3y^8)^2$.

ENRICHMENT

19 For what values of x is it true that:

a $x \times x \le x + x$? **b** $x \times x \times x \le x + x + x$?

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1B Expanding brackets

Expanding brackets is routine in arithmetic. For example, calculate 7×61 as

$$7 \times (60 + 1) = 7 \times 60 + 7 \times 1,$$

which quickly gives the result $7 \times 61 = 420 + 7 = 427$. The algebraic version of this procedure can be written as:

2 EXPANDING BRACKETS IN ALGEBRA

$a(x + y) = ax + ay$ and $(x + y)a = xa + ya$

There may then be like terms to collect.

Example 4 1B

Expand and simplify each expression.

a $3x(4x - 7)$

b $5a(3 - b) - 3b(1 - 5a)$

SOLUTION

a $3x(4x - 7) = 12x^2 - 21x$

b $5a(3 - b) - 3b(1 - 5a) = 15a - 5ab - 3b + 15ab$
$= 15a + 10ab - 3b$ (collecting like terms)

Expanding the product of two bracketed terms

Expand one pair of brackets, then expand the other pair of brackets. Then collect any like terms.

Example 5 1B

Expand and simplify each expression.

a $(x + 3)(x - 5)$

b $(3 + x)(9 + 3x + x^2)$

SOLUTION

a $(x + 3)(x - 5)$
$= x(x - 5) + 3(x - 5)$
$= x^2 - 5x + 3x - 15$
$= x^2 - 2x - 15$

b $(3 + x)(9 + 3x + x^2)$
$= 3(9 + 3x + x^2) + x(9 + 3x + x^2)$
$= 27 + 9x + 3x^2 + 9x + 3x^2 + x^3$
$= 27 + 18x + 6x^2 + x^3$

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Special expansions

These three identities are important and must be memorised. Examples of these expansions occur constantly, and knowing the formulae greatly simplifies the working. They are proven in the exercises.

3 SPECIAL EXPANSIONS

- Square of a sum: $(A + B)^2 = A^2 + 2AB + B^2$
- Square of a difference: $(A - B)^2 = A^2 - 2AB + B^2$
- Difference of squares: $(A + B)(A - B) = A^2 - B^2$

Example 6 1B

Use the three special expansions above to simplify:

a $(x + 4)^2$ **b** $(s - 3t)^2$ **c** $(x + 3y)(x - 3y)$

SOLUTION

a $(x + 4)^2 = x^2 + 8x + 16$ (the square of a sum)

b $(s - 3t)^2 = s^2 - 6st + 9t^2$ (the square of a difference)

c $(x + 3y)(x - 3y) = x^2 - 9y^2$ (the difference of squares)

Exercise 1B

FOUNDATION

1 Expand:

a $3(x - 2)$ **b** $2(x - 3)$ **c** $-3(x - 2)$
d $-2(x - 3)$ **e** $-3(x + 2)$ **f** $-2(x + 3)$
g $-(x - 2)$ **h** $-(2 - x)$ **i** $-(x + 3)$

2 Expand:

a $3(x + y)$ **b** $-2(p - q)$ **c** $4(a + 2b)$
d $x(x - 7)$ **e** $-x(x - 3)$ **f** $-a(a + 4)$
g $5(a + 3b - 2c)$ **h** $-3(2x - 3y + 5z)$ **i** $xy(2x - 3y)$

3 Expand and simplify:

a $2(x + 1) - x$ **b** $3a + 5 + 4(a - 2)$
c $2 + 2(x - 3)$ **d** $-3(a + 2) + 10$
e $3 - (x + 1)$ **f** $b + c - (b - c)$
g $(2x - 3y) - (3x - 2y)$ **h** $3(x - 2) - 2(x - 5)$
i $4(2a - 3b) - 3(a + 2b)$ **j** $4(s - t) - 6(s + t)$
k $2x(x + 6y) - x(x - 5y)$ **l** $5(2a - 5b) - 6(-a - 4b)$

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4 Expand and simplify:

a $(x + 2)(x + 3)$ **b** $(y + 4)(y + 7)$ **c** $(t + 6)(t - 3)$
d $(x - 4)(x + 2)$ **e** $(t - 1)(t - 3)$ **f** $(2a + 3)(a + 5)$
g $(u - 4)(3u + 2)$ **h** $(4p + 5)(2p - 3)$ **i** $(2b - 7)(b - 3)$
j $(5a - 2)(3a + 1)$ **k** $(6 - c)(c - 3)$ **l** $(2d - 3)(4 + d)$

DEVELOPMENT

5 **a** By expanding $(A + B)(A + B)$, prove the special expansion $(A + B)^2 = A^2 + 2AB + B^2$.
b Similarly, prove the special expansions:
i $(A - B)^2 = A^2 - 2AB + B^2$ **ii** $(A - B)(A + B) = A^2 - B^2$

6 Use the special expansions to expand:

a $(x + y)^2$ **b** $(x - y)^2$ **c** $(x - y)(x + y)$ **d** $(a + 3)^2$
e $(b - 4)^2$ **f** $(c + 5)^2$ **g** $(d - 6)(d + 6)$ **h** $(7 + e)(7 - e)$
i $(8 + f)^2$ **j** $(9 - g)^2$ **k** $(h + 10)(h - 10)$ **l** $(i + 11)^2$
m $(2a + 1)^2$ **n** $(2b - 3)^2$ **o** $(3c + 2)^2$ **p** $(2d + 3e)^2$
q $(2f + 3g)(2f - 3g)$ **r** $(3h - 2i)(3h + 2i)$ **s** $(5j + 4)^2$ **t** $(4k - 5\ell)^2$
u $(4 + 5m)(4 - 5m)$ **v** $(5 - 3n)^2$ **w** $(7p + 4q)^2$ **x** $(8 - 3r)^2$

7 Expand and simplify:

a $\left(t + \frac{1}{t}\right)^2$ **b** $\left(t - \frac{1}{t}\right)^2$ **c** $\left(t + \frac{1}{t}\right)\left(t - \frac{1}{t}\right)$

8 By writing 102 as $(100 + 2)$, and adopting a similar approach for the other two parts, use the special expansions to find (without using a calculator) the value of:

a 102^2 **b** 999^2 **c** 203×197

9 Expand and simplify:

a $(a - b)(a^2 + ab + b^2)$ **b** $(x + 2)^2 - (x + 1)^2$
c $(a - 3)^2 - (a - 3)(a + 3)$ **d** $(2x + 3)(x - 1) - (x - 2)(x + 1)$
e $(x - 2)^3$ **f** $(p + q + r)^2 - 2(pq + qr + rp)$

10 Expand and simplify:

a $(x - 2)^3$ **b** $(x + y + z)^2 - 2(xy + yz + zx)$
c $(x + y - z)(x - y + z)$ **d** $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

11 **a** Subtract $a(b + c - a)$ from the sum of $b(c + a - b)$ and $c(a + b - c)$.
b Subtract the sum of $2x^2 - 3(x - 1)$ and $2x + 3(x^2 - 2)$ from the sum of $5x^2 - (x - 2)$ and $x^2 - 2(x + 1)$.
c If $X = x - a$ and $Y = 2x + a$, find the product of $Y - X$ and $X + 3Y$ in terms of x and a.

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ENRICHMENT

12 Suppose that $x + \dfrac{1}{x} = 3$. Find the value of $x^2 + \dfrac{1}{x^2}$ without solving for x.

13 Prove these identities.

a $(a + b + c)(ab + bc + ca) - abc = (a + b)(b + c)(c + a)$

b $(ax + by)^2 + (ay - bx)^2 + c^2(x^2 + y^2) = (x^2 + y^2)(a^2 + b^2 + c^2)$

14 If $(a + b)^2 + (b + c)^2 + (c + d)^2 = 4(ab + bc + cd)$, prove that $a = b = c = d$.

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1C Factoring

Factoring is the reverse process of expanding brackets, and is needed routinely throughout the course. There are four basic methods, but in every situation, common factors should always be taken out first.

4 THE FOUR BASIC METHODS OF FACTORING

- **Highest common factor:** *Always try this first.*
- **Difference of squares:** This involves two terms.
- **Quadratics:** This involves three terms.
- **Grouping:** This involves four or more terms.

Factoring should continue until each factor is *irreducible*, meaning that it cannot be factored further.

Factoring by taking out the highest common factor

Always look first for any common factors of all the terms, and then take out the highest common factor (usually abbreviates to HCF).

Example 7 1C

Factor each expression by taking out the highest common factor.

a $4x^3 + 3x^2$ **b** $9a^2b^3 - 15b^3$

SOLUTION

a The HCF of $4x^3$ and $3x^2$ is x^2,
so $4x^3 + 3x^2 = x^2(4x + 3)$.

b The HCF of $9a^2b^3$ and $15b^3$ is $3b^3$,
so $9a^2b^3 - 15b^3 = 3b^3(3a^2 - 5)$.

Factoring by difference of squares

The expression must have two terms, both of which are squares. Sometimes a common factor must be taken out first.

Example 8 1C

Use the difference of squares to factor each expression.

a $a^2 - 36$ **b** $80x^2 - 5y^2$

SOLUTION

a $a^2 - 36 = (a + 6)(a - 6)$

b $80x^2 - 5y^2 = 5(16x^2 - y^2)$ (take out the highest common factor)
$= 5(4x - y)(4x + y)$ (use the difference of squares)

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Factoring monic quadratics

A quadratic is called *monic* if the coefficient of x^2 is 1. Suppose that we want to factor the monic quadratic expression $x^2 - 13x + 36$. Look for two numbers:

- whose sum is -13 (the coefficient of x), and
- whose product is $+36$ (the constant term).

Example 9 — 1C

Factor these monic quadratics.

a $x^2 - 13x + 36$ **b** $a^2 + 12a - 28$

SOLUTION

a The numbers with sum -13 and product $+36$ are -9 and -4,
so $x^2 - 13x + 36 = (x - 9)(x - 4)$.

b The numbers with sum $+12$ and product -28 are $+14$ and -2,
so $a^2 + 12a - 28 = (a + 14)(a - 2)$.

Factoring non-monic quadratics

In a *non-monic* quadratic such as $2x^2 + 11x + 12$, where the coefficient of x^2 is not 1, look for two numbers:

- whose sum is 11 (the coefficient of x), and
- whose product is $12 \times 2 = 24$ $\left(\text{the constant times the coefficient of } x^2\right)$.

Then split the middle term into two terms.

Example 10 — 1C

Factor these non-monic quadratics.

a $2x^2 + 11x + 12$ **b** $6s^2 - 11s - 10$

SOLUTION

a The numbers with sum 11 and product $12 \times 2 = 24$ are 8 and 3,

so $2x^2 + 11x - 12 = \left(2x^2 + 8x\right) + (3x + 12)$ (split $11x$ into $8x + 3x$)
$= 2x(x + 4) + 3(x + 4)$ (take out the HCF of each group)
$= (2x + 3)(x + 4)$. ($x + 4$ is a common factor)

b The numbers with sum -11 and product $-10 \times 6 = -60$ are -15 and 4,

so $6s^2 - 11s - 10 = \left(6s^2 - 15s\right) + (4s - 10)$ (split $-11s$ into $-15s + 4s$)
$= 3s(2s - 5) + 2(2s - 5)$ (take out the HCF of each group)
$= (3s + 2)(2s - 5)$. ($2s - 5$ is a common factor)

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Factoring by grouping

When there are four or more terms, it is sometimes possible to factor the expression by *grouping*.

- Split the expression into groups.
- Then factor each group in turn.
- Then factor the whole expression by taking out a common factor or by some other method.

Example 11 — 1C

Factor each expression by grouping.

a $12xy - 9x - 16y + 12$ **b** $s^2 - t^2 + s - t$

SOLUTION

a $12xy - 9x - 16y + 12 = 3x(4y - 3) - 4(4y - 3)$ (take out the HCF of each pair)
$= (3x - 4)(4y - 3)$ ($4y - 3$ is a common factor)

b $s^2 - t^2 + s - t = (s + t)(s - t) + (s - t)$ (factor $s^2 - t^2$ using difference of squares)
$= (s - t)(s + t + 1)$ ($s - t$ is a common factor)

Exercise 1C — FOUNDATION

1 Factor by taking out any common factors:

a $2x + 8$ **b** $6a - 15$ **c** $ax - ay$ **d** $20ab - 15ac$
e $x^2 + 3x$ **f** $p^2 + 2pq$ **g** $3a^2 - 6ab$ **h** $12x^2 + 18x$
i $20cd - 32c$ **j** $a^2b + b^2a$ **k** $6a^2 + 2a^3$ **l** $7x^3y - 14x^2y^2$

2 Factor by grouping in pairs:

a $mp + mq + np + nq$ **b** $ax - ay + bx - by$ **c** $ax + 3a + 2x + 6$
d $a^2 + ab + ac + bc$ **e** $z^3 - z^2 + z - 1$ **f** $ac + bc - ad - bd$
g $pu - qu - pv + qv$ **h** $x^2 - 3x - xy + 3y$ **i** $5p - 5q - px + qx$
j $2ax - bx - 2ay + by$ **k** $ab + ac - b - c$ **l** $x^3 + 4x^2 - 3x - 12$
m $a^3 - 3a^2 - 2a + 6$ **n** $2t^3 + 5t^2 - 10t - 25$ **o** $2x^3 - 6x^2 - ax + 3a$

3 Factor using the difference of squares:

a $a^2 - 1$ **b** $b^2 - 4$ **c** $c^2 - 9$ **d** $d^2 - 100$
e $25 - y^2$ **f** $1 - n^2$ **g** $49 - x^2$ **h** $144 - p^2$
i $4c^2 - 9$ **j** $9u^2 - 1$ **k** $25x^2 - 16$ **l** $1 - 49k^2$
m $x^2 - 4y^2$ **n** $9a^2 - b^2$ **o** $25m^2 - 36n^2$ **p** $81a^2b^2 - 64$

4 Factor each quadratic expression. They are all monic quadratics.

a $a^2 + 3a + 2$ **b** $k^2 + 5k + 6$ **c** $m^2 + 7m + 6$ **d** $x^2 + 8x + 15$
e $y^2 + 9y + 20$ **f** $t^2 + 12t + 20$ **g** $x^2 - 4x + 3$ **h** $c^2 - 7c + 10$
i $a^2 - 7a + 12$ **j** $b^2 - 8b + 12$ **k** $t^2 + t - 2$ **l** $u^2 - u - 2$
m $w^2 - 2w - 8$ **n** $a^2 + 2a - 8$ **o** $p^2 - 2p - 15$ **p** $y^2 + 3y - 28$
q $c^2 - 12c + 27$ **r** $u^2 - 13u + 42$ **s** $x^2 - x - 90$ **t** $x^2 + 3x - 40$
u $t^2 - 4t - 32$ **v** $p^2 + 9p - 36$ **w** $u^2 - 16u - 80$ **x** $t^2 + 23t - 50$

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5 Factor each quadratic expression. They are all non-monic quadratics.

a $3x^2 + 4x + 1$ **b** $2x^2 + 5x + 2$ **c** $3x^2 + 16x + 5$ **d** $3x^2 + 8x + 4$
e $2x^2 - 3x + 1$ **f** $5x^2 - 13x + 6$ **g** $5x^2 - 11x + 6$ **h** $6x^2 - 11x + 3$
i $2x^2 - x - 3$ **j** $2x^2 + 3x - 5$ **k** $3x^2 + 2x - 5$ **l** $3x^2 + 14x - 5$
m $2x^2 - 7x - 15$ **n** $2x^2 + x - 15$ **o** $6x^2 + 17x - 3$ **p** $6x^2 - 7x - 3$
q $6x^2 + 5x - 6$ **r** $5x^2 + 23x + 12$ **s** $5x^2 + 4x - 12$ **t** $5x^2 - 19x + 12$
u $5x^2 - 11x - 12$ **v** $5x^2 + 28x - 12$ **w** $9x^2 - 6x - 8$ **x** $3x^2 + 13x - 30$

DEVELOPMENT

6 Use the techniques of the previous questions to factor each expression.

a $a^2 - 25$ **b** $b^2 - 25b$ **c** $c^2 - 25c + 100$
d $2d^2 + 25d + 50$ **e** $e^3 + 5e^2 + 5e + 25$ **f** $16 - f^2$
g $16g^2 - g^3$ **h** $h^2 + 16h + 64$ **i** $i^2 - 16i - 36$
j $5j^2 + 16j - 16$ **k** $4k^2 - 16k - 9$ **l** $2k^3 - 16k^2 - 3k + 24$
m $2a^2 + ab - 4a - 2b$ **n** $6m^3n^4 + 9m^2n^5$ **o** $49p^2 - 121q^2$
p $t^2 - 14t + 40$ **q** $3t^2 + 2t - 40$ **r** $5t^2 + 54t + 40$
s $5t^2 + 33t + 40$ **t** $5t^3 + 10t^2 + 15t$ **u** $u^2 + 15u - 54$
v $3x^3 - 2x^2y - 15x + 10y$ **w** $(p + q)^2 - r^2$ **x** $4a^2 - 12a + 9$

7 Factor each expression. (Take out any common factors first.)

a $3a^2 - 12$ **b** $x^4 - y^4$ **c** $x^3 - x$
d $5x^2 - 5x - 30$ **e** $25y - y^3$ **f** $16 - a^4$
g $4x^2 + 14x - 30$ **h** $a^4 + a^3 + a^2 + a$ **i** $c^3 + 9c^2 - c - 9$
j $x^3 - 8x^2 + 7x$ **k** $x^4 - 3x^2 - 4$ **l** $ax^2 - a - 2x^2 + 2$

8 Factor as fully as possible:

a $4p^2 - (q + r)^2$ **b** $a^2 - b^2 - a + b$ **c** $a^3 - 10a^2b + 24ab^2$
d $6x^4 - x^3 - 2x^2$ **e** $4x^4 - 37x^2 + 9$ **f** $40 - 18x - 40x^2$
g $4x^3 - 12x^2 - x + 3$ **h** $x^2 + 2ax + a^2 - b^2$ **i** $x^4 - x^2 - 2x - 1$

ENRICHMENT

9 Factor fully:

a $a^2 + b(b + 1)a + b^3$ **b** $\left(x^2 + xy\right)^2 - \left(xy + y^2\right)^2$
c $\left(a^2 - b^2\right)^2 - (a - b)^4$ **d** $4x^4 - 2x^3y - 3xy^3 - 9y^4$
e $\left(a^2 - b^2 - c^2\right)^2 - 4b^2c^2$ **f** $(ax + by)^2 + (ay - bx)^2 + c^2\left(x^2 + y^2\right)$
g $a^4 + a^2b^2 + b^4$ **h** $a^4 + 4b^4$

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1D Algebraic fractions

An *algebraic fraction* is a fraction that contains pronumerals. Algebraic fractions are manipulated in the same way as arithmetic fractions, and factoring may play a major role.

Adding and subtracting algebraic fractions

A common denominator is needed. Finding the *lowest common denominator* may involve factoring each denominator.

5 ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

- First factor each denominator.
- Then work with the *lowest common denominator*.

Example 12 1D

Use a common denominator to simplify each algebraic fraction.

a $\dfrac{x}{2} - \dfrac{x}{3}$ **b** $\dfrac{5x}{6} + \dfrac{11x}{4}$ **c** $\dfrac{2}{3x} - \dfrac{3}{5x}$ **d** $\dfrac{1}{x-4} - \dfrac{1}{x}$

SOLUTION

a $\dfrac{x}{2} - \dfrac{x}{3} = \dfrac{3x}{6} - \dfrac{2x}{6}$
$= \dfrac{x}{6}$

b $\dfrac{5x}{6} + \dfrac{11x}{4} = \dfrac{10x}{12} + \dfrac{33x}{12}$
$= \dfrac{43x}{12}$

c $\dfrac{2}{3x} - \dfrac{3}{5x} = \dfrac{10}{15x} - \dfrac{9}{15x}$
$= \dfrac{1}{15x}$

d $\dfrac{1}{x-4} - \dfrac{1}{x} = \dfrac{x-(x-4)}{x(x-4)}$
$= \dfrac{4}{x(x-4)}$

Example 13 1D

Factor the denominators of $\dfrac{2+x}{x^2-x} - \dfrac{5}{x-1}$, then simplify the expression.

SOLUTION

$$\frac{2+x}{x^2-x} - \frac{5}{x-1} = \frac{2+x}{x(x-1)} - \frac{5}{x-1}$$
$$= \frac{2+x-5x}{x(x-1)}$$
$$= \frac{2-4x}{x(x-1)}$$

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Cancelling algebraic fractions

The key step here is to factor the numerator and denominator completely before cancelling factors.

6 CANCELLING ALGEBRAIC FRACTIONS

- First factor the numerator and denominator.
- Then cancel all common factors.

Example 14 1D

Simplify each algebraic fraction.

a $\dfrac{6x + 8}{6}$

b $\dfrac{x^2 - x}{x^2 - 1}$

SOLUTION

a
$$\frac{6x + 8}{6} = \frac{2(3x + 4)}{6} = \frac{3x + 4}{3}$$

(which could be written as $x + \frac{4}{3}$)

b
$$\frac{x^2 - x}{x^2 - 1} = \frac{x(x - 1)}{(x + 1)(x - 1)} = \frac{x}{x + 1}$$

Multiplying and dividing with algebraic fractions

These processes are done exactly as for arithmetic fractions.

7 MULTIPLYING AND DIVIDING WITH ALGEBRAIC FRACTIONS

Multiplying algebraic fractions:

- First factor all numerators and denominators completely.
- Then cancel common factors.

Dividing by an algebraic fraction:

- To divide by an algebraic fraction, multiply by its reciprocal. For example:
  $\dfrac{3}{x} \div \dfrac{4}{y} = \dfrac{3}{x} \times \dfrac{y}{4}$.
- The reciprocal of the fraction $\dfrac{4}{y}$ is $\dfrac{y}{4}$.

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Example 15 1D

Simplify these products and quotients of algebraic fractions.

a $\dfrac{2a}{a^2 - 9} \times \dfrac{a - 3}{5a}$

b $\dfrac{12x}{x + 1} \div \dfrac{6x}{x^2 + 2x + 1}$

SOLUTION

a $\dfrac{2a}{a^2 - 9} \times \dfrac{a - 3}{5a} = \dfrac{2a}{(a - 3)(a + 3)} \times \dfrac{a - 3}{5a}$ (factor $a^2 - 9$)

$= \dfrac{2}{5(a + 3)}$ (cancel $a - 3$ and a)

b $\dfrac{12x}{x + 1} \div \dfrac{6x}{x^2 + 2x + 1} = \dfrac{12x}{x + 1} \times \dfrac{x^2 + 2x + 1}{6x}$ (multiply by the reciprocal)

$= \dfrac{12x}{x + 1} \times \dfrac{(x + 1)^2}{6x}$ (factor $x^2 + 2x + 1$)

$= 2(x + 1)$ (cancel $x + 1$ and $6x$)

Simplifying compound fractions

A *compound fraction* is a fraction in which either the numerator or the denominator is itself a fraction.

8 SIMPLIFYING COMPOUND FRACTIONS

- Find the lowest common multiple of the denominators on the top and bottom.
- Multiply top and bottom by this lowest common multiple.

This will clear all the fractions from the top and bottom together.

Example 16 1D

Simplify each compound fraction.

a $\dfrac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{6}}$

b $\dfrac{\frac{1}{t} + 1}{\frac{1}{t} - 1}$

SOLUTION

a $\dfrac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{6}} = \dfrac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{6}} \times \dfrac{12}{12}$

$= \dfrac{6 - 4}{3 + 2}$

$= \dfrac{2}{5}$

b $\dfrac{\frac{1}{t} + 1}{\frac{1}{t} - 1} = \dfrac{\frac{1}{t} + 1}{\frac{1}{t} - 1} \times \dfrac{t}{t}$

$= \dfrac{1 + t}{1 - t}$

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Exercise 1D

FOUNDATION

1 Simplify:

a $\frac{x}{x}$ **b** $\frac{2x}{x}$ **c** $\frac{x}{2x}$ **d** $\frac{a}{a^2}$ **e** $\frac{3x^2}{9xy}$ **f** $\frac{12ab}{4a^2b}$

2 Simplify:

a $\frac{x}{3} \times \frac{3}{x}$ **b** $\frac{a}{4} \div \frac{a}{2}$ **c** $x^2 \times \frac{3}{x}$ **d** $\frac{1}{2b} \times b^2$

e $\frac{3x}{4} \times \frac{2}{x^2}$ **f** $\frac{5}{a} \div 10$ **g** $\frac{2ab}{3} \times \frac{6}{ab^2}$ **h** $\frac{8ab}{5} \div \frac{4ab}{15}$

3 Write as a single fraction:

a $x + \frac{x}{2}$ **b** $\frac{y}{4} + \frac{y}{2}$ **c** $\frac{m}{3} - \frac{m}{9}$ **d** $\frac{n}{2} + \frac{n}{5}$

e $\frac{x}{8} - \frac{y}{12}$ **f** $\frac{2a}{3} + \frac{3a}{2}$ **g** $\frac{7b}{10} - \frac{19b}{30}$ **h** $\frac{xy}{30} - \frac{xy}{12}$

4 Write as a single fraction:

a $\frac{1}{a} + \frac{1}{a}$ **b** $\frac{1}{x} - \frac{2}{x}$ **c** $\frac{1}{a} + \frac{1}{2a}$ **d** $\frac{1}{2x} - \frac{1}{3x}$ **e** $\frac{3}{4a} + \frac{4}{3a}$ **f** $\frac{5}{6x} - \frac{1}{3x}$

5 Simplify:

a $\frac{x+1}{2} + \frac{x+2}{3}$ **b** $\frac{2x-1}{5} + \frac{2x+3}{4}$ **c** $\frac{x+3}{6} + \frac{x-3}{12}$

d $\frac{x+2}{2} - \frac{x+3}{3}$ **e** $\frac{2x+1}{4} - \frac{2x-3}{5}$ **f** $\frac{2x-1}{3} - \frac{2x+1}{6}$

6 Factor where possible and then simplify:

a $\frac{2p+2q}{p+q}$ **b** $\frac{3t-12}{2t-8}$ **c** $\frac{x^2+3x}{3x+9}$

d $\frac{a}{ax+ay}$ **e** $\frac{3a^2-6ab}{2a^2b-4ab^2}$ **f** $\frac{x^2+2x}{x^2-4}$

g $\frac{a^2-9}{a^2+a-12}$ **h** $\frac{x^2+2x+1}{x^2-1}$ **i** $\frac{x^2+10x+25}{x^2+9x+20}$

DEVELOPMENT

7 Simplify:

a $\frac{1}{x} + \frac{1}{x+1}$ **b** $\frac{1}{x} - \frac{1}{x+1}$ **c** $\frac{1}{x+1} + \frac{1}{x-1}$

d $\frac{2}{x-3} + \frac{3}{x-2}$ **e** $\frac{3}{x+1} - \frac{2}{x-1}$ **f** $\frac{2}{x-2} - \frac{2}{x+3}$

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8 Simplify:

a $\dfrac{3x+3}{2x} \times \dfrac{x^2}{x^2-1}$

b $\dfrac{a^2+a-2}{a+2} \times \dfrac{a^2-3a}{a^2-4a+3}$

c $\dfrac{c^2+5c+6}{c^2-16} \div \dfrac{c+3}{c-4}$

d $\dfrac{x^2-x-20}{x^2-25} \times \dfrac{x^2-x-2}{x^2+2x-8} \div \dfrac{x+1}{x^2+5x}$

e $\dfrac{ax+bx-2a-2b}{3x^2-5x-2} \times \dfrac{9x^2-1}{a^2+2ab+b^2}$

f $\dfrac{2x^2+x-15}{x^2+3x-28} \div \dfrac{x^2+6x+9}{x^2-4x} \div \dfrac{6x^2-15x}{x^2-49}$

9 Simplify:

a $\dfrac{b-a}{a-b}$

b $\dfrac{v^2-u^2}{u-v}$

c $\dfrac{x^2-5x+6}{2-x}$

d $\dfrac{1}{a-b} - \dfrac{1}{b-a}$

e $\dfrac{m}{m-n} + \dfrac{n}{n-m}$

f $\dfrac{x-y}{y^2+xy-2x^2}$

10 Simplify:

a $\dfrac{1}{x^2+x} + \dfrac{1}{x^2-x}$

b $\dfrac{1}{x^2-4} + \dfrac{1}{x^2-4x+4}$

c $\dfrac{1}{x-y} + \dfrac{2x-y}{x^2-y^2}$

d $\dfrac{3}{x^2+2x-8} - \dfrac{2}{x^2+x-6}$

e $\dfrac{x}{a^2-b^2} - \dfrac{x}{a^2+ab}$

f $\dfrac{1}{x^2-4x+3} + \dfrac{1}{x^2-5x+6} - \dfrac{1}{x^2-3x+2}$

11 Study Example 16 on compound fractions and then simplify:

a $\dfrac{1-\frac{1}{2}}{1+\frac{1}{2}}$

b $\dfrac{2+\frac{1}{3}}{5-\frac{2}{3}}$

c $\dfrac{\frac{1}{2}-\frac{1}{5}}{1+\frac{1}{10}}$

d $\dfrac{\frac{17}{20}-\frac{3}{4}}{\frac{4}{5}-\frac{3}{10}}$

e $\dfrac{\frac{1}{x}}{1+\frac{2}{x}}$

f $\dfrac{t-\frac{1}{t}}{t+\frac{1}{t}}$

g $\dfrac{1}{\frac{1}{b}+\frac{1}{a}}$

h $\dfrac{\frac{x}{y}+\frac{y}{x}}{\frac{x}{y}-\frac{y}{x}}$

i $\dfrac{1-\frac{1}{x+1}}{\frac{1}{x}+\frac{1}{x+1}}$

j $\dfrac{\frac{3}{x+2}-\frac{2}{x+1}}{\frac{5}{x+2}-\frac{4}{x+1}}$

12 If $x = \dfrac{1}{t}$ and $y = \dfrac{1}{1-x}$ and $z = \dfrac{y}{y-1}$, show that $z = t$.

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ENRICHMENT

13 Simplify:

a $\dfrac{(a-b)^2-c^2}{ab-b^2-bc} \times \dfrac{c}{a^2+ab-ac} \div \dfrac{ac-bc+c^2}{a^2-(b-c)^2}$

b $\dfrac{8x^2+14x+3}{8x^2-10x+3} \times \dfrac{12x^2-6x}{4x^2+5x+1} \div \dfrac{18x^2-6x}{4x^2+x-3}$

c $\dfrac{4y}{x^2+2xy} - \dfrac{3x}{xy+2y^2} + \dfrac{3x-2y}{xy}$

d $\dfrac{1}{x-1} + \dfrac{2}{x+1} - \dfrac{3x-2}{x^2-1} - \dfrac{1}{x^2+2x+1}$

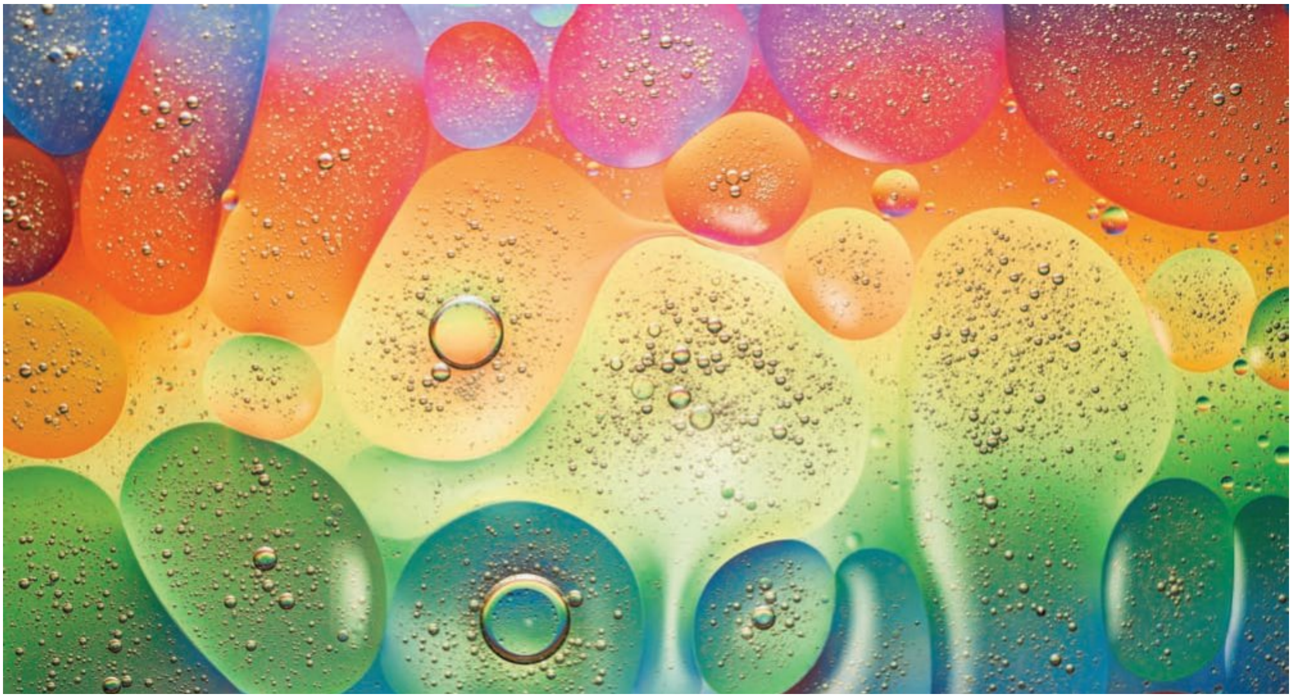
14 Simplify as fully as possible:

a $\dfrac{1}{(a-b)(a-c)} + \dfrac{1}{(b-c)(b-a)} + \dfrac{1}{(c-a)(c-b)}$

b $\left(1+\dfrac{45}{x-8}-\dfrac{26}{x-6}\right)\left(3-\dfrac{65}{x+7}+\dfrac{8}{x-2}\right)$

c $\left(2-\dfrac{3n}{m}+\dfrac{9n^2-2m^2}{m^2+2mn}\right) \div \left(\dfrac{1}{m}-\dfrac{1}{m-2n-\dfrac{4n^2}{m+n}}\right)$

d $\dfrac{1}{x+\dfrac{1}{x+2}} \times \dfrac{1}{x+\dfrac{1}{x-2}} \div \dfrac{x-\dfrac{4}{x}}{x^2-2+\dfrac{1}{x^2}}$



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1E Solving linear equations

The first principle in solving any equation is to simplify it by doing the same things to both sides.

9 SOLVING EQUATIONS — A GENERAL PRINCIPLE

- Any number can be added to or subtracted from both sides.
- Both sides can be multiplied or divided by any non-zero number.

An *linear equation* can be solved completely this way. Some equation involving algebraic fractions can often be reduced to a linear equation and solved using these steps.

Example 17 1E

Solve each equation.

a $6x + 5 = 4x - 9$

b $\dfrac{4 - 7x}{4x - 7} = 1$

SOLUTION

a
$$\begin{aligned} & & 6x + 5 &= 4x - 9 \\ &\boxed{-4x} & 2x + 5 &= -9 \\ &\boxed{-5} & 2x &= -14 \\ &\boxed{\div 2} & x &= -7 \end{aligned}$$

b
$$\begin{aligned} & & \frac{4 - 7x}{4x - 7} &= 1 \\ &\boxed{\times (4x - 7)} & 4 - 7x &= 4x - 7 \\ &\boxed{+ 7x} & 4 &= 11x - 7 \\ &\boxed{+ 7} & 11 &= 11x \\ &\boxed{\div 11} & x &= 1 \end{aligned}$$

Changing the subject of a formula

Similar sequences of operations often allow the subject of a formula to be changed from one pronumeral to another.

Example 18 1E

Change the subject of each formula to x.

a $y = 4x - 3$

b $y = \dfrac{x + 1}{x + 2}$

SOLUTION

a
$$\begin{aligned} & & y &= 4x - 3 \\ &\boxed{+ 3} & y + 3 &= 4x \\ &\boxed{\div 4} & \frac{y + 3}{4} &= x \\ & & x &= \frac{y + 3}{4} \end{aligned}$$

b
$$\begin{aligned} & & y &= \frac{x + 1}{x + 2} \\ &\boxed{\times (x + 2)} & xy + 2y &= x + 1 \\ &\text{Rearranging,} & xy - x &= 1 - 2y \\ &\text{Factoring,} & x(y - 1) &= 1 - 2y \\ &\boxed{\div (y - 1)} & x &= \frac{1 - 2y}{y - 1} \end{aligned}$$

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Exercise 1E

FOUNDATION

1 Solve:

a $2x + 1 = 7$ **b** $5p - 2 = -2$ **c** $\frac{a}{2} - 1 = 3$ **d** $3 - w = 4$

e $3x - 5 = 22$ **f** $4x + 7 = -13$ **g** $1 - \frac{x}{2} = 9$ **h** $11 - 6x = 23$

2 Solve:

a $3n - 1 = 2n + 3$ **b** $4b + 3 = 2b + 1$

c $5x - 2 = 2x + 10$ **d** $5 - x = 27 + x$

e $16 + 9a = 10 - 3a$ **f** $13y - 21 = 20y - 35$

g $13 - 12x = 6 - 3x$ **h** $3x + 21 = 18 - 2x$

3 Solve:

a $\frac{a}{12} = \frac{2}{3}$ **b** $\frac{y}{20} = \frac{4}{5}$ **c** $\frac{1}{x} = 3$ **d** $\frac{2}{a} = 5$

e $1 = \frac{3}{2y}$ **f** $\frac{2x + 1}{5} = -3$ **g** $\frac{3a - 5}{4} = 4$ **h** $\frac{7 - 4x}{6} = \frac{3}{2}$

i $\frac{20 + a}{a} = -3$ **j** $\frac{9 - 2t}{t} = 13$ **k** $\frac{3}{x - 1} = -1$ **l** $\frac{3x}{2x - 1} = \frac{5}{3}$

4 Solve:

a $y + \frac{y}{2} = 1$ **b** $\frac{x}{3} - \frac{x}{5} = 2$

c $\frac{a}{10} - \frac{a}{6} = 1$ **d** $\frac{x}{6} + \frac{2}{3} = \frac{x}{2} - \frac{5}{6}$

e $\frac{x}{3} - 2 = \frac{x}{2} - 3$ **f** $\frac{1}{x} - 3 = \frac{1}{2x}$

g $\frac{1}{2x} - \frac{2}{3} = 1 - \frac{1}{3x}$ **h** $\frac{x - 2}{3} = \frac{x + 4}{4}$

i $\frac{3}{x - 2} = \frac{4}{2x + 5}$ **j** $\frac{x + 1}{x + 2} = \frac{x - 3}{x + 1}$

DEVELOPMENT

5 a If $v = u + at$, find a when $t = 4$, $v = 20$ and $u = 8$.

b Given that $v^2 = u^2 + 2as$, find the value of s when $u = 6$, $v = 10$ and $a = 2$.

c Suppose that $\frac{1}{u} + \frac{1}{v} = \frac{1}{t}$. Find v, given that $u = -1$ and $t = 2$.

d If $S = -15$, $n = 10$ and $a = -24$, find ℓ, given that $S = \frac{n}{2}(a + \ell)$.

e The formula $F = \frac{9}{5}C + 32$ relates temperatures in degrees Fahrenheit and Celsius. Find the value of C that corresponds to $F = 95$.

f Suppose that c and d are related by the formula $\frac{1}{c + 1} = \frac{5}{d - 1}$. Find c when $d = 4$.

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6 Rearrange each formula so that the pronumeral written in square brackets is the subject.

a $a = bc - d$ $[b]$

b $t = a + (n - 1)d$ $[n]$

c $\dfrac{p}{q + r} = t$ $[r]$

d $u = 1 + \dfrac{3}{v}$ $[v]$

7 Expand the brackets on both sides of each equation, then solve it.

a $(x - 3)(x + 6) = (x - 4)(x - 5)$

b $(1 + 2x)(4 + 3x) = (2 - x)(5 - 6x)$

c $(x + 3)^2 = (x - 1)^2$

d $(2x - 5)(2x + 5) = (2x - 3)^2$

8 Solve:

a $\dfrac{a + 5}{2} - \dfrac{a - 1}{3} = 1$

b $\dfrac{3}{4} - \dfrac{x + 1}{12} = \dfrac{2}{3} - \dfrac{x - 1}{6}$

c $\dfrac{3}{4}(x - 1) - \dfrac{1}{2}(3x + 2) = 0$

d $\dfrac{4x + 1}{6} - \dfrac{2x - 1}{15} = \dfrac{3x - 5}{5} - \dfrac{6x + 1}{10}$

9 Solve each problem by forming and then solving a linear equation.

a Five more than twice a certain number is one more than the number itself. What is the number?

b I have \$175 in my wallet, consisting of \$10 and \$5 notes. If I have twice as many \$10 notes as \$5 notes, how many \$5 notes do I have?

c My father is 24 years older than me, and 12 years ago he was double my age. How old am I now?

d The fuel tank in my new car was 40% full. I added 28 litres and then found that it was 75% full. How much fuel does the tank hold?

e A certain tank has an inlet valve and an outlet valve. The tank can be filled via the inlet valve in 6 minutes and emptied (from full) via the outlet valve in 10 minutes. If both valves are operating, how long would it take to fill the tank if it was empty to begin with?

f A basketball player has scored 312 points in 15 games. How many points must he average per game in his next 3 games to take his overall average to 20 points per game?

g A cyclist rides for 5 hours at a certain speed and then for 4 hours at a speed 6 km/h greater than her original speed. If she rides 294 km altogether, what was her initial speed?

h Two trains travel at speeds of 72 km/h and 48 km/h respectively. If they start at the same time and travel towards each other from two places 600 km apart, how long will it be before they meet?

10 Rearrange each formula so that the pronumeral written in square brackets is the subject.

a $\dfrac{a}{2} - \dfrac{b}{3} = a$ $[a]$

b $\dfrac{1}{f} + \dfrac{2}{g} = \dfrac{5}{h}$ $[g]$

c $x = \dfrac{y}{y + 2}$ $[y]$

d $a = \dfrac{b + 5}{b - 4}$ $[b]$

ENRICHMENT

11 Solve:

a $\dfrac{x}{x - 2} + \dfrac{3}{x - 4} = 1$

b $\dfrac{3a - 2}{2a - 3} - \dfrac{a + 17}{a + 10} = \dfrac{1}{2}$

12 **a** Show that $\dfrac{x - 1}{x - 3} = 1 + \dfrac{2}{x - 3}$.

b Hence solve $\dfrac{x - 1}{x - 3} - \dfrac{x - 3}{x - 5} = \dfrac{x - 5}{x - 7} - \dfrac{x - 7}{x - 9}$.

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1F Solving quadratic equations

There are three approaches to solving a *quadratic equation*:

- factoring
- completing the square
- the quadratic formula.

This section reviews factoring and the quadratic formula. Completing the square is reviewed in Section 1H.

Solving a quadratic by factoring

This method is the simplest, but it only works in special cases.

10 SOLVING A QUADRATIC BY FACTORING

- Get all the terms on the left, then factor the left-hand side.
- Then use the principle that if $A \times B = 0$, then $A = 0$ or $B = 0$.

Example 19 **1F**

Solve the quadratic equation by factoring.

SOLUTION

$$5x^2 + 34x - 7 = 0$$

$$5x^2 + 35x - x - 7 = 0 \qquad \text{(35 and } -1 \text{ have sum 34 and product } -7 \times 5 = -35.)$$

$$5x(x + 7) - (x + 7) = 0$$

$$(5x - 1)(x + 7) = 0 \qquad \text{(the LHS is now factored)}$$

$$5x - 1 = 0 \text{ or } x + 7 = 0 \qquad \text{(one of the factors must be zero)}$$

$$x = \frac{1}{5} \text{ or } x = -7 \qquad \text{(there are two solutions)}$$

Solving a quadratic by the formula

This method works whether the solutions are rational numbers or involve surds. It will be proven in the last Development question of Exercise 3E.

11 THE QUADRATIC FORMULA

- The solutions of $ax^2 + bx + c = 0$ are:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

- Always calculate $b^2 - 4ac$ first.
  (Later, this quantity will be called the *discriminant* and given the symbol Δ.)

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Example 20 1F

Solve each quadratic equation using the quadratic formula.

a $5x^2 + 2x - 7 = 0$

b $3x^2 + 4x - 1 = 0$

SOLUTION

a For $5x^2 + 2x - 7 = 0$,
$a = 5, b = 2$ and $c = -7$.

$$\text{Hence } b^2 - 4ac = 2^2 + 140 = 144 = 12^2,$$

$$\text{so } x = \frac{-2 + 12}{10} \text{ or } \frac{-2 - 12}{10}$$

$$= 1 \text{ or } -1\frac{2}{5}.$$

b For $3x^2 + 4x - 1 = 0$,
$a = 3, b = 4$ and $c = -1$.

$$\text{Hence } b^2 - 4ac = 4^2 + 12 = 28 = 4 \times 7,$$

$$\text{so } x = \frac{-4 + 2\sqrt{7}}{6} \text{ or } \frac{-4 - 2\sqrt{7}}{6}$$

$$= \frac{-2 + \sqrt{7}}{3} \text{ or } \frac{-2 - \sqrt{7}}{3}.$$

Exercise 1F

FOUNDATION

1 Solve:

a $x^2 = 9$ **b** $y^2 = 25$ **c** $a^2 - 4 = 0$
d $c^2 - 36 = 0$ **e** $1 - t^2 = 0$ **f** $x^2 = \frac{9}{4}$
g $4x^2 - 1 = 0$ **h** $9a^2 - 64 = 0$ **i** $25y^2 = 16$

2 Solve by factoring:

a $x^2 - 5x = 0$ **b** $y^2 + y = 0$ **c** $c^2 + 2c = 0$
d $k^2 - 7k = 0$ **e** $t^2 = t$ **f** $3a = a^2$
g $2b^2 - b = 0$ **h** $3u^2 + u = 0$ **i** $4x^2 + 3x = 0$
j $2a^2 = 5a$ **k** $3y^2 = 2y$ **l** $3n + 5n^2 = 0$

3 Solve by factoring:

a $x^2 + 4x + 3 = 0$ **b** $x^2 - 3x + 2 = 0$ **c** $x^2 + 6x + 8 = 0$
d $a^2 - 7a + 10 = 0$ **e** $t^2 - 4t - 12 = 0$ **f** $c^2 - 10c + 25 = 0$
g $n^2 - 9n + 8 = 0$ **h** $p^2 + 2p - 15 = 0$ **i** $a^2 - 10a - 24 = 0$
j $y^2 + 4y = 5$ **k** $p^2 = p + 6$ **l** $a^2 = a + 132$
m $c^2 + 18 = 9c$ **n** $8t + 20 = t^2$ **o** $u^2 + u = 56$
p $k^2 = 24 + 2k$ **q** $50 + 27h + h^2 = 0$ **r** $a^2 + 20a = 44$

4 Solve by factoring:

a $2x^2 + 3x + 1 = 0$ **b** $3a^2 - 7a + 2 = 0$ **c** $4y^2 - 5y + 1 = 0$
d $2x^2 + 11x + 5 = 0$ **e** $2x^2 + x - 3 = 0$ **f** $3n^2 - 2n - 5 = 0$
g $3b^2 - 4b - 4 = 0$ **h** $2a^2 + 7a - 15 = 0$ **i** $2y^2 - y - 15 = 0$
j $3y^2 + 10y = 8$ **k** $5x^2 - 26x + 5 = 0$ **l** $4t^2 + 9 = 15t$
m $13t + 6 = 5t^2$ **n** $10u^2 + 3u - 4 = 0$ **o** $25x^2 + 1 = 10x$
p $6x^2 + 13x + 6 = 0$ **q** $12b^2 + 3 + 20b = 0$ **r** $6k^2 + 13k = 8$

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5 Solve each equation using the quadratic formula. Give exact answers, followed by approximations correct to four significant figures where appropriate.

a $x^2 - x - 1 = 0$ **b** $y^2 + y = 3$ **c** $a^2 + 12 = 7a$
d $u^2 + 2u - 2 = 0$ **e** $c^2 - 6c + 2 = 0$ **f** $4x^2 + 4x + 1 = 0$
g $2a^2 + 1 = 4a$ **h** $5x^2 + 13x - 6 = 0$ **i** $2b^2 + 3b = 1$
j $3c^2 = 4c + 3$ **k** $4t^2 = 2t + 1$ **l** $x^2 + x + 1 = 0$

DEVELOPMENT

6 Solve by factoring:

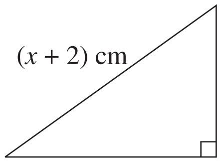
a $x = \dfrac{x + 2}{x}$ **b** $a + \dfrac{10}{a} = 7$
c $y + \dfrac{2}{y} = \dfrac{9}{2}$ **d** $(5b - 3)(3b + 1) = 1$

7 Find the exact solutions of:

a $x = \dfrac{1}{x} + 2$ **b** $\dfrac{4x - 1}{x} = x$ **c** $a = \dfrac{a + 4}{a - 1}$ **d** $\dfrac{5m}{2} = 2 + \dfrac{1}{m}$

8 **a** If $y = px - ap^2$, find p, given that $a = 2$, $x = 3$ and $y = 1$.
b Given that $(x - a)(x - b) = c$, find x when $a = -2$, $b = 4$ and $c = 7$.
c Suppose that $S = \dfrac{n}{2}\big(2a + (n - 1)d\big)$. Find the positive value of n that gives $S = 80$ when $a = 4$ and $d = 6$.

9 **a** Find a positive integer that, when increased by 30, is 12 less than its square.
b Two positive numbers differ by 3 and the sum of their squares is 117. Find the numbers.
c Find the value of x in the diagram to the right.



10 Find a in terms of b if:

a $a^2 - 5ab + 6b^2 = 0$ **b** $3a^2 + 5ab - 2b^2 = 0$

11 Find y in terms of x if:

a $4x^2 - y^2 = 0$ **b** $x^2 - 9xy - 22y^2 = 0$

12 Solve each equation.

a $\dfrac{5k + 7}{k - 1} = 3k + 2$ **b** $\dfrac{u + 3}{2u - 7} = \dfrac{2u - 1}{u - 3}$
c $\dfrac{y + 1}{y + 2} = \dfrac{3 - y}{y - 4}$ **d** $2(k - 1) = \dfrac{4 - 5k}{k + 1}$
e $\dfrac{2}{a + 3} + \dfrac{a + 3}{2} = \dfrac{10}{3}$ **f** $\dfrac{k + 10}{k - 5} - \dfrac{10}{k} = \dfrac{11}{6}$
g $\dfrac{3t}{t^2 - 6} = \sqrt{3}$ **h** $\dfrac{3m + 1}{3m - 1} - \dfrac{3m - 1}{3m + 1} = 2$

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ENRICHMENT

13 Solve each problem by forming and then solving a quadratic equation.

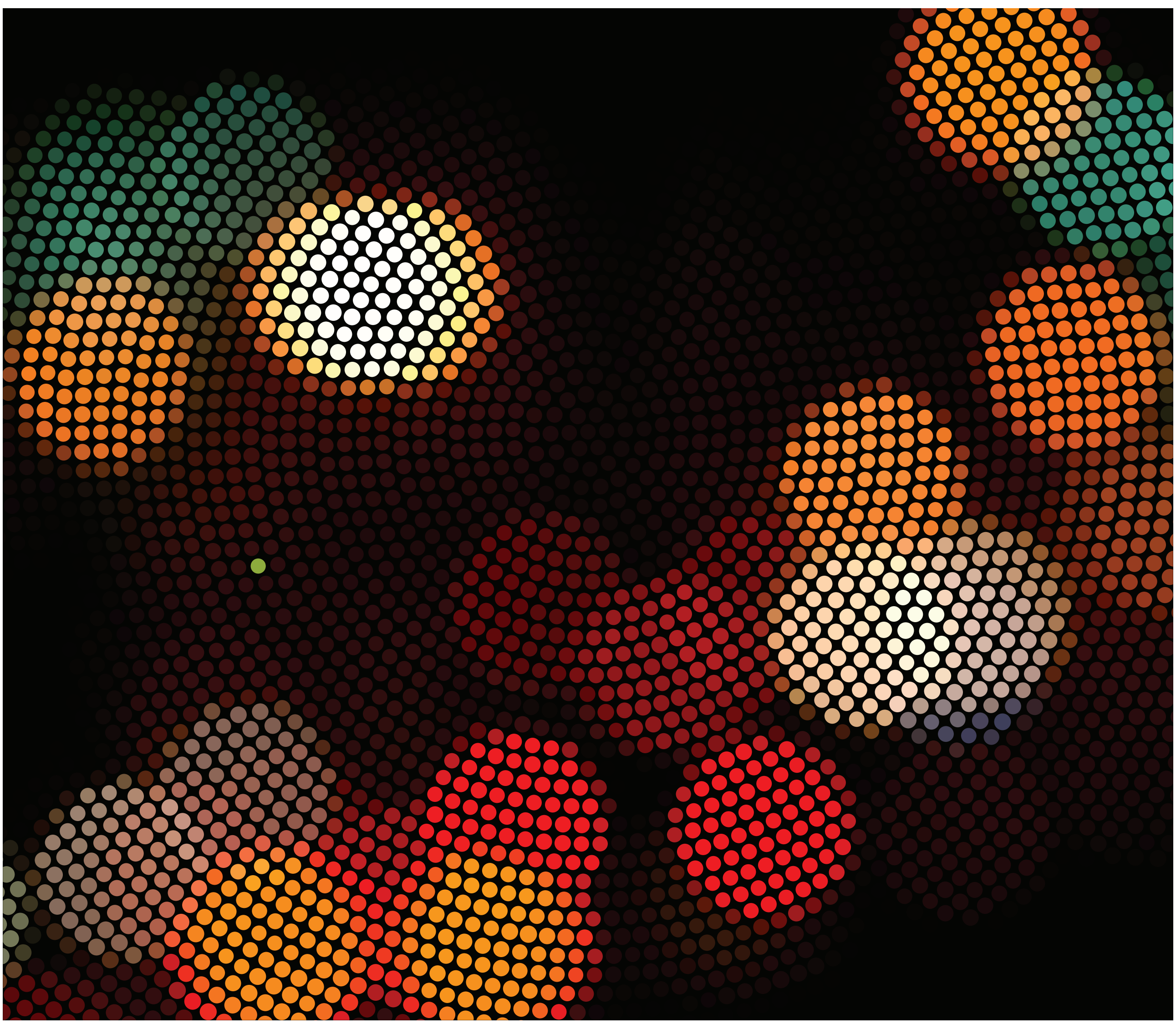
a A rectangular area can be completely tiled with 200 square tiles. If the side length of each tile was increased by 1 cm, it would take only 128 tiles to tile the area. Find the side length of each tile.

b A photograph is 18 cm by 12 cm. It is to be surrounded by a frame of uniform width whose area is equal to that of the photograph. Find the width of the frame.

c Two trains each make a journey of 330 km. One of the trains travels 5 km/h faster than the other and takes 30 minutes less. Find the speeds of the trains.

14 **a** Find x in terms of c, given that $\dfrac{2}{3x - 2c} + \dfrac{3}{2x - 3c} = \dfrac{7}{2c}$.

b Find x in terms of a and b if $\dfrac{a^2b}{x^2} + \left(1 + \dfrac{b}{x}\right)a = 2b + \dfrac{a^2}{x}$.



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1G Solving simultaneous equations

There are two algebraic approaches to solving *simultaneous equations* — substitution and elimination. They can be applied to both linear and non-linear simultaneous equations.

Solution by substitution

This method can be applied whenever one of the equations can be solved for one of the variables.

12 SOLVING SIMULTANEOUS EQUATIONS BY SUBSTITUTION

- Solve one of the equations for one of the variables.
- Then substitute it into the other equation.

Example 21 — 1G

Solve each pair of simultaneous equations by substitution.

a $3x - 2y = 29$ (1)
$4x + y = 24$ (2)

b $y = x^2$ (1)
$y = x + 2$ (2)

SOLUTION

a Solving (2) for y, $\quad y = 24 - 4x.$ (2A)

Substituting (2A) into (1), $\quad 3x - 2(24 - 4x) = 29$

$x = 7.$

Substituting $x = 7$ into (1), $\quad 21 - 2y = 29$

$y = -4.$

Hence $x = 7$ and $y = -4$. (This should be checked in the original equations.)

b Substituting (1) into (2),

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2 \text{ or } -1.$$

From (1), when $x = 2$, $y = 4$, and when $x = -1$, $y = 1$.

Hence $x = 2$ and $y = 4$, or $x = -1$ and $y = 1$. (Check in the original equations.)

Solution by elimination

This method, when it can be used, is more elegant, and usually involves less algebraic manipulation.

13 SOLVING SIMULTANEOUS EQUATIONS BY ELIMINATION

Take suitable multiples of the equations so that one variable is eliminated when the equations are added or subtracted.

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Example 22 1G

Solve each pair of simultaneous equations by elimination.

a $3x - 2y = 29$ (1)
$4x + 5y = 8$ (2)

b $x^2 + y^2 = 53$ (1)
$x^2 - y^2 = 45$ (2)

SOLUTION

a Taking $4 \times$ (1) and $3 \times$ (2),
$12x - 8y = 116$ (1A)
$12x + 15y = 24.$ (2A)
Subtracting (1A) from (2A),
$23y = -92$
$\boxed{\div 23}$ $y = -4.$
Substituting into (1),
$3x + 8 = 29$
$x = 7.$
Hence $x = 7$ and $y = -4$.

b Adding (1) and (2),
$2x^2 = 98$
$x^2 = 49.$
Subtracting (2) from (1),
$2y^2 = 8$
$y^2 = 4.$
Hence $x = 7$ and $y = 2$,
or $x = 7$ and $y = -2$,
or $x = -7$ and $y = 2$,
or $x = -7$ and $y = -2$.

Exercise 1G FOUNDATION

1 Solve by substituting the first equation into the second:

a $y = x$ and $2x + y = 9$
b $y = 2x$ and $3x - y = 2$
c $y = x - 1$ and $2x + y = 5$
d $a = 2b + 1$ and $a - 3b = 3$
e $p = 2 - q$ and $p - q = 4$
f $v = 1 - 3u$ and $2u + v = 0$

2 Solve by either adding or subtracting the two equations:

a $x + y = 5$ and $x - y = 1$
b $3x - 2y = 7$ and $x + 2y = -3$
c $2x + y = 9$ and $x + y = 5$
d $a + 3b = 8$ and $a + 2b = 5$
e $4c - d = 6$ and $2c - d = 2$
f $p - 2q = 4$ and $3p - 2q = 0$

3 Solve by substitution:

a $y = 2x$ and $3x + 2y = 14$
b $y = -3x$ and $2x + 5y = 13$
c $y = 4 - x$ and $x + 3y = 8$
d $x = 5y + 4$ and $3x - y = 26$
e $2x + y = 10$ and $7x + 8y = 53$
f $2x - y = 9$ and $3x - 7y = 19$
g $4x - 5y = 2$ and $x + 10y = 41$
h $2x + 3y = 47$ and $4x - y = 45$

4 Solve by elimination:

a $2x + y = 1$ and $x - y = -4$
b $2x + 3y = 16$ and $2x + 7y = 24$
c $3x + 2y = -6$ and $x - 2y = -10$
d $5x - 3y = 28$ and $2x - 3y = 22$
e $3x + 2y = 7$ and $5x + y = 7$
f $3x + 2y = 0$ and $2x - y = 56$
g $15x + 2y = 27$ and $3x + 7y = 45$
h $7x - 3y = 41$ and $3x - y = 17$
i $2x + 3y = 28$ and $3x + 2y = 27$
j $3x - 2y = 11$ and $4x + 3y = 43$

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DEVELOPMENT

5 Solve by substitution:

a $y = 2 - x$ and $y = x^2$

b $y = 2x - 3$ and $y = x^2 - 4x + 5$

c $y = 3x^2$ and $y = 4x - x^2$

d $x - y = 5$ and $y = x^2 - 11$

e $x - y = 2$ and $xy = 15$

f $3x + y = 9$ and $xy = 6$

6 Solve each problem by forming and then solving a pair of simultaneous equations.

a Find two numbers that differ by 16 and have a sum of 90.

b I paid 75 cents for a pen and a pencil. If the pen cost four times as much as the pencil, find the cost of each item.

c If 7 apples and 2 oranges cost \$4, and 5 apples and 4 oranges cost \$4.40, find the cost of each apple and each orange.

d Twice as many adults as children attended a certain concert. If adult tickets cost \$8 each, child tickets cost \$3 each, and the total takings were \$418, find the numbers of adults and children who attended.

e A man is 3 times as old as his son. In 12 years' time he will be twice as old as his son. How old is each of them now?

f At a meeting of the members of a certain club, a proposal was voted on. If 357 members voted and the proposal was carried by a majority of 21, how many voted for and how many voted against?

g Kathy paid \$320 in cash for a CD player. If she paid in \$20 and \$10 notes, and there were 23 notes altogether, how many of each type were there?

h Two people are 16 km apart on a straight road. They start walking at the same time. If they walk towards each other, they will meet in 2 hours, but if they walk in the same direction (so that the distance between them is decreasing), they will meet in 8 hours. Find their walking speeds.

7 Solve simultaneously:

a $\frac{y}{4} - \frac{x}{3} = 1$ and $\frac{x}{2} + \frac{y}{5} = 10$

b $4x + \frac{y - 2}{3} = 12$ and $3y - \frac{x - 3}{5} = 6$

8 Solve simultaneously:

a $x + y = 15$ and $x^2 + y^2 = 125$

b $x - y = 3$ and $x^2 + y^2 = 185$

c $2x + y = 5$ and $4x^2 + y^2 = 17$

d $x + y = 9$ and $x^2 + xy + y^2 = 61$

e $x + 2y = 5$ and $2xy - x^2 = 3$

f $3x + 2y = 16$ and $xy = 10$

ENRICHMENT

9 Solve simultaneously:

a $\frac{7}{x} - \frac{5}{y} = 3$ and $\frac{2}{x} + \frac{25}{2y} = 12$

b $9x^2 + y^2 = 52$ and $xy = 8$

10 Consider the equations $12x^2 - 4xy + 11y^2 = 64$ and $16x^2 - 9xy + 11y^2 = 78$.

a By letting $y = mx$, show that $7m^2 + 12m - 4 = 0$.

b Hence, or otherwise, solve the two equations simultaneously.

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1H Completing the square

Completing the square can be done with all quadratic equations, whereas factoring is only possible in special cases.

The review in this section is mostly restricted to monic quadratics, in which the coefficient of x^2 is 1. Chapter 3 will deal with non-monic quadratics. Chapter 3 will also require completing the square in a quadratic *function*, which is only slightly different from completing the square in a quadratic *equation*.

Perfect squares

The expansion of the quadratic $(x + 5)^2$ is

$$(x + 5)^2 = x^2 + 10x + 25.$$

Notice that the coefficient of x is twice 5, and the constant is the square of 5.

Reversing the process, the constant term in a perfect square can be found by taking half the coefficient of x and squaring the result.

14 COMPLETING THE SQUARE IN AN EXPRESSION $x^2 + bx + \cdots$

Halve the coefficient b of x, and square the result.

Example 23 **1H**

Complete the square in each expression.

a $x^2 + 16x + \cdots$ **b** $x^2 - 3x + \cdots$

SOLUTION

a The coefficient of x is 16, half of 16 is 8, and $8^2 = 64$,
so $x^2 + 16x + 64 = (x + 8)^2$.

b The coefficient of x is -3, half of -3 is $-1\frac{1}{2}$, and $\left(-1\frac{1}{2}\right)^2 = 2\frac{1}{4}$,
so $x^2 - 3x + 2\frac{1}{4} = \left(x - 1\frac{1}{2}\right)^2$.

Solving a quadratic equation by completing the square

This process always works.

15 SOLVING A QUADRATIC EQUATION BY COMPLETING THE SQUARE

Complete the square in the quadratic by adding the same to both sides.

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Example 24 — 1H

Solve each quadratic equation by completing the square.

a $t^2 + 8t = 20$ **b** $x^2 - x - 1 = 0$ **c** $x^2 + x + 1 = 0$

SOLUTION

a
$$t^2 + 8t = 20$$
$$\boxed{+16}\quad t^2 + 8t + 16 = 36$$
$$(t+4)^2 = 36$$
$$t + 4 = 6 \text{ or } t + 4 = -6$$
$$t = 2 \text{ or } -10$$

b
$$x^2 - x - 1 = 0$$
$$\boxed{+1}\quad x^2 - x = 1$$
$$\boxed{+\tfrac{1}{4}}\quad x^2 - x + \tfrac{1}{4} = 1\tfrac{1}{4}$$
$$\left(x - \tfrac{1}{2}\right)^2 = \tfrac{5}{4}$$
$$x - \tfrac{1}{2} = \tfrac{1}{2}\sqrt{5} \text{ or } x - \tfrac{1}{2} = -\tfrac{1}{2}\sqrt{5}$$
$$x = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} \text{ or } \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}$$

c
$$x^2 + x + 1 = 0$$
$$x^2 + x + \tfrac{1}{4} = -\tfrac{3}{4}$$
$$\left(x + \tfrac{1}{2}\right)^2 = -\tfrac{3}{4},$$ which is impossible, because a square can't be negative,

Hence the equation has no solutions.

The word 'algebra'

Al-Khwarizmi was a famous and influential Persian mathematician who worked in Baghdad during the early ninth century when the Baghdad Caliphate excelled in science and mathematics. The Arabic word 'algebra' comes from *al-jabr*, a word in the title of his most important work, and means 'the restoration of broken parts' — a reference to the balancing of terms on both sides of an equation. Al-Khwarizmi's own name came into modern European languages as 'algorithm'.

Exercise 1H

FOUNDATION

1 What constant must be added to each expression in order to create a perfect square?

a $x^2 + 2x$ **b** $y^2 - 6y$ **c** $a^2 + 10a$ **d** $m^2 - 18m$
e $c^2 + 3c$ **f** $x^2 - x$ **g** $b^2 + 5b$ **h** $t^2 - 9t$

2 Factor:

a $x^2 + 4x + 4$ **b** $y^2 + 2y + 1$ **c** $p^2 + 14p + 49$ **d** $m^2 - 12m + 36$
e $t^2 - 16t + 64$ **f** $x^2 + 20x + 100$ **g** $u^2 - 40u + 400$ **h** $a^2 - 24a + 144$

3 Copy and complete:

a $x^2 + 6x + \ldots = (x + \ldots)^2$ **b** $y^2 + 8y + \ldots = (y + \ldots)^2$
c $a^2 - 20a + \ldots = (a - \ldots)^2$ **d** $b^2 - 100b + \ldots = (b - \ldots)^2$
e $u^2 + u + \ldots = (u + \ldots)^2$ **f** $t^2 - 7t + \ldots = (t - \ldots)^2$
g $m^2 + 50m + \ldots = (m + \ldots)^2$ **h** $c^2 - 13c + \ldots = (c - \ldots)^2$

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4 Solve each quadratic equation by completing the square.

a $x^2 - 2x = 3$ **b** $x^2 - 6x = 0$ **c** $a^2 + 6a + 8 = 0$

d $x^2 + 4x + 1 = 0$ **e** $x^2 - 10x + 20 = 0$ **f** $y^2 + 3y = 10$

g $b^2 - 5b - 14 = 0$ **h** $y^2 - y + 2 = 0$ **i** $a^2 + 7a + 7 = 0$

DEVELOPMENT

5 Solve by dividing both sides by the coefficient of x^2 and then completing the square:

a $2x^2 - 4x - 1 = 0$ **b** $2x^2 + 8x + 3 = 0$

c $3x^2 + 6x + 5 = 0$ **d** $4x^2 + 4x - 3 = 0$

e $4x^2 - 2x - 1 = 0$ **f** $2x^2 - 10x + 7 = 0$

6 **a** If $x^2 + y^2 + 4x - 2y + 1 = 0$, show that $(x + 2)^2 + (y - 1)^2 = 4$.

b Show that the equation $x^2 + y^2 - 6x - 8y = 0$ can be written in the form

$$(x - a)^2 + (y - b)^2 = c,$$

where a, b and c are constants. Hence find a, b and c.

c If $x^2 + 1 = 10x + 12y$, show that $(x - 5)^2 = 12(y + 2)$.

d Find values for A, B and C if $y^2 - 6x + 16y + 94 = (y + C)^2 - B(x + A)$.

ENRICHMENT

7 **a** Write down the expansion of $(x + \alpha)^3$ and hence complete the cube in

$$x^3 + 12x^2 + \ldots = (x + \ldots)^3.$$

b Hence use a suitable substitution to change the equation $x^3 + 12x^2 + 30x + 4 = 0$ into a cubic equation of the form $u^3 + cu + d = 0$.

8 Given that $3a^2 + 4b^2 + 18c^2 - 4ab - 12ac = 0$, prove, by completing squares, that $a = 2b = 3c$.

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Chapter 1 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 1 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Simplify:

a $-8y + 2y$ **b** $-8y - 2y$ **c** $-8y \times 2y$ **d** $-8y \div 2y$

2 Simplify:

a $-2a^2 - a^2$ **b** $-2a^2 - (-a^2)$ **c** $-2a^2 \times (-a^2)$ **d** $-2a^2 \div (-a^2)$

3 Simplify:

a $3t - 1 - t$ **b** $-6p + 3q + 10p$

c $7x - 4y - 6x + 2y$ **d** $2a^2 + 8a - 13 + 3a^2 - 11a - 5$

4 Simplify:

a $-6k^6 \times 3k^3$ **b** $-6k^6 \div 3k^3$ **c** $(-6k^6)^2$ **d** $(3k^3)^3$

5 Expand and simplify:

a $4(x + 3) + 5(2x - 3)$ **b** $8(a - 2b) - 6(2a - 3b)$ **c** $-(a - b) - (a + b)$

d $-4x^2(x + 3) - 2x^2(x - 1)$ **e** $(n + 7)(2n - 3)$ **f** $(r + 3)^2$

g $(y - 5)(y + 5)$ **h** $(3x - 5)(2x - 3)$ **i** $(t - 8)^2$

j $(2c + 7)(2c - 7)$ **k** $(4p + 1)^2$ **l** $(3u - 2)^2$

6 Factor:

a $18a + 36$ **b** $20b - 36$ **c** $9c^2 + 36c$

d $d^2 - 36$ **e** $e^2 + 13e + 36$ **f** $f^2 - 12f + 36$

g $36 - 25g^2$ **h** $h^2 - 9h - 36$ **i** $i^2 + 5i - 36$

j $2j^2 + 11j + 12$ **k** $3k^2 - 7k - 6$ **l** $5\ell^2 - 14\ell + 8$

m $4m^2 + 4m - 15$ **n** $mn + m + pn + p$ **o** $p^3 + 9p^2 + 4p + 36$

p $qt - rt - 5q + 5r$ **q** $u^2w + vw - u^2x - vx$ **r** $x^2 - y^2 + 2x - 2y$

7 Simplify:

a $\frac{x}{2} + \frac{x}{4}$ **b** $\frac{x}{2} - \frac{x}{4}$ **c** $\frac{x}{2} \times \frac{x}{4}$ **d** $\frac{x}{2} \div \frac{x}{4}$

e $\frac{3a}{2b} + \frac{2a}{3b}$ **f** $\frac{3a}{2b} - \frac{2a}{3b}$ **g** $\frac{3a}{2b} \times \frac{2a}{3b}$ **h** $\frac{3a}{2b} \div \frac{2a}{3b}$

i $\frac{x}{y} + \frac{y}{x}$ **j** $\frac{x}{y} - \frac{y}{x}$ **k** $\frac{x}{y} \times \frac{y}{x}$ **l** $\frac{x}{y} \div \frac{y}{x}$

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Review

8 Simplify:

a $\frac{x+4}{5}+\frac{x-5}{3}$ b $\frac{5}{x+4}+\frac{3}{x-5}$ c $\frac{x+1}{2}-\frac{x-4}{5}$

d $\frac{2}{x+1}-\frac{5}{x-4}$ e $\frac{x}{2}-\frac{x+3}{4}$ f $\frac{2}{x}-\frac{4}{x+3}$

9 Factor each expression where possible, then simplify it.

a $\frac{6a+3b}{10a+5b}$ b $\frac{2x-2y}{x^2-y^2}$ c $\frac{x^2+2x-3}{x^2-5x+4}$

d $\frac{2x^2+3x+1}{2x^3+x^2+2x+1}$ e $\frac{a+b}{a^2+2ab+b^2}$ f $\frac{3x^2-19x-14}{9x^2-4}$

10 Solve each linear equation.

a $3x+5=17$ b $3(x+5)=17$

c $\frac{x+5}{3}=17$ d $\frac{x}{3}+5=17$

e $7a-4=2a+11$ f $7(a-4)=2(a+11)$

g $\frac{a-4}{7}=\frac{a+11}{2}$ h $\frac{a}{7}-4=\frac{a}{2}+11$

11 Solve each quadratic equation by factoring the left-hand side.

a $a^2-49=0$ b $b^2+7b=0$

c $c^2+7c+6=0$ d $d^2+6d-7=0$

e $e^2-5e+6=0$ f $2f^2-f-6=0$

g $2g^2-13g+6=0$ h $3h^2+2h-8=0$

12 Solve using the quadratic formula. Write the solutions in simplest exact form.

a $x^2-4x+1=0$ b $y^2+3y-3=0$

c $t^2+6t+4=0$ d $3x^2-2x-2=0$

e $2a^2+5a-5=0$ f $4k^2-6k-1=0$

13 Solve each quadratic equation by completing the square on the left-hand side.

a $x^2+4x=6$ b $y^2-6y+3=0$

c $x^2-2x=12$ d $y^2+10y+7=0$

The following questions are more difficult

14 If $x+y=5$ and $xy=7$, find the value of x^2+y^2 without attempting to find x or y.

15 Factor x^4+x^2+1 by adding and subtracting x^2.

16 Simplify $\frac{x^3+2x^2+4x+8}{2x^3-8x}\times\frac{x^3-4x^2+4x}{x^4-16}$.

17 Simplify $\frac{5x}{x^2+3x+2}+\frac{3x}{x^2+4x+3}-\frac{8x}{x^2+5x+6}$.

18 Solve the equation $\frac{7}{3x-1}-\frac{4}{x+1}=\frac{1}{4}$.

19 Solve $y=x^2-4$ and $x^2+y^2=4$ simultaneously.

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2 Numbers and surds

Arithmetic is the study of numbers and operations on them. This short chapter reviews whole numbers, integers, rational numbers and real numbers, with particular attention to the arithmetic of surds and their approximations. Most of this material will be familiar from earlier years.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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2A Whole numbers, integers and rationals

Our ideas about numbers arise from the two quite distinct sources:

- The *whole numbers*, the *integers* and the *rational numbers* are developed from counting.
- The *real numbers* are developed from geometry and the number line.

This section very briefly reviews the whole numbers, integers and rational numbers, with particular attention to percentages and recurring decimals.

The whole numbers

Counting is the first operation in arithmetic. Counting things such as people in a room requires *zero* (if the room is empty) and then the successive numbers 1, 2, 3, . . . , generating all the *whole numbers*:

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \dots$$

The number zero is the first number on this list, but there is no last number, because every number is followed by another number, distinct from all previous numbers. The list is therefore called *infinite*, which means that it never 'finishes'. The symbol $\mathbb{N}$ is usually used for the set of whole numbers.

A non-zero whole number can be factored, in one and only one way, into the product of prime numbers, where a *prime number* is a whole number greater than 1 whose only divisors are itself and 1. The primes form a sequence whose distinctive pattern has confused every mathematician since Greek times:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47,\ 53,\ 59,\ 61,\ 67,\ 71,\ \dots$$

The whole numbers greater than 1 and not prime are called *composite numbers*,

$$4,\ 6,\ 8,\ 9,\ 10,\ 12,\ 14,\ 15,\ 16,\ 18,\ 20,\ 21,\ 22,\ 24,\ 25,\ 26,\ 27,\ 28,\ 30,\ 32,\ \dots$$

The whole numbers 0 and 1 are special cases, being neither prime nor composite.

1 THE SET $\mathbb{N}$ OF WHOLE NUMBERS

- The *whole numbers* $\mathbb{N}$ are $0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \dots$
- Every whole number except 0 and 1 is either *prime* or *composite*, and every composite number can be factored into primes in one and only one way.
- When whole numbers are added or multiplied, the result is a whole number.

The integers

Any two whole numbers can be *added* or *multiplied*, and the result is another whole number. *Subtraction*, however, requires the *negative integers* as well:

$$\dots,\ -6,\ -5,\ -4,\ -3,\ -2,\ -1$$

so that calculations such as $5 - 7 = -2$ can be completed. The symbol $\mathbb{Z}$ (from German 'Zahlen' meaning 'numbers') is conventionally used for the set of integers.

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2 THE SET $\mathbb{Z}$ OF INTEGERS

- The *integers* $\mathbb{Z}$ are $\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots$
- When integers are added, subtracted, or multiplied, the result is an integer.

The rational numbers

A problem such as, 'Divide 7 cakes into 3 equal parts', leads naturally to *fractions*, where the whole is 'fractured' or 'broken' into pieces. Thus we have the system of *rational numbers*, which are numbers that can be written as the 'ratio' of two integers. Here are some examples of rational numbers written as single fractions:

$$2\tfrac{1}{3} = \frac{7}{3} \qquad -\frac{1}{3} = \frac{-1}{3} \qquad 30 \div 24 = \frac{5}{4} \qquad 3.72 = \frac{372}{100} \qquad 4 = \frac{4}{1}$$

Operations on the rational numbers

Addition, multiplication, subtraction and division (except by 0) can all be carried out within the rational numbers.

- Rational numbers are simplified by dividing top and bottom by their HCF (highest common factor). For example, 21 and 35 have HCF 7, so

$$\begin{aligned}\frac{21}{35} &= \frac{21 \div 7}{35 \div 7}\\ &= \frac{3}{5}\end{aligned}$$

- Rational numbers are added and subtracted using the LCM (lowest common multiple) of their denominators. For example, 6 and 8 have LCM 24, so

$$\begin{aligned}\frac{1}{6} + \frac{5}{8} &= \frac{1 \times 4}{24} + \frac{5 \times 3}{24}\\ &= \frac{19}{24}\end{aligned} \qquad \begin{aligned}\frac{1}{6} - \frac{5}{8} &= \frac{1 \times 4}{24} - \frac{5 \times 3}{24}\\ &= -\frac{11}{24}\end{aligned}$$

- Fractions are multiplied by multiplying the numerators and multiplying the denominators, after first cancelling out any common factors. To divide by a fraction, multiply by its reciprocal.

$$\begin{aligned}\frac{10}{21} \times \frac{9}{25} &= \frac{2}{7} \times \frac{3}{5}\\ &= \frac{6}{35}\end{aligned} \qquad \begin{aligned}\frac{8}{21} \div \frac{3}{4} &= \frac{8}{21} \times \frac{4}{3}\\ &= \frac{32}{63}\end{aligned}$$

The symbol $\mathbb{Q}$ for 'quotient' is conventionally used for the set of rational numbers.

3 THE SET $\mathbb{Q}$ OF RATIONAL NUMBERS

- The *rational numbers* $\mathbb{Q}$ are the numbers that can be written as *fractions* $\frac{a}{b}$, where a and b are integers and $b \neq 0$.
- Every integer a can be written as a fraction $\frac{a}{1}$, and so is a rational number.
- When rational numbers are added, subtracted, multiplied, and divided (but not by zero), the result is a rational number.

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Decimal notation — terminating or recurring decimals

Decimal notation extends place value to negative powers of 10. For example:

$$123.456 = 1 \times 10^2 + 2 \times 10^1 + 3 \times 10^0 + 4 \times 10^{-1} + 5 \times 10^{-2} + 6 \times 10^{-3}.$$

Such a number can be written as a fraction $\frac{123\,456}{1000}$, so is a rational number.

If a rational number can be written as a fraction whose denominator is a power of 10, then it can easily be written as a *terminating decimal*:

$$\tfrac{3}{25} = \tfrac{12}{100} = 0.12 \qquad \text{and} \qquad 578\tfrac{3}{50} = 578 + \tfrac{6}{100} = 578.06$$

If a rational number cannot be written with a power of 10 as its denominator, then repeated division will yield an infinite string of digits in its decimal representation. This string will cycle once the same remainder recurs, giving a *recurring decimal*.

$$\tfrac{2}{3} = 0.666666666666\ldots = 0.\dot{6} \qquad \text{(which has cycle length 1)}$$

$$6\tfrac{3}{7} = 6.428571428571\ldots = 6.\dot{4}2857\dot{1} \qquad \text{(which has cycle length 6)}$$

$$4\tfrac{7}{22} = 4.31818181818\ldots = 4.3\dot{1}\dot{8} \qquad \text{(which has cycle length 2)}$$

Conversely, every recurring decimal can be written as a fraction — such calculations are discussed in Year 12 in the context of geometric series.

Percentages

Many practical situations involving fractions, decimals and ratios are commonly expressed in terms of percentages. The symbol % evolved from the handwritten 'per centum', meaning 'per hundred' — interpret the symbol as /100, that is, 'over 100'.

4 PERCENTAGES

- To convert a fraction to a percentage, multiply by $\frac{100}{1}\%$:

  $$\frac{3}{20} = \frac{3}{20} \times \frac{100}{1}\% = 15\%$$

- To convert a percentage to a fraction, replace % by $\times \frac{1}{100}$:

  $$15\% = 15 \times \frac{1}{100} = \frac{3}{20}$$

Many problems are best solved by the *unitary method*, illustrated below.

Example 1 — 2A

a A table marked \$1400 has been discounted by 30%. How much does it now cost?

b A table discounted by 30% now costs \$1400. What was the original price?

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SOLUTION

a 100% is \$1400

÷ 10: 10% is \$140

× 7: 70% is \$980

so the discounted price is \$980.

b 70% is \$1400

÷ 7: 10% is \$200

× 10: 100% is \$2000

so the original price was \$2000.

Exercise 2A

FOUNDATION

Note: All Foundation and Development questions are non-calculator questions.

1 Write as a fraction in lowest terms:

a 30% **b** 80% **c** 75% **d** 5%

2 Write as a decimal:

a 60% **b** 27% **c** 9% **d** 16.5%

3 Write as a percentage:

a $\frac{1}{4}$ **b** $\frac{2}{5}$ **c** $\frac{6}{25}$ **d** $\frac{13}{20}$

4 Write as a percentage:

a 0.32 **b** 0.09 **c** 0.225 **d** 1.5

5 Cancel each fraction down to lowest terms.

a $\frac{4}{12}$ **b** $\frac{8}{10}$ **c** $\frac{10}{15}$ **d** $\frac{21}{28}$ **e** $\frac{16}{40}$

f $\frac{21}{45}$ **g** $\frac{24}{42}$ **h** $\frac{45}{54}$ **i** $\frac{36}{60}$ **j** $\frac{54}{72}$

6 Express each fraction as a decimal by rewriting it with denominator 10, 100 or 1000.

a $\frac{1}{2}$ **b** $\frac{1}{5}$ **c** $\frac{3}{5}$ **d** $\frac{3}{4}$

e $\frac{1}{25}$ **f** $\frac{7}{20}$ **g** $\frac{1}{8}$ **h** $\frac{5}{8}$

7 Express each terminating decimal as a fraction in lowest terms.

a 0.4 **b** 0.25 **c** 0.15 **d** 0.16

e 0.78 **f** 0.005 **g** 0.375 **h** 0.264

8 Express each fraction as a recurring decimal by dividing the numerator by the denominator.

a $\frac{1}{3}$ **b** $\frac{2}{3}$ **c** $\frac{1}{9}$ **d** $\frac{5}{9}$

e $\frac{3}{11}$ **f** $\frac{1}{11}$ **g** $\frac{1}{6}$ **h** $\frac{5}{6}$

9 Find the lowest common denominator, then simplify:

a $\frac{1}{2}+\frac{1}{4}$ **b** $\frac{3}{10}+\frac{2}{5}$ **c** $\frac{1}{2}+\frac{1}{3}$ **d** $\frac{2}{3}-\frac{2}{5}$

e $\frac{1}{6}+\frac{1}{9}$ **f** $\frac{5}{12}-\frac{3}{8}$ **g** $\frac{7}{10}+\frac{2}{15}$ **h** $\frac{2}{25}-\frac{1}{15}$

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10 Find the value of:

a $\frac{1}{4} \times 20$ **b** $\frac{2}{3} \times 12$ **c** $\frac{1}{2} \times \frac{1}{5}$ **d** $\frac{1}{3} \times \frac{3}{7}$ **e** $\frac{2}{5} \times \frac{5}{8}$

f $2 \div \frac{1}{3}$ **g** $\frac{3}{4} \div 3$ **h** $\frac{1}{3} \div \frac{1}{2}$ **i** $1\frac{1}{2} \div \frac{3}{8}$ **j** $\frac{5}{12} \div 1\frac{2}{3}$

11 Find the prime factorisations of:

a 24 **b** 60 **c** 72 **d** 126 **e** 104

f 135 **g** 189 **h** 294 **i** 315 **j** 605

DEVELOPMENT

12 **a** Find 12% of \$5. **b** Find 7.5% of 200 kg.

c Increase \$6000 by 30%. **d** Decrease $1\frac{1}{2}$ hours by 20%.

13 Express each fraction as a decimal.

a $\frac{33}{250}$ **b** $\frac{1}{40}$ **c** $\frac{5}{16}$ **d** $\frac{27}{80}$

e $\frac{7}{12}$ **f** $1\frac{9}{11}$ **g** $\frac{2}{15}$ **h** $\frac{13}{55}$

14 Express each fraction in lowest terms, without using a calculator.

a $\frac{588}{630}$ **b** $\frac{455}{1001}$ **c** $\frac{500}{1\,000\,000}$

15 **a** Steve's council rates increased by 5% this year to \$840. What were his council rates last year?

b Joanne received a 10% discount on a pair of shoes. If she paid \$144, what was the original price?

c Marko spent \$135 this year at the Easter Show, a 12.5% increase on last year. How much did he spend last year?

16 **a** Use your calculator to find the recurring decimals for $\frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \frac{4}{11}, \ldots, \frac{10}{11}$. Is there a pattern?

b Use your calculator to find the recurring decimals for $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}$ and $\frac{6}{7}$. Is there a pattern?

ENRICHMENT

17 (The numbers you obtain in this question may vary depending on the calculator used.)

a Use your calculator to express $\frac{1}{3}$ as a decimal by entering 1 ÷ 3.

b Subtract 0.33333333 from this, multiply the result by 10^8, and then take the reciprocal.

c Show arithmetically that the final answer in part **b** is 3. Is the answer on your calculator also equal to 3? What does this tell you about the way fractions are stored on a calculator?

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18 [These procedures can be implemented in any programming language or on a spreadsheet.]

a A number n is prime if it not divisible by any prime less than n. Explain why it is only necessary to test for divisibility by primes up to $\sqrt{n}$.

b If we are looking for all prime numbers less than 400, list the primes that we need to test divisibility by.

c Test whether 247, 241, 133, 367, 379 and 319 are prime.

d Using the list of primes up to 20, write a procedure that generates the list of primes up to 400. Use that list to generate the list of primes up to 160 000. Use that list to generate the list of primes up to 20 000 000. On your particular system, at what point do numeric errors begin to wreck the procedure?

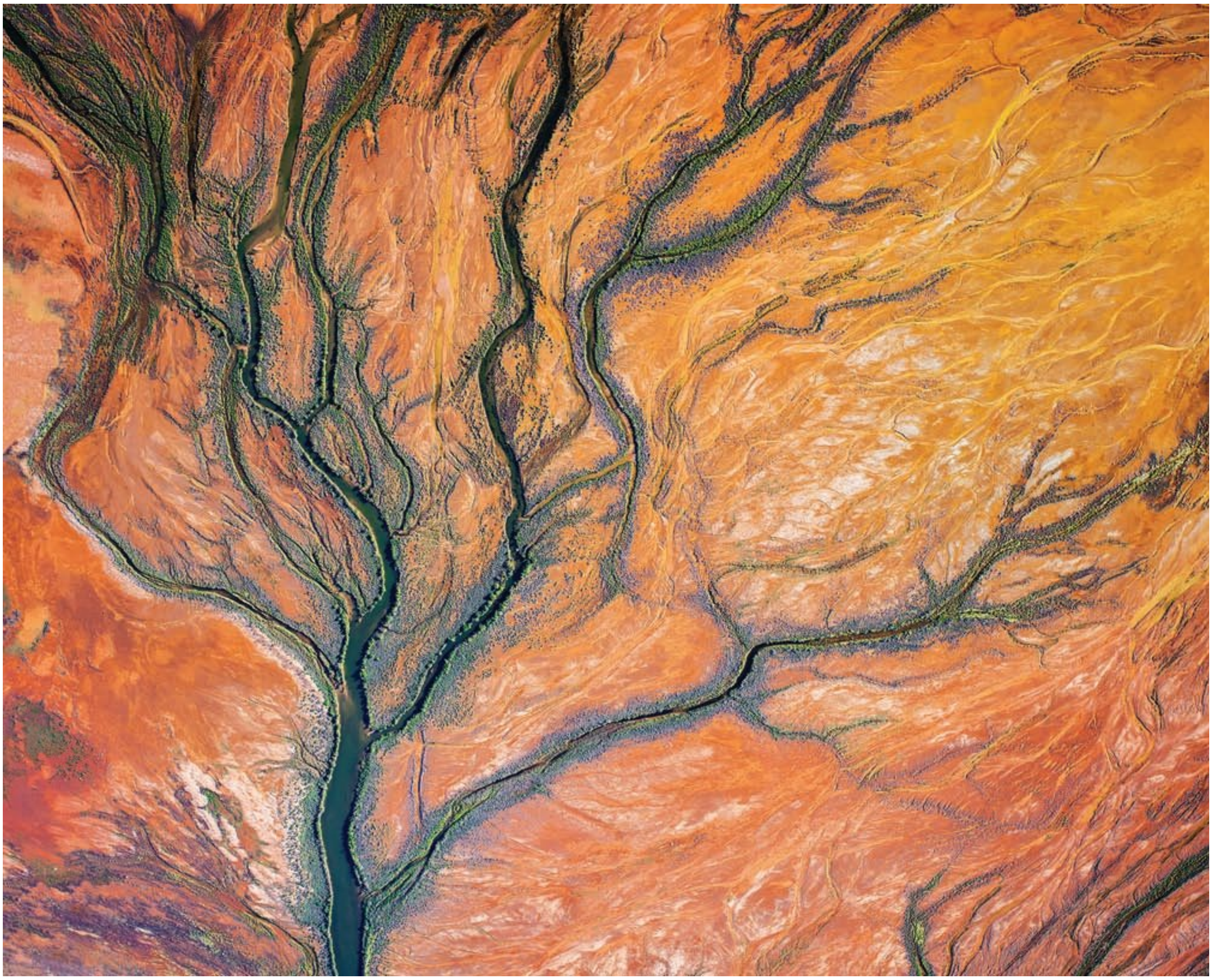
19 Two whole numbers are called *coprime* if their HCF is 1. The *Euler phi function* $\phi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n.

a Confirm the following by listing the whole numbers coprime to the given number:

i $\phi(9) = 6$ **ii** $\phi(25) = 20$ **iii** $\phi(32) = 16$ **iv** $\phi(45) = 24$

b It is known that $\phi(p^k) = p^k - p^{k-1}$ for a prime p and a positive integer k. Show that this is true for $p = 2$ and $k = 1, 2, 3, 4$.

c Prove that $\phi(3^k) = 2 \times 3^{k-1}$. Generalise this to a proof of the result for to $\phi(p^k)$, where p is prime and k is a positive integer.



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2B Real numbers and approximations

This section introduces the set $\mathbb{R}$ of real numbers, which are based not on counting, but on geometry — they are the points on the number line. They certainly include all rational numbers, but as we shall see, they also include many more numbers that cannot be written as fractions.

Dealing with real numbers that are not rational requires special symbols, such as $\sqrt{\ }$ and π, but when a real number needs to be approximated, a decimal is usually the best approach, written to as many decimal places as is necessary.

Decimals are used routinely in mathematics and science for two good reasons:
- Any two decimals can easily be compared with each other.
- Any quantity can be approximated 'as closely as we like' by a decimal.

Every measurement is only approximate, no matter how good the instrument, and rounding using decimals is a useful way of showing how accurate it is.

Rounding to a certain number of decimal places

The rules for rounding a decimal are:

5 RULES FOR ROUNDING A DECIMAL

To round a decimal, say to *two decimal places*, look at the *third* digit.
- If the *third* digit is 0, 1, 2, 3 or 4, leave the *second* digit alone.
- If the *third* digit is 5, 6, 7, 8 or 9, increase the *second* digit by 1.

Always use $\doteqdot$ rather than $=$ when a quantity has been rounded or approximated.

For example,

$3.8472 \doteqdot 3.85$, correct to two decimal places. (look at 7, the third digit)

$3.8472 \doteqdot 3.8$, correct to one decimal place. (look at 4, the second digit)

Scientific notation and rounding to a certain number of significant figures

The very large and very small numbers common in astronomy and atomic physics are easier to comprehend when they are written in *scientific notation*:

$1\,234\,000 = 1.234 \times 10^6$ (there are four significant figures)

$0.000065432 = 6.5432 \times 10^{-5}$ (there are five significant figures)

The digits in the first factor are called the *significant figures* of the number. It is often more sensible to round a quantity correct to a given number of significant figures rather than to a given number of decimal places.

To round, say to *three significant figures*, look at the *fourth* digit. If it is 5, 6, 7, 8 or 9, increase the *third* digit by 1. Otherwise, leave the *third* digit alone.

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$$3.0848 \times 10^9 \doteqdot 3.08 \times 10^9, \qquad \text{correct to three significant figures.}$$
$$2.789654 \times 10^{-29} \doteqdot 2.790 \times 10^{-29}, \qquad \text{correct to four significant figures.}$$

The number can be in normal notation and still be rounded this way:

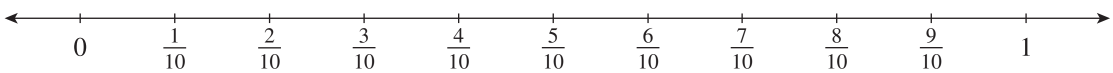
$31.203 \doteqdot 31.20$, correct to four significant figures.

Unfortunately, this may be ambiguous. For example, when we see a number such as 3200, we do not know whether it has been rounded to 2, 3 or 4 significant figures. That can only be conveyed by changing to scientific notation and writing

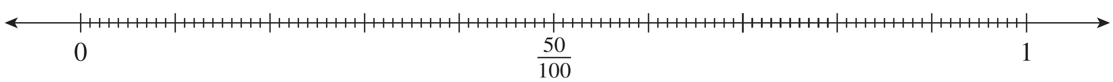
$$3.2 \times 10^3 \quad \text{or} \quad 3.20 \times 10^3 \quad \text{or} \quad 3.200 \times 10^3.$$

There are numbers that are not rational

At first glance, it would seem reasonable to believe that all the numbers on the number line are rational, because the rational numbers are clearly spread 'as finely as we like' along the whole number line. Between 0 and 1 there are 9 rational numbers with denominator 10:



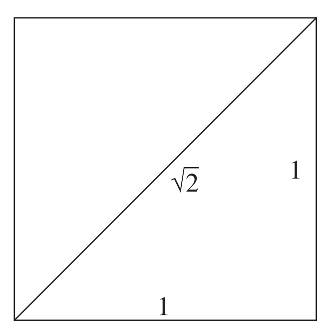
Between 0 and 1 there are 99 rational numbers with denominator 100:



Most points on the number line, however, represent numbers that cannot be written as fractions, and are called *irrational numbers*. Some of the most important numbers in this course are irrational, such as $\sqrt{2}$ and π, and the number e, which will be introduced in Chapter 11.

The square root of 2 is irrational

The number $\sqrt{2}$ is particularly important, because by Pythagoras' theorem, $\sqrt{2}$ is the diagonal of a unit square. Here is a proof by contradiction that $\sqrt{2}$ is an irrational number.



Suppose that $\sqrt{2}$ were a rational number.

Then $\sqrt{2}$ could be written as a fraction $\dfrac{a}{b}$ in lowest terms.

That is, $\sqrt{2} = \dfrac{a}{b}$, where a and b have no common factor,

and we know that $b > 1$ because $\sqrt{2}$ is not a whole number.

Squaring, $2 = \dfrac{a^2}{b^2}$, where $b^2 > 1$ because $b > 1$.

Because $\dfrac{a}{b}$ is in lowest terms, $\dfrac{a^2}{b^2}$ is also in lowest terms,

which is impossible, because $\dfrac{a^2}{b^2} = 2$, but $b^2 > 1$.

This is a contradiction, so $\sqrt{2}$ cannot be a rational number.

The Greek mathematicians were greatly troubled by the existence of irrational numbers. Their concerns can still be seen in modern English, where the word 'irrational' means both 'not a fraction' and 'not reasonable'.

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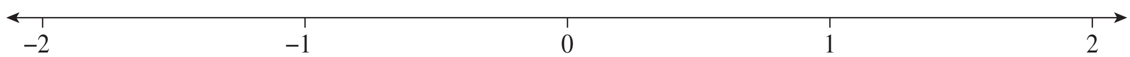
The real numbers and the number line

The whole numbers, the integers, and the rational numbers are based on *counting*. The existence of irrational numbers, however, means that this approach to arithmetic is inadequate, and a more general idea of number is needed. We have to turn away from counting and make use of *geometry*.

6 DEFINITION OF THE SET $\mathbb{R}$ OF REAL NUMBERS

- The *real numbers* $\mathbb{R}$ are defined to be all the points on the number line.
- All rational numbers are real, but real numbers such as $\sqrt{2}$ and π are irrational.



At this point, geometry replaces counting as the basis of arithmetic.

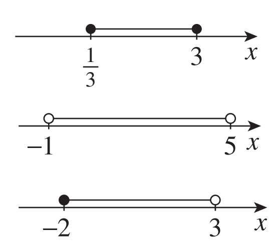
An irrational real number cannot be written as a fraction, or as a terminating or recurring decimal. In this course, such a number is usually specified in exact form, such as $x = \sqrt{2}$ or $x = \pi$, or as a decimal approximation correct to a certain number of significant figures, such as $x \doteqdot 1.4142$ or $x \doteqdot 3.1416$. Very occasionally a fractional approximation is useful or traditional, such as $\pi \doteqdot 3\frac{1}{7}$.

The real numbers are often referred to as the *continuum*, because the rationals, despite being dense, are scattered along the number line like specks of dust, but do not 'join up'. For example, the rational multiples of $\sqrt{2}$, which are all irrational, are just as dense on the number line as the rational numbers. It is only the real line itself that is completely joined up, to be the continuous line of geometry rather than falling apart into an infinitude of discrete points.

Open and closed intervals

Any connected part of the real number line is called an *interval*.

- An interval such as $\frac{1}{3} \le x \le 3$ is called a *closed interval* because it contains all its endpoints.
- An interval such as $-1 < x < 5$ is called an *open interval* because it does not contain any of its endpoints.
- An interval such as $-2 \le x < 3$ is neither open nor closed (the word *half-closed* is sometimes used).

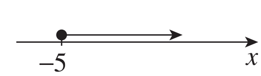


In diagrams, an endpoint is represented by a *closed circle* • if it is contained in the interval, and by an *open circle* ∘ if it is not contained in the interval.

Bounded and unbounded intervals

The three intervals above are *bounded* because they have two endpoints, which bound the interval. An *unbounded interval* in contrast is either open or closed, and the direction that continues towards ∞ or $-\infty$ is represented by an arrow.

- The unbounded interval $x \ge -5$ is a *closed interval* because it contains all its endpoints (it only has one).



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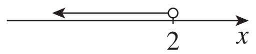
- The unbounded interval $x < 2$ is an *open interval* because it does not contain any of its endpoints (it only has one).
- The real line itself is an unbounded interval without any endpoints.



7 INTERVALS

- An *interval* is a connected part of the number line.
- A *closed interval* such as $\frac{1}{3} \le x \le 3$ contains all its endpoints.
- An *open interval* such as $-1 < x < 5$ does not contain any of its endpoints.
- An interval such as $-2 \le x < 3$ is neither open nor closed.
- A *bounded interval* has two endpoints, which bound the interval.
- An *unbounded interval* such as $x \ge -5$ continues towards ∞ or $-\infty$ (or both).

A single point is regarded as a *degenerate interval*, and is closed. The empty set is sometimes also regarded as a degenerate interval.
An alternative interval notation using round and square brackets will be introduced in Year 12.

Exercise 2B

FOUNDATION

1 Classify these real numbers as rational or irrational. Express those that are rational in the form $\frac{a}{b}$ in lowest terms, where a and b are integers.

a -3 **b** $1\frac{1}{2}$ **c** $\sqrt{3}$ **d** $\sqrt{4}$ **e** $\sqrt[3]{27}$

f $\sqrt[4]{8}$ **g** $\sqrt{\frac{4}{9}}$ **h** 0.45 **i** 12% **j** 0.333

k $0.\dot{3}$ **l** $3\frac{1}{7}$ **m** π **n** 3.14 **o** 0

2 Write each number correct to one decimal place.

a 0.32 **b** 5.68 **c** 12.75 **d** 0.05 **e** 3.03 **f** 9.96

3 Write each number correct to two significant figures.

a 0.429 **b** 5.429 **c** 5.029 **d** 0.0429 **e** 429 **f** 4290

4 Use a calculator to find each number correct to three decimal places.

a $\sqrt{10}$ **b** $\sqrt{47}$ **c** $\frac{9}{16}$

d $\frac{37}{48}$ **e** π **f** π^2

5 Use a calculator to find each number correct to three significant figures.

a $\sqrt{58}$ **b** $\sqrt[3]{133}$ **c** 62^2

d 14^5 **e** $\sqrt[4]{0.3}$ **f** 124^{-1}

6 To how many significant figures is each number accurate?

a 0.04 **b** 0.40 **c** 0.404

d 0.044 **e** 4.004 **f** 400

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DEVELOPMENT

7 **a** Classify each interval as open or closed or neither.

i $0 \le x \le 7$ **ii** $x > 5$ **iii** $x \le 7$ **iv** $5 < x \le 15$

v $x < -1$ **vi** $-4 < x < 10$ **vii** $x \ge 6$ **viii** $-4 \le x < -3$

b Classify each interval in part **a** as bounded or unbounded.

8 Write each interval in symbols, then sketch it on a separate number line.

a The real numbers greater than -2 and less than 5.

b The real numbers greater than or equal to -3 and less than or equal to 0.

c The real numbers less than 7.

d The real numbers less than or equal to -6.

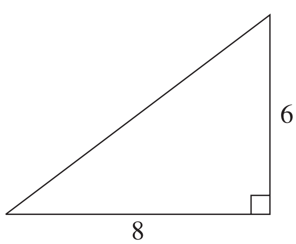
9 Use a calculator to evaluate each expression correct to three decimal places.

a $\dfrac{67 \times 29}{43}$ **b** $\dfrac{67 + 29}{43}$ **c** $\dfrac{67}{43 \times 29}$

d $\dfrac{67}{43 + 29}$ **e** $\dfrac{67 + 29}{43 + 71}$ **f** $\dfrac{67 + 71}{43 \times 29}$

10 Use Pythagoras' theorem to find the length of the unknown side in each triangle, and state whether it is rational or irrational.

a



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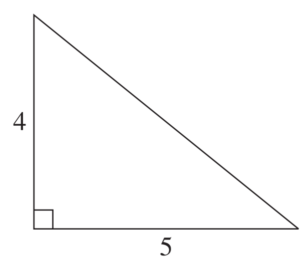

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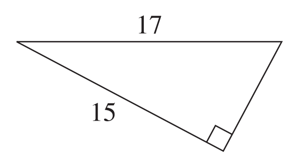

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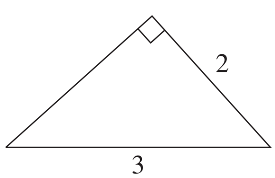
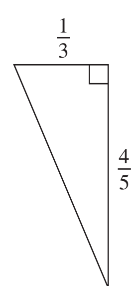
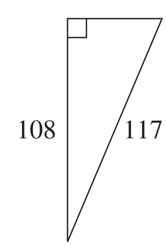


e



f



11 Calculate, correct to four significant figures:

a $10^{-0.4}$ **b** $\dfrac{1}{240 - 13 \times 17}$ **c** $\dfrac{\sqrt{6.5 + 8.3}}{2.7}$

d $\sqrt[3]{10.57 \times 12.83}$ **e** $\dfrac{3.5 \times 10^4}{2.3 \times 10^5}$ **f** $20\,000 \times (1.01)^{25}$

g $\dfrac{11.3}{\sqrt{19.5 - 14.7}}$ **h** $\dfrac{3\frac{2}{3} + 5\frac{1}{4}}{4\frac{1}{2} + 6\frac{4}{5}}$ **i** $\left(87.3 \times 10^4\right) \div \left(0.629 \times 10^{-8}\right)$

j $\dfrac{\sqrt{3} + \sqrt[3]{4}}{\sqrt[4]{5} + \sqrt[5]{6}}$ **k** $\dfrac{\left(\frac{2}{5}\right)^4 \times \left(\frac{3}{4}\right)^5}{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{3}\right)^3}$ **l** $\sqrt{\dfrac{36.41 - 19.57}{23.62 - 11.39}}$

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12 **a** Identify the approximation of π that seems to be used in 1 Kings 7 : 23, *'He made the Sea of cast metal, circular in shape, measuring ten cubits from rim to rim and five cubits high. It took a line of thirty cubits to measure around it.'*

b Many centuries later, in the 3rd century BC, Archimedes used regular polygons with 96 sides to prove that $\frac{223}{71} < \pi < \frac{22}{7}$. To how many significant figures is this correct?

c What is the current record for the computation of π?

d It is well known that a python has length about 3.14159 yards. How many pythons can be lined up between the wickets of a cricket pitch?

ENRICHMENT

Use a calculator to answer the next two questions. Write each answer in scientific notation.

13 The speed of light is approximately 2.997925×10^8 m/s.

a How many metres are there in a light-year (the distance that light travels in one year)? Assume that there are $365\frac{1}{4}$ days in a year and write your answer in metres, correct to three significant figures.

b The nearest large galaxy is Andromeda, which is estimated to be 2 200 000 light-years away. How far is that in metres, correct to two significant figures?

c The time since the Big Bang is estimated to be 13.6 billion years. How long is that in seconds, correct to three significant figures?

d How far has light travelled since the Big Bang? Give your answer in metres, correct to two significant figures.

14 The mass of a proton is 1.6726×10^{-27} kg and the mass of an electron is 9.1095×10^{-31} kg.

a Calculate, correct to four significant figures, the ratio of the mass of a proton to the mass of an electron.

b How many protons, correct to one significant figure, are there in 1 kg?

15 Prove that $\sqrt{3}$ is irrational. (Adapt the given proof that $\sqrt{2}$ is irrational.)

16 Suppose that a and b are positive irrational numbers, where $a < b$. Choose any positive integer n such that $\frac{1}{n} < b - a$, and let p be the greatest integer such that $\frac{p}{n} < a$.

a Prove that the rational number $\frac{p+1}{n}$ lies between a and b.

b If $a = \frac{1}{\sqrt{1001}}$ and $b = \frac{1}{\sqrt{1000}}$, find the least possible value of n and the corresponding value of p.

c Hence use part **a** to construct a rational number between $\frac{1}{\sqrt{1001}}$ and $\frac{1}{\sqrt{1000}}$.

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2C Surds and their arithmetic

Numbers such as $\sqrt{2}$ and $\sqrt{3}$ occur constantly in this course because they occur in the solutions of quadratic equations and when using Pythagoras' theorem. The last three sections of this chapter review various methods of dealing with them.

Square roots and positive square roots

The square of any real number is positive, except that $0^2 = 0$. Hence a negative number cannot have a square root, and the only square root of 0 is 0 itself.

A positive number, however, has two square roots, which are the opposites of each other. For example, the square roots of 9 are 3 and -3.

Note that the well-known symbol $\sqrt{x}$ does not mean 'the square root of x'. It is defined to mean the *positive* square root of x (or zero, if $x = 0$).

8 DEFINITION OF THE SYMBOL $\sqrt{x}$

- For $x > 0$, $\sqrt{x}$ means the *positive* square root of x.
- For $x = 0$, $\sqrt{0} = 0$.
- For $x < 0$, $\sqrt{x}$ is not defined.

For example, $\sqrt{25} = 5$, even though 25 has two square roots, -5 and 5. The symbol for the negative square root of 25 is $-\sqrt{25}$.

Cube roots

Cube roots are less complicated. Every number has exactly one cube root, so the symbol $\sqrt[3]{x}$ simply means 'the cube root of x'. For example,

$$\sqrt[3]{8} = 2 \qquad \text{and} \qquad \sqrt[3]{-8} = -2 \qquad \text{and} \qquad \sqrt[3]{0} = 0.$$

What is a surd?

The word *surd* is often used to refer to any expression involving a square or higher root. More precisely, however, surds do not include expressions such as $\sqrt{\frac{4}{9}}$ and $\sqrt[3]{8}$, which can be simplified to rational numbers.

9 SURDS

An expression $\sqrt[n]{x}$, where x is a rational number and $n \geq 2$ is an integer, is called a *surd* if it is not itself a rational number.

The word 'surd' is related to 'absurd' — surds are irrational.

It was proven in the last section that $\sqrt{2}$ was irrational, and in the same way, most roots of rational numbers are irrational. Here is the precise result for square roots, which won't be proven formally, and applies similarly to higher roots:

> 'If a and b are positive integers with no common factor, then $\sqrt{\dfrac{a}{b}}$ is rational if and only if both a and b are squares of integers.'

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Simplifying expressions involving surds

Here are some laws from earlier years for simplifying expressions involving square roots. The first pair restate the definition of the square root, and the second pair are easily proven by squaring.

10 LAWS CONCERNING SURDS

Let $a \geq 0$ and $b \geq 0$ be real numbers. Then:

$$\sqrt{a^2} = a \qquad\qquad \sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

and

$$(\sqrt{a})^2 = a \qquad\qquad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}, \text{ provided that } b \neq 0.$$

Taking out square divisors

A surd such as $\sqrt{500}$ is not regarded as being simplified, because 500 is divisible by the square number 100, so $\sqrt{500}$ can be written as $10\sqrt{5}$:

$$\sqrt{500} = \sqrt{100 \times 5} = \sqrt{100} \times \sqrt{5} = 10\sqrt{5}.$$

11 SIMPLIFYING A SURD

- Check the number inside the square root for divisibility by one of the squares
  4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, . . .
- Continue until the number inside the square root sign has no more square divisors (apart from 1).

Example 2 — 2C

Simplify these expressions involving surds.

a $\sqrt{108}$ **b** $5\sqrt{27}$ **c** $\sqrt{216}$

SOLUTION

a
$$\begin{aligned}\sqrt{108} &= \sqrt{36 \times 3}\\ &= \sqrt{36} \times \sqrt{3}\\ &= 6\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}5\sqrt{27} &= 5\sqrt{9 \times 3}\\ &= 5 \times \sqrt{9} \times \sqrt{3}\\ &= 15\sqrt{3}\end{aligned}$$

c
$$\begin{aligned}\sqrt{216} &= \sqrt{4} \times \sqrt{54}\\ &= \sqrt{4} \times \sqrt{9} \times \sqrt{6}\\ &= 6\sqrt{6}\end{aligned}$$

Example 3 — 2C

Simplify the surds in these expressions, then collect like terms.

a $\sqrt{44} + \sqrt{99}$ **b** $\sqrt{72} - \sqrt{50} + \sqrt{12}$

SOLUTION

a
$$\begin{aligned}\sqrt{44} + \sqrt{99} &= 2\sqrt{11} + 3\sqrt{11}\\ &= 5\sqrt{11}\end{aligned}$$

b
$$\begin{aligned}&\sqrt{72} - \sqrt{50} + \sqrt{12}\\ &= 6\sqrt{2} - 5\sqrt{2} + 2\sqrt{3}\\ &= \sqrt{2} + 2\sqrt{3}\end{aligned}$$

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Exercise 2C

FOUNDATION

1 Write down the value of:

a $\sqrt{16}$ **b** $\sqrt{36}$ **c** $\sqrt{81}$ **d** $\sqrt{121}$
e $\sqrt{144}$ **f** $\sqrt{400}$ **g** $\sqrt{2500}$ **h** $\sqrt{10000}$

2 Simplify:

a $\sqrt{12}$ **b** $\sqrt{18}$ **c** $\sqrt{20}$ **d** $\sqrt{27}$
e $\sqrt{28}$ **f** $\sqrt{40}$ **g** $\sqrt{32}$ **h** $\sqrt{99}$
i $\sqrt{54}$ **j** $\sqrt{200}$ **k** $\sqrt{60}$ **l** $\sqrt{75}$
m $\sqrt{80}$ **n** $\sqrt{98}$ **o** $\sqrt{800}$ **p** $\sqrt{1000}$

3 Simplify:

a $\sqrt{3} + \sqrt{3}$ **b** $5\sqrt{7} - 3\sqrt{7}$ **c** $2\sqrt{5} - \sqrt{5}$ **d** $-3\sqrt{2} + \sqrt{2}$
e $4\sqrt{3} + 3\sqrt{2} - 2\sqrt{3}$ **f** $-5\sqrt{5} - 2\sqrt{7} + 6\sqrt{5}$
g $7\sqrt{6} + 5\sqrt{3} - 4\sqrt{6} - 7\sqrt{3}$ **h** $-6\sqrt{2} - 4\sqrt{5} + 3\sqrt{2} - 2\sqrt{5}$
i $3\sqrt{10} - 8\sqrt{5} - 7\sqrt{10} + 10\sqrt{5}$

4 Simplify:

a $3\sqrt{8}$ **b** $5\sqrt{12}$ **c** $2\sqrt{24}$ **d** $4\sqrt{44}$
e $3\sqrt{45}$ **f** $6\sqrt{52}$ **g** $2\sqrt{300}$ **h** $2\sqrt{96}$

5 Write each expression as a single square root. For example, $3\sqrt{2} = \sqrt{9} \times \sqrt{2} = \sqrt{18}$.

a $2\sqrt{5}$ **b** $5\sqrt{2}$ **c** $8\sqrt{2}$ **d** $6\sqrt{3}$
e $5\sqrt{5}$ **f** $4\sqrt{7}$ **g** $2\sqrt{17}$ **h** $7\sqrt{10}$

DEVELOPMENT

6 Simplify:

a $\sqrt{8} + \sqrt{2}$ **b** $\sqrt{12} - \sqrt{3}$ **c** $\sqrt{50} - \sqrt{18}$
d $\sqrt{54} + \sqrt{24}$ **e** $\sqrt{45} - \sqrt{20}$ **f** $\sqrt{90} - \sqrt{40} + \sqrt{10}$
g $\sqrt{27} + \sqrt{75} - \sqrt{48}$ **h** $\sqrt{45} + \sqrt{80} - \sqrt{125}$ **i** $\sqrt{2} + \sqrt{32} + \sqrt{72}$
j $\sqrt[3]{125}$ **k** $\sqrt[4]{81}$ **l** $\sqrt[5]{32}$

7 Simplify fully:

a $\sqrt{600} + \sqrt{300} - \sqrt{216}$ **b** $4\sqrt{18} + 3\sqrt{12} - 2\sqrt{50}$ **c** $2\sqrt{175} - 5\sqrt{140} - 3\sqrt{28}$

8 Find the value of x if:

a $\sqrt{63} - \sqrt{28} = \sqrt{x}$ **b** $\sqrt{80} - \sqrt{20} = \sqrt{x}$
c $2\sqrt{150} - 3\sqrt{24} = \sqrt{x}$ **d** $\sqrt{150} + \sqrt{54} - \sqrt{216} = \sqrt{x}$

ENRICHMENT

9 **a** Without using a calculator, show that $\sqrt{7} < 3$.
b Without using a calculator, show that $\sqrt{7} + \sqrt[3]{7} + \sqrt[4]{7} < 7$.

10 Explain why, if we only used rational numbers when drawing graphs in the coordinate plane, the graph of $y = x^2 - 2$ would lie above and below the x-axis, but never intersect it.

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2D Further simplification of surds

This section deals with the simplification of more complicated surdic expressions. The usual rules of algebra, together with the methods of simplifying surds given in the last section, are all that is needed.

Simplifying products of surds

The product of two surds is found using the identity $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

It is important to check whether the answer needs further simplification.

Example 4 — 2D

Simplify each product.

a $\sqrt{15} \times \sqrt{5}$

b $5\sqrt{6} \times 7\sqrt{10}$

SOLUTION

a
$$\begin{aligned}\sqrt{15} \times \sqrt{5} &= \sqrt{75}\\ &= \sqrt{25 \times 3}\\ &= 5\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}5\sqrt{6} \times 7\sqrt{10} &= 35\sqrt{60}\\ &= 35\sqrt{4 \times 15}\\ &= 35 \times 2\sqrt{15}\\ &= 70\sqrt{15}\end{aligned}$$

Using binomial expansions

All the usual algebraic methods of expanding binomial products can be applied to surdic expressions.

Example 5 — 2D

Expand these products and then simplify them.

a $\left(\sqrt{15} + 2\right)\left(\sqrt{3} - 3\right)$

b $\left(\sqrt{15} - \sqrt{6}\right)^2$

SOLUTION

a
$$\begin{aligned}\left(\sqrt{15} + 2\right)\left(\sqrt{3} - 3\right) &= \sqrt{15}\left(\sqrt{3} - 3\right) + 2\left(\sqrt{3} - 3\right)\\ &= \sqrt{45} - 3\sqrt{15} + 2\sqrt{3} - 6\\ &= 3\sqrt{5} - 3\sqrt{15} + 2\sqrt{3} - 6\end{aligned}$$

b
$$\begin{aligned}\left(\sqrt{15} - \sqrt{6}\right)^2 &= 15 - 2\sqrt{90} + 6 \quad \left(\text{using the identity } (A - B)^2 = A^2 - 2AB + B^2\right)\\ &= 21 - 2 \times 3\sqrt{10}\\ &= 21 - 6\sqrt{10}\end{aligned}$$

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Exercise 2D

FOUNDATION

1 Simplify:

a $\left(\sqrt{3}\right)^2$ b $\sqrt{2} \times \sqrt{3}$ c $\sqrt{7} \times \sqrt{7}$ d $\sqrt{6} \times \sqrt{5}$

e $2 \times 3\sqrt{2}$ f $2\sqrt{5} \times 5$ g $2\sqrt{3} \times 3\sqrt{5}$ h $6\sqrt{2} \times 5\sqrt{7}$

i $\left(2\sqrt{3}\right)^2$ j $\left(3\sqrt{7}\right)^2$ k $5\sqrt{2} \times 3\sqrt{2}$ l $6\sqrt{10} \times 4\sqrt{10}$

2 Simplify:

a $\sqrt{15} \div \sqrt{3}$ b $\sqrt{42} \div \sqrt{6}$ c $3\sqrt{5} \div 3$ d $2\sqrt{7} \div \sqrt{7}$

e $3\sqrt{10} \div \sqrt{5}$ f $6\sqrt{33} \div 6\sqrt{11}$ g $10\sqrt{14} \div 5\sqrt{2}$ h $15\sqrt{35} \div 3\sqrt{7}$

3 Expand:

a $\sqrt{5}\left(\sqrt{5} + 1\right)$ b $\sqrt{2}\left(\sqrt{3} - 1\right)$ c $\sqrt{3}\left(2 - \sqrt{3}\right)$

d $2\sqrt{2}\left(\sqrt{5} - \sqrt{2}\right)$ e $\sqrt{7}\left(7 - 2\sqrt{7}\right)$ f $\sqrt{6}\left(3\sqrt{6} - 2\sqrt{5}\right)$

4 Simplify fully:

a $\sqrt{6} \times \sqrt{2}$ b $\sqrt{5} \times \sqrt{10}$ c $\sqrt{3} \times \sqrt{15}$

d $\sqrt{2} \times 2\sqrt{22}$ e $4\sqrt{12} \times \sqrt{3}$ f $3\sqrt{8} \times 2\sqrt{5}$

5 Expand and simplify:

a $\sqrt{2}\left(\sqrt{10} - \sqrt{2}\right)$ b $\sqrt{6}\left(3 + \sqrt{3}\right)$ c $\sqrt{5}\left(\sqrt{15} + 4\right)$

d $\sqrt{6}\left(\sqrt{8} - 2\right)$ e $3\sqrt{3}\left(9 - \sqrt{21}\right)$ f $3\sqrt{7}\left(\sqrt{14} - 2\sqrt{7}\right)$

DEVELOPMENT

6 Expand and simplify:

a $\left(\sqrt{3} + 1\right)\left(\sqrt{2} - 1\right)$ b $\left(\sqrt{5} - 2\right)\left(\sqrt{7} + 3\right)$ c $\left(\sqrt{5} + \sqrt{2}\right)\left(\sqrt{3} + \sqrt{2}\right)$

d $\left(\sqrt{6} - 1\right)\left(\sqrt{6} - 2\right)$ e $\left(\sqrt{7} - 2\right)\left(2\sqrt{7} + 5\right)$ f $\left(3\sqrt{2} - 1\right)\left(\sqrt{6} - \sqrt{3}\right)$

7 Use the special expansion $(a + b)(a - b) = a^2 - b^2$ to expand and simplify:

a $\left(\sqrt{5} + 1\right)\left(\sqrt{5} - 1\right)$ b $\left(3 - \sqrt{7}\right)\left(3 + \sqrt{7}\right)$ c $\left(\sqrt{3} + \sqrt{2}\right)\left(\sqrt{3} - \sqrt{2}\right)$

d $\left(3\sqrt{2} - \sqrt{11}\right)\left(3\sqrt{2} + \sqrt{11}\right)$ e $\left(2\sqrt{6} + 3\right)\left(2\sqrt{6} - 3\right)$ f $\left(7 - 2\sqrt{5}\right)\left(7 + 2\sqrt{5}\right)$

8 Expand and simplify each expression, using the special expansions

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \text{and} \quad (a - b)^2 = a^2 - 2ab + b^2.$$

a $\left(\sqrt{3} + 1\right)^2$ b $\left(\sqrt{5} - 1\right)^2$ c $\left(\sqrt{3} + \sqrt{2}\right)^2$

d $\left(\sqrt{7} - \sqrt{5}\right)^2$ e $\left(2\sqrt{3} - 1\right)^2$ f $\left(2\sqrt{5} + 3\right)^2$

g $2\left(\sqrt{7} + \sqrt{5}\right)^2$ h $\left(3\sqrt{2} - 2\sqrt{3}\right)^2$ i $\left(3\sqrt{5} + \sqrt{10}\right)^2$

9 Simplify fully:

a $\dfrac{\sqrt{40}}{\sqrt{10}}$ b $\dfrac{\sqrt{18}}{\sqrt{50}}$ c $\dfrac{2\sqrt{6} \times \sqrt{5}}{\sqrt{10}}$

d $\dfrac{5\sqrt{7} \times \sqrt{3}}{\sqrt{28}}$ e $\dfrac{\sqrt{15} \times \sqrt{20}}{\sqrt{12}}$ f $\dfrac{6\sqrt{3} \times 8\sqrt{2}}{\sqrt{32} \times \sqrt{27}}$

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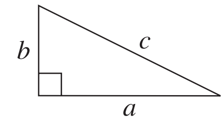
10 Use Pythagoras' theorem to find the hypotenuse c of the right-angled triangle in which the lengths a and b of the other two sides are:



a $\sqrt{2}$ and $\sqrt{7}$ **b** $\sqrt{5}$ and $2\sqrt{5}$

c $\sqrt{7} + 1$ and $\sqrt{7} - 1$ **d** $2\sqrt{3} - \sqrt{6}$ and $2\sqrt{3} + \sqrt{6}$

11 Simplify by forming the lowest common denominator:

a $\dfrac{1}{\sqrt{3}+1} + \dfrac{1}{\sqrt{3}-1}$ **b** $\dfrac{3}{2\sqrt{5}-\sqrt{7}} - \dfrac{3}{2\sqrt{5}+\sqrt{7}}$

12 Given that x and y are positive, simplify:

a $\sqrt{x^2y^3}$ **b** $x\sqrt{x^2y^6}$ **c** $\sqrt{x^2 + 6x + 9}$

d $\sqrt{x^3 + 2x^2 + x}$ **e** $\sqrt{x^2y^4(x^2 + 2x + 1)}$ **f** $\sqrt{x^4 + 2x^3 + x^2}$

13 **a** Find a pair of values a and b for which $\sqrt{a^2 + b^2} \neq a + b$.

b For what values of a and b is it true that $\sqrt{a^2 + b^2} = a + b$?

ENRICHMENT

14 Determine, without using a calculator, which is the greater number in each pair:

a $2\sqrt{3}$ or $\sqrt{11}$ **b** $7\sqrt{2}$ or $3\sqrt{11}$

c $3 + 2\sqrt{2}$ or $15 - 7\sqrt{2}$ **d** $2\sqrt{6} - 3$ or $7 - 2\sqrt{6}$

15 **a** Write down the expansion of $(a + b)^2$.

b Use the expansion in part **a** to square $\sqrt{6 + \sqrt{11}} - \sqrt{6 - \sqrt{11}}$.

c Hence simplify $\sqrt{6 + \sqrt{11}} - \sqrt{6 - \sqrt{11}}$.

16 Given that $x - y = 8\sqrt{2}$ and $xy = 137$, where x and y are both positive, find the value of $x + y$ without finding x or y.

17 Suppose that $a = 1 + \sqrt{2}$.

a Show that $a^2 - 2a - 1 = 0$.

b Hence show that $a = 2 + \dfrac{1}{a}$.

c Show that $a\sqrt{2} = a + 1$, and hence that $\sqrt{2} = 1 + \dfrac{1}{a}$.

d Deduce that $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$

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2E Rationalising the denominator

When dealing with surdic expressions, it is usual to remove any surds from the denominator, a process called *rationalising the denominator*. There are two cases.

The denominator has a single term

In the first case, the denominator is a surd or a multiple of a surd.

12 RATIONALISING A SINGLE-TERM DENOMINATOR

In an expression such as $\dfrac{\sqrt{7}}{2\sqrt{3}}$, multiply top and bottom by $\sqrt{3}$.

Example 6 2E

Simplify each expression by rationalising the denominator.

a $\dfrac{\sqrt{7}}{2\sqrt{3}}$

b $\dfrac{55}{\sqrt{11}}$

SOLUTION

a
$$\frac{\sqrt{7}}{2\sqrt{3}} = \frac{\sqrt{7}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{21}}{2 \times 3}$$
$$= \frac{\sqrt{21}}{6}$$

b
$$\frac{55}{\sqrt{11}} = \frac{55}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}}$$
$$= \frac{55\sqrt{11}}{11}$$
$$= 5\sqrt{11}$$

The denominator has two terms

The second case involves a denominator with two terms, one or both of which contain a surd. The method uses the difference of squares identity

$$(A + B)(A - B) = A^2 - B^2$$

to square the unwanted surds and convert them to integers.

13 RATIONALISING A BINOMIAL DENOMINATOR

- In an expression such as $\dfrac{3}{5 + \sqrt{3}}$, multiply top and bottom by $5 - \sqrt{3}$.
- Then use the difference of squares.

Example 7 2E

Rationalise the denominator in each expression.

a $\dfrac{3}{5 + \sqrt{3}}$

b $\dfrac{1}{2\sqrt{3} - 3\sqrt{2}}$

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SOLUTION

a $\dfrac{3}{5+\sqrt{3}} = \dfrac{3}{5+\sqrt{3}} \times \dfrac{5-\sqrt{3}}{5-\sqrt{3}}$

$= \dfrac{15-3\sqrt{3}}{25-3}$

$= \dfrac{15-3\sqrt{3}}{22}$

Using the difference of squares,

$(5+\sqrt{3})(5-\sqrt{3}) = 5^2 - (\sqrt{3})^2$

$= 25 - 3.$

b $\dfrac{1}{2\sqrt{3}-3\sqrt{2}} = \dfrac{1}{2\sqrt{3}-3\sqrt{2}} \times \dfrac{2\sqrt{3}+3\sqrt{2}}{2\sqrt{3}+3\sqrt{2}}$

$= \dfrac{2\sqrt{3}+3\sqrt{2}}{4\times 3 - 9\times 2}$

$= -\dfrac{2\sqrt{3}+3\sqrt{2}}{6}$

Using the difference of squares,

$(2\sqrt{3}-3\sqrt{2})(2\sqrt{3}+3\sqrt{2}) = (2\sqrt{3})^2 - (3\sqrt{2})^2$

$= 4\times 3 - 9\times 2.$

Exercise 2E

FOUNDATION

1 Rewrite each fraction with a rational denominator.

a $\dfrac{1}{\sqrt{3}}$ **b** $\dfrac{1}{\sqrt{7}}$ **c** $\dfrac{3}{\sqrt{5}}$ **d** $\dfrac{5}{\sqrt{2}}$

e $\dfrac{\sqrt{2}}{\sqrt{3}}$ **f** $\dfrac{\sqrt{5}}{\sqrt{7}}$ **g** $\dfrac{2\sqrt{11}}{\sqrt{5}}$ **h** $\dfrac{3\sqrt{7}}{\sqrt{2}}$

2 Simplify each expression by rationalising the denominator.

a $\dfrac{2}{\sqrt{2}}$ **b** $\dfrac{5}{\sqrt{5}}$ **c** $\dfrac{6}{\sqrt{3}}$ **d** $\dfrac{21}{\sqrt{7}}$

e $\dfrac{3}{\sqrt{6}}$ **f** $\dfrac{5}{\sqrt{15}}$ **g** $\dfrac{8}{\sqrt{6}}$ **h** $\dfrac{14}{\sqrt{10}}$

3 Rewrite each fraction with a rational denominator.

a $\dfrac{1}{2\sqrt{5}}$ **b** $\dfrac{1}{3\sqrt{7}}$ **c** $\dfrac{3}{5\sqrt{2}}$ **d** $\dfrac{2}{7\sqrt{3}}$

e $\dfrac{10}{3\sqrt{2}}$ **f** $\dfrac{9}{4\sqrt{3}}$ **g** $\dfrac{\sqrt{3}}{2\sqrt{10}}$ **h** $\dfrac{2\sqrt{11}}{5\sqrt{7}}$

4 Rewrite each fraction with a rational denominator.

a $\dfrac{1}{\sqrt{3}-1}$ **b** $\dfrac{1}{\sqrt{7}+2}$ **c** $\dfrac{1}{3+\sqrt{5}}$ **d** $\dfrac{1}{4-\sqrt{7}}$

e $\dfrac{1}{\sqrt{5}-\sqrt{2}}$ **f** $\dfrac{1}{\sqrt{10}+\sqrt{6}}$ **g** $\dfrac{1}{2\sqrt{3}+1}$ **h** $\dfrac{1}{5-3\sqrt{2}}$

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DEVELOPMENT

5 Rewrite each fraction with a rational denominator.

a $\dfrac{3}{\sqrt{5} + 1}$ b $\dfrac{4}{2\sqrt{2} - \sqrt{3}}$ c $\dfrac{\sqrt{7}}{5 - \sqrt{7}}$ d $\dfrac{3\sqrt{3}}{\sqrt{5} + \sqrt{3}}$

e $\dfrac{2\sqrt{7}}{2\sqrt{7} - 5}$ f $\dfrac{\sqrt{5}}{\sqrt{10} - \sqrt{5}}$ g $\dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$ h $\dfrac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}}$

i $\dfrac{3 - \sqrt{7}}{3 + \sqrt{7}}$ j $\dfrac{3\sqrt{2} + \sqrt{5}}{3\sqrt{2} - \sqrt{5}}$ k $\dfrac{\sqrt{10} - \sqrt{6}}{\sqrt{10} + \sqrt{6}}$ l $\dfrac{7 + 2\sqrt{11}}{7 - 2\sqrt{11}}$

6 Simplify each expression by rationalising the denominator.

a $\dfrac{\sqrt{3} - 1}{2 - \sqrt{3}}$ b $\dfrac{2\sqrt{5} - \sqrt{2}}{\sqrt{5} + \sqrt{2}}$

7 Show that each expression is rational by first rationalising the denominators.

a $\dfrac{3}{\sqrt{2}} + \dfrac{3}{2 + \sqrt{2}}$ b $\dfrac{1}{3 + \sqrt{6}} + \dfrac{2}{\sqrt{6}}$

c $\dfrac{4}{2 + \sqrt{2}} + \dfrac{1}{3 - 2\sqrt{2}}$ d $\dfrac{8}{3 - \sqrt{7}} - \dfrac{6}{2\sqrt{7} - 5}$

8 If $x = \dfrac{\sqrt{5} + 1}{2}$, show that $1 + \dfrac{1}{x} = x$.

9 The fraction $\dfrac{\sqrt{6} + 1}{\sqrt{3} + \sqrt{2}}$ can be written in the form $a\sqrt{3} + b\sqrt{2}$. Find a and b.

10 Evaluate $a + \dfrac{1}{a}$ for these values of a:

a $1 + \sqrt{2}$ b $2 - \sqrt{3}$ c $\dfrac{3 - \sqrt{3}}{3 + \sqrt{3}}$ d $\dfrac{\sqrt{x} + \sqrt{2 - x}}{\sqrt{x} - \sqrt{2 - x}}$

11 Rationalise the denominator of $\dfrac{1}{\sqrt{x + h} + \sqrt{x}}$, where $x \geq 0$ and $x + h \geq 0$.

12 a Expand $\left(x + \dfrac{1}{x}\right)^2$.

b Suppose that $x = \sqrt{7} + \sqrt{6}$.

i Show that $x + \dfrac{1}{x} = 2\sqrt{7}$.

ii Use the expansion in part **a** to find the value of $x^2 + \dfrac{1}{x^2}$.

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ENRICHMENT

13 The value of $\sqrt{17}$ is 4.12, correct to two decimal places.

a Substitute this value to determine an approximation for $\dfrac{1}{\sqrt{17} - 4}$.

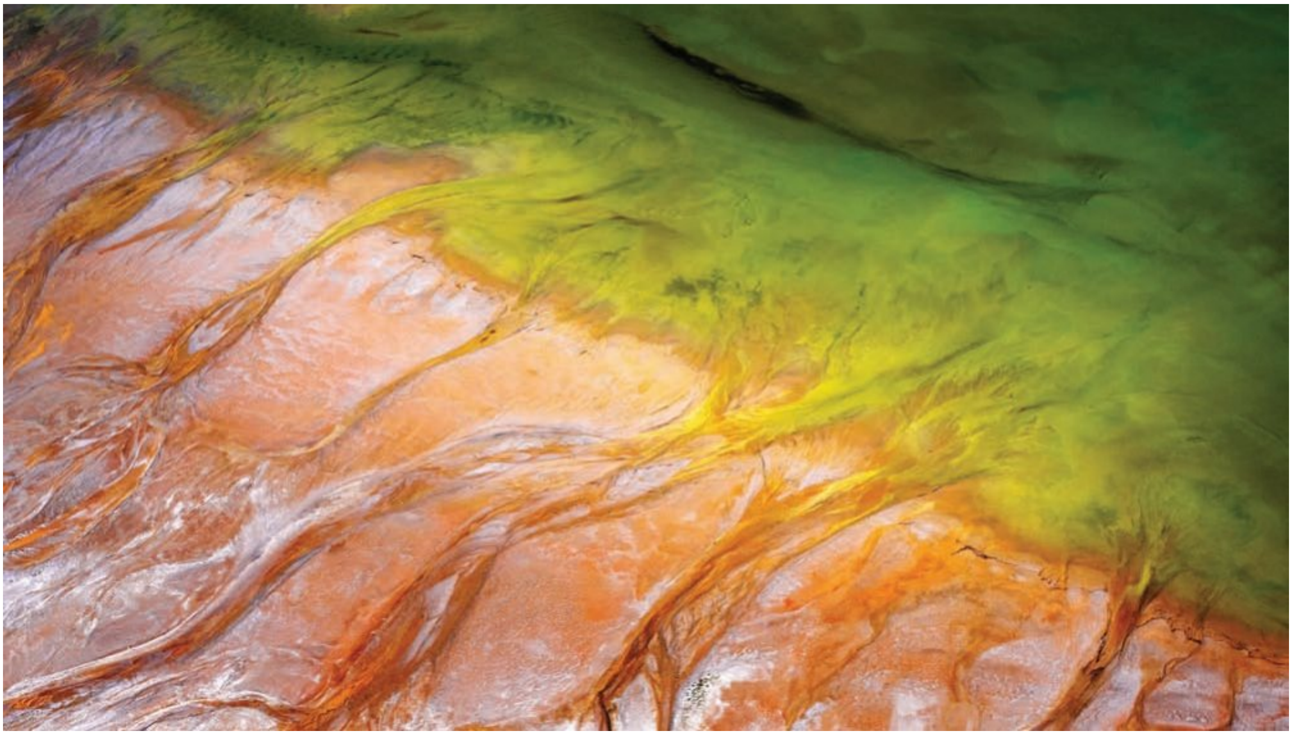
b Show that $\dfrac{1}{\sqrt{17} - 4} = \sqrt{17} + 4$, and that this last result gives a more accurate value for the approximation than that found in part **a**.

14 Express $\dfrac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$ with a rational denominator.

15 Suppose that a, b, c and d are positive integers and c is not a square.

a Given that $\dfrac{a}{b + \sqrt{c}} + \dfrac{d}{\sqrt{c}}$ is rational, prove that $b^2 d = c(a + d)$.

b Use part **a** to show that $\dfrac{a}{1 + \sqrt{c}} + \dfrac{d}{\sqrt{c}}$ is not rational.



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Chapter 2 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 2 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Classify each of these real numbers as rational or irrational. Express those that are rational in the form $\frac{a}{b}$, where a and b are integers.

a 7 **b** $-2\frac{1}{4}$ **c** $\sqrt{9}$ **d** $\sqrt{10}$

e $\sqrt[3]{15}$ **f** $\sqrt[4]{16}$ **g** -0.16 **h** π

2 Use a calculator to write each number correct to:

i two decimal places, **ii** two significant figures.

a $\sqrt{17}$ **b** $\sqrt[3]{102}$ **c** 1.16^7

d $\frac{49}{64}$ **e** 7.3^{-2} **f** $\pi^{5.5}$

3 Evaluate, correct to three significant figures:

a $\frac{7.93}{8.22 - 3.48}$ **b** $-4.9 \times (-5.8 - 8.5)$ **c** $\sqrt[4]{1.6 \times 2.6}$

d $\frac{13^5}{11^6 + 17^4}$ **e** $\frac{\frac{4}{9} - \frac{2}{7}}{\frac{5}{8} - \frac{3}{10}}$ **f** $\sqrt{2.4^{-1.6}}$

g $\sqrt{\frac{1.347}{2.518 - 1.679}}$ **h** $\frac{2.7 \times 10^{-2}}{1.7 \times 10^{-5}}$ **i** $\frac{\sqrt{\frac{1}{2}} + \sqrt[3]{\frac{1}{3}}}{\sqrt[4]{\frac{1}{4}} + \sqrt[5]{\frac{1}{5}}}$

4 Simplify:

a $\sqrt{24}$ **b** $\sqrt{45}$ **c** $\sqrt{50}$

d $\sqrt{500}$ **e** $3\sqrt{18}$ **f** $2\sqrt{40}$

5 Simplify:

a $\sqrt{5} + \sqrt{5}$ **b** $\sqrt{5} \times \sqrt{5}$ **c** $\left(2\sqrt{7}\right)^2$

d $2\sqrt{5} + \sqrt{7} - 3\sqrt{5}$ **e** $\sqrt{35} \div \sqrt{5}$ **f** $6\sqrt{55} \div 2\sqrt{11}$

g $\sqrt{8} \times \sqrt{2}$ **h** $\sqrt{10} \times \sqrt{2}$ **i** $2\sqrt{6} \times 4\sqrt{15}$

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Review

6 Simplify:

a $\sqrt{27} - \sqrt{12}$

b $\sqrt{18} + \sqrt{32}$

c $3\sqrt{2} + 3\sqrt{8} - \sqrt{50}$

d $\sqrt{54} - \sqrt{20} + \sqrt{150} - \sqrt{80}$

7 Expand:

a $\sqrt{7}(3 - \sqrt{7})$

b $\sqrt{5}(2\sqrt{6} + 3\sqrt{2})$

c $\sqrt{15}(\sqrt{3} - 5)$

d $\sqrt{3}(\sqrt{6} + 2\sqrt{3})$

8 Expand and simplify:

a $(\sqrt{5} + 2)(3 - \sqrt{5})$

b $(2\sqrt{3} - 1)(3\sqrt{3} + 5)$

c $(\sqrt{7} - 3)(2\sqrt{5} + 4)$

d $(\sqrt{10} - 3)(\sqrt{10} + 3)$

e $(2\sqrt{6} + \sqrt{11})(2\sqrt{6} - \sqrt{11})$

f $(\sqrt{7} - 2)^2$

g $(\sqrt{5} + \sqrt{2})^2$

h $(4 - 3\sqrt{2})^2$

9 Write with a rational denominator:

a $\dfrac{1}{\sqrt{5}}$

b $\dfrac{3}{\sqrt{2}}$

c $\dfrac{\sqrt{3}}{\sqrt{11}}$

d $\dfrac{1}{5\sqrt{3}}$

e $\dfrac{5}{2\sqrt{7}}$

f $\dfrac{\sqrt{2}}{3\sqrt{10}}$

10 Write with a rational denominator:

a $\dfrac{1}{\sqrt{5} + \sqrt{2}}$

b $\dfrac{1}{3 - \sqrt{7}}$

c $\dfrac{1}{2\sqrt{6} - \sqrt{3}}$

d $\dfrac{\sqrt{3}}{\sqrt{3} + 1}$

e $\dfrac{3}{\sqrt{11} + \sqrt{5}}$

f $\dfrac{3\sqrt{7}}{2\sqrt{5} - \sqrt{7}}$

11 Rationalise the denominator of each fraction.

a $\dfrac{\sqrt{7} - \sqrt{2}}{\sqrt{7} + \sqrt{2}}$

b $\dfrac{3\sqrt{3} + 5}{3\sqrt{3} - 5}$

12 Find the value of x if $\sqrt{18} + \sqrt{8} = \sqrt{x}$.

13 Simplify $\dfrac{3}{\sqrt{5} - 2} + \dfrac{2}{\sqrt{5} + 2}$ by forming the lowest common denominator.

14 Find the values of p and q such that $\dfrac{\sqrt{5}}{\sqrt{5} - 2} = p + q\sqrt{5}$.

15 Show that $\dfrac{2}{6 - 3\sqrt{3}} - \dfrac{1}{2\sqrt{3} + 3}$ is rational by first rationalising each denominator.

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Review

16 Suppose that $x = 3 + \sqrt{10}$.

a Find the value of $x + \dfrac{1}{x}$ in simplest form.

b Hence find the value of $x^2 + \dfrac{1}{x^2}$.

17 By considering the reciprocals of the left-hand and right-hand sides, prove without a calculator that $\sqrt{6} - \sqrt{5} < \sqrt{5} - 2$.

18 Without a calculator, find $\dfrac{1}{\sqrt{1} + \sqrt{2}} + \dfrac{1}{\sqrt{2} + \sqrt{3}} + \dfrac{1}{\sqrt{3} + \sqrt{4}} + \cdots + \dfrac{1}{\sqrt{15} + \sqrt{16}}$.

19 a Show that $\sqrt{4 + 2\sqrt{3}} = 1 + \sqrt{3}$.

b If $x = \dfrac{\sqrt{3}}{2}$, use part **a** to simplify $\dfrac{1 + \sqrt{1 + x}}{\sqrt{1 + x}}$.

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3 Functions and graphs

The principal purpose of this course is the study of functions. Now that the real numbers have been reviewed, this chapter develops the idea of functions and relations, and their graphs. A variety of known graphs are then discussed, with particular emphasis on quadratics and their parabolic graphs, on factor cubics and polynomials, and on graphs with asymptotes.

The final section uses the vertical and horizontal line tests to classify four types of relations. It also gives examples of relations as they occur in databases and spreadsheets, where the elements of the two sets are not restricted to numbers. This section is demanding, and could be delayed until later in the year.

Curve-sketching software is useful in emphasising the basic idea that a function has a graph. It is, of course, also useful in sketching quickly a large number of graphs and recognising their relationships. Nevertheless, readers must eventually be able to construct a graph from its equation on their own.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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3A Functions and function notation

Many of the graphs studied in previous years are examples of functions. This section will make the idea of functions more precise and introduce some new notation.

Functions

Here is a situation that naturally leads to a function. An electrician charges \$100 to visit a home, and then charges \$40 for each power point that he installs.

Let x be the number of power points he installs.
Let y be the total cost, in dollars.
Then $y = 100 + 40x$.

This is an example of a *function*. Informally we say that *y is a function of x* because the value of y is determined by the value of x, and we call x and y *variables* because they take many different values. The variable x is called the *independent variable* of the function, and the variable y is called the *dependent variable* because its value depends on x.

Thus a function is a rule. We input a value of x, and the rule produces an output y. In this example, x must be a whole number $0, 1, 2, \ldots,$ and we can add this restriction to the rule, describing the function as:

'The function $y = 100 + 40x$, where x is a whole number.'

A *table of values* is a useful tool — a few values of the function are selected and arranged in a table. Here is a representative table of values showing the total cost y dollars of installing x power points:

x	0	1	2	3	4	5	6
y	100	140	180	220	260	300	340

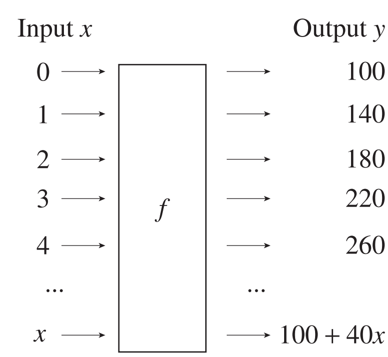
1 INFORMALLY, A FUNCTION IS A RULE

- *A variable y is a function of a variable x* when y is completely determined by x as a result of some rule.
- The variable x is called the *independent variable* of the function, and the variable y is called the *dependent variable* because it depends on x.
- The function rule is almost always an equation, possibly with a restriction. For example:

  'The function $y = 100 + 40x$, where x is a whole number.'

The function machine

The function and its rule can be regarded as a 'machine' with inputs and outputs. For example, the numbers in the right-hand column are the outputs from the function $y = 100 + 40x$, when the numbers 0, 1, 2, 3 and 4 are the inputs.



This model of a function has become far more intuitive in the last few decades because computers and calculators routinely produce output from a given input.

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Function notation

In the diagram above, we gave the name f to our function. We can now write the results of the input–output routines as follows:

$$f(0) = 100, \quad f(1) = 140, \quad f(2) = 180, \quad f(3) = 220, \quad \ldots$$

This is read aloud as 'f of zero is equal to one hundred', and so on. When x is the input, the output is $100 + 40x$. Thus, using the well-known notation introduced by Euler in 1735, we can write the *function rule* as

$$f(x) = 100 + 40x, \text{ where } x \text{ is a whole number.}$$

This $f(x)$ notation will be used throughout the course alongside $y = \ldots$ notation. The previous table of values can therefore also be written as

x	0	1	2	3	4	5	6
$f(x)$	100	140	180	220	260	300	340

Example 1 — 3A

A function is defined by the rule $f(x) = x^2 + 5x$. Find $f(3)$, $f(0)$ and $f(-3)$.

SOLUTION

$$\begin{aligned} f(3) &= 3^2 + 5 \times 3 \\ &= 9 + 15 \\ &= 24 \end{aligned}$$

$$\begin{aligned} f(0) &= 0^2 + 0 \times 3 \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(-3) &= (-3)^2 + 5 \times (-3) \\ &= 9 - 15 \\ &= -6 \end{aligned}$$

Example 2 — 3A

[This is an example of a function that is described verbally]

A function $g(x)$ is defined by the rule, 'Cube the number and subtract 7'. Write down its function rule as an equation, then draw up a table of values for -2, -1, 0, 1 and 2.

SOLUTION

The function rule is $g(x) = x^3 - 7$.

x	-2	-1	0	1	2
$g(x)$	-15	-8	-7	-6	1

Example 3 — 3A

[Other pronumerals, and even expressions, can be substituted into a function]

If $f(x) = x^2 + 5$, find and simplify $f(a)$, $f(a + 1)$ and $f(a + h)$.

SOLUTION

$$f(a) = a^2 + 5$$

$$\begin{aligned} f(a+1) &= (a+1)^2 + 5 \\ &= a^2 + 2a + 1 + 5 \\ &= a^2 + 2a + 6 \end{aligned}$$

$$\begin{aligned} f(a+h) &= (a+h)^2 + 5 \\ &= a^2 + 2ah + h^2 + 5 \end{aligned}$$

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Example 4 3A

[Many functions have natural restrictions on the variables. These restrictions are part of the function.]

Sadie the snail is crawling at a steady 10 cm per minute vertically up a wall 3 metres high, starting at the bottom. Write down the height y metres as a function of the time x minutes of climbing, adding the restriction on x.

SOLUTION

The snail's height after x minutes is $\frac{1}{10}x$ metres, and the snail will take 30 minutes to get to the top, so the function is

$$y = \frac{1}{10}x, \text{ where } 0 \le x \le 30.$$

Exercise 3A

FOUNDATION

1 Let $p(x) = x^2 - 2x - 3$. Determine:
   a $p(0)$ b $p(4)$ c $p(3)$ d $p(-2)$

2 Find the value of the function $y = 5 + 2x - x^2$ at each value of x.
   a 0 b 5 c -2 d -1

3 Find $f(2)$, $f(0)$ and $f(-2)$ for each function.
   a $f(x) = 3x - 1$ b $f(x) = 4 - x^2$ c $f(x) = x^3 + 8$ d $f(x) = 2^x$

4 Find $h(-3)$, $h(1)$ and $h(5)$ for each function.
   a $h(x) = 2x + 2$ b $h(x) = \frac{1}{x}$ c $h(x) = 3x - x^2$ d $h(x) = \sqrt{x + 4}$

5 Copy and complete the table of values for each function.
   a $y = x^2 - 2x$

x	-1	0	1	2	3
y					

   b $f(x) = x^3 - 4x$

x	-3	-2	-1	0	1	2	3
$f(x)$							

DEVELOPMENT

6 For the function $L(x) = 3x + 1$, determine:
   a $L(1) - 2$ b $3L(-1)$ c $L(1) + L(2)$ d $L(9) \div L(2)$

7 a Let $f(x) = \begin{cases} x, & \text{for } x \le 0, \\ 2 - x, & \text{for } x > 0. \end{cases}$ Create a table of values for $-3 \le x \le 3$.

   b Let $f(x) = \begin{cases} (x - 1)^2 - 1, & \text{for } x < 1, \\ (x - 1)^2, & \text{for } x \ge 1. \end{cases}$ Create a table of values for $-1 \le x \le 3$.

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8 Given that $f(x) = x^2 - 3x + 5$, find the value of:

a $\frac{1}{2}\big(f(2) + f(3)\big)$

b $\frac{1}{6}\big(f(0) + 4f(2) + f(4)\big)$

9 Let $P(x) = x^2 - 2x - 4$. Find the value of: a $P\left(1 + \sqrt{5}\right)$, b $P\left(\sqrt{3} - 1\right)$.

10 A restaurant offers a special deal to groups by charging a cover fee of \$50, then \$20 per person. Write down C, the total cost of the meal in dollars, as a function of x, the number of people in the group.

11 a Write the equation $3x + 4y + 5 = 0$ as a function with independent variable x.

b Write the equation $3x + 4y + 5 = 0$ as a function with independent variable y.

c Write the equation $4 + xy = 0$ as a function with dependent variable y.

d The volume of a cube with side length s is $V = s^3$, and its surface area is $A = 6s^2$. Write each formula as a function with dependent variable s.

e If a rectangle has area $100\,\text{m}^2$ and sides ℓ and b, then $\ell b = 100$. Write this formula as a function with:

i ℓ as the dependent variable,

ii ℓ as the independent variable.

12 In each case explain why the function value cannot be found.

a $F(0)$, where $F(x) = \sqrt{x - 4}$.

b $H(3)$, where $H(x) = \sqrt{1 - x^2}$.

c $g(-2)$, where $g(x) = \dfrac{1}{2 + x}$.

d $f(0)$, where $f(x) = \dfrac{1}{x}$.

13 Find $g(a)$, $g(-a)$ and $g(a + 1)$ for each function.

a $g(x) = 2x - 4$

b $g(x) = 2 - x$

c $g(x) = x^2$

d $g(x) = \dfrac{1}{x - 1}$

14 Find $F(t) - 2$ and $F(t - 2)$ for each function.

a $F(x) = 5x + 2$

b $F(x) = \sqrt{x}$

c $F(x) = x^2 + 2x$

d $F(x) = 2 - x^2$

15 If $f(x) = x^2 + 5x$, find in simplest form:

a $\dfrac{f(1 + h) - f(1)}{h}$

b $\dfrac{f(p) - f(q)}{p - q}$

c $\dfrac{f(x + h) - f(x)}{h}$

16 a If $f(x) = x^4 + 2x^2 + 3$, show that $f(-x) = f(x)$ for all values of x.

b If $g(x) = x^3 + \dfrac{4}{x}$, show that $g(-x) = -g(x)$ whenever $x \neq 0$.

c If $h(x) = \dfrac{x}{x^2 + 1}$, show that $h\left(\frac{1}{x}\right) = h(x)$ whenever $x \neq 0$.

ENRICHMENT

17 Given the functions $f(x) = x^2$ and $g(x) = x + 3$, find:

a $f\big(g(5)\big)$

b $g\big(f(5)\big)$

c $f\big(g(x)\big)$

d $g\big(f(x)\big)$

18 Evaluate $e(x) = \left(1 + \dfrac{1}{x}\right)^x$ on your calculator for $x = 1$, 10, 100, 1000 and 10000, giving your answer to two decimal places. What do you notice happens as x gets large?

19 Let $c(x) = \dfrac{3^x + 3^{-x}}{2}$ and $s(x) = \dfrac{3^x - 3^{-x}}{2}$. Show that $\big(c(x)\big)^2 - \big(s(x)\big)^2 = 1$.

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3B Functions, relations and graphs

A *graph* is the most important and helpful way to represent a function. Most things that we will discover about a function can be seen on its graph, and the sketching of graphs is consequently central to this course.

A function and its graph

A ball is thrown vertically upwards. While the ball is in the air, its height y metres, x seconds after it is thrown, is

$$y = 5x(6 - x).$$

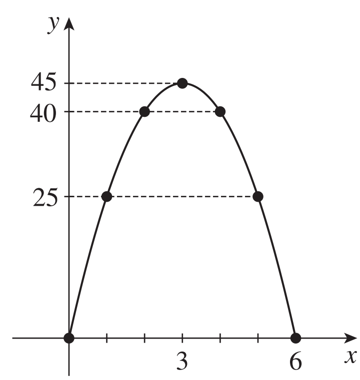
Here is a representative table of values of the function.

x (time)	0	1	2	3	4	5	6
y (height)	0	25	40	45	40	25	0

Each x-value and its corresponding y-value can be put into an ordered pair ready to plot on a graph of the function. The seven ordered pairs calculated here are:

$$(0, 0),\ (1, 25),\ (2, 40),\ (3, 45),\ (4, 40),\ (5, 25),\ (6, 0)$$

and the graph is sketched opposite. The seven representative points have been plotted, but there are infinitely many such ordered pairs, and they all join up to make the nice smooth curve shown to the right.



Like all graphs of functions, this graph has a crucial property — no two points have the same x-coordinate. This is because at any one time, the ball can only be in one position. In function-machine language, no input can have two outputs.

2 THE GRAPH OF A FUNCTION

- The *graph of a function* consists of all the ordered pairs (x, y) plotted on a pair of axes, where x and y are the values of the input and output variables.
- No two points on the graph ever have the same x-coordinate.
- In most graphs in this course, the points join up to make a smooth curve.

A function as a set of ordered pairs

These ideas of the graph and its ordered pairs allow a more formal definition of a function — a function can be defined simply as a set of ordered pairs satisfying the crucial property mentioned above.

3 MORE FORMALLY, A FUNCTION IS A SET OF ORDERED PAIRS SATISFYING A CONDITION

A *function* is a set of ordered pairs (x, y) in which:

- no two ordered pairs have the same x-coordinate.

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Domain and range

There are two restrictions on the time x in our example above:

- The time variable x cannot be negative, because the ball had not been thrown then.
- The time variable cannot be greater than 6, because the ball hits the ground after 6 seconds.

The *domain* of the function is the set of possible x-values, so the domain is the closed interval $0 \le x \le 6$. Thus the function is more correctly written as

$$y = 5x(6 - x), \text{ where } 0 \le x \le 6.$$

The endpoints of the graph are marked with *closed (filled-in) circles* • to indicate that these endpoints are included in the graph. If they were not included, they would be marked with *open circles* ∘. These are the same conventions that were used with intervals in Section 2B.

From the graph, we can see that the height of the ball ranges from 0 on the ground to 45 metres. The *range* is the set of possible y-values, so the range is the interval $0 \le y \le 45$, which includes the two endpoints $y = 0$ and $y = 45$.

4 THE DOMAIN AND RANGE OF A FUNCTION

- The *domain* of a function is the set of all possible x-coordinates.
- The *range* of a function is the set of all possible y-coordinates.

It is usually easier to find the range after the graph has been drawn.

Reading the domain and range from the graph

Sometimes we have the graph of a function, but not its equation or rule. We may be able to read the domain and range from such a graph.

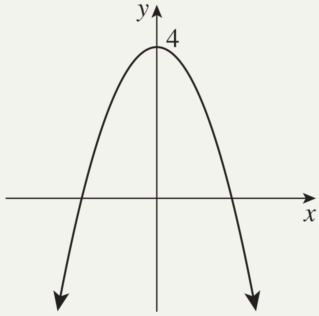
5 READING THE DOMAIN AND RANGE FROM THE GRAPH

- **Domain:** Take all the values on the x-axis that have graph points above or below them.
- **Range:** Take all the values on the y-axis that have graph points to the left or right of them.

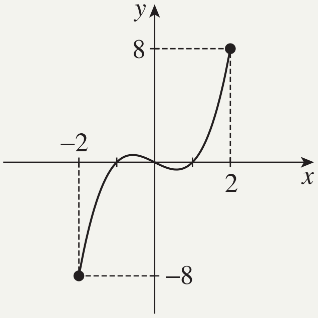
Example 5 3B

Write down the domain and the range of the functions whose graphs are sketched below.

a



b



SOLUTION

a Domain: all real x, range: $y \le 4$.

b Domain: $-2 \le x \le 2$, range: $-8 \le y \le 8$.

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The natural domain

When the equation of a function is given with no restriction, we assume *as a convention* that the domain is all the x-values that can validly be substituted into the equation. This is called the *natural domain*.

6 THE NATURAL DOMAIN

If no restriction is given, the domain is all x-values that can validly be substituted into the equation. This is called the *natural domain*.

There are many reasons why a number cannot be substituted into an equation. So far, the two most common reasons are:

- We cannot divide by zero.
- We cannot take square roots of negative numbers.

Example 6 3B

Find the natural domain of each function.

a $y = \dfrac{1}{x - 2}$

b $y = \sqrt{x - 2}$

SOLUTION

a We cannot divide by zero.
Hence the domain is $x - 2 \neq 0$
that is, $x \neq 2$.

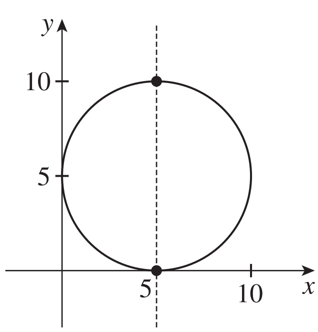
b We cannot take square roots of negative numbers.
Hence the domain is $x - 2 \geq 0$
that is, $x \geq 2$.

Relations

We shall often be dealing with graphs such as the circle sketched to the right. This graph is a set of ordered pairs. But it is not a function, because, for example,

the points $(5, 0)$ and $(5, 10)$ have the same x-coordinate,

so that the input $x = 5$ has the two outputs $y = 0$ and $y = 10$. The graph thus fails the crucial property that no input can have more than one output. The more general word 'relation' is used for any graph in the plane, whether it is a function or not. In terms of ordered pairs:



7 RELATIONS

A *relation* is any set of ordered pairs.

- Like a function, a relation has a graph.
- Like a function, a relation has a *domain* and a *range*.
- Unlike a function, a relation may have two or more points with the same x-coordinate.

A function is thus a special type of relation, just as a square is a special type of rectangle.

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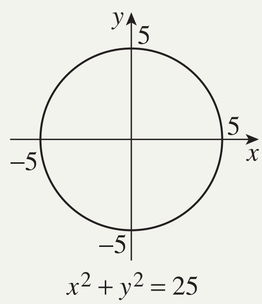
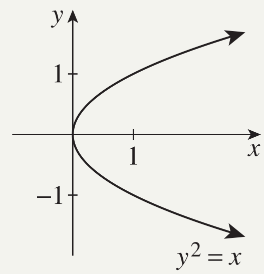
Example 7 3B

In each part, show that the relation is not a function by writing down two ordered pairs on the graph with the same x-coordinate. Illustrate this by connecting the two points by a vertical line. Then write down the domain and range.

a

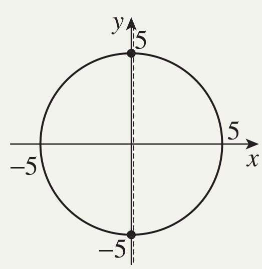


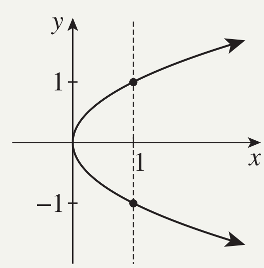
b



SOLUTION

a The points $(0, 5)$ and $(0, -5)$ on the graph have the same x-coordinate, $x = 0$.
Thus when $x = 0$ is the input, there are two outputs, $y = 5$ and $y = -5$.
The vertical line $x = 0$ meets the graph at $(0, 5)$ and at $(0, -5)$.
Domain: $-5 \le x \le 5$, range: $-5 \le y \le 5$





b The points $(1, 1)$ and $(1, -1)$ on the graph have the same x-coordinate, $x = 1$.
Thus when $x = 1$ is the input, there are two outputs, $y = -1$ and $y = 1$.
The vertical line $x = 1$ meets the graph at $(1, -1)$ and $(1, 1)$.
Domain: $x \ge 0$, range: all real y.

The vertical line test

Example 7 shows that we can easily use vertical lines on any graph to see whether or not it is a function.

To show that the graph is not a function, we need to identify just two points with the same x-coordinate. That means, we need to draw just one vertical line that crosses the graph twice.

8 THE VERTICAL LINE TEST

If at least one vertical line crosses the graph of a relation more than once, then the relation is not a function.

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The word ‘map’

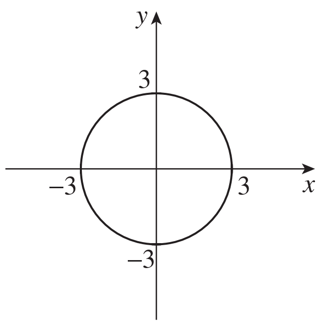
Functions (but not relations in general) are also called *maps* or *mappings*. The word *map* may also be used as a verb. Thus in the function $f(x) = 2^x$, we may say that ‘3 is mapped to 8’.

A map of NSW is a one-to-one correspondence from points on the surface of the Earth to points on a piece of paper.

Exercise 3B

FOUNDATION

1 In each case, copy the graph, then draw a vertical line to show that the curve does not represent a function.

a


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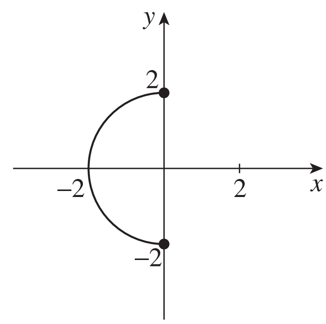

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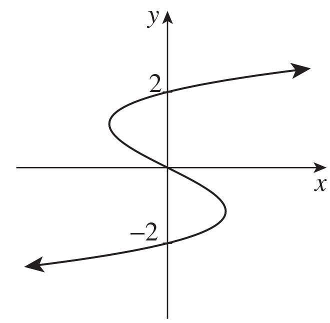

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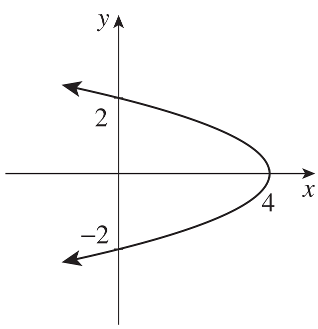
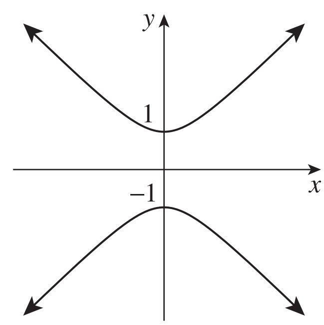
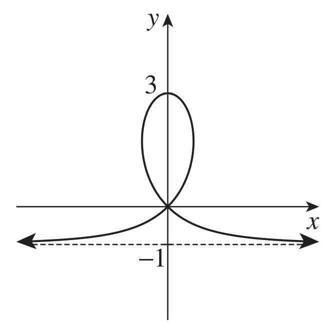
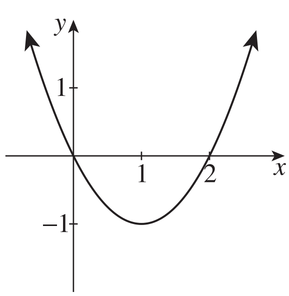


e


f


2 Use the vertical line test to find which of these graphs represent functions.

a


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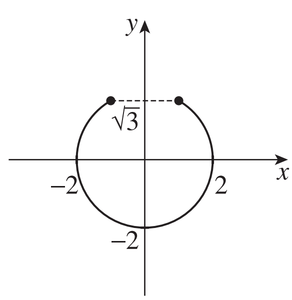

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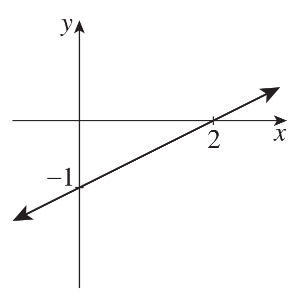

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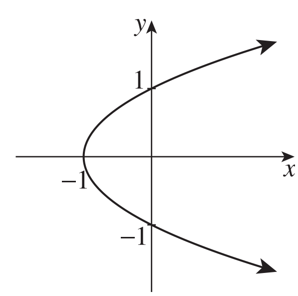

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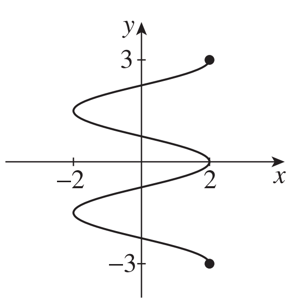

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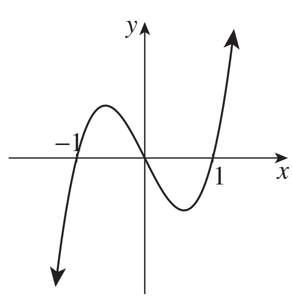

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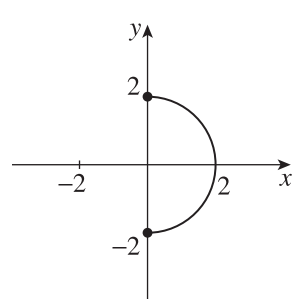
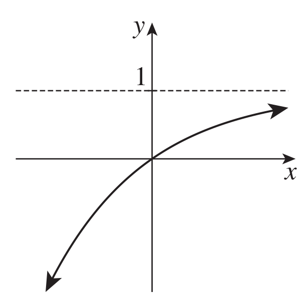


h


3 What are the domain and range of each relation in Question 2?

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4 Use the fact that division by zero is undefined to find the natural domain of each function.

a $f(x) = \dfrac{1}{x}$ **b** $f(x) = \dfrac{1}{x - 3}$ **c** $f(x) = \dfrac{1}{2 + x}$

5 Use the fact that a negative number does not have a square root to find the natural domain of each function.

a $f(x) = \sqrt{x}$ **b** $f(x) = \sqrt{x - 2}$ **c** $f(x) = \sqrt{5 + x}$

6 For each function:

i Copy and complete the table of values.

ii Plot the points in the table and hence sketch the function.

iii Then write down the domain and range.

a $y = x^2 + 2x + 1$

x	-3	-2	-1	0	1
y					

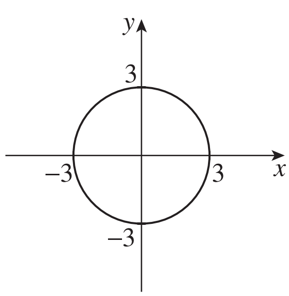
b $y = 2^x$

x	-2	-1	0	1	2
y					

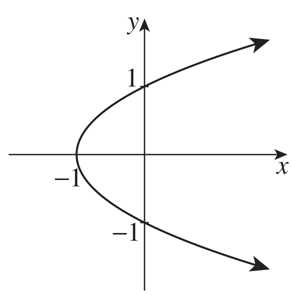
DEVELOPMENT

7 The relations graphed below are not functions. Write down the coordinates of two points on each graph that have the same x-coordinate.

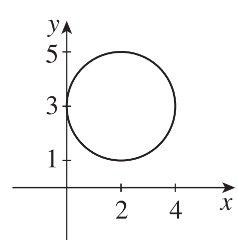
a



b



c



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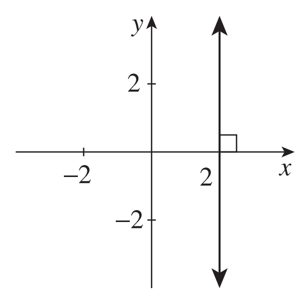

8 Find the natural domain of each function.

a $f(x) = 7 - 3x$ **b** $f(x) = \dfrac{3}{2x - 1}$ **c** $f(x) = \sqrt{x + 4}$

d $f(x) = \sqrt{4 - 2x}$ **e** $f(x) = \dfrac{2}{\sqrt{1 - x}}$ **f** $f(x) = \dfrac{1}{\sqrt{2x - 3}}$

9 Let $R(x) = \sqrt{x}$.

a What is the natural domain of $R(x)$?

b Copy and complete the table of values. Use a calculator to approximate values correct to one decimal place where necessary.

x	0	$\frac{1}{2}$	1	2	3	4	5
$R(x)$							

c Plot these points and join them with a smooth curve starting at the origin. This curve may look similar to a curve you know. Describe it.

10 Let $h(x) = \dfrac{2}{x}$.

a What is the natural domain of $h(x)$?

b Copy and complete the table of values. Why is there a star for the value where $x = 0$?

x	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4
$h(x)$					*				

c Plot these points and join them with a smooth curve in two parts. This curve is called a *rectangular hyperbola*.

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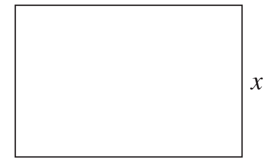
11 Jordan is playing with a 20 cm piece of copper wire, which he bends into the shape of a rectangle, as shown. Let x be the length of a side of the rectangle.



- **a** Find the area A of the rectangle as a function of x.
- **b** Use the fact that lengths must be positive to find the domain of A.
- **c** Sketch the graph of A for the domain you found in part **b**.

12 Solve each equation for y, thus showing that it represents a function.

- **a** $2x - y + 3 = 0$
- **b** $xy = 4$
- **c** $xy - 2y = 3$
- **d** $y + 2 = \sqrt{9 - x^2}$
- **e** $x = y^3 + 1$
- **f** $x = \dfrac{3 + y}{2 - y}$

13 State the natural domain of each function.

- **a** $f(x) = \dfrac{x}{\sqrt{x + 2}}$
- **b** $f(x) = \dfrac{2}{x^2 - 4}$
- **c** $f(x) = \dfrac{1}{x^2 + x}$
- **d** $f(x) = \dfrac{1}{x^2 - 5x + 6}$
- **e** $f(x) = \sqrt{x^2 - 4}$
- **f** $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$

14 Let $f(x) = \begin{cases} 2 + x, & \text{for } x \le 0, \\ 2 - x, & \text{for } x > 0. \end{cases}$

- **a** Create a table of values for $-3 \le x \le 3$
- **b** Hence sketch this function. (The graph is not smooth.)

ENRICHMENT

15 Let $f(x) = \begin{cases} 1 - x^2 & \text{for } x \le 1, \\ x - 2 & \text{for } x > 1. \end{cases}$

- **a** Create a table of values for $-2 \le x \le 4$ and carefully sketch the function.
- **b** What do you notice happens at $x = 1$?

16 **a** **i** What are the domain and range of the parabolic function $y = 4 - x^2$?

- **ii** Use part **i** and the properties of square roots to find the domain and range of $y = \sqrt{4 - x^2}$.
- **iii** Hence determine the domain and range for $y = \dfrac{1}{\sqrt{4 - x^2}}$.

b Likewise find the domain and range of each of these functions.

- **i** $\dfrac{1}{\sqrt{3 - 2x - x^2}}$
- **ii** $\dfrac{1}{\sqrt{x^2 + 2x + 3}}$

17 Consider the function $\operatorname{ath}(x) = \log_2\left(\dfrac{1 + x}{1 - x}\right)$.

Note: A function name does not have to be a single letter. In this case the function has been given the name 'ath' since it is related to the arc hyperbolic tangent, which is studied in some university courses.

- **a** What is the domain of $\operatorname{ath}(x)$?
- **b** Show that $\operatorname{ath}\left(\dfrac{2x}{1 + x^2}\right) = 2\operatorname{ath}(x)$.

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3C Review of linear graphs

The next few sections will review some functions and relations that have been introduced in previous years and the sketching of their graphs. Linear graphs are briefly reviewed in this section, and are the main subject of Chapter 7.

Linear functions

A function is called *linear* if its graph is a straight line. Its equation is then something like

$$y = 2x - 3,$$

with a term in x and a constant term. We often write a linear function with all its terms on the left — the equation of the function above then becomes

$$2x - y - 3 = 0,$$

where the coefficient of y cannot be zero, because we must be able to solve for y.

9 LINEAR FUNCTIONS

- A *linear function* has a graph that is a straight line.
- The equation of a linear function can be written in *gradient–intercept form*, $y = mx + b$.
- Alternatively, the equation of a linear function can be written in *general form*, $ax + by + c = 0$, where the coefficient of y is non-zero.

Sketching linear functions

When all three terms of the equation are non-zero, the easiest way to sketch a linear function is to find the two intercepts with the axes.

10 SKETCHING A LINEAR FUNCTION WHOSE EQUATION HAS THREE NON-ZERO TERMS

- Find the x-intercept by putting $y = 0$.
- Find the y-intercept by putting $x = 0$.

Example 8 3C

a Sketch each linear function by finding its x-intercept and y-intercept.

i $x + 2y - 6 = 0$ **ii** $y = x - 3$

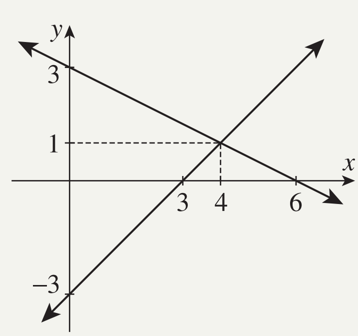
b Estimate from the graph where the two lines intersect.

SOLUTION

a i Consider $x + 2y - 6 = 0$.

When $y = 0$, $x - 6 = 0$
$x = 6$.

When $x = 0$, $2y - 6 = 0$
$y = 3$.



ii Consider $y = x - 3$.

When $y = 0$, $x - 3 = 0$
$x = 3$.

When $x = 0$, $y = -3$.

b From the diagram, the lines appear to meet at $(4, 1)$.

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Using simultaneous equations

The methods of solving simultaneous equations were reviewed in Section 1G.

Example 9 3C

Solve the simultaneous equations in Example 8, and check that the solution agrees with the estimate from the graph.

SOLUTION

The equations are $x + 2y - 6 = 0$ (1)

$x - y - 3 = 0.$ (2)

Subtracting (2) from (1), $3y - 3 = 0$

$y = 1.$

Substituting into (1), $x + 2 - 6 = 0$

$x = 4.$ Check this by substitution into (2).

Thus the lines meet at $(4, 1)$, as seen in the graph in Example 8.

Linear relations

A *linear relation* is a relation whose graph is a straight line. By the vertical line test, every linear relation is a function except for vertical lines, which fail the vertical line test dramatically. The equation of such a relation has no term in y, such as the relation:

$$x = 3.$$

Its graph consists of all points whose x-coordinate is 3, giving a vertical line, which is sketched in Example 10.

Three special cases

The two-intercept method won't work for sketching the graph of $ax + by + c = 0$ if any of the constants a, b or c is zero.

11 SKETCHING SPECIAL CASES OF LINEAR GRAPHS $ax + by + c = 0$

Horizontal lines: If $a = 0$, the equation can be put into the form $y = k$.

- Its graph is a horizontal line with y-intercept k.

Vertical lines: If $b = 0$, the equation can be put into the form $x = \ell$.

- Its graph is a vertical line with x-intercept ℓ.
- This is the only type of linear graph that is a relation, but not a function.

Lines through the origin: If $c = 0$, and the line is neither horizontal nor vertical, then the equation can be put into the form $y = mx$, where $m \neq 0$.

- Both intercepts are zero, so the graph passes through the origin.
- Find one more point on it, by substituting a value such as $x = 1$.

Example 10 3C

Sketch these three lines. Which of the three is not a function?

a $y + 2 = 0$ **b** $x - 3 = 0$ **c** $x + 2y = 0$

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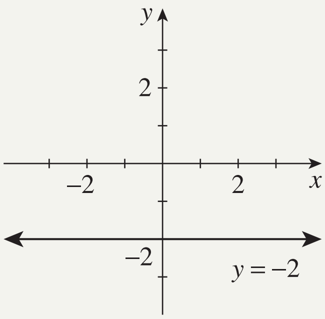
SOLUTION

a



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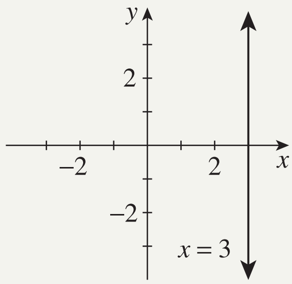
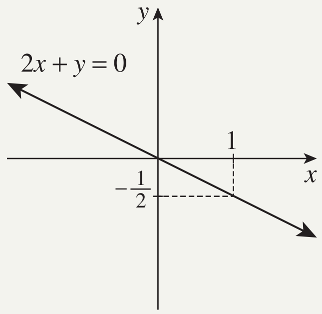


c



a The line $y + 2 = 0$, or $y = -2$, is horizontal with y-intercept -2.

b The line $x - 3 = 0$, or $x = 3$, is vertical with x-intercept 3.

c The line $x + 2y = 0$ passes through the origin, and when $x = 1$, $y = -\frac{1}{2}$.

The vertical line in part **b** is a relation, but not a function — it fails the vertical line test.

Exercise 3C

FOUNDATION

1 Follow these steps for the linear function $y = 2x - 2$.
- **a** Find the y-intercept by putting $x = 0$.
- **b** Find the x-intercept by putting $y = 0$.
- **c** Plot these intercepts and hence sketch the line.

2 Repeat the steps in the previous two questions for each line.
- **a** $y = x + 1$
- **b** $y = 4 - 2x$
- **c** $x + y - 1 = 0$
- **d** $x - 2y - 4 = 0$
- **e** $2x - 3y - 12 = 0$
- **f** $x + 4y + 6 = 0$

3 A linear function has equation $y = -2x$.
- **a** Show that the x-intercept and the y-intercept are both zero.
- **b** Substitute $x = 1$ to find a second point on the line, then sketch the line.

4 Repeat the steps in the previous question for each line.
- **a** $y = 3x$
- **b** $x + y = 0$
- **c** $x - 2y = 0$

5 Sketch these vertical and horizontal lines.
- **a** $x = 1$
- **b** $y = 2$
- **c** $x = -2$
- **d** $y = 0$
- **e** $2y = -3$
- **f** $3x = 5$

6
- **a** State which lines in the previous question are not functions.
- **b** For each line that is not a function, write down the coordinates of two points on it with the same x-coordinate.

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7 For each of parts **c** to **f** in Question 2, solve the given equation for y to show that it is a function.

8 Determine, by substitution, whether or not the given point lies on the given line.

a $(3, 1)$ $y = x - 2$

b $(7, 4)$ $y = 20 - 2x$

c $(1, -2)$ $y = -3x + 1$

d $(-5, 3)$ $2x + 3y + 1 = 0$

e $(-1, -4)$ $3x - 2y - 5 = 0$

f $(-6, -4)$ $4x - 5y - 4 = 0$

DEVELOPMENT

9 Consider the lines $x + y = 5$ and $x - y = 1$.

a Graph the lines on a number plane, using a scale of 1 cm to 1 unit on each axis.

b Read off the point of intersection of the two lines.

c Confirm your answer to part **b** by solving the two equations simultaneously.

10 Repeat the previous question for these pairs of lines.

a $x + y = 2$
$x - y = -4$

b $x - y = 3$
$2x + y = 0$

c $x + 2y = -4$
$2x - y = -3$

11 Looksmart Shirts charges \$60 for one shirt and \$50 for each shirt after this.

a Find, as a function of n, the cost C in dollars of n shirts.

b Delivery costs \$10 for one shirt and \$2 for each subsequent shirt.

i Find, as a function of n, the cost D in dollars of delivering n shirts.

ii Find, as a function of n, the total cost T in dollars of buying n shirts and having them delivered.

12 Consider the linear equation $y = \frac{1}{2}x + c$

a Sketch on one number plane the four lines corresponding to the following values of c:

i $c = -2$ **ii** $c = -1$ **iii** $c = 1$ **iv** $c = 2$

b What do you notice about all these lines?

13 Consider the linear equation $y - 2 = m(x - 1)$

a Sketch on one number plane the four lines corresponding to the following values of m:

i $m = 1$ **ii** $m = 2$ **iii** $m = -\frac{1}{2}$ **iv** $m = 0$

b Which point in the number plane do all these lines pass through?

c Now prove that the line $y - 2 = m(x - 1)$ passes through the point found in the previous part, regardless of the value of m.

ENRICHMENT

14 [Two-intercept form]

a Find the x- and y-intercepts of the equation $\dfrac{x}{a} + \dfrac{y}{b} = 1$. What do you notice?

b Use the result of part **a** to sketch these lines quickly.

i $\dfrac{x}{6} - \dfrac{y}{3} = 1$ **ii** $-x + \dfrac{y}{2} = 1$ **iii** $\dfrac{x}{2} + \dfrac{y}{5} = 1$

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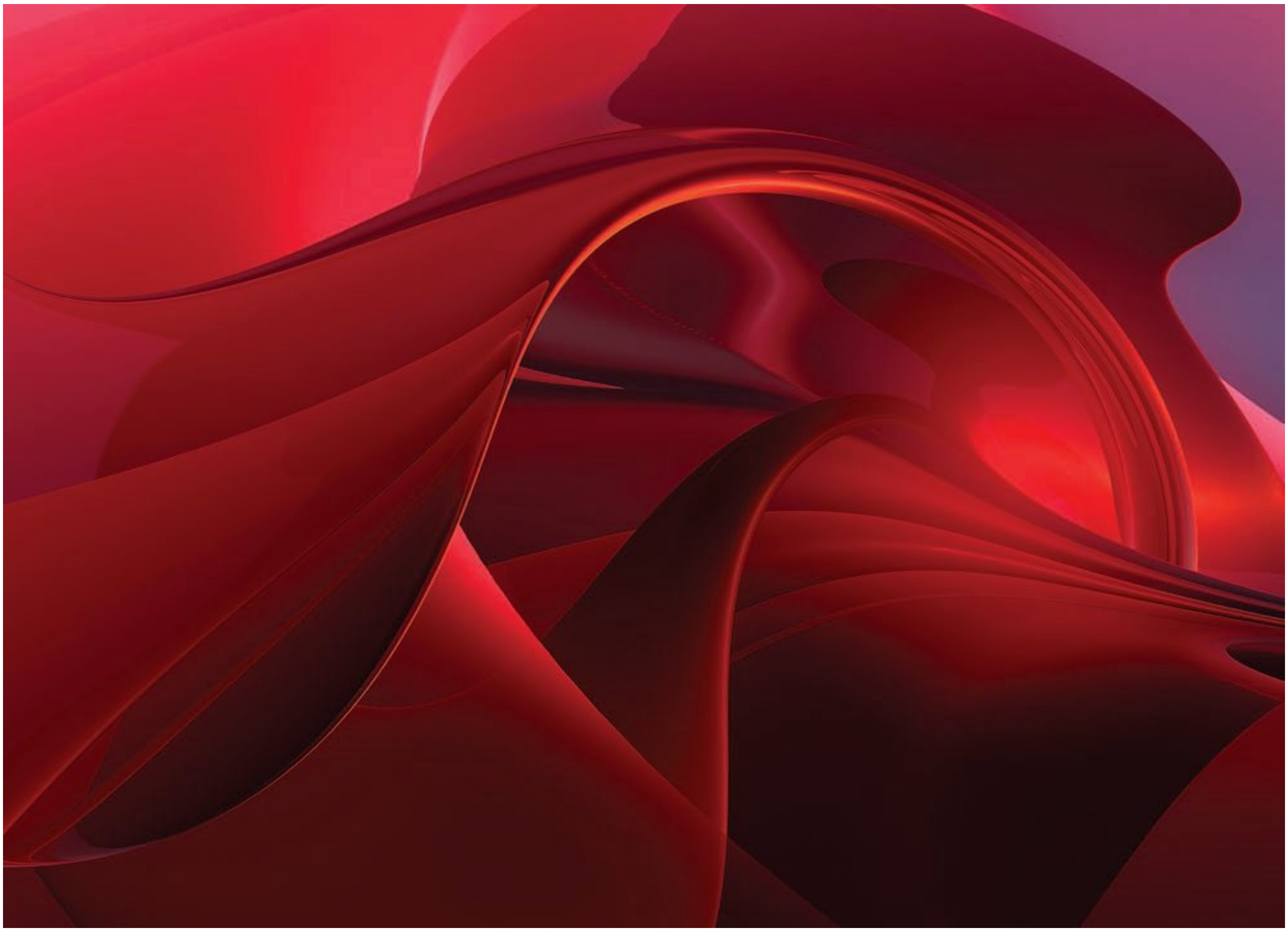
15 a Find the coordinates of the point M where $x + 2y - 6 = 0$ and $3x - 2y - 6 = 0$ meet.
 b Show that the new line $(x + 2y - 6) + k(3x - 2y - 6) = 0$ always passes through M, regardless of the value of k.
 c Hence find the equation of the line PM where $P = (2, -1)$.

16 The line $Ax + By + C = 0$ passes through the two fixed points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Let $R(x, y)$ be a variable point on the line.
 a Substitute the coordinates of P, Q and R into the equation of the line to get three equations.
 b Solve these equations simultaneously by eliminating A, B and C in order to show that the equation of the line can also be written as

$$(y_1 - y)(x_2 - x) = (y_2 - y)(x_1 - x).$$

 c Use this formula to write down the equation of the line through $P(1, 2)$ and $Q(3, -4)$.



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3D Quadratics functions — factoring and the graph

The material in Sections 3D–3F will be partly review and partly new work, with the emphasis on the graph. Readers confident with quadratics from earlier years will not need to work so thoroughly through the earlier questions in the exercises.

A *quadratic function* is a function that can be written in the form

$$f(x) = ax^2 + bx + c, \text{ where } a, b \text{ and } c \text{ are constants, and } a \neq 0.$$

A *quadratic equation* is an equation that can be written in the form

$$ax^2 + bx + c = 0, \text{ where } a, b \text{ and } c \text{ are constants, and } a \neq 0.$$

The requirement that $a \neq 0$ means that the term in x^2 cannot vanish. Thus linear functions and equations are not regarded as special cases of quadratics.

The word 'quadratic' comes from the Latin word *quadratus*, meaning 'square', and reminds us that quadratics tend to arise as the areas of regions in the plane.

Monic quadratics

A quadratic function $f(x) = ax^2 + bx + c$ is called *monic* if $a = 1$. Calculations are usually easier in monic quadratics than in non-monic quadratics.

12 MONIC QUADRATICS

A quadratic is called *monic* if the coefficient of x^2 is 1. For example,

$$y = x^2 - 8x + 15 \text{ is monic} \qquad \text{and} \qquad y = -x^2 + 8x - 15 \text{ is non-monic.}$$

Zeroes and roots

The solutions of a quadratic equation are called the *roots* of the equation, and the x-intercepts of a quadratic function are called the *zeroes* of the function. This distinction is often not strictly observed, however, because questions about quadratic functions and their graphs are so closely related to questions about quadratic equations.

Five questions about the graph of a quadratic

The graph of any quadratic function $y = ax^2 + bx + c$ is a *parabola*, as seen in earlier years. Before sketching the parabola, five questions need to be answered:

13 FIVE QUESTIONS ABOUT THE PARABOLA $y = ax^2 + bx + c$

1. Which way up is the parabola? **Answer:** Look at the sign of a.
2. What is the y-intercept? **Answer:** Put $x = 0$, and then $y = c$.
3. What are the x-intercepts, or zeroes, if there are any?
4. What is the axis of symmetry?
5. What is the vertex? **Method:** Substitute the axis back into the quadratic.

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The first two questions are easy to answer:

- If a is positive, the curve is concave up. If a is negative, it is concave down.
- To find the y-intercept, put $x = 0$, then $y = c$.

And once the axis of symmetry is found, the y-coordinate of the vertex can be found by substituting back into the quadratic.

But finding the x-intercepts, and finding the axis of symmetry, need careful working — there are three standard approaches:

- Factoring (discussed in this section)
- Completing the square (Section 3E)
- Formulae (Section 3F)

Factoring and the zeroes

Factoring of monic and non-monic quadratics was reviewed in Chapter 1. Most quadratics cannot easily be factored, but when straightforward factoring is possible, this is usually the quickest approach. After factoring, the zeroes can be found by putting $y = 0$ and using the principle:

$$\text{If } A \times B = 0 \quad \text{then} \quad A = 0 \quad \text{or} \quad B = 0.$$

For example, $y = x^2 - 2x - 3$ is a quadratic function.

1 Its graph is concave up because $a = 1$ is positive.

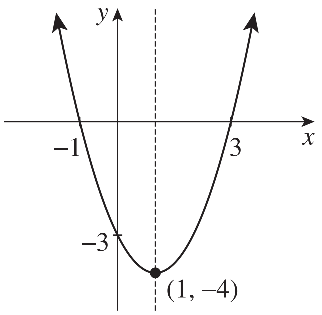
2 Substituting $x = 0$ gives $y = 0 + 0 - 3$
so the y-intercept is $y = -3$.

3 Factoring, $y = (x + 1)(x - 3)$,
Substituting $y = 0$ gives $0 = (x + 1)(x - 3)$
$x + 1 = 0$ or $x - 3 = 0$
so the x-intercepts are $x = -1$ and $x = 3$.

Finding the axis of symmetry and the vertex from the zeroes

The axis of symmetry is always the vertical line midway between the x-intercepts. Thus its x-intercept is the average of the zeroes.

Continuing with our example of $y = x^2 - 2x - 3$, which factors as $y = (x + 1)(x - 3)$, and so has zeroes -1 and 3:



4 Taking the average of the zeroes, the axis of symmetry is

$$x = \tfrac{1}{2}(-1 + 3)$$
$$x = 1$$

5 Substituting $x = 1$ into the factored quadratic,

$$y = (1 + 1)(1 - 3)$$
$$= -4,$$

so the vertex is $(1, -4)$.

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14 THE ZEROES AND INTERCEPTS OF A FACTORED QUADRATIC

Suppose that we have managed to factor a quadratic as $y = a(x - \alpha)(x - \beta)$.

- Its x-intercepts (zeroes) are $x = \alpha$ and $x = \beta$.
- Its axis of symmetry is the line $x = \frac{1}{2}(\alpha + \beta)$. Take the average of the zeroes.
- Substitute the axis into the factored form of the quadratic to find the y-coordinate of the vertex.

Example 11 — 3D

[This is an example of a non-monic quadratic.]
Sketch the curve $y = -x^2 - 2x + 3$.

SOLUTION

1 Because $a < 0$, the curve is concave down.

2 When $x = 0$, $\quad y = 3$.

3 Factoring, $\quad y = -(x^2 + 2x - 3)$

$\quad = -(x + 3)(x - 1)$.

When $y = 0$, $\quad x + 3 = 0 \quad$ or $\quad x - 1 = 0$,

so the zeroes are $x = -3 \quad$ and $\quad x = 1$.

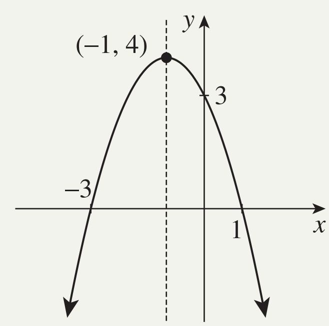
4 Taking the average of the zeroes, the axis of symmetry is

$$x = \tfrac{1}{2}(-3 + 1)$$
$$x = -1.$$

5 When $x = -1$, $\quad y = -(-1 + 3) \times (-1 - 1)$

$\quad = 4$,

so the vertex is $(-1, 4)$.

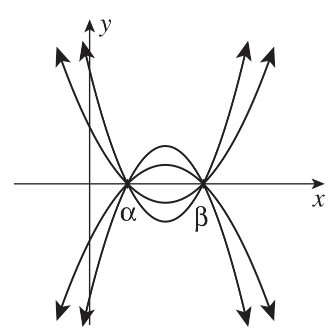


Quadratics with given zeroes

If a quadratic $f(x)$ has zeroes at $x = \alpha$ and $x = \beta$, then its equation has the form

$$f(x) = a(x - \alpha)(x - \beta),$$

where a is the coefficient of x^2. By taking different values of the coefficient a, this equation forms a *family* of quadratics, all with the same x-intercepts. The sketch to the right shows four of the curves in the family, two concave up with positive values of a, and two concave down with negative values of a.



15 THE FAMILY OF QUADRATICS WITH GIVEN ZEROES

The quadratics with zeroes at $x = \alpha$ and $x = \beta$ form a *family* of parabolas with equation

$$y = a(x - \alpha)(x - \beta), \text{ for some non-zero value of } a.$$

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Example 12 3D

a Write down the family of quadratics with zeroes $x = -2$ and $x = 4$.
b Then find the equation of such a quadratic if:
  i the y-intercept is 16, **ii** the curve passes through $(5, 1)$.

SOLUTION

a The family of quadratics with zeroes -2 and 4 is $y = a(x + 2)(x - 4)$.

b i Substituting the point $(0, 16)$ gives $16 = a \times 2 \times (-4)$
$a = -2$,
so the quadratic is $y = -2(x + 2)(x - 4)$.

ii Substituting the point $(5, 1)$ gives $1 = a \times 7 \times 1$
$a = \frac{1}{7}$,
so the quadratic is $y = \frac{1}{7}(x + 2)(x - 4)$.

Exercise 3D FOUNDATION

1 a The parabola with equation $y = (x - 1)(x - 3)$ is concave up.
  i Write down its y-intercept.
  ii Put $y = 0$ to find the x-intercepts.
  iii Hence determine the equation of the axis of symmetry.
  iv Use the axis of symmetry to find the coordinates of the vertex.
  v Sketch the parabola, showing these features.
b Follow the steps of part **a** to sketch these concave-up parabolas.
  i $y = (x - 1)(x + 3)$ **ii** $y = (x - 1)(x + 1)$

2 a The parabola $y = -x(x - 2)$ is concave-down. Follow similar steps to Question 1**a** in order to sketch it.
b Likewise, sketch these concave-down parabolas.
  i $y = (2 + x)(2 - x)$ **ii** $y = (x + 2)(4 - x)$

3 a The parabola $y = (x - 1)^2$ is a perfect square. Follow similar steps to Question 1**a** in order to sketch it.
b Likewise, sketch these parabolas involving perfect squares.
  i $y = (x + 1)^2$ **ii** $y = -(x - 2)^2$

4 [Technology]
a Use computer graphing software to plot accurately on the one number plane the parabola $y = a(x - 1)(x - 3)$ for the following values of a.
  i $a = 2$ **ii** $a = 1$ **iii** $a = -1$ **iv** $a = -2$
b Which two points do all these parabolas pass through?

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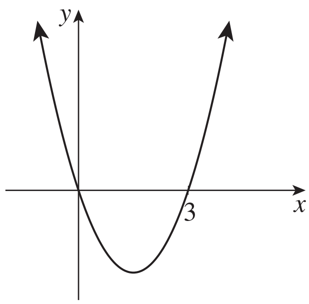
5 Write down, in factored form, the equation of the monic quadratic function with zeroes:

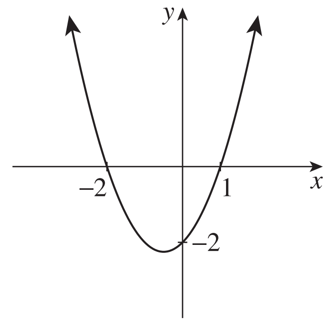
a 4 and 6 **b** 0 and 3 **c** −3 and 5 **d** −6 and −1

6 Write down, in factored form, the equation of each quadratic function sketched below, given that the coefficient of x^2 is either 1 or −1.

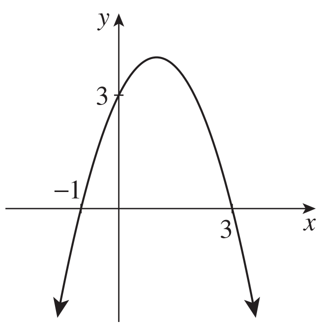
a



b



c



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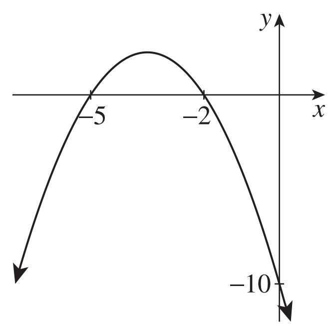

DEVELOPMENT

7 Use factoring to find the zeroes of each quadratic function. Hence sketch the graph of $y = f(x)$, showing all intercepts and the coordinates of the vertex.

a $f(x) = x^2 - 9$ **b** $f(x) = x^2 + 4x - 5$ **c** $f(x) = x^2 + 4x - 12$

d $f(x) = 4x - x^2$ **e** $f(x) = -x^2 + 2x + 3$ **f** $f(x) = 8 - 2x - x^2$

8 Sketch these parabolas involving perfect squares. Begin by factoring the quadratic. In each case, symmetry will be needed to find a third point on the parabola.

a $y = x^2 - 6x + 9$ **b** $y = -x^2 + 2x - 1$

9 Use factoring to sketch the graphs of these non-monic quadratic functions, clearly indicating the vertex and the intercepts with the axes.

a $y = 2x^2 + 7x + 5$ **b** $y = 2x^2 + 5x - 3$ **c** $y = 3x^2 + 2x - 8$

d $y = 2x^2 - 18$ **e** $y = 3x^2 + x - 4$ **f** $y = 7x - 3 - 4x^2$

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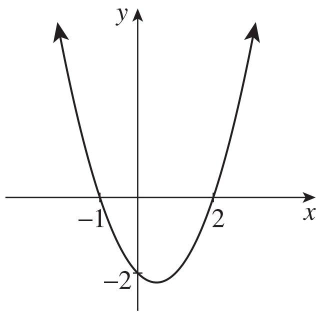
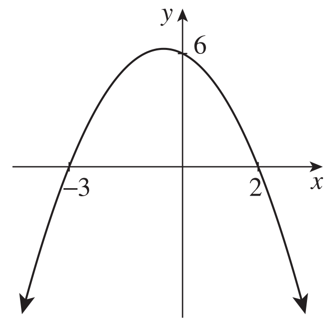
10 Find the equations of the quadratic functions sketched below.

a



b



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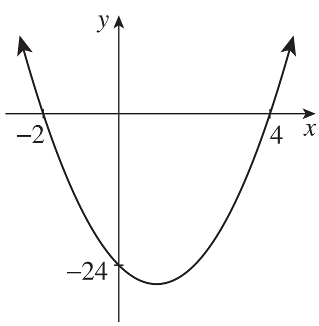

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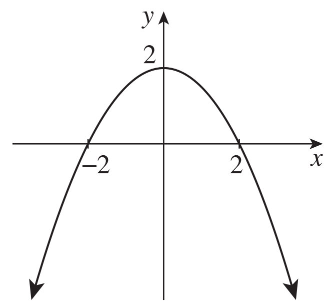

11 Find, in factored form, the equations of the parabolas with the given intercepts.

a $x = 1, 3$
$y = 6$

b $x = -2, 1$
$y = 4$

c $x = -1, 5$
$y = 15$

d $x = -2, -4$
$y = 2$

12 The general form of a quadratic with zeroes $x = 2$ and $x = 8$ is $y = a(x - 2)(x - 8)$. Find the equation of such a quadratic for which:

a the coefficient of x^2 is 3,

b the y-intercept is -16,

c the vertex is $(5, -12)$

d the curve passes through $(1, -20)$.

13 The graph of $y = ax(x - \alpha)$ passes through the origin. (Why?) Find the values of a and α in this quadratic given that:

a it is monic and passes through $(-3, 0)$

b there is one x-intercept and it passes through $(2, 6)$

c the vertex is $(1, 4)$

d the axis of symmetry is $x = -3$ and the coefficient of x is -12.

14 The general form of a quadratic with zeros α and β is $y = a(x - \alpha)(x - \beta)$. Find a in terms of α and β if:

a the y-intercept is c,

b the coefficient of x is b,

c the curve passes through $(1, 2)$.

15 Consider the quadratic function $f(x) = x^2 - 2x - 8$.

a Factor the quadratic and hence find the equation of the axis of symmetry.

b **i** Expand and simplify $f(1 + h)$ and $f(1 - h)$. What do you notice?

ii What geometric feature of the parabola does this result demonstrate?

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ENRICHMENT

16 In each case, factor the quadratic to find the x-intercepts. Hence find the equation of the axis of symmetry.

a $y = x^2 + 2x + 1 - p^2$ **b** $y = x^2 - 2px - 1 + p^2$ **c** $y = x^2 - 2x - 2p - p^2$

17 Let $f(x) = a(x - \alpha)(x - \beta)$. In each case, prove the given identity and explain how it demonstrates that $x = \frac{1}{2}(\alpha + \beta)$ is the axis of symmetry.

a $f\left(\frac{1}{2}(\alpha + \beta) + h\right) = f\left(\frac{1}{2}(\alpha + \beta) - h\right)$ **b** $f(\alpha + \beta - x) = f(x)$

18 Show that the quadratic equation $ax^2 + bx + c = 0$ cannot have more than two distinct roots when $a \neq 0$. (Hint: Suppose that the equation has three distinct roots α, β and γ. Substitute α, β and γ into the equation and conclude that $a = b = c = 0$.)

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3E Completing the square and the graph

Completing the square is the most general method of dealing with quadratics. It works in every case, whereas factoring really only works in exceptional cases.

We first review the algebra of completing the square, and extend it to non-monic quadratics. Once this has been done, sketching the graph follows easily.

Completing the square in a monic quadratic

The quadratic $y = x^2 + 6x + 5$ is not a perfect square, but it can be made into the sum of a perfect square and a constant. As explained in Chapter 1, the procedure is:

- Look just at the two terms in x, that is, $x^2 + 6x$.
- Halve the coefficient 6 of x to get 3, then square to get 9.
- Add and subtract 9 on the RHS to produce a perfect square plus a constant.

$$\begin{aligned} y &= x^2 + 6x + 5 \\ &= (x^2 + 6x + 9) - 9 + 5 \quad \text{(add and subtract 9)} \\ &= (x + 3)^2 - 4 \end{aligned}$$

16 COMPLETING THE SQUARE IN A MONIC QUADRATIC

- Take the coefficient of x, halve it, then square the result.
- Add and subtract this number to produce a perfect square plus a constant.

Example 13 3E

Complete the square in each quadratic.

a $y = x^2 - 4x - 5$ **b** $y = x^2 + x + 1$

SOLUTION

a Here $y = x^2 - 4x - 5$.

The coefficient of x is -4. Halve it to get -2, then square to get $(-2)^2 = 4$.

$$\begin{aligned} \text{Hence } y &= (x^2 - 4x + 4) - 4 - 5 \quad \text{(add and subtract 4)} \\ &= (x - 2)^2 - 9. \end{aligned}$$

b Here $y = x^2 + x + 1$.

The coefficient of x is 1. Halve it to get $\frac{1}{2}$, then square to get $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

$$\begin{aligned} \text{Hence } y &= \left(x^2 + x + \tfrac{1}{4}\right) - \tfrac{1}{4} + 1 \left(\text{add and subtract } \tfrac{1}{4}\right) \\ &= \left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}. \end{aligned}$$

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Completing the square in a non-monic quadratic

For a *non-monic quadratic* such as $y = 2x^2 - 12x + 16$, where the coefficient of x^2 is not 1, divide through by the coefficient of x^2 before completing the square. This slightly more difficult procedure was not covered in Chapter 1.

17 COMPLETING THE SQUARE IN A NON-MONIC QUADRATIC

- Divide through by the coefficient of x^2 so that the coefficient of x^2 is 1.
- Complete the square in the resulting monic quadratic.

Example 14 3E

Complete the square in each quadratic.

a $y = 2x^2 - 12x + 16$ **b** $y = -x^2 + 8x - 15$

SOLUTION

a
$$y = 2x^2 - 12x + 16$$
$$\frac{y}{2} = x^2 - 6x + 8 \quad \text{(divide through by the coefficient 2 of } x^2\text{)}$$
$$\frac{y}{2} = (x^2 - 6x + 9) - 9 + 8 \quad \text{(complete the square on the RHS)}$$
$$\frac{y}{2} = (x - 3)^2 - 1$$
$$y = 2(x - 3)^2 - 2 \quad \text{(multiply by 2 to make } y \text{ the subject again)}$$

b
$$y = -x^2 + 8x - 15$$
$$-y = x^2 - 8x + 15 \quad \text{(divide through by the coefficient } -1 \text{ of } x^2\text{)}$$
$$-y = (x^2 - 8x + 16) - 16 + 15 \quad \text{(complete the square on the RHS)}$$
$$-y = (x - 4)^2 - 1$$
$$y = -(x - 4)^2 + 1 \quad \text{(multiply by } -1 \text{ to make } y \text{ the subject again)}$$

Finding the vertex from the completed square

The completed square allows the vertex to be found using one fundamental fact about squares:

18 A SQUARE CAN NEVER BE NEGATIVE

- $x^2 = 0$, when $x = 0$,
- $x^2 > 0$, when $x \neq 0$.

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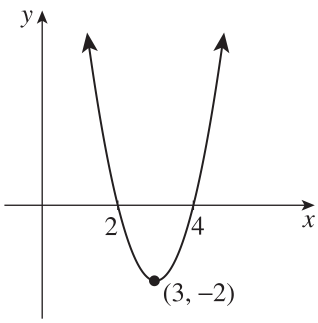
Thus in Example 14 part **a**, where

$$y = 2(x - 3)^2 - 2,$$

the term $2(x - 3)^2$ can never be negative.

Substituting $x = 3$ gives $y = 0 - 2$
$= -2,$
but if $x \neq 3$, then $y > -2.$



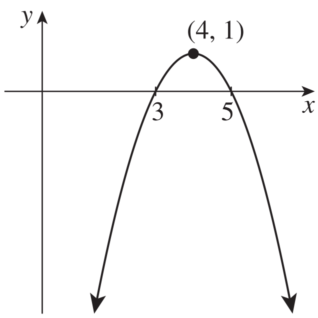
Hence the graph passes through the point $V(3, -2)$, but never goes below it. This means that $V(3, -2)$ is the vertex of the parabola, and $x = 3$ is its axis of symmetry.

But in Example 14 part **b**, where

$$y = -(x - 4)^2 + 1,$$

the term $-(x - 4)^2$ can never be positive.

Substituting $x = 4$ gives $y = 0 + 1$
$= 1,$
but if $x \neq 4$, then $y < 1$



Hence the graph passes through the point $V(4, 1)$, but never goes above it. This means that $V(4, 1)$ is the vertex of the parabola, and $x = 4$ is its axis of symmetry.

There is no need to repeat this argument every time. The result is simple:

19 FINDING THE AXIS AND VERTEX FROM THE COMPLETED SQUARE

For the quadratic $y = a(x - h)^2 + k$,

the axis is $x = h$ and the vertex is $V(h, k)$.

In Chapter 4, we will interpret this result as a translation — some readers may have done this already in earlier years.

Finding the zeroes from the completed square

The completed square also allows the zeroes to be found in the usual way:

20 FINDING THE ZEROES FROM THE COMPLETED SQUARE

To find the x-intercepts from the completed square, put $y = 0$.

- There may be two zeroes, in which case they may or may not involve surds.
- There may be no zeroes.
- There may be exactly one zero, in which case the quadratic is a *perfect square*. The x-axis is a tangent to the graph, and the zero is called a *double zero* of the quadratic.

These methods are illustrated in Examples 15–18 below.

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Example 15 3E

[Examples with and without surds]

Use completing the square to sketch the graphs of these quadratics.

a $y = x^2 - 4x - 5$ **b** $y = x^2 - 4x - 1$

SOLUTION

a $y = x^2 - 4x - 5$ is concave up, with y-intercept -5.

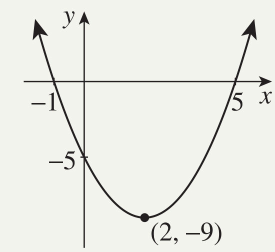
The square was completed earlier in this section,

$$y = (x - 2)^2 - 9,$$

so the axis is $x = 2$ and the vertex is $(2, -9)$.

Put $y = 0$, then $(x - 2)^2 = 9$

$$x - 2 = 3 \quad \text{or} \quad x - 2 = -3$$
$$x = 5 \quad \text{or} \quad x = -1.$$

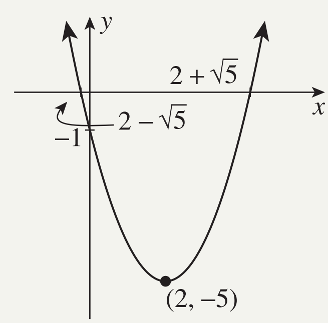


b $y = x^2 - 4x - 1$ is concave up, with y-intercept 1.

Completing the square, $y = (x^2 - 4x + 4) - 4 - 1,$
$$= (x - 2)^2 - 5,$$

so the axis is $x = 2$ and the vertex is $(2, -5)$.

Put $y = 0$, then $(x - 2)^2 = 5$
$$x - 2 = \sqrt{5} \quad \text{or} \quad x - 2 = -\sqrt{5}$$
$$x = 2 + \sqrt{5} \quad \text{or} \quad x = 2 - \sqrt{5}.$$



Example 16 3E

[An examples with no zeroes, and an example with one zero.]

Use completing the square to sketch the graphs of these quadratics.

a $y = x^2 + x + 1$ **b** $y = x^2 + 6x + 9$

SOLUTION

a $y = x^2 + x + 1$ is concave up, with y-intercept 1.

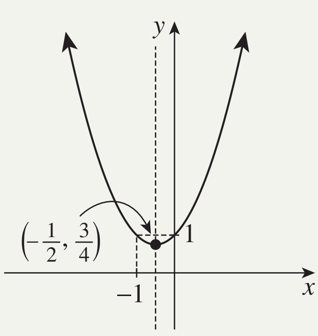
The square was completed earlier in this section,

$$y = \left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4},$$

so the axis is $x = -\frac{1}{2}$, and the vertex is $\left(-\frac{1}{2}, \frac{3}{4}\right)$.

Put $y = 0$, then $\left(x + \frac{1}{2}\right)^2 = -\frac{3}{4}$.

Because negative numbers do not have square roots, this equation has no solutions, so there are no x-intercepts.



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b $y = x^2 + 6x + 9$ is concave up, with y-intercept 9.
This quadratic is already a perfect square,

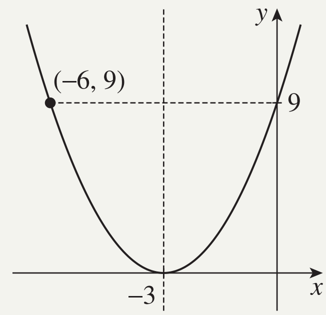
$$y = (x + 3)^2 + 0,$$

so the axis is $x = -3$ and the vertex is $(-3, 0)$.

Put $y = 0$, then $(x + 3)^2 = 0$

$$x + 3 = 0$$

$$x = -3 \quad \text{(a double zero).}$$



Notice that the symmetric point $(-6, 9)$ has been plotted so that the parabolic graph has at least three points on it.

Note: With this method, the axis of symmetry and the vertex are read directly off the completed-square form — the zeroes are then calculated afterwards. Compare this with factoring, where the zeroes are found first and the axis and vertex can then be calculated from them.

Example 17 — 3E

Use the previous completed squares to sketch the graphs of these quadratics.

a $y = 2x^2 - 12x + 16$ **b** $y = -x^2 + 8x - 15$

SOLUTION

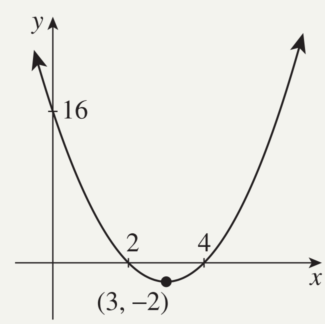
a $y = 2x^2 - 12x + 16$ is concave up, with y-intercept 16.
Completing the square, $y = 2(x - 3)^2 - 2$,
so the axis is $x = 3$ and the vertex is $(3, -2)$.

Put $y = 0$, then $2(x - 3)^2 = 2$

$$(x - 3)^2 = 1$$

$$x - 3 = 1 \quad \text{or} \quad x - 3 = -1$$

$$x = 4 \quad \text{or} \quad x = 2.$$

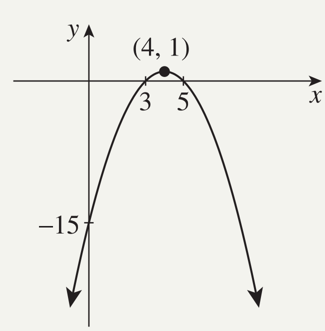


b $y = -x^2 + 8x - 15$ is concave down, with y-intercept -15.
Completing the square, $y = -(x - 4)^2 + 1$,
so the axis is $x = 4$ and the vertex is $(4, 1)$.

Put $y = 0$, then $(x - 4)^2 = 1$

$$x - 4 = 1 \quad \text{or} \quad x - 4 = -1$$

$$x = 5 \quad \text{or} \quad x = 3.$$



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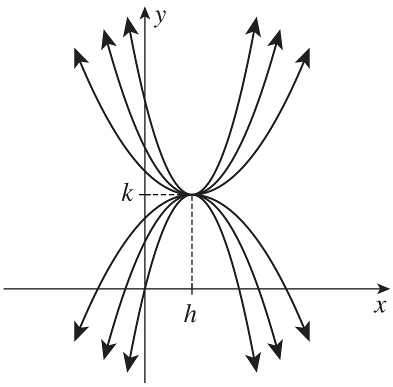
The family of quadratics with a common vertex

A quadratic with vertex (h, k) has an equation of the form

$$y = a(x - h)^2 + k$$

where a is the coefficient of x^2. This equation gives a *family* of parabolas all with vertex (h, k), as different values of a are taken. The sketch to the right shows six curves in the family, three with a positive, and three with a negative.



21 THE FAMILY OF QUADRATICS WITH A COMMON VERTEX

The quadratics with vertex (h, k) form a family of parabolas, all with equation
$y = a(x - h)^2 + k$, for some non-zero value of a.

Example 18 — 3E

Write down the family of quadratics with vertex $(-3, 2)$. Then find the equation of such a quadratic:

a if $x = 5$ is one of its zeroes,

b if the coefficient of x is equal to 1.

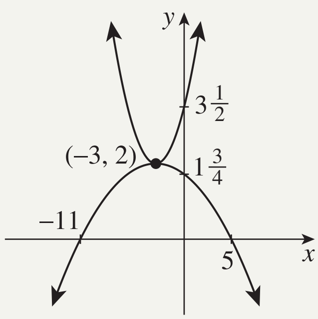
SOLUTION

The family of quadratics with vertex $(-3, 2)$ is

$$y = a(x + 3)^2 + 2. \quad (*)$$

a Substituting $(5, 0)$ into $(*)$ gives $0 = a \times 64 + 2$,
so $a = -\frac{1}{32}$, and the quadratic is $y = -\frac{1}{32}(x + 3)^2 + 2$.

b Expanding the equation $(*)$, $y = ax^2 + 6ax + (9a + 2)$,
so $6a = 1$.
Hence $a = \frac{1}{6}$, and the quadratic is $y = \frac{1}{6}(x + 3)^2 + 2$.



Exercise 3E

FOUNDATION

1 The equation of a particular parabola in completed square form is $y = (x - 2)^2 - 1$.
- **a** What is the concavity of this parabola?
- **b** Substitute $x = 0$ to find the y-intercept.
- **c** Put $y = 0$ to find the x-intercepts.
- **d** Use the results of Box 19 to write down the equation of the axis of symmetry and the coordinates of the vertex.
- **e** Hence sketch this parabola, showing this information.

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2 Repeat the steps of Question 1 in order to sketch the graphs of these quadratic functions.

a $y = (x + 1)^2 - 4$ **b** $y = (x - 1)^2 - 9$

c $y = -(x + 2)^2 + 4$ **d** $y = -(x - 2)^2 + 9$

3 Complete the square in each monic quadratic function.

a $f(x) = x^2 - 4x + 5$ **b** $f(x) = x^2 + 6x + 11$ **c** $f(x) = x^2 - 2x + 8$

d $f(x) = x^2 - 10x + 1$ **e** $f(x) = x^2 + 2x - 5$ **f** $f(x) = x^2 + 4x - 1$

4 Follow the steps in Question 1 to sketch these quadratics. The x-intercepts involve surds.

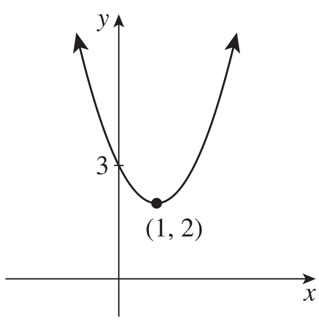
a $y = (x + 1)^2 - 3$ **b** $y = (x - 4)^2 - 7$ **c** $y = 2 - (x - 3)^2$ **d** $y = 5 - (x + 1)^2$

5 Find the zeroes of each quadratic function by completing the square. Then show that the same answer is obtained by factoring.

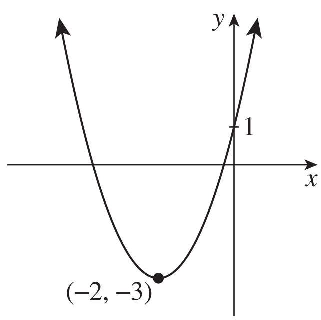
a $f(x) = x^2 - 4x + 3$ **b** $f(x) = x^2 + 2x - 3$ **c** $f(x) = x^2 - x - 2$

6 Use the formula in Box 21 to write down the equation of each quadratic function sketched below. In each case, the coefficient of x^2 is either 1 or -1.

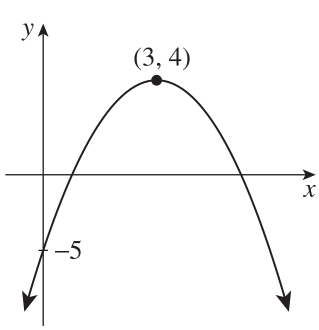
a



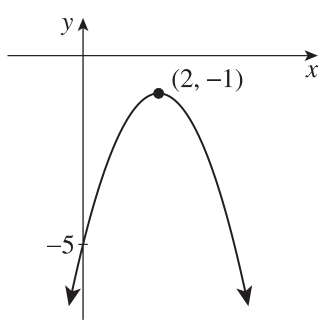
b



c



d



7 Write down the equation of the monic quadratic with vertex:

a $(2, 5)$ **b** $(0, -3)$ **c** $(-1, 7)$ **d** $(3, -11)$

8 [Technology]

a Use computer graphing software to plot accurately on the one number plane the parabola $y = a(x - 1)^2 - 2$ for the following values of a.

i $a = 2$ **ii** $a = 1$ **iii** $a = -1$ **iv** $a = -2$

b Which point do all these parabolas pass through?

c For which values of a does the parabola have x-intercepts?

d Explain your answer to part **c** geometrically.

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DEVELOPMENT

9 Complete the square in each quadratic. Then sketch the graph of each function, showing the vertex and the intercepts with the axes.

a $y = x^2 - 2x$ **b** $y = x^2 - 4x + 3$ **c** $y = x^2 - 2x - 5$
d $y = x^2 + 2x - 1$ **e** $y = x^2 + 2x + 2$ **f** $y = x^2 - 3x + 4$

10 Explain why $y = a(x + 4)^2 + 2$ is the general form of a quadratic with vertex $(-4, 2)$. Then find the equation of such a quadratic for which:

a the quadratic is monic, **b** the coefficient of x^2 is 3,
c the y-intercept is 16, **d** the curve passes through the origin.

11 Write down the coordinates of the vertex and the concavity for each parabola. Hence determine the number of x-intercepts.

a $y = 2(x - 3)^2 - 5$ **b** $y = 3 - (x + 1)^2$ **c** $y = -3(x + 2)^2 - 1$
d $y = 2(x - 4)^2 + 3$ **e** $y = 4(x + 1)^2$ **f** $y = -(x - 3)^2$

12 Complete the square for these non-monic quadratics. (In each case, notice that the coefficient of x^2 is not 1.) Then sketch each curve, showing the vertex and any intercepts.

a $y = -x^2 - 2x$ **b** $y = -x^2 + 4x + 1$ **c** $y = 2x^2 - 4x + 3$
d $y = 4x^2 - 8x + 1$ **e** $y = 2x^2 + 6x + 2$ **f** $y = -2x^2 - 8x - 11$
g $y = -3x^2 + 6x + 3$ **h** $y = 5x^2 - 20x + 23$ **i** $y = 3x^2 + 18x + 21$

13 Complete the square for each quadratic function. Hence write each quadratic in factored form. Your answers will involve surds.

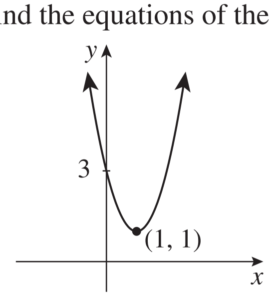
a $f(x) = x^2 + 2x - 1$ **b** $f(x) = x^2 - 4x + 1$ **c** $f(x) = -x^2 - 2x + 4$

14 Write down the general form of a monic quadratic whose axis of symmetry is $x = -2$. Hence find the equation of such a quadratic which also:

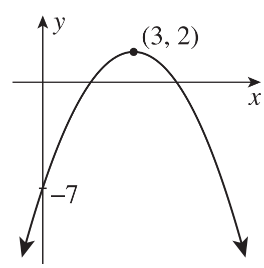
a passes through $(0, 0)$, **b** passes through $(5, 1)$, **c** has a zero at $x = 1$,
d has y-intercept -6, **e** touches the line $y = -2$, **f** has range $y \geq 7$.

15 Find the equations of the quadratic functions sketched below.

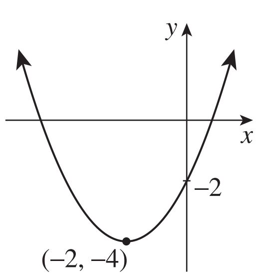
a



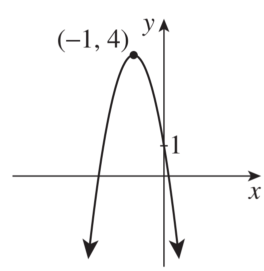
b



c



d



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16 **a** Find the zeroes of the monic quadratic $y = (x + d)^2 - e$, where $e > 0$.
b Find an expression for the difference between the two zeroes.
c Hence find the condition for the difference between the two zeroes to be 2, and describe geometrically the family of quadratics with this property.

17 The monic quadratics $y = (x - h_1)^2 + k_1$ and $y = (x - h_2)^2 + k_2$ do not intersect at all. Find the corresponding condition on the constants h_1, h_2, k_1 and k_2, and describe geometrically the relationship between the two curves.

18 Consider the general quadratic function $y = ax^2 + bx + c$.
a Complete the square in x in order to show that

$$y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}.$$

b Hence write down the coordinates of the vertex, and the equation of the axis.
c Put $y = 0$ in order to find the x-intercepts.
d Write the quadratic in factored form.

ENRICHMENT

19 Complete the square to find the vertex and x-intercepts of the function $y = x^2 + px + q$. Then sketch a possible graph of the function if:
a $p > 0$ and $p^2 > 4q$, **b** $p > 0$ and $p^2 < 4q$, **c** $p > 0$ and $p^2 = 4q$.

20 Consider the quadratic $f(x) = a(x - h)^2 + k$ with vertex (h, k). Prove the following identities and hence establish that $x = h$ is the axis of symmetry.
a $f(h + t) = f(h - t)$ **b** $f(2h - x) = f(x)$

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3F The quadratic formulae and the graph

Completing the square in the general quadratic yields formulae for its axis of symmetry and for its zeroes. These formulae are extremely useful, and like factoring and completing the square, allow the graph to be sketched.

The algebra of completing the square in a general quadratic was made into a structured question at the end of the Development section of Exercise 3E.

The formula for the axis of symmetry

The structured question at the end of Exercise 3E (Question 18) yields the formula for the axis of symmetry:

22 THE AXIS OF SYMMETRY OF $y = ax^2 + bx + c$

- The axis of symmetry is the line $x = -\dfrac{b}{2a}$.
- Substitute back into the quadratic to find the y-coordinate of the vertex.

Remember just the formula for the axis of symmetry, and find the y-coordinate of the vertex by substituting back into the quadratic.

The formula for the zeroes

Further working in Question 18 of Exercise 3E shows that putting $y = 0$ into $y = ax^2 + bx + c$ gives

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The quantity $b^2 - 4ac$ is very important in the theory of quadratics. It is called the *discriminant* because it discriminates, and has the symbol Δ (Greek uppercase delta, corresponding to 'D'). Using Δ in this formula makes it much easier to deal with and to remember.

23 THE ZEROES (x-INTERCEPTS) OF THE QUADRATIC $y = ax^2 + bx + c$

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}, \text{ where } \Delta = b^2 - 4ac.$$

- Always calculate the discriminant first when finding the zeroes of a quadratic.
- If $\Delta > 0$, there are two zeroes, because positives have two square roots.
- If $\Delta = 0$, there is only one zero because 0 is the only square root of 0, and in this situation, the x-axis is tangent to the graph.
- If $\Delta < 0$, there are no zeroes, because negatives don't have square roots.

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You will soon see from solving some quadratic equations that if $\Delta > 0$ and all three coefficients are rational numbers, then we can 'discriminate' further:

- If Δ is a perfect square, then the zeroes are rational.
- If Δ is not a perfect square, then the zeroes involve surds.

Example 19 3F

Use the quadratic formulae to sketch each quadratic. Give any irrational zeroes first in simplified surd form, then approximated correct to three decimal places.

a $y = -x^2 + 6x + 1$ **b** $y = 3x^2 - 6x + 4$

SOLUTION

a The curve $y = -x^2 + 6x + 1$ is concave down, with y-intercept 1.

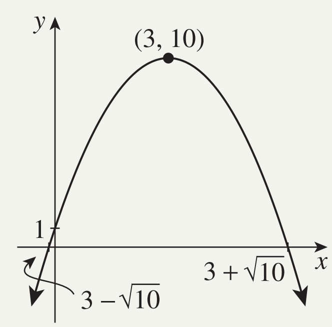
The formulae are now applied with $a = -1$, $b = 6$ and $c = 1$.

First, the axis of symmetry is $x = -\frac{b}{2a}$

$$x = -\left(\frac{6}{-2}\right)$$
$$x = 3.$$

When $x = 3$, $y = -9 + 18 + 1$
$$= 10,$$

so the vertex is $(3, 10)$.



Secondly, $\Delta = b^2 - 4ac$
$$= 40$$
$$= 4 \times 10 \quad \text{(take out square factors)},$$

so $y = 0$ when $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ or $\dfrac{-b - \sqrt{\Delta}}{2a}$

$$= \frac{-6 + 2\sqrt{10}}{-2} \text{ or } \frac{-6 - 2\sqrt{10}}{-2}$$
$$= 3 - \sqrt{10} \text{ or } 3 + \sqrt{10}$$
$$\doteqdot -0.162 \text{ or } 6.162.$$

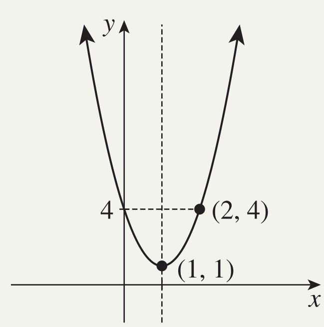
b The curve $y = 3x^2 - 6x + 4$ is concave up, with y-intercept 4.

Apply the formulae with $a = 3$, $b = -6$ and $c = 4$.

First, the axis of symmetry is $x = -\frac{b}{2a}$

$$x = -\left(\frac{-6}{6}\right)$$
$$x = 1,$$

and substituting $x = 1$, the vertex is $(1, 1)$.



Secondly, $\Delta = 36 - 48$
$$= -12$$

which is negative, so there are no zeroes, because negative numbers do not have square roots.

Notice that the symmetric point $(2, 4)$ has been plotted so that the parabolic graph has at least three points on it.

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Example 20 3F

a Use the discriminant to find the number of zeroes of $y = 5x^2 - 20x + 20$.
b What would be a better approach to this question?

SOLUTION

a $y = 5x^2 - 20x + 20$
$\Delta = 20^2 - 4 \times 5 \times 20$
$= 0,$
so there is exactly one zero.

b Better, take out the common factor:
$y = 5x^2 - 20x + 20$
$= 5(x^2 - 4x + 4)$
$= 5(x - 2)^2.$

Exercise 3F

FOUNDATION

1 Answer the following questions for the parabola $y = x^2 - 2x - 1$.
a **i** Use the value of a to determine the concavity of the parabola.
ii Write down the value of the y-intercept.
iii Use the formula $x = \frac{-b}{2a}$ to find the axis of symmetry.
iv Use the axis of symmetry to find the y-coordinate of the vertex.
v Calculate the discriminant $\Delta = b^2 - 4ac$.
vi Explain why this parabola has x-intercepts.
vii Find the x-intercepts using the formula $x = \dfrac{-b + \sqrt{\Delta}}{2a}$ or $\dfrac{-b - \sqrt{\Delta}}{2a}$.
b Sketch the parabola, showing these features.

2 Find the discriminant $\Delta = b^2 - 4ac$ of each quadratic function, then find the zeroes. Give the zeroes first in surd form, then correct to two decimal places.
a $f(x) = x^2 + 2x - 2$
b $f(x) = x^2 - 4x + 1$
c $f(x) = -x^2 + 3x + 2$
d $f(x) = -x^2 - 2x + 4$
e $f(x) = 3x^2 - 2x - 2$
f $f(x) = 2x^2 + 4x - 1$

3 Sketch a graph of each parabola by following similar steps to those outlined in Question 1.
a $y = x^2 + 6x + 4$
b $y = x^2 - 4x + 5$
c $y = -x^2 + 2x + 2$
d $y = -2x^2 + 4x - 3$
e $y = 3x^2 + 6x - 1$
f $y = 2x^2 + 2x - 1$
g $y = -x^2 - 2x - 4$
h $y = -x^2 + 2x + 5$
i $y = x^2 + 2x + 3$

4 In each case, find the zeroes of the quadratic function first by factoring, then by completing the square, and finally by using the quadratic formula. Observe that the answers to all three methods are the same for each function.
a $f(x) = x^2 - 3x - 4$
b $f(x) = x^2 - 5x + 6$
c $f(x) = -x^2 + 4x + 12$

5 **a** Consider the parabola $y = x^2 + 2$.
i Calculate Δ and explain why this parabola has no x-intercept.
ii What do you notice about the y-intercept and the vertex?

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iii Sketch the parabola showing the y-intercept and vertex. Then add to your sketch the point where $x = 1$.

iv Complete the sketch with another point found by symmetry.

b Follow similar working to sketch these parabolas.

i $y = -x^2 - 1$ ii $y = \frac{1}{2}x^2 + 1$

DEVELOPMENT

6 Use the quadratic formula to find the roots α and β of each quadratic equation. Hence show in each case that $\alpha + \beta = -\frac{b}{a}$ and that $\alpha\beta = \frac{c}{a}$.

a $x^2 - 6x + 1 = 0$ b $x^2 - 2x - 4 = 0$ c $-3x^2 + 10x - 5 = 0$

7 Find the discriminant $\Delta = b^2 - 4ac$ of each quadratic. Use this and the concavity to state how many zeroes the function has, without drawing its graph.

a $f(x) = x^2 + 3x - 2$ b $f(x) = 9x^2 - 6x + 1$ c $f(x) = -2x^2 + 5x - 4$

8 Consider the parabola with equation $y = -x^2 + 2x + 3$.

a Use algebra to find the x-coordinates of any intersection points of this parabola with each line.

i $y = 2$ ii $y = 4$ iii $y = 6$

b Graph the situation.

c For what values of k does the parabola intersect the line $y = k$ twice?

9 Use the quadratic formula to find the zeroes α and β of each quadratic function. Hence write the function in factored form, $f(x) = a(x - \alpha)(x - \beta)$.

a $f(x) = x^2 - 6x + 4$ b $f(x) = x^2 + 2x - 1$ c $f(x) = x^2 - 3x + 1$

d $f(x) = 3x^2 + 6x + 2$ e $f(x) = -x^2 + 3x + 1$ f $f(x) = -2x^2 - x + 1$

10 The parabola $y = ax^2 + bx + c$ has axis of symmetry $x = -\frac{b}{2a}$.

a Substitute this value into the parabola to show that the vertex is $V = \left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$.

b For each of the following quadratics, use the formula in part **a** to find the axis of symmetry. Then find the coordinates of the vertex, first by substitution into the quadratic, then using the formula in part **a**.

i $y = x^2 + 4x + 1$ ii $y = x^2 - 6x + 10$ iii $y = -3x^2 - 12x + 1$

11 The interval PQ has length p, and the point A lies between the points P and Q. Find PA when $PQ \times QA = PA^2$.

12 a Expand $f(x) = (x - h)^2 + k$ and show that $\Delta = -4k$. Then use the quadratic formulae to show that the axis is $x = h$ (as expected) and to find an expression for the zeroes.

b Expand $f(x) = (x - \alpha)(x - \beta)$ and show that $\Delta = (\alpha - \beta)^2$. Hence show that the zeroes are $x = \alpha$ and $x = \beta$ (as expected) and that the vertex is $\left(\frac{\alpha + \beta}{2}, -\left(\frac{\alpha - \beta}{2}\right)^2\right)$.

13 a Find the axis of symmetry and the vertex of $y = x^2 + bx + c$.

b Find the zeroes, and then find the difference between them.

c What condition on the constants b and c must be satisfied for the difference to be exactly 1?

d Hence show that the family of such quadratics is the family of parabolas with vertices on the line $y = -\frac{1}{4}$.

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ENRICHMENT

14 [The golden mean]

Sketch $y = x^2 - x - 1$, showing the vertex and all intercepts.

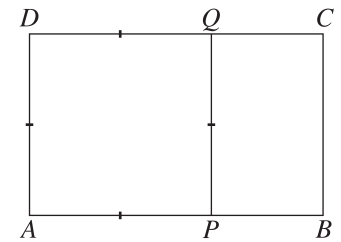
a Let $\alpha = \frac{1}{2}\left(\sqrt{5} + 1\right)$. Show that:

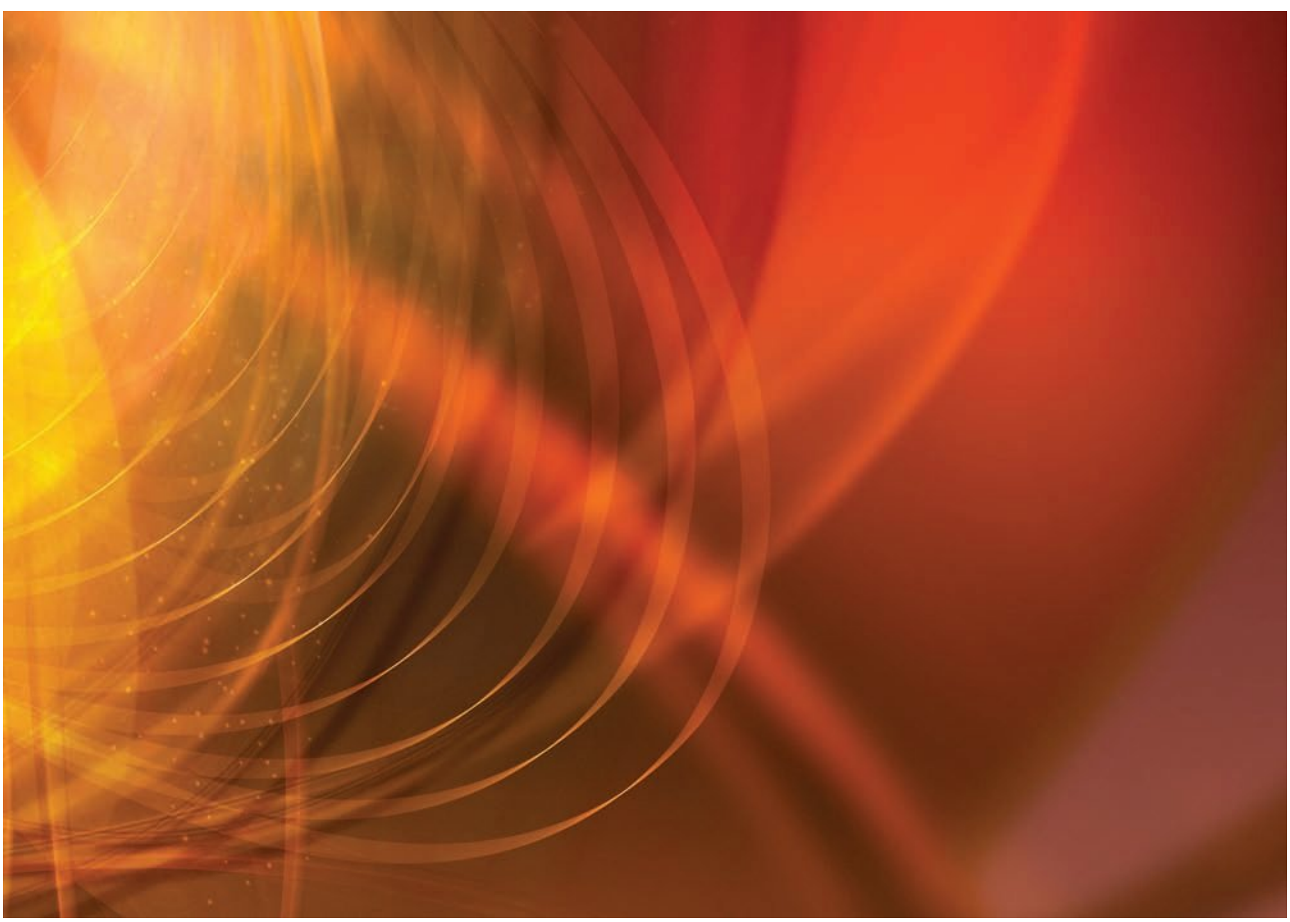
i $\alpha^2 = \alpha + 1$

ii $\dfrac{1}{\alpha} = \alpha - 1$

iii $\alpha^6 = 8\alpha + 5$

b $ABCD$ is a rectangle with length and breadth in the ratio $\alpha : 1$. It is divided into a square $APQD$ and a second rectangle $PBCQ$, as shown. Show that the length and breadth of rectangle $PBCQ$ are also in the ratio $\alpha : 1$.





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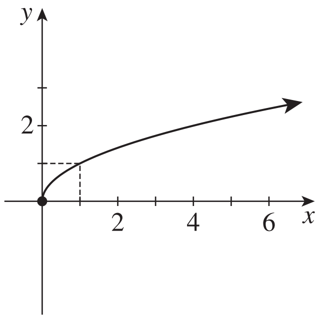
3G Powers, polynomials and circles

This section deals with the graphs of cubes, fourth powers, and higher powers of x, and of the square root of x, with particular attention to sketching cubics and polynomials that have been factored into linear factors. Then it reviews circles and semicircles.

The function $y = \sqrt{x}$

The graph of $y = \sqrt{x}$ is the upper half of a parabola on its side, as can be seen by squaring both sides to give $y^2 = x$. Remember that the symbol $\sqrt{x}$ means the *positive* square root of x, so the lower half of the parabola $y^2 = x$ is excluded:

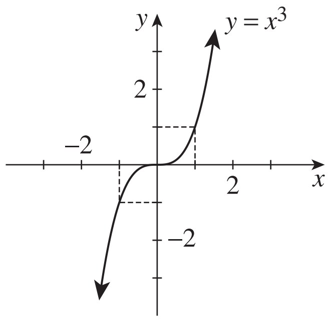


$y = \sqrt{x}$

x	0	$\frac{1}{4}$	1	2	4
y	0	$\frac{1}{2}$	1	$\sqrt{2}$	2

The cube of x

Graphed to the right is the cubic function $y = x^3$. It has a zero at $x = 0$, it is positive when x is positive, and negative when x is negative.



$y = x^3$

x	−2	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	−8	−1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8

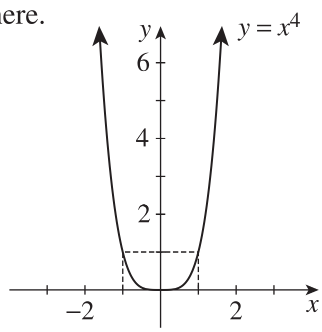
The curve becomes flat at the origin. You can see this by substituting a small number such as $x = 0.1$. The corresponding value of y is $y = 0.001$, which is far smaller.

The origin is called a *horizontal inflection* of the curve — 'inflection' means that the curve 'flexes' smoothly from concave down on the left to concave up on the right, and 'horizontal' means that the curve is momentarily horizontal. Inflections in general will be studied in Year 12.

The graphs of odd powers $y = x^3$, $y = x^5$, $y = x^7$, . . ., look similar. The origin is always a horizontal inflection. As the index increases, the curves become flatter near the origin, and steeper further away.

The fourth power of x

The graph to the right shows $y = x^4$. It has a zero at $x = 0$, and is positive elsewhere.



$y = x^4$

x	−2	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	16	1	$\frac{1}{16}$	0	$\frac{1}{16}$	1	16

All even powers $y = x^6$, $y = x^8$, . . . look similar. Like odd powers, as the index increases they become flatter near the origin and steeper further away.

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The origin is called a *turning point* of the curve because at the origin, the curve turns smoothly from decreasing on the left to increasing on the right. Turning points in general are an important topic in Year 12.

Polynomials

Polynomials are expressions such as

$$3x^4 + 4x^2 - 2x \qquad \text{and} \qquad x^3 - x^2 \qquad \text{and} \qquad 7 - 2x^3 + \tfrac{3}{4}x^6$$

that can be written as the sum of multiples of x, x^2, x^3, . . . and a constant.

A polynomial is usually written with the powers in descending or ascending order, as above. The index of the power with highest index is called the *degree* of the polynomial, thus the first polynomial above has degree 4, the second has degree 3, and the third has degree 6.

We have already been studying three types of polynomials.

- Quadratic expressions such as $3x^2 + 4x + 5$ are polynomials of degree 2.
- Linear expressions such as $-4x + 2$, where the coefficient of x is non-zero, are polynomials of degree 1.
- Numbers such as 7 and 12 are polynomials of degree 0, except that the *zero polynomial* 0 is regarded as having no terms, and thus no degree.

Polynomials of degree 3 are called *cubic polynomials* — the volume of a cube is x^3. Polynomials of degree 4 are called *quartic polynomials*, and further such names can be used for polynomials of higher degrees. The word 'polynomial' means 'many terms'.

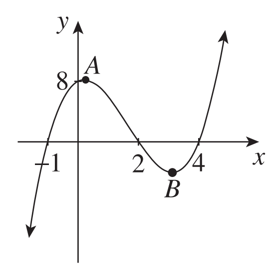
Sketching a cubic factored into linear factors

At this stage, little systematic graphing can be done of polynomial functions of degree 3 and higher. If, however, the polynomial has already been factored into linear factors, its basic shape can be established. This is done by drawing up a table of values to test its sign. For example, consider the cubic

$$y = (x + 1)(x - 2)(x - 4).$$

The function has zeroes at $x = -1$, $x = 2$ and $x = 4$.
Its domain is all real x, and it is continuous for all x,
so the zeroes are the only places where it can change sign.
We can draw up a table of values dodging these zeroes,

x	-2	-1	0	2	3	4	5
y	-24	0	8	0	-4	0	18
sign	$-$	0	$+$	0	$-$	0	$+$



We also know that, as with any cubic function, y becomes very large positive or negative for large positive or negative values of x, and combining this with the table above allows us to draw a sketch.

Be careful! We cannot yet find the two points A and B on the curve because that requires calculus. Notice that if we draw tangents there, the tangents are horizontal, which means that the curve is neither increasing nor decreasing there. Don't ever think that those points A and B are midway between the zeroes — this curve is not a parabola. Mark the zeroes and the y-intercept, and nothing else.

Cubics with square factors

The cubic in Example 21 has a square factor, so that there are only two zeroes.

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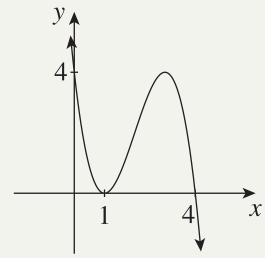
Example 21 3G

Sketch $y = -(x-1)^2(x-4)$. What is happening at $x = 1$?

SOLUTION

The cubic has zeroes at $x = 1$ and $x = 4$.
Here is a table of test values dodging around these zeroes,

x	0	1	2	4	5
y	4	0	2	0	-16
sign	$+$	0	$+$	0	$-$



At $x = 1$, the cubic has a zero, but does not cross the x-axis.
Instead, the x-axis is a tangent to the curve at $(1, 0)$.

24 SKETCHING A CUBIC FACTORED INTO LINEAR FACTORS

To sketch a cubic that has been factored into linear factors as

$$y = a(x-\alpha)(x-\beta)(x-\gamma).$$

- Draw up a table of values dodging around the zeroes to test the sign.
- If the curve meets the x-axis at $x = \alpha$ without crossing it, then the x-axis is tangent to the curve at $x = \alpha$.

Cubics such as $y = 3(x-2)^3$ will be sketched in Question 14 of Exercise 4A. Their basic shape has already been discussed at the start of this section.

Sketching a polynomial factored into linear factors

This same method can be applied to polynomials of higher degree that have been factored into linear factors.

The factoring may now involve a cubic or higher degree linear factor such as $(x-5)^3$, but we will not consider these situations yet because the behaviour at the corresponding x-intercept requires calculus to analyse.

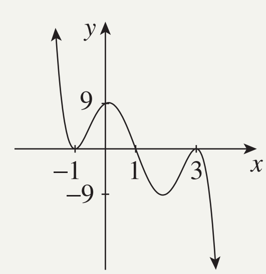
Example 22 3G

Sketch $y = -(x+1)^2(x-1)(x-3)^2$. What is happening at $x = -1$ and at $x = 3$?

SOLUTION

The polynomial has zeroes at $x = -1$, $x = 1$ and $x = 3$.
Here is a table of test values dodging around these zeroes,

x	-2	-1	0	1	2	3	4
y	75	0	9	0	-9	0	-75
sign	$+$	0	$+$	0	$-$	0	$-$



At $x = -1$ and $x = 3$, the curve meets the x-axis, without crossing it, so the x-axis is a tangent to the curve at those two places.

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25 SKETCHING A POLYNOMIAL FACTORED INTO LINEAR FACTORS

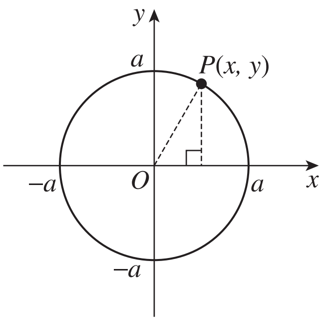
To sketch a polynomial that has been factored into linear factors as

$$y = a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)\ldots$$

- Draw up a table of values dodging around the zeroes to test the sign.
- If the curve meets the x-axis at $x = \alpha$ without crossing it, then the x-axis is tangent to the curve at $x = \alpha$.

Circles and semicircles

The equation of the circle with centre the origin and radius a can be found using Pythagoras' theorem, in the form of the distance formula.



A point $P(x, y)$ in the plane will lie on the circle if and only if its distance from the centre is the radius a.

That is, if and only if

$$\begin{aligned} OP &= a \\ OP^2 &= a^2 \\ (x - 0)^2 + (y - 0)^2 &= a^2 \\ x^2 + y^2 &= a^2. \end{aligned}$$

To put it very briefly, the equation of a circle is Pythagoras' theorem.

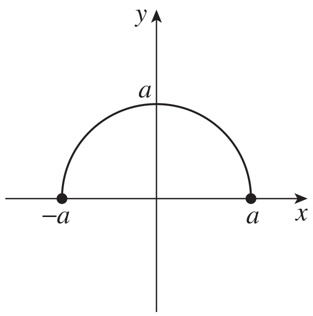
This graph fails the vertical line test, so it is not a function. This can also be seen algebraically — solving the equation for y yields

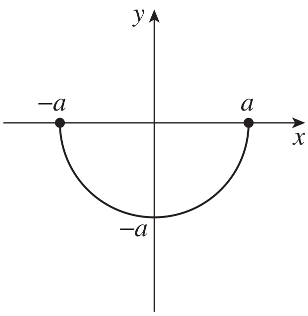
$$y = \sqrt{a^2 - x^2} \quad \text{or} \quad y = -\sqrt{a^2 - x^2},$$

giving two values of y for some values of x.

The *positive square root* $y = \sqrt{a^2 - x^2}$, however, is a function, whose graph is the *upper semicircle* on the left below.

Similarly, the *negative square root* $y = -\sqrt{a^2 - x^2}$ is also a function, whose graph is the *lower semicircle* on the right below.





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Exercise 3G

FOUNDATION

1 Write down the radius and the coordinates of the centre of each circle.

a $x^2 + y^2 = 16$ b $x^2 + y^2 = 49$ c $x^2 + y^2 = \frac{1}{9}$ d $x^2 + y^2 = 1.44$

2 Sketch graphs of these circles, marking all intercepts with the axes, then write down the domain and range of each.

a $x^2 + y^2 = 1$ b $x^2 + y^2 = 9$ c $x^2 + y^2 = \frac{1}{4}$ d $x^2 + y^2 = \frac{9}{4}$

3 Consider the curve $y = x^3$.

a Copy and complete the following table of values:

x	-1.5	-1	-0.5	0	0.5	1	1.5
y							

b Plot the points in the table, using a scale of 2 cm to 1 unit on each axis, and then join the points with a smooth curve.

4 Repeat the previous question for the curve $y = x^4$.

5 [Technology]

a Use computer graphing software to plot accurately on the one number plane $y = x$, $y = x^3$ and $y = x^5$.

b Which three points do all these graphs pass through?

c Which curve is nearest the x-axis for:

i $0 < x < 1$, ii $x > 1$?

d Which curve is nearest the x-axis for:

i $-1 < x < 0$, ii $x < -1$?

e Rotate each curve by 180° about the origin. What do you notice?

f Try finding other powers of x which have the same feature found in part **e**. What do you notice about the index of these functions?

6 [Technology]

a Use computer graphing software to plot accurately on the one number plane $y = x^2$, $y = x^4$ and $y = x^6$.

b Which three points do all these graphs pass through?

c Which curve is nearest the x-axis for:

i $0 < x < 1$, ii $x > 1$?

d Which curve is nearest the x-axis for:

i $-1 < x < 0$, ii $x < -1$?

e Reflect each curve in the y-axis. What do you notice?

f Try finding other powers of x which have the same feature found in part **e**. What do you notice about the index of these functions?

7 In each case, state whether or not the function is a polynomial. If it is a polynomial, write down its degree and the coefficient of x.

a $a(x) = 2x + 3$ b $b(x) = x^3 - 4x^2 + 5$

c $c(x) = 3x^2 - \dfrac{1}{x}$ d $d(x) = \sqrt{x} + x - 1$

e $e(x) = -\dfrac{x^3}{6} + \dfrac{x^2}{2} - x + 1$ f $f(x) = \sqrt{x^2 - 9} + 1$

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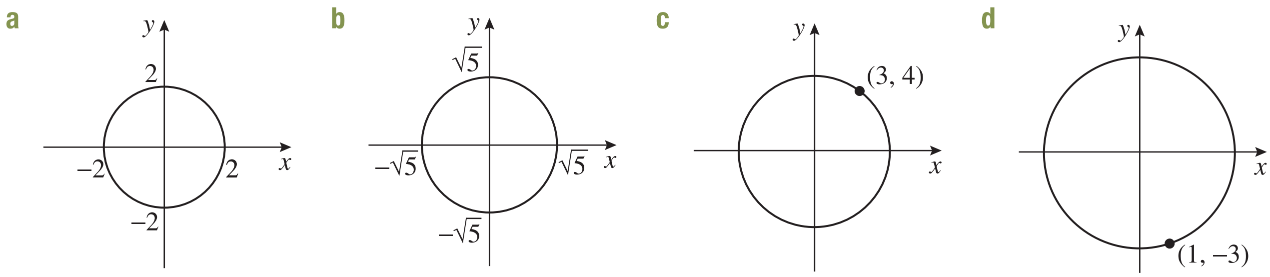
8 Write down the zeroes of each cubic, use a table of values to test its sign, then sketch it, showing the y-intercept.

a $y = (x - 1)(x - 3)(x - 5)$ **b** $y = -3(x + 4)x(x - 2)$ **c** $y = 2x^2(3 - x)$

9 Write down the equation of each circle.



DEVELOPMENT

10 Consider the circle $x^2 + y^2 = 25$.

a Copy and complete the following table of values, correct to one decimal place where necessary. (Remember that a positive number has two square roots.)

x	0	1	2	3	4	5
$y \geq 0$						
$y \leq 0$						

b Plot the points in the table, using a scale of 1 cm to 1 unit on each axis.

c Reflect the points plotted in part **b** in the y-axis, and so sketch the entire circle.

11 Consider the curve $y = \sqrt{x}$.

a Copy and complete the following table of values:

x	0	0.25	1	2.25	4	6.25
y						

b Plot the points in the table, using a scale of 2 cm to 1 unit on each axis, and then join the points with a smooth curve.

c Repeat the process for the function $y = -\sqrt{x}$.

12 **a** Use the results of Question 11 to sketch the graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ on the same number plane.

b What shape has been formed?

c Explain why this has happened.

13 Sketch each semicircle, and state the domain and range.

a $y = \sqrt{4 - x^2}$ **b** $y = -\sqrt{4 - x^2}$ **c** $y = -\sqrt{1 - x^2}$

d $y = \sqrt{\frac{25}{4} - x^2}$ **e** $y = -\sqrt{\frac{9}{4} - x^2}$ **f** $y = \sqrt{0.64 - x^2}$

14 Write down the zeroes of each polynomial, use a table of values to test its sign, then sketch it, showing the y-intercept.

a $y = (x + 2)(x + 1)x(x - 1)(x - 2)$

b $y = -(x - 3)^2(x + 2)^2$

c $y = 2x^2(x - 2)^4(x - 4)$

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15 [Technology]

a Graph these cubic polynomials accurately using computer graphing software.

i $y = \frac{1}{4}x^3 + 2$ **ii** $y = \frac{1}{2}(x^3 - 6x^2 + 9x)$ **iii** $y = \frac{1}{2}(x^3 - 2x^2 - 5x + 6)$

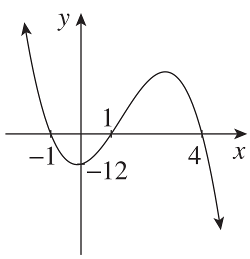
b In each case, check that the y-intercept is equal to the constant term.

c Read the x-intercepts from the screen and then check these values by substituting them into the corresponding polynomial.

d Write each cubic in parts **ii** and **iii** in the form $y = ax^3 + bx^2 + cx + d$. Then take the product of all the zeroes found in part **c** and compare the result with $\frac{d}{a}$ in each case. What do you notice?

16 These graphs are known to be polynomials, and the second is known to have degree 7. Write down their equations factored into linear factors.

a



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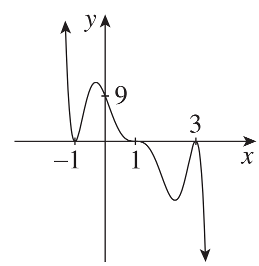
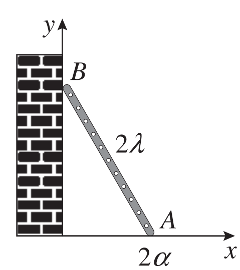


17 Write down the radius of each circle or semicircle. Also state any points on each curve whose coordinates are both integers.

a $x^2 + y^2 = 5$ **b** $y = -\sqrt{2 - x^2}$ **c** $x = \sqrt{10 - y^2}$ **d** $x^2 + y^2 = 17$

ENRICHMENT

18 The diagram shows a ladder of length 2λ leaning against a wall so that the foot of the ladder is distant 2α from the wall.



a Find the coordinates of B.

b Show that the midpoint P of the ladder lies on a circle with centre at the origin. What is the radius of this circle?

c Knowing that P lies on a circle, how could the radius have been more easily found?

19 [Technology]

As discovered in the previous sections, the graphs of quadratic functions fall into one of two categories, concave up and concave down. In contrast, cubic functions have six basic types.

a Use computer software or a table of values to plot accurately the cubic $y = c(x)$ for:

i $c(x) = x^3 + x$ **ii** $c(x) = x^3$ **iii** $c(x) = x^3 - x$

b These three curves have some similarities.

i For large positive and negative values of x, which two quadrants does the graph of $y = c(x)$ lie in?

ii Rotate each curve by 180° about the origin. What do you notice?

iii By considering your investigation in Question 5, explain why that might be.

c Look carefully at each curve as it passes through the origin. What distinguishes each graph there?

d Conclude this investigation by accurately graphing $y = -c(x)$ for each cubic function in part **a**. Then look for any similarities and differences.

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3H Two graphs that have asymptotes

This section reviews exponential graphs and rectangular hyperbolas. They are grouped together because both types of graphs have asymptotes, which need further discussion. Then direct and inverse variation are briefly reviewed.

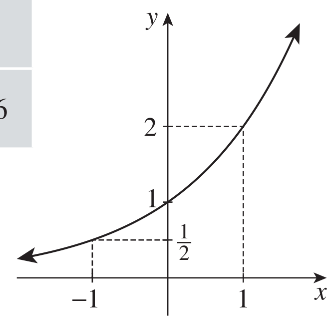
Exponential functions

Functions of the form $y = a^x$, where the base a is positive and $a \neq 1$, are called *exponential functions*, because the variable x is in the *exponent* or *index*.

Here is a sketch of the function $y = 2^x$.

$y = 2^x$

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16



Three key features should be shown when sketching this graph:

- The y-intercept is $y = 1$, because $2^0 = 1$.
- When $x = 1$, $y = 2$, which is the base 2, because $2^1 = 2$.
- The x-axis is a horizontal asymptote, as discussed below.

Limits and asymptotes of exponential functions

On the far left, as x becomes a very large negative number, $y = 2^x$ becomes very small. Indeed, we can make y 'as small as we like' by choosing sufficiently large negative values of x. We say that 'as x approaches negative infinity, y approaches the limit zero', and write

$$\text{As } x \to -\infty, y \to 0 \qquad \text{or} \qquad \lim_{x \to \infty} y = 0.$$

The x-axis is called an *asymptote* of the curve (from the Greek word *asymptotos*, meaning 'apt to fall together'), because the curve gets as close as we like to the x-axis for sufficiently large negative values of x.

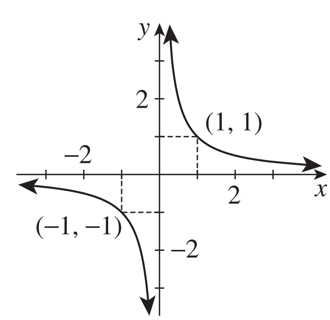
Rectangular hyperbolas

The *reciprocal function* $y = \dfrac{1}{x}$ can also be written as $xy = 1$, and has a graph that is called a *rectangular hyperbola*. This graph has two disconnected parts called *branches*.

$y = \dfrac{1}{x}$

x	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1	2	5	10
y	$*$	10	5	2	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$

x	-10	-5	-2	-1	$-\frac{1}{2}$	$-\frac{1}{5}$	$-\frac{1}{10}$
y	$-\frac{1}{10}$	$-\frac{1}{5}$	$-\frac{1}{2}$	-1	-2	-5	-10



The star ($*$) at $x = 0$ means that the function is not defined there.

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Limits and asymptotes of rectangular hyperbolas

The x-axis is an asymptote to this curve on both sides of the graph. We can make y 'as small as we like' by choosing sufficiently large positive or negative values of x. We say that 'as x approaches $-\infty$, y approaches the limit zero', and write

$$\text{As } x \to \infty,\ y \to 0 \quad \text{or} \quad \lim_{x \to \infty} y = 0 \qquad \text{and} \qquad \text{As } x \to -\infty,\ y \to 0 \quad \text{or} \quad \lim_{x \to -\infty} y = 0$$

The y-axis is a second asymptote to the graph. On the right-hand side of the origin, when x is a very small positive number, y becomes very large. We can make y 'as large as we like' by taking sufficiently small but still positive values of x. We say that 'as x approaches zero from the right, y approaches ∞', and write

$$\text{As } x \to 0^+,\ y \to \infty.$$

On the left-hand side of the origin, y is negative and can be made 'as large negative as we like' by taking sufficiently small negative values of x. We say that 'as x approaches zero from the left, y approaches $-\infty$', and write

$$\text{As } x \to 0^-,\ y \to -\infty.$$

Direct and inverse variation

Direct variation and inverse variation were introduced in previous years, and are easily summarised.

26 DIRECT AND INVERSE VARIATION

Direct variation: A variable y *varies directly* with a variable x if

$y = kx$, for some non-zero *constant k of proportionality*.

- The graph of y as a function of x is thus a line through the origin.

Inverse variation: A variable y *varies inversely* with a variable x if

$y = \dfrac{k}{x}$, for some non-zero *constant k of proportionality*.

- The graph of y as a function of x is thus a rectangular hyperbola whose asymptotes are the x-axis and the y-axis.

Inverse variation is also called 'indirect variation', and the word 'proportion' is often used instead of 'variation'. Most applications begin by finding the constant of proportionality, as in Examples 23 and 24.

Example 23 — 3H

[Direct variation]

a Garden mulch is sold in bulk, with the cost C proportional to the volume V in cubic metres. Write this algebraically.

b The shop quotes \$270 for 7.5 m^3. Find the constant of proportionality, and graph the function.

c How much does 12 m^3 cost?

d How much can I buy for \$600?

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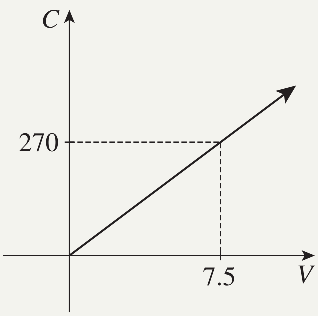
SOLUTION

a $C = kV$, for some constant k.

b Substituting, $270 = k \times 7.5$
$k = 36.$
(More precisely, $k = \$36/\text{m}^3$.)



c $C = 36 \times 12$
$= \$432$

d $600 = 36V$
$V = 16\frac{2}{3}\,\text{m}^3.$

Example 24 3H

[Inverse variation]

a The wavelength λ in metres of a musical tone is inversely proportional to its frequency f in vibrations per second. Write this algebraically.

b The frequency of middle C is about $260\,\text{s}^{-1}$ ('260 vibrations per second'), and its wavelength is about 1.319 m. Find the constant of proportionality.

c Find the wavelength of a sound wave with frequency $440\,\text{s}^{-1}$.

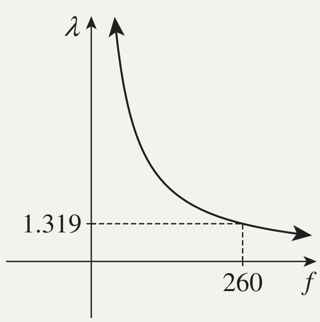
d Find the frequency of a sound wave with wave length 1 metre.

e What is the approximate speed of sound in air, and why?

SOLUTION

a $\lambda = \frac{k}{f}$, for some constant k.

b Substituting, $1.319 = \dfrac{k}{260}$
$k \doteqdot 343$
(More precisely, $k = 343\,\text{m/s}$.)



c $\lambda = \dfrac{k}{440}$
$\doteqdot 0.779\,\text{m}$

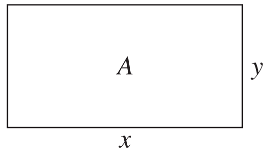
d $1 = \dfrac{k}{f}$
$f \doteqdot 343\,\text{s}^{-1}$

e About 343 m/s, because 343 waves, each 1 metre long, go past in 1 second.

Direct and inverse variation are closely related

Direct and inverse variation are very closely related, despite their contrasting graphs. For example, if a rectangle has area A and adjacent sides x and y, then

$$A = xy \qquad \text{and} \qquad y = \frac{A}{x}.$$



- If the side y is constant, the area A is directly proportional to the other side x.
- If the area A is constant, the side y is inversely proportional to the side x.

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Exercise 3H

FOUNDATION

Note: If computer graphing software is not available, the two technology questions can be completed using tables of values.

1 a Consider the hyperbola $y = \frac{2}{x}$.

i Copy and complete the table of values for this hyperbola.

x	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4
y					*				

ii Plot the points, using a scale of 1 cm to 1 unit on each axis, then sketch the hyperbola.

iii Which two quadrants do the branches of the curve lie in?

iv Write down the equations of the two asymptotes of the hyperbola.

v Write down the domain and range of the function.

b Do likewise for each of these hyperbolas.

i $y = \frac{4}{x}$ **ii** $y = \frac{3}{x}$

2 [Technology]
Use computer graphing software (if unavailable, tables of values will also do) to plot accurately on the one number plane

$$y = \frac{1}{x} \quad \text{and} \quad y = \frac{4}{x} \quad \text{and} \quad y = \frac{9}{x}.$$

a Which two quadrants do the branches of each hyperbola lie in?

b Write down the equations of the two asymptotes of each hyperbola.

c Write down the domain and range of each hyperbola.

d Write down the coordinates of the points on each hyperbola closest to the origin. What do you notice?

3 a Consider the exponential curve $y = 3^x$.

i Copy and complete the following table of values of the exponential function $y = 3^x$. Give answers correct to one decimal place where necessary.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y									

ii Plot these points, using scales of 1 cm to 1 unit on both axes, then sketch the curve.

iii What is the y-intercept of the function?

iv What is the y-coordinate when $x = 1$?

v Write down the equation of the asymptote.

vi Write down the domain and range of the function.

b Do likewise for each of these exponential functions.

i $y = 4^x$ **ii** $y = 1.5^x$

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4 [Technology]
Use computer graphing software to plot accurately on the one number plane
$y = 2^x$ and $y = 3^x$ and $y = 4^x$.
- **a** Which point is common to all three graphs?
- **b** Write down the equation of the asymptote of each exponential curve.
- **c** Write down the domain and range of each exponential function.
- **d** Confirm by observation that at $x = 1$, the y-coordinate equals the base.
- **e** Which curve increases more rapidly to the right of the y-axis, and why?
- **f** Which curve approaches the asymptote more quickly to the left of the y-axis? Why?

5 **a** Sketch each hyperbola. (A table of values may help.)
i $y = -\frac{2}{x}$ **ii** $y = -\frac{4}{x}$ **iii** $y = -\frac{3}{x}$
b Compare these equations and graphs with those in Question 1.
- **i** Which quadrants do the graphs of part **a** lie in?
- **ii** What has changed in the equation to cause this difference?

6 **a** Sketch each of these exponential functions. (A table of values may help.)
i 3^{-x} **ii** 4^{-x} **iii** 1.5^{-x}
b Compare these equations and graphs with those in Question 3.
- **i** Has the y-intercept changed?
- **ii** Has the asymptote changed?
- **iii** For the graphs in part **a**, at what value of x is the y-coordinate equal to the base?
- **iv** Heading to the right along each of these curves, describe how the y-coordinate changes.
- **v** What has changed in the equation to cause these differences?

DEVELOPMENT

7 Make y the subject of each equation, then sketch its graph.
a $xy = \frac{1}{2}$ **b** $xy = -6$

8 For each hyperbola in the previous question:
- **i** write down the coordinates of the points closest to the origin,
- **ii** list any points with integer coordinates.

9 Sketch these exponential graphs without resorting to a table of values. Ensure that the key features are shown.
a $y = 5^x$ **b** $y = 2^{-x}$

10 [Direct variation]
- **a** The amount of paint P, in litres, used to paint a building is directly proportional to the area A, in square metres, to be covered. Write this algebraically.
- **b** A certain building requires 48 L to cover an area of $576\,\text{m}^2$. Find the constant of proportionality, and graph the function.
- **c** A larger building has an area of $668\,\text{m}^2$ to be painted. How many litres of paint will this require?
- **d** A paint supplier sells 40 L buckets and 4 L tins of paint. How many buckets and tins must be bought in order to paint the building completely?

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11 [Inverse variation]
The owners of the Fizgig Manufacturing Company use a uniformly elastic demand curve to model their sales. Thus if the price of a Fizgig is p and the quantity sold per year is q, then the turnover from sales is constant. That is, $pq = T$, for some constant T.

- **a** Last year, the price of a Fizgig was $p = \$6$ and the quantity sold was $q = 400000$. Find T.
- **b** The company's board of directors want to raise the price to $p = \$8$ next year. How many Fizgigs can the company expect to sell if this happens?
- **c** Under this model, what will happen to the sales if the price is doubled instead?
- **d** Sketch the graph of the demand curve with q on the horizontal axis and p on the vertical axis.

12 This question requires the language of limits from pages 106–7 of this section.

- **a** In Question 3, the line $y = 0$ is an asymptote to $y = 3^x$. Write a statement using limits to justify this.
- **b** In Question 6 **a ii**, the line $y = 0$ is an asymptote to the exponential curve $y = 4^{-x}$. Write a statement using limits to justify this.
- **c** In Question 1, the lines $y = 0$ and $x = 0$ are asymptotes to the hyperbola $y = \frac{2}{x}$. Write four statements using limits to justify this.

13
- **a** Use the index laws to explain why $y = \left(\frac{1}{2}\right)^x$ has the same graph as $y = 2^{-x}$.
- **b** Hence sketch the graph of the function $y = \left(\frac{1}{2}\right)^x$.

14
- **a** Where does the hyperbola $xy = c^2$ intersect the line $y = x$?
- **b** Confirm your answer by plotting the situation when $c = 2$.

15 [Inverse variation]
An architect is designing a building. The client has insisted that one of the rooms in the building must have an area of $48\,\text{m}^2$. For ease of design, the length ℓ and the breadth b must each be a whole number of metres. Because of the furniture that must go into the room, no wall may be less than 4 m. What are the possible dimensions of the room?

16 Does the equation $xy = 0$ represent a hyperbola? Explain your answer.

17
- **a** Show that $(x + y)^2 - (x - y)^2 = 4$ is the equation of a hyperbola. Sketch it.
- **b** Show that $(x + y)^2 + (x - y)^2 = 4$ is the equation of a circle. Sketch it.
- **c** Solve these two equations simultaneously. Begin by subtracting the equation of the hyperbola from the equation of the circle.
- **d** Sketch both curves on the same number plane, showing the points of intersection.

ENRICHMENT

18 You will have observed that hyperbolas do not intersect their asymptotes. There are some curves, however, that cross their asymptotes.

- **a** Complete the following table of values for $y = \dfrac{2x}{x^2 + 1}$.

x	-8	-4	-2	-1	-0.5	0	0.5	1	2	4	8
y											

- **b** Plot the points, using a scale of 1 cm to 1 unit on each axis, and join them with a smooth curve.

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c What is the horizontal asymptote of this curve?

d Where does the curve cross its asymptote?

19 a The line $y = -\frac{1}{4}b^2x + b$ has intercepts at A and B. Find the coordinates of P, the midpoint of AB.

b Show that P lies on the hyperbola $y = \dfrac{1}{x}$.

c Show that the area of $\triangle OAB$, where O is the origin, is independent of the value of b.

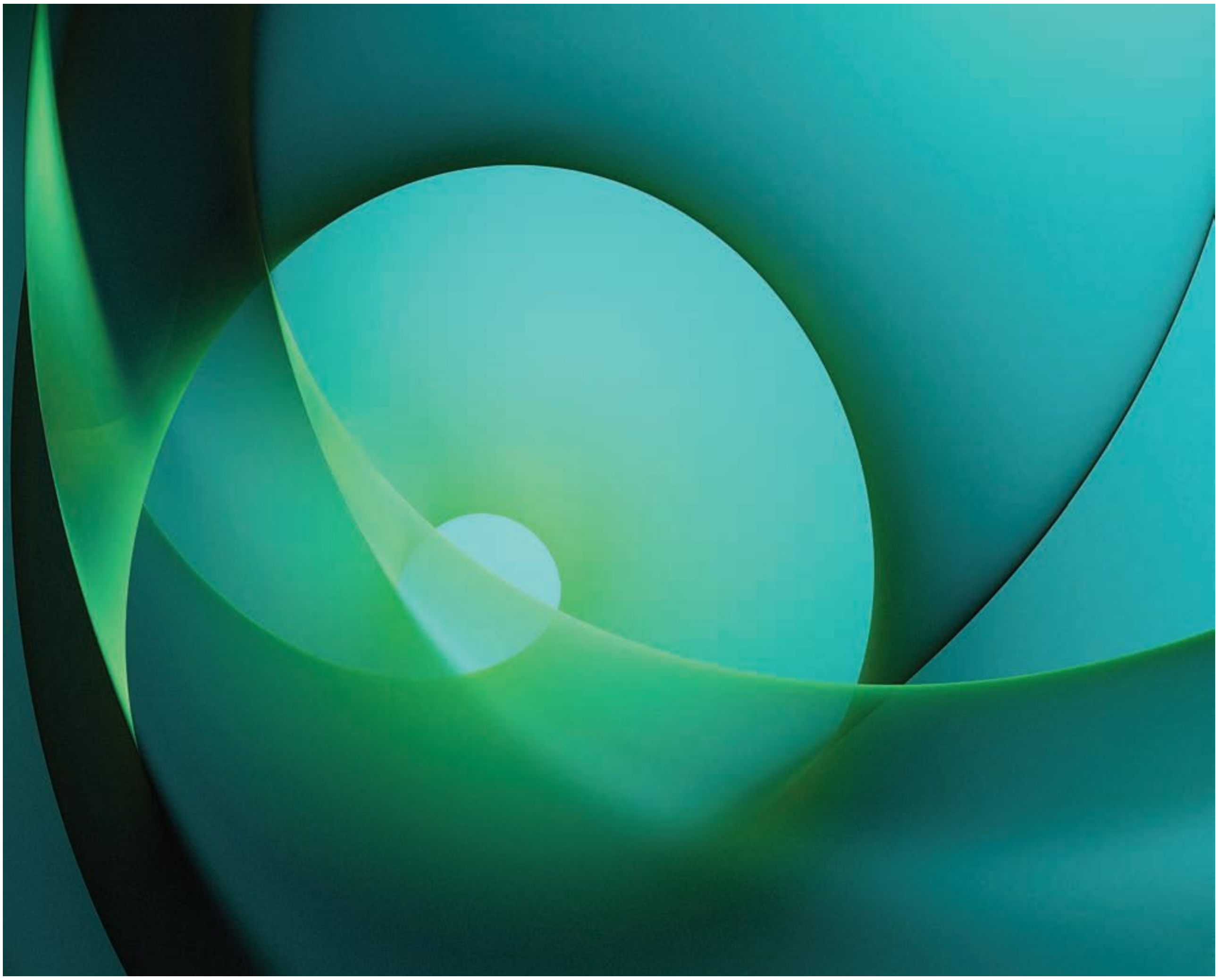
20 The curve $y = 2^x$ is approximated by the parabola $y = ax^2 + bx + c$ for $-1 \le x \le 1$. The values of the constants a, b and c are chosen so that the two curves intersect at $x = -1, 0, 1$.

a Find the values of the constant coefficients.

b Use this parabola to estimate the values of $\sqrt{2}$ and $\dfrac{1}{\sqrt{2}}$.

c Compare the values found in part **b** with the values obtained by a calculator. Show that the percentage errors are approximately 1.6% and 2.8% respectively.

Refer to Appendix A for new material on Bracket interval notation added due to syllabus changes requiring both notations for intervals to be presented in Year 11.



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3I Four types of relations

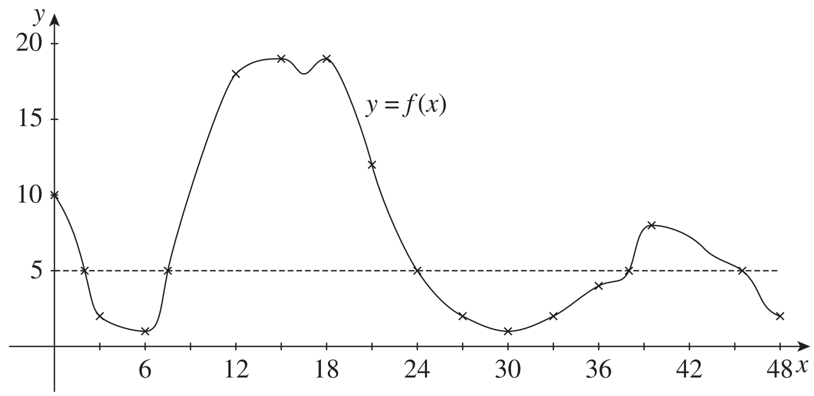
In Section 3B, functions were generalised to relations, and a relation was defined as any set of ordered pairs. The *vertical line test* was used to establish from the graph whether or not a relation is a function.

This section introduces the corresponding *horizontal line test*, and classifies relations into four types depending on whether their graphs pass or fail one or the other of these two tests. Reading the graph backwards is the key idea because it makes it clear why horizontal lines are so important on a graph.

There are also a few examples of relations and functions whose domain and range do not consist of numbers. Such relations do not appear very often in this course, but they are vitally important in the databases and spreadsheets that we use everyday on the internet and in our record-keeping.

Using horizontal lines to read a graph backwards

Here is a graph of the temperature y°C measured x hours after midnight over two days. Call the function $y = f(x)$.



The straightforward way to read this graph is to take a time, say $x = 2$, on the horizontal axis, draw the vertical line $x = 2$ to the graph, and read the temperature off the y-axis. This graph is a function — there are never two answers.

'After 2 hours, the temperature was 5°C.'

Algebraically, substituting $x = 2$ gives $f(2) = 5$.

We can also read the graph backwards. Take a temperature, say 5°C, on the vertical axis, draw a horizontal line to the graph, and read off the times when the temperature was 5°C.

'The temperature was 5°C after 2, $7\frac{1}{2}$, 24, 38 and $45\frac{1}{2}$ hours.'

Algebraically, the solutions of $f(x) = 5$ are $x = 2, 7\frac{1}{2}, 24, 38$ and $45\frac{1}{2}$.

27 USING HORIZONTAL LINES TO READ A GRAPH BACKWARDS

- To solve $f(x) = b$ from the graph, draw the horizontal line $y = b$ and read off the corresponding x-values.
- Thus we are *reading the graph backwards* — starting with the y-values and looking for the corresponding x-values.

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The horizontal line test

The graph above is called a *many-to-one* function, because many x-values all map to the one y-value. To formalise this, we introduce the *horizontal line test*. This test is the companion of the vertical line test — the two definitions simply exchange the words 'vertical' and 'horizontal'.

28 THE VERTICAL AND HORIZONTAL LINE TESTS

- **Vertical line test:**
  No vertical line meets the graph more than once.
- **Horizontal line test:**
  No horizontal line meets the graph more than once.

Using the horizontal line test with functions

The two graphs below are both functions, because they pass the vertical line test.

$y = 2^x$ is one-to-one $\qquad$ $y = x^2$ is many-to-one

The first graph of $y = 2^x$ also passes the horizontal line test. This means that the situation is symmetric in the following way:

- Every number in the domain corresponds to exactly one number in the range.
- Every number in the range corresponds to exactly one number in the domain.

This graph is therefore called *one-to-one*, and the function is a *one-to-one correspondence* between the domain, all real x, and the range, $y > 0$. When the graph is read backwards, there is never more than one answer.

The second graph of $y = x^2$ fails the horizontal line test. For example, the horizontal line $y = 4$ meets the curve twice. This is why 4 has two square roots, 2 and -2. The situation is no longer symmetric vertically and horizontally.

- Every number in the domain corresponds to exactly one number in the range.
- At least one number in the range (say 4) corresponds to more than one number in the domain (to 2 and -2).

The graph is therefore called *many-to-one*, because, for example, many x-values map to the one y-value 4. When we read the graph backwards, at least one value of y will give more than one answer.

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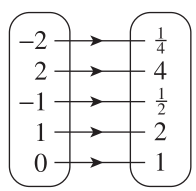
29 ONE-TO-ONE AND MANY-TO-ONE

Suppose that a graph passes the vertical line test, that is, it is a function.

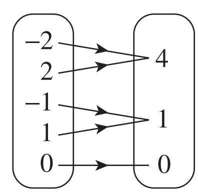
- If it also passes the horizontal line test, it is called *one-to-one*.
  - Whether the graph is read forwards or backwards, there will be no more than one answer.
  - It is a one-to-one correspondence between the domain and the range.
- If it fails the horizontal line test, it is called *many-to-one*.
  - When the graph is read forwards, there is never more than one answer.
  - When the graph is read backwards, at least one value of y gives more than one answer. Many values of x map to this one value of y.

Two diagrams of a different type may help to clarify the situation. Both functions are something-to-one because every number in the domain maps to exactly one number in the range.



$y = 2^x$ is one-to-one

For every input there is at most one output, and for every output there is at most one input.

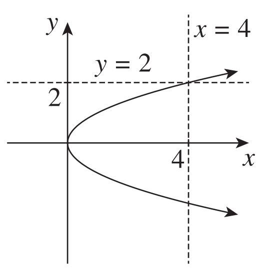


$y = x^2$ is many-to-one

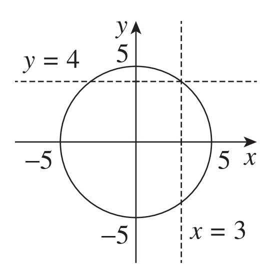
For every input there is at most one output, but for at least one output there is more than one input.

Using the horizontal line test with relations that are not functions

The two graphs below are relations, but not functions, because they fail the vertical line test.



$y^2 = x$ is one-to-many



$x^2 + y^2 = 25$ is many-to-many

The left-hand graph of $y^2 = x$ fails the vertical line test, but passes the horizontal line test. For example, the vertical line $x = 4$ meets the curve where $y = 2$ and where $y = -2$, but no horizontal line meets the graph more than once. The situation here is not symmetric vertically and horizontally.

- At least one number in the domain (say 4) corresponds to more than one number in the range (2 and -2).
- Every number in the range corresponds to exactly one number in the domain.

This graph is therefore called *one-to-many*, because, for example, the one number 4 in the domain corresponds to the two numbers 2 and -2 in the range.

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The right-hand graph of $x^2 + y^2 = 25$ fails both horizontal and vertical line tests. For example, the vertical line $x = 3$ meets the curve twice, and the horizontal line $y = 4$ meets the curve twice. The situation is again symmetric in that:

- At least one number in the domain corresponds to more than one number in the range.
- At least one number in the range corresponds to more than one number in the domain.

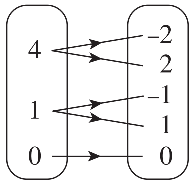
The graph is therefore called *many-to-many*, because, for example, the one number 3 in the domain corresponds to the two numbers 4 and -4 in the range, and the one number 4 in the range corresponds to the two numbers 3 and -3 in the domain.

30 ONE-TO-MANY AND MANY-TO-MANY

Suppose that a graph fails the vertical line test, that is, it is not a function.

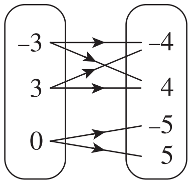
- If it passes the horizontal line test, it is called *one-to-many*.
  - — When the graph is read forwards, at least one value of x gives more than one answer. Many values of y correspond to this one value of x.
  - — When the graph is read backwards, there is never more than one answer.
- If it also fails the horizontal line test, it is called *many-to-many*.
  - — Whether the graph is read forwards or backwards, there will be values of x or of y that give more than one answer.

Again, two further diagrams help to clarify the situation. Both relations are something-to-many because at least one number in the domain maps to at least two numbers in the range



$y^2 = x$ is one-to-many

For at least one number in the domain, there is more than one corresponding number in the range, but for every number in the range, there is only one number in the domain.



$x^2 + y^2 = 25$ is many-to-many

For at least one number in the domain, there is more than one corresponding number in the range, and for at least one number in the range, there is more than one corresponding number in the domain.

The four types of relations

Here is a table of the four types of relations:

Type	Vertical line test	Horizontal line test
One-to-one	Passes (so a function)	Passes
Many-to-one	Passes (so a function)	Fails
One-to-many	Fails (so not a function)	Passes
Many-to-many	Fails (so not a function)	Fails

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Relations may involve objects other than numbers

Function and relations do not have to act on numbers. They can act on anything at all, and the four types of relations defined in this section are as common in database logic as in mathematics.

Example 25 3I

In a firm of 10 000 employees, a database records the postcode (y-value) that each person (x-value) has nominated as a home address.

a Explain why this is a function, and whether it is one-to-one or many-to-one.

b What change may there be for a firm of 10 employees?

SOLUTION

a It is a function because every person's nominated home has a postcode. It will not pass the horizontal line test because there are fewer than 10 000 postcodes in NSW.

b It is quite possible that the 10 employees all live in different postcodes, in which case the function would be one-to-one.

Example 26 3I

Many people (x-values) in the suburb of Blue Hills own a pet (y-value).

a What sort of relation is this?

b Council registers each pet (y-value) with only one owner (x-value). What sort of relation is this?

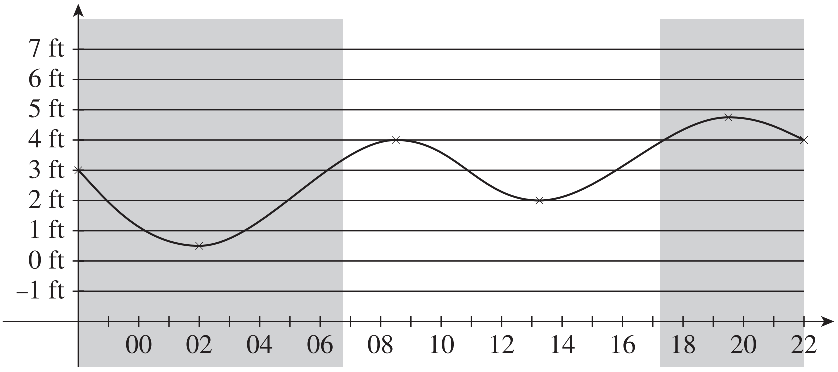
SOLUTION

a It fails the vertical line test because many residents have several pets. It fails the horizontal line test because many pets are owned by everyone in the family. Hence the relation is many-to-many.

b This relation passes the horizontal line test because each pet has no more than one person registering it. Hence the relation is one-to-many.

Exercise 3I FOUNDATION

1 The graph below shows the tides at Benicia, Carquinez Strait, California, on Sunday, 16 September 2016. It is in feet because the US uses imperial measures, and there are no vertical lines, which is inconvenient. Your numerical answers will only be approximate.



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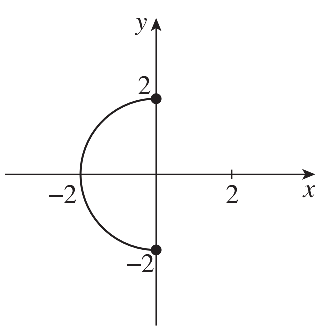
a Does the graph pass or fail the vertical line test? Is it a function?
b Does the graph pass or fail the horizontal line test? Is the graph one-to-one, many-to-one, one-to-many or many-to-many?
c The graph runs for 24 hours. What are the starting and finishing times?
d What were the tide heights at 6:00 am and 5:00 pm?
e When was the tide height:
i 3 ft? **ii** 2 ft? **iii** 6 ft?
f What are the possible numbers of solutions of the equation $f(x) = k$, as k varies, where $f(x)$ is the function?

2 Go back to the temperature graph printed on page 113 at the start of Section 3I.
a Why is the graph a function, and why is it classified as many-to-one?
b What was the temperature at 6:00 am on the second day?
c When was the temperature 20°C, and when was it 8°C?
d What are the possible numbers of solutions of the equation $f(x) = k$, as k varies?

3 **a** Say whether each relation sketched below passes the vertical line test, and whether it passes the horizontal line test.

i


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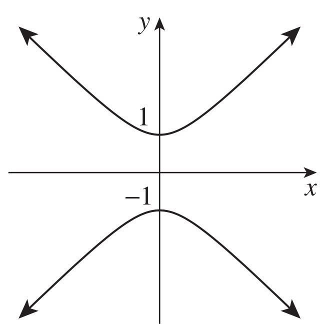

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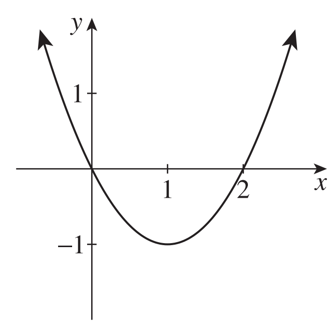

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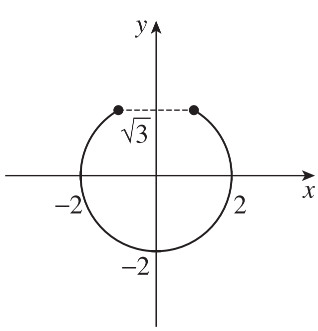

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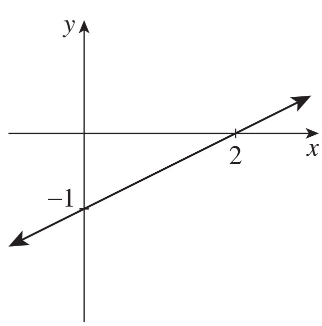

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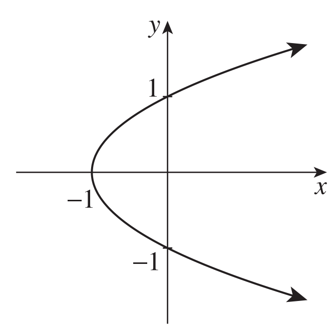

b Which graphs above give no more than one answer when read forwards from x-value to y-value(s)?
c Which graphs above give no more than one answer when read backwards from y-value to x-value(s)?
d Which graphs above are a one-to-one correspondence between domain and range?
e Classify each graph as one-to-one, many-to-one, one-to-many or many-to-many.

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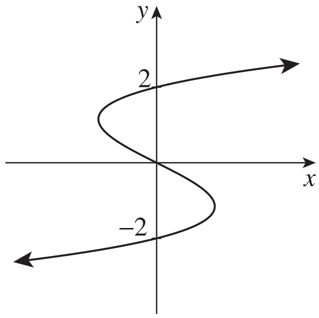
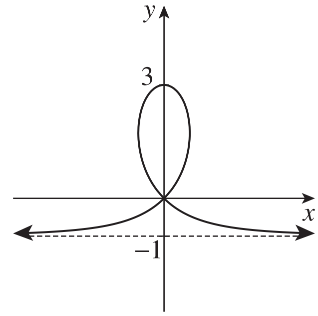
4 Classify each graph as one-to-one, many-to-one, one-to-many or many-to-many.

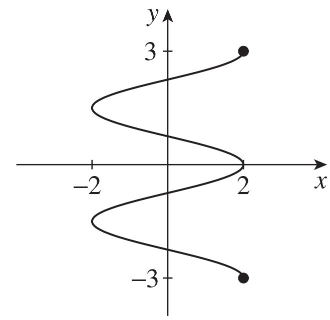
a



b



c



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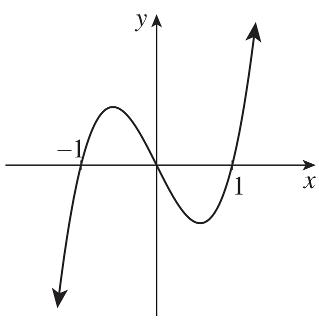
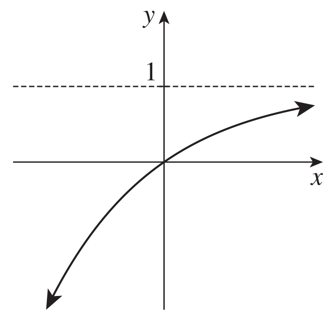


e



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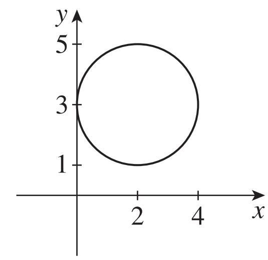

DEVELOPMENT

5 **a** Explain, with an example using a y-value, why each function is many-to-one.

i $y = x^2 - 4$ **ii** $y = (x - 1)x(x + 1)$ **iii** $y = x^4 + 1$

b Hence classify each relation below.

i $x = y^2 - 4$ **ii** $x = (y - 1)y(y + 1)$ **iii** $x = y^4 + 1$

6 **a** By solving for x, show that each function is one-to-one. (This method works only if x can be made the subject. Then the relation is one-to-one if there is never more than one answer, and many-to-one otherwise.)

i $y = 3x - 1$ **ii** $y = 5 - 2x$ **iii** $y = 8x^3$ **iv** $y = \dfrac{5}{x}$

b Hence classify each relation below.

i $x = 3y - 1$ **ii** $x = 5 - 2y$ **iii** $x = 8y^3$ **iv** $x = \dfrac{5}{y}$

7 By giving an example using an x-value, and an example using a y-value, show how each relation fails the horizontal and vertical line tests, and hence is many-to-many.

a $(x - 3)^2 + (y + 1)^2 = 25$ **b** $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ **c** $x^2 - y^2 = 1$

8 **a** A database records all the doctors that a person has visited. Does this relation pass the horizontal or vertical line tests, and how should it be classified? (The x-values are all people in Australia, and the y-values are the doctors.)

b The database is queried to report the last doctor (y-value) that a person (x-value) has visited. Does this change the answers in part **a**?

9 Every student (x-value) in a class has a preferred name (y-value).

a What types of relation could this be?

b Explain what extra condition could make the classification unambiguous?

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10 a A dancer points north, closes his eyes, and spins, ending up pointing east. Through how many degrees has the dancer turned (take clockwise as positive)?

b Regard the final position as the x-value, and the degrees the dancer turns as the y-value. Classify this relation.

c Now regard the degrees the dancer turns as the x-value, and the final position as the y-value. Classify this relation.

11 A person living in a block of flats is called a co-habitant of another person if they both live in the same flat.

a Classify this relation (the x-values and y-values are both any person in the block).

b Under what condition would this relation be one-to-one, and what would then be particularly special about the relation?

c What happens if the block is being renovated and no one is living there?

12 Classify each relation as one-to-one, many-to-one, one-to-many or many-to-many.

a $y = 4$ **b** $x = -3$ **c** $x + y = 0$

d $x^2 + y^2 = 0$ **e** $x^2 - y^2 = 0$ **f** $x = y^2 - 5y + 6$

g $y = x^3 - 7x^2 + 12x$ **h** $y = x^3 + 8$ **i** $y = (x + 8)^3$

j $xy = 36$ **k** $x^2y^2 = 36$ **l** $x^3y^3 = 1000$

13 Suppose that $f(x)$ and $g(x)$ are one-to-one functions.

a If a and b lie in the domain of $g\big(f(x)\big)$, and $g\big(f(a)\big) = g\big(f(b)\big)$, show that $a = b$.

b What does this prove about the composition of two one-to-one functions?

ENRICHMENT

14 Classify each function as one-to-one or many-to-one, giving reasons.

a $f(x) = \begin{cases} 2x & \text{for integers } x, \\ x, & \text{otherwise.} \end{cases}$

b $f(x) = \begin{cases} \frac{1}{2}x & \text{for integers } x, \\ x, & \text{otherwise.} \end{cases}$

c $f(x) = \begin{cases} x^3 & \text{for } x \text{ rational,} \\ x, & \text{for } x \text{ irrational.} \end{cases}$

d $f(x) = \begin{cases} x^3 & \text{for } x \text{ irrational,} \\ x, & \text{for } x \text{ rational.} \end{cases}$

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Chapter 3 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 3 Multiple-choice quiz

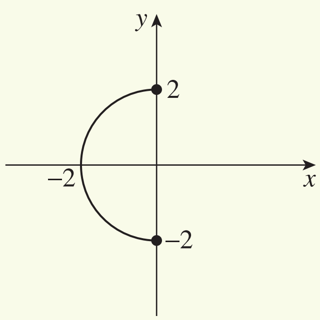
- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

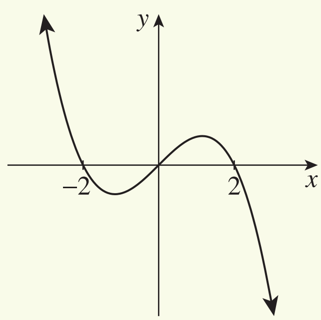
Chapter review exercise

1 Determine which of the relations sketched below are functions.

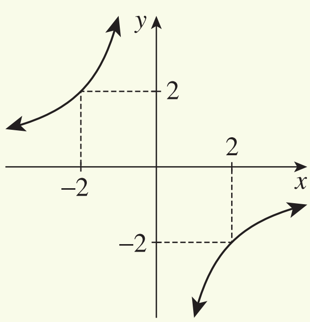
a



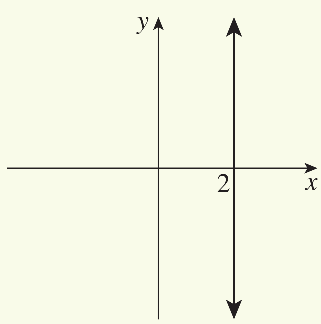
b



c



d



2 Write down the domain and range of each relation in the previous question.

3 Find $f(3)$ and $f(-2)$ for each function.

a $f(x) = x^2 + 4x$ **b** $f(x) = x^3 - 3x^2 + 5$

4 Find the natural domain of each function.

a $f(x) = \dfrac{1}{x - 2}$ **b** $f(x) = \sqrt{x - 1}$ **c** $f(x) = \sqrt{3x + 2}$ **d** $f(x) = \dfrac{1}{\sqrt{2 - x}}$

5 Find $F(a) - 1$ and $F(a - 1)$ for each function.

a $F(x) = 2x + 3$ **b** $F(x) = x^2 - 3x - 7$

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Review

6 In each case, sketch the graph of $y = f(x)$. A table of values may be helpful.

a $f(x) = \begin{cases} x + 1, & \text{for } x < 0, \\ 1 - x, & \text{for } x \geq 0. \end{cases}$ **b** $f(x) = \begin{cases} x^2 - 1, & \text{for } x < 1, \\ 2x - x^2, & \text{for } x \geq 1. \end{cases}$

7 Find the x-intercept and the y-intercept of each line, then sketch it.

a $y = 2x + 2$ **b** $x - 3y + 6 = 0$

8 Sketch each of these lines through the origin.

a $y = 2x$ **b** $x + 2y = 0$

9 **a** Sketch these two lines.

i $y = -1$ **ii** $x - 3 = 0$

b Where do the lines intersect?

10 Use factoring where necessary to find the zeroes of each quadratic function. Hence sketch the graph of $y = f(x)$, showing all intercepts and the coordinates of the vertex. Also state the domain and range.

a $f(x) = 16 - x^2$ **b** $f(x) = x(x + 2)$ **c** $f(x) = (x - 2)(x - 6)$
d $f(x) = -(x + 5)(x - 1)$ **e** $f(x) = x^2 + x - 6$ **f** $f(x) = -x^2 + 2x + 8$

11 Complete the square in each quadratic. Then sketch the graph of each function, showing the vertex and the intercepts with the axes.

a $y = x^2 + 2x - 5$ **b** $y = -x^2 + 6x - 6$
c $y = -x^2 + 2x - 3$ **d** $y = x^2 + 6x + 10$

12 Sketch the graph of each parabola. Use the discriminant to determine whether or not there are any x-intercepts. Show the vertex and the intercepts with the axes.

a $y = -x^2 - 2x + 1$ **b** $y = x^2 - 4x + 2$
c $y = x^2 - 4x + 8$ **d** $y = -x^2 + 6x - 15$

13 Use a table of values to test the sign, and then sketch each polynomial.

a $y = (x - 1)(x - 3)(x - 6)$ **b** $y = -x^2(x + 2)(x - 2)^2$

14 Sketch each circle.

a $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 100$

15 Sketch each semicircle, and write down the domain and range in each case.

a $y = \sqrt{16 - x^2}$ **b** $y = -\sqrt{25 - x^2}$

16 Construct a table of values for each hyperbola, then sketch it. Write down the domain and range in each case.

a $y = \dfrac{8}{x}$ **b** $y = -\dfrac{4}{x}$

17 Construct a table of values for each exponential function, then sketch it. Write down the domain and range in each case.

a $y = 2^x$ **b** $y = 3^{-x}$

18 Construct a table of values for each function, then sketch it.

a $y = x^3 - 3x^2$ **b** $y = x^4 - 4x^2$ **c** $y = \sqrt{x + 1}$

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Review

19 [A revision medley of curve sketches] Sketch each set of graphs on a single pair of axes, showing all significant points. A table of values may help in certain cases.

a $y = 2x$, $y = 2x + 3$, $y = 2x - 1$
b $y = -\frac{1}{2}x$, $y = -\frac{1}{2}x + 1$, $y = -\frac{1}{2}x - 2$
c $y = x^2$, $y = (x + 2)^2$, $y = (x - 1)^2$
d $x + y = 0$, $x + y = 2$, $x + y = -3$
e $y = x^2$, $y = 2x^2$, $y = \frac{1}{2}x^2$
f $x - y = 0$, $x - y = 1$, $x - y = -2$
g $x^2 + y^2 = 4$, $x^2 = 1 - y^2$, $y^2 = 25 - x^2$
h $y = 3x$, $x = 3y$, $y = 3x + 1$, $x = 3y + 1$
i $y = 2^x$, $y = 3^x$, $y = 4^x$
j $y = -x$, $y = 4 - x$, $y = x - 4$, $x = -4 - y$
k $y = x^2 - x$, $y = x^2 - 4x$, $y = x^2 + 3x$
l $y = x^2 - 1$, $y = 1 - x^2$, $y = 4 - x^2$, $y = -1 - x^2$
m $y = (x + 2)^2$, $y = (x + 2)^2 - 4$, $y = (x + 2)^2 + 1$
n $y = x^2 - 1$, $y = x^2 - 4x + 3$, $y = x^2 - 8x + 15$
o $y = \sqrt{9 - x^2}$, $y = -\sqrt{4 - x^2}$, $y = \sqrt{1 - x^2}$
p $y = \frac{1}{x}$, $y = \frac{2}{x}$, $y = -\frac{3}{x}$
q $y = \sqrt{x}$, $y = 2 - \sqrt{x}$, $y = \sqrt{1 - x}$
r $y = x^3$, $y = x^3 + 1$, $y = (x + 1)^3$
s $y = x^4$, $y = (x - 1)^4$, $y = x^4 - 1$
t $y = 2^{-x}$, $y = \left(\frac{1}{2}\right)^x$, $y = \frac{1}{2^x}$

20 Classify each relation below as one-to-one, many-to-one, one-to-many or many-to-many. It's probably best to work from a sketch of the curves, but you may want to approach the question algebraically.

a $y = 5x - 7$
b $(x - 2)^2 + (y - 3)^2 = 4$
c $y^2 = x - 2$
d $y = x^4 + 1$

21 a Twenty people (x-values) in an office have their country of birth (y-values) recorded in the office manager's spreadsheet. Is this relation a function? Classify it as one-to-one, many-to-one, one-to-many or many-to-many.

b What condition would make this relation one-to-one?

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4 Transformations and symmetry

In the previous chapter, various graphs of functions and relations were reviewed or introduced. This chapter deals with transformations of these graphs under vertical and horizontal translations, under reflections in the x-axis and y-axis, and under rotations of 180° about the origin (dilations are introduced in Year 12). These procedures allow a wide variety of new graphs to be obtained, and relationships amongst different graphs to be discovered.

Many graphs are unchanged under one or more of these transformations, which means that they are symmetric in some way. This chapter deals with line symmetry under reflection in the y-axis, and point symmetry under rotation of 180° about the origin. These transformations and symmetries are described geometrically and algebraically, and the theme remains the interrelationship between the algebra and the graphs.

The absolute value function is then introduced, together with its own transformations and reflection symmetry. The final section generalises the transformations of this chapter to the far more general idea of composite functions.

As always, computer sketching of curves is very useful in demonstrating how the features of a graph are related to the algebraic properties of its equation, and to gain familiarity with the variety of graphs and their interrelationships.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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4A Translations of known graphs

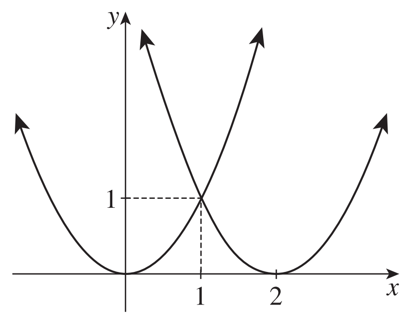
Once a graph is drawn, it can be *shifted* (or *translated*) vertically or horizontally to produce further graphs. These procedures work generally on all functions and relations, and greatly extend the range of functions and relations whose graphs can be quickly recognised and drawn.

In particular, translations are very helpful when dealing with parabolas and circles, where they are closely related to completing the square.

Shifting right and left

The graphs of $y = x^2$ and $y = (x - 2)^2$ are sketched opposite from their tables of values.

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$(x - 2)^2$	16	9	4	1	0	1	4



- The values for $(x - 2)^2$ in the third row are the values of x^2 in the second row shifted 2 steps to the right.
- Hence the graph of $y = (x - 2)^2$ is obtained by shifting the graph of $y = x^2$ to the right by 2 units.

1 SHIFTING (OR TRANSLATING) RIGHT AND LEFT

- To shift a graph h units to the *right*, replace x by $x - h$.
- Alternatively, if the graph is a function, the new function rule is $y = f(x - h)$.

Shifting a graph h units to the left means shifting it $-h$ units to the right, so x is replaced by $x - (-h) = x + h$.

Example 1 — 4A

a Draw up tables of values for $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x + 1}$.

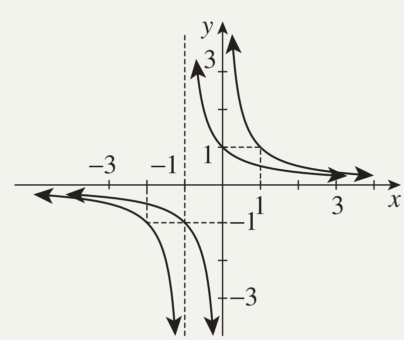
b Sketch the two graphs, and state the asymptotes of each graph.

c What transformation maps $y = \dfrac{1}{x}$ to $y = \dfrac{1}{x + 1}$?

SOLUTION

a

x	-3	-2	-1	0	1	2	3
$\frac{1}{x}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$	$\frac{1}{3}$
$\frac{1}{x+1}$	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$



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b $y = \frac{1}{x}$ has asymptotes $x = 0$ and $y = 0$.

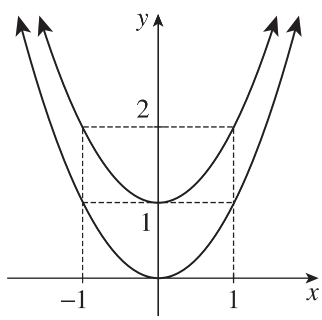
$y = \frac{1}{x + 1}$ has asymptotes $x = -1$ and $y = 0$.

c Because x is replaced by $x + 1 = x - (-1)$, it is a shift left of 1 unit.

Shifting up and down

The graphs of $y = x^2$ and $y = x^2 + 1$ are sketched to the right from their tables of values.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 1$	10	5	2	1	2	5	10



- The values for $x^2 + 1$ in the third row are each 1 more than the corresponding values of x^2 in the second row.
- Hence the graph of $y = x^2 + 1$ is produced by shifting the graph of $y = x^2$ upwards 1 unit.

Rewriting the transformed graph as $y - 1 = x^2$ makes it clear that the shifting has been obtained by replacing y by $y - 1$, giving a rule that is completely analogous to that for horizontal shifting.

2 SHIFTING (OR TRANSLATING) UP AND DOWN

- To shift a graph k units *upwards*, replace y by $y - k$.
- Alternatively, if the graph is a function, the new function rule is $y = f(x) + k$.

Shifting a graph k units down means shifting it $-k$ units up, so y is replaced by $y - (-k) = y + k$.

Example 2 4A

The graph of $y = 2^x$ is shifted down 2 units.

a Write down the equation of the shifted graph.
b Construct tables of values, and sketch the two graphs.
c State the asymptotes of the two graphs.

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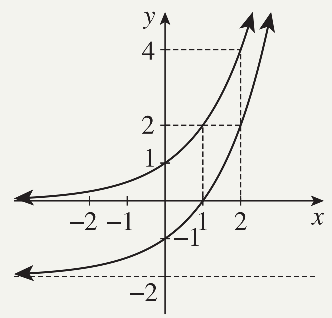
SOLUTION

a Replace y by $y - (-2) = y + 2$, so the new function is

$$y + 2 = 2^x, \qquad \text{that is,} \qquad y = 2^x - 2.$$

b

x	-2	-1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$2^x - 2$	$-1\frac{3}{4}$	$-1\frac{1}{2}$	-1	0	2



c $y = 2^x$ has asymptote $y = 0$.
$y = 2^x - 2$ has asymptote $y = -2$.

Combining horizontal and vertical translations

When a graph is shifted horizontally and vertically, the order in which the translations are applied makes no difference. Example 3 shows the effect of two translations on a cubic graph.

Example 3 4A

a How is the graph of $y = (x + 2)^3 - 4$ obtained from the graph of $y = x^3$ by a horizontal translation followed by a vertical translation?

b Draw up a table of values for the two functions and the intermediate function.

c Sketch the two curves, together with the intermediate graph.

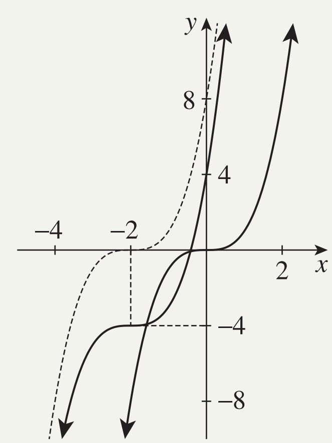
SOLUTION

a Shifting $y = x^3$ left 2 gives $\quad y = (x + 2)^3$.
Shifting $y = (x + 2)^3$ down 4 gives $\quad y + 4 = (x + 2)^3$,
which can be written as $\quad y = (x + 2)^3 - 4$.

b

x	-4	-3	-2	-1	0	1	2
x^3	-64	-27	-8	-1	0	1	8
$(x + 2)^3$	-8	-1	0	1	8	27	64
$(x + 2)^3 - 4$	-12	-5	-4	-3	4	23	60

c



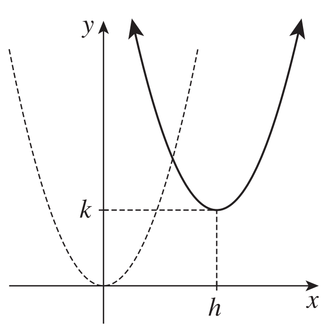
Translations and the vertex of a parabola

When we complete the square in a quadratic, it has the form

$$y = a(x - h)^2 + k, \qquad \text{that is,} \qquad y - k = a(x - h)^2.$$

This is a translation of the quadratic $y = ax^2$. The parabola has been shifted h units right and k units up.

This gives a clear and straightforward motivation for completing the square.



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3 THE COMPLETED SQUARE AND THE VERTEX OF A PARABOLA

- The completed square form of a quadratic
$$y = a(x-h)^2 + k \quad \text{or} \quad y - k = a(x-h)^2$$
displays its graph as the parabola $y = ax^2$ shifted right h units and up k units.
- Thus the vertex of the parabola is (h, k).

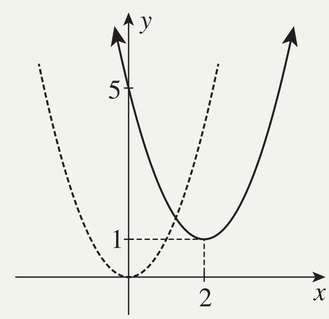
Example 4 — 4A

In each part, complete the square in the quadratic. Then identify its graph as a translation of a parabola with vertex at the origin, and sketch the two graphs.

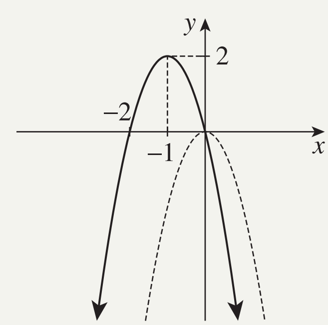
a $y = x^2 - 4x + 5$ **b** $y = -2x^2 - 4x$

SOLUTION

a $y = x^2 - 4x + 5$
$y = (x^2 - 4x + 4) - 4 + 5$
$y = (x-2)^2 + 1 \quad \text{or} \quad y - 1 = (x-2)^2$
This is $y = x^2$ shifted right 2 and up 1.

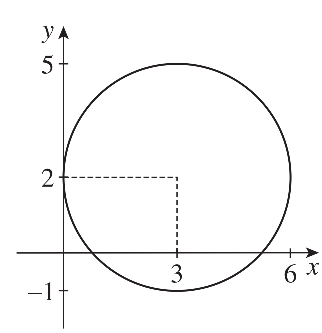


b $y = -2x^2 - 4x$
$-\dfrac{y}{2} = x^2 + 2x$
$-\dfrac{y}{2} = (x^2 + 2x + 1) - 1$
$-\dfrac{y}{2} = (x+1)^2 - 1$
$y = -2(x+1)^2 + 2 \quad \text{or} \quad y - 2 = -2(x+1)^2$
This is $y = -2x^2$ shifted left 1 and up 2.



Translations and the centre of a circle

The circle drawn to the right has centre $(3, 2)$ and radius 3. To find its equation, we start with the circle with centre the origin and radius 3,
$$x^2 + y^2 = 9,$$
then translate it 3 to the right and 2 up, to give
$$(x-3)^2 + (y-2)^2 = 9.$$



This formula can also be established directly by Pythagoras' theorem in the form of the distance formula, but as we saw with parabolas, translations make things clearer and more straightforward.

When the squares in the equation of a circle have both been expanded, the centre and radius can be found by completing the squares in x and in y.

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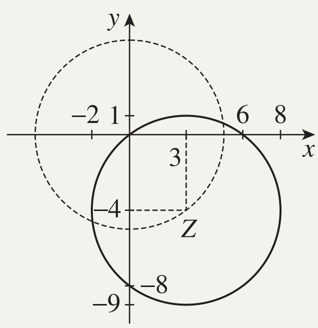
Example 5 4A

a Complete the squares in x and in y of the relation $x^2 + y^2 - 6x + 8y = 0$.
b Identify the circle with centre the origin that can be translated to it, and state the translations.
c Sketch both circles on the same diagram, and explain why each circle passes through the centre of the other circle.

SOLUTION

a Completing the squares in x and in y,

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 9 + 16$$
$$(x - 3)^2 + (y + 4)^2 = 25.$$



b It is the circle $x^2 + y^2 = 5^2$ shifted right 3 and down 4, so its centre is $Z(3, -4)$ and its radius is 5.

c Using the distance formula,

$$OZ^2 = 3^2 + 4^2$$
$$OZ = 5,$$

which is the radius of each circle.

Exercise 4A FOUNDATION

1 **a** Copy and complete the table of values for $y = x^2$ and $y = (x - 1)^2$.

x	-2	-1	0	1	2	3
x^2						
$(x - 1)^2$						

b Sketch the two graphs and state the vertex of each.
c What transformation maps $y = x^2$ to $y = (x - 1)^2$?

2 **a** Copy and complete the table of values for $y = \frac{1}{4}x^3$ and $y = \frac{1}{4}x^3 + 2$.

x	-3	-2	-1	0	1	2	3
$\frac{1}{4}x^3$							
$\frac{1}{4}x^3 + 2$							

b Sketch the two graphs and state the y-intercept of each.
c What transformation maps $y = \frac{1}{4}x^3$ to $y = \frac{1}{4}x^3 + 2$?

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3 In each case, the given equation is the result of shifting one of the curves $y = x^2$, $y = \dfrac{1}{x}$ or $y = 2^x$. State the direction of the shift and by how much. Use this information to help sketch the curve without resorting to a table of values.

a $y = x^2 + 2$ **b** $y = (x + 1)^2$ **c** $y = \dfrac{1}{x - 2}$

d $y = \dfrac{1}{x} + 1$ **e** $y = 2^{x-1}$ **f** $y = 2^x - 2$

4 Sketch each circle by shifting $x^2 + y^2 = 1$ either horizontally or vertically. Mark all intercepts with the axes.

a $(x - 1)^2 + y^2 = 1$ **b** $x^2 + (y - 1)^2 = 1$

c $x^2 + (y + 1)^2 = 1$ **d** $(x + 1)^2 + y^2 = 1$

5 Write down the new equation for each function or relation after the given translation has been applied. Then sketch the graph of the new curve.

a $y = x^2$: right 1 unit **b** $y = 2^x$: down 3 units

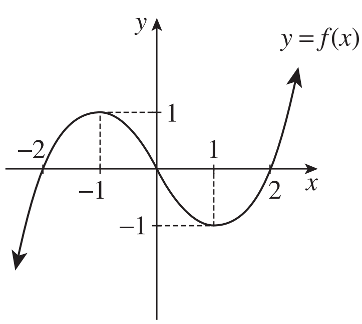
c $y = x^3$: left 1 unit **d** $y = \dfrac{1}{x}$: right 3 units

e $x^2 + y^2 = 4$: up 1 unit **f** $y = x^2 - 4$: left 1 unit

g $xy = 1$: down 1 unit **h** $y = \sqrt{x}$: up 2 units

6 In each case an unknown function has been drawn. Draw the functions specified below it.

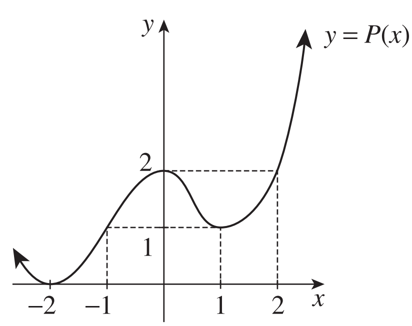
a



i $y = f(x - 2)$

ii $y = f(x + 1)$

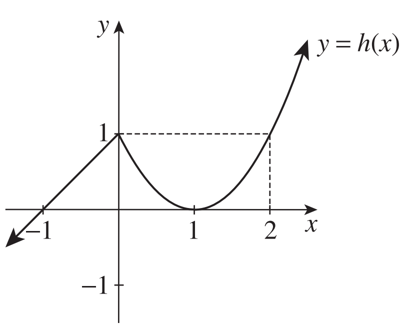
b



i $y = P(x + 2)$

ii $y = P(x + 1)$

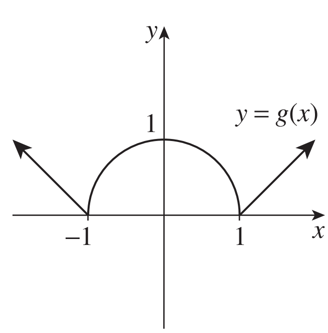
c



i $y - 1 = h(x)$

ii $y = h(x) - 1$

d



i $y - 1 = g(x)$

ii $y = g(x - 1)$

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DEVELOPMENT

7 Use shifting, and completion of the square where necessary, to determine the centre and radius of each circle.

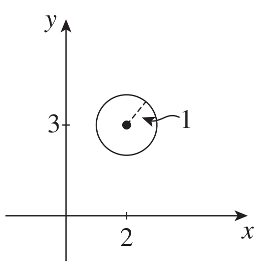
a $(x + 1)^2 + y^2 = 4$
b $(x - 1)^2 + (y - 2)^2 = 1$
c $x^2 - 2x + y^2 - 4y - 4 = 0$
d $x^2 + 6x + y^2 - 8y = 0$
e $x^2 - 10x + y^2 + 8y + 32 = 0$
f $x^2 + 14x + 14 + y^2 - 2y = 0$

8 In each part, complete the square in the quadratic. Then identify its graph as a translation of a parabola with vertex at the origin. Finally, sketch its graph.

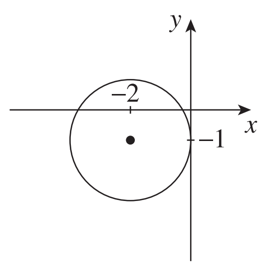
a $y = x^2 + 2x + 3$
b $y = x^2 - 2x - 2$
c $y = -x^2 + 4x + 1$
d $y = -x^2 - 4x - 5$
e $y = 2x^2 - 4x - 2$
f $y = \frac{1}{2}x^2 - x - 2$

9 Describe each graph below as the circle $x^2 + y^2 = r^2$ transformed by shifts, and hence write down its equation.

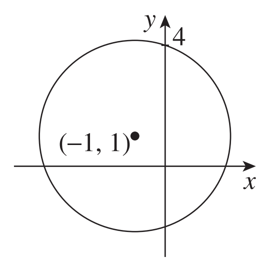
a



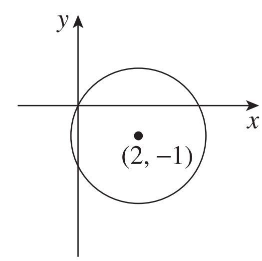
b



c



d



10 **a** Use a table of values to sketch $y = \frac{1}{2}x^3$. Then use translations to sketch:

i $y = \frac{1}{2}x^3 - 2$
ii $y = \frac{1}{2}(x - 2)^3$
iii $y = \frac{1}{2}(x + 3)^3 + 1$

b Use a table of values to sketch $y = -2x^3$. Then use translations to sketch:

i $y = 3 - 2x^3$
ii $y = -2(x + 3)^3$
iii $y = -2(x - 1)^3 - 2$

11 In each part, explain how the graph of each subsequent equation is a transformation of the first graph (there may be more than one answer), then sketch each function.

a From $y = 2x$:
i $y = 2x + 4$
ii $y = 2x - 4$

b From $y = x^2$:
i $y = x^2 + 9$
ii $y = x^2 - 9$
iii $y = (x - 3)^2$

c From $y = -x^2$:
i $y = 1 - x^2$
ii $y = -(x + 1)^2$
iii $y = -(x + 1)^2 + 2$

d From $y = \sqrt{x}$:
i $y = \sqrt{x + 4}$
ii $y = \sqrt{x} + 4$
iii $y = \sqrt{x + 4} - 2$

e From $y = \frac{2}{x}$:
i $y = \frac{2}{x} + 1$
ii $y = \frac{2}{x + 2}$
iii $y = \frac{2}{x + 2} + 1$

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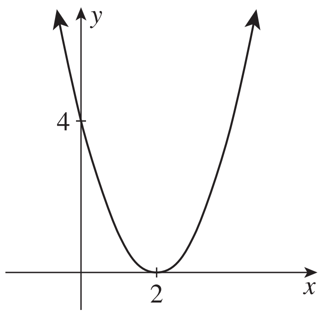
12 Sketch $y = \frac{1}{x}$, then use shifting to sketch the following graphs. Find any x-intercepts and y-intercepts, and mark them on your graphs.

a $y = \frac{1}{x - 2}$ **b** $y = 1 + \frac{1}{x - 2}$ **c** $y = \frac{1}{x - 2} - 2$

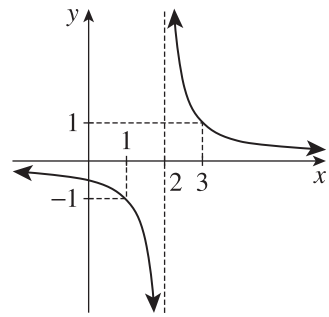
d $y = \frac{1}{x + 1} - 1$ **e** $y = 3 + \frac{1}{x + 2}$ **f** $y = \frac{1}{x - 3} + 4$

13 Describe each graph below as a standard curve transformed by shifts, and hence write down its equation.

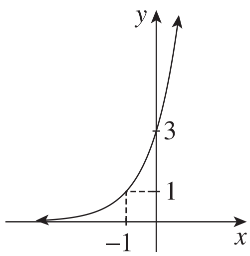
a



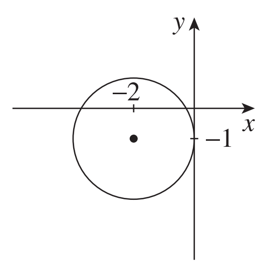
b



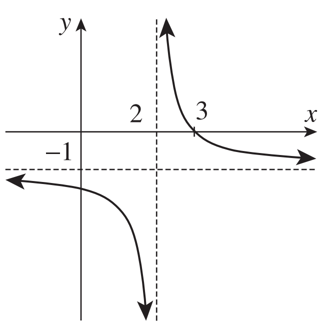
c



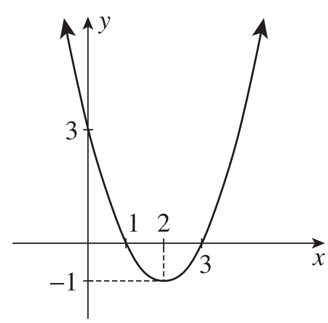
d



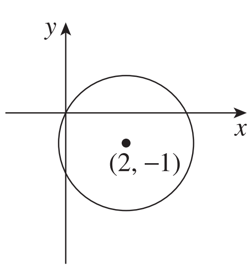
e



f



g



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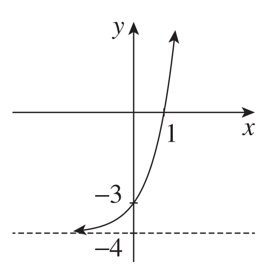

14 Complete squares, then sketch each circle, stating the centre and radius. By substituting $x = 0$ and then $y = 0$, find any intercepts with the axes.

a $x^2 - 4x + y^2 - 10y = -20$ **b** $x^2 + y^2 + 6y - 1 = 0$

c $x^2 + 4x + y^2 - 8y = 0$ **d** $x^2 - 2x + y^2 + 4y = 1$

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15 Consider the straight line equation $x + 2y - 4 = 0$.

- **a** The line is translated 2 units left. Find the equation of the new line.
- **b** The original line is translated 1 unit down. Find the equation of this third line.
- **c** Comment on your answers, and draw the lines on the same number plane.

ENRICHMENT

16 **a** The circle $x^2 + y^2 = r^2$ has centre the origin and radius r. This circle is shifted so that its centre is at $C(h, k)$. Write down its equation.

- **b** The point $P(x, y)$ lies on the circle with centre $C(h, k)$ and radius r. That is, P lies on the shifted circle in part **a**. This time use the distance formula for the radius PC to obtain the equation of the circle.

17 **a** Explain the point–gradient formula $y - y_1 = m(x - x_1)$ for a straight line in terms of shifts of the line $y = mx$.

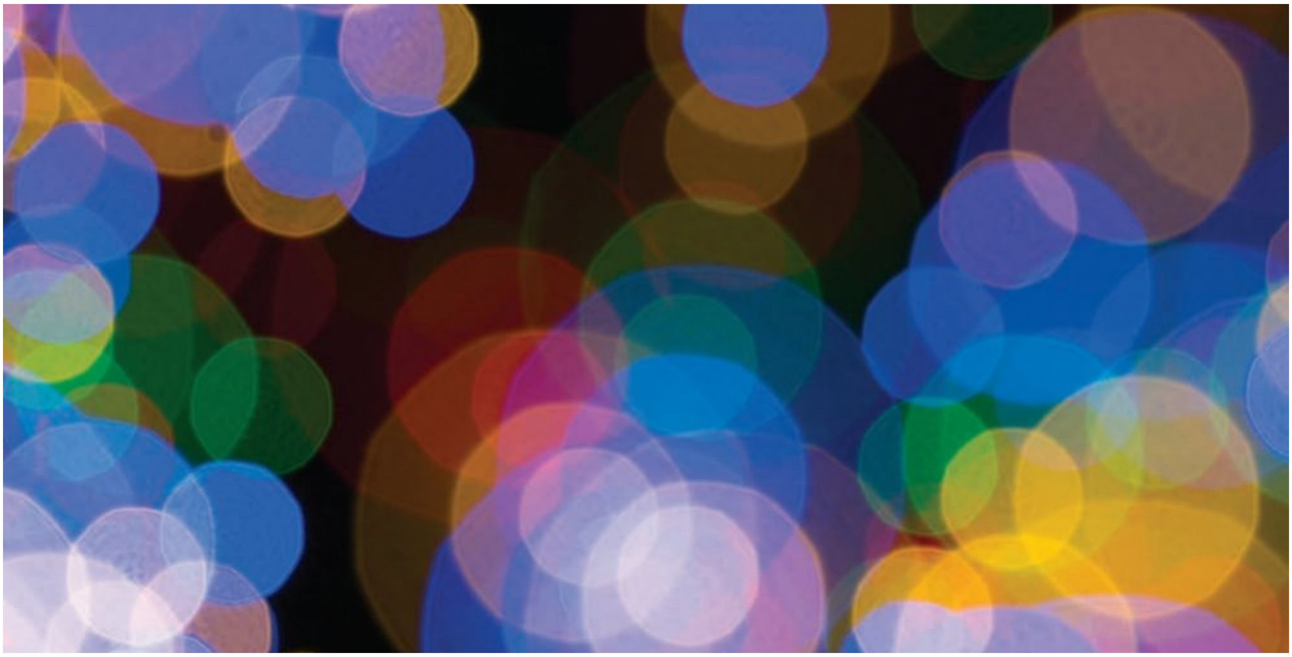
- **b** Hence explain why parallel lines must have the same gradient.

18 Suppose that the graph of $y = f(x)$ has been drawn. Let $\mathcal{H}$ be a shift to the right by a units, and let $\mathcal{V}$ be a shift upwards by b units.

- **a** Write down the equations of the successive equations obtained by applying $\mathcal{H}$ then $\mathcal{V}$.
- **b** Write down the equations of the successive equations obtained by applying $\mathcal{V}$ then $\mathcal{H}$.
- **c** What do you conclude about the order of applying horizontal and vertical shifts?

19 A certain circle has equation $(x - 1)(x + 3) + (y - 2)(y - 4) = 0$.

- **a** From this equation, write down four points that lie on the circle.
- **b** What shape do these four points form? Give its dimensions.
- **c** Hence find the centre and radius of the circle.
- **d** Confirm your answer to part **c** by expanding, then completing the squares.



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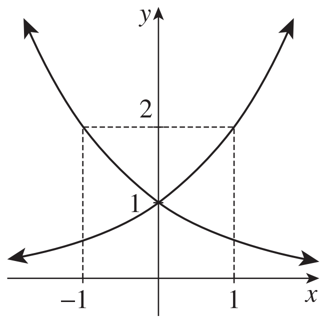
4B Reflections in the x-axis and y-axis

Reflecting only in the x-axis and the y-axis may seem an unnecessary restriction, but in fact these two transformations are the key to understanding many significant properties of functions, particularly the symmetry of graphs.

When these two reflections are combined, they produce a rotation of 180° about the origin, which again is the key to the symmetry of many graphs.

Reflection in the y-axis

The graphs of $y = 2^x$ and $y = 2^{-x}$ have been sketched to the right from their tables of values.



x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

- The second and third rows are the reverse of each other.
- Hence the graphs are reflections of each other in the y-axis.

4 REFLECTION IN THE y-AXIS

- To reflect a graph in the y-axis, replace x by $-x$.
- Alternatively, if the graph is a function, the new function rule is $y = f(-x)$.

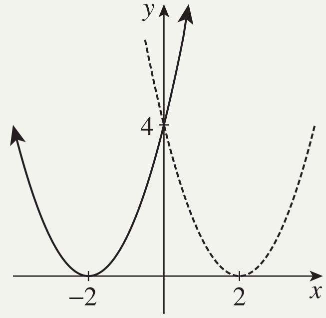
Reflection is mutual — it maps each graph to the other graph.

Example 6 4B

a Sketch the parabola $y = (x - 2)^2$, and on the same set of axes, sketch its reflection in the y-axis.
b Use the rule in the box above to write down the equation of the reflected graph.
c Why can this equation be written as $y = (x + 2)^2$?
d What are the vertices of the two parabolas?
e What other transformation would move the first parabola to the second?

SOLUTION

a



b Replacing x by $-x$ gives the equation

$$y = (-x - 2)^2.$$

c Taking out the factor -1 from the brackets,

$$y = (-x - 2)^2$$
$$y = (-1)^2 \times (x + 2)^2$$
$$y = (x + 2)^2.$$

d The vertices are $(2, 0)$ and $(-2, 0)$.

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e The second parabola is also the first parabola shifted left 4 units.
This replaces x by $x + 4$, so the new equation is the same as before,

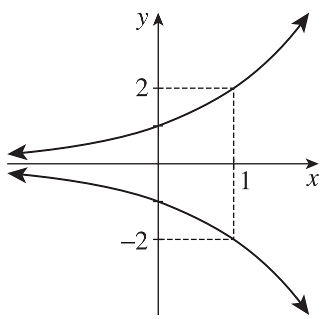
$$y = \big((x+4)-2\big)^2, \qquad \text{that is,} \qquad y = (x+2)^2.$$

The reason why there are two possible transformations is that the parabola has line symmetry in its axis of symmetry.

Reflection in the x-axis

The graphs of $y = 2^x$ and $y = -2^x$ have been sketched to the right from their tables of values.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
-2^x	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-8



- The values in the second and third rows are the opposites of each other.
- Hence the graphs are reflections of each other in the x-axis.

Rewriting the transformed graph as $-y = 2^x$ makes it clear that the reflection has been obtained by replacing y by $-y$, giving a rule that is completely analogous to that for reflection in the y-axis.

5 REFLECTION IN THE x-AXIS

- To reflect a graph in the x-axis, replace y by $-y$.
- Alternatively, if the graph is a function, the new function rule is $y = -f(x)$.

Again, reflection is mutual — it maps each graph to the other graph.

Example 7 — 4B

The graph of $y = (x - 2)^2$ is reflected in the x-axis.

a Write down the equation of the reflected graph.
b Construct tables of values, and sketch the two graphs.
c What are the vertices of the two parabolas?

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SOLUTION

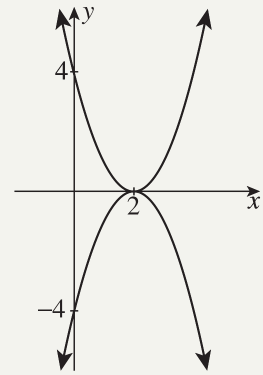
a Replace y by $-y$, so the new function is

$$-y = (x-2)^2, \quad \text{that is,} \quad y = -(x-2)^2.$$

b

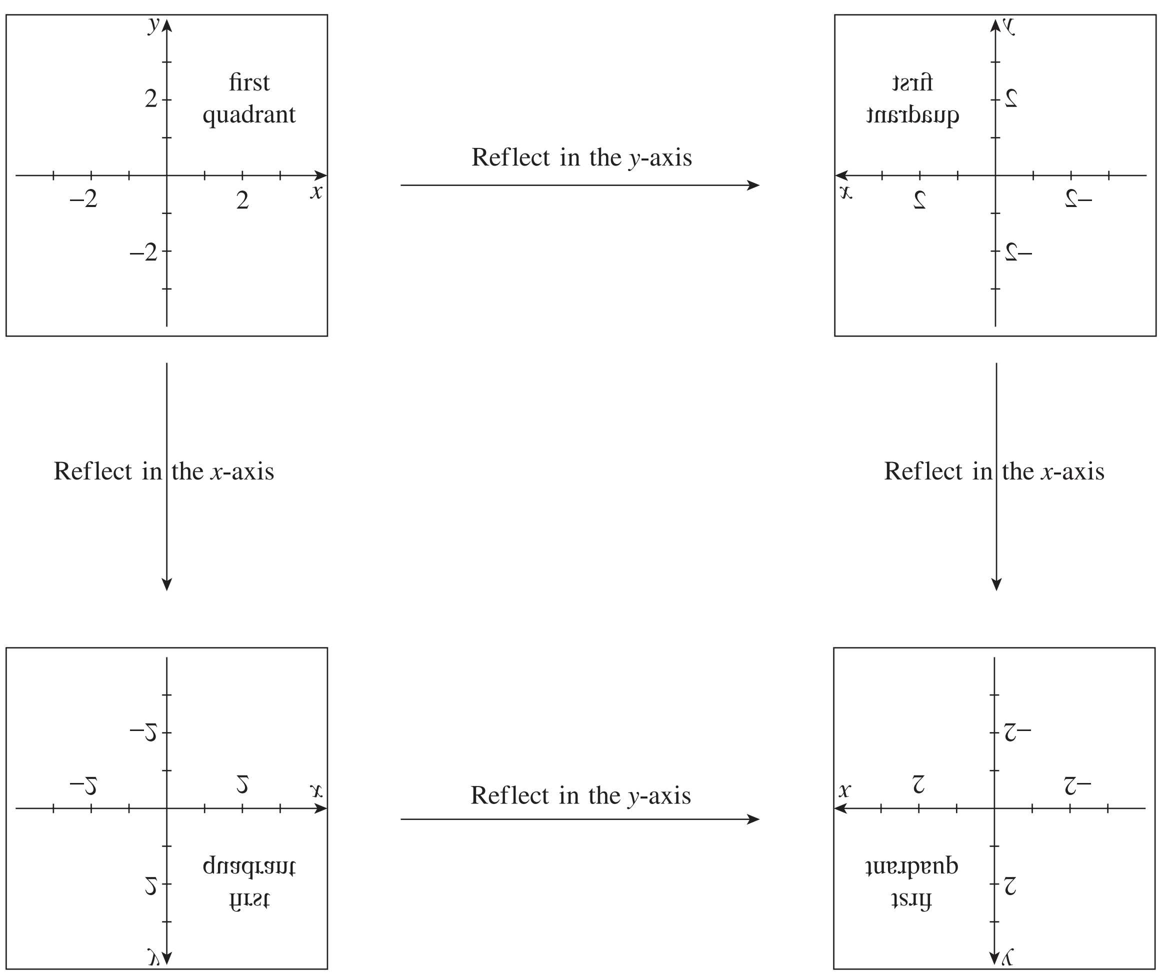
x	0	1	2	3	4
$(x-2)^2$	4	1	0	1	4
$-(x-2)^2$	−4	−1	0	−1	−4

c They both have vertex $(2, 0)$.



Combining the two reflections — rotating 180° about the origin

- Draw an x-axis and y-axis on a thin, semi-transparent sheet of paper.
- Hold the paper out flat, and regard it as a two-dimensional object.
- Reflect in the x-axis — do this by holding the sheet steady at the two ends of the x-axis and rotating it 180° so that you are now looking at the back of the sheet. Then reflect in the y-axis — do this by holding the sheet at the two ends of the y-axis and rotating it 180° so that you are looking at the front of the sheet again. What has happened?
- Reflect it in the y-axis, then in the x-axis. What happens?



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This little experiment should convince you of two things:

- Performing successive reflections in the x-axis and in the y-axis results in a rotation of 180° about the origin.
- The order in which these two reflections are done does not matter.

This rotation of 180° about the origin is sometimes called *reflection in the origin* — every point in the plane is moved along a line through the origin to a point the same distance from the origin on the opposite side.

6 ROTATION OF 180° ABOUT THE ORIGIN

- To rotate a graph 180° about the origin, replace x by $-x$ and y by $-y$.
- Successive reflections in the x-axis and the y-axis are the same as a rotation of 180° about the origin.
- The order in which these two successive reflections are done does not matter.
- Rotation of 180° about the origin is also called *reflection in the origin*, because every point is moved through the origin to a point the same distance from the origin on the opposite side.

Rotation of 180° about the origin is also mutual — it maps each graph to the other graph.

Example 8 4B

a From the graph of $y = \sqrt{x}$, deduce the graph of $y = -\sqrt{-x}$ using reflections.

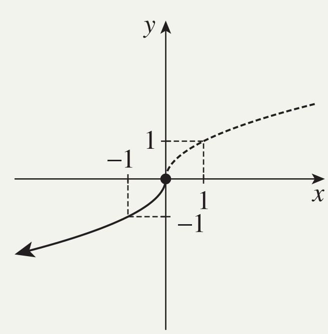
b What single transformation maps each graph to the other?

SOLUTION

a The equation $y = -\sqrt{-x}$ can be rewritten as

$$-y = \sqrt{-x}$$

so the graph is obtained from the graph of $y = \sqrt{x}$ by successive reflections in the x-axis and the y-axis, where the reflections may be done in either order.



b This is the same as rotation of 180° about the origin.

Exercise 4B FOUNDATION

1 Consider the parabola $y = x^2 - 2x$.

a Show that when y is replaced by $-y$, the equation becomes $y = 2x - x^2$.

b Copy and complete the table of values for $y = x^2 - 2x$ and $y = 2x - x^2$.

x	-2	-1	0	1	2	3	4
$x^2 - 2x$							
$2x - x^2$							

c Sketch the two parabolas and state the vertex of each.

d What transformation maps $y = x^2 - 2x$ to $y = 2x - x^2$?

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2 Consider the hyperbola $y = \dfrac{2}{x - 2}$.

a Show that when x is replaced by $-x$ the equation becomes $y = -\dfrac{2}{x + 2}$.

b Copy and complete the table of values for $y = \dfrac{2}{x - 2}$ and $y = -\dfrac{2}{x + 2}$.

x	-4	-3	-2	-1	0	1	2	3	4
$\dfrac{2}{x - 2}$							*		
$-\dfrac{2}{x + 2}$			*						

c Sketch the two hyperbolas and state the vertical asymptote of each.

d What transformation maps $y = \dfrac{2}{x - 2}$ to $y = -\dfrac{2}{x + 2}$?

3 a Sketch the graph of the quadratic function $y = x^2 - 2x - 3$, showing the intercepts and vertex.

b In each case, determine the equation of the result when this parabola is reflected as indicated. Then sketch the new function.

i in the y-axis **ii** in the x-axis **iii** in both axes

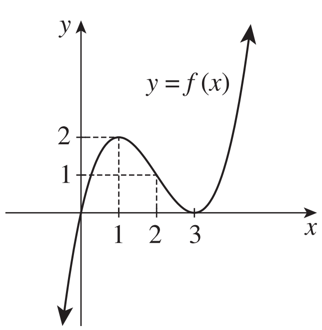
4 a Sketch the graph of the exponential function $y = 2^{-x}$, showing the y-intercept and the coordinates at $x = -1$, and clearly indicating the asymptote.

b In each case, find the equation of the curve when this exponential graph is reflected as indicated. Then sketch the new function.

i in the y-axis **ii** in the x-axis **iii** in both axes

5 In each case, an unknown function has been drawn. Draw the reflections of the function specified below it.

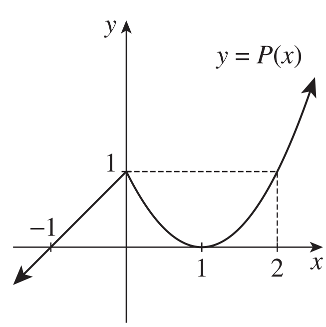
a



i $y = f(-x)$

ii $y = -f(x)$

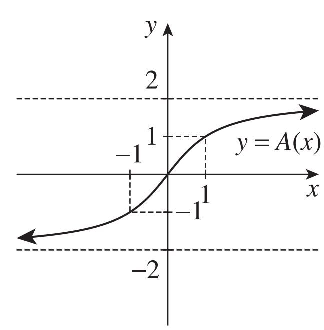
b



i $y = -P(x)$

ii $y = -P(-x)$

c



i $y = A(-x)$

ii $y = -A(-x)$

6 Write down the new equation for each function or relation after the given transformation has been applied. Then sketch the graph of the new curve.

a $y = x^2$: reflect in the x-axis

b $y = x^3$: reflect in the y-axis

c $y = 2^x$: rotate by 180°

d $y = 2x - x^2$: rotate by 180°

e $x^2 + y^2 = 9$: reflect in the y-axis

f $y = \dfrac{1}{x}$: reflect in the x-axis

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DEVELOPMENT

7 Consider the hyperbola $y = \dfrac{1}{x+2} - 1$.

a Sketch this hyperbola.

b In each case, determine the reflection or rotation required to achieve the specified result. Then write down the equation of the new hyperbola and sketch it.

i The vertical asymptote is unchanged, but the horizontal asymptote changes sign.

ii The intercepts with the axes are positive.

8 a Sketch the circles $(x-3)^2 + y^2 = 4$ and $(x+3)^2 + y^2 = 4$.

b What reflection maps each circle onto the other?

c Confirm your answer by making an appropriate substitution into the first equation.

d What translation maps the first circle onto the second?

e Confirm your answer by making an appropriate substitution into the first equation.

9 Consider $x^2 + y^2 = r^2$, the circle with centre the origin and radius r.

a Show that this equation is unchanged when reflected in either the x-axis or the y-axis.

b Explain this result geometrically.

10 In each part, explain how the graph of each subsequent equation is a reflection of the first graph or a rotation of 180°, then sketch each one.

a From $y = \frac{1}{2}x + 1$:

i $y = -\frac{1}{2}x + 1$ **ii** $y = -\frac{1}{2}x - 1$ **iii** $y = \frac{1}{2}x - 1$

b From $y = 4 - x$:

i $y = x - 4$ **ii** $y = x + 4$ **iii** $y = -x - 4$

c From $y = (x-1)^2$:

i $y = -(x+1)^2$ **ii** $y = (x+1)^2$ **iii** $y = -(x-1)^2$

d From $y = \sqrt{x}$:

i $y = -\sqrt{-x}$ **ii** $y = -\sqrt{x}$ **iii** $y = \sqrt{-x}$

e From $y = 3^x$:

i $y = -3^x$ **ii** $y = -3^{-x}$ **iii** $y = 3^{-x}$

f From $y = 1 + \dfrac{1}{x-1}$:

i $y = 1 - \dfrac{1}{x+1}$ **ii** $y = -1 + \dfrac{1}{x+1}$ **iii** $y = -1 + \dfrac{1}{1-x}$

11 Consider the two parabolas $y = x^2 - 4x + 3$ and $y = x^2 + 4x + 3$.

a Sketch both quadratic functions on the same set of axes.

b What reflection maps each parabola onto the other?

c How can the second parabola be obtained by shifting the first?

d Confirm your answer to part **c** algebraically.

e Investigate which parts of the previous question could also have been achieved by shifting instead.

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12 **a** Let $c(x) = \dfrac{2^x + 2^{-x}}{2}$. Show that $c(-x) = c(x)$, and explain this geometrically.

b Let $t(x) = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$. Show that $-t(-x) = t(x)$, and explain this geometrically.

c [Technology] Confirm your observations in parts **a** and **b** by plotting each function using graphing software.

13 In each case, explain how the graph of the second function can be obtained from the first by a combination of reflections and translations (there may be more than one correct answer), then sketch each pair.

a From $y = x$ to $y = 2 - x$.

b From $y = x^2$ to $y = 4 - x^2$.

c From $y = \dfrac{1}{x}$ to $y = \dfrac{1}{2 - x}$

d From $y = x^2$ to $y = -x^2 - 2x$

e From $y = 2^x$ to $y = 2^{1-x} - 2$

f From $y = \sqrt{x}$ to $y = -\sqrt{4 - x}$

14 Consider the parabola $y = (x - 1)^2$. Sketches or plots done on graphing software may help answer the following questions.

a **i** The parabola is shifted right 1 unit. What is the new equation?

ii This new parabola is then reflected in the y-axis. Write down the equation of the new function.

b **i** The original parabola is reflected in the y-axis. What is the new equation?

ii This fourth parabola is then shifted right 1 unit. What is the final equation?

c Parts **a** and **b** both used a reflection in the y-axis and a shift right 1 unit. Did the order of these affect the answer?

d Investigate other combinations of shifts and reflections. In particular, what do you notice if the shift is parallel with the axis of reflection?

ENRICHMENT

15 Suppose that $y = f(x)$ is a function whose graph has been drawn.

a Let $\mathcal{U}$ be shifting upwards a units and $\mathcal{H}$ be reflection in $y = 0$. Write down the equations of the successive graphs obtained by applying $\mathcal{U}$, then $\mathcal{H}$, then $\mathcal{U}$, then $\mathcal{H}$, and prove that the final graph is the same as the first. Confirm the equations by applying these operations successively to a square book or piece of paper.

b Let $\mathcal{R}$ be shifting right by a units. Write down the equations of the successive graphs obtained by applying $\mathcal{R}$, then $\mathcal{H}$, then $\mathcal{R}$, then $\mathcal{H}$, and describe the final graph. Confirm using the square book.

c Let $\mathcal{V}$ be reflection in $x = 0$ and $\mathcal{I}$ be reflection in $y = x$. Write down the equations of the successive graphs obtained by applying $\mathcal{I}$, then $\mathcal{V}$, then $\mathcal{I}$, then $\mathcal{H}$, and show that the final graph is the same as the first. Confirm using the square book.

d Write down the equations of the successive graphs obtained by applying the combination $\mathcal{I}$-followed-by-$\mathcal{V}$ once, twice, . . . , until the original graph returns. Confirm using the square book.

16 Use combinations of shifts and reflections in the axes to demonstrate that the graph of $y = f(2a - x)$ is the result of reflecting the graph of $y = f(x)$ in the line $x = a$.
(Hint: How could a reflection in $x = a$ be turned into a reflection in the y-axis?)

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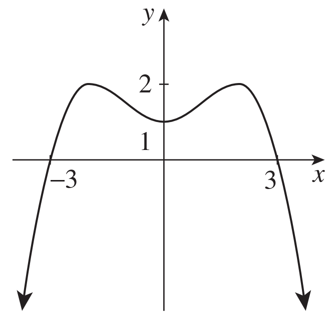
4C Even and odd symmetry

It has been said that all mathematics is the study of symmetry. Two simple types of symmetry occur so often in the functions of this course that every function should be tested routinely for them.

Even functions and line symmetry in the *y*-axis



A relation or a function is called *even* if its graph has *line symmetry in the y-axis*. This means that the graph is unchanged by reflection in the y-axis, as with the graph to the right.

As explained in Section 4B, when the graph of a function $y = f(x)$ is reflected in the y-axis, the new curve has equation $y = f(-x)$. Hence for a function to be *even*, the graphs of $y = f(x)$ and $y = f(-x)$ must coincide, that is,

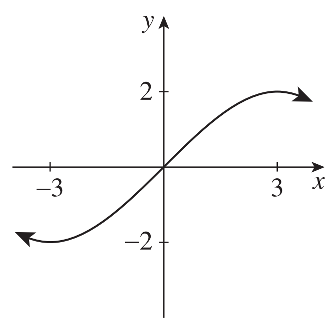
$$f(-x) = f(x), \text{ for all } x \text{ in the domain.}$$

7 EVEN FUNCTIONS AND LINE SYMMETRY IN THE *y*-AXIS

- A relation or function is called *even* if its graph has *line symmetry in the y-axis.*
- Algebraically, a function $f(x)$ is even if
  $$f(-x) = f(x), \text{ for all } x \text{ in the domain.}$$

More generally, a relation is even if its equation is unchanged when x is replaced by $-x$.

Odd functions and point symmetry in the origin



A relation or function is called *odd* if its graph has *point symmetry in the origin*. This means that the graph is unchanged by a rotation of 180° about the origin, or equivalently, by successive reflections in the x-axis and the y-axis.

When the graph of $y = f(x)$ is reflected in the x-axis and then in the y-axis, the new curve has equation $y = -f(-x)$. Hence for a function to be *odd*, the graph of $-f(-x)$ must coincide with the graph of $f(x)$, that is,

$$f(-x) = -f(x), \text{ for all } x \text{ in the domain.}$$

8 ODD FUNCTIONS AND POINT SYMMETRY IN THE ORIGIN

- A relation or function is called *odd* if its graph has *point symmetry in the origin*.
- Algebraically, a function $f(x)$ is odd if
  $$f(-x) = -f(x), \text{ for all } x \text{ in the domain.}$$
  More generally, a relation is odd if its equation is unchanged when x is replaced by $-x$ and y is replaced by $-y$.
- *Point symmetry in the origin* means that the graph is mapped onto itself by a rotation of 180° about the origin.
- Equivalently, it means that the graph is mapped onto itself by successive reflections in the x-axis and the y-axis. The order of these two reflections does not matter.

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Testing functions algebraically for evenness and oddness

A single test will pick up both these types of symmetry in functions.

9 TESTING FOR EVENNESS AND ODDNESS (OR NEITHER)

- Simplify $f(-x)$ and note whether it is $f(x)$, $-f(x)$ or neither.

Most functions are neither even nor odd.

Example 9 4C

Test each function for evenness or oddness, then sketch it.

a $f(x) = x^4 - 3$

b $f(x) = x^3$

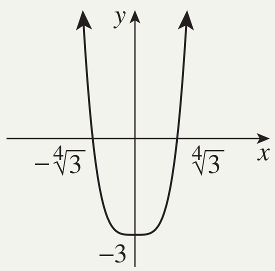
c $f(x) = x^2 - 2x$

SOLUTION

a Here $f(x) = x^4 - 3$.

Substituting $-x$ for x,
$$\begin{aligned} f(-x) &= (-x)^4 - 3 \\ &= x^4 - 3 \\ &= f(x). \end{aligned}$$

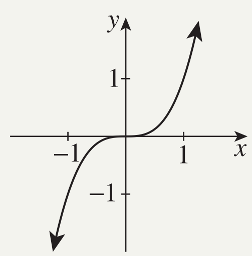
Hence $f(x)$ is an even function.



b Here $f(x) = x^3$.

Substituting $-x$ for x,
$$\begin{aligned} f(-x) &= (-x)^3 \\ &= -x^3 \\ &= -f(x). \end{aligned}$$

Hence $f(x)$ is an odd function.

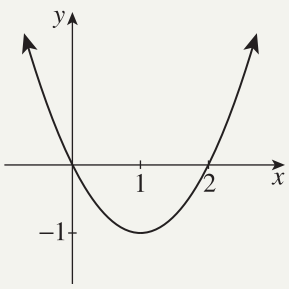


c Here $f(x) = x^2 - 2x$.

Substituting $-x$ for x,
$$\begin{aligned} f(-x) &= (-x)^2 - 2(-x) \\ &= x^2 + 2x. \end{aligned}$$

Because $f(-x)$ is equal neither to $f(x)$ nor to $-f(x)$, the function is neither even nor odd.

(The parabola does, however, have line symmetry, not in the y-axis, but in its axis of symmetry $x = 1$.)



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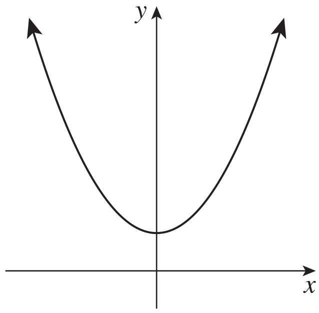
Exercise 4C

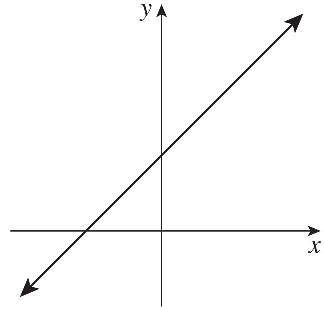
FOUNDATION

1 Classify each function $y = f(x)$ as even, odd or neither.

a



b



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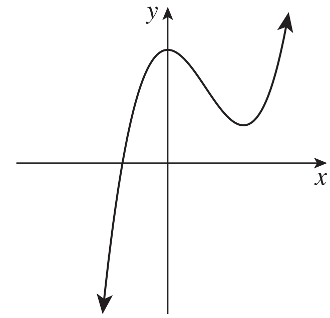
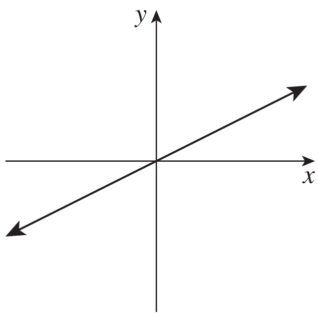
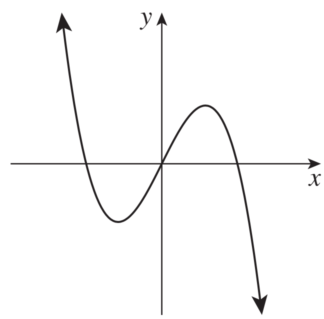


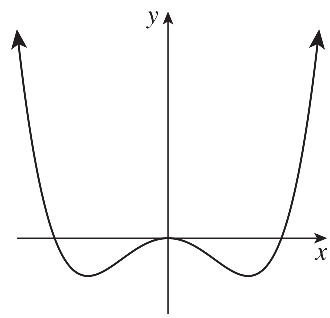
d



e



f

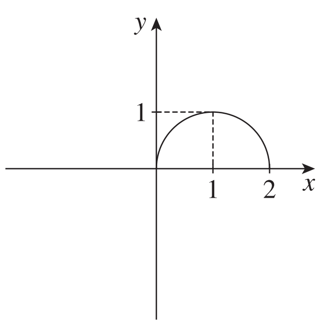


2 In each diagram below, complete the graph so that:

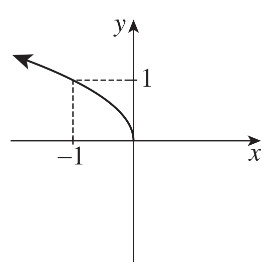
i $f(x)$ is even,

ii $f(x)$ is odd.

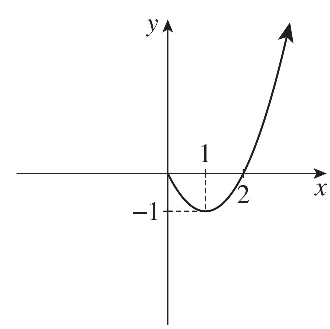
a



b



c



3 Consider the function $f(x) = x^4 - 2x^2 + 1$.

a Simplify $f(-x)$.

b Hence show that $f(x)$ is an even function.

4 Consider the function $g(x) = x^3 - 3x$.

a Simplify $g(-x)$.

b Hence show that $g(x)$ is an odd function.

5 Consider the function $h(x) = x^3 + 3x^2 - 2$.

a Simplify $h(-x)$.

b Hence show that $h(x)$ is neither even nor odd.

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6 Simplify $f(-x)$ for each function, and hence determine whether it is even, odd or neither.

a $f(x) = x^2 - 9$
b $f(x) = x^2 - 6x + 5$
c $f(x) = x^3 - 25x$
d $f(x) = x^4 - 4x^2$
e $f(x) = x^3 + 5x^2$
f $f(x) = x^5 - 16x$
g $f(x) = x^5 - 8x^3 + 16x$
h $f(x) = x^4 + 3x^3 - 9x^2 - 27x$

7 On the basis of the previous questions, copy and complete these sentences.

a 'A polynomial function is odd if . . .'.
b 'A polynomial function is even if . . .'.

DEVELOPMENT

8 Factor each polynomial in parts a–f of Question 6 above and write down its zeroes (that is, its x-intercepts). Then use a table of values to sketch its graph. Confirm that the graph exhibits the symmetry established above.

9 [Algebra and technology]
In Questions 3 to 8, the odd and even functions were all polynomials. Other functions can also be classified as odd or even. In each case following, simplify $f(-x)$ and compare it with $f(x)$ and $-f(x)$ to determine whether it is odd or even. Then confirm your answer by plotting the function on appropriate graphing software.

a $f(x) = \dfrac{2^x + 2^{-x}}{2}$
b $f(x) = \dfrac{2^x - 2^{-x}}{2}$
c $f(x) = \sqrt[3]{x}$
d $f(x) = \left(\sqrt[3]{x}\right)^2$
e $f(x) = \dfrac{x}{x^2 - 4}$
f $f(x) = \dfrac{2}{x^2 - 4}$
g $f(x) = \sqrt{9 - x^2}$
h $f(x) = x\sqrt{9 - x^2}$

10 Determine whether each function is even, odd or neither.

a $f(x) = 2^x$
b $f(x) = 2^{-x}$
c $f(x) = \sqrt{3 - x^2}$
d $f(x) = \dfrac{1}{x^2 + 1}$
e $f(x) = \dfrac{4x}{x^2 + 4}$
f $f(x) = 3^x + 3^{-x}$
g $f(x) = 3^x - 3^{-x}$
h $f(x) = 3^x + x^3$

11 a Explain why the relation $x^2 + (y - 5)^2 = 49$ is even.
b Explain why the relation $x^2 + y^2 = 49$ is both odd and even.

ENRICHMENT

12 a Show that $f(x) = \dfrac{1}{2} - \dfrac{1}{2^x + 1}$ is an odd function.
b Confirm the result by plotting the graph.

13 a Given that $h(x) = f(x) \times g(x)$, determine what symmetry $h(x)$ has if:
i both f and g are even,
ii both f and g are odd,
iii one is even and the other odd.

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b Given that $h(x) = f(x) + g(x)$, determine what symmetry $h(x)$ has if:

- **i** both f and g are even,
- **ii** both f and g are odd,
- **iii** one is even and the other odd.

14 **a** Prove that an odd function defined at $x = 0$ passes through the origin.

b Show that $f(x) = \dfrac{\sqrt{x^2}}{x}$ is odd, and explain why it does not pass through the origin. (Hint: What does its graph look like?)

15 For any function $f(x)$, define $g(x) = \frac{1}{2}\big(f(x) + f(-x)\big)$ and $h(x) = \frac{1}{2}\big(f(x) - f(-x)\big)$.

a Show that $f(x) = g(x) + h(x)$, that $g(x)$ is even and that $h(x)$ is odd.

b Hence write each function as the sum of an even and an odd function:

i $f(x) = 1 - 2x + x^2$ **ii** $f(x) = 2^x$

c Why is there a problem with this process if $f(x) = \dfrac{1}{x - 1}$ or $f(x) = \sqrt{x}$?

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4D The absolute value function

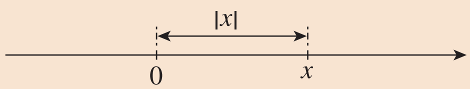
Often it is the size or magnitude of a number that is significant, rather than whether it is positive or negative. *Absolute value* is the mathematical name for this concept.

Absolute value as distance

Distance is the clearest way to define absolute value.

10 ABSOLUTE VALUE AS DISTANCE

- The absolute value $|x|$ of a number x is the distance from x to the origin on the number line.



  For example, $|-5| = 5$ and $|0| = 0$ and $|5| = 5$.
- Distance is always positive or zero, so
  $|x| \geq 0$, for all real numbers x.
- The numbers x and $-x$ are equally distant from the origin, so
  $|-x| = |x|$, for all real numbers x.

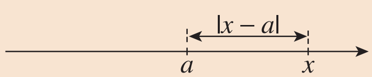
Thus absolute value is a measure of the *size* or *magnitude* of a number. In the examples above, the numbers -5 and $+5$ both have the same magnitude 5, and differ only in their signs.

Distance between numbers

Replacing x by $x - a$ in the previous definition gives a measure of the distance from x to a on the number line.

11 DISTANCE BETWEEN NUMBERS

- The distance from x to a on the number line is $|x - a|$.



  For example, the distance between 5 and -2 is $|5 - (-2)| = 7$.
- If follows that $|x - a| = |a - x|$, for all real numbers x and a.

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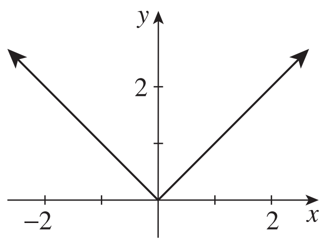
An expression for absolute value involving cases

If x is a negative number, then the absolute value of x is $-x$, the opposite of x. This gives an alternative definition:

12 ABSOLUTE VALUE INVOLVING CASES

For any real number x, define $|x| = \begin{cases} x, & \text{for } x \ge 0, \\ -x, & \text{for } x < 0. \end{cases}$

The two cases lead directly to the graph of $y = |x|$.



A table of values confirms the sharp point at the origin where the two branches meet at right angles:

x	-2	-1	0	1	2
$\lvert x\rvert$	2	1	0	1	2

- The domain is the set of all real numbers, and the range is $y \ge 0$.
- The function is even, because the graph has line symmetry in the y-axis.
- The function has a zero at $x = 0$, and is positive for all other values of x.

Graphing functions with absolute value

Transformations can now be applied to the graph of $y = |x|$ to sketch many functions involving absolute value. More complicated functions, however, require the approach involving cases.

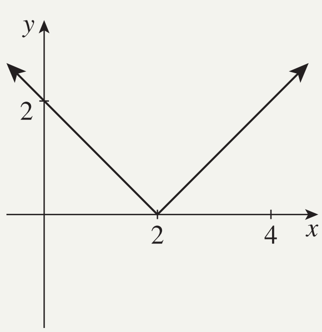
A short table of values is always an excellent safety check.

Example 10 4D

a Sketch $y = |x - 2|$ using shifting.
b Check the graph using a table of values.
c Write down the equations of the two branches.

SOLUTION

a This is $y = |x|$ shifted 2 units to the right.



b

x	0	1	2	3	4
y	2	1	0	1	2

c From the expression using cases, or from the graph:

$$y = \begin{cases} x - 2, & \text{for } x \ge 2, \\ -x + 2, & \text{for } x < 2. \end{cases}$$

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Example 11 4D

a Use cases to sketch $y = |x| - x$.

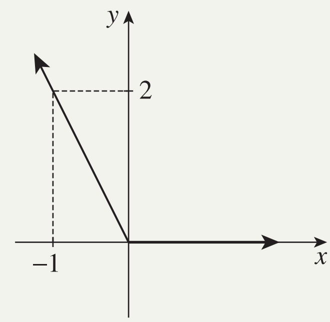
b Check using a table of values.

SOLUTION

a Considering separately the cases $x \geq 0$ and $x < 0$,

$$y = \begin{cases} x - x, & \text{for } x \geq 0, \\ -x - 2, & \text{for } x < 0. \end{cases}$$

$$\text{that is, } y = \begin{cases} 0, & \text{for } x \geq 0, \\ -2x, & \text{for } x < 0. \end{cases}$$



b Checking using a table of values:

x	-2	-1	0	1	2
y	4	2	0	0	0

Solving absolute value equations

Three observations should make everything clear:

- An equation such as $|3x + 6| = -21$ has no solutions, because an absolute value can never be negative.
- An equation such as $|3x + 6| = 0$ is true when $3x + 6 = 0$.
- An equation such as $|3x + 6| = 21$ can be solved by realising that:

$$|3x + 6| = 21 \quad \text{is true when} \quad 3x + 6 = 21 \text{ or } 3x + 6 = -21.$$

13 TO SOLVE THE EQUATION $|ax + b| = k$

- If $k < 0$, the equation has no solutions.
- If $k = 0$, the equation has one solution, found by solving
  $ax + b = 0$.
- If $k > 0$, the equation has two solutions, found by solving
  $ax + b = k \quad \text{or} \quad ax + b = -k$.

Example 12 4D

Solve these absolute value equations.

a $|3x + 6| = -21$ **b** $|3x + 6| = 0$ **c** $|3x + 6| = 21$

SOLUTION

a $|3x + 6| = -21$ has no solutions, because an absolute value cannot be negative.

b

$$\begin{aligned} |3x + 6| &= 0 \\ 3x + 6 &= 0 \\ \boxed{-6} \quad 3x &= -6 \\ \boxed{\div 3} \quad x &= -2 \end{aligned}$$

c

$$\begin{aligned} |3x + 6| &= 21 \\ 3x + 6 &= 21 \quad \text{or} \quad 3x + 6 = -21 \\ \boxed{-6} \quad 3x &= 15 \quad \text{or} \quad 3x = -27 \\ \boxed{\div 3} \quad x &= 5 \quad \text{or} \quad x = -9 \end{aligned}$$

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Example 13 **4D**

Solve each absolute value equation.

a $|x - 2| = 3$

b $\left|7 - \frac{1}{4}x\right| = 3$

SOLUTION

a

$$|x - 2| = 3$$
$$x - 2 = 3 \quad \text{or} \quad x - 2 = -3$$
$$\boxed{+2} \qquad x = 5 \quad \text{or} \quad x = -1$$

b

$$\left|7 - \tfrac{1}{4}x\right| = 3$$
$$7 - \tfrac{1}{4}x = 3 \quad \text{or} \quad 7 - \tfrac{1}{4}x = -3$$
$$\boxed{-7} \qquad \tfrac{1}{4}x = -4 \quad \text{or} \quad -\tfrac{1}{4}x = -10$$
$$\boxed{\times(-4)} \qquad x = 16 \quad \text{or} \quad x = 40$$

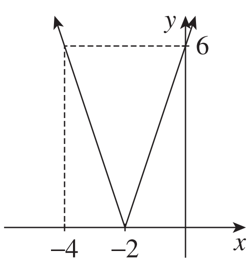
Sketching $y = |ax + b|$

To sketch a function such as $y = |3x + 6|$, the first step is always to find the x-intercept and y-intercept. As in Example 12:

Put $y = 0$, then $|3x + 6| = 0$
$3x + 6 = 0$
$x = -2.$

Put $x = 0$.
Then $y = |0 + 6|$
$= 6.$

Plot those two points $(-2, 0)$ and $(0, 6)$. The graph is symmetric about the vertical line $x = -2$, so the point $(-4, 6)$ also lies on the curve. Now join the points up in the characteristic V shape.



Alternatively, draw up a small table of values,

x	-4	-3	-2	-1	0
y	6	3	0	3	6

A good final check: The two branches of the curve should have gradients 3 and -3.

Absolute value as the square root of the square

Taking the absolute value of a number means stripping any negative sign from the number. We already have algebraic functions capable of doing this job — we can square the number, then apply the function $\sqrt{}$ that says 'take the positive square root (or zero)'.

14 ABSOLUTE VALUE AS THE POSITIVE SQUARE ROOT OF THE SQUARE

- For all real numbers x, $|x|^2 = x^2$ and $|x| = \sqrt{x^2}$.

For example, $|-3|^2 = 9 = (-3)^2$ and $|-3| = \sqrt{9} = \sqrt{(-3)^2}$.

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Identities involving absolute value

Here are some standard identities.

15 IDENTITIES INVOLVING ABSOLUTE VALUE

- $|-x| = |x|$, for all x.
- $|x - y| = |y - x|$, for all x and y.
- $|xy| = |x||y|$, for all x and y.
- $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, for all x, and for all $y \neq 0$.

Substitution of some positive and negative values for x and y should be sufficient to demonstrate these results.

Exercise 4D

FOUNDATION

1 Evaluate:

a $|3|$ **b** $|-3|$ **c** $|4 - 7|$ **d** $|7 - 4|$
e $|14 - 9 - 12|$ **f** $|-7 + 8|$ **g** $|3^2 - 5^2|$ **h** $|11 - 16| - 8$

2 Solve each absolute value equation, then graph the solution on a number line.

a $|x| = 1$ **b** $|x| = 3$ **c** $|4x| = 8$
d $|2x| = 10$ **e** $|2x| = -6$ **f** $|3x| = -12$

3 Solve each equation and graph the solution on a number line.

a $|x - 4| = 1$ **b** $|x - 3| = 7$ **c** $|x - 3| = -3$ **d** $|x - 7| = -2$
e $|x + 5| = 2$ **f** $|x + 2| = 2$ **g** $|x + 1| = 6$ **h** $|x + 3| = 1$

4 **a** Copy and complete the tables of values for the functions $y = |x - 1|$ and $y = |x| - 1$.

x	-2	-1	0	1	2	3
$\lvert x - 1\rvert$						

x	-2	-1	0	1	2	3
$\lvert x\rvert - 1$						

b Draw the graphs of the two functions on separate number planes, and observe the similarities and differences between them.

c Explain how each graph is obtained by shifting $y = |x|$.

5 Show that each statement is false when $x = -2$.

a $|x| = x$ **b** $|-x| = x$ **c** $|x + 2| = |x| + 2$
d $|x + 1| = x + 1$ **e** $|x - 1| < |x| - 1$ **f** $|x|^3 = x^3$

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6 Say whether these statements are true or false, and if false, give a counter-example (any difficulties will usually involve negative numbers):

a $|x| > 0$ b $|-x| = |x|$ c $-|x| \le x \le |x|$ d $|x + 2| = |x| + 2$
e $|5x| = 5|x|$ f $|x|^2 = x^2$ g $|x|^3 = x^3$ h $|x - 7| = |7 - x|$

DEVELOPMENT

7 In each part, use either shifting of $y = |x|$ or a table of values to help sketch the graph. Then write down the equations of the two branches.

a $y = |2x|$ b $y = \left|\frac{1}{2}x\right|$ c $y = |x - 3|$ d $y = |x + 2|$
e $y = |x| - 2$ f $y = |x| + 3$ g $y = |x - 2| - 1$ h $y = |x + 1| - 1$

8 In each case, use the rules of Box 13 to solve the equation for x.

a $|7x| = 35$ b $|2x + 1| = 3$ c $|2x - 1| = 11$
d $|7x - 3| = -11$ e $|3x + 2| = -8$ f $|5x + 2| = 0$
g $|5 - 3x| = 0$ h $|7 - 6x| = 5$ i $|5x + 4| = 6$

9 a Use cases to help sketch the branches of $y = |x|$.

b In each part, identify the shift or shifts of $y = |x|$ and hence sketch the graph. Then write down the equations of the two branches.

i $y = |x - 3|$ ii $y = |x + 2|$ iii $y = |x| - 2$
iv $y = |x| + 3$ v $y = |x - 2| - 1$ vi $y = |x + 1| - 1$

10 Sketch each function using cases. Check the graph with a table of values.

a $y = |2x|$ b $y = \left|\frac{1}{2}x\right|$

11 Find the x-intercept and y-intercept of each function. Then sketch the graph using symmetry, and confirm with a small table of values.

a $y = |2x - 6|$ b $y = |9 - 3x|$ c $y = |5x|$
d $y = |4x + 10|$ e $y = -|3x + 7|$ f $y = -|7x|$

12 [Technology]
Use suitable graphing software to help solve these problems.

a i Sketch $y = |x - 4|$ and $y = 1$ on the same set of axes, clearly showing the points of intersection.

ii Hence write down the solution of $|x - 4| = 1$.

b Now use similar graphical methods to solve each equation.

i $|x + 3| = 1$ ii $|2x + 1| = 3$
iii $|3x - 3| = -2$ iv $|2x - 5| = 0$

13 Consider the absolute value function $f(x) = |x|$.

a Use the result $f(x) = \sqrt{x^2}$ to help prove that the absolute value function is even.

b Why was this result obvious from the graph of $y = |x|$?

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14 Use the fact that $|-x| = |x|$ to decide whether these functions are odd, even or neither.

a $f(x) = |x| + 1$
b $f(x) = |x| + x$
c $f(x) = x \times |x|$
d $f(x) = |x^3 - x|$

15 **a** For what values of x is $y = \dfrac{|x|}{x}$ undefined?

b Use a table of values of x from -3 to 3 to sketch the graph.

c Hence write down the equations of the two branches of $y = \dfrac{|x|}{x}$.

16 Sketch each graph by drawing up a table of values for $-3 \le x \le 3$. Then use cases to determine the equation of each branch of the function.

a $y = |x| + x$
b $y = |x| - x$
c $y = 2(x + 1) - |x + 1|$
d $y = x^2 - |2x|$

ENRICHMENT

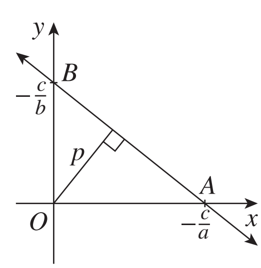
17 Prove that $f(|x|)$ is an even function, for all functions $f(x)$.

18 Sketch the relation $|y| = |x|$ by considering the possible cases.

19 Carefully write down the equations of the branches of each function, then sketch its graph:

a $y = |x + 1| - |x - 3|$
b $y = |x - 2| + |x + 1| - 4$
c $y = 2|x + 1| - |x - 1| - 1$

20 The diagram on the right shows the line $ax + by + c = 0$, which intersects the axes at $A\left(-\dfrac{c}{a}, 0\right)$ and $B\left(0, -\dfrac{c}{b}\right)$. Let p be the perpendicular distance from O to the line AB.



a Use distances $|OA|$ and $|OB|$ to find the area of $\triangle AOB$.

b The area of $\triangle AOB$ is also $\frac{1}{2}p \times |AB|$. Use Pythagoras' theorem to rewrite this in terms of a, b, c and p.

c Equate your answers to parts **a** and **b**, and so find p.

d Now use shifting to find a formula for the distance from $P(x_1, y_1)$ to the line $ax + by + c = 0$.

e Use this formula to find the distance from $(3, 2)$ to the line $2x - 5y + 3 = 0$.

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4E Composite functions

Shifting, reflecting, and taking absolute value, are all examples of a far more general procedure of creating a composite function from two given functions. The first example below shows how a translation left 3 and a translation up 3 are obtained using composites. In this section, however, attention is on the algebra rather than on the final graph.

Composition of functions

Suppose that we are given the two functions $f(x) = x + 3$ and $g(x) = x^2$. We can put them together by placing their function machines so that the output of the first function is the input of the second function

x		$f(x)$		$g(f(x))$
-2	$\longrightarrow$ Add 3 $\longrightarrow$	1	$\longrightarrow$ Square $\longrightarrow$	1
-1	$\longrightarrow$ Add 3 $\longrightarrow$	2	$\longrightarrow$ Square $\longrightarrow$	4
0	$\longrightarrow$ Add 3 $\longrightarrow$	3	$\longrightarrow$ Square $\longrightarrow$	9
1	$\longrightarrow$ Add 3 $\longrightarrow$	4	$\longrightarrow$ Square $\longrightarrow$	16
2	$\longrightarrow$ Add 3 $\longrightarrow$	5	$\longrightarrow$ Square $\longrightarrow$	25
x	$\longrightarrow$ Add 3 $\longrightarrow$	$x + 3$	$\longrightarrow$ Square $\longrightarrow$	$(x + 3)^2$

The middle column is the output of the first function 'Add 3'. This output is then the input of the second function 'Square'. The result is the *composite function*

$$g(f(x)) = (x + 3)^2 \qquad \text{'Add 3, then square.'}$$

If the functions are composed the other way around, the result is different,

$$f(g(x)) = x^2 + 3 \qquad \text{'Square, then add 3.'}$$

Notice how in this example, both ways around are examples of translations.

- The composite graph $y = g(f(x))$ is $y = g(x)$ shifted left 3.
- The composite graph $y = f(g(x))$ is $y = g(x)$ shifted up 3.

Example 14 **4E**

Find and simplify $k(h(x))$ and $h(k(x))$ when $h(x) = 2x + 3$ and $k(x) = 1 - 5x$.

SOLUTION

$$\begin{aligned} k(h(x)) &= k(2x + 3) \\ &= 1 - 5(2x + 3) \\ &= -10x - 14 \end{aligned}$$

$$\begin{aligned} h(k(x)) &= h(1 - 5x) \\ &= 2(1 - 5x) + 3 \\ &= -10x + 5 \end{aligned}$$

Domain and range of composite functions

In Example 14, $h(x)$ and $k(x)$ have domain and range all real numbers, so there are no problems, and the domains and ranges of both composites are all real numbers.

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In the original example with the function machines, $f(x)$ and $g(x)$ again both have domain all real numbers, and so do $g\big(f(x)\big)$ and $f\big(g(x)\big)$. But while $f(x)$ has range all real numbers, $g(x)$ has range $y \geq 0$. Thus the range of $g\big(f(x)\big) = (x + 3)^2$ is $y \geq 0$, and the range of $f\big(g(x)\big) = x^2 + 3$ is $y \geq 3$.

In general, neither the domain nor the range of $g\big(f(x)\big)$ are all real numbers.

- For a real number a to be in the domain of $g\big(f(x)\big)$, a must be in the domain of $f(x)$, and $f(a)$ must be in the domain of $g(x)$.
- The range of $g\big(f(x)\big)$ is the range of $g(x)$ when it is restricted just to the range of $f(x)$.

Example 15 4E

Find the domain and range of $g\big(f(x)\big)$ if $f(x) = \sqrt{x - 4}$ and $g(x) = \dfrac{1}{x}$.

SOLUTION

The domain of $f(x)$ is $x \geq 4$, and the domain of $g(x)$ is $x \neq 0$.
When $x = 4$, $f(4) = 0$, and when $x > 4$, $f(x) > 0$,
so $g\big(f(x)\big)$ is not defined at $x = 4$, but is defined for $x > 4$.
Thus the domain of $g\big(f(x)\big)$ is $x > 4$.
When restricted to $x > 4$, the range of $f(x)$ is $y > 0$,
and when $g(x)$ is restricted to $x > 0$, its range is $y > 0$
(note the change of variable as output becomes input).
Thus the range of $g\big(f(x)\big)$ is $y > 0$.

This approach, however, is rather elaborate. It is almost always enough to find the equation of the composite and look at it as a single function. In this example,

$$g\big(f(x)\big) = \frac{1}{\sqrt{x - 4}},$$

from which it is easily seen that the domain is $x > 4$ and the range is $y > 0$.

16 COMPOSITE FUNCTIONS

- The *composites* of two functions $f(x)$ and $g(x)$ are $g\big(f(x)\big)$ and $f\big(g(x)\big)$.
- For a real number a to be in the domain of $g\big(f(x)\big)$, a must be in the domain of $f(x)$, and then $f(a)$ must be in the domain of $g(x)$.
- The range of $g\big(f(x)\big)$ is the range of $g(x)$ when it is restricted just to the range of $f(x)$.

It is almost always enough to read the domain and range from the equation of the composite function.

The notations $(g \circ f)(x)$ for the composite $g\big(f(x)\big)$, and $(f \circ g)(x)$ for $f\big(g(x)\big)$, are widely used.

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The empty function

Let $f(x) = -x^2 - 1$ and $g(x) = \sqrt{x}$. Then

$$g\big(f(x)\big) = \sqrt{-x^2 - 1}.$$

This is a problem, because $\sqrt{-x^2 - 1}$ is undefined, whatever the value of x. The range of $f(x)$ is all real numbers less than or equal to -1, and $g(x)$ is undefined on all of these because you can't take the square root of a negative. Thus $g\big(f(x)\big)$ has domain the empty set, and its range is therefore also the empty set. It is *the empty function*.

The empty function has domain the empty set, and its range is therefore also the empty set. For those interested in trivialities, the empty function is one-to-one.

Exercise 4E

FOUNDATION

1 Consider the function $f(x) = x + 2$.

a Find the values of:

i $f\big(f(0)\big)$ **ii** $f\big(f(3)\big)$ **iii** $f\big(f(-1)\big)$ **iv** $f\big(f(-8)\big)$

b Find expressions for:

i $f\big(f(x)\big)$ **ii** $f\Big(f\big(f(x)\big)\Big)$

c Find the value of x for which $f\big(f(x)\big) = 0$.

2 Consider the function $F(x) = 2x$.

a Find the values of $F\big(F(0)\big)$, $F\big(F(7)\big)$, $F\big(F(-3)\big)$ and $F\big(F(-11)\big)$.

b Find expressions for $F\big(F(x)\big)$ and for $F\Big(F\big(F(x)\big)\Big)$

c Find the value of x for which $F\big(F(x)\big) = 32$.

3 Consider the function $g(x) = 2 - x$.

a Find the values of $g\big(g(0)\big)$, $g\big(g(4)\big)$, $g\big(g(-2)\big)$ and $g\big(g(-9)\big)$.

b Show that $g\big(g(x)\big) = x$.

c Show that $g\Big(g\big(g(x)\big)\Big) = g(x)$.

4 Consider the function $h(x) = 3x - 5$.

a Find the values of $h\big(h(0)\big)$, $h\big(h(5)\big)$, $h\big(h(-1)\big)$ and $h\big(h(-5)\big)$.

b Find expressions for $h\big(h(x)\big)$ and for $h\Big(h\big(h(x)\big)\Big)$.

5 Two linear functions are defined by $f(x) = x + 1$ and $g(x) = 2x - 3$.

a Find the values of $f\big(g(7)\big)$, $g\big(f(7)\big)$, $f\big(f(7)\big)$ and $g\big(g(7)\big)$.

b Find expressions for:

i $f\big(g(x)\big)$ **ii** $g\big(f(x)\big)$ **iii** $f\big(f(x)\big)$ **iv** $g\big(g(x)\big)$

c What transformation maps the graph of $y = g(x)$ to the graph of $y = g\big(f(x)\big)$?

d What transformation maps the graph of $y = g(x)$ to the graph of $y = f\big(g(x)\big)$?

6 Two functions $\ell(x)$ and $q(x)$ are defined by $\ell(x) = x - 3$ and $q(x) = x^2$.

a Find the values of $\ell\big(q(-1)\big)$, $q\big(\ell(-1)\big)$, $\ell\big(\ell(-1)\big)$ and $q\big(q(-1)\big)$.

b Find:

i $\ell\big(q(x)\big)$ **ii** $q\big(\ell(x)\big)$ **iii** $\ell\big(\ell(x)\big)$ **iv** $q\big(q(x)\big)$

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c Determine the domains and ranges of:

i $\ell\big(q(x)\big)$ **ii** $q\big(\ell(x)\big)$

d What transformation maps the graph of $y = q(x)$ to the graph of $y = q\big(\ell(x)\big)$?

e What transformation maps the graph of $y = q(x)$ to the graph of $y = \ell\big(q(x)\big)$?

DEVELOPMENT

7 Suppose that $F(x) = 4x$ and $G(x) = \sqrt{x}$.

a Find the values of $F\big(G(25)\big)$, $G\big(F(25)\big)$, $F\big(F(25)\big)$ and $G\big(G(25)\big)$.

b Find $F\big(G(x)\big)$.

c Find $G\big(F(x)\big)$.

d Hence show that $F\big(G(x)\big) = 2G\big(F(x)\big)$.

e State the domain and range of $F\big(G(x)\big)$.

8 Two functions f and h are defined by $f(x) = -x$ and $h(x) = \dfrac{1}{x}$.

a Find the values of $f\Big(h\big(-\tfrac{1}{4}\big)\Big)$, $h\Big(f\big(-\tfrac{1}{4}\big)\Big)$, $f\Big(f\big(-\tfrac{1}{4}\big)\Big)$ and $h\Big(h\big(-\tfrac{1}{4}\big)\Big)$.

b Show that for all $x \neq 0$:

i $f\big(h(x)\big) = h\big(f(x)\big)$ **ii** $f\big(f(x)\big) = h\big(h(x)\big)$

c Write down the domain and range of $f\big(h(x)\big)$.

d Describe how the graph of $h(x)$ is transformed to obtain the graph of $h\big(f(x)\big)$.

9 Suppose that $f(x) = -5 - |x|$ and $g(x) = \sqrt{x}$.

a Find $f\big(g(x)\big)$, state its domain and range, and sketch its graph.

b Explain why $g\big(f(x)\big)$ is the empty function.

10 **a** Show that if $f(x)$ and $g(x)$ are odd functions, then $g\big(f(x)\big)$ is odd.

b Show that if $f(x)$ is an odd function and $g(x)$ is even, then $g\big(f(x)\big)$ is even.

c Show that if $f(x)$ is an even function, then $g\big(f(x)\big)$ is even.

11 Find the composite functions $g\big(f(x)\big)$ and $f\big(g(x)\big)$.

a $f(x) = 4$, for all x, and $g(x) = 7$, for all x.

b $f(x) = x$, $g(x)$ any function.

12 **a** Let $f(x)$ be any function, and let $g(x) = x - a$, where a is a constant. Describe each composite graph as a transformation of the graph of $y = f(x)$.

i $y = g\big(f(x)\big)$ **ii** $y = f\big(g(x)\big)$

b Let $f(x)$ be any function, and let $g(x) = -x$. Describe each composite graph as a transformation of the graph of $y = f(x)$.

i $y = g\big(f(x)\big)$ **ii** $y = f\big(g(x)\big)$

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ENRICHMENT

13 Let $f(x) = 2x + 3$ and $g(x) = 5x + b$, where b is a constant.

a Find expressions for $g\big(f(x)\big)$ and $f\big(g(x)\big)$.

b Hence find the value of b so that $g\big(f(x)\big) = f\big(g(x)\big)$, for all x.

14 Let $f(x) = 2x + 3$ and $g(x) = ax + b$, where b is a constant.

a Find expressions for $g\big(f(x)\big)$ and $f\big(g(x)\big)$.

b Hence find the values of a and b so that $g\big(f(x)\big) = x$, for all x.

c Show that if a and b have these values, then $f\big(g(x)\big) = x$, for all x.

15 Let $f(x) = x^2 + x - 3$ and $g(x) = |x|$.

a Find $f\big(g(0)\big)$, $g\big(f(0)\big)$, $f\big(g(-2)\big)$ and $g\big(f(-2)\big)$.

b Write an expression for $f\big(g(x)\big)$ without any use of absolute value, given that:

i $x \geq 0$ **ii** $x < 0$

16 Let $L(x) = x + 1$ and $Q(x) = x^2 + 2x$.

a State the ranges of $L(x)$ and $Q(x)$.

b Find $L\big(Q(x)\big)$ and determine its range.

c Find $Q\big(L(x)\big)$ and determine its range.

d Find the zeroes of $Q\big(L(x)\big)$.

e Show that $Q\left(L\left(\dfrac{1}{x + 1}\right)\right) = \dfrac{(x + 2)(3x + 4)}{(x + 1)^2}$, provided that $x \neq -1$.

17 Let $f(x)$ be any function with domain D, and let $z(x)$ be the *zero function* defined by $z(x) = 0$ for all x. What are these functions (specify the domain of each)?

a $f(x) + z(x)$ **b** $f(x) \times z(x)$ **c** $z\big(f(x)\big)$ **d** $f\big(z(x)\big)$

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Chapter 4 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 4 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 **a** Copy and complete the table of values for $y = x^2$ and $y = (x - 2)^2$.

x	-2	-1	0	1	2	3	4
x^2							
$(x - 2)^2$							

b Sketch the two graphs and state the vertex of each.

c What transformation maps $y = x^2$ to $y = (x - 2)^2$?

2 Consider the parabola $y = x^2 - 2x$.

a Show that when this is reflected in the y-axis the equation becomes $y = x^2 + 2x$.

b Copy and complete the table of values for $y = x^2 - 2x$ and $y = x^2 + 2x$.

x	-3	-2	-1	0	1	2	3
$x^2 - 2x$							
$x^2 + 2x$							

c Sketch the two parabolas and state the vertex of each.

3 Evaluate:

a $|-7|$ **b** $|4|$ **c** $|3 - 8|$

d $|-2-(-5)|$ **e** $|-2| - |-5|$ **f** $|13 - 9 - 16|$

4 Solve for x:

a $|x| = 5$ **b** $|3x| = 18$ **c** $|x - 2| = 4$

d $|x + 3| = 2$ **e** $|2x - 3| = 5$ **f** $|3x - 4| = 7$

5 Explain how to shift the graph of $y = x^2$ to obtain each function.

a $y = x^2 + 5$ **b** $y = x^2 - 1$

c $y = (x - 3)^2$ **d** $y = (x + 4)^2 + 7$

6 Write down the equation of the monic quadratic with vertex:

a $(1, 0)$ **b** $(0, -2)$ **c** $(-1, 5)$ **d** $(4, -9)$

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Review

7 Write down the centre and radius of each circle. Shifting may help locate the centre.

a $x^2 + y^2 = 1$ **b** $(x + 1)^2 + y^2 = 4$

c $(x - 2)^2 + (y + 3)^2 = 5$ **d** $x^2 + (y - 4)^2 = 64$

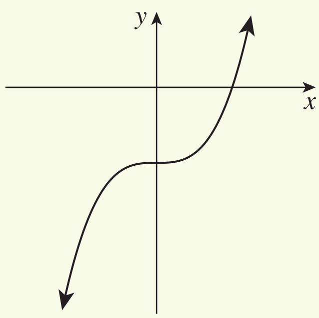
8 In each case, find the function obtained by the given reflection or rotation.

a $y = x^3 - 2x + 1$: reflect in the y-axis **b** $y = x^2 - 3x - 4$: reflect in the x-axis

c $y = 2^x - x$: rotate 180° about the origin **d** $y = \sqrt{9 - x^2}$: reflect in the y-axis

9 Classify each function $y = f(x)$ as odd, even or neither.

a



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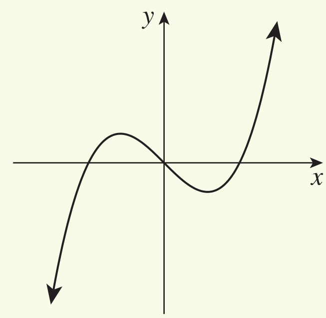

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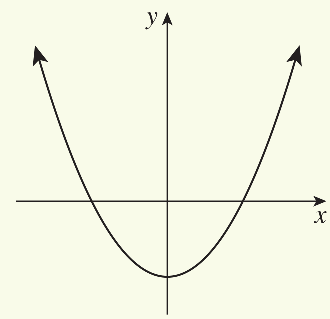
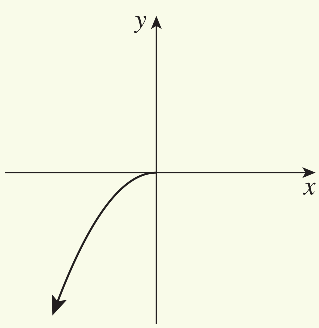


10 In each diagram below, complete the graph so that:

i $f(x)$ is odd,

ii $f(x)$ is even.

a



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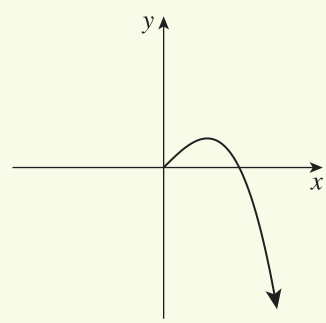

11 Sketch each graph by shifting $y = |x|$, or by using a table of values. Mark all x- and y-intercepts.

a $y = |x| - 2$ **b** $y = |x - 2|$ **c** $y = |x + 2|$ **d** $y = |x| + 2$

12 Sketch each graph by finding the x-intercept and y-intercept and then using symmetry. Perhaps also use a table of values to confirm the graph.

a $y = |3x + 9|$ **b** $y = -|2x - 8|$ **c** $y = |4x + 13|$

13 Solve these absolute value equations.

a $|3x| = 15$ **b** $|x + 4| = 5$ **c** $|x + 4| = -5$ **d** $|5 - x| = 7$

e $|2x + 7| = 9$ **f** $|3x - 8| = 4$ **g** $|7x + 2| = 0$ **h** $|x^2 - 25| = 0$

14 Find $f(-x)$ for each function, and then decide whether the function is odd, even or neither.

a $f(x) = x + 3$ **b** $f(x) = 2x^2 - 5$ **c** $f(x) = \dfrac{1}{x}$ **d** $f(x) = \dfrac{x}{x^2 + 1}$

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15 For each parabola, complete the square to find the coordinates of the vertex.

a $y = x^2 - 2x + 5$ **b** $y = x^2 + 4x - 3$
c $y = 2x^2 + 8x + 11$ **d** $y = -x^2 + 6x + 1$

16 Use completion of the square to help sketch the graph of each quadratic function. Indicate the vertex and all intercepts with the axes.

a $y = x^2 + 2x + 3$ **b** $y = x^2 - 4x + 1$
c $y = 2 + 2x - x^2$ **d** $y = x^2 - x - 1$

17 Complete the squares to find the centre and radius of each circle.

a $x^2 + y^2 - 2y = 3$ **b** $x^2 + 6x + y^2 + 8 = 0$
c $x^2 - 4x + y^2 + 6y - 3 = 0$ **d** $x^2 + y^2 - 8x + 14y = 35$

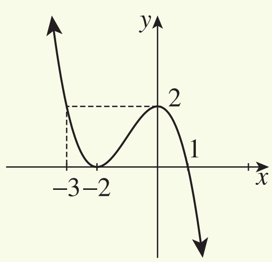
18 Consider the cubic with equation $y = x^3 - x$.

a Use an appropriate substitution to show that when the graph of this function is shifted right 1 unit the result is $y = x^3 - 3x^2 + 2x$.

b [Technology] Plot both cubics using graphing software to confirm the outcome.

19 The graph drawn to the right shows the curve $y = f(x)$. Use this graph to sketch these graphs.

a $y = f(x + 1)$
b $y = f(x) + 1$
c $y = f(x - 1)$
d $y = f(x) - 1$
e $y = f(-x)$
f $y = -f(x)$
g $y = -f(-x)$



20 [A revision medley of curve sketches]

Sketch each set of graphs on a single pair of axes, showing all significant points. Use transformations, tables of values, or any other convenient method.

a $y = 2x,\quad y = 2x + 3,\quad y = 2x - 2$

b $y = -\frac{1}{2}x,\quad y = -\frac{1}{2}x + 1,\quad y = -\frac{1}{2}x - 2$

c $y = x + 3,\quad y = 3 - x,\quad y = -x - 3$

d $y = (x - 2)^2 - 1,\quad y = (x + 2)^2 - 1,\quad y = -(x + 2)^2 + 1$

e $y = x^2,\quad y = (x + 2)^2,\quad y = (x - 1)^2$

f $(x - 1)^2 + y^2 = 1,\quad (x + 1)^2 + y^2 = 1,\quad x^2 + (y - 1)^2 = 1$

g $y = x^2 - 1,\quad y = 1 - x^2,\quad y = 4 - x^2$

h $y = (x + 2)^2,\quad y = (x + 2)^2 - 4,\quad y = (x + 2)^2 + 1$

i $y = -|x|,\quad y = -|x| + 1,\quad y = -|x - 2|$

j $y = \sqrt{x},\quad y = \sqrt{x} + 1,\quad y = \sqrt{x + 1}$

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Review

k $y = 2^x, \quad y = 2^x - 1, \quad y = 2^{x-1}$

l $y = \dfrac{1}{x}, \quad y = \dfrac{1}{x-2}, \quad y = \dfrac{1}{x+1}$

m $y = x^3, \quad y = x^3 - 1, \quad y = (x-1)^3$

n $y = x^4, \quad y = (x-1)^4, \quad y = x^4 + 1$

o $y = \sqrt{x}, \quad y = -\sqrt{x}, \quad y = 2 - \sqrt{x}$

p $y = 2^{-x}, \quad y = 2^{-x} - 2, \quad y = 2 - 2^x$

21 Prove that the function $f(x) = \dfrac{3^x}{3^{2x} + 1}$ is even.

22 Every function $f(x)$ defined for all real numbers can be written as the sum $f(x) = g(x) + h(x)$, where $g(x)$ is an even function and $h(x)$ is an odd function. Prove this statement by finding formulae for $g(x)$ and $h(x)$ in terms of $f(x)$.

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5 Further graphs

This chapter contains four Extension 1 topics on functions, loosely related by the graphing and transformation techniques that each of them involve.

- Further types of equations and inequations (also called ‘inequalities’) are solved using algebraic and graphical methods. Section 5A deals with linear, quadratic and simple absolute value inequations, then Section 5B develops the more general approach of a table of signs for any function.
- Translations and reflections of known graphs were introduced in Chapter 4, and several further transformations are discussed here. Section 5C sketches the reciprocal of a known graph, a procedure that requires a little more discussion of asymptotes. Section 5D sketches the sum, difference and product of known graphs. Section 5E sketches squares and square roots of known graphs, and graphs transformed using absolute value.
- Inverse functions and relations are introduced graphically in Section 5F with their corresponding reflections in the diagonal line $y = x$. This is yet another type of transformation of a known graph. Section 5G develops the formal notation for inverse functions.
- Parameters are introduced in Section 5H so that equations of functions, and relations in general, can be written and graphed in terms of functions x and y of a single parameter. The examples need the trigonometry of the general angle and Pythagorean identities, and it may be appropriate to delay this final section until after trigonometry is reviewed and extended in Chapter 6.

As always, computer sketching of curves is very useful in demonstrating how the features of a graph are related to the algebraic properties of its equation, and in gaining familiarity with the variety of graphs and their interrelationships.

The chapter is conceptually demanding, particularly Sections 5C–5E, and these three sections could be left until later in the year.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

5A Solving inequations

An equation such as $3x = 12$ has a solution $x = 4$. When the equals sign is replaced by $<$ or $\le$ or $>$ or $\ge$ the result is an *inequation*. We say that:

The inequation $3x > 12$ has solution $x > 4$

because substituting any number greater than 4 makes the statement true, and substituting any other number makes the statement false. In this section and the next, various inequations are solved using algebraic and graphical methods.

An *inequality* is an inequation such as $x^2 \ge 0$ that is true for all values of x. Inequalities are thus similar to *identities*, which are equations that are true for all values of x, such as $(x + 3)^2 = x^2 + 6x + 9$.

There are, however, different conventions about the words 'inequation' and 'inequality'. Often the word 'inequation' is not used at all, and the word 'inequality' is used for both objects. Don't be alarmed if you are asked to 'solve an inequality'.

The meaning of 'less than'

There are a geometric and an algebraic interpretation of the phrase 'less than'. Suppose that a and b are real numbers.

1 THE MEANING OF $a < b$

The geometric interpretation:
We say that $a < b$ if a is to the left of b on the number line:

The algebraic interpretation:
We say that $a < b$ if $b - a$ is positive.

The first interpretation is geometric, relying on the idea of a *number line* and of one point being *on the left-hand side of* another. The second interpretation requires that the term *positive number* be already understood. This second interpretation turns out to be very useful when solving inequations.

Solving linear inequations

As reviewed in Chapter 1, the rules for adding and subtracting from both sides, and for multiplying or dividing both sides, are exactly the same as for equations, with one qualification — the inequality symbol reverses when multiplying or dividing by a negative.

2 SOLVING A LINEAR INEQUATION

- Use the same methods as for linear equations, except that:
- When multiplying or dividing both sides by a negative number, the inequality symbol is reversed.

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Example 1 — 5A

a Solve and sketch $3x - 7 \le 8x + 18$.
b Solve and sketch $20 > 2 - 3x \ge 8$.

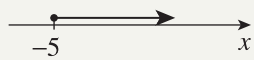
SOLUTION

a

$$3x - 7 \le 8x + 18$$

$\boxed{+(-8x + 7)}$ $\quad -5x \le 25$

$\boxed{\div\ (-5)}$ $\quad x \ge -5$

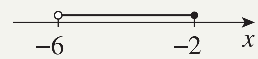


b

$$20 > 2 - 3x \ge 8$$

$\boxed{-2}$ $\quad 18 > -3x \ge 6$

$\boxed{\div\ (-3)}$ $\quad -6 < x \le -2$ (note the reversal)



Solving quadratic inequations

The most straightforward way to solve a quadratic inequation is to sketch the graph of the associated parabola.

3 SOLVING A QUADRATIC INEQUATION

- Move everything to the left-hand side.
- Sketch the graph of $y =$ LHS, showing the x-intercepts.
- Read the solution off the graph — y is positive when the graph is above the x-axis and negative when the graph is below the x-axis.

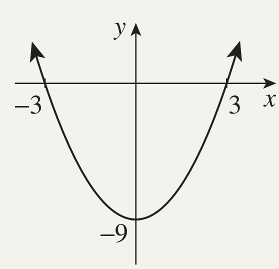
Example 2 — 5A

Solve each inequation by constructing a function and sketching it.

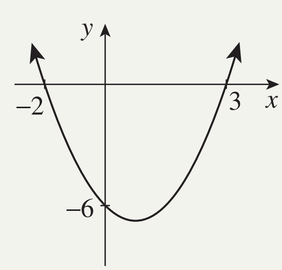
a $x^2 > 9$ **b** $x + 6 \ge x^2$

SOLUTION

a Moving everything onto the left, $x^2 - 9 > 0$,
then factoring, $(x - 3)(x + 3) > 0$.
We now sketch the graph of $y = (x - 3)(x + 3)$,
and examine where the graph is above the x-axis.
Thus the values of x for which $y > 0$ are
$x > 3$ or $x < -3$.
(This example $x^2 < 9$ is easy, and could be done at sight.)



b Moving everything onto the left, $x^2 - x - 6 \le 0$,
then factoring, $(x - 3)(x + 2) \le 0$.
We now sketch the graph of $y = (x - 3)(x + 2)$,
and examine where the graph is below or on the x-axis.
Thus the values of x for which $y \le 0$ are
$-2 \le x \le 3$.



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Solving absolute value equations and inequations on the number line

Most equations and inequations involving absolute values in the course are simple enough to be solved using distances on the number line.

4 SOLVING SIMPLE ABSOLUTE VALUE EQUATIONS AND INEQUATIONS

- Force the equation or inequation into one of the following forms:

  $|x - b| = a$, or $|x - b| < a$, or $|x - b| > a$, or $|x - b| \le a$, or $|x - b| \ge a$.
- Then find the solution using distance on a number line.

Example 3 5A

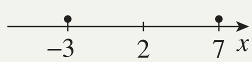
Solve these equations and inequations on the number line.

a $|x - 2| = 5$ **b** $|x + 3| = 4$ **c** $|3x + 7| < 3$ **d** $\left|7 - \frac{1}{4}x\right| \ge 3$

SOLUTION

a

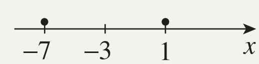
$$|x - 2| = 5$$
$$(\text{distance from } x \text{ to } 2) = 5$$



so $x = -3$ or $x = 7$.

b

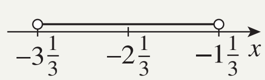
$$|x + 3| = 4$$
$$|x - (-3)| = 4$$
$$(\text{distance from } x \text{ to } -3) = 4$$



so $x = -7$ or $x = 1$.

c

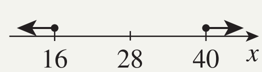
$$|3x + 7| < 3$$
$\boxed{\div 3}$ $$\left|x - \left(-2\tfrac{1}{3}\right)\right| < 1$$
$$\left(\text{distance from } x \text{ to } -2\tfrac{1}{3}\right) < 1$$



so $-3\frac{1}{3} < x < -1\frac{1}{3}$.

d

$$\left|7 - \tfrac{1}{4}x\right| \ge 3$$
$\boxed{\times 4}$ $$|28 - x| \ge 12$$
$$(\text{distance from } x \text{ to } 28) \ge 12$$



so $x \le 16$ or $x \ge 40$.

Solving absolute value equations and inequations algebraically

We saw in Section 4D how an absolute value equation of the form $|f(x)| = a$ can be solved algebraically by rewriting the equation.

Rewrite an equation $|f(x)| = a$ as $f(x) = a$ or $f(x) = -a$.

We can take a similar approach to solving an inequation $|f(x)| < a$ or $|f(x)| > a$.

Rewrite an inequation $|f(x)| < a$ as $-a < f(x) < a$.
Rewrite an inequation $|f(x)| > a$ as $f(x) < -a$ or $f(x) > a$.

The absolute value $|f(x)|$ cannot be negative. Thus if a is negative:

- $|f(x)| = a$ and $|f(x)| < a$ have no solutions, and
- $|f(x)| > a$ is true for all values of x in the domain of $f(x)$.

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5 SOLVING AN ABSOLUTE VALUE EQUATION OR INEQUATION ALGEBRAICALLY

- Rewrite an equation $|f(x)| = a$ as $f(x) = a$ or $f(x) = -a$.
- Rewrite an inequation $|f(x)| < a$ as $-a < f(x) < a$.
- Rewrite an inequation $|f(x)| > a$ as $f(x) < -a$ or $f(x) > a$.

Example 4 **5A**

a Solve $|9 - 2x| = 5$. **b** Solve $|9 - 2x| < 5$. **c** Solve $|9 - 2x| > 5$.

SOLUTION

a Using the first dotpoint of Box 5 above,

$$|9 - 2x| = 5$$
$$9 - 2x = 5 \text{ or } 9 - 2x = -5$$
$$\boxed{-9} \quad -2x = -4 \text{ or } -2x = -14$$
$$\boxed{\div (-2)} \quad x = 2 \text{ or } x = 7.$$

b Using the second dotpoint,

$$|9 - 2x| < 5$$
$$-5 < 9 - 2x < 5$$
$$\boxed{-9} \quad -14 < -2x < -4$$
$$\boxed{\div (-2)} \quad 7 > x > 2$$
that is, $\quad 2 < x < 7.$

c Using the third dotpoint,

$$|9 - 2x| > 5$$
$$9 - 2x < -5 \text{ or } 9 - 2x > 5$$
$$\boxed{-9} \quad -2x < -14 \text{ or } -2x > -4$$
$$\boxed{\div (-2)} \quad x > 7 \text{ or } x < 2$$
that is, $\quad x < 2$ or $x > 7.$

Solving inequations with a variable in the denominator

There is a problem with

$$\frac{5}{x - 4} \ge 1.$$

The denominator $x - 4$ is sometimes positive and sometimes negative. Thus if we were to multiply both sides by the denominator $x - 4$, the inequality symbol would reverse sometimes and not other times.

To avoid using cases, the most straightforward approach is to multiply through instead by the *square of the denominator*, which is positive or zero.

6 SOLVING AN INEQUATION WITH THE VARIABLE IN THE DENOMINATOR

- Multiply through by the *square of the denominator*.
- Be careful to exclude the zeroes of the denominator from the solutions.

Once the fractions have been cleared, there will usually be common factors on both sides. These should *not* be multiplied out, because the factoring will be easier if they are left unexpanded.

An alternative approach using a table of signs is presented in the next section.

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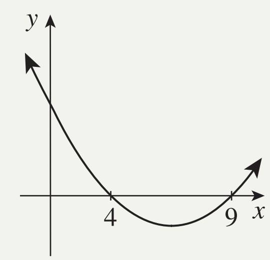
Example 5 5A

Solve $\dfrac{5}{x-4} \ge 1$.

SOLUTION

The key step is to multiply both sides by $(x-4)^2$.



$\boxed{\times\ (x-4)^2}$ $5(x-4) \ge (x-4)^2$, and $x \ne 4$,

$(x-4)^2 - 5(x-4) \le 0$, and $x \ne 4$,

$(x-4)(x-4-5) \le 0$, and $x \ne 4$,

$(x-4)(x-9) \le 0$, and $x \ne 4$.

From the diagram, $4 < x \le 9$.

Notice that $x = 4$ is not a solution of the original inequation because substituting $x = 4$ into the LHS of the original inequation gives $\dfrac{5}{4-4}$, which is undefined.

Exercise 5A

FOUNDATION

1 Solve, then sketch the solution on a number line:

a $x > 1$ **b** $x \le 2$ **c** $-2x < 4$

d $2x < 6$ **e** $x + 4 \ge 3$ **f** $3 - x > 1$

g $3x - 1 < 5$ **h** $5 - 2x \le -1$ **i** $5x - 5 \ge 10$

j $2 - 3x \ge 8$ **k** $\frac{1}{3}x - 1 > -\frac{1}{3}$ **l** $\frac{1}{4}x + 2 \le 1\frac{1}{2}$

2 Solve each double inequation, and sketch the solution on a number line.

a $-8 \le 4x < 12$ **b** $4 < 3x \le 15$

c $-2 \le 2x - 1 \le 3$ **d** $-1 \le 4x - 3 < 13$

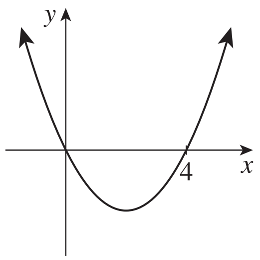
3 Solve each inequation.

a $2x + 3 > x + 7$ **b** $3x - 2 \le \frac{1}{2}x + 3$ **c** $2 - x > 2x - 4$

d $1 - 3x \ge 2 - 2x$ **e** $2 < 3 - x \le 5$ **f** $-4 \le 1 - \frac{1}{3}x \le 3$

4 Use the given graph of $y = $ LHS to help solve each inequation.

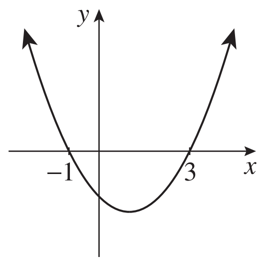
a



i $x(x-4) < 0$

ii $x(x-4) > 0$

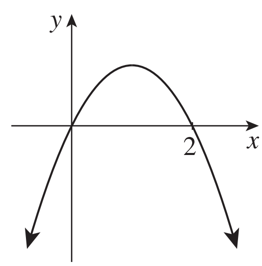
b



i $(x-3)(x+1) \le 0$

ii $(x-3)(x+1) \ge 0$

c



i $x(2-x) \le 0$

ii $x(2-x) > 0$

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5 Sketch the graph of $y = \text{LHS}$ and hence solve:

a $(x + 2)(x - 4) < 0$ **b** $(x - 3)(x + 1) > 0$
c $(x + 1)(x + 3) \geq 0$ **d** $(2x - 1)(x - 5) > 0$

6 In each part, factor the LHS, then sketch $y = \text{LHS}$ to solve:

a $x^2 - 9 < 0$ **b** $x^2 - 6x > 0$ **c** $x^2 - 100 \geq 0$ **d** $x^2 + 4x \leq 0$

7 Solve for x using distance on the number line.

a $|x| = 7$ **b** $|x| = 0$ **c** $|x| \leq 2$ **d** $|x| > 5$ **e** $|x| < \frac{1}{4}$ **f** $|x| \geq \frac{3}{2}$

8 Multiply both sides by x^2 and hence solve:

a $\frac{1}{x} > 1$ **b** $\frac{3}{x} < 1$ **c** $\frac{1}{x} \geq 2$ **d** $4 + \frac{3}{x} \geq 0$

DEVELOPMENT

9 Solve for x by drawing a graph of $y = \text{LHS}$:

a $x^2 + 2x - 3 < 0$ **b** $x^2 - 5x + 4 \geq 0$ **c** $x^2 + 6x + 8 > 0$
d $x^2 - x - 6 \leq 0$ **e** $2x^2 - x - 3 \leq 0$ **f** $4 + 3x - x^2 > 0$

10 Using distance on the number line, or any other suitable method, solve:

a $|x - 1| < 2$ **b** $|x - 5| \geq 4$ **c** $|x + 1| > 3$ **d** $|x + 8| \leq 6$

11 Multiply both sides of each inequation by the square of the denominator and hence solve it. Do not multiply out any common factor.

a $\frac{2}{x + 1} \leq 1$ **b** $\frac{2}{x - 3} > 1$ **c** $\frac{3}{x + 4} \geq 2$
d $\frac{5}{2x - 3} < 1$ **e** $\frac{2}{3 - x} > 1$ **f** $\frac{4}{5 - 3x} \leq -1$

12 Solve by any suitable method, and graph the solution on a number line.

a $|x - 2| < 3$ **b** $|3x - 5| \leq 4$ **c** $|x - 7| \geq 2$
d $|2x + 1| < 3$ **e** $|6x - 7| > 5$ **f** $|5x + 4| \geq 6$

13 Solve for x:

a $x^2 \leq 0$ **b** $x^2 > 0$ **c** $x^2 \geq 25$
d $x^2 > 25x$ **e** $2x^2 < x^2$ **f** $x^2 + 1 \leq 2x$

14 Solve for x:

a $\frac{5x}{2x - 1} \geq 3$ **b** $\frac{2x + 5}{x + 3} < 1$ **c** $\frac{x + 1}{x - 1} \leq 2$ **d** $\frac{4x + 7}{x - 2} > -3$

15 **a** Show that the double inequation $2 \leq |x| \leq 6$ has solution $2 \leq x \leq 6$ or $-6 \leq x \leq -2$.

b Similarly solve:

i $2 < |x + 4| < 6$ **ii** $1 \leq |2x - 5| < 4$

16 Say whether each statement is true or false. If it is false, give a counter-example.

a $x^2 > 0$ **b** $x^2 \geq x$ **c** $2^x > 0$ **d** $x \geq \frac{1}{x}$
e $2x \geq x$ **f** $x + 2 > x$ **g** $x \geq -x$ **h** $2x - 3 > 2x - 7$

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17 Solve these equations and inequations.

a $|4 - 5x| = -2$ **b** $|3x + 2| < -7$ **c** $|5 - x| \geq -6$ **d** $|3x - 5| \leq 0$

ENRICHMENT

18 Consider the inequation $\left|x + \dfrac{1}{x}\right| < 2x$.

a Explain why x must be positive.

b Hence solve the inequation.

19 Solve the inequation $1 + 2x - x^2 \geq \dfrac{2}{x}$.

20 Consider the inequation $|x - a| + |x - b| < c$, where $a < b$.

a If $a \leq x \leq b$, show, using distances on a number line, that there can only be a solution if $b - a < c$.

b If $b < x$, show, using distances on a number line, that $x < \dfrac{a + b + c}{2}$.

c If $x < a$, show, using distances on a number line, that $x > \dfrac{a + b - c}{2}$.

d Hence show that either $\left|x - \dfrac{a + b}{2}\right| < \dfrac{c}{2}$ or there is no solution to the original problem.

e Hence find the solution of $|x + 2| + |x - 6| < 10$.

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5B The sign of a function

We can always put all the terms of an inequation on the left. For example,

$$x^3 + 1 \ge x^2 + x \qquad \text{can be written as} \qquad x^3 - x^2 - x + 1 \ge 0.$$

This procedure reduces every inequation to one of the four forms

$$f(x) < 0 \quad \text{or} \quad f(x) \le 0 \quad \text{or} \quad f(x) \ge 0 \quad \text{or} \quad f(x) > 0,$$

so that the task of solving an inequation becomes the task of finding the sign of the function $f(x)$.

Where can a function change sign?

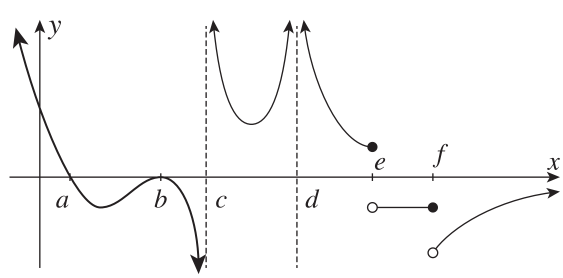
Sketching polynomials in Section 3G required a table of test values dodging around the zeroes to see where the function changed signs. The functions in this chapter, however, may also have breaks or *discontinuities*, and we need to dodge around them as well:

7 WHERE CAN A FUNCTION CHANGE SIGN?

The only places where a function may possibly change sign are zeroes and discontinuities.

Informally, a function is called *continuous at* $x = a$ if $f(x)$ is defined at $x = a$, and the curve $y = f(x)$ can be drawn through the point $\big(a, f(a)\big)$ without lifting the pen off the paper. Otherwise, the value $x = a$ is called a *discontinuity of* $f(x)$.

We will return to continuity in the last two sections of Chapter 9.



The graph above has discontinuities at $x = c$, $x = d$, $x = e$ and $x = f$, and has zeroes at $x = a$ and $x = b$. The function changes sign at the zero $x = a$ and at the discontinuities $x = c$ and $x = e$, and nowhere else. Notice that it does not change sign at the zero $x = b$ or at the discontinuities $x = d$ and $x = f$.

The statement in the box goes to the heart of what the real numbers are and what continuity means. In this course, the sketch above is sufficient justification.

A table of signs

As a consequence, we can examine the sign of a function using a table of test values, dodging around any zeroes and discontinuities. We add a third row for the sign, so that the table becomes a *table of signs*.

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8 EXAMINING THE SIGN OF A FUNCTION

To examine the sign of a function, draw up a a *table of signs* using test values that dodge around any zeroes and discontinuities.

Finding the zeroes of a function has been a constant concern ever since quadratics were introduced in earlier years. To find discontinuities, assume that all the functions in the course are continuous everywhere in their domains, except where there is an obvious problem.

Solving polynomial inequations

A simpler form of this table of signs was introduced in Section 3G to sketch polynomials. Polynomials do not have any discontinuities, so the test values only needed to dodge around their zeroes.

The attention in Section 3G was on the sketch of the function, whereas in this section our attention is on solving inequations, as can be seen in Examples 6–10. The graphs are not necessary for this purpose, but they are useful because they allow us to see the whole situation.

Example 6 5B

a Draw up a table of signs of the function $y = (x - 1)(x - 3)(x - 5)$.
b State where the function is positive and where it is negative.
c Solve the inequation $(x - 1)(x - 3)(x - 5) \le 0$.
d Confirm the answers by sketching the graph of the function.

SOLUTION

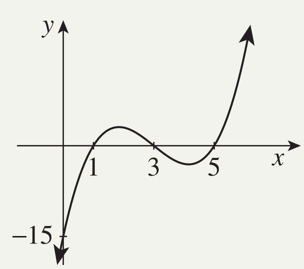
a There are zeroes at 1, 3 and 5, and no discontinuities.

x	0	1	2	3	4	5	6
y	-15	0	3	0	-3	0	15
sign	$-$	0	$+$	0	$-$	0	$+$

b Hence y is positive for $1 < x < 3$ or $x > 5$, and negative for $x < 1$ or $3 < x < 5$.

c $x \le 1$ or $3 \le x \le 5$.

d



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Example 7 5B

Solve $x^3 + 1 \le x^2 + x$ using a table of signs. Then confirm the answer by drawing a graph.

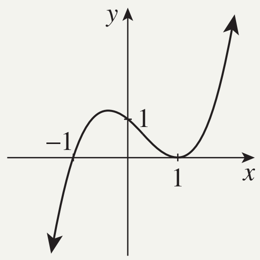
SOLUTION

Move all terms to the left, then factor by grouping,

$$x^3 - x^2 - x + 1 \le 0$$
$$x^2(x - 1) - (x - 1) \le 0$$
$$(x^2 - 1)(x - 1) \le 0$$
$$(x + 1)(x - 1)^2 \le 0.$$

The LHS has zeroes at 1 and -1, and no discontinuities.

x	-2	-1	0	1	2
y	-9	0	1	0	3
sign	$-$	0	$+$	0	$+$



Hence $x \le -1$ or $x = 1$.

Solving inequations involving discontinuities

When the function has discontinuities, the method is the same, except that the test values now need to dodge around discontinuities as well as zeroes.

Again, a sketch is very useful because it shows the whole situation. This may not always be possible, however, because some features, such as the horizontal asymptote in Example 8, will not be introduced until Year 12.

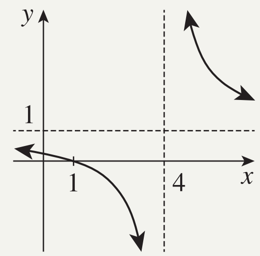
Example 8 5B

Examine the sign of $y = \dfrac{x - 1}{x - 4}$.

SOLUTION

There is a zero at $x = 1$, and a discontinuity at $x = 4$.

x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	*	4
sign	$+$	0	$-$	*	$+$



Hence y is positive for $x < 1$ or $x > 4$, and negative for $1 < x < 4$.

(The function is sketched here to illustrate the situation, but the methods used to sketch it will not be presented until Year 12.)

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Example 9 5B

Examine the sign of $y = \dfrac{1}{1 + x^2}$.

SOLUTION

The function is always positive because $1 + x^2$ is always at least 1. There is thus no need to use a table of signs.

A table of signs can still be used, however. The function has no zeroes because the numerator is never zero, and has no discontinuities because the denominator is never zero. Hence one test value $f(0) = 1$ establishes that the function is always positive.

x	0
y	1
sign	+

Comparing the methods of Sections 5A and 5B

We now have two ways to solve an inequation with x in the denominator. Examples 10 and 11 solve the same inequation in two ways, first using the table-of-sign approach of this section, then using the multiply-by-the-square-of-the-denominator approach of Section 5A.

Example 10 5B

[Using the table-of-signs method of this section]

Solve $\dfrac{3}{x + 2} \le x$ using a table of signs.

SOLUTION

Collecting everything on the left, $\dfrac{3}{x + 2} - x \le 0,$

using a common denominator, $\dfrac{3 - x^2 - 2x}{x + 2} \le 0,$

and factoring, $\dfrac{(3 + x)(1 - x)}{x + 2} \le 0.$

The LHS has zeroes at $x = -3$ and $x = 1$, and a discontinuity at $x = -2$.

x	-4	-3	$-2\frac{1}{2}$	-2	0	1	2
LHS	$2\frac{1}{2}$	0	$-3\frac{1}{2}$	*	$1\frac{1}{2}$	0	$-1\frac{1}{4}$
sign	+	0	−	*	+	0	−

So the solution is $x \ge 1$ or $-3 \le x < -2$.

(No sketch here — rely just on the table of signs.)

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Example 11 5B

[The same example using the method of Section 5A]

Solve $\frac{3}{x + 2} \le x$ by first multiplying though by the square of the denominator.

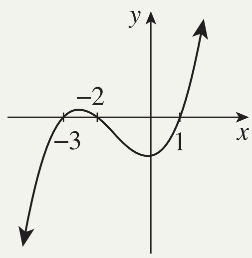
SOLUTION

Notice that the common factor $(x + 2)$ is never multiplied out. That would cause a serious problem, because of the effort required to re-factor the expanded cubic.

$$\frac{3}{x+2} \le x$$

$\boxed{\times\ (x+2)^2}$ $\quad 3(x + 2) \le x(x + 2)^2,$ and $x \ne -2,$

$$x(x + 2)^2 - 3(x + 2) \ge 0, \text{ and } x \ne -2,$$
$$(x + 2)(x^2 + 2x - 3) \ge 0, \text{ and } x \ne -2,$$
$$(x + 2)(x + 3)(x - 1) \ge 0, \text{ and } x \ne -2.$$



The LHS can now be sketched using a table of signs.

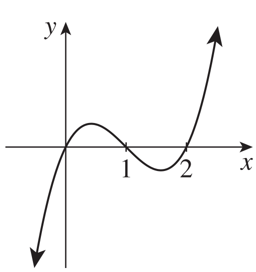
From the graph, $x \ge 1$ or $-3 \le x < -2$.

Exercise 5B FOUNDATION

The purpose of this exercise is to solve inequations using a table of signs. In the case of polynomials, this table also allows the sketch to be drawn, as in Section 3G. A sketch makes the situation clearer, but the sketch is not necessary for obtaining the solution.

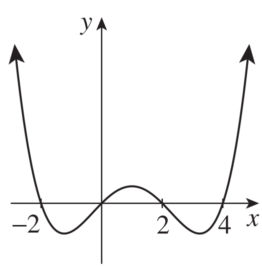
1 Use the given graph of the LHS to help solve each inequation.

a



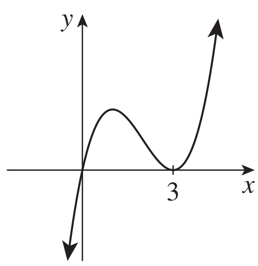
$x(x - 1)(x - 2) \le 0$

b



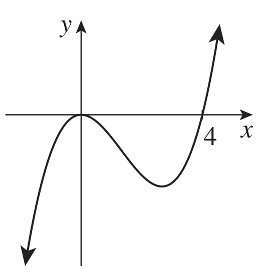
$x(x + 2)(x - 2)(x - 4) < 0$

c



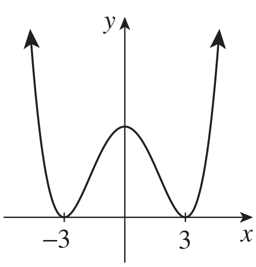
$x(x - 3)^2 > 0$

d



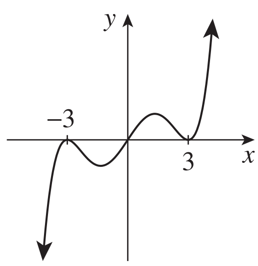
$x^2(x - 4) \ge 0$

e



$(x - 3)^2(x + 3)^2 \le 0$

f



$x(x - 3)^2(x + 3)^2 \ge 0$

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2 **a** Explain why the zeroes of $y = (x + 1)^2(1 - x)$ are $x = 1$ and $x = -1$. Then copy and complete the table of signs.

x	-2	-1	0	1	2
y sign					

b Use the table of signs to solve $(x + 1)^2(1 - x) \geq 0$.

c Sketch the graph to confirm the solution in part **b**.

3 Use the three steps of the previous question to solve each inequation.

a $(x + 1)(x + 3) < 0$ **b** $x(x - 2)(x - 4) \geq 0$ **c** $(x - 1)(x + 2)^2 \geq 0$

d $x(x - 2)(x + 2) \leq 0$ **e** $(x - 2)x(x + 2)(x + 4) > 0$ **f** $(x - 1)^2(x - 3)^2 \leq 0$

4 First factor each polynomial completely, then use the methods of the first two questions to sketch its graph (take out any common factors first).

a $f(x) = x^3 - 4x$ **b** $f(x) = x^3 - 5x^2$ **c** $f(x) = x^3 - 4x^2 + 4x$

5 From the graphs in the previous question, or from the tables of signs used to construct them, solve the following inequations. Begin by getting all terms onto the one side.

a $x^3 > 4x$ **b** $x^3 < 5x^2$ **c** $x^3 + 4x \leq 4x^2$

DEVELOPMENT

6 **a** Find the zeroes and discontinuities of $y = \dfrac{x^2}{x - 3}$ and construct a table of signs.

b Hence solve $\dfrac{x^2}{x - 3} < 0$.

7 If necessary, collect all terms on the LHS and factor. Then solve the inequation by finding any zeroes and discontinuities and drawing up a table of signs dodging around them.

a $(x - 1)(x - 3)(x - 5) < 0$ **b** $(x - 1)^2(x - 3)^2 > 0$. **c** $\dfrac{x - 4}{x + 2} \leq 0$

d $x^3 > 9x$ **e** $\dfrac{x + 3}{x + 1} < 0$ **f** $\dfrac{x^2}{x - 5} < 0$

g $x^4 \geq 5x^3$ **h** $\dfrac{x^2 - 4}{x} \geq 0$ **i** $\dfrac{x - 2}{x^2 + 3x} \leq 0$

8 **a** Factor each equation completely, and hence find the x-intercepts of the graph. Factor parts **ii** and **iii** by grouping in pairs.

i $y = x^3 - x$ **ii** $y = x^3 - 2x^2 - x + 2$ **iii** $y = x^3 + 2x^2 - 4x - 8$

b For each function in the previous question, examine the sign of the function around each zero, and hence draw a graph of the function.

9 Find all zeroes of these functions, and any values of x where the function is discontinuous. Then analyse the sign of the function by taking test points around these zeroes and discontinuities.

a $f(x) = \dfrac{x}{x - 3}$ **b** $f(x) = \dfrac{x - 4}{x + 2}$ **c** $f(x) = \dfrac{x + 3}{x + 1}$

10 Multiply through by the square of the denominator, collect all terms on one side and then factor to obtain a factored cubic. Sketch this cubic by examining the intercepts and the sign. Hence solve the original inequation.

a $\dfrac{4}{x + 3} \geq x$ **b** $\dfrac{2}{2x + 3} < x$ **c** $\dfrac{8}{2x - 3} \leq 2x - 1$

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11 Solve the inequations in the previous question by the alternative method of collecting everything on the LHS, finding a common denominator, identifying zeroes and discontinuities, and drawing up a table of signs.

ENRICHMENT

12 **a** Prove that $f(x) = 1 + x + x^2$ is positive for all x.

b Prove that $f(x) = 1 + x + x^2 + x^3 + x^4$ is positive for all x. Consider separately the three cases $x \geq 0, -1 < x < 0$ and $x \leq -1$. Group the five terms into pairs in different ways with the second and third cases.

c Use similar methods to prove that for all integers $n \geq 0$,

$$f(x) = 1 + x + x^2 + \cdots + x^{2n-1} + x^{2n} \text{ is positive for all } x.$$

d Prove that $x = -1$ is the only zero of $f(x) = 1 + x + x^2 + \cdots + x^{2n-1}$, for all positive integers n.

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5C Reciprocals and asymptotes

This section and the next three present several further ways to transform or combine graphs. Sometimes the equations of the original graphs will be given, and sometimes only the sketches of the graphs are known.

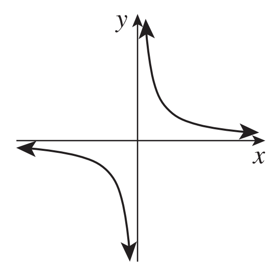
This section deals with sketches of the reciprocal $\dfrac{1}{f(x)}$ of a function. Taking reciprocals typically results in asymptotes, so we begin with a short discussion of asymptotes and the language used to describe them.

Asymptotes

Asymptotes occur naturally when taking reciprocals. There are two straightforward principles to keep in mind:

- The reciprocal of a very small number is a very large number.
  The reciprocal of a very large number is a very small number.
- The reciprocal of a positive number is positive.
  The reciprocal of a negative number is negative.

All this is well demonstrated by the rectangular hyperbola $y = \dfrac{1}{x}$, which has a vertical and a horizontal asymptote, as discussed in Section 3H.



x	-1	0	1
y	-1	*	1
sign	$-$	*	$+$

Horizontally, when x is a very large number, positive or negative, y is a very small number with the same sign, so the curve approaches the x-axis on the left and on the right. We described this situation earlier in section 3H by the statement:

‘As $x \to \infty, y \to 0$, and as $x \to -\infty, y \to 0$.’

Vertically, when x is a very small number, positive or negative, y is a very large number with the same sign, so the curve flies off to ∞ or to $-\infty$ near $x = 0$. We described this situation earlier in Section 3H by the statement:

‘As $x \to 0^-, y \to -\infty$, and as $x \to 0^+, y = +\infty$.’

‘As x approaches 0 from the left, . . . , and as x approaches 0 from the right, . . .’.

In more difficult situations than this, use a table of signs, as introduced in the previous section, to check the sign and distinguish between $y \to \infty$ and $y \to -\infty$.

9 TESTING FOR VERTICAL ASYMPTOTES

- If the denominator has a zero at $x = a$, and the numerator is *not* zero at $x = a$, then the vertical line $x = a$ is an asymptote.
- The choice between $y \to \infty$ and $y \to -\infty$ can be made by looking at a table of signs.

Once the asymptote has been identified, the behaviour of the curve near it can then be described using the notation $x \to a^+$ and $x \to a^-$.

A more systematic approach to horizontal asymptotes will be given in Year 12.

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Example 12 5C

a Find the vertical asymptote of the function $y = \dfrac{2}{3 - x}$.

b Construct a table of signs, and describe the behaviour of the curve near it.

c Name the horizontal asymptote, and describe the behaviour of y for large x.

d Sketch the curve.

SOLUTION

a When $x = 3$, the denominator vanishes, but the numerator does not, so $x = 3$ is an asymptote.

b There are no zeroes, and there is a discontinuity at $x = 3$:

From the table of signs,

As $x \to 3^+$, $y \to -\infty$,

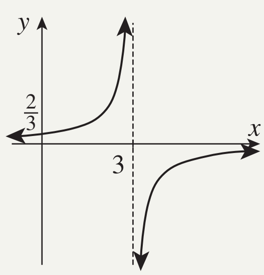
and as $x \to 3^-$, $y \to +\infty$.

x	0	3	4
y	$\frac{2}{3}$	*	-2
sign	$+$	*	$-$

c The x-axis is a horizontal asymptote, and:

As $x \to \infty$ or $x \to -\infty$, $y \to 0$.

d

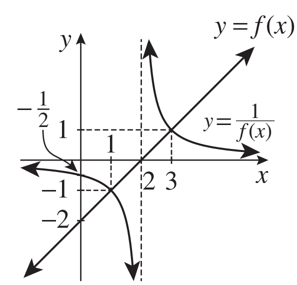


Sketching the reciprocal of a function

The main problem addressed in this section is to take some sketched graph $y = f(x)$, and from it draw the graph of the reciprocal function $y = \big(f(x)\big)^{-1}$. In order to develop the method, let us begin by taking the linear function $f(x) = x - 2$ and looking at the tables of values and the graphs of $y = f(x)$ and $y = \big(f(x)\big)^{-1}$.

$f(x) = x - 2$

x	0	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4
$f(x)$	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$\big(f(x)\big)^{-1}$	$-\frac{1}{2}$	-1	-2	*	2	1	$\frac{1}{2}$



1 The signs of $f(x)$ and $y = \big(f(x)\big)^{-1}$ are always the same.

2 If $x = a$ is a zero of $f(x)$, then the vertical line $x = a$ is a vertical asymptote of $\big(f(x)\big)^{-1}$.

3 As $f(x) \to \infty$, and as $f(x) \to -\infty$, $\big(f(x)\big)^{-1} \to 0$, so the x-axis is a horizontal asymptote.

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4 When $f(x) = 1$, then $\left(f(x)\right)^{-1} = 1$,
and when $f(x) = -1$, then $\left(f(x)\right)^{-1} = -1$.

5 When $f(x)$ is increasing, then $\left(f(x)\right)^{-1}$ is decreasing.
(And if $f(x)$ were ever decreasing, then $\left(f(x)\right)^{-1}$ would be increasing.)

The domain should probably have been thought about first. Every number in the domain of $f(x)$ is also in the domain of $\left(f(x)\right)^{-1}$, apart from the zeroes of $f(x)$, because we can take the reciprocal of every number except zero. In this case, the domain of $\left(f(x)\right)^{-1}$ is $x \neq 2$.

Here is the full menu for sketching $y = \left(f(x)\right)^{-1}$ from any graph $y = f(x)$.

10 SKETCHING THE RECIPROCAL OF A GIVEN GRAPH

Suppose that the graph of $y = f(x)$ is drawn, and we want to sketch $y = \left(f(x)\right)^{-1}$.

1 The new graph has the same sign, for all values x.

2 The zeroes of $y = f(x)$ correspond to vertical asymptotes of $y = \left(f(x)\right)^{-1}$.
The vertical asymptotes of $y = f(x)$ correspond to zeroes of $y = \left(f(x)\right)^{-1}$, except that the points themselves are removed.

3 If $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to \infty$ or $x \to -\infty$, the x-axis is a horizontal asymptote to $y = \left(f(x)\right)^{-1}$.
If $f(x) \to b \neq 0$ as $x \to \infty$ or $x \to -\infty$, then $y = b^{-1}$ is a horizontal asymptote to $y = \left(f(x)\right)^{-1}$.

4 When one curve intersects $y = 1$ or $y = -1$, so does the other.

5 When one curve is increasing, the other is decreasing.

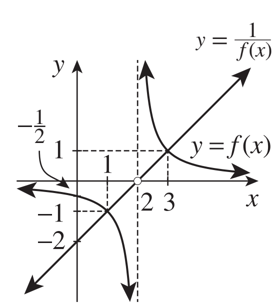
6 When one curve has a local maximum, the other has a local minimum.

The domain of $\left(f(x)\right)^{-1}$ is the domain of $f(x)$ with the zeroes of $f(x)$ removed.

The second assertions in points 2 and 3, and point 6, all need yet to be explained, because they were not necessary in the example above.

To explain the second assertion in point 2, we reverse the reciprocal transformation performed in the example above. Thus we start this time with the graph of $f(x) = \dfrac{1}{x - 2}$, and from it sketch the graph of $\left(f(x)\right)^{-1}$.

x	0	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	4
$f(x)$	$-\frac{1}{2}$	-1	-2	*	2	1	$\frac{1}{2}$
$\left(f(x)\right)^{-1}$	-2	-1	$-\frac{1}{2}$	*	$\frac{1}{2}$	1	2



This table is obtained by exchanging the second and third rows of the previous table, with one significant exception — when $f(x)$ is undefined at $x = 2$, so also is $\left(f(x)\right)^{-1}$. Thus the asymptote of $f(x)$ at $x = 2$ becomes a zero, except that the point itself is removed.

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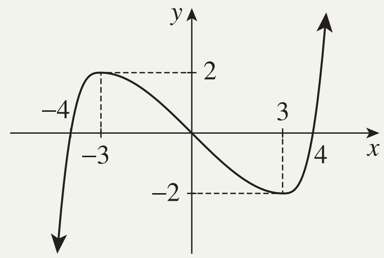
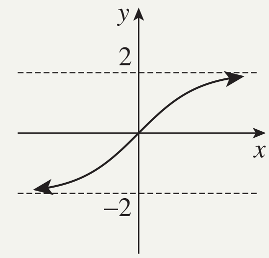
Example 13 5C

Given each sketch of $y = f(x)$ below, sketch $\left(f(x)\right)^{-1}$ and state its domain.

a (To explain point 6)

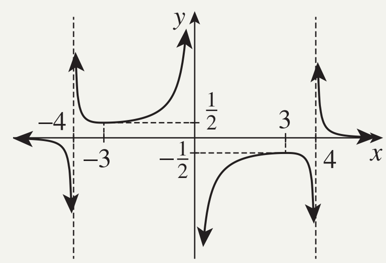


b (To explain the second assertion of point 3)



SOLUTION

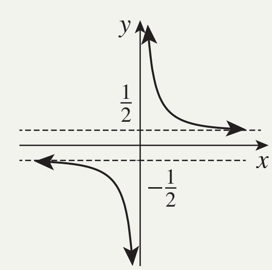
a This example illustrates point 6 of Box 10. The point $(-3, 2)$ is called a *local maximum* because for a small region around $x = -3$, the value of $f(x)$ is greatest at $x = -3$. Similarly, $(3, -2)$ is called a *local minimum*.



The value of $\left(f(x)\right)^{-1}$ at $x = -3$ is $\frac{1}{2}$, and $\left(f(x)\right)^{-1}$ is decreasing on the left of $x = -3$ and increasing on the right, so $\left(-3, \frac{1}{2}\right)$ is a local minimum. Similarly, $\left(3, -\frac{1}{2}\right)$ is a local maximum.

The domain of $\left(f(x)\right)^{-1}$ is $x \neq -4$, 0 or 4.

b This illustrates the second assertion of point 3 of Box 10.



As $x \to \infty$, $f(x) \to 2$, so $\left(f(x)\right)^{-1} \to \frac{1}{2}$.

Thus means that $y = \frac{1}{2}$ is a horizontal asymptote on the right.

Similarly, as $x \to -\infty$, $f(x) \to -2$, so $\left(f(x)\right)^{-1} \to -\frac{1}{2}$.

Thus $y = -\frac{1}{2}$ is a horizontal asymptote on the left.

The domain of $\left(f(x)\right)^{-1}$ is $x \neq 0$.

Exercise 5C FOUNDATION

1 a Sketch $y = \dfrac{1}{x - 1}$ after carrying out the following steps.

- **i** State the natural domain, and find the y-intercept.
- **ii** Find any points where $y = 1$ or $y = -1$.
- **iii** Explain why $y = 0$ is a horizontal asymptote.
- **iv** Draw up a table of values and examine the sign.
- **v** Identify any vertical asymptotes, and use the table of signs to write down its behaviour near any vertical asymptotes.

b Repeat for $y = \dfrac{2}{3 - x}$. **c** Repeat for $y = -\dfrac{2}{x + 2}$. **d** Repeat for $y = \dfrac{5}{2x + 5}$.

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2 Follow steps **i–v** of Question 1 part **a** to investigate the function $y = \dfrac{2}{(x-1)^2}$ and hence sketch its graph. Show any points where $y = 1$.

3 Investigate the domain, zeroes, sign and asymptotes of the function $y = -\dfrac{1}{(x-2)^2}$ and hence sketch its graph. Show any points where $y = -1$.

4 **a** Let $f(x) = x + 1$.
- **i** Graph $y = f(x)$ showing the intercepts with the axes.
- **ii** Also show the points where $f(x) = 1$ and $f(x) = -1$.
- **iii** Hence on the same number plane sketch $y = \dfrac{1}{f(x)}$.

b Follow similar steps for the function $g(x) = 2 - x$.

DEVELOPMENT

5 Let $y = f(x)$ where $f(x) = \frac{1}{3}(x+1)(x-3)$.
- **a** Show that $y = 1$ at $x = 1 - \sqrt{7}$ and $x = 1 + \sqrt{7}$. Plot these points.
- **b** Complete the graph of $y = f(x)$ showing the vertex, the intercepts with the axes and the points where $f(x) = -1$.
- **c** What is the range of $y = f(x)$?
- **d** Hence sketch $y = \dfrac{1}{f(x)}$ on the same number plane.
- **e** What is the range of $y = \dfrac{1}{f(x)}$?

6 Follow similar steps to Question 5 to sketch $y = \frac{1}{4}(4 - x^2)$ and $y = \dfrac{4}{4 - x^2}$.
- **a** What is the range of $y = \frac{1}{4}(4 - x^2)$?
- **b** What is the range of $y = \dfrac{4}{4 - x^2}$?

7 **a** Sketch $y = \frac{1}{2}(x^2 + 1)$ showing the points where $y = 1$.
- **b** What is the minimum value of $\frac{1}{2}(x^2 + 1)$?
- **c** Sketch $y = \dfrac{2}{x^2 + 1}$ on the same number plane.
- **d** Explain why the x-axis is an asymptote to $y = \dfrac{1}{f(x)}$.
- **e** What is the maximum value of $\dfrac{2}{x^2 + 1}$?

8 Let $f(x) = -x^2 + 2x - 3$.
- **a** What is the maximum of $f(x)$?
- **b** Sketch $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same number plane.
- **c** Explain why the x-axis is an asymptote for $y = \dfrac{1}{f(x)}$.
- **d** What is the minimum of $\dfrac{1}{f(x)}$?

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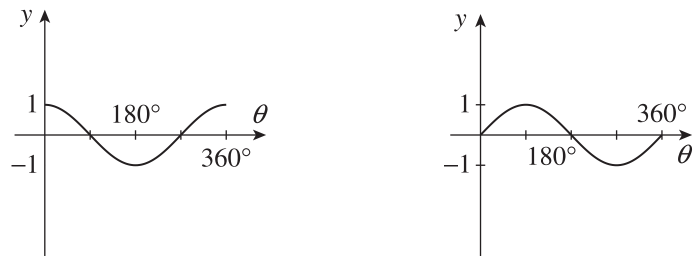
9 The diagrams show $y = \cos\theta$ (on the left) and $y = \sin\theta$ (on the right) for $0° \le \theta \le 360°$.



a **i** Copy the sketch of $y = \cos\theta$, and add to it the sketch of $y = \dfrac{1}{\cos\theta}$.

ii What are the domain and range of $y = \dfrac{1}{\cos\theta}$ in this interval?

b **i** Copy the sketch of $y = \sin\theta$, and add to it the sketch of $y = \dfrac{1}{\sin\theta}$.

ii What are the domain and range of $y = \dfrac{1}{\sin\theta}$ in this interval?

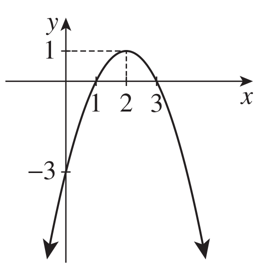
10 Prove that the symmetry of a function is preserved when taking reciprocals. That is, prove that the reciprocal of an even function is an even function, and prove that the reciprocal of an odd function is an odd function.

11 **a** Graph $y = \dfrac{2 + x}{x}$ by first noting that $y = 1 + \dfrac{2}{x}$.

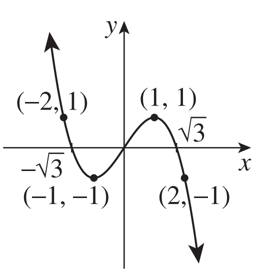
b Hence graph $y = \dfrac{x}{2 + x}$.

12 Sketch the reciprocal of each function shown, showing all significant features.

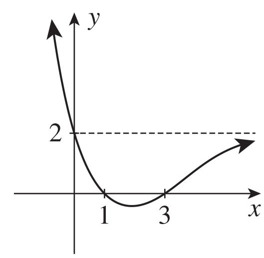
a



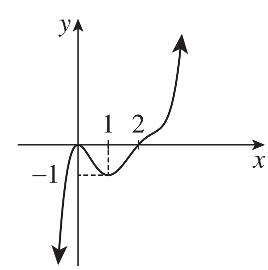
b



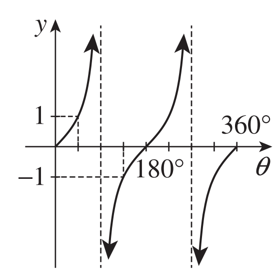
c



d



13 The sketch shows $y = \tan\theta$ for $0° \le \theta \le 360°$.



a What is the domain of $y = \tan\theta$ in this interval?

b Where is $\tan\theta = 0$ in this interval?

c Hence state the domain of $y = \dfrac{1}{\tan\theta}$ in this interval.

d What is $\lim\limits_{\theta \to 90°} \dfrac{1}{\tan\theta}$?

e Copy the graph of $y = \tan\theta$, and add to it the graph of $y = \dfrac{1}{\tan\theta}$.

f What is the range of the reciprocal function?

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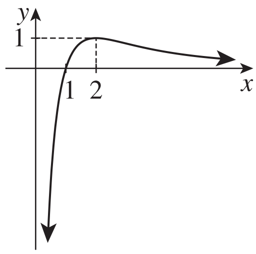
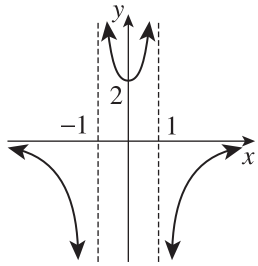
14 In each case the graph of $y = f(x)$ is given. Sketch the graph of $y = \dfrac{1}{f(x)}$, paying careful attention to the domain, any asymptotes, and any relevant limits.

a



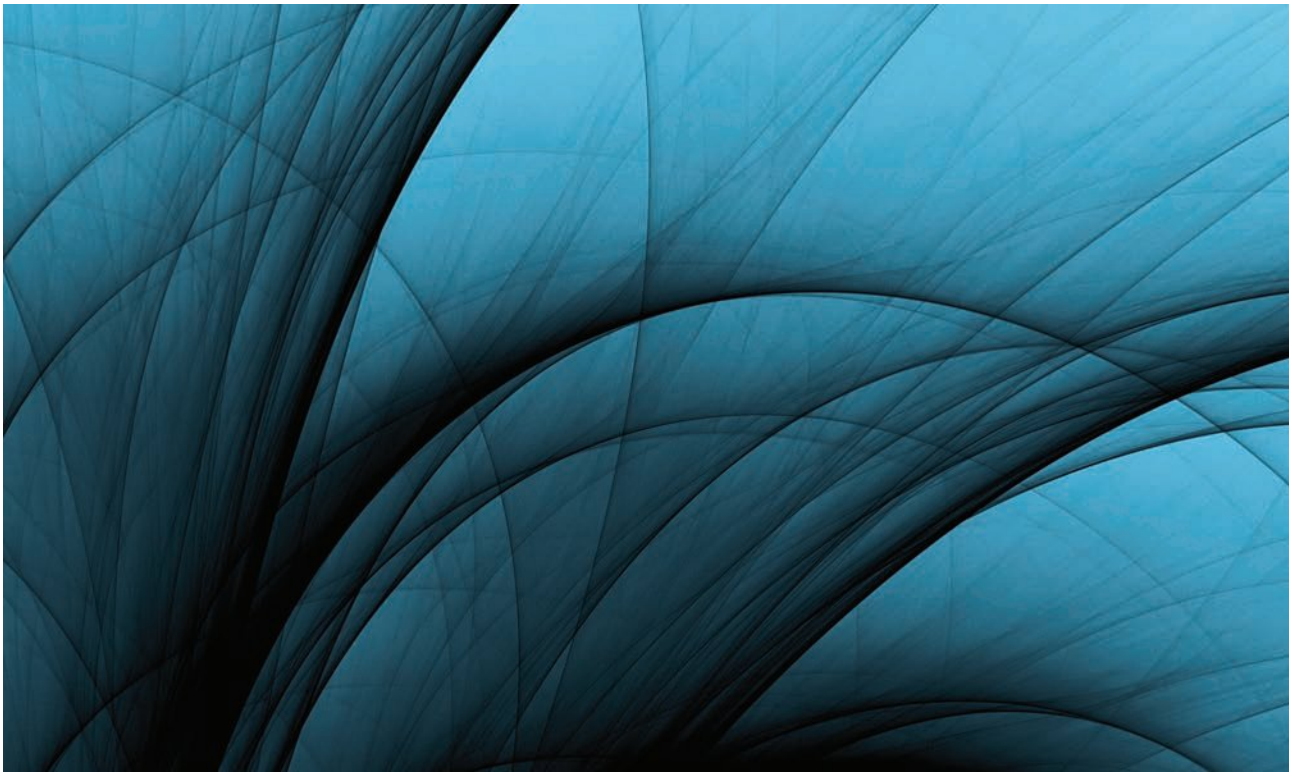
b



ENRICHMENT

15 Point 6 of Box 10 says about the graphs of $y = f(x)$ and $y = \big(f(x)\big)^{-1}$, 'When one curve has a local maximum, the other has a local minimum.' This is not strictly true. State the qualificaion that needs to be made in this statement, and give an example where the qualification is necessary.

16 Let $y = \dfrac{1}{x - 2}$. Write down precisely the equation of the reciprocal function.



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5D Graphing sums and products

The problem addressed in this section is to take the sketches of two functions $f(x)$ and $g(x)$, and working just from those sketches, sketch the sum, difference and product of the two functions:

$$s(x) = f(x) + g(x), \quad d(x) = f(x) - g(x), \quad p(x) = f(x) \times g(x).$$

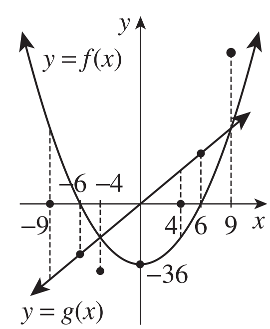
The pronumerals $s(x)$, $d(x)$ and $p(x)$ used here are not any mathematical convention, they are just convenient notation in this section.

The word 'ordinate'

The *ordinate* of a point is the y-coordinate, or more generally, the coordinate on the vertical axis. The operations in this section and the next routinely act on the y-coordinates of points, so this shorter word is useful. The coordinate on the horizontal axis is sometimes referred to as the 'abcissa', plural 'abcissae', but that word is not necessary in the course.

Sketching the sum of two sketched functions

The graphs of two functions $f(x)$ and $g(x)$ are sketched to the right, and we want to sketch the sum $s(x) = f(x) + g(x)$. The equations of the functions will usually not be given, but it is convenient to state them here while developing the method. They are:

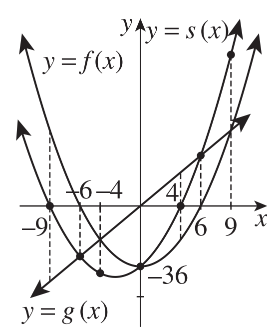


$$f(x) = x^2 - 36 \quad \text{and} \quad g(x) = 5x,$$

and here is a tables of values:

x	-9	-6	-4	-1	0	1	4	6	9
$f(x)$	45	0	-20	-35	-36	-35	-20	0	45
$g(x)$	-45	30	-20	-5	0	5	20	30	45
$s(x)$	0	-30	-40	-40	-36	-30	0	30	90

Each ordinate of $s(x)$ is the sum of the ordinates of $f(x)$ and $g(x)$. This is the key idea that leads to everything else. The sketch below adds $s(x)$ to the other two graphs.



Now let us consider how we could have sketched $s(x)$ if we had not had the equations.

0 Add the ordinates where possible — the key idea. This has been done in the top diagram.

1 If one curve, say $f(x)$, has a zero at $x = a$, then

$$s(a) = 0 + g(a) = g(a),$$

so $s(x)$ has the same ordinate as $g(x)$ at $x = a$, and the curves meet there.

This happens at both the zeroes $x = 6$ and $x = -6$ of $f(x) = x^2 - 36$, where $s(x)$ meets $g(x)$. It also happens at the zero $x = 0$ of $g(x) = 5x$, where $s(x)$ meets $f(x)$.

2 If the ordinates are opposites at $x = a$, then

$$s(a) = f(a) + g(a) = 0,$$

so $s(x)$ has a zero at $x = a$. This happens at $x = -9$ and at $x = 4$.

3 If the two curves meet at $x = a$, so that their ordinates are equal, then

$$s(a) = f(a) + g(a) = 2f(a) \qquad \big(\text{or } 2g(a)\big)$$

and the ordinate of $s(x)$ is twice the ordinate of $f(x)$ (or of $g(x)$). This happens at $x = -4$ and at $x = 9$.

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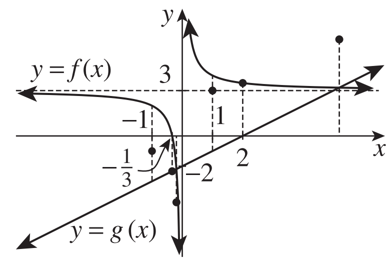
We cannot find from the graphs alone the precise position of the minimum, so in the absence of the equations, this is not required in the sketch. But once we know the equations, the minimum is the vertex of the parabola

$$s(x) = (x^2 - 36) + 5x = x^2 + 5x - 36 = (x + 9)(x - 4),$$

where the average of the zeroes is $x = \frac{1}{2}(4 - 9) = -2\frac{1}{2}$, and $s\left(-2\frac{1}{2}\right) = -42\frac{1}{4}$.

An example of adding graphs with asymptotes

Here is an example in which one curve has a horizontal and a vertical asymptote. We apply the steps set out in the previous example to sketch the sum $s(x) = f(x) + g(x)$.



Given also: $f(1) = 4$
$f(-1) = 2$

0 Add the ordinates at $x = 1$ and $x = -1$, so

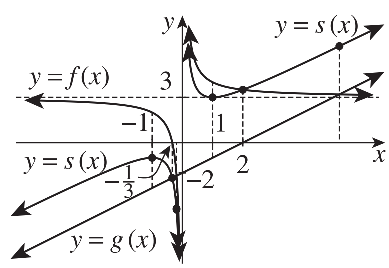
$$s(1) = f(1) + g(1) = 4 + (-1) = 3.$$
$$s(-1) = f(-1) + g(-1) = 2 + (-3) = -1.$$

1 There is a zero of $f(x)$ at $x = -\frac{1}{3}$, and of $g(x)$ at $x = 2$, so the ordinates at those values of x are the ordinates of the other curve there.

2 There are no values of x where ordinates are opposites, so $s(x)$ has no zeroes.

3 Double the ordinate at the two points where the curves meet.

4 There are no zeroes for $s(x)$, so it can only change sign at the asymptote $x = 0$ (next step). Hence $s(x)$ is negative for $x < 0$, and positive for $x > 0$.

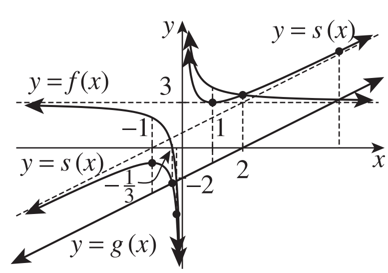


Dealing with the horizontal and vertical asymptotes of $f(x)$:

5 Vertically, as $x \to 0^+$, $f(x) \to +\infty$, so $s(x) \to +\infty$. Also as $x \to 0^-$, $f(x) \to -\infty$, so $s(x) \to -\infty$.
Thus the y-axis is a vertical asymptote to $y = s(x)$.

Horizontally, on the right and the left:
As $x \to \infty$, $f(x) \to 3$ and $g(x) \to \infty$, so $s(x) \to \infty$.
As $x \to -\infty$, $f(x) \to 3$ and $g(x) \to -\infty$, so $s(x) \to -\infty$.

The horizontal situation has a further detail that seems beyond what is required in the course, but is described here for completeness.
Look again at the top graph.



For large values of x, positive or negative, $f(x)$ is almost 3, so $y = s(x)$ is almost the same graph as $y = 3 + g(x)$.
Hence $y = s(x)$ eventually looks like a line parallel to $y = g(x)$.

This argument has problems when it come to products of functions. See Question 19 part **a** of Exercise 5D for another approach.

Having argued from the graphs alone, we will now reveal the equations of the two functions — they are $f(x) = \dfrac{1}{x} + 3$ and $g(x) = x - 2$. Their sum is

$$s(x) = x + 1 + \frac{1}{x} = \frac{x^2 + x + 1}{x}.$$

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This has no zeroes because the discriminant $\Delta = 1 - 4 = -3$ is negative, and it has a vertical asymptote at $x = 0$. The exact points of intersection are found by equating $f(x)$ and $g(x)$, but the answers are irrational, so this is omitted.

Summary of sketching the sum of two sketched functions

11 SKETCHING THE SUM OF TWO SKETCHED FUNCTIONS

- To sketch the sum $s(x) = f(x) + g(x)$ of two sketched functions:
  0 Add the ordinates wherever possible. This is the key idea.
- Some systematic approaches:
  1 If one curve has a zero, then $s(x)$ meets the other curve at that value of x.
  2 If the curves meet, then the ordinate of $s(x)$ is double the ordinate of $f(x)$.
  3 If the ordinates of $f(x)$ and $g(x)$ are opposites, then $s(x)$ has a zero.
  4 The sign of $s(x)$ everywhere will usually be clear now.
- To clarify any vertical or horizontal asymptotic behaviour:
  5 If $f(x) \to \infty$ or $g(x) \to \infty$, then so also does $s(x)$.
  If $f(x) \to -\infty$ or $g(x) \to -\infty$, then so also does $s(x)$.
  If, however, $f(x)$ and $g(x)$ go in opposite directions, we may not be able determine what happens with the sum.

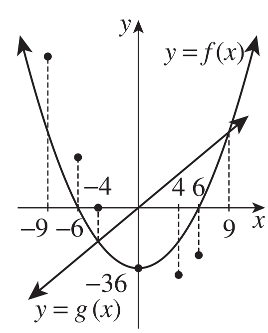
As discussed in Section 4A, a translation of $y = f(x)$ up 5 is $y = f(x) + 5$. This is $f(x) + g(x)$ where $g(x) = 5$ is a constant function, so translations up and down are special cases of the construction in this section.

Sketching the difference of two sketched functions

Now let us sketch the difference $d(x) = f(x) - g(x)$ of the same two functions

$$f(x) = x^2 - 36 \quad \text{and} \quad g(x) = 5x,$$

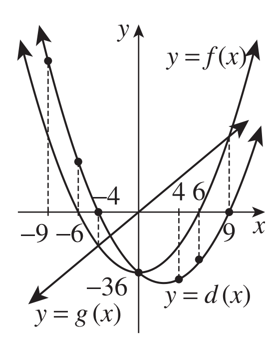
but this time we will work from the sketches alone and then confirm the sketch using a table of values. The difference is the sum of $f(x)$ and $-g(x)$, so it could be done by reflecting $g(x)$ in the x-axis and then adding the graphs, but it is easier to sketch it in one step.



0 Subtract the ordinates where possible — the key idea.

1 If $f(x)$ has a zero at $x = a$, then $d(a) = 0 - g(a) = -g(a)$, so the ordinate of $d(x)$ is the opposite of the ordinate of $f(x)$. This happens at $x = -6$ and at $x = 6$.

If $g(x)$ has a zero at $x = a$, then $d(a) = f(a) - 0 = f(a)$, so the ordinate of $d(x)$ is the same as the ordinate of $f(x)$, and the curve $d(x)$ meets the curve $f(x)$. This happens at $x = 0$.



2 If the ordinates of $f(x)$ and $g(x)$ are opposites, then the ordinate of $d(x)$ is double the ordinate of $f(x)$. This happens at $x = -9$ and at $x = 4$.

3 If the two curves meet at $x = a$, then they have equal ordinates there, so $d(x)$ has a zero at $x = a$. This happens at $x = -4$ and at $x = 9$.

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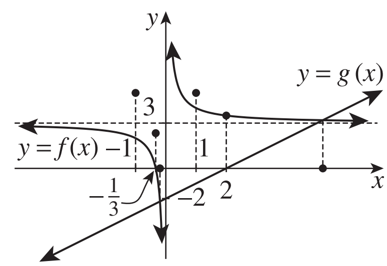
Here is a table of values to confirm these arguments:

x	-9	-6	-4	-1	0	1	4	6	9
$f(x)$	45	0	-20	-35	-36	-35	-20	0	45
$g(x)$	-45	-30	-20	-5	0	5	20	30	45
$d(x)$	90	30	0	-30	-36	-40	-40	-30	0

Again, we cannot find from the graphs alone the precise location of the minimum, so it is not needed in the sketch. It is the vertex of the parabola $y = d(x)$. Readers should complete the square for $d(x)$ and show that the vertex is $\left(2\tfrac{1}{2}, -42\tfrac{1}{4}\right)$.

An example of subtracting graphs with asymptotes

In the diagram to the right, $f(x)$ has two asymptotes. To sketch the difference $d(x) = f(x) - g(x)$, begin by applying the steps in the previous example:



Given also: $f(1) = 4$
$f(-1) = 2$

0 Subtract the ordinates at $x = 1$, so

$$d(1) = f(1) - g(1) = 4 - (-1) = 5.$$
$$d(-1) = f(-1) - g(-1) = 2 + 3 = 5.$$

1 There is a zero of $g(x)$ at $x = 2$, so $d(x)$ and $f(x)$ have the same ordinate there. There is a zero of $f(x)$ at $x = -\frac{1}{3}$, so $d(x)$ and $g(x)$ have opposite ordinates there.

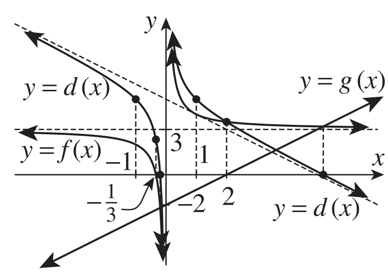
2 There are no values of x where ordinates are opposites.

3 The curves meet at two points, and the difference $d(x)$ has zeroes there.

4 The difference $d(x)$ can change sign at the two zeroes and at the asymptote. A glance at the relative positions of the two curves $f(x)$ and $g(x)$ shows that it does in fact change sign at all three places.

5 Vertically, as $x \to 0^+$, $f(x) \to +\infty$, so $d(x) \to +\infty$.
Also, as $x \to 0^-$, $f(x) \to -\infty$, so $d(x) \to -\infty$.
Thus the y-axis is a vertical asymptote to $y = d(x)$.

Horizontally, on the right and the left:
As $x \to \infty$, $f(x) \to 3$ and $g(x) \to \infty$, so $d(x) \to -\infty$.
As $x \to -\infty$, $f(x) \to 3$ and $g(x) \to -\infty$, so $d(x) \to +\infty$.



These are the same functions as in the second example of adding functions. The reader should now form the difference $d(x)$ of the two functions and find the zeroes. The diagram shows the line that the curve eventually looks like — this detail is in Question 19 part **b** of Exercise 5D, but it seems not to be required in the course.

12 SKETCHING THE DIFFERENCE OF TWO SKETCHED FUNCTIONS

- To sketch the difference $d(x) = f(x) - g(x)$ of two sketched functions.
  0 Subtract the ordinates wherever possible. This is the key idea.
- Some systematic approaches:
  1 If $f(x)$ has a zero at $x = a$, then $d(a) = -g(a)$.
  If $g(x)$ has a zero at $x = a$, then $d(a) = f(a)$.
  2 If the curves meet at $x = a$, then $d(a)$ has a zero there.
  3 If the ordinates are opposites at $x = a$, then $d(a) = 2f(a) = -2g(a)$.
  4 The sign of $d(x)$ everywhere will usually be clear now.

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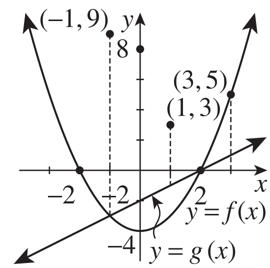
- To clarify any vertical or horizontal asymptotic behaviour:
  5 If $f(x) \to \infty$ or $g(x) \to -\infty$, then $d(x) \to \infty$.
  If $f(x) \to -\infty$ or $g(x) \to \infty$, then $d(x) \to -\infty$.
  If, however, $f(x)$ and $g(x)$ go in the same direction, we may not be able determine what happens with the difference.

Sketching the product of two sketched functions

We sketch the product $p(x) = f(x) \times g(x)$ of two functions

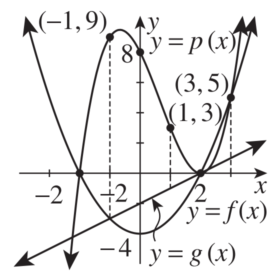
$$f(x) = x^2 - 4 \quad \text{and} \quad g(x) = x - 2,$$



To help develop the method, here is a table of values:

x	-3	-2	-1	0	1	2	3
$f(x)$	5	0	-3	-4	-3	0	5
$g(x)$	-5	-4	-3	-2	-1	0	1
$p(x)$	-25	0	9	8	3	0	5

Each ordinate of $p(x)$ is the product of the ordinates of $f(x)$ and $g(x)$ — this is the key idea. The sketch is to the right.



Here are the steps for sketching $p(x)$ from the graphs alone. Multiplying by 1 and -1 are important considerations.

0 Multiply ordinates where possible — this is the key idea. Here the y-intercept of $p(x)$ is $p(0) = (-4) \times (-2) = 8$.

1 If either curve has a zero, then $p(x)$ also has a zero there. This happens at $x = -2$ and at $x = 2$.

2 If one curve meets the horizontal line $y = 1$, then the ordinates of $p(x)$ is the ordinate of the other curve. This happens at $x = 3$, and also just left of $x = -2$ and just right of $x = 2$.

3 If one curve meets the horizontal line $y = -1$, then the ordinate of $p(x)$ is the opposite of the ordinate of the other curve. This happens at $x = 1$, and also just right of $x = -2$ and just left of $x = 2$.

4 When $f(x)$ and $g(x)$ are both positive or both negative, that is, the curves are both above or both below the x-axis, then $p(x)$ is positive, so is above the x-axis. This happens for $-2 < x < 2$ and for $x > 2$.
When $f(x)$ and $g(x)$ have opposite signs, that is, one curve is above the x-axis and the other is below, then $p(x)$ is negative and so is below the x-axis. This happens for $x < -2$.

The situation at $x = 2$ is interesting. The curve passes smoothly through the point $(2, 0)$, so the x-axis is a tangent to the curve here.

The maximum between -2 and 2 is inaccessible, just as in Section 3G, where factored cubics similar to this example $p(x) = (x - 2)^2(x + 2)$ were sketched.

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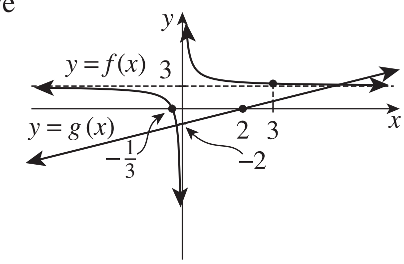
A product example with asymptotes

Using the same sketch from pages 185 and 187 and applying the steps above to begin the sketch of the product $p(x) = f(x) \times g(x)$:



Given also: $f(1) = 4$
$f(-1) = 2$

0 Multiplying ordinates, $p(-1) = 2 \times (-3) = -6$.

1 The product $p(x)$ has zeroes at $x = -\frac{1}{3}$ and $x = 2$ where one curve meets the x-axis.

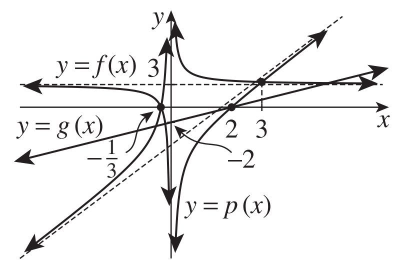
2 The line $y = 1$ meets $g(x)$ at $x = 3$, and meets $f(x)$ between $x = -1$ and $x = -\frac{1}{3}$, so at each point the other ordinate is the ordinate of $p(x)$.

3 The line $y = -1$ meets $g(x)$ at $x = 1$, and meets $f(x)$ just to the left of $x = -\frac{1}{3}$, so at each point the opposite of the other ordinate is the ordinate of $p(x)$.

4 When $f(x)$ and $g(x)$ are on the same side of the x-axis, then $p(x)$ is above the x-axis. When $f(x)$ and $g(x)$ are on opposite sides of the x-axis, then $p(x)$ is below it.

5 Vertically, as $x \to 0^+$, $f(x) \to +\infty$, so $p(x) \to -\infty$.
Also, as $x \to 0^-$, $f(x) \to -\infty$, so $p(x) \to +\infty$.
Thus the y-axis is a vertical asymptote to $y = p(x)$.

Horizontally, on the right and left:
As $x \to +\infty$, $f(x) \to 3$, and $g(x) \to +\infty$, so $p(x) \to +\infty$.
As $x \to -\infty$, $f(x) \to 3$, and $g(x) \to -\infty$, so $p(x) \to -\infty$.



These are the same functions as in the second example of adding functions. The reader should now form the product $p(x)$ of the two functions, find the zeroes, and perhaps complete Question 19 part **c** of Exercise 5D identifying the line that $p(x)$ eventually looks like.

13 SKETCHING THE PRODUCT OF TWO SKETCHED FUNCTIONS

- To sketch the product $p(x) = f(x) \times g(x)$ of two sketched functions:
  0 Multiply the ordinates wherever possible. This is the key idea.
- Some systematic approaches:
  1 If one curve has a zero, then $p(x)$ has a zero.
  2 If one curve meets $y = 1$, then the ordinate of $p(x)$ is the other ordinate.
  3 If one curve meets $y = -1$, then the ordinate of $p(x)$ is the opposite of the other ordinate.
  4 If $f(x)$ and $g(x)$ are on the same side of the x-axis, then $p(x)$ is above the x-axis. If $f(x)$ and $g(x)$ are on opposite side of the x-axis, then $p(x)$ is below the x-axis.
- To clarify any vertical or horizontal asymptotic behaviour:
  5 If $f(x) \to 0$ or $g(x) \to 0$, then $p(x) \to 0$.
  If $f(x) \to \infty$ or $-\infty$, or $g(x) \to \infty$ or $-\infty$, then $p(x) \to \infty$ or $-\infty$ depending on the sign of $g(x)$.
  If, however, $f(x) \to 0$ and $g(x) \to \infty$ or $-\infty$, we may not be able determine what happens with the product.

When $g(x)$ is a constant function, say $g(x) = 2$, then the product $f(x) \times g(x)$ stretches the graph of $y = f(x)$ vertically by a factor of 2. This will be discussed in Year 12.

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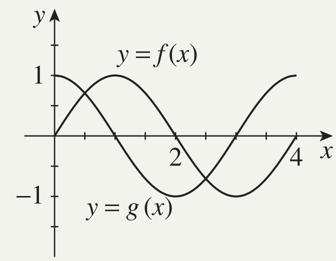
Example 14 5D

Sketch the sum, difference and product of the two functions sketched to the right.

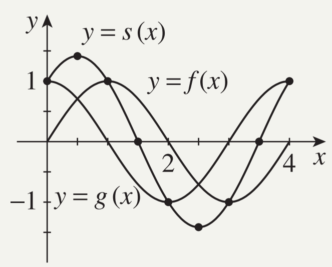


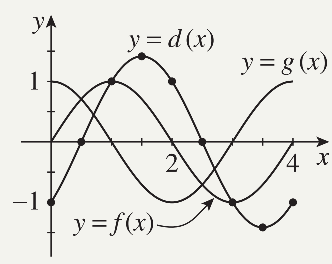
SOLUTION

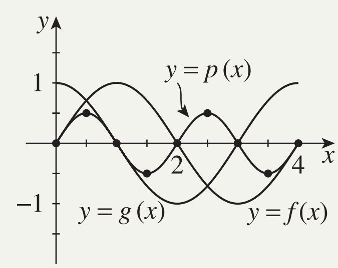
1 In each graph, plot the ordinate at each of the five zeroes.

2 Now look at the four points where the curves meet, and the points where the ordinates are opposite.

3 In the case of the product, use the fact that squaring a positive number less than 1 results in a smaller number. For example, $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.





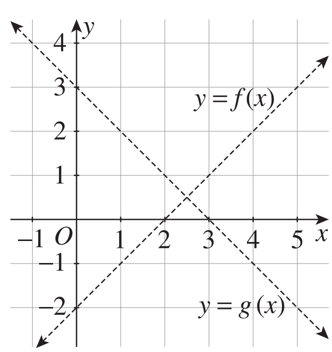


Readers who have sketched the trigonometric functions in earlier years will recognise these graphs as sin x and cos x, with a bit of stretching. These three operations on such graphs are vital in every part of science and engineering where there are wave-like periodic phenomena such as tides and alternating currents, and in the wave equations of particle physics.

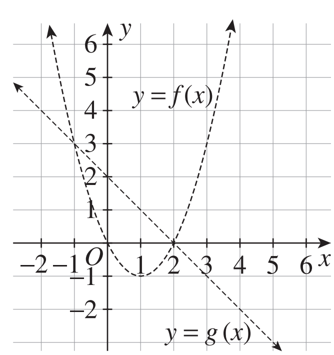
Exercise 5D FOUNDATION

1 Each diagram below shows the graph of two functions, $y = f(x)$ and $y = g(x)$. Copy each diagram to your book and, by adding ordinates, draw the graph of $y = f(x) + g(x)$. Try to distinguish the original graphs from the graph of the sum — use different colours, or dot the original as in the diagrams.

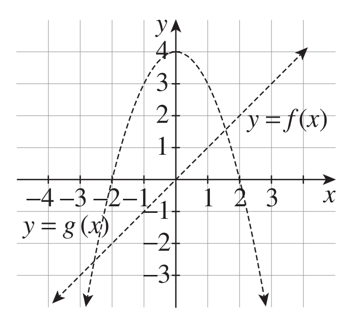
a



b



c



2 Copy each diagram in Question 1 to a fresh number plane. By subtracting ordinates, sketch the graphs of $y = f(x) - g(x)$.

3 Copy each diagram in Question 1 to a fresh number plane. By multiplying ordinates, sketch the graphs of $y = f(x) \times g(x)$.

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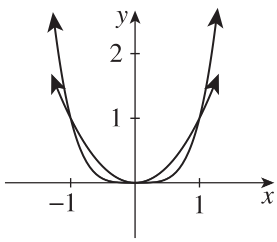
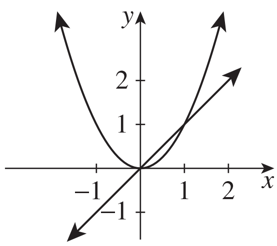
4 The diagram to the right shows the graphs of $y = f(x)$, where $f(x) = x^4$, and of $y = g(x)$, where $g(x) = x^2$.



a Copy the diagram to your book.

b On the same set of axes and in a different colour, sketch $y = f(x) - g(x)$ by subtracting ordinates. Pay careful attention to points where the graphs cross, and to the zeroes of $f(x)$ and $g(x)$.

5 The diagram to the right shows the graphs of $y = f(x)$, where $f(x) = x^2$, and of $y = g(x)$, where $g(x) = x$.



a Copy the diagram to your book.

b On the same set of axes and in a different colour, sketch $y = f(x) + g(x)$ by adding ordinates. Pay careful attention to points where $g(x) = -f(x)$, because at those points $f(x) + g(x) = 0$. Notice also the zeroes of $f(x)$ because $f(x) + g(x) = f(x)$ at those points, and the zeroes of $g(x)$, because $f(x) + g(x) = g(x)$ at those points.

6 a Sketch graphs of $y = f(x)$, where $f(x) = x^2$, and of $y = g(x)$, where $g(x) = x - 1$, on the same set of axes. Make sure that any points where $g(x) = 1$ are clear.

b Mark the zeroes of $y = f(x) \times g(x)$, which occur where $f(x) = 0$ or $g(x) = 0$.

c By noting the signs of $f(x)$ and $g(x)$, and hence of the product $y = f(x) \times g(x)$, add the graph of $y = f(x) \times g(x)$ to your diagram.

DEVELOPMENT

7 Sketch $y = x^4$ and $y = x(2 - x)$, then sketch the difference $y = x^4 - x(2 - x)$.

8 When sketching the sums of absolute value graphs in this question, it is helpful to remember that the sum of two linear functions is itself a linear function. Thus the following graphs will be made up of straight-line sections.

a Graph $f(x) = |x + 1|$ and $g(x) = |x - 1|$, then graph:

i $y = f(x) + g(x)$ ii $y = f(x) - g(x)$

b Graph $f(x) = |2x|$ and $g(x) = |x - 1|$, then graph:

i $y = f(x) + g(x)$ ii $y = f(x) - g(x)$

9 a Sketch graphs of $y = x^2$ and $y = |x - 1|$ on the same set of axes.

b Hence graph the product $y = x^2|x - 1|$.

c Explain why your graph of $y = x^2|x - 1|$ lies below both graphs $y = x^2$ and $y = |x - 1|$ in the interval $0 \le x \le 1$.

10 a Sketch $y = \sqrt{x}$ and $y = x$, paying careful attention to the domain and to the points where these graphs intersect.

b Hence sketch $y = x - \sqrt{x}$.

11 Use the same steps as in the previous question to sketch the product $y = (x - 1)\sqrt{x}$.

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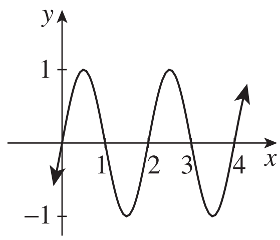
12 The techniques of sketching the product of two graphed functions extend readily to sketching a square $y = \big(f(x)\big)^2$ because $\big(f(x)\big)^2 = f(x) \times f(x)$. Always pay careful attention to the zeroes of $f(x)$ and to points where $f(x) = 1$ or $f(x) = -1$.

Sketch each function. Then sketch the square $y = \big(f(x)\big)^2$. In part **b** you may assume that $|f(x)| < 1$ for $0 < x < 2$.

a $f(x) = x^2 - 1$ **b** $f(x) = x(x - 1)(x - 2)$ **c** $f(x) = \dfrac{1}{x}$.

13 Sketch $y = f(x)^2$ for the function $y = f(x)$ sketched to the right. This is an example of a *wave function* that you may have met in physics or elsewhere in science, or previously in trigonometry.



14 For each pair of functions, first graph $y = f(x)$ and $y = g(x)$ on the same number plane. Then in a different colour graph the sum $y = f(x) + g(x)$.

a $f(x) = x$ and $g(x) = \dfrac{1}{x}$

b $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x + 2}$

c $f(x) = 2^x$ and $g(x) = x - 1$

d $y = 2^x$ and $y = 2^{-x}$

15 For each pair of functions in the previous question, sketch the difference $y = f(x) - g(x)$.

16 It may be that if $y = f(x)$ and $y = g(x)$ exhibit even or odd symmetry, then we know the symmetry of $s(x) = f(x) + g(x)$, of $d(x) = f(x) - g(x)$, and of $p(x) = f(x)g(x)$.

a Try to complete the following table, predicting the symmetry of the resulting functions, and test your claims on the examples in this exercise.

	$f(x)$ even; $g(x)$ even	$f(x)$ odd; $g(x)$ odd	$f(x)$ even; $g(x)$ odd
$s(x)$			
$d(x)$			
$p(x)$			

b Prove formally that if $f(x)$ and $g(x)$ are both odd, then $s(x)$ is also odd. That is, prove that $s(-x) = -s(x)$ for all x in its domain.

ENRICHMENT

17 The speed at which graphs tend to infinity, or to zero, is important in establishing questions of *dominance* and hence to sketching a curve.

a Sketch $x \times 2^x$. You may assume $x \times 2^x \to 0$ as $x \to -\infty$.

b Sketch $(x + 1) \times \dfrac{1}{x^2}$. You may assume $\dfrac{x + 1}{x^2} \to 0$ as $x \to \infty$ and as $x \to -\infty$.

c Sketch $x \times \dfrac{1}{x - 1}$. You may assume $\dfrac{x}{x - 1} \to 1$ as $x \to \infty$ and as $x \to -\infty$.

d Redraw the graph of part **c**, by writing it as $y = 1 + \dfrac{1}{x - 1}$ and using shifting.

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18 Consider the two functions $f(x) = x - 5$ and $g(x) = 2 + \frac{2}{x}$. The purpose of this question is to sketch the graph $y = f(x) + g(x)$.

a Find the coordinates of the two pairs of points where $f(x) = -g(x)$.

b On a half-page number plane, sketch the line $y = f(x)$ and the hyperbola $y = g(x)$.

c Show that as $x \to +\infty$ and as $x \to -\infty$, $f(x) + g(x) - (x - 3) \to 0$. Add the line $y = x - 3$ to your graph — this is an *oblique asymptote* to the curve $y = f(x) + g(x)$.

d Complete your sketch of the sum $y = f(x) + g(x)$.

19 In the text, we look the functions $f(x) = \frac{1}{x} + 3$ and $g(x) = x - 2$ and drew their sum $s(x) = f(x) + g(x)$, difference $d(x) = f(x) - g(x)$, and product $p(x) = f(x)\,g(x)$. We also drew, and commented on, their oblique asymptotes, but did not find them. This Enrichment question now finds the equations of the oblique asymptotes to the three curves. (The same method was used in Question 18 part **c** above.)

a Simplfy $s(x) = f(x) + g(x)$, and show that $s(x) - (x + 1) = \frac{1}{x}$. Hence expalin why $y = x + 1$ is an oblique asymptote to the curve $y = s(x)$.

b Simliarly simplify $d(x) = f(x) - g(x)$, and find its oblique asymptote.

c Simliarly simplify $p(x) = f(x)\,g(x)$, and find its oblique asymptote.

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5E Absolute value and square roots

The problem in this section is to take some graph $y = f(x)$, and from it sketch

$$y = |f(x)|, \qquad y = f(|x|), \qquad |y| = f(x), \qquad y = \sqrt{f(x)}, \qquad y^2 = f(x).$$

The first three graphs are constructed using only reflections in the two axes. The fourth needs care, and the fifth combines the methods of the third and fourth.

Summary of the definition of absolute value

The absolute value graphs rely on the algebraic definition using cases:

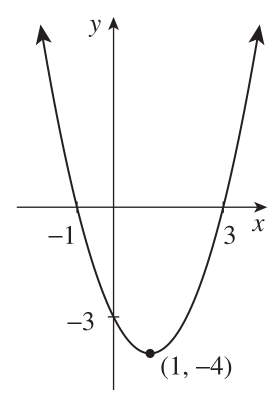
$$|x| = \begin{cases} x, & \text{for } x \ge 0 \quad (\text{for example, } |5| = 5 \text{ and } |0| = 0) \\ -x, & \text{for } x < 0 \quad (\text{for example, } |-5| = 5). \end{cases}$$

Expressed in words:

- The absolute value of a positive number or zero is unchanged.
- The absolute value of a negative number is the opposite, and so is positive.

Thus an absolute value can never be negative.

Sketching $y = |f(x)|$ from the sketch of $y = f(x)$

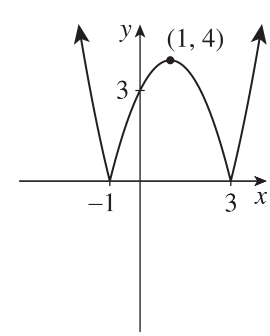


Each absolute value transformation will be illustrated in turn using the graph to the right, which is the parabola

$$f(x) = (x + 1)(x - 3).$$

Graphing $y = |f(x)|$ requires two arguments:

- When the graph is above or on the x-axis, $f(x)$ is positive or zero. Hence $|f(x)| = f(x)$, so the graph is unchanged.
- When the graph is below the x-axis, $f(x)$ is negative. Hence $|f(x)| = -f(x)$ is the opposite of $f(x)$, so replace any part of the graph below the x-axis by its reflection back above the x-axis.



A table of values helps to understand the situation:

x	-2	-1	0	1	2	3	4
$f(x)$	5	0	-3	-4	-3	0	5
$\lvert f(x)\rvert$	5	0	3	4	3	0	5

14 TO SKETCH $y = |f(x)|$ FROM THE SKETCH OF $y = f(x)$

- Everything above and on the x-axis stays the same.
- Replace everything below the x-axis by its reflection back above the x-axis.

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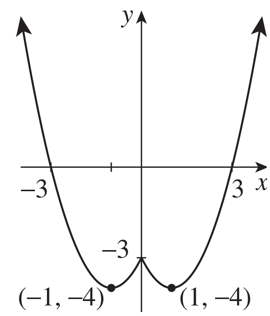
Sketching $y = f(|x|)$ from the sketch of $y = f(x)$



The procedure to graph $y = f(|x|)$ again has two arguments:

- To the right of or on the y-axis, x is positive or zero. Hence $|x| = x$ and the graph is unchanged.
- To the left of x-axis, x is negative, so $|x| = -x$, and $f(|x|) = f(-x)$. This means that the part of the graph to the left of the y-axis is removed, and replaced by duplicating by reflection everything to the right of the y-axis.

The table of values demonstrating this has a preliminary row for $|x|$:

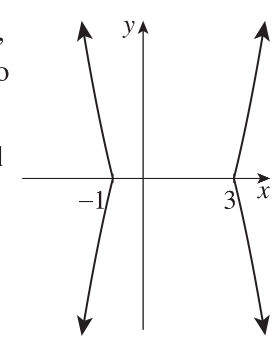
x	-4	-3	-2	-1	0	1	2	3	4
$\lvert x\rvert$	4	3	2	1	0	1	2	3	4
$f(\lvert x\rvert)$	5	0	-3	-4	-3	-4	-3	0	5

15 TO SKETCH $y = f(|x|)$ FROM THE SKETCH OF $y = f(x)$

- Everything to the right of the y-axis stays the same.
- Delete everything to the left of the y-axis, and replace it by duplicating, by reflection in the y-axis, everything to the right of y-axis.

The resulting function is even, that is, it has line symmetry in the y-axis. To demonstrate this, the table of values was constructed symmetrically about $x = 0$.

Sketching $|y| = f(x)$ from the sketch of $y = f(x)$



The steps for sketching $|y| = f(x)$ are the same as the steps for sketching $y = f(|x|)$, except that horizontal and vertical are reversed. Normally the graph of $|y| = f(x)$ is no longer a function.

- When the graph of $y = f(x)$ is below the x-axis, $f(x)$ is negative, so it cannot equal $|y|$. Thus we first remove all parts of the graph below the x-axis.
- When the graph of $y = f(x)$ is above the x-axis, then $|y| = f(x)$, so y has two possible values $f(x)$ and $-f(x)$. Thus below the x-axis we duplicate by reflection everything above the x-axis.

Here is a table of values. There are two values for y when $f(x)$ is positive, one value of y when $f(x)$ is zero, and no values of y when $f(x)$ is negative.

x	-2	-1	0	1	2	3	4
$f(x)$	5	0	-3	-4	-3	0	5
y	-5 or 5	0	*	*	*	0	-5 or 5

The result in this case is a many-to-many relation.

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16 TO SKETCH $|y| = f(x)$ FROM THE SKETCH OF $y = f(x)$

- Everything above the x-axis stays the same.
- Delete everything below the x-axis, and replace it by duplicating, by reflection in the x-axis, everything above the x-axis.

The resulting graph always has line symmetry in the x-axis.

Combining these transformations

These various transformation can be combined, as in the worked example below.

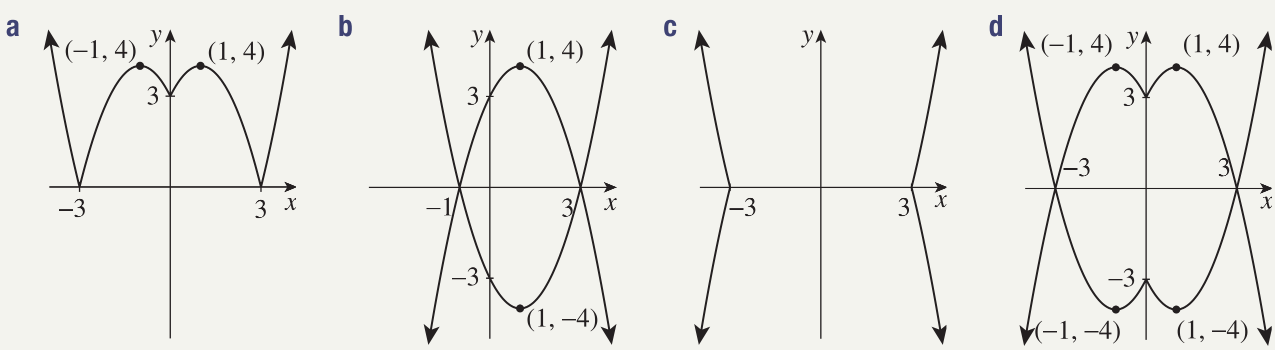
Example 15 5E

Earlier in this section, we sketched $f(x) = (x + 1)(x - 3)$ and its transformations $y = |f(x)|$ and $y = f(|x|)$. Use these graphs to sketch the further transformations:

a $y = \left|f(|x|)\right|$ **b** $|y| = |f(x)|$ **c** $|y| = f(|x|)$ **d** $|y| = \left|f(|x|)\right|$

SOLUTION

In part **a**, start with either the earlier sketch of $|f(x)|$ or the earlier sketch of $f(|x|)$.
In parts **b** and **c**, start with the earlier sketches of $|f(x)|$ and $f(|x|)$.
In part **d**, start with the sketch of $\left|f(|x|)\right|$ from part **a**.



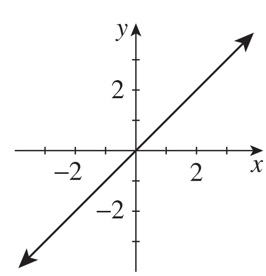
$y = \left|f(|x|)\right|$ $|y| = |f(x)|$ $|y| = f(|x|)$ $|y| = \left|f(|x|)\right|$

Sketching $y = \sqrt{f(x)}$ from the sketch of $y = f(x)$

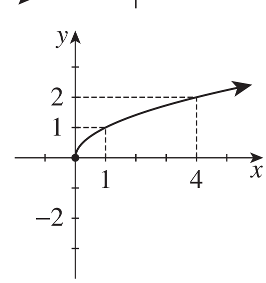
We start with a simple and familiar example:

$$f(x) = x$$

and investigate how to sketch $r(x) = \sqrt{f(x)}$. The resulting graph of $y = \sqrt{x}$ is well known (see Section 3G), but here is a table of values displaying what is happening:



x	-2	-1	0	$\frac{1}{4}$	1	4
$f(x)$	-2	-1	0	$\frac{1}{4}$	1	4
$r(x)$	*	*	0	$\frac{1}{2}$	1	2



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With only the graph and no equation, the steps are:

0 Take the square root of ordinates where possible — the key idea.

1 Negative numbers have no square roots — delete everything below the x-axis.

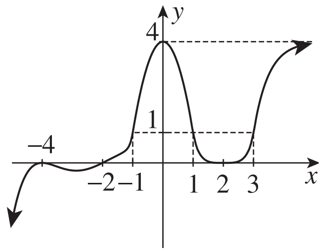
2 When $f(x)$ is zero or 1, $r(x)$ is also zero or 1. That is, the curves meet.

3 When $f(x)$ is above $y = 1$, the square root is smaller, so $r(x)$ is below $f(x)$.

4 When $f(x)$ is between the x-axis and $y = 1$, the square root is larger, so $r(x)$ is above $f(x)$.

5 As $x \to \infty$, $f(x) \to \infty$, so $r(x) \to \infty$, but more slowly.

An example for $y = \sqrt{f(x)}$ with a horizontal asymptote



This graph of $f(x)$ has no equation. Using the steps above to sketch $r(x) = \sqrt{f(x)}$:

0 At the y-intercept, $r(0) = \sqrt{4} = 2$.

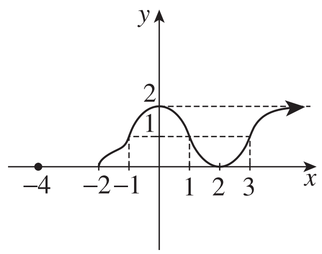
1 Delete everything below the x-axis.

2 There are six points on $f(x)$ where $y = 0$ or 1, and $r(x)$ meets $f(x)$ at those points: $(3, 1)$, $(2, 0)$, $(1, 1)$, $(-1, 1)$, $(-2, 0)$ and $(-4, 0)$ (where $r(x)$ has an isolated point).

3 For $-1 < x < 1$ and for $x > 3$, $r(x)$ is below $f(x)$.

4 For $-2 < x < -1$, for $1 < x < 2$, and for $2 < x < 3$, the graph of $r(x)$ is above the graph of $f(x)$.

5 As $x \to \infty$, $f(x) \to 4$, so $r(x) \to \sqrt{4} = 2$.



The behaviour at $x = -2$ requires calculus to explain — the rule is that if $y = f(x)$ crosses the x-axis obliquely, then $y = \sqrt{f(x)}$ emerges vertically from the x-axis. The behaviour at $x = 2$, on the other hand, is a complete guess — on each side, the curve could emerge flat as drawn, or it could emerge obliquely, or it could even emerge vertically.

17 SKETCHING THE SQUARE ROOT OF A SKETCHED FUNCTION

- To sketch the square root $r(x) = \sqrt{f(x)}$ from the sketch of $y = f(x)$:
  0 Take the square root of ordinates wherever possible. This is the key idea.
- Some systematic approaches:
  1 Delete everything below the x-axis.
  2 When $f(x)$ is zero or 1, $r(x)$ is also zero or 1.
  3 When $y = f(x)$ is above $y = 1$, the curve $y = r(x)$ is below $y = f(x)$.
  4 When $y = f(x)$ is between $y = 1$ and the x-axis, $r(x)$ is above $f(x)$.
- To clarify any asymptotic behaviour:
  5 If $f(x) \to \infty$ as $x \to \infty$ or as $x \to -\infty$, so also does $r(x)$.
  If $f(x) \to b \geq 0$ as $x \to \infty$ or as $x \to -\infty$, then $r(x) \to \sqrt{b}$.

We may not be able to determine the behaviour of the curve near a zero. Similarly, the concavity of the curve is often unclear.

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Sketching $y^2 = f(x)$ from the sketch of $y = f(x)$

Sketching this curve involves a combination of previous methods.

1 The LHS y^2 cannot be negative, so ignore everything below the x-axis.
2 Solving for y gives $y = \sqrt{f(x)}$ or $y = -\sqrt{f(x)}$, so graph $y = \sqrt{f(x)}$.
3 Duplicate $y = \sqrt{f(x)}$ below the x-axis by reflection below the x-axis.

In fact, the relation $y^2 = f(x)$ is the same relation as $|y| = \sqrt{f(x)}$.

18 SKETCHING $y^2 = f(x)$ FROM A SKETCHED FUNCTION $f(x)$

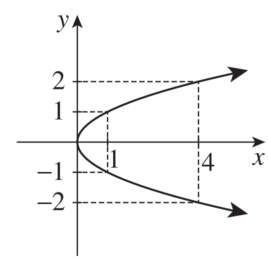
To sketch $y^2 = f(x)$ from the sketch of $y = f(x)$, combine the methods of sketching $y = \sqrt{f(x)}$ and the method of sketching $|y| = f(x)$:

1 Delete everything below the x-axis.
2 Sketch $y = \sqrt{f(x)}$ by the method in Box 17.
3 Duplicate $y = \sqrt{f(x)}$ below the x-axis by reflection in the x-axis.

Alternatively, the relation $y^2 = f(x)$ is exactly the same as $|y| = \sqrt{f(x)}$.

This can all be illustrated using the same simple example of $f(x) = x$. Here is a table of values for y when $y^2 = f(x)$:

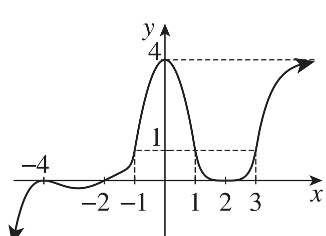
x	-2	-1	0	$\frac{1}{4}$	1	4
$f(x)$	-2	-1	0	$\frac{1}{4}$	1	4
y	*	*	0	$\frac{1}{2}$ or $-\frac{1}{2}$	1 or -1	2 or -2



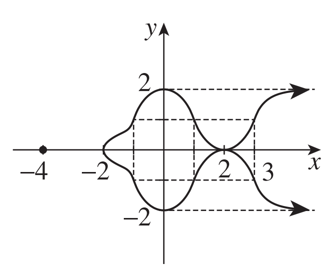
Notice how the curve passes smoothly through the origin — it is in fact the sideways parabola $y^2 = x$. This is why the earlier graph of $y = \sqrt{x}$ becomes vertical at the origin.

An example for $y^2 = f(x)$ with a horizontal asymptote

The diagrams below show the result when the steps are applied to the previous example of a function with no equation.



$y = f(x)$



$y^2 = f(x)$

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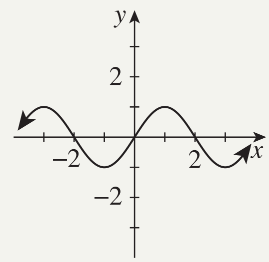
Example 16 5E

Using the sketched graph of $y = f(x)$, sketch:

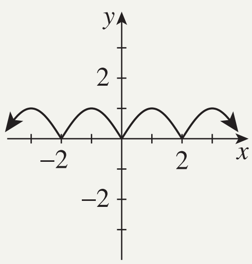
a $y = |f(x)|$ **b** $y = f(|x|)$ **c** $y = |f(|x|)|$

d $|y| = f(x)$ **e** $y = \sqrt{f(x)}$ **f** $y^2 = f(x)$



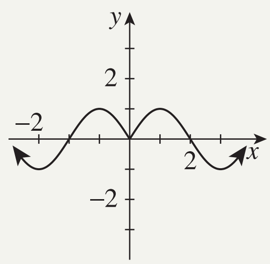
SOLUTION

a



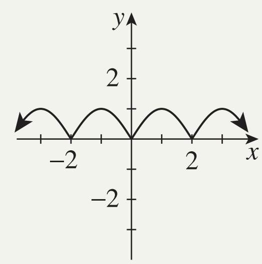
$y = |f(x)|$

b



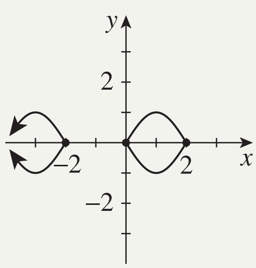
$y = f(|x|)$

c



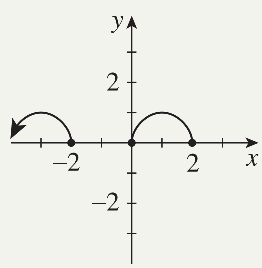
$y = |f(|x|)|$, same as part **a**

d



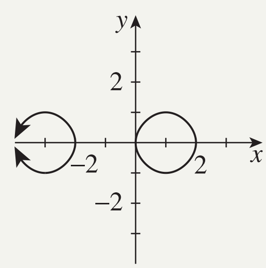
$|y| = f(x)$

e



$y = \sqrt{f(x)}$

f

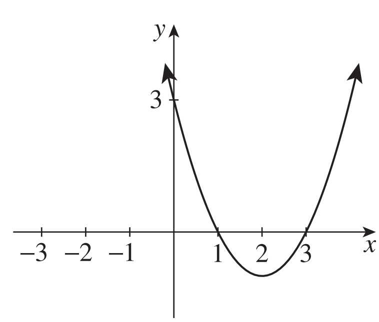


$y^2 = f(x)$

Exercise 5E

FOUNDATION

1 The graph of $y = f(x)$ is sketched to the right. To draw each transformation, copy the graph and draw the transformed graph in a different colour on the same axes.



a Replace any part of the graph below the x-axis by its reflection above the x-axis. This will give you the graph of $y = |f(x)|$.

b Delete any parts of the graph to the left of the y-axis, then make a copy by reflection in the y-axis of any parts to the right of the y-axis. This will give you the graph of $y = f(|x|)$. Notice the symmetry in the y-axis.

c Delete any parts of the graph below the x-axis, then make a copy by reflection in the x-axis of any parts above the x-axis. This will give you the graph of $|y| = f(x)$. Notice the symmetry in the x-axis.

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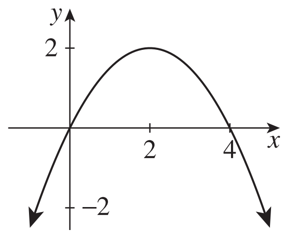
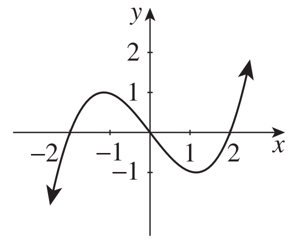
2 Use the given graph of $y = f(x)$ to sketch: **i** $y = |f(x)|$ **ii** $y = f(|x|)$ **iii** $|y| = f(x)$

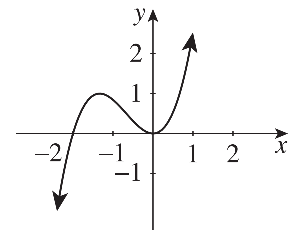
a



b

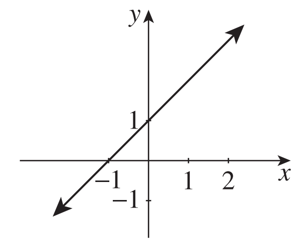


c



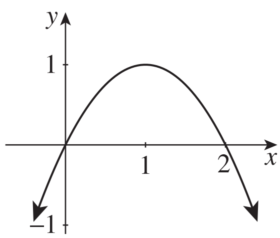
3 **a** Copy the graph of $y = f(x)$ sketched to the right and use Box 17 to sketch $y = \sqrt{f(x)}$. Remember that the transformed graph will be higher than the original where $f(x) < 1$, and lower where $f(x) > 1$.

b By adding the reflection of part **a** in the x-axis, draw the graph of $y^2 = f(x)$, that is, of $|y| = \sqrt{f(x)}$.

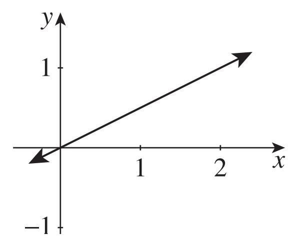


4 Use the given graph of $y = f(x)$ to sketch: **i** $y = \sqrt{f(x)}$ **ii** $y^2 = f(x)$.

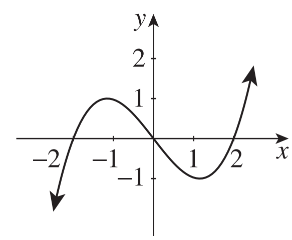
a



b



c



DEVELOPMENT

5 **a** Sketch $y = f(x)$ where $f(x) = \frac{8}{9}(x + 1)(x - 2)$, showing all x-intercepts and the vertex.

b Hence sketch:

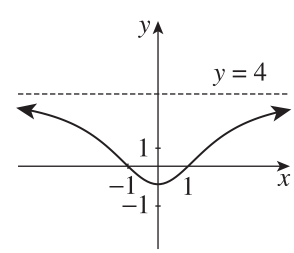
i $y = |f(x)|$ **ii** $y = f(|x|)$ **iii** $|y| = f(x)$

6 Repeat the steps of the previous question for the function $f(x) = |x - 1| - 1$.

7 The graph $y = f(x)$ to the right has a horizontal asymptote with equation $y = 4$.

a What is the equation of the horizontal asymptote of $y = \sqrt{f(x)}$?

b Sketch $y^2 = f(x)$.



8 **a** Graph $y = f(x)$, where $f(x) = \frac{1}{x - 2} + 1$. Be careful to identify any intercepts with the axes and any asymptotes.

b Hence sketch:

i $y = |f(x)|$ **ii** $y = f(|x|)$ **iii** $|y| = f(x)$

9 **a** Graph $y = f(x)$, where $f(x) = \frac{2}{x - 2} + 2$. Be careful to identify any intercepts with the axes and any asymptotes.

b Hence sketch:

i $y = \sqrt{f(x)}$ **ii** $y^2 = f(x)$

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10 **a** Sketch $y = f(x)$ for $f(x) = x(x - 2)$.
b Use repeated transformations to sketch: **i** $y = \left|f(|x|)\right|$ **ii** $|y| = f(|x|)$.
c Repeat parts **a** and **b** for $f(x) = (x + 1)(3 - x)$.

11 [The behaviour of $y = \sqrt{f(x)}$ at a zero of $y = f(x)$ is uncertain.]
a For each function $f(x)$, sketch the square root $y = \sqrt{f(x)}$ and comment on the behaviour of the transformed graph at the zero when $x = 1$.
i $f(x) = (x - 1)$ **ii** $f(x) = (x - 1)^2$ **iii** $f(x) = (x - 1)^4$
b Explain why $y = \sqrt[4]{x - 1}$ is known to be vertical at $x = 1$.

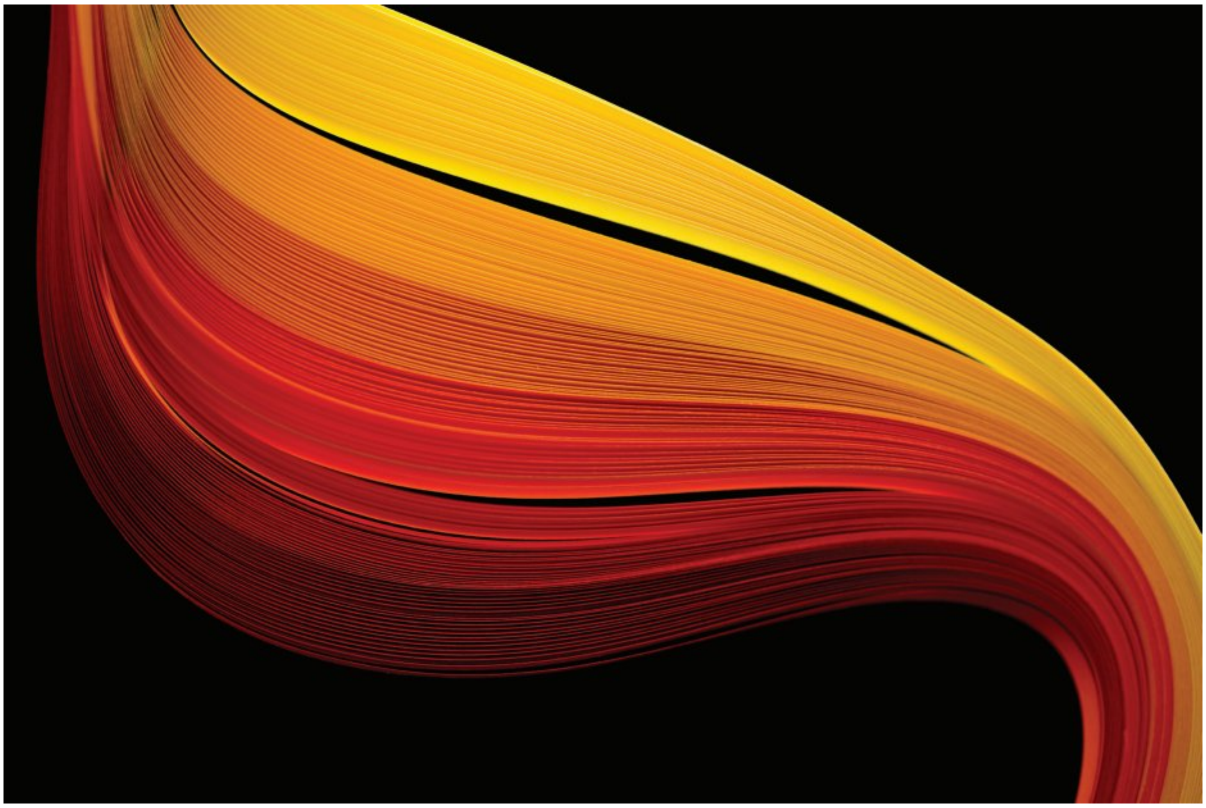
12 Sketch $|y| = |x|$.

13 Let $f(x)$ be any function. Explain why:
a $y = f(|x|)$ is even **b** $|y| = f(x)$ has line symmetry in the x-axis.

ENRICHMENT

14 **a** Sketch $f(x) = x(x - 2)$. Hence sketch $y = \left|f(x)\right|$ following the steps of Box 14, and from it sketch $|y| = \left|f(x)\right|$ following the steps of Box 16.
b **i** Sketch $f(x) = x(x - 2)$. Hence sketch $|y| = f(x)$ following the steps of Box 16.
ii Why do the steps of Box 14 applied to the graph in part **i** not give $|y| = \left|f(x)\right|$?

15 The original function in Example 16 has equation $f(x) = \sin 90x°$. The final graph of $y^2 = f(x)$ in part **f** may look very much like a sequence of circles, but they are not circles — appearances can be deceiving. Prove that this graph does not consist of circles.



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5F Inverse relations and functions

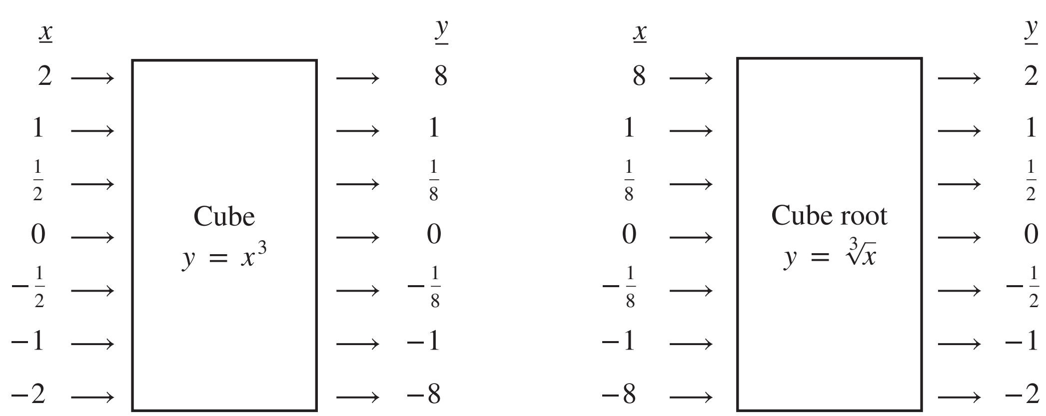
Mathematics is full of inverse processes:

- The inverse of multiplying by 7 is dividing by 7 — and the inverse of dividing by 7 is multiplying by 7.
- The inverse of shifting up 3 is shifting down 3 — and the inverse of shifting down 3 is shifting up 3.
- The inverse of reflecting in the y-axis is reflecting in the y-axis — this process is its own inverse.

In this section we use geometric and graphical methods to obtain the inverse of a relation, and find the condition for the inverse to be a function.

Inverse relations

Suppose that I cube the number 5 and get 125. The inverse process is taking the cube root, which sends 125 back to 5. I can do this with any number, positive, negative or zero, so the cubing function $y = x^3$ has a well-defined *inverse function* $y = \sqrt[3]{x}$ that sends any output back to its original input. When the two functions machines are put next to each other, the *composition* of the two functions is the identity function that maps every number to itself:



The exchanging of input and output can also be seen in the two tables of values, where the two rows are interchanged:

x	2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	-2
x^3	8	1	$\frac{1}{8}$	0	$-\frac{1}{8}$	-1	-8

x	8	1	$\frac{1}{8}$	0	$-\frac{1}{8}$	-1	-8
$\sqrt[3]{x}$	2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	-2

This exchanging of input and output means that the coordinates of each ordered pair are exchanged. Remembering that a relation was defined simply as a set of ordered pairs, we are led to a definition of inverse that can be applied to any relation, whether it is a function or not:

19 INVERSE RELATIONS

- The *inverse relation* of any relation is obtained by reversing each ordered pair.
- The inverse relation of the inverse relation is the original relation.

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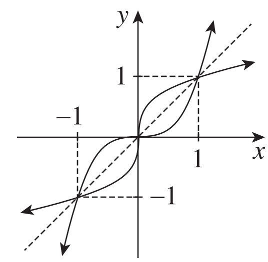
The second statement follows from the first because each ordered pair returns to its original state when reversed a second time. For example, the pair $(2, 8)$ in the original becomes the pair $(8, 2)$ in the inverse, and reversed goes back to $(2, 8)$.

Graphing the inverse relation

Reversing an ordered pair means that the original first coordinate is read off the vertical axis, and the original second coordinate is read off the horizontal axis. Geometrically, this exchanging can be done by reflecting the point in the diagonal line $y = x$, as can be seen by comparing the graphs of $y = x^3$ and $y = \sqrt[3]{x}$, which are drawn here on the same pair of axes.



20 THE GRAPH OF THE INVERSE RELATION

The graph of the inverse relation is obtained by reflecting the original graph in the diagonal line $y = x$.

Domain and range of the inverse relation

The exchanging of the x- and y-coordinates means that the domain and range are exchanged:

21 DOMAIN AND RANGE OF THE INVERSE RELATION

- The domain of the inverse is the range of the relation
- The range of the inverse is the domain of the relation.

Finding the equations and conditions of the inverse relation

When the coordinates are exchanged, the x-variable becomes the y-variable and the y-variable becomes the x-variable, so the method for finding the equation and conditions of the inverse is:

22 THE EQUATION OF THE INVERSE RELATION

- To find the equations and conditions of the inverse relation, write x for y and y for x every time each variable occurs.
- This process can be applied to any relation whose equations and conditions are known, whether or not it is a function.

For example, the inverse of the function $y = x^3$ is $x = y^3$. This particular equation can then be solved for y to give $y = \sqrt[3]{x}$, confirming that in this particular case, the inverse relation is again a function.

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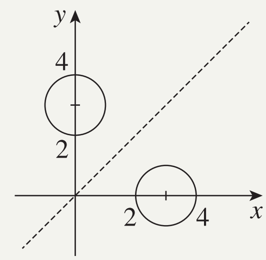
Example 17 5F

a Write down the inverse relation of $(x - 3)^2 + y^2 = 1$.
b Graph both relations on the same number plane, showing the reflection line.
c Write down the domain and range of both relations.
d Is the relation or its inverse a function?

SOLUTION

a Writing x for y and y for x,
the inverse is $(y - 3)^2 + x^2 = 1$.

b



c For the original, domain: $2 \le x \le 4$, range: $-1 \le y \le 1$.
For the inverse, domain: $-1 \le x \le 1$, range: $2 \le y \le 4$.
Notice how the domain and the range have been swapped.

d Neither relation is a function.

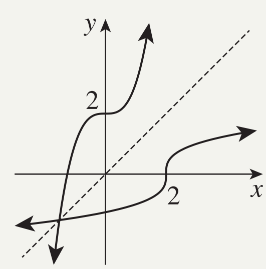
Example 18 5F

Repeat the previous questions for $y = x^3 + 2$.

SOLUTION

a Writing x for y and y for x,
the inverse is $x = y^3 + 2$,
which is $y = \sqrt[3]{x - 2}$.

b



c For both, domain and range are all real numbers.

d Both the relation and its inverse are functions.

Forming the inverse when there are restrictions

When there are any conditions, particularly restrictions, then x and y must be swapped in these as well, as in the next worked example.

Example 19 5F

Consider the function $y = 2x - 3$, where $x > 1$.
a Write down the equation and condition of the inverse relation.
b Rewrite the inverse as a function with y as the subject, changing the condition to a restriction on x.

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SOLUTION

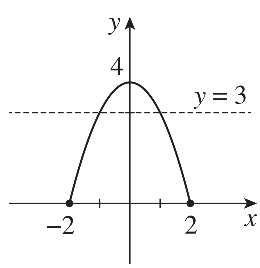
a The function is $y = 2x - 3$, where $x > 1$,
so the inverse relation is $x = 2y - 3$, where $y > 1$.

b Solving for y, $y = \frac{1}{2}(x + 3)$, where $y > 1$.

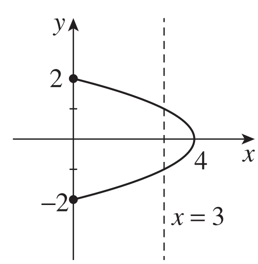
The condition $y > 1$ is
$$\begin{aligned} \tfrac{1}{2}(x+3) &> 1 \\ x + 3 &> 2 \\ x &> -1, \end{aligned}$$
so the inverse is the function $y = \frac{1}{2}(x + 3)$, where $x > -1$.

Testing graphically whether the inverse relation is a function

It is not true in general that the inverse of a function is a function. For example, the sketches below show the graphs of another function, with a restriction, and its inverse:



$y = 4 - x^2$, where $-2 \le x \le 2$



$x = 4 - y^2$, where $-2 \le y \le 2$

The first graph is a many-to-one function, passing the vertical line test, but failing the horizontal line test. When we read it backwards, the value $y = 3$ gives two answers, $x = 1$ and $x = -1$.

Reflection in $y = x$ exchanges vertical and horizontal lines. Thus the second graph is one-to-many, failing the vertical line test, but passing the horizontal line test. It is not a function — when the input to the second graph is $x = 3$, there are two outputs, $y = 1$ and $y = -1$. More generally, solving the equation of the inverse relation for y gives two answers, $y = \sqrt{4 - x}$ and $= -\sqrt{4 - x}$.

We didn't need to draw the second graph to know that the inverse is not a function. All we need to know is that the first graph fails the horizontal line test.

23 HORIZONTAL LINE TEST FOR WHETHER THE INVERSE IS A FUNCTION

- Geometrically, the inverse relation of a given relation is a function if and only if no horizontal line crosses the original graph more than once.
- Algebraically, if solving the equation of the inverse for y gives two answers, then the inverse is not a function.

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In Example 17, each circle fails both the horizontal line test and the vertical line test. In Example 18, however, each graph passes both the horizontal line test and the vertical line test.

The four types of relations

A function passes the vertical line test, so its inverse is also a function when it is one-to-one. A relation that is not a function fails the vertical line test, so its inverse is a function when it is one-to-many. Now we can complete the table from Section 3I.

Type	Vertical line test	Horizontal line test
One-to-one	Passes (it is a function)	Passes (the inverse is a function)
Many-to-one	Passes (it is a function)	Fails (the inverse is not a function)
One-to-many	Fails (it is not a function)	Passes (the inverse is a function)
Many-to-many	Fails (it is not a function)	Fails (the inverse is not a function)

As far as functions are concerned, the crucial ideas here are:

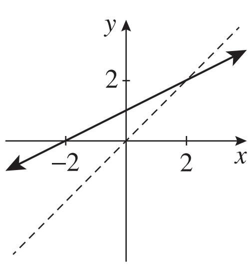
- A function is either one-to-one or many-to-one.
- If a function is one-to-one, then its inverse relation is again a function.
- If a function is many-to-one, then its inverse relation is not a function.

Exercise 5F

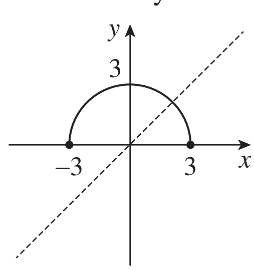
FOUNDATION

1 Draw the inverse relation of each relation by reflecting in the line $y = x$.

a



b



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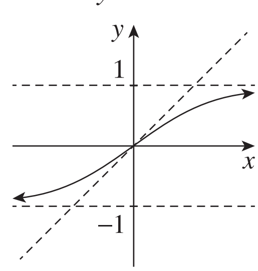

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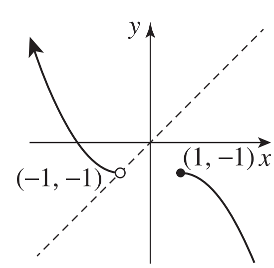
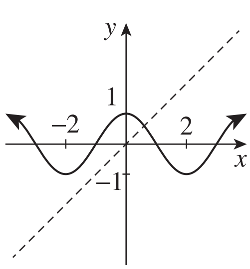
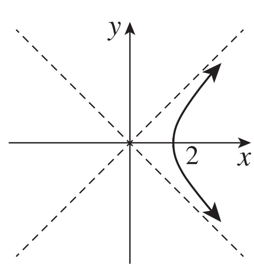


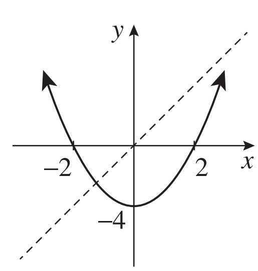
e



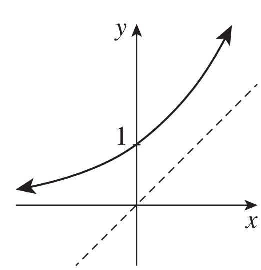
f



g



h



2 Use the vertical and horizontal line tests to determine which relations and which inverse relations drawn in Question 1 are also functions.

3 Classify each relation in Question 1 as one-to-one, many-to-one, one-to-many or many-to-many.

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4 Determine each inverse algebraically by swapping x and y and then making y the subject.

a $y = 3x - 2$ b $y = \frac{1}{2}x + 1$ c $y = 3 - \frac{1}{2}x$

d $x - y + 1 = 0$ e $2x + 5y - 10 = 0$ f $y = 2$

5 For each function in the previous question, draw a graph of the function and its inverse on the same number plane to verify the reflection property. Draw a separate number plane for each part.

6 a Find each inverse algebraically by swapping x and y and then making y the subject.

i $y = \dfrac{1}{x} + 1$ ii $y = \dfrac{1}{x + 1}$ iii $y = \dfrac{x + 2}{x - 2}$ iv $y = \dfrac{3x}{x + 2}$

b For parts i and iv above, find the domain and range of the function, and the domain and range of the inverse function.

7 Swap x and y and solve for y to find the inverse of each function. What do you notice, and what is the geometric significance of this?

a $y = \dfrac{1}{x}$ b $y = \dfrac{2x - 2}{x - 2}$ c $y = \dfrac{-3x - 5}{x + 3}$ d $y = -x$

DEVELOPMENT

8 a Graph each relation and its inverse relation, then find the equation of the inverse relation. Which of the four original relations are functions, and which inverse relations are functions?

i $(x - 3)^2 + y^2 = 4$ ii $(x + 1)^2 + (y + 1)^2 = 9$

iii $y = x^2 - 4$ iv $y = x^2 + 1$

b For parts i and iv above, find the domain and range of the relation, and the domain and range of the inverse relation.

9 Write down the inverse of each function, solving for y if it is a function. Sketch the function and the inverse on the same graph and observe the symmetry in the line $y = x$.

a $y = x^2$ b $y = 2x - x^2$ c $y = -\sqrt{x}$ d $y = -\sqrt{4 - x^2}$

10 Each function below has a restriction. Write down its inverse relation. Then attempt to solve it for y. If the inverse is a function, rewrite the restriction as a restriction on x. If the inverse is not a function, give a value of x that corresponds to two or more values of y.

a $y = 3x - 10$, where $x < 2$ b $y = 13 - 6x$, where $x \geq 3$

c $y = x^3 + 2$, where $x < 3$ d $y = x^2 - 3$, where $x \geq -2$

11 a Show that the inverse function of $y = \dfrac{ax + b}{x + c}$ is $y = \dfrac{b - cx}{x - a}$.

b Hence show that $y = \dfrac{ax + b}{x + c}$ is its own inverse if and only if $a + c = 0$.

12 a Classify each linear graph as one-to-one, many-to-one, one-to-many or many-to-many.

i $y = mx + b$, where $m \neq 0$ ii $y = b$ iii $x = a$

b Which of these three types of linear graphs have an inverse relation that is a function?

13 Classify the inverse relation of a relation that is:

a one-to-one b many-to-one c one-to-many d many-to-many

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ENRICHMENT

14 **a** Show that the inverse of $y = \dfrac{2^x + 2^{-x}}{2}$ is not a function.

b Show that the inverse of $y = \dfrac{2^x - 2^{-x}}{2}$ is a function.

15 **a** Show that if the domain of an even function contains a non-zero number, then its inverse is not a function.

b Is the inverse of an odd function always a function? If not, give a counter-example.

16 **a** Show geometrically that the inverse relation of an odd relation is odd.

b Is the inverse relation of an even relation always an even relation? If not, give a counter-example.

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5G Inverse function notation

The next step is to incorporate function notation into the theory of inverse functions.

Inverse function notation

If $f(x)$ is a one-to-one function, that is, its inverse is also a function, then that inverse function is written as $f^{-1}(x)$. The index -1 used here means 'inverse function' and must not be confused with its more common use for the reciprocal of a number. To return to the original example,

$$\text{If } f(x) = x^3, \text{ then } f^{-1}(x) = \sqrt[3]{x}. \qquad \left(\text{Be careful: } \big(f(x)\big)^{-1} = \frac{1}{f(x)}\right)$$

The inverse function sends each number back where it came from. Hence if the function and the inverse function are applied successively to a number, in either order, the result is the original number. For example, using cubes and cube roots,

$$\left(\sqrt[3]{8}\right)^3 = 2^3 = 8 \quad \text{and} \quad \sqrt[3]{8^3} = \sqrt[3]{512} = 8.$$

Thus each composite of $f(x)$ and $f^{-1}(x)$ is an *identity function* $I(x) = x$, mapping every number for which the composite is defined back to itself.

24 INVERSE FUNCTIONS

- If a function $f(x)$ is one-to-one, that is, its inverse relation is also a function, then the inverse function is written as $f^{-1}(x)$.
- Each composite of the function and its inverse sends every number for which it is defined back to itself:
$$f^{-1}\big(f(x)\big) = x, \quad \text{for all } x \text{ in the domain of } f(x)$$
$$f\big(f^{-1}(x)\big) = x, \quad \text{for all } x \text{ in the domain of } f^{-1}(x)\,.$$
so that both $f^{-1}\big(f(x)\big)$ and $f\big(f^{-1}(x)\big)$ are identity functions.
- An *identity function* is a function $I(x)$ whose output is the same as its input,
$$I(x) = x, \quad \text{for all } x \text{ in its domain.}$$
- To find $f^{-1}(x)$ from $f(x)$:
  — Convert to $y = \ldots$ notation to generate the inverse relation.
  — Write y for x and x for y in all the equations and conditions.
  — Solve for y, and convert back to the notation $f^{-1}(x) = \ldots$

Examples 20 and 21 demonstrate the method described in the final dotpoint.

Example 20 — 5G

Find the equation of $f^{-1}(x)$ for each function, then verify that

$$f^{-1}\big(f(x)\big) = x \quad \text{and} \quad f\big(f^{-1}(x)\big) = x.$$

a $f(x) = x^3 + 2$

b $f(x) = 6 - 2x$, where $x > 0$

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SOLUTION

a Let $y = x^3 + 2$.

Then the inverse has equation $x = y^3 + 2$ (the key step)

and solving for y, $y = \sqrt[3]{x - 2}$.

Hence $f^{-1}(x) = \sqrt[3]{x - 2}$.

Verifying, $f^{-1}\big(f(x)\big) = \sqrt[3]{(x^3 + 2) - 2} = \sqrt[3]{x^3} = x$ and $f\big(f^{-1}(x)\big) = \left(\sqrt[3]{x - 2}\right)^3 + 2 = (x - 2) + 2 = x$

b Let $y = 6 - 2x$, where $x > 0$.

Then the inverse has equation $x = 6 - 2y$, where $y > 0$ (the key step)

$y = 3 - \frac{1}{2}x$, where $y > 0$.

The condition $y > 0$ means $3 - \frac{1}{2}x > 0$

$x < 6$,

so $f^{-1}(x) = 3 - \frac{1}{2}x$, where $x < 6$.

Verifying, $f^{-1}\big(f(x)\big) = f^{-1}(6 - 2x) = 3 - \frac{1}{2}(6 - 2x) = 3 - 3 + x = x$ and $f\big(f^{-1}(x)\big) = f(3 - \frac{1}{2}x) = 6 - 2(3 - \frac{1}{2}x) = 6 - 6 + x = x$

Example 21 5G

Write down the inverse relations of each function. If the inverse is a function, find an expression for $f^{-1}(x)$, and verify that $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(x)\big) = x$.

a $f(x) = \dfrac{1 - x}{1 + x}$

b $f(x) = x^2 - 9$

What is surprising about the result of part **a**?

SOLUTION

a Let $y = \dfrac{1 - x}{1 + x}$.

Then the inverse has equation $x = \dfrac{1 - y}{1 + y}$ (the key step)

$\times (1 + y)$: $x + xy = 1 - y$

$y + xy = 1 - x$ (terms in y on one side)

$y(1 + x) = 1 - x$ (now y occurs only once)

$y = \dfrac{1 - x}{1 + x}$.

Hence $f^{-1}(x) = \dfrac{1 - x}{1 + x}$.

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Notice that this function $f(x)$ and its inverse $f^{-1}(x)$ are identical, so that if the function $f(x)$ is applied twice, each number is sent back to itself.

Thus $f(f(2)) = f\left(-\frac{1}{3}\right) = 1\frac{1}{3} \div \frac{2}{3} = 2$ and in general,

$$f(f(x)) = \frac{1 - \dfrac{1-x}{1+x}}{1 + \dfrac{1-x}{1+x}} = \frac{(1+x)-(1-x)}{(1+x)+(1-x)} = x$$

b The function $f(x) = x^2 - 9$ fails the horizontal line test. For example, $f(3) = f(-3) = 0$, which means that the x-axis meets the graph twice. Hence the inverse relation of $f(x)$ is not a function. Alternatively, the inverse relation is $x = y^2 - 9$, which on solving for y gives

$$y = \sqrt{x+9} \quad \text{or} \quad -\sqrt{x+9},$$

which is not unique, so the inverse relation is not a function.

Restricting the domain so the inverse is a function

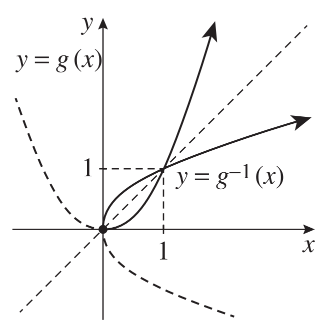
When a function is many-to-one, that is, its inverse is not a function, it is often convenient to restrict the domain of the function so that this new restricted function has an inverse function.

The clearest example is the squaring function $f(x) = x^2$, whose inverse relation is not a function because, for example, 49 has two square roots, 7 and -7.

If, however, we restrict the domain of $f(x) = x^2$ to $x \geq 0$ and define a new restricted function

$$g(x) = x^2, \quad \text{where } x \geq 0,$$

then the new restricted function $g(x)$ is one-to-one, and thus has an inverse function. This inverse function has equation $g^{-1}(x) = \sqrt{x}$, where as explained earlier, the symbol $\sqrt{\ }$ means 'take the positive square root (or zero)'.



To the right are the graphs of the restricted function and its inverse function, with the unrestricted function and its inverse relation shown dotted. These ideas will be developed a great deal further in the last chapter, where Section 17A generalises the ideas so that Section 17B can apply them to constructing the inverse trigonometric functions.

Exercise 5G

FOUNDATION

1 Let $f(x) = 2x - 8$ and $g(x) = \frac{1}{2}x + 4$.

a Verify by substitution that:

i $g(f(5)) = 5$

ii $f(g(5)) = 5$

iii $g(f(x)) = x$

iv $f(g(x)) = x$

b What do you conclude about the functions $f(x)$ and $g(x)$?

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2 Each pair of functions $f(x)$ and $g(x)$ are known to be mutual inverses. Show in each case that $f\big(g(2)\big) = 2$ and $g\big(f(2)\big) = 2$, and that $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$.

a $f(x) = x + 13$ and $g(x) = x - 13$

b $f(x) = 7x$ and $g(x) = \frac{1}{7}x$

c $f(x) = 2x + 6$ and $g(x) = \frac{1}{2}(x - 6)$

d $f(x) = x^3 - 6$ and $g(x) = \sqrt[3]{x + 6}$

3 **a** Find the inverse function $f^{-1}(x)$ of $f(x) = 2x + 5$. Begin 'Let $y = 2x + 5$', then swap x and y to find the inverse, then solve for y, then write down the equation of $f^{-1}(x)$ where $g = f^{-1}(x)$.

b Check your answer by calculating $g\big(f(x)\big)$ and $f\big(g(x)\big)$.

c Similarly find the inverse functions of each function, and check each answer.

i $f(x) = 4 - 3x$

ii $f(x) = x^3 - 2$

iii $f(x) = \dfrac{1}{x - 5}$

DEVELOPMENT

4 Explain whether the inverse relation is a function by testing whether it is one-to-one. If it is a function, find $f^{-1}(x)$, specifying its domain. Then verify the two identities $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(x)\big) = x$.

a $f(x) = x^2$

b $f(x) = \sqrt{x}$

c $f(x) = x^4$

d $f(x) = x^3 + 1$

e $f(x) = 9 - x^2$

f $f(x) = 9 - x^2, x \ge 0$

g $f(x) = 3^{-x^2}$

h $f(x) = \dfrac{1 - x}{3 + x}$

i $f(x) = x^2, x \le 0$

j $f(x) = x^2 - 2x, x \ge 1$

k $f(x) = x^2 - 2x, x \le 1$

l $f(x) = \dfrac{x + 1}{x - 1}$

5 Sketch on separate graphs:

a $y = -x^2$

b $y = -x^2$, for $x \ge 0$

Draw the inverse of each on the same graph, then comment on the similarities and differences between parts **a** and **b**.

6 **a** What is the gradient of the line $y = ax + b$?

b Write down the inverse relation of $y = ax + b$.

c What are the conditions for this inverse relation to be a function?

d When the inverse is a function, solve it for y, find its gradient, and explain why the gradients of the function and its inverse both have the same sign.

e Give an argument using reflection in the line $y = x$ for your answers in part **c**.

7 **a** Let $f(x)$ and $g(x)$ be one-to-one functions, and let $h(x) = g\big(f(x)\big)$. Show that the inverse function of $h(x)$ is $h^{-1}(x) = f^{-1}\big(g^{-1}(x)\big)$.

b Find the inverse function of $h(x) = \dfrac{1}{x - 3}$.

c Express $h(x)$ as the composition of the reciprocal function and a linear function, and hence use part **a** to find its inverse function.

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8 Let $f(x)$ be the restricted function $f(x) = 3x - 2$, where $1 \le x \le 4$.

a Find the inverse function $f^{-1}(x)$, being careful to add its restriction.

b Show that $f^{-1}\big(f(x)\big)$ and $f\big(f^{-1}(x)\big)$ are both identity functions, and find their respective domains. A sketch may make the situation clearer.

ENRICHMENT

9 Suggest restrictions on the domains of each function to produce a new function whose inverse is also a function (there may be more than one answer). Draw the restricted function and its inverse.

a $y = -\sqrt{4 - x^2}$

b $y = \dfrac{1}{x^2}$

c $y = x^3 - x$

d $y = \sqrt{x^2}$

10 **a** Let $f(x) = ax + b$ and $g(x) = \alpha x + \beta$. Find $g\big(f(x)\big)$, and hence prove that the condition for $f(x)$ and $g(x)$ to be inverse functions is

$$\alpha = a^{-1} \quad \text{and} \quad \beta = -a^{-1}b.$$

b Find three linear functions $f(x)$, $g(x)$ and $h(x)$, none of whose graphs pass through the origin, with no two graphs parallel, such that $h\Big(g\big(f(x)\big)\Big)$ is the identity function.

11 Does the empty function have an inverse function, and if so, what is it?

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5H Defining functions and relations parametrically

There is an ingenious way of handling curves by making each coordinate a function of a single variable, called a *parameter*. Each point on the curve is specified by a single number, rather than by a pair of coordinates.

The section uses some trigonometry that is only reviewed in Chapter 6. Angles of any magnitude and the Pythagorean identities are needed. Readers may prefer to delay studying this last section until Chapter 6 is completed.

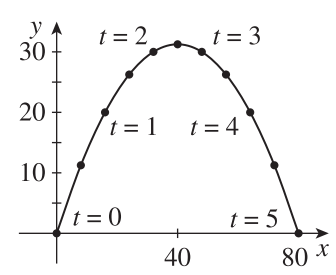
An example of parameters

SDB hits a six at the Sydney Cricket Ground.

- A cameraman in a distant stand, exactly behind the path of the ball, sees the ball rise and fall, with its height y in metres given by $y = -5t^2 + 25t$, where t is time in seconds after the strike.
- An observer filming the shot from a helicopter high above sees the ball move across the ground, with distance x from the batsman given by $x = 16t$.

Between them, the cameraman in the stand and the observer in the helicopter have a complete record of the ball's flight. (Ignore parallax here, and regard both observers as 'distant'). From their results, we can draw up a table of values of the (x, y) position of the ball at each time t:

t	0	1	2	$2\frac{1}{2}$	3	4	5
x	0	16	32	40	48	64	80
y	0	20	30	$31\frac{1}{4}$	30	20	0



The resulting (x, y)-graph shows the path of the ball, and each point on the graph can be labelled with the corresponding value of time t.
This variable t is called a *parameter*, and the equations

$$x = 16t, \quad y = -5t^2 + 25t$$

are called *parametric equations of the curve*.
It is possible to *eliminate the parameter t* from these two equations.

Solving the first equation for t, $\quad t = \frac{x}{16}$,

then substituting into the second, $\quad y = -\frac{5}{256}x^2 + \frac{25}{16}x$,

$$y = \frac{-5x^2 + 400x}{256}$$

and factoring displays the zeroes, $\quad y = \dfrac{5x(80 - x)}{256}$.

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25 PARAMETERS

- A curve in the (x, y)-plane may be *parametrically defined*, meaning that x and y are given as functions of a third variable t called a *parameter*.
  — These two equations for x and y in terms of t are called *parametric equations* of the curve.
- In many situations, the parameter t may be *eliminated* to give a single equation in x and y for the curve.
  — The single equation in x and y is called the *Cartesian equation* of the curve.

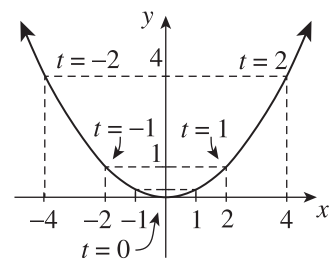
The letter t is often used for the parameter because it stands for 'time'. Other letters are often used, however, particularly θ and ϕ for an angle, and p and q.

Examples of parametrisation — a parabola

We can reverse the process of eliminating the parameter, and *parametrise* familiar curves. Some straightforward parametrisations are given below of a parabola, a circle, and a rectangular hyperbola. The parabola $x^2 = 4y$ can be parametrised by the pair of equations

$$x = 2t \quad \text{and} \quad y = t^2$$

because elimination of t gives $x^2 = 4y$. The variable point $(2t, t^2)$ now runs along the whole curve as the parameter t takes all the real numbers as its values:



t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	-4	-2	-1	0	1	2	4
y	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4

The sketch shows the curve with the seven plotted points labelled by their parameter. The curve can be seen as a 'bent and stretched number line.'

A parametrisation of the circle

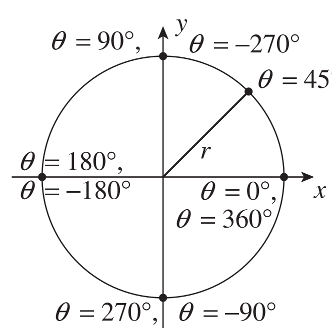
The circle $x^2 + y^2 = r^2$ can be parametrised using trigonometric functions by

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

This parametrisation uses the Pythagorean identity

$$r^2\cos^2\theta + r^2\sin^2\theta = r^2.$$

Notice from the table of values below that with these equations, each parameter corresponds to just one point, but each point corresponds to infinitely many different values of the parameter, all differing by multiples of 360°:



θ	$-360°$	$-270°$	$-180°$	$-90°$	0	$45°$	$90°$	$180°$	$270°$	$360°$
x	r	0	$-r$	0	r	$\frac{1}{2}r\sqrt{2}$	0	$-r$	0	r
y	0	r	0	$-r$	0	$\frac{1}{2}r\sqrt{2}$	r	0	$-r$	0

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In our previous examples, the map from parameters to points was always one-to-one, but in this parametrisation of the circle, the map is many-to-one.

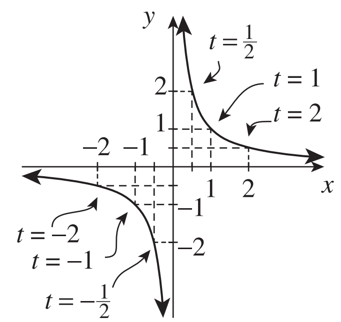
The circle is a relation, but not a function. The great advantage of parametrising the circle is that it is now described by a pair of *functions*, which in many situations are easier to handle than the original relation.

A parametrisation of the rectangular hyperbola

The rectangular hyperbola $xy = 1$ can be parametrised algebraically by

$$x = t \quad \text{and} \quad y = \frac{1}{t}.$$

There is a one-to-one correspondence between the points on the curve and the real numbers, with the one exception that $t = 0$ does not correspond to any point:



t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	$-\frac{1}{2}$	-1	-2	*	2	1	$\frac{1}{2}$

Example 22 — 5H

Find the Cartesian equations of the curves defined by the parametric equations:

a $x = 4p,\ y = p^2 + 1$

b $x = \sec\theta,\ y = \sin\theta$

Describe part **a** geometrically.

SOLUTION

a From the first, $p = \frac{1}{4}x$,

and substituting into the second,

$$y = \tfrac{1}{16}x^2 + 1$$
$$x^2 = 16(y - 1),$$

which is a parabola with vertex $(0, 1)$, and concave up.

b Squaring, $x^2 = \sec^2\theta$,

and $y^2 = \sin^2\theta$

$= 1 - \cos^2\theta$,

so $y^2 = 1 - \dfrac{1}{x^2}$

$x^2(1 - y^2) = 1.$

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Exercise 5H

FOUNDATION

Note: Some questions in this exercise use trigonometry reviewed in Chapter 6, in particular, angles of any magnitude and the Pythagorean identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

1 **a** Complete the table below for the curve $x = 4t$, $y = 2t^2$ and sketch its graph.

t	-6	-4	-2	-1	0	1	2	4	6
x									
y									

b Eliminate the parameter to find the Cartesian equation of the curve.

c The curve is a parabola. What value of t gives the coordinates of the vertex?

2 Repeat the previous question for the curve $x = t$, $y = \frac{1}{2}t^2$.

3 **a** Show that the point $\left(cp, \frac{c}{p}\right)$ lies on the hyperbola $xy = c^2$, where c is a constant.

b Complete the table of values below for $x = 2p$, $y = \dfrac{2}{p}$ and sketch the graph.

p	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
x											
y											

c Explain what happens as $p \to \infty$, $p \to -\infty$, $p \to 0^+$ and $p \to 0^-$.

4 **a** Show that the point $(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

b **i** Complete a table of values for the curve $x = 4\cos\theta$, $y = 3\sin\theta$, taking the values $\theta = 0°, 30°, 60°, 90°, 120°, \ldots, 360°$.

ii Sketch the curve and state its Cartesian equation.

DEVELOPMENT

5 Consider the parametric equations $y = 2t - 1$ and $x = t - 2$.

a Complete the table below.

b Explain from the table why the graph is a line.

c From the table, find the y-intercept and the gradient.

d Eliminate t to find the Cartesian equation, and check it from part **c**.

t	-2	-1	0	1	2
x					
y					

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6 **a** The parametric equations $x = 2t - 3$ and $y = 6t - 5$ represent a line.

i Find the points A and B with parameters $t = 0$ and $t = 1$, and hence find the gradient of the line.

ii Find the value of t that makes $x = 0$, and hence find the y-intercept.

iii Check your answers by eliminating t to form a Cartesian equation.

b Repeat these steps for the lines:

i $x = 2t - 3$ and $y = 3t - 2$

ii $x = at + b$ and $y = ct + d$

7 Eliminate the parameter and hence find the Cartesian equation of the curve.

a $x = 3 - p, y = 2p + 1$

b $x = 1 + 2\tan\theta, y = 3\sec\theta - 4$

c $x = p + \dfrac{1}{p}, y = p^2 + \dfrac{1}{p^2}$

d $x = \cos\theta + \sin\theta, y = \cos\theta - \sin\theta$

8 **a** Show that $x = a + r\cos\theta$ and $y = b + r\sin\theta$ define a circle with centre (a, b) and radius r.

b Hence sketch a graph of the curve $x = 1 + 2\cos\theta, y = -3 + 2\sin\theta$.

9 Show by elimination that $x = \dfrac{t^2 - 1}{t^2 + 1}$ and $y = \dfrac{2t}{t^2 + 1}$ almost represent the unit circle $x^2 + y^2 = 1$. What point is missing?

10 Different parametric representations may result in the same Cartesian equation. The graphical representation, however, may be different.

a Find the Cartesian equation of the curve $x = 2 - t, y = t - 1$ and sketch its graph.

b Find the Cartesian equation of the curve $(\sin^2 t, \cos^2 t)$. Explain why $0 \le x \le 1$ and $0 \le y \le 1$ and sketch a graph of the curve.

c Find the Cartesian equation of the curve $x = 4 - t^2, y = t^2 - 3$. Explain why $x \le 4$ and $y \ge -3$ and sketch a graph of the curve.

11 Find the Cartesian equation of the curve $x = 3 + r\cos\theta, y = -2 + r\sin\theta$, and describe it geometrically if:

a r is constant and θ is variable,

b θ is constant and r is variable.

ENRICHMENT

12 **a** Show by elimination that $x = \dfrac{2t + 1}{2t^2 + 2t + 1}$ and $y = \dfrac{2t^2 + 2t}{2t^2 + 2t + 1}$ almost represent the unit circle $x^2 + y^2 = 1$. What point is missing?

b Is this map from parameters to points one-to-one or many-to-one?

13 **a** Explain why the parametric equations $x = \cos t, y = \sin t, z = t$ describe a spiral.

b Is this map from parameters to points one-to-one or many-to-one?

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14 **a** Show that the point $(a \sec \theta, b \tan \theta)$ lies on the curve $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

b Complete a table of values for the curve $x = 4 \sec \theta$, $y = 3 \tan \theta$, where $0^\circ \le \theta \le 360^\circ$. What happens when $\theta = 90^\circ$ and $\theta = 270^\circ$?

c Sketch the curve (it has two asymptotes) and state its Cartesian equation.

d Is this map from parameters to points one-to-one or many-to-one?

15 After finding the Cartesian equation, sketch the curve whose parametric equations are

$$x = \tfrac{1}{2}\left(2^t + 2^{-t}\right) \quad \text{and} \quad y = \tfrac{1}{2}\left(2^t - 2^{-t}\right).$$

16 A relation is defined parametrically by $x = f(t)$ and $y = g(t)$.

a What transformation of the relation occurs when t is replaced by $-t$ if:

i $f(t)$ and $g(t)$ are both even,

ii $f(t)$ and $g(t)$ are both odd,

iii $f(t)$ is even and $g(t)$ is odd,

iv $f(t)$ is odd and $g(t)$ is even.

b What is the relationship between this relation and the relation defined by $x = g(t)$ and $y = f(t)$?

c Where is the graph of the relation $x = |f(t)|$ and $y = |g(t)|$ located?

d Where is the graph of the relation $x = f(t)$ and $y = f(t)$ located?

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Chapter 5 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 5 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

Note: Graphing software could be very helpful in this exercise.

1 Solve each inequation, and graph the solution on a number line.

a $17 - 3x \geq 5$ **b** $-15 \leq 9 - 4x < 25$ **c** $12 - \frac{1}{2}x < \frac{1}{3}x + 22$

2 Solve each inequation by sketching the associated parabola.

a $x^2 - 8x + 15 \leq 0$ **b** $6x < x^2$ **c** $3x^2 > 5x + 12$

3 Solve each inequation.

a $|x| < 3$ **b** $|x + 2| \geq 4$ **c** $|2x - 5| \leq 11$

4 Solve each inequation by multiplying both sides by the square of the denominator.

a $\frac{5}{x} > 1$ **b** $\frac{3}{x - 3} \leq 1$ **c** $\frac{x - 2}{x + 1} \geq 4$

5 Consider the function $f(x) = x(x + 2)(x - 3)$.

a Write down the zeroes of the function, and draw up a table of signs.

b Copy and complete: '$f(x)$ is positive for . . ., and negative for . . .'

c Write down the solution of the inequation $x(x + 2)(x - 3) \leq 0$.

d Sketch the graph of the function to confirm these results.

6 Consider the function $y = (1 - x)(x - 3)^2$.

a Write down the zeroes of the function and draw up a table of signs.

b Hence solve the inequation $(1 - x)(x - 3)^2 \geq 0$.

c Confirm the solution by sketching a graph of the function.

7 Solve each inequation in question 4 using the table-of-signs method. First move everything to the left-hand side, then make the LHS into a single fraction, then identify its zeroes and discontinuities, then draw up a table of signs, then read the solution from the table.

8 Consider the linear function $f(x) = x - 2$.

a Sketch $y = f(x)$, clearly indicating the x- and y-intercepts.

b Also show on your sketch the points where $y = 1$ and $y = -1$.

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c Hence sketch the graph of $y = \dfrac{1}{f(x)}$ on the same number plane.

d Write down the equation of the vertical asymptote of $y = \dfrac{1}{f(x)}$, then copy and complete the sentence, 'As $x \to 2^-$, $y \to \ldots$, and as $x \to 2^+$, $y \to \ldots$'

9 Consider the quadratic function $f(x) = 3 - x^2$.

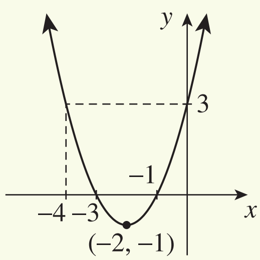
a Sketch $y = f(x)$, clearly indicating the x- and y-intercepts.

b Also show on your sketch the points where $y = 1$ and $y = -1$.

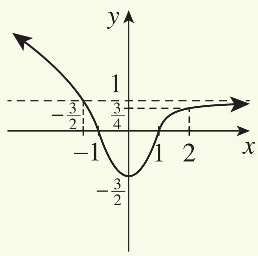
c Hence sketch the graph of $y = \dfrac{1}{f(x)}$ on the same number plane.

10 Sketch the reciprocal of each function graphed below, showing all the important features.

a



b



11 **a** Find the equations of the vertical asymptotes of each function.

i $y = \dfrac{2}{x + 1}$ **ii** $y = \dfrac{2x + 1}{x - 2}$ **iii** $y = \dfrac{4x}{x^2 - 25}$

b In part **iii** above, identify the zeroes and discontinuities and draw up a table of signs. Then describe the behaviour of the curve near each vertical asymptote by copying and completing, 'As $x \to 5^-$, $y \to \ldots$, and as $x \to 5^+$, $y \to \ldots$'

12 Consider the function $y = \dfrac{2x}{x^2 - 1}$.

a Show that it is an odd function.

b Find the zeroes and discontinuities, and draw up a table of signs.

c Identify any vertical and horizontal asymptotes.

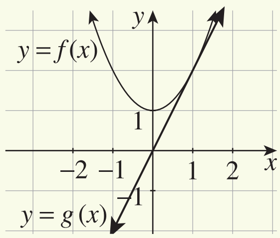
d Hence sketch the graph.

13 The graphs of $y = f(x)$ and $y = g(x)$ are sketched to the right. On separate number planes, sketch:

a $y = f(x) + g(x)$

b $y = f(x) - g(x)$

c $y = f(x) \times g(x)$



14 The graphs of $y = f(x)$ and $y = g(x)$ are sketched to the right.

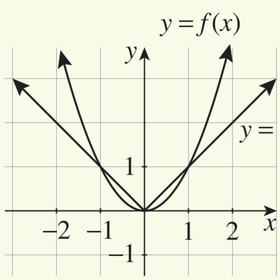
a On separate number planes, sketch:

i $y = f(x) + g(x)$

ii $y = f(x) - g(x)$

iii $y = f(x) \times g(x)$

b What symmetry should be evident in your sketches?



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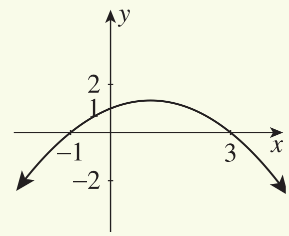
Review

15 The graph of $y = f(x)$ is sketched to the right. On four separate number planes, sketch:



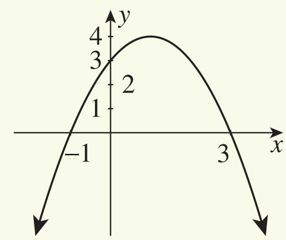
a $y = |f(x)|$

b $y = f(|x|)$

c $|y| = f(x)$

d $y = |f(|x|)|$

16 The quadratic function $y = f(x)$ has been sketched to the right. On two separate number planes, sketch:

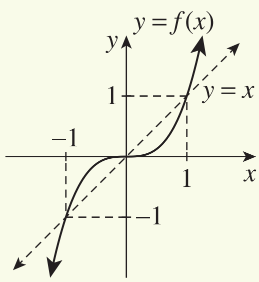


a $y = \sqrt{f(x)}$

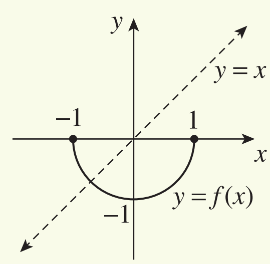
b $y^2 = f(x)$

17 Copy each diagram below, then sketch the inverse relation of the function. Also state whether or not the inverse relation is a function.

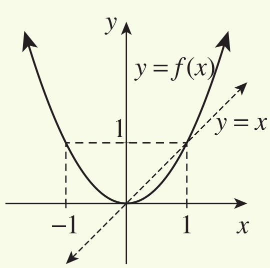
a



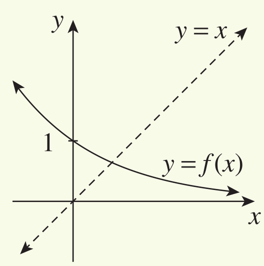
b



c



d



18 Find the equation of the inverse function for each function.

a $y = 5 - 3x$ **b** $y = \dfrac{5}{x - 3}$ **c** $y = \dfrac{5x}{x - 3}$ **d** $y = x^3 + 5$

19 Find the inverse function $f^{-1}(x)$ of each function, and then confirm algebraically that $f\left(f^{-1}(x)\right) = x$ and $f^{-1}\left(f(x)\right) = x$.

a $f(x) = \frac{1}{2}x + 4$ **b** $f(x) = (x + 2)^3$ **c** $f(x) = \dfrac{3}{x} - 6$

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20 A line is defined parametrically by the equations $x = t + 2$ and $y = 2t + 6$.

a Copy and complete the table below.

t	-5	-4	-3	-2	-1	0	1
x							
y							

b Use the table to sketch the line, marking each point with its t-value.

c Eliminate the parameter t to find the Cartesian equation of the line.

21 A parabola is defined parametrically by the equations $x = \frac{1}{2}t$ and $y = \frac{1}{4}t^2$.

a Copy and complete the table below.

t	-6	-4	-2	-1	0	1	2	4	6
x									
y									

b Use the table to sketch the parabola, marking each point with its t-value.

c Eliminate the parameter t to find the Cartesian equation of the parabola.

22 A curve is defined parametrically by the equations $x = \cos\theta - 1$, $y = \sin\theta + 1$.

a Use the identity $\cos^2\theta + \sin^2\theta = 1$ to find the Cartesian equation of the curve.

b Describe the curve, and then sketch it.

23 A curve is represented parametrically by the equations $x = 2t$, $y = \dfrac{1}{t + 1}$.
Find the Cartesian equation of the curve and hence sketch it.

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6 Trigonometry

Trigonometry is important in modern science principally because the graphs of the sine and cosine functions are waves. Waves appear everywhere in the natural world, for example as water waves, as sound waves, and as electromagnetic waves in various forms — radio waves, heat waves, light waves, ultraviolet radiation, X-rays and gamma rays. In quantum mechanics, a wave is associated with every particle.

Trigonometry began, however, in classical times as the study of the relationships between angles and lengths in geometrical figures. It used the relationships between the angles and the side lengths in a triangle, and its name comes from the Greek words *trigonon*, 'triangle', and *metron*, 'measure'. This introductory chapter establishes the geometric basis of the trigonometric functions and their graphs, developing them from the geometry of triangles and circles.

Some of this chapter will be new to many readers, in particular the extension of the trigonometric functions to angles of any magnitude, the graphs of these functions, trigonometric identities and equations, and three-dimensional problems.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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6A Trigonometry with right-angled triangles

This section and the next will review the definitions of the trigonometric functions for acute angles, and apply them to problems involving right-angled triangles.

Pythagoras' theorem and similarity

You will know from your previous study that the trigonometry of triangles begins with these two fundamental ideas.

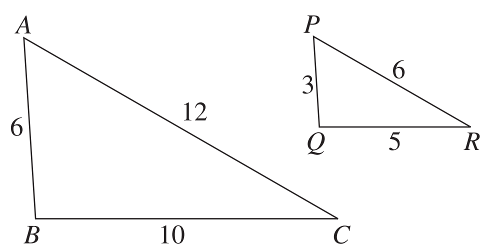
First, Pythagoras' theorem tells us how to find the third side of a right-angled triangle. The theorem is the best-known theorem in all of mathematics, and has been mentioned several times already in earlier chapters. Here is what it says:

The square on the hypotenuse of a right-angled triangle is the sum of the squares on the other two sides.

The diagrams below provide a very simple proof. Can you work out how the four shaded triangles have been pushed around inside the square to prove the theorem?

Secondly, similarity is required even to define the trigonometric functions, because each function is defined as the ratio of two sides of a triangle.

- Two figures are called *congruent* if one can be obtained from the other by translations, rotations and reflections.
- They are called *similar* if enlargements are allowed as well.



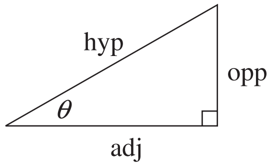
In two similar figures,

matching angles are equal, and matching sides are in ratio.

The trigonometric functions for acute angles

Let θ be any acute angle, that is, $0° < \theta < 90°$. Construct a right-angled triangle with an acute angle θ, and label the sides:

hyp — the *hypotenuse*, the side opposite the right angle,
opp — the side *opposite* the angle θ,
adj — the third side, *adjacent* to θ but not the hypotenuse.



1 THE TRIGONOMETRIC FUNCTIONS FOR AN ACUTE ANGLE θ

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\operatorname{cosec}\theta = \frac{\text{hyp}}{\text{opp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

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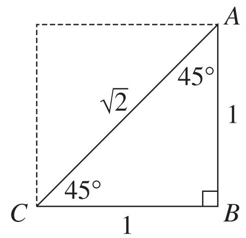
Any two triangles with angles of 90° and θ are similar, by the AA similarity test. Hence the values of the six trigonometric functions are the same, whatever the size of the triangle. The full names of the six trigonometric functions are:

sine, cosine, tangent, cosecant, secant, cotangent.

Question 19 in the next exercise gives some clues about these names, and about the way in which the functions were originally defined.

Special angles

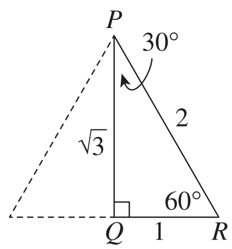
The values of the trigonometric functions for the three acute angles 30°, 45° and 60° can be calculated exactly, using half a square and half an equilateral triangle, and applying Pythagoras' theorem.



Take half a square with side length 1.

The resulting right-angled triangle ABC has two angles of 45°.

By Pythagoras' theorem, the hypotenuse AC has length $\sqrt{2}$.



Take half an equilateral triangle with side length 2 by dropping an altitude.

The resulting right-angled $\triangle PQR$ has angles of 60° and 30°.

By Pythagoras' theorem, $PQ = \sqrt{3}$.

Applying the definitions in Box 1 gives the values in the table below.

2 A TABLE OF EXACT VALUES

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$

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Trigonometric functions of other angles

The calculator is usually used to approximate trigonometric functions of other angles. Alway check first that the calculator is in degrees mode — there is usually a key labelled mode or DRG or something similar. Later, you will be swapping between degrees mode and radian mode (ignore the 'grads' unit).

Make sure also that you can enter angles in degrees, minutes and seconds, and that you can convert decimal output to degrees, minutes and seconds — there is usually a key labelled ° ' " or DMS or something similar. Check that you can perform these two procedures:

$$\sin 53°47' \doteqdot 0.8068 \qquad \text{and} \qquad \sin\theta = \tfrac{5}{8},\ \text{so}\ \theta \doteqdot 38°41'.$$

The reciprocal trigonometric functions

The functions cosecant, secant and cotangent can mostly be avoided by using the sine, cosine and tangent functions.

3 AVOIDING THE RECIPROCAL TRIGONOMETRIC FUNCTIONS

The three reciprocal trigonometric functions can mostly be avoided because

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

Finding an unknown side of a right triangle

Calculators only have the sine, cosine and tangent functions, so it is best to use only these three functions in problems.

4 TO FIND AN UNKNOWN SIDE OF A RIGHT-ANGLED TRIANGLE

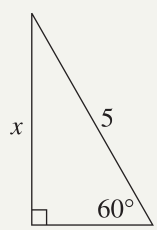
1 Start by writing $\dfrac{\text{unknown side}}{\text{known side}} = \ldots\ldots$ (place the unknown at top left)

2 Complete the RHS with sin, cos or tan, or the reciprocal of one of these.

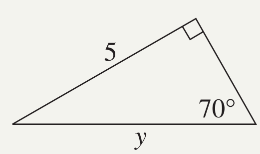
Example 1 **6A**

Find the side marked with a pronumeral in each triangle. Give the answer in exact form if possible, or else correct to five significant figures.

a



b



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SOLUTION

a $\dfrac{x}{5} = \sin 60^\circ \quad \left(\dfrac{\text{opposite}}{\text{hypotenuse}}\right)$

$\boxed{\times 5}$ $x = 5\sin 60^\circ$

$= \dfrac{5\sqrt{3}}{2}$

b $\dfrac{y}{5} = \dfrac{1}{\sin 70^\circ} \quad \left(\dfrac{\text{hypotenuse}}{\text{opposite}}\right)$

$\boxed{\times 5}$ $y = \dfrac{5}{\sin 70^\circ}$

$\doteqdot 5.3209$

Finding an unknown angle of a right triangle

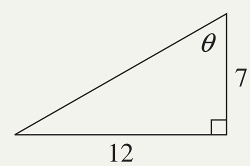
As before, use only sin, cos and tan.

5 FINDING AN UNKNOWN ANGLE, GIVEN TWO SIDES OF A RIGHT-ANGLED TRIANGLE

Work out from the known sides which one of $\cos\theta$, $\sin\theta$ or $\tan\theta$ to use.

Example 2 6A

Find θ in the triangle drawn to the right.



SOLUTION

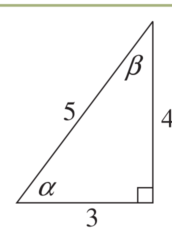
The given sides are the opposite and the adjacent sides, so $\tan\theta$ is known.

$$\tan\theta = \frac{12}{7} \quad \left(\frac{\text{opposite}}{\text{adjacent}}\right)$$

$$\theta \doteqdot 59^\circ 45'.$$

Exercise 6A

FOUNDATION



1 From the diagram to the right, write down the values of:

a $\cos\alpha$ **b** $\tan\beta$
c $\sin\alpha$ **d** $\cos\beta$
e $\sin\beta$ **f** $\tan\alpha$

2 Use your calculator to find, correct to four decimal places:

a $\sin 24^\circ$ **b** $\cos 61^\circ$ **c** $\tan 35^\circ$ **d** $\sin 87^\circ$
e $\tan 2^\circ$ **f** $\cos 33^\circ$ **g** $\sin 1^\circ$ **h** $\cos 3^\circ$

3 Use your calculator to find, correct to four decimal places:

a $\tan 57^\circ 30'$ **b** $\cos 32^\circ 24'$ **c** $\tan 78^\circ 40'$ **d** $\cos 16^\circ 51'$
e $\sin 43^\circ 6'$ **f** $\sin 5^\circ 50'$ **g** $\sin 8'$ **h** $\tan 57'$

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4 Use your calculator to find the acute angle θ, correct to the nearest degree, if:

a $\tan\theta = 4$ **b** $\cos\theta = 0.7$ **c** $\sin\theta = \frac{1}{5}$ **d** $\sin\theta = 0.456$

e $\cos\theta = 2$ **f** $\cos\theta = \frac{7}{9}$ **g** $\tan\theta = 1\frac{3}{4}$ **h** $\sin\theta = 1.1$

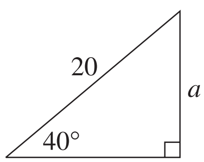
5 Use your calculator to find the acute angle α, correct to the nearest minute, if:

a $\cos\alpha = \frac{3}{4}$ **b** $\tan\alpha = 2$ **c** $\sin\alpha = 0.1$

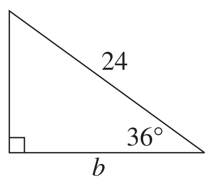
d $\tan\alpha = 0.3$ **e** $\sin\alpha = 0.7251$ **f** $\cos\alpha = \frac{7}{13}$

6 Find, correct to the nearest whole number, the value of each pronumeral.

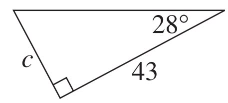
a



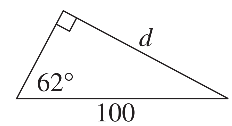
b



c

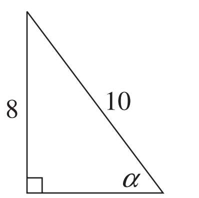


d

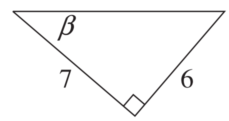


7 Find, correct to the nearest degree, the size of each angle marked with a pronumeral.

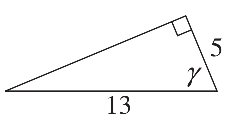
a



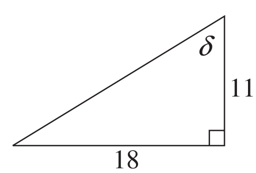
b



c



d

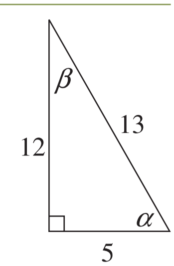


DEVELOPMENT

8 From the diagram to the right, write down the values of:

a $\sin\alpha$ **b** $\tan\beta$ **c** $\sec\beta$

d $\cot\alpha$ **e** $\operatorname{cosec}\alpha$ **f** $\sec\alpha$

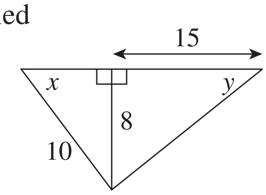


9 **a** Use Pythagoras' theorem to find the unknown side in each of the two right-angled triangles in the diagram to the right.

b Write down the values of:

i $\cos y$ **ii** $\sin x$ **iii** $\cot x$

iv $\operatorname{cosec} y$ **v** $\sec x$ **vi** $\cot y$



10 Draw the two special triangles containing the acute angles 30°, 60° and 45°. Hence write down the exact values of:

a $\sin 60°$ **b** $\tan 30°$ **c** $\cos 45°$

d $\sec 60°$ **e** $\operatorname{cosec} 45°$ **f** $\cot 30°$

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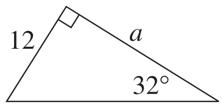
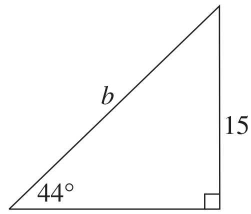
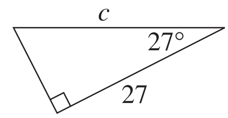
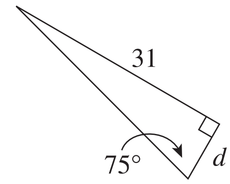
11 Find, correct to one decimal place, the lengths of the sides marked with pronumerals.

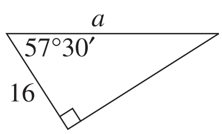
a


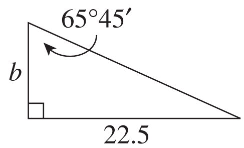
b


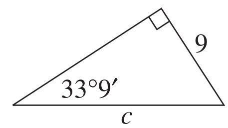
c


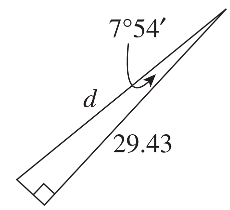
d


12 Find, correct to two decimal places, the lengths of the sides marked with pronumerals.

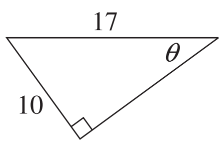
a


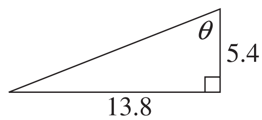
b


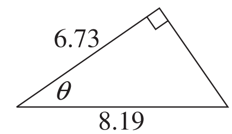
c


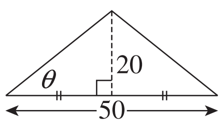
d


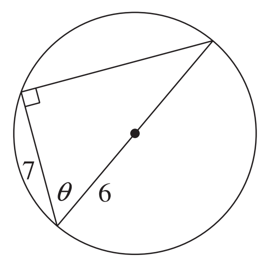
13 Find θ, correct to the nearest minute, in each diagram below.

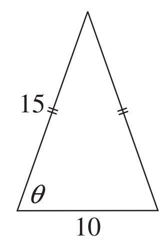
a


b


c


d


e


f


14 It is given that α is an acute angle and that $\tan \alpha = \dfrac{\sqrt{5}}{2}$.

a Draw a right-angled triangle (it may be of any size), one of whose angles is α, and show this information.

b Use Pythagoras' theorem to find the length of the unknown side.

c Hence write down the exact values of $\sin \alpha$ and $\cos \alpha$.

d Show that $\sin^2 \alpha + \cos^2 \alpha = 1$.

15 Suppose that β is an acute angle and $\sec \beta = \dfrac{\sqrt{11}}{3}$.

a Find the exact values of:

i $\operatorname{cosec} \beta$ **ii** $\cot \beta$

b Show that $\operatorname{cosec}^2 \beta - \cot^2 \beta = 1$.

16 Find, without using a calculator, the value of:

a $\sin 45° \cos 45° + \sin 30°$

b $\sin 60° \cos 30° - \cos 60° \sin 30°$

c $1 + \tan^2 60°$

d $\operatorname{cosec}^2 30° - \cot^2 30°$

17 Without using a calculator, show that:

a $1 + \tan^2 45° = \sec^2 45°$

b $2 \sin 30° \cos 30° = \sin 60°$

c $\cos^2 60° - \cos^2 30° = -\frac{1}{2}$

d $\dfrac{2 \tan 30°}{1 - \tan^2 30°} = \tan 60°$

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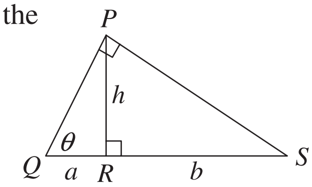
18 In the diagram to the right, $\triangle PQS$ is a right triangle, and PR is the altitude to the hypotenuse QS.



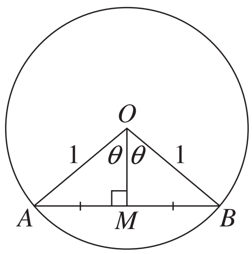
a Explain why $\angle RPS = \theta$.

b Find two expressions for $\tan\theta$.

c Hence show that $ab = h^2$.

19 **a** [An earlier definition of the sine function]

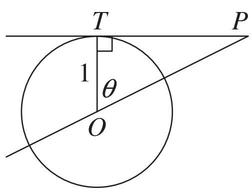
The value $\sin\theta$ used to be defined as the length of the *semichord* subtending an angle θ at the centre of a circle of radius 1.



In the diagram to the right, the chord AB of a circle of radius 1 subtends an angle 2θ at the centre O. We know that the perpendicular bisector of the chord passes through the centre O. Prove that $\sin\theta = \frac{1}{2}AB$.

b [The origin of the notations $\tan\theta$ and $\sec\theta$]

The word 'tangent' comes from the Latin *tangens* meaning 'touching', and a tangent to a circle is a line touching it at one point. The word 'secant' comes from the Latin *secans* meaning 'cutting', and a secant to a circle is a line cutting it at two points.

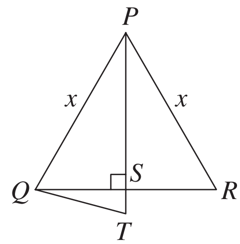


In the diagram to the right, P lies outside a circle of centre O and radius 1, and a tangent PT and secant PO have been drawn. Let the tangent PT subtend θ at the centre. We know that the radius OT is perpendicular to the tangent. Prove that

$PT = \tan\theta$ and $PO = \sec\theta$

ENRICHMENT

20 An equilateral triangle PQR has side length x, and PS is the perpendicular from P to QR. PS is produced to T so that $PT = x$.



a Show that $\angle PQT = 75°$ and hence that $\angle SQT = 15°$.

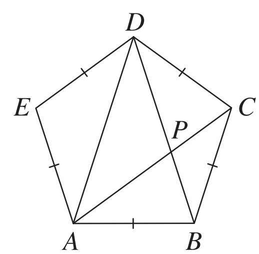
b Show that $QS = \frac{1}{2}x$ and that $PS = \frac{1}{2}x\sqrt{3}$.

c Show that $ST = \frac{1}{2}x(2 - \sqrt{3})$.

d Hence show that $\tan 15° = 2 - \sqrt{3}$.

21 [The regular pentagon and the exact value of $\sin 18°$]

The regular pentagon $ABCDE$ has sides of length 1 unit. The diagonals AD, BD and AC have been drawn, and the diagonals BD and AC meet at P.



a Find the size of each interior angle of the pentagon.

b Show that $\angle DAB = 72°$ and $\angle DAP = \angle BAP = 36°$.

c Show that the triangles DAB and ABP are similar.

d Let $BP = x$, and show that $AB = AP = DP = 1$ and $DA = \dfrac{1}{x}$.

e Show that $AD = \frac{1}{2}\left(\sqrt{5} + 1\right)$.

f Hence show that $\sin 18° = \cos 72° = \frac{1}{4}\left(\sqrt{5} - 1\right)$.

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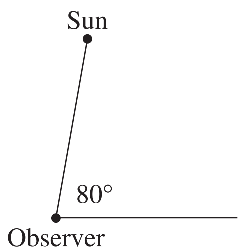
6B Problems involving right-angled triangles

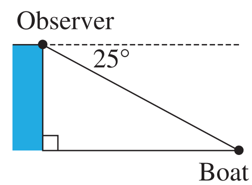
The trigonometry developed so far can be used to solve practical problems involving right-angled triangles. The examples below are typical of problems involving compass bearings and angle of elevation or depression.

When the figure has two or more triangles, always name the triangle that you are working in.

Angles of elevation and depression

Angles of elevation and depression are always measured from the horizontal, and are always acute angles.





The *angle of elevation* of the sun in the diagram above is 80°, because the angle at the observer between the sun and the horizontal is 80°.

For an observer on top of the cliff, the *angle of depression* of the boat is 25°, because the angle at the observer between boat and horizontal is 25°.

Example 3 — 6B

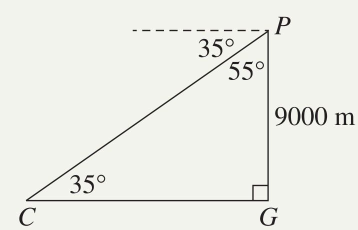
From a plane flying at 9000 metres above level ground, I can see a church at an angle of depression of 35°. Find how far the church is from the plane, correct to the nearest 100 metres:

a measured along the ground,

b measured along the line of sight.

SOLUTION

The situation is illustrated in the diagram by $\triangle PGC$.



a
$$\frac{GC}{9000} = \tan 55^\circ$$
$$GC = 9000 \tan 55^\circ$$
$$\doteqdot 12\,900 \text{ metres}$$

b
$$\frac{PC}{9000} = \frac{1}{\cos 55^\circ}$$
$$PC = \frac{9000}{\cos 55^\circ}$$
$$\doteqdot 15\,700 \text{ metres.}$$

Example 4 — 6B

A walker on level ground is 1 kilometre from the base of a 300-metre vertical cliff.

a Find, correct to the nearest minute, the angle of elevation of the top of the cliff.

b Find, correct to the nearest metre, the line-of-sight distance to the top of the cliff.

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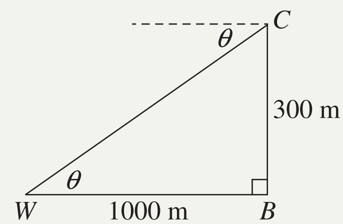
SOLUTION

The situation is illustrated by $\triangle CWB$ in the diagram to the right.



a $\tan\theta = \dfrac{CB}{WB}$

$= \dfrac{300}{1000}$

$= \dfrac{3}{10}$

$\theta \doteqdot 16°42'$.

b Using Pythagoras' theorem,

$CW^2 = CB^2 + WB^2$

$= 1000^2 + 300^2$

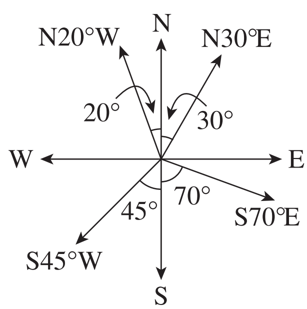
$= 1\,090\,000$

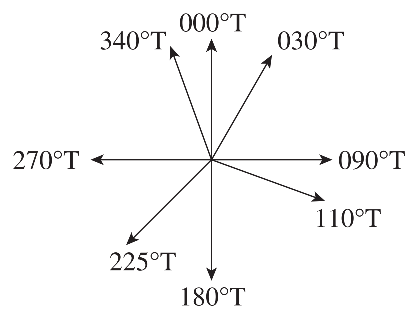
$CW \doteqdot 1044$ metres.

Compass bearings and true bearings

Compass bearings are based on north, south, east and west. Any other direction is specified by the deviation from north or south towards the east or west. The diagram to the left below gives four examples. Note that S45°W can also be written simply as SW (that is, south-west).

True bearings are measured clockwise from north (not anti-clockwise, as in the coordinate plane). The diagram to the right below gives the same four directions expressed as true bearings. Three digits are used, even for angles less than 100°.





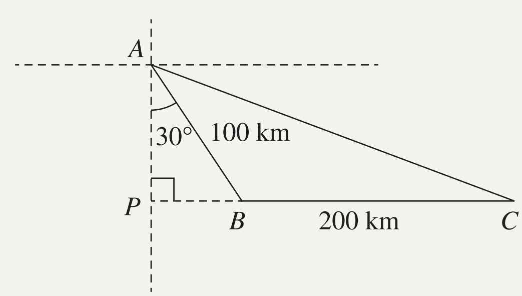
Example 5 — 6B

[Compass bearings and true bearings]

A plane flies at 400 km per hour, and flies from A to B in the direction S30°E for 15 minutes. The plane then turns sharply to fly due east for 30 minutes to C.

a Find how far south and east of A the point B is.

b Find the true bearing of C from A, correct to the nearest degree.



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SOLUTION

a The distances AB and BC are 100 km and 200 km respectively.
Working in $\triangle PAB$,

$$\frac{PB}{100} = \sin 30^\circ$$
$$PB = 100 \sin 30^\circ$$
$$= 50\text{ km},$$
and $$AP = 100 \cos 30^\circ$$
$$= 50\sqrt{3}\text{ km}.$$

b Using opposite over adjacent in $\triangle PAC$,

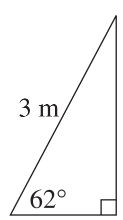
$$\tan \angle PAC = \frac{PC}{AP}$$
$$= \frac{50 + 200}{50\sqrt{3}}$$
$$= \frac{5}{\sqrt{3}}$$
$$\angle PAC \doteqdot 71^\circ.$$

Hence the bearing of C from A is about 109°T.

Exercise 6B

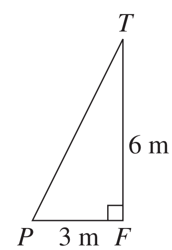
FOUNDATION

1



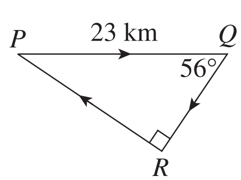
A ladder of length 3 metres is leaning against a wall and is inclined at 62° to the ground. How far does it reach up the wall? (Answer in metres correct to two decimal places.)

2



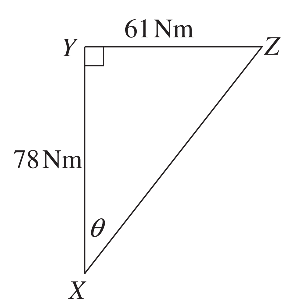
Determine, correct to the nearest degree, the angle of elevation of the top T of a 6-metre flagpole FT from a point P on level ground 3 metres from F.

3



Ben cycles from P to Q to R and then back to P in a road-race. Find, correct to the nearest kilometre, the distance he has ridden.

4



A ship sails 78 nautical miles due north from X to Y, then 61 nautical miles due east from Y to Z. Find θ, the bearing of Z from X, correct to the nearest degree.

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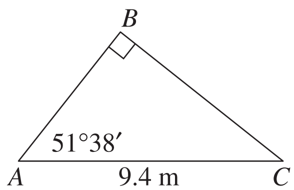
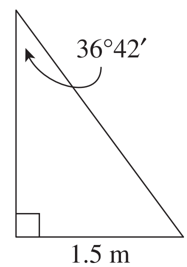
5



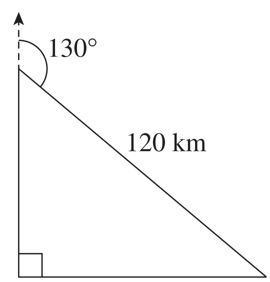
A tree snapped into two sections AB and BC in high winds and then fell. The section BA is inclined at $51°38'$ to the horizontal and AC is 9.4 metres long. Find the height of the original tree, in metres correct to one decimal place.

6



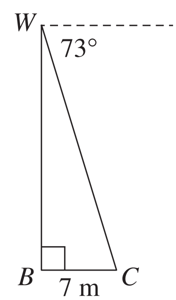
A ladder makes an angle of $36°42'$ with a wall, and its foot is 1.5 metres out from the base of the wall. Find the length of the ladder, in metres correct to one decimal place.

7



Eleni drives 120 km on a bearing of 130°T. She then drives due west until she is due south of her starting point. How far is she from her starting point, correct to the nearest kilometre?

8



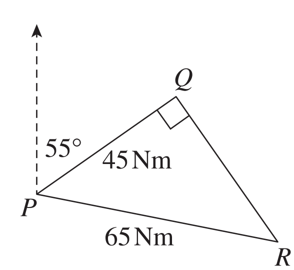
John is looking out a window W at a car C parked on the street below. If the angle of depression of C from W is 73° and the car is 7 metres from the base B of the building, find the height WB of the window, correct to the nearest metre.

DEVELOPMENT

9 A ladder of length 5 metres is placed on level ground against a vertical wall. If the foot of the ladder is 1.5 metres from the base of the wall, find, correct to the nearest degree, the angle at which the ladder is inclined to the ground.

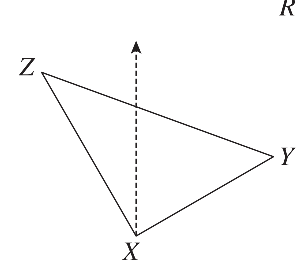
10 Find, correct to the nearest tenth of a metre, the height of a tower, if the angle of elevation of the top of the tower is $64°48'$ from a point on horizontal ground 10 metres from the base of the tower.

11 A boat is 200 metres out to sea from a vertical cliff of height 40 metres. Find, correct to the nearest degree, the angle of depression of the boat from the top of the cliff.



12 Port Q is 45 nautical miles from port P on a bearing of 055°T. Port R is 65 nautical miles from port P, and $\angle PQR = 90°$.

- **a** Find $\angle QPR$ to the nearest degree.
- **b** Hence find the bearing of R from P, correct to the nearest degree.



13 The bearings of towns Y and Z from town X are 060°T and 330°T respectively.

- **a** Show that $\angle ZXY = 90°$.
- **b** Given that town Z is 80 km from town X and that $\angle XYZ = 50°$, find, correct to the nearest kilometre, how far town Y is from town X.

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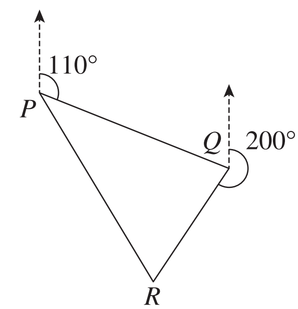
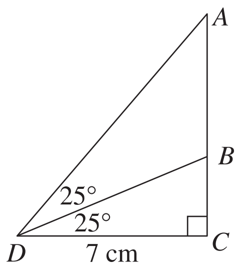
14 A ship leaves port P and travels 150 nautical miles to port Q on a bearing of 110°T. It then travels 120 nautical miles to port R on a bearing of 200°T.

a Explain why $\angle PQR = 90°$.

b Find, correct to the nearest degree, the bearing of port R from port P.

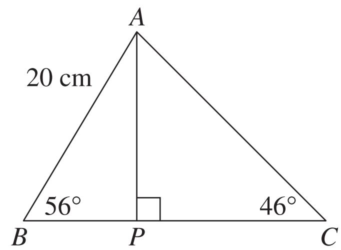


15 a



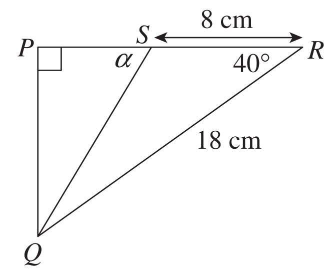
Show that $AC = 7 \tan 50°$ and $BC = 7 \tan 25°$, and hence find the length AB, correct to 1 mm.

b



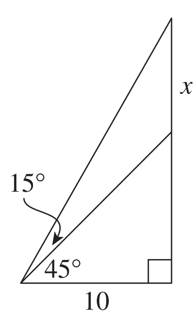
Show that $AP = 20 \sin 56°$, and hence find the length of PC, giving your answer correct to 1 cm.

c



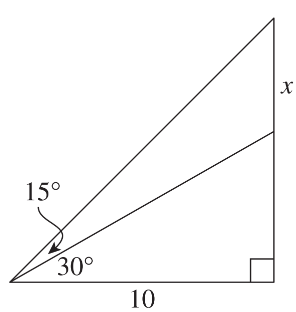
Show that $PR = 18 \cos 40°$, find an expression for PQ, and hence find the angle α, correct to the nearest minute.

16 a



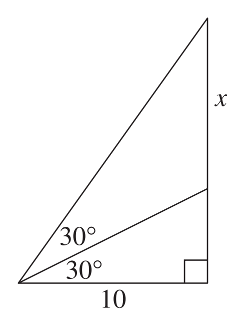
Show that $x = 10\left(\sqrt{3} - 1\right)$.

b



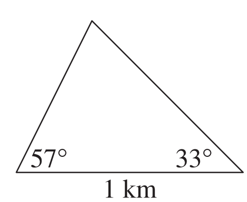
Show that $x = \frac{10}{3}\left(3 - \sqrt{3}\right)$.

c



Show that $x = \frac{20}{3}\sqrt{3}$.

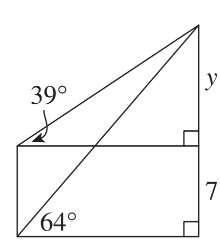
17 From the ends of a straight horizontal road 1 km long, a balloon directly above the road is observed to have angles of elevation of 57° and 33° respectively. Find, correct to the nearest metre, the height of the balloon above the road.



18 a Write down two equations involving x and y.

b By solving the equations simultaneously, show that

$$x = \frac{7}{\tan 64° - \tan 39°}.$$



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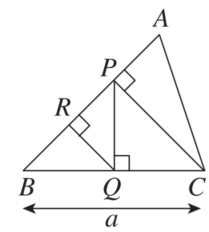
19 In triangle ABC, lines CP, PQ and QR are drawn perpendicular to AB, BC and AB respectively.

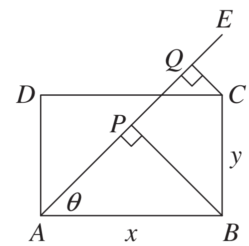
a Explain why $\angle RBQ = \angle RQP = \angle QPC$.

b Show that $QR = a \sin B \cos^2 B$.



20 In the diagram to the right, $ABCD$ is a rectangle in which $AB = x$ and $BC = y$. BP and CQ are drawn perpendicular to the interval AE, which is inclined at an angle θ to AB.

Show that $AQ = x \cos\theta + y \sin\theta$.



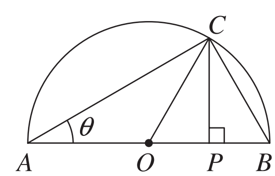
ENRICHMENT

21 In the diagram, O is the centre of the semicircle ACB, and P is the foot of the perpendicular from C to the diameter AB. Let $\angle OAC = \theta$.

a Show that $\angle POC = 2\theta$ and that $\angle PCB = \theta$.

b Using the two triangles $\triangle APC$ and $\triangle ABC$, show that $\sin\theta \cos\theta = \dfrac{PC}{AB}$.

c Hence show that $2 \sin\theta \cos\theta = \sin 2\theta$.

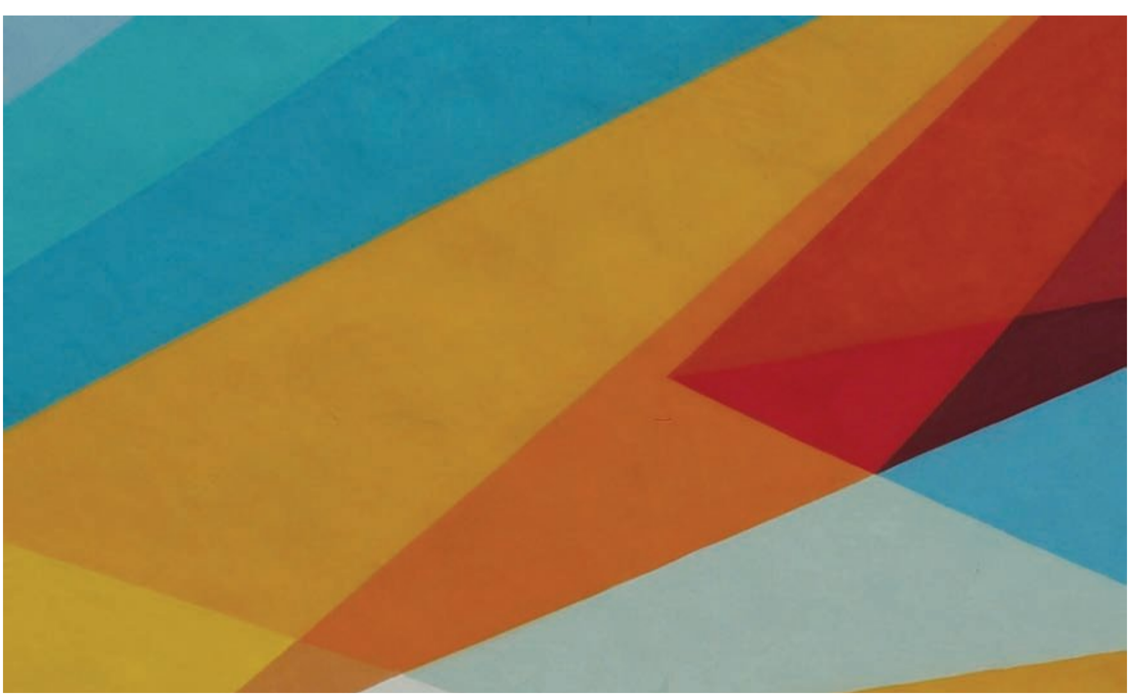


22 Using the same diagram as the previous question:

a Explain why $AP - PB = 2 \times OP$.

b Show that $\cos^2\theta - \sin^2\theta = \dfrac{AP}{AC} \times \dfrac{AC}{AB} - \dfrac{PB}{CB} \times \dfrac{CB}{AB}$.

c Hence show that $\cos^2\theta - \sin^2\theta = \cos 2\theta$.



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6C Three-dimensional trigonometry

Trigonometry is based on triangles, which are two-dimensional objects. When trigonometry is applied to a three-dimensional problem, the diagram must be broken up into a collection of triangles in space, and trigonometry applied to each triangle in turn. A carefully drawn diagram is always essential.

Two new ideas about angles are needed — the angle between a line and a plane, and the angle between two planes. Pythagoras' theorem remains fundamental.

Trigonometry and Pythagoras' theorem in three dimensions

Here are the steps in a successful approach to a three-dimensional problem.

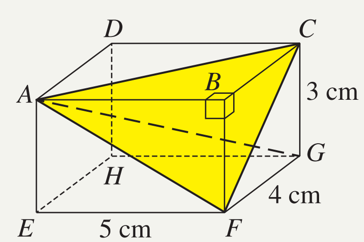
6 TRIGONOMETRY AND PYTHAGORAS' THEOREM IN THREE DIMENSIONS

1. Draw a careful sketch of the situation.
2. Note carefully all the triangles in the figure.
3. Mark, or note, all right angles in these triangles.
4. Always name the triangle you are working with.

Example 6 — 6C

The rectangular prism sketched below has dimensions:

$$EF = 5\,\text{cm} \quad \text{and} \quad FG = 4\,\text{cm} \quad \text{and} \quad CG = 3\,\text{cm}.$$



a Use Pythagoras' theorem in $\triangle CFG$ to find the length of the diagonal FC.

b Similarly find the lengths of the diagonals AC and AF.

c Use Pythagoras' theorem in $\triangle ACG$ to find the length of the space diagonal AG.

d Use trigonometry in $\triangle BAF$ to find $\angle BAF$ (nearest minute).

e Use trigonometry in $\triangle GAF$ to find $\angle GAF$ (nearest minute).

SOLUTION

a In $\triangle CFG$, $FC^2 = 3^2 + 4^2$, using Pythagoras,
$FC = 5\,\text{cm}$.

b In $\triangle ABC$, $AC^2 = 5^2 + 4^2$, using Pythagoras,
$AC = \sqrt{41}\,\text{cm}$.
In $\triangle ABF$, $AF^2 = 5^2 + 3^2$, using Pythagoras,
$AF = \sqrt{34}\,\text{cm}$.

c In $\triangle ACG$, the angle $\angle ACG$ is a right angle, and $AC = \sqrt{41}$ and $CG = 3$.
Hence $AG^2 = AC^2 + CG^2$, using Pythagoras,
$= 41 + 3^2$
$AG = \sqrt{50}$
$= 5\sqrt{2}\,\text{cm}$.

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d In $\triangle BAF$, the angle $\angle ABF$ is a right angle, and $AB = 5$ and $BF = 3$.

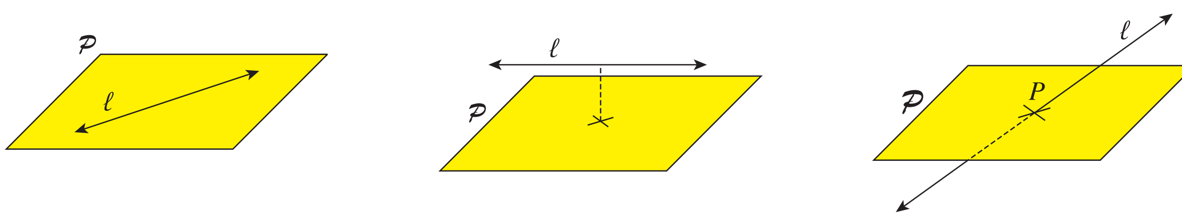
$$\text{Hence } \tan \angle BAF = \frac{BF}{AB}$$
$$= \tfrac{3}{5}$$
$$\angle BAF \doteqdot 30°58'.$$

e In $\triangle GAF$, the angle $\angle AFG$ is a right angle, and $AF = \sqrt{34}$ and $FG = 4$.

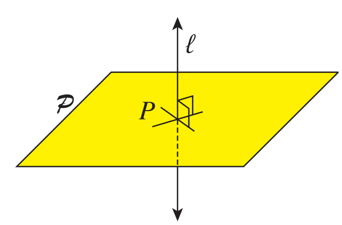
$$\text{Hence } \tan \angle GAF = \frac{FG}{AF}$$
$$= \frac{4}{\sqrt{34}}$$
$$\angle GAF \doteqdot 34°27'.$$

The angle between a line and a plane

In three-dimensional space, a plane $\mathcal{P}$ and a line ℓ can be related in three different ways:

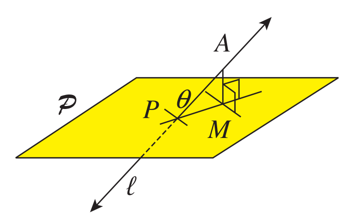


- In the first diagram above, the line lies wholly within the plane.
- In the second diagram, the line never meets the plane. We say that the line and the plane are *parallel*.
- In the third diagram, the line intersects the plane in a single point P.



When the line ℓ meets the plane $\mathcal{P}$ in the single point P, it can do so in two distinct ways.

In the upper diagram, the line ℓ is perpendicular to every line in the plane through P. We say that the line is *perpendicular* to the plane.



In the lower diagram, the line ℓ is not perpendicular to $\mathcal{P}$. To construct the angle θ between the line and the plane:

- Choose another point A on the line ℓ.
- Construct the point M in the plane $\mathcal{P}$ so that $AM \perp \mathcal{P}$.

Then $\angle APM$ is the angle between the plane and the line.

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Example 7 6C

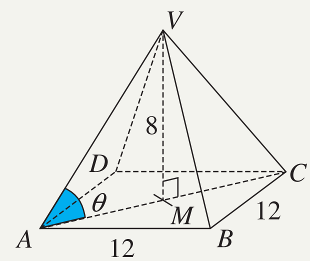
Find the angle between a slant edge and the base in a square pyramid of height 8 metres whose base has side length 12 metres.

SOLUTION

Using Pythagoras' theorem in the base $ABCD$,

$$AC^2 = 12^2 + 12^2$$
$$AC = 12\sqrt{2} \text{ metres.}$$

The perpendicular from the vertex V to the base meets the base at the midpoint M of the diagonal AC.



In $\triangle MAV$, $\tan \angle MAV = \dfrac{MV}{MA}$

$$= \frac{8}{6\sqrt{2}}$$

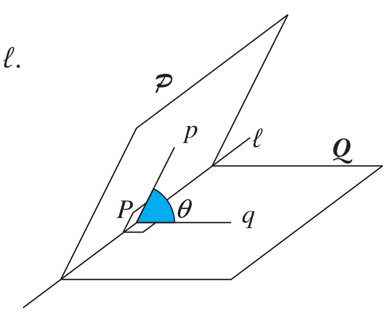
$\angle MAV \doteqdot 43°19'$,

and this is the angle between the slant edge AV and the base.

The angle between two planes

In three-dimensional space, two planes that are not parallel intersect in a line ℓ. To construct the angle between the planes:

- Take any point P on this line of intersection.
- Construct the line p through P perpendicular to ℓ lying in the plane $\mathcal{P}$.
- Construct the line q through P perpendicular to ℓ lying in the plane $\mathcal{Q}$.



The angle between the planes $\mathcal{P}$ and $\mathcal{Q}$ is the angle between these two lines p and q.

Example 8 6C

In the pyramid in Example 7, find the angle between an oblique face of the pyramid and the base.

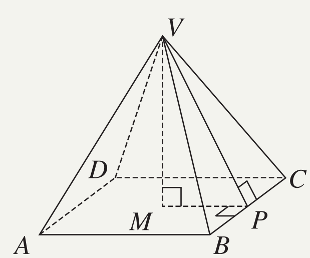
SOLUTION

Let P be the midpoint of the edge BC.
Then $VP \perp BC$ and $MP \perp BC$,
so $\angle VPM$ is the angle between the oblique face and the base.

In $\triangle VPM$, $\tan \angle VPM = \dfrac{VM}{PM}$

$$= \tfrac{8}{6}$$

$\angle VPM \doteqdot 53°8'$.



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Three-dimensional problems in which no triangle can be solved

In the following classic problem, there are four triangles forming a tetrahedron, but no triangle can be solved, because no more than two measurements are known in any one of these triangles. The method is to introduce a pronumeral for the height, then work around the figure until *four* measurements are known in terms of h in the base triangle — at this point an equation in h can be formed and solved.

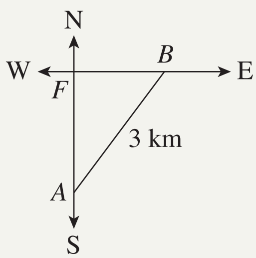
Example 9 — 6C

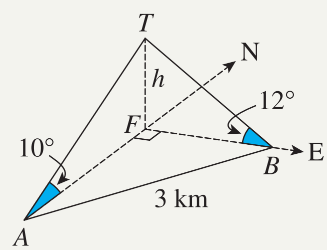
A motorist driving on level ground sees, due north of her, a tower whose angle of elevation is 10°. After driving 3 km further in a straight line, the tower is in the direction due west, with angle of elevation 12°.

a How high is the tower?

b In what direction is she driving?

SOLUTION





Let the tower be TF, and let the motorist be driving from A to B.

a There are four triangles, none of which can be solved.

Let h be the height of the tower.

In $\triangle TAF$, $\quad AF = h\cot 10^\circ$.

In $\triangle TBF$, $\quad BF = h\cot 12^\circ$.

We now have expressions for four measurements in $\triangle ABF$, so we can use Pythagoras' theorem to form an equation in h.

In $\triangle ABF$,

$$\begin{aligned} AF^2 + BF^2 &= AB^2 \\ h^2\cot^2 10^\circ + h^2\cot^2 12^\circ &= 3^2 \\ h^2(\cot^2 10^\circ + \cot^2 12^\circ) &= 9 \\ h^2 &= \frac{9}{\cot^2 10^\circ + \cot^2 12^\circ} \\ h &\doteqdot 0.407\text{ km}, \end{aligned}$$

so the tower is about 407 metres high.

b Let $\theta = \angle FAB$, then in $\triangle AFB$,

$$\begin{aligned} \sin\theta &= \frac{FB}{AB} \\ &= \frac{h\cot 12^\circ}{3} \\ \theta &\doteqdot 40^\circ, \end{aligned}$$

so her direction is about N40°E.

The general method of approach

Here is a summary of what has been said about three-dimensional problems (apart from the ideas of angles between lines and planes, and between planes and planes).

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7 THREE-DIMENSIONAL TRIGONOMETRY

1. Draw a careful diagram of the situation, marking all right angles.
2. A plan diagram, looking down, is usually a great help.
3. Identify every triangle in the diagram, to see whether it can be solved.
4. If one triangle can be solved, then work from it around the diagram until the problem is solved.
5. If no triangle can be solved, assign a pronumeral to what is to be found, then work around the diagram until an equation in that pronumeral can be formed and solved.

Exercise 6C

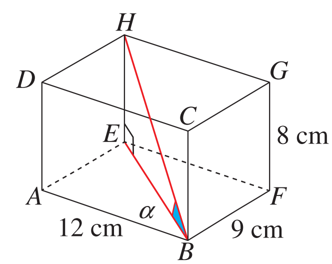
FOUNDATION

1 The diagram to the right shows a rectangular prism.

a Use Pythagoras' theorem to find the length of the base diagonal BE.

b Hence find the length of the prism diagonal BH.

c Find, correct to the nearest degree, the angle α that BH makes with the base of the prism.



2 The diagram to the right shows a cube.

a Write down the size of:

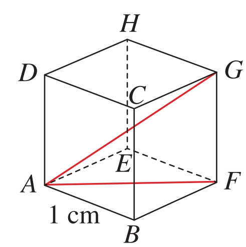
i $\angle ABF$ **ii** $\angle AFG$ **iii** $\angle ABG$

b Use Pythagoras' theorem to find the exact length of:

i AF **ii** AG

c Hence find, correct to the nearest degree:

i $\angle GAF$ **ii** $\angle AGB$



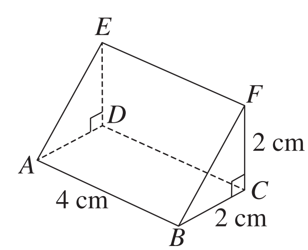
3 The diagram to the right shows a triangular prism.

a Find the exact length of:

i AC **ii** AF

b What is the size of $\angle ACF$?

c Find $\angle AFC$, correct to the nearest degree.



DEVELOPMENT

4 The diagram to the right shows a square pyramid. The point C is the centre of the base, and TC is perpendicular to the base.

a Write down the size of:

i $\angle CMQ$ **ii** $\angle TCM$ **iii** $\angle TCQ$

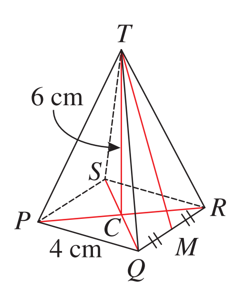
b Find the length of:

i CM **ii** CQ

c Find, correct to the nearest degree:

i the angle between a side face and the base,

ii the angle between a slant edge and the base.



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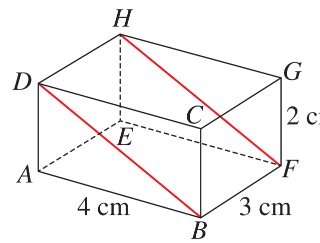
5 The diagram to the right shows a rectangular prism.

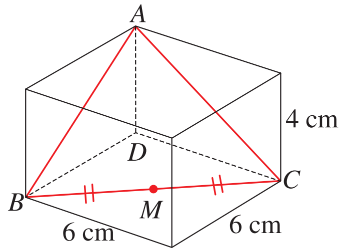


a Write down the size of:

i $\angle ABF$ **ii** $\angle DBF$

b Find, correct to the nearest degree, the angle that the diagonal plane *DBFH* makes with the base of the prism.

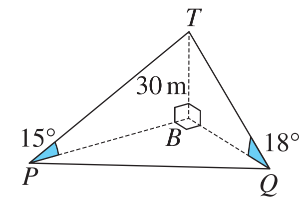
6 The diagram to the right shows a square prism. The plane *ABC* is inside the prism, and *M* is the midpoint of the base diagonal *BC*.



a Find the exact length of *MD*.

b Hence find, correct to the nearest degree, the angle that the plane *ABC* makes with the base of the prism.

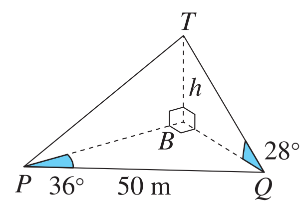
7 Two landmarks *P* and *Q* on level ground are observed from the top *T* of a vertical tower *BT* of height 30 m. Landmark *P* is due south of the tower, while landmark *Q* is due east of the tower. The angles of elevation of *T* from *P* and *Q* are 15° and 18° respectively.



a Show that $BP = 30\tan 75^\circ$ and find a similar expression for *BQ*.

b Find, correct to the nearest metre, the distance between the two landmarks.

8 A tree *BT* is due north of an observer at *P* and due west of an observer at *Q*. The two observers are 50 m apart and the bearing of *Q* from *P* is 36°. The angle of elevation of *T* from *Q* is 28°.

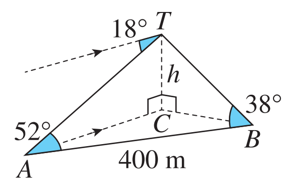


a Show that $BQ = 50\sin 36^\circ$.

b Hence find the height *h* of the tree correct to the nearest metre.

c Find, correct to the nearest degree, the angle of elevation of *T* from *P*.

9 Two monuments *A* and *B* are 400 m apart on a horizontal plane. The angle of depression of *A* from the top *T* of a tall building is 18°. Also, $\angle TAB = 52^\circ$ and $\angle TBA = 38^\circ$.

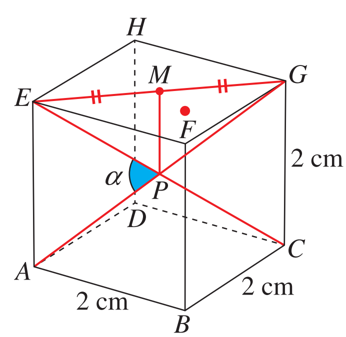


a Show that $TA = 400\cos 52^\circ$.

b Find the height *h* of the building, correct to the nearest metre.

c Find, correct to the nearest degree, the angle of depression of *B* from *T*.

10 The diagram shows a cube of side 2 cm, with diagonals *AG* and *CE* intersecting at *P*. The point *M* is the midpoint of the face diagonal *EG*. Let α be the acute angle between the diagonals *AG* and *CE*.



a What is the length of *PM*?

b Find the exact length of *EM*.

c Write down the exact value of $\tan\angle EPM$.

d Hence find α, correct to the nearest minute.

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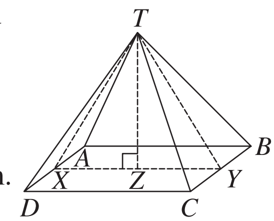
11 The diagram shows a rectangular pyramid. X and Y are the midpoints of AD and BC respectively and T is directly above Z. $TX = 15\,\text{cm}$, $TY = 20\,\text{cm}$, $AB = 25\,\text{cm}$ and $BC = 10\,\text{cm}$.

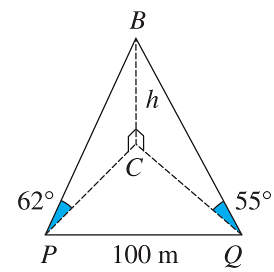


a Show that $\angle XTY = 90°$.

b Using either similar triangles or Pythagoras' theorem, show that $TZ = 12\,\text{cm}$.

c Hence find, correct to the nearest minute, the angle that the front face DCT makes with the base.

12 A balloon B is due north of an observer P and its angle of elevation is $62°$. From another observer Q 100 metres from P, the balloon is due west and its angle of elevation is $55°$. Let the height of the balloon be h metres and let C be the point on the level ground vertically below B.



a Show that $PC = h\cot 62°$, and write down a similar expression for QC.

b Explain why $\angle PCQ = 90°$.

c Use Pythagoras' theorem in $\triangle CPQ$ to show that

$$h^2 = \frac{100^2}{\cot^2 62° + \cot^2 55°}.$$

d Hence find h, correct to the nearest metre.

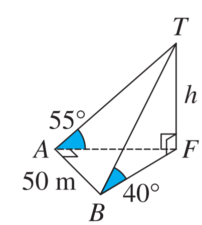
13 From a point P due south of a vertical tower, the angle of elevation of the top of the tower is $20°$. From a point Q situated 40 metres from P and due east of the tower, the angle of elevation is $35°$. Let h metres be the height of the tower.

a Draw a diagram to represent the situation.

b Show that $h = \dfrac{40}{\sqrt{\tan^2 70° + \tan^2 55°}}$, and evaluate h, correct to the nearest metre.

ENRICHMENT

14 The diagram shows a tower of height h metres standing on level ground. The angles of elevation of the top T of the tower from two points A and B on the ground nearby are $55°$ and $40°$ respectively. The distance AB is 50 metres and the interval AB is perpendicular to the interval AF, where F is the foot of the tower.



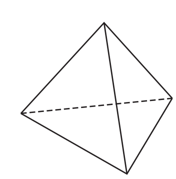
a Find AT and BT in terms of h.

b What is the size of $\angle BAT$?

c Use Pythagoras' theorem in ΔBAT to show that $h = \dfrac{50\sin 55°\sin 40°}{\sqrt{\sin^2 55° - \sin^2 40°}}$.

d Hence find the height of the tower, correct to the nearest metre.

15 The diagram shows a triangular pyramid, all of whose faces are equilateral triangles — such a solid is called a *regular tetrahedron*. Suppose that the slant edges are inclined at an angle θ to the base. Show that $\cos\theta = \frac{1}{\sqrt{3}}$.



16 A square pyramid has perpendicular height equal to the side length of its base. Show that the angle between a slant edge and a base edge it meets is $\tan^{-1}\sqrt{5}$.

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6D Trigonometric functions of a general angle

The definitions of the trigonometric functions given in Section 6A only apply to acute angles, because in a right-angled triangle, both other angles are acute angles.

This section introduces more general definitions based on circles in the coordinate plane (whose equations are Pythagoras' theorem, as we saw in Section 3G). The new definitions will apply to any angle, but will, of course, give the same values as the previous definitions for acute angles.

Putting a general angle on the coordinate plane

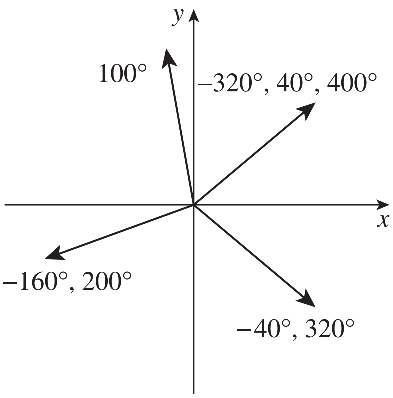
Let θ be any angle — possibly negative, possibly obtuse or reflex, possibly greater than 360°. We shall associate with θ a ray with vertex at the origin.

8 THE RAY CORRESPONDING TO θ

- The positive direction of the x-axis is the ray representing the angle 0°.
- For all other angles, rotate this ray anti-clockwise through an angle θ.

If the angle is negative, the ray is rotated backwards, which means clockwise.

Here are some examples of angles and the corresponding rays. The angles have been written at the ends of the arrows representing the rays.



Notice that one ray can correspond to many angles. For example, all the following angles have the same ray as 40°:

$$\ldots,\ -680^\circ, -320^\circ, 40^\circ, 400^\circ, 760^\circ,\ \ldots$$

A given ray thus corresponds to infinitely many angles, all differing by multiples of 360°. The relation from rays to angles is thus one-to-many.

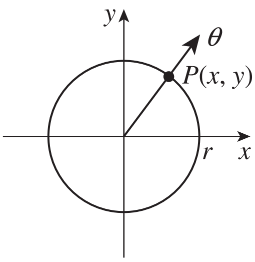
9 CORRESPONDING ANGLES AND RAYS

- To each angle, there corresponds exactly one ray.
- To each ray, there correspond infinitely many angles, all differing from each other by multiples of 360°.

Defining the trigonometric functions for general angles

Let θ be any angle, positive or negative.

Construct a circle with centre the origin and any positive radius r. Let the ray corresponding to θ intersect the circle at the point $P(x, y)$.



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The six trigonometric functions are now defined in terms of x, y and r as follows:

10 DEFINITIONS OF THE SIX TRIGONOMETRIC FUNCTIONS

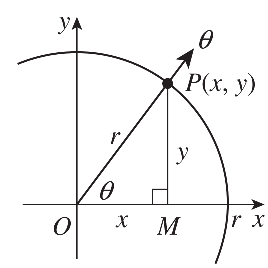
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\operatorname{cosec}\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

Note: We chose r to be 'any positive radius'. If a different radius were chosen, the two figures would be similar, so the lengths x, y and r would stay in the same ratio. Because the definitions depend only on the ratios of the lengths, the values of the trigonometric functions would not change.

In particular, one may use a circle of radius 1 in the definitions. When this is done, however, we lose the intuition that a trigonometric function is not a length, but is the ratio of two lengths.

Agreement with the earlier definition



Let θ be an acute angle. Construct the ray corresponding to θ.
Let the perpendicular from P meet the x-axis at M.
Then $\theta = \angle POM$, so relating the sides to the angle θ,

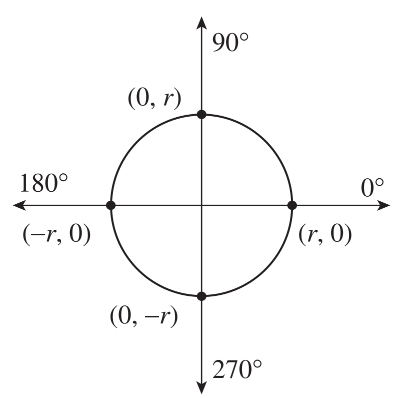
$$\text{hyp} = OP = r, \quad \text{opp} = PM = y, \quad \text{adj} = OM = x.$$

Hence the old and the new definitions are in agreement.

Note: Most people find that the diagram above is the easiest way to learn the new definitions of the trigonometric functions. Take the old definitions in terms of hypotenuse, opposite and adjacent sides, and make the replacements

$$\text{hyp} \longleftrightarrow r, \quad \text{opp} \longleftrightarrow y, \quad \text{adj} \longleftrightarrow x.$$

Boundary angles



Integer multiples of 90°, that is

$$\ldots, -90°, 0°, 90°, 180°, 270°, 360°, 450°, \ldots$$

are called *boundary angles* because they lie on the boundaries between quadrants.

The values of the trigonometric functions at these boundary angles are not always defined, and are 0, 1 or -1 when they are defined. The diagram to the right can be used to calculate them, and the results are shown in the table that follows (where the star $*$ indicates that the value is undefined).

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11 THE BOUNDARY ANGLES

θ	$0°$	$90°$	$180°$	$270°$
x	r	0	$-r$	0
y	0	r	0	$-r$
r	r	r	r	r
$\sin\theta$	0	1	0	-1
$\cos\theta$	1	0	-1	0
$\tan\theta$	0	*	0	*
$\operatorname{cosec}\theta$	*	1	*	-1
$\sec\theta$	1	*	-1	*
$\cot\theta$	*	0	*	0

In practice, the answer to any question about the values of the trigonometric functions at these boundary angles should be read off the graphs of the functions. These graphs need to be known very well indeed.

The domains of the trigonometric functions

The trigonometric functions are defined everywhere except where the denominator is zero.

12 DOMAINS OF THE TRIGONOMETRIC FUNCTIONS

- $\sin\theta$ and $\cos\theta$ are defined for all angles θ.
- $\tan\theta$ and $\sec\theta$ are undefined when $x = 0$, that is, when
  $\theta = \ldots, -90°, 90°, 270°, 450°, \ldots.$
- $\cot\theta$ and $\operatorname{cosec}\theta$ are undefined when $y = 0$, that is, when
  $\theta = \ldots, -180°, 0°, 180°, 360°, \ldots.$

Exercise 6D

FOUNDATION

1 On a number plane, draw rays representing the following angles.

a $40°$ b $110°$ c $190°$
d $290°$ e $420°$ f $500°$

2 On another number plane, draw rays representing the following angles.

a $-50°$ b $-130°$ c $-250°$
d $-350°$ e $-440°$ f $-550°$

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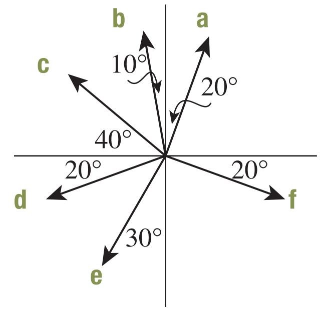
3 For each angle in Question 1, write down the size of the negative angle between $-360°$ and $0°$ that is represented by the same ray.

4 For each angle in Question 2, write down the size of the positive angle between $0°$ and $360°$ that is represented by the same ray.

5 Write down two positive angles between $0°$ and $720°$ and two negative angles between $-720°$ and $0°$ that are represented by each of the rays in the diagram to the right.



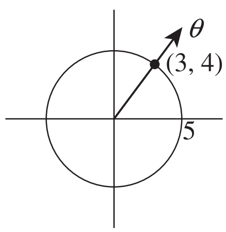
DEVELOPMENT

6 Use the definitions

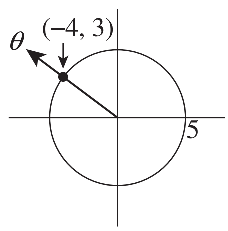
$$\sin\theta = \frac{y}{r} \quad \text{and} \quad \cos\theta = \frac{x}{r} \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

to write down the values of the six trigonometric ratios of the angle θ in each diagram.

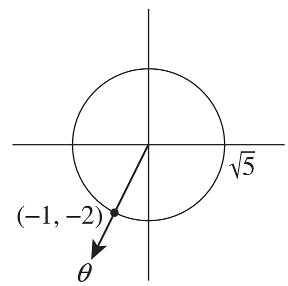
a



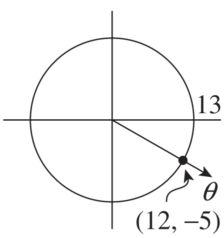
b



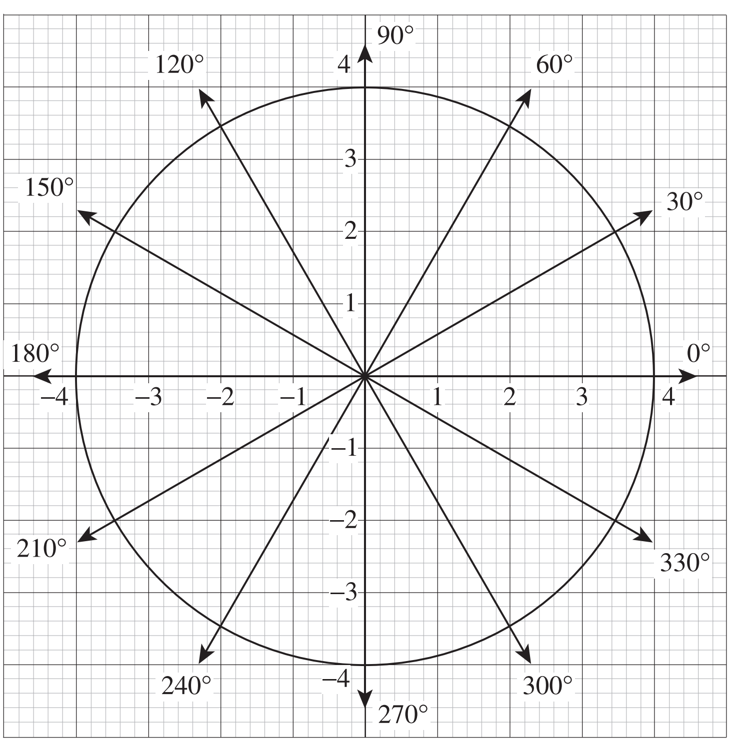
c



d



7 [The graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$]
The diagram shows angles from $0°$ to $360°$ at $30°$ intervals. The circle has radius 4 units.



a Use the diagram and the definitions of the three trigonometric ratios to copy and complete the following table. Measure the values of x and y correct to two decimal places, and use your calculator only to perform the necessary divisions.

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θ	$-30°$	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$	$390°$
x															
y															
r															
$\sin\theta$															
$\cos\theta$															
$\tan\theta$															

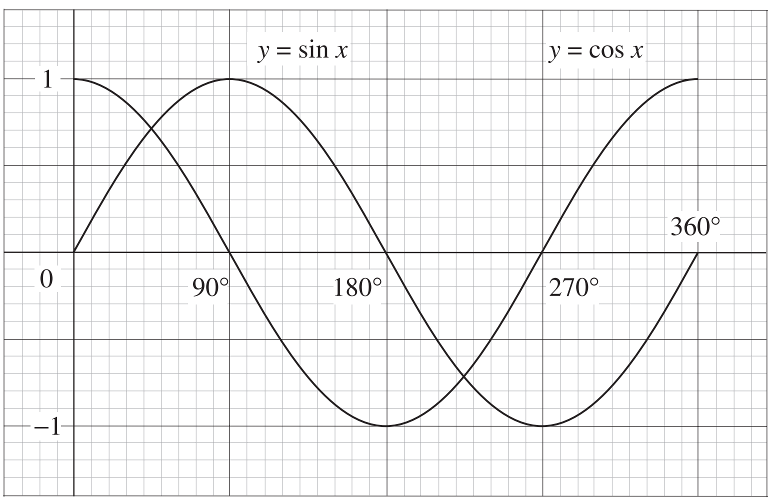
b Use your calculator to check the accuracy of the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ that you obtained in part **a**.

c Using the table of values in part **a**, graph the curves $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ as accurately as possible on graph paper. Use the following scales:

On the horizontal axis: let 2 mm represent $10°$.

On the vertical axis: let 2 cm represent 1 unit.

8



a Read off the diagram, correct to two decimal places where necessary, the values of:

i $\cos 60°$ **ii** $\sin 210°$ **iii** $\sin 72°$ **iv** $\cos 18°$ **v** $\sin 144°$

vi $\cos 36°$ **vii** $\cos 153°$ **viii** $\sin 27°$ **ix** $\sin 234°$ **x** $\cos 306°$

b Find from the graphs two values of x between $0°$ and $360°$ for which:

i $\sin x = 0.5$ **ii** $\cos x = -0.5$ **iii** $\sin x = 0.9$ **iv** $\cos x = 0.6$

v $\sin x = 0.8$ **vi** $\cos x = -0.8$ **vii** $\sin x = -0.4$ **viii** $\cos x = -0.3$

c Find two values of x between $0°$ and $360°$ for which $\sin x = \cos x$.

9 [The graphs of $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$]

From the definitions of the trigonometric functions,

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

a Explain why the graph of $y = \operatorname{cosec}\theta$ has vertical asymptotes wherever $\sin\theta = 0$. Explain why the upper branches of $y = \operatorname{cosec}\theta$ have a minimum of 1 wherever $y = \sin\theta$ has a maximum of 1, and the lower branches have a maximum of -1 wherever $y = \sin\theta$ has a minimum of -1. Hence sketch the graph of $y = \operatorname{cosec}\theta$.

b Use similar methods to produce the graph of $y = \sec\theta$ from the graph of $y = \cos\theta$, and the graph of $y = \cot\theta$ from the graph of $y = \tan\theta$.

ENRICHMENT

10 If $\sin\theta = k$ and θ is obtuse, find an expression for $\tan(\theta + 90°)$.

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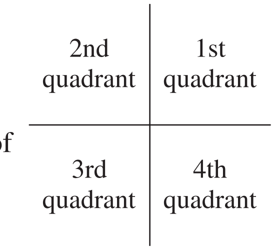
6E Quadrant, sign, and related acute angle

Symmetry is an essential aspect of trigonometric functions. This section uses symmetry to express the values of the trigonometric functions of any angle in terms of trigonometric functions of acute angles.

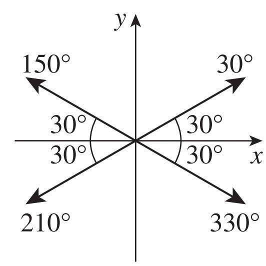


The diagram shows the conventional anti-clockwise numbering of the four quadrants of the coordinate plane. Acute angles are in the first quadrant, obtuse angles are in the second, and reflex angles are in the third and fourth.

The quadrant and the related acute angle

The diagram to the right shows the four rays corresponding to the four angles

$$30^\circ, 150^\circ, 210^\circ \text{ and } 330^\circ.$$



These four rays lie in each of the four quadrants of the plane, and they all make the same acute angle of 30° with the x-axis. The four rays are thus the reflections of each other in the two axes.

13 QUADRANT AND RELATED ACUTE ANGLE

Let θ be any angle.

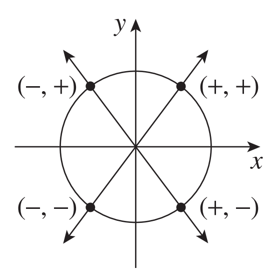
- The *quadrant* of θ is the quadrant (1, 2, 3 or 4) in which the ray lies.
- The *related acute angle* of θ is the acute angle between the ray and the x-axis.

Each of the four angles above has the same related acute angle 30°. Notice that θ and its related angle are only the same when θ is an acute angle.

The signs of the trigonometric functions

The signs of the trigonometric functions depend only on the signs of x and y. (The radius r is always positive.) The signs of x and y depend in turn only on the quadrant in which the ray lies. Thus we can easily compute the signs of the trigonometric functions from the accompanying diagram and the definitions.

quadrant	1st	2nd	3rd	4th	
x	+	−	−	+	
y	+	+	−	−	
r	+	+	+	+	
$\sin\theta$	+	+	−	−	
$\cos\theta$	+	−	−	+	
$\tan\theta$	+	−	+	−	
$\operatorname{cosec}\theta$	+	+	−	−	(same as $\sin\theta$)
$\sec\theta$	+	−	−	+	(same as $\cos\theta$)
$\cot\theta$	+	−	+	−	(same as $\tan\theta$)



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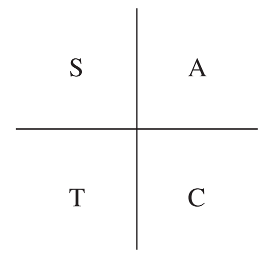
In NSW, these results are usually remembered by the phrase:

14 SIGNS OF THE TRIGONOMETRIC FUNCTIONS

'All Stations To Central'



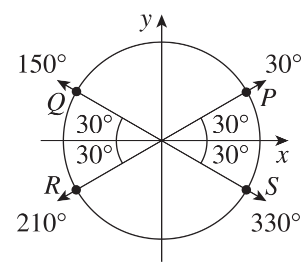
indicating that the four letters A, S, T and C are placed successively in the four quadrants as shown. The significance of the letters is:

A means *all* six functions are positive,
S means only *sine* (and cosecant) are positive,
T means only *tangent* (and cotangent) are positive,
C means only *cosine* (and secant) are positive.

Three of the graphs of the trigonometric functions were constructed in the previous exercise, and all six are drawn together at the end of this section. Study each of them to see how the table of signs above, and the ASTC rule, agree with your observations about when the graph is above the x-axis and when it is below.

The angle and the related acute angle

In the diagram to the right, a circle of radius r has been added to the earlier diagram showing the four angles 30°, 150°, 210° and 330°.



The four points P, Q, R and S where the four rays meet the circle are all reflections of each other in the x-axis and y-axis. Because of this symmetry, the coordinates of these four points are identical, apart from their signs.

Hence the trigonometric functions of these angles will all be the same too, except that the signs may be different.

15 THE ANGLE AND THE RELATED ACUTE ANGLE

- The trigonometric functions of any angle θ are the same as the trigonometric functions of its related acute angle, apart from a possible change of sign.
- The sign is best found using the ASTC diagram.

Evaluating the trigonometric functions of any angle

This gives a straightforward way of evaluating the trigonometric functions of any angle.

16 TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

Draw a quadrant diagram, then:

- Place the ray in the correct quadrant,
  then use the ASTC rule to work out the sign of the answer.
- Find the related acute angle,
  then work out the value of the trigonometric function at this related angle.

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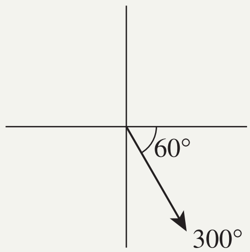
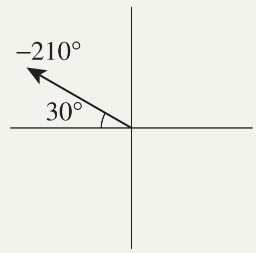
Example 10 — 6E

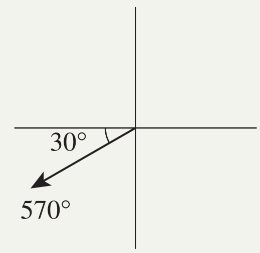
Find the exact value of:

a $\tan 300^\circ$ **b** $\sin(-210^\circ)$ **c** $\cos 570^\circ$

SOLUTION







a 300° is in quadrant 4, the related angle is 60°,

so $\tan 300^\circ = -\tan 60^\circ$
$= -\sqrt{3}$.

b -210° is in quadrant 2, the related angle is 30°,

so $\sin(-210^\circ) = +\sin 30^\circ$
$= \frac{1}{2}$.

c 570° is in quadrant 3, the related angle is 30°,

so $\cos 570^\circ = -\cos 30^\circ$
$= -\frac{\sqrt{3}}{2}$.

Note: The calculator will give approximate values of the trigonometric functions without any need to find the related acute angle. It will *not* give exact values, however, when these values involve surds.

General angles with pronumerals

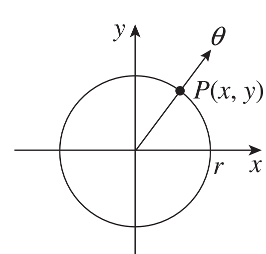
This quadrant-diagram method can be used to generate formulae for expressions such as $\sin(180^\circ + A)$ or $\cot(360^\circ - A)$. The trick is to place A on the quadrant diagram *as if it were acute*.

17 SOME FORMULAE WITH GENERAL ANGLES

$\sin(180^\circ - A) = \sin A$	$\sin(180^\circ + A) = -\sin A$	$\sin(360^\circ - A) = -\sin A$
$\cos(180^\circ - A) = -\cos A$	$\cos(180^\circ + A) = -\cos A$	$\cos(360^\circ - A) = \cos A$
$\tan(180^\circ - A) = -\tan A$	$\tan(180^\circ + A) = \tan A$	$\tan(360^\circ - A) = -\tan A$

Some people prefer to learn this list of identities to evaluate trigonometric functions, but this seems unnecessary when the quadrant-diagram method is so clear.

Specifying a point in terms of r and θ



The definitions of $\sin\theta$ and $\cos\theta$ are $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$. Rewriting these with x and y as the subject, we obtain:

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18 RECOVERING THE COORDINATES OF A POINT

$$x = r\cos\theta$$
$$y = r\sin\theta$$

This means that if a point P is specified in terms of its distance OP from the origin and the angle of the ray OP, then the x and y coordinates of P can be recovered by means of these formulae.

Example 11 — 6E

The circle $x^2 + y^2 = 36$ meets the positive direction of the x-axis at A. Find the coordinates of the points P on the circle such that $\angle AOP = 60^\circ$.

SOLUTION

The circle has radius 6, so $r = 6$, and the ray OP has angle 60° or -60°, so the coordinates (x, y) of P are

$$x = 6\cos 60^\circ = 3 \quad \text{or} \quad x = 6\cos(-60^\circ) = 3$$
$$y = 6\sin 60^\circ = 3\sqrt{3}, \quad \text{or} \quad y = 6\sin(-60^\circ) = -3\sqrt{3}.$$

So $P = \left(3, 3\sqrt{3}\right)$ or $P = \left(3, -3\sqrt{3}\right)$.

The graphs of the six trigonometric functions

The diagrams on the next page show the graphs of the six trigonometric functions over a domain extending beyond $-360^\circ \le x \le 360^\circ$. With this extended domain, it becomes clear how the graphs are built up by infinite repetition of a simple element.

The sine and cosine graphs are waves, and they are the basis of all mathematics that deals with waves. The later trigonometry in this course will mostly deal with these wave properties. These two graphs each repeat every 360°, and the graphs are therefore said to have *period* 360°.

The piece of the sine wave from $\theta = 0^\circ$ to $\theta = 90^\circ$ is enough to construct the whole sine wave and the whole cosine wave — use reflections, rotations and translations.

The other four graphs also repeat themselves periodically. The graphs of cosec x and sec x can be found from the graphs of sin x and cos x by the method of sketching reciprocals, and have already occurred in Question 9 of Exercise 6D. They each have period 360°. The graphs of tan x and cot x, on the other hand, each have period 180°. This will all be discussed in more detail in Chapter 11.

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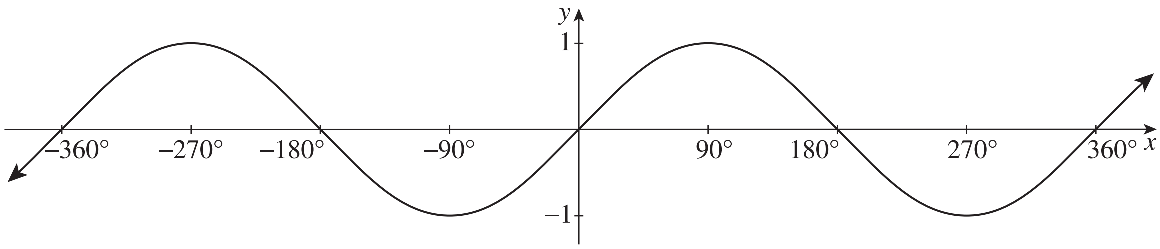
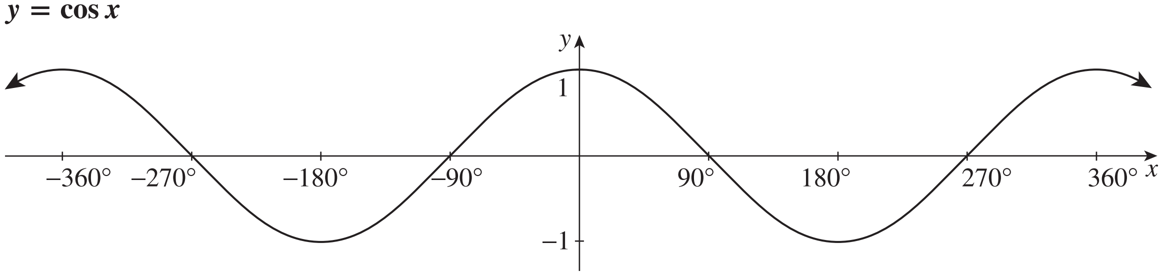
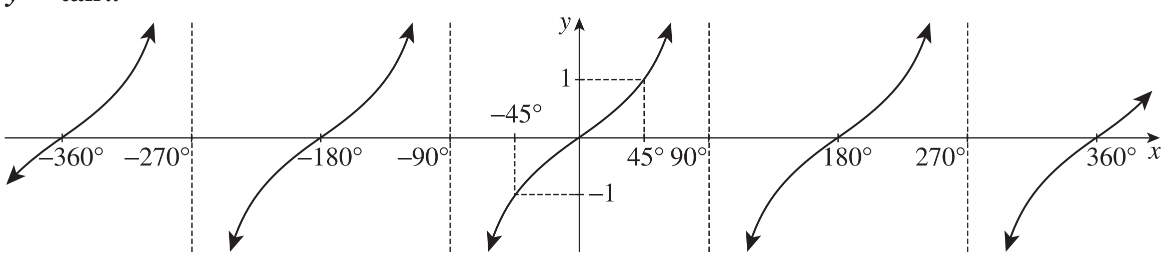
$y = \sin x$



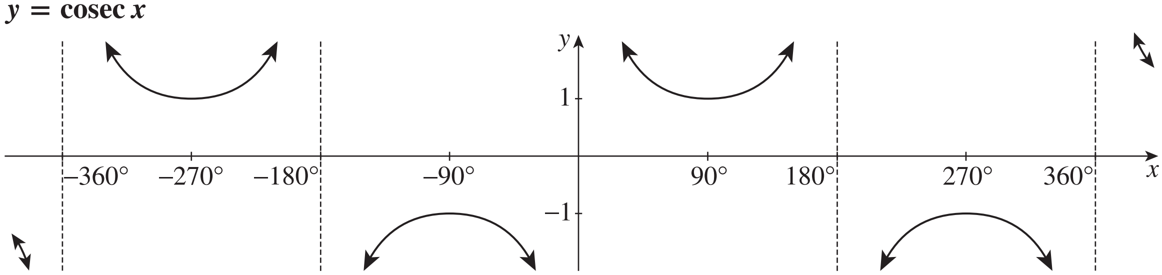
$y = \cos x$



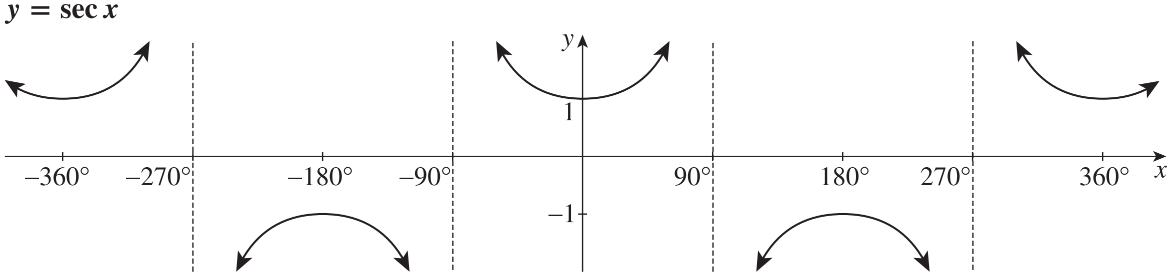
$y = \tan x$



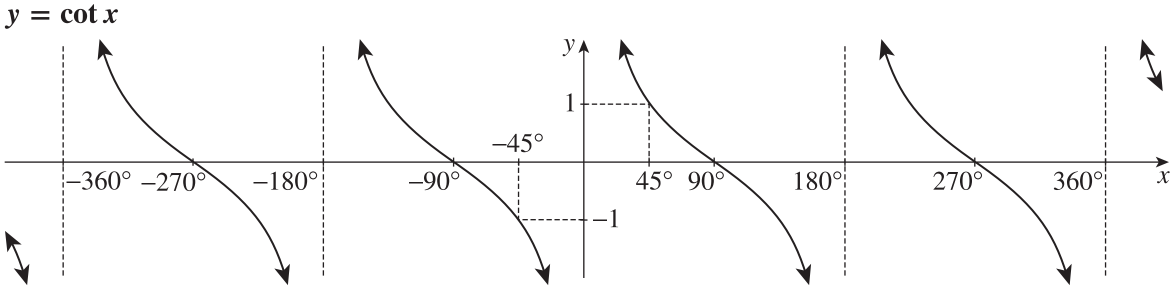
$y = \operatorname{cosec} x$



$y = \sec x$



$y = \cot x$



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Exercise 6E

FOUNDATION

1 Use the ASTC rule to determine the sign (+ or −) of each trigonometric ratio.

a $\sin 20°$ b $\cos 50°$ c $\cos 100°$ d $\tan 140°$
e $\tan 250°$ f $\sin 310°$ g $\sin 200°$ h $\cos 280°$
i $\sin 340°$ j $\cos 350°$ k $\tan 290°$ l $\cos 190°$
m $\tan 170°$ n $\sin 110°$ o $\tan 80°$ p $\cos 170°$

2 Write down the related acute angle of each angle.

a $10°$ b $150°$ c $310°$ d $200°$ e $80°$
f $250°$ g $290°$ h $100°$ i $350°$ j $160°$

3 Write each trigonometric ratio as the ratio of an acute angle with the correct sign attached.

a $\tan 130°$ b $\cos 310°$ c $\sin 220°$ d $\tan 260°$
e $\cos 170°$ f $\sin 320°$ g $\cos 185°$ h $\sin 125°$
i $\tan 325°$ j $\sin 85°$ k $\cos 95°$ l $\tan 205°$

4 Use the trigonometric graphs to find the values (if they exist) of these trigonometric ratios of boundary angles.

a $\sin 0°$ b $\cos 180°$ c $\cos 90°$ d $\tan 0°$
e $\sin 90°$ f $\cos 0°$ g $\sin 270°$ h $\tan 270°$
i $\sin 180°$ j $\cos 270°$ k $\tan 90°$ l $\tan 180°$

5 Find the exact value of:

a $\sin 60°$ b $\sin 120°$ c $\sin 240°$ d $\sin 300°$
e $\cos 45°$ f $\cos 135°$ g $\cos 225°$ h $\cos 315°$
i $\tan 30°$ j $\tan 150°$ k $\tan 210°$ l $\tan 330°$

DEVELOPMENT

6 Find the exact value of:

a $\cos 120°$ b $\tan 225°$ c $\sin 330°$ d $\sin 135°$
e $\tan 240°$ f $\cos 210°$ g $\tan 315°$ h $\cos 300°$
i $\sin 225°$ j $\cos 150°$ k $\sin 210°$ l $\tan 300°$

7 Find the exact value of:

a $\operatorname{cosec} 150°$ b $\sec 225°$ c $\cot 120°$
d $\cot 210°$ e $\sec 330°$ f $\operatorname{cosec} 300°$

8 Use the trigonometric graphs to find (if they exist):

a $\sec 0°$ b $\operatorname{cosec} 270°$ c $\sec 90°$
d $\operatorname{cosec} 180°$ e $\cot 90°$ f $\cot 180°$

9 Find the related acute angle of each angle.

a $-60°$ b $-200°$ c $-150°$ d $-300°$
e $430°$ f $530°$ g $590°$ h $680°$

10 Find the exact value of:

a $\cos(-60°)$ b $\sin(-120°)$ c $\tan(-120°)$ d $\sin(-315°)$
e $\tan(-210°)$ f $\cos(-225°)$ g $\tan 420°$ h $\cos 510°$
i $\sin 495°$ j $\sin 690°$ k $\cos 600°$ l $\tan 585°$

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11 Given that $\sin 25^\circ \doteqdot 0.42$ and $\cos 25^\circ \doteqdot 0.91$, write down approximate values, without using a calculator, for:

a $\sin 155^\circ$ **b** $\cos 205^\circ$ **c** $\cos 335^\circ$

d $\sin 335^\circ$ **e** $\sin 205^\circ - \cos 155^\circ$ **f** $\cos 385^\circ - \sin 515^\circ$

12 Given that $\tan 35^\circ \doteqdot 0.70$ and $\sec 35^\circ \doteqdot 1.22$, write down approximate values, without using a calculator, for:

a $\tan 145^\circ$ **b** $\sec 215^\circ$ **c** $\tan 325^\circ$

d $\tan 215^\circ + \sec 145^\circ$ **e** $\sec 325^\circ + \tan 395^\circ$ **f** $\sec(-145)^\circ - \tan(-215)^\circ$

13 Show by substitution into LHS and RHS that each trigonometric identity is satisfied by the given values of the angles.

a Show that $\sin 2\theta = 2\sin\theta\cos\theta$, when $\theta = 150^\circ$.

b Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, when $\theta = 225^\circ$.

c Show that $\sin(A + B) = \sin A\cos B + \cos A\sin B$, when $A = 300^\circ$ and $B = 240^\circ$.

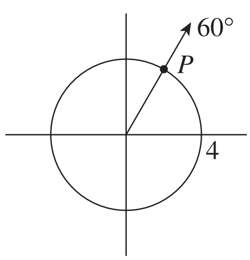
14 Write each expression as a trigonometric ratio of θ, with the correct sign attached.

a $\sin(-\theta)$ **b** $\cos(-\theta)$ **c** $\tan(-\theta)$ **d** $\sec(-\theta)$

e $\sin(180^\circ - \theta)$ **f** $\sin(360^\circ - \theta)$ **g** $\cos(180^\circ - \theta)$ **h** $\tan(180^\circ + \theta)$

15 Find the coordinates of the point P in each diagram.

a



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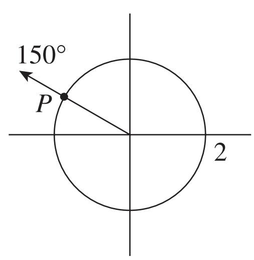

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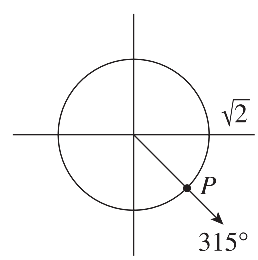

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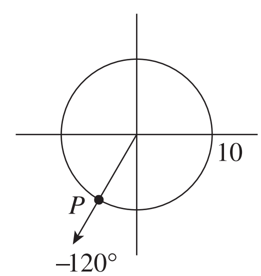
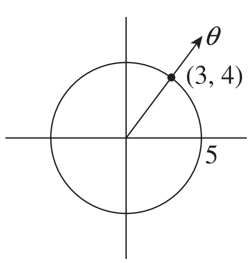
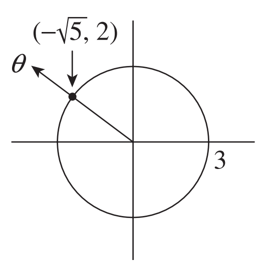


16 Find each angle θ, correct to the nearest minute where necessary, given that $0^\circ < \theta < 360^\circ$.

a



b



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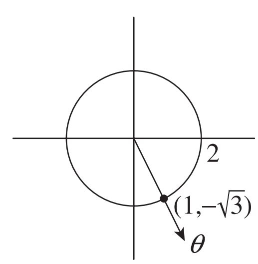
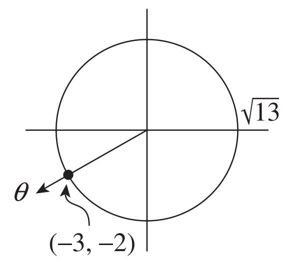


d



17 Classify the six trigonometric graphs as one-to-one, many-to-one, one-to-many or many-to-many.

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ENRICHMENT

18 Prove that:

a $\sin 330° \cos 150° - \cos 390° \sin 390° = 0$

b $\sin 420° \cos 405° + \cos 420° \sin 405° = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$

c $\dfrac{\sin 135° - \cos 120°}{\sin 135° + \cos 120°} = 3 + 2\sqrt{2}$

d $(\sin 150° + \cos 270° + \tan 315°)^2 = \sin^2 135° \cos^2 225°$

e $\dfrac{\sin 120°}{\tan 300°} - \dfrac{\cos 240°}{\cot 315°} = \tan^2 240° - \operatorname{cosec}^2 330°$

19 Examine the graphs of the six trigonometric functions drawn just before this exercise, then answer these questions.

a What are the ranges of the six functions?

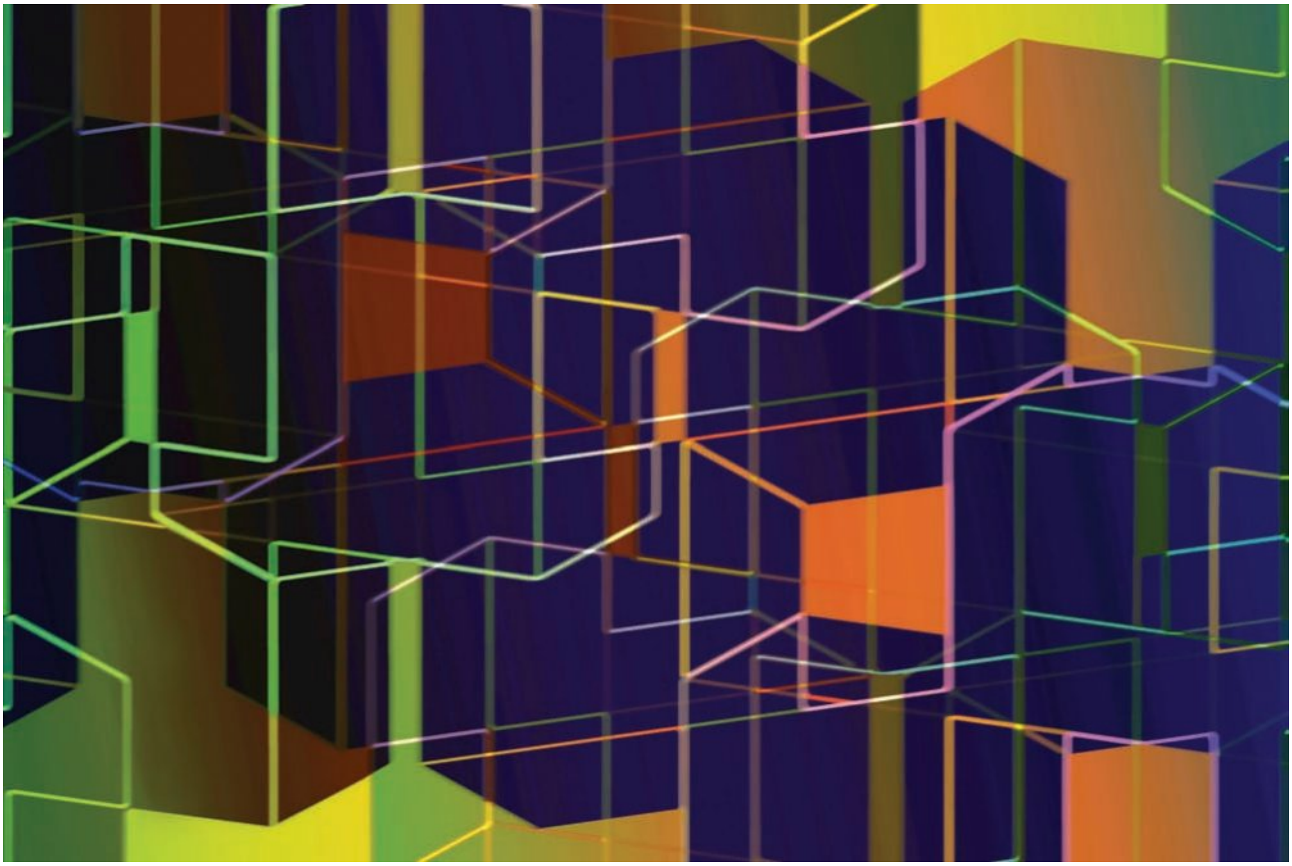
b What is the *period* of each function, that is, how far does one move on the horizontal axis before the graph repeats itself? How is this period related to the identities

$$\sin(\theta + 360°) = \sin\theta, \quad \sec(\theta + 360°) = \sec\theta, \quad \tan(\theta + 180°) = \tan\theta\,?$$

c Which functions are even and which are odd?

d More generally, about what points do the graphs have point symmetry? (That is, about what points are they unchanged by a rotation of 180°?)

e What are the axes of symmetry of the graphs?



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6F Given one trigonometric function, find another

When the exact value of one trigonometric function is known for an angle, the exact values of the other trigonometric functions can easily be found using the circle diagram and Pythagoras' theorem.

19 GIVEN ONE TRIGONOMETRIC FUNCTION, FIND ANOTHER

- Place a ray or rays on a circle diagram in the quadrants allowed in the question.
- Complete the triangle and use Pythagoras' theorem to find whichever of x, y and r is missing.

Example 12 6F

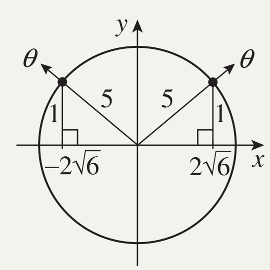
It is known that $\sin\theta = \frac{1}{5}$.

a Find the possible values of $\cos\theta$.

b Find $\cos\theta$ if it is also known that $\tan\theta$ is negative.

SOLUTION

a The angle must be in quadrant 1 or 2, because $\sin\theta$ is positive.



Because $\sin\theta = \frac{y}{r} = \frac{1}{5}$, we can take $y = 1$ and $r = 5$,

so by Pythagoras' theorem, $x = \sqrt{24}$ or $-\sqrt{24}$,

so $\cos\theta = \frac{2\sqrt{6}}{5}$ or $-\frac{2\sqrt{6}}{5}$.

b Because $\tan\theta$ is negative, θ can only be in quadrant 2,

so $\cos\theta = -\frac{2\sqrt{6}}{5}$.

Note: In the diagrams of this section, some of the 'side lengths' on the horizontal and vertical sides of triangles are marked as negative because they are really displacements from the origin rather than lengths. Always trust the quadrants to give you the correct sign.

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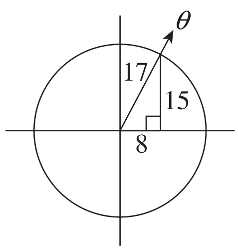
Exercise 6F

FOUNDATION

Note: Diagrams have been drawn for Questions 1–4, and similar diagrams should be drawn for the subsequent questions. Many answers will involve surds, but it is not important to rationalise denominators.

Do not use the calculator at all in this exercise, because you are looking for exact values, not approximations.

1 Write down the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in each part.

a


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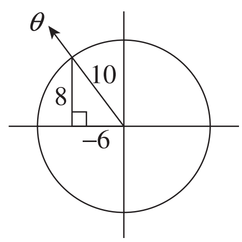
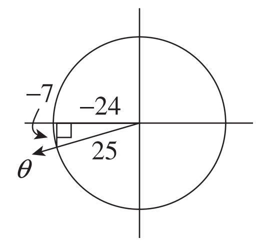


c


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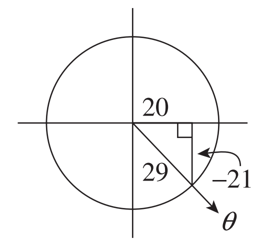
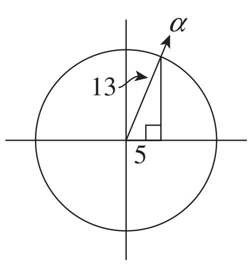


2 In each part use Pythagoras' theorem to find whichever of x, y or r is unknown. Then write down the values of $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$.

a


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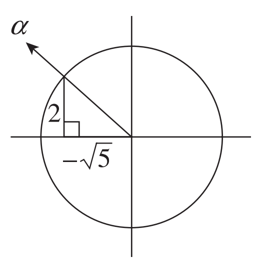

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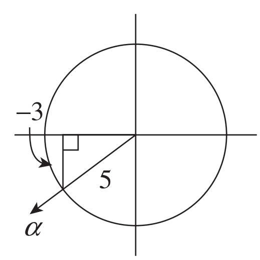

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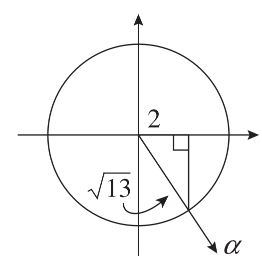
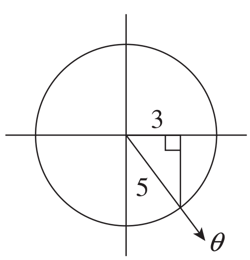
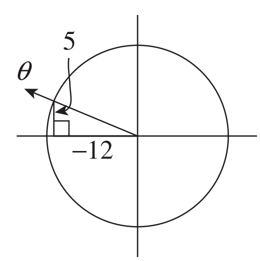


3 **a**


Let $\cos\theta = \frac{3}{5}$, where $270^\circ < \theta < 360^\circ$.

i Find $\sin\theta$.

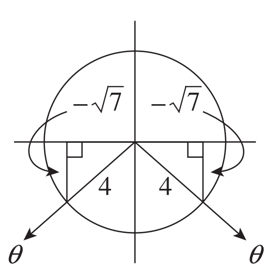
ii Find $\tan\theta$.

b


Let $\tan\theta = -\frac{5}{12}$, where θ is obtuse.

i Find $\sin\theta$.

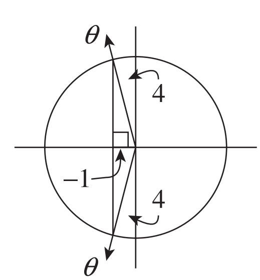
ii Find $\cos\theta$.

4 **a**


Suppose that $\sin\theta = -\frac{\sqrt{7}}{4}$.

i Find the possible values of $\cos\theta$.

ii Find the possible values of $\tan\theta$.

b


Suppose that $\cos\theta = -\frac{1}{4}$.

i Find the possible values of $\sin\theta$.

ii Find the possible values of $\tan\theta$.

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5 Draw similar diagrams for this question and subsequent questions.

a If $\cos\theta = \frac{1}{3}$ and θ is acute, find $\tan\theta$.

b If $\cos\theta = -\frac{4}{5}$ and θ is obtuse, find $\tan\theta$.

c If $\cos\theta = \frac{1}{2}$ and θ is reflex, find $\sin\theta$.

d Find $\cos\theta$ if $\tan\theta = -\frac{2}{3}$ and θ is reflex.

e Find $\sin\theta$, given that $\cos\theta = -\frac{40}{41}$ and $0° \le \theta \le 180°$.

f Find $\tan\theta$, given that $\sin\theta = \frac{1}{\sqrt{5}}$ and $-90° \le \theta \le 90°$.

DEVELOPMENT

6 In this question, each part has two possible answers.

a If $\tan\alpha = \frac{1}{3}$, find $\sin\alpha$.

b If $\cos\alpha = \frac{2}{\sqrt{5}}$, find $\sin\alpha$.

c If $\sin\alpha = \frac{3}{5}$, find $\cos\alpha$.

d Find $\tan\theta$, given that $\cos\theta = -\frac{2}{3}$.

e Find $\tan\theta$, given that $\sin\theta = -\frac{12}{13}$.

f Find $\cos\theta$, given that $\tan\theta = -\frac{2}{\sqrt{3}}$.

7 **a** If $\cos\alpha = \frac{4}{5}$ and $\sin\alpha < 0$, find $\tan\alpha$.

b If $\tan\alpha = -\frac{8}{15}$ and $\sin\alpha > 0$, find $\cos\alpha$.

c Find $\cos\alpha$, given that $\sin\alpha = \frac{1}{4}$ and $\tan\alpha < 0$.

d Find $\sin\theta$, given that $\tan\theta = \frac{35}{12}$ and $\cos\theta > 0$.

e Find $\tan\theta$, given that $\sin\theta = -\frac{21}{29}$ and $\cos\theta > 0$.

f Find $\sin\theta$, given that $\cos\theta = -\frac{5}{6}$ and $\tan\theta < 0$.

8 **a** Find $\sec\theta$, given that $\sin\theta = \frac{1}{\sqrt{2}}$.

b Find $\tan\theta$, given that $\sec\theta = -\frac{17}{8}$.

c If $\sec C = -\frac{\sqrt{7}}{\sqrt{3}}$, find $\cot C$.

d If $\cot D = \frac{\sqrt{11}}{5}$, find $\operatorname{cosec} D$.

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9 **a** Find $\sec\theta$, given that $\operatorname{cosec}\theta = \frac{3}{2}$ and θ is obtuse.

b Find $\sec\theta$, given that $\cot\theta = \frac{9}{40}$ and θ is reflex.

c Find $\tan\theta$, given that $\sec\theta = -\frac{17}{8}$ and $0^\circ \le \theta \le 180^\circ$.

d Find $\operatorname{cosec}\theta$, given that $\cot\theta = \frac{2}{\sqrt{3}}$ and $-90^\circ \le \theta \le 90^\circ$.

10 **a** If $\sin A = -\frac{1}{3}$ and $\tan A < 0$, find $\sec A$.

b If $\operatorname{cosec} B = \frac{7}{3}$ and $\cos B < 0$, find $\tan B$.

c Find $\cot\theta$, given that $\sec\theta = -\sqrt{2}$ and $\operatorname{cosec}\theta < 0$.

d Find $\cos\theta$, given that $\operatorname{cosec}\theta = -\frac{13}{5}$ and $\cot\theta < 0$.

11 Given that $\sin\theta = \dfrac{p}{q}$, with θ obtuse and p and q both positive, find $\cos\theta$ and $\tan\theta$.

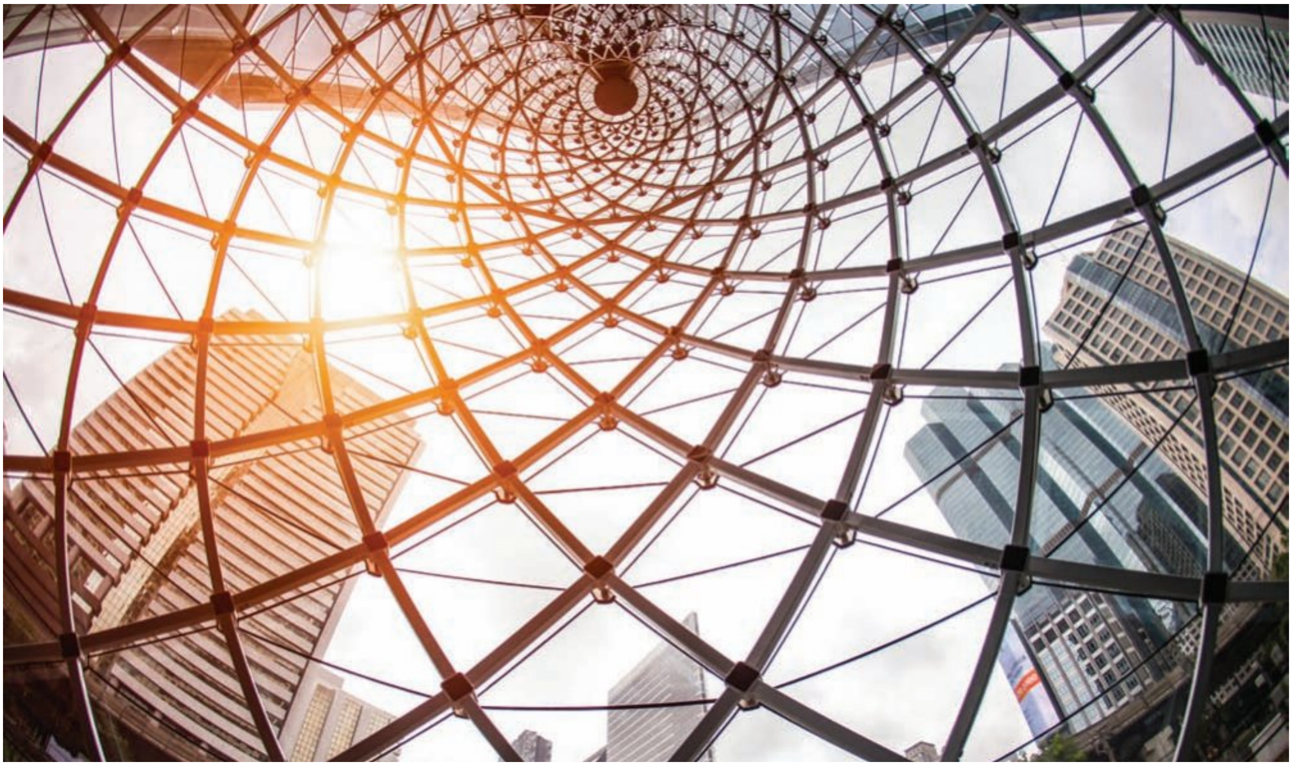
12 If $\tan\alpha = k$, where $k > 0$, find the possible values of $\sin\alpha$ and $\sec\alpha$.

13 **a** Prove the algebraic identity $(1 - t^2)^2 + (2t)^2 = (1 + t^2)^2$.

b If $\cos x = \dfrac{1 - t^2}{1 + t^2}$, where x is acute and t is positive, find expressions for $\sin x$ and $\tan x$.

ENRICHMENT

14 If $a > 0$ and $\sec\theta = a + \dfrac{1}{4a}$, prove that $\sec\theta + \tan\theta = 2a$ or $\dfrac{1}{2a}$.



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6G Trigonometric identities

Working with the trigonometric functions requires knowledge of a number of formulae called *trigonometric identities*, which relate trigonometric functions to each other. This section introduces eleven trigonometric identities in four groups:

- the three *reciprocal identities*,
- the two *ratio identities*,
- the three *Pythagorean identities*,
- the three *identities concerning complementary angles*.

The three reciprocal identities

It follows immediately from the definitions of the trigonometric functions in terms of x, y and r that:

20 THE RECIPROCAL IDENTITIES

For any angle θ:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad (\text{provided that } \sin\theta \neq 0)$$

$$\sec\theta = \frac{1}{\cos\theta} \quad (\text{provided that } \cos\theta \neq 0)$$

$$\cot\theta = \frac{1}{\tan\theta} \quad (\text{provided that } \tan\theta \neq 0 \text{ and } \cot\theta \neq 0)$$

Note: We cannot use a calculator to find $\cot 90^\circ$ or $\cot 270^\circ$ by first finding $\tan 90^\circ$ or $\tan 270^\circ$, because both are undefined. We already know, however, that

$$\cot 90^\circ = \cot 270^\circ = 0$$

The two ratio identities

Again using the definitions of the trigonometric functions:

21 THE RATIO IDENTITIES

For any angle θ:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad (\text{provided that } \cos\theta \neq 0)$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (\text{provided that } \sin\theta \neq 0)$$

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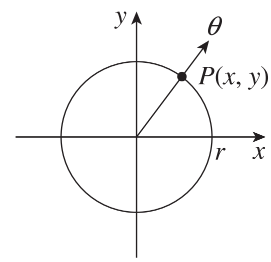
The three Pythagorean identities

The point $P(x, y)$ lies on the circle with centre O and radius r, so its coordinates satisfy

$$x^2 + y^2 = r^2.$$

Dividing through by r^2, $\quad \dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1,$

that is, $\quad \sin^2\theta + \cos^2\theta = 1.$



Dividing through by $\cos^2\theta$ and using the ratio and reciprocal identities,

$$\tan^2\theta + 1 = \sec^2\theta, \text{ provided that } \cos\theta \neq 0.$$

Dividing instead by $\sin^2\theta$, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$, provided that $\sin\theta \neq 0$.

These identities are called the *Pythagorean identities* because they rely on the circle equation $x^2 + y^2 = r^2$, which is a restatement of Pythagoras' theorem.

22 THE PYTHAGOREAN IDENTITIES

For any angle θ:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta \quad (\text{provided that } \cos\theta \neq 0)$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta \quad (\text{provided that } \sin\theta \neq 0)$$

The three identities for complementary angles

The angles θ and $90^\circ - \theta$ are called *complementary angles* because they add to a right angle. Three trigonometric identities relate the values of the trigonometric functions at an angle θ and at the complementary angle $90^\circ - \theta$.

23 THE COMPLEMENTARY ANGLE IDENTITIES

For any angle θ:

$$\cos(90^\circ - \theta) = \sin\theta$$
$$\cot(90^\circ - \theta) = \tan\theta \quad (\text{provided that } \tan\theta \text{ is defined})$$
$$\operatorname{cosec}(90^\circ - \theta) = \sec\theta \quad (\text{provided that } \sec\theta \text{ is defined})$$

For example,

$$\cos 20^\circ = \sin 70^\circ, \qquad \operatorname{cosec} 20^\circ = \sec 70^\circ, \qquad \cot 20^\circ = \tan 70^\circ.$$

Proof

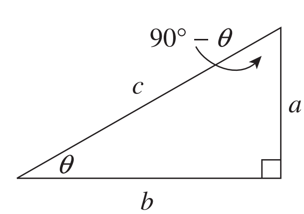
A [Acute angles]

The triangle to the right shows that when a right-angled triangle is viewed from $90^\circ - \theta$ instead of from θ, then the opposite side and the adjacent side are exchanged. Hence

$$\cos(90^\circ - \theta) = \frac{a}{c} = \sin\theta,$$

$$\cot(90^\circ - \theta) = \frac{a}{b} = \tan\theta,$$

$$\operatorname{cosec}(90^\circ - \theta) = \frac{c}{b} = \sec\theta.$$



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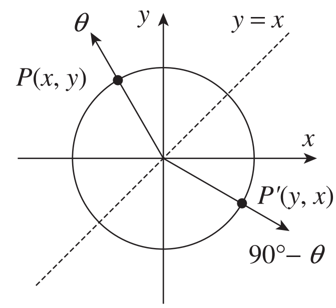
B [General angles]

For general angles, we take the full circle diagram, and reflect it in the diagonal line $y = x$. Let P' be the image of P under this reflection.

1 The image OP' of the ray OP corresponds to the angle $90^\circ - \theta$.

2 The image P' of $P(x, y)$ has coordinates $P'(y, x)$, because reflection in the line $y = x$ reverses the coordinates of each point.



Applying the definitions of the trigonometric functions to the angle $90^\circ - \theta$:

$$\cos(90^\circ - \theta) = \frac{y}{r} = \sin\theta,$$

$$\cot(90^\circ - \theta) = \frac{y}{x} = \tan\theta, \text{ provided that } x \neq 0,$$

$$\operatorname{cosec}(90^\circ - \theta) = \frac{r}{x} = \sec\theta, \text{ provided that } x \neq 0.$$

Cosine, cosecant and cotangent

The complementary identities are the origin of the names 'cosine', 'cosecant' and 'cotangent' — the prefix 'co-' has the same meaning as the prefix 'com-' of 'complementary' angle.

24 COSINE, COSECANT AND COTANGENT

<u>co</u>sine θ = sine (<u>com</u>plement of θ)
<u>co</u>tangent θ = tangent (<u>com</u>plement of θ)
<u>co</u>secant θ = secant (<u>com</u>plement of θ)

Proving identities

An *identity* is a statement that is true for all values of θ for which both sides are defined, and an identity needs to be proven. It is quite different from an *equation*, which needs to be solved and to have its solutions listed.

25 PROVING TRIGONOMETRIC IDENTITIES

- Work separately on the LHS and the RHS until they are the same.
- Use the four sets of identities in Boxes 20–23 above.

Mostly it is only necessary to work on one of the two sides. The important thing is never to treat it as an equation, moving terms from one side to the other.

Example 13 — 6G

Prove that $(1 - \cos\theta)(1 + \cos\theta) = \sin^2\theta$.

SOLUTION

$$\begin{aligned} \text{LHS} &= 1 - \cos^2\theta && \text{(use the difference of squares identity, from algebra)} \\ &= \sin^2\theta && \text{(use the Pythagorean identities in Box 22)} \\ &= \text{RHS}. \end{aligned}$$

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Example 14 6G

a Prove that $\sin A \sec A = \tan A$.

b Prove that $\dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta} = \sec^2\theta \operatorname{cosec}^2\theta$.

SOLUTION

a LHS $= \sin A \times \dfrac{1}{\cos A}$ (use the reciprocal identities in Box 20)

$= \tan A$ (use the ratio identities in Box 21)

$=$ RHS.

b LHS $= \dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta}$

$= \dfrac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta}$ (use a common denominator)

$= \dfrac{1}{\sin^2\theta\cos^2\theta}$ (use the Pythagorean identities in Box 22)

$= \sec^2\theta \operatorname{cosec}^2\theta$ (use the reciprocal identities in Box 20)

$=$ RHS.

Elimination

If x and y are given as functions of θ, then using the techniques of simultaneous equations, the variable θ can often be eliminated to give a relation (rarely a function) between x and y.

Example 15 6G

Eliminate θ from the following pair, and describe the graph of the relation:

$$x = 5\cos\theta$$
$$y = 5\sin\theta$$

SOLUTION

Squaring, $x^2 = 25\cos^2\theta$

and $y^2 = 25\sin^2\theta$,

and adding, $25\cos^2\theta + 25\sin^2\theta = x^2 + y^2$

Because $\cos^2\theta + \sin^2\theta = 1$, $x^2 + y^2 = 25$,

which is a circle of radius 5 and centre the origin.

Exercise 6G

FOUNDATION

1 Use your calculator to confirm that:

a $\sin 16^\circ = \cos 74^\circ$ **b** $\tan 63^\circ = \cot 27^\circ$ **c** $\sec 7^\circ = \operatorname{cosec} 83^\circ$

d $\sin^2 23^\circ + \cos^2 23^\circ = 1$ **e** $1 + \tan^2 55^\circ = \sec^2 55^\circ$ **f** $\operatorname{cosec}^2 32^\circ - 1 = \cot^2 32^\circ$

2 Simplify:

a $\dfrac{1}{\sin\theta}$ **b** $\dfrac{1}{\tan\alpha}$ **c** $\dfrac{\sin\beta}{\cos\beta}$ **d** $\dfrac{\cos\phi}{\sin\phi}$

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3 Simplify:

a $\sin\alpha \operatorname{cosec}\alpha$ b $\cot\beta \tan\beta$ c $\cos\theta \sec\theta$

4 Prove:

a $\tan\theta \cos\theta = \sin\theta$ b $\cot\alpha \sin\alpha = \cos\alpha$ c $\sin\beta \sec\beta = \tan\beta$

5 Use the complementary identities to simplify:

a $\sin(90^\circ - \theta)$ b $\sec(90^\circ - \alpha)$ c $\dfrac{1}{\cot(90^\circ - \beta)}$ d $\dfrac{\cos(90^\circ - \phi)}{\sin(90^\circ - \phi)}$

6 Use the Pythagorean identities to simplify:

a $\sin^2\alpha + \cos^2\alpha$ b $1 - \cos^2\beta$ c $1 + \tan^2\phi$ d $\sec^2 x - \tan^2 x$

7 Simplify:

a $1 - \sin^2\beta$ b $1 + \cot^2\phi$ c $\operatorname{cosec}^2 A - 1$ d $\cot^2\theta - \operatorname{cosec}^2\theta$

8 Use the reciprocal and ratio identities to simplify:

a $\dfrac{1}{\sec^2\theta}$ b $\dfrac{\sin^2\beta}{\cos^2\beta}$ c $\dfrac{\cos^2 A}{\sin^2 A}$ d $\sin^2\alpha \operatorname{cosec}^2\alpha$

9 Prove that:

a $\cos A \operatorname{cosec} A = \cot A$ b $\operatorname{cosec} x \cos x \tan x = 1$ c $\sin y \cot y \sec y = 1$

10 Simplify:

a $\dfrac{\cos\alpha}{\sec\alpha}$ b $\dfrac{\sin\alpha}{\operatorname{cosec}\alpha}$ c $\dfrac{\tan A}{\sec A}$ d $\dfrac{\cot A}{\operatorname{cosec} A}$

DEVELOPMENT

11 Prove the identities:

a $(1 - \sin\theta)(1 + \sin\theta) = \cos^2\theta$

b $(1 + \tan^2\alpha)\cos^2\alpha = 1$

c $(\sin A + \cos A)^2 = 1 + 2\sin A\cos A$

d $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

e $\tan^2\phi\cos^2\phi + \cot^2\phi\sin^2\phi = 1$

f $3\cos^2\theta - 2 = 1 - 3\sin^2\theta$

g $2\tan^2 A - 1 = 2\sec^2 A - 3$

h $1 - \tan^2\alpha + \sec^2\alpha = 2$

i $\cos^4 x + \cos^2 x\sin^2 x = \cos^2 x$

j $\cot\theta(\sec^2\theta - 1) = \tan\theta$

12 Prove the identities:

a $\tan\alpha \operatorname{cosec}\alpha = \sec\alpha$

b $\cot\beta \sec\beta = \operatorname{cosec}\beta$

c $\operatorname{cosec}^2\gamma + \sec^2\gamma = \operatorname{cosec}^2\gamma \sec^2\gamma$

d $\tan\delta + \cot\delta = \operatorname{cosec}\delta \sec\delta$

e $\operatorname{cosec}\phi - \sin\phi = \cos\phi \cot\phi$

f $\sec\theta - \cos\theta = \tan\theta \sin\theta$

13 Prove the identities:

a $\sin\theta \cos\theta \operatorname{cosec}^2\theta = \cot\theta$

b $(\cos\phi + \cot\phi)\sec\phi = 1 + \operatorname{cosec}\phi$

c $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$

d $\sin\beta + \cot\beta\cos\beta = \operatorname{cosec}\beta$

e $\dfrac{1 + \tan^2 x}{1 + \cot^2 x} = \tan^2 x$

f $\dfrac{1 + \cot x}{1 + \tan x} = \cot x$

g $\dfrac{1}{1 + \sin\theta} + \dfrac{1}{1 - \sin\theta} = 2\sec^2\theta$

h $\dfrac{1}{\sec\phi - \tan\phi} - \dfrac{1}{\sec\phi + \tan\phi} = 2\tan\phi$

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14 Prove that each expression is independent of θ by simplifying it.

a $x = a\cos\theta$ and $y = b\sin\theta$

b $x = a\tan\theta$ and $y = b\sec\theta$

c $x = 2 + \cos\theta$ and $y = 1 + \sin\theta$

d $x = \sin\theta + \cos\theta$ and $y = \sin\theta - \cos\theta$

15 Prove that each expression is independent of θ by simplifying it.

a $\dfrac{\cos^2\theta}{1 + \sin\theta} + \dfrac{\cos^2\theta}{1 - \sin\theta}$

b $\tan\theta\,(1 - \cot^2\theta) + \cot\theta\,(1 - \tan^2\theta)$

c $\dfrac{\tan\theta + \cot\theta}{\sec\theta\,\operatorname{cosec}\theta}$

d $\dfrac{\tan\theta + 1}{\sec\theta} - \dfrac{\cot\theta + 1}{\operatorname{cosec}\theta}$

ENRICHMENT

16 Prove the identities:

a $\dfrac{2\cos^3\theta - \cos\theta}{\sin\theta\cos^2\theta - \sin^3\theta} = \cot\theta$

b $\sec y + \tan y + \cot y = \dfrac{1 + \sin y}{\sin y\cos y}$

c $\dfrac{\cos A - \tan A\sin A}{\cos A + \tan A\sin A} = 1 - 2\sin^2 A$

d $(\sin\phi + \cos\phi)(\sec\phi + \operatorname{cosec}\phi) = 2 + \tan\phi + \cot\phi$

e $\dfrac{1}{1 + \tan^2 x} - \dfrac{1}{1 + \sec^2 x} = \dfrac{\cos^4 x}{1 + \cos^2 x}$

f $\dfrac{\cos\theta}{1 - \tan\theta} + \dfrac{\sin\theta}{1 - \cot\theta} = \sin\theta + \cos\theta$

g $(\tan\alpha + \cot\alpha - 1)(\sin\alpha + \cos\alpha) = \dfrac{\sec\alpha}{\operatorname{cosec}^2\alpha} + \dfrac{\operatorname{cosec}\alpha}{\sec^2\alpha}$

h $\dfrac{1}{\sec\theta + \tan\theta} = \sec\theta - \tan\theta = \dfrac{\cos\theta}{1 + \sin\theta}$

i $\sin^2 x\,(1 + n\cot^2 x) + \cos^2 x\,(1 + n\tan^2 x) = n + 1$
$= \sin^2 x\,(n + \cot^2 x) + \cos^2 x\,(n + \tan^2 x)$

j $\dfrac{1 + \operatorname{cosec}^2 A\tan^2 C}{1 + \operatorname{cosec}^2 B\tan^2 C} = \dfrac{1 + \cot^2 A\sin^2 C}{1 + \cot^2 B\sin^2 C}$

17 **a** If $\dfrac{a}{\sin A} = \dfrac{b}{\cos A}$, show that $\sin A\cos A = \dfrac{ab}{a^2 + b^2}$.

b If $\dfrac{a + b}{\operatorname{cosec} x} = \dfrac{a - b}{\cot x}$, show that $\operatorname{cosec} x\cot x = \dfrac{a^2 - b^2}{4ab}$.

18 Eliminate θ from each pair of equations.

a $x = \operatorname{cosec}^2\theta + 2\cot^2\theta$ and $y = 2\operatorname{cosec}^2\theta + \cot^2\theta$

b $x = \sin\theta - 3\cos\theta$ and $y = \sin\theta + 2\cos\theta$

c $x = \sin\theta + \cos\theta$ and $y = \tan\theta + \cot\theta$ (Hint: Find $x^2 y$.)

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6H Trigonometric equations

This piece of work is absolutely vital, because so many problems in later work end up with a trigonometric equation that has to be solved.

There are many small details and qualifications in the methods, and the subject needs a great deal of careful study.

Pay attention to the domain

To begin with a warning, before any other details:

26 THE DOMAIN

Always pay careful attention to the domain in which the angle can lie.

Equations involving boundary angles

Boundary angles are a special case because they do not lie in any quadrant.

27 THE BOUNDARY ANGLES

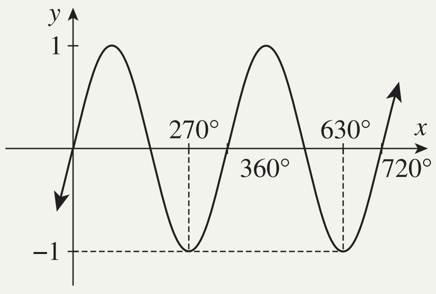
If the solutions are boundary angles, read the solutions off a sketch of the graph.

Example 16 6H

a Solve $\sin x = -1$, for $0^\circ \le x \le 720^\circ$.
b Solve $\sin x = 0$, for $0^\circ \le x \le 720^\circ$.

SOLUTION

a The graph of $y = \sin x$ is drawn to the right.
Examine where the curve touches the line $y = -1$, and read off the x-coordinates of these points.
The solution is $x = 270^\circ$ or 630°.



b Examine where the graph crosses the x-axis.
The solution is $x = 0^\circ, 180^\circ, 360^\circ, 540^\circ$ or 720°.

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The standard method — quadrants and the related acute angle

Most trigonometric equations eventually come down to one or more equations such as

$$\sin x = -\tfrac{1}{2}, \quad \text{where } -180^\circ \le x \le 180^\circ.$$

Provided that the angle is not a boundary angle, the method is:

28 THE QUADRANTS-AND-RELATED-ANGLE METHOD

1. Draw a quadrant diagram, then draw a ray in each quadrant that the angle could be in.
2. Find the related acute angle (only work with positive numbers here):
   - **a** using special angles, or
   - **b** using the calculator to find an approximation.

   Never enter a negative number into the calculator at this point.
3. Mark the angles on the ends of the rays, taking account of any restrictions on x, and write a conclusion.

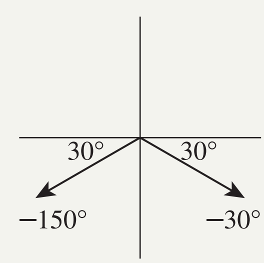
Example 17 — 6H

Find an exact solution if possible. Otherwise solve correct to the nearest degree.

a $\sin x = -\tfrac{1}{2}$, for $-180^\circ \le x \le 180^\circ$.

b $\tan x = -3$, for $0^\circ \le x \le 360^\circ$.

SOLUTION

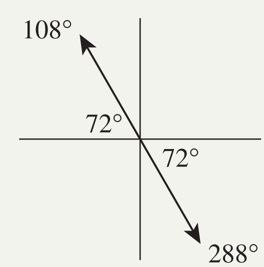


a Here $\sin x = -\tfrac{1}{2}$, where $-180^\circ \le x \le 180^\circ$.

Because $\sin x$ is negative, x is in quadrant 3 or 4.

The sine of the related acute angle is $+\tfrac{1}{2}$, so the related angle is 30°.

Hence $x = -15^\circ$ or -30°.



b Here $\tan x = -3$, where $0^\circ \le x \le 360^\circ$.

Because $\tan x$ is negative, x is in quadrant 2 or 4.

The tangent of the related acute angle is $+3$,

so the related angle is about 72°.

Hence $x \doteqdot 108^\circ$ or 288°.

Note: When using the calculator, *never enter a negative number and take an inverse trigonometric function of it*. In the example above, the calculator was used to find the *related acute angle* whose tan was 3, which is $71^\circ 34'$, correct to the nearest minute. The positive number 3 was entered, not -3.

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The three reciprocal functions

The calculator doesn't have specific keys for secant, cosecant and cotangent. These functions should be converted to sine, cosine and tangent as quickly as possible.

29 THE RECIPROCAL FUNCTIONS

Take reciprocals to convert the three reciprocal functions secant, cosecant and cotangent to the three more common functions.

Example 18 — 6H

a Solve $\operatorname{cosec} x = -2$, for $-180° \le x \le 180°$.

b Solve $\sec x = 0.7$, for $-180° \le x \le 180°$.

SOLUTION

a Taking reciprocals of both sides gives

$$\sin x = -\tfrac{1}{2},$$

which was solved in the previous worked example,

so $\quad x = -150° \text{ or } -30°$.

b Taking reciprocals of both sides gives

$$\cos x = \tfrac{10}{7},$$

which has no solutions, because $\cos\theta$ can never be greater than 1.

Equations with compound angles

In some equations, the angle is a function of x rather than simply x itself. For example,

$$\tan 2x = \sqrt{3}, \text{ where } 0° \le x \le 360°, \qquad \text{or}$$

$$\sin(x - 250°) = \frac{\sqrt{3}}{2}, \text{ where } 0° \le x \le 360°.$$

These equations are really trigonometric equations in the *compound angles* $2x$ and $(x - 250°)$ respectively. The secret lies in solving for the *compound angle*, and in *first calculating the domain for that compound angle*.

30 EQUATIONS WITH COMPOUND ANGLES

1. Let u be the compound angle.
2. Find the restrictions on u from the given restrictions on x.
3. Solve the trigonometric equation for u.
4. Hence solve for x.

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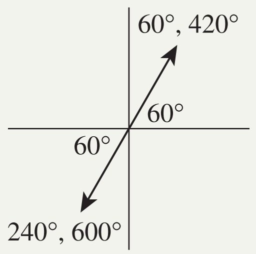
Example 19 6H

Solve $\tan 2x = \sqrt{3}$, where $0^\circ \le x \le 360^\circ$.

SOLUTION



Let $u = 2x$.

Then $\tan u = \sqrt{3}$.

The restriction on x is $0^\circ \le x \le 360^\circ$

$\boxed{\times 2}$ $0^\circ \le 2x \le 720^\circ$

and replacing $2x$ by u, $0^\circ \le u \le 720^\circ$.

(The restriction on u is the key step here.)

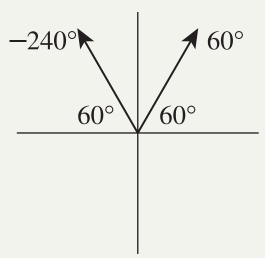
Hence from the diagram, $u = 60^\circ, 240^\circ, 420^\circ$ or 600°.

Because $x = \frac{1}{2}u$, $x = 30^\circ, 120^\circ, 210^\circ$ or 300°.

Example 20 6H

Solve $\sin(x - 250^\circ) = \dfrac{\sqrt{3}}{2}$, where $0^\circ \le x \le 360^\circ$.

SOLUTION



Let $u = x - 250^\circ$.

Then $\sin u = \dfrac{\sqrt{3}}{2}$.

The restriction on x is $0^\circ \le x \le 360^\circ$

$\boxed{-250^\circ}$ $-250^\circ \le x - 250^\circ \le 110^\circ$

and replacing $x - 250^\circ$ by u, $-250^\circ \le u \le 110^\circ$.

(Again, the restriction on u is the key step here.)

Hence from the diagram, $u = -240^\circ$ or 60°.

Because $x = u + 250^\circ$, $x = 10^\circ$ or 310°.

Equations with more than one trigonometric function

Some trigonometric equations involve more than one trigonometric function. For example,

$$\sin x + \sqrt{3}\cos x = 0.$$

The general approach is to use trigonometric identities to produce an equation in only one trigonometric function.

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Example 21 6H

Solve $\sin x + \sqrt{3}\cos x = 0$, where $0^\circ \le x \le 360^\circ$.

SOLUTION

$$\sin x + \sqrt{3}\cos x = 0$$

$\boxed{\div \cos x}$

$$\tan x + \sqrt{3} = 0$$

$$\tan x = -\sqrt{3}, \quad \text{where} \quad 0^\circ \le x \le 360^\circ.$$

Because $\tan x$ is negative, x is in quadrants 2 or 4.

The tan of the related acute angle is $\sqrt{3}$, so the related angle is 60°.

Hence $x = 120^\circ$ or 300°.

Exercise 6H

FOUNDATION

1 Solve each equation for $0^\circ \le \theta \le 360^\circ$. (Each related acute angle is 30°, 45° or 60°.)

a $\sin\theta = \dfrac{\sqrt{3}}{2}$ b $\sin\theta = \dfrac{1}{2}$ c $\tan\theta = 1$ d $\tan\theta = \sqrt{3}$

e $\cos\theta = -\dfrac{1}{\sqrt{2}}$ f $\tan\theta = -\sqrt{3}$ g $\sin\theta = -\dfrac{1}{2}$ h $\cos\theta = -\dfrac{\sqrt{3}}{2}$

2 Solve each equation for $0^\circ \le \theta \le 360^\circ$. (The trigonometric graphs are helpful here.)

a $\sin\theta = 1$ b $\cos\theta = 1$ c $\cos\theta = 0$

d $\cos\theta = -1$ e $\tan\theta = 0$ f $\sin\theta = -1$

3 Solve each equation for $0^\circ \le x \le 360^\circ$. Use your calculator to find the related acute angle in each case, and give solutions correct to the nearest degree.

a $\cos x = \frac{3}{7}$ b $\sin x = 0.1234$ c $\tan x = 7$

d $\sin x = -\frac{2}{3}$ e $\tan x = -\frac{20}{9}$ f $\cos x = -0.77$

4 Solve for $0^\circ \le \alpha \le 360^\circ$. Give solutions correct to the nearest minute where necessary.

a $\sin\alpha = 0.1$ b $\cos\alpha = -0.1$ c $\tan\alpha = -1$ d $\operatorname{cosec}\alpha = -1$

e $\sin\alpha = 3$ f $\sec\alpha = -2$ g $\sqrt{3}\tan\alpha + 1 = 0$ h $\cot\alpha = 3$

DEVELOPMENT

5 Solve for $-180^\circ \le x \le 180^\circ$. Give solutions correct to the nearest minute where necessary.

a $\tan x = -0.3$ b $\cos x = 0$ c $\sec x = \sqrt{2}$ d $\sin x = -0.7$

6 Solve each equation for $0^\circ \le \theta \le 720^\circ$.

a $2\cos\theta - 1 = 0$ b $\cot\theta = 0$ c $\operatorname{cosec}\theta + 2 = 0$ d $\tan\theta = \sqrt{2} - 1$

7 Solve each equation for $0^\circ \le x \le 360^\circ$. (Let $u = 2x$.)

a $\sin 2x = \frac{1}{2}$ b $\tan 2x = \sqrt{3}$ c $\cos 2x = -\frac{1}{\sqrt{2}}$ d $\sin 2x = -1$

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8 Solve each equation for $0^\circ \le \alpha \le 360^\circ$. (Let u be the compound angle.)

a $\tan(\alpha - 45^\circ) = \frac{1}{\sqrt{3}}$

b $\sin(\alpha + 30^\circ) = -\frac{\sqrt{3}}{2}$

c $\cos(\alpha + 60^\circ) = 1$

d $\cos(\alpha - 75^\circ) = -\frac{1}{\sqrt{2}}$

9 Solve each equation for $0^\circ \le \theta \le 360^\circ$.

a $\sin\theta = \cos\theta$

b $\sin\theta + \cos\theta = 0$

c $\sin\theta = \sqrt{3}\cos\theta$

d $\sqrt{3}\sin\theta + \cos\theta = 0$

ENRICHMENT

10 Solve for $0^\circ \le \theta \le 360^\circ$, giving solutions correct to the nearest minute where necessary:

a $\sin^2\theta - \sin\theta = 0$

b $2\cos^2\theta = \cos\theta$

c $2\sin\theta\cos\theta = \sin\theta$

d $2\sin^2\theta + \sin\theta = 1$

e $\sec^2\theta + 2\sec\theta = 8$

f $3\cos^2\theta + 5\cos\theta = 2$

g $4\operatorname{cosec}^2\theta - 4\operatorname{cosec}\theta - 15 = 0$

h $4\sin^3\theta = 3\sin\theta$

11 Use the quadratic formula to solve $4\cos^2\theta + 2\sin\theta = 3$ for $0^\circ \le \theta \le 360^\circ$.

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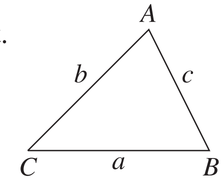
6I The sine rule and the area formula

The last three sections of this chapter review the sine rule, the area formula and the cosine rule. These three rules extend trigonometry to non-right-angled triangles, and are closely connected to the standard congruence tests of Euclidean geometry.

The usual statement of all three rules uses the convention shown in the diagram to the right. The vertices are named with upper-case letters, then each side takes as its name the lower-case letter of the opposite vertex.



Statement of the sine rule

The sine rule states that the ratio of each side of a triangle to the sine of its opposite angle is constant for the triangle:

31 THE SINE RULE

In any triangle ABC,

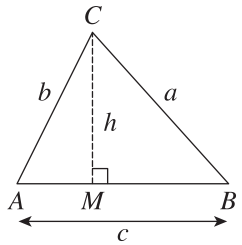
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

'The ratio of each side to the sine of the opposite angle is constant.'

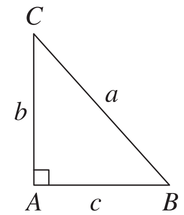
Proving the sine rule by constructing an altitude

Any proof of the sine rule must involve a construction with a right angle, because trigonometry so far has been restricted to right-angled triangles. Thus we construct an *altitude*, which is the perpendicular from one vertex to the opposite side. This breaks the triangle into two right-angled triangles, where previous methods can be used.

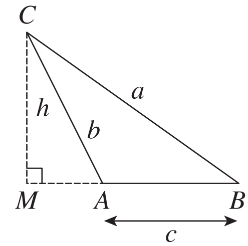
Given: Let ABC be any triangle. There are three cases, depending on whether $\angle A$ is an acute angle, a right angle, or an obtuse angle.



Case 1: $\angle A$ is acute



Case 2: $\angle A = 90°$



Case 3: $\angle A$ is obtuse

Aim: To prove that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

In case 2, $\sin A = \sin 90° = 1$, and $\sin B = \dfrac{b}{a}$, so the result is clear.

Construction: In the remaining cases 1 and 3, construct the altitude from C, meeting AB (produced if necessary) at M. Let h be the length of CM.

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Proof

Case 1 — Suppose that $\angle A$ is acute.

In $\triangle ACM$, $\quad \dfrac{h}{b} = \sin A$

$\boxed{\times\, b}\quad h = b \sin A.$

In $\triangle BCM$, $\quad \dfrac{h}{a} = \sin B$

$\boxed{\times\, a}\quad h = a \sin B.$

Equating these, $\quad b \sin A = a \sin B$

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

Case 3 — Suppose that $\angle A$ is obtuse.

In $\triangle ACM$, $\dfrac{h}{b} = \sin\,(180^\circ - A)$,

and because $\sin\,(180^\circ - A) = \sin A$,

$\boxed{\times\, b}\quad h = b \sin A.$

In $\triangle BCM$, $\quad \dfrac{h}{a} = \sin B$

$\boxed{\times\, a}\quad h = a \sin B.$

Equating these, $\quad b \sin A = a \sin B$

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

Using the sine rule to find a side — the AAS congruence situation

When using the sine rule to find a side, one side and two angles must be known. This is the situation described by the AAS congruence test from geometry, so we know that there will only be one solution.

32 USING THE SINE RULE TO FIND A SIDE

In the AAS congruence situation:

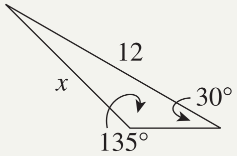
$$\frac{\text{unknown side}}{\text{sine of its opposite angle}} = \frac{\text{known side}}{\text{sine of its opposite angle}}.$$

Always place the unknown side at the top left of the equation.

If two angles of a triangle are known, so is the third, because the angles add to 180°.

Example 22 — 6I

Find the side x in the triangle drawn to the right.



SOLUTION

Using the sine rule, and placing the unknown at the top left,

$$\frac{x}{\sin 30^\circ} = \frac{12}{\sin 135^\circ}$$

$\boxed{\times\, \sin 30^\circ}\quad x = \dfrac{12 \sin 30^\circ}{\sin 135^\circ}.$

Using special angles, $\quad \sin 30^\circ = \dfrac{1}{2}$,

and $\quad \sin 135^\circ = +\sin 45^\circ \quad$ (sine is positive for obtuse angles)

$\qquad = \dfrac{1}{\sqrt{2}}. \quad$ (the related acute angle is 45°)

Hence $\quad x = 12 \times \dfrac{1}{2} \times \dfrac{\sqrt{2}}{1}$

$\qquad = 6\sqrt{2}.$

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The area formula

The well-known area formula, area $= \frac{1}{2} \times$ base $\times$ height, can be generalised to a formula involving two sides and the included angle.

33 THE AREA FORMULA

In any triangle ABC,

$$\text{area } \triangle ABC = \tfrac{1}{2} bc \sin A.$$

'The area of a triangle is half the product of any two sides times the sine of the included angle.'

Proof

Use the same three diagrams as in the proof of the sine rule.

In case 2, $\angle A = 90^\circ$ and $\sin A = 1$, so area $= \frac{1}{2} bc = \frac{1}{2} bc \sin A$, as required.

$$\begin{aligned} \text{Otherwise, area} &= \tfrac{1}{2} \times \text{base} \times \text{height} \\ &= \tfrac{1}{2} \times AB \times h \\ &= \tfrac{1}{2} \times c \times b \sin A, \text{ because we proved before that } h = b \sin A. \end{aligned}$$

Using the area formula — the SAS congruence situation

The area formula requires the SAS congruence situation in which two sides and the included angle are known.

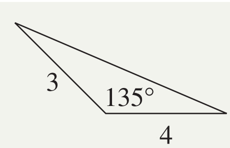
34 USING THE AREA FORMULA

In the SAS congruence situation:

area $=$ (half the product of two sides) $\times$ (sine of the included angle) .

Example 23 6I

Find the area of the triangle drawn to the right.



SOLUTION

Using the formula, area $= \dfrac{1}{2} \times 3 \times 4 \times \sin 135^\circ$.

Because $\sin 135^\circ = \dfrac{1}{\sqrt{2}}$, (as in Example 22)

$$\begin{aligned} \text{area} &= 6 \times \frac{1}{\sqrt{2}} \\ &= 6 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \quad \text{(rationalise the denominator)} \\ &= 3\sqrt{2} \text{ square units.} \end{aligned}$$

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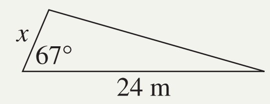
Using the area formula to find a side or an angle

Substituting into the area formula when the area is known may allow an unknown side or angle to be found.

When finding an angle, the formula will always give a single answer for sin θ. There will be two solutions for θ, however, one acute and one obtuse.

Example 24 6I

Find x, correct to four significant figures, given that the triangle to the right has area $72\,\text{m}^2$.



SOLUTION

Substituting into the area formula,

$$72 = \tfrac{1}{2} \times 24 \times x \times \sin 67^\circ$$

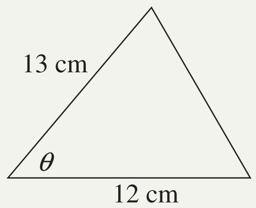
$$72 = 12 \times x \times \sin 67^\circ$$

$\boxed{\div\ 12\sin 67^\circ}$ $\quad x = \dfrac{6}{\sin 67^\circ}$

$$\doteqdot 6.518\ \text{metres}.$$

Example 25 6I

Find θ, correct to the nearest minute, given that the triangle to the right has area $60\,\text{cm}^2$.



SOLUTION

Substituting into the area formula,

$$60 = \tfrac{1}{2} \times 13 \times 12 \times \sin\theta$$

$$60 = 6 \times 13\sin\theta$$

$\boxed{\div\ (6 \times 13)}$ $\quad \sin\theta = \tfrac{10}{13}.$

Hence $\quad \theta \doteqdot 50^\circ 17'$ or $129^\circ 43'$.

Notice that the second angle is the supplement of the first.

Using the sine rule to find an angle — the ambiguous ASS situation

The SAS congruence test requires that the angle be included between the two sides. When two sides and a non-included angle are known, the resulting triangle may not be determined up to congruence, and two triangles may be possible. This situation may be referred to as ‘the ambiguous ASS situation’.

When the sine rule is applied in the ambiguous ASS situation, there is only one answer for the sine of an angle. There may be two possible solutions for the angle itself, however, one acute and one obtuse.

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35 USING THE SINE RULE TO FIND AN ANGLE

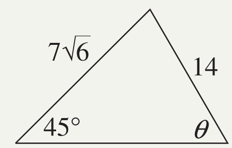
In the ambiguous ASS situation, where two sides and a non-included angle of the triangle are known,

$$\frac{\text{sine of unknown angle}}{\text{its opposite side}} = \frac{\text{sine of known angle}}{\text{its opposite side}}.$$

Always check the angle sum to see whether both answers are possible.

Example 26 6I

Find the angle θ in the triangle drawn to the right.



SOLUTION

$$\frac{\sin\theta}{7\sqrt{6}} = \frac{\sin 45^\circ}{14} \quad \text{(always place the unknown at the top left)}$$

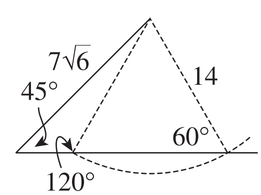
$$\sin\theta = 7 \times \sqrt{6} \times \frac{1}{14} \times \frac{1}{\sqrt{2}}, \text{ because } \sin 45^\circ = \frac{1}{\sqrt{2}},$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ \quad \text{or} \quad 120^\circ.$$

Note: There are *two* angles whose sine is $\frac{\sqrt{3}}{2}$, one of them acute and the other obtuse. Moreover,

$$120^\circ + 45^\circ = 165^\circ,$$

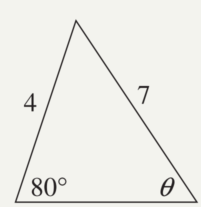


leaving just 15° for the third angle in the obtuse case, so it all seems to work. To the right is the ruler-and-compasses construction of the triangle, showing how two different triangles can be produced from the same given ASS measurements.

In many examples, however, the obtuse angle solution can be excluded using the fact that the angle sum of a triangle cannot exceed 180°.

Example 27 6I

Find the angle θ in the triangle drawn to the right, and show that there is only one solution.



SOLUTION

$$\frac{\sin\theta}{4} = \frac{\sin 80^\circ}{7} \quad \text{(always place the unknown at the top left)}$$

$$\sin\theta = \frac{4\sin 80^\circ}{7}$$

$$\theta \doteqdot 34^\circ 15' \quad \text{or} \quad 145^\circ 45'$$

But $\theta \doteqdot 145^\circ 45'$ is impossible, because the angle sum would then exceed 180°,
so $\theta \doteqdot 34^\circ 15'$ is the only solution.

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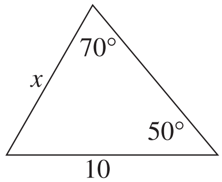
Exercise 6I

FOUNDATION

1 Find x in each triangle, correct to one decimal place.

a


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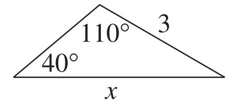

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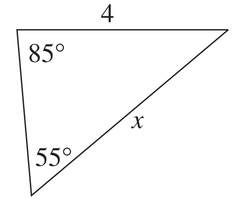

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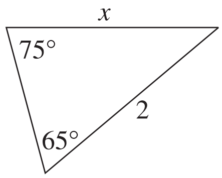

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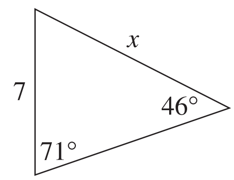

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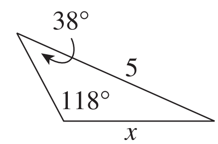
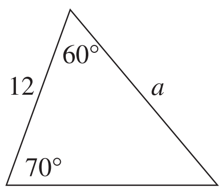


2 Find the value of the pronumeral in each triangle, correct to two decimal places.

a


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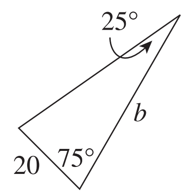

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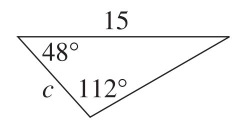

3 Find θ in each triangle, correct to the nearest degree.

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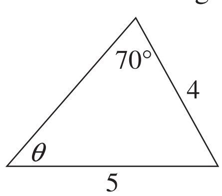

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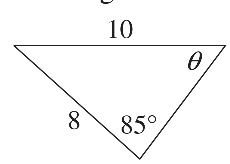

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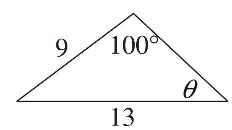

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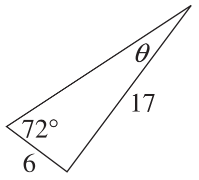

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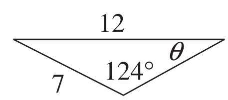

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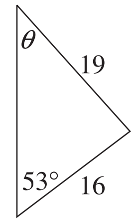
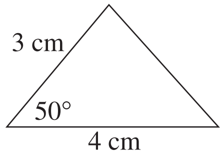


4 Find the area of each triangle, correct to the nearest square centimetre.

a


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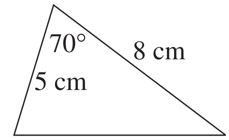

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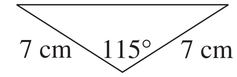

5 **a** Sketch $\triangle ABC$ in which $A = 43°$, $B = 101°$ and $a = 7.5$ cm.
b Find b and c, in cm correct to two decimal places.

6 Sketch $\triangle ABC$ in which $a = 2.8$ cm, $b = 2.7$ cm and $A = 52°21'$.
a Find B, correct to the nearest minute.
b Hence find C, correct to the nearest minute.
c Hence find the area of $\triangle ABC$ in cm^2, correct to two decimal places.

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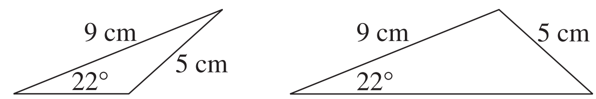
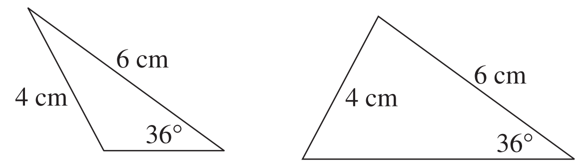
DEVELOPMENT

7 There are two triangles that have sides 9 cm and 5 cm, and in which the angle opposite the 5 cm side is 22°. Find, in each case, the size of the angle opposite the 9 cm side, correct to the nearest degree.



8 Two triangles are shown, with sides 6 cm and 4 cm, in which the angle opposite the 4 cm side is 36°. Find, in each case, the angle opposite the 6 cm side, correct to the nearest degree.

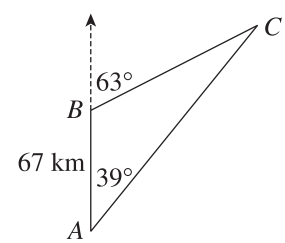


9 Sketch $\triangle PQR$ in which $p = 7$ cm, $q = 15$ cm and $\angle P = 25^\circ 50'$.

a Find the two possible sizes of $\angle Q$, correct to the nearest minute.

b For each possible size of $\angle Q$, find r in cm, correct to one decimal place.

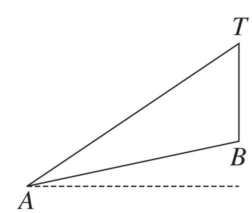
10 A travelling salesman drove from town A to town B, then to town C, and finally directly home to town A.
Town B is 67 km north of town A, and the bearings of town C from towns A and B are 039°T and 063°T respectively.
Find, correct to the nearest kilometre, how far the salesman drove.



11 Melissa is standing at A on a path that leads to the base B of a vertical flagpole. The path is inclined at 12° to the horizontal and the angle of elevation of the top T of the flagpole from A is 34°.

a Explain why $\angle TAB = 22^\circ$ and $\angle ABT = 102^\circ$.

b Given that $AB = 20$ metres, find the height of the flagpole, correct to the nearest metre.

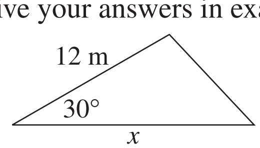


12 **a** In $\triangle ABC$, $\sin A = \frac{1}{4}$, $\sin B = \frac{2}{3}$ and $a = 12$. Find the value of b.

b In $\triangle PQR$, $p = 25$, $q = 21$ and $\sin Q = \frac{3}{5}$. Find the value of $\sin P$.

13 Substitute into the area formula to find the side length x, given that each triangle has area $48\,\text{m}^2$. Give your answers in exact form, or correct to the nearest centimetre.

a



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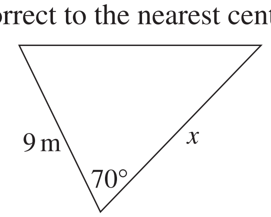

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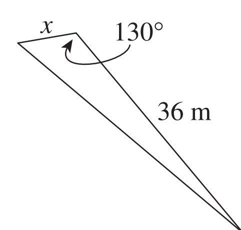
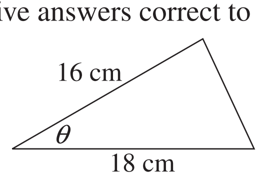


14 Substitute into the area formula to find the angle θ, given that each triangle has area $72\,\text{cm}^2$. Give answers correct to the nearest minute, where appropriate.

a



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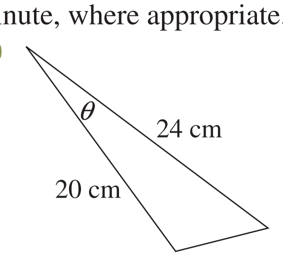

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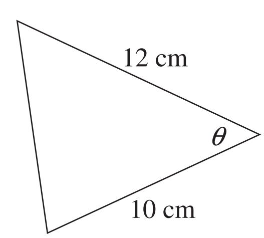

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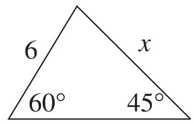
15 Find the exact value of x in each diagram.

a



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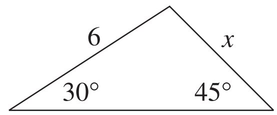

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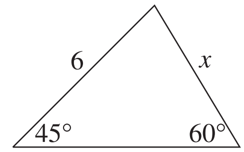

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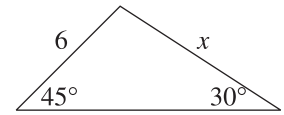
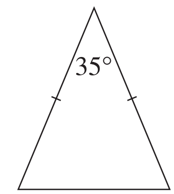
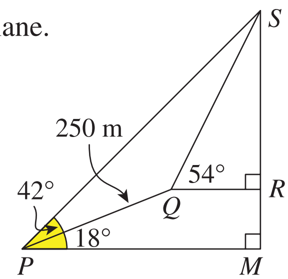


16 The diagram to the right shows an isosceles triangle in which the apex angle is 35°. Its area is $35\,\text{cm}^2$.
Find the length of the equal sides, correct to the nearest millimetre.



17 The summit S of a mountain is observed by climbers at two points P and Q that are 250 metres apart. The point Q is uphill and due north from P, and the summit is uphill and due north from Q — thus the whole diagram to the right all lies in one vertical plane.

Construct perpendicular lines PM and QR to the vertical line through S. The interval PQ is inclined at 18° to the horizontal, and the respective angles of elevation of S from P and Q are 42° and 54°.



a Explain why $\angle PSQ = 12^\circ$ and $\angle PQS = 144^\circ$.

b Show that $SP = \dfrac{250 \sin 144^\circ}{\sin 12^\circ}$.

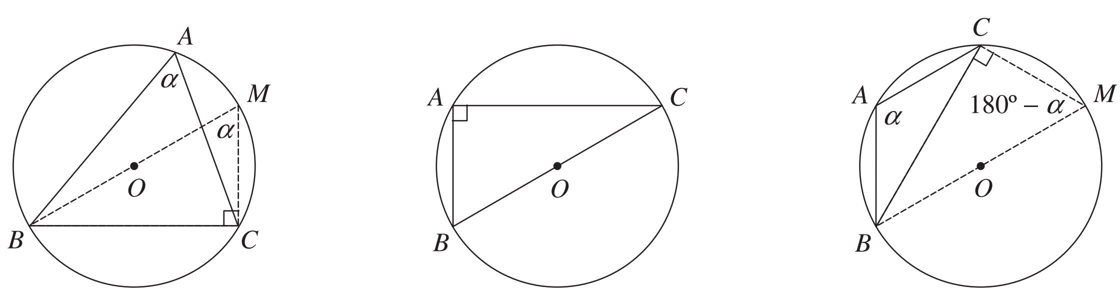
c Hence find the vertical height SM, correct to the nearest metre.

ENRICHMENT

18 **a** Suppose that we are using the sine rule in an ASS situation to find the angle θ in a triangle known to exist. Using the known angle β and the two sides, we have found the value of $\sin\theta$, and from that we have found the related angle α. Explain why the obtuse value $\theta = 180^\circ - \alpha$ is a solution if and only if the related angle α is greater than the known angle β.

b Wayan used the sine rule in the ASS situation to find $\angle A$ in $\triangle ABC$ where $a = 3$, $b = 2$ and $\angle B = 150^\circ$. He found that $\sin A = \frac{3}{4}$, and discarding the obtuse solution, he concluded that $\angle A \doteqdot 49^\circ$. What is wrong, and why did it go wrong?

19 [The circumcircle]
By the sine rule, the three ratios $\dfrac{a}{\sin A}$, $\dfrac{b}{\sin B}$ and $\dfrac{c}{\sin C}$ are equal. Each is a length over a ratio, so is a length, and in this question, you will show that this common length is the diameter of the *circumcircle* that passes through all three vertices.



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a In the left-hand diagram, the circumcentre lies inside the triangle ABC. Show that $\dfrac{a}{\sin A}$ is the diameter BM of the circumcircle. The proof requires a circle theorem that some will have proven in earlier years, that $\angle A = \angle M$ because they both stand on the same arc BC.

b Prove the result in the middle diagram, where the circumcentre lies on BC.

c Prove the result in the right-hand diagram, where the circumcentre lies outside the triangle. Here you will need the theorem that the opposite angles of the quadrilateral $ABMC$ whose vertices lie on the circle are supplementary.

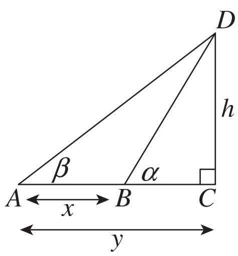
20 [The circumcircle]

We saw in the previous question that $\dfrac{a}{\sin A}$, $\dfrac{b}{\sin B}$ and $\dfrac{c}{\sin C}$ are each equal to the diameter of the circumcircle of $\triangle ABC$.

a In a particular triangle ABC, $\angle A = 60^\circ$ and $BC = 12$. Find the diameter D_C of the circumcircle.

b A triangle $\triangle PQR$ with $\angle RPQ = 150^\circ$ is inscribed inside a circle of diameter D_C. Find the ratio $D_C : RQ$.

21 **a** Show that $h = \dfrac{x \sin \alpha \sin \beta}{\sin(\alpha - \beta)}$ in the diagram to the right.



b Use the fact that $\tan \alpha = \dfrac{h}{y - x}$ and $\tan \beta = \dfrac{h}{y}$ to show that

$$h = \frac{x \tan \alpha \tan \beta}{\tan \alpha - \tan \beta}.$$

c Combine the expressions in parts **a** and **b** to show that

$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$.

d Hence find the exact value of $\sin 15^\circ$.

22 Three tourists T_1, T_2 and T_3 at ground level are observing a landmark L whose top we shall call B. T_1 is due north of L, T_3 is due east of L, and T_2 is on the line of sight from T_1 to T_3 and between them. The angles of elevation to L from T_1, T_2 and T_3 are 25°, 32° and 36° respectively.

a Show that $\tan \angle BT_1T_2 = \dfrac{\cot 36^\circ}{\cot 25^\circ}$.

b Use the sine rule in $\triangle BT_1T_2$ to find, correct to the nearest minute, the bearing of T_2 from L.

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6J The cosine rule

The cosine rule is a generalisation of Pythagoras' theorem to non-right-angled triangles. It gives a formula for the square of any side in terms of the squares of the other two sides and the cosine of the opposite angle.

36 THE COSINE RULE

In any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

'The square of any side of a triangle equals:
the sum of the squares of the other two sides, minus
twice the product of those sides and the cosine of their included angle.'

The formula should be understood as Pythagoras' theorem with an error term $-2bc\cos A$ that disappears when $A = 90°$, leaving Pythagoras' theorem.

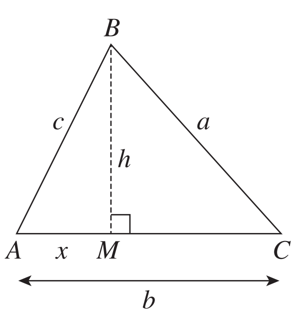
37 THE COSINE RULE AND PYTHAGORAS' THEOREM

- When $\angle A = 90°$, then $\cos A = 0$ and the cosine rule is Pythagoras' theorem.
- The last term is thus a correction to Pythagoras' theorem when $\angle A \neq 90°$.
- When $\angle A < 90°$, then $\cos A$ is positive, so $a^2 < b^2 + c^2$.
  When $\angle A > 90°$, then $\cos A$ is negative, so $a^2 > b^2 + c^2$.

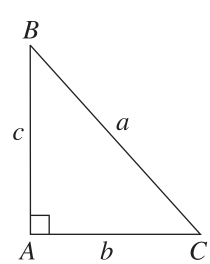
Proof of the cosine rule

The proof is based on Pythagoras' theorem, and again begins with the construction of an altitude.

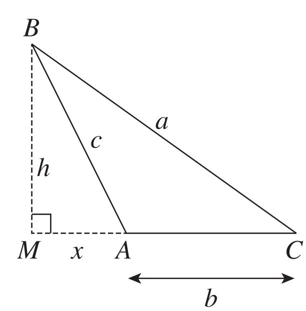
Given: Let ABC be any triangle. Again, there are three cases, according as to whether $\angle A$ is acute, obtuse, or a right angle.



Case 1: $\angle A$ is acute



Case 2: $\angle A = 90°$



Case 3: $\angle A$ is obtuse

Aim: To prove that $a^2 = b^2 + c^2 - 2bc\cos A$.

In case 2, $\cos A = 0$, and this is just Pythagoras' theorem.

Construction: In the remaining cases 1 and 3, construct the altitude from B, meeting AC (produced if necessary) at M. Let $BM = h$ and $AM = x$.

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Proof

Case 1 — Suppose that $\angle A$ is acute.

By Pythagoras' theorem in $\triangle BMC$,

$$a^2 = h^2 + (b - x)^2.$$

By Pythagoras' theorem in $\triangle BMA$,

$$h^2 = c^2 - x^2,$$

$$\text{so}\quad a^2 = c^2 - x^2 + (b - x)^2$$
$$= c^2 - x^2 + b^2 - 2bx + x^2$$
$$= b^2 + c^2 - 2bx. \qquad (*)$$

Using trigonometry in $\triangle ABM$,

$$x = c\cos A.$$

$$\text{So}\quad a^2 = b^2 + c^2 - 2bc\cos A.$$

Case 3 — Suppose that $\angle A$ is obtuse.

By Pythagoras' theorem in $\triangle BMC$,

$$a^2 = h^2 + (b + x)^2.$$

By Pythagoras' theorem in $\triangle BMA$,

$$h^2 = c^2 - x^2,$$

$$\text{so}\quad a^2 = c^2 - x^2 + (b + x)^2$$
$$= c^2 - x^2 + b^2 + 2bx + x^2$$
$$= b^2 + c^2 + 2bx. \qquad (*)$$

Using trigonometry in $\triangle ABM$,

$$x = c\cos(180^\circ - A)$$
$$= -c\cos A.$$

$$\text{So}\quad a^2 = b^2 + c^2 - 2bc\cos A.$$

Note: The identity $\cos(180^\circ - A) = -\cos A$ is the key step in Case 3 of the proof. The cosine rule appears in Euclid's geometry book, but without any mention of the cosine ratio — the form given there is approximately the two statements in the proof marked with $(*)$.

Using the cosine rule to find a side — the SAS situation

For the cosine rule to be applied to find a side, the other two sides and their included angle must be known. This is the SAS congruence situation.

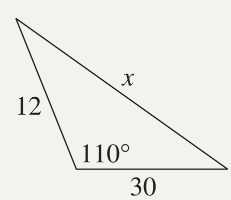
38 USING THE COSINE RULE TO FIND A SIDE

In the SAS congruence situation:

(square of any side) = (sum of squares of other two sides)
− (twice the product of those sides) × (cosine of their included angle).

Example 28 — 6J

Find x in the triangle drawn to the right.



SOLUTION

Applying the cosine rule to the triangle,

$$x^2 = 12^2 + 30^2 - 2 \times 12 \times 30 \times \cos 110^\circ$$
$$= 144 + 900 - 720\cos 110^\circ$$
$$= 1044 - 720\cos 110^\circ,$$

and because $\cos 110^\circ = -\cos 70^\circ$, (cosine is negative in the second quadrant)

$x^2 = 1044 + 720\cos 70^\circ$ (until this point, all calculations have been exact)

Using the calculator to approximate x^2, and then to take the square root,

$$x \doteqdot 35.92.$$

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Using the cosine rule to find an angle — the SSS situation

To use the cosine rule to find an angle, all three sides need to be known, which is the SSS congruence test. Finding the angle is done most straightforwardly by substituting into the usual form of the cosine rule:

39 USING THE COSINE RULE TO FIND AN ANGLE

In the SSS congruence situation:

- Substitute into the cosine rule and solve for $\cos\theta$.

There is an alternative approach. Solving the cosine rule for $\cos A$ gives a formula for $\cos A$. Some readers may prefer to remember and apply this second form of the cosine rule — but the triangle may then need to be relabelled.

40 THE COSINE RULE WITH $\cos A$ AS SUBJECT

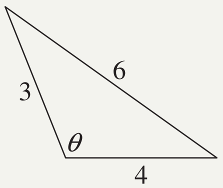
In any triangle ABC,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Notice that $\cos\theta$ is positive when θ is acute, and is negative when θ is obtuse. Hence there is only ever one solution for the unknown angle, unlike the situation for the sine rule, where there are often two possible angles.

Example 29 6J

Find θ in the triangle drawn to the right.



SOLUTION

Substituting into the cosine rule,

$$\begin{aligned} 6^2 &= 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos\theta \\ 24\cos\theta &= -11 \\ \cos\theta &= -\frac{11}{24} \\ \theta &\doteqdot 117^\circ 17'. \end{aligned}$$

OR

Using the boxed formula,

$$\begin{aligned} \cos\theta &= \frac{3^2 + 4^2 - 6^2}{2 \times 3 \times 4} \\ &= \frac{-11}{24} \\ \theta &\doteqdot 117^\circ 17'. \end{aligned}$$

Using the cosine rule in three-dimensional problems

The three-dimensional problem in Example 9 at the end of Section 6C involved four triangles, none of which could be solved. We assigned the pronumeral h to the height, then worked around the diagram until we knew four things in terms of h in the base triangle, and could therefore form an equation in h.

That final triangle was right-angled. The following problem has only one small change from the previous problem, but as a consequence, we need to apply the cosine rule instead of Pythagoras' theorem.

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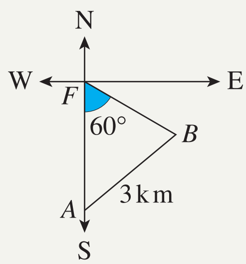
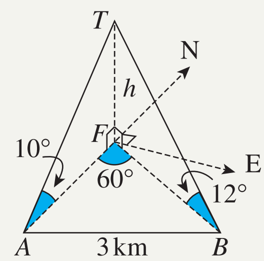
Example 30 6J

A motorist driving on level ground sees, due north of her, a tower whose angle of elevation is 10°. After driving 3 km further in a straight line, the tower is in the direction N60°W, with angle of elevation 12°.

a How high is the tower? **b** In what direction is she driving?

SOLUTION





Let the tower be TF, and let the motorist be driving from A to B.

a There are four triangles, none of which can be solved, so let h be the height of the tower.

In ΔTAF, $AF = h\cot 10°$.

In ΔTBF, $BF = h\cot 12°$.

We now have expressions for four measurements in ΔABF, so we can use the cosine rule to form an equation in h.

In $\triangle ABF$,
$$3^2 = h^2\cot^2 10° + h^2\cot^2 12° - 2h^2\cot 10°\cot 12° \times \cos 60°$$
$$9 = h^2(\cot^2 10° + \cot^2 12° - \cot 10°\cot 12°)$$
$$h^2 = \frac{9}{\cot^2 10° + \cot^2 12° - \cot 10°\cot 12°},$$
$$h \doteqdot 0.571\,\text{km}$$

so the tower is about 571 metres high.

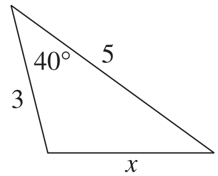
b Let $\theta = \angle FAB$, then in $\triangle ABF$, $\dfrac{\sin\theta}{h\cot 12°} = \dfrac{\sin 60°}{3}$
$$\sin\theta = h\cot 12° \times \frac{\sqrt{3}}{6}$$
$$\theta \doteqdot 51°,$$

so her direction is about N51°E.

Exercise 6J FOUNDATION

1 Find x in each triangle, correct to one decimal place.

a



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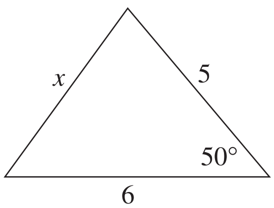

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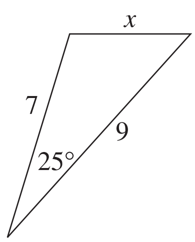

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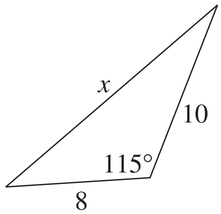

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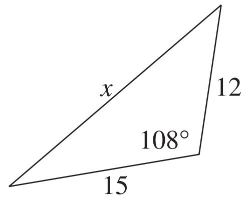

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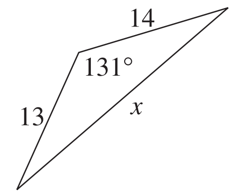
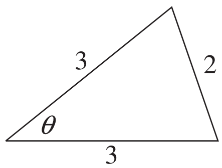


2 Find θ in each triangle, correct to the nearest degree.

a



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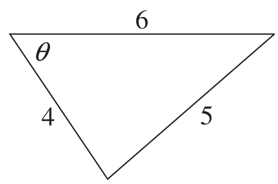

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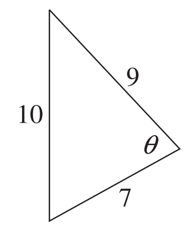

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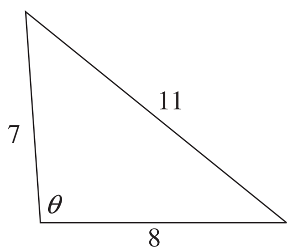

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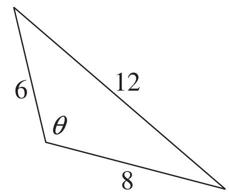

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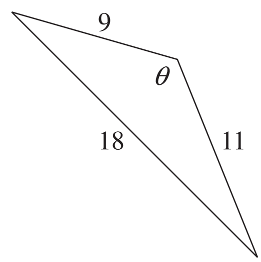
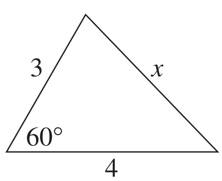
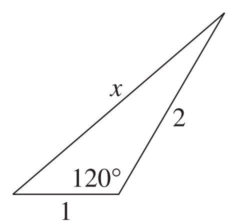


3 Using the fact that $\cos 60° = \frac{1}{2}$ and $\cos 120° = -\frac{1}{2}$, find x as a surd in each triangle.

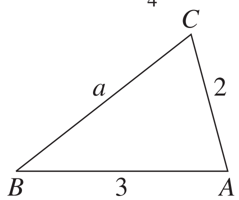
a



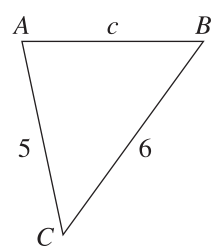
b



4 **a** If $\cos A = \frac{1}{4}$, find the exact value of a.

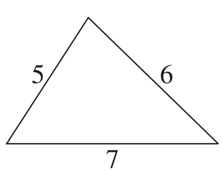


b If $\cos C = \frac{2}{3}$, find the exact value of c.



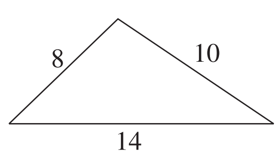
DEVELOPMENT

5 **a**



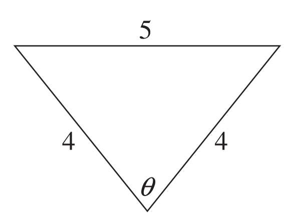
Find the smallest angle of the triangle, correct to the nearest minute.

b



Find the largest angle of the triangle, correct to the nearest minute.

c



Find the value of $\cos \theta$.

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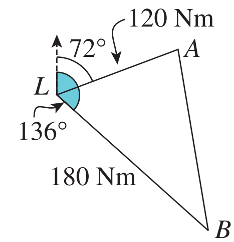
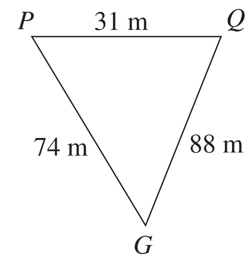
6 In the diagram to the right, ship A is 120 nautical miles from a lighthouse L on a bearing of 072°T, while ship B is 180 nautical miles from L on a bearing of 136°T. Calculate the distance between the two ships, correct to the nearest nautical mile.



7 A golfer at G wishes to hit a shot between two trees P and Q, as shown in the diagram to the right. The trees are 31 metres apart, and the golfer is 74 metres from P and 88 metres from P. Find the angle within which the golfer must play the shot, correct to the nearest degree.



8 The sides of a triangle are in the ratio 5 : 16 : 19. Find the smallest and largest angles of the triangle, correct to the nearest minute where necessary.

9 In $\triangle ABC$, $a = 31$ units, $b = 24$ units and $\cos C = \frac{59}{62}$.

a Show that $c = 11$ units. **b** Show that $A = 120°$.

10 In $\triangle PQR$, $p = 5\sqrt{3}$ cm, $q = 11$ cm and $R = 150°$.

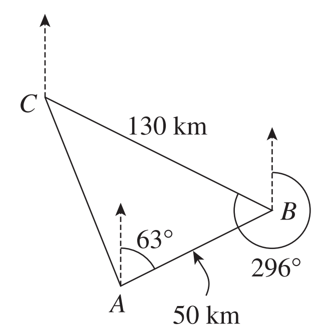
a Find r. **b** Find $\cos P$.

11 A ship sails 50 km from port A to port B on a bearing of 63°, then sails 130 km from port B to port C on a bearing of 296°.

a Show that $\angle ABC = 53°$.

b Find, to the nearest km, the distance of port A from port C.

c Use the cosine rule to find $\angle ACB$, and hence find the bearing of port A from port C, correct to the nearest degree.

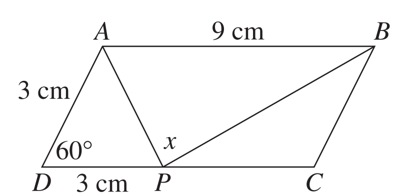


12 In a parallelogram $ABCD$, $\angle ADC = 60°$, $AB = 9$ cm and $AD = 3$ cm. The point P lies on DC such that $DP = 3$ cm.

a Explain why $\triangle ADP$ is equilateral, and hence find AP.

b Use the cosine rule in $\triangle BCP$ to find BP.

c Let $\angle APB = x$. Show that $\cos x = -\frac{\sqrt{7}}{14}$.



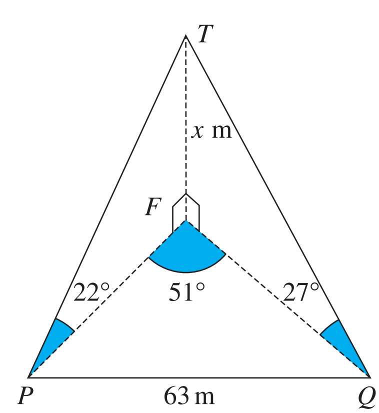
13 In the diagram, TF represents a vertical tower of height x metres standing on level ground. From P and Q at ground level, the angles of elevation of T are 22° and 27° respectively. $PQ = 63$ metres and $\angle PFQ = 51°$.

a Show that $PF = x \cot 22°$ and write down a similar expression for QF.

b Use the cosine rule to show that

$$x^2 = \frac{63^2}{\cot^2 22° + \cot^2 27° - 2\cot 22° \cot 27° \cos 51°}.$$

c Use a calculator to show that $x \doteqdot 32$.



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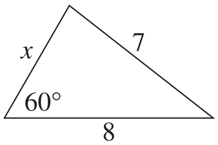
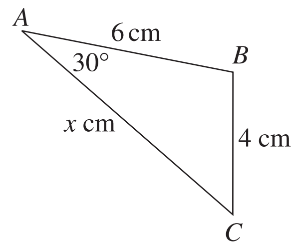
14 [Using the cosine rule in the ambiguous ASS situation]
Use the cosine rule to find the two possible values of x in the diagram to the right.



15 [Using the cosine rule in the ambiguous ASS situation]
The diagram shows $\triangle ABC$ in which $\angle A = 30°$, $AB = 6$ cm and $BC = 4$ cm. Let $AC = x$ cm.



a Use the cosine rule to show that $x^2 - 6\sqrt{3}x + 20 = 0$.

b Use the quadratic formula to show that AC has length $3\sqrt{3} + \sqrt{7}$ cm or $3\sqrt{3} - \sqrt{7}$ cm.

c Copy the diagram and indicate on it (approximately) the other possible position of the point C.

ENRICHMENT

16 The sides of a triangle are $n^2 + n + 1$, $2n + 1$ and $n^2 - 1$, where $n > 1$. Find the largest angle of the triangle.

17 Two identical rods BA and CA are hinged at A. When $BC = 8$ cm, $\angle BAC = 30°$ and when $BC = 4$ cm, $\angle BAC = \alpha$. Show that

$$\cos\alpha = \frac{6 + \sqrt{3}}{8}.$$

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6K Problems involving general triangles

A triangle has three lengths and three angles, and most triangle problems involve using three of these six measurements to calculate some of the others. The key to deciding which formula to use is to see which congruence situation applies.

Trigonometry and the congruence tests

There are four standard congruence tests — RHS, AAS, SAS and SSS. These tests can also be regarded as theorems about constructing triangles from given data.

If you know three measurements including one length, then apart from the ambiguous ASS situation, any two triangles with these three measurements will be congruent.

41 THE SINE, COSINE AND AREA RULES AND THE STANDARD CONGRUENCE TESTS

In a right-angled triangle, use simple trigonometry and Pythagoras. Otherwise:

AAS: Use the sine rule to find each of the other two sides.

SAS: Use the cosine rule to find the third side.
Use the area formula to find the area.

SSS: Use the cosine rule to find any angle.

ASS: [The ambiguous situation] Use the sine rule to find the unknown angle opposite a known side.
There may be a second solution.

In the ambiguous ASS situation, it is also possible to use the cosine rule to find the third side. See Questions 14 and 15 of the previous exercise.

Problems requiring two steps

Various situations with non-right-angled triangles require two steps for their solution, for example, finding the other two angles in an SAS situation, or finding the area given AAS, ASS or SSS situations.

Example 31 — 6K

A boat sails 6 km due north from the harbour H to A, and a second boat sails 10 km from H to B on a bearing of 120°T.

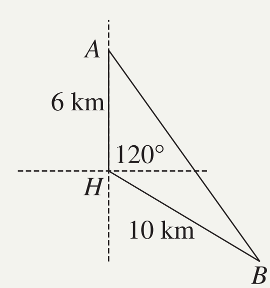
a What is the distance AB?

b What is the bearing of B from A, correct to the nearest minute?

SOLUTION

a This is an SAS situation,
so we use the cosine rule to find AB:

$$\begin{aligned} AB^2 &= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 120^\circ \\ &= 36 + 100 - 120 \times \left(-\tfrac{1}{2}\right) \\ &= 196 \\ AB &= 14\,\text{km}. \end{aligned}$$



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b Because AB is now known, this is an SSS situation,
so we use the cosine rule in reverse to find $\angle A$:

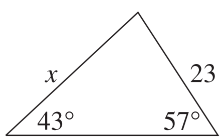
$$\begin{aligned} 10^2 &= 14^2 + 6^2 - 2 \times 14 \times 6 \times \cos A \\ 12 \times 14 \times \cos A &= 196 + 36 - 100 \\ \cos A &= \frac{132}{12 \times 14} \\ &= \frac{11}{14} \\ A &\doteqdot 38^\circ 13', \text{ and the bearing of } B \text{ from } A \text{ is about } 141^\circ 47'\text{T.} \end{aligned}$$

Exercise 6K

FOUNDATION

1 Use right-angled triangle trigonometry, the sine rule or the cosine rule to find x in each triangle, correct to one decimal place.

a



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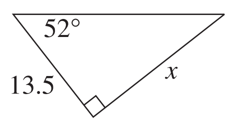

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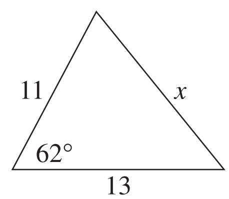

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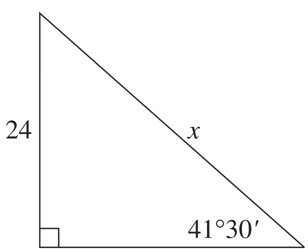

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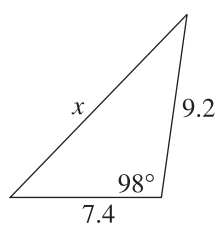

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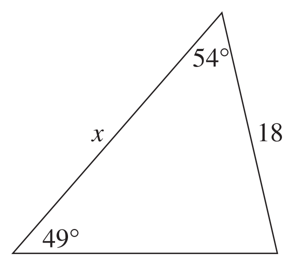
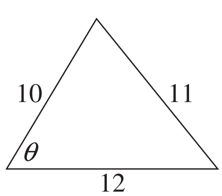


2 Use right-angled triangle trigonometry, the sine rule or the cosine rule to find θ in each triangle, correct to the nearest degree.

a



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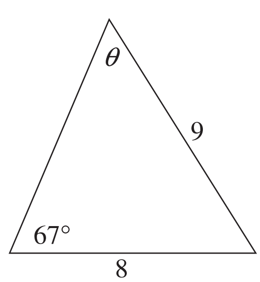

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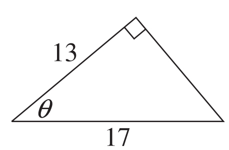

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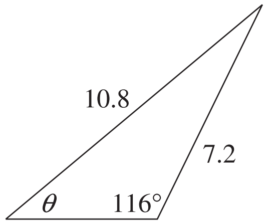

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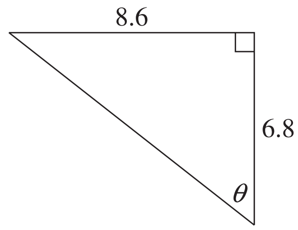

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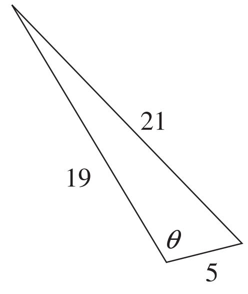

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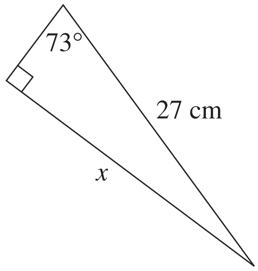
3 [This question is designed to show that the sine and cosine rules work in right-angled triangles, but are not the most efficient methods.]
In each part find the pronumeral (correct to the nearest cm or to the nearest degree), using either the sine rule or the cosine rule. Then check your answer using right-angled triangle trigonometry.

a


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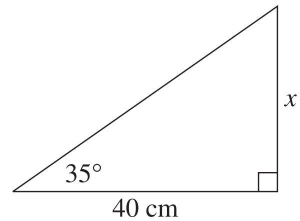

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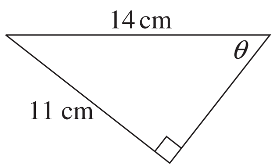

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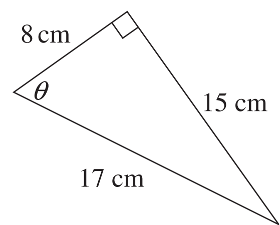
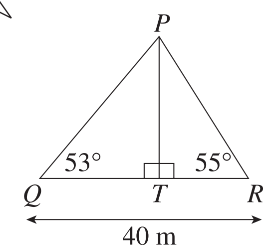


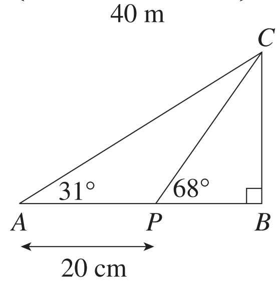
4 In $\triangle PQR$, $\angle Q = 53^\circ$, $\angle R = 55^\circ$ and $QR = 40$ metres. The point T lies on QR such that $PT \perp QR$.



a Use the sine rule in $\triangle PQR$ to show that $PQ = \dfrac{40 \sin 55^\circ}{\sin 72^\circ}$.

b Use $\triangle PQT$ to find PT, correct to the nearest metre.

5 In $\triangle ABC$, $\angle B = 90^\circ$ and $\angle A = 31^\circ$. The point P lies on AB such that $AP = 20\,\text{cm}$ and $\angle CPB = 68^\circ$.

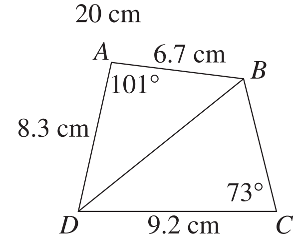


a Explain why $\angle ACP = 37^\circ$.

b Use the sine rule to show that $PC = \dfrac{20 \sin 31^\circ}{\sin 37^\circ}$.

c Hence find PB, correct to the nearest centimetre.

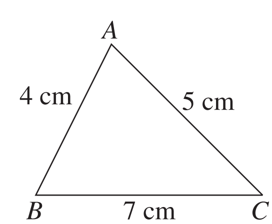
6 In the diagram to the right, $AB = 6.7\,\text{cm}$, $AD = 8.3\,\text{cm}$ and $DC = 9.2\,\text{cm}$. Also, $\angle A = 101^\circ$ and $\angle C = 73^\circ$.



a Use the cosine rule to find the diagonal BD, correct to the nearest millimetre.

b Hence use the sine rule to find $\angle CBD$, correct to the nearest degree.

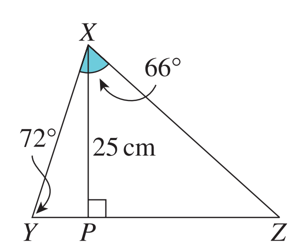
7 In $\triangle ABC$, $AB = 4\,\text{cm}$, $BC = 7\,\text{cm}$ and $CA = 5\,\text{cm}$.



a Use the cosine rule to find $\angle ABC$, correct to the nearest minute.

b Hence calculate the area of $\triangle ABC$, correct to the nearest square centimetre.

8 In triangle XYZ, $\angle Y = 72^\circ$ and $\angle YXZ = 66^\circ$. $XP \perp YZ$ and $XP = 25\,\text{cm}$.



a Use the sine ratio in $\triangle PXY$ to show that $XY \doteqdot 26.3\,\text{cm}$.

b Hence use the sine rule in $\triangle XYZ$ to find YZ, correct to the nearest centimetre.

c Check your answer to part **b** by using the tangent ratio in triangles PXY and PXZ to find PY and PZ.

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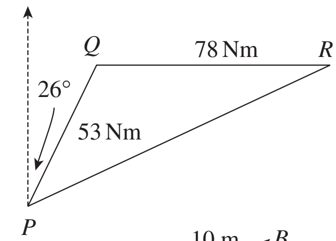
DEVELOPMENT

9 A triangle has sides 13 cm, 14 cm and 15 cm. Use the cosine rule to find one of its angles, and hence show that its area is 84 cm^2.

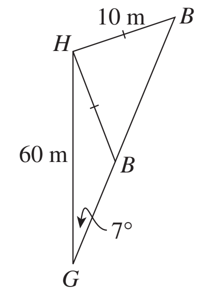
10 A ship sails 53 nautical miles from P to Q on a bearing of 026°T. It then sails 78 nautical miles due east from Q to R.

- **a** Explain why $\angle PQR = 116^\circ$.
- **b** How far apart are P and R, correct to the nearest nautical mile?



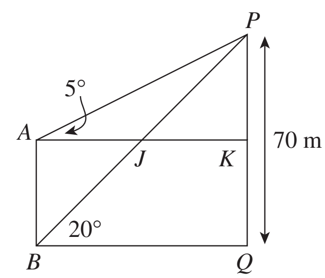
11 A golfer at G, 60 metres from the hole H, played a shot that landed at B, 10 metres from the hole. The direction of the shot was 7° away from the direct line between G and H.

- **a** Find, correct to the nearest minute, the two possible sizes of $\angle GBH$.
- **b** Hence find the two possible distances the ball has travelled. (Answer in metres correct to one decimal place.)



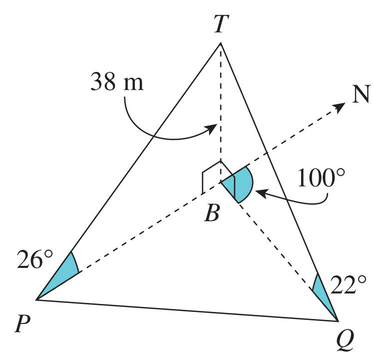
12 Two towers AB and PQ stand on level ground. The angles of elevation of the top of the taller tower from the top and bottom of the shorter tower are 5° and 20° respectively. The height of the taller tower is 70 metres.

- **a** Explain why $\angle APJ = 15^\circ$.
- **b** Show that $AB = \dfrac{BP \sin 15^\circ}{\sin 95^\circ}$.
- **c** Show that $BP = \dfrac{70}{\sin 20^\circ}$.
- **d** Hence find the height of the shorter tower, correct to the nearest metre.



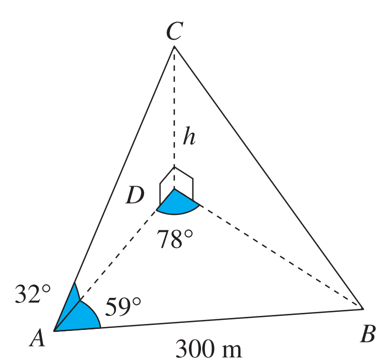
13 From two points P and Q on level ground, the angles of elevation of the top T of a 38 m tower are 26° and 22° respectively. Point P is due south of the tower, and the bearing of Q from the tower is 100°T.

- **a** Show that $PB = 38 \tan 64^\circ$, and find a similar expression for QB.
- **b** Hence determine, correct to the nearest metre, the distance between P and Q.



14 Two observers at A and B on horizontal ground are 300 m apart. From A, the angle of elevation of the top C of a tall building DC is 32°. It is also known that $\angle DAB = 59^\circ$ and $\angle ADB = 78^\circ$.

- **a** Show that $AD = \dfrac{300 \sin 43^\circ}{\sin 78^\circ}$.
- **b** Hence find the height of the building, correct to the nearest metre.



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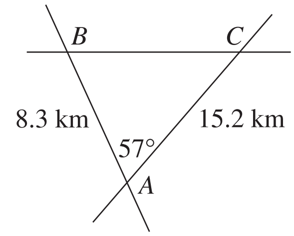
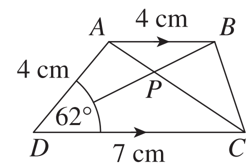
15 The diagram shows three straight roads, AB, BC and CA, where $AB = 8.3\,\text{km}$, $AC = 15.2\,\text{km}$, and the roads AB and AC intersect at $57°$.



Two cars, P_1 and P_2, leave A at the same instant. Car P_1 travels along AB and then BC at 80 km/h while P_2 travels along AC at 50 km/h. Which car reaches C first, and by how many minutes? (Answer correct to one decimal place.)

16 In the diagram to the right, $ABCD$ is a trapezium in which $AB \parallel DC$. The diagonals AC and BD meet at P. Also, $AB = AD = 4\,\text{cm}$, $DC = 7\,\text{cm}$ and $\angle ADC = 62°$.

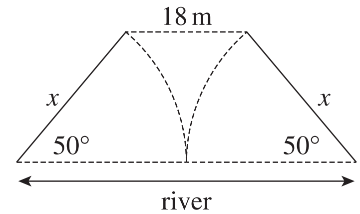


a Find $\angle ACD$, correct to the nearest minute.

b Explain why $\angle PDC = \frac{1}{2} \angle ADC$.

c Hence find, to the nearest minute, the acute angle between the diagonals of the trapezium.

17 A bridge spans a river, and the two identical sections of the bridge, each of length x metres, can be raised to allow tall boats to pass. When the two sections are fully raised, they are each inclined at $50°$ to the horizontal, and there is an 18-metre gap between them, as shown in the diagram. Calculate the width of the river in metres, correct to one decimal place.



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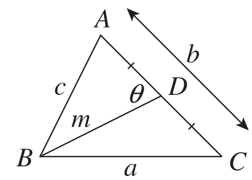
ENRICHMENT

18 Let ABC be a triangle and let D be the midpoint of AC.
Let $BD = m$ and $\angle ADB = \theta$.



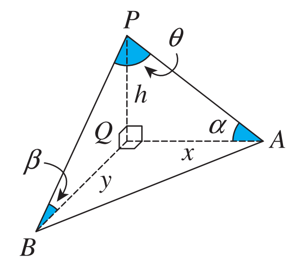
a Simplify $\cos(180° - \theta)$.

b Show that $\cos\theta = \dfrac{4m^2 + b^2 - 4c^2}{4mb}$. Write down a similar expression for $\cos(180° - \theta)$ in terms of a, b and m.

c Hence show that $a^2 + c^2 = 2m^2 + \frac{1}{2}b^2$.

19 In $\triangle ABC$, $a\cos A = b\cos B$.
Use the cosine rule to prove that the triangle is either isosceles or right-angled.

20 In the diagram of a triangular pyramid, $AQ = x$, $BQ = y$, $PQ = h$, $\angle APB = \theta$, $\angle PAQ = \alpha$ and $\angle PBQ = \beta$. Also, there are three right angles at Q.



a Show that $x = h\cot\alpha$ and write down a similar expression for y.

b Use Pythagoras' theorem and the cosine rule to show that

$$\cos\theta = \frac{h^2}{\sqrt{(x^2 + h^2)(y^2 + h^2)}}.$$

c Hence show that $\sin\alpha\sin\beta = \cos\theta$.

21 [Heron's formula for the area of a triangle in terms of the side lengths]

a By repeated application of factoring by the difference of squares, prove the identity

$$(2ab)^2 - (a^2 + b^2 - c^2)^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c).$$

b Let $\triangle ABC$ be any triangle, and let $s = \frac{1}{2}(a + b + c)$ be the *semiperimeter*. Prove that

$$(a + b + c)(a + b - c)(a - b + c)(-a + b + c) = 16s(s - a)(s - b)(s - c).$$

c Write down the formula for $\cos C$ in terms of the sides a, b and c, then use parts **a** and **b** and the Pythagorean identities to prove that $\sin C = \dfrac{2\sqrt{s(s - a)(s - b)(s - c)}}{ab}$.

d Hence show that the area A of the triangle is $A = \sqrt{s(s - a)(s - b)(s - c)}$.

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Chapter 6 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 6 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Find, correct to four decimal places:

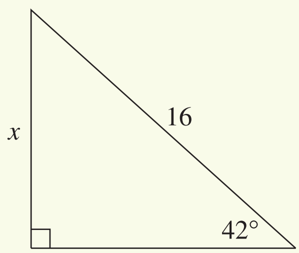
a $\cos 73°$ **b** $\tan 42°$ **c** $\sin 38°24'$ **d** $\cos 7°56'$

2 Find the acute angle θ, correct to the nearest minute, given that:

a $\sin\theta = 0.3$ **b** $\tan\theta = 2.36$ **c** $\cos\theta = \frac{1}{4}$ **d** $\tan\theta = 1\frac{1}{3}$

3 Find, correct to two decimal places, the side marked x in each triangle below.

a



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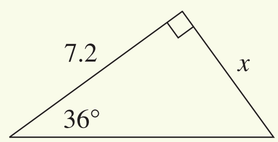

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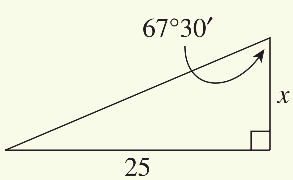

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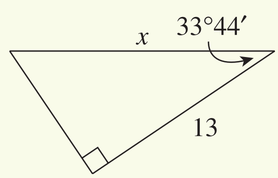
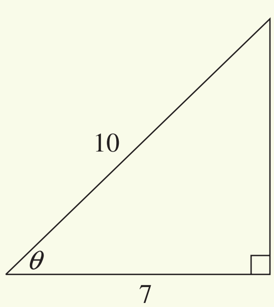


4 Find, correct to the nearest minute, the angle θ in each triangle below.

a



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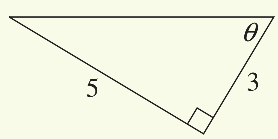

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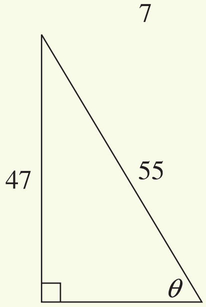

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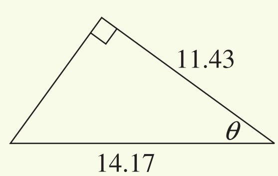

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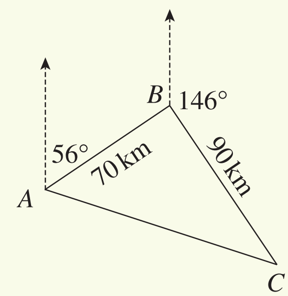
5 Use the special triangles to find the exact values of:

a $\tan 60^\circ$ **b** $\sin 45^\circ$ **c** $\cos 30^\circ$ **d** $\cot 45^\circ$ **e** $\sec 60^\circ$ **f** $\operatorname{cosec} 60^\circ$

6 A vertical pole stands on level ground. From a point on the ground 8 metres from its base, the angle of elevation of the top of the pole is 38°. Find the height of the pole, correct to the nearest centimetre.

7 At what angle, correct to the nearest degree, is a 6-metre ladder inclined to the ground if its foot is 2.5 metres out from the wall?

8 A motorist drove 70 km from town A to town B on a bearing of 056°T, and then drove 90 km from town B to town C on a bearing of 146°T.



a Explain why $\angle ABC = 90^\circ$.

b How far apart are the towns A and C, correct to the nearest kilometre?

c Find $\angle BAC$, and hence find the bearing of town C from town A, correct to the nearest degree.

9 Sketch each graph for $0^\circ \le x \le 360^\circ$.

a $y = \sin x$ **b** $y = \cos x$ **c** $y = \tan x$

10 Write each trigonometric ratio as the ratio of its related acute angle, with the correct sign attached.

a $\cos 125^\circ$ **b** $\sin 312^\circ$ **c** $\tan 244^\circ$ **d** $\sin 173^\circ$

11 Find the exact value of:

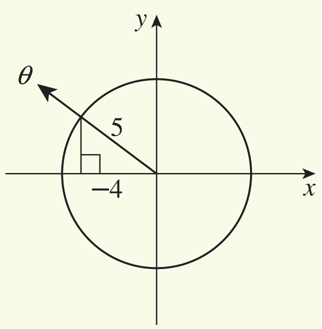
a $\tan 240^\circ$ **b** $\sin 315^\circ$ **c** $\cos 330^\circ$ **d** $\tan 150^\circ$

12 Use the graphs of the trigonometric functions to find these values, if they exist.

a $\sin 180^\circ$ **b** $\cos 180^\circ$ **c** $\tan 90^\circ$ **d** $\sin 270^\circ$

13 Use Pythagoras' theorem to find whichever of x, y or r is unknown. Then write down the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

a



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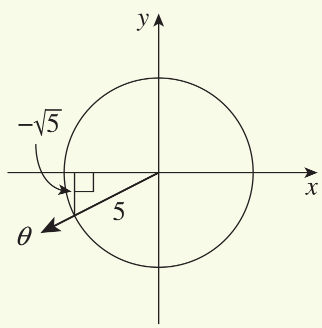

14 a If $\tan\alpha = \frac{12}{5}$ and α is acute, find the values of $\sin\alpha$ and $\cos\alpha$.

b If $\sin\beta = \frac{2\sqrt{6}}{7}$ and β is acute, find the values of $\cos\beta$ and $\tan\beta$.

15 a If $\tan\alpha = -\frac{9}{40}$ and $270^\circ < \alpha < 360^\circ$, find the values of $\sin\alpha$ and $\cos\alpha$.

b If $\sin\beta = \frac{2\sqrt{6}}{7}$ and $90^\circ < \beta < 180^\circ$, find the values of $\cos\beta$ and $\tan\beta$.

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16 Simplify:

a $\dfrac{1}{\cos\theta}$ **b** $\dfrac{1}{\cot\theta}$ **c** $\dfrac{\sin\theta}{\cos\theta}$

d $1 - \sin^2\theta$ **e** $\sec^2\theta - \tan^2\theta$ **f** $\operatorname{cosec}^2\theta - 1$

17 Prove these trigonometric identities.

a $\cos\theta\sec\theta = 1$ **b** $\tan\theta\operatorname{cosec}\theta = \sec\theta$ **c** $\dfrac{\cot\theta}{\cos\theta} = \operatorname{cosec}\theta$

d $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ **e** $4\sec^2\theta - 3 = 1 + 4\tan^2\theta$ **f** $\cos\theta + \tan\theta\sin\theta = \sec\theta$

18 Solve each trigonometric equation for $0^\circ \le x \le 360^\circ$.

a $\cos x = \frac{1}{2}$ **b** $\sin x = 1$ **c** $\tan x = -1$

d $\cos x = 0$ **e** $\sqrt{3}\tan x = 1$ **f** $\tan x = 0$

g $\sqrt{2}\sin x + 1 = 0$ **h** $2\cos x + \sqrt{3} = 0$ **i** $\cos^2 x = \frac{1}{2}$

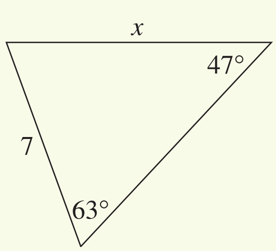
j $\cos 2x = \frac{1}{2}$ **k** $\cos\,(x - 75^\circ) = \frac{1}{2}$ **l** $\sin x = -\sqrt{3}\cos x$

19 Solve each equation for $0^\circ \le \theta \le 360^\circ$ by reducing it to a quadratic equation in u. Give your solutions correct to the nearest minute where necessary.

a $2\sin^2\theta + \sin\theta = 0$ **b** $\cos^2\theta - \cos\theta - 2 = 0$ **c** $2\tan^2\theta + 5\tan\theta - 3 = 0$

20 Use the sine rule or the cosine rule in each triangle to find x, correct to one decimal place.

a



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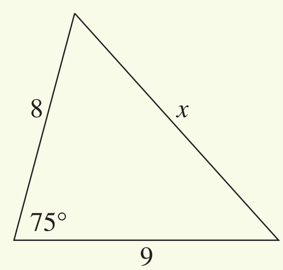

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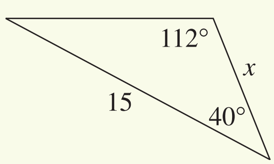

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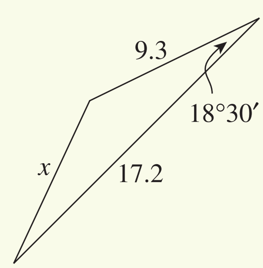
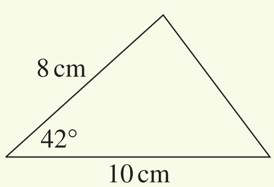


21 Calculate the area of each triangle, correct to the nearest cm^2.

a



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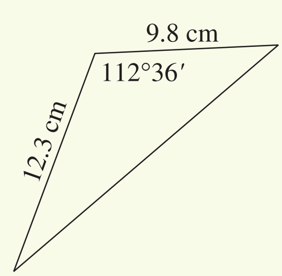

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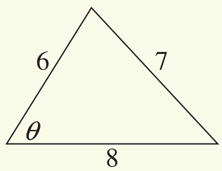
22 Use the sine rule or the cosine rule in each triangle to find θ, correct to the nearest minute.

a



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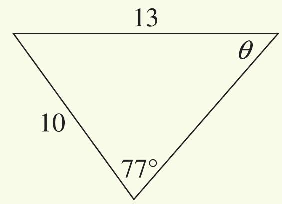

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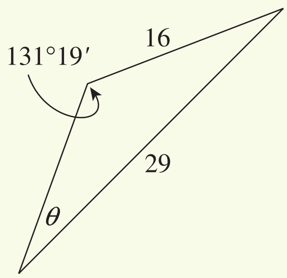

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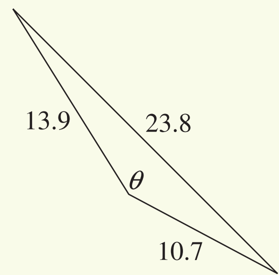
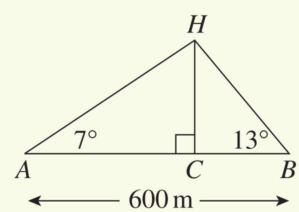


23 A triangle has sides 7 cm, 8 cm and 10 cm. Use the cosine rule to find one of its angles, and hence find the area of the triangle, correct to the nearest cm^2.

24 a Find the side a in $\triangle ABC$, where $\angle C = 60^\circ$, $b = 24$ cm and the area is 30 cm^2.

b Find the size of $\angle B$ in $\triangle ABC$, where $a = 9$ cm, $c = 8$ cm and the area is 18 cm^2.

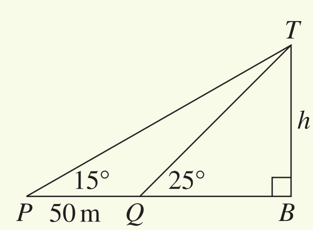
25 A helicopter H is hovering above a straight, horizontal road AB of length 600 metres. The angles of elevation of H from A and B are 7° and 13° respectively. The point C lies on the road directly below H.



a Use the sine rule to show that $HB = \dfrac{600 \sin 7^\circ}{\sin 160^\circ}$.

b Hence find the height CH of the helicopter above the road, correct to the nearest metre.

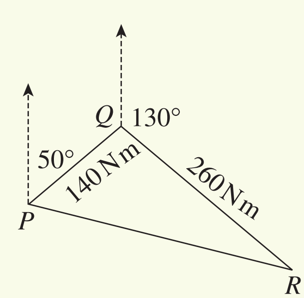
26 A man is sitting in a boat at P, where the angle of elevation of the top T of a vertical cliff BT is 15°. He then rows 50 metres directly towards the cliff to Q, where the angle of elevation of T is 25°.



a Show that $TQ = \dfrac{50 \sin 15^\circ}{\sin 10^\circ}$.

b Hence find the height h of the cliff, correct to the nearest tenth of a metre.

27 A ship sailed 140 nautical miles from port P to port Q on a bearing of 050°T. It then sailed 260 nautical miles from port Q to port R on a bearing of 130°T.



a Explain why $\angle PQR = 100^\circ$.

b Find the distance between ports R and P, correct to the nearest nautical mile.

c Find the bearing of port R from port P, correct to the nearest degree.

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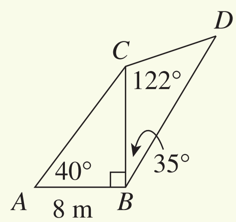
28 From two points P and Q on horizontal ground, the angles of elevation of the top T of a 10 m monument are 16° and 13° respectively. It is also known that $\angle PBQ = 70^\circ$, where B is the base of the monument.

a Show that $PB = 10 \tan 74^\circ$, and find a similar expression for QB.

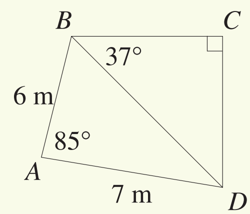
b Hence determine, correct to the nearest metre, the distance between P and Q.

29 In each diagram, find CD correct to the nearest centimetre:

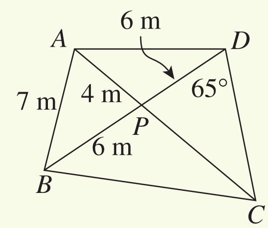
a



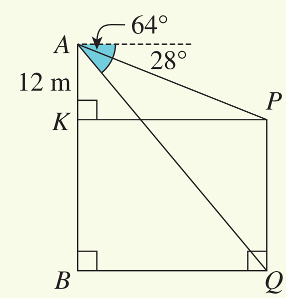
b



c



30 Two towers AB and PQ stand on level ground. Tower AB is 12 metres taller than tower PQ. From A, the angles of depression of P and Q are 28° and 64° respectively.



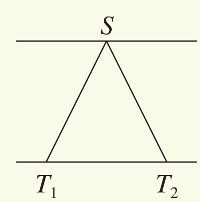
a Use $\triangle AKP$ to show that $KP = BQ = 12 \tan 62^\circ$.

b Use $\triangle ABQ$ to show that $AB = 12 \tan 62^\circ \tan 64^\circ$.

c Hence find the height of the shorter tower, correct to the nearest metre.

d Solve the problem again by finding AP using $\triangle AKP$ and then using the sine rule in $\triangle APQ$.

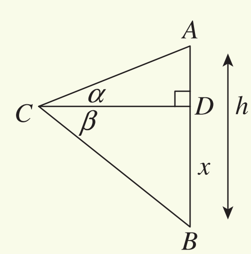
31 Two trees T_1 and T_2 on one bank of a river are 86 metres apart. A sign S on the opposite bank is between the trees and the angles ST_1T_2 and ST_2T_1 are $53^\circ 30'$ and $60^\circ 45'$ respectively.



a Find ST_1 in exact form.

b Hence find the width of the river, correct to the nearest metre.

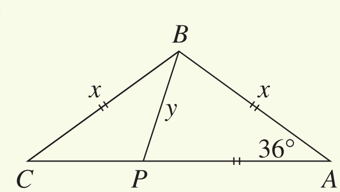
32 In the diagram to the right, $\angle ACD = \alpha$, $\angle BCD = \beta$, $AB = h$ and $BD = x$.



a Show that $BC = \dfrac{h \cos \alpha}{\sin (\alpha + \beta)}$.

b Hence show that $x = \dfrac{h \sin \beta \cos \alpha}{\sin (\alpha + \beta)}$.

33 In the diagram to the right, $\triangle ABC$ is isosceles with $BA = BC = x$ units and $\angle BAC = 36^\circ$. The point P is on AC such that $PA = BA$. Let $BP = y$ units.



a Use the cosine rule in $\triangle ABP$ to show that $\cos 72^\circ = \dfrac{y}{2x}$.

b Show that $\triangle BCP$ is isosceles.

c Hence show that $\cos 36 \cos 72 = \frac{1}{4}$.

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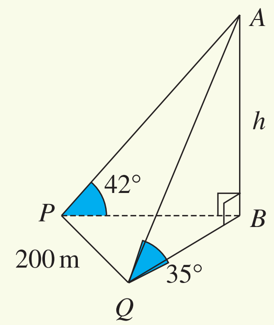
34 The points P, Q and B lie in a horizontal plane. From P, which is due west of B, the angle of elevation of the top of a tower AB of height h metres is 42°. From Q, which is on a bearing of 196° from the tower, the angle of elevation of the top of the tower is 35°. The distance PQ is 200 metres.

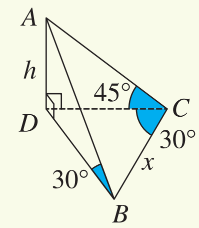


a Explain why $\angle PBQ = 74^\circ$.

b Show that $h^2 = \dfrac{200^2}{\cot^2 42^\circ + \cot^2 35^\circ - 2\cot 35^\circ \cot 42^\circ \cos 74^\circ}$.

c Hence find the height of the tower, correct to the nearest metre.

35 A triangular pyramid $ABCD$ has base BCD and perpendicular height AD.



a Find BD and CD in terms of h.

b Use the cosine rule to show that $2h^2 = x^2 - \sqrt{3}hx$.

c Let $u = \dfrac{h}{x}$. Write the result of the previous part as a quadratic equation in u, and hence show that $\dfrac{h}{x} = \dfrac{\sqrt{11} - \sqrt{3}}{4}$.

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7 The coordinate plane

The earlier chapters have used graphs to turn functions into geometric objects within the coordinate plane, thus allowing them to be visualised and studied by geometric as well as algebraic methods. In the coordinate plane:

- Points are represented by pairs of numbers.
- Lines are represented by linear equations.
- Circles, parabolas and other curves are represented by non-linear equations.

Points, lines and intervals are the main concern of this chapter. Its purpose is to prepare for Chapter 9, where the new topic of calculus will be introduced in the coordinate plane, heavily based on geometric ideas of tangents and areas.

This material may all have been covered in earlier years. Readers should do as much or as little of it as they need in order to master the skills.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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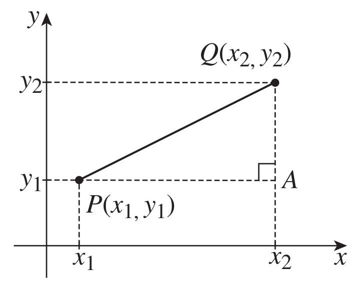
7A Lengths and midpoints of intervals

An *interval* is completely determined by its two endpoints (in this chapter, unlike Section 2B, the word *interval* will always mean *bounded interval*). There are simple formulae for the length of an interval and for the midpoint of an interval.

The distance formula

The formula for the length of an interval PQ is just Pythagoras' theorem in different notation.



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane.
Construct the right-angled triangle $\triangle PQA$, where $A(x_2, y_1)$ lies level with P and vertically above or below Q.
Then $PA = |x_2 - x_1|$ and $QA = |y_2 - y_1|$,
so by Pythagoras' theorem in $\triangle PQA$,

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

1 DISTANCE FORMULA

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane. Then

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

- First find the square PQ^2 of the distance.
- Then take the square root to find the distance PQ.

Example 1 — 7A

Find the lengths of the sides AB and AC of the triangle with vertices $A(1, -2)$, $B(-4, 2)$, and $C(5, -7)$, and say why $\triangle ABC$ is isosceles.

SOLUTION

First,
$$\begin{aligned} AB^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (-4 - 1)^2 + \big(2 - (-2)\big)^2 \\ &= (-5)^2 + 4^2 \\ &= 41, \end{aligned}$$
so $AB = \sqrt{41}$.

Secondly,
$$\begin{aligned} AC^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (5 - 1)^2 + \big(-7 - (-2)\big)^2 \\ &= 4^2 + (-5)^2 \\ &= 41, \end{aligned}$$
so $AC = \sqrt{41}$.

Because the two sides AB and AC are equal, the triangle is isosceles.

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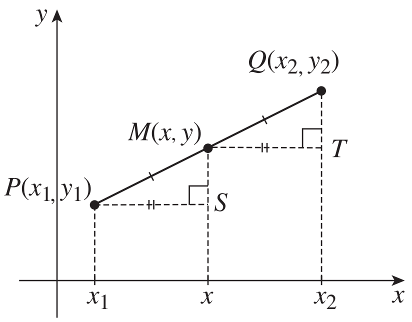
The midpoint formula

The midpoint of an interval is found by taking the averages of the coordinates of the two points. Congruence is the basis of the proof below.



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane, and let $M(x, y)$ be the midpoint of PQ.

Construct $S(x, y_1)$ and $T(x_2, y)$, as shown.

Then $\triangle PMS \equiv \triangle MQT$ (AAS).

Hence $PS = MT$ (matching sides of congruent triangles).

$$x - x_1 = x_2 - x$$
$$2x = x_1 + x_2$$
$$x = \frac{x_1 + x_2}{2}, \text{ which is the average of } x_1 \text{ and } x_2.$$

The calculation of the y-coordinate of M is similar.

2 MIDPOINT FORMULA

Let $M(x, y)$ be the midpoint of the interval joining $P(x_1, y_1)$ and $Q(x_2, y_2)$. Then

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2} \quad \text{(take the averages of the coordinates)}$$

Example 2 7A

The interval joining the points $A(3, -1)$ and $B(-7, 5)$ is a diameter of a circle. Find, for this circle:

a the centre M,
b the radius,
c the equation.

SOLUTION

a The centre of the circle is the midpoint $M(x, y)$ of the interval AB.

Using the midpoint formula, $x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$

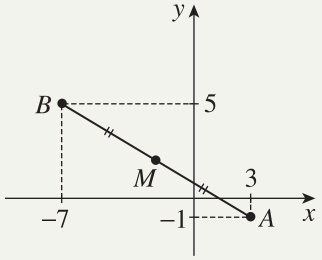
$= \frac{3 - 7}{2}$ $\qquad$ $= \frac{-1 + 5}{2}$

$= -2,$ $\qquad$ $= 2,$

so the centre is $M(-2, 2)$.

b Using the distance formula,

$$AM^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$= (-2 - 3)^2 + \big(2 - (-1)\big)^2$$
$$= 34,$$
$$AM = \sqrt{34}.$$



Hence the circle has radius $\sqrt{34}$.

c The equation of the circle is therefore $(x + 2)^2 + (y - 2)^2 = 34$.

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Testing for special quadrilaterals

Many questions in this chapter ask for a proof that a quadrilateral is of a particular type. The most obvious way is to test the definition itself.

3 DEFINITIONS OF THE SPECIAL QUADRILATERALS

- A *trapezium* is a quadrilateral with one pair of opposite sides parallel.
- A *parallelogram* is a quadrilateral with both pairs of opposite sides parallel.
- A *rhombus* is a parallelogram with a pair of adjacent sides equal.
- A *rectangle* is a parallelogram with one angle a right angle.
- A *square* is both a rectangle and a rhombus.

There are, however, several further standard tests that the exercises assume, and that were developed in earlier years. (Tests involving angles are omitted here, being irrelevant in this chapter.)

4 FURTHER STANDARD TESTS FOR SPECIAL QUADRILATERALS

A quadrilateral is a parallelogram:

- if the opposite sides are equal, or
- if one pair of opposite sides are equal and parallel, or
- if the diagonals bisect each other.

A quadrilateral is a rhombus:

- if all sides are equal, or
- if the diagonals bisect each other at right angles.

A quadrilateral is a rectangle:

- if the diagonals are equal and bisect each other.

Exercise 7A

FOUNDATION

Note: Diagrams should be drawn wherever possible.

1 Find the midpoint of each interval AB. Use $x = \dfrac{x_1 + x_2}{2}$ and $y = \dfrac{y_1 + y_2}{2}$.

a $A(3, 5)$ and $B(1, 9)$
b $A(4, 8)$ and $B(6, 4)$
c $A(-4, 7)$ and $B(8, -11)$
d $A(-3, 6)$ and $B(3, 1)$
e $A(0, -8)$ and $B(-11, -12)$
f $A(4, -7)$ and $B(4, 7)$

2 Find the length of each interval. Use $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, then find AB.

a $A(1, 4), B(5, 1)$
b $A(-2, 7), B(3, -5)$
c $A(-5, -2), B(3, 4)$
d $A(3, 6), B(5, 4)$
e $A(-4, -1), B(4, 3)$
f $A(5, -12), B(0, 0)$

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3 a Find the midpoint M of the interval joining $P(-2, 1)$ and $Q(4, 9)$.
b Find the lengths PM and MQ, and verify that $PM = MQ$.

4 a Find the length of each side of the triangle formed by $P(0, 3)$, $Q(1, 7)$ and $R(5, 8)$.
b Hence show that $\triangle PQR$ is isosceles.

5 The vertices of $\triangle ABC$ are given by $A(0, 0)$, $B(9, 12)$ and $C(25, 0)$.
a Find the lengths of the three sides AB, BC, AC.
b By checking that $AB^2 + BC^2 = AC^2$, show that $\triangle ABC$ is right-angled.

6 a Find the length of each side of $\triangle ABC$, where $A = (0, 5)$, $B = (3, -2)$ and $C = (-3, 4)$.
b Find the midpoint of each side of this triangle ABC.

DEVELOPMENT

7 a A circle with centre $O(0, 0)$ passes through $A(5, 12)$. What is its radius?
b A circle with centre $B(4, 5)$ passes through the origin. What is its radius?
c Find the centre of the circle with diameter CD, where $C = (2, 1)$ and $D = (8, -7)$.
d Show that $E(-12, -5)$ lies on the circle with centre the origin and radius 13.

8 a Find the midpoint of the interval joining $A(4, 9)$ and $C(-2, 3)$.
b Find the midpoint of the interval joining $B(0, 4)$ and $D(2, 8)$.
c What can you conclude about the diagonals of the quadrilateral $ABCD$?
d What sort of quadrilateral is $ABCD$? (Hint: See Box 4 above.)

9 The points $A(3, 1)$, $B(10, 2)$, $C(5, 7)$ and $D(-2, 6)$ are the vertices of a quadrilateral.
a Find the lengths of all four sides.
b What sort of quadrilateral is $ABCD$? (Hint: See Box 4 above.)

10 a Find the side lengths of the triangle with vertices $X(0, -4)$, $Y(4, 2)$ and $Z(-2, 6)$.
b Show that $\triangle XYZ$ is a right-angled isosceles triangle by showing that its side lengths satisfy Pythagoras' theorem.
c Hence find the area of $\triangle XYZ$.

11 a Find the distance of each point $A(1, 4)$, $B\left(2, \sqrt{13}\right)$, $C\left(3, 2\sqrt{2}\right)$ and $D(4, 1)$ from the origin O. Hence explain why the four points lie on a circle with centre the origin.
b What are the radius, diameter, circumference and area of this circle?

12 The circle with centre (h, k) and radius r has equation $(x - h)^2 + (y - k)^2 = r^2$. By identifying the centre and radius, find the equations of:
a the circle with centre $(5, -2)$ and passing through $(-1, 1)$,
b the circle with $K(5, 7)$ and $L(-9, -3)$ as endpoints of a diameter.

13 The point $M(3, 7)$ is the midpoint of the interval joining $A(1, 12)$ and $B(x_2, y_2)$. Find the coordinates x_2 and y_2 of B by substituting into the formulae

$$x = \frac{x_1 + x_2}{2} \qquad \text{and} \qquad y = \frac{y_1 + y_2}{2}.$$

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14 Solve each problem using the same methods as in the previous question.

a If $A(-1, 2)$ is the midpoint of $S(x, y)$ and $T(3, 6)$, find the coordinates of S.

b The midpoint of the interval PQ is $M(2, -7)$. Find the coordinates of P if:

i $Q = (0, 0)$ **ii** $Q = (5, 3)$ **iii** $Q = (-3, -7)$

c Find B, if AB is a diameter of a circle with centre $Q(4, 5)$, and $A = (8, 3)$.

d Given that $P(4, 7)$ is one vertex of the square $PQRS$, and that the centre of the square is $M(8, -1)$, find the coordinates of the opposite vertex R.

15 Each set of three points given below forms a triangle of one of these types:

A isosceles, **B** equilateral, **C** right-angled, **D** none of these.

Find the side lengths of each triangle below and hence determine its type.

a $A(-1, 0), B(1, 0), C\left(0, \sqrt{3}\right)$ **b** $P(-1, 1), Q(0, -1), R(3, 3)$

c $D(1, 1), E(2, -2), F(-3, 0)$ **d** $X(-3, -1), Y(0, 0), Z(-2, 2)$

16 **a** Given the point $A(7, 8)$, find the coordinates of three points P with integer coordinates such that $AP = \sqrt{5}$.

b Given that the distance from $U(3, 7)$ to $V(1, y)$ is $\sqrt{13}$, find the two possible values of y.

c Find a, if the distance from $A(a, 0)$ to $B(1, 4)$ is $\sqrt{18}$ units.

ENRICHMENT

17 **a** Given that $C(x, y)$ is equidistant from each of the points $P(1, 5)$, $Q(-5, -3)$ and $R(2, -2)$, use the distance formula to form two equations in x and y and solve them simultaneously to find the coordinates of C.

b Find the coordinates of the point $M(x, y)$ which is equidistant from each of the points $P(4, 3)$ and $Q(3, 2)$, and is also equidistant from $R(6, 1)$ and $S(4, 0)$.

18 Suppose that A, B and P are the points $(0, 0)$, $(3a, 0)$ and (x, y) respectively. Use the distance formula to form an equation in x and y for the point P, and describe the curve so found if:

a $PA = PB$, **b** $PA = 2PB$.

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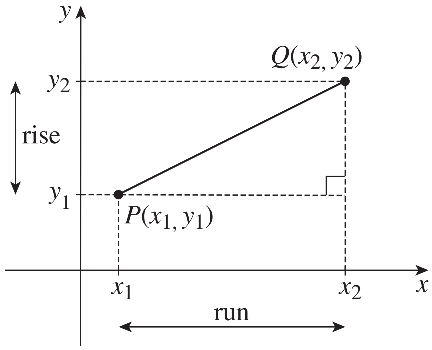
7B Gradients of intervals and lines

Gradient is the key idea that will be used in the next section to bring lines and their equations into the coordinate plane.

The gradient of an interval



Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane.
The *gradient* of the interval PQ is a measure of its steepness, as someone walks along the interval from P to Q.

The *rise* is the vertical difference $y_2 - y_1$, and the *run* is the horizontal difference $x_2 - x_1$.

The *gradient* of the interval PQ is defined to be the ratio of the rise and the run:

$$\begin{aligned}\text{gradient of } PQ &= \frac{\text{rise}}{\text{run}}\\ &= \frac{y_2 - y_1}{x_2 - x_1}.\end{aligned}$$

5 THE GRADIENT OF AN INTERVAL

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points. Then

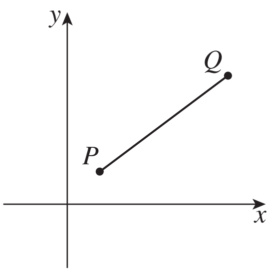
$$\text{gradient of } PQ = \frac{\text{rise}}{\text{run}}, \qquad \text{that is,} \qquad \text{gradient of } PQ = \frac{y_2 - y_1}{x_2 - x_1}.$$

- Horizontal intervals have gradient zero, because the rise is always zero.
- Vertical intervals don't have a gradient — the run is always zero, so the fraction is undefined.

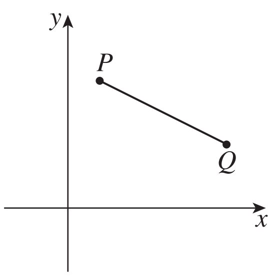
Positive and negative gradients

If the rise and the run have the same sign, then the gradient is *positive*, as in the first diagram below. In this case the interval slopes *upwards* as we move from left to right.

If the rise and run have opposite signs, then the gradient is *negative*, as in the second diagram. The interval slopes *downwards* as we move from left to right.



Positive gradient



Negative gradient

If the points P and Q are interchanged, in either diagram, then the rise and the run both change sign, but the gradient remains the same.

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Example 3 7B

Find the gradients of the sides of $\triangle XYZ$, where $X = (2, 5)$, $Y = (5, -2)$ and $Z = (-3, 4)$.

SOLUTION

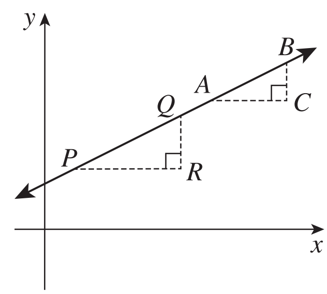
$$\text{gradient of } XY = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 5}{5 - 2} = \frac{-7}{3} = -\tfrac{7}{3}$$

$$\text{gradient of } YZ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-2)}{-3 - 5} = \frac{4 + 2}{-8} = -\tfrac{3}{4}$$

$$\text{gradient of } ZX = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 4}{2 - (-3)} = \frac{1}{2 + 3} = \tfrac{1}{5}$$

The gradient of a line

The *gradient of a line* is defined to be the gradient of any interval within the line. This definition makes sense because any two intervals on the same line always have the same gradient.



To prove this, suppose that PQ and AB are two intervals on the same line ℓ. Construct right triangles PQR and ABC underneath the intervals, with sides parallel to the axes. Because these two triangles are similar (by the AA similarity test), the ratios of their heights and bases are the same, which means that the two intervals AB and PQ have the same gradient.

6 THE GRADIENT OF A LINE

- The *gradient of a line* is found by taking any two distinct points P and Q on the line and finding the gradient of the interval PQ.
- It doesn't matter which two points on the line are taken, because the ratio of rise over run will be the same, as is easily seen using similar triangles.

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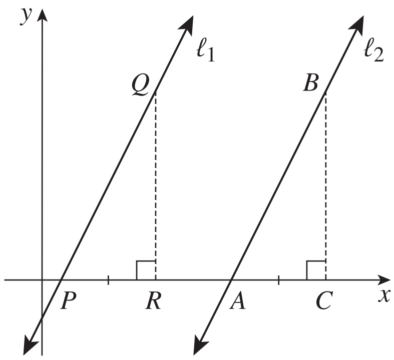
A condition for two lines to be parallel

Any two vertical lines are parallel. Otherwise, lines are parallel when their gradients are equal.

7 PARALLEL LINES

Two lines are parallel if and only if:

- they have the same gradient OR they are both vertical.



The phrase 'if and only if' means that

- if the condition holds, then the lines are parallel, and
- if the lines are parallel then the condition holds, and the two statements are called *converses* of each other.

'If and only if' is a very convenient abbreviation, but its logic needs attention.

Proof (assuming neither is vertical)
The two lines ℓ_1 and ℓ_2 meet the x-axis at P and A.
Construct $PR = AC$, then construct QR and BC perpendicular to the x-axis.

A If the lines ℓ_1 and ℓ_2 are parallel, then

$\angle QPR = \angle BAC$ (corresponding angles on parallel lines).

Hence $\triangle PQR \equiv \triangle ABC$ (AAS)

so $QR = BC$ (matching sides of congruent triangles)

$\dfrac{QR}{PR} = \dfrac{BC}{AC}$ (because $PR = AC$).

gradient ℓ_1 = gradient ℓ_2.

B Conversely, if the two gradients are equal, then

$\dfrac{QR}{PR} = \dfrac{BC}{AC}$

$QR = BC$ (because $PR = AC$).

Hence $\triangle PQR \equiv \triangle ABC$ (SAS)

so $\ell_1 = \ell_2$ (corresponding angles are equal).

Example 4 7B

Given the four points $A(3, 6)$, $B(7, -2)$, $C(4, -5)$ and $D(-1, 5)$, show that the quadrilateral $ABCD$ is a trapezium with $AB \parallel CD$.

SOLUTION

$$\text{gradient of } AB = \frac{-2 - 6}{7 - 3} = \frac{-8}{4} = -2,$$

$$\text{gradient of } CD = \frac{5 - (-5)}{-1 - 4} = \frac{10}{-5} = -2.$$

Hence $AB \parallel CD$ because their gradients are equal, so $ABCD$ is a trapezium.

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Testing for collinear points

Three points are called *collinear* if they all lie on one line.

8 TESTING FOR COLLINEAR POINTS

- To test whether three points A, B and C are collinear, test whether AB and BC are parallel. (Are they both vertical, or do they have the same gradient?)
- If AB and BC are parallel, then the three points are collinear, because then AB and BC are parallel lines passing through a common point B.

Example 5 7B

Test whether the three points $A(-2, 5)$, $B(1, 3)$ and $C(7, -1)$ are collinear.

SOLUTION

gradient of $AB = \dfrac{3 - 5}{1 + 2}$

$= -\frac{2}{3}$,

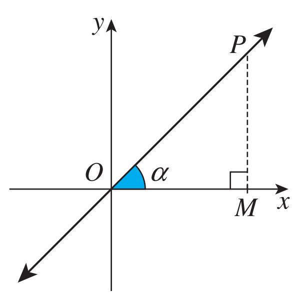
gradient of $BC = \dfrac{-1 - 3}{7 - 1}$

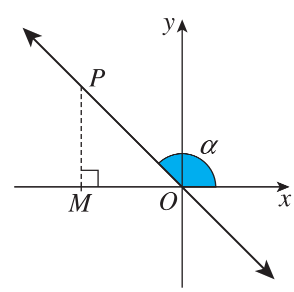
$= -\frac{2}{3}$.

Because the gradients are equal, the points A, B and C are collinear.

Gradient and the angle of inclination

The *angle of inclination* of a line is the angle between the upward direction of the line and the positive direction of the x-axis.





The two diagrams above show that lines with positive gradients have acute angles of inclination, and lines with negative gradients have obtuse angles of inclination. They also illustrate the trigonometric relationship between the gradient and the angle of inclination α.

9 ANGLE OF INCLINATION

Suppose that a line has angle of inclination α. Then:

- gradient of line $= \tan\alpha$ OR the line is vertical and $\alpha = 90^\circ$.

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Proof

A When α is acute, as in the first diagram, then the rise MP and the run OM are the opposite and adjacent sides of the triangle POM, so

$$\tan\alpha = \frac{MP}{OM} = \text{gradient of } OP.$$

B When α is obtuse, as in the second diagram, then $\angle POM = 180^\circ - \alpha$, so

$$\tan\alpha = -\tan\angle POM = -\frac{MP}{OM} = \text{gradient of } OP.$$

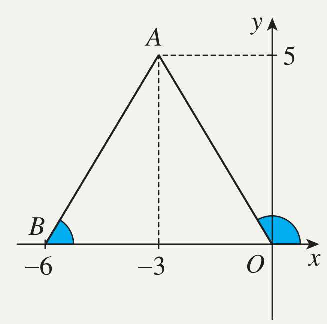
Example 6 — 7B

a Given the points $A(-3, 5)$, $B(-6, 0)$ and $O(0, 0)$, find the angles of inclination of the intervals AB and AO.

b What sort of triangle is $\triangle ABO$?

SOLUTION

a First,
$$\text{gradient of } AB = \frac{0-5}{-6+3} = \tfrac{5}{3},$$
and using a calculator to solve $\tan\alpha = \tfrac{5}{3}$,
angle of inclination of $AB \doteqdot 59^\circ$.



Secondly,
$$\text{gradient of } AO = \frac{0-5}{0+3} = -\tfrac{5}{3},$$
and using a calculator to solve $\tan\alpha = -\tfrac{5}{3}$,
angle of inclination of $AO \doteqdot 121^\circ$.

b Hence $\angle AOB = 59^\circ$ (straight angle).

Thus the base angles of $\triangle AOB$ are equal, so the triangle is isosceles.

A condition for lines to be perpendicular

A vertical and a horizontal line are perpendicular. Otherwise, the condition is that the product of their gradients is -1:

10 PERPENDICULAR LINES

Two lines are perpendicular if and only if:
- $m_1 m_2 = -1$, where m_1 and m_2 are the gradients of the two lines.

  OR
- one is vertical and the other is horizontal.

The condition $m_1 m_2 = -1$ can also be written as $m_2 = -\dfrac{1}{m_1}$.

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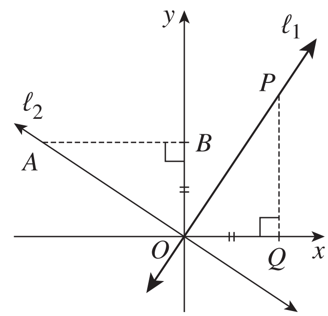
Proof (assuming that neither is vertical)
Shift each line sideways, without rotating it, so that the lines intersect at the origin.
One line must have positive gradient and the other negative gradient, otherwise one of the angles between them would be acute.
So let ℓ_1 be a line with positive gradient through the origin, and let ℓ_2 be a line with negative gradient through the origin.
Construct the two triangles POQ and AOB as shown in the diagram, with the run OQ of ℓ_1 equal to the rise OB of ℓ_2.



Then $m_1 \times m_2 = \dfrac{QP}{OQ} \times \left(-\dfrac{OB}{AB}\right) = -\dfrac{QP}{AB}$, because $OQ = OB$.

A If the lines are perpendicular, then $\angle AOB = \angle POQ$. (adjacent angles at O)
Hence $\triangle AOB \equiv \triangle POQ$ (AAS)
so $QP = AB$ (matching sides of congruent triangles)
so $m_1 \times m_2 = -1$.

B Conversely, if $m_1 \times m_2 = -1$, then $QP = AB$.
Hence $\triangle AOB \equiv \triangle POQ$ (SAS)
so $\angle AOB = \angle POQ$, (matching angles of congruent triangles)
and so ℓ_1 and ℓ_2 are perpendicular.

Example 7 — 7B

What is the gradient of a line perpendicular to a line with gradient $\frac{2}{3}$?

SOLUTION

Perpendicular gradient $= -\frac{3}{2}$ $\left(\text{take the opposite of the reciprocal of } \frac{2}{3}\right)$

Example 8 — 7B

Show that the diagonals of the quadrilateral $ABCD$ are perpendicular, where the vertices are $A(3, 7)$, $B(-1, 6)$, $C(-2, -3)$ and $D(11, 0)$.

SOLUTION

Gradient of $AC = \dfrac{y_2 - y_1}{x_2 - x_1}$
$= \dfrac{-3 - 7}{-2 - 3}$
$= 2,$

gradient of $BD = \dfrac{y_2 - y_1}{x_2 - x_1}$
$= \dfrac{0 - 6}{11 + 1}$
$= -\frac{1}{2}.$

Hence $AC \perp BD$, because the product of the gradients of AC and BD is -1.

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Example 9 7B

The interval joining the points $C(-6, 0)$ and $D(-1, a)$ is perpendicular to a line with gradient 10. Find the value of a.

SOLUTION

$$\text{The interval } CD \text{ has gradient } = \frac{a - 0}{-1 + 6} = \frac{a}{5}.$$

Because the interval CD is perpendicular to a line with gradient 10,

$$\frac{a}{5} \times \frac{10}{1} = -1 \quad \text{(the product of the gradients is } -1\text{)}$$

$$a \times 2 = -1$$

$$\boxed{\div 2} \quad a = -\tfrac{1}{2}.$$

Exercise 7B

FOUNDATION

Note: Diagrams should be drawn wherever possible.

1 **a** Write down the gradient of a line parallel to a line with gradient:

i 2 **ii** $\frac{3}{4}$ **iii** $-1\frac{1}{2}$

b Find the gradient of a line perpendicular to a line with gradient:

i 2 **ii** $\frac{3}{4}$ **iii** $-1\frac{1}{2}$

2 Use the formula gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$ to find the gradient of each interval AB below. Then find the gradient of a line perpendicular to it.

a $A(1, 4)$, $B(5, 0)$ **b** $A(-2, -7)$, $B(3, 3)$ **c** $A(-5, -2)$, $B(3, 2)$

d $A(3, 6)$, $B(5, 5)$ **e** $A(-1, -2)$, $B(1, 4)$ **f** $A(-5, 7)$, $B(15, -7)$

3 Each of the lines AB in this question are either horizontal (with gradient zero), vertical (with gradient undefined) or neither. State which in each case.

a $A(1, 2)$, $B(1, 3)$ **b** $A(4, 3)$, $B(7, 3)$ **c** $A(5, 1)$, $B(4, 5)$

d $A(6, -3)$, $B(1, -3)$ **e** $A(-2, 2)$, $B(2, -2)$ **f** $A(0, 4)$, $B(0, -4)$

4 The points $A(2, 5)$, $B(4, 11)$, $C(12, 15)$ and $D(10, 9)$ form a quadrilateral.

a Find the gradients of AB and DC, and hence show that $AB \parallel DC$.

b Find the gradients of BC and AD, and hence show that $BC \parallel AD$.

c What type of quadrilateral is $ABCD$? (Hint: Look at the definitions in Box 3 on page 305.)

5 **a** Show that $A(-2, -6)$, $B(0, -5)$, $C(10, -7)$ and $D(8, -8)$ form a parallelogram.

b Show that $A(2, 5)$, $B(3, 7)$, $C(-4, -1)$ and $D(-5, 2)$ do not form a parallelogram.

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6 Use the formula gradient $=\tan\alpha$ to find the gradient, correct to two decimal places where necessary, of a line with angle of inclination:

a 15° b 135° c $22\frac{1}{2}^\circ$ d 72°

7 Use the formula gradient $=\tan\alpha$ to find the angle of inclination, correct to the nearest degree where necessary, of a line with gradient:

a 1 b $-\sqrt{3}$ c 4 d $\dfrac{1}{\sqrt{3}}$

DEVELOPMENT

8 The quadrilateral $ABCD$ has vertices $A(-1, 1)$, $B(3, -1)$, $C(5, 3)$ and $D(1, 5)$. Use the definitions of the special quadrilaterals in Box 3 (page 305) to answer these questions.
a Show that the opposite sides are parallel, and hence that $ABCD$ is a parallelogram.
b Show that $AB \perp BC$, and hence that $ABCD$ is a rectangle.
c Show that $AB = BC$, and hence that $ABCD$ is a square.

9 Use gradients to show that each quadrilateral $ABCD$ below is a parallelogram. Then use the definitions in Box 3 (page 305) to show that it is:
a a rhombus, for the vertices $A(2, 1)$, $B(-1, 3)$, $C(1, 0)$ and $D(4, -2)$,
b a rectangle, for the vertices $A(4, 0)$, $B(-2, 3)$, $C(-3, 1)$ and $D(3, -2)$,
c a square, for the vertices $A(3, 3)$, $B(-1, 2)$, $C(0, -2)$ and $D(4, -1)$.

10 Find the gradients of PQ and QR, and hence determine whether P, Q and R are collinear.
a $P(-2, 7)$, $Q(1, 1)$, $R(4, -6)$ b $P(-5, -4)$, $Q(-2, -2)$, $R(1, 0)$

11 Show that the four points $A(2, 5)$, $B(5, 6)$, $C(11, 8)$ and $D(-16, -1)$ are collinear.

12 The triangle ABC has vertices $A(-1, 0)$, $B(3, 2)$ and $C(4, 0)$. Calculate the gradient of each side, and hence show that $\triangle ABC$ is a right-angled triangle.

13 Similarly, show that each triangle below is right-angled. Then find the lengths of the sides enclosing the right angle, and calculate the area of each triangle.
a $P(2, -1)$, $Q(3, 3)$, $R(-1, 4)$ b $X(-1, -3)$, $Y(2, 4)$, $Z(-3, 2)$

14 The interval PQ has gradient -3. A second line passes through $A(-2, 4)$ and $B(1, k)$.
a Find k if AB is parallel to PQ. b Find k if AB is perpendicular to PQ.

15 Find the points A and B where each line below meets the x-axis and y-axis respectively. Hence find the gradient of AB and its angle of inclination α (correct to the nearest degree).

a $y = 3x + 6$ b $y = -\frac{1}{2}x + 1$ c $3x + 4y + 12 = 0$ d $\dfrac{x}{3} - \dfrac{y}{2} = 1$

16 The quadrilateral $ABCD$ has vertices $A(1, -4)$, $B(3, 2)$, $C(-5, 6)$ and $D(-1, -2)$.
a Find the midpoints P of AB, Q of BC, R of CD, and S of DA.
b Prove that $PQRS$ is a parallelogram by showing that $PQ \parallel RS$ and $PS \parallel QR$.

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17 **a** Show that the points $A(-5, 0)$, $B(5, 0)$ and $C(3, 4)$ all lie on the circle $x^2 + y^2 = 25$.
b Explain why AB is a diameter of the circle.
c Show that $AC \perp BC$.

18 Given the points $X(-1, 0)$, $Y(1, a)$ and $Z(a, 2)$, find a if $\angle YXZ = 90°$.

19 For the four points $P(k, 1)$, $Q(-2, -3)$, $R(2, 3)$ and $S(1, k)$, it is known that PQ is parallel to RS. Find the possible values of k.

ENRICHMENT

20 **a** The points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x, y)$ are collinear. Use the gradient formula to show that
$$(x - x_1)(y - y_2) = (y - y_1)(x - x_2).$$
b If AB is the diameter of a circle and P another point on the circumference, then Euclidean geometry tells us that $\angle APB = 90°$. Use this fact to show that the equation of the circle whose diameter has endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ is
$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$

21 **a** Three points $A_1(a_1, b_1)$, $A_2(a_2, b_2)$, $A_3(a_3, b_3)$ form a triangle. By dropping perpendiculars to the x-axis and taking the areas of the resulting trapeziums, show that the area A of the triangle $A_1A_2A_3$ is
$$\text{A} = \frac{1}{2}|a_1b_2 - a_2b_1 + a_2b_3 - a_3b_2 + a_3b_1 - a_1b_3|,$$
with the expression inside the absolute value sign positive if and only if the vertices A_1, A_2 and A_3 are in anti-clockwise order.
b Use part **a** to generate a test for A_1, A_2 and A_3 to be collinear.
c Generate the same test by putting gradient A_1A_2 = gradient A_2A_3.

22 Consider the points $P(2p, p^2)$, $Q\left(-\frac{2}{p}, \frac{1}{p^2}\right)$ and $T(x, -1)$. Find the x-coordinate of T if:
a the three points are collinear,
b PT and QT are perpendicular.

23 The points $P(p, 1/p)$, $Q(q, 1/q)$, $R(r, 1/r)$ and $S(s, 1/s)$ lie on the curve $xy = 1$.
a If $PQ \parallel RS$, show that $pq = rs$.
b Show that $PQ \perp RS$ if and only if $pqrs = -1$.
c Use part **b** to conclude that if a triangle is drawn with its vertices on the rectangular hyperbola $xy = 1$, then the altitudes of the triangle intersect at a common point which also lies on the hyperbola (an altitude of a triangle is the perpendicular from a vertex to the opposite side).

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7C Equations of lines

In the coordinate plane, a line is represented by a linear equation in x and y. This section and the next develop various useful forms for the equation of a line.

Horizontal and vertical lines

All the points on a vertical line have the same x-coordinate, but the y-coordinate can take any value.

11 VERTICAL LINES

The vertical line through the point $P(a, b)$ has equation

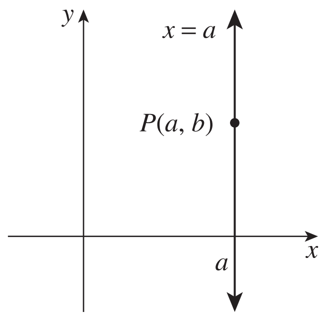
$$x = a.$$

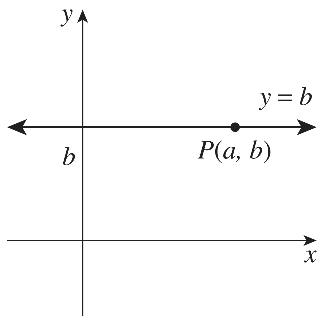
All the points on a horizontal line have the same y-coordinate, but the x-coordinate can take any value.

12 HORIZONTAL LINES

The horizontal line through the point $P(a, b)$ has equation

$$y = b.$$





Gradient–intercept form

There is a simple form of the equation of a line whose gradient and y-intercept are known.

Let ℓ have gradient m and y-intercept b, and pass through the point $B(0, b)$.

Let $Q(x, y)$ be any other point in the plane.

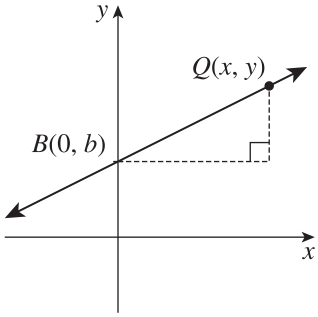
Then the condition for Q to lie on the line ℓ is

$$\text{gradient of } BQ = m,$$

that is, $\dfrac{y - b}{x - 0} = m$ (this is the formula for gradient)

$$y - b = mx.$$

$$y = mx + b.$$



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13 GRADIENT–INTERCEPT FORM

The line with gradient m and y-intercept b is

$$y = mx + b.$$

Example 10 7C

a Write down the gradient and the y-intercept of the line ℓ: $y = 3x - 2$.
b Find the equation of the line through $B(0, 5)$ parallel to ℓ.
c Find the equation of the line through $B(0, 5)$ perpendicular to ℓ.

SOLUTION

a The line ℓ: $y = 3x - 2$ has gradient 3 and y-intercept -2.

b The line through $B(0, 5)$ parallel to ℓ has gradient 3 and y-intercept 5, so its equation is $y = 3x + 5$.

c The line through $B(0, 5)$ perpendicular to ℓ has gradient $-\frac{1}{3}$ and y-intercept 5, so its equation is $y = -\frac{1}{3}x + 5$.

General form

It is often useful to write the equation of a line so that all the terms are on the LHS and only zero is on the RHS. This is called *general form*.

14 GENERAL FORM

The equation of a line is said to be in *general form* if it is

$$ax + by + c = 0, \qquad \text{where } a, b \text{ and } c \text{ are constants.}$$

When an equation is given in general form, it should usually be *simplified* by multiplying out all fractions and dividing through by all common factors.

Example 11 7C

a Put the equation of the line $3x + 4y + 5 = 0$ into gradient–intercept form.
b Hence write down the gradient and y-intercept of the line $3x + 4y + 5 = 0$.

SOLUTION

a Solving the equation for y, $\quad 4y = -3x - 5$

$\boxed{\div 4}$ $\quad y = -\frac{3}{4}x - \frac{5}{4}$, which is gradient–intercept form.

b Hence the line has gradient $-\frac{3}{4}$ and y-intercept $-1\frac{1}{4}$.

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Example 12 7C

Find, in general form, the equation of the line passing through $B(0, -2)$ and:

a perpendicular to a line ℓ with gradient $\frac{2}{3}$,

b having angle of inclination 60°.

SOLUTION

a The line through B perpendicular to ℓ has gradient $-\frac{3}{2}$ and y-intercept -2,

so its equation is $y = -\frac{3}{2}x - 2$ (this is gradient–intercept form)

$\boxed{\times 2}$ $2y = -3x - 4$

$3x + 2y + 4 = 0.$

b The line through B with angle of inclination 60° has gradient $\tan 60^\circ = \sqrt{3}$,

so its equation is $y = x\sqrt{3} - 2$ (this is gradient–intercept form)

$x\sqrt{3} - y - 2 = 0.$

Exercise 7C

FOUNDATION

1 Determine, by substitution, whether the point $A(3, -2)$ lies on the line:

a $y = 4x - 10$ **b** $8x + 10y - 4 = 0$ **c** $x = 3$

2 Find the x-intercept and y-intercept of each line.

a $3x + 4y = 12$ **b** $y = 4x - 6$ **c** $x - 2y = 8$

3 Write down the coordinates of any three points on the line $x + 3y = 24$.

4 Write down the equations of the vertical and horizontal lines through:

a $(1, 2)$ **b** $(0, -4)$ **c** $(5, 0)$

5 Write down the gradient and y-intercept of each line.

a $y = 4x - 2$ **b** $y = \frac{1}{5}x - 3$ **c** $y = 2 - x$ **d** $y = -\frac{5}{7}x$

6 Use the formula $y = mx + b$ to write down the equation of the line with gradient -3 and:

a y-intercept 5, **b** y-intercept $-\frac{2}{3}$, **c** y-intercept 0.

7 Use the formula $y = mx + b$ to write down the equation of the line with y-intercept -4 and:

a gradient 5, **b** gradient $-\frac{2}{3}$, **c** gradient 0.

8 Use the formula $y = mx + b$ to write down the equation of the line:

a with gradient 1 and y-intercept 3,

b with gradient -2 and y-intercept 5,

c with gradient $\frac{1}{5}$ and y-intercept -1,

d with gradient $-\frac{1}{2}$ and y-intercept 3.

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9 Solve each equation for y and hence write down its gradient m and y-intercept b.

a $x - y + 3 = 0$ **b** $y + x - 2 = 0$

c $x - 3y = 0$ **d** $3x + 4y = 5$

10 Write down the gradient m of each line. Then use the formula gradient $=$ $\tan\alpha$ to find its angle of inclination α, correct to the nearest minute where appropriate.

a $y = x + 3$ **b** $y = -x - 16$

c $y = 2x$ **d** $y = -\frac{3}{4}x$

DEVELOPMENT

11 Substitute $y = 0$ and $x = 0$ into the equation of each line below to find the points A and B where the line crosses the x-axis and y-axis respectively. Hence sketch the line.

a $5x + 3y - 15 = 0$ **b** $2x - y + 6 = 0$ **c** $3x - 5y + 12 = 0$

12 Find the gradient of the line through each pair of given points. Then find its equation, using gradient–intercept form. Give your final answer in general form.

a $(0, 4)$, $(2, 8)$ **b** $(0, 0)$, $(1, -1)$ **c** $(-9, -1)$, $(0, -4)$

13 Find the gradient of each line below. Hence find, in gradient–intercept form, the equation of a line passing through $A(0, 3)$ and: **i** parallel to it, **ii** perpendicular to it.

a $2x + y + 3 = 0$ **b** $5x - 2y - 1 = 0$ **c** $3x + 4y - 5 = 0$

14 In each part, find the gradients of the four lines, and hence state what sort of special quadrilateral they enclose.

a $3x + y + 7 = 0$, $x - 2y - 1 = 0$, $3x + y + 11 = 0$, $x - 2y + 12 = 0$

b $4x - 3y + 10 = 0$, $3x + 4y + 7 = 0$, $4x - 3y - 7 = 0$, $3x + 4y + 1 = 0$

15 Find the gradients of the three lines $5x - 7y + 5 = 0$, $2x - 5y + 7 = 0$ and $7x + 5y + 2 = 0$. Hence show that they enclose a right-angled triangle.

16 Draw a sketch of, then find the equations of the sides of:

a the rectangle with vertices $P(3, -7)$, $Q(0, -7)$, $R(0, -2)$ and $S(3, -2)$,

b the triangle with vertices $F(3, 0)$, $G(-6, 0)$ and $H(0, 12)$.

17 In each part below, the angle of inclination α and the y-intercept A of a line are given. Use the formula gradient $=$ $\tan\alpha$ to find the gradient of each line, then find its equation in general form.

a $\alpha = 45^\circ$, $A = (0, 3)$

b $\alpha = 60^\circ$, $A = (0, -1)$

c $\alpha = 30^\circ$, $A = (0, -2)$

d $\alpha = 135^\circ$, $A = (0, 1)$

18 A triangle is formed by the x-axis and the lines $5y = 9x$ and $5y + 9x = 45$.

a Find (correct to the nearest degree) the angles of inclination of the two lines.

b What sort of triangle has been formed?

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19 Consider the two lines $\ell_1 : 3x - y + 4 = 0$ and $\ell_2 : x + ky + \ell = 0$. Find the value of k if:

a ℓ_1 is parallel to ℓ_2,

b ℓ_1 is perpendicular to ℓ_2.

20 Show by substitution that the line $y = mx + b$ passes through $A(0, b)$ and $B(1, m + b)$. Then show that the gradient of AB, and hence of the line, is m.

ENRICHMENT

21 Find the equations of the four circles which are tangent to the x-axis, the y-axis, and the line $x + y = 2$.

22 a Show that the four lines $y = 2x - 1$, $y = 2x + 1$, $y = 3 - \frac{1}{2}x$ and $y = k - \frac{1}{2}x$ enclose a rectangle.

b Find the possible values of k if they enclose a square.

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7D Further equations of lines

This section introduces another standard form of the equation of a line, called *point–gradient form*. It also deals with lines through two given points, and the point of intersection of two lines.

People often ask what form an equation of a line should be put in when the question specifies no form. There are two forms that are acceptable in this situation:

gradient–intercept form and general form.

If general form is used, then in most circumstances it should be simplified by multiplying out any fractions, and dividing through by any common factor.

Point–gradient form

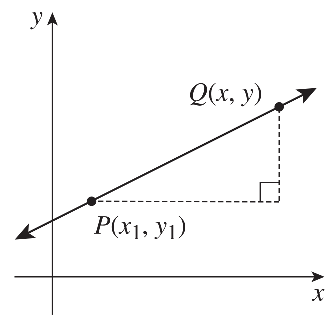
Point–gradient form gives the equation of a line with gradient m passing through a particular point $P(x_1, y_1)$.

Let $Q(x, y)$ be any other point in the plane.
Then the condition for Q to lie on the line is

gradient of $PQ = m$,

that is, $\dfrac{y - y_1}{x - x_1} = m$ (this is the formula for gradient)

$y - y_1 = m(x - x_1)$.



The proof doesn't work when the point Q coincides with P, that is, when $x = x_1$, first because PP does not have a gradient, and secondly because you can't divide by $x - x_1 = 0$. But (x_1, y_1) is still a point on the line $y - y_1 = m(x - x_1)$, as is obvious by direct substitution.

15 POINT–GRADIENT FORM

The line with gradient m through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

Example 13 7D

a Find the equation of the line through $(-2, -5)$ and parallel to $y = 3x + 2$.
b Express the answer in gradient–intercept form, and hence find its y-intercept.

SOLUTION

a The line $y = 3x + 2$ has gradient 3.
Hence the required line is $y - y_1 = m(x - x_1)$ (this is point–gradient form)

$$y + 5 = 3(x + 2)$$
$$y + 5 = 3x + 6$$
$$y = 3x + 1.$$ (this is gradient–intercept form)

b Hence the new line has y-intercept 1.

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The line through two given points

Given two distinct points, there is just one line passing through them both. Its equation is best found by a two-step approach.

16 THE LINE THROUGH TWO GIVEN POINTS

- First find the gradient of the line, using the gradient formula.
- Then find the equation of the line, using point–gradient form.

Example 14 7D

Find the equation of the line passing through $A(1, 5)$ and $B(4, -1)$.

SOLUTION

First, using the gradient formula,

$$\begin{aligned}\text{gradient of } AB &= \frac{-1 - 5}{4 - 1}\\ &= -2.\end{aligned}$$

Then, using point–gradient form for a line with gradient -2 through $A(1, 5)$,

the line AB is

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - 5 &= -2(x - 1)\\ y - 5 &= -2x + 2\\ y &= -2x + 7.\end{aligned}$$

Note: Using the coordinates of $B(4, -1)$ rather than $A(1, 5)$ would give the same equation.

Example 15 7D

Given the points $A(6, 0)$ and $B(0, 9)$, find, in general form, the equation of the perpendicular bisector of AB.

SOLUTION

First,

$$\begin{aligned}\text{gradient of } AB &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{9 - 0}{0 - 6}\\ &= -\tfrac{3}{2},\end{aligned}$$

so any line perpendicular to AB has gradient $\frac{2}{3}$.

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Secondly, midpoint of $AB = \left(\frac{6+0}{2}, \frac{0+9}{2}\right)$
$= \left(3, 4\tfrac{1}{2}\right)$.

Hence, using point–gradient form, the perpendicular bisector of AB is

$$
\begin{aligned}
y - y_1 &= m(x - x_1)\\
y - 4\tfrac{1}{2} &= \tfrac{2}{3}(x - 3)\\
\boxed{\times 6} \quad 6y - 27 &= 4x - 12\\
4x - 6y + 15 &= 0.
\end{aligned}
$$

Intersection of lines — concurrent lines

The point where two distinct lines intersect can be found using simultaneous equations, as discussed in Sections 1G and 3C.

Three distinct lines are called *concurrent* if they all pass through the same point.

17 TESTING FOR CONCURRENT LINES

To test whether three lines are concurrent:

- First find the point of intersection of two of them.
- Then test, by substitution, whether this point lies on the third line.

Example 16 — 7D

Test whether the following three lines are concurrent.

ℓ_1: $5x - y - 10 = 0$, $\quad \ell_2$: $x + y - 8 = 0$, $\quad \ell_3$: $2x - 3y + 9 = 0$.

SOLUTION

A Solve ℓ_1 and ℓ_2 simultaneously.

Adding ℓ_1 and ℓ_2, $\quad 6x - 18 = 0$
$x = 3$,

and substituting into ℓ_2, $3 + y - 8 = 0$
$y = 5$

so the lines ℓ_1 and ℓ_2 intersect at $(3, 5)$.

B Substituting the point $(3, 5)$ into the third line ℓ_3,

$$
\begin{aligned}
\text{LHS} &= 6 - 15 + 9\\
&= 0\\
&= \text{RHS},
\end{aligned}
$$

so the three lines are concurrent, meeting at $(3, 5)$.

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Exercise 7D

FOUNDATION

1 Find, in general form, the equation of the line:

a through $(1, 1)$ with gradient 2,

b with gradient -1 through $(3, 1)$,

c through $(0, 0)$ with gradient -5,

d through $(-1, 3)$ with gradient $-\frac{1}{3}$,

e with gradient $-\frac{4}{5}$ through $(3, -4)$.

2 Find, in gradient–intercept form, the equation of:

a the line through $(2, 5)$ and parallel to $y = 2x + 5$,

b the line through $(2, 5)$ and perpendicular to $y = 2x + 5$,

c the line through $(5, -7)$ and perpendicular to $y = -5x$,

d the line through $(-7, 6)$ and parallel to $y = \frac{3}{7}x - 8$,

e the line through $(-4, 0)$ and perpendicular to $y = -\frac{2}{5}x$.

3 **a** Find the gradient of the line through the points $A(4, 7)$ and $B(6, 13)$.

b Hence use point–gradient form to find, in general form, the equation of the line AB.

4 Find the gradient of the line through each pair of points, and hence find its equation.

a $(3, 4)$, $(5, 8)$ **b** $(-1, 3)$, $(1, -1)$ **c** $(5, 6)$, $(-1, 4)$

d $(-1, 0)$, $(0, 2)$ **e** $(0, -1)$, $(-4, 0)$ **f** $(0, -3)$, $(3, 0)$

5 **a** Find the gradient of the line through $A(1, -2)$ and $B(-3, 4)$.

b Hence find, in general form, the equation of:

i the line AB,

ii the line through A and perpendicular to AB.

6 Find the equation of the line parallel to $2x - 3y + 1 = 0$ and:

a passing through $(2, 2)$, **b** passing through $(3, -1)$.

7 Find the equation of the line perpendicular to $3x + 4y - 3 = 0$ and:

a passing through $(-1, -4)$, **b** passing through $(-2, 1)$.

DEVELOPMENT

8 **a** Find the point M of intersection of the lines ℓ_1: $x + y = 2$ and ℓ_2: $4x - y = 13$.

b Show that M lies on ℓ_3: $2x - 5y = 11$, and hence that ℓ_1, ℓ_2 and ℓ_3 are concurrent.

c Use the same method to test whether each set of lines is concurrent.

i $2x + y = -1$, $x - 2y = -18$ and $x + 3y = 15$

ii $6x - y = 26$, $5x - 4y = 9$ and $x + y = 9$

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9 Put the equation of each line into gradient–intercept form and hence write down the gradient. Then find, in gradient–intercept form, the equation of the line that is:

i parallel to it through $A(3, -1)$, **ii** perpendicular to it through $B(-2, 5)$.

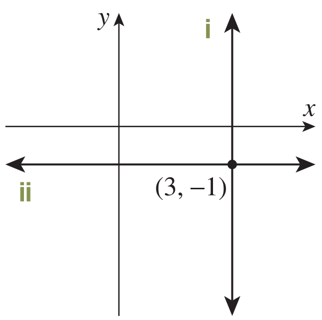
a $2x + y + 3 = 0$ **b** $5x - 2y - 1 = 0$ **c** $4x + 3y - 5 = 0$

10 The angle of inclination α and a point A on a line are given below. Use the formula gradient $= \tan\alpha$ to find the gradient of each line, then find its equation in general form.

a $\alpha = 45°$, $A = (1, 0)$ **b** $\alpha = 120°$, $A = (-1, 0)$
c $\alpha = 30°$, $A = (4, -3)$ **d** $\alpha = 150°$, $A = (-2, -5)$

11 Determine, in general form, the equation of each straight line sketched below.

a



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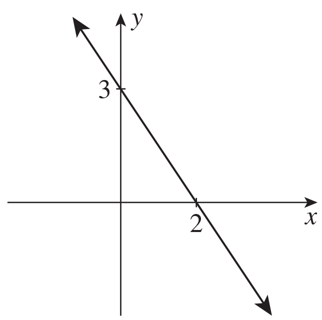

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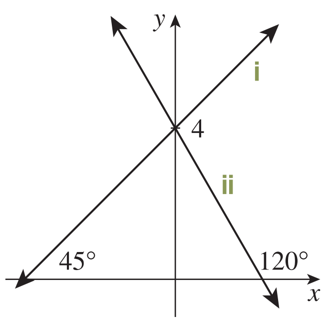

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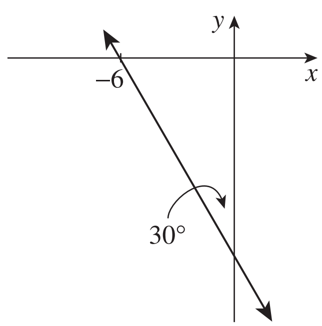
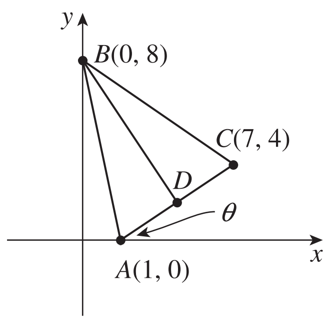


12 Explain why the four lines $\ell_1: y = x + 1$, $\ell_2: y = x - 3$, $\ell_3: y = 3x + 5$ and $\ell_4: y = 3x - 5$ enclose a parallelogram. Then find the vertices of this parallelogram.

13 Show that the triangle with vertices $A(1, 0)$, $B(6, 5)$ and $C(0, 2)$ is right-angled. Then find the equation of each side.

14 The three points $A(1, 0)$, $B(0, 8)$ and $C(7, 4)$ form a triangle. Let θ be the angle between AC and the x-axis.



a Find the gradient of the line AC and hence determine θ, correct to the nearest degree.
b Derive the equation of AC.
c Find the coordinates of the midpoint D of AC.
d Show that AC is perpendicular to BD.
e What type of triangle is ABC?
f Find the area of this triangle.
g Write down the coordinates of a point E such that the parallelogram $ABCE$ is a rhombus.

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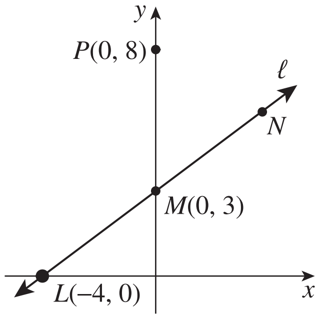
15 The line ℓ crosses the x- and y-axes at $L(-4, 0)$ and $M(0, 3)$. The point N lies on ℓ, and P is the point $P(0, 8)$.



- **a** Copy the sketch and find the equation of ℓ.
- **b** Find the lengths of ML and MP and hence show that LMP is an isosceles triangle.
- **c** If M is the midpoint of LN, find the coordinates of N.
- **d** Show that $\angle NPL = 90°$.
- **e** Write down the equation of the circle through N, P and L.

16 Find k if the lines ℓ_1: $x + 3y + 13 = 0$, ℓ_2: $4x + y - 3 = 0$ and ℓ_3: $kx - y - 10 = 0$ are concurrent. (Hint: Find the point of intersection of ℓ_1 and ℓ_2 and substitute into ℓ_3.)

17 Consider the two lines ℓ_1: $3x + 2y + 4 = 0$ and ℓ_2: $6x + \mu y + \lambda = 0$. Write down the value of μ if:

- **a** ℓ_1 is parallel to ℓ_2,
- **b** ℓ_1 is perpendicular to ℓ_2.

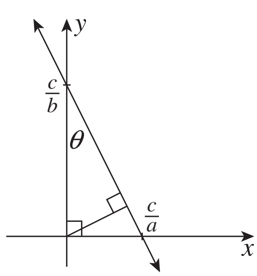
18 Write down the equation of the line with the intercepts $(a, 0)$ and $(0, b)$, then rewrite it in general form.

ENRICHMENT

19 The line passing through $M(a, b)$ intersects the x-axis at A and the y-axis at B. Find the equation of the line, given that M bisects AB.

20 The tangent to a circle is perpendicular to the radius at the point of contact. Use this fact to show that the tangent to $x^2 + y^2 = r^2$ at the point (a, b) has equation $ax + by = r^2$.

21 [Perpendicular form of a line]
Consider the line ℓ with equation $ax + by = c$ where, for the sake of convenience, the values of a, b and c are positive. Suppose that this line makes an acute angle θ with the y-axis as shown.



- **a** Show that $\cos\theta = \dfrac{a}{\sqrt{a^2 + b^2}}$ and $\sin\theta = \dfrac{b}{\sqrt{a^2 + b^2}}$.
- **b** The *perpendicular form* of the line ℓ is
$$\frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y = \frac{c}{\sqrt{a^2 + b^2}}.$$
Use part **a** to help show that the RHS of this equation is the perpendicular distance from the line to the origin.
- **c** Write these lines in perpendicular form and hence find their perpendicular distances from the origin:
  - **i** $3x + 4y = 5$
  - **ii** $3x - 2y = 1$

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7E Using pronumerals in place of numbers

Most problems of this chapter so far have used numbers for the coordinates of points, and for the coefficients of x and y. When constant pronumerals are used, however, far more general results can be obtained. For example, Chapter 3 used pronumeral constants systematically in the study of quadratics and their parabolic graphs. Accordingly, this final section consolidates the methods of the last four sections, but uses pronumerals, rather than numbers, for points and coefficients. Because of the generality, most of the questions have an interpretation as a geometric result.

When the French mathematician and philosopher René Descartes introduced the coordinate plane in the 17th century, he intended it as an alternative approach to geometry, in which all the ancient theorems of Euclid would be proven again using algebraic rather than geometric arguments. In our course, however, Descartes' methods are used with the reverse purpose — we are turning functions that have been defined algebraically into geometric objects that can be visualised and interpreted using the methods of geometry. Questions here should be interpreted both algebraically and geometrically.

The procedures in this section are more demanding and sustained than in the previous sections, and the whole section may be regarded as extension.

Example 17 7E

[This worked example has an introductory part **a** that uses numbers. Part **b** uses pronumerals, and is the important part.]

a A triangle has vertices at $A(1, -3)$, $B(3, 3)$ and $C(-3, 1)$.

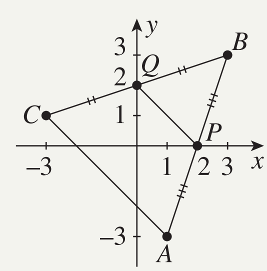
  i Find the coordinates of the midpoint P of AB and the midpoint Q of BC.

  ii Show that $PQ \parallel AC$ and that $PQ = \frac{1}{2}AC$.

b Repeat this procedure for a triangle with vertices $A(2a, 0)$, $B(2b, 2c)$ and $C(0, 0)$, where $a > 0$.

SOLUTION

a i Using the midpoint formula,



For P, $x = \dfrac{1+3}{2} = 2$ $y = \dfrac{-3+3}{2} = 0.$

For Q, $x = \dfrac{3-3}{2} = 0$ $y = \dfrac{3+1}{2} = 2.$

Hence $P = (2, 0)$ and $Q = (0, 2)$

ii The lines PQ and AC are parallel because they both have gradient -1:

$$\text{gradient } PQ = \frac{2-0}{0-2} = -1 \qquad \text{gradient } AC = \frac{1+3}{-3-1} = -1.$$

Using the distance formula,

$$PQ^2 = (0-2)^2 + (2-0)^2 = 8 \qquad AC^2 = (-3-1)^2 + (1+3)^2 = 32$$

so $PQ = 2\sqrt{2}$ and $AC = 4\sqrt{2} = 2PQ$.

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b **i** Using the midpoint formula,

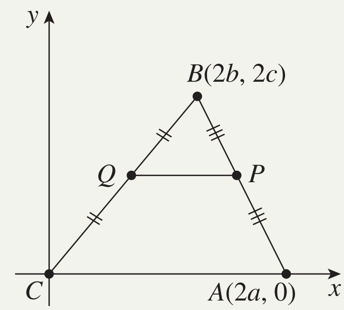
For P, $x = \dfrac{2a + 2b}{2}$ $\quad y = \dfrac{0 + 2c}{2}$

$= a + b$ $\quad = c.$

For Q, $x = \dfrac{2b + 0}{2}$ $\quad y = \dfrac{2c + 0}{2}$

$= b$ $\quad = c.$

Hence $P = (a + b, c)$ and $Q = (b, c)$.



ii The lines PQ and AC are parallel because they are both horizontal.

Using the distance formula,

$PQ^2 = (b - a - b)^2 + (c - c)^2$ $\qquad AC^2 = (0 - 2a)^2 + (0 - 0)^2$

$= a^2$ $\qquad = 4a^2$

so $PQ = a$ and $AC = 2a = 2PQ$

Any triangle can be translated and rotated into the position of the triangle in part **b**, so this worked example demonstrates that the interval joining the midpoints of two sides of a triangle is parallel to the base and half its length.

Exercise 7E

FOUNDATION

Note: Diagrams should be drawn wherever possible.

Many of these questions involve a particular triangle with numerical coordinates, before applying your understanding to a triangle where the coordinates include pronumerals.

1 **a** On a number plane, plot the points $O(0, 0)$, $A(6, 0)$, $B(6, 6)$ and $C(0, 6)$, which form a square.

i Find the gradients of the diagonals OB and AC.

ii Hence show that the diagonals OB and AC are perpendicular.

b Let the vertices of a square be $O(0, 0)$, $A(a, 0)$, $B(a, a)$ and $C(0, a)$.

i Find the gradients of the diagonals OB and AC.

ii Hence show that the diagonals OB and AC are perpendicular.

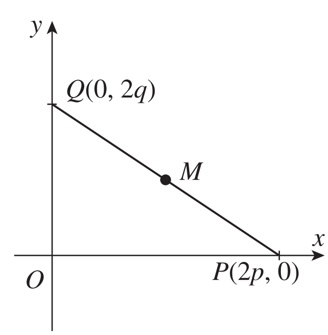
In this question you have shown that *the diagonals of a square are perpendicular*.

2 **a** The points $O(0, 0)$, $P(8, 0)$ and $Q(0, 10)$ form a right-angled triangle. Let M be the midpoint of PQ.

i Find the coordinates of M.

ii Find the distances OM, PM and QM, and show that M is equidistant from each of the vertices.

iii Explain why a circle with centre M can be drawn through the three vertices O, P and Q.



b Consider a right-angled triangle with vertices at $O(0, 0)$, $P(2p, 0)$ and $Q(0, 2q)$ and repeat the steps of part **a**.

In this question you have shown that *the midpoint of the hypotenuse of a right-angled triangle is the centre of a circle through all three vertices.*

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3 **a** A triangle has vertices at $A(3, 1)$, $B(7, 3)$ and $C(1, -1)$.

i Find the coordinates of the midpoint P of AB and the midpoint Q of BC.

ii Show that the equation of the line ℓ through P parallel to AC is $y = x - 3$.

iii Show that Q lies on the line found in part **ii**. That is, show that ℓ bisects BC.

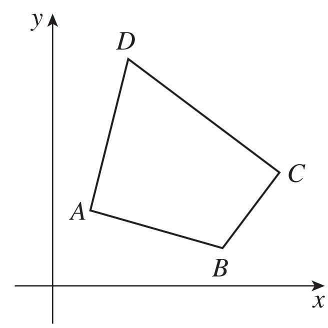
iv Show that $PQ = \frac{1}{2}AC$.

b Consider a triangle by placing its vertices at $A(2a, 0)$, $B(2b, 2c)$ and $C(0, 0)$, where $a > 0$, and repeat the steps in part **a**.

In this question you have shown that *the interval through the midpoint of a side and parallel to the base bisects the third side and is half the length of the base.*

DEVELOPMENT

4 Let the vertices of a quadrilateral be $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$ and $D(d_1, d_2)$, as in the diagram to the right.



a Find the midpoints P, Q, R and S of the sides AB, BC, CD and DA respectively. (The figure $PQRS$ is also a quadrilateral.)

b Find the midpoints of the diagonals PR and QS.

c Explain why this proves that $PQRS$ is a parallelogram.

In this question you have shown that *the midpoints of the sides of a quadrilateral form a parallelogram.*

5 The points $A(a, 0)$ and $Q(q, 0)$ are points on the positive x-axis, and the points $B(0, b)$ and $P(0, p)$ lie on the positive y-axis. Show that $AB^2 - AP^2 = QB^2 - QP^2$.

6 The triangle OBA has its vertices at the origin $O(0, 0)$, at $A(3, 0)$ and at $B(0, 4)$. The point C lies on AB, and OC is perpendicular to AB. Draw a diagram showing this information.

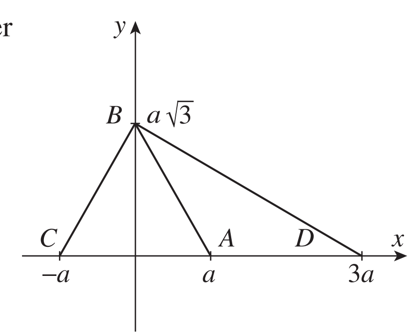
a Find the equations of AB and OC and hence find the coordinates of C.

b Find the lengths OA, AB, OC, BC and AC.

c Thus confirm these important results for a right-angled triangle:

i $OC^2 = AC \times BC$ **ii** $OA^2 = AC \times AB$

7 The diagram to the right shows the points A, B, C and D on the number plane.



a Show that $\triangle ABC$ is equilateral.

b Show that $\triangle ABD$ is isosceles, with $AB = AD$.

c Show that $AB^2 = \frac{1}{3}BD^2$.

8 Place three vertices of a parallelogram $ABCD$ at $A(0, 0)$, $B(2a, 2b)$ and $D(2c, 2d)$.

a Use gradients to show that $C = (2a + 2c, 2b + 2d)$ completes the parallelogram.

b Find the midpoints of the diagonals AC and BD.

c Explain why this proves that *the diagonals of a parallelogram bisect each other.*

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ENRICHMENT

9 a The points $A(1, -2)$, $B(5, 6)$ and $C(-3, 2)$ are the vertices of a triangle, and P, Q and R are the midpoints of BC, CA and AB respectively.
   i Find the equations of the three medians BQ, CR and AP.
   ii Find the intersection of BQ and CR, and show that it lies on the third median AP.
  b Now consider the more general situation of a triangle with vertices at $A(6a, 6b)$, $B(-6a, -6b)$ and $C(0, 6c)$, and follow these steps.
   i Find the midpoints P, Q and R of BC, CA and AB respectively.
   ii Show that the median through C is $x = 0$ and find the equations of the other two medians.
   iii Find the point where the median through C meets the median through A, and show that this point lies on the median through B.

  In this question you have shown that *the three medians of a triangle are concurrent.* (Their point of intersection is called the *centroid*.)

10 Place a triangle in the plane with vertices $A(2a, 0)$, $B(-2a, 0)$ and $C(2b, 2c)$.
  a Find the gradients of AB, BC and CA.
  b Hence find the equations of the three perpendicular bisectors.
  c Find the intersection M of any two bisectors, and show that it lies on the third.
  d Explain why the circumcentre is equidistant from each vertex.

  In this question you have shown that *the perpendicular bisectors of the sides of a triangle are concurrent*. Their point of intersection (called the *circumcentre*) is the centre of a circle through all three vertices (called the *circumcircle*).

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Chapter 7 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 7 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Let $X = (2, 9)$ and $Y = (14, 4)$. Use the standard formulae to find:

a the midpoint of XY, **b** the gradient of XY, **c** the length of XY.

2 A triangle has vertices $A(1, 4)$, $B(-3, 1)$ and $C(-2, 0)$.

a Find the lengths of all three sides of $\triangle ABC$.

b What sort of triangle is $\triangle ABC$?

3 A quadrilateral has vertices $A(2, 5)$, $B(4, 9)$, $C(8, 1)$ and $D(-2, -7)$.

a Find the midpoints P of AB, Q of BC, R of CD and S of DA.

b Find the gradients of PQ, QR, RS and SP.

c What sort of quadrilateral is $PQRS$?

4 A circle has diameter AB, where $A = (2, -5)$ and $B = (-4, 7)$.

a Find the centre C and radius r of the circle.

b Use the distance formula to test whether $P(6, -1)$ lies on the circle.

5 **a** Find the gradients of the sides of $\triangle LMN$, given $L(3, 9)$, $M(8, -1)$ and $N(-1, 7)$.

b Explain why $\triangle LMN$ is a right-angled triangle.

6 **a** Find the gradient of the interval AB, where $A = (3, 0)$ and $B = (5, -2)$.

b Find a if $AP \perp AB$, where $P = (a, 5)$.

c Find the point $Q(b, c)$ if B is the midpoint of AQ.

d Find d if the interval AD has length 5, where $D = (6, d)$.

7 Find, in general form, the equation of the line:

a with gradient -2 and y-intercept 5,

b with gradient $\frac{2}{3}$ through the point $A(3, 5)$,

c through the origin and perpendicular to $y = 7x - 5$,

d through $B(-5, 7)$ and parallel to $y = 4 - 3x$,

e with y-intercept -2 and angle of inclination 60°.

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8 Put the equation of each line into gradient–intercept form, and hence find its y-intercept b, its gradient m and its angle of inclination α (correct to the nearest minute when necessary).

a $5x - 6y - 7 = 0$

b $4x + 4y - 3 = 0$

9 Find the gradient of each line AB, then find its equation in general form.

a $A(3, 0)$ and $B(4, 8)$

b $A(5, -2)$ and $B(7, -7)$

10 **a** Are the points $L(7, 4)$, $M(13, 2)$ and $N(25, -3)$ collinear?

b Are the lines $2x + 5y - 29 = 0$, $4x - 2y + 2 = 0$ and $7x - 3y + 1 = 0$ concurrent?

11 **a** Determine whether the lines $8x + 7y + 6 = 0$, $6x - 4y + 3 = 0$ and $2x + 3y + 9 = 0$ enclose a right-angled triangle.

b Determine what sort of figure the lines $4x + 8y + 3 = 0$, $5x - 2y + 7 = 0$, $x + 2y - 6 = 0$ and $9x - 3y = 0$ enclose.

12 **a** Find the points where the line $5x + 4y - 30 = 0$ meets the x-axis and y-axis.

b Hence find the area of the triangle formed by the line, the x-axis and the y-axis.

13 A sketch is essential in this question.

a Find the gradient, length and midpoint M of the interval joining $A(10, 2)$ and $B(2, 8)$.

b Show that the perpendicular bisector of the interval AB has equation $4x - 3y - 9 = 0$.

c Find the point C where the perpendicular bisector meets the line $x - y + 2 = 0$.

d Use the distance formula to show that C is equidistant from A and B.

e Show that $CM = 15$ and hence find the area of $\triangle ABC$.

f Let $\theta = \angle ACB$. Use the area formula area $= \frac{1}{2} \times AC \times BC \times \sin\theta$ to find θ, correct to the nearest minute.

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8 Exponential and logarithmic functions

Suppose that the amount of mould on a piece of cheese is doubling every day.

- In one day, the mould increases by a factor of 2.
- In two days, the mould increases by a factor of $2^2 = 4$.
- In three days, the mould increases by a factor of $2^3 = 8$.
- In four days, the mould increases by a factor of $2^4 = 16$.

And so on, until the cheese is thrown out. Applying mathematics to situations such as this requires indices and logarithms, and their graphs. Logarithms may be quite unfamiliar to many readers.

The application questions throughout, and particularly the final exercise, indicate some of the extraordinary variety of situations where exponential and logarithmic functions are needed.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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8A Indices

This section reviews powers such as 2^3 and 5^{-2}, whose indices are integers. Then the various index laws for working with them are reviewed.

Power, base and index

Each of these three words means a different thing.

1 POWER, BASE AND INDEX

- An expression a^n is called a *power*.
- The number a is called the *base*.
- The number n is called the *index* or *exponent*.

Thus 2^3 is a *power* with *base 2* and *index 3*.

The words *exponent* and *index* (plural *indices*) mean exactly the same thing.

Powers whose indices are positive integers

Powers are defined differently depending on what sort of number the index is. When the index is a positive integer, the definition is quite straightforward:

2 POWERS WHOSE INDICES ARE POSITIVE INTEGERS

For any number a,

$$a^1 = a, \qquad a^2 = a \times a, \qquad a^3 = a \times a \times a, \qquad \ldots$$

In general, $a^n = \overbrace{a \times a \times a \times \cdots \times a}^{n \text{ factors}}$, for all positive integers n.

For example, $7^1 = 7$ $\qquad$ $3^4 = 3 \times 3 \times 3 \times 3 = 81$ $\qquad$ $9^5 = 9 \times 9 \times 9 \times 9 \times 9 = 59049$

although with a large number such as 9^5, the index form may be more convenient.

Scientific calculators can help to evaluate or approximate powers. The button for powers is labelled $\boxed{x^y}$ or $\boxed{\wedge}$, depending on the calculator. For example,

$$2^{26} = 67\,108\,864 \qquad 3^{50} \doteqdot 7.179 \times 10^{23} \qquad \pi^4 \doteqdot 97.41$$

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Index laws — combining powers with the same base

The first three index laws show how to combine powers when the base is fixed.

3 INDEX LAWS — PRODUCTS, QUOTIENTS AND POWERS OF POWERS

- To multiply powers with the same base, add the indices,
  $a^m \times a^n = a^{m+n}$
- To divide powers with the same base, subtract the indices,
  $a^m \div a^n = a^{m-n}$ $\left(\text{also written as } \dfrac{a^m}{a^n} = a^{m-n}\right)$
- To raise a power to a power, multiply the indices,
  $(a^m)^n = a^{mn}$

Demonstrating the results when $m = 5$ and $n = 3$ should make these laws clear.

- $$\begin{aligned} a^5 \times a^3 &= (a \times a \times a \times a \times a) \times (a \times a \times a) \\ &= a^8 \\ &= a^{5+3} \end{aligned}$$
- $$\begin{aligned} a^5 \div a^3 &= \frac{a \times a \times a \times a \times a}{a \times a \times a} \\ &= a^2 \\ &= a^{5-3} \end{aligned}$$
- $$\begin{aligned} (a^5)^3 &= a^5 \times a^5 \times a^5 \\ &= a^{5+5+5} \quad \text{(to multiply powers of the same base, add the indices)} \\ &= a^{5\times 3} \end{aligned}$$

Example 1 8A

Use the index laws above to simplify each expression.

a $x^3 \times x^7$ **b** $3^x \times 3^{5x}$ **c** $w^{12} \div w^3$

d $10^{a+b} \div 10^b$ **e** $(y^6)^9$ **f** $(2^{3x})^{2y}$

SOLUTION

a $x^3 \times x^7 = x^{10}$ **b** $3^x \times 3^{5x} = 3^{6x}$ **c** $w^{12} \div w^3 = w^9$

d $10^{a+b} \div 10^b = 10^a$ **e** $(y^6)^9 = y^{54}$ **f** $(2^{3x})^{2y} = 2^{6xy}$

Index laws — powers of products and quotients

The next two index laws show how to work with powers of products and powers of quotients.

4 INDEX LAWS — POWERS OF PRODUCTS AND QUOTIENTS

- The power of a product is the product of the powers,
  $(ab)^n = a^n \times b^n$
- The power of a quotient is the quotient of the powers,
  $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

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Demonstrating the results when $n = 3$ should make the general results obvious.

- $(ab)^3 = ab \times ab \times ab$
  $= a \times a \times a \times b \times b \times b$
  $= a^3 b^3$
- $\left(\frac{a}{b}\right)^3 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$
  $= \frac{a^3}{b^3}$

Example 2 — 8A

Expand the brackets in each expression.

a $(10a)^3$ **b** $(3x^2y)^3$ **c** $\left(\frac{x}{3}\right)^4$ **d** $\left(\frac{a^3}{3b}\right)^2$

SOLUTION

a $(10a)^3 = 10^3 \times a^3$
$= 1000a^3$

b $(3x^2y)^3 = 3^3 \times (x^2)^3 \times y^3$
$= 27x^6y^3$

c $\left(\frac{x}{3}\right)^4 = \frac{x^4}{3^4}$
$= \frac{x^4}{81}$

d $\left(\frac{a^3}{3b}\right)^2 = \frac{(a^3)^2}{3^2 \times b^2}$
$= \frac{a^6}{9b^2}$

Zero and negative indices

The index laws were demonstrated only when the indices were all positive integers. Powers with negative indices are defined in such a way that these laws are valid for negative indices as well.

Zero and negative indices involve division by the base a, so a cannot be zero.

1 We know that $a^3 \div a^3 = 1$.
If we use the index laws, however, $a^3 \div a^3 = a^{3-3} = a^0$.
Hence it is convenient to define $a^0 = 1$.

2 We know that $a^2 \div a^3 = \frac{1}{a}$.
If we used the index laws, however, $a^2 \div a^3 = a^{2-3} = a^{-1}$.
Hence it is convenient to define $a^{-1} = \frac{1}{a}$.

3 Similarly, we shall define $a^{-2} = \frac{1}{a^2}$, and $a^{-3} = \frac{1}{a^3}$, and so on.

5 ZERO AND NEGATIVE INDICES

Let the base a be any non-zero number.

- Define $a^0 = 1$.
- Define $a^{-1} = \frac{1}{a}$, define $a^{-2} = \frac{1}{a^2}$, define $a^{-3} = \frac{1}{a^3}$, ...
- In general, define $a^{-n} = \frac{1}{a^n}$, for all positive integers n.

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Thus the negative sign in the index says, 'Take the reciprocal'. Always do this first, and always write the reciprocal of $\frac{2}{3}$ as $\frac{3}{2}$, never as $\frac{1}{\frac{2}{3}}$. For example,

$$\left(\tfrac{2}{3}\right)^{-3} = \left(\tfrac{3}{2}\right)^{3} = \tfrac{27}{8}.$$

Example 3 8A

Write each expression using fractions instead of negative indices, then simplify it.

a 12^{-1}

b 7^{-2}

c $\left(\tfrac{2}{3}\right)^{-1}$

d $\left(\tfrac{2}{3}\right)^{-4}$

SOLUTION

a $12^{-1} = \tfrac{1}{12}$

b $7^{-2} = \dfrac{1}{7^2}$

$= \tfrac{1}{49}$

c $\left(\tfrac{2}{3}\right)^{-1} = \tfrac{3}{2}$ (the negative index says, 'take the reciprocal')

d $\left(\tfrac{2}{3}\right)^{-4} = \left(\tfrac{3}{2}\right)^{4}$ (take the reciprocal first)

$= \tfrac{81}{16}$ (then take the fourth power)

Example 4 8A

Write each expression using negative indices instead of fractions.

a $\dfrac{1}{2^{20}}$

b $\dfrac{1}{x^3}$

c $\dfrac{5}{a^7}$

d $\dfrac{w^2}{v^4}$

SOLUTION

a $\dfrac{1}{2^{20}} = 2^{-20}$

b $\dfrac{1}{x^3} = x^{-3}$

c $\dfrac{5}{a^7} = 5 \times \dfrac{1}{a^7}$

$= 5a^{-7}$

d $\dfrac{w^2}{v^4} = w^2 \times \dfrac{1}{v^4}$

$= w^2 v^{-4}$

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Example 5 — 8A

Use the index laws, extended to negative indices, to simplify these expressions, giving answers in fraction form without negative indices.

a $x^4 \times x^{-10}$ **b** $12a^2b^4 \div 6a^3b^4$ **c** $(2^n)^{-3}$

SOLUTION

a
$$\begin{aligned} x^4 \times x^{-10} &= x^{4-10} \\ &= x^{-6} \\ &= \frac{1}{x^6} \end{aligned}$$

b
$$\begin{aligned} 12a^2b^4 \div 6a^3b^4 &= \frac{12}{6}a^{2-3}b^{4-4} \\ &= 2a^{-1}b^0 \\ &= 2a^{-1} \\ &= \frac{2}{a} \end{aligned}$$

c
$$\begin{aligned} (2^n)^{-3} &= 2^{n\times(-3)} \\ &= 2^{-3n} \\ &= \frac{1}{2^{3n}} \end{aligned}$$

Solving index equations

The basic approach is to write everything in terms of primes.

Example 6 — 8A

a Solve $27^x = 9^{3x-6}$ **b** Solve $\left(\frac{1}{16}\right)^{2x+4} = 4^{-6x}$

SOLUTION

a
$$\begin{aligned} 27^x &= 9^{3x-6} \\ (3^3)^x &= (3^2)^{3x-6} \\ 3^{3x} &= 3^{6x-12} \\ 3x &= 6x - 12 \\ x &= 4 \end{aligned}$$

b
$$\begin{aligned} \left(\tfrac{1}{16}\right)^{2x+4} &= 4^{-6x} \\ (2^{-4})^{2x+4} &= (2^2)^{-6x} \\ 2^{-8x-16} &= 2^{-12x} \\ -8x - 16 &= -12x \\ x &= 4 \end{aligned}$$

Exercise 8A — FOUNDATION

Note: Do not use a calculator in this exercise at all, except for the very last question.

1 The town of Goldhope had a small population in 1970, but the population has been increasing by a factor of 3 every decade since then.

- **a** By what factor had the population increased from its 1970 value in 1980, 1990, 2000, 2010 and 2020?
- **b** If the town's population was 10 000 in 1970, during which decade did the population pass 1 000 000?

2 **Preparation:** The answers to this question will be needed for all the exercises in this chapter. Keep the results in a place where you can refer to them easily.

- **a** Write down all the powers of 2 from $2^0 = 1$ to $2^{12} = 4096$.
- **b** In the same way, write down:
  - **i** the powers of 3 to $3^6 = 729$,
  - **ii** the powers of 5 to $5^5 = 3125$,
  - **iii** the powers of 6 to $6^3 = 216$,
  - **iv** the powers of 7 to $7^3 = 343$,
- **c** From your list of powers of 2, read off:
  - **i** powers of 4,
  - **ii** powers of 8.

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3 Simplify:

a 2^6 **b** $\left(\frac{2}{3}\right)^2$ **c** $\left(\frac{2}{3}\right)^3$ **d** $\left(\frac{3}{10}\right)^4$ **e** $\left(\frac{4}{7}\right)^2$

f $\left(\frac{5}{9}\right)^1$ **g** 7^0 **h** 5^{-1} **i** 11^{-1} **j** 6^{-2}

k 10^{-2} **l** 3^{-3} **m** 5^{-3} **n** 2^{-5} **o** 10^{-6}

4 Simplify these expressions. With negative indices, first take the reciprocal, then take the power.

a $\left(\frac{1}{11}\right)^{-1}$ **b** $\left(\frac{2}{7}\right)^{-1}$ **c** $\left(\frac{7}{2}\right)^{-1}$ **d** $\left(\frac{10}{23}\right)^{-1}$ **e** $(0.1)^{-1}$

f $(0.01)^{-1}$ **g** $(0.02)^{-1}$ **h** 5^{-2} **i** $\left(\frac{1}{5}\right)^{-3}$ **j** $\left(\frac{1}{2}\right)^{-4}$

k $\left(\frac{1}{10}\right)^{-6}$ **l** $\left(\frac{2}{3}\right)^{-2}$ **m** $\left(\frac{3}{2}\right)^{-4}$ **n** $\left(\frac{2}{5}\right)^{-2}$ **o** $\left(\frac{3}{7}\right)^0$

5 Simplify these expressions, leaving your answers in index form.

a $2^9 \times 2^5$ **b** $a^8 \times a^7$ **c** $9^6 \times 9^{-6}$ **d** $x^7 \times x^{-5}$

e $a^5 \times a^{-5}$ **f** $8^4 \times 8 \times 8^{-4}$ **g** $7^8 \div 7^3$ **h** $a^5 \div a^7$

i $2^8 \div 2^{-8}$ **j** $2^8 \div 2^8$ **k** $x^{10} \div x^{-2}$ **l** $y^{12} \div y$

m $y \div y^{12}$ **n** $x^5 \times x^5 \times x^5$ **o** $(x^5)^3$ **p** $(z^2)^7$

q $a^{-2} \times a^{-2} \times a^{-2}$ **r** $(a^{-2})^3$ **s** $(5^4)^{-7}$ **t** $(2^{-4})^{-4}$

6 Expand the brackets in each expression.

a $(3x)^2$ **b** $(5a)^3$ **c** $(2c)^6$ **d** $(3st)^4$ **e** $(7xyz)^2$

f $\left(\frac{1}{x}\right)^5$ **g** $\left(\frac{3}{x}\right)^2$ **h** $\left(\frac{y}{5}\right)^2$ **i** $\left(\frac{7a}{5}\right)^2$ **j** $\left(\frac{3x}{2y}\right)^3$

7 Write each expression as a fraction without negative indices or brackets.

a 9^{-1} **b** x^{-1} **c** b^{-2} **d** $-a^{-4}$ **e** $(7x)^{-1}$

f $7x^{-1}$ **g** $-9x^{-1}$ **h** $(3a)^{-2}$ **i** $3a^{-2}$ **j** $4x^{-3}$

8 First change each mixed numeral or decimal to a fraction, then simplify the expression.

a $\left(1\frac{1}{2}\right)^{-1}$ **b** $\left(2\frac{2}{3}\right)^{-1}$ **c** $\left(2\frac{1}{2}\right)^{-2}$ **d** $\left(3\frac{1}{3}\right)^{-3}$

e 0.2^{-1} **f** 2.25^{-1} **g** 2.5^{-2} **h** 0.05^{-2}

DEVELOPMENT

9 A huge desert rock is entirely made up of coarse grains of sandstone, each about 1 mm^3 in size, and is very roughly 3 km long, 2 km wide, and 500 m high.

a Find the approximate volume of the rock in cubic kilometres.

b How many cubic millimetres are in a cubic kilometre? Answer in index notation.

c Find the approximate number of grains of sandstone in the rock.

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10 Write each fraction in index form.

a $\frac{1}{x}$ **b** $-\frac{1}{x^2}$ **c** $-\frac{12}{x}$ **d** $\frac{9}{x^2}$ **e** $-\frac{1}{x^3}$

f $\frac{12}{x^5}$ **g** $\frac{7}{x^3}$ **h** $-\frac{6}{x}$ **i** $\frac{1}{6x}$ **j** $-\frac{1}{4x^2}$

11 Write down the solutions of these index equations.

a $7^x = \frac{1}{7}$ **b** $2^x = \frac{1}{8}$ **c** $\left(\frac{1}{3}\right)^x = 3$ **d** $\left(\frac{5}{6}\right)^x = \frac{6}{5}$

e $9^x = 1$ **f** $\left(\frac{5}{8}\right)^x = \frac{25}{64}$ **g** $\left(\frac{5}{8}\right)^x = \frac{8}{5}$ **h** $\left(\frac{3}{5}\right)^x = \frac{25}{9}$

i $x^2 = \frac{100}{169}$ **j** $x^{-2} = \frac{1}{4}$ **k** $x^{-3} = 27$ **l** $x^{-2} = \frac{64}{81}$

m $3^{2x-8} = 81$ **n** $4^{7-x} = \frac{1}{4}$ **o** $\left(\frac{1}{3}\right)^{x+1} = 27$ **p** $\left(\frac{1}{5}\right)^{x+1} = \left(\frac{1}{125}\right)^{x-1}$

12 Simplify each expression, giving the answer without negative indices.

a $2^x \times 2^3$ **b** $3^x \times 3$ **c** $7^{4x} \times 7^{-5x}$ **d** $5^{2x} \div 5^3$

e $(10^{2x})^3$ **f** $(5^{4x})^{-2}$ **g** $(6^x)^4 \times (6^{2x})^5$ **h** $(2^x)^3 \div 2^4$

13 Simplify each expression, giving the answer without negative indices.

a $x^2y \times x^4y^3$ **b** $x^{-5}y^3 \times x^3y^{-2}$ **c** $3ax^{-2} \times 7a^2x$

d $\frac{1}{15}s^{-2}t^3 \times 5st^{-5}$ **e** $56x^2y^6 \div 8xy^8$ **f** $35a^{-2}b^4 \div 28a^4b^{-6}$

g $(s^2y^{-3})^3$ **h** $(5c^{-2}d^3)^{-1}$ **i** $(3x^2y^3)^3 \times (xy^4)^2$

j $(2a^{-2}y^3)^{-2} \times (2ay^{-3})^3$ **k** $(5st^2)^3 \div (5s^{-1}t^3)^2$ **l** $(10x^2y^{-2})^3 \div (2x^{-1}y^3)^2$

14 Expand and simplify, answering without using negative indices:

a $(x + x^{-1})^2$ **b** $(x - x^{-1})^2$ **c** $(x^2 - x^{-2})^2$

15 Write as a single fraction, without negative indices, and simplify:

a $a^{-1} - b^{-1}$ **b** $\frac{1 - y^{-1}}{1 - y^{-2}}$ **c** $(x^{-2} - y^{-2})^{-1}$

d $\frac{a^{-1} + b^{-1}}{a^{-2} - b^{-2}}$ **e** $x^{-2}y^{-2}(x^2y^{-1} - y^2x^{-1})$ **f** $\frac{(a^2 - 1)^{-1}}{(a - 1)^{-1}}$

16 Explain why $12^n = (2^2 \times 3)^n = 2^{2n} \times 3^n$. Using similar methods, write these expressions with prime bases and simplify them.

a $2^n \times 4^n \times 8^n$ **b** $\frac{9^{n+2} \times 3^{n+1}}{3 \times 27^n}$ **c** $6^x \times 4^x \div 3^x$

d $\frac{12^x \times 18^x}{3^x \times 2^x}$ **e** $100^{2n-1} \times 25^{-1} \times 8^{-1}$ **f** $\frac{24^{x+1} \times 8^{-1}}{6^{2x}}$

17 Explain why $3^n + 3^{n+1} = 3^n(1 + 3) = 4 \times 3^n$. Then use similar methods to simplify:

a $7^{n+2} + 7^n$ **b** $\frac{3^{n+3} - 3^n}{3^n}$ **c** $5^n - 5^{n-3}$

d $\frac{7^n + 7^{n+2}}{7^{n-1} + 7^{n+1}}$ **e** $2^{2n+2} - 2^{2n-1}$ **f** $\frac{2^{2n} - 2^{n-1}}{2^n - 2^{-1}}$

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18 Use the methods of the last two questions to simplify:

a $\dfrac{6^n + 3^n}{2^{n+1} + 2}$

b $\dfrac{12^x + 1}{6^{2x} + 3^x}$

c $\dfrac{12^n - 18^n}{3^n - 2^n}$

ENRICHMENT

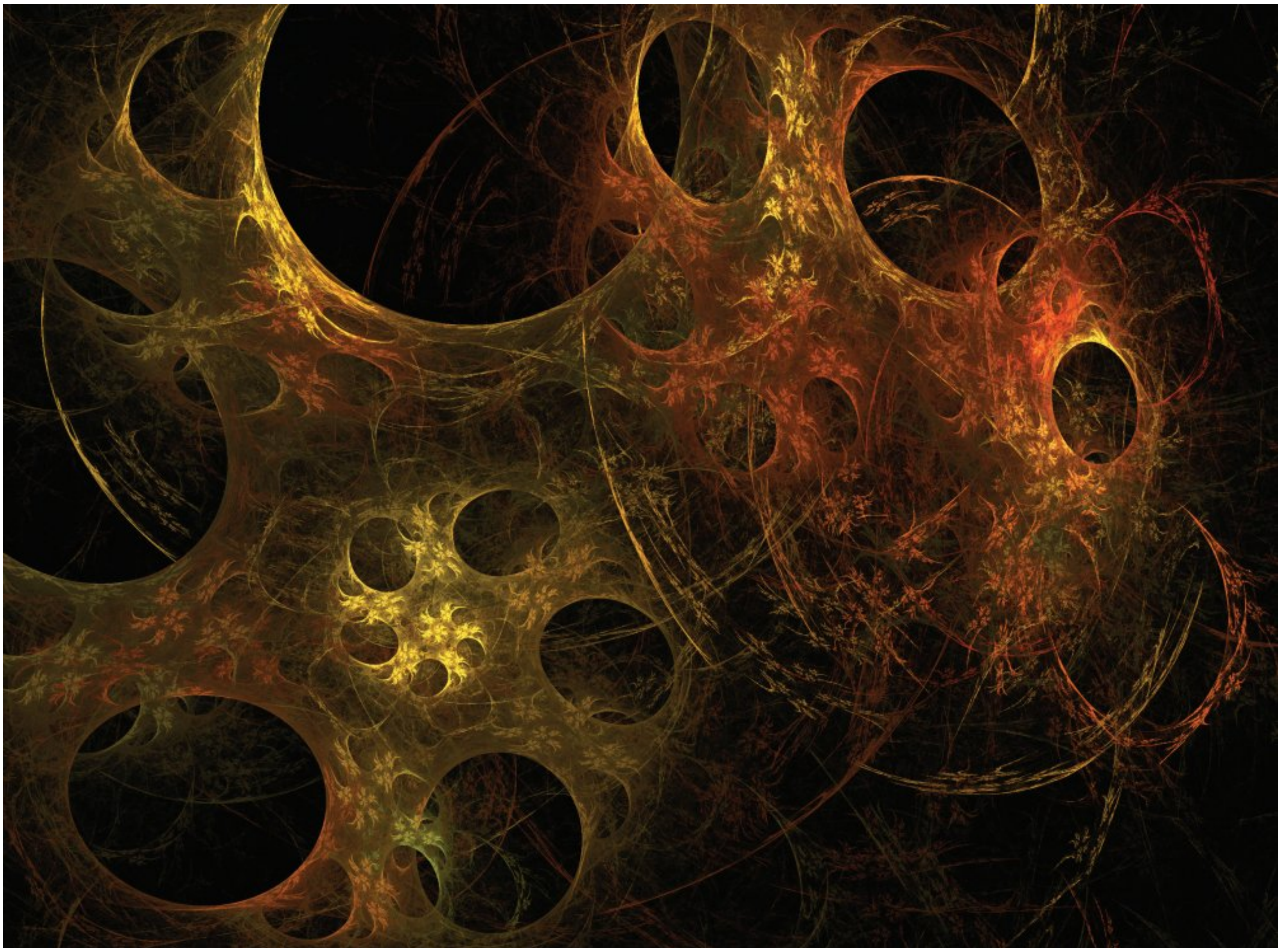
19 A calculator is required in this question — use scientific notation throughout.

a The mass of a neutron is about 1.675×10^{-27} kg. About how many neutrons are there in 1 kg of neutrons?

b The radius of a neutron is about 1.11×10^{-15} m. Use the volume-of-a-sphere formula $V = \frac{4}{3}\pi r^3$ to find its approximate volume.

c Use the formula density $= \dfrac{\text{mass}}{\text{volume}}$ to find its approximate density in kg/m^3.

Note: This question makes several extremely naive assumptions about what a neutron actually is, but the extraordinarily high density that this calculation gives is close to the density of neutron stars, which are the densest objects in the universe.



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8B Fractional indices

In this section, the definition of powers is extended to powers such as $4^{\frac{1}{2}}$ and $27^{-\frac{2}{3}}$, where the index is a positive or negative fraction. Again, the definitions are made so that the index laws work for all indices.

Numerical answers can be checked on the calculator using $\boxed{x^y}$ or $\boxed{\wedge}$.

Fractional indices with numerator 1

Let the base a be any real number $a \geq 0$.

We know that $\left(\sqrt{a}\right)^2 = a$.

If we use the index laws, however, $\left(a^{\frac{1}{2}}\right)^2 = a$.

Hence it is convenient to define $a^{\frac{1}{2}} = \sqrt{a}$.

(Remember that $\sqrt{a}$ means the non-negative square root of a.)

6 FRACTIONAL INDICES WITH NUMERATOR 1

Let the base a be any real number $a \geq 0$.

- Define $a^{\frac{1}{2}} = \sqrt{a}$, define $a^{\frac{1}{3}} = \sqrt[3]{a}$, define $a^{\frac{1}{4}} = \sqrt[4]{a}$, ...
- In general, define $a^{\frac{1}{n}} = \sqrt[n]{a}$, for all positive integers n, where in every case, the *non-negative root* of a is to be taken.

Example 7 — 8B

Write each expression using surd notation, then simplify it.

a $64^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $10000^{\frac{1}{4}}$ **d** $32^{\frac{1}{5}}$

SOLUTION

a $64^{\frac{1}{2}} = \sqrt{64} = 8$

b $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

c $10000^{\frac{1}{4}} = \sqrt[4]{10000} = 10$

d $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$

General fractional indices

Now let the the base a be any positive real number. If the index laws are to apply to a power such as $a^{\frac{2}{3}}$, then we must be able to write

$$\left(a^{\frac{1}{3}}\right)^2 = a^{\frac{2}{3}} \quad \text{and} \quad (a^2)^{\frac{1}{3}} = a^{\frac{2}{3}},$$

because to raise a power to a power, we multiply the indices.

Hence we shall define $a^{\frac{2}{3}}$ as $\left(a^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{a}\right)^2$, which is the same as $\left(a^2\right)^{\frac{1}{3}} = \sqrt[3]{a^2}$.

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7 GENERAL FRACTIONAL INDICES

Let the base a be any positive real number, and m and n be positive integers.

- Define $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$, which is the same as $\sqrt[n]{a^m}$.
- Define $a^{-\frac{m}{n}} = \dfrac{1}{a^{\frac{m}{n}}}$.

When there are negatives and fractions in the index, first deal with the negative sign, then deal with the fractional index:

8 DEAL WITH COMPLICATED INDICES IN THIS ORDER

1 If the index is negative, take the reciprocal.
2 If the index has a denominator, take the corresponding root.
3 Finally, take the power indicated by the numerator of the index.

For example, $\left(\frac{4}{9}\right)^{-\frac{3}{2}} = \left(\frac{9}{4}\right)^{\frac{3}{2}} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$.

Example 8 — 8B

Simplify each power. First take the root indicated by the denominator.

a $125^{\frac{2}{3}}$ **b** $100^{\frac{3}{2}}$ **c** $\left(\frac{16}{81}\right)^{\frac{3}{4}}$

SOLUTION

a $125^{\frac{2}{3}} = 5^2$
$= 25$

b $100^{\frac{3}{2}} = 10^3$
$= 1000$

c $\left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{2}{3}\right)^3$
$= \frac{8}{27}$

Example 9 — 8B

Simplify each power. First take the reciprocal as indicated by the negative index.

a $25^{-\frac{3}{2}}$ **b** $1000^{-\frac{2}{3}}$ **c** $\left(\frac{8}{27}\right)^{-\frac{4}{3}}$

SOLUTION

a $25^{-\frac{3}{2}} = \left(\frac{1}{25}\right)^{\frac{3}{2}}$
$= \left(\frac{1}{5}\right)^3$
$= \frac{1}{125}$

b $1000^{-\frac{2}{3}} = \left(\frac{1}{1000}\right)^{\frac{2}{3}}$
$= \left(\frac{1}{10}\right)^2$
$= \frac{1}{100}$

c $\left(\frac{8}{27}\right)^{-\frac{4}{3}} = \left(\frac{27}{8}\right)^{\frac{4}{3}}$
$= \left(\frac{3}{2}\right)^4$
$= \frac{81}{16}$

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Example 10 — 8B

Write each expression using a fractional index.

a $\sqrt{x^3}$ **b** $\dfrac{1}{\sqrt{x^3}}$ **c** $\sqrt[3]{x^2}$ **d** $\dfrac{1}{\sqrt[3]{x^2}}$

SOLUTION

a $\sqrt{x^3} = x^{\frac{3}{2}}$ **b** $\dfrac{1}{\sqrt{x^3}} = x^{-\frac{3}{2}}$ **c** $\sqrt[3]{x^2} = x^{\frac{2}{3}}$ **d** $\dfrac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}}$

Negative and zero bases, and irrational indices

Negative numbers do not have square roots, so negative bases are impossible if a square root or fourth root or any even root is involved. For example, $(-64)^{\frac{1}{2}}$ is undefined, but $(-64)^{\frac{1}{3}} = -4$.

Powers of zero are only defined for positive indices x, in which case $0^x = 0$ — for example, $0^3 = 0$. If the index x is negative, then 0^x is undefined, being the reciprocal of zero — for example, 0^{-3} is undefined. If the index is zero, then 0^0 is also undefined — see Question 24.

Irrational indices need only be defined informally. To define 2^π, for example, take the sequence 3, 3.1, 3.14, 3.141, 3.1415, . . . of truncated decimals for π, then form the sequence of rational powers 2^3, $2^{3.1}$, $2^{3.14}$, $2^{3.141}$, $2^{3.1415}$, . . . Everything then works as expected.

Exercise 8B

FOUNDATION

Note: Do not use a calculator in this exercise at all, unless the question asks for it or you are checking answers. Make sure that you can refer easily to the list of powers of 2, 3, 5, . . . from the previous exercise.

1 Simplify these powers. First take the root indicated by the denominator.

a $36^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $1\,000\,000^{\frac{1}{6}}$ **d** $25^{\frac{3}{2}}$ **e** $27^{\frac{2}{3}}$

f $16^{\frac{3}{4}}$ **g** $27^{\frac{4}{3}}$ **h** $64^{\frac{2}{3}}$ **i** $4^{2.5}$ **j** $32^{0.6}$

2 Simplify these powers. First convert mixed numerals and decimals to fractions.

a $\left(\frac{1}{8}\right)^{\frac{1}{3}}$ **b** $\left(\frac{25}{49}\right)^{\frac{1}{2}}$ **c** $\left(\frac{27}{8}\right)^{\frac{1}{3}}$ **d** $\left(\frac{27}{8}\right)^{\frac{4}{3}}$

e $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ **f** $0.25^{\frac{7}{2}}$ **g** $0.09^{\frac{3}{2}}$ **h** $0.04^{\frac{3}{2}}$

3 Simplify these powers. First take the reciprocal as indicated by the negative index.

a $25^{-\frac{1}{2}}$ **b** $100^{-\frac{1}{2}}$ **c** $125^{-\frac{1}{3}}$ **d** $16^{-\frac{3}{4}}$

e $27^{-\frac{2}{3}}$ **f** $81^{-\frac{3}{4}}$ **g** $\left(\frac{1}{16}\right)^{-\frac{1}{4}}$ **h** $\left(\frac{1}{125}\right)^{-\frac{1}{3}}$

i $\left(\frac{1}{16}\right)^{-\frac{3}{4}}$ **j** $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$ **k** $\left(2\frac{1}{4}\right)^{-\frac{3}{2}}$ **l** $\left(\frac{125}{8}\right)^{-\frac{2}{3}}$

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4 Use the index laws to simplify these expressions, leaving answers in index form.

a $x^{2\frac{1}{2}} \times x^{3\frac{1}{2}}$ **b** $x^4 \times x^{-\frac{1}{2}}$ **c** $3x^{-2}y \times 5x^4y^{-2}$

d $x^{4\frac{1}{2}} \div x^{3\frac{1}{2}}$ **e** $x^{-7} \div x^{-2\frac{1}{2}}$ **f** $14a^4b^{-5} \div 2a^5b^{-4}$

g $\left(x^{-\frac{2}{3}}\right)^6$ **h** $(x^9)^{\frac{2}{3}}$ **i** $(9s^{-4}t^5)^{1.5}$

5 Use the index laws to simplify these expressions, giving answers as integers or fractions.

a $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ **b** $2^{\frac{1}{2}} \times 2^{-\frac{1}{2}}$ **c** $2^{\frac{1}{2}} \times 2^{2\frac{1}{2}}$

d $3^{4\frac{1}{2}} \div 3^{5\frac{1}{2}}$ **e** $25 \div 25^{\frac{1}{2}}$ **f** $7^{\frac{2}{3}} \div 7^{\frac{2}{3}}$

g $3^3 \times 3^4 \div 3^{10}$ **h** $\left(3^{-\frac{1}{3}}\right)^6$ **i** $(9^8)^{\frac{1}{4}}$

6 Write down the solutions of these index equations.

a $9^x = 3$ **b** $121^x = 11$ **c** $81^x = 3$

d $64^x = 2$ **e** $\left(\frac{1}{25}\right)^x = \frac{1}{5}$ **f** $\left(\frac{1}{8}\right)^x = \frac{1}{2}$

7 Rewrite each expression using surds instead of fractional indices.

a $x^{\frac{1}{2}}$ **b** $x^{\frac{1}{3}}$ **c** $7x^{\frac{1}{2}}$ **d** $(7x)^{\frac{1}{2}}$

e $15x^{\frac{1}{4}}$ **f** $x^{\frac{3}{2}}$ **g** $6x^{\frac{5}{2}}$ **h** $x^{\frac{4}{3}}$

8 Rewrite each expression using fractional indices instead of surds.

a $\sqrt{x}$ **b** $3\sqrt{x}$ **c** $\sqrt{3x}$ **d** $12\sqrt[3]{x}$

e $9\sqrt[6]{x}$ **f** $\sqrt{x^3}$ **g** $\sqrt{x^9}$ **h** $25\sqrt[5]{x^6}$

9 Rewrite each expression as a single power of x using fractional indices.

a $x^2\sqrt{x}$ **b** $\dfrac{\sqrt{x}}{x^3}$ **c** $x^3\sqrt[3]{x^2}$ **d** $\dfrac{x}{\sqrt[3]{x^2}}$

DEVELOPMENT

10 Use the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$ on your calculator to approximate these powers. Express your answers in scientific notation correct to four significant figures.

a 7^8 **b** $565^{\frac{2}{5}}$ **c** $5^{-0.12}$ **d** $(0.001)^{0.7}$

e 2^{π} **f** $10^{\sqrt{2}}$ **g** $\left(\sqrt{7}\right)^{-\sqrt{7}}$ **h** $0.001^{-\frac{\pi}{2}}$

11 The cost of installing a standard swimming pool is rising exponentially according to the formula $C = \$6000 \times (1.03)^n$, where n is the number of years after 1 January 2015.

a What was the original cost on 1 January 2015?

b What was the cost on 1 January 2016?

c Use your calculator to find, correct to the nearest \$10, the cost of installing a pool on:

i 1 January 2020 **ii** 1 July 2015 **iii** 1 July 2018.

12 Given that $x = 16$ and $y = 25$, evaluate:

a $x^{\frac{1}{2}} + y^{\frac{1}{2}}$ **b** $x^{\frac{3}{4}} - y^{\frac{1}{2}}$ **c** $x^{-\frac{1}{2}} - y^{-\frac{1}{2}}$ **d** $(y - x)^{\frac{1}{2}} \times (4y)^{-\frac{1}{2}}$

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13 Simplify each expression, giving the answer in index form.

a $3x^{\frac{1}{2}}y \times 3x^{\frac{1}{2}}y^2$ **b** $5a^{\frac{1}{3}}b^{\frac{2}{3}} \times 7a^{-\frac{1}{3}}b^{\frac{1}{3}}$ **c** $\frac{1}{8}s^{2\frac{1}{2}} \times 24s^{-2}$

d $x^2y^3 \div x^{\frac{1}{2}}y^{\frac{1}{2}}$ **e** $a^{\frac{1}{2}}b^{\frac{1}{2}} \div a^{-\frac{1}{2}}b^{\frac{1}{2}}$ **f** $(a^{-2}b^4)^{\frac{1}{2}}$

g $(8x^3y^{-6})^{\frac{1}{3}}$ **h** $\left(p^{\frac{1}{5}}q^{-\frac{3}{5}}\right)^{10}$ **i** $\left(x^{\frac{3}{4}}\right)^4 \times \left(x^{\frac{4}{3}}\right)^3$

14 Rewrite these expressions, using fractional and negative indices.

a $\dfrac{1}{\sqrt{x}}$ **b** $\dfrac{12}{\sqrt{x}}$ **c** $-\dfrac{5}{\sqrt{x}}$ **d** $\dfrac{15}{\sqrt[3]{x}}$

e $-\dfrac{4}{\sqrt[3]{x^2}}$ **f** $x\sqrt{x}$ **g** $\dfrac{5}{x\sqrt{x}}$ **h** $8x^2\sqrt{x}$

15 Write down the solutions of these index equations.

a $49^x = \frac{1}{7}$ **b** $81^x = \frac{1}{3}$ **c** $8^x = 4$ **d** $8^x = \frac{1}{4}$ **e** $4^x = 8$

f $4^x = \frac{1}{8}$ **g** $81^x = 27$ **h** $27^x = \frac{1}{81}$ **i** $\left(\frac{1}{25}\right)^x = 5$ **j** $\left(\frac{8}{125}\right)^x = \frac{25}{4}$

16 Expand and simplify, giving answers in index form:

a $\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2$ **b** $\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)^2$ **c** $\left(x^{\frac{5}{2}} - x^{-\frac{5}{2}}\right)^2$

17 Expand and simplify, answering without using negative indices:

a $(x + 5x^{-1})^2$ **b** $(x^2 - 7x^{-2})^2$ **c** $\left(3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}\right)^2$

18 Explain why if $3^{3x-1} = 9$, then $3x - 1 = 2$, so $x = 1$. Similarly, by reducing both sides to powers of the same base, solve:

a $125^x = \frac{1}{5}$ **b** $25^x = \sqrt{5}$ **c** $8^x = \frac{1}{4}$

d $64^x = \sqrt{32}$ **e** $8^{x+1} = 2 \times 4^{x-1}$ **f** $\left(\frac{1}{9}\right)^x = 3^4$

19 By taking appropriate powers of both sides, solve:

a $b^{\frac{1}{3}} = \frac{1}{7}$ **b** $n^{-2} = 121$ **c** $x^{-\frac{3}{4}} = 27$

20 Solve simultaneously:

a $7^{2x-y} = 49$
$2^{x+y} = 128$

b $8^x \div 4^y = 4$
$11^{y-x} = \frac{1}{11}$

c $13^{x+4y} = 1$
$25^{x+5y} = 5$

21 a If $x = 2^{\frac{1}{3}} + 4^{\frac{1}{3}}$, show that $x^3 = 6(1 + x)$.

b If $x = \dfrac{1}{2} + \dfrac{1}{2}\sqrt{5}$, show that $\dfrac{x^2 + x^{-2}}{x - x^{-1}} = 3$.

c Show that $\dfrac{pq^{-1} - p^{-1}q}{p^2q^{-2} - p^{-2}q^2} = \dfrac{pq}{p^2 + q^2}$.

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ENRICHMENT

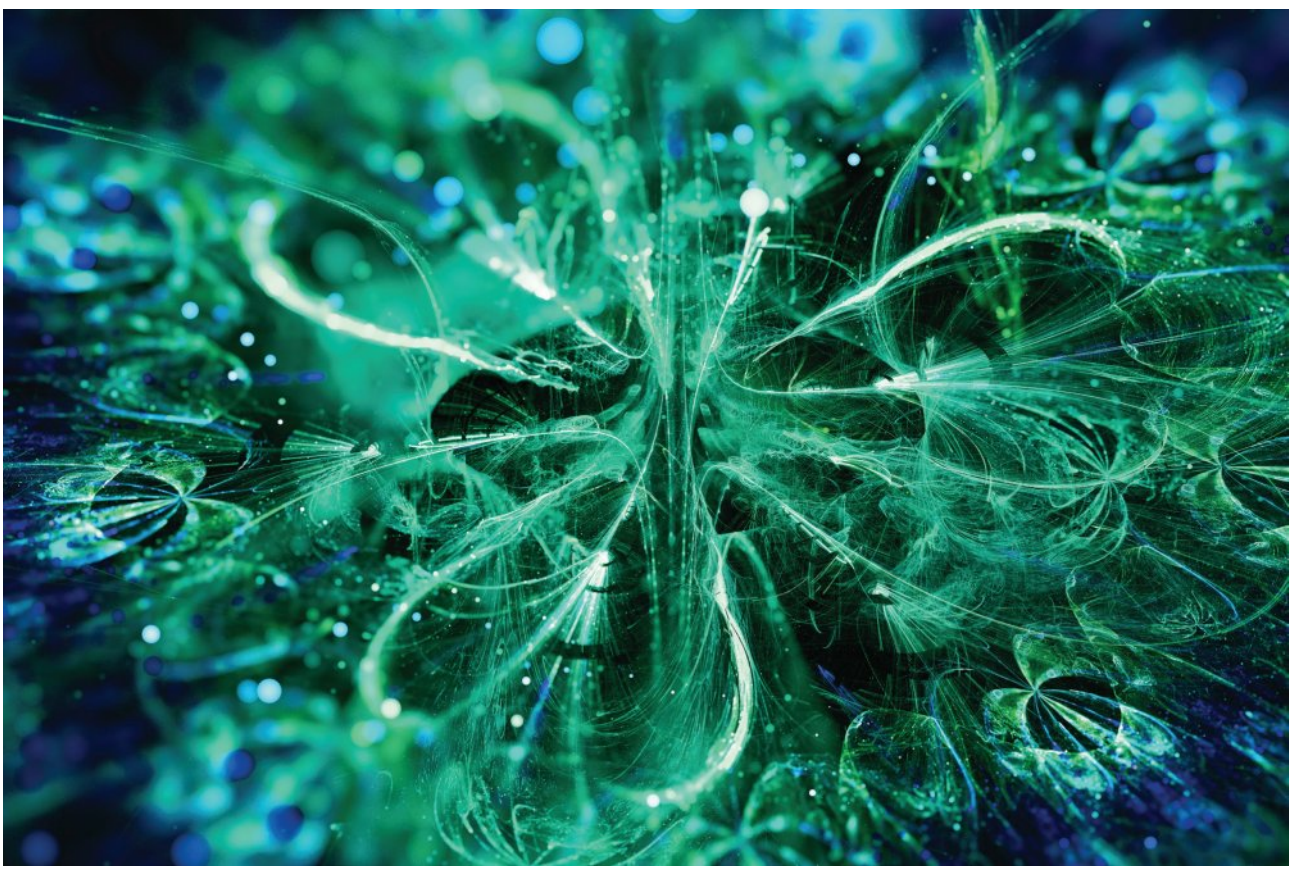
22 By taking 6th powers of both sides, show that $11^{\frac{1}{3}} < 5^{\frac{1}{2}}$. Using similar methods (followed perhaps by a check on the calculator), compare:

a $3^{\frac{1}{3}}$ and $2^{\frac{1}{2}}$ **b** $2^{\frac{1}{2}}$ and $5^{\frac{1}{5}}$ **c** $7^{\frac{3}{2}}$ and 20 **d** $5^{\frac{1}{5}}$ and $3^{\frac{1}{3}}$

23 Find (and check on the calculator) the smallest whole numbers m and n for which:

a $12 < 2^{\frac{m}{n}} < 13$ **b** $13 < 2^{\frac{m}{n}} < 14$

24 Find $\lim_{x \to 0^+} 0^x$ and $\lim_{x \to 0} x^0$. Explain what these two limits have to do with the remark made on page 345 that 0^0 is undefined.



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8C Logarithms

The introduction to this chapter mentioned that the mould growing on a piece of cheese was doubling every day. Thus in 2 days the amount of mould will increase by a factor of $2^2 = 4$, in 3 days by $2^3 = 8$, in 4 days by $2^4 = 16$, and so on.

Now suppose that someone asks the question backwards, asking how many days it takes for the mould to increase by a factor of 8. The answer is 3 days, because $8 = 2^3$. This index 3 is the *logarithm base 2 of 8*, and is written as $\log_2 8 = 3$:

$$\log_2 8 = 3 \quad \text{means that} \quad 8 = 2^3.$$

Read this as, 'The log base 2 of 8', mentioning the base first, and then the number.

Logarithms

The logarithm is the index. More precisely:

9 THE LOGARITHM IS THE INDEX

The *logarithm base a* of a positive number x is the index, when the number x is expressed as a power of the base a:

$$y = \log_a x \qquad \text{means that} \qquad x = a^y.$$

The base a must always be positive, and not equal to 1.

The basic skill with logarithms is converting between statements about indices and statements about logarithms. The next box should be learnt off by heart.

10 A SENTENCE TO COMMIT TO MEMORY

This one sentence should fix most problems:

'$\log_2 8 = 3 \qquad \text{means that} \qquad 8 = 2^3$.'

Examine the sentence and notice that:

- The base of the log is the base of the power (in this case 2).
- The log is the index (in this case 3).

Example 11 8C

Rewrite each index statement in logarithmic form.

a $10^3 = 1000$ **b** $3^4 = 81$

SOLUTION

a $10^3 = 1000$

$\log_{10} 1000 = 3$

b $3^4 = 81$

$\log_3 81 = 4$

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Example 12 — 8C

Rewrite each logarithmic statement in index form. Then state whether it is true or false.

a $\log_2 16 = 4$ **b** $\log_3 27 = 4$

SOLUTION

a $\log_2 16 = 4$
$16 = 2^4$, which is true.

b $\log_3 27 = 4$
$27 = 3^4$, which is false.

Finding logarithms

Change questions about logarithms to questions about indices.

Example 13 — 8C

a Find $\log_2 32$. **b** Find $\log_{10} 1\,000\,000$.

SOLUTION

a Let $x = \log_2 32$.
Then $2^x = 32$.
Hence $x = 5$.

b Let $x = \log_{10} 1\,000\,000$.
Then $10^x = 1\,000\,000$.
Hence $x = 6$.

Negative and fractional indices

A logarithmic equation can involve a negative or a fractional index.

Example 14 — 8C

Solve each logarithmic equation by changing it to an index equation.

a $x = \log_7 \frac{1}{7}$ **b** $\frac{1}{3} = \log_8 x$ **c** $\log_x 9 = -1$

SOLUTION

a $x = \log_7 \frac{1}{7}$
$\frac{1}{7} = 7^x$
$x = -1$

b $\frac{1}{3} = \log_8 x$
$x = 8^{\frac{1}{3}}$
$= 2$

c $\log_x 9 = -1$
$x^{-1} = 9$
$x = \frac{1}{9}$

Logarithms on the calculator

Most scientific calculators only allow direct calculations of logarithms base 10, using the button labelled [log]. (They also allow calculation of logarithms using the important mathematical constant e as the base — this uses the button marked [ln], as explained later in Chapter 11.)

The function [10^x] is usually on the same key as [log], and is reached by pressing [shift] followed by [log]. These two functions [log] and [10^x] are inverses of each other — when used one after the other, the original number returns.

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Example 15 8C

Write each statement below in logarithmic form, then use the function labelled log on your calculator to approximate x, correct to four significant figures. Then check your calculation using the button labelled 10^x.

a $10^x = 750$

b $10^x = 0.003$

SOLUTION

a $10^x = 750$
$x = \log_{10} 750$
$\doteqdot 2.875$, using log.

Using 10^x, $10^{2.875} \doteqdot 750$.

b $10^x = 0.003$
$x = \log_{10} 0.003$
$\doteqdot -2.523$, using log.

Using 10^x, $10^{-2.523} \doteqdot 0.003$.

Locating a logarithm between two integers

Consider again the cheese with a mould that doubles in amount every day. Suppose that someone asks how many days it takes for the amount of mould to increase by a factor of 10.

The answer is $\log_2 10$ days, but this is an awkward question, because the answer is not an integer. The change-of-base formula in the next section will allow $\log_2 10$ to be approximated on a calculator, but in the meantime, we can find which two integers the answer lies between.

— In three days the mould increases 8-fold.
— In four days it increases 16-fold.

Thus we can at least say that the answer lies between 3 days and 4 days.

Example 16 8C

Use a list of powers of 5 to explain why $\log_5 100$ is between 2 and 3.

SOLUTION

We know that $5^2 = 25$ and $5^3 = 125$, so $5^2 < 100 < 5^3$.
Hence taking logarithms base 5, $2 < \log_5 100 < 3$.

Exercise 8C FOUNDATION

Note: Do not use a calculator in this exercise unless the question asks for it. Make sure that you can refer easily to the list of powers of 2, 3, 5, . . . from Exercise 8A.

1 Copy and complete each sentence.

a '$\log_2 8 = 3$ because . . .'
b '$\log_5 25 = 2$ because . . .'
c '$\log_{10} 1000 = 3$ because . . .'
d '$7^2 = 49$, so $\log_7 49 = \ldots$'
e '$3^4 = 81$, so . . .'
f '$10^5 = 100000$, so . . .'

2 Copy and complete the following statements of the meaning of logarithms.

a '$y = \log_a x$ means that . . .'
b '$y = a^x$ means that . . .'

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3 Rewrite each equation in index form, then solve it for x.

a $x = \log_{10} 1000$ **b** $x = \log_{10} 10$ **c** $x = \log_{10} 1$ **d** $x = \log_{10} \frac{1}{100}$

e $x = \log_3 9$ **f** $x = \log_5 125$ **g** $x = \log_2 64$ **h** $x = \log_4 64$

i $x = \log_8 64$ **j** $x = \log_7 \frac{1}{7}$ **k** $x = \log_{12} \frac{1}{12}$ **l** $x = \log_{11} \frac{1}{121}$

m $x = \log_6 \frac{1}{36}$ **n** $x = \log_4 \frac{1}{64}$ **o** $x = \log_8 \frac{1}{64}$ **p** $x = \log_2 \frac{1}{64}$

4 Rewrite each equation in index form, then solve it for x.

a $\log_7 x = 2$ **b** $\log_5 x = 3$ **c** $\log_2 x = 5$ **d** $\log_{100} x = 3$

e $\log_7 x = 1$ **f** $\log_{11} x = 0$ **g** $\log_{13} x = -1$ **h** $\log_{12} x = -2$

i $\log_5 x = -3$ **j** $\log_7 x = -3$ **k** $\log_2 x = -5$ **l** $\log_3 x = -4$

5 Rewrite each equation in index form, then solve it for x. Remember that a base must be positive and not equal to 1.

a $\log_x 49 = 2$ **b** $\log_x 8 = 3$ **c** $\log_x 27 = 3$ **d** $\log_x 10\,000 = 4$

e $\log_x 10\,000 = 2$ **f** $\log_x 64 = 6$ **g** $\log_x 64 = 2$ **h** $\log_x 125 = 1$

i $\log_x 11 = 1$ **j** $\log_x \frac{1}{17} = -1$ **k** $\log_x \frac{1}{6} = -1$ **l** $\log_x \frac{1}{7} = -1$

m $\log_x \frac{1}{9} = -2$ **n** $\log_x \frac{1}{49} = -2$ **o** $\log_x \frac{1}{8} = -3$ **p** $\log_x \frac{1}{81} = -2$

6 Use the calculator buttons [log] and [10^x] to approximate each expression. Give answers correct to three significant figures, then check each answer using the other button.

a $\log_{10} 2$ **b** $\log_{10} 20$ **c** $10^{0.301}$ **d** $10^{1.301}$

e $10^{0.5}$ **f** $10^{1.5}$ **g** $\log_{10} 3.16$ **h** $\log_{10} 31.6$

i $\log_{10} 0.7$ **j** $\log_{10} 0.007$ **k** $10^{-0.155}$ **l** $10^{-2.15}$

DEVELOPMENT

7 Rewrite each equation in index form and then solve it for x (where a is a constant).

a $x = \log_a a$ **b** $\log_a x = 1$ **c** $\log_x a = 1$

d $x = \log_a \frac{1}{a}$ **e** $\log_a x = -1$ **f** $\log_x \frac{1}{a} = -1$

g $x = \log_a 1$ **h** $\log_a x = 0$ **i** $\log_x 1 = 0$

8 Given that a is a positive real number not equal to 1, evaluate:

a $\log_a a$ **b** $\log_a \frac{1}{a}$ **c** $\log_a a^3$ **d** $\log_a \frac{1}{a^2}$

e $\log_a \frac{1}{a^5}$ **f** $\log_a \sqrt{a}$ **g** $\log_a \frac{1}{\sqrt{a}}$ **h** $\log_a 1$

9 Use the list of powers of 2 prepared in Exercise 8A to find which two integers each expression lies between.

a $\log_2 3$ **b** $\log_2 1.8$ **c** $\log_2 13$ **d** $\log_2 50$

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10 Use the tables of powers of 2, 3, 5, . . . prepared in Exercise 8A to find which two integers each expression lies between.

a $\log_2 7$ **b** $\log_{10} 35$ **c** $\log_{10} 6$ **d** $\log_2 1000$ **e** $\log_{10} 2000$

f $\log_3 2$ **g** $\log_3 50$ **h** $\log_3 100$ **i** $\log_5 100$ **j** $\log_7 20$

k $\log_2 \frac{4}{5}$ **l** $\log_2 \frac{1}{3}$ **m** $\log_5 0.1$ **n** $\log_{10} 0.4$ **o** $\log_{10} 0.05$

11 Solve each equation for x. These questions involve fractional indices.

a $x = \log_7 \sqrt{7}$ **b** $x = \log_{11} \sqrt{11}$ **c** $\log_9 x = \frac{1}{2}$ **d** $\log_{144} x = \frac{1}{2}$

e $\log_x 3 = \frac{1}{2}$ **f** $\log_x 13 = \frac{1}{2}$ **g** $x = \log_6 \sqrt[3]{6}$ **h** $x = \log_9 3$

i $\log_{64} x = \frac{1}{3}$ **j** $\log_{16} x = \frac{1}{4}$ **k** $\log_x 2 = \frac{1}{3}$ **l** $\log_x 2 = \frac{1}{6}$

m $x = \log_8 2$ **n** $x = \log_{125} 5$ **o** $\log_7 x = \frac{1}{2}$ **p** $\log_7 x = -\frac{1}{2}$

q $\log_x \frac{1}{7} = -\frac{1}{2}$ **r** $\log_x \frac{1}{20} = -\frac{1}{2}$ **s** $x = \log_4 \frac{1}{2}$ **t** $x = \log_{27} \frac{1}{3}$

u $\log_{121} x = -\frac{1}{2}$ **v** $\log_{81} x = -\frac{1}{4}$ **w** $\log_x \frac{1}{2} = -\frac{1}{4}$ **x** $\log_x 2 = -\frac{1}{4}$

12 Problems arise when a logarithm is written down with too few significant figures.

a Find $\log_{10} 45$ correct to two significant figures.

b Then find 10 to that index, again correct to two significant figures.

c To how many significant figures should $\log_{10} 45$ be recorded for the inverse procedure to yield 45.00, that is, to be correct to four significant figures?

ENRICHMENT

13 A *googol* is 10^{100} — the company name ‘Google’ was an accidental misspelling of ‘googol’. A *googolplex* is 10^{googol}. A googol is about 100 billion times greater than the number of elementary particles in the observable universe, and a googolplex is very big.

a Find:

i $\log_{10}$ googol **ii** $\log_{100}$ googol **iii** $\log_{10}$ googolplex

iv $\log_{100}$ googolplex **v** $\log_{\text{googol}}$ googol **vi** $\log_{\text{googol}}$ googolplex

b Which whole numbers does $\log_2$ googol lie between? Do not use [log] or [ln].

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8D The laws for logarithms

Logarithms are indices, so the laws for manipulating indices can be rewritten as laws for manipulating logarithms. As always, the base a of a logarithm must be a positive number not equal to 1.

Logarithmic and exponential functions are inverse functions

Recall from Chapter 5 that two functions $f(x)$ and $g(x)$ are called *inverse functions* if

$$f\big(g(x)\big) = x \text{ and } g\big(f(x)\big) = x, \text{ for all } x \text{ where the calculations are possible.}$$

We have already seen that the functions [log] and [10^x] are inverse functions. That is, when the two functions are applied to a number one after the other, in any order, the original number returns. Check, for example, using the buttons [log] and [10^x] on your calculator, that

$$\log_{10} 10^3 = 3 \quad \text{and} \quad 10^{\log_{10} 3} = 3.$$

These relationships are true whatever the base of the logarithm, simply because of the definition of logarithms. For example:

$\log_2 2^6 = 6$, because 6 is the index, when 2^6 is written as a power of 2.

$2^{\log_2 64} = 64$, because $\log_2 64$ is the index, when 64 is written as a power of 2.

11 THE FUNCTIONS $y = a^x$ AND $y = \log_a x$ ARE INVERSE FUNCTIONS

Let the base a be any positive real number not equal to 1. Then

- $\log_a a^x = x$, for all real x and
- $a^{\log_a x} = x$, for all real $x > 0$.

Example 17 8D

a Simplify $\log_7 7^{12}$.

b Simplify $5^{\log_5 11}$.

SOLUTION

a Using the first identity above,

$\log_7 7^{12} = 12$

b Using the second identity above,

$5^{\log_5 11} = 11$

Example 18 8D

a Write 3 as a power of 10.

b Write 7 as a logarithm base 2.

SOLUTION

a Using the second identity above,

$3 = 10^{\log_{10} 3}$

b Using the first identity above,

$7 = \log_2 2^7$

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Three laws for logarithms

Three laws are particularly useful when working with logarithms:

12 THREE LAWS FOR LOGARITHMS

- The log of a product is the sum of the logs,
  $$\log_a xy = \log_a x + \log_a y$$
- The log of a quotient is the difference of the logs,
  $$\log_a \frac{x}{y} = \log_a x - \log_a y$$
- The log of a power is the multiple of the log,
  $$\log_a x^n = n \log_a x$$

The simple reason for these three laws is that logarithms are indices, and these laws therefore mirror the index laws.

- When multiplying powers, add the indices.
- When dividing powers, subtract the indices.
- And when raising a power to a power, multiply the indices.

Here is a more formal proof.

Proof: To prove each law, show that $a^{\text{LHS}} = a^{\text{RHS}}$, by using the second identity in the previous paragraph and the index laws.

$$\begin{aligned} a^{\log_a x + \log_a y} &= a^{\log_a x} \times a^{\log_a y} \quad \text{(to multiply powers, add the indices)} \\ &= xy \\ &= a^{\log_a (xy)} \end{aligned}$$

$$\begin{aligned} a^{\log_a x - \log_a y} &= a^{\log_a x} \div a^{\log_a y} \quad \text{(to divide powers, subtract the indices)} \\ &= x \div y \\ &= \frac{x}{y} \\ &= a^{\log_a \left(\frac{x}{y}\right)} \end{aligned}$$

$$\begin{aligned} a^{n(\log_a x)} &= \left(a^{\log_a x}\right)^n \quad \text{(to raise a power to a power, multiply the indices)} \\ &= x^n \\ &= a^{\log_a (x^n)} \end{aligned}$$

Example 19 — 8D

Suppose that it has been found that $\log_3 2 \doteqdot 0.63$ and $\log_3 5 \doteqdot 1.46$. Use the log laws to find approximations for:

a $\log_3 10$

b $\log_3 \frac{2}{5}$

c $\log_3 32$

d $\log_3 18$

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SOLUTION

a Because $10 = 2 \times 5$,

$$\log_3 10 = \log_3 2 + \log_3 5 \\ \doteqdot 0.63 + 1.46 \\ \doteqdot 2.09\,.$$

b $$\log_3 \tfrac{2}{5} = \log_3 2 - \log_3 5 \\ \doteqdot 0.63 - 1.46 \\ \doteqdot -0.83.$$

c Because $32 = 2^5$,

$$\log_3 32 = 5\log_3 2 \\ \doteqdot 3.15\,.$$

d Because $18 = 2 \times 3^2$,

$$\log_3 18 = \log_3 2 + 2\log_3 3\,. \\ = \log_3 2 + 2, \text{ because } \log_3 3 = 1, \\ \doteqdot 2.63.$$

Some particular values of logarithmic functions

Some particular values of logarithmic functions occur very often and are worth committing to memory.

13 SOME PARTICULAR VALUES AND IDENTITIES OF LOGARITHMIC FUNCTIONS

$$\log_a 1 = 0, \quad \text{because } 1 = a^0.$$

$$\log_a a = 1, \quad \text{because } a = a^1.$$

$$\log_a \sqrt{a} = \tfrac{1}{2}, \quad \text{because } \sqrt{a} = a^{\frac{1}{2}}.$$

$$\log_a \frac{1}{a} = -1, \quad \text{because } \frac{1}{a} = a^{-1}.$$

$$\log_a \frac{1}{x} = -\log_a x, \quad \text{because } \log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x.$$

Example 20 8D

Use the log laws to expand:

a $\log_3 7x^3$

b $\log_5 \frac{5}{x}$

SOLUTION

a $\log_3 7x^3 = \log_3 7 + \log_3 x^3$ (the log of a product is the sum of the logs)

$\quad = \log_3 7 + 3\log_3 x$ (the log of a power is the multiple of the log)

b $\log_5 \frac{5}{x} = \log_5 5 - \log_5 x^3$ (the log of a quotient is the difference of the logs)

$\quad = 1 - \log_5 x$ ($\log_5 5 = 1$, because $5 = 5^1$)

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Exercise 8D

FOUNDATION

Note: Do not use a calculator in this exercise. Make sure that you can refer easily to the list of powers of 2, 3, 5, . . . from Exercise 8A.

1 Use the three log laws in Box 12 to evaluate (each answer is a whole number):

a $\log_6 9 + \log_6 4$ b $\log_5 75 - \log_5 3$ c $\log_{15} 3 + \log_{15} 5$
d $\log_{12} 72 + \log_{12} 2$ e $\log_{10} 50 + \log_{10} 20$ f $\log_3 15 - \log_3 5$
g $\log_2 24 - \log_3 3$ h $\log_3 810 - \log_3 10$ i $\log_2 12 + \log_2 6 - \log_2 9$
j $\log_{30} 2 + \log_{30} 3 + \log_{30} 5$ k $\log_{12} 9 + \log_{12} 8 + \log_{12} 2$ l $\log_2 12 + \log_2 6 - \log_2 72$

2 Use the log laws to simplify:

a $\log_5 2 - \log_5 50$ b $\log_2 6 - \log_2 48$ c $\log_3 8 - \log_3 6 - \log_3 12$
d $\log_5 12 - \log_5 20 - \log_5 15$ e $\log_6 \frac{1}{3} - \log_6 12$ f $\log_7 \frac{1}{9} + \log_7 63$

3 Use the log law $\log_a x^n = n \log_a x$ to write each expression in terms of $\log_a 2$.

a $\log_a 8$ b $\log_a 16$ c $\log_a 64$ d $\log_a \frac{1}{2}$
e $\log_a \frac{1}{8}$ f $\log_a \frac{1}{32}$ g $\log_a \sqrt{2}$ h $\log_a \frac{1}{\sqrt{2}}$

4 Express each logarithm in terms of $\log_2 3$ and $\log_2 5$. Remember that $\log_2 2 = 1$.

a $\log_2 9$ b $\log_2 25$ c $\log_2 6$ d $\log_2 10$
e $\log_2 18$ f $\log_2 20$ g $\log_2 \frac{2}{3}$ h $\log_2 2\frac{1}{2}$

5 Given that $\log_2 3 \doteqdot 1.58$ and $\log_2 5 \doteqdot 2.32$, use the log laws to find approximations for:

a $\log_2 15$ b $\log_2 9$ c $\log_2 10$ d $\log_2 50$
e $\log_2 \frac{3}{2}$ f $\log_2 \frac{3}{5}$ g $\log_2 \frac{2}{3}$ h $\log_2 75$

6 Use the identity $\log_a a^x = x$ to simplify:

a $\log_{10} 10^3$ b $\log_7 7^5$ c $\log_{12} 12^{1.3}$ d $\log_8 8^n$

7 Use the identity $a^{\log_a x} = x$ to simplify:

a $10^{\log_{10} 100}$ b $3^{\log_3 7}$ c $4^{\log_4 3.6}$ d $6^{\log_6 y}$

DEVELOPMENT

8 Use the log law $\log_a x^n = n \log_a x$ and the identity $\log_a a = 1$ to simplify:

a $\log_a a^2$ b $5 \log_a a^3$ c $\log_a \frac{1}{a}$ d $12 \log_a \sqrt{a}$

9 Use the log law $\log_a x^n = n \log_a x$ to write each expression in terms of $\log_a x$.

a $\log_a x^3$ b $\log_a \frac{1}{x}$ c $\log_a \sqrt{x}$ d $\log_a \frac{1}{x^2}$
e $\log_a x^3 - \log_a x^5$ f $\log_a x^4 + \log_a \frac{1}{x^2}$ g $2 \log_a a^4 - \log_a x^8$ h $\log_a \frac{1}{\sqrt{x}} + 3 \log_a \sqrt{x}$

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10 Write these expressions in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

a $\log_a yz$ **b** $\log_a \frac{z}{y}$ **c** $\log_a y^4$ **d** $\log_a \frac{1}{x^2}$

e $\log_a xy^3$ **f** $\log_a \frac{x^2y}{z^3}$ **g** $\log_a \sqrt{y}$ **h** $\log_a \sqrt{xz}$

11 Given that $\log_{10} 2 \doteqdot 0.30$ and $\log_{10} 3 \doteqdot 0.48$, use the fact that $\log_{10} 5 = \log_{10} 10 - \log_{10} 2$ to find an approximation for $\log_{10} 5$. Then use the log laws to find approximations for:

a $\log_{10} 20$ **b** $\log_{10} 0.2$ **c** $\log_{10} 360$ **d** $\log_{10} \sqrt{2}$

e $\log_{10} \sqrt{8}$ **f** $\log_{10} \frac{1}{\sqrt{10}}$ **g** $\log_{10} \sqrt{12}$ **h** $\log_{10} \frac{1}{\sqrt{5}}$

12 If $x = \log_a 2$, $y = \log_a 3$ and $z = \log_a 5$, simplify:

a $\log_a 64$ **b** $\log_a \frac{1}{30}$ **c** $\log_a 27a^5$ **d** $\log_a \frac{100}{a}$

e $\log_a 1.5$ **f** $\log_a \frac{18}{25a}$ **g** $\log_a 0.04$ **h** $\log_a \frac{8}{15a^2}$

13 Using the identities $x = a^{\log_a x}$ and $x = \log_a a^x$, express:

a 10 as a power of 3 **b** 3 as a power of 10 **c** 0.1 as a power of 2

d 2 as a logarithm base 10 **e** -4 as a logarithm base 3 **f** $\frac{1}{2}$ as a logarithm base 7

14 Give exact values of.

a $\log_{25} 45 - \log_{25} 9$ **b** $\log_{81} 12 - \log_{81} 36$

c $\log_8 3 - \log_8 96$ **d** $\log_{\frac{1}{32}} 5 - \log_{\frac{1}{32}} 10$

15 Simplify these expressions.

a $5^{-\log_5 2}$ **b** $12^{2\log_{12} 7}$ **c** $2^{\log_2 3 + \log_2 5}$ **d** $a^{n\log_a x}$

e $7^{-\log_7 x}$ **f** $5^{x + \log_5 x}$ **g** $2^{x\log_2 x}$ **h** $3^{\frac{\log_3 x}{x}}$

16 Rewrite these relations in index form (that is, without using logarithms).

a $\log_a (x + y) = \log_a x + \log_a y$ **b** $\log_{10} x = 3 + \log_{10} y$

c $\log_3 x = 4\log_3 y$ **d** $2\log_2 x + 3\log_2 y - 4\log_2 z = 0$

e $x\log_a 2 = \log_a y$ **f** $\log_a x - \log_a y = n\log_a z$

g $\frac{1}{2}\log_2 x = \frac{1}{3}\log_2 y - 1$ **h** $2\log_3(2x + 1) = 3\log_3(2x - 1)$

ENRICHMENT

17 Prove by contradiction that $\log_2 3$ is irrational. Begin with the sentence, 'Suppose by way of contradiction that $\log_2 3 = \frac{a}{b}$, where a and b are positive whole numbers.'

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8E Equations involving logarithms and indices

The calculator allows approximations to logarithms base 10. This section explains how to obtain approximations to any other base. Index equations can then be solved approximately, whatever the base. As always, every base must be positive and not equal to 1.

The change-of-base formula

Finding approximations of logarithms to other bases requires a formula that converts logarithms from one base to another.

14 THE CHANGE-OF-BASE FORMULA

- To write logs base a in terms of logs base b, use the formula

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Remember this as 'the log of the number over the log of the base'.

- The calculator button log gives approximations for logs base 10, so to find logarithms to another base a, put $b = 10$ and use the formula

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}.$$

Proof

To prove this formula, let $y = \log_a x$.
Then by the definition of logs, $x = a^y$
and taking logs base b of both sides, $\log_b x = \log_b a^y$.
Using the log laws, $\log_b x = y \log_b a$
and rearranging, $y = \dfrac{\log_b x}{\log_b a}$, as required.

Example 21 — 8E

Find, correct to four significant figures:

a $\log_2 5$, **b** $\log_3 0.02$.

Then check your approximations using the button labelled x^y or ∧.

SOLUTION

In each part, use the change-of-base-formula to change to logs base 10.

a $\log_2 5 = \dfrac{\log_{10} 5}{\log_{10} 2}$
$\doteqdot 2.322$
Checking, $2^{2.322} \doteqdot 5$.

b $\log_3 0.02 = \dfrac{\log_{10} 0.02}{\log_{10} 3}$
$\doteqdot -3.561$
Checking, $3^{-3.561} \doteqdot 0.02$.

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Solving index equations

An index equation such as $5^x = 18$ is solved, in exact form, just by rewriting it in terms of logarithms,

$$\begin{aligned} 5^x &= 18 \\ x &= \log_5 18. \end{aligned}$$

To find an approximate solution, however, first use the change-of-base formula to convert to logs base 10,

$$\begin{aligned} x &= \frac{\log_{10} 18}{\log_{10} 5} \\ &\doteqdot 1.796 \end{aligned}$$

Example 22 8E

Solve these equations correct to four significant figures. Then check your approximations using the button labelled

$\boxed{x^y}$ or $\boxed{\wedge}$.

a $2^x = 7$

b $3^x = 0.05$

SOLUTION

a

$$2^x = 7$$

Solving for x, $x = \log_2 7$

Changing to base 10, $x = \dfrac{\log_{10} 7}{\log_{10} 2}$

$\doteqdot 2.807$

Checking, $2^{2.807} \doteqdot 7$.

b

$$3^x = 0.05$$

Solving for x, $x = \log_3 0.05$

Changing to base 10, $x = \dfrac{\log_{10} 0.05}{\log_{10} 3}$

$\doteqdot -2.727$

Checking, $3^{-2.727} \doteqdot 0.05$.

Example 23 8E

The rabbit population on Kanin Island is doubling every year. A few years ago, a study estimated that there were 500 rabbits there. How many years later will the rabbit population be 10 000? Answer in exact form and correct to the nearest tenth of a year.

SOLUTION

Let P be the rabbit population n years after the study.

Then $P = 500 \times 2^n$

Put $P = 10\,000$, then $10\,000 = 500 \times 2^n$

$$\begin{aligned} 2^n &= 20 \\ n &= \log_2 20 \quad \text{(the exact answer)} \\ &= \frac{\log_{10} 20}{\log_{10} 2} \\ &\doteqdot 4.3 \text{ years} \quad \text{(correct to 0.1 years)} \end{aligned}$$

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Exponential equations reducible to quadratics

Some exponential equations can be reduced to quadratics with a substitution. A similar procedure with trigonometric equations will be discussed in Section 11H.

Example 24 — 8E

Use the substitution $u = 2^x$ to solve the equation $4^x - 7 \times 2^x + 12 = 0$.

SOLUTION

Writing $4^x = (2^x)^2$, the equation becomes $(2^x)^2 - 7 \times 2^x + 12 = 0$.

Substituting $u = 2^x$,

$$u^2 - 7u + 12 = 0$$
$$(u - 4)(u - 3) = 0$$
$$u = 4 \text{ or } 3$$

and returning to x,

$$2^x = 4 \text{ or } 2^x = 3$$
$$x = 2 \text{ or } \log_2 3.$$

Solving exponential and logarithmic inequations

Provided that the base a is greater than 1, the exponential function $y = a^x$ and the logarithmic function $y = \log_a x$ both increase as x increases, so there are no difficulties solving inequations such as $2^x \le 64$ and $\log_{10} x > 4$.

Example 25 — 8E

Solve:

a $2^x \le 64$

b $\log_{10} x \ge 4$

SOLUTION

a
$$2^x \le 64$$
$$2^x \le 2^6$$
$$x \le 6$$

b
$$\log_{10} x \ge 4$$
$$x \ge 10^4$$
$$x \ge 10000$$

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Taking logarithms of both sides

When an approximate solution of an index equation is required, it is often quicker to take logarithms base 10 of both sides. This is particularly useful when powers with different bases are involved.

Example 26 8E

Solve these equations correct to four significant figures by taking logarithms base 10 of both sides.

a $5^{3x} = 3 \times 5^{x+1}$ **b** $2^{3x} = 3^{x+1}$

SOLUTION

a $5^{3x} = 3 \times 5^{x+1}$

Taking logs base 10 of both sides,

$$\begin{aligned} \log_{10} 5^{3x} &= \log_{10} 3 + \log_{10} 5^{x+1} \\ 3x \log_{10} 5 &= \log_{10} 3 + (x+1)\log_{10} 5 \\ 2x \log_{10} 5 &= \log_{10} 3 + \log_{10} 5 \\ x &= \frac{\log_{10} 3 + \log_{10} 5}{2\log_{10} 5} \\ &\doteqdot 0.8413 \end{aligned}$$

b $2^{3x} = 3^{x+1}$

Taking logs base 10 of both sides,

$$\begin{aligned} 3x \log_{10} 2 &= (x+1)\log_{10} 3 \\ x(3\log_{10} 2 - \log_{10} 3) &= \log_{10} 3 \\ x &= \frac{\log_{10} 3}{3\log_{10} 2 - \log_{10} 3} \\ &\doteqdot 1.1201. \end{aligned}$$

Exercise 8E

FOUNDATION

1 Use the change-of-base formula, $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$, and your calculator, to verify that:

a $\log_2 8 = 3$ **b** $\log_3 81 = 4$ **c** $\log_{100} 10000 = 2$

2 Use the change-of-base formula, $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$, to approximate these logarithms, correct to four significant figures. Check each answer using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

a $\log_2 7$ **b** $\log_2 26$ **c** $\log_2 0.07$ **d** $\log_3 4690$ **e** $\log_5 2$
f $\log_7 31$ **g** $\log_6 3$ **h** $\log_{12} 2$ **i** $\log_3 0.1$ **j** $\log_3 0.0004$
k $\log_{11} 1000$ **l** $\log_{0.8} 0.2$ **m** $\log_{0.03} 0.89$ **n** $\log_{0.99} 0.003$ **o** $\log_{0.99} 1000$

3 Rewrite each equation with x as the subject, using logarithms. Then use the change-of-base formula to solve it, giving your answers correct to four significant figures. Check each answer using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

a $2^x = 15$ **b** $2^x = 5$ **c** $2^x = 1.45$ **d** $2^x = 0.1$ **e** $2^x = 0.0007$
f $3^x = 10$ **g** $3^x = 0.01$ **h** $5^x = 10$ **i** $12^x = 150$ **j** $8^x = \frac{7}{9}$
k $6^x = 1.4$ **l** $30^x = 2$ **m** $0.7^x = 0.1$ **n** $0.98^x = 0.03$ **o** $0.99^x = 0.01$

4 Give exact solutions to these inequations. Do not use a calculator.

a $2^x > 32$ **b** $2^x \le 32$ **c** $2^x < 64$ **d** $3^x \ge 81$
e $5^x > 5$ **f** $4^x \le 1$ **g** $2^x < \frac{1}{2}$ **h** $10^x \le 0.001$

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DEVELOPMENT

5 a Use the substitution $u = 2^x$ to solve $4^x - 9 \times 2^x + 14 = 0$.
b Use the substitution $u = 3^x$ to solve $3^{2x} - 8 \times 3^x - 9 = 0$.
c Use similar substitutions to solve:
i $25^x - 26 \times 5^x + 25 = 0$
ii $9^x - 5 \times 3^x + 4 = 0$
iii $3^{2x} - 3^x - 20 = 0$
iv $7^{2x} + 7^x + 1 = 0$
v $2^{2x} - 64 = 0$
vi $2^x - 3 \times 2^{\frac{1}{2}x} + 2 = 0$

6 Give exact solutions to these inequations. Do not use a calculator.
a $\log_2 x < 3$
b $\log_2 x \geq 3$
c $\log_{10} x > 3$
d $\log_{10} x \geq 1$
e $\log_5 x > 0$
f $\log_6 x < 1$
g $\log_5 x \leq 3$
h $\log_6 x > 2$

7 Rewrite each inequation in terms of logarithms, with x as the subject. Then use the change-of-base formula to solve it, giving your answer correct to three significant figures.
a $2^x > 12$
b $2^x < 100$
c $2^x < 0.02$
d $2^x > 0.1$
e $5^x < 100$
f $3^x < 0.007$
g $1.2^x > 10$
h $1.001^x > 100$

8 A few years ago, Rahul and Fiona were interested in building a new garage, whose price is rising with inflation at 5% per annum. Its price then was \$12 000.
a Explain why the cost C in dollars can be modelled by $C = 12000 \times 1.05^n$, where n is the years since.
b Substitute into the formula to find, correct to the nearest 0.1 years, how many years later the cost will be \$18 000.

9 Use the change-of-base formula $\log_a x = \dfrac{\log_b x}{\log_b a}$ to prove that:
a $\log_8 x = \frac{1}{3}\log_2 x$
b $\log_{a^n} x = \frac{1}{n}\log_a x$

10 Solve these equation and inequations, correct to three significant figures if necessary.
a $2^{x+1} = 16$
b $3^{2x} = 81$
c $5^{x-1} < 1$
d $10^{\frac{1}{3}x} \leq 1000$
e $\left(\frac{1}{2}\right)^{x-3} = 8$
f $\left(\frac{1}{10}\right)^{5x} = \frac{1}{10}$
g $2^{x-2} < 7$
h $3^{x+5} > 10$

11 a Solve $2^x < 10^{10}$. How many positive integer powers of 2 are less than 10^{10}?
b Solve $3^x < 10^{50}$. How many positive integer powers of 3 are less than 10^{50}?

12 a Explain why $\log_{10} 300$ lies between 2 and 3.
b If x is a two-digit number, what two integers does $\log_{10} x$ lie between?
c If $\log_{10} x = 4.7$, how many digits does x have to the left of the decimal point?
d Find $\log_{10} 25^{20}$, and hence find the number of digits in 25^{20}.
e Find $\log_{10} 2^{1000}$, and hence find the number of digits in 2^{1000}.

13 a Explain why, if $a > 1$, then $\log_a x$ is
i positive for $x > 1$,
ii negative for $0 < x < 1$.
b Explain why, if $0 < a < 1$, then $\log_a x$ is
i negative for $x > 1$,
ii positive for $0 < x < 1$.
c Explain why $\log_a x$ is never defined for:
i $x < 0$,
ii $x = 0$.
d Explain why the base of logarithms
i cannot be 0,
ii cannot be 1.

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14 Use a substitution such as $u = 4^x$ to solve each equation. Give each solution as a rational number, or approximate to three decimal places.

a $2^{4x} - 7 \times 2^{2x} + 12 = 0$ **b** $100^x - 10^x - 1 = 0$ **c** $\left(\frac{1}{5}\right)^{2x} - 7 \times \left(\frac{1}{5}\right)^x + 10 = 0$

15 By taking logarithms base 10 of both sides, solve each index equation correct to four significant figures.

a $3^{x-4} = 47$ **b** $2^{x-5} = 5 \times 2^{2x}$ **c** $5^{2x} = 6^{x+1}$ **d** $7^{1-x} = 6 \times 5^{x-3}$

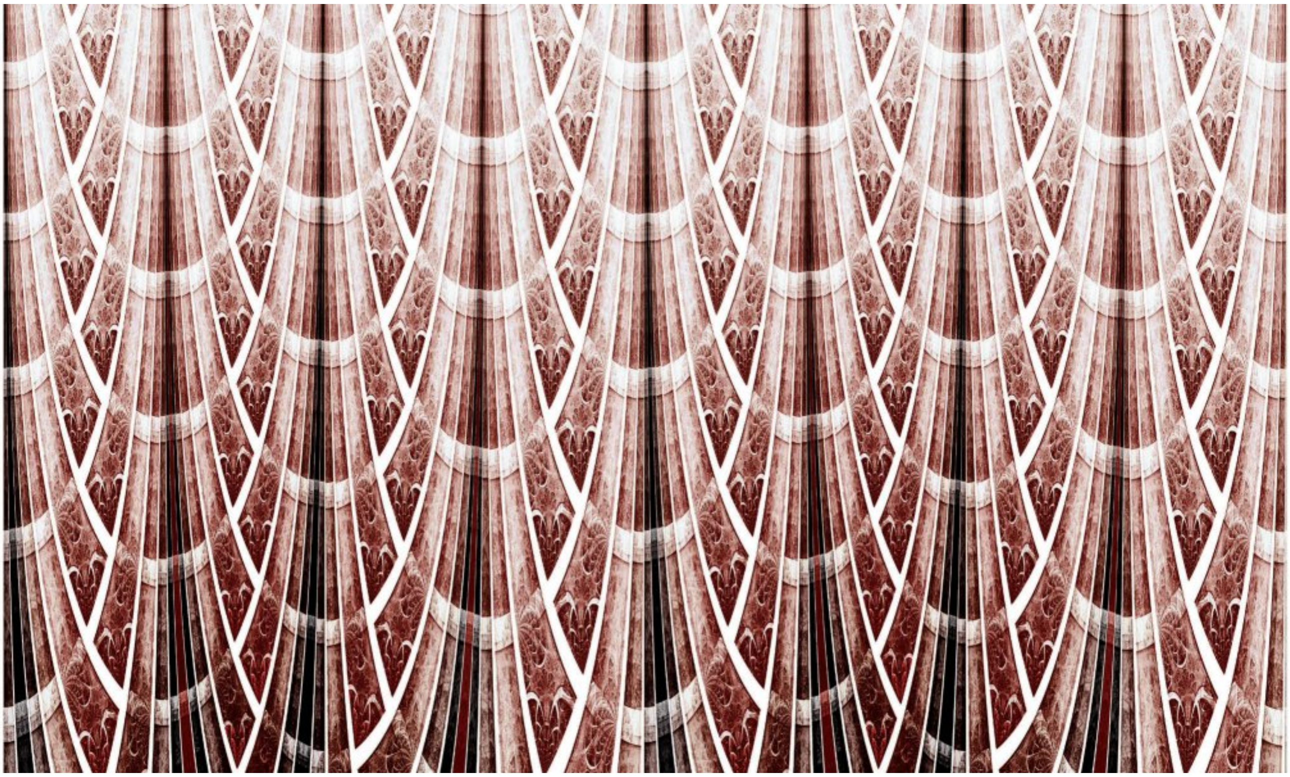
ENRICHMENT

16 Let $S = \frac{1}{2}(2^x + 2^{-x})$ and $D = \frac{1}{2}(2^x - 2^{-x})$.

a Simplify SD, $S + D$, $S - D$ and $S^2 - D^2$.

b Rewrite the formulae for S and D as quadratic equations in 2^x. Hence express x in terms of S, and in terms of D, in the case where $x > 1$.

c Show that $x = \frac{1}{2}\log_2 \dfrac{1 + y}{1 - y}$, where $y = DS^{-1}$.



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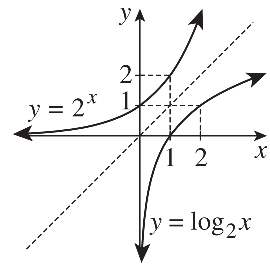
8F Exponential and logarithmic graphs

The function $y = a^x$ is an *exponential function*, because the variable x is in the *exponent* or *index*. The function $y = \log_a x$ is a *logarithmic function*.

The graphs of $y = 2^x$ and $y = \log_2 x$

These two graphs will demonstrate the characteristic features of all exponential and logarithmic graphs. Here are their tables of values:



$y = 2^x$

x	-2	-1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

$y = \log_2 x$

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	-2	-1	0	1	2

- The two graphs are reflections of each other in the diagonal line $y = x$. This is because they are *inverse functions of each other*, so their tables of values are the same, except that the x-values and y-values have been swapped.
- For $y = 2^x$, the domain is all real x and the range is $y > 0$.
  For $y = \log_2 x$, the domain is $x > 0$ and the range is all real y.
- For $y = 2^x$, the x-axis is a horizontal asymptote — as $x \to -\infty$, $y \to 0$.
  For $y = \log_2 x$, the y-axis is a vertical asymptote — as $x \to 0^+$, $y \to -\infty$.
- As x increases, $y = 2^x$ also increases, getting steeper all the time.
  As x increases, $y = \log_2 x$ also increases, but gets flatter all the time.
- The graph of $y = 2^x$ is concave up, but $y = \log_2 x$ is concave down.

15 EXPONENTIAL AND LOGARITHMIC FUNCTIONS ARE INVERSE FUNCTIONS

Let the base a be any positive number not equal to 1.

- The functions $y = a^x$ and $y = \log_a x$ are inverse functions, meaning that

$$\log_a (a^x) = x \qquad \text{and} \qquad a^{\log_a x} = x.$$

- The graphs of $y = a^x$ and $y = \log_a x$ are reflections of each other in the diagonal line $y = x$.
- The domains, the asymptotes, the steepness and the concavity of the two graphs are clearly seen from their graphs and this reflection property.

In the language of Section 3I, $y = 2^x$ is a *one-to-one function*, satisfying both the *vertical line test* and the *horizontal line test*. Taking logarithms base 2 is *reading the graph of* $y = 2^x$ *backwards*, and the inverse function $y = \log_2 x$ is also a *one-to-one function*.

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Transformations of exponential and logarithmic functions

The usual transformations of shifting and reflecting apply to exponential and logarithmic functions. Transformations of some exponential functions occurred earlier in Chapter 4, but logarithmic functions were only introduced in this chapter.

Example 27 [Reflections] 8F

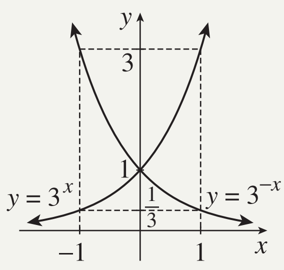
For each pair of functions:
— Draw up convenient tables of values for them.
— Sketch both functions on one set of axes.
— Explain what transformation transforms one graph onto the other, and why.
— State their domain and range.

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = -3^x$

SOLUTION

a

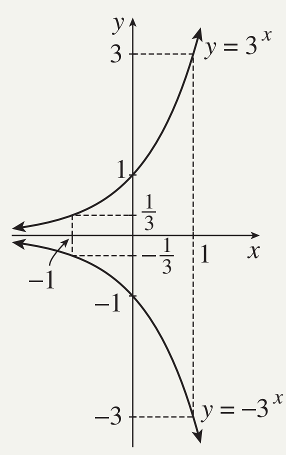
x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
3^{-x}	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$



The graphs are reflections of each other in the y-axis, because x has been replaced by $-x$.
For both functions, the domain is all real x, the range is $y > 0$.

b

x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
-3^x	$-\frac{1}{9}$	$-\frac{1}{3}$	-1	-3	-9



The graphs are reflections of each other in the x-axis, because y has been replaced by $-y$.
Both functions have domain all real x.
For $y = 3^x$, the range is $y > 0$.
For $y = -3^x$, the range is $y < 0$.

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Example 28 [Translations] 8F

Repeat Example 27 for:

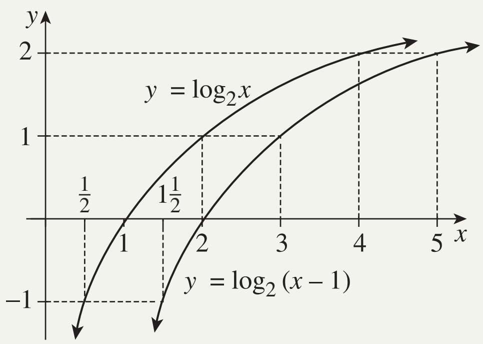
a $y = \log_2 x$ and $y = \log_2(x - 1)$ **b** $y = \log_2 x$ and $y = 3 + \log_2 x$

SOLUTION

a

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$	-2	-1	0	1	2

x	$1\frac{1}{4}$	$1\frac{1}{2}$	2	3	5
$\log_2(x - 1)$	-2	-1	0	1	2

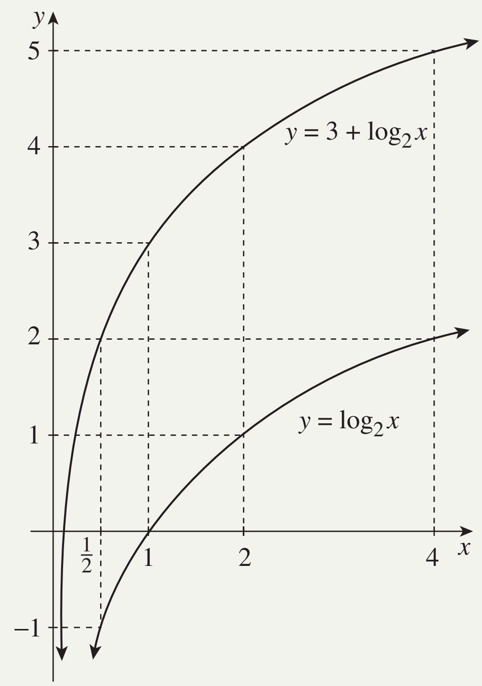


The second graph is the first shifted right by 1 unit, because x has been replaced by $x - 1$.
Both functions have range all real y.
$y = \log_2 x$ has domain $x > 0$, and $y = \log_2(x - 1)$ has domain $x > 1$.

b

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$	-2	-1	0	1	2
$3 + \log_2 x$	1	2	3	4	5

The second graph is the first shifted up by 3 units, because y has been replaced by $y - 3$
(or because 3 has been added to each value of y).
Both functions have domain $x > 0$ and range all real y.



The word 'logarithm'

The Scottish mathematician John Napier (1550–1617) formed his new word 'log|arithm' from the two Greek words 'logos', meaning 'ratio' or 'calculation', and 'arithmos', meaning 'number'. Until the invention of calculators, the routine method for performing difficult calculations in arithmetic was to use tables of logarithms, invented by Napier, to convert products to sums, quotients to differences, and powers to multiples.

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Exercise 8F

FOUNDATION

1 **a** Use the calculator button log to complete the following table of values, giving each entry correct to two significant figures where appropriate.

x	0.1	0.25	0.5	0.75	1	2	3	4	5	6	7	8	9	10
$\log_{10} x$														

b Hence sketch the graph of $y = \log_{10} x$. Use a large scale, with the same scale on both axes. Ideally use graph paper so that you can see the shape of the curve.

2 **a** Copy and complete these two tables of values.

i

x	-3	-2	-1	0	1	2	3
2^x							

ii

x	-3	-2	-1	0	1	2	3
2^{-x}							

b Hence sketch the graphs of $y = 2^x$ and $y = 2^{-x}$ on one set of axes.

c How are the two tables of values related to each other?

d What symmetry does the diagram of the two graphs display, and why?

e Write down the domains and ranges of:

i $y = 2^x$, **ii** $y = 2^{-x}$.

f Write down the equations of the asymptotes of:

i $y = 2^x$, **ii** $y = 2^{-x}$.

g Copy and complete:

i ‘As $x \to -\infty$, $2^x \to \ldots$’ **ii** ‘As $x \to \infty$, $2^x \to \ldots$’

h Copy and complete:

i ‘As $x \to -\infty$, $2^{-x} \to \ldots$’ **ii** ‘As $x \to \infty$, $2^{-x} \to \ldots$’

3 **a** Copy and complete these two tables of values.

i

x	-2	-1	0	1	2
3^x					

ii

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\log_3 x$					

b Hence sketch the graphs of $y = 3^x$ and $y = \log_3 x$ on one set of axes.

c How are the two tables of values related to each other?

d What symmetry does the diagram of the two graphs display, and why?

e Write down the domain and range of:

i $y = 3^x$, **ii** $y = \log_3 x$.

f Write down the equations of the asymptotes of:

i $y = 3^x$, **ii** $y = \log_3 x$.

g Copy and complete:

i ‘As $x \to -\infty$, $3^x \to \ldots$’ **ii** ‘As $x \to 0^+$, $\log_3 x \to \ldots$’

DEVELOPMENT

4 **a** Sketch on one set of axes the graphs of $y = 3^x$ and $y = 3^{-x}$.

b Sketch on one set of axes the graphs of $y = 10^x$ and $y = 10^{-x}$.

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5 Sketch the four graphs below on one set of axes.

a $y = 2^x$ **b** $y = -2^x$ **c** $y = 2^{-x}$ **d** $y = -2^{-x}$

6 Sketch each set of graphs on one set of axes, clearly indicating the asymptote, the y-intercept, and the x-intercept if it exists. Use shifting of the graphs in the previous questions, but also use a table of values to confirm your diagram.

a $y = 2^x$, $y = 2^x + 3$, $y = 2^x - 1$

b $y = -2^x$, $y = 2 - 2^x$, $y = -2 - 2^x$

c $y = 2^{-x}$, $y = 2^{-x} + 1$, $y = 2^{-x} - 2$

7 Use reflections in the x-axis and y-axis to sketch the four graphs below on one set of axes.

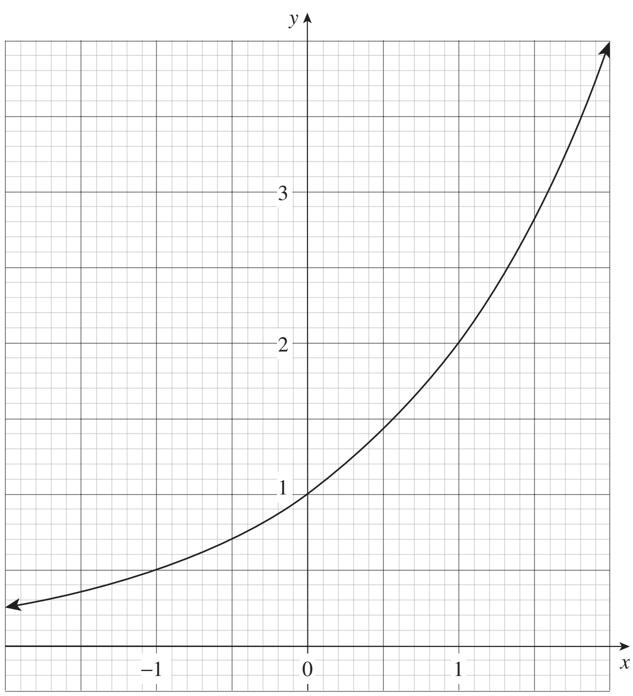
a $y = \log_2 x$ **b** $y = -\log_2 x$ **c** $y = \log_2(-x)$ **d** $y = -\log_2(-x)$

8 Use shifting to sketch each set of graphs on one set of axes, clearly indicating the asymptote and the intercepts with the axes.

a $y = \log_2 x$, $y = \log_2 x + 1$, $y = \log_2 x - 1$

b $y = -\log_2 x$, $y = 2 - \log_2 x$, $y = -2 - \log_2 x$

9



The diagram above shows the graph of $y = 2^x$. Use the graph to answer the questions below, giving your answers correct to no more than two decimal places.

a Read from the graph the values of:

i 2^2 **ii** 2^{-2} **iii** $2^{1.5}$ **iv** $2^{0.4}$ **v** $2^{-0.6}$

b Find x from the graph if:

i $2^x = 2$ **ii** $2^x = 3$ **iii** $2^x = 1.2$ **iv** $2^x = 0.4$

c Find the values of x for which:

i $1 \le 2^x \le 4$ **ii** $1 \le 2^x \le 2$ **iii** $1.5 \le 2^x \le 3$ **iv** $0.5 \le 2^x \le 2$

d Read the graph backwards to find:

i $\log_2 4$ **ii** $\log_2 3$ **iii** $\log_2 1.4$ **iv** $\log_2 0.8$

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10 Sketch, on separate axes, the graphs of:

a $y = 2^{x-2}$ **b** $y = 2^{x+1}$ **c** $y = \log_2(x + 1)$

d $y = \log_2(x - 1)$ **e** $y = \frac{1}{2}(2^x + 2^{-x})$ **f** $y = \frac{1}{2}(2^x - 2^{-x})$

11 Sketch, on separate axes, the graphs of:

a $y = |\log_2 x|$ **b** $y = |2^x|$ **c** $y = \log_2 |x|$

d $y = 2^{|x|}$ **e** $y = \big|\log_2 |x|\big|$ **f** $y = |2^{|x|}|$

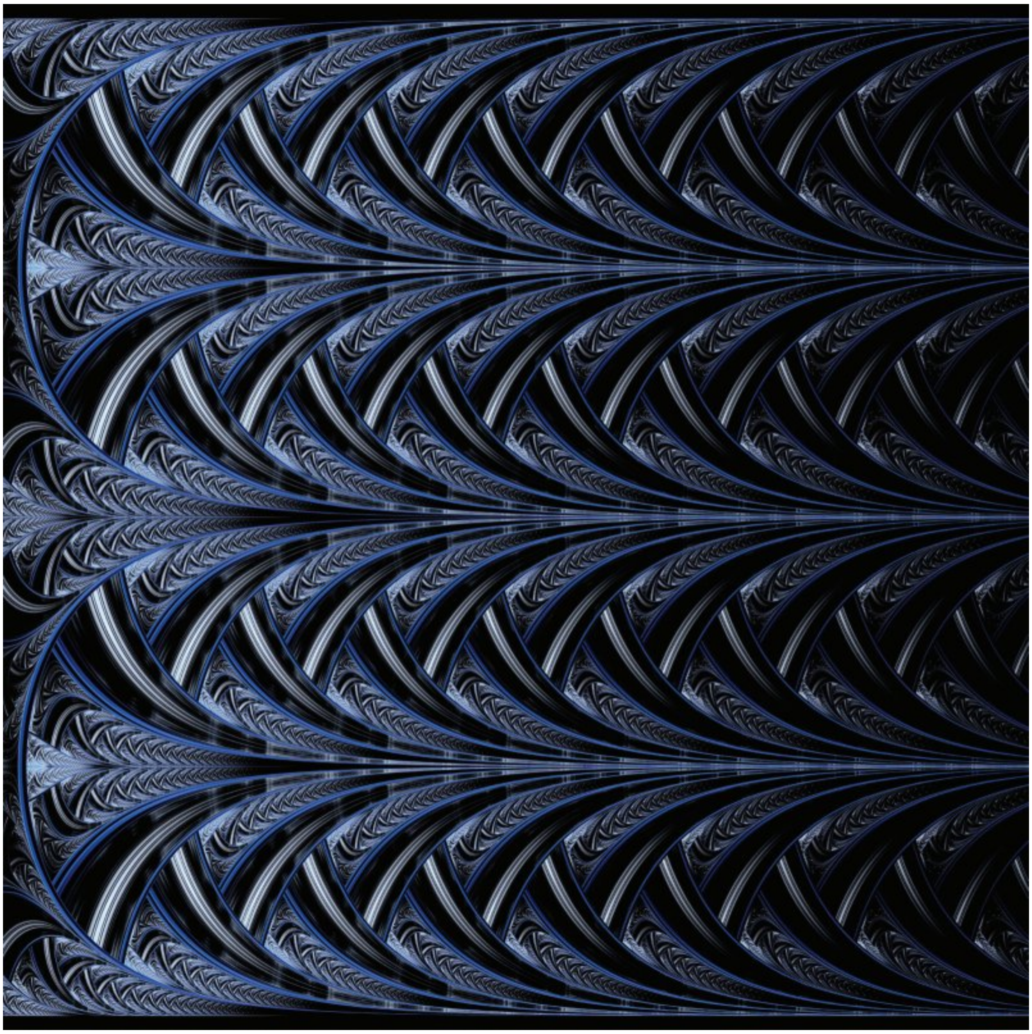
ENRICHMENT

12 Sketch, on separate axes, the graphs of:

a $|y| = \log_2 x$ **b** $|y| = |\log_2 x|$ **c** $|y| = \log_2 |x|$

d $|y| = \big|\log_2 |x|\big|$ **e** $|y| = 2^x$ **f** $|y| = 2^{|x|}$

13 Look at the graphs of $y = 2^x$ and $y = \log_2 x$ at the beginning of this section. Let the line $y = 2 - x$ meet the curves at A and B. Explain why the distance AB is less than $\sqrt{2}$.



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8G Applications of these functions

Some applications of these functions have been scattered through the preceding exercises — the applications in this exercise are longer and more sustained. The intention of all the questions is to indicate how diverse the applications of exponential and logarithmic functions are in various sciences and technologies.

The questions require conversion between a statement in exponential form and a statement in logarithmic form. The pattern for this is

$$2^3 = 8 \quad \text{means that} \quad 3 = \log_2 8\,.$$

The change-of-base formula is also needed to convert logs to base 10,

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}.$$

The later questions in this exercise are more difficult than in other exercises, but they are included because of their vital interest to readers studying computing, physics, geology and chemistry. Many of them could easily be adapted to projects that would go into the subjects in more detail. The exercise concludes with some suggestions for a project that would investigate web data to see whether it is exponential.

Exercise 8G

FOUNDATION

1 A quantity Q is varying over time t according to the formula $Q = 5 \times 10^{\frac{t}{2}}$. Give answers correct to four significant figures.

a Find Q when $t = 6$, and when $t = 5.43$.

b Rewrite the formula with t as the subject.

c Find t when $Q = 500$, and when $Q = 256$.

2 A quantity Q is varying over time t according to the formula $t = 20 \log_2 2Q$.

a Find t when $Q = 4$, and when $Q = 6$.

b Rewrite the formula with Q as the subject.

c Find Q when $t = 40$, and when $t = 45$.

DEVELOPMENT

3 The population of a country is doubling every 30 years, and was 3 000 000 at the last census.

a Explain why the population P after another n years is $P = 3\,000\,000 \times 2^{\frac{n}{30}}$.

b Find the population (nearest million) after:

i 90 years, ii 100 years.

c By substituting into the equation, find (nearest year) when the population will be:

i 48 000 000, ii 60 000 000.

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4 The price of a particular metal has been rising with inflation at 5% per annum, from a base price of \$100 per kilogram in 1900. Let \$$P$ be the price n years since 1900, so that $P = 100 \times (1.05)^n$.

a Copy and complete the following table of values, giving values correct to the nearest whole number:

n	0	20	40	60	80	100
P	100					

b Sketch the graph of the function.

c Now copy and complete this table for $\log_{10} P$, correct to two decimal places:

n	0	20	40	60	80	100
$\log_{10} P$	2					

d Draw a graph with n on the horizontal axis and $\log_{10} P$ on the vertical axis.

e Use the log laws to prove that $\log_{10} P = 2 + n \log_{10}(1.05)$, and hence explain the shape of the second graph.

5 [Moore's law]
Gordon Moore predicted in 1965 that over the next few decades, the number of transistors within a computer chip would very roughly double every two years — this is called *Moore's law*. Let D_0 be the density in 1975.

a Explain why the density D after n more years is predicted to be $D = D_0 2^{\frac{n}{2}}$.

b What was the predicted density for 2015?

c Substitute into the formula to find, correct to the nearest year, the prediction of the year when the density increases by a factor of 10000000.

6 [Newton's law of cooling]
A container of water, originally just below boiling point at 96°C, is placed in a fridge whose temperature is 0°C. The container is known to cool in such a way that its temperature halves every 20 minutes.

a Explain why the temperature $T°$ C after n hours in the fridge is given by the function
$T = 96 \times \left(\frac{1}{2}\right)^{3n}$.

b Draw up a table of values of the function and sketch its graph.

c What is the temperature after 2 hours?

d Write the equation with the time n in hours as the subject.

e How long does it take, correct to the nearest minute, for the temperature to fall to 1°C?

7 [Decay of uranium-235]
Uranium-235 is a naturally-occurring isotope of uranium. It is particularly important because it can be made to fission, when it releases vast amounts of energy in a nuclear power station or a nuclear bomb. It has a half-life of about 700 million years, which means that if you store a mass of uranium-235 for 700 million years, half of it will then have decayed into other elements.

a Explain why the mass of uranium-235 in the Earth n years after the present is given by
$M = M_0 \times \left(\frac{1}{2}\right)^{\frac{n}{700000000}}$, where M_0 is the mass in the Earth now.

b The Andromeda Galaxy will collide with our Milky Way galaxy in about 4 billion years. What percentage of the present uranium-235 will still be present in the Earth?

c The Earth itself is about 4.5 billion years old. How many times more uranium-235 was present in the Earth when it was formed?

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8 [Decibels]
Noise intensity I is measured in units of watts per square metre (W/m^2). The more common (absolute) decibel scale for noise intensity uses a reference level of $I_r = 10^{-12}\ \text{W/m}^2$, which is the auditory threshold of someone with perfect hearing. The (absolute) decibel level n of noise with intensity I is then defined as the ratio

$$n = 10 \log_{10}\left(\frac{I}{I_r}\right) \text{dB}.$$

(One decibel is one-tenth of a bel, which explains the initial 10 in the formula.)

a What decibel level corresponds to $4 \times 10^{-3}\ \text{W/m}^2$ (nearest decibel)?

b Write the formula with I as the subject, and find what intensity corresponds to 75 decibels (animated conversation).

c The sound level rises from 72 dB to 108 dB (rock concert level). By what multiple is the power increased?

d What does a decibel level of 70 dB fall to if the power decreases by a factor of 1600?

9 [The Richter scale]
Earthquake strengths are usually reported on the Richter scale, which has several minor variants. They are all based on the log base 10 of the 'shaking amplitude' of the earthquake wave at some standardised distance from the epicentre. (An earthquake occurs kilometres underground — the *epicentre* is the point on the surface above the place where the earthquake occurs.)

a An earthquake of strength 4.0–4.9 is classified as 'light', and will cause only minimal damage. An earthquake of strength 7.0 or above is classified as 'major', and will cause damage or total collapse to most buildings. What is the ratio of the shaking amplitude of the smallest major quake to the smallest light quake?

b The energy released by the quake is proportional to the $\frac{3}{2}$th power of the shaking amplitude. What is the ratio of the energies released by the quakes in part **a**?

c An earthquake of magnitude 9.0 will cause total destruction. Find the ratio of the shaking amplitudes in such an earthquake and the smallest light quake, and find also the ratio of the energies released.

ENRICHMENT

10 [pH of a solution]
The pH of a liquid is traditionally defined on a logarithmic scale as $\text{pH} = -\log_{10}[\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen ion concentration in units of moles per litre (mol/L). It is not necessary to understand the units, except to know that the greater the concentration of hydrogen ions, the greater the acidity.

a Rewrite the formula with $[\text{H}^+]$ as the subject.

b Pure water has a pH of about 7. What is its hydrogen ion concentration?

c Lemon juice typically has a pH of about 2.0. What is its hydrogen ion concentration? How many times more acidic is it than pure water?

d Sea water typically has a pH of about 8.1. What is its hydrogen ion concentration? How many times more alkaline (meaning 'less acidic') is it than pure water?

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11 [Keyboard tuning]
When a keyboard is tuned to modern concert pitch, the note A below middle C vibrates at 220 Hz (1 hertz is 1 vibration per second), and the note A′ above middle C vibrates at twice that frequency, which is 440 Hz (these names are not standard, but are convenient for this question). When modern *equal temperament* is used for the twelve semitones of the scale from A and A′, the frequency ratio of each semitone is $2^{\frac{1}{12}}$, so that on a logarithmic scale base 10, the log of all the semitones ratios is $\frac{1}{12}\log_{10} 2$.

- **a** Find, correct to two decimal places, the frequencies of C, C♯ and E, if the interval A C consists of three semitones, A C♯ of four semitones, and A E of seven semitones.
- **b** Unfortunately, intervals sound best when their ratios consist of small whole numbers, as most of them are in *meantone temperament*, the most common temperament from say 1500 to 1800. Using that temperament, find the frequencies of C, C♯ and E, if the interval A C is in the ratio 6 : 5 (meaning frequency of C : frequency of A = 6 : 5), A C♯ is in ratio 5 : 4, and A E is in ratio 3 : 2.
- **c** When two frequencies close together are sounded together, there is an ugly *beat frequency*, which is the difference between the frequencies. Calculate the beat frequencies between the equal-tempered and meantone versions of C, C♯ and E.
- **d** Meantone temperament is actually very complicated. Suppose that a careless tuner tuned A at 220 Hz, then tuned three major thirds so that A C♯ was in ratio 5 : 4, C♯ F was in ratio 5 : 4, and F A′ was in ratio 5 : 4. What would the resulting frequency of A′ be, and how would it beat with A′ tuned to 440 Hz on another instrument?
- **e** Meantone temperament gets its name from the fact that B between A and C♯ is tuned so that the log of its frequency is the mean of the logs of the frequencies of A and C♯ (B is two semitones above A, and two semitones below C♯). Find the meantone frequency of B, and the equal-temperament frequency of B.

---

12 **A possible project on exponential data**
Large amounts of the data collected in science and elsewhere are exponential, as the examples in this exercise have shown. But if one suspects that some particular data are exponential, what is the most straightforward way to demonstrate this, or to prove it wrong? There is another problem — data are messy. No set of collected data is likely to be exactly exponential, so it is a matter of identifying a trend and trying to quantify it.

This project uses logarithms to test whether given data are exponential. It would probably be useful to do at least Question 4 in this exercise before embarking on it.

- **a** Find on the web a table of data of some phenomenon that may be exponential. The data could be from science, or engineering, or geography (concerning the number of cars or houses), or economics, or history (concerning populations, or the number of archaeological remnants discovered).
- **b** Identify the quantity Q that you are interested in, and the independent variable x on which Q depends — the variable x will usually be time, but there are many other possibilities.
- **c** Draw up a suitable table of values, not of x and Q, but of x and $\log_{10} Q$. (Actually, you can use any base for the logarithmic function. Base 2 may be interesting because it easily identifies doubling, and base $e \doteqdot 2.7183$ is useful because it allows the rate to be calculated by differentiation once Chapter 9 has been mastered):

x	
$\log_{10} Q$	

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d Draw a graph with x on the horizontal axis and $\log_{10} Q$ on the vertical axis. Use graph paper or a software plotter. If the graph is roughly a straight line, you have found an example of exponential growth.

e Estimate the gradient m and the vertical intercept b of your graph. You can now argue:

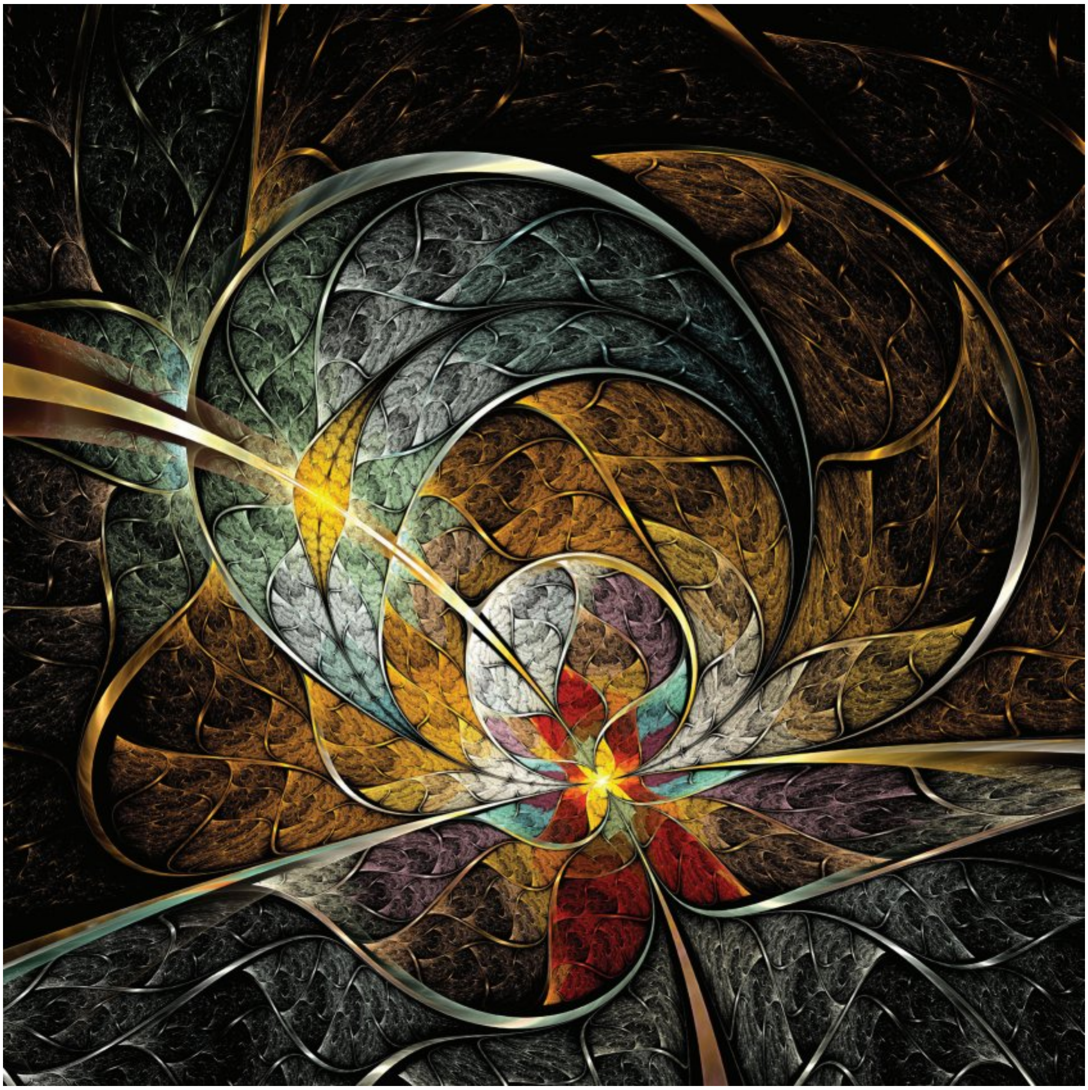
We know that $\log_{10} Q = mx + b$, where m and b are estimated from the graph.

Hence
$$Q = 10^{mx+b}$$
$$Q = 10^{mx} \times 10^{b}$$
$$Q = Q_0 10^{mx}, \text{ where } Q_0 = 10^b \text{ is the value of } Q \text{ when } x = 0.$$

f With this formula, and its log equivalent $x = \frac{1}{m}\log_{10}\frac{Q}{Q_0}$, you can make predictions, test other data, compare other data by treating them in the same way, and draw all sorts of impressive conclusions.

g If your graph is roughly a straight line, but with kinks, you may be able to identify factors that have caused the data to deviate from an ideal exponential function.

h If you analyse some data in this way, and find that the graph is nothing like a straight line, your time is not wasted — you can now conclude that the particular phenomenon is not exponential, and perhaps find some interesting reasons why this is so.



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Chapter 8 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 8 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

1 Write each expression as an integer or fraction.

a 5^3 **b** 2^8 **c** 10^9 **d** 17^{-1}

e 9^{-2} **f** 2^{-3} **g** 3^{-4} **h** 27^0

i $\left(\frac{2}{3}\right)^3$ **j** $\left(\frac{7}{12}\right)^{-1}$ **k** $\left(\frac{5}{6}\right)^{-2}$ **l** $36^{\frac{1}{2}}$

m $27^{\frac{1}{3}}$ **n** $8^{\frac{2}{3}}$ **o** $9^{\frac{5}{2}}$ **p** $\left(\frac{4}{49}\right)^{\frac{1}{2}}$

q $\left(\frac{14}{59}\right)^0$ **r** $\left(\frac{9}{25}\right)^{-\frac{1}{2}}$ **s** $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ **t** $\left(\frac{9}{100}\right)^{-\frac{3}{2}}$

2 Write each expression in index form.

a $\frac{1}{x}$ **b** $\frac{7}{x^2}$ **c** $-\frac{1}{2x}$ **d** $\sqrt{x}$

e $5\sqrt{36x}$ **f** $\frac{4}{\sqrt{x}}$ **g** $\frac{y}{x}$ **h** $2y\sqrt{x}$

3 Simplify:

a $(x^4)^5$ **b** $\left(\frac{3}{a^3}\right)^4$ **c** $(25x^6)^{\frac{1}{2}}$ **d** $\left(\frac{8r^3}{t^6}\right)^{\frac{1}{3}}$

4 Simplify each expression, leaving the answer in index form.

a $x^2y \times y^2x$ **b** $15xyz \times 4y^2z^4$ **c** $3x^{-2}y \times 6xy^{-3}$

d $4a^{-2}bc \times a^5b^2c^{-2}$ **e** $x^3y \div xy^3$ **f** $14x^{-2}y^{-1} \div 7xy^{-2}$

g $m^{\frac{1}{2}}n^{-\frac{1}{2}} \times m^{1\frac{1}{2}}n^{-\frac{1}{2}}$ **h** $(2st)^3 \times (3s^3)^2$ **i** $(4x^2y^{-2})^3 \div (2xy^{-1})^3$

5 Solve each equation for x.

a $3^x = 81$ **b** $5^x = 25$ **c** $7^x = \frac{1}{7}$ **d** $2^x = \frac{1}{32}$

e $\left(\frac{1}{3}\right)^x = \frac{1}{9}$ **f** $\left(\frac{2}{3}\right)^x = \frac{8}{27}$ **g** $25^x = 5$ **h** $8^x = 2$

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Review

6 Rewrite each logarithmic equation as an index equation and solve it for x.

a $x = \log_2 8$ **b** $x = \log_3 9$ **c** $x = \log_{10} 10000$ **d** $x = \log_5 \frac{1}{5}$

e $x = \log_7 \frac{1}{49}$ **f** $x = \log_{13} 1$ **g** $x = \log_9 3$ **h** $x = \log_2 \sqrt{2}$

i $2 = \log_7 x$ **j** $-1 = \log_{11} x$ **k** $\frac{1}{2} = \log_{16} x$ **l** $\frac{1}{3} = \log_{27} x$

m $2 = \log_x 36$ **n** $3 = \log_x 1000$ **o** $-1 = \log_x \frac{1}{7}$ **p** $\frac{1}{2} = \log_x 4$

7 Use the log laws and identities to simplify:

a $\log_{22} 2 + \log_{22} 11$ **b** $\log_{10} 25 + \log_{10} 4$ **c** $\log_7 98 - \log_7 2$

d $\log_5 6 - \log_5 150$ **e** $\log_3 54 + \log_3 \frac{1}{6}$ **f** $\log_{12} \frac{2}{7} + \log_{12} \frac{7}{2}$

8 Write each expression in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

a $\log_a xyz$ **b** $\log_a \frac{x}{y}$ **c** $\log_a x^3$ **d** $\log_a \frac{1}{z^2}$

e $\log_a x^2y^5$ **f** $\log_a \frac{y^2}{xz^2}$ **g** $\log_a \sqrt{x}$ **h** $\log_a \sqrt{xyz}$

9 Between what two consecutive integers do the following logarithms lie?

a $\log_{10} 34$ **b** $\log_7 100$ **c** $\log_3 90$ **d** $\log_2 35$

e $\log_{10} 0.4$ **f** $\log_{10} 0.007$ **g** $\log_2 0.1$ **h** $\log_7 0.1$

10 Use a calculator, and the change-of-base formula if necessary, to approximate these logarithms, correct to four significant figures.

a $\log_{10} 215$ **b** $\log_{10} 0.0045$ **c** $\log_7 50$ **d** $\log_2 1000$

e $\log_5 0.215$ **f** $\log_{1.01} 2$ **g** $\log_{0.5} 8$ **h** $\log_{0.99} 0.001$

11 Use logarithms to solve these index equations, correct to four significant figures.

a $2^x = 11$ **b** $2^x = 0.04$ **c** $7^x = 350$ **d** $3^x = 0.67$

e $1.01^x = 5$ **f** $1.01^x = 0.2$ **g** $0.8^x = \frac{1}{10}$ **h** $0.99^x = \frac{1}{100}$

12 **a** Sketch on one set of axes $y = 3^x$, $y = 3^{-x}$, $y = -3^x$ and $y = -3^{-x}$.

b Sketch on one set of axes $y = 2^x$ and $y = \log_2 x$.

c Sketch on one set of axes $y = 3^x$, $y = 3^x - 1$ and $y = 3^x + 2$

13 The number of bacteria growing in a Petri dish is doubling every four hours. Initially there were 100 bacteria.

a Explain why the formula for the population P after n hours is $P = 100 \times 2^{\frac{n}{4}}$.

b How many bacteria are there after:

i 12 hours, **ii** 13 hours?

c Write the formula with n as the subject.

d How long, correct to the nearest hour, before there are 10 000 000 bacteria?

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9 Differentiation

We are now ready to begin studying functions and their graphs using *calculus*. Calculus begins with two processes called *differentiation* and *integration*.

- *Differentiation* looks at the changing steepness of a curve.
- *Integration* looks at the areas of regions bounded by curves.

Both processes involve taking limits. They were well known to the Greeks, but it was not until the late 17th century that Gottfried Leibniz in Germany and Sir Isaac Newton in England independently gave systematic accounts of them.

This chapter deals with differentiation, and introduces the derivative as the gradient of the tangent to a curve. Integration will be introduced in Year 12. Section 9J on rates of change begins to show how essential calculus is for science, for economics, and for solving problems.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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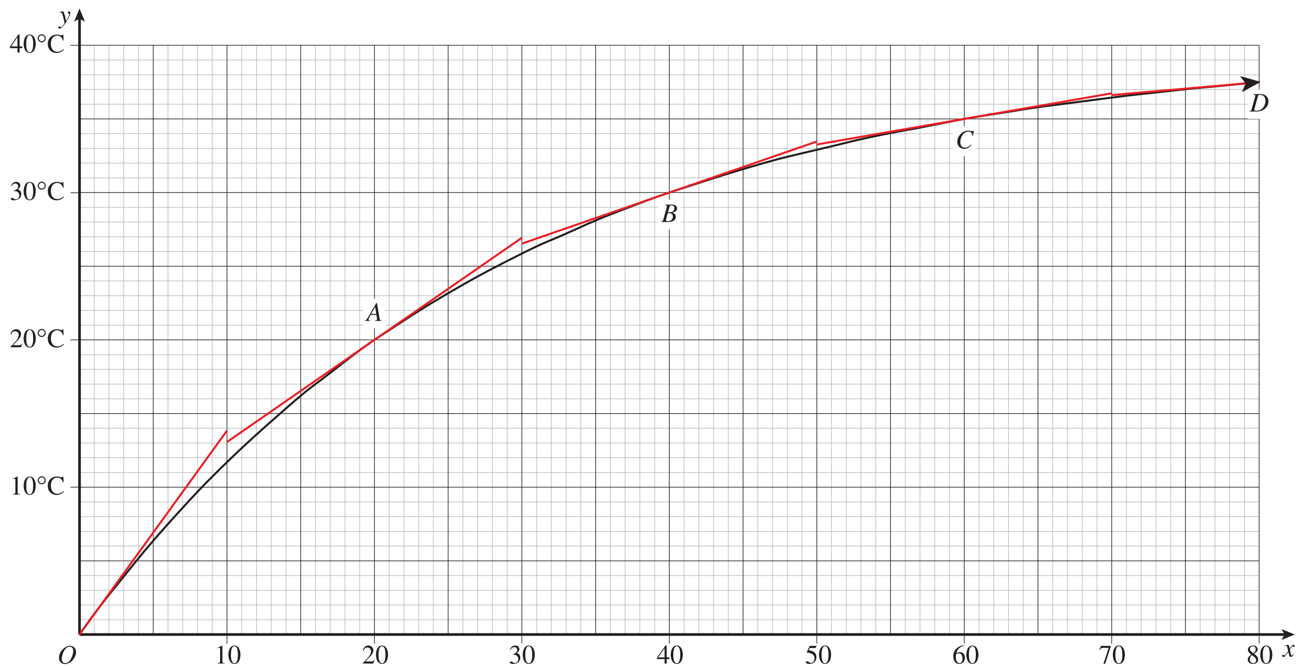
9A Tangents and the derivative



A bottle of water was taken out of a fridge on a hot day when the air temperature was 40°C. The graph $y = f(x)$ above shows how the temperature increased over the next 80 minutes. The horizontal axis gives the time x in minutes, and the vertical axis gives the temperature y°C.

The water temperature was originally 0°C, and 20 minutes later it was 20°C. Thus during the first 20 minutes, the temperature was rising at an *average rate* of 1°C per minute. This rate is the gradient of the chord OA.

Measuring the *instantaneous rate of temperature increase*, however, requires a tangent to be drawn. The gradient of the tangent is the instantaneous rate of increase at the time x — such gradients are easy to measure on the graph paper by counting little divisions and using the formula gradient $= \dfrac{\text{rise}}{\text{run}}$.

The gradient of the tangent to the curve $y = f(x)$ at any point is called the *derivative*, which is written as $f'(x)$. Measuring the gradients at the marked points O, A, B and C gives a table of values of the derivative:

x	0	20	40	60
$f'(x)$	1.39	0.69	0.35	0.17

Notice that the derivative $f'(x)$ is a new function. It has a table of values, and it can be sketched, just as the original function was. You will be sketching the derivative in the investigation questions at the beginning of Exercise 9A.

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Geometric definition of the derivative

Here is the essential definition of the derivative.

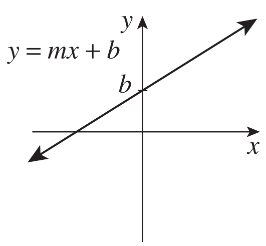
1 THE DERIVATIVE $f'(x)$ DEFINED GEOMETRICALLY

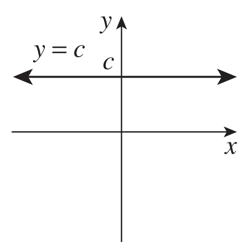
$f'(x)$ is the gradient of the tangent to $y = f(x)$ at each point on the curve.

At present, circles are the only curves whose tangents we know much about, so the only functions we can apply our definition to are constant functions, linear functions and semicircle functions.

Linear and constant functions

When a graph is a straight line, as in the diagram on the left below, the tangent at every point on the graph is just the line itself. Thus if $f(x) = mx + b$ is a line with gradient m, the derivative at every point is m. Hence the derivative is the constant function $f'(x) = m$.





In particular, a horizontal straight line has gradient zero, as in the diagram on the right above. Hence the tangent to the graph of a constant function $f(x) = c$ at any point is horizontal, so the derivative is the zero function $f'(x) = 0$.

2 THE DERIVATIVE OF LINEAR AND CONSTANT FUNCTIONS

- If $f(x) = mx + b$ is a linear function, then
  $f'(x) = m$ is a constant function.
- In particular, if $f(x) = c$ is a constant function, then
  $f'(x) = 0$ is the zero function.

Example 1 — 9A

Write down the derivative $f'(x)$ of each linear function.

a $f(x) = 3x + 2$ **b** $f(x) = \dfrac{5 - 2x}{3}$ **c** $f(x) = -4$

SOLUTION

In each case the derivative is just the gradient of the line:

a $f'(x) = 3$ **b** $f'(x) = -\frac{2}{3}$ **c** $f'(x) = 0$

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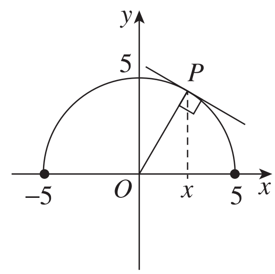
The derivative of a semicircle function

Let $f(x) = \sqrt{25 - x^2}$ be the upper semicircle with centre O and radius 5. We know from circle geometry that at any point $P\left(x, \sqrt{25 - x^2}\right)$ on the semicircle, the tangent at P is the line perpendicular to the radius OP.



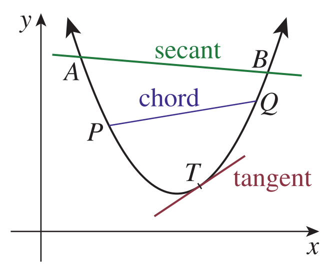
Now gradient of radius $OP = \dfrac{\sqrt{25 - x^2}}{x}$

so gradient of tangent at $P = -\dfrac{x}{\sqrt{25 - x^2}}$

and hence $f'(x) = -\dfrac{x}{\sqrt{25 - x^2}}$.

(This result is not to be memorised.)

Chords, secants and tangents



Some new terminology has been used in this section. The words 'chord', 'secant' and 'tangent' were probably introduced in earlier years only for circles, but are now being used in the context of graphs.

- A *chord* (meaning 'cord', such as a bowstring) is the interval joining two distinct points P and Q on the graph.
- A *secant* (meaning 'cutting') is the line through two distinct points A and B on the graph — it therefore contains the chord AB.
- Informally, a *tangent* (meaning 'touching') at a point T on the graph is the line that continues in the direction the curve is going at the point T — think of the ray of light from the headlights of a car going around a bend. The definition used with circles, 'meets the curve only at T', doesn't work for graphs, because if the curve twists around, the tangent may meet the curve again at one or more other points. Section 9B has a more formal definition.

Some preliminary investigations

The following exercise, Exercise 9A, begins with four investigations that apply the definition of the derivative directly to several functions. The investigations may be done on graph paper by photocopying the graphs, or by using graphing software.

Section 9B will then develop these ideas into a more rigorous treatment of the derivative.

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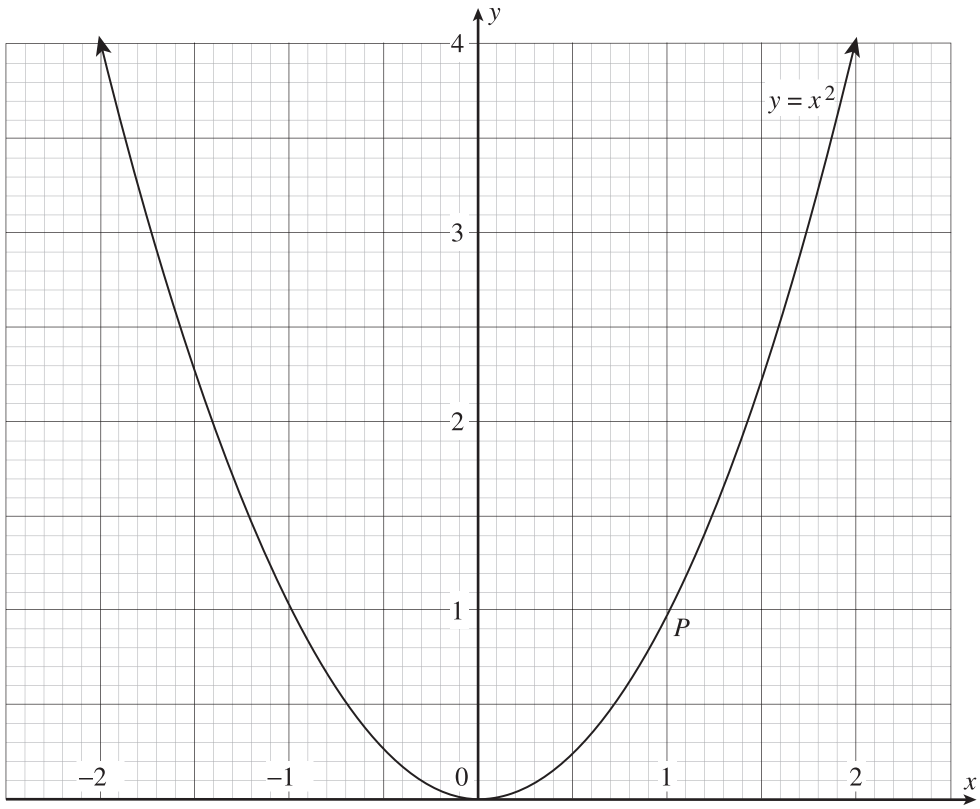
Exercise 9A

INVESTIGATION

1 [Graph paper]



a Photocopy the sketch above of $f(x) = x^2$.

b At the point $P(1, 1)$, construct the tangent. Place your pencil point on P, bring your ruler to the pencil, then rotate the ruler about P until it seems reasonably like a tangent.

c Use the definition gradient $= \dfrac{\text{rise}}{\text{run}}$ to measure the gradient of this tangent correct to at most two decimal places. Choose the run to be 10 little divisions, and count how many vertical divisions the tangent rises as it runs across the 10 horizontal divisions.

d Copy and complete the following table of values of the derivative $f'(x)$ by constructing a tangent at each of the nine points on the curve and measuring its gradient.

x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
$f'(x)$									

e On a separate set of axes, use your table of values to sketch the curve $y = f'(x)$.

f Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

2 [The same investigation using technology]

a Use graphing software to sketch the graph of $y = x^2$. Use the same scale on both axes so that gradients are represented correctly. The values on the y-axis should run from at least $y = 0$ to $y = 4$, and on the x-axis from at least $x = -2$ to $x = 2$.

b Construct the tangent at the point $P(1, 1)$.

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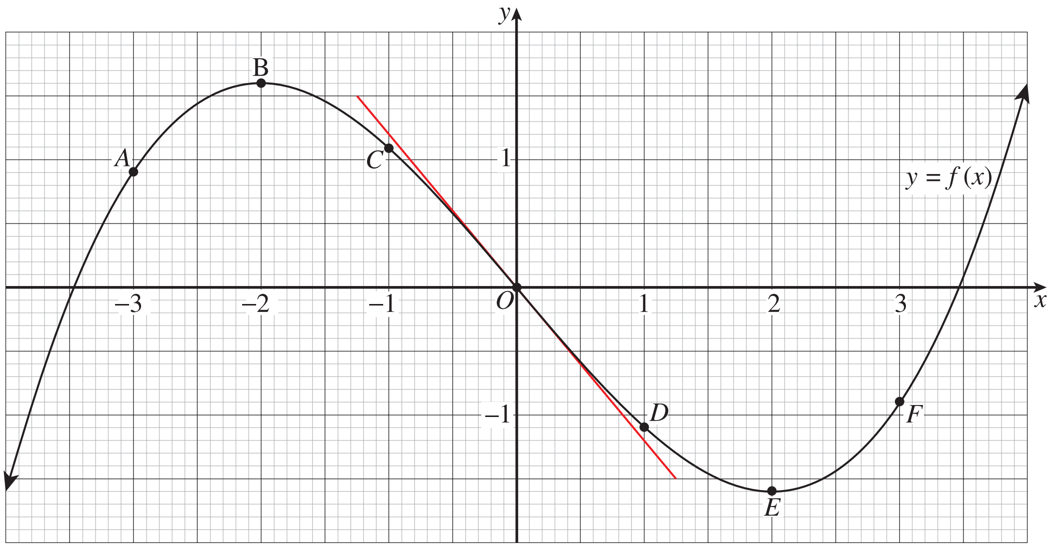
c Find the gradient of this tangent, either by asking the software, or by producing a graph-paper background and counting little squares as in Question 1.

d Copy and complete the table of values of $f'(x)$ as in Question 1 part **d**.

e On a new set of axes, plot these values of $f'(x)$, and join them up to form the graph of $y = f'(x)$.

f Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

3 [Graph paper, but easily adapted to technology]



a Photocopy the cubic graph above.

b The tangent at the origin $O(0, 0)$ has been drawn — notice that it crosses the curve at the origin.

Use the definition gradient $= \dfrac{\text{rise}}{\text{run}}$ to measure the gradient of this tangent.

c Copy and complete the following table of values of the derivative $f'(x)$ by constructing a tangent at each of the seven points on the curve and measuring its gradient.

x	-3	-2	-1	0	1	2	3
$f'(x)$							

d On a separate set of axes, use your table of values to sketch the curve $y = f'(x)$.

4 [Technology, but easily done on graph paper]

a Use graphing software to sketch the graph of $y = x^3$, using the same scale on both axes. The values on the y-axis should run from just $y = -1$ to $y = 1$, and on the x-axis from $x = -1$ to $x = 1$. (Larger values of x will send the graph off the screen.)

b Using the software to construct tangents and calculate their gradients, copy and complete the following table of values of the derivative $f'(x)$.

x	-1	-0.9	-0.8	-0.5	0	0.5	0.8	0.9	1
$f'(x)$									

c On a new set of axes, plot the graph of $y = f'(x)$.

d Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

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Conclusions of the four investigations

- Either of the first two investigations should demonstrate reasonably clearly that the derivative of the quadratic function $f(x) = x^2$ is the linear function $y = 2x$.
- Either of the last two investigations should demonstrate at least that the derivative of a cubic curve looks very much like a quadratic curve — the last investigation may have indicated that $f(x) = x^3$ has derivative $f'(x) = 3x^2$.

DEVELOPMENT

5 Write each function in the form $f(x) = mx + b$, and hence write down $f'(x)$.

a $f(x) = 2x + 3$ **b** $f(x) = 5 - 3x$ **c** $f(x) = \frac{1}{2}x - 7$

d $f(x) = -4$ **e** $f(x) = ax + b$ **f** $f(x) = \frac{2}{3}(x + 4)$

g $f(x) = \dfrac{3 - 5x}{4}$ **h** $f(x) = \frac{5}{2}(7 - \frac{4}{3}x)$ **i** $f(x) = \frac{1}{2} + \frac{1}{3}$

6 Write each function in the form $f(x) = mx + b$, and hence write down the derived function.

a $f(x) = \dfrac{3 + 5x}{2} - \dfrac{5 - 2x}{2}$ **b** $f(x) = (x + 3)^2 - (x - 3)^2$

c $f(x) = \dfrac{k - \ell x}{r} + \dfrac{k + \ell x}{r}$

7 Sketch graphs of these functions, draw tangents at the points where $x = -2, -1, 0, 1, 2$, estimate their gradients, and hence draw a reasonable sketch of the derivative.

a $f(x) = 4 - x^2$ **b** $f(x) = \dfrac{1}{x}$ **c** $f(x) = 2^x$

8 Sketch the upper semicircle $f(x) = \sqrt{25 - x^2}$, mark the points $(4, 3)$, $(3, 4)$, $(0, 5)$, $(-3, 4)$ and $(-4, 3)$ on it, and draw tangents and radii at these points. By using the fact that the tangent is perpendicular to the radius at the point of contact, find:

a $f'(4)$ **b** $f'(3)$ **c** $f'(0)$ **d** $f'(-4)$ **e** $f'(-3)$

9 Use the fact that the tangent to a circle is perpendicular to the radius at the point of contact to find the derived functions of the following. Begin with a sketch.

a $f(x) = \sqrt{1 - x^2}$ **b** $f(x) = -\sqrt{1 - x^2}$ **c** $f(x) = \sqrt{4 - x^2}$

ENRICHMENT

10 Use the radius-and-tangent theorem to find the derivatives of:

a $f(x) = \sqrt{9 - x^2} + 4$ **b** $f(x) = 3 - \sqrt{16 - x^2}$

c $f(x) = \sqrt{36 - (x - 7)^2}$ **d** $f(x) = 7 - \sqrt{2x - x^2}$

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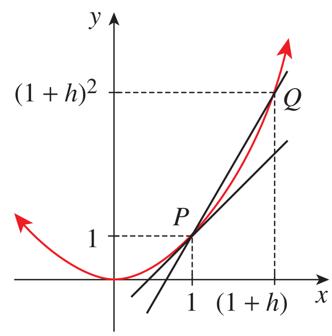
9B The derivative as a limit

This section introduces a limiting process to find the gradient of a tangent at a point P on a curve. This will be the formal definition of the derivative.

The tangent as the limit of secants

When the function is not linear, we need to look at secants through P that cross the curve again at another point Q near P, and then take the limit as Q moves towards P. The diagram below shows the graph of $f(x) = x^2$ and the tangent at the point $P(1, 1)$ on the curve.



Let Q be another point on the curve, and join the secant PQ.

Let the x-coordinate of Q be $1 + h$, where $h \neq 0$.

Then the y-coordinate of Q is $(1 + h)^2$.

$$\begin{aligned}\text{Hence gradient } PQ &= \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{(this is rise over run)}\\ &= \frac{(1 + h)^2 - 1}{(1 + h) - 1}\\ &= \frac{2h + h^2}{h}\\ &= 2 + h, \text{ because } h \neq 0.\end{aligned}$$

As Q moves along the curve, to the right or left of P, the secant PQ changes.

The closer Q is to the point P, the closer the secant PQ is to the tangent at P.

The gradient of the secant PQ becomes 'as close as we like' to the gradient of the tangent as Q moves sufficiently close to P.

This is called 'taking the limit as Q approaches P', with notation $\lim\limits_{Q \to P}$.

$$\begin{aligned}\text{gradient (tangent at } P) &= \lim_{Q \to P} (\text{gradient } PQ)\\ &= \lim_{h \to 0} (2 + h), \quad \text{because } h \to 0 \text{ as } Q \to P\\ &= 2, \qquad\qquad\quad \text{because } 2 + h \to 2 \text{ as } h \to 0.\end{aligned}$$

Thus the tangent at P has gradient 2, which means that $f'(1) = 2$.

Notice that the point Q cannot actually coincide with P, that is, h cannot be zero. Otherwise both rise and run would be zero, and the calculation would be invalid. The point Q may, however, be on the left or the right of P.

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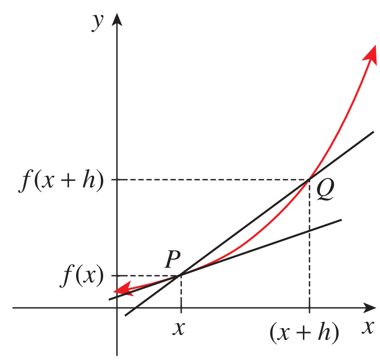
The derivative as a limit



This same process can be applied to any function $f(x)$.

Let $P\big(x, f(x)\big)$ be any point on the curve.

Let Q be any other point on the curve, to the left or right of P, and let Q have x-coordinate $x + h$, where $h \neq 0$, and y-coordinate $f(x + h)$.

Then (gradient of secant PQ) $= \dfrac{f(x + h) - f(x)}{h}$ (rise over run).

As $h \to 0$, the point Q moves 'as close as we like' to P, and the gradient of the secant PQ becomes 'as close as we like' to the gradient of the tangent at P.

Hence (gradient of tangent at P) $= \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$,

that is, $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

This last expression is the *limit formula for the derivative*.

3 THE DERIVATIVE $f'(x)$ AS A LIMIT

For each value of x, the derivative is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \quad \text{(if the limit exists).}$$

The expression $\dfrac{f(x + h) - f(x)}{h}$ is sometimes called a *difference quotient*, because the numerator $f(x + h) - f(x)$ is the difference between the heights at P and Q, and the denominator h is the difference between the x-coordinates of P and Q. Geometrically, it is the gradient of the secant PQ.

What is a tangent?

The careful reader will realise that the word 'tangent' was introduced without definition in Section 9A. Whereas tangents to circles are well understood, tangents to more general curves are not so easily defined. It is possible to define a tangent geometrically, but it is far easier to take the formula for the derivative as a limit as the definition. So our strict definition of the tangent at a point $P\big(x, f(x)\big)$ is that it is the line through P with gradient $f'(x)$.

4 THE FORMAL DEFINITION OF A TANGENT

The tangent at a point P on $y = f(x)$ is the line through P whose gradient is the derivative $f'(x)$ evaluated at the point P (if the derivative exists there).

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Using the definition of the derivative — first-principles differentiation

Using the limit formula above to find the derivative is called *first-principles differentiation*.

Example 2 9B

a For $f(x) = x^2$, use first-principles differentiation to show that $f'(5) = 10$.

b **i** Use first-principles differentiation to find the derivative $f'(x)$.

ii Substitute $x = 5$ to confirm that $f'(5) = 10$, as in part **a**.

SOLUTION

a For all $h \neq 0$,

$$\begin{aligned}\frac{f(5+h) - f(5)}{h} &= \frac{(5+h)^2 - 5^2}{h} \\ &= \frac{25 + 10h + h^2 - 25}{h} \\ &= \frac{10h + h^2}{h} \\ &= 10 + h, \quad \text{because } h \neq 0.\end{aligned}$$

Taking the limit as $h \to 0$, $f'(5) = 10$.

b **i** For all $h \neq 0$,

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= 2x + h, \quad \text{because } h \neq 0.\end{aligned}$$

Taking the limit as $h \to 0$, $f'(x) = 2x$.

ii Substituting $x = 5$, $f'(5) = 10$, as established in part **a**.

Finding points on a curve with a given gradient

Once the derivative has been found, we can find the points on a curve where the tangent has a particular gradient.

5 FINDING POINTS ON A CURVE WITH A GIVEN GRADIENT

- To find the points where the tangent has a given gradient m, solve the equation

  $$f'(x) = m.$$

- To find the y-coordinates of the points, substitute back into $f(x)$.

The points on the curve where the tangent is horizontal are particularly important.

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Example 3 9B

a Differentiate $f(x) = 6x - x^2$ by first-principles.

b Find the point on $y = 6x - x^2$ where the tangent is horizontal, then sketch the curve. Describe the curve and the point.

SOLUTION

a For all $h \neq 0$,

$$\frac{f(x+h) - f(x)}{h} = \frac{6(x+h) - (x+h)^2 - 6x + x^2}{h}$$
$$= \frac{6h - 2xh - h^2}{h}$$
$$= 6 - 2x + h, \quad \text{because } h \neq 0.$$

Taking the limit as $h \to 0$, $\quad f'(x) = 6 - 2x$.

b Put $\quad f'(x) = 0 \quad$ (because the tangent is horizontal).

Then $\quad 6 - 2x = 0$

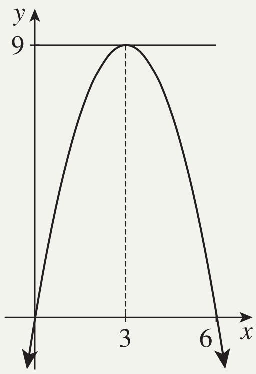
$\quad x = 3$.

Substituting, $\quad f(3) = 18 - 9$

$\quad = 9$,

so the tangent is horizontal at the point $(3, 9)$.

This point is the vertex of the parabola $y = 6x - x^2$.



Exercise 9B

FOUNDATION

Note: The questions in this exercise use the formula for the derivative as a limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

1 Consider the function $f(x) = 5x^2$.

a Show that $f(1) = 5$ and $f(1+h) = 5 + 10h + 5h^2$.

b Show that $\dfrac{f(1+h) - f(1)}{h} = 10 + 5h$.

c Take the limit as $h \to 0$ to show that $f'(1) = 10$.

2 Consider again the function $f(x) = 5x^2$.

a Show that $f(x+h) = 5x^2 + 10xh + 5h^2$.

b Show that $\dfrac{f(x+h) - f(x)}{h} = 10x + 5h$.

c Take the limit as $h \to 0$ to show that $f'(x) = 10x$.

d Substitute $x = 1$ into $f'(x)$ to confirm that $f'(1) = 10$, as found in Question 1.

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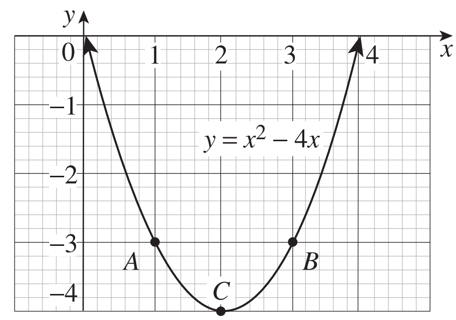
3 Consider the function $f(x) = x^2 - 4x$.



a Show that $\dfrac{f(x + h) - f(x)}{h} = 2x + h - 4$.

b Show that $f'(x) = 2x - 4$ by taking the limit as $h \to 0$.

c Evaluate $f'(1)$ to find the gradient of the tangent at $A(1, -3)$.

d Similarly, find the gradients of the tangents at $B(3, -3)$ and $C(2, -4)$.

e The function $f(x) = x^2 - 4x$ is graphed to the right. Place your ruler on the curve at A, B and C to check the reasonableness of the results obtained above.

4 In the previous section, you found by geometry that a linear function $f(x) = mx + b$ has derivative $f'(x) = m$. This question confirms that the limit formula for $f'(x)$ gives the same answer.

a Let $f(x) = 3x + 7$. Find $\dfrac{f(x + h) - f(x)}{h}$, and hence show that $f'(x) = 3$.

b Let $f(x) = mx + b$. Find $\dfrac{f(x + h) - f(x)}{h}$, and hence show that $f'(x) = m$.

c Let $f(x) = c$. Find $\dfrac{f(x + h) - f(x)}{h}$, and hence show that $f'(x) = 0$.

5 **a** **i** Simplify $\dfrac{f(x + h) - f(x)}{h}$ for the function $f(x) = x^2 + 10$, then find $f'(x)$.

ii Find the gradient of the tangent at the point P on the curve where $x = 2$.

iii Find the coordinates of any points on the curve where the tangent is horizontal.

b Repeat the steps of part **a** for $f(x) = x^2 + 6x + 2$.

c Repeat the steps of part **a** for $f(x) = 2x^2 - 20x$.

d Repeat the steps of part **a** for $f(x) = 9 - 4x^2$.

DEVELOPMENT

6 **a** Sketch $f(x) = x^2 - 10x$, then show that the derivative is $f'(x) = 2x - 10$.

b Hence find the gradient of the tangent at the points where:

i $x = 0$ **ii** $x = 10$ **iii** $x = 5$ **iv** $x = 4\frac{1}{2}$ **v** $x = 5\frac{1}{2}$

c What is the angle between the tangents in part **b iv** and **v**?

7 **a** Show that the derivative of $g(x) = x^2 - 5x$ is $g'(x) = 2x - 5$.

b Hence find the coordinates of the points on $y = x^2 - 5x$ where the tangent has gradient:

i 1 **ii** -1 **iii** 5 **iv** -5 **v** 0

c Then sketch the curve and the tangents.

8 **a** Sketch $f(x) = x^2 - 7x + 6$, then show that the derivative is $f'(x) = 2x - 7$.

b Find the two x-intercepts, find the gradients of the tangents there, and show that they are opposites.

c Find the coordinate of the y-intercept A and the gradient m at A, then find the point B on the curve where the tangent has the opposite gradient $-m$.

d Find the coordinates of the point where the tangent is horizontal.

9 **a** Show that the derivative of the quadratic $f(x) = ax^2 + bx + c$ is $f'(x) = 2ax + b$.

b Find the x-coordinate of the point V on $y = f(x)$ where the tangent is horizontal.

c What geometric significance does the result of part **b** have?

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10 The two *binomial expansions* in this question will be studied more systematically in Chapter 15 on binomial expansions.

a **i** Prove that $(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ by writing $(x + h)^3 = (x + h) \times (x + h)^2$ and then expanding brackets.

ii Hence find the derivative of $f(x) = x^3$.

iii Show that the tangents to $y = x^3$ at the points $A\left(a, a^3\right)$ and $B\left(-a, -a^3\right)$ are parallel.

b **i** Similarly prove that $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

ii Hence find the derivative of $f(x) = x^4$.

iii Show that the tangents to $y = x^4$ at the points $A\left(a, a^4\right)$ and $B\left(-a, a^4\right)$ have opposite gradients.

11 **a** For $f(x) = \dfrac{1}{x}$, show that $\dfrac{f(x + h) - f(x)}{h} = \dfrac{-1}{x(x + h)}$.

b Hence show that $f'(x) = -\dfrac{1}{x^2}$.

c Show from this formula, and from sketching the curve, that all tangents slope downwards (that is, have negative gradient). What happens to the slopes of the tangents as $x \to 0^+$ and as $x \to 0^-$? What happens as $x \to \infty$ and as $x \to -\infty$?

12 **a** If $f(x) = \sqrt{x}$, where $x \geq 0$, write down $\dfrac{f(x + h) - f(x)}{h}$.

b Use the method of *rationalising the numerator*:

$$\frac{\sqrt{x + h} - \sqrt{x}}{h} = \frac{\left(\sqrt{x + h} - \sqrt{x}\right)\left(\sqrt{x + h} + \sqrt{x}\right)}{h\left(\sqrt{x + h} + \sqrt{x}\right)}$$

to show that $\dfrac{f(x + h) - f(x)}{h} = \dfrac{1}{\sqrt{x + h} + \sqrt{x}}$.

c Hence find $f'(x)$.

d Show from this formula, and from sketching the curve, that all tangents slope upwards. What happens to the slopes of the tangents as the point of contact moves towards the origin? What happens as $x \to \infty$?

ENRICHMENT

13 Use the methods of Questions 11 and 12 above, find $\dfrac{f(x + h) - f(x)}{h}$ to show that:

a If $f(x) = \dfrac{1}{x^2}$, then $f'(x) = -\dfrac{2}{x^3}$.

b If $f(x) = \dfrac{1}{\sqrt{x}}$, then $f'(x) = -\dfrac{1}{2x\sqrt{x}}$.

14 [Algebraic differentiation of x^2]

Let $P\left(a, a^2\right)$ be any point on the curve $y = x^2$, then the line ℓ through P with gradient m has equation $y - a^2 = m(x - a)$. Show that the x-coordinates of the points where ℓ meets the curve are $x = a$ and $x = m - a$. Find the value of the gradient m for which these two points coincide, and explain why it follows that the derivative of x^2 is $2x$.

15 [An alternative algebraic approach]

Find the x-coordinates of the points where the line $\ell : y = mx + b$ meets the curve $y = x^2$, and hence deduce that the derivative of x^2 is $2x$.

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9C A rule for differentiating powers of x

The long calculations of $f'(x)$ in Exercise 9B had quite simple answers, as you will probably have noticed. Here is the pattern:

$f(x)$	1	x	x^2	x^3	x^4	x^5	$\cdots$
$f'(x)$	0	1	$2x$	$3x^2$	$4x^3$	$5x^4$	$\cdots$

These results are examples of a simple rule for differentiating any power of x:

6 THE DERIVATIVE OF ANY POWER OF x

Let $f(x) = x^n$, where n is any real number. Then the derivative is

$$f'(x) = nx^{n-1}.$$

This rule is usually memorised as:

'Take the index as a factor, and reduce the index by 1.'

The general proof has several stages — in this section we prove the general result only for whole-number indices n, then the proofs for rational indices are completed in Sections 9F and 9G. Here are four examples of the use of the general theorem.

Example 4 **9C**

Differentiate:

a $f(x) = x^8$ **b** $f(x) = x^{100}$ **c** $f(x) = x^{-4}$ **d** $f(x) = x^{\frac{2}{3}}$

SOLUTION

a $f(x) = x^8$
$f'(x) = 8x^7$

b $f(x) = x^{100}$
$f'(x) = 100x^{99}$

c $f(x) = x^{-4}$
$f'(x) = -4x^{-5}$

d $f(x) = x^{\frac{2}{3}}$
$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

Proof when n is a whole number

The result has already been proven for $n = 0, 1, 2$ and 3, so suppose that $n \geq 4$. Look carefully at the first two terms in the expansion of $(x + h)^4$:

$$\begin{aligned}(x + h)^4 &= (x + h)(x + h)(x + h)(x + h) \\ &= x^4 + (xxxh + xxhx + xhxx + hxxx) + \text{(terms with at least two } h\text{s)} \\ &= x^4 + 4x^3h + \text{(terms in } h^2 \text{ and above)}.\end{aligned}$$

Every term in the expansion has four factors. The terms in hx^3 are $xxxh$, $xxhx$, $xhxx$ and $hxxx$, being the four factors h, x, x and x in all four possible orders.

The general case of the expansion of $(x + h)^n$ is similar:

$$\begin{aligned}(x + h)^n &= (x + h)(x + h) \cdots (x + h)(x + h) \quad \text{(there are } n \text{ factors)} \\ &= x^n + (x^{n-1}h + x^{n-2}hx + x^{n-3}hx^2 + \cdots + hx^{n-1}) + \text{(terms with at least two } h\text{s)} \\ &= x^n + nx^{n-1}h + \text{(terms in } h^2 \text{ and above)}.\end{aligned}$$

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Every term in the expansion has n factors. The n factors of each term in hx^{n-1} consist of one h, and $n - 1$ copies of x, in any one of the n possible orders.

$$\begin{aligned}\text{Hence } f'(x) &= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \to 0} \frac{nx^{n-1}h + (\text{terms in } h^2 \text{ and above})}{h} \\ &= \lim_{h \to 0} \left(nx^{n-1} + (\text{terms in } h \text{ and above})\right) \\ &= nx^{n-1}.\end{aligned}$$

Later in the course, the binomial theorem, and mathematical induction, will each provide proofs that are more concise than what is possible now.

Linear combinations of functions

Functions formed by taking sums and multiples of simpler functions can be differentiated in the obvious way, one term at a time.

7 LINEAR COMBINATIONS OF FUNCTIONS

- If $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.
- If $f(x) = au(x)$, then $f'(x) = au'(x)$.

Proof

For the first:

$$\begin{aligned}f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{u(x+h) + v(x+h) - u(x) - v(x)}{h} \\ &= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} \\ &= u'(x) + v'(x).\end{aligned}$$

For the second:

$$\begin{aligned}f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{au(x+h) - au(x)}{h} \\ &= a \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \\ &= au'(x).\end{aligned}$$

Example 5 9C

Differentiate each function.

a $f(x) = 4x^2 - 3x + 2$ **b** $f(x) = \frac{1}{2}x^6 - \frac{1}{6}x^3$ **c** $f(x) = x^{100} - 2 \times x^{50} + 100$

SOLUTION

a $f'(x) = 8x - 3$ **b** $f'(x) = 3x^5 - \frac{1}{2}x^2$ **c** $f'(x) = 100x^{99} - 100x^{49}$

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Expanding products

Sometimes a product needs to be expanded before the function can be differentiated.

Example 6 9C

Differentiate each function after first expanding the brackets.

a $f(x) = x^3(x - 10)$

b $f(x) = (x + 2)(2x + 3)$

SOLUTION

a
$$\begin{aligned} f(x) &= x^3(x - 10) \\ &= x^4 - 10x^3 \\ f'(x) &= 4x^3 - 30x^2 \end{aligned}$$

b
$$\begin{aligned} f(x) &= (x + 2)(2x + 3) \\ &= 2x^2 + 7x + 6 \\ f'(x) &= 4x + 7 \end{aligned}$$

Gradient of a curve, and angle of inclination of a tangent

The *gradient of a curve* at a particular point P on it is defined to be the gradient of the tangent at P — the gradient of the curve at P is thus the value of the derivative at P.

The steepness of a tangent can be expressed either by giving its gradient, or by giving its angle of inclination. As we saw in Chapter 7, negative gradients correspond to obtuse angles of inclination.

8 GRADIENT OF A CURVE, AND ANGLE OF INCLINATION OF A TANGENT

- The *gradient of a curve* at a point P on it is the gradient of the tangent at P.
- The steepness of a tangent can also be expressed by its angle of inclination.

Example 7 9C

a Differentiate $f(x) = x^2 + 2x$.
b Find the gradient and angle of inclination of the curve at the origin.
c Find the gradient and angle of inclination of the curve at the point $A(-2, 0)$.
d Sketch the curve and the tangents, marking their angles of inclination.

SOLUTION

a Differentiating $f(x) = x^2 + 2x$ gives $f'(x) = 2x + 2$.

b At the origin $O(0, 0)$, gradient of tangent $= f'(0) = 2$.

Let α be the angle of inclination of the tangent.

Then $\tan \alpha = 2$, where $0° \le \alpha < 180°$,

so $\alpha \doteqdot 63°26'$.

c At $A(-2, 0)$, gradient of tangent
$$\begin{aligned} &= f'(-2) \\ &= -4 + 2 \\ &= -2. \end{aligned}$$

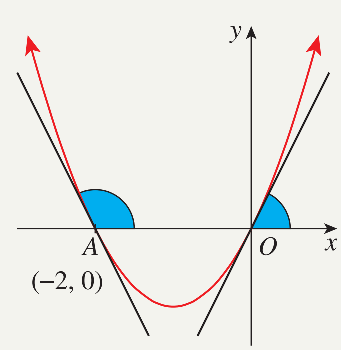
Let β be the angle of inclination of the tangent.

Then $\tan \beta = -2$, where $0° \le \beta < 180°$,

so
$$\begin{aligned} \beta &\doteqdot 180° - 63°26' \\ &\doteqdot 116°34'. \end{aligned}$$

Method: Enter 2 into the calculator, take $\tan^{-1} 2$, then take its supplement.

d



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Finding points on a curve with a given angle of inclination

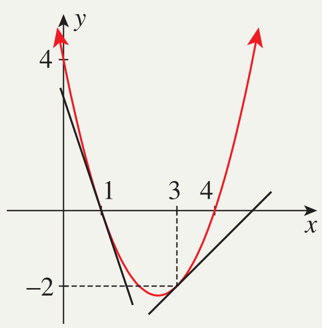
When the angle of inclination is given, use the tan θ function to find the gradient.

Example 8 9C

Find the points on the graph of $f(x) = x^2 - 5x + 4$ where:

a the tangent has gradient -3,

b the tangent has angle of inclination $45°$.



SOLUTION

Here $f(x) = x^2 - 5x + 4$,
so $f'(x) = 2x - 5$.

a Put $f'(x) = -3$.
Then $2x - 5 = -3$
$x = 1$.
Substituting $x = 1$ into the function,
$f(1) = 1 - 5 + 4$
$= 0$,
so the tangent has gradient -3 at the point $(1, 0)$.

b First, $\tan 45° = 1$,
so put $f'(x) = 1$.
Then $2x - 5 = 1$
$x = 3$.
Substituting $x = 3$ into the function,
$f(3) = 9 - 15 + 4$
$= -2$,
so the tangent has angle of inclination $45°$ at the point $(3, -2)$.

Tangents and normals to a curve

The *normal* to a curve $y = f(x)$ at a point P on it is the line through P perpendicular to the tangent. Its gradient m_1 is therefore found from the gradient m of the tangent using $mm_1 = -1$.

9 TANGENTS AND NORMALS TO A CURVE

- The *normal* at P is the line through P perpendicular to the tangent at P.
- The gradients m and m_1 of the tangent and the normal at P are related by $mm_1 = -1$
- Equations of tangents and normals are found using gradient–intercept form,
  $y - y_1 = m(x - x_1)$

Example 9 9C

For the curve $y = f(x)$, where $f(x) = (x + 1)^2$:

a Find the equation and angle of inclination of the tangent at $A(0, 1)$.

b Find the gradient and angle of inclination of the normal at A, and draw a sketch.

c Find the x-intercepts of the tangent and normal by finding their equations.

d Find the area of the triangle formed by the tangent, normal and x-axis.

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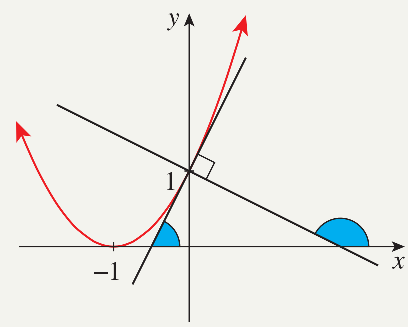
SOLUTION



a Expanding, $f(x) = x^2 + 2x + 1$,
so $f'(x) = 2x + 2$.
At $A(0, 1)$ $f'(0) = 2$,
so the tangent at $A(0, 1)$ is $y - 1 = 2(x - 0)$
$y = 2x + 1$.
Solving $\tan\alpha = 2$, where $0° \le \alpha < 180°$,
its angle of inclination is about $63°26$.

b The normal has gradient $-\frac{1}{2}$ (take the opposite of the reciprocal)
so the normal at $A(0, 1)$ is $y - 1 = -\frac{1}{2}(x - 0)$

$y = -\frac{1}{2}x + 1$.

Solving $\tan\beta = -\frac{1}{2}$, where $0° \le \beta < 180°$,

its angle of inclination is about $153°26$.
(The two angles differ by $90°$, by the exterior-angle-of-a-triangle theorem.)

c The tangent is $y = 2x + 1$ (using gradient–intercept form)
and substituting $y = 0$ gives $x = -\frac{1}{2}$.

The normal is $y = -\frac{1}{2}x + 1$

and substituting $y = 0$ gives $x = 2$.

d Hence area of triangle $= \frac{1}{2} \times 2\frac{1}{2} \times 1$
$= 1\frac{1}{4}$ square units.

Exercise 9C

FOUNDATION

Note: The next seven exercises, Exercises 9C–9I, contain exhaustive practice of differentiation skills, which are essential for most of what follows in the course. As is the case throughout the book, the exercises are there to support readers learning the relevant skills — once those skills are mastered, it is not necessary to do all the questions.

1 Use the rule for differentiating x^n to differentiate these functions.

a $f(x) = x^7$
b $f(x) = 9x^5$
c $f(x) = \frac{1}{3}x^6$
d $f(x) = 3x^2 - 5x$
e $f(x) = x^4 + x^3 + x^2 + x + 1$
f $f(x) = 2 - 3x - 5x^3$
g $f(x) = \frac{1}{3}x^6 - \frac{1}{2}x^4 + x^2 - 2$
h $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 1$

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2 Differentiate these functions by first expanding the products.

a $f(x) = 3x(4 - 2x)$ **b** $f(x) = x(x^2 + 1)$ **c** $f(x) = x^2(3 - 2x - 4x^2)$

d $f(x) = (x + 4)(x - 2)$ **e** $f(x) = (2x + 1)(2x - 1)$ **f** $f(x) = (x - 7)^2$

g $f(x) = (x^2 + 3)^2$ **h** $f(x) = x(7 - x)^2$ **i** $f(x) = (x^2 + 3)(x - 5)$

3 Differentiate these functions, given that a, b, c, k and ℓ are constants.

a $f(x) = ax^4 - bx^2 + c$ **b** $f(x) = (ax - 5)^2$

c $f(x) = k(ax + b)(ax - b)$ **d** $f(x) = x^{\ell}$

e $f(x) = x^{5a+1}$ **f** $f(x) = bx^{3b}$

4 Find the gradients of the tangent and normal at the point on $y = f(x)$ where $x = 3$.

a $f(x) = x^2 - 5x + 2$ **b** $f(x) = x^3 - 3x^2 - 10x$ **c** $f(x) = 2x^2 - 18x$

Then find the angles of inclination of these tangents and normals.

5 Find the equations of the tangent and normal to the graph of $f(x) = x^2 - 8x + 15$ at:

a $A(1, 8)$ **b** $B(6, 3)$ **c** the y-intercept **d** $C(4, -1)$

6 Find any points on the graph of each function where the tangent is parallel to the x-axis, and write down the equation of each horizontal tangent.

a $f(x) = 4 + 4x - x^2$ **b** $f(x) = x^3 - 12x + 24$

c $f(x) = 3x + 2$ **d** $f(x) = 4ax - x^2$

e $f(x) = x^4 - 2x^2$ **f** $f(x) = x^5 + x$

7 Differentiate $f(x) = x^3$. Hence show that the tangents to $y = x^3$ have positive gradient everywhere except at the origin, and show that the tangent there is horizontal. Explain the situation using a sketch.

DEVELOPMENT

8 Show that the line $y = 3$ meets the parabola $y = 4 - x^2$ at $D(1, 3)$ and $E(-1, 3)$. Find the equations of the tangents to $y = 4 - x^2$ at D and E, and find the point where these tangents intersect. Sketch the situation.

9 Find the equation of the tangent to $f(x) = 10x - x^3$ at the point $P(2, 12)$. Then find the points A and B where the tangent meets the x-axis and y-axis respectively, and find the length of AB and the area of $\triangle OAB$.

10 The tangent and normal to $f(x) = 9 - x^2$ at the point $K(1, 8)$ meet the x-axis at A and B respectively. Sketch the situation, find the equations of the tangent and normal, find the coordinates of A and B, and hence find the length AB and the area of $\triangle AKB$.

11 The tangent and normal to the cubic $f(x) = x^3$ at the point $U(1, 1)$ meet the y-axis at P and Q respectively. Sketch the situation and find the equations of the tangent and normal. Find the coordinates of P and Q, and the area of $\triangle QUP$.

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12 **a** Show that the tangents at the x-intercepts of $f(x) = x^2 - 4x - 45$ have opposite gradients.
b Find the vertex of the parabola, and use symmetry to explain geometrically why the result in part **a** has occurred.

13 Find the derivative of the cubic $f(x) = x^3 + ax + b$, and hence find the x-coordinates of the points where the tangent is horizontal. For what values of a and b do such points exist?

14 [Change of pronumeral]
a Find $G'(t)$ and $G'(3)$, if $G(t) = t^3 - 4t^2 + 6t - 27$.
b Given that $\ell(h) = 5h^4$, find $\ell'(h)$ and $\ell'(2)$.
c If $Q(k) = ak^2 - a^2k$, where a is a constant, find:
i $Q'(k)$ **ii** $Q'(a)$ **iii** $Q'(0)$ **iv** $|Q'(0) - Q'(a)|$

15 Sketch the graph of $f(x) = x^2 - 6x$ and find the gradient of the tangent and normal at the point $A(a, a^2 - 6a)$ on the curve. Hence find the value of a if:
a the tangent has gradient 2,
b the normal has gradient 4,
c the tangent has angle of inclination 135°,
d the normal has angle of inclination 30°,
e the tangent is parallel to $2x - 3y + 4 = 0$,
f the normal is parallel to $2x - 3y + 4 = 0$.

16 Show that the derivative of $f(x) = x^2 - ax$ is $f'(x) = 2x - a$, and find the value of a when:
a the tangent at the origin has gradient 7,
b $y = f(x)$ has a horizontal tangent at $x = 3$,
c the tangent at the point where $x = 1$ has an angle of inclination of 45°,
d the tangent at the non-zero x-intercept has gradient 5,
e the tangent at the vertex has y-intercept -9.

17 **a** The tangents to $y = x^2$ at two points $A(a, a^2)$ and $B(b, b^2)$ on the curve meet at K. Find the coordinates of K.
b When is K above the x-axis? Explain algebraically and geometrically.

ENRICHMENT

18 Let n be the number of horizontal tangents to the cubic $y = ax^3 + bx^2 + cx + d$. Find the condition on a, b, c and d for:
a $n = 2$ **b** $n = 1$ **c** $n = 0$.

19 Another formula for the derivative is $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h}$. Draw a diagram to justify this formula, then use it to find the derivatives of x^2 and x^3.

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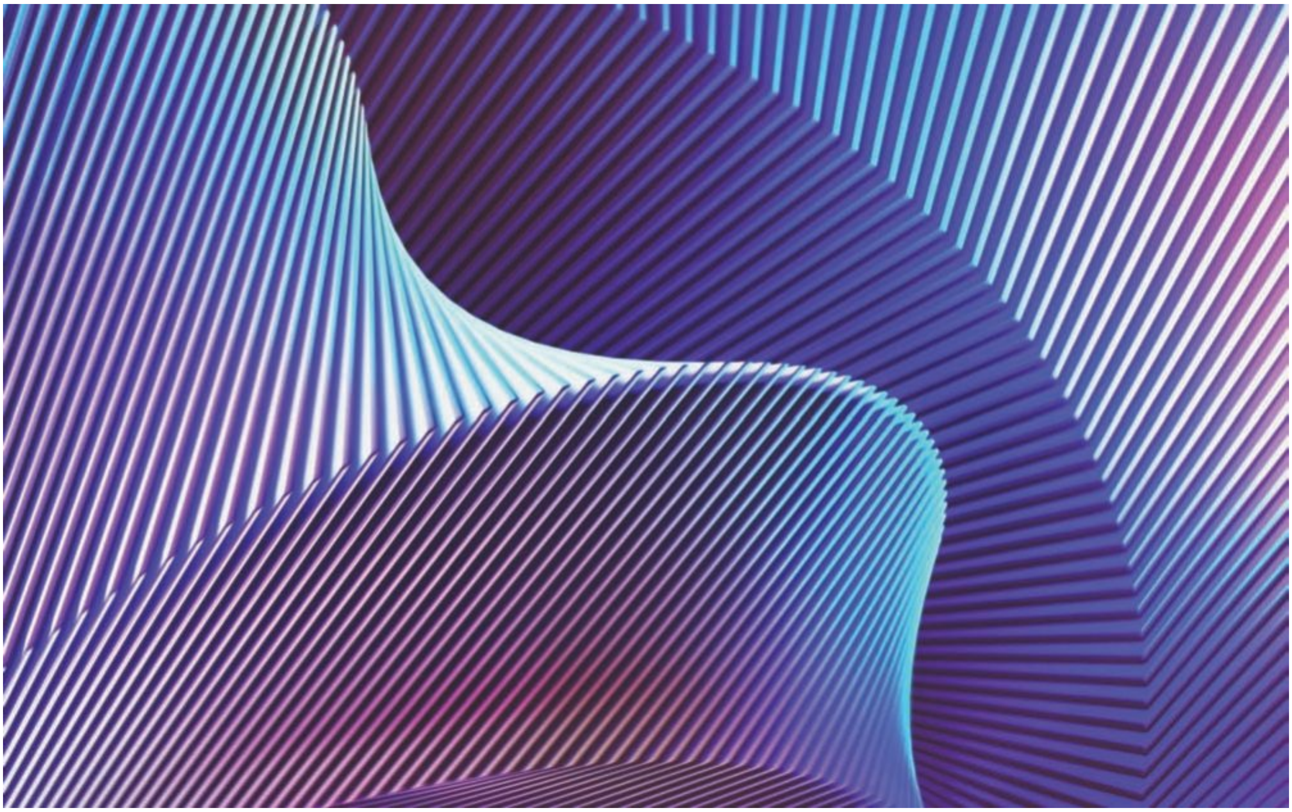
20 **a** Write down the equation of the tangent to the parabola $y = ax^2 + bx + c$ (where $a \neq 0$) at the point P where $x = t$, and show that the condition for the tangent at P to pass through the origin is $at^2 - c = 0$. Hence find the condition on a, b and c for such tangents to exist, and the equations of these tangents.

b Find the points A and B where the tangents from the origin touch the curve, and show that the y-intercept $C(0, c)$ is the midpoint of the interval joining the origin and the midpoint of the chord AB. Show also that the tangent at the y-intercept C is parallel to the chord AB.

c Hence show that the triangle $\triangle OAB$ has four times the area of the triangle $\triangle OCA$, and find the area of $\triangle OAB$.



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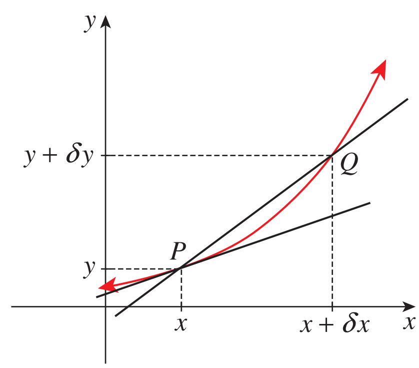
9D The notation $\frac{dy}{dx}$ for the derivative

Leibniz's original notation for the derivative remains the most widely used and best-known notation, particularly in science. It is even said that Dee Why Beach was named after the derivative $\frac{dy}{dx}$. The notation is extremely flexible, and clearly expresses the fact that the derivative behaves very much like a fraction.

Small changes in *x* and in *y*

The diagram used to explain $\frac{dy}{dx}$ is exactly the same as for $f'(x)$, but with different notation.



Let $P(x, y)$ be any point on the graph of a function.

Let x change by a small amount δx to $x + \delta x$,

and let y change by a corresponding amount δy to $y + \delta y$.

Thus $Q(x + \delta x, y + \delta y)$ is another point on the curve, and

$$\text{gradient } PQ = \frac{\delta y}{\delta x} \quad \text{(rise over run).}$$

When δx is small, the secant PQ is almost the same as the tangent at P, and the derivative is the limit of $\frac{\delta y}{\delta x}$ as Q gets 'as close as we like' to P, that is, as $\delta x \to 0$.

This is the basis for Leibniz's notation:

10 AN ALTERNATIVE NOTATION FOR THE DERIVATIVE

Let δy be the small change in y resulting from a small change δx in x. Then

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

The object dx is intuitively understood as an 'infinitesimal change' in x, and dy is understood as the corresponding 'infinitesimal change' in y. The derivative $\frac{dy}{dx}$ is then understood as the ratio of these two infinitesimal changes. 'Infinitesimal changes', however, are for the intuition only — the logic of the situation is:

11 FRACTION NOTATION AND THE DERIVATIVE $\frac{dy}{dx}$

The derivative $\frac{dy}{dx}$ is not a fraction, but is the limit of the fraction $\frac{\delta y}{\delta x}$ as $\delta x \to 0$.

The notation is very clever because the derivative is a gradient, the gradient is a fraction $\frac{\text{rise}}{\text{run}}$, and the notation $\frac{dy}{dx}$ preserves the intuition of fractions.

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The words 'differentiation' and 'calculus'

The *small differences* δx and δy, and the *infinitesimal differences* dx and dy, are the origins of the word 'differentiation'. The symbol δ is the lower-case Greek letter 'delta', standing for difference.

'Calculus' is a Latin word meaning 'stone'. An *abacus* consists of *stones* sliding on bars and was once used routinely to help with arithmetic — this is the origin of the modern word 'calculate'. The word 'calculus' in English can refer to any systematic method of calculation, but is most often used for the twin theories of differentiation and integration studied in this course.

Second and higher derivatives

The derivative of a function is another function, so it can in turn be differentiated to give the *second derivative* of the original function. This is written as $f''(x)$ in $f(x)$ notation, and as $\dfrac{d^2y}{dx^2}$ in the new $\dfrac{dy}{dx}$ notation.

Thus for a function such as $x^5 + 5x^3$, the two notations are:

$$y = x^4 + 5x^3 \qquad\qquad f(x) = x^4 + 5x^3$$

$$\frac{dy}{dx} = 4x^3 + 15x^2 \qquad\qquad f'(x) = 4x^3 + 15x^2$$

$$\frac{d^2y}{dx^2} = 12x^2 + 30x \qquad\qquad f''(x) = 12x^2 + 30x$$

This process can be continued indefinitely to *third and higher derivatives*:

$$\frac{d^3y}{dx^3} = 24x + 30 \qquad\qquad f'''(x) = 24x + 30$$

$$\frac{d^4y}{dx^4} = 24x \qquad\qquad f^{(4)}(x) = 24$$

and after that, the fifth and higher derivatives are all zero. Notice that to avoid counting multiple dashes, we usually write $f^{(4)}(x)$ rather than $f''''(x)$.

The second derivative $f''(x)$ is the *gradient function of the gradient function*. Next year, we will interpret the second derivative in terms of curvature, but there are no simple geometric interpretations of higher derivatives.

Operator notation

The derivative $\dfrac{dy}{dx}$ can also be regarded as the *operator* $\dfrac{d}{dx}$ operating on the function y. This gives an alternative notation for the derivative:

$$\frac{d}{dx}\left(x^5\right) = 5x^4 \qquad \text{and} \qquad \frac{d}{dx}\left(x^2 + x - 1\right) = 2x + 1.$$

(An *operator* acts on a function to give another function. It is thus a function whose domain and range are sets of functions rather than sets of numbers.)

Thus the standard rule for differentiating powers of x can be written as

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}, \text{ for all real numbers } n.$$

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Setting out using $\frac{dy}{dx}$ notation

Examples 10–12 show how the new notation is used in calculations on the geometry of a curve.

Example 10 9D

Find the equations of the tangent and normal to the curve $y = 4 - x^2$ at the point $P(1, 3)$ on the curve.

SOLUTION

Differentiating, $\frac{dy}{dx} = -2x$,

so at $P(1, 3)$, $\frac{dy}{dx} = -2$ (substitute $x = -1$ into the derivative)

so the tangent at P has gradient -2 and the normal has gradient $\frac{1}{2}$.

Hence the tangent is $y - 3 = -2(x - 1)$

$y = -2x + 5$,

and the normal is $y - 3 = \frac{1}{2}(x - 1)$

$y = \frac{1}{2}x + 2\frac{1}{2}$.

Finding the tangents passing through some fixed point

Sometimes we want to find the equation of a tangent to a curve $y = f(x)$ passing through a fixed point F in the plane. This problem can be solved by finding the equation of the tangent at a variable point $P\big(t, f(t)\big)$ on the curve, then substituting the coordinates of F into this equation. The method can be applied to other similar problems.

Example 11 9D

a Find the equation of the tangent to $y = x^2 + x + 1$ at the point P on the curve where $x = t$.

b Hence find the equations of the tangents to the curve passing through the origin, and the corresponding point of contact of each tangent.

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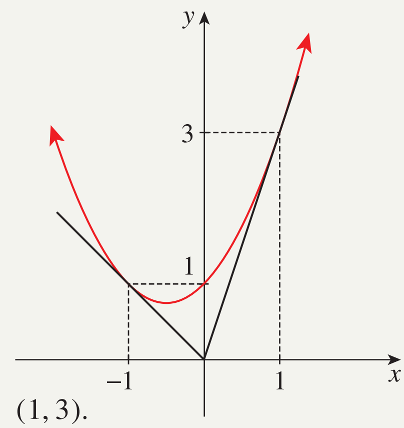
SOLUTION



a Differentiating, $\dfrac{dy}{dx} = 2x + 1,$

so at P, $\dfrac{dy}{dx} = 2t + 1.$

The coordinates of P are $(t, t^2 + t + 1)$,

so the tangent is $y - (t^2 + t + 1) = (2t + 1)(x - t)$

$$y = (2t + 1)x - t^2 + 1.$$

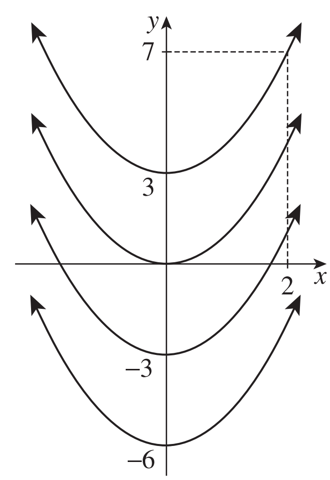
b Substituting $(0, 0)$, $0 = -t^2 + 1$

$$t = 1 \text{ or } -1.$$

When $t = 1$, the tangent is $y = 3x$ and the point of contact is $P = (1, 3)$.

When $t = -1$, the tangent is $y = -x$ and the point of contact is $P = (-1, 1)$.

Families of curves with the same derivative



The four functions $y = x^2$, $y = x^2 + 3$, $y = x^2 - 3$ and $y = x^2 - 6$ are all seen to be related when they are differentiated. In all four cases,

$$\frac{dy}{dx} = 2x$$

because the derivative of any constant term is zero. More generally, we can prove that the curves with derivative $2x$ form an infinite family of curves

$$y = x^2 + C, \text{ where } C \text{ is a constant.}$$

First, all the curves in this family have derivative $2x$.

Conversely, if some function y has derivative $\dfrac{dy}{dx} = 2x$, then

$$\frac{d}{dx}(y - x^2) = 2x - 2x = 0,$$

so $y - x^2$ is a constant function C, because its gradient is always zero.

That is, $y - x^2 = C,$

$$y = x^2 + C.$$

The diagram above shows four members of this family. Notice that no two curves in the family ever intersect. Notice also that we can pick out the curve that goes through the particular point $(2, 7)$, because a simple substitution shows that this curve is $y = x^2 + 3$.

12 FAMILIES OF CURVES WITH THE SAME DERIVATIVE

The functions in a *family of curves with the same derivative* all differ by a constant.

For example, the family of curves with derivative $\dfrac{dy}{dx} = 2x$ is

$$y = x^2 + C, \text{ for some constant } C.$$

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These remarks are the beginning of the very large subject of *integration*, which is the reverse process of *differentiation*, and will be a major concern in Year 12. This is not the place to formalise any rules — the table below contains all the forms that will be used this year. Just check that each entry in the second line below is the family of curves whose derivative is the entry above it.

$\frac{dy}{dx} = 1$	$\frac{dy}{dx} = x$	$\frac{dy}{dx} = x^2$	$\frac{dy}{dx} = x^3$	$\frac{dy}{dx} = x^4$
$y = x + C$	$y = \frac{x^2}{2} + C$	$y = \frac{x^3}{3} + C$	$y = \frac{x^4}{4} + C$	$y = \frac{x^5}{5} + C$

Example 12 — 9D

Use the table above to find the family of curves whose derivative is:

a $\frac{dy}{dx} = 9x^2 - 5$ **b** $\frac{dy}{dx} = x^3 + 6x$

SOLUTION

a Given that $\frac{dy}{dx} = 9x^2 - 5$

it follows that $y = \frac{9x^3}{3} - 5x + C,$ for some constant C

$y = 3x^3 - 5x + C.$ (Check by differentiating.)

b Given that $\frac{dy}{dx} = x^3 + 6x$

it follows that $y = \frac{x^4}{4} + \frac{6x^2}{2} + C,$ for some constant C

$y = \frac{x^4}{4} + 3x^2 + C.$ (Check by differentiating.)

Exercise 9D

FOUNDATION

1 Find the derivative $\frac{dy}{dx}$ of each function, and the value of $\frac{dy}{dx}$ when $x = -1$.

a $y = x^3 + 3x^2 + 6x + 8$ **b** $y = x^4 + x^2 + 8x$

c $y = 7$ **d** $y = (2x - 1)(x - 2)$

2 **a** Using $\frac{dy}{dx}$ notation, find the first, second and third derivatives of each function.

i $y = x^6 + 2x$ **ii** $y = 5x^2 - x^5$ **iii** $y = 4x$

b Using $f(x)$ notation, find the 1st, 2nd, 3rd and 4th derivatives of each function.

i $f(x) = 10x^3 + x$ **ii** $f(x) = 2x^4 + 2$ **iii** $f(x) = 5$

c **i** How many times must $y = x^5$ be differentiated to give the zero function?

ii How many times must $y = x^n$ be differentiated to give the zero function?

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3 Find any points on each graph where the tangent has gradient -1.

a $y = x^4 - 5x$ **b** $y = x^3 - 4x$ **c** $y = x^3 - 4$

4 Find the x-coordinates of any points on each curve where the normal is vertical.

a $y = 3 - 2x + x^2$ **b** $y = x^4 - 18x^2$ **c** $y = x^3 + x$

5 In each function, first divide through by the denominator, then differentiate. Then factor the derivative and state any values of x where the tangent is horizontal.

a $y = \dfrac{3x^4 - 9x^2}{x}$ **b** $y = \dfrac{5x^6 + 3x^5}{3x^3}$

6 Find the tangent and normal to each curve at the point indicated.

a $y = x^2 - 6x$ at $O(0, 0)$ **b** $y = x^2 - x^4$ at $J(-1, 0)$ **c** $y = x^3 - 3x + 2$ at $P(1, 0)$

7 **a** Show that each function below has the same derivative.

$y = x^3 + 7x + 4$, $y = x^3 + 7x - 6$, $y = x^3 + 7x - 3\frac{1}{2}$, $y = x^3 + 7x + 100$

What transformations map the first curve to each of the other three?

b Which function below is not in the same family as the other three?

$y = \frac{1}{2}x^4 + 3x^2 + 6$, $y = \frac{1}{2}x^4 - 7 + 3x^2$, $y = 3x^2 - \frac{1}{2}x^4 + 1$, $y = 3x^2 + \frac{1}{2}x^4 - 1$

DEVELOPMENT

8 Find, correct to the nearest minute, the angles of inclination of the tangents to the parabola $y = 1 - x^2$ at its x-intercepts.

9 Find any points on each curve where the tangent has the given angle of inclination.

a $y = \frac{1}{3}x^3 - 7$, $45°$ **b** $y = x^2 + \frac{1}{3}x^3$, $135°$ **c** $y = x^2 + 1$, $120°$

10 **a** Find the equation of the tangent to $y = x^2 + 9$ at the point $P(p, p^2 + 9)$.

b Hence find the points on the curve where the tangents pass through the origin, and the equation of those tangents. Draw a sketch of the situation.

11 For each curve below, find the equation of the tangent at the point P where $x = t$. Hence find the equations of any tangents passing through the origin.

a $y = x^2 - 10x + 9$ **b** $y = x^2 + 15x + 36$

12 **a** Find the equation of the tangent to $y = x^2 + 2x - 8$ at the point K on the curve with x-coordinate t.

b Hence find the points on the curve where the tangents from $H(2, -1)$ touch the curve.

13 Find the families of curves with derivative:

a $\dfrac{dy}{dx} = x + x^2$ **b** $\dfrac{dy}{dx} = 6x^2 - 7$

c $\dfrac{dy}{dx} = 5x^3 + 3x^2 - 4$ **d** $\dfrac{dy}{dx} = 10x^4 - 12x^2 - 24$

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14 Differentiate $y = x^2 + bx + c$, and hence find b and c given that:

a the parabola passes through the origin, and the tangent there has gradient 7.

b the parabola has y-intercept -3 and gradient -2 there,

c the parabola is tangent to the x-axis at the point $(5, 0)$,

d when $x = 3$ the gradient is 5, and $x = 2$ is a zero,

e the parabola is tangent to $3x + y - 5 = 0$ at the point $T(3, -4)$,

f the line $3x + y - 5 = 0$ is a normal at the point $T(3, -4)$.

15 Find the tangent to the curve $y = x^4 - 4x^3 + 4x^2 + x$ at the origin, and show that this line is also the tangent to the curve at the point $(2, 2)$.

16 Find the points where the line $x + 2y = 4$ cuts the parabola $y = (x - 1)^2$. Show that the line is the normal to the curve at one of these points.

17 **a** The tangent at $T(a, a^2)$ to $y = x^2$ meets the x-axis at U and the y-axis at V. Find the equation of this tangent, and show that $\triangle OUV$ has area $\left|\frac{1}{4}a^3\right|$ square units.

b Hence find the coordinates of T for which this area is $31\frac{1}{4}$.

18 Show that the line $x + y + 2 = 0$ is a tangent to $y = x^3 - 4x$, and find the point of contact. (Hint: Find the equations of the tangents parallel to $x + y + 2 = 0$, and show that one of them is this very line.)

19 **a** Find the points A and B where $y = -2x$ meets the parabola $y = (x + 2)(x - 3)$.

b Find the midpoint M of the chord AB, find the point T where the vertical line through M meets the parabola, and show that the tangent at T is parallel to the chord AB.

20 If $P = tx^2 + 3tu^2 + 3xu + t$, find $\frac{dP}{dx}$, $\frac{dP}{du}$ and $\frac{dP}{dt}$ (assuming that when differentiating with respect to one variable, the other pronumerals are constant).

21 [For discussion]
Sketch the graph of $y = x^3$. Then choose any point in the plane and check by examining the graph that at least one tangent to the curve passes through every point in the plane. What points in the plane have three tangents to the curve passing through them? This problem can also be solved algebraically, but that is considerably harder.

ENRICHMENT

22 **a** Show that the tangent to $y = ax^2 + bx + c$ with gradient m has y-intercept $c - \frac{(m - b)^2}{4a}$.

b Hence find the equations of any quadratics that pass through the origin and are tangent to both $y = -2x - 4$ and to $y = 8x - 49$.

c Find also any quadratics that are tangent to $y = -5x - 10$, to $y = -3x - 7$ and to $y = x - 7$.

23 Let $y = ax^3 + bx^2 + cx + d$ be a cubic (so that $a \neq 0$).

a Write down the equation of the tangent at the point T on the curve where $x = t$.

b Show that every point $P(x_0, y_0)$ in the plane lies on at least one tangent to the cubic.

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9E The chain rule

In Section 9C we showed that the derivative of a *sum* or *difference* of functions is the sum or difference of the derivatives, and that the derivative of a *multiple* of a function is the multiple of the derivative. These were easy rules. Differentiating the *composite* or *product* or *quotient* of two functions, however, is more complicated. Sections 9E, 9H and 9I develop rules for differentiating such compound functions.

Composition of functions — a chain of functions

We begin by reviewing composite functions from Section 4E. They are formed by putting two functions into a *chain* so that the output of the first function becomes the input of the second.

We now want to apply the process in reverse. For example, the semicircle function $y = \sqrt{25 - x^2}$ can be decomposed into a chain of two functions — 'square and subtract from 25', followed by 'take the positive square root'.

x				u				y
0	⟶	Square and subtract from 25	⟶	25	⟶	Take the positive square root	⟶	5
3	⟶		⟶	16	⟶		⟶	4
-4	⟶		⟶	9	⟶		⟶	3
x	⟶		⟶	$25 - x^2$	⟶		⟶	$\sqrt{25 - x^2}$

The middle column is the output of the first function 'Square and subtract from 25'. This output is then the input of the second function 'Take the positive square root'. The resulting decomposition of the original function $y = \sqrt{25 - x^2}$ into the chain of two functions may be expressed as follows:

'Let $u = 25 - x^2$. Then $y = \sqrt{u}$.'

The resulting function is called the *composite* of the two functions. We met such composite functions first in Section 4E, but this is the first time that we have reversed the process and decomposed a function into two parts.

The chain rule — differentiating a composite function

Suppose then that y is a function of u, where u is a function of x.

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x}\\ &= \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}\right) && \text{(multiplying top and bottom by } \delta u)\\ &= \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} && \text{(because } \delta u \to 0 \text{ as } \delta x \to 0)\\ &= \frac{dy}{du} \times \frac{du}{dx}.\end{aligned}$$

Although the proof uses limits, the usual attitude to this rule is that 'the du's cancel out'. The chain rule should be remembered in this form:

13 THE CHAIN RULE

Suppose that y is a function of u, where u is a function of x. Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

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Example 13 **9E**

Use the chain rule to differentiate each function.

a $(x^2 + 1)^6$

b $7(3x + 4)^5$

Note: The working in the right-hand column below is the recommended setting out of the calculation. The calculation should begin with that working, because the first step is the decomposition of the function into a chain of two functions.

SOLUTION

a Let $y = (x^2 + 1)^6$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 6(x^2 + 1)^5 \times 2x$

$= 12x(x^2 + 1)^5$.

Let $u = x^2 + 1$.

Then $y = u^6$.

Hence $\frac{du}{dx} = 2x$

and $\frac{dy}{du} = 6u^5$.

b Let $y = 7(3x + 4)^5$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 35(3x + 4)^4 \times 3$

$= 105(3x + 4)^4$.

Let $u = 3x + 4$.

Then $y = 7u^5$.

Hence $\frac{du}{dx} = 3$

and $\frac{dy}{du} = 35u^4$.

Powers of a linear function

Here is the derivative of $(ax + b)^n$.

Let $y = (ax + b)^n$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= n(ax + b)^{n-1} \times a$

$= an(ax + b)^{n-1}$.

Let $u = ax + b$.

Then $y = u^n$.

Hence $\frac{du}{dx} = a$

and $\frac{dy}{du} = nu^{n-1}$.

This result occurs so often that it should be remembered as a formula for differentiating any linear function of x raised to a power.

14 POWERS OF A LINEAR FUNCTION

$$\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$$

This formula is an extension of the standard form $\frac{d}{dx}(x^n) = nx^{n-1}$, which has been proven so far for whole numbers n, for $n = -1$, and for $n = \frac{1}{2}$.

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Example 14 — 9E

Use the formula $\dfrac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$ to differentiate each function.

a $y = (4x-1)^7$

b $y = 3(7-4x)^5$

SOLUTION

a $y = (4x-1)^7$

Here $a = 4$, $b = -1$ and $n = 7$.

$$\frac{dy}{dx} = 4 \times 7 \times (4x-1)^6$$
$$= 28(4x-1)^6$$

b $y = 3(7-4x)^5$

Here $a = -4$, $b = 7$ and $n = 5$.

$$\frac{dy}{dx} = 3 \times (-4) \times 5 \times (7-4x)^4$$
$$= -60(7-4x)^4$$

A shorter setting-out

It is important to practise the full setting-out — first, because that is the best way to handle many tricky calculations, and secondly, because the process will be reversed in Year 12 and must be clearly understood.

Many people like to shorten the setting-out. If so, it is safer to write down at least the function u on the right, and take at least one middle step in the working.

Example 15 — 9E

Differentiate $y = (3-5x^3)^7$ with a shorter setting-out.

SOLUTION

$$y = (3-5x^3)^7$$
$$\frac{dy}{dx} = 7(3-5x^3)^6 \times (-15x^2)$$
$$= -105x^2(3-5x^3)^6$$

Let $u = 3 - 5x^3$.
(further steps if it gets tricky)

Another approach to shortening the setting-out is to remember a new standard form for differentiating a power of a function:

$$\frac{d}{dx}\big(f(x)\big)^n = n\big(f(x)\big)^{n-1}f'(x)$$

Again, it is probably safer to write down the function $f(x)$ on the right.

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Example 16 9E

Differentiate $y = (3 - 5x^3)^7$ using the standard form above.

SOLUTION

$y = (3 - 5x^3)^7$

$\frac{dy}{dx} = 7(3 - 5x^3)^6 \times (-15x^2)$

$= -105x^2(3 - 5x^3)^6$

Let $f(x) = 3 - 5x^3$.
(further steps if it gets tricky)

The chain rule and tangents

The usual methods of dealing with tangents apply when the chain rule is used to find the derivative.

Example 17 9E

Differentiate $y = (1 - x^4)^4$, and hence find the points on the curve where the tangent is horizontal.

SOLUTION

Let $y = (1 - x^4)^4$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 4(1 - x^4)^3 \times (-4x^3)$

$= -16x^3(1 - x^4)^3$,

which is zero when $x = -1, 0$ or 1,

so the points are $(-1, 0)$, $(0, 1)$ and $(1, 0)$.

Let $u = 1 - x^4$.
Then $y = u^4$.
Hence $\frac{du}{dx} = -4x^3$
and $\frac{dy}{du} = 4u^3$.

Parametric differentiation

We have seen in Section 5H how a curve can be specified *parametrically* by two equations giving x and y in terms of some third variable t, called a *parameter*. For example,

$$x = t^2, \quad y = 2t$$

specifies the parabola $y^2 = 4x$, as is shown in Example 18 by eliminating t from the two equations. In this situation it is very simple to calculate $\frac{dy}{dx}$ directly using *parametric differentiation*. The formula below is another version of the chain rule, because 'the dt's just cancel out'.

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15 PARAMETRIC FUNCTIONS

Suppose that the x- and y-coordinates of a curve are each functions of a parameter t. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example 18 9E

A curve is defined parametrically by $x = t^2$ and $y = 2t$.

a Find the derivative at the point $T(t^2, 2t)$ on the curve.

b Find the equation of the tangent at T.

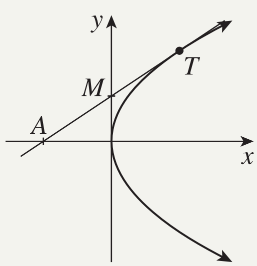
c Find the x-intercept A of the tangent at T.

d Find the midpoint M of AT. Why does M lie on the y-axis?

e Eliminate t to show that the curve is $y^2 = 4x$.

SOLUTION

a Using the chain rule,
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{2t} = \frac{1}{t}.$$



b The tangent at T is
$$y - 2t = \frac{1}{t}(x - t^2)$$
$$y = \frac{x}{t} + t.$$

c When $y = 0$, $x = -t^2$, so A is the point $(-t^2, 0)$.

d Taking averages, the midpoint of $T(t^2, 2t)$ and $A(-t^2, 0)$ is $M(0, t)$, which lies on the y-axis because its x-coordinate is zero.

e Substituting $t = \frac{1}{2}y$ into $x = t^2$

gives $x = \frac{1}{4}y^2$, that is, $y^2 = 4x$.

Differentiating inverse functions

Suppose that y is a function of x, and that the inverse relation is also a function, making x a function of y. Then by the chain rule,

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \quad \text{that is,} \quad \frac{dx}{dy} = \frac{1}{dy/dx}.$$

16 INVERSE FUNCTIONS

Suppose that y is a function of x, and x is a function of y. Then

$$\frac{dx}{dy} = \frac{1}{dy/dx} \quad \text{(provided that neither is zero).}$$

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Example 19 — 9E

Differentiate $y = \sqrt{x}$ by forming the inverse function and differentiating it.

SOLUTION

Solving for x, $\quad x = y^2.$

Differentiating, $\quad \dfrac{dx}{dy} = 2y,$

and taking reciprocals, $\quad \dfrac{dy}{dx} = \dfrac{1}{2y}$

$$= \frac{1}{2\sqrt{x}}.$$

Exercise 9E — FOUNDATION

Note: Use the standard form $\dfrac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ when the function is a power of a linear function. Otherwise use the full setting-out of the chain rule, or use a shorter form if you are quite confident to do so.

1 Copy and complete the setting out below to differentiate $(x^2 + 9)^5$ by the chain rule.

Let $y = (x^2 + 9)^5$.	Let $u = x^2 + 9$.
Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$	Then $y = u^5$.
$= \cdots \times \cdots$	Hence $\dfrac{du}{dx} = \cdots$
$= \cdots.$	and $\dfrac{dy}{du} = \cdots.$

2 Use the chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ to differentiate each function. Use the full setting-out above, first identifying u as a function of x, and y as a function of u.

a $y = (3x + 7)^4$ b $y = (5 - 4x)^7$ c $y = (x^2 + 1)^{12}$
d $y = 8(7 - x^2)^4$ e $y = (x^2 + 3x + 1)^9$ f $y = -3(x^3 + x + 1)^6$

3 Use the standard form $\dfrac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ to differentiate:

a $y = (5x - 7)^5$ b $y = (7x + 3)^7$ c $y = 9(5x + 3)^4$
d $y = (4 - 3x)^7$ e $y = 6\left(\tfrac{1}{2}x - 1\right)^4$ f $y = \tfrac{2}{3}\left(5 - \tfrac{1}{3}x\right)^4$

4 Find the first six derivatives of $y = (5x - 2)^4$.

5 a Differentiate $y = (x - 3)^2 + 12$ by expanding the RHS and differentiating each term.

b Differentiate $y = (x - 3)^2 + 12$ using the standard form $\dfrac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$.

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6 Find the x-coordinates of any points on $y = (4x - 7)^3$ where the tangent is:

a parallel to $y = 108x + 7$

b perpendicular to $x + 12y + 6 = 0$

7 Find the equations of the tangent and normal to each curve at the point where $x = 1$.

a $y = (5x - 4)^4$

b $y = (x^2 + 1)^3$

8 Differentiate each function, and hence find any points where the tangent is horizontal.

a $y = 4 + (x - 5)^6$

b $y = 24 - 7(x - 5)^2$

c $y = a(x - h)^2 + k$

d $y = (x^2 - 1)^3$

e $y = (x^2 - 4x)^4$

f $y = (2x + x^2)^5$

DEVELOPMENT

9 Find the value of a if:

a $y = (x - a)^3$ has gradient 12 when $x = 6$.

b $y = (5x + a)^4$ has gradient -160 when $x = 3$.

10 Find the values of a and b if the parabola $y = a(x + b)^2 - 8$:

a has tangent $y = 2x$ at the point $P(4, 8)$,

b has a common tangent with $y = 2 - x^2$ at the point $A(1, 1)$.

11 a Find the x-coordinates of the points P and Q on $y = (x - 7)^2 + 3$ such that the tangents at P and Q have gradients 1 and -1 respectively.

b Show that the square formed by the tangents and normals at P and Q has area $\frac{1}{2}$.

12 a Show that the equation of the tangent to $y = (x - 2)^4 - 16$ at the point T where $x = t$ is $y = (t - 2)^4 + 4(t - 2)^3(x - t) - 16$.

b Show that if the tangent passes through $P\left(\frac{1}{2}, -16\right)$, then $(t - 2)^4 + 4(t - 2)^3\left(\frac{1}{2} - t\right) = 0$.

c Solve this equation to find the tangents to the curve through P.

d Use translations, and your knowledge of the graph $y = x^4$, to sketch the curve and the tangents.

13 Use parametric differentiation to find $\frac{dy}{dx}$ in terms of t, then evaluate $\frac{dy}{dx}$ when $t = -1$:

a $x = 5t$, $y = 10t^2$

b $x = at + b$, $y = bt + a$

c $x = 2t^2$, $y = 3t^3$

14 Find the tangent to each curve at the point where $t = 3$:

a $x = 5t^2, y = 10t$

b $x = (t - 1)^2, y = (t - 1)^3$

15 Use the chain rule to show that:

a $\frac{d}{dx}(x^2)^3 = 6x^5$

b $\frac{d}{dx}(x^k)^\ell = k\ell x^{k\ell - 1}$

16 Differentiate $y = \sqrt[3]{x}$ by forming the inverse function and differentiating it.

17 The function $y = f(x)$ has an inverse function $y = g(x)$. There is a tangent at the point $P(a, b)$ on $y = f(x)$, and that tangent is neither horizontal nor vertical. Explain why the tangent to $y = f(x)$ at $P(a, b)$, and the tangent to $y = g(x)$ at $Q(b, a)$, both have positive gradient, or both have negative gradient.

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ENRICHMENT

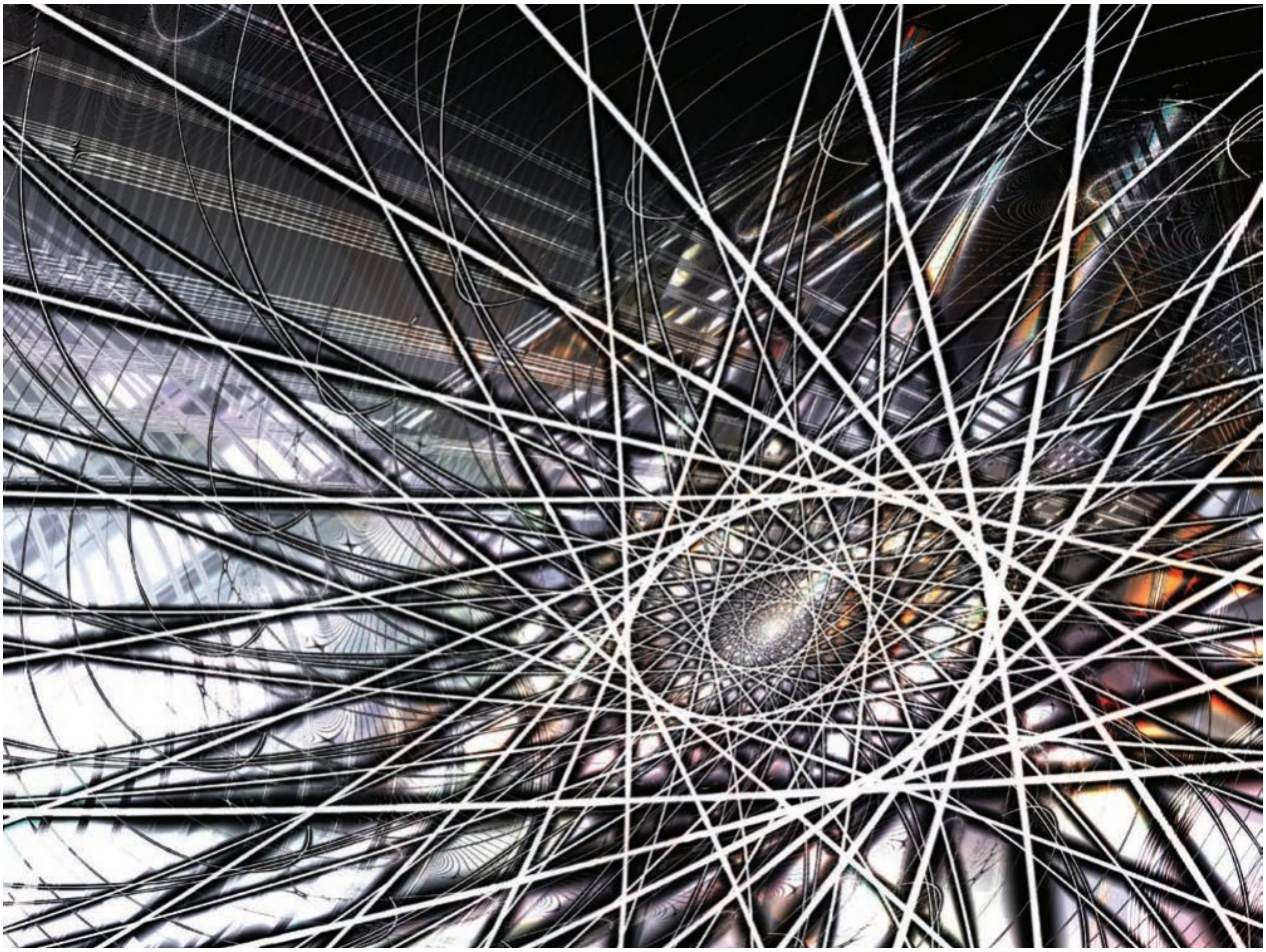
18 **a** Find the x-coordinates of the points P and Q on $y = (x - h)^2 + k$ such that the tangents at P and Q have gradients m and $-m$ respectively.

b Find the area of the quadrilateral formed by the tangents and normals at P and Q.

19 **a** Let T be the point on the parabola $y = a(x - h)^2 + k$ where $x = \alpha$. Show that the tangent at T has equation $y = 2a(\alpha - h)x + k - a(\alpha^2 - h^2)$.

b Let the vertical line through the vertex V meet the tangent at the point P. Show that VP is proportional to the square of the distance between $A(\alpha, 0)$ and the axis of symmetry.

c Use the equation in part **a** to find the equations of the tangents to the parabola through the origin, and the x-coordinates of the points of contact.



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9F Differentiating powers with negative indices

The chain rule allows us to extend the proof of the formula $\frac{d}{dx}(x^n) = nx^{n-1}$ to all rational indices n. This section deals with powers whose indices are negative integers, allowing us to differentiate functions such as $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$.

Proving the formula for $n = -1$

The first step is to use first-principles differentiation to prove the formula for the index $n = -1$, that is, for $y = \frac{1}{x}$. This was done earlier as Question 11 of Exercise 9B, but is repeated here for completeness.

Let $$f(x) = \frac{1}{x}$$

Then
$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)\\ &= \frac{x - (x+h)}{(x+h)xh} \quad \text{(with a common denominator)}\\ &= \frac{-h}{(x+h)xh}\\ &= -\frac{1}{(x+h)x}, \text{ provided that } h \neq 0.\end{aligned}$$

Taking the limit as $h \to 0$, $f'(x) = -\frac{1}{x^2}$.

This can be written as $-x^{-2}$, which is the result given by the formula.

Proving the formula for all negative integers

The formula can now be proven for all negative integers using the chain rule. The result for $n = -1$ is used at (*) in the fourth line on the right below.

Suppose then that $y = x^{-m}$, where $m \geq 2$ is an integer.

By the chain rule,
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= -\frac{1}{x^{2m}} \times mx^{m-1}\\ &= -mx^{-m-1}, \quad \text{as required.}\end{aligned}$$

Let $u = x^m$,

then $y = \frac{1}{u}$.

So $\frac{du}{dx} = mx^{m-1}$,

and $\frac{dy}{du} = -\frac{1}{u^2}$ (*)

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17 DIFFERENTIATING POWERS WITH NEGATIVE INDICES

- The rule $\dfrac{d}{dx}(x^n) = nx^{n-1}$ applies for all integers n. For example,

$$\frac{d}{dx}\left(x^{-4}\right) = -4x^{-5}.$$

- In particular, $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$.

The particular derivative of $\dfrac{1}{x}$ above is useful because hyperbolas occur so often. Readers may choose whether or not to memorise it as a special result.

Example 20 9F

Differentiate each function, giving answers in fraction form.

a $y = -5x^{-4}$. **b** $f(x) = \dfrac{5}{x^2}$ **c** $f(x) = \dfrac{1}{3x^5}$

SOLUTION

a
$$\begin{aligned} y &= -5x^{-4} \\ \frac{dy}{dx} &= 20x^{-5} \\ &= \frac{20}{x^5} \end{aligned}$$

b
$$\begin{aligned} f(x) &= \frac{5}{x^2} \\ &= 5x^{-2} \\ f'(x) &= -10x^{-3} \\ &= -\frac{10}{x^3} \end{aligned}$$

c
$$\begin{aligned} f(x) &= \frac{1}{3x^5} \\ &= \tfrac{1}{3}x^{-5} \\ f'(x) &= -\tfrac{5}{3}x^{-6} \\ &= -\frac{5}{3x^6} \end{aligned}$$

The chain rule and general indices

The chain rule can be combined with these results, as in the next worked example.

Example 21 9F

Differentiate $y = \dfrac{1}{4 - x^2}$.

SOLUTION

$$\begin{aligned} y &= \frac{1}{4 - x^2} \\ \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= -\frac{1}{(4 - x^2)^2} \times (-2x) \\ &= \frac{2x}{(4 - x^2)^2}. \end{aligned}$$

Let $u = 4 - x^2$.

Then $y = \dfrac{1}{u}$.

Hence $\dfrac{du}{dx} = -2x$

and $\dfrac{dy}{du} = -\dfrac{1}{u^2}$

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Dividing through by the denominator

If the denominator of a function is a single term, divide through by it first, and then differentiate.

Example 22 9F

In each function, first divide through by the denominator, and differentiate. Then find any points on the curve $y = f(x)$ where the tangent is horizontal.

a $f(x) = \dfrac{16 - 24x}{x^3}$ **b** $f(x) = \dfrac{5 - x^2 + 5x^4}{x^2}$

SOLUTION

a
$$\begin{aligned} f(x) &= \frac{16 - 24x}{x^3} \\ &= 16x^{-3} - 24x^{-2} \\ f'(x) &= -48x^{-4} + 48x^{-3} \\ &= -\frac{48}{x^4} + \frac{48}{x^3} \\ &= \frac{-48 + 48x}{x^4} \end{aligned}$$

Factoring, $f'(x) = \dfrac{48(x - 1)}{x^4}$

Hence $f'(x) = 0$ when $\dfrac{48(x - 1)}{x^4} = 0$

that is, when $x = 1$.

Substituting, $f(1) = 16 - 24 = -8$,

so the tangent is horizontal at the point $(1, -8)$.

b
$$\begin{aligned} f(x) &= \frac{5 - x^2 + 5x^4}{x^2} \\ &= 5x^{-2} - 1 + 5x^2 \\ f'(x) &= -10x^{-3} + 10x \\ &= \frac{-10 + 10x^4}{x^3}. \end{aligned}$$

Factoring, $f'(x) = \dfrac{10(x - 1)(x + 1)(x^2 + 1)}{x^3}$

Hence $f'(x) = 0$ when $x = 1$ or $x = -1$.

Substituting, $f(1) = 9$ and $f(-1) = 9$

so the tangent is horizontal at the point $(1, 9)$

and at the point $(-1, 9)$ (the function is even).

Exercise 9F

FOUNDATION

1 Use the rule $\dfrac{d}{dx}(x^n) = nx^{n-1}$ to find the derivative of each function in index notation.

a $f(x) = x^{-1}$ **b** $f(x) = x^{-5}$ **c** $f(x) = 3x^{-1}$

d $f(x) = 5x^{-2}$ **e** $f(x) = -\frac{4}{3}x^{-3}$ **f** $f(x) = 2x^{-2} + \frac{1}{2}x^{-8}$

2 Write each function using negative indices, differentiate it, and convert back to a fraction.

a $f(x) = \dfrac{1}{x}$ **b** $f(x) = \dfrac{1}{x^2}$ **c** $f(x) = \dfrac{1}{x^4}$ **d** $f(x) = \dfrac{3}{x}$

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3 Use the standard form $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ to differentiate each function, leaving results in index form.

a $y = (7x - 5)^{-6}$ b $y = \frac{1}{3 + 5x}$ c $y = \frac{4}{2x - 1}$ d $y = \frac{3}{4(5x + 6)^7}$

4 Write down, in index form, the first five derivatives of $y = \frac{1}{x}$.

5 Find any points on these curves where the tangent has gradient -1.

a $y = \frac{1}{x}$ b $y = \frac{1}{2}x^{-2}$

6 Find the equations of the tangent and normal to each curve at the point where $x = 1$.

a $y = \frac{3}{4x - 1}$ b $y = \frac{3}{(x - 2)^3}$

7 Use the fact that the derivative of $f(x) = \frac{1}{x}$ is $f'(x) = -\frac{1}{x^2}$ to differentiate:

a $f(x) = \frac{3}{x}$ b $f(x) = \frac{1}{3x}$ c $f(x) = -\frac{7}{3x}$ d $f(x) = \frac{a}{x}$

DEVELOPMENT

8 Expand brackets, then differentiate:

a $y = \left(x + \frac{1}{x}\right)^2$ b $y = \left(x + \frac{1}{x}\right)^2 - \left(x - \frac{1}{x}\right)^2$ c $y = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

9 Use the full setting out of the chain rule to differentiate each function. Then find the coordinates of any points where the tangent is horizontal.

a $y = \frac{1}{1 + x^2}$ b $y = \frac{-3}{x^4 - 2x^2 - 1}$

10 Consider the function $f(x) = x^{-1}$.

a Find $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$ and $f^{(5)}(x)$.

b Find $f'(1)$, $f''(1)$, $f'''(1)$, $f^{(4)}(1)$ and $f^{(5)}(1)$.

c Describe the pattern in the results of part **b**.

d Describe the pattern if -1 is substituted rather than 1.

11 Find a, if the tangent to $y = \frac{1}{x + a}$ at the point where $x = 6$ has gradient -1.

12 a Find the equation of the tangent to $y = \frac{1}{x - 4}$ at the point L where $x = b$.

b Hence find the equations of the tangents passing through:

i the origin, ii $W(6, 0)$.

13 a Show that the tangents to $f(x) = \frac{12}{x}$ at $M(2, 6)$ and $T(-2, -6)$ are parallel.

b Find the tangents at $M(2, 6)$ and $N(6, 2)$, and their point of intersection.

c Sketch the situation in parts **a** and **b**, and explain the particular symmetry of the rectangular hyperbola that is illustrated in each part.

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14 **a** Find the equation of the tangent to the hyperbola $xy = c$ at the point $T\left(t, \frac{c}{t}\right)$, find the points A and B where the tangent meets the x-axis and y-axis respectively, and show that A is independent of c.

b Find the area of ΔOAB and show that it is constant as T varies.

c Show that T bisects AB and that $OT = AT = BT$.

d Hence explain why the rectangle with diagonal OT and sides parallel to the axes has a constant area that is half the area of ΔOAB.

15 **a** Show that the hyperbola $xy = c^2$ can be described by the two parametric equations $x = ct$ and $y = \frac{c}{t}$.

b Differentiate parametrically to find the gradient at the point $T\left(ct, \frac{c}{t}\right)$ with parameter t.

c Find the equation of the tangent at T, and prove that the x-intercept A of the tangent, the origin O, and the original point T form an isosceles triangle.

ENRICHMENT

16 This question shows how to differentiate $f(x) = x^{-m}$, where m is a whole number, using only the definition of the derivative.

a Use a common denominator to show that $\dfrac{f(x+h) - f(x)}{h} = \dfrac{x^m - (x+h)^m}{hx^m(x+h)^m}$.

b Simplify this expression using the fact, proven in Section 9C, that

$$(x+h)^m = x^m + mhx^{m-1} + (\text{terms in } h^2 \text{ and above}).$$

c Hence show that $f'(x) = -mx^{-m-1}$, agreeing with the formula.

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9G Differentiating powers with fractional indices

The chain rule further allows us to prove the formula $\frac{d}{dx}(x^n) = nx^{n-1}$ for all rational indices n. This will allow us to differentiate functions such as $y = \sqrt{x}$ and $= x^{-\frac{2}{3}}$. The formula actually holds for all real indices n — see the note below for irrational indices.

Extending the proof to fractional indices requires two steps, each using the chain rule. The first step extends the result to the fractional indices $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$.

Proving the formula for the reciprocal of a positive integer

Consider the function $\quad y = x^{\frac{1}{k}}$, where $k \geq 2$ is an integer.

Forming the inverse function, $\quad x = y^k$.

Differentiating, $\quad \frac{dx}{dy} = ky^{k-1}$, because k is a positive integer.

Taking reciprocals, $\quad \frac{dy}{dx} = \frac{1}{ky^{k-1}}$, because $\left(\frac{dx}{dy}\right)^{-1} = \frac{dy}{dx}$,

$$= \frac{1}{kx^{\frac{k-1}{k}}}$$

$$= \tfrac{1}{k}x^{\frac{1}{k}-1}, \quad \text{as required.}$$

Proving the formula for any rational number

Consider the function $y = x^{\frac{m}{k}}$, where m and k are integers and $k \geq 2$.

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= mx^{\frac{m-1}{k}} \times \frac{1}{k}x^{\frac{1}{k}-1}$$

$$= \frac{m}{k}x^{\frac{m}{k}-1}.$$

Let $u = x^{\frac{1}{k}}$,

then $y = u^m$.

So $\frac{du}{dx} = \frac{1}{k}x^{\frac{1}{k}-1}$, by the first step

and $\frac{dy}{du} = mu^{m-1}$.

18 DIFFERENTIATING POWERS WITH FRACTIONAL INDICES

- The rule $\frac{d}{dx}(x^n) = nx^{n-1}$ applies for all real numbers n. For example,

  $$\frac{d}{dx}(x^{\frac{1}{4}}) = \tfrac{1}{4}x^{-\frac{3}{4}} \quad \text{and} \quad \frac{d}{dx}(x^{-\frac{5}{3}}) = -\tfrac{5}{3}x^{-\frac{8}{3}}.$$

- In particular, $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

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The calculation of the derivative of $\sqrt{x}$ by the rule is

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

This result is useful to know because functions involving square roots occur so often. Again, readers may choose whether to memorise it as a special result.

A note on irrational indices

We do not have a precise definition of powers such as x^{π} or $x^{\sqrt{2}}$ with irrational indices, so we cannot provide a rigorous proof of the fact that the derivative of $y = x^n$ is indeed $y = nx^{n-1}$ for irrational values of n. Nevertheless, every irrational is 'as close as we like' to a rational number for which the theorem is certainly true, so the result is intuitively clear for irrationals.

The domain of x^n

A power x^n does not always exist, because zero has no reciprocal, and negative numbers have no square roots.

- When n is negative, x cannot be zero.
- When n is a fraction $\frac{m}{k}$, where k is an even number, x cannot be negative.
- When n is irrational, x cannot be negative.

Example 23 9G

Write each function using a fractional index, then differentiate it, then convert the answer back to surd form.

a $y = \dfrac{3}{\sqrt{x}}$ **b** $f(x) = x^3\sqrt{x}$ **c** $f(x) = \dfrac{1}{x^3\sqrt{x}}$

SOLUTION

a
$$\begin{aligned} f(x) &= \frac{3}{\sqrt{x}} \\ &= 3x^{-\frac{1}{2}} \\ f'(x) &= -\tfrac{1}{2} \times 3 \times x^{-1\frac{1}{2}} \\ &= -\frac{3}{2x\sqrt{x}} \end{aligned}$$

b
$$\begin{aligned} f(x) &= x^3\sqrt{x} \\ &= x^3 \times x^{\frac{1}{2}} \\ &= x^{3\frac{1}{2}} \\ f'(x) &= \tfrac{7}{2}x^{2\frac{1}{2}} \\ &= \tfrac{7}{2}x^2\sqrt{x} \end{aligned}$$

c
$$\begin{aligned} f(x) &= \frac{1}{x^3\sqrt{x}} \\ &= \frac{1}{x^3 \times x^{\frac{1}{2}}} \\ &= x^{-3\frac{1}{2}} \\ f'(x) &= -\tfrac{7}{2}x^{-4\frac{1}{2}} \\ &= -\frac{7}{2x^4\sqrt{x}} \end{aligned}$$

The chain rule and general indices

The chain rule can be combined with these results, as in Examples 24 and 25.

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Example 24 9G

Differentiate these functions.

a $y = \sqrt{4 - x^2}$ (an upper semicircle)

b $y = \dfrac{1}{\sqrt{1 + 4x^2}}$

SOLUTION

a
$$y = \sqrt{4 - x^2}$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{2\sqrt{4 - x^2}} \times (-2x)$$
$$= \frac{-x}{\sqrt{4 - x^2}}.$$

Let $u = 4 - x^2$.
Then $y = \sqrt{u}$.
Hence $\dfrac{du}{dx} = -2x$
and $\dfrac{dy}{du} = \dfrac{1}{2\sqrt{u}}$

b
$$y = \frac{1}{\sqrt{1 + 4x^2}}$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -\tfrac{1}{2}(1 + 4x^2)^{-1\frac{1}{2}} \times 8x$$
$$= \frac{-4x}{(1 + 4x^2)^{1\frac{1}{2}}}.$$

Let $u = 1 + 4x^2$.
Then $y = u^{-\frac{1}{2}}$.
Hence $\dfrac{du}{dx} = 8x$
and $\dfrac{dy}{du} = -\tfrac{1}{2}u^{-1\frac{1}{2}}$

Dividing through by the denominator

If the denominator of a function is a single term, divide through by it first, and then differentiate.

Example 25 9G

In each function, divide through by the denominator and differentiate. Then find any values of x where the tangent is horizontal.

a $y = \dfrac{3x^2 + 4x - 7}{x}$

b $y = \dfrac{12x - 6}{\sqrt{x}}$

SOLUTION

a
$$y = \frac{3x^2}{x} + \frac{4x}{x} - \frac{7}{x}$$
$$= 3x + 4 - 7x^{-1}$$
$$\frac{dy}{dx} = 3 + 7x^{-2}$$
$$= \frac{3x^2 + 7}{x^2},$$

which is never zero.

b
$$y = \frac{12x^1}{x^{\frac{1}{2}}} - \frac{6}{x^{\frac{1}{2}}}$$
$$= 12x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = 6x^{-\frac{1}{2}} + 3x^{-1\frac{1}{2}}$$
$$= \frac{3(2x + 1)}{x^{1\frac{1}{2}}},$$

which is never zero $(x = -\tfrac{1}{2}$ is outside the domain).

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Exercise 9G

FOUNDATION

1 Use the rule $\frac{d}{dx}(x^n) = nx^{n-1}$ to find the derivative of each function in index notation.

a $y = x^{-\frac{1}{2}}$ **b** $y = x^{1\frac{1}{2}}$ **c** $y = 6x^{\frac{2}{3}}$

d $y = 12x^{-\frac{1}{3}}$ **e** $y = 4x^{\frac{1}{4}} + 8x^{-\frac{1}{2}}$ **f** $y = 7x^{2\frac{1}{3}}$

2 **a** Write $y = \sqrt{x}$ as a power of x, and use the rule to show that $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

b Explain why $y = x^2\sqrt{x}$ can be written as $y = x^{\frac{5}{2}}$, and hence why $\frac{dy}{dx} = \frac{5}{2}x\sqrt{x}$.

c Using similar procedures, differentiate each function and write the answer using surds.

i $y = x\sqrt{x}$ **ii** $y = \frac{1}{\sqrt{x}}$ **iii** $y = \frac{1}{x\sqrt{x}}$

3 Write down, in index form, the first five derivatives of $y = \sqrt{x}$.

4 Use the standard form $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ to differentiate each function, leaving results in index form.

a $y = (7 + 2x)^{1\frac{1}{3}}$ **b** $y = \sqrt{x + 4}$ **c** $y = \sqrt{5 - 3x}$ **d** $y = 4(2 - 5x)^{-2\frac{1}{4}}$

5 Find any points on these curves where the tangent has gradient -1.

a $y = -\sqrt{x}$ **b** $y = \sqrt{x}$

6 Find the equations of the tangent and normal to each curve at the point $T(4, 2)$.

a $y = \sqrt{x}$ **b** $y = \sqrt{8 - x}$

7 Use the fact that the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$ to differentiate:

a $f(x) = 3\sqrt{x}$ **b** $f(x) = 10\sqrt{x}$ **c** $f(x) = \sqrt{49x}$ **d** $f(x) = \sqrt{7x}$

DEVELOPMENT

8 Find any points on each curve where the tangent has the given angle of inclination.

a $y = 2\sqrt{x}$, $30°$ **b** $y = 3\sqrt[3]{x}$, $45°$

9 Divide each function through by the denominator, then differentiate.

a $y = \frac{x^3 - 3x + 8}{x^2}$ **b** $y = \frac{x^2 + 6x\sqrt{x} + x}{x}$ **c** $y = \frac{3x - 2\sqrt{x}}{\sqrt{x}}$

10 Use the full setting-out of the chain rule to differentiate each function. Then find the coordinates of any points where the tangent is horizontal.

a $y = \sqrt{x^2 - 2x + 5}$ **b** $y = \sqrt{x^2 - 2x}$

c $y = 7\sqrt{x^2 + 1}$ **d** $y = \frac{1}{1 + \sqrt{x}}$

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11 Show that the tangent to $y = x^{\frac{1}{3}}$ at $T(-1, -1)$ meets the curve again at $(8, 2)$.

12 Find a, if the normal to $y = \sqrt{x + a}$ at the point where $x = 4$ has gradient -6.

13 **a** Differentiate the semicircle $y = \sqrt{169 - x^2}$, and find the equations of the tangent and the normal at $P(12, 5)$. Show that the normal passes through the centre.

b Find the x-intercept and y-intercept of the tangent, and the area of the triangle enclosed by the tangent and the two axes.

c Find the perimeter of this triangle.

14 **a** Let the point $P(4, 3)$ lie on the semicircle $y = \sqrt{25 - x^2}$, and let $Q\left(4, \frac{9}{5}\right)$ lie on the curve $y = \frac{3}{5}\sqrt{25 - x^2}$ (which is half an ellipse). Find the equations of the tangents at P and at Q, and show that they intersect on the x-axis.

b Find the equation of the tangent at the point P with x-coordinate $x_0 > 0$ on the curve $y = \lambda\sqrt{25 - x^2}$ (again, half an ellipse). Let the tangent meet the x-axis at T, let the ellipse meet the x-axis at $A(5, 0)$, and let the vertical line through P meet the x-axis at M. Show that the point T is independent of λ, and show that OA is the geometric mean of OM and OT.

15 **a** If $y = Ax^n$, show that $x\dfrac{dy}{dx} = ny$.

b If $y = \dfrac{C}{x^n}$, show that $x\dfrac{dy}{dx} = -ny$.

c **i** If $y = a\sqrt{x}$, show that $y\dfrac{dy}{dx}$ is a constant.

ii Conversely, if $y = ax^n$ and $a \neq 0$, find $y\dfrac{dy}{dx}$ and show that it is constant if and only if $n = \frac{1}{2}$ or 0.

ENRICHMENT

16 **a** Develop a three-step chain rule for the derivative $\dfrac{dy}{dx}$, where y is a function of u, u is a function of v, and v is a function of x. Hence differentiate $y = \dfrac{1}{1 + \sqrt{1 - x^2}}$.

b Generalise the chain rule to n steps.

17 **a** Write down the first four derivatives of $f(x) = x^{-\frac{1}{2}}$, leaving the numerator and denominator of the coefficient in factored form.

b Write down a formula (using dots) for the nth derivative $f^{(n)}x$.

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9H The product rule

The product rule extends the methods for differentiation to functions that are products of two simpler functions. For example,

$$y = x(x - 10)^4 \quad \text{is the product} \quad y = x \times (x - 10)^4,$$

and we shall decompose y as

$$y = uv, \quad \text{where } u = x \quad \text{and} \quad v = (x - 10)^4.$$

Statement of the product rule

The product rule expresses the derivative in terms of the derivatives of the two simpler functions,

19 THE PRODUCT RULE

Suppose that the function

$$y = u \times v$$

is the *product* of two functions u and v, each of which is a function of x. Then

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} \quad \text{or more concisely,} \quad y' = vu' + uv'.$$

The second form uses the convention of the dash $'$ to represent differentiation with respect to x. That is, $y' = \dfrac{dy}{dx}$ and $u' = \dfrac{du}{dx}$ and $v' = \dfrac{dv}{dx}$.

Proof of the product rule

Suppose that x changes to $x + \delta x$, and that as a result:

u changes to $u + \delta u$, v changes to $v + \delta v$, and y changes to $y + \delta y$.

Then $\quad y = uv,$

and $\quad y + \delta y = (u + \delta u)(v + \delta v)$

$\quad = uv + v\delta u + u\delta v + \delta u \delta v,$

so subtracting, $\quad \delta y = v\delta u + u\delta v + \delta u \delta v.$

Dividing by δx, $\quad \dfrac{\delta y}{\delta x} = v\dfrac{\delta u}{\delta x} + u\dfrac{\delta v}{\delta x} + \dfrac{\delta u}{\delta x} \times \dfrac{\delta v}{\delta x} \times \delta x,$

and taking limits as $\delta x \to 0$, $\quad \dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx} + 0$, as required.

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Using the product rule

The working in the right-hand column in Example 26 below is the recommended setting out. The first step is to decompose the function into the product of two simpler functions.

Example 26 9H

Differentiate each function, writing the result in fully factored form. Then state for what value(s) of x the derivative is zero.

a $x(x-10)^4$ **b** $x^2(3x+2)^3$

SOLUTION

a Let $y = x(x-10)^4$.

Then $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

$= (x-10)^4 \times 1 + x \times 4(x-10)^3$

$= (x-10)^3(x-10+4x)$

$= (x-10)^3(5x-10)$

$= 5(x-10)^3(x-2)$,

so the derivative is zero for $x = 10$ and for $x = 2$.

Let $u = x$
and $v = (x-10)^4$.
Then $\dfrac{du}{dx} = 1$
and $\dfrac{dv}{dx} = 4(x-10)^3$
(using the chain rule).

b Let $y = x^2(3x+2)^3$.

Then $y' = vu' + uv'$

$= (3x+2)^3 \times 2x + x^2 \times 9(3x+2)^2$

$= x(3x+2)^2(6x+4+9x)$

$= x(3x+2)^2(15x+4)$,

Let $u = x^2$
and $v = (3x+2)^3$.
Then $u' = 2x$
and $v' = 9(3x+2)^2$.

so the derivative is zero for $x = 0$, $x = -\frac{2}{3}$ and for $x = -\frac{4}{15}$.

Note: The product rule can be difficult to use with the algebraic functions under consideration at present, because the calculations can easily become quite involved. The rule will seem far more straightforward later in the context of exponential, logarithmic and trigonometric functions.

Exercise 9H

FOUNDATION

1 Copy and complete the setting out below to differentiate $5x(x-2)^4$ by the product rule.

Let $y = 5x(x-2)^4$.

Then $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

$= \cdots \times \cdots + \cdots \times \cdots$

$= 5(x-2)^4 + 20x(x-2)^3$

$= \cdots$ (take out the common factor)

$= 5(x-2)^3(5x-2)$.

Let $u = 5x$
and $v = (x-2)^4$.
Then $\dfrac{du}{dx} = \cdots$
and $\dfrac{dv}{dx} = \cdots$.

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2 Differentiate each function:
i by expanding the product and then differentiating each term,
ii using the product rule.
a $y = x^3(x - 2)$ b $y = (2x + 1)(x - 5)$ c $y = (x^2 - 3)(x^2 + 3)$

3 Differentiate these functions using the product rule, identifying the factors u and v in each example. Express your answers in fully factored form, and state the values of x for which the derivative is zero.
a $y = x(3 - 2x)^5$ b $y = x^3(x + 1)^4$ c $y = x^5(1 - x)^7$
d $y = (x - 1)(x - 2)^3$ e $y = 2(x + 1)^3(x + 2)^4$ f $y = (2x - 3)^4(2x + 3)^5$

4 Find the tangents and normals to these curves at the indicated points.
a $y = x(1 - x)^6$ at the origin b $y = (2x - 1)^3(x - 2)^4$ at $A(1, 1)$

DEVELOPMENT

5 Differentiate each function using the product rule, giving your answer in fully factored form. One of the factors will require the chain rule to differentiate it.
a $y = x(x^2 + 1)^5$ b $y = 2\pi x^3(1 - x^2)^4$ c $y = -2x(x^2 + x + 1)^3$

6 a Find the first and second derivatives of $y = (x^2 - 1)^4$.
b What are the degrees of the three polynomials y, y' and y''?
c What values of x are zeroes of all three polynomials?

7 Differentiate $y = (x^2 - 10)^3x^4$, using the chain rule to differentiate the first factor. Hence find the points on the curve where the tangent is horizontal.

8 Show that the function $y = x^3(1 - x)^5$ has a horizontal tangent at a point P with x-coordinate $\frac{3}{8}$. Show that the y-coordinate of P is $\dfrac{3^3 \times 5^5}{2^{24}}$.

9 Differentiate each function using the product rule. Then combine terms using a common denominator and factoring the numerator completely. State the values of x for which the derivative is zero.
a $y = 6x\sqrt{x + 1}$ b $y = -4x\sqrt{1 - 2x}$ c $y = 10x^2\sqrt{2x - 1}$

10 a What is the domain of $y = x\sqrt{1 - x^2}$?
b Differentiate $y = x\sqrt{1 - x^2}$, using the chain rule to differentiate the second factor, then combine the terms using a common denominator.
c Find the points on the curve where the tangent is horizontal.
d Find the tangent and the normal at the origin.

11 a Differentiate $y = a(x - \alpha)(x - \beta)$ using the product rule.
b Show that the tangents at the x-intercepts have opposite gradients and meet at a point M whose x-coordinate is the average of the x-intercepts.
c Find the point V where the tangent is horizontal. Show that M is vertically above or below V and twice as far from the x-axis. Sketch the situation.

12 Show that if a polynomial $f(x)$ can be written as a product $f(x) = (x - a)^n q(x)$ of the polynomials $(x - a)^n$ and $q(x)$, where $n \geq 2$, then $f'(x)$ can be written as a multiple of $(x - a)^{n-1}$. What does this say about the shape of the curve near $x = a$?

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ENRICHMENT

13 **a** Show that the function $y = x^r(1 - x)^s$, where $r, s > 1$, has a horizontal tangent at a point P whose x-coordinate p lies between 0 and 1.

b Show that the vertical line through P divides the interval joining $O(0, 0)$ and $A(1, 0)$ in the ratio $r : s$, and find the y-coordinate of P. What are the coordinates of P if $s = r$?

14 Establish the rule for differentiating a product $y = uvw$, where u, v and w are functions of x. Hence differentiate each function, and find the zeroes of y'.

a $x^5(x - 1)^4(x - 2)^3$ **b** $x(x - 2)^4\sqrt{2x + 1}$

15 Establish the rule for differentiating a product $y = u_1u_2 \ldots u_n$ of n functions of x.

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9I The quotient rule

The quotient rule extends the formulae for differentiation to functions that are quotients of two simpler functions. For example,

$$y = \frac{2x + 1}{2x - 1} \text{ is the quotient of the two functions } 2x + 1 \quad \text{and} \quad 2x - 1.$$

Statement of the quotient rule

The quotient rule, like the product rule, expresses the derivative in terms of the derivatives of the two simpler functions,

20 THE QUOTIENT RULE

Suppose that the function

$$y = \frac{u}{v}$$

is the *quotient* of two functions u and v, each of which is a function of x. Then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad \text{or more concisely,} \qquad y' = \frac{vu' - uv'}{v^2}.$$

Proof of the quotient rule

Differentiate uv^{-1} using the product rule.

Let $y = uv^{-1}$.

Then $\dfrac{dy}{dx} = V\dfrac{dU}{dx} + U\dfrac{dV}{dx}$

$= v^{-1}\dfrac{du}{dx} - uv^{-2}\dfrac{dv}{dx}$.

Multiplying top and bottom of each expression by v^2,

$$= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Let $U = u$

and $V = v^{-1}$.

Then $\dfrac{dU}{dx} = \dfrac{du}{dx}$

and by the chain rule,

$$\frac{dV}{dx} = \frac{dV}{dv} \times \frac{dv}{dx}$$

$$= -v^{-2}\frac{dv}{dx}.$$

Example 27 9I

Differentiate each function, stating any values of x where the derivative is zero.

a $y = \dfrac{2x + 1}{2x - 1}$

b $y = \dfrac{x}{x^2 + 1}$

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SOLUTION

a Let $y = \frac{2x + 1}{2x - 1}$.

Then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{2(2x - 1) - 2(2x + 1)}{(2x - 1)^2}$

$= \frac{-4}{(2x - 1)^2}$, which is never zero.

Let $u = 2x + 1$
and $v = 2x - 1$.
Then $\frac{du}{dx} = 2$
and $\frac{dv}{dx} = 2$.

b Let $y = \frac{x}{x^2 + 1}$.

Then $y' = \frac{vu' - uv'}{v^2}$

$= \frac{(x^2 + 1) - 2x^2}{(x^2 + 1)^2}$

$= \frac{1 - x^2}{(x^2 + 1)^2}$

$= \frac{(1 - x)(1 + x)}{(x^2 + 1)^2}$, which is zero when $x = 1$ or $x = -1$.

Let $u = x$
and $v = x^2 + 1$.
Then $u' = 1$
and $v' = 2x$.

Note: Both these functions could have been differentiated using the product rule after writing them as $(2x + 1)(2x - 1)^{-1}$ and $x(x^2 + 1)^{-1}$. The quotient rule, however, makes the calculations much easier.

Exercise 9I

FOUNDATION

1 Copy and complete the setting out below to differentiate $\frac{2x + 3}{3x + 2}$ by the quotient rule.

Let $y = \frac{2x + 3}{3x + 2}$.

Then $\frac{dy}{dx} = \frac{v\,u' - u\,v'}{v^2}$

$= \frac{\cdots \times \cdots - \cdots \times \cdots}{\cdots}$

$= \frac{-5}{(3x + 2)^2}$.

Let $u = 2x + 3$
and $v = 3x + 2$.
Then $u' = \cdots$
and $v' = \cdots$

2 Differentiate each function using the quotient rule, taking care to identify u and v first. Express your answer in fully factored form, and state any values of x for which the tangent is horizontal.

a $y = \frac{x + 1}{x - 1}$ **b** $y = \frac{2x}{x + 2}$ **c** $y = \frac{3 - 2x}{x + 5}$ **d** $y = \frac{x^2}{1 - x}$

e $y = \frac{x^2 - 1}{x^2 + 1}$ **f** $y = \frac{x}{1 - x^2}$ **g** $y = \frac{x^3 - 1}{x^3 + 1}$ **h** $y = \frac{x^2 - 9}{x^2 - 4}$

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3 Differentiate $y = \dfrac{1}{3x - 2}$:

a using the chain rule with $u = 3x - 2$ and $y = \dfrac{1}{u}$. This is the better method.

b using the quotient rule with $u = 1$ and $v = 3x - 2$. This method is longer.

4 For each curve below, find the equations of the tangent and normal and their angles of inclination at the given point.

a $y = \dfrac{x}{5 - 3x}$ at $K(2, -2)$

b $y = \dfrac{x^2 - 4}{x - 1}$ at $L(4, 4)$

DEVELOPMENT

5 Differentiate these functions, where m, b, a and n are constants.

a $y = \dfrac{mx + b}{bx + m}$

b $y = \dfrac{x^2 - a}{x^2 - b}$

c $y = \dfrac{x^n - 3}{x^n + 3}$

6 a Find the value of c if $f'(c) = -3$, where $f(x) = \dfrac{x^2}{x + 1}$.

b Find the value of k if $f'(-3) = 1$, where $f(x) = \dfrac{x^2 + k}{x^2 - k}$.

7 a Differentiate $y = \dfrac{x - \alpha}{x - \beta}$.

b Show that for $\alpha > \beta$, all tangents have positive gradient, and for $\alpha < \beta$, all tangents have negative gradient.

c What happens when $\alpha = \beta$?

8 Differentiate $y = \dfrac{5 + 2x}{5 - 2x}$:

a using the quotient rule (the better method),

b using the product rule, with the function in the form $y = (5 + 2x)(5 - 2x)^{-1}$.

9 a Find the normal to the curve $x = \dfrac{t}{t + 1}$ and $y = \dfrac{t}{t - 1}$ at the point T where $t = 2$.

b Eliminate t from the two equations (by solving the first for t and substituting into the second). Then differentiate this equation to find the gradient of the normal at T.

10 Differentiate, stating any zeroes of the derivative:

a $\dfrac{\sqrt{x} + 1}{\sqrt{x} + 2}$

b $\dfrac{x - 3}{\sqrt{x + 1}}$

11 a Evaluate $f'(8)$ if $f(x) = \dfrac{\sqrt{x} + \sqrt{2}}{\sqrt{x} - \sqrt{2}}$.

b Evaluate $g'(5)$ if $g(r) = \dfrac{(r - 6)^3 - 1}{(r - 4)^3 + 1}$.

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12 **a** Sketch the hyperbola $y = \dfrac{x}{x+1}$, showing the horizontal and vertical asymptotes, and state its domain and range.

b Show that the tangent at the point P where $x = a$ is $x - (a+1)^2 y + a^2 = 0$.

c Let the tangent at $A\left(1, \frac{1}{2}\right)$ meet the x-axis at I, and let G be the point on the x-axis below A. Show that the origin bisects GI.

d Let $T(c, 0)$ be any point on the x-axis.

i Show that for $c > 0$, no tangents pass through T.

ii Show that for $c < 0$ and $c \neq -1$, there are two tangents through T, the x-coordinates of whose points of contact are opposites of each other. For what values of c are these two points of contact on the same and on different branches of the hyperbola?

13 **a** Suppose that $y = \dfrac{u}{x}$, where u is a function of x. Show that $y + x\dfrac{dy}{dx} = \dfrac{du}{dx}$.

b Suppose that $y = \dfrac{x}{u}$, where u is a function of x. Show that $y\dfrac{du}{dx} + u\dfrac{dy}{dx} = 1$.

14 **a** Find the first and second derivatives of:

i $y = \dfrac{x-1}{x+1}$ **ii** $y = \dfrac{2x+1}{x-1}$ **iii** $y = \dfrac{x^2}{x-1}$

b Repeat part **i** by writing $\dfrac{x-1}{x+1} = \dfrac{x+1}{x+1} - \dfrac{2}{x+1}$, and similarly for parts **ii**–**iii**.

15 [A difficult medley intended for later revision]

Differentiate these functions using the most appropriate method. Factor each answer completely, provided that surds are not involved.

a $(3x-7)^4$ **b** $\dfrac{x^2+3x-2}{x}$ **c** $(2x+3)(2x-3)$ **d** $\dfrac{1}{x^2-9}$

e $x(4-x)^3$ **f** $\dfrac{3-x}{3+x}$ **g** $(x^4-1)^5$ **h** $\dfrac{1}{\sqrt{2-x}}$

i $(x^3+5)^2$ **j** $\dfrac{x^2+x+1}{2\sqrt{x}}$ **k** $\frac{2}{3}x^2(x^3-1)$ **l** $\dfrac{x}{x+5}$

m $x\sqrt{x} + x^2\sqrt{x}$ **n** $\left(x - \dfrac{1}{x}\right)^2$ **o** $x^3(x-1)^8$ **p** $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$

ENRICHMENT

16 Sketch a point P on a curve $y = f(x)$ where x, $f(x)$ and $f'(x)$ are all positive. Let the tangent, normal and vertical at P meet the x-axis at T, N and M respectively. Let the (acute) angle of inclination of the tangent be $\theta = \angle PTN$, so that $y' = \tan\theta$.

a Using trigonometry, show that:

i $MN = yy'$ **ii** $TM = \dfrac{y}{y'}$ **iii** $\sec\theta = \sqrt{1 + y'^2}$

iv $\operatorname{cosec}\theta = \dfrac{\sqrt{1+y'^2}}{y'}$ **v** $PN = y\sqrt{1+y'^2}$ **vi** $PT = \dfrac{y\sqrt{1+y'^2}}{y'}$

b Hence find the four lengths when $x = 3$ and:

i $y = x^2$ **ii** $y = \dfrac{3x-1}{x+1}$

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9J Rates of change

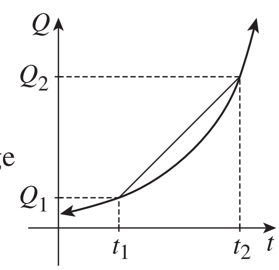
So far, differentiation has been developed geometrically in terms of tangents to curves, and many more geometric ideas will be involved as calculus is developed. Calculus has many interpretations, however. It has always been a very practical subject, and is an essential part of engineering, economics, and all the sciences.

This section will begin to interpret the derivative as a *rate of change*. In various practical situations, some variable quantity Q is a function of time t. Differentiation with respect to time will give the *rate* at which the quantity Q is changing over time.

Two important type of rates are velocity and acceleration, but we have left the discussion of motion until Year 12 because motion requires a great deal more theory. A couple of questions in the next exercise, however, indicate how velocity and acceleration are particular types of rates.

Average rates and instantaneous rates

Suppose that a quantity Q is given as a function of time t, as in the diagram to the right. There are two types of rates:



- An *average rate of change* corresponds to a chord. In the diagram, the quantity Q takes the value Q_1 at time t_1, and the value Q_2 at time t_2. The average rate of change from time t_1 to time t_2 is the usual gradient formula:

$$\text{average rate} = \frac{Q_2 - Q_1}{t_2 - t_1}.$$

- An *instantaneous rate of change* corresponds to a tangent. The instantaneous rate of change at time t_1 is the value of the derivative $\dfrac{dQ}{dt}$ at time $t = t_1$:

$$\text{instantaneous rate} = \frac{dQ}{dt}, \quad \text{evaluated at } t = t_1.$$

One can see from the diagram that as the time t_2 gets closer to t_1, the average rate from $t = t_1$ to $t = t_2$ gets closer to the instantaneous rate at t_1 (provided that the curve is continuous and smooth at $t = t_1$). This is exactly the way in which first-principles differentiation was defined using a limit in Section 9B.

21 AVERAGE AND INSTANTANEOUS RATES OF CHANGE

Suppose that a quantity Q is a function of time t.

- The *average rate of change* from the time $t = t_1$, when the value is Q_1, to the time $t = t_2$, when the value is Q_2, is the gradient of the chord,

$$\text{average rate} = \frac{Q_2 - Q_1}{t_2 - t_1}.$$

- The *instantaneous rate of change* at time $t = t_1$ is the gradient of the tangent, that is, the value of the derivative $\dfrac{dQ}{dt}$ evaluated at $t = t_1$:

$$\text{instantaneous rate} = \frac{dQ}{dt}, \quad \text{evaluated at } t = t_1.$$

- In this course, the unqualified phrase ‘rate of change’ will always mean the ‘instantaneous rate of change’.

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In many problems, it is useful to sketch two graphs, one of the quantity Q as a function of time t, the other of the rate of change $\frac{dQ}{dt}$ also as a function of time t.

The quantity Q is usually replaced by a more convenient pronumeral, such as P for population, or V for volume, or M for mass. Take particular care to use the correct units for the quantity, the time, and the rate, in final answers.

Example 28 9J

A cockroach plague hit the suburb of Berrawong last year, but was gradually brought under control. The council estimated that the cockroach population P in millions, t months after 1 January, was given by

$$P = 7 + 6t - t^2.$$

a Differentiate to find the rate of change $\frac{dP}{dt}$ of the cockroach population.

b Find the cockroach population on 1 January, and the rate at which the population was increasing at that time. Be careful with the units.

c Find the cockroach population on 1 April, and the rate at which the population was increasing at that time.

d When did the council manage to stop the cockroach population increasing any further, and what was the population then?

e When were the cockroaches finally eliminated?

f What was the average rate of increase from 1 January to 1 April?

g Draw the graphs of P and $\frac{dP}{dt}$ as functions of time t, and state their domains and ranges. Add to your graph of P the tangents and chords corresponding to parts **b**, **d** and **f**.

SOLUTION

a The population function was $P = 7 + 6t - t^2$.

Differentiating, the rate of change was $\frac{dP}{dt} = 6 - 2t$.

b When $t = 0$, $P = 7$,

and when $t = 0$, $\frac{dP}{dt} = 6$.

Thus on 1 January, the population was 7 million, and the population was increasing at 6 million per month.
(The time t is in months, so the rate is 'per month'.)

c On 1 April, $t = 3$, so $P = 16$ million,

and $\frac{dP}{dt} = 0$ cockroaches/month.

d We found in part **b** that the population stopped increasing on 1 April, and that the population then was 16 million.

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e To find the time when the population was zero,

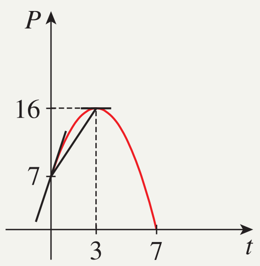
$$\begin{aligned} \text{put} \quad P &= 0, \\ \text{that is,} \quad 7 + 6t - t^2 &= 0 \\ \boxed{\times\ (-1)} \quad t^2 - 6t - 7 &= 0 \\ (t - 7)(t + 1) &= 0 \\ t &= 7 \text{ or } -1. \end{aligned}$$

Hence $t = 7$, and the cockroaches were finally eliminated on 1 August.
(The negative solution is irrelevant because council action started at $t = 0$.)

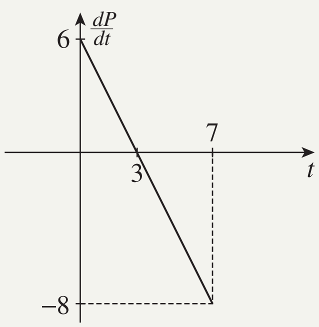
f From part **b**, the population was 7 million on 1 January, and from part **c**, the population was 16 million on 1 April.

$$\begin{aligned} \text{Hence average rate of increase} &= \frac{P_2 - P_1}{t_2 - t_1} \quad \text{(gradient of chord formula)} \\ &= \frac{16 - 7}{3} \\ &= 3 \text{ million per month.} \end{aligned}$$

g



Domain: $0 \le t \le 7$,
range: $0 \le P \le 16$



Domain: $0 \le t \le 7$,
range: $-8 \le \dfrac{dP}{dt} \le 6$

Increasing, decreasing and stationary

We can see from the left-hand graph above that the cockroach population was *increasing* when $0 < t < 3$ — every tangent to the left of $t = 3$ has positive gradient. For $3 < t < 7$, the cockroach population was *decreasing* — every tangent to the right of $t = 3$ has negative gradient.

At $t = 3$, the cockroach population was *stationary*, that is, it was neither increasing nor decreasing and the tangent is horizontal. These three words are used throughout calculus to describe functions, whether or not a rate is involved.

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22 INCREASING, DECREASING AND STATIONARY

Suppose that $y = f(x)$ is defined at $x = a$, with $f'(a)$ also defined.

- If $f'(a)$ is positive (that is, if the tangent slopes upwards), then the curve is called *increasing* at $x = a$.
- If $f'(a)$ is negative (that is, if the tangent slopes downwards), then the curve is called *decreasing* at $x = a$.
- If $f'(a)$ is zero (that is, if the tangent is horizontal), then the curve is called *stationary* at $x = a$.

The diagram on the right above is a graph of the rate at which the population is changing. It is a straight line with negative gradient — this means that *the rate of change of the cockroach population was decreasing at a constant rate*.

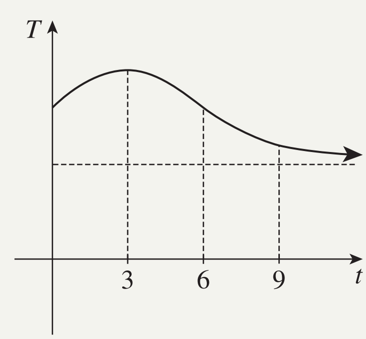
Questions with a diagram or a graph instead of an equation

In some problems about rates, the graph of some quantity Q over time is known, but its equation is unknown. Such problems require careful reading of the graph — pay attention to zeroes of the quantity and of the rate, and to where the quantity and rates are increasing, decreasing and stationary.

As always, it is often useful to draw an approximate sketch of the rate of change. The gradient of this second rate-of-change graph is the derivative of the rate of change $\frac{dQ}{dt}$, and is thus the second derivative $\frac{d^2Q}{dt^2}$ of the quantity.

Example 29 9J

The graph to the right shows the temperature T of a patient suffering from Symond's syndrome at time t hours after her admission to hospital at midnight.



a When did her temperature reach its maximum?

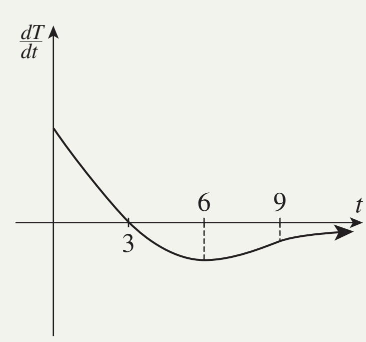
b When was her temperature increasing most rapidly,when was it decreasing most rapidly, and when was it stationary?

c What happened to her temperature eventually?

d Sketch the graph of the rate $\frac{dT}{dt}$ at which the temperature is changing.

e When was the rate of change decreasing, and when was it increasing?

SOLUTION



a The maximum temperature occurred when $t = 3$, that is, at 3:00 am.

b Her temperature was increasing most rapidly at the start when $t = 0$, that is, at midnight. It was decreasing most rapidly at 6:00 am. It was stationary at 3:00 am.

c The patient's temperature eventually stabilised.

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d The graph of $\frac{dT}{dt}$ is zero at $t = 3$, and minimum at $t = 6$.

As $t \to \infty$, $\frac{dT}{dt} \to 0$, so the t-axis is a horizontal asymptote.

e The rate of change of temperature was decreasing from midnight to 6:00 am, and increasing from 6:00 am onwards.

Exercise 9J

FOUNDATION

Note: Except in Questions 1–3, be careful to give the correct units in all final answers.

1 **a** If $Q = t^3 - 10t^2$, find the function $\frac{dQ}{dt}$ for the rate of change of Q.

b Hence find the quantity Q and the rate $\frac{dQ}{dt}$ when $t = 2$.

2 **a** If $Q = t^2 + 6t$, find the function $\frac{dQ}{dt}$ for the rate of change of Q.

b Hence find the quantity Q and the rate $\frac{dQ}{dt}$ when $t = 2$.

c When is the quantity Q:

i stationary, **ii** increasing, **iii** decreasing.

3 **a** If $Q = 8t - t^2$, find the values of Q when $t = 1$ and $t = 3$.

b Hence find the average rate of change from $t = 1$ to $t = 3$.

c Similarly find the average rate of change from $t = 5$ to $t = 7$.

4 Orange juice is being poured at a constant rate into a glass. After t seconds there are V mL of juice in the glass, where $V = 60t$.

a How much juice is in the glass after 3 seconds?

b Show that the glass was empty to begin with.

c If the glass takes 5 seconds to fill, what is its capacity?

d Differentiate to find the rate at which the glass is being filled.

e What sort of function is the derivative in part **d**?

5 The volume V litres of fuel in a tanker, t minutes after it has started to empty, is given by $V = 200\left(400 - t^2\right)$. Initially the tanker was full.

a Find the initial volume of fuel in the tanker.

b Find the volume of fuel in the tanker after 15 minutes.

c Find the time taken for the tanker to empty.

d Show that $\frac{dV}{dt} = -400t$, and hence find the rate at which the tanker is emptying after 5 minutes.

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DEVELOPMENT

6 Water surges into a rock pool, and then out again. The mass M in kilograms of water in the pool t seconds after time zero is given by the function $M = 10t - t^2$.

a Find the rate $\dfrac{dM}{dt}$ as a function of time t.

b Find (with units) the values of M and $\dfrac{dM}{dt}$ when $t = 4$.

c Find the value of M when $t = 2$, and hence find the average rate of change over the time interval from $t = 2$ to $t = 4$.

d At what times is M zero?

e For how long was there water in the pool?

f How long after time zero is $\dfrac{dM}{dt}$ zero?

g For how long was the water level in the pool rising, and for how long was it falling?

h Sketch the graphs of M and $\dfrac{dM}{dt}$ as functions of t, showing these results. Add the chord corresponding to the result in part **c** and the tangent corresponding to the result in part **f**.

7 The share price $\$P$ of the Eastcom Bank t years after it opened on 1 January 1970 was $P = -0.4t^2 + 4t + 2$.

a What was the initial share price?

b What was the share price after one year?

c At what rate was the share price increasing after two years?

d Find when the share price was stationary.

e Explain why the maximum share price was \$12, at the start of 1975.

f Explain why the rate of change of the price decreased at a constant rate.

g The directors decided to close the bank when the share price fell back to its initial value. When did this happen?

8 For a certain brand of medicine, the amount M present in the blood after t hours is given by $M = 3t^2 - t^3$, for $0 \le t \le 3$.

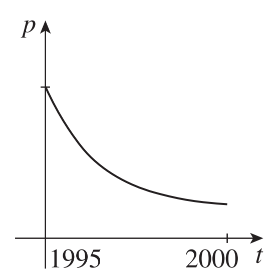
a Sketch a graph of M as a function of t.

b When is the amount of medicine in the blood a maximum?

c Differentiate to find the rate $\dfrac{dM}{dt}$, and find the axis of symmetry of the resulting parabola.

d When is the amount of medicine increasing most rapidly?

9 The graph shows the level p of pollution in a river between 1995 and 2000. In 1995, the local council implemented a scheme to reduce the level of pollution in the river.



Comment briefly on whether this scheme worked and how the level of pollution changed. Include mention of the rate of change.

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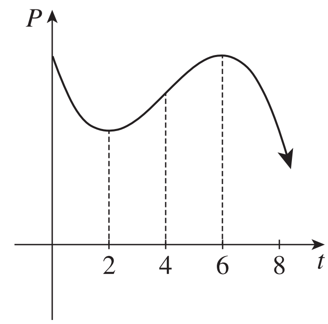
10 The graph to the right shows the share price P in Penn & Penn Stationery Suppliers t months after 1 January.



a When is the price stationary, neither increasing nor decreasing?
b When is the price maximum and when is it minimum?
c When is the price increasing and when is it decreasing?
d When is the share price increasing most rapidly?
e When is the share price increasing at an increasing rate?
f Sketch a possible graph of $\dfrac{dP}{dt}$ as a function of time t.

11 A lightning bolt hits the ground with a tremendous noise. The noise spreads out across the suburbs in a circle of area A, startling people.

a The speed of sound in air is about $\frac{1}{3}$ km/s. Show that the area A in which people are startled as a function of the time t in seconds after the strike is $A = \frac{\pi}{9}t^2$.
b Find the rate at which the area of startled people is increasing.
c Find, correct to four significant figures, the time when the area is $5\,\text{km}^2$, and the rate at which the area is increasing at that time.

12 [A question about motion]
A stone is dropped from the top of a tall building. The stone's height h metres above the ground t seconds later is $h = 80 - 5t^2$.

a How high is the building?
b How many seconds does the stone take to reach the ground?
c The velocity v is the rate of change $\dfrac{dh}{dt}$ of the height. Find the velocity function.
d How fast is the stone travelling when it hits the ground?
e Acceleration is the rate of change $\dfrac{dv}{dt}$ of the velocity. Find the acceleration of the stone (the units are 'metres-per-second, per second', usually written as m/s^2).

13 [A question about motion]
A truck driver was travelling on a straight, flat road. He later stated in court that at a certain 'time zero', he applied the brakes in such a way that his velocity decreased at a constant rate. His speed-monitoring equipment record showed that t seconds after time zero, his velocity was $v = -\frac{1}{2}t + 50$, in units of m/s.

a Acceleration is the rate of change of velocity — differentiate to find his acceleration as a function of time. Was the truck driver's statement in court correct, and why?
b His GPS recorded that his displacement x, relative to where the truck was at 'time zero', was $x = -\frac{1}{4}t^2 + 50t$. Velocity is the rate of change of displacement — differentiate this function, and say whether his GPS record agreed with his speed-monitoring equipment.
c The police car behind him tracked, by laser and GPS, his displacement y relative to where the police car was at 'time zero'. They found that $y = -\frac{1}{4}t^2 + 50t + 450$. Differentiate this function, and say whether the police car's record agreed with the truck's speed-monitoring equipment.
d How far behind the truck were the police at 'time zero'?
e How long did the truck take to stop, and how far did it go in that time?
f What was the truck's speed at 'time zero' in km/h, and why was he later in court?

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14 A 3000 cm long water trough has the shape of a triangular prism. Its cross-section is an isosceles triangle with apex 90°, and the trough is built with the apex at the bottom.

a **i** Find the area of an isosceles triangle with apex angle 90°, and height h centimetres.

ii Find the volume of water in the water trough when the water height is h cm.

b The water trough was initially empty, and then was filled with water at a constant rate of 27 litres per minute.

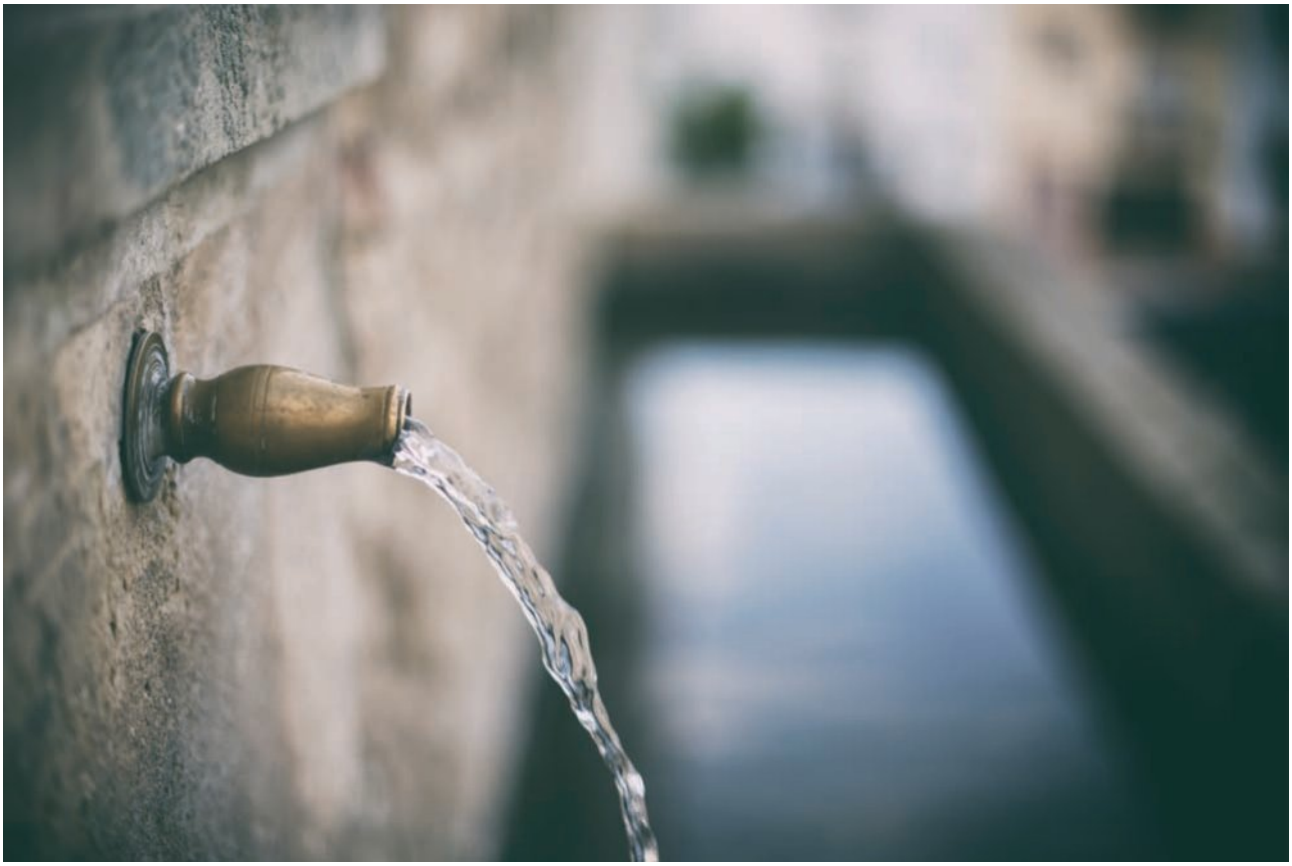
i Find the height h cm of the water after t minutes, and the rate the water was rising.

ii Find the water height, and the rate the water was rising, after 25 minutes.

iii Find when the water height was 30 cm, and the rate the water was then rising.

ENRICHMENT

15 A glass sphere with an opening at the top is being filled slowly with water in a controlled way so that the height rises at a constant rate. Show that the rate at which the surface area of the water is changing is decreasing at a constant rate.



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9K Continuity

A tangent can only be drawn at a point on a curve if:

- the curve has no break at the point (it is continuous at the point), and
- there is no sharp corner there (it is smooth at the point).

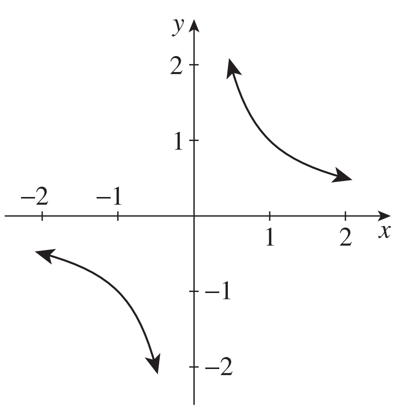
Sections 9K and 9L will make these two ideas a little more precise.

Continuity at a point

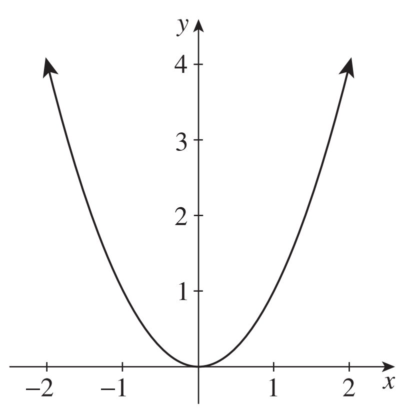
Continuity at a point means that there is no break in the curve at that point. This is an informal definition, but it will be enough for our purposes.

23 CONTINUITY AT A POINT — INFORMAL DEFINITION

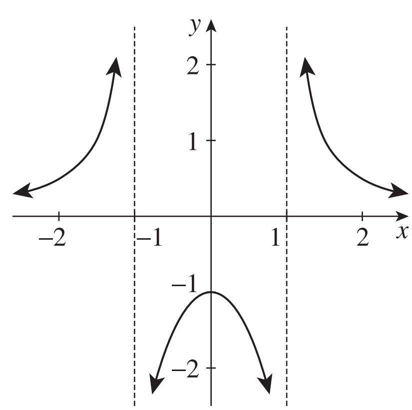
- A function $f(x)$ is called *continuous at* $x = a$ if the graph of $y = f(x)$ can be drawn *through* the point where $x = a$ without lifting the pencil off the paper.
- If there is a break in the curve, we say that there is a *discontinuity* at $x = a$.



$y = \dfrac{1}{x}$ has a discontinuity at $x = 0$, and is continuous everywhere else.



$y = x^2$ is continuous for all values of x.



$y = \dfrac{1}{x^2 - 1}$ has discontinuities at $x = 1$ and at $x = -1$, and is continuous everywhere else.

An assumption of continuity

It is intuitively obvious that a function such as $y = x^2$ is continuous for every value of x. Without further formality, we will make a general assumption of continuity, loosely stated as follows:

24 ASSUMPTION ABOUT CONTINUITY

The functions in this course are continuous for every value of x in their domain, except where there is an obvious problem.

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Piecewise-defined functions:

A *piecewise-defined function* has different definitions in different parts of its domain, for example:

$$f(x) = \begin{cases} 4 - x^2, & \text{for } x \le 0, \\ 4 + x, & \text{for } x > 0. \end{cases}$$

The two pieces of this graph are a parabola and line, but we need to find out whether these two parts join up or leave a gap. The most obvious way is to construct two tables of values, one for each line of the function's definition.

Example 30 — 9K

Let $f(x) = \begin{cases} 4 - x^2, & \text{for } x \le 0, \\ 4 + x, & \text{for } x > 0. \end{cases}$

a Find $f(0)$. Then construct tables of values of $4 - x^2$ and $4 + x$ to establish whether the graph is continuous at $x = 0$.

b Sketch the graph.

c Hence write down its domain and range.

SOLUTION

a First, $f(0) = 4 - 0^2 = 4$ (using the top line of the definition).

Secondly, the values of $4 - x^2$ on the left, and of $4 + x$ on the right, are

x	-2	-1	0
$4 - x^2$	0	3	4

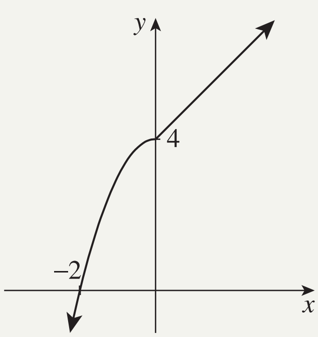
x	0	1	2
$4 + x$	4	5	6

Hence the curve is *continuous* at $x = 0$.

Reason: When $x = 0$, the values of $4 - x^2$ and $4 + x$ are 4, which is the same as $f(0)$, so the two pieces join up.

b The graph is made up of the parabolic piece on the left, and the linear piece on the right.

c The domain is all real x, and the range is all real y.



Testing whether piecewise-defined functions join up

In Example 30, all we really needed to do to test whether the pieces joined up at $x = 0$ was:

- Find $f(0)$.
- Substitute $x = 0$ into the formulae on the left.
- Substitute $x = 0$ into the formulae on the right.

This is the approach taken in Example 31.

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Example 31 9K

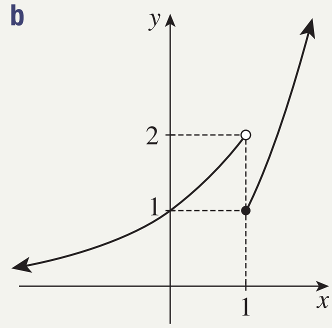
Let $f(x) = \begin{cases} 2^x, & \text{for } x < 1, \\ 1, & \text{for } x = 1, \\ x^2, & \text{for } x > 1. \end{cases}$

a Establish whether the graph is continuous at $x = 1$.
b Sketch the graph.
c Hence write down its domain and range.

SOLUTION

a First, $f(x) = 1$.
Secondly, $2^x = 2$, when $x = 1$.
Thirdly, $x^2 = 1$, when $x = 1$.
Hence the function is *not continuous* at $x = 1$.

b



c The domain is all real x, and the range is $y > 0$.

Note: When $x = 1$, the value of $f(x)$ is 1, so the endpoint $(1, 1)$ of the right-hand piece is included in the graph, as indicated by the closed circle. But the endpoint $(1, 2)$ of the left-hand piece is not included in the graph, as indicated by the open circle there.

25 OPEN AND CLOSED CIRCLES, AND ARROWS

- A *closed circle* marks an endpoint that is included in the graph.
- An *open circle* marks an endpoint that is not included in the graph.
- An *arrow* marks a curve that continues forever.

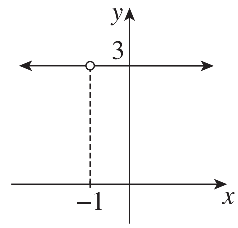
We used these same conventions in Section 2B when drawing intervals.

Functions in which numerator and denominator have the same zero

The function $f(x) = \dfrac{3x + 3}{x + 1}$ is tricky because $x = -1$ is a zero of the numerator and of the denominator. The function is not defined at $x = -1$, so it is not continuous there, but what does it look like elsewhere?

In this case, we can factor top and bottom. The factors will cancel everywhere except at the common zero:

$$f(x) = \frac{3(x + 1)}{x + 1}$$
$$= 3, \text{ provided that } x \neq -1.$$



The graph is drawn to the right. It is the horizontal line $y = 3$, with the single point $(-1, 3)$ removed. Its domain is $x \neq -1$, and is domain is $y = 3$.

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This is precisely the situation that occurred with first-principles differentiation. To find the gradient of $f(x) = x^2$ at $x = 1$, we calculated

$$\frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 - 1}{h} = \frac{h(2+h)}{h} = 2 + h, \quad \text{provided that} \quad h \neq 0,$$

so the limit as $h \to 0$ is 2. We cancelled because the value at $h = 2$ is irrelevant.

Exercise 9K

FOUNDATION

1 State the zeroes and discontinuities of each function.

a $f(x) = \dfrac{5}{6 - x}$

b $f(x) = \dfrac{3x}{(x-1)(x-3)(x-5)}$

c $f(x) = \dfrac{x(x+1)}{(x+2)(x+3)}$

2 Let $f(x) = \begin{cases} 1 - x, & \text{for } x < 0, \\ 1 + x^2, & \text{for } x \geq 0. \end{cases}$

a Find $f(0)$. Then copy and complete these tables of values.

x	-2	-1	0
$1 - x$			

x	0	1	2
$1 + x^2$			

b Is $f(x)$ continuous at $x = 0$?

c Sketch the graph, and write down its domain and range.

3 Let $y = \begin{cases} 2 - x, & \text{for } x \leq 1, \\ x - 1, & \text{for } x > 1. \end{cases}$

a Find $f(1)$. Then find the values of $2 - x$ and $x - 1$ when $x = 1$.

b Is the function continuous at $x = 1$?

c Sketch the graph, and write down its domain and range

DEVELOPMENT

4 Factor numerator and denominator, and hence write down the zeroes and discontinuities.

a $f(x) = \dfrac{1}{x^2 - 5x}$ **b** $f(x) = \dfrac{x}{x^2 - 5x + 6}$ **c** $f(x) = \dfrac{x^2 - 16}{x^2 - 9}$

5 Draw up a table of values for $y = \dfrac{|x|}{x}$, and explain whether the function is continuous at $x = 0$. Sketch the curve, and write down its domain and range.

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6 Find whether each function is continuous at $x = 2$. Then sketch the curve and state the domain and range.

a $f(x) = \begin{cases} x^3, & \text{for } x \le 2, \\ 10 - x, & \text{for } x > 2. \end{cases}$

b $f(x) = \begin{cases} 3^x, & \text{for } x < 2, \\ 13 - x^2, & \text{for } x > 2, \\ 4, & \text{for } x = 2. \end{cases}$

c $f(x) = \begin{cases} \dfrac{1}{x}, & \text{for } 0 < x < 2, \\ 1 - \dfrac{1}{4}x, & \text{for } x > 2, \\ \dfrac{1}{2}, & \text{for } x = 2. \end{cases}$

d $f(x) = \begin{cases} x, & \text{for } x < 2, \\ 2 - x, & \text{for } x > 2, \\ 2, & \text{for } x = 2. \end{cases}$

7 Cancel the algebraic fraction in each function, noting first the value of x for which the function is undefined. Then sketch the curve and state its domain and range.

a $y = \dfrac{x^2 + 2x + 1}{x + 1}$ **b** $y = \dfrac{x^4 - x^2}{x^2 - 1}$ **c** $y = \dfrac{x - 3}{x^2 - 4x + 3}$ **d** $y = \dfrac{3x + 3}{x + 1}$

8 Find for what values of a each function is continuous.

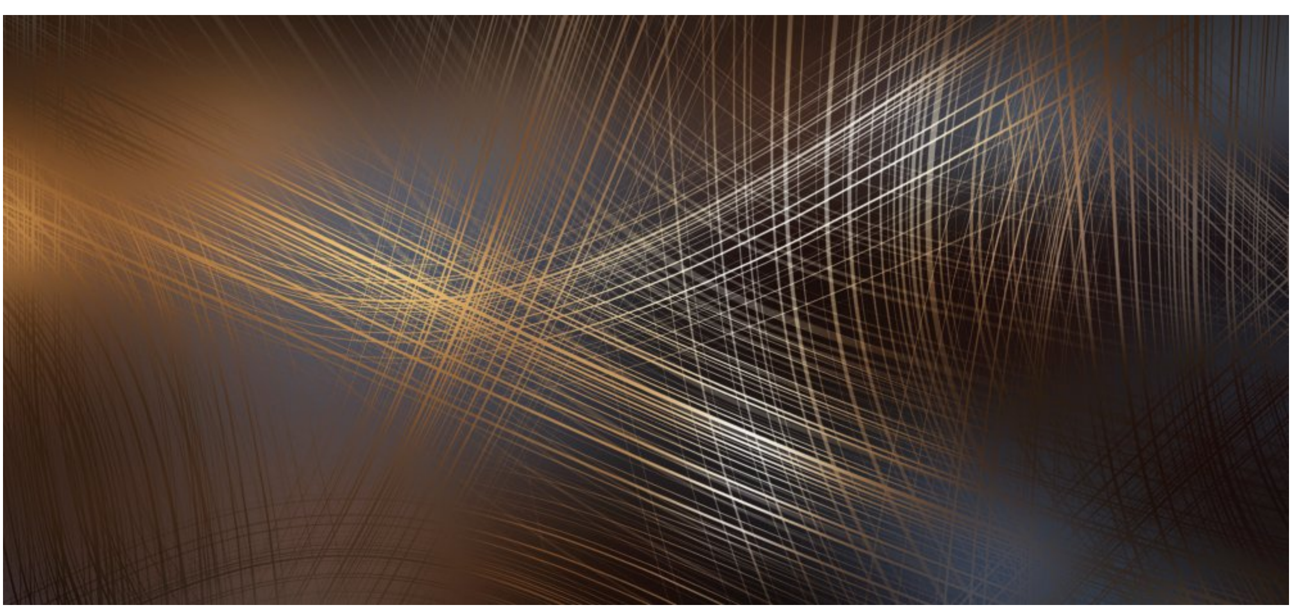
a $f(x) = \begin{cases} ax^2, & \text{for } x \le 1, \\ 6 - x, & \text{for } x > 1. \end{cases}$

b $g(x) = \begin{cases} \dfrac{a(x^2 - 9)}{x + 3}, & \text{for } x \ne -3, \\ 12, & \text{for } x = -3. \end{cases}$

ENRICHMENT

9 Find the zeroes and discontinuities of:

a $y = \dfrac{1}{\cos x° - 1}$ **b** $y = \dfrac{\cos x° + \sin x°}{\cos x° - \sin x°}$ **c** $y = \dfrac{\cos x° - \sin x°}{\cos x° + \sin x°}$



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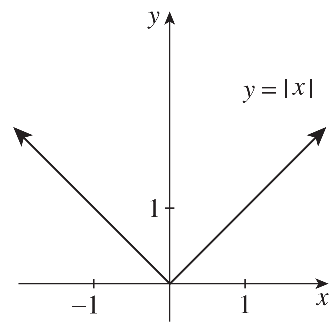
9L Differentiability

Look at the origin on the graph of $y = |x|$ to the right. The graph is continuous at $x = 0$ because the two pieces join up.



But this chapter is about differentiation, and there is no way that a tangent can be drawn at the origin, because it is a sharp corner where the gradient changes abruptly — from -1 on the left of the origin, to 1 on the right of the origin. So there is no tangent at the origin, and the derivative $f'(0)$ does not exist. We say that *the function is not differentiable at* $x = 0$.

A tangent can only be drawn at a point P on a curve if the curve is *smooth* at that point, meaning that the curve can be drawn through P without lifting the pencil off the paper, and without any sharp change of direction. The words *smooth* and *differentiable* are used interchangeably, and we have already used the word *smooth* on several occasions because it is intuitively obvious what it means.

26 DIFFERENTIABLE AT A POINT

A function $f(x)$ is called *differentiable* (or *smooth*) at $x = a$ if it satisfies two conditions there.

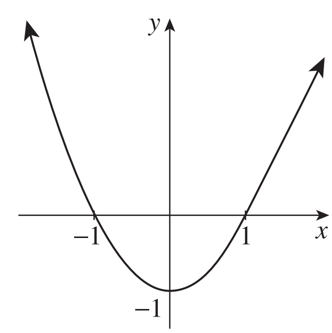
- The graph is continuous at $x = a$.
- The graph passes through the point where $x = a$ without any sharp corner, and the resulting tangent is not vertical there.

In this situation, $f'(a)$ is the gradient of the tangent at $x = a$.

If a tangent is vertical, its gradient, and therefore the derivative, is undefined.

Piecewise-defined functions

If the pieces of a piecewise-defined function don't join up at $x = a$, then the function is not continuous at $x = a$, so it is certainly not differentiable there. If it is continuous there, take the derivative in the two pieces, and see whether the gradients join up smoothly at $x = a$.



For example, the sketch opposite shows $f(x) = \begin{cases} x^2 - 1, & \text{for } x < 1, \\ 2x - 2, & \text{for } x \geq 1, \end{cases}$

A The graph is continuous, because the two pieces join at $P(1, 0)$:

$$f(1) = 0 \quad \text{and} \quad x^2 - 1 = 0 \text{ when } x = 1 \quad \text{and} \quad 2x - 2 = 0 \text{ when } x = 1.$$

B Now differentiate to find the gradient functions on the left and right of $x = 1$:

$$f'(x) = \begin{cases} 2x, & \text{for } x < 1, \\ 2, & \text{for } x > 1. \end{cases}$$

We can see from this that when the two pieces join, they do so with the same gradient 2, because

$$2x = 2 \text{ when } x = 1 \qquad \text{and} \qquad 2 = 2 \text{ when } x = 1.$$

The combined curve is therefore *smooth* or *differentiable* at the point $P(1, 0)$, with derivative $f'(1) = 2$, and a well-defined tangent there of gradient 2.

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27 DIFFERENTIABILITY OF A PIECEWISE-DEFINED FUNCTION

To test a function for differentiability at a join between pieces at $x = a$:

1 Test whether the function is continuous at $x = a$.

2 If it is, look at $f'(x)$ on the left and right to see whether the join is smooth.

Example 32 — 9L

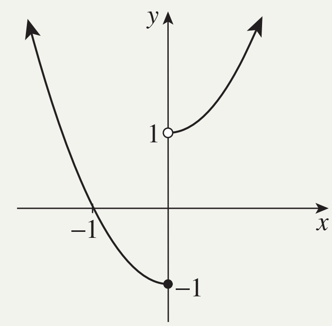
Test each function for continuity, and then for differentiability, at $x = 0$. Then sketch the curve, and write down its domain and range.

a $f(x) = \begin{cases} x^2 - 1, & \text{for } x \le 0, \\ x^2 + 1, & \text{for } x > 0. \end{cases}$

b $f(x) = \begin{cases} x, & \text{for } x \le 0, \\ x - x^2, & \text{for } x > 0. \end{cases}$

SOLUTION

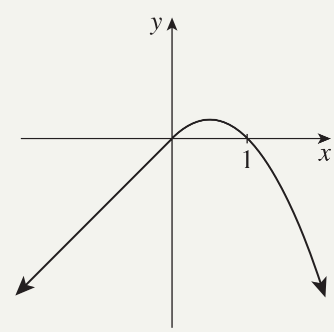
a First, $x^2 - 1 = -1$ when $x = 0$, and $x^2 + 1 = 1$ when $x = 0$,
so the function is not even continuous at $x = 0$,
let alone differentiable there.
Domain: all real x, range: $x \ge -1$



b First, $f(x) = f(0) = 0$, and $x = 0$ when $x = 0$,
and $x - x^2 = 0$ when $x = 0$
so the function is continuous at $x = 0$.

Secondly, $f'(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - 2x, & \text{for } x > 0, \end{cases}$

and $1 = 1$ when $x = 0$, and $1 - 2x = 1$ when $x = 0$,
so the function is differentiable at $x = 0$, with $f'(0) = 1$.
Domain: all real x, range: $x \le \frac{1}{4}$
The quadratic piece has vertex $\left(\frac{1}{2}, \frac{1}{4}\right)$ and a zero at $x = 1$.

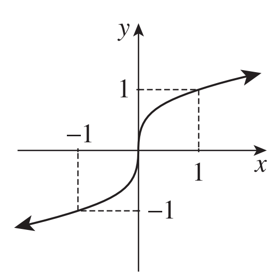


Vertical tangents

A graph $y = f(x)$ may have a vertical tangent at $x = a$, but as mentioned above, its gradient and $f'(a)$ are then undefined, and it is not differentiable there. For example, the curve on the right is

$$f(x) = x^{\frac{1}{3}}, \quad \text{whose derivative is} \quad f'(x) = \tfrac{1}{3}x^{-\frac{2}{3}}.$$

This function is the inverse function of $y = x^3$, so its graph is the graph of $y = x^3$ reflected in the diagonal line $y = x$. The curve is continuous at $x = 0$, and the y-axis is a vertical tangent there. The curve becomes infinitely steep on both sides of the origin, and it is not differentiable there. This is an example of a *vertical inflection* — discussed in Year 12.



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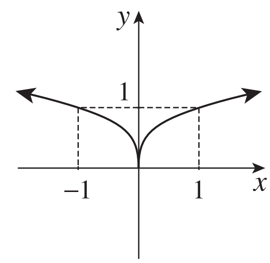
Cusps



A stranger picture is provided by the closely related function

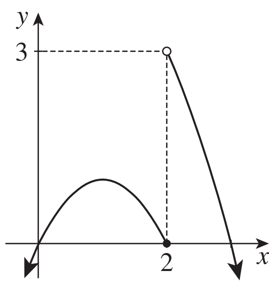
$$f(x) = x^{\frac{2}{3}}, \quad \text{whose derivative is} \quad f'(x) = \tfrac{2}{3}x^{-\frac{1}{3}}.$$

This function is also continuous at the origin, and again $f'(0)$ is undefined, but this time there is no tangent there. The function is never negative because it is a square, and the curve is infinitely steep on both sides of the origin, with one side sloping backwards and the other forwards. The point $(0, 0)$ is called a *cusp*.

Exercise 9L

FOUNDATION

1 State whether each function $f(x)$ is continuous at $x = 0$ and at $x = 2$, and whether it is differentiable (smooth) there.

a


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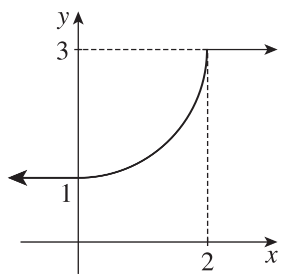

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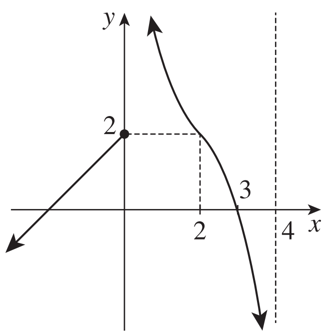

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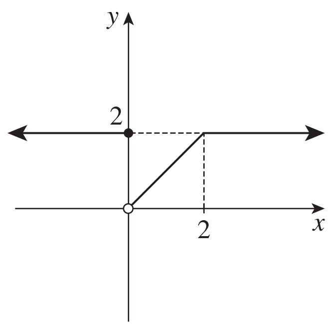

2 a Sketch the graph of the function $f(x) = \begin{cases} x^2, & \text{for } x \le 1, \\ 2x - 1, & \text{for } x > 1. \end{cases}$

b Show that the function is continuous at $x = 1$.

c Explain why $f'(x) = \begin{cases} 2x, & \text{for } x < 1, \\ 2, & \text{for } x > 1. \end{cases}$

d Hence explain why there is a tangent at $x = 1$, state its gradient, and state $f'(1)$.

DEVELOPMENT

3 Sketch each function. State any values of x where it is not continuous or not differentiable.

a $y = |x + 2|$

b $y = |x| + 2$

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4 Test each function for continuity at $x = 1$. If the function is continuous there, check for differentiability at $x = 1$. Then sketch the graph.

a $f(x) = \begin{cases} (x+1)^2, & \text{for } x \le 1, \\ 4x - 2, & \text{for } x > 1. \end{cases}$

b $f(x) = \begin{cases} 3 - 2x, & \text{for } x < 1, \\ 1/x, & \text{for } x \ge 1. \end{cases}$

c $f(x) = \begin{cases} 2 - x^2, & \text{for } x \le 1, \\ (x-2)^2, & \text{for } x > 1. \end{cases}$

d $f(x) = \begin{cases} (x-1)^3, & \text{for } x \le 1, \\ (x-1)^2, & \text{for } x > 1. \end{cases}$

5 Find whether each function is differentiable at $x = 0$.

a $y = |x^2|$ **b** $y = |x^3|$ **c** $y = \sqrt{|x|}$

6 Sketch each function, giving any values of x where it is not continuous or not differentiable.

a $y = |x + 2|$ **b** $y = \dfrac{1}{|x+2|}$ **c** $y = |x^2 - 4x + 3|$

d $y = \dfrac{1}{|x^2 - 4x + 3|}$ **e** $y = |x^2 + 2x + 2|$ **f** $y = \dfrac{1}{|x^2 + 2x + 2|}$

7 **a** Sketch $y = \begin{cases} \sqrt{1 - (x+1)^2}, & \text{for } -2 \le x \le 0, \\ \sqrt{1 - (x-1)^2}, & \text{for } 0 < x \le 2. \end{cases}$ Describe the situation at $x = 0$.

b Repeat part **a** for $f(x) = \begin{cases} \sqrt{1 - (x+1)^2}, & \text{for } -2 \le x \le 0, \\ -\sqrt{1 - (x-1)^2}, & \text{for } 0 < x \le 2. \end{cases}$

8 **a** Differentiate $f(x) = x^{\frac{1}{5}}$. Then sketch the curve and state whether it has a vertical tangent or a cusp at $x = 0$.

b Repeat for $f(x) = x^{\frac{2}{5}}$.

ENRICHMENT

9 Each example gives a curve and two points P and Q on the curve. Find the gradient of the chord PQ. Then find the coordinates of any points M on the curve between P and Q such that the tangent at M is parallel to PQ.

a $y = x^2 - 6x$, $P = (0, 0)$, $Q = (10, 40)$

b $y = x^3 - 9$, $P = (-1, -10)$, $Q = (2, -1)$

c $y = \dfrac{1}{x}$, $P = (1, 1)$, $Q = \left(4, \frac{1}{4}\right)$

d $y = \dfrac{1}{x}$, $P = (-1, -1)$, $Q = (1, 1)$

e $y = |x|$, $P = (-1, 1)$, $Q = (1, 1)$

f $y = x^2$, $P = (\alpha, \alpha^2)$, $Q = (\beta, \beta^2)$

Note: The existence of at least one such point is guaranteed by the *mean value theorem*, provided that the curve is differentiable everywhere between the two points.

10 Suppose that p and q are integers with no common factors and with $q > 0$. Write down the derivative of $f(x) = x^{\frac{p}{q}}$, and hence find the conditions on p and q for which:

a $f(x)$ is defined for $x < 0$,

b $f(x)$ is defined at $x = 0$,

c $f(x)$ is defined for $x > 0$,

d $f(x)$ is continuous at $x = 0$,

e $f(x)$ is continuous for $x \ge 0$,

f $f(x)$ is differentiable at $x = 0$,

g there is a vertical tangent at the origin,

h there is a cusp at the origin.

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Chapter 9 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 9 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

Review

1 Use the definition of the derivative $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find the derivative of each function by first principles.

a $f(x) = x^2 + 5x$ **b** $f(x) = 6 - x^2$ **c** $f(x) = 3x^2 - 2x + 7$

2 Write down the derivative of each function. You will need to expand any brackets.

a $y = x^3 - 2x^2 + 3x - 4$ **b** $y = x^6 - 4x^4$ **c** $y = 3x^2(x - 2x^3)$

d $y = (x + 3)(x - 2)$ **e** $y = (2x - 1)(2 - 3x)$ **f** $y = 3x^{-2} - 2x^{-1}$

g $y = 4x^3 - 4x^{-3}$ **h** $y = 3x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$ **i** $y = x^{-2}(x^2 - x + 1)$

3 Find the first and second derivatives of each function.

a $f(x) = x^4 + x^3 + x^2 + x + 1$

b $f(x) = 5x^{-2}$

c $f(x) = 8x^{-\frac{1}{2}}$

4 Find the family of curves whose derivative is:

a $\dfrac{dy}{dx} = 3x^2 + 4$ **b** $\dfrac{dy}{dx} = 7 - 12x - 12x^2$ **c** $\dfrac{dy}{dx} = 20x^4 - 12x^2 + 4$

5 Rewrite each function in index notation and then differentiate it. Write each final answer without negative or fractional indices.

a $y = \dfrac{3}{x}$ **b** $y = \dfrac{1}{6x^2}$ **c** $y = 7\sqrt{x}$

d $y = \sqrt{144x}$ **e** $y = -3x\sqrt{x}$ **f** $y = \dfrac{6}{\sqrt{x}}$

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Review

6 Divide each function through by the numerator and then differentiate it. Give each final answer without negative or fractional indices.

a $y = \dfrac{3x^4 - 2x^3}{x^2}$ **b** $y = \dfrac{x^3 - x^2 + 7x}{2x}$ **c** $y = \dfrac{5x^3 - 7}{x}$

d $y = \dfrac{x^2 + 2x + 1}{x^2}$ **e** $y = \dfrac{4x + 5\sqrt{x}}{\sqrt{x}}$ **f** $y = \dfrac{2x^2\sqrt{x} + 3x\sqrt{x}}{x}$

7 Use the formula $\dfrac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ to differentiate:

a $y = (3x + 7)^3$ **b** $y = (5 - 2x)^2$ **c** $y = \dfrac{1}{5x - 1}$

d $y = \dfrac{1}{(2 - 7x)^2}$ **e** $y = \sqrt{5x + 1}$ **f** $y = \dfrac{1}{\sqrt{1 - x}}$

8 Use the chain rule to differentiate:

a $y = (7x^2 - 1)^3$ **b** $y = (1 + x^3)^{-5}$ **c** $y = (1 + x - x^2)^8$

d $y = \dfrac{1}{(x^2 - 1)^3}$ **e** $y = \sqrt{9 - x^2}$ **f** $y = \dfrac{1}{\sqrt{9 - x^2}}$

9 Differentiate each function, using the product or quotient rule. Factor each answer completely.

a $y = x^9(x + 1)^7$ **b** $y = \dfrac{x^2}{1 - x}$ **c** $y = x^2(4x^2 + 1)^4$

d $y = \dfrac{2x - 3}{2x + 3}$ **e** $y = (x + 1)^5(x - 1)^4$ **f** $y = \dfrac{x^2 + 5}{x - 2}$

10 Differentiate $y = x^2 + 3x + 2$. Hence find the gradient and the angle of inclination, correct to the nearest minute when appropriate, of the tangents at the points where:

a $x = 0$ **b** $x = -1$ **c** $x = -2$

11 a Find the equations of the tangent and normal to $y = x^3 - 3x$ at the origin.

b Find the equation of the tangent and normal to the curve at the point $P(1, -2)$.

c By solving $f'(x) = 0$, find the points on the curve where the tangent is horizontal.

d Find the points on the curve where the tangent has gradient 9.

12 a Find the equations of the tangent and normal to $y = x^2 - 5x$ at the point $P(2, -6)$.

b The tangent and normal at P meet the x-axis at A and B respectively. Find the coordinates of A and B.

c Find the length of the interval AB and the area of the triangle PAB.

13 Find the tangents to the curve $y = x^4 - 4x^3 + 4x^2 + x$ at the origin and at the point where $x = 2$, and show that these two lines are the same.

14 Find any points on each curve where the tangent has the given angle of inclination.

a $y = \frac{1}{3}x^3 - 7$, 45° **b** $y = x^2 + \frac{1}{3}x^3$, 135°

15 a Find the points on $y = x^3 - 4x$ where the tangents are parallel to $\ell: x + y + 2 = 0$.

b Find the equations of these two tangents, and show that ℓ is one of them.

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16 Find the x-coordinates of any points on $y = (4x - 7)^3$ where the tangent is:

a parallel to $y = 108x + 7$

b perpendicular to $x + 12y + 6 = 0$

17 A huge fireball high in the air is exploding at the speed of sound, which is about $\frac{1}{3}$ km/s.

a Using the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere, show that the volume V of the fireball after t seconds is $V = \frac{4\pi}{81}t^3$.

b Differentiate to find the rate at which the volume is expanding.

c Find, correct to two significant figures, the volume and the rate after 0.1 seconds.

d Find, correct to two significant figures, when the volume is increasing by 1 km^3/s.

18 Let $f(x) = \begin{cases} x^2, & \text{for } x \le 0, \\ x^2 + 1, & \text{for } x > 0. \end{cases}$

a Is the curve continuous at $x = 0$?

b Write down the domain and range of the function.

19 Let $f(x) = \begin{cases} x^2 - 4, & \text{for } x \le 2, \\ 4x - 8, & \text{for } x > 2. \end{cases}$

a Is the curve continuous at $x = 2$?

b Is the curve differentiable at $x = 2$?

c What are the domain and range of the function?

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10 Polynomials

Differentiating a quadratic function produces a linear function, differentiating a cubic function produces a quadratic function, and so on, so ultimately the study even of linear functions involves the study of polynomial functions of arbitrary degree. In this course, linear and quadratic functions are discussed in detail, but this chapter begins the systematic study of polynomials of higher degree.

The terminology of polynomials was mentioned briefly in Chapter 3, and readers may already have some familiarity with their graphs, and with division of polynomials, the remainder theorem and the factor theorems. The later sections in the chapter deal with the relationships between the coefficients and the zeroes, and with multiple zeroes. The problem of factoring a given polynomial is a constant theme, and the final Section 10H applies the methods of the chapter to geometrical problems about polynomial curves, circles and rectangular hyperbolas.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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10A The language of polynomials

Polynomials are expressions such as the quadratic $x^2 - 5x + 6$ or the quartic $3x^4 - \frac{2}{3}x^3 + 4x + 7$. They have occurred already in the course, but now our language and notation needs to be a little more precise.

1 POLYNOMIALS

- A *polynomial* is an expression of the form
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$
where the coefficients $a_0, a_1, \ldots, a_n$ are constants, and n is a whole number.
- A *polynomial function* is a function that can be written as a polynomial.

We will seldom have any reason to distinguish between a polynomial expression and a polynomial function.

The term a_0 is called the *constant term*. This is the value of the polynomial at $x = 0$, so is the y-intercept of its graph. The constant term can also be written as $a_0 x^0$, then every term is a multiple $a_k x^k$ of a whole-number power of x.

(Careful readers may notice that $a_0 x^0$ is undefined at $x = 0$. This means that rewriting a quadratic $x^2 + 3x + 2$ as $x^2 + 3x^1 + 2x^0$ causes a problem at $x = 0$. To overcome this, the convention is made that the term $a_0 x^0$ is interpreted as a_0 before any substitution is performed.)

Leading term and degree

The term of highest index with non-zero coefficient is called the *leading term*. Its coefficient is called the *leading coefficient* of the polynomial, and its index is called the *degree*. For example, the polynomial

$$P(x) = -5x^6 - 3x^4 + 2x^3 + x^2 - x + 9$$

has leading term $-5x^6$ and leading coefficient -5. It has degree 6, written as

$$\deg P(x) = 6.$$

A *monic polynomial* is a polynomial whose leading coefficient is 1. For example, $P(x) = x^3 - 2x^2 - 3x + 4$ is monic. Every non-zero polynomial is a unique multiple of a monic polynomial:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n\left(x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \cdots + \frac{a_1}{a_n}x + \frac{a_0}{a_n}\right).$$

Some names of polynomials

Polynomials of low degree have standard names.

- The *zero polynomial* $Z(x) = 0$ is a special case. It has a constant term 0. But it has no term with a non-zero coefficient, and therefore has no leading term, no leading coefficient, and most importantly, no degree. It is also quite exceptional in that its graph is the x-axis, so that every real number is a zero of the zero polynomial.

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- A *constant polynomial* is a polynomial whose only term is the constant term, for example,

$$P(x) = 4, \quad Q(x) = -\tfrac{3}{5}, \quad R(x) = \pi, \quad Z(x) = 0.$$

Apart from the zero polynomial, all constant polynomials have degree 0, have no zeroes, and are equal to their leading term and to their leading coefficient.

- A *linear polynomial* is a polynomial of degree 1 such as

$$P(x) = x - 3, \quad Q(x) = 4x + 7, \quad R(x) = -\tfrac{1}{2}x.$$

A constant polynomial is not a linear polynomial. Be careful that constant functions are linear functions, but constant polynomials are not linear polynomials. This is for convenience — finding linear factors of a polynomial is a central concern, whereas taking out constant factors is trivial.

- A polynomial of degree 2 is called a *quadratic polynomial*:

$$P(x) = 3x^2 + 4x - 1, \quad Q(x) = -\tfrac{1}{2}x - x^2, \quad R(x) = 9 - x^2.$$

Notice that the coefficient of x^2 must be non-zero for the degree to be 2.

- Polynomials of higher degree are called *cubics* (degree 3), *quartics* (degree 4), *quintics* (degree 5), and so on.

Addition and subtraction

Any two polynomials can be added or subtracted, and the results are again polynomials:

$$(5x^3 - 4x + 3) + (3x^2 - 3x - 2) = 5x^3 + 3x^2 - 7x + 1$$
$$(5x^3 - 4x + 3) - (3x^2 - 3x - 2) = 5x^3 - 3x^2 - x + 5$$

The zero polynomial $Z(x) = 0$ is the zero for addition, in the usual sense that $P(x) + Z(x) = P(x)$, for all polynomials $P(x)$. The *opposite polynomial* $-P(x)$ of any polynomial $P(x)$ is obtained by taking the opposite of every coefficient. Then the sum of $P(x)$ and $-P(x)$ is the zero polynomial. For example,

$$(4x^4 - 2x^2 + 3x - 7) + (-4x^4 + 2x^2 - 3x + 7) = 0.$$

The degree of the sum or difference of two polynomials is usually the maximum of the degrees of the two polynomials, as in the first example above, where the two polynomials have degrees 2 and 3 and their sum has degree 3. If, however, the two polynomials have the same degree, then the leading terms may cancel out and disappear, for example,

$$(x^2 - 3x + 2) + (9 + 4x - x^2) = x + 11, \text{ which has degree 1,}$$

or the two polynomials may be opposites, in which case everything cancels out so that their sum is zero and thus has no degree.

2 DEGREE OF THE SUM AND DIFFERENCE

Let $P(x)$ and $Q(x)$ be non-zero polynomials of degree n and m respectively.

- If $n \neq m$, then $\deg\big(P(x) + Q(x)\big) = \text{maximum of } m \text{ and } n.$
- If $n = m$, then $\deg\big(P(x) + Q(x)\big) \le n \quad \text{or} \quad P(x) + Q(x) = 0.$

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Multiplication

Any two polynomials can be multiplied, giving another polynomial:

$$\begin{aligned}(3x^3 + 2x + 1) \times (x^2 - 1) &= (3x^5 + 2x^3 + x^2) - (3x^3 + 2x + 1)\\ &= 3x^5 - x^3 + x^2 - 2x - 1\end{aligned}$$

The constant polynomial $I(x) = 1$ is the identity for multiplication, in the sense that $P(x) \times I(x) = P(x)$, for all polynomials $P(x)$. Multiplication by the zero polynomial on the other hand always gives the zero polynomial.

If two polynomials are non-zero, then the degree of their product is the sum of their degrees, because the leading term of the product is always the product of the two leading terms.

3 DEGREE OF THE PRODUCT

If $P(x)$ and $Q(x)$ are non-zero polynomials, then

$$\deg\big(P(x) \times Q(x)\big) = \deg P(x) + \deg Q(x).$$

Identically equal polynomials

We need to be quite clear what is meant by saying that two polynomials are the same.

4 IDENTICALLY EQUAL POLYNOMIALS

- Two polynomials $P(x)$ and $Q(x)$ are called *identically equal*, often written as $P(x) \equiv Q(x)$, if they are equal for all values of x:

  $P(x) = Q(x)$, for all x.

- If two polynomials are identically equal, then the corresponding coefficients of the two polynomials are equal.

The second assertion in the box above is not completely obvious, and it will be convenient to delay the proof until the necessary machinery has been constructed in Section 10E. In the meantime, here is an example that uses the result.

Example 1 10A

Find a, b, c, d and e if $ax^4 + bx^3 + cx^2 + dx + e = (x^2 - 3)^2$ for all x.

SOLUTION

Expanding, $(x^2 - 3)^2 = x^4 - 6x^2 + 9$.
Now comparing coefficients, $a = 1$, $b = 0$, $c = -6$, $d = 0$ and $e = 9$.

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Factoring polynomials

The most significant problem of this chapter is the factoring of a given polynomial. For example,

$$x(x + 2)^2(x - 2)^2(x^2 + x + 1) = x^7 + x^6 - 7x^5 - 8x^4 + 8x^3 + 16x^2 + 16x$$

is a routine expansion of a factored polynomial, but it is not at all clear how to move in the other direction from the expanded form back to the factored form.

Polynomial equations

If $P(x)$ is a polynomial, then the equation formed by putting $P(x) = 0$ is a *polynomial equation*. For example, using the polynomial in the previous paragraph, we can form the polynomial equation

$$x^7 + x^6 - 7x^5 - 8x^4 + 8x^3 + 16x^2 + 16x = 0.$$

Solving polynomial equations and factoring polynomial functions are very closely related. Using the factoring of the previous paragraph,

$$x(x + 2)^2(x - 2)^2(x^2 + x + 1) = 0,$$

so the solutions are $x = 0$, $x = 2$ and $x = -2$. Notice that the quadratic factor $x^2 + x + 1$ has no zeroes, because its discriminant is $\Delta = -3$. Thus factoring a polynomial into linear and irreducible quadratic factors solves the corresponding polynomial equation.

The solutions of a polynomial equation are called *roots*, whereas the *zeroes* of a polynomial function are the values of x where the value of the polynomial is zero. The distinction between the two words is not always strictly observed.

Exercise 10A

FOUNDATION

1 State whether or not each expression is a polynomial.

a $3x^2 - 7x$ **b** $\dfrac{1}{x^2} + x$ **c** $\sqrt{x} - 2$ **d** $3x^{\frac{2}{3}} - 5x + 11$

e $\sqrt{3}x^2 + \sqrt{5}x$ **f** $2^x - 1$ **g** $(x + 1)^3$ **h** $\dfrac{7x^{13} + 3x}{4}$

i $\log_e x$ **j** $\frac{4}{3}x^3 - ex^2 + \pi x$ **k** 5 **l** $\dfrac{x - 2}{x + 1}$

2 For each polynomial, state: **i** the degree, **ii** the leading coefficient, **iii** the leading term, **iv** the constant term, **v** whether or not the polynomial is monic.
Expand the polynomial first where necessary.

a $4x^3 + 7x^2 - 11$ **b** $10 - 4x - 6x^3$ **c** 2

d x^{12} **e** $x^2(x - 2)$ **f** $(x^2 - 3x)(1 - x^3)$

g 0 **h** $x(x^3 - 5x + 1) - x^2(x^2 - 2)$

i $6x^7 - 4x^6 - (2x^5 + 1)(5 + 3x^2)$

3 If $P(x) = 5x + 2$ and $Q(x) = x^2 - 3x + 1$, find:

a $P(x) + Q(x)$ **b** $Q(x) + P(x)$ **c** $P(x) - Q(x)$

d $Q(x) - P(x)$ **e** $P(x) \times Q(x)$ **f** $Q(x) \times P(x)$

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4 If $P(x) = 5x + 2$, $Q(x) = x^2 - 3x + 1$ and $R(x) = 2x^2 - 3$, show, by simplifying the LHS and RHS separately, that:

a $\big(P(x) + Q(x)\big) + R(x) = P(x) + \big(Q(x) + R(x)\big)$

b $P(x)\big(Q(x) + R(x)\big) = P(x)Q(x) + P(x)R(x)$

c $\big(P(x)Q(x)\big)R(x) = P(x)\big(Q(x)R(x)\big)$

DEVELOPMENT

5 Factor each polynomial completely, and write down all its zeroes.

a $x^3 - 8x^2 - 20x$ **b** $2x^4 - x^3 - x^2$ **c** $x^4 - 81$ **d** $x^4 - 5x^2 - 36$

6 For each polynomial, determine: **i** the degree, **ii** the leading coefficient, **iii** the constant term.

a $(2x^3 - 3)^3$ **b** $(2x^2 + 1)(3x^3 - 2)(4x^4 + 3)(5x^5 - 4)$

7 a The polynomials $P(x)$ and $Q(x)$ have degrees p and q respectively, and $p \neq q$. What is the degree of: **i** $P(x)Q(x)$, **ii** $P(x) + Q(x)$?

b What differences would it make if $P(x)$ and $Q(x)$ both had the same degree p?

c Give an example of two polynomials, both of degree 2, which have a sum of degree 0.

8 Write down the monic polynomial whose degree, leading coefficient, and constant term are all equal.

9 Find a, b and c, given that the following pairs of polynomials are identically equal.

a $ax^2 + bx + c = 3x^2 - 4x + 1$, for all x.

b $(a - b)x^2 + (2a + b)x = 7x - x^2$, for all x.

c $a(x - 1)^2 + b(x - 1) + c = x^2$, for all x.

d $a(x + 2)^2 + b(x + 3)^2 + c(x + 4)^2 = 2x^2 + 8x + 6$, for all x.

10 a Suppose that $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is even, so that $P(-x) = P(x)$. Show that $b = d = 0$.

b Suppose that $Q(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is odd, so that $Q(-x) = -Q(x)$. Show that $b = d = f = 0$.

c Give a general statement of the situation in parts **a** and **b**.

11 Suppose that $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ are polynomials. Indicate whether the following statements are true or false. Provide a counter-example for any false statements.

a If $P(x)$ is even, then $P'(x)$ is odd. **b** If $Q'(x)$ is even, then $Q(x)$ is odd.

c If $R(x)$ is odd, then $R'(x)$ is even. **d** If $S'(x)$ is odd, then $S(x)$ is even.

ENRICHMENT

12 a Find a and b so that $x^4 + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$.

b Find a and b so that $x^4 + x^2 + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$.

c Find a and b so that $x^4 - x^2 + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$.

d Show that all the resulting quadratics in parts **a**, **b** and **c** are irreducible.

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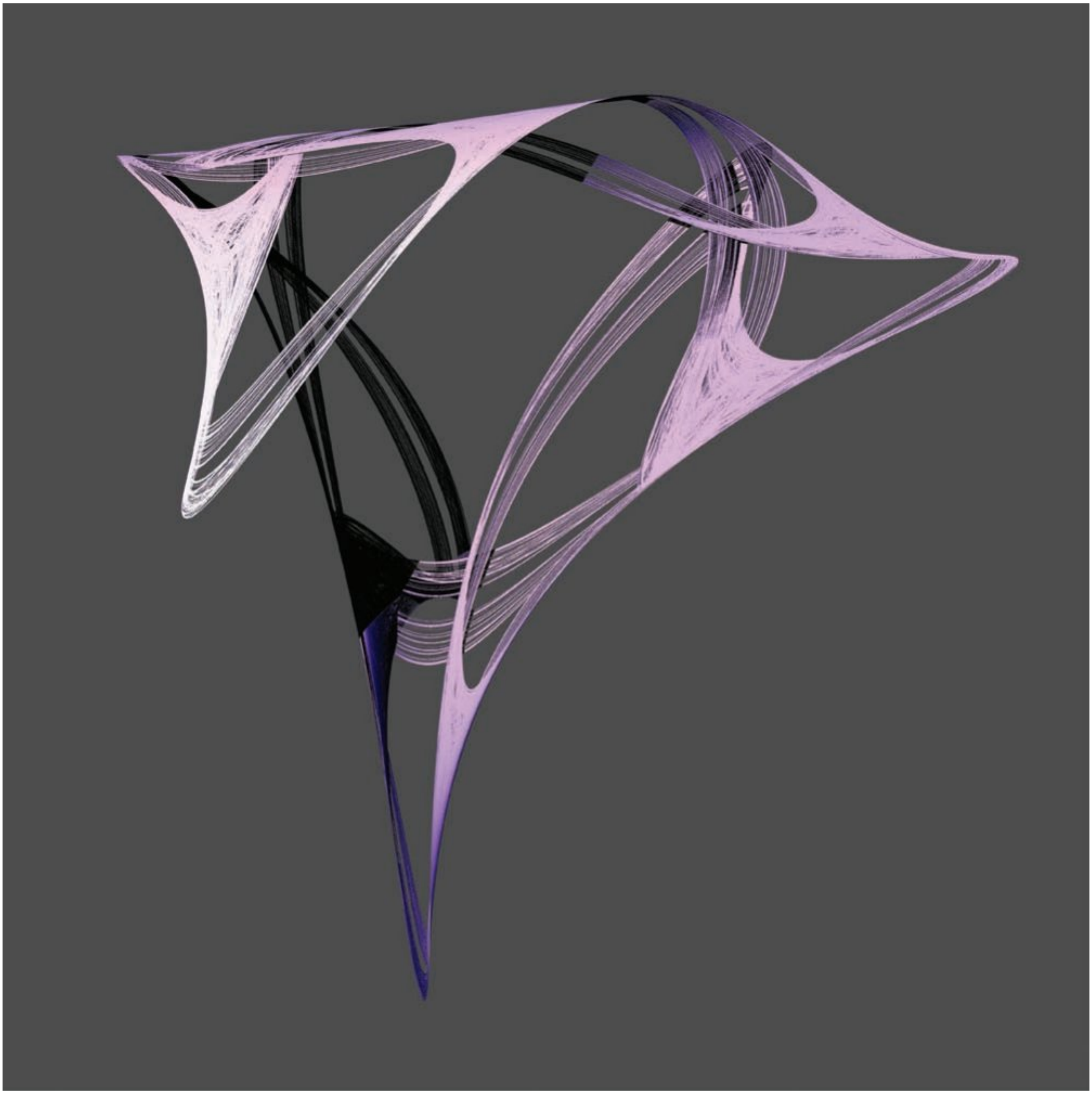
13 **a** What is the coefficient of x in the polynomial $B(x) = (1 + x)^n$?
b What is the coefficient of x^n in the polynomial $G(x) = (1 + x + x^2 + \cdots + x^n)^2$?

14 We claimed in Box 4 of the notes above that if two polynomials $P(x)$ and $Q(x)$ are equal for all values of x (that is, if their graphs are the same), then their degrees are equal and their corresponding coefficients are equal. Here is a proof using calculus.
a Explain why substituting $x = 0$ proves that the constant terms are equal.
b Explain why differentiating k times and substituting $x = 0$ proves that the coefficients of x^k are equal.



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10B Graphs of polynomial functions

A lot of work has already been done on sketching polynomial functions. We know already that the graph of any polynomial function will be a continuous and differentiable curve, whose domain is all real numbers, and which possibly intersects the x-axis at one or more points. This section will concentrate on two main concerns.

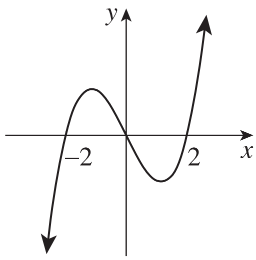
- First, how does the graph behave for large positive and negative values of x?
- Secondly, given the full factoring of the polynomial, how does the graph behave near its various x-intercepts?

We will not be concerned here with further questions about stationary points and inflections that are not also zeroes of the polynomial.

The graphs of polynomial functions

It should be intuitively obvious that for large positive and negative values of x, the behaviour of the curve is governed entirely by the sign of its leading term. For example, the cubic graph sketched on the right below is

$$P(x) = x^3 - 4x = x(x - 2)(x + 2).$$



For large positive values of x, the degree 1 term $-4x$ is negative, but is completely swamped by the positive values of the degree 3 term x^3. Hence $P(x) \to \infty$ as $x \to \infty$. Similarly, for large negative values of x, the term $-4x$ is positive, but is negligible compared with the far bigger negative values of the term x^3. Hence $P(x) \to -\infty$ as $x \to -\infty$.

In the same way, every polynomial of odd degree has a graph that disappears off diagonally opposite corners. Being continuous, it must therefore be zero somewhere. Our example actually has three zeroes, but however much it were raised, or lowered, or twisted by manipulating the coefficients, only two zeroes could ever be removed. Here is the general situation.

5 BEHAVIOUR OF POLYNOMIALS FOR LARGE x

Suppose that $P(x)$ is a polynomial of degree at least 1 with leading term $a_n x^n$.

- As $x \to \infty$, $P(x) \to \infty$ if a_n is positive, and $P(x) \to -\infty$ if a_n is negative.
- As $x \to -\infty$, $P(x)$ behaves the same as when $x \to \infty$ if the degree is even, but $P(x)$ behaves in the opposite way if the degree is odd.
- It follows that every polynomial of odd degree has at least one zero.

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Proof

As remarked above, the leading term dominates the behaviour as $x \to \infty$ and as $x \to -\infty$, but here is a more formal proof.

A Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$,

then $\dfrac{P(x)}{x^n} = a_n + \dfrac{a_{n-1}}{x} + \cdots + \dfrac{a_1}{x^{n-1}} + \dfrac{a_0}{x^n}$.

As $x \to \infty$ or $x \to -\infty$, $\dfrac{P(x)}{x^n} \to a_n$.

Hence for large positive x, $P(x)$ has the same sign as a_n. For large negative x, $P(x)$ has the same sign as a_n when n is even, and the opposite sign to a_n when n is odd.

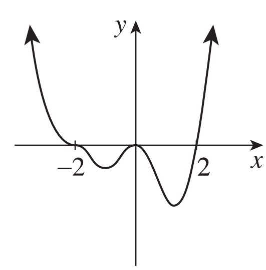
B If $P(x)$ is a polynomial of odd degree, then $P(x) \to \infty$ on either the left or right side, and $P(x) \to -\infty$ on the other side. Hence, being a continuous function, $P(x)$ must cross the x-axis somewhere.

Zeroes and sign

If the polynomial can be completely factored, then its zeroes can be read off very quickly, and we can construct a table of test values to decide its sign. Here, for example, is the table of test values and the sketch of

$$P(x) = (x+2)^3 x^2 (x-2).$$

x	-3	-2	-1	0	1	2	3
y	45	0	-3	0	-27	0	1125



The function changes sign around $x = -2$ and $x = 2$, where the associated factors $(x+2)^3$ and $(x-2)$ have odd degrees, but not around $x = 0$, where the factor x^2 has even degree.

Because the curve is smooth (differentiable) at $x = 0$, we know that the curve will be increasing on the left of $x = 0$, decreasing on the right of $x = 0$, and stationary at $x = 0$. This produces a *turning point* at the origin — the x-axis is a tangent there, and the curve *turns over* smoothly from increasing to stationary to decreasing without crossing the x-axis.

At $x = -2$, our table of values tells us that the curve crosses the x-axis. We shall see in Section 10G that the curve is momentarily flat there, with a stationary point, and hence has a *horizontal inflection* on the x-axis at $x = -2$, corresponding to the fact that the factor $(x+2)^3$ has odd degree greater than 1. Proving this will require calculus, although the result is fairly obvious by comparison with the known graphs of the very simple polynomial function $y = x^3$ that we first drew in Section 3G.

Multiple zeroes

Some machinery is needed to describe the situation. The zero $x = -2$ of the polynomial $P(x) = (x+2)^3 x^2 (x-2)$ is called a *triple zero*, the zero $x = 0$ is called a *double zero*, and the zero $x = 2$ is called a *simple zero*.

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6 MULTIPLE ZEROES

- Suppose that $x - \alpha$ is a factor of a polynomial $P(x)$, and
$$P(x) = (x - \alpha)^m Q(x), \text{ where } Q(x) \text{ is not divisible by } x - \alpha.$$
Then $x = \alpha$ is called a *zero of multiplicity m*.
- A zero of multiplicity 1 is called a *simple zero*, and a zero of multiplicity greater than 1 is called a *multiple zero*.

Behaviour at simple and multiple zeroes

Here is the statement of the way that a polynomial graph crosses the x-axis at its zeroes, to be proven in Section 10G.

7 MULTIPLE ZEROES AND THE SHAPE OF THE CURVE

Suppose that $x = \alpha$ is a zero of a polynomial $P(x)$.

- If $x = \alpha$ has even multiplicity, then the curve is tangent to the x-axis at $x = \alpha$, and does not cross the x-axis there. The point $(\alpha, 0)$ is a *turning point*.
- If $x = \alpha$ has odd multiplicity at least 3, then the curve is tangent to the x-axis at $x = \alpha$, but crosses the x-axis. The point $(\alpha, 0)$ is a *horizontal inflection*.
- If $x = \alpha$ is a simple zero, then the curve crosses the x-axis at $x = \alpha$ and is not tangent to the x-axis there.

It seemed more appropriate to sketch polynomial curves here at the start of the chapter rather than maintain strict logical order.

Example 2 10B

Sketch, showing the behaviour near any x-intercepts:

a $P(x) = (x - 1)^2(x - 2)$

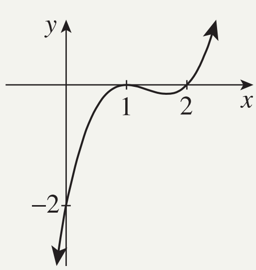
b $Q(x) = x^3(x + 2)^4(x^2 + x + 1)$

c $R(x) = -2(x - 2)^2(x + 1)^5(x - 1)$

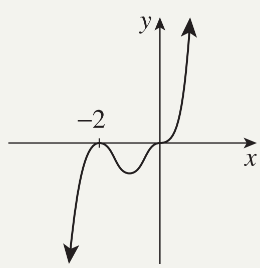
SOLUTION

In part **b**, $x^2 + x + 1$ is irreducible, because $\Delta = 1 - 4 < 0$.

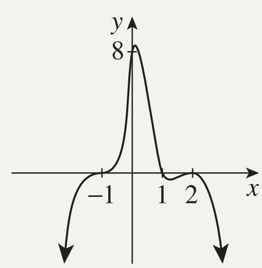
a



b



c



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Exercise 10B

FOUNDATION

1 Without the aid of calculus, sketch graphs of these linear polynomials, clearly indicating all intercepts with the axes.

a $P(x) = 2$ b $P(x) = x$ c $P(x) = x - 4$ d $P(x) = 3 - 2x$

2 Without the aid of calculus, sketch graphs of these quadratic polynomials, clearly indicating all intercepts with the axes.

a $P(x) = x^2$ b $P(x) = (x - 1)(x + 3)$
c $P(x) = (x - 2)^2$ d $P(x) = 9 - x^2$
e $P(x) = 2x^2 + 5x - 3$ f $P(x) = 4 + 3x - x^2$

3 Without the aid of calculus, sketch graphs of these cubic polynomials, clearly indicating all intercepts with the axes.

a $y = x^3$ b $y = x^3 + 2$
c $y = (x - 4)^3$ d $y = (x - 1)(x + 2)(x - 3)$
e $y = x(2x + 1)(x - 5)$ f $y = (1 - x)(1 + x)(2 + x)$
g $y = (2x + 1)^2(x - 4)$ h $y = x^2(1 - x)$
i $y = (2 - x)^2(5 - x)$

Classify each graph as one-to-one, many-to-one, one-to-many or many-to-many.

4 Without the aid of calculus, sketch graphs of these quartic polynomials, clearly indicating all intercepts with the axes.

a $F(x) = x^4$ b $F(x) = (x + 2)^4$
c $F(x) = x(3x + 2)(x - 3)(x + 2)$ d $F(x) = (1 - x)(x + 5)(x - 7)(x + 3)$
e $F(x) = x^2(x + 4)(x - 3)$ f $F(x) = (x + 2)^3(x - 5)$
g $F(x) = (2x - 3)^2(x + 1)^2$ h $F(x) = (1 - x)^3(x - 3)$
i $F(x) = (2 - x)^2(1 - x^2)$

DEVELOPMENT

5 These polynomials are not factored, but the positions of their zeroes can be found by trial and error. Copy and complete each table of values, then sketch the graph, and state how many zeroes there are, and between which integers they lie.

a $y = x^2 - 3x + 1$

x	-1	0	1	2	3	4
y						

b $y = 1 + 3x - x^3$

x	-2	-1	0	1	2	3
y						

6 Without the aid of calculus, sketch each polynomial function, clearly indicating all intercepts with the axes.

a $P(x) = x(x - 2)^3(x + 1)^2$ b $P(x) = (x + 2)^2(3 - x)^3$
c $P(x) = x(2x + 3)^3(1 - x)^4$ d $P(x) = (x + 1)(4 - x^2)(x^2 - 3x - 10)$

7 Use the graphs drawn in the previous question to solve these inequations.

a $x(x - 2)^3(x + 1)^2 > 0$ b $(x + 2)^2(3 - x)^3 \geq 0$
c $x(2x + 3)^3(1 - x)^4 \geq 0$ d $(x + 1)(4 - x^2)(x^2 - 3x - 10) < 0$

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8 Factor each polynomial by making a suitable substitution. Then sketch it without using calculus, showing all intercepts with the axes.

a $P(x) = x^4 - 13x^2 + 36$

b $P(x) = 4x^4 - 13x^2 + 9$

c $P(x) = (x^2 - 5x)^2 - 2(x^2 - 5x) - 24$

d $P(x) = (x^2 - 3x + 1)^2 - 4(x^2 - 3x + 1) - 5$

9 a Graph each polynomial, clearly indicating all intercepts with the axes.

i $F(x) = x(x - 4)(x + 1)$

ii $F(x) = (x - 1)^2(x + 3)$

iii $F(x) = x(x + 3)^2(5 - x)$

iv $F(x) = x^2(x - 3)^3(x - 7)$

b Without the aid of calculus, draw graphs of the derivatives of each polynomial in part a. (You will not be able to find the x-intercepts or y-intercepts.)

ENRICHMENT

10 Consider the polynomial $P(x) = x^4 - 5x^2 + 4x + 13$.

a Show that $P(x)$ can be expressed in the form $(x^2 - a)^2 + (x - b)^2$.

b How many x-intercepts does the graph of $P(x)$ have? Explain your answer.

11 At what points do the graphs of the polynomials $f(x) = (x + 1)^n$ and $g(x) = (x + 1)^m$ intersect, where $m \neq n$? (Hint: Consider the cases where m and n are odd and even.)

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10C Division of polynomials

The previous exercise had examples of adding, subtracting and multiplying polynomials, operations which are quite straightforward. The division of one polynomial by another, however, requires some explanation.

Division of polynomials

It can happen that the quotient of two polynomials is again a polynomial; for example,

$$\frac{6x^3 + 4x^2 - 9x}{3x} = 2x^2 + \frac{4}{3}x - 3 \quad \text{and} \quad \frac{x^2 + 4x - 5}{x + 5} = x - 1.$$

But usually, division results in rational functions, not polynomials:

$$\frac{x^4 + 4x^2 - 9}{x^2} = x^2 + 4 - \frac{9}{x^2} \quad \text{and} \quad \frac{x + 4}{x + 3} = 1 + \frac{1}{x + 3}.$$

There is a very close analogy here between the set $\mathbb{Z}$ of all integers and the set of all polynomials. In both cases, everything works nicely for addition, subtraction and multiplication, but the results of division do not usually lie within the set. For example, although $20 \div 5 = 4$ is an integer, the division of two integers usually results in a fraction rather than an integer, as in $23 \div 5 = 4\frac{3}{5}$.

In both cases, the best way to handle division is to use remainders.

The division algorithm for integers

On the right is an example of the well-known long division algorithm for integers, applied here to $197 \div 12$. The number 12 is called the *divisor*, 197 is called the *dividend*, 16 is called the *quotient*, and 5 is called the *remainder*.

$$\begin{array}{r|l} & \text{16 remainder 5} \\ \hline 12 & 197 \\ & \underline{12} \\ & \ \ 77 \\ & \ \ \underline{72} \\ & \ \ \ \ 5 \end{array}$$

The result of the division can be written as $\frac{197}{12} = 16\frac{5}{12}$, but we can avoid fractions completely by writing the result as:

$$197 = 12 \times 16 + 5, \qquad \text{dividend} = \text{divisor} \times \text{quotient} + \text{remainder}.$$

The remainder 5 must be less than 12, otherwise the division process could be continued. Thus the general result for division of integers can be expressed as:

8 DIVISION OF INTEGERS

- Let p (the *dividend*) and d (the *divisor*) be integers, with $d > 0$. Then there are unique integers q (the *quotient*) and r (the *remainder*) such that

  $$p = dq + r \quad \text{and} \quad 0 \le r < d.$$

- When the remainder r is zero, then d is a *divisor of* p, and the integer p *factors* into the product

  $$p = d \times q.$$

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The division algorithm for polynomials

The method of dividing one polynomial by another is similar to the method of dividing integers.

9 THE METHOD OF LONG DIVISION OF POLYNOMIALS

- At each step, divide the leading term of the remainder by the leading term of the divisor. Continue the process for as long as possible.
- Unless otherwise specified, express the answer in the form

  dividend = divisor × quotient + remainder.

Example 3 10C

Divide $3x^4 - 4x^3 + 4x - 8$ by:

a $x - 2$

b $x^2 - 2$

Give results first in the standard manner, then using rational functions.

SOLUTION

In each part, the steps have been annotated to explain the method.

a

$$
\begin{array}{r|rl}
 & 3x^3 + 2x^2 + 4x + 12 & \text{(leave a gap for the missing term in } x^2\text{)} \\ \hline
x - 2 & 3x^4 - 4x^3 \qquad + 4x - 8 & \text{(divide } x \text{ into } 3x^4 \text{, giving the } 3x^3 \text{ above)} \\
 & 3x^4 - 6x^3 \qquad\qquad\qquad & \text{(multiply } x - 2 \text{ by } 3x^3 \text{ and then subtract)} \\ \cline{2-2}
 & 2x^3 \qquad + 4x - 8 & \text{(divide } x \text{ into } 2x^3 \text{, giving the } 2x^2 \text{ above)} \\
 & 2x^3 - 4x^2 \qquad\qquad & \text{(multiply } x - 2 \text{ by } 2x^2 \text{ and then subtract)} \\ \cline{2-2}
 & 4x^2 + 4x - 8 & \text{(divide } x \text{ into } 4x^2 \text{, giving the } 4x \text{ above)} \\
 & 4x^2 - 8x \qquad & \text{(multiply } x - 2 \text{ by } 4x \text{ and then subtract)} \\ \cline{2-2}
 & 12x - 8 & \text{(divide } x \text{ into } 12x \text{, giving the 12 above)} \\
 & 12x - 24 & \text{(multiply } x - 2 \text{ by 12 and then subtract)} \\ \cline{2-2}
 & 16 & \text{(this is the final remainder)}
\end{array}
$$

Hence $3x^4 - 4x^3 + 4x - 8 = (x - 2)(3x^3 + 2x^2 + 4x + 12) + 16$,

or, writing the result using rational functions,

$$\frac{3x^4 - 4x^3 + 4x - 8}{x - 2} = 3x^3 + 2x^2 + 4x + 12 + \frac{16}{x - 2}.$$

b

$$
\begin{array}{r|rl}
 & 3x^2 - 4x + 6 & \\ \hline
x^2 - 2 & 3x^4 - 4x^3 \qquad + 4x - 8 & \text{(divide } x^2 \text{ into } 3x^4 \text{, giving the } 3x^2 \text{ above)} \\
 & 3x^4 \qquad - 6x^2 \qquad\qquad & \text{(multiply } x^2 - 2 \text{ by } 3x^2 \text{ and then subtract)} \\ \cline{2-2}
 & -4x^3 + 6x^2 + 4x - 8 & \text{(divide } x^2 \text{ into } -4x^3 \text{, giving the } -4x \text{ above)} \\
 & -4x^3 \qquad + 8x \qquad & \text{(multiply } x^2 - 2 \text{ by } -4x \text{ and then subtract)} \\ \cline{2-2}
 & 6x^2 - 4x - 8 & \text{(divide } x^2 \text{ into } 6x^2 \text{, giving the 6 above)} \\
 & 6x^2 \qquad - 12 & \text{(multiply } x^2 - 2 \text{ by 6 and then subtract)} \\ \cline{2-2}
 & -4x + 4 & \text{(this is the final remainder)}
\end{array}
$$

Hence $3x^4 - 4x^3 + 4x - 8 = (x^2 - 2)(3x^2 - 4x + 6) + (-4x + 4)$,

or $$\frac{3x^4 - 4x^3 + 4x - 8}{x^2 - 2} = 3x^2 - 4x + 6 + \frac{-4x + 4}{x^2 - 2}.$$

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The division theorem

The division process illustrated above can be continued until the remainder is zero or has degree less than the degree of the divisor. Thus the general result for polynomial division is:

10 DIVISION OF POLYNOMIALS

- Suppose that $P(x)$ (the *dividend*) and $D(x)$ (the *divisor*) are polynomials with $D(x) \neq 0$. Then there are unique polynomials $Q(x)$ (the *quotient*) and $R(x)$ (the *remainder*) such that
  1 $P(x) = D(x)Q(x) + R(x)$,
  2 either $\deg R(x) < \deg D(x)$, or $R(x) = 0$.
- When the remainder $R(x)$ is zero, then $D(x)$ is called a *divisor* of $P(x)$, and the polynomial $P(x)$ *factors* into the product $P(x) = D(x) \times Q(x)$.

For example, in the two long divisions in Example 3:

- In part **a**, the remainder after division by the degree 1 polynomial $x - 2$ was the constant polynomial 16.
- In part **b**, the remainder after division by the degree 2 polynomial $x^2 - 2$ was the linear polynomial $-4x + 4$.

Exercise 10C

FOUNDATION

1 Perform each integer division, and write the result in the form $p = dq + r$, where $0 \leq r < d$. For example, $30 \div 7 = 4$, remainder 2, so $30 = 4 \times 7 + 2$.
a $63 \div 5$ **b** $125 \div 8$ **c** $324 \div 11$ **d** $1857 \div 23$

2 Use long division to perform each division. Express each result in the form $P(x) = D(x)Q(x) + R(x)$.
a $(x^2 - 4x + 1) \div (x + 1)$
b $(x^2 - 6x + 5) \div (x - 5)$
c $(x^3 - x^2 - 17x + 24) \div (x - 4)$
d $(2x^3 - 10x^2 + 15x - 14) \div (x - 3)$
e $(4x^3 - 4x^2 + 7x + 14) \div (2x + 1)$
f $(x^4 + x^3 - x^2 - 5x - 3) \div (x - 1)$
g $(6x^4 - 5x^3 + 9x^2 - 8x + 2) \div (2x - 1)$
h $(10x^4 - x^3 + 3x^2 - 3x - 2) \div (5x + 2)$

3 Express the answers to parts **a**–**d** of the previous question in rational form, that is, as
$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}.$$

4 Use long division to perform each division. Express each result in the standard form $P(x) = D(x)Q(x) + R(x)$.
a $(x^3 + x^2 - 7x + 6) \div (x^2 + 3x - 1)$
b $(x^3 - 4x^2 - 2x + 3) \div (x^2 - 5x + 3)$
c $(x^4 - 3x^3 + x^2 - 7x + 3) \div (x^2 - 4x + 2)$
d $(2x^5 - 5x^4 + 12x^3 - 10x^2 + 7x + 9) \div (x^2 - x + 2)$

5 **a** If the divisor of a polynomial has degree 3, what are the possible degrees of the remainder?
b On division by $D(x)$, a polynomial has remainder $R(x)$ of degree 2. What are the possible degrees of $D(x)$?

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DEVELOPMENT

6 Use long division to perform each division. Take care to ensure that the columns line up correctly. Express each result in the form $P(x) = D(x)\,Q(x) + R(x)$.

a $(x^3 - 5x + 3) \div (x - 2)$

b $(2x^3 + x^2 - 11) \div (x + 1)$

c $(x^3 - 3x^2 + 5x - 4) \div (x^2 + 2)$

d $(2x^4 - 5x^2 + x - 2) \div (x^2 + 3x - 1)$

e $(2x^3 - 3) \div (2x - 4)$

f $(x^5 + 3x^4 - 2x^2 - 3) \div (x^2 + 1)$

Write the answers to parts **c** and **f** above in rational form, that is, in the form
$\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$.

7 a Use long division to show that $P(x) = x^3 + 2x^2 - 11x - 12$ is divisible by $x - 3$, and hence express $P(x)$ as the product of three linear factors.

b Find the values of x for which $P(x) > 0$.

8 a Use long division to show that $F(x) = 2x^4 + 3x^3 - 12x^2 - 7x + 6$ is divisible by $x^2 - x - 2$, and hence express $F(x)$ as the product of four linear factors.

b Find the values of x for which $F(x) \le 0$.

9 a Find the quotient and remainder when $x^4 - 2x^3 + x^2 - 5x + 7$ is divided by $x^2 + x - 1$.

b Hence find a and b so that $x^4 - 2x^3 + x^2 + ax + b$ is exactly divisible by $x^2 + x - 1$.

10 a Use long division to divide the polynomial $f(x) = x^4 - x^3 + x^2 - x + 1$ by the polynomial $d(x) = x^2 + 4$. Express your answer in the form $f(x) = d(x)q(x) + r(x)$.

b Hence find the values of c and d such that $x^4 - x^3 + x^2 + cx + d$ is divisible by $x^2 + 4$.

11 If $x^4 - 2x^3 - 20x^2 + mx + n$ is exactly divisible by $x^2 - 5x + 2$, find m and n.

12 Suppose that $P(x) = x^4 + x^3 - 5x^2 - 22x + 5$ and $D(x) = x^2 + 3x + 5$.

a Find the polynomials $Q(x)$ and $R(x)$, where $R(x)$ is of lower degree than $D(x)$, so that $P(x) = D(x)\,Q(x) + R(x)$.

b Hence explain why $P(x)$ and $D(x)$ cannot have a common zero.

ENRICHMENT

13 Consider the cubic equation $x^3 - kx + (k + 11) = 0$:

a Use long division to show that $k = x^2 + x + 1 + \dfrac{12}{x - 1}$.

b Hence find all the integer values of k for which the equation has at least one *positive* integer solution for x.

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10D The remainder and factor theorems

Long division of polynomials is a cumbersome process. It is therefore very useful to have two theorems, called the *remainder theorem* and the *factor theorem*, that provide information about the results of a division without the division actually being carried out. In particular, the factor theorem gives a simple test whether a particular linear polynomial is a factor.

The remainder theorem

The remainder theorem is a remarkable result which, in the case of linear divisors, allows the remainder to be calculated without the long division being performed.

11 THE REMAINDER THEOREM

Suppose that $P(x)$ is a polynomial and α is a constant.
Then the remainder after division of $P(x)$ by $x - \alpha$ is $P(\alpha)$.

Proof

Because $x - \alpha$ is a polynomial of degree 1, the division theorem tells us that there are unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = (x - \alpha)Q(x) + R(x),$$

and $$\text{either } R(x) = 0 \text{ or } \deg R(x) = 0.$$

Hence $R(x)$ is a zero or non-zero constant, which we can write more simply as r,

so that $$P(x) = (x - \alpha)Q(x) + r.$$

Substituting $x = \alpha$ gives $$P(\alpha) = (\alpha - \alpha)Q(\alpha) + r$$

and rearranging, $$r = P(\alpha), \text{ as required.}$$

Example 4 — 10D

Find the remainder when $3x^4 - 4x^3 + 4x - 8$ is divided by $x - 2$:

a by long division,

b by the remainder theorem.

SOLUTION

a In Example 3 of the previous section, performing the division showed that

$$3x^4 - 4x^3 + 4x - 8 = (x - 2)(3x^3 + 2x^2 + 4x + 12) + 16,$$

that is, that the remainder is 16.

b Alternatively, substituting $x = 2$ into $P(x)$,

$$\begin{aligned}\text{remainder} &= P(2) \qquad \text{(this is the remainder theorem)}\\ &= 48 - 32 + 8 - 8\\ &= 16, \text{ as expected.}\end{aligned}$$

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Example 5 10D

The polynomial $P(x) = x^4 - 2x^3 + ax + b$ has remainder 3 after division by $x - 1$, and has remainder -5 after division by $x + 1$. Find a and b.

SOLUTION

Applying the remainder theorem for each divisor,

$$\begin{aligned} P(1) &= 3 \\ 1 - 2 + a + b &= 3 \\ a + b &= 4. \qquad (1) \end{aligned}$$

Also

$$\begin{aligned} P(-1) &= -5 \\ 1 + 2 - a + b &= -5 \\ -a + b &= -8. \qquad (2) \end{aligned}$$

Adding (1) and (2), $2b = -4$,

subtracting them, $2a = 12$.

Hence $a = 6$ and $b = -2$.

The factor theorem

The remainder theorem tells us that the number $P(\alpha)$ is the remainder after division by $x - \alpha$. But $x - \alpha$ is a factor if and only if the remainder after division by $x - \alpha$ is zero, so:

12 THE FACTOR THEOREM

Suppose that $P(x)$ is a polynomial and α is a constant.
Then $x - \alpha$ is a factor if and only if $P(\alpha) = 0$.

This is a very quick and easy way to test whether $x - \alpha$ is a factor of $P(x)$.

Example 6 10D

Show that $x - 3$ is a factor of $P(x) = x^3 - 2x^2 + x - 12$, and $x + 1$ is not. Then use long division to factor the polynomial completely.

SOLUTION

$P(3) = 27 - 18 + 3 - 12 = 0$, so $x - 3$ is a factor.

$P(-1) = -1 - 2 - 1 - 12 = -16 \neq 0$, so $x + 1$ is not a factor.

Long division of $P(x) = x^3 - 2x^2 + x - 12$ by $x - 3$ (which we omit) gives

$P(x) = (x - 3)(x^2 + x + 4)$,

and because $\Delta = 1 - 16 = -15 < 0$ for the quadratic, this factoring is complete.

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Factoring polynomials — the initial approach

The factor theorem gives us the beginnings of an approach to factoring polynomials. This approach will be further refined in the next three sections.

13 FACTORING POLYNOMIALS — THE INITIAL APPROACH

- Use trial and error to find an integer zero $x = \alpha$ of $P(x)$.
- Then use long division to factor $P(x)$ in the form $P(x) = (x - \alpha)Q(x)$.

If the coefficients of $P(x)$ are all integers, then all the integer zeroes of $P(x)$ are divisors of the constant term.

Proof

We must prove the claim that if the coefficients of $P(x)$ are integers, then every integer zero of $P(x)$ is a divisor of the constant term.

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$,
where the coefficients $a_n, a_{n-1}, \ldots a_1, a_0$ are all integers,
and let $x = \alpha$ be an integer zero of $P(x)$.
Substituting into $P(\alpha) = 0$ gives $a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0$

$$\begin{aligned} a_0 &= -a_n\alpha^n - a_{n-1}\alpha^{n-1} - \cdots - a_1\alpha \\ &= \alpha(-a_n\alpha^{n-1} - a_{n-1}\alpha^{n-2} - \cdots - a_1), \end{aligned}$$

so a_0 is an integer multiple of α.

Example 7 — 10D

Factor $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

SOLUTION

Because all the coefficients are integers, any integer zero is a divisor of the constant term -4.
Thus we test $1, 2, 4, -1, -2$ and -4.

$$\begin{aligned} P(1) &= 1 + 1 - 9 + 11 - 4 \\ &= 0, \text{ so } x - 1 \text{ is a factor.} \end{aligned}$$

After long division (omitted), $P(x) = (x - 1)(x^3 + 2x^2 - 7x + 4)$.

Let $Q(x) = x^3 + 2x^2 - 7x + 4$, then

$$\begin{aligned} Q(1) &= 1 + 2 - 7 + 4 \\ &= 0, \text{ so } x - 1 \text{ is a factor.} \end{aligned}$$

Again after long division (omitted), $P(x) = (x - 1)(x - 1)(x^2 + 3x - 4)$.
Factoring the quadratic, $P(x) = (x - 1)^3(x + 4)$.

Note: In the next three sections we will develop methods that will often allow long division to be avoided.

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Exercise 10D

FOUNDATION

1 Without division, find the remainder when $P(x) = x^3 - x^2 + 2x + 1$ is divided by:

a $x - 1$
b $x - 3$
c $x + 2$
d $x + 1$
e $x - 5$
f $x + 3$

2 Without division, find which of the following are factors of $F(x) = x^3 + 4x^2 + x - 6$.

a $x - 1$
b $x + 1$
c $x - 2$
d $x + 2$
e $x - 3$
f $x + 3$

3 a Find k, if $x - 1$ is a factor of $P(x) = x^3 - 3x^2 + kx - 2$.

b Find m, if -2 is a zero of the function $F(x) = x^3 + mx^2 - 3x + 4$.

c When the polynomial $P(x) = 2x^3 - x^2 + px - 1$ is divided by $x - 3$, the remainder is 2. Find p.

d For what value of a is $3x^4 + ax^2 - 2$ divisible by $x + 1$?

4 Factor each polynomial and sketch its graph, indicating all intercepts with the axes. Remember that any integer zeroes are divisors of the constant term.

a $P(x) = x^3 + 2x^2 - 5x - 6$
b $P(x) = x^3 + 3x^2 - 25x + 21$
c $P(x) = -x^3 + x^2 + 5x + 3$
d $P(x) = x^4 - x^3 - 19x^2 - 11x + 30$

5 Solve these equations by first factoring the LHS.

a $x^3 + 3x^2 - 6x - 8 = 0$
b $x^3 - 4x^2 - 3x + 18 = 0$
c $6x^3 - 5x^2 - 12x - 4 = 0$
d $2x^4 + 11x^3 + 19x^2 + 8x - 4 = 0$

DEVELOPMENT

6 a Show that $P(x) = x^3 - 8x^2 + 9x + 18$ is divisible by both $x - 3$ and $x + 1$.

b By considering the leading term and constant term, express $P(x)$ as a product of three linear factors.

7 a Show that $P(x) = 2x^3 - x^2 - 13x - 6$ is divisible by both $x - 3$ and $2x + 1$.

b By considering the leading term and constant term, express $P(x)$ as a product of three linear factors.

8 a When the polynomial $P(x) = 2x^3 - x^2 + ax + b$ is divided by $x - 1$ the remainder is 16, and when it is divided by $x + 2$ the remainder is -17. Find a and b.

b The polynomial $P(x)$ is given by $P(x) = x^3 + ax^2 + bx - 18$. Find a and b given that $x + 2$ is a factor of $P(x)$, and -24 is the remainder when $P(x)$ is divided by $x - 1$.

9 Without division, find the remainder when $P(x) = x^3 + 2x^2 - 4x + 5$ is divided by:

a $2x - 1$
b $2x + 3$
c $3x - 2$

10 a If $P(x) = 2x^3 + x^2 - 13x + 6$, evaluate $P\left(\frac{1}{2}\right)$. Hence use long division to express $P(x)$ in fully factored form.

b Given $P(x) = 6x^3 + x^2 - 5x - 2$, evaluate $P\left(-\frac{2}{3}\right)$ then express $P(x)$ in factored form.

11 Is either $x + 1$ or $x - 1$ a factor of $x^n + 1$, where n is a positive integer? (Hint: Consider the cases where n is even or odd separately.)

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12 **a** $P(x)$ is an odd polynomial of degree 3. It has $x + 4$ as a factor, and when it is divided by $x - 3$ the remainder is 21. Find $P(x)$.

b Find p so that $x - p$ is a factor of $4x^3 - (10p - 1)x^2 + (6p^2 - 5)x + 6$.

13 When the polynomial $P(x)$ is divided by $(x - 1)(x + 3)$, the quotient is $Q(x)$ and the remainder is $2x + 5$.

a Write down a division identity based on this information.

b Hence, by evaluating $P(1)$, find the remainder when $P(x)$ is divided by $x - 1$.

c What is the remainder when $P(x)$ is divided by $x + 3$?

14 The polynomial $P(x)$ is divided by $(x - 1)(x + 2)$. Suppose that the quotient is $Q(x)$ and the remainder is $R(x)$.

a Explain why the general form of $R(x)$ is $ax + b$, where a and b are constants.

b If $P(1) = 2$ and $P(-2) = 5$, find a and b. (Hint: Use the division identity.)

15 The polynomial $P(x)$ is divided by $(x + 4)(x - 3)$. If $P(-4) = 11$ and $P(3) = -3$, use the same approach as the previous question to find the remainder.

16 **a** When a polynomial is divided by $(2x + 1)(x - 3)$, the remainder is $3x - 1$. What is the remainder when the polynomial is divided by $2x + 1$?

b When $x^5 + 3x^3 + ax + b$ is divided by $x^2 - 1$, the remainder is $2x - 7$. Find a and b.

c When a polynomial $P(x)$ is divided by $x^2 - 5$, the remainder is $x + 4$. Find the remainder when $P(x) + P(-x)$ is divided by $x^2 - 5$. (Hint: Write down the division identity.)

ENRICHMENT

17 When the polynomial $P(x)$ is divided by $x^2 - k^2$, the remainder is $ax + b$.

a Show that $a = \frac{1}{2k}\big(P(k) - P(-k)\big)$ and $b = \frac{1}{2}\big(P(k) + P(-k)\big)$.

b Given that $P(x) = 8x^5 - 4x^4 + 6x^3 - 11x^2 - 2x + 3$ and $k = \frac{1}{2}$, find a and b, and hence factor $P(x)$ fully.

18 **a** Use the factor theorem to prove that $a + b + c$ is a factor of $a^3 + b^3 + c^3 - 3abc$. Then find the other factor. (Hint: Regard it as a polynomial in a.)

b Factor $ab^3 - ac^3 + bc^3 - ba^3 + ca^3 - cb^3$.

19 **a** If all the coefficients of a monic polynomial are integers, prove that all the rational zeroes are integers. (Hint: Look carefully at the proof under Box 13.)

b If all the coefficients of a polynomial are integers, prove that the denominators of all the rational zeroes (in lowest terms) are divisors of the leading coefficient.

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10E Consequences of the factor theorem

The factor theorem has a number of straightforward but very useful consequences. They are presented here as six successive theorems.

A. Several distinct zeroes

Suppose that several distinct zeroes of a polynomial have been found, probably using test substitutions into the polynomial.

14 DISTINCT ZEROES

Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_s$ are distinct zeroes of a polynomial $P(x)$.

Then $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_s)$ is a factor of $P(x)$.

Proof

Because α_1 is a zero, $x-\alpha_1$ is a factor, and $P(x) = (x-\alpha_1)p_1(x)$.

Because $P(\alpha_2) = 0$ but $\alpha_2 - \alpha_1 \neq 0$, $p_1(\alpha_2)$ must be zero.

Hence $x - \alpha_2$ is a factor of $p_1(x)$, and $p_1(x) = (x-\alpha_2)p_2(x)$,

so $P(x) = (x-\alpha_1)(x-\alpha_1)p_2(x)$.

Continuing similarly for s steps, $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_s)$ is a factor of $P(x)$.

B. All distinct zeroes

If n distinct zeroes of a polynomial of degree n can be found, then the factoring is complete, and the polynomial is the product of distinct linear factors.

15 ALL DISTINCT ZEROES

Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n distinct zeroes of a polynomial $P(x)$ of degree n. Then

$$P(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n),$$

where a is the leading coefficient of $P(x)$.

Proof

By the previous theorem, $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ is a factor of $P(x)$,
so $P(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)Q(x)$, for some polynomial $Q(x)$.

But $P(x)$ and $(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ both have degree n, so $Q(x)$ is a constant.

Equating coefficients of x^n, the constant $Q(x)$ must be the leading coefficient.

Factoring polynomials — finding several zeroes first

If we can find more than one zero of a polynomial, then we have found a quadratic or cubic factor, and the long divisions required can be reduced or even avoided completely.

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16 FACTORING POLYNOMIALS — FINDING SEVERAL ZEROES FIRST

- Use trial and error to find as many integer zeroes of $P(x)$ as possible.
- Using long division, divide $P(x)$ by the product of the known factors.

If the coefficients of $P(x)$ are all integers, then any integer zero of $P(x)$ must be one of the divisors of the constant term.

When this procedure is applied to the polynomial factored in the previous section, one rather than two long divisions is required.

Example 8 10E

Factor $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

SOLUTION

As before, all the coefficients are integers, so any integer zero is a divisor of the constant term -4, that is we test $1, 2, 4, -1, -2$ and -4.

$$P(1) = 1 + 1 - 9 + 11 - 4 = 0, \quad \text{so } x - 1 \text{ is a factor.}$$
$$P(-4) = 256 - 64 - 144 - 44 - 4 = 0, \quad \text{so } x + 4 \text{ is a factor.}$$

After long division by $(x - 1)(x + 4) = x^2 + 3x - 4$ (omitted),

$$P(x) = (x^2 + 3x - 4)(x^2 - 2x + 1).$$

Factoring both quadratics, $P(x) = (x - 1)(x - 4) \times (x - 1)^2$

$$= (x - 1)^3(x + 4).$$

Note: The methods of the next section will allow this particular factoring to be done with no long divisions. Example 9 involves a polynomial that factors into distinct linear factors, so that nothing more than the factor theorem is required to complete the task.

Example 9 10E

Factor $P(x) = x^4 - x^3 - 7x^2 + x + 6$ completely.

SOLUTION

The divisors of the constant term 6 are $1, 2, 3, 6, -1, -2, -3$ and -6.

$$P(1) = 1 - 1 - 7 + 1 + 6 = 0, \quad \text{so } x - 1 \text{ is a factor.}$$
$$P(-1) = 1 + 1 - 7 - 1 + 6 = 0, \quad \text{so } x + 1 \text{ is a factor.}$$
$$P(2) = 16 - 8 - 28 + 2 + 6 = -12 \neq 0, \quad \text{so } x - 2 \text{ is not a factor.}$$
$$P(-2) = 16 + 8 - 28 - 2 + 6 = 0, \quad \text{so } x + 2 \text{ is a factor.}$$
$$P(3) = 81 - 27 - 63 + 3 + 6 = 0, \quad \text{so } x - 3 \text{ is a factor.}$$

We now have four distinct zeroes of a polynomial of degree 4.

Hence $\quad P(x) = (x - 1)(x + 1)(x + 2)(x - 3) \quad$ (notice that $P(x)$ is monic).

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C. The maximum number of zeroes

If a polynomial of degree n were to have $n + 1$ zeroes, then by the first theorem above, it would be divisible by a polynomial of degree $n + 1$, which is impossible.

17 MAXIMUM NUMBER OF ZEROES

A polynomial of degree n has at most n zeroes.

D. A vanishing condition

The previous theorem translates easily into a condition for a polynomial to be the zero polynomial.

18 A VANISHING CONDITION

- Suppose that $P(x)$ is a polynomial in which no term has degree more than n, yet which is zero for at least $n + 1$ distinct values of x. Then $P(x)$ is the zero polynomial.
- In particular, the only polynomial that is zero for all values of x is the zero polynomial.

Proof

Suppose that $P(x)$ had a degree. This degree must be at most n because there is no term of degree more than n. But the degree must also be at least $n + 1$ because there are $n + 1$ distinct zeroes. This is a contradiction, so $P(x)$ has no degree, and is therefore the zero polynomial.

Note: Once again, the zero polynomial $Z(x) = 0$ is seen to be quite different in nature from all other polynomials. It is the only polynomial with an infinite number of zeroes — in fact every real number is a zero of $Z(x)$. Associated with this is the fact that $x - \alpha$ is a factor of $Z(x)$ for all real values of α, because $Z(x) = (x - \alpha)Z(x)$ (which is trivially true, because both sides are zero for all x). It is no wonder then that the zero polynomial does not have a degree.

E. A condition for two polynomials to be identically equal

A most important consequence of this last theorem is a condition for two polynomials $P(x)$ and $Q(x)$ to be identically equal — meaning that $P(x) = Q(x)$ for all values of x.

19 AN IDENTICALLY EQUAL CONDITION

Suppose that $P(x)$ and $Q(x)$ are degree n polynomials that have the same values for at least $n + 1$ values of x.
Then the polynomials $P(x)$ and $Q(x)$ are identically equal, that is, they are equal for all values of x.

This result was foreshadowed in Box 4 of Section 10A, page 455.

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Proof

Let $F(x) = P(x) - Q(x)$.

Because $F(x)$ is zero whenever $P(x)$ and $Q(x)$ have the same value,

it follows that $F(x)$ is zero for at least $n + 1$ values of x,

so by the previous theorem, $F(x)$ is the zero polynomial,

so $P(x) = Q(x)$ for all values of x.

Example 10 — 10E

Find a, b, c and d, if $x^3 - x = a(x - 2)^3 + b(x - 2)^2 + c(x - 2) + d$ for at least four values of x.

SOLUTION

Because they are equal for four values of x, they are identically equal.

Substituting $x = 2$, $\quad 6 = d$.

Equating coefficients of x^3, $\quad 1 = a$.

Substituting $x = 0$, $\quad 0 = -8 + 4b - 2c + 6$

$2b - c = 1$.

Substituting $x = 1$, $\quad 0 = -1 + b - c + 6$

$b - c = -5$.

Hence $b = 6$ and $c = 11$.

F. Identically equal polynomials have the same coefficients

We can now justify the claim made at the beginning of the chapter (Box 4 of Section 10A, page 455).

20 IDENTICALLY EQUAL POLYNOMIALS HAVE THE SAME COEFFICIENTS

If two polynomials $P(x)$ and $Q(x)$ are identically equal, that is, $P(x) = Q(x)$ for all values of x, then the corresponding coefficients of the polynomials are all equal.

Proof

Let $F(x) = P(x) - Q(x)$.

Then $F(x)$ has infinitely many zeroes, so is the zero polynomial.

All the coefficients of the zero polynomial are zero,

so the corresponding coefficients of $P(x)$ and $Q(x)$ are equal.

G. Geometrical implications of the factor theorem

Here are some of the geometrical versions of the factor theorem — they are translations of the consequences given above into the language of coordinate geometry. You will already have seen them in operation when dealing with graphs of quadratics.

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21 GEOMETRICAL IMPLICATIONS OF THE FACTOR THEOREM

1. The graph of a polynomial function of degree n is completely determined by any $n + 1$ points on the curve.
2. The graphs of two distinct polynomial functions cannot intersect in more points than the maximum of the two degrees.
3. A line cannot intersect the graph of a polynomial of degree n in more than n points.

In parts **2** and **3**, points where the two curves are tangent to each other are to be counted according to their multiplicity.

Example 11 10E

By factoring the difference $F(x) = P(x) - Q(x)$, describe the intersections between the curves $P(x) = x^4 + 4x^3 + 2$ and $Q(x) = x^4 + 3x^3 + 3x$, and find where $P(x)$ is above $Q(x)$.

SOLUTION

Subtracting, $F(x) = x^3 - 3x + 2.$

Substituting, $F(1) = 1 - 3 + 2 = 0,$ so $x - 1$ is a factor.

$F(-2) = -8 + 6 + 2 = 0,$ so $x + 2$ is a factor.

After long division by $(x - 1)(x + 2) = x^2 + x - 2,$

$$F(x) = (x - 1)^2(x + 2).$$

Hence $y = P(x)$ and $y = Q(x)$ are tangent at $x = 1$, but do not cross there,

and intersect also at $x = -2$, where they cross at an angle.

Because $F(x)$ is positive for $-2 < x < 1$ or $x > 1$, and negative for $x < -2$,

$P(x)$ is above $Q(x)$ for $-2 < x < 1$ or $x > 1$, and below it for $x < -2$.

A note for Extension 2 students

The *fundamental theorem of algebra* cannot be proven in the Extension 2 course, but the theorem helps to understand the importance of complex numbers for polynomials. It tells us that every polynomial equation of degree n has exactly n roots, provided first that roots are counted according to their multiplicity, and secondly that complex roots are also counted. For example:

- $x^3 = 0$ has one root $x = 0$, but this root has multiplicity 3.
- $x^3 - 1 = 0$, which factors to $(x - 1)(x^2 + x + 1) = 0$, has root $x = 1$, but also has the two complex roots of $x^2 + x + 1 = 0$.

This means that the graph of a polynomial of degree n intersects every line in exactly n points, provided first that points where the line is a tangent are counted according to their multiplicity, and secondly that complex points of intersection are also counted. This theorem provides the fundamental link between the algebra of polynomials and the geometry of their graphs, and allows the degree of a polynomial to be defined algebraically as the highest index, or geometrically as the number of times every line crosses it.

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Exercise 10E

FOUNDATION

1 Use the factor theorem to write down in factored form:
 a a monic cubic polynomial with zeroes -1, 3 and 4.
 b a monic quartic polynomial with zeroes 0, -2, 3 and 1.
 c a cubic polynomial with leading coefficient 6 and zeroes at $\frac{1}{3}$, $-\frac{1}{2}$ and 1.

2 a Show that 2 and 5 are zeroes of $P(x) = x^4 - 3x^3 - 15x^2 + 19x + 30$.
 b Hence explain why $(x - 2)(x - 5)$ is a factor of $P(x)$.
 c Divide $P(x)$ by $(x - 2)(x - 5)$ and hence express $P(x)$ as the product of four linear factors.

3 Use trial and error to find as many integer zeroes of $P(x)$ as possible. Use long division to divide $P(x)$ by the product of the known factors and hence express $P(x)$ in factored form.
 a $P(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$
 b $P(x) = 2x^4 - 5x^3 - 5x^2 + 20x - 12$
 c $P(x) = 6x^4 - 25x^3 + 17x^2 + 28x - 20$
 d $P(x) = 9x^4 - 51x^3 + 85x^2 - 41x + 6$

4 Refer to Box 18 to answer parts a and b.
 a The polynomial $(a - 2)x^2 + (1 - 3b)x + (5 - 2c)$ has two zeroes. What are the values of a, b and c?
 b The polynomial $(a + 1)x^3 + (b - 3)x^2 + (2c - 1)x + (5 - 4d)$ has three zeroes. What are the values of a, b, c and d?

DEVELOPMENT

5 a If $3x^2 - 4x + 7 = a(x + 2)^2 + b(x + 2) + c$ for all x, find a, b and c.
 b If $2x^3 - 8x^2 + 3x - 4 = a(x - 1)^3 + b(x - 1)^2 + c(x - 1) + d$ for all x, find a, b, c and d.
 c Use similar methods to express $x^3 + 2x^2 - 3x + 1$ as a polynomial in $(x + 1)$.
 d If the polynomials $2x^2 + 4x + 4$ and $a(x + 1)^2 + b(x + 2)^2 + c(x + 3)^2$ are equal for three values of x, find a, b and c.

6 a A polynomial of degree 3 has a double zero at 2. When $x = 1$ it takes the value 6 and when $x = 3$ it takes the value 8. Find the polynomial.
 b Two zeroes of a polynomial of degree 3 are 1 and -3. When $x = 2$ it takes the value -15 and when $x = -1$ it takes the value 36. Find the polynomial.

7 Show that $x^2 - 3x + 2$ is a factor of $P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + (2^n - 2^m)$, where m and n are positive integers.

8 If two polynomials have degrees m and n, where $m > n$, what is the maximum number of intersection points of their graphs?

9 Explain why a cubic with three distinct zeroes must have two stationary points.

10 The line $y = k$ meets the curve $y = ax^3 + bx^2 + cx + d$ four times. Find the values of a, b, c and d in term of k.

11 Find, in expanded form, the monic degree five polynomial whose zeroes are 0, -1, 1, $2 - \sqrt{2}$ and $2 + \sqrt{2}$.

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12 By factoring the difference $F(x) = P(x) - Q(x)$, describe the intersections between the curves $P(x)$ and $Q(x)$.

a $P(x) = 2x^3 - 4x^2 + 3x + 1, Q(x) = x^3 + x^2 - 8$
b $P(x) = x^4 + x^3 + 10x - 4, Q(x) = x^4 + 7x^2 - 6x + 8$
c $P(x) = -2x^3 + 3x^2 - 25, Q(x) = -3x^3 - x^2 + 11x + 5$
d $P(x) = x^4 - 3x^2 - 2, Q(x) = x^3 - 5x$
e $P(x) = x^4 + 4x^3 - x + 5, Q(x) = x^3 - 3x^2 - 2x + 5$

13 If a and b are non-zero, and $a + b = 0$, prove that the polynomials $A(x) = x^3 + ax^2 - x + b$ and $B(x) = x^3 + bx^2 - x + a$ have a common factor of degree 2 but are not identical polynomials. What is the common factor?

ENRICHMENT

14 Suppose that $P(x) = x^5 + x^2 + 1$ and $Q(x) = x^2 - 2$. If r_1, r_2, r_3, r_4 and r_5 are the five zeroes of $P(x)$, find the value of $Q(r_1) \times Q(r_2) \times Q(r_3) \times Q(r_4) \times Q(r_5)$.

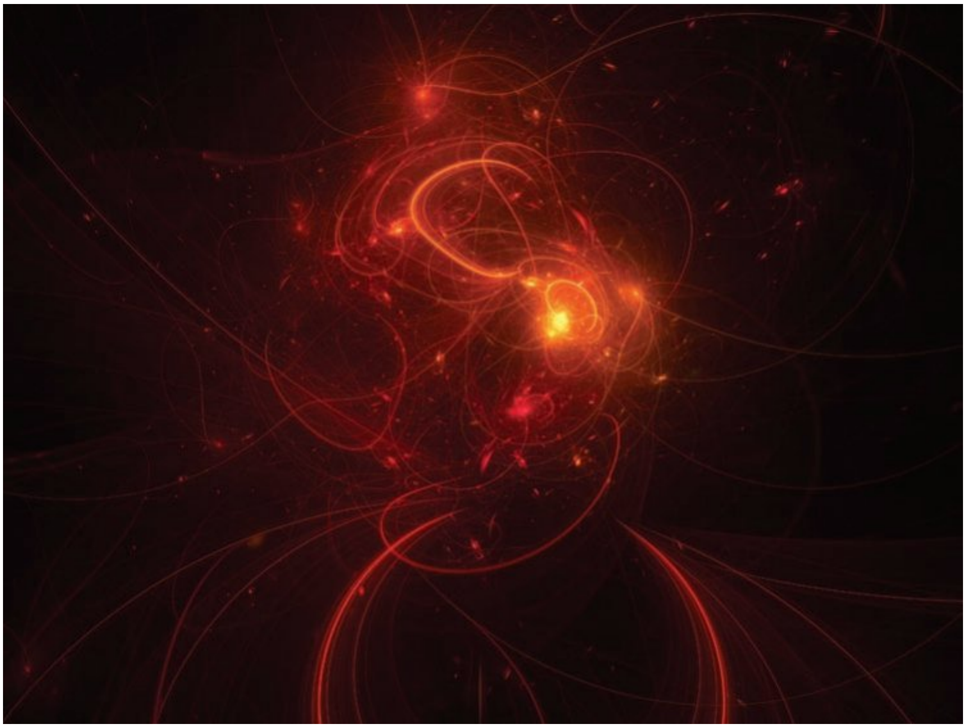
15 Suppose that $P(x)$ is a polynomial of *odd degree n*.

It is known that $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$.

a Write down the zeroes of the polynomial $(x + 1)P(x) - x$.
b Let A be the leading coefficient of the polynomial $(x + 1)P(x) - x$.
Factor the polynomial, and hence show that

$$A = \frac{1}{1 \times 2 \times 3 \times \cdots \times n \times (n+1)} = \frac{1}{(n+1)!}.$$

c Find $P(n + 1)$.



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10F Sums and products of zeroes

When expanding a monic polynomial with only linear factors, such as

$$P(x) = (x - 2)(x - 3)(x - 5)(x - 7) = x^4 - 17x^3 + 101x^2 - 247x + 210,$$

the five coefficients 1, −17, 101, −247 and 210 are clearly determined by the four zeroes 2, 3, 5 and 7 of the polynomial. But what are the formulae?

If we study the coefficient −17 of the term in x^3, and the constant term 210, it does not take long to realise that

$$\begin{aligned} -17 &= -(2 + 3 + 5 + 7) \\ &= -(\text{sum of the zeroes}) \end{aligned} \qquad \text{and} \qquad \begin{aligned} 210 &= +2 \times 3 \times 5 \times 7 \\ &= +(\text{product of the zeroes}). \end{aligned}$$

It is not at all clear, however, how the coefficients 101 of x^2, and −247 of x, are related to the zeroes. (We invite readers to try working it out before continuing.)

This section answers these questions for monic and non-monic polynomials of every degree, starting with quadratics. Knowing these formulae involving the sums and products of zeroes is a very helpful technique when trying to factor a polynomial, and has other more general applications.

We will not state every time that $\alpha, \beta, \gamma, \delta, \ldots$ are the zeroes of the polynomial, and that $a, b, c, d, \ldots$ are its successive coefficients.

The zeroes of a quadratic

Let $P(x) = ax^2 + bx + c$ be a quadratic with zeroes α and β and leading coefficient a. Then $P(x) = a(x - \alpha)(x - \beta)$ by the factor theorem, and expanding,

$$P(x) = a(x - \alpha)(x - \beta) = ax^2 - a(\alpha + \beta)x + a\alpha\beta,$$

Hence the coefficient of x is $-a$ times the sum of the zeroes, and the constant is $+a$ times the product of the zeroes,

$$b = -a(\alpha + \beta) \qquad \text{and} \qquad c = +a\alpha\beta.$$

This is just what happened with the quartic above, except for multiplication by the leading coefficient a. Solving for the sum and product of the zeroes,

22 SUM AND PRODUCT OF ZEROES OF A QUADRATIC

Let $P(x) = ax^2 + bx + c$ have zeroes α and β. Then

$$\alpha + \beta = -\frac{b}{a} \qquad \text{and} \qquad \alpha\beta = +\frac{c}{a}.$$

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Example 12 — 10F

a **i** Show by substitution that $x = 5$ is a zero of $P(x) = 3x^2 - 30x + 75$.

ii Use sum-of-zeroes to find the other zero.

iii Use product-of-zeroes to find the other zero.

b Find the sum of the zeroes of $Q(x) = x^2 + 7x - 11$, and hence find its axis of symmetry.

SOLUTION

a **i** $P(5) = 3 \times 25 - 30 \times 5 + 75$
$= 0.$

ii $\alpha + \beta = +\frac{30}{3}$
$\alpha + 5 = 10$
$\alpha = 5$, so 5 is a double zero.

iii $\alpha\beta = \frac{75}{3}$
$\alpha \times 5 = 25$
$\alpha = 5$, so again, 5 is a double zero.

b $\alpha + \beta = -\frac{7}{1}$
$= -7.$

The axis of symmetry is midway between zeroes, so it is $x = -3\frac{1}{2}$

(This calculation is exactly the same as the formula $x = -\dfrac{b}{2a}$ for the axis.)

The zeroes of a cubic

Let $P(x) = ax^3 + bx^2 + cx + d$ be a cubic with zeroes α, β and γ and leading coefficient a. Then $P(x) = a(x - \alpha)(x - \beta)(x - \gamma)$ by the factor theorem, and expanding,

$$\begin{aligned} P(x) &= a(x - \alpha)(x - \beta)(x - \gamma) \\ &= ax^3 - a(\alpha + \beta + \gamma)x^2 + a(\alpha\beta + \beta\gamma + \gamma\alpha)x - a\alpha\beta\gamma. \end{aligned}$$

So as we saw before with the quartic and the quadratic,

$$b = -a(\alpha + \beta + \gamma) \quad \text{and} \quad d = -a\alpha\beta\gamma.$$

where the negative sign of the product comes from the cube of -1. But the new phenomenon here is the coefficient of x,

$$c = a(\alpha\beta + \beta\gamma + \gamma\alpha) = +(\text{sum of products of pairs of zeroes}).$$

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Solving for the sums and products of the zeroes,

23 SUMS AND PRODUCTS OF ZEROES OF A CUBIC

Let $P(x) = ax^3 + bx^2 + cx + d$ have zeroes α, β and γ. Then

$$\alpha + \beta + \gamma = -\frac{b}{a} \quad \text{(sum of the zeroes)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = +\frac{c}{a} \quad \text{(sum of products of pairs of zeroes)}$$

$$\alpha\beta\gamma = -\frac{d}{a} \quad \text{(product of the zeroes)}$$

Example 13 — 10F

a Show that -6 is a zero of $P(x) = x^3 - 4x^2 - 39x + 126$.
b Use sum and product of roots to find the other two zeroes.
c Check the formula for the sum of products of pairs of zeroes.

SOLUTION

a $P(-6) = -216 - 144 + 234 + 126 = 0$.

b $\alpha + \beta - 6 = -\frac{-4}{1}$ and $\alpha\beta \times (-6) = -\frac{126}{1}$

$\alpha + \beta = 10$ and $\alpha\beta = 21$

Hence α and β are 3 and 7.

c $\alpha\beta + \beta\gamma + \gamma\alpha = 3 \times 7 + 7 \times (-6) + (-6) \times 3$
$= -39$, which is the coefficient of x.

The zeroes of a quartic

Suppose that the four zeroes of the quartic polynomial

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e$$

are α, β, γ and δ. By the factor theorem (see Box 15 in Section 10E, page 473), $P(x)$ is a multiple of the product $(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$:

$$\begin{aligned} P(x) &= a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) \\ &= ax^4 - a(\alpha + \beta + \gamma + \delta)x^3 + a(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 \\ &\quad - a(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)x + a\alpha\beta\gamma\delta. \end{aligned}$$

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Equating coefficients of terms in x^3, x^2, x and constants now gives:

24 ZEROES AND COEFFICIENTS OF A QUARTIC

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} \quad \text{(sum of the zeroes)}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = +\frac{c}{a} \quad \text{(sum of products of pairs of zeroes)}$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{a} \quad \text{(sum of products of triples of zeroes)}$$

$$\alpha\beta\gamma\delta = +\frac{e}{a} \quad \text{(product of the zeroes)}$$

The second formula gives 'the sum of the products of pairs of zeroes', and the third formula gives 'the sum of the products of triples of zeroes'.

The general case

The method is the same for the general case. Notation is unfortunately a major difficulty here, and the results are better written in words. Suppose that $\alpha_1, \alpha_2, \ldots \alpha_n$ are the n zeroes of the degree n polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

25 ZEROES AND COEFFICIENTS OF A POLYNOMIAL

$$\text{sum of the zeroes} = \alpha_1 + \alpha_2 + \cdots + \alpha_n = -\frac{a_{n-1}}{a_n}$$

$$\text{sum of products of pairs of zeroes} = +\frac{a_{n-2}}{a_n}$$

$$\text{sum of products of triples of zeroes} = -\frac{a_{n-3}}{a_n}$$

$$\ldots\ \ldots$$

$$\text{product of the zeroes} = \alpha_1\alpha_2 \cdots \alpha_n = (-1)^n\frac{a_0}{a_n}$$

Notice the alternating signs of the successive results. It is unlikely that anything apart from the first and last formulae would be required.

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Example 14 10F

Let α, β and γ be the roots of the cubic equation $x^3 - 3x + 2 = 0$. Use the formulae above to find:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha\beta + \beta\gamma + \gamma\alpha$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **e** $\alpha^2 + \beta^2 + \gamma^2$ **f** $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$

Check the result with the factoring of $x^3 - 3x + 2 = (x - 1)^2(x + 2)$ obtained in the solution of Example 11 on page 477.

SOLUTION

a $\alpha + \beta + \gamma = \frac{-0}{1} = 0$

b $\alpha\beta\gamma = -\frac{2}{1} = -2$

c $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-3}{1} = -3$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{-3}{-2} = \frac{3}{2}$

e $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$,

so $0^2 = \alpha^2 + \beta^2 + \gamma^2 + 2 \times (-3)$

$\alpha^2 + \beta^2 + \gamma^2 = 6$.

f $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$

$= \alpha\beta(\alpha + \beta + \gamma) + \beta\gamma(\beta + \gamma + \alpha) + \gamma\alpha(\gamma + \alpha + \beta) - 3\alpha\beta\gamma$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma$

$= (-3) \times 0 - 3 \times (-2)$

$= 6$

Because $x^3 - 3x + 2 = (x - 1)^2(x + 2)$, the actual roots are 1, 1 and -2, hence

a $\alpha + \beta + \gamma = 1 + 1 - 2 = 0$

b $\alpha\beta\gamma = 1 \times 1 \times (-2) = -2$

c $\alpha\beta + \beta\gamma + \gamma\alpha = 1 - 2 - 2 = -3$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1 + 1 - \frac{1}{2} = 1\frac{1}{2}$

e $\alpha^2 + \beta^2 + \gamma^2 = 1 + 1 + 4 = 6$

f $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$

$= 1 \times 1 + 1 \times 1 + 1 \times (-2)$

$+ 1 \times 4 + 4 \times 1 + (-2) \times 1$

$= 6$,

all of which agree with the previous calculations.

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Factoring polynomials using the factor theorem and the sum and product of zeroes

Long division can be avoided in many situations by applying the sum and product of zeroes formulae after one or more zeroes have been found. Here are the steps that we have developed so far for factoring polynomials:

26 FACTORING POLYNOMIALS — THE STEPS SO FAR

- Use trial and error to find as many integer zeroes of $P(x)$ as possible.
- Use sum and product of zeroes to find the other zeroes.
- Alternatively, use long division of $P(x)$ by the product of the known factors.

If the coefficients of $P(x)$ are all integers, then any integer zero of $P(x)$ must be one of the divisors of the constant term.

In Example 15, we factor a polynomial factored twice already, but this time there is no need for any long division.

Example 15 10F

Factor $F(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

SOLUTION

As before, $F(1) = 1 + 1 - 9 + 11 - 4 = 0$,
and $F(-4) = 256 - 64 - 144 - 44 - 4 = 0$.
Let the zeroes be 1, −4, α and β.

Then $\alpha + \beta + 1 - 4 = -1$
$$\alpha + \beta = 2. \qquad (1)$$
Also $\alpha\beta \times 1 \times (-4) = -4$
$$\alpha\beta = 1. \qquad (2)$$
From (1) and (2), $\alpha = \beta = 1$, so $F(x) = (x - 1)^3(x + 4)$.

Example 16 10F

Factor completely the cubic $G(x) = x^3 - x^2 - 4$.

SOLUTION

First, $G(2) = 8 - 4 - 4 = 0$.
Let the zeroes be 2, α and β.

Then $2 + \alpha + \beta = 1$
$$\alpha + \beta = -1, \qquad (1)$$
and $2 \times \alpha \times \beta = 4$
$$\alpha\beta = 2. \qquad (2)$$

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Substituting (1) into (2), $\alpha(-1 - \alpha) = 2$

$$\alpha^2 + \alpha + 2 = 0$$

This is an irreducible quadratic, because $\Delta = -7$,

so the complete factoring is $G(x) = (x - 2)(x^2 + x + 2)$.

Note: This procedure — developing the irreducible quadratic factor from the sum and product of zeroes — is really little easier than the long division it avoids.

Forming identities with the coefficients

If some information can be gained about the roots of a polynomial equation, it may be possible to form an identity with the coefficients of the polynomial.

Example 17 10F

If one zero of the cubic $f(x) = ax^3 + bx^2 + cx + d$ is the opposite of another, prove that $ad = bc$.

SOLUTION

We know that one zero is the opposite of another.
The following sentence expresses this symbolically:
Let the zeroes be α, $-\alpha$ and β.
Using the formula for the sum of the roots,

$$\alpha - \alpha + \beta = -\frac{b}{a}$$

$$a\beta = -b. \qquad (1)$$

Using the formula for the product of the roots,

$$-\alpha^2\beta = -\frac{d}{a}$$

$$a\alpha^2\beta = d. \qquad (2)$$

These are three products of pairs of roots, namely $-\alpha^2$ and $-\alpha\beta$ and $\beta\alpha$, so using the sum of products of pairs of roots,

$$-\alpha^2 - \alpha\beta + \beta\alpha = \frac{c}{a}$$

$$a\alpha^2 = -c. \qquad (3)$$

Combine the three identities by taking $(1) \times (3) \div (2)$,

$$\frac{a\beta \times a\alpha^2}{a\alpha^2\beta} = \frac{(-b) \times (-c)}{d}$$

$$a = \frac{bc}{d}$$

$$ad = bc\text{, as required.}$$

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Exercise 10F

FOUNDATION

1 If α and β are the roots of the quadratic equation $x^2 - 4x + 2 = 0$, find:

a $\alpha + \beta$
b $\alpha\beta$
c $\alpha^2\beta + \alpha\beta^2$
d $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$
e $(\alpha + 2)(\beta + 2)$
f $\alpha^2 + \beta^2$
g $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$
h $\alpha\beta^3 + \alpha^3\beta$
i $\left(\alpha + \dfrac{1}{\alpha}\right)\left(\beta + \dfrac{1}{\beta}\right)$

2 If α, β and γ are the roots of the equation $x^3 + 2x^2 - 11x - 12 = 0$, find:

a $\alpha + \beta + \gamma$
b $\alpha\beta + \alpha\gamma + \beta\gamma$
c $\alpha\beta\gamma$
d $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$
e $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\beta\gamma}$
f $(\alpha + 1)(\beta + 1)(\gamma + 1)$
g $(\alpha\beta)^2\gamma + (\alpha\gamma)^2\beta + (\beta\gamma)^2\alpha$
h $\alpha^2 + \beta^2 + \gamma^2$
i $(\alpha\beta)^{-2} + (\alpha\gamma)^{-2} + (\beta\gamma)^{-2}$

Now find the roots of the equation $x^3 + 2x^2 - 11x - 12 = 0$ by factoring the LHS. Hence check your answers for the expressions in parts a–i.

3 If α, β, γ and δ are the roots of the equation $x^4 - 5x^3 + 2x^2 - 4x - 3 = 0$, find:

a $\alpha + \beta + \gamma + \delta$
b $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$
c $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$
d $\alpha\beta\gamma\delta$
e $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$
f $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\alpha\delta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\beta\delta} + \dfrac{1}{\gamma\delta}$
g $\dfrac{1}{\alpha\beta\gamma} + \dfrac{1}{\alpha\beta\delta} + \dfrac{1}{\alpha\gamma\delta} + \dfrac{1}{\beta\gamma\delta}$
h $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

4 If α and β are the roots of the equation $2x^2 + 5x - 4 = 0$, find:

a $\alpha + \beta$
b $\alpha\beta$
c $\alpha^2 + \beta^2$
d $|\alpha - \beta|$ (Hint: Square it.)

DEVELOPMENT

5 a The polynomial $P(x) = 2x^3 - 5x^2 - 14x + 8$ has zeroes at -2 and 4. Use the sum of the zeroes to find the other zero.

b Suppose that $x - 3$ and $x + 1$ are factors of $P(x) = x^3 - 6x^2 + 5x + 12$. Use the product of the zeroes to find the other factor of $P(x)$.

6 Consider the polynomial $P(x) = x^3 - x^2 - x + 10$.

a Show that -2 is a zero of $P(x)$.
b Suppose that the zeroes of $P(x)$ are -2, α and β. Show that $\alpha + \beta = 3$ and $\alpha\beta = 5$.
c By solving the two equations in part b simultaneously, show that $\alpha^2 - 3\alpha + 5 = 0$.
d Hence show that there are no such real numbers α and β.
e Hence state how many times the graph of the cubic crosses the x-axis.

7 Show that $x = 1$ and $x = -2$ are zeroes of $P(x)$, and use the sum and product of zeroes to find the other one or two zeroes. Note any multiple zeroes.

a $P(x) = x^3 - 2x^2 - 5x + 6$
b $P(x) = 2x^3 + 3x^2 - 3x - 2$
c $P(x) = x^4 + 3x^3 - 3x^2 - 7x + 6$
d $P(x) = 3x^4 - 5x^3 - 10x^2 + 20x - 8$

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8 **a** Find the values of a and b for which $x^3 + ax^2 - 10x + b$ is exactly divisible by $x^2 + x - 12$, and then factor the cubic.

b Find the values of a and b for which $x^2 - x - 20$ is a factor of $x^4 + ax^3 - 23x^2 + bx + 60$, and then find all the zeroes.

9 **a** If one of the roots of the equation $x^2 + bx + c = 0$ is twice the other root, show that $2b^2 = 9c$. (Hint: Let the roots be α and 2α.)

b If one of the roots of the equation $x^2 + px + q = 0$ is one more than the other root, show that $p^2 = 4q + 1$. (Hint: Let the roots be α and $\alpha + 1$.)

10 **a** Find the roots of the equation $3x^3 - x^2 - 48x + 16 = 0$ given that the sum of two of them is zero. (Hint: Let the roots be α, $-\alpha$ and β.)

b Find the roots of the equation $2x^3 - 5x^2 - 46x + 24 = 0$ given that the product of two of them is 3. (Hint: Let the roots be α, $\dfrac{3}{\alpha}$ and β.)

c Find the roots of the equation $x^3 - 27x - 54 = 0$ given that two of them are equal. (Hint: Let the roots be α, α and β.)

d Find the zeroes of the polynomial $P(x) = 2x^3 - 13x^2 + 22x - 8$ given that one of them is the product of the other two. (Hint: Let the zeroes be α, β and $\alpha\beta$.)

11 **a** Two of the roots of the equation $x^3 + 3x^2 - 4x + a = 0$ are opposites. Find the value of a and the three roots.

b Two of the roots of the equation $4x^3 + ax^2 - 47x + 12 = 0$ are reciprocals. Find the value of a and the three roots.

12 Find the zeroes of the polynomial $x^4 - 3x^3 - 8x^2 + 12x + 16$ if they are α, 2α, β and 2β.

13 **a** If one root of the equation $x^3 - bx^2 + cx - d = 0$ is equal to the product of the other two, show that $(c + d)^2 = d(b + 1)^2$.

b The polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ has two zeroes that are opposites and another two zeroes that are reciprocals. Show that:

i $b = 1 + d$ **ii** $c = ad$

14 **a** The cubic equation $2x^3 - x^2 + x - 1 = 0$ has roots α, β and γ. Evaluate:

i $\alpha + \beta + \gamma$ **ii** $\alpha\beta + \alpha\gamma + \beta\gamma$

iii $\alpha\beta\gamma$ **iv** $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

b The equation $2\cos^3\theta - \cos^2\theta + \cos\theta - 1 = 0$ has roots $\cos a$, $\cos b$ and $\cos c$. Show that $\sec a + \sec b + \sec c = 1$.

15 The cubic equation $x^3 - Ax^2 + 3A = 0$, where $A > 0$, has roots α, β and $\alpha + \beta$.

a Use the sum of the roots to show that $\alpha + \beta = \frac{1}{2}A$.

b Use the sum of the products of pairs of roots to show that $\alpha\beta = -\frac{1}{4}A^2$.

c Show that $A = 2\sqrt{6}$.

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16 The polynomial $P(x) = x^3 - Lx^2 + Lx - M$ has zeroes α, $\dfrac{1}{\alpha}$ and β.

a Show that:

i $\alpha + \dfrac{1}{\alpha} + \beta = L$ **ii** $1 + \alpha\beta + \dfrac{\beta}{\alpha} = L$ **iii** $\beta = M$

b Show that either $M = 1$ or $M = L - 1$.

17 Suppose that the polynomial $P(x) = ax^3 + bx^2 + cx + d$ has zeroes α, β and γ.

a Show that $\alpha^2 + \beta^2 + \gamma^2 = \dfrac{b^2 - 2ac}{a^2}$.

b Hence explain why the polynomial $2x^3 - 3x^2 + 5x - 8$ cannot have three real zeroes.

ENRICHMENT

18 If α, β and γ are the roots of the equation $x^3 + 5x - 4 = 0$, evaluate $\alpha^3 + \beta^3 + \gamma^3$. (Hint: $x = \alpha$, $x = \beta$ and $x = \gamma$ satisfy the given equation.)

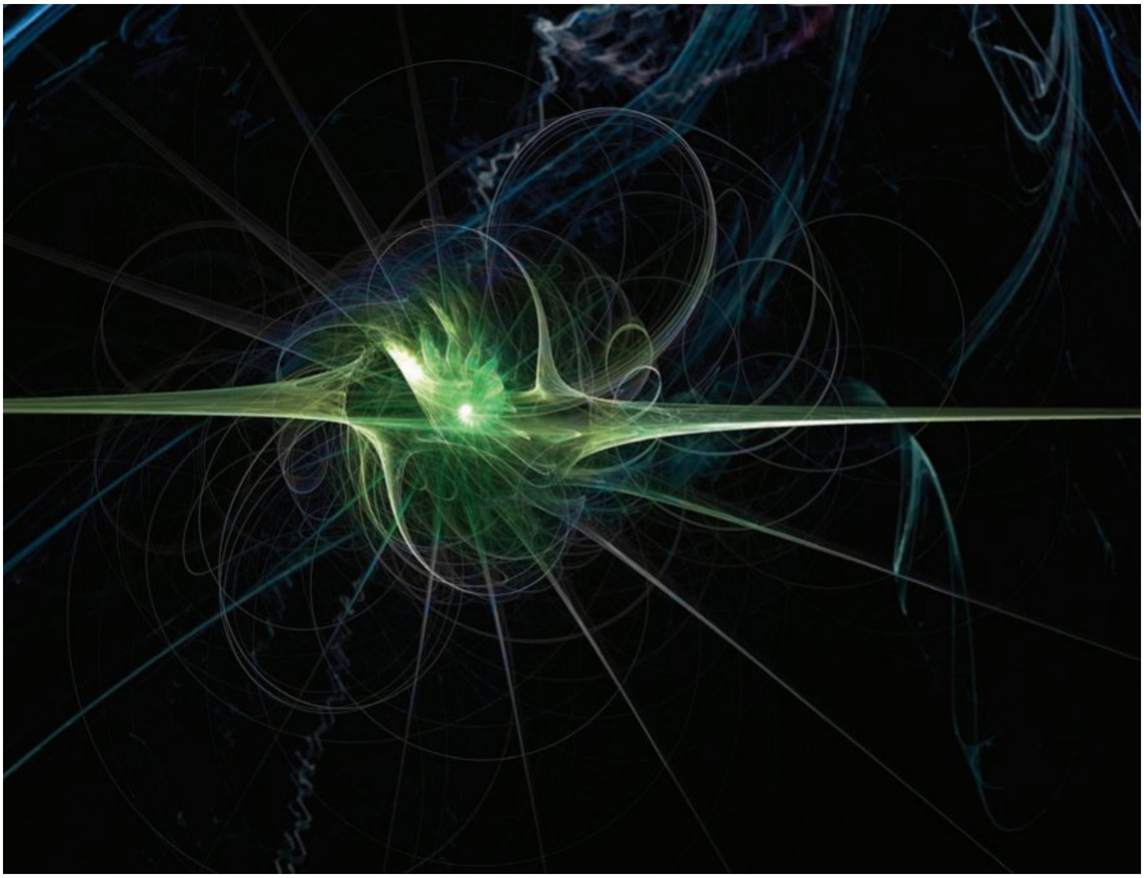
19 If $x^n - 1 = 0$ has n roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, find the value of $(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_n)$.

20 Suppose that the equation $x^2 + bx + c = 0$ has roots α and β.

a Show that b and c are the roots of the equation $x^2 + (\alpha + \beta - \alpha\beta)x - \alpha\beta(\alpha + \beta) = 0$.

b Find the non-zero values of b and c for which the roots of the equation in part **a** are:

i α and β **ii** α^2 and β^2



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10G Multiple zeroes

It was claimed in Box 7 of Section 10B, page 461, that if α is a multiple root of a polynomial $P(x)$, then the graph of $y = P(x)$ has a stationary point at $x = \alpha$, which is either a horizontal inflection or a turning point, depending on whether the curve changes sign around $x = \alpha$. In this section we prove the necessary theorem.

Once a zero is found, this theorem allows its multiplicity to be found very quickly. The technique provides yet another approach to the factoring of polynomials.

Multiple zeroes and the factor theorem

Divisibility was used earlier (Box 6 of Section 10B, page 460) when we defined the multiplicity m of a zero $x = \alpha$ of a polynomial $P(x)$ by

$$P(x) = (x - \alpha)^m Q(x), \text{ where } Q(x) \text{ is not divisible by } x - \alpha.$$

We now know from the factor theorem that $Q(x)$ is divisible by $x - \alpha$ if and only if $Q(\alpha) = 0$, so we can rewrite the condition of multiplicity m as

27 MULTIPLICITY OF A ZERO

- A value $x = \alpha$ is a zero of multiplicity m of the polynomial $P(x)$ if
  $$P(x) = (x - \alpha)^m Q(x), \text{ where } Q(\alpha) \neq 0.$$
- Interpret a 'zero of multiplicity 0' as a value of x that is not a zero. Then this statement also applies when $x = \alpha$ is not a zero of $P(x)$.

The second assertion is trivial because when $m = 0$, then $P(x) = Q(x)$.

Multiple zeroes are stationary points

We claimed in Section 10B that the graph of $y = P(x)$ has a stationary point at any multiple zero $x = \alpha$. This follows from a more general theorem about the derivative of polynomials at its zeroes.

28 MULTIPLE ZEROES AND THE DERIVATIVE

- Let $x = \alpha$ be a zero of multiplicity $m \geq 1$ of a polynomial $P(x)$.
  Then $x = \alpha$ is a zero of multiplicity $m - 1$ of the derivative $P'(x)$.
- In particular:
  — If $x = \alpha$ is a multiple zero of $P(x)$, then it is a zero of $P'(x)$.
  — If $x = \alpha$ is a single zero of $P(x)$, then it is not a zero of $P'(x)$.

Proof

The proof uses the product rule to differentiate $(x - \alpha)^m Q(x)$.

Because $x = \alpha$ has multiplicity m, we can write

$$P(x) = (x - \alpha)^m Q(x), \text{ where } Q(\alpha) \neq 0.$$

so
$$\begin{aligned} P'(x) &= m(x - \alpha)^{m-1} Q(x) + (x - \alpha)^m Q'(x) \\ &= (x - \alpha)^{m-1}\big(mQ(x) + (x - \alpha)Q'(x)\big) \\ &= (x - \alpha)^{m-1} R(x), \text{ where } R(x) = mQ(x) + (x - \alpha)Q'(x). \end{aligned}$$

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We know that $Q(\alpha) \neq 0$, so substituting $x = \alpha$ into $R(x)$,

$$\begin{aligned} R(\alpha) &= Q(\alpha) + 0 \\ &\neq 0. \end{aligned}$$

Hence $x = \alpha$ is a zero of multiplicity $m - 1$ of $P'(x)$.

Behaviour at simple and multiple zeroes

We can now give a satisfactory proof of the theorem stated in Box 7 of Section 10B, which is restated below:

29 MULTIPLE ZEROES AND THE SHAPE OF THE CURVE

Suppose that $x = \alpha$ is a zero of a polynomial $P(x)$.

- If $x = \alpha$ has even multiplicity, then the curve is tangent to the x-axis at $x = \alpha$, and does not cross the x-axis there. The point $(\alpha, 0)$ is a *turning point*.
- If $x = \alpha$ has odd multiplicity at least 3, then the curve is tangent to the x-axis at $x = \alpha$, but crosses the x-axis. The point $(\alpha, 0)$ is a *horizontal inflection*.
- If $x = \alpha$ is a simple zero, then the curve crosses the x-axis at $x = \alpha$ and there is no stationary point there.

Proof

The result follows quickly from the previous theorem, and the fact that

$$P(x) = (x - \alpha)^m Q(x), \text{ where } Q(\alpha) \neq 0.$$

Consider the cases separately, beginning with simple zeroes.

A Let $x = \alpha$ be a simple zero. Then $P'(\alpha) \neq 0$ by the previous result, so $x = \alpha$ is not a stationary point, and the curve crosses the x-axis there.

B Let $x = \alpha$ be an even-order zero. Then $P'(\alpha) = 0$ by the previous result, so there is a stationary point at $x = \alpha$, and the x-axis is a tangent there. The factor $(x - \alpha)^m$ has the same sign on both sides of $x = \alpha$ because m is even, and $Q(x)$ has the same sign for a small distance on both sides because $Q(\alpha) \neq 0$, so the curve does not cross at $x = \alpha$, and the x-intercept is a turning point.

C Let $x = \alpha$ be an odd-order zero, with $m \geq 3$. Then $P'(\alpha) = 0$, so there is a stationary point at $x = \alpha$, and the x-axis is a tangent there. The factor $(x - \alpha)^m$ has opposite signs on both sides of $x = \alpha$ because m is odd, and $Q(x)$ has the same sign for a small distance on both sides because $Q(\alpha) \neq 0$, so the curve crosses at $x = \alpha$, and the x-intercept is a horizontal inflection.

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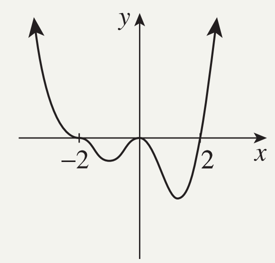
Example 18 10G

The curve $y = (x + 2)^3 x^2 (x - 2)$, used as an explanatory example in Section 10B (page 460), can now be sketched much more easily.

SOLUTION

Begin as always with a table of test points:

x	-3	-2	-1	0	1	2	3
y	45	0	-3	0	-27	0	1125



- The curve has a horizontal inflection at the triple zero $x = -2$.
- It has a turning point at the double zero $x = 0$.
- It crosses the curve with no stationary point at the simple zero $x = 2$.

Factoring using the theory of multiple zeroes

From now on, once a zero of a polynomial $P(x)$ is found, it should immediately be checked whether it is a double zero by substituting into $P'(x)$, and if so, a triple zero by substituting into $P''(x)$, and so on.

30 FACTORING AND MULTIPLE ZEROES

Once a zero of a polynomial $P(x)$ is found:

- Substitute into $P'(x)$ to check whether it is a double zero, and if so:
- Substitute into $P''(x)$ to check whether it is a triple zero, and so on.

Example 19 10G

Factor the polynomial $P(x) = x^5 - 8x^4 + 25x^3 - 38x^2 + 28x - 8$ completely.

SOLUTION

The polynomial is monic, and the positive factors of the constant term are $1, 2, 4, 8$, so we need only test $1, 2, 4, 8, -1, -2, -4$ and -8.

Testing, $P(1) = 1 - 8 + 25 - 38 + 28 - 8 = 0,$

$P(2) = 32 - 128 + 200 - 152 + 56 - 8 = 0,$

so $x = 1$ and $x = 2$ are zeroes.

Differentiating, $P'(x) = 5x^4 - 32x^3 + 75x^2 - 76x + 28.$

Testing, $P'(1) = 5 - 32 + 75 - 76 + 28 = 0$

$P'(2) = 80 - 256 + 300 - 152 + 28 = 0,$

so $x = 1$ and $x = 2$ are zeroes of multiplicity at least 2.

Differentiating, $P''(x) = 20x^3 - 96x^2 + 150x - 76.$

Testing, $P''(1) = 20 - 96 + 150 - 76 \neq 0$

$P''(2) = 160 - 384 + 300 - 76 = 0,$

so $x = 2$ is a triple zero (the whole degree is only 5) and $x = 1$ is a double zero.

The polynomial is monic, so $P(x) = (x - 1)^2 (x - 2)^3$.

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The full factoring menu

Combining all the methods of factoring presented so far gives the following menu for factoring a polynomial (and one should remember that it is only special polynomials that can be factored using these techniques).

31 A MENU FOR FACTORING A POLYNOMIAL

- Test whether some low positive and negative whole numbers are zeroes.
  If all coefficients are integers, every integer zero is a divisor of the constant term.
- Test each zero successively in $P'(x)$, $P''(x)$, …to find its multiplicity.
- Use sum and product of zeroes to find other zeroes.
- If all else fails, multiply the known linear factors together and use long division.

If the coefficients of $P(x)$ are all integers, then any integer zero of $P(x)$ must be one of the divisors of the constant term.

Exercise 10G

FOUNDATION

1 Consider the polynomial $P(x) = x^3 - 4x^2 - 3x + 18$.
  a i Show that $P(3)$ and $P'(3)$ are both zero.
    ii What can be deduced from the results in part i?
  b Use part a and the sum of zeroes to find all the zeroes of $P(x)$.
  c Hence factor $P(x)$.

2 Consider the polynomial $P(x) = x^4 + 8x^3 + 18x^2 + 16x + 5$.
  a i Show that $P(-1)$, $P'(-1)$ and $P''(-1)$ are all zero.
    ii What can be deduced from the results in part i?
  b Use part a and the product of zeroes to find all the zeroes of $P(x)$.
  c Hence factor $P(x)$.

3 The polynomial $P(x) = x^3 - 27x + 54$ has a double zero.
  a Find the zeroes of $P'(x)$.
  b Determine which of the zeroes of $P'(x)$ is the double zero of $P(x)$.
  c Find the remaining simple zero of $P(x)$.

4 The polynomial $P(x) = x^4 + 5x^3 - 75x^2 - 625x - 1250$ has a triple zero.
  a Find the zeroes of $P''(x)$.
  b Determine which of the zeroes of $P''(x)$ is the triple zero of $P(x)$.
  c Find the remaining simple zero of $P(x)$.

5 The polynomial $P(x) = 2x^3 + 5x^2 - 4x - 12$ has a double zero.
  a Find the double zero.
  b Find the remaining simple zero, and hence factor $P(x)$.

6 The polynomial $P(x) = 8x^4 - 28x^3 + 30x^2 - 13x + 2$ has a triple zero.
  a Find the triple zero.
  b Find the remaining simple zero, and hence factor $P(x)$.

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DEVELOPMENT

7 Consider the polynomial equation $x^4 - 10x^3 + 34x^2 - 42x + 9 = 0$.
- a Show that $x = 3$ is a double root of the equation.
- b Hence solve the equation.

8 The polynomial $P(x) = x^3 - 3x^2 - 9x + k$ has a double zero.
- a Find the two possible values of k.
- b For each of the possible values of k, factor $P(x)$.

9 The coefficients of the polynomial $P(x) = ax^3 + bx + c$ are real and $P(x)$ has a multiple zero at $x = 1$. When $P(x)$ is divided by $x + 1$ the remainder is 4. Find the values of a, b and c.

10 The polynomial $P(x) = x^4 + 7x^3 + 9x^2 - 27x + c$ has a triple zero.
- a Determine the value of the triple zero.
- b Hence find the value of c.
- c Factor $P(x)$.

11
- a Find the values of b and c if $x = 1$ is a double root of the equation $x^4 + bx^3 + cx^2 - 5x + 1 = 0$.
- b Find the other roots of the equation.

12 It is known that $(x - 1)^2$ is a factor of the polynomial $P(x) = ax^{n+1} + bx^n + 1$. Show that $a = n$ and $b = -(1 + n)$.

13 The equation $Ax^3 + Bx^2 + D = 0$, where A, B and D are non-zero, has a double root.
- a Show that the double root is $-\dfrac{2B}{3A}$.
- b Hence show that $27A^2D + 4B^3 = 0$.

ENRICHMENT

14 Prove that $P(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$, where $n \geq 2$, has no multiple zeroes.

15 Let $P(x) = x^4 + mx^2 + n = 0$, where m and n are non-zero.
- a Prove that the equation $P(x) = 0$ cannot have a root of multiplicity greater than 2.
- b Let $x = \alpha$ be a double root of $P(x) = 0$.
  - i Prove that $x = -\alpha$ is also a double root.
  - ii What range of values can m take?
  - iii Prove that $n = \frac{1}{4}m^2$, and write down the roots of the equation in terms of m.

16 The polynomial $P(x) = x^3 + 3px^2 + 3qx + r$ has a double zero.
- a Prove that the double zero is $\alpha = \dfrac{pq - r}{2(q - p^2)}$.
- b Hence, or otherwise, prove that $4(p^2 - q)(q^2 - pr) = (pq - r)^2$.

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10H Geometry using polynomial techniques

This final section adds the methods of the preceding sections, particularly the sum and product of zeroes, to the available techniques for studying the geometry of various curves. The standard technique is to examine the roots of the equation formed in the process of solving two curves simultaneously.

This section is rather demanding, and could all be regarded as Enrichment.

Midpoints and tangents

When two curves intersect, we can form the equation whose solutions are the x- or y-coordinates of points of intersection of the two curves. The midpoint of two points of intersection can then be found using the average of the roots. Tangents can be identified as corresponding to double roots.

Example 20 could also be done using quadratic equations, but it is a very clear example of the use of sum and product of roots.

Example 20 — 10H

The line $y = 2x$ meets the parabola $y = x^2 - 2x - 8$ at the two points $A(\alpha, 2\alpha)$ and $B(\beta, 2\beta)$.

a Show that α and β are roots of $x^2 - 4x - 8 = 0$, and hence find the coordinates of the midpoint M of AB.

b Use the identity $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to find the horizontal distance $|\alpha - \beta|$ from A to B. Then use Pythagoras' theorem and the gradient of the line to find the length of AB.

c Find the value of b for which $y = 2x + b$ is a tangent to the parabola, and find the point T of contact.

SOLUTION

a Solving the line and the parabola simultaneously,

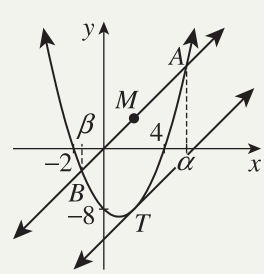
$$x^2 - 2x - 8 = 2x$$
$$x^2 - 4x - 8 = 0.$$

Hence $\alpha + \beta = 4$,

and $\alpha\beta = -8$.

Averaging the roots, M has x-coordinate $x = 2$,

and substituting into the line, $M = (2, 4)$.



b We know that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$= 16 + 32$$
$$= 48$$

so $|\alpha - \beta| = 4\sqrt{3}$.

Because the line has gradient 2, the vertical distance is $8\sqrt{3}$,

so using Pythagoras, $AB^2 = (4\sqrt{3})^2 + (8\sqrt{3})^2$

$$= 16 \times 15$$
$$AB = 4\sqrt{15}.$$

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c Solving $y = x^2 - 2x - 8$ and $y = 2x + b$ simultaneously,

$$x^2 - 4x - (8 + b) = 0$$

Because the line is a tangent, let the roots be θ and θ.

Then using the sum of roots, $\theta + \theta = 4$,

so $\theta = 2$,

Using the product of roots, $\theta^2 = -8 - b$

and because $\theta = 2$, $b = -12$.

So the line $y = 2x - 12$ is a tangent at $T(2, -8)$.

Geometric problems using sum and product of roots

The sum and product of roots can make some interesting geometric problems quite straightforward.

Example 21 10H

A line with gradient m through the point $P(-1, 0)$ on the cubic $y = x^3 - x$ crosses the curve at two further points A and B with x-coordinates α and β respectively.

a Sketch the situation.

b Show that α and β satisfy the cubic equation $x^3 - (m + 1)x - m = 0$.

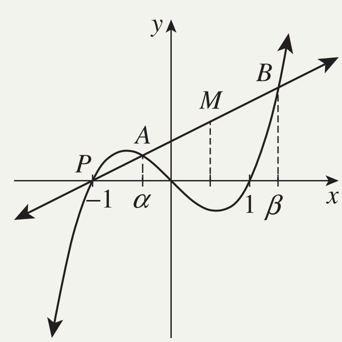
c Show that the midpoint M of AB lies on the vertical line $x = \frac{1}{2}$.

d Find the line through P tangent to the cubic at a point distinct from P.

SOLUTION

a The cubic factors as $y = x(x - 1)(x + 1)$,

so the zeros are $x = -1$, $x = 0$ and $x = 1$.



b The line through $P(-1, 0)$ has equation

$$y = m(x + 1).$$

Solving the line simultaneously with the cubic,

$$x^3 - x = mx + m$$

$$x^3 - (m + 1)x - m = 0.$$

c The cubic has roots α, β and 1.

Using the sum of roots, $\alpha + \beta + (-1) = 0$ (there is no term in x^2)

so $\alpha + \beta = 1$.

Hence the midpoint M of AB has x-coordinate $\frac{1}{2}(\alpha + \beta) = \frac{1}{2}$,

so M lies on the line $x = \frac{1}{2}$.

d The line PM is a tangent at another point on the curve when the points A, M and B coincide, that is, when $\alpha = \beta$,

and because $\alpha + \beta = 1$, this means that $\alpha = \beta = \frac{1}{2}$.

Using the product of roots, $\alpha \times \beta \times (-1) = m$, (the constant term in $-m$)

$$\tfrac{1}{2} \times \tfrac{1}{2} \times (-1) = m$$

$$m = -\tfrac{1}{4}$$

so the tangent through P is $y = -\frac{1}{4}(x + 1)$.

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Exercise 10H

FOUNDATION

Note: Sketches should be drawn in all these questions to make the situation clear.

1 a Show that the x-coordinates of the points of intersection of the parabola $y = x^2 - 6x$ and the line $y = 2x - 16$ satisfy the equation $x^2 - 8x + 16 = 0$.

b Solve this equation, and hence show that the line is a tangent to the parabola. Find the point T of contact.

2 a Show that the x-coordinates of the points of intersection of the line $y = -2x + b$ and the parabola $y = x^2 - 6x$ satisfy the quadratic equation $x^2 - 4x - b = 0$.

b Suppose that the line is a tangent to the parabola, so that the roots of the quadratic equation are equal. Let these roots be α and α.

i Using the sum of roots, show that $\alpha = 2$.

ii Using the product of roots, show that $\alpha^2 = -b$, and hence find b.

iii Find the equation of the tangent and its point T of contact.

3 The line $y = x + 1$ meets the parabola $y = x^2 - 3x$ at A and B.

a Show that the x-coordinates α and β of A and B satisfy the equation $x^2 - 4x - 1 = 0$.

b Find $\alpha + \beta$, and hence find the coordinates of the midpoint M of AB.

DEVELOPMENT

4 a Show that the x-coordinates of the points of intersection of the line $y = 3 - x$ and the cubic $y = x^3 - 5x^2 + 6x$ satisfy the equation $x^3 - 5x^2 + 7x - 3 = 0$.

b Show that $x = 1$ and $x = 3$ are roots of the equation, and use the sum of the roots to find the third root.

c Explain why the line is a tangent to the cubic, then find the point of contact and the other point of intersection.

5 a Show that the x-coordinates of the points of intersection of the line $y = mx$ and the cubic $y = x^3 - 5x^2 + 6x$ satisfy the equation $x^3 - 5x^2 + (6 - m)x = 0$.

b Suppose now that the line is a tangent to the cubic at a point other than the origin, so that the roots of the equation are 0, α and α.

i Using the sum of roots, show that $\alpha = \frac{5}{2}$.

ii Using the product of pairs of roots, show that $\alpha^2 = 6 - m$, and hence find m.

iii Find the equation of the tangent and its point T of contact.

6 The line $y = x - 2$ meets the cubic $y = x^3 - 5x^2 + 6x$ at $F(2, 0)$, and also at A and B.

a Show that the x-coordinates α and β of A and B satisfy $x^3 - 5x^2 + 5x + 2 = 0$.

b Find $\alpha + \beta$, and hence find the coordinates of the midpoint M of AB.

c Show that $\alpha\beta = -1$, then use the identity $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to show that the horizontal distance $|\alpha - \beta|$ between A and B is $\sqrt{13}$.

d Hence use Pythagoras' theorem to find the length AB.

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7 A line passes through the point $A(-1, -7)$ on the curve $y = x^3 - 3x^2 + 4x + 1$. Suppose that the line has gradient m and is tangent to the curve at another point P on the curve whose x-coordinate is α.

a Show that the line has equation $y = mx + (m - 7)$.

b Show that the cubic equation whose roots are the x-coordinates of the points of intersection of the line and the curve is $x^3 - 3x^2 + (4 - m)x + (8 - m) = 0$.

c Explain why the roots of this equation are -1, α and α, and hence find the point P and the value of m.

8 The point $P(p, p^3)$ lies on the curve $y = x^3$. A line through P with gradient m intersects the curve again at A and B.

a Find the equation of the line through P.

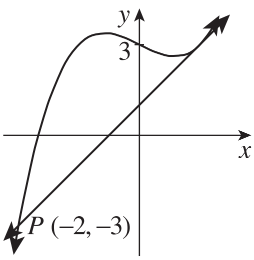
b Show that the x-coordinates of A and B satisfy the equation $x^3 - mx + mp - p^3 = 0$.

c Hence find the x-coordinate of the midpoint M of AB, and show that for fixed p, M always lies on a line that is parallel to the y-axis.

9 **a** The equation $x^3 - (m + 1)x + (6 - 2m) = 0$ has a root at $x = -2$ and a double root at $x = \alpha$. Find α and m.

b Write down the equation of the line ℓ passing through the point $P(-2, -3)$ with gradient m.

c The diagram shows the curve $y = x^3 - x + 3$ and the point $P(-2, -3)$ on the curve. The line ℓ cuts the curve at P, and is tangent to the curve at another point A on the curve. Find the equation of the line ℓ.



10 **a** Use the factor theorem to factor the polynomial $y = x^4 - 4x^3 - 9x^2 + 16x + 20$, given that there are four distinct zeroes, then sketch the curve.

b The line $\ell: y = mx + b$ touches the quartic $y = x^4 - 4x^3 - 9x^2 + 16x + 20$ at two distinct points A and B. Explain why the x-coordinates α and β of A and B are double roots of the equation $x^4 - 4x^3 - 9x^2 + (16 - m)x + (20 - b) = 0$.

c Use the theory of the sum and product of roots to write down four equations involving α, β, m and b.

d Hence find m and b, and write down the equation of ℓ.

Note: If two curves touch each other at P, then they are tangent to each other at P. This means that the two curves have a common tangent at P.

11 **a** Find k and the points of contact if the parabola $y = x^2 - k$ touches the quartic $y = x^4$ at two points.

b Find k and the point T of contact if the parabola $y = x^2 - k$ touches the cubic $y = x^3$.

c Find k and the points of contact if the parabola $y = x^2 - k$ touches the circle $x^2 + y^2 = 1$ at two points.

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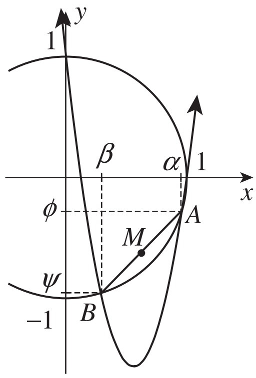
ENRICHMENT

12 A circle passing through the origin O is tangent to the hyperbola $xy = 1$ at A, and intersects the hyperbola again at two distinct points B and C. Prove that $OA \perp BC$.

13 The diagram to the right shows the circle $x^2 + y^2 = 1$ and the parabola $y = (\lambda x - 1)(x - 1)$, where λ is a constant. The circle and parabola meet in the four points

$$P(1, 0), \qquad Q(0, 1), \qquad A(\alpha, \phi), \qquad B(\beta, \psi).$$



The point M is the midpoint of the chord AB.

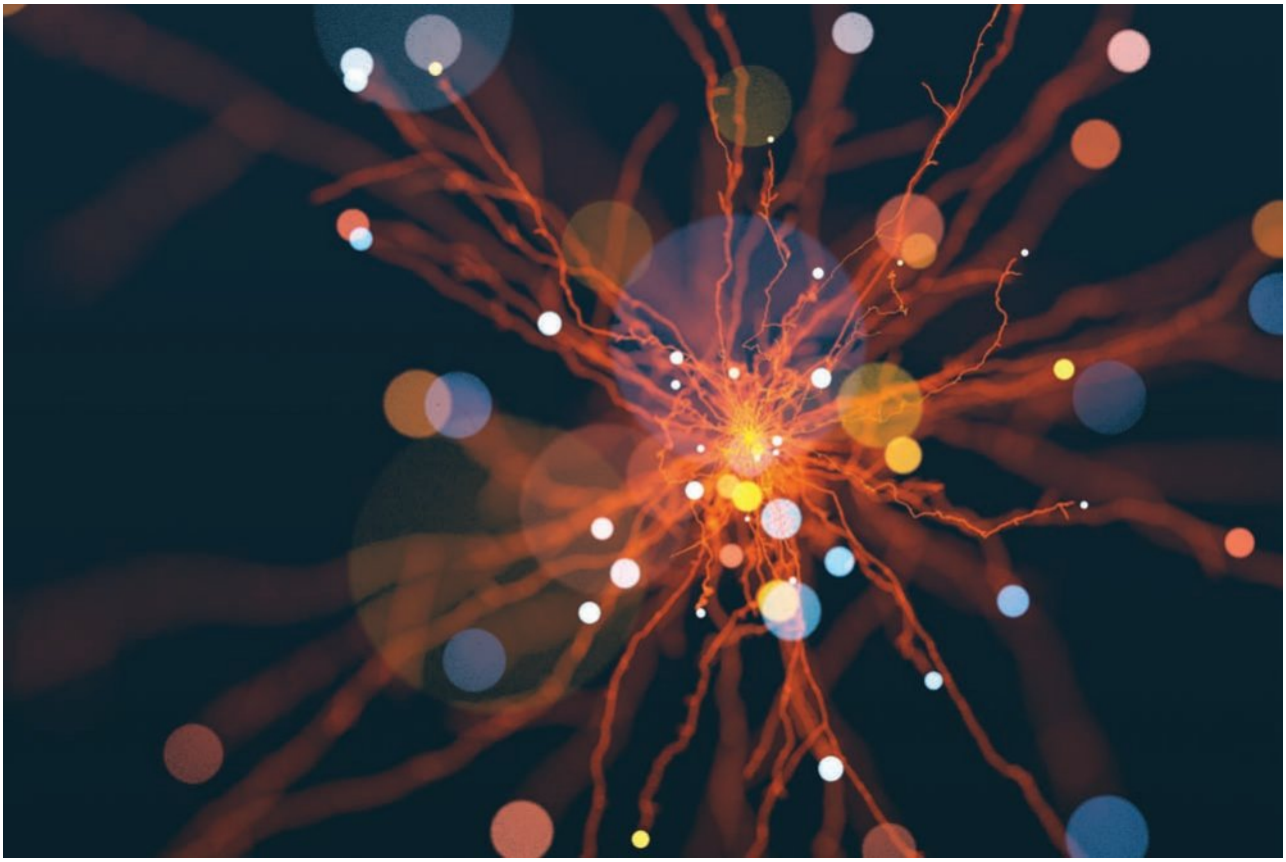
a Show that the x-coordinates of the points of intersection of the two curves satisfy the equation
$\lambda^2 x^4 - 2\lambda(1 + \lambda)x^3 + (\lambda^2 + 4\lambda + 2)x^2 - 2(1 + \lambda)x = 0.$

b Use the formula for the sum of the roots to show that the x-coordinate of M is $\dfrac{\lambda + 2}{2\lambda}$.

c Use a similar method to find the y-coordinate of M, and hence show that M lies on the line through the origin O parallel to PQ.

d For what values of λ is the parabola tangent to the circle in the fourth quadrant?

e For what values of λ are the four points P, Q, A and B distinct, with real numbers as coordinates?



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Chapter 10 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 10 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Consider the polynomial $P(x) = 2x^3 - 5x^2 - 6x - 11$. State:

a the degree of $P(x)$,
b the leading coefficient of $P(x)$,
c the leading term of $P(x)$,
d the constant term of $P(x)$,

2 The polynomial $P(x)$ has degree 3. Write down the degree of the polynomial:

a $3P(x)$ b $(P(x))^3$

3 Find the coefficient of x^2 in the polynomial $P(x) = (x^2 - 3x - 7)(2x^2 + 4x - 9)$.

4 a Sketch the graph of the polynomial function $y = (x + 2)^2(x - 1)(x - 3)$, showing all intercepts with the coordinate axes.
b Hence find the values of x for which $(x + 2)^2(x - 1)(x - 3) < 0$.

5 Sketch the graph of the polynomial $P(x) = x^3 - x^5$.

6 Suppose that the polynomial $P(x) = 2x^3 + 7x^2 - 4x + 5$ is divided by $D(x) = x - 3$.

a Find the quotient $Q(x)$ and the remainder $R(x)$.
b Write down a division identity using the information above.

7 Without long division, find the remainder when $P(x) = x^3 - 5x^2 + 1$ is divided by:

a $x - 3$ b $x + 2$

8 a Use the factor theorem to show that $x - 2$ is a factor of $P(x) = x^3 - 19x + 30$.
b Hence factor $P(x)$ fully.

9 Find the value of k given that $x + 3$ is a factor of $P(x) = x^3 + 4x^2 + kx - 12$.

10 Find the values of b and c given that $x + 1$ is a factor of $P(x) = x^3 + bx^2 + cx - 7$, and the remainder is -12 when $P(x)$ is divided by $x - 5$.

11 Find the values of h and k given that $x + 2$ is a factor of $Q(x) = (x + h)^2 + k$, and the remainder is 16 when $Q(x)$ is divided by x.

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12 The polynomial $P(x)$ is divided by $(x + 1)(x - 2)$. Suppose that the quotient is $Q(x)$ and the remainder is $R(x)$.

a Explain why the general form of $R(x)$ is $ax + b$, where a and b are constants.

b When $P(x)$ is divided by $x + 1$ the remainder is 10, and when $P(x)$ is divided by $x - 2$ the remainder is -8. Find a and b. (Hint: Use the division identity.)

13 Suppose that the polynomial $Q(x) = x^2 - 6x - 4$ has zeroes α and β. Without finding the zeroes, find the value of:

a $\alpha + \beta$ **b** $\alpha\beta$ **c** $\alpha^2\beta + \beta^2\alpha$

d $\frac{1}{\alpha} + \frac{1}{\beta}$ **e** $(\alpha - 3)(\beta - 3)$ **f** $\alpha^2 + \beta^2$

14 If α, β and γ are the roots of the equation $x^3 + 10x^2 + 5x - 20 = 0$, find:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta + \alpha\gamma + \beta\gamma$ **c** $\alpha\beta\gamma$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **e** $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$ **f** $(\alpha + 2)(\beta + 2)(\gamma + 2)$

g $\alpha^2\beta^2\gamma + \alpha^2\gamma^2\beta + \beta^2\gamma^2\alpha$ **h** $\alpha^2 + \beta^2 + \gamma^2$ **i** $\frac{1}{\alpha^2\beta^2} + \frac{1}{\alpha^2\gamma^2} + \frac{1}{\beta^2\gamma^2}$

15 The equation $x^3 + 5x^2 + cx + d = 0$ has roots -3, 7 and α.

a Use the sum of the roots to find α.

b Use the product of the roots to find d.

c Use the sum of the roots in pairs to find c.

16 The equation $6x^3 - 17x^2 - 5x + 6 = 0$ has roots α, β and γ, where $\alpha\beta = -2$.

a Use the product of the roots to find γ.

b Use the sum of the roots to find the other two roots.

17 One root of the equation $ax^2 + 2bx + c = 0$ is the reciprocal of the square of the other root. Show that $a^3 + c^3 + 2abc = 0$.

18 Solve the equation $9x^3 - 27x^2 + 11x + 7 = 0$ given that the roots are $\alpha - \beta$, α and $\alpha + \beta$.

19 Find the zeroes of the polynomial $P(x) = 8x^3 - 14x^2 + 7x - 1$ given that they are $\frac{\alpha}{\beta}$, α and $\alpha\beta$.

20 The polynomial $P(x) = x^3 - x^2 - 16x - 20$ has a double zero.

a Find $P'(x)$ and hence find the double zero.

b Find the remaining zero, and hence factor $P(x)$.

21 The polynomial $P(x) = 3x^4 - 11x^3 + 15x^2 - 9x + 2$ has a triple zero.

a Find the zeroes of $P''(x)$.

b Determine which of the zeroes of $P''(x)$ is the triple zero of $P(x)$.

c Find the remaining zero of $P(x)$.

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Review

22 The polynomial $P(x) = x^3 + 3x^2 - 24x + k$ has a double zero.

a Find the two possible values of k.

b For each of the possible values of k, factor $P(x)$.

23 The line $y = 9x + 5$ is the tangent to the curve $y = x^3 - 3x^2$ at the point $A(-1, -4)$. The line intersects the curve at another point B. Suppose that the x-coordinate of B is α.

a Write down the cubic equation whose roots are the x-coordinates of A and B.

b Explain why the roots of this equation are -1, -1 and α.

c Hence find the point B.

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11 Extending calculus

So far, calculus has been developed for algebraic functions such as

$$f(x) = x^3 + 8x \quad \text{and} \quad f(x) = \sqrt{x} \quad \text{and} \quad f(x) = x^2 - \frac{1}{x^2}$$

that involve only powers and the four operations of arithmetic. Some of the most important functions in science, however, cannot be written in this way. This chapter begins to extend calculus to two important groups of non-algebraic functions — exponential and logarithmic functions, and trigonometric functions.

Exponential functions are functions such as $y = 2^x$ and $y = 10^x$, where the variable is in the exponent (index). They were introduced in Chapter 8. These functions model many very common natural phenomena where something is dying away, such as radioactive decay or the noise of a plucked guitar string, or where something is growing, such as populations or inflation.

Trigonometric functions include functions such as $\sin x$ and $\cos x$, whose graphs are waves. They were introduced in Chapter 6. These functions model all the many wave-like phenomena in science, such as sound waves, light and radio waves, vibrating strings, tides, and economic cycles.

Differentiating trigonometric functions requires a new unit for measuring angles, based on $\pi \doteqdot 3.1416$ — this is hardly surprising, because trigonometric functions are based on circles. Differentiating exponential functions requires a new base $e \doteqdot 2.7183$ — this new number *e* is just as important in mathematics as π, but only makes its first appearance in this chapter, in the exponential function $y = e^x$.

Both numbers π and e are real numbers. But neither number is a fraction, although the proofs of this are difficult.

This chapter provides only an introduction, in preparation for much further development next year. First, the differentiation of exponential functions is introduced (Sections 11A–11F). Secondly, radian measure of angles is introduced (Sections 11G–11J) in preparation for the calculus of the trigonometric functions in Year 12.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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11A The exponential function base e

As indicated in the Introduction, differentiating exponential functions requires a new base $e \doteqdot 2.7183$. The fundamental result of this section is surprisingly simple — the function $y = e^x$ is its own derivative:

$$\frac{d}{dx}e^x = e^x.$$

An investigation of the graph of $y = 2^x$

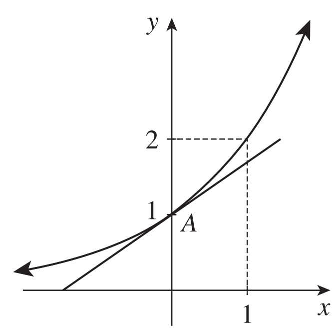
We begin by examining the tangents to the familiar exponential function $y = 2^x$. Before reading the following argument, carry out the investigation in Question 1 of Exercise 11A. The investigation looks at the tangent to $y = 2^x$ at each of several points on the curve, and relates its gradient to the height of the curve at the point. The procedures demonstrate graphically what the argument proves.

Differentiating $y = 2^x$

Below is a sketch of $y = 2^x$, with the tangent drawn at its y-intercept $A(0, 1)$. Differentiating $y = 2^x$ requires first-principles differentiation, because the theory so far hasn't provided any rule for differentiating 2^x.

The formula for first-principles differentiation is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$



Applying this formula to the function $f(x) = 2^x$,

$$f'(x) = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h}$$
$$= \lim_{h \to 0} \frac{2^x \times 2^h - 2^x}{h}, \text{ because } 2^x \times 2^h = 2^{x+h},$$

and taking out the common factor 2^x in the numerator,

$$f'(x) = 2^x \times \lim_{h \to 0} \frac{2^h - 1}{h}$$
$$f'(x) = 2^x \times m, \text{ where } m = \lim_{h \to 0} \frac{2^h - 1}{h}.$$

This limit $m = \lim_{h \to 0} \dfrac{2^h - 1}{h}$ cannot be found by algebraic methods.

But substituting $x = 0$ gives a very simple geometric interpretation of the limit:

$$f'(0) = 2^0 \times m$$
$$= m, \text{ because } 2^0 = 1,$$

so m is just the gradient of the tangent to $y = 2^x$ at its y-intercept $(0, 1)$.

The conclusion of all this is that

$$\frac{d}{dx}2^x = 2^x \times m, \quad \text{where } m \text{ is the gradient of } y = 2^x \text{ at its } y\text{-intercept.}$$

The investigation in Question 1 of Exercise 11A below shows that $m \doteqdot 0.7$

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This argument applied to the base 2. But exactly the same argument can be applied to any exponential function $y = a^x$, whatever the base a, simply by replacing 2 by a in the argument above:

1 DIFFERENTIATING $y = a^x$

For all positive real numbers a,

$$\frac{d}{dx}a^x = a^x \times m, \text{ where } m \text{ is the gradient of } y = a^x \text{ at its } y\text{-intercept.}$$

That is, the derivative of an exponential function is a multiple of itself.

The gradient m of the curve at its y-intercept will of course change as the base a changes.

The definition of e

It now makes sense to choose the base that will make the gradient exactly 1 at the y-intercept, because the value of m will then be exactly 1. This base is given the symbol e, and has the value $e \doteqdot 2.7183$.

2 THE DEFINITION OF e

Define e to be the number such that the exponential function $y = e^x$ has gradient exactly 1 at its y-intercept. Then

$$e \doteqdot 2.7183.$$

The function $y = e^x$ is called *<u>the</u> exponential function* to distinguish it from all other exponential functions $y = a^x$.

Investigations to approximate e

Question 4 of Exercise 11A uses a simple argument with no technology to prove that e is between 2 and 4. The investigation in Question 5 of Exercise 11A uses technology to find ever closer approximations to e. Both questions are based on the definition of e given above.

The derivative of e^x

The fundamental result of this section then follows immediately from the previous two boxed results.

$$\begin{aligned}\frac{d}{dx}e^x &= e^x \times m, \quad \text{where } m \text{ is the gradient of } y = e^x \text{ at its } y\text{-intercept,}\\ &= e^x \times 1, \quad \text{by the definition of } e,\\ &= e^x.\end{aligned}$$

3 THE EXPONENTIAL FUNCTION $y = e^x$ IS ITS OWN DERIVATIVE

$$\frac{d}{dx}e^x = e^x.$$

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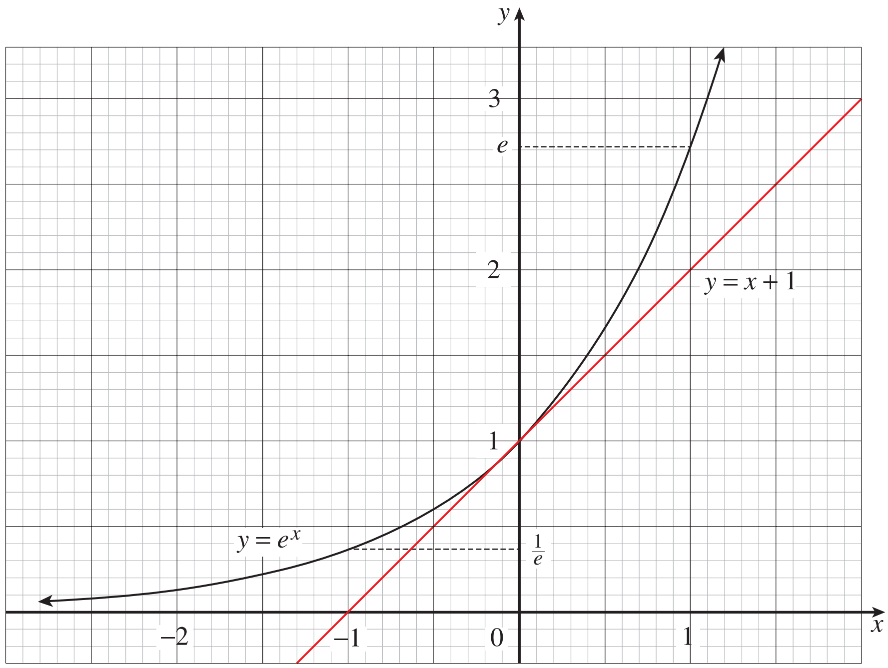
The graph of e^x

The graph of $y = e^x$ has been drawn below on graph paper. The tangent has been drawn at the y-intercept $(0, 1)$ — it has gradient exactly 1.

This graph of $y = e^x$ is one of the most important graphs in the whole course, and its shape and properties need to be memorised.



- The domain is all real x.
  The range is $y > 0$.
- There are no zeroes.
  The curve is always above the x-axis.
- The x-axis $y = 0$ is a horizontal asymptote to the curve on the left.
  That is, as $x \to -\infty$, $y \to 0$ and $\dfrac{dy}{dx} \to 0$.
- On the right-hand side, the curve rises steeply.
  That is, as $x \to \infty$, $y \to \infty$ and $\dfrac{dy}{dx} \to \infty$.
- The curve has gradient 1 at its y-intercept $(0, 1)$.
- The curve is always increasing — increasing at an increasing rate — and is always concave up.

Gradient equals height

The fact that the derivative of the exponential function e^x is the same function has a striking geometrical interpretation in terms of its graph. If $y = e^x$, then $\dfrac{dy}{dx} = e^x$, which means that for this function

$$\frac{dy}{dx} = y, \qquad \text{that is,} \qquad \text{gradient} = \text{height}.$$

Thus at each point on the curve $y = e^x$, the gradient $\dfrac{dy}{dx}$ of the curve is equal to the height y of the curve above the x-axis. We have already seen this happening at the y-intercept $(0, 1)$, where the gradient is 1 and the height is also 1.

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4 GRADIENT EQUALS HEIGHT

At each point on the graph of the exponential function $y = e^x$,

$$\frac{dy}{dx} = y.$$

That is, the gradient of the curve is always equal to its height above the x-axis.

This property of the exponential function is the reason why the function is so important in calculus.

An investigation of the graph of $y = e^x$

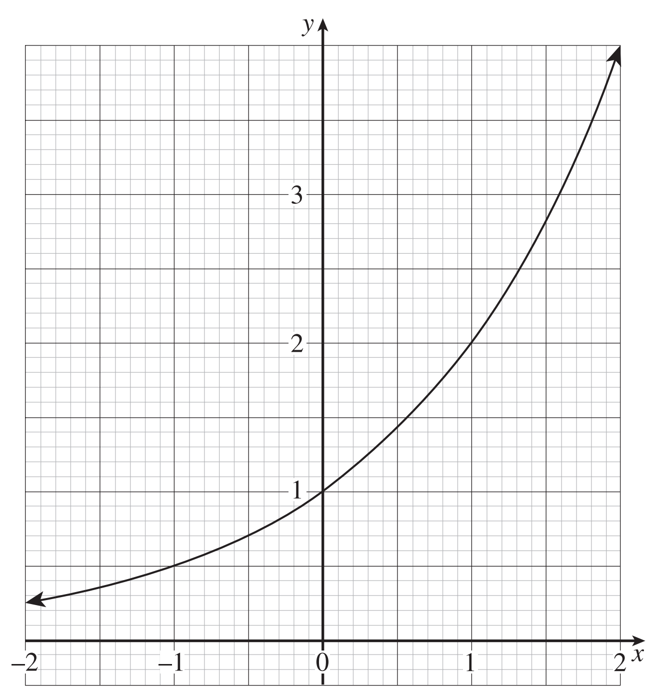
The investigation in Question 2 of Exercise 11A confirms these properties of the graph of $y = e^x$, particularly the fact that the gradient and the height are equal at every point on the curve.

Exercise 11A

INVESTIGATION

Technology: The first two investigations are written as graph-paper exercises, so that they are independent of any device. The instructions are identical if graphing software is used. When using the graph-paper method, the diagrams should first be photocopied.

1 [Graph paper, but easily transferred to graphing software]
The graph is the function $y = 2^x$.



a Photocopy the graph of $y = 2^x$, and on it draw the tangent at the point $(0, 1)$. Extend the tangent across the diagram.

b Use 'rise over run' to measure the gradient $\frac{dy}{dx}$ of this tangent. The run should be chosen as 10 or 20 little divisions, then count how many little divisions the rise is.

c Similarly, draw tangents at four more points on the curve where $x = -2, -1, 1$ and 2. Measure the gradient $\frac{dy}{dx}$ of each tangent.

d Copy and complete the table of values to the right (the values will only be rough).

e What do you notice about the ratios of gradient to height?

f Hence copy and complete:

$$\frac{dy}{dx} = \ldots y$$

x	-2	-1	0	1	2
height y					
gradient $\frac{dy}{dx}$					
$\frac{\text{gradient}}{\text{height}}$					

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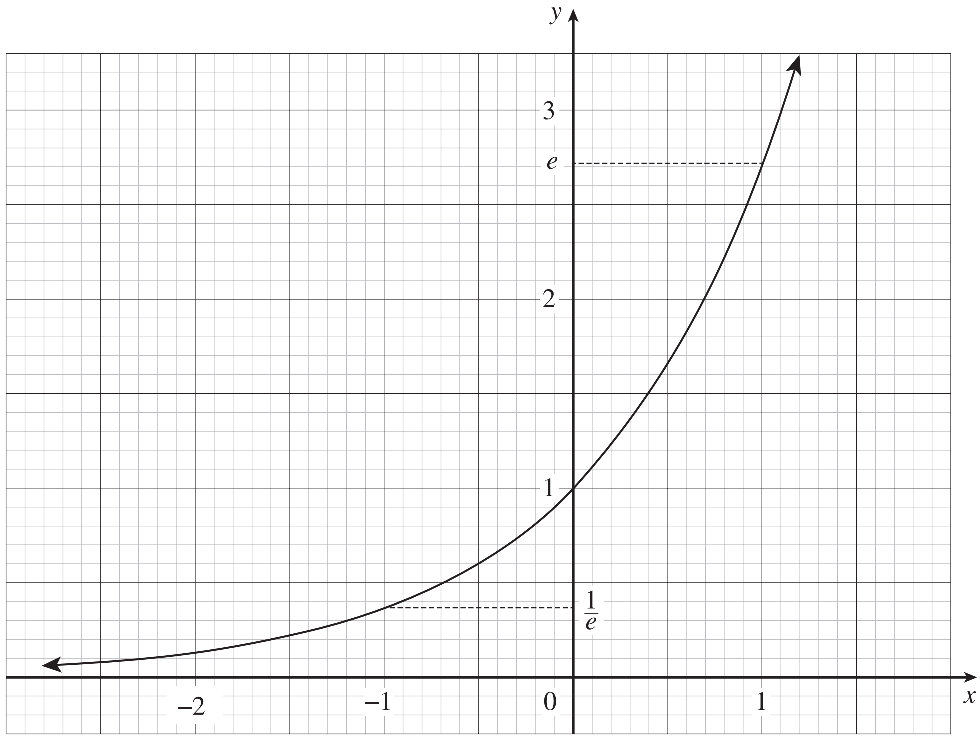
2 [Graph paper, but easily transferred to graphing software]



These questions refer to the graph of $y = e^x$ drawn above.

a Photocopy the graph of $y = e^x$ above and on it draw the tangent at the point $(0, 1)$ where the height is 1. Extend the tangent across the diagram.

b Measure the gradient of this tangent and confirm that it is equal to the height of the exponential graph at the point of contact.

c Copy and complete the table of values to the right by measuring the gradient $\frac{dy}{dx}$ of the tangent at the points where the height y is $\frac{1}{2}$, 1, 2 and 3.

height y	$\frac{1}{2}$	1	2	3
gradient $\frac{dy}{dx}$				
$\frac{\text{gradient}}{\text{height}}$				

d What do you notice about the ratios of gradient to height?

e What does this tell you about the derivative of $y = e^x$?

3 **a** Photocopy again the graph of $y = e^x$ in Question 2 (or use graphing software).

b Draw the tangents at the points where $x = -2, -1, 0$ and 1, extending each tangent down to the x-axis.

c Measure the gradient of each tangent and confirm that it is equal to the height of the graph at the point.

d What do you notice about the x-intercepts of all the tangents?

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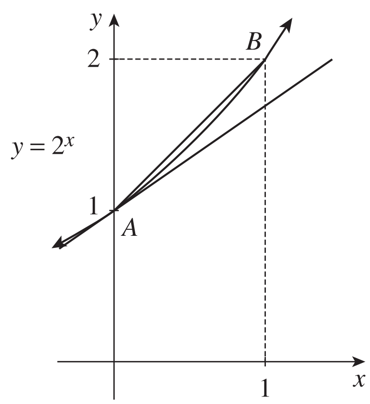
DEVELOPMENT

4 Approximating e to many significant figures is difficult. The argument below, however, at least shows very quickly that the number e lies between 2 and 4.

Here are tables of values and sketches of $y = 2^x$ and $y = 4^x$. On each graph, the tangent at the y-intercept $A(0, 1)$ has been drawn.

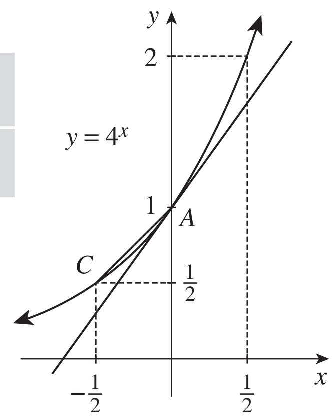
For $y = 2^x$

x	0	1
2^x	1	2



For $y = 4^x$

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
2^x	$\frac{1}{2}$	1	2



a **i** In the first diagram, find the gradient of the chord AB.
ii Hence explain why the tangent at A has gradient less than 1.
b **i** In the second diagram, find the gradient of the chord CA.
ii Hence explain why the tangent at A has gradient greater than 1.
c Use these two results to explain why $2 < e < 4$.

5 [Technology]
This question requires graphing software.
a Use graphing software to graph $y = 2^x$ and $y = x + 1$ on the same number plane, and hence observe that the gradient of $y = 2^x$ at $(0, 1)$ is less than 1.
b Similarly, graph $y = 3^x$ and $y = x + 1$ on the same number plane and hence observe that the gradient of $y = 3^x$ at $(0, 1)$ is greater than 1.
c Choose a sequence of bases between 2 and 3 that make the line $y = x + 1$ more and more like a tangent to the curve at $(0, 1)$ — that is, the gradient at $(0, 1)$ becomes closer and closer to 1. In this way a reasonable approximation for e can be obtained.

ENRICHMENT

6 [Technology]
At the start of this section, it was shown by first-principles differentiation that the gradient of $y = 2^x$ at its y-intercept is given by $\lim_{h \to 0} \dfrac{2^h - 1}{h}$. The investigation in Question 1 showed that the limit is about 0.7. Use a calculator or a spreadsheet to evaluate $\dfrac{2^h - 1}{h}$, correct to five decimal places, at least for the following values of h:

a 1 **b** 0.1 **c** 0.01 **d** 0.001 **e** 0.0001 **f** 0.00001

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11B Transformations of exponential functions

Transformations of exponential functions with bases other than e were discussed in Chapter 8. This short exercise first practises using the calculator to evaluate powers of e, and is then concerned with transformations of $y = e^x$.

Using the calculator to approximate powers of *e*

A calculator will provide approximate values of e^x correct to about ten significant figures. Most calculators label the function [e^x] and locate it above the button labelled [ln]. Pressing [shift] first may be required.

Example 1 — 11B

Use your calculator to find, correct to four significant figures:

a e^2 **b** e^{-4} **c** $e^{\frac{1}{3}}$

SOLUTION

Using the function labelled [e^x] on the calculator:

a $e^2 \doteqdot 7.389$ **b** $e^{-4} \doteqdot 0.01832$ **c** $e^{\frac{1}{3}} \doteqdot 1.396$

The next three examples must first be put into index form before using the calculator. In particular, e itself must be written as e^1, so that an approximation for the number e can be found using the function [e^x] with input $x = 1$.

Example 2 — 11B

Write each as a power of e, then approximate it correct to five decimal places.

a e **b** $\dfrac{1}{e}$ **c** $\sqrt{e}$

SOLUTION

a $e = e^1 \doteqdot 2.71828$

b $\dfrac{1}{e} = e^{-1} \doteqdot 0.36788$

c $\sqrt{e} = e^{\frac{1}{2}} \doteqdot 1.64872$

Transformations of $y = e^x$

The usual methods of shifting and reflecting graphs can be applied to $y = e^x$. When the graph is shifted vertically, the horizontal asymptote at $y = 0$ will be shifted also. A small table of approximate values can be a very useful check.

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Example 3 11B

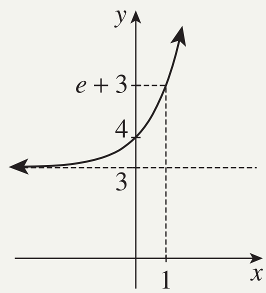
Use transformations of the graph of $y = e^x$, confirmed by a table of values, to generate a sketch of each function. Show and state the y-intercept and the horizontal asymptote, and state the range.

a $y = e^x + 3$ **b** $y = e^{-x}$ **c** $y = e^{x-2}$

SOLUTION

a Graph $y = e^x + 3$ by shifting $y = e^x$ up 3 units.

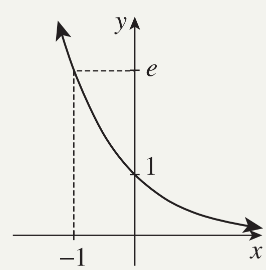
x	-1	0	1
y	$e^{-1} + 3$	4	$e + 3$
approximation	3.37	4	5.72



y-intercept: $(0, 4)$
asymptote: $y = 3$
range: $y > 3$

b Graph $y = e^{-x}$ by reflecting $y = e^x$ in the y-axis.

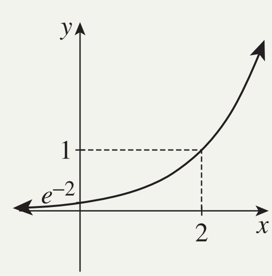
x	-1	0	1
y	e	1	e^{-1}
approximation	2.72	1	0.37



y-intercept: $(0, 1)$
asymptote: $y = 0$ (the x-axis)
range: $y > 0$

c Graph $y = e^{x-2}$ by shifting $y = e^x$ to the right by 2 units.

x	0	1	2	3
y	e^{-2}	e^{-1}	1	e
approximation	0.14	0.37	1	2.72



y-intercept: $(0, e^{-2})$
asymptote: $y = 0$ (the x-axis)
range: $y > 0$

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Exercise 11B

FOUNDATION

Technology: The transformations of the graph of the exponential function $y = e^x$ in Questions 4–7 and 9, and the procedure in Question 8, can be confirmed using graphing software, after which experimentation with further graphs can be done.

1 Use the function labelled e^x on your calculator to approximate the following correct to four decimal places:

a e^2 **b** e^{10} **c** e^0 **d** e^1 **e** e^{-1}
f e^{-2} **g** $e^{\frac{1}{2}}$ **h** $e^{-\frac{1}{2}}$ **i** $e^{-0.001}$ **j** e^{-6}

2 Write each expression as a power of e. Then approximate it correct to four significant figures.

a $\dfrac{1}{e}$ **b** $\dfrac{1}{e^4}$ **c** $\sqrt[3]{e}$ **d** $\dfrac{1}{\sqrt{e}}$ **e** $\dfrac{1}{e^{20}}$ **f** e^{30}

3 Write each expression as a power of e. Then approximate it correct to four significant figures.

a $5e^2$ **b** $\frac{1}{64}e^6$ **c** $7\sqrt{e}$ **d** $\frac{3}{5}\sqrt{e}$ **e** $\dfrac{4}{e}$ **f** $\dfrac{5}{7e^4}$

4 Sketch the graph of $y = e^x$, then use your knowledge of transformations to graph these functions, showing the horizontal asymptote. For each function, state the transformation, give the equation of the asymptote, and state the range.

a $y = e^x + 1$ **b** $y = e^x + 2$ **c** $y = e^x - 1$ **d** $y = e^x - 2$

5 **a** Copy and complete the following tables of values for the functions $y = e^x$ and $y = e^{-x}$, giving your answers correct to two decimal places.

x	-2	-1	0	1	2
e^x					

x	-2	-1	0	1	2
e^{-x}					

b Sketch both graphs on one number plane, and draw the tangents at each y-intercept.

c What transformation exchanges these graphs?

d We saw that the tangent to $y = e^x$ at its y-intercept has gradient 1. What is the gradient of $y = e^{-x}$ at its y-intercept? Explain why the two tangents are perpendicular.

e Hence use point–gradient form to find the equations of the tangents to $y = e^x$ and $y = e^{-x}$ at the point $A(0, 1)$ where they intersect.

6 **a** Sketch the graph of $y = e^{-x}$. State the horizontal asymptote and the range.

b Then use your knowledge of transformations to graph each function below. State the transformation of $y = e^{-x}$, give the equation of the asymptote, and state the range.

i $y = e^{-x} + 1$ **ii** $y = e^{-x} + 2$ **iii** $y = e^{-x} - 1$ **iv** $y = e^{-x} - 2$

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DEVELOPMENT

7 **a** What transformation maps $y = e^x$ to $y = e^{x-1}$?
b Sketch $y = e^{x-1}$.
c Similarly sketch these functions:
i $y = e^{x-3}$ **ii** $y = e^{x+1}$ **iii** $y = e^{x+2}$

8 Let A, B and C be the points on $y = e^x$ with x-coordinates 0, 1 and 2.
a Find the y-coordinates of A, B and C, and sketch the situation.
b Find the gradient of the chord AB, find its equation, then show that its x-intercept is $-\dfrac{1}{e-1}$.
c Similarly show that the x-intercept of the chord BC is $1 - \dfrac{1}{e-1}$.
d Let P and Q be the points on $y = e^x$ with x-coordinates a and $a + 1$. Show similarly that the x-intercept of the chord PQ is $a - \dfrac{1}{e-1}$.

9 Use the graph of $y = e^x$ and your knowledge of transformations to graph these functions. Show the horizontal asymptote and state the range in each case.
a $y = -e^x$ **b** $y = 1 - e^x$ **c** $y = 3 - e^x$
d $y = -e^{-x}$ **e** $y = 1 - e^{-x}$ **f** $y = e^{-|x|}$

ENRICHMENT

10 [A calculator or spreadsheet investigation]
The function e^x may be approximated by adding up a few terms of the infinite power series

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \times 2} + \frac{x^3}{1 \times 2 \times 3} + \frac{x^4}{1 \times 2 \times 3 \times 4} + \cdots.$$

Use this power series to approximate each power of e, correct to two decimal places. Then compare your answers with those given by your calculator.
a e **b** e^{-1} **c** e^2 **d** e^{-2} **e** e^0

When x is large, many more terms are needed to approximate e^x correct to say two decimal places. Why is this so? Use your spreadsheet to investigate this.

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11C Differentiation of exponential functions

Now that the new standard form $\frac{d}{dx}e^x = e^x$ has been established, the familiar chain, product and quotient rules can be applied to functions involving e^x. Those procedures, however, will be developed in Year 12. At this stage, all that is necessary is a simple chain-rule extension to the standard form

$$\frac{d}{dx}e^{ax+b} = ae^{ax+b}.$$

Standard forms for differentiating e^{ax+b}

The chain rule can be applied to differentiating functions such as $y = e^{3x+4}$, where the index $3x + 4$ is a linear function.

Example 4 11C

Use the chain rule to differentiate

a $y = e^{3x+4}$

b $y = e^{ax+b}$

SOLUTION

a Let $y = e^{3x+4}$.
Then using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 3e^{3x+4}.$$

Let $u = 3x + 4$.
Then $y = e^u$.
Hence $\frac{du}{dx} = 3$
and $\frac{dy}{du} = e^u$.

b Let $y = e^{ax+b}$.
Then using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= ae^{ax+b}.$$

Let $u = ax + b$.
Then $y = e^u$.
Hence $\frac{du}{dx} = a$
and $\frac{dy}{du} = e^u$.

This situation occurs so often that the result should be learnt as a standard form.

5 STANDARD FORMS FOR DIFFERENTIATING EXPONENTIAL FUNCTIONS

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^{ax+b} = ae^{ax+b}$$

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Example 5 11C

Use the standard forms above to find the derivatives of:

a $y = e^{4x-7}$ **b** $y = 2e^{-9x}$ **c** $y = e^{\frac{1}{2}(6-x)}$

SOLUTION

Each function needs to be written in the form $y = e^{ax+b}$.

a $y = e^{4x-7}$

$\frac{dy}{dx} = 4e^{4x-7}$

b $y = 2e^{-9x}$

$\frac{dy}{dx} = -18e^{-9x}$

c $y = e^{\frac{1}{2}(6-x)}$

$y = e^{3-\frac{1}{2}x}$

$\frac{dy}{dx} = -\frac{1}{2}e^{3-\frac{1}{2}x}$

Exercise 11C FOUNDATION

Technology: Programs that perform algebraic differentiation can be used to confirm the answers to many of these exercises.

1 Use the standard form $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = e^{2x}$ **b** $y = e^{7x}$ **c** $y = e^{-x}$ **d** $y = -e^{5x}$

e $y = e^{\frac{1}{2}x}$ **f** $y = 6e^{\frac{1}{3}x}$ **g** $y = e^{-\frac{1}{3}x}$ **h** $y = 5e^{\frac{1}{5}x}$

2 Using the standard form $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $f(x) = e^{x+2}$ **b** $f(x) = e^{x-3}$ **c** $f(x) = e^{5x+1}$ **d** $f(x) = e^{2x-1}$

e $f(x) = e^{-4x+1}$ **f** $f(x) = e^{-3x+4}$ **g** $f(x) = e^{-3x-6}$ **h** $f(x) = 2e^{\frac{1}{2}x+4}$

3 Differentiate:

a $f(x) = e^{x} + e^{-x}$ **b** $f(x) = e^{2x} - e^{-3x}$ **c** $f(x) = \frac{e^{2x}}{2} + \frac{e^{3x}}{3}$

d $f(x) = \frac{e^{4x}}{4} + \frac{e^{5x}}{5}$ **e** $f(x) = \frac{e^{x} - e^{-x}}{2}$ **f** $f(x) = \frac{e^{x} + e^{-x}}{3}$

4 **a** Differentiate $y = e^{2x}$.

b Hence find $\frac{dy}{dx}$ in exact form when $x = 0$ and when $x = 4$.

5 **a** Differentiate $f(x) = e^{-x+3}$.

b Hence find $f'(x)$ in exact form when $x = 0$ and when $x = 4$.

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DEVELOPMENT

6 For each function, first find $\frac{dy}{dx}$. Then evaluate $\frac{dy}{dx}$ at $x = 2$, giving your answer first in exact form and then correct to two decimal places.

a $y = e^{3x}$ **b** $y = e^{-2x}$ **c** $y = e^{\frac{3}{2}x}$

7 **a** For the function $f(x) = e^{-x}$, write down $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$. What is the pattern in these derivatives?

b For the function $f(x) = e^{2x}$, write down $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$. What is the pattern in these derivatives?

c For the function $f(x) = e^{x}$, write down $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$. What is the pattern in these derivatives?

d What happens when $y = e^x + x^2 + x + 1$ is differentiated successively?

8 Differentiate:

a $f(x) = e^{5x} + e^{7x}$ **b** $f(x) = e^{4x+2} + e^{5+8x}$ **c** $f(x) = 4e^{-x} + 4e^{-3x}$

d $f(x) = 6e^{-2x-3} - 7e^{5-6x}$ **e** $f(x) = 5x^2 - 4x - 3e^{-x}$ **f** $f(x) = e^{\frac{1}{2}x} + x^{\frac{1}{2}}$

9 Write each expression as a simple power of e, and then differentiate it. Write your answer without fractional indices.

a $y = \sqrt{e^x}$ **b** $y = \sqrt[3]{e^x}$ **c** $y = \frac{1}{\sqrt{e^x}}$ **d** $y = \frac{1}{\sqrt[3]{e^x}}$

10 Use the standard form $\frac{d}{dx}e^{ax+b} = ae^{ax+b}$ to differentiate:

a $y = e^{ax}$ **b** $y = e^{-kx}$ **c** $y = Ae^{kx}$ **d** $y = Be^{-\ell x}$

e $y = e^{px+q}$ **f** $y = Ce^{px+q}$ **g** $y = \frac{e^{px} + e^{-qx}}{r}$ **h** $y = \frac{e^{ax}}{a} + \frac{e^{-px}}{p}$

11 **a** Show by substitution that the function $y = e^{5x}$ satisfies the equation $\frac{dy}{dx} = 5y$.

b Show by substitution that the function $y = 3e^{2x}$ satisfies the equation $\frac{dy}{dx} = 2y$.

c Show by substitution that the function $y = 5e^{-4x}$ satisfies the equation $\frac{dy}{dx} = -4y$.

ENRICHMENT

12 Define two new functions, $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$.

a Show that $\frac{d}{dx}\cosh x = \sinh x$ and $\frac{d}{dx}\sinh x = \cosh x$.

b Find the second derivative of each function, and show that they both satisfy $y'' = y$.

c Show that $\cosh^2 x - \sinh^2 x = 1$.

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11D Differentiation and the graph

Differentiation can now be applied as usual to the graphs of functions whose equations involve e^x. This section uses the derivative to deal with the tangents and normals to such graphs.

The graphs of e^x and e^{-x}

The graphs of $y = e^x$ and $y = e^{-x}$ are fundamental to this whole course. Before dealing with further graphs, it is worth reviewing the most basic results obtained so far in this chapter.

6 REVIEW OF THE MOST BASIC FACTS ABOUT $y = e^x$

- The real number $e \doteqdot 2.7183$ is defined to be the base so that the exponential function $y = e^x$ has gradient 1 at its y-intercept.
- The function $y = e^x$ is its own derivative. That is,
$$\frac{d}{dx}e^x = e^x.$$
- We have obtained the further standard form
$$\frac{d}{dx}e^{ax+b} = ae^{ax+b}.$$

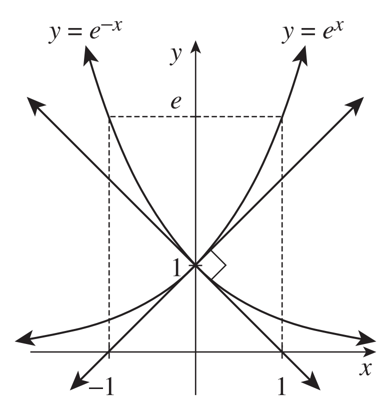
Because x has been replaced by $-x$ in the second equation, the two graphs $y = e^x$ and $y = e^{-x}$ are reflections of each other in the y-axis.

For $y = e^x$:

x	-2	-1	0	1	2
y	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2

For $y = e^{-x}$:

x	-2	-1	0	1	2
y	e^2	e	1	$\frac{1}{e}$	$\frac{1}{e^2}$



The two curves cross at $(0, 1)$.

By the definition of e, the gradient of $y = e^x$ at its y-intercept $(0, 1)$ is 1.

Hence by reflection, the gradient of $y = e^{-x}$ at its y-intercept $(0, 1)$ is -1.

Thus the curves are perpendicular at their point of intersection.

Note: The function $y = e^{-x}$ is as important as $y = e^x$ in applications, or even more important. It describes a great many physical situations where a quantity 'dies away exponentially', such as the dying away of the sound of a plucked string.

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Example 6 11D

a Differentiate $y = e^x - x$.
b For what value of x is the curve stationary?

SOLUTION

a Differentiating, $\dfrac{dy}{dx} = e^x - 1$ (the derivative of e^x is e^x)

b Put $\dfrac{dy}{dx} = 0$ to find where the curve is stationary,

$$\begin{aligned} e^x - 1 &= 0 \\ e^x &= 1 \\ x &= 0. \end{aligned}$$

The geometry of tangents and normals

Example 7 shows how to work with tangents and normals of exponential functions.
Remember that the word 'normal' means 'perpendicular'. The *normal* to a curve at a point P on it is the line through P perpendicular to the tangent at P.

Example 7 11D

Let A be the point on the curve $y = 2e^x$ where $x = 1$.
a Find the equation of the tangent to the curve at the point A.
b Show that the tangent at A passes through the origin.
c Find the equation of the normal at the point A.
d Find the point B where this normal crosses the x-axis.
e Find the area of $\triangle AOB$.

SOLUTION

a The given function is $y = 2e^x$
and differentiating, $y' = 2e^x$.
When $x = 1$, $y = 2e^1$
$= 2e$ (this is the y-coordinate of A)
and also when $x = 1$, $y' = 2e^1$
$= 2e$ (this is the gradient of the tangent at A)

so A has coordinates $A(1, 2e)$ and the tangent at A has gradient $2e$.

Hence, using point–gradient form, the tangent at A is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2e &= 2e(x - 1) \\ y &= 2ex. \end{aligned}$$

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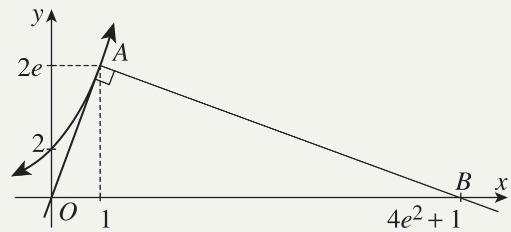
b Because its y-intercept is zero, the tangent passes through the origin.

c From part **a**, the tangent at A has gradient $2e$,

so the normal has gradient $-\dfrac{1}{2e}$ (it is perpendicular to the tangent).

Thus the normal at A is $y - 2e = -\dfrac{1}{2e}(x - 1)$

$$2ey - 4e^2 = -x + 1$$
$$x + 2ey = 4e^2 + 1.$$



d To find the x-intercept,

put $y = 0$,

thus $x = 4e^2 + 1$,

so B has coordinates $(4e^2 + 1, 0)$.

e Hence area $\triangle AOB = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \tfrac{1}{2} \times (4e^2 + 1) \times 2e$$
$$= e(4e^2 + 1) \text{ square units.}$$

Exercise 11D

FOUNDATION

1 **a** Use calculus to find the gradient of the tangent to $y = e^x$ at $Q(0, 1)$.

b Hence find the equation of the tangent at Q, and prove that it passes through $A(-1, 0)$.

2 **a** Use calculus to find the gradient of the tangent to $y = e^x$ at $P(1, e)$.

b Hence find the equation of the tangent at P, and prove that it passes through O.

3 **a** Use calculus to find the gradient of the tangent to $y = e^x$ at $R\left(-1, \dfrac{1}{e}\right)$.

b Hence find the equation of the tangent at R, and prove that it passes through $B(-2, 0)$.

4 **a** Find the y-coordinate of the point A on the curve $y = e^{2x-1}$ where $x = \frac{1}{2}$.

b Find the derivative of $y = e^{2x-1}$, and show that the gradient of the tangent at A is 2.

c Hence find the equation of the tangent at A, and prove that it passes through O.

5 **a** Explain why $y = e^x$ is always increasing.

b Explain why $y = e^{-x}$ is always decreasing.

6 **a** What is the y-coordinate of the point P on the curve $y = e^x - 1$ where $x = 1$?

b Find $\dfrac{dy}{dx}$ for this curve, and the value of $\dfrac{dy}{dx}$ when $x = 1$.

c Hence find the equation of the tangent at P.

d Find the values of x for which the curve is:

i stationary, **ii** increasing, **iii** decreasing.

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7 **a** Differentiate $y = x - e^x$.
b Find the gradient of the tangent to $y = x - e^x$ at the point where $x = 1$.
c Write down the equation of the tangent, and say why it passes through the origin.
d Find the values of x for which the curve is:
i stationary, **ii** increasing, **iii** decreasing.

DEVELOPMENT

8 **a** Write down the coordinates of the point R on the curve $y = e^{3x+1}$ where $x = -\frac{1}{3}$.
b Find $\dfrac{dy}{dx}$ and hence show that the gradient of the tangent at R is 3.
c What is the gradient of the normal at R?
d Hence find the equation of the normal at R in general form.

9 **a** Find the gradient of the tangent to $y = e^{-x}$ at the point $P(-1, e)$.
b Thus write down the gradient of the normal at this point.
c Hence determine the equation of this normal.
d Find the x- and y-intercepts of the normal.
e Find the area of the triangle whose vertices lie at the intercepts and the origin.

10 **a** Use the derivative to find the gradient of the tangent to $y = e^x$ at $B(0, 1)$.
b Hence find the equation of this tangent and show that it meets the x-axis at $F(-1, 0)$.
c Use the derivative to find the gradient of the tangent to $y = e^{-x}$ at $B(0, 1)$.
d Hence find the equation of this tangent and show that it meets the x-axis at $G(1, 0)$.
e Sketch $y = e^x$ and $y = e^{-x}$ on the same set of axes, showing the two tangents.
f What sort of triangle is $\triangle BFG$, and what is its area?

11 Find the gradient, and the angle of inclination correct to the nearest minute, of the tangent to $y = e^x$ at the points where:
a $x = 0$ **b** $x = 1$ **c** $x = -2$ **d** $x = 5$

12 A curve is defined by $y = e^{2x-4}$. The points A and B on it have x-coordinates 1 and 2. Find the coordinates of A and B. Then find the gradients of:
a the tangent at A, **b** the tangent at B, **c** the chord AB.

13 **a** Find the equation of the tangent to $y = e^x$ at $x = t$.
b Hence show that the x-intercept of this tangent is $t - 1$.
c Compare this result with those of Questions 2 and 3 above.

14 **a** Show that the equation of the tangent to $y = 1 - e^{-x}$ at the origin is $y = x$.
b Deduce the equation of the normal at the origin without further use of calculus.
c What is the equation of the asymptote of this curve?
d Sketch the curve, showing the points T and N where the tangent and normal respectively cut the asymptote.
e Find the area of $\triangle OTN$.

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15 The point $P(1, e)$ lies on the graph of $y = e^x$.

a Find the equation of the tangent at P, and show that is passes through the origin O.

b Find the normal at P, and find its x-intercept A and its y-intercept B.

c Show that area $\triangle OPA$: area $\triangle OPB = e^2 : 1$.

d What does this tell you about the intervals of AP and PB?

ENRICHMENT

16 Two new functions $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$ appeared in Exercise 11C. Find the gradients at their y-intercepts, then sketch on one set of axes the curves

$$y = \cosh x, \quad y = -\cosh x, \quad y = \sinh x, \quad y = -\sinh x.$$

Find all points of intersection of these four curves.

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11E The logarithmic function base *e*

We have seen that $e \doteqdot 2.7183$ is the most natural base to use for exponential functions in calculus. Work with the exponential function $y = e^x$ necessarily involves work with the logarithmic function $\log_e x$, because it is the inverse function of $y = e^x$. Thus e is also the most natural base to use for logarithms in calculus.

This section introduces $y = \log_e x$ and its graph, describing its relationship with the graph of $y = e^x$, and the transformations of its graph.

The logarithmic function

Because e is the natural base for logarithms, logarithms base e are called *natural logarithms*, and the function $y = \log_e x$ is called *the logarithmic function* to distinguish it from other logarithmic functions such as $\log_2 x$ and $\log_{10} x$ that have other bases.

7 THE LOGARITHMIC FUNCTION

- Logarithms base e are called *natural logarithms*.
- The logarithmic function with base e,

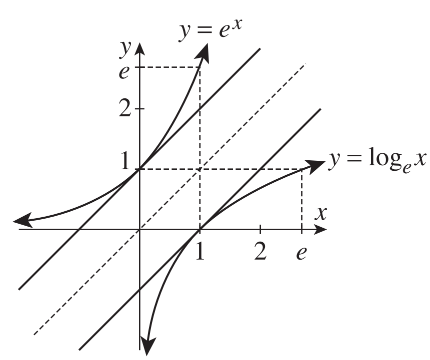
$$y = \log_e x$$

is called *the logarithmic function* to distinguish it from all other logarithmic functions $y = \log_a x$.

Below are tables of values and the graphs of the inverse functions $y = e^x$ and $y = \log_e x$. You can see that they are inverse functions because the rows are swapped. These are two of the most important graphs in the course.

x	-2	-1	0	1	2
e^x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2

x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2
$\log_e x$	-2	-1	0	1	2



We know already that the tangent to $y = e^x$ at its y-intercept $(0, 1)$ has gradient 1. When this graph is reflected in the line $y = x$, this tangent is reflected to a tangent to $y = \log_e x$ at its x-intercept $(1, 0)$. Both these tangents therefore have gradient 1.

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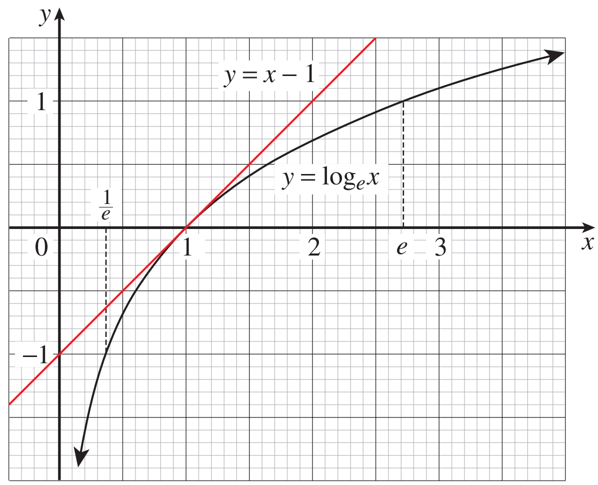
Properties of the graph of $y = \log_e x$

The graph of $y = \log_e x$ alone is drawn below on graph paper. The tangent has been drawn at the x-intercept $(1, 0)$ to show that the gradient of the curve there is exactly 1.



The graph of $y = \log_e x$ and its properties must be thoroughly known. Its properties correspond to the properties listed earlier of its inverse function $y = e^x$.

- The domain is $x > 0$.
- The range is all real y.
- The y-axis $x = 0$ is a vertical asymptote to the curve.
- As $x \to 0^+$, $y \to -\infty$.
- As $x \to \infty$, $y \to \infty$.
- The curve has gradient 1 at its x-intercept $(1, 0)$.
- The curve is always concave down.

The notations for the logarithmic function — $\log_e x$, $\log x$ and $\ln x$

In calculus, and in mathematics generally, $\log_e x$ is the only logarithmic function that matters, and is more often written simply as $\log x$. The problem is that many calculators, and even some mathematical software, contradict this mathematics convention, and use an engineering convention, where '$\log x$' means $\log_{10} x$.

The function $\log_e x$ is also written as $\ln x$, where the 'n' stands for 'natural logarithms'. The 'n' also stands for 'Napierian logarithms', in honour of the Scottish mathematician John Napier (1550–1617), who first developed tables of logarithms for calculations (first published in 1614).

The most sensible way to deal with this confusion is:

- Never write '$\log x$' without a base — always write either $\log_{10} x$ or $\log_e x$.
- Use either $\log_e x$ or $\ln x$ for logarithms base e.
  But if you do see $\log x$, interpret it as $\log_e x$ — except on calculators.

Note: Be careful of the different convention on calculators, where [log] means $\log_{10} x$, and the function labelled [ln] is used to find logarithms base e. The function [e^x] is usually located on the same button as [ln] because the two functions $y = e^x$ and $y = \log_e x$ are inverses of each other.

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8 THE NOTATION FOR THE LOGARITHMIC FUNCTION

- In this text, logarithms base e are written as $\log_e x$, and less often as $\ln x$.
- On calculators, however:
  — [ln] is used to approximate $\log_e x$. It is the inverse function of [e^x].
  — [log] is used to approximate $\log_{10} x$. It is the inverse function of [10^x].

Example 8 — 11E

a Use your calculator to find, correct to four significant figures:

i $\log_e 10$ **ii** $\log_e \frac{1}{10}$ **iii** $\log_e 100$

b How are the answers to parts **ii** and **iii** related to the answer to part **i**?

SOLUTION

a Using the function labelled [ln] on the calculator,

i $\log_e 10 \doteqdot 2.303$ **ii** $\log_e \frac{1}{10} \doteqdot -2.303$ **iii** $\log_e 100 \doteqdot 4.605$

b Using the log laws,

$$\log_e \frac{1}{10} = \log_e 10^{-1} = -\log_e 10 \quad \text{and} \quad \log_e 10^2 = 2\log_e 10,$$

and these relationships are clear from the approximations above.

Combining the logarithmic and exponential functions

As with any base, when the logarithmic and exponential functions base e are applied successively to a number, the result is the original number.

9 THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS ARE INVERSE FUNCTIONS

$$\log_e e^x = x \qquad \text{and} \qquad e^{\log_e x} = x.$$

These identities follow immediately from the definition of the logarithmic function as the inverse function of the exponential function.

Example 9 — 11E

Use the functions labelled [ln] and [e^x] on your calculator to demonstrate that:

a $\log_e e^{10} = 10$ **b** $e^{\log_e 10} = 10$

SOLUTION

a $\log_e e^{10} = \log_e 22026.46\ldots$
$\doteqdot 10$

b $e^{\log_e 10} = e^{2.302585\ldots}$
$\doteqdot 10$

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Transformations of the logarithmic graph

The usual methods of transforming graphs can be applied to $y = \log_e x$. When the graph is shifted sideways, the vertical asymptote at $x = 0$ will also be shifted.

A small table of approximate values can be a very useful check, particularly when a sequence of transformations is involved. Remember that $y = \log_e x$ has domain $x > 0$ and that the vertical asymptote is at $x = 0$.

Example 10 — 11E

Use transformations of the graph of $y = \log_e x$, confirmed by a table of values, to generate a sketch of each function. State the transformation used, and write down the domain, the x-intercept, and the vertical asymptote.

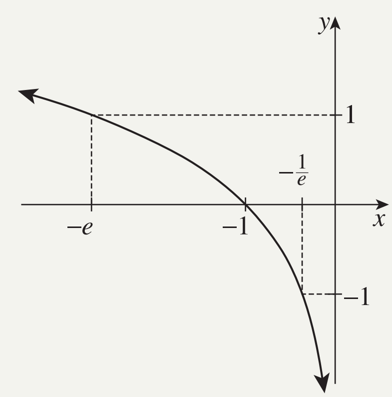
a $y = \log_e(-x)$ **b** $y = \log_e x - 2$ **c** $y = \log_e(x + 3)$

SOLUTION

a Graph $y = \log_e(-x)$ by reflecting $y = \log_e x$ in the y-axis.

x	$-e$	-1	$-\frac{1}{e}$
y	1	0	-1

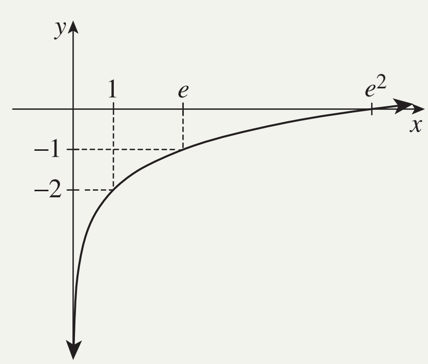
domain: $x < 0$
x-intercept: $(-1, 0)$
asymptote: $x = 0$ (the y-axis)



b Graph $y = \log_e x - 2$, which is $y + 2 = \log_e x$, by shifting $y = \log_e x$ down 2 units.

x	$\frac{1}{e}$	1	e	e^2
y	-3	-2	-1	0

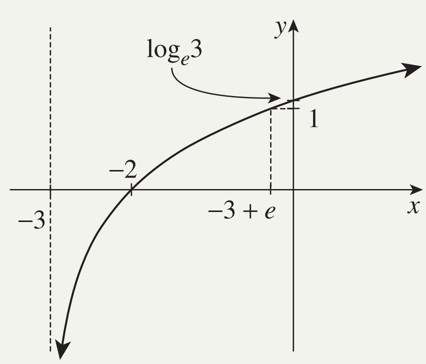
domain: $x > 0$
x-intercept: $(e^2, 0)$
asymptote: $x = 0$ (the y-axis)



c Graph $y = \log_e(x + 3)$ by shifting $y = \log_e x$ left 3 units.

x	$\frac{1}{e} - 3$	-2	$e - 3$
y	-1	0	1

domain: $x > -3$
x-intercept: $(-2, 0)$
asymptote: $x = -3$



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Exercise 11E

FOUNDATION

Note: Remember that $\ln x$ means $\log_e x$, the natural logarithm. On the calculator, the key [ln] means $\log_e x$, and the key [log] unfortunately means $\log_{10} x$.

1 Use your calculator to approximate the following, correct to four decimal places where necessary. Read the note above and remember to use the [ln] key on the calculator.

a $\log_e 1$ **b** $\log_e 2$ **c** $\ln 3$ **d** $\ln 8$

e $\log_e \frac{1}{2}$ **f** $\log_e \frac{1}{3}$ **g** $\ln \frac{1}{8}$ **h** $\ln \frac{1}{10}$

2 Use the functions labelled [ln] and [e^x] on your calculator to demonstrate that:

a $\log_e e^2 = 2$ **b** $\log_e e^3 = 3$ **c** $\log_e e^1 = 1$

d $\log_e e^{-2} = -2$ **e** $\log_e e^{-3} = -3$ **f** $\log_e e^{-1} = -1$

3 Use the functions labelled [ln] and [e^x] on your calculator to demonstrate that:

a $e^{\log_e 2} = 2$ **b** $e^{\log_e 3} = 3$ **c** $e^{\log_e 1} = 1$

d $e^{\log_e 10} = 10$ **e** $e^{\log_e \frac{1}{2}} = \frac{1}{2}$ **f** $e^{\log_e \frac{1}{10}} = \frac{1}{10}$

4 Solve each equation of x by first rewriting it as an index statement using the definition

$$x = \log_e a \quad \text{means} \quad e^x = a.$$

a $x = \log_e 1$ **b** $x = \log_e e$ **c** $x = \log_e e^2$

d $x = \log_e \frac{1}{e}$ **e** $x = \log_e \frac{1}{e^2}$ **f** $x = \log_e \sqrt{e}$

5 Simplify each expression using the logarithm law $\log_e a^n = n \log_e a$, then evaluate it using the fact that $\log_e e = 1$.

a $\log_e e^2$ **b** $\log_e e^5$ **c** $\log_e e^{200}$ **d** $\log_e e^{-6}$ **e** $\log_e \frac{1}{e^6}$

f $\log_e e^{-1}$ **g** $\log_e \frac{1}{e}$ **h** $\log_e e^{\frac{1}{2}}$ **i** $\log_e \sqrt{e}$ **j** $\log_e \frac{1}{\sqrt{e}}$

DEVELOPMENT

6 Sketch the graph of $y = \log_e x$, then use your knowledge of transformations to graph these functions. Note that in each case the y-axis is a vertical asymptote and the domain is $x > 0$.

a $y = \log_e x + 1$ **b** $y = \log_e x + 2$ **c** $y = \ln x - 1$ **d** $y = \ln x - 2$

7 a Copy and complete the following tables of values for the functions $y = \log_e x$ and $y = -\log_e x$, giving your answers correct to two decimal places.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_e x$					

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$-\log_e x$					

b Sketch both graphs on the same number plane, and draw the tangent to each at the x-intercept.

c It was shown at the beginning of this section that the tangent to $y = \log_e x$ at its y-intercept has gradient 1. Use this fact to find the gradient of $y = -\log_e x$ at its y-intercept, and hence explain why the two curves meet at right angles.

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8 Use the log laws to simplify:

a $e\log_e e$ **b** $\frac{1}{e}\log_e \frac{1}{e}$ **c** $3\log_e e^2$

d $\log_e \sqrt{e}$ **e** $e\log_e e^3 - e\log_e e$ **f** $\log_e e + \log_e \frac{1}{e}$

g $\log_e e^e$ **h** $\log_e(\log_e e^e)$ **i** $\log_e\big(\log_e(\log_e e^e)\big)$

9 Express as a single logarithm:

a $\log_e 3 + \log_e 2$ **b** $\log_e 100 - \log_e 25$

c $\log_e 2 - \log_e 3 + \log_e 6$ **d** $\log_e 54 - \log_e 10 + \log_e 5$

10 Use the graph of $y = \log_e x$ and your knowledge of transformations to graph these functions. Show the vertical asymptote and state the domain in each case.

a $y = \log_e(x - 1)$ **b** $y = \log_e(x - 3)$ **c** $y = \log_e(x + 1)$

d $y = \log_e(x + 2)$ **e** $y = -\log_e x$ **f** $y = \log_e(-x)$

11 Use the log laws and the change-of-base formula to prove:

a $\log_e \frac{a}{b} = -\log_e \frac{b}{a}$. **b** $\log_{\frac{1}{e}} x = -\log_e x$. **c** $\log_{\frac{1}{e}} x^{-1} = \log_e x$.

12 Sketch the graph of $y = -\log_e x$ and use your knowledge of transformations to graph these functions. Note that in each case the y-axis is a vertical asymptote and the domain is $x > 0$.

a $y = -\log_e x + 1$ **b** $y = -\log_e x + 2$ **c** $y = -\log_e x - 1$ **d** $y = -\log_e x - 2$

ENRICHMENT

13 [Spreadsheets]

The function $\log_e(1 + x)$ may be approximated using the *power series*

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \text{ for } -1 < x \le 1.$$

Use this power series to approximate each logarithm, correct to two decimal places. Then compare your answers with those given by your calculator.

a $\log_e 1\frac{1}{2}$ **b** $\log_e \frac{5}{4}$ **c** $\log_e \frac{1}{2}$ **d** $\log_e \frac{1}{3}$ **e** $\log_e 1$

Investigate on a spreadsheet how many terms must be taken to approximate $\log_e 2$ correct to say three, four or five decimal places.

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11F Exponential rates using the base *e*

Now that we have e^x, we can differentiate an exponential function and find the rate at which a quantity is changing at a particular time.

This short exercise is only concerned with rates involving e^{kx}, where k is a given constant. The topic will be developed further in Chapter 16.

Example 11 11F

The number B of bacteria in a laboratory culture is growing according to the formula $B = 4000e^{0.1t}$, where t is the time in hours after the experiment was started. Answer these questions correct to three significant figures.

a How many bacteria are there after 5 hours?

b Find the rate $\frac{dB}{dt}$ at which the number of bacteria is increasing, and show that $\frac{dB}{dt} = 0.1 \times B$.

c At what rate are the bacteria increasing after 5 hours?

d What is the average rate of increase over the first five hours?

e When are there 10 000 bacteria?

f When are the bacteria increasing by 10 000 per hour?

SOLUTION

a When $t = 5$, $B = 4000 \times e^{0.5}$
$\doteqdot 6590$ bacteria.

b Differentiating, $\frac{dB}{dt} = 400e^{0.1t}$.
$= 0.1 \times B$

c When $t = 5$, $\frac{dB}{dt} = 400 \times e^{0.5}$
$\doteqdot 659$ bacteria per hour.

d When $t = 0$, $B = 4000 \times e^0 = 4000$, and when $t = 5$, $B \doteqdot 6590$.

Hence average rate $= \frac{B_2 - B_1}{t_2 - t_1}$

$\doteqdot \frac{2590}{5}$

$\doteqdot 518$ bacteria per hour.

e Put $B = 10000$.

Then $4000e^{0.1t} = 10000$

$e^{\frac{t}{10}} = \frac{5}{2}$

$\frac{t}{10} = \log_e 2.5$

$t = 10\log_e 2.5$

$\doteqdot 9.16$ hours.

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f Put $\dfrac{dB}{dt} = 10000.$

$$\begin{aligned} \text{Then } 400\,e^{0.1t} &= 10000 \\ e^{\frac{t}{10}} &= 25 \\ \frac{t}{10} &= \log_e 25 \\ t &= 10\log_e 25 \\ &\doteqdot 32.2 \text{ hours.} \end{aligned}$$

Exercise 11F

FOUNDATION

Note: Give approximations correct to four significant figures unless otherwise stated.

1 **a** A quantity Q is given by $Q = 300e^{3t}$. Find the rate of change $\dfrac{dQ}{dt}$ of Q at time t, and show that $\dfrac{dQ}{dt} = 3Q$.

b Evaluate Q and $\dfrac{dQ}{dt}$ when $t = 2$.

c Find the average rate of change from $t = 0$ to $t = 2$.

2 **a** A quantity Q is given by $Q = 10000\,e^{-2t}$. Find the rate of change $\dfrac{dQ}{dt}$ of Q at time t, and show that $\dfrac{dQ}{dt} = -2Q$.

b Evaluate Q and $\dfrac{dQ}{dt}$ when $t = 4$.

c Find the average rate of change from $t = 0$ to $t = 4$.

3 **a** A quantity Q is given by $Q = 5e^{2t}$. Find the rate of change $\dfrac{dQ}{dt}$ of Q at time t.

b Find when the quantity Q is 1000.

c Find when the rate $\dfrac{dQ}{dt}$ is 1000.

DEVELOPMENT

4 A population P is varying with time t in hours according to the formula $P = 2000\,e^{0.3t}$.

a Find the population when $t = 5$.

b Find, as a function of t, the rate $\dfrac{dP}{dt}$ at which the population is changing, and show that $\dfrac{dP}{dt} = 0.3 \times P$.

c Find the rate at which the population is changing when $t = 5$.

d Find the average rate of change over the first 5 hours.

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5 The concentration $C = 2000\,e^{-2t}$ of a chemical is varying with time t in years.
  a Find the concentration when $t = 2$.
  b Find, as a function of t, the rate $\dfrac{dC}{dt}$ at which the concentration is changing.
  c Find the rate at which the concentration is changing when $t = 2$.
  d Find the average rate of change over the first 2 years.

6 The price \$$P$ of an item is rising with inflation according to the formula $P = 150\,e^{0.04t}$, where t is the time in years.
  a When will the price be \$300?
  b Find the rate $\dfrac{dP}{dt}$ at which the price is rising.
  c When will the price be rising at \$300 per year?

7 The population P of cats on Snake Island is $P = 1000\,e^{0.4t}$, where t is the time in years after time zero when the population was first estimated.
  a Find the rate at which the population is increasing.
  b Find the population, and the rate of increase, when $t = 5$, both correct to two significant figures.
  c When will the cat population reach 20 000 cats, correct to one decimal place?
  d When will the rate of increase reach 20 000 cats per year, correct to one decimal place?

8 A radioactive substance decays in such a way that the mass M of a sample t years after it has been measured and stored is $M = M_0 e^{-0.1t}$, where M_0 is its original mass.
  a How long until half the sample is gone?
  b Find the rate $\dfrac{dM}{dt}$ at which the sample is decaying, and show that $\dfrac{dM}{dt} = -0.1 \times M$.
  c What percentage of the remaining sample decays each year?
  d When will the rate of decay be 1% of the original mass M_0 per year?

9 a Explain why $Q = e^t$ is always increasing at an increasing rate.
  b Explain why $Q = e^{-t}$ is always decreasing at a decreasing rate.
  c For what values of A and k is $Q = Ae^{kt}$:
    i increasing,
    ii decreasing?
  d What happens to $Q = Ae^{kt}$ when A or k is zero?

ENRICHMENT

10 Suppose that a quantity Q varies with time according to $\dfrac{dQ}{dt} = kQ$, where k is a constant.
  a Let $Q = Ae^{kt}$, where we suppose for the moment that A is a function of t, and use the product rule to show that

$$\frac{dQ}{dt} = Ake^{kt} + e^{kt}\frac{dA}{dt}.$$

  b Hence show that A is a constant, equal to the value of Q at time zero.

11 a Find the possible values of λ that make $y = e^{\lambda x}$ a solution of $ay'' + by' + cy = 0$.
  b When will there be no real solution for λ?
  c Find two families of solutions of the differential equation $y'' - 7y' + 10y = 0$.

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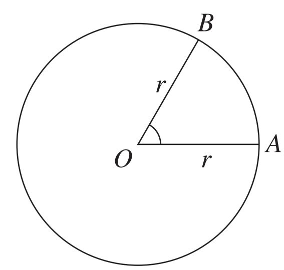
11G Radian measure of angle size

This section begins the second part of the chapter. It introduces a new way of measuring angle size in *radians*, based on the number π. Radian measure of angles will be needed next year for the calculus of the trigonometric functions.

The use of degrees to measure angle size is based on astronomy, not on mathematics. There are 360 days in the year, correct to the nearest convenient whole number, so $1°$ is the angle through which the Sun moves against the fixed stars each day. (After the work of Copernicus and Galileo, $1°$ is the angle swept out by the Earth each day in its orbit around the Sun.) Mathematics is far too general a discipline to be tied to the particularities of our solar system, so it is quite natural to develop a new system for measuring angles based on mathematics alone.

Radian measure of angle size



The size of an angle in radians is defined as the ratio of two lengths in a circle.

Given an angle with vertex O, construct a circle with centre O meeting the two arms of the angle at A and B.

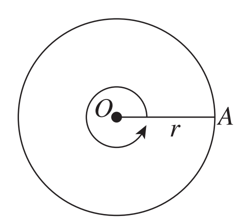
The size of $\angle AOB$ in radians is the ratio of the arc length AB and the radius OA.

10 RADIAN MEASURE

$$\text{Size of } \angle AOB = \frac{\text{arc length } AB}{\text{radius } OA}$$

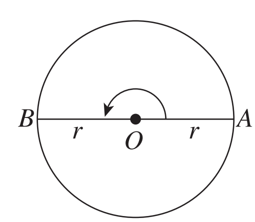
This definition gives the same angle size, whatever the radius of the circle, because all circles are similar to one another.

The arc subtended by a revolution is the whole circumference,



$$\begin{aligned}\text{so} \qquad 1 \text{ revolution} &= \frac{\text{circumference}}{\text{radius}} \\ &= \frac{2\pi r}{r} \\ &= 2\pi.\end{aligned}$$

A straight angle subtends a semicircle,



$$\begin{aligned}\text{so} \qquad 1 \text{ straight angle} &= \frac{\text{arc length of semicircle}}{\text{radius}} \\ &= \frac{\pi r}{r} \\ &= \pi.\end{aligned}$$

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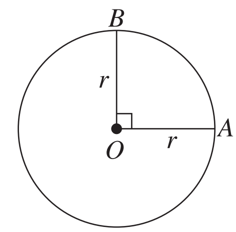
A right angle subtends a quarter-circle,



$$\text{so} \quad 1 \text{ right angle} = \frac{\text{arc length of quarter} - \text{circle}}{\text{radius}}$$

$$= \frac{\frac{1}{2}\pi r}{r}$$

$$= \frac{\pi}{2}.$$

These three basic conversions should be memorised very securely.

11 BASIC CONVERSIONS BETWEEN DEGREES AND RADIAN MEASURE

$360^\circ = 2\pi \qquad 180^\circ = \pi \qquad 90^\circ = \frac{\pi}{2}$

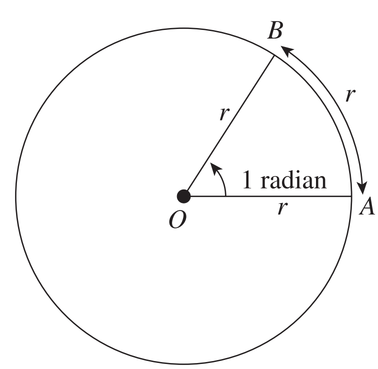
Because $180^\circ = \pi$ radians, an angle size in radians can be converted to an angle size in degrees by multiplying by $\frac{180^\circ}{\pi}$.

Conversely, degrees are converted to radians by multiplying by $\frac{\pi}{180}$.

12 CONVERTING BETWEEN DEGREES AND RADIANS

- To convert from radians to degrees,
  multiply by $\dfrac{180^\circ}{\pi}$.
- To convert from degrees to radians,
  multiply by $\dfrac{\pi}{180}$.
- One radian and one degree:
  $1 \text{ radian} = \dfrac{180^\circ}{\pi} \doteqdot 57^\circ 18' \qquad \text{and} \qquad 1 \text{ degree} = \dfrac{\pi}{180} \doteqdot 0.0175.$

One radian is the angle subtended at the centre of a circle by an arc of length equal to the radius. Notice that the sector *OAB* in the diagram to the right is almost an equilateral triangle, so 1 radian is about 60°. This makes sense of the value given above, that 1 radian is about 57°.



Note: The size of an angle in radians is a ratio of lengths, so is a *dimensionless real number*. It is therefore unnecessary to mention radians. For example, 'an angle of size 1.3' means an angle of 1.3 radians.

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This definition of angle size is very similar to the definitions of the six trigonometric functions, which are also ratios of lengths and so are also pure numbers.

Example 12 **11G**

Express these angle sizes in radians.

a $60°$ **b** $270°$ **c** $495°$ **d** $37°$

SOLUTION

a $$60° = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$$

b $$270° = 270 \times \frac{\pi}{180} = \frac{3\pi}{2}$$

c $$495° = 495 \times \frac{\pi}{180} = \frac{11\pi}{4}$$

d $$37° = 37 \times \frac{\pi}{180} = \frac{37\pi}{180}$$

Example 13 **11G**

Express these angle sizes in degrees. Give exact answers, and then where appropriate give answers correct to the nearest degree.

a $\frac{\pi}{6}$ **b** 0.3 **c** $\frac{3\pi}{4}$ **d** 20

SOLUTION

a $$\frac{\pi}{6} = \frac{\pi}{6} \times \frac{180°}{\pi} = 30°$$

b $$0.3 = \frac{3}{10} \times \frac{180°}{\pi} = \frac{54°}{\pi} \doteqdot 17°$$

c $$\frac{3\pi}{4} = \frac{3\pi}{4} \times \frac{180°}{\pi} = 135°$$

d $$20 = 20 \times \frac{180°}{\pi} = \frac{3600°}{\pi} \doteqdot 1146°$$

Evaluating trigonometric functions of special angles

The trigonometric function of an angle is the same whether the angle size is given in degrees or radians. With angles whose related angle is one of the three special angles, it is a matter of recognising the special angles

$$\frac{\pi}{6} = 30° \quad \text{and} \quad \frac{\pi}{4} = 45° \quad \text{and} \quad \frac{\pi}{3} = 60°.$$

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Example 14 11G

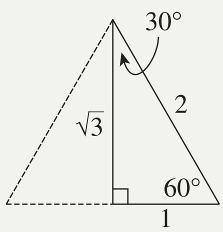
Evaluate these trigonometric functions, using the special triangles sketched below.

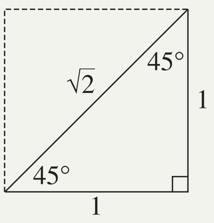
a $\sin \frac{\pi}{6}$ **b** $\operatorname{cosec} \frac{\pi}{4}$

SOLUTION

a $\sin \frac{\pi}{6} = \sin 30°$
$= \frac{1}{2}$

b $\operatorname{cosec} \frac{\pi}{4} = \frac{1}{\sin \frac{\pi}{4}}$
$= \frac{1}{\sin 45°}$
$= \sqrt{2}$





Example 15 11G

[Angles whose related angle is a special angle]

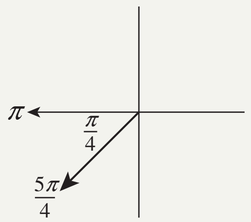
Use special angles to evaluate these trigonometric functions.

a $\sin \frac{5\pi}{4}$ **b** $\sec \frac{11\pi}{6}$

SOLUTION

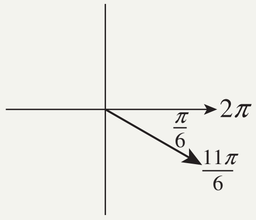
a Because $\frac{5\pi}{4}$ is in the third quadrant, with related angle $\frac{\pi}{4}$,

$\sin \frac{5\pi}{4} = -\sin \frac{\pi}{4}$
$= -\sin 45°$
$= -\frac{1}{\sqrt{2}}.$



b Because $\frac{11\pi}{6}$ is in the fourth quadrant, with related angle $\frac{\pi}{6}$,

$\sec \frac{11\pi}{6} = +\sec \frac{\pi}{6}$
$= \frac{1}{\cos \frac{\pi}{6}}$
$= \frac{1}{\cos 30°}$
$= \frac{2}{\sqrt{3}}.$



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Approximating trigonometric functions of other angles

For other angles, exact values of the trigonometric functions cannot normally be found. When calculator approximations are required, *it is vital to set the calculator to radians mode first*.

Your calculator has a key labelled mode or something similar to make the change — calculators set to the wrong mode routinely cause havoc at this point!

13 SETTING THE CALCULATOR TO RADIANS MODE OR DEGREES MODE

From now on, always decide whether the calculator should be in radians mode or degrees mode before using any of the trigonometric functions.

Example 16 — 11G

Evaluate correct to four decimal places:

a $\cos 1$ **b** $\cot 1.3$

SOLUTION

Here the calculator must be set to radians mode.

a $\cos 1 \doteqdot 0.5403$

b $\cot 1.3 = \dfrac{1}{\tan 1.3}$

$\doteqdot 0.2776$

Exercise 11G — FOUNDATION

Note: Be very careful throughout the remainder of this chapter whether your calculator is set in radians or degrees. The button used to make the change is usually labelled mode.

1 Express these angles in radians.

a $90°$ **b** $45°$ **c** $30°$ **d** $60°$
e $120°$ **f** $150°$ **g** $135°$ **h** $225°$
i $360°$ **j** $300°$ **k** $270°$ **l** $210°$

2 Express these angles in degrees.

a π **b** 2π **c** 4π **d** $\dfrac{\pi}{2}$ **e** $\dfrac{\pi}{3}$ **f** $\dfrac{\pi}{4}$ **g** $\dfrac{2\pi}{3}$
h $\dfrac{5\pi}{6}$ **i** $\dfrac{3\pi}{4}$ **j** $\dfrac{3\pi}{2}$ **k** $\dfrac{4\pi}{3}$ **l** $\dfrac{7\pi}{4}$ **m** $\dfrac{11\pi}{6}$

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3 Use your calculator in radians mode to evaluate, correct to two decimal places:
a $\sin 1$ b $\cos 2$ c $\tan 3$
d $\sin 0.7$ e $\cos 1.23$ f $\tan 5.678$

4 Use your calculator to express in radians, correct to three decimal places:
a $73°$ b $14°$ c $168°$
d $21°36'$ e $95°17'$ f $211°12'$

5 Use your calculator to express in degrees and minutes, correct to the nearest minute:
a 2 radians b 0.3 radians c 1.44 radians
d 0.123 radians e 3.1985 radians f 5.64792 radians

DEVELOPMENT

6 Using the two special triangles, find the exact value of:
a $\sin\frac{\pi}{6}$ b $\sin\frac{\pi}{4}$ c $\cos\frac{\pi}{6}$ d $\tan\frac{\pi}{3}$
e $\tan\frac{\pi}{4}$ f $\cos\frac{\pi}{3}$ g $\sec\frac{\pi}{4}$ h $\cot\frac{\pi}{3}$

7 Express in radians in terms of π:
a $20°$ b $22.5°$ c $36°$
d $100°$ e $112.5°$ f $252°$

8 Express in degrees:
a $\dfrac{\pi}{12}$ b $\dfrac{2\pi}{5}$ c $\dfrac{20\pi}{9}$
d $\dfrac{11\pi}{8}$ e $\dfrac{17\pi}{10}$ f $\dfrac{23\pi}{15}$

9 a Find the complement of $\frac{\pi}{6}$.
b Find the supplement of $\frac{\pi}{6}$.

10 Two angles of a triangle are $\frac{\pi}{3}$ and $\frac{2\pi}{9}$. Find, in radians, the third angle.

11 Using the two special triangles and your knowledge of angles of any magnitude, find the exact value of:
a $\sin\frac{2\pi}{3}$ b $\cos\frac{2\pi}{3}$ c $\cos\frac{5\pi}{6}$ d $\tan\frac{4\pi}{3}$
e $\tan\frac{3\pi}{4}$ f $\cos\frac{5\pi}{3}$ g $\sin\frac{5\pi}{4}$ h $\tan\frac{7\pi}{6}$

12 Give each answer in degrees and then in radians.
a What angle do the hour and minute hands of a clock move through in 1 hour?
b Find the non-reflex angle between the hour and minute hands of a clock at:
i 2:00 pm ii 2:15 pm iii 2:30 pm iv 2:45 pm

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13 Find, correct to three decimal places, the angle in radians through which:

a the second hand of a clock turns in 7 seconds,

b the hour hand of a clock turns between 6:00 am and 6:40 am,

c the second hand turns in a week.

14 If $f(x) = \sin x$, $g(x) = \cos 2x$ and $h(x) = \tan 3x$, find, correct to three significant figures:

a $f(1) + g(1) + h(1)$

b $f\big(g(h(1))\big)$

ENRICHMENT

15 a Explain why $n = 0$ is the only integer solution of $\sin n = 0$.

b Use your calculator to find the first positive integer n for which $|\sin n| < 0.01$.

c Use the fact that $\pi \doteqdot \frac{22}{7}$ to explain your answer to part **b**.

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11H Solving trigonometric equations

Once radians have been introduced into trigonometric functions, trigonometric equations can be solved in radians as well as in degrees.

Solving trigonometric equations in radians

Solving a trigonometric equation is done the same way whether the solution is to be given in radians or degrees.

14 SOLVING A TRIGONOMETRIC EQUATION IN RADIANS

- First establish on a diagram which quadrants the angle can lie in.
- Then find the related angle — but use radian measure, not degrees.
- Use the diagram and the related angle to find all the answers, taking account of any restrictions on the angle.

Alternatively, solve the equation first in degrees, then convert each answer to radians.

Example 17 11H

[Acute angles]

Solve each trigonometric equation in radians, where the angle x is an acute angle. Give answers in exact form, or correct to five significant figures.

a $\sin x = \frac{1}{2}$ **b** $\tan x = 3$

SOLUTION

With acute angles, a quadrants diagram is not necessary.

a $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ (30° changed to radians)

b $\tan x = 3$

$x \doteqdot 1.2490$ (use the calculator)

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Example 18 11H

[General angles]

Solve these trigonometric equations. Give answers in exact form if possible, otherwise correct to five significant figures.

a $\cos x = -\frac{1}{2}$, where $0 \le x \le 2\pi$

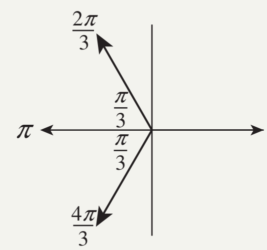
b $\cos 3x = -\frac{1}{2}$, where $0 \le x \le 2\pi$

c $\sin x = -\frac{1}{3}$, where $0 \le x \le 2\pi$

Note: When using the calculator's inverse trigonometric functions, *do not work with a negative number*. Always enter the absolute value of the number in order to find the related angle.

SOLUTION

a $\cos x = -\frac{1}{2}$, where $0 \le x \le 2\pi$.



Because $\cos x$ is negative, x is in quadrant 2 or 3.

The acute angle whose cosine is $+\frac{1}{2}$ is 60°, or $\frac{\pi}{3}$.

Hence $\quad x = \pi - \frac{\pi}{3}$ or $\pi + \frac{\pi}{3}$

$\quad = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

b $\cos 3x = -\frac{1}{2}$, where $0 \le x \le 2\pi$.

To find the domain of $3x$, multiply the condition by 3,

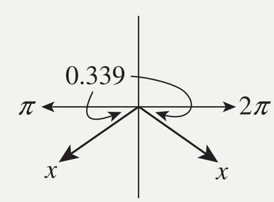
$\cos 3x = -\frac{1}{2}$, where $0 \le 3x \le 6\pi$.

Working with the diagram in part **a**,

$3x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}$ or $\frac{16\pi}{3}$

$x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}$ or $\frac{16\pi}{9}$

c $\sin x = -\frac{1}{3}$, where $0 \le x \le 2\pi$.



Because $\sin x$ is negative, x is in quadrant 3 or 4.

With the calculator in radians mode, enter $+\frac{1}{3}$,

then the related angle 0.339836 ... (store this in memory).

Hence $\quad x = \pi + 0.339836\ldots$ or $2\pi - 0.339836\ldots$

$\quad \doteqdot 3.4814$ or 5.9433.

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Solving trigonometric equations reducible to quadratics

Some trigonometric equations are quadratic equations in a trigonometric function. Some such equations can be solved directly, but it is often useful to make an algebraic substitution, as we did in Section 8E with exponential equations. Both methods are used in Example 19.

Example 19 — 11H

Solve these trigonometric equations in radians, for $0 \le x \le 2\pi$.

a $\sin^2 x = \frac{3}{4}$

b $2\cos^2 x - \cos x - 1 = 0$

SOLUTION

a This is easily done without any substitution.

$$\sin^2 x = \tfrac{3}{4}$$

$$\sin x = \tfrac{\sqrt{3}}{2} \text{ or } \sin x = -\tfrac{\sqrt{3}}{2}$$

The related angle is $\frac{\pi}{3}$, and the angle can be in any of the four quadrants,

so $\quad x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

b Because of the factoring required, substitution is a better approach here.

$$2\cos^2 x - \cos x - 1 = 0$$

Let $\quad u = \cos x$.

Then $\quad 2u^2 - u - 1 = 0$, thus 'reducing' it to a quadratic equation,

$$2u^2 - 2u + u - 1 = 0$$
$$2u(u - 1) + (u - 1) = 0$$
$$(2u + 1)(u - 1) = 0$$
$$u = -\tfrac{1}{2} \text{ or } u = 1$$
$$\cos x = -\tfrac{1}{2} \text{ or } \cos x = 1.$$

When $\cos x = -\frac{1}{2}$, the related angle is $\frac{\pi}{3}$, and the angle is quadrant 2 or 3,

so $\quad x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

When $\cos x = 1$, the solutions are boundary angles, so we use the graph.

From the graphs in Chapter 6, the answers are 0° and 360°.

Converting to radians, or using the graphs in radians in the later Section 11J,

$$x = 0 \text{ or } 2\pi.$$

Hence the solutions are $\quad x = 0$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or 2π .

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Exercise 11H

FOUNDATION

1 Find, in radians, the acute angle θ that satisfies:

a $\tan\theta = 1$ **b** $\sin\theta = \frac{1}{2}$ **c** $\cos\theta = \frac{1}{\sqrt{2}}$

d $\tan\theta = \frac{1}{\sqrt{3}}$ **e** $\sin\theta = \frac{\sqrt{3}}{2}$ **f** $\cos\theta = \frac{1}{2}$

2 Find, correct to three decimal places, the value of x between 0 and $\frac{\pi}{2}$ that satisfies each equation. (Your calculator needs to be in radians mode.)

a $\tan x = 3$ **b** $\sin x = 0.8$ **c** $\cos x = 0.4$

d $\sin x = 0.234$ **e** $\cos x = 0.987$ **f** $\tan x = 100$

3 Solve for x over the domain $0 \le x \le 2\pi$:

a $\sin x = \frac{1}{2}$ **b** $\cos x = -\frac{1}{2}$ **c** $\tan x = -1$

d $\sin x = 1$ **e** $2\cos x = \sqrt{3}$ **f** $\sqrt{3}\tan x = 1$

g $\cos x + 1 = 0$ **h** $\sqrt{2}\sin x + 1 = 0$

DEVELOPMENT

4 Solve each equation for $0 \le \theta \le 2\pi$. Remember that a positive number has two square roots.

a $\sin^2\theta = 1$ **b** $\tan^2\theta = 1$ **c** $\cos^2\theta = \frac{1}{4}$ **d** $\cos^2\theta = \frac{3}{4}$

5 Consider the equation $\cos^2\theta - \cos\theta = 0$, for $0 \le \theta \le 2\pi$.

a Write the equation as a quadratic equation in u by letting $u = \cos\theta$.

b Solve the quadratic equation for u.

c Hence find the values of θ that satisfy the original equation.

6 Consider the equation $\tan^2\theta - \tan\theta - 2 = 0$, for $0 \le \theta \le 2\pi$.

a Write the equation as a quadratic equation in u by letting $u = \tan\theta$.

b Solve the quadratic equation for u.

c Hence find the values of θ that satisfy the original equation, giving the solutions correct to two decimal places where necessary.

7 Solve each equation for $0 \le \theta \le 2\pi$ by transforming it into a quadratic equation in u. Give your solutions correct to two decimal places where necessary.

a $\tan^2\theta + \tan\theta = 0$ (Let $u = \tan\theta$.) **b** $2\sin^2\theta - \sin\theta = 0$ (Let $u = \sin\theta$.)

c $\sin^2\theta + \sin\theta - 2 = 0$ (Let $u = \sin\theta$.) **d** $\tan^2\theta + \tan\theta - 6 = 0$ (Let $u = \tan\theta$.)

e $2\cos^2\theta + \cos\theta - 1 = 0$ **f** $2\sin^2\theta - \sin\theta - 1 = 0$

g $3\sin^2\theta + 8\sin\theta - 3 = 0$ **h** $3\cos^2\theta - 8\cos\theta - 3 = 0$

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8 Use the trigonometric identities from Chapter 6 to transform each equation so that it only involves one trigonometric function. Then solve it for $0 \le x \le 2\pi$, giving solutions correct to two decimal places where necessary.

a $2\sin^2 x + \cos x = 2$
b $\sec^2 x - 2\tan x - 4 = 0$
c $8\cos^2 x = 2\sin x + 7$
d $6\tan^2 x = 5\sec x$

9 Solve each equation for $0 \le \theta \le 2\pi$, giving solutions correct to two decimal places where necessary. Again, you will need trigonometric identities.

a $3\sin\alpha = \operatorname{cosec}\alpha + 2$
b $3\tan\alpha - 2\cot\alpha = 5$

10 Solve each equation for $0 \le x \le 2\pi$ by first dividing through by $\cos^2 x$. Give solutions correct to two decimal places where necessary.

a $\sin^2 x + \sin x\cos x = 0$
b $\sin^2 x - 5\sin x\cos x + 6\cos^2 x = 0$

11 Solve for $-\pi \le x \le \pi$:

a $\sin 2x = \frac{1}{2}$
b $\cos 3x = -1$
c $\tan\left(x - \frac{\pi}{6}\right) = \sqrt{3}$
d $\sec\left(x + \frac{\pi}{4}\right) = -\sqrt{2}$
e $\operatorname{cosec}\left(x - \frac{3\pi}{4}\right) = 1$
f $\cot\left(x + \frac{5\pi}{6}\right) = \sqrt{3}$

12 [Technology]
Graphing programs provide an excellent way to see what is happening when an equation has many solutions. The equations in Question 2 are quite simple to graph, because $y = \text{LHS}$ is a single trigonometric function and $y = \text{RHS}$ is a horizontal line. Every one of the infinitely many points of intersection corresponds to a solution.

ENRICHMENT

13 Solve the equation $4\sin^2\theta + 2\cos\theta = 3$ for $0 \le \theta \le 2\pi$, giving exact solutions.

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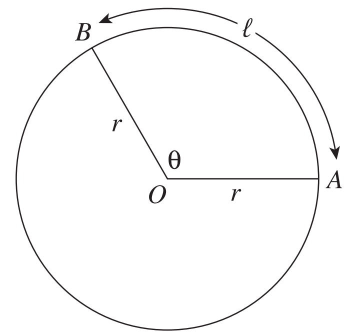
11I Arcs and sectors of circles

The lengths of arcs and the areas of sectors and segments can already be calculated using fractions of circles, but radian measure allows the formulae to be expressed in more elegant forms.

Arc length



In the diagram to the right, the *arc AB* has length ℓ and subtends an angle θ at the centre O of a circle with radius r.

The definition of angle size in radians is $\theta = \dfrac{\ell}{r}$.

Multiplying through by r, $\qquad \ell = r\theta$.

This is the standard formula for arc length:

15 ARC LENGTH

An arc subtending an angle θ at the centre of a circle of radius r has length

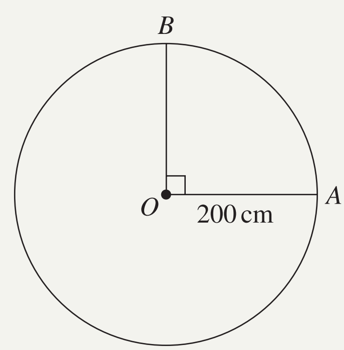
$$\ell = r\theta$$

Note: When the radius is 1, the arc length formula becomes $\ell = \theta$. Thus in a circle of radius 1, we can identify the arc length and the angle size in radians.

This is often taken as the definition of angle size in radians, but if this is done, it is important to remember that *an angle size in radians is still a pure number without units*, because it is the ratio of a length and a unit length.

Example 20 **11I**

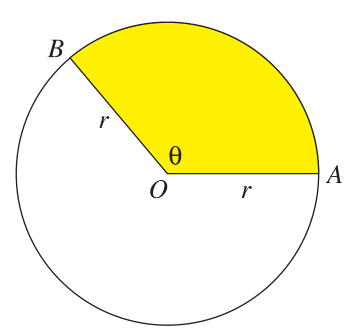
What is the length of an arc subtending a right angle at the centre of a circle of radius 200 cm?



SOLUTION

$$\begin{aligned}\text{Arc length} &= r\theta \\ &= 200 \times \frac{\pi}{2} \quad \left(\text{a right angle has size } \tfrac{\pi}{2}\right) \\ &= 100\pi \text{ cm}.\end{aligned}$$

Area of a sector



In the diagram to the right, the *sector AOB*, as shaded, is bounded by the arc AB and the radii OA and OB. Its area can be calculated as a fraction of the total area:

$$\begin{aligned}\text{area of sector} &= \frac{\theta}{2\pi} \times \text{area of circle} \\ &= \frac{\theta}{2\pi} \times \pi r^2 \\ &= \tfrac{1}{2} r^2 \theta.\end{aligned}$$

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16 AREA OF A SECTOR

An arc subtending an angle θ at the centre of a circle of radius r has area

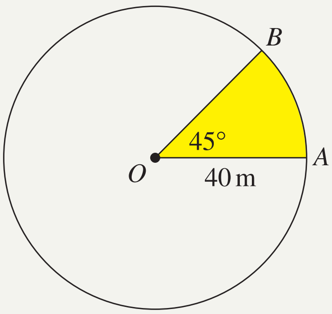
$$\text{Area} = \tfrac{1}{2}r^2\theta$$

Example 21 11I

What are the area and the perimeter of a sector subtending an angle of 45° at the centre of a circle of radius 40 metres?

SOLUTION

$$\begin{aligned}
\text{Area of sector} &= \tfrac{1}{2}r^2\theta \\
&= \frac{1}{2} \times 1600 \times \frac{\pi}{4} \qquad \left(45^\circ \text{ in radians is } \tfrac{\pi}{4}\right) \\
&= 200\pi \text{ square metres.} \\
\text{Perimeter} &= \text{arc length} + 2r \\
&= \tfrac{\pi}{4} \times 40 + 2 \times 40 \\
&= 10\pi + 80 \text{ metres.}
\end{aligned}$$

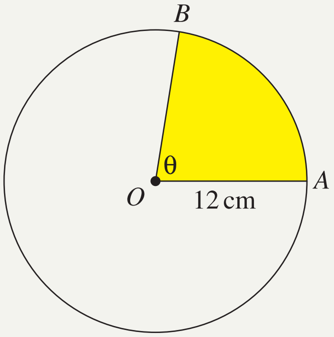


Example 22 11I

A circular cake has radius 12 cm. What angle at the centre is subtended by a sector of area 100 cm^2? Answer correct to the nearest degree.

SOLUTION

$$\begin{aligned}
\text{Area of sector} &= \tfrac{1}{2}r^2\theta\,. \\
100 &= \tfrac{1}{2} \times 144 \times \theta \\
\boxed{\div 72} \qquad \theta &= \frac{100}{72} \times \frac{180^\circ}{\pi} \qquad \text{(converting to degrees)} \\
&\doteqdot 80^\circ.
\end{aligned}$$



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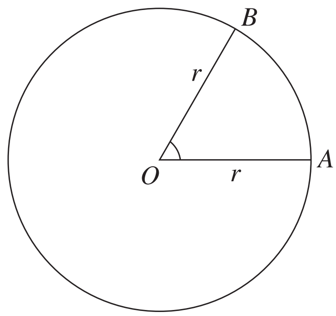
Major and minor arcs and sectors

In the diagram to the right, the phrase 'the arc AB' is ambiguous, because there are two arcs AB.



- The *minor arc AB* is the arc subtending the marked angle at the centre. This angle is less than a straight angle, and the minor arc is less than half the circumference
- The *major arc AB* is the *opposite arc* subtending the unmarked reflex angle at the centre. This angle is more than a straight angle, and the major arc is more than half the circumference.

The words 'major' and 'minor' are originally Latin words that simply mean 'greater' and 'lesser'. They apply also to sectors in the obvious way — there are two *opposite sectors AOB*, the *minor sector* containing the marked angle, and the *major sector* containing the unmarked reflex angle.

The length of a major arc and the area of a major sector can be calculated with the usual arc length and sector formulae, but using the reflex angle that they subtend, as in the example below. Alternatively, they can be calculated by subtraction from the circumference or area of the whole circle.

Example 23 11I

[The opposite arc and sector of an earlier example]

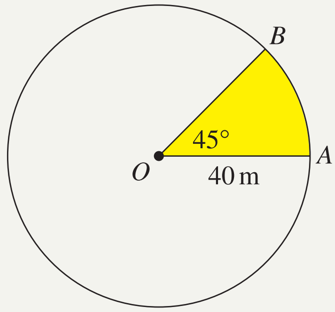
Find the area and the perimeter of the major sector AOB in the diagram below.

SOLUTION

The major arc and sector each subtend $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ at the centre, and are unshaded in the diagram.

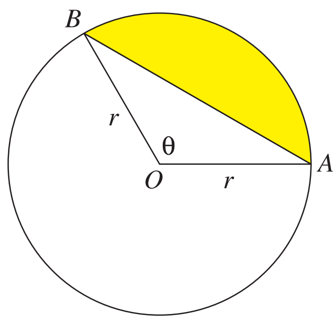
$$\begin{aligned}\text{Area of sector} &= \tfrac{1}{2}r^2\theta \\ &= \frac{1}{2} \times 1600 \times \frac{7\pi}{4} \quad \left(315° \text{ in radians is } \tfrac{7\pi}{4}\right) \\ &= 1400\pi \text{ square metres.}\end{aligned}$$

$$\begin{aligned}\text{Perimeter} &= \text{arc length} + 2r \\ &= \tfrac{7\pi}{4} \times 40 + 2 \times 40 \\ &= 70\pi + 80 \text{ metres.}\end{aligned}$$



Area of a segment

In the diagram to the right, the chord AB divides the circle into two *segments*. The *minor segment* has been shaded, and the rest of the circle is the *major segment*.



Drawing the two radii OA and OB produces an isosceles triangle AOB, together with two opposite sectors. The areas of the two segments can now be found by adding or subtracting appropriate areas.

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17 AREA OF A SEGMENT

To find the area of a segment:

- Construct the radii from the ends of the chord, to form an isosceles triangle and two opposite sectors.
- Add or subtract the appropriate areas.

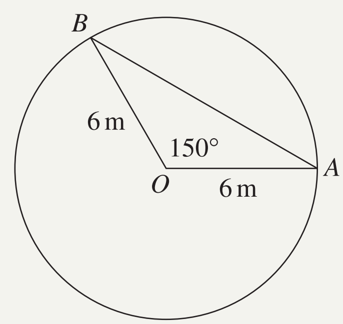
Example 24 11I

a Find the lengths of the minor and major arcs formed by two radii of a circle of radius 6 metres meeting at 150°.

b Find the areas of the minor and major sectors.

c Find the area of $\triangle AOB$.

d Find the areas of the major and minor segments.



SOLUTION

The minor arc subtends 150° at the centre, which in radians is $\frac{5\pi}{6}$, and the major arc subtends 210° at the centre, which in radians is $\frac{7\pi}{6}$.

a Minor arc $= r\theta$
$= 6 \times \frac{5\pi}{6}$
$= 5\pi$ metres.

Major arc $= r\theta$
$= 6 \times \frac{7\pi}{6}$
$= 7\pi$ metres.

b Minor sector $= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 6^2 \times \frac{5\pi}{6}$
$= 15\pi\ \text{m}^2$.

Major sector $= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 6^2 \times \frac{7\pi}{6}$
$= 21\pi\ \text{m}^2$.

c Area of $\triangle AOB = \frac{1}{2}r^2\sin\theta$
$= \frac{1}{2} \times 6^2 \times \sin 150^\circ \quad \left(\text{alternatively } \sin\frac{5\pi}{6}\right)$
$= \frac{1}{2} \times 36 \times \frac{1}{2}$
$= 9\ \text{m}^2$.

d The minor segment area is obtained by subtraction.

Minor segment $=$ minor sector $- \triangle AOB$
$= (15\pi - 9)\ \text{m}^2$.

The major segment area is obtained by addition.

Major segment $=$ major sector $+ \triangle AOB$
$= (21\pi + 9)\ \text{m}^2$.

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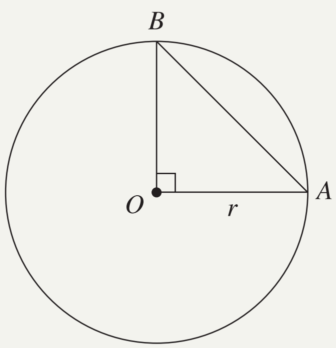
Example 25 11I

Find, correct to the nearest mm, the radius of a circle in which:

a a sector, **b** a segment,

of area 1 square metre subtends an angle of 90° at the centre of the circle.

SOLUTION



Let the radius of the circle be r metres.

Then area of sector $AOB = \frac{1}{2}r^2 \times \frac{\pi}{2}$

$= \frac{\pi}{4}r^2$

and area of $\triangle AOB = \frac{1}{2}r^2$ (it is half a square).

a Substituting into the formula for the area of a sector,

$\frac{\pi}{4} \times r^2 = 1$ (the area is $1\,m^2$)

$\boxed{\times \frac{4}{\pi}}$ $r^2 = \frac{4}{\pi}$

$r \doteqdot 1.128$ metres.

b Subtracting, area of segment $= \frac{\pi}{4}r^2 - \frac{1}{2}r^2$

$= \frac{1}{4}r^2(\pi - 2)$.

Hence $\frac{1}{4}r^2(\pi - 2) = 1$ (the area is $1\,m^2$)

$\boxed{\times \frac{4}{\pi - 2}}$ $r^2 = \frac{4}{\pi - 2}$

$r \doteqdot 1.872$ metres.

Exercise 11I FOUNDATION

Note: Are you working with radians or degrees? Remember the button labelled mode.

1 A circle has radius 6 cm. Find the length of an arc of this circle that subtends an angle at the centre of:

a 2 radians **b** 0.5 radians **c** $\frac{\pi}{3}$ radians **d** $\frac{\pi}{4}$ radians

2 A circle has radius 8 cm. Find the area of a sector of this circle that subtends an angle at the centre of:

a 1 radian **b** 3 radians **c** $\frac{\pi}{4}$ radians **d** $\frac{3\pi}{8}$ radians

3 What is the radius of the circle in which an arc of length 10 cm subtends an angle of 2.5 radians at the centre?

4 If a sector of a circle of radius 4 cm has area $12\,cm^2$, find the angle at the centre in radians.

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5 A circle has radius 3.4 cm. Find, correct to the nearest millimetre, the length of an arc of this circle that subtends an angle at the centre of:

a $40°$ **b** $73°38'$

(Hint: Remember that θ must be in radians.)

6 Find, correct to the nearest square metre, the area of a sector of a circle of radius 100 metres if the angle at the centre is $100°$.

7 A circle has radius 12 cm. Find, in exact form:

a the length of an arc that subtends an angle of $120°$ at the centre,

b the area of a sector in which the angle at the centre is $40°$.

8 An arc of a circle of radius 7.2 cm is 10.6 cm in length. Find the angle subtended at the centre by this arc, correct to the nearest degree.

9 A sector of a circle has area $52\,\text{cm}^2$ and subtends an angle of $44°16'$ at the centre. Find the radius in cm, correct to one decimal place.

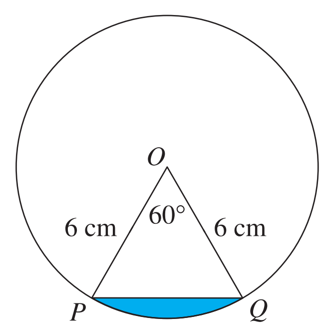
DEVELOPMENT

10 Consider the diagram to the right.

a Find the exact area of the sector OPQ.

b Find the exact area of $\triangle OPQ$.

c Hence find the exact area of the shaded minor segment.



11 A chord of a circle of radius 4 cm subtends an angle of $150°$ at the centre.

a Use the same method as the previous question to show that the area of the minor segment cut off by the chord is $\frac{4}{3}(5\pi - 3)\,\text{cm}^2$.

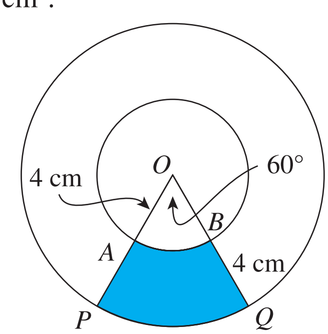
b By subtracting the area of the minor segment from the area of the circle, show that the area of the major segment cut off by the chord is $\frac{4}{3}(7\pi + 3)\,\text{cm}^2$.

12 A circle has centre C and radius 5 cm, and an arc AB of this circle has length 6 cm. Find the area of the sector ACB.

13 The diagram to the right shows two concentric circles with common centre O.

a Find the exact perimeter of the region $APQB$.

b Find the exact area of the region $APQB$.



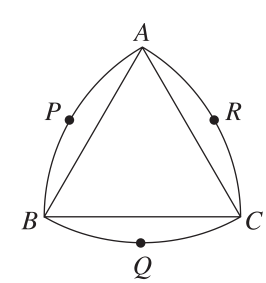
14 Triangle ABC is equilateral with side length 2 cm. Circular arcs AB, BC and CA have centres C, A and B respectively. Answer each part in exact form.

a Find the length of the arc AB.

b Find the area of the sector $CAPBC$.

c Find the length of the perimeter $APBQCRA$.

d Find the area of $\triangle ABC$ and hence find the area enclosed by the perimeter $APBQCRA$.



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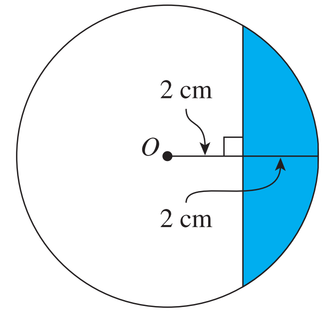
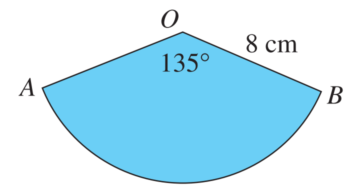
15 Find the exact area of the shaded region of the circle shown in the diagram to the right.

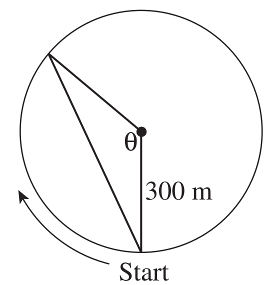


16 A piece of paper is cut in the shape of a sector of a circle. The radius is 8 cm and the angle at the centre is $135°$. The straight edges of the sector are placed together so that a cone is formed.



- **a** Show that the base of the cone has radius 3 cm.
- **b** Show that the cone has perpendicular height $\sqrt{55}$ cm.
- **c** Hence find, in exact form, the volume of the cone.
- **d** Find the curved surface area of the cone.

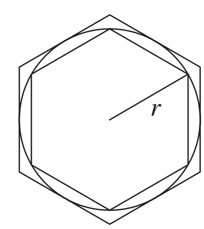
17 An athlete runs at a steady 4 m/s around a circular track of radius 300 metres. She runs clockwise, starting at the southernmost point.



- **a** How far has she run after 3 minutes?
- **b** What angle does this distance subtend at the centre?
- **c** How far is she from her start, in a direct line across the field, correct to the nearest 0.01 metre?
- **d** What is her true bearing from the centre then, correct to the nearest minute?

18 [A proof that $3 < \pi < 2\sqrt{3}$, which are significant bounds on the number π]

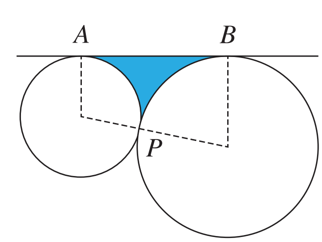
- **a** Prove that an equilateral triangle of side length s has height $\frac{1}{2}s\sqrt{3}$ and area $\frac{1}{4}s^2\sqrt{3}$.
- **b** In the diagram to the right, two regular hexagons have been drawn inside and outside a circle of radius r.



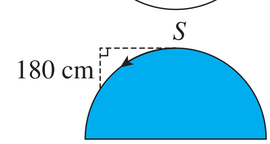
  - **i** Explain why the perimeter of the inner hexagon is $6r$, and why its perimeter is less than the circumference of the circle. Hence show that $3 < \pi$.
  - **ii** Explain why the area of the outer hexagon is $2r^2\sqrt{3}$, and why its area is greater than the area of the circle. Hence show that $\pi < 2\sqrt{3}$.

ENRICHMENT

19 Two circles of radii 2 cm and 3 cm touch externally at P. AB is a common tangent. Calculate, in cm^2 correct to two decimal places, the area of the region bounded by the tangent and the arcs AP and BP.



20 A certain hill is represented by a hemisphere of radius 1 km. A man 180 cm tall walks down the hill from the summit S at 6 km/h. How long (correct to the nearest second) will it be before he is invisible to a person lying on the ground at S?



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11J Trigonometric graphs in radians

Now that angle size has been defined as a ratio, that is, as a pure number, the trigonometric functions can be drawn in their true shapes. On the next page, the graphs of the six functions have been drawn using the same scale on the x-axis and y-axis. This means that the gradient of the tangent at each point now equals the true value of the derivative there.

For example, place a ruler on the graph of $y = \sin x$ so that it makes a tangent to the curve at the origin. The ruler should lie along the line $y = x$, indicating that the tangent at the origin has gradient 1. In the language of calculus, this means that the derivative of $\sin x$ has value 1 when $x = 0$. This is where we will begin when the topic is taken up again in Year 12.

Amplitude of the sine and cosine functions

The *amplitude* of a wave is the maximum height of the wave above the mean position. Both $y = \sin x$ and $y = \cos x$ have a maximum value of 1, a minimum value of -1 and a mean value of 0 (the *mean value* is the average of the maximum value and the minimum value). Thus both waves have amplitude 1.

18 THE AMPLITUDES OF THE SINE AND COSINE FUNCTIONS

- $y = \sin x$ and $y = \cos x$ both have amplitude 1.

The other four trigonometric functions increase without bound near their asymptotes, so the idea of amplitude makes no sense. We can conveniently tie down the vertical scale of $y = \tan x$, however, by using the fact that $\tan \frac{\pi}{4} = 1$.

The periods of the trigonometric functions

The trigonometric functions are called *periodic functions* because each graph repeats itself exactly over and over again. The *period* of such a function is the length of the smallest repeating unit.

The graphs of $y = \sin x$ and $y = \cos x$ on the previous page are waves, with a pattern that repeats every revolution. Thus they both have period 2π.

The graph of $y = \tan x$, on the other hand, has a pattern that repeats every half-revolution. Thus it has period π.

19 THE PERIODS OF THE SINE, COSINE AND TANGENT FUNCTIONS

- $y = \sin x$ and $y = \cos x$ both have period 2π (that is, a full revolution).
- $y = \tan x$ has period π (that is, half a revolution).

The secant and cosecant functions are reciprocals of the cosine and sine functions and so have the same period 2π as they do. Similarly, the cotangent function has the same period π as the tangent function.

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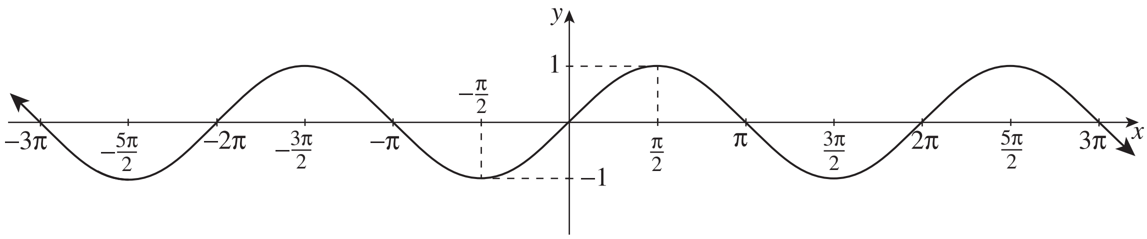
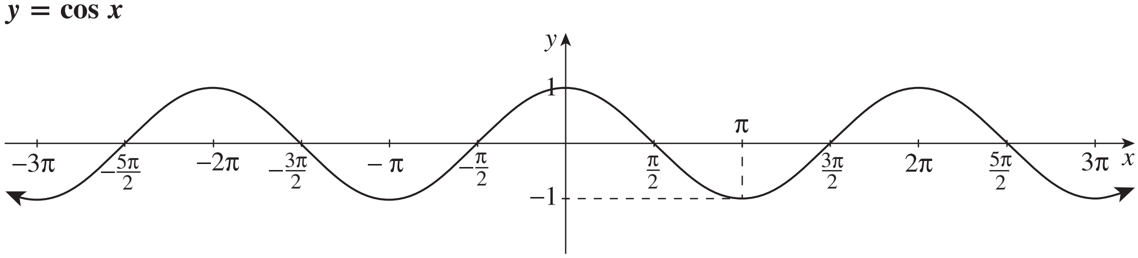
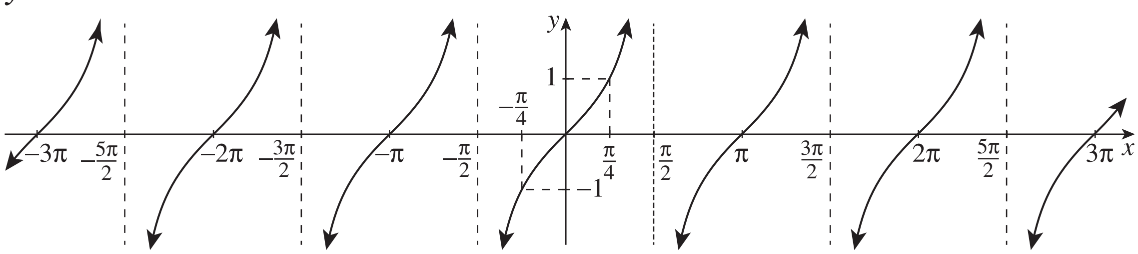
$y = \sin x$



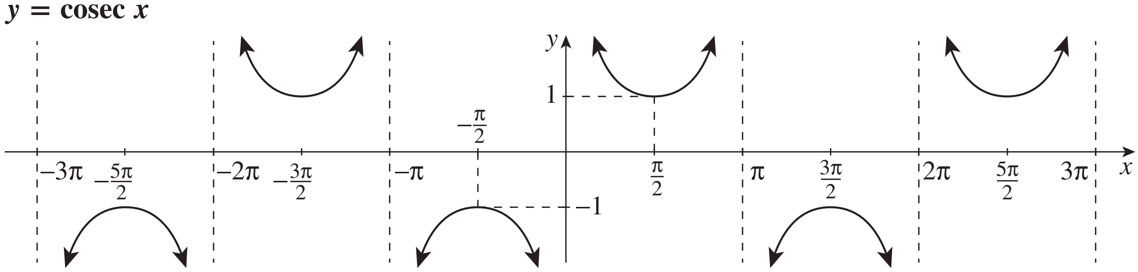
$y = \cos x$



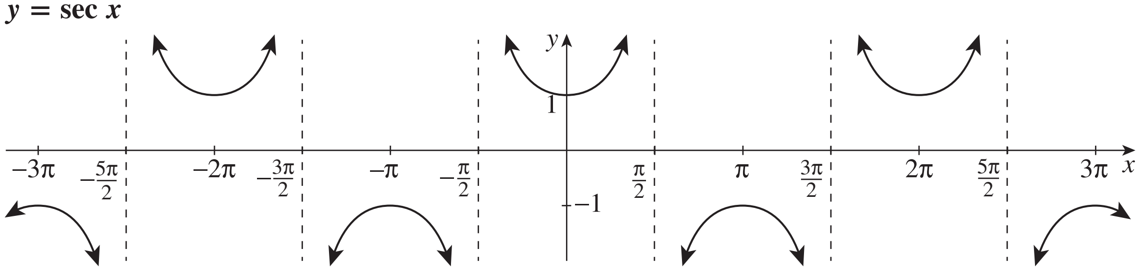
$y = \tan x$



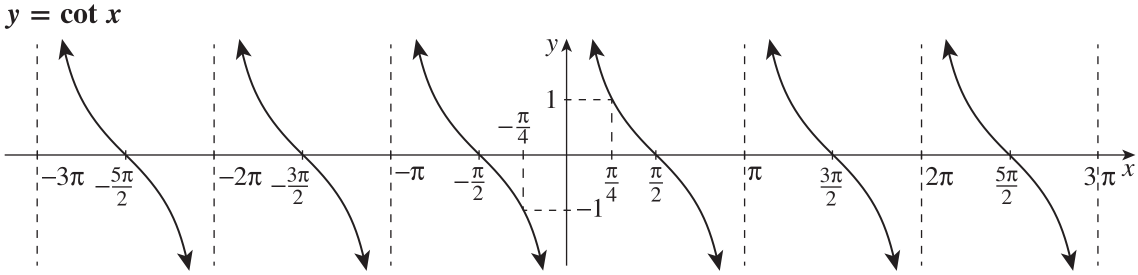
$y = \text{cosec}\, x$



$y = \sec x$



$y = \cot x$



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Oddness and evenness of the trigonometric functions

The graphs of $y = \sin x$ and $y = \tan x$ have point symmetry in the origin, as can easily be seen from their graphs on the previous page. This means that the functions $\sin x$ and $\tan x$ are odd functions. Algebraically, $\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$.

The graph of $y = \cos x$, however, has line symmetry in the y-axis. This means that the function $\cos x$ is an even function. Algebraically, $\cos(-x) = \cos x$.

20 ODDNESS AND EVENNESS OF THE TRIGONOMETRIC FUNCTIONS

- The functions $\sin x$ and $\tan x$ are odd functions. Thus
  $$\sin(-x) = -\sin x \quad \text{and} \quad \tan(-x) = -\tan x.$$
- The function $\cos x$ is an even function. Thus
  $$\cos(-x) = \cos x.$$

The functions $\operatorname{cosec} x$ and $\cot x$ are odd because their reciprocal functions $\sin x$ and $\tan x$ are odd. The function $\sec x$ is even because its reciprocal function $\cos x$ is even. This detail was mentioned in Section 5C on graphing the reciprocal of a function whose graph is given.

A preview of what is to come

Now that radian measure has been introduced, the way is clear in year 12 to differentiate the trigonometric functions $\sin x$ and $\cos x$. The results are surprisingly simple:

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

These results are also reasonably obvious from the graphs.

- Look at the graph of $y = \sin x$ on the previous page. Look at the places where the tangent is horizontal, where the tangent has its greatest positive slope, and where the tangent has its greatest negative slope. Then on a separate set of axes, begin a sketch of $y = \frac{d}{dx}(\sin x)$. This second sketch looks very much like $y = \cos x$.
- Do the same with the graph of $y = \cos x$. This time, the second sketch looks very much like $y = -\sin x$.

So differentiating a wave gives another wave shifted backwards by $\frac{\pi}{2}$, which is a quarter-revolution. But all that is next year's story.

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Exercise 11J

INVESTIGATION

This exercise is an investigation. Its purpose is familiarity with the six trigonometric functions, and particularly familiarity with their symmetries.

Except for Question 1, refer all the time to page 551, where all six trigonometric graphs have been drawn in radians with the same scale on both axes.

An accurate drawing and an accurate gradient:

1 a On graph paper, using a scale on both axes of 1 unit = 2 cm, draw an accurate graph of $y = \sin x$ for $-\pi \le x \le \pi$. Use your calculator to obtain a table of values, and in the interval from $x = -0.5$ to $x = 0.5$, plot points every 0.1 units. Also plot the important points at $x = \frac{\pi}{6}$, $x = \frac{\pi}{2}$, $x = \frac{5\pi}{6}$, $x = \pi$, and the opposites of these values.

b Place your ruler on the graph to confirm that the gradient of the tangent at the origin appears to be 1.

Reading the graphs backwards: Examine the graphs of the six trigonometric functions.

2 a Classify each graph as one-to-one, many-to-one, one-to-many or many-to-many.

b Use horizontal lines on the graph of $y = \sin x$ to find, if possible, at least six positive solutions of:

i $\sin x = 0$ ii $\sin x = \frac{1}{2}$ iii $\sin x = 1$ iv $\sin x = 2$

Line symmetries: Examine the graphs of the six trigonometric functions.

3 a List all the vertical lines in which the graph of $y = \sin x$ is symmetric.

b Which other trigonometric graphs are symmetric in the same vertical lines, and why?

c In which vertical lines are the graphs of $y = \tan x$ and $y = \cot x$ symmetric?

4 a List all the vertical lines in which the graph of $y = \cos x$ is symmetric.

b Which of these lines is related to the fact that $\cos x$ is even, that is, $\cos(-x) = \cos x$?

c Which other trigonometric graphs are symmetric in the same vertical lines, and why?

Point symmetries: Examine the graphs of the six trigonometric functions.

5 a List all the points in which the graph of $y = \sin x$ has point symmetry.

b Which of these points is related to the fact that $\sin x$ is odd, that is, $\sin(-x) = -\sin x$?

c Which other graph has the same point symmetries as $y = \sin x$, and why?

6 a List all the points in which the graph of $y = \cos x$ has point symmetry.

b Which other graph has the same point symmetries as $y = \cos x$, and why?

7 a About which points do $y = \tan x$ and $y = \cot x$ have point symmetry?

b Relate these symmetries to the oddness or evenness of $y = \tan x$ and $y = \cot x$.

Translation symmetries: Examine the graphs of the six trigonometric functions.

8 a What translation symmetry does $y = \sin x$ have? That is, what translations map the graph of $y = \sin x$ onto itself?

b What other trigonometric graphs have exactly these same translation symmetries?

c What translation symmetries do the other trigonometric graphs have?

d What periods do the six trigonometric functions have?

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Transformations: Examine the graphs of the six trigonometric functions.

9 **a** Reflection in some vertical lines transforms the graph of $y = \sin x$ onto the graph of $y = \cos x$. Identify all such vertical lines.

b For what other pair trigonometric functions is the answer to the question in part **a** the same?

c Identify all vertical lines that reflect $y = \tan x$ onto $y = \cot x$.

10 **a** Identify all translations that shift $y = \sin x$ onto the graph of $y = \cos x$.

b For what values of θ is it true that $\sin\,(x - \theta) = \cos x$?

c Identify all translations that shift $y = \tan x$ onto the graph of $y = \cot x$.

11 Identify any rotations about the origin that rotate the graph of one trigonometric function onto another.

Intersections: Examine the graphs of the six trigonometric functions.

12 **a** Sketch $y = \sin x$ and $y = \cos x$ on one set of axes. Where do they intersect?

b Write down the corresponding trigonometric equation and solve it algebraically.

13 **a** Sketch $y = \sin x$ and $y = \tan x$ on one set of axes. Where do they intersect?

b Write down the corresponding trigonometric equation and solve it algebraically.

14 **a** Sketch $y = \cos x$ and $y = \tan x$ together. Very approximately, what is the smallest positive value of x where they intersect?

15 **a** Describe the situation when $y = \sin x$ and $y = \operatorname{cosec} x$ are drawn together.

b What other pair of graphs have the same relationship?

c Name all the pairs of trigonometric functions whose graphs do not intersect when drawn together.

A harder question:

16 **a** Write down the trigonometric equation corresponding to the intersections of $y = \cos x$ and $y = \tan x$, and solve it algebraically.

b Write down the trigonometric equation corresponding to the intersections of $y = \tan x$ and $y = \sec x$, and show algebraically that it has no solutions.

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Chapter 11 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 11 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 Use the index laws to simplify the following. Leave your answers in index form.

a $3^4 \times 3^5$ **b** $\left(3^6\right)^2$ **c** $\dfrac{3^{10}}{3^5}$ **d** $2^5 \times 3^5$

2 Write as fractions:

a 5^{-1} **b** 10^{-2} **c** x^{-3} **d** 3^{-x}

3 Simplify:

a $9^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $8^{\frac{2}{3}}$

d $16^{-\frac{1}{2}}$ **e** $27^{-\frac{2}{3}}$ **f** $100^{-\frac{3}{2}}$

4 Simplify:

a $2^x \times 2^{2x}$ **b** $\dfrac{2^{6x}}{2^{2x}}$ **c** $\left(2^{3x}\right)^2$

d $2^x \times 5^x$ **e** $2^{x+1} \times 2^{x+2}$ **f** $\dfrac{2^{3x+2}}{2^{x+3}}$

5 Sketch the graphs of $y = 2^x$ and $y = 2^{-x}$ on the same number plane. Then write down the equation of the line that reflects each graph onto the other graph.

6 Use your calculator to approximate the following, correct to four significant figures:

a e **b** e^4 **c** e^{-2} **d** $e^{\frac{3}{2}}$

7 Simplify:

a $e^{2x} \times e^{3x}$ **b** $e^{7x} \div e^x$ **c** $\dfrac{e^{2x}}{e^{6x}}$ **d** $\left(e^{3x}\right)^3$

8 Sketch the graph of each function on a separate number plane, and state its range.

a $y = e^x$ **b** $y = e^{-x}$ **c** $y = e^x + 1$ **d** $y = e^{-x} - 1$

9 Differentiate:

a $y = e^x$ **b** $y = e^{3x}$ **c** $y = e^{x+3}$ **d** $y = e^{2x+3}$

e $y = e^{-x}$ **f** $y = e^{-3x}$ **g** $y = e^{3-2x}$ **h** $y = 3e^{2x+5}$

i $y = 4e^{\frac{1}{2}x}$ **j** $y = e^{x^3}$ **k** $y = e^{x^3-3x}$ **l** $y = \dfrac{2e^{6x-5}}{3}$

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Review

10 Write each function as a single power of e, and then differentiate it.

a $y = e^{3x} \times e^{2x}$ **b** $y = \frac{e^{7x}}{e^{3x}}$ **c** $y = \frac{e^{x}}{e^{4x}}$ **d** $y = \left(e^{-2x}\right)^3$

11 Find the gradient of the tangent to the curve $y = e^{2x}$ at the point $(0, 1)$.

12 Find the equation of the tangent to the curve $y = e^x$ at the point where $x = 2$, and find the x-intercept and y-intercept of this tangent.

13 Use the appropriate button on your calculator to approximate these expressions correct to four decimal places.

a $\log_{10} 27$ **b** $\log_{10} \frac{1}{2}$ **c** $\log_e 2$ **d** $\log_e 14$

14 Write each equation below in logarithmic form. Then use the appropriate button on your calculator to approximate x correct to four decimal places.

a $10^x = 15$ **b** $10^x = 3$ **c** $e^x = 7$ **d** $e^x = \frac{1}{3}$

15 Use the fact that $y = e^x$ and $y = \log_e x$ are inverse functions to simplify:

a $\log_e e^5$ **b** $\log_e e^{-\frac{1}{4}}$ **c** $e^{\log_e 3}$ **d** $e^{\log_e \frac{1}{5}}$

16 Use the log laws to simplify:

a $e \log_e e$ **b** $\log_e e^3$ **c** $\log_e \frac{1}{e}$ **d** $2e \log_e \sqrt{e}$

17 Sketch graphs of these functions, clearly indicating the vertical asymptote in each case.

a $y = \log_2 x$ **b** $y = -\log_2 x$ **c** $y = \log_2 (x - 1)$ **d** $y = \log_2 (x + 3)$

18 Sketch graphs of these functions, clearly indicating the vertical asymptote in each case.

a $y = \log_e x$ **b** $y = \log_e (-x)$ **c** $y = \log_e (x - 2)$ **d** $y = \log_e x + 1$

19 The lizards on Goanna Island are gradually dying out. The predicted population P of lizards, t years after the first observations were made, is $P = P_0 e^{-0.01t}$, where P_0 is the initial population.

a At what rate is the population changing?

b At what rate is the population changing 45 years later? Answer correct to two significant figures, in terms of P_0.

c What percentage of the original population remains 45 years later, correct to the nearest 1%?

d When will the population decline to 10% of the original population, correct to the nearest year?

20 Express in radians in terms of π:

a 180° **b** 20° **c** 240° **d** 315°

21 Express in degrees:

a $\frac{\pi}{6}$ **b** $\frac{3\pi}{5}$ **c** 3π **d** $\frac{5\pi}{3}$

22 Find the exact value of:

a $\sin \frac{\pi}{3}$ **b** $\tan \frac{5\pi}{6}$

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23 Solve for $0 \le x \le 2\pi$:

a $\cos x = \frac{1}{\sqrt{2}}$ **b** $\tan x = -\sqrt{3}$

24 Solve each equation for $0 \le \theta \le 2\pi$ by reducing it to a quadratic equation in u. Give your solutions in terms of π, or approximated correct to two decimal places, as appropriate.

a $2\sin^2\theta + \sin\theta = 0$ **b** $\cos^2\theta - \cos\theta - 2 = 0$ **c** $2\tan^2\theta + 5\tan\theta - 3 = 0$

25 A circle has radius 12 cm. Find, in exact form:

a the length of an arc that subtends an angle at the centre of 45°,

b the area of a sector in which the angle at the centre is 60°.

26 A chord of a circle of radius 8 cm subtends an angle at the centre of 90°. Find, correct to three significant figures, the area of the minor segment cut off by the chord.

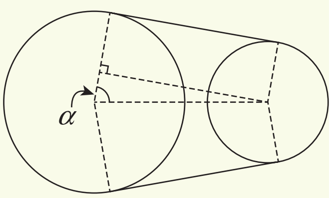
27 Find, correct to the nearest minute, the angle subtended at the centre of a circle of radius 5 cm by an arc of length 13 cm.

28 Find, correct to three significant figures, the radius of the circle in which a sector of area $20\,\text{cm}^2$ subtends an angle of 50° at the centre.

29 The diameters of two circular pulleys are 6 cm and 12 cm, and their centres are 10 cm apart.

a Calculate the angle α in radians, correct to four decimal places.

b Hence find, in centimetres correct to one decimal place, the length of a taut belt around the two pulleys.



30 **a** Which of the six trigonometric graphs have amplitudes, and what are they?

b Which of the six trigonometric graphs are periodic, and what are their periods?

c Which of the six trigonometric functions are odd, and which are even?

31 **a** Give the smallest positive value of θ for which $\sin(x - \theta) = \cos x$.

b Give the smallest positive value of θ for which $\cos(x - \theta) = \sin x$.

c What is the smallest positive value of x for which $\sin x = \cos x$?

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12 Probability

Probability arises when one performs an experiment that has various possible outcomes, but there is insufficient information to predict which of these outcomes will occur. The classic examples of this are tossing a coin, throwing a die, and drawing a card from a pack. Probability, however, is involved in almost every experiment done in science, and is fundamental to understanding statistics.

This chapter reviews some basic ideas of probability and develops a more systematic approach to solving probability problems. It concludes with the new topic of conditional probability.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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12A Probability and sample spaces

The first task is to develop a workable formula for probability that can serve as the foundation for the topic. We will do this only when the possible results of an experiment can be divided into a finite number of equally likely possible outcomes.

Equally likely possible outcomes

The idea of equally likely possible outcomes is well illustrated by the experiment of throwing a die and recording the number shown. (A *die*, plural *dice*, is a cube with its corners and edges rounded so that it rolls easily, and with the numbers 1–6 printed on its six sides.) There are six *possible outcomes*: $1,\ 2,\ 3,\ 4,\ 5,\ 6$. This is a complete list of the possible outcomes, because each time the die is rolled, one and only one of these outcomes will occur.

Provided that we believe the die to be completely symmetric and not *biased* in any way, there is no reason for us to expect that any one outcome is more likely to occur than any of the other five. We may thus regard these six possible outcomes as *equally likely possible outcomes*.

With the results of the experiment now divided into six equally likely possible outcomes, the probability $\frac{1}{6}$ is assigned to each of these six outcomes. Notice that these six probabilities are equal, and that they all add up to 1.

1 EQUALLY LIKELY POSSIBLE OUTCOMES

Suppose that the possible results of an experiment can be divided into a finite number n of *equally likely possible outcomes*. This means that

- one and only one of these n outcomes will occur, and
- we have no reason to expect one outcome to be more likely than another.

Then the probability $\dfrac{1}{n}$ is assigned to each of these equally likely possible outcomes.

Randomness

The explanations above assumed that the terms ‘equally likely’ and ‘more likely’ already have a meaning in the mind of the reader. There are many ways of interpreting these words. In the case of a thrown die, one could interpret the phrase ‘equally likely’ as a statement about the natural world, in this case that the die is perfectly symmetric. Alternatively, we could interpret it as saying that we lack entirely the knowledge to make any statement of preference for one outcome over another.

The word *random* can be used here. In the context of equally likely possible outcomes, saying that a die is thrown ‘randomly’ means that we are justified in assigning the same probability to each of the six possible outcomes. In a similar way, a card can be drawn ‘at random’ from a pack, or a queue of people can be formed in a ‘random order’.

The basic formula for probability

Suppose that a throw of at least 3 on a die is needed to win a game. Getting at least 3 is called the particular *event* under discussion. The outcomes 3, 4, 5 and 6 are called *favourable* for this event, and the other two possible outcomes 1 and 2 are called *unfavourable*. The probability assigned to getting a score of at least 3 is then

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$$P(\text{scoring at least 3}) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} = \frac{4}{6} = \frac{2}{3}.$$

Again, this is easily generalised.

2 THE BASIC FORMULA FOR PROBABILITY

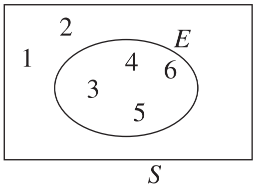
If the results of an experiment can be divided into a finite number of equally likely possible outcomes, some of which are favourable for a particular event and the others unfavourable, then:

$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

An *experiment* always includes what is recorded. For example, 'tossing a coin and recording heads or tails' is a different experiment from 'tossing a coin and recording its distance from the wall'.

The sample space and the event space

Venn diagrams and the language of sets are a great help when explaining probability or solving probability problems. Section 12C will present a short account of sets and Venn diagrams, but at this stage the diagrams should be self-explanatory.



The Venn diagram to the right shows the six equally likely possible outcomes when a die is thrown. The set of all these possible outcomes is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

This set S is called a *uniform sample space*, and is represented by the outer rectangular box. The word 'uniform' indicates that the six possible outcomes are all equally likely, and the word 'sample' refers to the fact that running the experiment can be regarded as 'sampling'. The event 'scoring at least 3' is identified with the set

$$E = \{3, 4, 5, 6\},$$

which is called the *event space*. It is represented by the ellipse, which is inside the rectangle because E is a subset of S.

Sample spaces and uniform sample spaces

In general, a sample space need not consist of equally likely possible outcomes. The only condition is that one and only one of the possible outcomes must occur when the experiment is performed.

A finite sample space is called *uniform* if it does consist of equally likely possible outcomes, as when a die is thrown.

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3 THE SAMPLE SPACE AND THE EVENT SPACE

Sample space:
A set of possible outcomes of an experiment is called a *sample space* if:
- one and only one of these outcomes will occur.

Uniform sample space:
A finite sample space is called *uniform* if:
- all its possible outcomes are equally likely.

Event space:
The set of all favourable outcomes is called the *event space*.

The assumptions of the chapter

First, the sample spaces in this chapter are assumed to be finite — this qualification applies without further mention throughout the chapter. Infinite sample spaces appear in the next chapter and in Year 12.

Secondly, the discussions are restricted to situations where the results of experiment can be reduced to a set of equally likely possible outcomes, that is, to a uniform sample space (assumed finite, as discussed above).

Thus the basic probability formula can now be restated in set language as:

4 THE BASIC FORMULA FOR PROBABILITY

Suppose that an event E is a subset of a uniform sample space S. Then

$$P(E) = \frac{|E|}{|S|},$$

where the symbols $|E|$ and $|S|$ mean the number of members of E and S.

Probabilities involving playing cards

So many questions in probability involve a pack of playing cards that anyone studying probability needs to be familiar with them. You are encouraged to acquire some cards and play some simple games with them. A pack of cards consists of 52 cards organised into four *suits*, each containing 13 cards. The four suits are:

two black suits:	♣ clubs	♠ spades
two red suits:	♢ diamonds	♡ hearts

Each of the four suits contains 13 cards:

A (ace), 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king).

An ace is also regarded as a 1. Three cards in each suit are called *court cards* (or *picture cards* or *face cards*) because they depict people in the royal court:

J (Jack), Q (Queen), K (King).

It is assumed that when a pack of cards is shuffled, the order is totally *random*, meaning that there is no reason to expect any one ordering of the cards to be more likely to occur than any other.

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Example 1 12A

A card is drawn at random from a standard pack of 52 playing cards. Find the probability that the card is:

a the seven of hearts,
b any heart,
c any seven,
d any red card,
e any court card (that is, a jack, a queen, or a king),
f any green card,
g any red or black card.

SOLUTION

In each case, there are 52 equally likely possible outcomes.

a There is 1 seven of hearts, so $P(7\heartsuit) = \frac{1}{52}$.

b There are 13 hearts, so $P(\text{heart}) = \frac{13}{52} = \frac{1}{4}$.

c There are 4 sevens, so $P(\text{seven}) = \frac{4}{52} = \frac{1}{13}$.

d There are 26 red cards, so $P(\text{red card}) = \frac{26}{52} = \frac{1}{2}$.

e There are 12 court cards, so $P(\text{court card}) = \frac{12}{52} = \frac{3}{13}$.

f No card is green, so $P(\text{green card}) = \frac{0}{52} = 0$.

g All 52 cards are red or black, so $P(\text{red or black card}) = \frac{52}{52} = 1$.

Impossible and certain events

Parts **f** and **g** of Example 1 were intended to illustrate the probabilities of events that are impossible or certain.

- Getting a green card is impossible because there are no green cards. Hence there are no favourable outcomes, and the probability is 0.
- Getting a red or black card is certain because all the cards are either red or black. Hence all possible outcomes are favourable, and the probability is 1.
- For the other five events, the probability lies between 0 and 1.

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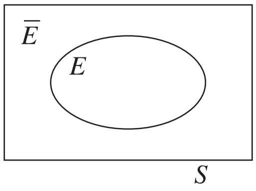
5 IMPOSSIBLE AND CERTAIN EVENTS

- An impossible event has probability zero.
- A certain event has probability 1.
- The probability of any other event E lies in the interval $0 < P(E) < 1$.

Complementary events and the word 'not'



It is often easier to find the probability that an event does *not* occur than the probability that it does occur. The *complement* of an event E is the event 'E does *not* occur'. It is written as $\overline{E}$, or sometimes as E' or E^c.

Let S be a uniform sample space of the experiment. The complementary event $\overline{E}$ is represented by the region outside the circle in the Venn diagram to the right. Because $|\overline{E}| = |S| - |E|$, it follows that

$$P\left(\overline{E}\right) = \frac{|\overline{E}|}{|S|} = \frac{|S| - |E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|} = 1 - P(E).$$

6 COMPLEMENTARY EVENTS AND THE WORD 'NOT'

- The *complement* $\overline{E}$ of an event E is the event 'E does *not* occur', normally read as 'not E'.
- $P\left(\overline{E}\right) = 1 - P(E)$.
- The symbols E' and E^c are also used for the complementary event.
- Always consider using complementary events in problems, particularly when the word 'not' occurs.

In Section 12C, the *complement* $\overline{E}$ of a set E will be defined to be the set of things in S but *not* in E. The notation $\overline{E}$ for complementary event is quite deliberately the same notation as that for the complement of a set.

Example 2 12A

A card is drawn at random from a pack of playing cards. Find the probability:

a that it is not a spade,

b that it is not a court card (jack, queen or king),

c that it is neither a red two nor a black six.

SOLUTION

a Thirteen cards are spades, so

$$P(\text{spade}) = \tfrac{13}{52} = \tfrac{1}{4}.$$

Hence, using complementary events,

$$P(\text{not a spade}) = 1 - \tfrac{1}{4} = \tfrac{3}{4}.$$

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b Twelve cards are court cards, so $P\,(\text{court card}) = \frac{12}{52}$

$= \frac{3}{13}.$

Hence, using complementary events, $P\,(\text{court card}) = 1 - \frac{3}{13}$

$= \frac{10}{13}.$

c There are two red 2s and two black 6s, so $P\,(\text{red 2 or black 6}) = \frac{4}{52}$

$= \frac{1}{13}.$

Hence, using complementary events, $P\,(\text{neither}) = 1 - \frac{1}{13}$

$= \frac{12}{13}.$

Invalid arguments

Arguments offered in probability theory can be invalid for all sorts of subtle reasons, and it is common for a question to ask for comment on a given argument. It is most important in such a situation that any fallacy in the given argument be explained — it is not sufficient to offer only an alternative argument with a different conclusion.

Example 3 12A

Comment on the validity of this argument.

‘Brisbane is one of 16 League teams, so the probability that Brisbane wins the premiership is $\frac{1}{16}$.’

SOLUTION

[Identifying the fallacy]

The division into 16 possible outcomes is correct (assuming that a tie for first place is impossible), but no reason has been offered as to why each team is equally likely to win, so the argument is invalid.

[Offering a replacement argument]

If a team is selected at random from the 16 teams, then the probability that it is the premiership-winning team is $\frac{1}{16}$. Also, if someone knows nothing whatsoever about League, then for him the statement is correct. But these are different experiments.

Note: It is difficult to give a satisfactory account of this situation, indeed the idea of exact probabilities seems to have no meaning. Those with knowledge of the game would have some idea of ranking the 16 teams in order from most likely to win to least likely to win. If there is an organised system of betting, one may, or may not, agree to take this as an indication of the community’s collective wisdom on Brisbane’s chance of winning, and nominate it as ‘the probability’.

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Experimental probability — relative frequency

When a drawing pin is thrown, there are two possible outcomes, point-up and point-down. But these two outcomes are not equally likely, and there seems to be no way to analyse the results of the experiment into equally likely possible outcomes.

In the absence of a fancy argument from physics about rotating pins falling on a smooth surface, however, some estimate of probability can be gained by performing the experiment many times. We can then use the *relative frequencies* of the drawing pin landing pin-up and pin-down as estimates of the probabilities, as in Example 4 below (relative frequencies may have been introduced in earlier years). But an exact value is inaccessible by these methods, indeed the very idea of an exact value may well have no meaning.

The questions in Example 4 could raise difficult issues beyond this course, but the intention here is only that the questions be answered briefly in a common-sense manner. Chapter 13 will pursue these issues in a little more detail, particularly in Section 13D.

Example 4 — 12A

A drawing pin was thrown 400 times and fell point-up 362 times.

a What were the relative frequencies of the drawing pin falling point-up, and of falling point-down?

b What probability does this experiment suggest for the result 'point-up'?

c A machine later repeated the experiment 1 000 000 times, and the pin fell point-up 916 203 times. Does this change your estimate in part **b**?

SOLUTION

a $\text{Relative frequency of point-up} = \dfrac{362}{400} = 0.905$

$\text{Relative frequency of point-down} = \dfrac{38}{400} = 0.095$

b These results suggest that $P(\text{point-up}) \doteqdot 0.905$, but with only 400 trials, there would be little confidence in this result past the second decimal place, or even the first decimal place, because different *runs* of the same experiment would be expected to differ by small numbers. The safest conclusion would be that $P(\text{point-up}) \doteqdot 0.9$.

c The new results suggest that the estimate of the probability can now be refined to $P(\text{point-up}) \doteqdot 0.916$ — we can now be reasonably sure that the rounding to 0.9 in part **a** gave a value that was too low. (Did the machine throw the pin in a random manner, whatever that may mean?)

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Exercise 12A

FOUNDATION

Note: The letter Y is normally classified as a consonant.

1 A pupil has 3 tickets in the class raffle. If there are 60 tickets in the raffle and one ticket is drawn, find the probability that the pupil:
- a wins,
- b does not win.

2 A coin is tossed. Write down the probability that it shows:
- a a head,
- b a tail,
- c either a head or a tail,
- d neither a head nor a tail.

3 If a die is rolled, find the probability that the uppermost face is:
- a a three,
- b an even number,
- c a number greater than four,
- d a multiple of three.

4 A bag contains five red and seven blue marbles. If one marble is drawn from the bag at random, find the probability that it is:
- a red,
- b blue,
- c green.

5 A bag contains eight red balls, seven yellow balls and three green balls. A ball is selected at random. Find the probability that it is:
- a red,
- b yellow or green,
- c not yellow.

6 In a bag there are four red capsicums, three green capsicums, six red apples and five green apples. One item is chosen at random. Find the probability that it is:
- a green,
- b red,
- c an apple,
- d a capsicum,
- e a red apple,
- f a green capsicum.

7 A letter is chosen at random from the word TASMANIA. Find the probability that it is:
- a an A,
- b a vowel,
- c a letter of the word HOBART.

8 A letter is randomly selected from the 26 letters in the English alphabet (remember that we are regarding Y as a consonant). Find the probability that the letter is:
- a the letter S,
- b a vowel,
- c a consonant,
- d the letter γ,
- e either C, D or E,
- f one of the letters of the word MATHS.

9 A student has a 22% chance of being chosen as a prefect. What is the chance that he will not be chosen as a prefect?

10 When breeding labradors, the probability of breeding a black dog is $\frac{3}{7}$.
- a What is the probability of breeding a dog that is not black?
- b If you bred 56 dogs, how many would you expect not to be black?

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11 A box containing a light bulb has a chance of $\frac{1}{15}$ of holding a defective bulb.

- **a** If 120 boxes were checked, how many would you expect to hold defective bulbs?
- **b** What is the probability that the box holds a bulb that works?

12 When a dice is rolled, the theoretical probability of throwing a six is $\frac{1}{6}$, which is about 17%.

- **a** How many sixes would you expect to get if a die is thrown 60 times?
- **b** In an experiment, a certain die was thrown 60 times and a six turned up 18 times.
  - **i** What was the relative frequency of throwing a six?
  - **ii** What does this experiment suggest is the experimental probability of throwing a six?
  - **iii** Comment on whether the die is *biased* or *fair*.

13 Every day the principal goes early to the tuckshop to buy a sandwich for his lunch. He knows that the canteen makes 400 sandwiches a day — equal numbers of four types: bacon-lettuce-tomato, chicken, ham and cheese, and Vegemite. He always chooses a sandwich at random.

- **a** What is his theoretical probability of obtaining a chicken sandwich?
- **b** Over 20 days, the principal found he had bought 8 chicken sandwiches. What was his relative frequency, and therefore his experimental probability, of obtaining a chicken sandwich?
- **c** Comment on why your answers for the theoretical and experimental probability might differ.

14 A number is selected at random from the integers 1, 2, 3, . . . , 19, 20. Find the probability of choosing:

- **a** the number 4,
- **b** a number greater than 15,
- **c** an even number,
- **d** an odd number,
- **e** a prime number,
- **f** a square number,
- **g** a multiple of 4,
- **h** the number e,
- **i** a rational number.

15 From a standard pack of 52 cards, one card is drawn at random. Find the probability that:

- **a** it is black,
- **b** it is red,
- **c** it is a king,
- **d** it is the jack of hearts,
- **e** it is a club,
- **f** it is a picture card,
- **g** it is a heart or a spade,
- **h** it is a red five or a black seven,
- **i** it is less than a four.

16 A book has 150 pages. The book is randomly opened at a page. Find the probability that the page number is:

- **a** greater than 140,
- **b** a multiple of 20,
- **c** an odd number,
- **d** a number less than 25,
- **e** either 72 or 111,
- **f** a three-digit number.

DEVELOPMENT

17 An integer x, where $1 \leq x \leq 200$, is chosen at random. Determine the probability that it:

- **a** is divisible by 5,
- **b** is a multiple of 13,
- **c** has two digits,
- **d** is a square number,
- **e** is greater than 172,
- **f** has three equal digits.

18 A bag contains three times as many yellow marbles as blue marbles, and no other colours. If a marble is chosen at random, find the probability that it is:

- **a** yellow,
- **b** blue.

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19 Fifty tagged fish were released into a dam known to contain fish. Later a sample of 30 fish was netted from this dam, of which eight were found to be tagged. Estimate the total number of fish in the dam just prior to the sample of 30 being removed.

20 Comment on the following arguments. Identify precisely any fallacies in the arguments, and, if possible, give some indication of how to correct them.

a 'On every day of the year, it either rains or it doesn't. Therefore the chance that it will rain tomorrow is $\frac{1}{2}$.

b 'When the Sydney Swans play Hawthorn, either Hawthorn wins, the Swans win, or the game is a draw. Therefore the probability that the next game between these two teams results in a draw is $\frac{1}{3}$.'

c 'When answering a multiple-choice test in which there are four possible answers to each question, the chance that Peter answers a question correctly is $\frac{1}{4}$.'

d 'A bag contains a number of red, white and black beads. If you choose one bead at random from the bag, the probability that it is black is $\frac{1}{3}$.'

e 'Four players play in a knockout tennis tournament resulting in a single winner. A man with no knowledge of the game or the players declares that one particular player will win his semi-final, but lose the final. The probability that he is correct is $\frac{1}{4}$.'

ENRICHMENT

21 A rectangular field is 60 metres long and 30 metres wide. A cow wanders randomly around the field. Find the probability that the cow is:

a more than 10 metres from the edge of the field,

b not more than 10 metres from a corner of the field.

22 Let $O(0, 0)$, $A(6, 0)$, $B(6, 6)$, $C(0, 6)$ be the vertices of a square $OABC$, and let M be the midpoint of OB. Find the probability that a point chosen at random from the square is:

a farther from O than from M,

b more than twice as far from O as from M.

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12B Sample space graphs and tree diagrams

Many experiments consist of several *stages*. For example, when a die is thrown twice, the two throws can be regarded as two separate stages of the one experiment. This section develops two approaches — graphing and tree diagrams — to investigating the sample space of a multi-stage experiment.

Graphing the sample space

The word ‘sample space’ is used in probability, rather than ‘sample set’, because the sample space of a multi-stage experiment takes on some of the characteristics of a space. In particular, the sample space of a two-stage experiment can be displayed on a two-dimensional graph, and the sample space of a three-stage experiment can be displayed in a three-dimensional graph.

The following example shows how a two-dimensional dot diagram can be used for calculations with the sample space of a die thrown twice.

Example 5 12B

A die is thrown twice. Find the probability that:

a the pair is a double,

b at least one number is four,

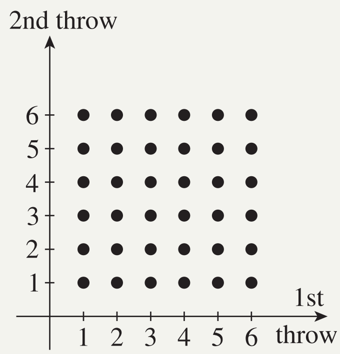
c both numbers are greater than four,

d both numbers are even,

e the sum of the two numbers is six,

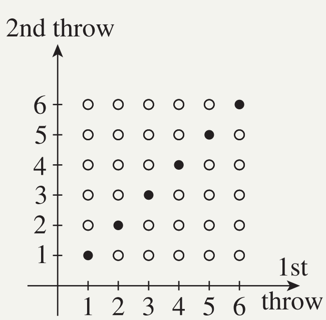
f the sum is at most four.

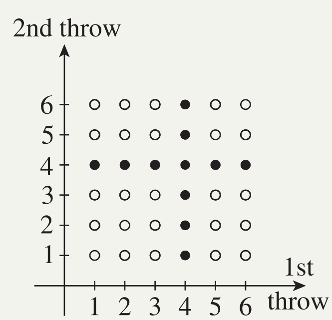
SOLUTION



The horizontal axis in the diagram to the right represents the six possible outcomes of the first throw, and the vertical axis represents the six possible outcomes of the second throw. Each dot is an ordered pair, such as $(3, 5)$, representing the first and second outcomes. The 36 dots therefore represent all 36 different possible outcomes of the two-stage experiment, all equally likely. This is the full sample space.

Parts **a**–**f** can now be answered by counting the dots representing the various event spaces.





a There are 6 doubles,

$$\text{so } P(\text{double}) = \frac{6}{36} \quad \text{(count the dots)}$$
$$= \frac{1}{6}.$$

b 11 pairs contain a 4,

$$\text{so } P(\text{at least one is a 4}) = \frac{11}{36}.$$

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c 4 pairs consist only of 5 or 6, so $P(\text{both greater than } 4) = \frac{4}{36}$

$= \frac{1}{9}.$

d 9 pairs have two even members, so $P(\text{both even}) = \frac{9}{36}$

$= \frac{1}{4}.$

e 5 pairs have sum 6, so $P(\text{sum is } 6) = \frac{5}{36}.$

f 6 pairs have sum 2, 3 or 4, so $P(\text{sum at most } 4) = \frac{6}{36}$

$= \frac{1}{6}.$

7 GRAPHING THE SAMPLE SPACE OF A TWO-STAGE EXPERIMENT

- The horizontal axis represents the first stage of the experiment.
- The vertical axis represents the second stage.

Complementary events in multi-stage experiments

Example 6 continues the two-stage experiment in Example 5, but this time it involves working with complementary events.

Example 6 12B

A die is thrown twice.

a Find the probability of failing to throw a double six.

b Find the probability that the sum is not seven.

SOLUTION

The same sample space can be used as in worked Example 5.

a The double six is one outcome amongst 36,

so $P(\text{double six}) = \frac{1}{36}.$

Hence $P(\text{not throwing a double six}) = 1 - P(\text{double six})$

$= \frac{35}{36}.$

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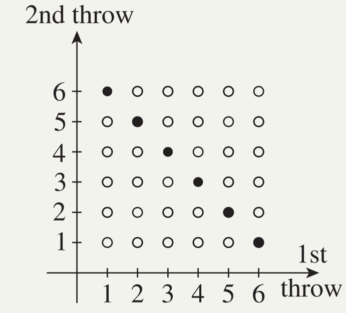
b The graph to the right shows that there are six outcomes giving a sum of seven,

so $\quad P(\text{sum is seven}) = \frac{6}{36}$

$= \frac{1}{6}.$

Hence $P(\text{sum is not seven}) = 1 - P(\text{sum is seven})$

$= \frac{5}{6}.$



Tree diagrams

Listing the sample space of a multi-stage experiment can be difficult, and the dot diagrams of the previous paragraph are hard to draw in more than two dimensions. Tree diagrams provide a very useful alternative way to display the sample space. Such diagrams have a column for each stage, plus an initial column labelled 'Start' and a final column listing the possible outcomes.

Example 7 — 12B

A three-letter word is chosen in the following way. The first and last letters are chosen from the three vowels 'A', 'O' and 'U', with repetition not allowed, and the middle letter is chosen from 'L' and 'M'. List the sample space, then find the probability that:

a the word is 'ALO',

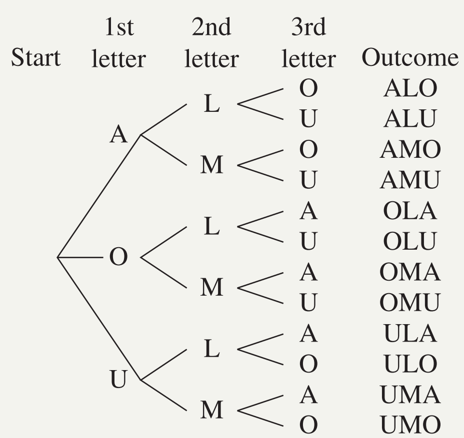
b the letter 'O' does not occur,

c 'M' and 'U' do not both occur,

d the letters are in alphabetical order.

SOLUTION

The tree diagram to the right lists all 12 equally likely possible outcomes. The two vowels must be different, because repetition was not allowed.



a $P(\text{'ALO'}) = \frac{1}{12}$

b $P(\text{no 'O'}) = \frac{4}{12}$

$= \frac{1}{3}$

c $P(\text{not both 'M' and 'U'}) = \frac{8}{12}$

$= \frac{2}{3}$

d $P(\text{alphabetical order}) = \frac{4}{12}$

$= \frac{1}{3}$

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8 A TREE DIAGRAM OF A MULTI-STAGE EXPERIMENT

- The first column is headed 'Start'.
- The last column is headed 'Output', and lists all the paths through the tree.
- Each other column represents all the equally likely possible outcomes in the particular stage of the experiment.

The meaning of 'word'

In Example 7 and throughout this chapter, 'word' simply means an arrangement of letters — the arrangement doesn't have to have any meaning or be a word in the dictionary. Thus 'word' simply becomes a convenient device for discussing arrangements of things in particular orders.

Invalid arguments

Example 8 illustrates another invalid argument in probability. As always, the solution first offers an explanation of the fallacy, before then offering an alternative argument with a different conclusion.

Example 8 12B

Comment on the validity of this argument.

'When two coins are tossed together, there are three outcomes: two heads, two tails, and one of each. Hence the probability of getting one of each is $\frac{1}{3}$.'

SOLUTION

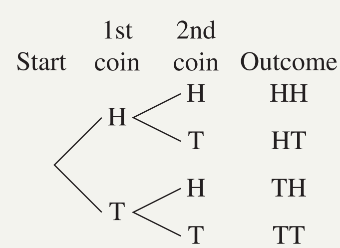
[Identifying the fallacy]

The division of the results into the three given outcomes is correct, but no reason is offered as to why these outcomes are equally likely.

[Supplying the correct argument]

On the right is a tree diagram of the sample space. It divides the results of the experiment into four *equally likely* possible outcomes. Two of these outcomes, HT and TH, are favourable to the event 'one of each', so

$$P(\text{one of each}) = \tfrac{2}{4} = \tfrac{1}{2}.$$



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Exercise 12B

FOUNDATION

1 A fair coin is tossed twice.
  a Use a tree diagram to list the four possible outcomes.
  b Hence find the probability that the two tosses result in:
    i two heads,
    ii a head and a tail,
    iii a head on the first toss and a tail on the second.

2 A coin is tossed and a die is thrown.
  a Use a tree diagram to list all 12 possible outcomes.
  b Hence find the probability of obtaining:
    i a head and an even number,
    ii a tail and a number greater than 4,
    iii a tail and a number less than 4,
    iv a head and a prime number.

3 Two tiles are chosen at random, one after the other, from three Scrabble tiles T, O and E, and placed in order on the table.
  a List the six possible outcomes in the sample space.
  b Find the probability of choosing:
    i two vowels,
    ii a consonant then a vowel,
    iii a T.

4 From a group of four students, Anna, Bill, Charlie and David, two are chosen at random, one after the other, to be on the Student Representative Council.
  a List the 12 possible outcomes in the sample space.
  b Hence find the probability that:
    i Bill and David are chosen,
    ii Anna is chosen,
    iii Charlie is chosen but Bill is not,
    iv neither Anna nor David is selected.
    v Bill is chosen first.
    vi Charlie is not chosen second.

5 From the digits 2, 3, 8 and 9, a two-digit number is formed in which no digit is repeated.
  a Use a tree diagram to list the possible outcomes.
  b If the number was formed at random, find the probability that it is:
    i the number 82,
    ii a number greater than 39,
    iii an even number,
    iv a multiple of 3,
    v a number ending in 2,
    vi a perfect square.

6 A captain and vice-captain of a cricket team are to be chosen from Amanda, Belinda, Carol, Dianne and Emma.
  a Use a tree diagram to list the possible pairings, noting that order is important.
  b If the choices were made at random, find the probability that:
    i Carol is captain and Emma is vice-captain,
    ii Belinda is either captain or vice-captain,
    iii Amanda is not selected for either position,
    iv Emma is vice-captain.

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7 A coin is tossed three times. Draw a tree diagram to illustrate the possible outcomes. Then find the probability of obtaining:

a three heads, **b** a head and two tails, **c** at least two tails,
d at most one head, **e** more heads than tails, **f** a head on the second toss.

8 A green die and a red die are thrown simultaneously. List the set of 36 possible outcomes on a two-dimensional graph and hence find the probability of:

a obtaining a three on the green die, **b** obtaining a four on the red die,
c a double five, **d** a total score of seven,
e a total score greater than nine, **f** an even number on both dice,
g at least one two, **h** neither a one nor a four appearing,
i a five and a number greater than three, **j** the same number on both dice.

DEVELOPMENT

9 Suppose that the births of boys and girls are equally likely.

a In a family of two children, determine the probability that there are:
i two girls, **ii** no girls, **iii** one boy and one girl.

b In a family of three children, determine the probability that there are:
i three boys, **ii** two girls and one boy, **iii** more boys than girls.

10 A coin is tossed four times. Find the probability of obtaining:

a four heads, **b** exactly three tails, **c** at least two heads,
d at most one head, **e** two heads and two tails, **f** more tails than heads.

11 A hand of five cards contains a ten, jack, queen, king and ace. From the hand, two cards are drawn in succession, the first card not being replaced before the second card is drawn. Find the probability that:

a the ace is drawn,
b the king is not drawn,
c the queen is the second card drawn.

12 Three-digit numbers are formed from the digits 2, 5, 7 and 8, without repetition.

a Use a tree diagram to list all the possible outcomes. How many are there?
b Hence find the probability that the number is:
i greater than 528, **ii** divisible by 3,
iii divisible by 13, **iv** prime.

ENRICHMENT

13 If a coin is tossed n times, where $n > 1$, find the probability of obtaining:

a n heads,
b at least one head and at least one tail.
c at most half of them as heads, assuming the coin is tossed an odd number of times.

14 A coin with radius 1 cm is thrown down on a grid whose lines are 4 centimetres apart. What is the probability that the coin does not intersect with any grid line?

15 What is the probability that the third card drawn from a pack is a diamond? Can you find a simple solution to this problem?

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12C Sets and Venn diagrams

This section is a brief account of sets and Venn diagrams for those who have not met these ideas already. The three key ideas needed in probability are the intersection of sets, the union of sets, and the complement of a set.

Logic is very close to the surface when we talk about sets and Venn diagrams. The three ideas of intersection, union and complement mentioned above correspond very precisely to the words 'and', 'or' and 'not'.

Listing sets and describing sets

A *set* is a collection of things called *elements* or *members*. When a set is specified, it needs to be made absolutely clear what things are its members. This can be done by *listing* the members inside curly brackets. For example,

$$S = \{1, 3, 5, 7, 9\},$$

which is read as 'S is the set whose members are 1, 3, 5, 7 and 9'.

It can also be done by *writing a description* of the members inside curly brackets. For example,

$$T = \{\text{odd integers from 0 to 10}\},$$

read as 'T is the set of odd integers from 0 to 10'.

Equal sets

Two sets are called *equal* if they have exactly the same members. Hence the sets S and T in the previous paragraph are equal, which is written as $S = T$. The order in which the members are written doesn't matter at all, and neither does repetition. For example,

$$\{1, 3, 5, 7, 9\} = \{3, 9, 7, 5, 1\} = \{5, 9, 1, 3, 7\} = \{1, 3, 1, 5, 1, 7, 9\}.$$

The size of a set

A set may be *finite*, like the set above of positive odd numbers less than 10, or *infinite*, like the set of all integers. Only finite sets are needed here.

If a set S is finite, then the symbol $|S|$ is used to mean the number of members of S. For example:

If $A = \{5, 6, 7, 8, 9, 10\}$, then $|A| = 6$.
If $B = \{\text{letters in the alphabet}\}$, then $|B| = 26$.
If $C = \{12\}$, then $|C| = 1$.
If $D = \{\text{odd numbers between 1.5 and 2.5}\}$, then $|D| = 0$.

The empty set

The last set D in the list above is called the *empty set*, because it has no members at all. The usual symbol for the empty set is $\varnothing$. There is only one empty set, because any two empty sets have exactly the same members (that is, none at all) and so are equal.

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9 SET TERMINOLOGY AND NOTATION

- A *set* is a collection of *elements*. For example,
  $S = \{1, 3, 5, 7, 9\}$ and $T = \{\text{odd integers from 0 to 10}\}$.
- Two sets are called *equal* if they have exactly the same members.
- The number of elements in a finite set S is written as $|S|$.
- The *empty set* is written as $\varnothing$ — there is only one empty set.

Intersection and union

There are two obvious ways of combining two sets A and B. The *intersection* $A \cap B$ of A and B is the set of everything belonging to A <u>*and*</u> B. The *union* $A \cup B$ of A and B is the set of everything belonging to A <u>*or*</u> B. For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 3, 6\}$, then

$$A \cap B = \{1, 3\}$$
$$A \cup B = \{0, 1, 2, 3, 6\}$$

Two sets A and B are called *disjoint* if they have no elements in common, that is, if $A \cap B = \varnothing$. For example, if $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7\}$, then

$$A \cap B = \varnothing,$$

so A and B are disjoint.

'And' means intersection, 'or' means union

The important mathematical words 'and' and 'or' can be interpreted in terms of union and intersection:

$$A \cap B = \{\text{elements that are in } A \text{ \underline{and} in } B\}$$
$$A \cup B = \{\text{elements that are in } A \text{ \underline{or} in } B\}$$

Note: The word 'or' in mathematics always means 'and/or'. Correspondingly, all the elements of $A \cap B$ are members of $A \cup B$.

10 THE INTERSECTION AND UNION OF SETS

Intersection of sets:

- The *intersection* of two sets A and B is the set of elements in A *and* in B:
  $A \cap B = \{\text{elements in } A \text{ \underline{and} in } B\}$,
- The sets A and B are called *disjoint* when $A \cap B = \varnothing$.

Union of sets:

- The *union* of A and B is the set of elements in A *or* in B:
  $A \cup B = \{\text{elements in } A \text{ \underline{or} in } B\}$.
- The word 'or' in mathematics always means 'and/or'.

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Subsets of sets

A set A is called a *subset* of a set B if every member of A is a member of B. This relation is written as $A \subset B$. For example,

$$\{\text{teenagers in Australia}\} \subset \{\text{people in Australia}\}$$
$$\{2, 3, 4\} \subset \{1, 2, 3, 4, 5\}$$
$$\{\text{vowels}\} \subset \{\text{letters in the alphabet}\}$$

Because of the way subsets have been defined, every set is a subset of itself. Also, the empty set is a subset of every set. For example,

$$\{1, 3, 5\} \subset \{1, 3, 5\}, \text{ and}$$
$$\varnothing \subset \{1, 3, 5\}.$$

The universal set and the complement of a set

A *universal set* is the set of everything under discussion in a particular situation. For example, if $A = \{1, 3, 5, 7, 9\}$, then possible universal sets could be the set of all positive integers less than 11, or the set of all integers, or even the set of all real numbers.

Once a universal set E is fixed, then the *complement* $\overline{A}$ of any set A is the set of all members of that universal set which are <u>not</u> in A. For example,

If $A = \{1, 3, 5, 7, 9\}$ and $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
then $\overline{A} = \{2, 4, 6, 8, 10\}$.

Every member of the universal set is either in A or in $\overline{A}$, but never in both A and $\overline{A}$. This means that

$A \cap \overline{A} = \varnothing$, the empty set, and
$A \cup \overline{A} = E$, the universal set.

'Not' means complement

As mentioned in Section 12A, the important mathematical word 'not' can be interpreted in terms of the complementary set:

$$\overline{A} = \{\text{members of } E \text{ that are } \underline{not} \text{ members of } A\}.$$

11 SUBSETS, THE UNIVERSAL SET, AND COMPLEMENTS

- A set A is a *subset* of a set S, written as $A \subset S$, if every element of A is also an element of S.
- If A and B are sets, then $A \cap B \subset A \cup B$.
- A *universal set* is a conveniently chosen set that contains all the elements relevant to the situation.
- Let A be a subset of a universal set E. The *complement* of A is the set of all elements of E that are <u>not</u> in A,

$$\overline{A} = \{\text{elements of } E \text{ that are } \underline{not} \text{ in } A\}.$$

The notations A' and A^c are also used for the complement of A.

Venn diagrams

A *Venn diagram* is a diagram used to represent the relationship between sets. For example, the four diagrams below represent four different possible relationships between two sets A and B. In each case, the universal set is again $E = \{1, 2, 3, \ldots, 10\}$.

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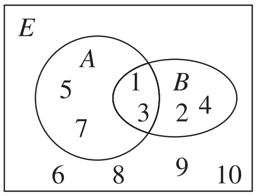
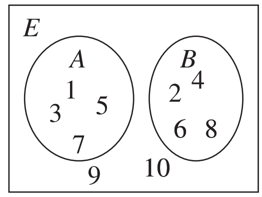
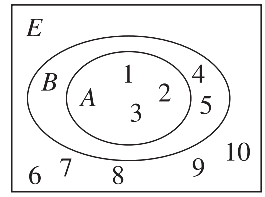


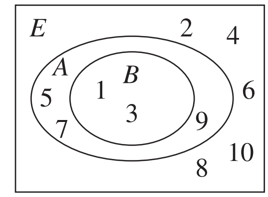
$A = \{1, 3, 5, 7\}$
$B = \{1, 2, 3, 4\}$



$A = \{1, 3, 5, 7\}$
$B = \{2, 4, 6, 8\}$

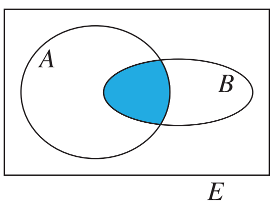


$A = \{1, 2, 3\}$
$B = \{1, 2, 3, 4, 5\}$

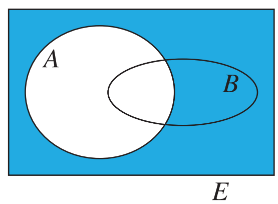


$A = \{1, 3, 5, 7, 9\}$
$B = \{1, 3\}$

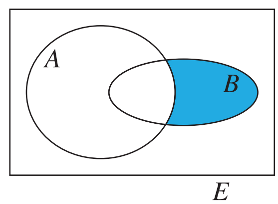
Sets can also be visualised by shading regions of the Venn diagram, as in the following examples:



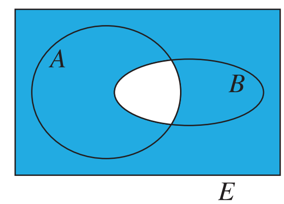
$A \cap B$



$\overline{A}$



$\overline{A} \cap B$

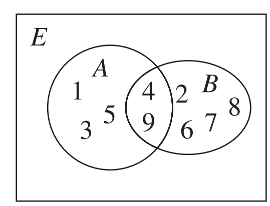


$\overline{A \cap B}$

The counting rule for finite sets

To calculate the size of the union $A \cup B$ of two finite sets, adding the sizes of A and of B will not do, because the members of the intersection $A \cap B$ would be counted twice. Hence $|A \cap B|$ needs to be subtracted again, and the rule is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$



For example, the Venn diagram to the right shows the sets

$$A = \{1, 3, 4, 5, 9\} \quad \text{and} \quad B = \{2, 4, 6, 7, 8, 9\}.$$

From the diagram, $|A \cup B| = 9$, $|A| = 5$, $|B| = 6$ and $|A \cap B| = 2$, and the formula works because

$$9 = 5 + 6 - 2.$$

When two sets are disjoint, there is no overlap between A and B to cause any double counting. With $|A \cap B| = 0$, the counting rule becomes

$$|A \cup B| = |A| + |B|.$$

12 THE COUNTING RULE FOR FINITE SETS

- Let A and B be finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

- Let A and B be disjoint finite sets. Then $|A \cap B| = 0$, so

$$|A \cup B| = |A| + |B|.$$

Problem solving using Venn diagrams

A Venn diagram is a very convenient way to sort out problems involving overlapping sets. But once there are more than a handful of elements, listing the elements is most inconvenient. Instead, we work only with the number of elements in each region, and show that number, in brackets, in the corresponding region.

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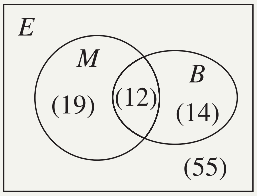
Example 9 12C

100 Sydneysiders were surveyed to find out how many of them had visited the cities of Melbourne and Brisbane. The survey showed that 31 people had visited Melbourne, 26 people had visited Brisbane and 12 people had visited both cities. Find how many people had visited:

a Melbourne or Brisbane,
b Brisbane but not Melbourne,
c only one of the two cities,
d neither city.

SOLUTION

Let M be the set of people who have visited Melbourne, let B be the set of people who have visited Brisbane, and let E be the universal set of all people surveyed. Calculations should begin with the 12 people in the intersection of the two regions. Then the numbers shown in the other three regions of the Venn diagram can easily be found.



a Number visiting Melbourne or Brisbane $= 19 + 14 + 12$
$= 45$

b Number visiting Brisbane but not Melbourne $= 14$

c Number visiting only one city $= 19 + 14$
$= 33$

d Number visiting neither city $= 100 - 45$
$= 55$

Exercise 12C FOUNDATION

1 Write down the following sets by listing their members.
- **a** Odd positive integers less than 10.
- **b** The first six positive multiples of 6.
- **c** The numbers on a die.
- **d** The factors of 20.

2 Find $A \cup B$ and $A \cap B$ for each pair of sets.
- **a** $A = \{1, 3, 5\}$, $B = \{3, 5, 7\}$
- **b** $A = \{1, 3, 4, 8, 9\}$, $B = \{2, 4, 5, 6, 9, 10\}$
- **c** $A = \{\text{h, o, b, a, r, t}\}$, $B = \{\text{b, i, c, h, e, n, o}\}$
- **d** $A = \{\text{j, a, c, k}\}$, $B = \{\text{e, m, m, a}\}$
- **e** $A = \{\text{prime numbers less than } 10\}$, $B = \{\text{odd numbers less than } 10\}$

3 If $A = \{1, 4, 7, 8\}$ and $B = \{1, 2, 4, 5, 7\}$, state whether the following statements are true or false.
- **a** $A = B$
- **b** $|A| = 4$
- **c** $|B| = 6$
- **d** $A \subset B$
- **e** $A \cup B = \{1, 2, 4, 5, 7, 8\}$
- **f** $A \cap B = \{1, 4, 7\}$

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4 If $A = \{1, 3, 5\}$, $B = \{3, 4\}$ and $E = \{1, 2, 3, 4, 5\}$, find:

a $|A|$ **b** $|B|$ **c** $A \cup B$ **d** $|A \cup B|$
e $A \cap B$ **f** $|A \cap B|$ **g** $\overline{A}$ **h** $\overline{B}$

5 If $A = \{$students who study Japanese$\}$ and $B = \{$students who study History$\}$, carefully describe each of the following sets.

a $A \cap B$ **b** $A \cup B$

6 If $A = \{$students with blue eyes$\}$ and $B = \{$students with blond hair$\}$, with universal set $E = \{$students at Clarence High School$\}$, carefully describe each of the following sets.

a $\overline{A}$ **b** $\overline{B}$ **c** $A \cup B$ **d** $A \cap B$

7 List all the subsets of each set.

a $\{$a$\}$ **b** $\{$a, b$\}$ **c** $\{$a, b, c$\}$ **d** $\varnothing$

8 State in each case whether or not $A \subset B$ (that is, whether A is a subset of B).

a $A = \{2, 3, 5\}$, $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
b $A = \{3, 6, 9, 12\}$, $B = \{3, 5, 9, 11\}$
c $A = \{$d, a, n, c, e$\}$, $B = \{$e, d, u, c, a, t, i, o, n$\}$
d $A = \{$a, m, y$\}$, $B = \{$s, a, r, a, h$\}$
e $A = \varnothing$, $B = \{51, 52, 53, 54\}$

DEVELOPMENT

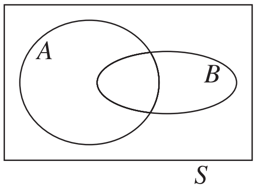
9 Let $A = \{1, 3, 7, 10\}$ and $B = \{4, 6, 7, 9\}$, and take the universal set to be the set $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. List the members of:

a $\overline{A}$ **b** $\overline{B}$ **c** $A \cap B$
d $\overline{A \cap B}$ **e** $A \cup B$ **f** $\overline{A \cup B}$

10 Let $A = \{1, 3, 6, 8\}$ and $B = \{3, 4, 6, 7, 10\}$, and take the universal set to be the set $E = \{1, 2, 3, \dots, 10\}$. List the members of:

a $\overline{A}$ **b** $\overline{B}$ **c** $\overline{A} \cup \overline{B}$ **d** $\overline{A} \cap \overline{B}$
e $A \cup B$ **f** $\overline{A \cup B}$ **g** $A \cap B$ **h** $\overline{A \cap B}$

11 In each part, draw a copy of the diagram to the right and shade the corresponding region.



a $A \cap B$ **b** $\overline{A} \cap B$ **c** $A \cap \overline{B}$
d $A \cup B$ **e** $\overline{A \cup B}$ **f** $\overline{\varnothing}$ ($\varnothing$ is the empty set)

12 Answer true or false.

a If $A \subset B$ and $B \subset A$, then $A = B$.
b If $A \subset B$ and $B \subset C$, then $A \subset C$.

13 Copy and complete:

a If $P \subset Q$, then $P \cup Q = \dots$
b If $P \subset Q$, then $P \cap Q = \dots$

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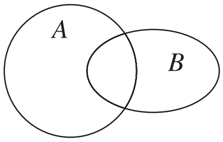
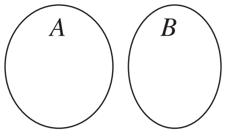
14 Select the Venn diagram that best shows the relationship between each pair of sets A and B:



I



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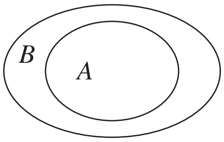
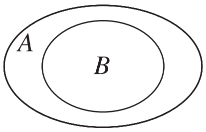


III



IV

- **a** $A = \{\text{Tasmania}\}, B = \{\text{states of Australia}\}$
- **b** $A = \{7, 1, 4, 8, 3, 5\}, B = \{2, 9, 0, 7\}$
- **c** $A = \{\text{l, e, a, r, n}\}, B = \{\text{s, t, u, d, y}\}$
- **d** $A = \{\text{politicians in Australia}\}, B = \{\text{politicians in NSW}\}$.

15 State whether each set is finite or infinite. If it is finite, state its number of members.

- **a** $\{1, 3, 5, \ldots\}$
- **b** $\{0, 1, 2, \ldots, 9\}$
- **c** $\varnothing$
- **d** $\{\text{points on a line}\}$
- **e** $\{n : n \text{ is a positive integer and } 1 < n < 20\}$
- **f** $\{x : 3 \le x \le 5\}$
- **g** $\{\text{a, l, g, e, b, r, a}\}$
- **h** $\{\text{positive multiples of } 7 \text{ that are less than } 100\}$

16 Decide whether each of the following statements is true or false.

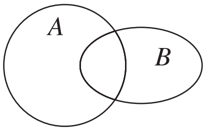
- **a** If two sets have the same number of members, then they are equal.
- **b** If two sets are equal, then they have the same number of members.
- **c** If $A = \{0, 0\}$, then $|A| = 1$.
- **d** $|\{0\}| = 0$
- **e** $1000000 \in \{1, 2, 3, \ldots\}$
- **f** $|\{40, 41, 42, \ldots, 60\}| = 20$

17 In each of the following, A and B represent sets of real numbers. For each part, graph on separate number lines:

i A, **ii** B, **iii** $A \cup B$, **iv** $A \cap B$.

- **a** $A = \{x : x > 0\}, B = \{x : x \le 3\}$
- **b** $A = \{x : x \le -1\}, B = \{x : x > 2\}$
- **c** $A = \{x : -3 \le x < 1\}, B = \{x : -1 \le x \le 4\}$

18 a Explain the counting rule $|A \cup B| = |A| + |B| - |A \cap B|$ by making reference to the Venn diagram to the right.



- **b** If $|A \cup B| = 17$, $|A| = 12$ and $|B| = 10$, find $|A \cap B|$.
- **c** Show that the relationship in part **a** is satisfied when

$$A = \{3, 5, 6, 8, 9\} \quad \text{and} \quad B = \{2, 3, 5, 6, 7, 8\}.$$

19 Use a Venn diagram to solve each problem.

- **a** In a group of 20 people, there are 8 who play the piano, 5 who play the violin and 3 who play both. How many people play neither instrument?
- **b** Each person in a group of 30 plays either tennis or golf. 17 play tennis, while 9 play both. How many play golf?
- **c** In a class of 28 students, there are 19 who like geometry and 16 who like trigonometry. How many like both if there are 5 students who don't like either?

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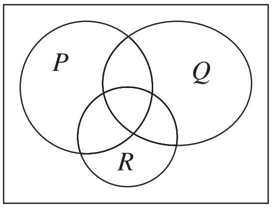
20 Shade each of the following regions on the given diagram. Use a separate copy of the diagram for each part.



a $P \cap Q \cap R$

b $(P \cap R) \cup (Q \cap R)$

c $\overline{P} \cup \overline{Q} \cup \overline{R}$

21 A group of 80 people was surveyed about their approaches to keeping fit. It was found that 20 jog, 22 swim and 18 go to the gym on a regular basis. Further questioning found that 10 people both jog and swim, 11 people both jog and go to the gym, and 6 people both swim and go to the gym. Finally, 43 people do none of these activities. How many of the people do all three?

22 43 people were surveyed and asked whether they drank Coke™, Sprite™ or Fanta™. Three people drank all of these, while four people did not drink any of them. 19 drank Coke™, 21 drank Sprite™ and 17 drank Fanta™. One person drank Coke™ and Fanta™ but not Sprite™. Find the probability that a person selected at random from the group drank Sprite™ only.

ENRICHMENT

23 [The inclusion–exclusion principle]

This rule allows you to count the number of elements contained in the union of the sets without counting any element more than once.

a Given three finite sets A, B and C, find a rule for calculating $|A \cup B \cup C|$. (Hint: Use a Venn diagram and pay careful to attention to the elements in $A \cap B \cap C$.)

b A new car dealer offers three options to his customers: power steering, air conditioning and a CD player. He sold 72 cars without any options, 12 with all three options, 38 included power steering and air conditioning, 25 included power steering and a CD player, 22 included air conditioning and a CD player, 83 included power steering, 55 included air conditioning and 70 included a CD player. Using the formula established in part **a**, how many cars did he sell?

c Extend the formula to the unions of four sets, each with finitely many elements. Is it possible to draw a sensible Venn diagram of four sets?

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12D Venn diagrams and the addition theorem

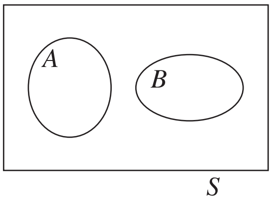
In probability situations where an event is described using the logical words 'and', 'or' and 'not', Venn diagrams and the language of sets are a useful way to visualise the sample space and the event space.

A uniform sample space S is taken as the universal set because it includes all the equally likely possible outcomes. The event spaces are then subsets of S.

Mutually exclusive events and disjoint sets

Two events A and B are called *mutually exclusive* if they cannot both occur. For example:

> If a die is thrown, the events 'throwing a number less than three' and 'throwing a number greater than four' cannot both occur and so are mutually exclusive.
>
> If a card is drawn at random from a pack, the events 'drawing a red card' and 'drawing a spade' cannot both occur and so are mutually exclusive.



In the Venn diagram of such a situation, the two events A and B are represented as disjoint sets (*disjoint* means that their intersection is empty). The event 'A and B' is impossible, and therefore has probability zero.

On the other hand, the event 'A or B' is represented on the Venn diagram by the union $A \cup B$ of the two sets. Because $|A \cup B| = |A| + |B|$ for disjoint sets, it follows that

$$\begin{aligned} P(A \text{ or } B) &= \frac{|A \cup B|}{|S|} \\ &= \frac{|A| + |B|}{|S|} \qquad \text{(because } A \text{ and } B \text{ are disjoint)} \\ &= P(A) + P(B). \end{aligned}$$

13 MUTUALLY EXCLUSIVE EVENTS

Suppose that two mutually exclusive events A and B are subsets of the same uniform sample space S. Then the event 'A or B' is represented by $A \cup B$, and

$$P(A \text{ or } B) = P(A) + P(B).$$

The event 'A and B' cannot occur, and has probability zero.

Example 10 12D

If a die is thrown, find the probability that the result is less than three or greater than four.

SOLUTION

The events 'throwing a number less than three' and 'throwing a number greater than four' are mutually exclusive,

$$\begin{aligned} \text{so } P(\text{less than three or greater than four}) &= P(\text{less than three}) + P(\text{greater than four}) \\ &= \tfrac{1}{3} + \tfrac{1}{3} \\ &= \tfrac{2}{3}. \end{aligned}$$

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Example 11 12D

If a card is drawn at random from a standard pack, find the probability that it is a red card or a spade.

SOLUTION

'Drawing a red card' and 'drawing a spade' are mutually exclusive,

so $P(\text{a red card or a spade}) = P(\text{a red card}) + P(\text{a spade})$

$$= \tfrac{1}{2} + \tfrac{1}{4}$$

$$= \tfrac{3}{4}.$$

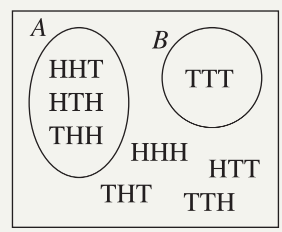
Example 12 12D

[Mutually exclusive events in multi-stage experiments]
If three coins are tossed, find the probability of throwing an odd number of tails.

SOLUTION

Let A be the event 'one tail' and B the event 'three tails'. Then A and B are mutually exclusive, with

$$A = \{\text{HHT}, \text{HTH}, \text{THH}\} \quad \text{and} \quad B = \{\text{TTT}\}.$$



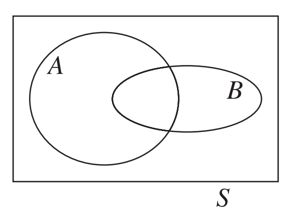
The full sample space has eight members altogether (Question 7 in Exercise 12B lists them all), so

$$P(A \text{ or } B) = \tfrac{3}{8} + \tfrac{1}{8}$$

$$= \tfrac{1}{2}.$$

The events '*A* and *B*' and '*A* or *B*' — the addition rule

More generally, suppose that two events A and B are subsets of the same uniform sample space S, not necessarily mutually exclusive. The Venn diagram of the situation now represents the two events A and B as overlapping sets within the same universal set S.



The event 'A and B' is represented by the intersection $A \cap B$ of the two sets, and the event 'A or B' is represented by the union $A \cup B$.

The general counting rule for sets is $|A \cup B| = |A| + |B| - |A \cap B|$, because the members of the intersection $A \cap B$ are counted in A, and are counted again in B, and so have to be subtracted. It follows then that

$$P(A \text{ or } B) = \frac{|A \cup B|}{|S|}$$

$$= \frac{|A| + |B| - |A \cap B|}{|S|}$$

$$= P(A) + P(B) - P(A \text{ and } B).$$

This rule is the *addition rule* of probability.

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14 THE EVENTS '*A* OR *B*' AND '*A* AND *B*', AND THE ADDITION RULE

Suppose that two events A and B are subsets of the same uniform sample space S.

- The event 'A and B' is represented by the intersection $A \cap B$.
- The event 'A or B' is represented by the union $A \cup B$.

The addition rule:

- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

The symbols ∩ and ∪, and the words 'and', 'or' and 'but'

As explained in the previous section, the word 'and' is closely linked with the intersection of event spaces. For this reason, the event A and B is also written as $A \cap B$, and the following two notations are used interchangeably:

$$P(A \text{ and } B) \qquad \text{means the same as} \qquad P(A \cap B).$$

Similarly, the word 'or' is closely linked with the union of event spaces — the word 'or' always means 'and/or' in logic and mathematics. Thus the event A or B is also written as $A \cup B$, and the following two notations are used interchangeably:

$$P(A \text{ or } B) \qquad \text{means the same as} \qquad P(A \cup B).$$

The word 'but' has the same logical meaning as 'and' — the difference in meaning between 'and' and 'but' is rhetorical, not logical. Using this notation, the addition rule now looks much closer to the counting rule for finite sets:

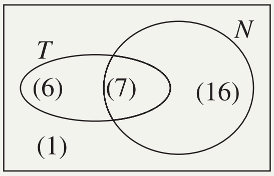
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Example 13 12D

In a class of 30 girls, 13 play tennis and 23 play netball. If 7 girls play both sports, what is the probability that a girl chosen at random plays neither sport?

SOLUTION

Let T be the event 'she plays tennis',
and let N be the event 'she plays netball'.



Then $P(T) = \frac{13}{30}$

$P(N) = \frac{23}{30}$

and $P(N \cap T) = \frac{7}{30}$ (meaning $P(N \text{ and } T)$).

Hence $P(N \cup T) = \frac{13}{30} + \frac{23}{30} - \frac{7}{30}$ (meaning $P(N \text{ or } T)$)

$= \frac{29}{30}$,

and $P(\text{neither sport}) = 1 - P(N \text{ or } T)$

$= \frac{1}{30}$.

Note: An alternative approach is shown in the diagram above. Starting with the 7 girls in the intersection, the numbers 6 and 16 can then be written into the respective regions 'tennis but not netball' and 'netball but not tennis'. These numbers add to 29, leaving only one girl playing neither tennis nor netball.

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Complementary events and the addition rule

In Example 14, the addition rule has to be applied in combination with the idea of complementary events. Some careful thinking is required when the words 'and' and 'or' are combined with 'not'.

Example 14 12D

A card is drawn at random from a pack.

a Find the probability that it is not an ace, but also not a two.

b Find the probability that it is an even number, or a court card, or a red card (the court cards are jack, queen and king).

Note: The word 'or' always means 'and/or' in logic and mathematics. Thus in part **b**, there is no need to add 'or any two of these, or all three of these'.

SOLUTION

a The complementary event $\overline{E}$ is drawing a card that is an ace or a two.

There are eight such cards, so $P(\text{ace or two}) = \frac{8}{52}$

$= \frac{2}{13}$.

Hence $P(\text{not an ace, but also not a two}) = 1 - \frac{2}{13}$

$= \frac{11}{13}$.

(The words 'and' and 'but' have different emphasis, but they have the same logical meaning.)

b The complementary event $\overline{E}$ is drawing a card that is a black odd number less than ten. This complementary event has 10 members:

$$\overline{E} = \{A♣,\ 3♣,\ 5♣,\ 7♣,\ 9♣,\ A♠,\ 3♠,\ 5♠,\ 7♠,\ 9♠\}.$$

There are 52 possible cards to choose, so

$$P(\overline{E}) = \frac{10}{52} = \frac{5}{26}.$$

Hence, using the complementary event formula,

$$P(E) = 1 - P(\overline{E}) = \frac{21}{26}.$$

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Exercise 12D

FOUNDATION

1 A die is rolled. If n denotes the number on the uppermost face, find:

a $P(n = 5)$

b $P(n \neq 5)$

c $P(n = 4 \text{ or } n = 5)$

d $P(n = 4 \text{ and } n = 5)$

e $P(n \text{ is even or odd})$

f $P(n \text{ is neither even nor odd})$

g $P(n \text{ is even and divisible by three})$

h $P(n \text{ is even or divisible by three})$

2 A card is selected from a standard pack of 52 cards. Find the probability that the card:

a is a jack,

b is a ten,

c is a jack or a ten,

d is a jack and a ten,

e is neither a jack nor a ten,

f is black,

g is a picture card,

h is a black picture card,

i is black or a picture card,

j is neither black nor a picture card.

3 Show that the following events A and B are disjoint by showing that $A \cap B = \varnothing$, and check that $P(A \text{ or } B) = P(A) + P(B)$.

a Two coins are thrown and $A = \{\text{two heads}\}$, $B = \{\text{one tail}\}$

b A committee of two is formed by choosing at random from two men, Ricardo and Steve, and one woman, Tania. Suppose $A = \{\text{committee of two men}\}$, $B = \{\text{committee includes Tania}\}$.

4 A die is thrown. Let A be the event that an even number appears. Let B be the event that a number greater than 2 appears.

a Are A and B mutually exclusive?

b Find:

i $P(A)$

ii $P(B)$

iii $P(A \cap B)$

iv $P(A \cup B)$

5 Two dice are thrown. Let a and b denote the numbers rolled. Find:

a $P(a \text{ is odd})$

b $P(b \text{ is odd})$

c $P(a \text{ and } b \text{ are odd})$

d $P(a \text{ or } b \text{ is odd})$

e $P(\text{neither } a \text{ nor } b \text{ is odd})$

f $P(a = 1)$

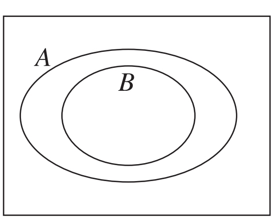
g $P(b = a)$

h $P(a = 1 \text{ and } b = a)$

i $P(a = 1 \text{ or } b = a)$

j $P(a \neq 1 \text{ and } a \neq b)$

6 a



$P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$.

Find:

i $P(\overline{A})$

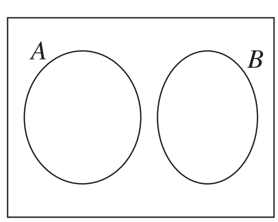
ii $P(\overline{B})$

iii $P(A \text{ and } B)$

iv $P(A \text{ or } B)$

v $P(\text{neither } A \text{ nor } B)$

b



$P(A) = \frac{2}{5}$ and $P(B) = \frac{1}{5}$.

Find:

i $P(\overline{A})$

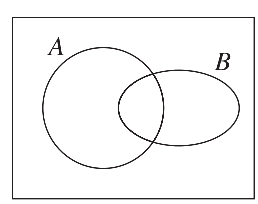
ii $P(\overline{B})$

iii $P(A \text{ or } B)$

iv $P(A \text{ and } B)$

v $P(\text{not both } A \text{ and } B)$

c



$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \text{ and } B) = \frac{1}{6}$.

Find:

i $P(\overline{A})$

ii $P(\overline{B})$

iii $P(A \text{ or } B)$

iv $P(\text{neither } A \text{ nor } B)$

v $P(\text{not both } A \text{ and } B)$

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7 Use the addition rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to answer the following questions.

a If $P(A) = \frac{1}{5}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{15}$, find $P(A \cup B)$.

b If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{5}{6}$, find $P(A \cap B)$.

c If $P(A \cup B) = \frac{9}{10}$, $(A \cap B) = \frac{1}{5}$ and $P(A) = \frac{1}{2}$, find $P(B)$.

d If A and B are mutually exclusive and $P(A) = \frac{1}{7}$ and $P(B) = \frac{4}{7}$, find $P(A \cup B)$.

DEVELOPMENT

8 An integer n is picked at random, where $1 \leq n \leq 20$. The events A, B, C and D are:

A: an even number is chosen, B: a number greater than 15 is chosen,

C: a multiple of 3 is chosen, D: a one-digit number is chosen.

a i Are the events A and B mutually exclusive?

ii Find $P(A)$, $P(B)$ and $P(A \text{ and } B)$, and hence evaluate $P(A \text{ or } B)$.

b i Are the events A and C mutually exclusive?

ii Find $P(A)$, $P(C)$ and $P(A \text{ and } C)$, and hence evaluate $P(A \text{ or } C)$.

c i Are the events B and D mutually exclusive?

ii Find $P(B)$, $P(D)$ and $P(B \text{ and } D)$, and hence evaluate $P(B \text{ or } D)$.

9 In a group of 50 students, there are 26 who study Latin and 15 who study Greek and 8 who study both languages. Draw a Venn diagram and find the probability that a student chosen at random:

a studies only Latin,

b studies only Greek,

c does not study either language.

10 During a game, all 21 members of an Australian Rules football team consume liquid. Some players drink only water, some players drink only Gatorade™ and some players drink both. There are 14 players who drink water and 17 players who drink Gatorade™.

a How many drink both water and Gatorade™?

b If one team member is selected at random, find the probability that:

i he drinks water but not Gatorade™,

ii he drinks Gatorade™ but not water.

11 Each student in a music class of 28 studies either the piano or the violin or both. It is known that 20 study the piano and 15 study the violin. Find the probability that a student selected at random studies both instruments.

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ENRICHMENT

12 List the 25 primes less than 100 . A number is drawn at random from the integers from 1 to 100. Find the probability that:

a it is prime,

b it has remainder 1 after division by 4,

c it is prime and it has remainder 1 after division by 4,

d it is either prime or it has remainder 1 after division by 4.

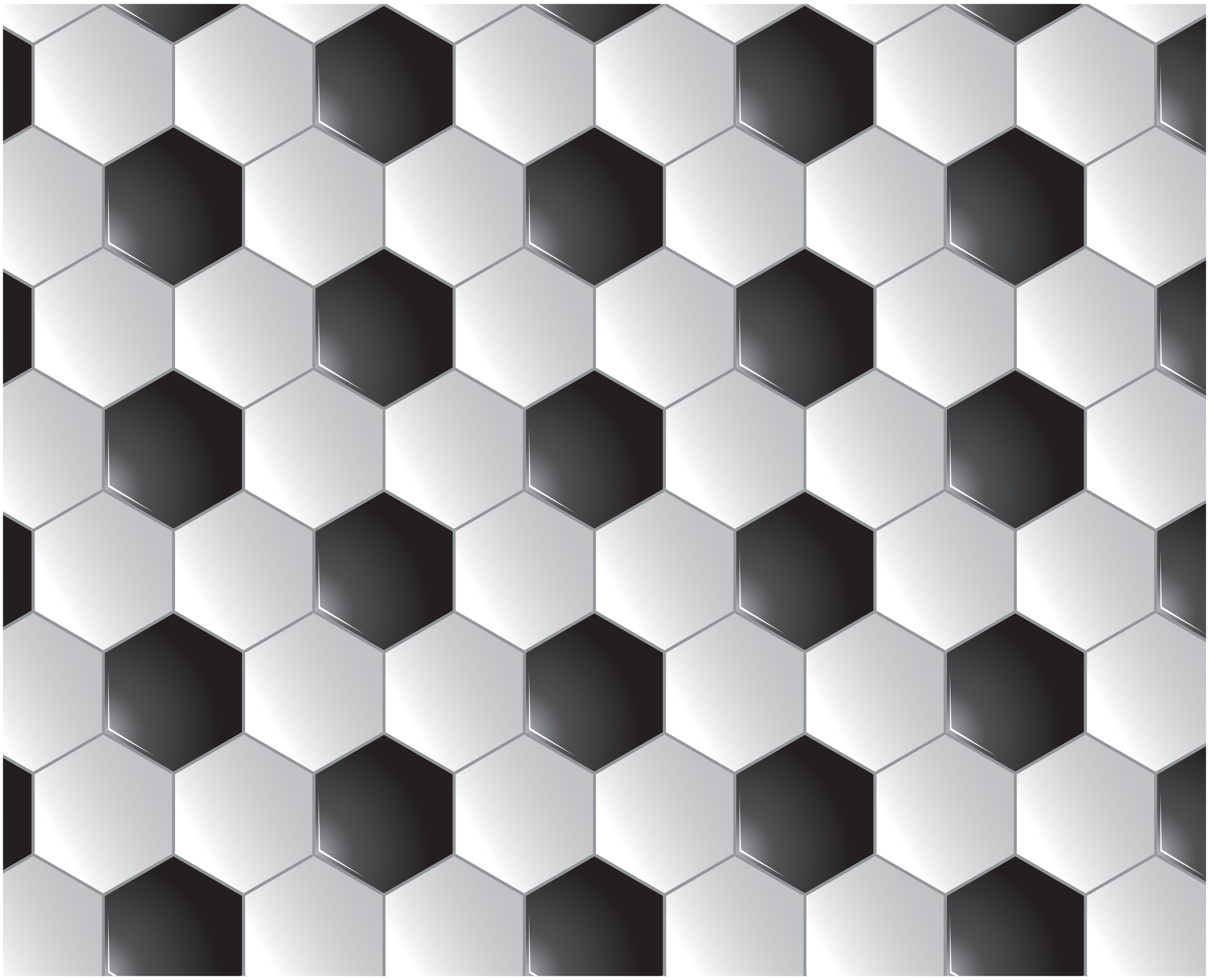
13 A group of 60 students was invited to try out for three sports: rugby, soccer and cross country. Of these, 32 tried out for rugby, 29 tried out for soccer, 15 tried out for cross country, 11 tried out for rugby and soccer, 9 tried out for soccer and cross country, 8 tried out for rugby and cross country, and 5 tried out for all three sports. Draw a Venn diagram and find the probability that a student chosen at random:

a tried out for only one sport,

b tried out for exactly two sports,

c tried out for at least two sports,

d did not try out for a sport.



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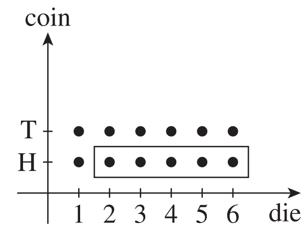
12E Multi-stage experiments and the product rule

This section deals with experiments that have a number of stages. The full sample space of such an experiment can quickly become too large to be conveniently listed, but instead a rule can be developed for multiplying together the probabilities associated with each stage.

Two-stage experiments — the product rule

Here is a simple question about a two-stage experiment:

'Throw a die, then toss a coin. What is the probability of obtaining at least 2 on the die, followed by a head?'



Graphed to the right are the twelve possible outcomes of the experiment, all equally likely, with a box drawn around the five favourable outcomes. Thus

$$P\,(\text{at least 2 and a head}) = \tfrac{5}{12}.$$

Now consider the two stages separately. The first stage is throwing a die, and the desired outcome is A = 'getting at least 2' — here there are six possible outcomes and five favourable outcomes, giving probability $\frac{5}{6}$. The second stage is tossing a coin, and the desired outcome is B = 'tossing a head' — here there are two possible outcomes and one favourable outcome, giving probability $\frac{1}{2}$.

The full experiment then has $6 \times 2 = 12$ possible outcomes and there are $5 \times 1 = 5$ favourable outcomes. Hence

$$P(AB) = \tfrac{5 \times 1}{6 \times 2} = \tfrac{5}{6} \times \tfrac{1}{2} = P(A) \times P(B).$$

Thus the probability of the compound event 'getting at least 2 and a head' can be found by multiplying together the probabilities of the two stages. The argument here can easily be generalised to any two-stage experiment.

15 TWO-STAGE EXPERIMENTS

If A and B are events in successive independent stages of a two-stage experiment, then

$$P(AB) = P(A) \times P(B),$$

where the word 'independent' means that the outcome of one stage does not affect the probabilities of the other stage.

Independent events

The word 'independent' needs further discussion. In the example above, the throwing of the die clearly does not affect the tossing of the coin, so the two events are independent.

Here is a very common and important type of experiment where the two stages are not independent:

'Choose an elector at random from the NSW population. First note the elector's gender. Then ask the elector if he or she voted Labor or non-Labor in the last State election.'

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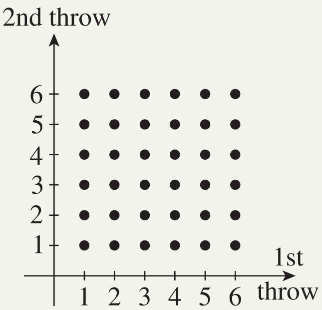
In this example, one might suspect that the gender and the political opinion of a person may not be independent and that there is *correlation* between them. This is often the case, as many opinion polls have shown over the years. More on this in Year 12.

Example 15 — 12E

A pair of dice is thrown twice. What is the probability that the first throw is a double, and the second throw gives a sum of at least 4?

SOLUTION

We saw in Section 12B that when two dice are thrown, there are 36 possible outcomes, graphed in the diagram to the right.



There are six doubles amongst the 36 possible outcomes,

so $$P(\text{double}) = \tfrac{6}{36} = \tfrac{1}{6}.$$

All but the pairs $(1, 1)$, $(2, 1)$ and $(1, 2)$ give a sum at least 4,

so $$P(\text{sum is at least } 4) = \tfrac{33}{36} = \tfrac{11}{12}.$$

Because the two stages are independent,

$$P\ (\text{double, sum at least } 4) = \tfrac{1}{6} \times \tfrac{11}{12} = \tfrac{11}{72}.$$

Multi-stage experiments — the product rule

The same arguments clearly apply to an experiment with any number of stages.

16 MULTI-STAGE EXPERIMENTS

If $A_1, A_2, \ldots, A_n$ are events in successive independent stages of a multi-stage experiment, then

$$P(A_1A_2 \ldots A_n) = P(A_1) \times P(A_2) \times \ldots \times P(A_n).$$

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Example 16 12E

A coin is tossed five times. Find the probability that:

a every toss is a head, **b** no toss is a head, **c** there is at least one head.

SOLUTION

The five tosses are independent events.

a $P(\text{HHHHH}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{32}$

b $P(\text{no heads}) = P(\text{TTTTT})$
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{32}$

c $P(\text{at least one head}) = 1 - P(\text{no heads})$
$= 1 - \frac{1}{32}$
$= \frac{31}{32}$

Sampling without replacement — an extension of the product rule

The product rule can be extended to the following question, where the two stages of the experiment are not independent.

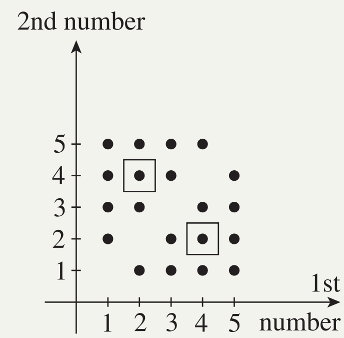
Example 17 12E

A box contains five discs numbered 1, 2, 3, 4 and 5.
Two numbers are drawn in succession, without replacement.
What is the probability that both are even?

SOLUTION

The probability that the first number is even is $\frac{2}{5}$.
When this even number is removed, one even and three odd numbers remain, so the probability that the second number is also even is $\frac{1}{4}$.

Hence $P(\text{both even}) = \frac{2}{5} \times \frac{1}{4}$
$= \frac{1}{10}$.



Note: The graph to the right allows the calculation to be checked by examining its full sample space.

Because doubles are not allowed (that is, there is no replacement), there are only 20 possible outcomes.

The two boxed outcomes are the only outcomes that consist of two even numbers, giving the same probability of $\frac{2}{20} = \frac{1}{10}$.

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Listing the favourable outcomes

The product rule is often combined with a listing of the favourable outcomes. A tree diagram may help in producing that listing, although this is hardly necessary in the straightforward example below.

Example 18 12E

A coin is tossed four times. Find the probability that:

a the first three coins are heads,

b the middle two coins are tails.

c Copy and complete this table:

Number of heads	0	1	2	3	4
Probability					

SOLUTION

a $P(\text{the first three coins are heads}) = P(\text{HHHH}) + P(\text{HHHT})$

(notice that the two events HHHH and HHHT are mutually exclusive)

$= \frac{1}{16} + \frac{1}{16}$

$\left(\text{because each of these two probabilities is } \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$

$= \frac{1}{8}.$

b $P(\text{middle two are tails}) = P(\text{HTTH}) + P(\text{HTTT}) + P(\text{TTTH}) + P(\text{TTTT})$

$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$

$= \frac{1}{4}.$

c $P(\text{no heads}) = P(\text{TTTT})$

$= \frac{1}{16}$

$P(\text{one head}) = P(\text{HTTT}) + P(\text{THTT}) + P(\text{TTHT}) + P(\text{TTTH})$

$= \frac{4}{16}$

$P(\text{two heads}) = P(\text{HHTT}) + P(\text{HTHT}) + P(\text{THHT}) + P(\text{HTTH}) + P(\text{THTH}) + P(\text{TTHH})$

(these are all six possible orderings of H, H, T and T).

$= \frac{6}{16}$, and the last two results follow by symmetry.

Number of heads	0	1	2	3	4
Probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Describing the experiment in a different way

Sometimes, the manner in which an experiment is described makes calculation difficult, but the experiment can be described in a different way so that the probabilities are the same but the calculations are much simpler.

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Example 19 12E

Wes is sending Christmas cards to ten friends. He has two cards with angels, two with snow, two with reindeer, two with trumpets and two with Santa Claus.

What is the probability that Harry and Helmut get matching cards?

SOLUTION

Describe the process in a different way as follows:

'Wes decides that he will choose Harry's card first and Helmut's card second. Then he will choose the cards for his remaining eight friends.'

All that matters now is whether the card that Wes chooses for Helmut is the same as the card that he has already chosen for Harry. After he chooses Harry's card, there are nine cards remaining, of which only one matches Harry's card. Thus the probability that Helmut's card matches is $\frac{1}{9}$.

Exercise 12E

FOUNDATION

1 Suppose that A, B, C and D are independent events, with $P(A) = \frac{1}{8}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$ and $P(D) = \frac{2}{7}$. Use the product rule to find:

a $P(AB)$
b $P(AD)$
c $P(BC)$
d $P(ABC)$
e $P(BCD)$
f $P(ABCD)$

2 A coin and a die are tossed. Use the product rule to find the probability of obtaining:

a a three and a head,
b a six and a tail,
c an even number and a tail,
d a number less than five and a head.

3 One set of cards contains the numbers 1, 2, 3, 4 and 5, and another set contains the letters A, B, C, D and E. One card is drawn at random from each set. Use the product rule to find the probability of drawing:

a 4 and B,
b 2 or 5, then D,
c 1, then A or B or C,
d an odd number and C,
e an even number and a vowel,
f a number less than 3, and E,
g the number 4, followed by a letter from the word MATHS.

4 Two marbles are picked at random, one from a bag containing three red and four blue marbles, and the other from a bag containing five red and two blue marbles. Find the probability of drawing:

a two red marbles,
b two blue marbles,
c a red marble from the first bag and a blue marble from the second.

5 A box contains five light bulbs, two of which are faulty. Two bulbs are selected, one at a time without replacement. Find the probability that:

a both bulbs are faulty,
b neither bulb is faulty,
c the first bulb is faulty and the second one is not,
d the second bulb is faulty and the first one is not.

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6 A die is rolled twice. Using the product rule, find the probability of throwing:

a a six and then a five,
b a one and then an odd number,
c a double six,
d two numbers greater than four,
e a number greater than four and then a number less than four.

7 A box contains twelve red and ten green discs. Three discs are selected, one at a time without replacement.

a What is the probability that the discs selected are red, green, red in that order?
b What is the probability of this event if the disc is replaced after each draw?

8 **a** From a standard pack of 52 cards, two cards are drawn at random without replacement. Find the probability of drawing:

i a spade and then a heart,
ii two clubs,
iii a jack and then a queen,
iv the king of diamonds and then the ace of clubs.

b Repeat the question if the first card is replaced before the second card is drawn.

DEVELOPMENT

9 A coin is weighted so that it is twice as likely to fall heads as it is tails.

a Write down the probabilities that the coin falls:

i heads,
ii tails.

b If you toss the coin three times, find the probability of:

i three heads,
ii three tails,
iii head, tail, head in that order.

10 [Valid and invalid arguments]
Identify any fallacies in the following arguments. If possible, give some indication of how to correct them.

a 'The probability that a Year 12 student chosen at random likes classical music is 50%, and the probability that a student plays a classical instrument is 20%. Therefore the probability that a student chosen at random likes classical music and plays a classical instrument is 10%.'

b 'The probability of a die showing a prime is $\frac{1}{2}$, and the probability that it shows an odd number is $\frac{1}{2}$. Hence the probability that it shows an odd prime number is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.'

c 'I choose a team at random from an eight-team competition. The probability that it wins any game is $\frac{1}{2}$, so the probability that it defeats all of the other seven teams is $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$.'

d 'A normal coin is tossed and shows heads eight times. Nevertheless, the probability that it shows heads the next time is still $\frac{1}{2}$.'

11 In basketball, the chance of a girl making a basket from the free-throw line is 0.7 and the chance of a boy making the basket is 0.65. Therefore if a boy and a girl are selected at random, the chance that at least one of them will shoot a basket is 1.35. Explain the problem with this argument.

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12 A die is rolled twice. Using the product rule, find the probability of rolling:

a a double two,
b any double,
c a number greater than three, then an odd number,
d a one and then a four,
e a four and then a one,
f a one and a four in any order,
g an even number, then a five,
h a five and then an even number,
i an even number and a five in any order.

13 An archer fires three shots at a bullseye. He has a 90% chance of hitting the bullseye. Using H for hit and M for miss, list all eight possible outcomes. Then, assuming that successive shots are independent, use the product rule to find the probability that he will:

a hit the bullseye three times,
b miss the bullseye three times,
c hit the bullseye on the first shot only,
d hit the bullseye exactly once,
e miss the bullseye on the first shot only,
f miss the bullseye exactly once.

(Hint: Part **d** requires adding the probabilities of HMM, MHM and MMH, and part **f** requires a similar calculation.)

14 There is a one-in-five chance that you will guess the correct answer to a multiple-choice question. The test contains five such questions — label the various possible results of the test as CCCCC, CCCCI, CCCIC, ... Find the chance that you will answer:

a all five correctly,
b all five incorrectly,
c the first, third and fifth correctly, and the second and fourth incorrectly,
d the first correctly and the remainder incorrectly,
e exactly one correctly, (Hint: Add the probabilities of CIIII, ICIII, IICII, IIICI, IIIIC.)
f exactly four correctly. (Hint: List the possible outcomes first.)

15 A die is thrown six times.

a What is the probability that the nth throw is n on each occasion?
b What is the probability that the nth throw is n on exactly five occasions?

16 From a bag containing two red and two green marbles, marbles are drawn one at a time without replacement until two green marbles have been drawn. Find the probability that:

a exactly two draws are required,
b at least three draws are required,
c exactly four draws are required,
d exactly three draws are required.

(Hint: In part **c**, consider the colour of the fourth marble drawn.)

17 Sophia, Gabriel and Elizabeth take their driving test. The chances that they pass are $\frac{1}{2}$, $\frac{5}{8}$ and $\frac{3}{4}$ respectively.

a Find the probability that Sophia passes and the other two fail.
b By listing the possible outcomes for one of the girls passing and the other two failing, find the probability that exactly one of the three passes.
c If only one of them passes, find the probability that it is Gabriel.

18 **a** If a coin is tossed repeatedly, find the probability of obtaining at least one head in:

i two tosses,
ii five tosses,
iii ten tosses.

b Write down the probability of obtaining at least one head in n tosses.
c How many times would you need to toss a coin so that the probability of tossing at least one head is greater than 0.9999?

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19 One layer of tinting material on a window cuts out $\frac{1}{5}$ of the sun's UV rays at a certain frequency.

a What fraction would be cut out by using two layers?

b How many layers would be required to cut out at least $\frac{9}{10}$ of the sun's UV rays?

20 In a lottery, the probability of the jackpot being won in any draw is $\frac{1}{60}$.

a What is the probability that the jackpot prize will be won in each of four consecutive draws?

b How many consecutive draws need to be made for there to be a greater than 98% chance that at least one jackpot prize will have been won?

ENRICHMENT

21 [This question is best done by retelling the story of the experiment, as explained in the notes above.] Nick has five different pairs of socks to last the working week, and they are scattered loose in his drawer. Each morning, he gets up before light and chooses two socks at random. Find the probability that he wears a matching pair:

a on the first morning,
b on the last morning,
c on the third morning,
d the first two mornings,
e every morning,
f every morning but one.

22 Kia and Abhishek are two of twelve guests at a tennis party, where people are playing doubles on three courts. The twelve have been divided randomly into three groups of four. Find the probability that:

a Kia and Abhishek play on the same court,

b Kia and Abhishek both play on River court,

c Kia is on River court and Abhishek is on Rose court.

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12F Probability tree diagrams

In more complicated problems, and particularly in unsymmetric situations, a probability tree diagram can be very useful in organising the various cases, in preparation for the application of the product rule.

Constructing a probability tree diagram

A *probability tree diagram* differs from the tree diagrams used in Section 12B for counting possible outcomes, in that the relevant probabilities are written on the branches and then multiplied together in accordance with the product rule. An example will demonstrate the method.

As before, these diagrams have one column labelled 'Start', a column for each stage, and a column listing the outcomes, but there is now an extra column labelled 'Probability' at the end.

Example 20 — 12F

One card is drawn from each of two packs. Use a probability tree diagram to find the probability that:

a both cards are court cards (jack, queen or king),

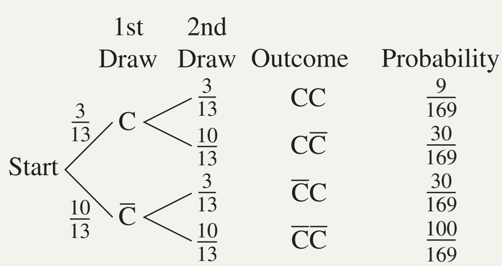
b neither card is a court card,

c one card is a court card and the other is not.

SOLUTION

In each pack of cards, there are 12 court cards out of 52 cards,

so $P(\text{court card}) = \frac{3}{13}$ and $P(\text{not a court card}) = \frac{10}{13}$.



Multiply the probabilities along each arm because they are successive stages.
Add the probabilities in the final column because they are mutually exclusive.

a $P(\text{two court cards}) = \frac{3}{13} \times \frac{3}{13}$
$= \frac{9}{169}$

b $P(\text{no court cards}) = \frac{10}{13} \times \frac{10}{13}$
$= \frac{100}{169}$

c $P(\text{one court card})$
$= \frac{3}{13} \times \frac{10}{13} + \frac{10}{13} \times \frac{3}{13}$
$= \frac{30}{169} + \frac{30}{169}$
$= \frac{60}{169}$

Note: The four probabilities in the last column of the tree diagram add exactly to 1, which is a useful check on the working. The three answers also add to 1.

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17 PROBABILITY TREE DIAGRAMS

In a probability tree diagram:

- Probabilities are written on the branches.
- There is a final column giving the probability of each outcome.

Example 21 12F

[A more complicated experiment]

A bag contains six white marbles and four blue marbles. Three marbles are drawn in succession. At each draw, if the marble is white it is replaced, and if it is blue it is not replaced.
Find the probabilities of drawing:

a no blue marbles,

b one blue marble,

c two blue marbles,

d three blue marbles.

SOLUTION

With the ten marbles all in the bag,

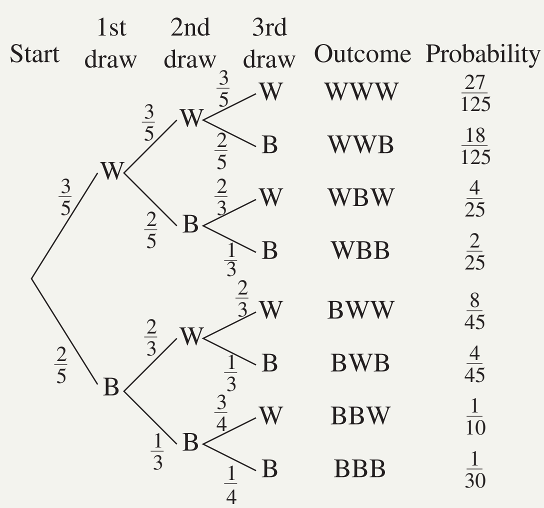
$$P(\text{W}) = \tfrac{6}{10} = \tfrac{3}{5} \quad \text{and} \quad P(\text{B}) = \tfrac{4}{10} = \tfrac{2}{5}.$$

If one blue marble is removed, there are six white and three blue marbles, so

$$P(\text{W}) = \tfrac{6}{9} = \tfrac{2}{3} \quad \text{and} \quad P(\text{B}) = \tfrac{3}{9} = \tfrac{1}{3}.$$

If two blue marbles are removed, there are six white and two blue marbles, so

$$P(\text{W}) = \tfrac{6}{8} = \tfrac{3}{4} \quad \text{and} \quad P(\text{B}) = \tfrac{2}{8} = \tfrac{1}{4}.$$



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In each part, multiply the probabilities along each arm and then add the cases.

a $P(\text{no blue marbles}) = \frac{27}{125}$

b $P(\text{one blue marble}) = \frac{18}{125} + \frac{4}{25} + \frac{8}{45}$
$= \frac{542}{1125}$

c $P(\text{two blue marbles}) = \frac{2}{25} + \frac{4}{45} + \frac{1}{10}$
$= \frac{121}{450}$

d $P(\text{three blue marbles}) = \frac{1}{30}$

Note: Again, as a check on the working, your calculator will show that the eight probabilities in the last column of the diagram add exactly to 1 and that the four answers above also add to 1.

Exercise 12F

FOUNDATION

1 A bag contains three black and four white discs. A disc is selected from the bag, its colour is noted, and it is then returned to the bag before a second disc is drawn.

a By multiplying along the branches of the tree, find:

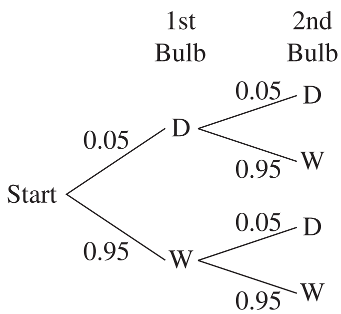
i $P(\text{BB})$ **ii** $P(\text{BW})$
iii $P(\text{WB})$ **iv** $P(\text{WW})$

b Hence, by adding, find the probability that:

i both discs drawn have the same colour,
ii the discs drawn have different colours.

c Draw your own tree diagram, and repeat part **b** if the first ball is not replaced before the second one is drawn.

2 Two light bulbs are selected at random from a large batch of bulbs in which 5% are defective.



a By multiplying along the branches of the tree, find:

i $P(\text{both bulbs work})$,
ii $P(\text{the first works, but the second is defective})$,
iii $P(\text{the first is defective, but the second works})$,
iv $P(\text{both bulbs are defective})$.

b Hence find the probability that at least one bulb works.

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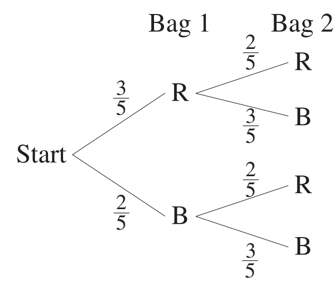
3 One bag contains three red and two blue balls and another bag contains two red and three blue balls. A ball is drawn at random from each bag.



a By multiplying along the branches of the tree, find:

i $P(\text{RR})$ **ii** $P(\text{RB})$

iii $P(\text{BR})$ **iv** $P(\text{BB})$

b Hence, by adding, find the probability that:

i the balls have the same colour,

ii the balls have different colours.

4 In group A there are three girls and seven boys, and in group B there are six girls and four boys. One person is chosen at random from each group. Draw a probability tree diagram.

a By multiplying along the branches of the tree, find:

i $P(\text{GG})$ **ii** $P(\text{GB})$ **iii** $P(\text{BG})$ **iv** $P(\text{BB})$

b Hence, by adding, find the probability that:

i two boys or two girls are chosen,

ii one boy and one girl are chosen.

5 There is an 80% chance that Garry will pass his driving test and a 90% chance that Emma will pass hers. Draw a probability tree diagram, and find the chance that:

a Garry passes and Emma fails,

b Garry fails and Emma passes,

c only one of Garry and Emma passes,

d at least one fails.

6 The probability that a set of traffic lights will be green when you arrive at them is $\frac{3}{5}$. A motorist drives through two sets of lights. Assuming that the two sets of traffic lights are not synchronised, find the probability that:

a both sets of lights will be green,

b at least one set of lights will be green.

7 A factory assembles calculators. Each calculator requires a chip and a battery. It is known that 1% of chips and 4% of batteries are defective. Find the probability that a calculator selected at random will have at least one defective component.

8 The probability of a woman being alive at 80 years of age is 0.2, and the probability of her husband being alive at 80 years of age is 0.05. Find the probability that:

a they will both live to be 80 years of age,

b only one of them will live to be 80.

9 Alex and Julia are playing in a tennis tournament. They will play each other twice, and each has an equal chance of winning the first game. If Alex wins the first game, his probability of winning the second game is increased to 0.55. If he loses the first game, his probability of winning the second game is reduced to 0.25. Find the probability that Alex wins exactly one game.

10 One bag contains four red and three blue discs, and another bag contains two red and five blue discs. A bag is chosen at random and then a disc is drawn from it. Find the probability that the disc is blue.

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11 In a raffle in which there are 200 tickets, the first prize is drawn and then the second prize is drawn without replacing the winning ticket. If you buy 15 tickets, find the probability that you win:

a both prizes,

b at least one prize.

DEVELOPMENT

12 A box contains 10 chocolates, all of identical appearance. Three of the chocolates have caramel centres and the other seven have mint centres. Hugo randomly selects and eats three chocolates from the box. Find the probability that:

a the first chocolate Hugo eats is caramel,

b Hugo eats three mint chocolates,

c Hugo eats exactly one caramel chocolate.

13 In an aviary there are four canaries, five cockatoos and three budgerigars. If two birds are selected at random, find the probability that:

a both are canaries,

b neither is a canary,

c one is a canary and one is a cockatoo,

d at least one is a canary.

14 Max and Jack each throw a die. Find the probability that:

a they do not throw the same number,

b the number thrown by Max is greater than the number thrown by Jack,

c the numbers they throw differ by three.

15 A game is played in which two coloured dice are rolled once. The six faces of the black die are numbered 5, 7, 8, 10, 11, 14. The six faces of the white die are numbered 3, 6, 9, 12, 13, 15. The player wins if the number on the black die is bigger than the number on the white die.

a Calculate the probability of a player winning the game.

b Calculate the probability that a player will lose at least once in two consecutive games.

c How many games must be played before you have a 90% chance of winning at least one game?

16 In a large co-educational school, the population is 47% female and 53% male. Two students are selected from the school population at random. Find, correct to two decimal places, the probability that:

a both are male,

b a girl and a boy are selected.

17 The numbers 1, 2, 3, 4 and 5 are each written on a card. The cards are shuffled and one card is drawn at random. The number is noted and the card is then returned to the pack. A second card is selected, and in this way a two-digit number is recorded. For example, a 2 on the first draw and a 3 on the second results in the number 23.

a What is the probability of the number 35 being recorded?

b What is the probability of an odd number being recorded?

18 A twenty-sided die has the numbers from 1 to 20 on its faces.

a If it is rolled twice, what is the probability that the same number appears on the uppermost face each time?

b If it is rolled three times, what is the probability that the number 15 appears on the uppermost face exactly twice?

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19 An interviewer conducts a poll in Sydney and Melbourne on the popularity of the prime minister. In Sydney, 52% of the population approve of the prime minister, and in Melbourne her approval rating is 60%. If one of the two capital cities is selected at random and two electors are surveyed, find the probability that:

a both electors approve of the prime minister,

b at least one elector approves of the prime minister.

20 In a bag there are four green, three blue and five red discs.

a Two discs are drawn at random, and the first disc is not replaced before the second disc is drawn. Find the probability of drawing:

i two red discs,

ii one red and one blue disc,

iii at least one green disc,

iv a blue disc on the first draw,

v two discs of the same colour,

vi two differently coloured discs.

b Repeat part **a** if the first disc is replaced before the second disc is drawn.

21 In a game, two dice are rolled and the score given is the maximum of the two numbers on the uppermost faces. For example, if the dice show a three and a five, the score is a five.

a Find the probability that you score a one in a single throw of the two dice.

b What is the probability of scoring three consecutive ones in three rolls of the dice?

c Find the probability that you score a six in a single roll of the dice.

22 In each game of Sic Bo, three standard six-sided dice are thrown once.

a In a single game, what is the probability that all three dice show six?

b What is the probability that exactly two of the dice show six?

c What is the probability that exactly two of the dice show the same number?

d What is the probability of rolling three different numbers on the dice?

ENRICHMENT

23 A bag contains two green and two blue marbles. Marbles are drawn at random, one by one without replacement, until two green marbles have been drawn. What is the probability that exactly three draws will be required?

24 Prasad and Wilson are going to enlist in the Australian Army. The recruiting officer will be in town for seven consecutive days, starting on Monday and finishing the following Sunday. The boys must nominate three consecutive days on which to attend the recruitment office. They do this randomly and independently of one another.

a By listing all the ways in which to choose the three consecutive days, find the probability that they both go on Monday.

b What is the probability that they meet at the recruitment office on Tuesday for the first time?

c Find the probability that Prasad and Wilson will not meet at the recruitment office.

d Hence find the probability that they will meet on at least one day at the recruitment office.

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12G Conditional probability

Sometimes our knowledge of the results of an experiment change over time, and as we gain more information, our probabilities change. Conditional probability allows us to calculate these changing probabilities.

Reduced sample space and reduced event space

Drago will win a certain game if an odd number is thrown on a die. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$, and the event space is $A = \{1, 3, 5\}$, so

$$\begin{aligned} P(\text{he wins}) &= \tfrac{3}{6} \\ &= \tfrac{1}{2}. \end{aligned}$$

Drago nervously turns away as the die is thrown, and someone calls out, 'It's a prime number!' What, for him, is now the probability that he wins the game?

- First, the sample space has now been reduced to the set $B = \{2, 3, 5\}$ of prime numbers on the die. This is called the *reduced sample space*.
- Secondly, the event space has now been reduced to the set $\{3, 5\}$ of odd prime numbers on the die. This *reduced event space* is the intersection $A \cap B$ of the event space A and the reduced sample space B. Hence

$$\begin{aligned} P(\text{he wins, given that the number is prime}) &= \frac{|\,\text{reduced event space}\,|}{|\,\text{reduced sample space}\,|} \\ &= \tfrac{2}{3}. \end{aligned}$$

Alternatively, with B as the reduced sample space and $A \cap B$ as the reduced event space, we can write the working using a concise formula,

$$\begin{aligned} P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \tfrac{2}{3}, \end{aligned}$$

where $P(A|B)$ is the symbol for conditional probability, and means 'the probability of A, given that B has occurred'.

(We are, of course, assuming that the person calling out was telling the truth.)

Conditional probability

It is only a matter of convenience whether we work with the reduced sample space and reduced event space, or with formulae:

18 CONDITIONAL PROBABILITY

Suppose that two events A and B are subsets of the same uniform sample space S. Then the *conditional probability* $P(A|B)$ of A, given that B has occurred, is obtained by removing all the elements not in B from the sample space S, and from the event space A. Thus there are two equivalent formulae,

$$P(A|B) = \frac{|\,\text{reduced event space}\,|}{|\,\text{reduced sample space}\,|} \quad \text{and} \quad P(A|B) = \frac{|A \cap B|}{|B|}.$$

Read $P(A|B)$ as 'the probability of A, given that B has occurred'.

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Example 22 — 12G

A card is drawn at random from a standard pack of 52 cards. Find the probability that it is a 9, 10, jack or queen if

a Nothing more is known.

b It is known to be a court card (jack, queen or king)

c It is known to be an 8 or a 10.

d It is known not to be a 2, 3, 4 or 5.

SOLUTION

Let the sample space S be the set of all 52 cards, and let
$A = \{9\clubsuit, 9\diamondsuit, 9\heartsuit, 9\spadesuit, 10\clubsuit, 10\diamondsuit, 10\heartsuit, 10\spadesuit, J\clubsuit, J\diamondsuit, J\heartsuit, J\spadesuit, Q\clubsuit, Q\diamondsuit, Q\heartsuit, Q\spadesuit\}$

a There is no reduction of the sample space or the event space,

$$\begin{aligned} \text{so } P(A) &= \tfrac{|A|}{|S|} \\ &= \tfrac{16}{52} \\ &= \tfrac{4}{13}. \end{aligned}$$

b The reduced sample space is

$$B = \{\text{court cards}\},$$

and the reduced event space is

$$A \cap B = \{\text{jacks and queens}\},$$

$$\begin{aligned} \text{so } P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \tfrac{8}{12} \\ &= \tfrac{2}{3}. \end{aligned}$$

c The reduced sample space is

$$B = \{\text{8s and 10s}\},$$

and the reduced event space is

$$A \cap B = \{\text{10s}\},$$

$$\begin{aligned} \text{so } P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \tfrac{4}{8} \\ &= \tfrac{1}{2}. \end{aligned}$$

d The reduced sample space is

$$B = \{\text{cards not 2, 3, 4 or 5}\},$$

and the reduced event space is

$$A \cap B = A,$$

$$\begin{aligned} \text{so } P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \tfrac{16}{36} \\ &= \tfrac{4}{9}. \end{aligned}$$

Another formula for conditional probability

Suppose as before that two events A and B are subsets of the same uniform sample space S. The previous formula gave $P(A|B)$ in terms of the sizes of the sets $A \cap B$ and B. By dividing through by $|S|$, we obtain a formula for $P(A|B)$ in terms of the probabilities of $A \cap B$ and B:

$$\begin{aligned} P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \frac{|A \cap B|}{|S|} \div \frac{|B|}{|S|} \\ &= \frac{P(A \cap B)}{P(B)} \end{aligned}$$

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19 ANOTHER FORMULA FOR CONDITIONAL PROBABILITY

For two events A and B,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

The great advantage of this formula is that it can be used when nothing is known about the actual sizes of the sample space and the event space, as in Example 23.

Example 23 **12G**

In a certain population, 35% have blue eyes, 15% have blond hair, and 10% have blue eyes and blond hair. A person is chosen from this population at random.

a Find the probability that they have blond hair, given that they have blue eyes.

b Find the probability that they have blue eyes, given that they have blond hair.

SOLUTION

From the given percentages, we know that

$$P(\text{blue eyes}) = 0.35, \quad P(\text{blond hair}) = 0.15, \quad P(\text{blue eyes and blond hair}) = 0.1.$$

a $$\begin{aligned} P(\text{blond hair, given blue eyes}) &= \frac{P(\text{blond hair and blue eyes})}{P(\text{blue eyes})} \\ &= \frac{0.1}{0.35} \\ &\doteqdot 0.29 \end{aligned}$$

b $$\begin{aligned} P(\text{blue eyes, given blond hair}) &= \frac{P(\text{blond hair and blue eyes})}{P(\text{blond hair})} \\ &= \frac{0.1}{0.15} \\ &\doteqdot 0.67 \end{aligned}$$

Independent events

We say that an event A is *independent* of an event B if knowing whether or not B has occurred does not affect our probability of A. That is,

$$A \text{ and } B \text{ are independent} \quad \text{means that} \quad P(A|B) = P(A).$$

20 INDEPENDENT EVENTS

- Two events A and B are called *independent* if the probability of A is not affected by knowing whether or not B has occurred

$$P(A|B) = P(A)$$

- If A is independent of B, then B is independent of A.

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The second dotpoint may seem trivial, but it needs proving.
Suppose that A is independent of B, meaning that $P(A|B) = P(A)$.
Then to prove that B is independent of A,

$$\begin{aligned}
P(B|A) &= \frac{P(B \cap A)}{P(A)} && \text{(using the formula for } P(B|A)\text{)} \\
&= \frac{P(A \cap B)}{P(B)} \times \frac{P(B)}{P(A)} && \text{(multiply top and bottom by } P(B)\text{)} \\
&= P(A|B) \times \frac{P(B)}{P(A)} && \text{(using the formula for } P(A|B)\text{)} \\
&= P(A) \times \frac{P(B)}{P(A)} && \text{(we are assuming that } P(A|B) = P(A)\text{)} \\
&= P(B), \quad \text{as required.}
\end{aligned}$$

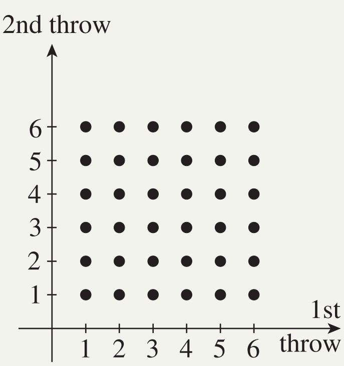
The consequence of this is that we can just say that 'the two events A and B are independent', which is what we would expect.

Example 24 12G

Two dice are thrown one after the other.

- Let A be the event 'the first die is odd'.
- Let B be the event 'the second die is 1, 2 or 3'.
- Let C be the event 'the sum is five'.

Which of the three pairs of events are independent?



SOLUTION

The sample space S has size 36, and

$$P(A) = \tfrac{18}{36} = \tfrac{1}{2}, \quad P(B) = \tfrac{18}{36} = \tfrac{1}{2}, \quad P(C) = \tfrac{4}{36} = \tfrac{1}{9}.$$

The last result follows because $5 = 1 + 4 = 2 + 3 = 3 + 2 = 4 + 1$ are all the ways that a sum of five can be obtained. Taking intersections,

$$P(A \cap B) = \tfrac{9}{36} = \tfrac{1}{4}, \quad P(B \cap C) = \tfrac{3}{36} = \tfrac{1}{12}, \quad P(C \cap A) = \tfrac{2}{36} = \tfrac{1}{18}.$$

Testing the definition of independence on the three pairs of events,

$$\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \tfrac{1}{4} \div \tfrac{1}{2} \\
&= \tfrac{1}{2} \\
&= P(A)
\end{aligned}
\qquad
\begin{aligned}
P(B|C) &= \frac{P(B \cap C)}{P(C)} \\
&= \tfrac{1}{12} \div \tfrac{1}{9} \\
&= \tfrac{3}{4} \\
&\neq P(B)
\end{aligned}
\qquad
\begin{aligned}
P(C|A) &= \frac{P(C \cap A)}{P(A)} \\
&= \tfrac{1}{18} \div \tfrac{1}{2} \\
&= \tfrac{1}{9} \\
&= P(C)
\end{aligned}$$

Hence A and B are independent, and C and A are independent, but B and C are not independent.

Note: The word 'independent' here means 'mathematically independent', and carries no scientific implication. A scientist who makes a statement about the natural world, on the basis of such mathematical arguments, is doing science, not mathematics.

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Note: The word 'independent' was introduced in Section 12E with multi-stage experiments. Its use in this section is a generalisation of its use in the multi-stage situation. For example, the events A and B above are independent in both senses, but the independence of C and A is a new idea.

The product rule

The formula $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ can be rearranged as

$$P(A \cap B) = P(A|B) \times P(B).$$

If the events are independent, then $P(A|B) = P(A)$, so

$$P(A \cap B) = P(A) \times P(B).$$

Conversely, if $P(A \cap B) = P(A) \times P(B)$, then $P(A|B) = P(A)$, meaning that the events are independent. Thus we have both a formula that applies to independent events, and another test for independence:

21 THE PRODUCT RULE AND INDEPENDENT EVENTS

- For any events A and B,

  $$P(A \text{ and } B) = P(A|B) \times P(B).$$

- If A and B are independent events, then

  $$P(A \text{ and } B) = P(A) \times P(B).$$

- Conversely, if $P(A \text{ and } B) = P(A) \times P(B)$, then the events are independent.

Remember that $P(A \text{ and } B)$ and $P(A \cap B)$ mean exactly the same thing

All this is a generalisation of the product rule introduced with the multi-stage experiments and tree diagrams in Section 12E.

Example 25 12G

Calculate $P(A) \times P(B)$ and $P(B) \times P(C)$ and $P(C) \times P(A)$ in Example 24. Then use the third dot point of Box 21 above to confirm that these calculations give the same results for the independence of A and B, of B and C, and of C and A.

SOLUTION

$$\begin{aligned} P(A) \times P(B) &= \tfrac{1}{2} \times \tfrac{1}{2} \\ &= \tfrac{1}{4} \\ &= P(A \cap B) \end{aligned}$$

$$\begin{aligned} P(B) \times P(C) &= \tfrac{1}{2} \times \tfrac{1}{9} \\ &= \tfrac{1}{18} \\ &\neq P(B \cap C) \end{aligned}$$

$$\begin{aligned} P(C) \times P(A) &= \tfrac{1}{9} \times \tfrac{1}{2} \\ &= \tfrac{1}{18} \\ &= P(C \cap A) \end{aligned}$$

again showing that A and B are independent, that C and A are independent, but that B and C are not independent.

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Example 26 12G

Example 23 described the incidence of blue eyes and blond hair in a population.

a Why does the product rule formula show immediately that 'blue eyes' and 'blond hair' are not independent?

b What would be the probability of having blue eyes and blond hair if the two characteristics were independent?

SOLUTION

a Box 21 says that 'blue eyes' and 'blond hair' are independent if and only if

$$P(\text{blue eyes and blond hair}) = P(\text{blue eyes}) \times P(\text{blond hair}).$$

But according to the data given in Example 23, LHS $= 0.1$ and RHS $= 0.35 \times 0.15 = 0.0525$, so the events are not independent.

b If they were independent, then $P(\text{blue eyes and blond hair}) \doteqdot 0.67$ would be the product 0.35×0.15 of the two probabilities, which it clearly is not.

c If the events were independent, then the product rule formula would hold, so

$$\begin{aligned} P(\text{blue eyes and blond hair}) &= P(\text{blue eyes}) \times P(\text{blond hair}) \\ &= 0.35 \times 0.15 \\ &= 0.0525. \end{aligned}$$

Using the sample space without any formulae

Sometimes, however, it is easier not to use any of the machinery in this section, but to go back to basics and deal directly with the sample space, with little or no notation.

Example 27 12G

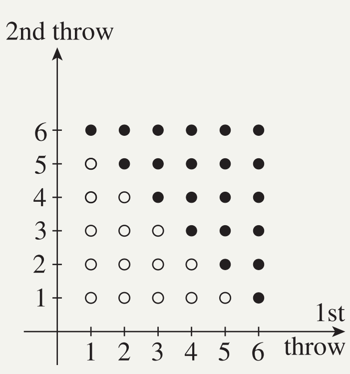
Yanick and Oskar can't agree which movie to watch, so they play a game throwing two dice.

- If the sum is 7 or 8, Yanick chooses the movie.
- If the sum is 9, 10, 11 or 12, Oskar chooses the movie.
- If the sum is less than 7, they throw the dice again.

Who has the higher probability of choosing the movie?

SOLUTION

There are 36 possible throws in the original sample space. But we can ignore the throws whose sum is less than 7, represented in the diagram by the open circles, because these 15 throws are discarded if they occur.



This leaves a sample space with only the 21 closed circles. There are $6 + 5 = 11$ throws with sum 7 or 8, so

$$P(\text{Yanick chooses}) = \frac{11}{21}$$

and $$P(\text{Oskar chooses}) = \frac{10}{21}.$$

Hence Yanick has the higher probability of choosing the movie.

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A final note about general sample spaces

The theory developed in this chapter can be summarised in four results:

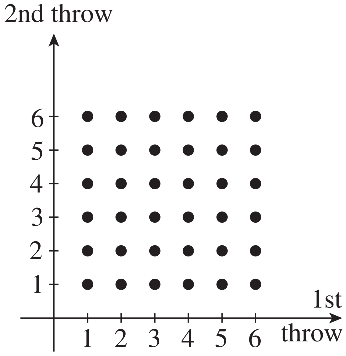
- $P(\text{not } A) = 1 - P(A)$
- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$
- $P(A \text{ and } B) = P(A) \times P(B)$ if and only if A and B are independent events.

These four results were all developed using equally likely possible outcomes, which restricts them to experiments where there is a finite sample space that is uniform. The four results, however, hold in all theories of probability, either as axioms and definitions, or as results proven by different methods.

Exercise 12G

FOUNDATION

Note: Several questions in this exercise, and elsewhere, require the assumptions that a baby is equally likely to be a boy or a girl, that these are the only two possibilities, and that the sex of a child is independent of the sex of any previous children. These assumptions are simplifications of complex scientific and social considerations, and they do not describe the real situation.



1 Two dice are thrown. The sample space of this experiment is shown in the dot diagram to the right. Following the throw, it is revealed that the first die shows an even number.

- **a** Copy the diagram and circle the reduced sample space.
- **b** Find the conditional probability of getting two sixes.
- **c** Find the conditional probability of getting at least one six.
- **d** Find the conditional probability that the sum of the two numbers is five.

2 A poll is taken amongst 1000 people to determine their voting patterns in the last election.

	Coalition	Labor	Other	Total
Male	130	340	110	580
Female	210	190	20	420
Total	340	530	130	1000

- **a** Determine the probability that a particular person in the group voted Coalition.
- **b** What is the probability that a particular person voted Labor, given that they were female?
- **c** What is the probability that a particular member of the group was male, if it is known that they voted Coalition?
- **d** Robin voted neither Coalition nor Labor. What is the probability that Robin was female?

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3 A student is investigating if there is any relationship between those who choose Extension 1 Mathematics (M) and those who choose Extension 1 English (E) at his school.

	M	$\overline{M}$	Total
E	29	27	
$\overline{E}$	95	42	
Total			

a Copy and complete the table by filling in the totals.
b What is the probability that a particular student chose
 i neither Extension 1 Maths nor Extension 1 English,
 ii Extension 1 English, given that they chose Extension 1 Maths,
 iii Extension 1 Maths, if it is known that they chose Extension 1 English,
 iv Extension 1 Maths, given that they did not choose Extension 1 English.

4 Two cards are drawn in turn from a standard pack — the first card is replaced and the pack shuffled before the second card is drawn — and the suit of each card (spades, hearts, diamonds or clubs) is noted by the game master. A player wants to know the probability that both cards are hearts.
a What is the probability if nothing else is known?
b The first card is known to be a heart. List the reduced sample space. What is the conditional probability of two hearts?
c List the reduced sample space if at least one of the cards is known to be a heart. What is the probability of two hearts?
d List the reduced sample space if the first card is known to be red. What is the probability of two hearts?

5 In a certain game, the player tosses two coins and then throws a die. The sample space is shown in the table below.

	1	2	3	4	5	6
HH						
HT						
TH						
TT						

The rules of the game assign one point for each head, and zero points for each tail. This score is then added to the score on the die.
a Copy the table and fill in the point score for each outcome.
b Find the probability that a player gets more than 7 points.
c Suppose it is known that he has thrown two heads. Find the probability that a player gets more than 7 points.
d Find the probability that a player got an odd number of heads, given that their score was odd.

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6 Use the formula $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ to answer the following questions.

- **a** Find $P(A|B)$ if $P(A \cap B) = 0.5$ and $P(B) = 0.7$.
- **b** Find $P(A|B)$ if $P(A \cap B) = 0.15$ and $P(B) = 0.4$.
- **c** Find $P(E|F)$ if $P(E \cap F) = 0.8$ and $P(F) = 0.95$.

7 The two events A and B in the following experiments are known to be independent.

- **a** $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A \cap B)$.
- **b** $P(A) = 0.3$ and $P(B) = 0.5$. Find $P(A \cap B)$.
- **c** $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A|B)$.
- **d** $P(A) = 0.7$ and $P(B) = 0.2$. Find $P(A|B)$.

8 Each of the following experiments involves two events, A and B. State in each case whether they are dependent or independent.

- **a** $P(A|B) = 0.5$ and $P(A) = 0.4$ and $P(B) = 0.5$
- **b** $P(A|B) = 0.3$ and $P(A) = 0.3$ and $P(B) = 0.6$
- **c** $P(A|B) = \frac{3}{4}$ and $P(A) = \frac{2}{5}$ and $P(B) = \frac{3}{10}$
- **d** $P(A) = 0.3$ and $P(B) = 0.7$ and $P(A \cap B) = 0.21$
- **e** $P(A) = 0.2$ and $P(B) = 0.4$ and $P(A \cap B) = 0.8$
- **f** $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{3}$

DEVELOPMENT

9 **a** Draw a table showing the sample space if two dice are thrown in turn and their sum is recorded.

b Highlight the reduced sample space if the sum of the two dice is 5 . Given that the sum of the two dice is 5, what is the probability that:

- **i** the first die shows a 1,
- **ii** at least one die shows a 1,
- **iii** at least one of the dice shows an odd number.

10 **a** For two events A and B it is known that $P(A \cup B) = 0.6$ and $P(A) = 0.4$ and $P(B) = 0.3$.

- **i** Use the addition formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to find the probability $P(A \cap B)$.
- **ii** Use the formula for conditional probability to find $P(A|B)$.
- **iii** Find $P(B|A)$.

b Suppose that $P(V \cup W) = 0.7$ and $P(V) = 0.5$ and $P(W) = 0.35$. Find $P(V|W)$.

c Find $P(X|Y)$ if $P(X \cap Y) = 0.2$ and $P(X) = 0.3$ and $P(Y) = 0.4$.

d Find $P(A|B)$ if $P(A \cup B) = \frac{1}{3}$ and $P(A) = \frac{1}{5}$ and $P(B) = \frac{3}{10}$.

11 Two dice are rolled. A three appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than seven.

12 A cricket team knows that in half of the games they played last season, they won the game and their star player Arnav was playing. Arnav consistently plays in 80% of the games. What is the probability that they will win this Saturday if Arnav is playing?

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13 a A couple has two children, the older of which is a boy. What is the probability that they have two boys?
b A couple has two children and at least one is a boy. What is the probability that they have two boys?

14 A couple has three children, each being either a boy or a girl.
a List the sample space.
b Given that at least one is a boy, what is the probability that the oldest is male?
c Given that at least one of the first two children is a boy, what is the probability that the oldest is male?

15 A card is drawn from a standard pack. The dealer tells the players that it is a court card (jack, queen or king).
a What is the probability that it is a jack?
b What is the probability that it is either a jack or a red card?
c What is the probability that the next card drawn is a jack? Assume that the first card was not replaced.

16 Two dice are tossed in turn and the outcomes recorded. Let A be the probability that the first die is odd. Let S be the probability that the sum is odd. Let M be the probability that the product is odd.
a Use the definition of independence in Box 20 to find which of the three pairs of events are independent.
b Confirm your conclusions using the product-rule test for independence from Box 21.

17 The two events A and B in the following experiments are known to be independent.
a $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A \cup B)$.
b The probability of event A occurring is 0.6 and the probability of event B occurring is 0.3. What is the probability that either A or B occurs?

18 A set of four cards contains two jacks, a queen and a king. Bob selects one card and then, without replacing it, selects another. Find the probability that:
a both Bob's cards are jacks,
b at least one of Bob's cards is a jack,
c given that one of Bob's cards is a jack, the other one is also a jack.

19 A small committee of two is formed from a group of 4 boys and 8 girls. If at least one of the members is a girl, what is the probability that both are girls?

20 Jack and Ben have been tracking their success rate of converting goals in their rugby games. Jack converts 70% of his goals and Ben converts 60%. At a recent home game, both get a kick, but only one converts his goal. What is the probability that it was Ben?

21 Susan picks two notes at random from four \$5, three \$10 and two \$20 notes. Given that at least one of the notes was \$10, what is the probability that Susan has picked up a total of \$20 or more?

ENRICHMENT

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22 Researchers are investigating a potential new test for a disease, because although the usual test is totally reliable, it is painful and expensive.

The new test is intended to show a positive result when the disease is present. Unfortunately the test may show a *false positive*, meaning that the test result is positive even though the disease is not present.

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The test may also show a *false negative*, meaning that the test result is negative even though the disease is in fact present.

The researchers tested a large sample of people who had symptoms that vaguely suggested that it was worth testing for the disease. They used the new test, and checked afterwards for the disease with the old totally reliable test, and came up with the following results:

- Of this sample group, 1% had the disease.
- Of those who had the disease, 80% tested positive and 20% tested negative.
- Of those who did not have the disease, 5% tested positive and 95% tested negative.

a What percentage of this group would test positive to this new test?

b Use the conditional probability formula $P(A \cap B) = P(A|B) \times P(B)$ to find the probability that a person tests positive, but does not have the disease.

c Find the probability that the new test gives a false positive. That is, find the probability the patient does not have the disease, given that the patient has tested positive.

d Find the probability that the new test gives a false negative. That is, find the probability the patient has the disease, given that the patient has tested negative.

e Comment on the usefulness of the new test.

23 a Prove that for events A and B,

$$P(A|B) = \frac{P(B|A)}{P(B)} \times P(A).$$

b Use this formula to produce an alternative solution to the probability of a false positive or negative in the previous question.

24 Prove the *symmetry of independence*, that is, if B is independent of A then A is independent of B.

25 [A notoriously confusing question]
In a television game show, the host shows the contestant three doors, only one of which conceals the prize, and the game proceeds as follows. First, the contestant chooses a door. Secondly, the host opens one of the other two doors, showing the contestant that it is not the prize door. Thirdly, the host invites the contestant to change her choice, if she wishes. Analyse the game, and advise the contestant what to do.

26 A family has two children. Given that at least one of the children is a girl who was born on a Monday, what is the probability that both children are girls?

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Chapter 12 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 12 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

Review

1 If a die is rolled, find the probability that the uppermost face is:

a a two, **b** an odd number,
c a number less than two, **d** a prime number.

2 A number is selected at random from the integers 1, 2, . . . , 10. Find the probability of choosing:

a the number three, **b** an even number, **c** a square number,
d a negative number, **e** a number less than 20, **f** a multiple of three.

3 From a standard pack of 52 cards, one card is drawn at random. Find the probability that the chosen card is:

a black, **b** red, **c** a queen,
d the ace of spades, **e** a club or a diamond, **f** not a seven.

4 A student has a 63% chance of passing his driving test. What is the chance that he does not pass?

5 A fair coin is tossed twice. Find the probability that the two tosses result in:

a two tails, **b** a head followed by a tail, **c** a head and a tail.

6 Two dice are thrown. Find the probability of:

a a double three, **b** a total score of five,
c a total greater than nine, **d** at least one five,
e neither a four nor a five appearing, **f** a two and a number greater than four,
g the same number on both dice, **h** a two on at least one die.

7 In a group of 60 tourists, 31 visited Queenstown, 33 visited Strahan and 14 visited both places. Draw a Venn diagram and find the probability that a tourist:

a visited Queenstown only, **b** visited Strahan only, **c** did not visit either place.

8 A die is thrown. Let A be the event that an odd number appears. Let B be the event that a number less than five appears.

a Are A and B mutually exclusive?
b Find

i $P(A)$ **ii** $P(B)$ **iii** $P(A \text{ and } B)$ **iv** $P(A \text{ or } B)$

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Review

9 The events A, B and C are independent, with $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{3}$ and $P(C) = \frac{3}{5}$. Use the product rule to find:

a $P(AB)$ **b** $P(BC)$ **c** $P(AC)$ **d** $P(ABC)$

10 a From a standard pack of 52 cards, two cards are drawn at random without replacement. Find the probability of drawing:

i a club then a diamond, **ii** two hearts,

iii a seven then an ace, **iv** the queen of hearts then the eight of diamonds.

b Repeat part **a** if the first card is replaced before the second card is drawn.

11 There is a 70% chance that Harold will be chosen for the boys' debating team, and an 80% chance that Grace will be chosen for the girls' team. Draw a probability tree diagram and find the chance that:

a Harold is chosen, but Grace is not, **b** Grace is chosen, but Harold is not,

c only one of Harold and Grace is chosen, **d** neither Harold nor Grace is chosen.

12 In a park there are four Labradors, six German shepherds and five beagles. If two dogs are selected at random, find the probability that:

a both are beagles, **b** neither is a Labrador,

c at least one is a Labrador, **d** a beagle and German shepherd are chosen.

13 There are 500 tickets sold in a raffle. The winning ticket is drawn and then the ticket for second prize is drawn, without replacing the winning ticket. If you buy 20 tickets, find the probability that you win:

a both prizes, **b** at least one prize.

14 In a certain experiment, the probability of events A and B are $P(A) = 0.6$ and $P(B) = 0.3$ respectively. In which of these cases are the events independent:

a if $P(A \cap B) = 0.18$ **b** if $P(A|B) = 0.3$ **c** if $P(A \cup B) = 0.72$

15 Two dice are rolled. A five appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than nine.

16 [A harder question]

When Xavier started work, he bought five shirts with matching ties, but he has since thrown the ties into the same drawer and hung the shirts at random in his wardrobe. Each day he picks a shirt and a tie at random to wear and then throws them in the used pile at the end of the day. Find the probability that:

a he wears a matching shirt and tie on Monday,

b he wears a matching shirt and tie on Thursday,

c he wears a matching shirt and tie every day of the week,

d he wears a matching shirt and tie every day of the week, except Tuesday.

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13 Discrete probability distributions

In the previous chapter, the probabilities of individual outcomes of an experiment were calculated. In this chapter, we look at the *probability distribution* of the experiment, which consists of all the probabilities of all the possible outcomes. This provides an overview of the whole experiment.

Having established the probability distribution, we can calculate the *expected value* or *mean* of the experiment, which is a measure of central tendency. We can also calculate the *variance* and the *standard deviation*, which are measures of spread. The standard deviation is the square root of the variance.

The last section on sampling invites readers to perform some experiments of their own. This should give them some feeling for how the theory of probability distributions, expected value, and variance is reflected in practical experiments that they carry out themselves. It also has suggestions about analysing data from the internet. The relationship between theoretical probabilities and practical experiments is at the centre of the much-discussed subject of *statistics*.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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13A The language of probability distributions

This section explains what a probability distribution is and introduces some necessary language. After a distinction is made between discrete and continuous probability distributions, the rest of the chapter deals only with discrete distributions, and continuous probability distributions are left until Year 12.

Discrete probability distributions

Four coins are tossed and the number of heads is recorded. There are five possible outcomes 0, 1, 2, 3 and 4, with different probabilities. Using the methods of the previous chapter, we can calculate all five probabilities and arrange them in a table — this particular example was done in Example 18 of Section 12E:

number of heads	0	1	2	3	4
probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

The set $\{0, 1, 2, 3, 4\}$ is the sample space of the experiment, because each time the experiment is performed, the result will be one, and only one, of these five outcomes. It is not a uniform sample space, however, because these possible outcomes are not equally likely.

The set of these possible outcomes, together with the corresponding probabilities given in the table, is a *probability distribution* — in the context of probability distributions, we usually use the word *values* for the possible outcomes in the sample space.

This particular probability distribution is a *numeric distribution* because all its values are numbers, and it is a *discrete distribution* because it is numeric and its values can be *listed*, which means that they can be written down in some order (in this case we listed them as 0, 1, 2, 3, 4).

1 DISCRETE PROBABILITY DISTRIBUTIONS

Suppose that the sample space of an experiment can be *listed* — meaning that it can be written down in some order. Suppose also that the possible outcomes, usually called *values*, are *numeric*.

Then these possible outcomes, together with their probabilities, are called a *discrete probability distribution*.

The five probabilities in this example were originally calculated by dividing the results of tossing four coins into 16 equally likely possible outcomes. Many of our examples, however, will have probabilities that are obtained without any appeal to such a uniform finite sample space — some probabilities may be obtained by other forms of calculation, and some may be estimates obtained empirically.

Random experiments and random variables

Throwing four coins, and recording the number of heads, is a *random experiment* because there is more than one possible outcome. Throwing four coins into an empty bucket, and counting the number of coins in the bucket, is a *deterministic experiment* because 4 is the only possible outcome.

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Denote by X the number of heads when four coins are thrown. This variable X is called a *random variable* because it is the result of a random experiment, and it can take the values 0, 1, 2, 3 and 4. Using the standard probability notation from the previous chapter, we can write

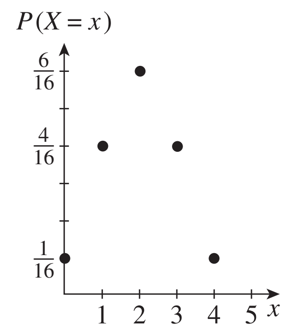
$$P(X = 3) = \tfrac{4}{16}.$$

This is read as 'the probability that X takes the value 3', and means that the probability that there are three heads is $\frac{4}{16}$. Similarly,

$$P(X > 2) = \tfrac{5}{16} \quad \text{and} \quad P(X \text{ is odd}) = \tfrac{8}{16} \quad \text{and} \quad P(X \text{ is prime}) = \tfrac{10}{16}.$$

Using x (lower case) for the number of heads, the table of probabilities can now be written as:

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$



The distribution is graphed to the right, with the values on the horizontal axis and the probabilities on the vertical axis.

Categorical data

The values of a probability distribution don't have to be numbers. They may be names, or objects, or any verbal description. The general word for this is 'category', and such data is called *categorical*.

For example, the table to the right was claimed on the web to be the world's population by continent in 2016.

Continent	Population	Per cent
Asia	4 436 224 000	59.7%
Africa	1 216 130 000	16.4%
Europe	738 849 000	9.9%
North America	579 024 000	7.8%
South America	422 535 000	5.7%
Oceania	39 901 000	0.5%
Total	7 432 663 000	100%

Now consider the following experiment. A person is chosen at random from the world's population, and the continent he or she lives in is recorded as the value of a random variable X. This experiment would be very hard indeed to carry out without bias, but we can imagine it. The percentages now become probabilities, which we can arrange in a probability distribution whose values are the six continents:

x	Asia	Africa	Europe	NA	SA	Oceania
$P(X = x)$	0.597	0.164	0.099	0.078	0.057	0.005

and, for example, $P(X = \text{Europe or North America}) = 0.099 + 0.078 = 0.177$.

2 RANDOM VARIABLE

- A *deterministic experiment* is an experiment with one possible outcome.
- A *random experiment* is an experiment with more than one possible outcome.
- A *random variable*, usually denoted by an upper-case letter such as X, is the result of running a random experiment. It may be *numeric* or *categorical*.

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The calculations in the next two sections require the values to be numeric, so distributions with categorical values will not greatly concern us. A word of warning, however. A probability distribution whose values are categorical and can be listed is usually classified as discrete — but in this course, such a distribution is definitely not discrete because it is not numeric.

Uniform probability distributions

A discrete or categorical probability distribution is called *uniform* if all its values have the same probability. This means that the values are *equally likely possible outcomes*, and its sample space is a *uniform sample space*. This is what we were dealing with throughout the previous chapter.

3 UNIFORM PROBABILITY DISTRIBUTION

A discrete or categorical probability distribution is called *uniform* if the probabilities of all its values are the same. That is, the values are equally likely possible outcomes, and the sample space is a uniform sample space.

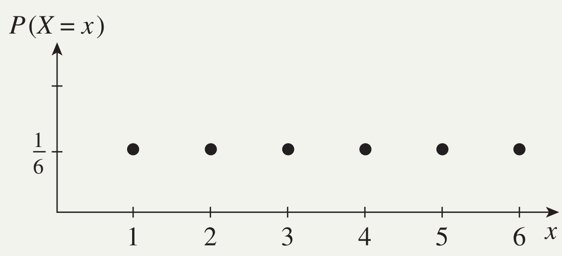
Example 1 13A

A die is thrown. Write out the probability distribution for the number shown on the die, and graph the distribution.

SOLUTION

Each value has probability $\frac{1}{6}$, so the table of the distribution is

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

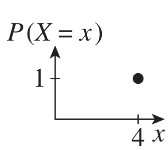


The probability distribution of a deterministic experiment

x	4
$P(X = x)$	1

To the right is the trivial probability distribution of the deterministic experiment, 'Throw four coins into an empty bucket and record the number of coins in the bucket.'

The sample space has one possible outcome 4, which is certain to occur, and so has probability 1. The graph is drawn underneath. Every deterministic experiment has a similar trivial distribution, and trivially it is a uniform distribution.



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What is a list?

Our definition of discrete probability distribution requires that the values can be listed. Obviously any finite set can be listed, but some infinite sets can also be listed. The best example is the set of whole numbers, which can be listed as $0, 1, 2, 3, 4, 5, \dots$. The list will not terminate, but every whole number will eventually appear once and once only. For this reason, an infinite set that can be listed is called *countably infinite*.

The integers can be listed, for example as $0,\ 1, -1, 2, -2, 3, -3, \dots$. But the set of real numbers in an interval such as $0 \le x \le 1$ cannot be listed, as the German mathematician Georg Cantor proved in the late 19th century.

4 LISTING THE VALUES OF THE SAMPLE SPACE

- Any finite set can be listed.
- The whole numbers, and the integers, do not form finite sets, but can be listed:
  $$0, 1, 2, 3, 4, \dots \quad \text{and} \quad 0, 1, -1, 2, -2, 3, -3, \dots$$
- The real numbers in an interval such as $0 \le x \le 1$ cannot be listed.

Example 2 13A

[An infinite sample space that can be listed]

Wulf is a determined person. He has decided to keep tossing a coin until it shows heads. Adolfa is counting how many times he tosses the coin. What is the probability distribution for this experiment?

SOLUTION

The result of this experiment is how many times Wulf tosses the coin to get a head. If he is very, very unlucky, he may have to toss the coin many, many times.

$$P\,(\text{Wulf requires 1 toss}) = P\,(\text{H}) = \tfrac{1}{2}$$
$$P\,(\text{Wulf requires 2 tosses}) = P\,(\text{TH}) = \tfrac{1}{4}$$
$$P\,(\text{Wulf requires 3 tosses}) = P\,(\text{TTH}) = \tfrac{1}{8}$$
$$P\,(\text{Wulf requires 4 tosses}) = P\,(\text{TTTH}) = \tfrac{1}{16}, \quad \text{and so on, giving the table:}$$

x	1	2	3	4	5	$\cdots$	n	$\cdots$
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$

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Continuous probability distributions

In a continuous probability distribution, the sample space is typically a closed interval such as $0 \le x \le 100$ on the real number line. Examples are the heights or weights of people, and speeds of cars on an expressway.

Suppose that the police have set up a speed camera to measure the speeds of oncoming cars. If we regard the speed as a real number, then we are dealing with a continuous probability distribution. If, however, we take into account that the speed camera only records speeds correct to the nearest 0.01 km/h, then we are dealing with a discrete probability distribution. Such distinctions arise all the time in statistics, because any measurement, no matter how accurate, will only be correct to some number of decimal places.

In a continuous probability distribution, the probability of any one particular value occurring is zero. For example, a speed of 56.0123456789 km/h has almost no chance of ever being recorded, even if it could be measured. But worse, the decimal expansion of most real numbers never terminates or repeats. The probabilities involved in a continuous distribution must therefore be recorded as the probabilities that the random variable lies within an interval, for example as $P(55 \le X \le 60)$. The required machinery for this is *integration*, which is the other process in calculus, and is not covered until Year 12, so continuous probability distributions will not concern us in this chapter.

5 CONTINUOUS PROBABILITY DISTRIBUTIONS

In a continuous probability distribution, the sample space is typically a closed interval such as $0 \le x \le 100$ on the real number line.

The sample space is therefore infinite and cannot even be listed.

Exercise 13A

FOUNDATION

1 State whether each probability distribution is *numeric* or *categorical*. If it is numeric, state whether it is *discrete* or *continuous*.

- **a** The number showing when a die is thrown.
- **b** The weight of a randomly chosen adult male in Australia.
- **c** Whether it rains or not on a spring day in Sydney.
- **d** The daily rainfall in Sydney on a September day.
- **e** The colour of a ball drawn from a bag containing four red and three green balls.
- **f** The colours of two balls drawn together from a bag containing four red and three green balls.
- **g** The shoe size of a randomly chosen adult female in Australia.
- **h** ATAR results for a particular year.

2 **a** Complete the table for the probability distribution obtained when two coins are thrown one after the other and the successive results are recorded.

outcome	HH	HT	TH	TT
probability				

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Look at the probabilities you have obtained. What sort of distribution is this?

b Complete the table for the probability distribution obtained when two coins are thrown and the numbers of heads and tails recorded.

outcome	2 heads	1 head and 1 tail	2 tails
probability			

3 Construct tables for these probability distributions.

a A ball is drawn from a bag containing four red and three green balls, and its colour is noted.

b One hundred tickets are sold in the class lottery. Three friends Jack, Kylie and Lochlan buy six, eight and four tickets respectively. The winner is the person whose ticket is drawn first. Construct the distribution table of the probability that the winner is Jack, Kylie, Lochlan, or Other.

c A letter is chosen at random from the word 'Parramatta'.

d A whole number is chosen at random between 1 and 1000 inclusive, and the number of digits is recorded.

e A whole number between 10 and 19 inclusive is selected at random. A record is made whether it is even, prime, or neither.

4 Each experiment below returns a numerical result. Define a random variable X for each experiment and record its distribution in a table.

a The word lengths in the sentence, 'The ginger cat ran off with the meat.'

b Two coins are thrown and the number of heads recorded.

c A digit is chosen at random from the 12-digit number 1.41421356237.

d Jeff has a collection of marbles with digits on them. He puts marbles numbered 1, 1, 2, 4 into one bag, then puts marbles numbered 2, 3, 3, 5 into a second bag. He randomly selects one of the bags, then randomly selects a marble from that bag and records its number.

5 Amy scoops up coins at random from a pile containing one 10 c coin and two 5 c coins.

a Name the 10 c coin T and the two 5 c coins F1 and F2, and write down as sets the seven possible non-empty scoops.

b Let the random variable X be the value of the money that she picks up. Assuming that she is equally likely to pick up any of these seven non-empty scoops, draw up a probability distribution table for X.

6 Which of these tables are probability distributions? Remember that the probabilities must be all non-negative and add to 1.

a

x	1	2	3	4
$P(X = x)$	0.1	0.6	0.2	0.1

b

x	1	2	3	4
$P(X = x)$	0.5	0.3	−0.2	0.4

c

x	1	2	3	4
$P(X = x)$	0.25	0.25	0.25	0.25

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d

x	1	2	3	4
$P(X = x)$	30%	20%	40%	10%

e

x	1	2	3	4
$P(X = x)$	0.7	0.2	0.4	0.2

f

x	1	2	3	4
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$

7 A discrete probability distribution is tabulated below.

x	0	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.3	0.25	0.05	0.1

Find:

a $P(X = 1)$ **b** $P(2 \le X \le 4)$ **c** $P(1 \le X < 4)$
d $P(X = 6)$ **e** $P(X \le 2)$ **f** $P(X < 4)$
g $P(X \ge 1)$ **h** $P(X > 1)$ **i** $P(X \text{ is even})$

DEVELOPMENT

8 These questions all require probability tree diagrams with identical stages.

a A pack of cards has 52 cards, of which 12 are court cards (jack, queen or king). A card is drawn at random, and it is recorded whether or not it is a court card. The card is replaced, then a second card is drawn and again it is recorded whether or not it is a court card.

i Draw up a probability tree diagram with two identical stages.

ii Let X be the number of court cards drawn. Use the results of part **i** to draw up a probability distribution table for X.

b Three dice are thrown, and the number of even numbers is recorded.

i Draw up a probability tree diagram with three identical stages, each stage being, 'Throw a die and record whether the result is even or odd'.

ii Let X be the number of even numbers when the three dice are thrown. Use the results of part **i** to draw up a probability distribution table for X.

c A class has 10 girls and 15 boys. Three times the teacher chooses a student at random to answer a question, not caring whether he has called that name before. Use a similar method to draw up a probability distribution table for the number of girls called.

d The Spring Hill Zoo has 20 friendly wallabies. Six are from Snake Gully, five are from Dingo Ridge, and nine are from Acacia Flat. On three days last week the zookeepers selected a wallaby at random and took it to breakfast with them, then returned it to the enclosure. Draw up a probability distribution table for the number X of Snake Gully wallabies taken to breakfast.

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9 Find the unknown constant a in these probability distributions. Use the facts that $0 \le P(X = x) \le 1$ for each value x, and that the sum of the probabilities is 1.

a

x	1	2	3	4	5
$P(X = x)$	$4a$	$2a$	$9a$	$3a$	$7a$

b

x	1	2	3	4	5
$P(X = x)$	$3a$	a	$4a$	a	$5a$

c

x	-2	-1	0	1	2
$P(X = x)$	a	$3a$	$5a$	$7a$	$11a$

d

x	10	20	30	40	50
$P(X = x)$	$1 - 3a$	a	$1 - 9a$	$1 - 10a$	a

e

x	1	2	3	4	5
$P(X = x)$	$0.2a$	$0.1a$	$0.5a$	$0.1a$	$0.1a$

10 These questions all involve probability tree diagrams in which the stages are different because the sampling has been done 'without replacement'.

a A bag contains six marbles numbered 1, 2, 3, 4, 5, 6. A marble is drawn, and it is recorded whether the number is odd or even. Without replacing the marble, a second marble is drawn, and it is recorded again whether the number is odd or even.

i Draw up a probability tree diagram for the four outcomes.

ii Draw up a probability distribution table for the number X of even numbers chosen.

b Two students are chosen from a group of four boys and two girls. Use a similar method to draw up a probability distribution table for the number X of girls chosen.

c Five tiles marked E, E, E, R, T are turned upside down, and two are selected at random one after the other.

i Draw a probability tree diagram for the seven possible selections of two tiles.

ii Draw up a probability distribution table for the number X of Es selected.

11 A small pack of cards consists of three 4s, two 2s and one 5 (the suit is not important for this experiment). A card is selected at random and the value recorded. It is then returned to the pack. This is repeated. Complete the probabilities for the outcomes of the categorical random variable X in the following table.

x	22	44	55	24 or 42	25 or 52	45 or 54
$P(X = x)$						

[This is an example of a *multinomial distribution* — a multi-stage experiment with identical stages, each with multiple outcomes. Here there are two stages, because we select a card twice, and the stages are identical because the card is replaced.]

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12 Construct tables for the following distributions. In each case the outcome is categorical, namely a pair of colours or a pair of suits.

- **a** A ball is drawn from a bag containing four red and three green balls, and the colour is recorded. The ball is returned, a second ball is drawn, and the colour also recorded.
- **b** A ball is drawn from a bag containing four red and three green balls, and the colour is recorded. A second ball is drawn without replacement and the colour also recorded.
- **c** A pair of cards are drawn from a pack and the suits are noted (but not their order).

13 [Simulation experiment]
Cut five identical pieces of paper or cardboard. Label three of them *Red* and the other two *Green*. Place the pieces in a hat or bag.

In this experiment, two pieces of paper are drawn out, the number of greens is recorded, and the pieces are then returned. Let X be the random variable for the number of greens.

- **a** Copy the table below for recording the results of your experiment.

x	0	1	2
Tally			

- **b** Repeat the experiment 40 times, recording each outcome in the tally column.
- **c** Add two rows to your table, as below. Complete the table to calculate the experimental probabilities of each outcome.

x	0	1	2
Tally			
Frequency f			
Relative frequency $f_r = \dfrac{f}{40}$			

- **d** Compare your results with other members of the class. Does this suggest that the results are accurate? Can you explain any differences in the results obtained?
- **e** Calculate the theoretical probabilities for this experiment and add this as a further row to your table. How do your results compare?
- **f** Comment on any aspects of your experimental design that assist the accuracy of the results. Is there any way you could have improved the reliability and randomness of your experiment?

ENRICHMENT

14 A bag has six marbles marked $1, 2, 3, 4, 5, 6$. Three marbles are drawn in succession, without being replaced, and the number of even-numbered marbles is recorded. Let X be the number of marbles with even numbers chosen. Draw up a probability distribution table for X.

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15 Find the unknown constant a in these probability distributions.

a

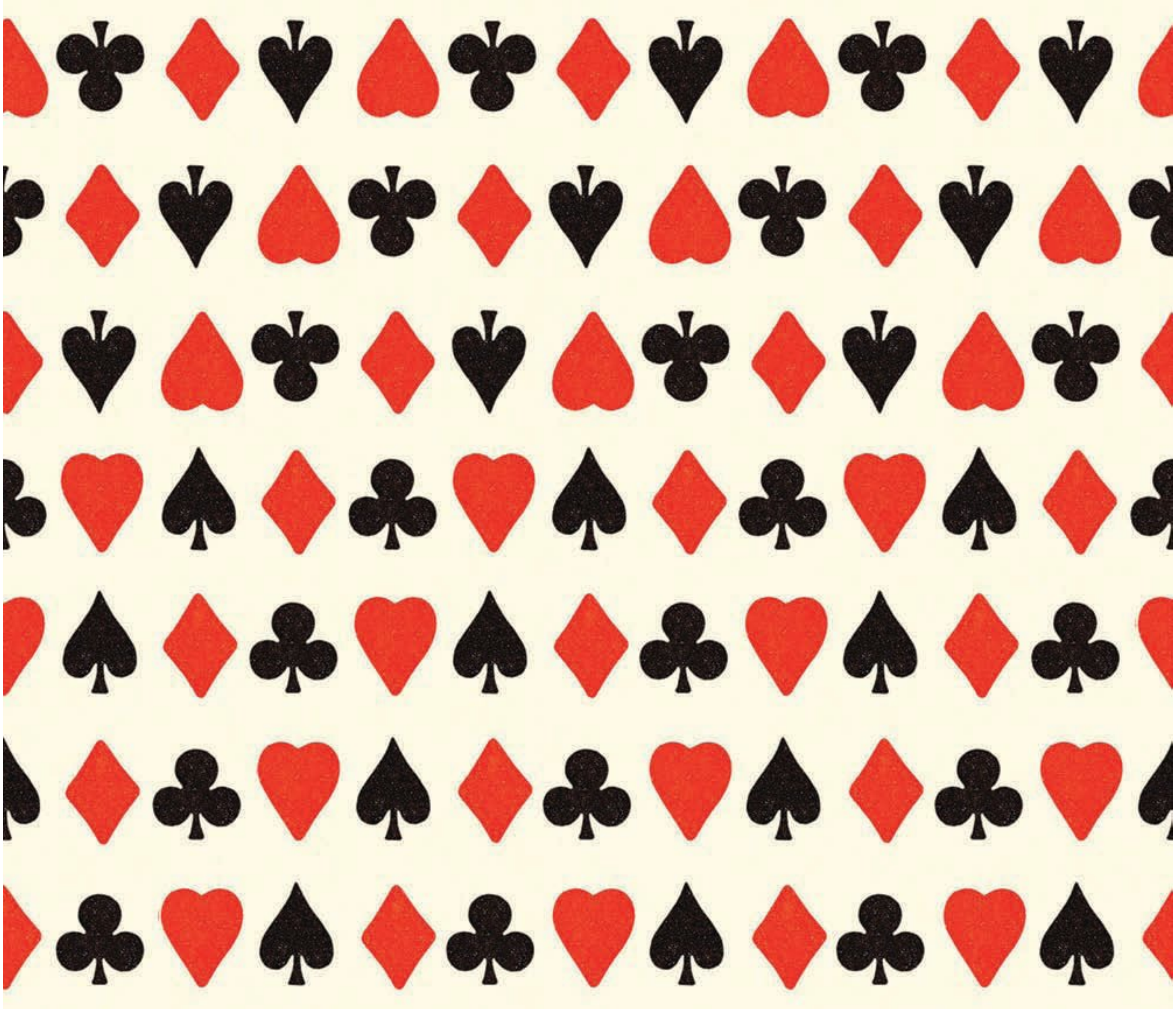
x	1	2	3	4	5
$P(X = x)$	$a(a + 1)$	$3a^2$	$1 - 3a$	$1 - 4a$	a

b

x	1	2	3	4
$P(X = x)$	$\frac{1}{6}(a + 1)$	$\frac{1}{4}a^2$	$\frac{1}{8}(5 - 3a)$	$\frac{1}{6}(3 - 2a)$

16 **a** Kylie has a hand of four cards: the 7 of hearts, 7 of diamonds, 6 of clubs and 8 of spades. She takes three of the cards at random, places them on the table, and adds the cards' values. Construct a table showing the probability distribution of the sum.

b Repeat this experiment if Kylie has a hand of five cards, including three 7s, a 6 and an 8, and still chooses three cards at random.



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13B Expected value

The first experiment in the previous section was, 'Toss four coins and count the number of heads':

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

The notation $P(X = x)$ will be too unwieldy for what follows, so we will write it instead using standard function notation as $p(x)$. For each value x in the distribution, $p(x)$ means 'the probability that $X = x$'.

$p(x) = P(X = x)$, for all values x in the distribution.

With this more concise notation, the table becomes:

x	0	1	2	3	4
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Where is the centre of the distribution?

Someone looking at this table may want to say,

'I would expect to get the result two heads.'

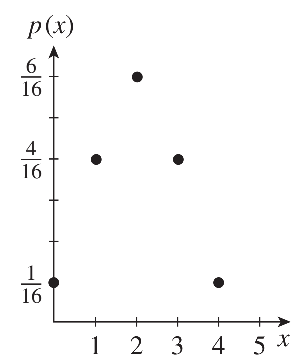
When asked, she may try to explain what she means by saying,

'When we do the experiment a large number of times, the average number of heads will be about 2, and the more times we do it, the closer the average will be to two'.

This is a good explanation, because it is based on the interpretation of probability as running the experiment a large number of times and averaging things out.

She may also try to explain it by symmetry:

'If I plot the five probabilities, the pattern is symmetric about the value 2'.



This is also a good explanation, but is completely different. It ignores the idea of running trials, and is based solely on the symmetry of the distribution's graph. It observes that each smaller number of heads on the left of $x = 2$ is balanced out by a larger number of heads on the right of $x = 2$ with the same probability, so $x = 2$ can be regarded as the centre of the distribution.

This section introduces a clearer and more general way of dealing with this 'expected value', which is better described as the 'mean' of the distribution. From now on, values must be numeric and not categorical.

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Expected value

Everyone is familiar with the idea of the mean or average of a set of n numbers — add them up and divide by n,

$$\text{mean} = \frac{0 + 1 + 2 + 3 + 4}{5} = \frac{10}{5} = 2.$$

Alternatively, multiply each number by $\frac{1}{n}$ and add the results:

$$\text{mean} = 0 \times \tfrac{1}{5} + 1 \times \tfrac{1}{5} + 2 \times \tfrac{1}{5} + 3 \times \tfrac{1}{5} + 4 \times \tfrac{1}{5} = \tfrac{10}{5} = 2.$$

But to calculate the mean of a discrete probability distribution, we need a *weighted mean* of the values, giving more *weight* to more likely values, which will occur more often, and less weight to the less likely values, which will occur less often. So instead of multiplying each value by $\frac{1}{n}$, we *weight* each value by its probability,

$$\begin{aligned}\text{weighted mean} &= 0 \times \tfrac{1}{16} + 1 \times \tfrac{4}{16} + 2 \times \tfrac{6}{16} + 3 \times \tfrac{4}{16} + 4 \times \tfrac{1}{16}\\ &= 0 + \tfrac{4}{16} + \tfrac{12}{16} + \tfrac{12}{16} + \tfrac{4}{16}\\ &= \tfrac{32}{16}\\ &= 2, \text{ which is called the } \textit{expected value } \mathrm{E}(X).\end{aligned}$$

This process is called *weighting the values by their probabilities*, and is the way the expected value $\mathrm{E}(X) = 2$ of this distribution is calculated. Here the result 2 happens to be the same as the ordinary mean of the five values, but this is only because of the mirror symmetry of the probabilities about $x = 2$.

The formula for expected value

The usual notation for this procedure will be completely unfamiliar. In any distribution, the weighting of a value x by its probability $p(x)$ means that we take the product of the value x and its probability $p(x)$, written as the product

$$xp(x).$$

Now we need to add up all these products, for all the values in the distribution. The symbol $\sum$ is the mathematical symbol for 'take the sum'. It is the Greek upper-case sigma corresponding to the Latin letter 'S' and standing for 'sum'. Thus the formula for the expected value $\mathrm{E}(X)$ is written as

$$\mathrm{E}(X) = \sum xp(x), \text{ summing over all values of the distribution.}$$

The phrase, 'summing over all values of the distribution' means that we take all the products $xp(x)$, for all the values of the distribution, and add them up.

Setting out the calculation of expected value

The actual calculation is best set out not as shown above, nor by substituting into the formula, but in tabular form, extending the table of values and probabilities of the distribution with an extra row and an extra column:

x	0	1	2	3	4	Sum
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1
$xp(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2

as we obtained before.

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The probabilities must add to 1 — summing the middle row is only a check.

Question 3 of Exercise 13B is an investigation that involves throwing four coins a large number of times and seeing how close the average value is to the expected value 2 calculated above.

An alternative setting out using columns rather than rows

Many people prefer to use columns rather than rows for these calculations. There is a very good reason for this — it is easier to add up a column of numbers than to add up a row of numbers.

If you use columns rather than rows, then your table will be *transposed*, meaning that the rows becomes columns and the columns become rows.

This setting out is shown to the right.

x	$p(x)$	$xp(x)$
0	$\frac{1}{16}$	0
1	$\frac{4}{16}$	$\frac{4}{16}$
2	$\frac{6}{16}$	$\frac{12}{16}$
3	$\frac{4}{16}$	$\frac{12}{16}$
4	$\frac{1}{16}$	$\frac{4}{16}$
Sum	1	2

Expectation is a measure of central tendency

As mentioned earlier, the expected value $\text{E}(X)$ of a distribution is also called the *mean* of the distribution, because it is obtained as the weighted mean of the values. It is a *measure of central tendency*, meaning that it is a measure of where the middle of the distribution lies.

For this reason, the expected value $\text{E}(X)$ is often assigned the pronumeral μ, a Greek lower-case letter corresponding to Latin 'm', and standing for 'mean'.

6 EXPECTED VALUE OF A DISCRETE PROBABILITY DISTRIBUTION

The *expected value* of a discrete probability distribution with numeric data is the weighted mean of the values x weighted by the probabilities $p(x)$,

$$\text{E}(X) = \sum xp(x), \text{ summing over all values of the distribution.}$$

- The symbol $\sum$ is the Greek upper-case sigma, and stands for 'sum'.
- Each term in the sum is the product $xp(x)$ of a value and its probability.
- Take all these products $xp(x)$, for all the values of the distribution, and add them up.
- The expected value is also called the *mean* of the distribution. It is usually assigned the pronumeral μ — the Greek letter 'mu' stands for 'mean'.
- The expected value is a *measure of central tendency*.

The example above is symmetric, which is why the mean of the values turned out to be the same as the expected value. Example 3 shows how well the idea of expected value is able to pick out the central tendency in quite unsymmetric data.

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Example 3 13B

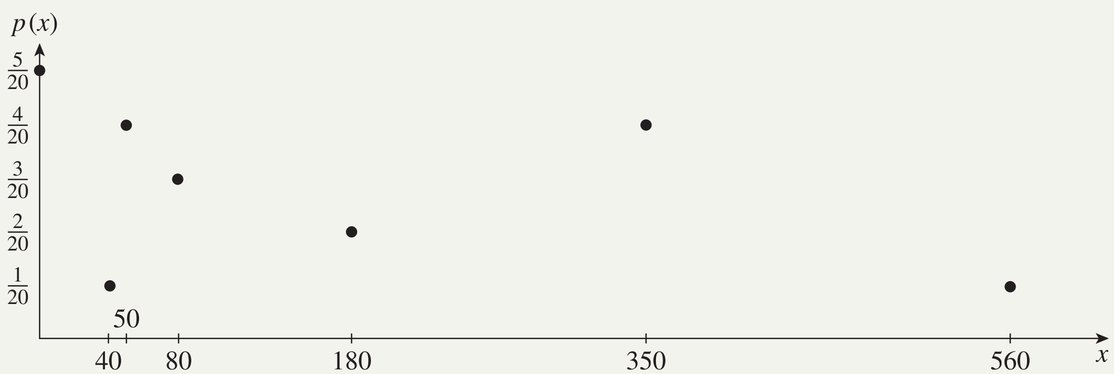
Twenty friends go out to dinner at a Vietnamese restaurant. One has $560 cash, four have $350 cash, two have $180 cash, three have $80 cash, four have $50 cash, and one has $40 cash. Five have no cash, and are expecting to borrow from the others. An armed robber bursts in and seizes one of the friends at random. She threatens him, grabs all his cash, and runs away. Graph the distribution, and find her expected criminal gain.

SOLUTION

Let $\$x$ be the amount the criminal seizes. The probability distribution with the added row and column is

x	0	40	50	80	180	350	560	Sum
$p(x)$	$\frac{5}{20}$	$\frac{1}{20}$	$\frac{4}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{1}{20}$	1
$xp(x)$	0	2	10	12	18	70	28	140

Hence her expected criminal gain is $140.



Expected value of a uniform distribution

In a uniform distribution, all the values have the same probability. We could add the values and then divide by the number of values, but it is better to continue with the same setting out. There are two things to notice here:

- The sample space of a uniform distribution consists of equally likely possible outcomes. These were the sorts of sample spaces that most of the calculations in Chapter 12 were based on.
- In a uniform distribution with n values, each value has probability $\frac{1}{n}$. Thus to weight each value by its probability, we multiply it by $\frac{1}{n}$, so that calculating the expected value is the same as calculating the mean of the values.

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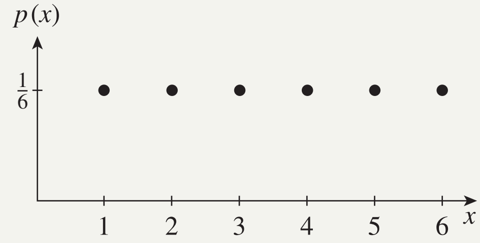
Example 4 13B

A die is thrown. Graph the distribution and find its expected value.

SOLUTION

Each value has probability $\frac{1}{6}$, so the table is

x	1	2	3	4	5	6	Sum
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1
$xp(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{21}{6}$



Hence the expected value on the die is $\mu = 3\frac{1}{2}$.

Do you find this answer just a little unsettling? How can we 'expect' an answer that we will never see on the die?

The same thing happened in the restaurant robbery in Example 3 — the expected value was \$140, but no one amongst the friends had \$140 cash. There is actually no problem here, because the expected value is just a fancy name (and perhaps a slightly misleading name) for the mean of the distribution, and we know from countless examples that the mean of a set of numbers is usually not one of the numbers.

Exercise 13B

FOUNDATION

1 Complete each table and find the expected value $E(X)$ for the distribution, where the function $p(x)$ is defined as usual by $p(x) = P(X = x)$, for all values x.

a

x	0	1	2	3	Sum
$p(x)$	0.4	0.1	0.2	0.3	
$x\,p(x)$					

b

x	2	4	6	8	Sum
$p(x)$	0.1	0.4	0.4	0.1	
$x\,p(x)$					

c

x	−50	−20	0	30	100	Sum
$p(x)$	0.1	0.35	0.4	0.1	0.05	
$x\,p(x)$						

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2 A simple gambling game involves the throw of a die. Players are charged 40 cents if the die shows 1, 2 or 3, they are charged nothing for 4, and they receive 30 cents or 60 cents respectively for 5 or 6. Let the random variable X be the payout to the player.

x	$p(x)$	$x\,p(x)$
-40		
0		
30		
60		
Sum	1	

a Copy and complete the table to the right written this time in columns instead of rows.

b Calculate the expected value by summing the third column.

c What does this expected value represent?

d How much profit would the casino expect to make on 100 games?

3 Four coins are tossed and the number of heads recorded. In the theory for this section, we constructed the table reproduced below. The expected value was calculated to be 2.

x	0	1	2	3	4	Sum
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1
$x\,p(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2

In this question we shall simulate the experiment and see if we have experimental agreement with these results.

a Toss four coins and record the number of heads, using a tally row and filling in a copy of the table below. Repeat this experiment 32 times.

x	0	1	2	3	4	Sum
Tally						—
Frequency f						
Relative frequency $f_r = \frac{f}{32}$						

b Check whether your frequencies agree reasonably with the results 2, 8, 12, 8, 2 that the theoretical distribution would predict. Check also whether the relative frequencies agree reasonably with the corresponding probabilities 0.0625, 0.25, 0.375, 0.25, 0.0625. If your results do not agree closely, you might like to repeat the experiment a further 16 or 32 times (and divide by 48 or 64 to calculate the relative frequencies.)

c Using your calculator, or by hand using the table below, calculate the mean for this data using the values x and their frequencies.

x	0	1	2	3	4	Sum
Frequency f						
$x \times f$						

$$\text{Mean} = \frac{\text{sum of } xf}{\text{sum of frequencies}}$$
$$= \cdots$$

d Does your answer for the mean approximate the theoretical expected value $E(X) = 2$?

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4 [An alternative notation]
Sometimes the values of a discrete probability experiment are indexed $x_1, x_2, x_3, \ldots$, with corresponding probabilities $p_1, p_2, p_3, \ldots$. These values and probabilities can be abbreviated to x_i and p_i, where i is called the *index* variable because it *indexes* the values and probabilities. The calculations then proceed as before.

Find the expected value for these distributions.

a

x_i	2	4	6	8	10	Sum
p_i	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	
$x_i p_i$						

b

x_i	−3	1	2	5	6	Sum
p_i	0.1	0.3	0.2	0.3	0.1	
$x_i p_i$						

DEVELOPMENT

5 Fiona's mother tells her that she may buy a pencil, eraser or pen from the stationery shop. A pencil costs \$1.50, an eraser costs \$2.10 and a pen costs \$2.40. There are five boxes of pencils, four boxes of erasers and three boxes of pens.

a If she chooses a box at random and takes one item from the box, what is the expected cost?

b If Fiona and her 99 friends each choose at random, what would be the expected total cost?

6 In this question we investigate what happens to the expected value if we transform the random variable, such as by doubling all the values, or increasing them all by 1. Because we are dealing with more than one random variable, we have retained the notation $P(X = x)$, $P(Y = y)$, and so on from Section 11A.

A random variable X records the outcomes of a spinner with sectors labelled 1, 2, 3, 4. The spinner is biased because it has been weighted.

x	1	2	3	4
$P(X = x)$	0.1	0.1	0.5	0.3
$x \times P(X = x)$				

y	2	4	6	8
$P(Y = y)$	0.1	0.1	0.5	0.3
$y \times P(Y = y)$				

z	2	3	4	5
$P(Z = z)$	0.1	0.1	0.5	0.3
$z \times P(Z = z)$				

a Copy and complete the table to calculate $E(X)$.

b A second random variable defined by $Y = 2X$ records twice the outcome of the weighted spinner.

i Calculate $E(Y)$ from this table.

ii Does your result agree with the result $E(aX) = aE(X)$?

c A third random variable defined by $Z = X + 1$ records the outcome of the weighted spinner plus one.

i Calculate $E(Z)$ from this table.

ii Does your result agree with the result $E(X + b) = E(X) + b$?

The general result that these examples illustrate is

$$E(aX + b) = aE(X) + b, \quad \text{for all constants } a \text{ and } b.$$

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7 A random variable X is known to have the property that $\mathrm{E}(X) = 5$. Use the formula $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ to calculate:

a $\mathrm{E}(3X)$ b $\mathrm{E}(X + 5)$ c $\mathrm{E}\left(\frac{1}{2}X\right)$

d $\mathrm{E}(X - 2)$ e $\mathrm{E}(10 - 2X)$ f $\mathrm{E}(4X - 2)$

8 A coin is tossed three times and the number of heads is recorded. Construct a table showing the probability distribution, and calculate the expected value.

(This is an example of a *binomial distribution* — a multi-stage experiment with identical stages, where at each stage there are two possible outcomes. Here there are three stages, because we toss the coin three times and there are two possible outcomes, heads or tails, for each throw.)

9 Two cards are selected at random from a standard pack, and the number of hearts is recorded. Note that this is equivalent to selecting two cards without replacement. Construct a table showing the probability distribution and calculate the expected value.

(This is an example of a hypergeometric distribution — a multi-stage experiment involving two possible outcomes, where at each stage the object is selected without replacement. Here there are two stages because we select two cards.)

10 [Simulation experiment]
Two dice are thrown. Let X be the difference between the two resulting numbers, so that the sample space consists of the integers 0, 1, 2, 3, 4, 5.

a Conduct an experiment to determine the experimental probability of the six outcomes. You should conduct the experiment 36 times and record your results in a copy of the table below.

x	0	1	2	3	4	5	Sum
Tally							—
Frequency f							
Relative frequency $f_r = \dfrac{f}{36}$							
$x \times f_r$							

b Use the relative frequencies as estimates for the probabilities of each outcome. Calculate the experimental expected value (the symbol for this experimental expected value is $\bar{x}$ to distinguish it from the theoretical expected value μ).

c Compare your results with other members of the class.

d i Calculate the theoretical probabilities, using the normal 6×6 array of dots.

ii Also calculate the theoretical expected value by copying and completing the table below.

x	0	1	2	3	4	5	Sum
$p(x)$							
$xp(x)$							

e How do your results compare?

f If you think your results are inaccurate, consider any design faults in your experiment.

g Do your results agree more closely with the theoretical probabilities if you increase the number of trials?

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ENRICHMENT

11 The backers of the game in Question 2 have weighted the die so that it is 50% more likely to turn up each of the results 1, 2 or 3 as to turn up 4, 5 or 6. (This is illegal.)

a What are the probabilities now of each outcome?

b Find the expected value now.

12 A company is designing a new gambling slot machine for a casino. A player pulls a lever and the machine randomly rolls three outcomes from Orange, Strawberry and Apple, for example AOS or SSO. The probabilities of the various fruits turning up are in the ratio 1 : 2 : 3, so that Apple is three times as likely to occur in a given position as Orange. The machine pays out in the ratio 11 : 2 : 1 if it turns up three Oranges, three Strawberries or three Apples, respectively. No other combination pays the player. A player initiates a roll by feeding a \$1 coin into the machine.

a Find the probabilities that Orange, Strawberry or Apple turn up in a given position.

b Suppose the triple AAA pays \$$k$. Construct a probability distribution table showing the outcomes OOO, SSS, AAA and Other for the random variable X representing the payout.

c Determine how much a player would get for the highest paying outcome OOO, if the machine is designed so that it will just break even (this is unlikely).

13 [Expected value when the sample space is infinite, but can be listed]
In Example 2 in Section 11A, Wulf tosses a coin repeatedly until it shows a head. The probability table for the number X of tosses is infinite:

x	1	2	3	4	5	$\cdots$	n	$\cdots$
$p(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$

Thus to calculate the expected value we need to calculate the infinite sum:

$$\mu = 1 \times \tfrac{1}{2} + 2 \times \tfrac{1}{4} + 3 \times \tfrac{1}{8} + 4 \times \tfrac{1}{16} + 5 \times \tfrac{1}{32} + 6 \times \tfrac{1}{64} + \cdots$$

a Write down the sum for 2μ and by carefully subtracting like fractions, show that

$$2\mu - \mu = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \cdots$$

and hence write down a second infinite sum for μ.

b Repeat the process of doubling and subtracting with this new infinite sum for μ, and hence calculate μ.

c Explain what this expected value means practically and design an experiment to test your theoretical result.

Note: The operations on infinite series used here are valid for these particular series because they are convergent (this concept is explained in Year 12, and even then the proofs are difficult). The operations are certainly not true for all infinite series.

14 [St Petersburg Paradox]
Patrons at a casino may play a game involving a single coin, which is tossed until a head turns up. A jackpot is set aside. It initially contains \$2, but its value is doubled on each toss of the coin turning up a tail. The winner receives the contents of the pot when the first head is tossed. Thus the player would win \$2 if the initial toss is a head, \$4 if the second toss is the first head, \$8 if the third toss is the first head, and so on.

A manager at the casino suggests that a player should be charged \$40 to play the game. Calculate the player's expected return, and comment on the manager's advice.

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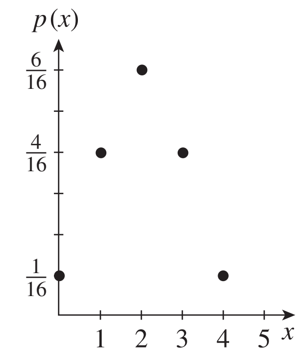
13C Variance and standard deviation

After discussing E(X), which is a measure of the central tendency of a discrete probability distribution, we now turn to variance and standard deviation, which are two measures of spread. Variance is easier to define and calculate, so we will work first with the variance. But standard deviation, which is just the square root of the variance, is important because it has the same units as the values of the distribution, and increases in proportion when all the values are increased by some factor.

The variance — first formula

Return again to the number of heads when four coins are tossed:

x	0	1	2	3	4
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$



In the last section, the expected value of this distribution — its mean — was found to be $\mu = 2$. We now want a measure of how spread out the values are from the mean. The usual measure of this is the *variance* Var(X).

To find the variance, take the difference $x - \mu$ of each value from the mean μ. This difference is called the *deviation* of the value x from the mean μ. Then we square the deviation to give $(x - \mu)^2$. This squared deviation is a good measure of how far x is from the mean for two reasons:

- The square $(x - \mu)^2$ is always a positive number or zero, whether the value x is on the left or the right of μ, because of the squaring.
- The square $(x - \mu)^2$ gets larger as x moves away from the mean, and the square means it gets larger very quickly. For example, doubling the distance of x from the mean has four times the effect on the square.

Then, as with expected value, take the weighted mean of these squared deviations $(x - \mu)^2$, weighted as before by the probabilities. Using sigma notation,

$$\text{Var}(X) = \sum (x - \mu)^2 p(x), \quad \text{summed over the distribution.}$$

As with expected value, the calculation is best done by adding more rows to the probability distribution table. Here are the calculations for the number of heads on four coins. The table calculates the expected value and variance together:

x	0	1	2	3	4	Sum	
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1	(a check)
$xp(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2	(the mean μ)
$(x - \mu)^2$	4	1	0	1	4	—	
$(x - \mu)^2 p(x)$	$\frac{4}{16}$	$\frac{4}{16}$	0	$\frac{4}{16}$	$\frac{4}{16}$	1	(the variance)

Thus the mean is 2 and the variance is 1.

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Using columns instead of rows

To the right is the alternative setting out that uses columns instead of rows.

Take your pick — use the layout that you find more convenient.

x	$p(x)$	$xp(x)$	$(x-\mu)^2$	$(x-\mu)^2 p(x)$
0	$\frac{1}{16}$	0	4	$\frac{4}{16}$
1	$\frac{4}{16}$	$\frac{4}{16}$	1	$\frac{4}{16}$
2	$\frac{6}{16}$	$\frac{12}{16}$	0	0
3	$\frac{4}{16}$	$\frac{12}{16}$	1	$\frac{4}{16}$
4	$\frac{1}{16}$	$\frac{4}{16}$	4	$\frac{4}{16}$
Sum	1	2	—	1

The variance as an expected value

The variance is the weighted mean of $(x-\mu)^2$, weighted by the probabilities. Thus the variance is the expected value of $(x-\mu)^2$,

$$\text{Var}(X) = \text{E}\big((X-\mu)^2\big).$$

This is a very useful form of the variance formula, because it will apply also to the continuous probability distributions introduced in Year 12.

Example 5 — 13C

A die is thrown, producing a uniform distribution. Find the mean and variance of the distribution using the formula $\text{Var}(X) = \text{E}\big((X-\mu)^2\big)$.

SOLUTION

x	1	2	3	4	5	6	Sum	
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1	(a check)
$xp(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$3\frac{1}{2}$	(the mean μ)
$(x-\mu)^2$	$\frac{25}{4}$	$\frac{9}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{9}{4}$	$\frac{25}{4}$	—	
$(x-\mu)^2 p(x)$	$\frac{25}{24}$	$\frac{9}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{9}{24}$	$\frac{25}{24}$	$\frac{70}{24}$	(the variance)

Hence the mean is $3\frac{1}{2}$ and the variance is $\frac{70}{24} = 2\frac{11}{12}$.

The variance — alternative formula

The method of calculation in the four-tossed-coins table above worked well because the mean was a whole number. Things became a little turgid in the subsequent throwing-a-die experiment in Example 5, because the mean $3\frac{1}{2}$ was a fraction. If the denominator had been a larger number such as 287 rather than 2, things would have looked a great deal messier.

Fortunately, there is another formula for variance, again using weighted means, and this alternative formula makes calculations much easier:

$$\text{Var}(X) = \sum x^2 p(x) - \mu^2.$$

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This formula uses the squares of the values, and takes the weighted mean of these squares, weighted as always by the probabilities. Thus, like the first formula, it can be written in terms of expected values, this time in terms of the expected value of X^2,

$$\text{Var}(X) = \text{E}(X^2) - \mu^2.$$

The alternative formula makes the calculations more straightforward, whether or not μ is a whole number. The following table uses this alternative formula to recalculate the variance for the distribution obtained by throwing four coins and counting the heads.

x	0	1	2	3	4	Sum	
$p(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	1	(a check)
$xp(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$	2	(the mean μ)
$x^2p(x)$	0	$\frac{4}{16}$	$\frac{24}{16}$	$\frac{36}{16}$	$\frac{16}{16}$	5	(this is $\text{E}(X^2)$)

From the third row, $\text{E}(X) = 2$, which is μ.

From the last row,
$$\begin{aligned}\text{Var}(X) &= \text{E}(X^2) - \mu^2\\ &= 5 - 2^2\\ &= 1, \text{ as we obtained before.}\end{aligned}$$

Example 6 — 13C

A die is thrown, producing a uniform distribution. Find the mean and variance of the distribution using the alternative formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$.

SOLUTION

x	1	2	3	4	5	6	Sum	
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1	(a check)
$xp(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$3\frac{1}{2}$	(the mean μ)
$x^2p(x)$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{9}{6}$	$\frac{16}{6}$	$\frac{25}{6}$	$\frac{36}{6}$	$\frac{91}{6}$	(this is $\text{E}(X^2)$)

From the third row, $\text{E}(X) = 3\frac{1}{2}$.

From the last row,
$$\begin{aligned}\text{Var}(X) &= E(X^2) - \mu^2\\ &= \frac{91}{6} - \frac{49}{4}\\ &= 15\tfrac{1}{6} - 12\tfrac{1}{4}\\ &= 2\tfrac{11}{12}, \quad \text{as we obtained before.}\end{aligned}$$

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7 VARIANCE OF A DISCRETE PROBABILITY DISTRIBUTION

- The *variance* of a discrete probability distribution with numeric data is
$$\text{Var}(X) = \text{E}\left((X - \mu)^2\right) \text{ or equivalently } \text{Var}(X) = \text{E}(X^2) - \mu^2.$$
- Written in terms of weighted means,
$$\text{Var}(X) = \sum (x - \mu)^2 p(x) \text{ or } \sum x^2 p(x) - \mu^2$$
where the sums are taken over all the values in the distribution.
- The first of each pair gives the best intuitive understanding of the variance.
- The second of each pair is usually easier for calculation.
- Each difference $x - \mu$ is called the *deviation* of x from the mean μ.

Extension — proof of the alternative formula

The proof looks complicated only because of sigma notation. To prove the alternative formula for $\text{Var}(X)$, start with the first formula, and move to the second.

$$\begin{aligned}\text{Var}(X) &= \sum (x - \mu)^2 p(x)\\ &= \sum (x^2 - 2\mu x + \mu^2)p(x) \quad \text{(expand the square)}\\ &= \sum x^2 p(x) - \sum 2\mu x p(x) + \sum \mu^2 p(x) \quad \text{(expand the brackets)}\\ &= \sum x^2 p(x) - 2\mu \sum x p(x) + \mu^2 \sum p(x) \quad \text{(take out common factors)}\end{aligned}$$

Using the fact that $\sum p(x) = 1$ and $\sum x p(x) = \mu$,

$$\begin{aligned}\text{Var}(X) &= \sum x^2 p(x) - 2\mu^2 + \mu^2\\ &= \sum x^2 p(x) - \mu^2.\end{aligned}$$

Standard deviation

The units of variance are the square of whatever units the values have. The *standard deviation* is the square root of the variance, and therefore has the same units as the values. Its usual pronumeral is σ, the lower-case Greek letter sigma (Latin ‘s’), standing for ‘standard’,

$$\sigma = \sqrt{\text{Var}(X)} \quad \text{or equivalently} \quad \sigma^2 = \text{Var}(X)$$

In the four-coins example at the start of this section, the units of the values and of the standard deviation are heads, and in Example 7 below, their units are dollars.

The term *standard deviation* implies that the deviations from the mean have been taken into account across all values of the distribution.

The other nice thing about the standard deviation is its proportionality. Standard deviation, like variance, is a measure of spread. If we stretch out all the values by doubling them, then we correspondingly double the standard deviation. If we spread out all the values by a factor of k, the standard deviation is multiplied by k. This will be proven in the Enrichment section of Exercise 13C (Questions 12–14).

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Example 7 13C

Find the standard deviation of the distribution in Example 3 Section 13B.

SOLUTION

We use the alternative formula for variance, then take the square root.

x	0	40	50	80	180	350	560	Sum
$p(x)$	$\frac{5}{20}$	$\frac{1}{20}$	$\frac{4}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{1}{20}$	1
$xp(x)$	0	2	10	12	18	70	28	140
$x^2p(x)$	0	80	500	960	3240	24500	15680	44960

From the last row,

$$\begin{aligned}\text{Var}(X) &= E(X^2) - \mu^2\\ &= 44960 - 140^2\\ &= 25360\end{aligned}$$

Taking the square root,

$$\begin{aligned}\sigma &= \sqrt{\text{Var}(X)}\\ &\doteqdot \$159.25\end{aligned}$$

giving a spread of \$159.25 about the mean of \$140.

8 STANDARD DEVIATION OF A DISCRETE PROBABILITY DISTRIBUTION

- The *standard deviation* is the square root of the variance. It is usually assigned the pronumeral σ — the Greek letter sigma stands for 'standard'.

$$\sigma = \sqrt{\text{Var}(X)} \qquad \text{or equivalently} \qquad \sigma^2 = \text{Var}(X)$$

- The pronumeral for variance is thus σ^2.
- The standard deviation has the same units as the values.
- The variance and the standard deviation are *measures of spread*.

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Exercise 13C

FOUNDATION

1 Consider a random variable X whose probability distribution is given in the table to the right.

x	1	2	3	4	Sum
$p(x)$	0.3	0.5	0.1	0.1	
$xp(x)$					
$(x - \mu)^2$					—
$(x - \mu)^2 p(x)$					

a Copy and complete the table to calculate the mean $\mathrm{E}(X) = \mu$ and the variance $\mathrm{Var}(X)$ using the definition

$$\mathrm{Var}(X) = \mathrm{E}\big((X - \mu)^2\big).$$

b Calculate the standard deviation

$$\sigma = \sqrt{\mathrm{Var}(X)}.$$

2 This question uses the alternative formula for $\mathrm{Var}(X)$, rather than the definition, to calculate the variance for the random variable in the previous question.

x	1	2	3	4	Sum
$p(x)$	0.3	0.5	0.1	0.1	
$xp(x)$					
x^2					—
$x^2 p(x)$					

a Copy and complete the table.

b Now calculate the variance using the alternative formula

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2.$$

3 For each random variable, calculate $\mu = \mathrm{E}(X)$. Then calculate the variance $\mathrm{Var}(X)$ twice, first using the definition $\mathrm{Var}(X) = \mathrm{E}\big((X - \mu)^2\big)$, then using the alternative formula $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$. Use columns instead of rows in parts **c** and **d**.

a

x	0	1	2	3	4
$p(x)$	0.2	0.2	0.2	0.2	0.2

b

x	0	1	2	3	4
$p(x)$	0.0	0.1	0.2	0.3	0.4

c

x	$p(x)$
−2	0.3
−1	0.1
0	0.2
1	0.1
2	0.3

d

x	$p(x)$
1	0.1
2	0.4
3	0.2
4	0.2
5	0.1

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DEVELOPMENT

4 **a** Calculate the expected value, variance and standard deviation for each random variable. For the variance, choose whether to use definition $\text{Var}(X) = \text{E}\left((X - \mu)^2\right)$ or the alternative formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$.

i

y	0	1	2	3	4
$P(Y = y)$	0	0.5	0	0.5	0

ii

z	0	1	2	3	4
$P(Z = z)$	0.5	0	0	0	0.5

iii

v	$P(V = v)$
0	0.3
1	0.5
2	0.1
3	0.1
4	0

iv

w	$p(W = w)$
0	0
1	0.1
2	0.1
3	0.5
4	0.3

b Use the idea that the expected value measures the centre of the data, and the variance and standard deviation measure its spread, to comment on:

i how $\text{E}(Y)$ and $\text{Var}(Y)$ compare with $\text{E}(Z)$ and $\text{Var}(Z)$,

ii how $\text{E}(V)$ and $\text{Var}(V)$ compare with $\text{E}(W)$ and $\text{Var}(W)$.

5 A distribution that takes a single value is called *deterministic*, because it is no longer random. Our formulae for expected value and variance may still be calculated and the results are not a surprise, as this question demonstrates.

Calculate $\text{E}(X)$ and $\text{Var}(X) = \text{E}\left((X - \mu)^2\right)$ for the distribution:

x	1	2	3	4	5
$P(X = x)$	0	1	0	0	0

6 John and Liam are keen basketballers and keep track of the number of baskets they score in games. Using the data from a large number of games, they have estimated the probability of scoring in any one game. Let the random variables J and L be the number of baskets scored by John and Liam respectively in a game. Their probability data is recorded in the tables below.

j	0	1	2	3	4
$P(J = j)$	0.35	0.2	0.1	0.25	0.1

ℓ	0	1	2	3	4
$P(L = \ell)$	0.2	0.3	0.4	0.1	0

a Calculate the expected value and variance for J and L using the alternative form $\text{Var}(X) = \text{E}(X^2) - \mu^2$.

b With reference to expected value, comment on who is the better player.

c With reference to variance, comment on who is the more consistent player.

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7 The random variable X records the number on a spinner with sectors of equal size marked 1, 2, 3.

a What is the probability of each outcome, and what sort of distribution is this?

b Calculate the expected value $\mathrm{E}(X)$.

c Calculate the variance $\mathrm{Var}(X)$.

8 The *deviation of a score x from the mean* is often expressed in terms of how many standard deviations x lies from the mean μ. The formula for this is:

$$\text{number of standard deviations from the mean} = \frac{x - \mu}{\sigma},$$

where a negative sign means that the score x is below the mean.

a Englebert's score in his English test was 55. The test mean was $\mu = 65$ and the standard deviation was $\sigma = 5$. How many standard deviations was his score below the mean?

b Matthew's score in his Mathematics test was 54. The test mean was $\mu = 72$ and the standard deviation was $\sigma = 12$. How many standard deviations was his score below the mean?

c Comment on which score was more impressive, by noting which score was furthest from the mean in terms of the number of standard deviations.

9 Using the method of the previous question, that is, how many standard deviations a score is from the mean, decide which of each pair of test scores below is better.

a A score of 45 for Visual Arts (mean 60, standard deviation 15) or a score of 46 for Music (mean of 67, standard deviation of 12).

b A score of 88 for Earth Science (mean 70, standard deviation 9) or a score of 90 for Biology (mean of 75, standard deviation of 10).

c A score of 62 for Chinese (mean 50, standard deviation 6) or a score of 63 for Sanskrit (mean of 55, standard deviation of 4).

10 Jasmine is practising her accuracy with bow and arrow over 10 shots. The random variable X is the number of bull's-eyes obtained. She repeats this experiment twenty times and records the relative frequency as an estimate to the probability of each outcome. Her results are tabulated below.

x	0	1	2	3	4	5	6	7	8
$p(x)$	0	0.1	0.15	0.3	0.4	0	0	0	0.05

a Calculate the expected value and standard deviation for the experiment.

b An *outlier* is informally a value that is a long way away from the mean, and from the rest of the data, of the distribution. What value(s) would you think are outliers in this distribution?

c There are various quantitative ways of defining outliers — one such definition is *any value three or more standard deviations from the mean*. Are there any outliers using this definition?

d Jasmine realises that due to poor handwriting, her results have been wrongly interpreted. The corrected table is below. Recalculate the expected value and standard deviation for this new table.

x	0	1	2	3	4	5
$p(x)$	0	0.1	0.15	0.3	0.4	0.05

e Because of their distance from the mean, outliers can have a big influence on the value of the mean and standard deviation. Comment on the differences in the expected value and variance between the two tables above.

f Would it be valid in general to discard or 'correct' any outliers?

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11 For some value of k, a random variable X with values 1, 2, 3, 4 is defined by

$$P(X = x) = kx, \text{ for } x = 1, 2, 3, 4.$$

Find k, then find the expected value and the standard deviation.

ENRICHMENT

12 A distribution is said to be *uniform* if every outcome has the same probability. Consider the random variable X of a uniform distribution with values $1, 2, \ldots, n$.
To complete this question you will find these formulae useful:

$$1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n + 1) \quad 1 + 4 + 9 + \cdots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1)$$

- **a** What is the probability of $P(X = k)$ for integers $1 \le k \le n$?
- **b** Calculate the expected value $\mathrm{E}(X)$.
- **c** Calculate the variance $\mathrm{Var}(X)$.
- **d** Compare your answers for expected value and variance with those for Question 7 involving a three-valued spinner, and with the example in the theory about throwing a standard six-sided die.

13 The expected value $\mathrm{E}(X)$ of a discrete probability distribution is μ.

- **a** A constant a is added to all the values in the distribution. Let the random variable of this new distribution be Z, so that $Z = X + a$. Show that the expected value $\mathrm{E}(Z)$ of the new distribution is $\mu + a$.
- **b** Each value in the distribution is multiplied by a constant k. Let the random variable of this new distribution be Z, so that $Z = kX$. Show that the expected value $\mathrm{E}(Z)$ of the new distribution is $k\mu$.

You have now proven that for all constants k and a,

$$\mathrm{E}(kX + a) = k\mathrm{E}(X) + a.$$

14 The mean and standard deviation of a discrete probability distribution are μ and σ. Use the formula $\mathrm{Var}(X) = \mathrm{E}\big((X - \mu)^2\big)$ to solve these two problems.

- **a** A constant a is added to all the values in the distribution. Let the random variable of this new distribution be Z, so that $Z = X + a$. Show that the standard deviation $\sqrt{\mathrm{Var}(Z)}$ of the new distribution is also σ.
- **b** Each value in the distribution is multiplied by a constant $k > 0$. Let the random variable of this new distribution be Z, so that $Z = kX$. Show that the standard deviation $\sqrt{\mathrm{Var}(Z)}$ of the new distribution is $k\sigma$.

You have now proven that for all constants $k > 0$ and a,

$$(\text{standard deviation of } kX + a) = k\sigma.$$

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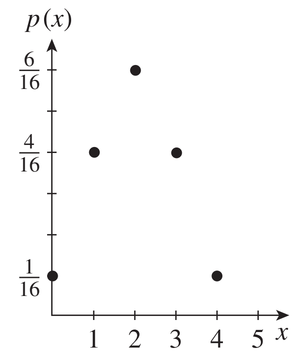
13D Sampling

The word *sampling* is used when we perform an experiment a number of times and record the results. The set of results is a *sample*. One example of sampling is the large numbers of surveys from researchers, the government, and marketers. These surveys are constantly asking what we think about some product or service, how many people live in our house, whether we agree with the latest government policy, how many holidays we intend to take in the next five years, and so forth. The purpose of most of these surveys is to gain some idea of probabilities, usually so that future behaviours can be predicted.

Now that discrete probability distributions, expected value and variance have been introduced, this section discusses the relationship between sampling and probabilities. The subject is fundamental to statistics, and extremely complicated. Only a few very basic ideas are accessible here.

Sampling of a theoretical probability distribution

We return once again to the experiment of throwing four coins and recording the number of heads.



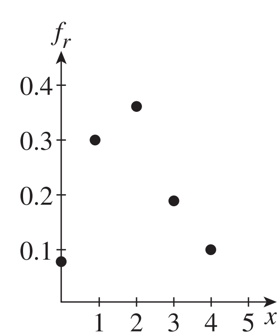
x	0	1	2	3	4	Sum
$p(x)$	0.06	0.25	0.38	0.25	0.06	1
$x\,p(x)$	0	0.25	0.75	0.75	0.25	2
$x^2 p(x)$	0	0.25	1.50	2.25	1.00	5

We concluded that $E(X) = 2$ and $\text{Var}(X) = 5^2 - 2^2 = 1$.

Each fraction in the earlier table has been rounded correct to two decimal places for comparison with the sample below — the last two rows were not recalculated. Notice that because of rounding errors, the first row may no longer add to 1, and the entry 1.50 in the middle of the bottom row is not $2^2 \times 0.38$. In this distribution, the errors could have been removed by rounding to four decimal places, but in general, rounding errors are a constant issue in statistics.

Can we confirm these probabilities and the subsequent calculations empirically? We could do a number of *simulations* or *trials*, which are independent *runs* of the experiment. It would be most unlikely that the resulting frequencies would be in the exact proportions of the theoretical table above, but it may be interesting to perform say 100 simulations and see what happens. The authors have done this using random numbers in an Excel spreadsheet rather than coins, because they only had three coins between them. Here are the results:

x	0	1	2	3	4	Sum
f	7	29	34	21	9	100
f_r	0.07	0.29	0.34	0.21	0.09	1
xf_r	0	0.29	0.68	0.63	0.36	$1.96 = \bar{x}$
$x^2 f_r$	0	0.29	1.36	1.89	1.44	4.98



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The resulting *sample distribution* or *frequency table* has been structured as closely as possible to the theoretical table above. Its graph is drawn to the right above.
In the table:

- The second line is the *frequency* f. There is no corresponding line in the probability distribution table.
- The third line is the *relative frequency* f_r, obtained by dividing each frequency by 100, which is the number of simulations performed. These relative frequencies correspond to the probabilities in the theoretical distribution. The graph of the sample distribution has been drawn, and should be compared with the theoretical graph above it.

Did you expect the relative frequencies to be closer to the probabilities, and the sample graph to be closer to the theoretical graph? That is a very difficult question, and beyond the course, but at least we can begin to see what may happen in practice.

- The fourth line gives us the *sample mean*, which is written as $\bar{x} = 1.96$ — placing a bar over the pronumeral x is the usual notation for the sample mean. It is the mean of the values weighted by their relative frequencies, just as the *theoretical mean* $\mu = 2$ (the expected value) is the mean of the values weighted by their probabilities.
- The last line gives us the mean of the squares of the values, again weighted by the relative frequency, corresponding to $E(X^2)$ in the probability distribution table. It allows us to calculate the *sample variance* s^2 and the *sample standard deviation* s in exactly the same way that the *theoretical variance* σ^2 and the *theoretical standard deviation* σ were calculated:

$$\begin{aligned} s^2 &= \sum x^2 f_r - \bar{x}^2 \\ &= 4.98 - (1.96)^2 \\ &\doteqdot 1.14 \quad \text{(compare with 1)} \end{aligned} \qquad \begin{aligned} s &= \sqrt{s^2} \\ &\doteqdot 1.07 \text{ heads} \quad \text{(compare with 1)} \end{aligned}$$

Although this course does not have the machinery to characterise whether the sample distribution differs significantly from the theoretical distribution, at least we have the basic data to begin the comparison.

9 SAMPLE DISTRIBUTION

Suppose that a number of independent *runs* (or *simulations* or *trials*) are performed of an experiment whose possible outcomes are discrete.

- The resulting *sample* then yields a *sample distribution* or *frequency table*.
- The relative frequencies f_r in the sample distribution correspond to the probabilities $p(x)$ in a probability distribution, and can be graphed the same way.

Now suppose also that the possible outcomes are numeric.

- The *mean* of the sample is denoted by $\bar{x}$, and corresponds to the expected value of the theoretical probability distribution:

$$\bar{x} = \sum x f_r$$

- The *sample variance* s^2 and the *sample standard deviation* s are given by:

$$s^2 = \sum (x - \bar{x})^2 f_r \qquad \text{or} \qquad s^2 = \sum x^2 f_r - \bar{x}^2$$

- The calculation of the mean and sample variance can be done in tabular form in the same way as the expected value and variance of a probability distribution.

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The sample mean $\bar{x}$ will seldom be the same as the theoretical mean μ, which is why different symbols must be used. Similarly, the sample standard deviation s will seldom be the same as the theoretical standard deviation σ.

Populations — census and survey

A school asks all its 1000 students to report how many brothers and sisters the student has — this is a *census*, because everyone is asked. Another similar-sized school chooses 100 students at random and asks them the same question — this is a *survey* or *sample* because not everyone is asked. When a marketer or the government wants to know something about the Australian population, almost always a survey is used because it is quicker and much cheaper than a census. Every five years, however, the Federal Government performs a compulsory census of the whole Australian population, and the results are eagerly awaited and discussed.

In any one census question, the relative frequencies are exact probabilities of a particular result when a person or thing is chosen at random. A *population distribution* is therefore akin to a probability distribution that has been calculated exactly using the laws of probability, so the pronumerals μ and σ are assigned to its mean and standard deviation.

When designing a survey, it is vitally important to know whether the sample is truly random. For example, many websites offer internet surveys, but such a sample is in no way random, because it includes only people visiting that website.

Also, the people chosen in the survey may be a significant proportion of the population — in the school survey above, 100 students were chosen out of only 1000. That should be taken into account, but it is beyond the present course.

Everyone wants to use surveys to approximate the statistical parameters of a population, but there are many problems, as Example 8 shows.

Example 8 13D

In 1992, astronomers confirmed the first discovery of an *exoplanet*, which is a planet orbiting a star other than our Sun. They soon found examples of multiple exoplanets orbiting the same star. Up to 28th November 2017, the website http://www.openexoplanetcatalogue.com reported that 3653 planets had been discovered, in 2657 nearby star systems in our galaxy, including our Sun. Here is the sample distribution for the number of stars that have x known planets orbiting them — multiple-star systems are excluded because they are hard to classify, and only stars with at least one discovered planet are included. The frequency table below thus involves 3457 known planets orbiting 2527 single stars.

x	1	2	3	4	5	6	7	8	9	Sum
f	1921	401	133	47	14	5	3	1	2	2527 suns
xf	1921	802	399	188	70	30	21	8	18	3457 planets
f_r	0.7602	0.1587	0.0526	0.0186	0.0055	0.0020	0.0012	0.0004	0.0008	1
xf_r	0.7602	0.3174	0.1579	0.0744	0.0277	0.0119	0.0083	0.0032	0.0071	1.3680
x^2f_r	0.7602	0.6347	0.4737	0.2976	0.1385	0.0712	0.0582	0.0253	0.0641	2.5235

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$$\bar{x} = \sum x f_r \qquad s^2 = \sum x^2 f_r - \bar{x}^2 \qquad s = \sqrt{s^2}$$

$$= 1.37 \text{ planets} \qquad = 2.5235 - 1.3680^2 \qquad \doteqdot 0.81 \text{ planets.}$$

$$\doteqdot 0.65$$

This is a sample of the population of all the single-star-system stars in our Milky Way galaxy. Why would scientists never use this table alone to infer the distribution of exoplanets across all the single-star systems in the galaxy?

SOLUTION

[Some of the more obvious reasons.]

- Only larger planets are counted because smaller planets are hard to find. The number of planets observed to orbit each star in the table will probably be less than the actual number of planets orbiting that star.
- We can't yet show that a star has no planets, so it would be most confusing to include the value 0.
- These stars are near Earth, so they may not represent the average situation in the galaxy. In particular, in denser regions, stars are closer, and planets may perhaps be more easily stripped away by near collisions with other stars.
- This is an extremely small sample of the 100–400 billion stars in the galaxy.

10 POPULATION — CENSUS AND SURVEY

- A *census* performs an experiment (perhaps by asking a question) on everyone or everything in a population. A *survey* samples only some of the population.
- The relative frequencies in a population distribution are the exact probabilities of a particular result when a person or thing is chosen at random from the population.
- The population mean is therefore assigned the pronumeral μ, and the population standard deviation is assigned the pronumeral σ.

Sampling when there are no exact probabilities

Until now, almost everything has been based on equally likely possible outcomes. The four-coins probabilities were based on 16 equally likely outcomes when four coins are tossed. The number-of-exoplanets discussion was based on the unknown population of stars in the galaxy, with the equally likely possible outcomes being choosing a single-star-system star in the galaxy at random and counting its planets (not that we have any idea how such an experiment could be performed).

In 1903, a German researcher recorded the number x of baby mice in each litter of mice in his laboratory (Yule & Kendall 1950, *An introduction to the theory of statistics*, Charles Griffin, page 121).

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x	1	2	3	4	5	6	7	8	9	Sum
f	7	11	16	17	26	31	11	1	1	121 litters
xf	7	22	48	68	130	186	77	8	9	555 babies
f_r	$\frac{7}{121}$	$\frac{11}{121}$	$\frac{16}{121}$	$\frac{17}{121}$	$\frac{26}{121}$	$\frac{31}{121}$	$\frac{11}{121}$	$\frac{1}{121}$	$\frac{1}{121}$	1
xf_r	$\frac{7}{121}$	$\frac{22}{121}$	$\frac{48}{121}$	$\frac{68}{121}$	$\frac{130}{121}$	$\frac{186}{121}$	$\frac{77}{121}$	$\frac{8}{121}$	$\frac{9}{121}$	$\frac{555}{121} = \bar{x}$
x^2f_r	$\frac{7}{121}$	$\frac{44}{121}$	$\frac{144}{121}$	$\frac{272}{121}$	$\frac{650}{121}$	$\frac{1116}{121}$	$\frac{539}{121}$	$\frac{64}{121}$	$\frac{81}{121}$	$\frac{2917}{121}$

$$\bar{x} = \sum xf_r = \frac{555}{121} \doteqdot 4.59 \text{ baby mice}$$

$$s^2 = \sum x^2 f_r - \bar{x}^2 = \frac{2917}{121} - \frac{308\,025}{14\,641} \doteqdot 3.0689$$

$$s = \sqrt{s^2} \doteqdot 1.75 \text{ baby mice.}$$

The sample statistics were easily calculated. The obvious intention, however, is to talk about all mice, and this time there are no theoretical probabilities based on equally likely possible outcomes, and no population. We would normally interpret the relative frequencies as probabilities that a mouse litter, chosen at random, has this many baby mice, but what would such a probability mean?

- Is such a probability a statement about the physical world, in this case mice?
- Or is it a statement about our lack of knowledge about the size of a litter?

In the second case, the probability is never fixed, but changes all the time as further information about mouse litters comes in.

Then there are also the usual questions about the experiment. For example:

- Are these laboratory mice better fed and housed than wild mice?
- Is there some subtle genetic difference between these mice and other mice?
- Does weather affect litter size, and were weather patterns over the last few years normal, whatever that means?

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Predicting the future

The most important, and the most tricky, use of sampling is to predict the future. From the website http://www.bom.gov.au, here are the numbers of years from 1913 to 2012 when there were x rainy days in April, recorded at Observatory Hill, Sydney (where 'rainy day' means here a day when the rainfall was greater than zero.)

x	3	4	5	6	7	8	9	10	11	12	13
f	1	1	3	2	7	9	3	9	10	9	4
f_r	0.01	0.01	0.03	0.02	0.07	0.09	0.03	0.09	0.10	0.09	0.04
xf_r	0.03	0.04	0.15	0.12	0.49	0.72	0.27	0.90	1.10	1.08	0.52
x^2f_r	0.09	0.16	0.75	0.72	3.43	5.76	2.43	9.00	12.10	12.96	6.76

x	14	15	16	17	18	19	20	21	22	23	Sum
f	9	2	4	9	7	5	2	1	2	1	100
f_r	0.09	0.02	0.04	0.09	0.07	0.05	0.02	0.01	0.02	0.01	1
xf_r	1.26	0.30	0.64	1.53	1.26	0.95	0.40	0.21	0.44	0.23	12.64
x^2f_r	17.64	4.50	10.24	26.01	22.68	18.05	8.00	4.41	9.68	5.29	180.66

$$\bar{x} = \sum x f_r = 12.64 \text{ rainy days}$$

$$s^2 = \sum x^2 f_r - \bar{x}^2 = 180.66 - 12.64^2 \doteqdot 20.89$$

$$s = \sqrt{s^2} \doteqdot 4.57 \text{ rainy days.}$$

In the five subsequent years from 2013 to 2017, the successive numbers of rainy days in April were 16, 15, 23, 13 and 10. Here are some questions that the reader may raise, but the course cannot answer.

- Are these subsequent results consistent with the probabilities suggested by the relative frequencies in the table?
- Droughts last for many years, so perhaps rainy Aprils come in groups? If there are cycles in weather with periods of say 15 years, or 1100 years, or 125 000 years, then how could we use this table as a predictor of future weather? Are the last five years' results consistent with a permanent and unprecedented shift in climate?
- What does the probability of 10 rainy days in April actually mean? Is it a statement about the physical world, in this case the climate, or is it statement about our lack of knowledge? If such probabilities exist, do they vary over time, in which case how does one define them? There are certainly no equally likely possible outcomes in sight here to base a theoretical approach on.

Clearly a single sample distribution such as this does nothing to answer the urgent questions people are asking. Far more data on all sorts of different phenomena, together with sophisticated statistical tests that can quantify exceptional results and take into account possible cyclical data, are required before any reliable conclusions can be made. Climate science is extremely complicated.

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Extension — a correction factor for the sample variance

When we know the theoretical mean μ and we are sampling to find the variance, there is no problem with the formulae for the sample variance. When, however, we are sampling both to find the mean and to find the variance, then the sample mean will drift very slightly towards the sample results, with the effect that the sample variance will tend to be slightly smaller than it should be.

This phenomenon is much discussed, and the standard answer is that the sample variance should be multiplied by a correction factor $\frac{n}{n-1}$, where n is the size of the sample.

Thus in the example above about rainy days in April, there were 100 results, and we were using a sample mean rather than a theoretical mean, so the correction factor is $\frac{100}{99}$. Using the correction factor would yield

$$\bar{x} = 12.64 \text{ rainy days (as before)} \qquad s^2 = 20.89 \times \frac{100}{99} = 21.10 \qquad s = \sqrt{s^2} \doteqdot 4.59 \text{ rainy days.}$$

The larger the size n of the sample, the less difference the correction makes.

Your calculator may have a button $\boxed{\sigma_n}$ or something equivalent. This gives the standard deviation without the correction factor, and is all that is required in this course. It may also have a button labelled $\boxed{\sigma_{n-1}}$ or equivalent, which applies the correction factor — this button is not required in this course.

Approaching the following exercise

The following exercise contains many suggestions for experiments that will generate data. In each situation you should be asking a number of important questions:

- How can you ensure that the data are randomly generated?
- Are there any biases due to the population used for your experiment? For example, measuring the heights of students in the class will give results different from measuring the heights of students in the school.
- How can you complete the experiment efficiently and accurately, including recording the data? Tally tables, and accurate record-keeping, are two important skills in experiments.
- What special equipment is needed, such as stopwatches, tape measures, metre rulers?
- How big does the sample size need to be to get a reasonable estimate? This might become clearer as the experiment proceeds, depending on whether the data appear stable.
- Is your data from a population, with mean μ and standard deviation σ, or from a sample, with sample mean $\bar{x}$ and sample standard deviation s?
- What are the mean and standard deviation measuring — what is their interpretation in the context of the experiment? Are there theoretical or population means and standard deviations to compare the data with? If so, what conclusions can you draw?

Using a spreadsheet

The probabilities of a distribution need to be calculated, and the frequencies of a sample need to be tallied. Once this is done, however, the calculations of mean, variance and standard deviation can be automated on a spreadsheet, provided that you know how to do two things:

Write formulae into cells and Fill down and fill right.

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You are invited to use a spreadsheet in this exercise to save time in the calculations. Once a calculation has been set up, it can be copied and adapted.

Many simulations can be done using *random numbers* — see Question 5, for example, and see the table of random numbers after Exercise 13D. Excel has two very useful random number functions:

RANDBETWEEN (a, b) generates a random integer in the interval $a \le x \le b$. For example, RANDBETWEEN (11, 20) may generate 14, or 20, or 11, or 15.

RAND () generates a random number in the interval $0 < x < 1$. For example, it may generate or 0.865348256572775, or 0.44037476507047, or 0.0174096864227915.

Once you have a random number in a cell, the commands fill down (Ctrl+D) and fill right (Ctrl+R) will give you as many random numbers as you like.

Exercise 13D

FOUNDATION

This is not a normal exercise with a normal set of questions. Apart from Question 1, it is a set of suggestions for activities and investigations, some of which could easily be turned into projects. Most questions ask for a number of simulations (or trials or runs) of an experiment in order to create a sample, followed by an analysis of the results.

First do Question 1. After that, there is no expectation that anyone should simply work through the exercise, because the subsequent questions all takes some time, however they are done. They are intended to be adapted to the needs of the class or the individual.

These simulations could be done by individual students, but they could also be done in groups, or by the class as a whole. Most could also be done on a spreadsheet using random numbers — this requires setting up a spreadsheet, which also takes time.

1 Three dice are thrown. Let X be the number of dice showing a 5 or 6.

a The probability distribution for this experiment is shown below (it can easily be calculated using a probability tree diagram).

x	0	1	2	3
$p(x)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

Graph the distribution. Then calculate the theoretical mean μ, the theoretical variance σ^2, and the theoretical standard deviation σ.

b The frequency table below gives the results when the experiment was done 100 times:

x	0	1	2	3	Sum
f	33	47	16	4	100

Calculate the relative frequencies, and graph them. Then calculate the sample mean $\bar{x}$, the sample variance s^2, and the sample standard deviation s.

c Do the sample results appear to be consistent with the theoretical results?

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2 Two dice are thrown. Let X be the sum of the two numbers on the dice.

a Here is the theoretical probability distribution of the experiment, together with the calculations for obtaining the mean and variance. From the table, write down the mean μ, the variance σ^2, and the standard deviation σ.

x	2	3	4	5	6	7	8	9	10	11	12	Sum
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	1
$x\,p(x)$	$\frac{2}{36}$	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{20}{36}$	$\frac{30}{36}$	$\frac{42}{36}$	$\frac{40}{36}$	$\frac{36}{36}$	$\frac{30}{36}$	$\frac{22}{36}$	$\frac{12}{36}$	7
$(x-\mu)^2$	25	16	9	4	1	0	1	4	9	16	25	—
$(x-\mu)^2\,p(x)$	$\frac{25}{36}$	$\frac{32}{36}$	$\frac{27}{36}$	$\frac{16}{36}$	$\frac{5}{36}$	0	$\frac{5}{36}$	$\frac{16}{36}$	$\frac{27}{36}$	$\frac{32}{36}$	$\frac{25}{36}$	$\frac{210}{36}$

b Throw a pair of dice 72 times and record your sample distribution in a relative frequency table.

c How do your probabilities in part **a** compare with the relative frequencies obtained in your experiment? Draw the two graphs and compare them.

d Use the sample distribution table to calculate the sample mean $\bar{x}$, the sample variance s^2, and the sample standard deviation s. The sample mean will not be a whole number, so use the alternative formula for variance.

e How do the sample statistics $\bar{x}$ and s compare with the theoretical statistics μ and σ?

f To improve your estimation, combine your results with those from other members of your class.

3 Design some censuses in your class. Draw up a population distribution table for each, and calculate the population mean μ, the population variance σ^2, and the population standard deviation σ. Here are some suggestions.

a The month number of each student's birthday.

b The number of brothers and sisters of each student.

c Each student's writing speed — count the number of times in one minute that a student can fully write the four words, 'The quick brown fox'.

d The number of sports they do, or the number of musical instruments they play.

e The number of pets at home — if they have goldfish, decide whether that result is an outlier or should be rejected.

f How many days in the last week did each student cook a meal?

Why is, or is not, each class census an unbiased sample of the school population?

4 Many experiments in this chapter have involved throwing a die. It is important to know *how random* the results will be that are obtained from throwing a die.

a Throw a die 40 times and write down the results one after the other.

b Are the probabilities approximately *uniform*, that is, the same for each outcome?

c We now investigate whether each outcome on the die is *independent* of the previous roll. Perhaps the way you roll the die affects things here?

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i Write down the 39 *differences* between successive rolls, discarding any minus sign.

ii Draw up a sample distribution table, and calculate the sample mean $\bar{x}$ and sample standard deviation s.

iii Question 10 of Exercise 13B asked for the probability distribution table of an experiment equivalent to this, namely 'Throw two dice and record their difference'. Copy and complete the table to find the standard deviation σ.

x	0	1	2	3	4	5	Sum
$p(x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	1
$x\,p(x)$	0	$\frac{10}{36}$	$\frac{16}{36}$	$\frac{18}{36}$	$\frac{16}{36}$	$\frac{10}{36}$	$\frac{70}{36}$

iv Check whether your experimental probabilities agree with the theoretical results. Draw the two graphs and compare them.

5 Many calculators have a random-number generator. One calculator has a button labelled `Ran#` (returning a three-digit number between 0 and 1) and a button labelled `ranint ()` (for example, `ranint (1, 13)` returns a random integer between 1 and 13 inclusive). Use your manual, or ask others, to find what functions are available on your calculator.

Excel's `RAND` function is a random-number generator. Many websites provide random numbers. There is also a list of 10 000 random numbers at the end of this exercise. If your method generates a three-digit number and only one digit is required, simply discard the unwanted digits.

a Generate a single-digit random number 50 times, keeping the numbers. Calculate the sample mean $\bar{x}$ and the sample standard deviation s. Compare them with the uniform distribution on the numbers 0, 1, . . . , 9, where $\mu = 4\frac{1}{2}$ and $\sigma^2 = 8\frac{1}{4}$ (these results were calculated in general in Question 12 of Exercise 13C).

b Discard the results of part **a** if the digit is 0 or 9, calculate the new sample mean $\bar{x}$ and sample standard deviation s, and compare them with the corresponding uniform distribution of the numbers 1, 2, . . . , 8, where $\mu = 4\frac{1}{2}$ and $\sigma^2 = 5\frac{1}{4}$. This models an eight-sided die used in some games.

DEVELOPMENT

6 This question models a simple lottery. You will be using random-number simulations to estimate the probabilities and payouts — theoretical probabilities are not required.

Write down any four distinct single-digit numbers between 0 and 9 inclusive. These will be your winning numbers for the next four parts.

a Generate, by any method, four distinct random single-digit numbers — if you get a number that has already occurred, just discard it and generate a new number.

b Record the number of matches you have between your four random numbers and your four winning numbers. The number of matches is your random variable X.

c Repeat this experiment say 30 times and tabulate your results as a frequency table for X. These are your experimental estimates of the probability of each outcome.

d Calculate your expected payout if 4 matches wins you \$100 and 3 matches wins you \$10. Then calculate your expected profit or loss if entering the game costs \$2.

[A longer investigation]
You may change the range of numbers, the number of numbers, the payouts, and the cost of entering the game, to generate other results. How much of the procedures in parts **a–d** can you automate if you write your simulations in Excel?

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7 [Benford's law]
This question models *Benford's law*, which is used, amongst other things, in forensic analysis of large data sets, for example to spot tax fraud. It models the frequency distribution of the leading digit in each score of the data by the following table:

x	1	2	3	4	5	6	7	8	9
$p(x)$	30.1%	17.6%	12.5%	9.7%	7.9%	6.7%	5.8%	5.1%	4.6%

a Copy and complete the table to find the mean μ and standard deviation σ.

b We can test Benford's law using the first digit of the powers of 2 from 2^0 to 2^{99}. Generate a random two-digit number d (whose leading digit may be 0). Define a random variable X to be the leading digit of 2^d. For example,

$$\text{If } d = 36, \text{ then } 2^d \doteqdot 6.8719 \times 10^{10}, \text{ so } X = 6.$$

Repeat this experiment 40 times, record your results in a frequency table, and calculate the sample mean $\bar{x}$ and the sample standard deviation s. Do your results agree reasonably with Benford's law?

Applying Benford's law to internet data: Obtain a large set of numerical data, such as stock prices, house prices, population numbers, death rates, lengths of rivers. Choose data that varies widely in its range across several orders of magnitude — a distribution with range from say 10 to 100 would not be a good set of data for this experiment. See if the distribution of the leading digits in the data agrees reasonably with Benford's law.

A possible project: What formula generated these nine percentages in the table above? How can it be proven, and what assumptions are being made in that proof? What is the history of Benford's law? What are some typical situations, in all sorts of contrasting fields, where Benford's law is useful? In what situations does Benford's law begin to break down, or break down completely?

8 This experiment estimates throwing accuracy. First, you need an object that you can throw and will not bounce much, such as a small bag filled with sand, or tightly rolled paper. Secondly, find a clear wall that you can stand some distance away from. On the wall at ground-level, mark a bull's-eye that a thrower is to aim for. Use tape to place successive marks at 10 cm intervals from the bull's-eye, horizontally to the left and to the right, at the base of the wall. Label these marks $\ldots, -2, -1, 0, 1, 2, \ldots$. Later, if people are missing the bull's-eye badly, you may have to increase the number of marks. Thirdly, mark a position quite some distance away to throw from.

Throw the projectile 50 times and record the integer nearest to where the projectile lands — the successive throws can be done to test an individual's accuracy, or a group's accuracy. Calculate the sample mean and sample standard deviation of the experiment.

This experiment can also be done as a continuous probability experiment. If the results are graphed, the result might be expected to be the bell-shaped curve of the *normal distribution* that you will study in Year 12. A discrete version of this curve may be evident if you graph your results.

9 **a** Open a novel on a random page, and choose a word at random. Let X be the random variable recording the number of letters in the word.

b Compare results obtained from a 'light' novel and a more serious text or classic, such as you might study in English.

c Will your results be different if you use a textbook from history or science? Perhaps different groups in the class could use different types of books.

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10 The distribution of letters in English is well documented and may be used in code cracking, as well as having application to games such as Scrabble.

a Use the internet to find the expected distribution of letters in an English document. (A search for 'distribution of letters in English' should return a suitable result.)

b Again using a random page in a novel, count the number of occurrences of each letter within a block of 200 letters, and hence make your own estimate of the frequencies of each letter.

c Do the results depend on the type of document selected?

11 Many irrational mathematical constants are called *normal*, meaning that the distribution of the digits in their decimal expansion is uniform, with every digit being equally likely. Use the internet to find the decimal expansions of constants such as π and $\sqrt{2}$, and find the frequency of each digit in their first 50 digits.

12 [Experiments involving estimation]
These experiments involve certain difficulties with randomisation that you will need to minimise in your experimental design.

a First read the list of experiments below and discuss what issues may occur.

b To be completed in groups or with the whole class:

i Draw a line on the board. Each member of the group is to estimate the length of the line, correct to the nearest centimetre.

ii Each member is asked to draw a 10 cm line on the board without a ruler. Then measure the answer correct to the nearest centimetre.

iii Each member is to draw a circle of radius 20 cm on the board. Then estimate its roundness by measuring the difference between the longest and shortest diagonal, correct to the nearest 1 cm.

Again, these experiments could also be done as continuous probability experiments.

13 **Applying the theory to internet data:** The internet is a vast source of data. Here are some suggestions of how some internet data can be investigated — they should prompt many more similar investigations.

a Most countries in the world have land borders with other countries. Let X be the number of adjacent countries of a country, and draw up a frequency table for X. Then comment on your results, with the help of people with a knowledge of geography.

You could do the same analysis of the states of one country, such as USA, Germany or Australia (decide first whether to regard the Northern Territory and ACT as states).

b There are less than 100 naturally occurring elements, but over 300 naturally occurring isotopes of these elements. Let X be the number of isotopes of an element, and draw up a frequency table for X, with mean and standard deviation. Then comment on your results, with the help of people with a knowledge of physics and chemistry.

You will soon run into huge problems of classification. What shall we class as 'naturally occurring'? Do we include radioactive isotopes that have extremely short half-lives? Do we include isotopes not found on Earth? Do we include isotopes and elements produced in nuclear reactors?

c Natural disasters are rich sources of data. Let X be the number of major earthquakes worldwide in a particular year. Draw up a frequency table for X, and comment on your results. You will need to define the term 'major earthquake', probably after looking at some data, and also decide whether several large earthquakes close together in time and space count as one earthquake or many. You may also like to consider smaller earthquakes and restrict the study to some region of the globe.

Major hurricanes could be the subject of a similar study.

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ENRICHMENT

14 [The German tank problem]
The enemy has N tanks, each labelled with a unique serial number from 1 to N. Unfortunately the number of tanks is unknown, but that would be useful information to the war effort. Tanks are occasionally captured and their serial numbers read. Counter-intelligence operatives want to use these serial numbers to estimate the total number N of tanks.

They begin with the assumption that these serial numbers lie randomly within the interval from 1 to N. For the purposes of modelling this problem, we will work as if the serial numbers are equally spaced.

Suppose that k serial numbers have been discovered, with m being the largest of them. The case $k = 3$ and $m = 9$ is shown in the diagram.

a Explain why the average number of undiscovered serials between the k samples on the interval from 1 to m is $\dfrac{m - k}{k}$.

b If this same gap extends past the largest discovered serial number, explain why

$$N = m + \frac{m}{k} - 1 .$$

c Test this estimate by modelling the tank problem.

i Working in pairs, the enemy partner chooses a two-digit number N, representing the number of tanks they have. They do not share this.

ii The enemy tank owner generates a random number from 1 and N and allows the counter-intelligence partner to 'discover' this serial number. The counter-intelligence operative tabulates the number k of discovered serials, the maximum serial number m discovered so far, and their estimate of N using the previous formula.

iii How quickly and accurately does the scheme estimate the correct value of N? How many tanks must be discovered before the estimate is within 10% of the correct value?

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A table of random numbers

Some cautions about using these or any table of random numbers:

- Any printed set of numbers such as these is only a simulation of random numbers. Once they are written down they are no longer random.
- If you are repeating an experiment, never take your random numbers from a place on this table where you have taken numbers before.
- It may be safer to search for 'table of random numbers' on the internet than to use these.
- Excel generates random numbers with the functions RAND and RANDBETWEEN, and most calculators generate them. How do they generate them?

37743	65520	55183	10968	38039	67295	66872	49849	24546	23324	77131	09038	02082	50408
41801	02262	85716	34590	09006	66392	21083	70211	01972	13007	29700	23977	26832	04955
79271	24018	05443	57916	79007	73955	33197	66964	99935	91957	61833	58130	20211	74807
07252	64966	95430	86966	82324	30105	53945	74630	54709	31824	17172	04797	29660	35607
44907	29330	57843	70710	05639	74016	68147	31718	71858	52629	25494	05112	40372	05998
13655	20830	77618	66061	43060	28829	89975	44934	83615	00648	19074	48971	22506	55971
62113	17825	47021	14189	17171	88516	52356	70962	53516	84486	74430	99783	06920	29668
47698	78539	99974	87544	87961	41074	87394	25210	00183	53751	26340	02438	11297	17016
71726	85738	28023	91959	96532	87421	19680	51877	37115	29660	25994	58836	63527	33556
89917	65236	74820	33196	15434	40370	19366	14720	50378	21987	40163	80380	94243	92858
45152	37604	30995	82334	02529	13058	86885	92913	61053	91709	13261	51796	95365	80887
94383	87457	76296	99349	80233	31202	66181	91339	44328	43728	39172	65651	66531	66694
26550	59696	40868	99275	87036	28556	08335	99728	02106	59054	14716	11957	96302	08814
93391	17072	82546	70322	73149	09950	32449	59990	11004	00805	71630	32220	50003	41740
00019	18462	02044	78320	37180	91282	55680	26126	92615	53419	08942	72322	61082	51852
27142	02218	42903	16862	25693	50829	89282	26090	77915	79441	72611	93606	47853	77171
81790	83966	48086	87531	42531	51278	22542	59990	62679	03263	54306	26536	02516	90318
98440	03241	70082	19577	39788	16819	41482	93977	75793	56193	06199	76316	24307	49053
37407	78320	56407	63424	87287	74701	54201	34224	07333	42224	93650	14828	93670	23765
52371	53632	57363	17378	42455	95610	19732	90354	68279	08886	32663	18284	09316	36288
23921	49125	27951	71912	67462	92257	02289	62690	78746	32550	83974	34946	35050	69424
80994	45961	20780	80912	65499	60536	97782	12724	06612	52555	00657	92307	28239	31544
57072	67909	91887	94239	33074	72642	04470	01144	03400	13543	51126	78094	46792	85639
53028	42821	66307	11004	50198	56513	34633	02830	09316	00376	64396	54481	56535	77200
03481	53562	35294	37918	55334	20976	89131	33172	24283	74992	69490	89872	85848	20303
68542	12761	82050	62092	76990	10975	41729	45090	66887	49608	25854	17644	99446	26275
97832	20775	57850	72669	18829	02244	05069	81133	68073	55611	66672	56212	88173	84300
38636	91223	55429	90662	20162	69815	75145	90034	71484	92532	01942	49805	96687	28373
85068	53478	51514	91358	38715	89730	76181	42089	37326	66477	53189	14313	35914	49938
90136	66483	15368	28691	27533	58321	52769	02744	71611	52382	16702	97590	52136	40617
87569	03275	05223	95931	02900	38457	81146	48751	27668	39470	77154	88585	49599	74388
95278	90584	99483	82082	25149	67110	96494	36290	51806	44290	18492	34675	88514	63587
97345	12886	34569	08356	39236	83625	58118	55409	69976	00707	03967	51579	05318	48122
26393	15280	46270	54100	46962	63071	99405	30169	37334	61219	22175	35222	27080	02257
93225	64579	36496	06554	07701	64046	34410	48872	97568	68980	20999	09930	76920	68352
35601	52966	12289	93810	89525	25301	17966	33739	23866	84899	76304	72323	92317	51934
83037	53612	68980	85346	17591	70582	81182	84410	38393	79195	67283	15293	03191	37464

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59490	62620	14356	69518	91673	06038	89207	54660	32172	21772	55311	41429	41614	09193
70806	86669	55723	42381	69705	19586	93131	46682	32216	30809	88198	00277	80048	86458
48813	60741	96728	67204	39217	16605	72911	32330	87809	63919	76696	08795	19962	07818
48931	05297	15832	89078	23535	51333	93176	35684	72804	61267	15158	39390	77725	23384
23167	57283	23289	13961	94625	38343	18805	36323	25804	75008	43210	84912	39633	79735
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35035	10254	98654	85034	89902	27024	29227	75663	95969	99961	62004	47571	94526	86443
81097	79671	55086	59182	24623	88864	33197	32783	05015	69100	53255	15310	54334	55445
16316	84845	94584	13138	32907	91051	33021	51660	80844	81737	82469	63789	88037	54421
06631	89667	40308	32886	90400	05865	14201	68751	35174	39438	34796	68027	79168	06994
58519	42683	42723	78754	80957	87208	98625	67616	69671	80173	67412	43213	22006	10792
37689	70982	47903	30488	11438	76221	77716	29452	74337	33157	95020	38207	91062	79001
94427	54174	25610	53018	64393	93225	86605	60584	95038	66257	97613	62254	10952	80745
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04928	21118	41927	15114	57461	38447	90159	94759	70504	68385	18143	82414	17721	21192
01349	32918	90842	52928	77845	13319	44975	29518	56107	79434	68921	15811	47838	77411
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33740	60037	11609	59254	27984	30552	65324	27333	61075	44293	10061	60237	86020	23485
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53891	71142	38621	72237	81979	64735	49272	90992	98589	21276	85534	89556	05852	15624
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87455	58024	90255	55495	92727	25923	60412	48467	28484	99634	21334	44533	28789	58216
92096	17721	77803	30436	42649	89425	17599	71328	64138	17463	80969	63760	64210	10142
49244	44994	87330	65321	35101	58249								

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Chapter 13 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 13 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 State whether each random variable is numeric or categorical. If it is numeric, say whether it is discrete or continuous.

a The maximum temperature in Sydney on a given day.
b The number of test cricket games in Australia in a given year.
c A die is thrown until it shows a six, and the number of throws required is recorded.
d The state of origin of a Rugby League player.

2 Which of these tables are probability distributions?

a

x	1	2	3	4
$P(X = x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

b

x	1	2	3	4
$P(X = x)$	0.4	0.1	0.5	0.2

c

x	1	2	3	4
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{6}$	$\frac{1}{8}$

3 Give three reasons why this table is not a valid probability distribution.

x	1	2	3	4
$P(X = x)$	0.2	-0.1	0.5	1.3

4 Copy and complete each table to find the expected value $E(X)$ of the distribution.

a

x	0	1	2	3	Sum
$p(x)$	0.2	0.3	0.4	0.1	
$xp(x)$					

b

x	-2	-1	0	1	Sum
$p(x)$	0.5	0.1	0.1	0.3	
$xp(x)$					

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Review

5 When Jack first visited the Thai Pin Restaurant, he read the menu and assigned each meal a probability indicating how likely he was to order it in the future — this was determined by how much the meal interested him. The fish cost \$27 and he rated it $\frac{4}{9}$, the steak cost \$32 and he rated it $\frac{2}{9}$, the vegetarian option cost \$23 and he rated it $\frac{1}{9}$, the chicken cost \$25 and he rated it $\frac{2}{9}$.

a What was Jack's expected cost in buying a meal at the restaurant?

b This is now Jack's favourite restaurant, and he visits it once a week (52 times a year). What is his expected cost over the next year, assuming his interest ratings do not change and the prices remain constant?

6 Copy and complete the probability distribution tables below to calculate $\text{Var}(X)$ using the definition $\text{Var}(X) = \left(\text{E}(X - \mu)^2\right)$. Also write down σ.

a

x	1	2	3	4	Sum
$p(x)$	0.4	0.3	0.2	0.1	
$x\,p(x)$					
$(x - \mu)^2$					—
$(x - \mu)^2\,p(x)$					

b

x	4	5	6	7	Sum
$p(x)$	0.2	0.6	0.1	0.1	
$x\,p(x)$					
$(x - \mu)^2$					—
$(x - \mu)^2\,p(x)$					

7 Now use the alternative formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$ for the variance for the distributions of the previous question. Copy and complete the tables, then calculate the variance.

a

x	1	2	3	4	Sum
$p(x)$	0.4	0.3	0.2	0.1	
$x\,p(x)$					
$x^2\,p(x)$					

b

x	4	5	6	7	Sum
$p(x)$	0.2	0.6	0.1	0.1	
$x\,p(x)$					
$x^2\,p(x)$					

8 Calculate the mean, variance and standard deviation of each probability distribution.

a

x	0	1	2	3	4
$p(x)$	0.05	0.1	0.8	0	0.05

b

x	0	1	2	3	4
$p(x)$	0.3	0.1	0.2	0.1	0.3

9 Explain briefly the meaning and significance of the expected value of a probability distribution.

10 Explain briefly the meaning and significance of the variance and standard deviation of a probability distribution.

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11 For the random variable X, it is known that $E(X) = 6$ and $Var(X) = 2$

- **a** Write down $E(2X)$, $Var(2X)$ and σ for the new distribution $2X$.
- **b** Write down $E(X + 5)$, $Var(X + 5)$ and σ for the new distribution $X + 5$.
- **c** Write down $E(3X - 1)$, $Var(3X - 1)$ and σ for the new distribution $3X - 1$.

12 a The sample space for a certain random variable X is $\{5, 6, 7, 8, 9\}$. The probability distribution is known to be uniform. Write down the table for this probability distribution and find the expected value μ and the standard deviation σ.

- **b** Use random numbers to simulate this experiment. First decide how you will obtain the random numbers, how you will use them, and how many simulations your sample will have. Then draw up a relative frequency table and calculate the sample mean $\bar{x}$ and the sample standard deviation s.

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14 Combinatorics

Arithmetic begins with counting, and throughout all branches of mathematics, counting continues to be important. How many possible Tasmanian car number plates with the pattern letter + letter + number + number + number + number are there? How many different hydrocarbons have a chain of 20 carbon atoms? Counting is particularly important in probability. What is the probability that a five-digit number without any 7s is divisible by 3? This chapter introduces some of the standard methods of counting and applies them to calculating probabilities.

Counting ordered selections with repetition is easily done using powers. Counting permutations — ordered selections without repetition — requires the use of factorials such as $4! = 4 \times 3 \times 2 \times 1$ and the symbol ${}^{n}\mathrm{P}_{r}$. Counting unordered selections requires the use of the symbol ${}^{n}\mathrm{C}_{r}$, also written as $\binom{n}{r}$.

The next chapter examines binomial expansions $(x + y)^n$. It turns out that basic counting methods, and particularly the numbers ${}^{n}\mathrm{C}_{r}$, are essential for understanding these expansions.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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14A Factorial notation

How many ways can five people form a queue at a bus stop? Solutions of such problems will be formalised in the next section as the *multiplication principle*, but this particular problem can be solved straightforwardly, and the boxes below help to visualise the solution.

- Choose the person at the head of the queue in five ways.
- Four people remain, so the person in second place can be chosen in four ways.
- Three people remain, so the person in third place can be chosen in three ways.
- Two people remain, so the person in fourth place can be chosen in two ways.
- And there is now only one way to choose the person in the last place.

1st place	2nd place	3rd place	4th place	5th place
5	4	3	2	1

Number of possible queues $= 5 \times 4 \times 3 \times 2 \times 1$.
What is the fifth derivative of $y = x^5$? Differentiating five times,

$$\begin{aligned} y' &= 5x^4 \\ y'' &= 5 \times 4 \times x^3 \\ y''' &= 5 \times 4 \times 3 \times x^2 \\ y^{(4)} &= 5 \times 4 \times 3 \times 2 \times x \\ y^{(5)} &= 5 \times 4 \times 3 \times 2 \times 1. \end{aligned}$$

Such products of descending sequences of whole numbers occur in many areas of mathematics, particularly combinatorics, and it is useful to have a name and a notation for them. The product above is written as 5! and is called 5 *factorial*,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 \quad \text{(which evaluates to 120).}$$

The definition of *n* factorial

Informally, the number $n!$ is thought of as the product of all the positive whole numbers from n down to 1:

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1.$$

But the better definition of $n!$ is *recursive*. First define 0! and then for each successive value of n, say exactly how to define $n!$ in terms of $(n - 1)!$

$$\begin{cases} 0! = 1, \\ n! = n \times (n - 1)!, \quad \text{for } n \ge 1. \end{cases}$$

This definition has three advantages. It avoids the dots . . . in the first formula, it gives 0! a meaning, and as we shall soon see, it gives a better insight into how to manipulate factorial notation

1 THE DEFINITION OF *n* FACTORIAL

- Informally, for each whole number n, the number $n!$ (*n factorial*) is the product of all positive whole numbers from n down to 1:

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1.$$

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- The better definition of the function $n!$ is recursive:

$$\begin{cases} 0! = 1, \\ n! = n \times (n-1)!, \text{ for all whole numbers } n \geq 1. \end{cases}$$

The fact that 0! is defined to be 1 needs some explanation.

- An empty product is regarded as being 1, because if nothing has yet been multiplied, the register remains at 1 where it was originally set in preparation for performing multiplication.
- In a similar way, an empty sum is 0, because if nothing has yet been added, the register remains at 0 where it was originally set in preparation for performing addition.

Using the recursive definition,

$$\begin{array}{lll} 0! = 1 & 4! = 4 \times 3! = 24 & 8! = 8 \times 7! = 40\,320 \\ 1! = 1 \times 0! = 1 & 5! = 5 \times 4! = 120 & 9! = 9 \times 8! = 362\,880 \\ 2! = 2 \times 1! = 2 & 6! = 6 \times 5! = 720 & 10! = 10 \times 9! = 3\,628\,800 \\ 3! = 3 \times 2! = 6 & 7! = 7 \times 6! = 5040 & 11! = 11 \times 10! = 39\,916\,800 \end{array}$$

and so on, increasing very quickly indeed. Calculators have a factorial button labelled $\boxed{x!}$ or $\boxed{n!}$. Use it straight away to see that at least the *calculator* believes that $0! = 1$. Notice also the error message if n is not a whole number — the domain of the function $n!$ is the whole numbers 0, 1, 2, ...

Unrolling factorials

The recursive definition of $n!$ given and used above is the key idea in many calculations. Successive applications of the definition can be thought of as *unrolling* the factorial further and further:

$$\begin{aligned} 8! &= 8 \times 7! \quad \text{(unrolling once)} \\ &= 8 \times 7 \times 6! \quad \text{(unrolling twice)} \\ &= 8 \times 7 \times 6 \times 5! \quad \text{(unrolling three times)} \end{aligned}$$

and so on. This idea is vital when there are fractions involved.

Example 1 14A

Simplify each expression using unrolling techniques:

a $\dfrac{10!}{7!}$ **b** $\dfrac{(n+2)!}{(n-1)!}$ **c** $\dfrac{n!}{(n-r)!}$

SOLUTION

a $$\begin{aligned} \frac{10!}{7!} &= \frac{10 \times 9 \times 8 \times 7!}{7!} \\ &= 10 \times 9 \times 8 \\ &= 720 \end{aligned}$$

b $$\begin{aligned} \frac{(n+2)!}{(n-1)!} &= \frac{(n+2)(n+1)n(n-1)!}{(n-1)!} \\ &= (n+2)(n+1)n \end{aligned}$$

c $$\begin{aligned} \frac{n!}{(n-r)!} &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!} \\ &= \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ factors}} \end{aligned}$$

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Example 2 — 14A

Simplify each expression using a common denominator.

a $\dfrac{1}{8!} - \dfrac{1}{10!}$

b $\dfrac{1}{n!} + \dfrac{1}{(n+1)!}$

SOLUTION

a $\dfrac{1}{8!} - \dfrac{1}{10!} = \dfrac{10 \times 9}{10!} - \dfrac{1}{10!}$

$= \dfrac{89}{10!}$

b $\dfrac{1}{n!} + \dfrac{1}{(n+1)!} = \dfrac{n+1}{(n+1)!} + \dfrac{1}{(n+1)!}$

$= \dfrac{n+2}{(n+1)!}$

2 CALCULATING BY UNROLLING FACTORIALS

- A factorial can be unrolled one step at a time:

  $7! = 7 \times 6! = 7 \times 6 \times 5! = 7 \times 6 \times 5 \times 4! = \cdots$

- Cancel fractions with factorials by unrolling until the factorials cancel:

  $\dfrac{7!}{4!} = \dfrac{7 \times 6 \times 5 \times 4!}{4!} = 7 \times 6 \times 5 = 210.$

Exercise 14A

FOUNDATION

1 Use arguments similar to the start of this section to express the following in factorial notation.

a The number of ways to form a queue with 8 people.

b The number of ways to arrange 4 distinct books on a bookself.

c The number of ways 3 particular people can be placed first, second and third in a competition, assuming that each is placed in one of these three positions.

d Sam drops his portable keyboard and all 101 keys fall off. How many ways can he put the keys back on the keyboard, assuming that they are all interchangable?

e When the teacher calls on students in alphabetical order, Andrzej Zywiec is upset that he is always called last. In a class of 20, how many ways could the roll be called, if the alphabetical ordering restriction were dropped?

2 Use the definition of $n!$ and methods of unrolling factorials to evaluate each expression. Do not use a calculator.

a $3!$ **b** $5!$ **c** $1!$ **d** $\dfrac{15!}{14!}$

e $\dfrac{10!}{8! \times 2!}$ **f** $\dfrac{15!}{13! \times 3!}$ **g** $\dfrac{12!}{3! \times 9!}$ **h** $\dfrac{8!}{4! \times 4!}$

3 Use the factorial button $\boxed{n!}$ on your calculator to evaluate each expression.

a $7!$ **b** $10!$ **c** $0!$ **d** $\dfrac{9!}{4!}$

e $\dfrac{8!}{3!}.$ **f** $\dfrac{10!}{5! \times 3! \times 2!}$ **g** $\dfrac{15!}{3! \times 5! \times 9!}$ **h** $\dfrac{12!}{2! \times 3! \times 4! \times 5!}$

4 If $f(x) = x^6$, find expressions for:

a $f'(x)$ **b** $f''(x)$ **c** $f'''(x)$ **d** $f''''(x)$ **e** $f^{(5)}(x)$ **f** $f^{(6)}(x)$ **g** $f^{(7)}(x)$

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5 Simplify by unrolling factorials appropriately:

a $\dfrac{n!}{(n-1)!}$ **b** $n \times (n-1)!$ **c** $\dfrac{n(n-1)!}{n!}$ **d** $\dfrac{(n+1)!}{(n-1)!}$

e $\dfrac{(n+2)!}{n!}$ **f** $\dfrac{(n-2)!}{n!}$ **g** $\dfrac{(n-2)!\,(n-1)!}{n!\,(n-3)!}$ **h** $\dfrac{n!\,(n-1)!}{(n+1)!}$

6 Simplify by taking out a common factor:

a $8! - 7!$ **b** $(n+1)! - n!$ **c** $8! + 6!$

d $(n+1)! + (n-1)!$ **e** $9! + 8! + 7!$ **f** $(n+1)! + n! + (n-1)!$

DEVELOPMENT

7 Write each expression as a single fraction.

a $\dfrac{1}{n!} + \dfrac{1}{(n-1)!}$ **b** $\dfrac{1}{n!} - \dfrac{1}{(n+1)!}$ **c** $\dfrac{1}{(n+1)!} - \dfrac{1}{(n-1)!}$

8 **a** If $f(x) = x^n$, find:

i $f'(x)$ **ii** $f''(x)$

iii $f^{(n)}(x)$ **iv** $f^{(k)}(x)$, where $k \le n$.

b If $f(x) = \dfrac{1}{x}$, find:

i $f'(x)$ **ii** $f''(x)$

iii $f^{(5)}(x)$ **iv** $f^{(n)}(x)$.

9 **a** Show that $k \times k! = (k+1)! - k!$

b Hence by considering each individual term as a difference of two terms, sum the series $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n!$

10 **a** Find the largest power of:

i 2, **ii** 10, that is a divisor of 10!

b Find the largest power of:

i 2, **ii** 5,

iii 7, **iv** 13, that is a divisor of 100!

11 [A relationship between higher derivatives of polynomials and factorials]

a If $f(x) = 11x^3 + 7x^2 + 5x + 3$, show that:

i $f(0) = 3 \times 0!$ **ii** $f'(0) = 5 \times 1!$

iii $f''(0) = 7 \times 2!$ **iv** $f'''(0) = 11 \times 3!$

v $f^{(k)}(0) = 0$, for all $k \ge 4$.

Hence explain why $f(x)$ can be written $f(x) = \dfrac{f(0)}{0!} + \dfrac{f'(0)\,x}{1!} + \dfrac{f''(0)\,x^2}{2!} + \dfrac{f'''(0)\,x^3}{3!}$.

b Show that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is any polynomial, then

$$f^{(k)}(0) = \begin{cases} a_k\,k!, & \text{for } k = 0, 1, 2, \ldots, n, \\ 0, & \text{for } k > n, \end{cases}$$

and hence explain why $f(x) = \dfrac{f(0)}{0!} + \dfrac{f'(0)\,x}{1!} + \dfrac{f''(0)\,x^2}{2!} + \dfrac{f'''(0)\,x^3}{3!} + \cdots$.

Because $f(x)$ is a polynomial, this *power series* has only finitely many terms. There are ways of generalising this to some other functions that are not polynomials.

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12 **a** Evaluate $\dfrac{k}{(k+1)!}$, for $k = 1, 2, 3, 4$ and 5.

b Evaluate $\dfrac{1}{(1+1)!} + \dfrac{2}{(2+1)!} + \cdots + \dfrac{n}{(n+1)!}$, for $n = 1, 2, 3, 4$ and 5.

c Write the expression in part **b** as S_n, to indicate that it is a sum whose value depends on n. That is,

$$S_n = \frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \cdots + \frac{n}{(n+1)!}.$$

Make a reasonable guess about the value of S_n. Hence find $\lim\limits_{n\to\infty} S_n$.

d Prove that $\dfrac{k}{(k+1)!} = \dfrac{1}{k!} - \dfrac{1}{(k+1)!}$, and hence produce a proof of part **c** using a collapsing sum.

ENRICHMENT

13 Express using factorial notation:

a $30 \times 28 \times 26 \times \cdots \times 2$ **b** $29 \times 27 \times 25 \times \cdots \times 1$ **c** $\dfrac{30 \times 28 \times 26 \times \cdots \times 2}{29 \times 27 \times 25 \times \cdots \times 1}$

14 [Stirling's formula]

The following formula is too difficult to prove at this stage, but it is most important because it provides a continuous function that approximates $n!$ for integer values of n:

$$n! \doteqdot \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}, \text{ in the sense that the percentage error } \to 0 \text{ as } n \to \infty.$$

Show that the formula has an error of approximately 2.73% for 3! and 0.83% for 10! Find the percentage error for 60!

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14B Counting ordered selections

It should be clear from Chapter 12 that probability problems would become easier if we could develop greater sophistication in counting methods. This chapter develops a more systematic approach to counting, then applies the methods to probability. Two questions dominate:

- Are the selections we are counting ordered or unordered?
- If they are ordered, is repetition allowed or not?

Sections 14B–14D develop the theory of counting ordered selections, with and without repetition, then Section 14E deals with unordered selections.

The multiplication principle

This morning, Stefan chose a shirt to wear from the twelve that he owns. Then without worrying about a match, he chose a pair of trousers from the six pairs he owns. How many possible outfits could he have walked out of the house wearing?

The answer is $12 \times 6 = 72$. Each choice of trousers was part of twelve outfits, depending on the choice of shirt. There were six choices of trousers, so

$$\text{number of outfits} = 12 + 12 + 12 + 12 + 12 + 12 = 12 \times 6.$$

The multiplication principle may be applied to more than two successive choices. Stefan also chose a pair of shoes from his eight pairs, and a tie from his collection of 87 ties. Taking these choices into account,

$$\text{number of outfits} = 12 \times 6 \times 8 \times 87 = 50\,112.$$

3 THE MULTIPLICATION PRINCIPLE

Suppose that a selection is to be made in r stages. Suppose that the first stage can be chosen in n_1 ways, the second in n_2 ways, . . . , the rth in n_r ways. Then

$$\text{number of ways of choosing the complete selection} = n_1 \times n_2 \times \cdots \times n_r.$$

Using boxes for multiplication principle questions

An ordered selection can usually be regarded as a sequence of choices made one after the other. An efficient setting-out here is to use a box diagram to keep track of these successive choices. In the situation above with Stefan's outfits:

shirt	trousers	shoes	tie
12	6	8	87

Number of outfits $= 12 \times 6 \times 8 \times 87 = 50\,112.$

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Example 3 14B

How many five-letter words can be formed in which the second and fourth letters are vowels and the other three letters are consonants?

SOLUTION

We can select each letter in order:

1st letter	2nd letter	3rd letter	4th letter	5th letter
21	5	21	5	21

Number of words $= 21 \times 5 \times 21 \times 5 \times 21 = 231\,525$.

Note: Unless otherwise indicated, always take the letter 'y' as a consonant.

Ordered selections with repetition

Suppose that r-letter words are to be formed from n distinct letters, where any letter can be used any number of times. Then each successive letter in the word can be chosen in n ways:

1st letter	2nd letter	3rd letter	. . .	rth letter
n	n	n	. . .	n

giving n^r distinct words altogether.

4 ORDERED SELECTIONS WITH REPETITION

The number of r-letter words formed from n distinct letters, with repetition, is n^r.

Example 4 14B

a How many six-digit numbers can be formed entirely from odd digits?
b How many of these numbers contain at least one seven?

SOLUTION

a There are five odd digits, so the number of such numbers is 5^6.

b We first count the number of these six-digit numbers not containing 7.
Such numbers are formed from the digits 1, 3, 5 and 9, so there are 4^6 of them.
Subtracting this from the answer to part **a**, number of numbers $= 5^6 - 4^6 = 11\,529$.

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Ordered selections without repetition

Counting ordered selections without repetition typically involves factorials, because as each stage is completed, the number of objects to choose from diminishes by 1.

Example 5 14B

In how many ways can a class of 16 select a committee consisting of a president, a vice-president, a treasurer and a secretary?

SOLUTION

Select in order the president, the vice-president, the treasurer and the secretary (we assume that the same person cannot fill two roles).

president	vice-president	treasurer	secretary
16	15	14	13

Hence there are $16 \times 15 \times 14 \times 13 = \frac{16!}{12!}$ possible committees.

Note: The phrases 'with replacement' and 'with repetition' are used almost interchangeably. Similarly, the phrases 'without replacement' and 'without repetition' are also used almost interchangeably.

The general case — permutations

A *permutation* or *ordered set* is an arrangement of objects chosen from a certain set. For example, the words ABC, CED, EAB and DBC are some of the permutations of three letters taken from the 5-member set $\{ A, B, C, D, E \}$.

The symbol ${}^nP_r$ is used to denote the number of permutations of r letters chosen without repetition from a set of n distinct letters. Example 5 is easily generalised to show that there are $\frac{n!}{(n-r)!}$ such permutations, so this becomes the formula for ${}^nP_r$:

1st letter	2nd letter	3rd letter	4th letter	$\cdots$	rth letter
n	$n-1$	$n-2$	$n-3$	$\cdots$	$n-r+1$

Hence
$$\begin{aligned} {}^nP_r &= n(n-1)(n-2)(n-3)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)(n-3)\times\cdots\times 2\times 1}{(n-r)(n-r-1)\times\cdots\times 2\times 1} \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

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5 PERMUTATIONS—ORDERED SELECTIONS WITHOUT REPETITION

- A *permutation* is an arrangement of objects chosen from a certain set without repetition, that is, without replacement.
- We often imagine permutations as words formed from distinct letters chosen from a larger word consisting only of distinct letters.
- The number of permutations with r letters that can be formed without repetition from a set of n distinct letters, is

$$^{n}\mathrm{P}_r = \frac{n!}{(n-r)!}.$$

The phrases 'without repetition' and 'without replacement' are used almost interchangeably. The term 'distinct objects' is also used to describe this situation.

For example, the number of permutations of three distinct letters taken from the 5-member set $\{A, B, C, D, E\}$ is $^{5}\mathrm{P}_3 = \frac{5!}{2!} = 60$.

Scientific calculators have a button labelled $\boxed{^{n}\mathrm{P}_r}$ which will find values of $^{n}\mathrm{P}_r$. For low values of n and r, the answers are exact, but for high values they are only approximations. You should, however, make a practice of evaluating these number by hand when it is reasonable to do so, because such calculations greatly help the intuition.

Example 6 14B

How many eight-digit numbers, and how many nine-digit numbers, can be formed from the nine non-zero digits if no repetition is allowed?

SOLUTION

Number of 8-digit numbers $= {}^{9}\mathrm{P}_8$
$= \frac{9!}{1!}$
$= 9!$

Number of 9-digit numbers $= {}^{9}\mathrm{P}_9$
$= \frac{9!}{0!}$
$= 9!$

The permutations of a set

A *permutation of a set* is an arrangement of all the members of the set in a particular order. If the set has n members, then the number of permutations is $^{n}\mathrm{P}_n = \frac{n!}{0!} = n!$

6 PERMUTATIONS OF A SET

The number of permutations of an n-member set, that is, the number of distinct orderings of the set, is $^{n}\mathrm{P}_n = n!$

Box 6 summarises the arguments used at the start of Section 14A and in Question 1 of Exercise 14A.

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Example 7 14B

In how many ways can 20 people form a queue? Will the number of ways double with 40 people?

SOLUTION

With 20 people, number of ways $= 20!$ $(\doteqdot 2.4 \times 10^{18})$.
With 40 people, number of ways $= 40!$ $(\doteqdot 8.2 \times 10^{47})$.
Hence 40 people can form a queue in about 3×10^{29} times more ways than 20 people.

A general counting principle — deal with the restrictions first

Many problems have some restrictions in the way things can be arranged. These restrictions should be dealt with first. It is also important to keep in mind that the order of the boxes represents the order in which the choices are made, not the final ordering of the objects, and that they can be used in surprisingly flexible ways.

7 DEAL WITH THE RESTRICTIONS FIRST

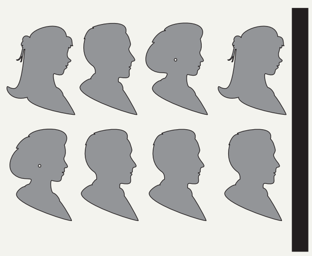
- When counting ordered selections, deal with any restrictions first.
- Place the boxes in the order in which the selections are made.

Example 8 14B

Eight people form two queues, each with four people. Albert will only stand in the left-hand queue, Beth will only stand in the right-hand queue, and Charles and Diana insist on standing in the same queue. In how many ways can the two queues be formed?

SOLUTION

Place Albert in any of the 4 possible positions in the left-hand queue,
Then place Beth in any of the 4 positions in the right-hand queue.
Place Charles in any of the remaining 6 positions.
Place Diana in one of the 2 remaining positions in the same queue.
There remain 4 unfilled positions, which can be filled in 4! ways:



Albert	Beth	Charles	Diana	Last four positions
4	4	6	2	4!

Hence number of ways $= 4 \times 4 \times 6 \times 2 \times 4!$
$= 4608.$

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Exercise 14B

FOUNDATION

1 List all the permutations of the letters of the word DOG. How many are there?

2 List all the permutations of the letters EFGHI, beginning with F, taken three at a time.

3 Find how many arrangements of the letters of the word FRIEND are possible if the letters are taken:
a four at a time,
b six at a time.

4 Find how many four-digit numbers can be formed using the digits 5, 6, 7, 8 and 9 if:
a no digit is to be repeated,
b any of the digits can occur more than once.

5 How many three-digit numbers can be formed using the digits 2, 3, 4, 5 and 6 if no digit can be repeated? How many of these are greater than 400?

6 In how many ways can seven people be seated in a row of seven different chairs?

7 The symbol ${}^nP_r$ is the number of ways of arranging r objects selected without replacement from n objects. Use this expression, and the ${}^nP_r$ button on your calculator, to answer the following questions.
a From a group of 10, three people line up to buy tickets. How many ways can this happen?
b Five cards are each labelled uniquely with one of the digits 1, 2, 3, 4, 5. Three of the five cards are placed down in a row. How many ways can the cards be arranged?
c One hundred people each buy one ticket in a lottery. How many ways can the first three places be awarded?

8 The expression n^r is the number of ways of arranging r objects selected with replacement from n objects. Use this expression to answer the following questions.
a How many three-digit numbers can be written down from the digits 1 to 9? The digits need not be distinct.
b One hundred people each buy one ticket in a lottery. Tickets are selected one by one at random from a barrel and then replaced between selections. How many ways can the first three places be awarded?
c A computer sends a string of ten binary digits, that is, each symbol can only be 0 or 1. How many such ten-digit strings are possible?

9 Eight runners are participating in a 400-metre race.
a In how many ways can they finish?
b In how many ways can the gold, silver and bronze medals be awarded?

10 a If you toss a coin and roll a die, how many outcomes are possible?
b If you toss two coins and roll three dice, how many outcomes are possible?

11 A woman has four hats, three blouses, five skirts, two handbags and six pairs of shoes. In how many ways can she be attired, assuming that she wears one of each item?

12 Jack has six different football cards and Meg has another eight different football cards. In how many ways can one of Jack's cards be exchanged for one of Meg's cards?

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DEVELOPMENT

13 In Sydney, landline phone numbers, ignoring the area code, consist of eight digits. Let us consider those starting with the digit 9.

- a How many such phone numbers are possible?
- b How many of these end in an odd number?
- c How many consist of odd digits only?
- d How many are there that do not contain a zero, and in which the consecutive digits alternate between odd and even?

14 Users of automatic teller machines are required to enter a four-digit PIN (personal identification number). Find how many PINs:

- a are possible,
- b consist of four distinct digits,
- c consist of odd digits only,
- d start and end with the same digit.

15
- a If repetitions are not allowed, how many four-digit numbers can be formed from the digits $1, 2, \ldots 8, 9$?
- b How many of these end in 1?
- c How many of these are even?
- d How many are divisible by 5?
- e How many are greater than 7000?

16 Repeat the previous question if repetitions are allowed.

17 In Tasmania, a car licence plate consists of two letters followed by four digits. Find how many of these are possible:

- a if there are no restrictions,
- b if there is no repetition of letters or digits,
- c if the second letter is X and the third digit is 3,
- d if the letters are D and Q and the digits are $3, 6, 7$ and 9.

18
- a In how many ways can the letters of the word NUMBER be arranged?
- b How many begin with N?
- c How many begin with N and end with U?
- d In how many is the N somewhere to the left of the U?

19 In how many ways can a boat crew of eight women be arranged if three of the women can only row on the bow side and two others can only row on the stroke side?

20 A motor bike can carry three people: the driver, one passenger behind the driver and one in the sidecar. If among five people, only two can drive, in how many ways can the bike be filled?

21
- a How many five-digit numbers can be formed from the digits $2, 3, 4, 5$ and 6?
- b How many of these numbers are greater than 56 432?
- c How many of these numbers are less than 56 432?

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22 **a** Integers are formed from the digits 2, 3, 4 and 5, with repetitions not allowed.

i How many such numbers are there?

ii How many of them are even?

b Repeat the two parts to this question if repetitions are allowed.

23 **a** How many five-digit numbers can be formed from the digits 0, 1, 2, 3 and 4 if repetitions are not allowed?

b How many of these are odd?

c How many are divisible by 5?

24 **a** If ${}^{8}\mathrm{P}_{r} = 336$, find the value of r.

b If $7 \times {}^{2n}\mathrm{P}_{n} = 4 \times {}^{2n+1}\mathrm{P}_{n}$, find n.

c Using the result ${}^{n}\mathrm{P}_{r} = \dfrac{n!}{(n - r)!}$, prove that:

i ${}^{n+1}\mathrm{P}_{r} = {}^{n}\mathrm{P}_{r} + r \times {}^{n}\mathrm{P}_{r-1}$

ii ${}^{n}\mathrm{P}_{r} = {}^{n-2}\mathrm{P}_{r} + 2r \times {}^{n-2}\mathrm{P}_{r-1} + r(r - 1) \times {}^{n-2}\mathrm{P}_{r-2}$

ENRICHMENT

25 Recall that a whole number is divisible by 3 if and only if the sum of the digits is divisible by 3.

a How many 5-digit whole numbers are divisible by 3?

b How many 5-digit whole numbers with no 0s are divisible by 3?

c How many 5-digit whole numbers with no 0s are divisible by 2?

d How many 5-digit whole numbers with no 0s are divisible by 6?

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14C Ordered selections and grouping

We have already dealt with one important counting principle — deal with the difficulties first. This section deals with three other principles that are often useful in counting.

A general counting principle — grouping

In some ordering problems, particular members must be grouped together. This produces a *compound ordering*, in which the various groups must first be ordered, and then the individuals ordered within each group.

8 GROUPING

- First order the groups, then order the individuals within each group.
- A group may sometimes consist of a single individual.

Example 9 14C

Four boys and four girls form a queue at the bus stop. There is one couple who want to stand together, the other three girls want to stand together, but the other three boys don't care where they stand. How many acceptable ways are there of forming the queue?

SOLUTION

There are five groups — the couple, the group of three girls, and the three groups each consisting of one individual boy. First order the five groups, then order the couple, and the three girls.

order the 5 groups	order the couple	order the 3 girls
5!	2!	3!

Hence number of ways $= 5! \times 2! \times 3! = 1440$.

A general counting principle — deal with the complementary situation

Many probability questions have already been solved using complementary events. The same principle applies to counting — count the unacceptable orderings, then subtract them from the total number of orerings. The word 'not' may or may not be there to prompt you.

Example 10 14C

How many seven-letter words can be formed from the letters A, B, C, D, E, F, G if the vowels must be separated by at least one consonant?

SOLUTION

$$\text{Number of orderings without restriction} = 7!$$
$$\text{Number of orderings with A and E together} = 2! \times 6!$$

(Order the A and E and glue them together, then order the 6 groups.)

$$\begin{aligned}\text{Hence number of orderings with A and E apart} &= 7! - 2 \times 6! \\ &= 3600.\end{aligned}$$

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A general counting principle — using cases

Sometimes the fiddly conditions of the problem mean that one set of boxes is not enough to solve it, and separate cases need to be considered. In this situation, the cases should not overlap, or if they do, the number in the overlap needs to be subtracted.

9 USING CASES IN COUNTING

- The cases should not overlap.
- If the cases do overlap, the overlap needs to be subtracted.

This same principle was expressed in Section 12C, Box 12, by the formula

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example 11 — 14C

a How many whole numbers less than 1000 are odd and greater than 500, or multiples of 5 and less than 200?

b How many whole numbers less than 1000 are odd and greater than 200, or multiples of 5 and less than 500?

SOLUTION

a There are 250 odd numbers between 500 and 1000, and there are 40 multiples of 5 less than 200 (including zero). The two cases do not overlap, so the total number is 290.

b There are 400 odd numbers between 200 and 1000, and there are 100 multiples of 5 less than 500 (including zero). But the two cases overlap — from 200 to 500 there are 30 odd numbers that are multiples of 5. Hence

$$\text{Total} = 400 + 100 - 30 = 470.$$

Exercise 14C

FOUNDATION

1 How many rearrangements are there of the letters in each word, if the vowels must be together?
a BOARDS **b** RIO **c** QUIT **d** TROUNCE

2 How many arrangements of the word MATHS are possible, if:
a the T and H must remain together?
b the TH must remain together and in this order?

3 How many ways can Andrew, Becky, Courtney, Dion and Ellie sit in a row, if Andrew and Becky sit together and Dion and Ellie sit together?

4 In how many ways can three different Maths books, six different Science books and four different English books be placed on a shelf, if the books relating to each subject are to be kept together?

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5 A class is asked to determine how many ways the letters of the word SOLAR can be arranged, if the first two positions cannot both be vowels.

a Jack decides to use cases (the word either starts with a vowel or it does not). Use Jack's method to answer the question.

b Jill decides to consider the complementary situation (both first positions are vowels). Use Jill's method to answer the question.

6 A family of two parents and two children are going on a car trip. Only the parents can drive. If the father drives, then the mother will sit in the back seat where she feels safer. The father always sits in a front seat of the car. How many arrangements are possible, if two sit in the front and two in the back?

7 How many numbers can be written down using each digit of 789 at most once, if the result must be at least 80?

DEVELOPMENT

8 Find how many arrangements of the letters of the word UNIFORM are possible:

a if the vowels must occupy the first, middle and last positions,

b if the word must start with U and end with M,

c if all the consonants must be in a group at the end of the word,

d if the M is somewhere to the right of the U.

9 Find how many arrangements of the letters of the word BEHAVING:

a end in NG,

b begin with three vowels,

c have three vowels occurring together.

10 A Maths test is to consist of six questions. In how many ways can it be arranged so that:

a the shortest question is first and the longest question is last,

b the shortest and longest questions are next to one another?

11 In Morse code, letters are formed by a sequence of dashes and dots. How many different letters is it possible to represent if a maximum of ten symbols are used?

12 Four boys and four girls are to sit in a row. Find how many ways this can be done if:

a the boys and girls alternate,

b the boys and girls sit in distinct groups.

13 Five-letter words are formed without repetition from the letters of PHYSICAL.

a How many consist only of consonants?

b How many begin with P and end with S?

c How many begin with a vowel?

d How many contain the letter Y?

e How many have the two vowels occurring next to one another?

f How many have the letter A immediately following the letter L?

14 a How many seven-letter words can be formed without repetition from the letters of the word INCLUDE?

b How many of these do not begin with I?

c How many end in L?

d How many have the vowels and consonants alternating?

e How many have the C immediately following the D?

f How many have the letters N and D separated by exactly two letters?

g How many have the letters N and D separated by more than two letters?

15 Repeat parts a–d of the previous question if repetition is allowed.

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16 **a** In how many ways can ten people be arranged in a line:

- **i** without restriction,
- **ii** if one particular person must sit at either end,
- **iii** if two particular people must sit next to one another,
- **iv** if neither of two particular people can sit on either end of the row?

b In how many ways can n people be placed in a row of n chairs:

- **i** if one particular person must be on either end of the row,
- **ii** if two particular people must sit next to one another,
- **iii** if two of them are not permitted to sit at either end?

17 Five boys and four girls form a queue at the cinema. There are two brothers who want to stand together, the remaining three boys wish to stand together, and the four girls don't mind where they stand. In how many ways can the queue be formed?

18 Eight people are to form two queues of four. In how many ways can this be done if:

a there are no restrictions,
b Jim will only stand in the left-hand queue,
c Sean and Liam must stand in the same queue?

19 There are eight swimmers in a race. In how many ways can they finish if there are no dead heats and the swimmer in Lane 2 finishes:

a immediately after the swimmer in Lane 5,
b any time after the swimmer in Lane 5?

20 Five backpackers arrive in a city where there are five youth hostels.

a How many different accommodation arrangements are there if there are no restrictions on where the backpackers stay?

b How many different accommodation arrangements are there if each backpacker stays at a different youth hostel?

c Suppose that two of the backpackers are brother and sister and wish to stay in the same youth hostel. How many different accommodation arrangements are there if the other three can go to any of the other youth hostels?

21 Numbers less than 4000 are formed from the digits $1,\ 3,\ 5,\ 8$ and 9, without repetition.

a How many such numbers are there?
b How many of them are odd?
c How many of them are divisible by 5?
d How many of them are divisible by 3?

ENRICHMENT

22 [Derangements]

A *derangement* of n distinct letters is a permutation of them so that no letter appears in its original position. For example, DABC is a derangement of ABCD, but DACB is not. Denote the number of derangements of n letters by $D(n)$.

a By listing all the derangements of A, AB, ABC and ABCD, find the values of $D(1)$, $D(2)$, $D(3)$ and $D(4)$.

b Suppose that we have formed a derangement of the five letters ABCDE. Let the last letter in the derangement be X, and exchange X with E — this puts E back to its original position. Either X is now also in its original position so that three letters are away from their original positions, or X is not in its original position so that four letters are away from their original positions. Hence explain why

$$D(5) = 4 \times D(4) + 4 \times D(3).$$

c Use this formula to evaluate $D(5)$. Then apply the corresponding arguments and formulae to evaluate $D(6)$, $D(7)$ and $D(8)$.

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14D Ordered selections with identical elements

So far, our ordered sets have not permitted any element to be repeated. This section deals with permutations of all the letters of a word containing several copies of the one letter.

Counting with identical elements

Finding the number of different words formed using all the letters of the word 'PRESSES' is complicated by the fact that there are three Ss and two Es. If the seven letters were all different, we would conclude that

$$\text{number of ways} = 7!$$

But we have *overcounted by a factor of* $2! = 2$, because the Es can be interchanged without changing the word. We have also *overcounted by a factor of* $3! = 6$, because the three Ss can be permuted amongst themselves in 3! ways without changing the word. Taking account of both overcountings,

$$\text{number of ways} = \frac{7!}{2! \times 3!} = 420.$$

This method is easily generalised. In the language of 'words':

10 COUNTING WITH IDENTICAL ELEMENTS

Suppose that a word of n letters has r_1 alike of one type, r_2 alike of another type, . . . , r_k alike of a final type. Then the number of distinct words that can be formed by rearranging the letters is

$$\text{number of words} = \frac{n!}{r_1! \times r_2! \times \cdots \times r_k!}.$$

If $r_i = 1$, that is, there is only one letter of the ith type, then $r_i! = 1$, so it doesn't matter whether we include it in the formula or not.

Example 12 — 14D

Three identical wine glasses and five identical tumblers are to be arranged in a row across the front of a cupboard.

a In how many ways can this be done (counting only the patterns)?

b How does this change if one of the wine glasses becomes clearly chipped, and two tumblers break and are replaced by two identical tumblers different from the other three?

SOLUTION

a There are 8 glasses, 3 alike of one type and 2 alike of another,

$$\text{so number of ways} = \frac{8!}{3! \times 5!} = 56.$$

b There are now 2 alike of one type, 1 of another, 3 of another, 2 of another,

$$\text{so number of ways} = \frac{8!}{2! \times 1! \times 3! \times 2!} = 1680 \text{ (the 1! can be omitted).}$$

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Words containing only two different letters

The theory about counting with identical letters assumes particular importance when there are only two types of letters, as in part **a** of Example 12. The answer to part **a** was $\dfrac{8!}{3! \times 5!}$. This formula is true in general, and will be vital in the next section:

11 PERMUTATIONS OF WORDS CONTAINING ONLY TWO DIFFERENT LETTERS

Suppose that an n-letter word consists of r Ys and $n - r$ Ns. Then the number of possible distinct words is

$$\text{number of words} = \frac{n!}{r! \times (n - r)!}.$$

After the next section, we will be able to use the concise symbol ${}^{n}C_{r}$ for this expression $\dfrac{n!}{r! \times (n - r)!}$.

Example 13 — 14D

a Six women and four men form a queue at the bus stop. How many patterns of men and women are there?

b There is space at the bus stop for a queue of ten adults. How many patterns of men and women are possible?

SOLUTION

a We can use the formula above, or we can think of arranging ten letters, six alike of one type and four of another.

$$\text{Number of patterns} = \frac{10!}{6! \times 4!} = 210.$$

b Each of the 10 positions can be filled in two possible ways:

1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
2	2	2	2	2	2	2	2	2	2

Hence number of ways $= 2^{10} = 1024$.

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Using cases

As mentioned in the previous section, many counting problems are too complicated to be analysed completely by a single box diagram. In such situations, the use of cases is unavoidable. Attention should be given, however, to minimising the number of different cases that need to be considered.

Example 14 — 14D

How many six-letter words can be formed by using the letters of the word 'PRESSES'?

SOLUTION

We omit in turn each of the four letters 'P', 'R', 'E' and 'S'.
This leaves six letters which we must then arrange in order.

1 If an S is omitted, there are then 2 Es and 2 Ss,

$$\text{so number of words } = \frac{6!}{2! \times 2!} = 180.$$

2 If an E is omitted, there are then 3 Ss,

$$\text{so number of words } = \frac{6!}{3!} = 120.$$

3 If P or R is omitted (2 cases), there are then 2 Es and 3 Ss,

$$\text{so number of words } = \frac{6!}{3! \times 2!} \times 2 \quad \text{(doubling for the two cases)}$$

$$= 120.$$

Hence the total number of words is $180 + 120 + 120 = 420$.

Exercise 14D

FOUNDATION

1 Find the number of permutations of the following words if all the letters are used.

- **a** BOB
- **b** ALAN
- **c** SNEEZE
- **d** TASMANIA
- **e** BEGINNER
- **f** FOOTBALLS
- **g** EQUILATERAL
- **h** COMMITTEE
- **i** WOOLLOOMOOLOO

2 The six digits 1, 1, 1, 2, 2, 3 are used to form a six-digit number. How many numbers can be formed?

3 Six coins are lined up on a table. Find how many patterns are possible if there are:

- **a** five tails and one head,
- **b** four heads and two tails,
- **c** three tails and three heads.

4 Eight balls, identical except for colour, are arranged in a line. Find how many different arrangements are possible if:

- **a** all balls are of a different colour,
- **b** there are seven red balls and one white ball,
- **c** there are six red balls, one white ball and one black ball,
- **d** there are three red balls, three white balls and two black balls.

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5 Five identical green chairs and three identical red chairs are arranged in a row. Find how many arrangements are possible:

a if there are no restrictions,

b if there must be a green chair on either end.

DEVELOPMENT

6 A motorist travels through eight sets of traffic lights, each of which is red or green. He is forced to stop at three sets of lights.

a In how many ways could this happen?

b What other number of red lights would give an identical answer to part a?

7 In how many ways can the letters of the word SOCKS be arranged in a line:

a without restriction,

b so that the two Ss are together,

c so that the two Ss are separated by at least one other letter,

d so that the K is somewhere to the left of the C?

8 a Find the number of arrangements of the letters in SLOOPS if:

i there are no restrictions,

ii the two Os are together,

iii the two Os are to be separated,

iv the Os are together and the Ss are together.

b In how many arrangements of the letters in TATTOO are the two Os separated?

9 In how many ways can the letters of the word DECISIONS be arranged:

a without restriction,

b so that the vowels and consonants alternate,

c so that the vowels come first followed by the consonants,

d so that the N is somewhere to the right of the D?

10 In how many ways can the letters of the word PROPORTIONALITY be arranged so that the vowels and consonants still occupy the same places?

11 A form has ten questions in order, each of which requires the answer 'Yes' or 'No'. Find the number of ways the form can be filled in:

a without restriction,

b if the first and last answers are 'Yes',

c if two are 'Yes' and eight are 'No',

d if five are 'Yes' and five are 'No',

e if more than seven answers are 'Yes',

f if an odd number of answers are 'Yes',

g if exactly three answers are 'Yes', and they occur together,

h if the first and last answers are 'Yes' and exactly four more are 'Yes'.

12 Containers are identified by a row of coloured dots on their lids. The colours used are yellow, green and purple. In any arrangement, there are to be no more than three yellow dots, no more than two green dots and no more than one purple dot.

a If six dots are used, what is the number of possible codes?

b What is the number of different codes possible if only five dots are used?

13 a How many five-letter words can be formed by using the letters of the word STRESS?

b How many five-letter words can be formed by using the letters of the word BANANA?

14 Find how many arrangements of the letters of the word TRANSITION are possible if:

a there are no restrictions,

b the Is are together,

c the Is are together, and so are the Ns, and so are the Ts,

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d the Ns occupy the end positions,
e an N occupies the first but not the last position,
f the letter N is not at either end,
g the vowels are together.

15 Ten coloured marbles are placed in a row.
a If they are all of different colours, how many arrangements are possible?
b What is the minimum number of colours needed to guarantee at least 10 000 different patterns? (Hint: This will need a guess-and-check approach.)

ENRICHMENT

16 Find how many arrangements there are of the letters of the word GUMTREE if:
a there are no restrictions,
b the Es are together,
c the Es are separated by:
i one, **ii** two, **iii** three, **iv** four, **v** five letters,
d the G is somewhere between the two Es,
e the M is somewhere to the left of both Es and the U is somewhere between them,
f the G is somewhere to the left of the U and the M is somewhere to the right of the U.

17 If the letters of the word GUMTREE and the letters of the word KOALA are combined and arranged into a single twelve-letter word, in how many of these arrangements do the letters of KOALA appear in their correct order, but not necessarily together?

18 Bob is about to hang his eight shirts in the wardrobe. He has four different styles of shirt, with two identical shirts in each style. How many different arrangements are possible if no two identical shirts are next to one another?

19 [Derangements and the number e] A *derangement* was defined in question 22 of Exercise 14C as a permutation that leaves no letter unmoved, and $D(n)$ was defined as the number of derangements of n letters. Another approach to finding a formula for $D(n)$ uses the *inclusion–exclusion principle* introduced in the Enrichment section of Exercise 12C. Consider, for example, the derangements of ABCD.
a How many permutations are there of ABCD?
b How many of these leave in its original position:
i A, **ii** B, **iii** C, **iv** D?
Each of these numbers will need to be subtracted from the answer to part **a**.
c In doing part B, however, we have subtracted some of the permutations twice. For example, some of them would leave both A and B unmoved. Thus we need to add back the number of permutations that leave any two particular letters unmoved. How many of these are there?
d Now you will need to subtract the number of permutations that leave three letters unmoved, and so on. Hence find an expression for $D(4)$.
e Rearrange your expression into the form $D(4) = 4!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$.
f Explain how this can be generalised to $D(n) = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}\right)$.
g We shall prove in an Enrichment question of the Year 12 volume that the sequence in the brackets of part **f** above converges to $\frac{1}{e}$ as $n \to \infty$. Give both a combinatorial and a probabilistic interpretation of this result in terms of the ratio of the numbers of permutations and derangements of n letters.

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14E Counting unordered selections

This section turns attention from the ordered selections of the previous three sections to the counting of unordered selections. An unordered selection of distinct objects chosen from a certain set is just a subset of the set. So given a set S with n members, there are two basic questions to ask:

- How many subsets does S have altogether?
- How many r-member subsets does S have, for $r = 0, 1, \ldots, n$?

An example of the result

Here is an example to illustrate the situation. Let S be the five-member set

$$S = \{A, B, C, D, E\}.$$

and here is the list of all the subsets of S, arranged by the number of members:

1 0-member subset: $\varnothing$

5 1-member subsets: $\{A\}, \{B\}, \{C\}, \{D\}, \{E\}$

10 2-member subsets: $\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\},$
$\{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}$

10 3-member subsets: $\{A, B, C\}, \{A, B, D\}, \{A, B, E\}, \{A, C, D\}, \{A, C, E\},$
$\{A, D, E\}, \{B, C, D\}, \{B, C, E\}, \{B, D, E\}, \{C, D, E\}$

5 4-member subsets: $\{A, B, C, D\}, \{A, B, C, E\}, \{A, B, D, E\}, \{A, C, D, E\}, \{B, C, D, E\}$

1 5-member subset: $\{A, B, C, D, E\}$

making $1 + 5 + 10 + 10 + 5 + 1 = 32$ subsets altogether. We will now establish the general results by finding clear reasons for these results.

How many subsets does *S* have altogether?

Choosing a subset of S requires looking at each member of S in turn and deciding whether to include it in the subset or not. Let us write the code Y for 'Yes' if it is included, and N for 'No' if it is not included. Using this code, each subset of S corresponds to a five-letter word made up of Ys and Ns. For example,

$$\text{YYNYN} \longleftrightarrow \{A, B, D\} \qquad \text{and} \qquad \text{NNNNY} \longleftrightarrow \{E\}.$$

We can now use this code to count the total number of subsets. Thus we are counting the number of 5-letter words made up only of Ys and Ns:

$$\begin{aligned}\text{Total number of subsets} &= \text{number of 5-letter words made up of Ys and Ns}\\ &= 2^5\\ &= 32.\end{aligned}$$

How many 3-member subsets does *S* have?

We can also use the code to count the number of 3-member subsets, because that is the number of 5-letter words consisting of 3 Ys and 2 Ns:

$$\begin{aligned}\text{Number of 3-member subsets} &= \text{number of 5-letter words with 3 Ys and 2 Ns}\\ &= \frac{5!}{3! \times 2!}.\\ &= 10.\end{aligned}$$

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The general formulae

These two results are easily generalised to any finite set S with n elements:

$$\text{total number of subsets} = 2^n$$

$$\text{number of } r\text{-member subsets} = \frac{n!}{r!\,(n-r)!}, \text{ for } r = 0, 1, \ldots, n.$$

The notation ${}^nC_r$ or $\binom{n}{r}$

The notation ${}^nC_r$, or alternatively $\binom{n}{r}$ — both are spoken as '*n choose r*' — is used for the number of r-member subsets of S. Summarising all this:

12 COUNTING THE SUBSETS OF A SET

Suppose that S is a finite set with n elements, and suppose that $0 \le r \le n$.

- Total number of subsets of $S = 2^n$.
- Number of r-member subsets of $S = {}^nC_r = \dfrac{n!}{r!\,(n-r)!}$.
- ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$.

The symbol ${}^nC_r$ or $\binom{n}{r}$ — both are spoken as '*n choose r*' — denotes the number of r-subsets of an n-member set, where $0 \le r \le n$.

The last bullet point follows because LHS and RHS are both counting the total number of subsets of S.
The notations ${}^nP_r$ and ${}^nC_r$ are intended to be similar. The letter C stands for 'combination' (which is an old term for 'unordered selection'), just as the letter P stands for 'permutation'. By a convenient, but false, etymology, the letter C also stands for 'Choose', which is the origin of the more recent convention of saying 'n choose r' for ${}^nC_r$.

Scientific calculators have a button labelled ${}^nC_r$ which will find values of ${}^nC_r$. For low values of n and r, the answers are exact, but for high values they are only approximations. As with ${}^nP_r$, however, you should make a practice of evaluating these number by hand when it is reasonable to do so, because such calculations greatly help the intuition.

A proof moving from ordered selections to unordered selections

There is another way to prove the formula for ${}^nC_r$. In Section 14B we saw that the number of three-letter words formed without repetition from the five letters A, B, C, D and E is ${}^5P_3 = 5 \times 4 \times 3 = 60$. When we turn to unordered selections, however, there are six distinct words which all correspond, for example, to the three-member subset { B, C, E }:

BCE, BEC, CBE, CEB, EBC, ECB $\longleftrightarrow$ { B, C, E }

The reason for this is that there are ${}^3P_3 = 3 \times 2 \times 1 = 6$ ways of ordering the subset { B, C, E }. Thus the correspondence between three-letter words and three-member subsets is many-to-one, with a six-fold overcounting. Hence the number of three-member subsets is $60 \div 6 = 10$, as required.

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In general, ${}^n\mathrm{P}_r = \dfrac{n!}{(n-r)!}$ words of r letters can be formed without repetition from the members of an n-member set S. But every r-member subset can be ordered in ${}^r\mathrm{P}_r = r!$ ways, so the correspondence between the ordered selections and the unordered selections is many-to-one with overcounting by a factor of $r!$. Hence the number of (unordered) r-member subsets of the set S is

$${}^n\mathrm{P}_r \div {}^r\mathrm{P}_r = \frac{n!}{(n-r)!} \div r! = \frac{n!}{r! \times (n-r)!} = {}^n\mathrm{C}_r.$$

Calculations of ${}^n\mathrm{C}_r$

Here are some examples of using the formula to calculate ${}^n\mathrm{C}_r$ for some values of n and r.

Example 15 — 14E

Use the formula for ${}^nC_r$ to evaluate:

a ${}^8\mathrm{C}_5$

b ${}^n\mathrm{C}_3$

SOLUTION

a $${}^8\mathrm{C}_5 = \frac{8!}{3! \times 5!} = \frac{8 \times 7 \times 6 \times \cancel{5}!}{3 \times 2 \times 1 \times \cancel{5}!} = 56$$

b $${}^n\mathrm{C}_3 = \frac{n(n-1)(n-2)(n-3)!}{(n-3)! \times 3!} = \frac{n(n-1)(n-2)}{6}$$

Example 16 — 14E

Find ${}^{16}\mathrm{C}_5$, leaving your answer factored into primes.

SOLUTION

$$\begin{aligned}{}^{16}\mathrm{C}_5 &= \frac{16 \times 15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4 \times 5} \\ &= 2 \times 14 \times 13 \times 12 \\ &= 2^4 \times 3 \times 7 \times 13 \quad \text{(Check this on the calculator.)}\end{aligned}$$

A natural (or canonical) correspondence — ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$

Suppose that two people are to be chosen from five to make afternoon tea. This task can be done in two ways:

- Choose the two people out of five who will *make the tea*.
- Choose the three people out of five who will *not make the tea*.

Thus for every choice of two people out of five, the remaining three people is a corresponding choice of three people out of five. This confirms that ${}^5\mathrm{C}_2 = {}^5\mathrm{C}_3$.

But it also gives a one-to-one correspondence between the two-member subsets and the three-member subsets of a five-member set:

$\{A, B\} \longleftrightarrow \{C, D, E\}$ $\quad$ $\{B, D\} \longleftrightarrow \{A, C, E\}$

$\{A, C\} \longleftrightarrow \{B, D, E\}$ $\quad$ $\{B, E\} \longleftrightarrow \{A, C, D\}$

$\{A, D\} \longleftrightarrow \{B, C, E\}$ $\quad$ $\{C, D\} \longleftrightarrow \{A, B, E\}$

$\{A, E\} \longleftrightarrow \{B, C, D\}$ $\quad$ $\{C, E\} \longleftrightarrow \{A, B, D\}$

$\{B, C\} \longleftrightarrow \{A, D, E\}$ $\quad$ $\{D, E\} \longleftrightarrow \{A, B, C\}$

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In this *natural* or *canonical correspondence* between the 2-member subsets and the 3-member subsets, every subset T is paired with its complement $\overline{T}$:

$$T \longleftrightarrow \overline{T}$$

The situation is easily generalised:

13 A CANONICAL CORRESPONDENCE

Let n and r be whole numbers with $o \le r \le n$, and let S be an n-member set.

- ${}^{n}\mathrm{C}_{r} = {}^{n}\mathrm{C}_{n-r}$
- The r-member subsets of S are *naturally* or *canonically* paired up with the $(n - r)$-subsets of S by pairing each subset with its complement.

The correspondence is by no means restricted to mathematics. In normal language, a situation can often be described just as well by saying what it is not as by saying what it is — we are all familiar with 'days when no rain fell' or 'unforgivable actions' or 'invisible enemies'.

Example 17 — 14E

Write ${}^{7}\mathrm{C}_{2}$ and ${}^{7}\mathrm{C}_{5}$ in factorial notation, showing that they are equal.

SOLUTION

$${}^{7}\mathrm{C}_{2} = \frac{7!}{2! \times 5!} \qquad {}^{7}\mathrm{C}_{5} = \frac{7!}{5! \times 2!}$$

Using ${}^{n}\mathrm{C}_{r}$ in counting problems

Here are a number of examples of counting problems. As always, words such as 'at least', 'at most', 'not' and 'excluding' should always be regarded as warnings that the problem may best be solved by considering the complementary situation.

Example 18 — 14E

Ten people meet to play doubles tennis.

a In how many ways can four people be selected from this group to play the first game? (Ignore the subsequent organisation into pairs.)

b How many of these ways will include Maria and exclude Alex?

c If there are four women and six men, in how many ways can two men and two women be chosen for this game?

d Again with four women and six men, in how many ways will women be in the majority?

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SOLUTION

a Number of ways $= {}^{10}C_4 = 210$.

b Because Maria is included, three further people must be chosen, and because Alex is excluded, there are now eight people to choose these three from.
Hence number of ways $= {}^{8}C_3 = 56$.

c Number of ways of choosing the women $= {}^{4}C_2 = 6$,
number of ways of choosing the men $= {}^{6}C_2 = 15$,
so number of ways of choosing all four $= 15 \times 6 = 90$.

d Number of ways with one man and three women $= {}^{6}C_1 \times {}^{4}C_3 = 24$,
number of ways with four women $= 1$,
so number of ways with a majority of women $= 24 + 1 = 25$.

Example 19 — 14E

Let $S = \{2, 4, 6, 8, 10, 12\}$ be the set consisting of the first six positive even numbers.

a How many subsets of S contain at least two numbers?

b How many subsets with at least two numbers do not contain 8?

c How many subsets with at least two numbers do not contain 8 but do contain 10?

SOLUTION

a Number of 1-member and 0-member subsets $= {}^{6}C_1 + {}^{6}C_0 = 7$,
so number with at least 2 members $= 2^6 - 7 = 57$.

b We consider now the 5-member set $T = \{2, 4, 6, 10, 12\}$.
Number of 1-member and 0-member subsets $= {}^{5}C_1 + {}^{5}C_0 = 6$,
so number with at least 2 members $= 2^5 - 6 = 26$.

c Because 10 has already been chosen, we need to choose subsets with at least one member from the four-member set $U = \{2, 4, 6\ 12\}$.
Number of 0-member subsets $= 1$ (the empty set),
so number of such subsets $= 2^4 - 1 = 15$.

Example 20 — 14E

[A harder question]

Ten points $A_1, A_2, \dots, A_{10}$ are arranged in order around a circle.

a How many triangles can be drawn with these points as vertices?

b How many pairs of such triangles can be drawn, if the vertices of the two triangles are distinct?

c In how many such pairs will the triangles:

i not overlap, **ii** overlap?

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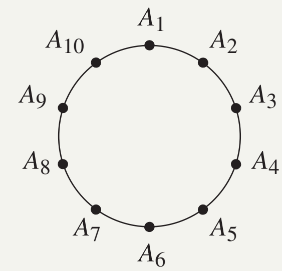
SOLUTION



a To form a triangle, we must choose 3 points out of 10,
so number of triangles $= {}^{10}C_3 = 120$.

b To form two triangles, first choose 6 points out of 10,
which can be done in ${}^{10}C_6 = 210$ ways.
Take any one of those 6 points, then choose the other 2 points in its triangle,
which can be done in ${}^{5}C_2 = 10$ ways.
The second triangle is then drawn using the other 3 points.
Hence number of pairs of triangles $= 210 \times 10 = 2100$.

c To form two non-overlapping triangles, we first choose 6 points out of 10,
which again can be done in ${}^{10}C_6 = 210$ ways.
These 6 points can be made into two non-overlapping triangles in 3 ways,
by arranging the 6 points in cyclic order, and choosing 3 adjacent points.
i Hence, the number of non-overlapping pairs $= 210 \times 3 = 630$.
ii By subtraction, number of overlapping pairs $= 2100 - 630 = 1470$.

Exercise 14E

FOUNDATION

1 Two people are chosen from a group of five people called P, Q, R, S and T. List all possible combinations, and find how many there are.

2 Find in how many ways you can form a group of:
a two people from a choice of seven,
b three people from a choice of seven,
c two people from a choice of six,
d five people from a choice of nine.

3 **a** Find how many possible combinations there are if, from a group of ten people:
i two people are chosen,
ii eight people are chosen.
b Why are the answers identical?

4 From a party of twelve men and eight women, find how many groups can be chosen consisting of:
a five men and three women,
b four women and four men.

5 Four numbers are to be selected from the set of the first eight positive integers. Find how many possible combinations there are if:
a there are no restrictions,
b there are two odd numbers and two even numbers,
c there is exactly one odd number,
d all the numbers must be even,
e there is at least one odd number.

6 Four balls are simultaneously drawn from a bag containing three identical green balls and six identical blue balls. Find how many ways there are of drawing the four balls if:
a the balls may be of any colour,
b there are exactly two green balls,
c there are at least two green balls,
d there are more blue balls than green balls.

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7 A committee of five is to be chosen from six men and eight women. Find how many committees are possible if:

a there are no restrictions,
b all members are to be female,
c all members are to be male,
d there are exactly two men,
e there are four women and one man,
f there is a majority of women,
g a particular man must be included,
h a particular man must not be included.

DEVELOPMENT

8 a What is the number of combinations of the letters of the word EQUATION taken four at a time (without repetition)?
b How many of the four-letter combinations contain four vowels?
c How many of the four-letter combinations contain the letter Q?

9 A team of seven netballers is to be chosen from a squad of twelve players A, B, C, D, E, F, G, H, I, J, K and L. In how many ways can they be chosen:

a with no restrictions,
b if the captain C is to be included,
c if J and K are both to be excluded,
d if A is included but H is not,
e if one of F and L is to be included and the other excluded?

10 a Consider the digits 9, 8, 7, 6, 5, 4, 3, 2, 1 and 0. Find how many five-digit numbers are possible if the digits are to be in:
i descending order,
ii ascending order.
b Why do these two questions involve unordered selections?

11 Twelve people arrive at a restaurant. There is one table for six, one table for four and one table for two. In how many ways can they be assigned to a table?

12 Twenty students, ten male and ten female, are to travel from school to the sports ground. Eight of them go in a minibus, six of them in cars, four of them on bikes and two walk.

a In how many ways can they be distributed for the trip?
b In how many ways can they be distributed if none of the boys walk?

13 Ten points $P_1, P_2, \ldots, P_{10}$ are chosen in a plane, no three of the points being collinear.

a How many lines can be drawn through pairs of the points?
b How many triangles can be drawn using the given points as vertices?
c How many of these triangles have P_1 as one of their vertices?
d How many of these triangles have P_1 and P_2 as vertices?

14 Ten points are chosen in a plane. Five of the points are collinear, but no other set of three of the points is collinear.

a How many sets of three points can be selected from the five that are collinear?
b How many triangles can be formed using three of the ten points as vertices?

15 From a standard deck of 52 playing cards, find how many five-card hands can be dealt:

a consisting of black cards only,
b consisting of diamonds only,
c containing all four kings,
d consisting of three diamonds and two clubs,
e consisting of three twos and another pair,
f consisting of one pair and three of a kind.

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16 **a** In how many ways can a group of six people be divided into:

  i two unequal groups (neither group being empty),

  ii two equal groups?

b Repeat part **a** for four people.

c Repeat part **a** for eight people.

17 Find how many diagonals there are in:

a a quadrilateral,

b a pentagon,

c a decagon,

d a polygon with n sides.

18 Twelve points are arranged in order around a circle.

a How many triangles can be drawn with these points as vertices?

b In how many pairs of such triangles are the vertices of the two triangles distinct?

c In how many such pairs will the triangles:

  i not overlap,

  ii overlap?

19 Let $S = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ be the set of the first ten positive odd integers.

a How many subsets does S have?

b How many subsets of S contain at least three numbers?

c How many subsets with at least three numbers do not contain 7?

d How many subsets with at least three numbers do not contain 7 but do contain 19?

20 In how many ways can two numbers be selected from $1, 2, \ldots, 8, 9$ so that their sum is:

a even,

b odd,

c divisible by 3,

d divisible by 5,

e divisible by 6?

21 There are ten basketballers in a team. Find in how many ways:

a the starting five can be chosen,

b they can be split into two teams of five.

22 Nine players are to be divided into two teams of four and one umpire.

a In how many ways can the teams be formed?

b If two particular people cannot be on the same team, how many different combinations are possible?

23 By considering their prime factorisations, find the number of positive divisors of:

a $2^3 \times 3^2$

b 1 000 000

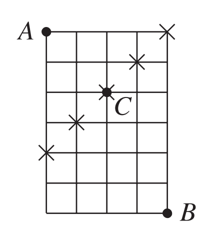
c 315 000

d $2^a \times 5^b \times 13^c$

24 **a** The six faces of a number of identical cubes are painted in six distinct colours. How many different cubes can be formed?

b A die fits perfectly into a cubical box. How many ways are there of putting the die into the box?

25 The diagram shows a 6×4 grid. The aim is to walk from the point A in the top left-hand corner to the point B in the bottom right-hand corner by walking along the black lines either downwards or to the right. A single move is defined as walking along one side of a single small square, thus it takes you ten moves to get from A to B.



a Find how many different routes are possible:

  i without restriction,

  ii if you must pass through C,

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iii if you cannot move along the top line of the grid,
iv if you cannot move along the second row from the top of the grid.

b Notice that every route must pass through one and only one of the five crossed points. Hence prove that ${}^{10}C_4 = {}^{4}C_0 \times {}^{6}C_4 + {}^{4}C_1 \times {}^{6}C_3 + {}^{4}C_2 \times {}^{6}C_2 + {}^{4}C_3 \times {}^{6}C_1 + {}^{4}C_4 \times {}^{6}C_0$.

c Draw another suitable diagonal and, using a method similar to that in part b, prove that ${}^{10}C_4 = {}^{5}C_0 \times {}^{5}C_4 + {}^{5}C_1 \times {}^{5}C_3 + {}^{5}C_2 \times {}^{5}C_2 + {}^{5}C_3 \times {}^{5}C_1 + {}^{5}C_4 \times {}^{5}C_0$.

d Draw up a similar 6×6 grid, then using the same idea as that used in parts b and c, prove that ${}^{12}C_6 = \left({}^{6}C_0\right)^2 + \left({}^{6}C_1\right)^2 + \left({}^{6}C_2\right)^2 + \left({}^{6}C_3\right)^2 + \left({}^{6}C_4\right)^2 + \left({}^{6}C_5\right)^2 + \left({}^{6}C_6\right)^2$.

ENRICHMENT

26 A piece of art receives an integer mark from zero to 100 for each of the categories design, technique and originality. In how many ways is it possible to score a total mark of 200?

27 How many different combinations are there of three different integers between one and thirty inclusive such that their sum is divisible by three?

28 a How many doubles tennis games are possible, given a group of four players?

b In how many ways can two games of doubles tennis be arranged, given a group of eight players?

c Six men and six women are to play in three games of doubles tennis. Find how many ways the pairings can be arranged if:

i there are no restrictions,
ii each game is to be a game of mixed doubles.

29 Referring to Question 18 of Exercise 14D, Bob's interest in shirts has matured recently — he now has $2n$ shirts, with two identical shirts in each of n distinct styles. He still wants to hang his shirts in the wardrobe so that no two identical shirts are next to one another.

a Show that the number of allowed arrangements is ${}^{n}C_0\,(2n)!\,2^{-n} - {}^{n}C_1\,(2n-1)!\,2^{1-n} + {}^{n}C_2\,(2n-2)!\,2^{2-n} + \cdots + (-1)^n\,{}^{n}C_n n!\,2^0$.

b Evaluate this number for $n = 1, 2, 3, 4, 5$ and 6, and find the corresponding ratio of the total number of arrangements to the number of allowed arrangements.

30 Computing is based on *binary strings*, which are sequences of 0s and 1s, such as 00101 and 1001110. The problem in this question is to find the number N of binary strings that contain exactly a 0s and at most b 1s, where a and b are whole numbers. We will count such strings in two different ways, and hence prove an interesting identity.

a Using ${}^{n}C_r$ notation, find how many binary strings with exactly a 0s contain:

i no 1s ii one 1 iii two 1s iv three 1s v r 1s.

b Hence prove that $N = {}^{a}C_0 + {}^{a+1}C_1 + {}^{a+2}C_2 + \cdots + {}^{a+b}C_b$.

c Let S be a binary string consisting of a 0s and r 1s, where r is at most b. Extend S to a longer binary string $S01 \cdots 1$ by adding 0 on the right, and then adding $b - r$ 1s. How many 0s and 1s are there in the resulting string, and what is its total length?

d Describe the inverse process by which a string of $(a + 1)$ 0s and b 1s can be converted to a string of a 0s and at most b 1s, and show that the two processes are one-to-one correspondences.

e Hence prove that $N = {}^{a+b+1}C_b$.

f Show that ${}^{a}C_0 + {}^{a+1}C_1 + {}^{a+2}C_2 + \cdots + {}^{a+b}C_b = {}^{a+b+1}C_b$.

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14F Using counting in probability

The purpose of this section is to apply the counting procedures of the last four sections to questions about probability. In these more complicated questions, counting procedures are required for counting both the sample space and the event space.

Counting the sample space and the event space

As always in this topic, the two questions that need to be asked are:

- Are the selections we are counting ordered or unordered?
- If they are ordered, is repetition allowed or not?

If the questions can be done with ordered selections or with unordered selections, it is usually easier to use unordered selections because the numbers are smaller.

Example 21 — 14F

Three cards are dealt from a pack of 52.

a Find the probability that one club and two hearts are dealt, in any order.

b Find the probability that one club and two hearts are dealt in that order.

SOLUTION

a Let the sample space be the set of all unordered selections of 3 cards from 52,

so number of unordered hands $= {}^{52}C_3 = \dfrac{52 \times 51 \times 50}{3 \times 2 \times 1}$.

We can now choose the hand by choosing 1 club from 13 in ${}^{13}C_1 = 13$ ways,
and choosing the 2 hearts from 13 in ${}^{13}C_2 = 78$ ways,
so the hand can be chosen in 13×78 ways.

Hence $P(\text{1 club and 2 hearts}) = 13 \times 78 \times \dfrac{3 \times 2 \times 1}{52 \times 51 \times 50} = \dfrac{39}{850}$.

b Let the sample space be the set of all ordered selections of 3 cards from 52,

so number of ordered hands $= {}^{52}P_3 = 52 \times 51 \times 50$,

and number of such hands in the order $\clubsuit\heartsuit\heartsuit = 13 \times 13 \times 12$.

Hence $P(\clubsuit\heartsuit\heartsuit) = \dfrac{13 \times 13 \times 12}{52 \times 51 \times 50} = \dfrac{13}{850}$.

Note: The answer to part **b** must be $\frac{1}{3}$ of the answer to part **a**, because in a hand with one club and two hearts, the club can be any one of three positions. This indicates that it would be quite reasonable to do part **a** using ordered selections, and to do part **b** using unordered selections, although the methods chosen above are more natural to the way in which each question was worded.

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Problems requiring a variety of methods

The sample spaces in the next two examples are easily found, but a variety of methods is needed to establish the sizes of the various event spaces.

Example 22 14F

A five-digit number is chosen at random. Find the probability:

a that it is at least 60 000,

b that it consists only of even digits,

c that the digits are distinct,

d that the digits are distinct and in increasing order.

SOLUTION

The first digit of a five-digit number cannot be zero, giving nine choices, but the other digits can be any one of the ten digits.

Hence the number of five-digit numbers $= 9 \times 10 \times 10 \times 10 \times 10 = 90000$.

a To be at least 60 000, the first digit can be 6, 7, 8 or 9,

so the number of favourable numbers is $4 \times 10 \times 10 \times 10 \times 10 = 40000$.

Hence $P(\text{at least } 60000) = \frac{4}{9}$.

b If all the digits are even, there are four choices for the first digit (it cannot be zero) and five choices for each of the other four.

Hence number of such numbers $= 4 \times 5 \times 5 \times 5 \times 5 = 2500$,

and $P(\text{all digits are even}) = \dfrac{2500}{90000} = \dfrac{1}{36}$.

c This is counting without replacement:

1st digit	2nd digit	3rd digit	4th digit	5th digit
9	9	8	7	6

so number of such numbers $= 9 \times 9 \times 8 \times 7 \times 6$

and $P(\text{digits are distinct}) = \dfrac{9 \times 9 \times 8 \times 7 \times 6}{90000} = \dfrac{189}{625}$.

d Every *unordered* five-member subset of the set of nine non-zero digits can be arranged in exactly one way into a five-digit number with the digits in increasing order. (Note that the digit zero cannot be used, because a number can't begin with the digit zero.)

Hence number of such numbers

$= \text{number of unordered 5-member subsets of } \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$= {}^9C_5 = 126$,

so $P(\text{digits are distinct and in increasing order}) = \dfrac{126}{90000} = \dfrac{7}{5000}$.

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Example 23 14F

[A harder question]

Continuing with Example 22, find the probability that:

a the number contains at least one four,

b the number contains at least one four and at least one five,

c the number contains exactly three sevens,

d the number contains at least three sevens.

SOLUTION

a This can be approached using the complementary event:

$$\text{number of five-digit numbers without a } 4 = 8 \times 9 \times 9 \times 9 \times 9 = 52488,$$

so $$\text{number of five-digit numbers with a } 4 = 90000 - 52488 = 37512.$$

Hence $$P(\text{at least one } 4) = \frac{37512}{90000} = \frac{521}{1250}.$$

b This can be approached using the addition theorem:

$$\text{number without a } 4 = 52488,$$

similarly $$\text{number without a } 5 = 52488,$$

and $$\text{number with no 5 and no } 4 = 7 \times 8 \times 8 \times 8 \times 8 = 28672.$$

Hence $$\text{number with no 5 or no } 4 = 52488 + 52488 - 28672 = 76304,$$

and $$\text{number with at least one 5 and at least one } 4 = 90000 - 76304 = 13696.$$

Hence $$P(\text{at least one 4 and at least one } 5) = \frac{13696}{90000} = \frac{856}{5625}.$$

c Counting the number of five-digit numbers with exactly three 7s requires cases. First we count the five-digit strings with exactly three 7s, by first placing the three 7s and then choosing the first and second non-7 digits:

position of the three 7s	choose first non-7	choose second non-7
${}^5C_3 = 10$	9	9

giving 810 such strings. Secondly, we must subtract the number of five-digit strings with exactly three 7s and beginning with zero:

position of the three 7s	choose the other non-7
${}^4C_3 = 4$	9

giving 36 such strings. Hence there are $810 - 36 = 774$ such numbers, and

$$P(\text{number has exactly three 7s}) = \frac{774}{90000} = \frac{43}{5000}.$$

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d The number 77 777 is the only five-digit number with five 7s. Any five-digit number with exactly four 7s has one of the five forms

$$*7777, \quad 7*777, \quad 77*77, \quad 777*7, \quad 7777*,$$

where the $*$ in $*7777$ is a non-zero digit. There are eight numbers of the first form, and nine of the other four forms, giving 44 numbers altogether.

Hence the number with at least three 7s $= 774 + 44 + 1 = 819$,

and $$P(\text{at least three 7s}) = \frac{819}{90000} = \frac{91}{10000}.$$

Exercise 14F

FOUNDATION

1 A committee of three is to be selected from the nine members in a club.
- **a** How many different committees can be formed?
- **b** If there are five men in the club, what is the probability that the selected committee consists entirely of males?

2 The integers from 1 to 10 inclusive are written on ten separate pieces of paper. Four pieces of paper are drawn at random. Find the probability that:
- **a** the four numbers drawn are 1, 2, 3 and 6,
- **b** the number 9 is one of the numbers drawn,
- **c** the number 8 is not drawn,
- **d** the number 7 is drawn but the number 1 is not.

3 A bag contains three red, seven yellow and five blue balls. If three balls are drawn from the bag simultaneously, find the probability that:
- **a** all three balls are yellow,
- **b** all the balls are of the same colour,
- **c** there are two red balls and one blue ball,
- **d** all the balls are of different colours.

4 A sports committee of five members is to be chosen from eight AFL footballers and seven soccer players. Find the probability that the committee will contain:
- **a** only AFL footballers,
- **b** only soccer players,
- **c** three soccer players and two AFL footballers,
- **d** at least one soccer player,
- **e** at most one soccer player,
- **f** Ian, a particular soccer player.

5 From a standard pack of 52 cards, three are selected at random. Find the probability that:
- **a** they are the jack of spades, the two of clubs and the seven of diamonds,
- **b** all three are aces,
- **c** they are all diamonds,
- **d** they are all of the same suit,
- **e** they are all picture cards,
- **f** two are red and one is black,
- **g** one is a seven, one is an eight and one is a nine,
- **h** two are sevens and one is a six,
- **i** exactly one is a diamond,
- **j** at least two of them are diamonds.

6 Repeat the previous question if the cards are selected from the pack one at a time, and each card is replaced before the next one is drawn.

7 Three boys and three girls are to sit in a row. Find the probability that:
- **a** the boys and girls alternate,
- **b** the boys sit together and the girls sit together,
- **c** two specific girls sit next to one another.

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8 A family of five are seated in a row at the cinema. Find the probability that:
- a the parents sit on the end and the three children are in the middle,
- b the parents sit next to one another.

9 Six people, of whom Patrick and Jessica are two, arrange themselves in a row. Find the probability that:
- a Patrick and Jessica occupy the end positions,
- b Patrick and Jessica are not next to each other.

DEVELOPMENT

10 The letters of PROMISE are arranged randomly in a row. Find the probability that:
- a the word starts with R and ends with S,
- b the letters P and R are next to one another,
- c the letters P and R are separated by at least three letters,
- d the vowels and the consonants alternate,
- e the vowels are together.

11 The digits 3, 3, 4, 4, 4 and 5 are placed in a row to form a six-digit number. If one of these numbers is selected at random, find the probability that:
- a it is even,
- b it ends in 5,
- c the 4s occur together,
- d the number starts with 5 and then the 4s and 3s alternate,
- e the 3s are separated by at least one other number.

12 The letters of the word PRINTER are arranged in a row. Find the probability that:
- a the word starts with the letter E,
- b the letters I and P are next to one another,
- c there are three letters between N and T,
- d there are at least three letters between N and T.

13 The letters of KETTLE are arranged randomly in a row. Find the probability that:
- a the two letters E are together,
- b the two letters E are not together,
- c the two letters E are together and the two letters T are together,
- d the Es and Ts are together in one group.

14 The letters of ENTERTAINMENT are arranged in a row. Find the probability that:
- a the letters E are together,
- b two Es are together and one is apart,
- c all the letters E are apart,
- d the word starts and ends with E.

15 A tank contains 20 tagged fish and 80 untagged fish. On each day, four fish are selected at random, and after noting whether they are tagged or untagged, they are returned to the tank. Answer the following questions, correct to three significant figures.
- a What is the probability of selecting no tagged fish on a given day?
- b What is the probability of selecting at least one tagged fish on a given day?
- c Calculate the probability of selecting no tagged fish on every day for a week.
- d What is the probability of selecting no tagged fish on exactly three of the seven days during the week?

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16 A bag contains seven white and five black discs. Three discs are chosen from the bag. Find the probability that all three discs are black, if the discs are chosen:

a without replacement,

b with replacement,

c so that after each draw the disc is replaced with one of the opposite colour.

17 Six people are to be divided into two groups, each with at least one person. Find the probability that:

a there will be three in each group,

b there will be two in one group and four in the other,

c there will be one group of five and an individual.

18 A three-digit number is formed from the digits 3, 4, 5, 6 and 7 (no repetitions allowed). Find the probability that:

a the number is 473,

b the number is odd,

c the number is divisible by 5,

d the number is divisible by 3,

e the number starts with 4 and ends with 7,

f the number contains the digit 3,

g the number contains the digits 3 and 5,

h the number contains the digit 3 or 5,

i all digits in the number are odd,

j the number is greater than 500.

19 The digits 1, 2, 3 and 4 are used to form numbers that may have 1, 2, 3 or 4 digits in them. If one of the numbers is selected at random, find the probability that:

a it has three digits,

b it is even,

c it is greater than 200,

d it is odd and greater than 3000,

e it is divisible by 3.

20 **a** A senate committee of five members is to be selected from six Labor and five Liberal senators. What is the probability that Labor will have a majority on the committee?

b The senate committee is to be selected from N Labor and five Liberal senators. Use trial and error to find the minimum value of N, given that the probability of Labor having a majority on the committee is greater than $\frac{3}{4}$.

21 Four basketball teams A, B, C and D each consist of ten players, and in each team, the players are numbered 1, 2, . . . 9, 10. Five players are to be selected at random from the four teams. Find the probability that of the five players selected:

a three are numbered 4 and two are numbered 9,

b at least four are from the same team.

22 A poker hand of five cards is dealt from a standard pack of 52. Find the probability of obtaining:

a one pair,

b two pairs,

c three of a kind,

d four of a kind,

e a full house (one pair and three of a kind),

f a straight (five cards in sequence regardless of suit, ace high or low),

g a flush (five cards of the same suit),

h a royal flush (ten, jack, queen, king and ace in a single suit).

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23 Four adults are standing in a room that has five exits. Each adult is equally likely to leave the room through any one of the five exits.

- **a** What is the probability that all four adults leave the room via the same exit?
- **b** What is the probability that three particular adults use the same exit and the fourth adult uses a different exit?
- **c** What is the probability that any three of the four adults come out the same exit, and the remaining adult comes out a different exit?
- **d** What is the probability that no more than two adults come out any one exit?

24 **a** Five diners in a restaurant choose randomly from a menu featuring five main courses. Find the probability that exactly one of the main courses is not chosen by any of the diners.

- **b** Repeat the question if there are n diners and a choice of n main courses.

25 [The birthday problem]

- **a** Assuming a 365-day year, find the probability that in a group of three people there will be at least one birthday in common. Answer correct to two significant figures.
- **b** If there are n people in the group, find an expression for the probability of at least one common birthday.
- **c** By choosing a number of values of n, plot a graph of the probability of at least one common birthday against n for $n \le 50$.
- **d** How many people need to be in the group before the probability exceeds 0.5?
- **e** How many people need to be in the group before the probability exceeds 90%?

ENRICHMENT

26 During the seven games of the football season, Max and Bert must each miss three consecutive games. The games to be missed by each player are randomly and independently selected.

- **a** What is the probability that they both have the first game off together?
- **b** What is the probability that the second game is the first one where Max and Bert are both missing?
- **c** What is the probability that Max and Bert miss at least one of the same games?

27 Eight players make the quarter-finals at Wimbledon. The winner of each of the quarter-finals plays a semi-final to see who enters the final.

- **a** Assuming that all eight players are equally likely to win a match, show that the probability that any two particular players will play each other is $\frac{1}{4}$.
- **b** What is the probability that two particular people will play each other if the tournament starts with 16 players?
- **c** What is the probability that two particular players will meet in a similar knockout tournament if 2^n players enter?

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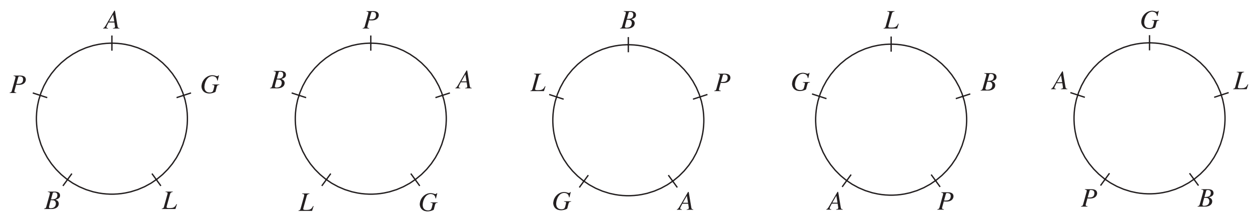
14G Arrangements in a circle

Arrangements in a circle or around a round table are complicated by the fact that two arrangements are regarded as equivalent if one can be rotated to produce the other. For example, all the five round-table seatings below of King Arthur, Queen Guinevere, Sir Lancelot, Sir Bors and Sir Percival are to be regarded as the same:



The basic algorithm

The most straightforward way of counting arrangements in a circle is to seat the people in order, dealing with the restrictions first as always, but reckoning that there is essentially only one way to seat the first person who sits down, because until that time, all the seats are identical.

14 COUNTING ARRANGEMENTS IN A CIRCLE

There is essentially only one way to seat the first person, because until then, all the seats are equivalent.

Example 24 14G

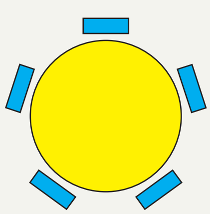
King Arthur, Queen Guinevere, Sir Lancelot, Sir Bors and Sir Percival sit around a round table. Find in how many ways this can be done:

a without restriction,

b if Queen Guinevere sits at King Arthur's right hand,

c if Queen Guinevere sits between Sir Lancelot and Sir Bors.

d if King Arthur and Sir Lancelot do not sit together.



SOLUTION

a

Seat Arthur	Seat Guinevere	Seat Lancelot	Seat Bors	Seat Percival
1	4	3	2	1

Number of ways $= 24$.

b

Seat Arthur	Seat Guinevere	Seat Lancelot	Seat Bors	Seat Percival
1	1	3	2	1

Number of ways $= 6$.

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c

Seat Guinevere	Seat Lancelot	Seat Bors	Seat Arthur	Seat Percival
1	2	1	2	1

Number of ways $= 4$.

d

Seat Arthur	Seat Lancelot	Seat Guinevere	Seat Bors	Seat Percival
1	2	3	2	1

Number of ways $= 12$.

Arranging groups around a circle

When arranging groups around a circle, the principle is the same as the principle for compound orderings established in Section 14C.

15 ARRANGING GROUPS AROUND A CIRCLE

- First choose an order for each group.
- Then arrange the groups around the circle, reckoning that there is essentially only one way to place the first group.

Example 25 — 14G

Five boys and five girls are to sit around a table. Find in how many ways this can be done:

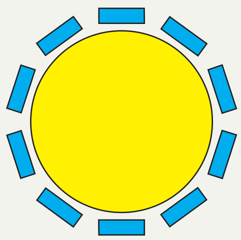
a without restriction,

b if the boys and girls alternate,

c if there are five couples, all of whom sit together,

d if the boys sit together and the girls sit together,

e if four couples sit together, but Walter and Maude do not.



SOLUTION

a

1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
1	9	8	7	6	5	4	3	2	1

Number of ways $= 9! = 362\,880$.

b

1st boy	2nd boy	3rd boy	4th boy	5th boy	1st girl	2nd girl	3rd girl	4th girl	5th girl
1	4	3	2	1	5	4	3	2	1

Number of ways $= 5! \times 4! = 2880$.

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c Each couple can be ordered in 2 ways, giving 2^5 orderings of the five couples.
Then seat the five couples around the table:

1st couple	2nd couple	3rd couple	4th couple	5th couple
1	4	3	2	1

Number of ways $= 2^5 \times 4! = 768$.

d The boys can be ordered in 5! ways, and the girls in 5! ways also.
Then seat the two groups around the table:

group of boys	group of girls
1	1

Number of ways $= 5! \times 5! \times 1 = 14400$.

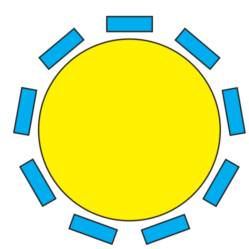
e Order each of the four couples in 2 ways, giving 16 orderings of the couples.
There are now four couples and two individuals to seat around the table, with the restriction that Maude does not sit next to Walter:

Walter	Maude	1st couple	2nd couple	3rd couple	4th couple
1	3	4	3	2	1

Number of ways $= 2^4 \times 3 \times 4! = 1152$.

Probability in arrangements around a circle

As always, counting allows probability problems to be solved by counting the sample space and the event space.



Example 26 — 14G

Three Tasmanians, three New Zealanders and three people from NSW are seated at random around a round table. What is the probability that the three groups are seated together?

SOLUTION

Using the same boxes as before, there are $1 \times 8!$ possible orderings.
To find the number of favourable orderings,
first order each group in $3! = 6$ ways,
then order the three groups around the table in $1 \times 2 \times 1 = 2$ ways,
so the total number of favourable orderings is $6 \times 6 \times 6 \times 2$.

$$\text{Hence } P(\text{groups are together}) = \frac{6 \times 6 \times 6 \times 2}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{3}{280}.$$

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Exercise 14G

FOUNDATION

1 a In how many ways can five people be arranged:
  i in a line,
  ii in a circle?

  b In how many ways can ten people be arranged:
  i in a line,
  ii in a circle?

2 Eight people are arranged in:
  a a straight line,
  b a circle.

  In how many ways can they be arranged so that two particular people sit together?

3 Bob, Betty, Ben, Brad and Belinda are to be seated at a round table. In how many ways can this be done:
  a if there are no restrictions,
  b if Betty sits on Bob's right-hand side,
  c if Brad is to sit between Bob and Ben,
  d if Belinda and Betty sit apart,
  e if Ben and Belinda sit apart, but Betty sits next to Bob?

4 Four boys and four girls are arranged in a circle. In how many ways can this be done:
  a if there are no restrictions,
  b if the boys and the girls alternate,
  c if the boys and girls are in distinct groups,
  d if a particular boy and girl wish to sit next to one another,
  e if two particular boys do not wish to sit next to one another,
  f if one particular boy wants to sit between two particular girls?

DEVELOPMENT

5 The letters A, E, I, P, Q and R are arranged in a circle. Find the probability that:
  a the vowels are together,
  b A is opposite R,
  c the vowels and consonants alternate,
  d at least two vowels are next to one another.

6 In how many ways can the integers $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8$ be placed in a circle if:
  a there are no restrictions,
  b all the even numbers are together,
  c the odd and even numbers alternate,
  d at least three odd numbers are together,
  e the numbers 1 and 7 are adjacent,
  f the numbers 3 and 4 are separated?

7 A committee of three women and seven men is to be seated randomly at a round table.
  a What is the probability that the three females will sit together?
  b The committee elects a president and a vice-president. What is the probability that they are sitting opposite one another?

8 Find how many arrangements of n people around a circle are possible if:
  a there are no restrictions,
  b two particular people must sit together,
  c two particular people sit apart,
  d three particular people sit together.

9 Twelve marbles are to be placed in a circle. In how many ways can this be done if:
  a all the marbles are of different colours,
  b there are eight red, three blue and one green marble?

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10 There are two round tables, one oak and one mahogany, each with five seats. In how many ways may a group of ten people be seated?

11 A sports committee consisting of four rowers, three basketballers and two cricketers sits at a circular table.

a How many different arrangements of the committee are possible if the rowers and basketballers both sit together in groups, but no rower sits next to a basketballer?

b One rower and one cricketer are related. If the conditions in **a** apply, what is the probability that these two members of the committee will sit next to one another?

ENRICHMENT

12 A group of n men and $n + 1$ women sit around a circular table. What is the probability that no two men sit next to one another?

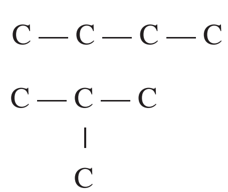
13 **a** Consider a necklace of six differently coloured beads. Because the necklace can be turned over, clockwise and anti-clockwise arrangements of the beads do not yield different orders. Hence find how many different arrangements there are of the six beads on the necklace.

b In how many ways can ten different keys be placed on a key ring?

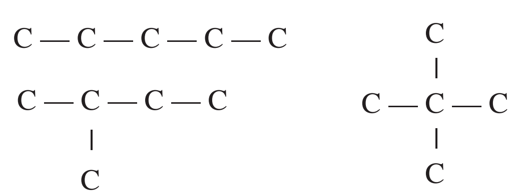
c In how many ways can one yellow, two red and four green beads be placed on a bracelet if the beads are identical apart from colour? (Hint: This will require a listing of patterns to see if they are identical when turned over.)

Isomers — a possible computing project

Because of branching, it is difficult to count the number of possible hydrocarbons with say 20 carbon atoms. For example, for butane C_4H_{10} there are two carbon chains, and for pentane C_5H_{12} there are three carbon chains:



butane



pentane

This is a hard problem. If you can write computer programmes, you could write your own programme to investigate the number of possible carbon chains for hydrocarbons with increasing numbers of carbon atoms. Perhaps your programme can display diagrams of all the chains, as with butane and pentane above.

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14H The pigeonhole principle

Even pigeons cannot break the iron laws of logic — if 11 pigeons fly into 10 pigeonholes, there will be at least one pigeonhole with at least two pigeons. This is a fundamental principle of counting, with its own name, *the pigeonhole principle*.

There is a more general form of the pigeonhole principle — if 31 pigeons fly into 10 pigeonholes, there will be at least one pigeonhole with at least 4 pigeons.

The more general form can be asked backwards — if there are 10 pigeonholes, then the least number of pigeons that will guarantee that there is at least one pigeonhole with at least 8 pigeons is 71.

Statement and explanation of the principle

The principle is better stated at first using these three examples rather than algebra:

16 THE PIGEONHOLE PRINCIPLE

- If 11 pigeons fly into 10 pigeonholes, there will be at least one pigeonhole with at least two pigeons.
- If 31 pigeons fly into 10 pigeonholes, there will be at least one pigeonhole with at least 4 pigeons.
- If there are 10 pigeonholes, then the least number of pigeons that will guarantee that there is at least one pigeonhole with at least 8 pigeons is 71.

- We can prove the first statement by contradiction. Suppose that every pigeonhole had fewer than two pigeons. Then every one of the 10 pigeonhole would have at most one pigeon, so there would be at most 10 pigeons in the holes. This contradicts the fact that there are 11 pigeons, thus proving the result.
- The second statement is similarly proven by contradiction. Suppose that every pigeonhole had fewer than 4 pigeons. Then every one of the 10 pigeonholes would have at most 3 pigeons, so total number of pigeons $\le 3 \times 10 = 30$. This contradicts the fact that there are 31 pigeons, thus proving the result.
- To prove the third statement, note first that 71 pigeons guarantees that at least one pigeonhole has at least eight pigeons. If there were only 70 pigeons, however, we could arrange the pigeons
$$7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 = 70$$
so that no pigeonhole has at least 8 pigeons.

Example 27 **14H**

a Seventy guests sit at a restaurant with 23 tables. Prove that there must be at least one table with at least four guests.

b Explain how many guests there must be to guarantee that at least one of the 23 tables has at least 11 guests?

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SOLUTION

a If no table had more than 3 diners, then there would be at most $3 \times 23 = 69$ diners in the restaurant. Hence there must be a table with at least four diners.

b With 230 guests, the guests could be organised 10 to each table. With 231 guests, at least one table must have at least 11 guests.

Using division with remainder in the pigeonhole principle

The pigeonhole principle may seem ludicrously simple at the start, but as the number of pigeons increases, the logic become less intuitive. You have probably noticed that division with remainder is the important idea in behind these problems.

Example 28 — 14H

A suburb has 15 large apartment blocks.

a If a shop-keeper knows 200 people from these blocks, explain why he must know at least 14 people from at least one block.

b How many people from these blocks must he know in order to know at least 25 people from at least one block?

SOLUTION

a Using division with remainder, $200 \div 15 = 13$ remainder 5. Here the pigeons are the people, and the pigeonholes are the apartment blocks. Because the remainder is greater than zero, the shopkeeper must know at least 14 people from at least one block.

b $15 \times 24 = 360$, so with 360 people, it is just possible that he knows exactly 24 people from each block. There must be 361 people to guarantee that he knows at least 25 people from at least one block. Notice that

$$360 \div 15 = 24 \text{ remainder } 0 \qquad \text{and} \qquad 361 \div 15 = 24 \text{ remainder } 1.$$

Here is a statement of the principle for algebraic pigeons skilled in division.

17 THE PIGEONHOLE PRINCIPLE AND THE REMAINDER AFTER DIVISION

Suppose that n pigeons fly into d pigeonholes. Performing the division, let

$$n = qd + r, \text{ where the remainder is } r = 0, 1, 2, \ldots, d - 1.$$

- If $r = 0$, there must be at least one pigeonhole with at least q pigeons.
- If $r > 0$, there must be at least one pigeonhole with at least $q + 1$ pigeons.
- The least number of pigeons that will guarantee that there is at least one pigeonhole with at least $q + 1$ pigeons is $qd + 1$.

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Example 29 14H

Every person has fewer than 500 000 hairs on their head. Prove that in Sydney, with a population of more than 5 000 000, there are at least eleven people with exactly the same number of hairs on their heads.

SOLUTION

The pigeons are the people, the pigeonholes are the number of hairs on someone's head, and $5000000 \div 500000 = 10$ remainder 0. Each of the more than 5 000 000 people is assigned a number less than 500 000, so at least eleven people are assigned the same number.

Example 30 14H

Vikram has 30 distinct ties. Every day, including weekends, he selects a tie at random to wear. How many successive dates are needed to guarantee that there is at least one day of the week on which he has worn the same tie on at least 6 occasions?

SOLUTION

There are $30 \times 7 = 210$ tie–day pairs (where 'day' here means 'day of the week'), and we are looking for repeats of these tie–day pairs — they are the pigeonholes. The pigeons are the successive dates on which Vikram wears a tie. It follows that $210 \times 5 + 1 = 1051$ dates guarantee that at least one tie–day pair occurs on at least 6 dates.
With 1050 dates, however, this is not guaranteed — in 1050 consecutive days, each day occurs exactly 1050/7 = 150 times, allowing for each tie to be worn 150/30 = 5 times on that day. Thus the smallest number that guarantees at least 6 occurrences of at least one tie–day pair is 1051 days.

Some more advanced examples of the pigeonhole principle

The pigeonhole principle is notorious for its many surprising and rather difficult applications. Here are three such 'surprising and rather difficult' worked examples.

Example 31 14H

Fifteen people come into a room, and there are many handshakes as they meet — no pair shakes hands twice. Prove that there will be at least two people who have made the same number of handshakes.

SOLUTION

The number of handshakes that each person makes is one of the 15 numbers from 0 to 14. But if someone shakes hands 0 times, then no one shakes 14 times, and if someone shakes hands 14 times, then no one shakes 0 times. Hence there are two cases. Either each person's number of handshakes is one of 14 numbers from 0 to 13, or each person's number of handshakes is one of 14 numbers from 1 to 14. Thus by the pigeonhole principle, there at least two people whose numbers of handshakes are the same.

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Example 32 14H

How many odd numbers less than 20 must be chosen to guarantee that two of the chosen numbers add to 20?

SOLUTION

The answer must be at least 6, because every pair of numbers from the 5 odd numbers $1,\ 3,\ 5,\ 7,\ 9$ has a sum less than 20. But if 6 odd numbers less than 20 are chosen, then two of the numbers add to 20 — to prove this, partition the set S of odd numbers less than 20 into the five 2-member subsets adding to 20,

$$S = \{1, 19\} \cup \{3, 17\} \cup \{5, 15\} \cup \{7, 13\} \cup \{9, 11\}.$$

When we choose 6 numbers from S, two of the numbers must be from the same 2-member subset, and so add to 20.

Example 33 14H

Use the pigeonhole principle to prove that there are two powers of 2 that differ by a multiple of 8479.

SOLUTION

Take the 8480 powers of 2 up to 2^{8479}, that is, $2^0, 2^1, 2^2, \dots, 2^{8479}$. Divide each power by 8479, and look at its remainder. There are 8479 possible remainders for the 8480 powers, so by the pigeonhole principle, there must be two powers with the same remainder after division by 8479. The difference of these two powers is a multiple of 8479.

Notice that we have not found two such powers of 2 — we have only proven their existence. Notice also that 8479 was just an arbitrarily chosen whole number.

Exercise 14H

FOUNDATION

1 A die is thrown repeatedly.

a What is the smallest number of times the die must be thrown to ensure at least one number turns up at least twice?

b What is the smallest number of times the die must be thrown to ensure that a six is rolled at least twice?

2 Omar writes three numbers on a piece of paper. Why can he guarantee that at least 2 are odd or at least 2 are even?

3 Jemima the duck hides each egg she lays in one of five different locations at random. If she lays 16 eggs, can she be sure that some location holds at least 4 eggs?

4 A pair of coins is tossed repeatedly. Each time the pair is tossed, there are three possible outcomes if order is ignored,

two heads, two tails, a head and a tail.

a What is the smallest number of tosses of the pair required to get at least one outcome occurring at least twice?

b What is the smallest number of tosses of the pair required to get at least one outcome occurring at least three times?

c What is the smallest number of tosses of the pair required to get at least one outcome occurring at least n times?

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5 When a die is thrown 13 times, explain why at least one of the numbers 1 to 6 inclusive must turn up at least three times.

6 In a promotion, organisers of the school dance gave out ten tickets at random to a group of boys. 'At least someone is going to get at least two tickets', realised Ben. How many boys were there in the group?

7 Chef Gustav throws dirty pots, pans and casserole dishes into three different sinks for washing. He always delays washing up until he has four items in one of the sinks. All of his recipes use exactly one of these three items.
   a What is the maximum number of recipes he can make before he must wash up?
   b What is the minimum number of recipes he might make before he must wash up?

8 Suppose there are 100 points scattered around a $7\,\text{m} \times 7\,\text{m}$ square. Show that there exists a $1\,\text{m} \times 1\,\text{m}$ square covering at least three points.

DEVELOPMENT

9 Baruch has two spinners, each with three sides labelled with the numbers $1, 2, 3$. He throws the pair of spinners and records their sum. What is the minimum number of throws of the pair required to ensure that the same total occurs at least twice?

10 Show that in a group of 100, at least nine have a birthday in the same month.

11 A player is dealt a hand of ten cards. Show that at least one suit occurs three times in his hand.

12 Suppose that 567 chairs are divided amongst 23 classrooms. Can we be sure that there is a classroom with at least 25 chairs?

13 Jackie and her 19 friends turn up to soccer practice and discover that they don't know anyone else there. Everyone who turns up is allocated to one of three groups.
   a How many friends can she guarantee to be in her group?
   b Show that there is a group containing at least seven of her friends.
   c What is the maximum number of friends who could be in her group?

14 a What is the largest number of kings that can be placed on an 8×8 chess board so that no two of them are in squares that are adjacent horizontally, vertically or diagonally?
   b What is the largest number of rooks that can be placed on a chess board so that no two of them are in the same row or column?

15 Nine thin territorial cows are placed in an $8\,\text{m} \times 12\,\text{m}$ field. ('Thin' means that they are point-cows.)
   a Why do we know that two of them are at most 5 metres apart?
   b Show that if only eight cows are placed in the field, it is possible to separate each pair by at least 5 metres.

16 Show that amongst 100 numbers, at least 34 must leave the same remainder when divided by 3.

17 Prove that there exist powers of two that differ by a multiple of 2019.

18 Show that in any set of 14 different positive odd numbers, all less than 52, there is a pair whose sum is 52.

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19 Consider the set of odd whole numbers less than 200, and select a subset. How big must the subset be to guarantee that it contains two whole numbers with sum 200?

20 Suppose that every point in the plane is coloured in one of two colours. Prove that no matter how this is done, there must be two points of the same colour exactly 1 unit apart.

ENRICHMENT

21 A flock of 41 pigeons fly into 10 different pigeonholes.

- **a** Show that there is at least one pigeonhole with at least 5 pigeons.
- **b** Show that there is at least one pair of pigeonholes in which there are at least 9 pigeons.

22 A school is planning a new school email system and plans to make every student's email address their first and last initial. Thus Hans Olo would have email address HO. Email addresses are not case sensitive, so that HO and ho are the same.

- **a** Someone immediately notes that there will certainly be two people with the same email address — what does this tell you about the number of students attending the school?
- **b** Not all initials are equally common. Suppose it is known that at least 50% of the students have first name starting with one of 8 different letters. What is the size of the school population, if it is certain that two students will share an email address?
- **c** Another school decides to borrow this system of assigning email addresses, but additionally adds either no third symbol or a digit 0–9. Thus HO, HO0, HO1, . . ., HO9 are all possible. If the school has a population of 1200, and 200 new students join every year, how long until an email address is definitely re-used? Assume that email addresses are not reassigned when a student departs and no assumption is made about how common certain initials are.

23 Show that amongst 6 people at a dinner table, there are two that have an identical number of friends at the table. (Assume that if A is a friend of B then B is a friend of A.)

24 A regular octahedron has 6 vertices. Each vertex is connected to each other vertex by a rod that is coloured yellow or blue.

- **a** Explain why there are 15 rods.
- **b** Each set of three vertices, together with the rods joining them, forms a triangle. Explain why there are 20 such triangles.
- **c** Explain why there will be at least one triangle whose rods all have the same colour.

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Chapter 14 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 14 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 How many ways can 8 people line up in a queue?

2 By unrolling the factorial (see Box 2 in Section 14A) simplify:

a $\dfrac{9!}{7!}$ b $\dfrac{(n + 1)!}{(n - 1)!}$ c $(k + 1)! - k!$

3 Use your calculator to evaluate:

a ${}^{12}C_7$ b ${}^{10}C_3 \div {}^{6}C_3$ c $\binom{20}{17}$

4 How many four-letter words, with no repeated letters, can be formed from the letters of JACKSON?

5 A number plate in a certain country consists of 3 letters A–Z followed by 4 digits 0–9. How many number plates are possible?

6 Find how many arrangements of the letters of the word FOUNDER are possible:

a if the vowels and consonants must alternate,

b if the word must start with N and end with D,

c if all the consonants must be in a group at the end of the word,

d if the R is somewhere to the right of the U.

7 Five boys and five girls are to sit in a row. Find how many ways this can be done if:

a there are no restrictions,

b the boys and girls sit in distinct groups,

c a particular boy and girl must sit together.

8 How many ways can the letters of REPORTER be arranged?

9 How many words of three or four letters may be formed using the letters of SAMUEL?

10 A quiz consists of ten questions, each taking the answer Yes or No. How many ways is it possible to get 6 correct and 4 wrong answers?

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11 A committee of seven is to be chosen from six men and ten women. Find how many committees are possible if:

a there are no restrictions,
b all members are to be female,
c all members are to be male,
d there are to be exactly two men,
e there are to be four women and three men,
f there is to be a majority of women,
g a particular man must be included,
h a particular man must not be included,
i Mustafa refuses to be on a committee with Ying Yue.

12 Eight people arrive at a restaurant. Find how many ways can they be assigned to:

a a large table for five and a smaller table for three,
b two quite different tables for four,
c two indistinguishable tables for four.

13 What is the probability that if a committee of six is formed at random from four men and three women, it will have more men than women?

14 From a standard pack of 52 cards, three are selected at random. Find the probability that:

a they are the queen of spades, the three of clubs and the nine of hearts,
b all three are kings,
c they are all clubs,
d they are all of the same suit,
e one is red and two are black,
f one is a three, one is a five, and one is an eight,
g two are fives and one is a seven,
h at least two of them are spades.

15 Three boys and three girls are arranged in a circle. In how many ways can this be done:

a if there are no restrictions,
b if the boys and the girls alternate,
c if the boys and girls are in distinct groups,
d if a particular boy and girl wish to sit next to one another,
e if two particular boys do not wish to sit next to one another,
f if a particular boy wants to sit opposite a particular girl?

16 Seventeen peanuts are shared amongst six monkeys. Show that there is at least one monkey who receives at least three peanuts.

17 Amongst 1500 people, show that there is guaranteed to be a birthday shared by at least 5 people.

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15 The binomial expansion and Pascal's triangle

In Chapter 10 we discussed the factoring of a polynomial into irreducible factors, so that it could be written in a form such as

$$P(x) = (x - 4)^2(x + 1)^3(x^2 + x + 1).$$

In this chapter we will now study in more detail the individual binomial power factors such as $(x - 4)^2$ and $(x + 1)^3$ that appear in such a factoring and their expansions. For example, we have already seen that

$$(x + 1)^3 = x^3 + 3x^2 + 3x + 1.$$

The coefficients in the general expansion of $(x + y)^n$ will be investigated through the patterns they form when they are written down in *Pascal's triangle*. It turns out that basic counting methods of the previous chapter are essential for understanding these expansions.

Pascal's triangle displays in a clear visual form the interrelationships between the counting methods of Chapter 14 and the binomial expansions of this chapter. It has an extraordinary collection of symmetries and properties. Section 15D investigates these patterns using various combinatoric and algebraic methods. The final Section 15E is marked as Enrichment because it requires more abstract algebraic methods using the general term in the binomial expansion.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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15A Binomial expansions and Pascal's triangle

A *binomial expansion* is the expansion of a power of a binomial, for example

$$(1 - x)^3 = 1 - 3x + 3x^2 - x^3.$$

A *binomial* is a polynomial with two terms, such as $1 - x$ or $3x^4 - \frac{1}{2}x^2$.

This first section is restricted to the expansion of $(1 + x)^n$ and to the various techniques arising from such expansions. The techniques are based on Pascal's triangle and its three most basic properties.

Some expansions of $(1 + x)^n$

Here are the expansions of $(1 + x)^n$ for low values of n. The calculations have been carried out using two rows so that like terms can be written above each other in columns. In this way, the process by which the coefficients build up can be followed better.

Examine the last expansion in particular, and work out why the boxed $\boxed{4}$ is the sum of the boxed $\boxed{1}$ and $\boxed{3}$.

$$\begin{aligned}
(1 + x)^0 &= 1 \\
(1 + x)^2 &= 1 + x \\
(1 + x)^2 &= 1(1 + x) + x(1 + x) \\
&= 1 + x \\
&\quad + x + x^2 \\
&= 1 + 2x + x^2
\end{aligned}$$

$$\begin{aligned}
(1 + x)^3 &= 1(1 + x)^2 + x(1 + x)^2 \\
&= 1 + 2x + x^2 \\
&\quad + x + 2x^2 + x^3 \\
&= 1 + 3x + 3x^2 + x^3 \\
(1 + x)^4 &= 1(1 + x)^3 + x(1 + x)^3 \\
&= 1 + 3x + 3x^2 + \boxed{1}x^3 \\
&\quad + x + 3x^2 + \boxed{3}x^3 + x^4 \\
&= 1 + 4x + 6x^2 + \boxed{4}x^3 + x^4
\end{aligned}$$

Notice how the expansion of $(1 + x)^2$ has 3 terms, that of $(1 + x)^3$ has 4 terms, and so on. In general, the expansion of $(1 + x)^n$ has $n + 1$ terms, from the constant term in $x^0 = 1$ to the term in x^n. Be careful — this is inclusive counting — there are $n + 1$ numbers from 0 to n inclusive.

Pascal's triangle and the addition property

When the coefficients in the expansions of $(1 + x)^n$ are arranged in a table, the result is known as *Pascal's triangle*. The table below contains the first five rows of the triangle, copied from the expansions above, plus the next four rows, obtained by continuing these calculations up to $(1 + x)^8$.

	Coefficient of:								
n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
0	1								
1	1	1							
2	1	2	1						
3	1	3	$\boxed{3}$	$\boxed{1}$					
4	1	4	6	$\boxed{4}$	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

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Three basic properties of this triangle should quickly become obvious. They will be used in this section, and proven formally later.

1 THREE BASIC PROPERTIES OF PASCAL'S TRIANGLE

1. Each row starts and ends with 1.
2. Each row is reversible. That is, the rows are symmetric.
3. [The addition property]
   Every number in the triangle, apart from the 1s, is the sum of the number directly above, and the number above and to the left.

The first two properties should be reasonably obvious after looking at the expansions at the start of the section. The third property, called the *addition property*, however, needs attention. Three numbers in Pascal's triangle above have been boxed as an Example of this — notice how

$$3 + 1 = 4.$$

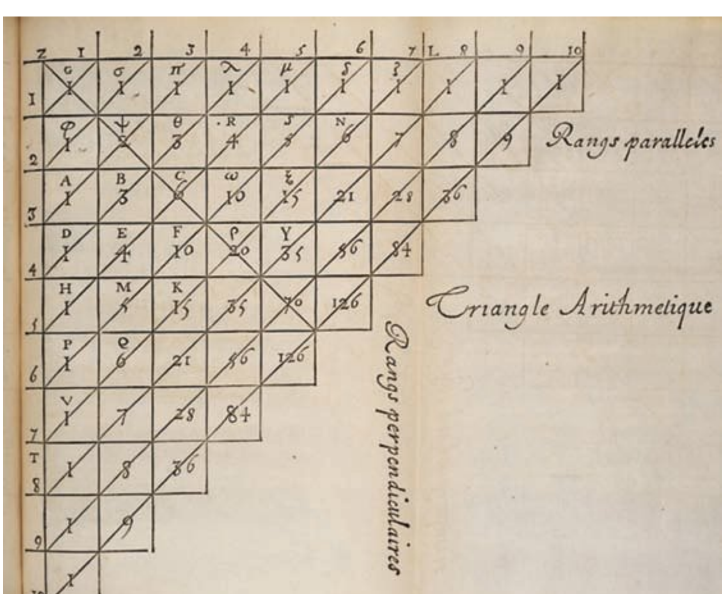
The binomial expansions on the previous page were written with the columns aligned, and these particular coefficients also boxed, to make this property stand out. The sum $3 + 1 = 4$ arises because the coefficient of x^3 in the expansion of $(1 + x)^4$ is the sum of the coefficients of x^3 and x^2 in the expansion of $(1 + x)^3$.

Pascal's triangle can be constructed using these rules, and the first question in Exercise 15A asks for the first thirteen rows to be calculated.

Pascal's triangle is often drawn in the layout to the right below, as an equilateral triangle. This layout displays the left-right symmetry well, but it is less useful for our purposes because we want the columns to line up with the powers of x. Nevertheless, you should be able to work from both layouts of the triangle.

Pascal did not discover the triangle, although he did write an important treatise on it in 1653. It was known to the ancient Indian mathematician Pingala in the 2nd century BC, and later to mediaeval Persian and Chinese mathematicians, but the first known occurrence in Europe is in a 1527 book on business calculations by Petrus Adrianus. Here is Pascal's version of the triangle from his treatise, with yet another layout.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1



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Using Pascal's triangle

Examples 1–5 illustrate various calculations involving the coefficients of $(1+x)^n$ for low values of n.

Example 1 15A

Use Pascal's triangle to write out the expansions of:

a $(1+2a)^6$ **b** $(1-x)^4$ **c** $\left(1-\frac{2}{3}x\right)^5$

SOLUTION

a
$$\begin{aligned}(1+2a)^6 &= 1+6(2a)+15(2a)^2+20(2a)^3+15(2a)^4+6(2a)^5+(2a)^6\\ &= 1+12a+60a^2+160a^3+240a^4+192a^5+64a^6\end{aligned}$$

b Be particularly careful with the alternating signs in this expansion.

$$\begin{aligned}(1-x)^4 &= 1+4(-x)+6(-x)^2+4(-x)^3+(-x)^4\\ &= 1-4x+6x^2-4x^3+x^4\end{aligned}$$

c The signs also alternate in this expansion.

$$\begin{aligned}\left(1-\tfrac{2}{3}x\right)^5 &= 1+5\left(-\tfrac{2}{3}x\right)+10\left(-\tfrac{2}{3}x\right)^2+10\left(-\tfrac{2}{3}x\right)^3+5\left(-\tfrac{2}{3}x\right)^4+\left(-\tfrac{2}{3}x\right)^5\\ &= 1-\tfrac{10}{3}x+\tfrac{40}{9}x^2-\tfrac{80}{27}x^3+\tfrac{80}{81}x^4-\tfrac{32}{243}x^5\end{aligned}$$

Example 2 15A

By expanding the first few terms of $(1+0.02)^8$, find an approximation of 1.02^8 correct to five decimal places.

SOLUTION

$$\begin{aligned}(1+0.02)^8 &= 1+8\times 0.02+28\times(0.02)^2+56\times(0.02)^3+70\times(0.02)^4+\cdots\\ &= 1+0.16+0.0112+0.000448+0.00001120+\cdots\\ &\doteqdot 1.17166.\end{aligned}$$

The remaining four terms are too small to affect the fifth decimal place.

Example 3 15A

a Write out the expansion of $\left(1+\dfrac{5}{x}\right)^2$, then write out the first four terms in the expansion of $(1-x)^8$.

b Hence find, in the expansion of $\left(1+\dfrac{5}{x}\right)^2(1-x)^8$:

i the term independent of x, **ii** the term in x.

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SOLUTION

a $\left(1 + \frac{5}{x}\right)^2 = 1 + 10x^{-1} + 25x^{-2}$

$(1 - x)^8 = 1 - 8x + 28x^2 - 56x^3 + \cdots$

b Hence in the expansion of $\left(1 + \frac{5}{x}\right)^2 (1 - x)^8$:

i $$\begin{aligned}\text{constant term} &= 1 \times 1 + (10x^{-1}) \times (-8x) + (25x^{-2}) \times (28x^2)\\ &= 1 - 80 + 700\\ &= 621.\end{aligned}$$

ii $$\begin{aligned}\text{term in } x &= 1 \times (-8x) + (10x^{-1}) \times (28x^2) + (25x^{-2}) \times (-56x^3)\\ &= -8x + 280x - 1400x\\ &= -1128x.\end{aligned}$$

Example 4 — 15A

a Write down the terms in x^4 and x^3 in the expansion of $(1 + 2kx)^6$.

b Find k if these terms in x^4 and x^3 have coefficients in the ratio 2 : 3.

SOLUTION

a $$\begin{aligned}(1 + 2kx)^6 &= \cdots + 15(2kx)^2 + 20(2kx)^3 + 15(2kx)^4 + \cdots\\ &= \cdots + 60k^2x^2 + 160k^3x^3 + 240k^4x^4 + \cdots\end{aligned}$$

(Alternatively, just write down the two terms.)

b Put $\dfrac{240k^4}{160k^3} = \dfrac{2}{3}$.

Then $\frac{3}{2}k = \frac{2}{3}$

$k = \frac{4}{9}$.

Example 5 — 15A

[A harder example]

Expand $(1 + x + x^2)^4$ using Pascal's triangle, by writing $1 + x + x^2 = 1 + (x + x^2)$, and writing $x + x^2 = x(1 + x)$.

SOLUTION

$$\begin{aligned}(1 + x + x^2)^4 &= \big(1 + (x(1 + x))\big)^4\\ &= 1 + 4x(1 + x) + 6x^2(1 + x)^2 + 4x^3(1 + x)^3 + x^4(1 + x)^4\\ &= 1 + 4x(1 + x) + 6x^2(1 + 2x + x^2) + 4x^3(1 + 3x + 3x^2 + x^3)\\ &\quad + x^4(1 + 4x + 6x^2 + 4x^3 + x^4)\\ &= 1 + (4x + 4x^2) + (6x^2 + 12x^3 + 6x^4) + (4x^3 + 12x^4 + 12x^5 + 4x^6)\\ &\quad + (x^4 + 4x^5 + 6x^6 + 4x^7 + x^8)\\ &= 1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8\end{aligned}$$

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Exercise 15A

FOUNDATION

1 Complete all the rows of Pascal's triangle for $n = 0, 1, 2, 3, \dots, 12$. Keep this in a prominent place for use in the rest of this chapter.

2 Using Pascal's triangle to provide the binomial coefficients, give the expansions of:

a $(1+x)^6$ **b** $(1-x)^6$ **c** $(1+x)^9$ **d** $(1-x)^9$

e $(1+c)^5$ **f** $(1+2y)^4$ **g** $\left(1+\frac{x}{3}\right)^7$ **h** $(1-3z)^3$

i $\left(1-\frac{1}{x}\right)^8$ **j** $\left(1+\frac{2}{x}\right)^5$ **k** $\left(1+\frac{y}{x}\right)^5$ **l** $\left(1+\frac{3x}{y}\right)^4$

3 Continue the calculations of the expansions of $(1+x)^n$ at the beginning of this section, expanding $(1+x)^5$ and $(1+x)^6$ in the same manner. Keep your work in columns, so that the addition property of Pascal's triangle is clear.

4 Find the specified term in each expansion.

a For $(1+x)^{11}$:
- **i** find the term in x^2,
- **ii** find the term in x^8.

b For $(1-x)^7$:
- **i** find the term in x^3,
- **ii** find the term in x^5.

c For $(1+2x)^6$:
- **i** find the term in x^4,
- **ii** find the term in x^5.

d For $\left(1-\frac{3}{x}\right)^4$:
- **i** find the term in x^{-1},
- **ii** find the term in x^{-2}.

5 Expand $(1+x)^9$ and $(1+x)^{10}$, and show that the sum of the coefficients in the second expansion is twice the sum of the coefficients in the first expansion.

DEVELOPMENT

6 Without expanding, simplify:

a $1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$

b $1 - 6(x+1) + 15(x+1)^2 - 20(x+1)^3 + 15(x+1)^4 - 6(x+1)^5 + (x+1)^6$

7 Find the coefficient of x^4 in the expansion of $(1-x)^4 + (1-x)^5 + (1-x)^6$.

8 Find integers a and b such that:

a $\left(1+\sqrt{3}\right)^5 = a + b\sqrt{3}$ **b** $\left(1-\sqrt{5}\right)^3 = a + b\sqrt{5}$

9 Verify by direct expansion, and by taking out the common factor, that:

a $(1+x)^4 - (1+x)^3 = x(1+x)^3$ **b** $(1+x)^7 - (1+x)^6 = x(1+x)^6$

10 Do not use a calculator in this question.

a Expand the first few terms of $(1+x)^6$, hence evaluate 1.003^6 to five decimal places.

b Similarly, expand $(1-4x)^5$, and hence evaluate 0.96^5 to five decimal places.

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11 **a** **i** Expand $(1 + x)^4$ as far as the term in x^2.
ii Hence find the coefficient of x^2 in the expansion of $(1 - 5x)(1 + x)^4$.
b **i** Expand $(1 + 2x)^5$ as far as the term in x^3.
ii Hence find the coefficient of x^3 in the expansion of $(2 - 3x)(1 + 2x)^5$.
c **i** Expand $(1 - 3x)^4$ as far as the term in x^3.
ii Hence find the coefficient of x^3 in the expansion of $(2 + x)^2(1 - 3x)^4$.

12 **a** When $(1 + 2x)^5$ is expanded in increasing powers of x, the third and fourth terms in the expansion are equal. Find the value of x.
b When $(1 + x)^7$ is expanded in increasing powers of x, the sixth term is the average of the fifth and seventh terms in the expansion. Find the value of x.

13 Find the coefficient of:
a x^3 in $(3 - 4x)(1 + x)^4$
b x in $(1 + 3x + x^2)(1 - x)^3$
c x^4 in $(5 - 2x^3)(1 + 2x)^5$
d x^0 in $\left(1 - \frac{x}{3}\right)^3\left(1 + \frac{2}{x}\right)^2$

14 Determine the value of the term independent of x in the expansion of:
a $(1 + 2x)^4\left(1 - \frac{1}{x^2}\right)^6$
b $\left(1 - \frac{x}{3}\right)^5\left(1 + \frac{2}{x}\right)^3$

15 **a** In the expansion of $(1 + x)^6$:
i find the term in x^2,
ii find the term in x^3,
iii find the ratio of the term in x^2 to the term in x^3,
iv find the values of **i**, **ii** and **iii** when $x = 3$.
b In the expansion of $\left(1 + \frac{2}{3x}\right)^7$:
i find the term in x^{-5},
ii find the term in x^{-6},
iii find the ratio of the term in x^{-5} to the term in x^{-6},
iv find the values of **i**, **ii** and **iii** when $x = 2$.

16 **a** Find the coefficients of x^4 and x^5 in the expansion of $(1 + kx)^8$. Hence find k if these coefficients are in the ratio 1 : 4.
b Find the coefficients of x^5 and x^6 in the expansion of $\left(1 - \frac{3}{4}kx\right)^9$. Hence find k if these coefficients are equal.

17 [Patterns in Pascal's triangle]
Check the following results using the triangle you constructed in Question 1. (Some of these will be proven later.)
a The sum of the numbers in the row beginning1, n, . . . is equal to 2^n.
b If the second member of a row is a prime number, all the numbers in that row excluding the 1s are divisible by it.

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c [The hockey-stick pattern]
Start at the top 1 of any column of Pascal's triangle, and go vertically down any number of rows. Then the sum of these numbers is the number in the next row down and to the right. For example, if you start at the 1 in the x^2 column, and add downwards:

$$1 + 3 + 6 + 10 + 15 + 21 = 56, \text{ which is one row down in the } x^3 \text{ column.}$$

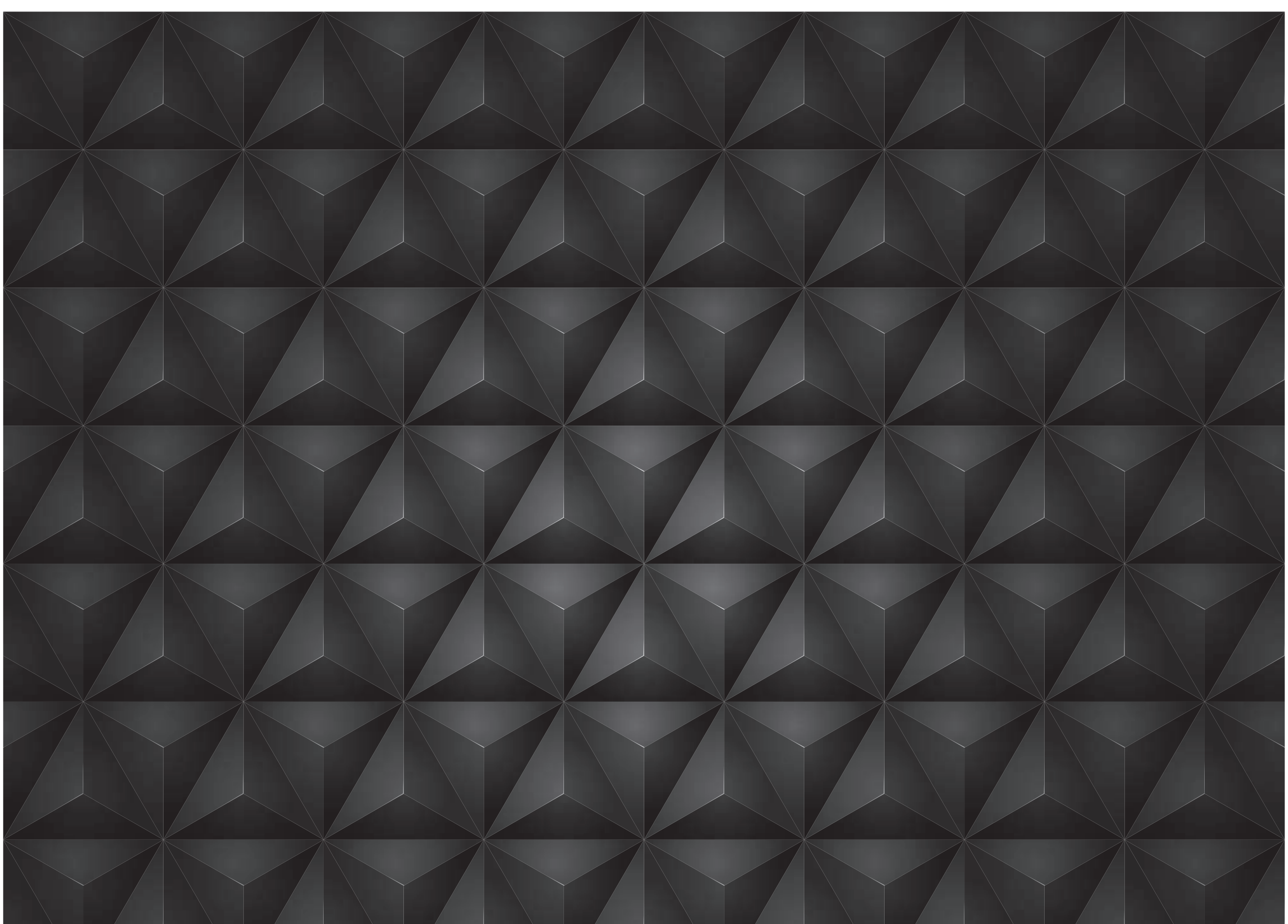
d [The powers of 11]
If a row is made into a single number by using each element as a digit of the number, the number is a power of 11 (except that after the row 1, 4, 6, 4, 1, the pattern gets confused by carrying).

e Find the diagonal and the column containing the triangular numbers, and show that adding adjacent pairs gives the square numbers.

ENRICHMENT

18 By writing $(1 + x + 3x^2)^6$ as $(1 + A)^6$, where $A = x + 3x^2$, expand $(1 + x + 3x^2)^6$ as far as the term in x^3. Hence evaluate $(1.0103)^6$ to four decimal places.

19 [The Pascal pyramid]
By considering the expansion of $(1 + x + y)^n$, where $0 \le n \le 4$, calculate the first five layers of the Pascal pyramid.



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15B Further binomial expansions

The more general case of the expansion of $(x+y)^n$ is very similar. Because x and y are both variables, however, the symmetries of the expansion will be more obvious.

A minor point about language — we are now widening the term *polynomial* to include terms made up of powers of any number of variables. Thus $5xy^4$ is a monomial, $x+y$ is a binomial, and $x^2+2xy+y^2$ and $x+y+z$ are trinomials.

Some expansions of $(x+y)^n$

Here are the expansions of $(x+y)^n$ for low values of n. Again, the calculations have been carried out with like terms written in the same column so that the addition property is clear — see the boxed coefficients.

$$\begin{aligned}
(x+y)^0 &= 1\\
(x+y)^1 &= x+y\\
(x+y)^2 &= x(x+y)+y(x+y)\\
&= x^2 + xy\\
&\qquad + xy + y^2\\
&= x^2+2xy+y^2
\end{aligned}$$

$$\begin{aligned}
(x+y)^3 &= x(x+y)^2 + y(x+y)^2\\
&= x^3 + 2x^2y + xy^2\\
&\qquad + x^2y + 2xy^2 + y^3\\
&= x^3+3x^2y+3xy^2+y^3\\
(x+y)^4 &= x(x+y)^3+y(x+y)^3\\
&= x^4+3x^3y+3x^2y^2+\boxed{1}xy^3\\
&\qquad + x^3y+3x^2y^2+\boxed{3}xy^3+y^4\\
&= x^4+4x^3y+6x^2y^2+\boxed{4}xy^3+y^4
\end{aligned}$$

First, the coefficients in the expansions are exactly the same as before, so as before they can be taken from Pascal's triangle.

Secondly, the pattern for the indices of x and y is straightforward.

- The expansion of $(x+y)^3$ has four terms, and in each term the indices of x and y are whole numbers adding to 3.
- Similarly the expansion of $(x+y)^4$ has five terms, and in each term the indices of x and y are whole numbers adding to 4.

More generally, in each successive expansion, beginning with $(x+y)^0=1$, the terms of the previous expansion are multiplied first by x, and then by y, so the sum of the two indices goes up by 1.

2 THE EXPANSION OF $(x+y)^n$:

- The expansion of $(x+y)^n$ has $n+1$ terms, and in each term the indices of x and y are whole numbers adding to n.
- The coefficients in the expansion of $(x+y)^n$ are the same as the coefficients in the expansion of $(1+x)^n$.

The expansion of $(x+y)^n$ is called *homogeneous of degree n in x and y together*, because in each term, the sum of the indices of x and y is n.

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Using the general expansion

The general expansion of $(x + y)^n$ is applied in the same way as the expansion of $(1 + x)^n$.

Example 6 15B

Use Pascal's triangle to write out the expansions of

a $(2 - 3x)^4$ **b** $\left(5x + \frac{1}{5}a\right)^5$

SOLUTION

a $(2 - 3x)^4 = 2^4 + 4 \times 2^3 \times (-3x) + 6 \times 2^2 \times (-3x)^2$
$+ 4 \times 2 \times (-3x)^3 + (-3x)^4$
$= 16 - 96x + 216x^2 - 216x^3 + 81x^4$

b $\left(5x + \frac{1}{5}a\right)^5 = (5x)^5 + 5 \times (5x)^4 \times \frac{1}{5}a + 10 \times (5x)^3 \times \left(\frac{1}{5}a\right)^2$
$+ 10 \times (5x)^2 \times \left(\frac{1}{5}a\right)^3 + 5 \times (5x) \times \left(\frac{1}{5}a\right)^4 + \left(\frac{1}{5}a\right)^5$
$= 3125x^5 + 625ax^4 + 50a^2x^3 + 2a^3x^2 + \frac{1}{25}a^4x + \frac{1}{3125}a^5$

Example 7 15B

Use Pascal's triangle to write out the expansion of $(2x + x^{-2})^6$, leaving the terms unsimplified.
Hence find:

a the term independent of x, **b** the term in x^{-3}.

SOLUTION

$(2x + x^{-2})^6 = (2x)^6 + 6 \times (2x)^5 \times (x^{-2}) + 15 \times (2x)^4 \times (x^{-2})^2 + 20 \times (2x)^3 \times (x^{-2})^3$
$+ 15 \times (2x)^2 \times (x^{-2})^4 + 6 \times (2x) \times (x^{-2})^5 + (x^{-2})^6$

a Constant term $= 15 \times (2x)^4 \times (x^{-2})^2$
$= 15 \times 2^4 \times x^4 \times x^{-4}$
$= 240$

b Term in $x^{-3} = 20 \times (2x)^3 \times (x^{-2})^3$
$= 20 \times 2^3 \times x^3 \times x^{-6}$
$= 160x^{-3}$

Example 8 15B

Expand $(2 - 3x)^7$ as far as the term in x^2, and hence find the term in x^2 in the expansion of $(5 + x)(2 - 3x)^7$.

SOLUTION

$(2 - 3x)^7 = 2^7 + 7 \times 2^6 \times (-3x) + 21 \times 2^5 \times (-3x)^2 + \cdots$
$= 128 - 1344x + 6048x^2 - \cdots$

Hence the term in x^2 in the expansion of $(5 + x)(2 - 3x)^7$
$= 5 \times 6048x^2 - x \times 1344x$
$= 28\,896x^2$.

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Exercise 15B

FOUNDATION

1 Use Pascal's triangle to expand:

a $(x + y)^4$ **b** $(x - y)^4$ **c** $(r - s)^6$ **d** $(p + q)^{10}$

e $(a - b)^9$ **f** $(2x + y)^5$ **g** $(p - 2q)^7$ **h** $(3x + 2y)^4$

i $\left(a - \frac{1}{2}b\right)^3$ **j** $\left(\frac{1}{2}r + \frac{1}{3}s\right)^5$ **k** $\left(x + \frac{1}{x}\right)^6$

2 Use Pascal's triangle to expand:

a $(1 + x^2)^4$ **b** $(1 - 3x^2)^3$ **c** $(x^2 + 2y^3)^6$

d $\left(x - \frac{1}{x}\right)^9$ **e** $\left(\sqrt{x} + \sqrt{y}\right)^7$ **f** $\left(\frac{2}{x} + 3x^2\right)^5$

3 Simplify these expressions without expanding the brackets.

a $y^5 + 5y^4(x - y) + 10y^3(x - y)^2 + 10y^2(x - y)^3 + 5y(x - y)^4 + (x - y)^5$

b $a^4 - 4a^3(a - b) + 6a^2(a - b)^2 - 4a(a - b)^3 + (a - b)^4$

c $x^3 + 3x^2(2y - x) + 3x(2y - x)^2 + (2y - x)^3$

d $(x + y)^6 - 6(x + y)^5(x - y) + 15(x + y)^4(x - y)^2 - 20(x + y)^3(x - y)^3$
$+15(x + y)^2(x - y)^4 - 6(x + y)(x - y)^5 + (x - y)^6$

4 **a** **i** Expand $(4 + x)^5$ as far as the term in x^3.

ii Hence find the coefficient of x^3 in the expansion of $(3 - x)(4 + x)^5$.

b **i** Expand $(1 - 2x)^6$ as far as the term in x^4.

ii Hence find the coefficient of x^4 in the expansion of $(1 - 3x)(1 - 2x)^6$.

c **i** Expand $(3 - y)^7$ as far as the term in y^4.

ii Hence find the coefficient of y^4 in the expansion of $(1 - y)^2(3 - y)^7$.

DEVELOPMENT

5 **a** Expand and simplify $(x + y)^6 + (x - y)^6$.

b Hence (and without a calculator) prove that

$$5^6 + 5^5 \times 3^3 + 5^3 \times 3^5 + 3^6 = 2^5(2^{12} + 1).$$

6 Find the coefficient of:

a x^3 in $(2 - 5x)(x^2 - 3)^4$ **b** x^5 in $(x^2 - 3x + 11)(4 + x^3)^3$

c x^0 in $(3 - 2x)^2\left(x + \frac{2}{x}\right)^5$ **d** x^9 in $(x + 2)^3(x - 2)^7$

7 **a** **i** Use Pascal's triangle to expand $(x + h)^3$.

ii If $f(x) = x^3$, simplify $f(x + h) - f(x)$.

iii Hence use the definition $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ to differentiate x^3.

b Similarly, differentiate x^5 from first principles.

8 **a** Show that $\left(3 + \sqrt{5}\right)^6 + \left(3 - \sqrt{5}\right)^6 = 20608$.

b Show that $\left(2 + \sqrt{7}\right)^4 + \left(2 - \sqrt{7}\right)^4$ is rational.

c If $\left(\sqrt{6} + \sqrt{3}\right)^3 - \left(\sqrt{6} - \sqrt{3}\right)^3 = a\sqrt{3}$, where a is an integer, find the value of a.

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9 Show that $\dfrac{1}{(\sqrt{3}-1)^4}+\dfrac{1}{(\sqrt{3}+1)^4}=\dfrac{(\sqrt{3}+1)^4+(\sqrt{3}-1)^4}{(\sqrt{3}-1)^4(\sqrt{3}+1)^4}$ by putting the LHS over a common denominator. Then simplify the expression using Pascal's triangle.

10 Expand $(x+2y)^5$ and hence evaluate:

a $(1.02)^5$ correct to to five decimal places,

b $(0.98)^5$ correct to to five decimal places.

11 **a** Expand:

i $\left(x+\dfrac{1}{x}\right)^3$ **ii** $\left(x+\dfrac{1}{x}\right)^5$ **iii** $\left(x+\dfrac{1}{x}\right)^7$

b Hence, if $x+\dfrac{1}{x}=2$, evaluate:

i $x^3+\dfrac{1}{x^3}$ **ii** $x^5+\dfrac{1}{x^5}$ **iii** $x^7+\dfrac{1}{x^7}$

12 Find the coefficients of x and x^{-3} in the expansion of $\left(3x-\dfrac{a}{x}\right)^5$. Hence find the values of a if these coefficients are in the ratio 2 : 1.

13 The coefficients of the terms in a^3 and a^{-3} in the expansion of $\left(ma+\dfrac{n}{a^2}\right)^6$ are equal, where m and n are non-zero real numbers. Prove that $m^2 : n^2 = 10 : 3$.

ENRICHMENT

14 **a** Expand $\left(x+\dfrac{1}{x}\right)^6$.

b If $U=x+\dfrac{1}{x}$, express $x^6+\dfrac{1}{x^6}$ in the form $U^6+AU^4+BU^2+C$. State the values of A, B and C.

15 **a** By starting with $\big((x+y)+z\big)^3$, expand $(x+y+z)^3$.

b Find the term independent of x in the expansion of $(x+1+x^{-1})^4$.

16 [The Sierpinski triangle fractal]

a Draw an equilateral triangle of side length 1 unit on a piece of white paper. Join the midpoints of the sides of this triangle to form a smaller triangle. Colour it black. Repeat this process on all white triangles that remain. What do you notice?

b Draw up Pascal's triangle in the shape of an equilateral triangle, then colour all the even numbers black and leave the odd numbers white. What do you notice? This pattern will be more evident if you take at least the first 16 rows — perhaps use a computer program to generate 100 rows of Pascal's triangle.

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15C The binomial theorem

The reader may well have realised already that the entries in Pascal's triangle are exactly the numbers ${}^{n}\mathrm{C}_r$ that arose in the previous chapter when counting unordered selections of r elements from an n-member set. Once this has been proven below, we can write the general binomial expansion out explicitly as

$$(x + y)^n = {}^{n}\mathrm{C}_0\, x^n + {}^{n}\mathrm{C}_1\, x^{n-1}y + {}^{n}\mathrm{C}_2\, x^{n-2}y^2 + \cdots + {}^{n}\mathrm{C}_n\, y^n,$$

or using the alternative notation $\binom{n}{r}$ for 'n choose r',

$$(x + y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n,$$

This result is known as the *binomial theorem*. It gives a striking connection between the counting methods of the previous chapter and the algebraic structures in this chapter, showing yet again the unity between algebra and arithmetic.

Proving the binomial theorem

We have already established that

$$(x + y)^n = *x^n + *x^{n-1}y + *x^{n-2}y^2 + \cdots + *y^n,$$

where $*$ denotes the different coefficients. What remains to be proven is that the coefficient of $x^{n-r}y^r$ is ${}^{n}\mathrm{C}_r$. As usual, the most straightforward way to establish this is by looking at how things work in a special case. For example, let us establish why the coefficient of x^2y^2 in the expansion of $(x + y)^4$ is ${}^{4}\mathrm{C}_2 = 6$.

Expanding $(x + y)^4 = (x + y)(x + y)(x + y)(x + y)$ without collecting like terms, or even applying the commutative law of multiplication, gives $2^4 = 16$ terms:

$$\begin{aligned}(x + y)^4 = {} & xxxx \quad \text{(the term with 4 } x\text{s)}\\ & + xxxy + xxyx + xyxx + yxxx \quad \text{(the terms with 3 } x\text{s and 1 } y\text{)}\\ & + xxyy + xyxy + yxxy + xyyx + yxyx + yyxx \quad \text{(2 } x\text{s and 2 } y\text{s)}\\ & + xyyy + yxyy + yyxy + yyyx \quad \text{(the term with 1 } x \text{ and 3 } y\text{s)}\\ & + yyyy \quad \text{(the term with 4 } y\text{s)}\end{aligned}$$

which then becomes $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ after collecting like terms.

The middle row above shows that the coefficient of x^2y^2 is 6 because there are six terms with two xs and two ys. Why are there six terms? The reason is:

- these six terms are all the words formed with 4 letters, 2 alike of one kind, and 2 of another.

We showed in the last chapter that the number of such words is $\dfrac{4!}{2! \times 2!} = 6$.

More generally, when we expand $(x + y)^n$, the coefficient of x^ry^{n-r} is the number of words formed with n letters, r alike of one kind and $n - r$ alike of another.

Now we can bring in the theory of the last chapter, where we used the symbol ${}^{n}\mathrm{C}_r$ or $\binom{n}{r}$ for the number of r-member subsets of an n-member set. Otherwise expressed, this is the number of unordered selections of r objects chosen from n objects. We showed there that

$${}^{n}\mathrm{C}_r = \frac{n!}{r! \times (n - r)!},$$

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and we also showed that this was equal to the number of n-letter words with r letters alike of one kind and $n - r$ letters alike of another. This means that the coefficient of $x^r y^{n-r}$ can be written as ${}^nC_r$, giving a concise form of the expansion:

3 THE BINOMIAL THEOREM

$$(x + y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}y^2 + \dots + {}^nC_n\, y^n,$$

for all whole numbers n, where ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$, for $r = 0, 1, \dots, n$.

Alternatively, we can write ${}^nC_r = \dfrac{n \times (n-1) \times \dots \times (n-r+1)}{1 \times 2 \times \dots \times r}$. Remember that the notations $\binom{n}{r}$ and ${}^nC_r$ mean exactly the same thing. But don't mix the two notations up in the same calculation because it looks dreadful.

Examples of the binomial theorem

We calculated ${}^nC_r$ in Section 14E. These worked examples use the formula for ${}^nC_r$ to calculate coefficients in binomial expansions.

Example 9 — 15C

Use the binomial theorem to calculate:

a the coefficient of x^8 in $(1 + x)^{12}$,
b the term in x^3y^4 in $(x + y)^7$,
c the coefficient of A^5B^5 in $(2A - 3B)^{10}$ (factored into primes),
d the term in $a^{n-4}b^4$ in $(a - 2b)^n$.

SOLUTION

a
$$(1 + x)^{12} = \dots + {}^{12}C_8 \times 1^4 \times x^8 + \dots,$$
$$\text{so coefficient of } x^8 = \frac{12!}{8! \times 4!}$$
$$= \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1}$$
$$= 495.$$

b
$$(x + y)^7 = \dots + \binom{7}{4} \times x^3 \times y^4 + \dots,$$
$$\text{so term in } x^3y^4 = \frac{7!}{4! \times 3!}x^3y^4$$
$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1}x^3y^4$$
$$= 35x^3y^4.$$

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c $$(2A - 3B)^{10} = \cdots + {}^{10}\mathrm{C}_5 \times (2A)^5 \times (-3B)^5 + \cdots,$$

$$\begin{aligned}\text{so coefficient of } A^5B^5 &= -\frac{10!}{5! \times 5!} \times 2^5 \times 3^5\\ &= -\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} \times 2^5 \times 3^5\\ &= -9 \times 4 \times 7 \times 2^5 \times 3^5\\ &= -2^7 \times 3^7 \times 7.\end{aligned}$$

d $$(a - 2b)^n = \cdots + \binom{n}{4} \times a^{n-4} \times (-2b)^4 + \ldots,$$

$$\begin{aligned}\text{so term in } a^{n-4}b^4 &= \frac{n!}{4! \times (n-4)!} \times 2^4 \times a^{n-4}b^4\\ &= \frac{n \times (n-1) \times (n-2) \times (n-3)}{4 \times 3 \times 2 \times 1} \times 2^4 \times a^{n-4}b^4\\ &= \frac{2n(n-1)(n-2)(n-3)}{3}a^{n-4}b^4.\end{aligned}$$

The values of ${}^n\mathrm{C}_r$ for $r = 0, 1$ and 2

The particular formulae for ${}^n\mathrm{C}_r$ for $n = 0, 1$ and 2 are important enough to be memorised:

$$\begin{aligned}{}^n\mathrm{C}_0 &= \frac{n!}{0!\,n!}\\ &= 1\end{aligned} \qquad \begin{aligned}{}^n\mathrm{C}_1 &= \frac{n!}{1!\,(n-1)!}\\ &= n\end{aligned} \qquad \begin{aligned}{}^n\mathrm{C}_2 &= \frac{n!}{2!\,(n-2)!}\\ &= \tfrac{1}{2}n(n-1)\end{aligned}$$

We saw in Chapter 14 that ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$ for $r = 0, 1, 2, \ldots, n$. It follows therefore that ${}^n\mathrm{C}_n = 1$, ${}^n\mathrm{C}_{n-1} = n$ and ${}^n\mathrm{C}_{n-2} = \frac{1}{2}n(n-1)$.

4 SOME PARTICULAR VALUES OF ${}^n\mathrm{C}_r$

- For all whole numbers n,
  ${}^n\mathrm{C}_0 = {}^n\mathrm{C}_n = 1,$
  ${}^n\mathrm{C}_1 = {}^n\mathrm{C}_{n-1} = n,$
  ${}^n\mathrm{C}_2 = {}^n\mathrm{C}_{n-2} = \frac{1}{2}n(n-1).$
  In particular, the rows begin and end with 1.
- For all whole numbers n, and for $r = 0, 1, 2, \ldots, n$,
  ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}.$
  That is, the rows are reversible.

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Example 10 15C

Find the value of n if:

a ${}^nC_2 = 55$

b ${}^nC_2 + {}^nC_1 + {}^nC_0 = 29$

SOLUTION

We know that ${}^nC_0 = 1$ and ${}^nC_1 = n$ and ${}^nC_2 = \frac{1}{2}n(n-1)$.

a $\frac{1}{2}n(n-1) = 55$

$n^2 - n - 110 = 0$

$(n-11)(n+10) = 0$

Because $n \geq 0$, $n = 11$.

b ${}^nC_2 + {}^nC_1 + {}^nC_0 = 29$

$\frac{1}{2}n(n-1) + n + 1 = 29$

$n^2 - n + 2n + 2 = 58$

$n^2 + n - 56 = 0$

$(n-7)(n+8) = 0$

Because $n \geq 0$, $n = 7$.

Exercise 15C

FOUNDATION

1 Use the result ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$ to evaluate the following. Do not use a calculator — you will need to unroll the factorial symbol. Check your answers against Pascal's triangle.

a ${}^4C_3$ **b** ${}^6C_3$ **c** ${}^9C_1$ **d** ${}^7C_3$

2 Repeat the previous question for these binomial coefficients. Remember the alternative notation $\binom{n}{r} = {}^nC_r$.

a $\binom{5}{2}$ **b** $\binom{8}{8}$ **c** $\binom{11}{10}$ **d** $\binom{10}{4}$

3 Use your calculator button ${}^nC_r$ to evaluate:

a ${}^{15}C_{10}$ **b** $\binom{13}{5}$ **c** ${}^{12}C_7$

d $\binom{12}{3} \div \binom{5}{2}$ **e** $\dfrac{{}^{15}C_8}{{}^6C_4}$ **f** $\dfrac{{}^{19}C_6}{{}^7C_5}$

4 **a** Expand $(1+x)^4$, and hence write down the values of ${}^4C_0$, ${}^4C_1$, ${}^4C_2$, ${}^4C_3$ and ${}^4C_4$.

b Hence find:

i ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$ **ii** ${}^4C_0 - {}^4C_1 + {}^4C_2 - {}^4C_3 + {}^4C_4$

5 Use the values of ${}^nC_r$ from Pascal's triangle to find:

a ${}^6C_0 + {}^6C_2 + {}^6C_4 + {}^6C_6$

b ${}^6C_1 + {}^6C_3 + {}^6C_5$

c ${}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2$

d $({}^5C_0)^2 + ({}^5C_1)^2 + ({}^5C_2)^2 + ({}^5C_3)^2 + ({}^5C_4)^2 + ({}^5C_5)^2$

6 **a** Evaluate:

i $\binom{8}{3}$ and $\binom{8}{5}$, **ii** $\binom{7}{4}$ and $\binom{7}{3}$.

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b If $\binom{n}{3} = \binom{n}{2}$, find the value of n.

c If $\binom{12}{4} = \binom{12}{n}$, find the value of n.

7 **a** By evaluating the LHS and RHS, verify the following result for $n = 8$ and $r = 3$:

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

b Use this identity to solve each equation for n.

i $\binom{5}{3} + \binom{5}{4} = \binom{n}{4}$ **ii** $\binom{n}{7} + \binom{n}{8} = \binom{11}{8}$

8 Find the specified term or coefficient in each expansion.

a For $(1 + x)^{18}$, find the term in x^5.

b For $(1 + 6x)^{12}$, find the term in x^9.

c For $(1 - 7x)^8$, find the term in x^3.

d For $(1 + 3x)^7$, find the coefficient of x^2.

9 Find the specified terms in each expansion.

a For $(2 + x)^7$, find the term in x^2.

b For $\left(x + \frac{1}{2}y\right)^{14}$, find the term in x^9y^5.

c For $\left(\frac{1}{2}x - 3y^2\right)^{11}$, find the term in $x^{10}y^2$.

d For $\left(a - b^{\frac{1}{2}}\right)^{20}$, find the term in a^2b^9.

10 **a** Find $x \neq 0$ if the terms in x^{10} and x^{11} in the expansion of $(5 + 2x)^{15}$ are equal.

b Find $x \neq 0$ if the terms in x^{13} and x^{14} in the expansion of $(2 - 3x)^{17}$ are equal.

DEVELOPMENT

11 **a** In the expansion of $(1 + x)^{16}$, find the ratio of the term in x^{13} to the term in x^{11}.

b Find the ratio of the coefficients of x^{14} and x^5 in the expansion of $(1 + x)^{20}$.

c In the expansion of $(2 + x)^{18}$, find the ratio of the coefficients of x^{10} and x^{16}.

12 **a** Use the binomial theorem to obtain formulae for:

i ${}^nC_0$ **ii** ${}^nC_1$ **iii** ${}^nC_2$ **iv** ${}^nC_3$

b Hence solve each equation for n.

i ${}^9C_2 - {}^nC_1 = {}^6C_3$ **ii** ${}^nC_2 = 36$ **iii** ${}^nC_2 + {}^6C_2 = {}^7C_2$

iv ${}^nC_2 + {}^nC_1 = 22 - {}^nC_0$ **v** ${}^nC_1 + {}^nC_2 = {}^5C_2$ **vi** ${}^nC_3 + {}^nC_2 = 8\,{}^nC_1$

c Use the formula for ${}^nC_2$ to show that ${}^nC_2 + {}^{n+1}C_2 = n^2$, and verify the result on the third column of Pascal's triangle.

13 The expression $(1 + ax)^n$ is expanded in increasing powers of x. Find the values of a and n if the first three terms are:

a $1 + 28x + 364x^2 + \dots$ **b** $1 - \frac{10}{3}x + 5x^2 - \dots$

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14 **a** In the expansion of $(2 + 3x)^n$, the coefficients of x^5 and x^6 are in the ratio 4 : 9. Find the value of n.
b In the expansion of $(1 + 3x)^n$, the coefficients of x^8 and x^{10} are in the ratio 1 : 2. Find the value of n.

15 In the expansion of $\left(x + \frac{1}{x}\right)^{40}\left(x - \frac{1}{x}\right)^{40}$, find the term independent of x. Give your answer in the form ${}^nC_r$, and also correct to four significant figures.

16 [Divisibility problems]
a Use the binomial theorem to show that $7^n + 2$ is divisible by 3, where n is a positive integer. (Hint: Write $7 = 6 + 1$.)
b Use the binomial theorem to show that $5^n + 3$ is divisible by 4, where n is a positive integer.
c Suppose that b, c and n are positive integers, and $a = b + c$. Use the binomial expansion of $(b + c)^n$ to show that $a^n - b^{n-1}(b + cn)$ is divisible by c^2. Hence show that $5^{42} - 2^{48}$ is divisible by 9.

17 **a** Use the binomial theorem to expand $(x + h)^n$.
b Hence use the definition $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ to differentiate x^n from first principles.

ENRICHMENT

18 Use the formula ${}^nC_r = \dfrac{n!}{r!\,(n - r)!}$ to prove that if the second member of a row is a prime number, all the numbers in that row, excluding the 1s, are divisible by it.

19 [These geometrical results should be related to the numbers in Pascal's triangle.]
a Place three points on the circumference of a circle. How many line segments and triangles can be formed using these three points?
b Place four points on the circumference of a circle. How many segments, triangles and quadrilaterals can be formed using these four points?
c What happens if five points are placed on the circle?
d How many pentagons could you form if you placed seven points on the circumference of a circle?

20 [A more general form of the binomial theorem]
a Show that the binomial expansion can be written as

$$(1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!}x^2 + \frac{n(n - 1)(n - 2)}{3!}x^3 + \cdots.$$

b In this form, it can be shown that the expansion is true for negative or fractional values of n, provided that the RHS is regarded as the limit of an infinite sum of powers of x. This is called the *power series expansion* of $(1 + x)^n$. Assuming this, generate the binomial expansions of:

i $\dfrac{1}{1 - x}$ **ii** $\dfrac{1}{(1 - x)^2}$ **iii** $\dfrac{1}{(1 + x)^2}$ **iv** $\sqrt{1 + x}$

c Verify, using your expansions in parts **b i** and **ii**, that $\dfrac{d}{dx}\left(\dfrac{1}{1 - x}\right) = \dfrac{1}{(1 - x)^2}$.
(This assumes that a power series can be differentiated term-by-term.)

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15D Identities in Pascal's triangle

There are a great number of patterns in Pascal's triangle. Some are quite straightforward to recognise and to prove, others are more complicated. They are very important in all applications of the binomial theorem, and in particular they will be important in Year 12 binomial probability.

Each pattern in Pascal's triangle is described by an identity on the binomial coefficients ${}^nC_r$. These identities sometimes have a rather forbidding appearance, and it is important to take the time to interpret each identity as some sort of pattern in Pascal's triangle. Use small values of n such as $n = 3$, $n = 4$ and $n = 5$ to work out what the identity is saying.

Here again is the first part of Pascal's triangle. Each identity that is obtained should be interpreted as a pattern in the triangle and verified there, either before or after the proof is completed.

$n \backslash r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Methods of proof

Methods of proof as well as the identities themselves are the subject of this section. Each proof begins with some form of the binomial expansion. We will begin with three basic approaches — many identities can be proven using two or even all three of these methods.

- Use combinatorics based on the definition of ${}^nC_r$ as the number of r-member subsets of an n-member set.
- Use algebraic methods based on substitution into the binomial expansion
  $$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \cdots + {}^nC_n\, x^0 y^n.$$
  or into the simpler expansion of $(1 + x)^n$.
- The addition property can be proven by both methods, and it in turn can be used to prove some important identities.

As an example, here are all these three methods used to prove that the row indexed by n has sum 2^n, that is,

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n.$$

To illustrate this identity, the row indexed by $n = 5$ adds to

$$1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5.$$

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Proving the result using combinatorics

Let S be an n-member set. The left-hand side ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$ is the sum of the numbers of 0-member subsets, 1-members subsets, 2-members subsets, . . . , n-members subsets of S. That takes account of all the subsets of S, so the LHS is just the number of subsets of S.

We can count the subsets of S a different way. To form a subset, take each element in turn and choose whether it is in or out of the subset. The multiplication principle allows us to calculate the total number of ways of doing this:

1st element	2nd element	3rd element	...	nth element
2	2	2	...	2

$$\begin{aligned}\text{Hence number of subsets} &= 2 \times 2 \times 2 \times \cdots \times 2 \\ &= 2^n, \quad \text{which proves the result.}\end{aligned}$$

Proving the result using algebraic substitution

Start with the binomial expansion

$$(x + y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \dots + {}^nC_n\, x^0 y^n.$$

Now substitute $x = y = 1$, and because $1 + 1 = 2$,

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n.$$

Proving the result using the addition property

The entries in the row indexed by n are the sum of two entries in the row above, and every entry in the row above is used in exactly two sums to calculate entries in the row below. (We can regard the 1s on the end as being $0 + 1$ and $1 + 0$.) Thus the sum of the row indexed by n is double the sum of the row above.

The first row is a single 1, which is 2^0, and the row is indexed by $n = 0$. Thus continuing down the rows one-by-one, the result is clear.

Some further approaches

There are many ways to prove identities in Pascal's triangle. Here are three more:

- Differentiate both sides of the binomial expansion of $(1 + x)^n$, and then substitute. This is done in Questions 1, 2, and 3 below, and in Question 13 in the Review exercise using double differentiation. These results will be needed in Year 12 binomial probability.
- Use the formula for ${}^nC_r$ to express the LHS and RHS in terms of factorials, then simplify. This method is used in Questions 7, 10 and 14 below.
- Equate coefficients on both sides of an algebraic identity. This method is more elaborate, and seems beyond the course, but a couple of examples have been included in the Enrichment section.

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Exercise 15D

FOUNDATION

1 [Substitution and differentiation]
Begin with the binomial expansion for $n = 4$,
$$(1 + x)^4 = {}^4C_0 + {}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3 + {}^4C_4 x^4, \quad (*)$$
and use it to prove the following results. Explain each result in terms of the row 1 4 6 4 1 indexed by $n = 4$ in Pascal's triangle.

a **i** By substituting $x = 1$, show that ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4$.
ii By substituting $x = -1$, show that ${}^4C_0 + {}^4C_2 + {}^4C_4 = {}^4C_1 + {}^4C_3$.
iii Hence, by using the result of part **i**, show that ${}^4C_0 + {}^4C_2 + {}^4C_4 = 2^3$.

b **i** Differentiate both sides of the binomial expansion $(*)$ above.
ii By substituting $x = 1$, show that ${}^4C_1 + 2\,{}^4C_2 + 3\,{}^4C_3 + 4\,{}^4C_4 = 4 \times 2^3$.
iii By substituting $x = -1$, show that ${}^4C_1 - 2\,{}^4C_2 + 3\,{}^4C_3 - 4\,{}^4C_4 = 0$.

2 [Substitution and differentiation]
Use the general binomial expansion,
$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + \cdots + {}^nC_n x^n \quad (*)$$
to prove the following results. Explain each result in terms of the row 1 5 10 10 5 1 indexed by $n = 5$ in Pascal's triangle.

a **i** By substituting $x = 1$, show that ${}^nC_0 + {}^nC_1 + \cdots + {}^nC_n = 2^n$.
ii By substituting $x = -1$, show that ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$
iii Hence, by using the result of part **i**, show that ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = 2^{n-1}$.

b **i** Differentiate both sides of the binomial expansion $(*)$ above.
ii By substituting $x = 1$, show that $1 \times {}^nC_1 + 2 \times {}^nC_2 + \cdots + n \times {}^nC_n = n2^{n-1}$.
iii By substituting $x = -1$, show that $1 \times {}^nC_1 - 2 \times {}^nC_2 + \cdots + (-1)^{n-1} n \times {}^nC_n = 0$.

3 [Substitution and differentiation]
This question follows the same steps as Question 2, and uses the expansion
$$(1 + x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + \cdots + {}^{2n}C_{2n} x^{2n}. \quad (*)$$

a **i** Show that ${}^{2n}C_0 + {}^{2n}C_1 + \cdots + {}^{2n}C_{2n} = 2^{2n}$.
ii Show that ${}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + \cdots + {}^{2n}C_{2n-1} = 2^{2n-1}$.
Check both results in Pascal's triangle, using $n = 3$ and $n = 4$.

b By differentiating both sides of the identity, show that:
i $1 \times {}^{2n}C_1 + 2 \times {}^{2n}C_2 + \cdots + 2n \times {}^{2n}C_{2n} = n2^{2n}$.
ii $1 \times {}^{2n}C_1 - 2 \times {}^{2n}C_2 + \cdots + (-1)^{2n-1} 2n \times {}^{2n}C_{2n} = 0$.
Check both results in Pascal's triangle, using $n = 3$ and $n = 4$.

DEVELOPMENT

For the next three questions, you should use the definition for the symbol ${}^nC_r$ as the number of ways of choosing an r-member subsets from an n-member set.

4 [Combinatorics]
Consider the set $S = \{A, B, C, D, E\}$.

a List all the subsets of $\{A, B, C, D, E\}$ containing 2 letters.
b List all the subsets of $\{A, B, C, D, E\}$ containing 3 letters.
c Consider the subset $\{A, B\}$. What letters have been omitted from the subset?

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d Use the definition of ${}^nC_r$ to explain why ${}^5C_2 = {}^5C_3$.

e Explain why ${}^nC_r = {}^nC_{n-r}$ for any whole numbers n and r with $r \le n$.

5 [Combinatorics]
Consider the set $S = \{A, B, C, D\}$.

a Write down all the subsets of S, then explain why there are exactly 2^4 of them.

b Hence explain why $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4$.

c Explain why $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$, for any whole number n.

6 [A combinatorics proof of the addition property]
Consider the sets

$$S = \{A, B, C, D, E\} \quad \text{and} \quad U = \{A, B, C, D\}.$$

a **i** Write down all the 3-letter subsets of S that do not contain E.

ii Explain why this is a list of all the 3-letter subsets of U.

b **i** Write down all the 3-letter subsets of S that contain E.

ii Explain how to pair them up with all the 2-letter subsets of U.

c Hence explain why ${}^4C_2 + {}^4C_3 = {}^5C_3$.

d Explain why ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$, for any whole numbers n and r with $1 \le r \le n$.

7 [The addition property and the formula for ${}^nC_r$]
In this question, you will prove that if a, b, c and d are any four consecutive terms in any row of Pascal's triangle, then

$$\frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}.$$

a Consider the row 1, 7, 21, 35, 35, 21, 7, 1 indexed by $n = 7$. Show that the identity holds for each sequence a, b, c, d of four consecutive terms from this row.

b Choose four consecutive terms from any other row and show that the identity holds.

c Prove the identity by letting $a = {}^nC_{r-1}$, $b = {}^nC_r$, $c = {}^nC_{r+1}$ and $d = {}^nC_{r+2}$. You will need to use the addition property, then the formula ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$.

8 [Two combinatorics proofs]
We have seen in Question **2a** that substituting $x = -1$ and $x = 1$ into the binomial expansion proves

$${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots = 2^n. \qquad (1)$$

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots = 0, \text{ for } n \ge 1. \qquad (2)$$

Here are combinatorics proofs of these results.

a Let S be an n-member set, and interpret each ${}^nC_r$ as the number of r-member subsets of S. Hence prove the first identity (1).

b To prove the second identity (2), choose a fixed element A in the set S. Pair up each subset U not containing A with the unique subset $U \cup \{A\}$ containing A.

i Explain why the procedure arranges all the subsets of S uniquely into pairs.

ii Explain why one member of each pair has an even number of members, and the other has an odd number of members.

iii Hence prove that ${}^nC_0 + {}^nC_2 + \cdots = {}^nC_1 + {}^nC_3 + \cdots$.

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9 [The hockey-stick identity]

Look at the column indexed by $r = 2$ in Pascal's triangle:

$$^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2 + {}^6C_2 = 1 + 3 + 6 + 10 + 15 = 35 = {}^7C_3. \quad (*)$$

The general form of this well-known *hockey-stick identity* is

$$^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^nC_r = {}^{n+1}C_{r+1} \qquad (**)$$

where n and r are whole numbers with $0 \le r \le n$. Here are proofs of this identity using the addition property, and using a combinatorial proof.

a [A proof using the addition property]

i To prove the particular result $(*)$, start with the right-hand side and write

$$^7C_3 = {}^6C_3 + {}^6C_2$$

then keep expanding the term with $r = 3$. To complete the proof, you will need to use the fact that $^3C_3 = {}^2C_2 = 1$.

ii Generalise this to a proof of the identity $(**)$.

b [A combinatorics proof]

i The number $^7C_3$ is the number of 3-member subsets of a 7-member set S. To choose a 3-member subset U of $S = \{1, 2, 3, 4, 5, 6, 7\}$, make the choice in the following way:

- First choose from S the greatest number k that will be in the subset U. This must be one of the numbers 3, 4, 5, 6 or 7, because U is to have three members, so it will have two numbers smaller than k.
- Then choose the remaining two numbers in U from the $k - 1$ possible numbers $1, 2, \ldots, k - 1$.

Explain why this method of choosing the subset yields the identity $(*)$.

ii Generalise this to a proof of the identity $(**)$.

10 [The formula for ^nC_r]

This question involves the formula $^nC_r = \dfrac{n!}{r!\,(n-r)!}$.

a Prove that $\dfrac{n!}{r!\,(n-r)!} + \dfrac{n!}{(r-1)!\,(n-r+1)!} = \dfrac{(n+1)!}{r!\,(n-r+1)!}$.

b What result have you proven?

ENRICHMENT

11 [Comparing coefficients]

By comparing coefficients of x^n on both sides of the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$, show that

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + ({}^nC_3)^2 + \cdots + ({}^nC_n)^2 = {}^{2n}C_n.$$

Check this identity in Pascal's triangle by adding the squares of the rows indexed by $n = 1, 2, 3, 4, 5$ and 6.

12 [A combinatorial approach to Question 11]

In this question we consider binary words consisting only of the letters A and B.

a Consider a binary word consisting of $a + b$ letters, with A occuring a times and B occuring b times. Show that there are $^{a+b}C_a = {}^{a+b}C_b$ permutations of such a word.

b How many possible permutations are there of a binary word with $2n$ letters, if A and B both occur n times?

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c A word with $2n$ letters may be split down the middle into two words of n letters. Consider the example where two As fall in the first n-letter word.

i How many arrangements are there of the first n-letter binary word with two As?

ii How many arrangements are there of the second n-letter binary word with $n - 2$ of the As and 2 Bs?

iii How many arrangements are there of a ten-letter binary word with two As in the first half and two Bs in the second half?

d Hence prove that

$$({}^{2n}C_0)^2 + ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 + ({}^{2n}C_3)^2 + \cdots + ({}^{2n}C_{2n})^2 = {}^{2n}C_n.$$

13 [Comparing coefficients to prove the addition property]

Prove the addition property in Pascal's triangle by comparing coefficients of x^r on both sides of the identity

$$(1 + x)(1 + x)^n = (1 + x)^{n+1}.$$

14 [An inequality proven using the formula for ${}^nC_r$]

a Prove that ${}^{2n}C_r < {}^{2n}C_{r+1}$, for all whole numbers n and r with $r < n$.

b Prove that ${}^{2n+1}C_r < {}^{2n+1}C_{r+1}$, for all whole numbers n and r with $r < n$.

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15E Enrichment — using the general term

This section involves problems on the general term of a binomial expansion, and should be regarded as Enrichment. Readers wanting more experience working with the binomial expansion will be interested in attempting these problems.

Example 11 15E

a Find the general term in the expansion of $(2x^2 - x^{-1})^{20}$.

b **i** Find the term in x^{34}.

ii Find the term in x^{-5}.

Give each coefficient as a numeral, and factored into primes.

SOLUTION

a
$$\begin{aligned}\text{General term} &= {}^{20}C_r \times (2x^2)^{20-r} \times (-x^{-1})^r \\ &= {}^{20}C_r \times 2^{20-r} \times x^{40-2r} \times (-1)^r \times x^{-r} \\ &= {}^{20}C_r \times 2^{20-r} \times (-1)^r \times x^{40-3r}.\end{aligned}$$

b **i** To obtain the term in x^{34}, $40 - 3r = 34$
$r = 2.$

$$\begin{aligned}\text{Hence the term in } x^{34} &= {}^{20}C_2 \times (2x^2)^{20-2} \times (-x^{-1})^2 \\ &= \frac{20 \times 19}{1 \times 2} \times 2^{18} \times x^{36} \times x^{-2} \\ &= 2^{19} \times 5 \times 19 \times x^{34} \\ &= 49\,807\,360x^{34}.\end{aligned}$$

(Check this on the calculator using ${}^{20}C_2 \times 2^{18}$.)

ii To obtain the term in x^{-5}, $40 - 3r = -5$
$r = 15.$

$$\begin{aligned}\text{Hence the term in } x^{-5} &= {}^{20}C_{15} \times (2x^2)^{20-15} \times (-x^{-1})^{15} \\ &= -\frac{20 \times 19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4 \times 5} \times 2^5 \times x^{10} \times x^{-15} \\ &= -19 \times 3 \times 17 \times 16 \times 2^5 \times x^{-5} \\ &= -2^9 \times 3 \times 17 \times 19 \times x^{-5} \\ &= -496\,128x^{-5}.\end{aligned}$$

(Check this on the calculator using ${}^{20}C_{15} \times 2^5$.)

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Exercise 15E

FOUNDATION

1 Write down the general term for each binomial expansion.

a $(1 + x)^{13}$
b $(1 + 2x)^7$
c $(5 + 7x)^{12}$
d $(2x - y)^9$
e $(x + 2x^{-1})^5$
f $\left(6x - \dfrac{2}{x}\right)^8$

2 Consider the expansion $\left(x^2 + \dfrac{1}{x}\right)^9$.

a Show that each term in the expansion of $\left(x^2 + \dfrac{1}{x}\right)^9$ can be written as ${}^9C_i\, x^{18-3i}$.

b Hence find the coefficients of:

i x^3
ii x^{-3}
iii x^0

3 In the expansion of $(2x^3 + 3x^{-2})^{10}$, the general term is ${}^{10}C_k\,(2x^3)^{10-k}(3x^{-2})^k$.

a Show that this general term can be written as ${}^{10}C_k\, 2^{10-k}\, 3^k\, x^{30-5k}$.

b Hence find the coefficients of these terms, giving your answers factored into primes:

i x^{10}
ii x^{-5}
iii x^0

4 a Show that the general term in the expansion of $\left(\dfrac{x}{2} - \dfrac{5}{x}\right)^{15}$ can be written as

$${}^{15}C_j\,(-1)^j\, 5^j\, 2^{j-15}\, x^{15-2j}.$$

b Hence find, without simplifying, the coefficients of:

i x^{11}
ii x
iii x^{-5}

DEVELOPMENT

5 Find the term independent of x in each expansion.

a $\left(x + \dfrac{3}{x}\right)^8$
b $\left(2x^3 - \dfrac{1}{x}\right)^{12}$
c $(5x^4 - 2x^{-1})^{10}$
d $\left(ax^{-2} + \tfrac{1}{2}x\right)^6$

6 Find the coefficient of the power of x specified in each expansion.

a x^{15} in $\left(x^3 - \dfrac{2}{x}\right)^9$
b x^{-14} in $(x - 3x^{-4})^{11}$
c the constant term in $\left(\dfrac{1}{2x^3} + x\right)^{20}$
d x^7 in $\left(5x^2 + \dfrac{1}{2x}\right)^8$
e x^{-1} in $\left(x^{-1} + \dfrac{2}{7}x\right)^5$

7 Determine the coefficients of the specified term in each expansion.

a For $(3 + x)(1 - x)^{15}$, find the coefficient of the term in x^4.
b For $(2 - 5x + x^2)(1 + x)^{11}$, find the coefficient of the term in x^9.
c For $(x - 3)(x + 2)^{15}$, find the coefficient of the term in x^{12}.
d For $(1 - 2x - 4x^2)\left(1 - \dfrac{3}{x}\right)^9$, find the coefficient of the term in x^0.

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8 **a** Find the coefficient of x in the expansion of $\left(x + \frac{1}{x}\right)^5 \left(x - \frac{1}{x}\right)^4$.

b Find the coefficient of x^2 in the expansion of $\left(x - \frac{1}{x}\right)^9 \left(x + \frac{1}{x}\right)^5$.

c Find the coefficient of y^{-3} in the expansion of $\left(y + \frac{1}{y}\right)^{10} \left(y - \frac{1}{y}\right)^7$.

9 **a** In the expansion of $(2 + ax + bx^2)(1 + x)^{13}$, the coefficients of x^0, x^1 and x^2 are all equal to 2. Find the values of a and b.

b In the expansion of $(1 + x)^n$, the coefficient of x^5 is 1287. Find the value of n by trial and error, and hence find the coefficient of x^{10}.

ENRICHMENT

10 Show that there will always be a term independent of x in the expansion of $\left(x^p + \frac{1}{x^{2p}}\right)^{3n}$, where n is a positive integer, and find that term.

11 **a** Write down the term in x^r in the expansion of $(a - bx)^{12}$.

b In the expansion of $(1 + x)(a - bx)^{12}$, the coefficient of x^8 is zero. Find the value of the ratio $\frac{a}{b}$ in simplest form.

12 **a** By writing it as $\left((1 - x) + x^2\right)^4$, expand $(1 - x + x^2)^4$ in ascending powers of x as far as the term containing x^4.

b In the expansion of $(1 + 3x + ax^2)^n$, where n is a positive integer, the coefficient of x^2 is 0. Find, in terms of n, the value of:

i a,

ii the coefficient of x^3.

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Chapter 15 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 15 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Chapter review exercise

Review

1 Write out the first five rows of Pascal's triangle.

2 Use your answer to the previous question to expand:

a $(1+x)^5$ **b** $(1+2x)^5$ **c** $(1-3x)^3$ **d** $(1-xy)^4$

3 **a** Expand $(1+7x)^5$ as far as the term in x^2.

b Hence find the coefficient of x^2 in the expansion of $(1-5x)(1+7x)^5$.

4 **a** Expand $(1+x)^7$.

b Hence find the first decimal place of 1.02^7.

5 Expand:

a $(3+2x)^4$ **b** $(5-x)^3$ **c** $(2x+4y)^5$ **d** $\left(x-\frac{1}{x}\right)^4$

6 Use the result ${}^n\mathrm{C}_r = \dfrac{n!}{r!\,(n-r)!}$ to evaluate the following. Do not use a calculator — You will need to unroll the factorial symbol. Check your answers against Pascal's triangle you developed in Exercise 16A.

a ${}^5\mathrm{C}_3$ **b** ${}^7\mathrm{C}_4$ **c** ${}^8\mathrm{C}_5$ **d** ${}^{120}\mathrm{C}_{60} \div {}^{119}\mathrm{C}_{60}$

7 Use your calculator button ${}^n\mathrm{C}_r$ to evaluate the following.

a ${}^{10}\mathrm{C}_8$ **b** $\binom{12}{7}$ **c** ${}^9\mathrm{C}_6$

d $\binom{10}{3} \div \binom{3}{2}$ **e** $\dfrac{{}^{10}\mathrm{C}_5}{{}^6\mathrm{C}_2}$ **f** $\dfrac{{}^{16}\mathrm{C}_6}{{}^{10}\mathrm{C}_5}$

8 Use your understanding of the patterns in Pascal's triangle to simplify the following expressions.

a ${}^n\mathrm{C}_0$ **b** ${}^9\mathrm{C}_4 - {}^9\mathrm{C}_5$

c ${}^5\mathrm{C}_0 + {}^5\mathrm{C}_1 + {}^5\mathrm{C}_2 + {}^5\mathrm{C}_3 + {}^5\mathrm{C}_4 + {}^5\mathrm{C}_5$ **d** ${}^{2n+1}\mathrm{C}_n - {}^{2n}\mathrm{C}_n - {}^{2n}\mathrm{C}_{n-1}$

9 Solve ${}^n\mathrm{C}_2 = 28$.

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Review

10 Find the coefficient of x^8 in the expansion of $(1 - 3x)(1 + 5x)^{14}$.

11 **a** Write out the first few terms in the expansion of $(1 + x)^n$.

b By an appropriate substitution, prove that:

$$\binom{n}{0} + 2 \times \binom{n}{1} + 4 \times \binom{n}{2} + 8 \times \binom{n}{3} + \cdots = 3^n$$

c Verify this result for the row indexed by $n = 4$ in Pascal's triangle.

12 **a** Prove that ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots = 2^n$.

b What is the significance of this result for Pascal's triangle?

c How may this result be interpreted as a sum of combinations?

13 **a** Differentiate the identity $(1 + x)^n = {}^nC_0 + {}^nC_1 x + \cdots + {}^nC_n x^n$, then make a substitution, to prove that

$$1 \times {}^nC_1 + 2 \times {}^nC_2 + \cdots + n \times {}^nC_n = n2^{n-1}.$$

b Differentiate the identity a second time, then make a substitution, to prove that

$$1 \times 2 \times {}^nC_2 + 2 \times 3 \times {}^nC_3 + 3 \times 4 \times {}^nC_4 + \cdots + (n-1) \times n \times {}^nC_n$$
$$= (n-1) \times n \times 2^{n-2}.$$

c Check that the row 1, 5, 10, 10, 5, 1 indexed by $n = 5$ obeys these two identities.

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16 Further rates

This short chapter extends the theory of rates in two ways. First, when a problem involves two rates, the chain rule can often be used to generate a relation between the two rates, resulting in clearer and more elegant solutions to problems.

Secondly, *differential equations* are used to place the theory of exponential rates on a surer foundation.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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16A Related rates

Suppose that water is flowing into a large balloon. The radius and volume are both increasing, and if we know the rate at which one is increasing, we should be able to calculate the rate of increase of the other. Such problems were done in Section 9J, but the chain rule provides an easier and more elegant approach.

The rate at which the water is flowing in, and the rate at which the radius is increasing, are called *related rates* because one determines the other.

Using the chain rule to compare rates

We use the chain rule to differentiate with respect to time. This will establish a relation between two rates.

1 RELATED RATES

- Express one quantity as a function of the other quantity.
- Then differentiate with respect to time t using the chain rule.

Be careful to include the correct units when giving the final solution to problems.

Example 1 16A

Water is flowing into a large spherical balloon of radius r and volume V.

a Use the chain rule to find $\dfrac{dV}{dt}$ as a function of r and $\dfrac{dr}{dt}$.

b Suppose that the flow rate is constant at $50\,\text{cm}^3/\text{s}$.

i At what rate is the radius r increasing when the radius is $7\,\text{cm}$?

ii At what rate is the radius increasing when the volume is $4500\pi\,\text{cm}^3$?

c Suppose instead that the flow rate is being continuously adjusted so that the rate of increase of the radius is constant at $2\,\text{cm/s}$.

i Find $\dfrac{dV}{dt}$ as a function of r.

ii What is the radius when the flow rate is $100\,\text{cm}^3/\text{s}$?

SOLUTION

a The volume of a sphere is $V = \dfrac{4}{3}\pi r^3$.

Differentiating with respect to time t, $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$ (chain rule)

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

b i Substituting the known rate $\dfrac{dV}{dt} = 50$ and the radius $r = 7$,

$$50 = 4\pi \times 49 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{25}{98\pi}\ \text{cm/s}$$ ($\doteqdot 0.81\,\text{mm/s}$), the rate of increase of the radius.

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ii When $V = 4500\pi$, $\frac{4}{3}\pi r^3 = 4500\pi$

$$r^3 = 3375$$

$$r = 15,$$

so substituting again, $50 = 4\pi \times 225 \times \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{1}{18\pi} \quad (\doteqdot 0.177\,\text{mm/s}).$$

c **i** Substituting $\frac{dr}{dt} = 2$ into part **a**, $\frac{dV}{dt} = 8\pi r^2$.

ii Substituting $\frac{dV}{dt} = 100$, $100 = 8\pi r^2$

$$r^2 = \frac{25}{2\pi}$$

$$r = \frac{5}{\sqrt{2\pi}}\,\text{cm} \quad (\doteqdot 1.995\,\text{cm}).$$

Some formulae for solids

These formulae were developed in earlier years, and are needed in many of the problems in this section.

2 VOLUME AND SURFACE AREA OF SOLIDS

A sphere:	A cylinder:	A cone:	A pyramid:
$V = \frac{4}{3}\pi r^3$	$V = \pi r^2 h$	$V = \frac{1}{3}\pi r^2 h$	$V = \frac{1}{3} \times \text{base} \times \text{height}$
$A = 4\pi r^2$	$A = 2\pi r^2 + 2\pi rh$	$A = \pi r^2 + \pi r\ell$	$A = \text{sum of faces}$

In the formula for the surface area of a cone, ℓ is the slant height.

The next example uses the formulae for the volume and base area of a cone.

Example 2 16A

Sand is being poured onto the top of a pile at the rate of $3\,\text{m}^3/\text{min}$. The pile always remains in the shape of a cone with semi-vertical angle 45°. Find the rate at which:

a the height,

b the base area,

is changing when the height is 2 metres.

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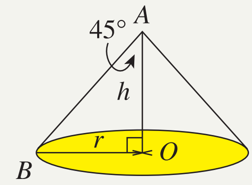
SOLUTION



Let the cone have volume V, height h and base radius r.

The semi-vertical angle is 45°, so $r = h$ (isosceles $\triangle AOB$).

The rate of change of volume is known to be $\dfrac{dV}{dt} = 3\,\text{m}^3/\text{min}$.

a We know that $V = \frac{1}{3}\pi r^2 h$,

and because $r = h$, $V = \frac{1}{3}\pi h^3$.

Differentiating with respect to time, $\dfrac{dV}{dt} = \dfrac{dV}{dh} \times \dfrac{dh}{dt}$ (chain rule)

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}.$$

Substituting, $3 = \pi \times 2^2 \times \dfrac{dh}{dt}$

$$\frac{dh}{dt} = \frac{3}{4\pi}\,\text{m/min} \quad (\doteqdot 0.239\,\text{m/min}).$$

b The base area is $A = \pi h^2$, because $r = h$.

Differentiating, $\dfrac{dA}{dt} = \dfrac{dA}{dh} \times \dfrac{dh}{dt}$

$$= 2\pi h \frac{dh}{dt}.$$

Substituting, $\dfrac{dA}{dt} = 2 \times \pi \times 2 \times \dfrac{3}{4\pi}$

$$= 3\,\text{m}^2/\text{min}.$$

Exercise 16A

FOUNDATION

1 Given that $y = x^3 + x$, differentiate with respect to time, using the chain rule.

a If $\dfrac{dx}{dt} = 5$, find $\dfrac{dy}{dt}$ when $x = 2$.

b If $\dfrac{dy}{dt} = -6$, find $\dfrac{dx}{dt}$ when $x = -3$.

2 A circular oil stain of radius r and area A is spreading on water. Differentiate the area formula with respect to time to show that $\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$. Hence find:

a the rate of increase of area when $r = 40\,\text{cm}$ if the radius is increasing at $3\,\text{cm/s}$,

b the rate of increase of the radius when $r = 60\,\text{cm}$ if the area is increasing at $10\,\text{cm}^2/\text{s}$.

3 A spherical bubble of radius r is shrinking so that its volume V is decreasing at a constant rate of $200\,\text{cm}^3/\text{s}$. Show that $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.

a At what rate is its radius decreasing when the radius is $5\,\text{cm}$?

b What is the radius when the radius is decreasing at $2\pi\,\text{cm/s}$?

c At what rate is the radius decreasing when the volume is $36\pi\,\text{cm}^3$? (Hint: First find the radius when the volume is $36\pi\,\text{cm}^3$.)

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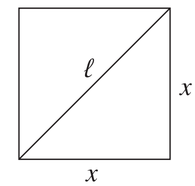
4 The side length x of a square shadow is increasing at 6 cm/s. If the area is A and the length of the diagonal is ℓ, show that

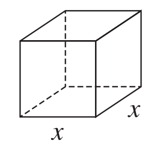
$$\frac{dA}{dt} = 2x\,\frac{dx}{dt} \quad \text{and} \quad \frac{d\ell}{dt} = \sqrt{2}\,\frac{dx}{dt}.$$



Hence find the rates of increase of the area and the diagonal when:

a the side length is 70 cm, **b** the area is $1\,\text{m}^2$.

5 The side length x of a shrinking cube is decreasing at a constant rate of 5 mm per minute.



a Show that the rates of change of volume V, surface area A, and the total edge length ℓ are

$$\frac{dV}{dt} = 3x^2\,\frac{dx}{dt} \quad \text{and} \quad \frac{dA}{dt} = 12x\,\frac{dx}{dt} \quad \text{and} \quad \frac{d\ell}{dt} = 12\,\frac{dx}{dt}.$$

b Find the rate at which volume, surface area and edge length are decreasing when:

i the side length is 30 cm, **ii** the volume is $8000\,\text{cm}^3$.

c Find the side length when the volume is decreasing at $300\,\text{cm}^3/\text{min}$.

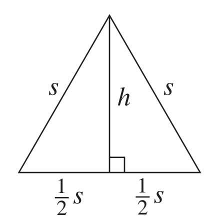
6 A lathe is used to shave down the radius of a cylindrical piece of wood 500 mm long. The radius is decreasing at a rate of 3 mm/min.

a Show that the rate of change of volume is $\frac{dV}{dt} = -3000\pi r$, and find how fast the volume is decreasing when the radius is 30 mm.

b How fast is the circumference decreasing when the radius is:

i 20 mm, **ii** 37 mm?

7 Show that in an equilateral triangle of side length s, the area is $A = \frac{1}{4}s^2\sqrt{3}$ and the height is $h = \frac{1}{2}s\sqrt{3}$.



a Find formulae for the rates of change of area and height in terms of the rate of change of side length.

b Hence find the rate at which the area and the height are increasing when the side length is 12 cm and is increasing at 3 mm/s.

DEVELOPMENT

8 A spherical balloon is to be filled with water so that its surface area increases at a constant rate of $1\,\text{cm}^2/\text{s}$.

a Find, when the radius is 3 cm:

i the required rate of increase of the radius,

ii the rate at which the water is flowing in at that time.

b Find the volume when the volume is increasing at $10\,\text{cm}^3/\text{s}$.

9 Sand being poured from a conveyor belt forms a cone with height h and semi-vertical angle 60°. Show that the volume of the pile is $V = \pi h^3$, and differentiate with respect to t.

a Suppose that the sand is being poured at a constant rate of $0.3\,\text{m}^3/\text{min}$, and let A be the area of the base. Find the rate at which the height is increasing:

i when the height is 4 metres,

ii when the radius is 4 metres.

b Show that $\frac{dA}{dt} = 6\pi h\,\frac{dh}{dt}$, and find the rate of increase of the base area at these times.

c At what rate must the sand be poured if it is required that the height increase at 8 cm/min, when the height is 4 metres?

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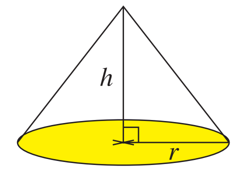
10 Sand is poured at a rate of $0.5\,\text{m}^3/\text{s}$ onto the top of a pile in the shape of a cone, as shown in the diagram. Let the base have radius r, and let the height of the cone be h. The pile always remains in the same shape, with $r = 2h$.



a Find the cone's volume in terms of h, and show that it is the same as the volume of a sphere whose radius is equal to the cone's height.

b Find the rate at which the height is increasing when the radius of the base is 4 metres.

11 A factory has a sliding door 4 metres high. The door can be slid open mechanically at a varying rate. For safety reasons, the rectangular opening is always blocked by a retractable rope that stretches diagonally across the opening from one corner to the opposite corner.

a Let x be the width of the opening, and ℓ the length of the stretched rope. Use Pythagoras' theorem to show that $\ell = \sqrt{16 + x^2}$.

b Use the chain rule to show that $\dfrac{d\ell}{dt} = \dfrac{x}{\sqrt{16 + x^2}}\dfrac{dx}{dt}$, and hence that $\ell\dfrac{d\ell}{dt} = x\dfrac{dx}{dt}$.

c Show that when the opening is a square, $\dfrac{dx}{dt} = \sqrt{2} \times \dfrac{d\ell}{dt}$.

d Find x and ℓ when the rate of increase of ℓ is half the rate of increase of x.

e Explain why $\dfrac{d\ell}{dt}$ is always less than $\dfrac{dx}{dt}$.

f What happens to the ratio $\dfrac{d\ell}{dt}\Big/\dfrac{dx}{dt}$ as $x \to \infty$?

12 An upturned cone of semi-vertical angle 45° is being filled with water at a constant rate of $20\,\text{cm}^3/\text{s}$. Find the rate at which the height of the water, the area of the water surface, and the area of the cone wetted by the water, are increasing when the water height is 50 cm.

13 A square pyramid has height twice its side length s.

a Show that the volume V and the surface area A are $V = \frac{2}{3}s^3$ and $A = \left(\sqrt{17} + 1\right)s^2$.

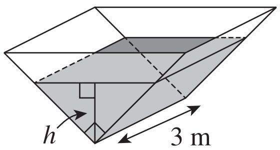
b Hence find the rate at which V and A are decreasing when the side length is 4 metres if the side length is shrinking at $3\,\text{mm/s}$.

14 A ladder 13 metres long rests against a wall, with its base x metres from the wall and its top y metres high. Explain why $x^2 + y^2 = 169$, solve for y, and differentiate with respect to t. Hence find, when the base is 5 metres from the wall:

a the rate at which the top is slipping down when the base is slipping out at $1\,\text{cm/s}$,

b the rate at which the base is slipping out when the top is slipping down at $5\,\text{mm/s}$.

15 The water trough in the diagram is in the shape of an isosceles right triangular prism, 3 metres long. A jackaroo is filling the trough with a hose at the rate of 2 litres per second.



a Show that the volume of water in the trough when the depth is h cm is $V = 300h^2\,\text{cm}^3$.

b Given that 1 litre is $1000\,\text{cm}^3$, find the rate at which the depth of the water is changing when $h = 20\,\text{cm}$.

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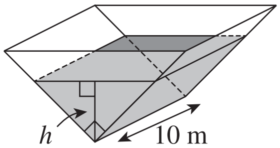
16 A water trough is 10 metres long, with cross section a right isosceles triangle.



a Show that when the water has depth h cm, its volume is $V = 1000h^2$ and its surface area is $A = 2000h$.

b Find the rates at which depth and surface area are increasing when the depth is 60 cm if the trough is filling at 5 litres per minute (remember that 1 litre is $1000\,\text{cm}^3$).

c Find the rates at which the volume and the surface area must increase when the depth is 40 cm, if the depth is required to increase at a constant rate of 0.1 cm/min.

17 [This question shows how much easier and more elegant the approach of this section is compared to the approach taken in Section 9J.]
A balloon of volume $V\,\text{cm}^3$ and radius r cm is being inflated at the rate of $40\,\text{cm}^3/\text{s}$.

a Show that $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$, and hence find $\dfrac{dr}{dt}$ in exact form when $r = 10$ cm.

b Using instead the methods of Section 9J, we begin with $V = 40t$.

i Write r as a function of t, and hence write $\dfrac{dr}{dt}$ as a function of t.

ii Find the time when $r = 10$ cm, and hence find $\dfrac{dr}{dt}$ when $r = 10$ cm.

ENRICHMENT

18 Show that the volume V of a regular tetrahedron, all of whose side lengths are s, is $V = \frac{1}{12}s^3\sqrt{2}$ (all four faces of a regular tetrahedron are equilateral triangles). Hence find the rate of increase of the surface area when the volume is $144\sqrt{2}\,\text{cm}^3$ and is increasing at a rate of $12\,\text{cm}^3/\text{s}$.

19 A large vase has a square base of side length 6 cm, and flat sides sloping outwards at an angle of 120° with the base. Water is flowing in at $12\,\text{cm}^3/\text{s}$. Find, correct to three significant figures, the rate at which the height of water is rising when the water has been flowing in for 3 seconds.

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16B Exponential growth and decay

This section and the next will develop the approaches to exponential growth and decay introduced in Section 11F. The key idea here is the very first property of the exponential function $y = e^x$ with base e that we established — its derivative is again e^x, so the function is equal to its derivative:

$$\text{If } y = e^x, \quad \text{then } \frac{dy}{dx} = y.$$

Because this section concerns rates, replace y by Q and x by t. We also make the statement more general by adding the constants k and A and considering the function $Q = Ae^{kt}$. The derivative of this function is kAe^{kt}, which is k times Q:

$$\text{If } Q = Ae^{kt}, \quad \text{then } \frac{dQ}{dt} = kQ.$$

Thus the rate $\dfrac{dQ}{dt}$ of change of Q is proportional to the value of Q, with k as the constant of proportionality. Thus k is called the *growth constant* of the equation.

The other constant A is the value that Q takes at time zero, because when $t = 0$, $Q = A \times e^0 = A$. Thus A is called the *initial value*, and is often written as Q_0.

3 EXPONENTIAL GROWTH — DIFFERENTIATING

Suppose that a quantity Q is described by an exponential function $Q = Ae^{kt}$.

- Then $\dfrac{dQ}{dt} = kQ$, so that the rate of change of Q is proportional to Q.
- The constant k of proportionality is called the *growth constant*.
- The other constant A is the value of Q at time zero, and is therefore called the *initial value*. It is often denoted by Q_0.

Exponential growth is often called *natural growth* because so many different phenomena in the natural world are described by these equations. The exercises in this book present only a very few of the multitude of such situations.

Exponential decay

The behaviour of $Q = Ae^{kt}$ as time increases depends on the sign of k. For the sake of the following argument, we assume that the initial value A is positive. The situation when A is negative is similar.

- Suppose first that k is positive. Then Q increases without bound as $t \to \infty$, and because $\dfrac{dQ}{dt} = kQ$, we know that Q increases at an increasing rate. This behaviour is *exponential growth*.
- Now suppose that k is negative. Then Q *decays to zero* as $t \to \infty$. This behaviour is *exponential decay*. See Example 3 below.

In practice, when we know that the growth constant is negative, it is usually convenient to write the function as $Q = Ae^{-kt}$, where k is a positive constant. In most of what follows, the term ‘growth’ may mean both growth and decay.

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The exponential growth model

The crucial result of this section is that the converse of Box 3 is also true. That is, if the rate of change of some quantity Q is proportional to Q, then Q is described by a simple exponential function:

4 EXPONENTIAL GROWTH AND DECAY — THE DIFFERENTIAL EQUATION

Suppose conversely that the rate of change of a quantity Q is proportional to Q,

$$\frac{dQ}{dt} = kQ, \quad \text{where } k \text{ is a constant of proportionality.}$$

Then $Q = Q_0 e^{kt}$, where Q_0 is the value of Q at time $t = 0$.

This equation $\frac{dQ}{dt} = kQ$ is our first example of a *differential equation*, which is an equation — it involves the function and its derivative function, and its solutions are functions rather than numbers. In general, differential equations may also involve second and higher derivatives, and relations rather than just function. We will meet another differential equation in the next section, and several more in Year 12.

Extension — proof of the exponential growth model

The proof of this result requires the product rule, which has not yet been used in the context of exponential functions, but which is nevertheless straightforward. We start, however, in an unexpected way by multiplying Q by e^{-kt}, which is the reciprocal of e^{kt}.

Let $y = Qe^{-kt}$.

$$\begin{aligned} \text{Then } \frac{dy}{dt} &= v\frac{du}{dt} + u\frac{dv}{dt} \\ &= e^{-kt} \times kQ + Q \times \left(-ke^{-kt}\right) \\ &= e^{-kt}kQ - kQe^{-kt} \\ &= 0. \end{aligned}$$

Let $u = Q$

and $v = e^{-kt}$.

$$\begin{aligned} \text{Then } \frac{du}{dt} &= \frac{dQ}{dt} \\ &= kQ \quad \text{(given)} \end{aligned}$$

and $\frac{dv}{dt} = -ke^{-kt}$.

Hence y has derivative zero. Thus its graph is a horizontal line, which means that y is a constant function, which we can write as $y = Q_0$, where Q_0 is a constant.

Thus $\quad Qe^{-kt} = Q_0$, and multiplying both sides by e^{kt} gives

$$Q = Q_0 e^{kt}, \text{ as required.}$$

You have now solved your first differential equation. The Enrichment section of Exercise 11F (Question 10) has a contrasting approach to this same method of solution.

What is expected in problems

It is unlikely that a problem would ever ask for the proof above to be reproduced — differential equations are regarded as difficult. What is asked routinely, however, is to prove by substitution that some function satisfies some differential equation, and that the constant is the value at time zero.

Study part **a** of Example 3 very carefully.

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Example 3 16B

The value V of some machinery is depreciating according to the law of exponential decay $\frac{dV}{dt} = -kV$, for some positive constant k. Each year its value drops by 15%.

a Show that $V = V_0 e^{-kt}$ satisfies this differential equation, where V_0 is the initial cost of the machinery.

b Find the value of k, in exact form, and correct to four significant figures.

c Find, correct to four significant figures, the percentage drop in value over five years.

d Find, correct to the nearest 0.1 years, when the value has dropped by 90%.

SOLUTION

a First, substituting $V = V_0 e^{-kt}$ into $\frac{dV}{dt} = -kV$,

$$\begin{aligned} \text{LHS} &= \frac{d}{dt}\left(V_0 e^{-kt}\right) \\ &= -kV_0 e^{-kt}, \end{aligned} \qquad \begin{aligned} \text{RHS} &= -k \times V_0 e^{-kt} \\ &= \text{LHS}. \end{aligned}$$

Secondly, substituting $t = 0$ gives $V = V_0 e^0 = V_0$, as required.

b When $t = 1$, $V = 0.85\,V_0$, so

$$\begin{aligned} 0.85\,V_0 &= V_0 e^{-k} \\ e^{-k} &= 0.85 \\ k &= -\log_e 0.85 \\ &\doteqdot 0.1625. \end{aligned}$$

c When $t = 5$,

$$\begin{aligned} V &= V_0 e^{-5k} \\ &\doteqdot 0.4437\,V_0, \end{aligned}$$

so the value has dropped by about 55.63% over the 5 years.

d Put $V = 0.1\,V_0$.

Then

$$\begin{aligned} V_0 e^{-kt} &= 0.1\,V_0 \\ -kt &= \log_e 0.1 \\ t &\doteqdot 14.2 \text{ years}. \end{aligned}$$

The 'growth constant'

Suppose that an insect population P is growing according to the equation $P = P_0 e^{3t}$, where time is in years. We have seen that

$$\frac{dP}{dt} = 3P_0 e^{3t}, \qquad \text{so that } \frac{dP}{dt} = 3P,$$

which means that the growth constant $k = 3$ is a constant of proportionality in the differential equation describing exponential growth. Because $\frac{dP}{dt}$ is the instantaneous growth rate, this growth constant $k = 3$ can be described as the *instantaneous proportional growth rate*. Its units are 'per year'.

Be careful not to interpret $k = 3$ as the *average proportional growth rate over the year*. The table of values to the right shows that the population, which is described by $P = P_0 e^{3t}$, grows by a factor of about 20 over the year, not 3.

t	0	1
P	P_0	$e^3 P_0 \doteqdot 20P_0$

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Average rates of growth are represented by chords on the exponential graph, and instantaneous rates of growth are represented by tangents.

Neither is $k = 3$ the ordinary *instantaneous growth rate* — if there were 1 million insects at time zero, the growth rate at time zero would be 3 million insects per year. And it is certainly not the ordinary *average growth rate* — that average growth rate is about 19 million insects per year.

Thus there are four different rates — two instantaneous rates, one ordinary and one proportional, and two average rates, one ordinary and one proportional. Example 4 on inflation asks for all four of these rates.

Example 4 16B

[Four different rates associated with exponential growth]

The cost C of building an average house is rising according to the exponential growth equation $C = 150\,000e^{0.08t}$, where t is time in years since 1st January 2000.

a Show that $\frac{dC}{dt}$ is proportional to C, and find the constant of proportionality (the 'growth constant', or the 'instantaneous proportional growth rate').

b Find the instantaneous rates at which the cost is increasing on 1st January 2000, 2001, 2002 and 2003, correct to the nearest dollar per year, and show that each rate is about 0.0833 times the preceding rate.

c Find the value of C when $t = 1$, $t = 2$ and $t = 3$, and the average increases in cost over the first year, the second year and the third year, correct to the nearest dollar per year, and show that each rate is about 0.0833 times the preceding rate.

d Show that the average increase in cost over the first year, the second year and the third year, expressed as a proportion of the cost at the start of that year, is constant.

SOLUTION

a Differentiating, $\frac{dC}{dt} = 0.08 \times 150\,000 \times e^{0.08t} = 0.08C$,

so $\frac{dC}{dt}$ is proportional to C, with constant of proportionality 0.08.

b Substituting into $\frac{dC}{dt} = 12\,000e^{0.08t}$,

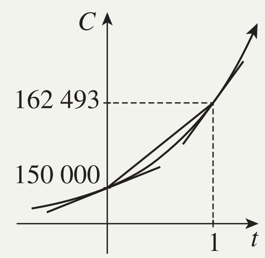
on 1st January 2000, $\frac{dC}{dt} = 12\,000e^{0} = \$12\,000$ per year,

on 1st January 2001, $\frac{dC}{dt} = 12\,000e^{0.08} \doteqdot \$12\,999$ per year,

on 1st January 2002, $\frac{dC}{dt} = 12\,000e^{0.16} \doteqdot \$14\,082$ per year,

on 1st January 2003, $\frac{dC}{dt} = 12\,000e^{0.24} \doteqdot \$15\,255$ per year.

Each is $e^{0.08} \doteqdot 1.0833$ times the preceding rate.



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c The values of C when $t = 0, t = 1, t = 2$ and $t = 3$ are respectively

$$\$150000,\ 150000e^{0.08},\ 150000e^{0.16}\ \text{and}\ 150000e^{0.24},$$

so over the first year, increase $= 150000(e^{0.08} - 1) \doteqdot \$12493$,

over the second year, increase $= 150000(e^{0.16} - e^{0.08})$

$= 150\,000 \times e^{0.08}(e^{0.08} - 1) \doteqdot \$13\,534$,

over the third year, increase $= 150000(e^{0.24} - e^{0.16})$

$= 150000 \times e^{0.16}(e^{0.08} - 1) \doteqdot \$14\,661$.

Again, each is $e^{0.08} \doteqdot 1.0833$ times the preceding rate.

d The three proportional increases are

over the first year, $\dfrac{150000(e^{0.08} - 1)}{150000} = e^{0.08} - 1$,

over the second year, $\dfrac{150000 \times e^{0.08} \times (e^{0.08} - 1)}{150000 \times e^{0.08}} = e^{0.08} - 1$,

over the third year, $\dfrac{150000 \times e^{0.16} \times (e^{0.08} - 1)}{150000 \times e^{0.16}} = e^{0.08} - 1$,

so the proportional increases are all equal to $e^{0.08} - 1 \doteqdot 8.33\%$.

Exercise 16B

FOUNDATION

1 A quantity Q varies with time according to the function $Q = 400e^{3t}$.

a Show that Q satisfies the differential equation $\dfrac{dQ}{dt} = 3Q$, and find the value of Q at time zero.

b **i** Find Q and $\dfrac{dQ}{dt}$ when $t = 2$, in exact form and correct to four significant figures.

ii Find t when $Q = 20000$, in exact form and correct to four significant figures.

iii Find t when $\dfrac{dQ}{dt} = 20000$, in exact form and correct to four significant figures.

2 A quantity Q varies with time according to the function $Q = 1000e^{-2t}$

a Prove that Q satisfies the differential equation $\dfrac{dQ}{dt} = -2Q$, and find the value of Q at time zero.

b **i** Find Q and $\dfrac{dQ}{dt}$ when $t = 0.4$, in exact form and correct to the nearest unit.

ii Find t when $Q = 40$, in exact form and correct to three decimal places.

iii Find t when $\dfrac{dQ}{dt} = -40$, in exact form and correct to four significant figures.

3 It is found that under certain conditions, the number of bacteria in a sample grows exponentially according to the differential equation $\dfrac{dB}{dt} = \frac{1}{10}B$, where t is measured in hours.

a Show that $B = B_0e^{0.1t}$ satisfies the differential equation and the initial condition.

b Initially, the number of bacteria is estimated to be 1000. Find how many bacteria there are after three hours. Answer correct to the nearest bacterium.

c Use parts **a** and **b** to find how fast the number of bacteria is growing after three hours.

d By solving $1000e^{0.1t} = 10000$, find, correct to the nearest hour, when there will be 10 000 bacteria.

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4 Twenty grams of salt is gradually dissolved in hot water. Assume that the amount S left undissolved after t minutes satisfies the exponential decay model, that is, $\frac{dS}{dt} = -kS$, for some positive constant k.

a Show that $S = 20e^{-kt}$ satisfies the differential equation.

b Given that only half the salt is left after three minutes, show that $k = \frac{1}{3}\log_e 2$.

c Find how much salt is left after five minutes, and how fast the salt is dissolving then. Answer correct to two decimal places.

d After how long, correct to the nearest second, will there be 4 grams of salt left undissolved?

e Find the amounts of undissolved salt when $t = 0, 1, 2,$ and 3 minutes, correct to the nearest 0.01 g, and show that each amount is a fixed multiple of the previous amount.

5 The population P of a rural town has been declining over the last few years. Five years ago the population was estimated at 30 000 and today it is estimated at 21 000.

a Assume that the population obeys the exponential decay model $\frac{dP}{dt} = -kP$, for some positive constant k, where t is time in years from the first estimate, and show that $P = 30000e^{-kt}$ satisfies this differential equation and the initial condition.

b Find the value of the positive constant k.

c Estimate the population ten years from now.

d The local bank has estimated that it will not be profitable to stay open once the population falls below 16 000. When will the bank close?

6 A radioactive substance decays with a half-life of 1 hour. The initial mass is 80 g.

a Write down the mass when $t = 0, 1, 2$ and 3 hours (no need for calculus here).

b Write down the average loss of mass during the 1st, 2nd and 3rd hour, then show that the percentage loss of mass per hour during each of these hours is the same.

c The mass M at any time satisfies the equation of exponential decay $M = M_0 e^{-kt}$, where k is a constant. Find the values of M_0 and k.

d Show that $\frac{dM}{dt} = -kM$, and find the instantaneous rates of mass loss when $t = 0, t = 1, t = 2$ and $t = 3$.

e Sketch the graph of M as a function of t, and add the relevant chords and tangents.

DEVELOPMENT

7 A chamber is divided into two identical parts by a porous membrane. The left part of the chamber is initially more full of a liquid than the right. The liquid is let through at a rate proportional to the difference x in the levels, measured in centimetres. Thus $\frac{dx}{dt} = -kx$.

a Show that $x = Ae^{-kt}$ is a solution of this equation.

b Given that the initial difference in heights is 30 cm, find the value of A.

c The level in the right compartment has risen 2 cm in five minutes, and the level in the left has fallen correspondingly by 2 cm.

i What is the value of x at this time?

ii Hence find the value of k.

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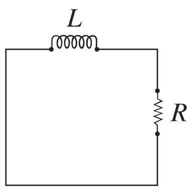
8 A current i_0 is established in the circuit shown on the right. When the source of the current is removed, the current in the circuit decays according to the equation $L\dfrac{di}{dt} = -iR$ where t is time in seconds.



a Show that $i = i_0 e^{-\frac{R}{L}t}$ is a solution of this equation.

b Given that the resistance is $R = 2$ and that the current in the circuit decays to 37% of the initial current in a quarter of a second, find L. $\left(\text{Notice that } 37\% \doteqdot \frac{1}{e}.\right)$

9 A tank in the shape of a vertical hexagonal prism with base area A is filled to a depth of 25 metres. The liquid inside is leaking through a small hole in the bottom of the tank, and it is found that the change in volume at any instant t hours after the tank starts leaking is proportional to the depth h metres, that is, $\dfrac{dV}{dt} = -kh$.

a Show that $\dfrac{dh}{dt} = -\dfrac{kh}{A}$.

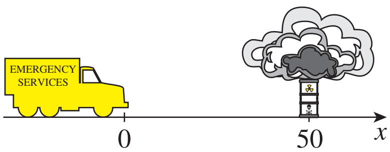
b Show that $h = h_0 e^{-\frac{k}{A}t}$ is a solution of this equation.

c What is the value of h_0?

d Given that the depth in the tank is 15 metres after 2 hours, find $\dfrac{k}{A}$.

e How long will it take to empty to a depth of just 5 metres? Answer correct to the nearest minute.

10 The emergency services are dealing with a toxic gas cloud around a leaking gas cylinder 50 metres away. The prevailing conditions mean that the concentration C in parts per million (ppm) of the gas increases proportionally to the concentration as one moves towards the cylinder. That is, $\dfrac{dC}{dx} = kC$, where x is the distance in metres towards the cylinder from their current position.



a Show that $C = C_0 e^{kx}$ is a solution of the above equation.

b At the truck, where $x = 0$, the concentration is $C = 20\,000$ ppm. Five metres closer, the concentration is $C = 22\,500$ ppm. Use this information to find the values of the constants C_0 and k. Give k exactly, then correct to three decimal places.

c Find the gas concentration at the cylinder, correct to the nearest part per million.

d The accepted safe level for this gas is 30 parts per million. The emergency services calculate how far back from the cylinder they should keep the public, rounding their answer up to the nearest 10 metres.

- **i** How far do they keep the public back?
- **ii** Why do they round their answer up and not round it in the normal way?

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11 **a** The price of shares in Bravo Company rose in one year from \$5.25 to \$6.10.

i Assuming the exponential growth model, show that the share price in cents is given by $B = 525e^{kt}$, where t is measured in months.

ii Find the value of k.

b A new information technology company, ComIT, enters the stock market at the same time with shares at \$1, and by the end of the year these are worth \$2.17.

i Again assuming the exponential growth model, show that the share price in cents is given by $C = 100e^{\ell t}$.

ii Find the value of ℓ.

c During which month will the share prices in both companies be equal?

d What will be the (instantaneous) rate of increase in ComIT shares at the end of that month, correct to the nearest cent per month?

12 Given that $y = A_0e^{kt}$, it is found that at $t = 1$, $y = \frac{3}{4}A_0$.

a Show that it is not necessary to evaluate k in order to find y when $t = 3$.

b Find $y(3)$ in terms of A_0.

ENRICHMENT

13 The growing population of rabbits on Br'er Island can initially be modelled by the exponential growth model, with $N = N_0e^{\frac{t}{2}}$. When the population reaches a critical value, $N = N_c$, the model changes to $N = \dfrac{B}{C + e^{-t}}$, with the constants B and C chosen so that both models predict the same rate of growth at that time.

a Find the values of B and C in terms of N_c and N_0.

b Show that the population reaches a limit, and find that limit in terms of N_c.

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16C Modified exponential growth and decay

In many situations, the rate of change of a quantity Q is proportional not to Q itself, but to the amount $Q - B$ by which Q exceeds some fixed value B. Mathematically, this means shifting the graph upwards by B, which is easily done using theory previously established.

The general case

Here is the general statement of the situation.

5 MODIFIED EXPONENTIAL GROWTH

Suppose that the rate of change of a quantity Q is proportional to the difference $Q - B$, where B is some fixed value of Q:

$$\frac{dQ}{dt} = k(Q - B), \text{ where } k \text{ is a constant of proportionality.}$$

Then $Q = B + Ae^{kt}$, where A is the value of $Q - B$ at time zero.

Extension — proof of the modified exponential growth model

The proof of the modified exponential growth model in the box above uses the ordinary exponential growth model from the previous section.

Let $y = Q - B$ be the difference between Q and B.

Then $\frac{dy}{dt} = \frac{dQ}{dt} - 0$, because B is a constant.

so $\frac{dy}{dt} = k(Q - B)$, because we are given that $\frac{dQ}{dt} = k(Q - B)$,

that is, $\frac{dy}{dt} = ky$, because we defined y by $y = Q - B$.

Hence, using the previous theory of exponential growth,

$$y = y_0 e^{kt}, \text{ where } y_0 \text{ is the value of } y \text{ at time zero,}$$

and substituting $y = Q - B$,

$$Q = B + Ae^{kt}, \text{ where } A \text{ is the value of } Q - B \text{ at time zero.}$$

What is expected in problems

It is unlikely that a problem would ask for the proof above to be reproduced. Questions routinely ask to prove by substitution that some function satisfies some differential equation, and to establish the value of the constant in terms of the initial value.

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Example 5 16C

The large French tapestries that are hung in the permanently air-conditioned La Châtille Hall have a normal water content W of 8 kg. When the tapestries were removed for repair, they dried out in the workroom atmosphere. When they were returned, the rate of increase of the water content was proportional to the difference from the normal 8 kg, that is,

$$\frac{dW}{dt} = k(8 - W), \text{ for some positive constant } k \text{ of proportionality.}$$

a Prove that for any constant A, the function $W = 8 - Ae^{-kt}$ is a solution of the differential equation.

b Weighing established that $W = 4$ initially, and $W = 6.4$ after 3 days.

i Find the values of A and k.

ii Find when the water content has risen to 7.9 kg.

iii Find the rate of absorption of the water after 3 days.

iv Sketch the graph of water content against time.

SOLUTION

a Substituting $W = 8 - Ae^{-kt}$ into $\frac{dW}{dt} = k(8 - W)$,

$$\text{LHS} = \frac{dW}{dt} = kAe^{-kt}, \qquad \text{RHS} = k\left(8 - 8 + Ae^{-kt}\right) = \text{LHS, as required.}$$

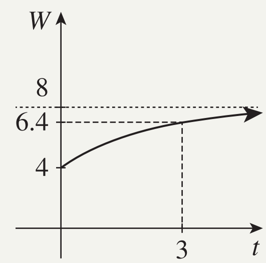
b i When $t = 0$, $W = 4$, so
$$4 = 8 - A$$
$$A = 4.$$

When $t = 3$, $W = 6.4$, so
$$6.4 = 8 - 4e^{-3k}$$
$$e^{-3k} = 0.4$$
$$k = -\tfrac{1}{3}\log 0.4 \quad \text{(calculate and leave in memory).}$$

ii Put $W = 7.9$, then $7.9 = 8 - 4e^{-kt}$
$$e^{-kt} = 0.025$$
$$t = -\frac{1}{k}\log 0.025$$
$$\doteqdot 12 \text{ days.}$$

iii We know that $\frac{dW}{dt} = k(8 - W)$.

When $t = 3$, $W = 6.4$, so $\frac{dW}{dt} = k \times 1.6$
$$\doteqdot 0.49 \text{ kg/day.}$$



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Newton's law of cooling

Newton's law of cooling is a well-known example of exponential decay. When a hot object is placed in a cool environment, the rate at which the temperature decreases is proportional to the difference between the temperature T of the object and the temperature E of the environment:

$$\frac{dT}{dt} = -k(T - E), \text{ where } k \text{ is a constant of proportionality.}$$

The same law applies to a cold body placed in a warmer environment.

Example 6 16C

In a kitchen where the temperature is 20°C, Stanley takes a kettle of boiling water off the stove at time zero. Five minutes later, the temperature of the water is 70°C.

a Show that $T = 20 + 80e^{-kt}$ satisfies the cooling equation $\frac{dT}{dt} = -k(T - 20)$, and gives the correct value of 100°C at $t = 0$. Then find k.

b How long will it take for the water temperature to drop to 25°C?

c Graph the temperature–time function.

SOLUTION

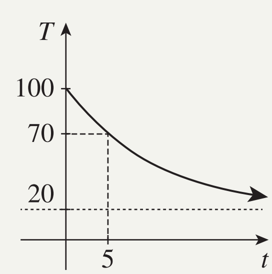
a Substituting $T = 20 + 80e^{-kt}$ into $\frac{dT}{dt} = -k(T - 20)$,

$$\text{LHS} = \frac{dT}{dt} \qquad\qquad \text{RHS} = -k(20 + 80e^{kt} - 20)$$
$$= -80ke^{-kt}, \qquad\qquad = \text{LHS}, \quad \text{as required.}$$

Substituting $t = 0$, $T = 20 + 80 \times 1 = 100$, as required.

When $t = 5$, $T = 70$, so
$$70 = 20 + 80e^{-5k}$$
$$e^{-5k} = \tfrac{5}{8}$$
$$k = -\tfrac{1}{5}\log\tfrac{5}{8}.$$

b Substituting $T = 25$,
$$25 = 20 + 80e^{-kt}$$
$$e^{-kt} = \tfrac{1}{16}$$
$$t = -\frac{1}{k}\log\tfrac{1}{16}$$
$$\doteqdot 29\tfrac{1}{2} \text{ minutes.}$$



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Exercise 16C

FOUNDATION

1 a Suppose that $P = 10000 + 2000e^{0.1t}$.

i Show that $\frac{dP}{dt} = \frac{1}{10}(P - 10000)$.

ii Find the value of P when $t = 0$, and state what happens as $t \to \infty$.

b Suppose that $P = 10000 + 2000e^{-0.1t}$.

i Show that $\frac{dP}{dt} = -\frac{1}{10}(P - 10000)$.

ii Find the value of P when $t = 0$, and state what happens as $t \to \infty$.

c Suppose that $P = 10000 - 2000e^{-0.1t}$.

i Show that $\frac{dP}{dt} = -\frac{1}{10}(P - 10000)$.

ii Find the value of P when $t = 0$, and state what happens as $t \to \infty$.

2 The rate of increase of a population P of green and purple flying bugs is proportional to the excess of the population over 2000, that is, $\dfrac{dP}{dt} = k(P - 2000)$, for some constant k.

Initially the population is 3000, and three weeks later the population is 8000.

a Show that $P = 2000 + Ae^{kt}$ satisfies the differential equation, where A is constant.

b By substituting $t = 0$ and $t = 3$, find the values of A and k.

c Find the population after seven weeks, correct to the nearest ten bugs.

d Find when the population reaches 500000, correct to the nearest 0.1 weeks.

3 During the autumn, the rate of decrease of the fly population F in Wanzenthal is proportional to the excess over 30000, that is, $\dfrac{dF}{dt} = -k(F - 30000)$, for some positive constant k. Initially there are 1000000 flies in the valley, and ten days later the number has halved.

a Show that $F = 30000 + Be^{-kt}$ satisfies the differential equation, where B is constant.

b Find the values of B and k.

c Find the population after four weeks, correct to the nearest 1000 flies.

d Find when the population reaches 35000, correct to the nearest day.

4 A hot cup of coffee loses heat in a colder environment according to Newton's law of cooling, $\dfrac{dT}{dt} = -k(T - T_e)$, where T is the temperature of the coffee in degrees Celsius at time t minutes, T_e is the temperature of the environment, and k is a positive constant.

a Show that $T = T_e + Ae^{-kt}$ is a solution of this equation, for any constant A.

b I make myself a cup of coffee and find that it has already cooled from boiling to 90°C. The temperature of the air in the office is 20°C. What are the values of T_e and A?

c The coffee cools from 90°C to 50°C after six minutes. Find k.

d Find how long, correct to the nearest second, it will take for the coffee to reach 30°C.

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5 A piece of meat is taken out of the freezer at $-9°$C into the air at $25°$C. The rate at which the meat warms follows Newton's law of cooling $\dfrac{dT}{dt} = -k(T - 25)$, with time t measured in minutes.

a Show that $T = 25 - Ae^{-kt}$ is a solution of this equation, and find the value of A.

b The meat reaches $8°$C in 45 minutes. Find the value of k.

c Find the temperature it reaches after another 45 minutes.

6 A 1 kilogram weight falls from rest through the air. When both gravity and air resistance are taken into account, it is found that its velocity is given by $v = 160\left(1 - e^{-\frac{t}{16}}\right)$. The velocity v is measured in metres per second, and downwards has been taken as positive.

a Confirm that the initial velocity is zero. Show that the velocity is always positive for $t > 0$, and explain this physically.

b Show that $\dfrac{dv}{dt} = \dfrac{1}{16}(160 - v)$, and explain what this represents.

c What velocity does the body approach?

d How long does it take to reach one eighth of this speed?

DEVELOPMENT

7 **a** Suppose that $P = B + Ae^{-kt}$, where B, A and k are constants with $A > 0$ and $k > 0$. What happens to P as $t \to \infty$ and as $t \to -\infty$?

b Suppose that $P = B + Ae^{kt}$, where B, A and k are constants with $A > 0$ and $k > 0$. What happens to P as $t \to \infty$ and as $t \to -\infty$?

8 A chamber 30 cm high is divided into two identical parts by a porous membrane. The left compartment is initially full and the right compartment is empty. The liquid is let through from left to right at a rate proportional to the difference between the level x cm in the left compartment and the average level. The time t is measured in minutes.

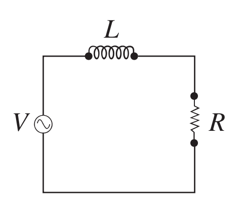
a Explain why $\dfrac{dx}{dt} = k(15 - x)$, for some positive constant k of proportionality.

b Show that $x = 15 + Ae^{-kt}$ is a solution of this equation and find the value of A.

c What value does the level in the left compartment approach?

d The level in the right compartment has risen 6 cm in 5 minutes. Find the value of k.

9 The diagram shows a simple circuit containing an inductor L and a resistor R with a constant applied voltage V. Circuit theory tells us that $V = RI + L\dfrac{dI}{dt}$, where I is the current at time t seconds (we will ignore the complicated units).



a Prove that $I = \dfrac{V}{R} + Ae^{-\frac{R}{L}t}$ is a solution of the differential equation, for any constant A.

b Given that initially the current is zero, find A in terms of V and R.

c Find the limiting value of the current in the circuit.

d Given that $R = 12$ and $L = 8 \times 10^{-3}$, find how long it takes for the current to reach half its limiting value. Give your answer correct to three significant figures.

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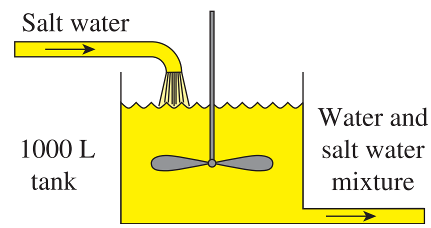
10 When a person takes a pill, the medicine is absorbed into the bloodstream at a rate given by $\frac{dM}{dt} = -k(M - a)$, where M is the concentration of the medicine in the blood t minutes after taking the pill, and a and k are constants.

- **a** Show that $M = a(1 - e^{-kt})$ satisfies the given equation, and gives an initial concentration of zero.
- **b** What is the limiting value of the concentration?
- **c** Find k if the concentration reaches 99% of the limiting value after 2 hours.
- **d** The patient starts to notice relief when the concentration reaches 10% of the limiting value. When will this occur, correct to the nearest second?

11 In the diagram, a tank initially contains 1000 litres of pure water. Salty water begins pouring into the tank from a pipe, and a stirring blade ensures that it is always completely mixed with the pure water. A second pipe draws the mixture off at the same rate, so that there is always a total of 1000 litres in the tank.



- **a** If the salty water entering the tank contains 2 grams of salt per litre, and is flowing in at the constant rate of w litres/min, how much salt is entering the tank per minute?
- **b** If there are Q grams of salt in the tank at time t, how much salt is in 1 litre at time t?
- **c** Hence write down the amount of salt leaving the tank per minute.
- **d** Use the previous parts to show that $\frac{dQ}{dt} = -\frac{w}{1000}(Q - 2000)$.
- **e** Show that $Q = 2000 + Ae^{-\frac{wt}{1000}}$ is a solution of this differential equation.
- **f** Determine the value of A.
- **g** What happens to Q as $t \to \infty$?
- **h** If there is 1 kg of salt in the tank after $5\frac{3}{4}$ hours, find w.

ENRICHMENT

12 [Alternative proof of the modified exponential growth theorem]

Suppose that a quantity Q varies with time according to $\frac{dQ}{dt} = k(Q - B)$, where k is a constant and B is some fixed value of Q.

- **a** Let $Q - B = Ae^{kt}$, where A is a function of t. Use the product rule to show that
$$\frac{d}{dt}(Q - B) = kAe^{kt} + e^{kt}\frac{dA}{dt}.$$
- **b** Hence show that A is a constant, equal to the value of $Q - B$ at time zero.

13 It is assumed that the population of a newly introduced species on an island will usually grow or decay in proportion to the difference between the current population P and the ideal population I, that is, $\frac{dP}{dt} = k(P - I)$, where k may be positive or negative.

- **a** Prove that $P = I + Ae^{kt}$ is a solution of this equation.
- **b** Initially 10 000 animals are released. A census is taken 7 weeks later and again at 14 weeks, and the population grows to 12 000 and then to 18 000. Use these data to find the values of I, A and k.
- **c** Predict the population after 21 weeks.

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14 [The coffee drinkers' problem]
Two coffee drinkers pour themselves a cup of coffee each just after the kettle has boiled. Mia adds milk from the fridge, stirs it in and then waits for it to cool. Adam waits for the coffee to cool first, then just before drinking adds the milk and stirs. If they both begin drinking at the same time, whose coffee is cooler? Justify your answer mathematically. Assume that the room temperature is colder than the coffee and that the milk is colder still. Also assume that when the milk is added and stirred, the temperature drops by a fixed percentage of the difference between the temperatures of the coffee and the milk.

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Chapter 16 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 16 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 The side lengths of a square are increasing at 3 mm per hour.

a Find the rate at which the area is increasing when the side length is 10 cm.

b Find the rate at which the diagonal is increasing when the area is $64\,\text{cm}^2$.

2 Coal is pouring from a high conveyor belt onto a conical pile. The conical pile has a semi-vertical angle of 30°, and at time t the radius of the base is r.

a Find the height of the cone in terms of r, and hence show that the volume V of the cone is

$V = \dfrac{\pi r^3\sqrt{3}}{3}$. (See Box 2 for the various formulae.)

b Differentiate to express $\dfrac{dV}{dt}$ in terms of $\dfrac{dr}{dt}$.

c Find the slant height ℓ in terms of r, and hence show that the area A of the curved surface of the cone is $A = 2\pi r^2$. (See Box 2.)

d Differentiate to express $\dfrac{dA}{dt}$ in terms of $\dfrac{dr}{dt}$.

e Hence show that $\dfrac{dA}{dt} = \dfrac{4}{r\sqrt{3}}\dfrac{dV}{dt}$.

f If the coal is pouring onto the pile at a rate of $5\,\text{m}^3/\text{s}$, find the rate at which the radius and area are increasing when the radius is 4 metres.

3 Mice are multiplying in Mosman. Their population M was estimated to be 80000 at the start of 2010 and 130000 at the start of 2015. Assume an exponential growth model,

$$\frac{dM}{dt} = kM, \quad \text{where } k \text{ is a constant and } t \text{ is time in years after 2010.}$$

a Show that $M = 80000e^{kt}$ satisfies the differential equation, and has initial value 80 000.

b Find k, and predict the population at the start of 2028, correct to the nearest 1000.

c Predict in what year the population will first exceed 1000000.

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Review

4 Caesium-137 has a half-life of about 30.2 years, meaning that half its mass decays into something else in 30.2 years. It was deposited in very tiny, but highly toxic, glass-like grains around Fukushima in Japan as a result of the nuclear power-plant disaster there on 11th March 2010. Radioactive decay follows the exponential decay model $\frac{dM}{dt} = -kM$, where M is the mass remaining after time t years from an initial mass M_0.

a Prove that $M = M_0e^{-kt}$ satisfies the differential equation and has initial mass M_0.

b Use the half-life given above to calculate k.

c Find what percentage of the original caesium-137 remains after 100 years, correct to the nearest 0.01%.

d How long will it take for the mass of caesium-137 to decrease to 1% of its original level, correct to the nearest year?

5 A steel bar is taken out of a fire that has a temperature of 500°C. Newton's law of cooling tells us that the temperature T after t minutes satisfies the differential equation

$$\frac{dT}{dt} = k(T - E), \quad \text{where } k \text{ is a constant and } E \text{ is room temperature.}$$

a Explain why the constant k is negative.

b Show that $T = E + Ae^{kt}$ satisfies the differential equation, and that $A = 500 - E$.

c After 6 minutes, the bar has cooled to 250°C.

i If room temperature is 0°C, find A and k, and find the temperature after 15 minutes, correct to the nearest degree.

ii If room temperature is 40°C, find A and k, and find the temperature after 15 minutes, correct to the nearest degree.

6 Goats have been introduced for the second time onto Goat Island, and their population P is growing. It is known from earlier years that their population is limited by lack of resources to an estimated maximum of $M = 10000$. Their numbers are therefore being modelled by the differential equation

$$\frac{dP}{dt} = -k(P - M), \quad \text{where } k \text{ is a constant and } t \text{ is time in years.}$$

a Explain why the constant k is positive.

b Show that $P = M - Ae^{-kt}$ satisfies the differential equation, has initial value $M - A$, and has limit 10000.

c The population at the start of 2010 was 500, and at the start of 2020 is 2000.

i Find A and k.

ii What is the predicted population, correct to the nearest 10 goats, at the start of 2030?

iii In what year is the population predicted to reach 8000?

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17 Further trigonometry

A proper understanding of how to solve trigonometric equations requires a theory of inverse trigonometric functions. This theory is complicated by the fact that the trigonometric functions are periodic functions — they therefore fail the horizontal line test quite seriously, in that some horizontal lines cross their graphs infinitely many times. Understanding inverse trigonometric functions therefore requires that we first discuss more generally the procedures for restricting the domain of a function so that the inverse relation is also a function.

The second half of this chapter deals with identities involving compound angles in trigonometry. We begin with the formulae for $\sin(x + y)$, $\cos(x + y)$ and $\tan(x + y)$, then develop other identities based on these formulae — the double-angle formulae, the *t*-formulae, and the products-to-sums formulae.

All this work is required in Year 12 calculus.

- Compound angles are needed for the differentiation of the trigonometric functions, the sketching of trigonometric functions, and the solutions of many problems.
- Inverse trigonometric functions are needed to integrate functions such as $y = \dfrac{1}{\sqrt{1 - x^2}}$ and $y = \dfrac{1}{1 + x^2}$.

Digital Resources are available for this chapter in the **Interactive Textbook** and **Online Teaching Suite**. See the *overview* at the front of the textbook for details.

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17A Restricting the domain

Sections 5F and 5G discussed how the inverse relation of a function may or may not be a function, and briefly mentioned that if the inverse is not a function, then the domain can be restricted so that the inverse of this restricted function is a function. This section develops a more systematic approach to restricting the domain, in preparation for constructing the inverse trigonometric functions.

Inverse relations and functions

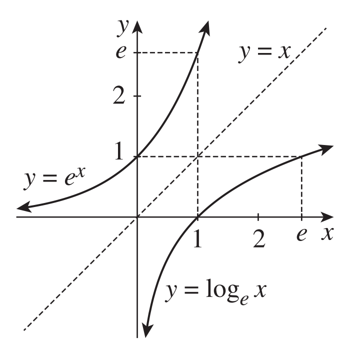
Here are the basic ideas from Section 5F and Section 3I on inverse relations and functions.

1 INVERSE RELATIONS AND FUNCTIONS

Suppose that the graph of a relation has been drawn.

- The graph of the inverse relation is obtained by reflecting the original graph in the diagonal line $y = x$.
- The inverse relation of a given relation is a function if and only if no horizontal line crosses the original graph more than once.
- In particular, suppose that the original relation is a function.
  — If the function is one-to-one, then the relation is also a function.
  — If the function is many-to-one, then the inverse is not a function.
- The domain and range of the inverse relation are the range and domain respectively of the original function.
- To find the equation and conditions of the inverse relation, write x for y and y for x every time each variable occurs. If the resulting equation can be solved for y, then the inverse is a function.

Example — the logarithmic and exponential functions



The two functions $y = e^x$ and $y = \log_e x$ provide a particularly clear example of a function and its inverse function.

- The function $y = e^x$ passes the horizontal line test, so we know that its inverse is also a function.
- The inverse relation is $x = e^y$, which we can solve for y as $y = \log_e x$, again showing that it is a function.
- The graphs are each one-to-one, and are reflections of each other in $y = x$.
- The domain of $y = \log_e x$ is all positive numbers, and its range is all real numbers — these are the range and domain of the original function.
- Taking the composition of the function and its inverse, in both orders,

$$\log_e e^x = x, \text{ for all real } x \qquad \text{and} \qquad e^{\log_e x} = x, \text{ for all } x > 0.$$

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Inverse functions

Section 5G introduced notation that could be used when the inverse of a function is again a function, and began dealing with restrictions to a function.

2 INVERSE FUNCTIONS

A function $f(x)$ is either one-to-one or many-to-one.

- If $f(x)$ is one-to-one, then the inverse of $y = f(x)$ is also a function, and is written as $f^{-1}(x)$. The composition of the function and its inverse, in either order, is an identity function, leaving every number in their domains unchanged:
$$f^{-1}\big(f(x)\big) = x \qquad \text{and} \qquad f\big(f^{-1}(x)\big) = x.$$
- If $f(x)$ is many-to-one, then the inverse is not a function. We can usually restrict the domain of the original function so that the inverse of the new restricted function is a function.

Caution: The notation $f^{-1}(x)$ means the inverse function of $f(x)$. Never confuse it with the reciprocal of $f(x)$, which is written as $\dfrac{1}{f(x)}$ or as $\big(f(x)\big)^{-1}$.

Example 1 shows how to manipulate inverse function notation.

Example 1 17A

a Write down the inverse relation of $f(x) = \dfrac{x-2}{x+2}$.

b By solving for y, show that this inverse is a function, and hence write down the equation of $f^{-1}(x)$.

c Show directly that $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(x)\big) = x$.

SOLUTION

a Let $y = \dfrac{x-2}{x+2}$.

Then the inverse relation is $x = \dfrac{y-2}{y+2}$ (writing y for x. and x for y)

b Solving for y,
$$xy + 2x = y - 2$$
$$y(x-1) = -2x - 2$$
$$y = \frac{2+2x}{1-x}.$$

Because there is only one solution for y, the inverse relation is a function,

and $f^{-1}(x) = \dfrac{2+2x}{1-x}$.

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c $f\left(f^{-1}(x)\right) = f\left(\dfrac{2+2x}{1-x}\right)$

$= \dfrac{\dfrac{2+2x}{1-x} - 2}{\dfrac{2+2x}{1-x} + 2} \times \dfrac{1-x}{1-x}$

$= \dfrac{(2+2x) - 2(1-x)}{(2+2x) + 2(1-x)}$

$= \dfrac{4x}{4}$

$= x$, as required.

$f^{-1}\left(f(x)\right) = f^{-1}\left(\dfrac{x-2}{x+2}\right)$

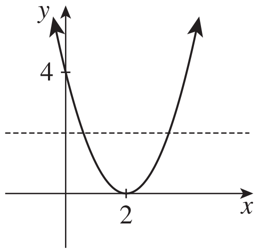
$= \dfrac{2 + \dfrac{2(x-2)}{x+2}}{1 - \dfrac{x-2}{x+2}} \times \dfrac{x+2}{x+2}$

$= \dfrac{2(x+2) + 2(x-2)}{(x+2) - (x-2)}$

$= \dfrac{4x}{4}$

$= x$, as required.

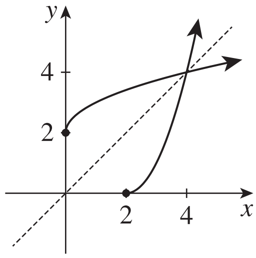
Restricting the domain so that the inverse is a function



The diagram to the right shows the parabola $y = (x-2)^2$. It clearly fails the horizontal line test, and is many-to-one.

If, however, we start at the stationary point $V(2, 0)$ and restrict to the interval $x \ge 2$ to the right of V, then we have a new function that passes the horizontal line test,

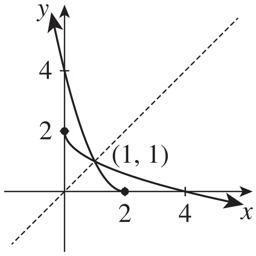
$$y = (x-2)^2, \quad \text{where } x \ge 2.$$



The inverse of this new one-to-one function is

$$x = (y-2)^2, \quad \text{where } y \ge 2$$
$$y - 2 = \sqrt{x} \text{ or } -\sqrt{x}, \quad \text{where } y \ge 2$$
$$y = 2 + \sqrt{x} \text{ or } 2 - \sqrt{x}, \quad \text{where } y \ge 2.$$

The condition $y \ge 2$ excludes the second alternative, so the inverse function is $y = 2 + \sqrt{x}$.



Alternatively, we could restrict to the interval $x \le 2$ to the left of the stationary point, giving a new function

$$y = (x-2)^2, \quad \text{where } x \le 2.$$

The inverse of this new one-to-one function is

$$x = (y-2)^2, \quad \text{where } y \le 2$$
$$y - 2 = \sqrt{x} \text{ or } -\sqrt{x}, \quad \text{where } y \le 2$$
$$y = 2 + \sqrt{x} \text{ or } 2 - \sqrt{x}, \quad \text{where } y \le 2.$$

The condition $y \le 2$ excludes the first alternative, so the inverse function is $y = 2 - \sqrt{x}$.

Notice that when the first graph meets the mirror $y = x$, at $(4, 4)$, it also meets the other graph there too. Similarly, when the second graph meets the mirror $y = x$, at $(1, 1)$, it also meets the other graph there too. This is because a point on the mirror is left unchanged by reflection in the mirror.

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Which restriction should be chosen?

There are no rules, except for convenience.

- Always look first at the stationary points of the curve, if possible (the stationary points are the points where the tangent is horizontal).
- It is usually helpful to choose an interval as the domain so that the domain is connected.
- It is usually helpful to extend the restricted domain as far as possible.
- It may be helpful to choose a domain that includes zero.

Examples 2 and 3 use function notation to describe the procedures.

Example 2 17A

a Explain why the inverse relation of $f(x) = (x-1)^2 + 2$ is not a function.

b Define $g(x)$ to be the restriction of $f(x)$ to the largest possible connected domain containing $x = 0$ so that $g(x)$ has an inverse function.

c Write down the equation of $g^{-1}(x)$.

d Sketch $g(x)$ and $g^{-1}(x)$ on one set of axes.

SOLUTION

a The graph of $y = f(x)$ is a parabola with vertex $(1, 2)$.
This fails the horizontal line test, so the inverse is not a function.
(Alternatively, $f(0) = f(2) = 3$, so $y = 3$ meets the curve twice.)

b Restricting $f(x)$ to the domain $x \le 1$ gives the function

$$g(x) = (x-1)^2 + 2, \quad \text{where } x \le 1,$$

which is sketched opposite, and includes the value at $x = 0$.

c Because $g(x)$ passes the horizontal line test, it has an inverse

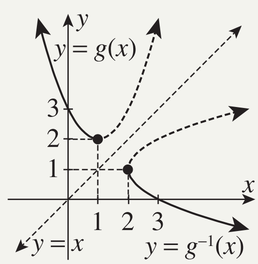
$$x = (y-1)^2 + 2, \quad \text{where } y \le 1.$$

Solving for y, $(y-1)^2 = x - 2$, where $y \le 1$,

$$y = 1 + \sqrt{x-2} \text{ or } 1 - \sqrt{x-2}, \quad \text{where } y \le 1.$$

Hence $g^{-1}(x) = 1 - \sqrt{x-2}$, because $y \le 1$.

d



Example 3 17A

a Find the stationary points of $y = (x-2)^2(x+1)$, then sketch the curve.

b Explain why the restriction $f(x)$ of this function to the part of the curve between the two stationary points has an inverse function.

c Sketch $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on one set of axes.

d Write down an equation satisfied by the x-coordinate of the point M where the function and its inverse intersect.

SOLUTION

a For $y = (x-2)^2(x+1) = x^3 - 3x^2 + 4$,
$y' = 3x^2 - 6x = 3x(x-2)$,

So there are zeroes at $x = 2$ and $x = -1$, and stationary points at $(0, 4)$ and $(2, 0)$.

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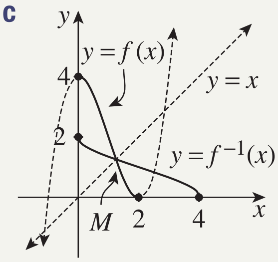
b The part of the curve between the stationary points satisfies the horizontal line test, so the function

$$f(x) = (x - 2)^2(x + 1), \quad \text{where } 0 \le x \le 2,$$

has an inverse function $f^{-1}(x)$.

c



d The curves $y = f(x)$ and $y = f^{-1}(x)$ intersect on $y = x$, and substituting $y = x$ into the function,

$$x = x^3 - 3x^2 + 4,$$

so the x-coordinate of M satisfies the cubic $x^3 - 3x^2 - x + 4 = 0$.

Exercise 17A

FOUNDATION

1 The function $f(x) = x + 3$ is defined over the domain $0 \le x \le 2$.

a State the range of $f(x)$.

b State the domain and range of $f^{-1}(x)$.

c Write down the rule for $f^{-1}(x)$.

2 The function $F(x)$ is defined by $F(x) = \sqrt{x}$ over the domain $0 \le x \le 4$.

a State the range of $F(x)$.

b State the domain and range of $F^{-1}(x)$.

c Write down the rule for $F^{-1}(x)$.

d Graph $y = F(x)$ and $y = F^{-1}(x)$.

3 Sketch the graph of each function. Then use reflection in the line $y = x$ to sketch the inverse relation. State whether the inverse is a function, and find its equation if it is.

a $f(x) = 2x$ **b** $f(x) = x^3 + 1$ **c** $f(x) = \sqrt{1 - x^2}$

d $f(x) = x^2 - 4$ **e** $f(x) = 2^x$ **f** $f(x) = \sqrt{x - 3}$

4 Consider the functions $f(x) = 3x + 2$ and $g(x) = \frac{1}{3}(x - 2)$.

a Find $f\big(g(x)\big)$ and $g\big(f(x)\big)$.

b What is the relationship between $f(x)$ and $g(x)$?

5 Each function $g(x)$ is defined over a restricted domain so that $g^{-1}(x)$ exists. Find $g^{-1}(x)$ and write down its domain and range. (Sketches of $g(x)$ and $g^{-1}(x)$ will prove helpful.)

a $g(x) = x^2, x \ge 0$

b $g(x) = x^2 + 2, x \le 0$

c $g(x) = -\sqrt{4 - x^2},\ 0 \le x \le 2$

6 **a** Write down $\dfrac{dy}{dx}$ for the function $y = x^3 - 1$.

b Make x the subject and hence find $\dfrac{dx}{dy}$.

c Hence show that $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$.

7 Repeat the previous question for $y = \sqrt{x}$.

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DEVELOPMENT

8 The function $F(x) = x^2 + 2x + 4$ is defined over the domain $x \geq -1$.
a Sketch the graphs of $F(x)$ and $F^{-1}(x)$ on the same diagram.
b Classify each function as one-to-one or many-to-one.
c Find the equation of $F^{-1}(x)$ and state its domain and range.

9 a Solve the equation $1 - \ln x = 0$.
b Sketch the graph of $f(x) = 1 - \ln x$ by suitably transforming the graph of $y = \ln x$.
c Hence sketch the graph of $f^{-1}(x)$ on the same diagram.
d Find the equation of $f^{-1}(x)$ and state its domain and range.
e Classify $f(x)$ and $f^{-1}(x)$ as increasing, decreasing or neither.

10 a Carefully sketch the function defined by $g(x) = \dfrac{x+2}{x+1}$, for $x > -1$.
b Classify $g(x)$ as one-to-one or many-to-one.
c Find $g^{-1}(x)$ and sketch it on the same diagram.
d Find any values of x for which $g(x) = g^{-1}(x)$.
(Hint: In this case, the easiest way is to solve $g(x) = x$. Why does this work?)

11 The previous question seems to imply that the graphs of a function and its inverse can only intersect on the line $y = x$. This is not true, as the following example demonstrates.
a Find the equation of the inverse of $y = -x^3$.
b At what points do the graphs of the function and its inverse meet?
c Sketch the situation.

12 a Show that any linear function $f(x) = mx + b$ has an inverse function if $m \neq 0$.
b Does the constant function $F(x) = b$ have an inverse function?

13 a Explain how the graph of $f(x) = x^2$ must be transformed to obtain the graph of $g(x) = (x+2)^2 - 4$.
b Hence sketch the graph of $g(x)$, showing the x- and y-intercepts and the vertex.
c Classify $g(x)$ as one-to-one or many-to-one.
d What is the largest connected domain containing $x = 0$ so that the restriction of $g(x)$ has an inverse function?
e Let $g^{-1}(x)$ be the inverse function corresponding to the restricted domain of $g(x)$ in part **c**. What is the domain of $g^{-1}(x)$? Is $g^{-1}(x)$ increasing or decreasing?
f Find the equation of $g^{-1}(x)$, and sketch it on your diagram in part **b**.

14 a Show that $F(x) = x^3 - 3x$ is an odd function.
b Sketch the graph of $F(x)$, showing the x-intercepts and the coordinates of the two stationary points.
c Classify $F(x)$ as one-to-one or many-to-one.
d What is the largest connected domain containing $x = 0$ so that the restriction of $F(x)$ has an inverse function?
e Let $F^{-1}(x)$ be the inverse function of the restriction of $F(x)$ in part **d.** State the domain of $F^{-1}(x)$, and sketch it on the same diagram as part **b**.

15 a State the domain of $f(x) = \dfrac{e^x}{1 + e^x}$.
b Use the quotient rule to show that $f'(x) = \dfrac{e^x}{(1+e^x)^2}$.
c Hence explain why $f(x)$ is increasing for all x.
d Explain why $f(x)$ has an inverse function, and find its equation.

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16 The function $f(x)$ is defined by $f(x) = x - \frac{1}{x}$, for $x > 0$.

a By considering the graphs of $y = x$ and $y = \frac{1}{x}$ for $x > 0$, sketch $y = f(x)$.

b Sketch $y = f^{-1}(x)$ on the same diagram.

c By completing the square or using the quadratic formula, show that

$$f^{-1}(x) = \tfrac{1}{2}\left(x + \sqrt{4 + x^2}\right).$$

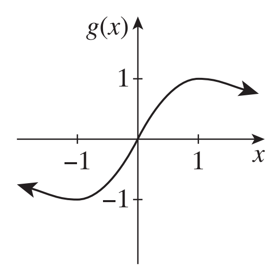
17 **a** Prove geometrically that the inverse relation of an odd relation is also odd.

b Prove algebraically that if an odd function has an inverse function, then that inverse function is also odd.

c What sort of even functions have inverse functions?

ENRICHMENT

18 The diagram shows the function $g(x) = \dfrac{2x}{1 + x^2}$, whose domain is all real x.



a Show that $g\left(\frac{1}{a}\right) = g(a)$, for all $a \neq 0$.

b Hence explain why the inverse of $g(x)$ is not a function.

c **i** What is the largest connected domain of $g(x)$ containing $x = 0$ for which $g^{-1}(x)$ exists?

ii Sketch $g^{-1}(x)$ for this restricted domain of $g(x)$.

iii Find the equation of $g^{-1}(x)$ for this restricted domain of $g(x)$.

d Repeat part **c** using the complement of the domain in part **c**.

e Show that the two expressions for $g^{-1}(x)$ in parts **c** and **d** are reciprocals of each other. Why could we have anticipated this?

19 Consider the function $f(x) = \frac{1}{6}\left(x^2 - 4x + 24\right)$.

a Sketch the parabola $y = f(x)$, showing the vertex and any x- or y-intercepts.

b State the largest connected domain containing only positive numbers for which $f(x)$ has an inverse function.

c Sketch $f^{-1}(x)$ corresponding to this restriction on your diagram from part **a**, and state its domain.

d Find any points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

e Let N be a negative real number. Find $f^{-1}\left(f(N)\right)$.

20 [The hyperbolic sine function]

The function $\sinh x$ is defined by $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$.

a State the domain of $\sinh x$.

b Find the value of $\sinh 0$.

c Show that $y = \sinh x$ is an odd function.

d Find $\dfrac{d}{dx}(\sinh x)$ and hence show that $\sinh x$ is increasing for all x.

e To which curve is $y = \sinh x$ asymptotic for large values of x?

f Sketch $y = \sinh x$, and explain why the function has an inverse function $\sinh^{-1} x$.

g Sketch the graph of $\sinh^{-1} x$ on the same diagram as part **f**.

h Show that $\sinh^{-1} x = \log_e\left(x + \sqrt{x^2 + 1}\right)$, by treating the equation $x = \frac{1}{2}\left(e^y - e^{-y}\right)$ as a quadratic equation in e^y.

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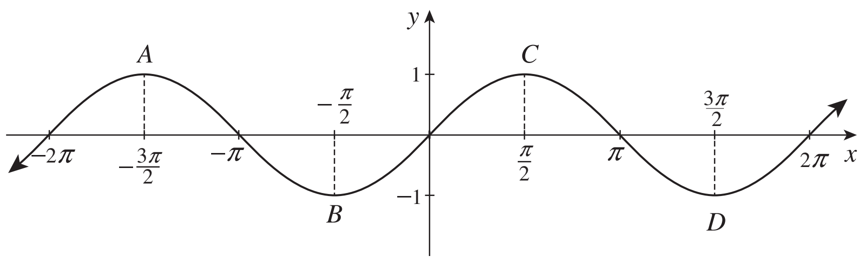
17B Defining the inverse trigonometric functions

Each of the six trigonometric functions fails the horizontal line test completely, in that there are horizontal lines that cross each of their graphs infinitely many times. For example, $y = \sin x$ is graphed below, and clearly every horizontal line between $y = 1$ and $y = -1$ crosses it infinitely many times.



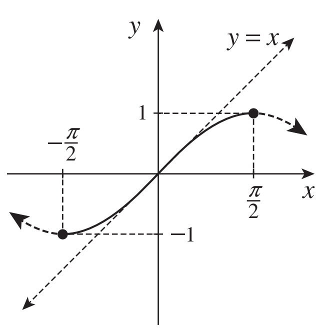
Thus $y = \sin x$ is many-to-one, not one-to-one, and its inverse relation is not a function. To create an inverse function from $y = \sin x$, we need to restrict the domain to a piece of the curve between two stationary points. For example, the pieces AB, BC and CD all satisfy the horizontal line test. Because acute angles should be included, the obvious choice is the arc BC from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.

The definition of $\sin^{-1} x$

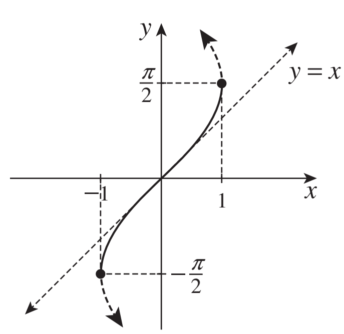
The function $y = \sin^{-1} x$ (read this as 'inverse sine ex') is accordingly *defined to be the inverse function of the restricted function*

$$y = \sin x, \qquad \textit{where } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}. \qquad \text{(This is a closed interval.)}$$

The two curves are sketched below. Notice, when sketching the graphs, that $y = x$ is a tangent to $y = \sin x$ at the origin, as we shall prove in Year 12. Thus when the graph is reflected in $y = x$, the line $y = x$ does not move, so it is also the tangent to $y = \sin^{-1} x$ at the origin. Notice also that $y = \sin x$ is horizontal at its stationary points, so $y = \sin^{-1} x$ is vertical at its endpoints.



$y = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



$y = \sin^{-1} x$

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3 THE DEFINITION OF $y = \sin^{-1}x$

- $y = \sin^{-1} x$ is not the inverse relation of $y = \sin x$, it is the inverse function of the restriction of $y = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
- $y = \sin^{-1} x$ has domain $-1 \le x \le 1$ and range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
- $y = \sin^{-1} x$ is an increasing function.
- $y = \sin^{-1} x$ has tangent $y = x$ at the origin, and is vertical at its endpoints.

Radian measure

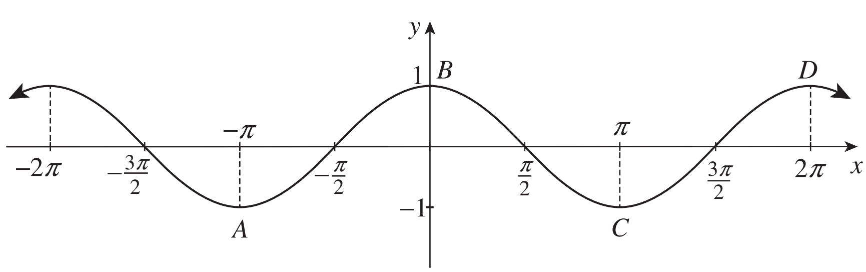
In this course, radian measure is used exclusively when dealing with the inverse trigonometric functions (except on the calculator, if you leave it in degrees mode). Calculations using degrees should be avoided, or at least not included in the formal working of problems.

4 RADIAN MEASURE

Use radians when dealing with inverse trigonometric functions.

The definition of $\cos^{-1}x$

The many-to-one function $y = \cos x$ is graphed below. To create a satisfactory inverse function from $y = \cos x$, we need to restrict the domain to a piece of the curve between two stationary points. Because acute angles should be included, the obvious choice is the arc BC from $x = 0$ to $x = \pi$.



Thus the function $y = \cos^{-1}x$ (read this as 'inverse cos ex') is defined to be the inverse function of the restricted function

$$y = \cos x, \quad \text{where} \quad 0 \le x \le \pi. \qquad \text{(This is a closed interval.)}$$

The two curves are sketched below. Notice that the tangent to $y = \cos x$ at its x-intercept $\left(\frac{\pi}{2}, 0\right)$ is the line $t: x + y = \frac{\pi}{2}$ with gradient -1. Reflection in $y = x$ reflects this line onto itself, so t is also the tangent to $y = \cos^{-1}x$ at its y-intercept $\left(0, \frac{\pi}{2}\right)$. Like $y = \sin^{-1} x$, the graph is vertical at its endpoints.

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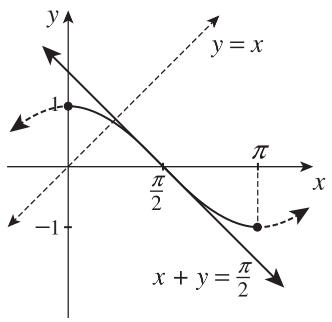
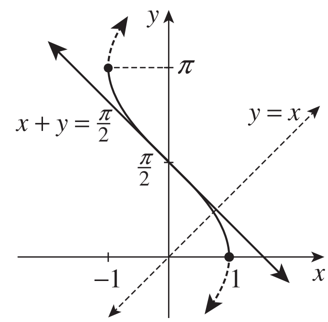


$y = \cos x, 0 \le x \le \pi$



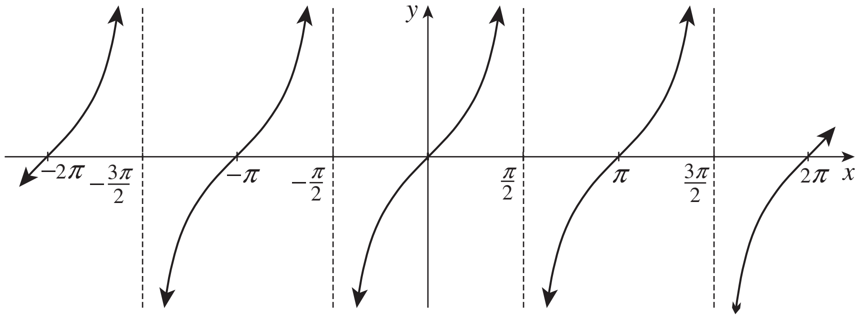
$y = \cos^{-1} x$

5 THE DEFINITION OF $y = \cos^{-1} x$

- $y = \cos^{-1} x$ is not the inverse relation of $y = \cos x$, it is the inverse function of the restriction of $y = \cos x$ to $0 \le x \le \pi$.
- $y = \cos^{-1} x$ has domain $-1 \le x \le 1$ and range $0 \le y \le \pi$.
- $y = \cos^{-1} x$ is a decreasing function.
- $y = \cos^{-1} x$ has gradient -1 at its y-intercept, and is vertical at its endpoints.

The definition of $\tan^{-1} x$

The graph of $y = \tan x$ below consists of a collection of disconnected branches, and is many-to-one, not one-to-one. Because acute angles should be included, the most satisfactory inverse function is formed by choosing the branch in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



Thus the function $y = \tan^{-1} x$ is defined to be the inverse function of

$$y = \tan x, \quad \text{where } -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}. \qquad \text{(Here we are using an open interval.)}$$

The line of reflection $y = x$ is the tangent to both curves at the origin. Notice also that the vertical asymptotes $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ of $y = \tan x$ are reflected onto the horizontal asymptotes $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ of $y = \tan^{-1} x$.

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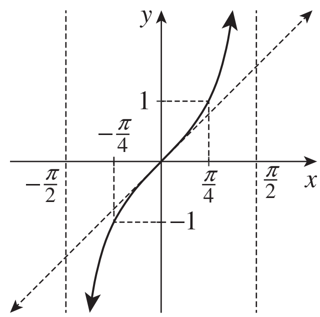
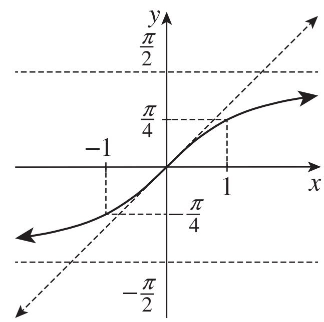


$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



$y = \tan^{-1}x$

6 THE DEFINITION OF $y = \tan^{-1}x$

- $y = \tan^{-1}x$ is not the inverse relation of $y = \tan x$, it is the inverse function of the restriction of $y = \tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
- $y = \tan^{-1}x$ has domain the real line and range $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
- $y = \tan^{-1}x$ is an increasing function.
- $y = \tan^{-1}x$ has gradient 1 at its y-intercept.
- The lines $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes.

Inverse functions of cosec x, sec x and cot x

It is not appropriate in this course to define the functions $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ because of difficulties associated with their discontinuities and because they are not needed.

Alternative notations for inverse trigonometric functions

An alternative notation for the inverse trigonometric functions uses the prefix 'arc',

$$\arcsin x = \sin^{-1} x \qquad \arccos x = \cos^{-1} x \qquad \arctan x = \tan^{-1} x$$

This notation arises from the arc length formula in Section 11I. In a circle of radius 1, the arc length ℓ subtended by an angle θ is just $\ell = \theta$, allowing angle and arc length to be identified. Thus $\arcsin x$ can be understood either as 'the angle whose sine is x', or as 'the arc length whose sine is x'.

The advantage of the $\arcsin x$ notation is that because $\sin^2 x$ means $(\sin x)^2$, the inverse function $\sin^{-1} x$ could possibly be confused with the reciprocal function $(\sin x)^{-1} = \frac{1}{\sin x}$. On the other hand, the advantage of the $\sin^{-1} x$ notation is that it is consistent with the notation used for other inverse functions, and for this reason, we will use $\sin^{-1} x$ most of the time.

A shorter form abbreviating 'arc' to 'a' is also standard, particularly in computing, but is best not used in this course,

$$\operatorname{asin} x = \sin^{-1} x \qquad \operatorname{acos} x = \cos^{-1} x \qquad \operatorname{atan} x = \tan^{-1} x.$$

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Calculations with the inverse trigonometric functions

The key to calculations is to *include the restriction every time an expression involving the inverse trigonometric functions is rewritten using trigonometric functions.*

7 INTERPRETING THE RESTRICTIONS

- $y = \sin^{-1} x$ means $x = \sin y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
- $y = \cos^{-1} x$ means $x = \cos y$ where $0 \le y \le \pi$.
- $y = \tan^{-1} x$ means $x = \tan y$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Example 4 17B

Find:

a $\cos^{-1}\left(-\frac{1}{2}\right)$

b $\tan^{-1}(-1)$

SOLUTION

a Let $\alpha = \cos^{-1}\left(-\frac{1}{2}\right)$.

Then $\cos \alpha = -\frac{1}{2}$, where $0 \le \alpha \le \pi$.

Hence α is in the second quadrant,

and the related angle is $\frac{\pi}{3}$,

so $\alpha = \frac{2\pi}{3}$.

b Let $\alpha = \tan^{-1}(-1)$.

Then $\tan \alpha = -1$, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

Hence α is in the fourth quadrant,

and the related angle is $\frac{\pi}{4}$,

so $\alpha = -\frac{\pi}{4}$.

Example 5 17B

Evaluate these expressions exactly, rationalising denominators where necessary.

a $\tan\left(\sin^{-1}\left(-\frac{1}{5}\right)\right)$

b $\sin\left(\cos^{-1}\frac{4}{5}\right)$

c $\sin\left(\sin^{-1}\frac{4}{5}\right)$

SOLUTION

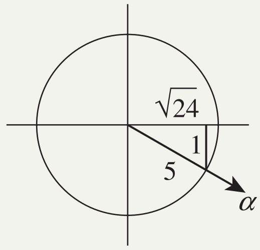
a Let $\alpha = \sin^{-1}\left(-\frac{1}{5}\right)$.

Then $\sin \alpha = -\frac{1}{5}$, where $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.

Hence α is in the fourth quadrant,

$$\text{and } \tan \alpha = \frac{-1}{\sqrt{24}} \times \frac{\sqrt{6}}{\sqrt{6}}$$

$$= -\frac{1}{12}\sqrt{6}.$$



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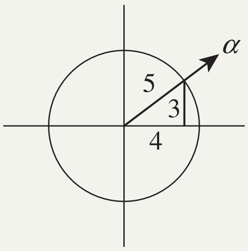
b Let $\alpha = \cos^{-1}\frac{4}{5}$.

Then $\cos\alpha = \frac{4}{5}$, where $0 \le \alpha \le \pi$.

Hence α is in the first quadrant,

and $\sin\alpha = \frac{3}{5}$.



c Let $\alpha = \sin^{-1}\frac{4}{5}$.

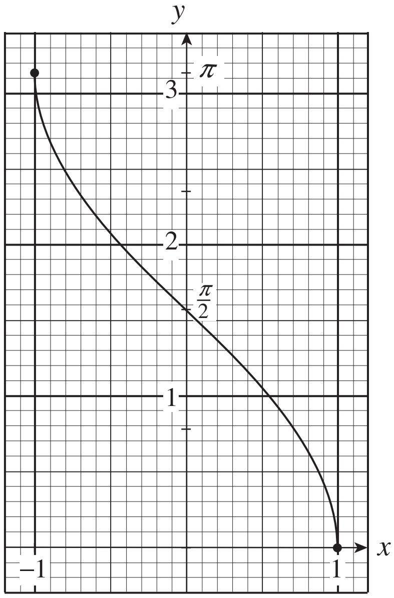
Then $\sin\alpha = \frac{4}{5}$ (where the condition $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ is now irrelevant).

Exercise 17B

FOUNDATION

1 Read off the graph the values of the following correct to two decimal places.

a $\cos^{-1}0.4$
b $\cos^{-1}0.8$
c $\cos^{-1}0.25$
d $\arccos(-0.1)$
e $\arccos(-0.4)$
f $\arccos(-0.75)$



2 Find the exact value of:

a $\sin^{-1}0$ **b** $\sin^{-1}\frac{1}{2}$ **c** $\arccos 1$
d $\arctan 1$ **e** $\sin^{-1}(-1)$ **f** $\cos^{-1}0$
g $\arctan 0$ **h** $\arctan(-1)$ **i** $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
j $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ **k** $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ **l** $\arccos(-1)$

3 Use your calculator to find, correct to three decimal places, the value of:

a $\cos^{-1}0.123$ **b** $\arccos(-0.123)$ **c** $\sin^{-1}\frac{2}{3}$
d $\arcsin\left(-\frac{2}{3}\right)$ **e** $\tan^{-1}5$ **f** $\arctan(-5)$

4 Find the exact value of:

a $\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$ **b** $\sin(\cos^{-1}0)$ **c** $\tan(\arctan 1)$
d $\cos^{-1}\left(\sin\frac{\pi}{3}\right)$ **e** $\sin\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$ **f** $\arccos\left(\cos\frac{3\pi}{4}\right)$
g $\tan^{-1}\left(-\tan\frac{\pi}{6}\right)$ **h** $\cos\left(2\tan^{-1}(-1)\right)$ **i** $\arctan\left(\sqrt{6}\sin\frac{\pi}{4}\right)$

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DEVELOPMENT

5 Find the exact value of:

a $\sin^{-1}\left(\sin\frac{4\pi}{3}\right)$

b $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$

c $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$

d $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$

e $\arcsin\left(2\sin\left(-\frac{\pi}{6}\right)\right)$

f $\arctan\left(3\tan\frac{7\pi}{6}\right)$

6 By evaluating LHS and RHS, show that:

a $\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{2} - \tan^{-1}\sqrt{3}$

b $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\frac{1}{2}$

c $\tan^{-1}\left(-\sin\frac{\pi}{2}\right) = -\tan^{-1}\left(\sin\frac{\pi}{2}\right)$

d $\arcsin\left(-\cos\frac{\pi}{6}\right) = \frac{\pi}{2} - \arccos\left(-\cos\frac{\pi}{6}\right)$

7 a In each part use a right-angled triangle to find the exact value of:

i $\sin\left(\cos^{-1}\frac{3}{5}\right)$

ii $\tan\left(\sin^{-1}\frac{5}{13}\right)$

iii $\cos\left(\sin^{-1}\frac{2}{3}\right)$

iv $\sin\left(\cos^{-1}\left(-\frac{15}{17}\right)\right)$

v $\cos\left(\tan^{-1}\left(-\frac{1}{3}\right)\right)$

vi $\tan\left(\cos^{-1}\left(-\frac{3}{4}\right)\right)$

b Use a right-angled triangle in each part to show that:

i $\sin\left(\cos^{-1}x\right) = \sqrt{1 - x^2}$

ii $\sin^{-1}x = \tan^{-1}\left(\dfrac{x}{\sqrt{1 - x^2}}\right)$

8 [Algebraic proof that $\sin^{-1}x$ is odd]

Let $\theta = \sin^{-1}(-x)$.

a Use the fact that $\sin(-\theta) = -\sin\theta$ to show that $\theta = -\sin^{-1}x$.

b Deduce that $\sin^{-1}(-x) = -\sin^{-1}x$.

9 Prove similarly that $\tan^{-1}(-x) = -\tan^{-1}x$.

10 [Algebraic proof that $\cos^{-1}(-x) = \pi - \cos^{-1}x$]

Let $\theta = \cos^{-1}(-x)$.

a Use the fact that $\cos(\pi - \theta) = -\cos\theta$ to show that $\cos^{-1}x = \pi - \theta$.

b Deduce that $\cos^{-1}(-x) = \pi - \cos^{-1}x$.

11 [Algebraic proof that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$]

Let $\theta = \sin^{-1}x$.

a Use the fact that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ to show that $\cos^{-1}x = \frac{\pi}{2} - \theta$.

b Deduce that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$.

12 [Algebraic proof that $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$ for $x > 0$]

a Let $x > 0$. Use the identity $\tan\left(\dfrac{\pi}{2} - \theta\right) = \cot\theta$, with $\theta = \tan^{-1}x$ to show that $\tan^{-1}x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$.

b Let $x < 0$. Use the fact that $\tan^{-1}x$ is odd to find the value of $\tan^{-1}x + \tan^{-1}\dfrac{1}{x}$ for $x < 0$.

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ENRICHMENT

13 Determine the range of each function.

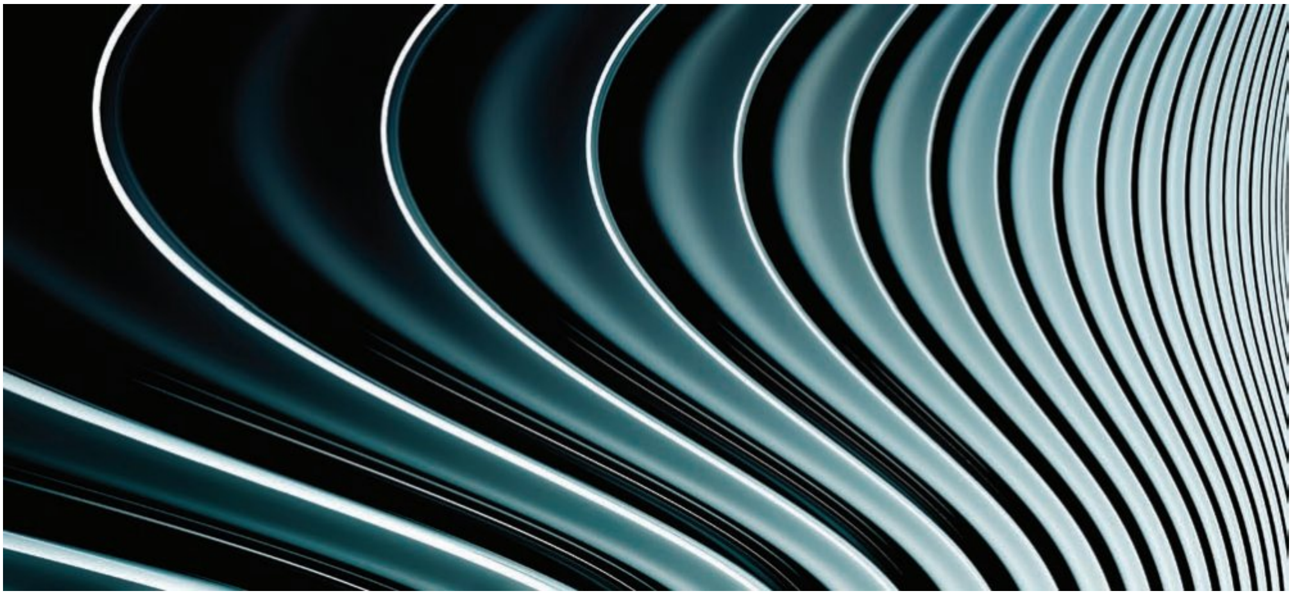
a $y = \tan^{-1}(x^2)$

b $y = \tan^{-1}\left(\dfrac{1}{1 + x^2}\right)$

14 **a** Explain why $\cos^{-1}(\cos 2) = 2$ but $\sin^{-1}(\sin 2) \neq 2$.

b Sketch the curve $y = \sin x$ for $0 \leq x \leq \pi$, and use symmetry to explain why $\sin 2 = \sin(\pi - 2)$.

c What is the exact value of $\sin^{-1}(\sin 2)$?



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17C Graphs involving inverse trigonometric functions

This section deals mostly with graphs that can be obtained using transformations of the graphs of the three inverse trigonometric functions.

Graphs involving shifting and reflecting

Reflecting in either axes, and shifting, can both be applied, but as always substitution of key values should be used to confirm the graph. In the case of $\tan^{-1}x$, it is wise to take limits so as to confirm the horizontal asymptotes.

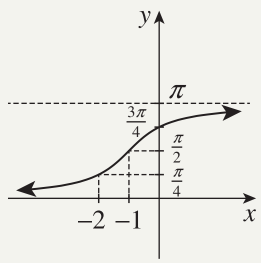
Example 6 — 17C

Sketch these graphs, stating their domain and range, and any asymptotes:

a $y - \frac{\pi}{2} = \tan^{-1}(x + 1)$ **b** $y = \sin^{-1}(-x) - \frac{\pi}{2}$

SOLUTION

a $y - \frac{\pi}{2} = \tan^{-1}(x + 1)$ is $y = \tan^{-1}x$ shifted left 1 unit and up $\frac{\pi}{2}$ units.

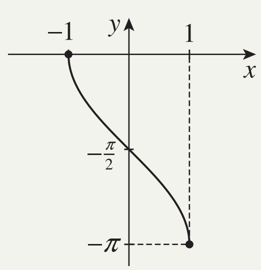


Confirm this by a small table of values:

x	-2	-1	0
y	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$

The domain is all real numbers, and the range is $0 < y < \pi$.
As $x \to \infty$, $y \to \pi$, and as $x \to -\infty$, $y \to 0$,
so the horizontal asymptotes are $y = 0$ and $y = \pi$.

b We can rewrite $y = \sin^{-1}(-x) - \frac{\pi}{2}$ as $y + \frac{\pi}{2} = \sin^{-1}(-x)$,
so the graph is $y = \sin^{-1}x$ reflected in the y-axis
then shifted down $\frac{\pi}{2}$ units.



Using a small table of values to confirm the graph:

x	-1	0	1
y	0	$-\frac{\pi}{2}$	$-\pi$

The domain is $-1 \le 0 \le 1$, and the range is $-\pi \le y \le 0$.

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Symmetries of the inverse trigonometric functions

The functions $y = \sin^{-1}x$ and $y = \tan^{-1}x$ are both odd. The function $y = \cos^{-1}x$ also has odd symmetry, not about the origin, but about its y-intercept $\left(0, \frac{\pi}{2}\right)$.

8 SYMMETRIES OF THE INVERSE TRIGONOMETRIC FUNCTIONS

- $y = \sin^{-1}x$ is odd, that is, $\sin^{-1}(-x) = -\sin^{-1}x$.
- $y = \tan^{-1}x$ is odd, that is, $\tan^{-1}(-x) = -\tan^{-1}x$.
- $y = \cos^{-1}x$ has odd symmetry about its y-intercept $\left(0, \frac{\pi}{2}\right)$, that is,

$$\cos^{-1}(-x) = \pi - \cos^{-1}x$$

These were proven in Questions 8, 9 and 10 of Exercise 17B. The last identity is a little more difficult, and its proof is repeated here.

Proof

$$\begin{aligned}
\text{Let}\quad \alpha &= \cos^{-1}(-x)\\
\text{Then}\quad -x &= \cos\alpha,\ \text{ where } 0 \le \alpha \le \pi,\\
\text{so}\quad \cos(\pi - \alpha) &= x,\ \text{ because } \cos(\pi - \alpha) = -\cos\alpha,\\
\pi - \alpha &= \cos^{-1}x,\ \text{ because } 0 \le \pi - \alpha \le \pi,\\
\alpha &= \pi - \cos^{-1}x,\ \text{ as required.}
\end{aligned}$$

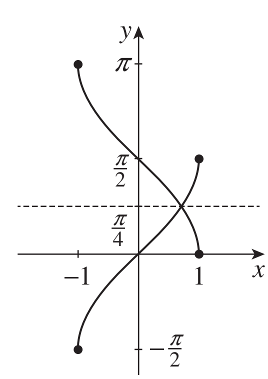
The identity $\sin^{-1}x + \cos^{-1}x = \pi/2$

The graphs of $y = \sin^{-1}x$ and $y = \cos^{-1}x$ are reflections of each other in the horizontal line $y = \frac{\pi}{4}$. Hence adding the graphs pointwise, it should be clear that

9 COMPLEMENTARY ANGLES

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \quad \text{for } -1 \le x \le 1$$

This was proven algebraically in Question 11 of the previous Exercise 17B, where it was clear that the identity is really only another form of the complementary angle identity $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$. The result is important, and the algebraic proof is repeated here.



Proof

$$\begin{aligned}
\text{Let}\quad \alpha &= \cos^{-1}x.\\
\text{Then}\quad x &= \cos\alpha,\ \text{ where } 0 \le \alpha \le \pi,\\
\sin\left(\tfrac{\pi}{2} - \alpha\right) &= x,\ \text{ because } \sin\left(\tfrac{\pi}{2} - \alpha\right) = \cos\alpha,\\
\sin^{-1}x &= \tfrac{\pi}{2} - \alpha,\ \text{ because } -\tfrac{\pi}{2} \le \tfrac{\pi}{2} - \alpha \le \tfrac{\pi}{2},\\
\sin^{-1}x + \alpha &= \tfrac{\pi}{2}\\
\sin^{-1}x + \cos^{-1}\alpha &= \tfrac{\pi}{2}.
\end{aligned}$$

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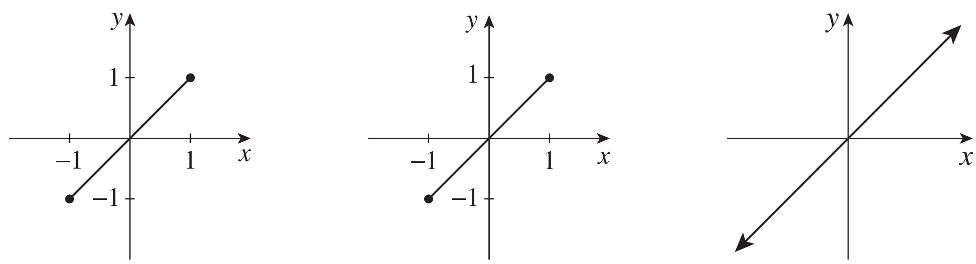
The graphs of $\sin(\sin^{-1} x)$, $\cos(\cos^{-1} x)$ and $\tan(\tan^{-1} x)$

The composite function defined by $y = \sin(\sin^{-1} x)$ has the same domain as $\sin^{-1} x$, that is, $-1 \leq x \leq 1$. Because it is the function $y = \sin^{-1} x$ followed by the function $y = \sin x$, the composite is therefore the identity function $y = x$ restricted to $-1 \leq x \leq 1$.

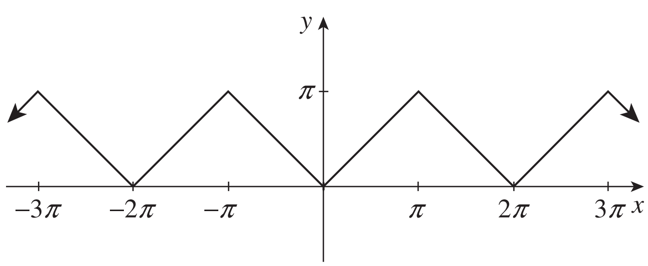


$y = \sin(\sin^{-1} x)$ $\quad$ $y = \cos(\cos^{-1} x)$ $\quad$ $y = \tan(\tan^{-1} x)$

The same remarks apply to $y = \cos(\cos^{-1} x)$ and $y = \tan(\tan^{-1} x)$, except that the domain of $y = \tan(\tan^{-1} x)$ is the whole real number line.

The graph of $\cos^{-1}(\cos x)$

The domain of this function is the whole real number line, and the graph is far more complicated. Constructing a simple table of values is probably the surest approach, but under the graph is an argument based on symmetries.



A For $0 \leq x \leq \pi$, $\cos^{-1}(\cos x) = x$, and the graph follows $y = x$.

B Because $\cos x$ is an even function, the graph in the interval $-\pi \leq x \leq 0$ is the reflection of the graph in the interval $0 \leq x \leq \pi$.

C We now have the shape of the graph in the interval $-\pi \leq x \leq \pi$. Because the graph has period 2π, the rest of the graph is just a repetition of this section.

Enrichment questions in the next exercise deal with the other confusing functions $\sin^{-1}(\sin x)$ and $\tan^{-1}(\tan x)$, and also with functions such as $y = \sin^{-1}(\cos x)$.

Exercise 17C

FOUNDATION

1 Sketch each function, stating the domain and range and whether it is even, odd or neither.

a $y = \tan^{-1} x$ $\quad$ **b** $y = \cos^{-1} x$ $\quad$ **c** $y = \sin^{-1} x$

2 Sketch each function, using appropriate translations of $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$. State the domain and range, and whether it is even, odd or neither.

a $y = \sin^{-1}(x - 1)$ $\quad$ **b** $y = \cos^{-1}(x + 1)$ $\quad$ **c** $y - \frac{\pi}{2} = \tan^{-1} x$

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3 Sketch each function by reflecting in the x- or y-axis as appropriate. State the domain and range, and whether it is even, odd or neither.

a $y = -\cos^{-1}x$ **b** $y = \tan^{-1}(-x)$ **c** $y = -\sin^{-1}(-x)$

DEVELOPMENT

4 a i Sketch the graphs of $y = \cos^{-1}x$ and $y = \sin^{-1}x - \frac{\pi}{2}$ on the same set of axes.

ii Hence show graphically that $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$.

b Use a graphical approach to show that:

i $\tan^{-1}(-x) = -\tan^{-1}x$ **ii** $\cos^{-1}x + \cos^{-1}(-x) = \pi$

5 a Determine the domain of the function $y = \sin^{-1}(1 - x)$ by solving $-1 \le 1 - x \le 1$.

b State the range of the function.

c Complete the table to the right, and hence sketch the graph of the function.

x	0	1	2
y			

d About which line are the graphs of $y = \sin^{-1}(1 - x)$ and $y = \sin^{-1}x$ symmetrical?

6 Find the domain and range, then use a table of values or transformations (or both) to sketch:

a $y = \cos^{-1}(1 - x)$ **b** $y = -\sin^{-1}(x + 1)$ **c** $y = -\tan^{-1}(1 - x)$

7 Sketch these graphs, stating whether each function is even, odd or neither.

a $y = \sin(\sin^{-1}2x)$ **b** $y = \cos\left(\cos^{-1}\frac{x}{2}\right)$ **c** $y = \tan\left(\tan^{-1}(x - 1)\right)$

8 a What is the domain of $y = \sin(\cos^{-1}x)$? Is it even, odd or neither?

b By considering the range of $\cos^{-1}x$, explain why $\sin(\cos^{-1}x) \ge 0$, for all x in its domain.

c By squaring both sides of $y = \sin(\cos^{-1}x)$ and using the identity $\sin^2\theta + \cos^2\theta = 1$, show that $y = \sqrt{1 - x^2}$.

d Hence sketch $y = \sin(\cos^{-1}x)$.

e Use similar methods to sketch the graph of $y = \cos(\sin^{-1}x)$.

ENRICHMENT

9 Consider the function $y = \tan^{-1}(\tan x)$.

a State its domain and range, and whether it is even, odd or neither.

b Simplify $\tan^{-1}(\tan x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

c What is the period of the function?

d Use the above information and a table of values if necessary to sketch the function.

10 On page 787, just above this exercise, $y = \cos^{-1}(\cos x)$ is sketched. Use the identity $\sin^{-1}t = \frac{\pi}{2} - \cos^{-1}t$ and simple transformations to sketch $y = \sin^{-1}(\cos x)$. State its symmetry.

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11 Consider the function $y = \sin^{-1}(\sin x)$.

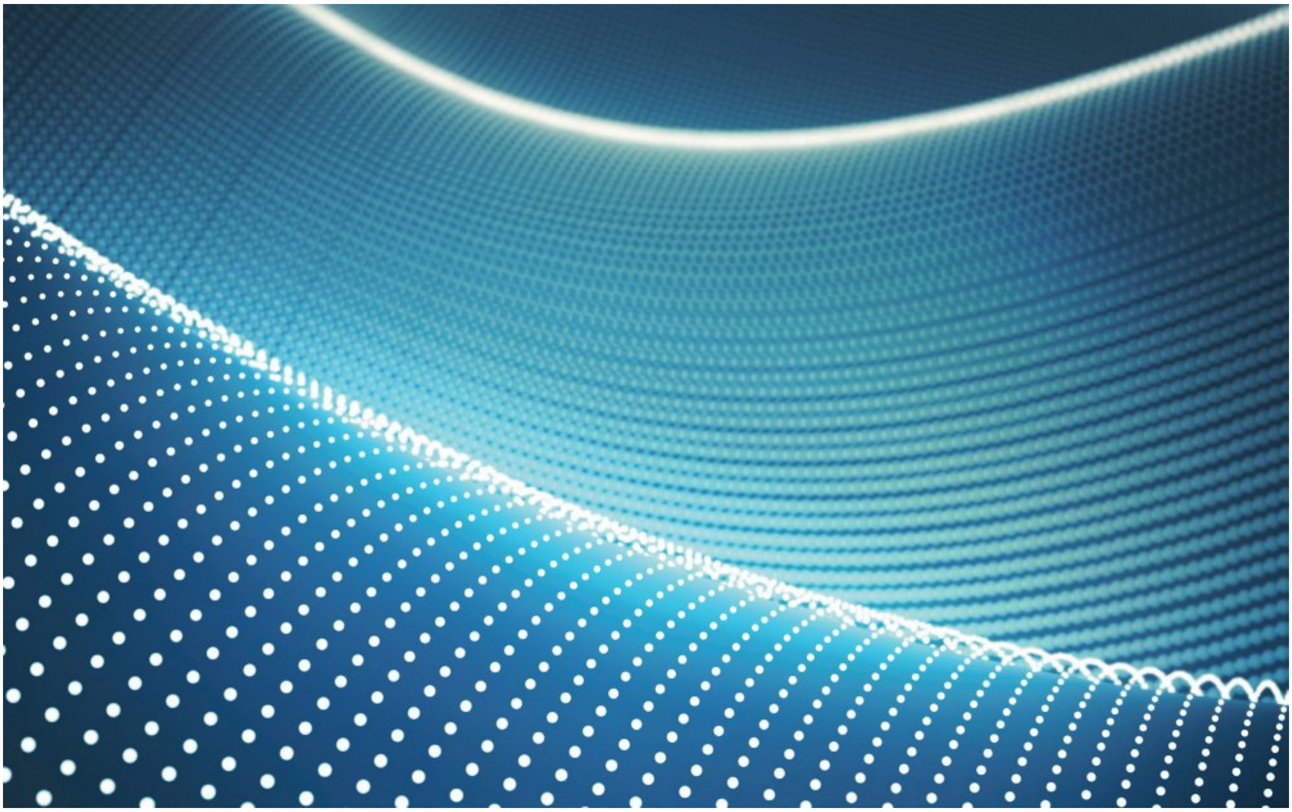
a State its domain, range and period, and whether it is even, odd or neither.

b Simplify $\sin^{-1}(\sin x)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, and sketch the function in this region.

c Use the symmetry of $y = \sin x$ in $x = \frac{\pi}{2}$ to continue the sketch for $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$.

d Use the above information and a table of values if necessary to sketch the function.

e Hence sketch $y = \cos^{-1}(\sin x)$ by making use of the fact that $\cos^{-1} t = \frac{\pi}{2} - \sin^{-1} t$.



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17D Trigonometric functions of compound angles

The derivatives of the trigonometric functions will be found in Year 12. Before that can be done, however, various formulae involving compound angles need to be established, beginning with the expansion of objects such as $\sin(x+h)$, $\tan(x-y)$ and $\cos 2x$. These trigonometric identities are most important for all sorts of other reasons as well, and must be thoroughly memorised. Their development is the concern of the rest of this chapter.

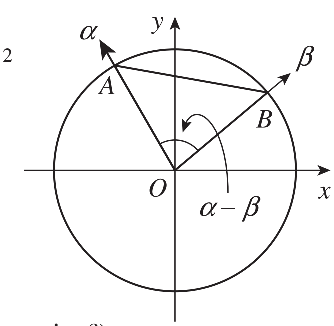
As with all fundamental results in trigonometry, these formulae require an appeal to geometry, in this case Pythagoras' theorem in the form of the distance formula, the cosine rule, and the Pythagorean trigonometric identity. The approach given here begins with the expansion of $\cos(\alpha-\beta)$ and uses that result to derive the other expansions. There are many alternative approaches.

The expansion of $\cos(\alpha-\beta)$

We shall prove that for all real numbers α and β,

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Proof



Let the rays corresponding to the angles α and β intersect the circle $x^2+y^2=r^2$ at the points A and B respectively. Then by the definitions of the trigonometric functions for general angles,

$$A = (r\cos\alpha,\ r\sin\alpha) \quad \text{and} \quad B = (r\cos\beta,\ r\sin\beta).$$

Now we can use the distance formula to find AB^2:

$$\begin{aligned} AB^2 &= r^2(\cos\alpha-\cos\beta)^2 + r^2(\sin\alpha-\sin\beta)^2 \\ &= r^2(\cos^2\alpha+\cos^2\beta+\sin^2\alpha+\sin^2\beta-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta) \\ &= 2r^2(1-\cos\alpha\cos\beta-\sin\alpha\sin\beta), \quad \text{because } \sin^2\theta+\cos^2\theta=1. \end{aligned}$$

But the angle $\angle AOB$ is $\alpha-\beta$, so by the cosine rule,

$$\begin{aligned} AB^2 &= r^2+r^2-2r^2\cos(\alpha-\beta) \\ &= 2r^2\big(1-\cos(\alpha-\beta)\big). \end{aligned}$$

Equating these two expressions for AB^2,

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Note: It was claimed in the proof that $\angle AOB = \alpha-\beta$. This is not necessarily the case, because it's also possible that $\angle AOB = \beta-\alpha$, or that $\angle AOB$ differs from either of these two values by a multiple of 2π. But the cosine function is even, and it is periodic with period 2π. So it will still follow in every case that $\cos\angle AOB = \cos(\alpha-\beta)$, which is all that is required in the proof.

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The six compound-angle formulae

Here are the compound-angle formulae for the sine, cosine and tangent functions, followed by the remaining five proofs.

10 THE COMPOUND-ANGLE FORMULAE

A $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

B $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

C $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

D $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

E $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

F $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

These formulae are of course true whether the angle is expressed in degrees or radians.

Proof

We proceed from formula E, which has already been proven.

B Replacing β by $-\beta$ in E, which is the expansion of $\cos(\alpha - \beta)$,

$$\begin{aligned}\cos(\alpha + \beta) &= \cos\big(\alpha - (-\beta)\big)\\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad \text{because cosine is even and sine is odd.}\end{aligned}$$

A Using the identity $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$,

$$\begin{aligned}\sin(\alpha + \beta) &= \cos\left(\tfrac{\pi}{2} - (\alpha + \beta)\right)\\ &= \cos\left(\left(\tfrac{\pi}{2} - \alpha\right) - \beta\right)\\ &= \cos\left(\tfrac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\tfrac{\pi}{2} - \alpha\right)\sin\beta\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.\end{aligned}$$

D Replacing β by $-\beta$, and noting that cosine is even and sine is odd,

$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

C Because $\tan\theta$ is the ratio of $\sin\theta$ and $\cos\theta$,

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}\\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}, \quad \text{dividing top and bottom by } \cos\alpha\cos\beta.\end{aligned}$$

E Replacing β by $-\beta$, and noting that the tangent function is odd, $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$.

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Example 7 17D

Express $\sin\left(x + \frac{2\pi}{3}\right)$ in the form $a\cos x + b\sin x$.

SOLUTION

$$\begin{aligned}\sin\left(x + \tfrac{2\pi}{3}\right) &= \sin x\cos\tfrac{2\pi}{3} + \cos x\sin\tfrac{2\pi}{3} \\ &= -\tfrac{1}{2}\sin x + \tfrac{1}{2}\sqrt{3}\cos x\end{aligned}$$

Example 8 17D

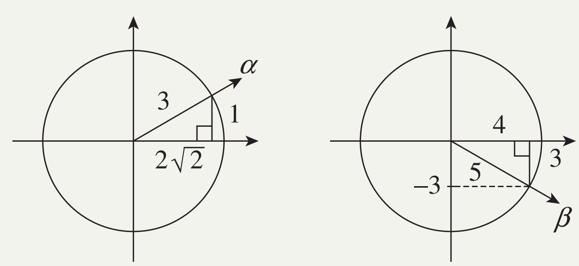
Given that $\sin\alpha = \frac{1}{3}$ and $\cos\beta = \frac{4}{5}$, where α is acute and $-\frac{\pi}{2} < \beta < 0$, find $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$.

SOLUTION

First, using the diagrams on the right, $\cos\alpha = \frac{2}{3}\sqrt{2}$ and $\sin\beta = -\frac{3}{5}$.

$$\begin{aligned}\text{So}\quad \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ &= \tfrac{4}{15} + \tfrac{6}{15}\sqrt{2} \\ &= \tfrac{2}{15}\left(2 + 3\sqrt{2}\right),\end{aligned}$$

$$\begin{aligned}\text{and}\quad \cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \tfrac{8}{15}\sqrt{2} + \tfrac{3}{15} \\ &= \tfrac{1}{15}\left(8\sqrt{2} + 3\right).\end{aligned}$$



Further exact values of trigonometric functions

The various compound-angle formulae can be used to find expressions in surds for trigonometric functions of many angles other than ones whose related angles are the standard 30°, 45° and 60°.

Example 9 17D

Find exact values of:

a $\sin 75^\circ$

b $\cos 75^\circ$

SOLUTION

There are many alternative methods.

a
$$\begin{aligned}\sin 75^\circ &= \sin(30^\circ + 45^\circ) \\ &= \sin 30^\circ\cos 45^\circ + \cos 30^\circ\sin 45^\circ \\ &= \tfrac{1}{2} \times \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{3} \times \tfrac{1}{2}\sqrt{2} \\ &= \tfrac{1}{4}\left(\sqrt{2} + \sqrt{6}\right)\end{aligned}$$

b
$$\begin{aligned}\cos 75^\circ &= \cos(135^\circ - 60^\circ) \\ &= \cos 135^\circ\cos 60^\circ + \sin 135^\circ\sin 60^\circ \\ &= -\tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2} + \tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2}\sqrt{3} \\ &= \tfrac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)\end{aligned}$$

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Exercise 17D

FOUNDATION

1 Expand using the formulae in Box 10 (one step only):

a $\sin(x - y)$ **b** $\cos(2A + 3B)$ **c** $\sin(3\alpha + 5\beta)$

d $\cos\left(\theta - \frac{\phi}{2}\right)$ **e** $\tan(A + 2B)$ **f** $\tan(3\alpha - 4\beta)$

2 Express as a single trigonometric function:

a $\cos x \cos y - \sin x \sin y$ **b** $\sin 3\alpha \cos 2\beta + \cos 3\alpha \sin 2\beta$

c $\dfrac{\tan 40^\circ - \tan 20^\circ}{1 + \tan 40^\circ \tan 20^\circ}$ **d** $\sin 5A \cos 2A - \cos 5A \sin 2A$

e $\cos 70^\circ \cos 20^\circ + \sin 70^\circ \sin 20^\circ$ **f** $\dfrac{\tan\alpha + \tan 10^\circ}{1 - \tan\alpha \tan 10^\circ}$

3 Use the compound-angle formulae to prove:

a $\sin(90^\circ + A) = \cos A$ **b** $\cos(90^\circ - A) = \sin A$ **c** $\tan(360^\circ - A) = -\tan A$

d $\tan(180^\circ + A) = \tan A$ **e** $\cos(270^\circ - A) = -\sin A$ **f** $\sin(360^\circ - A) = -\sin A$

4 Prove the identities:

a $\sin(A + 45^\circ) = \frac{1}{\sqrt{2}}(\sin A + \cos A)$ **b** $2\cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta - \sqrt{3}\sin\theta$

c $\tan\left(\frac{\pi}{4} - A\right) = \dfrac{1 - \tan A}{1 + \tan A}$ **d** $\sin(A - 30^\circ) = \frac{1}{2}\left(\sqrt{3}\sin A - \cos A\right)$

5 a If $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$, find $\tan(\alpha + \beta)$.

b If $\cos A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$, where A and B are both acute, find $\sin(A + B)$.

c If $\sin\theta = \frac{2}{3}$ and $\cos\phi = \frac{1}{4}$, where θ and ϕ are both acute, find $\cos(\theta - \phi)$.

6 Prove each identity.

a $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$

b $\cos(x - y) - \cos(x + y) = 2\sin x\sin y$

c $\sin(x + y) + \cos(x - y) = (\sin x + \cos x)(\sin y + \cos y)$

DEVELOPMENT

7 a By expressing 15° as $(45^\circ - 30^\circ)$, show that:

i $\sin 15^\circ = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$ **ii** $\cos 15^\circ = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$ **iii** $\tan 15^\circ = 2 - \sqrt{3}$

b Hence find surd expressions for:

i $\sin 75^\circ$ **ii** $\cos 75^\circ$ **iii** $\tan 75^\circ$

8 If $\sin A = \frac{2}{3}$, where $\frac{\pi}{2} < A < \pi$, and $\tan B = \frac{2}{3}$, where $\pi < B < \frac{3\pi}{2}$, show that

$$\cos(A + B) = \frac{3\sqrt{5} + 4}{3\sqrt{13}}.$$

9 Use compound-angle formulae to find the exact value of:

a $\cos 105^\circ$ **b** $\sin\dfrac{13\pi}{12}$ **c** $\cot 285^\circ$

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10 Use compound-angle formulae to find the exact value of:

a $\tan\left(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5}\right)$

b $\sin\left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{12}{13}\right)$

c $\tan\left(\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13}\right)$

d $\cos\left(\tan^{-1}\frac{1}{2} + \sin^{-1}\frac{1}{4}\right)$

11 **a** If $\alpha = \tan^{-1}x$ and $\beta = \tan^{-1}2x$, show that $\tan(\alpha + \beta) = \dfrac{3x}{1 - 2x^2}$.

b Hence solve the equation $\tan^{-1}x + \tan^{-1}2x = \tan^{-1}3$.

12 Use a similar approach to the previous question to solve for x:

a $\tan^{-1}x + \tan^{-1}2 = \tan^{-1}7$

b $\tan^{-1}3x - \tan^{-1}x = \tan^{-1}\frac{1}{2}$

13 **a** Show that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.

b Hence simplify $\sin^2 75° - \sin^2 15°$.

14 Show that:

a $\sin\left(\frac{\pi}{4} - \theta\right)\cos\left(\frac{\pi}{4} + \theta\right) + \cos\left(\frac{\pi}{4} - \theta\right)\sin\left(\frac{\pi}{4} + \theta\right) = 1$

b $\tan 35° + \tan 10° + \tan 35° \tan 10° = 1$

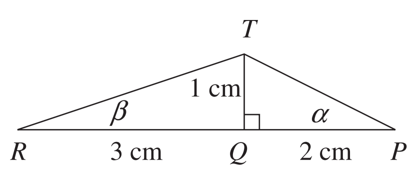
15 Prove each identity:

a $\dfrac{2\sin(x - y)}{\cos(x + y) - \cos(x - y)} = \cot x - \cot y$

b $\dfrac{\sin(\theta + \phi)}{\cos(\theta - \phi)} = \dfrac{\tan\theta + \tan\phi}{1 + \tan\theta\tan\phi}$

c $\dfrac{\sin(\alpha - \beta)}{\sin\alpha\sin\beta} + \dfrac{\sin(\beta - \gamma)}{\sin\beta\sin\gamma} + \dfrac{\sin(\gamma - \alpha)}{\sin\gamma\sin\alpha} = 0$

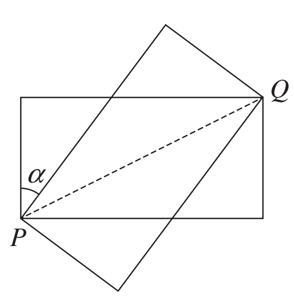
16



a Write down the formula for $\sin(\alpha + \beta)$.

b Hence show that the angles α and β in the diagram above have sum 45°.

17



The diagram shows two rectangles. Each rectangle is 6 cm long and 3 cm wide, and they share a common diagonal PQ. Show that $\tan\alpha = \frac{3}{4}$.

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ENRICHMENT

18 Solve for x:

$$\tan^{-1}\frac{x}{1+x} + \tan^{-1}\frac{x}{1-x} = \tan^{-1}\frac{6}{7}$$

19 Suppose that $\alpha = \sin^{-1}x$ and $\beta = \tan^{-1}x$.

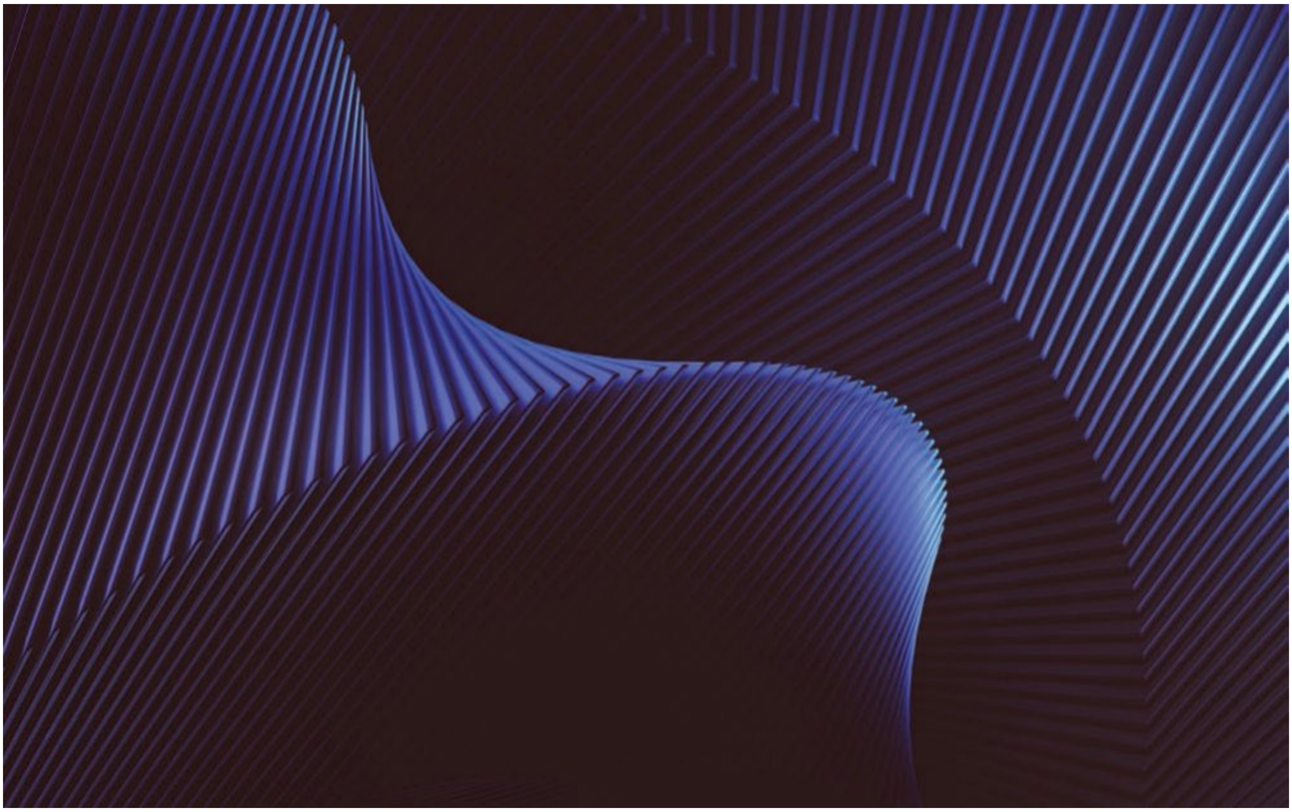
a Show that $\cos(\alpha + \beta) = \dfrac{\sqrt{1-x^2} - x^2}{\sqrt{1+x^2}}$.

b If $\alpha + \beta = \dfrac{\pi}{2}$, show that $x^2 = \dfrac{\sqrt{5}-1}{2}$.

20 a Prove the trigonometric identity

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

b Hence show that in any triangle ABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.



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17E The double-angle formulae

The double-angle formulae

Replacing both α and β by θ in the compound-angle formula $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ gives

$$\sin 2\theta = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta.$$

The same process gives expansions of $\cos 2\theta$ and $\tan 2\theta$.

11 THE DOUBLE-ANGLE FORMULAE

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

The expansion of $\cos 2\theta$ can then be combined with the Pythagorean identity to give two other forms of the expansion, first by using $\sin^2\theta = 1 - \cos^2\theta$, then by using $\cos^2\theta = 1 - \sin^2\theta$.

12 THE COS 2θ FORMULAE

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

Example 10 17E

Prove that $(\sin x + \cos x)^2 = 1 + \sin 2x$.

SOLUTION

$$\begin{aligned}\text{LHS} &= (\sin x + \cos x)^2 \\ &= \sin^2 x + \cos^2 x + 2\sin x\cos x \\ &= 1 + \sin 2x \\ &= \text{RHS}.\end{aligned}$$

Example 11 17E

Express $\cot 2x$ in terms of $\cot x$.

SOLUTION

$$\begin{aligned}\cot 2x &= \frac{1}{\tan 2x} \\ &= \frac{1 - \tan^2 x}{2\tan x} \\ &= \frac{\cot^2 x - 1}{2\cot x}, \quad \text{after division of top and bottom by } \tan^2 x.\end{aligned}$$

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Further exact values using the double-angle formulae

The double-angle formulae can be used to find exact values of the trigonometric functions at angles such as $22\frac{1}{2}^\circ$.

Example 12 17E

Find the exact value of $\tan 22\frac{1}{2}^\circ$.

SOLUTION

$$\tan 45^\circ = \frac{2\tan 22\frac{1}{2}^\circ}{1 - \tan^2 22\frac{1}{2}^\circ}$$

$$1 \times \left(1 - \tan^2 22\tfrac{1}{2}^\circ\right) = 2\tan 22\tfrac{1}{2}^\circ$$

$$\tan^2 22\tfrac{1}{2}^\circ + 2\tan 22\tfrac{1}{2}^\circ - 1 = 0.$$

$$\Delta = 4 + 4$$
$$= 4 \times 2$$

$$\tan 22\tfrac{1}{2}^\circ = \frac{-2 + 2\sqrt{2}}{2} \quad \text{or} \quad \frac{-2 - 2\sqrt{2}}{2}$$
$$= \sqrt{2} - 1, \text{ because } \tan 22\tfrac{1}{2}^\circ > 0$$

Combining double-angle formulae and inverse trigonometric functions

Example 13 combines the double-angle formulae with the methods developed in Section 17B to handle inverse trigonometric functions.

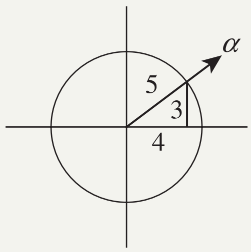
Example 13 17E

Find $\sin\left(2\cos^{-1}\frac{4}{5}\right)$.

SOLUTION

Let $\alpha = \cos^{-1}\frac{4}{5}$.

Then $\cos\alpha = \frac{4}{5}$, where $0 \le \alpha \le \pi$



Hence α is in the first quadrant,

and $\sin 2\alpha = 2\sin\alpha\cos\alpha$
$= 2 \times \frac{3}{5} \times \frac{4}{5}$
$= \frac{24}{25}$.

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Exercise 17E

FOUNDATION

1 Use the double-angle formulae to simplify:

a $2\sin x\cos x$

b $\cos^2\theta - \sin^2\theta$

c $\dfrac{2\tan\alpha}{1-\tan^2\alpha}$

d $2\sin 20^\circ\cos 20^\circ$

e $2\cos^2 50^\circ - 1$

f $\dfrac{2\tan 70^\circ}{1-\tan^2 70^\circ}$

g $2\sin 3\theta\cos 3\theta$

h $1 - 2\sin^2 2A$

i $\dfrac{2\tan 4x}{1-\tan^2 4x}$

2 Prove that:

a $(\cos A - \sin A)(\cos A + \sin A) = \cos 2A$

b $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$

c $\sin 2\theta = 2\sin\theta\sin\left(\dfrac{\pi}{2} - \theta\right)$

d $\dfrac{1}{1-\tan\theta} - \dfrac{1}{1+\tan\theta} = \tan 2\theta$

3 a If $\cos\alpha = \frac{4}{5}$, find $\cos 2\alpha$.

b If $\sin x = \frac{2}{3}$, find $\cos 2x$.

c If $\sin\theta = \frac{5}{13}$ and θ is acute, find $\sin 2\theta$.

d If $\tan A = \frac{1}{2}$, find $\tan 2A$.

DEVELOPMENT

4 If $\sin x = \frac{3}{4}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of $\sin 2x$.

5 Prove each identity.

a $\cos^4\alpha - \sin^4\alpha = \cos 2\alpha$

b $\cos 2x + \cos x = (\cos x + 1)(2\cos x - 1)$

c $\dfrac{\sin 2A}{1-\cos 2A} = \cot A$

d $\cos\theta - \sin\theta\sin 2\theta = \cos\theta\cos 2\theta$

e $\tan\left(\frac{\pi}{4} + \alpha\right) - \tan\left(\frac{\pi}{4} - \alpha\right) = 2\tan 2\alpha$

f $\dfrac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta + 1} = \tan\theta$

6 a If $\theta = \sin^{-1}\frac{3}{5}$, show that $\cos 2\theta = \frac{7}{25}$.

b Hence show that $\cos^{-1}\frac{7}{25} = 2\sin^{-1}\frac{3}{5}$.

7 Use techniques similar to the previous question to show that:

a $\tan^{-1}\frac{3}{4} = 2\tan^{-1}\frac{1}{3}$

b $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$, for $0 \le x \le 1$

8 Use a double-angle formula to find the exact value of:

a $\cos\left(2\cos^{-1}\frac{1}{3}\right)$

b $\sin\left(2\cos^{-1}\frac{6}{7}\right)$

c $\tan\left(2\tan^{-1}(-2)\right)$

9 a Show that $\dfrac{1-\cos 2x}{1+\cos 2x} = \tan^2 x$.

b Hence find the exact value of $\tan\frac{\pi}{8}$.

10 Eliminate θ from each pair of parametric equations.

a $x = 2 + \cos\theta,\ y = \cos 2\theta$

b $x = \tan\theta + 1,\ y = \tan 2\theta$

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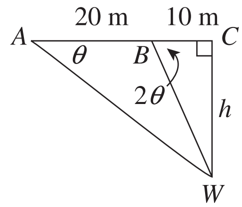
11 Points A, B, C and W lie in the same vertical plane. A bird at A observes a worm at W at an angle of depression θ. After flying 20 metres horizontally to B, the angle of depression of the worm is 2θ. If the bird flew another 10 metres horizontally it would be directly above the worm. Let $WC = h$.



a Write $\tan 2\theta$ in terms of $\tan\theta$.

b Use the two right-angled triangles to write two equations in h and θ.

c Use parts **a** and **b** to show that $\dfrac{1}{10} = \dfrac{60}{900 - h^2}$.

d Hence show that $h = 10\sqrt{3}$ metres.

e How could h have been found without trigonometry?

12 **a** By writing 3θ as $2\theta + \theta$ and using compound-angle and double-angle results, prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

b Hence show that $\cos\frac{2\pi}{9}$ is a root of the equation $8x^3 - 6x + 1 = 0$.

13 Use double-angle formulae to show that:

a $2\sin\frac{4\pi}{5}\cos\frac{\pi}{5} = \sin\frac{2\pi}{5}$

b $\cos^2\frac{4\pi}{7} - \sin^2\frac{3\pi}{7} = \cos\frac{6\pi}{7}$

14 Prove each identity.

a $\cot 2\alpha + \tan\alpha = \operatorname{cosec} 2\alpha$

b $\dfrac{\sin 3A}{\sin A} + \dfrac{\cos 3A}{\cos A} = 4\cos 2A$

c $\tan 2x \cot x = 1 + \sec 2x$

d $\dfrac{\sin 2\theta - \cos 2\theta + 1}{\sin 2\theta + \cos 2\theta - 1} = \tan\left(\theta + \dfrac{\pi}{4}\right)$

e $\dfrac{1 + \sin 2\alpha}{1 + \cos 2\alpha} = \dfrac{1}{2}(1 + \tan\alpha)^2$

f $\operatorname{cosec} 4A + \cot 4A = \frac{1}{2}(\cot A - \tan A)$

ENRICHMENT

15 Prove that $2\tan^{-1}2 = \pi - \cos^{-1}\frac{3}{5}$ (Hint: Use the fact that $\tan(\pi - x) = -\tan x$.)

16 **a** Write down the exact value of $\cos 45^\circ$.

b Hence show that:

i $\cos 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2 + \sqrt{2}}$

ii $\cos 11\frac{1}{4}^\circ = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}$

17 **a** Show that $\sqrt{8 - 4\sqrt{3}} = \sqrt{6} - \sqrt{2}$.

b Show that $\tan 165^\circ = \sqrt{3} - 2$.

c Hence show that $\tan 82\frac{1}{2}^\circ = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$.

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17F The *t*-formulae

The t-formulae express $\sin\theta$, $\cos\theta$ and $\tan\theta$ as algebraic functions of the single trigonometric function $\tan\frac{1}{2}\theta$. In the proliferation of trigonometric identities, this can often provide a systematic approach that does not rely on seeing some clever trick.

The *t*-formulae

The third t-formula in the box below is a restatement of the double-angle formula for the tangent function. The other two formulae follow quickly from it.

13 THE *t*-FORMULAE

Let $t = \tan\frac{1}{2}\theta$. Then

$$\sin\theta = \frac{2t}{1+t^2} \qquad \cos\theta = \frac{1-t^2}{1+t^2} \qquad \tan\theta = \frac{2t}{1-t^2}$$

Proof

Let $t = \tan\frac{1}{2}\theta$. We seek to express $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of t.

First, $\tan\theta = \dfrac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}$, by the double-angle formula,

$$= \frac{2t}{1-t^2}. \qquad (1)$$

Secondly, $\cos\theta = \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta$, by the double-angle formula,

$= \dfrac{\cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta}$, by the Pythagorean identity,

$= \dfrac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$, dividing through by $\cos^2\frac{1}{2}\theta$,

$$= \frac{1-t^2}{1+t^2}. \qquad (2)$$

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Thirdly, $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$, by the double-angle formula,

$$= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta}, \text{ by the Pythagorean identity,}$$

$$= \frac{2\tan\frac{1}{2}\theta}{1 + \tan^2\frac{1}{2}\theta}, \text{ dividing through by } \cos^2\frac{1}{2}\theta,$$

$$= \frac{2t}{1 + t^2}. \qquad (3)$$

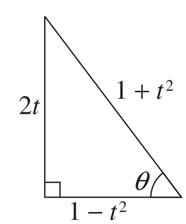
Note: The proofs given above for these identities rely heavily on the idea of expressions that are *homogeneous* of degree 2 in $\sin x$ and $\cos x$, meaning that the sum of the indices of $\sin x$ and $\cos x$ in each term is 2 — such expressions are easily converted into expressions in $\tan^2 x$ alone. Homogeneous expressions and equations will be dealt with again in Year 12.

An algebraic identity, and a way to memorise the t-formulae

On the right is a right-angled triangle that demonstrates the relationship amongst the three formulae when θ is acute. The three sides are related by Pythagoras' theorem, and the algebra rests on the quadratic identity in t^2:

$$(1 - t^2)^2 + (2t)^2 = (1 + t^2)^2.$$

This diagram may help you to memorise the t-formulae.



Example 14 — 17F

Use the t-formulae to prove:

a $\dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$

b $\sec 2x + \tan 2x = \tan\left(x + \dfrac{\pi}{4}\right)$

SOLUTION

a Let $t = \tan\frac{1}{2}x$.

$$\text{LHS} = \left(1 - \frac{1 - t^2}{1 + t^2}\right) \div \frac{2t}{1 + t^2}$$

$$= \frac{1 + t^2 - 1 + t^2}{1 + t^2} \times \frac{1 + t^2}{2t}$$

$$= \frac{2t^2}{2t}$$

$$= t$$

$$\text{RHS} = \frac{2t}{1 + t^2} \times \left(1 + \frac{1 - t^2}{1 + t^2}\right)^{-1}$$

$$= \frac{2t}{1 + t^2} \times \frac{1 + t^2}{1 + t^2 + 1 - t^2}$$

$$= \frac{2t}{2}$$

$$= t$$

$$= \text{LHS}$$

Notice that we have proven the further identity

$$\frac{1 - \cos\theta}{\sin\theta} = \frac{\sin\theta}{1 + \cos\theta} = \tan\frac{1}{2}\theta.$$

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b Let $t = \tan x$.

$$\text{LHS} = \frac{1 + t^2}{1 - t^2} + \frac{2t}{1 - t^2}$$
$$= \frac{1 + 2t + t^2}{(1 + t)(1 - t)}$$
$$= \frac{(1 + t)^2}{(1 + t)(1 - t)}$$
$$= \frac{1 + t}{1 - t}$$

$$\text{RHS} = \frac{\tan x + 1}{1 - \tan x \times 1}$$
$$= \frac{1 + t}{1 - t}$$
$$= \text{LHS}$$

Exercise 17F

FOUNDATION

1 Write in terms of t, where $t = \tan \frac{1}{2}\theta$:

a $\sin \theta$ **b** $\cos \theta$ **c** $\tan \theta$

d $\sec \theta$ **e** $1 - \cos \theta$ **f** $\dfrac{1 - \cos \theta}{\sin \theta}$

2 Write in terms of t, where $t = \tan \theta$:

a $\cos 2\theta$ **b** $1 - \sin 2\theta$ **c** $\tan 2\theta + \sec 2\theta$

3 Use the $t = \tan \frac{1}{2}\theta$ results to simplify:

a $\dfrac{2 \tan 10^\circ}{1 - \tan^2 10^\circ}$ **b** $\dfrac{2 \tan 10^\circ}{1 + \tan^2 10^\circ}$ **c** $\dfrac{1 - \tan^2 10^\circ}{1 + \tan^2 10^\circ}$

d $\dfrac{2 \tan 2x}{1 + \tan^2 2x}$ **e** $\dfrac{2 \tan 2x}{1 - \tan^2 2x}$ **f** $\dfrac{1 - \tan^2 2x}{1 + \tan^2 2x}$

4 Use the $t = \tan \frac{1}{2}\theta$ results to find the exact value of:

a $\dfrac{2 \tan 15^\circ}{1 - \tan^2 15^\circ}$ **b** $\dfrac{2 \tan 15^\circ}{1 + \tan^2 15^\circ}$ **c** $\dfrac{1 - \tan^2 75^\circ}{1 + \tan^2 75^\circ}$

d $\dfrac{2 \tan 112\frac{1}{2}^\circ}{1 + \tan^2 112\frac{1}{2}^\circ}$ **e** $\dfrac{1 - \tan^2 \frac{3\pi}{8}}{1 + \tan^2 \frac{3\pi}{8}}$ **f** $\dfrac{2 \tan \frac{11\pi}{12}}{1 - \tan^2 \frac{11\pi}{12}}$

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DEVELOPMENT

5 Prove each identity using the $t = \tan \frac{1}{2}\theta$ results.

a $\cos \theta \,(\tan \theta - \tan \frac{1}{2}\theta) = \tan \frac{1}{2}\theta$

b $\dfrac{1 - \cos 2x}{\sin 2x} = \tan x$

c $\dfrac{1 - \cos \theta}{1 + \cos \theta} = \tan^2 \frac{1}{2}\theta$

d $\dfrac{1 + \operatorname{cosec} \theta}{\cot \theta} = \dfrac{1 + \tan \frac{1}{2}\theta}{1 - \tan \frac{1}{2}\theta}$

e $\dfrac{\tan \theta \tan \frac{1}{2}\theta}{\tan \theta - \tan \frac{1}{2}\theta} = \sin \theta$

f $\dfrac{\cos \theta + \sin \theta - 1}{\cos \theta - \sin \theta + 1} = \tan \frac{1}{2}\theta$

g $\dfrac{\tan 2\alpha + \cot \alpha}{\tan 2\alpha - \tan \alpha} = \cot^2 \alpha$

h $\tan\left(\frac{1}{2}x + \frac{\pi}{4}\right) + \tan\left(\frac{1}{2}x - \frac{\pi}{4}\right) = 2 \tan x$

6 Consider the equation $\cos x - \sin x = 1$, for $0 \le x \le 2\pi$.

a Show that the equation can be written as $t^2 + t = 0$, where $t = \tan \frac{1}{2}x$.

b Hence show that $\tan \frac{1}{2}x = 0$ or -1, where $0 \le \frac{1}{2}x \le \pi$.

c Hence solve the given equation for x.

7 Consider the equation $\sqrt{3} \sin A + \cos A = 1$.

a Show that the equation can be written as $t^2 = \sqrt{3}t$, where $t = \tan \frac{1}{2}A$.

b Hence solve the equation, for $0 \le A \le 2\pi$.

8 **a** If $t = \tan 22\frac{1}{2}^\circ$, show that $\dfrac{2t}{1 - t^2} = 1$.

b **i** Hence show that $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$.

ii What does the other root of the equation represent?

9 Use the methods of the previous question to show that:

a $\tan 67\frac{1}{2}^\circ = \sqrt{2} + 1$

b $\tan 15^\circ = 2 - \sqrt{3}$

10 Suppose that $\tan \alpha = -\frac{1}{3}$ and $\dfrac{\pi}{2} < \alpha < \pi$. Find the exact value of:

a $\tan 2\alpha$

b $\sin 2\alpha$

c $\cos 2\alpha$

d $\tan \frac{1}{2}\alpha$

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11 [Alternative derivations of the t-formulae]

Let $t = \tan \frac{1}{2}\theta$.

a **i** Express $\cos \theta$ in terms of $\cos \frac{1}{2}\theta$.

ii Write $\cos^2 \frac{1}{2}\theta$ as $\dfrac{1}{\sec^2 \frac{1}{2}\theta}$, and hence show that $\cos \theta = \dfrac{1 - t^2}{1 + t^2}$

b **i** Write $\sin \theta$ in terms of $\sin \frac{1}{2}\theta$ and $\cos \frac{1}{2}\theta$.

ii Write $\sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$ as $\dfrac{\sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta} \cos^2 \frac{1}{2}\theta$, and hence show that $\sin \theta = \dfrac{2t}{1 + t^2}$.

ENRICHMENT

12 **a** If $x = \tan \theta + \sec \theta$, use the t-formulae to show that $\dfrac{x^2 - 1}{x^2 + 1} = \sin \theta$.

b If $x = \cos 2\theta$, use the t-formulae to show that $\sqrt{\dfrac{1 + x}{1 - x}} = |\cot \theta|$.

13 If $\cos x = \dfrac{5 \cos y - 3}{5 - 3 \cos y}$, prove that $\tan^2 \frac{1}{2}x = 4 \tan^2 \frac{1}{2}y$.

$\left(\text{Hint: Let } t_1 = \tan \frac{1}{2}x \text{ and } t_2 = \tan \frac{1}{2}y.\right)$

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17G Products to sums

This section develops a set of identities that convert the product of two sine or cosine functions to the sum of two sine or cosine functions. For example, we shall show that

$$2\sin 7x\cos 4x = \sin 3x + \sin 11x.$$

The product form on the left is important for purposes such as finding the zeroes of the function. The sum form on the right will be used in Year 12 for integration.

Products to sums

We begin with the four compound-angle formulae involving sine and cosine:

$$\sin(A + B) = \sin A\cos B + \cos A\sin B \qquad (1A)$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B \qquad (1B)$$
$$\cos(A + B) = \cos A\cos B - \sin A\sin B \qquad (1C)$$
$$\cos(A - B) = \cos A\cos B + \sin A\sin B \qquad (1D)$$

Adding and subtracting equations (1A) and (1B), then adding and subtracting equations (1C) and (1D), gives the four products-to-sums formulae:

14 PRODUCTS TO SUMS

$$2\sin A\cos B = \sin(A + B) + \sin(A - B)$$
$$2\cos A\sin B = \sin(A + B) - \sin(A - B)$$
$$2\cos A\cos B = \cos(A + B) + \cos(A - B)$$
$$-2\sin A\sin B = \cos(A + B) - \cos(A - B)$$

Example 15 17G

a Convert $\sin 7x\cos 4x$ to a sum of sine functions.
b Convert $\sin 7x\sin 4x$ to a sum of cosine functions.

SOLUTION

a Using the first identity above, with $A = 7x$ and $B = 4x$,

$$\sin 7x\cos 4x = \tfrac{1}{2}\sin\big((A + B) + \sin(A - B)\big)$$
$$= \tfrac{1}{2}(\sin 11x + \sin 3x).$$

b Using the fourth identity above, with $A = 7x$ and $B = 4x$,

$$\sin 7x\sin 4x = -\tfrac{1}{2}\big(\cos(A + B) - \cos(A - B)\big)$$
$$= -\tfrac{1}{2}(\cos 11x - \cos 3x).$$

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Sums to products

The previous formulae can be reversed to become formulae for sums to products by making a simple pair of substitutions. These formulae are used in Year 12 when finding the derivative of $y = \sin x$, but they are outside the course, and have therefore been placed in the Enrichment section of Exercise 17G.

Exercise 17G

FOUNDATION

1 **a** Establish each identity by expanding the RHS.

i $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

ii $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

iii $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

iv $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

b Hence express as a sum or difference of two trigonometric functions:

i $2\cos 35^\circ\cos 15^\circ$

ii $2\cos 48^\circ\sin 32^\circ$

iii $\sin 49^\circ\sin 22^\circ$

iv $\cos 61^\circ\sin 25^\circ$

v $2\sin 3\alpha\cos\alpha$

vi $\cos 2\theta\cos\theta$

vii $2\sin(x+y)\sin(x-y)$

viii $\sin(2A+B)\cos(2A-B)$

2 Show, without a calculator, that:

a $2\cos 75^\circ\cos 15^\circ = \frac{1}{2}$

b $2\sin 25^\circ\sin 35^\circ = \cos 10^\circ - \frac{1}{2}$

c $4\sin 40^\circ\cos 10^\circ = 1 + 2\sin 50^\circ$

d $2\sin 45^\circ\cos 35^\circ = \cos 10^\circ + \sin 10^\circ$

e $2\sin 54^\circ\sin 18^\circ = \sin 54^\circ - \sin 18^\circ$

f $4\cos 59^\circ\cos 61^\circ = 2\cos 2^\circ - 1$

DEVELOPMENT

3 Use the products-to-sums identities to prove that:

a $2\sin 3\theta\cos 2\theta + 2\cos 6\theta\sin\theta = \sin 7\theta + \sin\theta$

b $2\sin 4\alpha\sin 3\alpha + 2\cos 5\alpha\cos 2\alpha = \cos 3\alpha + \cos\alpha$

4 Show, without a calculator, that:

a $2\cos 40^\circ\cos 20^\circ - 2\sin 25^\circ\sin 5^\circ = \frac{1}{2}\left(\sqrt{3}+1\right)$

b $2\sin 19^\circ\cos 11^\circ + 2\sin 71^\circ\sin 11^\circ = 1$

5 Prove these identities.

a $4\cos 4x\cos 2x\cos x = \cos 7x + \cos 5x + \cos 3x + \cos x$

b $4\sin 5\alpha\cos 3\alpha\cos\alpha = \sin 9\alpha + \sin 7\alpha + \sin 3\alpha + \sin\alpha$

6 Show, without a calculator, that:

a $8\sin\frac{4\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{9} = \sqrt{3}$

b $\cos\frac{4\pi}{9}\cos\frac{2\pi}{9}\cos\frac{\pi}{9} = \frac{1}{8}$

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7 **a** Use a products-to-sums identity to prove that:

$$2\sin x(\cos 2x + \cos 4x + \cos 6x) = \sin 7x - \sin x$$

b **i** Deduce that $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$.

ii Hence evaluate $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7}$.

8 Use products to sums to solve $\sin x + 2\cos 2x\sin x = \frac{1}{2}$, for $0 \le x \le 2\pi$.

9 Use a products-to-sums identity to prove that:

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin(2n - 1)x = \frac{\sin^2 nx}{\sin x}$$

ENRICHMENT

10 Substitute $P = A + B$ and $Q = A - B$ into the products-to-sums identities to establish these sums-to-products identities.

a $\sin P + \sin Q = 2\sin\frac{1}{2}(P + Q)\cos\frac{1}{2}(P - Q)$

b $\sin P - \sin Q = 2\cos\frac{1}{2}(P + Q)\sin\frac{1}{2}(P - Q)$

c $\cos P + \cos Q = 2\cos\frac{1}{2}(P + Q)\cos\frac{1}{2}(P - Q)$

d $\cos P - \cos Q = -2\sin\frac{1}{2}(P + Q)\sin\frac{1}{2}(P - Q)$

11 Use the sums-to-products identities to prove that:

a $\sin 35^\circ + \sin 25^\circ = \sin 85^\circ$

b $\cos 36^\circ - \cos 84^\circ = \sqrt{3}\sin 24^\circ$

c $\dfrac{\sin 3\theta + \sin\theta}{\cos 3\theta + \cos\theta} = \tan 2\theta$

d $\dfrac{\sin 6\alpha - \sin 4\alpha + \sin 2\alpha}{\cos 6\alpha - \cos 4\alpha + \cos 2\alpha} = \tan 4\alpha$

12 Use the sums-to-products identities to solve these equations for $0 \le x \le 2\pi$.

a $\sin 4x - \sin 2x = 0$

b $\sin 7x + \sin 3x = 0$

c $\cos 5x + \cos 3x = 0$

d $\cos 3x - \cos 2x = 0$

13 If A, B and C are the three angles of a triangle, prove that:

a $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$

b $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$

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Chapter 17 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 17 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF worksheet version is also available there.

Review

Chapter review exercise

1 The function $f(x) = \frac{1}{2}x + 1$ is defined for $-4 \le x \le 6$.

a Find the range of $f(x)$.

b Find the equation of $f^{-1}(x)$.

c State the domain and range of $f^{-1}(x)$.

d Sketch the graphs of $f(x)$ and $f^{-1}(x)$ on the same diagram.

e About which line are the two graphs symmetrical?

2 Consider the function $F(x) = \ln(x + 1)$.

a State the domain and range of $F(x)$.

b Find the equation of $F^{-1}(x)$.

c State the domain and range of $F^{-1}(x)$.

d Sketch $F(x)$ and $F^{-1}(x)$ on the same diagram.

e Classify $F(x)$ and $F^{-1}(x)$ as increasing, decreasing or neither.

3 Consider the function $Q(x) = (x - 2)^2$.

a Sketch the graph of $Q(x)$.

b What is the largest domain containing $x = 0$ for which $Q(x)$ has an inverse function?

c Find the equation of $Q^{-1}(x)$ corresponding to the restricted domain of $Q(x)$ in part **b**.

d Sketch $Q(x)$, with its restricted domain, and $Q^{-1}(x)$ on the same diagram.

4 Write down the exact value of:

a $\sin^{-1}1$ **b** $\cos^{-1}\frac{\sqrt{3}}{2}$ **c** $\tan^{-1}\sqrt{3}$

d $\tan^{-1}(-1)$ **e** $\cos^{-1}\left(-\frac{1}{2}\right)$ **f** $\sin^{-1}\left(-\frac{1}{2}\right)$

5 Find the exact value of:

a $\cos\left(\cos^{-1}1\right)$ **b** $\sin\left(\tan^{-1}1\right)$ **c** $\cos\left(\tan^{-1}(-\sqrt{3})\right)$

d $\tan^{-1}\left(-\tan\frac{\pi}{6}\right)$ **e** $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ **f** $\tan^{-1}\left(-2\sin\frac{2\pi}{3}\right)$

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Review

6 Use a right-angled triangle to find the exact value of:

a $\sin\left(\cos^{-1}\frac{1}{3}\right)$ **b** $\cos\left(\tan^{-1}\left(-\frac{\sqrt{7}}{3}\right)\right)$

7 Sketch the graph of each function, stating the domain and range.

a $y = -\tan^{-1}x$ **b** $y = \sin^{-1}(x + 1)$ **c** $y = \cos^{-1}(x - 1) - \pi$

8 Simplify, using the compound-angle results:

a $\cos 3\theta \cos \theta + \sin 3\theta \sin \theta$ **b** $\sin 50^\circ \cos 10^\circ - \cos 50^\circ \sin 10^\circ$

c $\dfrac{\tan 41^\circ + \tan 9^\circ}{1 - \tan 41^\circ \tan 9^\circ}$ **d** $\cos 15^\circ \cos 55^\circ - \sin 15^\circ \sin 55^\circ$

e $\sin 4\alpha \cos 2\alpha + \cos 4\alpha \sin 2\alpha$ **f** $\dfrac{1 + \tan 2\theta \tan \theta}{\tan 2\theta - \tan \theta}$

9 Simplify, using the double-angle results:

a $2 \sin 2\theta \cos 2\theta$ **b** $\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x$ **c** $2\cos^2 3\alpha - 1$

d $\dfrac{2 \tan 35^\circ}{1 - \tan^2 35^\circ}$ **e** $1 - 2\sin^2 25^\circ$ **f** $\dfrac{2 \tan 4x}{1 - \tan^2 4x}$

10 Given that the angles A and B are acute, and that $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$, find:

a $\cos A$ **b** $\cos 2A$ **c** $\cos(A + B)$

d $\sin 2B$ **e** $\tan 2A$ **f** $\tan(B - A)$

11 a By writing 75° as $45^\circ + 30^\circ$, show that:

i $\sin 75^\circ = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$ **ii** $\cos 75^\circ = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$

b Hence show that:

i $\sin 75^\circ \cos 75^\circ = \frac{1}{4}$ **ii** $\sin 75^\circ - \cos 75^\circ = \sin 45^\circ$

iii $\sin^2 75^\circ - \cos^2 75^\circ = \sin 60^\circ$ **iv** $\sin^2 75^\circ + \cos^2 75^\circ = 1$

12 Use the compound-angle and double-angle results to find the exact value of:

a $2 \sin 15^\circ \cos 15^\circ$ **b** $\cos 35^\circ \cos 5^\circ + \sin 35^\circ \sin 5^\circ$

c $\dfrac{\tan 110^\circ + \tan 25^\circ}{1 - \tan 110^\circ \tan 25^\circ}$ **d** $1 - 2\sin^2 \frac{\pi}{8}$

e $\cos \frac{\pi}{12} \sin \frac{\pi}{12}$ **f** $\sin \frac{8\pi}{9} \cos \frac{2\pi}{9} - \cos \frac{8\pi}{9} \sin \frac{2\pi}{9}$

13 Prove each identity.

a $(\sin \alpha - \cos \alpha)^2 = 1 - \sin 2\alpha$ **b** $\cos A - \sin 2A \sin A = \cos A \cos 2A$

c $\sin 2\theta(\tan \theta + \cot \theta) = 2$ **d** $\cot \alpha \sin 2\alpha - \cos 2\alpha = 1$

e $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$ **f** $\dfrac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \cos 2\theta$

g $\dfrac{1}{1 - \tan A} - \dfrac{1}{1 + \tan A} = \tan 2A$ **h** $\tan 2A(\cot A - \tan A) = 2$, (provided $\cot A \neq \tan A$)

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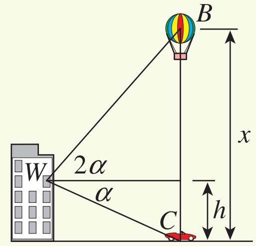
14 An office-worker is looking out a window W of a building standing on level ground. From W, a car C has an angle of depression α, while a balloon B directly above the car has an angle of elevation 2α. The height of the balloon above the car is x, and the height of the window above the ground is h.



a Show that $\dfrac{\tan \alpha}{h} = \dfrac{\tan 2\alpha}{x - h}$.

b Hence show that $\dfrac{h}{x} = \dfrac{1 - \tan^2 \alpha}{3 - \tan^2 \alpha}$.

15 **a** If $\alpha = \tan^{-1}\frac{1}{2}$ and $\beta = \tan^{-1}\frac{1}{3}$, show that $\tan\,(\alpha + \beta) = 1$.

b Hence find the exact value of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

16 Show without a calculator that:

a $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}} = \frac{\pi}{4}$

b $\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{2}{9}$

17 Show without a calculator that:

a $2\sin^{-1}\frac{2}{3} = \sin^{-1}\frac{4\sqrt{5}}{9}$

b $2\cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = \cos^{-1}\left(-\frac{3}{5}\right)$

18 Solve each equation for $0 \le x \le 2\pi$.

a $\sin 2x + \cos x = 0$

b $\cos 2x = \sin x$

c $2\cos 2x + 8\cos x + 5 = 0$

d $\tan 2x = 3\tan x$

19 Write in terms of t, where $t = \tan\frac{1}{2}\theta$:

a $\sin\theta$

b $\cos\theta$

c $\tan\theta$

d $\sec\theta$

e $1 + \cos\theta$

f $\dfrac{\sin\theta}{1 + \cos\theta}$

20 Use the $t = \tan\frac{1}{2}\theta$ results to find the exact value of:

a $\dfrac{2\tan 67\frac{1}{2}^\circ}{1 - \tan^2 67\frac{1}{2}^\circ}$

b $\dfrac{2\tan 67\frac{1}{2}^\circ}{1 + \tan^2 67\frac{1}{2}^\circ}$

c $\dfrac{1 - \tan^2 67\frac{1}{2}^\circ}{1 + \tan^2 67\frac{1}{2}^\circ}$

21 Prove each identity using the $t = \tan\frac{1}{2}\theta$ results.

a $\sin\theta\tan\frac{1}{2}\theta + \cos\theta = 1$

b $\operatorname{cosec}\theta - \cot\theta = \tan\frac{1}{2}\theta$

c $\sin 2\alpha\,(\tan 2\alpha - \tan\alpha) = \tan 2\alpha\tan\alpha$

d $\dfrac{\cos 2\alpha - \sin 2\alpha + 1}{\cos 2\alpha + \sin 2\alpha - 1} = \cot\alpha$

22 **a** If $t = \tan 75^\circ$, show that $\dfrac{2t}{1 - t^2} = -\frac{1}{\sqrt{3}}$.

b **i** Hence show that $\tan 75^\circ = 2 + \sqrt{3}$.

ii What does the other root of the equation represent?

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23 Use the products-to-sums identities to express as a sum or difference of two trigonometric functions:

a $2\cos 2A\cos A$

b $2\cos 5A\sin 3A$

c $2\sin(3A + 2B)\cos(2A - B)$

d $2\sin(2A + 5B)\sin(A + 3B)$

24 Use the products-to-sums identities to show that:

a $2\cos 58^\circ\cos 32^\circ = \cos 26^\circ$

b $2\cos 45^\circ\sin 35^\circ = \cos 10^\circ - \sin 10^\circ$

c $2\sin 70^\circ\sin 50^\circ = \frac{1}{2} + \cos 20^\circ$

d $2\sin 55^\circ\cos 40^\circ = \sin 15^\circ + \cos 5^\circ$

25 Prove these identities using products-to-sums formulae.

a $4\cos 5\alpha\cos 2\alpha\sin\alpha = \sin 8\alpha - \sin 6\alpha + \sin 4\alpha - \sin 2\alpha$

b $4\sin 5\theta\cos 3\theta\sin\theta = \cos\theta - \cos 3\theta + \cos 7\theta - \cos 9\theta$

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Answers to exercises

Answers are not provided for certain questions of the type 'show that' or 'prove that'. Please see worked solutions in these cases for a model.

Chapter 1

Exercise 1A

1 a $4x$ **b** $2x$ **c** $-2x$ **d** $-4x$
2 a $5a$ **b** $-a$ **c** $-9a$ **d** $-3a$
3 a 0 **b** $-y$ **c** $-10a$ **d** $-3b$
e $7x$ **f** $-3ab$ **g** $4pq$ **h** $-3abc$
4 a $-6a$ **b** $12a^2$ **c** a^5 **d** a^6
5 a $-2a$ **b** 3 **c** a^6 **d** a
6 a $2t^2$ **b** 0 **c** t^4 **d** 1
7 a $-3x$ **b** $-9x$ **c** $-18x^2$ **d** -2
8 a -4 **b** -12 **c** 18 **d** 2
9 a $x + 3$ **b** $2y - 3$ **c** $2a - 3$
d $8x + 4y$ **e** $-10t - 5$ **f** $4a - 3a^2$
g $-5x^2 - 12x - 3$ **h** $9a - 3b - 5c$
10 a 5 **b** $7m^2$ **c** $-12a$ **d** $-3p^3q^4r$
11 a $2x$ **b** $4x$ **c** $-6a$ **d** $-4b$
12 a $10a$ **b** $-18x$ **c** $-3a^2$
d $6a^3b$ **e** $-8x^5$ **f** $-6p^3q^4$
13 a -2 **b** $3x$ **c** xy
d $-a^4$ **e** $-7ab^3$ **f** $5ab^2c^6$
14 a $6a^5b^6$ **b** $-24a^4b^8$ **c** $9a^6$ **d** $-8a^{12}b^3$
15 a 0 **b** -1 **c** 59 **d** 40
16 a $3a^2$ **b** $5c^4$ **c** a^2bc^6
17 a $2x^5$ **b** $9xy^5$ **c** b^4 **d** $2a^3$
18 a $-x^3 + 3x^2 + 7x - 8$ **b** $-b + 11c$
c $8d - 14c - 2b$ **d** $-18x^{25}y^{22}$
19 a $0 \le x \le 2$
b $x \le -\sqrt{3}$ or $0 \le x \le \sqrt{3}$

Exercise 1B

1 a $3x - 6$ **b** $2x - 6$ **c** $-3x + 6$
d $-2x + 6$ **e** $-3x - 6$ **f** $-2x - 6$
g $-x + 2$ **h** $-2 + x$ **i** $-x - 3$
2 a $3x + 3y$ **b** $-2p + 2q$ **c** $4a + 8b$
d $x^2 - 7x$ **e** $-x^2 + 3x$ **f** $-a^2 - 4a$
g $5a + 15b - 10c$ **h** $-6x + 9y - 15z$
i $2x^2y - 3xy^2$
3 a $x + 2$ **b** $7a - 3$ **c** $2x - 4$
d $4 - 3a$ **e** $2 - x$ **f** $2c$
g $-x - y$ **h** $x + 4$ **i** $5a - 18b$
j $-2s - 10t$ **k** $x^2 + 17xy$ **l** $16a - b$
4 a $x^2 + 5x + 6$ **b** $y^2 + 11y + 28$
c $t^2 + 3t - 18$ **d** $x^2 - 2x - 8$
e $t^2 - 4t + 3$ **f** $2a^2 + 13a + 15$
g $3u^2 - 10u - 8$ **h** $8p^2 - 2p - 15$
i $2b^2 - 13b + 21$ **j** $15a^2 - a - 2$
k $-c^2 + 9c - 18$ **l** $2d^2 + 5d - 12$
6 a $x^2 + 2xy + y^2$ **b** $x^2 - 2xy + y^2$
c $x^2 - y^2$ **d** $a^2 + 6a + 9$
e $b^2 - 8b + 16$ **f** $c^2 + 10c + 25$
g $d^2 - 36$ **h** $49 - e^2$
i $64 + 16f + f^2$ **j** $81 - 18g + g^2$
k $h^2 - 100$ **l** $i^2 + 22i + 121$
m $4a^2 + 4a + 1$ **n** $4b^2 - 12b + 9$
o $9c^2 + 12c + 4$ **p** $4d^2 + 12de + 9e^2$
q $4f^2 - 9g^2$ **r** $9h^2 - 4i^2$
s $25j^2 + 40j + 16$ **t** $16k^2 - 40k\ell + 25\ell^2$
u $16 - 25m^2$ **v** $25 - 30n + 9n^2$
w $49p^2 + 56pq + 16q^2$ **x** $64 - 48r + 9r^2$
7 a $t^2 + 2 + \frac{1}{t^2}$ **b** $t^2 - 2 + \frac{1}{t^2}$ **c** $t^2 - \frac{1}{t^2}$
8 a $10\,404$ **b** $998\,001$ **c** $39\,991$
9 a $a^3 - b^3$ **b** $2x + 3$
c $18 - 6a$ **d** $x^2 + 2x - 1$
e $x^3 - 6x^2 + 12x - 8$ **f** $p^2 + q^2 + r^2$
10 a $x^3 - 6x^2 + 12x - 8$ **b** $x^2 + y^2 + z^2$
c $x^2 - y^2 - z^2 + 2yz$ **d** $a^3 + b^3 + c^3 - 3abc$
11 a $a^2 - b^2 - c^2 + 2bc$ **b** $x^2 - 2x + 3$
c $7x^2 + 16ax + 4a^2$
12 7

Exercise 1C

1 a $2(x + 4)$ **b** $3(2a - 5)$
c $a(x - y)$ **d** $5a(4b - 3c)$
e $x(x + 3)$ **f** $p(p + 2q)$
g $3a(a - 2b)$ **h** $6x(2x + 3)$
i $4c(5d - 8)$ **j** $ab(a + b)$
k $2a^2(3 + a)$ **l** $7x^2y(x - 2y)$
2 a $(p + q)(m + n)$ **b** $(x - y)(a + b)$
c $(x + 3)(a + 2)$ **d** $(a + b)(a + c)$
e $(z - 1)(z^2 + 1)$ **f** $(a + b)(c - d)$

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g $(p - q)(u - v)$ **h** $(x - 3)(x - y)$
i $(p - q)(5 - x)$ **j** $(2a - b)(x - y)$
k $(b + c)(a - 1)$ **l** $(x + 4)(x^2 - 3)$
m $(a - 3)(a^2 - 2)$ **n** $(2t + 5)(t^2 - 5)$
o $(x - 3)(2x^2 - a)$

3 a $(a - 1)(a + 1)$ **b** $(b - 2)(b + 2)$
c $(c - 3)(c + 3)$ **d** $(d - 10)(d + 10)$
e $(5 - y)(5 + y)$ **f** $(1 - n)(1 + n)$
g $(7 - x)(7 + x)$ **h** $(12 - p)(12 + p)$
i $(2c - 3)(2c + 3)$ **j** $(3u - 1)(3u + 1)$
k $(5x - 4)(5x + 4)$ **l** $(1 - 7k)(1 + 7k)$
m $(x - 2y)(x + 2y)$ **n** $(3a - b)(3a + b)$
o $(5m - 6n)(5m + 6n)$ **p** $(9ab - 8)(9ab + 8)$

4 a $(a + 1)(a + 2)$ **b** $(k + 2)(k + 3)$
c $(m + 1)(m + 6)$ **d** $(x + 3)(x + 5)$
e $(y + 4)(y + 5)$ **f** $(t + 2)(t + 10)$
g $(x - 1)(x - 3)$ **h** $(c - 2)(c - 5)$
i $(a - 3)(a - 4)$ **j** $(b - 2)(b - 6)$
k $(t + 2)(t - 1)$ **l** $(u - 2)(u + 1)$
m $(w - 4)(w + 2)$ **n** $(a + 4)(a - 2)$
o $(p - 5)(p + 3)$ **p** $(y + 7)(y - 4)$
q $(c - 3)(c - 9)$ **r** $(u - 6)(u - 7)$
s $(x - 10)(x + 9)$ **t** $(x + 8)(x - 5)$
u $(t - 8)(t + 4)$ **v** $(p + 12)(p - 3)$
w $(u - 20)(u + 4)$ **x** $(t + 25)(t - 2)$

5 a $(3x + 1)(x + 1)$ **b** $(2x + 1)(x + 2)$
c $(3x + 1)(x + 5)$ **d** $(3x + 2)(x + 2)$
e $(2x - 1)(x - 1)$ **f** $(5x - 3)(x - 2)$
g $(5x - 6)(x - 1)$ **h** $(3x - 1)(2x - 3)$
i $(2x - 3)(x + 1)$ **j** $(2x + 5)(x - 1)$
k $(3x + 5)(x - 1)$ **l** $(3x - 1)(x + 5)$
m $(2x + 3)(x - 5)$ **n** $(2x - 5)(x + 3)$
o $(6x - 1)(x + 3)$ **p** $(2x - 3)(3x + 1)$
q $(3x - 2)(2x + 3)$ **r** $(5x + 3)(x + 4)$
s $(5x - 6)(x + 2)$ **t** $(5x - 4)(x - 3)$
u $(5x + 4)(x - 3)$ **v** $(5x - 2)(x + 6)$
w $(3x - 4)(3x + 2)$ **x** $(3x - 5)(x + 6)$

6 a $(a - 5)(a + 5)$ **b** $b(b - 25)$
c $(c - 5)(c - 20)$ **d** $(2d + 5)(d + 10)$
e $(e + 5)(e^2 + 5)$ **f** $(4 - f)(4 + f)$
g $g^2(16 - g)$ **h** $(h + 8)^2$
i $(i - 18)(i + 2)$ **j** $(j + 4)(5j - 4)$
k $(2k + 1)(2k - 9)$ **l** $(k - 8)(2k^2 - 3)$
m $(2a + b)(a - 2)$ **n** $3m^2n^4(2m + 3n)$
o $(7p - 11q)(7p + 11q)$ **p** $(t - 4)(t - 10)$
q $(3t - 10)(t + 4)$ **r** $(5t + 4)(t + 10)$
s $(5t + 8)(t + 5)$ **t** $5t(t^2 + 2t + 3)$
u $(u + 18)(u - 3)$ **v** $(3x - 2y)(x^2 - 5)$
w $(p + q - r)(p + q + r)$ **x** $(2a - 3)^2$

7 a $3(a - 2)(a + 2)$
b $(x - y)(x + y)(x^2 + y^2)$
c $x(x - 1)(x + 1)$ **d** $5(x + 2)(x - 3)$
e $y(5 - y)(5 + y)$
f $(2 - a)(2 + a)(4 + a^2)$
g $2(2x - 3)(x + 5)$ **h** $a(a + 1)(a^2 + 1)$
i $(c + 1)(c - 1)(c + 9)$
j $x(x - 1)(x - 7)$
k $(x - 2)(x + 2)(x^2 + 1)$
l $(x - 1)(x + 1)(a - 2)$

8 a $(2p - q - r)(2p + q + r)$
b $(a - b)(a + b - 1)$
c $a(a - 4b)(a - 6b)$ **d** $x^2(3x - 2)(2x + 1)$
e $(2x - 1)(2x + 1)(x - 3)(x + 3)$
f $2(4 - 5x)(5 + 4x)$
g $(2x - 1)(2x + 1)(x - 3)$
h $(x + a - b)(x + a + b)$
i $(x^2 - x - 1)(x^2 + x + 1)$

9 a $(a + b)(a + b^2)$ **b** $(x - y)(x + y)^3$
c $4ab(a - b)^2$
d $(2x^2 + 3y^2)(2x - 3y)(x + y)$
e $(a - b - c)(a + b + c)(a - b + c)(a + b - c)$
f $(x^2 + y^2)(a^2 + b^2 + c^2)$
g Add and subtract a^2b^2.
$(a^2 - ab + b^2)(a^2 + ab + b^2)$
h Add and subtract $4a^2b^2$.
$(a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$

Exercise 1D

1 a 1 **b** 2 **c** $\frac{1}{2}$ **d** $\frac{1}{a}$ **e** $\frac{x}{3y}$ **f** $\frac{3}{a}$

2 a 1 **b** $\frac{1}{2}$ **c** $3x$ **d** $\frac{b}{2}$
e $\frac{3}{2x}$ **f** $\frac{1}{2a}$ **g** $\frac{4}{b}$ **h** 6

3 a $\frac{3x}{2}$ **b** $\frac{3y}{4}$ **c** $\frac{2m}{9}$ **d** $\frac{7n}{10}$
e $\frac{3x - 2y}{24}$ **f** $\frac{13a}{6}$ **g** $\frac{b}{15}$ **h** $-\frac{xy}{20}$

4 a $\frac{2}{a}$ **b** $-\frac{1}{x}$ **c** $\frac{3}{2a}$ **d** $\frac{1}{6x}$ **e** $\frac{25}{12a}$ **f** $\frac{1}{2x}$

5 a $\frac{5x + 7}{6}$ **b** $\frac{18x + 11}{20}$ **c** $\frac{x + 1}{4}$
d $\frac{x}{6}$ **e** $\frac{2x + 17}{20}$ **f** $\frac{2x - 3}{6}$

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6 a 2 **b** $\frac{3}{2}$ **c** $\frac{x}{3}$ **d** $\frac{1}{x+y}$ **e** $\frac{3}{2b}$

f $\frac{x}{x-2}$ **g** $\frac{a+3}{a+4}$ **h** $\frac{x+1}{x-1}$ **i** $\frac{x+5}{x+4}$

7 a $\frac{2x+1}{x(x+1)}$ **b** $\frac{1}{x(x+1)}$

c $\frac{2x}{(x+1)(x-1)}$ **d** $\frac{5x-13}{(x-2)(x-3)}$

e $\frac{x-5}{(x+1)(x-1)}$ **f** $\frac{10}{(x+3)(x-2)}$

8 a $\frac{3x}{2(x-1)}$ **b** a **c** $\frac{c+2}{c+4}$

d x **e** $\frac{3x-1}{a+b}$ **f** $\frac{x-7}{3(x+3)}$

9 a -1 **b** $-u-v$ **c** $3-x$

d $\frac{2}{a-b}$ **e** 1 **f** $\frac{-1}{2x+y}$

10 a $\frac{2}{x^2-1}$ **b** $\frac{2x}{(x-2)^2(x+2)}$

c $\frac{3x}{x^2-y^2}$

d $\frac{x+1}{(x-2)(x+3)(x+4)}$

e $\frac{bx}{a(a-b)(a+b)}$

f $\frac{x}{(x-1)(x-2)(x-3)}$

11 a $\frac{1}{3}$ **b** $\frac{7}{13}$ **c** $\frac{3}{11}$ **d** $\frac{1}{5}$

e $\frac{1}{x+2}$ **f** $\frac{t^2-1}{t^2+1}$ **g** $\frac{ab}{a+b}$

h $\frac{x^2+y^2}{x^2-y^2}$ **i** $\frac{x^2}{2x+1}$ **j** $\frac{x-1}{x-3}$

13 a $\frac{a-b+c}{ab}$ **b** $\frac{2x+3}{3x-1}$

c $\frac{4}{x+2y}$ **d** $\frac{2}{(x+1)^2(x-1)}$

14 a 0 **b** 3 **c** $\frac{3n-m}{2}$ **d** $\frac{1}{x}$

Exercise 1E

1 a $x=3$ **b** $p=0$ **c** $a=8$
d $w=-1$ **e** $x=9$ **f** $x=-5$
g $x=-16$ **h** $x=-2$

2 a $n=4$ **b** $b=-1$ **c** $x=4$
d $x=-11$ **e** $a=-\frac{1}{2}$ **f** $y=2$
g $x=\frac{7}{9}$ **h** $x=-\frac{3}{5}$

3 a $a=8$ **b** $y=16$ **c** $x=\frac{1}{3}$
d $a=\frac{2}{5}$ **e** $y=\frac{3}{2}$ **f** $x=-8$
g $a=7$ **h** $x=-\frac{1}{2}$ **i** $a=-5$
j $t=\frac{3}{5}$ **k** $x=-2$ **l** $x=5$

4 a $y=\frac{2}{3}$ **b** $x=15$ **c** $a=-15$
d $x=\frac{9}{2}$ **e** $x=6$ **f** $x=\frac{1}{6}$
g $x=\frac{1}{2}$ **h** $x=20$ **i** $x=-\frac{23}{2}$
j $x=-\frac{7}{3}$

5 a $a=3$ **b** $s=16$ **c** $v=\frac{2}{3}$
d $\ell=21$ **e** $C=35$ **f** $c=-\frac{2}{5}$

6 a $b=\frac{a+d}{c}$ **b** $n=\frac{t-a+d}{d}$

c $r=\frac{p-qt}{t}$ **d** $v=\frac{3}{u-1}$

7 a $x=\frac{19}{6}$ **b** $x=\frac{3}{14}$ **c** $x=-1$ **d** $x=\frac{17}{6}$

8 a $a=-11$ **b** $x=2$ **c** $x=-\frac{7}{3}$ **d** $x=-\frac{5}{2}$

9 a -4 **b** 7 **c** 36 **d** 80 litres
e 15 min **f** 16 **g** 30 km/h **h** 5 hours

10 a $a=-\frac{2b}{3}$ **b** $g=\frac{2fh}{5f-h}$

c $y=\frac{2x}{1-x}$ **d** $b=\frac{4a+5}{a-1}$

11 a $x=\frac{14}{5}$ **b** $a=4$

12 a $x=6$

Exercise 1F

1 a $x=3$ or -3 **b** $y=5$ or -5
c $a=2$ or -2 **d** $c=6$ or -6
e $t=1$ or -1 **f** $x=\frac{3}{2}$ or $-\frac{3}{2}$
g $x=\frac{1}{2}$ or $-\frac{1}{2}$ **h** $a=2\frac{2}{3}$ or $-2\frac{2}{3}$
i $y=\frac{4}{5}$ or $-\frac{4}{5}$

2 a $x=0$ or 5 **b** $y=0$ or -1
c $c=0$ or -2 **d** $k=0$ or 7
e $t=0$ or 1 **f** $a=0$ or 3
g $b=0$ or $\frac{1}{2}$ **h** $u=0$ or $-\frac{1}{3}$
i $x=-\frac{3}{4}$ or 0 **j** $a=0$ or $\frac{5}{2}$
k $y=0$ or $\frac{2}{3}$ **l** $n=0$ or $-\frac{3}{5}$

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3 a $x = -3$ or -1 **b** $x = 1$ or 2
c $x = -4$ or -2 **d** $a = 2$ or 5
e $t = -2$ or 6 **f** $c = 5$
g $n = 1$ or 8 **h** $p = -5$ or 3
i $a = -2$ or 12 **j** $y = -5$ or 1
k $p = -2$ or 3 **l** $a = -11$ or 12
m $c = 3$ or 6 **n** $t = -2$ or 10
o $u = -8$ or 7 **p** $k = -4$ or 6
q $h = -25$ or -2 **r** $a = -22$ or 2

4 a $x = -\frac{1}{2}$ or -1 **b** $a = \frac{1}{3}$ or 2
c $y = \frac{1}{4}$ or 1 **d** $x = -5$ or $-\frac{1}{2}$
e $x = -1\frac{1}{2}$ or 1 **f** $n = -1$ or $1\frac{2}{3}$
g $b = -\frac{2}{3}$ or 2 **h** $a = -5$ or $1\frac{1}{2}$
i $y = -2\frac{1}{2}$ or 3 **j** $y = -4$ or $\frac{2}{3}$
k $x = \frac{1}{5}$ or 5 **l** $t = \frac{3}{4}$ or 3
m $t = -\frac{2}{5}$ or 3 **n** $u = -\frac{4}{5}$ or $\frac{1}{2}$
o $x = \frac{1}{5}$ **p** $x = -\frac{2}{3}$ or $-\frac{3}{2}$
q $b = -\frac{3}{2}$ or $-\frac{1}{6}$ **r** $k = -\frac{8}{3}$ or $\frac{1}{2}$

5 a $x = \frac{1 + \sqrt{5}}{2}$ or $\frac{1 - \sqrt{5}}{2}$, $x \doteqdot 1.618 \doteqdot$ or -0.6180
b $x = \frac{-1 + \sqrt{13}}{2}$ or $\frac{-1 - \sqrt{13}}{2}$, $x \doteqdot 1.303$ or -2.303
c $a = 3$ or 4
d $u = -1 + \sqrt{3}$ or $-1 - \sqrt{3}$, $u \doteqdot 0.7321$ or -2.732
e $c = 3 + \sqrt{7}$ or $3 - \sqrt{7}$, $c \doteqdot 5.646$ or 0.3542
f $x = -\frac{1}{2}$
g $a = \frac{2 + \sqrt{2}}{2}$ or $\frac{2 - \sqrt{2}}{2}$, $a \doteqdot 1.707$ or 0.2929
h $x = -3$ or $\frac{2}{5}$
i $b = \frac{-3 + \sqrt{17}}{4}$ or $\frac{-3 - \sqrt{17}}{4}$, $b \doteqdot 0.2808$ or -1.781
j $c = \frac{2 + \sqrt{13}}{3}$ or $\frac{2 - \sqrt{13}}{3}$, $c \doteqdot 1.869$ or -0.5352
k $t = \frac{1 + \sqrt{5}}{4}$ or $\frac{1 - \sqrt{5}}{4}$, $t \doteqdot 0.8090$ or -0.3090
l no solutions

6 a $x = -1$ or 2 **b** $a = 2$ or 5
c $y = \frac{1}{2}$ or 4 **d** $b = -\frac{2}{5}$ or $\frac{2}{3}$

7 a $x = 1 + \sqrt{2}$ or $1 - \sqrt{2}$
b $x = 2 + \sqrt{3}$ or $2 - \sqrt{3}$
c $a = 1 + \sqrt{5}$ or $1 - \sqrt{5}$
d $m = \frac{2 + \sqrt{14}}{5}$ or $\frac{2 - \sqrt{14}}{5}$

8 a $p = \frac{1}{2}$ or 1 **b** $x = -3$ or 5 **c** $n = 5$
9 a 7 **b** 6 and 9 **c** $x = 15$
10 a $a = 2b$ or $a = 3b$ **b** $a = -2b$ or $a = \frac{b}{3}$
11 a $y = 2x$ or $y = -2x$ **b** $y = \frac{x}{11}$ or $y = -\frac{x}{2}$
12 a $k = -1$ or 3 **b** $u = \frac{4}{3}$ or 4
c $y = 1 + \sqrt{6}$ or $1 - \sqrt{6}$
d $k = \frac{-5 + \sqrt{73}}{4}$ or $\frac{-5 - \sqrt{73}}{4}$
e $a = -\frac{7}{3}$ or 3 **f** $k = -4$ or 15
g $t = 2\sqrt{3}$ or $-\sqrt{3}$
h $m = \frac{1 + \sqrt{2}}{3}$ or $\frac{1 - \sqrt{2}}{3}$
13 a 4 cm **b** 3 cm
c 55 km/h and 60 km/h
14 a $x = 2c$ or $x = \frac{11c}{14}$, where $c \neq 0$.
b $x = a$ or $x = \frac{ab}{a - 2b}$, provided that $a \neq 2b$.

Exercise 1G

1 a $x = 3, y = 3$ **b** $x = 2, y = 4$
c $x = 2, y = 1$ **d** $a = -3, b = -2$
e $p = 3, q = -1$ **f** $u = 1, v = -2$
2 a $x = 3, y = 2$ **b** $x = 1, y = -2$
c $x = 4, y = 1$ **d** $a = -1, b = 3$
e $c = 2, d = 2$ **f** $p = -2, q = -3$
3 a $x = 2, y = 4$ **b** $x = -1, y = 3$
c $x = 2, y = 2$ **d** $x = 9, y = 1$
e $x = 3, y = 4$ **f** $x = 4, y = -1$
g $x = 5, y = 3\frac{3}{5}$ **h** $x = 13, y = 7$
4 a $x = -1, y = 3$ **b** $x = 5, y = 2$
c $x = -4, y = 3$ **d** $x = 2, y = -6$
e $x = 1, y = 2$ **f** $x = 16, y = -24$
g $x = 1, y = 6$ **h** $x = 5, y = -2$
i $x = 5, y = 6$ **j** $x = 7, y = 5$
5 a $x = 1$ & $y = 1$, or $x = -2$ & $y = 4$
b $x = 2$ & $y = 1$, or $x = 4$ & $y = 5$
c $x = 0$ & $y = 0$, or $x = 1$ & $y = 3$
d $x = -2$ & $y = -7$, or $x = 3$ & $y = -2$
e $x = -3$ & $y = -5$, or $x = 5$ & $y = 3$
f $x = 1$ & $y = 6$, or $x = 2$ & $y = 3$

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6 a 53 and 37
b The pen cost 60c, the pencil cost 15c.
c Each apple cost 40c, each orange cost 60c.
d 44 adults, 22 children
e The man is 36, the son is 12.
f 189 for, 168 against
g 9 \$20 notes, 14 \$10 notes
h 5km/h, 3km/h

7 a $x = 12, y = 20$
b $x = 3, y = 2$

8 a $x = 5 \,\&\, y = 10$, or $x = 10 \,\&\, y = 5$
b $x = -8 \,\&\, y = -11$, or $x = 11 \,\&\, y = 8$
c $x = \frac{1}{2} \,\&\, y = 4$, or $x = 10 \,\&\, y = 5$
d $x = 4 \,\&\, y = 5$, or $x = 5 \,\&\, y = 4$
e $x = 1 \,\&\, y = 2$, or $x = \frac{3}{2} \,\&\, y = \frac{7}{4}$
f $x = 2 \,\&\, y = 5$, or $x = \frac{10}{3} \,\&\, y = 3$

9 a $x = 1 \,\&\, y = \frac{5}{4}$
b $x = 2 \,\&\, y = 4$, or $x = -2 \,\&\, y = -4$, or $x = \frac{4}{3} \,\&\, y = 6$, or $x = -\frac{4}{3} \,\&\, y = -6$

10 a $x = 1 \,\&\, y = -2$, or $x = -1 \,\&\, y = 2$, or $x = \frac{7}{3} \,\&\, y = \frac{2}{3}$, or $x = -\frac{7}{3} \,\&\, y = -\frac{2}{3}$

Exercise 1H

1 a 1 **b** 9 **c** 25 **d** 81
e $\frac{9}{4}$ **f** $\frac{1}{4}$ **g** $\frac{25}{4}$ **h** $\frac{81}{4}$

2 a $(x + 2)^2$ **b** $(y + 1)^2$
c $(p + 7)^2$ **d** $(m - 6)^2$
e $(t - 8)^2$ **f** $(x + 10)^2$
g $(u - 20)^2$ **h** $(a - 12)^2$

3 a $x^2 + 6x + 9 = (x + 3)^2$
b $y^2 + 8y + 16 = (y + 4)^2$
c $a^2 - 20a + 100 = (a - 10)^2$
d $b^2 - 100b + 2500 = (b - 50)^2$
e $u^2 + u + \frac{1}{4} = \left(u + \frac{1}{2}\right)^2$
f $t^2 - 7t + \frac{49}{4} = \left(t - \frac{7}{2}\right)^2$
g $m^2 + 50m + 625 = (m + 25)^2$
h $c^2 - 13c + \frac{169}{4} = \left(c - \frac{13}{2}\right)^2$

4 a $x = -1$ or 3 **b** $x = 0$ or 6
c $a = -4$ or -2
d $x = -2 + \sqrt{3}$ or $-2 - \sqrt{3}$
e $x = 5 + \sqrt{5}$ or $5 - \sqrt{5}$
f $y = -5$ or 2 **g** $b = -2$ or 7
h no solution for y
i $a = \frac{-7 + \sqrt{21}}{2}$ or $\frac{-7 - \sqrt{21}}{2}$

5 a $x = \frac{2 + \sqrt{6}}{2}$ or $\frac{2 - \sqrt{6}}{2}$
b $x = \frac{-4 + \sqrt{10}}{2}$ or $\frac{-4 - \sqrt{10}}{2}$
c no solution for x **d** $x = -\frac{3}{2}$ or $\frac{1}{2}$
e $x = \frac{1 + \sqrt{5}}{4}$ or $\frac{1 - \sqrt{5}}{4}$
f $x = \frac{5 + \sqrt{11}}{2}$ or $\frac{5 - \sqrt{11}}{2}$

6 a $a = 3, b = 4$ and $c = 25$
b $A = -5, B = 6$ and $C = 8$

7 a $x^3 + 12x^2 + 48x + 64 = (x + 4)^3$
b $u = x + 4, u^3 - 18u + 12 = 0$

Chapter 1 review exercise

1 a $-6y$ **b** $-10y$ **c** $-16y^2$ **d** -4
2 a $-3a^2$ **b** $-a^2$ **c** $2a^4$ **d** 2
3 a $2t - 1$ **b** $4p + 3q$
c $x - 2y$ **d** $5a^2 - 3a - 18$
4 a $-18k^9$ **b** $-2k^3$ **c** $36k^{12}$ **d** $27k^9$
5 a $14x - 3$ **b** $-4a + 2b$
c $-2a$ **d** $-6x^3 - 10x^2$
e $2n^2 + 11n - 21$ **f** $r^2 + 6r + 9$
g $y^2 - 25$ **h** $6x^2 - 19x + 15$
i $t^2 - 16t + 64$ **j** $4c^2 - 49$
k $16p^2 + 8p + 1$ **l** $9u^2 - 12u + 4$
6 a $18(a + 2)$ **b** $4(5b - 9)$
c $9c(c + 4)$ **d** $(d - 6)(d + 6)$
e $(e + 4)(e + 9)$ **f** $(f - 6)^2$
g $(6 - 5g)(6 + 5g)$ **h** $(h - 12)(h + 3)$
i $(i + 9)(i - 4)$ **j** $(2j + 3)(j + 4)$
k $(3k + 2)(k - 3)$ **l** $(5\ell - 4)(\ell - 2)$
m $(2m - 3)(2m + 5)$ **n** $(n + 1)(m + p)$
o $(p + 9)(p^2 + 4)$ **p** $(q - r)(t - 5)$
q $(u^2 + v)(w - x)$ **r** $(x - y)(x + y + 2)$
7 a $\frac{3x}{4}$ **b** $\frac{x}{4}$ **c** $\frac{x^2}{8}$ **d** 2
e $\frac{13a}{6b}$ **f** $\frac{5a}{6b}$ **g** $\frac{a^2}{b^2}$ **h** $\frac{9}{4}$
i $\frac{x^2 + y^2}{xy}$ **j** $\frac{x^2 - y^2}{xy}$ **k** 1 **l** $\frac{x^2}{y^2}$
8 a $\frac{8x - 13}{15}$ **b** $\frac{8x - 13}{(x + 4)(x - 5)}$
c $\frac{3x + 13}{10}$ **d** $\frac{-3x - 13}{(x + 1)(x - 4)}$

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e $\frac{x-3}{4}$ **f** $\frac{-2x+6}{x(x+3)}$

9 a $\frac{3}{5}$ **b** $\frac{2}{x+y}$ **c** $\frac{x+3}{x-4}$

d $\frac{x+1}{x^2+1}$ **e** $\frac{1}{a+b}$ **f** $\frac{x-7}{3x-2}$

10 a $x = 4$ **b** $x = \frac{2}{3}$ **c** $x = 46$ **d** $x = 36$

e $a = 3$ **f** $a = 10$ **g** $a = -17$ **h** $a = -42$

11 a $a = -7$ or 7 **b** $b = -7$ or 0

c $c = -6$ or -1 **d** $d = -7$ or 1

e $e = 2$ or 3 **f** $f = -\frac{3}{2}$ or 2

g $g = \frac{1}{2}$ or 6 **h** $h = -2$ or $\frac{4}{3}$

12 a $x = 2 + \sqrt{3}$ or $2 - \sqrt{3}$

b $y = \frac{-3 + \sqrt{21}}{2}$ or $\frac{-3 - \sqrt{21}}{2}$

c $y = -3 + \sqrt{5}$ or $-3 - \sqrt{5}$

d $y = \frac{1 + \sqrt{7}}{3}$ or $\frac{1 - \sqrt{7}}{3}$

e $y = \frac{-5 + \sqrt{65}}{4}$ or $\frac{-5 - \sqrt{65}}{4}$

f $y = \frac{3 + \sqrt{13}}{4}$ or $\frac{3 - \sqrt{13}}{4}$

13 a $x = -2 + \sqrt{10}$ or $-2 - \sqrt{10}$

b $x = 3 + \sqrt{6}$ or $3 - \sqrt{6}$

c $x = 1 + \sqrt{13}$ or $1 - \sqrt{13}$

d $x = -5 + 3\sqrt{2}$ or $-5 - 3\sqrt{2}$

14 11

15 $(x^2 - x + 1)(x^2 + x + 1)$

16 $\frac{1}{2(x+2)}$

17 $\frac{13x}{(x+1)(x+2)(x+3)}$

18 $x = -9$ or $\frac{5}{3}$

19 $x = 2$ & $y = 0$, or $x = -2$ & $y = 0$, or $x = \sqrt{3}$ & $y = -1$, or $x = -\sqrt{3}$ & $y = -1$

Chapter 2

Exercise 2A

1 a $\frac{3}{10}$ **b** $\frac{4}{5}$ **c** $\frac{3}{4}$ **d** $\frac{1}{20}$

2 a 0.6 **b** 0.27 **c** 0.09 **d** 0.165

3 a 25% **b** 40% **c** 24% **d** 65%

4 a 32% **b** 9% **c** 22.5% **d** 150%

5 a $\frac{1}{3}$ **b** $\frac{4}{5}$ **c** $\frac{2}{3}$ **d** $\frac{3}{4}$ **e** $\frac{2}{5}$

f $\frac{7}{15}$ **g** $\frac{4}{7}$ **h** $\frac{5}{6}$ **i** $\frac{3}{5}$ **j** $\frac{3}{4}$

6 a 0.5 **b** 0.2 **c** 0.6 **d** 0.75

e 0.04 **f** 0.35 **g** 0.125 **h** 0.625

7 a $\frac{2}{5}$ **b** $\frac{1}{4}$ **c** $\frac{3}{20}$ **d** $\frac{4}{25}$

e $\frac{39}{50}$ **f** $\frac{1}{200}$ **g** $\frac{3}{8}$ **h** $\frac{33}{125}$

8 a $0.\dot{3}$ **b** $0.\dot{6}$ **c** $0.\dot{1}$ **d** $0.\dot{5}$

e $0.\dot{2}\dot{7}$ **f** $0.\dot{0}\dot{9}$ **g** $0.1\dot{6}$ **h** $0.8\dot{3}$

9 a $\frac{3}{4}$ **b** $\frac{7}{10}$ **c** $\frac{5}{6}$ **d** $\frac{4}{15}$

e $\frac{5}{18}$ **f** $\frac{1}{24}$ **g** $\frac{5}{6}$ **h** $\frac{1}{75}$

10 a 5 **b** 8 **c** $\frac{1}{10}$ **d** $\frac{1}{7}$ **e** $\frac{1}{4}$

f 6 **g** $\frac{1}{4}$ **h** $\frac{2}{3}$ **i** 4 **j** $\frac{1}{4}$

11 a $2^3 \times 3$ **b** $2^2 \times 3 \times 5$ **c** $2^3 \times 3^2$

d $2 \times 3^2 \times 7$ **e** $2^3 \times 13$ **f** $3^3 \times 5$

g $3^3 \times 7$ **h** $2 \times 3 \times 7^2$

i $3^2 \times 5 \times 7$ **j** 5×11^2

12 a 60c **b** 15 kg

c \$7800 **d** 72 min or $1\frac{1}{5}$h

13 a 0.132 **b** 0.025 **c** 0.3125 **d** 0.3375

e $0.58\dot{3}$ **f** $1.\dot{8}\dot{1}$ **g** $0.1\dot{3}$ **h** $0.2\dot{3}\dot{6}$

14 a $\frac{14}{15}$ **b** $\frac{5}{11}$ **c** $\frac{1}{2000}$

15 a \$800 **b** \$160 **c** \$120

16 a $\frac{1}{11} = 0.\dot{0}\dot{9}$, $\frac{2}{11} = 0.\dot{1}\dot{8}$, . . . , $\frac{5}{11} = 0.\dot{4}\dot{5}$, $\frac{6}{11} = 0.\dot{5}\dot{4}$, . . . , $\frac{10}{11} = 0.\dot{9}\dot{0}$. The first digit runs from 0 to 9, the second runs from 9 to 0.

b $\frac{1}{7} = 0.\dot{1}4285\dot{7}$, $\frac{2}{7} = 0.\dot{2}8571\dot{4}$, etc. The digits of each cycle are in the same order but start at a different place in the cycle.

17 c $3.0000003 \neq 3$, showing that some fractions are not stored exactly.

18 a If n is divisible by a prime p larger than $\sqrt{n}$, then n is divisible by $\frac{n}{p}$. Hence n is divisible by a prime less than or equal to $\frac{n}{p}$, which is less than $\sqrt{n}$.

b 2, 3, 5, 7, 11, 13, 17, 19 because $\sqrt{400} = 20$.

c $247 = 13 \times 19$, 241 is prime, $133 = 7 \times 19$, 367 is prime, 379 is prime, $319 = 11 \times 29$

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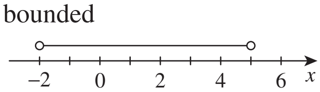
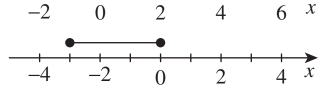
19 a i 1, 2, 4, 5, 7, 8
ii 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24
iii 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31
iv 1, 2, 4, 7, 8, 11, 13, 14, 16, 17, 19, 22, 23, 26, 28, 29, 31, 32, 34, 37, 38, 41, 43, 44

Exercise 2B

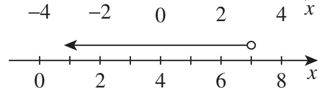
1 a rational, $\frac{-3}{1}$ **b** rational, $\frac{3}{2}$ **c** irrational
d rational, $\frac{2}{1}$ **e** rational, $\frac{3}{1}$ **f** irrational
g rational, $\frac{2}{3}$ **h** rational, $\frac{9}{20}$ **i** rational, $\frac{3}{25}$
j rational, $\frac{333}{1000}$ **k** rational, $\frac{1}{3}$ **l** rational, $\frac{22}{7}$
m irrational **n** rational, $3\frac{7}{50}$ **o** rational, $\frac{0}{1}$
2 a 0.3 **b** 5.7 **c** 12.8 **d** 0.1 **e** 3.0 **f** 10.0
3 a 0.43 **b** 5.4 **c** 5.0
d 0.043 **e** 430 **f** 4300
4 a 3.162 **b** 6.856 **c** 0.563
d 0.771 **e** 3.142 **f** 9.870
5 a 7.62 **b** 5.10 **c** 3840
d 538 000 **e** 0.740 **f** 0.00806
6 a 1 **b** 2 **c** 3 **d** 2 **e** 4
f either 1, 2 or 3
7 a i closed **ii** open **iii** closed
iv neither open nor closed
v open **vi** open **vii** closed
viii neither open nor closed
b i bounded **ii** unbounded **iii** unbounded
iv bounded **v** unbounded **vi** bounded
vii unbounded **viii** bounded
8 a $-2 < x < 5$



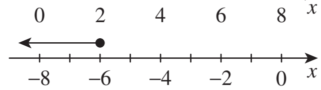
b $-3 \le x \le 0$



c $x < 7$



d $x \le -6$



9 a 45.186 **b** 2.233 **c** 0.054
d 0.931 **e** 0.842 **f** 0.111
10 a 10, rational **b** $\sqrt{41}$, irrational
c 8, rational **d** $\sqrt{5}$, irrational
e $\frac{13}{15}$, rational **f** 45, rational
11 a 0.3981 **b** 0.05263 **c** 1.425
d 5.138 **e** 0.1522 **f** 25 650
g 5.158 **h** 0.7891 **i** 1.388×10^{14}
j 1.134 **k** 0.005892 **l** 1.173
12 a The passage seems to take $\pi \doteqdot 3$.
b 3 significant figures
c Ask the internet.
d 7, with a gap of about 0.3 inches
13 a 9.46×10^{15} m **b** 2.1×10^{22} m
c 4.29×10^{17} seconds **d** 1.3×10^{26} m
14 a 1.836×10^3 **b** 6×10^{26}
16 a Clearly $\dfrac{p+1}{n} > a$.

$$\frac{p+1}{n} = \frac{p}{n} + \frac{1}{n} < a + b - a = b$$

b $n = 63\,293$, $p = 2000$ **c** $\frac{2001}{63\,293}$

Exercise 2C

1 a 4 **b** 6 **c** 9 **d** 11
e 12 **f** 20 **g** 50 **h** 100
2 a $2\sqrt{3}$ **b** $3\sqrt{2}$ **c** $2\sqrt{5}$ **d** $3\sqrt{3}$
e $2\sqrt{7}$ **f** $2\sqrt{10}$ **g** $4\sqrt{2}$ **h** $3\sqrt{11}$
i $3\sqrt{6}$ **j** $10\sqrt{2}$ **k** $2\sqrt{15}$ **l** $5\sqrt{3}$
m $4\sqrt{5}$ **n** $7\sqrt{2}$ **o** $20\sqrt{2}$ **p** $10\sqrt{10}$
3 a $2\sqrt{3}$ **b** $2\sqrt{7}$ **c** $\sqrt{5}$ **d** $-2\sqrt{2}$
e $2\sqrt{3} + 3\sqrt{2}$ **f** $\sqrt{5} - 2\sqrt{7}$ **g** $3\sqrt{6} - 2\sqrt{3}$
h $-3\sqrt{2} - 6\sqrt{5}$ **i** $-4\sqrt{10} + 2\sqrt{5}$
4 a $6\sqrt{2}$ **b** $10\sqrt{3}$ **c** $4\sqrt{6}$ **d** $8\sqrt{11}$
e $9\sqrt{5}$ **f** $12\sqrt{13}$ **g** $20\sqrt{3}$ **h** $8\sqrt{6}$
5 a $\sqrt{20}$ **b** $\sqrt{50}$ **c** $\sqrt{128}$ **d** $\sqrt{108}$
e $\sqrt{125}$ **f** $\sqrt{112}$ **g** $\sqrt{68}$ **h** $\sqrt{490}$
6 a $3\sqrt{2}$ **b** $\sqrt{3}$ **c** $2\sqrt{2}$ **d** $5\sqrt{6}$
e $\sqrt{5}$ **f** $2\sqrt{10}$ **g** $4\sqrt{3}$ **h** $2\sqrt{5}$
i $11\sqrt{2}$ **j** 5 **k** 3 **l** 2
7 a $4\sqrt{6} + 10\sqrt{3}$ **b** $2\sqrt{2} + 6\sqrt{3}$ **c** $4\sqrt{7} - 10\sqrt{35}$
8 a 7 **b** 20 **c** 96 **d** 24
9 b Show that $\sqrt[3]{7} < 2$ and $\sqrt[4]{7} < 2$.
10 a The graph intersects the x-axis at $x = \sqrt{2}$ and $x = -\sqrt{2}$, which are both irrational.

Exercise 2D

1 a 3 **b** $\sqrt{6}$ **c** 7 **d** $\sqrt{30}$
e $6\sqrt{2}$ **f** $10\sqrt{5}$ **g** $6\sqrt{15}$ **h** $30\sqrt{14}$
i 12 **j** 63 **k** 30 **l** 240
2 a $\sqrt{5}$ **b** $\sqrt{7}$ **c** $\sqrt{5}$ **d** 2
e $3\sqrt{2}$ **f** $\sqrt{3}$ **g** $2\sqrt{7}$ **h** $5\sqrt{5}$

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3 a $5+\sqrt{5}$ **b** $\sqrt{6}-\sqrt{2}$ **c** $2\sqrt{3}-3$
d $2\sqrt{10}-4$ **e** $7\sqrt{7}-14$ **f** $18-2\sqrt{30}$

4 a $2\sqrt{3}$ **b** $5\sqrt{2}$ **c** $3\sqrt{5}$
d $4\sqrt{11}$ **e** 24 **f** $12\sqrt{10}$

5 a $2\sqrt{5}-2$ **b** $3\sqrt{6}+3\sqrt{2}$ **c** $5\sqrt{3}+4\sqrt{5}$
d $4\sqrt{3}-2\sqrt{6}$ **e** $27\sqrt{3}-9\sqrt{7}$ **f** $21\sqrt{2}-42$

6 a $\sqrt{6}-\sqrt{3}+\sqrt{2}-1$
b $\sqrt{35}+3\sqrt{5}-2\sqrt{7}-6$
c $\sqrt{15}+\sqrt{10}+\sqrt{6}+2$
d $8-3\sqrt{6}$ **e** $4+\sqrt{7}$ **f** $7\sqrt{3}-4\sqrt{6}$

7 a 4 **b** 2 **c** 1 **d** 7 **e** 15 **f** 29

8 a $4+2\sqrt{3}$ **b** $6-2\sqrt{5}$ **c** $5+2\sqrt{6}$
d $12-2\sqrt{35}$ **e** $13-4\sqrt{3}$ **f** $29+12\sqrt{5}$
g $33+4\sqrt{35}$ **h** $30-12\sqrt{6}$ **i** $55+30\sqrt{2}$

9 a 2 **b** $\frac{3}{5}$ **c** $2\sqrt{3}$ **d** $\frac{5\sqrt{3}}{2}$ **e** 5 **f** 4

10 a 3 **b** 5 **c** 4 **d** 6

11 a $\sqrt{3}$ **b** $\frac{6\sqrt{7}}{13}$

12 a $xy\sqrt{y}$ **b** x^2y^3 **c** $x+3$
d $(x+1)\sqrt{x}$ **e** $x(x+1)y^2$ **f** $x(x+1)$

13 a If $a=3$ and $b=4$, then LHS $=5$, but RHS $=7$.
b Squaring both sides gives $2ab=0$. Thus the statement is true when one of a or b is zero and the other is not negative.

14 a $2\sqrt{3}$ **b** $3\sqrt{11}$ **c** $3+2\sqrt{2}$ **d** $7-2\sqrt{6}$

15 a $a^2+2ab+b^2$
b 2
c $\sqrt{2}$

16 26

Exercise 2E

1 a $\frac{\sqrt{3}}{3}$ **b** $\frac{\sqrt{7}}{7}$ **c** $\frac{3\sqrt{5}}{5}$ **d** $\frac{5\sqrt{2}}{2}$
e $\frac{\sqrt{6}}{3}$ **f** $\frac{\sqrt{35}}{7}$ **g** $\frac{2\sqrt{55}}{5}$ **h** $\frac{3\sqrt{14}}{2}$

2 a $\sqrt{2}$ **b** $\sqrt{5}$ **c** $2\sqrt{3}$ **d** $3\sqrt{7}$
e $\frac{\sqrt{6}}{2}$ **f** $\frac{\sqrt{15}}{3}$ **g** $\frac{4\sqrt{6}}{3}$ **h** $\frac{7\sqrt{10}}{5}$

3 a $\frac{\sqrt{5}}{10}$ **b** $\frac{\sqrt{7}}{21}$ **c** $\frac{3\sqrt{2}}{10}$ **d** $\frac{2\sqrt{3}}{21}$
e $\frac{5\sqrt{2}}{3}$ **f** $\frac{3\sqrt{3}}{4}$ **g** $\frac{\sqrt{30}}{20}$ **h** $\frac{2\sqrt{77}}{35}$

4 a $\frac{\sqrt{3}+1}{2}$ **b** $\frac{\sqrt{7}-2}{3}$ **c** $\frac{3-\sqrt{5}}{4}$
d $\frac{4+\sqrt{7}}{9}$ **e** $\frac{\sqrt{5}+\sqrt{2}}{3}$ **f** $\frac{\sqrt{10}-\sqrt{6}}{4}$
g $\frac{2\sqrt{3}-1}{11}$ **h** $\frac{5+3\sqrt{2}}{7}$

5 a $\frac{3\sqrt{5}-3}{4}$ **b** $\frac{8\sqrt{2}+4\sqrt{3}}{5}$ **c** $\frac{5\sqrt{7}+7}{18}$
d $\frac{3\sqrt{15}-9}{2}$ **e** $\frac{28+10\sqrt{7}}{3}$ **f** $\sqrt{2}+1$
g $2-\sqrt{3}$ **h** $\frac{7+2\sqrt{10}}{3}$ **i** $8-3\sqrt{7}$
j $\frac{23+6\sqrt{10}}{13}$ **k** $4-\sqrt{15}$ **l** $\frac{93+28\sqrt{11}}{5}$

6 a $\sqrt{3}+1$ **b** $4-\sqrt{10}$

7 a 3 **b** 1 **c** 7 **d** 2

9 $a=-1, b=2$

10 a $2\sqrt{2}$ **b** 4 **c** 4 **d** $\dfrac{2}{x-1}$

11 $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

12 a $x^2+2+\dfrac{1}{x^2}$
b ii $x^2+\dfrac{1}{x^2}=\left(x+\dfrac{1}{x}\right)^2-2=28-2=26$

13 a 8.33 **b** 8.12

14 $\frac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}$

Chapter 2 review exercise

1 a rational, $\frac{7}{1}$ **b** rational, $\frac{-9}{4}$ **c** rational, $\frac{3}{1}$
d irrational **e** irrational **f** rational, $\frac{2}{1}$
g rational, $\frac{-4}{25}$ **h** irrational

2 a 4.12, 4.1 **b** 4.67, 4.7 **c** 2.83, 2.8
d 0.77, 0.77 **e** 0.02, 0.019 **f** 542.41, 540

3 a 1.67 **b** 70.1 **c** 1.43
d 0.200 **e** 0.488 **f** 0.496
g 1.27 **h** 1590 **i** 0.978

4 a $2\sqrt{6}$ **b** $3\sqrt{5}$ **c** $5\sqrt{2}$
d $10\sqrt{5}$ **e** $9\sqrt{2}$ **f** $4\sqrt{10}$

5 a $2\sqrt{5}$ **b** 5 **c** 28
d $\sqrt{7}-\sqrt{5}$ **e** $\sqrt{7}$ **f** $3\sqrt{5}$
g 4 **h** $2\sqrt{5}$ **i** $24\sqrt{10}$

6 a $\sqrt{3}$ **b** $7\sqrt{2}$
c $4\sqrt{2}$ **d** $8\sqrt{6}-6\sqrt{5}$

7 a $3\sqrt{7}-7$ **b** $2\sqrt{30}+3\sqrt{10}$
c $3\sqrt{5}-5\sqrt{15}$ **d** $3\sqrt{2}+6$

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8 a $\sqrt{5} + 1$ **b** $13 + 7\sqrt{3}$
c $2\sqrt{35} + 4\sqrt{7} - 6\sqrt{5} - 12$
d 1 **e** 13 **f** $11 - 4\sqrt{7}$
g $7 + 2\sqrt{10}$ **h** $34 - 24\sqrt{2}$

9 a $\frac{\sqrt{5}}{5}$ **b** $\frac{3\sqrt{2}}{2}$ **c** $\frac{\sqrt{33}}{11}$
d $\frac{\sqrt{3}}{15}$ **e** $\frac{5\sqrt{7}}{14}$ **f** $\frac{\sqrt{5}}{15}$

10 a $\frac{\sqrt{5} - \sqrt{2}}{3}$ **b** $\frac{3 + \sqrt{7}}{2}$ **c** $\frac{2\sqrt{6} + \sqrt{3}}{21}$
d $\frac{3 - \sqrt{3}}{2}$ **e** $\frac{\sqrt{11} - \sqrt{5}}{2}$ **f** $\frac{6\sqrt{35} + 21}{13}$

11 a $\frac{9 - 2\sqrt{14}}{5}$ **b** $26 + 15\sqrt{3}$

12 $x = 50$ **13** $5\sqrt{5} + 2$

14 $p = 5, q = 2$ **15** $\frac{7}{3}$

16 a $2\sqrt{10}$ **b** 38

18 By rationalising the denominators, the series 'telescopes' to just $\sqrt{16} - \sqrt{1} = 3$.

19 a It is sufficient to show that $\text{LHS}^2 = \text{RHS}^2$, because both sides are positive.
b $\sqrt{3}$

Chapter 3

Exercise 3A

1 a -3 **b** 5 **c** 0 **d** 5

2 a 5 **b** -10 **c** -3 **d** 2

3 a $5, -1, -7$ **b** $0, 4, 0$
c $16, 8, 0$ **d** $4, 1, \frac{1}{4}$

4 a $-4, 4, 12$ **b** $-\frac{1}{3}, 1, \frac{1}{5}$
c $-18, 2, -10$ **d** $1, \sqrt{5}, 3$

5 a $p(x)$: $3, 0, -1, 0, 3$
b $c(x)$: $-15, 0, 3, 0, -3, 0, 15$

6 a 2 **b** -6 **c** 11 **d** 4

7 a $-3, -2, -1, 0, 1, 0, -1$
b $3, 0, 0, 1, 4$

8 a 4 **b** $4\frac{1}{3}$

9 a 0 **b** $2 - 4\sqrt{3}$

10 $C = 50 + 20x$

11 a $y = -\frac{3}{4}x - \frac{5}{4}$ **b** $x = -\frac{4}{3}y - \frac{5}{3}$
c $y = -\frac{4}{x}$ **d** $s = \sqrt[3]{V}, s = \sqrt{\frac{A}{6}}$
e i $\ell = \frac{100}{b}$ **ii** $b = \frac{100}{\ell}$

12 a The square root of a negative is undefined.
b The square root of a negative is undefined.
c Division by zero is undefined.
d Division by zero is undefined.

13 a $2a - 4, -2a - 4, 2a - 2$
b $2 - a, 2 + a, 1 - a$
c $a^2, a^2, a^2 + 2a + 1$
d $\frac{1}{a - 1}, \frac{1}{-a - 1} = -\frac{1}{a + 1}, \frac{1}{a}$

14 a $5t, 5t - 8$ **b** $\sqrt{t} - 2, \sqrt{t - 2}$
c $t^2 + 2t - 2, t^2 - 2t$ **d** $-t^2, -t^2 + 4t - 2$

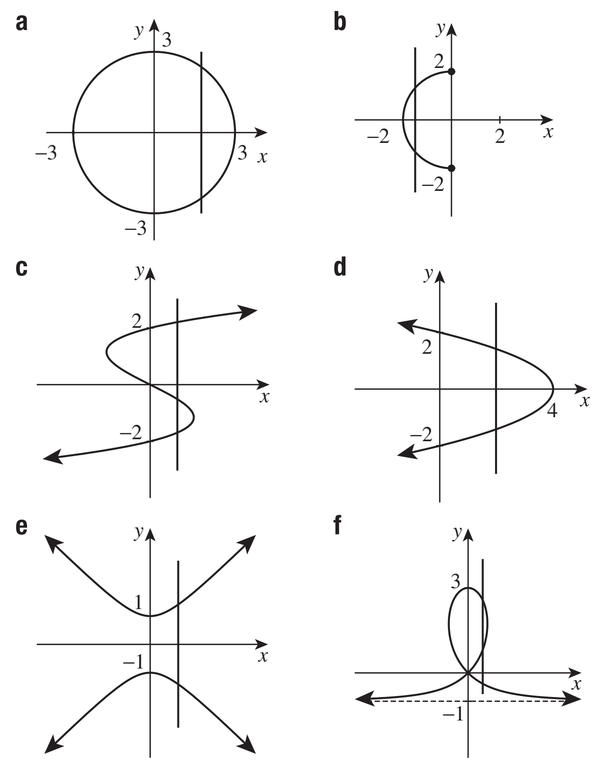
15 a $7 + h$ **b** $p + q + 5$ **c** $2x + h + 5$

17 a 64 **b** 28 **c** $(x + 3)^2$ **d** $x^2 + 3$

18 It approaches 2.72.

Exercise 3B

1 Notice that the y-axis is such a line in every case. Shown below are some other vertical lines that intersect at least twice.



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2 a, c, f, h

3 a domain: all real x, range: $y \geq -1$

b domain: $-2 \leq x \leq 2$, range: $-2 \leq y \leq \sqrt{3}$

c domain: all real x, range: all real y

d domain: $-1 \leq x$, range: all real y

e domain: $-2 \leq x \leq 2$, range: $-3 \leq y \leq 3$

f domain: all real x, range: all real y

g domain: $0 \leq x \leq 2$, range: $-2 \leq y \leq 2$

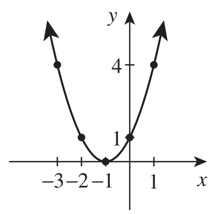
h domain: all real x, range: $y < 1$

4 a $x \neq 0$ **b** $x \neq 3$ **c** $x \neq -2$

5 a $x \geq 0$ **b** $x \geq 2$ **c** $x \geq -5$

6 a i 4, 1, 0, 1, 4

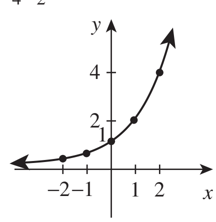
ii



iii domain: all real x, range: $y \geq 0$

b i $\frac{1}{4}, \frac{1}{2}$, 1, 2, , 4

ii



iii domain: all real x, range: $y > 0$

7 a $(0, 3)$ and $(0, -3)$ **b** $(0, 1)$ and $(0, -1)$

c $(2, 1)$ and $(2, 5)$ **d** $(2, 2)$ and $(2, -2)$

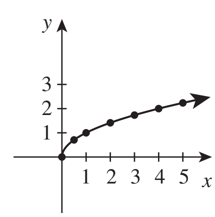
8 a all real x **b** $x \neq \frac{1}{2}$ **c** $x \geq -4$

d $x \leq 2$ **e** $x < 1$ **f** $x > 1\frac{1}{2}$

9 a $x \geq 0$

b 0, 0.7, 1, 1.4, 1.7, 2, 2.2

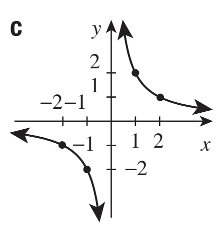
c It is the top half of a concave right parabola.



10 a $x \neq 0$

b $-\frac{1}{2}$, -1, -2, -4, *, 4, 2, 1, $\frac{1}{2}$
Division by zero is undefined.

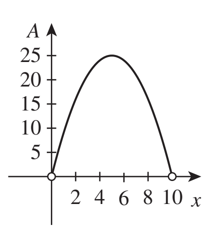
c



11 a $A = x(10 - x)$

b Both $10 - x > 0$ and $x > 0$. Thus $10 > x$ and $x > 0$. Hence the domain is $0 < x < 10$.

c



12 a $y = 2x + 3$

b $y = \dfrac{4}{x}$

c $y = \dfrac{3}{x - 2}$ **d** $y = -2 + \sqrt{9 - x^2}$

e $y = \sqrt[3]{x - 1}$ **f** $y = \dfrac{2x - 3}{x + 1}$

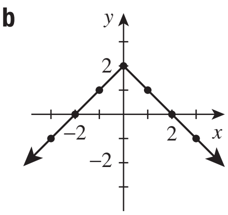
13 a $x > -2$ **b** $x \neq 2$ and $x \neq -2$

c $x \neq -1$ and $x \neq 0$ **d** $x \neq 2$ and $x \neq 3$

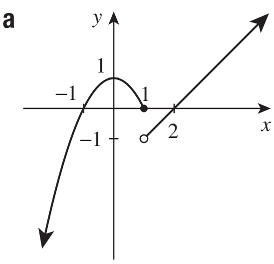
e $x \leq -2$ or $x \geq 2$ **f** $-1 < x < 1$

14 a $-1, 0, 1, 2, 1, 0, -1$

b



15 a



b There is a break. The proper way to say this is that $f(x)$ is not continuous at $x = 1$.

16 a i domain: all real x, range: $y \leq 4$

ii domain: $-2 \leq x \leq 2$, range: $0 \leq y \leq 2$

iii domain: $-2 < x < 2$, range: $y \geq \frac{1}{2}$

b i domain: $-3 < x < 1$, range $y > \frac{1}{2}$

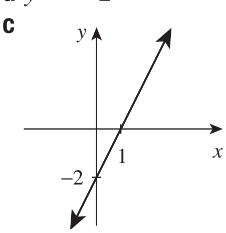
ii domain: all real x, range: $0 < y \leq \frac{1}{\sqrt{2}}$

17 a $-1 < x < 1$

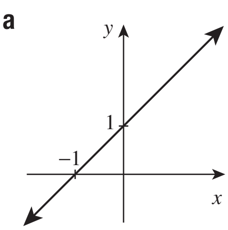
Exercise 3C

1 a $y = -2$ **b** $x = 1$

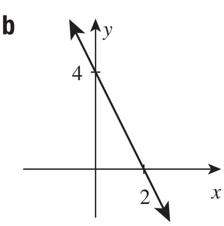
c



2 a



b



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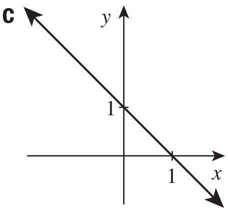
c



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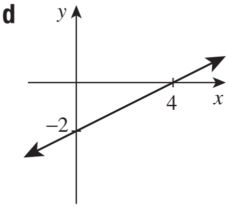
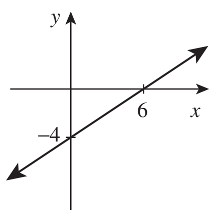
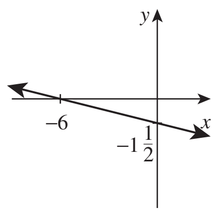


e

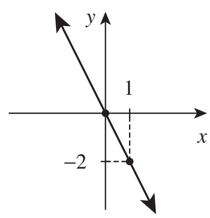


f

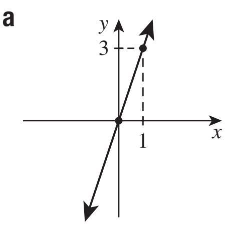


3 a When $x = 0$, $y = 0$.

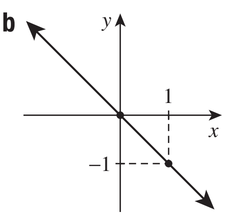
b $(1, -2)$



4 a



b



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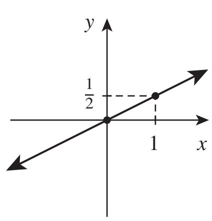
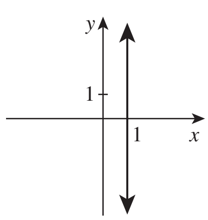


5 a



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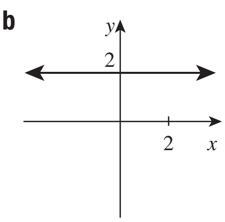
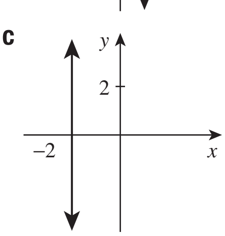


c



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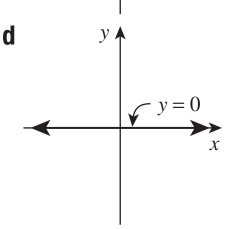
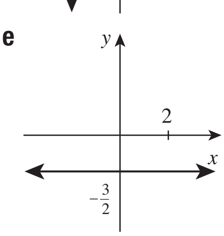


e



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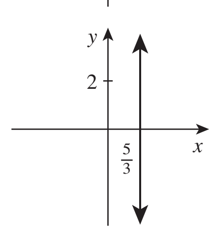


6 a a, c, f

b $(1, 0)$ and $(1, 1)$ are on $x = 1$.

c $(-2, 0)$ and $(-2, 1)$ are on $x = -2$.

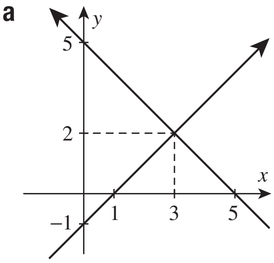
d $\left(\frac{5}{3}, 0\right)$ and $\left(\frac{5}{3}, 1\right)$ are on $3x = 5$.

7 c $y = 1 - x$ **d** $y = \frac{1}{2}x - 2$

e $y = \frac{2}{3}x - 4$ **f** $y = -\frac{1}{4}x - \frac{3}{2}$

8 a yes **b** no **c** yes **d** yes **e** yes **f** no

9 a



b $(3, 2)$

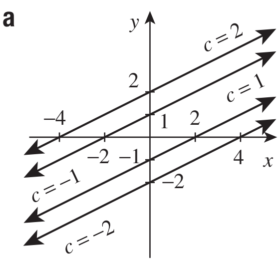
10 a $(-1, 3)$ **b** $(1, -2)$ **c** $(-2, -1)$

11 a $C(n) = 10 + 50n$

b i $D(n) = 8 + 2n$

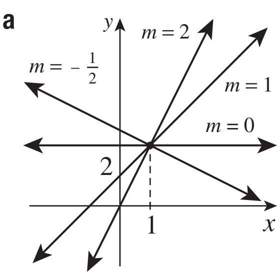
ii $T = C + D$ so $T(n) = 18 + 52n$

12 a



b They are parallel. The value of c gives the y-intercept.

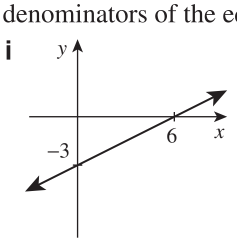
13 a



b $(1, 2)$

14 a $(a, 0)$ and $(0, b)$. The intercepts appear in the denominators of the equation.

b i



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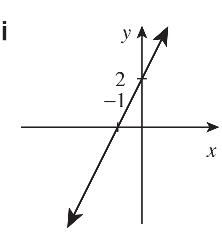

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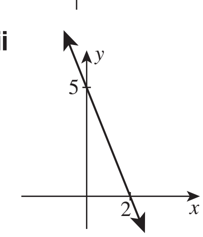

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15 a $\left(3, 1\frac{1}{2}\right)$

c $5x - 2y - 12 = 0$

16 a $Ax_1 + By_1 + C = 0, Ax_2 + By_2 + C = 0,$
$Ax + By + C = 0$

b See worked solutions

c $(2 - y)(3 - x) = (4 + y)(x - 1)$

Exercise 3D

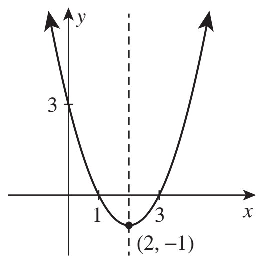
1 a i $y = 3$

ii $x = 1, 3$

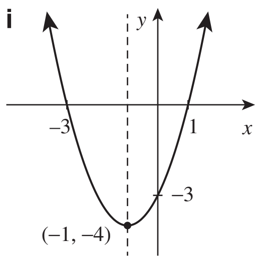
iii $x = 2$

iv $(2, -1)$

v



b i



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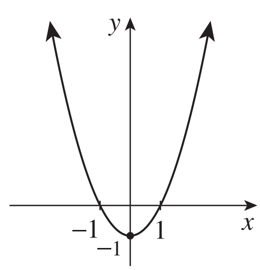
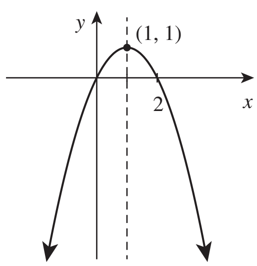
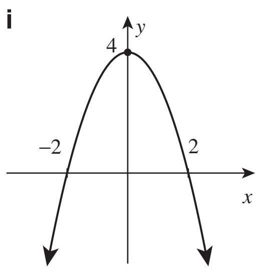


2 a



b i



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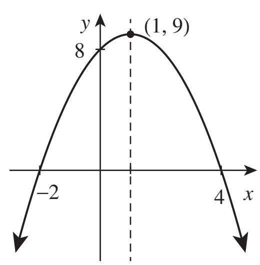
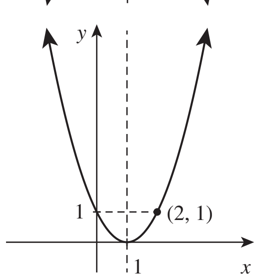
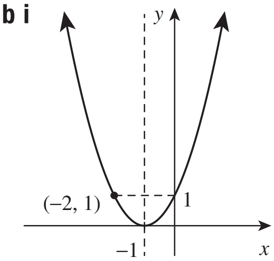


3 a



b i



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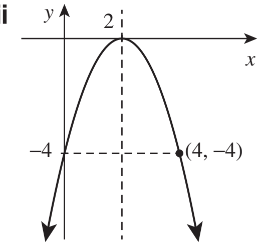
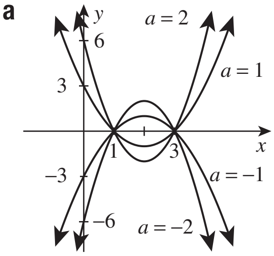


4 a



b $(1, 0)$ and $(3, 0)$

5 a $y = (x - 4)(x - 6)$

b $y = x(x - 3)$

c $y = (x + 3)(x - 5)$

d $y = (x + 6)(x + 1)$

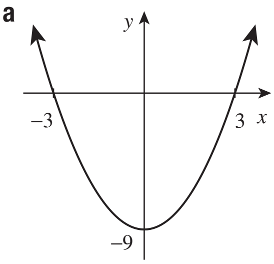
6 a $y = x(x - 3)$

b $y = (x + 2)(x - 1)$

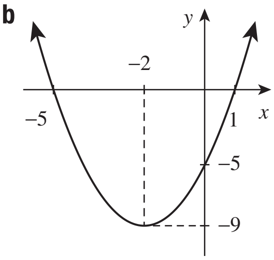
c $y = -(x + 1)(x - 3)$

d $y = -(x + 2)(x + 5)$

7 a



b



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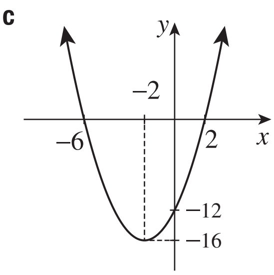

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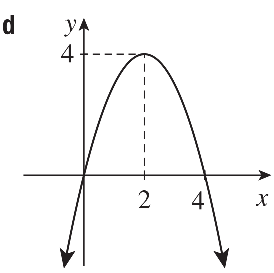
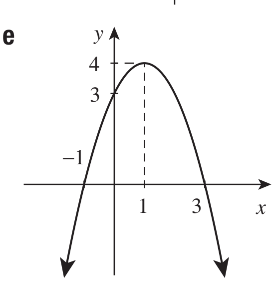


e



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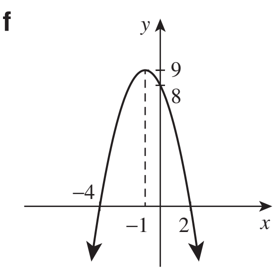

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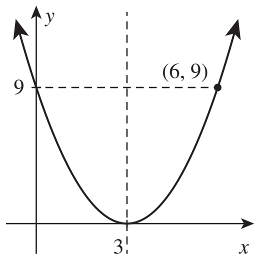
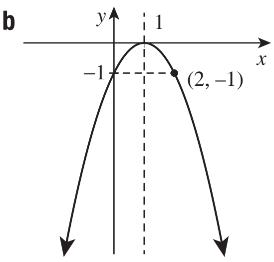


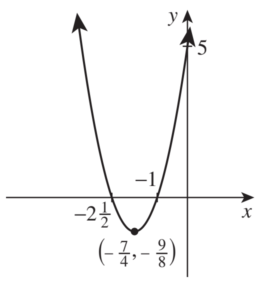
$y = (x - 3)^2$

b



$y = -(x - 1)^2$

9 a



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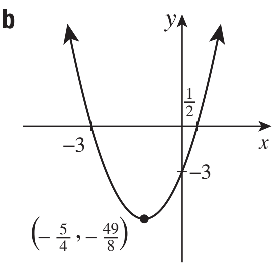

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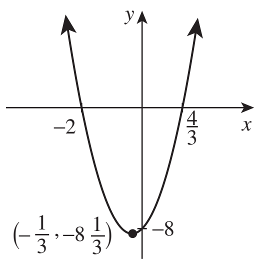

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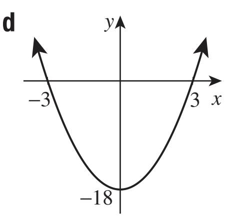

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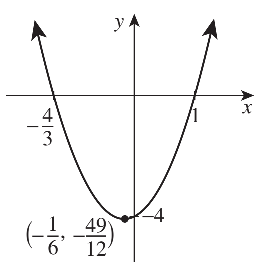
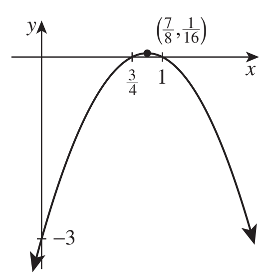


f



10 a $y = (x + 1)(x - 2)$
b $y = -(x + 3)(x - 2)$
c $y = 3(x + 2)(x - 4)$
d $y = -\frac{1}{2}(x - 2)(x + 2)$

11 a $y = 2(x - 1)(x - 3)$
b $y = -2(x + 2)(x - 1)$
c $y = -3(x + 1)(x - 5)$
d $y = \frac{1}{4}(x + 2)(x + 4)$

12 a $y = 3(x - 2)(x - 8)$
b $y = -(x - 2)(x - 8)$
c $y = \frac{4}{3}(x - 2)(x - 8)$
d $y = -\frac{20}{7}(x - 2)(x - 8)$

13 a $y = x(x + 3)$
b $y = \frac{3}{2}x^2$
c $y = -4x(x - 2)$
d $y = -2x(x + 6)$

14 a $a = \dfrac{c}{\alpha\beta}$
b $a = -\dfrac{b}{\alpha + \beta}$
c $a = \dfrac{2}{(1 - \alpha)(1 - \beta)}$

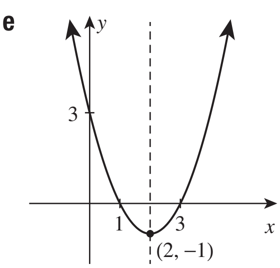
15 a $f(x) = (x - 4)(x + 2)$, so the axis is $x = 1$.
b i Both $f(1 + h) = h^2 - 9$ and $f(1 - h) = h^2 - 9$.
ii The parabola is symmetric in the line $x = 1$.

16 a $(-1 + p), (-1 - p), x = -1$
b $(p - 1), (p + 1), x = p$
c $(2 + p), -p, x = 1$

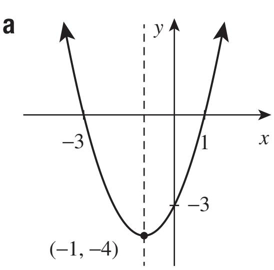
17 a The value of the function is the same h units right $\left(\frac{1}{2}(\alpha + \beta) + h\right)$ or left $\left(\frac{1}{2}(\alpha + \beta) - h\right)$ of the axis.
b The result follows from part **a** by putting $h = \left(\frac{1}{2}(\alpha + \beta) + x\right)$

Exercise 3E

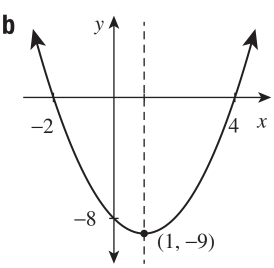
1 a $a = 1$, concave up
b $y = 3$
c $x = 1, 3$
d $x = 2, V(2, -1)$
e



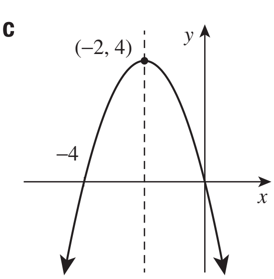
2 a



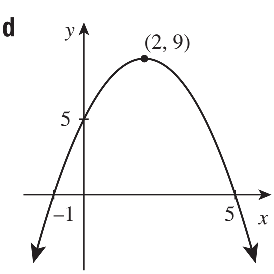
b



c



d



3 a $f(x) = (x - 2)^2 + 1$
b $f(x) = (x + 3)^2 + 2$
c $f(x) = (x - 1)^2 + 7$
d $f(x) = (x - 5)^2 - 24$
e $f(x) = (x + 1)^2 - 6$
f $f(x) = (x + 2)^2 - 5$

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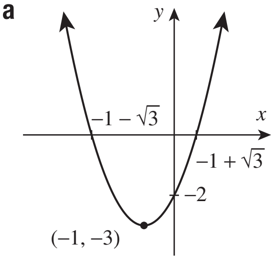

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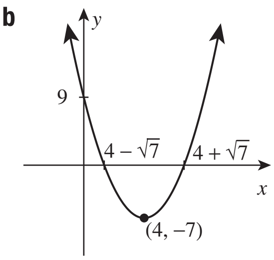

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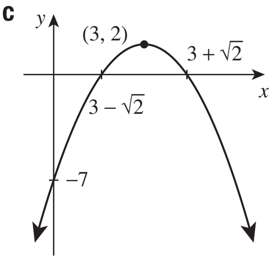

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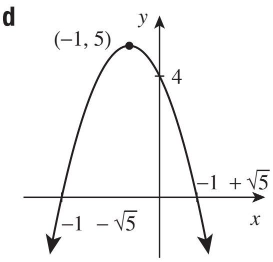


5 a $x = 1, 3$ **b** $x = -3, 1$ **c** $x = -1, 2$

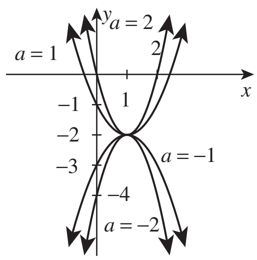
6 a $y = (x - 1)^2 + 2$ **b** $y = (x + 2)^2 - 3$

c $y = -(x - 3)^2 + 4$ **d** $y = -(x - 2)^2 - 1$

7 a $y = (x - 2)^2 + 5$ **b** $y = x^2 - 3$

c $y = (x + 1)^2 + 7$ **d** $y = (x - 3)^2 - 11$

8 a

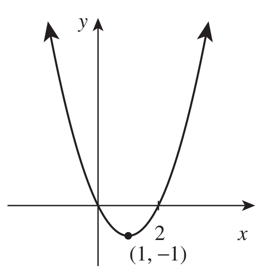


b $(1, -2)$

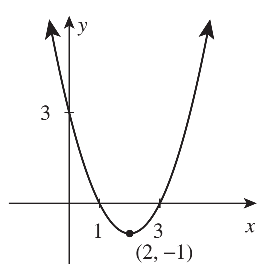
c $a > 0$

d The vertex is below the x-axis. Thus the parabola will only intersect the x-axis if it is concave up.

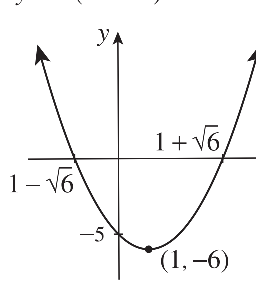
9 a $y = (x - 1)^2 - 1$



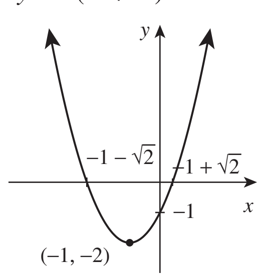
b $y = (x - 2)^2 - 1$



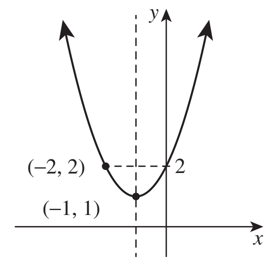
c $y = (x - 1)^2 - 6$



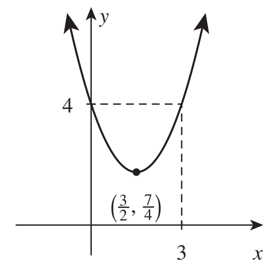
d $y = (x + 1)^2 - 2$



e $y = (x + 1)^2 + 1$



f $y = \left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$



10 Put $h = -4$ and $k = 2$ into the formula $y = a(x - h)^2 + k$.

a $y = (x + 4)^2 + 2$ **b** $y = 3(x + 4)^2 + 2$

c $y = \frac{7}{8}(x + 4)^2 + 2$ **d** $y = -\frac{1}{8}(x + 4)^2 + 2$

11 a $V = (3, -5)$, concave up, two x-intercepts.

b $V = (-1, 3)$, concave down, two x-intercepts.

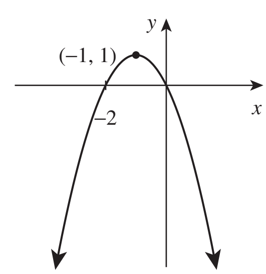
c $V = (-2, -1)$, concave down, no x-intercepts.

d $V = (4, 3)$, concave up, no x-intercepts.

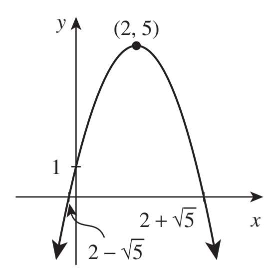
e $V = (-1, 0)$, concave up, one x-intercept.

f $V = (3, 0)$, concave down, one x-intercept.

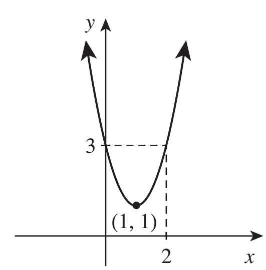
12 a $y = -(x + 1)^2 + 1$



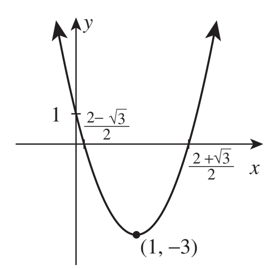
b $y = -(x - 2)^2 + 5$



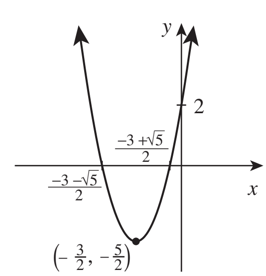
c $y = 2(x - 1)^2 + 1$



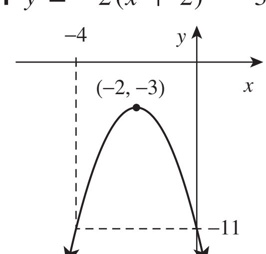
d $y = 4(x - 1)^2 - 3$



e $y = 2\left(x + \frac{3}{2}\right)^2 - \frac{5}{2}$



f $y = -2(x + 2)^2 - 3$



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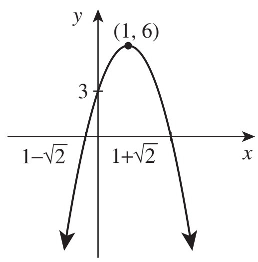
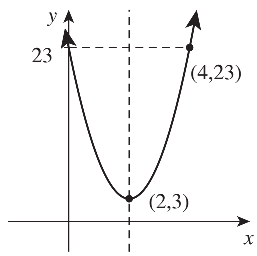
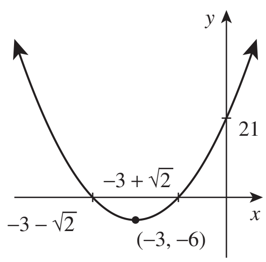
g $y = -3(x - 1)^2 + 6$



h $y = 5(x - 2)^2 + 3$



i $y = 3(x + 3)^2 - 6$



13 a $f(x) = \left(x + 1 + \sqrt{2}\right)\left(x + 1 - \sqrt{2}\right)$

b $f(x) = \left(x - 2 + \sqrt{3}\right)\left(x - 2 - \sqrt{3}\right)$

c $f(x) = -\left(x + 1 + \sqrt{5}\right)\left(x + 1 - \sqrt{5}\right)$

14 $y = (x + 2)^2 + k$

a $y = (x + 2)^2 - 4$ **b** $y = (x + 2)^2 - 48$

c $y = (x + 2)^2 - 9$ **d** $y = (x + 2)^2 - 10$

e $y = (x + 2)^2 - 2$ **f** $y = (x + 2)^2 + 7$

15 a $y = 2(x - 1)^2 + 1$ **b** $y = -(x - 3)^2 + 2$

c $y = \frac{1}{2}(x + 2)^2 - 4$ **d** $y = -3(x + 1)^2 + 4$

16 a $-d - \sqrt{e}, -d + \sqrt{e}$ **b** $2\sqrt{e}$

c $e = 1$. They have vertex on the line $y = -1$.

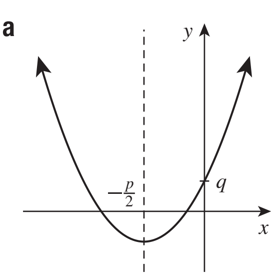
17 $h_1 = h_2$, but $k_1 \neq k_2$. The two curves have the same axis of symmetry, but different vertices.

18 b The vertex is $\left(-\dfrac{b}{2a}, -\dfrac{b^2 - 4ac}{4a}\right)$ and the axis of symmetry is $x = -\dfrac{b}{2a}$.

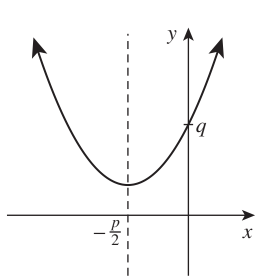
c $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

d $y = a\left(x + \dfrac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \dfrac{b - \sqrt{b^2 - 4ac}}{2a}\right)$

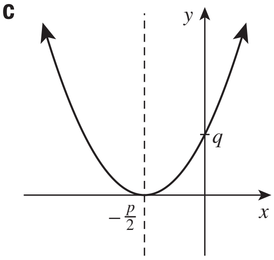
19 a



b



c



20 a The value of the function is the same t units right or left of the axis.

b The result follows from part **a** by putting $t = h - x$.

Exercise 3F

1 a i concave up

ii $y = -1$

iii $x = 1$

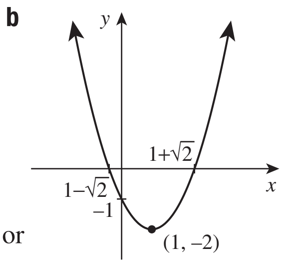
iv $= (1, -2)$

v $\Delta = 8$

vi $\Delta > 0$

vii $x = 1 - \sqrt{2} \doteqdot -0.41$, or $1 + \sqrt{2} \doteqdot 2.41$.

b



2 a $-1 - \sqrt{3}$ or $-1 + \sqrt{3}$, -2.73 or 0.73

b $2 - \sqrt{3}$ or $2 + \sqrt{3}$, 0.27 or 3.73

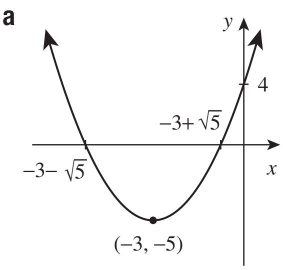
c $\frac{1}{2}\left(3 - \sqrt{17}\right)$ or $\frac{1}{2}\left(3 + \sqrt{17}\right)$, -0.56 or 3.56

d $-1 - \sqrt{5}$ or $-1 + \sqrt{5}$, -3.24 or 1.24

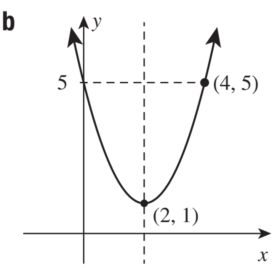
e $\frac{1}{3}\left(1 - \sqrt{7}\right)$ or $\frac{1}{3}\left(1 + \sqrt{7}\right)$, -0.55 or 1.22

f $\frac{1}{2}\left(-2 - \sqrt{6}\right)$ or $\frac{1}{2}\left(-2 + \sqrt{6}\right)$, -2.22 or 0.22

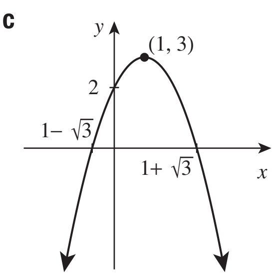
3 a



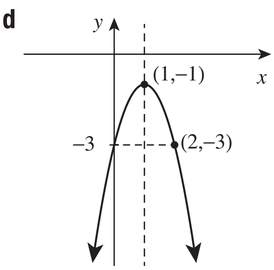
b



c



d



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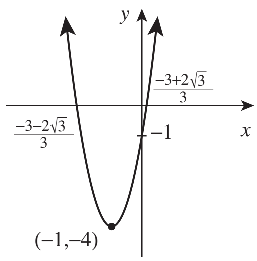
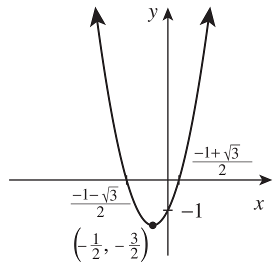
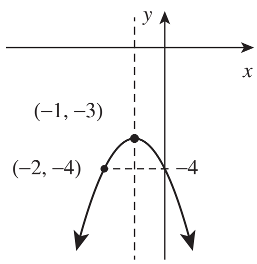
e



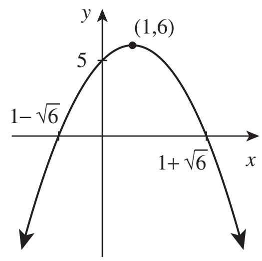
f



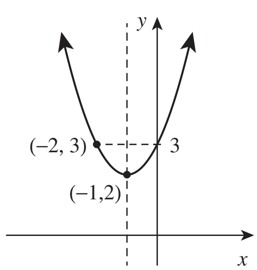
g



h



i



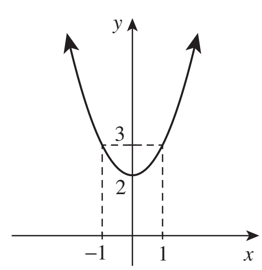
4 a $x = -1, 4$ **b** $x = 2, 3$ **c** $x = -2, 6$

5 a i $\Delta = -8 < 0$

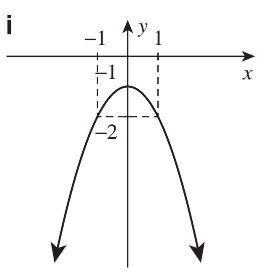
ii Both equal $(0, 2)$.

iii $(1, 3)$

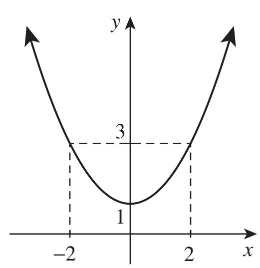
iv $(-1, 3)$



b i



ii



6 a $3 - 2\sqrt{2}, 3 + 2\sqrt{2}$

b $1 - \sqrt{5}, \ + \sqrt{5}$

c $\frac{5 - \sqrt{10}}{3}, \frac{5 + \sqrt{10}}{3}$

7 a $\Delta = 17$, two zeroes

b $\Delta = 0$, one zero

c $\Delta = -7$, no zeroes

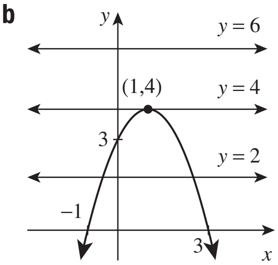
8 a i $x = 1 - \sqrt{2}$ and $x = 1 + \sqrt{2}$

ii $x = 1$

iii There are none.

c $k < 4$

b



9 a $f(x) = (x - 3 + \sqrt{5})(x - 3 - \sqrt{5})$

b $f(x) = (x + 1 + \sqrt{2})(x + 1 - \sqrt{2})$

c $f(x) = \left(x - \frac{3 - \sqrt{5}}{2}\right)\left(x - \frac{3 + \sqrt{5}}{2}\right)$

d $f(x) = 3\left(x + \frac{3 + \sqrt{3}}{3}\right)\left(x + \frac{3 - \sqrt{3}}{3}\right)$

e $f(x) = -\left(x - \frac{3 - \sqrt{13}}{2}\right)\left(x - \frac{3 + \sqrt{13}}{2}\right)$

f $f(x) = -2(x + 1)\left(x - \frac{1}{2}\right)$

10 b i axis: $x = -2$, vertex: $(-2, -3)$

ii axis: $x = 3$, vertex: $(3, 1)$

iii axis: $x = -2$, vertex: $(-2, 13)$

11 $\frac{1}{2}p\left(-1 + \sqrt{5}\right)$

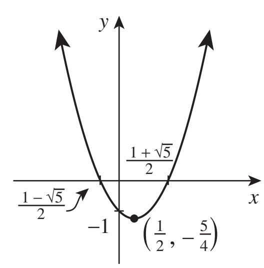
12 a $x = h + \sqrt{-k}$ or $h - \sqrt{-k}$

13 a $x = -\frac{b}{2}$, vertex $\left(-\frac{b}{2}, \frac{1}{4}(4c - b^2)\right)$

b Difference between zeroes is $\sqrt{b^2 - 4c}$.

c $b^2 - 4c = 1$

14

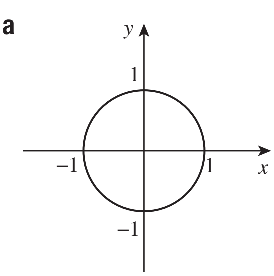


Exercise 3G

1 a $(0, 0)$, 4 units **b** $(0, 0)$, 7 units

c $(0, 0)$, $\frac{1}{3}$ units **d** $(0, 0)$, 1.2 units

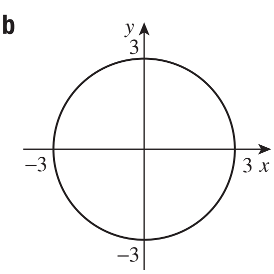
2 a



$-1 \le x \le 1$

$-1 \le y \le 1$

b



$-3 \le x \le 3$

$-3 \le y \le 3$

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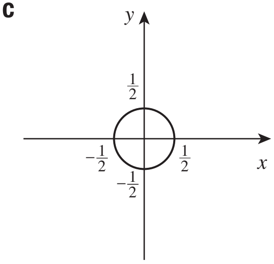
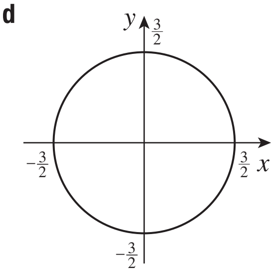
c



$-\frac{1}{2} \le x \le \frac{1}{2}$

$-\frac{1}{2} \le y \le \frac{1}{2}$

d

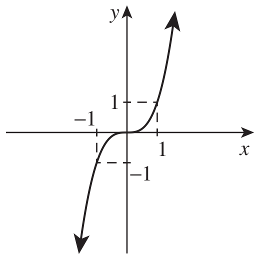


$-\frac{3}{2} \le x \le \frac{3}{2}$

$-\frac{3}{2} \le y \le \frac{3}{2}$

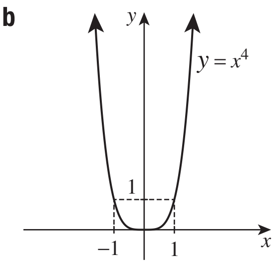
3 a −3.375, −1, −0.125, 0, 0.125, 1, 3.375

b

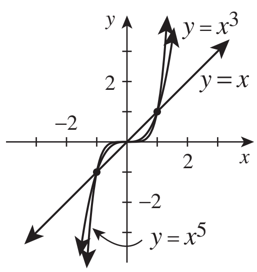


4 a 5.0625, 1, 0.0625, 0, 0.0625, 1, 5.0625

b



5 a



b $(-1, -1)$, $(0, 0)$ and $(1, 1)$

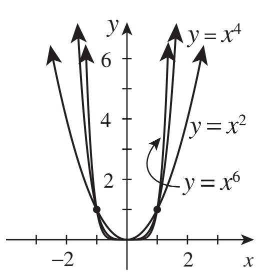
c i $y = x^5$ **ii** $y = x$

d i $y = x^5$ **ii** $y = x$

e In each case, the result is the same curve.

f Every index is odd.

6 a



b $(-1, 1)$, $(0, 0)$ and $(1, 1)$

c i $y = x^6$ **ii** $y = x^2$

d i $y = x^6$ **ii** $y = x^2$

e In each case, the result is the same curve.

f Every index is even.

7 a degree 1, coefficient 2

b degree 3, coefficient 0

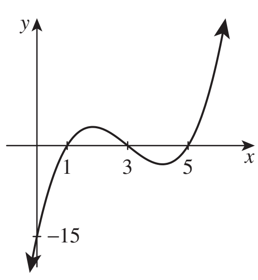
c not a polynomial

d not a polynomial

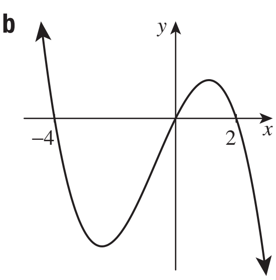
e degree 3, coefficient −1

f not a polynomial

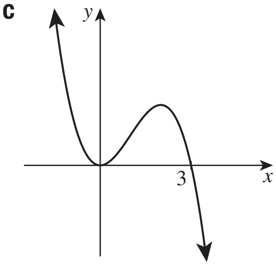
8 a



b



c



9 a $x^2 + y^2 = 4$

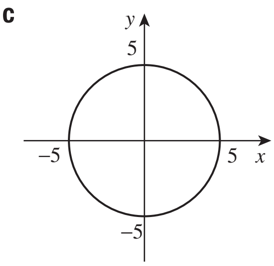
b $x^2 + y^2 = 5$

c $x^2 + y^2 = 25$

d $x^2 + y^2 = 10$

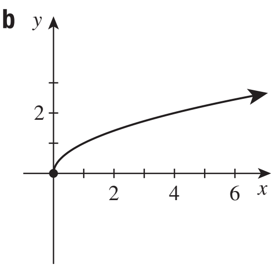
10 a 5 or −5, 4.9 or −4.9, 4.6 or −4.6, 4 or −4, 3 or −3, 0.

c

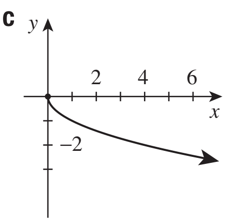


11 a 0, 0.5, 1, 1.5, 2, 2.5

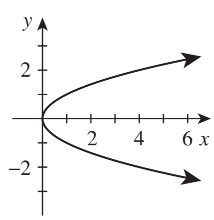
b



c



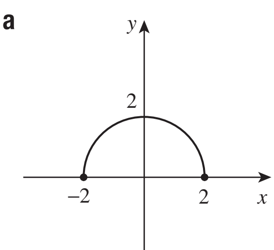
12 a



b It is a concave-right parabola.

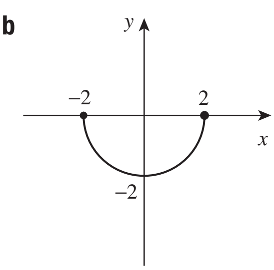
c In both cases, squaring gives $x = y^2$. This is the result of swapping x and y in $y = x^2$.

13 a



domain: $-2 \le x \le 2$, range: $0 \le y \le 2$

b



domain: $-2 \le x \le 2$, range: $-2 \le y \le 0$

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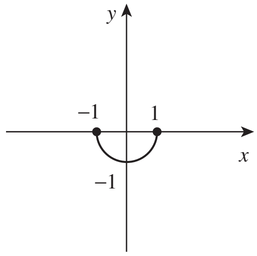
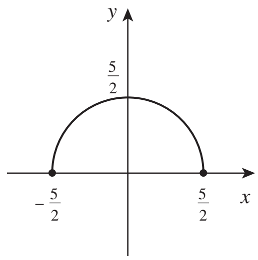
c



domain: $-1 \le x \le 1$,

range: $-1 \le y \le 0$

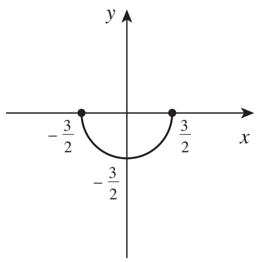
d



domain: $-\frac{5}{2} \le x \le \frac{5}{2}$,

range: $0 \le y \le \frac{5}{2}$

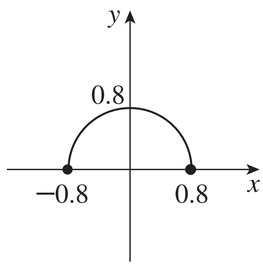
e



domain: $-\frac{3}{2} \le x \le \frac{3}{2}$,

range: $-\frac{3}{2} \le y \le 0$

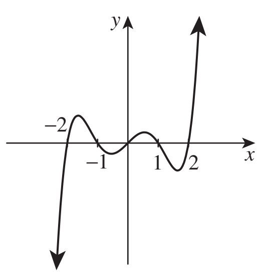
f



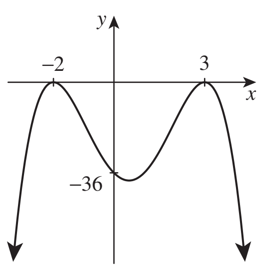
domain: $-0.8 \le x \le 0.8$,

range: $0 \le y \le 0.8$

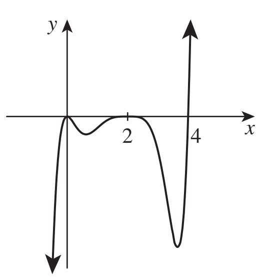
14 a



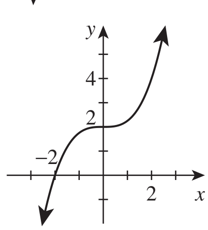
b



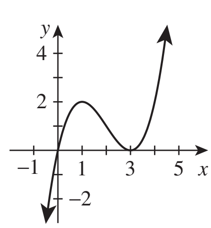
c



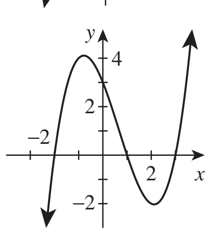
15 a i



ii



iii



d The product of the zeroes is $-\frac{d}{a}$.

16 a $y = -3(x+1)(x-1)(x-4)$

b $y = -(x+1)^2(x-1)^3(x-3)^2$

17 a $r = \sqrt{5}$, $(2, 1)$, $(1, 2)$, $(-1, 2)$, $(-2, 1)$, $(-2, -1)$, $(-1, -2)$, $(1, -2)$, $(2, -1)$

b $r = \sqrt{2}$, $(1, -1)$, $(-1, -1)$

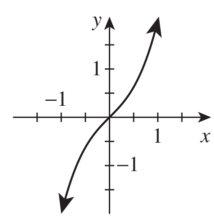
c $r = \sqrt{10}$, $(3, 1)$, $(1, 3)$, $(1, -3)$, $(3, -1)$

d $r = \sqrt{17}$, $(4, 1)$, $(1, 4)$, $(-1, 4)$, $(-4, 1)$, $(-4, -1)$, $(-1, -4)$, $(1, -4)$, $(4, -1)$

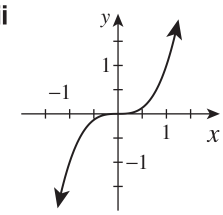
18 a $\left(0, 2\sqrt{\lambda^2 - \alpha^2}\right)$ **b** $r = \lambda$

c Lie the ladder on the ground and the midpoint is λ from the wall.

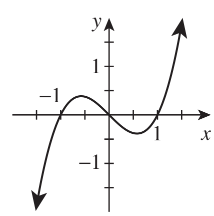
19 a i



ii



iii



b i 1st and 3rd

ii In each case, the result is the same curve.

iii Every index is odd.

c The slope:

$x^3 + x$ is upwards,

x^3 is horizontal,

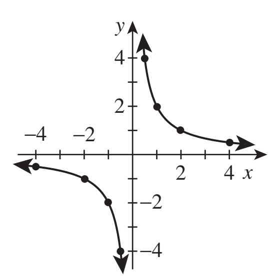
$x^3 - x$ is downwards.

Exercise 3H

1 a i $-\frac{1}{2}, -1, -2,$

$-4, 4, 2, 1, \frac{1}{2}$

ii



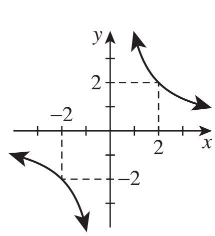
iii 1st and 3rd

iv the x-axis ($y = 0$) and the y-axis ($x = 0$)

v domain: $x \neq 0$, range: $y \neq 0$

b In each case, the domain is $x \neq 0$, the range is $y \neq 0$. The asymptotes are $y = 0$ and $x = 0$. The branches are in quadrants 1 and 3.

i



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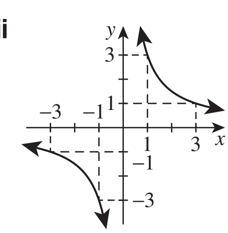

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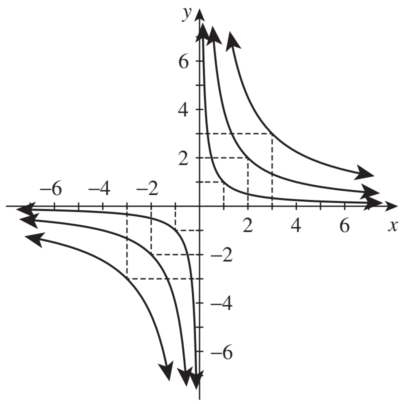


a 1st and 3rd

b the x-axis ($y = 0$) and the y-axis ($x = 0$)

c $x \neq 0,\ y \neq 0$

d $(1, 1)$ and $(-1, -1)$ on $y = \dfrac{1}{x}$

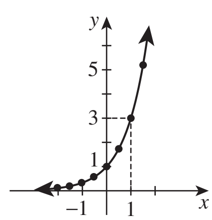
$(2, 2)$ and $(-2, -2)$ on $y = \dfrac{4}{x}$

$(3, 3)$ and $(-3, -3)$ on $y = \dfrac{9}{x}$

The values are the square roots of the numerator.

3 a i 0.1, 0.2, 0.3, 0.6, 1, 1.7, 3, 5.2, 9

ii



iii $(0, 1)$

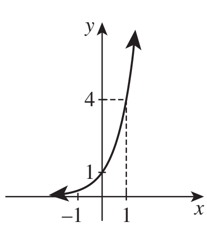
iv 3, the base

v the x-axis ($y = 0$)

vi domain: all real x, range: $y > 0$

b In each case, the domain is all real x, the range is $y > 0$. The asymptote is $y = 0$. The y-intercept is $(0, 1)$. At $x = 1$, $y =$ the base.

i



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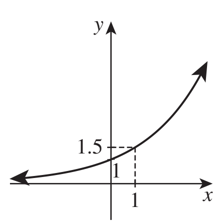

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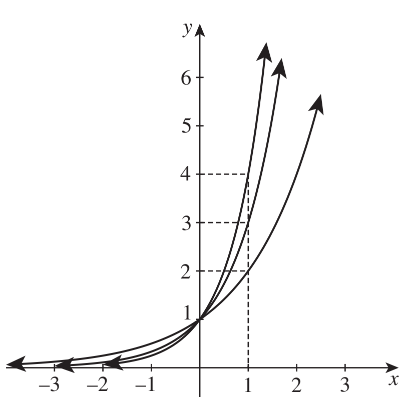


a $(0, 1)$

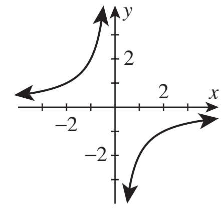
b the x-axis ($y = 0$)

c all real x, $y > 0$

e $y = 4^x$, it has the greater base.

f $y = 4^x$ again, it has the greater base.

5 a i



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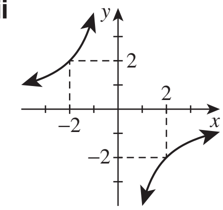

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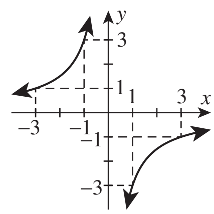
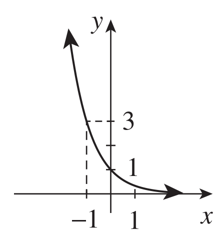


b i quadrants 2 and 4

ii The minus sign has caused the quadrants to change.

6 a i



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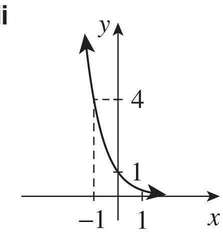

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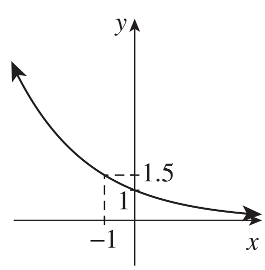


b i No: it is $(0, 1)$.

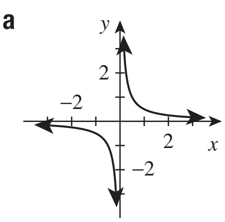
ii No: it is the x-axis.

iii $x = -1$

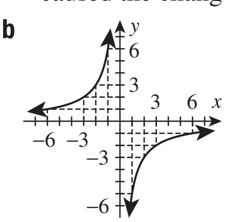
iv In Questions 4 and 5, the y-values grow. In these questions they decay away.

v The minus sign has caused the changes.

7 a



b



8 a i $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

ii There are no points with integer coordinates.

b i $\left(-\sqrt{6}, \sqrt{6}\right)$ and $\left(\sqrt{6}, -\sqrt{6}\right)$

ii $(-6, 1)$, $(-3, 2)$, $(-2, 3)$, $(-1, 6)$, $(1, -6)$, $(2, -3)$, $(3, -2)$, $(6, -1)$

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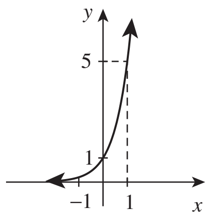
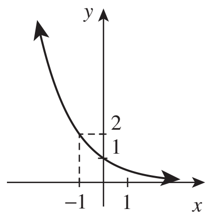
9 a



b

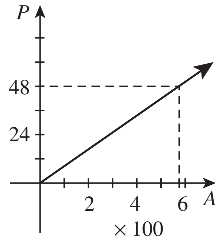


10 a $P = kA$

b $k = \frac{1}{12}$

c $55\frac{2}{3}$ L

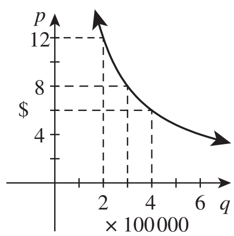
d 1 bucket, 4 tins



11 a $T = 2400000$

b 300000

c Sales will halve.



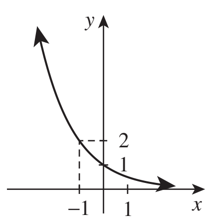
12 a $y \to 0$ as $x \to -\infty$. **b** $y \to 0$ as $x \to \infty$.

c $y \to 0$ as $x \to \infty$ and as $x \to -\infty$,
$y \to \infty$ as $x \to 0$, $y \to -\infty$ as $x \to 0^-$.

13 a $\left(\frac{1}{2}\right)^x = (2^{-1})^x$

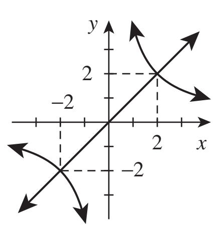
so $\left(\frac{1}{2}\right)^x = 2^{-x}$

b



14 a (c, c) and $(-c, -c)$

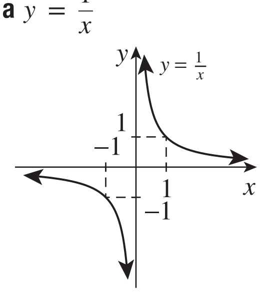
b



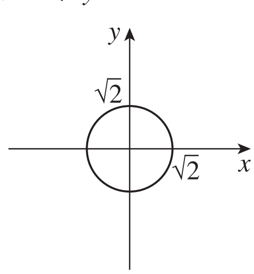
15 4 m × 12 m or 6 m × 8 m

16 No, because the only points that satisfy the equation lie on the x and y axes. The equation represents the two coordinate axes.

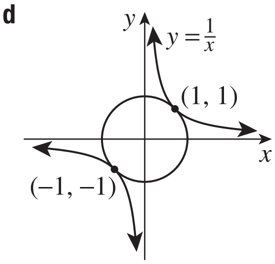
17 a $y = \frac{1}{x}$



b $x^2 + y^2 = 2$



d

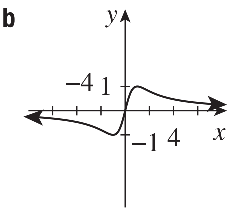


18 a $-\frac{16}{65}, -\frac{8}{17}, -\frac{4}{5}, -1, -\frac{4}{5}, 0, \frac{4}{5}, 1, \frac{4}{5}, \frac{8}{17}, \frac{16}{65}$

c x-axis ($y = 0$)

d (0, 0)

b



19 a $P\left(\frac{2}{b}, \frac{b}{2}\right)$

c 2 units^2

20 a $a = \frac{1}{4}, b = \frac{3}{4}, c = 1$

b $\sqrt{2} \doteqdot \frac{23}{16}, \frac{1}{\sqrt{2}} \doteqdot \frac{11}{16}$

Exercise 3I

1 a Vertical line test: Yes. It is a function.

b Horizontal line test: No. Many-to-one

c 10:00 pm on Saturday to 10:00 pm on Sunday

d 3 ft and 4 ft

e i 10:00 pm, 6:00 am, 10:30 am and 3:30 pm

ii 11:00 pm, 4:45 am and 1:00 pm

iii Never

f 0, 1, 2, 3 and 4

2 a It passes the vertical line test, so it is a function. Also, it fails the horizontal line test, so it is many-to-one.

b 1°C

c It was never 20°C. It was 8°C at 1:00 am, 8:00 am and 10:30 pm on the first day, and at about 3:30 pm on the second day.

d 0, 2, 3, 4, 5 (Whether 1 is omitted depends on how accurately you are supposed to read the graph.)

3 a and **e i** Vertical line test: No. Horizontal line test: Yes. One-to-many

ii Vertical line test: No. Horizontal line test: No. Many-to-many

iii Vertical line test: Yes. Horizontal line test: No. Many-to-one

iv Vertical line test: No. Horizontal line test: No. Many-to-many

v Vertical line test: Yes. Horizontal line test: Yes. One-to-one

vi Vertical line test: No. Horizontal line test: Yes. One-to-many

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b parts **iii**, **v** **c** parts **i**, **v**, **vi** **d** part **v**

4 a one-to-many **b** many-to-many
c one-to-many **d** many-to-one
e one-to-one **f** many-to-many

5 a **i** When $y = 0, x = 2$ or -2
ii When $y = 0, x = 1$ or 0 or -1
iii When $y = 2, x = 1$ or -1
b They are all one-to-many, because x and y are reversed.

6 a **i** $x = \frac{1}{3}y + \frac{1}{3}$ **ii** $x = -\frac{1}{2}y + \frac{5}{2}$
iii $x = \frac{1}{2}\sqrt[3]{y}$ **iv** $x = \dfrac{5}{y}$
b **iv** They are all one-to-one also, because x and y are reversed.

7 a When $x = 3, y = 4$ or -6. When $y = -1$, $x = 8$ or -2
b When $x = 0, y = 3$ or -3. When $y = 0$, $x = 2$ or -2
c When $x = 2, y = \sqrt{3}$ or $-\sqrt{3}$. When $y = 0$, $x = 1$ or -1

8 a It passes neither test, and is thus many-to-many.
b Vertical line test: Yes. Horizontal line test: No. It is many-to-one, and therefore a function.

9 a It is a function, but it may be one-to-one or many-to-one.
b If there are two or more students with the same preferred name, it is many-to-one. Otherwise it is one-to-one.

10 a $\dots, -270°, 90°, 450°, \dots$
b one-to-many **c** many-to-one

11 a Probably many-to-many, but just possibly one-to-one.
b The condition to be one-to-one is that every flat has no more than one occupant, and in this case, every inhabitant is mapped to himself or herself, that is, $f(x) = x$, for every inhabitant x. Otherwise the relation is many-to-many.
c The relation is then the *empty relation*, which is discussed later in Section 4E. This empty relation is a one-to-one function, because it trivially passes the vertical and horizontal line tests.

12 a many-to-one **b** one-to-many **c** one-to-one
d one-to-one (trivially because the graph has only one point)
e many-to-many
f one-to-many (factor as $x = (y - 2)(y - 3)$)
g many-to-one (factor as $y = x(x - 3)(x - 4)$)
h one-to-one **i** one-to-one
j one-to-one **k** many-to-many
l one-to-one

13 a $f(a) = f(b)$ because $g(x)$ is one-to-one. Hence $a = b$ because $f(x)$ is one-to-one.
b The composition of two one-to-one functions is one-to-one.

14 a One-to-one. Every even integer n is $f\left(\frac{1}{2}n\right)$, and is not the image of any other number.
Odd integers are not the image of anything.
Every other real number is only the image of itself.
b Many-to-one, $f(3) = 1\frac{1}{2} = f\left(1\frac{1}{2}\right)$.
c One-to-one. Every rational number x is cubed, and x^3 is again a rational number, and the cubes of two distinct numbers are never equal.
d Many-to-one, $f\left(\sqrt[3]{2}\right) = 2 = f(2)$.

Chapter 3 review exercise

1 a not a function **b** function
c function **d** not a function

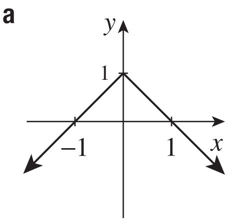
2 a $-2 \le x \le 0, -2 \le y \le 2$
b all real x, all real y
c $x \ne 0, y \ne 0$ **d** $x = 2$, all real y

3 a $21, -4$ **b** $5, -15$

4 a $x \ne 2$ **b** $x \ge 1$
c $x \ge -\dfrac{2}{3}$ **d** $x < 2$

5 a $2a + 2, 2a + 1$
b $a^2 - 3a - 8, a^2 - 5a - 3$

6 a



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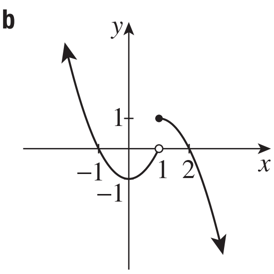

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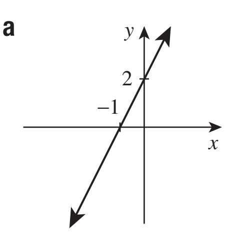

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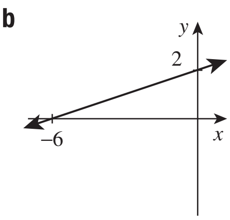

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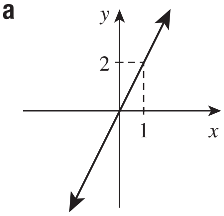
8 a



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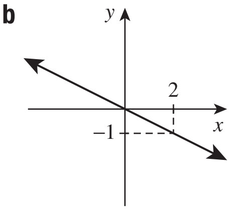
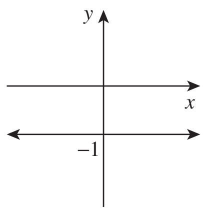


9 a i



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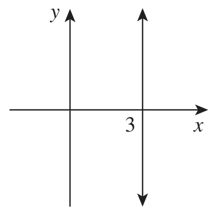
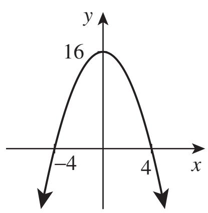


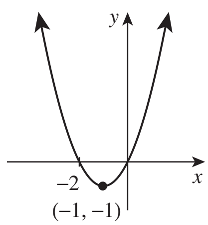
b (3, −1)

10 a



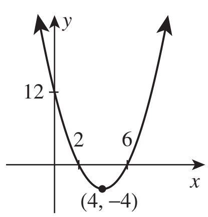
domain: all real x,
range: $y \le 16$

b



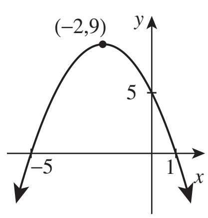
domain: all real x,
range: $y \ge -1$

c



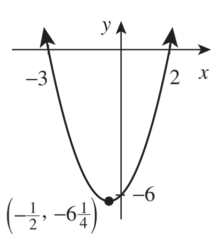
domain: all real x,
range: $y \ge -4$

d



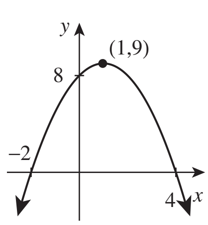
domain: all real x,
range: $y \le 9$

e



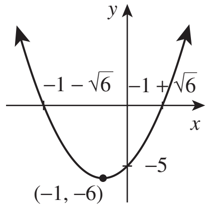
domain: all real x,
range: $y \ge -6\frac{1}{4}$

f



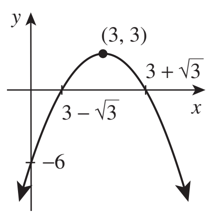
domain: all real x,
range: $y \le 9$

11 a



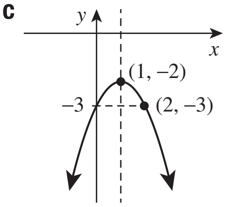
$y = (x + 1)^2 - 6$

b



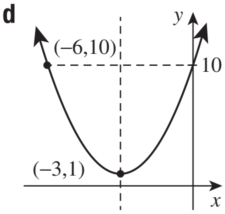
$y = -(x - 3)^2 + 3$

c



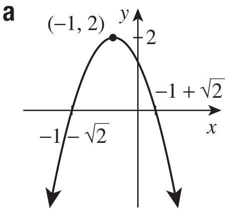
$y = -(x - 1)^2 - 2$

d

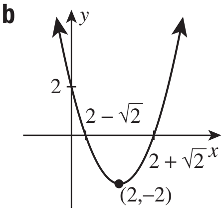


$y = (x + 3)^2 + 1$

12 a



b



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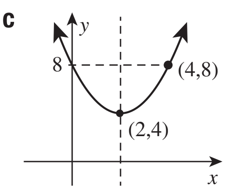

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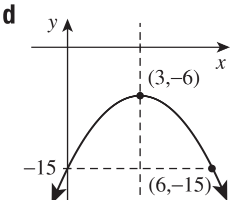
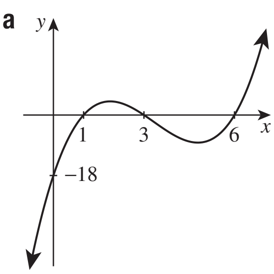


13 a



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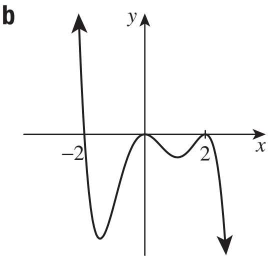
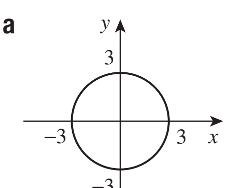
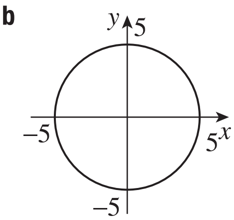


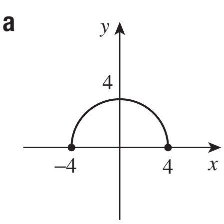
14 a



b

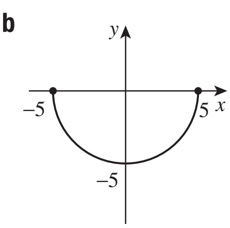


15 a



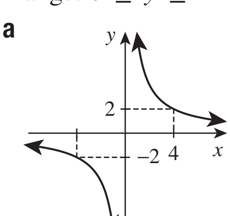
domain: $-4 \le x \le 4$,
range: $0 \le y \le 4$

b



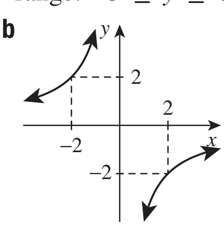
domain: $-5 \le x \le 5$,
range: $-5 \le y \le 0$

16 a



domain: $x \ne 0$,
range: $y \ne 0$

b



domain: $x \ne 0$,
range: $y \ne 0$

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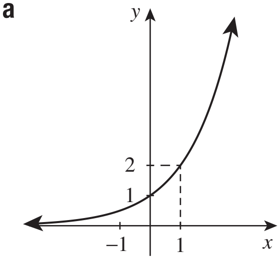
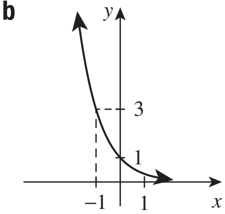
17 a



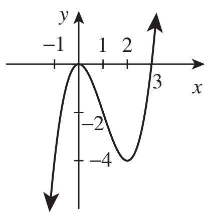
domain: all real x,
range: $y > 0$

b



domain: all real x,
range: $y > 0$

18 a



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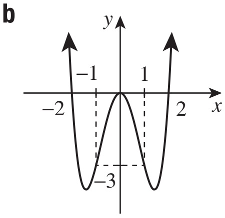

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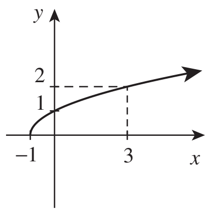
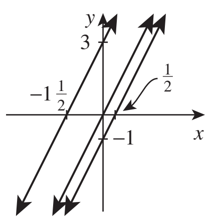


19 a



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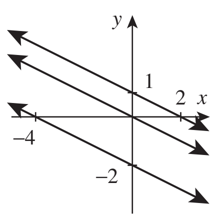

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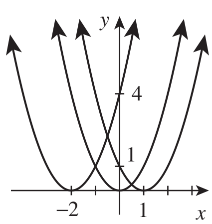

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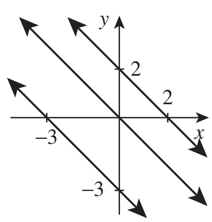

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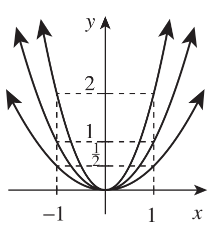

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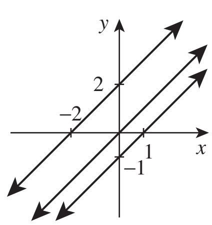

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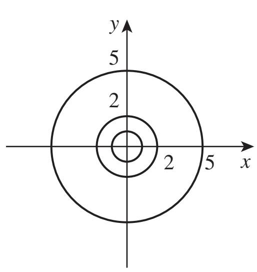

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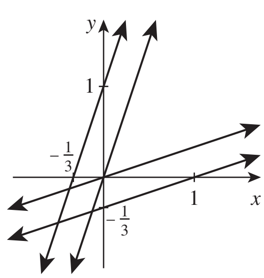

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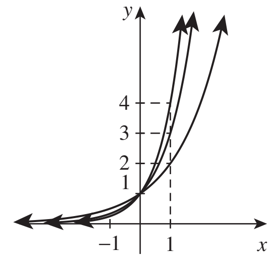
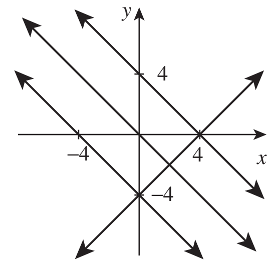


j



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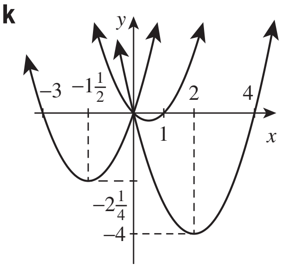
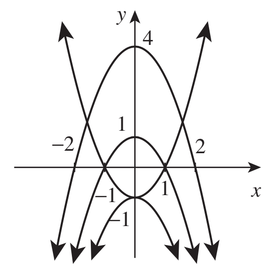


Top vertex: $\left(\frac{1}{2}, -\frac{1}{4}\right)$

l



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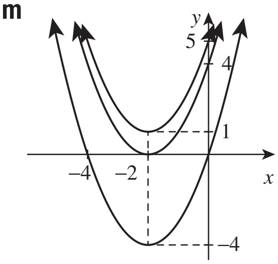
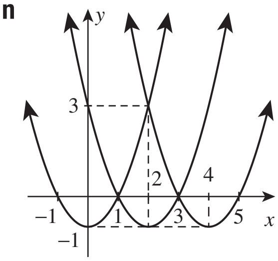
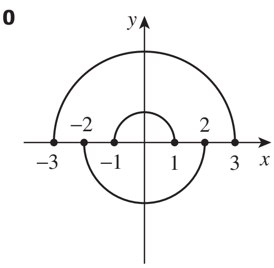


n



o



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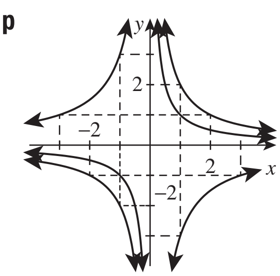

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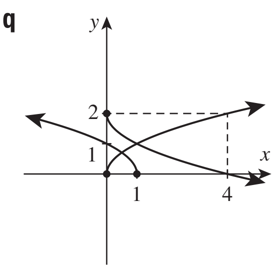

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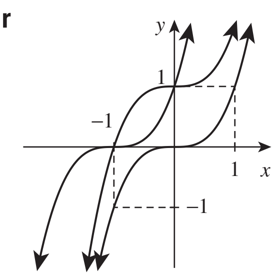

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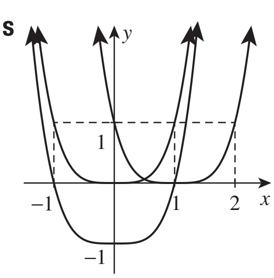
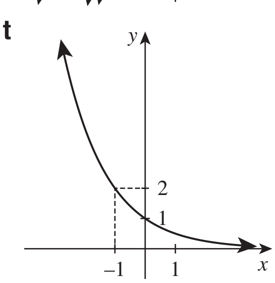


t



All three are the same.

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20 a one-to-one **b** many-to-many
c one-to-many **d** many-to-one

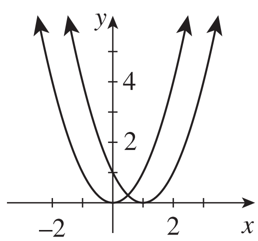
21 a It is probably a many-to-one function, but it is possibly a one-to-one function
b If every person was born in a different country, the function is one-to-one. Otherwise it is many-to-one.

Chapter 4

Exercise 4A

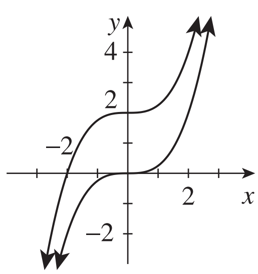
1 a x^2: 4, 1, 0, 1, 4, 9
$(x - 1)^2$: 9, 4, 1, 0, 1, 4
b $y = x^2$, $V = (0, 0)$.
$y = (x - 1)^2$, $V = (1, 0)$.
c Here x is replaced by $(x - 1)$, so it is a shift right by 1 unit.

b

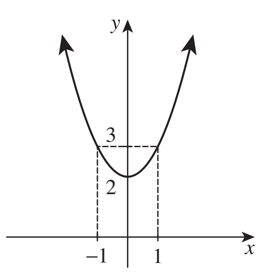


2 a $\frac{1}{4}x^3$: $-6\frac{3}{4}$, -2, $-\frac{1}{4}$, 0, $\frac{1}{4}$, 2, $6\frac{3}{4}$
$\left(\frac{1}{4}x^3 + 2\right)$: $-4\frac{3}{4}$, 0, $1\frac{3}{4}$, 2, $2\frac{1}{4}$, 4, $8\frac{3}{4}$
b $(0, 0)$ and $(0, 2)$
c The second equation is also $y - 2 = \frac{1}{4}x^3$.
Here y is replaced by $(y - 2)$, so it is a shift up by 2 units.

b

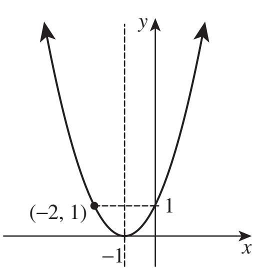


3 a



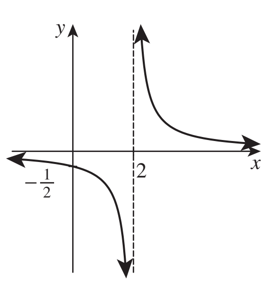
up 2 units

b



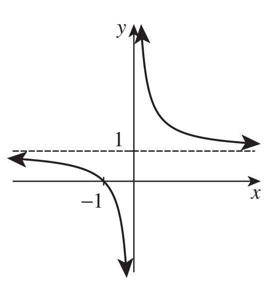
left 1 unit

c



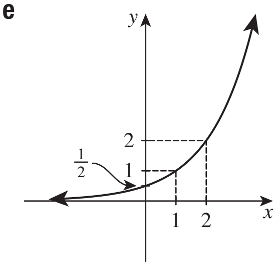
right 2 units

d



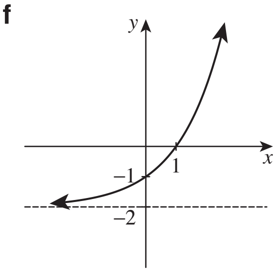
up 1 unit

e



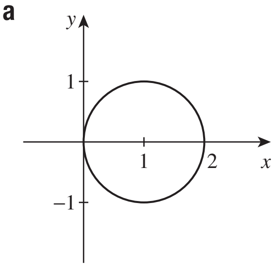
right 1 unit

f

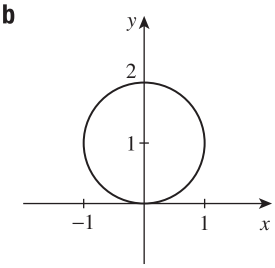


down 2 units

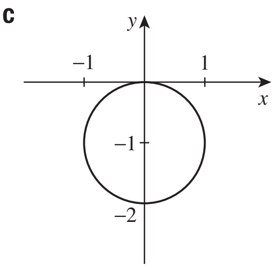
4 a



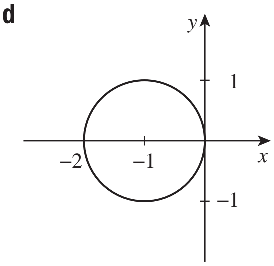
b



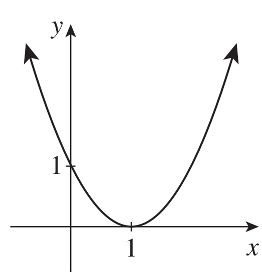
c



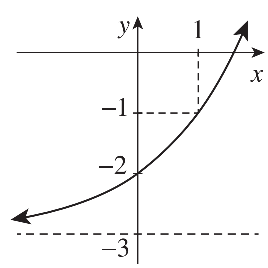
d



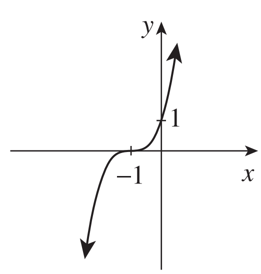
5 a $y = (x - 1)^2$



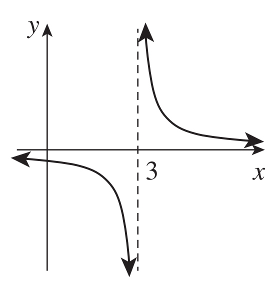
b $y = 2^x - 3$



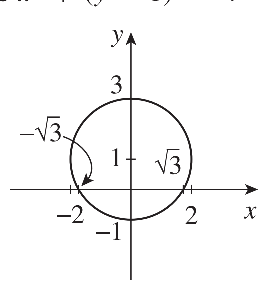
c $y = (x + 1)^3$



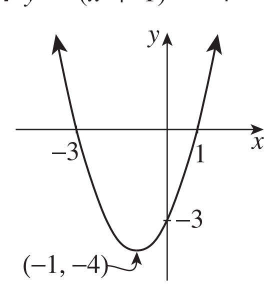
d $y = \dfrac{1}{x - 3}$



e $x^2 + (y - 1)^2 = 4$



f $y = (x + 1)^2 - 4$



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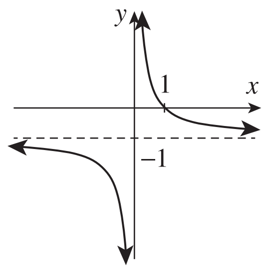
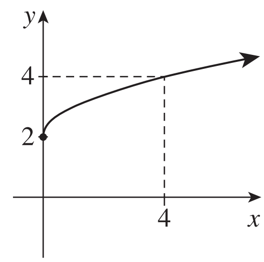
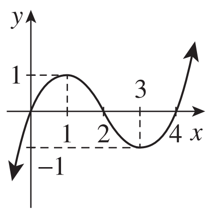
g $x(y+1)=1$



h $y=\sqrt{x}+2$



6 a i



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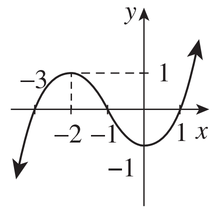
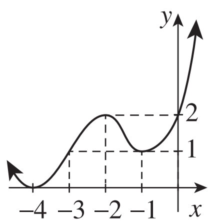


b i



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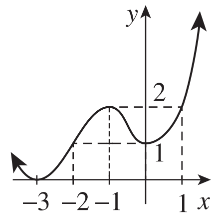
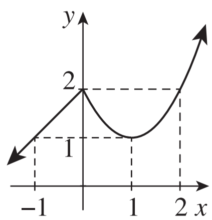


c i



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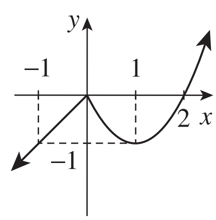
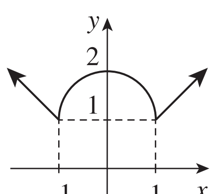
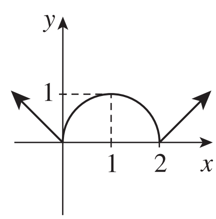


d i



ii



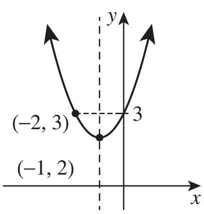
7 a $r=2,\ (-1,0)$ **b** $r=1,\ (1,2)$

c $r=3,\ (1,2)$ **d** $r=5,\ (-3,4)$

e $r=3,\ (5,\ -4)$ **f** $r=6,\ (-7,1)$

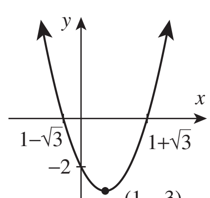
8 a $y=(x+1)^2+2$

This is $y=x^2$ shifted left 1 and up 2.



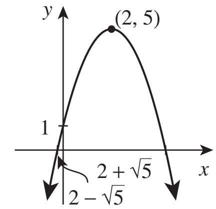
b $y=(x-1)^2-3$

This is $y=x^2$ shifted right 1 and down 3.



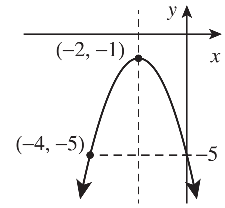
c $y=-(x-2)^2+5$

This is $y=-x^2$ shifted right 2 and up 5.



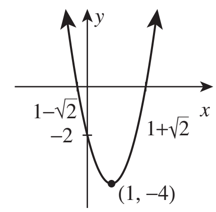
d $y=-(x+2)^2-1$

This is $y=-x^2$ shifted left 2 and down 1.



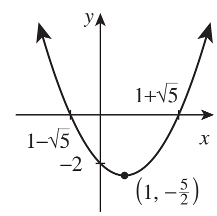
e $y=2(x-1)^2-4$

This is $y=2x^2$ shifted right 1 and down 4.



f $y=\frac{1}{2}(x-1)^2-\frac{5}{2}$

This is $y=\frac{1}{2}x^2$ shifted right 1 and down $\frac{5}{2}$.



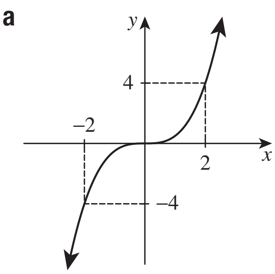
9 a the circle $x^2+y^2=1$ translated right 2, up 3, $(x-2)^2+(y-3)^2=1$

b the circle $x^2+y^2=4$ translated left 2, down 1, $(x+2)^2+(y+1)^2=4$

c the circle $x^2+y^2=10$ translated left 1, up 1, $(x+1)^2+(y-1)^2=10$

d the circle $x^2+y^2=5$ translated right 2, down 1, $(x-2)^2+(y+1)^2=5$

10 a



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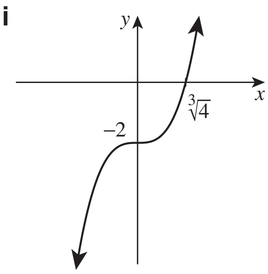

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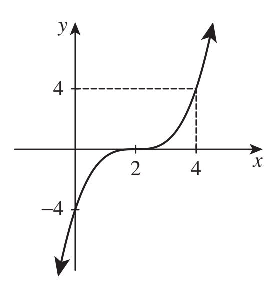

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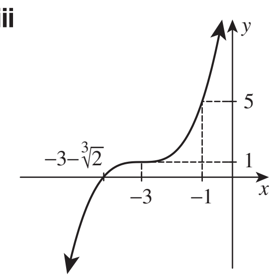

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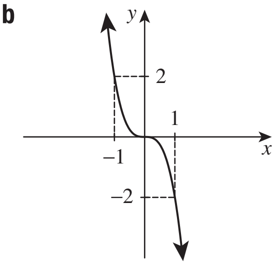
b



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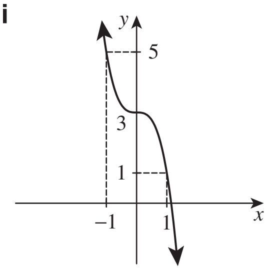

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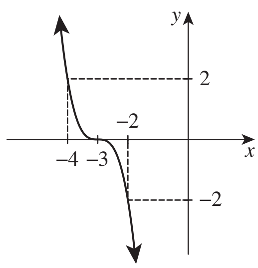

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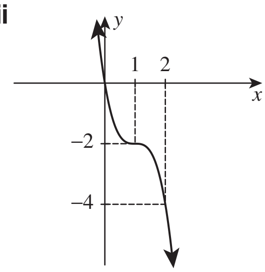
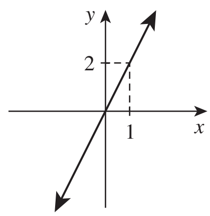


11 a From $y = 2x$:

i shift up 4 (or left 2)

ii shift down 4 (or right 2)



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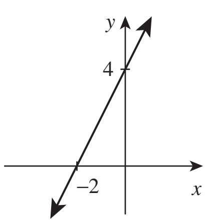

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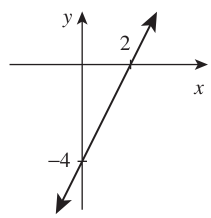
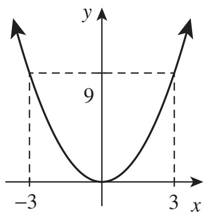


b From $y = x^2$:

i shift up 9

ii shift down 9

iii shift right 3



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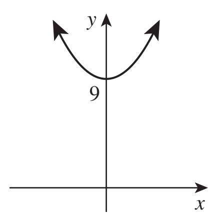

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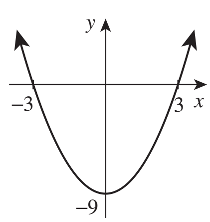

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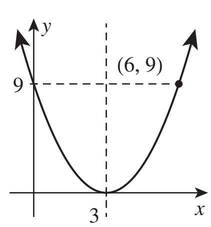
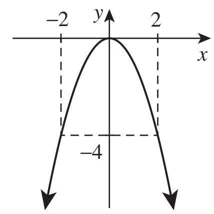


c From $y = -x^2$:

i shift up 1

ii shift left 1

iii shift left 1 and up 2



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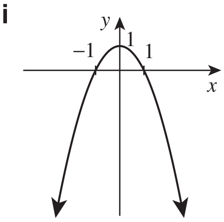

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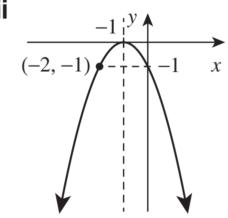

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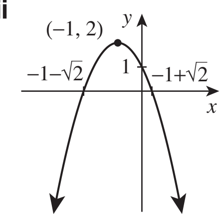
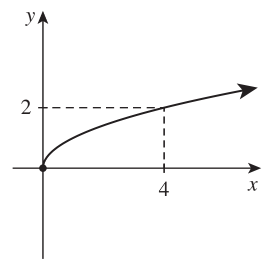


d From $y = \sqrt{x}$:

i shift left 4

ii shift up 4

iii shift left 4 and down 2



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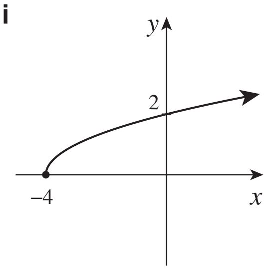

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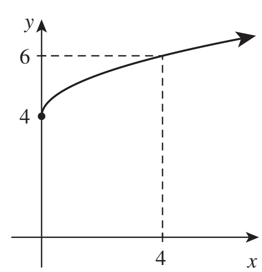

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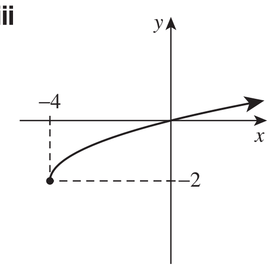
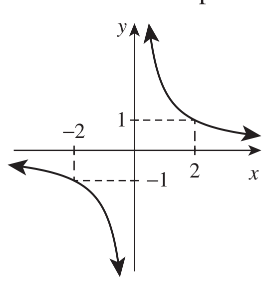


e From $y = \frac{2}{x}$:

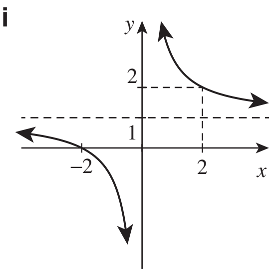
i shift up 1

ii shift left 2

iii shift left 2 and up 1



i



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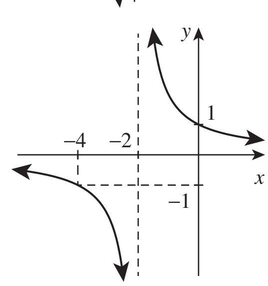

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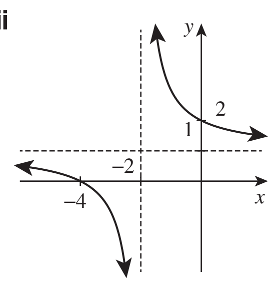

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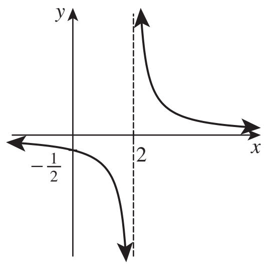
12 a



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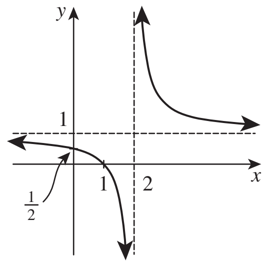

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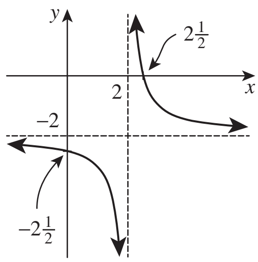

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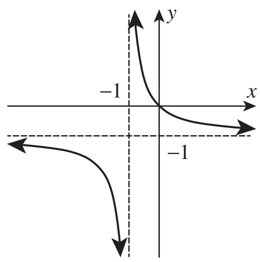

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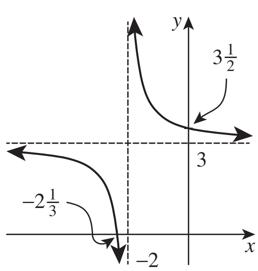

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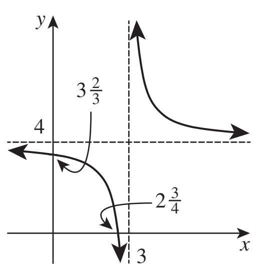


13 a the parabola $y = x^2$ translated right 2, $y = (x - 2)^2$

b the hyperbola $xy = 1$ translated right 2, $y = \dfrac{1}{x - 2}$

c the exponential $y = 3^x$ translated left 1, $y = 3^{x+1}$

d the circle $x^2 + y^2 = 4$ translated left 2, down 1, $(x + 2)^2 + (y + 1)^2 = 4$

e the hyperbola $xy = 1$ translated right 2, down 1, $y + 1 = \dfrac{1}{x - 2}$

f the parabola $y = x^2$ translated right 2, down 1, $y + 1 = (x - 2)^2$

g the circle $x^2 + y^2 = 5$ translated right 2, down 1, $(x - 2)^2 + (y + 1)^2 = 5$

h the exponential $y = 4^x$ translated down 4, $y = 4^x - 4$

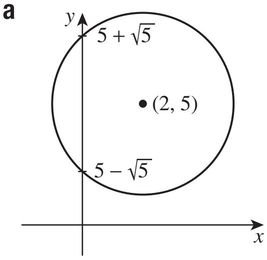
14 a $(x - 2)^2 + (y - 5)^2 = 9$, $r = 3$, centre $(2, 5)$, intercepts $(0, -\sqrt{5})$, $(0, \sqrt{5})$

b $x^2 + (y + 3)^2 = 10$, $r = \sqrt{10}$, centre $(0, -3)$, intercepts $(0, -3-\sqrt{10})$, $(0, -3+\sqrt{10})$, $(-1, 0)$, $(1, 0)$

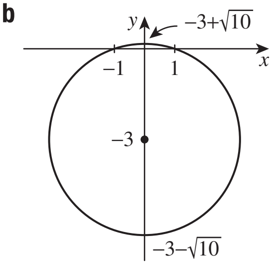
c $(x + 2)^2 + (y - 4)^2 = 20$, with $r = 2\sqrt{5}$, and centre $(-2, 4)$, intercepts, $(0, 0)$, $(0, 8)$, $(-4, 0)$

d $(x - 1)^2 + (y + 2)^2 = 6$, $r = \sqrt{6}$, centre $(1, -2)$, intercepts $(0, -2-\sqrt{5})$, $(0, -2 + \sqrt{5})$, $(1-\sqrt{2}, 0)$, $(1+\sqrt{2}, 0)$

a



b



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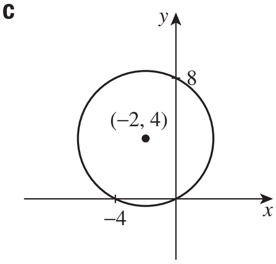

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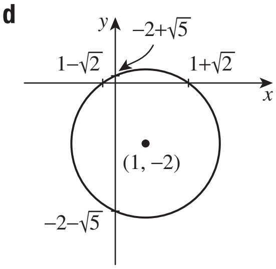


15 a $x + 2y - 2 = 0$ **b** $x + 2y - 2 = 0$

c Both translations yield the same result.

16 a $(x - h)^2 + (y - k)^2 = r^2$

17 a $y - y_1 = m(x - x_1)$ is the line $y = mx$ shifted right by x_1 and up by y_1.

b Because only shifts are involved, the lines in part **a** are parallel. Thus parallel lines have the same gradient m.

18 a $y = f(x - a)$, $y - b = f(x - a)$

b $y - b = f(x)$, $y - b = f(x - a)$

c The final transformed function is the same in both cases. Thus, the order of shifts is irrelevant.

19 a $(1, 2), (1, 4), (-3, 2), (-3, 4)$

b It is a 4×2 rectangle.

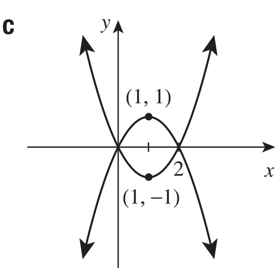
c $C = (-1, 3)$, $r = \sqrt{5}$

d $(x + 1)^2 + (y - 3)^2 = 5$

Exercise 4B

1 b $y = x^2 - 2x$:
8, 3, 0, −1, 0, 3, 8
$y = 2x - x^2$:
−8, −3, 0, 1, 0, −3, −8
$y = x^2 - 2x$:
$V = (1, -1)$.
$y = 2x - x^2$: $V = (1, 1)$.

c



d Here y is replaced with $-y$, so it is a reflection in the x-axis.

2 b $y = \dfrac{2}{x - 2}$: $-\frac{1}{3}$, $-\frac{2}{5}$, $-\frac{1}{2}$, $-\frac{2}{3}$, −1, −2, *, 2, 1

$y = -\dfrac{2}{x + 2}$: 1, 2, *, −2, −1, $-\frac{2}{3}$, $-\frac{1}{2}$, $-\frac{2}{5}$, $-\frac{1}{3}$

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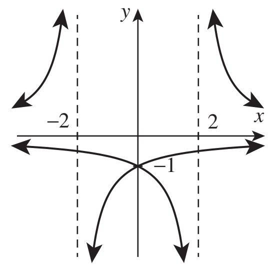
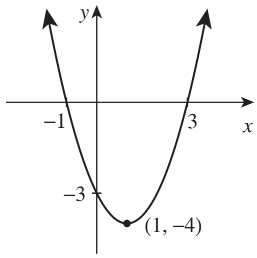
c $y = \dfrac{2}{x-2}: x = 2.$

$y = -\dfrac{2}{x+2}: x = -2.$

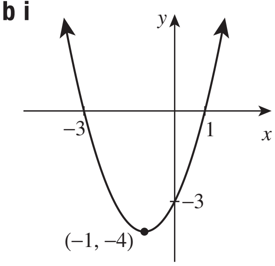
d Here x is replaced with $-x$, so it is a reflection in the y-axis.



3 a

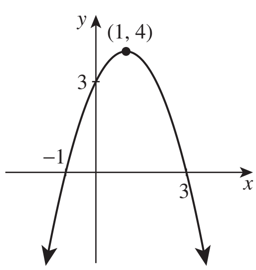


b i



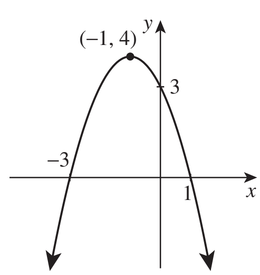
$y = x^2 + 2x - 3$

ii



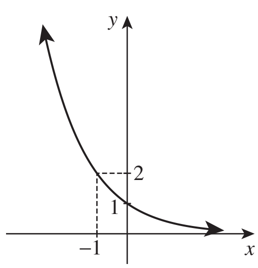
$y = -x^2 + 2x + 3$

iii

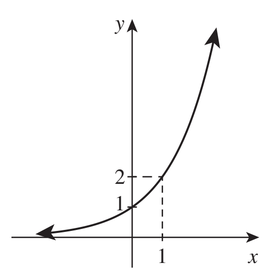


$y = -x^2 - 2x + 3$

4 a

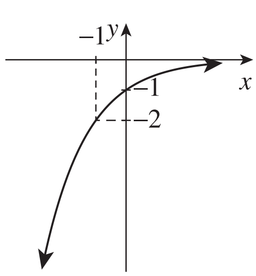


b i



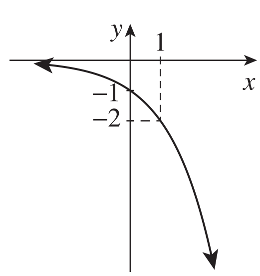
$y = 2^x$

ii



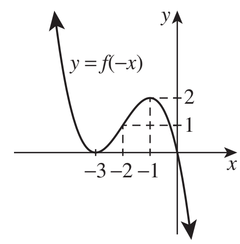
$y = -2^{-x}$

iii



$y = -2^x$

5 a i



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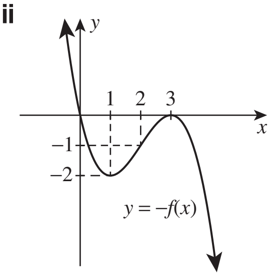
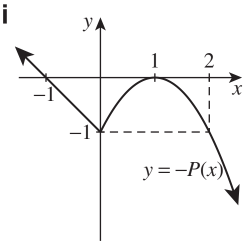


b i



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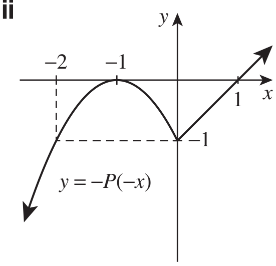
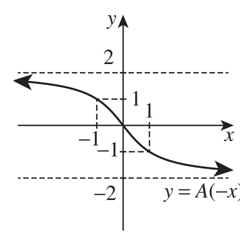


c i



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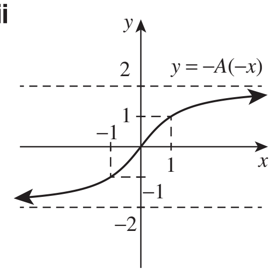
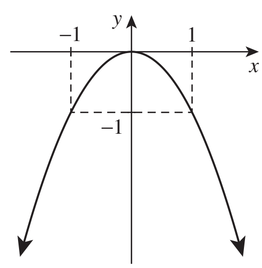
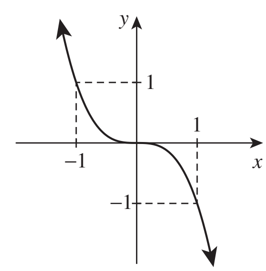


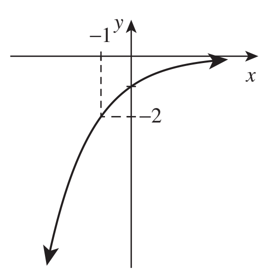
6 a $y = -x^2$



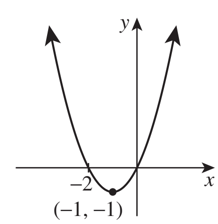
b $y = -x^3$



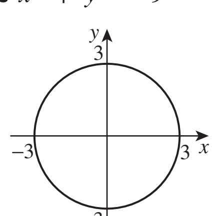
c $y = -2^{-x}$



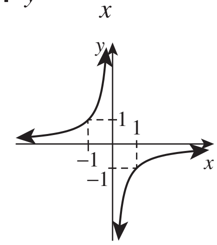
d $y = x^2 + 2x$



e $x^2 + y^2 = 9$



f $y = -\dfrac{1}{x}$



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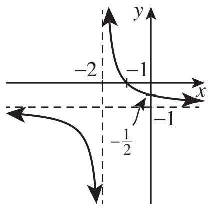
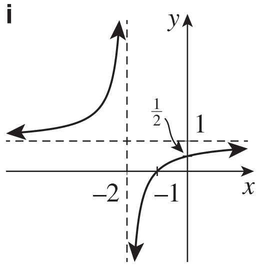
7 a



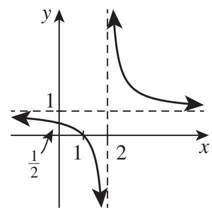
b i



Reflect in the x-axis:

$y = 1 - \dfrac{1}{x + 2}$.

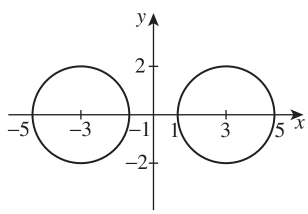
ii



Rotate by 180°:

$y = 1 - \dfrac{1}{2 - x}$.

8 a



b Reflect in the y-axis.

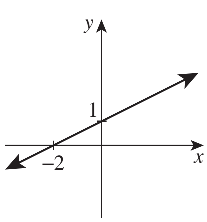
c You will need to use $(-x - 3)^2 = (x + 3)^2$.

d Shift left by 6 units.

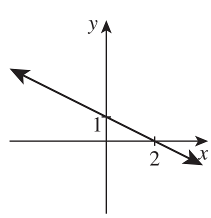
e Replace x with $(x + 6)$.

9 a The circle is symmetric in both axes.

10 a From $y = \frac{1}{2}x + 1$: **i** reflect in the y-axis **ii** reflect in the x-axis **iii** rotate by 180°



i



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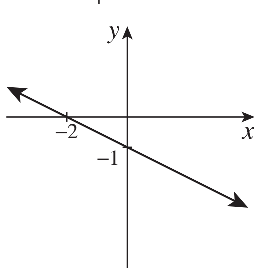

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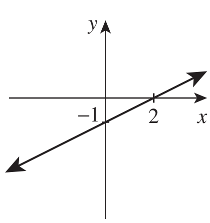
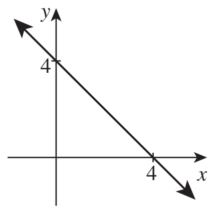
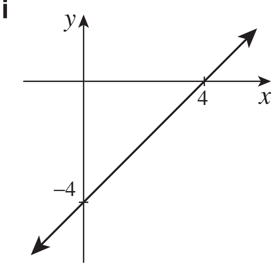


b From $y = 4 - x$: **i** reflect in the x-axis **ii** reflect in the y-axis **iii** rotate by 180°



i



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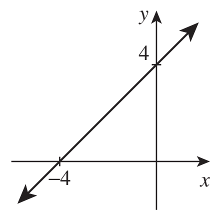

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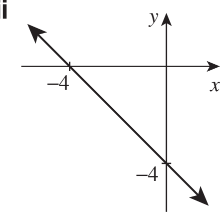
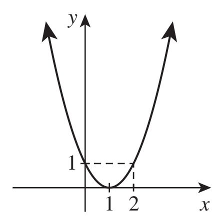
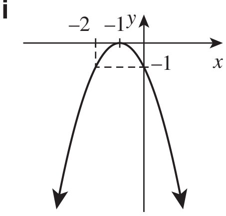


c From $y = (x - 1)^2$: **i** rotate by 180° **ii** reflect in the y-axis **iii** reflect in the x-axis



i



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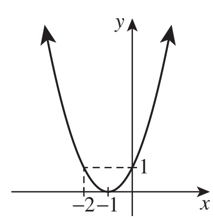

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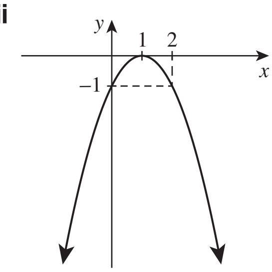
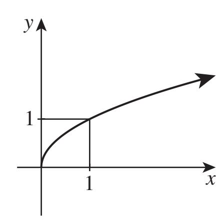
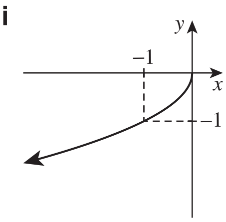


d From $y = \sqrt{x}$: **i** rotate by 180° **ii** reflect in the x-axis **iii** reflect in the y-axis



i



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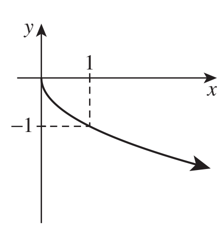

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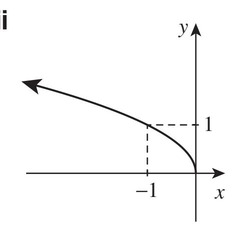

e From $y = 3^x$: **i** reflect in the x-axis **ii** rotate by 180° **iii** reflect in the y-axis

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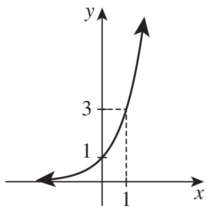

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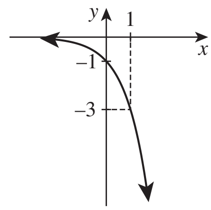

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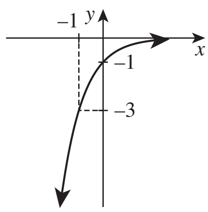

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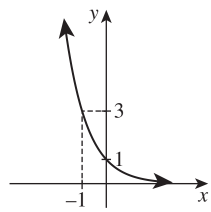
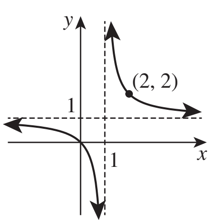


f From $y = 1 + \dfrac{1}{x - 1}$:

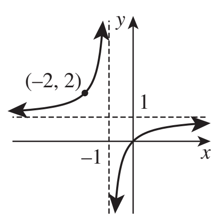
i reflect in the y-axis

ii rotate by 180°

iii reflect in the x-axis



i



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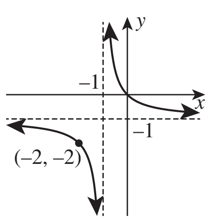

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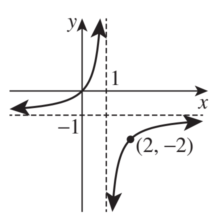
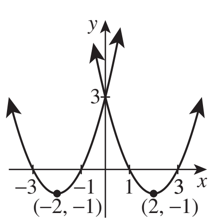


11 a



b Reflect in the y-axis.

c Shift left 4 units.

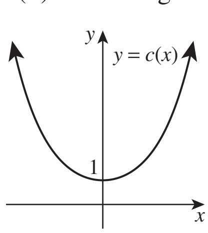
d $(x + 4)^2 - 4(x + 4) + 3 = x^2 + 4x + 3$

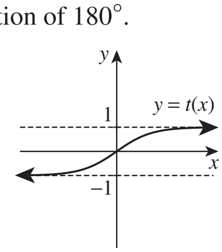
e part **b**, part **c**, part **f**

12 a $c(x)$ is the same when reflected in the y-axis.

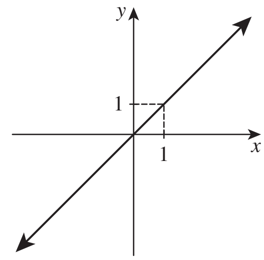
b $t(x)$ is unchanged by a rotation of 180°.

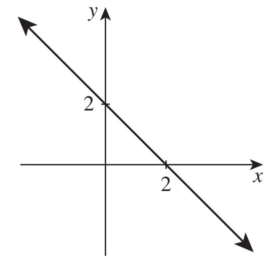
c



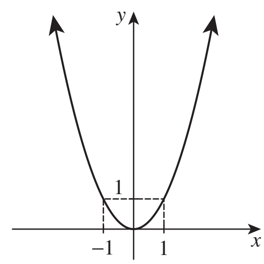


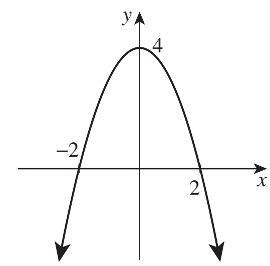
13 a Reflect in the y-axis then shift up 2.



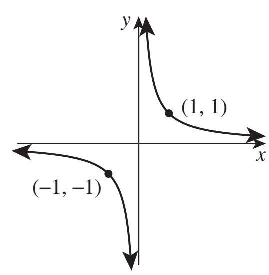


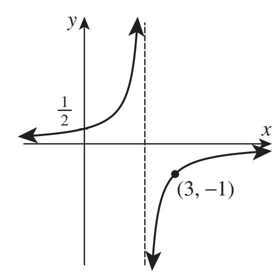
b Reflect in the x-axis then shift up 4.



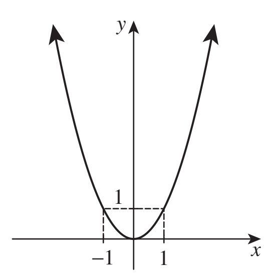


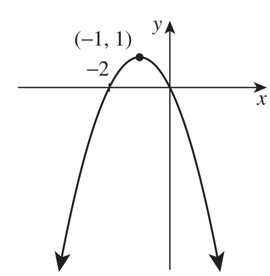
c Shift left 2 then reflect in the y-axis.



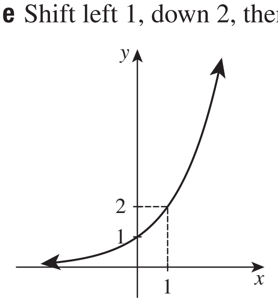


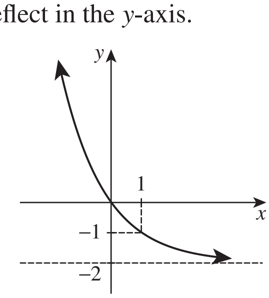
d Reflect in the x-axis then shift left 1, up 1.





e Shift left 1, down 2, then reflect in the y-axis.





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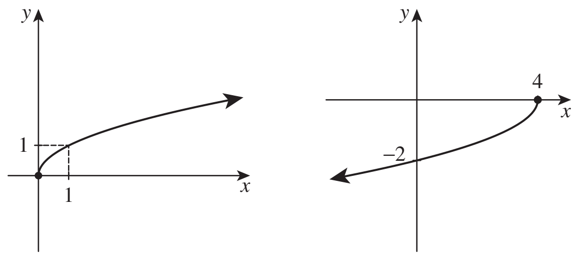
f Shift left 4, then reflect in both axes.



14 a i $y = (x - 2)^2$ **ii** $y = (x + 2)^2$

b i $y = (x + 1)^2$ **ii** $y = x^2$

c Yes: the answer depends on the order.

d The order is irrelevant when the shift is parallel with the axis of reflection.

15 a $y - a = f(x)$, $-y - a = f(x)$, $-y = f(x)$, $y = f(x)$

b $y = f(x - a)$, $-y = f(x - a)$, $-y = f(x - 2a)$, $y = f(x - 2a)$

c $x = f(y)$, $-x = f(y)$, $-y = f(x)$, $y = f(x)$

d $-x = f(y)$, $-y = f(-x)$, $x = f(-y)$, $y = f(x)$

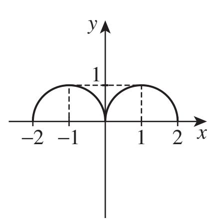
16 Shift left by a to get $y = f(x + a)$, then reflect in the y-axis to get $y = f(a - x)$, finally shift right by a to get $y = f(2a - x)$.

Exercise 4C

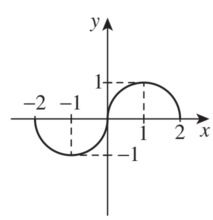
1 a even **b** neither **c** neither

d odd **e** odd **f** even

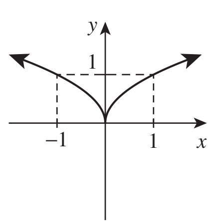
2 a i



ii



b i



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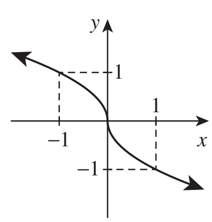
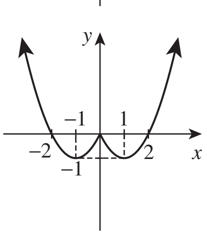


c i



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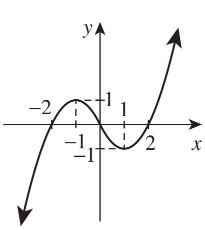


3 a $f(-x) = x^4 - 2x^2 + 1$

b $f(-x) = f(x)$, so it is even.

4 a $g(-x) = -x^3 + 3x$

b $-g(x) = -(x^3 - 3x) = g(-x)$, so it is odd.

5 a $h(-x) = -x^3 + 3x^2 - 2$

b $-h(x) = -x^3 - 3x^2 + 2$. Because $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, it is neither.

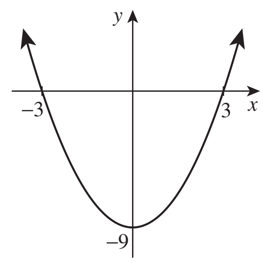
6 a even **b** neither **c** odd **d** even

e neither **f** odd **g** odd **h** neither

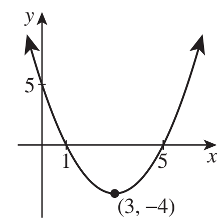
7 a . . . if all powers of x are odd.

b . . . if all powers of x are even.

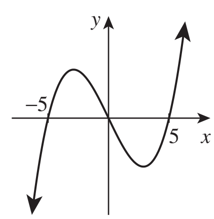
8 a $y = (x + 3)(x - 3)$



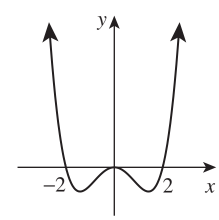
b $y = (x - 1)(x - 5)$



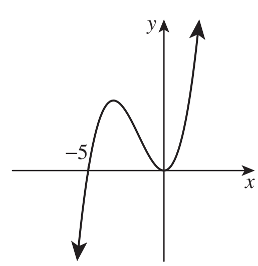
c $y = x(x - 5)(x + 5)$



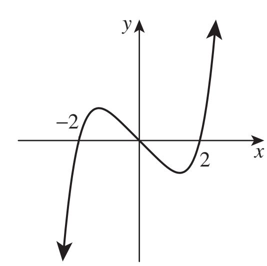
d $y = x^2(x - 2)(x + 2)$.



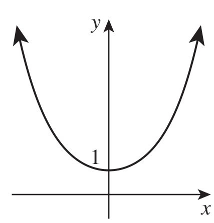
e $y = x^2(x + 5)$



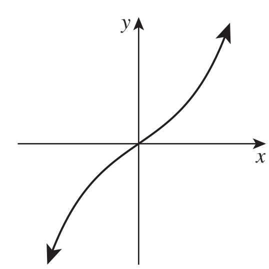
f $y = x(x - 2)(x + 2)(x^2 + 4)$



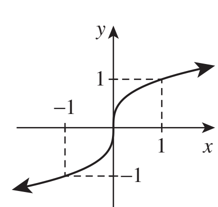
9 a even



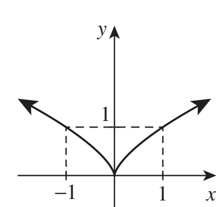
b odd



c odd



d even



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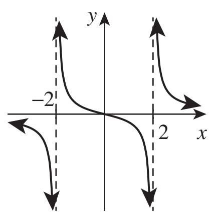
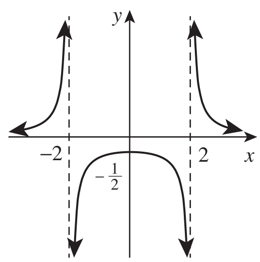
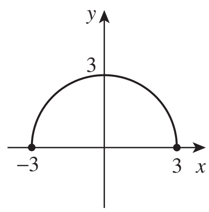
e odd



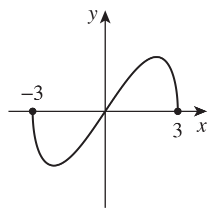
f even



g even



h odd



10 a neither **b** neither **c** even **d** even
e odd **f** even **g** odd **h** neither

11 a It is symmetric in the y-axis.
b It is symmetric in both axes.

13 a i even **ii** even **iii** odd
b i even **ii** odd
iii in general, neither

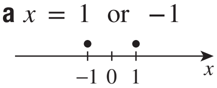
14 a Suppose $f(0) = c$. Then because $f(x)$ is odd, $f(0) = -f(0) = -c$. So $c = -c$, and hence $c = 0$.
b It is not defined at the origin (it is 1 for $x > 0$, and -1 for $x < 0$).

15 b i $g(x) = 1 + x^2$ and $h(x) = -2x$
ii $g(x) = \dfrac{2^x + 2^{-x}}{2}$ and $h(x) = \dfrac{2^x - 2^{-x}}{2}$
c In the first, $g(x)$ and $h(x)$ are not defined for all x in the natural domain of $f(x)$, specifically at $x = -1$. In the second, $x = 0$ is the only place at which $g(x)$ and $h(x)$ are defined.

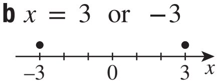
Exercise 4D

1 a 3 **b** 3 **c** 3 **d** 3
e 7 **f** 1 **g** 16 **h** −3

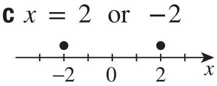
2 a $x = 1$ or -1



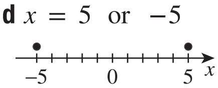
b $x = 3$ or -3



c $x = 2$ or -2

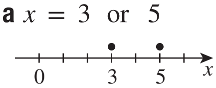


d $x = 5$ or -5

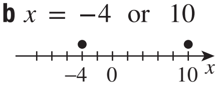


e no solutions **f** no solutions

3 a $x = 3$ or 5

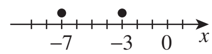


b $x = -4$ or 10

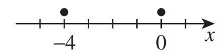


c no solutions **d** no solutions

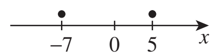
e $x = -3$ or -7



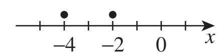
f $x = -4$ or 0



g $x = -7$ or 5

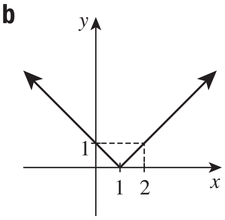


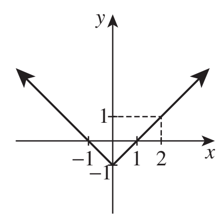
h $x = -4$ or -2



4 a For $|x - 1|$: 3, 2, 1, 0, 1, 2.
For $|x| - 1$: 1, 0, −1, 0, 1, 2.

b



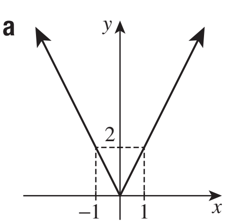


The two graphs overlap for $x \geq 1$.
c The first is $y = |x|$ shifted right 1 unit, the second is $y = |x|$ shifted down 1 unit.

5 a LHS = 2, RHS = −2
b LHS = 2, RHS = −2
c LHS = 0, RHS = 4
d LHS = 1, RHS = −1
e LHS = 3, RHS = 1
f LHS = 8, RHS = −8

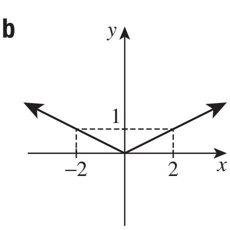
6 a false: $x = 0$ **b** true
c true **d** false: $x = -2$
e true **f** true
g false: $x = -2$ **h** true

7 a



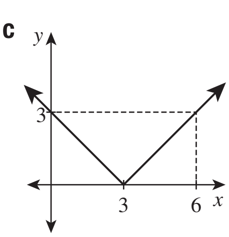
$y = \begin{cases} 2x, & \text{for } x \geq 0, \\ -2x, & \text{for } x < 0. \end{cases}$

b



$y = \begin{cases} \frac{1}{2}x, & \text{for } x \geq 0, \\ -\frac{1}{2}x, & \text{for } x < 0. \end{cases}$

c



shift right 3,
$y = \begin{cases} x - 3, & \text{for } x \geq 3, \\ 3 - x, & \text{for } x < 3. \end{cases}$

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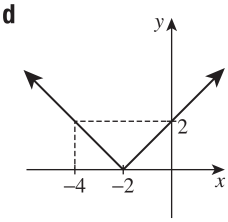
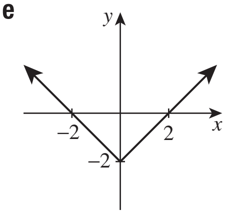
d



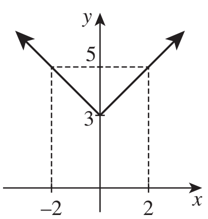
shift left 2,
$y = \begin{cases} x + 2, & \text{for } x \geq -2, \\ -x - 2, & \text{for } x < -2. \end{cases}$

e



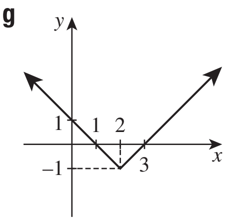
shift down 2,
$y = \begin{cases} x - 2, & \text{for } x \geq 0, \\ -x - 2, & \text{for } x < 0. \end{cases}$

f



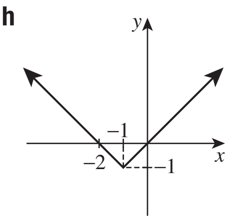
shift up 3,
$y = \begin{cases} x + 3, & \text{for } x \geq 0, \\ 3 - x, & \text{for } x < 0. \end{cases}$

g



shift right 2, down 1,
$y = \begin{cases} x - 3, & \text{for } x \geq 2, \\ 1 - x, & \text{for } x < 2. \end{cases}$

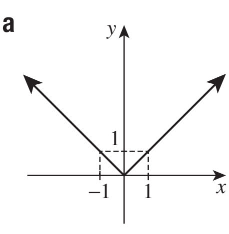
h



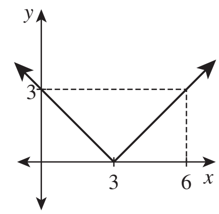
shift left 1, down 1,
$y = \begin{cases} x, & \text{for } x \geq -1, \\ -x - 2, & \text{for } x < -1. \end{cases}$

8 a $x = 5$ or -5
b $x = -2$ or 1
c $x = 6$ or -5
d no solution
e no solution
f $x = -\frac{2}{5}$
g $x = \frac{5}{3}$
h $x = \frac{1}{3}$ or 2
i $x = -2$ or $\frac{2}{5}$

9 a

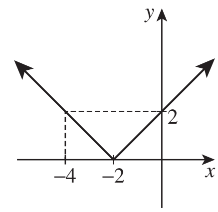


b i



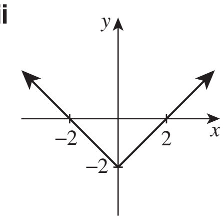
shift right 3,
$y = \begin{cases} x - 3, & \text{for } x \geq 3, \\ 3 - x, & \text{for } x < 3. \end{cases}$

ii



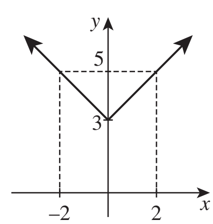
shift left 2,
$y = \begin{cases} x + 2, & \text{for } x \geq -2, \\ -x - 2, & \text{for } x < -2. \end{cases}$

iii



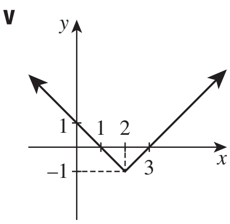
shift down 2,
$y = \begin{cases} x - 2, & \text{for } x \geq 0, \\ -x - 2, & \text{for } x < 0. \end{cases}$

iv



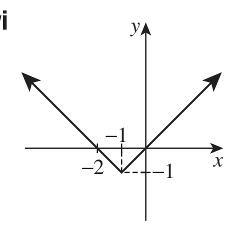
shift up 3,
$y = \begin{cases} x + 3, & \text{for } x \geq 0, \\ 3 - x, & \text{for } x < 0. \end{cases}$

v



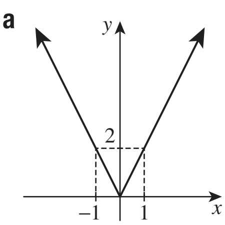
shift right 2, down 1,
$y = \begin{cases} x - 3, & \text{for } x \geq 2, \\ 1 - x, & \text{for } x < 2. \end{cases}$

vi



shift left 1, down 1,
$y = \begin{cases} x, & \text{for } x \geq -1, \\ -x - 2, & \text{for } x < -1. \end{cases}$

10 a



$y = \begin{cases} 2x, & \text{for } x \geq 0, \\ -2x, & \text{for } x < 0. \end{cases}$

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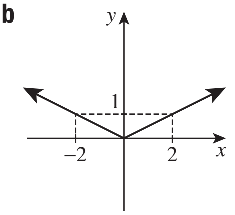
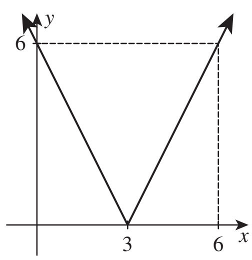
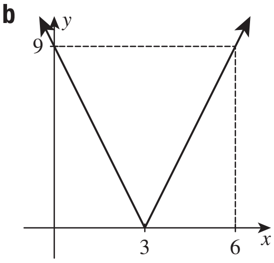
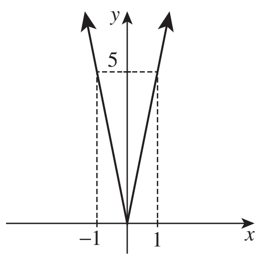
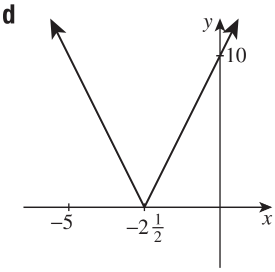
b 

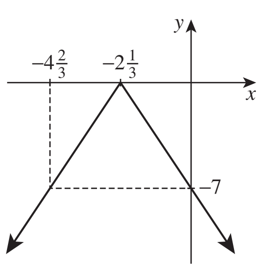
$$y = \begin{cases} \frac{1}{2}x, & \text{for } x \ge 0, \\ -\frac{1}{2}x, & \text{for } x < 0. \end{cases}$$

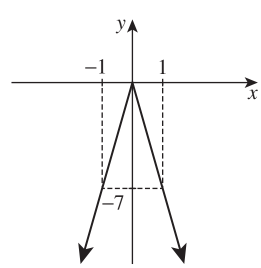
11 a 

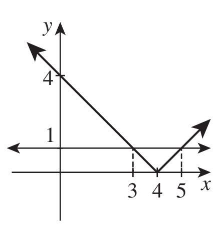
b 

c 

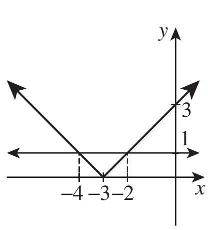
d 

e 

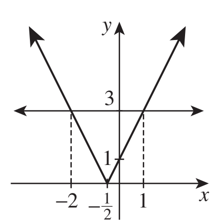
f 

12 a i 

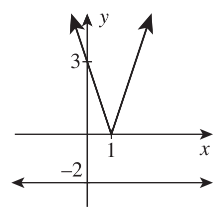
ii The x-coordinates of the points of intersection give:
$x = 3$ or 5

b i 

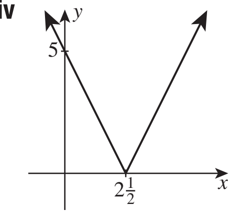
$x = -4$ or -2

ii 

$x = -2$ or 1

iii 

no solution

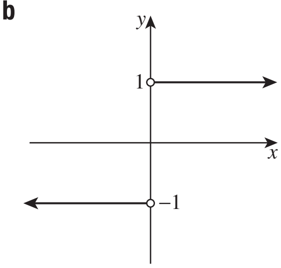
iv 

$x = 2\frac{1}{2}$

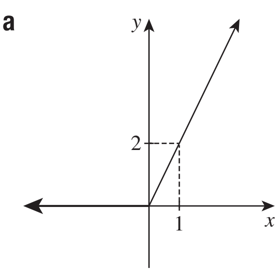
13 b The graph is symmetric in the y-axis.

14 a even **b** neither **c** odd **d** even

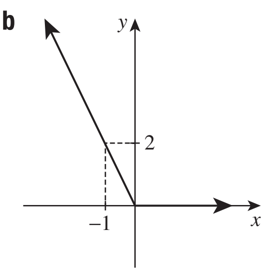
15 a $x = 0$

b 

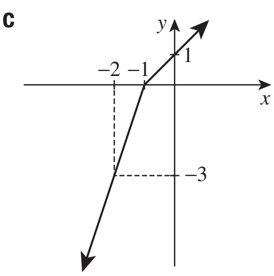
c $y = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0. \end{cases}$

16 a 

$$y = \begin{cases} 2x, & \text{for } x \ge 0, \\ 0, & \text{for } x < 0. \end{cases}$$

b 

$$y = \begin{cases} 0, & \text{for } x \ge 0, \\ -2x, & \text{for } x < 0. \end{cases}$$

c 

$$y = \begin{cases} x + 1, & \text{for } x \ge -1, \\ 3x + 3, & \text{for } x < -1. \end{cases}$$

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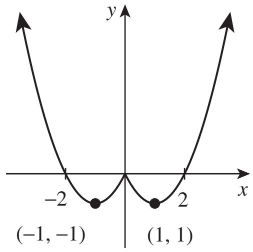
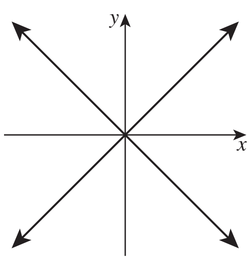
d



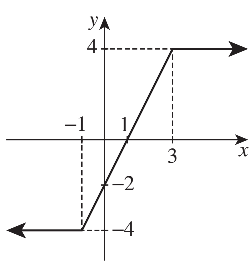
$$y = \begin{cases} x^2 - 2x, & \text{for } x \ge 0, \\ x^2 + 2x, & \text{for } x < 0. \end{cases}$$

17 $f(|-x|) = f(|x|)$

18 $y = x$ or $y = -x$

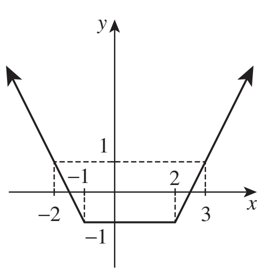


19 a



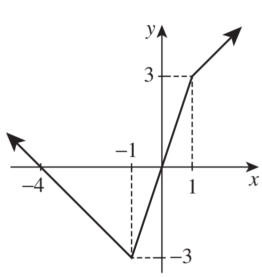
$$y = \begin{cases} -4, & \text{for } x < -1, \\ 2x - 2, & \text{for } -1 \le x < 3, \\ 4, & \text{for } x \ge 3. \end{cases}$$

b



$$y = \begin{cases} -2x - 3, & \text{for } x < -1, \\ -1, & \text{for } -1 \le x < 2, \\ 2x - 5, & \text{for } x \ge 2. \end{cases}$$

c



$$y = \begin{cases} -x - 4, & \text{for } x < -1, \\ 3x, & \text{for } -1 \le x < 1, \\ x + 2, & \text{for } x \ge 1. \end{cases}$$

20 a $|\Delta AOB| = \frac{1}{2}\left|\frac{c^2}{ab}\right|$

b $|\Delta AOB| = \frac{1}{2}p\left|\frac{c}{ab}\right|\sqrt{a^2 + b^2}$

c $p = \dfrac{|c|}{\sqrt{a^2 + b^2}}$

d $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ **e** $\frac{|6 - 10 + 3|}{\sqrt{2^2 + (-5)^2}} = \frac{1}{\sqrt{29}}$

Exercise 4E

1 a i 4 **ii** 7 **iii** 3 **iv** −4

b i $x + 4$ **ii** $x + 6$ **c** $x = -4$

2 a $F(F(0)) = 0$, $F(F(7)) = 28$, $F(F(F(x))) = 8x$

$F(F(-3)) = -12$, $F(F(F(-11))) = -44$

b $F(F(x)) = 4x$, $F(F(F(x))) = 8x$

c $x = 8$

3 a $g(g(0)) = 0$, $g(g(4)) = 4$, $g(g(-2)) = -2$,

$g(g(-9)) = -9$

b $g(g(x)) = 2 - (2 - x) = x$

c $g(g(g(x))) = g(x)$

4 a $h(h(0)) = -20$, $h(h(5)) = 25$,

$h(h(-1)) = -29$, $h(h(-5)) = -65$

b $h(h(x)) = 9x - 20$, $h(h(h(x))) = 27x - 65$

5 a $f(g(7)) = 12$, $g(f(7)) = 13$, $f(f(7)) = 9$,

$g(g(7)) = 19$

b i $2x - 2$ **ii** $2x - 1$ **iii** $x + 2$ **iv** $4x - 9$

c Shift 1 unit to the left (or shift two units up).

d Shift 1 unit up (or shift $\frac{1}{2}$ left).

6 a $\ell(q(-1)) = -2$, $q(\ell(-1)) = 16$,

$\ell(\ell(-1)) = -7$, $q(q(-1)) = 1$

b i $x^2 - 3$ **ii** $(x - 3)^2$ **iii** $x - 6$ **iv** x^4

c i Domain: all real x, range: $y \ge -3$

ii Domain: all real x, range: $y \ge 0$

d It is shifted 3 units to the right.

e It is shifted 3 units down.

7 a $F(G(25)) = 20$, $G(F(25)) = 10$,

$F(F(25)) = 400$, $G(G(25)) = \sqrt{5}$

b $4\sqrt{x}$ **c** $\sqrt{4x} = 2\sqrt{x}$

e Domain: $x \ge 0$, range: $y \ge 0$

8 a $f\left(h\left(-\frac{1}{4}\right)\right) = 4$, $h\left(f\left(-\frac{1}{4}\right)\right) = 4$,

$f\left(f\left(-\frac{1}{4}\right)\right) = -\frac{1}{4}$, $h\left(h\left(-\frac{1}{4}\right)\right) = -\frac{1}{4}$

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b i Both sides equal $-\frac{1}{x}$, for all $x \neq 0$.

ii Both sides equal x, for all $x \neq 0$.

c Domain: $x \neq 0$, range $y \neq 0$

d It is reflected in the y-axis (or in the x-axis).

9 a $f\big(g(x)\big) = -5 - \sqrt{x}$.

Domain: $x \geq 0$, range: $y \leq -5$. Take the graph of $y = \sqrt{x}$, reflect it in the y-axis, then shift down 5.

b $f(x) = -5 - |x|$, which is negative for all x, so $\big(f(x)\big) = \sqrt{-5 - |x|}$ is never defined.

10 a $\big(f(-x)\big) = g\big(f(-x)\big) = -g\big(f(x)\big)$

b $\big(f(-x)\big) = g\big(f(-x)\big) = g\big(f(x)\big)$

c $g\big(f(-x)\big) = g\big(f(x)\big)$

11 a $g\big(f(x)\big) = 7$ for all x, $f\big(g(x)\big) = 4$ for all x

b $g\big(f(x)\big) = g(x)$, $f\big(g(x)\big) = g(x)$

12 a i Translation down a **ii** Translation right a

b i Reflection in the x-axis

ii Reflection in the y-axis

13 a $g\big(f(x)\big) = 10x + 15 + b$,
$f\big(g(x)\big) = 10x + 2b + 3$

b $b = 12$

14 a $g\big(f(x)\big) = 2ax + 3a + b$,
$f\big(g(x)\big) = 2ax + 2b + 3$

b First, $2a = 1$, so $a = \frac{1}{2}$. Secondly, $2b + 3 = 0$, so $b = -1\frac{1}{2}$.

15 a $f\big(g(0)\big) = -3$, $g\big(f(0)\big) = 3$, $f\big(g(-2)\big) = 3$, $g\big(f(-2)\big) = 1$

b i $x^2 + x - 3$ **ii** $x^2 - x - 3$

16 a All real y and $y \geq -1$.

b $x^2 + 2x + 1 = (x + 1)^2$, range: $y \geq 0$

c $x^2 + 4x + 3 = (x + 1)(x + 3)$, range: $y \geq -1$

d -1 and -3.

17 a $f(x)$ **b** $z(x)$, with domain D

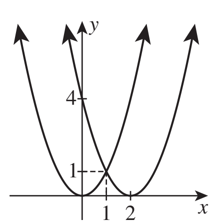
c $z(x)$, with domain D

d If $f(0)$ exists, it is the function $f(x) = f(0)$ with domain $\mathbb{R}$. Otherwise it is the empty function, with domain the empty set.

Chapter 4 review exercise

1a x^2: 4, 1, 0, 1, 4, 9, 16
$(x - 2)^2$: 16, 9, 4, 1, 0, 1, 4

b $y = x^2$, $V = (0, 0)$.
$y = (x - 2)^2$, $V = (2, 0)$.

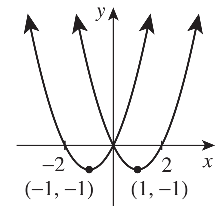


c Here x is replaced by $(x - 2)$, so it is a shift right by 2 units.

2 a Replace x with $-x$.

b $y = x^2 - 2x$:
15, 8, 3, 0, -1, 0, 3
$y = x^2 + 2x$:
3, 0, -1, 0, 3, 8, 15

c $y = x^2 - 2x$: $(1, -1)$.
$y = x^2 + 2x$: $(-1, -1)$.



3 a 7 **b** 4 **c** 5 **d** 3 **e** -3 **f** 12

4 a $x = -5$ or 5 **b** $x = -6$ or 6

c $x = -2$ or 6 **d** $x = -5$ or -1

e $x = -1$ or 4 **f** $x = -1$ or $3\frac{2}{3}$

5 a Shift $y = x^2$ up by 5 units.

b Shift $y = x^2$ down by 1 unit.

c Shift $y = x^2$ right by 3 units.

d Shift $y = x^2$ left by 4 units and up by 7 units.

6 a $y = (x - 1)^2$ **b** $y = x^2 - 2$

c $y = (x + 1)^2 + 5$ **d** $y = (x - 4)^2 - 9$

7 a $C(0, 0)$, $r = 1$ **b** $C(-1, 0)$, $r = 2$

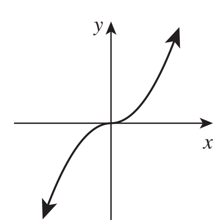
c $C(2, -3)$, $r = \sqrt{5}$ **d** $C(0, 4)$, $r = 8$

8 a $y = -x^3 + 2x + 1$ **b** $y = -x^2 + 3x + 4$

c $y = -2^{-x} - x$ **d** $y = \sqrt{9 - x^2}$

9 a neither **b** odd **c** even

10 a i



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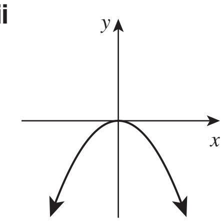
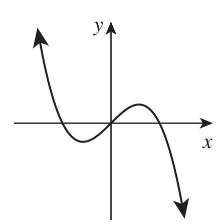


b i



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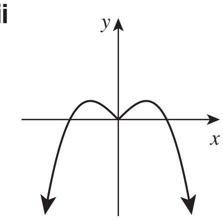
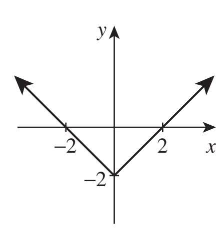
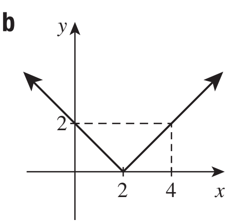


11 a i



b



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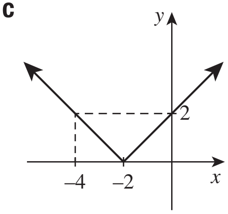
c



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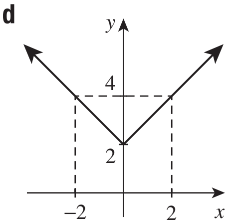
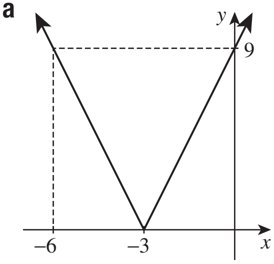


12 a



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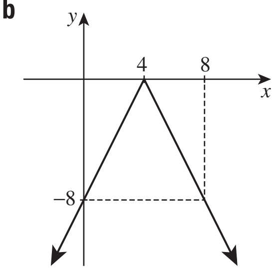

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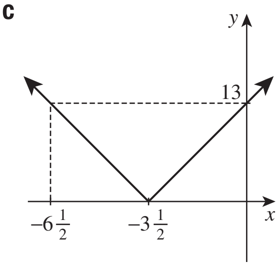


13 a 5 or -5 **b** 1 or -9 **c** no solutions

d 12 or -2 **e** 1 or -8 **f** 4 or $\frac{4}{3}$

g $-\frac{2}{7}$ **h** 5 or -5

14 a neither **b** even **c** odd **d** odd

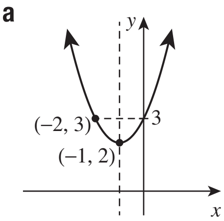
15 a $y = (x - 1)^2 + 4, V = (1, 4)$

b $y = (x + 2)^2 - 7, V = (-2, -7)$

c $y = 2(x + 2)^2 + 3, V = (-2, 3)$

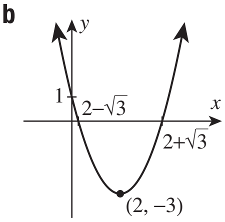
d $y = -(x - 3)^2 + 10, V = (3, 10)$

16 a



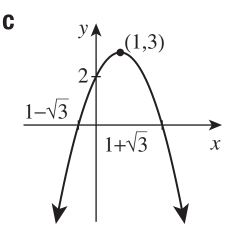
$y = (x + 1)^2 + 2$

b



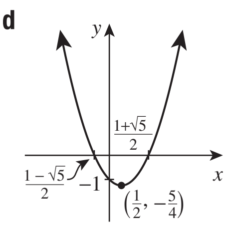
$y = (x - 2)^2 - 3$

c



$y = 3 - (x - 1)^2$

d



$y = \left(x - \frac{1}{2}\right)^2 - \frac{5}{4}$

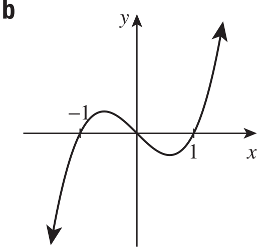
17 a $C(0, 1), r = 2$

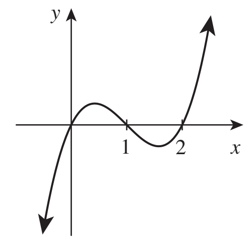
b $C(-3, 0), r = 1$

c $C(2, -3), r = 4$

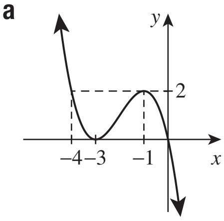
d $C(4, -7), r = 10$

18 b





19 a



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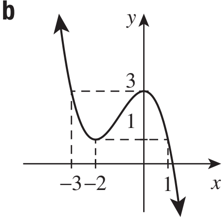

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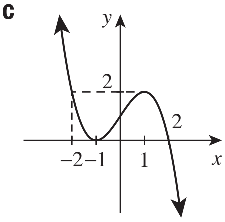

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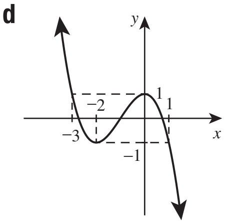

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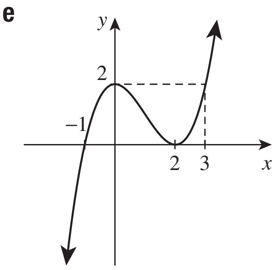

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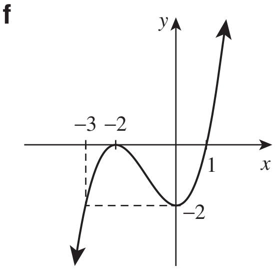

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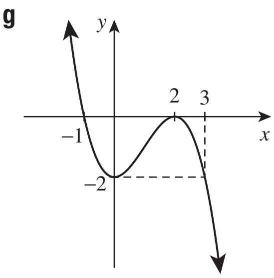
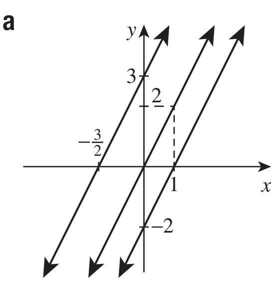


20 a



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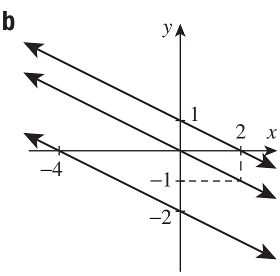

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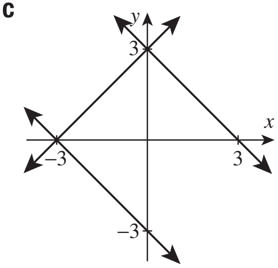

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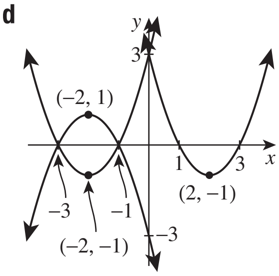

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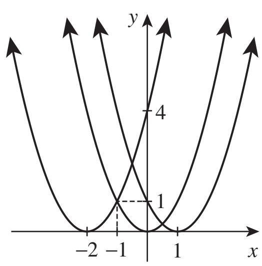

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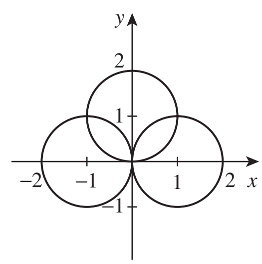

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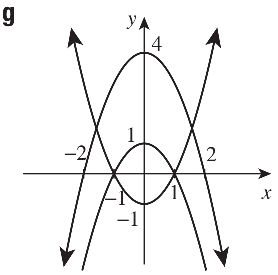

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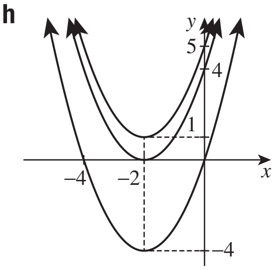

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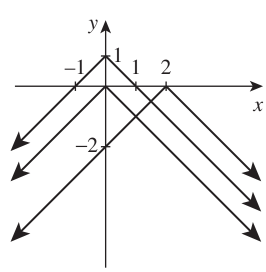

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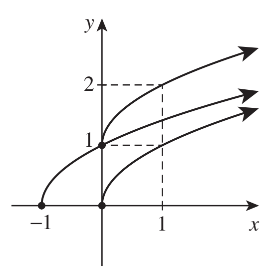

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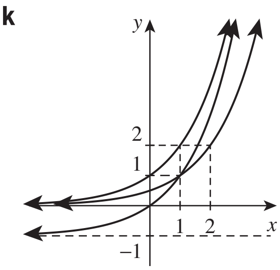

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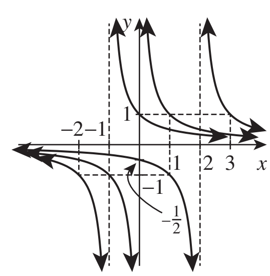

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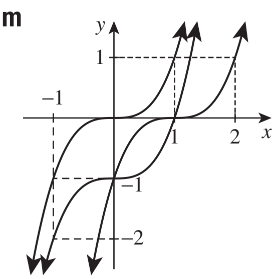

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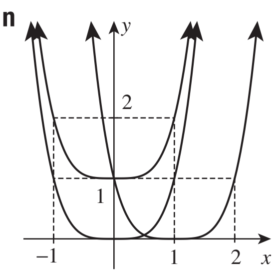

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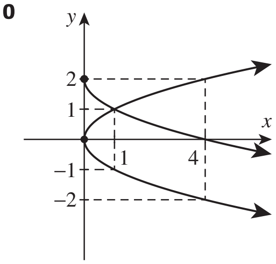

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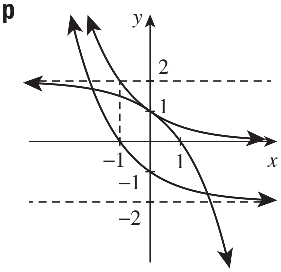


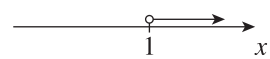
21 $g(x) = \frac{1}{2}\big(f(x) + f(-x)\big)$

$h(x) = \frac{1}{2}\big(f(x) - f(-x)\big)$

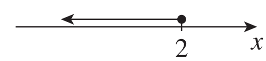
Chapter 5

Exercise 5A

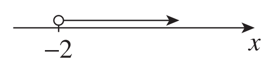
1 a $x > 1$



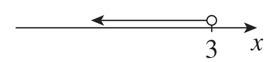
b $x \le 2$



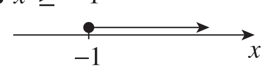
c $x > -2$



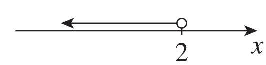
d $x < 3$



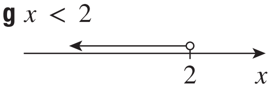
e $x \ge -1$



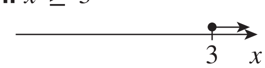
f $x < 2$



g $x < 2$

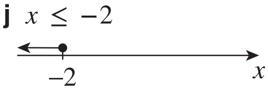


h $x \ge 3$



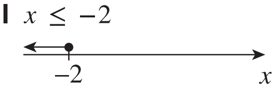
i $x \ge 3$

j $x \le -2$

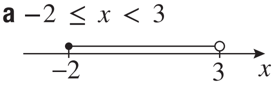


k $x > 2$

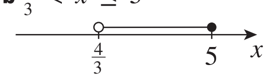
l $x \le -2$



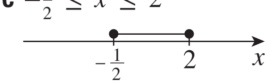
2 a $-2 \le x < 3$



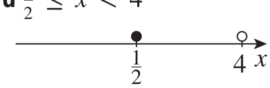
b $\frac{4}{3} < x \le 5$



c $-\frac{1}{2} \le x \le 2$



d $\frac{1}{2} \le x < 4$



3 a $x > 4$ **b** $x \le 2$

c $x < 2$ **d** $x \le -1$

e $-2 \le x < 1$ **f** $-6 \le x \le 15$

4 a i $0 < x < 4$ **ii** $x < 0$ or $x > 4$

b i $-1 \le x \le 3$ **ii** $x \le -1$ or $x \ge 3$

c i $x \le 0$ or $x \ge 2$ **ii** $0 < x < 2$

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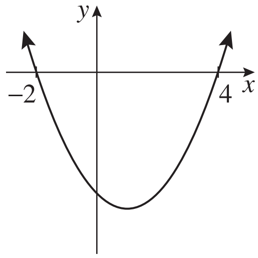
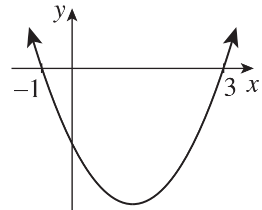
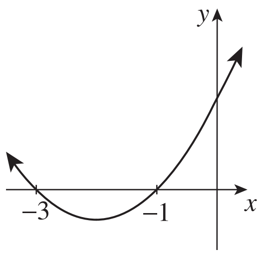
5 a $-2 < x < 4$



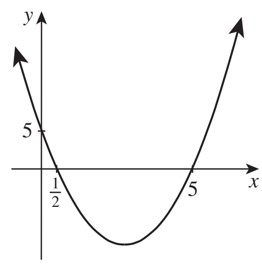
b $x < -1$ or $x > 3$



c $x \le -3$ or $x \ge -1$



d $x < \frac{1}{2}$ or $x > 5$

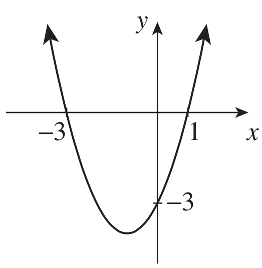


6 a $-3 < x < 3$ **b** $x < 0$ or $x > 6$
c $x \le -10$ or $x \ge 10$ **d** $-4 \le x \le 0$

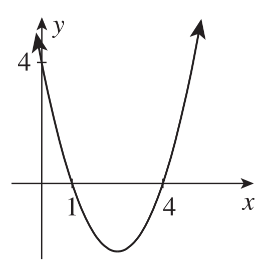
7 a $x = 7$ or -7 **b** $x = 0$
c $-2 \le x \le 2$ **d** $x < -5$ or $x > 5$
e $-\frac{1}{4} < x < \frac{1}{4}$ **f** $x \le -\frac{3}{2}$ or $x \ge \frac{3}{2}$

8 a $0 < x < 1$ **b** $x < 0$ or $x > 3$
c $0 < x \le \frac{1}{2}$ **d** $x \le -\frac{3}{4}$ or $x > 0$

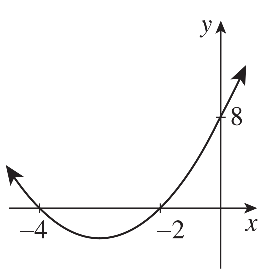
9 a $-3 < x < 1$



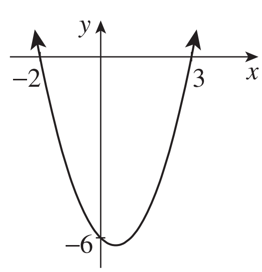
b $x \le 1$ or $x \ge 4$



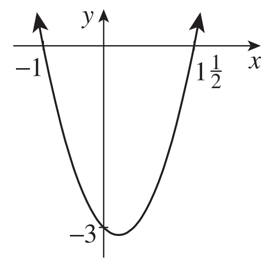
c $x < -4$ or $x > -2$



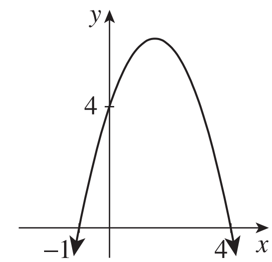
d $-2 \le x \le 3$



e $-1 \le x \le 1\frac{1}{2}$



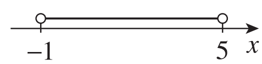
f $-1 < x < 4$



10 a $-1 < x < 3$ **b** $x \le 1$ or $x \ge 9$
c $x < -4$ or $x > 2$ **d** $-14 \le x \le -2$

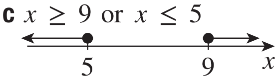
11 a $x < -1$ or $x \ge 1$ **b** $3 < x < 5$
c $-4 < x \le -2\frac{1}{2}$ **d** $x < \frac{3}{2}$ or $x > 4$
e $1 < x < 3$ **f** $\frac{5}{3} < x \le 3$

12 a $-1 < x < 5$

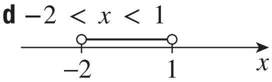


b $\frac{1}{3} \le x \le 3$

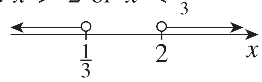
c $x \ge 9$ or $x \le 5$



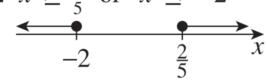
d $-2 < x < 1$



e $x > 2$ or $x < \frac{1}{3}$



f $x \ge \frac{2}{5}$ or $x \le -2$



13 a $x = 0$
b $x < 0$ or $x > 0$ (or simply $x \ne 0$)
c $x \le -5$ or $x \ge 5$
d $x < 0$ or $x > 25$
e No solution for x.
f $x = 1$

14 a $\frac{1}{2} < x \le 3$ **b** $-3 < x < -2$
c $x < 1$ or $x \ge 3$ **d** $x < -\frac{1}{7}$ or $x > 2$

15 a The first holds when x is positive, the second when x is negative.
b **i** $-2 < x < 2$ or $-10 < x < -6$
ii $3 \le x < 4\frac{1}{2}$ or $\frac{1}{2} < x \le 2$

16 a false: $x = 0$ **b** false: $x = \frac{1}{2}$ **c** true
d false: $x = \frac{1}{2}$ or $x = -2$
e false: $x = -1$ **f** true
g false: $x = -1$ **h** true

17 a No solutions **b** No solutions
c All real x **d** $x = \frac{5}{3}$

18 a An absolute value must be positive.
b $x > 1$

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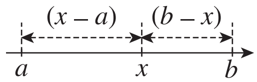
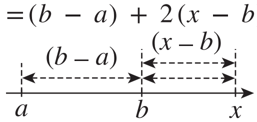
19 $-1 \le x < 0$ or $1 \le x \le 2$

20 a $|x - a| + |x - b| = (x - a) + (b - x) < c$



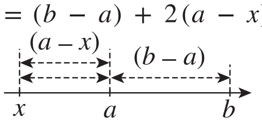
b $|x - a| + |x - b| = (x - a) + (x - b)$

$= (b - a) + 2(x - b) < c$



c $|x - a| + |x - b| = (a - x) + (b - x)$

$= (b - a) + 2(a - x) < c$



d The result follows directly from parts **a**, **b** and **c**.

e $-3 < x < 7$

Exercise 5B

1 a $x \le 0$ or $1 \le x \le 2$

b $-2 < x < 0$ or $2 < x < 4$

c $0 < x < 3$ or $x > 3$

d $x = 0$ or $x \ge 4$

e $x = -3$ or $x = 3$

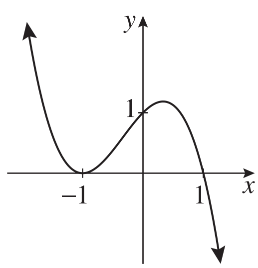
f $x = -3$ or $x \ge 0$

2 a

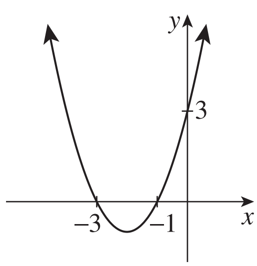
x	-2	-1	0	1	2
y	3	0	1	0	-9
sign	$+$	0	$+$	0	$-$

b Solution: $x \le 1$

c

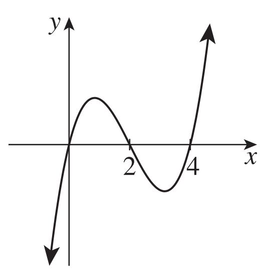


3 a



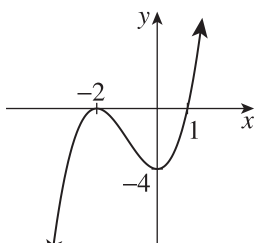
$-3 < x < -1$

b



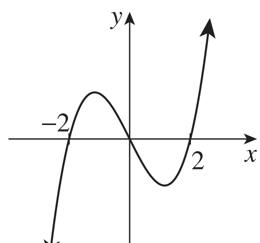
$0 \le x \le 2$ or $x \ge 4$

c



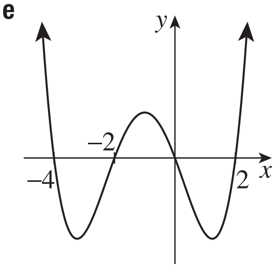
$x = -2$ or $x \ge 1$

d



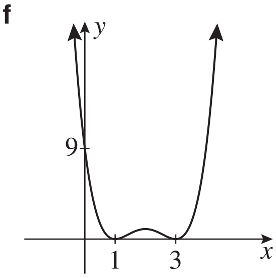
$x \le -2$ or $0 \le x \le 2$

e



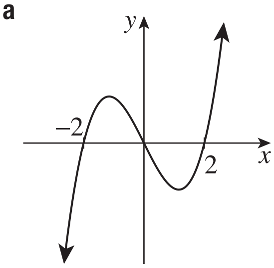
$x < -4$ or
$-2 < x < 0$ or $x > 2$

f



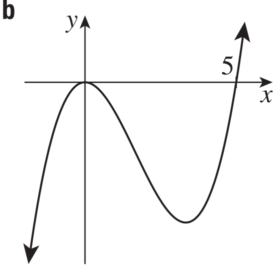
$x = 1$ or $x = 3$

4 a



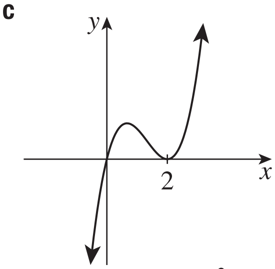
$f(x) = x(x - 2)(x + 2)$

b



$f(x) = x^2(x - 5)$

c



$f(x) = x(x - 2)^2$

5 a $-2 < x < 0$ or $x > 2$

b $x < 0$ or $0 < x < 5$

c $x \le 0$ or $x = 2$

6 a

x	-1	0	1	3	4
y	$-\frac{1}{4}$	0	$-\frac{1}{2}$	*	16
sign	$-$	0	$-$	*	$+$

b $x < 0$ or $0 < x < 3$

7 a $x < 1$ or $3 < x < 5$

b $x \ne 1$ and $x \ne 3$ (alternatively, $x < 1$ or $1 < x < 3$ or $x > 3$)

c $-2 < x \le 4$

d $-3 < x < 0$ or $x > 3$

e $-3 < x < -1$

f $x < 0$ or $0 < x < 5$

g $x \le 0$ or $x \ge 5$

h $-2 \le x < 0$ or $x \ge 2$

i $x < -3$ or $0 < x \le 2$

8 a i $y = x(x + 1)(x - 1)$, $x = -1, 0$ or 1

ii $y = (x - 2)(x - 1)(x + 1)$, $x = -1, 1$ or 2

iii $y = (x + 2)^2(x - 2)$, $x = -2$ or 2

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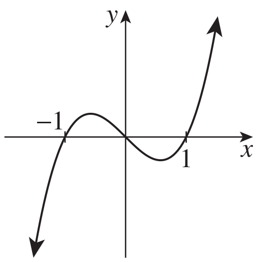
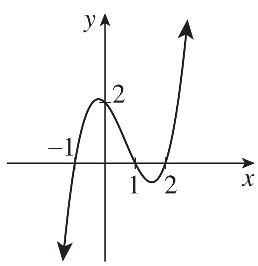
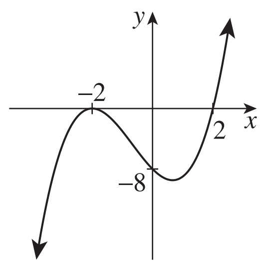
b i



ii



iii



9 a zero for $x = 0$, undefined at $x = 3$, positive for $x < 0$ or $x > 3$, negative for $0 < x < 3$

b zero for $x = 4$, undefined at $x = -2$, positive for $x < -2$ or $x > 4$, negative for $-2 < x < 4$

c zero for $x = -3$, undefined at $x = -1$, positive for $x < -3$ or $x > -1$, negative for $-3 < x < -1$

10 a $x \leq -4$ or $-3 < x \leq 1$

b $-2 < x < -1\frac{1}{2}$ or $x > \frac{1}{2}$

c $-\frac{1}{2} \leq x < 1\frac{1}{2}$ or $x \geq 2\frac{1}{2}$

Exercise 5C

1 In each case $y \to 0$ as $x \to \infty$ and as $x \to -\infty$.

a i Domain: $x \neq 1$. When $x = 0$, $y = -1$.

ii When $y = 1$, $x = 2$. When $y = -1$, $x = 0$.

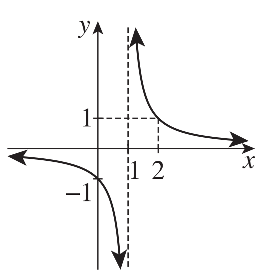
v Vertical asymptote: $x = 1$. As $x \to 1^+$, $y > 0$ so $y \to \infty$, and as $x \to 1^-$, $y < 0$ so $y \to -\infty$.

b i Domain: $x \neq 3$. When $x = 0$, $y = \frac{2}{3}$.

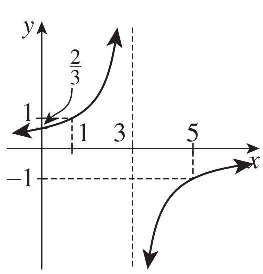
ii When $y = 1$, $x = 1$. When $y = -1$ at $x = 5$.

v Vertical asymptote: $x = 3$. As $x \to 3^+$, $y < 0$ so $y \to -\infty$, and as $x \to 3^-$, $y > 0$ so $y \to \infty$.

a



b



c i Domain: $x \neq -2$. When $x = 0$, $y = -1$.

ii When $y = 1$, $x = -4$. When $y = -1$, $x = 0$.

v Vertical asymptote: $x = -2$. As $x \to -2^+$, $y < 0$ so $y \to -\infty$, and as $x \to -2^-$, $y > 0$ so $y \to \infty$.

d i Domain: $x \neq -2\frac{1}{2}$. When $x = 0$, $y = 1$.

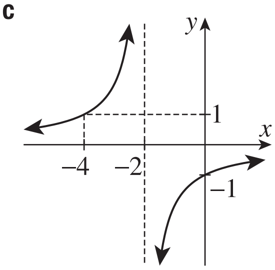
ii When $y = 1$, $x = 0$. When $y = -1$, $x = -5$.

v Vertical asymptote: $x = -2\frac{1}{2}$.

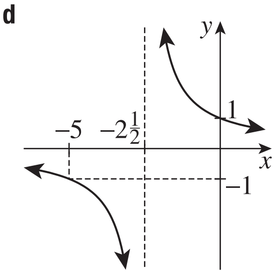
As $x \to -2\frac{1}{2}^+$, $y > 0$ so $y \to \infty$,

and as $x \to -2\frac{1}{2}^-$, $y < 0$ so $y \to -\infty$.

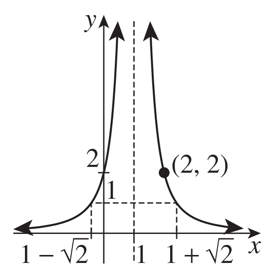
c



d



2



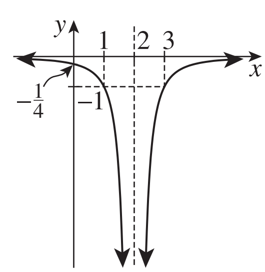
i Domain: $x \neq 1$.

ii When $y = 1$, $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$.

iii Horizontal asymptote $y = 0$, $y \to 0$ as $x \to \infty$ and as $x \to -\infty$.

v Vertical asymptote $x = 1$. As $x \to 1^+$, $y > 0$ so $y \to \infty$ and as $x \to 1^-$, $y > 0$ so $y \to \infty$.

3



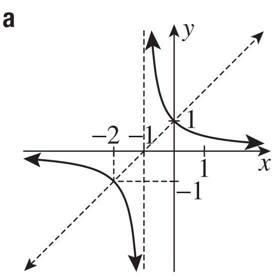
i Domain: $x \neq 2$.

ii When $y = -1$, $x = 1$ or 3.

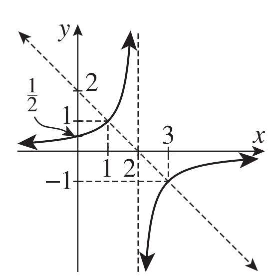
iii Horizontal asymptote $y = 0$, $y \to 0$ as $x \to \infty$ and as $x \to -\infty$.

v Vertical asymptote $x = 2$. As $x \to 2^+$, $y < 0$ so $y \to -\infty$, and as $x \to 2^-$, $y < 0$ so $y \to -\infty$.

4 a



b



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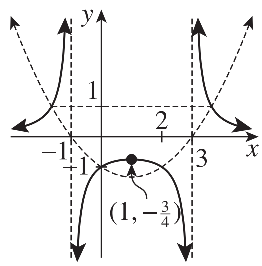
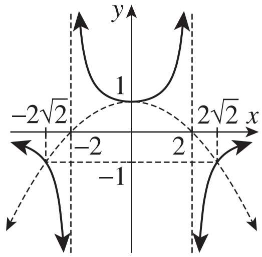
5 

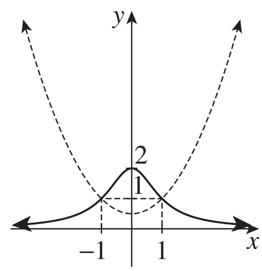
The curves also meet where $x = 1 - \sqrt{7}$ and $x = 1 + \sqrt{7}$.

c Range: $y \geq -\frac{4}{3}$.

e Range: $y \leq -\frac{3}{4}$ or $y > 0$.

6 a 

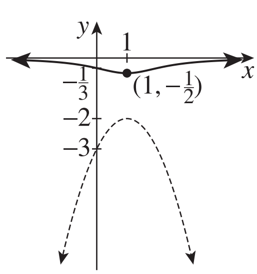
b Range: $y \leq 1$.

7 

b $\frac{1}{2}$

d As $x \to \infty$ or $x \to -\infty$, $y \to 0$.

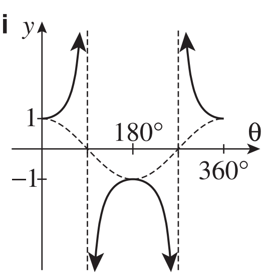
e 2

8 

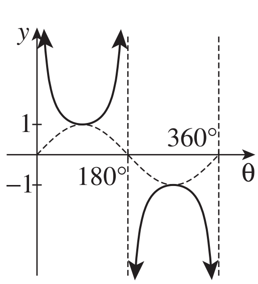
a −2

c As $x \to \infty$ or $x \to -\infty$, $y \to 0$.

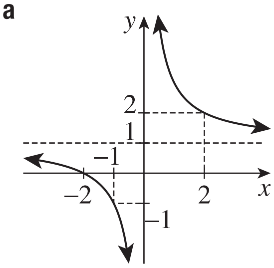
d $-\frac{1}{2}$

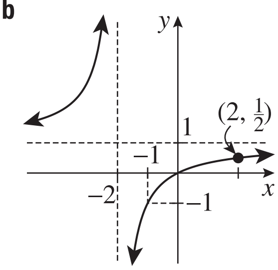
9 a i 

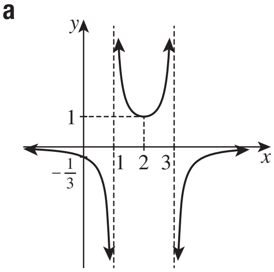
ii Domain: $0^\circ \leq \theta \leq 360^\circ$, except that $\theta \neq 90^\circ$ and $\theta \neq 270^\circ$.
Range: $y \leq -1$ or $y \geq 1$.

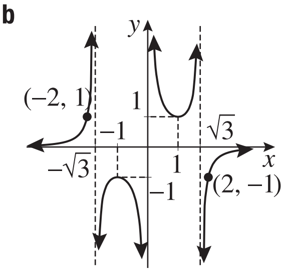
b i 

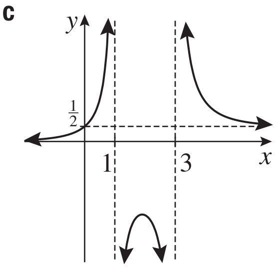
ii Domain: $0^\circ \leq \theta \leq 360^\circ$, except that $\theta \neq 0^\circ$, $\theta \neq 180^\circ$ and $\theta \neq 360^\circ$.
Range: $y \leq -1$ or $y \geq 1$.

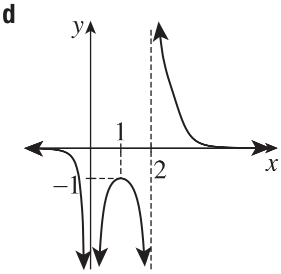
11 a 

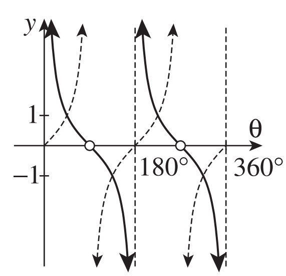
b 

12 a 

b 

c 

d 

13 

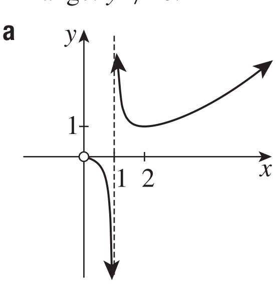
a Domain: $0^\circ \leq \theta \leq 360^\circ$ except that $\theta \neq 90^\circ$ and $\theta \neq 270^\circ$.

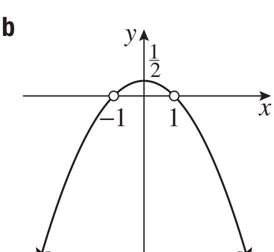
b $\tan\theta = 0$ at $\theta = 0^\circ$, $\theta = 180^\circ$ and $\theta = 360^\circ$.

c Domain: $0^\circ < \theta < 360^\circ$, except that $\theta \neq 90^\circ$, $\theta \neq 180^\circ$ and $\theta \neq 270^\circ$

d 0

f Range: $y \neq 0$.

14 a 

b 

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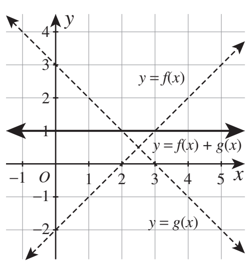
15 The problem is that zero does not have a reciprocal. For example, $y = -x^2$ has a maximum of 0 when $x = 0$, and $y = \dfrac{-1}{x^2}$ has an asymptote at $x = 0$, not a minimum. The statement should be, 'When one curve has a non-zero local maximum, the other curve has a non-zero local minimum.'

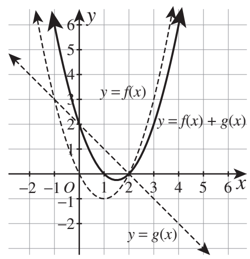
16 $y = x - 2$, for $x \neq 2$

Exercise 5D

1 a



b



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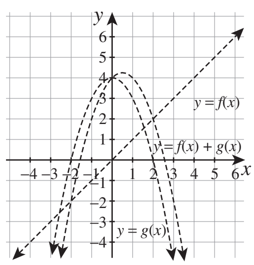
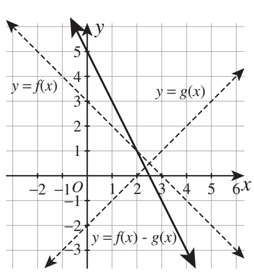


2 a



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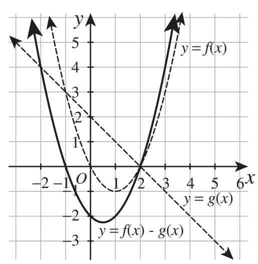

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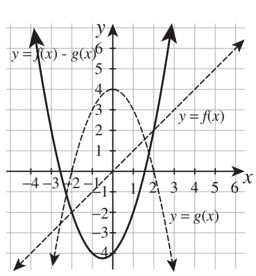
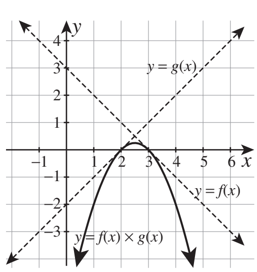


3 a



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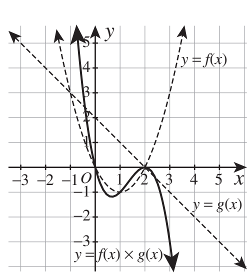

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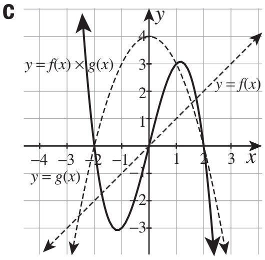

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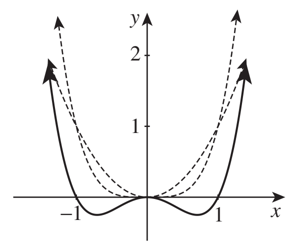

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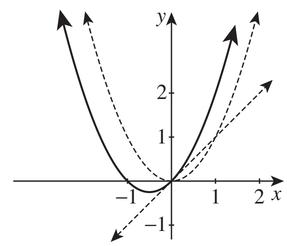
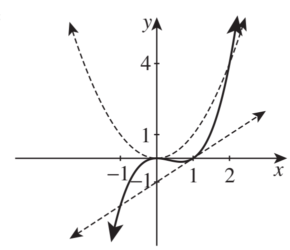


6 c



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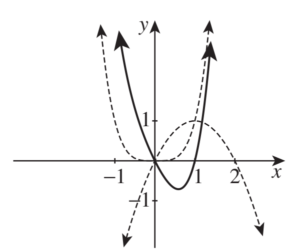
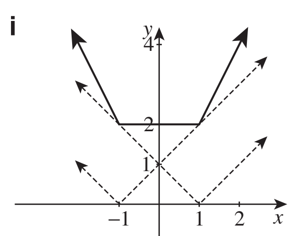


8 a i



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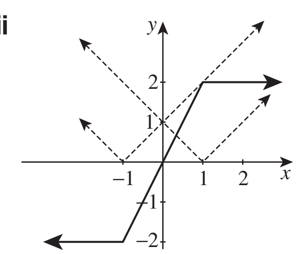
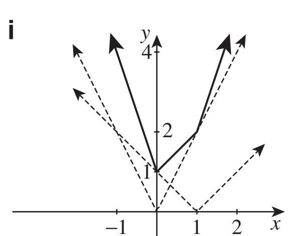


b i



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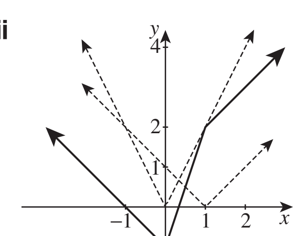

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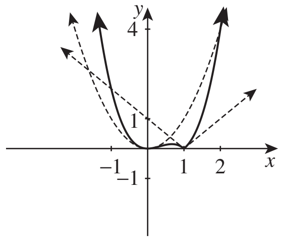
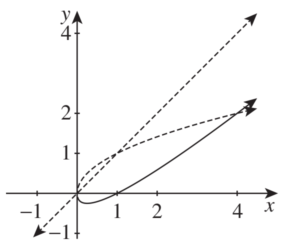
9 b



c Because $0 \le x^2 \le 1$ and $0 \le x - 1 \le 1$, the product will also lie between 0 and 1 inclusive.

10 b



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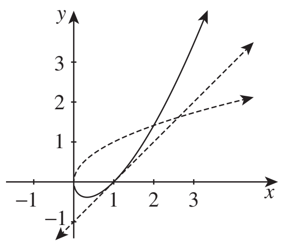
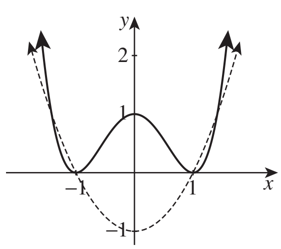


12 a



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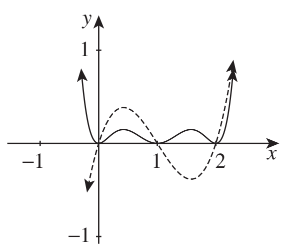

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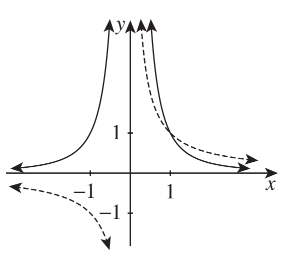

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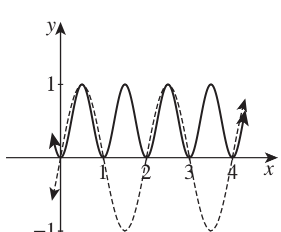

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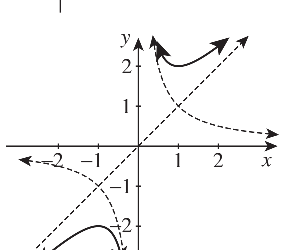

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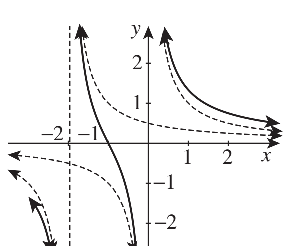

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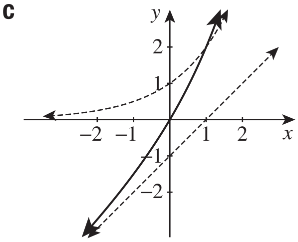
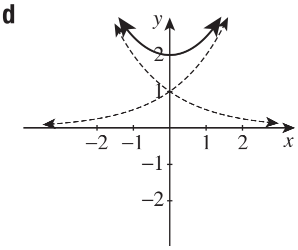
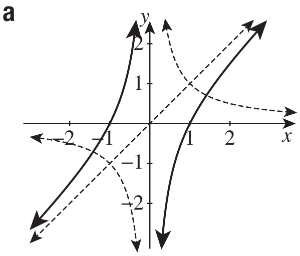


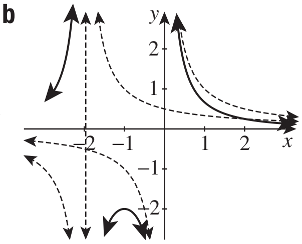
d



15 a



b



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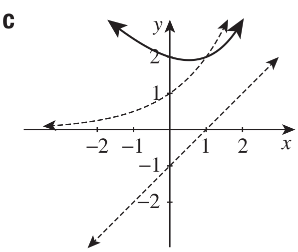

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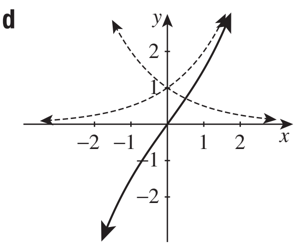


16 a

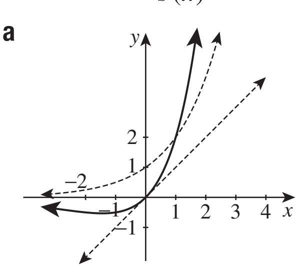
	$f(x)$ even, $g(x)$ even
$s(x)$	even
$d(x)$	even
$p(x)$	even

	$f(x)$ odd, $g(x)$ odd
$s(x)$	odd
$d(x)$	odd
$p(x)$	even

	$f(x)$ even, $g(x)$ odd
$s(x)$	neither
$d(x)$	neither
$p(x)$	odd

b
$$\begin{aligned} s(-x) &= f(-x) + g(-x) \\ &= -f(x) - g(x) \\ &= -\big(f(x) + g(x)\big) \\ &= -s(x) \end{aligned}$$

17 a



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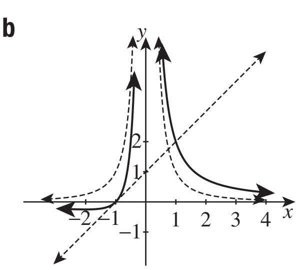

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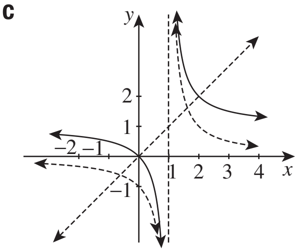
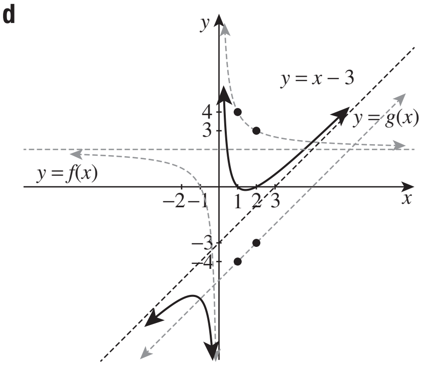
c



18 a $(1, -4)$, $(1, 4)$ and $(2, -3)$, $(2, 3)$

d

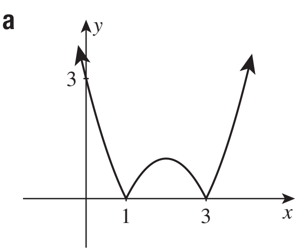


19 a As $x \to \infty$ and as $x \to -\infty$, $s(x) - (x + 1) \to 0$.

b $y = -x + 5$ **c** $y = 3x - 5$

Exercise 5E

1 a



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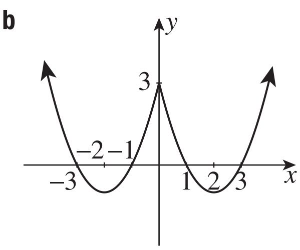

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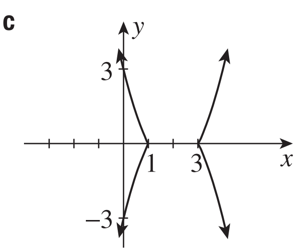
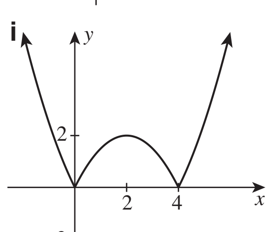


2 a i



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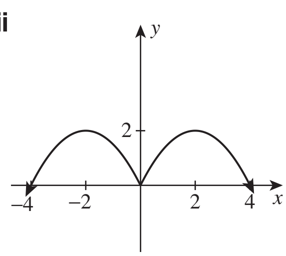

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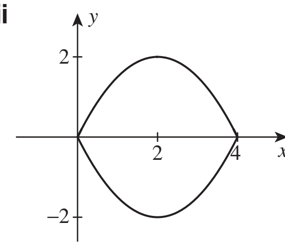
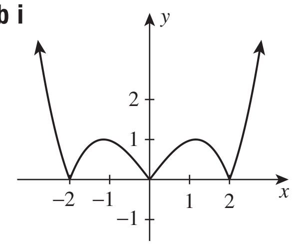


b i



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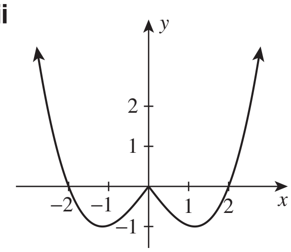

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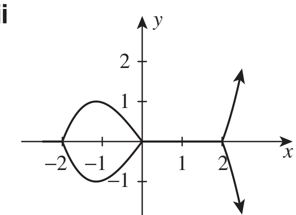
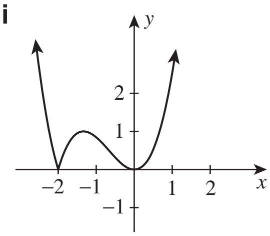


c i



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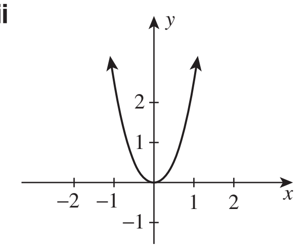

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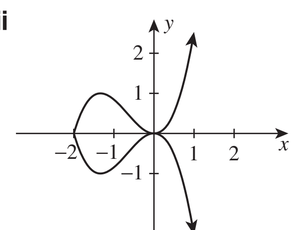
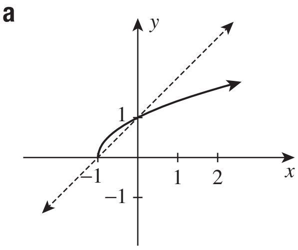


3 a



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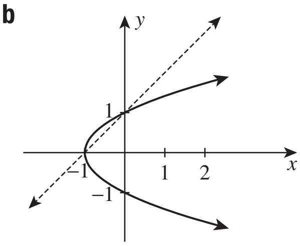
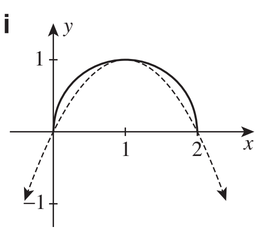


4 a i



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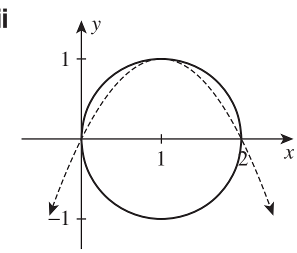

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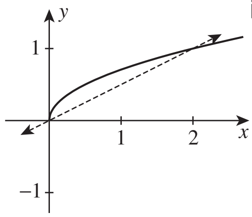
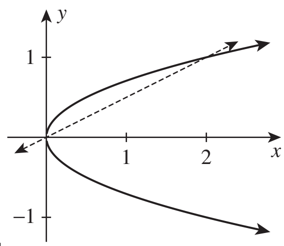
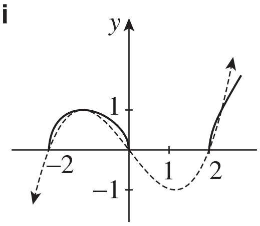
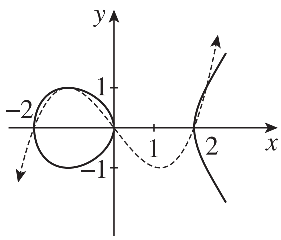
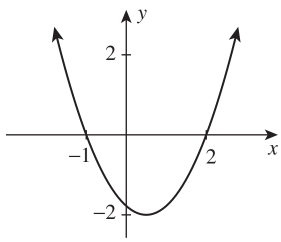
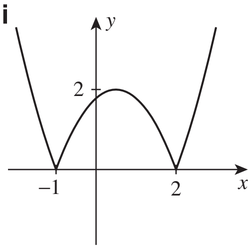
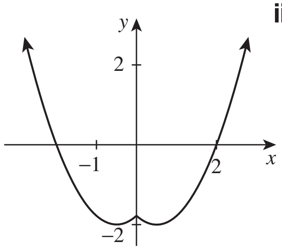
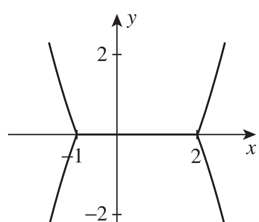
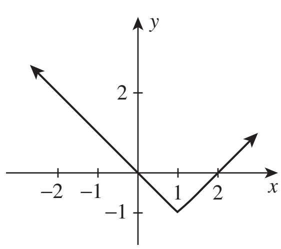
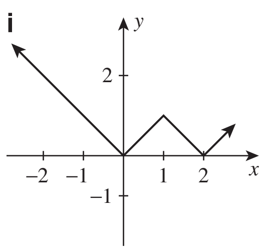
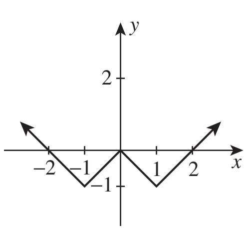
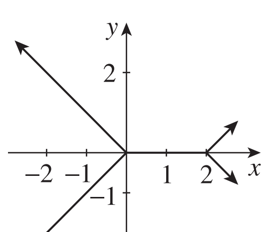
b i  **ii** 

c i  **ii** 

5 a  **b i** 

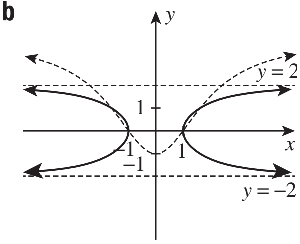
ii  **iii** 

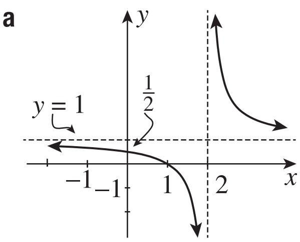
6 a  **b i** 

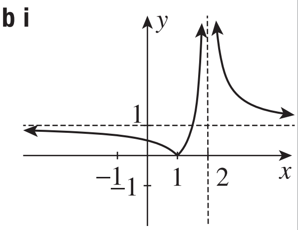
ii  **iii** 

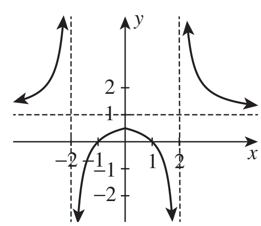
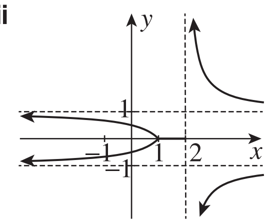
7 a As $x \to \pm\infty$, $\sqrt{f(x)} \to 2$, hence $y = 2$ will be the horizontal asymptote of the transformed graph.

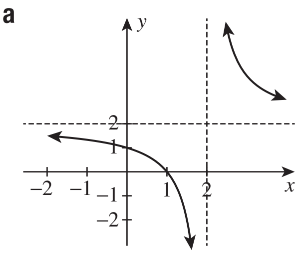
b

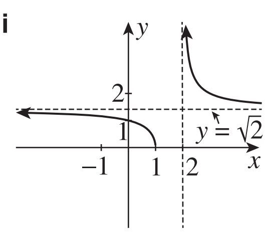


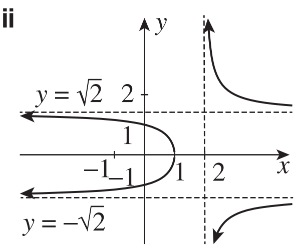
8 a 

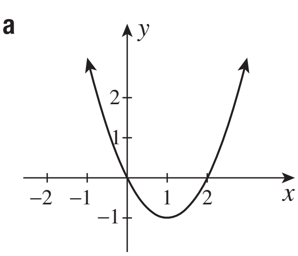
b i 

ii  **iii** 

9 a 

b i 

ii 

10 a 

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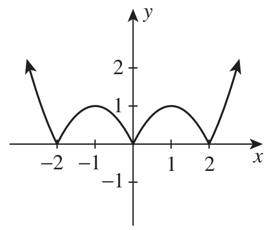
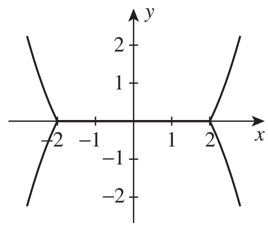
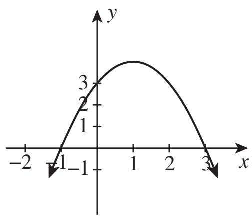
b i



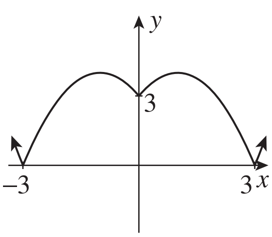
ii



c



i



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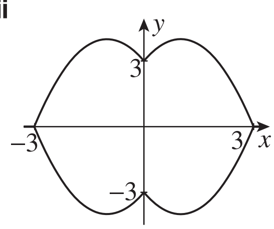


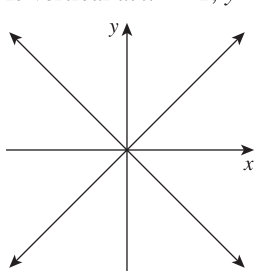
11a i The transformed graph is $y = \sqrt{x - 1}$, which is vertical at $x = 1$ (it is the graph $y = \sqrt{x}$ shifted 1 unit right).

ii The transformed graph is $y = |x - 1|$, which meets the axis at 45°.

iii The transformed graph is $y = (x - 1)^2$, which is horizontal at $x = 1$.

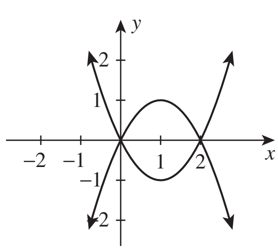
b When $f(x) < 1$, we know that $\sqrt{f(x)} > f(x)$, so that $y = \sqrt{f(x)}$ is always steeper than $y = f(x)$ at a zero of the original function. Because $y = \sqrt{x - 1}$ is vertical at $x = 1$, $y = \sqrt[4]{x - 1}$ must be also.

12

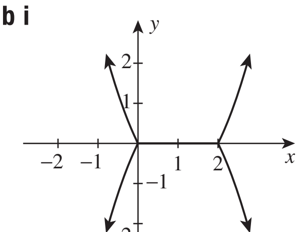


The two lines of the graph are inclined at 45° to the axes.

14 a



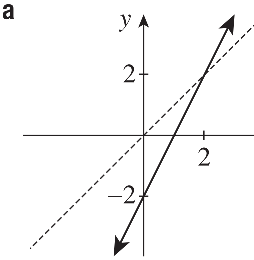
b i



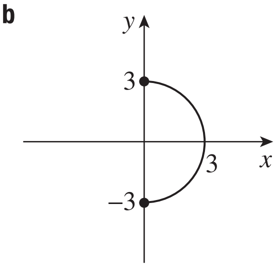
ii First, the parts of the original graph $y = f(x)$ below the x-axis were lost when sketching the function $|y| = f(x)$. Secondly, the parts of the graph of $|y| = f(x)$ below the x-axis will be lost in the steps of Box 14.

Exercise 5F

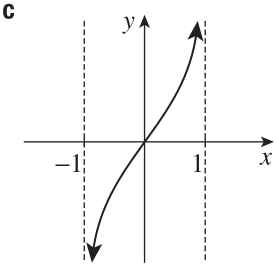
1 a



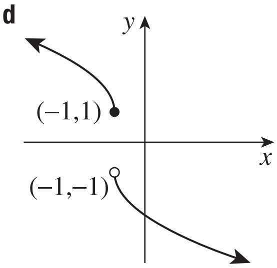
b



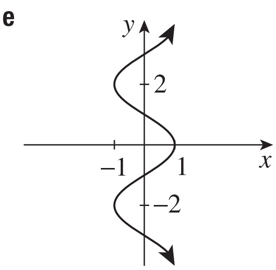
c



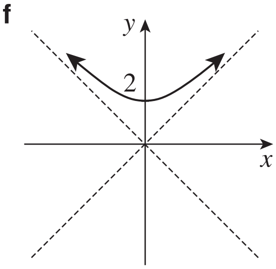
d



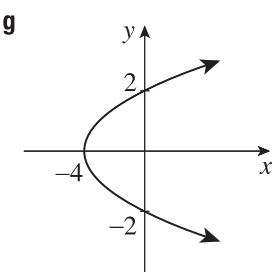
e



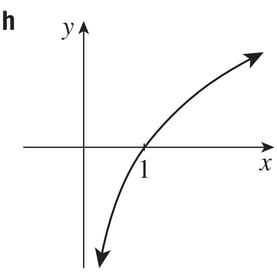
f



g



h



2 Original is a function: all except part **f**. Inverse is a function: part **a**, **c**, **d**, **f**, **h**

3 One-to-one: part **a**, **c**, **d**, **h**. Many-to-one: part **e**, **g**. One-to-many: part **b**, **f**

4 a $y = \dfrac{x + 2}{3}$ **b** $y = 2x - 2$ **c** $y = 6 - 2x$

d $y = x - 1$ **e** $y = -\frac{5}{2}x + 5$ **f** $x = 2$

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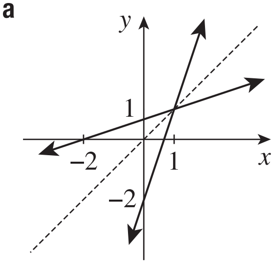
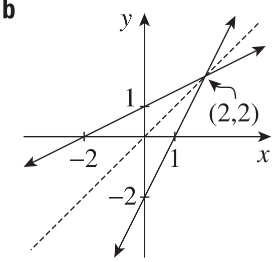
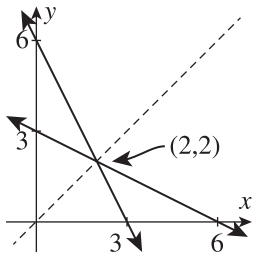
5 a



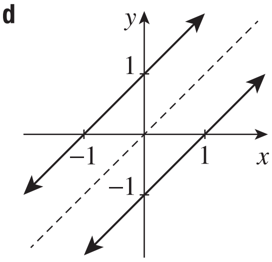
b



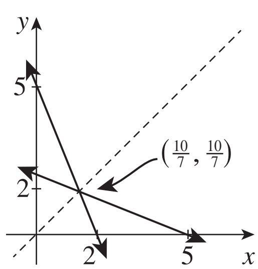
c



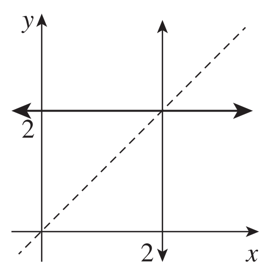
d



e



f



6 a i $y = \dfrac{1}{x - 1}$ **ii** $y = \dfrac{1}{x} - 1$ **iii** $y = \dfrac{2x + 2}{x - 1}$

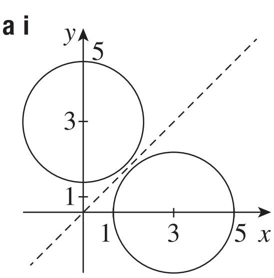
iv $y = \dfrac{2x}{3 - x}$

b i For the function, domain: $x \neq 0$, range: $y \neq 1$. For the inverse function, domain: $x \neq 1$, range: $y \neq 0$.

iv For the function, domain: $x \neq -2$, range: $y \neq 3$. For the inverse function, domain: $x \neq 3$, range: $y \neq -2$.

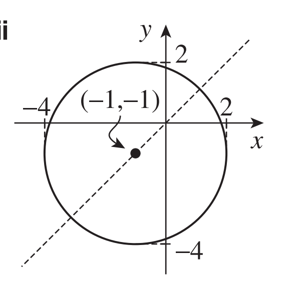
7 Each inverse is identical to the original function. Therefore the graph is symmetric about the line $y = x$.

8 a i



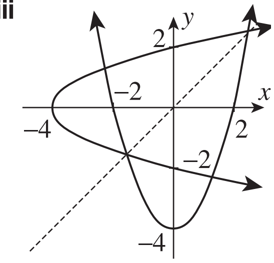
$x^2 + (y - 3)^2 = 4$. Neither the original relation nor its inverse is a function.

ii



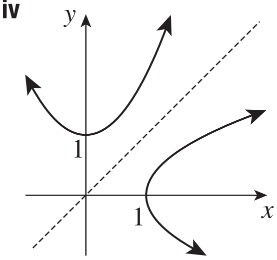
$(x + 1)^2 + (y + 1)^2 = 9$
The inverse relation is the same as the original relation, and is not a function.

iii



$x = y^2 - 4$.
The original relation is a function, but its inverse is not.

iv

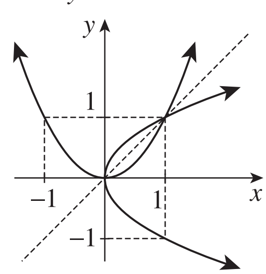


$x = y^2 + 1$.
The original relation is a function, but its inverse is not.

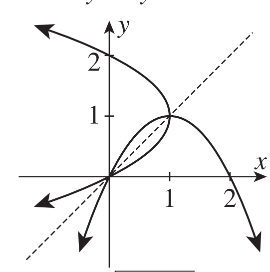
b i For the original, domain: $1 \leq x \leq 7$, range: $-2 \leq y \leq 2$. For the inverse, domain: $-2 \leq x \leq 2$, range: $1 \leq y \leq 7$.

iv For the function, domain: all real x, range: $y \geq 1$. For the inverse function, domain: $x \geq 1$, range: all real y.

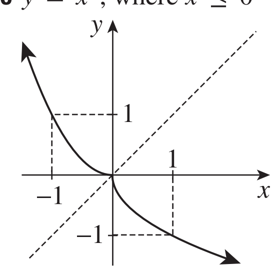
9 a $x = y^2$



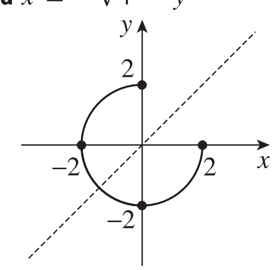
b $x = 2y - y^2$



c $y = x^2$, where $x \leq 0$



d $x = -\sqrt{4 - y^2}$



10 a Inverse: $x = 3y - 10$, where $y < 2$.
Hence $y = \dfrac{1}{6}(x + 10)$, where $x < -4$.

b Inverse: $x = 13 - 6y$, where $y \geq 3$.
Hence $y = \dfrac{1}{6}(13 - x)$, where $x \leq -5$.

c Inverse: $x = y^3 + 2$, where $y < 3$.
Hence $y = \sqrt[3]{x - 2}$, where $x < 29$.

d Inverse: $x = y^2 - 3$, where $y \geq -2$.
Hence $y^2 = 3 + x$, where $y \geq -2$, which is not a function because $x = -2$ corresponds to $y = 1$ and also to $y = -1$

12 a i One-to-one **ii** Many-to-one **iii** One-to-many

b Parts **i** and **iii**

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13 a One-to-one **b** One-to-many

c Many-to-one **d** Many-to-many

16 b No. Look at $y = x^2$, which is even. Its inverse is $x = y^2$, which is not even.

Exercise 5G

1 b They are inverse functions, that is, $g(x) = f^{-1}(x)$ and $f(x) = g^{-1}(x)$.

3 a Let $y = 2x + 5$.

The inverse is $x = 2y + 5$

$2y = x - 5$

$y = \frac{1}{2}(x - 5)$

so $f^{-1}(x) = \frac{1}{2}(x - 5)$

c i $f^{-1}(x) = \dfrac{1}{3}(4 - x)$ **ii** $f^{-1}(x) = \sqrt[3]{x + 2}$

iii $f^{-1}(x) = \dfrac{1}{x} + 5$

4 a It fails the horizontal line test, for example $f(1) = f(-1) = 1$, so the inverse is not a function.

b $f^{-1}(x) = x^2$, where $x \geq 0$.

c It fails the horizontal line test, for example $f(1) = f(-1) = 1$, so the inverse is not a function.

d $f^{-1}(x) = (x - 1)^{\frac{1}{3}}$

e It fails the horizontal line test, for example $f(1) = f(-1) = 8$, so the inverse is not a function.

f $f^{-1}(x) = \sqrt{9 - x}$

g It fails the horizontal line test, for example $f(1) = f(-1) = \dfrac{1}{3}$, so the inverse is not a function.

h $f^{-1}(x) = \dfrac{1 - 3x}{1 + x}$ **i** $f^{-1}(x) = -\sqrt{x}$

j $f^{-1}(x) = 1 + \sqrt{1 + x}$ **k** $f^{-1}(x) = 1 - \sqrt{1 + x}$

l $f^{-1}(x) = \dfrac{x + 1}{x - 1}$

5 b The inverse of the first, $x = -y^2$, is not a function. The second is a natural restriction of the domain of the first so that its inverse $y = \sqrt{-x}$ is a function.

6 a gradient $= a$ **b** $x = ay + b$

c The equation can be solved for y when $a \neq 0$. or The graph is a non-horizontal line when $a \neq 0$.

d $y = \dfrac{x}{a} - \dfrac{b}{a}$, gradient $= \dfrac{1}{a}$. A non-zero number and its reciprocal have the same sign.

e Reflection in $y = x$ exchanges the rise and run in every gradient construction.

7 a Show that $h^{-1}\big(h(x)\big) = x$ and $h\big(h^{-1}(x)\big) = x$.

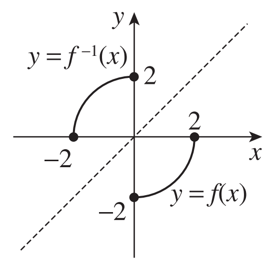
b $h^{-1}(x) = \dfrac{1}{x} + 3$

c $h(x) = g\big(f(x)\big)$, where $f(x) = x - 3$ and $g(x) = \dfrac{1}{x}$.

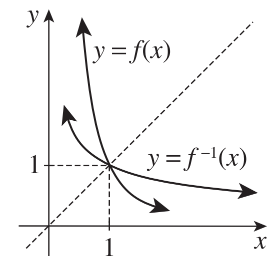
8 a $f^{-1}(x) = \frac{1}{3}x + \frac{2}{3}$, where $1 \leq x \leq 10$.

b $f^{-1}\big(f(x)\big)$ has domain $1 \leq x \leq 4$, and $f\big(f^{-1}(x)\big)$ has domain $1 \leq x \leq 10$.

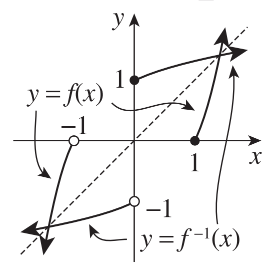
9 a $0 \leq x \leq 2$



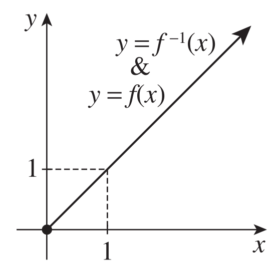
b $x > 0$



c $x < -1$ or $x \geq 1$



d $x \geq 0$



10 a $g\big(f(x)\big) = a\alpha x + b\alpha + \beta$. Put $a\alpha = 1$ and $b\alpha + \beta = 0$

b One example is $f(x) = x + 1$, $g(x) = 2x + 1$, $h(x) = \frac{1}{2}x - \frac{3}{2}$

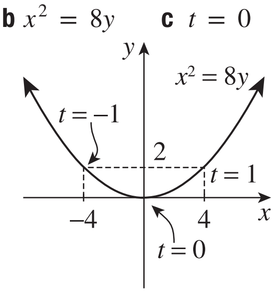
11 The empty function has no ordered pairs, so its inverse relation also has no ordered pairs, and is therefore the empty function. Thus the empty function is the inverse function of itself.

Exercise 5H

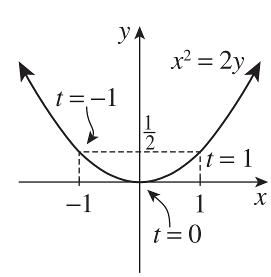
1 a

t	−6	−4	−2	−1	0	1	2	4	6
x	−24	−16	−8	−4	0	4	8	16	24
y	72	32	8	2	0	2	8	32	72

b $x^2 = 8y$ **c** $t = 0$



2 a $x^2 = 2y$ **b** $t = 0$



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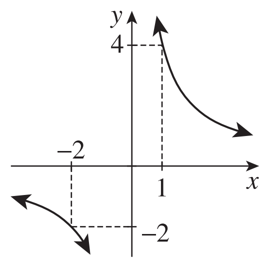
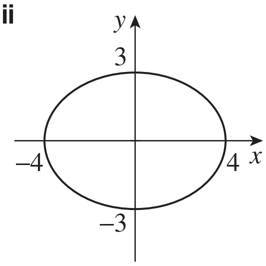
3 c As $p \to \infty$, $x \to \infty$ and $y \to 0$.
As $p \to -\infty$, $x \to -\infty$ and $y \to 0$.
As $p \to 0^+$, $x \to 0$ and $y \to \infty$.
As $p \to 0^-$ $x \to 0$ and $y \to -\infty$



4 b ii



$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

5 a

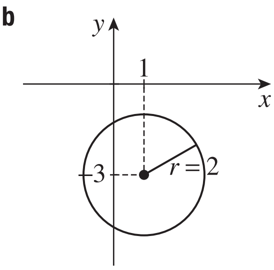
t	-2	-1	0	1	2
x	-4	-3	-2	-1	0
y	-5	-3	-1	-1	3

b When x increases by 1, y increases by 2, so it is a line with gradient 2.
c From the last column, when $x = 0$, $y = 3$.
d $y = 2x + 3$

6 a i $A = (-3, -5)$, $B = (-1, 1)$, gradient $= 3$
ii When $x = 0$, $t = 1\frac{1}{2}$, so $y = 4$
iii $y = 3x + 4$
b i $y = \frac{3}{2}x + \frac{5}{2}$ **ii** $y = \dfrac{cx}{a} + \dfrac{ad - bc}{a}$

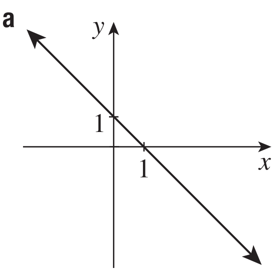
7 a $2x + y - 7 = 0$
b $4(y + 4)^2 - 9(x - 1)^2 = 36$
c $y = x^2 - 2$
d $x^2 + y^2 = 2$

8 b

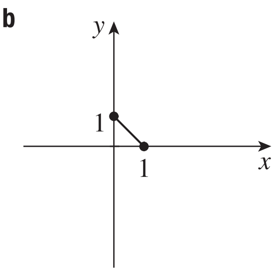


9 The point $(1, 0)$ is missing, because when $y = 0$, $t = 0$, so $x = -1$.

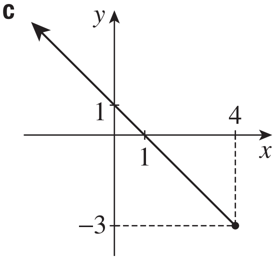
10 a



b



c



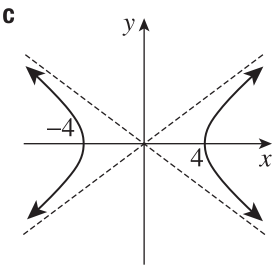
11 a $(x - 3)^2 + (y + 2)^2 = r^2$, circle with centre $(3, -2)$ and radius r
b $y = x\tan\theta - (3\tan\theta + 2)$, straight line with gradient $\tan\theta$

12 a The point $(0, 1)$ is missing, because when $x = 0$, $t = -\frac{1}{2}$, so $y = -1$. **b** One-to-one

13 a Without the variable z and the third equation, the curve would be a circle. Because of the third equation, as t increases, the height z of the curve in the third dimension increases, so the curve never meets back up with itself. Instead it describes a spiral heading upwards (and downwards as $t \to -\infty$), with the curve remaining distant 1 unit from the z-axis.
b One-to-one

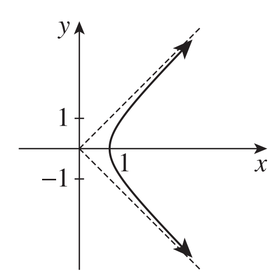
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

14 c



d Many-to-one

15 Cartesian equation:
$x^2 - y^2 = 1$,
where $x > 0$.



16 a i Nothing **ii** Rotation of 180° about O
iii Reflection in the x-axis
iv Reflection in the y-axis

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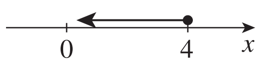
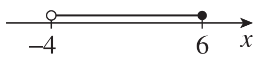
b They are inverse relations.
c The graph is all in the first quadrant.
d The graph is a subset of the line $y = x$.

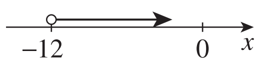
Chapter 5 review exercise

1 a 

$x \le 4$

b 

$-4 < x \le 6$

c 

$x > -12.$

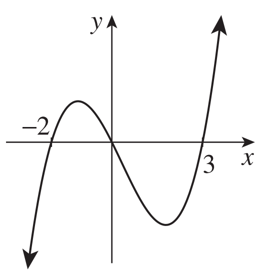
2 a $3 \le x \le 5$ **b** $x < 0$ or $x > 6$
c $x < -\frac{4}{3}$ or $x > 3$

3 a $-3 < x < 3$ **b** $x \le -6$ or $x \ge 2$
c $-3 \le x \le 8$

4 a $0 < x < 5$ **b** $x < 3$ or $x \ge 6$
c $-2 \le x < -1$

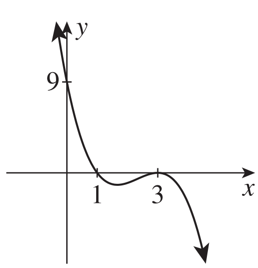
5 a The zeroes are -2, 0 and 3.

x	-3	-2	-1	0	1	3	4
y	-18	0	4	0	-6	0	24
sign	$-$	0	$+$	0	$-$	0	$+$

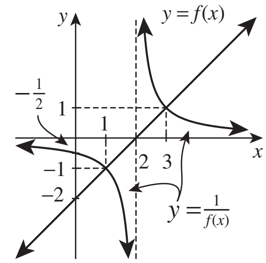
b $f(x)$ is positive for $-2 < x < 0$ and for $x > 3$, and negative for $x < -2$ and for $0 < x < 3$.
c $x \le -2$ or $0 \le x \le 3$
d


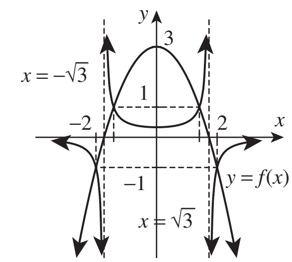
6 a Zeroes are 1 and 3.

x	0	1	2	3	4
y	9	0	-1	0	-3
sign	$+$	0	$-$	0	$-$

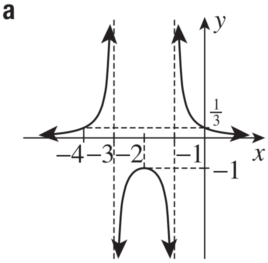
b $x \le 1$ or $x = 3$
c


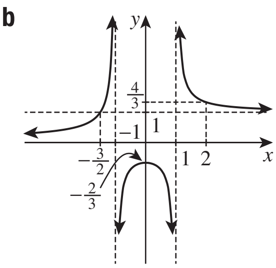
7 a $0 < x < 5$ **b** $x < 3$ or $x \ge 6$
c $-2 \le x < -1$

8


9


d As $x \to 2^-$, $y \to -\infty$, and as $x \to 2^+$, $y \to +\infty$.

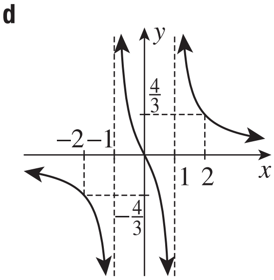
10 a


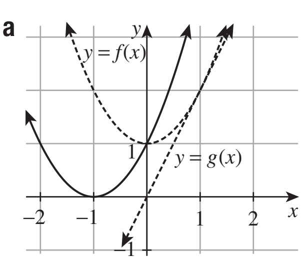
b


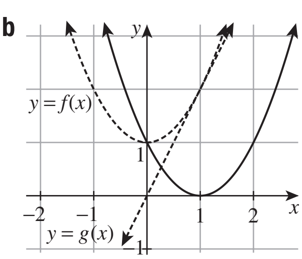
11 a i Vertical asymptote: $x = -1$.
ii Vertical asymptote: $x = 2$.
iii Vertical asymptotes: $x = 5$ and $x = -5$.
b Zero: $x = 0$, discontinuities: $x = -5$ and $x = 5$.

x	-6	-5	-1	0	1	5	6
y	$-\frac{24}{11}$	*	$\frac{1}{6}$	0	$-\frac{1}{6}$	*	$\frac{24}{11}$
sign	$-$	*	$+$	0	$-$	*	$+$

As $x \to (-5)^-$, $y \to -\infty$, and as $x \to (-5)^+$, $y \to \infty$. As $x \to 5^-$, $y \to -\infty$, and as $x \to 5^+$, $y \to \infty$.

12 d


13 a


b


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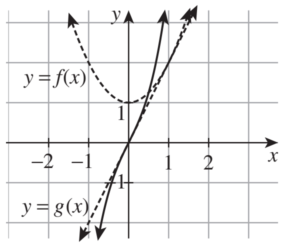
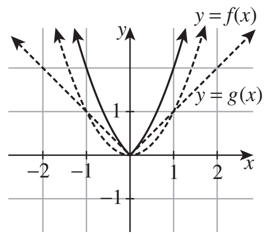
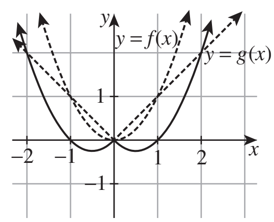
c



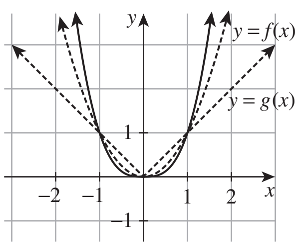
14 a i



ii

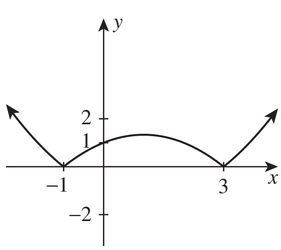


iii

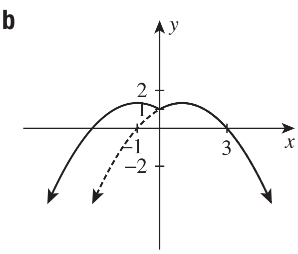


b The original graphs and your answers should be even.

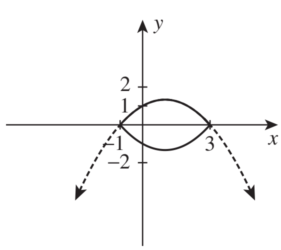
15 a



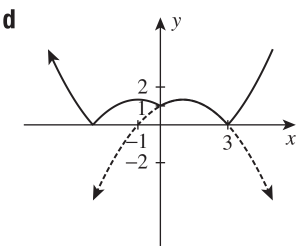
b



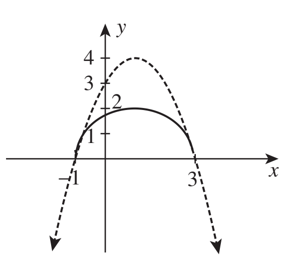
c



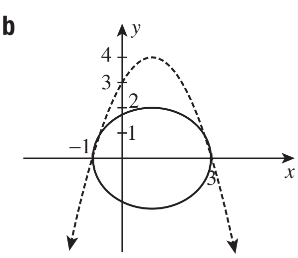
d



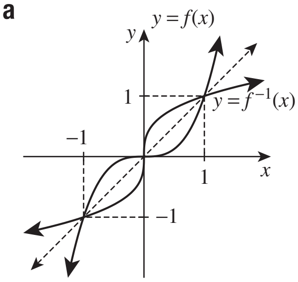
16 a



b

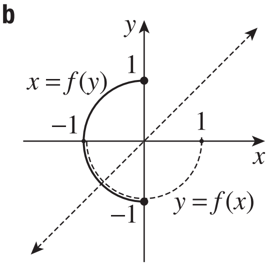


17 a



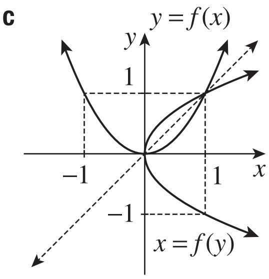
Inverse is a function

b



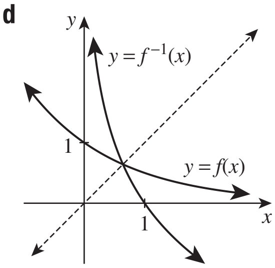
Inverse is not a function

c



Inverse is not a function

d



Inverse is a function

18 a $y = \frac{1}{3}(5 - x)$ **b** $y = \frac{5}{x} + 3$ **c** $y = \frac{3x}{x - 5}$

d $y = \sqrt[3]{x - 5}$

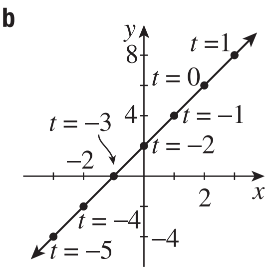
19 a $f^{-1}(x) = 2(x - 4)$ **b** $f^{-1}(x) = \sqrt[3]{x} - 2$

c $f^{-1}(x) = \frac{3}{x + 6}$

20 a

t	−5	−4	−3	−2	−1	0	1
x	−3	−2	−1	0	1	2	3
y	−4	−2	0	2	4	6	8

b



c $y = 2x + 2$

21 a

t	−6	−4	−2	−1	0	1	2	4	6
x	−3	−2	−1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y	9	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4	9

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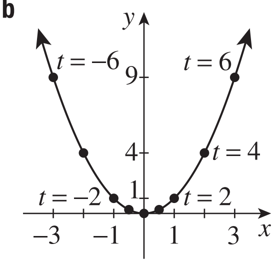
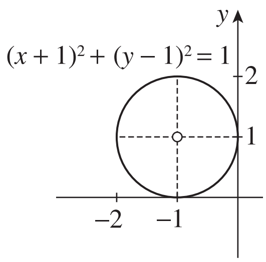
b



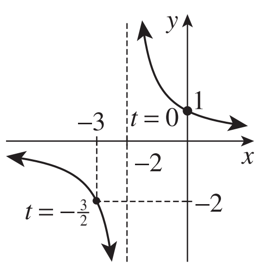
c $y = x^2$

22



a $(x + 1)^2 + (y - 1)^2 = 1$
b It is a circle with centre $(-1, 1)$ and radius 1 unit.

23



$y = \dfrac{2}{x + 2}$

Chapter 6

Exercise 6A

1 a $\frac{3}{5}$ **b** $\frac{3}{4}$ **c** $\frac{4}{5}$ **d** $\frac{4}{5}$ **e** $\frac{3}{5}$ **f** $\frac{4}{3}$
2 a 0.4067 **b** 0.4848 **c** 0.7002 **d** 0.9986
e 0.0349 **f** 0.8387 **g** 0.0175 **h** 0.9986
3 a 1.5697 **b** 0.8443 **c** 4.9894 **d** 0.9571
e 0.6833 **f** 0.1016 **g** 0.0023 **h** 0.0166
4 a 76° **b** 46° **c** 12° **d** 27°
e No such angle — cos θ cannot exceed 1.
f 39° **g** 60°
h No such angle — sin θ cannot exceed 1.
5 a 41°25′ **b** 63°26′ **c** 5°44′ **d** 16°42′
e 46°29′ **f** 57°25′
6 a 13 **b** 19 **c** 23 **d** 88
7 a 53° **b** 41° **c** 67° **d** 59°
8 a $\frac{12}{13}$ **b** $\frac{5}{12}$ **c** $\frac{13}{12}$ **d** $\frac{5}{12}$ **e** $\frac{13}{12}$ **f** $\frac{13}{5}$
9 a 6 and 17
b i $\frac{15}{17}$ **ii** $\frac{4}{5}$ **iii** $\frac{3}{4}$ **iv** $\frac{17}{8}$ **v** $\frac{5}{3}$ **vi** $\frac{15}{8}$
10 a $\frac{\sqrt{3}}{2}$ **b** $\frac{1}{\sqrt{3}}$ **c** $\frac{1}{\sqrt{2}}$ **d** 2 **e** $\sqrt{2}$ **f** $\sqrt{3}$
11 a 19.2 **b** 21.6 **c** 30.3 **d** 8.3
12 a 29.78 **b** 10.14 **c** 16.46 **d** 29.71
13 a 36°2′ **b** 68°38′ **c** 34°44′ **d** 38°40′
e 54°19′ **f** 70°32′
14 b 3 **c** $\frac{1}{3}\sqrt{5}, \frac{2}{3}$
15 a i $\frac{1}{2}\sqrt{22}$ **ii** $\frac{3}{2}\sqrt{2}$
16 a 1 **b** $\frac{1}{2}$ **c** 4 **d** 1
18 a $\angle QPR = 90° - \theta$, so $\angle RPS = \theta$.
b $\dfrac{h}{a}$ and $\dfrac{b}{h}$
21 a 108°

Exercise 6B

1 2.65 metres
2 63°
3 55 km
4 038°T
5 13.2 metres
6 2.5 metres
7 77 km
8 23 metres
9 73°
10 21.3 metres
11 11°
12 a 46° **b** 101°T
13 b 67 km
14 a $\angle PQR = 360° - (200° + 70°) = 90°$ (using co-interior angles on parallel lines and the fact that a revolution is 360°)
b $110° + 39° = 149°$T
15 a 5.1 cm **b** 16 cm
c $PQ = 18 \sin 40°$, 63°25′
17 457 metres
18 a $y = x \tan 39°$ and $y + 7 = x \tan 64°$
19 a If $\angle RBQ = \alpha$, then $\angle RQB = 90° - \alpha$ (angle sum of ΔBQR) and so $\angle RQP = \alpha$ (complementary angles). Therefore $\angle QPR = 90° - \alpha$ (angle sum of ΔPQR) and so $\angle QPC = \alpha$ (complementary angles). Thus $\angle RBQ = \angle RQP = \angle QPC$.
22 a If $OA = OB = x$ and $OP = y$, then $AP - PB = (x + y) - (x - y) = 2y = 2 \times OP$.

Exercise 6C

1 a 15 cm **b** 17 cm **c** 28°
2 a i 90° **ii** 90° **iii** 90° **b i** $\sqrt{2}$ **ii** $\sqrt{3}$
c i 35° **ii** 35°
3 a i $2\sqrt{5}$ cm **ii** $2\sqrt{6}$ cm **b** 90° **c** 66°
4 a i 90° **ii** 90° **iii** 90° **b i** 2 cm **ii** $2\sqrt{2}$ cm
c i 72° **ii** 65°
5 a i 90° **ii** 90° **b** 27°
6 a $3\sqrt{2}$ cm **b** 43°
7 a $BQ = 30 \tan 72°$ **b** 145 m

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8 b 16 m **c** 21°

9 b 76 m **c** 14°

10 a 1 cm **b** $\sqrt{2}$ cm **c** $\sqrt{2}$ **d** $70^\circ 32'$

11 c $67^\circ 23'$

12 a $h \cot 55^\circ$

b It is the angle between south and east.

d 114 metres

13 b 13 metres

14 a $AT = h\,\text{cosec}\,55^\circ$, $BT = h\,\text{cosec}\,40^\circ$

b 90° **d** 52 metres

Exercise 6D

3 a -320° **b** -250° **c** -170°
d -70° **e** -300° **f** -220°

4 a 310° **b** 230° **c** 110°
d 10° **e** 280° **f** 170°

5 a $70^\circ, 430^\circ, -290^\circ, -650^\circ$
b $100^\circ, 460^\circ, -260^\circ, -620^\circ$
c $140^\circ, 500^\circ, -220^\circ, -580^\circ$
d $200^\circ, 560^\circ, -160^\circ, -520^\circ$
e $240^\circ, 600^\circ, -120^\circ, -480^\circ$
f $340^\circ, 700^\circ, -20^\circ, -380^\circ$

6 a $\sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}, \tan\theta = \frac{4}{3}, \text{cosec}\,\theta = \frac{5}{4}, \sec\theta = \frac{5}{3}, \cot\theta = \frac{3}{4}$

b $\sin\theta = \frac{3}{5}, \cos\theta = -\frac{4}{5}, \tan\theta = -\frac{3}{4}, \text{cosec}\,\theta = \frac{5}{3}, \sec\theta = -\frac{5}{4}, \cot\theta = -\frac{4}{3}$

c $\sin\theta = -\frac{2}{\sqrt{5}}, \cos\theta = -\frac{1}{\sqrt{5}}, \tan\theta = 2, \text{cosec}\,\theta = -\frac{\sqrt{5}}{2}, \sec\theta = -\sqrt{5}, \cot\theta = \frac{1}{2}$

d $\sin\theta = -\frac{5}{13}, \cos\theta = \frac{12}{13}, \tan\theta = -\frac{5}{12}, \text{cosec}\,\theta = -\frac{13}{5}, \sec\theta = \frac{13}{12}, \cot\theta = -\frac{12}{5}$

7 All six trigonometric functions are sketched in Section 6E.

8 a i 0.5 **ii** −0.5 **iii** 0.95 **iv** 0.95 **v** 0.59
vi 0.81 **vii** −0.89 **viii** 0.45 **ix** −0.81 **x** 0.59

b i $30^\circ, 150^\circ$ **ii** $120^\circ, 240^\circ$ **iii** $64^\circ, 116^\circ$
iv $53^\circ, 307^\circ$ **v** $53^\circ, 127^\circ$ **vi** $143^\circ, 217^\circ$
vii $204^\circ, 336^\circ$ **viii** $107^\circ, 253^\circ$

c $45^\circ, 225^\circ$

10 $\tan(\theta + 90^\circ) = \dfrac{\sqrt{1-k^2}}{k}$

Exercise 6E

1 a + **b** + **c** − **d** − **e** + **f** − **g** −
h + **i** − **j** + **k** − **l** − **m** − **n** +
o + **p** −

2 a 10° **b** 30° **c** 50° **d** 20° **e** 80°
f 70° **g** 70° **h** 80° **i** 10° **j** 20°

3 a $-\tan 50^\circ$ **b** $\cos 50^\circ$ **c** $-\sin 40^\circ$ **d** $\tan 80^\circ$
e $-\cos 10^\circ$ **f** $-\sin 40^\circ$ **g** $-\cos 5^\circ$ **h** $\sin 55^\circ$
i $-\tan 35^\circ$ **j** $\sin 85^\circ$ **k** $-\cos 85^\circ$ **l** $\tan 25^\circ$

4 a 0 **b** −1 **c** 0
d 0 **e** 1 **f** 1
g −1 **h** undefined **i** 0
j 0 **k** undefined **l** 0

5 a $\frac{\sqrt{3}}{2}$ **b** $\frac{\sqrt{3}}{2}$ **c** $-\frac{\sqrt{3}}{2}$ **d** $-\frac{\sqrt{3}}{2}$ **e** $\frac{1}{\sqrt{2}}$ **f** $-\frac{1}{\sqrt{2}}$
g $-\frac{1}{\sqrt{2}}$ **h** $\frac{1}{\sqrt{2}}$ **i** $\frac{1}{\sqrt{3}}$ **j** $-\frac{1}{\sqrt{3}}$ **k** $\frac{1}{\sqrt{3}}$ **l** $-\frac{1}{\sqrt{3}}$

6 a $-\frac{1}{2}$ **b** 1 **c** $-\frac{1}{2}$ **d** $\frac{1}{\sqrt{2}}$ **e** $\sqrt{3}$ **f** $-\frac{\sqrt{3}}{2}$
g −1 **h** $\frac{1}{2}$ **i** $-\frac{1}{\sqrt{2}}$ **j** $-\frac{\sqrt{3}}{2}$ **k** $-\frac{1}{2}$ **l** $-\sqrt{3}$

7 a 2 **b** $-\sqrt{2}$ **c** $-\frac{1}{\sqrt{3}}$ **d** $\sqrt{3}$ **e** $\frac{2}{\sqrt{3}}$ **f** $-\frac{2}{\sqrt{3}}$

8 a 1 **b** −1 **c** undefined
d undefined **e** 0 **f** undefined

9 a 60° **b** 20° **c** 30° **d** 60°
e 70° **f** 10° **g** 50° **h** 40°

10 a $\frac{1}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $\sqrt{3}$ **d** $\frac{1}{\sqrt{2}}$ **e** $-\frac{1}{\sqrt{3}}$ **f** $-\frac{1}{\sqrt{2}}$
g $\sqrt{3}$ **h** $-\frac{\sqrt{3}}{2}$ **i** $\frac{1}{\sqrt{2}}$ **j** $-\frac{1}{2}$ **k** $-\frac{1}{2}$ **l** 1

11 a 0.42 **b** −0.91 **c** 0.91
d −0.42 **e** 0.49 **f** 0.49

12 a −0.70 **b** −1.22 **c** −0.70
d −0.52 **e** 1.92 **f** −0.52

14 a $-\sin\theta$ **b** $\cos\theta$ **c** $-\tan\theta$ **d** $\sec\theta$
e $\sin\theta$ **f** $-\sin\theta$ **g** $-\cos\theta$ **h** $\tan\theta$

15 a $\left(2, 2\sqrt{3}\right)$ **b** $\left(-\sqrt{3}, 1\right)$
c $(1, -1)$ **d** $\left(-5, -5\sqrt{3}\right)$

16 a $53^\circ 8'$ **b** $138^\circ 11'$ **c** 300° **d** $213^\circ 41'$

17 All six graphs are many-to-one.

19 a $y = \sin\theta$ and $y = \cos\theta$ have range $-1 \le y \le 1$, $y = \tan\theta$ and $y = \cot\theta$ have range $\mathbb{R}$, $y = \sec\theta$ and $y = \text{cosec}\,\theta$ have range $y \ge 1$ or $y \le -1$.

b $\sin\theta$, $\cos\theta$, $\text{cosec}\,\theta$ and $\sec\theta$ have period 360°; $\tan\theta$ and $\cot\theta$ have period 180°.

c $\sin\theta$, $\text{cosec}\,\theta$, $\tan\theta$ and $\cot\theta$ are odd; $\cos\theta$ and $\sec\theta$ are even.

d The graphs have point symmetry about every θ-intercept, and about every point where an asymptote crosses the θ axis.

e $\sin\theta$, $\cos\theta$, $\text{cosec}\,\theta$ and $\sec\theta$ have line symmetry in every vertical line through a maximum or minimum; $\tan\theta$ and $\cot\theta$ have no axes of symmetry.

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Exercise 6F

1 a $\sin\theta = \frac{15}{17}$, $\cos\theta = \frac{8}{17}$, $\tan\theta = \frac{15}{8}$
b $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$, $\tan\theta = -\frac{4}{3}$
c $\sin\theta = -\frac{7}{25}$, $\cos\theta = -\frac{24}{25}$, $\tan\theta = \frac{7}{24}$
d $\sin\theta = -\frac{21}{29}$, $\cos\theta = \frac{20}{29}$, $\tan\theta = -\frac{21}{20}$

2 a $y = 12$, $\sin\alpha = \frac{12}{13}$, $\cos\alpha = \frac{5}{13}$, $\tan\alpha = \frac{12}{5}$
b $r = 3$, $\sin\alpha = \frac{2}{3}$, $\cos\alpha = -\frac{\sqrt{5}}{3}$, $\tan\alpha = -\frac{2}{\sqrt{5}}$
c $x = -4$, $\sin\alpha = -\frac{3}{5}$, $\cos\alpha = -\frac{4}{5}$, $\tan\alpha = \frac{3}{4}$
d $y = -3$, $\sin\alpha = -\frac{3}{\sqrt{13}}$, $\cos\alpha = \frac{2}{\sqrt{13}}$, $\tan\alpha = -\frac{3}{2}$

3 a i $\sin\theta = -\frac{4}{5}$ **ii** $\tan\theta = -\frac{4}{3}$
b i $\sin\theta = \frac{5}{13}$ **ii** $\cos\theta = -\frac{12}{13}$

4 a $\cos\theta = -\frac{3}{4}$ and $\tan\theta = \frac{\sqrt{7}}{3}$, or $\cos\theta = \frac{3}{4}$ and $\tan\theta = -\frac{\sqrt{7}}{3}$
b $\sin\theta = \frac{\sqrt{15}}{4}$ and $\tan\theta = -\sqrt{15}$, or $\sin\theta = -\frac{\sqrt{15}}{4}$ and $\tan\theta = \sqrt{15}$

5 a $2\sqrt{2}$ **b** $-\frac{3}{4}$ **c** $-\frac{\sqrt{3}}{2}$ **d** $\frac{3}{\sqrt{13}}$ **e** $\frac{9}{41}$ **f** $\frac{1}{2}$

6 a $\frac{1}{\sqrt{10}}$ or $-\frac{1}{\sqrt{10}}$ **b** $\frac{1}{\sqrt{5}}$ or $-\frac{1}{\sqrt{5}}$ **c** $\frac{4}{5}$ or $-\frac{4}{5}$
d $\frac{\sqrt{5}}{2}$ or $-\frac{\sqrt{5}}{2}$ **e** $\frac{12}{5}$ or $-\frac{12}{5}$ **f** $\frac{\sqrt{3}}{\sqrt{7}}$ or $-\frac{\sqrt{3}}{\sqrt{7}}$

7 a $-\frac{3}{4}$ **b** $-\frac{15}{17}$ **c** $-\frac{\sqrt{15}}{4}$
d $\frac{35}{37}$ **e** $-\frac{21}{20}$ **f** $\frac{\sqrt{11}}{6}$

8 a $\sqrt{2}$ or $-\sqrt{2}$ **b** $\frac{15}{8}$ or $-\frac{15}{8}$
c $\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$ **d** $\frac{6}{5}$ or $-\frac{6}{5}$

9 a $-\frac{3}{\sqrt{5}}$ **b** $-\frac{41}{9}$ **c** $-\frac{15}{8}$ **d** $\frac{\sqrt{7}}{\sqrt{3}}$

10 a $\frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$ **b** $-\frac{3}{2\sqrt{10}} = -\frac{3\sqrt{10}}{20}$
c 1 **d** $\frac{12}{13}$

11 $\cos\theta = -\dfrac{\sqrt{q^2 - p^2}}{q}$, $\tan\theta = -\dfrac{p}{\sqrt{q^2 - p^2}}$

12 $\sin\alpha = \dfrac{k}{\sqrt{1 + k^2}}$ or $-\dfrac{k}{\sqrt{1 + k^2}}$,
$\sec\alpha = \sqrt{1 + k^2}$ or $-\sqrt{1 + k^2}$

13 b $\sin x = \dfrac{2t}{1 + t^2}$, $\tan x = \dfrac{2t}{1 - t^2}$

14 Note that $\tan\theta$ could be positive or negative.

Exercise 6G

2 a $\operatorname{cosec}\theta$ **b** $\cot\alpha$ **c** $\tan\beta$ **d** $\cot\phi$
3 a 1 **b** 1 **c** 1
5 a $\cos\theta$ **b** $\operatorname{cosec}\alpha$ **c** $\cot\beta$ **d** $\tan\phi$
6 a 1 **b** $\sin^2\beta$ **c** $\sec^2\phi$ **d** 1
7 a $\cos^2\beta$ **b** $\operatorname{cosec}^2\phi$ **c** $\cot^2 A$ **d** -1
8 a $\cos^2\theta$ **b** $\tan^2\beta$ **c** $\cot^2 A$ **d** 1
10 a $\cos^2\alpha$ **b** $\sin^2\alpha$ **c** $\sin A$ **d** $\cos A$
14 a $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **b** $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$
c $(x - 2)^2 + (y - 1)^2 = 1$ **d** $x^2 + y^2 = 2$
15 a 2 **b** 0 **c** 1 **d** 0
18 a $y - x = 1$ **b** $x^2 + 2xy + 2y^2 = 5$
c $x^2y = y + 2$

Exercise 6H

1 a $\theta = 60°$ or $120°$ **b** $\theta = 30°$ or $150°$
c $\theta = 45°$ or $225°$ **d** $\theta = 60°$ or $240°$
e $\theta = 135°$ or $225°$ **f** $\theta = 120°$ or $300°$
g $\theta = 210°$ or $330°$ **h** $\theta = 150°$ or $210°$
2 a $\theta = 90°$ **b** $\theta = 0°$ or $360°$
c $\theta = 90°$ or $270°$ **d** $\theta = 180°$
e $\theta = 0°$ or $180°$ or $360°$ **f** $\theta = 270°$
3 a $x \doteqdot 65°$ or $295°$ **b** $x \doteqdot 7°$ or $173°$
c $x \doteqdot 82°$ or $262°$ **d** $x \doteqdot 222°$ or $318°$
e $x \doteqdot 114°$ or $294°$ **f** $x \doteqdot 140°$ or $220°$
4 a $\alpha \doteqdot 5°44'$ or $174°16'$ **b** $\alpha \doteqdot 95°44'$ or $264°16'$
c $\alpha = 135°$ or $315°$ **d** $\alpha = 270°$
e no solutions **f** $\alpha = 120°$ or $240°$
g $\alpha = 150°$ or $330°$ **h** $\alpha \doteqdot 18°26'$ or $198°26'$
5 a $x \doteqdot -16°42'$ or $163°18'$ **b** $x = 90°$ or $-90°$
c $x = 45°$ or $-45°$
d $x \doteqdot -135°34'$ or $-44°26'$
6 a $\theta = 60°, 300°, 420°$ or $660°$
b $\theta = 90°, 270°, 450°$ or $630°$
c $\theta = 210°, 330°, 570°$ or $690°$
d $\theta = 22°30', 202°30', 382°30'$ or $562°30'$
7 a $x = 15°, 75°, 195°$ or $255°$
b $x = 30°, 120°, 210°$ or $300°$
c $x = 67°30', 112°30', 247°30'$ or $292°30'$
d $x = 135°$ or $315°$
8 a $\alpha = 75°$ or $255°$ **b** $\alpha = 210°$ or $270°$
c $\alpha = 300°$ **d** $\alpha = 210°$ or $300°$
9 a $\theta = 45°$ or $225°$ **b** $\theta = 135°$ or $315°$
c $\theta = 60°$ or $240°$ **d** $\theta = 150°$ or $330°$
10 a $\theta = 0°, 90°, 180°$ or $360°$
b $\theta = 60°, 90°, 270°$ or $300°$
c $\theta = 0°, 60°, 180°, 300°$ or $360°$
d $\theta = 30°, 150°$ or $270°$
e $\theta = 60°$ or $300°$, or $\theta \doteqdot 104°29'$ or $255°31'$

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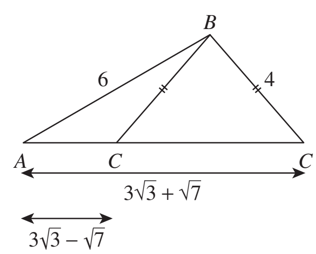
f $\theta \doteqdot 70°32'$ or $289°28'$
g $\theta \doteqdot 23°35', 156°25', 221°49'$ or $318°11'$
h $\theta = 0°, 60°, 120°, 180°, 240°, 300°$ or $360°$
11 Show that $\sin\theta = \frac{1 \pm \sqrt{5}}{4}$,
then $\theta = 54°, 126°, 198°$ or $342°$.

Exercise 6I

1 a 8.2 **b** 4.4 **c** 4.9 **d** 1.9 **e** 9.2 **f** 3.5
2 a 14.72 **b** 46.61 **c** 5.53
3 a 49° **b** 53° **c** 43° **d** 20° **e** 29° **f** 42°
4 a 5 cm^2 **b** 19 cm^2 **c** 22 cm^2
5 a $b \doteqdot 10.80$ cm, $c \doteqdot 6.46$ cm
6 a $49°46'$ **b** $77°53'$ **c** 3.70 cm^2
7 42°, 138°
8 62°, 118°
9 a $69°2'$ or $110°58'$ **b** 16.0 cm or 11.0 cm
10 317 km
11 b 9 metres
12 a 32 **b** $\frac{5}{7}$
13 a 16 metres **b** 11.35 metres **c** 3.48 metres
14 a 30° or 150° **b** $17°27'$ or $162°33'$
c No solutions, because $\sin\theta = 1.2$ is impossible.
15 a $3\sqrt{6}$ **b** $3\sqrt{2}$ **c** $2\sqrt{6}$ **d** $6\sqrt{2}$
16 11.0 cm
17 a $\angle QSM = 36°$ (angle sum of ΔQRS) and $\angle PSM = 48°$ (angle sum of ΔPSM), so $\angle PSQ = 48° - 36° = 12°$. $\angle SPQ = 24°$. So $\angle PQS = 180° - 24° - 12° = 144°$ (angle sum of ΔPQS).
c 473 metres
18 a Adding the known angle β and the obtuse solution,
$$(180° - \alpha) + \beta = 180° - (\alpha - \beta),$$
so the third angle is $\alpha - \beta$. If $\alpha \le \beta$, then the third angle is zero or negative, which is impossible . If $\alpha > \beta$, then there is room for the third angle.
b Two angles add to more than 180°. It is an impossible triangle, because the longest side b should be opposite the largest angle $\angle B = 150°$.
20 a $8\sqrt{3}$ **b** 2:1
21 d $\frac{\sqrt{3} - 1}{2\sqrt{2}}$
22 b $13°41'$

Exercise 6J

1 a 3.3 **b** 4.7 **c** 4.0 **d** 15.2 **e** 21.9 **f** 24.6
2 a 39° **b** 56° **c** 76° **d** 94° **e** 117° **f** 128°
3 a $\sqrt{13}$ **b** $\sqrt{7}$
4 a $\sqrt{10}$ **b** $\sqrt{21}$
5 a $44°25'$ **b** $101°32'$ **c** $\frac{7}{32}$
6 167 nautical miles
7 20°
8 $13°10'$, 120°
10 a 19 cm **b** $\frac{37}{38}$
11b 108 km **c** $\angle ACB \doteqdot 22°$, bearing $\doteqdot 138°$
12 a $\angle DAP = \angle DPA = 60°$ (angle sum of isosceles triangle), so ΔADP is equilateral.
Hence $AP = 3$ cm.
b $3\sqrt{7}$ cm
13 a $x \cot 27°$
14 3 or 5
15c



16 120°

Exercise 6K

1 a 28.3 **b** 17.3 **c** 12.5
d 36.2 **e** 12.6 **f** 23.2
2 a 59° **b** 55° **c** 40° **d** 37° **e** 52° **f** 107°
3 a 26 cm **b** 28 cm **c** 52° **d** 62°
4b 28 metres
5 a $\angle ACP + 31° = 68°$ (exterior angle of ΔACP)
c 6 cm
6 a 11.6 cm **b** 49°
7 a $44°25'$ **b** 10 cm^2
8 b 36 cm
10 a PQ is inclined at 26° to a north–south line through Q, because of alternate angles on parallel lines.
Then $\angle PQR = 26° + 90°$.
b 112 nautical miles
11 a $46°59'$ or $133°1'$
b 66.4 metres or 52.7 metres
12 a $\angle PJK = \angle PBQ = 20°$ (corresponding angles on parallel lines),
but $\angle PJK = \angle PAJ + \angle APJ$ (exterior angle of triangle), so $\angle APJ = 20° - 5° = 15°$.
d 53 metres
13 a 38 tan 68° **b** 111 m
14 b 131 m
15 P_1 by 2.5 min

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16 a $34°35'$

b $\angle PDA = \angle ABP$ (base angles of isosceles ΔABD) and $\angle ABP = \angle PDC$ (alternate angles on parallel lines), so $\angle PDA = \angle PDC$ and $\angle PDC = \frac{1}{2}\angle ADC$.

c $65°35'$

17 50.4 metres

18 a $-\cos\theta$

20 a $y = h\cot\beta$

Chapter 6 review exercise

1 a 0.2924 **b** 0.9004 **c** 0.6211 **d** 0.9904

2 a $17°27'$ **b** $67°2'$ **c** $75°31'$ **d** $53°8'$

3 a 10.71 **b** 5.23 **c** 10.36 **d** 15.63

4 a $45°34'$ **b** $59°2'$ **c** $58°43'$ **d** $36°14'$

5 a $\sqrt{3}$ **b** $\frac{1}{\sqrt{2}}$ **c** $\frac{\sqrt{3}}{2}$ **d** 1 **e** 2 **f** $\frac{2}{\sqrt{3}}$

6 6.25 metres

7 $65°$

8 b 114 km **c** $108°$T

9 All six trigonometric graphs are drawn just before Exercise 6E.

10 a $-\cos 55°$ **b** $-\sin 48°$ **c** $\tan 64°$ **d** $\sin 7°$

11 a $\sqrt{3}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $\frac{\sqrt{3}}{2}$ **d** $-\frac{1}{\sqrt{3}}$

12 a 0 **b** -1

c undefined **d** -1

13 a $y = 3$, $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$

b $x = -2\sqrt{5}$, $\sin\theta = -\frac{\sqrt{5}}{5}$, $\cos\theta = -\frac{2\sqrt{5}}{5}$, $\tan\theta = \frac{1}{2}$

14 a $\sin\alpha = \frac{12}{13}$, $\cos\alpha = \frac{5}{13}$

b $\cos\beta = \frac{5}{7}$, $\tan\beta = \frac{2\sqrt{6}}{5}$

15 a $\sin\alpha = -\frac{9}{41}$, $\cos\alpha = \frac{40}{41}$

b $\cos\beta = -\frac{5}{7}$, $\tan\beta = -\frac{2\sqrt{6}}{5}$

16 a $\sec\theta$ **b** $\tan\theta$ **c** $\tan\theta$

d $\cos^2\theta$ **e** 1 **f** $\cot^2\theta$

18 a $x = 60°$ or $300°$ **b** $x = 90°$

c $x = 135°$ or $315°$ **d** $x = 90°$ or $270°$

e $x = 30°$ or $210°$ **f** $x = 0°, 180°$ or $360°$

g $x = 225°$ or $315°$ **h** $x = 150°$ or $210°$

i $x = 45°, 135°, 225°$ or $315°$

j $x = 30°, 150°, 210°$ or $330°$

k $x = 15°$ or $135°$

l $\tan x = -\sqrt{3}$, $x = 120°$ or $300°$

19 a $\sin\theta = 0$ or $-\frac{1}{2}$, $\theta = 0°, 180°, 210°, 330°$ or $360°$

b $\cos\theta = -1$ or 2, $\theta = 180°$

c $\tan\theta = \frac{1}{2}$ or -3, $\theta = 26°34', 108°26', 206°34'$ or $288°26'$

20 a 8.5 **b** 10.4 **c** 7.6 **d** 8.9

21 a 27 cm^2 **b** 56 cm^2

22 a $57°55'$ **b** $48°33'$ **c** $24°29'$ **d** $150°26'$

23 28 cm^2

24 a $\frac{5\sqrt{3}}{3}$cm **b** $30°$ or $150°$

25 b 48 metres

26 b 31.5 metres

27 b 316 nautical miles **c** $104°$T

28 a $10\tan 77°$ **b** 45 m

29 a 9.85 metres **b** 5.30 metres **c** 12.52 metres

30 c 34 metres

31 a $\dfrac{86\sin 60°45'}{\sin 65°45'}$ **b** 66 metres

34 c 129 metres

35 a $BD = \sqrt{3}h$, $CD = h$

Chapter 7

Exercise 7A

1 a $(2, 7)$ **b** $(5, 6)$ **c** $(2, -2)$

d $\left(0, 3\frac{1}{2}\right)$ **e** $\left(-5\frac{1}{2}, -10\right)$ **f** $(4, 0)$

2 a 5 **b** 13 **c** 10

d $\sqrt{8} = 2\sqrt{2}$ **e** $\sqrt{80} = 4\sqrt{5}$ **f** 13

3 a $M(1, 5)$ **b** $PM = MQ = 5$

4 a $PQ = QR = \sqrt{17}$, $PR = \sqrt{50} = 5\sqrt{2}$

5 a $AB = 15$, $BC = 20$ and $AC = 25$

b LHS $= AB^2 + BC^2 = 15^2 + 20^2 = 625 =$ RHS

6 a $AB = \sqrt{58}$, $BC = \sqrt{72} = 6\sqrt{2}$, $CA = \sqrt{10}$

b AB: $\left(1\frac{1}{2}, 1\frac{1}{2}\right)$, BC: $(0, 1)$, CA: $\left(-1\frac{1}{2}, 4\frac{1}{2}\right)$

7 a 13 **b** $\sqrt{41}$ **c** $(5, -3)$

8 a $(1, 6)$ **b** $(1, 6)$

c The diagonals bisect each other.

d parallelogram

9 a All sides are $5\sqrt{2}$. **b** rhombus

10 a $XY = YZ = \sqrt{52} = 2\sqrt{13}$, $ZX = \sqrt{104} = 2\sqrt{26}$

b $XY^2 + YZ^2 = 104 = ZX^2$

c 26 square units

11 a Each point is $\sqrt{17}$ from the origin.

b $\sqrt{17}$, $2\sqrt{17}$, $2\pi\sqrt{17}$, 17π

12 a $(x - 5)^2 + (y + 2)^2 = 45$

b $(x + 2)^2 + (y - 2)^2 = 74$

13 $(5, 2)$

14 a $S(-5, -2)$ **b i** $P = (4, -14)$

ii $P = (-1, -17)$ **iii** $P = (7, -7)$

c $B = (0, 7)$ **d** $R = (12, -9)$

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15 a ABC is an equilateral triangle.
b PQR is a right triangle.
c DEF is none of these.
d XYZ is an isosceles triangle.
16 a Check the results using the distance formula — there are eight such points.
b $y = 4$ or 10
c $a = 1 + \sqrt{2}$ or $1 - \sqrt{2}$
17 a $(-2, 1)$ **b** $M = \left(4\frac{1}{2}, 1\frac{1}{2}\right)$
18 a $x = \frac{3}{2}a$, a vertical straight line through the midpoint of AB.
b $(x - 4a)^2 + y^2 = (2a)^2$, a circle with centre $(4a, 0)$ and radius $2a$.

Exercise 7B

1 a i 2 **ii** $\frac{3}{4}$ **iii** $-1\frac{1}{2}$
b i $-\frac{1}{2}$ **ii** $-\frac{4}{3}$ **iii** $\frac{2}{3}$
2 a $-1, 1$ **b** $2, -\frac{1}{2}$ **c** $\frac{1}{2}, -2$
d $-\frac{1}{2}, 2$ **e** $3, -\frac{1}{3}$ **f** $-\frac{7}{10}, \frac{10}{7}$
3 a vertical **b** horizontal **c** neither
d horizontal **e** neither **f** vertical
4 a 3 **b** $\frac{1}{2}$
c parallelogram
5 a $m_{AB} = m_{CD} = \frac{1}{2}, m_{BC} = m_{DA} = -\frac{1}{5}$.
b $m_{AB} = 2, m_{CD} = -3$
6 a 0.27 **b** −1.00 **c** 0.41 **d** 3.08
7 a 45° **b** 120° **c** 76° **d** 30°
8 a $m_{AB} = m_{CD} = -\frac{1}{2}, m_{BC} = m_{DA} = 2$
b $m_{AB} \times m_{BC} = -1$ **c** $AB = BC = 2\sqrt{5}$
9 In each case, show that each pair of opposite sides is parallel.
a Show also that two adjacent sides are equal.
b Show also that two adjacent sides are perpendicular.
c Show that it is both a rhombus and a rectangle.
10 a $-2, -\frac{7}{3}$, non-collinear **b** $\frac{2}{3}, \frac{2}{3}$, collinear
11 The gradients of AB, BC and CD are all $\frac{1}{3}$.
12 $m_{AB} = \frac{1}{2}, m_{BC} = -2$ and $m_{AC} = 0$, so $AB \perp BC$.
13 a $m_{PQ} = 4, m_{QR} = -\frac{1}{4}$ and $m_{PR} = -\frac{5}{3}$, so $PQ \perp QR$.
Area $= 8\frac{1}{2}$ square units
b $m_{XY} = \frac{7}{3}, m_{YZ} = \frac{2}{5}$ and $m_{XZ} = -\frac{5}{2}$, so $XZ \perp YZ$.
Area $= 14\frac{1}{2}$ square units
14 a -5 **b** 5
15 a $A(-2, 0), B(0, 6)$ $m = 3, \alpha \doteqdot 72°$
b $A(2, 0), B(0, 1), m = -\frac{1}{2}, \alpha \doteqdot 153°$
c $A(-4, 0), B(0, -3), m = -\frac{3}{4}, \alpha \doteqdot 143°$
d $A(3, 0), B(0, -2), m = \frac{2}{3}, \alpha \doteqdot 34°$
16 a $P = (2, -1), Q = (-1, 4), R = (-3, 2), S = (0, -3)$
b $m_{PQ} = m_{RS} = -\frac{5}{3}$ and $m_{PS} = m_{QR} = 1$
17 a They all satisfy the equation, or they all lie 5 units from O.
b The centre $O(0, 0)$ lies on AB.
c $m_{AC} = \frac{1}{2}, m_{BC} = -2$
18 $a = -\frac{1}{2}$
19 $k = 2$ or -1
21 a They are collinear if and only if $\Delta = 0$, that is $a_1b_2 + a_2b_3 + a_3b_1 = a_2b_1 + a_3b_2 + a_1b_3$.
22 a $x = \dfrac{4p}{1 - p^2}$ **b** $x = p - \dfrac{1}{p}$

Exercise 7C

1 a not on the line
b on the line
c on the line
2 a $(4, 0)$ and $(0, 3)$
b $(1.5, 0)$ and $(0, -6)$
c $(8, 0)$ and $(0, -4)$
3 Check the points in your answer by substitution. $(0, 8)$, $(3, 7)$ and $(6, 6)$ will do.
4 a $x = 1, y = 2$
b $x = 0, y = -4$
c $x = 5, y = 0$
5 a $m = 4, b = -2$ **b** $m = \frac{1}{5}, b = -3$
c $m = -1, b = 2$ **d** $m = -\frac{5}{7}, b = 0$
6 a $y = -3x + 5$ **b** $y = -3x - \frac{2}{3}$ **c** $y = -3x$
7 a $y = 5x - 4$ **b** $y = -\frac{2}{3}x - 4$ **c** $y = -4$
8 a $x - y + 3 = 0$ **b** $2x + y - 5 = 0$
c $x - 5y - 5 = 0$ **d** $x + 2y - 6 = 0$
9 a $m = 1, b = 3$ **b** $m = -1, b = 2$
c $m = \frac{1}{3}, b = 0$ **d** $m = -\frac{3}{4}, b = \frac{5}{4}$
10 a $m = 1, \alpha = 45°$ **b** $m = -1, \alpha = 135°$
c $m = 2, \alpha \doteqdot 63°26'$ **d** $m = -\frac{3}{4}, \alpha \doteqdot 143°8'$

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11 The sketches required are clear from the intercepts.
a $A(3, 0), B(0, 5)$
b $A(-3, 0), B(0, 6)$
c $A(-4, 0), B\left(0, 2\frac{2}{5}\right)$

12 a $y = 2x + 4, 2x - y + 4 = 0$
b $y = -x, x + y = 0$
c $y = -\frac{1}{3}x - 4, x + 3y + 12 = 0$

13 a $y = -2x + 3, y = \frac{1}{2}x + 3$
b $y = \frac{5}{2}x + 3, y = -\frac{2}{5}x + 3$
c $y = -\frac{3}{4}x + 3, y = \frac{4}{3}x + 3$

14 a $-3, \frac{1}{2}, -3, \frac{1}{2}$, parallelogram
b $\frac{4}{3}, -\frac{3}{4}, \frac{4}{3}, -\frac{3}{4}$, rectangle

15 The gradients are $\frac{5}{7}, \frac{2}{5}$ and $-\frac{7}{5}$, so the first and last are perpendicular.
a $A(-3, 0), B(0, 3)$ **b** $A(2, 0), B(0, 2)$
c $A\left(2\frac{1}{2}, 0\right), B(0, -5)$ **d** $A(-6, 0), B(0, 2)$
e $A\left(1\frac{2}{3}, 0\right), B\left(0, 1\frac{1}{4}\right)$ **f** $A\left(1\frac{1}{3}, 0\right), B(0, -2)$

16 a $x = 3, x = 0, y = -7, y = -2$
b $y = 0, y = -4x + 12, y = 2x + 12$

17 a $x - y + 3 = 0$ **b** $-\sqrt{3}x + y + 1 = 0$
c $x - \sqrt{3}y - 2\sqrt{3} = 0$ **d** $x + y - 1 = 0$

18 a They are about 61° and 119°.
b It is isosceles. (The two interior angles with the x-axis are equal.)

19 a $k = -\frac{1}{3}$ **b** $k = 3$

21 $(x - a)^2 + (y - a)^2 = a^2$,
where $a = 2 - \sqrt{2}$ or $a = 2 + \sqrt{2}$,
$\left(x - \sqrt{2}\right)^2 + \left(y + \sqrt{2}\right)^2 = 2$,
$\left(x + \sqrt{2}\right)^2 + \left(y - \sqrt{2}\right)^2 = 2$

22 a From their gradients, two pairs of lines are parallel and two lines are perpendicular.
b The distance between the x-intercepts of one pair of lines must equal the distance between the y-intercepts of the other pair. Thus $k = 2$ or 4.

Exercise 7D

1 a $2x - y - 1 = 0$ **b** $x + y - 4 = 0$
c $5x + y = 0$ **d** $x + 3y - 8 = 0$
e $4x + 5y + 8 = 0$

2 a $y = 2x + 1$ **b** $y = -\frac{1}{2}x + 6$
c $y = \frac{1}{5}x - 8$ **d** $y = \frac{3}{7}x + 9$
e $y = \frac{5}{2}x + 10$

3 a 3 **b** $3x - y - 5 = 0$

4 a $2, 2x - y - 2 = 0$ **b** $-2, 2x + y - 1 = 0$
c $\frac{1}{3}, x - 3y + 13 = 0$ **d** $2, 2x - y + 2 = 0$
e $-\frac{1}{4}, x + 4y + 4 = 0$ **f** $1, x - y - 3 = 0$

5 a $-\frac{3}{2}$
b i $3x + 2y + 1 = 0$ **ii** $2x - 3y - 8 = 0$

6 a $2x - 3y + 2 = 0$ **b** $2x - 3y - 9 = 0$

7 a $4x - 3y - 8 = 0$ **b** $4x - 3y + 11 = 0$

8 a $M(3, -1)$
c i No, the first two intersect at $(-4, 7)$, which does not lie on the third.
ii They all meet at $(5, 4)$.

9 a $y = -2x + 5, y = \frac{1}{2}x + 6$
b $y = 2\frac{1}{2}x - 8\frac{1}{2}, y = -\frac{2}{5}x + 4\frac{1}{5}$
c $y = -1\frac{1}{3}x + 3, y = \frac{3}{4}x + 6\frac{1}{2}$

10 a $x - y - 1 = 0$
b $\sqrt{3}x + y + \sqrt{3} = 0$
c $x - y\sqrt{3} - 4 - 3\sqrt{3} = 0$
d $x + \sqrt{3}y + 2 + 5\sqrt{3} = 0$

11 a i $x - 3 = 0$ **ii** $y + 1 = 0$
b $3x + 2y - 6 = 0$
c i $x - y + 4 = 0$ **ii** $\sqrt{3}x + y - 4 = 0$
d $x\sqrt{3} + y + 6\sqrt{3} = 0$

12 $\ell_1 \parallel \ell_2$, and $\ell_3 \parallel \ell_4$, so there are two pairs of parallel sides. The vertices are $(-2, -1), (-4, -7), (1, -2), (3, 4)$.

13 $m_{BC} \times m_{AC} = -1$ so $BC \perp AC$.
$AB: y = x - 1, BC: y = \frac{1}{2}x + 2,$
$AC: y = 2 - 2x$

14 a $m_{AC} = \frac{2}{3}, \theta \doteqdot 34°$
b $2x - 3y - 2 = 0$
c $D(4, 2)$
d $m_{AC} \times m_{BD} = \frac{2}{3} \times -\frac{3}{2} = -1$, hence they are perpendicular.
e isosceles
f area $= \frac{1}{2} \times AC \times BD = \frac{1}{2} \times \sqrt{52} \times \sqrt{52} = 26$
g $E(8, -4)$

15 a $4y = 3x + 12$ **b** $ML = MP = 5$
c $N(4, 6)$ **d** $x^2 + (y - 3)^2 = 25$

16 $k = 2\frac{1}{2}$

17 a $\mu = 4$ **b** $\mu = -9$

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18 $bx + ay = ab$

19 $bx + ay = 2ab$

21 c i 1 **ii** $\frac{1}{13}\sqrt{13}$

Exercise 7E

1 a i 1, −1

ii The product of their gradients is −1.

b i 1, −1

ii The product of their gradients is −1.

2 a i $M = (4, 5)$

ii $OM = PM = QM = \sqrt{41}$

iii OM, PM and QM are three radii of the circle.

b $M = (p, q)$, $OM = PM = QM = \sqrt{p^2 + q^2}$

3 a i $P(5, 2)$ and $Q(4, 1)$.

iv $AC = 2\sqrt{2}$ and $PQ = \sqrt{2}$

b $P(a + b, c)$, $Q(b, c)$, $y = c$ and so $Q(b, c)$ lies on $y = c$. Also, $AC = 2a$ and $PQ = a$ so $PQ = \frac{1}{2}AC$.

4 a $P = \left(\frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2)\right)$,

$Q = \left(\frac{1}{2}(b_1 + c_1), \frac{1}{2}(b_2 + c_2)\right)$,

$R = \left(\frac{1}{2}(c_1 + d_1), \frac{1}{2}(c_2 + d_2)\right)$,

$S = \left(\frac{1}{2}(d_1 + a_1), \frac{1}{2}(d_2 + a_2)\right)$

b Both midpoints are $\left(\frac{1}{4}(a_1 + b_1 + c_1 + d_1), \frac{1}{4}(a_2 + b_2 + c_2 + d_2)\right)$.

c Part **b** shows that its diagonals bisect each other, so (using Box 4) it is a parallelogram.

6 a $\frac{x}{3} + \frac{y}{4} = 1$ and $4y = 3x$, thus $C = \left(\frac{48}{25}, \frac{36}{25}\right)$.

b $OA = 3$, $AB = 5$, $OC = \frac{12}{5}$, $BC = \frac{16}{5}$, $AC = \frac{9}{5}$

7 a $AB = BC = CA = 2a$

b $AB = AD = 2a$

c $BD = 2a\sqrt{3}$

8 a AB and DC have gradient $\frac{b}{a}$; AD and BC have gradient $\frac{d}{c}$.

b Both the midpoints are $(a + c, b + d)$.

c The midpoints coincide.

9 a i $P = (1, 4)$, $Q = (-1, 0)$ and $R = (3, 2)$, $BQ: x - y + 1 = 0$, $CR: y - 2 = 0$, $AP: x = 1$

ii The medians intersect at $(1, 2)$.

b i $P(-3a, 3c - 3b)$, $Q(3a, 3c + 3b)$, $R(0, 0)$

ii The median passing through B is $3a(y + 6b) = (c + 3b)(x + 6a)$. The median passing through A is $-3a(y - 6b) = (c - 3b)(x - 6a)$.

iii The medians intersect at $(0, 2c)$.

10 a gradient $AB = 0$, gradient $BC = \frac{c}{b + a}$, gradient $CA = \frac{c}{b - a}$.

b perpendicular bisector of AB: $x = 0$, of BC: $c(c - y) = (b + a)(x - b + a)$, of AC: $c(c - y) = (b - a)(x - b - a)$

c They all meet at $\left(0, \frac{c^2 + b^2 - a^2}{c}\right)$.

d Any point on the perpendicular bisector of an interval is equidistant from the endpoints of that interval.

Chapter 7 review exercise

1 a $\left(8, 6\frac{1}{2}\right)$ **b** $-\frac{5}{12}$ **c** 13

2 a $AB = 5$, $BC = \sqrt{2}$, $CA = 5$

b isosceles

3 a $P(3, 7)$, $Q(6, 5)$, $R(3, -3)$, $S(0, -1)$

b PQ and RS have gradient $-\frac{2}{3}$, QR and SP have gradient $\frac{8}{3}$.

c parallelogram

4 a $C = (-1, 1)$, $r = \sqrt{45} = 3\sqrt{5}$

b $PC = \sqrt{53}$, no

5 a $m_{LM} = -2$, $m_{MN} = -\frac{8}{9}$, $m_{NL} = \frac{1}{2}$

b $m_{LM} \times m_{NL} = -1$

6 a −1 **b** $a = 8$

c $Q = (7, -4)$

d $d^2 = 16$, so $d = 4$ or -4.

7 a $2x + y - 5 = 0$ **b** $2x - 3y + 9 = 0$

c $x + 7y = 0$ **d** $3x + y + 8 = 0$

e $x\sqrt{3} - y - 2 = 0$

8 a $b = -\frac{7}{6}$, $m = \frac{5}{6}$, $\alpha \doteqdot 39°48'$

b $b = \frac{3}{4}$, $m = -1$, $\alpha = 135°$

9 a $8x - y - 24 = 0$ **b** $5x + 2y - 21 = 0$

10 a No; $m_{LM} = -\frac{1}{3}$ and $m_{MN} = -\frac{5}{12}$.

b Yes; they all pass through $(2, 5)$.

11 a Yes; the 2nd and 3rd lines have gradients $\frac{3}{2}$ and $-\frac{2}{3}$ and so are perpendicular.

b Trapezium; the 1st and 3rd lines are parallel.

12 a $A = (6, 0)$, $B = \left(0, 7\frac{1}{2}\right)$

b $22\frac{1}{2}$ square units

13 a $m_{AB} = -\frac{3}{4}$, $AB = 10$, $M = (6, 5)$

c $C = (15, 17)$ **d** $AC = BC = 5\sqrt{10}$

e $75\,u^2$

f $\sin\theta = \frac{3}{5}$, $\theta \doteqdot 36°52'$

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Chapter 8

Exercise 8A

1 a The factors are $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243$.
b Population in 2010 = 810000, population in 2020 = 2430000, so the decade was 2010–2020.

2 a 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096
b i 1, 3, 9, 27, 81, 243, 729
ii 1, 5, 25, 125, 625, 3125
iii 1, 6, 36, 216 **iv** 1, 7, 49, 343
c i 1, 4, 16, 64, 256, 1024, 4096
ii 1, 8, 64, 512, 4096

3 a 64 **b** $\frac{4}{9}$ **c** $\frac{8}{27}$
d $\frac{81}{10000}$ **e** $\frac{16}{49}$ **f** $\frac{5}{9}$
g 1 **h** $\frac{1}{5}$ **i** $\frac{1}{11}$
j $\frac{1}{36}$ **k** $\frac{1}{100}$ **l** $\frac{1}{27}$
m $\frac{1}{125}$ **n** $\frac{1}{32}$ **o** $\frac{1}{1000000}$

4 a 11 **b** $\frac{7}{2}$ or $3\frac{1}{2}$ **c** $\frac{2}{7}$
d $\frac{23}{10}$ or $2\frac{3}{10}$ **e** 10 **f** 100
g 50 **h** $\frac{1}{25}$ **i** 125
j 16 **k** 1000000 **l** $\frac{9}{4}$
m $\frac{16}{81}$ **n** $\frac{25}{4}$ **o** 1

5 a 2^{14} **b** a^{15} **c** $9^0 = 1$ **d** x^2
e $a^0 = 1$ **f** 8 **g** 7^5 **h** a^{-2}
i 2^{16} **j** 1 **k** x^{12} **l** y^{11}
m y^{-11} **n** x^{15} **o** x^{15} **p** z^{14}
q a^{-6} **r** a^{-6} **s** 5^{-28} **t** 2^{16}

6 a $9x^2$ **b** $125a^3$ **c** $64c^6$
d $81s^4t^4$ **e** $49x^2y^2z^2$ **f** $\frac{1}{x^5}$
g $\frac{9}{x^2}$ **h** $\frac{y^2}{25}$ **i** $\frac{49a^2}{25}$
j $\frac{27x^3}{8y^3}$

7 a $\frac{1}{9}$ **b** $\frac{1}{x}$ **c** $\frac{1}{b^2}$
d $-\frac{1}{a^4}$ **e** $\frac{1}{7x}$ **f** $\frac{7}{x}$
g $-\frac{9}{x}$ **h** $\frac{1}{9a^2}$ **i** $\frac{3}{a^2}$
j $\frac{4}{x^3}$

8 a $\frac{2}{3}$ **b** $\frac{3}{8}$ **c** $\frac{4}{25}$ **d** $\frac{27}{1000}$
e 5 **f** $\frac{4}{9}$ **g** $\frac{4}{25}$ **h** 400

9 a $3\,\text{km}^3$
b $(10^3 \times 10^3)^3 = 10^{18}$
c 3×10^{18}

10 a x^{-1} **b** $-x^{-2}$
c $-12x^{-1}$ **d** $9x^{-2}$
e $-x^{-3}$ **f** $12x^{-5}$
g $7x^{-3}$ **h** $-6x^{-1}$
i $\frac{1}{6}x^{-1}$ **j** $-\frac{1}{4}x^{-2}$

11 a $x = -1$ **b** $x = -3$ **c** $x = -1$ **d** $x = -1$
e $x = 0$ **f** $x = 2$ **g** $x = -1$ **h** $x = -2$
i $x = \frac{10}{13}$ or $-\frac{10}{13}$ **j** $x = 2$ or -2
k $x = \frac{1}{3}$ **l** $x = \frac{9}{8}$ or $-\frac{9}{8}$
m $x = 6$ **n** $x = 8$ **o** $x = -4$ **p** $x = 2$

12 a 2^{x+3} **b** 3^{x+1} **c** $\frac{1}{7^x}$ **d** $\frac{5^{2x}}{5^3}$
e 10^{6x} **f** $\frac{1}{5^{8x}}$ **g** 6^{14x} **h** $\frac{2^{3x}}{2^4}$

13 a x^6y^4 **b** $\frac{y}{x^2}$ **c** $\frac{21a^3}{x}$ **d** $\frac{1}{3st^2}$
e $\frac{7x}{y^2}$ **f** $\frac{5b^{10}}{4a^6}$ **g** $\frac{s^6}{y^9}$ **h** $\frac{c^2}{5d^3}$
i $27x^8y^{17}$ **j** $\frac{2a^7}{y^{15}}$ **k** $5s^5$ **l** $\frac{250x^8}{y^{12}}$

14 a $x^2 + 2 + \frac{1}{x^2}$ **b** $x^2 - 2 + \frac{1}{x^2}$ **c** $x^4 - 2 + \frac{1}{x^4}$

15 a $\frac{b - a}{ab}$ **b** $\frac{y}{y + 1}$ **c** $\frac{x^2y^2}{y^2 - x^2}$
d $\frac{ab}{b - a}$ **e** $\frac{x^3 - y^3}{x^3y^3}$ **f** $\frac{1}{a + 1}$

16 a 2^{6n} **b** 81 **c** 2^{3x}
d $2^{2x}3^{2x}$ (or 6^{2x}) **e** $5^{4n-4}2^{4n-5}$ **f** 2^x3^{1-x}

17 a 50×7^n **b** 26 **c** $124 \times 5^{n-3}$
d 7 **e** $7 \times 2^{2n-1}$ **f** 2^n

18 $\frac{3^n}{2}$ **b** $\frac{1}{3^x}$ **c** -2^n3^n

19 a Take the reciprocal, 5.97×10^{26}
b $5.73 \times 10^{-45}\,\text{m}^3$
c $2.9 \times 10^{17}\,\text{kg/m}^3$

Exercise 8B

1 a 6 **b** 4 **c** 10 **d** 125 **e** 9
f 8 **g** 81 **h** 16 **i** 32 **j** 8

2 a $\frac{1}{2}$ **b** $\frac{5}{7}$ **c** $\frac{3}{2}$ **d** $\frac{81}{16}$
e $\frac{4}{9}$ **f** $\frac{1}{128}$ **g** $\frac{27}{1000}$ **h** $\frac{1}{125}$

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e $\frac{1}{9}$ **f** $\frac{1}{27}$ **g** 2 **h** 5
i 8 **j** $\frac{27}{8}$ **k** $\frac{8}{27}$ **l** $\frac{4}{25}$

4 a x^6 **b** $x^{3\frac{1}{2}}$ **c** $15x^2y^{-1}$
d x **e** $x^{-4\frac{1}{2}}$ **f** $7a^{-1}b^{-1}$
g x^{-4} **h** x^6 **i** $27s^{-6}t^{7\frac{1}{2}}$

5 a $2^1 = 2$ **b** $2^0 = 1$ **c** $2^3 = 8$
d $3^{-1} = \frac{1}{3}$ **e** $25^{\frac{1}{2}} = 5$ **f** $7^0 = 1$
g $3^{-3} = \frac{1}{27}$ **h** $3^{-2} = \frac{1}{9}$ **i** $9^2 = 81$

6 a $x = \frac{1}{2}$ **b** $x = \frac{1}{2}$ **c** $x = \frac{1}{4}$
d $x = \frac{1}{6}$ **e** $x = \frac{1}{2}$ **f** $x = \frac{1}{3}$

7 a $\sqrt{x}$ **b** $\sqrt[3]{x}$
c $7\sqrt{x}$ **d** $\sqrt{7x}$
e $15\sqrt[4]{x}$ **f** $\sqrt{x^3}$ or $\left(\sqrt{x}\right)^3$
g $6\sqrt{x^5}$ or $6\left(\sqrt{x}\right)^5$ **h** $\sqrt[3]{x^4}$ or $\left(\sqrt[3]{x}\right)^4$

8 a $x^{\frac{1}{2}}$ **b** $3x^{\frac{1}{2}}$ **c** $(3x)^{\frac{1}{2}}$ **d** $12x^{\frac{1}{3}}$
e $9x^{\frac{1}{6}}$ **f** $x^{\frac{3}{2}}$ **g** $x^{\frac{9}{2}}$ **h** $25x^{\frac{6}{5}}$

9 a $x^{2\frac{1}{2}}$ **b** $x^{-2\frac{1}{2}}$ **c** $x^{3\frac{2}{3}}$ **d** $x^{\frac{1}{3}}$

10 a 5.765×10^6 **b** 1.261×10^1
c 8.244×10^{-1} **d** 7.943×10^{-3}
e 8.825×10^0 **f** 2.595×10^1
g 7.621×10^{-2} **h** 5.157×10^4

11 a $\$6000 \times (1.03)^0 = \$6000$
b $\$6000 \times (1.03)^1 = \$6180$
c i $\$6000 \times (1.03)^5 \doteqdot \$6960$
ii $\$6000 \times (1.03)^{\frac{1}{2}} \doteqdot \$6090$
iii $\$6000 \times (1.03)^{\frac{7}{2}} \doteqdot \$6650$

12 a 9 **b** 3 **c** $\frac{1}{20}$ **d** $\frac{3}{10}$

13 a $9xy^3$ **b** $35b$ **c** $3s^{\frac{1}{2}}$
d $x^{1\frac{1}{2}}y^{2\frac{1}{2}}$ **e** a **f** $a^{-1}b^2$
g $2xy^{-2}$ **h** p^2q^{-6} **i** x^7

14 a $x^{-\frac{1}{2}}$ **b** $12x^{-\frac{1}{2}}$ **c** $-5x^{-\frac{1}{2}}$ **d** $15x^{-\frac{1}{3}}$
e $-4x^{-\frac{2}{3}}$ **f** $x^{1\frac{1}{2}}$ **g** $5x^{-1\frac{1}{2}}$ **h** $8x^{2\frac{1}{2}}$

15 a $x = -\frac{1}{2}$ **b** $x = -\frac{1}{4}$
c $x = \frac{2}{3}$ **d** $x = -\frac{2}{3}$
e $x = \frac{3}{2}$ **f** $x = -\frac{3}{2}$
g $x = \frac{3}{4}$ **h** $x = -\frac{4}{3}$
i $x = -\frac{1}{2}$ **j** $x = -\frac{2}{3}$

16 a $x + 2 + x^{-1}$ **b** $x - 2 + x^{-1}$ **c** $x^5 - 2 + x^{-5}$

17 a $x^2 + 10 + \frac{25}{x^2}$ **b** $x^4 - 14 + \frac{49}{x^4}$ **c** $9x - 12 + \frac{4}{x}$

18 a $x = -\frac{1}{3}$ **b** $x = \frac{1}{4}$ **c** $-\frac{2}{3}$
d $x = \frac{5}{12}$ **e** $x = -4$ **f** $x = -2$

19 a $b = \frac{1}{343}$ **b** $\frac{1}{11}$ **c** $x = \frac{1}{81}$

20 a $x = 3$ and $y = 4$ **b** $x = 0$ and $y = -1$
c $x = -2$ and $y = \frac{1}{2}$

22 a $3^{\frac{1}{3}} > 2^{\frac{1}{2}}$ **b** $2^{\frac{1}{2}} > 5^{\frac{1}{5}}$ **c** $7^{\frac{3}{2}} < 20$ **d** $5^{\frac{1}{5}} < 3^{\frac{1}{3}}$

23 a $12 < 2^{\frac{11}{3}} < 13$ **b** $13 < 2^{\frac{15}{4}} < 14$

24 $\lim_{x \to 0^+} 0^x = 0$ and $\lim_{x \to 0} x^0 = 1$, so there is no sensible way to define 0^0.

Exercise 8C

1 a because $2^3 = 8$. **b** because $5^2 = 25$.
c because $10^3 = 1000$. **d** so $\log_7 49 = 2$.
e so $\log_3 81 = 4$. **f** so $\log_{10} 100\,000 = 5$.

2 a . . . $x = a^y$ **b** . . . $x = \log_a y$

3 a $10^x = 1000, x = 3$ **b** $10^x = 10, x = 1$
c $10^x = 1, x = 0$ **d** $10^x = \frac{1}{100}, x = -2$
e $3^x = 9, x = 2$ **f** $5^x = 125, x = 3$
g $2^x = 64, x = 6$ **h** $4^x = 64, x = 3$
i $8^x = 64, x = 2$ **j** $7^x = \frac{1}{7}, x = -1$
k $12^x = \frac{1}{12}, x = -1$ **l** $11^x = \frac{1}{121}, x = -2$
m $6^x = \frac{1}{36}, x = -2$ **n** $4^x = \frac{1}{64}, x = -3$
o $8^x = \frac{1}{64}, x = -2$ **p** $2^x = \frac{1}{64}, x = -6$

4 a $x = 7^2 = 49$ **b** $x = 5^3 = 125$
c $x = 2^5 = 32$ **d** $x = 100^3 = 1\,000\,000$
e $x = 7^1 = 7$ **f** $x = 11^0 = 1$
g $x = 13^{-1} = \frac{1}{13}$ **h** $x = 12^{-2} = \frac{1}{144}$
i $x = 5^{-3} = \frac{1}{125}$ **j** $x = 7^{-3} = \frac{1}{343}$
k $x = 2^{-5} = \frac{1}{32}$ **l** $x = 3^{-4} = \frac{1}{81}$

5 a $x^2 = 49, x = 7$ **b** $x^3 = 8, x = 2$
c $x^3 = 27, x = 3$ **d** $x^4 = 10\,000, x = 10$
e $x^2 = 10\,000, x = 100$ **f** $x^6 = 64, x = 2$
g $x^2 = 64, x = 8$ **h** $x^1 = 125, x = 125$
i $x^1 = 11, x = 11$ **j** $x^{-1} = \frac{1}{17}, x = 17$
k $x^{-1} = \frac{1}{6}, x = 6$ **l** $x^{-1} = \frac{1}{7}, x = 7$
m $x^{-2} = \frac{1}{9}, x = 3$ **n** $x^{-2} = \frac{1}{49}, x = 7$
o $x^{-3} = \frac{1}{8}, x = 2$ **p** $x^{-2} = \frac{1}{81}, x = 9$

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6 a 0.301 **b** 1.30 **c** 2.00
d 20.0 **e** 3.16 **f** 31.6
g 0.500 **h** 1.50 **i** −0.155
j −2.15 **k** 0.700 **l** 0.00708

7 a $a^x = a, x = 1$ **b** $x = a^1 = a$
c $x^1 = a, x = a$ **d** $a^x = \frac{1}{a}, x = -1$
e $x = a^{-1} = \frac{1}{a}$ **f** $x^{-1} = \frac{1}{a}, x = a$
g $a^x = 1, x = 0$ **h** $x = a^0 = 1$
i $x^0 = 1$, where x can be any positive number.

8 a 1 **b** −1 **c** 3 **d** −2
e −5 **f** $\frac{1}{2}$ **g** $-\frac{1}{2}$ **h** 0

9 a 1&2 **b** 0&1 **c** 3&4 **d** 5&6

10 a 2 & 3 **b** 1 & 2 **c** 0 & 1
d 9 & 10 **e** 3 & 4 **f** 0 & 1
g 3 & 4 **h** 4 & 5 **i** 2 & 3
j 1 & 2 **k** −1 & 0 **l** −2 & −1
m −2 & −1 **n** −1 & 0 **o** −2 & −1

11 a $7^x = \sqrt{7}, x = \frac{1}{2}$ **b** $11^x = \sqrt{11}, x = \frac{1}{2}$
c $x = 9^{\frac{1}{2}} = 3$ **d** $x = 144^{\frac{1}{2}} = 12$
e $x^{\frac{1}{2}} = 3, x = 9$ **f** $x^{\frac{1}{2}} = 13, x = 169$
g $6^x = \sqrt[3]{6}, x = \frac{1}{3}$ **h** $9^x = 3, x = \frac{1}{2}$
i $x = 64^{\frac{1}{3}} = 4$ **j** $x = 16^{\frac{1}{4}} = 2$
k $x^{\frac{1}{3}} = 2, x = 8$ **l** $x^{\frac{1}{6}} = 2, x = 64$
m $8^x = 2, x = \frac{1}{3}$ **n** $125^x = 5, x = \frac{1}{3}$
o $x = 7^{\frac{1}{2}}$ or $\sqrt{7}$ **p** $x = 7^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{7}}$
q $x^{-\frac{1}{2}} = \frac{1}{7}, x = 49$ **r** $x^{-\frac{1}{2}} = \frac{1}{20}, x = 400$
s $4^x = \frac{1}{2}, x = -\frac{1}{2}$ **t** $27^x = \frac{1}{3}, x = -\frac{1}{3}$
u $x = 121^{-\frac{1}{2}} = \frac{1}{11}$ **v** $x = 81^{-\frac{1}{4}} = \frac{1}{3}$
w $x^{-\frac{1}{4}} = \frac{1}{2}, x = 16$ **x** $x^{-\frac{1}{4}} = 2, x = \frac{1}{16}$

12 a $\log_{10} 45 \doteqdot 1.7$ **b** $10^{1.7} \doteqdot 50$
c 5 significant figures. $10^{1.653} \doteqdot 44.98$ and $10^{1.6532} \doteqdot 45.00$

13 a i 100 **ii** 50 **iii** 10^{100}
iv 5×10^{99} **v** 1 **vi** 10^{98}
b 332 and 333

Exercise 8D

1 a $\log_6 36 = 2$ **b** $\log_5 25 = 2$
c $\log_{15} 15 = 1$ **d** $\log_{12} 144 = 2$
e $\log_{10} 1000 = 3$ **f** $\log_3 3 = 1$
g $\log_2 8 = 3$ **h** $\log_3 81 = 4$
i $\log_2 8 = 3$ **j** 1
k 2 **l** 0

2 a −2 **b** −3 **c** −2 **d** −2 **e** −2 **f** 1

3 a $3 \log_a 2$ **b** $4 \log_a 2$
c $6 \log_a 2$ **d** $-\log_a 2$
e $-3 \log_a 2$ **f** $-5 \log_a 2$
g $\frac{1}{2} \log_a 2$ **h** $-\frac{1}{2} \log_2 2$

4 a $2 \log_2 3$ **b** $2 \log_2 5$
c $1 + \log_2 3$ **d** $1 + \log_2 5$
e $1 + 2\log_2 3$ **f** $2 + \log_2 5$
g $1 - \log_2 3$ **h** $-1 + \log_2 5$

5 a 3.90 **b** 3.16 **c** 3.32 **d** 5.64
e 0.58 **f** −0.74 **g** −0.58 **h** 6.22

6 a 3 **b** 5 **c** 1.3 **d** n

7 a 100 **b** 7 **c** 3.6 **d** y

8 a 2 **b** 15 **c** −1 **d** 6

9 a $3 \log_a x$ **b** $-\log_a x$
c $\frac{1}{2} \log_a x$ **d** $-2 \log_a x$
e $-2 \log_a x$ **f** $2 \log_a x$
g $8 - 8 \log_a x$ **h** $\log_a x$

10 a $\log_a y + \log_a z$ **b** $\log_a z - \log_a y$
c $4 \log_a y$ **d** $-2 \log_a x$
e $\log_a x + 3 \log_a y$
f $2 \log_a x + \log_a y - 3 \log_a z$
g $\frac{1}{2} \log_a y$ **h** $\frac{1}{2} \log_a x + \frac{1}{2} \log_a z$

11 a 1.30 **b** −0.70 **c** 2.56 **d** 0.15
e 0.45 **f** −0.50 **g** 0.54 **h** −0.35

12 a $6x$ **b** $-x - y - z$
c $3y + 5$ **d** $2x + 2z - 1$
e $y - x$ **f** $x + 2y - 2z - 1$
g $-2z$ **h** $3x - y - z - 2$

13 a $10 = 3^{\log_3 10}$ **b** $3 = 10^{\log_{10} 3}$
c $0.1 = 2^{\log_2 0.1}$ **d** $2 = \log_{10} 100$
e $-4 = \log_3 3^{-4}$ or $\log_3 \frac{1}{81}$
f $\frac{1}{2} = \log_7 7^{\frac{1}{2}}$ or $\log_7 \sqrt{7}$

14 a $\log_{25} 5 = \frac{1}{2}$ **b** $\log_{81} \frac{1}{3} = -\frac{1}{4}$
c $\log_8 \frac{1}{32} = -\frac{5}{3}$ **d** $\log_{\frac{1}{32}} \frac{1}{2} = \frac{1}{5}$

15 a $\frac{1}{2}$ **b** 49 **c** 15 **d** x^n
e $\frac{1}{x}$ **f** $x \times 5^x$ **g** x^x **h** $x^{1/x}$

16 a $x + y = xy$ **b** $x = 1000y$
c $x = y^4$ **d** $x^2y^3 = z^4$
e $2^x = y$ **f** $x = yz^n$
g $64x^3 = y^2$ **h** $(2x + 1)^2 = (2x - 1)^3$

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17 Let $\log_2 3 = \frac{a}{b}$, where a and b are positive whole numbers.
Then $b\log_2 3 = a$
$\log_2 3^b = a$
$3^b = 2^a$.
This is impossible because 3^b is odd and 2^a is even.

Exercise 8E

2 a 2.807 **b** 4.700 **c** −3.837
d 7.694 **e** 0.4307 **f** 1.765
g 0.6131 **h** 0.2789 **i** −2.096
j −7.122 **k** 2.881 **l** 7.213
m 0.03323 **n** 578.0 **o** −687.3

3 a $x = \log_2 15 \doteqdot 3.907$ **b** $x = \log_2 5 \doteqdot 2.322$
c $x = \log_2 1.45 \doteqdot 0.5361$ **d** $x = \log_2 0.1 \doteqdot -3.322$
e $x = \log_2 0.0007 \doteqdot -10.48$ **f** $x = \log_3 10 \doteqdot 2.096$
g $x = \log_3 0.01 \doteqdot -4.192$ **h** $x = \log_5 10 \doteqdot 1.431$
i $x = \log_{12} 150 \doteqdot 2.016$ **j** $x = \log_8 \frac{7}{9} \doteqdot -0.1209$
k $x = \log_6 1.4 \doteqdot 0.1878$ **l** $x = \log_{30} 2 \doteqdot 0.2038$
m $x = \log_{0.7} 0.1 \doteqdot 6.456$ **o** $x = \log_{0.99} 0.01 \doteqdot 458.2$
n $x = \log_{0.98} 0.03 \doteqdot 173.6$

4 a $x > 5$ **b** $x \le 5$
c $x < 6$ **d** $x \ge 4$
e $x > 1$ **f** $x \le 0$
g $x < -1$ **h** $x \le -3$

5 a $x = 1$ or $x = \log_2 7$
b $x = 2$ ($3^x = -1$ has no solutions.)
c i $x = 2$ or $x = 0$ **ii** $x = 0$ or $x = \log_3 4$
iii $x = \log_3 5$ ($3^x = -4$ has no solutions.)
iv The quadratic has no solutions
v $x = 3$
vi $x = 2$ or $x = 0$

6 a $0 < x < 8$ **b** $x \ge 8$
c $x > 1000$ **d** $x \ge 10$
e $x > 1$ **f** $0 < x < 6$
g $0 < x \le 125$ **h** $x > 36$

7 a $x > \log_2 12 \doteqdot 3.58$ **b** $x < \log_2 100 \doteqdot 6.64$
c $x < \log_2 0.02 \doteqdot -5.64$ **d** $x > \log_2 0.1 \doteqdot -3.32$
e $x < \log_5 100 \doteqdot 2.86$ **f** $x < \log_3 0.007 \doteqdot -4.52$
g $x > \log_{1.2} 10 \doteqdot 12.6$ **h** $x > \log_{1.001} 100 \doteqdot 4610$

8 a After 1 year, the price is 1.05 times greater, after 2 years, it is $(1.05)^2$ times greater, and so on.
b $\log_{1.05} 1.5 \doteqdot 8.3$ years

9 a $\log_8 x = \dfrac{\log_2 x}{\log_2 8} = \dfrac{1}{3}\log_2 x$
b $\log_{a^n} x = \dfrac{\log_a x}{\log_a a^n} = \dfrac{1}{n}\log_a x$

10 a $x = 3$ **b** $x = 2$
c $x < 1$ **d** $x \le 9$
e $x = 0$ **f** $x = \frac{1}{5}$
g $x < 4.81$ **h** $x > -2.90$

11 a $x < 33.2$, 33 powers
b $x < 104.8$, 104 powers

12 a $10^2 < 300 < 10^3$ **b** $1 \le \log_{10} x < 2$
c 5 digits **d** 27.96, 28 digits
e $1000\log_{10} 2 = 301.03$, 302 digits

14 a $x = 1$ or $x = \log_4 3 \doteqdot 0.792$
b $x = \log_{10}\frac{1+\sqrt{5}}{2} \doteqdot 0.209$. $\log_{10}\frac{1-\sqrt{5}}{2}$ does not exist because $\frac{1-\sqrt{5}}{2}$ is negative.
c $x = -1$ or $x = \log_{\frac{1}{5}} 2 \doteqdot -0.431$

15 a $\dfrac{\log_{10}47 + 4\log_{10}3}{\log_{10}3} \doteqdot 7.505$
b $\dfrac{-5\log_{10}2 - \log_{10}5}{\log_{10}2} \doteqdot -7.322$
c $\dfrac{\log_{10}6}{2\log_{10}5 - \log_{10}6} \doteqdot 1.256$
d $\dfrac{\log_{10}7 - \log_{10}6 + 3\log_{10}5}{\log_{10}5 + \log_{10}7} \doteqdot 1.401$

16 a $SD = \frac{1}{4}(2^{2x} - 2^{-2x})$, $S + D = 2^x$, $S - D = 2^{-x}$, $S^2 - D^2 = 1$
b $x = \log_2(S + \sqrt{S^2 - 1})$,
$x = \log_2(D + \sqrt{D^2 + 1})$

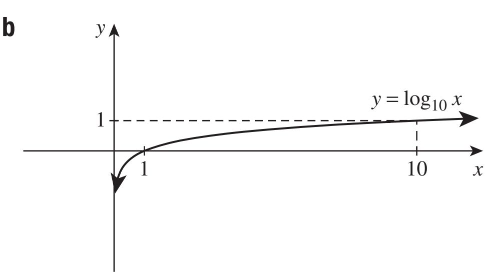
Exercise 8F

1 a

x	0.1	0.25	0.5	0.75	1	2
$\log_{10}x$	−1	−0.60	−0.30	−0.12	0	0.30

x	3	4	5	6	7	8	9	10
$\log_{10}x$	0.48	0.60	0.70	0.78	0.85	0.90	0.95	1

b



2 a i

x	−3	−2	−1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

ii

x	−3	−2	−1	0	1	2	3
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

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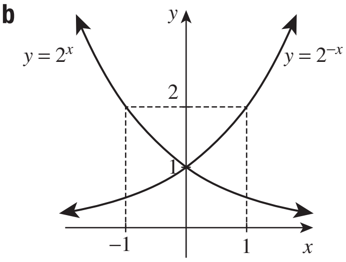
b



c The values of $y = 2^{-x}$ are the values of $y = 2^x$ in reverse order.

d The two graphs are reflections of each other in the y-axis, because x has been replaced with $-x$.

e For both, domain: all real x, range: $y > 0$

f For both, the asymptote is $y = 0$ (the x-axis).

g i 'As $x \to -\infty$, $2^x \to 0$.'

ii 'As $x \to \infty$, $2^x \to \infty$.'

h i 'As $x \to -\infty$, $2^{-x} \to \infty$.'

ii 'As $x \to \infty$, $2^{-x} \to 0$.'

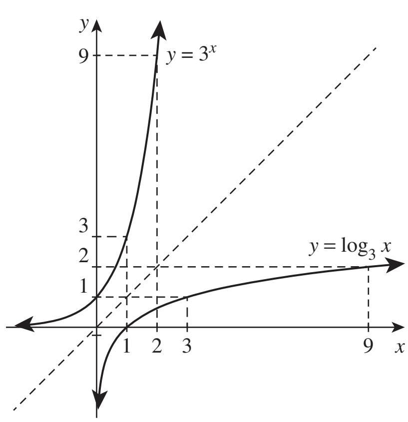
3 i

x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

ii

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
$\log_3 x$	-2	-1	0	1	2

b



c The two rows have been exchanged.

d The two graphs are reflections of each other in the diagonal line $y = x$, because the two functions are inverses of each other.

e i domain: all real x, range: $y > 0$

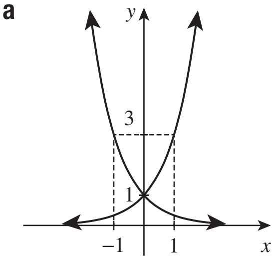
ii domain: $x > 0$, range: all real y

f i $y = 0$ (the x-axis) **ii** $x = 0$ (the y-axis)

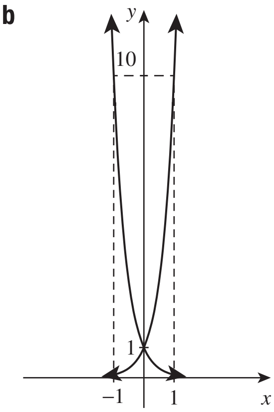
g i 'As $x \to -\infty$, $3^x \to 0$'.

ii 'As $x \to 0^+$, $\log_3 x \to -\infty$'.

4 a



b



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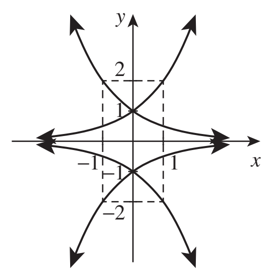
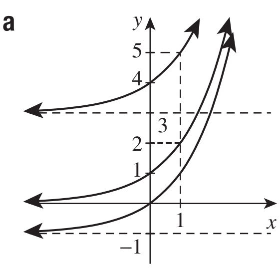


6 a



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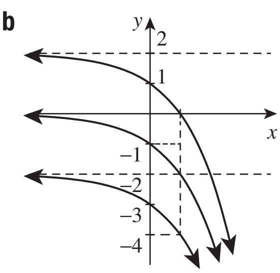

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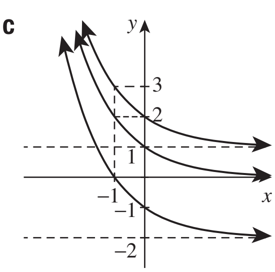

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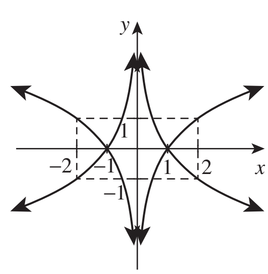
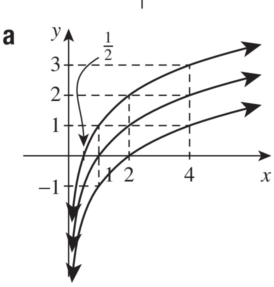


8 a



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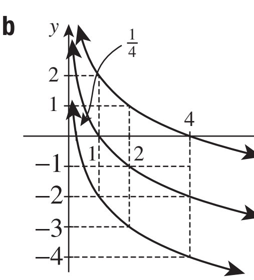

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9 a **i** 4 **ii** $\frac{1}{4}$ **iii** 2.83 **iv** 1.32 **v** 0.66

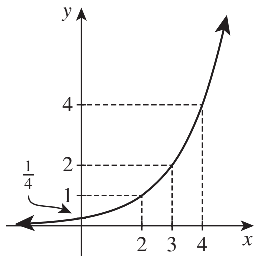
b **i** 1 **ii** 1.58 **iii** 0.26 **iv** -1.32

c **i** $0 \le x \le 2$ **ii** $0 \le x \le 1$

iii $0.58 \le x \le 1.58$ **iv** $-1 \le x \le 1$

d **i** 2 **ii** 1.58 **iii** 0.49 **iv** -0.32

10 a



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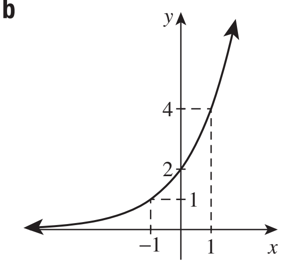

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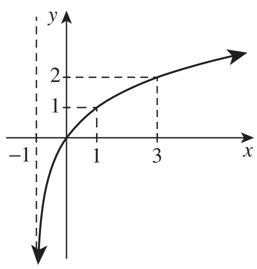

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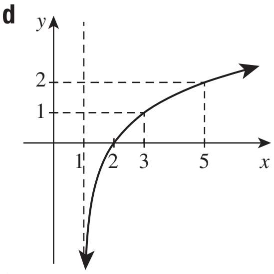

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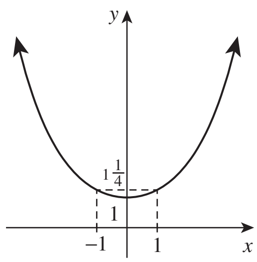

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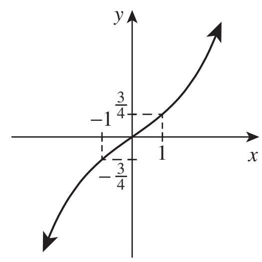
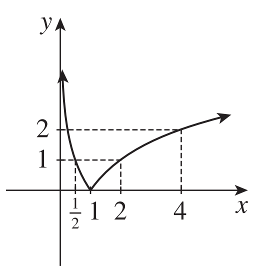


11 a



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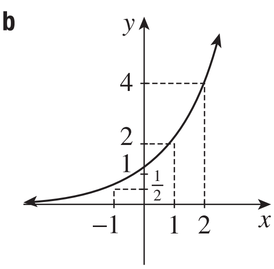

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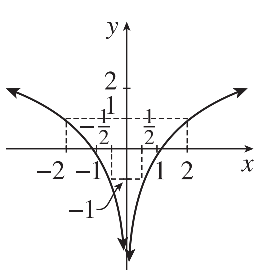

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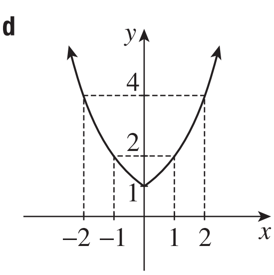

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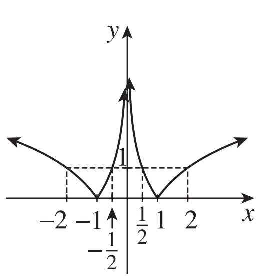

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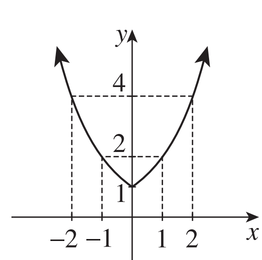
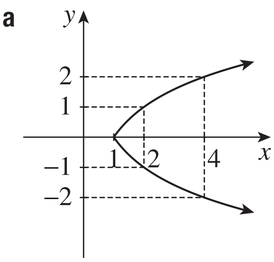


12 a



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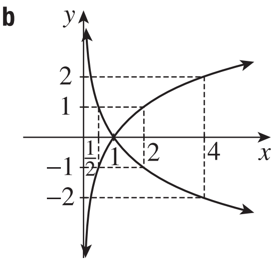

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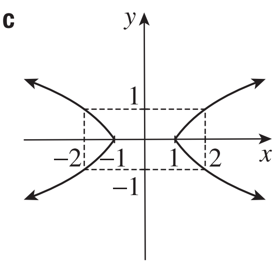

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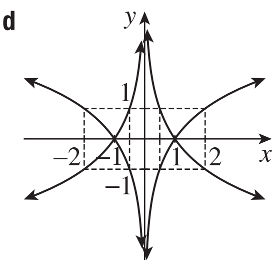
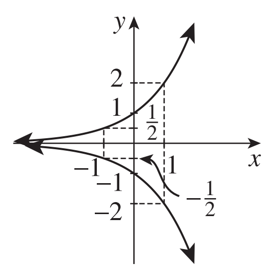


e v



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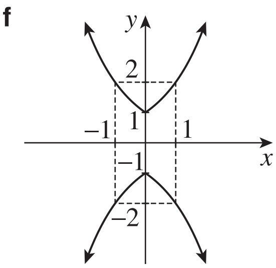


Exercise 8G

1 a 5000, 2594

b $\frac{t}{2} = \log_{10}\frac{Q}{5}$, so $t = 2\log_{10}\frac{Q}{5}$

c 4, 3.419

2 a 60, $20\log_{10}12 = \dfrac{20\log_{10}12}{\log_{10}2} \doteqdot 71.70$

b $\frac{t}{20} = \log_2 2Q$, so $2Q = 2^{\frac{t}{20}}$, so $Q = \frac{1}{2} \times 2^{\frac{t}{20}}$

c 2, 2.378

3 a There are $\frac{n}{30}$ thirty-year intervals in n years.

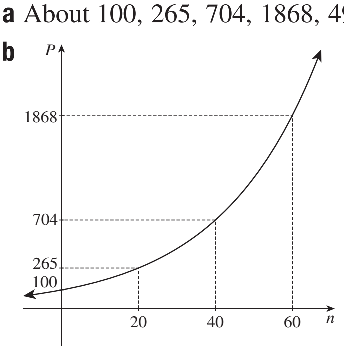
b **i** 24 000 000 **ii** 30 000 000

c **i** 120 years

ii $30\log_2 20 \doteqdot 130$ years

4 a About 100, 265, 704, 1868, 4956, 13 150

b



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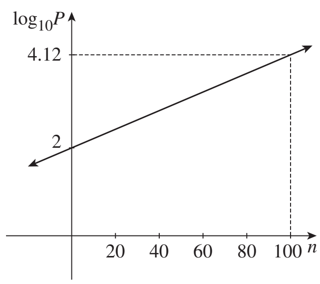
c The values are about 2, 2.42, 2.85, 3.27, 3.70, 4.12

d



e The new graph is a straight line, and $\log_{10}P$ is a linear function of n.

5 a $\frac{n}{2}$ is the number of 2-year periods.

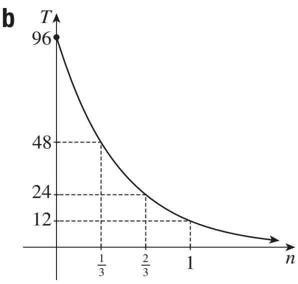
b $D = 2^{20}D_0 \doteqdot 1\,050\,000\,D_0$

c $2^{\frac{n}{2}} = 10^7$, so $\frac{n}{2} = \log_2 10^7$,

so $n = 2\log_2 10^7 \doteqdot 47$ years, that is, in 2022.

6 a $3n$ is the number of 20-minutes periods in n hours.

b



c $96 \times \left(\frac{1}{2}\right)^6 = 1\frac{1}{2}^\circ$C

d $3n = \log_{\frac{1}{2}}\frac{T}{96}$, so $n = \frac{1}{3}\log_{\frac{1}{2}}\frac{T}{96}$. (Alternatively, $n = -\frac{1}{3}\log_2\frac{T}{96}$.)

e $n = \frac{1}{3}\log_{\frac{1}{2}}\frac{1}{96} = 2.1949\ldots$ hours

$\doteqdot$ 2 hours and 12 minutes

7 a The mass halves every 700 000 000 years.

b When $n = 4$ billion, $\frac{n}{700\,000\,000} = \frac{40}{7}$, so

$M = M_0 \times \left(\frac{1}{2}\right)^{\frac{40}{7}} \doteqdot 1.9\%$ of M_0

c When $n = -4.5$ billion, $\frac{n}{700\,000\,000} = -\frac{45}{7}$, so

$M = M_0 \times \left(\frac{1}{2}\right)^{-\frac{45}{7}} \doteqdot 86\,M_0$

8 a 96 dB

b $I = I_r \times 10^{\frac{n}{10}}$, 3.16×10^{-5} W/m^2

c $10^{3.6} \doteqdot 3980$ times

d $70 - 10\log_{10}1600 \doteqdot 38$ dB

9 a 1000 **b** $1000^{\frac{3}{2}} \doteqdot 32\,000$

c Ratio of shaking amplitudes is 10^5, ratio of energies released is about 3.2×10^7.

10 a $[\mathrm{H}^+] = 10^{-\mathrm{pH}}$ **b** About 10^{-7} mol/L

c About 10^{-2} mol/L, about 100 000 times more acidic than water

d About 7.94×10^{-9} mol/L, about 12.6 times more alkaline than water

11 a C 261.63 Hz, C♯ 277.18 Hz, E 329.63 Hz

b C 264 Hz, C♯275 Hz, E 330 Hz

c C 2.37 Hz, C♯2.18 Hz, E 0.37 Hz

d A′ would be $220 \times \left(\frac{5}{4}\right)^3 \doteqdot 429.69$ Hz, beating at 10.31 Hz with A′440

e Meantone:245.97 Hz, equal temperament: 246.94 Hz, beating at about 1 Hz

Chapter 8 review exercise

1 a 125	**b** 256	**c** 1 000 000 000	**d** $\frac{1}{17}$
e $\frac{1}{81}$	**f** $\frac{1}{8}$	**g** $\frac{1}{81}$	**h** 1
i $\frac{8}{27}$	**j** $\frac{12}{7}$	**k** $\frac{36}{25}$	**l** 6
m 3	**n** 4	**o** 243	**p** $\frac{2}{7}$
q 1	**r** $\frac{5}{3}$	**s** $\frac{4}{9}$	**t** $\frac{1000}{27}$
2 a x^{-1}	**b** $7x^{-2}$	**c** $-\frac{1}{2}x^{-1}$	**d** $x^{\frac{1}{2}}$
e $30x^{\frac{1}{2}}$	**f** $4x^{-\frac{1}{2}}$	**g** yx^{-1}	**h** $2yx^{\frac{1}{2}}$
3 a x^{20}	**b** $\frac{81}{a^{12}}$	**c** $5x^3$	**d** $\frac{2r}{t^2}$

4 a x^3y^3	**b** $60xy^3z^5$	**c** $18x^{-1}y^{-2}$
d $4a^3b^3c^{-1}$	**e** x^2y^{-2}	**f** $2x^{-3}y$
g m^2n^{-1}	**h** $72s^9t^3$	**i** $8x^3y^{-3}$

5 a 4	**b** 2	**c** −1	**d** −5
e 2	**f** 3	**g** $\frac{1}{2}$	**h** $\frac{1}{3}$

6 a $2^x = 8, x = 3$	**b** $3^x = 9, x = 2$
c $10^x = 10\,000, x = 4$	**d** $5^x = \frac{1}{5}, x = -1$
e $7^x = \frac{1}{49}, x = -2$	**f** $13^x = 1, x = 0$
g $9^x = 3, x = \frac{1}{2}$	**h** $2^x = \sqrt{2}, x = \frac{1}{2}$
i $7^2 = x, x = 49$	**j** $11^{-1} = x, x = \frac{1}{11}$
k $16^{\frac{1}{2}} = x, x = 4$	**l** $27^{\frac{1}{3}} = x, x = 3$
m $x^2 = 36, x = 6$	**n** $x^3 = 1000, x = 10$
o $x^{-1} = \frac{1}{7}, x = 7$	**p** $x^{\frac{1}{2}} = 4, x = 16$

7 a 1 **b** 2 **c** 2 **d** −2 **e** 2 **f** 0

8 a $\log_a x + \log_a y + \log_a z$ **b** $\log_a x - \log_a y$

c $3\log_a x$ **d** $-2\log_a z$

e $2\log_a x + 5\log_a y$

f $2\log_a y - \log_a x - 2\log_a z$

g $\frac{1}{2}\log_a x$

h $\frac{1}{2}\log_a x + \frac{1}{2}\log_a y + \frac{1}{2}\log_a z$

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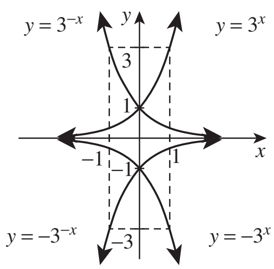
9 a 1 & 2 **b** 2 & 3 **c** 4 & 5 **d** 5 & 6
e -1 & 0 **f** -3 & -2 **g** -4 & -3 **h** -2 & -1

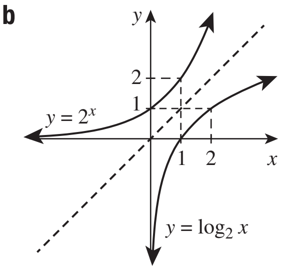
10 a 2.332 **b** -2.347 **c** 2.010 **d** 9.966
e -0.9551 **f** 69.66 **g** -3 **h** 687.3

11 a 3.459 **b** -4.644 **c** 3.010 **d** -0.3645
e 161.7 **f** -161.7 **g** 10.32 **h** 458.2

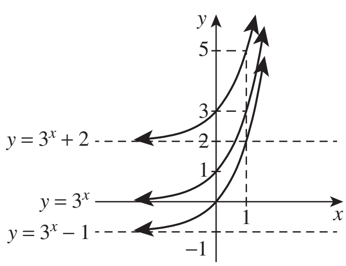
12 a



b



c



13 a There are $\frac{n}{4}$ four-hour periods in n hours.

b i 800 **ii** $100 \times 2^{3.25} \doteqdot 950$

c $\frac{n}{4} = \log_2 \frac{P}{100}$, so $n = 4 \log_2 \frac{P}{100}$.

d $4 \log_2 100000 \doteqdot 66$ hours

Chapter 9

Exercise 9A

1 The values of $f'(x)$ should be about $-4, -3, -2, -1, 0, 1, 2, 3, 4$. The graph of $y = f'(x)$ should approximate a line of gradient 2 through the origin; its exact equation is $f'(x) = 2x$.

2 Answers the same as Question 1

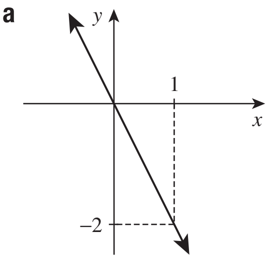
3 The values of $f'(x)$ should be about $1\frac{1}{2}, 0, -0.9, -1.2, -0.9, 0, 1\frac{1}{2}$. The graph of $y = f'(x)$ is a parabola crossing the x-axis at $x = -2$ and $x = 2$.

4 The eventual graph of $f'(x)$ is a parabola with its vertex at the origin. Depending on the software, you may be able to see that it is $y = 3x^2$.

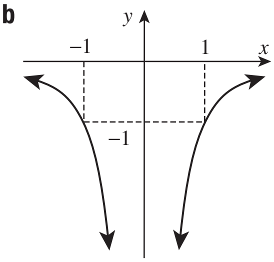
5 a 2 **b** -3 **c** $\frac{1}{2}$ **d** 0 **e** a
f $\frac{2}{3}$ **g** $-\frac{5}{4}$ **h** $-\frac{10}{3}$ **i** 0

6 a $\frac{7}{2}$ **b** 12 **c** 0

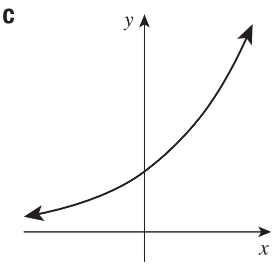
7 a



b



c



8 a $-\frac{4}{3}$ **b** $-\frac{3}{4}$ **c** 0 **d** $\frac{4}{3}$ **e** $\frac{3}{4}$

9 a $\frac{-x}{\sqrt{1 - x^2}}$ **b** $\frac{x}{\sqrt{1 - x^2}}$ **c** $\frac{-x}{\sqrt{4 - x^2}}$

10 a $\frac{-x}{\sqrt{9 - x^2}}$ **b** $\frac{x}{\sqrt{16 - x^2}}$

c $\frac{7 - x}{\sqrt{36 - (x - 7)^2}}$ **d** $\frac{x - 1}{\sqrt{2x - x^2}}$

Exercise 9B

3 c At A, $f'(1) = -2$.

d At B, $f'(3) = 2$; at C, $f'(2) = 0$.

4 a $\frac{f(x + h) - f(x)}{h} = 3$. Trivially this has limit 3 as $h \to 0$.

b $\frac{f(x + h) - f(x)}{h} = m$. Trivially this has limit m as $h \to 0$.

c $\frac{f(x + h) - f(x)}{h} = 0$. Trivially this has limit 0 as $h \to 0$.

5 a i $2x + h, 2x$ **ii** 4 **iii** $(0, 10)$

b i $2x + h + 6, 2x + 6$
ii 10
iii $(-3, -7)$

c i $4x + 2h - 20, 4x - 20$
ii -12
iii $(5, -50)$

d i $-8x - 4h, -8x$
ii -16
iii $(0, 9)$

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6 b **i** -10 **ii** 10 **iii** 0 **iv** -1 **v** $+1$

c 90°

7 b **i** $(3, -6)$ **ii** $(2, -6)$ **iii** $(5, 0)$

iv $(0, 0)$ **v** $\left(2\frac{1}{2}, -6\frac{1}{4}\right)$

8 b At $(6, 0)$, $f'(6) = 5$. At $(1, 0)$, $f'(1) = -5$.

c $A = (0, 6)$, $m = f'(0) = -7$, $B = (7, 6)$.

d $\left(3\frac{1}{2}, -6\frac{1}{4}\right)$

9 b $x = -\dfrac{b}{2a}$

c It is the axis of symmetry of the parabola.

10 a ii $f'(x) = 3x^2$

b ii $f'(x) = 4x^3$

11 b $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-2x - h}{(x+h)^2x^2}$

c As $x \to 0^+$ and as $x \to 0^-$, the gradient decreases without bound, so the tangents slope more and more steeply backwards. As $x \to \infty$ and as $x \to -\infty$, the gradient approaches zero, so the tangents become more and more horizontal.

12 c $f'(x) = \dfrac{1}{2\sqrt{x}}$

d As $x \to 0^+$, the gradient increases without bound, so the tangents slope more and more steeply.
As $x \to \infty$, the gradient approaches zero, so the tangents become more and more horizontal.

13 a $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-2x - h}{(x+h)^2x^2}$

b $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-1}{\sqrt{x(x+h)}\left(\sqrt{x} + \sqrt{x+h}\right)}$

14 The line is a tangent when the two points coincide, that is, when $m = 2a$, so the gradient of the tangent is twice the x-coordinate.

15 They meet at $x = \frac{1}{2}\left(m + \sqrt{m^2 + 4b}\right)$ and at $x = \frac{1}{2}\left(m - \sqrt{m^2 + 4b}\right)$. The line is a tangent when these coincide, that is, when $m^2 + 4b = 0$, in which case the tangent at $x = \frac{1}{2}m$ has gradient m, which is twice the x-coordinate.

Exercise 9C

1 a $7x^6$ **b** $45x^4$

c $2x^5$ **d** $6x - 5$

e $4x^3 + 3x^2 + 2x + 1$ **f** $-3 - 15x^2$

g $2x^5 - 2x^3 + 2x$ **h** $x^3 + x^2 + x + 1$

2 a $12 - 12x$ **b** $3x^2 + 1$

c $6x - 6x^2 - 16x^3$ **d** $2x + 2$

e $8x$ **f** $2x - 14$

g $4x^3 + 12x$ **h** $3x^2 - 28x + 49$

i $3x^2 - 10x + 3$

3 a $4ax^3 - 2bx$ **b** $2a^2x - 10a$

c $2ka^2x$ **d** $\ell x^{\ell - 1}$

e $(5a + 1)x^{5a}$ **f** $3b^2x^{3b-1}$

4 a $1, -1, 45^\circ, 135^\circ$ **b** $-1, 1, 135^\circ, 45^\circ$

c $-6, \frac{1}{6}$, about $99^\circ 28'$, $9^\circ 28'$

5 a $y = -6x + 14$, $x - 6y + 47 = 0$

b $y = 4x - 21$, $x + 4y - 18 = 0$

c $y = -8x + 15$, $x - 8y + 120 = 0$

d $y = -1$, $x = 4$

6 a $(2, 8)$ and $y = 8$

b $(2, 8)$ and $y = 8$, $(-2, 40)$ and $y = 40$

c None

d $(2a, 4a^2)$ and $y = 4a^2$

e $(0, 0)$ and $y = 0$, $(1, -1)$ and $y = -1$, $(-1, -1)$ and $y = -1$

f None, because $5x^4 + 1$ is always positive.

7 $f'(x) = 3x^2$, which is positive for $x \neq 0$ and zero for $x = 0$. The tangent crosses the curve at the origin.

8 $y = -2x + 5$, $y = 2x + 5$, $(0, 5)$

9 $2x + y = 16$, $A = (8, 0)$, $B = (0, 16)$, $AB = 8\sqrt{5}$, $|\Delta OAB| = 64$ square units

10 $y = -2x + 10$, $x - 2y + 15 = 0$, $A = (5, 0)$, $B = (-15, 0)$, $AB = 20$, $|\Delta AKB| = 80$ square units

11 $y = 3x - 2$, $x + 3y = 4$, $P = (0, -2)$, $Q = \left(0, 1\frac{1}{3}\right)$, $|\Delta QUP| = 1\frac{2}{3}$ square units

12 a $f'(9) = 14$, $f'(-5) = -14$

b The vertex is $(2, -49)$, where 2 is the mean of 9 and -5. The parabola has line symmetry in the vertical line through the vertex, and this symmetry exchanges the two x-intercepts and reflects a line with gradient m to a line with gradient $-m$.

13 $f'(x) = 3x^2 + a$, $x = \sqrt{\dfrac{-a}{3}}$ and $x = -\sqrt{\dfrac{-a}{3}}$, $a \le 0$ (but no restriction on b)

14 a $G'(t) = 3t^2 - 8t + 6$, $G'(3) = 9$

b $\ell'(h) = 20h^3$, $\ell'(2) = 160$

c **i** $2ak - a^2$ **ii** a^2

iii $-a^2$ **iv** $2a^2$

15 The tangent has gradient $2a - 6$, the normal has gradient $\dfrac{1}{6 - 2a}$.

a 4 **b** $2\frac{7}{8}$ **c** $2\frac{1}{2}$

d $3 - \frac{1}{2}\sqrt{3}$ **e** $3\frac{1}{3}$ **f** $2\frac{1}{4}$

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16 a -7 **b** 6 **c** 1
d 5 **e** 6 or -6

17 a The tangents are $y = 2ax - a^2$ and $y = 2bx - b^2$. They meet at $K = \left(\frac{1}{2}(a+b), ab\right)$.
b The y-coordinate ab of K is positive when a and b have the same sign, that is, when A and B are both on the right of the y-axis, or both on the left of the y-axis. The sketch of the parabola should make this result obvious.

18 a $b^2 > 3ac$ **b** $b^2 = 3ac$ **c** $b^2 < 3ac$

20 a $y = (2at + b)x - at^2 + c$. a and c must have the same sign, or $c = 0$ (b is arbitrary).
$y = (2\sqrt{ac} + b)x$ and $y = (-2\sqrt{ac} + b)x$
b Points of contact: $\left(\sqrt{\frac{c}{a}}, 2c + b\sqrt{\frac{c}{a}}\right)$ and $\left(-\sqrt{\frac{c}{a}}, 2c - b\sqrt{\frac{c}{a}}\right)$, whose midpoint is $(0, 2c)$.
c $2\sqrt{\frac{c^3}{a}}$ square units

Exercise 9D

1 a $3x^2 + 6x + 6, 3$ **b** $4x^3 + 2x + 8, 2$
c 0, 0 **d** $4x - 5, -9$

2 a i $\frac{dy}{dx} = 6x^5 + 2, \frac{d^2y}{dx^2} = 30x^4, \frac{d^3y}{dx^3} = 120x^3$
ii $\frac{dy}{dx} = 10x - 5x^4, \frac{d^2y}{dx^2} = 10 - 20x^3,$
$\frac{d^3y}{dx^3} = -60x^2$
iii $\frac{dy}{dx} = 4, \frac{d^2y}{dx^2} = 0, \frac{d^3y}{dx^3} = 0$
b i $f'(x) = 30x^2 + 1, f''(x) = 60x, f'''(x) = 60,$
$f^{(4)}(x) = 0$
ii $f'(x) = 8x^3, f''(x) = 24x^2, f'''(x) = 48x,$
$f^{(4)}(x) = 48$
iii $f'(x) = 0, f''(x) = 0, f'''(x) = 0, f^{(4)}(x) = 0$
c i 6 times **ii** $n + 1$ times

3 a $(1, -4)$
b $(1, -3)$ and $(-1, 3)$
c none

4 a 1 **b** 0, 3, -3 **c** none

5 a $\frac{dy}{dx} = 9x^2 - 9 = 9(x - 1)(x + 1)$, which is zero when $x = 1$ or $x = -1$
b $\frac{dy}{dx} = 5x^2 + 2x = x(5x + 2)$, which is zero when $x = -\frac{2}{5}$ (y) is undefined when ($x = 0$)

6 a $y = -6x, y = \frac{1}{6}x$
b $y = 2x + 2, x + 2y + 1 = 0$
c $y = 0, x = 1$

7 a They all have derivative $3x^2 + 7$. First to second, shift down 10. First to third, shift down $7\frac{1}{2}$. First to fourth, shift up 96.
b The third has derivative $-2x^3 + 6x$. The other three have derivative $2x^3 + 6x$.

8 $63°26'$ at $(-1, 0)$, $116°34'$ at $(1, 0)$

9 a $\left(1, -6\frac{2}{3}\right), \left(-1, -7\frac{1}{3}\right)$
b $\left(-1, \frac{2}{3}\right)$ **c** $\left(-\frac{1}{2}\sqrt{3}, 1\frac{3}{4}\right)$

10 a $y = 2px + 9 - p^2$
b Substitute $(0, 0)$. At $(3, 18)$ the tangent is $y = 6x$, and at $(-3, 18)$ the tangent is $y = -6x$.

11 a $y = (2t - 10)x - t^2 + 9, t = 3$ and $y = -4x$, or $t = -3$ and $y = -16x$
b $y = (2t + 15)x - t^2 + 36, t = 6$ and $y = 27x$, or $t = -6$ and $y = 3x$

12 a $y = 2(t + 1)x - t^2 - 8$
b $(1, -5), (3, 7)$

13 a $y = \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$
b $y = 2x^3 - 7x + C$
c $y = \frac{5}{4}x^4 + x^3 - 4x + C$
d $y = 2x^5 - 4x^3 - 24x + C$

14 a $b = 7, c = 0$ **b** $b = -2, c = -3$
c $b = -10, c = 25$ **d** $b = -1, c = -2$
e $b = -9, c = 14$ **f** $b = -\frac{17}{3}, c = 4$

15 The tangent is $y = x$.

16 At $(2, 1)$ the gradient is 2, which is perpendicular to $x + 2y = 4$; at $\left(-\frac{1}{2}, \frac{9}{4}\right)$ the gradient is -3.

17 a $y = 2ax - a^2, U = \left(\frac{1}{2}a, 0\right), V = (0, -a^2)$
b $T = (5, 25)$ or $(-5, 25)$

18 At $(1, -3)$ the tangent is $x + y + 2 = 0$, at $(-1, 3)$ the tangent is $x + y - 2 = 0$. The first tanget is the line given in the question.

19 a A and B are $(2, -4)$ and $(-3, 6)$.
b $M = \left(-\frac{1}{2}, 1\right)$, gradients of tangent at T and chord AB are both -2.

20 $\frac{dP}{dx} = 2tx + 3u, \frac{dP}{du} = 6tu + 3x,$
$\frac{dP}{dt} = x^2 + 3u^2 + 1$

22 b $y = x^2 - 6x$ and $y = \frac{25}{81}x^2 + \frac{2}{9}x$
c $y = x^2 - x - 6$

23 a $y - (at^3 + 2bt^2 + ct + d)$
$= (3at^2 + 2bt + c)(x - t)$

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b The condition for P to lie on the tangent at T is
$y_0 - (at^3 + bt^2 + ct + d)$
$= (3at^2 + 2bt + c)(x_0 - t)$.
This is a cubic in t, and every cubic has at least one solution. (Why?)

Exercise 9E

1 $\dfrac{du}{dx} = 2x, \dfrac{dy}{du} = 5u^4,$
$\dfrac{dy}{dx} = 5(x^2 + 9)^4 \times 2x = 10x(x^2 + 9)^4$

2 a $12(3x + 7)^3$
b $-28(5 - 4x)^6$
c $24x(x^2 + 1)^{11}$
d $-64x(7 - x^2)^3$
e $9(2x + 3)(x^2 + 3x + 1)^8$
f $-18(3x^2 + 1)(x^3 + x + 1)^5$

3 a $25(5x - 7)^4$
b $49(7x + 3)^6$
c $180(5x + 3)^3$
d $-21(4 - 3x)^6$
e $12\left(\frac{1}{2}x - 1\right)^3$
f $-\frac{8}{9}\left(5 - \frac{1}{3}x\right)^3$

4 $20(5x - 2)^3, 300(5x - 2)^2, 3000(5x - 2),$ $15000, 0, 0$

5 $2(x - 3)$

6 a $2\frac{1}{2}$ and 1 **b** 2 and $1\frac{1}{2}$

7 a $y = 20x - 19, x + 20y = 21$
b $y = 24x - 16, x + 24y = 193$

8 a $6(x - 5)^5$, $(5, 4)$
b $-14(x - 5)$, $(5, 24)$
c $2a(x - h)$, (h, k)
d $6x(x^2 - 1)^2$, $(0, -1)$, $(1, 0)$, $(-1, 0)$
e $8(x - 2)(x^2 - 4x)^3$, $(0, 0)$, $(2, 256)$, $(4, 0)$
f $10(x + 1)(2x + x^2)^4$, $(0, 0)$, $(-2, 0)$, $(-1, -1)$

9 a 4 or 8
b -17

10 a $a = \frac{1}{16}, b = 12$
b $a = \frac{1}{9}, b = -10$

11 a $P = \left(7\frac{1}{2}, 3\frac{1}{4}\right), Q = \left(6\frac{1}{2}, 3\frac{1}{4}\right)$
b area $= \frac{1}{2}PQ^2 = \frac{1}{2}$

12 c $y = -16, y = -32x$

13 a 4t, -4
b $\dfrac{b}{a}, \dfrac{b}{a}$
c $\frac{9}{4}t, -\frac{9}{4}$

14 a $y = \frac{1}{3}x + 15$ **b** $y = 3x - 4$

17 The second tangent is the first tangent reflected in the line $y = x$, which exchanges the rise and run and thus does not change the sign of the gradient.
Alternatively, $g'(b) = \dfrac{1}{f'(a)}$ by the formula for differentiating inverse functions, so they have the same sign.

18 a At P, $x = h + \frac{1}{2}m$. At Q, $x = h - \frac{1}{2}m$.
b $\frac{1}{4}m(m^2 + 1)$

19 b The distances are $(\alpha - h)$ and $a(\alpha - h)^2$.
c $\alpha = \sqrt{h^2 + \dfrac{k}{a}}$ or $-\sqrt{h^2 + \dfrac{k}{a}}$

Exercise 9F

1 a $-x^{-2}$ **b** $-5x^{-6}$
c $-3x^{-2}$ **d** $-10x^{-3}$
e $4x^{-4}$ **f** $-4x^{-3} - 4x^{-9}$

2 a $f(x) = x^{-1}, f'(x) = -x^{-2} = -\dfrac{1}{x^2}$
b $f(x) = x^{-2}, f'(x) = -2x^{-3} = -\dfrac{2}{x^3}$
c $f(x) = x^{-4}, f'(x) = -4x^{-5} = -\dfrac{4}{x^5}$
d $f(x) = 3x^{-1}, f'(x) = -3x^{-2} = -\dfrac{3}{x^2}$

3 a $-42(7x - 5)^{-7}$ **b** $-5(3 + 5x)^{-2}$
c $-8(2x - 1)^{-2}$ **d** $-\dfrac{105}{4}(5x + 6)^{-8}$

4 $-x^{-2}, 2x^{-3}, -6x^{-4}, 24x^{-5}, -120x^{-6}$

5 a $(1, 1)$ and $(-1, -1)$
b $\left(1, \frac{1}{2}\right)$

6 a $y = -\frac{4}{3}x + \frac{7}{3}, 3x - 4y + 1 = 0$
b $y = -9x + 6, x - 9y - 28 = 0$

7 a $-\dfrac{3}{x^2}$ **b** $-\dfrac{1}{3x^2}$
c $\dfrac{7}{3x^2}$ **d** $-\dfrac{a}{x^2}$

8 a $2x - 2x^{-3}$ **b** 0 **c** $1 - \dfrac{1}{x^2}$

9 a $\dfrac{-2x}{(1 + x^2)^2}$, $(0, 1)$
b $\dfrac{3(4x^3 - 4x)}{(x^4 - 2x^2 - 1)^2}$, $(0, 3)$ and $\left(1, \frac{1}{2}\right)$

10 a $f'(x) = -x^{-2}, f''(x) = 2x^{-3}, f'''(x) = -6x^{-4},$ $f^{(4)}(x) = 24x^{-5}, f^{(5)}(x) = -120x^{-6}$
b $f'(1) = -1, f''(1) = 2, f'''(1) = -6,$ $f^{(4)}(1) = 24, f^{(5)}(1) = -120$
c Start with -1, then multiply by $-n$ to get each next term.
d Same as before, except that all the terms are negative.

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11 $a = -5$ or $a = -7$

12 a $x + y(b - 4)^2 = 2b - 4$

b i $x + 4y = 0$ **ii** $x + y = 6$

13 a They both have gradient -3.

b At M: $y = -3x + 12$. At N: $y = -\frac{1}{3}x + 4$. They intersect at $(3, 3)$.

c Part **a** follows from the curve's odd symmetry in the origin — the point $M(2, 6)$ and its tangent corresponds to $T(-2, -6)$ and its tangent — a rotation of 180° maps any line to a parallel line (going in the other direction). Part **b** follows from the curve's line symmetry in $y = x$.

14 a $cx + t^2y = 2ct$, $A = (2t, 0)$, $B = \left(0, 2\frac{c}{t}\right)$

b $2|c|$

15 b $\frac{dy}{dx} = -\frac{c^2}{x^2} = -\frac{1}{t^2}$

c tangent: $y = -\frac{x}{t^2} + \frac{2c}{t}$,

$A(2ct, 0)$, $AT^2 = OT^2 = (ct)^2 + \left(\frac{c}{t}\right)^2$.

Exercise 9G

1 a $-\frac{1}{2}x^{-1\frac{1}{2}}$ **b** $\frac{3}{2}x^{\frac{1}{2}}$ **c** $4x^{-\frac{1}{3}}$

d $-4x^{-1\frac{1}{3}}$ **e** $x^{-\frac{3}{4}} - 2x^{-\frac{5}{4}}$ **f** $\frac{49}{3}x^{1\frac{1}{3}}$

2 a $y = x^{\frac{1}{2}}$, $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

b $y = x^2\sqrt{x} = x^2 \times x^{\frac{1}{2}} = x^{2\frac{1}{2}} = x^{\frac{5}{2}}$,

$\frac{dy}{dx} = \frac{5}{2}x^{1\frac{1}{2}} = \frac{5}{2}x\sqrt{x}$

c i $y = x\sqrt{x} = x^1 \times x^{\frac{1}{2}} = x^{1\frac{1}{2}} = x^{\frac{3}{2}}$,

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

ii $y = x^{-\frac{1}{2}}$, $\frac{dy}{dx} = -\frac{1}{2}x^{-1\frac{1}{2}} = -\frac{1}{2x\sqrt{x}}$

iii $y = \frac{1}{x^1 \times x^{\frac{1}{2}}} = x^{-1\frac{1}{2}} = x^{-\frac{3}{2}}$,

$\frac{dy}{dx} = -\frac{3}{2}x^{-2\frac{1}{2}} = -\frac{3}{2x^2\sqrt{x}}$

3 $\frac{1}{2}x^{-\frac{1}{2}}$, $\frac{1}{4}x^{-\frac{3}{2}}$, $\frac{3}{8}x^{-\frac{5}{2}}$, $-\frac{15}{16}x^{-\frac{7}{2}}$, $\frac{105}{32}x^{-\frac{9}{2}}$

4 a $\frac{8}{3}(7 + 2x)^{\frac{1}{3}}$ **b** $\frac{1}{2}(x + 4)^{-\frac{1}{2}}$

c $-\frac{3}{2}(5 - 3x)^{-\frac{1}{2}}$ **d** $45(2 - 5x)^{-3\frac{1}{4}}$

5 a $\left(\frac{1}{4}, -\frac{1}{2}\right)$ **b** none

6 a Tangent: $y = \frac{1}{4}x + 1$, Normal: $y = -4x + 18$

b Tangent: $y = -\frac{1}{4}x + 3$, Normal: $y = 4x - 14$

7 a $\frac{3}{2\sqrt{x}}$ **b** $\frac{5}{\sqrt{x}}$ **c** $\frac{7}{2\sqrt{x}}$ **d** $\frac{\sqrt{7}}{2\sqrt{x}}$

8 a $(3, 2\sqrt{3})$ **b** $(1, 3)$ and $(-1, -3)$

9 a $1 + 3x^{-2} - 16x^{-3}$ **b** $1 + 3x^{-\frac{1}{2}}$

c $\frac{3}{2}x^{-\frac{1}{2}}$

10 a $\frac{x - 1}{\sqrt{x^2 - 2x + 5}}$, $(1, 2)$

b $\frac{x - 1}{\sqrt{x^2 - 2x}}$, none ($x = 1$ is outside the domain)

c $\frac{7x}{\sqrt{x^2 + 1}}$, $(0, 7)$ **d** $\frac{-1}{2\sqrt{x}(1 + \sqrt{x})^2}$, none

12 $a = 5$

13 a $12x + 5y = 169$, $y = \frac{5}{12}x$

b The intercepts are $\left(\frac{169}{12}, 0\right)$, $\left(0, \frac{169}{5}\right)$, the area is $\frac{169^2}{120}$.

c $\frac{13^3}{60} + \frac{169}{12} + \frac{169}{5} = \frac{169}{2}$

14 a $4x + 3y = 25$, $4x + 5y = 25$, they intersect at $\left(6\frac{1}{4}, 0\right)$

b $\lambda x_0 x + y\sqrt{25 - x_0^2} = 25\lambda$, $T = \left(\frac{25}{x_0}, 0\right)$,
$OM \times OT = 25 = OA^2$

15 c i $y\frac{dy}{dx} = \frac{1}{2}a^2$ **ii** $y\frac{dy}{dx} = na^2x^{2n-1}$

16 a $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$, $\frac{x}{(1 + \sqrt{1 - x^2})^2\sqrt{1 - x^2}}$

b $\frac{dy}{dx} = \frac{dy}{du_1} \times \frac{du_1}{du_2} \times \cdots \times \frac{du_{n-1}}{dx}$

17 a $-\frac{1}{2}x^{-\frac{3}{2}}$, $\frac{1 \times 3}{2^2}x^{-\frac{5}{2}}$, $-\frac{1 \times 3 \times 5}{2^3}x^{-\frac{7}{2}}$, $\frac{1 \times 3 \times 5 \times 7}{2^4}x^{-\frac{9}{2}}$

b $(-1)^n \times \frac{1 \times 3 \times 5 \times \cdots \times (2n - 1)}{2^n}x^{-\frac{2n+1}{2}}$

Exercise 9H

1 Let $u = 5x$
and $v = (x - 2)^4$.

Then $\frac{du}{dx} = 5$

and $\frac{dv}{dx} = 4(x - 2)^3$.

Let $y = 5x(x - 2)^4$.

Then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$= (x - 2)^4 \times 5 + 5x \times 4(x - 2)^3$

$= 5(x - 2)^4 + 20x(x - 2)^3$

$= 5(x - 2)^3((x - 2) + 4x)$

$= 5(x - 2)^3(5x - 2)$.

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2 a $2x^2(2x-3)$ **b** $4x-9$ **c** $4x^3$

3 a $3(3-2x)^4(1-4x), 1\frac{1}{2}, \frac{1}{4}$

b $x^2(x+1)^3(7x+3), 0, -1, -\frac{3}{7}$

c $x^4(1-x)^6(5-12x), 0, 1, \frac{5}{12}$

d $(x-2)^2(4x-5), 2, \frac{5}{4}$

e $2(x+1)^2(x+2)^3(7x+10), -1, -2, -\frac{10}{7}$

f $6(2x-3)^3(2x+3)^4(6x-1), 1\frac{1}{2}, -1\frac{1}{2}, \frac{1}{6}$

4 a $y=x, y=-x$

b $y=2x-1, x+2y=3$

5 a $(x^2+1)^4(11x^2+1)$

b $2\pi x^2(1+x)^3(1-x)^3(3-11x^2)$

c $-2(x^2+x+1)^2(7x^2+4x+1)$

6 a $y'=8x(x^2-1)^3, y''=8(x^2-1)^2(7x^2-1)$

b 8, 7 and 6

c $x=1$ and $x=-1$

7 $10x^3(x^2-10)^2(x^2-4)$, $(0, 0)$, $(\sqrt{10}, 0)$, $(-\sqrt{10}, 0)$, $(2, -3456)$, $(-2, -3456)$

8 $y'=x^2(1-x)^4(3-8x)$

9 a $\dfrac{3(3x+2)}{\sqrt{x+1}}, -\dfrac{2}{3}$ **b** $\dfrac{4(3x-1)}{\sqrt{1-2x}}, \dfrac{1}{3}$

c $\dfrac{10x(5x-2)}{\sqrt{2x-1}}$, none (0 and $\frac{2}{5}$ are outside the domain).

10 a $-1 \le x \le 1$ **b** $\dfrac{1-2x^2}{\sqrt{1-x^2}}$

c $\left(\sqrt{\tfrac{1}{2}}, \tfrac{1}{2}\right)$ and $\left(-\sqrt{\tfrac{1}{2}}, -\tfrac{1}{2}\right)$

d $y=x, y=-x$

11 a $y'=a(2x-\alpha-\beta)$

b $y'(\alpha)=a(\alpha-\beta), y'(\beta)=a(\beta-\alpha)$, $M=\left(\tfrac{1}{2}(\alpha+\beta), -\tfrac{1}{2}a(\alpha-\beta)^2\right)$

c $V=\left(\tfrac{1}{2}(\alpha+\beta), -\tfrac{1}{4}a(\alpha-\beta)^2\right)$

12 $f'(x)=(x-a)^{n-1}\left(nq(x)+(x-a)q'(x)\right)$. The x-axis is a tangent to the curve at $x=a$.

13 a $P=\left(\dfrac{r}{r+s}, \dfrac{r^r s^s}{(r+s)^{r+s}}\right)$.

b When $r=s$, $P=\left(\tfrac{1}{2}, 2^{-2r}\right)$.

14 $y'=u'vw+uv'w+uvw'$

a $2x^4(x-1)^3(x-2)^2(3x-5)(2x-1)$, 0, 1, 2, $\frac{1}{2}$ and $\frac{5}{3}$

b $\dfrac{(x-2)^3(11x^2-x-2)}{\sqrt{2x+1}}, 2, \frac{1}{22}(1+\sqrt{89})$, $\frac{1}{22}(1-\sqrt{89})$

15 $y'=u_1'u_2\cdots u_n+u_1u_2'\cdots u_n+\cdots+u_1u_2\cdots u_n'$

Exercise 9I

1 Let $u=2x+3$
and $v=3x+2$.
Then $u'=2$
and $v'=3$.

Let $y=\dfrac{2x+3}{3x+2}$.

$$\text{Then } \frac{dy}{dx}=\frac{vu'-uv'}{v^2}$$
$$=\frac{(3x+2)\times 2-(2x+3)\times 3}{(3x+2)^2}$$
$$=\frac{6x+4-6x-9}{(3x+2)^2}$$
$$=\frac{-5}{(3x+2)^2}.$$

2 a $\dfrac{-2}{(x-1)^2}$, none **b** $\dfrac{4}{(x+2)^2}$, none

c $\dfrac{-13}{(x+5)^2}$, none **d** $\dfrac{x(2-x)}{(1-x)^2}$, 0, 2

e $\dfrac{4x}{(x^2+1)^2}$, 0 **f** $\dfrac{1+x^2}{(1-x^2)^2}$, none

g $\dfrac{6x^2}{(x^3+2)^2}, x=0$ **h** $\dfrac{10x}{(x^2-4)^2}, x=0$

3 $\dfrac{-3}{(3x-2)^2}$

4 a $y'=\dfrac{5}{(5-3x)^2}, y=5x-12, 78^\circ 41'$, $x+5y+8=0, 168^\circ 41'$

b $y'=\dfrac{x^2-2x+4}{(x-1)^2}, 4x-3y=4, 53^\circ 8'$, $3x+4y=28, 143^\circ 8'$

5 a $\dfrac{m^2-b^2}{(bx+m)^2}$ **b** $\dfrac{2x(a-b)}{(x^2-b)^2}$ **c** $\dfrac{6nx^{n-1}}{(x^n+3)^2}$

6 a $\dfrac{c^2+2c}{(c+1)^2}=-3, c=-\frac{1}{2}$ or $-1\frac{1}{2}$

b $\dfrac{12k}{(9-k)^2}=1, k=3$ or 27

7 a $y'=\dfrac{\alpha-\beta}{(x-\beta)^2}$

b The denominator is positive, being a square, so the sign of y' is the sign of $\alpha-\beta$.

c When $\alpha=\beta$, the curve is the horizontal line $y=1$, and $y'=0$ (except that y is undefined at $x=\beta$).

8 $\dfrac{20}{(5-2x)^2}$

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9 a $\dfrac{dy}{dx} = \dfrac{-(t+1)^2}{(t-1)^2}, T = \left(\tfrac{2}{3}, 2\right), 3x - 27y + 52 = 0$

b $y = \dfrac{x}{2x-1}, \dfrac{dy}{dx} = \dfrac{-1}{(2x-1)^2}, \tfrac{1}{9}$

10 a $\dfrac{1}{2\sqrt{x}(\sqrt{x}+2)^2}$, none

b $\dfrac{x+5}{2(x+1)^{\frac{3}{2}}}$, none ($x = -5$ is outside the domain.)

11 a $f'(x) = \dfrac{-\sqrt{2}}{\sqrt{x}(\sqrt{x}-\sqrt{2})^2}, f'(8) = -\tfrac{1}{4}$

b 3

12 a domain: $x \neq -1$, range: $y \neq 1$

c $I = (-1, 0), G = (1, 0)$

d ii Substitute $(c, 0)$, then $c + a^2 = 0$, so $a = \sqrt{-c}$ or $-\sqrt{-c}$. For $-1 < c < 0$, they are both on the right-hand branch. For $c < -1$, they are on different branches.

14 a i $y' = \dfrac{2}{(x+1)^2}, y'' = \dfrac{-4}{(x+1)^3}$

ii $y' = \dfrac{-3}{(x-1)^2}, y'' = \dfrac{6}{(x-1)^3}$

iii $y' = \dfrac{x^2 - 2x}{(x-1)^2}, y'' = \dfrac{2}{(x-1)^3}$

15 a $12(3x-7)^3$ **b** $\dfrac{x^2+2}{x^2}$

c $8x$ **d** $\dfrac{-2x}{(x-3)^2(x+3)^2}$

e $4(1-x)(4-x)^2$ **f** $\dfrac{-6}{(3+x)^2}$

g $20x^3(x^2+1)^4(x+1)^4(x-1)^4$

h $\dfrac{1}{2(2-x)^{\frac{3}{2}}}$ **i** $6x^2(x^3+5)$

j $\dfrac{3x^2+x-1}{4x\sqrt{x}}$ **k** $\tfrac{2}{3}x(5x^3-2)$

l $\dfrac{5}{(x+5)^2}$ **m** $\tfrac{1}{2}\sqrt{x}(3+5x)$

n $\dfrac{2(x-1)(x+1)(x^2+1)}{x^3}$ **o** $x^2(x-1)^7(11x-3)$

p $\dfrac{(x+1)(x-1)}{x^2}$

16 b i $54, \tfrac{3}{2}, 9\sqrt{37}, \tfrac{3}{2}\sqrt{37}$ **ii** $\tfrac{1}{2}, 8, \tfrac{1}{2}\sqrt{17}, 2\sqrt{17}$

Exercise 9J

1 a $\dfrac{dQ}{dt} = 3t^2 - 20t$

b When $t = 2, Q = -32, \dfrac{dQ}{dt} = -28$

2 a $\dfrac{dQ}{dt} = 2t + 6$

b When $t = 2, Q = 16, \dfrac{dQ}{dt} = 10$

c i $t = -3$ **ii** $t > -3$ **iii** $t < -3$

3 a 7 and 15 **b** $\frac{15-7}{3-1} = 4$ **c** $\frac{7-15}{7-5} = -4$

4 a 180 mL **b** When $t = 0, V = 0$.

c 300 mL **d** 60 mL/s

e The derivative is a constant function.

5 a 80 000 litres **b** 35 000 litres

c 20 min **d** 2000 litres/min

6 a $\dfrac{dM}{dt} = 10 - 2t$

b $M = 24$ kg, $\dfrac{dM}{dt} = 2$ kg/s

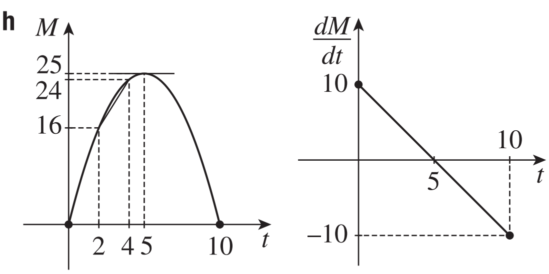
c $M = 16$ kg, average rate $= \frac{24-16}{4-2} = 4$ kg/s

d 0 seconds and 10 seconds

e 10 seconds **f** 5 seconds

g 5 seconds and 5 seconds

h



7 a \$2 **b** \$5.60

c $\dfrac{dP}{dt} = -0.8t + 4$, \$2.40 per annum

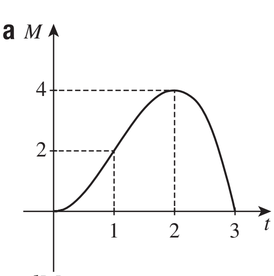
d $t = 5$, the start of 1975

e The price was increasing before then, and decreasing afterwards.

f $\dfrac{dP}{dt}$ is linear with negative gradient -0.8.

g At the start of 1980.

8 a



b $t = 2$

c $\dfrac{dM}{dt} = 6t - 3t^2, t = 1$ **d** $t = 1$

9 The scheme appears to have worked initially and the level of pollution decreased, but the rate at which the pollution decreased gradually slowed down and was almost zero in 2000. A new scheme would have been required to remove the remaining pollution.

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10 a The graph is stationary on 1 July and 1 March.

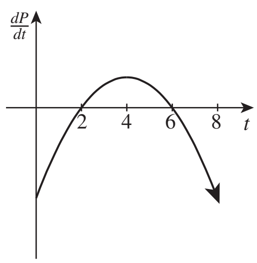
b It is maximum on 1 July and on 1 January. The price is locally minimum 1 March, but globally the graph has no minimum.

c It is increasing from 1 March to 1 July. It is decreasing from 1 January to 1 March and after 1 July.

d on 1 May

e from 1 March to 1 May

f



11 a $A = \pi r^2 = \pi\left(\frac{t}{3}\right)^2 = \frac{\pi}{9}t^2$ **b** $\frac{dA}{dt} = \frac{2\pi}{9}t$

c When $A = 5$, $t = \sqrt{\frac{45}{\pi}} \doteqdot 3.785$ seconds and $\frac{dA}{dt} = \frac{2\pi}{9}\sqrt{\frac{45}{\pi}} \doteqdot 2.642\,\text{km}^2/\text{s}$

12 a When $t = 0$, $h = 80$, so the building is 80 metres tall.

b When $h = 0$, $t = 4$, so it takes 4 seconds.

c $v = -10t$

d When $t = 4$, $v = -40$, so the stone hits the ground at 40 m/s.

e 10 m/s^2 downwards

13 a Yes. $\frac{dv}{dt} = -\frac{1}{2}$, meaning his velocity decreased at a constant rate of $\frac{1}{2}$m/s every second, just as he said.

b Yes. $\frac{dx}{dt} = -\frac{1}{2}t + 50$, which is what the truck's speed monitor recorded.

c Yes. $\frac{dy}{dt} = -\frac{1}{2}t + 50$, which also agrees with the truck's speed monitor.

d When $t = 0$, $x = 0$ and $y = 450$, so the truck was 450 metres ahead.

e Solving $-\frac{1}{2}t + 50 = 0$ gives 100 seconds. When $t = 100$, $x = 2500$ m or 2.5 km.

f When $t = 0$, $v = 50$ m/s, which is 180 km/h. He was in court for speeding.

14 a **i** Area $= h^2\text{cm}^2$ **ii** Volume $= 3000\,h^2\text{cm}^3$

b **i** $h = 3\sqrt{t}$, $\frac{dh}{dt} = \frac{3}{2\sqrt{t}}$

ii $h = 15\text{cm}$, $\frac{dh}{dt} = \frac{3}{10}$ cm/min

iii 100 minutes, $\frac{3}{20}$ cm/min

Exercise 9K

1 a Zeroes: none, discontinuities:, $x = 6$

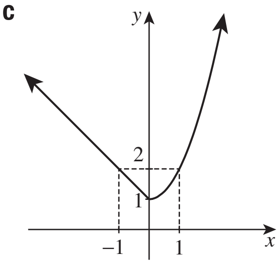
b Zeroes: $x = 0$, discontinuities:, $x = 1$, $x = 3$, $x = 5$.

c Zeroes: $x = 0$, $x = -1$, discontinuities:, $x = -2$, $x = -3$

2 a $f(0) = 1$. First table: 3, 2, 1. Second table: 1, 2, 5

b Yes

c

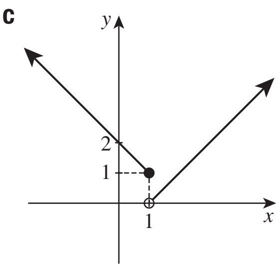


domain: all real x, range: $y \geq 1$

3 a $f(1) = 1$. First table: 3, 2, 1. Second table: 0, 1, 2

b No

c



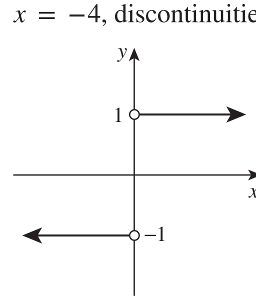
d domain: all real x, range: $y > 0$

4 a $f(x) = \frac{1}{x(x-5)}$, Zeroes: none, discontinuities: $x = 0$, $x = 5$

b $f(x) = \frac{x}{(x-2)(x-3)}$, Zeroes: $x = 0$, discontinuities: $x = 2$, $x = 3$

c $f(x) = \frac{(x-4)(x+4)}{(x-3)(x+3)}$, Zeroes: $x = 4$, $x = -4$, discontinuities: $x = -3$, $x = 3$

5



The table of values should make it clear that

$$y = \begin{cases} 1, & \text{for } x > 0, \\ -1, & \text{for } x < 0, \\ \text{undefined}, & \text{for } x = 0. \end{cases}$$

The curve is not continuous at $x = 0$ — it is not even defined there. domain: $x \neq 0$, range: $y = 1$ or -1

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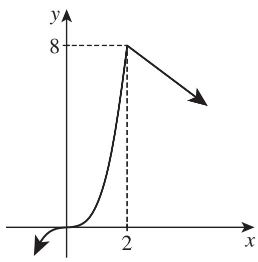
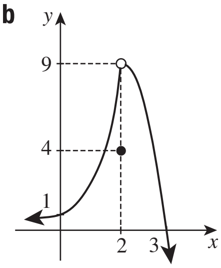
6 a



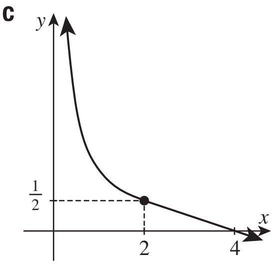
b



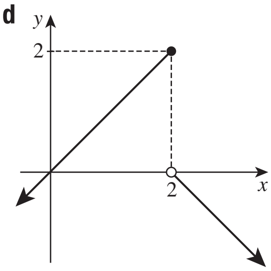
a $f(2) = 8$. When $x = 2$, $x^3 = 8$ and $10 - x = 8$. Thus $f(x)$ is continuous at $x = 2$. Domain: all real x, range: $y \le 8$

b $f(2) = 4$. When $x = 2$, $3^x = 9$ and $13 - x^2 = 9$. Thus $f(x)$ is not continuous at $x = 2$. domain: all real x, range: $y < 9$

c



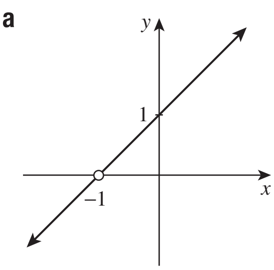
d



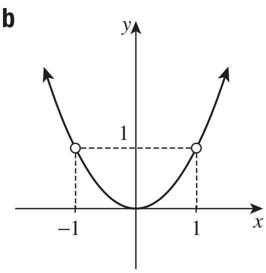
c $f(2) = \frac{1}{2}$. When $x = 2$, $\dfrac{1}{x} = \frac{1}{2}$ and $1 - \frac{1}{4}x = \frac{1}{2}$. Thus $f(x)$ is continuous at $x = 2$. domain: $x > 0$, range: all real y

d $f(2) = 2$. When $x = 2$, $x = 2$, but $2 - x = 0$. Thus $f(x)$ is not continuous at $x = 2$. domain: all real x, range: $y \le 2$

7 a



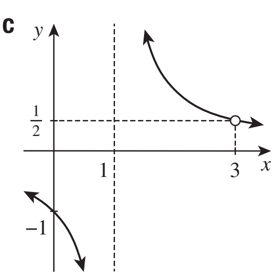
b



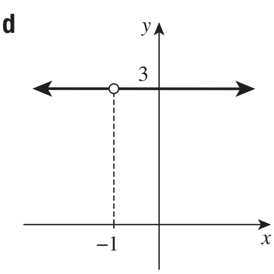
a $y = x + 1$, where $x \ne -1$, domain: $x \ne -1$, range: $y \ne 0$

b $y = x^2$, where $x \ne -1$ or 1, domain: $x \ne -1$ or 1, range: $y \ge 0$, $y \ne 1$

c



d



c $y = \dfrac{1}{x - 1}$, where $x \ne 1$ or 3, domain: $x \ne 1$ or 3, range: $y \ne 0$ or $\frac{1}{2}$

d $y = 3$, where $x \ne -1$, domain: $x \ne -1$, range: $y = 3$

8 a $a = 5$ **b** $a = -2$

9 a zeroes: none, discontinuities: $360n^\circ$, where $n \in \mathbb{Z}$

b zeroes: $135^\circ + 180n^\circ$, where $n \in \mathbb{Z}$, discontinuities: $45^\circ + 180n^\circ$, where $n \in \mathbb{Z}$

c zeroes: $45^\circ + 180n^\circ$, where $n \in \mathbb{Z}$, discontinuities: $135^\circ + 180n^\circ$, where $n \in \mathbb{Z}$

Exercise 9L

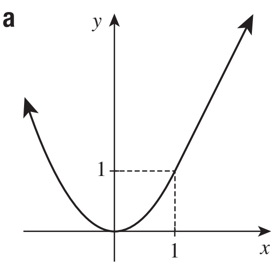
1 a continuous and differentiable at $x = 0$, neither at $x = 2$

b continuous and differentiable at $x = 0$, continuous but not differentiable at $x = 2$

c neither at $x = 0$, continuous and differentiable at $x = 2$

d neither at $x = 0$, continuous but not differentiable at $x = 2$

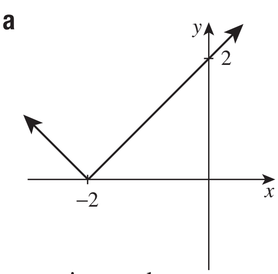
2 a



b $f(1) = 1$, $x^2 = 1$ when $x = 1$, $2x - 1 = 1$ when $x = 1$

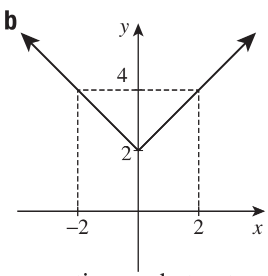
d $2x = 2$ when $x = 1$, and $2 = 2$ when $x = 1$. The tangent at $x = 1$ has gradient 2, so $f'(1) = 2$.

3 a



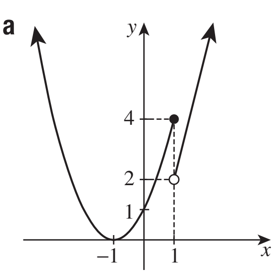
continuous but not differentiable at $x = -2$

b



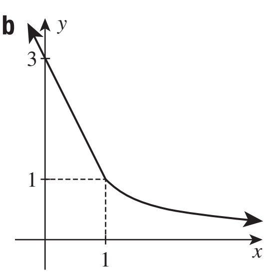
continuous but not differentiable at $x = 0$

4 a



not even continuous at $x = 1$

b



continuous but not differentiable at $x = 1$

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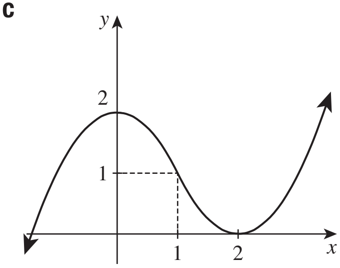
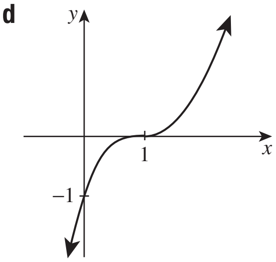
c



differentiable at $x = 1$,
$f'(1) = -2$

d



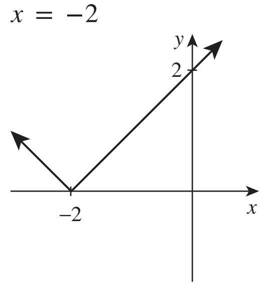
differentiable at $x = 1$,
$f'(1) = 0$

5 a Differentiable at $x = 0$. x^2 is never negative, so $|x^2| = x^2$ for all x.

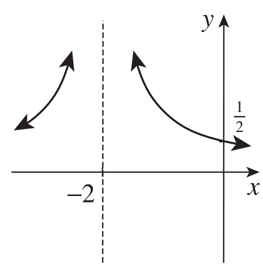
b Differentiable at $x = 0$. x^3 is flat at $x = 0$, so $|x^3|$ is also flat at $x = 0$.

c Continuous, but not differentiable, at $x = 0$. The graph of $y = \sqrt{x}$ becomes vertical near $x = 0$, producing a cusp.

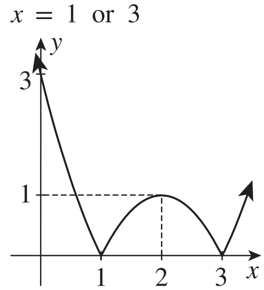
6 a not differentiable at $x = -2$



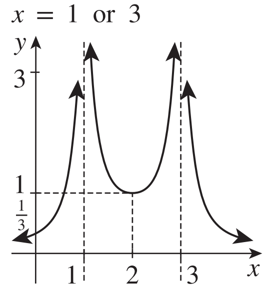
b not continuous at $x = -2$



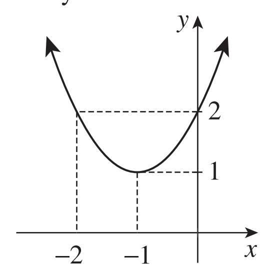
c not differentiable at $x = 1$ or 3



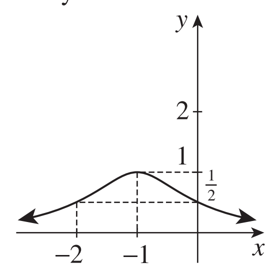
d not continuous at $x = 1$ or 3



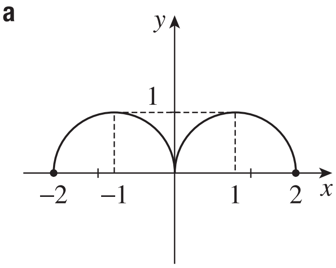
e differentiable everywhere



f differentiable everywhere

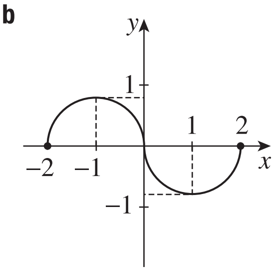


7 a



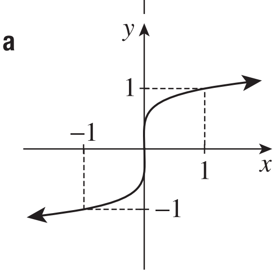
There is a cusp at the origin because the curve becomes infinitely steep on both sides. It slopes downwards on the left and upwards on the right.

b



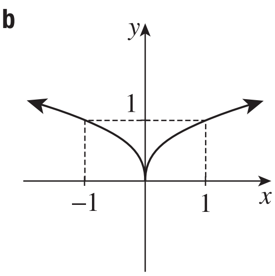
There is a vertical tangent at $x = 0$.

8 a



$f'(x) = \frac{1}{5}x^{-\frac{4}{5}}$. There is a vertical tangent at $(0, 0)$.

b



$f'(x) = \frac{2}{5}x^{-\frac{3}{5}}$. There is a cusp at $(0, 0)$.

9 a 4, $(5, -5)$ **b** 3, $(1, -8)$ **c** $-\frac{1}{4}$, $\left(2, \frac{1}{2}\right)$

d 1. There are none, because all the tangents have negative gradients.

e 0. There are none, because the tangents have gradient 1 for $x > 0$ and gradient -1 for $x < 0$.

f $\alpha + \beta$, $\left(\frac{1}{2}(\alpha + \beta), \frac{1}{4}(\alpha + \beta)^2\right)$

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10 a q must be odd.

b $p \geq 0$ (When $p = 0$ it is reasonable to take $f(0) = 1$ and ignore the problem of 0^0, because $\lim_{x \to 0} x^0 = 1$; thus when $p = 0$ the function is $y = 1$.)

c no conditions on p and q

d $p \geq 0$ and q is odd.

e $p \geq 0$ ($p = 0$ requires the qualification above.)

f $p \geq q$ and q is odd.

g $0 < p < q$ and q is odd and p is odd.

h $0 < p < q$ and q is odd and p is even.

Chapter 9 review exercise

1 a $2x + 5$ **b** $-2x$ **c** $6x - 2$

2 a $3x^2 - 4x + 3$ **b** $6x^5 - 16x^3$

c $9x^2 - 30x^4$ **d** $2x + 1$

e $-12x + 7$ **f** $-6x^{-3} + 2x^{-2}$

g $12x^2 + 12x^{-4}$ **h** $\frac{3}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-1\frac{1}{2}}$

i $x^{-2} - 2x^{-3}$

3 a $f'(x) = 4x^3 + 3x^2 + 2x + 1$, $f''(x) = 12x^2 + 6x + 2$

b $f'(x) = -10x^{-3}$, $f''(x) = 30x^{-4}$

c $f'(x) = -4x^{-\frac{3}{2}}$, $f''(x) = 6x^{-\frac{5}{2}}$

4 a $y = x^3 + 4x + C$

b $y = 7x - 6x^2 - 4x^3 + C$

c $y = 4x^5 - 4x^3 + 4x + C$

5 a $-\dfrac{3}{x^2}$ **b** $-\dfrac{1}{3x^3}$ **c** $\dfrac{7}{2\sqrt{x}}$

d $\dfrac{6}{\sqrt{x}}$ **e** $-\frac{9}{2}\sqrt{x}$ **f** $-\dfrac{3}{x\sqrt{x}}$

6 a $6x - 2$ **b** $x - \frac{1}{2}$

c $10x + \dfrac{7}{x^2}$ **d** $-\dfrac{2}{x^2} - \dfrac{2}{x^3}$

e $\dfrac{2}{\sqrt{x}}$ **f** $3\sqrt{x} + \dfrac{3}{2\sqrt{x}}$

7 a $9(3x + 7)^2$ **b** $-4(5 - 2x)$

c $-\dfrac{5}{(5x - 1)^2}$ **d** $\dfrac{14}{(2 - 7x)^3}$

e $\dfrac{5}{2\sqrt{5x + 1}}$ **f** $\dfrac{1}{2(1 - x)^{\frac{3}{2}}}$

8 a $42x(7x^2 - 1)^2$ **b** $-15x^2(1 + x^3)^{-6}$

c $8(1 - 2x)(1 + x - x^2)^7$

d $-6x(x^2 - 1)^{-4}$ **e** $-\dfrac{x}{\sqrt{9 - x^2}}$

f $\dfrac{x}{(9 - x^2)^{\frac{3}{2}}}$

9 a $x^8(x + 1)^6(16x + 9)$ **b** $\dfrac{x(2 - x)}{(1 - x)^2}$

c $2x(4x^2 + 1)^3(20x^2 + 1)$ **d** $\dfrac{12}{(2x + 3)^2}$

e $(9x - 1)(x + 1)^4(x - 1)^3$ **f** $\dfrac{(x - 5)(x + 1)}{(x - 2)^2}$

10 $\dfrac{dy}{dx} = 2x + 3$

a 3, 71°34′ **b** 1, 45° **c** −1, 135°

11 a tangent: $y = -3x$, normal: $3y = x$

b tangent: $y = -2$, normal: $x = 1$

c $(1, -2)$ and $(-1, 2)$ **d** $(2, 2)$ and $(-2, -2)$

12 a $y = -x - 4$, $y = x - 8$

b $A(-4, 0)$, $B(8, 0)$

c $AB = 12$, $|\Delta ABP| = 36$ square units

13 The tangent is $y = x$

14 a $\left(1, -6\frac{2}{3}\right)$, $\left(-1, -7\frac{1}{3}\right)$ **b** $\left(-1, \frac{2}{3}\right)$

15 At $(1, -3)$ the tangent is $\ell: x + y + 2 = 0$, at $(-1, 3)$ the tangent is $x + y - 2 = 0$.

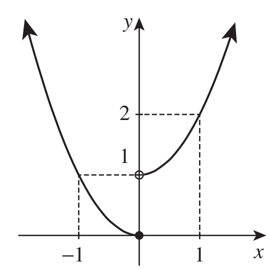
16 a $2\frac{1}{2}$ and 1 **b** 2 and $1\frac{1}{2}$

17 a $V = \frac{4}{3}\pi \times \left(\dfrac{t}{3}\right)^3 = \frac{4\pi}{81}t^3$ **b** $\dfrac{dV}{dt} = \frac{4\pi}{27}t^2$

c $V \doteqdot \frac{4\pi}{81} \times 0.001 \doteqdot 0.00016\,\text{km}^3$, $\dfrac{dV}{dt} \doteqdot \frac{4\pi}{27} \times 0.01 \doteqdot 0.0047\,\text{km}^3/\text{s}$

d $t^2 = \frac{27}{4\pi}$, $t \doteqdot 1.5\,\text{s}$

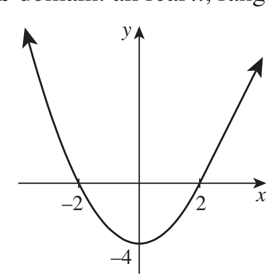
18



a $f(0) = 0$, $x^2 = 0$ when $x = 0$, $x^2 + 1 = 1$ when $x = 0$, so it is not continuous at $x = 0$.

b domain: all real x, range: $y \geq 0$

19



a $f(0) = 2$, $x^2 - 4 = 0$ when $x = 2$, $4x - 8 = 0$ when $x = 0$, so it is continuous at $x = 2$.

b $f'(2) = 4$ when $x < 2$ (substitute into $2x$), $f'(2) = 4$ when $x > 2$ (substitute into 4), so it is differentiable at $x = 2$, with $f'(2) = 4$

c domain: all real x, range: $y \geq 4$

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Chapter 10

Exercise 10A

1 a yes **b** no **c** no **d** no
e yes **f** no **g** yes **h** yes
i no **j** yes **k** yes **l** no

2 a 3, 4, $4x^3$, -11, not monic
b 3, -6, $-6x^3$, 10, not monic
c 0, 2, 2, 2, not monic
d 12, 1, x^{12}, 0, monic
e 3, 1, x^3, 0, monic
f 5, -1, $-x^5$, 0, not monic
g no degree, no leading coefficient, no leading term, 0, not monic
h 2, -3, $-3x^2$, 0, not monic
i 6, -4, $-4x^6$, -5, not monic

3 a $x^2 + 2x + 3$ **b** $x^2 + 2x + 3$
c $-x^2 + 8x + 1$ **d** $x^2 - 8x - 1$
e $5x^3 - 13x^2 - x + 2$ **f** $5x^3 - 13x^2 - x + 2$

5 a $x(x - 10)(x + 2)$, 0, 10, -2
b $x^2(2x + 1)(x - 1)$, 0, 1, $-\frac{1}{2}$
c $(x - 3)(x + 3)(x^2 + 9)$, 3, -3
d $(x - 3)(x + 3)(x^2 + 4)$, 3, -3

6 a 9, 8, -27 **b** 14, 120, 24

7 a i $p + q$
ii the maximum of p and q
b $P(x)Q(x)$ still has degree $p + p = 2p$, but $P(x) + Q(x)$ may have degree less than p (if the leading terms cancel out), or it could be the zero polynomial.
c $x^2 + 2$ and $-x^2 + 3$. Do not choose two opposite polynomials, such as $x^2 + 1$ and $-x^2 - 1$, because their sum is the zero polynomial, which does not have a degree.

8 $x + 1$

9 a $a = 3, b = -4$ and $c = 1$
b $a = 2$ and $b = 3$
c $a = 1, b = 2$ and $c = 1$
d $a = 1, b = 2$ and $c = -1$

10 c A polynomial is even if and only if the coefficients of the odd powers of x are zero. A polynomial is odd if and only if the coefficients of the even powers of x are zero.

11 a True. If $P(x)$ is even, then the terms are of the form $a_n x^{2n}$, where $n \geq 0$ is an integer. Therefore $P'(x)$ has terms of the form $2na_n x^{2n-1}$, so all powers of x will be odd.
b False. For example, $Q(x) = x^3 + 1$ is not odd but $Q'(x) = 3x^2$ is even.
c True. If $R(x)$ is odd, then the terms are of the form $a_n x^{2n+1}$, where $n \geq 0$ is an integer. Therefore $R'(x)$ has terms of the form $(2n + 1)a_n x^{2n}$, so all powers of x will be even.
d True. As $S'(x)$ is odd, it has no constant term, and all powers of x are odd. Therefore all the terms in $S(x)$ will have even powers.

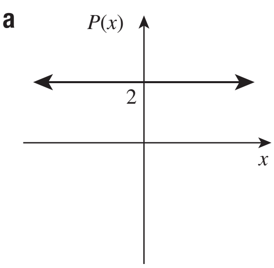
12 a $\sqrt{2}$ and $-\sqrt{2}$ **b** 1 and -1 **c** $\sqrt{3}$ and $-\sqrt{3}$
d Their discriminants are all negative.

13 a n **b** $n + 1$

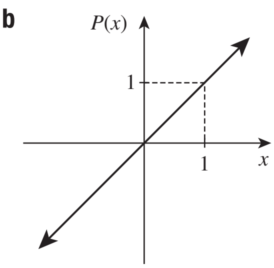
14 b The kth derivative of $a_k x^k$ is $k!a_k$ which is a constant. Substituting $x = 0$ proves that the coefficients of x^k are equal, since the common factor $k!$ cancels.

Exercise 10B

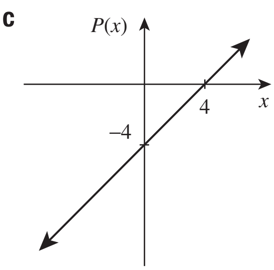
1 a



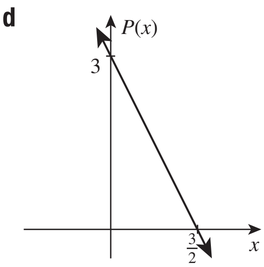
b



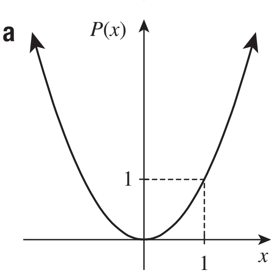
c



d



2 a



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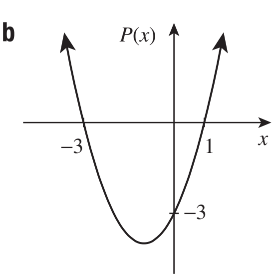

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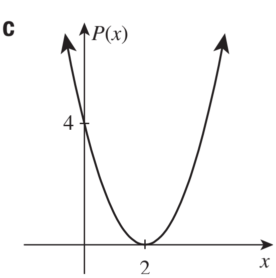

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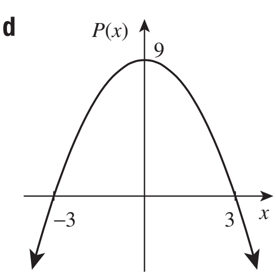

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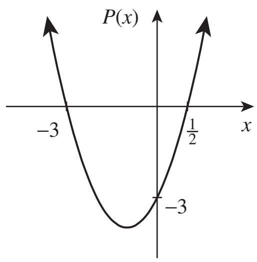
e



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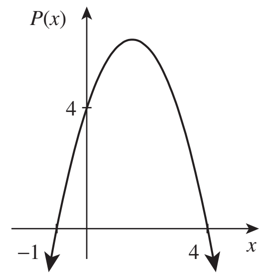
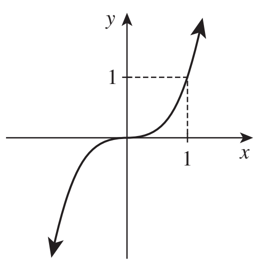


3 a



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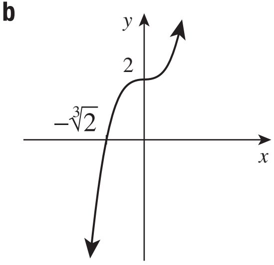

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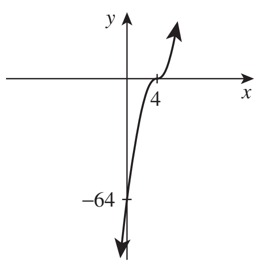

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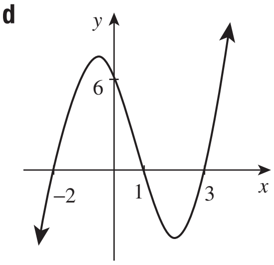

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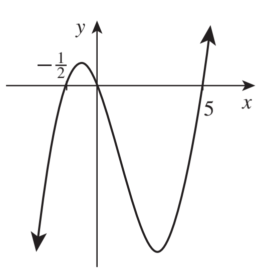

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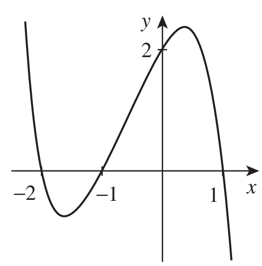

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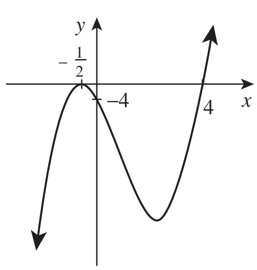

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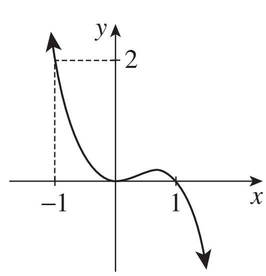
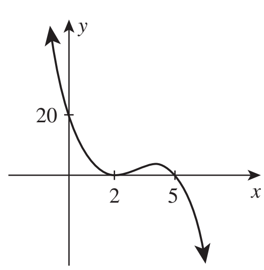
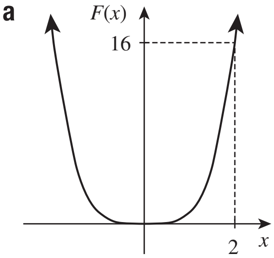


i

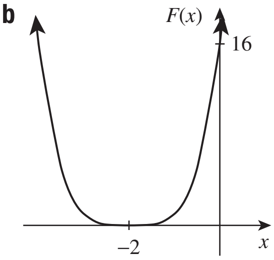


Parts **a**, **b** and **c** are one-to-one, the others are all many-to-one.

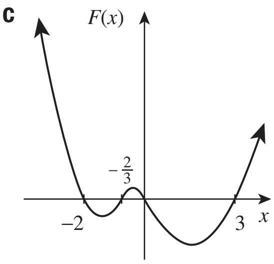
4 a



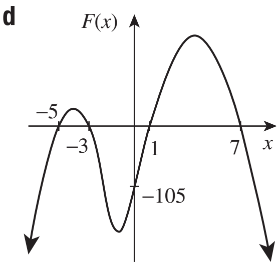
b



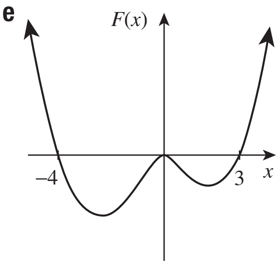
c



d



e



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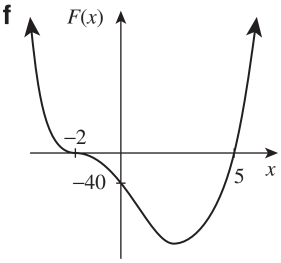

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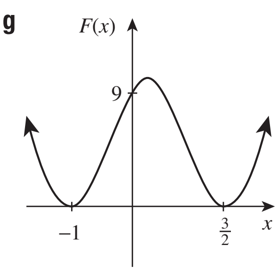

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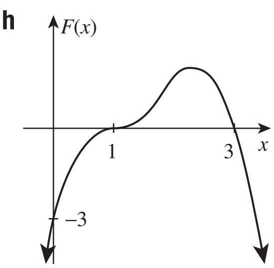

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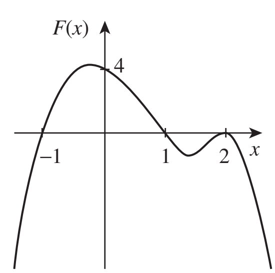

5 a There are two zeroes, one between 0 and 1, and one between 2 and 3.

b There are three zeroes, one between -2 and -1, one between -1 and 0, and one between 1 and 2.

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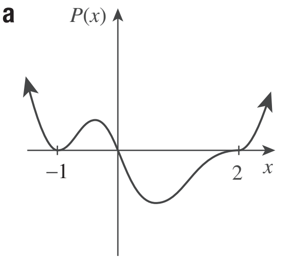
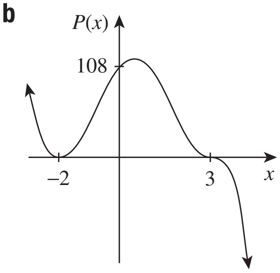
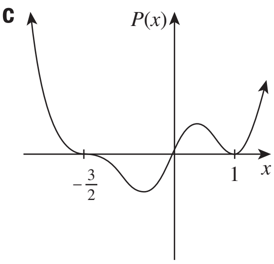
6 a



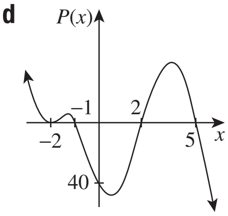
b



c



d



7 a $x > 2$ or $x < 0$, $x \neq -1$

b $x \leq 3$

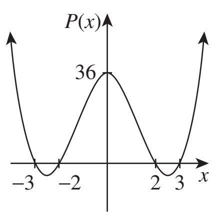
c $x \leq -\frac{3}{2}$ or $x \geq 0$

d $x > 5$ or $-1 < x < 2$

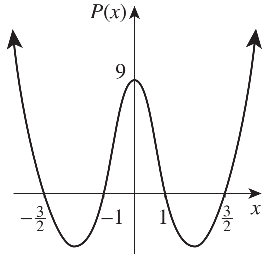
8 a $(x + 3)(x - 3)(x + 2)(x - 2)$

b $(2x - 3)(2x + 3)(x + 1)(x - 1)$

a



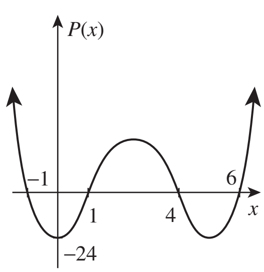
b



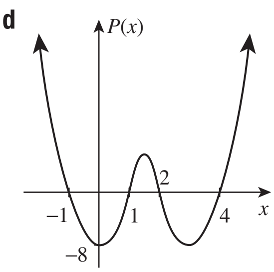
c $(x - 6)(x + 1)(x - 4)(x - 1)$

d $(x - 4)(x + 1)(x - 1)(x - 2)$

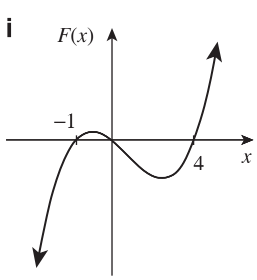
c



d



9 a i



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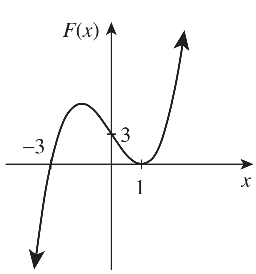

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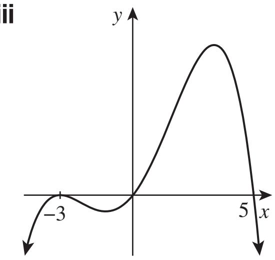

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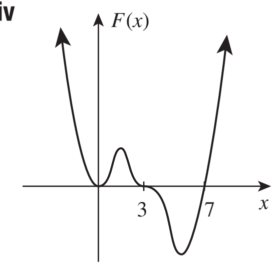
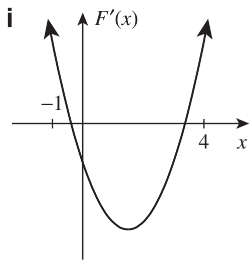


b i



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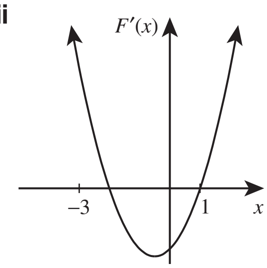

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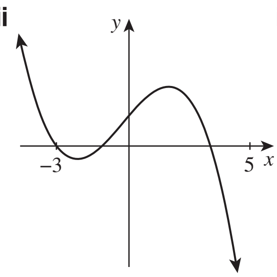
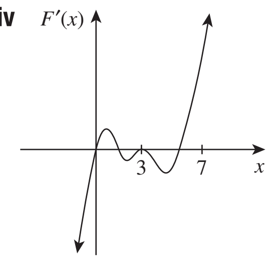


iv



10 a $P(x) = (x^2 - 3)^2 + (x + 2)^2$

b None, because $P(x) > 0$ for all x, so its graph lies above the x-axis.

11 The graphs always intersect at $(0, 1)$ and at $(-1, 0)$. If m and n are both even, they also intersect at $(-2, 1)$, and if m and n are both odd, they also intersect at $(-2, -1)$.

Exercise 10C

1 a $63 = 5 \times 12 + 3$

b $125 = 8 \times 15 + 5$

c $324 = 11 \times 29 + 5$

d $1857 = 23 \times 80 + 17$

2 a $x^2 - 4x + 1 = (x + 1)(x - 5) + 6$

b $x^2 - 6x + 5 = (x - 5)(x - 1)$

c $x^3 - x^2 - 17x + 24$
$= (x - 4)(x^2 + 3x - 5) + 4$

d $2x^3 - 10x^2 + 15x - 14$
$= (x - 3)(2x^2 - 4x + 3) - 5$

e $4x^3 - 4x^2 + 7x + 14$
$= (2x + 1)(2x^2 - 3x + 5) + 9$

f $x^4 + x^3 - x^2 - 5x - 3$
$= (x - 1)(x^3 + 2x^2 + x - 4) - 7$

g $6x^4 - 5x^3 + 9x^2 - 8x + 2$
$= (2x - 1)(3x^3 - x^2 + 4x - 2)$

h $10x^4 - x^3 + 3x^2 - 3x - 2$
$= (5x + 2)(2x^3 - x^2 + x - 1)$

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3 a $\dfrac{x^2 - 4x + 1}{x + 1} = x - 5 + \dfrac{6}{x + 1}$

b $\dfrac{x^2 - 6x + 5}{x - 5} = x - 1$

c $\dfrac{x^3 - x^2 - 17x + 24}{x - 4} = x^2 + 3x - 5 + \dfrac{4}{x - 4}$

d $\dfrac{2x^3 - 10x^2 + 15x - 14}{x - 3} = 2x^2 - 4x + 3 - \dfrac{5}{x - 3}$

4 a $x^3 + x^2 - 7x + 6 = (x^2 + 3x - 1)(x - 2) + 4$

b $x^3 - 4x^2 - 2x + 3 = (x^2 - 5x + 3)(x + 1)$

c $x^4 - 3x^3 + x^2 - 7x + 3$
$= (x^2 - 4x + 2)(x^2 + x + 3) + (3x - 3)$

d $2x^5 - 5x^4 + 12x^3 - 10x^2 + 7x + 9$
$= (x^2 - x + 2)(2x^3 - 3x^2 + 5x + 1)$
$+ (7 - 2x)$

5 a 0, 1 or 2

b $D(x)$ has degree 3 or higher.

6 a $x^3 - 5x + 3 = (x - 2)(x^2 + 2x - 1) + 1$

b $2x^3 + x^2 - 11 = (x + 1)(2x^2 - x + 1) - 12$

c $x^3 - 3x^2 + 5x - 4$
$= (x^2 + 2)(x - 3) + (3x + 2)$,
$\dfrac{x^3 - 3x^2 + 5x - 4}{x^2 + 2} = x - 3 + \dfrac{3x + 2}{x^2 + 2}$

d $2x^4 - 5x^2 + x - 2$
$= (x^2 + 3x - 1)(2x^2 - 6x + 15) + (13 - 50x)$

e $2x^3 - 3 = (2x - 4)(x^2 + 2x + 4) + 13$

f $x^5 + 3x^4 - 2x^2 - 3$
$= (x^2 + 1)(x^3 + 3x^2 - x - 5) + (x + 2)$,
$\dfrac{x^5 + 3x^4 - 2x^2 - 3}{x^2 + 1}$
$= x^3 + 3x^2 - x - 5 + \dfrac{x + 2}{x^2 + 1}$

7 a $P(x) = (x - 3)(x + 1)(x + 4)$

b $x > 3$ or $-4 < x < -1$

8 a $(x - 2)(x + 1)(2x - 1)(x + 3)$

b $-3 \le x \le -1$ or $\dfrac{1}{2} \le x \le 2$

9 a quotient: $x^2 - 3x + 5$, remainder: $12 - 13x$

b $a = 8$ and $b = -5$

10 a $x^4 - x^3 + x^2 - x + 1$
$= (x^2 + 4)(x^2 - x - 3) + (3x + 13)$

b $c = -4$ and $d = -12$

11 $m = 41$ and $n = -14$

12 a $Q(x) = x^2 - 2x - 4$ and $R(x) = 25$

13 b $k = 19, 25, 34, 59$ or 184

Exercise 10D

1 a 3 **b** 25 **c** −15 **d** −3 **e** 111 **f** −41

2 a yes **b** no **c** no **d** yes **e** no **f** yes

3 a $k = 4$ **b** $m = -\frac{1}{2}$ **c** $p = -14$ **d** $a = -1$

4 a $(x - 2)(x + 1)(x + 3)$

b $(x - 1)(x - 3)(x + 7)$

c $(x + 1)^2(3 - x)$

d $(x - 1)(x + 2)(x + 3)(x - 5)$

5 a $-1, -4$ or 2 **b** 3 or -2

c $2, -\frac{2}{3}$ or $-\frac{1}{2}$ **d** $-2, \frac{1}{4}\left(-3 \pm \sqrt{17}\right)$

6 b $P(x) = (x - 3)(x + 1)(x - 6)$

7 a $P(x) = (x - 3)(2x + 1)(x + 2)$

8 a $a = 4$ and $b = 11$ **b** $a = 2$ and $b = -9$

9 a $\frac{29}{8}$ **b** $\frac{97}{8}$ **c** $\frac{95}{27}$

10 a $(2x - 1)(x + 3)(x - 2)$

b $(3x + 2)(2x + 1)(x - 1)$

11 $x + 1$ is a factor when n is odd.

12 a $P(x) = -x^3 + 16x$ **b** $p = 2$ or $p = 3$

13 a $P(x) = (x - 1)(x + 3)Q(x) + (2x + 5)$

b 7 **c** −1

14 a The divisor has degree 2, so the remainder is zero or has degree 1 or 0.

b $a = -1$ and $b = 3$

15 $3 - 2x$

16 a $-2\frac{1}{2}$ **b** $a = -2$ and $b = -7$

c 8

17 b $P(x) = (2x + 1)(2x - 1)(x - 1)(2x^2 + x + 3)$

18 a $a^2 + b^2 + c^2 - ab - bc - ca$

b $(a - b)(b - c)(c - a)(a + b + c)$

Exercise 10E

1 a $(x + 1)(x - 3)(x - 4)$

b $x(x + 2)(x - 3)(x - 1)$

c $(3x - 1)(2x + 1)(x - 1)$

2 a $(x - 2)(x + 3)(x + 1)(x - 5)$

3 a $P(x) = (x - 1)(x + 1)(x - 3)(2x + 1)$

b $P(x) = (x - 1)(x - 2)(x + 2)(2x - 3)$

c $P(x) = (2x - 5)(3x - 2)(x + 1)(x - 2)$

d $P(x) = (x - 2)(x - 3)(3x - 1)^2$

4 a $a = 2, b = \frac{1}{3}$ and $c = \frac{5}{2}$

b $a = -1, b = 3, c = \frac{1}{2}$ and $d = \frac{5}{4}$

5 a $a = 3, b = -16$ and $c = 27$

b $a = 2, b = -2, c = -7$ and $d = -7$

c $(x + 1)^3 - (x + 1)^2 - 4(x + 1) + 5$

d $a = 3, b = -2$ and $c = 1$

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6 a $P(x) = (x - 2)^2(x + 5)$
b $P(x) = (x - 1)(x + 3)(2x - 7)$
8 m
9 There must be a stationary point between each of the consecutive zeroes, where the curve turns around and returns to the x-axis.
10 $a = b = c = 0, d = k$
11 $x^5 - 4x^4 + x^3 + 4x^2 - 2x$
12 a The curves are tangent at $x = 3$ and cross at $x = -1$.
b The curves are tangent at $x = 2$ and cross at $x = 3$.
c The curves cross at $x = -5$, $x = -2$ and $x = 3$.
d The curves are tangent to one another and cross at $x = 1$, and cross at $x = -2$.
e The curves cross at the origin, and cross and are tangent to each other at $x = -1$.
13 $x^2 - 1$
14 -23
15 a $0, 1, 2, \dots, n$ **c** 1

Exercise 10F

1 a 4 **b** 2 **c** 8 **d** 2 **e** 14
f 12 **g** 6 **h** 24 **i** $\frac{17}{2}$
2 a -2 **b** -11 **c** 12 **d** $-\frac{11}{12}$
e $-\frac{1}{6}$ **f** 0 **g** -132 **h** 26
i $\frac{13}{72}$
The roots are -1, -4 and 3.
3 a 5 **b** 2 **c** 4 **d** -3
e $-\frac{4}{3}$ **f** $-\frac{2}{3}$ **g** $-\frac{5}{3}$ **h** 21
4 a $-\frac{5}{2}$ **b** -2 **c** $\frac{41}{4}$ **d** $\frac{1}{2}\sqrt{57}$
5 a The other zero is $\frac{1}{2}$.
b The other factor is $(x - 4)$.
6 d The discriminant of the quadratic is negative.
e once
7 a 3 **b** $-\frac{1}{2}$
c $-3, 1$. Hence 1 is a double zero.
d $\frac{2}{3}, 2$
8 a $a = 3$ and $b = -24$, $(x - 3)(x + 4)(x + 2)$
b $a = -1$ and $b = 3$, zeroes are $5, -4, \sqrt{3}, -\sqrt{3}$.
10 a $\frac{1}{3}, -4$ and 4 **b** $6, \frac{1}{2}$ and -4
c -3 (double root) and 6 **d** $4, \frac{1}{2}$ and 2
11 a $a = -12$ and the roots are $-2, 2$ and -3.
b $a = -5$ and the roots are $4, \frac{1}{4}$ and -3.
12 $-1, -2, 2$ and 4
14 a i $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$ **iv** $\frac{1}{4}$

17 b $\alpha^2 + \beta^2 + \gamma^2 = -\frac{11}{4} < 0$ which is impossible if α, β and γ are all real numbers, because squares are never negative.
18 12
19 0, because 1 is one of the roots, so 0 is one of the factors of the expression.
20 b i $b = 1$ and $c = -2$
ii $b = 4$ and $c = 4$

Exercise 10G

1 a i 3 is at least a double zero of $P(x)$
b $3, 3, -2$
c $P(x) = (x - 3)^2(x + 2)$
2 a ii -1 is at least a triple zero of $P(x)$
b $-1, -1, -1, -5$
c $P(x) = (x + 1)^3(x + 5)$
3 a -3 and 3 **b** 3 **c** -6
4 a $\frac{5}{2}$ and -5 **b** -5 **c** 10
5 a -2
b $\frac{3}{2}$, $P(x) = (x + 2)^2(2x - 3)$
6 a $\frac{1}{2}$
b 2, $P(x) = (2x - 1)^3(x - 2)$
7 b $x = 3, 2 + \sqrt{3}$ or $2 - \sqrt{3}$
8 a $k = 27$ or -5
b When $k = 27$, $P(x) = (x - 3)^2(x + 3)$ and when $k = -5$, $P(x) = (x + 1)^2(x - 5)$.
9 $a = 1, b = -3, c = 2$
10 a -3 **b** $c = -54$
c $P(x) = (x + 3)^3(x - 2)$
11 a $b = -5$ and $c = 8$
b $x = \frac{1}{2}\left(3 - \sqrt{5}\right)$ or $\frac{1}{2}\left(3 + \sqrt{5}\right)$
14 Hint: consider $P(x) - P'(x)$
15 b ii $m < 0$
iii $x = -\sqrt{-\frac{m}{2}}$ or $\sqrt{-\frac{m}{2}}$

Exercise 10H

1 b The equation is $(x - 4)^2 = 0$, so $x = 4$ is a double root, and the line is a tangent at $T(4, -8)$.
2 b i $\alpha + \alpha = 4$ **ii** $b = -4$
iii $y = -4 - 2x$, $T = (2, -8)$.
3 b $\alpha + \beta = 4$, $M = (2, 3)$
4 b The roots are 1, 1 and 3.
c The line is a tangent at $(1, 2)$ because $x = 1$ is a double root of the equation. The other point is $(3, 0)$.

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5 b i $\alpha + \alpha + 0 = 5$
ii $m = -\frac{1}{4}$.
iii $y = -\frac{1}{4}x$, $T = \left(\frac{5}{2}, -\frac{5}{8}\right)$.

6 b $\alpha + \beta + 2 = 5$, $M = \left(\frac{3}{2}, -\frac{1}{2}\right)$
d $\sqrt{26}$

7 c The line intersects the curve at $x = -1$ and is tangent to the curve at $x = \alpha$. $\alpha = 2$, $P = (2, 5)$, $m = 4$

8 a $y = mx - mp + p^3$
c $x = -\frac{1}{2}p$, so M lies on $x = -\frac{1}{2}p$.

9 a $\alpha = 1$ and $m = 2$
b $y + 3 = m(x + 2)$
c $y = 2x + 1$

10 a $y = (x + 1)(x - 2)(x - 5)(x + 2)$
b Because the line ℓ is tangent to the curve at A and B.
c $\alpha + \beta = 2$, $\alpha^2 + \beta^2 + 4\alpha\beta = -9$, $2\alpha^2\beta + 2\alpha\beta^2 = m - 16$, $\alpha^2\beta^2 = 20 - b$
d $m = -10$, $b = -22\frac{1}{4}$, $y = -10x - 22\frac{1}{4}$

11 a $k = \frac{1}{4}$, $\left(\frac{\sqrt{2}}{2}, \frac{1}{4}\right)$ and $\left(-\frac{\sqrt{2}}{2}, \frac{1}{4}\right)$
b $k = 0$ and $T(0, 0)$ or $k = \frac{4}{27}$ and $T\left(\frac{2}{3}, \frac{8}{27}\right)$
c $k = \frac{5}{4}$, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

13 c $M = \left(\frac{\lambda + 2}{2\lambda}, -\frac{\lambda + 2}{2\lambda}\right)$, locus: $y = -x$
d $\lambda = 2(\sqrt{2} + 1)$
e $\lambda < -2(\sqrt{2} - 1)$ or $\lambda > 2(\sqrt{2} + 1)$, but $\lambda \neq -1$

Chapter 10 review exercise

1 a 3 b 2 c $2x^3$ d -11
2 a 3 b 9
3 -35
4 b $1 < x < 3$
6 a $Q(x) = 2x^2 + 13x + 35$, $R(x) = 110$
b $2x^3 + 7x^2 - 4x + 5$
$= (x - 3)(2x^2 + 13x + 35) + 110$
7 a -17 b -27
8 b $P(x) = (x - 2)(x - 3)(x + 5)$
9 $k = -1$
10 $b = -3$ and $c = -11$
11 $h = 5$ and $k = -9$
12 a The divisor has degree 2, so the remainder has degree 1 or 0.
b $a = -6$ and $b = 4$
13 a 6 b -4 c -24 d $-\frac{3}{2}$ e -13 f 44
14 a -10 b 5 c 20 d $\frac{1}{4}$ e $-\frac{1}{2}$
f -2 g 100 h 90 i $\frac{9}{40}$
15 a $\alpha = -9$ b $d = -189$ c $c = -57$
16 a $\gamma = \frac{1}{2}$ b $-\frac{2}{3}$ and 3
18 $x = -\frac{1}{3}$, 1 or $\frac{7}{3}$
19 $\frac{1}{4}, \frac{1}{2}$ and 1
20 a -2
b 5, $P(x) = (x + 2)^2(x - 5)$
21 a $\frac{5}{6}$ and 1 b 1 c $\frac{2}{3}$
22 a $k = 28$ or -80
b When $k = 28$, $P(x) = (x - 2)^2(x + 7)$ and when $k = -80$, $P(x) = (x + 4)^2(x - 5)$.
23 a $x^3 - 3x^2 - 9x - 5 = 0$
b The line is tangent to the curve at $x = -1$ and intersects the curve at $x = \alpha$. So -1 is a double root and α is a single root.
c $B = (5, 50)$

Chapter 11

Exercise 11A

1 e All the ratios are about 0.7.
f $\frac{dy}{dx} \doteqdot 0.7y$

x	-2	-1	0	1	2
height y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
gradient $\frac{dy}{dx}$	0.17	0.35	0.69	1.39	2.77
$\frac{\text{gradient}}{\text{height}}$	0.69	0.69	0.69	0.69	0.69

2 b Both are equal to 1.
c

height y	$\frac{1}{2}$	1	2	3
gradient $\frac{dy}{dx}$	$\frac{1}{2}$	1	2	3
$\frac{\text{gradient}}{\text{height}}$	1	1	1	1

d They are all equal to 1. e $\frac{dy}{dx} = y$.

3 c The values are: 0.14, 0.37, 1, 2.72.
d The x-intercept is always 1 unit to the left of the point of contact.

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4 a **i** AB has gradient 1

ii The curve is concave up, so the chord is steeper than the tangent.

b **i** CA has gradient 1

ii The curve is concave up, so the chord is not as steep as the tangent.

c As the base increases, the gradient at the y-intercept increases. With $y = 2^x$, the gradient at the y-intercept is less than 1, and with $y = 4^x$, the gradient at the y-intercept is greater than 1. Hence the base e for which the y-intercept is exactly 1 is between 2 and 4.

6 The values get closer and closer to the limit $\log_e 2 \doteqdot 0.69315$

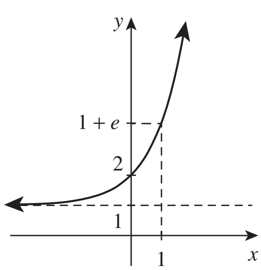
Exercise 11B

1 a 7.3891 **b** 22 026.4658 **c** 1.0000 **d** 2.7183
e 0.3679 **f** 0.1353 **g** 1.6487 **h** 0.6065
i 0.9990 **j** 0.0025

2 a $e^{-1} \doteqdot 0.3679$ **b** $e^{-4} \doteqdot 0.01832$
c $e^{\frac{1}{3}} \doteqdot 1.396$ **d** $e^{-\frac{1}{2}} \doteqdot 0.6065$
e 2.061×10^{-9} **f** 1.069×10^{13}

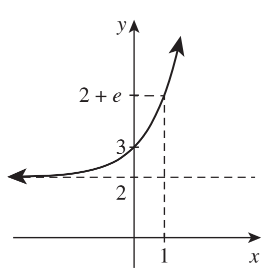
3 a $5e^2 \doteqdot 36.95$ **b** $\frac{1}{64}e^6 \doteqdot 6.304$
c $7e^{\frac{1}{2}} \doteqdot 11.54$ **d** $\frac{3}{5}e^{\frac{1}{2}} \doteqdot 0.9892$
e $4e^{-1} \doteqdot 1.472$ **f** $\frac{5}{7}e^{-4} \doteqdot 0.01308$

4 a



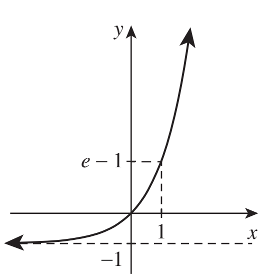
Shift up 1 unit, asymptote: $y = 1$, range: $y > 1$

b



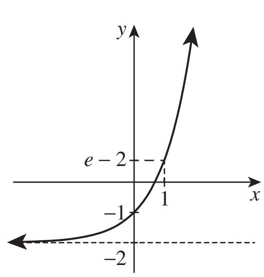
Shift up 2 units, asymptote: $y = 2$, range: $y > 2$

c



Shift down 1 unit, asymptote: $y = -1$, range: $y > -1$

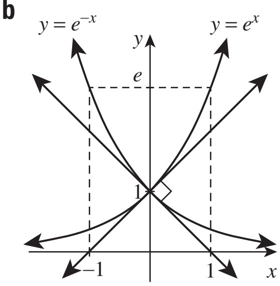
d



Shift down 2 units, asymptote: $y = -2$, range: $y > -2$

5 a For e^x, 0.14, 0.37, 1.00, 2.72, 7.39.
For $y = e^{-x}$, 7.39, 2.72, 1.00, 0.37, 0.14

b

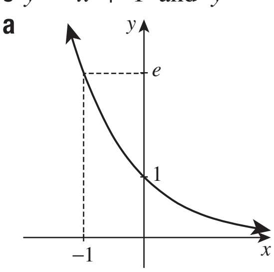


c Reflection in the y-axis.

d The graph of $y = e^{-x}$ is the reflection of $y = e^x$ in the y-axis, so its gradient at the y-intercept is -1. Hence the two tangents are perpendicular because the product of their gradients is -1 (or because $45° + 45° = 90°$).

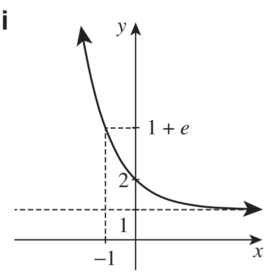
e $y = x + 1$ and $y = -x + 1$

6 a



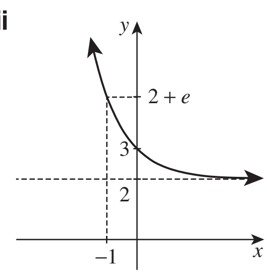
Asymptote: $y = 0$, range: $y > 0$

b **i**



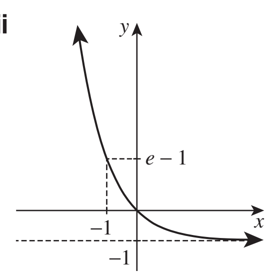
Shift up 1 unit, asymptote: $y = 1$, range: $y > 1$

ii



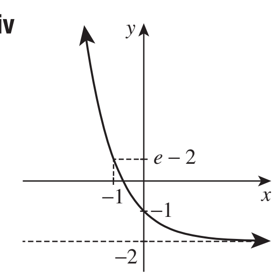
Shift up 2 units, asymptote: $y = 2$, range: $y > 2$

iii



Shift down 1 unit, asymptote: $y = -1$, range: $y > -1$

iv



Shift down 2 units, asymptote: $y = -2$, range: $y > -2$

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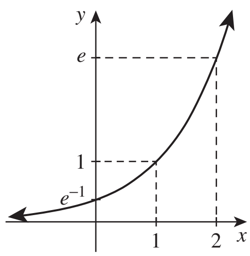
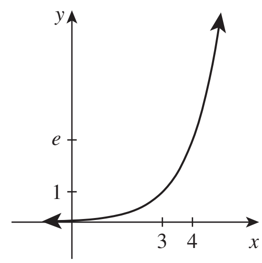
7 a Shift right 1 unit

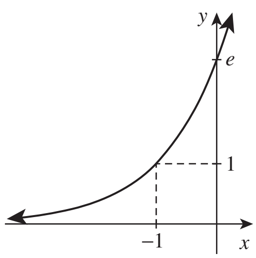
b



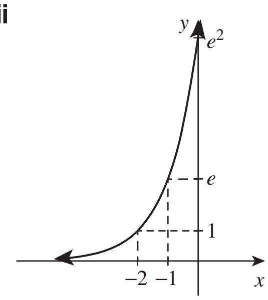
c i



ii



iii



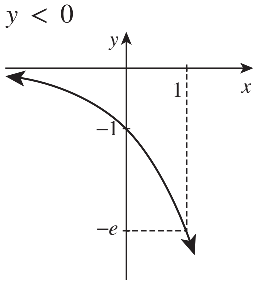
8 a $1, e, e^2$

b grad $AB = e - 1$, AB: $y - 1 = (e - 1)x$

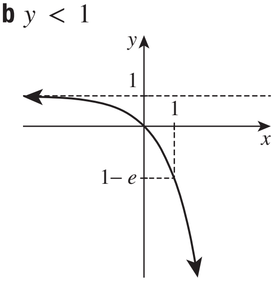
c grad $BC = e(e - 1)$,
BC: $y - e = e(e - 1)(x - 1)$

d grad $PQ = e^a(e - 1)$,
PQ: $y - e^a = e^a(e - 1)(x - a)$

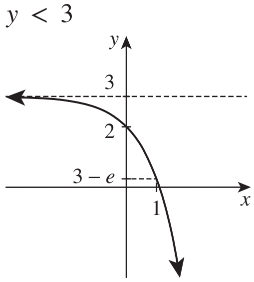
9 a $y < 0$



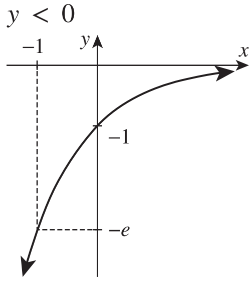
b $y < 1$



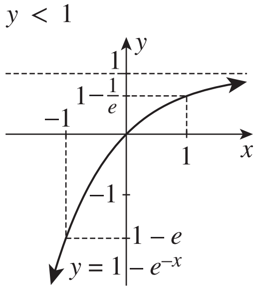
c $y < 3$

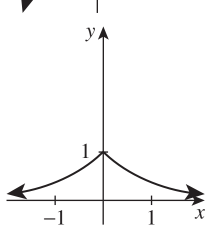


d $y < 0$



e $y < 1$





f $0 < y \leq 1$

Exercise 11C

1 a $2e^{2x}$ **b** $7e^{7x}$ **c** $-e^{-x}$ **d** $-5e^{5x}$
e $\frac{1}{2}e^{\frac{x}{2}}$ **f** $2e^{\frac{1}{3}x}$ **g** $-\frac{1}{3}e^{-\frac{x}{3}}$ **h** $e^{\frac{x}{5}}$

2 a $f'(x) = e^{x+2}$ **b** $f'(x) = e^{x-3}$
c $f'(x) = 5e^{5x+1}$ **d** $f'(x) = 2e^{2x-1}$
e $f'(x) = -4e^{-4x+1}$ **f** $f'(x) = -3e^{-3x+4}$
g $f'(x) = -3e^{-3x-6}$ **h** $f'(x) = e^{\frac{x}{2}} + 4$

3 a $e^x - e^{-x}$ **b** $2e^{2x} + 3e^{-3x}$
c $e^{2x} + e^{3x}$ **d** $e^{4x} + e^{5x}$
e $\dfrac{e^x + e^{-x}}{2}$ **f** $\dfrac{e^x - e^{-x}}{3}$

4 a $y' = 2e^{2x}$
b When $x = 0$, $y' = 2$. When $x = 4$, $y' = 2e^8$.

5 a $f'(x) = -e^{-x+3}$
b When $x = 0$, $f'(x) = -e^3$.
c When $x = 4$, $f'(x) = -e^{-1}$.

6 a $y' = 3e^{3x}$, $y'(2) = 3e^6 \doteqdot 1210.29$
b $y' = -2e^{-2x}$, $y'(2) = -2e^{-4} \doteqdot -0.04$
c $y' = \frac{3}{2}e^{\frac{3x}{2}}$, $y'(2) = \frac{3}{2}e^3 \doteqdot 30.13$

7 a $-e^{-x}, e^{-x}, -e^{-x}, e^{-x}$. Successive derivatives alternate in sign. More precisely,
$$f^{(n)}(x) = \begin{cases} e^{-x}, & \text{if } n \text{ is even,} \\ -e^{-x}, & \text{if } n \text{ is odd.} \end{cases}$$
b $2e^{2x}, 4e^{2x}, 8e^{2x}, 16e^{2x}$. Each derivative is twice the previous one. More precisely,
$f^{(n)}(x) = 2^n e^{2x}$.
c e^x, e^x, e^x, e^x All derivatives are the same, and are equal to the original function.
d $y' = e^x + 2x + 1$, $y'' = e^x + 2$,
y''' and all subsequent derivatives are e^x.

8 a $5e^{5x} + 7e^{7x}$ **b** $4e^{4x+2} + 8e^{5+8x}$
c $-4e^{-x} - 12e^{-3x}$ **d** $-12e^{-2x-3} + 42e^{5-6x}$
e $10x - 4 + 3e^{-x}$ **f** $\frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{2}x^{-\frac{1}{2}}$

9 a $y' = \frac{1}{2}\sqrt{e^x}$ **b** $y' = \frac{1}{3}\sqrt[3]{e^x}$
c $y' = -\dfrac{1}{2\sqrt{e^x}}$ **d** $y' = -\dfrac{1}{3\sqrt[3]{e^x}}$

10 a $y' = ae^{ax}$ **b** $y' = -ke^{-kx}$
c $y' = Ake^{kx}$ **d** $y' = -B\ell e^{-\ell x}$
e $y' = pe^{px+q}$ **f** $y' = pCe^{px+q}$
g $y' = \dfrac{pe^{px} - qe^{-qx}}{r}$ **h** $e^{ax} - e^{-px}$

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Exercise 11D

1 a 1 **b** $y = x + 1$

2 a e **b** $y = ex$

3 a $\frac{1}{e}$ **b** $y = \frac{1}{e}(x + 2)$

4 a $A = \left(\frac{1}{2}, 1\right)$ **b** $y' = 2e^{2x-1}$ **c** $y = 2x$

5 a $y' = e^x$, which is always positive.

b $y' = -e^{-x}$, which is always negative.

6 a $e - 1$

b $\frac{dy}{dx} = e^x$. When $x = 1$, $\frac{dy}{dx} = e$.

c $y = ex - 1$ **d i** never **ii** all real x **iii** never

7 a $y' = 1 - e^x$ **b** $1 - e$

c $y = (1 - e)x$. When $x = 0$, $y = 0$.

d i $x = 0$ **ii** $x < 0$ **iii** $x > 0$

8 a $R = \left(-\frac{1}{3}, 1\right)$ **b** $y' = 3e^{3x+1}$

c $-\frac{1}{3}$ **d** $3x + 9y - 8 = 0$.

9 a $-e$ **b** $\frac{1}{e}$

c $x - ey + e^2 + 1 = 0$

d $x = -e^2 - 1$, $y = e + e^{-1}$

e $\frac{1}{2}(e^3 + 2e + e^{-1})$

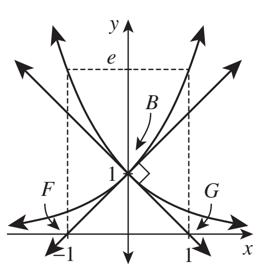
10 a 1

b $y = x + 1$

c -1

d $y = -x + 1$

e



f isosceles right-angled triangle, 1 square unit

11 a 1, $45°$ **b** e, $69°48'$

c e^{-2}, $7°42'$ **d** e^5, $89°37'$

12 $A = (1, e^{-2})$, $B = (2, 1)$, $y' = 2e^{2x-4}$

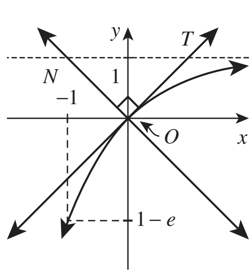
a $y' = 2e^{-2}$ **b** $y' = 2$ **c** $1 - e^{-2}$

13 a $y = e^t(x - t + 1)$

14 b $y = -x$

c $y = 1$

d



e 1 square unit

15 a $y = ex$. When $x = 0$, $y = 0$.

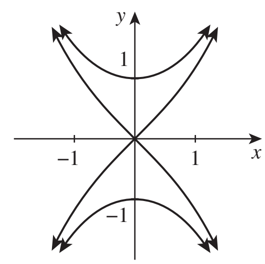
b $x + ey = 1 + e^2$, $A = (1 + e^2, 0)$,

$B = \left(0, \frac{1 + e^2}{e}\right)$

c The two triangles are similar, and their perpendicular heights are in the ratio $(e : 1)$ (or use the coordinates of (A) and (B)).

d $AP : PB = e^2 : 1$ because both triangles can be regarded as having perpendicular height OP.

16 For $y = \cosh x$, $y' = 0$ when $x = 0$. For $y = \sinh x$, $y' = 1$ when $x = 0$. The only point of intersection is the origin.



Exercise 11E

1 a 0 **b** 0.6931 **c** 1.0986 **d** 2.0794

e -0.6931 **f** -1.0986 **g** -2.0794 **h** -2.3026

4 a $e^x = 1$, $x = 0$ **b** $e^x = e$, $x = 1$

c $e^x = e^2$, $x = 2$ **d** $e^x = \frac{1}{e}$, $x = -1$

e $e^x = \frac{1}{e^2}$, $x = -2$ **f** $e^x = \sqrt{e}$, $x = \frac{1}{2}$

5 a $2 \log_e e = 2$ **b** $5 \log_e e = 5$

c $200 \log_e e = 200$ **d** $-6 \log_e e = -6$

e $\log_e e^{-6} = -6 \log_e e = -6$

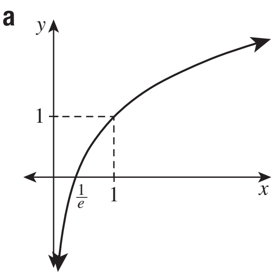
f $-\log_e e = -1$

g $\log_e e^{-1} = -\log_e e = -1$

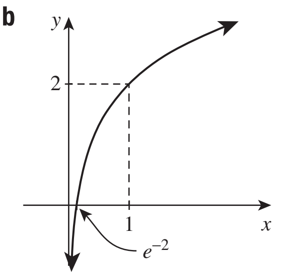
h $\frac{1}{2} \log_e e = \frac{1}{2}$ **i** $\frac{1}{2} \log_e e = \frac{1}{2}$

j $\log_e e^{-\frac{1}{2}} = -\frac{1}{2} \log_e e = -\frac{1}{2}$

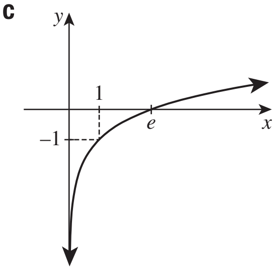
6 a



b



c



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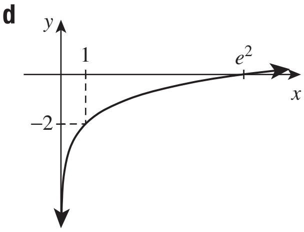

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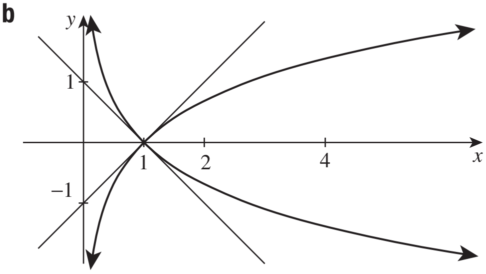
7 b 

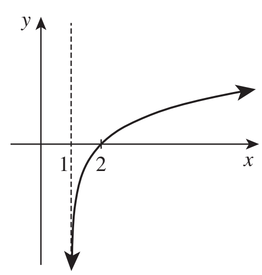
c The graph of $y = -\log_e x$ is obtained by reflecting the first in the x-axis. Hence its tangent has gradient -1, and the two are perpendicular.

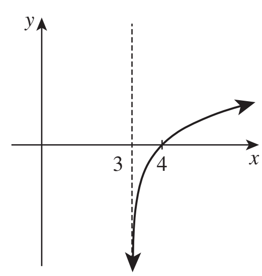
8 a e **b** $-\dfrac{1}{e}$ **c** 6 **d** $\frac{1}{2}$ **e** $2e$

f 0 **g** e **h** 1 **i** 0

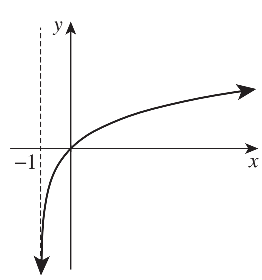
9 a $\log_e 6$ **b** $\log_e 4$ **c** $\log_e 4$ **d** $\log_e 27$

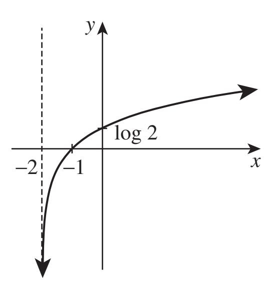
10 a $x > 1$ **b** $x > 3$



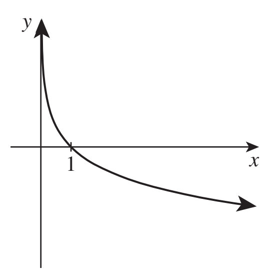


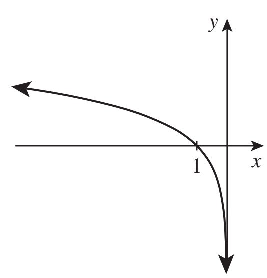
c $x > -1$ **d** $x > -2$





e $x > 0$ **f** $x < 0$



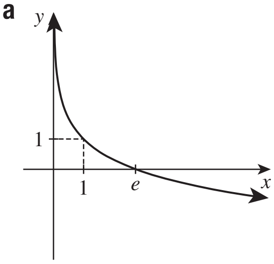


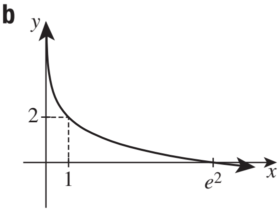
11 a $\log_e \dfrac{a}{b} = \log_e a - \log_e b$ and

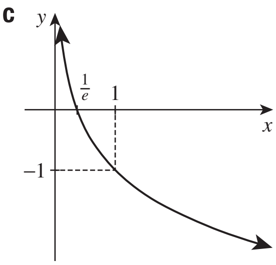
$-\log_e \dfrac{b}{a} = -\log_e b + \log_e a$

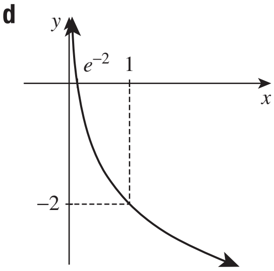
b $\log_{\frac{1}{e}} x = \dfrac{\log_e x}{\log_e \frac{1}{e}} = \dfrac{\log_e x}{-1} = -\log_e x$

c Using part **b**, $\log_{\frac{1}{e}} x^{-1} = -\log_e x^{-1} = +\log_e x$

12 a 

b 

c 

d 

Exercise 11F

1 a $\dfrac{dQ}{dt} = 900e^{3t}$ **b** $Q = 300e^6 \doteqdot 121\,000$. $\dfrac{dQ}{dt} = 900e^6 \doteqdot 363\,100$ **c** 60 360

2 a $\dfrac{dQ}{dt} = -20\,000e^{-2t}$ **b** $Q = 10\,000e^{-8} \doteqdot 3.355$ $\dfrac{dQ}{dt} = -20\,000e^{-8} \doteqdot -6.709$ **c** -2499

3 a $\dfrac{dQ}{dt} = 10e^{2t}$

b Put $1000 = 5e^{2t}$, $t = \frac{1}{2}\log_e 200 \doteqdot 2.649$

c Put $1000 = 10e^{2t}$, $t = \frac{1}{2}\log_e 100 \doteqdot 2.303$

4 a $P = 2000e^{1.5} \doteqdot 8963$ individuals

b $\dfrac{dP}{dt} = 600e^{0.3t}$

c $\dfrac{dP}{dt} = 600e^{1.5} \doteqdot 2689$ individuals per hour

d 1393 individuals per hour

5 a $C = 2000e^{-4} \doteqdot 36.63$ **b** $\dfrac{dC}{dt} = -4000e^{-2t}$

c $\dfrac{dC}{dt} = -4000e^{-4} \doteqdot -73.26$ per year

d -981.7 per year

6 a $t = 25\log_e 2 \doteqdot 17.33$ years **b** $\dfrac{dP}{dt} = 6e^{0.04t}$

c $t = 25\log_e 50 \doteqdot 97.80$ years

7 a $\dfrac{dP}{dt} = 400e^{0.4t}$ **b** $P = 1000e^2 \doteqdot 7400$ cats, $\dfrac{dP}{dt} = 400e^2 \doteqdot 3000$ cats per year

c $t = \frac{5}{2}\log_e 20 \doteqdot 7.5$ years

d $t = \frac{5}{2}\log_e 50 \doteqdot 9.8$ years

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8 a $t = -10 \log_e\left(\frac{1}{2}\right) = 10 \log_e 2 \doteq 6.931$ years

b $\dfrac{dM}{dt} = -\frac{1}{10}M_0 e^{-0.1t}$

c $(1 - e^{-0.1}) \times 100\% \doteq 9.516\%$

d When $\dfrac{dM}{dt} = -\frac{1}{100}M_0$,

$t = -10 \log_e\left(\frac{1}{10}\right) = 10 \log_e 10 \doteq 23.03$ years

9 a $\dfrac{dQ}{dt} = e^t$, which is always positive, so Q is increasing. Also $\dfrac{dQ}{dt}$ is increasing, so Q is increasing at an increasing rate.

b $\dfrac{dQ}{dt} = -e^{-t}$, which is always negative, so Q is decreasing. Also $\dfrac{dQ}{dt}$ is increasing, so the rate of change of Q is increasing, thus Q is decreasing at a decreasing rate. (The language here is not entirely satisfactory — more on this in Year 12.)

c i A and k both positive or both negative.

ii One positive and one negative.

d If $A = 0$, Q is the zero function. If $k = 0$, Q is the constant function $Q = A$.

11 a $\lambda = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $\lambda = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

b When $b^2 - 4ac < 0$ **c** $y = Ae^{2x}$ and $y = Ae^{5x}$

Exercise 11G

1 a $\frac{\pi}{2}$ **b** $\frac{\pi}{4}$ **c** $\frac{\pi}{6}$ **d** $\frac{\pi}{3}$ **e** $\frac{2\pi}{3}$ **f** $\frac{5\pi}{6}$
g $\frac{3\pi}{4}$ **h** $\frac{5\pi}{4}$ **i** 2π **j** $\frac{5\pi}{3}$ **k** $\frac{3\pi}{2}$ **l** $\frac{7\pi}{6}$

2 a 180° **b** 360° **c** 720° **d** 90° **e** 60°
f 45° **g** 120° **h** 150° **i** 135° **j** 270°
k 240° **l** 315° **m** 330°

3 a 0.84 **b** −0.42 **c** −0.14 **d** 0.64
e 0.33 **f** −0.69

4 a 1.274 **b** 0.244 **c** 2.932 **d** 0.377
e 1.663 **f** 3.686

5 a 114°35′ **b** 17°11′ **c** 82°30′ **d** 7°3′
e 183°16′ **f** 323°36′

6 a $\frac{1}{2}$ **b** $\frac{1}{\sqrt{2}}$ **c** $\frac{\sqrt{3}}{2}$ **d** $\sqrt{3}$
e 1 **f** $\frac{1}{2}$ **g** $\sqrt{2}$ **h** $\frac{1}{\sqrt{3}}$

7 a $\frac{\pi}{9}$ **b** $\frac{\pi}{8}$ **c** $\frac{\pi}{5}$ **d** $\frac{5\pi}{9}$ **e** $\frac{5\pi}{8}$ **f** $\frac{7\pi}{5}$

8 a 15° **b** 72° **c** 400° **d** 247.5°
e 306° **f** 276°

9 a $\frac{\pi}{3}$ **b** $\frac{5\pi}{6}$

10 $\frac{4\pi}{9}$

11 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{2}$ **c** $-\frac{\sqrt{3}}{2}$ **d** $\sqrt{3}$
e -1 **f** $\frac{1}{2}$ **g** $-\frac{1}{\sqrt{2}}$ **h** $\frac{1}{\sqrt{3}}$

12 a Hour hand: 30° or $\frac{\pi}{6}$ radians, minute hand: 360° or 2π radians.

b i 60° or $\frac{\pi}{3}$ radians **ii** $22\frac{1}{2}$° or $\frac{\pi}{8}$ radians
iii 105° or $\frac{7\pi}{12}$ radians **iv** $172\frac{1}{2}$° or $\frac{23\pi}{24}$ radians

13 a 0.733 **b** 0.349 **c** 63 334.508

14 a 0.283 **b** 0.819

15 a $k\pi$ is never an integer when k is an integer, except when $k = 0$.

b $n = 22$ **c** $\sin 22 \doteq \sin 7\pi = 0$

Exercise 11H

1 a $\frac{\pi}{4}$ **b** $\frac{\pi}{6}$ **c** $\frac{\pi}{4}$ **d** $\frac{\pi}{6}$ **e** $\frac{\pi}{3}$ **f** $\frac{\pi}{3}$

2 a $x \doteq 1.249$ **b** $x \doteq 0.927$ **c** $x \doteq 1.159$
d $x \doteq 0.236$ **e** $x \doteq 0.161$ **f** $x \doteq 1.561$

3 a $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ **b** $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$
c $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ **d** $x = \frac{\pi}{2}$
e $x = \frac{\pi}{6}$ or $\frac{11\pi}{6}$ **f** $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$
g $x = \pi$ **h** $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

4 a $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ **b** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$
c $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ or $\frac{5\pi}{3}$ **d** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

5 a $u^2 - u = 0$ **b** $u = 0$ or $u = 1$
c $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$ or 2π

6 a $u^2 - u - 2 = 0$ **b** $u = -1$ or $u = 2$
c $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$, or $\theta \doteq 1.11$ or 4.25

7 a $\theta = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$ or 2π
b $\theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$ or 2π
c $\theta = \frac{\pi}{2}$
d $\theta \doteq 1.11, 1.89, 4.25$ or 5.03
e $\theta = \frac{\pi}{3}, \pi$ or $\frac{5\pi}{3}$ **f** $\theta = \frac{\pi}{2}, \frac{7\pi}{6}$ or $\frac{11\pi}{6}$
g $\theta \doteq 0.34$ or 2.80 **h** $\theta \doteq 1.91$ or 4.37

8 a $\theta = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}$ or $\frac{5\pi}{3}$

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b $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$, or $x \doteq 1.25$ or 4.39

c $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$, or $x \doteq 0.25$ or 2.89

d $x \doteq 0.84$ or 5.44

9 a $\alpha = \frac{\pi}{2}$, or $\alpha \doteq 3.48$ or 5.94

b $\alpha \doteq 1.11, 2.82, 4.25$ or 5.96

10 a $x = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$ or 2π

b $x \doteq 1.11, 1.25, 4.25$ or 4.39

11 a $x = -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}$ or $\frac{5\pi}{12}$

b $x = -\pi, -\frac{\pi}{3}, \frac{\pi}{3}$ or π **c** $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$

d $x = -\pi$ or $\frac{\pi}{2}$ or π **e** $x = -\frac{3\pi}{4}$

f $x = -\frac{2\pi}{3}$ or $\frac{\pi}{3}$

13 $\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}$ or $\frac{9\pi}{5}$

Exercise 11I

1 a 12 cm **b** 3 cm **c** 2π cm **d** $\frac{3\pi}{2}$ cm

2 a $32\,\text{cm}^2$ **b** $96\,\text{cm}^2$ **c** $8\pi\ \text{cm}^2$ **d** $12\pi\ \text{cm}^2$

3 4 cm

4 1.5 radians

5 a 2.4 cm **b** 4.4 cm

6 $8727\,\text{m}^2$

7 a 8π cm **b** $16\pi\ \text{cm}^2$

8 84°

9 11.6 cm

10 a $6\pi\ \text{cm}^2$ **b** $9\sqrt{3}\ \text{cm}^2$ **c** $3(2\pi - 3\sqrt{3})\ \text{cm}^2$

12 $15\ \text{cm}^2$

13 a $4(\pi + 2)$ cm **b** $8\pi\ \text{cm}^2$

14 a $\frac{2\pi}{3}$ cm **b** $\frac{2\pi}{3}\ \text{cm}^2$ **c** 2π cm

d $\sqrt{3}\ \text{cm}^2, 2(\pi - \sqrt{3})\ \text{cm}^2$

15 $\frac{4}{3}(4\pi - 3\sqrt{3})\ \text{cm}^2$

16 c $3\sqrt{55}\,\pi\ \text{cm}^3$ **d** $24\pi\ \text{cm}^2$

17 a 720 metres **b** 2.4 radians (about 137°31′)

c 559.22 metres **d** 317°31′T

18 a By Pythagoras, $h^2 = r^2 - \left(\frac{r}{2}\right)^2 = \frac{3}{4}r^2$. By the area formula, $A = \frac{1}{2}r^2\sin 60° = \frac{1}{2}r^2 \times \frac{\sqrt{3}}{2}$.

b i Partition the hexagon into six equilateral triangles. An interval is the shortest distance between two points, so
$6r$ = perimeter < circumference = $2\pi r$, so
$3 < \pi$.

ii Each equilateral triangle of the outer hexagon has height r, and hence by part **a** has side length $s = \frac{2r}{\sqrt{3}}$ and area $\frac{1}{4} \times \frac{4r^2}{3} \times \sqrt{3} = \frac{1}{3}r^2\sqrt{3}$. The circle lies inside the outer hexagon,

so πr^2 = area of circle < area of outer hexagon

$= 6 \times \frac{1}{3}r^2\sqrt{3} = 2r^2\sqrt{3}$, so $\pi < 2\sqrt{3}$.

19 $2.54\,\text{cm}^2$

20 36 seconds

Exercise 11J

2 a All six graphs are many-to-one.

b i $\pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$

ii $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$

iii $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}, \frac{21\pi}{2}$

iv There are no solutions.

3 a $x = \frac{\pi}{2}, x = -\frac{\pi}{2}, x = \frac{3\pi}{2}, x = -\frac{3\pi}{2}, x = \frac{5\pi}{2}, x = -\frac{5\pi}{2}, \ldots$

b $y = \operatorname{cosec} x$, the reciprocal of $y = \sin x$.

c Neither graph has any line symmetries.

4 a $x = 0, x = \pi, x = -\pi, x = 2\pi, x = -2\pi, \ldots$

b Line symmetry in the y-axis $x = 0$

c $y = \sec x$, the reciprocal of $y = \cos x$.

5 a $(0, 0), (\pi, 0), (-\pi, 0), (2\pi, 0), (-2\pi, 0), \ldots$

b Point symmetry in the origin $(0, 0)$

c $y = \operatorname{cosec} x$, the reciprocal of $y = \sin x$.

6 a $\left(\frac{\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(-\frac{3\pi}{2}, 0\right), \ldots$

b $y = \sec x$, the reciprocal of $y = \cos x$.

7 a $(0, 0), \left(\frac{\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), (\pi, 0), (-\pi, 0), \left(\frac{3\pi}{2}, 0\right), \left(-\frac{3\pi}{2}, 0\right), \ldots$

b Both functions are odd, because both have point symmetry in the origin. Neither is even, because neither have line symmetry in the y-axis.

8 a Translations left or right by multiples of 2π.

b $y = \cos x$, $y = \operatorname{cosec} x$ and $y = \sec x$.

c $y = \tan x$ and $y = \cot x$ can each be mapped onto themselves by translations left or right by multiples of π.

d $y = \sin x, y = \cos x, y = \operatorname{cosec} x, y = \sec$ each has period 2π. $y = \tan x, y = \cot x$ each has period π.

9 a $x = \frac{\pi}{4}, x = -\frac{3\pi}{4}, x = \frac{5\pi}{4}, x = -\frac{7\pi}{4}, \ldots$

b $y = \operatorname{cosec} x$ and $y = \sec x$

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c $x = \frac{\pi}{4}, x = -\frac{\pi}{4}, x = \frac{3\pi}{4}, x = -\frac{3\pi}{4}, x = \frac{5\pi}{4},$

$x = -\frac{5\pi}{4}, \ldots$

10 a Translations left $\frac{\pi}{2}, \frac{5\pi}{2}, \ldots$, and right $\frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

b $y = \sin(x - \theta)$ is $y = \sin x$ shifted right by θ, so $\sin(x - \theta) = \cos x$ for $\theta = \frac{3\pi}{2}, -\frac{\pi}{2}, \frac{7\pi}{2}, -\frac{5\pi}{2}, \frac{11\pi}{2}, -\frac{9\pi}{2}, \ldots$

c There are none.

11 There are none.

12 a $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(-\frac{3\pi}{4}, -\frac{1}{\sqrt{2}}\right), \left(\frac{5\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(-\frac{7\pi}{4}, -\frac{1}{\sqrt{2}}\right), \ldots$

b $\sin x = \cos x$, so $\tan x = 1$.

13 a $(0, 0), (\pi, 0), (-\pi, 0), (2\pi, 0), (-2\pi, 0), \ldots$

b $\sin x = \frac{\sin x}{\cos x}$, so $\sin x \cos x = \sin x$, so $\sin x(\cos x - 1) = 0$, so $\sin x = 0$ or $\cos x = 1$.

14 a Roughly 0.7 (radians).

15 a They touch each other at their maxima and minima.

b $y = \cos x$ and $y = \sec x$.

c $y = \sin x$ & $y = \sec x$, $y = \cos x$ & $y = \operatorname{cosec} x$, $y = \tan x$ & $y = \sec x$, $y = \cot x$ & $y = \operatorname{cosec} x$

16 a

$$\cos x = \frac{\sin x}{\cos x}$$

$\boxed{\times \cos x}$ $\cos^2 x = \sin x$ and $\cos x \neq 0$

$$1 - \sin^2 x = \sin x$$
$$\sin^2 x + \sin x - 1 = 0$$
$$\Delta = 1 + 4 = 5$$
$$\sin x = \frac{-1 + \sqrt{5}}{2}$$

giving solutions in the first and second quadrants.

$\left(\frac{-1 - \sqrt{5}}{2} < -1,\right.$ so $\sin x = \frac{-1 - \sqrt{5}}{2}$ has no solutions.$\left.\right)$

b

$$\frac{\sin x}{\cos x} = \frac{1}{\cos x}$$

$\boxed{\times \cos x}$ $\sin x = 1$ and $\cos x \neq 0$

There are no solutions, because if $\sin x = 1$, then $\cos x = 0$.

Chapter 11 review exercise

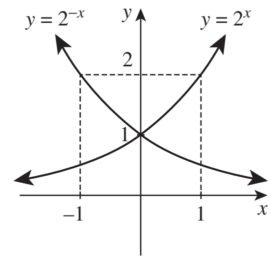
1 a 3^9 **b** 3^{12} **c** 3^5 **d** 6^5

2 a $\frac{1}{5}$ **b** $\frac{1}{100}$ **c** $\frac{1}{x^3}$ **d** $\frac{1}{3^x}$

3 a 3 **b** 3 **c** 4 **d** $\frac{1}{4}$ **e** $\frac{1}{9}$ **f** $\frac{1}{1000}$

4 a 2^{3x} **b** 2^{4x} **c** 2^{6x} **d** 10^x **e** 2^{2x+3} **f** 2^{2x-1}

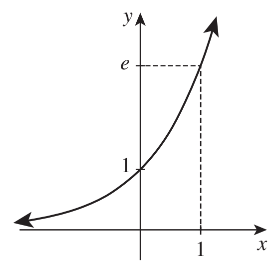
5 Each graph is reflected onto the other graph in the line $x = 0$.



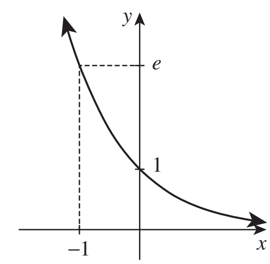
6 a 2.718 **b** 54.60 **c** 0.1353 **d** 4.482

7 a e^{5x} **b** e^{6x} **c** e^{-4x} **d** e^{9x}

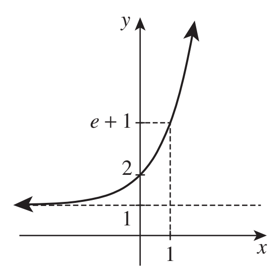
8 a $y > 0$



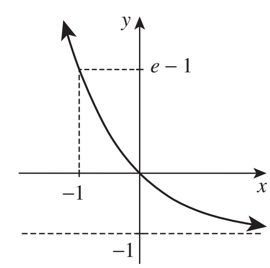
b $y > 0$



c $y > 1$



d $y > -1$



9 a e^x **b** $3e^{3x}$ **c** e^{x+3} **d** $2e^{2x+3}$ **e** $-e^{-x}$ **f** $-3e^{-3x}$ **g** $-2e^{3-2x}$ **h** $6e^{2x+5}$ **i** $2e^{\frac{1}{2}x}$ **j** $3x^2e^{x^3}$ **k** $(2x - 3)e^{x^2-3x}$ **l** $4e^{6x-5}$

10 a $5e^{5x}$ **b** $4e^{4x}$ **c** $-3e^{-3x}$ **d** $-6e^{-6x}$

11 2

12 $y = e^2x - e^2$, x-intercept 1, y-intercept $-e^2$.

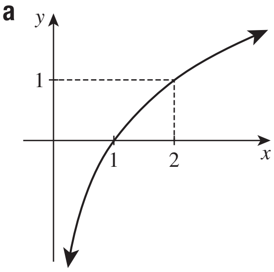
13 a 1.4314 **b** −0.3010 **c** 0.6931 **d** 2.6391

14 a 1.1761 **b** 0.4771 **c** 1.9459 **d** −1.0986

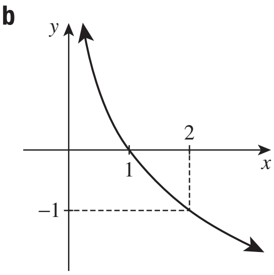
15 a 5 **b** $-\frac{1}{4}$ **c** 3 **d** $\frac{1}{5}$

16 a e **b** 3 **c** −1 **d** e

17 a



b



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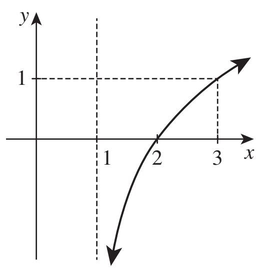
c



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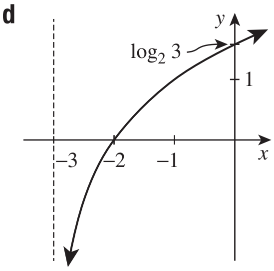
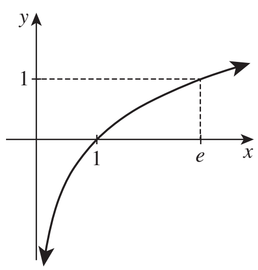


18 a



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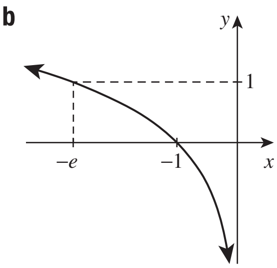

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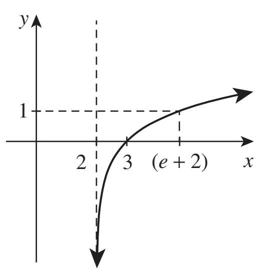

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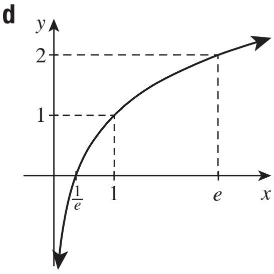

19 a $\frac{dP}{dt} = -\frac{1}{100}P_0 e^{-0.01t}$

b $\frac{dP}{dt} = -\frac{1}{100}P_0 e^{-0.45} = -0.0064\, P_0$ lizards per year.

c $P = P_0 e^{-0.45} \doteqdot 64\%$ of the original population.

d $e^{-0.01t} = \frac{1}{10}$, so $t = 100 \log_e 10 \doteqdot 230$ years

20 a π **b** $\frac{\pi}{9}$ **c** $\frac{4\pi}{3}$ **d** $\frac{7\pi}{4}$

21 a 30° **b** 108° **c** 540° **d** 300°

22 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{3}}$

23 a $x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$ **b** $x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$

24 a $\sin\theta = 0$ or $-\frac{1}{2}, \theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$ or 2π

b $\cos\theta = -1$ or $2, \theta = \pi$ ($\cos\theta = 2$ has no solutions.)

c $\tan\theta = \frac{1}{2}$ and $\theta \doteqdot 0.46$ or 3.61,

or $\tan\theta = -3$ and $\theta \doteqdot 1.89$ or 5.03

25 a 3π cm **b** 12π cm^2

26 $16(\pi - 2) \doteqdot 18.3\,\text{cm}^2$

27 148°58′

28 6.77 cm

29 a 1.2661 radians **b** 49.2 cm

30 a $y = \sin x$ and $y = \cos x$ both have amplitude 1.

b $y = \sin x$, $y = \cos x$, $y = \operatorname{cosec} x$ and $y = \sec x$ all have period 2π, $y = \tan x$ and $y = \cot x$ both have period π.

c $y = \sin x$, $y = \tan x$, $y = \operatorname{cosec} x$ and $y = \cot x$ are all odd, $y = \cos x$, and $y = \sec x$ are both even.

31 a $\theta = \frac{3\pi}{2}$ **b** $\theta = \frac{\pi}{2}$ **c** $x = \frac{\pi}{4}$

Chapter 12

Exercise 12A

1 a $\frac{1}{20}$ **b** $\frac{19}{20}$

2 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** 1 **d** 0

3 a $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** $\frac{1}{3}$ **d** $\frac{1}{3}$

4 a $\frac{5}{12}$ **b** $\frac{7}{12}$ **c** 0

5 a $\frac{4}{9}$ **b** $\frac{5}{9}$ **c** $\frac{11}{18}$

6 a $\frac{4}{9}$ **b** $\frac{5}{9}$ **c** $\frac{11}{18}$ **d** $\frac{7}{18}$ **e** $\frac{1}{3}$ **f** $\frac{1}{6}$

7 a $\frac{3}{8}$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$

8 a $\frac{1}{26}$ **b** $\frac{5}{26}$ **c** $\frac{21}{26}$ **d** 0 **e** $\frac{3}{26}$ **f** $\frac{5}{26}$

9 78%

10 a $\frac{4}{7}$ **b** 32

11 a 8 **b** $\frac{14}{15}$

12 a 10 sixes

b i $\frac{18}{60} = 30\%$

ii The experiment suggest a probability of about 30%.

iii The theoretical probability suggests that for an unbiased die, we would expect to get a six on one-sixth of the throws, that is, 10 times. The large number of sixes turning up suggests that this die is biased.

13 a $\frac{100}{400} = \frac{1}{4} = 25\%$ **b** $\frac{8}{20} = \frac{2}{5} = 40\%$

c We would expect him to get chicken one-quarter of the time, that is, on 5 occasions. He may have got more chicken sandwiches because of the way the canteen makes or sells the sandwiches, for example making the chicken sandwiches early and placing them at the front of the display, or making more Vegemite sandwiches as they sell out. Possibly also the sample is too small and the result would approach $\frac{1}{4}$ if the experiment were continued over a longer time. The experimental probability is only an estimate, and in fact it is possible he may have got no chicken sandwiches over the twenty days.

14 a $\frac{1}{20}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$ **d** $\frac{1}{2}$ **e** $\frac{2}{5}$

f $\frac{1}{5}$ **g** $\frac{1}{4}$ **h** 0 **i** 1

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15 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{13}$ **d** $\frac{1}{52}$
e $\frac{1}{4}$ **f** $\frac{3}{13}$ **g** $\frac{1}{2}$ **h** $\frac{1}{13}$
i $\frac{3}{13}$ (counting an ace as a one)

16 a $\frac{1}{15}$ **b** $\frac{7}{150}$ **c** $\frac{1}{2}$ **d** $\frac{4}{25}$ **e** $\frac{1}{75}$ **f** $\frac{17}{50}$

17 a $\frac{1}{5}$ **b** $\frac{3}{40}$ **c** $\frac{9}{20}$ **d** $\frac{7}{100}$ **e** $\frac{7}{50}$ **f** $\frac{1}{200}$

18 a $\frac{3}{4}$ **b** $\frac{1}{4}$

19 187 or 188

20 a The argument is invalid, because on any one day the two outcomes are not equally likely. The argument really can't be corrected.
b The argument is invalid. One team may be significantly better than the other, the game may be played in conditions that suit one particular team, and so on. Even when the teams are evenly matched, the high-scoring nature of the game makes a draw an unlikely event. The three outcomes are not equally likely. The argument really can't be corrected.
c The argument is invalid, because we would presume that Peter has some knowledge of the subject, and is therefore more likely to choose one answer than another. The argument would be valid if the questions were answered at random.
d The argument is only valid if there are equal numbers of red, white and black beads, otherwise the three outcomes are not equally likely.
e The argument is missing, but the conclusion is correct. Exactly one of the four players will win his semi-final and then lose the final. Our man is as likely to pick this player as he is to pick any of the other three players.

21 a $\frac{2}{9}$ **b** $\frac{\pi}{18}$

22 a $\frac{7}{8}$ **b** $\frac{\pi + 2}{9}$

Exercise 12B

1 a HH, HT, TH, TT
b i $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{4}$

2 a H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6
b i $\frac{1}{4}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{4}$ **iv** $\frac{1}{4}$

3 a TO, OT, OE, EO, ET, TE
b i $\frac{1}{3}$ **ii** $\frac{1}{3}$ **iii** $\frac{2}{3}$

4 a AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC
b i $\frac{1}{6}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{3}$ **iv** $\frac{1}{6}$ **v** $\frac{1}{4}$ **vi** $\frac{3}{4}$

5 a 23, 32, 28, 82, 29, 92, 38, 83, 39, 93, 89, 98
b i $\frac{1}{12}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$ **iv** $\frac{1}{6}$ **v** $\frac{1}{4}$ **vi** 0

6 a The captain is listed first and the vice-captain second: AB, AC, AD, AE, BC, BD, BE, CD, CE, DE, BA, CA, DA, EA, CB, DB, EB, DC, EC, ED
b i $\frac{1}{20}$ **ii** $\frac{2}{5}$ **iii** $\frac{3}{5}$ **iv** $\frac{1}{5}$

7 HHH, HHT, HTH, HTT, THH, THT, TTH, TTT
a $\frac{1}{8}$ **b** $\frac{3}{8}$ **c** $\frac{1}{2}$ **d** $\frac{1}{2}$ **e** $\frac{1}{2}$ **f** $\frac{1}{2}$

8 11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66
a $\frac{1}{6}$ **b** $\frac{1}{6}$ **c** $\frac{1}{36}$ **d** $\frac{1}{6}$ **e** $\frac{1}{6}$ **f** $\frac{1}{4}$
g $\frac{11}{36}$ **h** $\frac{4}{9}$ **i** $\frac{5}{36}$ **j** $\frac{1}{6}$

9 a i $\frac{1}{4}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{2}$ **b i** $\frac{1}{8}$ **ii** $\frac{3}{8}$ **iii** $\frac{1}{2}$

10 a $\frac{1}{16}$ **b** $\frac{1}{4}$ **c** $\frac{11}{16}$ **d** $\frac{5}{16}$ **e** $\frac{3}{8}$ **f** $\frac{5}{16}$

11 a $\frac{2}{5}$ **b** $\frac{3}{5}$ **c** $\frac{1}{5}$

12 a 24 **b i** $\frac{2}{3}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{12}$ **iv** $\frac{1}{6}$

13 a $\frac{1}{2^n}$ **b** $1 - 2^{1-n}$
c $\frac{1}{2}$ That is, half the time there will be more tails than heads.

14 $\frac{1}{4}$

15 $\frac{1}{4}$. The experiment is the same as asking the probability that the first card is a diamond.

Exercise 12C

1 a $\{1, 3, 5, 7, 9\}$ **b** $\{6, 12, 18, 24, 30, 36\}$
c $\{1, 2, 3, 4, 5, 6\}$ **d** $\{1, 2, 4, 5, 10, 20\}$

2 a $A \cup B = \{1, 3, 5, 7\}$, $A \cap B = \{3, 5\}$
b $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 9, 10\}$, $A \cap B = \{4, 9\}$
c $A \cup B = \{$h, o, b, a, r, t, i, c, e, n$\}$, $A \cap B = \{$h, o, b$\}$
d $A \cup B = \{$j, a, c, k, e, m$\}$, $A \cap B = \{$a$\}$
e $A \cup B = \{1, 2, 3, 5, 7, 9\}$, $A \cap B = \{3, 5, 7\}$

3 a false **b** true **c** false
d false **e** true **f** true

4 a 3 **b** 2 **c** $\{1, 3, 4, 5\}$ **d** 4
e $\{3\}$ **f** 1 **g** $\{2, 4\}$ **h** $\{1, 2, 5\}$

5 a students who study both Japanese and History
b students who study either Japanese or History or both

6 a students at Clarence High School who do not have blue eyes
b students at Clarence High School who do not have blond hair
c students at Clarence High School who have blue eyes or blond hair or both
d students at Clarence High School who have blue eyes and blond hair

7 a $\varnothing$, $\{$a$\}$
b $\varnothing$, $\{$a$\}$, $\{$b$\}$, $\{$a, b$\}$
c $\varnothing$, $\{$a$\}$, $\{$b$\}$, $\{$c$\}$, $\{$a, b$\}$, $\{$a, c$\}$, $\{$b, c$\}$, $\{$a, b, c$\}$
d $\varnothing$

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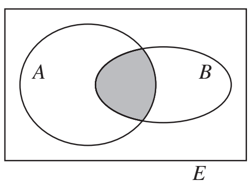
8 a true **b** false **c** true **d** false **e** true

9 a $\{2, 4, 5, 6, 8, 9\}$ **b** $\{1, 2, 3, 5, 8, 10\}$
c $\{7\}$ **d** $\{1, 2, 3, 4, 5, 6, 8, 9, 10\}$
e $\{1, 3, 4, 6, 7, 9, 10\}$ **f** $\{2, 5, 8\}$

10 a $\{2, 4, 5, 7, 9, 10\}$ **b** $\{1, 2, 5, 8, 9\}$
c $\{1, 2, 4, 5, 7, 8, 9, 10\}$ **d** $\{2, 5, 9\}$
e $\{1, 3, 4, 6, 7, 8, 10\}$ **f** $\{2, 5, 9\}$
g $\{3, 6\}$ **h** $\{1, 2, 4, 5, 7, 8, 9, 10\}$

11 a


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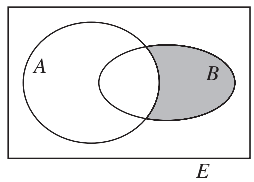

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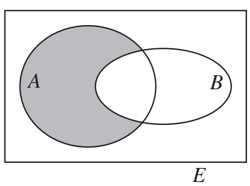

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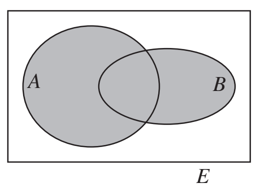

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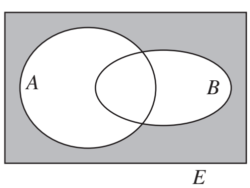

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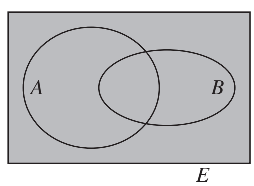


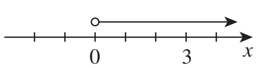
12 a true **b** true

13 a Q **b** P

14 a III **b** I **c** II **d** IV

15 a infinite **b** finite, 10 members
c finite, 0 members **d** infinite
e finite, 18 members **f** infinite
g finite, 6 members **h** finite, 14 members

16 a false **b** true **c** true
d false **e** true **f** false

17 a i


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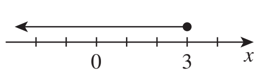

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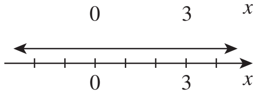

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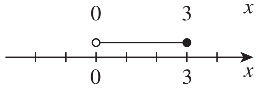
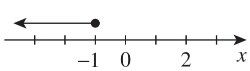


b i


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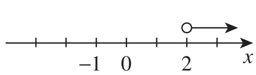

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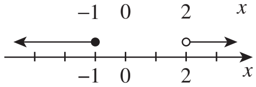

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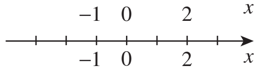

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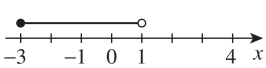

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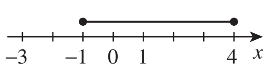

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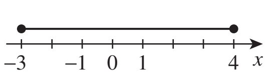

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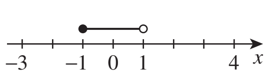
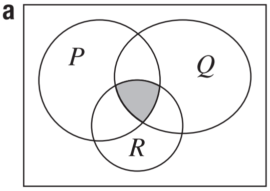


18 a $|A \cap B|$ is subtracted so that it is not counted twice.
b 5 **c** LHS $= 7$, RHS $= 5 + 6 - 4 = 7$

19 a 10 **b** 22 **c** 12

20 a


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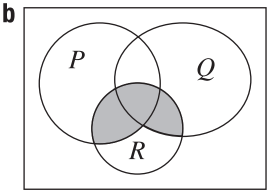

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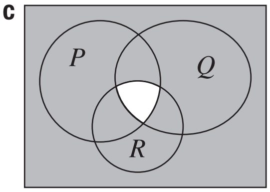

21 4

22 $\frac{7}{43}$

23 a $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
b 207
c $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$.
It is possible to draw a Venn diagram with four sets, but only if the fourth set is represented not by a circle, but by a complicated loop — the final diagram must have 16 regions.

Exercise 12D

1 a $\frac{1}{6}$ **b** $\frac{5}{6}$ **c** $\frac{1}{3}$ **d** 0
e 1 **f** 0 **g** $\frac{1}{6}$ **h** $\frac{2}{3}$

2 a $\frac{1}{13}$ **b** $\frac{1}{13}$ **c** $\frac{2}{13}$ **d** 0 **e** $\frac{11}{13}$ **f** $\frac{1}{2}$
g $\frac{3}{13}$ **h** $\frac{3}{26}$ **i** $\frac{8}{13}$ **j** $\frac{5}{13}$

3 a $A = \{\text{HH}\}, B = \{\text{HT, TH}\}, P(A \text{ or } B) = \frac{3}{4}$, $P(A) = \frac{1}{4}, P(B) = \frac{2}{4}$
b $A = \{\text{RS}\}, B = \{\text{RT, ST}\}, P(A \text{ or } B) = \frac{3}{3}$, $P(A) = \frac{1}{3}, P(B) = \frac{2}{3}$

4 a no **b i** $\frac{1}{2}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{3}$ **iv** $\frac{5}{6}$

5 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{4}$ **d** $\frac{3}{4}$ **e** $\frac{1}{4}$
f $\frac{1}{6}$ **g** $\frac{1}{6}$ **h** $\frac{1}{36}$ **i** $\frac{11}{36}$ **j** $\frac{25}{36}$

6 a i $\frac{1}{2}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{3}$ **iv** $\frac{1}{2}$ **v** $\frac{1}{2}$
b i $\frac{3}{5}$ **ii** $\frac{4}{5}$ **iii** $\frac{3}{5}$ **iv** 0 **v** 1
c i $\frac{1}{2}$ **ii** $\frac{2}{3}$ **iii** $\frac{2}{3}$ **iv** $\frac{1}{3}$ **v** $\frac{5}{6}$

7 a $\frac{7}{15}$ **b** 0 **c** $\frac{3}{5}$ **d** $\frac{5}{7}$

8 a i no **ii** $\frac{1}{2}, \frac{1}{4}, \frac{3}{20}, \frac{3}{5}$ **b i** no **ii** $\frac{1}{2}, \frac{3}{10}, \frac{3}{20}, \frac{13}{20}$
c i yes **ii** $\frac{1}{4}, \frac{9}{20}, 0, \frac{7}{10}$

9 a $\frac{9}{25}$ **b** $\frac{7}{50}$ **c** $\frac{17}{50}$

10 a 10 **b i** $\frac{4}{21}$ **ii** $\frac{1}{3}$

11 $\frac{1}{4}$

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12 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

a $\frac{1}{4}$ **b** $\frac{1}{4}$ **c** $\frac{11}{100}$ **d** $\frac{39}{100}$

13 a $\frac{7}{12}$ **b** $\frac{13}{60}$ **c** $\frac{3}{10}$ **d** $\frac{7}{60}$

Exercise 12E

1 a $\frac{1}{24}$ **b** $\frac{1}{28}$ **c** $\frac{1}{12}$ **d** $\frac{1}{96}$ **e** $\frac{1}{42}$ **f** $\frac{1}{336}$

2 a $\frac{1}{12}$ **b** $\frac{1}{12}$ **c** $\frac{1}{4}$ **d** $\frac{1}{3}$

3 a $\frac{1}{25}$ **b** $\frac{2}{25}$ **c** $\frac{3}{25}$ **d** $\frac{3}{25}$ **e** $\frac{4}{25}$ **f** $\frac{2}{25}$ **g** $\frac{1}{25}$

4 a $\frac{15}{49}$ **b** $\frac{8}{49}$ **c** $\frac{6}{49}$

5 a $\frac{1}{10}$ **b** $\frac{3}{10}$ **c** $\frac{3}{10}$ **d** $\frac{3}{10}$

6 a $\frac{1}{36}$ **b** $\frac{1}{12}$ **c** $\frac{1}{36}$ **d** $\frac{1}{9}$ **e** $\frac{1}{6}$

7 a $\frac{1}{7}$ **b** $\frac{180}{1331}$

8 a i $\frac{13}{204}$ **ii** $\frac{1}{17}$ **iii** $\frac{4}{663}$ **iv** $\frac{1}{2652}$

b $\frac{1}{16}, \frac{1}{16}, \frac{1}{169}, \frac{1}{2704}$

9 a i $\frac{2}{3}$ **ii** $\frac{1}{3}$ **b i** $\frac{8}{27}$ **ii** $\frac{1}{27}$ **iii** $\frac{4}{27}$

10 a The argument is invalid, because the events 'liking classical music' and 'playing a classical instrument' are not independent. One would expect that most of those playing a classical instrument would like classical music, whereas a smaller proportion of those not playing a classical instrument would like classical music. The probability that a student does both cannot be discovered from the given data — one would have to go back and do another survey.

b The argument is invalid, because the events 'being prime' and 'being odd' are not independent — two out of the three odd numbers less than 7 are prime, but only one out of the three such even numbers is prime. The correct argument is that the odd prime numbers amongst the numbers 1, 2, 3, 4, 5 and 6 are 3 and 5, hence the probability that the die shows an odd prime number is $\frac{2}{6} = \frac{1}{3}$.

c The teams in the competition may not be of equal ability, and factors such as home-ground advantage may also affect the outcome of a game, hence assigning a probability of $\frac{1}{2}$ to winning each of the seven games is unjustified. Also, the outcomes of successive games are not independent — the confidence gained after winning a game may improve a team's chances in the next one, a loss may adversely affect their chances, or a team may receive injuries in one game leading to a depleted team in the next. The argument really can't be corrected.

d This argument is valid. The coin is normal, not biased, and tossed coins do not remember their previous history, so the next toss is completely unaffected by the previous string of heads.

11 The chance that at least one of them will shoot a basket is $1 - P$(they both miss). The boy missing and the girl missing are independent events. The correct answer is 0.895.

12 a $\frac{1}{36}$ **b** $\frac{1}{6}$ **c** $\frac{1}{4}$

d $\frac{1}{36}$ **e** $\frac{1}{36}$ **f** $\frac{1}{18}$

g $\frac{1}{12}$ **h** $\frac{1}{12}$ **i** $\frac{1}{6}$

13 HHH, HHM, HMH, MHH, HMM, MHM, MMH, MMM

a $P(\text{HHH}) = 0.9^3 = 0.729$ **b** 0.001

c $P(\text{HMM}) = 0.9 \times 0.1^2 = 0.009$

d $P(\text{HMM}) + P(\text{MHM}) + P(\text{MMH}) = 3 \times 0.009 = 0.027$

e 0.081 **f** 0.243

14 a $P(\text{CCCCC}) = \left(\frac{1}{5}\right)^5 = \frac{1}{3125}$ **b** $\frac{1024}{3125}$

c $\frac{16}{3125}$ **d** $\frac{256}{3125}$ **e** $\frac{256}{625}$ **f** $\frac{4}{625}$

15 a $\frac{1}{46656}$ **b** $\frac{5}{7776}$

16 a $\frac{1}{6}$ **b** $\frac{5}{6}$ **c** $\frac{1}{2}$ **d** $\frac{1}{3}$

17 a $\frac{3}{64}$ **b** $\frac{17}{64}$ **c** $\frac{5}{17}$

18 a i $\frac{3}{4}$ **ii** $\frac{31}{32}$ **iii** $\frac{1023}{1024}$

b $1 - \dfrac{1}{2^n} = \dfrac{2^n - 1}{2^n}$ **c** 14

19 a $\frac{9}{25}$ **b** 11

20 a $\frac{1}{12\,960\,000}$ **b** 233

21 a $\frac{1}{9}$

b $\frac{1}{9}$. Retell as 'Nick begins by picking out two socks for the last morning and setting them aside'.

c $\frac{1}{9}$. Retell as 'Nick begins by picking out two socks for the third morning and setting them aside'.

d $\frac{1}{63}$ **e** $\frac{1}{9 \times 7 \times 5 \times 3}$ **f** zero

22 a In each part, retell the process of selection as 'First choose a court for Kia, then choose one of the remaining 11 positions for Abhishek'.

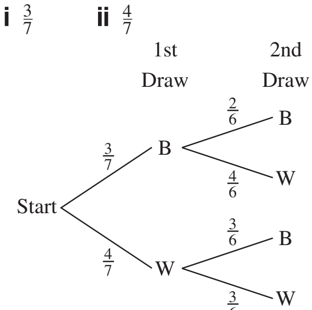
a $\dfrac{3}{11}$ **b** $\dfrac{1}{11}$ **c** $\dfrac{4}{33}$

Exercise 12F

1 a i $\frac{9}{49}$ **ii** $\frac{12}{49}$ **iii** $\frac{12}{49}$ **iv** $\frac{16}{49}$

b i $\frac{25}{49}$ **ii** $\frac{24}{49}$

c i $\frac{3}{7}$ **ii** $\frac{4}{7}$



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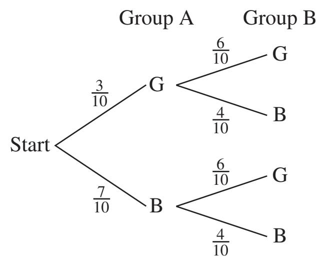
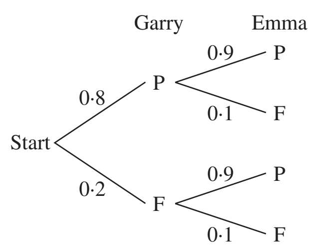
2 a i 90.25% **ii** 4.75% **iii** 4.75% **iv** 0.25%
b 99.75%
3 a i $\frac{6}{25}$ **ii** $\frac{9}{25}$ **iii** $\frac{4}{25}$ **iv** $\frac{6}{25}$
b i $\frac{12}{25}$ **ii** $\frac{13}{25}$
4 a i $\frac{9}{50}$ **ii** $\frac{3}{25}$ **iii** $\frac{21}{50}$ **iv** $\frac{7}{25}$
b i $\frac{23}{50}$ **ii** $\frac{27}{50}$



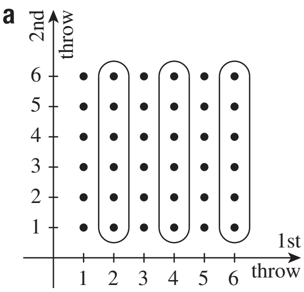
5 a 8% **b** 18% **c** 26% **d** 28%



6 a $\frac{9}{25}$ **b** $\frac{21}{25}$
7 4.96%
8 a 0.01 **b** 0.23
9 0.35
10 $\frac{4}{7}$
11 a $\frac{21}{3980}$ **b** $\frac{144}{995}$
12 a $\frac{3}{10}$ **b** $\frac{7}{24}$ **c** $\frac{21}{40}$
13 a $\frac{1}{11}$ **b** $\frac{14}{33}$ **c** $\frac{10}{33}$ **d** $\frac{19}{33}$
14 a $\frac{5}{6}$ **b** $\frac{5}{12}$ **c** $\frac{1}{6}$
15 a $\frac{4}{9}$ **b** $\frac{65}{81}$ **c** 4
16 The term 'large school' is code for saying that the probabilities do not change for the second choice because the sample space hardly changes.
a 0.28 **b** 0.50
17 a $\frac{1}{25}$ **b** $\frac{3}{5}$
18 a $\frac{1}{20}$ **b** $\frac{57}{8000}$
19 a 31.52% **b** 80.48%
20 a i $\frac{5}{33}$ **ii** $\frac{5}{22}$ **iii** $\frac{19}{33}$ **iv** $\frac{1}{4}$ **v** $\frac{19}{66}$ **vi** $\frac{47}{66}$
b $\frac{25}{144}, \frac{5}{24}, \frac{5}{9}, \frac{1}{4}, \frac{25}{72}, \frac{47}{72}$
21 a $\frac{1}{36}$ **b** $\frac{1}{46656}$ **c** $\frac{11}{36}$
22 a $\frac{1}{216}$ **b** $\frac{5}{72}$ **c** $\frac{5}{12}$ **d** $\frac{5}{9}$
23 $\frac{1}{3}$
24 a $\frac{1}{25}$ **b** $\frac{3}{25}$ **c** $\frac{6}{25}$ **d** $\frac{19}{25}$

Exercise 12G

1 a



b $\frac{1}{18}$ **c** $\frac{4}{9}$ **d** $\frac{1}{9}$
2 a $\frac{340}{1000} = \frac{17}{50} = 0.34$ **b** $\frac{190}{420} = \frac{19}{42} \doteqdot 0.45$
c $\frac{130}{340} = \frac{13}{34} \doteqdot 0.38$ **d** $\frac{20}{130} = \frac{2}{13} \doteqdot 0.15$
3 a Totals in last column: 56, 137, 193. Totals in last row: 124, 69, 193.
b i $\frac{42}{193} \doteqdot 0.22$ **ii** $\frac{29}{124} \doteqdot 0.23$
iii $\frac{29}{56} \doteqdot 0.52$ **iv** $\frac{95}{137} \doteqdot 0.69$
4 a $\frac{1}{16}$ **b** HH, HD, HC, HS; $\frac{1}{4}$
c HH, HD, HC, HS, DH, CH, SH; $\frac{1}{7}$
d HH, HD, HC, HS, DH, DD, DC, DS; $\frac{1}{8}$
5 a

	1	2	3	4	5	6
HH	3	4	5	6	7	8
HT	2	3	4	5	6	7
TH	2	3	4	5	6	7
TT	1	2	3	4	5	6

b $\frac{1}{24}$ **c** $\frac{1}{6}$ **d** $\frac{1}{2}$
6 a $\frac{5}{7}$ **b** $\frac{3}{8}$ **c** $\frac{16}{19}$
7 a $P(A \cap B) = 0.24$ **b** $P(A \cap B) = 0.15$
c $P(A|B) = 0.4$ **d** $P(A|B) = 0.7$
8 a dependent **b** independent
c dependent **d** independent
e impossible—$P(A \cap B)$ cannot be bigger than $P(A)$ or $P(B)$.
f independent
9 a

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

b The cases 1 + 4, 2 + 3, and 4 + 1 make up the reduced sample space.
i $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** 1

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10 a i 0.1 **ii** $\frac{1}{3}$ **iii** $\frac{1}{4}$
b $\frac{3}{7}$ **c** $\frac{1}{2}$ **d** $\frac{5}{9}$

11 $\frac{4}{11}$

12 $\frac{5}{8}$ or 62.5%

13 a $\frac{1}{2}$ **b** $\frac{1}{3}$

14 a BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG
b $\frac{4}{7}$ **c** $\frac{2}{3}$

15 a $\frac{1}{3}$ **b** $\frac{2}{3}$ **c** $\frac{11}{153}$

16 Draw up a 6×6 sample space and mark the points that are in A, in S and in M.
First, $P(A) = P(S) = \frac{1}{2}$ and $P(M) = \frac{1}{4}$.
a A and S are independent because $P(S|A) = \frac{1}{2} = P(S)$. A and M are not independent because $P(M|A) = \frac{1}{2}$, but $P(M) = \frac{1}{4}$. S and M are not independent because $P(M|S) = 0$, but $P(M) = \frac{1}{4}$.
b $P(A \cap S) = \frac{1}{4} = P(A) \times P(S)$, so A and S are independent. $P(A \cap M) = \frac{1}{4}$, but $P(A) \times P(M) = \frac{1}{8}$, so A and M are not independent. $P(S \cap M) = 0$, but $P(S) \times P(M) = \frac{1}{8}$, so A and M are not independent.

17 a $P(A \cup B) = 0.76$ **b** $P(A \cup B) = 0.72$

18 a $\frac{1}{6}$ **b** $\frac{5}{6}$ **c** $\frac{1}{5}$

19 $\frac{7}{15}$

20 $\frac{9}{23}$

21 $\frac{3}{7}$

22 a 5.75% **b** 4.95% **c** 86% **d** 0.21%
e It is most important that the number of false negatives is low — that almost all patients with the disease are picked up. False positives are worrying for the patient, but further tests should determine that they do not have the disease.

23 a
$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(B \cap A)}{P(B)} \\ &= \frac{P(B|A)P(A)}{P(B)} \\ &= \frac{P(B|A)}{P(B)} \times P(A) \end{aligned}$$

24 If B is independent of A then,
$$\begin{aligned} P(A|B) &= \frac{P(B|A)}{P(B)} \times P(A) \\ &= \frac{P(B)}{P(B)} \times P(A) \\ &= P(A) \end{aligned}$$
which states that A is dependent of B.

25 Suppose first that the contestant changes her choice. If her original choice was correct, she loses, otherwise she wins, so her chance of winning is $\frac{2}{3}$. Suppose now that the contestant does not change her choice. If her original choice was correct, she wins, otherwise she loses, so her chance of winning is $\frac{1}{3}$. Thus the strategy of changing will double her chance of winning.

26 Let $G1$ be, 'A girl is born on a Sunday', let $B1$ be, 'A boy is born on a Sunday', let $G2$ be, 'A girl is born on a Monday', …, giving 14 equally likely events at the birth of every child.

In this particular family, there are two children, giving $14^2 = 196$ equally likely possible outcomes for the two successive births in this family.

Draw up the 2×2 sample space, showing at least all the entries in the row indexed by $G2$ and the column indexed by $G2$.

Let A be, 'At least one child is a girl born on a Monday.' There are 27 favourable outcomes for A.

Let B be, 'Both children are girls.' There are 13 favourable outcomes for the event $A \cap B$.
Hence $P(B|A) = |A \cap B|/|A| = \frac{13}{27}$.

Chapter 12 review exercise

1 a $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** $\frac{1}{6}$ **d** $\frac{1}{2}$

2 a $\frac{1}{10}$ **b** $\frac{1}{2}$ **c** $\frac{3}{10}$ **d** 0 **e** 1 **f** $\frac{3}{10}$

3 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{13}$ **d** $\frac{1}{52}$ **e** $\frac{1}{2}$ **f** $\frac{12}{13}$

4 37%

5 a $\frac{1}{4}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$

6 a $\frac{1}{36}$ **b** $\frac{1}{9}$ **c** $\frac{1}{6}$ **d** $\frac{11}{36}$
e $\frac{4}{9}$ **f** $\frac{1}{9}$ **g** $\frac{1}{6}$ **h** $\frac{11}{36}$

7 a $\frac{17}{60}$ **b** $\frac{19}{60}$ **c** $\frac{1}{6}$

8 a w **b i** $\frac{1}{2}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{3}$ **iv** $\frac{5}{6}$

9 a $\frac{1}{12}$ **b** $\frac{1}{5}$ **c** $\frac{3}{20}$ **d** $\frac{1}{20}$

10 a i $\frac{13}{204}$ **ii** $\frac{1}{17}$ **iii** $\frac{4}{663}$ **iv** $\frac{1}{2652}$
b i $\frac{1}{16}$ **ii** $\frac{1}{16}$ **iii** $\frac{1}{169}$ **iv** $\frac{1}{2704}$

11 a 14% **b** 24% **c** 38% **d** 6%

12 a $\frac{2}{21}$ **b** $\frac{11}{21}$ **c** $\frac{10}{21}$ **d** $\frac{2}{7}$

13 a $\frac{19}{12475}$ **b** $\frac{979}{12475}$

14 a Independent **b** Dependent
c Independent, with $P(A \cap B) = 0.18$

15 $\frac{3}{11}$

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16 a $\frac{1}{5}$

b $\frac{1}{5}$. The answer is independent of the day of the week.

c $\frac{1}{120}$

d 0. There cannot be only 1 day where the short and tie do not match.

Chapter 13

Exercise 13A

1 a Numeric, discrete **b** Numeric, continuous

c Categorical **d** Numeric, continuous

e Categorical **f** Categorical

g On a standard scale of shoes sizes, this is numeric and discrete. The length of a person's foot would be a numeric, continuous distribution.

h Numeric, discrete. Reported ATAR scores are between 30 and 99.95 in steps of 0.05. There are around 1400 different scores awarded.

2 a

Outcome	HH	HT	TH	TT
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Uniform distribution (and categorical).

b

Outcome	2 heads	1 head & 1 tail	2 tails
Probability	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

3 a

Outcome	red	green
Probability	$\frac{4}{7}$	$\frac{3}{7}$

b

Outcome	J	K	L	O
Probability	0.06	0.08	0.04	0.82

c

Outcome	P	A	R	M	T
Probability	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$

d

Outcome	1	2	3	4
Probability	$\frac{9}{1000}$	$\frac{90}{1000}$	$\frac{900}{1000}$	$\frac{1}{1000}$

e

Outcome	even	prime	neither
Probability	$\frac{5}{10}$	$\frac{4}{10}$	$\frac{1}{10}$

4 a Let X be the number of letters in a randomly-chosen word.

Outcome x	3	4	6
Probability $P(X = x)$	$\frac{5}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

b Let X be the number of heads recorded when 2 coins are thrown.

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

c Let X be the digits recorded from the first 12 digits of $\sqrt{2}$

x	1	2	3	4	5	6	7
$P(X = x)$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

d Let X be the number selected.

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

(Note that the answer is the same if the sets are amalgamated. Why?)

5 a {T}, {F1}, {F2}, {T, F1}, {T, F2}, {F1, F2}, {T, F1, F2}

b

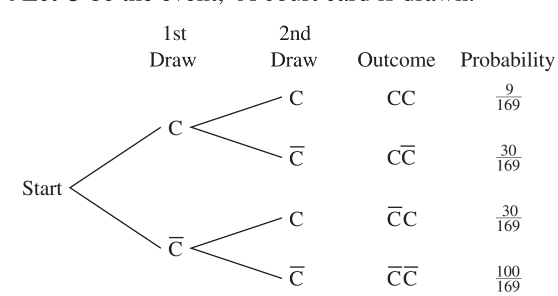
x	5	10	15	20
$P(X = x)$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{1}{7}$

6 a Yes **b** No **c** Yes **d** Yes **e** No **f** Yes

7 a 0.2 **b** 0.6 **c** 0.75 **d** 0 **e** 0.6

f 0.85 **g** 0.9 **h** 0.7 **i** 0.45

8 a i Let C be the event, 'A court card is drawn.'



ii

x	0	1	2
$P(X = x)$	$\frac{100}{169}$	$\frac{60}{169}$	$\frac{9}{169}$

b i The eight outcomes EEE, EEO, EOE, EOO, OEE, OEO, OOE, OOO each have probability $\frac{1}{8}$.

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ii

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

c GGG has probability $\frac{8}{125}$, GGB, GBG, BGG each have probability $\frac{12}{125}$, GBB, BGB, BBG each have probability $\frac{18}{125}$, BBB has probability $\frac{27}{125}$.

x	0	1	2	3
$P(X = x)$	$\frac{8}{125}$	$\frac{36}{125}$	$\frac{54}{125}$	$\frac{27}{125}$

d Let S be the event, 'A wallaby from Snake Ridge was selected'. SSS has probability 0.027, $\overline{S}SS$, $S\overline{S}S$, $SS\overline{S}$ each have probability 0.063, $\overline{S}\,\overline{S}S$, $\overline{S}S\overline{S}$, $S\overline{S}\,\overline{S}$ each have probability 0.147, $\overline{S}\,\overline{S}\,\overline{S}$ has probability 0.343.

x	0	1	2	3
$P(X = x)$	0.343	0.441	0.189	0.027

9 a $a = \frac{1}{25}$ **b** $a = \frac{1}{14}$ **c** $a = \frac{1}{27}$
d $a = \frac{1}{10}$ **e** $a = 1$

10 a i EE and OO each have probability $\frac{1}{5}$, EO and OE each have probability $\frac{3}{10}$.

ii

x	0	1	2
$P(X = x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

b BB has probability $\frac{2}{5}$, BG and GB each have probability $\frac{4}{15}$, GG has probability $\frac{1}{15}$.

x	0	1	2
$P(X = x)$	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{15}$

c i EE has probability $\frac{3}{10}$, ER, RE, ET, TE each have probability $\frac{3}{20}$, RT and TR each have probability $\frac{1}{20}$.

ii

x	0	1	2
$P(X = x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

11

x	22	44	55	24 or 42
$P(X = x)$	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{1}{36}$	$\frac{1}{3}$
	25 or 52	45 or 54		
	$\frac{1}{9}$	$\frac{1}{6}$		

12 a

Outcome	RR	RG	GR	GG
Probability	$\frac{16}{49}$	$\frac{12}{49}$	$\frac{12}{49}$	$\frac{9}{49}$

b

Outcome	RR	RG	GR	GG
Probability	$\frac{12}{42}$	$\frac{12}{42}$	$\frac{12}{42}$	$\frac{6}{42}$

c

Outcome	HH	DD	SS	CC
Probability	$\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{17}$	$\frac{1}{17}$

HS or SH	HC or CH	HD or DH
$\frac{13}{102}$	$\frac{13}{102}$	$\frac{13}{102}$
SC or CS	SD or DS	CD or DC
$\frac{13}{102}$	$\frac{13}{102}$	$\frac{13}{102}$

13a There is no guarantee that their results will be identical, though you would expect more *trials* (repeats of the experiment) would bring your results closer to each other and to the theoretical probabilities.

b Theoretical results: $P(X = 0) = 0.3$, $P(X = 1) = 0.6$, $P(X = 2) = 0.1$

c It might be easier to perform the experiment with coloured balls or tokens. Running the experiment in pairs with a nominated recorder also helps. The paper pieces need to be indistinguishable and well mixed in the bag. You could increase the number of trials or combine the class results.

14 EEE and OOO each have probability $\frac{1}{20}$, the other six possible outcomes each have probability $\frac{3}{20}$,

x	0	1	2	3
$P(X = x)$	$\frac{1}{20}$	$\frac{9}{20}$	$\frac{9}{20}$	$\frac{1}{20}$

15 a The condition that the sum of the probabilities is 1 gives $a = \frac{1}{4}$ or $a = 1$. But $a = 1$ gives probabilities outside the interval $0 \le p \le 1$, and the only valid answer is $a = \frac{1}{4}$.

b $a = 1$ or $\frac{7}{6}$ (both are valid)

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16 a Let X be the sum of the numbers on the three cards. This question is best done by asking what card is *discarded*.

x	20	21	22
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b

x	20	21	22
$P(X = x)$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{3}{10}$

Exercise 13B

1 a

x	0	1	2	3	Sum
$p(x)$	0.4	0.1	0.2	0.3	1
$x\,p(x)$	0	0.1	0.4	0.9	1.4

Hence $E(X) = 1.4$.

b

x	2	4	6	8	Sum
$p(x)$	0.1	0.4	0.4	0.1	1
$x\,p(x)$	0.2	1.6	2.4	0.8	5

Hence $E(X) = 5$.

c

x	-50	-20	0	30	100	Sum
$p(x)$	0.1	0.35	0.4	0.1	0.05	1
$xp(x)$	-5	-7	0	3	5	-4

Hence $E(X) = -4$.

2 a

x	-40	0	30	60	Sum
$p(x)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1
$x\,p(x)$	-20	0	5	10	-5

b Expected value $= -5$

c The average cost to the player per game is 5 cents.

d $100 \times (-5) = -500$ cents. Thus the player expects to lose 500 cents and the casino expects to make 500 cents profit. This is an expected average value, not guaranteed.

4 a

x	2	4	6	8	10	Sum
p_i	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1
$x_i p_i$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{8}{5}$	$\frac{10}{5}$	6

So $E(X) = 6$.

b

x	-3	1	2	5	6	Sum
p_i	0.1	0.3	0.2	0.3	0.1	1
$x_i p_i$	-0.3	0.3	0.4	1.5	0.6	2.5

So $E(X) = 2.5$.

5 a

x	1.50	2.10	2.40	Sum
$p(x)$	$\frac{5}{12}$	$\frac{4}{12}$	$\frac{3}{12}$	1
$x\,p(x)$	0.625	0.7	0.60	1.925

The expected value is \$1.925.

b If 100 purchases are made at random the expected cost is \$192.50.

6 a $E(X) = 3$

b i $E(Y) = 6$ **ii** Yes

c i $E(Z) = 4$ **ii** Yes

7 a 15 **b** 10 **c** $\frac{5}{2}$ **d** 3 **e** 0 **f** 18

8

x	0	1	2	3	Sum
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1
$x\,p(x)$	0	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{3}{8}$	$\frac{12}{8}$

The expected value is $1\frac{1}{2}$, as might be expected from the symmetry of the table of probabilities.

9

x	0	1	2	Sum
$p(x)$	$\frac{19}{34}$	$\frac{13}{34}$	$\frac{1}{17}$	1
$x\,p(x)$	0	$\frac{13}{34}$	$\frac{2}{17}$	$\frac{17}{34}$

The expected value is $\frac{1}{2}$.

10 d

x	0	1	2	3	4	5	Sum
$p(x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	1
$x\,p(x)$	0	$\frac{10}{36}$	$\frac{16}{36}$	$\frac{18}{36}$	$\frac{16}{36}$	$\frac{10}{36}$	$\frac{70}{36}$

Hence $E(X) = \frac{35}{18}$.

f In any dice experiment, it is important to check the randomness of your dice rolls. This can depend on your rolling technique. Try throwing a die 12 times and see if every outcome is equally likely. Does each outcome seem independent of the last?

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11 a $\frac{3}{15}, \frac{3}{15}, \frac{3}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}$

b -12, so the bank expects to make 12 cents each game, on average.

12 a $P(\text{Orange}) = \frac{1}{6}$, $P(\text{Strawberry}) = \frac{2}{6}$, $P(\text{Apple}) = \frac{3}{6}$

b

outcome	OOO	SSS	AAA	Other	Sum
x	$11k$	$2k$	k	0	—
$p(x)$	$\frac{1}{216}$	$\frac{8}{216}$	$\frac{27}{216}$	$\frac{180}{216}$	1

c The payout will be \$44 and their profit would be \$43, accounting for the \$1 entry fee.

13 b $\mu = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 4 \times \frac{1}{16} + 5 \times \frac{1}{32} + 6 \times \frac{1}{64} + \cdots \quad (1)$

Doubling,

$2\mu = 1 \times 1 + 2 \times \frac{1}{2} + 3 \times \frac{1}{4} + 4 \times \frac{1}{8} + 5 \times \frac{1}{16} + 6 \times \frac{1}{32} + \cdots \quad (2)$

Subtracting (1) from (2),

$\mu = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \quad (3)$

Doubling,

$2\mu = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \quad (4)$

Subtracting (3) from (4), $\mu = 2$.

c On average, we would expect to get a head on the second throw. You could test this by recording how many throws it takes over say 50 trials and averaging the results.

14 $E(X) = 2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} + 16 \times \frac{1}{16} + \cdots$
$= 1 + 1 + 1 + 1 + \cdots$

The expected value 'increases without bound', that is, $\mathrm{E}(X) \to \infty$ as the game continues. This suggests that there is no reasonable price the casino could put on this game and expect to break even. There are various issues with this scenario in real life. Casinos would not provide a game which had no upper limit to the payout. Patrons would also be unwilling to pay a large price for a game with such low apparent probabilities for the later stages of the game. The calculation of a simple expected value may not be the best way to analyse this game.

Exercise 13C

1 a

x	1	2	3	4	Sum
$p(x)$	0.3	0.5	0.1	0.1	1
$xp(x)$	0.3	1	0.3	0.4	2
$(x - \mu)^2$	1	0	1	4	—
$(x - \mu)^2 p(x)$	0.3	0	0.1	0.4	0.8

$\mu = 2$, $\mathrm{Var}(X) = 0.8$, $\sigma = \sqrt{0.8} \doteqdot 0.89$

2 a

x	1	2	3	4	Sum
$p(x)$	0.3	0.5	0.1	0.1	1
$xp(x)$	0.3	0.1	0.3	0.4	2
x^2	1	4	9	16	—
$x^2 p(x)$	0.3	2	0.9	1.6	4.8

b $\mathrm{Var}(X) = 4.8 - 2^2 = 0.8$.

3 a $\mathrm{E}(X) = 2$, $\mathrm{Var}(X) = 2$

b $\mathrm{E}(X) = 3$, $\mathrm{Var}(X) = 1$

c $\mathrm{E}(X) = 0$, $\mathrm{Var}(X) = 2.6$

d $\mathrm{E}(X) = 2.8$, $\mathrm{Var}(X) = 1.36$

4 a i $\mathrm{E}(Y) = 2$, $\mathrm{Var}(Y) = 1$, $\sigma = 1$

ii $\mathrm{E}(Z) = 2$, $\mathrm{Var}(Z) = 4$, $\sigma = 2$

iii $\mathrm{E}(V) = 1$, $\mathrm{Var}(V) = 0.8$, $\sigma \doteqdot 0.89$

iv $\mathrm{E}(W) = 3$, $\mathrm{Var}(W) = 0.8$, $\sigma \doteqdot 0.89$

b i Both sets of data are centred around 2 and the expected value of each is unsurprisingly 2. The second data set is more spread out — in fact in moving from Y to Z the distances from the mean to each data point have been doubled and the standard deviation is doubled.

ii The data has been 'flipped over', but is no more spread out than before — the variance is unchanged. You may notice that $W = 4 - V$.

5 $\mathrm{E}(X) = 2$, $\mathrm{Var}(X) = 0$.

6 a $\mathrm{E}(J) = 1.55$, $\mathrm{Var}(J) = 2.05$, $\mathrm{E}(L) = 1.4$, $\mathrm{Var}(L) = 0.84$.

b Over the season John might be expected to score more baskets, because his expected value is higher.

c Liam is the more consistent player, with the lower variance. Coaches may prefer a more consistent player, particularly if it is more important to score *some* goals, rather than the maximum number. This may also be a sign that John needs to work on the consistency of his game.

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7 a Each outcome has probability $\frac{1}{3}$. This is a uniform distribution.

b $\mathrm{E}(X) = 2$ **c** $\mathrm{Var}(X) = \frac{2}{3}$

8 a Two standard deviations

b One and a half standard deviations below the mean.

c The English score was more standard deviations below the mean than the Mathematics result, so it may be considered less impressive.

9 a Visual Arts is 1 standard deviation below the mean, Music is 1.75 standard deviations below the mean, hence the Visual Arts score is better.

b Earth Science is 2 standard deviations above the mean, Biology is 1.5 standard deviations above the mean, hence the Earth Science score is more impressive.

c Chinese is 2 standard deviations above the mean, Sanskrit is also 2 standard deviations above the mean, hence the scores are equally impressive.

10 a $\mathrm{E}(X) = 3.3, \sigma = 1.45$

b 8 appears to be a long way from 3.3 and well removed from the rest of the data.

c 8 is 3.2 standard deviations above the mean and thus would be an outlier by this definition.

d $\mathrm{E}(X) = 3.15, \sigma = 1.06.$

e The mean and standard deviation have changed significantly, especially the standard deviation.

f Outliers are interesting values in any distribution and should be a flag to investigate more closely. Were results recorded correctly? Was there an error in the experiment, for example, did Jasmine use a more powerful bow with greater range, or maybe she used a new set of arrows with better fletching? It may, however, be that Jasmine is inconsistent, occasionally getting much better results, but often getting fairly poor results — in this case the large standard deviation is warranted as a measure of this distribution. Over 20 trials, a probability of 0.05 only represents one set of 10 shots, so a larger set of results may give a better picture of her long-term accuracy and reduce the impact of one strong result amongst many other weaker scores.

11 $k = \frac{1}{10}, \mathrm{E}(X) = 3, \sigma = 1$

12 a $\dfrac{1}{n}$ **b** $\dfrac{n+1}{2}$ **c** $\frac{1}{12}(n^2 - 1)$

13 a Because $Z = X + a$,

$$\begin{aligned}\mathrm{E}(Z) &= \sum zP(Z = z)\\ &= \sum (x + a)\,P(X + a = x + a)\\ &= \sum (x + a)\,P(X = x)\\ &= \sum xP(X = x) + \sum aP(X = x)\\ &= \sum xP(X = x) + a\sum P(X = x)\\ &= \mu + a,\end{aligned}$$

because $\sum P(X = x) = 1$.

b Because $Z = kX$,

$$\begin{aligned}\mathrm{E}(Z) &= \sum zP(Z = z)\\ &= \sum (kx)\,P(kX = kx)\\ &= \sum (kx)\,P(X = x),\\ &= k \times \sum xP(X = x)\\ &= k\mu.\end{aligned}$$

14 a The mean of Z is $\mu + a$, by the previous question. Hence

$$\begin{aligned}\mathrm{Var}(Z) &= \mathrm{E}\big((Z - (\mu + a))^2\big)\\ &= \mathrm{E}\big((Z - a - \mu)^2\big)\\ &= \mathrm{E}\big((X - \mu)^2\big)\\ &= \mathrm{Var}(X)\end{aligned}$$

Hence the standard deviation of the new distribution remains σ. This is to be expected, because the distribution is no more spread out than previously.

b The mean of Z is $k\mu$, by the previous question. Hence

$$\begin{aligned}\mathrm{Var}(Z) &= \mathrm{E}\big((Z - k\mu)^2\big)\\ &= \mathrm{E}\big((kX - k\mu)^2\big)\\ &= k^2 \times \mathrm{E}\big((X - \mu)^2\big)\\ &= k^2\mathrm{Var}(X)\end{aligned}$$

Hence the standard deviation of the new distribution is $\sqrt{k^2\sigma^2} = k\sigma$.

Exercise 13D

1 a

x	0	1	2	3	Sum
$p(x)$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{8}{27}$	$\frac{1}{27}$	
$xp(x)$	0	$\frac{12}{27}$	$\frac{12}{27}$	$\frac{3}{27}$	1
$x^2p(x)$	0	$\frac{12}{27}$	$\frac{24}{27}$	$\frac{9}{27}$	$1\frac{2}{3}$

$\mu = 1, \sigma^2 = 1\frac{2}{3} - 1^2 = \frac{2}{3}, \sigma \doteqdot 0.82$

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b

x	0	1	2	3	Sum
f	33	47	16	4	100
f_r	0.33	0.47	0.16	0.04	1
xf_r	0	0.47	0.32	0.12	0.91
x^2f_r	0	0.47	0.64	0.36	1.47

$\bar{x} = 0.91, s^2 = 1.47 - (0.91)^2 = 0.6419,$
$s \doteqdot 0.80$

c The sample results are a little below what is predicted by the theoretical probabilities.

2 a $\mu = 7, \sigma^2 = \frac{35}{6}, \sigma \doteqdot 2.42$

4 c

x	0	1	2	3	4	5	Sum
$p(x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	1
$xp(x)$	0	$\frac{10}{36}$	$\frac{16}{36}$	$\frac{18}{36}$	$\frac{16}{36}$	$\frac{10}{36}$	$\frac{70}{36}$
$x^2p(x)$	0	$\frac{10}{36}$	$\frac{32}{36}$	$\frac{54}{36}$	$\frac{64}{36}$	$\frac{50}{36}$	$\frac{210}{36}$

$\mu \doteqdot 1.94, \sigma^2 = \frac{210}{36} - \left(\frac{70}{36}\right)^2, \sigma \doteqdot 1.43$

7 a $\mu = 3.441, \sigma \doteqdot 2.46$

12 a Later people taking part in the experiment will be influenced by earlier guesses, particularly if the previous guesses have been measured for accuracy. Perhaps students could record their estimate, or draw their estimated shape, at the same time and before any measuring occurs. Perhaps students go into a separate room for the experiment.

14 a $m - k$ is the number of serial numbers not yet discovered in the range from 1 to m. If these serial numbers are spread between the k gaps, the average size of the gap (number of undiscovered serials) is $\frac{m-k}{k}$.

b The gap of $\frac{m-k}{k}$ integers should extend past m to $m + \frac{m-k}{k}$. Using this estimate the last serial will be:

$$N = m + \frac{m-k}{k} = m + \frac{m}{k} - 1$$

Chapter 13 review exercise

1a Numeric, continuous
b Numeric, discrete
c Numeric, discrete (and infinite)
d Categorical

2 a Yes **a** No **b** No

3 The probabilities are not all positive, do not sum to 1, and are not all less than 1.

4 a $E(X) = 1.4$ **b** $E(X) = -0.8$

5 a $E(X) = 27.22$
b His expected cost is $\$27.22 \times 52 = \$1415.56$.

6 a $E(X) = 2, Var(X) = 1, \sigma = 1.$
b $E(X) = 5.1, Var(X) = 0.69, \sigma \doteqdot 0.83.$

7 a $E(X) = 2, E(X^2) = 5, Var(X) = 1$
b $E(X) = 5.1, E(X^2) = 26.70, Var(X) = 0.69$

8 a $E(X) = 1.9, Var(X) = 0.49, \sigma = 0.7$
b $E(X) = 2, Var(X) = 2.6, \sigma \doteqdot 1.61$

9 Expected value is a measure of central tendency — it measures the centre of the data set. It may also be thought of as a weighted mean (weighted by the probabilities of the distribution). If the experiment is carried out experimentally a large number of times we would expect that the average of the outcomes would approach the expected value.

10 The standard deviation is the square root of the variance. Both measure the spread of the data, so that a distribution with a larger standard deviation is more spread out than a distribution with a smaller standard deviation. Both are zero if the distribution only takes one value — that is, if it is not spread out at all. If the distribution is stretched (multiplied) by a constant k the standard deviation also increases by a factor k.

11 a $12, 8, 2\sqrt{2}$ **b** $11, 2, \sqrt{2}$ **c** $17, 18, 3\sqrt{2}$

12 a

x	5	6	7	8	9
$p(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

$E(X) = 7, Var(X) = 2, \sigma = \sqrt{2}.$

Chapter 14

Exercise 14A

1 a 8! **b** 4! **c** 3! **d** 101! **e** 20!

2 a 6 **b** 120 **c** 1 **d** 15
e 45 **f** 35 **g** 220 **h** 70

3 a 5040 **b** 3 628 800 **c** 1 **d** 15 120
e 6720 **f** 2520 **g** 5005 **h** 13 860

4 a $6x^5$ **b** $30x^4$ **c** $120x^3$ **d** $360x^2$
e $720x$ **f** 720 **g** 0

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5 a n **b** $n!$
c 1 **d** $n(n+1)$
e $(n+1)(n+2)$ **f** $\dfrac{1}{n(n-1)}$
g $\dfrac{n-2}{n}$ **h** $\dfrac{(n-1)!}{n+1}$
6 a $7 \times 7!$
b $n \times n!$
c $57 \times 6!$
d $(n^2+n+1) \times (n-1)!$
e $9^2 \times 7!$
f $(n+1)^2 \times (n-1)!$
7 a $\dfrac{1+n}{n!}$ **b** $\dfrac{n}{(n+1)!}$ **c** $\dfrac{1-n-n^2}{(n+1)!}$
8 a i nx^{n-1} **ii** $n(n-1)x^{n-2}$ **iii** $n!$
iv $n(n-1)(n-2)\cdots(n-k+1)x^{n-k}$
$= \dfrac{n!}{(n-k)!}x^{n-k}$
b i $-1! \times x^{-2}$
ii $2! \times x^{-3}$
iii $-5! \times x^{-6}$
iv $(-1)^n \times n! \times x^{-(n+1)}$
9 b $(n+1)! - 1$
11 a i 2^8 **ii** 10^2
b i 2^{97} **ii** 5^{24} **iii** 7^{16} **iv** 13^7
12 a $\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \frac{1}{30}, \frac{1}{144}$ **b** $\frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}, \frac{719}{720}$
c $S_n = 1 - \dfrac{1}{(n+1)!}$. The limit is 1.
d The sum can be written as
$$\left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \cdots + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right).$$
13 a $2^{15} \times 15!$ **b** $\dfrac{30!}{2^{15} \times 15!}$ or $\dfrac{29!}{2^{14} \times 14!}$
c $\dfrac{2^{30} \times (15!)^2}{30!}$
14 0.14%

Exercise 14B

1 There are 6: DOG, DGO, ODG, OGD, GOD, GDO
2 FEG, FGE, FEH, FHE, FEI, FIE, FGH, FHG, FGI, FIG, FHI, FIH
3 a 360 **b** 720
4 a 120 **b** 625
5 60, 36
6 5040
7 a ${}^{10}P_3 = 720$ **b** ${}^{5}P_3 = 60$ **c** ${}^{100}P_3 = 970\,200$
8 a $9^3 = 729$ **b** $100^3 = 1\,000\,000$ **c** $2^{10} = 1024$
9 a 40 320 **b** 336
10 a 12 **b** 864
11 720 **12** 48
13 a 10^7 **b** 5×10^6 **c** 5^7 **d** 32 000
14 a 10 000 **b** 5040 **c** 625 **d** 1000
15 a 3024 **b** 336 **c** 1344 **d** 336 **e** 1008
16 a 6561 **b** 729 **c** 2916 **d** 729 **e** 2187
17 a 6 760 000 **b** 3 276 000 **c** 26 000 **d** 48
18 a 720 **b** 120
c 24 **d** 360 (half of them)
19 1728 **20** 24
21 a 120 **b** 24 **c** 95
22 a i 64 **ii** 32
b i 340 **ii** 170
23 a 96 **b** 36 **c** 24
24 a 3 **b** 3
25 a 30 000
b $9 \times 9 \times 9 \times 9 \times 3 = 19\,683$ (Choose the last digit so that the sum is a multiple of 3.)
c $9 \times 9 \times 9 \times 9 \times 4 = 26\,244$
d $4 \times 9 \times 9 \times 9 \times 3 = 8748$ (Choose the last digit first and the first digit last.)

Exercise 14C

1 a $5! \times 2! = 240$ **b** $2! \times 2! = 4$
c $3! \times 2! = 12$ **d** $5! \times 3! = 720$
2 a $4! \times 2! = 48$ **b** $4! = 24$
3 $3! \times 2! \times 2! = 24$
4 622 080
5 a $2 \times 3 \times 3 \times 2 \times 1 + 3 \times 4! = 36 + 72 = 108$
b $5! - 2 \times 3! = 120 - 12 = 108$
6 If the father drives, there are $2 \times 2 \times 1$ ways to arrange the seating. If the mother drives, there are $1 \times 2 \times 1$ ways to arrange the seating. Thus there are 6 ways in total.
7 Number of three-digit numbers $= 3 \times 2 \times 1 = 6$.
Number of two-digit numbers $= 2 \times 2 = 4$.
The total number of numbers is 10.
8 a 144 **b** 120
c 144 **d** 2520 (half of the total)
9 a 720 **b** 720 **c** 4320
10 a 24 **b** 240
11 $2 + 4 + 8 + \cdots + 1024 = 2046$
12 a 1152 **b** 1152
13 a 720 **b** 120 **c** 1680
d 4200 **e** 960 **f** 480

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14 a 5040 **b** 4320 **c** 720 **d** 144
e 720 **f** 960 **g** 1440
15 a 7^7 **b** 6×7^6 **c** 7^6
d $3^4 \times 4^3 + 4^4 \times 3^3 = 7 \times 12^3$
16 a i 3 628 800 **ii** 725 760
iii 725 760 **iv** 2 257 920
b i $2(n - 1)!$ **ii** $2(n - 1)!$
iii $(n - 2)(n - 3)(n - 2)!$
17 8640
18 a 40 320 **b** 20 160 **c** 17 280
19 a 5040 **b** 20 160
20 a 5^5 ways **b** $5! = 120$ ways
c $5 \times 4^3 = 320$ ways
21 a 133 **b** 104 **c** 29 **d** 56
22 a $D(1) = 0, D(2) = 1, D(3) = 2, D(4) = 9$
c $D(5) = 44, D(6) = 265, D(7) = 1854,$
$D(8) = 14833$

Exercise 14D

1 a 3 **b** 12 **c** 120
d 6720 **e** 10 080 **f** 90 720
g 4 989 600 **h** 45 360 **i** 25 740
2 60
3 a 6 **b** 15 **c** 20
4 a 40 320 **b** 8 **c** 56 **d** 560
5 a 56 **b** 20
6 a 56 **b** 5
7 a 60 **b** 24
c 36 **d** 30 (half of them)
8 a i 180 **ii** 60 **iii** 120 **iv** 24
b 40
9 a 90 720 **b** 720
c 720 **d** 45 360 (half of them)
10 2 721 600
11 a 1024 **b** 256 **c** 45 **d** 252
e 56 **f** 512 **g** 8 **h** 70
12 a 60 **b** 60
13 a 120 **b** 60
14 a 453 600 **b** 90 720 **c** 5040 **d** 10 080
e 80 640 **f** 282 240 **g** 15 120
15 a 3 628 800 **b** 4
16 a 2520 **b** 720
c i 600 **ii** 480 **iii** 360 **iv** 240 **v** 120
d 840. Insert the letters U, M, T and R successively into the word EGE. Alternatively, the answer is one third of all arrangements.
e 210 **f** 420
17 1 995 840

18 864. The problem can be done by applying the inclusion–exclusion principle from the Extension section of Exercise 12C, or by considering separately the various different patterns.
19 a 4!
b Each is $3! = 6$, so subtract $4 \times 3! = 24$.
c 2! permutations leave A and B unmoved, and there are ${}^4C_2$ pairs of letters, so add $6 \times 2! = 12$.
d $D(4) = 1 \times 4! - 4 \times 3! + 6 \times 2!$
$- 4 \times 1! + 1 \times 0! = 9$
g The ratio of the number of permutations of n distinct letters to the number of derangements of them converges to e as $n \to \infty$. Thus, for example, if a long queue is formed at random and then rearranged into alphabetical order, the probability that no-one remains in his or her original position is $\frac{1}{e}$.

Exercise 14E

1 There are ${}^5C_2 = 10$ possible combinations:
PQ, PR, PS, PT, QR, QS, QT, RS, RT and ST.
2 a 21 **b** 35 **c** 15 **d** 126
3 a i 45 **ii** 45
b ${}^{10}C_2 = {}^{10}C_8$, and in general ${}^nC_r = {}^nC_{n-r}$.
4 a 44 352 **b** 34 650
5 a 70 **b** 36 **c** 16 **d** 1 **e** 69
6 a 126 **b** 45 **c** 51 **d** 75
7 a 2002 **b** 56 **c** 6 **d** 840
e 420 **f** 1316 **g** 715 **h** 1287
8 a 70 **b** 5 **c** 35
9 a 792 **b** 462 **c** 120 **d** 210 **e** 420
10 a i 252
ii 126. The number cannot begin with a zero.
b In each part, once the five numbers have been selected, they can only be arranged in one way.
11 13 860
12 a 1 745 944 200 **b** 413 513 100
13 a 45 **b** 120 **c** 36 **d** 8
14 a 10 **b** 110
15 a 65 780 **b** 1287 **c** 48
d 22 308 **e** 288 **f** 3744
16 a i ${}^6C_1 + {}^6C_2 = 21$
ii ${}^5C_2 = 10$ (choose the two people to go in the same group as Laura)
b i 4 **ii** 3
c i 92 **ii** 35
17 a 2 **b** 5 **c** 35 **d** ${}^nC_2 - n$
18 a 220 **b** 9240
c i 2772 **ii** 6468

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19 a 1024 **b** 968 **c** 466 **d** 247

20 a 16 **b** 20 **c** 12 **d** 8 **e** 5

21 a 252 **b** 126

22 a 315 **b** 210

23 a 12 **b** 49 **c** 120
d $(a + 1)(b + 1)(c + 1)$

24 a 30 **b** 24

25 a i 210 **ii** 90 **iii** 126 **iv** 126

26 5151

27 1360

28 a 3 **b** 315
c i 155 925 **ii** 10 800

29 b 0 (undefined), 2 (3), 30 (3), 864 (2.917), 39480 (2.872), 2631600 (2.844)

30 a i ${}^{a}C_0$ **ii** ${}^{a+1}C_1$ **iii** ${}^{a+2}C_2$ **iv** ${}^{a+3}C_3$ **v** ${}^{a+r}C_r$
b Add them up.
c $a + 1$ 0s and b 1s, total length $a + b + 1$.
d Go to the last 0 in the string, and remove it and any 1s that follow it. What remains is a string with a 0s and at most b 1s. When the process in part **c** is applied to the truncated string, the original string returns.
e The one-to-one correspondence in parts **c** and **d** show this.
f This follows from parts **b** and **e**.

Exercise 14F

1 a 84 **b** $\frac{5}{42}$

2 a $\frac{1}{210}$ **b** $\frac{2}{5}$ **c** $\frac{3}{5}$ **d** $\frac{4}{15}$

3 a $\frac{1}{13}$ **b** $\frac{46}{455}$ **c** $\frac{3}{91}$ **d** $\frac{3}{13}$

4 a $\frac{8}{429}$ **b** $\frac{1}{143}$ **c** $\frac{140}{429}$ **d** $\frac{421}{429}$ **e** $\frac{2}{11}$ **f** $\frac{1}{3}$

5 a $\frac{1}{22\,100}$ **b** $\frac{1}{5525}$
c $\frac{11}{850}$ **d** $\frac{22}{425}$
e $\frac{11}{1105}$ **f** $\frac{13}{34}$
g $\frac{16}{5525}$ **h** $\frac{6}{5525}$
i $\frac{741}{1700}$ **j** $\frac{64}{425}$

6 a $\frac{3}{70\,304}$ **b** $\frac{1}{2197}$
c $\frac{1}{64}$ **d** $\frac{1}{16}$
e $\frac{27}{2197}$ **f** $\frac{3}{8}$
g $\frac{6}{2197}$ **h** $\frac{3}{2197}$
i $\frac{27}{64}$ **j** $\frac{5}{32}$

7 a $\frac{1}{10}$ **b** $\frac{1}{10}$ **c** $\frac{1}{3}$

8 a $\frac{1}{10}$ **b** $\frac{2}{5}$

9 a $\frac{1}{15}$ **b** $\frac{2}{3}$

10 a $\frac{1}{42}$ **b** $\frac{2}{7}$ **c** $\frac{2}{7}$ **d** $\frac{1}{35}$ **e** $\frac{1}{7}$

11 a $\frac{1}{2}$ **b** $\frac{1}{6}$ **c** $\frac{1}{5}$ **d** $\frac{1}{60}$ **e** $\frac{2}{3}$

12 a $\frac{1}{7}$ **b** $\frac{2}{7}$ **c** $\frac{1}{7}$ **d** $\frac{2}{7}$

13 a $\frac{1}{3}$ **b** $\frac{2}{3}$ **c** $\frac{2}{15}$ **d** $\frac{1}{5}$

14 a $\frac{1}{26}$ **b** $\frac{5}{13}$ **c** $\frac{15}{26}$ **d** $\frac{1}{26}$

15 a 0.403 **b** 0.597 **c** 0.00174 **d** 0.291

16 a $\frac{1}{22}$ **b** $\frac{125}{1728}$ **c** $\frac{5}{144}$

17 a $\frac{10}{31}$ **b** $\frac{15}{31}$ **c** $\frac{6}{31}$

18 a $\frac{1}{60}$ **b** $\frac{3}{5}$ **c** $\frac{1}{5}$ **d** $\frac{2}{5}$ **e** $\frac{1}{20}$
f $\frac{3}{5}$ **g** $\frac{3}{10}$ **h** $\frac{9}{10}$ **i** $\frac{1}{10}$ **j** $\frac{3}{5}$

19 a $\frac{3}{8}$ **b** $\frac{1}{2}$ **c** $\frac{21}{32}$ **d** $\frac{3}{32}$ **e** $\frac{17}{64}$

20 a $\frac{281}{462}$ **b** 8

21 a $\frac{1}{27\,417}$ **b** $\frac{28}{703}$

22 In each part, the sample space has ${}^{52}C_5$ members.
a $\frac{352}{833}$. Choose the value of the pair in 13 ways, then choose the cards in the pair in ${}^{4}C_2 = 6$ ways, then choose the three values of the three remaining cards in ${}^{12}C_3$ ways, then choose the suits of those three cards in 4^3 ways.
b $\frac{198}{4165}$. Choose the values of the two pairs in ${}^{13}C_2$ ways, then choose the suits of the cards in the two pairs in ${}^{4}C_2 \times {}^{4}C_2$ ways, then choose the remaining card in 44 ways.
c $\frac{88}{4165}$ **d** $\frac{1}{4165}$ **e** $\frac{6}{4165}$
f $\frac{128}{32\,487}$. Choose the lowest card in 10 ways, then choose the suits of the five cards in 4^5 ways.
g $\frac{33}{16\,660}$ **h** $\frac{1}{649\,740}$

23 a $\frac{1}{125}$ **b** $\frac{4}{125}$ **c** $\frac{16}{125}$ **d** $\frac{108}{125}$

24 a $\frac{48}{125}$ **b** $\frac{n^2(n-1)^2(n-2)!}{2n^n}$

25 a 0.0082 **b** $1 - \frac{{}^{365}P_n}{365^n}$
d 23 **e** 41

26 a $\frac{1}{25}$ **b** $\frac{3}{25}$ **c** $\frac{19}{25}$

27 b $\frac{1}{8}$ **c** 2^{1-n}

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Exercise 14G

1 a i 120 **ii** 24

b i 3 628 800 **ii** 362 880

2 a 10 080 **b** 1440

3 a 24 **b** 6 **c** 4 **d** 12 **e** 4

4 a 5040 **b** 144 **c** 576

d 1440 **e** 3600 **f** 240

5 a $\frac{3}{10}$ **b** $\frac{1}{5}$ **c** $\frac{1}{10}$ **d** $\frac{9}{10}$

6 a 5040 **b** 576 **c** 144

d 2304 **e** 1440 **f** 3600

7 a $\frac{1}{12}$ **b** $\frac{1}{9}$

8 a $(n - 1)!$ **b** $2 \times (n - 2)!$

c $(n - 3) \times (n - 2)!$ **d** $6 \times (n - 3)!$

9 a 39 916 800 **b** 165

10 145 152

11 a 288 **b** $\frac{1}{4}$

12 $\dfrac{n!\,(n + 1)!}{(2n)!}$

13 a 60 **b** 181 440 **c** 9

Exercise 14H

1 a 7

b There is no guarantee — it is ‘possible’ (but unlikely) that 6 never turns up, even in 1 000 000 throws, or in any number of throws.

2 Pigeonhole the numbers as either odd or even. Because there are three numbers, at least one pigeonhole must have 2 numbers in it.

3 $16 \div 5 = 3$ remainder 1, so at least one location must have 4 eggs (maybe more).

4 a 4 **b** 7

c $3(n - 1) + 1 = 3n - 2$

5 Imagine that the pigeonholes are labelled with the numbers 1 to 6, and on each throw a token (pigeon) is put in the corresponding pigeonhole. Because $13 \div 6 = 2$ remainder 1, then when filling the 6 pigeonholes following each draw, at least one pigeonhole has three tokens.

6 Less than ten.

7 a 10

b Four — all might use the same item.

8 Divide the grid into 49 one-metre squares. Because $100 \div 49 = 2$ remainder 2, there must be a one-metre square covering at least three points.

9 The possible totals are 2, 3, 4, 5 and 6. After 6 throws one of these sums must have occured at least twice.

10 $100 \div 12 = 8$ remainder 4, so there must be at least one birth month shared by at least nine people in the group.

11 The pigeonholes correspond to the four suits. As $10 \div 4 = 2$ remainder 2, so one suit must occur three times.

12 Yes, because $567 \div 23 = 24$ remainder 15.

13 a Zero — they might all be in another group.

b $19 \div 3 = 6$ remainder 1.

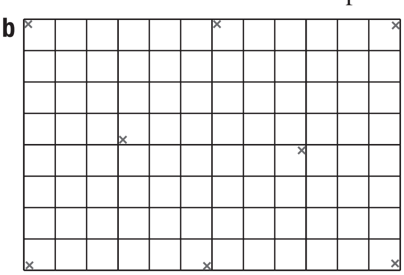
c All 19.

14 a Divide the board into sixteen 2×2 squares, and place a king in each square. If each king is in a corresponding place, the arrangement is permissible, so 16 kings can be arranged as required. If, however, a second king is placed into any square, the two kings in that square will be adjacent, so 16 is the maximum.

b 8 — lay them along the main diagonal.

15 a Divide the field into 4×3 rectangles — two rows and four columns of them. There must be a rectangle containing two cows which, by Pythagoras' theorem, must be closer than 5 metres apart.

b



16 The remainder on division by 3 is either 0, 1 or 2. If 100 numbers are placed in these three pigeonholes, at least one pigeonhole must contain 34 entries, because $100 \div 3 = 33$ remainder 1.

17 Consider the powers of 2 from 2^1 up to 2^{2020} and pigeonhole them by their remainders on division by 2019. The remainder must be a number from 0 to 2018. By the pigeonhole principle, there must be two powers that leave the same remainder, thus their difference is a multiple of 2019.

18 Consider the 13 pigeonholes $\{1, 51\}$, $\{3, 49\}$, . . ., $\{25, 27\}$. When the 14 odd numbers are distributed amongst these pigeonholes at least one must have two members; the two members of this pigeonhole add to 52.

19 Pair the numbers to form 50 pigeonholes labelled $\{1, 199\}$, $\{3, 197\}$, . . ., $\{99, 101\}$. Given 51 odd numbers less than 200, two must fall in the same pigeonhole and add to 200.

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20 Draw an equilateral triangle in the plane with side length 1 unit. Then two of the three vertices must be the same colour (three pigeons (vertices) must lie in the same pigeonhole (colour)). This theorem may be generalised to three colours but it is an open problem for the case of four, five and six colours. The result is false for seven colours.

21 a $41 \div 10 = 4$ remainder 1

b Arrange the 10 pigeonholes as $A, B, C \ldots, J$ in descending order $|A| \geq |B| \geq |C| \geq \cdots \geq |J|$, where $|X|$ is the number of pigeons in pigeonhole X. Then $|A| \geq 5$ by part **a**.
If $|B| \geq 4$, then $|A| + |B| \geq 9$, as required, so suppose that $|B| \leq 3$. Then also $|C| \leq 3, |D| \leq 3, \ldots, |J| \leq 3$, so $|B| + |C| + \cdots + |J| \leq 3 \times 9 = 27$, in which case $|A| \geq 14$, so $|A| + |B| \geq 14$.

22 a There must be more than $26 \times 26 = 676$ students.

b Half the school share at most $8 \times 26 = 208$ addresses, thus it is known that the school has at least 417 students

c We need to know when one of the 26×26 pigeonholes has more than 11 members. This must happen when 7437 addresses are assigned (but may happen sooner), thus in the thirty-third year of this scheme's operation, because $1200 + 32 \times 200 > 7437$.

23 The pigeonholes are the number of friends, 0–5. We need to reduce the number of categories if we are going to apply the pigeonhole principle usefully. If there is someone with no friends at the table, then there is no one with five friends. In this case there are five pigeonholes to place the six people and at least one pigeonhole has two of the people, that is, they have the same number of friends.
If there is someone with five friends at the table, then there is no one with no friends. As before, we have six people and five pigeonholes and thus two people must have the same number of friends.

24 a The number of rods is ${}^6C_2 = 15$.

b The number of triangles is ${}^6C_3 = 20$.

c Choose any vertex O. Five rods are joined to O, so there must be at least three rods OA, OB and OC of the same colour. If any one of the rods AB, BC or CA has that same colour, then that rod and O form a mono-coloured triangle. If all the rods have the other colour, then ABC is a mono-coloured triangle.

Chapter 14 review exercise

1 $8! = 40320$

2 a 72 **b** $n(n + 1)$ **c** $k \times k!$

3 a 792 **b** 6 **c** 1140

4 ${}^7P_4 = 840$

5 $26^3 \times 10^4 = 175760000$

6 a $4! \times 3! = 144$ **b** $5! = 120$ **c** $3! \times 4! = 144$

d $7! \div 2 = 2520$ (half of the total)

7 a $10! = 3628800$

b $2 \times 5! \times 5! = 28800$

c $9! \times 2! = 725760$

8 $\dfrac{8!}{3! \times 2!} = 3360$

9 ${}^6P_3 + {}^6P_4 = 480$

10 $\dfrac{10!}{6! \times 4!} = 210$

11 a ${}^{16}C_7 = 11440$

b ${}^{10}C_7 = 120$

c 0 (there are no such committees of 7)

d ${}^6C_2 \times {}^{10}C_5 = 3780$

e ${}^{10}C_4 \times {}^6C_3 = 4200$

f By adding cases: 9360

g ${}^{15}C_6 = 5005$

h The complement of part **g**, that is, 6435

i The complement of the case where both are members, that is, 9438.

12 a ${}^8C_5 = 56$ **b** ${}^8C_4 = 70$ **c** 35

13 Choose the four men members, then choose the two women members, $\dfrac{{}^4C_4 \times {}^3C_2}{{}^7C_6} = \frac{3}{7}$. Alternatively, only one of the seven is not on the committee, and there will be a majority of men, when this person is a women. Thus the probability is $\frac{3}{7}$.

14 a $\frac{1}{22100}$ **b** $\frac{1}{5525}$ **c** $\frac{11}{850}$ **d** $\frac{22}{425}$

e $\frac{13}{34}$ **f** $\frac{16}{5525}$ **g** $\frac{6}{5525}$ **h** $\frac{64}{425}$

15 a 120 **b** 12

c 36 **d** 48

e $120 - 48 = 72$ **f** 24

16 $17 \div 6 = 2$ remainder 5, so there must be a monkey who receives at least three peanuts.

17 Place the 1500 pigeons in pigeonholes labelled by the 366 possible days of a year. $1500 \div 366 = 4$ remainder 36, so there is at least one day shared by at least 5 people.

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Chapter 15

Exercise 15A

2 a $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$
b $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$
c $1 + 9x + 36x^2 + 84x^3 + 126x^4 + 126x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$
d $1 - 9x + 36x^2 - 84x^3 + 126x^4 - 126x^5 + 84x^6 - 36x^7 + 9x^8 - x^9$
e $1 + 5c + 10c^2 + 10c^3 + 5c^4 + c^5$
f $1 + 8y + 24y^2 + 32y^3 + 16y^4$
g $1 + \frac{7x}{3} + \frac{7x^2}{3} + \frac{35x^3}{27} + \frac{35x^4}{81} + \frac{7x^5}{81} + \frac{7x^6}{729} + \frac{1x^7}{2187}$
h $1 - 9z + 27z^2 - 27z^3$
i $1 - \frac{8}{x} + \frac{28}{x^2} - \frac{56}{x^3} + \frac{70}{x^4} - \frac{56}{x^5} + \frac{28}{x^6} - \frac{8}{x^7} + \frac{1}{x^8}$
j $1 + \frac{10}{x} + \frac{40}{x^2} + \frac{80}{x^3} + \frac{80}{x^4} + \frac{32}{x^5}$
k $1 + \frac{5y}{x} + \frac{10y^2}{x^2} + \frac{10y^3}{x^3} + \frac{5y^4}{x^4} + \frac{y^5}{x^5}$
l $1 + \frac{12x}{y} + \frac{54x^2}{y^2} + \frac{108x^3}{y^3} + \frac{81x^4}{y^4}$

4 a i $55x^2$ **ii** $165x^8$
b i $-35x^3$ **ii** $-21x^5$
c i $240x^4$ **ii** $192x^5$
d i $-\frac{12}{x}$ **ii** $\frac{54}{x^2}$

6 a $\big(1 + (x - 1)\big)^3 = x^3$
b $\big(1 - (x + 1)\big)^6 = (-x)^6 = x^6$

7 21

8 a $a = 76, b = 44$ **b** $a = 16, b = -8$

10 a 1.018 14 **b** 0.815 37

11 a i $1 + 4x + 6x^2 + \cdots$ **ii** -14
b i $1 + 10x + 40x^2 + 80x^3 + \cdots$
ii 40
c i $1 - 12x + 54x^2 - 108x^3 + \cdots$
ii -228

12 a $x = 0$ or $\frac{1}{2}$ **b** $x = 0, 1$ or 5

13 a -12 **b** 0 **c** 380 **d** $-\frac{5}{3}$

14 a 97 **b** $1\frac{10}{27}$

15 a i $15x^2$ **ii** $20x^3$
iii $3 : 4x$ **iv** $135, 540, 1 : 4$
b i $\frac{224}{81x^5}$ **ii** $\frac{448}{729x^6}$
iii $9x : 2$ **iv** $\frac{7}{81}, \frac{7}{729}, 9 : 1$

16 a $k = 5$ **b** $k = -2$ or 0

18 1.0634

19 $(1 + x + y)^0 = 1$, $(1 + x + y)^1 = 1 + x + y$,
$(1 + x + y)^2 = 1 + 2x + 2y + 2xy + x^2 + y^2$,
$(1 + x + y)^3 = 1 + 3x + 3y + 6xy + 3x^2 + 3y^2 + 3x^2y + 3xy^2 + x^3 + y^3$,
$(1 + x + y)^4 = 1 + 4x + 4y + 12xy + 6x^2 + 6y^2 + 6x^2y^2 + 12x^2y + 12xy^2 + 4x^3 + 4y^3 + 4x^3y + 4xy^3 + x^4 + y^4$

The coefficients form a triangular pyramid, with 1s on the edges, and each face a copy of Pascal's triangle.

Exercise 15B

1 a $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
b $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$
c $r^6 - 6r^5s + 15r^4s^2 - 20r^3s^3 + 15r^2s^4 - 6rs^5 + s^6$
d $p^{10} + 10p^9q + 45p^8q^2 + 120p^7q^3 + 210p^6q^4 + 252p^5q^5 + 210p^4q^6 + 120p^3q^7 + 45p^2q^8 + 10pq^9 + q^{10}$
e $a^9 - 9a^8b + 36a^7b^2 - 84a^6b^3 + 126a^5b^4 - 126a^4b^5 + 84a^3b^6 - 36a^2b^7 + 9ab^8 - b^9$
f $32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$
g $p^7 - 14p^6q + 84p^5q^2 - 280p^4q^3 + 560p^3q^4 - 672p^2q^5 + 448pq^6 - 128q^7$
h $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$
i $a^3 - \frac{3a^2b}{2} + \frac{3ab^2}{4} - \frac{1b^3}{8}$
j $\frac{1r^5}{32} + \frac{5r^4s}{48} + \frac{5r^3s^2}{36} + \frac{5r^2s^3}{54} + \frac{5rs^4}{162} + \frac{1s^5}{243}$
k $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

2 a $1 + 4x^2 + 6x^4 + 4x^6 + x^8$
b $1 - 9x^2 + 27x^4 - 27x^6$
c $x^{12} + 12x^{10}y^3 + 60x^8y^6 + 160x^6y^9 + 240x^4y^{12} + 192x^2y^{15} + 64y^{18}$
d $x^9 - 9x^7 + 36x^5 - 84x^3 + 126x - \frac{126}{x} + \frac{84}{x^3} - \frac{36}{x^5} + \frac{9}{x^7} - \frac{1}{x^9}$
e $x^3\sqrt{x} + 7x^3\sqrt{y} + 21x^2y\sqrt{x} + 35x^2y\sqrt{y} + 35xy^2\sqrt{x} + 21xy^2\sqrt{y} + 7y^3\sqrt{x} + y^3\sqrt{y}$
f $\frac{32}{x^5} + \frac{240}{x^2} + 720x + 1080x^4 + 810x^7 + 243x^{10}$

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3 a $\left(y + (x - y)\right)^5 = x^5$

b $\left(a - (a - b)\right)^4 = b^4$

c $\left(x + (2y - x)\right)^3 = (2y)^3 = 8y^3$

d $\left((x + y) - (x - y)\right)^6 = (2y)^6 = 64y^6$

4 a i $1024 + 1280x + 640x^2 + 160x^3 + \cdots$

ii -160

b i $1 - 12x + 60x^2 - 160x^3 + 240x^4 - \cdots$

ii 720

c i $2187 - 5103y + 5103y^2 - 2835y^3 + 945y^4 - \cdots$

ii 11 718

5 a $2x^6 + 30x^4y^2 + 30x^2y^4 + 2y^6$

6 a 540 **b** 48 **c** -960 **d** -8

7 a i $x^3 + 3x^2h + 3xh^2 + h^3$

ii $3x^2h + 3xh^2 + h^3$

iii $3x^2$ **b** $5x^4$

8 b 466 **c** 42

9 $\frac{7}{2}$

10 a 1.104 08 **b** 0.903 92

11 a i $\left(x^3 + \dfrac{1}{x^3}\right) + 3\left(x + \dfrac{1}{x}\right)$

ii $\left(x^5 + \dfrac{1}{x^5}\right) + 5\left(x^3 + \dfrac{1}{x^3}\right) + 10\left(x + \dfrac{1}{x}\right)$

iii $\left(x^7 + \dfrac{1}{x^7}\right) + 7\left(x^5 + \dfrac{1}{x^5}\right) + 21\left(x^3 + \dfrac{1}{x^3}\right) + 35\left(x + \dfrac{1}{x}\right)$

b i 2 **ii** 2 **iii** 2

12 $a = 3$ or $a = -3$

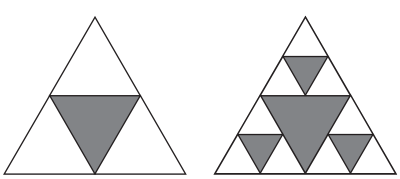
14 a $x^6 + 6x^4 + 15x^2 + 20 + \dfrac{15}{x^2} + \dfrac{6}{x^4} + \dfrac{1}{x^6}$

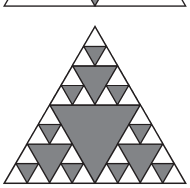
b $A = -6, B = 9$ and $C = -2$.

15 a $x^3 + y^3 + z^3 + 6xyz + 3x^2y + 3xy^2 + 3xz^2 + 3x^2z + 3y^2z + 3yz^2$

b 19

16 a The limiting figure for this process is called the Sierpinski Gasket. It is one of the classic regular fractals.





b Sierpinski's triangle is formed.

Exercise 15C

1 a 4 **b** 20 **c** 9 **d** 35

2 a 10 **b** 1 **c** 11 **d** 210

3 a 3003 **b** 1287 **c** 792 **d** 22 **e** 429 **f** 1292

4 a ${}^4C_0 = 1, {}^4C_1 = 4, {}^4C_2 = 6, {}^4C_3 = 4, {}^4C_4 = 1$

b i 16 **ii** 0

5 a 32 **b** 32 **c** 20 **d** 252

6 a i 56 **ii** 35

b 5 **c** $n = 4$ or $n = 8$

7 b i 6 **ii** 10

8 a $8568x^5$ **b** $2\,217\,093\,120x^9$

c $-19208x^3$ **d** 189

9 a $672x^2$ **b** $\dfrac{1001x^9y^5}{16}$ **c** $-\dfrac{33x^{10}y^2}{1024}$ **d** $190a^2b^9$

10 a $x = \frac{11}{2}$ **b** $x = -\frac{7}{3}$

11 a $5x^2 : 39$ **b** $5 : 2$ **c** $18\,304 : 1$

12 a i 1 **ii** n

iii $\frac{1}{2}n(n - 1)$ **iv** $\frac{1}{6}n(n - 1)(n - 2)$

b i 16 **ii** 9 **iii** 4

iv 6 **v** 4 **vi** 7

13 a $a = 2$ and $n = 14$ **b** $a = -\frac{1}{3}$ and $n = 10$

14 a $n = 14$ **b** $n = 13$

15 ${}^{40}C_{20} \doteqdot 1.378 \times 10^{11}$

17 a ${}^nC_0\, x^n + {}^nC_1\, x^{n-1}h + {}^nC_2\, x^{n-2}h^2 + \cdots + {}^nC_n\, h^n$

b nx^{n-1}

18 The second member is ${}^nC_1 = n$, so suppose that n is prime. Then n is coprime to every number less than n, so is coprime to $r!$ and $(n - r)!$ for all whole numbers $r = 2, 3, \ldots, r - 1$.

19 a 3 points, 3 segments, 1 triangle

b 4 points, 6 segments, 4 triangles, 1 quadrilateral

c 5 points, 10 segments, 10 triangles, 5 quadrilaterals, 1 pentagon

d 21

20 b i $1 + x + x^2 + x^3 + \cdots$

ii $1 + 2x + 3x^2 + 4x^3 + \cdots$

iii $1 - 2x + 3x^2 - 4x^3 + \cdots$

iv $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$

c Using part **b i**,

$$\begin{aligned} \text{LHS} &= \frac{d}{dx}\left(\frac{1}{1 - x}\right) \\ &= \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) \\ &= 1 + 2x + 3x^2 + \cdots \\ &= \text{RHS by part } \textbf{b ii}. \end{aligned}$$

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Exercise 15D

1 a **i** $1 + 4 + 6 + 4 + 1 = 16 = 2^4$.

ii The sum $1 + 6 + 1$ of the first, third and fifth terms on the row equals the sum $4 + 4$ of the second and fourth terms.

iii The sum of the first, third and fifth terms on the row is half the sum of the whole row.

b **i** $4(1 + x)^3 = {}^4C_1 + 2\,{}^4C_2 x + 3\,{}^4C_3 x^2 + 4\,{}^4C_4 x^3$

ii $1 \times 4 + 2 \times 6 + 3 \times 4 + 4 \times 1 = 32 = 4 \times 2^3$.

iii $1 \times 4 - 2 \times 6 + 3 \times 4 - 4 \times 1 = 0$.

4 a There are 10. **b** There are 10.

c C, D and E.

d Given 5 letters, choose 2. Those that are left form a set of 3. Thus for every set of 2, there is a corresponding set of 3. Thus ${}^5C_2 = {}^5C_3$.

e Given n people, choose r. Those that are left form a set of $n - r$. Thus for every set of r, there is a corresponding set of $n - r$. Thus ${}^nC_r = {}^nC_{n-r}$.

5 a To form a subset of S, take each of A, B, C and D in turn and decide whether it is in or out. Thus the total number of subsets of S is $2 \times 2 \times 2 \times 2 = 2^4$.

b The LHS is the sum of the numbers of 0-member, 1-member, 2-member, 3-member and 4-member subsets, and so is also the number of all subsets.

c Generalise the previous argument to an n-member set.

6 a **i** There are four of them.

b **i** There are six of them. **ii** Omit E from each set.

c Let T be a 3-letter subset of S. If T does not contain E, then T is one of the the 3-letter subsets of U. If Tdoes contain E, then remove E, and the remaining 2-letter subset pairs with one of the 2-letter subsets of U.

d Generalise the previous argument.

7 a Using the addition property,

$$\text{LHS} = \frac{{}^nC_{r-1}}{{}^{n+1}C_r} + \frac{{}^nC_{r+1}}{{}^{n+1}C_{r+2}} \text{ and } \text{RHS} = \frac{2 \times {}^nC_r}{{}^{n+1}C_{r+1}}.$$

Now use the formula for ${}^nC_r$.

8 a As explained in Question 5, the LHS counts all the subsets of S. We can also count the subsets by choosing whether each element in turn goes into a subset of not, giving 2^n subsets.

b **i** First, every subset of S either contains A or does not contain A. Secondly, a subset containing A is paired up with the unique subset obtained by removing A from the subset.

ii Adding A to a subset without A changes the number of members from odd to even or from even to odd.

iii The LHS is the total number of even-order subsets, and the RHS is the total number of odd-order subsets. We have seen that they are paired with each other, so the LHS and RHS are equal.

9 a **i**
$$\begin{aligned} {}^7C_3 &= {}^6C_3 + {}^6C_2 = {}^5C_3 + {}^5C_2 + {}^6C_2 \\ &= {}^4C_3 + {}^4C_2 + {}^5C_2 + {}^6C_2 \\ &= {}^3C_3 + {}^3C_2 + {}^4C_2 + {}^5C_2 + {}^6C_2, \end{aligned}$$
and ${}^3C_3 = {}^2C_2 = 1$

b **i** There are ${}^2C_2 = 1$ subsets with highest element 3.
There are ${}^3C_2 = 3$ subsets with highest element 4.
There are ${}^4C_2 = 6$ subsets with highest element 5.
There are ${}^5C_2 = 10$ subsets with highest element 6.
There are ${}^6C_2 = 15$ subsets with highest element 7. This makes 353-member subsets.

10 a A common denominator for the two fractions is required:

$$\begin{aligned} \text{LHS} &= \frac{n!}{r! \times (n-r)!} + \frac{n!}{(r-1)! \times (n-r+1)!} \\ &= \frac{n!}{r \times (r-1)! \times (n-r)!} \\ &\quad + \frac{n!}{(r-1)! \times (n-r+1) \times (n-r)!} \\ &= \frac{(n-r+1) \times n! + r \times n!}{r \times (r-1)! \times (n-r+1) \times (n-r)!} \\ &= \frac{\big((n-r+1) + r\big) \times n!}{r! \times (n-r+1)!} \\ &= \frac{(n+1) \times n!}{r! \times (n-r+1)!} \\ &= \frac{(n+1)!}{r! \times (n-r+1)!} \\ &= \text{RHS}. \end{aligned}$$

b This is the addition property of Pascal's triangle.

12 a Regarding this as an $(a + b)$-letter word with a identical As and b identical Bs, the number of permutations is $\dfrac{(a+b)!}{a!b!} = {}^{a+b}C_a = {}^{a+b}C_b$

b ${}^{2n}C_n$

c **i** ${}^nC_2$ **ii** ${}^nC_{n-2} = {}^nC_2$

d Consider a $2n$-letter binary word with n As and n Bs, split into two equal halves, namely the first and second half. Consider the $n + 1$ cases where $0, 1, 2, 3, \ldots, n$ As fall in the first half and the rest fall in the second half. Using the arguments of part **b** and **c**, we have

$$\begin{aligned} {}^{2n}C_n &= ({}^{2n}C_0)({}^{2n}C_{2n}) + ({}^{2n}C_1)({}^{2n}C_{2n-1}) \\ &\quad + ({}^{2n}C_2)({}^{2n}C_{2n-2}) + \cdots + ({}^{2n}C_{2n})({}^{2n}C_0) \\ &= ({}^{2n}C_0)({}^{2n}C_0) + ({}^{2n}C_1)({}^{2n}C_1) \\ &\quad + ({}^{2n}C_2)({}^{2n}C_2) + \cdots + ({}^{2n}C_{2n})({}^{2n}C_{2n}) \end{aligned}$$

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Exercise 15E

1 a ${}^{13}C_k\, x^k$ **b** ${}^{7}C_k\, 2^k\, x^k$

c ${}^{12}C_k\, 5^{12-k}\, 7^k\, x^k$

d ${}^{9}C_k\, 2^{9-k}\,(-1)^k\, x^{9-k}\, y^k$

e ${}^{5}C_k\, x^{5-k}\, 2^k\, x^{-k} = {}^{5}C_k\, 2^k\, x^{5-2k}$

f ${}^{8}C_k\,(6x)^{8-k}\,(-2)^k\, x^{-k} = {}^{8}C_k\, 3^{8-k}\,(-1)^k\, 2^8\, x^{8-2k}$

2 b i 126 **ii** 36 **iii** 84

3 b i ${}^{10}C_4 \times 2^6 \times 3^4 = 2^7 \times 3^5 \times 5 \times 7$

ii ${}^{10}C_7 \times 2^3 \times 3^7 = 2^6 \times 3^8 \times 5$

iii ${}^{10}C_6 \times 2^4 \times 3^6 = 2^5 \times 3^7 \times 5 \times 7$

4 b i ${}^{15}C_2 \times 5^2 \times 2^{-13}$

ii $-{}^{15}C_7 \times 5^7 \times 2^{-8}$

iii ${}^{15}C_{10} \times 5^{10} \times 2^{-5}$

5 a ${}^{8}C_4 \times 3^4 = 5670$

b $-{}^{12}C_9 \times 2^3 = -1760$

c ${}^{10}C_8 \times 5^2 \times 2^8 = 288\,000$

d ${}^{6}C_4 \times a^2 \times \left(\frac{1}{2}\right)^4 = \frac{15}{16}a^2$

6 a -672 **b** $-112\,266$ **c** $\frac{969}{2}$

d 21 875 **e** $\frac{40}{49}$

7 a 3640 **b** -385 **c** 10 920 **d** -1241

8 a 6 **b** 45 **c** 84

9 a $a = -24, b = 158$ **b** $n = 13, 286$

10 ${}^{3n}C_n\,(= {}^{3n}C_{2n})$

11 a ${}^{12}C_r\,(-1)^r\, a^{12-r}\, b^r x^r$ **b** $\frac{5}{8}$

12 a $1 - 4x + 10x^2 - 16x^3 + 19x^4 - \cdots$

b i $\dfrac{9 - 9n}{2}$

ii $\dfrac{-9n(n-1)(2n-1)}{2}$

Chapter 15 review exercise

1 The answer is in the theory of 15A.

2 a $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

b $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

c $1 - 9x + 27x^2 - 27x^3$

d $1 - 4xy + 6x^2y^2 - 4x^3y^3 + x^4y^4$

3 a $1 + 35x + 490x^2 + \cdots$

b $(1 - 5x)(1 + 35x + 490x^2 + \cdots)$
$= (490 - 175)x^2 + \cdots.$
The coefficient of x^2 is 315.

4 a $1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

b $(1 + 0.02)^7 = 1 + 7 \times 0.2 + 21 \times 0.0004 + \cdots$
$= 1 + 0.14 +$ much smaller terms
The first decimal place will be 1.

5 a $81 + 216x + 216x^2 + 96x^3 + 16x^4$

b $125 - 75x + 15x^2 - x^3$

c $32x^5 + 320x^4y + 1280x^3y^2 + 2560x^2y^3 + 2560xy^4 + 1024y^5$

d $x^4 - 4x^2 + 6 - \dfrac{4}{x^2} + \dfrac{1}{x^4}$

6 a 10 **b** 35 **c** 56 **d** 2

7 a 45 **b** 792 **c** 84

d 40 **e** $\frac{84}{5}$ **f** $\frac{286}{9}$

8 a 1 (first entry in a row)

b 0 (reversibility of rows)

c $2^5 = 32$ (sum of a row)

d 0 (addition formula)

9 $n = 8$

10 $(1 - 3x)(1 + 5x)^{14}$
$= (1 - 3x)(\cdots + {}^{14}C_7\,(5x)^7 + {}^{14}C_8\,(5x)^8 + \cdots)$
The coefficient of x^8 will be
${}^{14}C_8 \times 5^8 - 3 \times {}^{14}C_7 \times 5^7 = 368\,671\,875.$

11 a $(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots.$

b Substitute $x = 2$:
$(1 + 2)^n = \binom{n}{0} + \binom{n}{1} \times 2 + \binom{n}{2} \times 4 + \binom{n}{3} \times 8 + \cdots.$

c When $n = 4$,
$$\begin{aligned}\text{LHS} &= 1 + 4 \times 2 + 6 \times 4 + 4 \times 8 + 1 \times 16\\ &= 1 + 8 + 24 + 32 + 16\\ &= 81\\ &= 3^4.\end{aligned}$$

12 b This important result is proven in the theory of Exercise 15D. It tells us that the sum of the row in Pascal's triangle indexed by n, that is, containing the coefficients of $(1 + x)^n$, is 2^n.

c The list of all subsets of a set of n objects may be partitioned into sets of sizes $0, 1, 2, \ldots, n$. The sum of the number of sets of each type will be the total number of all subsets, which is 2^n.

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Chapter 16

Exercise 16A

1 $\frac{dy}{dt} = (3x^2 + 1)\frac{dx}{dt}$

a 65 **b** $-\frac{3}{14}$

2 a $240\pi \text{ cm}^2/\text{s}$

b $\frac{1}{12\pi}$ cm/s

3 a $\frac{2}{\pi}$ cm/s **b** $\frac{5}{\pi}$ cm **c** $\frac{50}{9\pi}$ cm/s

4 a $840 \text{ cm}^2/\text{s}$, $6\sqrt{2}$ cm/s **b** $1200 \text{ cm}^2/\text{s}$, $6\sqrt{2}$ cm/s

5 b i $1350 \text{ cm}^3/\text{min}$, $180 \text{ cm}^2/\text{min}$, 6 cm/min

ii $600 \text{ cm}^3/\text{min}$, $120 \text{ cm}^2/\text{min}$, 6 cm/min

c $10\sqrt{2}$ cm

6 a $90\,000\pi \text{ mm}^3/\text{min}$

b The rate is constant at 6π mm/min.

7 a $\frac{dA}{dt} = \frac{1}{2}s\sqrt{3}\,\frac{ds}{dt}$, $\frac{dh}{dt} = \frac{1}{2}\sqrt{3}\frac{ds}{dt}$

b $\frac{9}{5}\sqrt{3} \text{ cm}^2/\text{s}$, $\frac{3}{20}\sqrt{3}$ cm/s

8 a i $\frac{1}{24\pi}$ cm/s **ii** $1\frac{1}{2} \text{ cm}^3/\text{s}$

b $\frac{32\,000\pi}{3} \text{ cm}^3$

9 $\frac{dV}{dt} = 3\pi h^2\frac{dh}{dt}$

a i $\frac{1}{160\pi}$ m/min **ii** $\frac{3}{160\pi}$ m/min

b $\frac{dA}{dt} = 6\pi h\frac{dh}{dt}$, $\frac{3}{20} \text{ m}^2/\text{min}$, $\frac{3}{20}\sqrt{3} \text{ m}^2/\text{min}$

c $\frac{96\pi}{25} \text{ m}^3/\text{min}$

10 a $V = \frac{4\pi}{3}h^3$ **b** $\frac{1}{32\pi}$ m/s

11 c When it is a square, $\ell = x\sqrt{2}$.

d Put $\frac{d\ell}{dt} = \frac{1}{2}\frac{dx}{dt}$, then $\ell = 2x$, so $x = \frac{4}{\sqrt{3}}$ m and $\ell = \frac{8}{\sqrt{3}}$ m.

e It follows from part **a** because ℓ is always greater than x. Alternatively, $\frac{d\ell}{dt}\Big/\frac{dx}{dt} = \frac{x}{\sqrt{16 + x^2}}$ is always less than 1 because the denominator is greater than the numerator.

f $\frac{d\ell}{dt}\Big/\frac{dx}{dt} = \frac{x}{\sqrt{16 + x^2}} \to 1$ as $x \to \infty$.

12 $\frac{1}{125\pi}$ cm/s, $\frac{4}{5} \text{ cm}^2/\text{s}$, $\frac{4}{5}\sqrt{2} \text{ cm}^2/\text{s}$

13 b $0.096 \text{ m}^3/\text{s}$, $\frac{3}{125}\left(\sqrt{17} + 1\right) \text{m}^2/\text{s}$

14 $\frac{dy}{dt} = \frac{-x}{\sqrt{169 - x^2}}\frac{dx}{dt}$

a $\frac{5}{12}$ cm/s **b** 12 mm/s

15 b $\frac{1}{6}$ cm/s

16 b $\frac{1}{24}$ cm/min, $83\frac{1}{3} \text{ cm}^2/\text{min}$

c 8 litre/min, $200 \text{ cm}^2/\text{min}$

17 a When $r = 10$ cm, $\frac{dr}{dt} = \frac{1}{10\pi}$ cm/s.

b i $r = \left(\frac{30t}{\pi}\right)^{\frac{1}{3}}$ and $\frac{dr}{dt} = \frac{1}{3} \times \left(\frac{30}{\pi}\right)^{\frac{1}{3}}t^{-\frac{2}{3}}$

ii When $r = 10$, $t = \frac{100\pi}{3}$ s and $\frac{dr}{dt} = \frac{1}{10\pi}$ cm/s.

18 $4\sqrt{6} \text{ cm}^2/\text{s}$

19 0.246 cm/s. Some exact forms are $\frac{3}{(27 + 9\sqrt{3})^{\frac{2}{3}}}$ and $\frac{1}{(3 + 3\sqrt{3})^{\frac{2}{3}}}$.

Exercise 16B

1 a $\frac{dQ}{dt} = 1200e^{3t} = 3Q$. When $t = 0$, $Q = 400e^0 = 400$.

b i $Q = 400e^6 \doteqdot 1.614 \times 10^5$, $\frac{dQ}{dt} = 1200e^6 \doteqdot 4.841 \times 10^5$

ii $t = \frac{1}{3}\log_e 50 \doteqdot 1.304$ **iii** $t = \frac{1}{3}\log_e \frac{50}{3} \doteqdot 0.9378$

2 a $\frac{dQ}{dt} = -2000e^{-2t} = -2Q$. When $t = 0$, $Q = 1000e^0 = 1000$.

b i $Q = 1000e^{-0.8} \doteqdot 449$, $\frac{dQ}{dt} = -2000e^{-0.8} \doteqdot -899$

ii $t = \frac{1}{2}\log_e 25 \doteqdot 1.609$ **iii** $t = \frac{1}{2}\log_e 50 \doteqdot 1.956$

3 b 1350 **c** 135 per hour **d** 23 hours

4 c 6.30 grams, 1.46 grams per minute

d 6 minutes 58 seconds

e 20 g, $20e^{-k} \doteqdot 15.87$ g, $20e^{-2k} \doteqdot 12.60$ g, $20e^{-3k} = 10$ g, the common ratio is $e^{-k} = 2^{-\frac{1}{3}} \doteqdot 0.7937$

5 b $-\frac{1}{5}\log_e \frac{7}{10}$

c 10 290

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d At $t \doteqdot 8.8$, that is, some time in the fourth year from now.

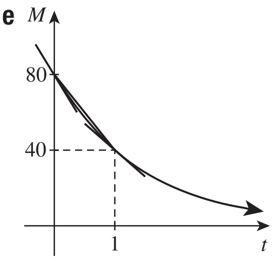
6 a 80 g, 40 g, 20 g, 10 g

b 40 g, 20 g, 10 g. During each hour, the average mass loss is 50%.

c $M_0 = 80$, $k = \log_e 2 \doteqdot 0.693$

d 55.45 g/h, 27.73 g/h, 13.86 g/h, 6.93 g/h

e



7 b 30

c i 26 **ii** $\frac{1}{5}\log_e \frac{15}{13}\left(\text{or} -\frac{1}{5}\log_e \frac{13}{15}\right)$

8 b $L \approx \frac{1}{2}$

9 c 25

d $\frac{k}{A} = \frac{1}{2}\log_e \frac{5}{3}\left(\text{or } -\frac{1}{2}\log_e \frac{3}{5}\right)$

e 6 hours 18 minutes

10 b $C_0 = 20000$, $k = \frac{1}{5}\log_e \frac{9}{8} \doteqdot 0.024$

c 64 946 ppm

d i 330 metres from the cylinder

ii If it had been rounded down, then the concentration would be above the safe level.

11 a ii $k = \frac{1}{12}\log_e \frac{122}{105}$ **b ii** $\ell = \frac{1}{12}\log_e \frac{217}{100}$

c At $t = \dfrac{\log_e \frac{525}{100}}{\ell - k} \doteqdot 31.85$, that is, in the 32nd month.

d $\ell C = \ell \times 100 \times e^{32\ell} \doteqdot 51$ cents per month

12 a $y(3) = A_0 e^{3k} = A_0\left(e^k\right)^3$ and we know that $e^k = \dfrac{3}{4}$.

b $y(3) = \frac{27}{64}A_0$

13 a $B = \dfrac{2N_0^2}{N_c}$ and $C = \left(\dfrac{N_0}{N_c}\right)^2$

b $\dfrac{B}{C} = 2N_c$

Exercise 16C

1 a ii 12 000, $P \to \infty$ as $t \to \infty$

b ii 12 000, $P \to 10000$ as $t \to \infty$

c ii 8000, $P \to 10000$ as $t \to \infty$

2 b $A = 1000$, $k = \frac{1}{3}\log_e 6$

c 67 420 bugs **d** 10.4 weeks

3 b $B = 970000$, $k = -\frac{1}{10}\log_e \frac{47}{97} = \frac{1}{10}\log_e \frac{97}{47}$

c 158 000 flies **d** 73 days

4 b $T_e = 20$, $A = 70$

c $k = \frac{1}{6}\log_e \frac{7}{3}$ $\left[\text{Alternatively, } k = -\frac{1}{6}\log_e \frac{3}{7}.\right]$

d 13 minutes 47 seconds

5 a $A = 34$ **b** $\frac{1}{45}\log_e 2\left(\text{or } -\frac{1}{45}\log_e \frac{1}{2}\right)$ **c** 16.5°C

6 a $1 - e^{-\frac{1}{16}t}$ is always positive for $t > 0$. The body is falling.

b It is the acceleration of the body.

c -160 m/s **d** $16\log_e \frac{8}{7} \doteqdot 2.14$ s

7 a As $t \to \infty$, $P \to B$. As $t \to -\infty$, $P \to \infty$.

b As $t \to \infty$, $P \to \infty$. As $t \to -\infty$, $P \to B$.

8 a The average level is 15 cm, so $x - 15$ is the difference between the left level and the average level. We write $x - 15$ so that k is positive.

b Substituting $t = 0$ shows that $A = 15$.

c 15 cm **d** $k = \frac{1}{5}\log_e \frac{5}{3}$

9 b $-\dfrac{V}{R}$ **c** $I \to \dfrac{V}{R}$ **d** 4.62×10^{-4} s

10 b $M \to a$ as $t \to \infty$ **c** $k = \dfrac{1}{120}\log_e 100$

d 2 minutes 45 seconds

11 a $2w$ g/min **b** $\dfrac{Q}{1000}$ g/L **c** $\dfrac{Qw}{1000}$ g/min

f -2000 **g** $Q \to 2000$

h $w = \frac{1000}{345}\log_e 2 \doteqdot 2$ L/min

13 b $A = 1000$, $I = 9000$ and $k = \dfrac{1}{7}\log_e 3$

c 36 000

14 Adam's coffee is cooler.

Chapter 16 review exercise

1 a 600 mm^2/h.

b $3\sqrt{2}$ mm/h (The rate is constant.)

2 a $h = r\sqrt{3}$ **b** $\dfrac{V}{t} = \pi r^2\sqrt{3}\dfrac{r}{t}$

c $\ell = 2r$ **d** $\dfrac{A}{t} = 4\pi r\dfrac{r}{t}$ **f** $\frac{5}{\sqrt{3}}\text{m}^2$/s

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3 b $k = \frac{1}{5}\log_e \frac{13}{8}$, $80000e^{18k} \doteqdot 459000$

c $t = \frac{1}{k}\log_e 12.5$, year 2036

4 b $k = \frac{\log_e 2}{30.2}$ **c** 10.07% **d** about 201 years

5 a The temperature is dropping, but $T - E$ is positive.

c i $A = 500$, $k = -\frac{1}{6}\log_e 2$, $T = 500e^{15k} \doteqdot 88°C$

ii $A = 460$, $k = -\frac{1}{6}\log_e \frac{46}{21}$,
$T = 40 + 460e^{15k} \doteqdot 105°C$

6 a The population is growing, and $P - M$ is negative.

b As $t \to \infty$, $P \to M - 0 = M$.

c i $A = 9500$, $k = \frac{1}{10}\log_e \frac{19}{16}$

ii $p = 10000 - 9500e^{-20k} \doteqdot 3260$

iii $t = \frac{1}{k}\log_e \frac{19}{4} \doteqdot 91$ years, year 2100.

Chapter 17

Exercise 17A

1 a $3 \le y \le 5$

b domain: $3 \le x \le 5$, range: $0 \le y \le 2$

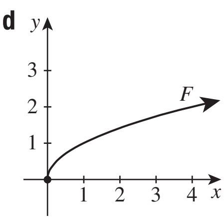
c $f^{-1}(x) = x - 3$

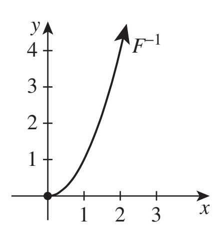
2 a $0 \le y \le 2$

b domain: $0 \le x \le 2$, range: $0 \le y \le 4$

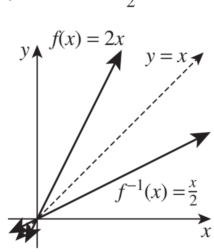
c $F^{-1}(x) = x^2$

d

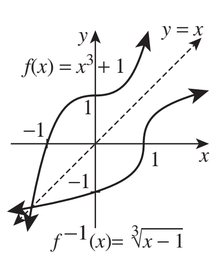




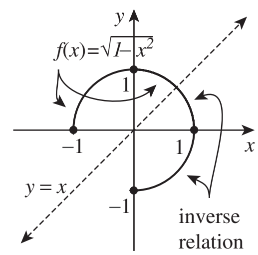
3 a The inverse is a function.
$f^{-1}(x) = \frac{1}{2}x$



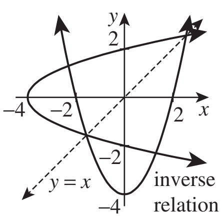
b The inverse is a function.
$f^{-1}(x) = \sqrt[3]{x - 1}$



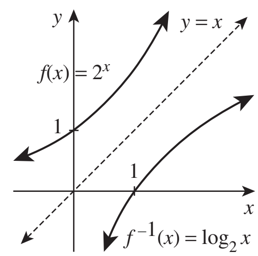
c The inverse is not a function.



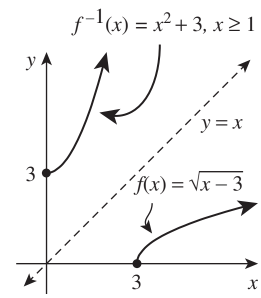
d The inverse is not a function.
$f(x) = x^2 - 4$



e The inverse is a function.
$f^{-1}(x) = \log_2 x$



f The inverse is a function.
$f^{-1}(x) = x^2 + 3, x \ge 0$



4 a Both x.

b They are inverse functions.

5 a $g^{-1}(x) = \sqrt{x}$, domain: $x \ge 0$, range: $y \ge 0$

b $g^{-1}(x) = -\sqrt{x - 2}$, domain: $x \ge 2$, range: $y \le 0$

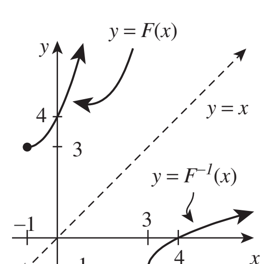
c $g^{-1}(x) = \sqrt{4 - x^2}$, $-2 \le x \le 0$,
domain: $-2 \le x \le 0$, range: $0 \le y \le 2$

6 a $3x^2$ **b** $\frac{1}{3}(y + 1)^{-\frac{2}{3}}$

7 $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, $\frac{dx}{dy} = 2y$

8 b They are both one-to-one.

c $F^{-1}(x) = -1 + \sqrt{x - 3}$,
domain: $x \ge 3$,
range: $y \ge -1$

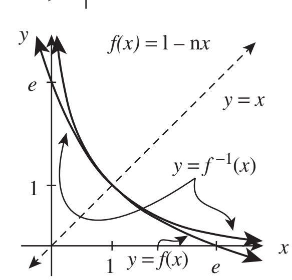


9 a $x = e$

b Reflect $y = \ln x$ in the x-axis, then shift it one unit up.

d $f^{-1}(x) = e^{1-x}$, domain: all real x, range: $y > 0$

e Both are decreasing.



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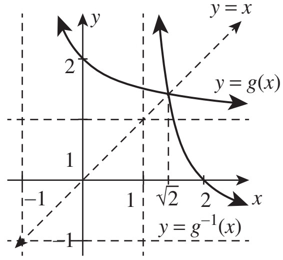
10 b One-to-one.

c $g^{-1}(x) = \dfrac{2 - x}{x - 1}$, for $x > 1$

d $x = \sqrt{2}$. It works because the graphs meet on the line of symmetry $y = x$.



11 a $y = \sqrt[3]{-x}$

b $(-1, 1)$, $(0, 0)$ and $(1, -1)$

12 a No. The graph of the inverse is a vertical line, which is not a function.

13 a Shift two units left and four units down.

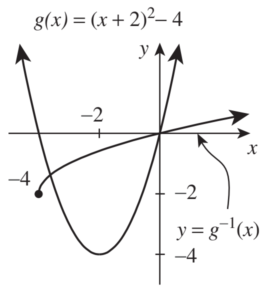
b x-intercepts: -4, 0, vertex: $(-2, -4)$.

c Many-to-one.

d $x \geq -2$

e $x \geq -4$, increasing

f $g^{-1}(x) = -2 + \sqrt{x + 4}$

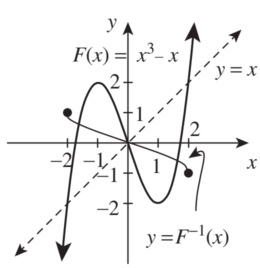


14 a x-intercepts: 0, $\sqrt{3}$, $-\sqrt{3}$, stationary points: $(-1, 2)$, $(1, -2)$

b Many-to-one.

c $-1 \leq x \leq 1$

d $-2 \leq x \leq 2$



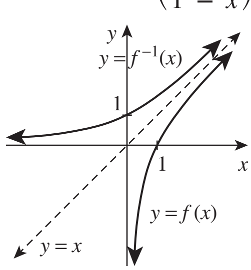
15 a all real x

c $f'(x) > 0$ for all x.

d Because $f(x)$ is always increasing, the graph of $f(x)$ passes the horizontal line test.

$f^{-1}(x) = \ln\left(\dfrac{x}{1 - x}\right)$

16



17 a Suppose that (a, b) lies on the graph of the inverse relation. Then (b, a) lies on the graph of the relation. Because the relation is odd, $(-b, -a)$ lies on the graph of the relation. Hence $(-a, -b)$ lies on the graph of the inverse relation.

b Let $y = f^{-1}(-x)$. Then $-x = f(y)$, so $x = -f(y) = f(-y)$ because f is odd. Hence $-y = f^{-1}(x)$, $y = -f^{-1}(x)$. This proves that $f^{-1}(-x) = -f^{-1}(x)$.

c Functions whose domain is $x = 0$ alone, because if $f(a) = b$, then $f(-a) = b$, so the graph fails the horizontal line test unless $a = -a$, that is, unless $a = 0$ (the empty function is also even).

18 b From part **a** we see, for example, that $g\left(\frac{1}{2}\right) = g(2)$, so the inverse is not a function.

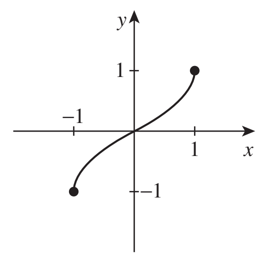
c i $-1 \leq x \leq 1$

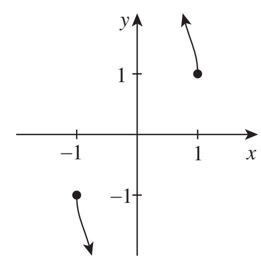
iii $g^{-1}(x) = \dfrac{1 - \sqrt{1 - x^2}}{x}$

d domain: $x \leq -1$ or $x \geq 1$,

$g^{-1}(x) = \dfrac{1 + \sqrt{1 - x^2}}{x}$

e Because of the result in part **a**.





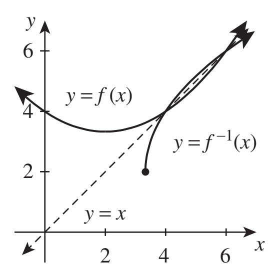
19 a vertex: $\left(2, \frac{10}{3}\right)$, y-intercept: 4

b $x \geq 2$

c $x \geq \frac{10}{3}$

d The easy way is to solve $y = f(x)$ simultaneously with $y = x$. They intersect at $(4, 4)$ and $(6, 6)$.

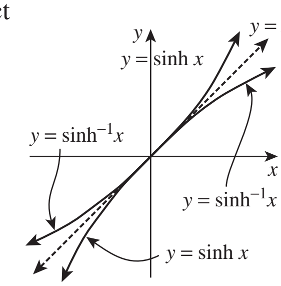
e $4 - N$



20 a all real x

b 0

d $\frac{1}{2}(e^x + e^{-x})$, which is positive for all real x.



e To $y = \frac{1}{2}e^x$ on the right. To $y = -\frac{1}{2}e^{-x}$ on the left.

f sinh$x$ is a one-to-one function.

Exercise 17B

1 a 1.16	**b** 0.64	**c** 1.32
d 1.67	**e** 1.98	**f** 2.42

2 a 0	**b** $\frac{\pi}{6}$	**c** 0	**d** $\frac{\pi}{4}$
e $-\frac{\pi}{2}$	**f** $\frac{\pi}{2}$	**g** 0	**h** $-\frac{\pi}{4}$
i $-\frac{\pi}{3}$	**j** $\frac{3\pi}{4}$	**k** $-\frac{\pi}{6}$	**l** π

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3 a 1.447 **b** 1.694 **c** 0.730
d −0.730 **e** 1.373 **f** −1.373

4 a $\frac{\pi}{2}$ **b** 1 **c** 1
d $\frac{\pi}{6}$ **e** $\frac{1}{2}$ **f** $\frac{3\pi}{4}$
g $-\frac{\pi}{6}$ **h** 0 **i** $\frac{\pi}{3}$

5 a $-\frac{\pi}{3}$ **b** $\frac{\pi}{4}$ **c** $-\frac{\pi}{6}$
d $\frac{3\pi}{4}$ **e** $-\frac{\pi}{2}$ **f** $\frac{\pi}{3}$

7 a i $\frac{4}{5}$ **ii** $\frac{5}{12}$ **iii** $\frac{1}{3}\sqrt{5}$
iv $\frac{8}{17}$ **v** $\frac{3}{10}\sqrt{10}$ **vi** $-\frac{1}{3}\sqrt{7}$

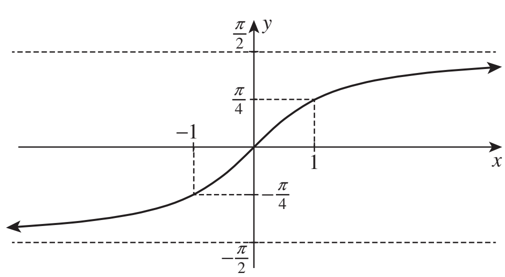
12 c $-\frac{\pi}{2}$

13 a $0 \le y < \frac{\pi}{2}$ **b** $0 < y \le \frac{\pi}{4}$

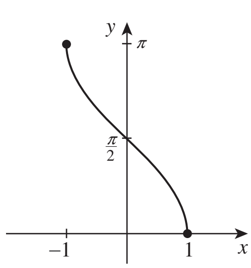
14 a 2 is within the range of the inverse cosine function, which is $0 \le y \le \pi$. However, 2 is outside the range of the inverse sine function, which is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.
b It is because the sine curve is symmetrical about $x = \frac{\pi}{2}$.
c $\pi - 2$

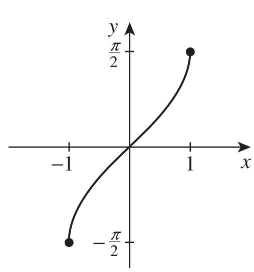
Exercise 17C

1 a domain: all real x, range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, odd

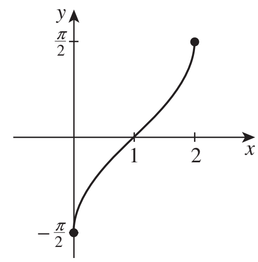


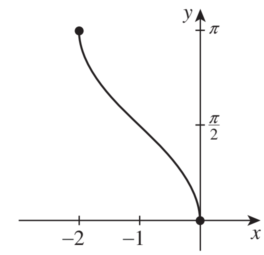
b domain: $-1 \le x \le 1$, range: $0 \le y \le \pi$, neither
c domain: $-1 \le x \le 1$, range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, odd



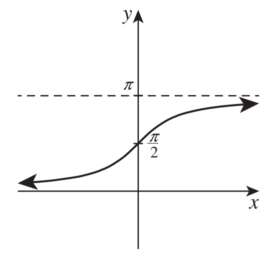


2 a domain: $0 \le x \le 2$, range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, neither
b domain: $-2 \le x \le 0$, range: $0 \le y \le \pi$, neither

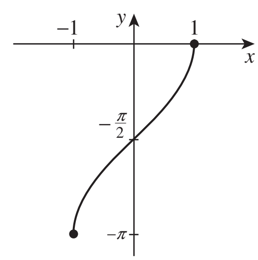


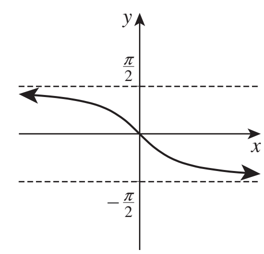


c domain: all real x, range: $0 < y < \pi$, neither

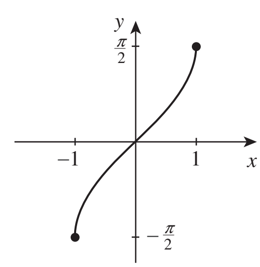


3 a domain: $-1 \le x \le 1$, range: $-\pi \le y \le 0$, neither
b domain: all real x, range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, odd

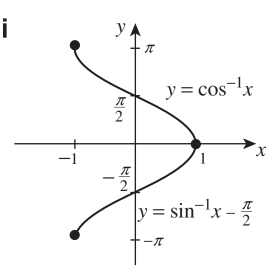




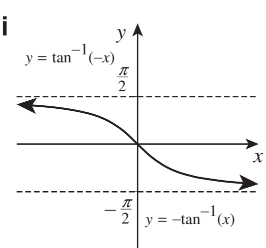
c domain: $-1 \le x \le 1$, range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, odd



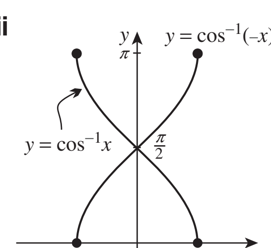
4 a i



b i



ii



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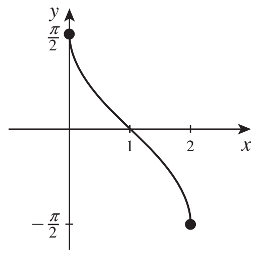
5 a $0 \le x \le 2$

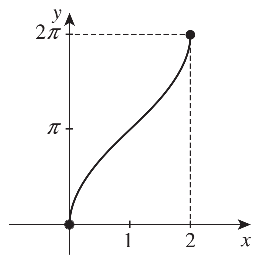
b $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

c $y = \frac{\pi}{2}, 0, -\frac{\pi}{2}$

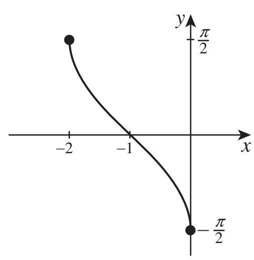
d $x = \frac{1}{2}$



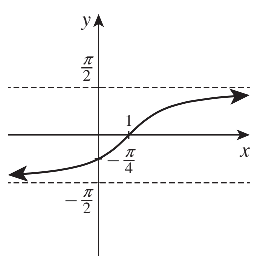
6 a domain: $0 \le x \le 2$, range: $0 \le y \le \pi$



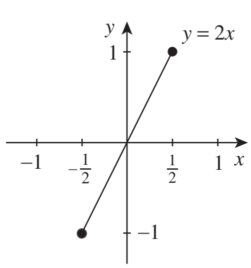
b domain: $-2 \le x \le 0$, range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



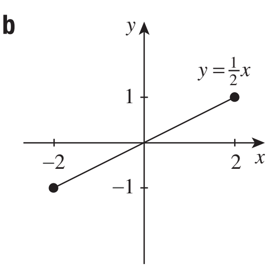
c domain: all real x, range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



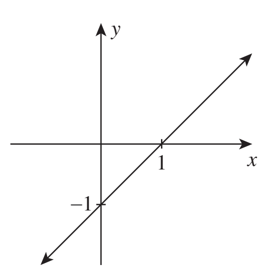
7 a



b



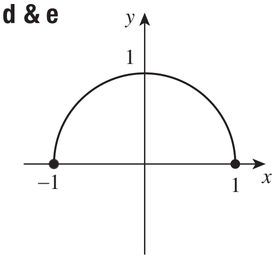
c The first two are odd, and the third is neither even nor odd.



8 a $-1 \le x \le 1$, even

b $0 \le \cos^{-1}x \le \pi$, so $\sin(\cos^{-1}x) \ge 0$.

d & e

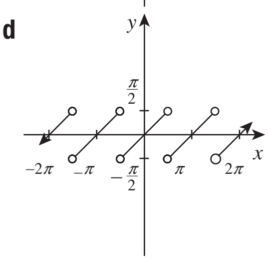


9 a domain: all real x, $x \ne \dfrac{(2n+1)\pi}{2}$, where n is an integer, range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$, odd

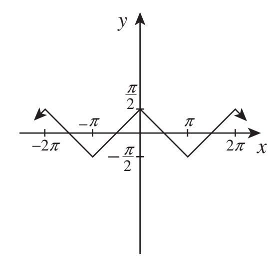
b x

c π

d



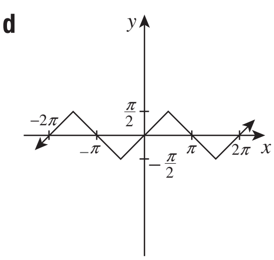
10 $\sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x)$, so we reflect in the x-axis and then shift $\frac{\pi}{2}$ units up. It is even.



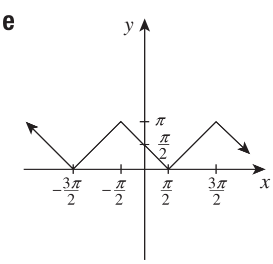
11 a domain: all real x, range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, period: 2π, odd

b x

d



e



For part e, $\cos^{-1}(\sin x) = \frac{\pi}{2} - \sin^{-1}(\sin x)$, so we reflect in the x-axis and then shift $\frac{\pi}{2}$ units up.

Exercise 17D

1 a $\sin x \cos y - \cos x \sin y$

b $\cos 2A \cos 3B - \sin 2A \sin 3B$

c $\sin 3\alpha \cos 5\beta + \cos 3\alpha \sin 5\beta$

d $\cos\theta \cos\dfrac{\phi}{2} + \sin\theta \sin\dfrac{\phi}{2}$

e $\dfrac{\tan A + \tan 2B}{1 - \tan A \tan 2B}$

f $\dfrac{\tan 3\alpha - \tan 4\beta}{1 + \tan 3\alpha \tan 4\beta}$

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2 a $\cos(x + y)$ **b** $\sin(3\alpha + 2\beta)$ **c** $\tan 20°$
d $\sin 3A$ **e** $\cos 50°$ **f** $\tan(\alpha + 10°)$

5 a 1 **b** $\frac{63}{65}$ **c** $\frac{\sqrt{5}(1 + 2\sqrt{3})}{12}$

7 b i $\frac{\sqrt{3} + 1}{2\sqrt{2}}$ **ii** $\frac{\sqrt{3} - 1}{2\sqrt{2}}$ **iii** $2 + \sqrt{3}$

9 a $\frac{1 - \sqrt{3}}{2\sqrt{2}}$ **b** $\frac{1 - \sqrt{3}}{2\sqrt{2}}$ **c** $\sqrt{3} - 2$

10 a 1 **b** $\frac{56}{65}$
c $\frac{56}{33}$ **d** $\frac{10\sqrt{3} - \sqrt{5}}{20}$

11 b $x = \frac{1}{2}$ (note that $x \neq -1$)

12 a $x = \frac{1}{3}$ **b** $x = \frac{1}{3}$ or 1

13 b $\frac{\sqrt{3}}{2}$

16 a $\sin\alpha\cos\beta + \cos\alpha\sin\beta$

18 $x = -\frac{3}{2}$ or $\frac{1}{3}$

Exercise 17E

1 a $\sin 2x$ **b** $\cos 2\theta$ **c** $\tan 2\alpha$
d $\sin 40°$ **e** $\cos 100°$ **f** $\tan 140°$
g $\sin 6\theta$ **h** $\cos 4A$ **i** $\tan 8x$

3 a $\frac{7}{25}$ **b** $\frac{1}{9}$ **c** $\frac{120}{169}$ **d** $\frac{4}{3}$

4 $-\frac{3\sqrt{7}}{8}$

8 a $-\frac{7}{9}$ **b** $\frac{12\sqrt{13}}{49}$ **c** $\frac{4}{3}$

9 b $\sqrt{2} - 1$

10 a $y = 2x^2 - 8x + 7$ **b** $y = \dfrac{2x - 2}{2x - x^2}$

11 a $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

b $\tan\theta = \dfrac{h}{30}$, $\tan 2\theta = \dfrac{h}{10}$

e $\angle AWB = \theta$ using the exterior angle $\angle WBC$ of ΔABW. Hence ΔABW is isosceles with $BW = BA = 20$ m, because the base angles are equal. Now use Pythagoras' theorem in ΔBCW.

16 a $\frac{1}{2}\sqrt{2}$

Exercise 17F

1 a $\dfrac{2t}{1 + t^2}$ **b** $\dfrac{1 - t^2}{1 + t^2}$ **c** $\dfrac{2t}{1 - t^2}$

d $\dfrac{1 + t^2}{1 - t^2}$ **e** $\dfrac{2t^2}{1 + t^2}$ **f** t

2 a $\dfrac{1 - t^2}{1 + t^2}$ **b** $\dfrac{(1 - t)^2}{1 + t^2}$ **c** $\dfrac{1 + t}{1 - t}$

3 a $\tan 20°$ **b** $\sin 20°$ **c** $\cos 20°$
d $\sin 4x$ **e** $\tan 4x$ **f** $\cos 4x$

4 a $\tan 30° = \frac{1}{\sqrt{3}}$ **b** $\sin 30° = \frac{1}{2}$

c $\cos 150° = -\frac{\sqrt{3}}{2}$ **d** $\sin 225° = -\frac{1}{\sqrt{2}}$

e $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ **f** $\tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$

6 c $x = 0, \frac{3\pi}{2}, 2\pi$

7 b $A = 0, \frac{2\pi}{3}, 2\pi$

8 a ii $-\sqrt{2} - 1 = \tan 112\frac{1}{2}°$,

because $\tan 225° = \tan 45° = 1$.

10 a $-\frac{3}{4}$ **b** $-\frac{3}{5}$ **c** $\frac{4}{5}$ **d** $3 + \sqrt{10}$

11 a i $\cos\theta = 2\cos^2\frac{1}{2}\theta - 1$

b ii $\sin\theta = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$

Exercise 17G

1 b i $\cos 50° + \cos 20°$ **ii** $\sin 80° - \sin 16°$
iii $\frac{1}{2}(\cos 27° - \cos 71°)$ **iv** $\frac{1}{2}(\sin 86° - \sin 36°)$
v $\sin 4\alpha + \sin 2\alpha$ **vi** $\frac{1}{2}(\cos 3\theta + \cos\theta)$
vii $\cos 2y - \cos 2x$ **viii** $\frac{1}{2}(\sin 4A + \sin 2B)$

7 b ii $\frac{1}{2}$

8 $\sin 3x = \frac{1}{2}$, $x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

12 a $\cos 3x = 0$ or $\sin x = 0$, $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 0, \pi, 2\pi$

b $\sin 5x = 0$ or $\cos 2x = 0$, $x = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

c $\cos 4x = 0$ or $\cos x = 0$, $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}, \frac{\pi}{2}, \frac{3\pi}{2}$

d $\sin\dfrac{5x}{2} = 0$ or $\sin\dfrac{x}{2} = 0$, $x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, 2\pi$

Chapter 17 review exercise

1 a $-1 \le f(x) \le 4$
b $f^{-1}(x) = 2x - 2$
c $-1 \le x \le 4$, $-4 \le f^{-1}(x) \le 6$
d $y = x$

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2 a $x > -1$, all real y

b $F^{-1}(x) = e^x - 1$

c All real x, $y > -1$

e Both functions are increasing.

3 b $x \le 2$

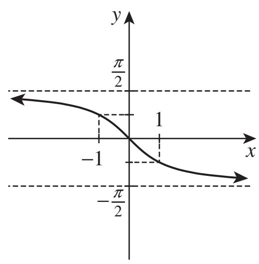
c $Q^{-1}(x) = 2 - \sqrt{x}$, because $Q^{-1}(x) \le 2$.

4 a $\frac{\pi}{2}$ **b** $\frac{\pi}{6}$ **c** $\frac{\pi}{3}$ **d** $-\frac{\pi}{4}$ **e** $\frac{2\pi}{3}$ **f** $-\frac{\pi}{6}$

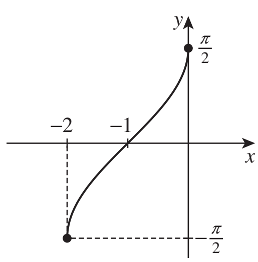
5 a 1 **b** $\frac{1}{\sqrt{2}}$ **c** $\frac{1}{2}$ **d** $-\frac{\pi}{6}$ **e** $\frac{2\pi}{3}$ **f** $-\frac{\pi}{3}$

6 a $\frac{2\sqrt{2}}{3}$ **b** $\frac{3}{4}$

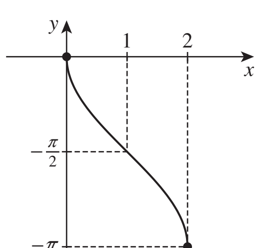
7 a All real x,
$-\frac{\pi}{2} < y < \frac{\pi}{2}$



b $-2 \le x \le 0$,
$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



c $0 \le x \le 2$,
$-\pi \le y \le 0$



8 a $\cos 2\theta$ **b** $\sin 40°$ **c** $\tan 50°$

d $\cos 70°$ **e** $\sin 6\alpha$ **f** $\frac{1}{\tan\theta} = \cot\theta$

9 a $\sin 4\theta$ **b** $\cos x$ **c** $\cos 6\alpha$

d $\tan 70°$ **e** $\cos 50°$ **f** $\tan 8x$

10 a $\frac{4}{5}$ **b** $\frac{7}{25}$ **c** $-\frac{16}{65}$ **d** $\frac{120}{169}$ **e** $\frac{24}{7}$ **f** $\frac{33}{56}$

12 a $\sin 30° = \frac{1}{2}$ **b** $\cos 30° = \frac{\sqrt{3}}{2}$

c $\tan 135° = -1$ **d** $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

e $\frac{1}{2}\sin\frac{\pi}{6} = \frac{1}{4}$ **f** $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

15 b $\frac{\pi}{4}$

18 a $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$ or $\frac{11\pi}{6}$

b $x = \frac{\pi}{6}, \frac{5\pi}{6}$ or $\frac{3\pi}{2}$

c $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

d $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$ or 2π

19 a $\frac{2t}{1 + t^2}$ **b** $\frac{1 - t^2}{1 + t^2}$ **c** $\frac{2t}{1 - t^2}$

d $\frac{1 + t^2}{1 - t^2}$ **e** $\frac{2}{1 + t^2}$ **f** t

20 a $\tan 135° = -1$

b $\sin 135° = \frac{1}{\sqrt{2}}$

c $\cos 135° = -\frac{1}{\sqrt{2}}$

22 b ii $-2 + \sqrt{3} = \tan 165°$, because
$\tan 330° = -\tan 30° = -\frac{1}{\sqrt{3}}$.

23 a $\cos 3A + \cos A$

b $\sin 8A - \sin 2A$

c $\sin(5A + B) + \sin(A + 3B)$

d $\cos(A + 2B) - \cos(3A + 8B)$

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Appendix A: Bracket interval notation

Section 2B discussed intervals, using inequalities such as $\frac{1}{3} \le x \le 3$ to describe them. This extra section for Chapter 3 introduces an alternative notation for intervals that can often make the notation a little more concise. The notation encloses the endpoints of the interval in brackets, using a square bracket if the endpoint is included and a round bracket if the endpoint is not included.

Here are the five examples from Section 2B written in both notations:

Diagram	Using inequalities	Using brackets
	$\frac{1}{3} \le x \le 3$	$\left[\frac{1}{3}, 3\right]$
Read this as, 'The closed interval from $\frac{1}{3}$ to 3'.		
	$-1 < x < 5$	$(-1, 5)$
Read this as, 'The open interval from -1 to 5'.		
	$-2 \le x < 3$	$[-2, 3)$
Read this as, 'The interval from -2 to 3, including -2 but excluding 3'.		
	$x \ge -5$	$[-5, \infty)$
Read this as, 'The closed ray from -5 to the right'.		
	$x < 2$	$(-\infty, 2)$
Read this as, 'The open ray from 2 to the left'.		

The first interval $\left[\frac{1}{3}, 3\right]$ is *closed*, meaning that it contains all its endpoints.

The second interval $(-1, 5)$ is *open*, meaning that it does not contain any of its endpoints.

The third interval $[-2, 3)$ is neither open nor closed — it contains one of its endpoints, but does not contain the other endpoint.

The fourth interval $[-5, \infty)$ is *unbounded on the right*, meaning that it continues towards infinity. It only has one endpoint -5, which it contains, so it is closed.

The fifth interval $(-\infty, 2)$ is *unbounded on the left*, meaning that it continues towards negative infinity. It only has one endpoint 2, which it does not contain, so it is open.

'Infinity' and 'negative infinity', with their symbols ∞ and $-\infty$, are not numbers. They are ideas used in specific situations and phrases to make language and notation more concise. Here, they indicate that an interval is unbounded on the left or right, and the symbol $(-\infty, 2)$ means 'all real numbers less than 2'.

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Bracket interval notation has some details that need attention.

- The variable x or y or whatever is missing. This can be confusing when we are talking about domain and range, or solving an inequation for some variable. When, however, we are just thinking about 'all real numbers greater than 100', no variable is involved, so the notation $(100, \infty)$ is more satisfactory than $x > 100$.
- The notation can be dangerously ambiguous. For example, the open interval $(-1, 5)$ can easily be confused with the point $(-1, 5)$ in the coordinate plane.
- Infinity and negative infinity are not numbers, as remarked above.
- The set **R** of all real numbers can be written as $(-\infty, \infty)$.
- The notation $[4, 4]$ is the one-member set $\{4\}$, called a *degenerate interval* because it has length zero.
- Notations such as $(4, 4)$, $(4, 4]$, $[7, 3]$ and $[7, 3)$ all suggest the empty set, if they mean anything at all, and should be avoided in this course.

1 BRACKET INTERVAL NOTATION

- A square bracket means that the endpoint is included, and a round bracket means that the endpoint is not included.
- For $a < b$, we can form the four *bounded intervals* below. The first is closed, the last is open, and the other two are neither open nor closed.

$$[a, b] \quad \text{and} \quad [a, b) \quad \text{and} \quad (a, b] \quad \text{and} \quad (a, b).$$

- For any real number a, we can form the four *unbounded intervals* below. The first two are closed, and the last two are open.

$$[a, \infty) \quad \text{and} \quad (-\infty, a] \quad \text{and} \quad (a, \infty) \quad \text{and} \quad (-\infty, a).$$

- The notation $(-\infty, \infty)$ means the whole real number line **R**.
- The notation $[a, a]$ is the one-member set $\{a\}$, called a *degenerate interval*.
- An interval is called *closed* if it contains all its endpoints, and *open* if it doesn't contain any of its endpoints.

For those who enjoy precision, the unbounded interval $(-\infty, \infty)$ is both open and closed (it has no endpoints), and a degenerate interval $[a, a]$ is closed.

Using bracket interval notation for domain and range

When the domain or range of a function or relation is an interval, bracket interval notation is a straightforward alternative notation for the domain or range, as in the next worked exercise.

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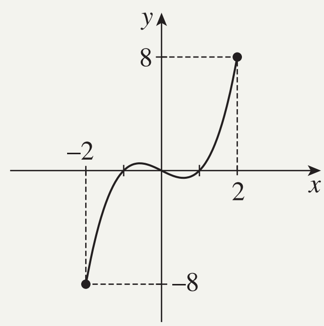
Example 1

Write down the domain and range of these relations:

- **i** using inequality interval notation,
- **ii** using bracket interval notation.

a



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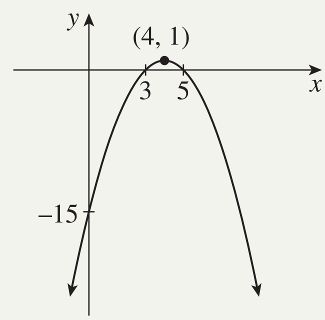

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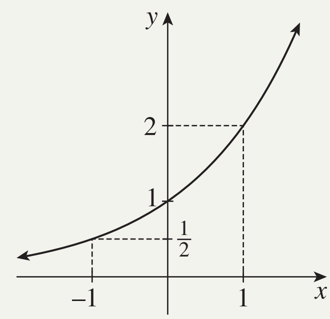

SOLUTION

a **i** domain: $-2 \le x \le 2$
range: $-8 \le y \le 8$

ii domain $= [-2, 2]$
range $= [-8, 8]$

b **i** domain: all real x
range: $y \le 1$

ii domain $= (-\infty, \infty)$
range $= (-\infty, 1]$

c **i** domain: all real x
range: $y > 0$

ii domain $= (-\infty, \infty)$
range $= (0, \infty)$

Using bracket interval notation for solutions of inequaltions

When the solutions of an inequation form an interval, bracket interval notation is a straightforward alternative notation.

Remember to reverse the inequality when multiplying or dividing by a negative.

Example 2

Write down the solution of these inequations, first using inequality interval notation, then using bracket interval notation.

a $-15 \le 5x < 30$

b $-2x < 10$

SOLUTION

a
$$-15 \le 5x < 30$$
$\boxed{\div 5}$
$$-3 \le x < 6$$

The solution is $[-3, 6)$.

b
$$-2x < 10$$
$\boxed{\div (-2)}$
$$x > -5$$

The solution is $(-5, \infty)$.

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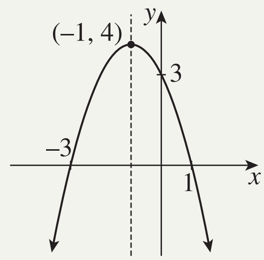
Example 3

State where $y \geq 0$ for each function, using both interval notations.

a



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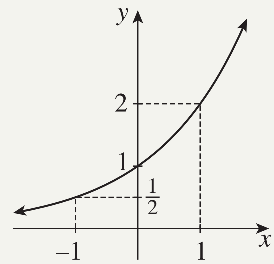
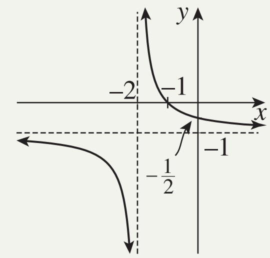


c



SOLUTION

a The solution is $-3 \leq x \leq 1$, or in the bracket notation, $[-3, 1]$.

b The solution is all real x, or in the bracket notation, $(-\infty, \infty)$.

c The solution is $-2 < x \leq -1$, or in the bracket notation, $(-2, -1]$.

Notation when the domain or range or set of solutions is not an interval

The domain of the function $f(x) = \frac{1}{x}$ is $x \neq 0$, which is not an interval. This domain is, however, the union of the two intervals $(-\infty, 0)$ and $(0, \infty)$. Notation in such situations is discussed in Section 3A of the Year 12 book, after the union and intersection of sets have been reviewed in in the context of probability (Section 12C in the Year 11 book).

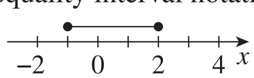
Exercise Appendix A

FOUNDATION

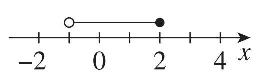
1 For each number line, write the graphed interval using:

i inequality interval notation, **ii** bracket interval notation.

a



b



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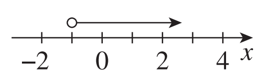

2 For each interval given by inequality interval notation:

i draw the interval on a number line,

ii write it using bracket interval notation.

a $-1 \leq x < 2$ **b** $x \leq 2$ **c** $x < 2$

3 For each interval given by bracket interval notation:

i draw the interval on a number line,

ii write it using inequality interval notation.

a $[-1, \infty)$ **b** $(-1, 2)$ **c** $(-\infty, \infty)$

4 Solve each inequation, writing the solution using both interval notations.

a $7x < 35$ **b** $5x \geq -45$ **c** $-3x \leq 15$

d $-6 < 2x < 8$ **e** $10 < -5x \leq 35$ **f** $-180 \leq -10x \leq 260$

5 There are fifteen intervals in Q1–Q4.

a Which are open? **b** Which are closed? **c** Which are unbounded?

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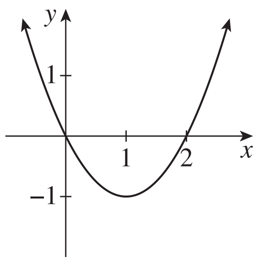
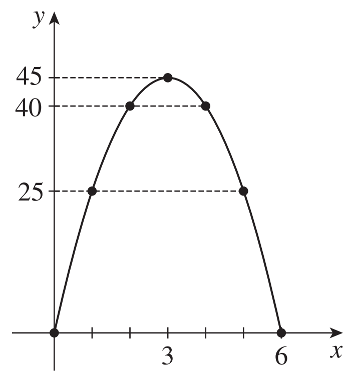
DEVELOPMENT

6 Write down the domains and ranges of each function using both interval notations.

a



b



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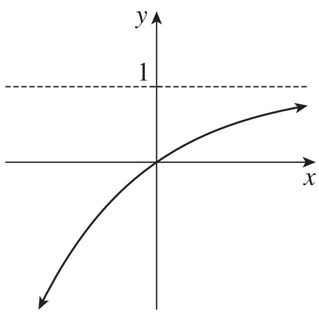
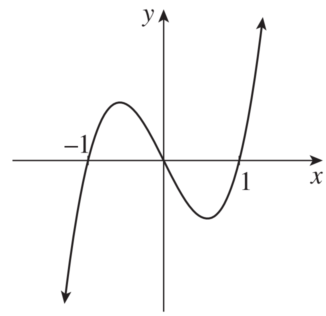
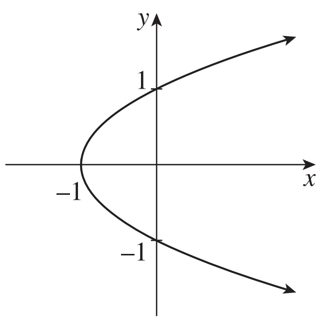


d



7 Write down the domains and ranges of each relation using both interval notations.

a



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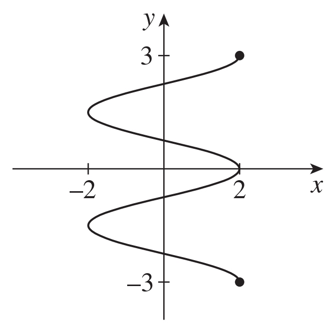

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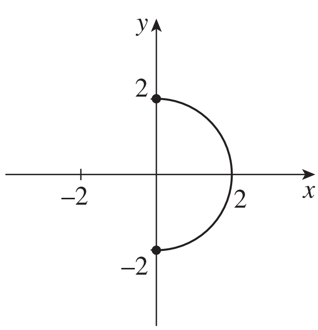

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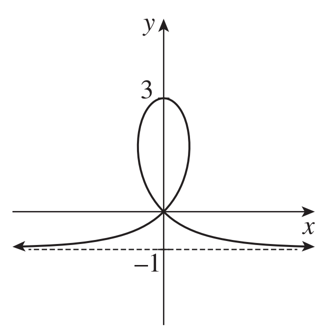

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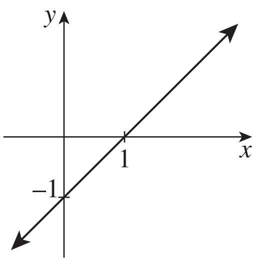
8 For each function, state the values of x for which $y < 0$, using both interval notations.

a



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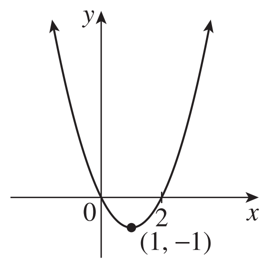

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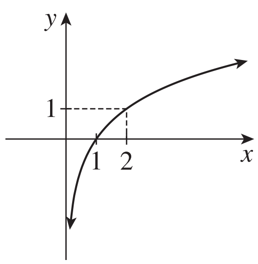

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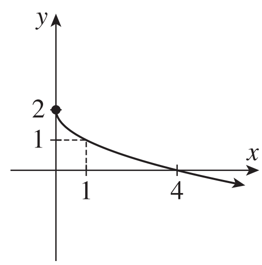
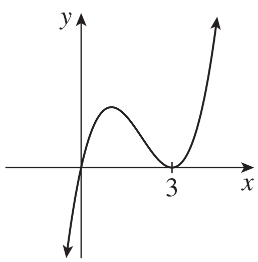


9 Repeat the previous question, finding the values of x for which $y \le 0$.

10 Use the given graph of the LHS to help solve each inequation using both interval notations.

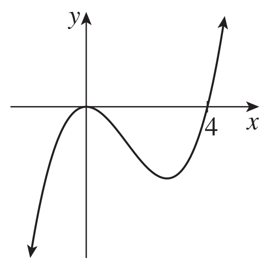
a



i $x(x-3)^2 \ge 0$

ii $x(x-3)^2 < 0$

b



i $x^2(x-4) > 0$

ii $x^2(x-4) \le 0$

CHALLENGE

11 Use bracket interval notation in this question, where $f(x) = 2^{x-3} + 1$.

a Write down the domain and range of $f(x)$.

b Write down the solutions of:

i $f(x) \ge 1$ **ii** $f(x) \ge 2$ **iii** $f(x) < 9$ **iv** $1\frac{1}{2} \le f(x) < 3$

12 Write the solutions of each inequation using bracket interval notation.

a $x^2 \le 0$ **b** $x^3 \le 0$ **c** $|x| \le 0$ **d** $|x| + x \le 0$

13 As defined in the notes above, an interval is called *closed* if it contains all its endpoints, and *open* if it doesn't contain any of its endpoints.

a Explain why the degenerate interval $[5, 5]$ is closed.

b Explain why the interval $(-\infty, \infty)$ is closed.

c Explain why the same interval $(-\infty, \infty)$ is also open.

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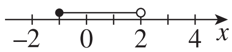
Answers

Exercise Appendix A

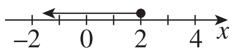
1 a i $-1 \le x \le 2$ **ii** $[-1, 2]$

b i $-1 < x \le 2$ **ii** $(-1, 2]$

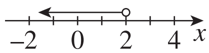
c i $x > -1$ **ii** $(-1, \infty)$

2 a i 

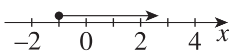
ii $[-1, 2)$

b i 

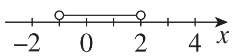
ii $(-\infty, 2]$

c i 

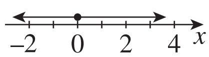
ii $(-\infty, 2)$

3 a i 

ii $x \ge -1$

b i 

ii $-1 < x < 2$

c i 

ii all real x

(The interval can't be written using inequalities.)

4 a $x < 5$ OR $(-\infty, 5)$

b $x \ge -9$ OR $[-9, \infty)$

c $x \ge -5$ OR $[-5, \infty)$

d $-3 < x < 4$ OR $(-3, 4)$

e $-7 \le x < -2$ OR $[-7, -2)$

f $-26 \le x \le 18$ OR $[-26, 18]$

5 a Open: Q1(c), Q2(c), Q3(b) & (c), Q4(a) & (d)

b Closed: Q1(a), Q2(b), Q3(a) & (c), Q4(b), (c) & (f)

c Unbounded: Q1(c), Q2(b) & (c), Q3(a) & (c), Q4(a), (b) & (c)

6 a domain: all real x OR $(-\infty, \infty)$
range: $y \ge -1$ OR $[-1, \infty)$

b domain: $0 \le x \le 6$ OR $[0, 6]$
range: $0 \le y \le 45$ OR $[0, 45]$

c domain: all real x OR $(-\infty, \infty)$
range: $y < 1$ OR $(-\infty, 1)$

d domain: all real x OR $(-\infty, \infty)$
range: all real y OR $(-\infty, \infty)$

7 a domain: $x \ge -1$ OR $[-1, \infty)$
range: all real y OR $(-\infty, \infty)$

b domain: $-2 \le x \le 2$ OR $[-2, 2]$
range: all real y OR $(-\infty, \infty)$

c domain: $0 \le x \le 2$ OR $[0, 2]$
range: $-2 \le y \le 2$ OR $[-2, 2]$

d domain: all real x OR $(-\infty, \infty)$
range: $-1 < y \le 3$ OR $(-1, 3]$

8 a $x < 1$ OR $(-\infty, 1)$

b $0 < x < 2$ OR $(0, 2)$

c $0 < x < 1$ OR $(0, 1)$

d $x > 4$ OR $(4, \infty)$

9 a $x \le 1$ OR $(-\infty, 1]$

b $0 \le x \le 2$ OR $[0, 2]$

c $0 < x \le 1$ OR $(0, 1]$

d $x \ge 4$ OR $[4, \infty)$

10 a i $x \ge 0$ OR $[0, \infty)$

ii $x < 0$ OR $(-\infty, 0)$

b i $x > 4$ OR $(4, \infty)$

ii $x \le 4$ OR $(-\infty, 4]$

11 a domain: $(-\infty, \infty)$, range: $(1, \infty)$

b i $(-\infty, \infty)$ **ii** $[3, \infty)$

iii $(-\infty, 6)$ **iv** $[2, 4)$

12 a $[0, 0]$ **b** $(-\infty, 0]$ **c** $[0, 0]$ **d** $(-\infty, 0]$

13 a It has one endpoint 5, which it contains.

b It contains all its endpoints (there are none).

c It does not contain any of its endpoints (there are none).

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