FUNCTIONS

1.

ALGEBRAIC TECHNIQUES

Ths chapter revses and extends the algebrac technques that you wll need for ths course These nclude ndces algebrac expressons expanson factorsaton algebrac fractons and surds

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- dentfy and use ndex rules ncludng fractonal and negatve ndces
- smplfy algebrac expressons
- remove groupng symbols ncludng perfect squares and the ifference of 2 squares
- factorse expressons ncludng bnomals and specal factors
- smplfy algebrac fractons
- use algebra to substtute nto formulas
- smplfy and use surds ncludng ratonalsng the denomnator

TERMINOLOGY

binomial A mathematical expression consisting of 2 terms for example $x + 3$ and $3x - 1$

binomial product The product of binomial expressions for example $(x + 3)(2x - 1)$

expression A mathematical statement involving numbers pronumerals and symbol; for example $2x - 3$

factor A whole number that divides exactly into another number. For exampe, 4 is a factor of 28

factorise To write an expression as a product of its factors that is take out the highest common factor in an expression and place the rest in brackets For exampl, $2y - 8 = 2(y - 4)$

index The power or exponent of a numbe. For example 2^3 has a base number of 2 and an index of 3 The plural of index is **indices**

power The index or exponent of a numbe. For example 2^3 has a base number of 2 and a power of 3

root A number that when multiplied by itself a given number of times equals another number. For example $\sqrt{25} = 5$ because $5^2 = 25$

surd A root that ca't be simplifid; for example $\sqrt{3}$

term A part of an expression containing pronumerals and/or numbers separated by an operation such as $+$ $-$ $\times$ or $\div$ For example in $2x - 3$ the terms are $2x$ and 3

trinomial An expression with 3 term; for example $3x^2 - 2x + 1$

1.01 Index laws

An **index** (or **power** or **exponent**) of a number shows how many times a number is multiplied by itself A **root** of a number is the inverse of the power.

For example

- $4^3 = 4 \times 4 \times 4 = 64$
- $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$
- $\sqrt{36} = 6$ since $6^2 = 36$
- $\sqrt[3]{8} = 2$ since $2^3 = 8$
- $\sqrt[6]{64} = 2$ since $2^6 = 64$

Note In 4^3 the 4 s called the base number and the 3 s called the ndex or power.

There are some general laws that simplify calculations with indices These laws work for any m and n including fractions and negative number.

Index laws

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

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EXAMPLE 1

Simplify

a $m^9 \times m^7 \div m^2$ **b** $(2y^4)^3$ **c** $\dfrac{(y^6)^3 \times y^{-4}}{y^5}$

Solution

a $m^9 \times m^7 \div m^2 = m^{9+7-2}$
$= m^{14}$

b $(2y^4)^3 = 2^3(y^4)^3$
$= 2^3 y^{4\times 3}$
$= 8y^{12}$

c $\dfrac{(y^6)^3 \times y^{-4}}{y^5} = \dfrac{y^{18} \times y^{-4}}{y^5}$
$= \dfrac{y^{18+(-4}}{y^5}$
$= \dfrac{y^{14}}{y^5}$
$= y^{14-5}$
$= y^9$

Exercise 1.01 Index laws

1 Evaluate without using a calculator

a $5^3 \times 2^2$ **b** $3^4 + 8^2$ **c** $\left(\dfrac{1}{4}\right)^3$

d $\sqrt[3]{27}$ **e** $\sqrt[4]{16}$

2 Evaluate correct to 1 decimal place

a $37^{\ 2}$ **b** $106^{\ 15}$ **c** $23^{\ -02}$

d $\sqrt[3]{19}$ **e** $\sqrt[3]{34\,8 - 1\,2 \times 43\,1}$ **f** $\dfrac{1}{\sqrt[3]{099\ +.61}}$

3 Simplify

a $a^6 \times a^9 \times a^2$ **b** $y^3 \times y^{-8} \times y^5$ **c** $a^{-} \times a^{-3}$

d $w^{\bar{2}} \times w^{\bar{2}}$ **e** $x^6 \div x$ **f** $p^3 \div p^{-7}$

g $\dfrac{y^{11}}{y^5}$ **h** $(x^7)^3$ **i** $(2x^5)^2$

j $(3y^{-2})^4$ **k** $a^3 \times a^5 \div a^7$ $\left(\dfrac{x^2}{y^9}\right)^5$

m $\dfrac{w^6 \times w^7}{w^3}$ **n** $\dfrac{p^2 \times (p^3)^4}{p^9}$ **o** $\dfrac{x^6 \div x^7}{x^2}$

p $\dfrac{a^2 \times (b^2)^6}{a^4 \times b^9}$ **q** $\dfrac{(x^2)^{-3} \times (y^3)^2}{x^{-1} \times y^4}$

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4 Simplify

a $x^5 \times x^9$ **b** $a^{-} \times a^{-6}$ **c** $\dfrac{m^7}{m^3}$

d $k^{13} \times k^6 \div k^9$ **e** $a^{-5} \times a^4 \times a^{-7}$ **f** $x^{\frac{2}{5}} \times x^{\frac{3}{5}}$

g $\dfrac{m^5 \times n^4}{m^4 \times n^2}$ **h** $\dfrac{p^{\frac{}{2}} \times p^{\frac{}{2}}}{p^2}$ **i** $(3x^{11})^2$

j $\dfrac{(x^4)^6}{x^3}$

5 Expand each expression and simplify where possible

a $(pq^3)^5$ **b** $\left(\dfrac{a}{b}\right)^8$ **c** $\left(\dfrac{4a}{b^4}\right)^3$

d $(7a^5b)^2$ **e** $\dfrac{(2m^7)^3}{m^4}$ **f** $\dfrac{xy^3 \times (xy^2)^4}{xy}$

g $\dfrac{(2k^8)^4}{(6k^3)^3}$ **h** $(2y^5)^7 \times \dfrac{y^{12}}{8}$ **i** $\left(\dfrac{a^6 \times a^4}{a^{11}}\right)^{-3}$

j $\left(\dfrac{5xy^9}{x^8 \times y^3}\right)^3$

6 Evaluate a^3b^2 when $a = 2$ and $b = \dfrac{3}{4}$

7 If $x = \dfrac{2}{3}$ and $y = \dfrac{1}{9}$ find the value of $\dfrac{x^3y^2}{xy^5}$

8 If $a = \dfrac{1}{2}$ $b = \dfrac{1}{3}$ and $c = \dfrac{1}{4}$ evaluate $\dfrac{a^2b^3}{c^4}$ as a fraction

9 **a** Simplify $\dfrac{a^{11}b^8}{a^8b^7}$

b Hence evaluate $\dfrac{a^{11}b^8}{a^8b^7}$ as a fraction when $a = \dfrac{2}{5}$ and $b = \dfrac{5}{8}$

10 **a** Simplify $\dfrac{p^5q^8r^4}{p^4q^6r^2}$

b Hence evaluate $\dfrac{p^5q^8r^4}{p^4q^6r^2}$ as a fraction when $p = \dfrac{7}{8}$ $q = \dfrac{2}{3}$ and $r = \dfrac{3}{4}$

11 Evaluate $(a^4)^3$ when $a = \left(\dfrac{2}{3}\right)^{\frac{}{6}}$

12 Evaluate $\dfrac{a^3b^6}{b^4}$ when $a = \dfrac{1}{2}$ and $b = \dfrac{2}{3}$

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13 Evaluate $\frac{x^4 y^7}{x^5 y^5}$ when $x = \frac{1}{3}$ and $y = \frac{2}{9}$

14 Evaluate $\frac{k^{-5}}{k^{-9}}$ when $k = \frac{1}{3}$

15 Evaluate $\frac{a^4 b^6}{a^3 (b^2)^2}$ when $a = \frac{3}{4}$ and $b = \frac{1}{9}$

16 Evaluate $\frac{a^6 \times b^3}{a^5 \times b^2}$ as a fraction when $a = \frac{1}{9}$ and $b = \frac{3}{4}$

1.02 Zero and negative indices

Zero and negative indices

$$x^0 = 1$$

$$x^{-n} = \frac{1}{x^n}$$

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Review o index law

EXAMPLE 2

a Simplify $\left(\frac{ab^5c}{abc^4}\right)^0$

b Evaluate 2^{-3}

c Write in index for:

i $\frac{1}{x^2}$ **ii** $\frac{3}{x^5}$ **iii** $\frac{1}{5x}$ **v** $\frac{1}{x+1}$

d Write a^{-3} without the negative index

Solution

a $\left(\frac{ab^5c}{abc^4}\right)^0 = 1$

b $2^{-3} = \frac{1}{2^3}$

$= \frac{1}{8}$

c **i** $\frac{1}{x^2} = x^{-2}$

ii $\frac{3}{x^5} = 3 \times \frac{1}{x^5}$

$= 3x^{-5}$

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iii $\frac{1}{5x} = \frac{1}{5} \times \frac{1}{x}$

$= \frac{1}{5}x^{-}$

v $\frac{1}{x+1} = \frac{1}{(x+1)}$

$= (x+1)^{-}$

d $a^{-3} = \frac{1}{a^3}$

Exercise 1.02 Zero and negative indices

1 Evaluate as a fraction or whole number

a	3^{-3}	**b**	4^{-}	**c**	7^{-3}	**d**	10^{-4}	**e**	2^{-8}
f	6^0	**g**	2^{-5}	**h**	3^{-4}	**i**	7^{-}	**j**	9^{-2}
k	2^{-6}		3^{-2}	**m**	4^0	**n**	6^{-2}	**o**	5^{-3}
p	10^{-5}	**q**	2^{-7}	**r**	2^0	**s**	8^{-2}	**t**	4^{-3}

2 Evaluate

a	2^0	**b**	$\left(\frac{1}{2}\right)^{-4}$	**c**	$\left(\frac{2}{3}\right)^{-}$	**d**	$\left(\frac{5}{6}\right)^{-2}$	**e**	$\left(\frac{x+2y}{3x-y}\right)^0$
f	$\left(\frac{1}{5}\right)^{-3}$	**g**	$\left(\frac{3}{4}\right)^{-}$	**h**	$\left(\frac{1}{7}\right)^{-2}$	**i**	$\left(\frac{2}{3}\right)^{-3}$	**j**	$\left(\frac{1}{2}\right)^{-5}$
k	$\left(\frac{3}{7}\right)^{-}$		$\left(\frac{8}{9}\right)^0$	**m**	$\left(\frac{6}{7}\right)^{-2}$	**n**	$\left(\frac{9}{10}\right)^{-2}$	**o**	$\left(\frac{6}{11}\right)^0$
p	$\left(-\frac{1}{4}\right)^{-2}$	**q**	$\left(-\frac{2}{5}\right)^{-3}$	**r**	$\left(-3\frac{2}{7}\right)^{-}$	**s**	$\left(-\frac{3}{8}\right)^0$	**t**	$\left(-1\frac{1}{4}\right)^{-2}$

3 Change into index form

a	$\frac{1}{m^3}$	**b**	$\frac{1}{x}$	**c**	$\frac{1}{p^7}$	**d**	$\frac{1}{d^9}$	**e**	$\frac{1}{k^5}$
f	$\frac{1}{x^2}$	**g**	$\frac{2}{x^4}$	**h**	$\frac{3}{y^2}$	**i**	$\frac{1}{2z^6}$	**j**	$\frac{3}{5t^8}$
k	$\frac{2}{7x}$		$\frac{5}{2m^6}$	**m**	$\frac{2}{3y^7}$	**n**	$\frac{1}{(3x+4)^2}$	**o**	$\frac{1}{(a+b)^8}$
p	$\frac{1}{x-2}$	**q**	$\frac{1}{(5p+1)^3}$	**r**	$\frac{2}{(4t-9)^5}$	**s**	$\frac{1}{4(x+1)^{11}}$	**t**	$\frac{5}{9(a+3b)^7}$

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4 Write without negative indice:

a	t^{-5}	**b**	x^{-6}	**c**	y^{-3}	**d**	n^{-8}	**e**	w^{-10}
f	$2x^{-}$	**g**	$3m^{-4}$	**h**	$5x^{-7}$	**i**	$(2x)^{-3}$	**j**	$(4n)^{-}$
k	$(x+1)^{-6}$		$(8y+z)^{-}$	**m**	$(k-3)^{-2}$	**n**	$(3x+2y)^{-9}$	**o**	$\left(\frac{1}{x}\right)^{-5}$
p	$\left(\frac{1}{y}\right)^{-10}$	**q**	$\left(\frac{2}{p}\right)^{-}$	**r**	$\left(\frac{1}{a+b}\right)^{-2}$	**s**	$\left(\frac{x+y}{x-y}\right)^{-}$	**t**	$\left(\frac{2w-z}{3x+y}\right)^{-7}$

1.03 Fractional indices

INVESTIGATION

FRACTIONAL INDICES

Consider the following examples

$$\left(x^{\frac{}{2}}\right)^2 = x \quad \text{(by index laws)}$$
$$= x$$

$$\left(\sqrt{x}\right)^2 = x$$

$$\text{So} \left(x^{\frac{}{2}}\right)^2 = \left(\sqrt{x}\right)^2$$
$$= x$$

$$\therefore \quad x^{\frac{}{2}} = \sqrt{x}$$

Now simplify these expressions

1	$\left(x^2\right)^{\frac{}{2}}$	**2**	$\sqrt{x^2}$	**3**	$\left(x^{\frac{}{3}}\right)^3$	**4**	$\left(x^3\right)^{\frac{}{3}}$	**5**	$\left(\sqrt[3]{x}\right)^3$
6	$\sqrt[3]{x^3}$	**7**	$\left(x^{\frac{}{4}}\right)^4$	**8**	$\left(x^4\right)^{\frac{}{4}}$	**9**	$\left(\sqrt[4]{x}\right)^4$	**10**	$\sqrt[4]{x^4}$

Use your results to complete

$$x^{\frac{}{n}} =$$

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Power of $\frac{1}{n}$

$$a^{\frac{}{n}} = \sqrt[n]{a}$$

Proof

$$\left(a^{\frac{}{n}}\right)^n = a \quad \text{(by index laws)}$$

$$(\sqrt[n]{a})^n = a$$

$$\therefore a^{\frac{}{n}} = \sqrt[n]{a}$$

EXAMPLE 3

a Evaluate

i $49^{\frac{}{2}}$ **ii** $27^{\frac{}{3}}$

b Write $\sqrt{3x-2}$ in index form

c Write $(a+b)^{\frac{}{7}}$ without fractional indices

Solution

a **i** $49^{\frac{}{2}} = \sqrt{49} = 7$ **ii** $27^{\frac{}{3}} = \sqrt[3]{27} = 3$

b $\sqrt{3x-2} = (3x-2)^{\frac{}{2}}$ **c** $(a+b)^{\frac{}{7}} = \sqrt[7]{a+b}$

Further fractional indices

$$a^{-\frac{}{n}} = \frac{1}{\sqrt[n]{a}}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ or } \left(\sqrt[n]{a}\right)^m$$

Proof

$$a^{\frac{m}{n}} = \left(a^{\frac{}{n}}\right)^m$$
$$= \left(\sqrt[n]{a}\right)^m$$

$$a^{\frac{m}{n}} = \left(a^m\right)^{\frac{}{n}}$$
$$= \sqrt[n]{a^m}$$

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$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

Proof

$$\begin{aligned}\left(\frac{a}{b}\right)^{-n} &= \frac{1}{\left(\frac{a}{b}\right)^{n}} \\ &= \frac{1}{\frac{a^n}{b^n}} \\ &= 1 \div \frac{a^n}{b^n} \\ &= 1 \times \frac{b^n}{a^n} \\ &= \frac{b^n}{a^n} \\ &= \left(\frac{b}{a}\right)^{n}\end{aligned}$$

EXAMPLE 4

a Evaluate

i $8^{\frac{4}{3}}$ **ii** $125^{-\frac{}{3}}$ **iii** $\left(\frac{2}{3}\right)^{-3}$

b Write in index for:

i $\sqrt{x^5}$ **ii** $\dfrac{1}{\sqrt[3]{(4x^2-1)^2}}$

c Write $r^{-\frac{3}{5}}$ without the negative and fractional indices

Solution

a **i** $8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4$ (or $\sqrt[3]{8^4}$)
$= 2^4$
$= 16$

ii $125^{-\frac{}{3}} = \dfrac{1}{125^{\frac{}{3}}}$
$= \dfrac{1}{\sqrt[3]{125}}$
$= \dfrac{1}{5}$

iii $\left(\dfrac{2}{3}\right)^{-3} = \left(\dfrac{3}{2}\right)^3$
$= \dfrac{27}{8}$
$= 3\dfrac{3}{8}$

b **i** $\sqrt{x^5} = x^{\frac{5}{2}}$

ii $\dfrac{1}{\sqrt[3]{(4x^2-1)^2}} = \dfrac{1}{(4x^2-1)^{\frac{2}{3}}}$
$= (4x^2-1)^{-\frac{2}{3}}$

c $r^{-\frac{3}{5}} = \dfrac{1}{r^{\frac{3}{5}}} = \dfrac{1}{\sqrt[5]{r^3}}$

DID YOU KNOW?

Fractional indices

Nicole Oresme (1323–82) was the first mathematician to use fractional indices

John Wallis (1616–1703) was the first person to explain the significance of zero negative and fractional indices He also introduced the symbol ∞ for infinit.

Research these mathematicians and find out more about their work and backgrouds. You could use keywords such as indices and infinity as well as their names to find this information

Exercise 1.03 Fractional indices

1 Evaluate

a $81^{\frac{}{2}}$	**b** $27^{\frac{}{3}}$	**c** $16^{\frac{}{2}}$	**d** $8^{\frac{}{3}}$	**e** $49^{\frac{}{2}}$
f $1000^{\frac{}{3}}$	**g** $16^{\frac{}{4}}$	**h** $64^{\frac{}{2}}$	**i** $64^{\frac{}{3}}$	**j** $1^{\frac{}{7}}$
k $81^{\frac{}{4}}$	$32^{\frac{}{5}}$	**m** $0^{\frac{}{8}}$	**n** $125^{\frac{}{3}}$	**o** $343^{\frac{}{3}}$
p $128^{\frac{}{7}}$	**q** $256^{\frac{}{4}}$	**r** $125^{\frac{2}{3}}$	**s** $4^{\frac{5}{2}}$	**t** $8^{\frac{2}{}}$
u $9^{\frac{3}{2}}$	**v** $8^{-\frac{}{3}}$	**w** $9^{-\frac{}{2}}$	**x** $16^{-\frac{}{4}}$	**y** $64^{-\frac{2}{3}}$

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2 Evaluate correct to 2 decimal places

a $23^{\overline{4}}$ **b** $\sqrt[4]{45\ 8}$ **c** $\sqrt[7]{124 +.\ \ 3^2}$

d $\dfrac{1}{\sqrt[5]{12\ 9}}$ **e** $\sqrt[8]{\dfrac{3\ 6-14}{1\ 5+3\ 7}}$ **f** $\dfrac{\sqrt[4]{5\ 9\times 3\ 7}}{8.79-1.4}$

3 Write without fractional or negative indice:

a $y^{\overline{3}}$ **b** $x^{\overline{6}}$ **c** $a^{\overline{2}}$ **d** $t^{\overline{9}}$ **e** $y^{\frac{2}{3}}$

f $x^{\frac{3}{4}}$ **g** $b^{\frac{2}{5}}$ **h** $a^{\frac{4}{7}}$ **i** $x^{-\overline{\ 2}}$ **j** $d^{-\overline{\ 3}}$

k $x^{-\overline{\ 8}}$ $y^{-\overline{\ 3}}$ **m** $a^{-\overline{\ 4}}$ **n** $z^{-\frac{3}{4}}$ **o** $y^{-\frac{3}{5}}$

p $(2x+5)^{\overline{2}}$ **q** $(6q+r)^{\overline{3}}$ **r** $(a+b)^{\overline{9}}$ **s** $(3x-1)^{-\overline{\ 2}}$ **t** $(x+7)^{-\frac{2}{5}}$

4 Write in index for:

a $\sqrt{t}$ **b** $\sqrt[5]{y}$ **c** $\sqrt{x^3}$ **d** $\sqrt[3]{9-x}$ **e** $\sqrt{4s+1}$

f $\sqrt{(3x+1)^5}$ **g** $\dfrac{1}{\sqrt{2t+3}}$ **h** $\dfrac{1}{\sqrt{(5x-y)^3}}$ **i** $\dfrac{1}{\sqrt[3]{(x-2)^2}}$ **j** $\dfrac{1}{2\sqrt{y+7}}$

k $\dfrac{5}{\sqrt[3]{x+4}}$ $\dfrac{1}{3\sqrt{y^2-1}}$ **m** $\dfrac{3}{5\sqrt[4]{(x^2+2)^3}}$

5 Write in index form and simplif:

a $x\sqrt{x}$ **b** $\dfrac{\sqrt{x}}{x}$ **c** $\dfrac{x}{\sqrt[3]{x}}$ **d** $\dfrac{x^2}{\sqrt[3]{x}}$ **e** $x\sqrt[4]{x}$

6 Write without fractional or negative indice:

a $(a-2b)^{-\overline{\ 3}}$ **b** $(y-3)^{-\frac{2}{3}}$ **c** $4(6a+1)^{-\frac{4}{7}}$ **d** $\dfrac{(x+y)^{-\frac{5}{4}}}{3}$ **e** $\dfrac{6(3x+8)^{-\frac{2}{9}}}{7}$

DID YOU KNOW?

The beginnings of algebra

One of the earliest mathematicians to use algebra was **Diophantus of Alexandria** in Greece It is not known when he live, but it is thought this may have been around 250 CE

In Persia around 700–800 CE a mathematician named **Muhammad ibn Musa al-Khwarizmi** wrote books on algebra and Hindu numerals One of his books was named *Al-Jabr wa'l Muqabala* and the word **algebra** comes from the first word in this title

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1.04 Simplifying algebraic expressions

EXAMPLE 5

Simplify

a $4x^2 - 3x^2 + 6x^2$ **b** $x^3 - 3x - 5x + 4$ **c** $3a - 4b - 5a - b$

Solution

a $4x^2 - 3x^2 + 6x^2 = x^2 + 6x^2$
$= 7x^2$

b $x^3 - 3x - 5x + 4 = x^3 - 8x + 4$

Only lke terms can be added or subtracted

c $3a - 4b - 5a - b = 3a - 5a - 4b - b$
$= -2a - 5b$

EXAMPLE 6

Simplify

a $-5x \times 3y \times 2x$ **b** $\frac{5a^3b}{15ab^2}$

Solution

a $-5x \times 3y \times 2x = -30xyx$
$= -30x^2y$

b $\frac{5a^3b}{15ab^2} = \frac{1}{3}a^{3-1}b^{1-2}$
$= \frac{1}{3}a^2b^-$
$= \frac{a^2}{3b}$

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Exercise 1.04 Simplifying algebraic expressions

1 Simplify

a $9a - 6a$ **b** $5z - 4z$ **c** $4b - b$

d $2r - 5r$ **e** $-4y + 3y$ **f** $-2x - 3x$

g $2a - 2a$ **h** $-4k + 7k$ **i** $3t + 4t + 2t$

j $8w - w + 3w$ **k** $4m - 3m - 2m$ $x + 3x - 5x$

m $8h - h - 7h$ **n** $3b - 5b + 4b + 9b$ **o** $-5x + 3x - x - 7x$

p $6x - 5y - y$ **q** $8a + b - 4b - 7a$ **r** $xy + 2y + 3xy$

s $2ab^2 - 5ab^2 - 3ab^2$ **t** $m^2 - 5m - m + 12$ **u** $p^2 - 7p + 5p - 6$

v $ab + 2b - 3ab + 8b$ **w** $ab + bc - ab - ac + bc$

x $a^5 - 7x^3 + a^5 - 2x^3 + 1$ **y** $x^3 - 3xy^2 + 4x^2y - x^2y + xy^2 + 2y^3$

2 Simplify

a $5 \times 2b$ **b** $2x \times 4y$ **c** $5p \times 2p$

d $-3z \times 2w$ **e** $-5a \times -3b$ **f** $x \times 2y \times 7z$

g $8ab \times 6c$ **h** $4d \times 3d$ **i** $3a \times 4a \times a$

j $(-3y)^3$ **k** $(2x^2)^5$ $2ab^3 \times 3a$

m $5a^2b \times -2ab$ **n** $7pq^2 \times 3p^2q^2$ **o** $5ab \times a^2b^2$

p $4h^3 \times -2h^7$ **q** $k^3p \times p^2$ **r** $(-3t^3)^4$

s $7m^6 \times -2m^5$ **t** $-2x^2 \times 3x^3y \times -4xy^2$

3 Simplify

a $30x \div 5$ **b** $2y \div y$ **c** $\frac{8a^2}{2}$

d $\frac{8a^2}{a}$ **e** $\frac{8a^2}{2a}$ **f** $\frac{xy}{2x}$

g $12p^3 \div 4p^2$ **h** $\frac{3a^2b^2}{6ab}$ **i** $\frac{20x}{15xy}$

j $\frac{-9x^7}{3x^4}$ **k** $-15ab \div -5b$ $\frac{2ab}{6a^2b^3}$

m $\frac{-8p}{4pqs}$ **n** $14cd^2 \div 21c^3d^3$ **o** $\frac{2xy^2z^3}{4x^3y^2z}$

p $\frac{42p^5q^4}{7pq^3}$ **q** $5a^9b^4c^{-2} \div 20a^5b^{-3}c^{-}$ **r** $\frac{2(a^{-5})^2b^4}{4a^{-9}(b^2)^{-}}$

s $-5x^4y^7z \div 15xy^8z^{-2}$ **t** $-9(a^4b^{-}\ )^3 \div -18a^{-}\ b^3$

WS

Expanding algebaic expeion

1.05 Expansion

When we remove grouping symbols we say that we are **expanding** an expression

Expanding expressions

To expand an expressio, use the distributive lw:

$$a(b + c) = ab + ac$$

EXAMPLE 7

Expand and simplify

a $5a^2(4 + 3ab - c)$ **b** $5 - 2(y + 3)$ **c** $2(b - 5) - (b + 1)$

Solution

a $5a^2(4 + 3ab - c) = 5a^2 \times 4 + 5a^2 \times 3ab - 5a^2 \times c$

$= 20a^2 + 15a^3b - 5a^2c$

b $5 - 2(y + 3) = 5 - 2 \times y - 2 \times 3$

$= 5 - 2y - 6$

$= -2y - 1$

c $2(b - 5) - (b + 1) = 2 \times b + 2 \times -5 - 1 \times b - 1 \times 1$

$= 2b - 10 - b - 1$

$= b - 11$

Exercise 1.05 Expansion

Expand and simplify each expression

1 $2(x - 4)$ **2** $3(2h + 3)$ **3** $-5(a - 2)$

4 $x(2y + 3)$ **5** $x(x - 2)$ **6** $2a(3a - 8b)$

7 $ab(2a + b)$ **8** $5n(n - 4)$ **9** $3x^2y(xy + 2y^2)$

10 $3 + 4(k + 1)$ **11** $2(t - 7) - 3$ **12** $y(4y + 3) + 8y$

13 $9 - 5(b + 3)$ **14** $3 - (2x - 5)$ **15** $5(3 - 2m) + 7(m - 2)$

16 $2(h + 4) + 3(2h - 9)$ **17** $3(2d - 3) - (5d - 3)$ **18** $a(2a + 1) - (a^2 + 3a - 4)$

19 $x(3x - 4) - 5(x + 1)$ **20** $2ab(3 - a) - b(4a - 1)$ **21** $5x - (x - 2) - 3$

22 $8 - 4(2y + 1) + y$ **23** $(a + b) - (a - b)$ **24** $2(3t - 4) - (t + 1) + 3$

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1.06 Binomial products

Expanding binomial poduc

A **binomial expression** consists of 2 **terms** for example $x + 3$.

A set of 2 binomial expressions multiplied together is called a **binomial product** for example $(x + 3)(x - 2)$

Each term in the first bracket is multiplied by each term in the second bracket

Binomial product

$$(x + a)(x + b) = x^2 + bx + ax + ab$$

EXAMPLE 8

Expand and simplify

a $(p + 3)(q - 4)$ **b** $(a + 5)^2$ **c** $(x + 4)(2x - 3y - 1)$

Solution

a $(p + 3)(q - 4) = pq - 4p + 3q - 12$

b $(a + 5)^2 = (a + 5)(a + 5)$

$= a^2 + 5a + 5a + 25$

$= a^2 + 10a + 25$

c $(x + 4)(2x - 3y - 1) = 2x^2 - 3xy - x + 8x - 12y - 4$

$= 2x^2 - 3xy + 7x - 12y - 4$

Exercise 1.06 Binomial products

Expand and simplify

1 $(a + 5)(a + 2)$ **2** $(x + 3)(x - 1)$ **3** $(2y - 3)(y + 5)$ **4** $(m - 4)(m - 2)$

5 $(x + 4)(x + 3)$ **6** $(y + 2)(y - 5)$ **7** $(2x - 3)(x + 2)$ **8** $(h - 7)(h - 3)$

9 $(x + 5)(x - 5)$ **10** $(5a - 4)(3a - 1)$ **11** $(2y + 3)(4y - 3)$ **12** $(x - 4)(y + 7)$

13 $(x^2 + 3)(x - 2)$ **14** $(n + 2)(n - 2)$ **15** $(2x + 3)(2x - 3)$ **16** $(4 - 7y)(4 + 7y)$

17 $(a + 2b)(a - 2b)$ **18** $(3x - 4y)(3x + 4y)$ **19** $(x + 3)(x - 3)$ **20** $(y - 6)(y + 6)$

21 $(3a + 1)(3a - 1)$ **22** $(2z - 7)(2z + 7)$ **23** $(x + 9)(x - 2y + 2)$

24 $(b - 3)(2a + 2b - 1)$ **25** $(x + 2)(x^2 - 2x + 4)$ **26** $(a - 3)(a^2 + 3a + 9)$

27 $(a + 9)^2$ **28** $(k - 4)^2$ **29** $(x + 2)^2$ **30** $(y - 7)^2$

31 $(2x + 3)^2$ **32** $(2t - 1)^2$ **33** $(3a + 4b)^2$ **34** $(x - 5y)^2$

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35 $(2a + b)^2$ **36** $(a - b)(a + b)$ **37** $(a + b)^2$ **38** $(a - b)^2$

39 $(a + b)(a^2 - ab + b^2)$ **40** $(a - b)(a^2 + ab + b^2)$

1.07 Special products

WS Expanding expeion

Special binomial poduc

Some binomial products have special results and can be simplified quickly using their special properties Did you notice some of these in Exercise .06 ?

Difference of two squares

$$(a + b)(a - b) = a^2 - b^2$$

Perfect squares

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

EXAMPLE 9

Expand and simplify

a $(2x - 3)^2$

b $(3y - 4)(3y + 4)$

Solution

a $(2x - 3)^2 = (2x)^2 - 2(2x)3 + 3^2$
$= 4x^2 - 12x + 9$

b $(3y - 4)(3y + 4) = (3y)^2 - 4^2$
$= 9y^2 - 16$

Exercise 1.07 Special products

Expand and simplify

1 $(t + 4)^2$ **2** $(z - 6)^2$ **3** $(x - 1)^2$

4 $(y + 8)^2$ **5** $(q + 3)^2$ **6** $(k - 7)^2$

7 $(n + 1)^2$ **8** $(2b + 5)^2$ **9** $(3 - x)^2$

10 $(3y - 1)^2$ **11** $(x + y)^2$ **12** $(3a - b)^2$

13 $(4d + 5e)^2$ **14** $(t + 4)(t - 4)$ **15** $(x - 3)(x + 3)$

16 $(p + 1)(p - 1)$ **17** $(r + 6)(r - 6)$ **18** $(x - 10)(x + 10)$

19 $(2a + 3)(2a - 3)$ **20** $(x - 5y)(x + 5y)$ **21** $(4a + 1)(4a - 1)$

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22 $(7-3x)(7+3x)$
23 $(x^2+2)(x^2-2)$
24 $(x^2+5)^2$
25 $(3ab-4c)(3ab+4c)$
26 $\left(x+\frac{2}{x}\right)^2$
27 $\left(a-\frac{1}{a}\right)\left(a+\frac{1}{a}\right)$
28 $[x+(y-2)][x-(y-2)]$
29 $[(a+b)+c]^2$
30 $[(x+1)-y]^2$
31 $(a+3)^2-(a-3)^2$
32 $16-(z-4)(z+4)$
33 $2x+(3x+1)^2-4$
34 $(x+y)^2-x(2-y)$
35 $(4n-3)(4n+3)-2n^2+5$
36 $(x-4)^3$
37 $\left(x-\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)^2+2$
38 $(x^2+y^2)^2-4x^2y^2$
39 $(2a+5)^3$

1.08 Factorisation

WS
Facoiing algebaic expeion

Factors divide exactly into an equal or larger number or term without leaving a remainde.

Factorising

To **factorise** an expression we use the distributive law in the opposite way from when we expand brackets

$$ax+bx=x(a+b)$$

EXAMPLE 10

Factorise

a $3x+12$
b y^2-2y
c x^3-2x^2
d $5(x+3)+2y(x+3)$
e $8a^3b^2-2ab^3$

Solution

a The highest common factor is 3 — $3x+12=3(x+4)$

b The highest common factor is y — $y^2-2y=y(y-2)$

c x and x^2 are both common factors
Take out the highest common facto, which is x^2 — $x^3-2x^2=x^2(x-2)$

d The highest common factor is $x+3$. — $5(x+3)+2y(x+3)=(x+3)(5+2y)$

e The highest common factor is $2ab^2$ — $8a^3b^2-2ab^3=2ab^2(4a^2-b)$

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Exercise 1.08 Factorisation

Factorise

1 $2y + 6$

2 $5x - 10$

3 $3m - 9$

4 $8x + 2$

5 $24 - 18y$

6 $x^2 + 2x$

7 $m^2 - 3m$

8 $2y^2 + 4y$

9 $15a - 3a^2$

10 $ab^2 + ab$

11 $4x^2y - 2xy$

12 $3mn^3 + 9mn$

13 $8x^2z - 2xz^2$

14 $6ab + 3a - 2a^2$

15 $5x^2 - 2x + xy$

16 $3q^5 - 2q^2$

17 $5b^3 + 15b^2$

18 $6a^2b^3 - 3a^3b^2$

19 $x(m + 5) + 7(m + 5)$

20 $2(y - 1) - y(y - 1)$

21 $4(7 + y) - 3x(7 + y)$

22 $6x(a - 2) + 5(a - 2)$

23 $x(2t + 1) - y(2t + 1)$

24 $a(3x - 2) + 2b(3x - 2) - 3c(3x - 2)$

25 $6x^3 + 9x^2$

26 $3pq^5 - 6q^3$

27 $15a^4b^3 + 3ab$

28 $4x^3 - 24x^2$

29 $35m^3n^4 - 25m^2n$

30 $24a^2b^5 + 16ab^2$

31 $2\pi r^2 + 2\pi rh$

32 $(x - 3)^2 + 5(x - 3)$

33 $y^2(x + 4) + 2(x + 4)$

34 $a(a + 1) - (a + 1)^2$

1.09 Factorisation by grouping in pairs

Factorising by grouping in pairs

If an expression has 4 terms it can sometimes be factorised in pair.

$$\begin{aligned} ax + bx + ay + by &= x(a + b) + y(a + b) \\ &= (a + b)(x + y) \end{aligned}$$

EXAMPLE 11

Factorise

a $x^2 - 2x + 3x - 6$ **b** $2x - 4 + 6y - 3xy$

Solution

a
$$\begin{aligned} x^2 - 2x + 3x - 6 &= x(x - 2) + 3(x - 2) \\ &= (x - 2)(x + 3) \end{aligned}$$

b
$$\begin{aligned} 2x - 4 + 6y - 3xy &= 2(x - 2) + 3y(2 - x) \\ &= 2(x - 2) - 3y(x - 2) \\ &= (x - 2)(2 - 3y) \end{aligned}$$

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Exercise 1.09 Factorisation by grouping in pairs

Factorise

1 $2x + 8 + bx + 4b$

2 $ay - 3a + by - 3b$

3 $x^2 + 5x + 2x + 10$

4 $m^2 - 2m + 3m - 6$

5 $ad - ac + bd - bc$

6 $x^3 + x^2 + 3x + 3$

7 $5ab - 3b + 10a - 6$

8 $2xy - x^2 + 2y^2 - xy$

9 $ay + a + y + 1$

10 $x^2 + 5x - x - 5$

11 $y + 3 + ay + 3a$

12 $m - 2 + 4y - 2my$

13 $2x^2 + 10xy - 3xy - 15y^2$

14 $a^2b + ab^3 - 4a - 4b^2$

15 $5x - x^2 - 3x + 15$

16 $x^4 + 7x^3 - 4x - 28$

17 $7x - 21 - xy + 3y$

18 $4d + 12 - de - 3e$

19 $3x - 12 + xy - 4y$

20 $2a + 6 - ab - 3b$

21 $x^3 - 3x^2 + 6x - 18$

22 $pq - 3p + q^2 - 3q$

23 $3x^3 - 6x^2 - 5x + 10$

24 $4a - 12b + ac - 3bc$

25 $xy + 7x - 4y - 28$

26 $x^4 - 4x^3 - 5x + 20$

27 $4x^3 - 6x^2 + 8x - 12$

28 $3a^2 + 9a + 6ab + 18b$

29 $5y - 15 + 10xy - 30x$

30 $\pi r^2 + 2\pi r - 3r - 6$

1.10 Factorising trinomials

Facoiing quadaic expeion

A **trinomial** is an expression with 3 terms for example $x^2 - 4x + 3$ Factorising a trinomial usually gives a **binomial product**

We know tha: $(x + a)(x + b) = x^2 + bx + ax + ab$

$$= x^2 + (a + b)x + ab$$

Factorising trinomials

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$

Find values for a and b so that the sum $a + b$ is the middle term and the product ab is the last term

EXAMPLE 12

Factorise

a $m^2 - 5m + 6$

b $y^2 + y - 2$

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Solution

a $a + b = -5$ and $ab = 6$

To have $a + b = -5$ at least one number must be negativ.

To have $ab = 6$ both numbers have the same sig. So both are negatie.

For $ab = 6$ we could have $-6 \times (-1)$ or $-3 \times (-2)$

$-3 + (-2) = -5$ so $a = -3$ and $b = -2$

So $m^2 - 5m + 6 = (m - 3)(m - 2)$

Check $(m - 3)(m - 2) = m^2 - 2m - 3m + 6$

$= m^2 - 5m + 6$

b $a + b = 1$ and $ab = -2$

To have $ab = -2$ the numbers must have opposite sign. So one is positive and one is negative

For $ab = -2$ we could have -2×1 or -1×2

$-1 + 2 = 1$ so $a = -1$ and $b = 2$.

So $y^2 + y - 2 = (y - 1)(y + 2)$

Check $(y - 1)(y + 2) = y^2 + 2y - y - 2$

$= y^2 + y - 2$

Exercise 1.10 Factorising trinomials

Factorise

1 $x^2 + 4x + 3$

2 $y^2 + 7y + 12$

3 $m^2 + 2m + 1$

4 $t^2 + 8t + 16$

5 $z^2 + z - 6$

6 $x^2 - 5x - 6$

7 $v^2 - 8v + 15$

8 $t^2 - 6t + 9$

9 $x^2 + 9x - 10$

10 $y^2 - 10y + 21$

11 $m^2 - 9m + 18$

12 $y^2 + 9y - 36$

13 $x^2 - 5x - 24$

14 $a^2 - 4a + 4$

15 $x^2 + 14x - 32$

16 $y^2 - 5y - 36$

17 $n^2 - 10n + 24$

18 $x^2 - 10x + 25$

19 $p^2 + 8p - 9$

20 $k^2 - 7k + 10$

21 $x^2 + x - 12$

22 $m^2 - 6m - 7$

23 $q^2 + 12q + 20$

24 $d^2 - 4d - 5$

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1.11 Further trinomials

When the coefficient of the first term is not 1 for example $5x^2 - 13x + 6$, we need to use a different method to factorise the trinomial

The coeffcent of the frst termis the numberin front of the x^2.

Facoiing quadaic expeion (Advanced)

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This method still involves finding 2 numbers that give a required sum and product but it also involves grouping in pairs

EXAMPLE 13

Factorise

a $5x^2 - 13x + 6$

b $4y^2 + 4y - 3$

Solution

a First multiply the coefficient of the first term by the last ter: $5 \times 6 = 30$

Now $a + b = -13$ and $ab = 30$

Since the sum is negative and the product is positive a and b must be both negative

2 numbers with product 30 and sum -13 are -10 and -3

Now write the trinomial with the middle term split into 2 terms $-10x$ and $-3x$ and then factorise by grouping in pairs

$$5x^2 - 13x + 6 = 5x^2 - 10x - 3x + 6$$
$$= 5x(x - 2) - 3(x - 2)$$

If you factorse correctly, you should always ind a common factor remanng such as $x - 2)$ here

$$= (x - 2)(5x - 3)$$

b First multiply the coefficient of the first term by the last ter: $4 \quad (-3) = -12$

Now $a + b = 4$ and $ab = -12$

Since the product is negative a and b have opposite signs (one positive and one negative)

2 numbers with product -12 and sum 4 are 6 and -2

Now write the trinomial with the middle term split into 2 terms $6y$ and $-2y$ and then factorise by grouping in pairs

$$4y^2 + 4y - 3 = 4y^2 + 6y - 2y - 3$$
$$= 2y(2y + 3) - 1(2y + 3)$$
$$= (2y + 3)(2y - 1)$$

There are other ways of factorising these trinomials Your teacher may show you some of these

Exercise 1.11 Further trinomials

Factorise

1 $2a^2 + 11a + 5$	**2** $5y^2 + 7y + 2$	**3** $3x^2 + 10x + 7$
4 $3x^2 + 8x + 4$	**5** $2b^2 - 5b + 3$	**6** $7x^2 - 9x + 2$
7 $3y^2 + 5y - 2$	**8** $2x^2 + 11x + 12$	**9** $5p^2 + 13p - 6$
10 $6x^2 + 13x + 5$	**11** $2y^2 - 11y - 6$	**12** $10x^2 + 3x - 1$
13 $8t^2 - 14t + 3$	**14** $6x^2 - x - 12$	**15** $6y^2 + 47y - 8$
16 $4n^2 - 11n + 6$	**17** $8t^2 + 18t - 5$	**18** $12q^2 + 23q + 10$
19 $4r^2 + 11r - 3$	**20** $4x^2 - 4x - 15$	**21** $6y^2 - 13y + 2$
22 $6p^2 - 5p - 6$	**23** $8x^2 + 31x + 21$	**24** $12b^2 - 43b + 36$
25 $6x^2 - 53x - 9$	**26** $9x^2 + 30x + 25$	**27** $16y^2 + 24y + 9$
28 $25k^2 - 20k + 4$	**29** $36a^2 - 12a + 1$	**30** $49m^2 + 84m + 36$

1.12 Perfect squares

You have looked at expanding $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$
These are called **perfect squares**

When factorising use these results the other way aroun.

EXAMPLE 14

Factorise

a $x^2 - 8x + 16$

b $4a^2 + 20a + 25$

Solution

a $x^2 - 8x + 16 = x^2 - 2(4)x + 4^2$
$= (x - 4)^2$

b $4a^2 + 20a + 25 = (2a)^2 + 2(2a)(5) + 5^2$
$= (2a + 5)^2$

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Exercise 1.12 Perfect squares

Factorise

1 $y^2 - 2y + 1$

2 $x^2 + 6x + 9$

3 $m^2 + 10m + 25$

4 $t^2 - 4t + 4$

5 $x^2 - 12x + 36$

6 $4x^2 + 12x + 9$

7 $16b^2 - 8b + 1$

8 $9a^2 + 12a + 4$

9 $25x^2 - 40x + 16$

10 $49y^2 + 14y + 1$

11 $9y^2 - 30y + 25$

12 $16k^2 - 24k + 9$

13 $25x^2 + 10x + 1$

14 $81a^2 - 36a + 4$

15 $49m^2 + 84m + 36$

16 $t^2 + t + \frac{1}{4}$

17 $x^2 - \frac{4x}{3} + \frac{4}{9}$

18 $9y^2 + \frac{6y}{5} + \frac{1}{25}$

19 $x^2 + 2 + \frac{1}{x^2}$

20 $25k^2 - 20 + \frac{4}{k^2}$

1.13 Difference of two squares

Difference of two squares

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLE 15

Factorise

a $d^2 - 36$ **b** $1 - 9b^2$ **c** $(a + 3)^2 - (b - 1)^2$

Solution

a $d^2 - 36 = d^2 - 6^2$

$= (d + 6)(d - 6)$

b $1 - 9b^2 = 1^2 - (3b)^2$

$= (1 + 3b)(1 - 3b)$

c $(a + 3)^2 - (b - 1)^2 = [(a + 3) + (b - 1)][(a + 3) - (b - 1)]$

$= (a + 3 + b - 1)(a + 3 - b + 1)$

$= (a + b + 2)(a - b + 4)$

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Exercise 1.13 Difference of two squares

Factorise

1 $a^2 - 4$	**2** $x^2 - 9$	**3** $y^2 - 1$
4 $x^2 - 25$	**5** $4x^2 - 49$	**6** $16y^2 - 9$
7 $1 - 4z^2$	**8** $25t^2 - 1$	**9** $9t^2 - 4$
10 $9 - 16x^2$	**11** $x^2 - 4y^2$	**12** $36x^2 - y^2$
13 $4a^2 - 9b^2$	**14** $x^2 - 100y^2$	**15** $4a^2 - 81b^2$
16 $(x+2)^2 - y^2$	**17** $(a-1)^2 - (b-2)^2$	**18** $z^2 - (1+w)^2$
19 $x^2 - \frac{1}{4}$	**20** $\frac{y^2}{9} - 1$	**21** $(x+2)^2 - (2y+1)^2$
22 $x^4 - 1$	**23** $9x^6 - 4y^2$	**24** $x^4 - 16y^4$

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Facoiing expeion

1.14 Mixed factorisation

EXAMPLE 16

Factorise $5x^2 - 45$

Solution

Using simple factors $\quad 5x^2 - 45 = 5(x^2 - 9)$

The difference of 2 squares $\quad = 5(x+3)(x-3)$

Exercise 1.14 Mixed factorisation

Factorise

1 $4a^3 - 36a$	**2** $2x^2 - 18$	**3** $3p^2 - 3p - 36$
4 $5y^2 - 5$	**5** $5a^2 - 10a + 5$	**6** $3z^3 + 27z^2 + 60z$
7 $9ab - 4a^3b^3$	**8** $x^3 - x$	**9** $6x^2 + 8x - 8$
10 $y^2(y+5) - 16(y+5)$	**11** $x^4 + 8x^3 - x^2 - 8x$	**12** $y^6 - 4$
13 $x^3 - 3x^2 - 10x$	**14** $x^3 - 3x^2 - 9x + 27$	**15** $4x^2y^3 - y$
16 $24 - 6b^2$	**17** $18x^2 + 33x - 30$	**18** $3x^2 - 6x + 3$

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19 $x^3 + 2x^2 - 25x - 50$

20 $z^3 + 6z^2 + 9z$

21 $3y^2 + 30y + 75$

22 $ab^2 - 9a$

23 $4k^3 + 40k^2 + 100k$

24 $3x^3 + 9x^2 - 3x - 9$

25 $4a^3b + 8a^2b^2 - 4ab^2 - 2a^2b$

1.15 Simplifying algebraic fractions

EXAMPLE 17

Simplify

a $\frac{4x+2}{2}$

b $\frac{2x^2 - 3x - 2}{x^2 - 4}$

Solution

a $\frac{4x+2}{2} = \frac{2(2x+1)}{2}$

$= 2x + 1$

b Factorise both top and bottom

$\frac{2x^2 - 3x - 2}{x^2 - 4} = \frac{(2x+1)(x-2)}{(x-2)(x+2)}$

$= \frac{2x+1}{x+2}$

Exercise 1.15 Simplifying algebraic fractions

Simplify

1 $\frac{5a+10}{5}$

2 $\frac{6t-3}{3}$

3 $\frac{8y+2}{6}$

4 $\frac{8}{4d-2}$

5 $\frac{x^2}{5x^2-2x}$

6 $\frac{y-4}{y^2-8y+16}$

7 $\frac{2ab-4a^2}{a^2-3a}$

8 $\frac{s^2+s-2}{s^2+5s+6}$

9 $\frac{b^4-1}{b^2-1}$

10 $\frac{2p^2+7p-15}{6p-9}$

11 $\frac{a^2-1}{a^2+2a-3}$

12 $\frac{3(x-2)+y(x-2)}{x^2-4}$

13 $\frac{x^3+3x^2-9x-27}{x^2+6x+9}$

14 $\frac{2p^2-3p-2}{2p^2+p}$

15 $\frac{ay-ax+by-bx}{2ay-by-2ax+bx}$

1.16 Operations with algebraic fractions

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Algebaic acion

EXAMPLE 18

Simplify

a $\frac{x-1}{5}-\frac{x+3}{4}$ **b** $\frac{2a^2b+10ab}{b^2-9}\div\frac{a^2-25}{4b+12}$ **c** $\frac{2}{x-5}+\frac{1}{x+2}$ **d** $\frac{2}{x+1}-\frac{1}{x^2-1}$

Solution

a
$$\begin{aligned}\frac{x-1}{5}-\frac{x+3}{4}&=\frac{4(x-1)-5(x+3)}{20}\\&=\frac{4x-4-5x-15}{20}\\&=\frac{-x-19}{20}\end{aligned}$$

b
$$\begin{aligned}\frac{2a^2b+10ab}{b^2-9}\div\frac{a^2-25}{4b+12}&=\frac{2a^2b+10ab}{b^2-9}\times\frac{4b+12}{a^2-25}\\&=\frac{2ab(a+5)}{(b+3)(b-3)}\times\frac{4(b+3)}{(a+5)(a-5)}\\&=\frac{8ab}{(a-5)(b-3)}\end{aligned}$$

c
$$\begin{aligned}\frac{2}{x-5}+\frac{1}{x+2}&=\frac{2(x+2)+1(x-5)}{(x-5)(x+2)}\\&=\frac{2x+4+x-5}{(x-5)(x+2)}\\&=\frac{3x-1}{(x-5)(x+2)}\end{aligned}$$

d
$$\begin{aligned}\frac{2}{x+1}-\frac{1}{x^2-1}&=\frac{2}{x+1}-\frac{1}{(x+1)(x-1)}\\&=\frac{2(x-1)}{(x+1)(x-1)}-\frac{1}{(x+1)(x-1)}\\&=\frac{2x-2}{(x+1)(x-1)}-\frac{1}{(x+1)(x-1)}\\&=\frac{2x-2-1}{(x+1)(x-1)}\\&=\frac{2x-3}{(x+1)(x-1)}\end{aligned}$$

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Exercise 1.16 Operations with algebraic fractions

1 Simplify

a $\frac{x}{2}+\frac{3x}{4}$ **b** $\frac{y+1}{5}+\frac{2y}{3}$ **c** $\frac{a+2}{3}-\frac{a}{4}$

d $\frac{p-3}{6}+\frac{p+2}{2}$ **e** $\frac{x-5}{2}-\frac{x-1}{3}$

2 Simplify

a $\frac{3x+6}{5}\times\frac{10}{x+2}$ **b** $\frac{a^2-4}{3}\times\frac{5b}{a+2}$

c $\frac{t^2+3t-10}{xy^2}\div\frac{5t-10}{2xy}$ **d** $\frac{2a-6}{2x+4}\times\frac{5x+10}{4}$

e $\frac{5x+10-xy-2y}{15}\div\frac{7x+14}{3}$ **f** $\frac{3}{b+2}\times\frac{b^2+2b}{6a-3}$

g $\frac{3ab^2}{5xy}\div\frac{12ab-6a}{x^2y+2xy^2}$ **h** $\frac{ax-ay+bx-by}{x^2-y^2}\times\frac{x^2y+xy^2}{ab^2+a^2b}$

i $\frac{x^2-6x+9}{x^2-25}\div\frac{x^2-5x+6}{x^2+4x-5}$ **j** $\frac{p^2-4}{q^2+2q+1}\times\frac{5q+5}{3p+6}$

3 Simplify

a $\frac{2}{x}+\frac{3}{x}$ **b** $\frac{1}{x-1}-\frac{2}{x}$ **c** $1+\frac{3}{a+b}$

d $x-\frac{x^2}{x+2}$ **e** $p-q+\frac{1}{p+q}$ **f** $\frac{1}{x+1}+\frac{1}{x-3}$

g $\frac{2}{x^2-4}-\frac{3}{x+2}$ **h** $\frac{1}{a^2+2a+1}+\frac{1}{a+1}$

4 Simplify

a $\frac{a^2-5a}{y^2-4y+4}\div\frac{3a-15}{y^2-4}\times\frac{y^2-y-2}{5ay}$ **b** $\frac{3}{x-3}+\frac{2x+8}{x^2-9}\times\frac{x^2+3x}{4x-16}$

c $\frac{5b}{2b+6}\div\frac{b^2}{b^2+b-6}-\frac{b}{b+1}$ **d** $\frac{x^2-8x+15}{5x^2+10x}\div\frac{x^2-9}{10x^2}\times\frac{x^2+5x+6}{2x-10}$

5 Simplify

a $\frac{5}{x^2-4}-\frac{3}{x-2}-\frac{2}{x+2}$ **b** $\frac{2}{p^2+pq}+\frac{3}{pq-q^2}$ **c** $\frac{a}{a+b}-\frac{b}{a-b}+\frac{1}{a^2-b^2}$

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1.17 Substitution

Algebra is used for writing general formulas or rules and we substitute numbers into these formulas to solve a problem

EXAMPLE 19

a $V = \pi r^2 h$ is the formula for finding the volume of a cylinder with radius r and height h
Find V (correct to 1 decimal place) when $r = 21$ and $h = 87$

b If $F = \frac{9C}{5} + 32$ is the formula for converting degrees Celsius (°C) into degrees Fahrenheit (°F) find F when $C = 25$

Solution

a When $r = 21$, $h = 87$

$$V = \pi r^2 h$$
$$= \pi(21)^2(87)$$
$$= 120533$$
$$\approx 1205$$

b When $C = 25$

$$F = \frac{9C}{5} + 32$$
$$= \frac{9(25)}{5} + 32$$
$$= 77$$

Ths means that 25 °C s the same as 77 °F.

Exercise 1.17 Substitution

1 Given $a = 31$ and $b = -23$ find correct to 1 decimal plac:

a ab **b** $3b$ **c** $5a^2$ **d** ab^3
e $(a + b)^2$ **f** $\sqrt{a - b}$ **g** $-b^2$

2 For the formula $T = a + (n - 1)d$ find T when $a = -4$, $n = 18$ and $d = 3$.

3 Given $y = mx + c$ the equation of a straight lin, find y if $m = 3$, $x = -2$ and $c = -1$

4 If $h = 100t - 5t^2$ is the height of a particle at time t find h when $t = 5$.

5 Given vertical velocity $v = -gt$ find v when $g = 98$ and $t = 20$

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6 If $y = 2^x + 3$ is the equation of a function find y when $x = 13$ correct to 1 decimal plac.

7 $S = 2\pi r(r + h)$ is the formula for the surface area of a cylinder. Find S when $r = 5$ and $h = 7$ correct to the nearest whole numbe.

8 $A = \pi r^2$ is the area of a circle with radius r Find A when $r = 95$ correct to 3 significant figures

9 For the formula $u = ar^{n-1}$ find u if $a = 5, r = -2$ and $n = 4$

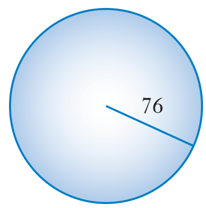
10 Given $V = \frac{1}{3} lbh$ is the volume formula for a rectangular pyramid find V if $l = 47$, $b = 5.1$ and $h = 65$

11 The gradient of a straight line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$ Find m if $x \ = 3, x_2 = -1, y \ = -2$ and $y_2 = 5$.

12 If $A = \frac{1}{2}h(a + b)$ gives the area of a trapezium find A when $h = 7, a = 25$ and $b = 3.9$.

13 $V = \frac{4}{3}\pi r^3$ is the volume formula for a sphere with radius r

Find V to 1 decimal place for a sphere with radius $r = 76$



14 The velocity of an object at time t is given by the formula $v = u + at$

Find v when $u = \frac{1}{4}$ $a = \frac{3}{5}$ and $t = \frac{5}{6}$

15 Given $S = \frac{a}{1-r}$ find S if $a = 5$ and $r = \frac{2}{3}$ S is the sum to infinity of a geometric series

16 $c = \sqrt{a^2 + b^2}$ according to Pythagora' theore. Find the value of c if $a = 6$ and $b = 8$.

17 Given $y = \sqrt{16 - x^2}$ is the equation of a semicircle find the exact value of y when $x = 2$.

18 Find the value of E in the energy equation $E = mc^2$ if $m = 83$ and $c = 1.7$.

19 $A = P\left(1 + \frac{r}{100}\right)^n$ is the formula for finding compound interest Find A correct to 2 decimal places when $P = 200, r = 12$ and $n = 5$.

20 If $S = \frac{a(r^n - 1)}{r - 1}$ is the sum of a geometric series find S if $a = 3, r = 2$ and $n = 5$.

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1.18 Simplifying surds

An **irrational number** is a number that cannot be written as a ratio or fraction

Surds such as $\sqrt{2}$ $\sqrt{3}$ and $\sqrt{5}$ are special types of irrational numbers

If a question involving surds asks for an exact answer, then leave it as a sud.

Properties of surds

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

$$\left(\sqrt{x}\right)^2 = \sqrt{x^2} = x \text{ for } x \geq 0$$

EXAMPLE 20

a Express $\sqrt{45}$ in simplest surd form

b Simplify $3\sqrt{40}$

c Write $5\sqrt{2}$ as a single surd

Solution

a

$$\begin{aligned}\sqrt{45} &= \sqrt{9 \times 5} \\ &= \sqrt{9} \times \sqrt{5} \\ &= 3 \times \sqrt{5} \\ &= 3\sqrt{5}\end{aligned}$$

b

$$\begin{aligned}3\sqrt{40} &= 3 \times \sqrt{4} \times \sqrt{10} \\ &= 3 \times 2 \times \sqrt{10} \\ &= 6\sqrt{10}\end{aligned}$$

c

$$\begin{aligned}5\sqrt{2} &= \sqrt{25} \times \sqrt{2} \\ &= \sqrt{50}\end{aligned}$$

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Exercise 1.18 Simplifying surds

1 Express these surds in simplest surd form

a $\sqrt{12}$ **b** $\sqrt{63}$ **c** $\sqrt{24}$ **d** $\sqrt{50}$ **e** $\sqrt{72}$

f $\sqrt{200}$ **g** $\sqrt{48}$ **h** $\sqrt{75}$ **i** $\sqrt{32}$ **j** $\sqrt{54}$

k $\sqrt{112}$ $\sqrt{300}$ **m** $\sqrt{128}$ **n** $\sqrt{243}$ **o** $\sqrt{245}$

p $\sqrt{108}$ **q** $\sqrt{99}$ **r** $\sqrt{125}$

2 Simplify

a $2\sqrt{27}$ **b** $5\sqrt{80}$ **c** $4\sqrt{98}$ **d** $2\sqrt{28}$ **e** $8\sqrt{20}$

f $4\sqrt{56}$ **g** $8\sqrt{405}$ **h** $15\sqrt{8}$ **i** $7\sqrt{40}$ **j** $8\sqrt{45}$

3 Write as a single sur:

a $3\sqrt{2}$ **b** $2\sqrt{5}$ **c** $4\sqrt{11}$ **d** $8\sqrt{2}$ **e** $5\sqrt{3}$

f $4\sqrt{10}$ **g** $3\sqrt{13}$ **h** $7\sqrt{2}$ **i** $11\sqrt{3}$ **j** $12\sqrt{7}$

4 Evaluate x if

a $\sqrt{x}=3\sqrt{5}$ **b** $2\sqrt{3}=\sqrt{x}$ **c** $3\sqrt{7}=\sqrt{x}$ **d** $5\sqrt{2}=\sqrt{x}$ **e** $2\sqrt{11}=\sqrt{x}$

f $\sqrt{x}=7\sqrt{3}$ **g** $4\sqrt{19}=\sqrt{x}$ **h** $\sqrt{x}=6\sqrt{23}$ **i** $5\sqrt{31}=\sqrt{x}$ **j** $\sqrt{x}=8\sqrt{15}$

1.19 Operations with surds

EXAMPLE 21

Simplify $\sqrt{3}-\sqrt{12}$

Solution

First change into like surd.

$$\begin{aligned}\sqrt{3}-\sqrt{12}&=\sqrt{3}-\sqrt{4}\times\sqrt{3}\\&=\sqrt{3}-2\sqrt{3}\\&=-\sqrt{3}\end{aligned}$$

Multiplication and division as in algebr, are easier to do than adding and subtractig.

EXAMPLE 22

Simplify

a $4\sqrt{2}\times 5\sqrt{18}$

b $\dfrac{2\sqrt{14}}{4\sqrt{2}}$

c $\left(\sqrt{\dfrac{10}{3}}\right)^2$

Solution

a $$\begin{aligned}4\sqrt{2}\times 5\sqrt{18} &= 20\sqrt{36}\\ &= 20\times 6\\ &= 120\end{aligned}$$

b $$\begin{aligned}\frac{2\sqrt{14}}{4\sqrt{2}} &= \frac{2\times\sqrt{7}}{4}\\ &= \frac{\sqrt{7}}{2}\end{aligned}$$

c $$\begin{aligned}\left(\sqrt{\frac{10}{3}}\right)^2 &= \frac{10}{3}\\ &= 3\frac{1}{3}\end{aligned}$$

EXAMPLE 23

Expand and simplify

a $3\sqrt{7}(2\sqrt{3}-3\sqrt{2})$

b $(\sqrt{2}+3\sqrt{5})(\sqrt{3}-\sqrt{2})$

c $(\sqrt{5}+2\sqrt{3})(\sqrt{5}-2\sqrt{3})$

Solution

a $$\begin{aligned}3\sqrt{7}(2\sqrt{3}-3\sqrt{2}) &= 3\sqrt{7}\times 2\sqrt{3}-3\sqrt{7}\times 3\sqrt{2}\\ &= 6\sqrt{21}-9\sqrt{14}\end{aligned}$$

b $$\begin{aligned}(\sqrt{2}+3\sqrt{5})(\sqrt{3}-\sqrt{2}) &= \sqrt{2}\times\sqrt{3}-\sqrt{2}\times\sqrt{2}+3\sqrt{5}\times\sqrt{3}-3\sqrt{5}\times\sqrt{2}\\ &= \sqrt{6}-2+3\sqrt{15}-3\sqrt{10}\end{aligned}$$

c Using the difference of 2 squares $$\begin{aligned}(\sqrt{5}+2\sqrt{3})(\sqrt{5}-2\sqrt{3}) &= \left(\sqrt{5}\right)^2-\left(2\sqrt{3}\right)^2\\ &= 5-4\times 3\\ &= -7\end{aligned}$$

Exercise 1.19 Operations with surds

1 Simplify

a $\sqrt{5}+2\sqrt{5}$

b $3\sqrt{2}-2\sqrt{2}$

c $\sqrt{3}+5\sqrt{3}$

d $7\sqrt{3}-4\sqrt{3}$

e $\sqrt{5}-4\sqrt{5}$

f $4\sqrt{6}-\sqrt{6}$

g $\sqrt{2}-8\sqrt{2}$

h $\sqrt{5}+4\sqrt{5}+3\sqrt{5}$

i $\sqrt{2}-2\sqrt{2}-3\sqrt{2}$

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j $\sqrt{5}+\sqrt{45}$ **k** $\sqrt{8}-\sqrt{2}$ $\sqrt{3}+\sqrt{48}$

m $\sqrt{12}-\sqrt{27}$ **n** $\sqrt{50}-\sqrt{32}$ **o** $\sqrt{28}+\sqrt{63}$

p $2\sqrt{8}-\sqrt{18}$ **q** $3\sqrt{54}+2\sqrt{24}$ **r** $\sqrt{90}-5\sqrt{40}-2\sqrt{10}$

s $4\sqrt{48}+3\sqrt{147}+5\sqrt{12}$ **t** $3\sqrt{2}+\sqrt{8}-\sqrt{12}$ **u** $\sqrt{63}-\sqrt{28}-\sqrt{50}$

v $\sqrt{12}-\sqrt{45}-\sqrt{48}-\sqrt{5}$

2 Simplify

a $\sqrt{7}\times\sqrt{3}$ **b** $\sqrt{3}\times\sqrt{5}$ **c** $\sqrt{2}\times3\sqrt{3}$

d $5\sqrt{7}\times2\sqrt{2}$ **e** $-3\sqrt{3}\times2\sqrt{2}$ **f** $5\sqrt{3}\times2\sqrt{3}$

g $-4\sqrt{5}\times3\sqrt{11}$ **h** $2\sqrt{7}\times\sqrt{7}$ **i** $2\sqrt{3}\times5\sqrt{12}$

j $\sqrt{6}\times\sqrt{2}$ **k** $\left(\sqrt{2}\right)^2$ $\left(2\sqrt{7}\right)^2$

m $\sqrt{3}\times\sqrt{5}\times\sqrt{2}$ **n** $2\sqrt{3}\times\sqrt{7}\times-\sqrt{5}$ **o** $\sqrt{2}\times\sqrt{6}\times3\sqrt{3}$

3 Simplify

a $\dfrac{4\sqrt{12}}{2\sqrt{2}}$ **b** $\dfrac{12\sqrt{18}}{3\sqrt{6}}$ **c** $\dfrac{5\sqrt{8}}{10\sqrt{2}}$ **d** $\dfrac{16\sqrt{2}}{2\sqrt{12}}$

e $\dfrac{10\sqrt{30}}{5\sqrt{10}}$ **f** $\dfrac{2\sqrt{2}}{6\sqrt{20}}$ **g** $\dfrac{4\sqrt{2}}{8\sqrt{10}}$ **h** $\dfrac{\sqrt{3}}{3\sqrt{15}}$

i $\dfrac{\sqrt{2}}{\sqrt{8}}$ **j** $\dfrac{3\sqrt{15}}{6\sqrt{10}}$ **k** $\dfrac{5\sqrt{12}}{5\sqrt{8}}$ $\dfrac{15\sqrt{18}}{10\sqrt{10}}$

m $\dfrac{\sqrt{15}}{2\sqrt{6}}$ **n** $\left(\sqrt{\dfrac{2}{3}}\right)^2$ **o** $\left(\sqrt{\dfrac{5}{7}}\right)^2$

4 Expand and simplify

a $\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)$ **b** $\sqrt{3}\left(2\sqrt{2}-\sqrt{5}\right)$ **c** $4\sqrt{3}\left(\sqrt{3}+2\sqrt{5}\right)$

d $\sqrt{7}\left(5\sqrt{2}-2\sqrt{3}\right)$ **e** $-\sqrt{3}\left(\sqrt{2}-4\sqrt{6}\right)$ **f** $\sqrt{3}\left(5\sqrt{11}+3\sqrt{7}\right)$

g $-3\sqrt{2}\left(\sqrt{2}+4\sqrt{3}\right)$ **h** $\sqrt{5}\left(\sqrt{5}-5\sqrt{3}\right)$ **i** $\sqrt{3}\left(\sqrt{12}+\sqrt{10}\right)$

j $2\sqrt{3}\left(\sqrt{18}+\sqrt{3}\right)$ **k** $-4\sqrt{2}\left(\sqrt{2}-3\sqrt{6}\right)$ $-7\sqrt{5}\left(-3\sqrt{20}+2\sqrt{3}\right)$

m $10\sqrt{3}\left(\sqrt{2}-2\sqrt{12}\right)$ **n** $-\sqrt{2}\left(\sqrt{5}+2\right)$ **o** $2\sqrt{3}\left(2-\sqrt{12}\right)$

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5 Expand and simplify

a $\left(\sqrt{2}+3\right)\left(\sqrt{5}+3\sqrt{3}\right)$ **b** $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{2}-\sqrt{7}\right)$ **c** $\left(\sqrt{2}+5\sqrt{3}\right)\left(2\sqrt{5}-3\sqrt{2}\right)$

d $\left(3\sqrt{10}-2\sqrt{5}\right)\left(4\sqrt{2}+6\sqrt{6}\right)$ **e** $\left(2\sqrt{5}-7\sqrt{2}\right)\left(\sqrt{5}-3\sqrt{2}\right)$ **f** $\left(\sqrt{5}+6\sqrt{2}\right)\left(3\sqrt{5}-\sqrt{3}\right)$

g $\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)$ **h** $\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)$ **i** $\left(\sqrt{6}+3\sqrt{2}\right)\left(\sqrt{6}-3\sqrt{2}\right)$

j $\left(3\sqrt{5}+\sqrt{2}\right)\left(3\sqrt{5}-\sqrt{2}\right)$ **k** $\left(\sqrt{8}-\sqrt{5}\right)\left(\sqrt{8}+\sqrt{5}\right)$ $\left(\sqrt{2}+9\sqrt{3}\right)\left(\sqrt{2}-9\sqrt{3}\right)$

m $\left(2\sqrt{11}+5\sqrt{2}\right)\left(2\sqrt{11}-5\sqrt{2}\right)$ **n** $\left(\sqrt{5}+\sqrt{2}\right)^2$

o $\left(2\sqrt{2}-\sqrt{3}\right)^2$ **p** $\left(3\sqrt{2}+\sqrt{7}\right)^2$ **q** $\left(2\sqrt{3}+3\sqrt{5}\right)^2$

r $\left(\sqrt{7}-2\sqrt{5}\right)^2$ **s** $\left(2\sqrt{8}-3\sqrt{5}\right)^2$ **t** $\left(3\sqrt{5}+2\sqrt{2}\right)^2$

6 If $a=3\sqrt{2}$ simplif:

a a^2 **b** $2a^3$ **c** $(2a)^3$

d $(a+1)^2$ **e** $(a+3)(a-3)$

7 Evaluate a and b if

a $\left(2\sqrt{5}+1\right)^2=a+\sqrt{b}$ **b** $\left(2\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}-3\sqrt{5}\right)=a+b\sqrt{10}$

8 Expand and simplify

a $\left(\sqrt{a+3}-2\right)\left(\sqrt{a+3}+2\right)$ **b** $\left(\sqrt{p-1}-\sqrt{p}\right)^2$

9 Evaluate $\left(2\sqrt{7}-\sqrt{3}\right)\left(2\sqrt{7}+\sqrt{3}\right)$

10 Simplify $\left(2\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-3\sqrt{y}\right)$

11 If $\left(2\sqrt{3}-\sqrt{5}\right)^2=a-\sqrt{b}$ evaluate a and b

12 Evaluate a and b if $\left(7\sqrt{2}-3\right)^2=a+b\sqrt{2}$

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1.20 Rationalising the denominator

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Rationalising the denominator of a fractional surd means writing it with a rational number (not a surd) in the denominator. For exampe, after rationalising the denominator $\frac{3}{\sqrt{5}}$ becomes $\frac{3\sqrt{5}}{5}$

To rationalise the denominato, multiply top and bottom by the same surd as in the denominaor:

Rationalising the denominator

$$\frac{a}{\sqrt{b}}\times\frac{\sqrt{b}}{\sqrt{b}}=\frac{a\sqrt{b}}{b}$$

EXAMPLE 24

Rationalise the denominator of $\frac{2}{5\sqrt{3}}$

Solution

$$\begin{aligned}\frac{2}{5\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}&=\frac{2\sqrt{3}}{5\sqrt{9}}\\&=\frac{2\sqrt{3}}{5\times 3}\\&=\frac{2\sqrt{3}}{15}\end{aligned}$$

When there is a binomial denominator, we use the difference of 2 squares to rationalise t.

Rationalising a binomial denominator

To rationalise the denominator of $\frac{b}{\sqrt{c}+\sqrt{d}}$ multiply by $\frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}-\sqrt{d}}$

To rationalise the denominator of $\frac{b}{\sqrt{c}-\sqrt{d}}$ multiply by $\frac{\sqrt{c}+\sqrt{d}}{\sqrt{c}+\sqrt{d}}$

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EXAMPLE 25

a Write with a rational denominato:

i $\dfrac{\sqrt{5}}{\sqrt{2}-3}$ **ii** $\dfrac{2\sqrt{3}+\sqrt{5}}{\sqrt{3}+4\sqrt{2}}$

b Evaluate a and b if $\dfrac{3\sqrt{3}}{\sqrt{3}-\sqrt{2}}=a+\sqrt{b}$

c Evaluate $\dfrac{2}{\sqrt{3}+2}+\dfrac{\sqrt{5}}{\sqrt{3}-2}$ as a fraction with rational denominator.

Solution

a i

$$\begin{aligned}\frac{\sqrt{5}}{\sqrt{2}-3}\times\frac{\sqrt{2}+3}{\sqrt{2}+3}&=\frac{\sqrt{5}\left(\sqrt{2}+3\right)}{\left(\sqrt{2}\right)^2-3^2}\\&=\frac{\sqrt{10}+3\sqrt{5}}{2-9}\\&=-\frac{\sqrt{10}+3\sqrt{5}}{7}\end{aligned}$$

ii

$$\begin{aligned}\frac{2\sqrt{3}+\sqrt{5}}{\sqrt{3}+4\sqrt{2}}\times\frac{\sqrt{3}-4\sqrt{2}}{\sqrt{3}-4\sqrt{2}}&=\frac{\left(2\sqrt{3}+\sqrt{5}\right)\left(\sqrt{3}-4\sqrt{2}\right)}{\left(\sqrt{3}\right)^2-\left(4\sqrt{2}\right)^2}\\&=\frac{2\times3-8\sqrt{6}+\sqrt{15}-4\sqrt{10}}{3-16\times2}\\&=\frac{6-8\sqrt{6}+\sqrt{15}-4\sqrt{10}}{-29}\\&=\frac{-6+8\sqrt{6}-\sqrt{15}+4\sqrt{10}}{29}\end{aligned}$$

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b $\dfrac{3\sqrt{3}}{\sqrt{3}-\sqrt{2}} \times \dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \dfrac{3\sqrt{3}\left(\sqrt{3}+\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)}$

$$= \frac{3\sqrt{9}+3\sqrt{6}}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{3\times3+3\sqrt{6}}{3-2}$$

$$= \frac{9+3\sqrt{6}}{1}$$

$$= 9+3\sqrt{6}$$

$$= 9+\sqrt{9}\times\sqrt{6}$$

$$= 9+\sqrt{54}$$

So $a = 9$ and $b = 54$

c $\dfrac{2}{\sqrt{3}+2} + \dfrac{\sqrt{5}}{\sqrt{3}-2} = \dfrac{2\left(\sqrt{3}-2\right)+\sqrt{5}\left(\sqrt{3}+2\right)}{\left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right)}$

$$= \frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{\left(\sqrt{3}\right)^2-2^2}$$

$$= \frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{3-4}$$

$$= \frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{-1}$$

$$= -2\sqrt{3}+4-\sqrt{15}-2\sqrt{5}$$

Exercise 1.20 Rationalising the denominator

1 Express with a rational denominator

a $\dfrac{1}{\sqrt{7}}$ **b** $\dfrac{\sqrt{3}}{2\sqrt{2}}$ **c** $\dfrac{2\sqrt{3}}{\sqrt{5}}$ **d** $\dfrac{6\sqrt{7}}{5\sqrt{2}}$

e $\dfrac{1+\sqrt{2}}{\sqrt{3}}$ **f** $\dfrac{\sqrt{6}-5}{\sqrt{2}}$ **g** $\dfrac{\sqrt{5}+2\sqrt{2}}{\sqrt{5}}$ **h** $\dfrac{3\sqrt{2}-4}{2\sqrt{7}}$

i $\dfrac{8+3\sqrt{2}}{4\sqrt{5}}$ **j** $\dfrac{4\sqrt{3}-2\sqrt{2}}{7\sqrt{5}}$

2 Express with a rational denominator

a $\frac{4}{\sqrt{3}+\sqrt{2}}$ **b** $\frac{\sqrt{3}}{\sqrt{2}-7}$ **c** $\frac{2\sqrt{3}}{\sqrt{5}+2\sqrt{6}}$

d $\frac{\sqrt{3}-4}{\sqrt{3}+4}$ **e** $\frac{\sqrt{2}+5}{\sqrt{3}-\sqrt{2}}$ **f** $\frac{3\sqrt{3}+\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}$

3 Express as a single fraction with a rational denominator

a $\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{2}-1}$ **b** $\frac{\sqrt{2}}{\sqrt{2}-\sqrt{3}}-\frac{3}{\sqrt{2}+\sqrt{3}}$

c $t+\frac{1}{t}$ where $t=\sqrt{3}-2$ **d** $z^2-\frac{1}{z^2}$ where $z=1+\sqrt{2}$

e $\frac{\sqrt{2}+3}{\sqrt{2}}+\frac{1}{\sqrt{3}}$ **f** $\frac{\sqrt{3}}{\sqrt{2}+3}+\frac{\sqrt{2}}{\sqrt{3}}$

g $\frac{\sqrt{5}}{\sqrt{6}+2}-\frac{2}{5\sqrt{3}}$ **h** $\frac{\sqrt{2}+7}{4+\sqrt{3}}-\frac{\sqrt{2}}{4-\sqrt{3}}$

i $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\frac{2+\sqrt{3}}{\sqrt{3}+1}$

4 Find a and b if

a $\frac{3}{2\sqrt{5}}=\frac{\sqrt{a}}{b}$ **b** $\frac{\sqrt{3}}{4\sqrt{2}}=\frac{a\sqrt{6}}{b}$ **c** $\frac{2}{\sqrt{5}+1}=a+b\sqrt{5}$

d $\frac{2\sqrt{7}}{\sqrt{7}-4}=a+b\sqrt{7}$ **e** $\frac{\sqrt{2}+3}{\sqrt{2}-1}=a+\sqrt{b}$

5 Show that $\frac{\sqrt{2}-1}{\sqrt{2}+1}+\frac{4}{\sqrt{2}}$ is rational

6 If $x=\sqrt{3}+2$ simplif:

a $x+\frac{1}{x}$ **b** $x^2+\frac{1}{x^2}$ **c** $\left(x+\frac{1}{x}\right)^2$

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1. TEST YOURSELF

For Questions **1** to **8** select the correct answer **A B C** or **D**

Qz
Pacice quiz

1 Rationalise the denominator of $\frac{\sqrt{3}}{2\sqrt{7}}$ (there may be more than one answer)

A $\frac{\sqrt{21}}{28}$ **B** $\frac{2\sqrt{21}}{28}$ **C** $\frac{\sqrt{21}}{14}$ **D** $\frac{\sqrt{21}}{7}$

2 Simplify $\frac{x-3}{5} - \frac{x+1}{4}$

A $\frac{-(x+7)}{20}$ **B** $\frac{x+7}{20}$ **C** $\frac{x+17}{20}$ **D** $\frac{-(x+17)}{20}$

3 Factorise $x^3 - 4x^2 - x + 4$ (there may be more than one answer)

A $(x^2 - 1)(x - 4)$ **B** $(x^2 + 1)(x - 4)$

C $x^2(x - 4)$ **D** $(x - 4)(x + 1)(x - 1)$

4 Simplify $3\sqrt{2} + 2\sqrt{98}$

A $5\sqrt{2}$ **B** $5\sqrt{10}$ **C** $17\sqrt{2}$ **D** $10\sqrt{2}$

5 Simplify $\frac{3}{x^2-4} + \frac{2}{x-2} - \frac{1}{x+2}$

A $\frac{x+5}{(x+2)(x-2)}$ **B** $\frac{x+1}{(x+2)(x-2)}$ **C** $\frac{x+9}{(x+2)(x-2)}$ **D** $\frac{x-3}{(x+2)(x-2)}$

6 Simplify $5ab - 2a^2 - 7ab - 3a^2$

A $2ab + a^2$ **B** $-2ab - 5a^2$ **C** $-13a^3b$ **D** $-2ab + 5a^2$

7 Simplify $\sqrt{\frac{80}{27}}$

A $\frac{4\sqrt{5}}{3\sqrt{3}}$ **B** $\frac{4\sqrt{5}}{9\sqrt{3}}$ **C** $\frac{8\sqrt{5}}{9\sqrt{3}}$ **D** $\frac{8\sqrt{5}}{3\sqrt{3}}$

8 Expand and simplify $(3x - 2y)^2$

A $3x^2 - 12xy - 2y^2$ **B** $9x^2 - 12xy - 4y^2$

C $3x^2 - 6xy + 2y^2$ **D** $9x^2 - 12xy + 4y^2$

9 Evaluate as a fraction

a 7^{-2} **b** 5^{-} **c** $9^{-\overline{\ }2}$

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10 Simplify

a $x^5 \times x^7 \div x^3$ **b** $(5y^3)^2$ **c** $\dfrac{(a^5)^4 b^7}{a^9 b}$ **d** $\left(\dfrac{2x^6}{3}\right)^3$ **e** $\left(\dfrac{ab^4}{a^5 b^6}\right)^0$

11 Evaluate

a $36^{\frac{}{2}}$ **b** 4^{-3} as fraction **c** $8^{\frac{2}{3}}$

d $49^{-\frac{}{2}}$ as a fraction **e** $16^{\frac{}{4}}$ **f** $(-3)^0$

12 Simplify

a $a^{14} \div a^9$ **b** $(x^5 y^3)^6$ **c** $p^6 \times p^5 \div p^2$

d $(2b^9)^4$ **e** $\dfrac{(2x^7)^3 y^2}{x^{10} y}$

13 Write in index for:

a $\sqrt{n}$ **b** $\dfrac{1}{x^5}$ **c** $\dfrac{1}{x+y}$ **d** $\sqrt[4]{x+1}$ **e** $\sqrt[7]{a+b}$

f $\dfrac{2}{x}$ **g** $\dfrac{1}{2x^3}$ **h** $\sqrt[3]{x^4}$ **i** $\sqrt[7]{(5x+3)^9}$ **j** $\dfrac{1}{\sqrt[4]{m^3}}$

14 Write without fractional or negative indice:

a a^{-5} **b** $n^{\frac{}{4}}$ **c** $(x+1)^{\frac{}{2}}$ **d** $(x-y)^{-}$ **e** $(4t-7)^{-4}$

f $(a+b)^{\frac{}{5}}$ **g** $x^{-\frac{}{3}}$ **h** $b^{\frac{3}{4}}$ **i** $(2x+3)^{\frac{4}{3}}$ **j** $x^{-\frac{3}{2}}$

15 Evaluate $a^2 b^4$ when $a = \dfrac{9}{25}$ and $b = 1\dfrac{2}{3}$

16 If $a = \left(\dfrac{1}{3}\right)^4$ and $b = \dfrac{3}{4}$ evaluate ab^3 as a fraction

17 Write in index for:

a $\sqrt{x}$ **b** $\dfrac{1}{y}$ **c** $\sqrt[6]{x+3}$ **d** $\dfrac{1}{(2x-3)^{11}}$ **e** $\sqrt[3]{y^7}$

18 Write without the negative inde:

a x^{-3} **b** $(2a+5)^{-}$ **c** $\left(\dfrac{a}{b}\right)^{-5}$

19 Simplify

a $5y - 7y$ **b** $\dfrac{3a+12}{3}$ **c** $-2k^3 \times 3k^2$ **d** $\dfrac{x}{3} + \dfrac{y}{5}$

e $4a - 3b - a - 5b$ **f** $\sqrt{8} + \sqrt{32}$ **g** $3\sqrt{5} - \sqrt{20} + \sqrt{45}$

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20 Factorise

a $x^2 - 36$ **b** $a^2 + 2a - 3$ **c** $4ab^2 - 8ab$

d $5y - 15 + xy - 3x$ **e** $4n - 2p + 6$

21 Expand and simplify

a $b + 3(b - 2)$ **b** $(2x - 1)(x + 3)$ **c** $5(m + 3) - (m - 2)$

d $(4x - 3)^2$ **e** $(p - 5)(p + 5)$ **f** $7 - 2(a + 4) - 5a$

g $\sqrt{3}\left(2\sqrt{2}-5\right)$ **h** $\left(3+\sqrt{7}\right)\left(\sqrt{3}-2\right)$

22 Simplify

a $\frac{4a-12}{5b^3}\times\frac{10b}{a^2-9}$ **b** $\frac{5m+10}{m^2-m-2}\div\frac{m^2-4}{3m+3}$

23 The volume of a cube is $V = s^3$ Evaluate V when $s = 54$

24 **a** Expand and simplify $\left(2\sqrt{5}+\sqrt{3}\right)\left(2\sqrt{5}-\sqrt{3}\right)$

b Rationalise the denominator of $\frac{3\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$

25 Simplify $\frac{3}{x-2}+\frac{1}{x+3}-\frac{2}{x^2+x-6}$

26 If $a = 4$, $b = -3$ and $c = -2$ find the value o:

a ab^2 **b** $a - bc$ **c** $\sqrt{a}$ **d** $(bc)^3$ **e** $c(2a + 3b)$

27 Simplify

a $\frac{3\sqrt{12}}{6\sqrt{15}}$ **b** $\frac{4\sqrt{32}}{2\sqrt{2}}$

28 The formula for the distance an object falls is given by $d = 5t^2$ Find d when $t = 1.5$.

29 Rationalise the denominator of

a $\frac{2}{5\sqrt{3}}$ **b** $\frac{1+\sqrt{3}}{\sqrt{2}}$

30 Expand and simplify

a $\left(3\sqrt{2}-4\right)\left(\sqrt{3}-\sqrt{2}\right)$ **b** $\left(\sqrt{7}+2\right)^2$

31 Factorise fully

a $3x^2 - 27$ **b** $6x^2 - 12x - 18$ **c** $5y^2 - 30y + 45$

32 Simplify

a $\frac{3x^4y}{9xy^5}$ **b** $\frac{5}{15x-5}$

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33 Simplify

a $\left(3\sqrt{11}\right)^2$ **b** $\left(2\sqrt{3}\right)^3$

34 Expand and simplify

a $(a+b)(a-b)$ **b** $(a+b)^2$

35 Factorise

a $a^2 - 2ab + b^2$ **b** $a^2 - b^2$

36 If $x = \sqrt{3} + 1$ simplify $x + \dfrac{1}{x}$ and give your answer with a rational denominator.

37 Simplify

a $\dfrac{4}{a} + \dfrac{3}{b}$ **b** $\dfrac{x-3}{2} - \dfrac{x-2}{5}$

38 Simplify $\dfrac{3}{\sqrt{5}+2} - \dfrac{\sqrt{2}}{2\sqrt{2}-1}$ writing your answer with a rational denominato.

39 Simplify

a $3\sqrt{8}$ **b** $-2\sqrt{2} \times 4\sqrt{3}$ **c** $\sqrt{108} - \sqrt{48}$ **d** $\dfrac{8\sqrt{6}}{2\sqrt{18}}$

e $5a \times -3b \times -2a$ **f** $\dfrac{2m^3n}{6m^2n^5}$ **g** $3x - 2y - x - y$

40 Expand and simplify

a $2\sqrt{2}\left(\sqrt{3}+\sqrt{2}\right)$ **b** $\left(5\sqrt{7}-3\sqrt{5}\right)\left(2\sqrt{2}-\sqrt{3}\right)$ **c** $\left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right)$

d $\left(4\sqrt{3}-\sqrt{5}\right)\left(4\sqrt{3}+\sqrt{5}\right)$ **e** $\left(3\sqrt{7}-\sqrt{2}\right)^2$

41 Rationalise the denominator of

a $\dfrac{3}{\sqrt{7}}$ **b** $\dfrac{\sqrt{2}}{5\sqrt{3}}$ **c** $\dfrac{2}{\sqrt{5}-1}$ **d** $\dfrac{2\sqrt{2}}{3\sqrt{2}+\sqrt{3}}$ **e** $\dfrac{\sqrt{5}+\sqrt{2}}{4\sqrt{5}-3\sqrt{3}}$

42 Simplify

a $\dfrac{3x}{5} - \dfrac{x-2}{2}$ **b** $\dfrac{a+2}{7} + \dfrac{2a-3}{3}$ **c** $\dfrac{1}{x^2-1} - \dfrac{2}{x+1}$

d $\dfrac{4}{k^2+2k-3} + \dfrac{1}{k+3}$ **e** $\dfrac{\sqrt{3}}{\sqrt{2}+\sqrt{5}} - \dfrac{5}{\sqrt{3}-\sqrt{2}}$

43 Evaluate n if

a $\sqrt{108} - \sqrt{12} = \sqrt{n}$ **b** $\sqrt{112} + \sqrt{7} = \sqrt{n}$ **c** $2\sqrt{8} + \sqrt{200} = \sqrt{n}$

d $4\sqrt{147} + 3\sqrt{75} = \sqrt{n}$ **e** $2\sqrt{245} + \dfrac{\sqrt{180}}{2} = \sqrt{n}$

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1. CHALLENGE EXERCISE

1 Write $64^{-\frac{2}{3}}$ as a rational number.

2 Show that $2(2^k - 1) + 2^{k+1} = 2(2^{k+1} - 1)$.

3 Find the value of $\frac{a}{b^3c^2}$ in index form if $a = \left(\frac{2}{5}\right)^4$ $b = \left(-\frac{1}{3}\right)^3$ and $c = \left(\frac{3}{5}\right)^2$

4 Expand and simplify

a $4ab(a - 2b) - 2a^2(b - 3a)$ **b** $(y^2 - 2)(y^2 + 2)$ **c** $(2x - 5)^3$

5 Find the value of $x + y$ with rational denominator if $x = \sqrt{3} + 1$ and $y = \frac{1}{2\sqrt{5} - 3}$

6 Simplify $\frac{2\sqrt{3}}{7\sqrt{6} - \sqrt{54}}$

7 Factorise

a $(x + 4)^2 + 5(x + 4)$ **b** $x^4 - x^2y - 6y^2$ **c** $a^2b - 2a^2 - 4b + 8$

8 Simplify $\frac{2xy + 2x - 6 - 6y}{4x^2 - 16x + 12}$

9 Simplify $\frac{(a + 1)^3}{a^2 - 1}$

10 Factorise $\frac{4}{x^2} - \frac{a^2}{b^2}$

11 **a** Expand $(2x - 1)^3$

b Hence or otherwis, simplify $\frac{6x^2 + 5x - 4}{8x^3 - 12x^2 + 6x - 1}$

12 If $V = \pi r^2 h$ is the volume of a cylinder, find the exact value of r when $V = 9$ and $h = 16$.

13 If $s = u + \frac{1}{2}at^2$ find the exact value of s when $u = 2$, $a = \sqrt{3}$ and $t = 2\sqrt{3}$

14 Expand and simplify, and write in index fom:

a $\left(\sqrt{x} + x\right)^2$ **b** $\left(\sqrt[3]{a} + \sqrt[3]{b}\right)\left(\sqrt[3]{a} - \sqrt[3]{b}\right)$

c $\left(p + \frac{1}{\sqrt{p}}\right)^2$ **d** $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

15 Find the value of $\frac{a^3b^2}{c^2}$ if $a = \left(\frac{3}{4}\right)^2$ $b = \left(\frac{2}{3}\right)^3$ and $c = \left(\frac{1}{2}\right)^4$

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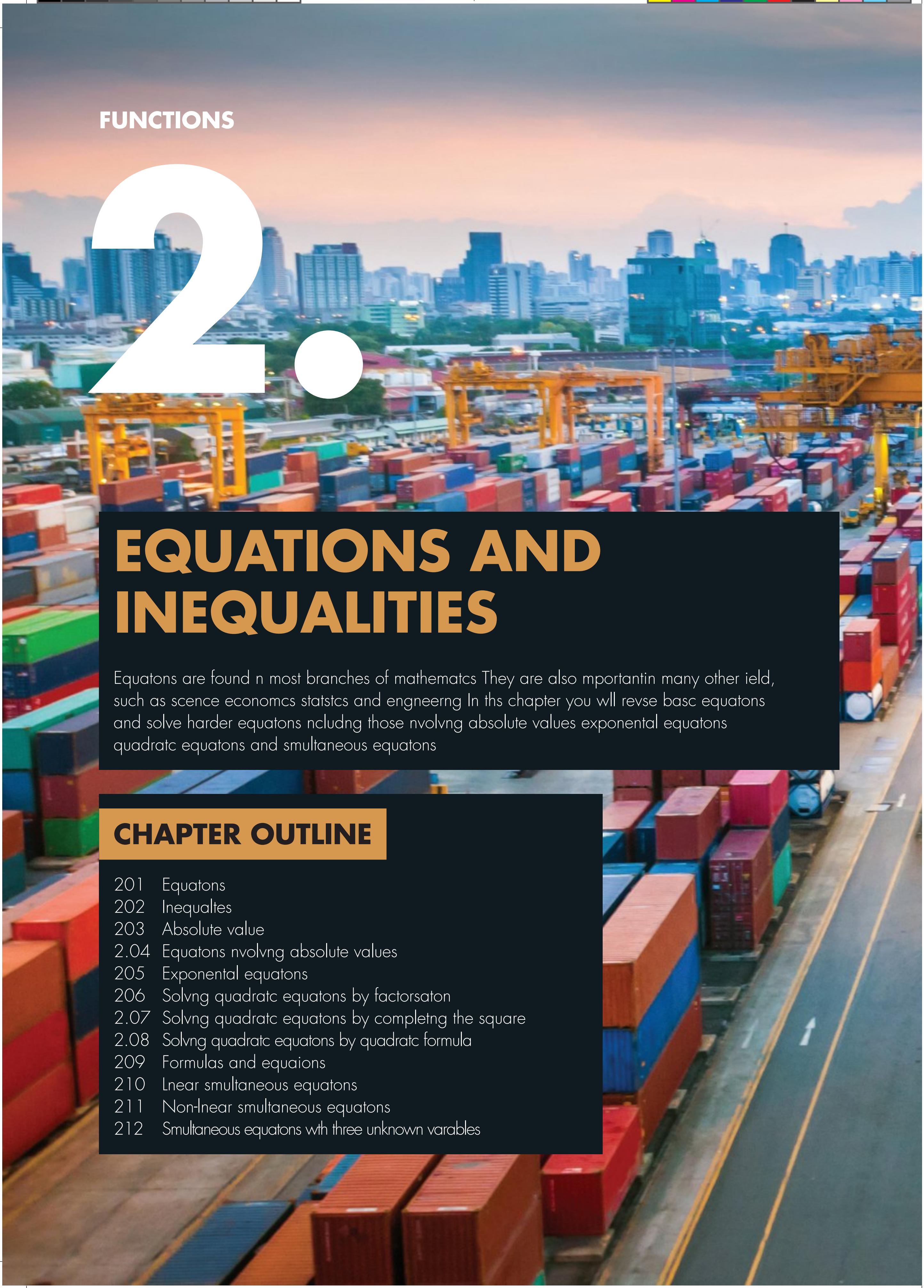

FUNCTIONS

2. EQUATIONS AND INEQUALITIES

Equatons are found n most branches of mathematcs They are also mportantin many other ield, such as scence economcs statstcs and engneerng In ths chapter you wll revse basc equatons and solve harder equatons ncludng those nvolvng absolute values exponental equatons quadratc equatons and smultaneous equatons

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- solve equatons and nequaltes
- understand and use absolute values n equatons
- solve smple exponental equatons
- solve quadratc equatons usng 3 dfferent methods
- understand how to substtute nto and rearrange formulas
- solve lnear and non-lnear smultaneous equatons

TERMINOLOGY

absolute value $|x|$ is the absolute value of x its size without sign or directio. Also the distance of x from 0 on the number line in either direction

equation A mathematical statement that has a pronumeral or unknown number and an equal sign An equation can be solved to find the value of the unknown number, for exampe, $3x + 1 = 7$

exponential equation An equation where the unknown pronumeral is the power or index for example $2^x = 8$

inequality A mathematical statement involving an inequality sign with an unknown pronumeral for exampl, $x - 7 \leq 12$

quadratic equation An equation involving x^2 in which the highest power of x is 2

simultaneous equations 2 or more equations that can be solved together to produce a solution that makes each equation true at the same time

PROBLEM

The age of Diophantus at his death can be calculated from his epitaph

Diophantus passed one-sixth of his life in childhood one-twelft h in youth and one-seventh more as a bachelor; five years after his marriage a son was born who died four years before his father at half his fathers final ae. How old was Diop hantus?

2.01 Equations

WS Equaion

EXAMPLE 1

Solve each equation

a $4y - 3 = 8y + 21$

b $2(3x + 7) = 6 - (x - 1)$

Solution

a

$$\begin{aligned} 4y - 3 &= 8y + 21 \\ 4y - 4y - 3 &= 8y - 4y + 21 \\ -3 &= 4y + 21 \\ -3 - 21 &= 4y + 21 - 21 \\ -24 &= 4y \\ \frac{-24}{4} &= \frac{4y}{4} \\ -6 &= y \\ y &= -6 \end{aligned}$$

b

$$\begin{aligned} 2(3x + 7) &= 6 - (x - 1) \\ 6x + 14 &= 6 - x + 1 \\ &= 7 - x \\ 6x + x + 14 &= 7 - x + x \\ 7x + 14 &= 7 \\ 7x + 14 - 14 &= 7 - 14 \\ 7x &= -7 \\ \frac{7x}{7} &= \frac{-7}{7} \\ x &= -1 \end{aligned}$$

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When an equation involves fractions multiply both sides of the equation by the common denominator of the fractions

EXAMPLE 2

Solve

a $\frac{m}{3} - 4 = \frac{1}{2}$

b $\frac{x+1}{3} + \frac{x}{4} = 5$

Solution

a

$$\frac{m}{3} - 4 = \frac{1}{2}$$
$$6\left(\frac{m}{3}\right) - 6(4) = 6\left(\frac{1}{2}\right)$$
$$2m - 24 = 3$$
$$2m - 24 + 24 = 3 + 24$$
$$2m = 27$$
$$\frac{2m}{2} = \frac{27}{2}$$
$$m = \frac{27}{2}$$
$$= 13\tfrac{1}{2}$$

b

$$\frac{x+1}{3} + \frac{x}{4} = 5$$
$$12\left(\frac{x+1}{3}\right) + 12\left(\frac{x}{4}\right) = 12(5)$$
$$4(x + 1) + 3x = 60$$
$$4x + 4 + 3x = 60$$
$$7x + 4 = 60$$
$$7x + 4 - 4 = 60 - 4$$
$$7x = 56$$
$$\frac{7x}{7} = \frac{56}{7}$$
$$x = 8$$

DID YOU KNOW?

History of algebra

Algebra was known in ancient civilisations Many equations were known in Babylon although general solutions were difficult because symbols were not used in those times

Diophantus around 250 CE first used algebraic notation and symbols (.. the minus sig). He wrote a treatise on algebra in his *Arithmetica* comprising 13 book. Only six of these books survived About 400 CE Hypatia of Alexandria wrote a commentary on thm.

Hypatia was the first female mathematician on record and was a philosopher and teacher. She was the daughter of Thon, who was also a mathematician and who ensured that she had the best education

In 1799 **Carl Friedrich Gauss** proved the Fundamental Theorem of Algebra: that every algebraic equation involving a power of x has at least one solution which may be a real number or a non-real number.

Exercise 2.01 Equations

Solve each equation

1 $t + 4 = -1$

2 $z + 1.7 = -39$

3 $y - 3 = -2$

4 $w - 26 = 41$

5 $5 = x - 7$

6 $15x = 6$

7 $5y = \frac{1}{3}$

8 $\frac{b}{7} = 5$

9 $-2 = \frac{n}{8}$

10 $\frac{r}{6} = \frac{2}{3}$

11 $2y + 1 = 19$

12 $33 = 4k + 9$

13 $7d - 2 = 12$

14 $-2 = 5x - 27$

15 $\frac{y}{3} + 4 = 9$

16 $\frac{x}{2} - 3 = 7$

17 $\frac{m}{5} + 7 = 11$

18 $3x + 5 = 17$

19 $4a + 7 = -21$

20 $7y - 1 = 20$

21 $3(x + 2) = 15$

22 $-2(3a + 1) = 8$

23 $7t + 4 = 3t - 12$

24 $x - 3 = 6x - 9$

25 $2(a - 2) = 4 - 3a$

26 $5b + 2 = -3(b - 1)$

27 $3(t + 7) = 2(2t - 9)$

28 $2 + 5(p - 1) = 5p - (p - 2)$

29 $37x + 12 = 54x - 63$

30 $\frac{b}{5} = \frac{2}{3}$

31 $\frac{5x}{4} = \frac{11}{7}$

32 $\frac{x}{3} - 4 = 8$

33 $\frac{5 + x}{7} = \frac{2}{7}$

34 $\frac{y}{2} = -\frac{3}{5}$

35 $\frac{x}{9} - \frac{2}{3} = 7$

36 $\frac{w - 3}{2} = 5$

37 $\frac{2t}{5} - \frac{t}{3} = 2$

38 $\frac{x}{4} + \frac{1}{2} = 4$

39 $\frac{x}{5} - \frac{x}{2} = \frac{3}{10}$

40 $\frac{x + 4}{3} + \frac{x}{2} = 1$

41 $\frac{p - 3}{2} + \frac{2p}{3} = 2$

42 $\frac{t + 3}{7} + \frac{t - 1}{3} = 4$

43 $\frac{x + 5}{9} - \frac{x + 2}{5} = 1$

44 $\frac{q - 1}{3} - \frac{q - 2}{4} = 2$

45 $\frac{x + 3}{5} + 2 = \frac{x + 7}{2}$

COULD THIS BE TRUE?

Half full = half empty

$\therefore$ full = empty

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2.02 Inequalities

$>$ means greater than

$<$ means less than

$\geq$ means greater than or equal to

$\leq$ means less than or equal to

Solving inequalities

The **inequality sign reverses** when

- multiplying by a negative
- dividing by a negative
- taking the reciprocal of both sides

On the number plane we graph inequalities using arrows and circles (open for greater than and less than and closed in for greater than or equal to and less than or equal to)

Inequalities on a number line

WS

Inequaliies on a numbe line

EXAMPLE 3

Solve each inequality and show its solution on a number line

a $5x + 7 \geq 17$ **b** $3t - 2 > 5t + 4$ **c** $1 < 2z + 7 \leq 11$

Solution

a

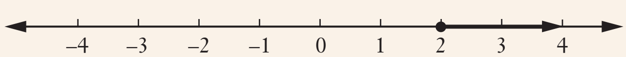
$$5x + 7 \geq 17$$

$$5x + 7 - 7 \geq 17 - 7$$

$$5x \geq 10$$

$$\frac{5x}{5} \geq \frac{10}{5}$$

$$x \geq 2$$



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b $$3t - 2 > 5t + 4$$

$$3t - 5t - 2 > 5t - 5t + 4$$

$$-2t - 2 > 4$$

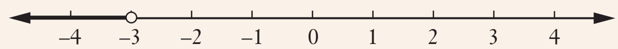
$$-2t - 2 + 2 > 4 + 2$$

$$-2t > 6$$

$$\frac{-2t}{-2} < \frac{6}{-2}$$

$$t < -3$$

Remember to change the inequality sign when dividing by –2

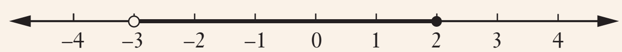


c $$1 < 2z + 7 \le 11$$

$$1 - 7 < 2z + 7 - 7 \le 11 - 7$$

$$-6 < 2z \le 4$$

$$-3 < z \le 2$$



Exercise 2.02 Inequalities

1 Solve each equation and plot the solution on a number line

a $x + 4 > 7$ **b** $y - 3 \le 1$

2 Solve

a $5t > 35$ **b** $3x - 7 \ge 2$ **c** $2(p + 5) > 8$

d $4 - (x - 1) \le 7$ **e** $3y + 5 > 2y - 4$ **f** $2a - 6 \le 5a - 3$

g $3 + 4y \ge -2(1 - y)$ **h** $2x + 9 < 1 - 4(x + 1)$ **i** $\frac{a}{2} \le -3$

j $8 > \frac{2y}{3}$ **k** $\frac{b}{2} + 5 < -4$ $\frac{x}{3} - 4 > 6$

3 Solve and plot each solution on a number line

a $3 < x + 2 < 9$ **b** $-4 \le 2p < 10$ **c** $2 < 3x - 1 < 11$

d $-6 \le 5y + 9 \le 34$ **e** $-2 < 3(2y - 1) < 7$

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2.03 Absolute value

The **absolute value** of a number is the size of the number without the sign or direction
So absolute value is always positive or zero

We write the absolute value of x as $|x|$

For example $|4| = 4$ and $|-3| = 3$.

We can also define $|x|$ as the distance of x from 0 on the number line

If x is positive then its absolute value is itsel.

If $x = 0$ then its absolute value is .

If x is negative then its absolute value is its opposit, $-x$ Because x is already negative the effect of the negative sign in front of it is to make it positive for example $-(-5) = 5$.

Absolute value

$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

$|4| = 4$ since $4 \geq 0$

$|-3| = -(-3)$ since $-3 < 0$

$= 3$

Properties of absolute value

Property	Example
$\lvert ab\rvert = \lvert a\rvert \times \lvert b\rvert$	$\lvert 2 \times -3\rvert = \lvert 2\rvert \times \lvert -3\rvert = 6$
$\lvert a\rvert^2 = a^2$	$\lvert -3\rvert^2 = (-3)^2 = 9$
$\sqrt{a^2} = \lvert a\rvert$	$\sqrt{(-5)^2} = \lvert -5\rvert = 5$
$\lvert -a\rvert = \lvert a\rvert$	$\lvert -7\rvert = \lvert 7\rvert = 7$
$\lvert a - b\rvert = \lvert b - a\rvert$	$\lvert 2 - 3\rvert = \lvert 3 - 2\rvert = 1$
$\lvert a + b\rvert \leq \lvert a\rvert + \lvert b\rvert$	$\lvert 2 + 3\rvert = \lvert 2\rvert + \lvert 3\rvert$ but $\lvert -3 + 4\rvert < \lvert -3\rvert + \lvert 4\rvert$

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EXAMPLE 4

a Evaluate $|2| - |-1| + |-3|^2$

b Show that $|a + b| \le |a| + |b|$ when $a = -2$ and $b = 3$.

c Write expressions for $|2x - 4|$ without the absolute value signs

Solution

a $|2| - |-1| + |-3|^2 = 2 - 1 + 3^2$
$= 10$

b LHS $= |a + b|$
$= |-2 + 3|$
$= |1|$
$= 1$

RHS $= |a| + |b|$
$= |-2| + |3|$
$= 2 + 3$
$= 5$

Note LHS means left-hand side and RHS means right-hand side

Since $1 < 5$,

$|a + b| \le |a| + |b|$

c $|2x - 4| = 2x - 4$ when $2x - 4 \ge 0$ ie when $2x \ge 4$

ie when $x \ge 2$

$|2x - 4| = -(2x - 4)$ when $2x - 4 < 0$ ie when $2x < 4$

$= -2x + 4$ ie when $x < 2$

CLASS DISCUSSION

ABSOLUTE VALUE

Are these statements true? If so are there some values for which the expression is undefined (values of x or y that the expression cannot have)?

1 $\frac{x}{|x|} = 1$

2 $|2x| = 2x$

3 $|2x| = 2|x|$

4 $|x| + |y| = |x + y|$

5 $|x|^2 = x^2$

6 $|x|^3 = x^3$

7 $|x + 1| = |x| + 1$

8 $\frac{|3x - 2|}{3x - 2} = 1$

9 $\frac{|x|}{x^2} = 1$

10 $|x| \ge 0$

Discuss absolute value and its definition in relation to these statements

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Exercise 2.03 Absolute value

1 Evaluate

a $|7|$ **b** $|-5|$ **c** $|-6|$ **d** $|0|$
e $|2|$ **f** $|-11|$ **g** $|-2\ \ 3|$ **h** $3|-8|$
i $|-5|^2$ **j** $|-5|^3$

2 Evaluate

a $|3| + |-2|$ **b** $|-3| - |4|$ **c** $|-5 + 3|$
d $|2 \times -7|$ **e** $|-3| + |-1|$ **f** $5 - |-2| \times |6|^2$
g $|-2 + 5 \times -1|$ **h** $3|-4|$ **i** $2|-3| - 3|-4|$
j $|5 - 7| + 4|-2|$

3 Evaluate $|a - b|$ if

a $a = 5$ and $b = 2$ **b** $a = -1$ and $b = 2$ **c** $a = -2$ and $b = -3$
d $a = 4$ and $b = 7$ **e** $a = -1$ and $b = -2$

4 Write an expression fo:

a $|a|$ when $a > 0$ **b** $|a|$ when $a < 0$ **c** $|a|$ when $a = 0$
d $|3a|$ when $a > 0$ **e** $|3a|$ when $a < 0$ **f** $|3a|$ when $a = 0$
g $|a + 1|$ when $a > -1$ **h** $|a + 1|$ when $a < -1$ **i** $|x - 2|$ when $x > 2$

5 Show that $|a + b| \le |a| + |b|$ when

a $a = 2$ and $b = 4$ **b** $a = -1$ and $b = -2$ **c** $a = -2$ and $b = 3$
d $a = -4$ and $b = 5$ **e** $a = -7$ and $b = -3$

6 Show that $\sqrt{x^2} = |x|$ when

a $x = 5$ **b** $x = -2$ **c** $x = -3$
d $x = 4$ **e** $x = -9$

7 Use the definition of absolute value to write each expression without the absolute value signs

a $|x + 5|$ **b** $|b - 3|$ **c** $|a + 4|$
d $|2y - 6|$ **e** $|3x + 9|$ **f** $|4 - x|$
g $|2k + 1|$ **h** $|5x - 2|$ **i** $|a + b|$

8 Find values of x for which $|x| = 3$.

9 Simplify $\dfrac{|n|}{n}$ where $n \neq 0$

10 Simplify $\dfrac{x-2}{|x-2|}$ and state which value x cannot be

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Abolue value equaion and inequaliies

2.04 Equations involving absolute values

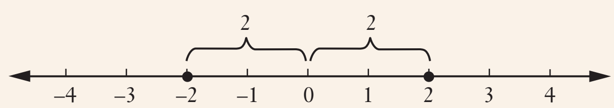
On a number line $|x|$ means the distance of x from 0 in either direction

EXAMPLE 5

Solve $|x| = 2$.

Solution

$|x| = 2$ means the distance of x from zero is 2 (in either direction)



$x = \pm 2$

CLASS DISCUSSION

ABSOLUTE VALUE AND THE NUMBER LINE

What does $|a - b|$ mean as a distance along the number line?

Select different values of a and b to help with this discussion

EXAMPLE 6

Solve

a $|x + 4| = 7$ **b** $|2x - 3| = 9$

Solution

a This means that the distance from $x + 4$ to 0 is 7 in either direction

So $x + 4 = \pm 7$

$$x + 4 = 7 \qquad \text{or} \qquad x + 4 = -7$$

$$x + 4 - 4 = 7 - 4 \qquad\qquad x + 4 - 4 = -7 - 4$$

$$x = 3 \qquad\qquad x = -11$$

So $x = 3$ or -11

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Checking your anser:

$\text{LHS} = |3 + 4|$
$= |7|$
$= 7$
$= \text{RHS}$

$\text{LHS} = |-11 + 4|$
$= |-7|$
$= 7$
$= \text{RHS}$

b $|2x - 3| = 9$

$2x - 3 = 9$ or $2x - 3 = -9$

$2x = 12$ $\quad$ $2x = -6$

$x = 6$ $\quad$ $x = -3$

So $x = 6$ or -3

Checking your anser:

$\text{LHS} = |2 \times 6 - 3|$
$= |9|$
$= 9$
$= \text{RHS}$

$\text{LHS} = |2 \times (-3) - 3|$
$= |-9|$
$= 9$
$= \text{RHS}$

Exercise 2.04 Equations involving absolute values

1 Solve

a $|x| = 5$ **b** $|y| = 8$ **c** $|x| = 0$

2 Solve

a $|x + 2| = 7$ **b** $|n - 1| = 3$ **c** $9 = |2x + 3|$

d $|7x - 1| = 34$ **e** $\left|\frac{x}{3}\right| = 4$

3 Solve

a $|8x - 5| = 11$ **b** $|5 - 3n| = 1$ **c** $16 = |5t + 4|$

d $21 = |9 - 2y|$ **e** $|3x + 2| - 7 = 0$

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WS
Exponenial equaion

2.05 Exponential equations

The word **exponent** means the power or index of a number.

So an **exponential equation** involves an unknown index or power for example $2^{x} = 8$.

EXAMPLE 7

Solve

a $3^x = 81$ **b** $5^{2k-1} = 25$ **c** $8^n = 4$

Solution

a
$$3^x = 81$$
$$3^x = 3^4$$
$$\therefore x = 4$$

b
$$5^{2k-1} = 25$$
$$5^{2k-1} = 5^2$$
$$\therefore 2k - 1 = 2$$
$$2k = 3$$
$$\frac{2k}{2} = \frac{3}{2}$$
$$k = 1\tfrac{1}{2}$$

c It is hard to write 8 as a power of 4 or 4 as a power of 8 but both can be written as powers of 2
$$8^n = 4$$
$$(2^3)^n = 2^2$$
$$2^{3n} = 2^2$$
$$\therefore 3n = 2$$
$$\frac{3n}{3} = \frac{2}{3}$$
$$n = \frac{2}{3}$$

To solve other equations involving indice, we do the opposite or inverse operatin. For examle, squares and square roots are inverse operations and cubes and cube roots are inverse operation.

EXAMPLE 8

Solve

a $x^2 = 9$ **b** $5n^3 = 40$

Solution

a There are two possible numbers whose square is 9
$$x^2 = 9$$
$$x = \pm\sqrt{9}$$
$$\therefore x = \pm 3$$

b
$$5n^3 = 40$$
$$\frac{5n^3}{5} = \frac{40}{5}$$
$$n^3 = 8$$
$$n = \sqrt[3]{8}$$
$$n = 2$$

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INVESTIGATION

SOLUTIONS FOR EQUATIONS INVOLVING x^n

Investigate equations of the type $x^n = k$ where k is a constant for example $x^n = 9$

Look at these questions

1 What is the solution when $n = 0$?

2 What is the solution when $n = 1$?

3 How many solutions are there when $n = 2$?

4 How many solutions are there when $n = 3$?

5 How many solutions are there when n is even?

6 How many solutions are there when n is odd?

Exercise 2.05 Exponential equations

1 Solve

a $2^n = 16$ **b** $3^y = 243$ **c** $2^m = 512$ **d** $10^x = 100\ 000$

e $6^m = 1$ **f** $4^x = 64$ **g** $4^x + 3 = 19$ **h** $5(3^x) = 45$

i $4^x = 4$ **j** $\frac{6^k}{2} = 18$

2 Solve

a $3^{2x} = 81$ **b** $2^{5x-1} = 16$ **c** $4^{x+3} = 4$ **d** $3^{n-2} = 1$

e $7^{2x+1} = 7$ **f** $3^{x-3} = 27$ **g** $5^{3y+2} = 125$ **h** $7^{3x-4} = 49$

i $2^{4x} = 256$ **j** $9^{3a+1} = 9$

3 Solve

a $4^m = 2$ **b** $27^x = 3$ **c** $125^x = 5$ **d** $\left(\frac{1}{49}\right)^k = 7$

e $\left(\frac{1}{1000}\right)^k = 100$ **f** $16^n = 8$ **g** $25^x = 125$ **h** $64^n = 16$

i $\left(\frac{1}{4}\right)^{3k} = 2$ **j** $8^{x-1} = 4$

4 Solve

a $2^{4x+1} = 8^x$ **b** $3^{5x} = 9^{x-2}$ **c** $7^{2k+3} = 7^{k-1}$

d $4^{3n} = 8^{n+3}$ **e** $6^{x-5} = 216^x$ **f** $16^{2x-1} = 4^{x-4}$

g $27^{x+3} = 3^x$ **h** $\left(\frac{1}{2}\right)^x = \left(\frac{1}{64}\right)^{2x+3}$ **i** $\left(\frac{3}{4}\right)^x = \left(\frac{27}{64}\right)^{2x-3}$

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5 Solve

a $4^m = \sqrt{2}$ **b** $\left(\frac{9}{25}\right)^{k+3} = \sqrt{\frac{3}{5}}$ **c** $\frac{1}{\sqrt{2}} = 4^{2x-5}$

d $3^k = 3\sqrt{3}$ **e** $\left(\frac{1}{27}\right)^{3n+1} = \frac{\sqrt{3}}{81}$ **f** $\left(\frac{2}{5}\right)^{3n+1} = \left(\frac{5}{2}\right)^{-n}$

g $32^{-x} = \frac{1}{16}$ **h** $9^{2b+5} = 3^b\sqrt{3}$ **i** $81^{x+1} = \sqrt{3^x}$

6 Solve giving exact answer:

a $x^3 = 27$ **b** $y^2 = 64$ **c** $n^4 = 16$

d $x^2 = 20$ **e** $p^3 = 1000$ **f** $2x^2 = 50$

g $6y^4 = 486$ **h** $w^3 + 7 = 15$ **i** $6n^2 - 4 = 92$

7 Solve and give the answer correct to 2 decimal places

a $p^2 = 45$ **b** $x^3 = 100$ **c** $n^5 = 240$

d $2x^2 = 70$ **e** $4y^3 + 7 = 34$ **f** $\frac{d^4}{3} = 14$

g $\frac{k^2}{2} - 3 = 7$ **h** $\frac{x^3 - 1}{5} = 2$ **i** $2y^2 - 9 = 20$

8 Solve

a $x^{-} = 5$ **b** $a^{-3} = 8$ **c** $y^{-5} = 32$ **d** $x^{-2} + 1 = 50$

e $2n^{-} = 3$ **f** $a^{-3} = \frac{1}{8}$ **g** $x^{-2} = \frac{1}{4}$ **h** $b^{-} = \frac{1}{9}$

i $x^{-2} = 2\frac{1}{4}$ **j** $b^{-4} = \frac{16}{81}$

PUZZLE

Test your logical thinking and that of your friend.

1 How many months have 28 days?

2 If I have 128 sheep and take away all but 10 how many do I have left ?

3 A bottle and its cork cost $110 to make If the bottle costs $1 more than the cor, how much does each cost?

4 What do you get if you add 1 to 15 four times?

5 On what day of the week does Good Friday fall in 2030?

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2.06 Solving quadratic equations by factorisation

Quadaic equaion by acoiing

A **quadratic equation** is an equation involving a square For exampl, $x^2 - 4 = 0$

When solving quadratic equations by factorising we use a property of zer.

For any real numbers a and b, if $ab = 0$ then $a = 0$ or $b = 0$

EXAMPLE 9

Solve

a $x^2 + x - 6 = 0$ **b** $y^2 - 7y = 0$ **c** $3a^2 - 14a = -8$

Solution

a

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$\therefore\ x + 3 = 0$ or $x - 2 = 0$

$x = -3$ or $x = 2$

So the solution is $x = -3$ or 2

b

$y^2 - 7y = 0$

$y\,(y - 7) = 0$

$\therefore\ y = 0$ or $y - 7 = 0$

$y = 7$

So the solution is $y = 0$ or 7

c First we make the equation equal to zero so we can factorise and use the rule for zero

$3a^2 - 14a = -8$

$3a^2 - 14a + 8 = -8 + 8$

$3a^2 - 14a + 8 = 0$

$(3a - 2)(a - 4) = 0$

$\therefore\ 3a - 2 = 0$ or $a - 4 = 0$

$3a = 2$ or $a = 4$

$\frac{3a}{3} = \frac{2}{3}$

$a = \frac{2}{3}$

So the solution is $a = \frac{2}{3}$ or 4

Exercise 2.06 Solving quadratic equations by factorisation

Solve each quadratic equation

1 $y^2 + y = 0$	**2** $b^2 - b - 2 = 0$	**3** $p^2 + 2p - 15 = 0$
4 $t^2 - 5t = 0$	**5** $x^2 + 9x + 14 = 0$	**6** $q^2 - 9 = 0$
7 $x^2 - 1 = 0$	**8** $a^2 + 3a = 0$	**9** $2x^2 + 8x = 0$
10 $4x^2 - 1 = 0$	**11** $3x^2 + 7x + 4 = 0$	**12** $2y^2 + y - 3 = 0$
13 $8b^2 - 10b + 3 = 0$	**14** $x^2 - 3x = 10$	**15** $3x^2 = 2x$
16 $2x^2 = 7x - 5$	**17** $5x - x^2 = 0$	**18** $y^2 = y + 2$
19 $8n = n^2 + 15$	**20** $12 = 7x - x^2$	**21** $m^2 = 6 - 5m$
22 $x(x + 1)(x + 2) = 0$	**23** $(y - 1)(y + 5)(y + 2) = 0$	**24** $(x + 3)(x - 1) = 32$
25 $(m - 3)(m - 4) = 20$		

WS
Compleing he quae

2.07 Solving quadratic equations by completing the square

Not all trinomials will factorise so other methods need to be used to solve quadratic equations

EXAMPLE 10

Solve

a $(x + 3)^2 = 11$

b $(y - 2)^2 = 7$

Solution

a

$$\begin{aligned}(x+3)^2 &= 11\\ x+3 &= \pm\sqrt{11}\\ x+3-3 &= \pm\sqrt{11}-3\\ x &= \pm\sqrt{11}-3\end{aligned}$$

b

$$\begin{aligned}(y-2)^2 &= 7\\ y-2 &= \pm\sqrt{7}\\ y-2+2 &= \pm\sqrt{7}+2\\ y &= \pm\sqrt{7}+2\end{aligned}$$

To solve a quadratic equation such as $x^2 - 6x + 3 = 0$ which will not factoris, we can use the method of **completing the square**

We use the perfect squar:

$$a^2 + 2ab + b^2 = (a + b)^2$$

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EXAMPLE 11

Complete the square on $a^2 + 6a$

Solution

Compare with $a^2 + 2ab + b^2$ $2\ ab = 6a$

$$b = 3$$

To complete the squar: $a^2 + 2ab + b^2 = (a + b)^2$

$$a^2 + 2a(3) + 3^2 = (a + 3)^2$$

$$a^2 + 6x + 9 = (a + 3)^2$$

Completing the square

To complete the square on $a^2 \pm pa$ divide p by 2 and square it

$$a^2 \pm pa + \left(\frac{p}{2}\right)^2 = \left(a \pm \frac{p}{2}\right)^2$$

EXAMPLE 12

Solve by completing the square

a $x^2 - 6x + 3 = 0$

b $y^2 + 2y - 7 = 0$ (correct to 3 significant figures)

Solution

a

$$x^2 - 6x + 3 = 0$$

$$x^2 - 6x = -3$$

$$x^2 - 6x + 9 = -3 + 9 \qquad \left(\frac{6}{2}\right)^2 = 3^2 = 9$$

$$(x - 3)^2 = 6$$

$$\therefore x - 3 = \pm\sqrt{6}$$

$$x = \pm\sqrt{6} + 3$$

The 3rd lne shows the completng the square step n both solutons

b

$$y^2 + 2y - 7 = 0$$

$$y^2 + 2y = 7$$

$$y^2 + 2y + 1 = 7 + 1 \qquad \left(\frac{2}{2}\right)^2 = 1^2 = 1$$

$$(y + 1)^2 = 8$$

$$\therefore y + 1 = \pm\sqrt{8}$$

$$y = \pm\sqrt{8} - 1$$

$$y \approx 183 \text{ or } -383$$

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Exercise 2.07 Solving quadratic equations by completing the square

1 Solve and give exact solutions

a $(x+1)^2 = 7$	**b** $(y+5)^2 = 5$	**c** $(a-3)^2 = 6$
d $(x-2)^2 = 13$	**e** $(2y+3)^2 = 2$	

2 Solve and give solutions correct to one decimal place

a $(h+2)^2 = 15$	**b** $(a-1)^2 = 8$	**c** $(x-4)^2 = 17$
d $(y+7)^2 = 21$	**e** $(3x-1)^2 = 12$	

3 Solve by completing the square giving exact solutions in simplest surd for:

a $x^2 + 4x - 1 = 0$	**b** $a^2 - 6a + 2 = 0$	**c** $y^2 - 8y - 7 = 0$
d $x^2 + 2x - 12 = 0$	**e** $p^2 + 14p + 5 = 0$	**f** $x^2 - 10x - 3 = 0$
g $y^2 + 20y + 12 = 0$	**h** $x^2 - 2x - 1 = 0$	**i** $n^2 + 24n + 7 = 0$

4 Solve by completing the square and writing answers correct to 3 significant figures

a $x^2 - 2x - 5 = 0$	**b** $x^2 + 12x + 34 = 0$	**c** $q^2 + 18q - 1 = 0$
d $x^2 - 4x - 2 = 0$	**e** $b^2 + 16b + 50 = 0$	**f** $x^2 - 24x + 112 = 0$
g $r^2 - 22r - 7 = 0$	**h** $x^2 + 8x + 5 = 0$	**i** $a^2 + 6a - 1 = 0$

2.08 Solving quadratic equations by quadratic formula

WS Quadaic omula

WS Quadaic equaion

WS Poblem involving quadaic equaion

The quadaic omula

Completing the square is difficult with harder quadratic equations such as $2x^2 - x - 5 = 0$. Completing the square on a general quadratic equation gives the following formula

The quadratic formula

For the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof

Solve $ax^2 + bx + c = 0$ by completing the square

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

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Completing the square

$$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$=\frac{-4ac+b^2}{4a^2}$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{-4ac+b^2}{4a^2}}$$

$$=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

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Solving algebaic equaion

EXAMPLE 13

a Solve $x^2 - x - 2 = 0$ by using the quadratic formula

b Solve $2y^2 - 9y + 3 = 0$ by formula and give your answer correct to 2 decimal places

Solution

a $a = 1, b = -1, c = -2$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-2)}}{2(1)}$$

$$=\frac{1\pm\sqrt{1+8}}{2}$$

$$=\frac{1\pm\sqrt{9}}{2}$$

$$=\frac{1\pm3}{2}$$

$$=2 \text{ or } -1$$

b $a = 2, b = -9 \quad c = 3$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-9)\pm\sqrt{(-9)^2-4(2)(3)}}{2(2)}$$

$$=\frac{9\pm\sqrt{81-24}}{4}$$

$$=\frac{9\pm\sqrt{57}}{4}$$

$$\approx 414 \text{ or } 036$$

Exercise 2.08 Solving quadratic equations by quadratic formula

1 Solve by formula correct to 3 significant figures where necessar:

a	$y^2 + 6y + 2 = 0$	**b**	$2x^2 - 5x + 3 = 0$	**c**	$b^2 - b - 9 = 0$
d	$2x^2 - x - 1 = 0$	**e**	$-8x^2 + x + 3 = 0$	**f**	$n^2 + 8n - 2 = 0$
g	$m^2 + 7m + 10 = 0$	**h**	$x^2 - 7x = 0$	**i**	$x^2 + 5x = 6$

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2 Solve by formula leaving the answer in simplest surd for:

a $x^2 + x - 4 = 0$ **b** $3x^2 - 5x + 1 = 0$ **c** $q^2 - 4q - 3 = 0$

d $4h^2 + 12h + 1 = 0$ **e** $3s^2 - 8s + 2 = 0$ **f** $x^2 + 11x - 3 = 0$

g $6d^2 + 5d - 2 = 0$ **h** $x^2 - 2x = 7$ **i** $t^2 = t + 1$

CLASS INVESTIGATION

FAULTY PROOF

Here is a proof that 1 = 2 Can you see the fault in the proof?

$$x^2 - x^2 = x^2 - x^2$$

$$x(x - x) = (x + x)(x - x)$$

$$x = x + x$$

$$x = 2x$$

$$\therefore \quad 1 = 2$$

2.09 Formulas and equations

Sometimes substituting values into a formula involves solving an equation

EXAMPLE 14

a The formula for the surface area of a rectangular prism is given by $S = 2(lb + bh + lh)$. Find the value of b when $S = 180$, $l = 9$ and $h = 6$

b The volume of a cylinder is given by $V = \pi r^2 h$ Evaluate the radius r correct to 2 decimal places when $V = 350$ and $h = 6.5$.

Solution

a

$$S = 2(lb + bh + lh)$$

$$180 = 2(9b + 6b + 9 \times 6)$$

$$= 2\,(15b + 54)$$

$$= 30b + 108$$

$$72 = 30b$$

$$\frac{72}{30} = \frac{30b}{30}$$

$$24 = b$$

b

$$V = \pi r^2 h$$

$$350 = \pi r^2(65)$$

$$\frac{350}{6\ 5\pi} = \frac{\pi r^2 (6\ 5)}{6\ 5\pi}$$

$$\frac{350}{6\ 5\pi} = r^2$$

$$\sqrt{\frac{350}{6\ 5\pi}} = r$$

$$414 = r$$

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Exercise 2.09 Formulas and equations

1 Given that $v = u + at$ is the formula for the velocity of a particle at time t find the value of t when $u = 17.3$, $v = 1006$ and $a = 98$

2 The sum of an arithmetic series is given by $S = \frac{n}{2}(a + l)$ Find l if $a = 3$, $n = 26$ and $S = 1625$

3 The formula for finding the area of a triangle is $A = \frac{1}{2}bh$ Find b when $A = 36$ and $h = 9$

4 The area of a trapezium is given by $A = \frac{1}{2}h(a + b)$ Find the value of a when $A = 120$, $h = 5$ and $b = 7$.

5 Find the value of y when $x = 3$ given the straight line equation $5x - 2y - 7 = 0$

6 The area of a circle is given by $A = \pi r^2$ Find r correct to 3 significant figures if $A = 140$

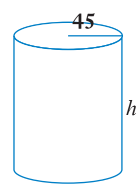
7 The area of a rhombus is given by the formula $A = \frac{1}{2}xy$ where x and y are its diagonals

Find the value of x correct to 2 decimal places when $y = 78$ and $A = 25.1$.

8 The simple interest formula is $I = Prn$ Find n if $r = 0145$, $P = 150$ and $I = 32625$

9 The gradient of a straight line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$ Find y when $m = -\frac{5}{6}$ $y_2 = 7$, $x_2 = -3$ and $x = 1$.

10 The surface area of a cylinder is given by the formula $S = 2\pi r(r + h)$.
Evaluate h correct to 1 decimal place if $S = 232$ and $r = 45$



11 The formula for body mass index is $\text{BMI} = \frac{w}{h^2}$ Evaluat:

- **a** the BMI when $w = 65$ and $h = 1.6$
- **b** w when BMI $= 215$ and $h = 1.8$
- **c** h when BMI $= 197$ and $w = 738$

12 A formula for depreciation is $D = P(1 - r)^n$ Find r if $D = 12\,000$, $P = 15\,000$ and $n = 3$.

13 The x value of the midpoint is given by $x = \frac{x_1 + x_2}{2}$ Find x when $x = -2$ and $x_2 = 5$.

14 Given the height of a particle at time t is $h = 5t^2$ evaluate t when $h = 23$.

15 If $y = x^2 + 1$ evaluate x when $y = 5$.

16 If the surface area of a sphere is $S = 4\pi r^2$ evaluate r to 3 significant figures when $S = 563$

17 The area of a sector of a circle is $A = \frac{1}{2}r^2\theta$ Evaluate r when $A = 246$ and $\theta = 045$

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18 If $y = \frac{2}{x^3 - 1}$ find the value of x when $y = 3$.

19 Given $y = \sqrt{2x+5}$ evaluate x when $y = 4$

20 The volume of a sphere is $V = \frac{4}{3}\pi r^3$ Evaluate r to 1 decimal place when $V = 150$.

INVESTIGATION

BODY MASS INDEX

Body mass index (BMI) is a formula that is used by health professionals to screen for weight categories that may lead to health problems

The formula for BMI is BMI $= \frac{m}{h^2}$ where m is the mass of a person in kg and h is the height in metres

For adults over 20 a BMI under 1.5 means that the person is underweight and over 25 is overweight Over 30 is considered obes.

The BMI may not always be a reliable measurement of bod fat. Can you think of some reasons?

Is it important where the body fat is stored? Does it make a difference if it is on the hips or the stomach?

Research more about BMI generaly.

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2.10 Linear simultaneous equations

You can solve two equations together to find one solution that satisfies both equation. Such equations are called **simultaneous equations** and there are two ways of solving them The **elimination method** adds or subtracts the equations The **substitution method** substitutes one equation into the other.

EXAMPLE 15

Solve simultaneously using the elimination method

a $3a + 2b = 5$ and $2a - b = -6$

b $5x - 3y = 19$ and $2x - 4y = 16$

Solution

a

$3a + 2b = 5$ [1]

$2a - b = -6$ [2]

[2] × 2: $4a - 2b = -12$ [3]

[1] + [3] $3a + 2b = 5$ [1]

$7a = -7$

$a = -1$

Substitute $a = -1$ in [1] $3(-1) + 2b = 5$

$-3 + 2b = 5$

$2b = 8$

$b = 4$

Check that the soluton s correct by substtutng backinto *boh* equatons

$\therefore$ Solution is $a = -1, b = 4$

b

$5x - 3y = 19$ [1]

$2x - 4y = 16$ [2]

[1] × 4 $20x - 12y = 76$ [3]

[2] × 3: $6x - 12y = 48$ [4]

[3] − [4] $14x = 28$

$x = 2$

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Substitute $x = 2$ in [2]

$$2(2) - 4y = 16$$
$$4 - 4y = 16$$
$$-4y = 12$$
$$y = -3$$

$\therefore$ Solution is $x = 2, y = -3$

Exercise 2.10 Linear simultaneous equations

Solve each pair of simultaneous equations

1 $a - b = -2$ and $a + b = 4$

2 $5x + 2y = 12$ and $3x - 2y = 4$

3 $4p - 3q = 11$ and $5p + 3q = 7$

4 $y = 3x - 1$ and $y = 2x + 5$

5 $2x + 3y = -14$ and $x + 3y = -4$

6 $7t + v = 22$ and $4t + v = 13$

7 $4x + 5y + 2 = 0$ and $4x + y + 10 = 0$

8 $2x - 4y = 28$ and $2x - 3y = -11$

9 $5x - y = 19$ and $2x + 5y = -14$

10 $5m + 4n = 22$ and $m - 5n = -13$

11 $4w \ + 3w_2 = 11$ and $3w \ + w_2 = 2$

12 $3a - 4b = -16$ and $2a + 3b = 12$

13 $5p + 2q + 18 = 0$ and $2p - 3q + 11 = 0$

14 $7x \ + 3x_2 = 4$ and $3x \ + 5x_2 = -2$

15 $9x - 2y = -1$ and $7x - 4y = 9$

16 $5s - 3t - 13 = 0$ and $3s - 7t - 13 = 0$

17 $3a - 2b = -6$ and $a - 3b = -2$

18 $3k - 2h = -14$ and $2k - 5h = -13$

PROBLEM

A group of 39 people went to see a play. There were both adults and children in the group The total cost of the tickets was $99, with children paying $17 each and adults paying $29 each How many in the group were adults and how many were children ? (Hint let x be the number of adults and y the number of children)

2.11 Non-linear simultaneous equations

In simultaneous equations involving **non-linear equations** there may be more than one set of solutions When solving thee, you need to use the substitution metod.

Nonlinea imulaneou equaion

EXAMPLE 16

Solve each pair of equations simultaneously using the substitution method

a $xy = 6$ and $x + y = 5$

b $x^2 + y^2 = 16$ and $3x - 4y - 20 = 0$

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Solution

a

$$xy = 6 \quad [1]$$

$$x + y = 5 \quad [2]$$

From [2] $\quad y = 5 - x \quad [3]$

Substitute [3] in [1] $\quad x(5 - x) = 6$

$$5x - x^2 = 6$$

$$0 = x^2 - 5x + 6$$

$$0 = (x - 2)(x - 3)$$

$$\therefore x = 2 \text{ or } x = 3$$

Substitute $x = 2$ in [3] $\quad y = 5 - 2 = 3$

Substitute $x = 3$ in [3] $\quad y = 5 - 3 = 2$

Solutions are $x = 2, y = 3$ and $x = 3, y = 2$

b

$$x^2 + y^2 = 16 \quad [1]$$

$$3x - 4y - 20 = 0 \quad [2]$$

From [2] $\quad 3x - 20 = 4y$

$$\frac{3x - 20}{4} = y \quad [3]$$

Substitute [3] into [1] $\quad x^2 + \left(\frac{3x - 20}{4}\right)^2 = 16$

$$x^2 + \left(\frac{9x^2 - 120x + 400}{16}\right) = 16$$

$$16x^2 + 9x^2 - 120x + 400 = 256$$

$$25x^2 - 120x + 144 = 0$$

$$(5x - 12)^2 = 0$$

$$\therefore 5x - 12 = 0$$

$$x = 24$$

Substitute $x = 24$ into [3] $\quad y = \frac{3(2\ 4) - 20}{4}$

$$= -3\ 2$$

So the solution is $x = 24,\ y = -32$

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Exercise 2.11 Non-linear simultaneous equations

Solve each pair of simultaneous equations

1. $y = x^2$ and $y = x$
2. $y = x^2$ and $2x + y = 0$
3. $x^2 + y^2 = 9$ and $x + y = 3$
4. $x - y = 7$ and $xy = -12$
5. $y = x^2 + 4x$ and $2x - y - 1 = 0$
6. $y = x^2$ and $6x - y - 9 = 0$
7. $x = t^2$ and $x + t - 2 = 0$
8. $m^2 + n^2 = 16$ and $m + n + 4 = 0$
9. $xy = 2$ and $y = 2x$
10. $y = x^3$ and $y = x^2$
11. $y = x - 1$ and $y = x^2 - 3$
12. $y = x^2 + 1$ and $y = 1 - x^2$
13. $y = x^2 - 3x + 7$ and $y = 2x + 3$
14. $xy = 1$ and $4x - y + 3 = 0$
15. $h = t^2$ and $h = (t + 1)^2$
16. $x + y = 2$ and $2x^2 + xy - y^2 = 8$
17. $y = x^3$ and $y = x^2 + 6x$
18. $y = |x|$ and $y = x^2$
19. $y = x^2 - 7x + 6$ and $24x + 4y - 23 = 0$
20. $x^2 + y^2 = 1$ and $5x + 12y + 13 = 0$

2.12 Simultaneous equations with three unknown variables

WS
Simulaneou equaion

Three equations can be solved simultaneously to find 3 unknown pronumerals

EXAMPLE 17

Solve simultaneously $a - b + c = 7$, $a + 2b - c = -4$ and $3a - b - c = 3$.

Solution

$$a - b + c = 7 \quad [1]$$
$$a + 2b - c = -4 \quad [2]$$
$$3a - b - c = 3 \quad [3]$$

[1] + [2]

$$\begin{array}{r} a - \ \ b + c = 7 \\ a + 2b - c = -4 \\ \hline 2a + \ \ b \qquad = 3 \end{array} \quad [4]$$

[1] + [3]

$$\begin{array}{r} a - \ \ b + c = 7 \\ 3a - \ \ b - c = 3 \\ \hline 4a - 2b \qquad = 10 \end{array}$$

or

$$2a - b = 5 \quad [5]$$

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[4] + [5]	$2a + b = 3$
	$2a - b = 5$
	$4a = 8$
	$a = 2$
Substitute $a = 2$ in [4]	$2(2) + b = 3$
	$4 + b = 3$
	$b = -1$
Substitute $a = 2$ and $b = -1$ in [1]	$2 - (-1) + c = 7$
	$2 + 1 + c = 7$
	$3 + c = 7$
	$c = 4$
	$\therefore$ solution is $a = 2, b = -1, c = 4$

Exercise 2.12 Simultaneous equations with three unknown variables

Solve each set of simultaneous equations

1 $x = -2$, $2x - y = 4$ and $x - y + 6z = 0$

2 $a = -2$, $2a - 3b = -1$ and $a - b + 5c = 9$

3 $2a + b + c = 1$, $a + b = -2$ and $c = 7$

4 $a + b + c = 0$, $a - b + c = -4$ and $2a - 3b - c = -1$

5 $x + y - z = 7$, $x + y + 2z = 1$ and $3x + y - 2z = 19$

6 $2p + 5q - r = 25$, $2p - 2q - r = -24$ and $3p - q + 5r = 4$

7 $2x - y + 3z = 9$, $3x + y - 2z = -2$ and $3x - y + 5z = 14$

8 $x - y - z = 1$, $2x + y - z = -9$ and $2x - 3y - 2z = 7$

9 $3h + j - k = -3$, $h + 2j + k = -3$ and $5h - 3j - 2k = -13$

10 $2a - 7b + 3c = 7$, $a + 3b + 2c = -4$ and $4a + 5b - c = 9$

2. TEST YOURSELF

Qz
Pacice quiz

For Questions **1** to **3** select the correct answer **A B C** or **D**

1 Find the exact solution of $x^2 - 5x - 1 = 0$

A $\frac{-5\pm\sqrt{29}}{2}$ **B** $\frac{5\pm\sqrt{21}}{2}$ **C** $\frac{5\pm\sqrt{29}}{2}$ **D** $\frac{-5\pm\sqrt{21}}{2}$

2 If $S = 4\pi r^2$ find the value of r when $S = 200$ (there may be more than one answer)

A $5\sqrt{\frac{2}{\pi}}$ **B** $\sqrt{\frac{200}{\pi}}$ **C** $10\sqrt{\frac{2}{\pi}}$ **D** $\sqrt{\frac{50}{\pi}}$

3 Solve the simultaneous equations $x - y = 7$ and $x + 2y = 1$.

A $x = 5, y = 2$ **B** $x = 5, y = -2$ **C** $x = -5, y = -2$ **D** $x = -5, y = 2$

4 Solve

a $8 = 3b - 22$ **b** $\frac{a}{4} - \frac{a+2}{3} = 9$ **c** $4(3x + 1) = 11x - 3$

d $3p + 1 \le p + 9$

5 The compound interest formula is $A = P(1 + r)^n$ Fin, correct to 2 decimal placs:

a A when $P = 1000$ $r = 006$ and $n = 4$

b P when $A = 12\,450, r = 0055$ and $n = 7$.

6 Solve each pair of simultaneous equations

a $x - y + 7 = 0$ and $3x - 4y + 26 = 0$ **b** $xy = 4$ and $2x - y - 7 = 0$

7 Solve

a $3^{x+2} = 81$ **b** $16^y = 2$

8 Solve $|3b - 1| = 5$.

9 The area of a trapezium is given by $A = \frac{1}{2}h(a + b)$ Fin:

a A when $h = 6, a = 5$ and $b = 7$

b b when $A = 40$ $h = 5$ and $a = 4$

10 Solve $2x^2 - 3x + 1 = 0$

11 Solve $-2 < 3y + 1 \le 10$ and plot the solution on a number line

12 Solve correct to 3 significant figure:

a $x^2 + 7x + 2 = 0$ **b** $y^2 - 2y - 9 = 0$ **c** $3n^2 + 2n - 4 = 0$

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13 The surface area of a sphere is given by $A = 4\pi r^2$ Evaluate to 1 decimal plac:

a A when $r = 78$ **b** r when $A = 1029$

14 Solve the simultaneous equations $x^2 + y^2 = 16$ and $3x + 4y - 20 = 0$

15 The volume of a sphere is $V = \frac{4}{3}\pi r^3$ Evaluate to 2 significant figure:

a V when $r = 8$ **b** r when $V = 250$

16 For each equation decide if it ha:

A 2 solutions **B** 1 solution **C** no solutions

a $x^2 - 6x + 9 = 0$ **b** $|2x - 3| = 7$ **c** $x^2 - x - 5 = 0$

d $2x^2 - x + 4 = 0$ **e** $3x + 2 = 7$

17 Solve simultaneously $a + b = 5$, $2a + b + c = 4$, $a - b - c = 5$.

18 Solve $9^{2x+1} = 27^x$

19 Solve

a $2(3y - 5) > y + 5$ **b** $3^{2x-1} = 27$ **c** $5x^3 - 1 = 39$

d $|5x - 4| = 11$ **e** $8^{x+1} = 4^x$ **f** $27^{2x-1} = 9$

2. CHALLENGE EXERCISE

1 Find the value of y if $a^{3y-5} = \frac{1}{a^2}$

2 The solutions of $x^2 - 6x - 3 = 0$ are in the form $a + b\sqrt{3}$ Find the values of a and b

3 **a** Factorise $x^5 - 9x^3 - 8x^2 + 72$

b Hence or otherwise solve $x^5 - 9x^3 - 8x^2 + 72 = 0$

4 Solve the simultaneous equations $y = x^3 + x^2$ and $y = x + 1$.

5 Find the value of b if $x^2 - 8x + b$ is a perfect square Hence solve $x^2 - 8x - 1 = 0$ by completing the square

6 Considering the definition of absolute value solve $\frac{|x-3|}{3-x} = x$ where $x \neq 3$.

7 Solve $x^{\frac{3}{2}} = \frac{1}{8}$

8 Find the solutions of $x^2 - 2ax - b = 0$ by completing the square

9 Solve $3x^2 = 8(2x - 1)$ and write the solution in the simplest surd form

10 Solve $|2x - 1| = 5 - x$ and check solutions

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Practice set 1

In Questions **1** to **6** select the correct answer **A B C** or **D**

1 Write $\dfrac{1}{3\sqrt{(x-2)^5}}$ in index form

A $(x-2)^{-\frac{5}{3}}$ **B** $\dfrac{(x-2)^{-\frac{5}{2}}}{3}$ **C** $3(x-2)^{-\frac{5}{2}}$ **D** $\dfrac{1}{(x-2)^{\frac{5}{3}}}$

2 Simplify $\dfrac{(2a^3b)^3}{(ab)^2}$

A $8a^7b$ **B** $8a^8b$ **C** $2a^7b$ **D** $2a^8b$

3 Evaluate $4^{-\frac{3}{2}}$

A -8 **B** $\dfrac{1}{8}$ **C** $\dfrac{1}{6}$ **D** -6

4 Simplify $\dfrac{a^2-6a+9}{a^2-9}$

A $\dfrac{1}{a+3}$ **B** $\dfrac{a-3}{a+3}$

C $\dfrac{a+3}{a-3}$ **D** $\dfrac{-6a+9}{a-9}$

5 Factorise $a^2-\dfrac{b^2}{4}$

A $\left(a-\dfrac{b}{2}\right)^2$ **B** $\left(a+\dfrac{b}{4}\right)\left(a-\dfrac{b}{4}\right)$

C $\left(a+\dfrac{b}{2}\right)^2$ **D** $\left(a+\dfrac{b}{2}\right)\left(a-\dfrac{b}{2}\right)$

6 The solution to $x^2+2x-6=0$ is

A $x=-1\pm2\sqrt{7}$ **B** $x=\dfrac{2\pm\sqrt{28}}{2}$

C $x=\dfrac{-2\pm\sqrt{-20}}{2}$ **D** $x=-1\pm\sqrt{7}$

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7 Solve

a $3x - 7 = 23$ **b** $5(b - 3) = 15$ **c** $\frac{x}{3} + 4 = 5$

d $4y - 7 = 3y + 9$ **e** $8z + 1 = 11z - 17$ **f** $2^x = 32$

g $9^{y-1} = 3$ **h** $x^2 - 3x = 0$ **i** $|x + 2| = 5$

j $|5a - 2| = 8$

8 Solve for p $\frac{p-3}{2} - \frac{p+1}{5} = 1$.

9 Simplify $2\sqrt{12}$

10 Factorise fully $10x + 2xy - 10y - 2y^2$

11 Write in index for:

a $\frac{1}{x}$ **b** $\sqrt[3]{x^4}$

12 Simplify the expression $8y - 2(y + 5)$

13 Rationalise the denominator of $\frac{5}{5 - \sqrt{2}}$

14 Solve $2x^2 - 3x - 1 = 0$ correct to 3 significant figures

15 Simplify $\frac{x+1}{5} \div \frac{x^2 - 2x - 3}{10}$

16 Evaluate $(39)^4$ correct to 1 decimal place

17 Simplify $2\sqrt{3} - \sqrt{27}$

18 Expand and simplify $(x - 3)(x^2 + 5x - 1)$.

19 Expand and simplify $\sqrt{2}(3\sqrt{5} - 2\sqrt{2})$

20 Simplify $\frac{2x + 6}{2}$

21 Solve $4a - 5 < 7a + 4$

22 The radius r of a circle with area A is given by $r = \sqrt{\frac{A}{\pi}}$ Find r correct to 2 decimal places if $A = 759$

23 Solve each set of simultaneous equations

a $3a - b = 7$ and $2a + b = 8$

b $a + b - c = 8$, $b + c = 5$ and $a + 2c = 3$

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24 Solve $5 - 2x < 3$ and show the solution on a number line

25 Solve the equation $x^2 - 4x + 1 = 0$ giving exact solutions in simplest surd for.

26 Write 7^{-2} as a rational number.

27 Solve the simultaneous equations $y = 3x - 1$ and $y = x^2 - 5$.

28 Find integers x and y such that $\frac{\sqrt{3}}{2\sqrt{3}+3} = x + y\sqrt{3}$

29 Evaluate $|-2|^2 - |-1| + |4|$

30 Factorise $8x^2 - 32$.

31 Rationalise the denominator of $\frac{2\sqrt{3}}{3\sqrt{5}-\sqrt{2}}$

32 Simplify $2|-4| - |3| + |-2|$

33 Rationalise the denominator of $\frac{\sqrt{5}+1}{2\sqrt{2}+3}$

34 Simplify $\frac{(a^{-4})^3 \times b^6}{a^9 \times (b^{-1})^4}$

35 Evaluate $4^{-\frac{3}{2}}$ as a rational number.

36 Simplify $2(x - 5) - 3(x - 1)$.

37 Solve $4^{2x+1} = 8$.

38 Write $\frac{1}{x+3}$ in index form

39 Find the value of a^3b^{-2} in index form if $a = \left(\frac{1}{2}\right)^3$ and $b = \left(\frac{4}{5}\right)^2$

40 Write $(3x+2)^{-\frac{}{2}}$ without an index

41 Simplify

a $8x - 7y - y + 4x$

b $\sqrt{124}$

c $\dfrac{x^2 - 9}{2x^2 + 5x - 3}$

d $\dfrac{1}{\sqrt{2}+1} + \dfrac{2}{\sqrt{2}-1}$

e $\dfrac{3}{x+1} + \dfrac{2}{x^2-1} - \dfrac{4}{x-1}$

f $x - \dfrac{1}{x}$ when $x = 2\sqrt{3}$

g $\dfrac{(x^{-2})^5 y^4 z^{-3}}{x^4 (y^3)^{-1} (z^{-4})^{-2}}$

h $\dfrac{a+b}{5a - 20ab^2} \div \dfrac{a^2 + 2ab + b^2}{3 - 6b}$

i $8\sqrt{5} - 3\sqrt{20} + 2\sqrt{45}$

j $\dfrac{a^3 b^2 (c^4)^2}{(a^2)^2 bc^5}$ if $a = \left(\dfrac{1}{2}\right)^2$ $b = \left(\dfrac{2}{3}\right)^3$ and $c = \left(\dfrac{4}{9}\right)^{-}$

42 The volume of a sphere is given by the formula $V = \dfrac{4}{3}\pi r^3$ Find the exact radius r if the volume V is $10\frac{2}{3}$ cm^3

43 Find the value of k if $(2x + 5)^2 = 4x^2 + kx + 25$

44 Simplify $\sqrt{81x^2 y^3}$

45 Factorise

a $5(a - 2)^2 + 40(a - 2)$

b $(2a - b + c)^2 - (a + 5b - c)^2$

46 Solve $-2 \le \dfrac{8x - 1}{5} < 9$

47 Simplify $\dfrac{x+1}{5} - \dfrac{x+2}{3}$

48 Solve $x^2 - 5x = 0$

49 Solve $x^2 - 5x - 1 = 0$ and write the solutions correct to 2 decimal places

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50 Simplify $\sqrt{8}+\sqrt{98}$

51 Write $\frac{3}{x^2+5x}-\frac{4}{x}+\frac{2}{x+5}$ as a single fraction

52 Solve for x $4^{2x-1}=\frac{1}{8}$

53 Factorise

a x^2-2x-8 **b** a^2-9 **c** y^2+6y+9

d $t^2+8t+16$ **e** $3x^2-11x+6$

54 Solve

a $5x-4=2x+11$ **b** $y^2-2y-13=0$ (correct to 2 decimal places)

c $4^{2x}=8$ **d** $|2b+3|=7$

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3.

FUNCTIONS

Functons and ther graphs are used n many areas such as mathematcs scence and economcs In ths chapter you wll explore what functons are and how to sketch some types of graphs ncludng straght lnes parabolas and cubcs

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- understand the defnton of a functon and use functon notaton
- test a functon usng the verical ine test
- dentfy a one-to-one functon usng the horzontal lne test
- fnd the doman and range of functons ncludng composte functons usng nterval notaion
- dentfy even and odd functons
- understand a lnear functon ts graph and properie,incluing the graient and axesintercepts
- graph stuatons nvolvng drect lnear varaton
- fnd the equaton of a lne ncludng parallel and perpendcular lnes
- dentfy a quadratc functon ts graph and properie,incluingits ais of symmetry, trnin point and axes ntercepts
- solve quadratc equatons and use the dscrmnant to dentfy the numbers and types of solutons
- fnd the quadratc equaton of a parabola
- dentfy a cubc functon ts graph and properie,incluing the shap, hoizontal pint of nflecton and axes ntercepts
- fnd a cubc equaton
- dentfy a polynomal and ts characterstcs
- draw the graph of a polynomal showng ntercepts
- solve smultaneous equatons nvolvng lnear and quadratc equatons both algebracally and graphcally, and solve problemsinvolingintersecion of graphs of funcions(for exampl, break-even ponts

TERMINOLOGY

angle of inclination The angle a straight line makes with the positive x-axis measured anticlockwise

axis of symmetry A line that divides a shape into halves that are mirror-images of each other

break-even point The point at which a busines' income equals its costs making neither a profit nor a loss

coefficient A constant multiplied by a pronumeral in an algebraic term For exampl, in ax^3 the a is the coefficient

constant term The term in a polynomial function that is independent of x

cubic function A function with x^3 as its highest power or degree

degree The highest power of x in a polynomial

dependent variable A variable whose value depends on another (independent) variable such as y (depending on x)

direct variation A relationship between two variables such that as one variable increases so does the other, or as one variable decreases so does the other. One variable is a multiple of the other, with equation $y = kx$ Also called **direct proportion**

discriminant The expression $b^2 - 4ac$ that shows how many roots the quadratic equation $ax^2 + bx + c = 0$ has

domain The set of all possible values of x for a function or relation the set of 'inpu' values

even function A function $f(x)$ that has the property $f(-x) = f(x)$ its graph is symmetrical about the y-axis

function A relation where every x value in the domain has a unique y value in the range

gradient The steepness of a graph at a point on the graph measured by the ratio $\frac{\text{rise}}{\text{run}}$ or the change in y values as x values change

horizontal line test A test that checks if a function is one-to-one whereby any horizontal line drawn on the graph of a function should cut the graph at most once If the horizontal line cuts the graph more than once it is not one-to-one

independent variable A variable whose value does not depend on another variable for example x in $y = f(x)$

intercepts The values where a graph cuts the x- and y- axes

interval notation A notation that represents an interval by writing its endpoints in square brackets [ ] when they are included and in parentheses ( ) when they are not included

leading coefficient The coefficient of the highest power of x For exampl, $2\ x^4 - x^3 + 3x + 1$ has a leading coefficient of 2

leading term The term with the highest power of x For exampl, $2\ x^4 - x^3 + 3x + 1$ has a leading term of $2x^4$

linear function A function with x as its highest power or degree

monic polynomial A polynomial whose leading coefficient is 1

odd function A function $f(x)$ that has the property $f(-x) = -f(x)$ its graph has point symmetry about the origin (0 0)

one-to-one function A function in which every y value in the range corresponds to exactly one x value in the domain

parabola The graph of a quadratic function

piecewise function A function that has different functions defined on different intervals

point of inflection A point on a curve where the concavity changes such as the turning point on the graph of a cubic function

polynomial An expression in the form $P(x) = a_n x^n + \quad + a_2 x^2 + a\ x + a_0$ where n is a positive integer or zero

quadratic function A function with x^2 as the highest power of x

range The set of all possible y values of a function or relation the set of 'outpu' values

root A solution of an equation

turning point Where a graph changes from increasing to decreasing or vice versa sometimes a turning point (horizontal inflection) where concavity changes

vertex A turning point

vertical line test A test that checks if a relation is a function whereby any vertical line drawn on the graph of a relation should cut the graph at most once If the vertical line cuts the graph more than once it is not a function

zero An x value of a function or polynomial for which the y value is zero that i, $f(x) = 0$

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3.01 Functions

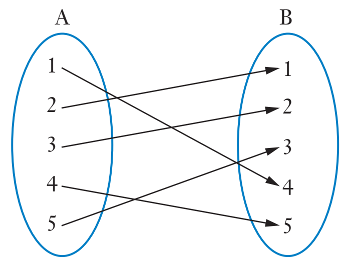
Funcions and elaion

A **relation** is a set of **ordered pairs** (x y) where the **variables** x and y are related according to some pattern or rule The x is called the **independent variable** and the y is called the **dependent variable** because the value of y depends on the value of x We usually choose a value of x and use it to find the corresponding value of y

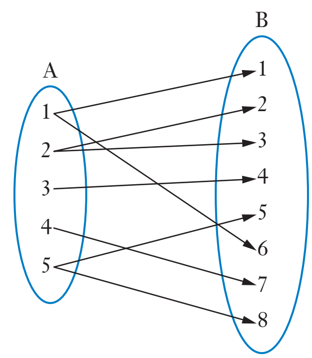
A relation can also be described as a mapping between 2 sets of numbers with the set of x values A, on the left and the set of y values B on the righ.

Types of relations

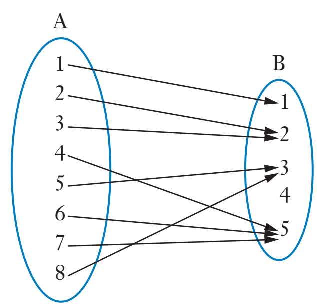
A **one-to-one** relation is a mapping where every element of A corresponds with exactly one element of B and every element of B corresponds with exactly one element of . Each element has its own unique match



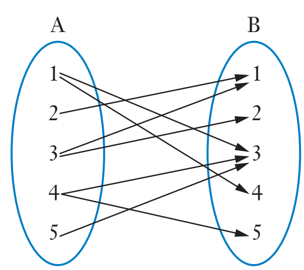
A **one-to-many** relation is a mapping where an element of A corresponds with 2 or more elements of B For exampl, 5 in set A matches with 5 and 8 in set .



A **many-to-one** relation is a mapping where 2 or more elements of A correspond with the same one element of . For example , 6 and 7 in set A match with 5 in se B.



A **many-to-many** relation is a mapping where 2 or more elements of A correspond with 2 or more elements of . This is a combination of the one-to-many and many-to-one relations



Function

A **function** is a special type of relation where for every value of x there is a unique value of y

The **domain** is the set of all values of x for which a function is defined

The **range** is the set of all values of y as x varies

A function could be a one-to-one or many-to-one relation

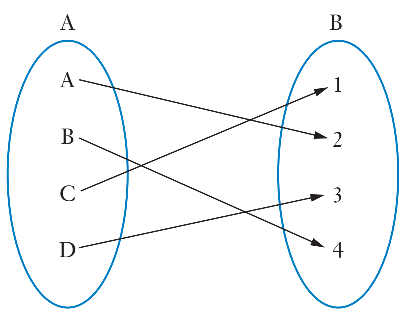
For example this table matches a group of people with their eye colour.

Person	Anne	Jacquie	Donna	Hien	Marco	Russell	Trang
Colour	Blue	Brown	Grey	Brown	Green	Brown	Brown

The ordered pairs are (Anne Blue, (Jacque, Bron), (Dnna, rey),(Hien,Brown), (Marco, Green) (Russel, Brown) and (Trag, Bron).

This table represents a function since for every person there is a unique eye colou. The domain is the set of people the range is the set of eye colours It is a many-to-one function since more than one person can correspond to one eye colour.

Here is a different function



Set A is the domai, set B is the rane.

The ordered pairs are (A 2, B,4), (C, 1) ad (, 3).

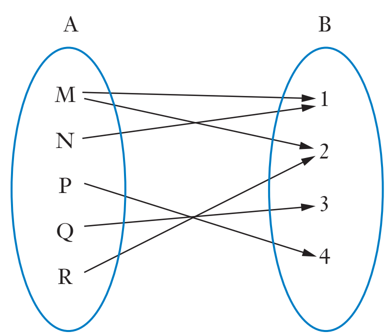
It is a function because every x value in A corresponds to exactly one x value in B

It is a one-to-one function because every y value in B corresponds to exactly one x value in .

Here is an example of a relation that is **not** a function Can you see why?

In this example the ordered pairs are (M 1, M,2), (N, 1), (P, 4), (Q, 3) and(R, 2).

Notice that M corresponds to 2 values in set B 1 and 2 This means that it is **not** a function Notice also that M and R both correspond with the same value 2 This is a many-to-many relation



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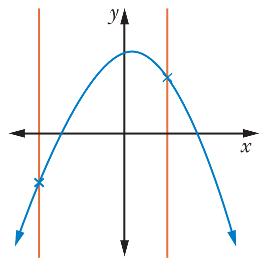
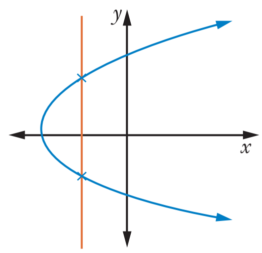
The vertical line test

Relations can also be described by algebraic rules or equations such as $y = x^2 + 1$ and $x^2 + y^2 = 4$, and hence graphed on a number plane There is a very simple test called the **vertical line test** to test if a graph represents a function

If any vertical line crosses a graph at only one point the graph represents a function This shows tht, for every value of x there is only one value of y



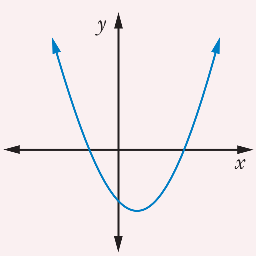
If any vertical line crosses a graph at more than one point the graph does not represent a function This shows tht, for some value of x there is more than one value of y



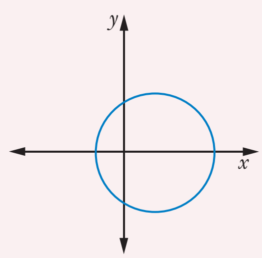
EXAMPLE 1

Does each graph or set of ordered pairs represent a function?

a

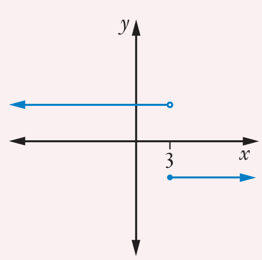


b



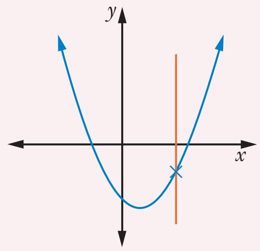
c (–2, 3), (–1, 4), (0, 5), (1, 3), (2, 4)

d



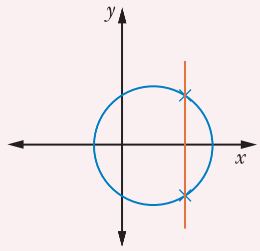
Solution

a



A vertical line only cuts the graph once So the graph represents a function

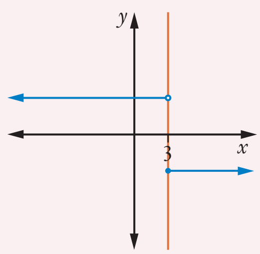
b



A vertical line can cut the curve in more than one place So the circle does not represent a function

c For each x value there is only one y value so this set of ordered pairs is a functio.

d



The open circle at $x = 3$ on the top line means that $x = 3$ is not included while the closed circle on the bottom line means that $x = 3$ is included on this line

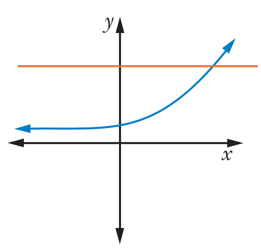
So a vertical line only touches the graph once at $x = 3$.

The graph represents a function

The horizontal line test

The **horizontal line test** is used on the graph of a function to test whether the function is **one-to-one**

If any horizontal line crosses a graph at only one point there is only one x value for every y value The graph represents a one-to-one function



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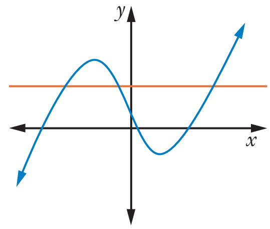
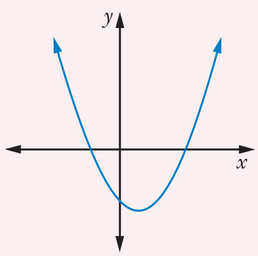
If any horizontal line crosses a graph at more than one point this means that there are 2 or more x values that have the same y value The graph does not represent a one-to-one function



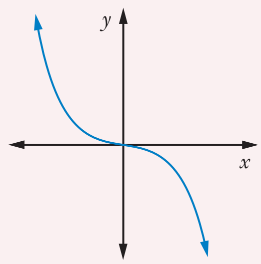
EXAMPLE 2

Does each graph represent a one-to-one function?

a

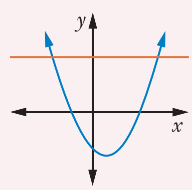


b

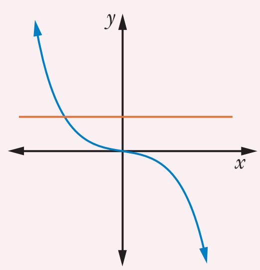


Solution

a A horizontal line cuts the curve in more than one place The function is not one-to-one



b A horizontal line cuts the curve in only one place The function is one-to-one



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DID YOU KNOW?

René Descartes

The number plane is called the **Cartesian plane** after René Descartes (1596–1650) Descartes used the number plane to develop analytical geometry. He discovered that any equation with two unknown variables can be represented by a line The points in the number plane can be called Cartesian coordinates

Descartes used letters at the beginning of the alphabet to stand for numbers that are known and letters near the end of the alphabet for unknown number. This is why we still use x and y so often

Research Descartes to find out more about his life and wok.

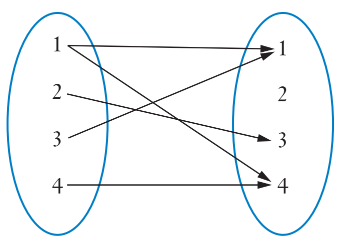
Exercise 3.01 Functions

1 List the ordered pairs for each relation then state whether the relation is a one-to-on, one-to-many, many-to-one or many-to-may.

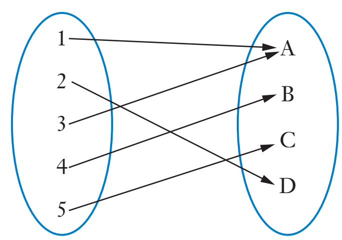
a

Name	Wade	Scott	Geoff	Deng	Mila	Stevie
Hair colour	Black	Blond	Grey	Black	Brown	Blond

b



c



d

x	3	5	8	9	5	8
y	5	±2	−7	3	6	0

e

x	y
1	9
2	15
3	27
4	33
5	45

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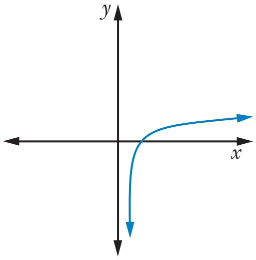
2 Does each graph or set of ordered pairs represent a function? If it does state whether it is one-to-one

a



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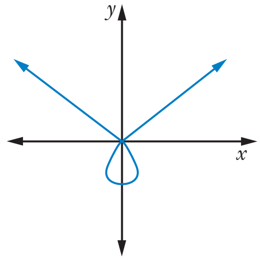
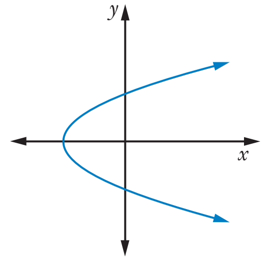
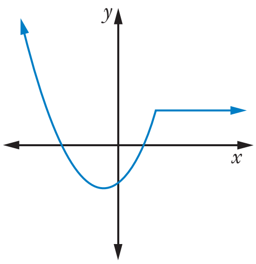


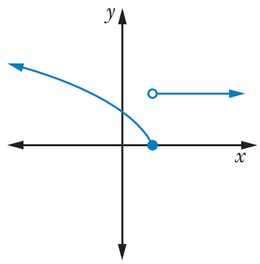
c



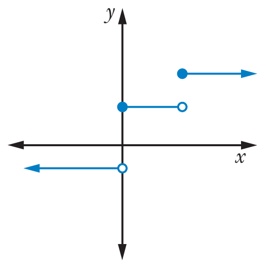
d



e



f



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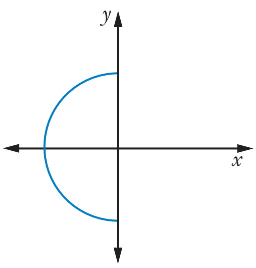

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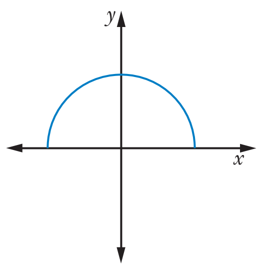
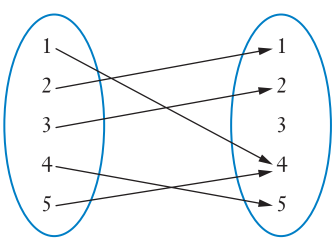
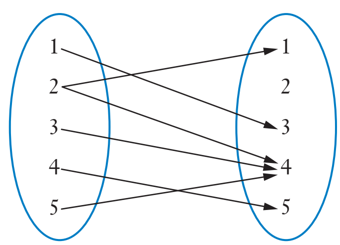


i (1, 3), (2, −1), (3, 3), (4, 0)

j (1, 3), (2, −1), (2, 7), (4, 0)

k





m (2, 5), (3, −1), (4, 0), (−1, 3), (−2, 7)

n

Person	Ben	Paula	Pierre	Hamish	Jacob	Leanne	Pierre	Lien
Sport	Tennis	Football	Tennis	Football	Football	Badminton	Football	Badminton

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o

A	3
B	4
C	7
D	3
E	5
F	7
G	4

3 A relation consists of the ordered pairs (–3, 4), (–1, 5), (0, –2), (1, 4) and (6,8).

a Write the set of independent variable, x

b Write the set of dependent variable, y

c Describe the relation as one-to-one one-to-man, many-to-one or many-to-mny.

d Is the relation a function?

WS

Funcion noaion

3.02 Function notation

Since the value of y depends on the value of x we say that y is a function of x We write this using **function notation** as $y = f(x)$

EXAMPLE 3

a Find the value of y when $x = 3$ in the equation $y = 2x - 1$.

b Evaluate $f(3)$ given $f(x) = 2x - 1$.

Solution

a When $x = 3$:

$$\begin{aligned} y &= 2(3) - 1 \\ &= 6 - 1 \\ &= 5 \end{aligned}$$

b

$$\begin{aligned} f(x) &= 2x - 1 \\ f(3) &= 2(3) - 1 \\ &= 6 - 1 \\ &= 5 \end{aligned}$$

Both questions in Example 3 are the same but the second one looks different because it uses function notation

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EXAMPLE 4

a If $f(x) = x^2 + 3x + 1$, find $f(-2)$

b If $f(x) = x^3 - x^2$ find the value of $f(-1)$

c Find the values of x for which $f(x) = 0$ given that $f(x) = x^2 + 3x - 10$.

Solution

a

$$\begin{aligned} f(x) &= x^2 + 3x + 1 \\ f(-2) &= (-2)^2 + 3(-2) + 1 \\ &= 4 - 6 + 1 \\ &= -1 \end{aligned}$$

b

$$\begin{aligned} f(x) &= x^3 - x^2 \\ f(-1) &= (-1)^3 - (-1)^2 \\ &= -1 - 1 \\ &= -2 \end{aligned}$$

c

$$\begin{aligned} f(x) &= 0 \\ x^2 + 3x - 10 &= 0 \\ (x + 5)(x - 2) &= 0 \\ x = -5,\ x &= 2 \end{aligned}$$

A **piecewise function** is a function made up of 2 or more functions defined on different intervas.

EXAMPLE 5

a

$$f(x) = \begin{cases} 3x + 4 & \text{when } x \geq 2 \\ -2x & \text{when } x < 2 \end{cases}$$

Find $f(3)$ $f(2)$ $f(0)$ and $f(-4)$

b

$$g(x) = \begin{cases} x^2 & \text{when } x > 2 \\ 2x - 1 & \text{when } -1 \leq x \leq 2 \\ 5 & \text{when } x < -1 \end{cases}$$

Find $g(1) + g(-2) - g(3)$

Solution

a

$$\begin{aligned} f(3) &= 3(3) + 4 && \text{since } 3 \geq 2 \\ &= 13 \\ f(2) &= 3(2) + 4 && \text{since } 2 \geq 2 \\ &= 10 \\ f(0) &= -2(0) && \text{since } 0 < 2 \\ &= 0 \\ f(-4) &= -2(-4) && \text{since } -4 < 2 \\ &= 8 \end{aligned}$$

b

$$\begin{aligned} g(1) &= 2(1) - 1 && \text{since } -1 \leq 1 \leq 2 \\ &= 1 \\ g(-2) &= 5 && \text{since } -2 < -1 \\ g(3) &= 3^2 && \text{since } 3 > 2 \\ &= 9 \end{aligned}$$

$$\text{So } g(1) + g(-2) - g(3) = 1 + 5 - 9 = -3$$

You can also substitute pronumerals instead of numbers into function.

EXAMPLE 6

Find $f(h + 1)$ given $f(x) = 5x + 4$

Solution

Substitute $h + 1$ for x

$$f(h + 1) = 5(h + 1) + 4$$
$$= 5h + 5 + 4$$
$$= 5h + 9$$

DID YOU KNOW?

Leonhard Euler

Leonhard Euler (1707–83) from Switzerlan, studied functions and invented the function notation $f(x)$ He studied theolog, astronmy, medcine, physics and oriental languages as well as mathematics and wrote more than 500 books and articles on mathematics He found time between books to marry and have 13 childre, and even when he went blind he kept on having books published

Exercise 3.02 Function notation

1 Given $f(x) = x + 3$ find $f(1)$ and $f(-3)$

2 If $h(x) = x^2 - 2$ find $h(0)$ $h(2)$ and $h(-4)$

3 If $f(x) = -x^2$ find $f(5)$ $f(-1)$ $f(3)$ and $f(-2)$

4 Find the value of $f(0) + f(-2)$ if $f(x) = x^4 - x^2 + 1$.

5 Find $f(-3)$ if $f(x) = 2x^3 - 5x + 4$

6 If $f(x) = 2x - 5$ find x when $f(x) = 13$.

7 Given $f(x) = x^2 + 3$ find any values of x for which $f(x) = 28$

8 If $f(x) = 3^x$ find x when $f(x) = \frac{1}{27}$

9 Find values of z for which $f(z) = 5$ given $f(z) = |2z + 3|$

10 If $f(x) = 2x - 9$ find $f(p)$ and $f(x + h)$

11 Find $g(x - 1)$ when $g(x) = x^2 + 2x + 3$.

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12 If $f(x) = x^2 - 1$, find $f(k)$ as a product of factors

13 Given $f(t) = t^2 - 2t + 1$, fin:

a t when $f(t) = 0$ **b** any values of t for which $f(t) = 9$

14 Given $f(t) = t^4 + t^2 - 5$ find the value of $f(b) - f(-b)$

15 $f(x) = \begin{cases} x^3 & \text{for } x > 1 \\ x & \text{for } x \circ 1 \end{cases}$

Find $f(5)$ $f(1)$ and $f(-1)$

16 $f(x) = \begin{cases} 2x - 4 & \text{if } x > 1 \\ x + 3 & \text{if } -1 \leq x \leq 1 \\ x^2 & \text{if } x < -1 \end{cases}$

Find the value of $f(2) - f(-2) + f(-1)$

17 Find $g(3) + g(0) + g(-2)$ if $g(x) = \begin{cases} x + 1 & \text{when } x \circ 0 \\ -2x + 1 & \text{when } x < 0 \end{cases}$

18 Find the value of $f(3) - f(2) + 2f(-3)$ when $f(x) = \begin{cases} x & \text{for } x > 2 \\ x^2 & \text{for } -2 \leq x \leq 2 \\ 4 & \text{for } x < -2 \end{cases}$

19 Find the value of $f(-1) - f(3)$ if $f(x) = \begin{cases} x^3 - 1 & \text{for } x \circ 2 \\ 2x^2 + 3x - 1 & \text{for } x < 2 \end{cases}$

20 If $f(x) = x^2 - 5x + 4$ find $f(x + h) - f(x)$ in its simplest form

21 Simplify $\dfrac{f(x+h) - f(h)}{h}$ where $f(x) = 2x^2 + x$

22 If $f(x) = 5x - 4$ find $f(x) - f(c)$ in its simplest form

23 Find the value of $f(k^2)$ if $f(x) = \begin{cases} 3x + 5 & \text{for } x \circ 0 \\ x^2 & \text{for } x < 0 \end{cases}$

24 If $f(x) = \begin{cases} x^3 & \text{when } x \geq 3 \\ 5 & \text{when } 0 < x < 3 \\ x^2 - x + 2 & \text{when } x \leq 0 \end{cases}$

evaluate

a $f(0)$ **b** $f(2) - f(1)$ **c** $f(-n^2)$

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25 If $f(x) = \dfrac{x^2 - 2x - 3}{x - 3}$

a evaluate $f(2)$

b explain why the function does not exist for $x = 3$.

c by taking several x values close to 3 find the value of y that the function is moving towards as x moves towards 3

WS

Funcion noaion

3.03 Properties of functions

We can use the properties of function, such as their **intercepts** to draw their graph.

Intercepts

The ***x*-intercept** of a graph is the value of x where the graph crosses the x-axis

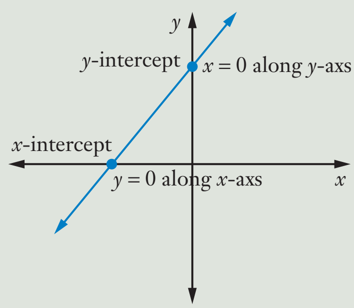
The ***y*-intercept** of a graph is the value of y where the graph crosses the y-axis

Intercepts of the graph of a function

For x-intercept(s) substitute $y = 0$

For y-intercept substitute $x = 0$

For the graph of $y = f(x)$ solving $f(x) = 0$ gives the x-intercepts and evaluating $f(0)$ gives the y-intercept



EXAMPLE 7

Find the x- and y-intercepts of the function $f(x) = x^2 + 7x - 8$.

Solution

For x-intercepts $y = f(x) = 0$

$0 = x^2 + 7x - 8$

$= (x + 8)(x - 1)$

$x = -8, x = 1$

So x-intercepts are -8 and 1

For y-intercept $x = 0$

$f(0) = 0^2 + 7(0) - 8$

$= -8$

So the y-intercept is -8

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Domain and range

The **domain** of a function $y = f(x)$ is the set of all x values for which $f(x)$ is defined

The **range** of a function $y = f(x)$ is the set of all y values for which $f(x)$ is defined

Interval notation

- $[a \quad b]$ means the interval is between a and b including a and b
- $(a \quad b)$ means the interval is between a and b excluding a and b
- $[a \quad b)$ means the interval is between a and b including a but excluding b
- $(a \quad b]$ means the interval is between a and b excluding a but including b
- $(-\infty \quad \infty)$ means that the interval includes the set of all real numbers R

EXAMPLE 8

Domain and ange

Find the domain and range of each function

a $f(x) = x^2$ **b** $y = \sqrt{x-1}$

Solution

a You can find the domain and range from the equation or the graph

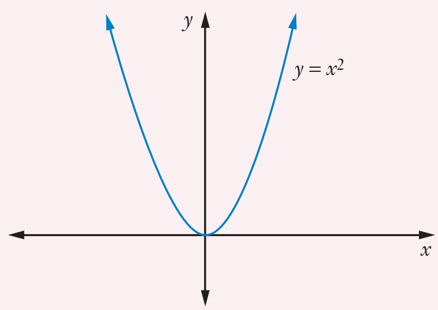
For $f(x) = x^2$ you can substitute any value for x The y values will be 0 or positive

So the domain is all real values of x and the range is all $y \geq 0$

We can write this using interval notatio:

Domain $(-\infty \quad \infty)$

Range $[0 \ \infty)$



b The function $y = \sqrt{x-1}$ is only defined if $x - 1 \geq 0$ because we can only evaluate the square root of a positive number or 0

For example $x = 0$ gives $y = \sqrt{-1}$ which is undefined for real number.

So $x - 1 \geq 0$

$x \geq 1$

Domain $[1 \quad \infty)$

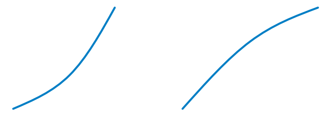
The value of $\sqrt{x-1}$ is always positive or zero So $y \geq 0$

Range $[0 \quad \infty)$

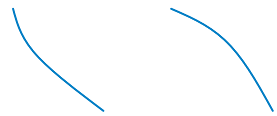
Increasing and decreasing graphs

When you draw a graph it helps to know whether the function is increasing or decreasing on an interval

If a graph is **increasing** y increases as x increases and the graph is moving upwards



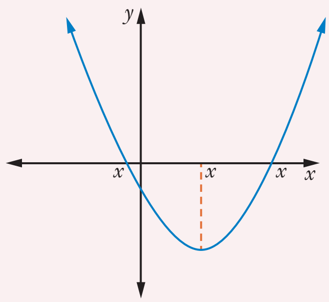
If a graph is **decreasing** then y decreases as x increases and the curve moves downwards



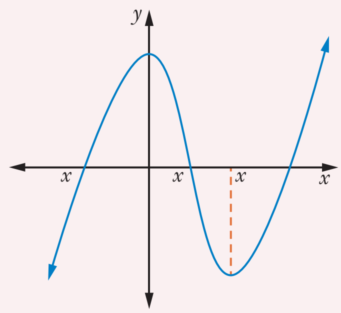
EXAMPLE 9

State the domain over which each curve is increasing

a



b



Solution

a The curve is decreasing to the left of x_2 and increasing to the right of x_2 that i, when $x > x_2$

So the domain over which the graph is increasing is $(x_2 \ \infty)$

b The curve is increasing on the left of the y-axis ($x = 0$) decreasing from $x = 0$ to $x = x_3$ then increasing again from $x = x_3$

So the curve is increasing for $x < 0, x > x_3$

So the domain over which the graph is increasing is $(-\infty, 0) \cup (x_3 \ \infty)$

The symbol $\cup$ is for 'unio' and means'a'. It stands for the union or joining of 2 separate parts You will meet this symbol again in probabilty.

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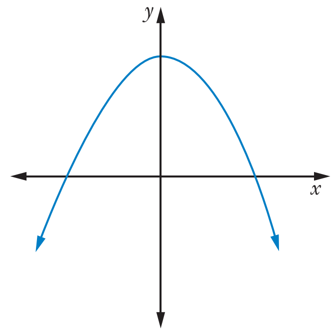
Even and odd functions

Even functions have graphs that are symmetrical about the y-axis The graph has line symmetry about the y-axis The left and right halves are mirror-images of each othe.



Even functions

A function is even if $f(x) = f(-x)$ for all values of x in the domain

EXAMPLE 10

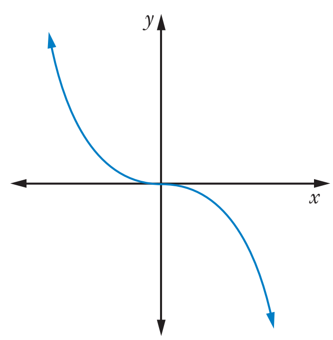
Show that $f(x) = x^2 + 3$ is an even function

Solution

$$f(-x) = (-x)^2 + 3$$
$$= x^2 + 3$$
$$= f(x)$$

So $f(x) = x^2 + 3$ is an even function

Odd functions have graphs that have point symmetry about the origin A graph rotated 180 ° about the origin gives the original graph



Odd functions

A function is odd if $f(-x) = -f(x)$ for all values of x in the domain

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Odd and even uncion

EXAMPLE 11

Show that $f(x) = x^3 - x$ is an odd function

Solution

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) \\ &= -f(x) \end{aligned}$$

So $f(x) = x^3 - x$ is an odd function

INVESTIGATION

EVEN AND ODD FUNCTIONS

Explore the family of graphs of $f(x) = kx^n$ the **power functions**

For what values of n is the function even?

For what values of n is the function odd?

Does the value of k change this?

Are these families of functions below even or odd? Does the value of k change this?

1 $f(x) = x^n + k$ **2** $f(x) = (x + k)^n$

Exercise 3.03 Properties of functions

1 Find the x- and y-intercepts of each function

- **a** $y = 3x - 2$
- **b** $2x - 5y + 20 = 0$
- **c** $x + 3y - 12 = 0$
- **d** $f(x) = x^2 + 3x$
- **e** $f(x) = x^2 - 4$
- **f** $p(x) = x^2 + 5x + 6$
- **g** $y = x^2 - 8x + 15$
- **h** $p(x) = x^3 + 5$
- **i** $y = \dfrac{x+3}{x}$
- **j** $g(x) = 9 - x^2$

2 $f(x) = 3x - 6$

- **a** Solve $f(x) = 0$.
- **b** Find the x- and y-intercepts

3 Show that $f(x) = f(-x)$ where $f(x) = x^2 - 2$ What type of function is it ?

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4 $f(x) = x^3 + 1$

a Find $f(x^2)$. **b** Find $[f(x)]^2$ **c** Find $f(-x)$

d Is $f(x) = x^3 + 1$ an even or odd function?

e Solve $f(x) = 0$

f Find the intercepts of the function

5 Show that $g(x) = x^8 + 3x^4 - 2x^2$ is an even function

6 Show that $f(x)$ is odd given $f(x) = x$

7 Show that $f(x) = x^2 - 1$ is an even function

8 Show that $f(x) = 4x - x^3$ is an odd function

9 **a** Prove that $f(x) = x^4 + x^2$ is an even function

b Find $f(x) - f(-x)$

10 Are these functions even odd or neither ?

a $y = \dfrac{x^3}{x^4 - x^2}$ **b** $f(x) = \dfrac{1}{x^3 - 1}$ **c** $f(x) = \dfrac{3}{x^2 - 4}$

d $y = \dfrac{x-3}{x+3}$ **e** $f(x) = \dfrac{x^3}{x^5 - x^2}$

11 If n is a positive integer, for what values of n is the power function $f(x) = kx^n$

a even? **b** odd?

12 Can the function $f(x) = x^n + x$ ever be

a even? **b** odd?

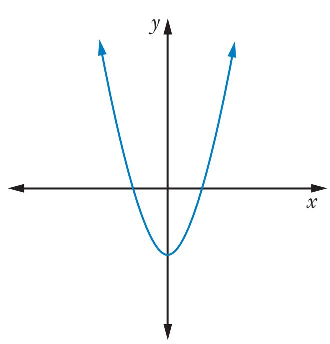
13 For the functions below, stae:

i the domain over which the graph is increasing

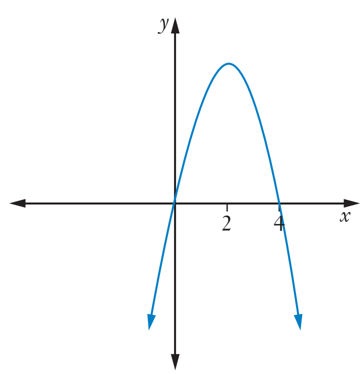
ii the domain over which the graph is decreasing

iii whether the graph is odd even or neithe.

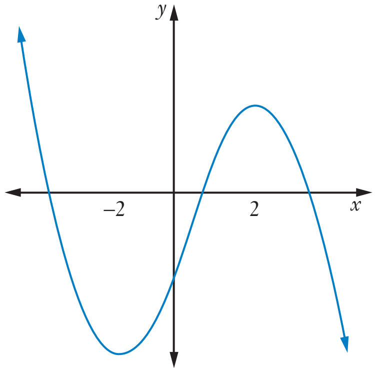
a



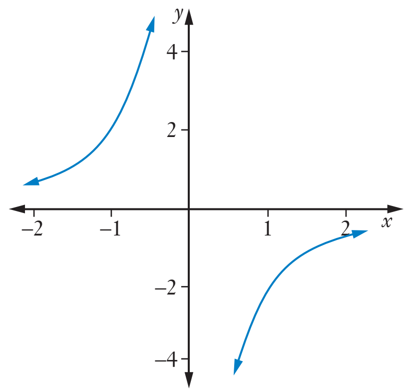
b



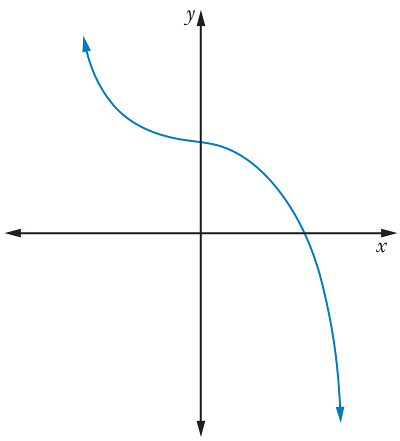
c



d



e



14 State the domain and range for each function

a $f(x) = x^2 + 1$ **b** $y = x^3$ **c** $y = \sqrt{x}$

d $f(x) = \sqrt{x + 5}$ **e** $y = -\sqrt{2x - 6}$

15 $f(x) = (x - 2)^2$

a Find $f(3)$

b Find $f(-5)$

c Solve $f(x) = 0$

d Find the x- and y-intercepts

e State the domain and range of $f(x)$

f Find $f(-x)$

g Is $f(x)$ even odd or neither ?

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3.04 Linear functions

Linear functions

A **linear function** has an equation of the form $y = mx + c$ or $ax + by + c = 0$

Its graph is a straight line with one x-intercept and one y-intercept

A page o numbe plane

Gaphing linea uncions

x and yinecep

Direct variation

When one variable is in **direct variation** (or **direct proportion**) with another variable one is a constant multiple of the other. This means that as one increaes, so does the oher.

Direct variation

If variables x and y are in direct proportion we can write the equation $y = kx$ where k is called the **proportionality constant**

EXAMPLE 12

Huang earns \$20 an hour. Find an equation for Huag's income (I) for working x hours and draw its graph

Solution

Income for 1 hour is \$20

Income for 2 hours is $\$20 \times 2$ or \$40

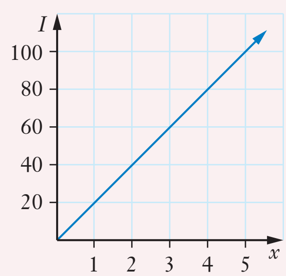
Income for 3 hours is $\$20 \times 3$ or \$60

Income for x hours is $\$20 \times x$ or $\$20x$

We can write the equation as $I = 20x$

We can graph the equation using a table of value.

x	1	2	3
I	20	40	60



$I = 20x$ is an example of direct variation Direct variation graphs are always straight lines passing through the origin

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Graphing linear functions

EXAMPLE 13

a Find the x- and y-intercepts of the graph of $y = 2x - 4$ and draw its graph on the number plane

b Find the x- and y-intercepts of the line with equation $x + 2y + 6 = 0$ and draw its graph

Solution

a For x-intercept $y = 0$

$0 = 2x - 4$

$4 = 2x$

$2 = x$

So the x-intercept is 2

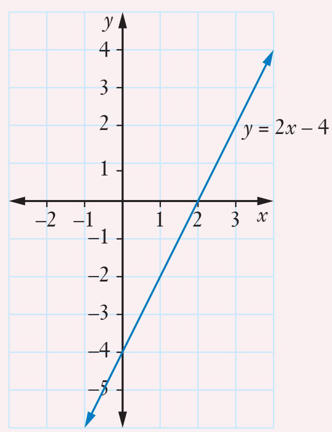
For y-intercept $x = 0$

$y = 2(0) - 4$

$= -4$

So the y-intercept is -4

Use the intercepts to graph the line



b For x-intercept $y = 0$

$x + 2(0) + 6 = 0$

$x + 6 = 0$

$x = -6$

So the x-intercept is -6

For y-intercept $x = 0$

$0 + 2y + 6 = 0$

$2y = -6$

$y = -3$

So the y-intercept is -3

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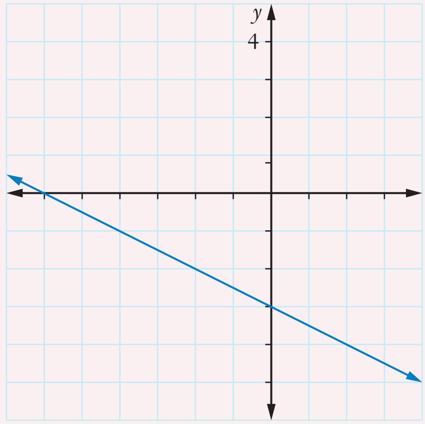


The domain of a linear function is (−∞ ∞) all real number.

– ll.

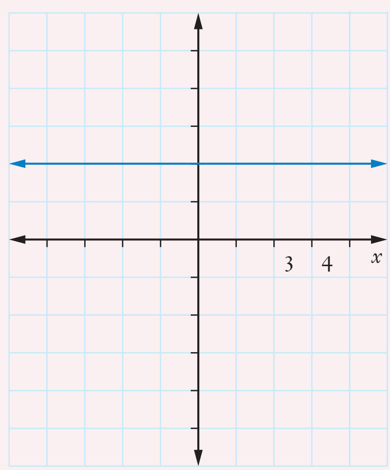
ls

EXAMPLE 14

aitinge

–

,1 a(2).



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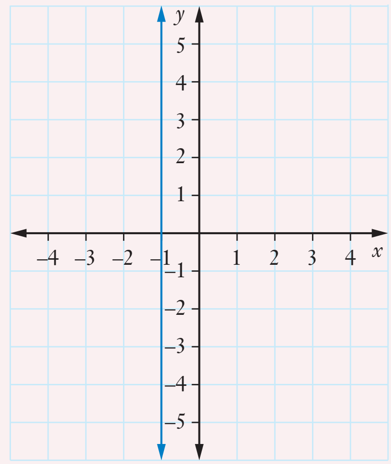
b y can have any value and x is always -1

Some of the points on the line will be $(-1, 0)$, $(-1, 1)$ and $(-1, 2$.

This gives a vertical line with x-intercept -1

Domain $[-1]$ Range $(-\infty \;\; \infty)$



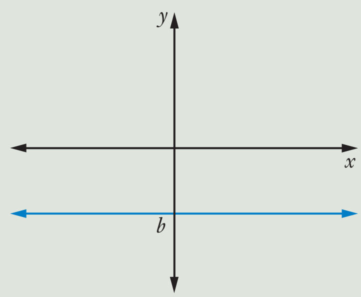
Horizontal lines

$y = b$ is a horizontal line with y-intercept b

$y = b$ is a many-to-one function

Domain $(-\infty \;\; \infty)$

Range $[b]$



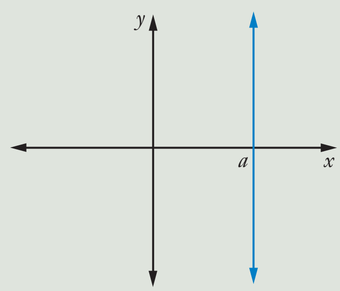
Vertical lines

$x = a$ is a vertical line with x-intercept a

$x = a$ is not a function

Domain $[a]$

Range $(-\infty \;\; \infty)$



Exercise 3.04 Linear functions

1 Write an equation fo:

- **a** the number of months (N) in x years
- **b** the amount of juice (A) in n lots of 2 litre bottles
- **c** the cost (c) of x litres of petrol at \$150 per litre
- **d** the number (y) of people in x debating teams if there are 4 people in each team
- **e** the weight (w) of x lots of 400 g cans of peaches

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2 Find the equation and draw the graph of the cost (c) of x refrigerators if each refrigerator costs \$850

3 Find the x- and y-intercepts of the graph of each function

a $y = x - 2$	**b** $y = 3x + 9$	**c** $y = 4 - 2x$
d $f(x) = 2x + 3$	**e** $f(x) = 5x - 4$	**f** $f(x) = 10x + 5$
g $x + y - 2 = 0$	**h** $2x - y + 4 = 0$	**i** $x - y + 3 = 0$
j $3x - 6y - 2 = 0$		

4 Draw the graph of each linear function

a $y = x + 4$	**b** $f(x) = 2x - 1$	**c** $f(x) = 3x + 2$
d $x + y = 3$	**e** $x - y - 1 = 0$	

5 Find the domain and range of each equation

a $3x - 2y + 7 = 0$	**b** $y = 2$	**c** $x = -4$
d $x - 2 = 0$	**e** $3 - y = 0$	

6 Sketch each equations graph and state its domain and rang.

a $x = 4$	**b** $x - 3 = 0$
c $y = 5$	**d** $y + 1 = 0$

7 A supermarket has boxes containing cans of dog food The number of cans of dog food is directly proportional to the number of boxes

- **a** If there are 144 cans in 4 boxes find an equation for the number of cans (N) in x boxes
- **b** How many cans are in 28 boxes?
- **c** How many boxes would be needed for 612 cans of dog food?

8 By sketching the graphs of $x - y - 4 = 0$ and $2x + 3y - 3 = 0$ on the same set of axes find the point where they cross

3.05 The gradient of a straight line

The **gradient** of a line measures its slope It compares the vertical rise with the horizontal ru.

Gadien and y inecep o a line

The gradient of a line

$$\text{Gradient} = \frac{\text{rise}}{\text{run}}$$

Positive gradient leans to the right

Negative gradient leans to the left

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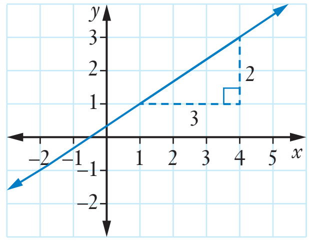


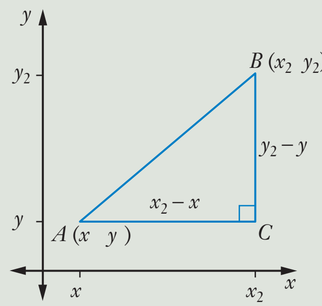
$$\text{Gradient} = \frac{\text{rise}}{\text{run}}$$
$$= \frac{2}{3}$$

On the number plane gradient is a measure of the rate of change of y with respect to x

Gradient formula

The gradient of the line joining points (x y) and (x_2 y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



EXAMPLE 15

Find the gradient of the line joining points (2 3) and (−3 4.

Solution

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4-3}{-3-2}$$
$$= \frac{1}{-5}$$
$$= -\frac{1}{5}$$

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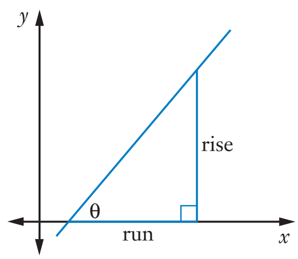
The angle of inclination of a line

The **angle of inclination** θ is the angle a straight line makes with the positive x-axis measured anticlockwise

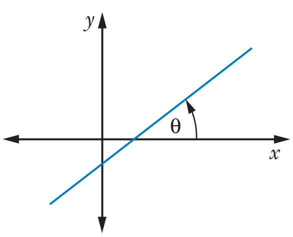
$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{\text{opposite}}{\text{adjacent}}$$
$$= \tan\theta$$



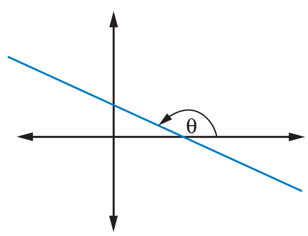
Gradient and angle of inclination of a line

$$m = \tan\theta$$

where m is the gradient and θ is the **angle of inclination**



For an acute angle $\tan\theta > 0$



For an obtuse angle $\tan\theta < 0$

DISCUSSION

ANGLES AND GRADIENTS

1 What type of angles give a positive gradient?

2 What type of angles give a negative gradient? Why?

3 What is the gradient of a horizontal line? What angle does it make with the x-axis?

4 What angle does a vertical line make with the x-axis? Can you find its gradient?

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EXAMPLE 16

a Find the gradient of the line that makes an angle of inclination of 135°

b Find correct to the nearest minute the angle of inclination of a straight line whose gradient is

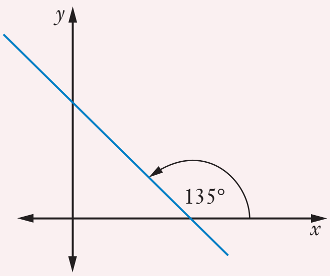
i 05 **ii** −3

Solution

a $m = \tan\theta$

$= \tan 135°$

$= -1$



b **i** $m = \tan\theta$

$\therefore \tan\theta = 05$

$\theta = \tan^{-}\ (05)$

$= 26°33'5418''$

$\approx 26°34'$

See page 172 of Chapter 4 f you need to revse roundng an angle to the nearest mnute

Operation	Casio scientific	Sharp scientific
Enter data	SHFT tan− 05 =	2ndF tan− 05 =
Change to degrees and minutes	°’ ”	2ndF DM'S

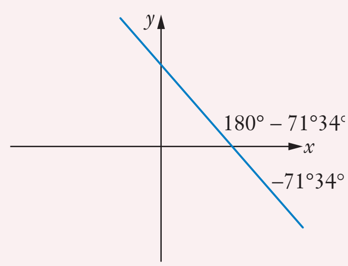
ii $m = \tan\theta$

$\therefore \tan\theta = -3$

$\theta = \tan^{-}\ (-3)$

$= -71°33'5418''$

$\approx -71°34'$



A negative gradient means that the angle of inclination is obtuse

To find this angl, subtract the acute angle from 180 °

$\theta = 180° - 71°34'$

$= 108°26'$

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INVESTIGATION

GRAPHING $y = mx + c$

Graph each linear function using a graphics calculator or graphing software Find the gradient of each function What do you notice ?

1 $y = x$	**2** $y = 2x$	**3** $y = 3x$	**4** $y = 4x$
5 $y = -x$	**6** $y = -2x$	**7** $y = -3x$	**8** $y = -4x$

Graph each function and find the y-intercept

9 $y = x$	**10** $y = x + 1$	**11** $y = x + 2$	**12** $y = x + 3$
13 $y = x - 1$	**14** $y = x - 2$	**15** $y = x - 3$	

The gradient–intercept equation of a straight line

The linear function with equation $y = mx + c$ has gradient m and y-intercept c

WS

y $mx + c$

EXAMPLE 17

a Find the gradient and y-intercept of the linear function $y = 7x - 5$.

b Find the gradient of the straight line with equation $2x + 3y - 6 = 0$

Solution

a Gradient = 7, y-intercept = –5

b First change the equation into the form $y = mx + c$

$$2x + 3y - 6 = 0$$
$$2x + 3y = 6$$
$$3y = 6 - 2x$$
$$= -2x + 6$$

$$y = \frac{-2x}{3} + \frac{6}{3}$$
$$= -\frac{2}{3}x + 2$$

So the gradient is $-\frac{2}{3}$

Exercise 3.05 The gradient of a straight line

1 Find the gradient of the line joining the points

a (3, 2) and (1, –2)	**b** (0, 2) and (3, 6)	**c** (–2, 3) and (4, –5)
d (2 –5) and (–3, 7)	**e** (2, 3) and (–1, 1)	**f** (–5, 1) and (3, 0)
g (–2, –3) and (–4 6)	**h** (–1, 3) and (–7, 7)	**i** (1, –4) and (5 5)

2 Find the gradient of the straight line correct to 1 decimal plac, whose angle of inclination is

a 25°	**b** 82°	**c** 68°
d 100°	**e** 130°	**f** 164°

3 For each linear function fin:

i the gradient **ii** the y-intercept

a $y = 3x + 5$	**b** $f(x) = 2x + 1$	**c** $y = 6x - 7$
d $y = -x$	**e** $y = -4x + 3$	**f** $y = x - 2$
g $f(x) = 6 - 2x$	**h** $y =1 - x$	**i** $y = 9x$

4 Find the gradient of the linear function

a with x-intercept 3 and y-intercept –1
b passing through (2 4) and x-intercept 5
c passing through (1 1) and (–2, 7)
d with x-intercept –3 and passing through (2 3)
e passing through the origin and (–3, –1)

5 Find the angle of inclination to the nearest minut, of a line with gradiet:

a 2	**b** 17	**c** 6
d –5	**e** –085	**f** –12

6 For each linear function fin:

i the gradient **ii** the y-intercept

a $2x + y - 3 = 0$	**b** $5x + y + 6 = 0$	**c** $6x - y - 1 = 0$
d $x - y + 4 = 0$	**e** $4x + 2y - 1 = 0$	**f** $6x - 2y + 3 = 0$
g $x + 3y + 6 = 0$	**h** $4x + 5y - 10 = 0$	**i** $7x - 2y - 1 = 0$

7 Find the gradient of each linear function

a $y = -2x - 1$	**b** $y = 2$	**c** $x + y + 1 = 0$
d $3x + y = 8$	**e** $2x - y + 5 = 0$	**f** $x + 4y - 12 = 0$
g $3x - 2y + 4 = 0$	**h** $5x - 4y = 15$	**i** $y = \frac{2}{3}x + 3$

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j $y=\frac{x}{5}-1$

k $y=\frac{2x}{7}+5$

$y=-\frac{3x}{5}-2$

m $2y=-\frac{x}{7}+\frac{1}{3}$

n $3x-\frac{y}{5}=8$

o $\frac{x}{2}+\frac{y}{3}=1$

8 If the gradient of the line joining (8 y) and (–1 3) is , find the value of y

9 The gradient of the line through (2 –1) and (x, 0) is –5 Find the value of x

10 The gradient of a line is –1 and the line passes through the points (4 2) and (x –3) Find the value of x

11 The number of frequent flyer points that Mario earns on his credit card is directly proportional to the amount of money he spends on his card

- **a** If Mario earns 150 points when he spends \$450 find an equation for the number of points (P) he earns when spending d dollars
- **b** Find the number of points Mario earns when he spends \$840
- **c** If Mario earns 57 points how much did he spend ?

12 The points A(–1, 2), B(1, 5), C(6 5) and D(4 2) form a parallelogra. Find the gradients of all 4 sides of the parallelogram What do you notice ?

3.06 Finding a linear equation

Linea uncions code puzzle

inea modelling

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EXAMPLE 18

Find the equation of the line with gradient 3 and y-intercept –1

Solution

The equation is $y = mx + c$ where m = gradient and c = y-intercept

$m = 3$ and $c = -1$

Equation is $y = 3x - 1$.

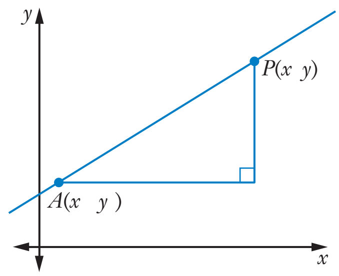
There is a formula you can use if you know the gradient and the coordinates of a point on the line

The point–gradient equation of a straight line

The linear function with equation $y - y = m(x - x)$ has gradient m and the point (x y) lies on the line

Proof

Let $P(x \;\; y)$ be a general point on the line with gradient m that passes through $A(x \quad y\ )$



Then line AP has gradient

$$m = \frac{y - y}{x - x}$$

$$m(x - x\ ) = y - y$$

EXAMPLE 19

Find the equation of the line

a with gradient −4 and x-intercept 1

b passing through (2 3) and (−1, 4.

Solution

a The x-intercept of 1 means the line passes through the point (1 0.

Substituting $m = -4$, $x \;\; = 1$ and $y \;\; = 0$ into the formula

$$y - y \;\; = m(x - x\ )$$

$$y - 0 = -4(x - 1)$$

$$y = -4x + 4$$

b First find the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{4 - 3}{-1 - 2}$$

$$= -\frac{1}{3}$$

Substitute the gradient and one of the points say (,), into the formla.

$$y - y \;\; = m(x - x\ )$$

$$y - 3 = -\frac{1}{3}(x - 2)$$

$$3 \times (y - 3) = 3 \times -\frac{1}{3}(x - 2)$$

$$3y - 9 = -(x - 2)$$

$$= -x + 2$$

$$x + 3y - 9 = 2$$

$$x + 3y - 11 = 0$$

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Applications of linear functions

EXAMPLE 20

A solar panel company has fixed overhead costs of \$3000 per day and earns \$150 for each solar cell sold

a Write the amount (\$$A$) that the company earns on selling x solar cells each day.

b Find the amount the company earns on a day when it sells 54 solar cells

c If the company earns \$2850 on another day, how many solar cells did it sell that day?

d What is the **break-even point** for this company (where income and costs of production are equal)?

Solution

a The company earns \$150 per cell so it earns \$150 x for x cells

Daily amount earned = value of solar cells sold – overhead costs

So $A = 150x - 3000$

b Substitute $x = 54$

$A = 150(54) - 3000 = 5100$

The company earns \$5100 when it sells 54 solar cells

c Substitute $A = 2850$

$2850 = 150x - 3000$

$5850 = 150x$

$39 = x$

The company sold 39 solar cells that day.

d At the break-even point

Income = overhead costs

$150x = 3000$ (or $A = 0$)

$x = 20$

So the break-even point is where the company sells 20 solar cells

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Exercise 3.06 Finding a linear equation

1 Find the equation of the straight line

a with gradient 4 and y-intercept -1

b with gradient -3 and passing through (0 4)

c passing through the origin with gradient 5

d with gradient 4 and x-intercept -5

e with x-intercept 1 and y-intercept 3

f with x-intercept 3 y-intercept -4

2 Find the equation of the straight line passing through the points

a (2, 5) and (–1, 1) **b** (0, 1) and (–4, –2) **c** (–2, 1) and (3, 5)

d (3 4) and (–1, 7) **e** (–4, –1) and (–2 0.

3 What is **a** the gradient and **b** the equation of the line with x-intercept 2 that passes through (3 –4)?

4 Find the equation of the line

a parallel to the x-axis and passing through (2 3)

b parallel to the y-axis and passing through (–1, 2.

5 A straight line passing through the origin has a gradient of –2 Fin:

a the y-intercept **b** its equation

6 In a game each person starts with 20 point, then earns 15 points for every level completed

a Write an equation for the number of points earned (P) for x levels completed

b Find the number of points earned for completing

i 24 levels **ii** 55 levels **iii** 247 levels

c Find the number of levels completed if the number of points earned is

i 2195 **ii** 7700 **iii** 12 665

7 A TV manufacturing business has fixed costs of \$1500 renta, \$3000 wages and other costs of \$2500 each week It costs \$250 to produce each V.

a Write an equation for the cost (c) of producing n TVs each wee.

b From the equation find the cost of producin:

i 100 TVs **ii** 270 TVs **iii** 1200 TVs

c From the equation find the number of TVs produced if the cost i:

i \$52 000 **ii** \$78 250 **iii** \$367 000

d If each TV sells for \$95, find the number of TVs needed to sell to break even

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8 There are 450 litres of water in a pond and 8 litres of water evaporate out of the pond every hour.

a Write an equation for the amount of water in the pond (A) after h hours

b Find the amount of water in the pond after

i 3 hours **ii** a day.

c After how many hours will the pond be empty?

9 Geordie has a $20 iTunes credi. He uses the credit to buy singles at 1.69 eah.

a Write an equation for the amount of credit (C) left if Geordie buys x singles

b How many songs can Geordie buy before his credit runs out?

10 Emily-Rose owes $20 000 and she pays back $320 a month

a Write an equation for the amount of money she owes (A) after x months

b How much does Emily-Rose owe after

i 5 months? **ii** 1 year? **iii** 5 years?

c How long will it take for Emily-Rose to pay all the money back?

11 Acme Party Supplies earns $5 for every helium balloon it sells

a If overhead costs are $100 each day, find an equation for the profit (P) of selling x balloons

b How much profit does Acme make if it sells 300 balloons?

c How many balloons does it sell if it makes a profit of $1055?

d What is the break-even point for this business?

3.07 Parallel and perpendicular lines

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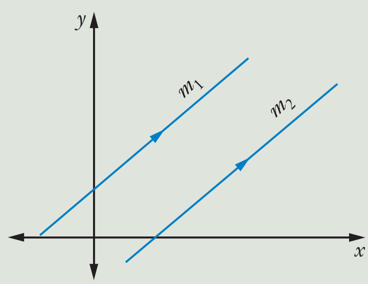
Gradients of parallel lines

If 2 lines are parallel then they have the same gradient That s, $m \ = m_2$



EXAMPLE 21

a Prove that the straight lines with equations $5x - 2y - 1 = 0$ and $5x - 2y + 7 = 0$ are parallel

b Find the equation of a straight line parallel to the line $2x - y - 3 = 0$ and passing through (1 −5)

Solution

a First change the equation into the form $y = mx + c$

$$5x - 2y - 1 = 0$$
$$5x - 1 = 2y$$
$$\frac{5}{2}x - \frac{1}{2} = y$$
$$\therefore \quad m = \frac{5}{2}$$

$$5x - 2y + 7 = 0$$
$$5x + 7 = 2y$$
$$\frac{5}{2}x + \frac{7}{2} = y$$
$$\therefore \quad m_2 = \frac{5}{2}$$

$$m = m_2 = \frac{5}{2}$$

$\therefore$ the lines are parallel

b $2x - y - 3 = 0$

$$2x - 3 = y$$
$$\therefore m = 2$$

For parallel lines $m = m_2$

$$\therefore m_2 = 2$$

Substitute this and (1 −5) into $y - y = m(x - x)$

$$y - (-5) = 2(x - 1)$$
$$y + 5 = 2x - 2$$
$$y = 2x - 7$$

CLASS INVESTIGATION

PERPENDICULAR LINES

Sketch each pair of straight lines on the same number plane

1 $3x - 4y + 12 = 0$ and $4x + 3y - 8 = 0$

2 $2x + y + 4 = 0$ and $x - 2y + 2 = 0$

What do you notice about each pair of lines?

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Gradients of perpendicular lines

If 2 lines with gradients m and m_2 are perpendicular, then $m\ m_2 = -1$,

that is $m_2 = -\frac{1}{m}$

EXAMPLE 22

a Show that the lines with equations $3x + y - 11 = 0$ and $x - 3y + 1 = 0$ are perpendicular.

b Find the equation of the straight line through (2 3) that is perpendicular to the line passing through (–1, 7) and (3,).

Solution

a

$$3x + y - 11 = 0$$

$$y = -3x + 11$$

$$\therefore m = -3$$

$$x - 3y + 1 = 0$$

$$x + 1 = 3y$$

$$\frac{1}{3}x + \frac{1}{3} = y$$

$$\therefore m_2 = \frac{1}{3}$$

$$m\ m_2 = -3 \times \frac{1}{3} = -1$$

$\therefore$ the lines are perpendicular.

b Line through (–1, 7) and (3,):

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{7-3}{-1-3}$$

$$= \frac{4}{-4}$$

$$= -1$$

For perpendicular lines $m\ m_2 = -1$

$$-1m_2 = -1$$

$$m_2 = 1$$

Substitute $m = 1$ and the point (2 3) into $y - y\ = m(x - x\ )$

$$y - 3 = 1(x - 2)$$

$$= x - 2$$

$$y = x + 1$$

Exercise 3.07 Parallel and perpendicular lines

1 Find the gradient of the straight line

a parallel to the line $3x + y - 4 = 0$

b perpendicular to the line $3x + y - 4 = 0$

c parallel to the line joining (3 5) and (−1, 2)

d perpendicular to the line with x-intercept 3 and y-intercept 2

e perpendicular to the line that has an angle of inclination of 135°

f perpendicular to the line $6x - 5y - 4 = 0$

g parallel to the line $x - 3y - 7 = 0$

h perpendicular to the line passing through (4 −2) and (3, 3.

2 Find the equation of the straight line

a passing through (2 3) and parallel to the line $y = x + 6$

b through (–1 5) and parallel to the line $x - 3y - 7 = 0$

c with x-intercept 5 and parallel to the line $y = 4 - x$

d through (3 −4) and perpendicular to the line $y = 2x$

e through (–2 1) and perpendicular to the line $2x + y + 3 = 0$

f through (7 −2) and perpendicular to the line $3x - y - 5 = 0$

g through (–3, −1) and perpendicular to the line $4x - 3y + 2 = 0$

h passing through the origin and parallel to the line $x + y + 3 = 0$

i through (3 7) and parallel to the line $5x - y - 2 = 0$

j through (0 −2) and perpendicular to the line $x - 2y = 9$

k perpendicular to the line $3x + 2y - 1 = 0$ and passing through the point (–2 4.

3 Show that the lines with equations $y = 3x - 2$ and $6x - 2y - 9 = 0$ are parallel

4 Show that lines $x + 5y = 0$ and $y = 5x + 3$ are perpendicular.

5 Show that lines $6x - 5y + 1 = 0$ and $6x - 5y - 3 = 0$ are parallel

6 Show that lines $7x + 3y + 2 = 0$ and $3x - 7y = 0$ are perpendicular.

7 If the lines $3x - 2y + 5 = 0$ and $y = kx - 1$ are perpendicular, find the value of k

8 Show that the line joining (3 −1) and (2 −5) is parallel to the line $8x - 2y - 3 = 0$

9 Show that the points A(–3, –2) B(–1, 4), C(7 −1) and D(5 −7) are the vertices of a parallelogram

10 The points A(–2, 0), B(1, 4), C(6 4) and D(3 0) form a rhombu. Show that the diagonals are perpendicular.

11 Find the equation of the straight line passing through (6 −3) that is perpendicular to the line joining (2 −1) and (–5, –7)

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3.08 Quadratic functions

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Quadratic functions

A **quadratic function** has an equation in the form $y = ax^2 + bx + c$ where the highest power of x is 2 The graph of a quadratic function is a **parabola**

EXAMPLE 23

Graph the quadratic function $y = x^2 - x$

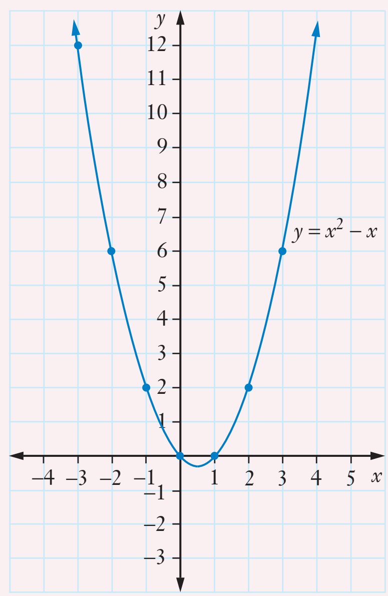
Solution

Draw up a table of values for $y = x^2 - x$

x	−3	−2	−1	0	1	2	3
y	12	6	2	0	0	2	6

Plot (−3, 12), (−2, 6), (−1, 2), (0, 0), (1, 0), (2, 2) and (3, 6) and draw a parabola through them

Label the graph with its equation



TECHNOLOGY

Transforming quadratic functions

Use a graphics calculator or graphing software to graph these quadratic functions Look for any patterns

$y = x^2$	$y = x^2 + 1$	$y = x^2 + 2$	$y = x^2 + 3$
$y = x^2 - 1$	$y = x^2 - 2$	$y = x^2 - 3$	$y = 2x^2$
$y = 3x^2$	$y = x^2 + x$	$y = x^2 + 2x$	$y = x^2 + 3x$
$y = x^2 - x$	$y = x^2 - 2x$	$y = x^2 - 3x$	$y = -x^2$
$y = -x^2 + 1$	$y = -x^2 + 2$	$y = -x^2 + 3$	$y = -x^2 - 1$
$y = -x^2 - 2$	$y = -x^2 - 3$	$y = -2x^2$	$y = -3x^2$
$y = -x^2 + x$	$y = -x^2 + 2x$	$y = -x^2 - x$	$y = -x^2 - 2x$

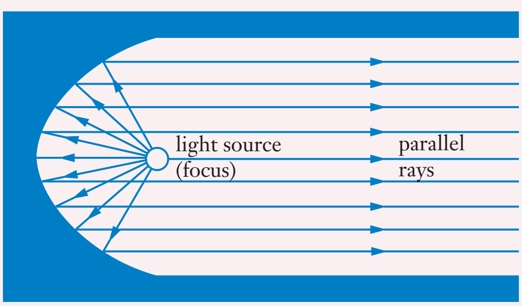
Could you predict where the graphs $y = x^2 + 9$, $y = 5x^2$ or $y = x^2 + 6x$ would lie?

Is the parabola always a function? Can you find an example of a parabola that is not a function?

DID YOU KNOW?

The parabola

The parabola shape has special properties that are very useful For exampl, if a light is placed inside a parabolic mirror at a special place called the focus then all light rays coming from this point and reflecting off the parabola shape will radiate out parallel to each other, giving a strong ligt. This is how car headlights ork. The dishes of radio telescopes also use this property of the parabola because radio signals coming in to the dish will reflect back to the focus



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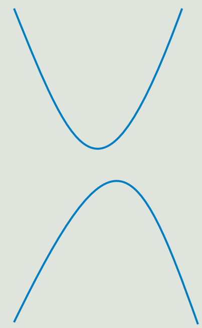
Concavity and turning points

For the parabola $y = ax^2 + bx + c$

- if $a > 0$ the parabola is **concave upwards** and has a **minimum turning point**
- if $a < 0$ the parabola is **concave downwards** and has a **maximum turning point**



The **turning point** is also called the **vertex** or **stationary point** of the parabola

Notice also that the parabola is always symmetrical

EXAMPLE 24

a **i** Sketch the graph of $y = x^2 - 1$ showing intercept.

ii State the domain and range

b **i** Find the x- and y-intercepts of the quadratic function $f(x) = -x^2 + 4x + 5$.

ii Sketch a graph of the function

iii Find the maximum value of the function

v State the domain and range

Solution

a **i** Since $a > 0$ the graph is concave upward.

For x-intercepts $y = 0$

$0 = x^2 - 1$

$1 = x^2$

$x = \pm 1$

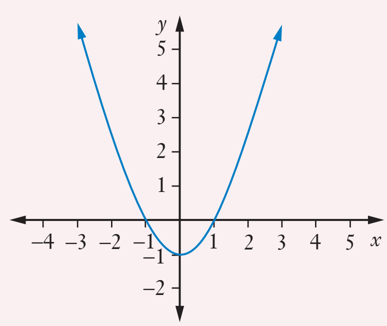
For y-intercept $x = 0$

$y = 0^2 - 1$

$= -1$

Since the parabola is symmetrical the turning point is at $x = 0$ halfway between the x-intercepts -1 and 1

When $x = 0, y = -1$ Vertex is 0, -1)



ii From the equation and the graph x can have any value

Domain $(-\infty\ \ \infty)$

The values of y are greater than or equal to -1

Range $[-1, \infty)$

b **i** For x-intercepts $f(x) = 0$

$0 = -x^2 + 4x + 5$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

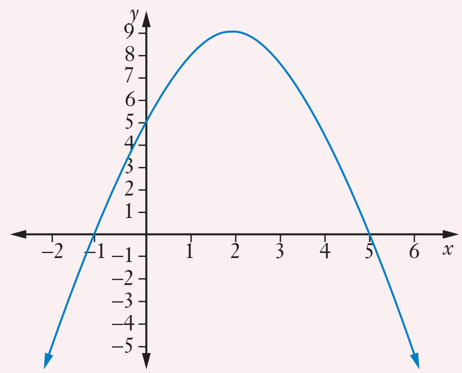
$x = 5, x = -1$

For y-intercept $x = 0$

$f(0) = -(0)^2 + 4(0) + 5$

$= 5$

ii Since $a < 0$ the quadratic function is concave downwards



iii The turning point is halfway between $x = -1$ and $x = 5$.

$$x = \frac{-1+5}{2}$$

$$= 2$$

$$f(2) = -(2)^2 + 4(2) + 5$$

$$= 9$$

The maximum value of $f(x)$ is 9

v For the domain the function can take on all real numbers for x

Domain $(-\infty\ \ \infty)$

For the range $y \le 9$

Range $(-\infty\ \ 9]$

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Exercise 3.08 Quadratic functions

1 Find the x- and y-intercepts of the graph of each quadratic function

a $y = x^2 + 2x$ **b** $y = -x^2 + 3x$ **c** $f(x) = x^2 - 1$

d $y = x^2 - x - 2$ **e** $y = x^2 - 9x + 8$

2 Sketch each parabola and find its maximum or minimum value

a $y = x^2 + 2$ **b** $y = -x^2 + 1$ **c** $f(x) = x^2 - 4$

d $y = x^2 + 2x$ **e** $y = -x^2 - x$ **f** $f(x) = (x - 3)^2$

g $f(x) = (x + 1)^2$ **h** $y = x^2 + 3x - 4$ **i** $y = 2x^2 - 5x + 3$

j $f(x) = -x^2 + 3x - 2$

3 For each parabola fin:

i the x- and y-intercepts **ii** the domain and range

a $y = x^2 - 7x + 12$ **b** $f(x) = x^2 + 4x$ **c** $y = x^2 - 2x - 8$

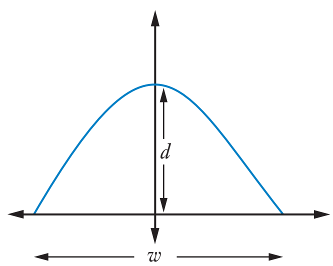
d $y = x^2 - 6x + 9$ **e** $f(x) = 4 - x^2$

4 Find the domain and range of

a $y = x^2 - 5$ **b** $f(x) = x^2 - 6x$ **c** $f(x) = x^2 - x - 2$

d $y = -x^2$ **e** $f(x) = (x - 7)^2$

5 A satellite dish is in the shape of a parabola with equation $y = -3x^2 + 6$ and all dimensions are in metres



a Find d the depth of the dis.

b Find w the width of the dis, to 1 decimal place

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3.09 Axis of symmetry

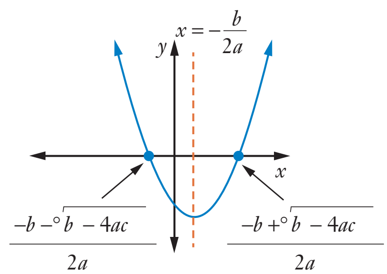
Axis of symmetry of a parabola

The **axis of symmetry** of a parabola with the equation $y = ax^2 + bx + c$ is the vertical line with equation

$$x = -\frac{b}{2a}$$

Proof

The axis of symmetry of a parabola lies halfway between the x-intercepts



For the x-intercepts $y = 0$

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The x-coordinate of the axis of symmetry is the average of the x-intercepts

$$x = \frac{\frac{-b - \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a}}{2} = \frac{\frac{-2b}{2a}}{2} = \frac{-2b}{4a} = -\frac{b}{2a}$$

Turning point of a parabola

The quadratic function $y = ax^2 + bx + c$ has a minimum value if $a > 0$ and a maximum value if $a < 0$

The minimum or maximum value of the quadratic function is $f\left(-\frac{b}{2a}\right)$

The turning point or vertex of a parabola is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

EXAMPLE 25

a Find the equation of the axis of symmetry and the minimum value of the quadratic function $y = x^2 - 5x + 1$.

b Find the equation of the axis of symmetry, the maximum value and the turning point of the quadratic function $y = -3x^2 + x - 5$.

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Solution

a Axis of symmetry

$x = -\frac{b}{2a}$

$= -\frac{(-5)}{2(1)}$

$= \frac{5}{2}$

$= 2\frac{1}{2}$

$\therefore$ Axis of symmetry is the line $x = 2\frac{1}{2}$

Minimum value $y = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 1$

$= \frac{25}{4} - \frac{25}{2} + 1$

$= -5\frac{1}{4}$

b $x = -\frac{b}{2a}$

$= -\frac{1}{2(-3)}$

$= \frac{1}{6}$

$\therefore$ Axis of symmetry is the line $x = \frac{1}{6}$

Maximum value $y = -3\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right) - 5$

$= -\frac{1}{12} + \frac{1}{6} - 5$

$= -4\frac{11}{12}$

The turning point is $\left(\frac{1}{6}, -4\frac{11}{12}\right)$

EXAMPLE 26

Determine whether each function is even

a $f(x) = x^2 + 3$ **b** $y = -x^2 + 3x$

Solution

a $f(x) = x^2 + 3$

$f(-x) = (-x)^2 + 3$

$= x^2 + 3$

$= f(x)$

So $f(x) = x^2 + 3$ is an even function

b Let $f(x) = -x^2 + 3x$

$f(-x) = -(-x)^2 + 3(-x)$

$= -x^2 - 3x$

$\neq f(x)$

So $y = -x^2 + 3x$ is not an even function

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Exercise 3.09 Axis of symmetry

1 For the parabola $y = x^2 + 2x$ find the equation of its axis of symmetry and the minimum value

2 Find the equation of the axis of symmetry and the minimum value of the parabola $y = x^2 - 4$

3 Find the equation of the axis of symmetry and the minimum turning point of the parabola $y = 4x^2 - 3x + 1$.

4 Find the equation of the axis of symmetry and the maximum value of the parabola $y = -x^2 + 2x - 7$.

5 Find the equation of the axis of symmetry and the vertex of the parabola $y = -2x^2 - 4x + 5$.

6 Find the equation of the axis of symmetry and the minimum value of the parabola $y = x^2 + 3x + 2$.

7 Find the equation of the axis of symmetry and the coordinates of the vertex for each parabola

a $y = x^2 + 6x - 3$ **b** $y = -x^2 - 8x + 1$ **c** $y = 3x^2 + 18x + 4$
d $y = -2x^2 + 5x$ **e** $y = 4x^2 + 10x - 7$

8 For each parabola fin:

i the equation of the axis of symmetry
ii the minimum or maximum value
iii the vertex

a $y = x^2 + 2x - 2$ **b** $y = -2x^2 + 4x - 1$

9 Find the turning point of each function and state whether it is a maximum or minimum

a $y = x^2 + 2x + 1$ **b** $y = x^2 - 8x - 7$ **c** $f(x) = x^2 + 4x - 3$
d $y = x^2 - 2x$ **e** $f(x) = x^2 - 4x - 7$ **f** $f(x) = 2x^2 + x - 3$
g $y = -x^2 - 2x + 5$ **h** $y = -2x^2 + 8x + 3$ **i** $f(x) = -3x^2 + 3x + 7$

10 For each quadratic function

i find x-intercepts using the quadratic formula
ii state whether the function has a maximum or minimum value and find this value
iii sketch the graph of the function on a number plane
v solve the quadratic equation $f(x) = 0$ graphically

a $f(x) = x^2 + 4x + 4$ **b** $f(x) = x^2 - 2x - 3$ **c** $y = x^2 - 6x + 1$
d $f(x) = -x^2 - 2x + 6$ **e** $f(x) = -x^2 - x + 3$

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11 **a** Find the minimum value of the parabola with equation $y = x^2 - 2x + 5$.

b How many solutions does the quadratic equation $x^2 - 2x + 5 = 0$ have?

c Sketch the parabola

12 **a** Find the maximum value of the quadratic function $f(x) = -2x^2 + x - 4$

b How many solutions are there to the quadratic equation $-2x^2 + x - 4 = 0$?

c Sketch the graph of the quadratic function

13 Show that $f(x) = -x^2$ is an even function

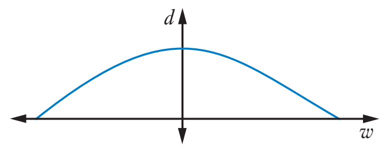
14 Determine which of these functions are even

a $y = x^2 + 1$	**b** $f(x) = x^2 - 3$	**c** $y = -2x^2$
d $f(x) = x^2 - 3x$	**e** $f(x) = x^2 + x$	**f** $y = x^2 - 4$
g $y = x^2 - 2x - 3$	**h** $y = x^2 - 5x + 4$	**i** $p(x) = (x + 1)^2$

15 A bridge has a parabolic span as shown

with equation $d = -\dfrac{w^2}{800} + 200$

where d is the depth of the arch in metres



a Show that the quadratic function is even

b Find the depth of the arch from the top of the span

c Find the total width of the span

d Find the depth of the arch at a point 10 m from its widest span

e Find the width across the span at a depth of 100 m

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3.10 The discriminant

The solutions of an equation are also called the **roots** of the equation

In the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ the expression $b^2 - 4ac$ is called the **discriminant**

It gives us information about the roots of the quadratic equation $ax^2 + bx + c = 0$

EXAMPLE 27

Use the quadratic formula to find how many real roots each quadratic equation has

a $x^2 + 5x - 3 = 0$ **b** $x^2 - x + 4 = 0$ **c** $x^2 - 2x + 1 = 0$

Solution

a

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times (-3)}}{2 \times 1} \\ &= \frac{-5 \pm \sqrt{25 + 12}}{2} \\ &= \frac{-5 \pm \sqrt{37}}{2} \end{aligned}$$

There are 2 real roots

$$x = \frac{-5 + \sqrt{37}}{2}, \frac{-5 - \sqrt{37}}{2}$$

b

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times 4}}{2 \times 1} \\ &= \frac{1 \pm \sqrt{-15}}{2} \end{aligned}$$

There are no real roots since $\sqrt{-15}$ has no real value

c

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 1}}{2 \times 1} \\ &= \frac{2 \pm \sqrt{0}}{2} \\ &= 1 \end{aligned}$$

There are 2 real roots

$x = 1, 1$

However, these are equal roos.

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The discriminant

The value of the **discriminant** $\Delta = b^2 - 4ac$ tells us information about the roots of the quadratic equation $ax^2 + bx + c = 0$

When $\Delta \geq 0$ there are 2 real root.

- If Δ is a perfect square the roots are rationa.
- If Δ is not a perfect square the roots are irrationa.

When $\Delta = 0$ there are 2 equal rational roots (or 1 rational root.

When $\Delta < 0$ there are no real root.

EXAMPLE 28

a Show that the equation $2x^2 + x + 4 = 0$ has no real roots

b Describe the roots of the equation

i $2x^2 - 7x - 1 = 0$ **ii** $x^2 + 6x + 9 = 0$

c Find the values of k for which the quadratic equation $5x^2 - 2x + k = 0$ has real roots

Solution

a $\Delta = b^2 - 4ac$

$= 1^2 - 4(2)(4)$

$= -31$

< 0

$\Delta < 0$ so the equation has no real root.

b **i** $\Delta = b^2 - 4ac$

$= (-7)^2 - 4(2)(-1)$

$= 57$

> 0

$\Delta > 0$ so there are 2 real irrational root.

Roots are rratonal because 57 s not a perfect squar.

ii $\Delta = b^2 - 4ac$

$= (6)^2 - 4(1)(9)$

$= 0$

$\Delta = 0$ so there are 2 real equal rational roots

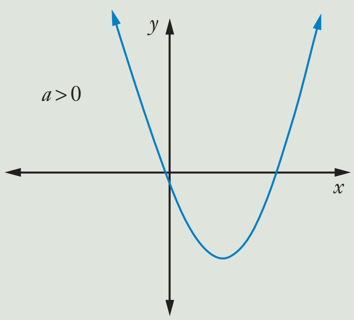
c For real roots $\Delta \geq 0$

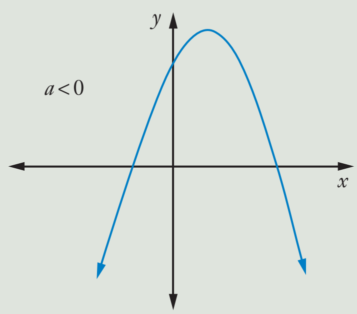
$$b^2 - 4ac \geq 0$$
$$(-2)^2 - 4(5)(k) \geq 0$$
$$4 - 20k \geq 0$$
$$4 \geq 20k$$
$$k \leq \frac{1}{5}$$

The discriminant and the parabola

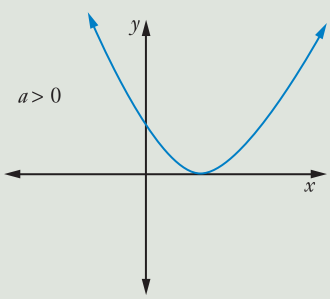
The roots of the quadratic equation $ax^2 + bx + c = 0$ give the x-intercepts of the parabola $y = ax^2 + bx + c$

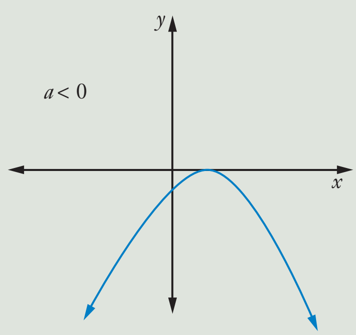
If $\Delta > 0$ then the quadratic equation has 2 real roots and the parabola has 2 x-intercepts





If $\Delta = 0$ then the quadratic equation has 1 real root or 2 equal roots and the parabola has one x-intercept





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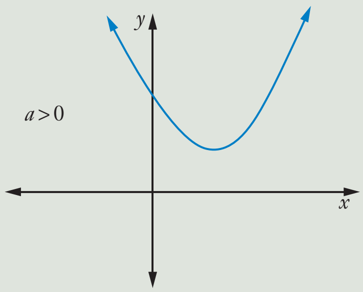
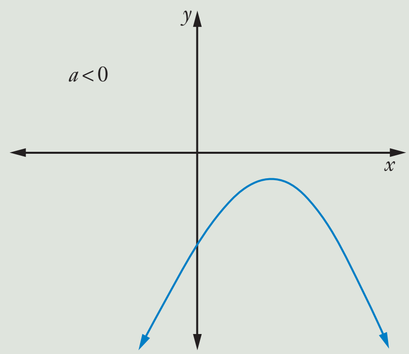
If $\Delta < 0$ then the quadratic equation has no real roots and the parabola has no x-intercepts



If $\Delta < 0$ and $a > 0$ then $ax^2 + bx + c > 0$ for all x



If $\Delta < 0$ and $a < 0$ then $ax^2 + bx + c < 0$ for all x

EXAMPLE 29

a Show that the parabola $f(x) = x^2 - x - 2$ has 2 x-intercepts

b Show that $x^2 - 2x + 4 > 0$ for all x

Solution

a $\Delta = b^2 - 4ac$

$= (-1)^2 - 4(1)(-2)$

$= 9$

> 0

So there are 2 real roots and the parabola has 2 x-intercepts

b If $a > 0$ and $\Delta < 0$ then $ax^2 + bx + c > 0$ for all x

$a = 1 > 0$

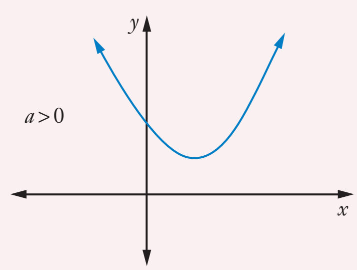
$\Delta = b^2 - 4ac$

$= (-2)^2 - 4(1)(4)$

$= -12$

< 0

Since $a > 0$ and $\Delta < 0$, $x^2 - 2x + 4 > 0$ for all x



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Exercise 3.10 The discriminant

1 Find the discriminant of each quadratic equation

a $x^2 - 4x - 1 = 0$ **b** $2x^2 + 3x + 7 = 0$ **c** $-4x^2 + 2x - 1 = 0$

d $6x^2 - x - 2 = 0$ **e** $-x^2 - 3x = 0$ **f** $x^2 + 4 = 0$

g $x^2 - 2x + 1 = 0$ **h** $-3x^2 - 2x + 5 = 0$ **i** $-2x^2 + x + 2 = 0$

2 Find the discriminant and state whether the roots of the quadratic equation are real or not real If the roots are rea, state whether they are equal or unequl, rational or irrational

a $x^2 - x - 4 = 0$ **b** $2x^2 + 3x + 6 = 0$ **c** $x^2 - 9x + 20 = 0$

d $x^2 + 6x + 9 = 0$ **e** $2x^2 - 5x - 1 = 0$ **f** $-x^2 + 2x - 5 = 0$

g $-2x^2 - 5x + 3 = 0$ **h** $-5x^2 + 2x - 6 = 0$ **i** $-x^2 + x = 0$

3 Find the value of p for which the quadratic equation $x^2 + 2x + p = 0$ has equal roots

4 Find any values of k for which the quadratic equation $x^2 + kx + 1 = 0$ has equal roots

5 Find all the values of b for which $2x^2 + x + b + 1 = 0$ has real roots

6 Evaluate p if $px^2 + 4x + 2 = 0$ has no real roots

7 Find all values of k for which $(k + 2)x^2 + x - 3 = 0$ has 2 real unequal roots

8 Prove that $3x^2 - x + 7 > 0$ for all real x

9 Show that the line $y = 2x + 6$ cuts the parabola $y = x^2 + 3$ in 2 points

10 Show that the line $3x + y - 4 = 0$ cuts the parabola $y = x^2 + 5x + 3$ in 2 points

11 Show that the line $y = -x - 4$ does not touch the parabola $y = x^2$

12 Show that the line $y = 5x - 2$ is a tangent to the parabola $y = x^2 + 3x - 1$.

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3.11 Finding a quadratic equation

WS

A page o paabola

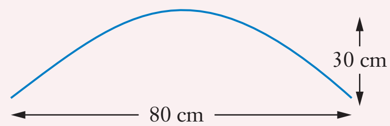
EXAMPLE 30

a Find the equation of the parabola that passes through the points (–1, –3), (0, 3) and (2, 21.

b A parabolic satellite dish is built so it is 30 cm deep and 80 cm wide as show.

i Find an equation for the parabola

ii Find the depth of the dish 10 cm out from the vertex



Solution

a The parabola has equation in the form $y = ax^2 + bx + c$

Substitute the points into the equation

(–1, –3)

$$-3 = a(-1)^2 + b(-1) + c$$
$$= a - b + c$$
$$\therefore\ a - b + c = -3 \qquad [1]$$

(0, 3:

$$3 = a(0)^2 + b(0) + c$$
$$= c$$
$$\therefore\ c = 3 \qquad [2]$$

(2, 21:

$$21 = a(2)^2 + b(2) + c$$
$$= 4a + 2b + c$$
$$\therefore\ 4a + 2b + c = 21 \qquad [3]$$

Solve simultaneous equations to find a b and c

Substitute [2] into [1]

$$a - b + 3 = -3$$
$$a - b = -6 \qquad [4]$$

Substitute [2] into [3]

$$4a + 2b + 3 = 21$$
$$4a + 2b = 18 \qquad [5]$$

[4] × 2

$$2a - 2b = -12 \qquad [6]$$

[5] + [6]

$$6a = 6$$
$$a = 1$$

Substitute $a = 1$ into [5]

$$4(1) + 2b = 18$$
$$4 + 2b = 18$$
$$2b = 14$$
$$b = 7$$

$$\therefore\ a = 1, b = 7, c = 3$$

Thus the parabola has equation $y = x^2 + 7x + 3$.

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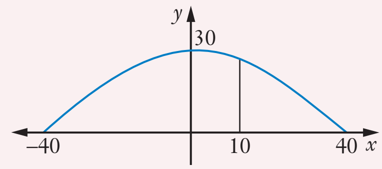
b **i** We can put the dish onto a number plane as shown Since the parabola is symmetrical the width of 80 cm means 40 cm either side of the y-axis



The parabola passes through points (0 30, (0, 0) and (–40 0.

Substitute these points into $y = ax^2 + bx + c$

(0, 30: $30 = a(0)^2 + b(0) + c = c$

So $y = ax^2 + bx + 30$

Substitute (40 0) into $y = ax^2 + bx + 30$

$0 = a(40)^2 + b(40) + 30$

$0 = 1600a + 40b + 30$ [1]

Substitute (–40 0) into $y = ax^2 + bx + 30$

$0 = a(-40)^2 + b(-40) + 30$

$0 = 1600a - 40b + 30$ [2]

[1] + [2]

$$0 = 3200a + 60$$

$$a = \frac{-60}{3200}$$

$$= -\frac{3}{160}$$

Substitute a into [1]

$$0 = 1600\left(-\frac{3}{160}\right) + 40b + 30$$

$$= -30 + 40b + 30$$

$$= 40b$$

$$0 = b$$

So $y = -\frac{3}{160}x^2 + 30$

ii Substitute $x = 10$

$$y = -\frac{3}{160}(10)^2 + 30$$

$$= 28125$$

So the depth of the dish at 10 cm is 28125 cm

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Exercise 3.11 Finding a quadratic equation

1 The braking distance of a car travelling at 100 km/h is 40 metres The formula for braking distance (d) in metres is $d = kx^2$ where k is a constant and x is speed in km/h

- **a** Find the value of k
- **b** Find the braking distance at 80 km/h
- **c** A dog runs out onto the road 15 m in front of a car travelling at 50 km/h Will the car be able to stop in time without hitting the dog?
- **d** If the dog was 40 m in front of a car travelling at 110 km/h would the car stop in time?

2 The area (A) of a figure is directly proportional to the square of its length (x). When $x = 5$ cm, its area is 125 cm^2

- **a** Find the equation for the area
- **b** Find the area when the length is 42 cm
- **c** Find the length correct to 1 decimal place when the area is 250 cm^2

3 The volume of a cylinder is given by $V = \pi r^2 h$ where r is the radius and h is height

- **a** Find the equation for volume if the height is fixed at 8 cm
- **b** Find the volume of a cylinder with radius 5 cm
- **c** Find the radius if the volume is 100 cm^3

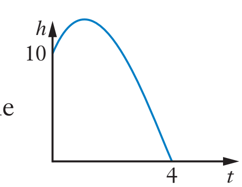
4 A rectangle has sides x and $3 - x$

- **a** Write an equation for its are.
- **b** Draw the graph of the area
- **c** Find the value of x that gives the maximum area
- **d** Find the maximum area of the rectangle

5 Find the equation of the parabola that passes through the points

- **a** (0 −5), (2, −3) and (−3, 7)
- **b** (1, −2), (3, 0) and (−2, 10)
- **c** (−2, 21), (1, 6) and (−1, 12)
- **d** (2, 3), (1, −4) and (−1, −12)
- **e** (0, 1), (−2, 1) and (2, −7)

6 Grania throws a ball off a 10 m high cliff After 1 s it is 2.5 m above ground and it reaches the ground after 4 s



- **a** Find the equation for the height (h metres) of the ball after time t seconds
- **b** Find the height of the ball after 2 seconds
- **c** Find when the ball is in line with the cliff

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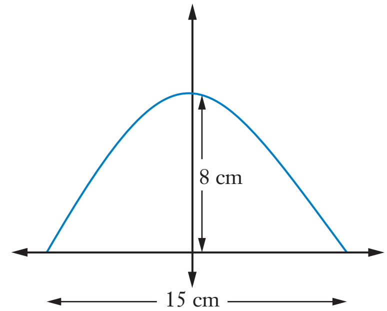
7 A parabolic shaped headlight is 15 cm wide and 8 cm deep as shown



a Find an equation for the parabola

b Find the depth of the headlight at a point 3 cm out from its axis of symmetry.

c At what width from the axis of symmetry does the headlight have a depth of 5 cm?

8 **a** Find the equation of the parabola passing through (0 0, 3,   −3) and (−1, 5.

b Find the value of y when

i $x = 5$ **ii** $x = -4$

c Find values of x when $y = -4$

d Find exact values of x when $y = 2$.

9 **a** Find the equation of the quadratic function $f(x)$ that passes through points (1 10, (0 7) and (−1, 6.

b Evaluate $f(-5)$

c Show that $f(x) > 0$ for all x

10 Find the equation of a parabola with axis of symmetry $x = 1$ minimum value −2 and passing through (0 0.

11 Find the equation of the quadratic function with axis $x = 3$ maximum value 13 and passing through (0 4.

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3.12 Cubic functions

A **cubic function** has an equation where the highest power of x is 3, such as $f(x) = kx^3$ $f(x) = k(x - b)^3 + c$ and $f(x) = k(x - a)(x - b)(x - c)$ where a b c and k are constants

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EXAMPLE 31

a Sketch the graph of the cubic function $f(x) = x^3 + 2$.

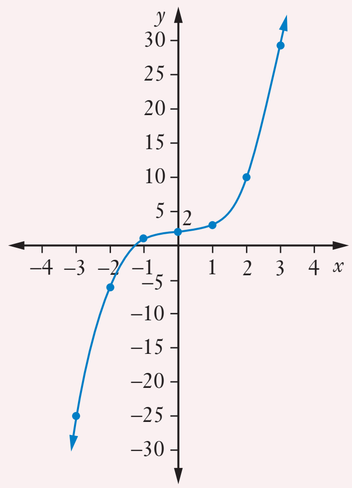
b State its domain and range

c Solve the equation $x^3 + 2 = 0$ graphically.

Solution

a Draw up a table of values

x	−3	−2	−1	0	1	2	3
y	−25	−6	1	2	3	10	29



b The function can have any real x or y value

Domain $(-\infty\ \ \infty)$

Range $(-\infty\ \ \infty)$

c From the graph the x-intercept is approximately −13

So the root of $x^3 + 2 = 0$ is approximately $x = -13$

INVESTIGATION

TRANSFORMING CUBIC FUNCTIONS

Use a graphics calculator or graphing software to sketch the graphs of some cubic functions such a:

$y = x^3$	$y = x^3 + 1$	$y = x^3 + 3$	$y = x^3 - 1$
$y = x^3 - 2$	$y = 2x^3$	$y = 3x^3$	$y = -x^3$
$y = -2x^3$	$y = -3x^3$	$y = 2x^3 + 1$	$y = (x + 1)^3$
$y = (x + 2)^3$	$y = (x - 1)^3$	$y = 2(x - 2)^3$	$y = 3(x + 2)^3 + 1$

$y = (x - 1)(x - 2)(x - 3)$ $\quad$ $y = x(x + 1)(x + 4)$ $\quad$ $y = 2(x + 1)(x - 2)(x + 5)$

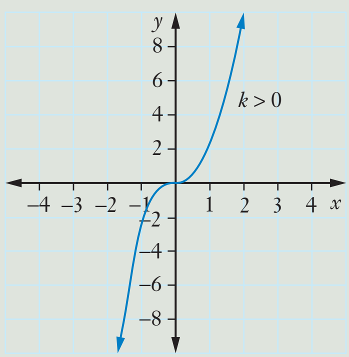
Can you see any patterns? Could you describe the shape of the cubic function? Could you predict where the graphs of different cubic functions would lie?

Is the cubic graph always a function? Can you find an example of a cubic that is not a function?

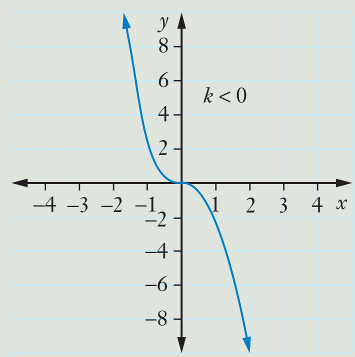
Point of inflection

The flat turning point of the cubic function $y = kx^3$ is called a **point of inflection** which is where the concavity of the curve changes

The graph of $y = kx^3$



This cubic curve is increasing and has a point of inflection at (0 0) where the curve changes from concave downwards to concave upwards



This cubic curve is decreasing and has a point of inflection at (0 0) where the curve changes from concave upwards to concave downwards

The graph of $y = k(x \quad b)^3 + c$

The graph of $y = k(x - b)^3 + c$ is the graph of $y = kx^3$ shifted so that its point of inflection is at $(b \;\; c)$

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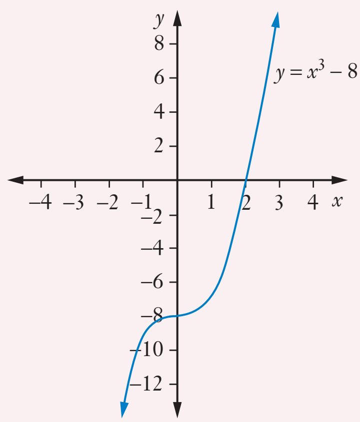
EXAMPLE 32

a Sketch the graph of $y = x^3 - 8$ showing intercept.

b Sketch the graph of $f(x) = -2(x - 3)^3 + 2$.

Solution

a This is the graph of $y = x^3$ shifted downwards 8 units so that its point of inflection is at (0 −8) Since $k > 0$, the function is increasing



For x-intercepts $y = 0$

$0 = x^3 - 8$

$8 = x^3$

$x = 2$

For y-intercept $x = 0$

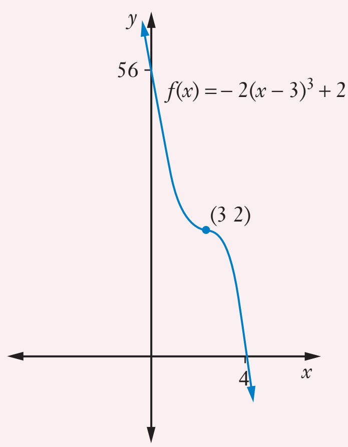
$y = 0^3 - 8$

$= -8$

The point of inflection is at (0 −8) where the curve changes from concave downwards to concave upwards

b Since $k < 0$, $f(x)$ is decreasing

This is the graph of $y = -2x^3$ shifted upwards and to the right so that its point of inflection is at (3 2.



For x-intercepts $f(x) = 0$

$0 = -2(x - 3)^3 + 2$

$-2 = -2(x - 3)^3$

$1 = (x - 3)^3$

$1 = x - 3$

$x = 4$

For y-intercept $x = 0$

$y = -2(0 - 3)^3 + 2$

$= -2(-27) + 2$

$= 56$

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EXAMPLE 33

Show that $y = 2x^3$ is an odd function

Solution

Let $f(x) = 2x^3$

$$f(-x) = 2(-x)^3$$
$$= -2x^3$$
$$= -f(x)$$

So $y = 2x^3$ is an odd function

A cubic function has one y-intercept and up to 3 x-intercepts We can sketch the graph of a more general cubic function using intercepts This will not give a very accurate graph but it will show the shape and important features

The graph of $y = k(x \quad a)(x \quad b)(x \quad c)$

The graph of $y = k(x - a)(x - b)(x - c)$ has x-intercepts at a b and c

EXAMPLE 34

a **i** Sketch the graph of the cubic function $f(x) = x(x + 3)(x - 2)$

ii Describe the shape of the graph and state its domain and range

b Sketch the graph of the cubic function $f(x) = (x - 3)(x + 1)^2$ and describe its shape

Solution

a **i** For x-intercepts $f(x) = 0$

$0 = x(x + 3)(x - 2)$

$x = 0, -3, 2$

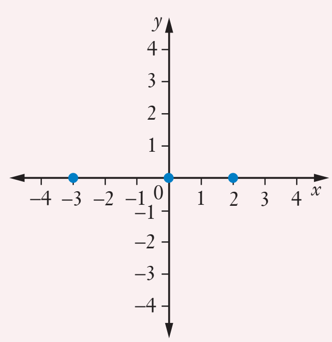
Plot x-intercepts on graph

For y-intercept $x = 0$

$f(0) = 0(0 + 3)(0 - 2)$

$= 0$

So y-intercept is 0



We look at which parts of the graph are above and which are below the x-axis between the x-intercepts

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Test $x < -3$ say $x = -4$

$f(-4) = -4(-4 + 3)(-4 - 2) = -24 < 0$

So here the curve is below the x-axis

Test $-3 < x < 0$ say $x = -1$

$f(-1) = -1(-1 + 3)(-1 - 2) = 6 > 0$

So here the curve is above the x-axis

Test $0 < x < 2$, say $x = 1$:

$f(1) = 1(1 + 3)(1 - 2) = -4 < 0$

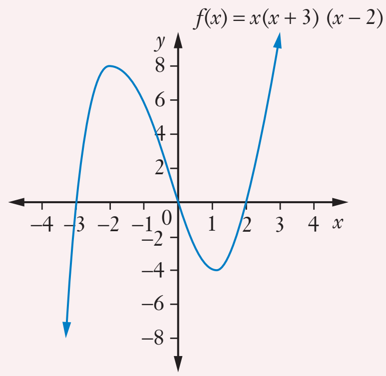
So here the curve is below the x-axis

Test $x > 2$, say $x = 3$:

$f(3) = 3(3 + 3)(3 - 2) = 18 > 0$

So here the curve is above the x-axis

We can sketch the cubic curve as shown



ii The graph increases to a maximum turning point then decreases to a minimum turning point Then it increases again

Domain $(-\infty\ \infty)$

Range $(-\infty\ \infty)$

b For x-intercepts $f(x) = 0$

$0 = (x - 3)(x + 1)^2$

$x = 3, x = -1$

So x-intercepts are -1 and 3

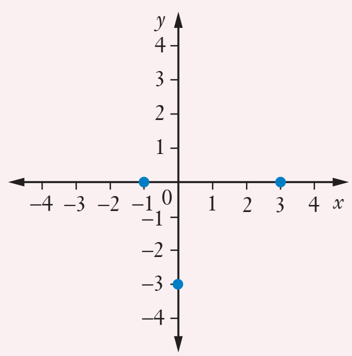
For y-intercept $x = 0$

$f(0) = (0 - 3)(0 + 1)^2$

$= (-3)(1)$

$= -3$

So y-intercept is -3



We look at which parts of the graph are above and below the x-axis

Test $x < -1$ say $x = -2$

$f(-2) = (-2 - 3)(-2 + 1)^2 = -5 < 0$

So here the curve is below the x-axis

Test $-1 < x < 3$, say $x = 0$

$f(0) = (0 - 3)(0 + 1)^2 = -3 < 0$

So here the curve is below the x-axis

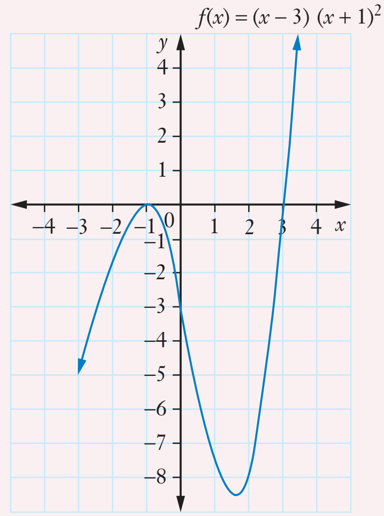
Test $x > 3$, say $x = 4$

$f(4) = (4 - 3)(4 + 1)^2 = 25 > 0$

So here the curve is above the x-axis

We can sketch the cubic curve as show.

The graph increases to a maximum turning point then decreases to a minimum turning point then increase.



Finding a cubic equation

EXAMPLE 35

a Find the equation of the cubic function $y = kx^3 + c$ if it passes through (0 16) and (4 0.

b Find the equation of the cubic function $f(x) = k(x - a)(x - b)(x - c)$ if it has x-intercepts –1 3 and 4 and passes through (, 1).

Solution

a Substitute (0 16) into $y = kx^3 + c$

$16 = k(0)^3 + c$

$\quad = c$

So $y = kx^3 + 16$.

Substitute (4 0) into $y = kx^3 + 16$.

$0 = k(4)^3 + 16$

$\quad = 64k + 16$

$-16 = 64k$

$k = -\frac{16}{64}$

$\quad = -\frac{1}{4}$

So the equation is $y = -\frac{1}{4}x^3 + 16$.

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b $f(x) = k(x - a)(x - b)(x - c)$ has x-intercepts when $f(x) = 0$

$0 = k(x - a)(x - b)(x - c)$

$x = a \;\; b \;\; c$

But we know x-intercepts are at -1, 3 and .

So $a = -1$, $b = 3$ and $c = 4$ (in any order)

So $f(x) = k(x - (-1))(x - 3)(x - 4)$

$= k(x + 1)(x - 3)(x - 4)$

To find k substitute (, 1):

$12 = k(1 + 1)(1 - 3)(1 - 4)$

$= k(2)(-2)(-3)$

$= 12k$

$1 = k$

So the cubic function is $f(x) = (x + 1)(x - 3)(x - 4)$

Exercise 3.12 Cubic functions

1 Find the x- and y-intercept(s) of the graph of each cubic function

a $y = x^3 - 1$	**b** $f(x) = -x^3 + 8$	**c** $y = (x + 5)^3$
d $f(x) = -(x - 4)^3$	**e** $f(x) = 3(x + 7)^3 - 3$	**f** $y = (x - 2)(x - 1)(x + 5)$

2 Draw each graph on a number plane

a $y = -x^3$	**b** $p(x) = 2x^3$	**c** $g(x) = x^3 + 1$
d $y = (x + 2)^3$	**e** $y = -(x - 3)^3 + 1$	**f** $f(x) = -x(x + 2)(x - 4)$
g $y = (x + 2)(x - 3)(x + 6)$	**h** $y = x^2(x - 2)$	**i** $f(x) = (x - 1)(x + 3)^2$

3 Find the point of inflection of the graph of each cubic function by sketching each graph

a $y = 8x^3 + 1$	**b** $y = -x^3 + 27$	**c** $f(x) = (x + 2)^3$
d $y = 2(x - 1)^3 - 16$	**e** $f(x) = -(x + 1)^3 + 1$	

4 Find the x-intercept of the graph of each cubic function correct to one decimal place

a $y = 2x^3 - 5$	**b** $f(x) = (x - 1)^3 + 2$	**c** $f(x) = -3x^3 + 1$
d $y = 2(x + 3)^3 - 3$	**e** $y = -3(2x - 1)^3 + 2$	

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5 Describe the shape of each cubic function

a $y = x^3 - 64$ **b** $f(x) = -(x - 3)^3$ **c** $y = x(x + 2)(x + 4)$

d $f(x) = -2(x + 3)(x + 1)(x - 4)$ **e** $y = x(x + 5)^2$

6 Solve graphically

a $x^3 - 5 = 0$ **b** $x^3 + 2 = 0$ **c** $2x^3 - 9 = 0$

d $3x^3 + 4 = 0$ **e** $(x - 1)^3 + 6 = 0$ **f** $x(x + 2)(x - 1) = 0$

7 The volume of a certain solid has equation $V = kx^3$ where x is the length of its side in cm

a Find the equation if $V = 120$ when $x = 3.5$.

b Find the volume when $x = 6$

c Find x when $V = 250$

8 The volume of a solid is directly proportional to the cube of its radius

a If radius $r = 12$ mm when the volume V is 7238 mm^3 find an equation for the volum.

b Find the volume if the radius is 25 mm

c Find the radius if the volume is 7000 mm^3

9 Show that $f(x) = -x^3$ is an odd function

10 Determine whether each function is odd

a $y = 3x^3$ **b** $y = (x + 1)^3$ **c** $f(x) = -2x^3 - 1$

d $y = -5x^3$ **e** $y = (x - 2)^3 + 3$

11 A cubic function is in the form $y = kx^3 + c$ Find its equation if it passes throug:

a (0, 0) and (1, 2) **b** (0, 5) and (2, –3) **c** (1, –4) and (–2, 23)

d (1, –2) and (2, 33) **e** (2 –29) and (–3, 111)

12 A cubic function is in the form $y = k(x - a)(x - b)(x - c)$ Find its equation i:

a it has x-intercepts 2 3 and −5 and passes through the point (–2 –120)

b it has x-intercepts −1 4 and 6 and passes through the point (, 96)

c it has x-intercepts 1 and 3 y-intercept –27 and $k = -3$

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3.13 Polynomial functions

Gaphing powe uncion

A **polynomial** is a function defined for all real x involving powers of x in the form

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \quad + a_2x^2 + a\ x + a_0$$

where n is a positive integer or zero and a_0 a $a_2, \ldots, a_n$ are real numbers

We generally write polynomials from the highest power down to the lowes, for example $P(x) = x^2 - 5x + 4$ We have already studied some polynomial functins, as liear, quadratic and cubic functions are all polynomials

Polynomial terminology

$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \quad + a_2x^2 + a\ x + a_0$ is called a **polynomial expression**

$P(x)$ has **degree** n (where n is the highest power of x)

a_n a_{n-1} $a_{n-2}, \ldots$ a_2 a and a_0 are called **coefficients**

a_nx^n is called the **leading term**

a_n is the **leading coefficient**

a_0 is called the **constant term**

If $a_n = 1$, $P(x)$ is called a **monic polynomial**

EXAMPLE 36

a Which of the following are polynomial expressions?

A $4 - x + 3x^2$ **B** $3x^4 - x^2 + 5x - 1$ **C** $x^2 - 3x + x^{-1}$

b $P(x) = x^6 - 2x^4 + 3x^3 + x^2 - 7x - 3$.

i Find the degree of $P(x)$

ii Is the polynomial monic?

iii State the leading term

v What is the constant term?

v Find the coefficient of x^4

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Solution

a **A** and **B** are polynomials but **C** is not because it has a term of x^{-1} that is not a positive integer power of x

b **i** Degree is 6 since x^6 is the highest power.

ii Ye, the polynomial is monic because the coefficient of x^6 is 1.

iii The leading term is x^6

v The constant term is -3

v The coefficient of x^4 is -2

Polynomial equations

$P(x) = 0$ is a **polynomial equation** of degree n

The values of x that satisfy the equation are called the **roots** of the equation or the **zeros** of the polynomial $P(x)$

EXAMPLE 37

a Find the zeros of the polynomial $P(x) = x^2 - 5x$

b Show that the polynomial $p(x) = x^2 - x + 4$ has no real zeros

Solution

a To find the zeros of the polynomia, solve $P(x) = 0$

$x^2 - 5x = 0$

$x(x - 5) = 0$

$x = 0, 5$

So the zeros are 0 .

b Solve $p(x) = 0$

$x^2 - x + 4 = 0$

The discriminant will show whether the polynomial has real zeros

$b^2 - 4ac = (-1)^2 - 4(1)(4)$

$= -15$

< 0

So the polynomial has no real zeros

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Graphing polynomials

EXAMPLE 38

a Write the polynomial $P(x) = x^4 + 2x^3 - 3x^2$ as a product of its factors

b Sketch the graph of the polynomial

Solution

a $P(x) = x^4 + 2x^3 - 3x^2$

$= x^2(x^2 + 2x - 3)$

$= x^2(x + 3)(x - 1)$

b For x-intercepts $P(x) = 0$

$0 = x^4 + 2x^3 - 3x^2$

$= x^2(x + 3)(x - 1)$

$x = 0, -3, 1$

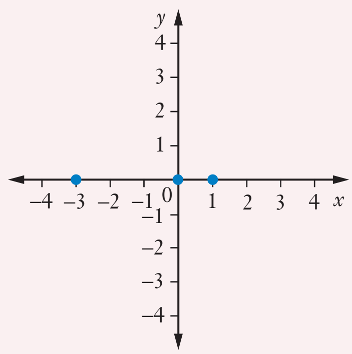
So the x-intercepts are $-3, 0, 1$.

For y-intercepts $x = 0$

$P(0) = 0^4 + 2(0)^3 - 3(0)^2$

$= 0$

So y-intercept is 0



Test $x < -3$ say $x = -4$

$P(-4) = (-4)^4 + 2(-4)^3 - 3(-4)^2 = 80 > 0$

So here the curve is above the x-axis

Test $0 < x < 1$, say $x = 05$

$P(05) = (05)^4 + 2(05)^3 - 3(05)^2$

$= -04375 < 0$

So here the curve is below the x-axis

Test $-3 < x < 0$ say $x = -1$

$P(-1) = (-1)^4 + 2(-1)^3 - 3(-1)^2 = -4 < 0$

So here the curve is below the x-axis

Test $x > 1$, say $x = 3$:

$P(3) = 3^4 + 2(3)^3 - 3(3)^2 = 108 > 0$

So here the curve is above the x-axis

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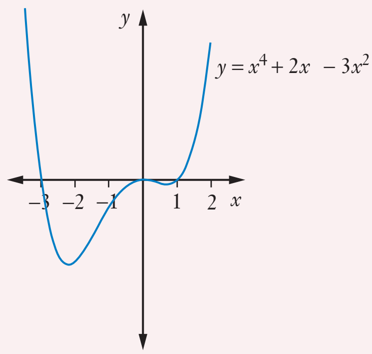
We can sketch the graph of the polynomial as show.



DID YOU KNOW?

'Poly' means many

The word 'polynomia' means an expression with many term. (A binomial has 2 terms and a trinomial has 3 terms) 'Pol' means'ma', and is used in many wods, for example polygamy, polyglt, polyon, polyheron, poymer, polphonic, polypod and polytechnic Do you know what all these words mean ? Do you know any others with poly- ?

Exercise 3.13 Polynomial functions

1 Write down the degree of each polynomia:

a $5x^7 - 3x^5 + 2x^3 - 3x + 1$ **b** $3 + x + x^2 - x^3 + 2x^4$ **c** $3x + 5$

d $x^{11} - 5x^8 + 4$ **e** $2 - x - 5x^2 + 3x^3$ **f** 3

2 For the polynomial $P(x) = x^3 - 7x^2 + x - 1$, fin:

a $P(2)$ **b** $P(-1)$ **c** $P(0)$

3 Given $P(x) = x + 5$ and $Q(x) = 2x - 1$, fin:

a $P(-11)$ **b** $Q(3)$ **c** $P(2) + Q(-2)$

d the degree of $P(x) + Q(x)$ **e** the degree of $P(x)Q(x)$

4 For the polynomial $P(x) = x^5 - 3x^4 - 5x + 4$ fin:

a the degree of $P(x)$ **b** the constant term

c the coefficient of x^4 **d** the coefficient of x^2

5 Find the zeros of each polynomial

a $P(x) = x^2 - 9$ **b** $p(x) = x + 5$ **c** $f(x) = x^2 + x - 2$

d $P(x) = x^2 - 8x + 16$ **e** $g(x) = x^3 - 2x^2 + 5x$

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6 Which of the following are not polynomials?

a $5x^4 - 3x^2 + x + \frac{1}{x}$ **b** $x^2 + 3^x$ **c** $x^2 + 3x - 7$

d $3x + 5$ **e** 0 **f** $4x^3 + 7x^{-2} + 5$

7 For the polynomial $P(x) = (a + 1)x^3 + (b - 7)x^2 + c + 5$ find values for a b or c if

a $P(x)$ is monic **b** the coefficient of x^2 is 3

c the constant term is -1 **d** $P(x)$ has degree 2

e the leading term has a coefficient of 5

8 Given $P(x) = 2x + 5$, $Q(x) = x^2 - x - 2$ and $R(x) = x^3 + 9x$ fin:

a any zeros of $P(x)$ **b** the roots of $Q(x) = 0$

c the degree of $P(x) + R(x)$ **d** the degree of $P(x)Q(x)$

e the leading term of $Q(x)R(x)$

9 Given $f(x) = 3x^2 - 2x + 1$ and $g(x) = 3x - 3$:

a show $f(x)$ has no zeros **b** find the leading term of $f(x)g(x)$

c find the constant term of $f(x) + g(x)$ **d** find the coefficient of x in $f(x)g(x)$

e find the roots of $f(x) + g(x) = 0$

10 State how many real roots there are for each polynomial equation $P(x) = 0$

a $P(x) = x^2 - 9$ **b** $P(x) = x^2 + 4$

c $P(x) = x^2 - 3x - 7$ **d** $P(x) = 2x^2 + x + 3$

e $P(x) = 3x^2 - 5x - 2$ **f** $P(x) = x(x - 1)(x + 4)(x + 6)$

11 Sketch the graph of each polynomial by finding its zeros and showing the x- and y-intercepts

a $f(x) = (x + 1)(x - 2)(x - 3)$ **b** $P(x) = x\,(x + 4)(x - 2)$

c $p(x) = -x(x - 1)(x - 3)$ **d** $f(x) = x(x + 2)^2$

e $g(x) = (5 - x)(x + 2)(x + 5)$

12 **i** Write each polynomial as a product of its factor.

ii Sketch the graph of the polynomial and describe its shape

a $P(x) = x^3 - 2x^2 - 8x$ **b** $f(x) = -x^3 - 4x^2 + 5x$

c $P(x) = x^4 + 3x^3 + 2x^2$ **d** $A(x) = 2x^3 + x^2 - 15x$

e $P(x) = -x^4 + 2x^3 + 3x^2$

13 **a** Find the x-intercepts of the polynomial $P(x) = x(x - 1)(x + 2)^2$

b Sketch the graph of the polynomial

14 **a** Show that $(x - 3)(x - 2)(x + 2) = x^3 - 3x^2 - 4x + 12$.

b Sketch the graph of the polynomial $P(x) = x^3 - 3x^2 - 4x + 12$.

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3.14 Intersection of graphs

Solving equations graphically

EXAMPLE 39

a Sketch $y = x^2$ and $y = 1$ on the same set of axes and hence solve $x^2 = 1$ graphically.

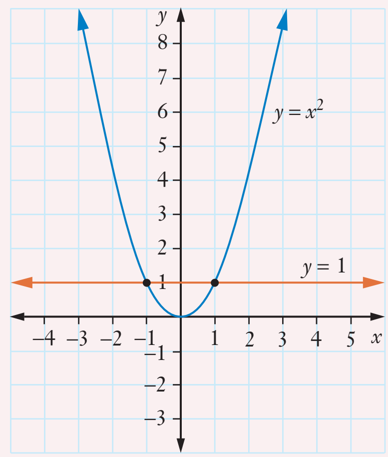
b Sketch $y = x^2 - x$ and $y = 2$ on the same set of axes and hence solve $x^2 - x = 2$ graphically.

Solution

a $y = x^2$ is a parabola and $y = 1$ is a horizontal line as show.

To solve $x^2 = 1$ graphically, find the x values where the 2 graphs $y = x^2$ and $y = 1$ intersect

The solution is $x = \pm 1$

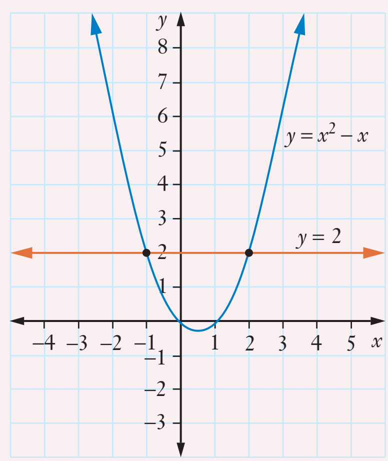


b $y = x^2 - x$ is a parabola with x-intercepts 0 1 and y-intercept 0 Since $a > 0$ it is concave upward.

$y = 2$ is a horizontal line

The solutions of $x^2 - x = 2$ are the x values at the intersection of the 2 graphs

$x = -1, .$



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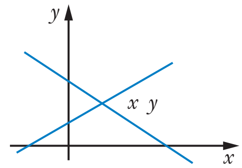
Intersecting lines

Two straight lines intersect at a single point (x y)



The point of intersection can be found graphically or algebraically using simultaneous equations

EXAMPLE 40

Find the point of intersection between lines $2x - 3y - 3 = 0$ and $5x - 2y - 13 = 0$

Solution

Solve simultaneous equations

$2x - 3y - 3 = 0$ [1]

$5x - 2y - 13 = 0$ [2]

[1] × 2

$4x - 6y - 6 = 0$ [3]

[2] × 3

$15x - 6y - 39 = 0$ [4]

[3] – [4]

$-11x + 33 = 0$

$33 = 11x$

$3 = x$

Substitute $x = 3$ into [1]

$2(3) - 3y - 3 = 0$

$-3y + 3 = 0$

$3 = 3y$

$1 = y$

So the point of intersection is (3 1.

Beakeven
poin

Break-even points

EXAMPLE 41

A company that manufactures cables sells them for \$2 each It costs 50 cents to produce each cable and the company has fixed costs of \$1500 per week

a Find the equation for the income \$ I on x cables per week

b Find the equation for the costs \$ C of manufacturing x cables per week

c Find the break-even point (where income = costs)

d Find the profit on 1450 cables

Solution

a $I = 2x$

b $C = 05\ x + 1500$

c Solving simultaneous equations

$I = 2x$ [1]

$C = 05\ x + 1500$ [2]

Substitute [1] into [2]

$2x = 05\ x + 1500$

$15\ x = 1500$

$x = 1000$

1000 cables is where income = costs
Substitute $x = 1000$ into [1] (or [2])

$I = 2(1000)$

$= 2000$

So the break-even point is (1000 2000. 1000 cables gives an income and cost of \$2000

d Profit = income − costs = $I - C$

Substitute $x = 1450$ into both equations

$I = 2x$

$= 2(1450)$

$= 2900$

So income is \$2900

$C = 05\ x + 1500$

$= 05(1450) + 1500$

$= 2225$

So costs are \$2225

Profit = \$2900 − \$2225

= \$675

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Break-even point

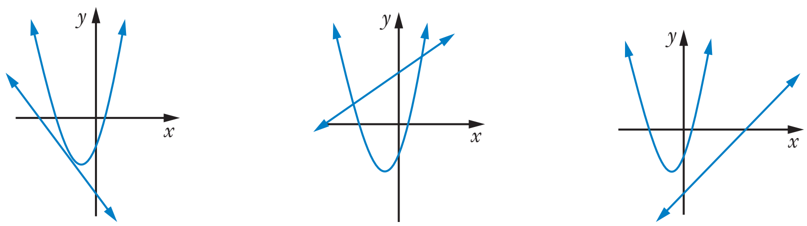
In business the **break-even point** is the point where the **income** (or **revenue**) **equals costs**

If income > costs the business makes a **profit**

If income < costs the business makes a **loss**

Intersecting lines and parabolas

A line and a parabola can intersect at 1 or 2 points or they may not intersect at al.



EXAMPLE 42

Find the points of intersection of the line $y = x - 1$ with the parabola $y = x^2 + 4x + 1$.

Solution

Solve simultaneous equations

$y = x - 1$ [1]

$y = x^2 + 4x + 1$ [2]

Substitute [1] into [2]

$x - 1 = x^2 + 4x + 1$

$0 = x^2 + 3x + 2$

$= (x + 2)(x + 1)$

$x = -2 \quad -1$

Substitute $x = -2$ into [1]

$y = -2 - 1$

$= -3$

Substitute $x = -1$ into [1]

$y = -1 - 1$

$= -2$

So the 2 points of intersection are (–2, –3) and (–1, –2)

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Exercise 3.14 Intersection of graphs

1 **a** Given $f(x) = 2x - 4$ solve graphicall:

- **i** $f(x) = 0$
- **ii** $f(x) = -2$
- **iii** $f(x) = 4$

b By sketching the graph of $f(x) = x^2 - 2x$ solve graphicall:

- **i** $f(x) = 0$
- **ii** $f(x) = 3$

c Use the sketch of $f(x) = x^3 - 1$ to solve graphically

- **i** $f(x) = 0$
- **ii** $f(x) = 7$
- **iii** $f(x) = -2$

2 Find the point of intersection between

- **a** $y = x + 3$ and $y = 2x + 2$
- **b** $y = 3x - 1$ and $y = 5x + 1$
- **c** $x + 2y - 4 = 0$ and $2x - y + 2 = 0$
- **d** $3x + y - 2 = 0$ and $2x - 3y - 5 = 0$
- **e** $4x - 3y - 5 = 0$ and $7x - 2y - 12 = 0$

3 Find points of intersection between

- **a** $y = x^2$ and $y = x$
- **b** $y = x^2$ and $y = 4$
- **c** $y = x^2$ and $y = x + 2$
- **d** $y = x^2$ and $y = -2x + 3$
- **e** $y = x^2 - 5$ and $y = 4x$

4 **a** Draw the graphs of $f(x) = x^2$ and $f(x) = (x - 2)^2$ on the same number plane

b From the graph find the number of points of intersection of the function.

c From the graph or by using algebra find any points of intersectio.

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5 Find any points of intersection between the functions $f(x) = x^2$ and $f(x) = (x + 2)^2$

6 Find any points of intersection between the curves $y = x^2 - 5$ and $y = 2x^2 + 5x + 1$.

7 Find any points of intersection between $y = 3x^2 - 4x - 4$ and $y = 5x^2 - 2$.

8 **a** If Paulas Posie' income on x roses is given by $y = 10x$ and the costs are $y = 3x + 980$ find the break-even poin.

b Find the profit on 189 roses

c Find the loss on 45 roses

9 Find the number of calculators that a company needs to sell to break even each week if it costs $3 to make each calculator and they are sold for $15 each Fixed overheads are $852 a week

10 Cupcakes Online sells cupcakes at $5 each The cost of making each cupcake is $1 and the company has fixed overheads of $264 a day.

a Find the equations for daily income and costs

b Find how many cupcakes the company needs to sell daily to break even

c What is the profit on 250 cupcakes?

d What is the loss on 50 cupcakes?

11 **a** The perimeter of a figure is in direct proportion to its side x Find an equation for perimeter if the perimeter $y = 90$ cm when side $x = 5$ cm

b The area of the figure is in direct proportion to the square of its side x If the area of the figure is $y = 108$ cm^2 when $x = 3$ cm find its equatio.

c Find any x values for the side for which the perimeter and area will have the same y value

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3. TEST YOURSELF

Qz

Pacice quiz

For Questions **1** to **5** select the correct answer **A B C** or **D**

1 Which polynomial below is a monic polynomial with constant term 5 and degree 6?

A $P(x) = -x^6 + 5$ **B** $P(x) = 6x^5 - 3x^4 + 5$

C $P(x) = x^6 - 3x^4 + 5$ **D** $P(x) = 5x^6 - 3x^4 + 1$

2 The axis of symmetry and turning point of the quadratic function $f(x) = 1 + 2x - x^2$ are respectively

A $x = 1, (1, 2)$ **B** $x = -1, (-1\ 4)$

C $x = 2, (2, 5)$ **D** $x = -2, (-2, 5)$

3 The linear function $2x - 3y - 6 = 0$ has x- and y-intercepts respectivel:

A -3 and 2 **B** 3 and -2 **C** -3 and -2 **D** 3 and 2

4 The domain and range of the straight line with equation $x = -2$ are

A Domain $(-\infty\ \infty)$ range $[-2]$ **B** Domain $[-2]$ range $(-\infty\ \infty)$

C Domain $(-\infty\ \infty)$ range $(-\infty\ \infty)$ **D** Domain $[-2]$ range $[-2]$

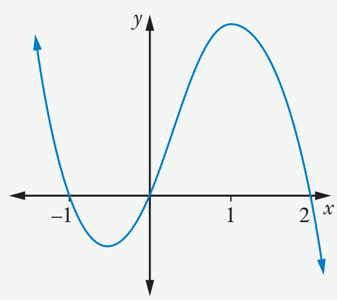
5 Which cubic function has this graph?

A $y = x(x + 1)(x - 2)$

B $y = -x(x - 1)(x + 2)$

C $y = x(x - 1)(x + 2)$

D $y = -x(x + 1)(x - 2)$



6 If $f(x) = x^2 - 3x - 4$ fin:

a $f(-2)$ **b** $f(a)$ **c** x when $f(x) = 0$

7 Sketch each graph and find its domain and range

a $y = x^2 - 3x - 4$ **b** $f(x) = x^3$ **c** $2x - 5y + 10 = 0$

d $x = 3$ **e** $y = (x + 1)^3$ **f** $y = -2$

g $f(x) = -x^2 + x$ **h** $f(x) = x^2 + 4x + 4$

8 If $f(x) = 3x - 4$ fin:

a $f(2)$ **b** x when $f(x) = 7$ **c** x when $f(x) = 0$

9 Sketch the graph of $P(x) = 2x^3 - 2x^2 - 4x$

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10 Find the gradient of the straight line

a passing through (3 −1) and (−2, 5)

b with equation $2x - y + 1 = 0$

c perpendicular to the line $5x + 3y - 8 = 0$

d making an angle of inclination of 45°

11 For the parabola $y = x^2 - 4x + 1$, fin:

a the equation of the axis of symmetry

b the minimum value

12 Sketch the graph of $f(x) = (x - 2)(x + 3)^2$ showing the intercept.

13 For the polynomial $P(x) = x^3 + 2x^2 - 3x$ fin:

a the degree

b the coefficient of x

c the zeros

d the leading tem.

14 Find the x- and y-intercepts of

a $2x - 5y + 20 = 0$

b $y = x^2 - 5x - 14$

c $y = (x + 2)^3$

d $2x - 5y - 10 = 0$

15 Find the point of intersection between lines $y = 2x + 3$ and $x - 5y + 6 = 0$

16 For the quadratic function $y = -2x^2 - x + 6$ fin:

a the equation of the axis of symmetry

b the maximum value

17 Find the domain and range of $y = -2x^2 - x + 6$

18 For each quadratic equation select the correct property of its roots **A B C** or **D**

A real different and rational

B real different and irrational

C equal

D unreal

a $2x^2 - x + 3 = 0$

b $x^2 - 10x - 25 = 0$

c $x^2 - 10x + 25 = 0$

d $3x^2 + 7x - 2 = 0$

e $6x^2 - x - 2 = 0$

19 Find the equation of the line

a passing through (2 3) and with gradient 7

b parallel to the line $5x + y - 3 = 0$ and passing through (1 1)

c through the origin and perpendicular to the line $2x - 3y + 6 = 0$

d through (3 1) and (−2 4)

e with x-intercept 3 and y-intercept −1

20 The polynomial $f(x) = ax^2 + bx + c$ has zeros 4 and 5 and $f(-1) = 60$ Evaluate a b and c

21 Determine whether each function is even odd or neithe.

a $y = x^2 - 1$

b $y = x + 1$

c $y = x^3$

d $y = (x + 1)^2$

e $y = -5x^3$

22 Show that $f(x) = x^3 - x$ is odd

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23 Prove that the line between (–1 4) and (, 3) is perpendicular to the line $4 \quad x - y - 6 = 0$

24 Show that $-4 + 3x - x^2 < 0$ for all x

25 For each pair of equations state whether their graphs have , 1 or 2 points of intersectin.

a $xy = 7$ and $3x - 5y - 1 = 0$ **b** $x^2 + y^2 = 9$ and $y = 3x - 3$

c $x^2 + y^2 = 1$ and $x - 2y - 3 = 0$ **d** $y = x^2$ and $y = 4x - 4$

26 Prove that the lines with equations $y = 5x - 7$ and $10x - 2y + 1 = 0$ are parallel

27 Find the zeros of $g(x) = -x^2 + 9x - 20$

28 Sketch the graph of $P(x) = 2x(x - 3)(x + 5)$ showing intercept.

29 Solve $P(x) = 0$ when $P(x) = x^3 - 4x^2 + 4x$

30 Find x if the gradient of the line through (3 –4) and (x, 2) is –.

31 If $f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x^2 - 3 & \text{if } x < 1 \end{cases}$ find $f(5) - f(0) + f(1)$

32 Given $f(x) = \begin{cases} 3 & \text{if } x > 3 \\ x^2 & \text{if } 1^\circ\ x^\circ\ 3 \\ 2 - x & \text{if } x < 1 \end{cases}$

find

a $f(2)$ **b** $f(-3)$ **c** $f(3)$

d $f(5)$ **e** $f(0)$

33 Find the equation of the parabola

a that passes through the points (–2 18, 3, –2) and(1, 0)

b with x-intercepts 3 and –2 and y-intercept 12

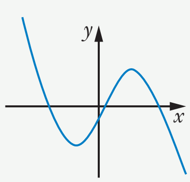
34 The area (A) of a certain shape is in direct proportion to the square of its length x If the area is 448 cm^2 when $x = 8$ fin:

a the equation for area **b** the area when $x = 10$

c x when the area is 109375 cm ^{2}

35 For each graph and set of ordered pairs state whether it represents a functio, and for those that do whether it represents a one-to-one functio.

a



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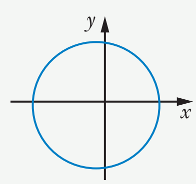

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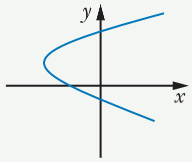
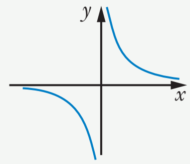
c



d



e (1, 2), (2, 5), (–1, 4), (1, 3), (3, 4)

36 Find the equation of a cubic function $f(x) = kx^3 + c$ if it passes through the point (1 2) and has y-intercept 5

37 A company has costs given by $y = 7x + 15$ and income $y = 12x$ Find the break-even point

38 **a** Find the equation of the straight line that is perpendicular to the line $y = \frac{1}{2}x - 3$ and passes through (1 –1.

b Find the x-intercept of this line

39 Find values of m such that $mx^2 + 3x - 4 < 0$ for all x

40 Find any points of intersection of the graphs of

a $y = 3x - 4$ and $y = 1 - 2x$

b $y = x^2 - x$ and $y = 2x - 2$

c $y = x^2$ and $y = 2x^2 - 9$

41 Find the equation of the straight line passing through the origin and parallel to the line with equation $3x - 4y + 5 = 0$

42 Find the equation of the line with y-intercept –2 and perpendicular to the line passing through (3 –2) and (,).

43 The amount of petrol used in a car is directly proportional to the distance travelled

a If the car uses 108 litres of petrol for an 87 km trip find the equation for the amount of petrol used (A) over a distance of d km

b Find the amount of petrol used for a 250 km trip

c Find how far the car travelled if it used 355 L of petrol

44 A function has equation $f(x) = x^3 - x^2 - 4x + 4$

a Solve $f(x) = 0$

b Find its x- and y-intercepts

c Sketch the graph of the function

d From the graph state how many solutions there are fo:

i $f(x) = 1$

ii $f(x) = -2$

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3. CHALLENGE EXERCISE

1 Find the values of b if $f(x) = 3x^2 - 7x + 1$ and $f(b) = 7$.

2 Sketch the graph of $y = (x + 2)^2 - 1$ in the domain [–3 0.

3 If points $(-3k, 1)$, $(k - 1, k - 3)$ and $(k - 4, k - 5)$ are collinear (lie on a straight line) find the value of k

4 Find the equation of the line that passes through the point of intersection of the lines $2x + 5y + 19 = 0$ and $4x - 3y - 1 = 0$ and is perpendicular to the line $3x - 2y + 1 = 0$

5 If $ax - y - 2 = 0$ and $bx - 5y + 11 = 0$ intersect at the point (3 4, find the values of a and b

6 By writing each as a quadratic equation solv:

a $(3x - 2)^2 - 2(3x - 2) - 3 = 0$ **b** $5^{2x} - 26(5^x) + 25 = 0$

c $2^{2x} - 10(2^x) + 16 = 0$ **d** $2^{2x+1} - 5(2^x) + 2 = 0$

7 Find the equation of the straight line through (1 3) that passes through the intersection of the lines $2x - y + 5 = 0$ and $x + 2y - 5 = 0$

8 $$f(x) = \begin{cases} 2x+3 & \text{when } x > 2 \\ 1 & \text{when } -2 \le x \le 2 \\ x^2 & \text{when } x < -2 \end{cases}$$

Find $f(3)$ $f(-4)$ $f(0)$ and sketch the graph of the function

9 If $h(t) = \begin{cases} 1-t^2 & \text{if } t > 1 \\ t^2-1 & \text{if } t \le 1 \end{cases}$ find the value of $h(2) + h(-1) - h(0)$ and sketch the curve

10 If $f(x) = 2x^3 - 2x^2 - 12x$ find x when $f(x) = 0$

11 Show that the quadratic equation $2x^2 - kx + k - 2 = 0$ has real rational roots

12 Find the values of p for which $x^2 - x + 3p - 2 > 0$ for all x

13 If $f(x) = 2x - 1$ show that $f(a^2) = f[(-a)^2]$ for all real a

14 Find the equation of the straight line through (3 –4) that is perpendicular to the line with x-intercept –2 and y-intercept 5

15 Find any points of intersection between $y = x^2$ and $y = x^3$

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16 Find the equation of a cubic function $y = ax^3 + bx^2 + cx + d$ if it passes through (0 1, (1, 3), (–1, 3) and (2,15).

17 Show that the quadratic equation $x^2 - 2px + p^2 = 0$ has equal roots

18 A monic polynomial $P(x)$ of degree 3 has zeros –2 1 and . Write down the equation of the polynomial

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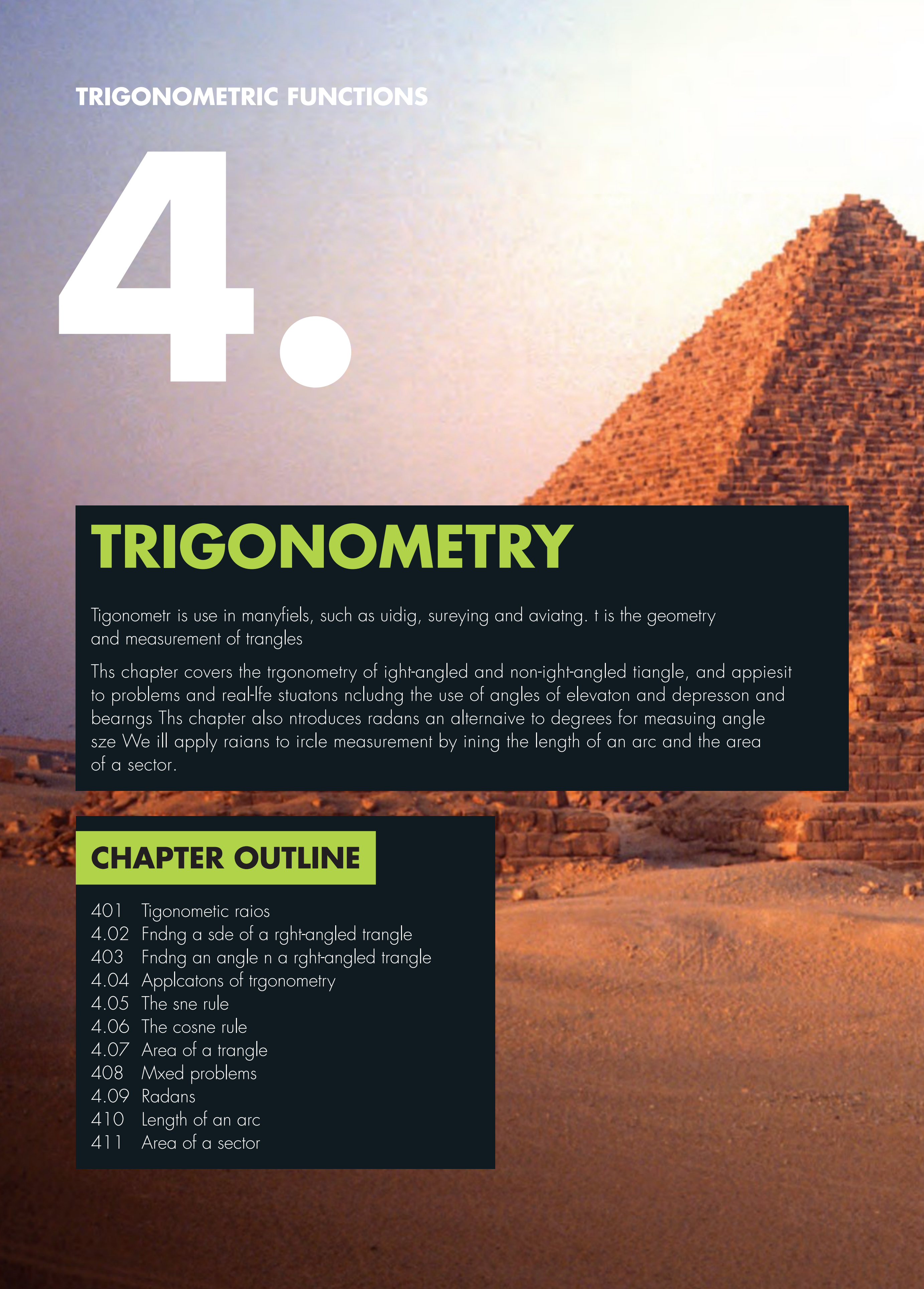


4.

TRIGONOMETRY

Tigonometr is use in manyfiels, such as uidig, sureying and aviatng. t is the geometry and measurement of trangles

Ths chapter covers the trgonometry of ight-angled and non-ight-angled tiangle, and appiesit to problems and real-lfe stuatons ncludng the use of angles of elevaton and depresson and bearngs Ths chapter also ntroduces radans an alternaive to degrees for measuing angle sze We ill apply raians to ircle measurement by ining the length of an arc and the area of a sector.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- dentfy the trgonometrc ratos
- solve rght-angled trangle problems
- apply trgonometry to angles of elevaion and depresion and beaings
- understand and apply the sne and cosne rules
- fnd the area of a trangle gven the length of two sdes and the sze of ther ncluded angle
- understand radans and convert between degrees and raians
- fnd the length of an arc and area of a sector of a crcle

TERMINOLOGY

ambiguous case When using the sine rule to find an angle there may be 2 possible angles – one acute and one obtuse

angle of depression The angle between the horizontal and the line of sight when looking down to an object below

angle of elevation The angle between the horizontal and the line of sight when looking up to an object above

bearing A direction from one point on Earths surface to anothe, measured in degrees. Bearings may be written as true bearings (clockwise from north) or as compass bearings (using N,S, E and W)

compass bearing Angles specified as either side of north or south for example N 20 ° W or S 67° E

cosine rule In any triangle $c^2 = a^2 + b^2 - 2ab \cos C$

radian A unit of angle measurement equal to the size of the angle subtended at the centre of a unit circle by an arc of length 1 unit

sine rule In any triangle $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

true bearing True or three-figure bearings are measured from north and turning clockwise

DID YOU KNOW?

Ptolemy

Ptolemy (Claudius Ptolemaeus) in the second centur, wrote *Hē mathē matikē syntaxis* (or *Almagest* as it is now known) on astronomy. This is considered to be the first treatise on trigonometry, but it was based on circles and spheres rather than on triangls. The notation 'chord of an angl' was used rather than si, cos or tn.

Ptolemy constructed a table of sines from 0° to 90° in steps of a quarter of a degree He also calculated a value of π to 5 decimal places and established the relationship for $\sin(x \pm y)$ and $\cos(x \pm y)$

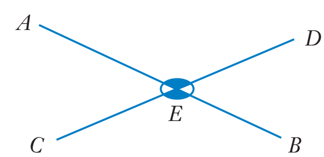
Geometry results

You will need to use some geometry when solving trigonometry problem. Here is a summary of the rules you may need

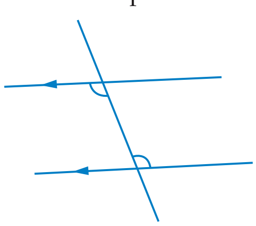
$\angle AEC$ and $\angle DEB$ are **vertically opposite angles**

$\angle AED$ and $\angle CEB$ are also vertically opposite

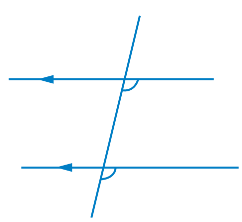
Vertically opposite angles are equa.



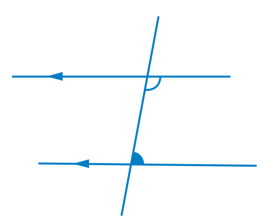
If lines are parallel the:



alternate angles are equal



corresponding angles are equal



cointerior angles are **supplementary** (their sum is 180°)

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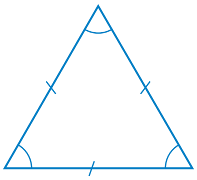
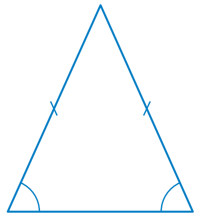
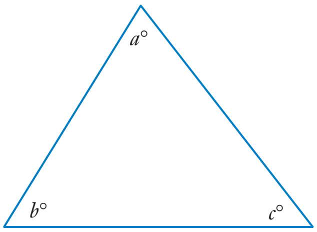


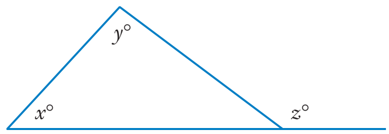
An **equilateral triangle** has 3 equal sides and 3 equal angles of size 60°



An **isosceles triangle** has 2 equal sides and 2 equal angles



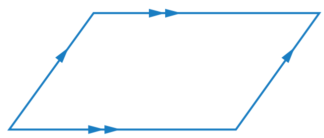
The **sum of the interior angles** in any triangle is 180°, that is,
$a + b + c = 180$



The **exterior angle** in any triangle is equal to the sum of the 2 opposite interior angles
That is $x + y = z$

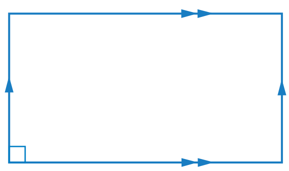
A **parallelogram** is a quadrilateral with opposite sides parallel

- Opposite sides are equal
- Opposite angles are equal
- Diagonals bisect each other.



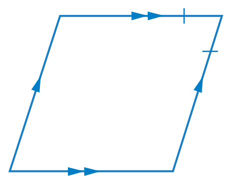
A **rectangle** is a parallelogram with one angle a right angle

- Opposite sides are equal
- All angles are right angles
- Diagonals are equal and bisect each other.



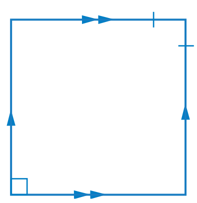
A **rhombus** is a parallelogram with a pair of adjacent sides equal

- All sides are equal
- Opposite angles are equal
- Diagonals bisect each other at right angles



A **square** is a rectangle with a pair of adjacent sides equal

- All sides are equal
- All angles are right angles
- Diagonals are equal and bisect each other at right angles
- Diagonals make angles of 45° with the sides



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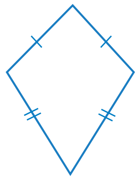
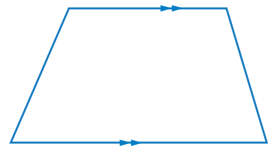
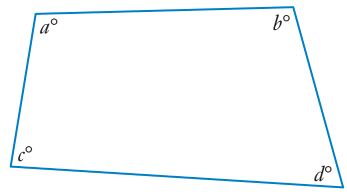
A **kite** is a quadrilateral with 2 pairs of adjacent sides equal



A **trapezium** is a quadrilateral with one pair of sides parallel



The **sum of the interior angles** in any quadrilateral is 360° that is $a + b + c + d = 360$

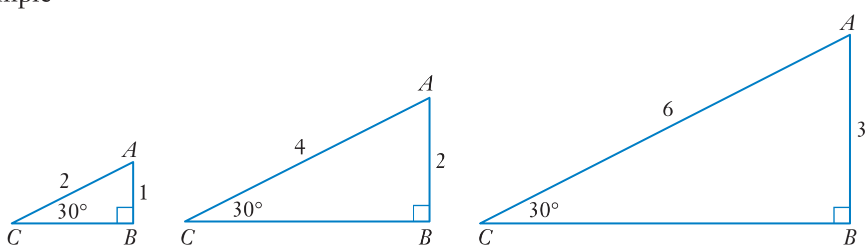


Trigonomric aio

Trigonomry calculaions

4.01 Trigonometric ratios

In similar triangles pairs of corresponding angles are equal and sides are in proportio. For example

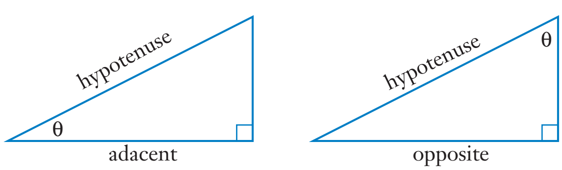


In any triangle containing an angle of 30° the ratio $AB : AC = 1 \ \ 2$ Similarl, the ratios of other corresponding sides will be equal These ratios of sides form the basis of the trigonometric ratios

The sides of a right-angled triangle

- The **hypotenuse** is the longest side and is always opposite the right angl.
- The **opposite** side is opposite the angle marked in the triangle
- The **adjacent** side is next to the angle marked

The opposite and adjacent sides vary according to where the angle is marked For exampl:



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The trigonometric ratios

Sine $\sin \theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

Cosine $\cos \theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

Tangent $\tan \theta = \dfrac{\text{opposite}}{\text{adjacent}}$

DID YOU KNOW?

The origins of trigonometry

Trigonometr, or **triangle measurement** progressed from the study of geometry in ancient Greece Trigonometry was seen as applied mathematcs. It gave a tool for the measurement of planets and their motion It was also used extensively in navigatio, surveying and mapping and it is still used in these fields toda.

Trigonometry was crucial in setting up an accurate calenda, since this involved measuring the distances between the Earth Sun and Moo.

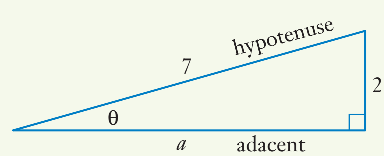
EXAMPLE 1

If $\sin \theta = \frac{2}{7}$ find the exact ratios of cos θ and tan θ

Solution

$\sin \theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{2}{7}$

First draw a triangle with opposite side 2 and hypotenuse 7 then use Pythagora' theorem to find the adjacent side



$c^2 = a^2 + b^2$

$7^2 = a^2 + 2^2$

$49 = a^2 + 4$

$45 = a^2$

$a = \sqrt{45}$

$\cos \theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{\sqrt{45}}{7}$

$\tan \theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{2}{\sqrt{45}}$

Degrees, minutes, seconds

Angles are measured in degrees minutes and second.

$$60 \text{ minutes} = 1 \text{ degree } (60' = 1^\circ)$$

$$60 \text{ seconds} = 1 \text{ minute } (60'' = 1')$$

When rounding numbers you round up if the digit to the right is 5 or mor. Howevr, with angles you round up to the next degree if there are 30 minutes or mor.

Similarly, round angles up to the nearest minute if there are 30 seconds or moe.

EXAMPLE 2

a Round to the nearest degree

i $54°17'45''$ **ii** $29°32'52''$

b Round to the nearest minute

i $23°12'22''$ **ii** $57°34'41''$ **iii** $84°19'30''$

Solution

a **i** $17'$ is less than $30'$ so rounding gives $54°$

ii $32'$ is more than $30'$ so rounding gives $30°$

b **i** $22''$ is less than $30''$ so rounding gives $23°12'$

ii $41''$ is more than $30''$ so rounding gives $57°35'$

iii $30''$ is exactly halfway so round up to $84°20'$

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Decimal degrees and degrees-minutes-seconds

Scientific calculators have a °’ ” or DM'S key for converting between decimal degrees and degrees minute, secons.

EXAMPLE 3

a Change 58°19′ into a decimal

b Change 45236 ° into degrees and minutes

Solution

a

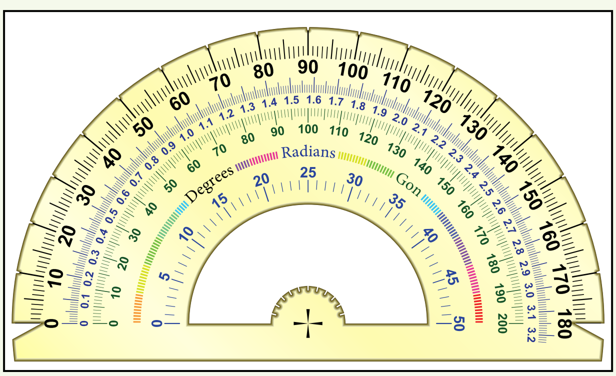
Operation	Casio scientific	Sharp scientific
Make sure the calculator is in degrees	SHFT SET UP deg	Press DRG until deg is on the screen
Enter data	58 °’ ” 19 °’ ” =	58 DM'S 19 DM'S
Change to a decimal	°’ ”	2ndF DM'S

So 58°19′ = 5831666 ≈ 5832

b

Operation	Casio scientific	Sharp scientific
Enter data	45236 =	45236 =
Change to degrees and minutes	°’ ”	2ndF DM'S

So 45236 ° = 45°14′96 ″ ≈ 45°14′



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EXAMPLE 4

a Find cos 58°19′ correct to 3 decimal places

b If tan θ = 0348 find θ in degrees and minutes

Solution

a

Operation	Casio scientific	Sharp scientific
Enter data	cos 58 °’ ” 19 °’ ” =	cos 58 DM'S 19 DM'S =

So cos 58°19′ = 052522   ≈ 0525

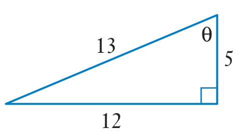
b To find the angle given the rati, use the inverse key (tan ⁻)

Operation	Casio scientific	Sharp scientific
Enter data	SHFT tan⁻ 0348 =	2ndF tan⁻ 0348 =
Change to degrees and minutes	°’ ”	2ndF DM'S

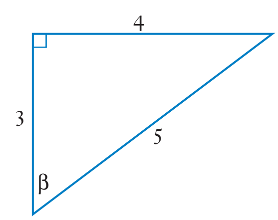
So θ = 19°11′1643 ″ ≈ 19°11′

Exercise 4.01 Trigonometric ratios

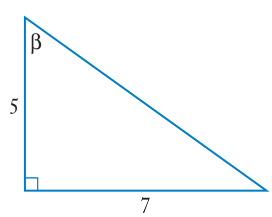
1 Write down the ratios of cos θ sin θ and tan θ



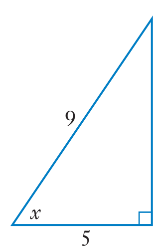
2 Find sin β tan β and cos β



3 Find the exact ratios of sin β tan β and cos β



4 Find exact values for cos x tan x and sin x



5 If tan θ = $\frac{4}{3}$ find cos θ and sin θ

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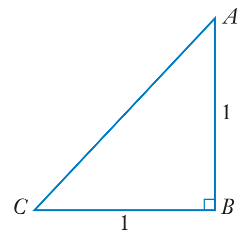
6 If $\cos\theta = \frac{2}{3}$ find exact values for $\tan\theta$ and $\sin\theta$

7 If $\sin\theta = \frac{1}{6}$ find the exact ratios of $\cos\theta$ and $\tan\theta$

8 If $\cos\theta = 07$ find exact values for $\tan\theta$ and $\sin\theta$

9 $\triangle ABC$ is a right-angled isosceles triangle with $\angle ABC = 90°$ and $AB = BC = 1$.



a Find the exact length of AC

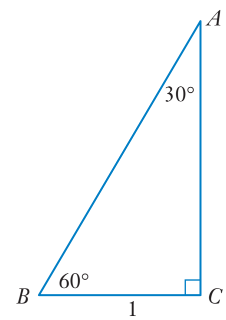
b Find $\angle BAC$

c From the triangle write down the exact ratios of $\sin 45°$ $\cos 45°$ and $\tan 45°$

10 **a** Using Pythagoras theorem find the exact length of AC

b Write down the exact ratios of $\sin 30°$, $\cos 30°$ and $\tan 30°$

c Write down the exact ratios of $\sin 60°$ $\cos 60°$ and $\tan 60°$



11 Round each angle to the nearest degree

a 47°13′12″ **b** 81°45′43″ **c** 19°25′34″ **d** 76°37′19″ **e** 52°29′54″

12 Round each angle to the nearest minute

a 47°13′12″ **b** 81°45′43″ **c** 19°25′34″ **d** 76°37′19″ **e** 52°29′54″

13 Change to a decimal

a 77°45′ **b** 65°30′ **c** 24°51′ **d** 68°21′ **e** 82°31′

14 Change into degrees and minutes

a 5953 ° **b** 72231 ° **c** 85887 ° **d** 469 ° **e** 73213 °

15 Find correct to 3 decimal places

a sin 39°25′ **b** cos 45°51′ **c** tan 18°43′ **d** sin 68°06′ **e** tan 54°20′

16 Find θ in degrees and minutes if

a $\sin\theta = 0298$ **b** $\tan\theta = 0683$ **c** $\cos\theta = 0827$

d $\tan\theta = 1056$ **e** $\cos\theta = 0188$

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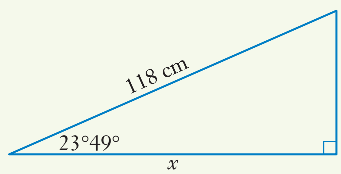
Finding an unknown ide

4.02 Finding a side of a right-angled triangle

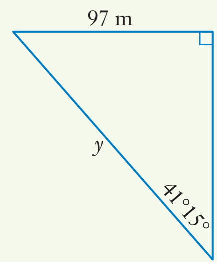
We can use trigonometry to find an unknown side of a triangl.

EXAMPLE 5

a Find the value of x correct to 1 decimal place



b Find the value of y correct to 3 significant figures



Solution

a

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos\ 23°49' = \frac{x}{11\ 8}$$

$$11\ 8\cos\ 23°49' = x$$

$$x \approx 10\ 8\text{ cm}$$

b

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 41°15' = \frac{9\ 7}{y}$$

$$y\sin 41°15' = 9\ 7$$

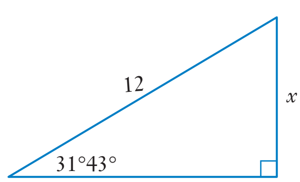
$$y = \frac{9\ 7}{\sin 41°15'}$$

$$\approx 147\text{ m}$$

Exercise 4.02 Finding a side of a right-angled triangle

1 Find the values of all pronumerals correct to 1 decimal plac:

a



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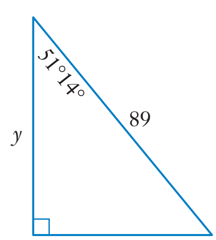

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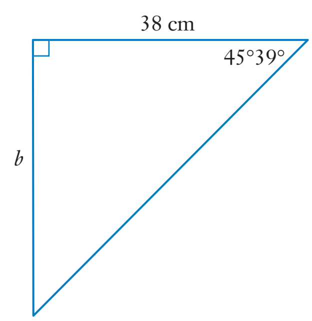

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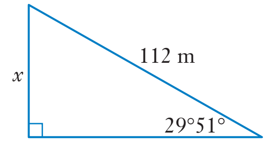

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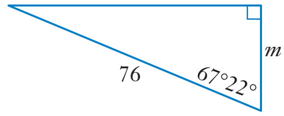

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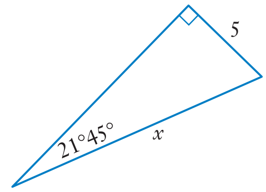

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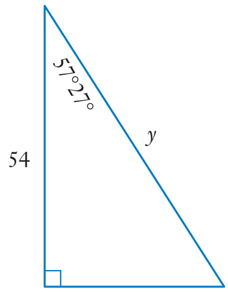

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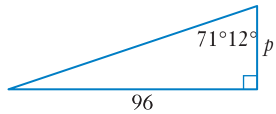

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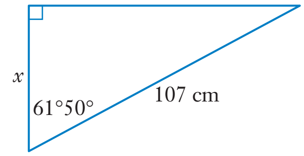

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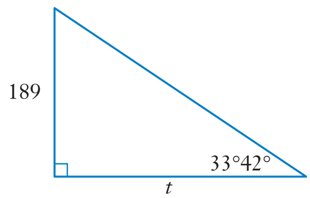

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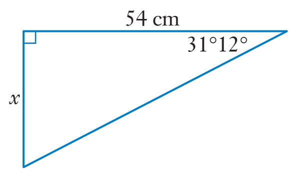

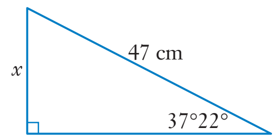

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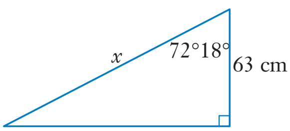

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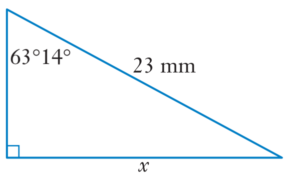

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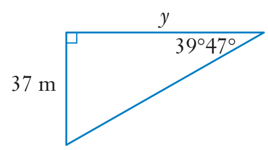

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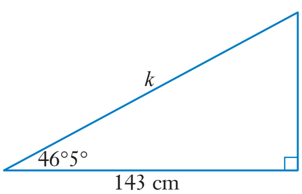

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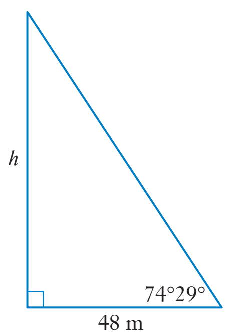

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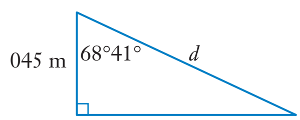

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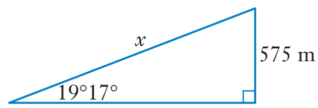

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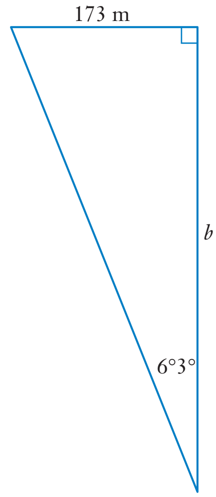
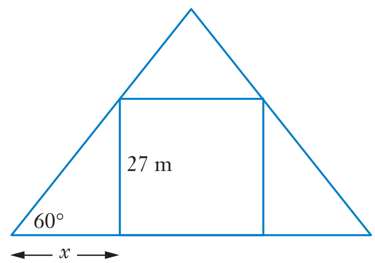
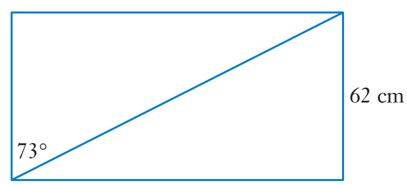


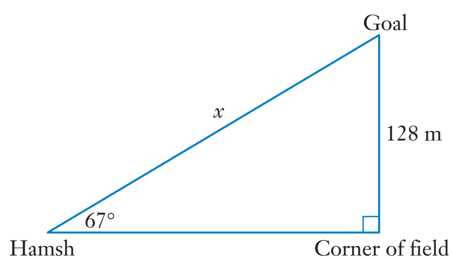
2 A roof is pitched at 60° A room built inside the roof space is to have a 27 m high ceiling How far in from the side of the roof will the wall for the room go?



3 A diagonal in a rectangle with width 62 cm makes an angle of 73° with the vertex as shown Find the length of the rectangle correct to 1 decimal place



4 Hamish is standing on the sideline of a soccer field and the goal is at an angle of 67 ° from his position as shown The goal is 2.8 m from the corner of the field How far does he need to kick a ball for it to reach the goal?



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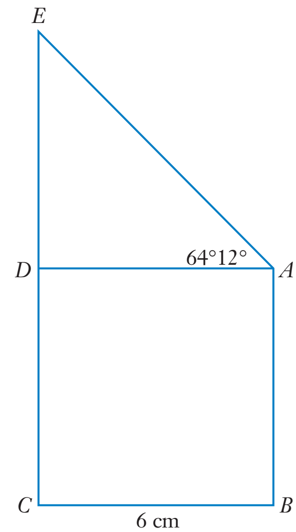
5 Square $ABCD$ with side 6 cm has line CD produced to E as shown so that $\angle EAD = 64°12'$ Evaluate the lengt, correct to 1 decimal place o:

a CE **b** AE



6 A right-angled triangle with hypotenuse 145 cm long has one interior angle of $43°36'$ Find the lengths of the other two sides of the triangle

7 A right-angled triangle ABC with the right angle at A has $\angle B = 56°44'$ and $AB = 26$ mm Find the length of the hypotenuse

8 A triangular fence is made for a garden inside a park Three holes A B and C for fence posts are made at the corners so that A and B are 102 m apart AB and CB are perpendicular, and angle CAB is $59°54'$ How far apart are A and C?

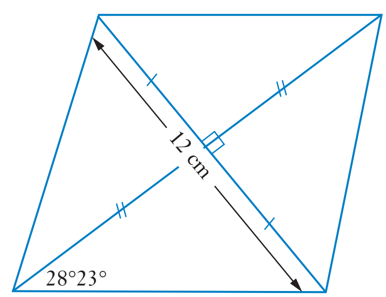
9 Triangle ABC has $\angle BAC = 46°$and $\angle ABC = 54°$ An altitude (perpendicular line) is drawn from C to meet AB at point D If the altitude is .3 cm lon, fid, correct to 1 decimal place the length o:

a AC **b** BC **c** AB

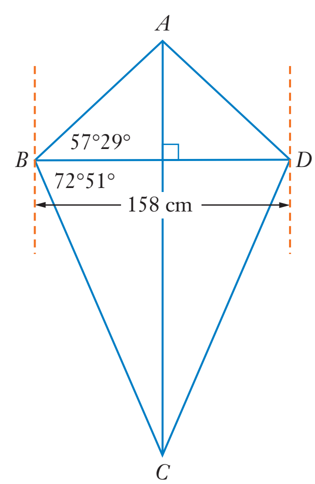
10 A rhombus has one diagonal 12 cm long and the other diagonal makes an angle of $28°23'$ with the side of the rhombus

a Find the length of the side of the rhombus

b Find the length of the other diagonal



11 Kite $ABCD$ has diagonal $BD = 158$ cm as shown If $\angle ABD = 57°29'$ and $\angle DBC = 72°51'$ find the length of the other diagonal AC



4.03 Finding an angle in a right-angled triangle

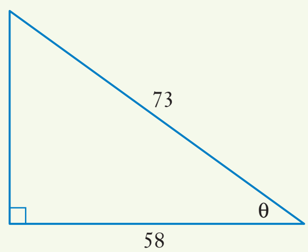
Finding an unknown angle

We can use trigonometry to find an unknown angle in a triangl.

EXAMPLE 6

Find the value of the pronumeral in degrees and minute.

a



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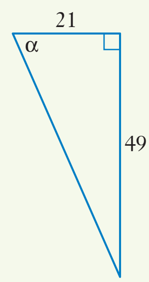


Solution

a $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

$= \frac{58}{73}$

$\therefore\ ° = \cos^{-}\left(\frac{58}{73}\right)$

$\approx 37°23$

b $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

$= \frac{49}{21}$

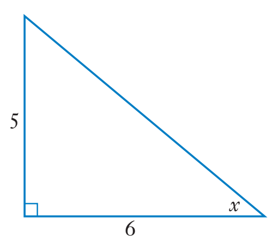
$\therefore\ ° = \tan^{-}\left(\frac{49}{21}\right)$

$= 66°48'$

Exercise 4.03 Finding an angle in a right-angled triangle

1 Find the value of each pronumeral in degrees and minute:

a



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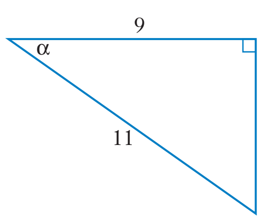

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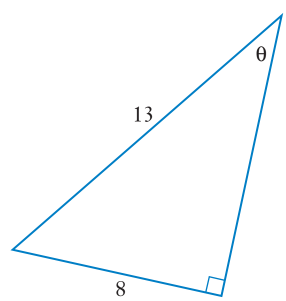

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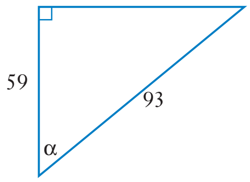

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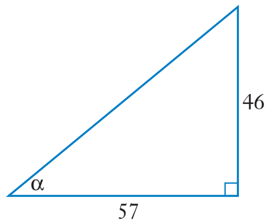

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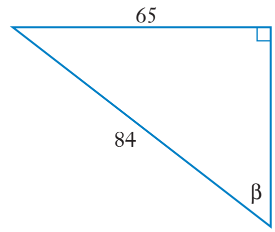

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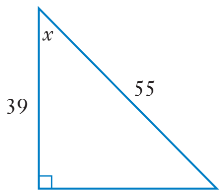

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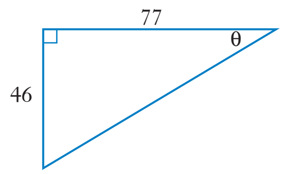

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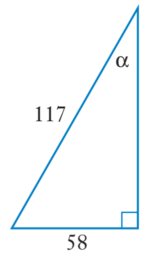

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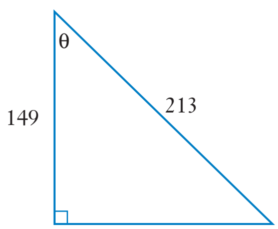

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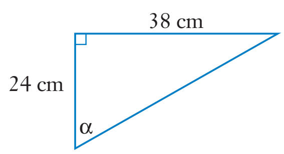

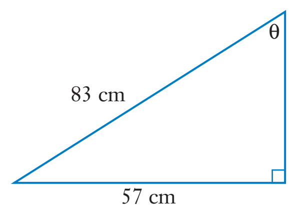

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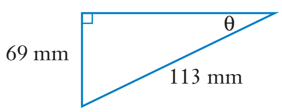

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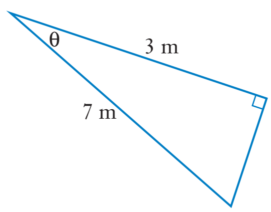

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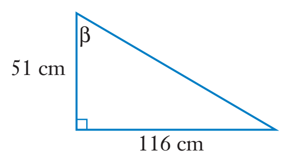

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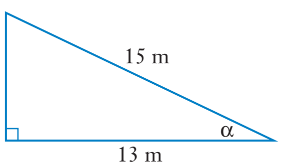

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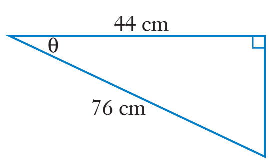

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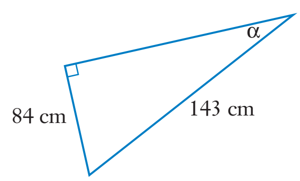

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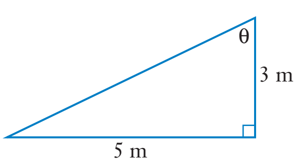

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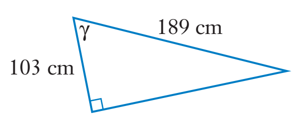
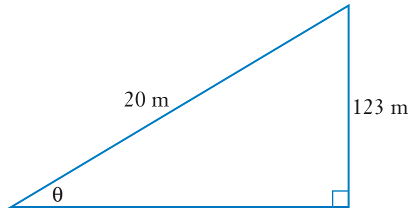
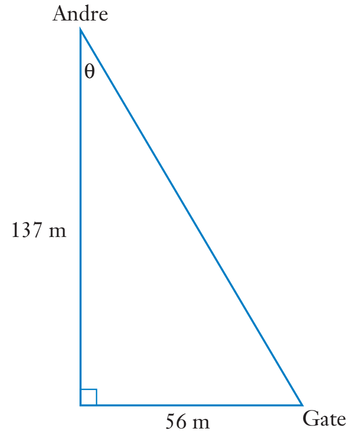


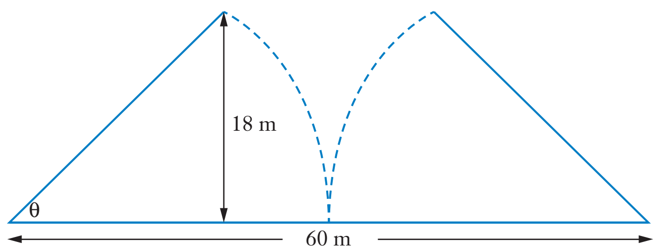
2 A kite is flying at an angle of θ above the ground as shown If the kite is 1.3 m above the ground and has 20 m of string find angle θ



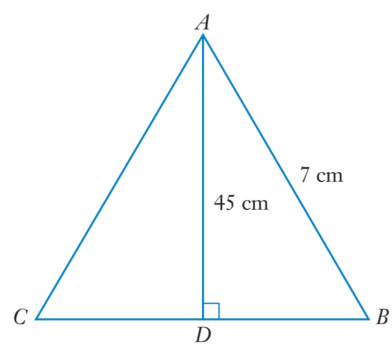
3 A field is 137 m wide and Andre is on one side There is a gate on the opposite side and 56 m along from where Andre is At what angle will he walk to get to the gate ?



4 A 60 m long bridge has an opening in the middle and both sides open up to let boats pass underneath The two parts of the bridge floor rise up to a height of 18 m Through what angle do they move?



5 An equilateral triangle *ABC* with side 7 cm has an altitude *AD* 45 cm long Evaluate the angle the altitude makes with vertex *A* (∠*DAB*)



6 Rectangle *ABCD* has dimensions 18 cm × 7 cm A line *AE* is drawn so that *E* bisects *DC*

a How long is line *AE*? (Answer to 1 decimal place)

b Evaluate ∠*DEA*

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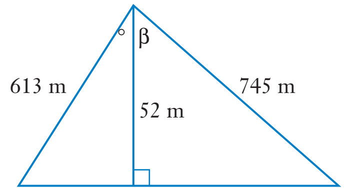
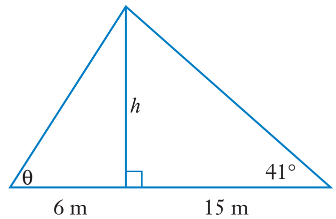
7 A 52 m tall tower has wire stays on either side to minimise wind movement One stay is 6.3 m long and the other is 745 m long as show. Find the angles that the tower makes with each stay.



8 The angle up from the ground to the top of a pole is 41° from a position 15 m to one side

a Find the height h of the pole to the nearest metr.

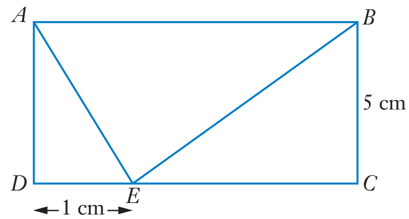
b If Sarah stands 6 m away on the other side find the angle of elevation θ from Sarah to the top of the pole



9 Rectangle $ABCD$ has a line BE drawn so that $\angle AEB = 90°$ and $DE = 1$ cm The width of the rectangle is 5 cm

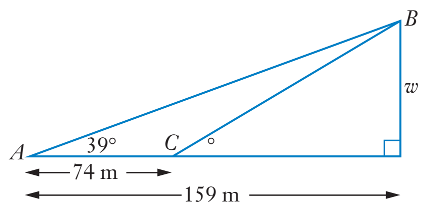
a Find $\angle BEC$

b Find the length of the rectangle



10 a Frankie is standing at the side of a road at point A 1.9 m away from an inter section She is at an angle of 39° from point B on the other side of the road What is the width w of the road?

b Frankie walks 74 m to point C At what angle is she from point B?



INVESTIGATION

LEANING TOWER OF PISA

The Tower of Pisa was built as a belltower for the cathedral neary. Work started in 1174 but when it was half-completed the soil underneath one side of it san. This made the tower lean to one side Work stoped, and it asn't until 100 years later that architects found a way of completing the tower. The third and fifth storeys were built close to the vertical to compensate for the lean Later a vertical top storey was adde. The tower is about 55 m tall and 16 m in diameter. It is tilted about 5 m from the vertical at the top and tilts by an extra 6 mm each yea.

Discuss some of the problems with the Leaning Tower of Pia.

1 Find the angle at which it is tilted from the vertical

2 Work out how far it will be tilted in 10 year.

3 Use research to find out if the tower will fall over, and if o, wen.

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Righangled igonomey

Angles o elevaion and depeion

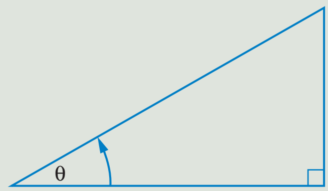
4.04 Applications of trigonometry

Angle of elevation

The **angle of elevation** can be used to measure the height of tall objects that cannot be measured directly for example a tre, clif, tower or buildng. Stand outside a tall building and look up to the top of the building Think about what angle your eyes pass through to look up to the top of the building

Angle of elevation

The angle of elevation θ is the angle measured when looking from the ground up to the top of the object We assume that the ground is horizonta.



EXAMPLE 7

The angle of elevation of a tree from a point 50 m out from its base is 38°14′

Find the height of the tree to the nearest metre

Solution

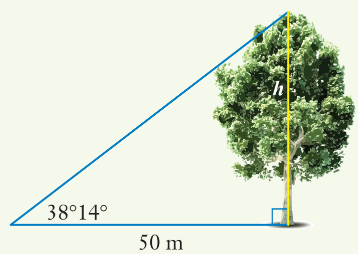
We assume that the tree is vertica.

$$\tan 38°14° = \frac{h}{50}$$

$$50 \tan 38°14° = h$$

$$39 \approx h$$

So the tree is 39 m tall



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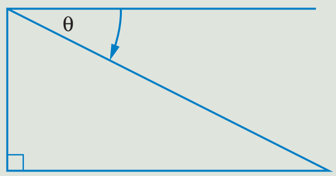
Angle of depression

The **angle of depression** is the angle formed when looking down from a high place to an object below. Find a tall buildig, hill or other high plce, and look down to something below. Through what angle do your eyes pass as you look down?

Angle o depeion

Angle of depression

The angle of depression θ is the angle measured when looking down from the horizontal to an object below.



Angle o depeion

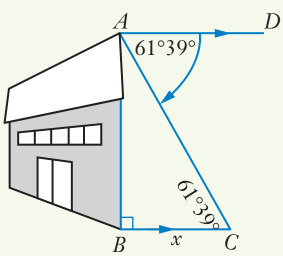
EXAMPLE 8

a The angle of depression from the top of a 20 m building to Gina below is 61°39′ How far is Gina from the building to 1 decimal place ?

b A bird sitting on top of an 8 m tall tree looks down at a possum 35 m out from the base of the tree Find the angle of depression to the nearest minut.

Solution

a



$\angle DAC = \angle ACB = 61°39'$ (alternate angles, $AD \parallel BC$)

$$\tan 61°39' = \frac{20}{x}$$

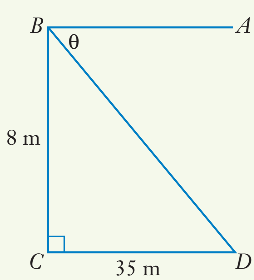
$$x \tan 61°39' = 20$$

$$x = \frac{20}{\tan 61°39'}$$

$$\approx 108$$

So Gina is 108 m from the building

b



The angle of depression is θ

$\angle ABD = \angle BDC = \theta$ (alternate angles AB DC)

$$\tan\theta = \frac{8}{3\ 5}$$

$$\therefore \theta \approx 66°22$$

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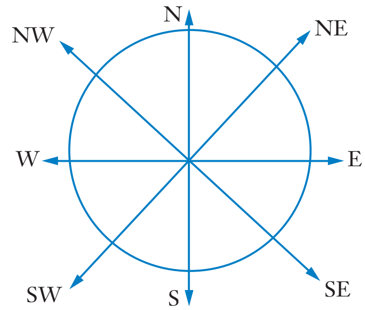
Compa
beaing

Compass bearings

A **bearing** is a direction according to a compass The main points on a compass are north (N) south (S, east (E) and west (W) Halfway between these are N, NW, E,SW. We write **compass bearings** with north or south first followed by an angle and then east or wes.



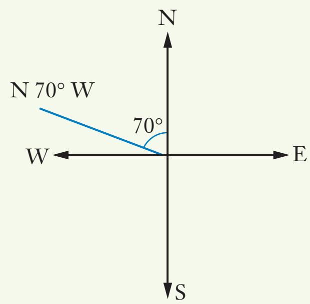
EXAMPLE 9

a Draw a compass bearing of N 70° W

b Eli walks from his house on a bearing of S 25° E If he walks .7 k, how far south is he from his house?

Solution

a Start at north and turn 70° towards west



b Start at south and turn 25° towards east

The hypotenuse is 57 and we want to measure the adjacent side (x)

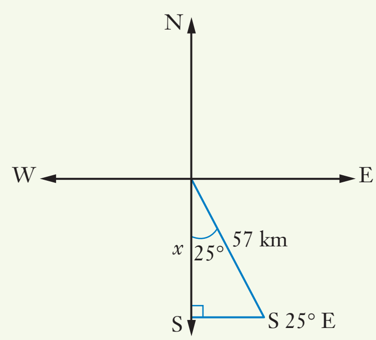
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 25° = \frac{x}{5\ 7}$$

$$5\ 7 \cos 25° = x$$

$$x \approx 5\ 2$$

So Eli is 52 km south of his house



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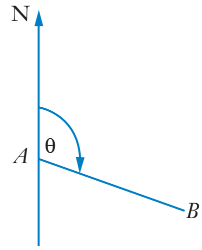
True bearings

True bearings measure angles clockwise from north

We say B is on a bearing of θ from A

A true bearing uses 3 digits from 000° to 360°



Beaing

True berins

A page o beaing

Beaing machup

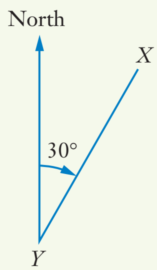
Elevaion and beaing

EXAMPLE 10

- **a** X is on a bearing of 030° from Y Sketch this diagra.
- **b** A house is on a bearing of 305° from a school What is the bearing of the school from the house?
- **c** A plane leaves Sydney and flies 100 km due east then 125 km due nort. Find the bearing of the plane from Sydney, to the nearest degre.
- **d** A ship sails on a bearing of 140° from Sydney for 250 km How far east of Sydney is the ship now, to the nearest km?

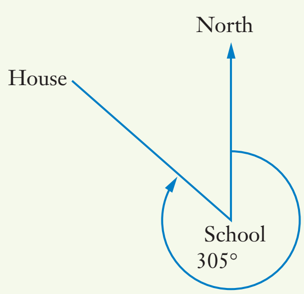
Solution

a



Note Bearings use 3 digits so a bearing of 030 ° is a 30° angle

b



The diagram below shows the bearing of the house from the school

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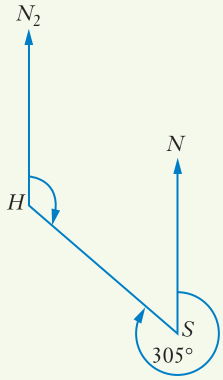
To find the bearing of the school from the hous, draw in north from the house and use geometry to find the bearing as follows



The bearing of the school from the house is $\angle N_2HS$

$\angle N\ SH = 360° - 305°$ (angles in a revolution)

$= 55°$

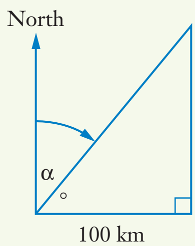
With parallel line, the sum of cointerior angles is 180 °

$\angle N_2HS = 180° - 55°$

$= 125°$

So the bearing of the school from the house is 125°

c



$\tan\theta = \frac{125}{100}$

$\theta \approx 51°$

$\alpha = 90° - 51° = 39°$

So the bearing of the plane from Sydney is 039°

d

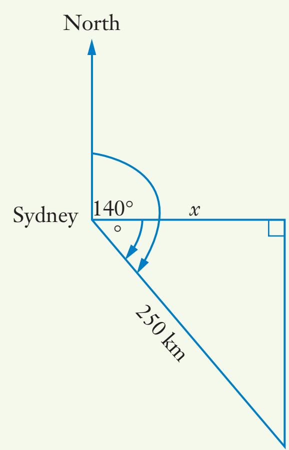
$\theta = 140° - 90° = 50°$

$\cos 50° = \frac{x}{250}$

$250\cos 50° = x$

$x \approx 161$

So the ship is 161 km east of Sydney.



Note A navigator on a ship uses a sextant to measure angle. A clinometer measures angles of elevation and depression

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Exercise 4.04 Applications of trigonometry

1 Draw a diagram to show the bearing in each question

a N 50° E **b** S 60° W **c** S 80° E **d** N 40° W

e A boat is on a bearing of 100° from a beach house

f Jamie is on a bearing of 320° from a campsite

g A seagull is on a bearing of 200° from a jetty.

h Alistair is on a bearing of 050° from the bus stop

i A plane is on a bearing of 285° from Broken Hill

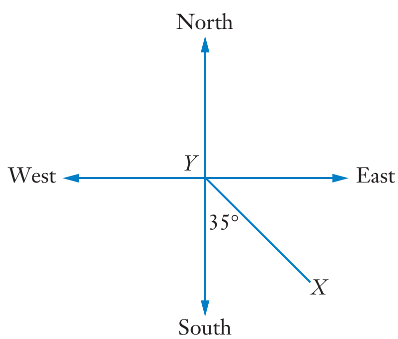
j A farmhouse is on a bearing of 012° from a dam

k Mohammed is on a bearing of 160° from his house

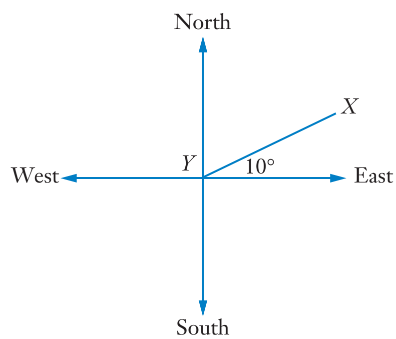
2 Find the bearing of X from Y in each question using

i compass bearings **ii** true bearings

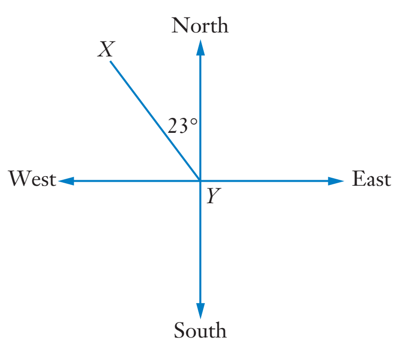
a



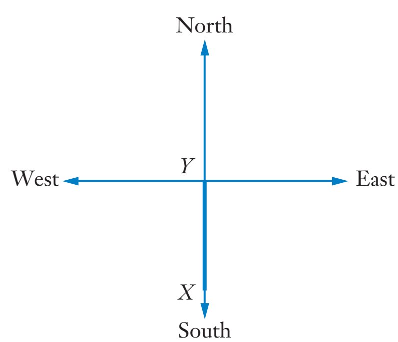
b



c



d



3 Jack is on a bearing of 260° from Jill What is Jil's bearing from Jack ?

4 A tower is on a bearing of 030° from a house What is the bearing of the house from the tower ?

5 Tamworth is on a bearing of 340° from Newcastle What is the bearing of Newcastle from Tamworth?

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6 The angle of elevation from a point 115 m away from the base of a tree up to the top of the tree is 42°12′ Find the height of the tree to one decimal plac.

7 Geoff stands 258 m away from the base of a tower and measures the angle of elevation as 39°20′ Find the height of the tower to the nearest metr.

8 A wire is suspended from the top of a 100 m tall bridge tower down to the bridge at an angle of elevation of 52° How long is the wir, to 1 decimal place ?

9 A cat crouches at the top of a 42 m high cliff and looks down at a mouse 13 m out from the foot (base) of the cliff What is the angle of depressin, to the nearest minute ?

10 A plane leaves Melbourne and flies on a bearing of 065° for 2500 km

a How far north of Melbourne is the plane?

b How far east of Melbourne is it?

c What is the bearing of Melbourne from the plane?

11 The angle of elevation of a tower is 39°44′ when measured at a point 100 m from its base Find the height of the towe, to 1 decimal plce.

12 Kim leaves her house and walks for 2 km on a bearing of 155° How far south is Kim from her house now, to 1 decimal place?

13 The angle of depression from the top of an 8 m tree down to a rabbit is 43°52′ If an eagle is perched in the top of the tree how far does it need to fly to reach the rabbi, to the nearest metre?

14 Sanjay rides a motorbike through his property, starting at his houe. If he rides south for 13 km then rides west for .4 k, what is his bearing from the houe, to the nearest degree?

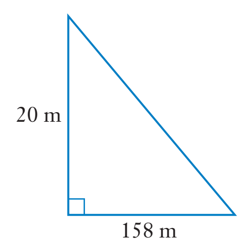
15 A plane flies north from Sydney for 560 km then turns and flies east for 390 k. What is its bearing from Sydney, to the nearest degree?

16 Find the height of a pole correct to 1 decimal plac, if a 10 m rope tied to it at the top and stretched out straight to reach the ground makes an angle of elevation of 67°13′

17 The angle of depression from the top of a cliff down to a boat 100 m out from the foot of the cliff is 59°42′ How high is the clif, to the nearest metre ?

18 A group of students are bushwalking They walk north from their camp for7.5 m, then walk west until their bearing from camp is 320° How far are they from cam, to 1 decimal place?

19 A 20 m tall tower casts a shadow 158 m long at a certain time of day. What is the angle of elevation from the edge of the shadow up to the top of the tower at this time?



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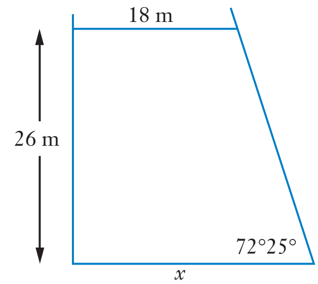
20 A flat verandah roof 18 m deep is 26 m up from the ground At a certain time of dy, the sun makes an angle of elevation of 72°25′ How much shade is provided on the ground by the verandah roof at that time to 1 decimal place?



21 Find the angle of elevation of a 159 m cliff from a point 100 m out from its base

22 A plane leaves Sydney and flies for 2000 km on a bearing of 195° How far due south of Sydney is it?

23 The angle of depression from the top of a 15 m tree down to a pond is 25°41′ If a bird is perched in the top of the tree how far does it need to fly to reach the pon, to the nearest metre?

24 Robin starts at her house walks south for .7 km then walks east for .6 k. What is her bearing from the house to the nearest degree ?

25 The angle of depression from the top of a tower down to a car 250 m out from the foot of the tower is 38°19′ How high is the towe, to the nearest metre ?

26 A blimp flies south for 36 km then turns and flies east until it is on a bearing of 127 ° from where it started How far east does it fly ?

27 A 24 m wire is attached to the top of a pole and runs down to the ground where the angle of elevation is 22°32′ Find the height of the pol.

28 A train depot has train tracks running north for 78 km where they meet another set of tracks going east for 58 km into a station What is the bearing of the depot from the station to the nearest degree ?

29 Jessica leaves home and walks for 47 km on a bearing of 075 ° She then turns and walks for 29 km on a bearing of 115 ° and she is then due east of her home

a What is the furthest north that Jessica walks?

b How far is she from home?

30 Builder Jo stands 45 m out from the foot of a building and looks up to the top of the building where the angle of elevation is 71° Builder Ben stands at the top of the building looking down at his wheelbarrow that is 108 m out from the foot of the building on the opposite side from where Jo is standing

a Find the height of the building

b Find the angle of depression from Ben down to his wheelbarrow.

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WS

Sine ule poblem

4.05 The sine rule

The sin cos and tan of angles greater than 90 ° give some interesting results You will explore these in Chapter 9 *Trigonometric functions* For no, we just need to know about **obtuse angles** (between 90° and 180°) so we can solve problems involving obtuse-angled triangles

INVESTIGATION

LARGER ANGLES

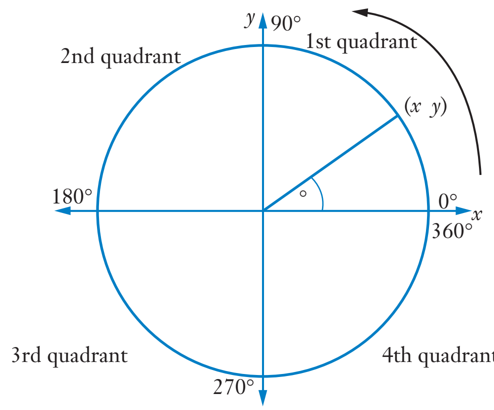
1 Use your calculator to find the sin cos and tan of some angles greater than 90 ° What do you notice?

2 Can you see a pattern for angles between 90° and 180° for

i sin? **ii** cos? **iii** tan?

We can use a circle to show angle, starting with 0 ° at the x-axis and turning anticlockwise to show other angles We divide the number plane into 4 **quadrants** as shown

1st quadrant	0° to 90°
2nd quadrant	90° to 180°
3rd quadrant	180° to 270°
4th quadrant	270° to 360°



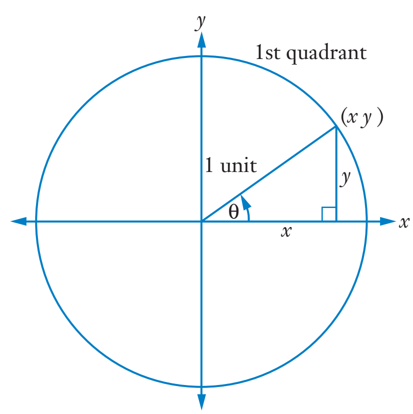
To make it easier to explore these result, we use a **unit circle** with radius 1

We can find the trigonometric ratios for angle θ

$$\sin\theta = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{1} = x$$

$$\tan\theta = \frac{y}{x}$$



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In the 2nd quadrant notice that x values are negative and y values are positive

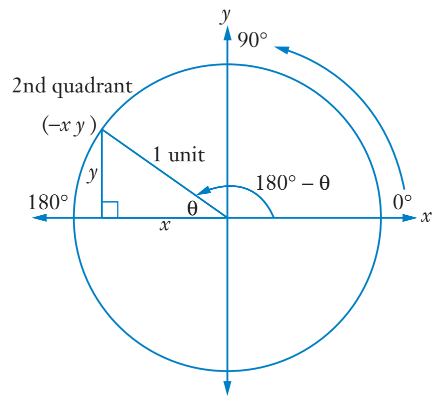
So the point in the 2nd quadrant will be $(-x\ y)$

Since $\sin\theta = y$ sin will be **positive** in the 2nd quadrant

Since $\cos\theta = -x$ cos will **negative** in the 2nd quadrant

Since $\tan\theta = \dfrac{y}{-x}$ tan will be negative in the 2nd quadrant (positive divided by negative)

To have an angle of θ in the triangle the obtuse angle in the 2nd quadrant is $180° - \theta$



Trigonometric ratios of obtuse angles

$$\sin(180° - \theta) = \sin\theta$$

$$\cos(180° - \theta) = -\cos\theta$$

$$\tan(180° - \theta) = -\tan\theta$$

EXAMPLE 11

a If cos 80° = 0174 evaluate cos 100 °

b If sin 55° = 0819 find the value of sin 125 °

Solution

a

$$\cos(180° - \theta) = -\cos\theta$$

$$\text{So } \cos(180° - 80°) = -\cos 80°$$

$$\cos 100° = -\cos 80°$$

$$= -0174$$

b

$$\sin(180° - \theta) = \sin\theta$$

$$\text{So } \sin(180° - 55°) = \sin 55°$$

$$\sin 125° = \sin 55°$$

$$= 0819$$

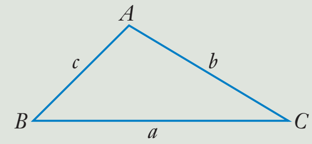
You can check that tisis true by fndng both ratos on the calculator.

Naming the sides and angles of a triangle

Side a is opposite angle A side b is opposite angle B and side c is opposite angle C

The shortest side is opposite the smallest angle

The longest side is opposite the largest angle



The sine rule

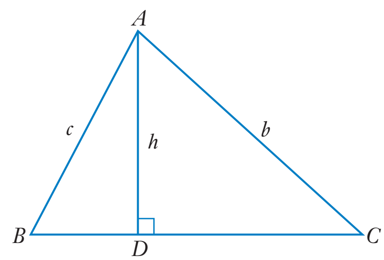
The **sine rule** is used to find unknown sides and angles in non-right-angled triangles

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof

In $\triangle ABC$ draw perpendicular AD and call it h



From $\triangle ABD$

$\sin B = \frac{h}{c}$

$\therefore h = c \sin B$ [1]

From $\triangle ACD$

$\sin C = \frac{h}{b}$

$\therefore h = b \sin C$ [2]

From [1] and [2]

$c \sin B = b \sin C$

$\frac{\sin B}{b} = \frac{\sin C}{c}$

Similarly, by drawing a perpendicular from C it can be proved that

$\frac{\sin A}{a} = \frac{\sin B}{b}$

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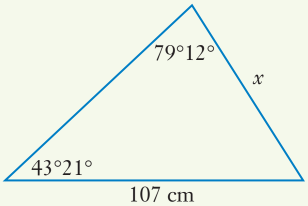
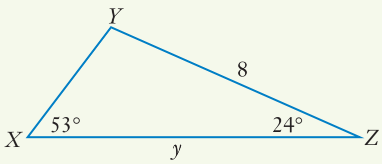
EXAMPLE 12

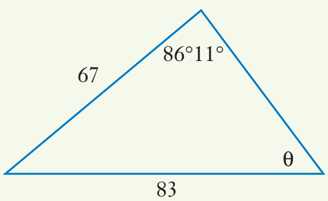
a Find the value of x correct to 1 decimal plac.



b Find the value of y to the nearest whole numbe.



c Find the value of θ in degrees and minute, given θ is acute



Solution

a Name the sides a and b and opposite angles A and B

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{x}{\sin 43°21'} = \frac{10\ 7}{\sin 79°12'}$$

$$x = \frac{10\ 7 \sin 43°21'}{\sin 79°12'}$$

$$\approx .5 \text{ cm}$$

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b First we need to find angle Y since it is opposite side y

$$\angle Y = 180° - (53° + 24°) = 103°$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{y}{\sin 103°} = \frac{8}{\sin 53°}$$

$$y = \frac{8 \sin 103°}{\sin 53°}$$

$$\approx 10$$

c

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{6\ 7} = \frac{\sin 86°11'}{8\ 3}$$

$$\sin \theta = \frac{6\ 7 \sin 86°11'}{8\ 3}$$

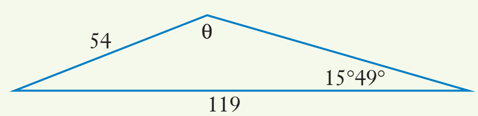
$$= 08054$$

$$\theta = \sin^{-}(08054\quad)$$ SHIFT sin ANS

$$\approx 53°39'$$

EXAMPLE 13

Find the value of θ in degrees and minutes given θ is obtuse



Solution

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{11\ 9} = \frac{\sin 15°49'}{5\ 4}$$

$$\sin \theta = \frac{11\ 9 \sin 15°49'}{5\ 4}$$

$$= 06006$$

$$\theta = \sin^{-}(06006\quad)$$

$$\approx 36°55'$$

But θ is obtuse

$$\therefore \theta = 180° - 36°55'$$

$$= 143°05'$$

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Ambiguous case

When using the sine rule to find an unknown angle there are 2 possible solution: one acute and one obtuse This is called the **ambiguous case** of the sine rule

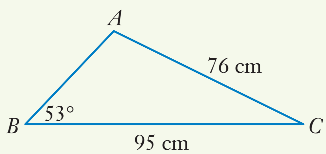
EXAMPLE 14

a Triangle ABC has $\angle B = 53°$ $AC = 76$ cm and $BC = 95$ cm Find $\angle A$ to the nearest degree

b In triangle XYZ $\angle Y = 118°35'$ $YZ = 125$ mm and $XZ = 143$ mm Find $\angle X$ in degrees and minutes

Solution

a Draw a diagram



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{9\ 5} = \frac{\sin 53°}{7\ 6}$$

$$\sin A = \frac{9\ 5 \sin 53°}{7\ 6}$$

$$= 0998$$

$$A = \sin^{-}\ (0998)$$

$$\approx 87°$$

But $\angle A$ could be obtuse

So $\angle A = 180° - 87°$

$= 93°$

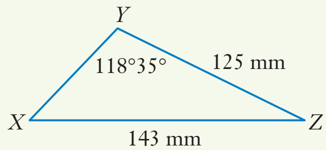
Checking angle sum of a triangle

$53° + 87° = 140° < 180°$ so $87°$ is a possible answe.

$53° + 93° = 146° < 180°$ so $93°$ is a possible answe.

So $\angle A = 87°$ or $93°$

b



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin X}{12\ 5} = \frac{\sin 118°35'}{14\ 3}$$

$$\sin X = \frac{12\ 5 \sin 118°35°}{14\ 3}$$

$$= 0768$$

$$X = \sin^{-}\ (0768)$$

$$\approx 50°8'$$

But $\angle X$ could be obtuse

So $\angle X = 180° - 50°8'$

$= 129°52'$

Checking angle sum of a triangle

$118°35' + 50°8' = 168°43' < 180°$, so a possible answer.

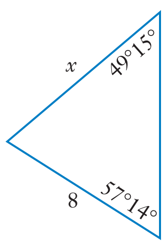
$118°35' + 129°52' = 248°27' > 180°$, so an impossible answer.

So $\angle X = 50°8'$

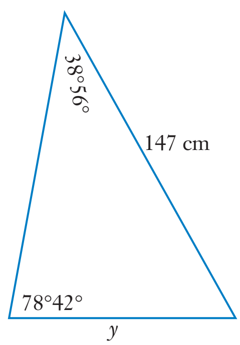
Exercise 4.05 The sine rule

1 Evaluate each pronumeral correct to 1 decimal plac:

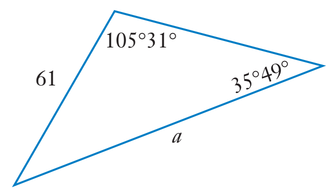
a



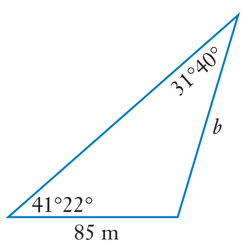
b



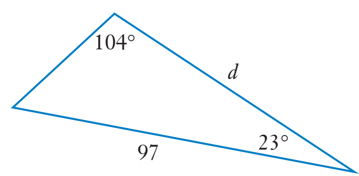
c



d



e



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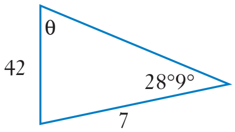
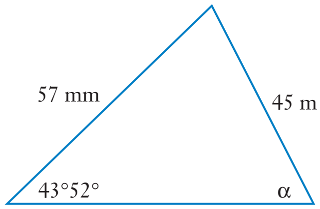
2 Find the value of all pronumerals in degrees and minutes (triangles not to scale:

a



b



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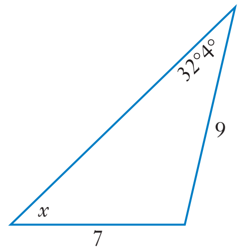

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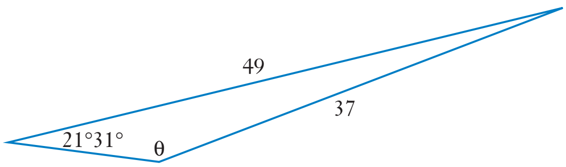
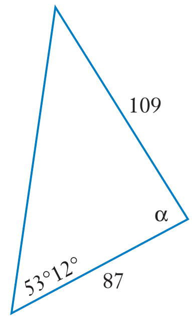


e



3 Triangle ABC has an obtuse angle at A Evaluate this angle to the nearest minute if $AB = 32$ cm, $BC = 46$ cm and $\angle ACB = 33°47'$

4 Triangle EFG has $\angle FEG = 48°$ $\angle EGF = 32°$ and $FG = 189$ mm Find the length o:

a the shortest side **b** the longest side

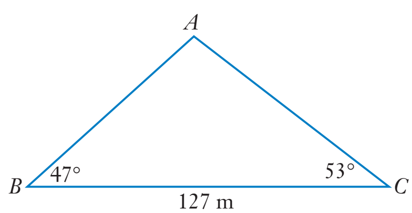
5 Triangle XYZ has $\angle XYZ = 51°$ $\angle YXZ = 86°$ and $XZ = 21$ m Find the length o:

a the shortest side **b** the longest side

6 Triangle XYZ has $XY = 54$ cm, $\angle ZXY = 48°$ and $\angle XZY = 63°$ Find the length of XZ

7 Triangle ABC has $BC = 127$ m, $\angle ABC = 47°$and $\angle ACB = 53°$ as shown Find the length o:

a AB **b** AC



8 Triangle PQR has sides $PQ = 15$ mm, $QR = 147$ mm and $\angle PRQ = 62°29'$ Find to the nearest minute

a $\angle QPR$ **b** $\angle PQR$

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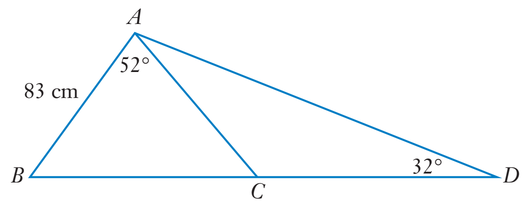
9 Triangle ABC is isosceles with $AB = AC$ BC is produced to D as shown

If $AB = 83$ cm, $\angle BAC = 52°$ and $\angle ADC = 32°$ find the length o:

a AD **b** BD



10 Triangle ABC is equilateral with side 63 mm A line is drawn from A to BC where it meets BC at D and $\angle DAB = 26°15'$ Find the length o:

a AD **b** DC

11 In triangle ABC find $\angle B$ to the nearest degree given

a $\angle C = 67°$ $AB = 72$, $AC = 75$

b $\angle A = 92°$ $BC = 107$, $AC = 84$

c $\angle A = 29°$ $BC = 49$, $AC = 83$

4.06 The cosine rule

WS Cosine ule poblem

WS The sine and cosine ules

WS Finding an unknown ide

WS Finding an unknown angle

The cosine rule

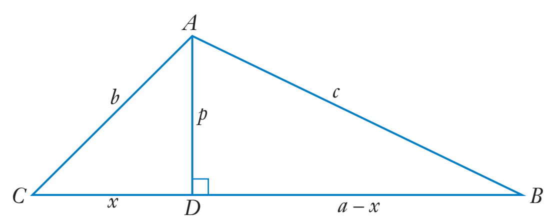
The **cosine rule** is also used to find unknown sides and angles in non-right-angled triangles

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Proof

In triangle ABC draw perpendicular AD with length p and let $CD = x$

Since $BC = a$ $BD = a - x$



From triangle ACD

$$b^2 = x^2 + p^2 \qquad [1]$$

$$\cos C = \frac{x}{b}$$

$$\therefore b \cos C = x \qquad [2]$$

From triangle DAB

$$c^2 = p^2 + (a - x)^2$$

$$= p^2 + a^2 - 2ax + x^2$$

$$= p^2 + x^2 + a^2 - 2ax \qquad [3]$$

Substitute [1] into [3]

$$c^2 = b^2 + a^2 - 2ax \qquad [4]$$

Substituting [2] into [4]

$$c^2 = b^2 + a^2 - 2ab \cos C$$

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DID YOU KNOW?

The cosine rule for right-angled triangles

Pythagoras theorem is a special case of the cosine rule when the triangle is right-angled

$c^2 = a^2 + b^2 - 2ab \cos C$

When $C = 90°$

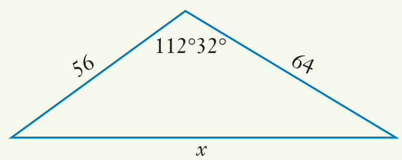
$c^2 = a^2 + b^2 - 2ab \cos 90°$

$\quad = a^2 + b^2 - 2ab \times 0$

$\quad = a^2 + b^2$

EXAMPLE 15

Find the value of x correct to the nearest whole number.



Solution

$c^2 = a^2 + b^2 - 2ab \cos C$

$x^2 = 56^2 + 64^2 - 2(56)(64) \cos 112°32'$

$\quad = 997892$

$x = \sqrt{997892}$

$\quad = 99894$

$\quad \approx 10$

When using the cosine rule to find an unknown angle it may be more convenient to change the subject of this formula to cos C

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$2ab \cos C = a^2 + b^2 - c^2$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

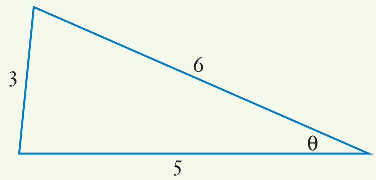
The cosine rule for angles

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

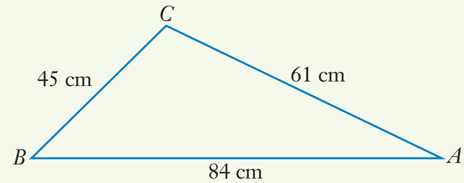
The coine ule o angle

EXAMPLE 16

a Find θ in degrees and minute.



b Evaluate $\angle BCA$ in degrees and minutes



Solution

Naming sides and opposite angles side c is opposite the unknown angle C

a

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\theta = \frac{5^2 + 6^2 - 3^2}{2(5)(6)}$$

$$= \frac{52}{60}$$

$$\theta = \cos^{-}\left(\frac{52}{60}\right)$$

$$\approx 29°56'$$

b

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\angle BCA = \frac{4\ 5^2 + 6\ 1^2 - 8\ 4^2}{2(4\ 5)(6\ 1)}$$

$$= -02386\ \ldots$$

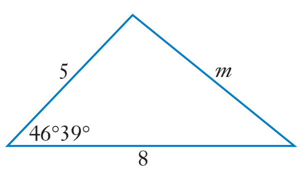
$$\angle BCA = \cos^{-}\ (-02386\ldots)$$

$$\approx 103°48'$$

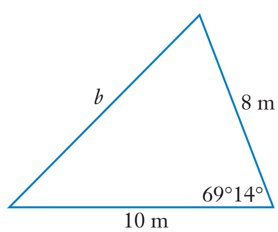
Exercise 4.06 The cosine rule

1 Find the value of each pronumeral correct to 1 decimal plac:

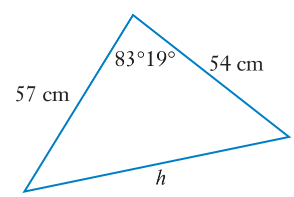
a



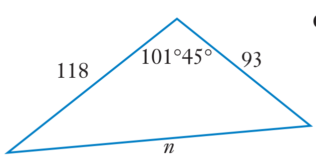
b



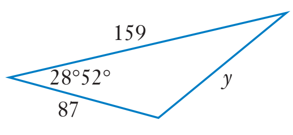
c



d



e



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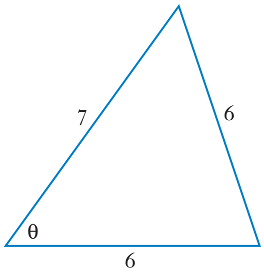
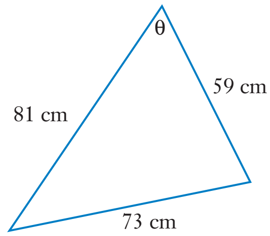
2 Evaluate each pronumeral correct to the nearest minut:

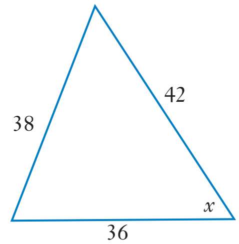
a



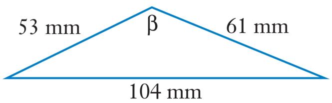
b



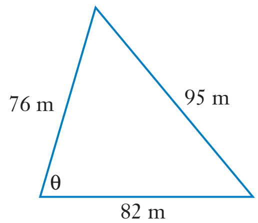
c



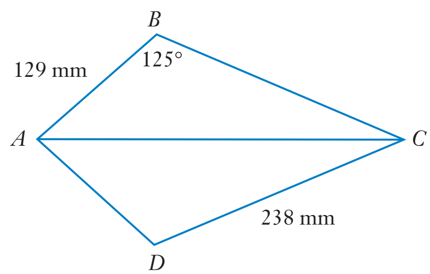
d



e



3 Kite $ABCD$ has $AB = 129$ mm, $CD = 238$ mm and $\angle ABC = 125°$ as shown Find the length of diagonal AC



4 Parallelogram $ABCD$ has sides 11 cm and 5 cm and one interior angle $79°25'$ Find the length of the diagonals

5 Quadrilateral $ABCD$ has sides $AB = 12$ cm, $BC = 104$ cm, $CD = 84$ cm and $AD = 97$ cm with $\angle ABC = 63°57'$ Fin:

a the length of diagonal AC **b** $\angle DAC$ **c** $\angle ADC$

6 Triangle XYZ is isosceles with $XY = XZ = 73$ cm and $YZ = 59$ cm Find the value of all angles to the nearest minut.

7 Quadrilateral $MNOP$ has $MP = 12$ mm, $NO = 127$ mm, $MN = 89$ mm $OP = 156$ mm and $\angle NMP = 119°15'$ Fin:

a the length of diagonal NP **b** $\angle NOP$

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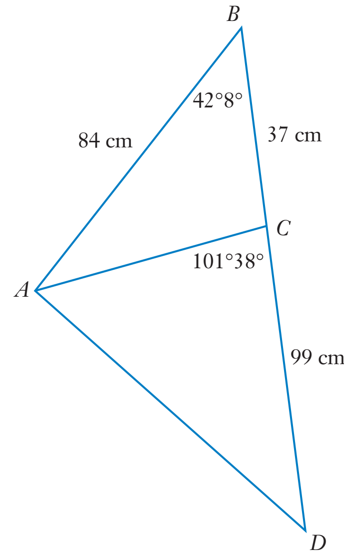
8 Given the figure find the length o:

a AC

b AD



9 In a regular pentagon $ABCDE$ with sides 8 cm find the length of diagonal AD

10 A regular hexagon $ABCDEF$ has sides 55 cm Fin:

a the length of AD **b** $\angle ADF$

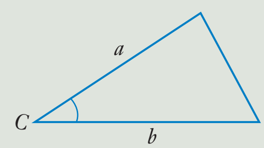
WS
Aea o iangle

4.07 Area of a triangle

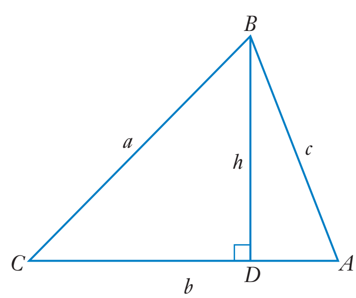
Trigonometry allows us to find the area of a triangle if we know 2 sides and their included angle

Sine formula for the area of a triangle

$$A = \frac{1}{2}ab\sin C$$



Proof



From $\triangle BCD$

$$\sin C = \frac{h}{a}$$

$$\therefore h = a\sin C$$

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}ba\sin C$$

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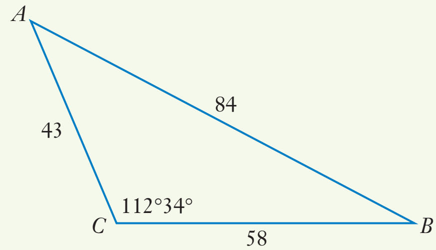
EXAMPLE 17

Find the area of $\triangle ABC$ correct to 2 decimal places



Solution

$$A = \frac{1}{2}ab \sin C$$

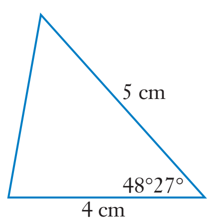
$$= \frac{1}{2}(43)(58) \sin 112°34'$$

$$\approx 1152 \text{ units}^2$$

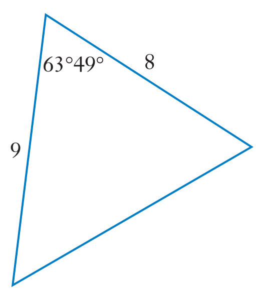
Exercise 4.07 Area of a triangle

1 Find the area of each triangle correct to 1 decimal place

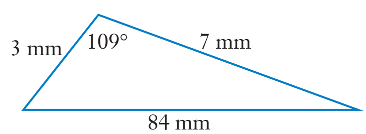
a



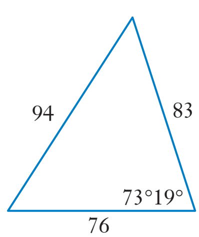
b



c



d



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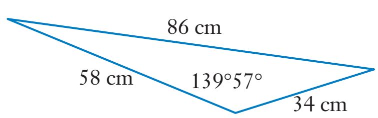

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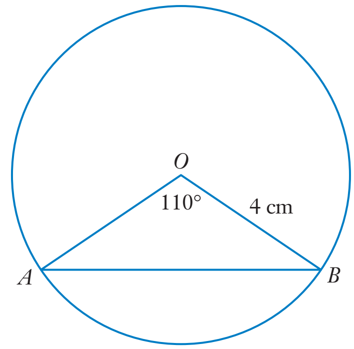
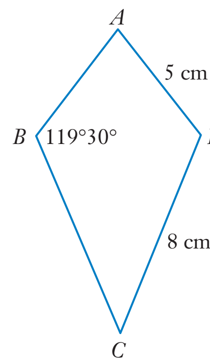
2 Find the area of $\triangle OAB$ correct to 1 decimal place (O is the centre of the circle)

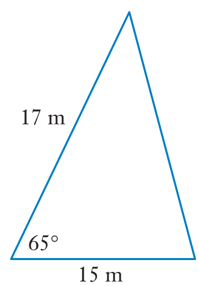


3 Find the area of a parallelogram with sides 35 cm and 48 cm and with one of its internal angles $67°13'$ correct to 1 decimal plac.

4 Find the area of kite $ABCD$ correct to 3 significant figures



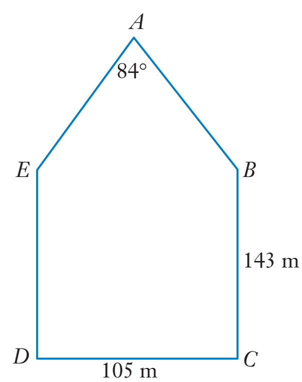
5 Find the area of this sail correct to 1 decimal plac:



6 This pentagon is made from a rectangle and isosceles triangle with $AE = AB$ as show. Fid:

a the length of AE

b the area of the figure



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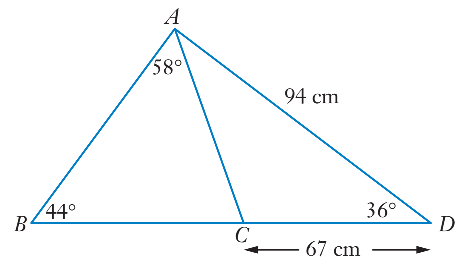
7 For this figure fin:

a the length of AC

b the area of triangle ACD

c the area of triangle ABC



8 Find the exact area of an equilateral triangle with sides 5 cm

4.08 Mixed problems

The sine and cosine rules

Use the **sine rule** to find

- a side given one side and 2 angles
- an angle given 2 sides and one angle

Use the **cosine rule** to find

- a side given 2 sides and one angle
- an angle given 3 sides

EXAMPLE 18

a The angle of elevation of a tower from point A is 72° From point B 50 m further away from the tower than A the angle of elevation is 47°

  i Find the exact length of AT the distance from A to the top of the tower.

  ii Hence or otherwis, find the height h of the tower to 1 decimal place

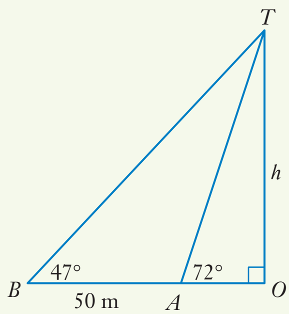
b A ship sails from Sydney for 200 km on a bearing of 040° then sails on a bearing of 157° for 345 km

  i How far from Sydney is the ship to the nearest km ?

  ii What is the bearing of the ship from Sydney, to the nearest degree?

Solution

a



i $\angle BAT = 180° - 72° = 108°$ (straight angle)

$\angle BTA = 180° - (47° + 108°)$ (angle sum of $\triangle BTA$)

$= 25°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{AT}{\sin 47°} = \frac{50}{\sin 25°}$$

$$\therefore AT = \frac{50 \sin 47°}{\sin 25°}$$

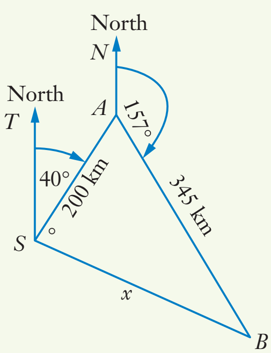
ii $\sin 72° = \dfrac{h}{AT}$

$\therefore h = AT \sin 72°$

$$= \frac{50 \sin 47°}{\sin 25°} \times \sin 72°$$

≈ 823 m

b



i $\angle SAN = 180° - 40° = 140°$ (cointerior angles)

$\therefore \angle SAB = 360° - (140° + 157°)$ (angle of revolution)

$= 63°$

$c^2 = a^2 + b^2 - 2ab \cos C$

$x^2 = 200^2 + 345^2 - 2(200)(345) \cos 63°$

$= 96\ 3743110\ldots$

$x = \sqrt{9637.3110}$

$= 3104421$

≈ 310

So the ship is 310 km from Sydney.

ii $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

$\dfrac{\sin °}{345} = \dfrac{\sin 63°}{3104421}$

$\therefore \sin \theta = \dfrac{345 \sin 63°}{3104421}$

$= 09901\ldots$

$\theta \approx 82°$

The bearing from Sydney $= 40° + 82° = 122°$

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We can also use trigonometry to solve 3-dimensional problem.

WS

3D
igonomey

EXAMPLE 19

a From point X 25 m due south of the base of a towe, the angle of elevation is 47 °
Point Y is 15 m due east of the tower. Fid:

i the height h of the towe, correct to 1 decimal place

ii the angle of elevation θ of the tower from point Y

b A cone has a base diameter of 18 cm and a slant height of 15 cm Find the vertical angle at the top of the cone

Solution

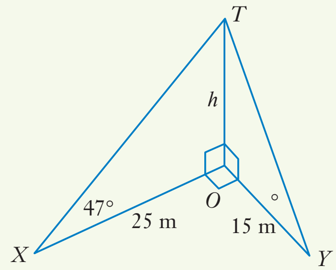
a **i** From $\triangle XTO$

$$\tan 47° = \frac{h}{25}$$

$$25 \tan 47° = h$$

$$268 = h$$

So the tower is 268 m high



ii From $\triangle YTO$

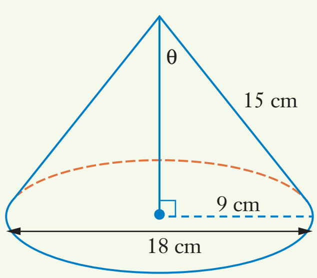
$$\tan \theta = \frac{26\ 8}{15}$$

$$\therefore \theta = \tan^{-} \frac{26\ 8}{15}$$

$$= 60°46'$$

So the angle of elevation from Y is $60°46'$

b The radius of the base is 9 cm



$$\sin \theta = \frac{9}{15}$$

$$\therefore \theta = \sin^{-} \frac{9}{15}$$

$$= 36°52'$$

$$\text{Vertical angle} = 2\theta$$

$$= 73°44'$$

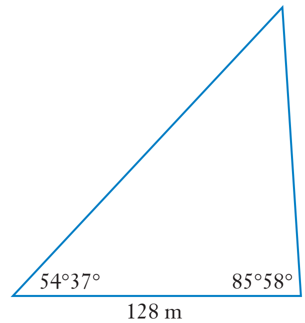
The verical angleis the angle at the vertex of the con.

Exercise 4.08 Mixed problems

1 A car is broken down to the north of 2 towns The car is 39 km from town A and 52 km from town B If A is due west of B and the 2 towns are 68 km apat, what is the bearng, to the nearest degree of the car fro:

a town A **b** town B?

2 The angle of elevation to the top of a tower is 54°37′ from a point 128 m out from its base The tower is leaning at an angle of 85°58′ as shown Find the height of the towe.



3 Rugby league goal posts are 55 m apart If a footballer is standing 8 m from one post and 11 m from the other, find the angle within which the ball must be kicked to score a goal to the nearest degre.

4 A boat is sinking 13 km out to sea from a marina Its bearing is 041 ° from the marina and 324° from a rescue boat The rescue boat is due east of the maria.

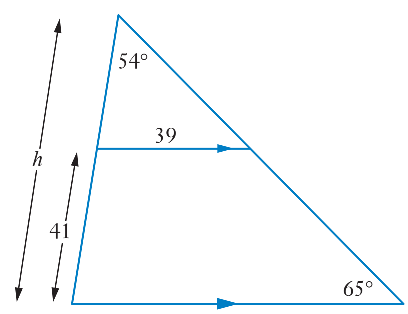
a How far, correct to 2 decimal placs, is the rescue boat from the sinking boat ?

b How long will it take the rescue boat to the nearest minut, to reach the other boat if it travels at 80 km/h?

5 The angle of elevation of the top of a flagpole is 20° from where Thuy stands a certain distance away from its base After walking 80 m towards the flagpoe, Thuy finds the angle of elevation is 75° Find the height of the flagpol, to the nearest mete.

6 A triangular field ABC has sides $AB = 85$ m and $AC = 50$ m If B is on a bearing of 065° from A and C is on a bearing of 166° from A find the length of BC correct to the nearest metre

7 Find the value of h correct to 1 decimal plac.



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8 A motorbike and a car leave a service station at the same time The motorbike travels on a bearing of 080° and the car travels for 157 km on a bearing of 108 ° until the bearing of the motorbike from the car is 310° How fa, correct to 1 decimal plce, has the motorbike travelled?

9 A submarine is being followed by two ships A andB, 3.8 km aprt, with A due eastof B. If A is on a bearing of 165° from the submarine and B is on a bearing of 205° from the submarine find the distance from the submarine to both ship.

10 A plane flies from Dubbo on a bearing of 139° for 852 km then turns and flies on a bearing of 285° until it is due west of Dubbo How far from Dubbo is the plan, to the nearest km?

11 Rhombus *ABCD* with side 8 cm has diagonal *BD* 113 cm long Find $\angle DAB$

12 Zeke leaves school and runs for 87 km on a bearing of 338 ° then turns and runs on a bearing of 061° until he is due north of school How far north of school is he ?

13 A car drives due east for 837 km then turns and travels for 1056 km on a bearing of 029° How far is the car from its starting point?

14 A plane leaves Sydney and flies for 1280 km on a bearing of 050° It then turns and flies for 3215 km on a bearing of 149° How far is the plane from Sydne, to the nearest km ?

15 Trapezium *ABCD* has *AD* *BC* with $AB = 46$ cm $BC = 113$ cm, $CD = 64$ cm, $\angle DAC = 23°30'$ and $\angle ABC = 78°$ Fin:

a the length of *AC*

b $\angle ADC$ to the nearest minute

16 A plane leaves Adelaide and flies for 875 km on a bearing of 056° It then turns and flies on a bearing of θ for 630 km until it is due east of Adelaid. Evaluate θ to the nearest degree

17 Quadrilateral *ABCD* has $AB = AD = 72$ cm, $BC = 89$ cm and $CD = 104$ cm with $\angle DAB = 107°$ Fin:

a the length of diagonal *BD*

b $\angle BCD$

18 A wall leans inwards and makes an angle of 88° with the floor.

a A 4 m long ladder leans against the wall with its base 23 m out from the wall Find the angle that the top of the ladder makes with the wall

b A longer ladder is placed the same distance out from the wall and its top makes an angle of 31° with the wall

i How long is this ladder?

ii How much further does it reach up the wall than the first ladder?

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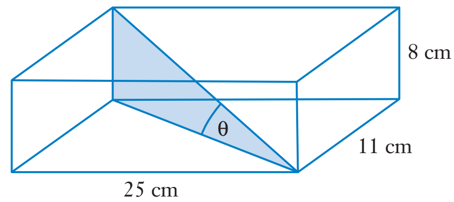
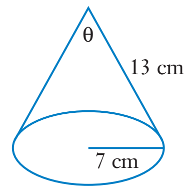
19 A 25 cm × 11 cm × 8 cm cardboard box contains an insert (the shaded area) made of foam

a Find the area of foam in the insert to the nearest cm^2

b Find θ the angle that the insert makes at the corner of the box



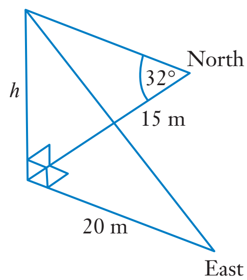
20 A cone has radius 7 cm and a slant height of 13 cm Find the vertical angle θ at the top of the con, in degrees and minuts.



21 From a point 15 m due north of a tower, the angle of elevation of the tower is 32°

a Find the height of the tower, correct to 2 decimal placs.

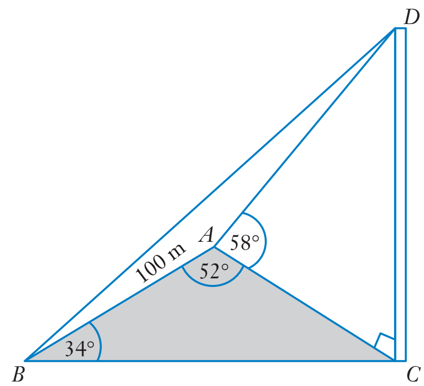
b Find correct to the nearest degree the angle of elevation of the tower at a point 20 m due east of the tower.



22 A pole DC is seen from two points A and B The angle of elevation from A is 58° If $\angle CAB = 52°$ $\angle ABC = 34°$ and A and B are 100 m apart fin:

a how far A is from the foot of the pole to the nearest metre

b the height of the pole to 1 decimal plac.



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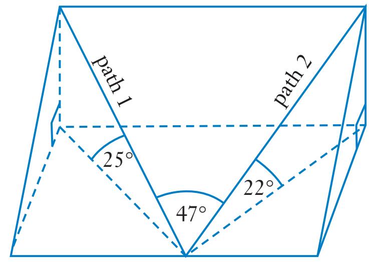
23 Two straight paths to the top of a cliff are inclined at angles of 25° and 22° to the horizontal

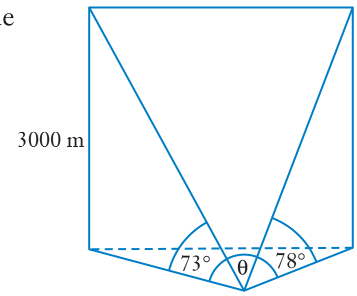
a If path 1 is 114 m long find the height of the cliff to the nearest metr.

b Find the length of path 2 to 1 decimal plac.

c If the paths meet at 47° at the base of the cliff find their distance apart at the top of the cliff correct to 1 decimal place



24 A hot-air balloon floating at 950 m/h at a constant altitude of 3000 m is observed to have an angle of elevation of 78° After 20 minuts, the angle of elevation is 73 ° Calculate the angle through which the observer has turned during those 20 minutes



4.09 Radians

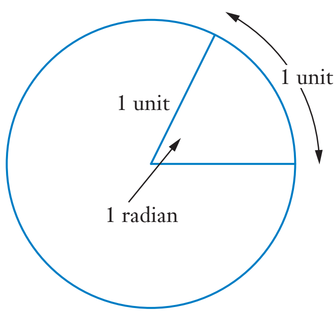
WS Radian

WS Conveting degee and adian

We use degrees to measure angles in geometry and trigonometr, but there are other units for measuring angles

A **radian** is a unit for measuring angles based on the length of an arc in a circle

One radian is the angle subtended by an arc with length 1 unit in a unit circle (of radius 1)



Conversions

We can change between radians and degrees using this equatio:

Radians and degrees

$$\pi \text{ radians} = 180°$$

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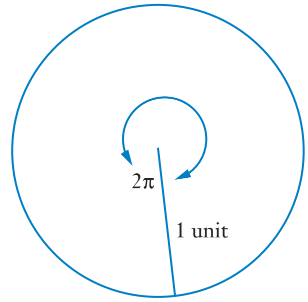
Proof

The circumference of a circle with radius 1 unit is

$C = 2\pi r$

$\quad = 2\pi(1)$

$\quad = 2\pi$



The arc length of the whole circle is 2π

$\therefore$ there are 2π radians in a whole circle

But there are 360° in a whole circle (angle of revolution)

So $2\pi = 360°$

$\quad \pi = 180°$

Converting between radians and degrees

To change from radians to degree: multiply by $\dfrac{180}{\pi}$

To change from degrees to radian: multiply by $\dfrac{\pi}{180}$

Notice that $1° = \dfrac{\pi}{180} \approx 0017$ radians

Also 1 radian $= \dfrac{180}{\pi} \approx 57°18'$

Degee and adian

EXAMPLE 20

a Convert $\dfrac{3\pi}{2}$ into degrees

b Change 60° to radians leaving your answer in terms of π

c Convert 50° into radians correct to 2 decimal place.

d Change 1145 radians into degrees to the nearest minut.

e Convert 38°41′ into radians correct to 3 decimal place.

f Evaluate cos 1145 correct to 2 decimal places

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Solution

a Since $\pi = 180°$

$$\frac{3\pi}{2} = \frac{3(180°)}{2} = 270°$$

b $180° = \pi$ radians

So $1° = \frac{\pi}{180}$ radians

$$60° = \frac{\pi}{180} \times 60$$
$$= \frac{60\pi}{180}$$
$$= \frac{\pi}{3}$$

c $180° = \pi$ radians

So $1° = \frac{\pi}{180}$ radians

$$50° = \frac{\pi}{180} \times 50$$
$$= \frac{50\pi}{180}$$
$$\approx 087$$

d π radians $= 180°$

$\therefore$ 1 radian $= \frac{180°}{\circ}$

$$1145 \text{ radians} = \frac{180°}{\circ} \times 1145$$
$$\approx 656\ °$$
$$= 65°36'$$

e $180° = \pi$ radians

$$1° = \frac{\pi}{180°}$$
$$38°41' = \frac{\pi}{180°} \times 38°41'$$
$$= 0675$$

f

Operation	Casio scientific	Sharp scientific
Make sure the calculator is in radians	SHIFT SET UP Rad	Press DRG until rad is on the screen
Enter data	cos 1145 =	cos 1145 =

$\cos 1145 = 04130$

≈ 041

Special angles

$$30° = \frac{\circ}{6} \qquad 45° = \frac{\circ}{4} \qquad 60° = \frac{\circ}{3} \qquad 90° = \frac{\circ}{2}$$

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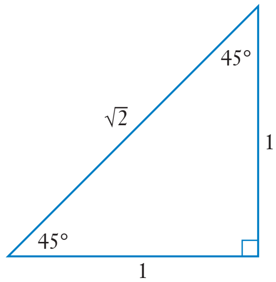
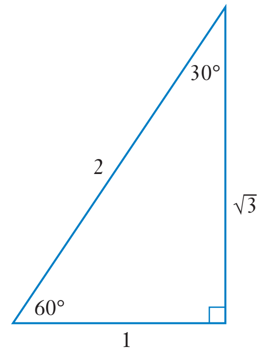
The angles 30° 45 ° and 60° give exact results in trigonometry using 2 special triangles
You looked at these in Exercise .0, Questions 9 and 0, on page 75.





From these triangles we have the exact trigonometric ratios

Exac value

Exac value 2

Exa igonomeic value

The exact ratios

$\sin 30° = \frac{1}{2}$ $\sin 45° = \frac{1}{\sqrt{2}}$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\cos 45° = \frac{1}{\sqrt{2}}$ $\cos 60° = \frac{1}{2}$

$\tan 30° = \frac{1}{\sqrt{3}}$ $\tan 45° = 1$ $\tan 60° = \sqrt{3}$

We can write these same results in radian:

The exact ratios in radians

$\sin\frac{\pi}{6} = \frac{1}{2}$ $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\cos\frac{\pi}{3} = \frac{1}{2}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\tan\frac{\pi}{4} = 1$ $\tan\frac{\pi}{3} = \sqrt{3}$

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EXAMPLE 21

a **i** Convert $\frac{\pi}{3}$ to degrees

ii Find the exact value of $\tan\frac{\pi}{3}$

b Find the exact value of $\cos\frac{\pi}{4}$

Solution

a **i** $\frac{\pi}{3} = \frac{180°}{3}$

$= 60°$

ii $\tan\frac{\pi}{3} = \tan 60°$

$= \sqrt{3}$

b $\cos\frac{\pi}{4} = \cos 45°$

$= \frac{1}{\sqrt{2}}$

Exercise 4.09 Radians

1 Convert to degrees

a $\frac{\pi}{5}$ **b** $\frac{2\pi}{3}$ **c** $\frac{5\pi}{4}$ **d** $\frac{7\pi}{6}$ **e** 3π

f $\frac{7\pi}{9}$ **g** $\frac{4\pi}{3}$ **h** $\frac{7\pi}{3}$ **i** $\frac{\pi}{9}$ **j** $\frac{5\pi}{18}$

2 Convert to radians in terms of π

a 135° **b** 30° **c** 150° **d** 240° **e** 300°

f 63° **g** 15° **h** 450° **i** 225° **j** 120°

3 Change to radians correct to 2 decimal place:

a 56° **b** 68° **c** 127° **d** 289° **e** 312°

4 Change to radians correct to 2 decimal place:

a 18°34′ **b** 35°12′ **c** 101°56′ **d** 88°29′ **e** 50°39′

5 Convert each radian measure into degrees and minutes to the nearest minut:

a 109 **b** 0768 **c** 116 **d** 099 **e** 032

f 32 **g** 27 **h** 431 **i** 56 **j** 011

6 Find correct to 2 decimal places

a sin 0342 **b** cos 15 **c** tan 0056 **d** cos 0589 **e** tan 229

f sin 28 **g** tan 53 **h** cos 477 **i** cos 39 **j** sin 298

7 Find the exact value of

a $\sin\frac{\pi}{4}$ **b** $\cos\frac{\pi}{3}$ **c** $\tan\frac{\pi}{6}$ **d** $\sin\frac{\pi}{3}$ **e** $\tan\frac{\pi}{4}$

f $\sin\frac{\pi}{6}$ **g** $\cos\frac{\pi}{4}$ **h** $\cos\frac{\pi}{6}$ **i** $\tan\frac{\pi}{3}$

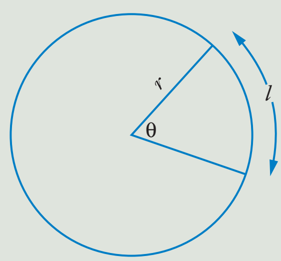
4.10 Length of an arc

Since radians are defined from the length of an arc of a circle we can use radians to find the arc length of a circle

You can find formulas for these using degree, but they are not as simpe. All the work on circles in this chapter uses radians

Length of an arc

$l = r\theta$



Proof

$$\frac{\text{arc length } l}{\text{circumference}} = \frac{\text{angle}^\circ}{\text{whole revolution}}$$

$$\frac{l}{2\pi r} = \frac{^\circ}{2\pi}$$

$$\therefore l = \frac{\theta 2\pi r}{2\pi}$$

$$= r^\circ$$

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EXAMPLE 22

a Find the length of the arc formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m

b Find the length of the arc formed given the angle subtended is 30° and the radius is 9 cm

c The area of a circle is 450 cm^2 Fin, in degrees and minuts, the angle subtended at the centre of the circle by a 27 cm arc

Solution

a

$$l = r\theta$$
$$= 5\left(\frac{\pi}{4}\right)$$
$$= \frac{5\pi}{4} \text{ m}$$

b First change 30° into radians

$$\theta = \frac{\pi}{6}$$
$$l = r\theta$$
$$= 9\left(\frac{\pi}{6}\right)$$
$$= \frac{3\pi}{2} \text{ cm}$$

c

$$A = \pi r^2$$
$$450 = \pi r^2$$
$$\frac{450}{\pi} = r^2$$
$$\sqrt{\frac{450}{\pi}} = r$$
$$119682\ldots = r$$

Now $l = r\theta$

$$27 = 119682 \quad \theta$$
$$\frac{27}{119682} = \theta$$
$$02255\ldots = \theta$$

$$\pi \text{ radians} = 180°$$
$$1 \text{ radian} = \frac{180°}{\pi}$$
$$02255 \text{ radians} = \frac{180°}{\pi} \times 02255$$
$$= 129257\ °$$
$$\approx 12°56'$$

So $\theta = 12°56'$

Exercise 4.10 Length of an arc

1 Find the exact arc length of a circle with

a radius 4 cm and angle subtended π

b radius 3 m and angle subtended $\frac{\pi}{3}$

c radius 10 cm and angle subtended $\frac{5\pi}{6}$

d radius 3 cm and angle subtended $30°$

e radius 7 mm and angle subtended $45°$

2 Find the arc length correct to 2 decimal place, givn:

a radius 15 m and angle subtended 043

b radius 321 cm and angle subtended 122

c radius 72 mm and angle subtended 55 °

d radius 59 cm and angle subtended 23 °12′

e radius 21 m and angle subtended 82 °35′

3 The angle subtended at the centre of a circle of radius 34 m is 29 °51′ Find the length of the arc cut off by this angle correct to 1 decimal plac.

4 The arc length when a sector of a circle is subtended by an angle of $\frac{\pi}{5}$ at the centre is $\frac{3\pi}{2}$ m Find the radius of the circl.

5 The radius of a circle is 3 cm and an arc is $\frac{2\pi}{7}$ cm long Find the angle subtended at the centre of the circle by the arc

6 The circumference of a circle is 300 mm Find the length of the arc that is formed by an angle of $\frac{\pi}{6}$ subtended at the centre of the circle

7 A circle with area 60 cm^2 has an arc 8 cm long Find the angle that is subtended at the centre of the circle by the arc

8 A circle with circumference 124 mm has a chord cut off it that subtends an angle of 40° at the centre Find the length of the arc cut off by the chor.

9 A circle has a chord of 25 mm with an angle of $\frac{\pi}{6}$ subtended at the centre
Find to 1 decimal plac:

a the radius

b the length of the arc cut off by the chord

10 A sector of a circle with radius 5 cm and an angle of $\frac{\pi}{3}$ subtended at the centre is cut out of cardboard It is then curved around to form an open con. Find its exact volue.

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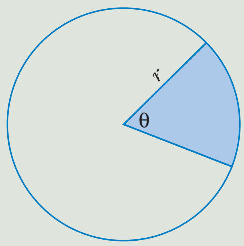
4.11 Area of a sector

Area of a sector

$$A = \frac{1}{2}r^2\theta$$



Proof

$$\frac{\text{area of sector } A}{\text{area of circle}} = \frac{\text{angle }^\circ}{\text{whole revolution}}$$

$$\frac{A}{\pi r^2} = \frac{^\circ}{2\pi}$$

$$\therefore A = \frac{\theta\pi r^2}{2\pi} = \frac{1}{2}r^{2\circ}$$

EXAMPLE 23

a Find the area of the sector formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m

b The area of the sector of a circle with radius 4 cm is $\frac{6\pi}{5}$ cm^2 Find the angl, in degrees that is subtended at the centre of the circl.

Solution

a

$$A = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}(5)^{2\circ}\left(\frac{\pi}{4}\right)$$

$$= \frac{25\pi}{8}\text{ m}^2$$

b

$$A = \frac{1}{2}r^2\theta$$

$$\frac{6\pi}{5} = \frac{1}{2}(4)^2\theta$$

$$= 8\theta$$

$$\frac{6\pi}{40} = \theta$$

$$\theta = \frac{3\pi}{20} = \frac{3(180^\circ)}{20} = 27^\circ$$

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Exercise 4.11 Area of a sector

1 Find the exact area of the sector of a circle whose radius is

a 4 cm and the subtended angle is π

b 3 m and the subtended angle is $\frac{\pi}{3}$

c 10 cm and the subtended angle is $\frac{5\pi}{6}$

d 3 cm and the subtended angle is 30°

e 7 mm and the subtended angle is 45°

2 Find the area of the sector, correct to 2 decimal placs, given the radiusis:

a 15 m and the subtended angle is 043

b 321 cm and the subtended angle is 122

c 72 mm and the subtended angle is 55 °

d 59 cm and the subtended angle is 23 °12′

e 21 m and the subtended angle is 82 °35′

3 Find the area correct to 3 significant figure, of the sector of a circle with radius4.3 m and an angle of 18 subtended at the centre

4 The area of a sector of a circle is 20 cm^2 If the radius of the circle is 3 c, find the angle subtended at the centre of the circle by the sector.

5 The area of the sector of a circle that is subtended by an angle of $\frac{\pi}{3}$ at the centre is 6π m^2 Find the radius of the circl.

6 A circle with radius 7 cm has a sector cut off by an angle of 30° subtended at the centre of the circle Fin:

a the arc length

b the area of the setor.

7 A circle has a circumference of 185 mm Find the area of the sector cut off by an angle of $\frac{\pi}{5}$ subtended at the centre

8 If the area of a circle is 200 cm^2 and a sector is cut off by an angle of $\frac{3\pi}{4}$ at the centre find the area of the sector.

9 Find the area of the sector of a circle with radius 57 cm if the length of the arc formed by this sector is 42 cm

10 The area of a sector is $\frac{3\pi}{10}$ cm^2 and the arc length cut off by the sector is $\frac{\pi}{5}$ cm
Find the angle subtended at the centre of the circle and the radius of the circle

11 If an angle of $\frac{\pi}{7}$ is subtended at the centre of a circle with radius 3 cm fin:

a the exact arc length

b the exact area of the sector.

12 An angle of $\frac{\pi}{6}$ is subtended at the centre of a circle with radius 5 cm Fin:

a the length of the arc

b the area of the sector

c the length of the chord

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13 A chord 8 mm long is formed by an angle of 45° subtended at the centre of a circle Find correct to 1 decimal plac:

a the radius of the circle

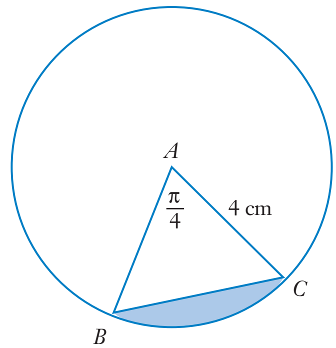
b the area of the sector cut off by the angle

14 **a** Find the area of the sector of a circle with radius 4 cm if the angle subtended at the centre is $\frac{\pi}{4}$

b Find the length of BC to 1 decimal place

c Find the exact area of triangle ABC

d Hence find the exact area of the shaded minor segment of the circle



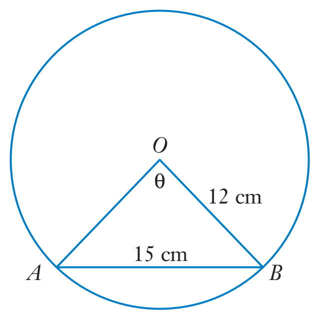
15 A triangle OAB is formed where O is the centre of a circle of radius 12 cm and A and B are endpoints of a 15 cm chord

a Find the angle subtended at the centre of the circle in degrees and minutes

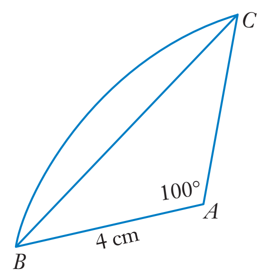
b Find the area of ΔOAB correct to 1 decimal place

c Find the area of the minor segment cut off by the chord correct to 2 decimal place.

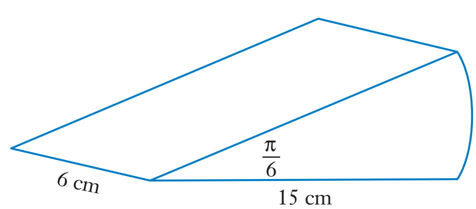
d Find the area of the major segment cut off by the chord correct to 2 decimal place.



16 Arc BC subtends an angle of 100° at the centre A of a circle with radius 4 cm Find the perimeter of sector ABC



17 A wedge is cut so that its cross-sectional area is a sector of a circle with radius 15 cm and subtending an angle of $\frac{\pi}{6}$ at the centre Find the exact volume of the wedge



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4. TEST YOURSELF

Qz

Pacice quiz

For Questions **1** to **3** select the correct answer **A B C** or **D**

1 Find the exact length of the radius of a circle if the arc length cut off by an angle of $\frac{5\pi}{4}$ is $\frac{25\pi}{8}$ cm

A 5π cm **B** 5 cm **C** 25 cm **D** $\frac{5\pi}{2}$ cm

2 The cosine rule is (there is more than one answer)

A $c^2 = a^2 + b^2 - 2ab \cos C$

B $c^2 = a^2 + c^2 - 2ac \cos C$

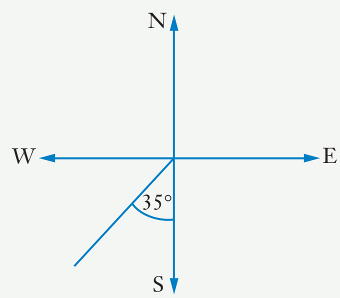
C $a^2 = b^2 + c^2 - 2bc \cos A$

D $a^2 = b^2 + c^2 - 2ab \cos A$

3 What bearing is shown on the diagram (there may be more than one answer)?

A 035° **B** W 35° S

C S 35° W **D** 215°



4 Find the exact value of cos θ and sin θ if $\tan \theta = \frac{3}{5}$

5 Evaluate to 2 decimal places

a sin 39°54′ **b** tan 61°30′ **c** cos 19°2′ **d** sin 014 **e** tan 35

6 Find θ to the nearest minute if

a sin θ = 072 **b** cos θ = 0286 **c** $\tan \theta = \frac{5}{7}$

7 A ship sails on a bearing of 215° from port until it is 100 km due south of port How far does it sail to the nearest km ?

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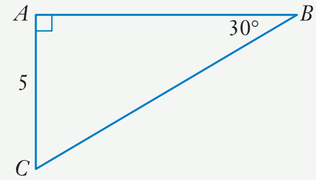
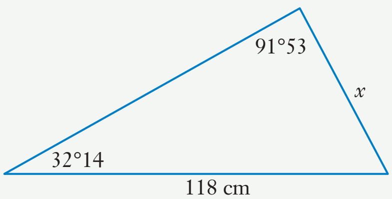
8 Find the length of AB as a surd

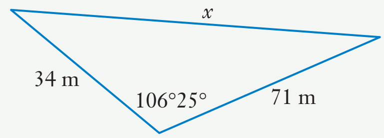


9 Evaluate x correct to 2 significant figure.

a



b

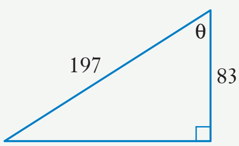


10 Convert each radian measure to degrees and minutes

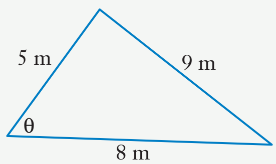
a 075 **b** 13 **c** 395 **d** 42 **e** 566

11 Evaluate θ to the nearest minute

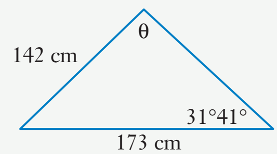
a



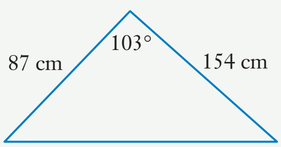
b



c



12 Find the area of this triangle



13 Jacquie walks south from home for 32 km then turns and walks west for .8 k. What is the bearing to the nearest degre, f:

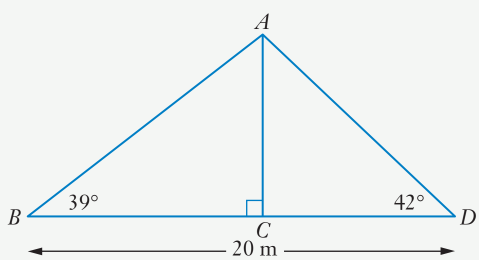
a Jacquie from her home?

b her home from where Jacquie is now?

14 The angle of elevation from point B to the top of a pole AC is 39° and the angle of elevation from D on the other side of the pole is 42 ° B and D are 20 m apart

a Find an expression for the length of AD

b Find the height of the pole to 1 decimal place



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15 A plane flies from Orange for 1800 km on a bearing of 300° It then turns and flies for 2500 km on a bearing of 205° How far is the plane from Orang, to the nearest km ?

16 Convert to radians leaving in terms of π

a 60° **b** 45° **c** 150° **d** 180° **e** 20°

17 A circle with radius 5 cm has an angle of $\frac{\pi}{6}$ subtended at the centre Fin:

a the exact arc length

b the exact area of the sector.

18 Find the exact value of

a $\tan\frac{\pi}{3}$ **b** $\cos\frac{\pi}{6}$ **c** $\sin\frac{\pi}{4}$ **d** $\tan\frac{\pi}{6}$ **e** $\cos\frac{\pi}{4}$

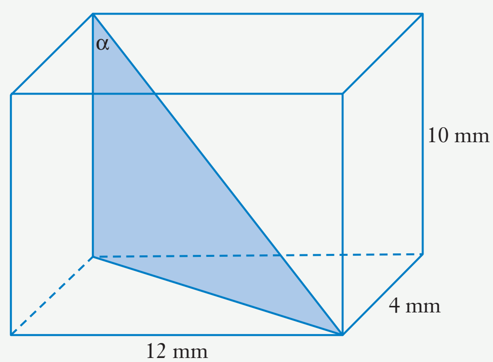
f $\sin\frac{\pi}{6}$ **g** $\tan\frac{\pi}{4}$ **h** $\cos\frac{\pi}{3}$ **i** $\sin\frac{\pi}{3}$

19 A circle has a circumference of 8π cm If an angle of $\frac{\pi}{7}$ is subtended at the centre of the circle fin:

a the exact area of the sector

b the area of the minor segment to 2 decimal place.

20 Evaluate α in this figure



21 In triangle MNP $NP = 149$ cm, $MP = 127$ cm and $\angle N = 43°49'$ Find $\angle M$ in degrees and minutes

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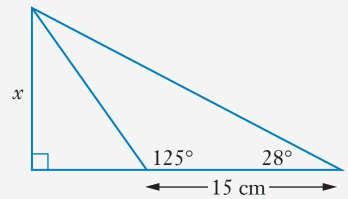
4. CHALLENGE EXERCISE

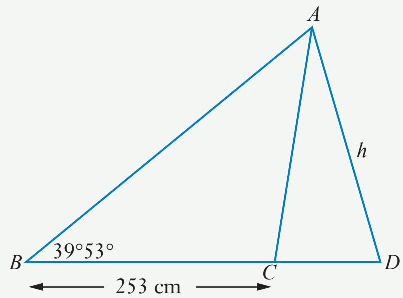
1 Two cars leave an intersection at the same tim, one travelling at 70 km/h along one straight road and the other car travelling at 80 km/h along another straight road After 2 hours they are 218 km apart At what ange, to the nearest minte, do the roads meet at the intersection?

2 Evaluate x correct to 3 significant figures



3 **a** Find an exact expression for the length of AC

b Hence or otherwis, find the value of h correct to 1 decimal place



4 From the top of a vertical pole the angle of depression to Ian standing at the foot of the pole is 43° Liam is on the other side of the pol, and the angle of depression from the top of the pole to Liam is 52° The boys are standing 58 m apat. Find the height of the pole to the nearest metr.

5 From point A 93 m due south of the base of a towe, the angle of elevation is 35 ° Point B is 124 m due east of the tower. Fid:

a the height of the tower, to the nearest metre

b the angle of elevation of the tower from point B

6 A cable car 100 m above the ground is seen to have an angle of elevation of 65° when it is on a bearing of 345° After a minue, it has an angle of elevation of 69 ° and is on a bearing of 025° Fin:

a how far it travels in that minute

b its speed in $m\ s^{-}$

7 Find the area of a regular hexagon with sides 4 cm to the nearest cm^2

8 Calculate correct to one decimal place the area of a regular pentagon with sides 12 mm

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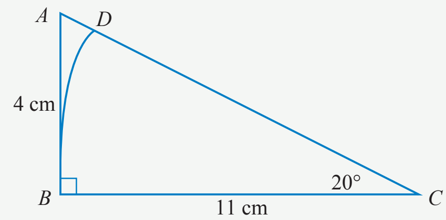
9 The length of an arc is 89 cm and the area of the sector is 243 cm^2 when an angle of θ is subtended at the centre of a circle Find the area of the minor segment cut off by θ correct to 1 decimal place

10 *BD* is the arc of a circle with centre *C*
Find correct to 2 decimal place:

a the length of arc *BD*

b the area of region *ABD*

c the perimeter of sector *BDC*

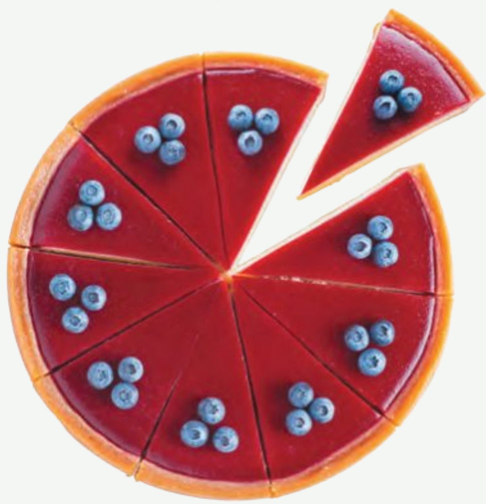


11 David walks along a straight road At one point he notices a tower on a bearing of 053° with an angle of elevation of 21° After David walks 230m, the tower is on a bearing of 342° with an angle of elevation of 26° Find the height of the tower correct to the nearest metre

12 The hour hand of a clock is 12 cm long Fin:

a the length of the arc through which the hand would turn in 5 hours

b the area through which the hand would pass in 2 hours



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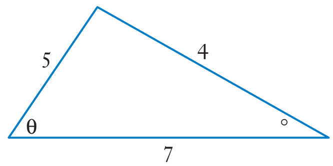
Practice set 2

In Questions **1** to **6** select the correct answer **A B C** or **D**

1 Find an expression involving θ for this triangle (there may be more than one answer)



A $\cos\theta = \dfrac{5^2 + 4^2 - 7^2}{2 \times 5 \times 4}$

B $\dfrac{\sin\theta}{4} = \dfrac{\sin\alpha}{5}$

C $\dfrac{\sin\theta}{5} = \dfrac{\sin\alpha}{4}$

D $\cos\theta = \dfrac{5^2 + 7^2 - 4^2}{2 \times 5 \times 7}$

2 If $f(x) = \begin{cases} 8x & \text{if } x > 3 \\ 3x^2 - 2 & \text{if } 0 \le x \le 3 \\ 9 & \text{if } x < 0 \end{cases}$ evaluate $f(3) + f(1) + f(-1)$

A 35 **B** 226 **C** 233 **D** 53

3 The linear function with equation $4x - 2y + 3 = 0$ has

A gradient -2, y-intercept $-1\frac{1}{2}$

B gradient $\frac{1}{2}$ y-intercept $\frac{3}{4}$

C gradient 2 y-intercept $1\frac{1}{2}$

D gradient 4 y-intercept 3

4 For the quadratic function $y = ax^2 + bx + c > 0$ for all x

A $a > 0, b^2 - 4ac > 0$ **B** $a < 0, b^2 - 4ac > 0$

C $a > 0, b^2 - 4ac < 0$ **D** $a < 0, b^2 - 4ac < 0$

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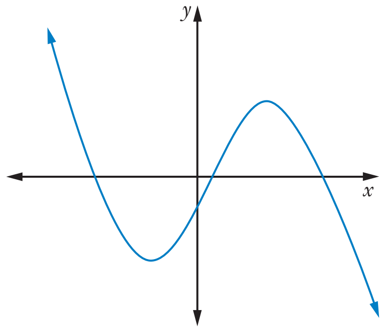
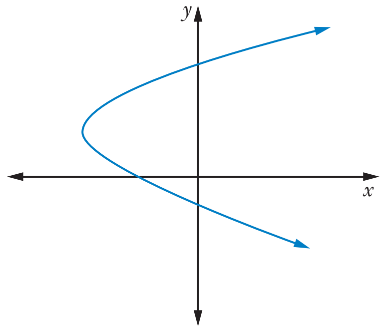
5 Which of the following is not the graph of a function?

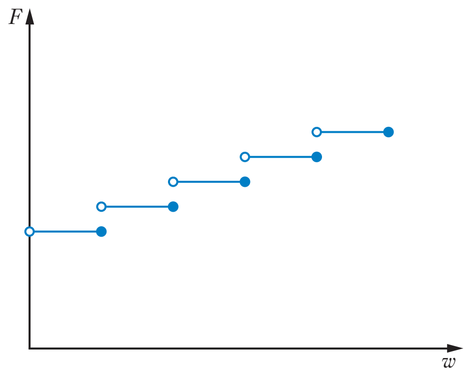
A



B



C



D (0, 3), (1, 3), (2, 5), (3, 1)

6 The polynomial $P(x) = x^3 - 5x^2 + 3x - 8$ (there is more than one answer)

A is monic

B has degree 3

C has leading coefficient –8

D has constant term –8

7 A triangle has sides of length 51 m .5 m and .2 .

a Find the size of the angle opposite the 65 m side correct to the nearest minut.

b Find the area of the triangle correct to one decimal place

8 Find the equation of the straight line

a with gradient –2 and y-intercept 3

b with x-intercept 5 and y-intercept –1

c passing through (2 0) and (–3, –4)

d through (5 –4) and parallel to the line through (7 4) and (, –1)

e through (3 –1) perpendicular to the line $3x - 2y - 7 = 0$

f through (1 2) parallel to the line through (–3, 4) and (5, 5)

g through (1 3) and an angle of inclination of 135.

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9 Simplify

a $\dfrac{6x}{2x-8}$ **b** $\dfrac{5y+10}{xy^2} \div \dfrac{y^2-4}{x^2y}$ **c** $\dfrac{4a-3}{5} - \dfrac{a+1}{4}$

10 Convert these angles into radians in terms of π

a 60° **b** 150° **c** 90°

d 10° **e** 315°

11 Sketch the graph of

a $5x - 2y - 10 = 0$ **b** $x = 2$ **c** $f(x) = (x - 3)^2$

d $y = x^2 - 5x + 4$ **e** $y = (x - 1)^3 + 2$

12 Convert each value in radians into degrees and minutes

a 17 **b** 036 **c** 254

13 The lines AB and AC have equations $3x - 4y + 9 = 0$ and $8x + 6y - 1 = 0$ respectively.

a Show that the lines are perpendicular.

b Find the coordinates of A

14 Find the gradient of the line through the origin and (–3, 5.

15 If $g(x) = \begin{cases} 3-x & \text{if } x > 1 \\ 2x & \text{if } x \le 1 \end{cases}$

a find $g(2)$ and $g(-3)$

b sketch the graph of $y = g(x)$

16 Find the value of x if $f(x) = 7$ where $f(x) = 2^x - 1$.

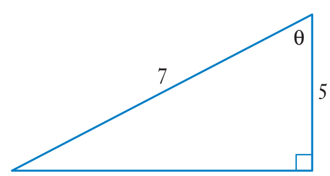
17 If $f(x) = 9 - 2x^2$ find the value of $f(-1)$

18 Show that $3x - 4y + 10 = 0$ is a tangent to the circle $x^2 + y^2 = 4$

19 Change each value in radians into degrees

a $\dfrac{\pi}{4}$ **b** $\dfrac{3\pi}{2}$ **c** $\dfrac{\pi}{5}$ **d** $\dfrac{7\pi}{8}$ **e** 6π

20 Given the triangle ABC find exact values of cos θ sin θ and tan θ



21 Show that

a $-x^2 + x - 9 < 0$ for all x

b $x^2 - x + 3 > 0$ for all x

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22 The distance travelled by a runner is directly proportional to the time she takes If Vesna runs 12 km in 2 hours 30 minuts, fnd:

a an equation for distance d in terms of time t

b how far Vesna runs n:

i 2 hours **ii** 5 hours

c how long it takes Vesna to rn:

i 30 km **ii** 19 km

d her average speed

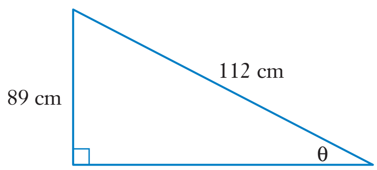
23 Find the equation of the parabola with x-intercepts 3 and -1 and y-intercept -3

24 Show that the quadratic equation $6x^2 + x - 15 = 0$ has 2 real rational root.

25 The area of a circle is 5π and an arc 3 cm long cuts off a sector with an angle of θ subtended at the centre Find θ in degrees and minutes

26 A soccer goal is 8 m wide Tim shoots for goal when he is 9 m from one post and 11 m from the other. Within what angle must a shot be made in order to score a goal?

27 Evaluate θ in degrees and minutes to the nearest minut:



28 **a** Find the equation of the straight line l through $(-1\ 2)$ that is perpendicular to the line $3x + 6y - 7 = 0$

b Line l cuts the x-axis at P and the y-axis at Q Find the coordinates of P and Q

29 Show that $f(x) = x^6 - x^2 - 3$ is an even function

30 Find the angle of depression from the top of a 56 m tall cliff down to a boat that is 150 m out from the base of the cliff

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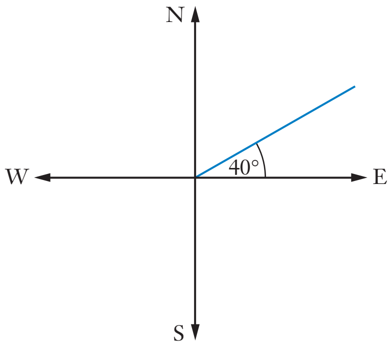
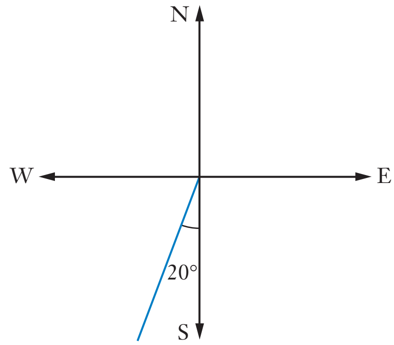
31 Write each direction shown a:

i a compass bearing **ii** a true bearing

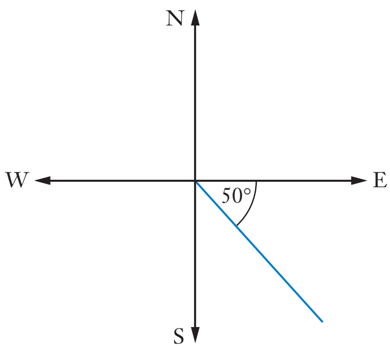
a



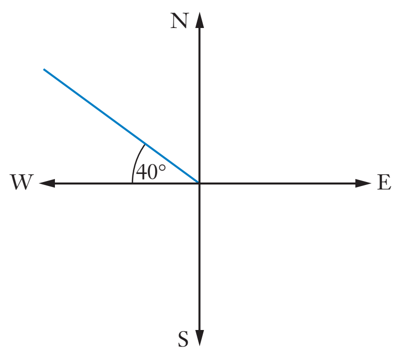
b



c



d



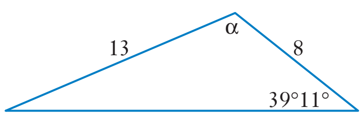
32 An angle of 30° is subtended at the centre of a circle with radius 5 cm Find the exact

a arc length **b** area of the sector.

33 Factorise

a $x^2 - 4x + 4$ **b** $9x^2 - 1$

34 Find α in degrees and minutes



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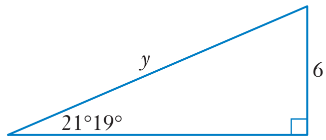
35 Find the value of y correct to 3 significant figures



36 Find the intersection of the graphs

a $x + 3y - 1 = 0$ and $x - 2y - 6 = 0$

b $y = x^2$ and $x - 2y + 15 = 0$

37 For each quadratic function

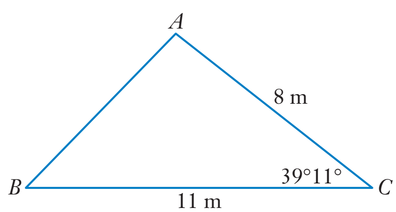
i find the equation of the axis of symmetry

ii state whether it has a maximum or minimum turning point and find its coordinates

a $y = x^2 - 6x + 1$ **b** $y = -2x^2 - 4x - 3$

38 A hawk at the top of a 10 m tree sees a mouse on the ground If the angle of depression is 34°51′ how far, to 1 decimal plae, does the bird need to fly to reach the mouse?

39



a Find AB to the nearest metr.

b Find the area of ΔABC to 3 significant figure.

40 Two points A and B are 100 m apart on the same side of a tower. The angle of elevation of A to the top of the tower is 20° and the angle of elevation from B is 27° Find the height of the tower, to the nearest mete.

41 The length of an arc in a circle of radius 6 cm is 7π cm Find the area of the sector cut off by this arc

42 Jordan walks for 31 km due west then turns and walks for .7 km on a bearing of 205° How far is he from his starting point ?

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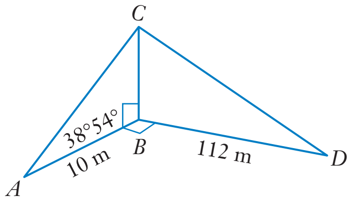
43 The angle of elevation from a point A to the top of a tower BC is 38°54′ A is 10 m due south of the tower.



a Find the height of the tower, to 1 decimal plae.

b If point D is 112 m due east of the tower, find the angle of elevation from D to the tower.

44 Find the domain and range of

a $f(x) = \dfrac{3}{x+4}$ **b** $y = |x| + 2$

c $y = 4$ **d** $y = x^2 - 3$

45 Nalini leaves home and cycles west for 125 km then turns and rides south for 113 km

a How far is Nalini from home?

b Find the bearing of Nalini from home

46 Show that $f(x) = x^3 - 5x$ is an odd function

47 Sketch the graph of

a $3x - 2y + 6 = 0$ **b** $y = x^2 - x - 2$

c $y = x^3 - 1$ **d** $y = x(x + 2)(x - 3)$

48 The length of an arc in a circle of radius 2 cm is 16 cm Find the area of the secto.

49 Change each angle size from radians into degrees

a 2π **b** $\dfrac{\pi}{6}$ **c** $\dfrac{9\pi}{4}$

50 A plane flies on a bearing of 034° from Sydney for 875 km How far due east of Sydney is the plane?

51 Solve

a $5b - 3 \geq 7$ **b** $x^2 - 3x = 0$ **c** $|2n + 5| = 9$

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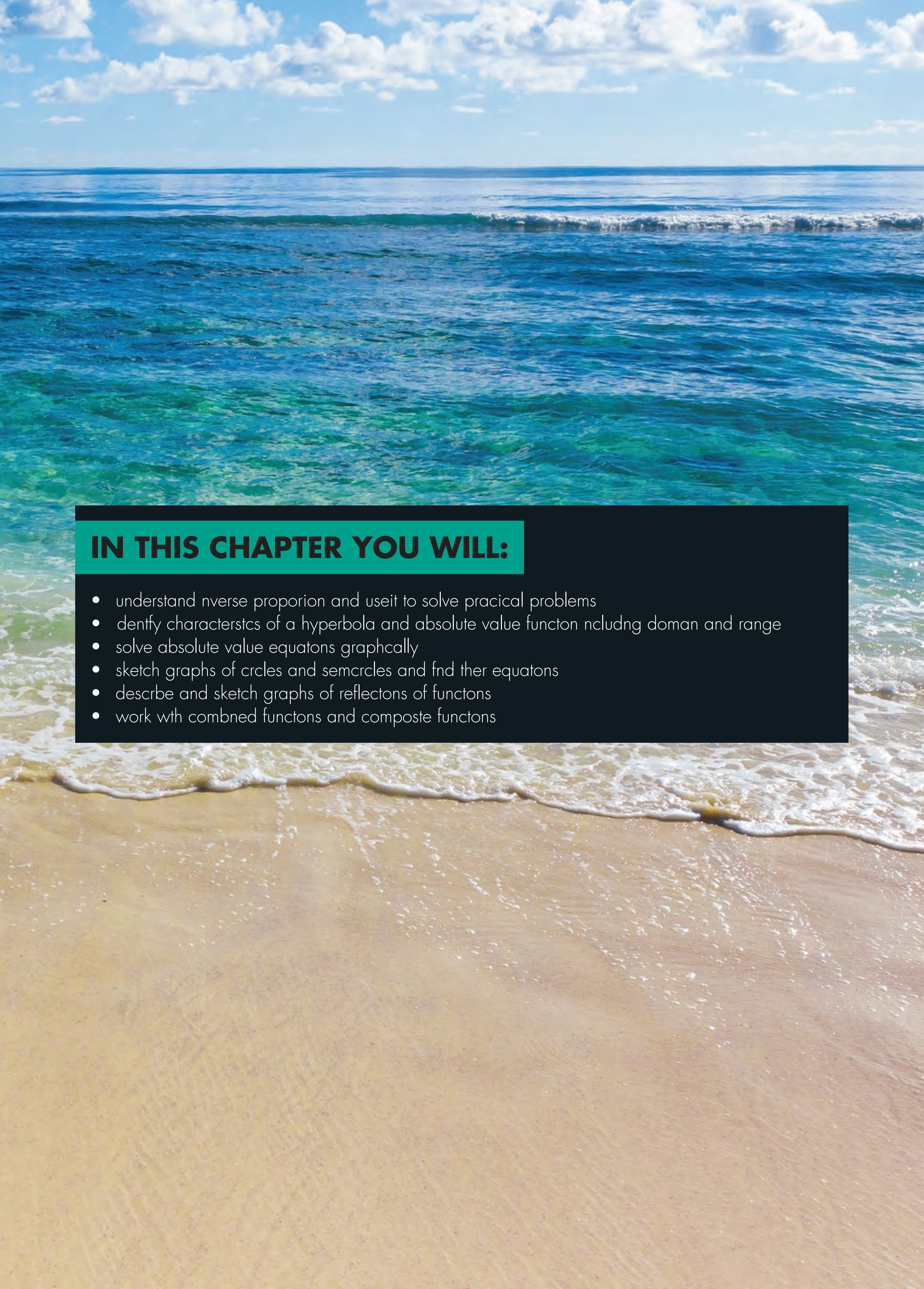
5.

FURTHER FUNCTIONS

In ths chapter, we look at funcions and relaions that are not polynoial, incluing the hyperbol, absolute value crcles and semcrcles We ill also study reflecions and relaionsips between functons ncludng combned functons and composte functons

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- understand nverse proporion and useit to solve pracical problems
- dentfy characterstcs of a hyperbola and absolute value functon ncludng doman and range
- solve absolute value equatons graphcally
- sketch graphs of crcles and semcrcles and fnd ther equatons
- descrbe and sketch graphs of reflectons of functons
- work wth combned functons and composte functons

TERMINOLOGY

asymptote A line that a curve approaches but doesnt touch

composite function A function of a functio, where the output of one function becomes the input of a second function written as $f(g(x))$ For example if $f(x) = x^2$ and $g(x) = 3x + 1$ then $f(g(x)) = (3x + 1)^2$

continuous function A function whose graph is smooth and does not have gaps or breaks

discontinuous function A function whose graph has a gap or break in it for example $f(x) = \frac{1}{x}$ whose graph is a hyperbola

hyperbol: The graph of the function $y = \frac{k}{x}$ which is made up of 2 separate curves

inverse variaion: A relationship between 2 variables such that as one variable increases the other variable decreases or as one variable decreases the other variable increases One variable is a multiple of the reciprocal of the other, with equation $y = \frac{k}{x}$ Also called **inverse proportion**

5.01 The hyperbola

Inverse variation

WS The hypebola

Gaphing hypebola

WS Gaphing $y \frac{0}{x}$

We looked at direct variation and the equation $y = kx$ in Chapter 3 *Functions* When one variable is in **inverse variation** (or **inverse proportion**) with another variable one is a constant multiple of the **reciprocal** of the other. This means that as one variable increaes, the other decreases and when one decreases the other increase.

For example

- The more slices you cut a pizza into the smaller the size of each slice
- The more workers there are on a project the less time it takes to complete
- The fewer people sharing a house the higher the rent each person pay.

Inverse variation

If variables x and y are in inverse variation we can write the equation $y = \frac{k}{x}$ where k is called the **constant of variation**

EXAMPLE 1

a Building a shed in 12 hours requires 3 builders If the number of builder, N, is in inverse variation to the amount of time t hours

i find the equation for N in terms of t

ii find the number of builders it would take to build the shed in 9 hours

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iii find how long it would take 2 builders to build the shed

v graph the equation for N after completing the table below.

t	1	2	3	4	5	6	7	8	9
N									

b The faster a car travels the less time it takes to travel a certain distanc. It takes the car 2 hours to travel this distance at a speed of 80 km/h If the time take, t hours is in inverse proportion to the speed s km/h the:

i find the equation for t in terms of s

ii find the time it would take if travelling at 100 km/h

iii find the speed at which the trip would take $2\frac{1}{2}$ hours

v graph the equation

Solution

a **i** For inverse variation the equation is in the form $N = \frac{k}{t}$

Substitute $t = 12, N = 3$ to find the value of k

$$3 = \frac{k}{12}$$

$$36 = k$$

$$\therefore N = \frac{36}{t}$$

ii Substitute $t = 9$

$$N = \frac{36}{9}$$

$$= 4$$

So it takes 4 builders to build the shed in 9 hours

iii Substitute $N = 2$.

$$2 = \frac{36}{t}$$

$$2t = 36$$

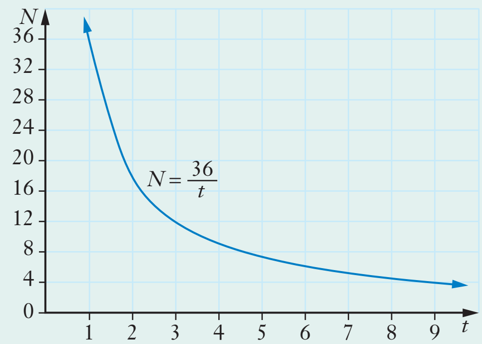
$$t = 18$$

So it takes 18 hours for 2 builders to build the shed

v $N = \frac{36}{t}$

Completing a table of values

t	1	2	3	4	5	6	7	8	9
N	36	18	12	9	7.2	6	5.1	4.5	4



b **i** For inverse proportion the equation is in the form $t = \frac{k}{s}$

Substitute $s = 80$, $t = 2$ to find k

$$2 = \frac{k}{80}$$

$$k = 160$$

$$\therefore \quad t = \frac{160}{s}$$

ii Substitute $s = 100$

$$t = \frac{160}{100}$$

$$= 16 \text{ hours}$$

$$= 1 \text{ h } 36 \text{ min}$$

So the car takes 1 h 36 min to travel the distance if travelling at 100 km/h

iii Substitute $t = 2.5$.

$$25 \ = \frac{160}{s}$$

$$25\ s = 160$$

$$s = \frac{160}{2\ 5}$$

$$= 64$$

So the car travels at 64 km/h if the trip takes $2\frac{1}{2}$ hours

v To graph this functio, complete a table of values for $t = \frac{160}{s}$

s	20	40	60	80	100	120
t	8	4	2.667	2	1.6	1.333

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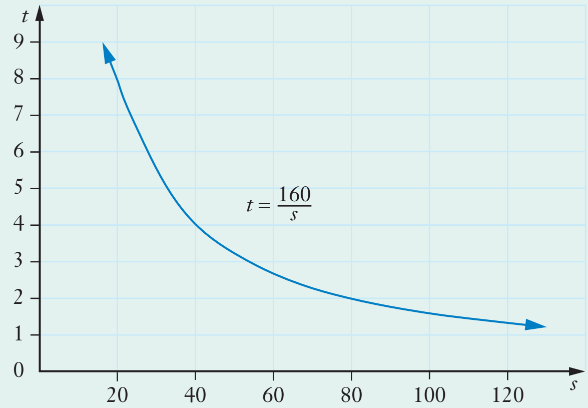


The graph of the function $y = \frac{k}{x}$ is a **hyperbola**

Hyperbolas

A hyperbola is the graph of a function of the form $y = \frac{k}{x}$

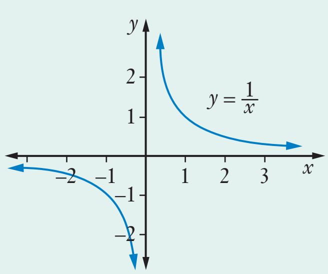
EXAMPLE 2

Sketch the graph of $y = \frac{1}{x}$ What is the domain and range?

Solution

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	–	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$

When $x = 0$ the value o y s undefned



Domain x can be any real number except 0
$(-\infty, 0) \cup (0, \infty)$

Range y can be any real number except 0
$(-\infty, 0) \cup (0, \infty)$

CLASS DISCUSSION

LIMITS OF THE HYPERBOLA

What happens to the graph as x becomes closer to 0? What happens as x becomes very large in both positive and negative directions? The value of y is never 0 Why?

Continuity

Most functions have graphs that are smooth unbroken curves (or lines) They are called **continuous functions** Howeve, some functions have discontinuites, meaning that their graphs have gaps or breaks These are called **discontinuous functions**

The hyperbola is discontinuous because there is a gap in the graph and it has two separate parts. The graph of $y = \frac{1}{x}$ also does not touch the x- or y-axes but it gets closer and closer to them We call the x- and y-axes **asymptotes** lines that the curve approaches but never touches

To find the shape of the graph close to the asymptotes or as $x \to \pm\infty$ we can check points nearby.

EXAMPLE 3

Find the domain and range of $f(x) = \frac{3}{x-3}$ and sketch the graph of the function

Solution

To find the domai, we notice that $x - 3 \neq 0$ So $x \neq 3$.

Domain $(-\infty, 3) \cup (3, \infty)$

Also y cannot be zero $y \neq 0$

Range $(-\infty, 0) \cup (0, \infty)$

The lines $x = 3$ and $y = 0$ (the x-axis) are the asymptotes of the hyperbola

To find the limiting behaviour of the grap, look at what is happening as $x \to \pm\infty$

As x increases and approaches ∞ $\frac{3}{x-3}$ becomes closer to 0 and is positive
Substitute large values of x into the function for exampl, $x = 1000$

As $x \to \infty$ $y \to 0^+$ (as x approaches infinity, y approaches 0 from above the positive side.

Similarly, as x decreases and approaches $-\infty$ y becomes closer to 0 and is negative
Substitute $x = -1000$ for exampl.

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As $x \to -\infty$ $y \to 0^-$ (as x approaches negative infinity, y approaches 0 from below, the negative side)

To see the behaviour of the function near the asymptote $x = 3$ we can test values either side

LHS When $x = 2999$

$$\frac{3}{2999 - 3} = -3000$$
$$< 0$$

As $x \to 3^-$ $y \to -\infty$

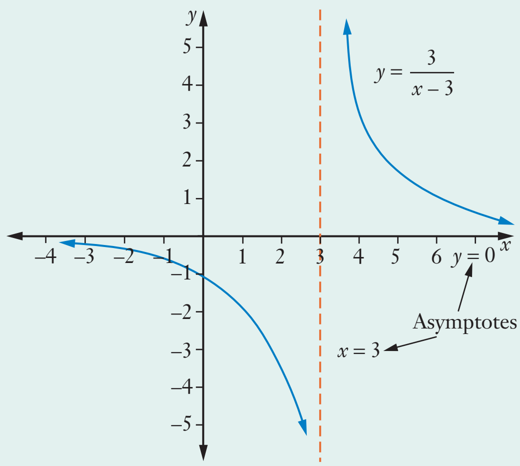
RHS When $x = 3001$,

$$\frac{3}{3001 - 3} = 3000$$
$$> 0$$

As $x \to 3^+$ $y \to \infty$

For y-intercept $x = 0$

$$y = \frac{3}{0 - 3}$$
$$= -1$$



EXAMPLE 4

Sketch the graph of $y = -\dfrac{1}{2x + 4}$

Solution

To find the domai, notice tht:

$2x + 4 \neq 0$

$2x \neq -4$

$x \neq -2$

Domain $(-\infty\ \ -2) \cup (-2, \infty)$

For the range $y \neq 0$

Range $(-\infty, 0) \cup (0, \infty)$

So there are asymptotes at $x = -2$ and $y = 0$

Limiting behaviour

Substtute say, $x = 5000$ and $x = -5000$

As $x \to \infty$ $y \to 0^-$

As $x \to -\infty$ $y \to 0^+$

To see the shape of the graph near the asymptote $x = -2$ we can test values either sid.

LHS When $x = -20001$

$$y = -\frac{1}{2(-20001) + 4} = 5000 > 0$$

As $x \to 2^- \; y \to \infty$

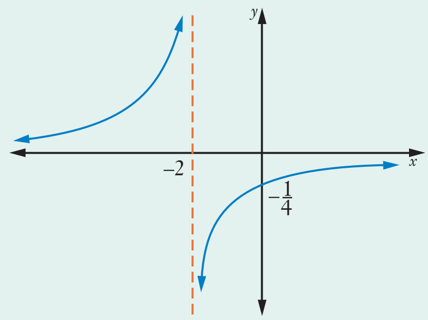
RHS When $x = -19999$

$$y = -\frac{1}{2(-19999) + 4} = -5000 < 0$$

As $x \to 2^+ \; y \to -\infty$

For y-intercept $x = 0$

$$y = -\frac{1}{2(0) + 4}$$
$$= -\frac{1}{4}$$



The hyperbola

The hyperbola $y = \frac{k}{bx + c}$ is a discontinuous function with 2 parts separated by vertical and horizontal asymptotes

Exercise 5.01 The hyperbola

1 The diameter of a balloon varies inversely with the thickness of the rubber. The diameter of the balloon is 80 mm when the rubber is 2 mm thick

- **a** Find an equation for the diameter D in terms of the thickness x
- **b** Find the diameter when the thickness is 08 mm
- **c** Find the thickness correct to one decimal place when the diameter is 1153 mm
- **d** Sketch the graph showing this information

2 The more boxes a factory produces the less it costs to produce each bo. When 128 boxes are produced it costs $2 per bo.

- **a** Write an equation for the cost c to produce each box when manufacturing n boxes
- **b** Find the cost of each box when 100 boxes are produced
- **c** Find how many boxes must be produced for the cost for each box to be 50 cents
- **d** Sketch the graph of this information

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3 For each function

- **i** state the domain and range
- **ii** find the y-intercept if it exists
- **iii** sketch the graph

a $y = \frac{2}{x}$ **b** $y = -\frac{1}{x}$ **c** $f(x) = \frac{1}{x+1}$

d $f(x) = \frac{3}{x-2}$ **e** $y = \frac{1}{3x+6}$ **f** $f(x) = -\frac{2}{x-3}$

g $f(x) = \frac{4}{x-1}$ **h** $y = -\frac{2}{x+1}$ **i** $f(x) = \frac{2}{6x-3}$

4 Show that $f(x) = \frac{2}{x}$ is an odd function

5 **a** Is the hyperbola $y = -\frac{2}{x+1}$

i a function? **ii** even odd or neither ? **iii** continuous?

b What are the equations of the asymptotes?

c State its domain and range

5.02 Absolute value functions

WS

Abolue value uncion

An absolute value function is an example of a piecewise function with 2 sections
We were introduced to absolute value in Chapter , *Functions*

The absolute value function

$$|x| = \begin{cases} x & \text{if } x \circ 0 \\ -x & \text{if } x < 0 \end{cases}$$

EXAMPLE 5

Sketch the graph of $y = |x|$ and state its domain and range

Solution

$y = |x|$ gives the piecewise function

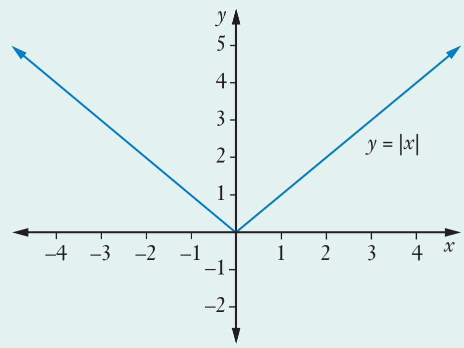
$$y = \begin{cases} x & \text{for } x \text{ ° } 0 \\ -x & \text{for } x < 0 \end{cases}$$

We can draw $y = x$ for $x \geq 0$ and $y = -x$ for $x < 0$ on the same set of axes

From the graph notice that x can be any real number while $y \geq 0$

Domain $(-\infty \;\; \infty)$

Range $[0 \;\; \infty)$



Abolue value gaphs

EXAMPLE 6

a Sketch the graph of $f(x) = |x| - 1$ and state its domain and range

b Sketch the graph of $y = |x + 2|$

Solution

a Using the definition of absolute value

$$y = \begin{cases} x - 1 & \text{for } x \geq 0 \\ -x - 1 & \text{for } x < 0 \end{cases}$$

Draw $y = x - 1$ for $x \geq 0$ and $y = -x - 1$ for $x < 0$

For x-intercepts $y = 0$

$y = x - 1$	$y = -x - 1$
$0 = x - 1$	$0 = -x - 1$
$1 = x$	$x = -1$

For y-intercept $x = 0$

$$\begin{aligned} y &= x - 1 \text{ for } x = 0 \\ &= 0 - 1 \\ &= -1 \end{aligned}$$

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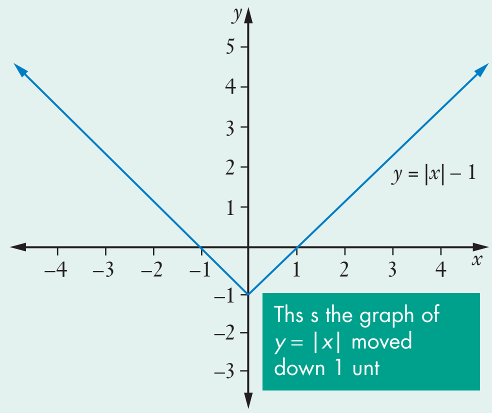
From the graph notice that x can be any real number while $y \geq -1$

Domain $(-\infty \quad \infty)$

Range $[-1, \infty)$



b Using the definition of absolute value

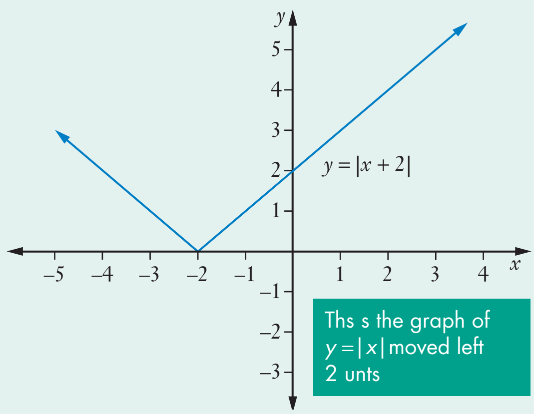
$$y = \begin{cases} x+2 & \text{for } x+2 \geq 0 \\ -(x+2) & \text{for } x+2 < 0 \end{cases}$$

Simplifying this gives

$$y = \begin{cases} x+2 & \text{for } x \geq -2 \\ -x-2 & \text{for } x < -2 \end{cases}$$

For x-intercepts $y = 0$

$y = x + 2$	$y = -x - 2$
$0 = x + 2$	$0 = -x - 2$
$-2 = x$	$x = -2$



For y-intercept $x = 0$

$y = x + 2$ for $x = 0$

$= 0 + 2$

$= 2$

INVESTIGATION

TRANSFORMATIONS OF THE ABSOLUTE VALUE FUNCTION

Use a graphics calculator or graphing software to explore each absolute value graph

1 $y = \|x\|$	**2** $y = 2\|x\|$	**3** $y = 3\|x\|$
4 $y = -\|x\|$	**5** $y = -2\|x\|$	**6** $y = \|x\| + 1$
7 $y = \|x\| + 2$	**8** $y = \|x\| - 1$	**9** $y = \|x\| - 2$
10 $y = \|x + 1\|$	**11** $y = \|x + 2\|$	**12** $y = \|x + 3\|$
13 $y = \|x - 1\|$	**14** $y = \|x - 2\|$	**15** $y = \|x - 3\|$

Are graphs that involve absolute value always functions? Can you find an example of one that is not a function?

Are any of them odd or even? Are they continuous? Could you predict what the graph $y = 2|x - 7|$ would look like?

Equations involving absolute values

We learned how to solve equations involving absolute values using algebra in Chapter , *Equations and inequalities* We can also solve these equations graphicaly.

EXAMPLE 7

Solve $|2x - 1| = 3$ graphically.

Solution

Sketch the graphs of $y = |2x - 1|$ and $y = 3$ on the same number plane

$$y = \begin{cases} 2x-1 & \text{for } 2x-1 \geq 0 \\ -(2x-1) & \text{for } 2x-1 < 0 \end{cases}$$

Simplifying this gives

$$y = \begin{cases} 2x-1 & \text{for } x \geq \frac{1}{2} \\ -2x+1 & \text{for } x < \frac{1}{2} \end{cases}$$

For x-intercepts $y = 0$

$y = 2x - 1$ $\quad\quad$ $y = -2x + 1$

$0 = 2x - 1$ $\quad\quad$ $0 = -2x + 1$

$1 = 2x$ $\quad\quad$ $2x = 1$

$\frac{1}{2} = x$ $\quad\quad$ $x = \frac{1}{2}$

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For y-intercept $x = 0$

$y = -2x + 1$ for $x = 0$

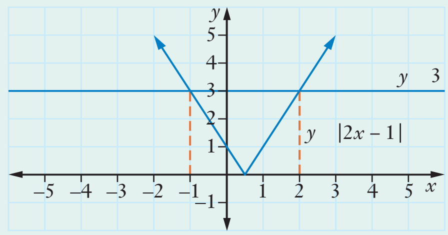
$= -2(0) + 1$

$= 1$

The graph of $y = 3$ is a horizontal line through 3 on the y-axis

The solutions of $|2x - 1| = 3$ are the values of x at the point of intersection of the graphs

$x = -1, 2.$



We can check that our soluions are correct by substtutng them back nto the equaton

Exercise 5.02 Absolute value functions

1 Find the x- and y-intercepts of the graph of each function

a $f(x) = |x| + 7$ **b** $f(x) = |x| - 2$ **c** $y = 5|x|$

d $f(x) = -|x| + 3$ **e** $y = |x + 6|$ **f** $f(x) = |3x - 2|$

g $y = |5x + 4|$ **h** $y = |7x - 1|$ $f(x) = |2x| + 9$

2 Sketch the graph of each function

a $y = |x|$ **b** $f(x) = |x| + 1$ **c** $f(x) = |x| - 3$

d $y = 2|x|$ **e** $f(x) = -|x|$ **f** $y = |x + 1|$

g $f(x) = -|x - 1|$ **h** $y = |2x - 3|$ $f(x) = |3x + 1$

3 Find the domain and range of each function

a $y = |x - 1|$ **b** $f(x) = |x| - 8$ **c** $f(x) = |2x + 5|$

d $y = 2|x| - 3$ **e** $f(x) = -|x - 3|$

4 Solve each equation graphically.

a $|x| = 3$ **b** $|x + 2| = 1$ **c** $|x - 3| = 0$

d $|2x - 3| = 1$ **e** $|2x + 3| = 11$ **f** $|5b - 2| = 8$

g $|3x + 1| = 2$ **h** $5 = |2x + 1|$ **i** $0 = |6t - 3|$

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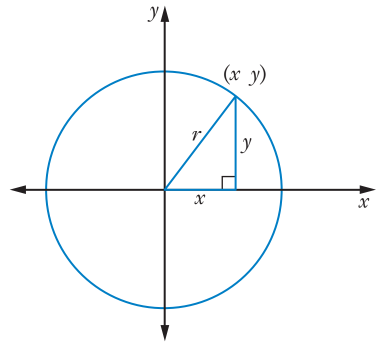
Equaion o cicle

5.03 Circles and semicircles

The circle is not a function It does not pass the vertical line tes.

Circle with centre (0, 0)

We can use Pythagora' theorem to find the equation of a circle using a general point (x y) on a circle with centre (0 0) and radius r



$$c^2 = a^2 + b^2$$

$$\therefore \quad r^2 = x^2 + y^2$$

Equation of a circle with centre (0, 0)

The equation of a circle with centre (0 0) and radius r is $x^2 + y^2 = r^2$

EXAMPLE 8

a Sketch the graph of $x^2 + y^2 = 4$

b Why is it not a function?

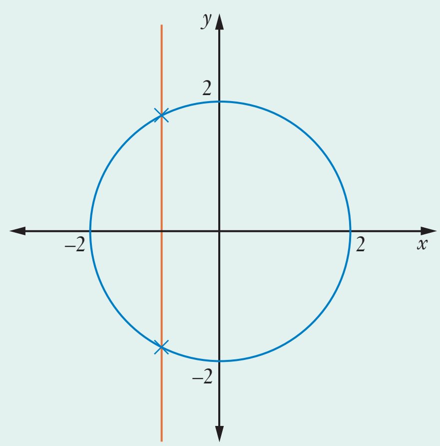
c State its domain and range

Solution

a The equation is in the form $x^2 + y^2 = r^2$ where $r^2 = 4$

Radius $r = \sqrt{4} = 2$

This is a circle with radius 2 and centre (0 0.



b The circle is not a function because a vertical line will cut the graph in more than one place

c The x values for this graph lie between −2 and 2 and the y values also lie between −2 and 2

Domain [−2, 2]

Range [−2, 2]

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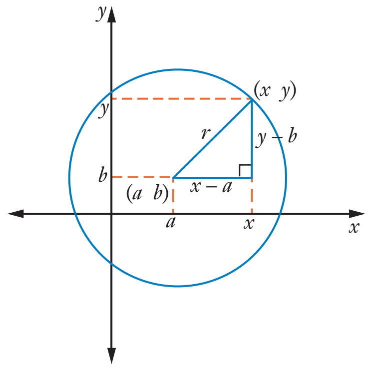
Circle with centre (a, b)

We can use Pythagora’ theorem to find the equation of a circle using a general point $(x \;\; y)$ on a circle with centre $(a \;\; b)$ and radius r

The smaller sides of the triangle are $x - a$ and $y - b$ and the hypotenuse is r the radiu.

$c^2 = a^2 + b^2$

$r^2 = (x - a)^2 + (y - b)^2$



Equation of a circle with centre (a, b)

The equation of a circle with centre $(a \;\; b)$ and radius r is $(x - a)^2 + (y - b)^2 = r^2$

EXAMPLE 9

a **i** Sketch the graph of the circle $(x - 1)^2 + (y + 2)^2 = 4$

ii State its domain and range

b Find the equation of a circle with radius 3 and centre $(-2 \;\; 1)$ in expanded for.

c Find the centre and radius of the circle with equation $x^2 + 2x + y^2 - 6y - 6 = 0$

Solution

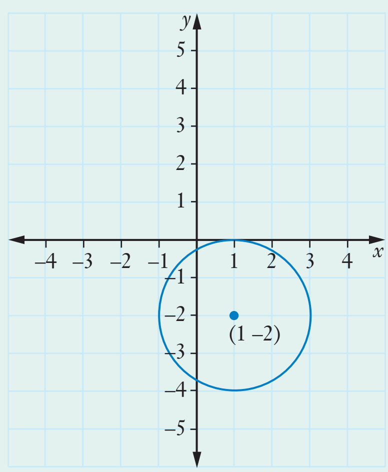
a **i** The equation is in the form $(x - a)^2 + (y - b)^2 = r^2$

$(x - 1)^2 + (y + 2)^2 = 4$

$(x - 1)^2 + (y - (-2))^2 = 2^2$

So $a = 1, b = -2$ and $r = 2$.

This is a circle with centre $(1 \;\; -2)$ and radius 2



ii From the graph we can see that all x values lie between −1 and 3 and all y values lie between −4 and 0

Domain [−1, 3]

Range [−4 0]

b Centre is (−2, 1) so $a = -2$ and $b = 1$.

Radius is 3 so $r = 3$.

$$(x - a)^2 + (y - b)^2 = r^2$$
$$(x - (-2))^2 + (y - 1)^2 = 3^2$$
$$(x + 2)^2 + (y - 1)^2 = 9$$

Expanding

$$x^2 + 4x + 4 + y^2 - 2y + 1 = 9$$
$$x^2 + 4x + y^2 - 2y - 4 = 0$$

c The equation of a circle is $(x - a)^2 + (y - b)^2 = r^2$

We need to complete the square to put the equation into this for.

To complete the square on $x^2 + 2x$ we add $\left(\frac{2}{2}\right)^2 = 1$

To complete the square on $y^2 - 6y$ we add $\left(\frac{6}{2}\right)^2 = 9$

$$x^2 + 2x + y^2 - 6y - 6 = 0$$
$$x^2 + 2x + y^2 - 6y = 6$$
$$x^2 + 2x + 1 + y^2 - 6y + 9 = 6 + 1 + 9$$
$$(x + 1)^2 + (y - 3)^2 = 16$$
$$(x - (-1))^2 + (y - 3)^2 = 4^2$$

This is in the form $(x - a)^2 + (y - b)^2 = r^2$ where $a = -1$, $b = 3$ and $r = 4$

So it is a circle with centre (−1 3) and radius 4 unit.

Semicircles

By rearranging the equation of a circle we can find the equations of 2 semicircle.

$$x^2 + y^2 = r^2$$
$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

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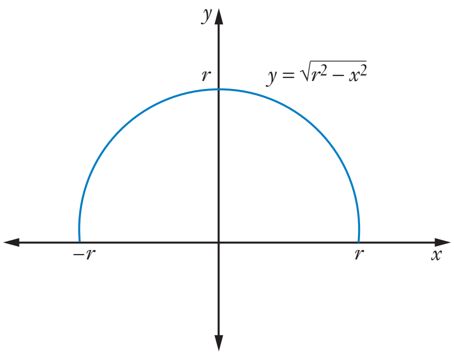
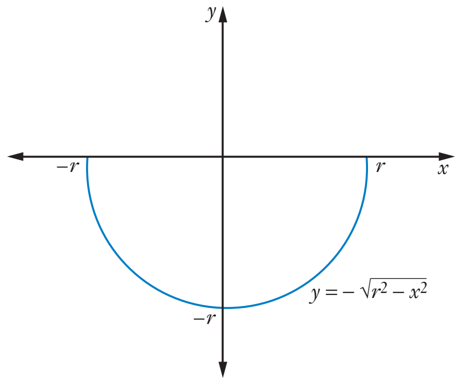
This gives 2 separate functions





$y = \sqrt{r^2 - x^2}$ is the semicircle above the x-axis since $y \geq 0$

$y = -\sqrt{r^2 - x^2}$ is the semicircle below the x-axis since $y \leq 0$

Equations of a semicircle with centre (0, 0)

The equation of a semicircle above the x-axis with centre (0 0) and radius r is

$$y = \sqrt{r^2 - x^2}$$

The equation of a semicircle below the x-axis with centre (0 0) and radius r is

$$y = -\sqrt{r^2 - x^2}$$

EXAMPLE 10

Sketch the graph of each function and state the domain and range

a $f(x) = \sqrt{9 - x^2}$ **b** $y = -\sqrt{4 - x^2}$

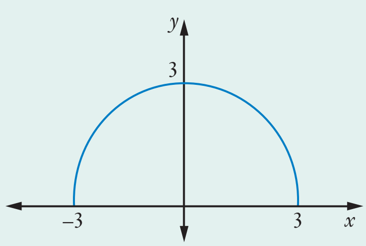
Solution

a This is in the form $f(x) = \sqrt{r^2 - x^2}$ where $r^2 = 9$, so $r = 3$.

It is a semicircle above the x-axis with centre (0 0) and radius .

Domain $[-3, 3]$

Range [0 3]



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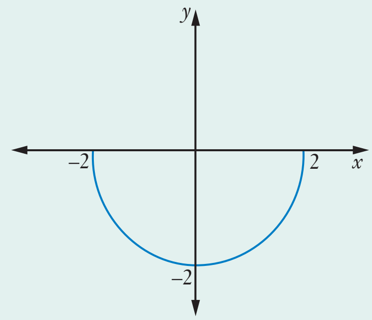
b This is in the form $y = -\sqrt{r^2 - x^2}$ where $r^2 = 4$, so $r = 2$.

It is a semicircle below the x-axis with centre (0 0) and radius .

Domain [−2, 2]

Range [−2 0]



Exercise 5.03 Circles and semicircles

1 For each equation

i sketch the graph

ii state the domain and range

a $x^2 + y^2 = 9$

b $x^2 + y^2 - 16 = 0$

c $(x - 2)^2 + (y - 1)^2 = 4$

d $(x + 1)^2 + y^2 = 9$

e $(x + 2)^2 + (y - 1)^2 = 1$

2 For each semicircle

i state whether it is above or below the x-axis

ii sketch the graph

iii state the domain and range

a $y = -\sqrt{25 - x^2}$

b $y = \sqrt{1 - x^2}$

c $y = \sqrt{36 - x^2}$

d $y = -\sqrt{64 - x^2}$

e $y = -\sqrt{7 - x^2}$

3 Find the radius and the centre of each circle

a $x^2 + y^2 = 100$

b $x^2 + y^2 = 5$

c $(x - 4)^2 + (y - 5)^2 = 16$

d $(x - 5)^2 + (y + 6)^2 = 49$

e $x^2 + (y - 3)^2 = 81$

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4 Find the equation of each circle in expanded form

a centre (0 0) and radius 4 **b** centre (3 2) and radius 5

c centre (–1 5) and radius 3 **d** centre (2 3) and radius 6

e centre (–4 2) and radius 5 **f** centre (0 –2) and radius 1

g centre (4 2) and radius 7 **h** centre (–3, –4) and radius 9

i centre (–2 0) and radius $\sqrt{5}$ **j** centre (–4, –7) and radius $\sqrt{3}$

5 Find the radius and the centre of each circle

a $x^2 - 4x + y^2 - 2y - 4 = 0$ **b** $x^2 + 8x + y^2 - 4y - 5 = 0$

c $x^2 + y^2 - 2y = 0$ **d** $x^2 - 10x + y^2 + 6y - 2 = 0$

e $x^2 + 2x + y^2 - 2y + 1 = 0$ **f** $x^2 - 12x + y^2 = 0$

g $x^2 + 6x + y^2 - 8y = 0$ **h** $x^2 + 20x + y^2 - 4y + 40 = 0$

i $x^2 - 14x + y^2 + 2y + 25 = 0$ **j** $x^2 + 2x + y^2 + 4y - 5 = 0$

6 Find the centre and radius of the circle with equation

a $x^2 - 6x + y^2 + 2y - 6 = 0$ **b** $x^2 - 4x + y^2 - 10y + 4 = 0$

c $x^2 + 2x + y^2 + 12y - 12 = 0$ **d** $x^2 - 8x + y^2 - 14y + 1 = 0$

7 Sketch the circle whose equation is given by $x^2 + 4x + y^2 - 2y + 1 = 0$

5.04 Reflections of functions

The graph of $y = -f(x)$

WS Advanced graph

WS Maching graph (Advanced)

EXAMPLE 11

For each function sketch the graph of $y = f(x)$ and $y = -f(x)$ on the same number plane

a $f(x) = x^2 - 2x$ **b** $f(x) = x^3$

Solution

a $f(x) = x^2 - 2x$ is a concave upwards parabola

For x-intercepts $f(x) = 0$

$x^2 - 2x = 0$

$x(x - 2) = 0$

$x = 0, 2$

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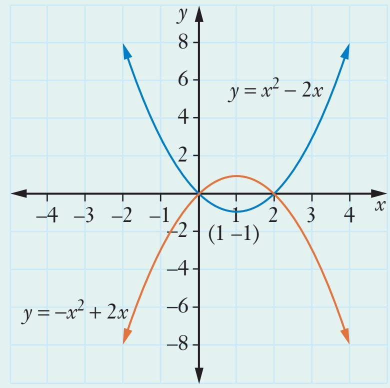
For y-intercept $x = 0$

$f(0) = 0^2 - 2(0) = 0$

Axis of symmetry at $x = 1$ (halfway between 0 and 2)

$f(1) = 1^2 - 2(1) = -1$

Minimum turning point at (1 −1)



$y = -f(x)$

$= -(x^2 - 2x)$

$= -x^2 + 2x$

$y = -x^2 + 2x$ is a concave downwards parabola also with x-intercepts 0 2 with y-intercept 0 and axis of symmetry at $x = 1$.

$f(1) = -1^2 + 2(1) = 1$

Maximum turning point at (1 1.

Draw both graphs on the same set of axes

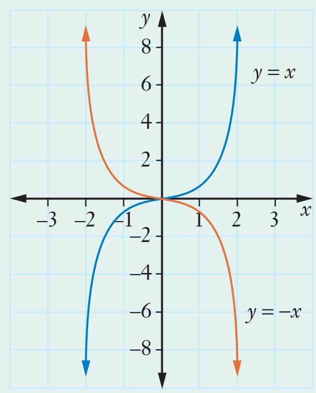
b $f(x) = x^3$ is a cubic curve with a point of inflection at (0 0.

x	−3	−2	−1	0	1	2	3
y	−27	−8	−1	0	1	8	27

$y = -f(x)$

$= -x^3$

x	−3	−2	−1	0	1	2	3
y	27	8	1	0	−1	−8	−27



$y = -f(x)$ changes the sign of the y values of the original function from positive to negative or negative to positive On the number plan, this means reflecting the graph in the x-axis

The graph of $y =\ \ f(x)$

The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis

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The graph of $y = f(-x)$

We have already seen that some functions are even or odd by finding $f(-x)$ We can see the relationship between $f(x)$ and $f(-x)$ by drawing their graphs

EXAMPLE 12

For each function sketch the graph of $y = f(x)$ and $y = f(-x)$ on a number plane

a $f(x) = x^3 + 1$ **b** $f(x) = \dfrac{1}{x-2}$

Solution

a $f(x) = x^3 + 1$ is a cubic curve with point of inflection at (0 1.

x	−3	−2	−1	0	1	2	3
y	−26	−7	0	1	2	9	28

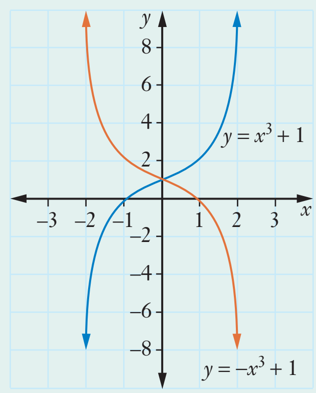
$y = f(-x)$

$= (-x)^3 + 1$

$= -x^3 + 1$

x	−3	−2	−1	0	1	2	3
y	28	9	2	1	0	−7	−26

Draw $y = x^3 + 1$ and $y = -x^3 + 1$ on the same set of axes



b $f(x) = \dfrac{1}{x-2}$ is a hyperbola with asymptotes at $x = 2$ and $y = 0$

y-intercept $x = 0$

$f(0) = \dfrac{1}{0-2} = -\dfrac{1}{2}$

$y = f(-x)$

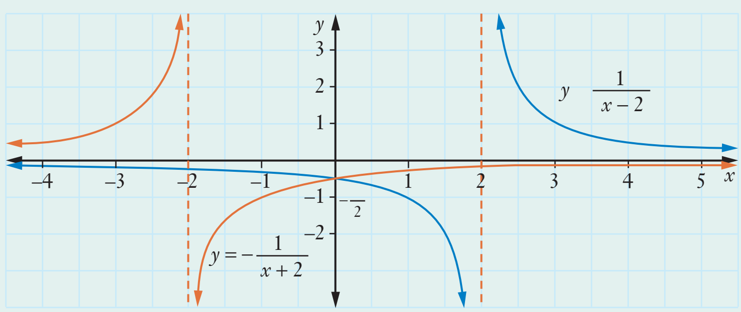
$= \dfrac{1}{-x-2}$

$= -\dfrac{1}{x+2}$

Asymptotes at $x = -2$ and $y = 0$

y-intercept $x = 0$

$f(0) = -\dfrac{1}{0+2} = -\dfrac{1}{2}$



Draw $y = \dfrac{1}{x-2}$ and $y = -\dfrac{1}{x+2}$ on the same set of axes

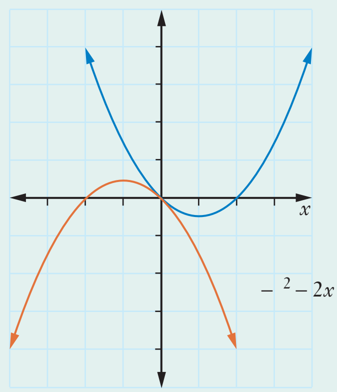
$y = f(-x)$ changes the sign of the x value of the original function from positive to negative or negative to positive On the number plan, this means reflecting the graph in the y-axis

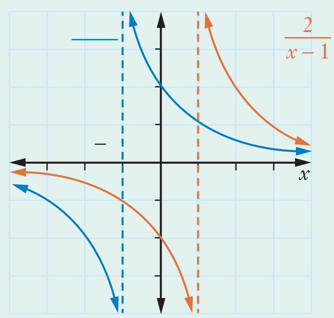
The graph of $y = f(-x)$

The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis

EXAMPLE 13

hehf – – lane





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The graph of $y =\ \ f(\ \ x)$

$y = -f(-x)$ is a reflection of the graph of $y = f(x)$ in both the x- and y-axes

CLASS DISCUSSION

REFLECTIONS OF FUNCTIONS

Use a graphics calculator or graphing software to draw the graphs of different functions $y = f(x)$ together with

1 $y = -f(x)$

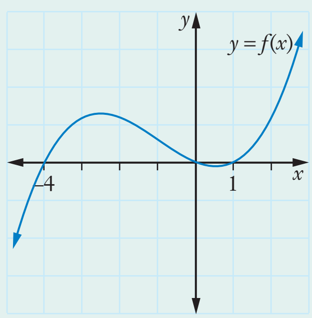
2 $y = f(-x)$

3 $y = -f(-x)$

Are any of these functions the same as $y = f(x)$ if $f(x)$ is an even or odd function? Why?

EXAMPLE 14

The graph of $y = f(x)$ is shown below.

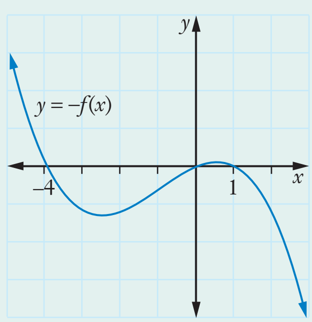


Sketch the graph of

a $y = -f(x)$ **b** $y = f(-x)$ **c** $y = -f(-x)$

Solution

a $y = -f(x)$ is a reflection in the x-axis



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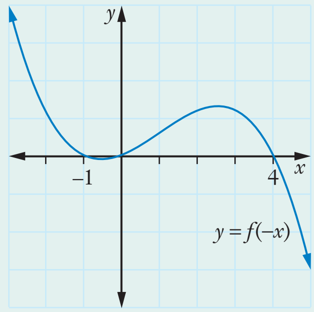
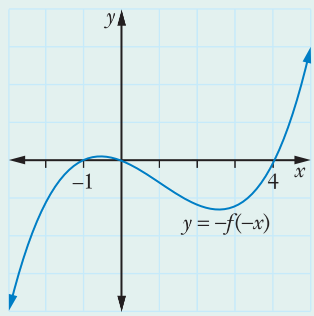
b $y = f(-x)$ is a reflection in the y-axis



c $y = -f(-x)$ is a reflection in both the x-and y-axes Using the graph from **a** that has been reflected in the x-axis and reflecting it in the y-axis gives the graph below.



Exercise 5.04 Reflections of functions

1 For each function find the equation o:

i $y = -f(x)$ **ii** $y = f(-x)$ **iii** $y = -f(-x)$

a $f(x) = x^2 - 2$ **b** $f(x) = (x + 1)^3$ **c** $y = 5x - 3$

d $y = 2x + 5|$ **e** $f(x) = \dfrac{1}{x - 1}$

2 Describe the type of reflection that each function has on $y = f(x)$

a $y = -f(x)$ **b** $y = f(-x)$ **c** $y = -f(-x)$

3 Sketch the graphs of the function $f(x) = (x - 1)^2$ and $y = -f(-x)$ on the same number plane

4 Sketch the graphs of the function $f(x) = 1 - x^3$ and $y = -f(x)$ on the same number plane

5 For the function $f(x) = x^2 + 2x$ sketch the graph o:

a $y = f(x)$ **b** $y = -f(x)$ **c** $y = f(-x)$ **d** $y = -f(-x)$

6 **a** Show that $f(x) = 2x^2$ is an even function

b Find the equation of

i $y = f(-x)$ **ii** $y = -f(x)$

c Sketch the graph of $y = -f(-x)$

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7 **a** Show that $f(x) = -x^3$ is an odd function

b Find the equation of

i $y = -f(x)$ **ii** $y = -f(-x)$

c Sketch the graph of $y = f(-x)$

8 **a** Find the x- and y-intercepts of the graph of $f(x) = x^3 - 7x^2 + 12x$ and sketch the graph

b Sketch the graph of

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = -f(-x)$

5.05 Combined and composite functions

Sometimes we use different operations to combine 2 different functions

WS

Cicles and compoie uncion

EXAMPLE 15

For $f(x) = 2x^2 - x + 1$ and $g(x) = x^3 - 2$ write each combined function below as a polynomial and find its degree and constant term

a $y = f(x) + g(x)$ **b** $y = f(x) - g(x)$ **c** $y = f(x)g(x)$

Solution

a $y = f(x) + g(x)$

$= 2x^2 - x + 1 + x^3 - 2$

$= x^3 + 2x^2 - x - 1$

This polynomial has degree 3 and constant term –1

b $y = f(x) - g(x)$

$= 2x^2 - x + 1 - (x^3 - 2)$

$= 2x^2 - x + 1 - x^3 + 2$

$= -x^3 + 2x^2 - x + 3$

This polynomial has degree 3 and constant term 3

c $y = f(x)g(x)$

$= (2x^2 - x + 1)(x^3 - 2)$

$= 2x^5 - 4x^2 - x^4 + 2x + x^3 - 2$

$= 2x^5 - x^4 + x^3 - 4x^2 + 2x - 2$

This polynomial has degree 5 and constant term –2

> We could also ind the degree by multplyng ust the 2 leadng term: $2x^2 \times x^3 = 2x^5$ and fnd the constant term by muliplingjust the 2 constant term: 1 × (–2 = (–2

EXAMPLE 16

a Find the domain and range of each function below given $f(x) = x^2 - x - 2$ and $g(x) = x - 2$.

i $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$ **iii** $y = f(x)g(x)$

b Find the domain of $y = \frac{f(x)}{g(x)}$ if $f(x) = x^3 + 1$ and $g(x) = x^2 - x - 6$

Solution

a **i** $y = f(x) + g(x)$

$= x^2 - x - 2 + x - 2$

$= x^2 - 4$

This is a quadratic function with a minimum turning point at (0 –4)

Domain (–∞ ∞) range [–4, ∞)

ii $y = f(x) - g(x)$

$= x^2 - x - 2 - (x - 2)$

$= x^2 - x - 2 - x + 2$

$= x^2 - 2x$

This is a quadratic function with x-intercepts 0 .

Axis of symmetry $x = 1$

Minimum value

$f(1) = 1^2 - 2(1)$

$= -1$

Domain (–∞ ∞) range [–1, ∞)

iii $y = f(x)g(x)$

$= (x^2 - x - 2)(x - 2)$

$= x^3 - 2x^2 - x^2 + 2x - 2x + 4$

$= x^3 - 3x^2 + 4$

This is a cubic function

Domain (–∞ ∞) range (–∞ ∞)

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b $y = \dfrac{f(x)}{g(x)}$

$= \dfrac{x^3+1}{x^2-x-6}$

For domain $x^2 - x - 6 \neq 0$

$(x-3)(x+2) \neq 0$

$x \neq 3, -2$

So domain is $(-\infty\ -2) \cup (-2, 3) \cup (3, \infty)$

Composite functions

A **composite function** $f(g(x))$ is a relationship between functions where the output of one function $g(x)$ becomes the input of a second function $f(x)$

EXAMPLE 17

a Find the composite function $f(g(x))$ given

i $f(x) = x^2$ and $g(x) = 2x - 5$

ii $f(x) = x^3$ and $g(x) = x^2 + 3$

iii $f(x) = 5x - 3$ and $g(x) = x^3 + 2$

b Given $f(x) = 5x + 2$ and $g(x) = \dfrac{1}{x}$ fin:

i $f(g(x))$ **ii** $g(f(x))$

c Find the domain and range of $f(g(x))$ given $f(x) = \sqrt{x}$ and $g(x) = 9 - x^2$

Solution

a **i** $f(g(x)) = (2x-5)^2$

$= 4x^2 - 20x + 25$

ii $f(g(x)) = (x^2+3)^3$

$= (x^2+3)(x^2+3)^2$

$= (x^2+3)(x^4+6x^2+9)$

$= x^6 + 9x^4 + 27x^2 + 27$

iii $f(g(x)) = 5(x^3+2) - 3$

$= 5x^3 + 10 - 3$

$= 5x^3 + 7$

b **i** $f(g(x)) = 5\left(\dfrac{1}{x}\right) + 2$

$= \dfrac{5}{x} + 2$

ii $g(f(x)) = \dfrac{1}{5x+2}$

c $f(g(x)) = \sqrt{9 - x^2}$

This is a semicircle above the x-axis with centre (0 0) and radius .

Domain [−3, 3], range [0, 3]

Exercise 5.05 Combined and composite functions

1 For each pair of functions find the combined functio:

i $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$

iii $y = f(x)g(x)$ **v** $y = \dfrac{f(x)}{g(x)}$

a $f(x) = 4x + 1$ and $g(x) = 2x^2 + x$

b $f(x) = x^4 + 5x - 4$ and $g(x) = x^3 + 5$

c $f(x) = x^2 + 3$ and $g(x) = 5x^2 - 7x - 2$

d $f(x) = 3x^2 + 2x - 1$ and $g(x) = x^2 - x + 5$

e $f(x) = 4x^5 + 7$ and $g(x) = 3x - 4$

2 For each pair of functions find the degree o:

i $f(x) + g(x)$ **ii** $f(x) - g(x)$ **iii** $f(x)g(x)$ without expanding

a $f(x) = 2x + 1$ and $g(x) = 5x - 7$ **b** $f(x) = x^2$ and $g(x) = 3x + 4$

c $f(x) = (x - 3)^2$ and $g(x) = x^2 - 6x + 1$ **d** $f(x) = 2x^3$ and $g(x) = x - 2$

3 For each pair of functions find the constant term o:

i $f(x) + g(x)$ **ii** $f(x) - g(x)$ **iii** $f(x)g(x)$ without expanding

a $f(x) = 5x^2 + 4$ and $g(x) = x - 7$ **b** $f(x) = 3x^2 + 1$ and $g(x) = 2x - 5$

c $f(x) = (2x - 5)^2$ and $g(x) = 4x - 3$ **d** $f(x) = x^3 + 7$ and $g(x) = 2x^2$

4 Find the domain and range of $y = f(x) + g(x)$ given

a $f(x) = x + 2$ and $g(x) = x - 4$ **b** $f(x) = 2x^2 + x - 1$ and $g(x) = -x - 1$

c $f(x) = x^3$ and $g(x) = x + 2$ **d** $f(x) = x^2 - 1$ and $g(x) = x - 1$

5 Find the domain and range of $y = f(x) - g(x)$ given

a $f(x) = 3x + 2$ and $g(x) = x - 1$ **b** $f(x) = x^2 - 1$ and $g(x) = x - 1$

c $f(x) = x^3 + x$ and $g(x) = x + 2$ **d** $f(x) = 3x^2 - x -1$ and $g(x) = x^2 + x + 3$

6 Find the domain and range of $y = f(x)g(x)$ given

a $f(x) = x + 2$ and $g(x) = x - 4$ **b** $f(x) = x - 5$ and $g(x) = x + 5$

c $f(x) = x^2$ and $g(x) = x$

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7 Find the domain of $y = \dfrac{f(x)}{g(x)}$ given

a $f(x) = 5$ and $g(x) = x - 4$

b $f(x) = x - 1$ and $g(x) = x + 1$

c $f(x) = 2x$ and $g(x) = x - 3$

d $f(x) = x + 3$ and $g(x) = x^3$

8 Find the composite function $f(g(x))$ given

a $f(x) = x^2$ and $g(x) = x^2 + 1$

b $f(x) = x^3$ and $g(x) = 5x - 3$

c $f(x) = x^7$ and $g(x) = x^2 - 3x + 2$

d $f(x) = \sqrt{x}$ and $g(x) = 2x - 1$

e $f(x) = \sqrt[3]{x}$ and $g(x) = x^4 + 7x^2 - 4$

f $f(x) = 3x$ and $g(x) = 2x + 1$

g $f(x) = 2x - 7$ and $g(x) = x^3$

h $f(x) = 6x - 5$ and $g(x) = x^2$

i $f(x) = 2x^2$ and $g(x) = 3x$

j $f(x) = 4x^2 + 1$ and $g(x) = x^2 + 3$

9 Find the domain and range of the composite function $f(g(x))$ given that

a $f(x) = x^2$ and $g(x) = x - 1$

b $f(x) = x^3$ and $g(x) = x + 5$

c $f(x) = \sqrt{x}$ and $g(x) = x - 2$

d $f(x) = -\sqrt{x}$ and $g(x) = 3x + 9$

e $f(x) = \sqrt{x}$ and $g(x) = 4 - x^2$

f $f(x) = -\sqrt{x}$ and $g(x) = 1 - x^2$

10 If $f(x) = \sqrt{x}$ and $g(x) = x^3$ fin:

a $f(g(x))$

b $g(f(x))$

11 If $f(x) = \dfrac{1}{x}$ and $g(x) = x^2 + 3$, fin:

a $y = f(x)g(x)$

b $y = f(g(x))$

c $y = \dfrac{f(x)}{g(x)}$

d $y = \dfrac{g(x)}{f(x)}$

5. TEST YOURSELF

Qz

Pacice quiz

For Questions **1** to **3** select the correct answer **A B C** or **D**

1 The domain of $y = -\frac{3}{x-4}$ is

A (-4) **B** $(-\infty\ 4) \cup (4, \infty)$

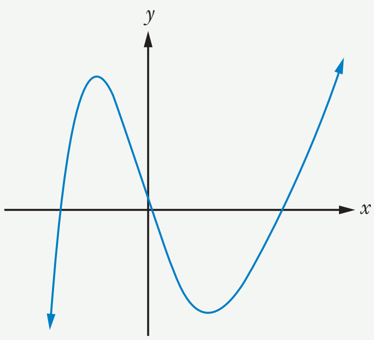
C $(-\infty\ -4) \cup (-4, \infty)$ **D** $(-\infty\ 4)$

2 The equation of a circle with radius 3 and centre $(1\ -2)$ is

A $(x-1)^2 + (y+2)^2 = 9$ **B** $(x+1)^2 + (y-2)^2 = 9$

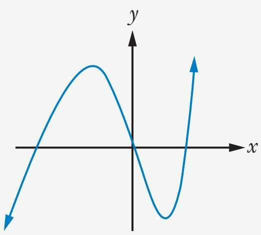
C $(x-1)^2 + (y+2)^2 = 3$ **D** $(x+1)^2 + (y-2)^2 = 3$

3 The graph of $y = f(x)$ is shown below.

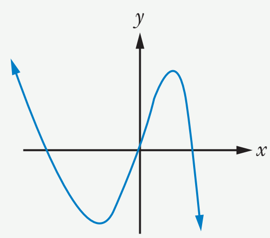


Which one of these is the graph of $y = -f(-x)$?

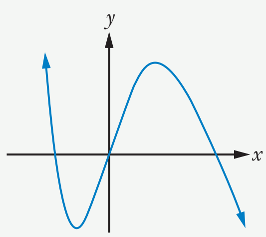
A



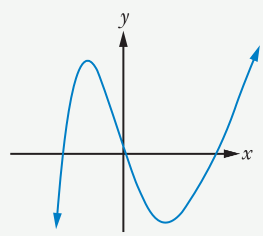
B



C



D



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4 The area of a pizza slice decreases as the number of people sharing it evenly increases When 5 people share the pizza the area of each slice is 30 cm ^{2}

a Find the equation of the area A of a pizza slice in terms of the number of people sharing n

b What is the area of one pizza slice when

i 10 people share? **ii** 8 people share?

c How many people are sharing the pizza when each slice has an area of

i 16 67 cm ^{2}? **ii** 25 cm^2?

5 Sketch the graph of each function or relation

a $x^2 + y^2 = 1$ **b** $y = \frac{2}{x}$ **c** $y = |x + 2|$

d $y = -\sqrt{4 - x^2}$ **e** $y = f(-x)$ given $f(x) = \frac{2}{x-1}$

f $y = -f(x)$ if $f(x) = 3x - 6$ **g** $y = -f(-x)$ given $f(x) = x^2 + x$

6 Find the radius and centre of the circle $x^2 - 6x + y^2 - 2y - 6 = 0$

7 If $f(x) = x^3$ and $g(x) = 3x - 1$ find the equation o:

a $y = f(x) + g(x)$ **b** $y = f(x)g(x)$

c $y = f(g(x))$ **d** $y = g(f(x))$

8 **a** Is the circle $x^2 + y^2 = 1$ a function?

b Change the subject of the equation to y in terms of x

c Sketch the graphs of 2 separate functions that together make up the circle $x^2 + y^2 = 1$.

9 Find the domain and range of each relation

a $x^2 + y^2 = 16$ **b** $y = \frac{1}{x+2}$

c $f(x) = |x| + 3$ **d** $y = \sqrt{9 - x^2}$

10 Find the domain and range of $y = f(x) + g(x)$ given $f(x) = x^2 - 4x$ and $g(x) = 2x - 3$.

11 **a** Write down the domain and range of the curve $y = \frac{2}{x-3}$

b Sketch the graph of $y = \frac{2}{x-3}$

12 **a** Sketch the graph of $y = |x + 1|$

b From the graph solve $|x + 1| = 3$

13 Solve graphically $|x - 3| = 2$.

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14 Find the centre and radius of the circle with equation

a $x^2 + y^2 = 100$

b $(x - 3)^2 + (y - 2)^2 = 121$

c $x^2 + 6x + y^2 + 2y + 1 = 0$

15 Find the x- and y-intercepts (where they exist) of

a $P(x) = x^3 - 4x$ **b** $y = -\frac{2}{x+1}$ **c** $x^2 + y^2 = 9$

d $y = \sqrt{25 - x^2}$ **e** $f(x) = |x - 2| + 3$

16 If $f(x) = 2x^2 + x - 6$ and $g(x) = 5x^3 + 1$, fin:

a the degree of $y = f(x) + g(x)$

b the leading term of $y = f(x)g(x)$

c the constant term of $y = f(x) - g(x)$

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5. CHALLENGE EXERCISE

1 Solve $|2x + 1| = 3x - 2$ graphically.

2 Given $f(x) = |x| + 3x - 4$ sketch the graph o:

a $y = f(x)$ **b** $y = -f(x)$

3 A variable a is inversely proportional to the square of b When $b = 3$, $a = 2$.

a Find the equation of a in terms of b

b Evaluate a when $b = 2$.

c Evaluate b when $a = 10$ correct to 2 decimal place, if $b > 0$

4 Find the centre and radius of the circle with equation given by $x^2 + 3x + y^2 - 2y - 3 = 0$

5 Find the equation of the straight line through the centres of the circles with equations $x^2 + 4x + y^2 - 8y - 5 = 0$ and $x^2 - 2x + y^2 + 10y + 10 = 0$

6 Sketch the graph of $y = \frac{|x|}{x^2}$

7 **a** Show that $\frac{2x+7}{x+3} = 2 + \frac{1}{x+3}$

b Find the domain and range of $y = \frac{2x+7}{x+3}$

c Hence sketch the graph of $y = \frac{2x+7}{x+3}$

8 Show that $x^2 - 2x + y^2 + 4y + 1 = 0$ and $x^2 - 2x + y^2 + 4y - 4 = 0$ are concentric

9 Sketch the graph of $f(x) = 1 - \frac{1}{x^2}$

10 **a** Sketch the graph of $f(x) = \begin{cases} x & \text{for } x < -2 \\ x^2 & \text{for } -2 \le x \le 0 \\ 2 & \text{for } x > 0 \end{cases}$

b Find any x values for which the function is discontinuous

c Find the domain and range of the function

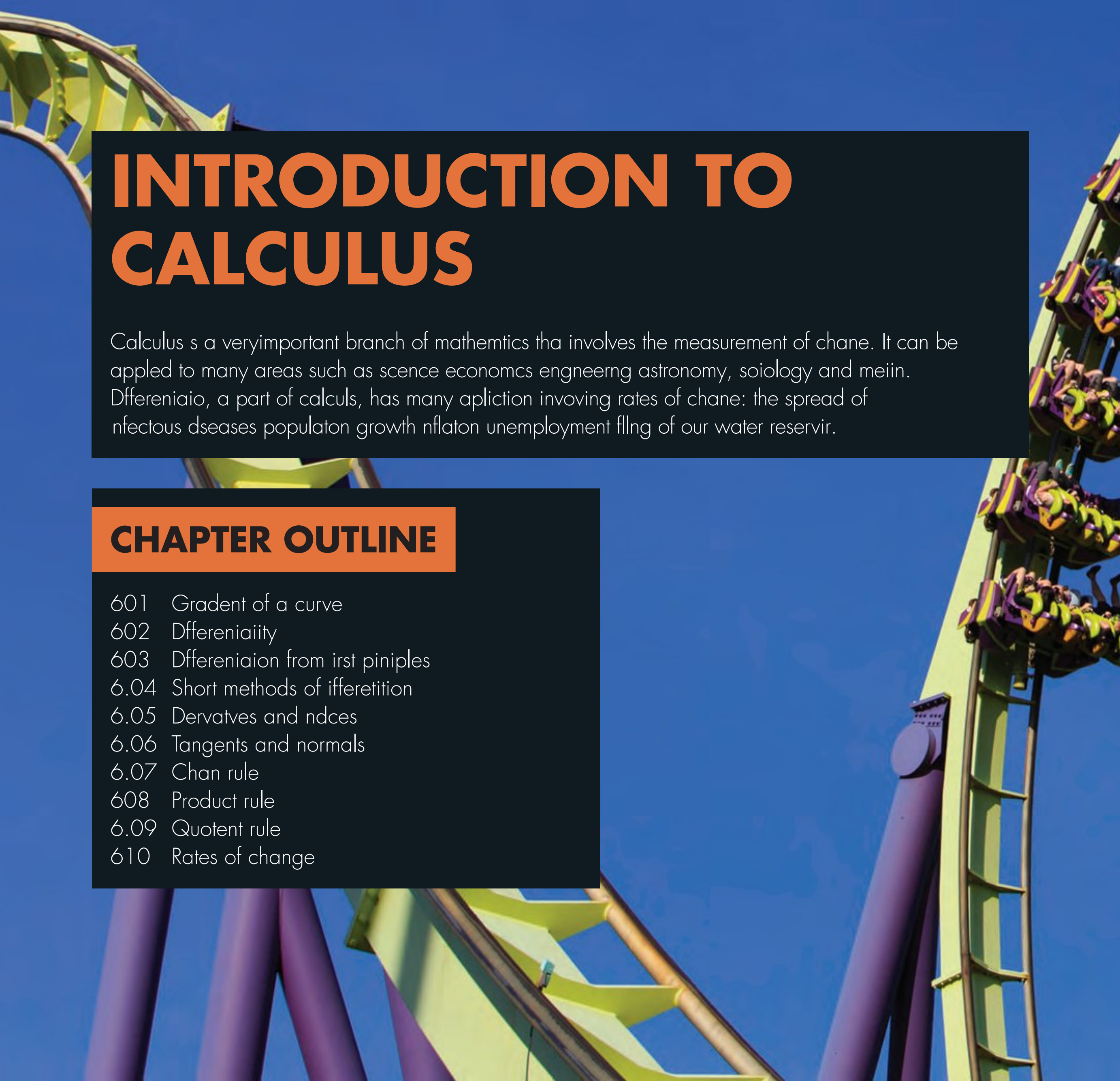
6.

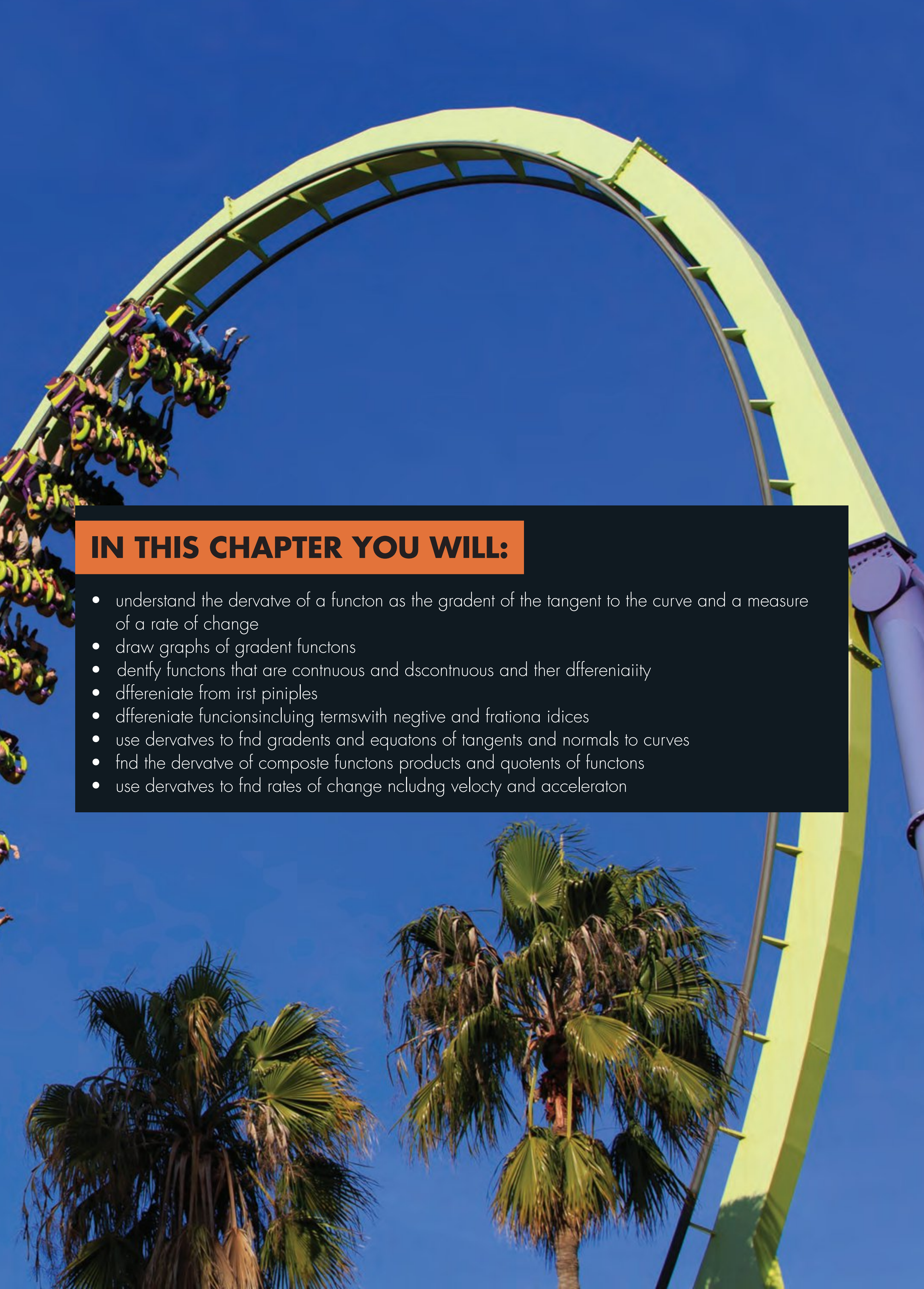
INTRODUCTION TO CALCULUS

Calculus s a veryimportant branch of mathemtics tha involves the measurement of chane. It can be appled to many areas such as scence economcs engneerng astronomy, soiology and meiin. Dffereniaio, a part of calculs, has many apliction invoving rates of chane: the spread of nfectous dseases populaton growth nflaton unemployment fllng of our water reservir.

CHAPTER OUTLINE

601 Gradent of a curve
602 Dffereniaiity
603 Dffereniaion from irst piniples
6.04 Short methods of ifferetition
6.05 Dervatves and ndces
6.06 Tangents and normals
6.07 Chan rule
608 Product rule
6.09 Quotent rule
610 Rates of change





IN THIS CHAPTER YOU WILL:

- understand the dervatve of a functon as the gradent of the tangent to the curve and a measure of a rate of change
- draw graphs of gradent functons
- dentfy functons that are contnuous and dscontnuous and ther dfferenaiity
- dfferenate from irst pniples
- dfferenate funcionsincluing termswith negtive and frationa idices
- use dervatves to fnd gradents and equatons of tangents and normals to curves
- fnd the dervatve of composte functons products and quotents of functons
- use dervatves to fnd rates of change ncludng velocty and acceleraton

TERMINOLOGY

acceleratio: The rate of change of velocity with respect to time

average rate of cange: The rate of change between 2 points on a function the gradient of the line (secant) passing through those points

chain rule A method for differentiating composite functions

derivative function The gradient function $y = f\ (x)$ of a function $y = f(x)$ obtained through differentiation

differentiability A function is differentiable wherever its gradient is defined

differentiation The process of finding the gradient function

differentiation from first principles The process of finding the gradient of a tangent to a curve by finding the gradient of the secant between 2 points and finding the limit as the secant becomes a tangent

displacement The distance and direction of an object in relation to the origin

gradient of a secant The gradient (slope) of the line between 2 points on a function measures the average rate of change between the 2 points

gradient of a tangent The gradient of a line that is a tangent to the curve at a point on a functio; measures the instantaneous rate of change of the function at that point

instantaneous rate of chane: The rate of change at a particular point on a function the gradient of the tangent at this point

limit The value that a function approaches as the independent variable approaches some value

normal A line that is perpendicular to the tangent at a given point on a curve

product rule A method for differentiating the product of 2 functions

quotient rule A method for differentiating the quotient of 2 functions

secant A straight line passing through 2 points on the graph of a function

stationary point A point on the graph of $y = f(x)$ where the tangent is horizontal and its gradient $f\ (x) = 0$ It could be a maximum point minimum point or a horizontal point of inflection

tangent A straight line that just touches a curve at one point The curve has the same gradient or direction as the tangent at that point

turning point A maximum or minimum point on a curve where the curve turns around

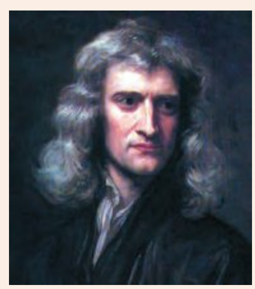
velocity The rate of change of displacement of an object with respect to time involves speed and direction

DID YOU KNOW?

Newton and Leibniz

Calculus comes from the Latin meaning 'pebbl' or'small sto'. In many ancient civilisations stones were used for counting but the mathematics they practised was quite sophisticated

It was not until the 17th century that there was a breakthrough in calculus when scientists were searching for ways of measuring motion of objects such as planets pendulums and projectile.


Isaac Newon

Isaac Newton (1642–1727) an Englishma, discovered the main principles of calculus when he was 23 years old At this time an epidemic of bubonic plague had closed Cambridge University where he was studying so many of his discoveries were made at hom. He first wrote about his calculus methods which he called fluxion, in 161, but his *Method of fluxions* was not published until 1704

Gottfried Leibniz (1646–1716) in German, was studying the same methods and there was intense rivalry between the two countries over who was first to discover calculus

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6.01 Gradient of a curve

The **gradient** of a straight line measures the **rate of change** of y (the dependent variable) with respect to the change in x (the independent variable)

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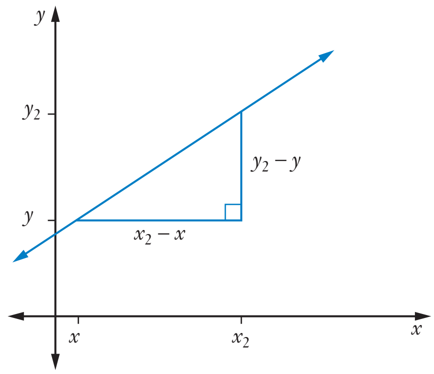

Gradient

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Notice that when the gradient of a straight line is positive the line is increasing and when the gradient is negative the line is decreasing Straight lines increase or decrease at a constant rate and the gradient is the same everywhere along the line

CLASS DISCUSSION

Remember that an **increasing** line has a positive gradient and a **decreasing** line has a negative gradient

positive negatve

What is the gradient of a horizontal line?

Can you find the gradient of a vertical line? Why?

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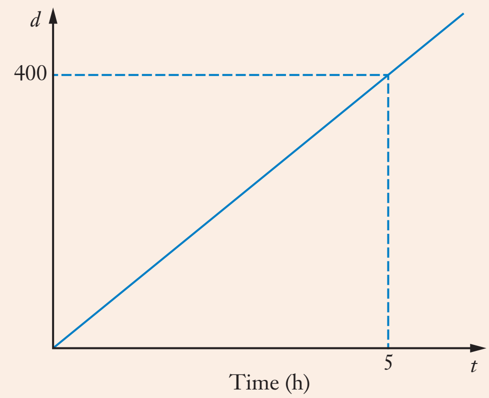
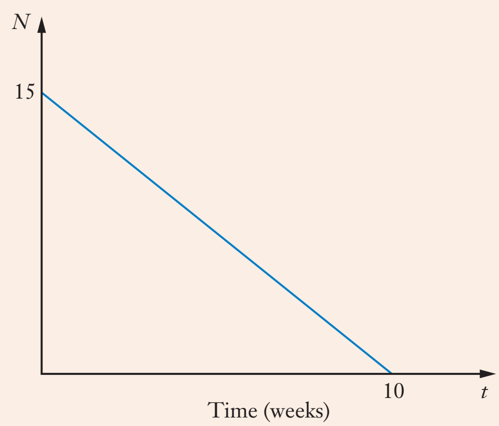
EXAMPLE 1

a The graph shows the distance travelled by a car over time Find the gradient and describe it as a rate



b The graph shows the number of cases of flu reported in a town over several weeks Find the gradient and describe it as a rate



Solution

a $m = \frac{\text{rise}}{\text{run}}$

The lne s ncreasng, so t has a postve gradent

$= \frac{400}{5}$

$= 80$

This means that the car is travelling at a constant rate (speed) of 80 km/h

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b $m = \dfrac{\text{rise}}{\text{run}}$

$= \dfrac{-1500}{10}$

$= -150$

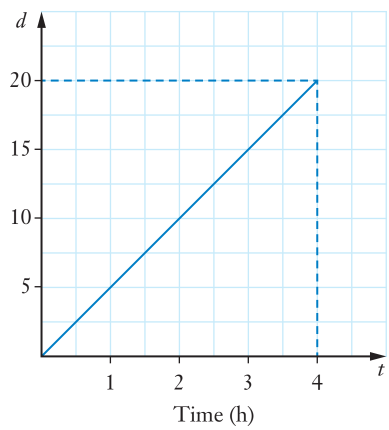
The lne s decreasng, so t wll have a negatve gradent

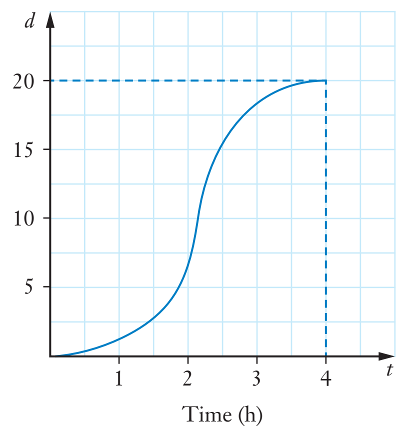
The ‘ris’ is a drop so i’s negatie.

This means that the rate is –150 cases/week or the number of cases reported is decreasing by 150 cases/week

CLASS DISCUSSION

The 2 graphs below show the distance that a bicycle travels over time One is a straight line and the other is a curve





Is the average speed of the bicycle the same in both cases? What is different about the speed in the 2 graphs?

How could you measure the speed in the second graph at any one time? Does it change? If so how does it change ?

We can start finding rates of change along a curve by looking at its shape and how it behave. We started looking at this in Chapter , *Functions*

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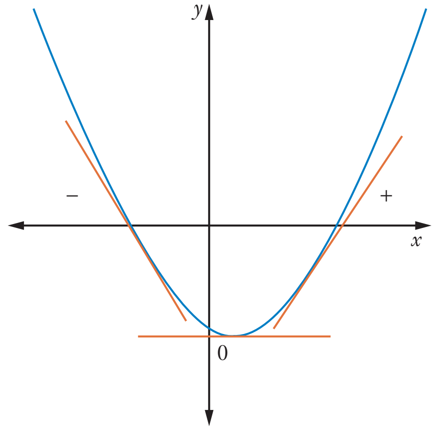
The gradient of a curve shows the **rate of change of *y*** as *x* change. A **tangent** to a curve is a straight line that just touches the curve at one point We can see where the gradient of a curve is positive negative or zero by drawing **tangents to the curve** at different places around the curve and finding the gradients of the tangents

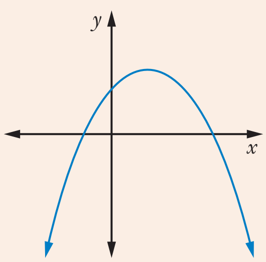
Notice that when the curve increases it has a positive gradient when it decreases it has a negative gradient and when it is a **turning point** the gradient is zero



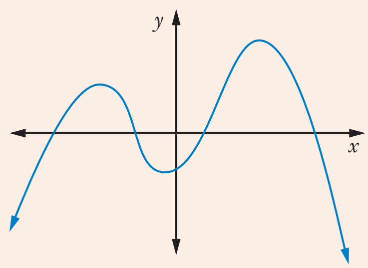
EXAMPLE 2

Copy each curve and write the sign of its gradient along the curve

a



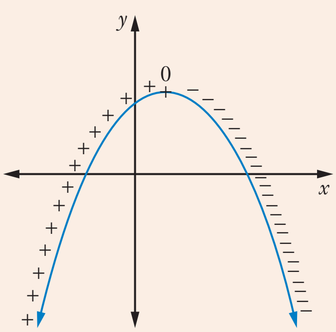
b



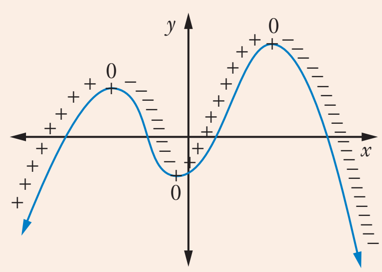
Solution

Where the curve increases the gradient is positiv. Where it decreaes, it is negaive. Where it is a turning point it has a zero gradien.

a



b



We find the gradient of a curve by measuring the **gradient of a tangent** to the curve at different points around the curve

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We can then sketch the graph of these gradient value, which we call $y = f'(x)$ the **gradient function** or the **derivative function**

EXAMPLE 3

a Make an accurate sketch of $f(x) = x^2$ on graph paper, or use graphing softwae.

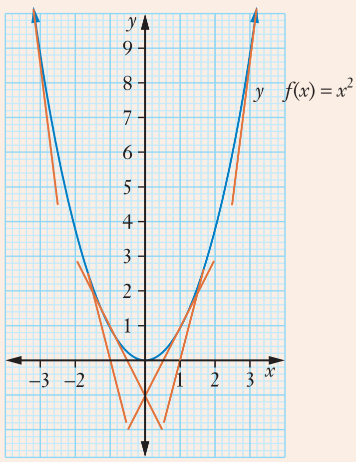
b Draw tangents to this curve at the points where $x = –3$, $x = –2$ $x = –1$, $x = 0$, $x = 1$, $x = 2$ and $x = 3$.

c Find the gradient of each of these tangents

d Draw the graph of $y = f'(x)$ (the derivative or gradient function)

Solution

a and **b**



c At $x = –3$, $m = –6$

At $x = –2$ $m = –4$

At $x = –1$, $m = –2$

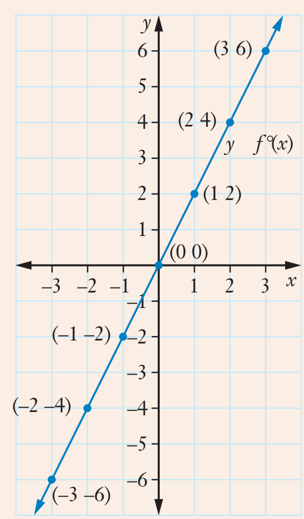
At $x = 0$, $m = 0$

At $x = 1$, $m = 2$

At $x = 2$, $m = 4$

At $x = 3$, $m = 6$

d Using the values from part **c**
$y = f'(x)$ is a linear function



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Notice in Example 3 that where $m > 0$ the gradient function is above the x-axis where $m = 0$, the gradient function is on the x-axis and where $m < 0$ the gradient function is below the x-axis Since $m = f'(x)$ we can write the followin:

Sketching gradient (derivative) functions

$f'(x) > 0$ gradient function is above the x-axis

$f'(x) < 0$ gradient function is below the x-axis

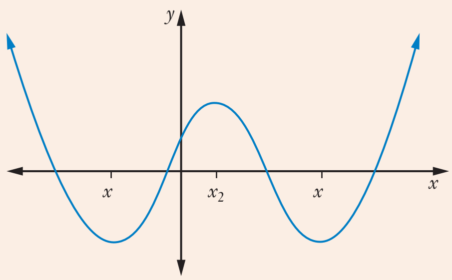
$f'(x) = 0$ gradient function is on the x-axis

Skeching gadien uncion

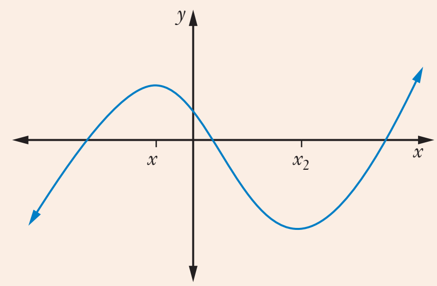
EXAMPLE 4

Sketch a gradient function for each curve

a

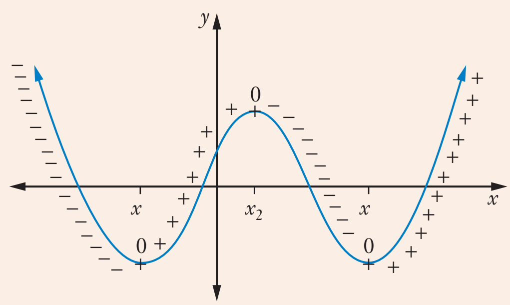


b

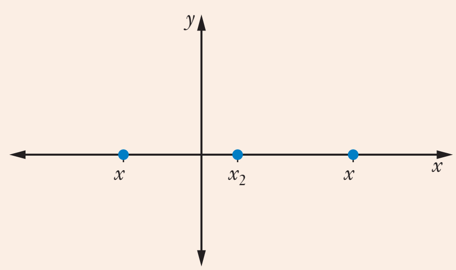


Solution

a First we mark in where the gradient is positive negative and zer.



$f'(x) = 0$ at x x_2 and x_3 so on the gradient graph these points will be on the x-axis (the x-intercepts of the gradient graph)



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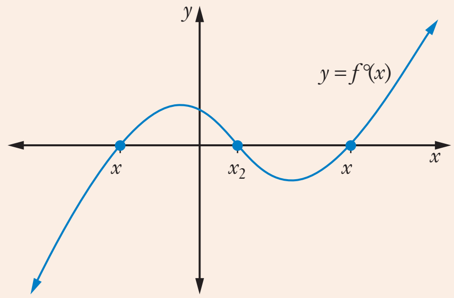
$f'(x) < 0$ to the left of x so this part of the gradient graph will be below the x-axis

$f'(x) > 0$ between x and x_2 so the graph will be above the x-axis here

$f'(x) < 0$ between x_2 and x_3 so the graph will be below the x-axis here



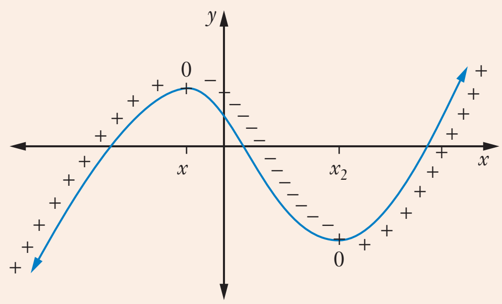
$f'(x) > 0$ to the right of x_3 so this part of the graph will be above the x-axis

Sketching this information gives the graph of the gradient function $y = f'(x)$. Note that this is only a rough graph that shows the shape and sign rather than precise values

b First mark in where the gradient is positive negative and zer.

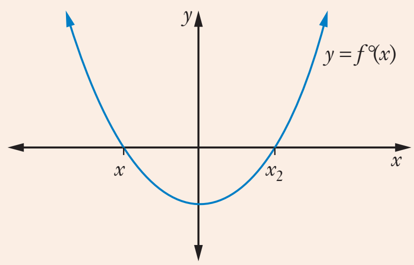
$f'(x) = 0$ at x and x_2 These points will be the x-intercepts of the gradient function graph

$f'(x) > 0$ to the left of x so the graph will be above the x-axis here



$f'(x) < 0$ between x and x_2 so the graph will be below the x-axis here

$f'(x) > 0$ to the right of x_2 so the graph will be above the x-axis here



TECHNOLOGY

Tangents to a curve

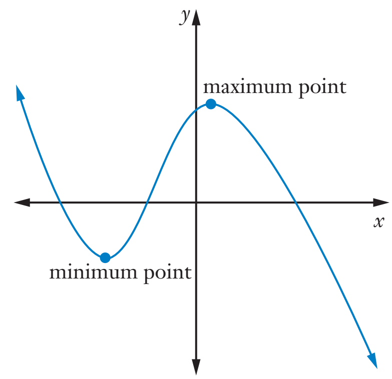
There are some excellent graphing software online apps and websites that will draw tangents to a curve and sketch the gradient function

Explore how to sketch gradient functions from the previous examples

Stationary points

The points on a curve where the gradient $f'(x) = 0$ are called **stationary points** because the gradient there is neither increasing nor decreasing

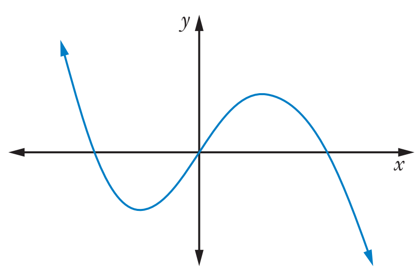
For example the curve shown decreases to a **minimum turning point** which is a type of **stationary point** It then increases to a **maximum turning point** (also a stationary point) and then decreases again



Exercise 6.01 Gradient of a curve

Sketch a gradient function for each curve

1



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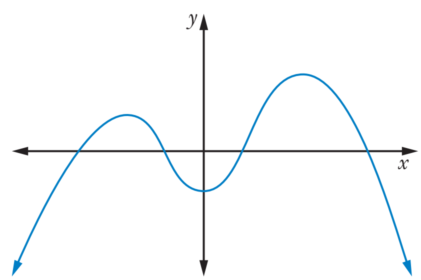

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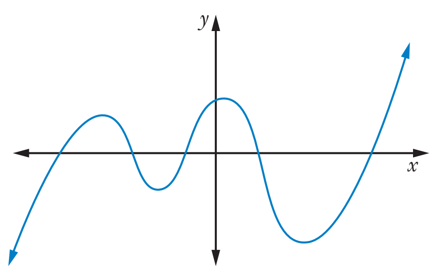

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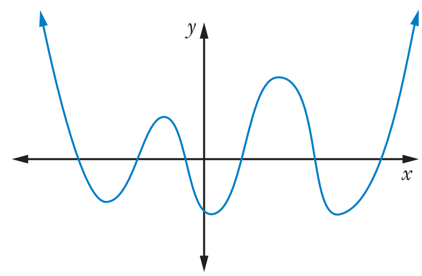

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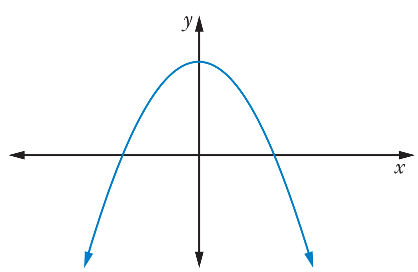

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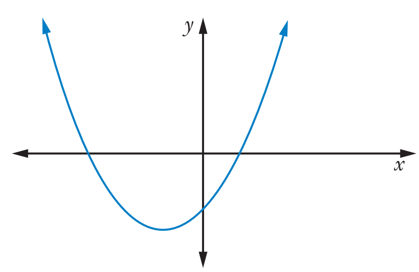

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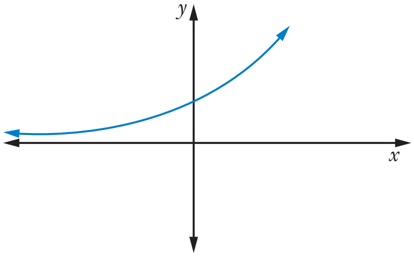

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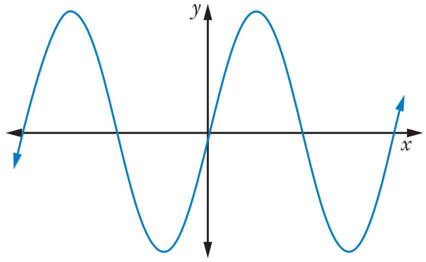

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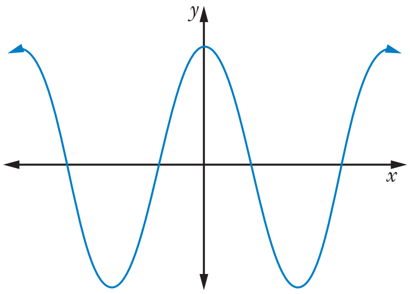

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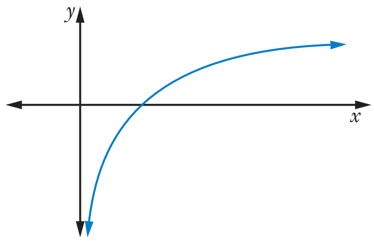


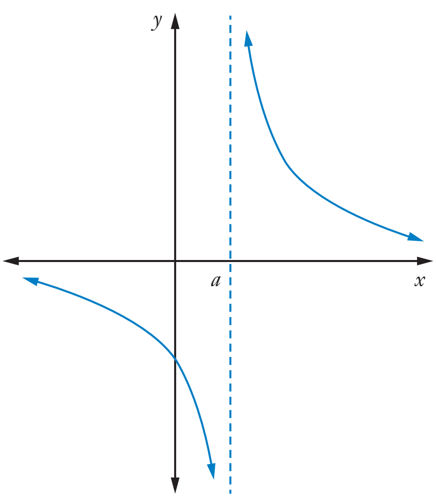
6.02 Differentiability

The process of finding the gradient function $y = f'(x)$ is called **differentiation**

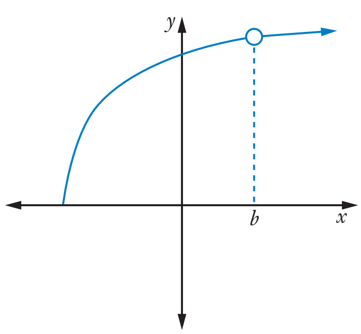
$y = f'(x)$ is called the **derivative function** or just the **derivative**

A function is **differentiable** at any point where it is continuous because we can find its gradient at that point Linea, quadraic, cubic and other polynomial functions are differentiable at all points because their graphs are smooth and unbroken A function is **not differentiable** at any point where it is **discontinuous** where there is a gap or break in its grap.

This hyperbola is not differentiable at $x = a$ because the curve is discontinuous at this point



This function is not differentiable at $x = b$ because the curve is discontinuous at this point



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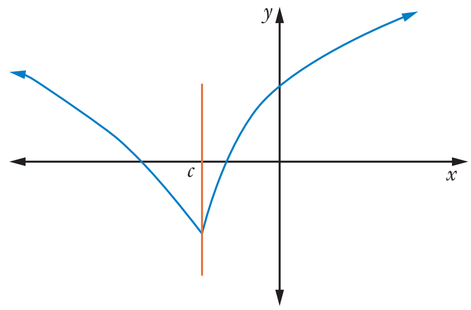
A function is also **not differentiable** where it is not smooth

This function is not **differentiable** at $x = c$ since it is not smooth at that point We cannot draw a unique tangent there so we cannot find the gradient of the function at that point

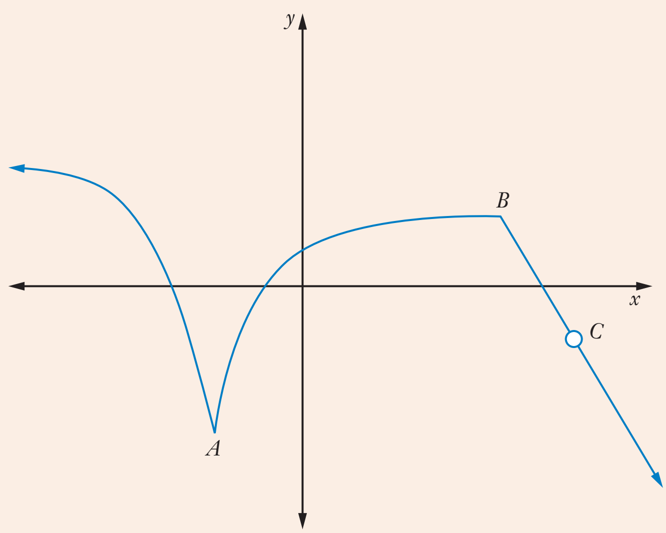


Differentiability at a point

A function $y = f(x)$ is **differentiable** at the point $x = a$ if its graph is **continuous** and **smooth** at $x = a$

EXAMPLE 5

a Find all points where the function below is not differentiable



b Is the function $f(x) = \begin{cases} x^2 & \text{for } x \geq 1 \\ 3x - 2 & \text{for } x < 1 \end{cases}$ differentiable at all points?

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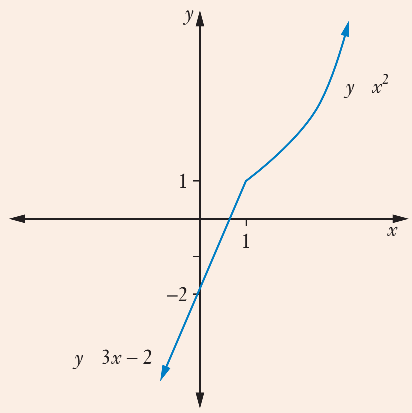
Solution

a The function is not differentiable at points A and B because the curve is not smooth at these points

It is not differentiable at point C because the function is discontinuous at this point

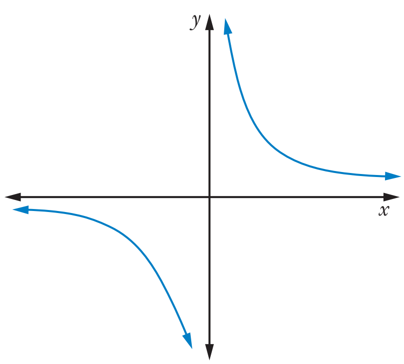
b Sketching this piecewise function shows that it is not smooth where the 2 parts meet so it is not differentiable at $x = 1$.



Exercise 6.02 Differentiability

For each graph of a function state any x values where the function is not differentiable

1



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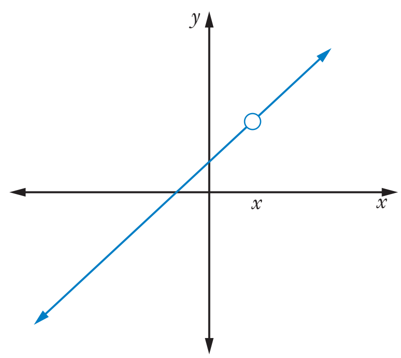

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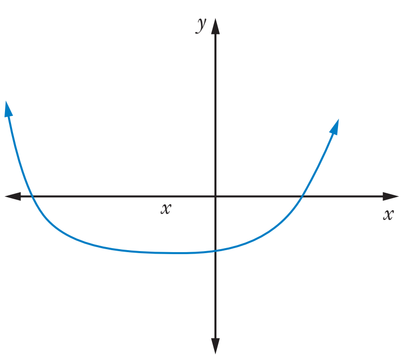

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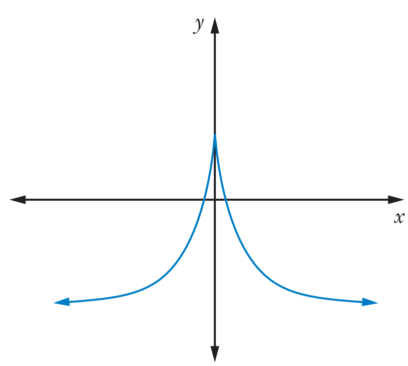

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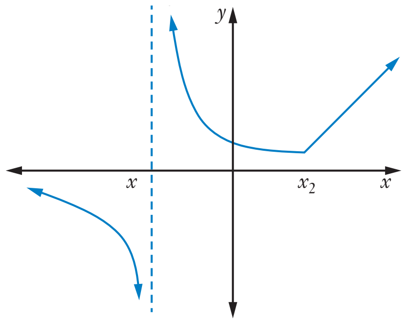


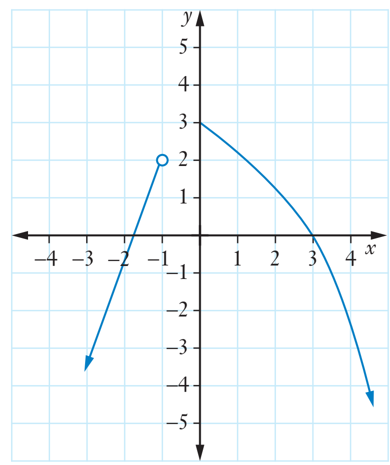
6 $f(x) = \frac{4}{x}$

7 $y = -\frac{1}{x+3}$

8 $f(x) = \begin{cases} x^3 & \text{for } x > 2 \\ x+1 & \text{for } x \le 2 \end{cases}$

9 $f(x) = \begin{cases} 2x & \text{for } x > 3 \\ 3 & \text{for } -2 \le x \le 3 \\ 1 - x^2 & \text{for } x < -2 \end{cases}$

10



11 $f(x) = \frac{|x|}{x}$

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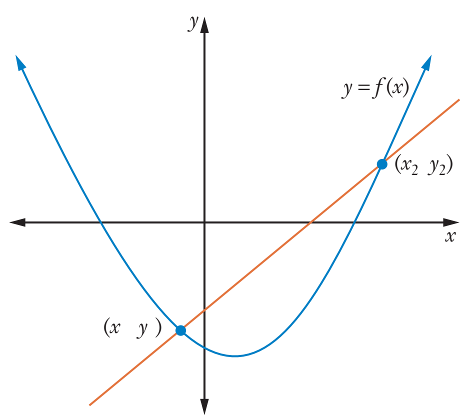
6.03 Differentiation from first principles

Gradient of a secant



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The line passing through the 2 points (x y) and (x_2 y_2) on the graph of a function $y = f(x)$ is called a **secant**

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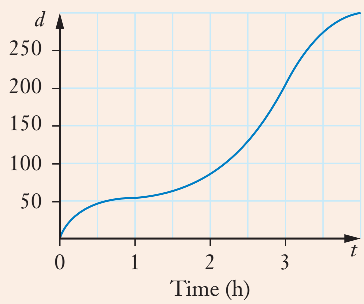
Gradient of the secant

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The **gradient of a secant** gives the **average rate of change** between the 2 points

EXAMPLE 6

a This graph shows the distance d in km that a car travels over time t in hours After 1 hour the car has travelled 55 km and after 3 hours the car has travelled 205 km Find the average speed of the car.



b Given the function $f(x) = x^2$ find the average rate of change between $x = 1$ and $x = 1.1$.

Solution

a Speed is the change in distance over time

The gradient of the secant will give the average speed

Average rate of chang:

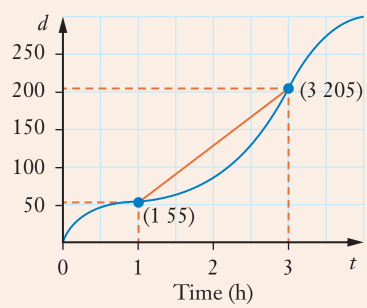
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{205 - 55}{3 - 1}$$

$$= \frac{150}{2}$$

$$= 75$$

So the average speed is 75 km/h



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b When $x = 1$:

$f(1) = 1^2$

$= 1$

When $x = 1.1$:

$f(11) = 1.1^2$

$= 1.21$

So points are (1, 1) and (.1,1.2).

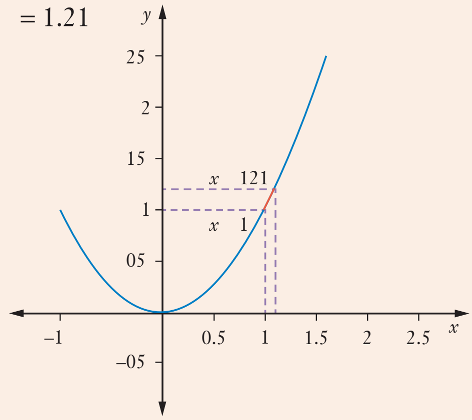
Average rate of chang:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{121 - 1}{1\ 1 - 1}$$

$$= 2\ 1$$

So the average rate of change is 21



Notice that the secant (orange interval) is very close to the shape of the curve itself This is because the 2 points chosen are close together.

Estimating the gradient of a tangent

By taking 2 points close together, the **average rate of change** is quite close to the gradient of the tangent to the curve at one of those points which is called the **instantaneous rate of change** at that point

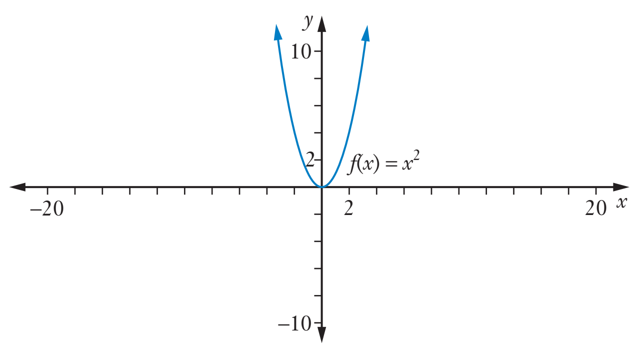
If you look at a close-up of a graph you can get some idea of this concep. When the curve is magnified any 2 points close together appear to be joined by a straight lin. We say the curve is **locally straight**

TECHNOLOGY

Locally straight curves

Use a graphics calculator or graphing software to sketch a curve and then zoom in on a section of the curve to see that it is locally straight

For example here is the parabola $y = x^2$



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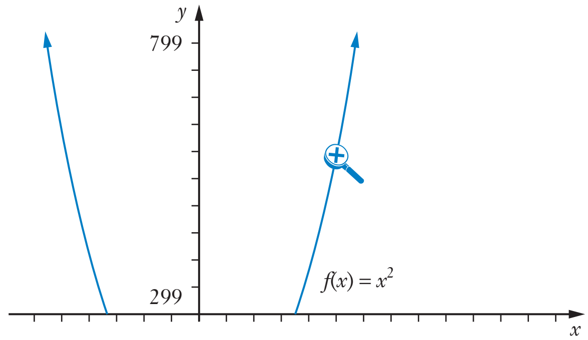
Notice how it looks straight when we zoom in on a point on the parabola

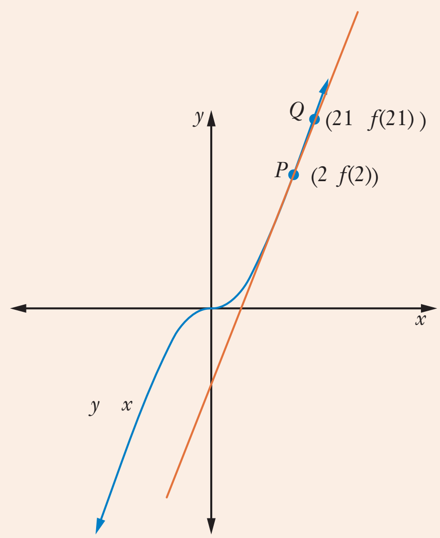


Use technology to sketch other curves and zoom in to show that they are locally straight

We can calculate an approximate value for the gradient of the tangent at a point on a curve by taking another point close by, then calculating the gradient of the secant joining those 2 points

EXAMPLE 7

a For $f(x) = x^3$ find the gradient of the secant PQ where P is the point on the curve where $x = 2$ and Q is another point on the curve where $x = 21$ Then choose different values for Q and use these results to estimate $f'(2)$ the gradient of the tangent to the curve at P



b For the curve $y = x^2$ find the gradient of the secant AB where A is the point on the curve where $x = 5$ and point B is close to A Find an estimate of the gradient of the tangent to the curve at A by using 3 different values for B

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Solution

a P is $(2, f(2))$ Take different values of x for point Q starting with $x = 21$ and find the gradient of the secant using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Point Q	Gradient of secant PQ	Point Q	Gradient of secant PQ
(21 f(21))	$m = \dfrac{f(2\ 1) - f(2)}{2\ 1 - 2}$ $= \dfrac{2\ 1^3 - 2^3}{0\ 1}$ $= 1261$	(19 f(19))	$m = \dfrac{f(1\ 9) - f(2)}{1\ 9 - 2}$ $= \dfrac{1\ 9^3 - 2^3}{-0\ 1}$ $= 1141$
(201 f(201))	$m = \dfrac{f(2\ 0\ 1) - f(2)}{201 - 2}$ $= \dfrac{201^3 - 2^3}{001}$ $= 120601$	(199 f(199))	$m = \dfrac{f(1\ 9\ 9) - f(2)}{199 - 2}$ $= \dfrac{199^3 - 2^3}{-001}$ $= 119401$
(2001 f(2001))	$m = \dfrac{f(2\ 001) - f(2)}{2001 - 2}$ $= \dfrac{2001^3 - 2^3}{0001}$ $= 12006\ 001$	(1999 f(1999))	$m = \dfrac{f(1\ 999) - f(2)}{1999 - 2}$ $= \dfrac{1999^3 - 2^3}{-0001}$ $= 11994\ 001$

From these results we can see that a good estimate for $f'(2)$ the gradient at P, is 12.

As $x \to 2, f'(2) \to 12$.

We use a special notation for **limits** to show this

$$f'(2) = \lim_{x^\circ\ 2} \frac{f(x) - f(2)}{x - 2}$$
$$= 12$$

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b $A = (5, f(5))$

Take 3 different values of x for point B for example $x = 49$, $x = 51$ and $x = 5.01$.

$B = (49 \quad f(49))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(49) - f(5)}{49 - 5}$$
$$= \frac{49^2 - 5^2}{-01}$$
$$= 99$$

$B = (5.1, f(51))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(51) - f(5)}{51 - 5}$$
$$= \frac{51^2 - 5^2}{01}$$
$$= 101$$

$B = (501, f(501))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(501) - f(5)}{501 - 5}$$
$$= \frac{501^2 - 5^2}{001}$$
$$= 1001$$

As $x \to 5, f'(5) \to 10$.

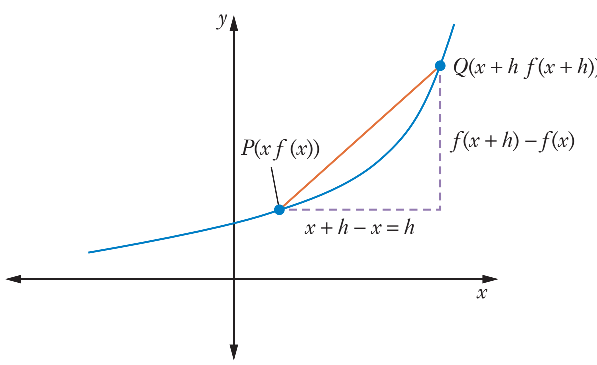
$$f'(5) = \lim_{x° \ 5} \frac{f(x) - f(5)}{x - 5}$$
$$= 10$$

The difference quotient

We measure the instantaneous rate of change of any point on the graph of a function by using limits to find the gradient of the tangent to the curve at that point This is called **differentiation from first principles** Using the method from the examples abov, we can find a general formula for the derivative function $y = f'(x)$

We want to find the instantaneous rate of change or gradient of the tangent to a curve $y = f(x)$ at point $P(x \ f(x))$

We choose a second point Q close to P with coordinates $(x + h \ f(x + h))$ where h is small



Now find the gradient of the secant PQ

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x+h) - f(x)}{x + h - x}$$

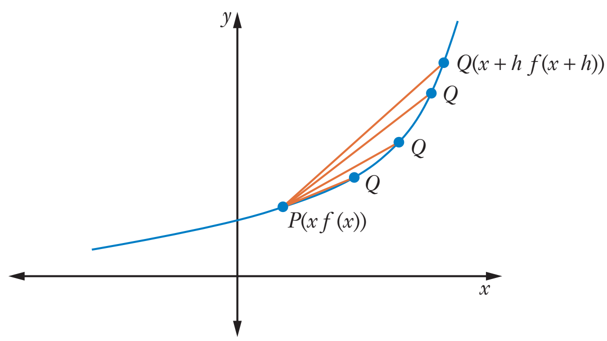
$$= \frac{f(x+h) - f(x)}{h}$$

$\frac{f(x+h) - f(x)}{h}$ is called the **difference quotient** and it gives an **average rate of change**

To find the gradient of the tangent at P we make h smaller as shown so that Q becomes closer and closer to P

As h approaches 0 the gradient of the tangent becomes $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

We call this $f'(x)$ or $\frac{dy}{dx}$ or y'



Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

INVESTIGATION

CALCULUS NOTATION

On p274 we learned about the mathematicians Isaac Newton and Gottfried Leibni. Newton used the notation $f'(x)$ for the derivative function while Leibniz used the notation $\frac{dy}{dx}$ where d stood for 'differenc. Can you see why he would have used this ?

Use the Internet to explore the different notations used in calculus and where they came from

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EXAMPLE 8

a Differentiate from first principles to find the gradient of the tangent to the curve $y = x^2 + 3$ at the point where $x = 1$.

b Differentiate $f(x) = 2x^2 + 7x - 3$ from first principles

Solution

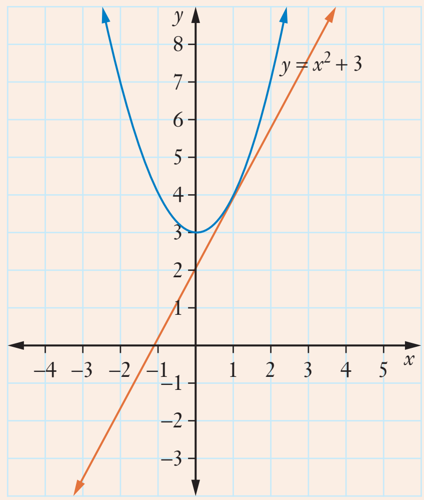
a

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^2 + 3$$

$$f(x + h) = (x + h)^2 + 3$$

$$= x^2 + 2xh + h^2 + 3$$



Substitute $x = 1$:

$$f(1) = 1^2 + 3$$

$$= 4$$

$$f(1 + h) = 1^2 + 2(1)h + h^2 + 3$$

$$= 4 + 2h + h^2$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{4 + 2h + h^2 - 4}{h}$$

$$= \lim_{h \to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2 + h)}{h}$$

$$= \lim_{h \to 0} (2 + h)$$

$$= 2 + 0$$

$$= 2$$

So the gradient of the tangent to the curve $y = x^2 + 3$ at the point (1, 4) is .

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b

$$f(x) = 2x^2 + 7x - 3$$
$$f(x+h) = 2(x+h)^2 + 7(x+h) - 3$$
$$= 2(x^2 + 2xh + h^2) + 7x + 7h - 3$$
$$= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3$$
$$f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - (2x^2 + 7x - 3)$$
$$= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - 2x^2 - 7x + 3$$
$$= 4xh + 2h^2 + 7h$$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{4xh + 2h^2 + 7h}{h}$$
$$= \lim_{h \to 0} \frac{h(4x + 2h + 7)}{h}$$
$$= \lim_{h \to 0} (4x + 2h + 7)$$
$$= 4x + 0 + 7$$
$$= 4x + 7$$

So the gradient function (derivative) of $f(x) = 2x^2 + 7x - 3$ is $f'(x) = 4x + 7$.

Exercise 6.03 Differentiation from first principles

1 **a** For the curve $y = x^4 + 1$ find the gradient of the secant between the point (, 2) and the point where $x = 1.01$.

b Find the gradient of the secant between (1 2) and the point where $x = 0999$ on the curve

c Use these results to find an approximation to the gradient of the tangent to the curve $y = x^4 + 1$ at the point (1, 2.

2 For the function $f(x) = x^3 + x$ find the average rate of change between the point (, 10) and the point on the curve where

a $x = 2.1$ **b** $x = 201$ **c** $x = 199$

Hence find an approximation to the gradient of the tangent at the point (2 10.

3 For the function $f(x) = x^2 - 4$ find the gradient of the tangent at point P where $x = 3$ by selecting points near P and finding the gradient of the secant

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4 A function is given by $f(x) = x^2 + x + 5$.

a Find $f(2)$

b Find $f(2 + h)$

c Find $f(2 + h) - f(2)$

d Show that $\dfrac{f(2+h)-f(2)}{h} = 5 + h$

e Find $f'(2)$

5 Given the curve $f(x) = 4x^2 - 3$ fin:

a $f(-1)$ **b** $f(-1 + h) - f(-1)$

c the gradient of the tangent to the curve at the point where $x = -1$

6 For the parabola $y = x^2 - 1$, fin:

a $f(3)$ **b** $f(3 + h) - f(3)$ **c** $f'(3)$

7 For the function $f(x) = 4 - 3x - 5x^2$ fin:

a $f'(1)$ **b** the gradient of the tangent at the point (–2 –10.

8 If $f(x) = x^2$

a find $f(x + h)$ **b** show that $f(x + h) - f(x) = 2xh + h^2$

c show that $\dfrac{f(x+h)-f(x)}{h} = 2x + h$ **d** show that $f'(x) = 2x$

9 A function is given by $f(x) = 2x^2 - 7x + 3$.

a Show that $f(x + h) = 2x^2 + 4xh + 2h^2 - 7x - 7h + 3$.

b Show that $f(x + h) - f(x) = 4xh + 2h^2 - 7h$

c Show that $\dfrac{f(x+h)-f(x)}{h} = 4x + 2h - 7$.

d Find $f'(x)$

10 Differentiate from first principles to find the gradient of the tangent to the curve

a $f(x) = x^2$ at the point where $x = 1$

b $y = x^2 + x$ at the point (2 6)

c $f(x) = 2x^2 - 5$ at the point where $x = -3$

d $y = 3x^2 + 3x + 1$ at the point where $x = 2$

e $f(x) = x^2 -7x - 4$ at the point (–1 4.

11 Find the derivative function for each function from first principles

a $f(x) = x^2$ **b** $y = x^2 + 5x$

c $f(x) = 4x^2 - 4x - 3$ **d** $y = 5x^2 - x - 1$

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Deivaive o linea podu

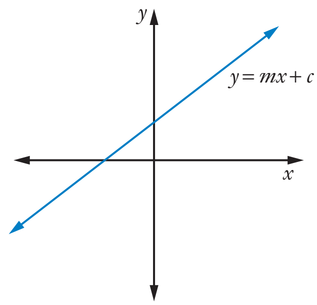
Deivaive o polynomial

6.04 Short methods of differentiation

Derivative of x^n

Remember that the gradient of a straight line $y = mx + c$ is m The tangent to the line is the line itsef, so the gradient of the tangent is m everywhere along the line

So if $y = mx$ $\frac{dy}{dx} = m$

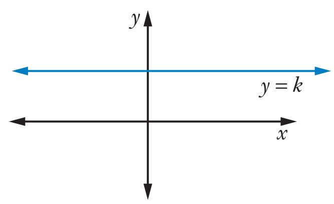


Derivative of *kx*

$$\frac{d}{dx}(kx) = k$$

A horizontal line $y = k$ has a gradient of zero

So if $y = k$ $\frac{dy}{dx} = 0$



Derivative of *k*

$$\frac{d}{dx}(k) = 0$$

TECHNOLOGY

Differentiation of powers of *X*

Find an approximation to the derivative of power functions such as $y = x^2$ $y = x^3$ $y = x^4$ $y = x^5$ by drawing the graph of $y = \frac{f(x+001) - f(x)}{001}$ You could use a graphics calculator or graphing software/website to sketch the derivative for these functions and find its equation Can you find a pattern ? Could you predict what the result would be for x^n?

When differentiating $y = x^n$ from first principles a simple pattern appear:

- For $y = x$ $f'(x) = 1x^0 = 1$
- For $y = x^2$ $f'(x) = 2x = 2x$
- For $y = x^3$ $f'(x) = 3x^2$
- For $y = x^4$ $f'(x) = 4x^3$
- For $y = x^5$ $f'(x) = 5x^4$

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Derivative of x^n

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$

There are some more properties of differentiation

Derivative of kx^n

$$\frac{d}{dx}(kx^n) = knx^{n-1}$$

More generally

Derivative of a constant multiple of a function

$$\frac{d}{dx}\left(kf(x)\right) = kf'(x)$$

EXAMPLE 9

a Find the derivative of $3x^8$

b Differentiate $f(x) = 7x^3$

Solution

a $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\frac{d}{dx}(3x^8) = 3 \times 8x^{8-}$

$= 24x^7$

b $f'(x) = knx^{n-}$

$f'(x) = 7 \times 3x^{3-1}$

$= 21x^2$

If there are several terms in an expression we differentiate each one separatel.

Derivative of a sum of functions

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

EXAMPLE 10

a Differentiate $x^3 + x^4$

b Find the derivative of $7x$

c Differentiate $f(x) = x^4 - x^3 + 5$.

d Find the derivative of $y = 4x^7$

e If $f(x) = 2x^5 - 7x^3 + 5x - 4$ evaluate $f'(-1)$

f Find the derivative of $f(x) = 2x^2(3x - 7)$

g Find the derivative of $\dfrac{3x^2 + 5x}{2x}$

h Differentiate $S = 6r^2 - 12r$ with respect to r

Solution

a $\dfrac{d}{dx}(x^3 + x^4) = 3x^2 + 4x^3$

b $\dfrac{d}{dx}(7x) = 7$

c
$$\begin{aligned} f'(x) &= 4x^3 - 3x^2 + 0 \\ &= 4x^3 - 3x^2 \end{aligned}$$

d
$$\begin{aligned} \frac{dy}{dx} &= 4 \times 7x^6 \\ &= 28x^6 \end{aligned}$$

e
$$\begin{aligned} f'(x) &= 10x^4 - 21x^2 + 5 \\ f'(-1) &= 10(-1)^4 - 21(-1)^2 + 5 \\ &= -6 \end{aligned}$$

f Expand first
$$\begin{aligned} f(x) &= 2x^2(3x - 7) \\ &= 6x^3 - 14x^2 \\ f'(x) &= 18x^2 - 28x \end{aligned}$$

g Simplify first
$$\begin{aligned} \frac{3x^2 + 5x}{2x} &= \frac{3x^2}{2x} + \frac{5x}{2x} \\ &= \frac{3x}{2} + \frac{5}{2} \\ \frac{d}{dx}\left(\frac{3x^2 + 5x}{2x}\right) &= \frac{3}{2} \\ &= 1\frac{1}{2} \end{aligned}$$

h Differentiating with respect to r rather than x
$$\begin{aligned} S &= 6r^2 - 12r \\ \frac{dS}{dr} &= 12r - 12 \end{aligned}$$

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INVESTIGATION

FAMILIES OF CURVES

1 Differentiate

a $x^2 + 1$ **b** $x^2 - 3$ **c** $x^2 + 7$
d x^2 **e** $x^2 + 20$ **f** $x^2 - 100$

What do you notice?

2 Differentiate

a $x^3 + 5$ **b** $x^3 + 11$ **c** $x^3 - 1$
d $x^3 - 6$ **e** x^3 **f** $x^3 + 15$

What do you notice?

These groups of functions are families because they have the same derivatives
Can you find others?

Exercise 6.04 Short methods of differentiation

1 Differentiate

a $x + 2$ **b** $5x - 9$ **c** $x^2 + 3x + 4$
d $5x^2 - x - 8$ **e** $x^3 + 2x^2 - 7x - 3$ **f** $2x^3 - 7x^2 + 7x - 1$
g $3x^4 - 2x^2 + 5x$ **h** $x^6 - 5x^5 - 2x^4$ **i** $2x^5 - 4x^3 + x^2 - 2x + 4$
j $4x^{10} - 7x^9$

2 Find the derivative of

a $x(2x + 1)$ **b** $(2x - 3)^2$ **c** $(x + 4)(x - 4)$
d $(2x^2 - 3)^2$ **e** $(2x + 5)(x^2 - x + 1)$

3 Find the derivative of

a $\frac{x^2}{6} - x$ **b** $\frac{x^4}{2} - \frac{x^3}{3} + 4$ **c** $\frac{1}{3}x^6(x^2 - 3)$

d $\frac{2x^3 + 5x}{x}$ **e** $\frac{x^2 + 2x}{4x}$ **f** $\frac{2x^5 - 3x^4 + 6x^3 - 2x^2}{3x^2}$

4 Find $f'(x)$ when $f(x) = 8x^2 - 7x + 4$

5 If $y = x^4 - 2x^3 + 5$ find $\frac{dy}{dx}$ when $x = -2$

6 Find $\frac{dy}{dx}$ if $y = 6x^{10} - 5x^8 + 7x^5 - 3x + 8$.

7 If $s = 5t^2 - 20t$ find $\frac{ds}{dt}$

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8 Find $g'(x)$ given $g(x) = 5x^4$

9 Find $\frac{dv}{dt}$ when $v = 15t^2 - 9$

10 If $h = 40t - 2t^2$ find $\frac{dh}{dt}$

11 Given $V = \frac{4}{3}\pi r^3$ find $\frac{dV}{dr}$

12 If $f(x) = 2x^3 - 3x + 4$ evaluate $f'(1)$

13 Given $f(x) = x^2 - x + 5$ evaluat:

a $f'(3)$ **b** $f'(-2)$ **c** x when $f'(x) = 7$

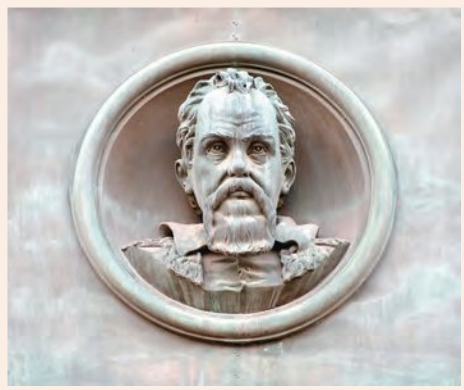
14 If $y = x^3 - 7$ evaluat:

a the derivative when $x = 2$ **b** x when $\frac{dy}{dx} = 12$

15 Evaluate $g'(2)$ when $g(t) = 3t^3 - 4t^2 - 2t + 1$.

DID YOU KNOW?

Motion and calculus


Galileo

Galileo (1564–1642) was very interested in the behaviour of bodies in motion He dropped stones from the Leaning Tower of Pisa to try to prove that they would fall with equal speed He rolled balls down slopes to prove that they move with uniform speed until friction slows them down He showed that a body moving through the air follows a curved path at a fairly constant speed

John Wallis (1616–1703) continued this study with his publication *Mechanica sive Tractatus de Motu Geometricus* He applied mathematical principles to the laws of motion and stimulated interest in the subject of mechanics

Soon after Walls' publicatin, **Christiaan Huygens** (1629–1695) wrote *Horologium Oscillatorium sive de Motu Pendulorum* in which he described various mechanical principles He invented the pendulum cloc, improved the telescope and investigated circular motion and the descent of heavy bodies

These three mathematicians provided the foundations of mechanics **Sir Isaac Newton** (1642–1727) used calculus to increase the understanding of the laws of motion He also used these concepts as a basis for his theories on gravity and inertia

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6.05 Derivatives and indices

INVESTIGATION

DERIVATIVES AND INDICES

1 **a** Show that $\frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)}$

b Hence differentiate $y = \frac{1}{x}$ from first principles

c Differentiate $y = x^{-}$ using the formula Do you get the same answer as in part **b**?

2 **a** Show that $\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right) = h$

b Hence differentiate $y = \sqrt{x}$ from first principles

c Differentiate $y = x^{\frac{}{2}}$ and show that this gives the same answer as in part **b**

EXAMPLE 11

a Differentiate $f(x) = 7\sqrt[3]{x}$

b Find the derivative of $y = \frac{4}{x^2}$ at the point where $x = 2$.

Solution

a $f(x) = 7\sqrt[3]{x} = 7x^{\frac{}{3}}$ ← Convert the funcion to a power of x frst

$f'(x) = 7 \times \frac{1}{3} x^{\frac{}{3}-}$

$= \frac{7}{3} x^{-\frac{2}{3}}$

$= \frac{7}{3} \times \frac{1}{x^{\frac{2}{3}}}$

$= \frac{7}{3} \times \frac{1}{\sqrt[3]{x^2}}$

$= \frac{7}{3\sqrt[3]{x^2}}$

b $y = \frac{4}{x^2}$

$= 4x^{-2}$

$\frac{dy}{dx} = -8x^{-3}$

$= -\frac{8}{x^3}$

When $x = 2$:

$\frac{dy}{dx} = -\frac{8}{2^3}$

$= -1$

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Exercise 6.05 Derivatives and indices

1 Differentiate

a x^{-3} **b** x^{14} **c** $6x^{02}$ **d** $x^{\frac{}{2}}$

e $2x^{\frac{}{2}} - 3x^{-}$ **f** $3x^{\frac{}{3}}$ **g** $8x^{\frac{3}{4}}$ **h** $-2x^{-\frac{}{2}}$

2 Find the derivative function

a $\frac{1}{x}$ **b** $5\sqrt{x}$ **c** $\sqrt[6]{x}$ **d** $\frac{2}{x^5}$

e $-\frac{5}{x^3}$ **f** $\frac{1}{\sqrt{x}}$ **g** $\frac{1}{2x^6}$ **h** $x\sqrt{x}$

i $\frac{2}{3x}$ **j** $\frac{1}{4x^2} + \frac{3}{x^4}$

3 Find the derivative of $y = \sqrt[3]{x}$ at the point where $x = 27$

4 If $x = \frac{12}{t}$ find $\frac{dx}{dt}$ when $t = 2$.

5 A function is given by $f(x) = \sqrt[4]{x}$ Evaluate $f'(16)$

6 Find the derivative of $y = \frac{3}{2x^2}$ at the point $\left(1, 1\frac{1}{2}\right)$

7 Find $\frac{dy}{dx}$ if $y = \left(x + \sqrt{x}\right)^2$

8 A function $f(x) = \frac{\sqrt{x}}{2}$ has a tangent at (4 1. Find its gradiet.

9 **a** Differentiate $\frac{\sqrt{x}}{x}$

b Hence find the derivative of $y = \frac{\sqrt{x}}{x}$ at the point where $x = 4$

10 The function $f(x) = 3\sqrt{x}$ has $f'(x) = \frac{3}{4}$ at $x = a$ Find a

11 The hyperbola $y = \frac{2}{x}$ has 2 tangents with gradient $-\frac{2}{25}$ Find the points where these tangents touch the hyperbola

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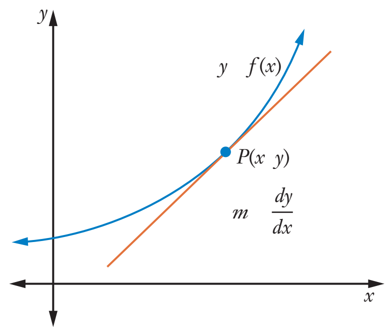
6.06 Tangents and normals

Tangents to a curve

Remember that the derivative is a function that gives the instantaneous rate of change or gradient of the tangent to the curve

A tangent is a line so we can use the formula $y = mx + c$ or $y - y\ \ = m(x - x\ )$ to find its equation



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EXAMPLE 12

a Find the gradient of the tangent to the parabola $y = x^2 + 1$ at the point (1, 2.

b Find values of x for which the gradient of the tangent to the curve $y = 2x^3 - 6x^2 + 1$ is equal to 18

c Find the equation of the tangent to the curve $y = x^4 - 3x^3 + 7x - 2$ at the point (2, 4.

Solution

a The gradient of a tangent to a curve is $\frac{dy}{dx}$

$$\frac{dy}{dx} = 2x + 0$$
$$= 2x$$

Substitute $x = 1$ from the point (1 2:

$$\frac{dy}{dx} = 2(1)$$
$$= 2$$

So the gradient of the tangent at (1 2) is .

b $\frac{dy}{dx} = 6x^2 - 12x$

Gradient is 18 so $\frac{dy}{dx} = 18$.

$$18 = 6x^2 - 12x$$
$$0 = 6x^2 - 12x - 18$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$\therefore x = 3, -1$$

c $\frac{dy}{dx} = 4x^3 - 9x^2 + 7$

At (2 4, $\frac{dy}{dx} = 4(2)^3 - 9(2)^2 + 7$

$= 3$

So the gradient of the tangent at (2 4) is .

Equation of the tangent

$y - y \;\; = m(x - x \;)$

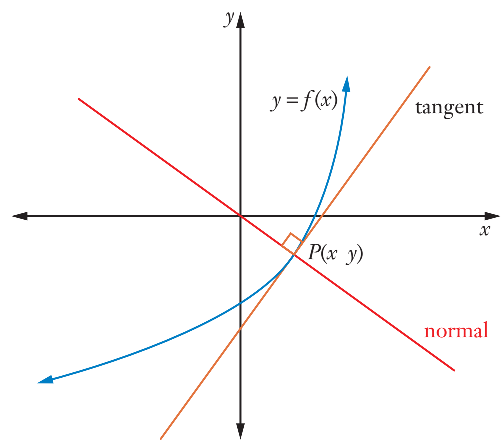
$y - 4 = 3(x - 2)$

$= 3x - 6$

$y = 3x - 2$ or $3x - y - 2 = 0$

Normals to a curve

The **normal** is a straight line **perpendicular** to the tangent at the same point of contact with the curve



Remember the rule for perpendicular lines from Chapter 3 *Functions*

Gradients of perpendicular lines

If 2 lines with gradients m and m_2 are perpendicular, then $m \; m_2 = -1$ or $m_2 = -\frac{1}{m}$

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EXAMPLE 13

a Find the gradient of the normal to the curve $y = 2x^2 - 3x + 5$ at the point where $x = 4$

b Find the equation of the normal to the curve $y = x^3 + 3x^2 - 2x - 1$ at $(-1, 3)$.

Solution

a $\frac{dy}{dx} = 4x - 3$

When $x = 4$

$\frac{dy}{dx} = 4 \times 4 - 3$

$= 13$

So $m \;\; = 13$

The normal is perpendicular to the tangent so $m \; m_2 = -1$

$13m_2 = -1$

$m_2 = -\frac{1}{13}$

So the gradient of the normal is $-\frac{1}{13}$

b $\frac{dy}{dx} = 3x^2 + 6x - 2$

When $x = -1$

$\frac{dy}{dx} = 3(-1)^2 + 6(-1) - 2$

$= -5$

So $m \;\; = -5$

The normal is perpendicular to the tangent so $m \; m_2 = -1$

$-5m_2 = -1$

$m_2 = \frac{1}{5}$

So the gradient of the normal is $\frac{1}{5}$

Equation of the normal $y - y \;\; = m(x - x \;)$

$y - 3 = \frac{1}{5}(x - (-1))$

$5y - 15 = x + 1$

$x - 5y + 16 = 0$

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Exercise 6.06 Tangents and normals

1 Find the gradient of the tangent to the curve

a $y = x^3 - 3x$ at the point where $x = 5$

b $f(x) = x^2 + x - 4$ at the point (–7 38)

c $f(x) = 5x^3 - 4x - 1$ at the point where $x = -1$

d $y = 5x^2 + 2x + 3$ at (–2, 19)

e $y = 2x^9$ at the point where $x = 1$

f $f(x) = x^3 - 7$ at the point where $x = 3$

g $v = 2t^2 + 3t - 5$ at the point where $t = 2$

h $Q = 3r^3 - 2r^2 + 8r - 4$ at the point where $r = 4$

i $h = t^4 - 4t$ where $t = 0$

j $f(t) = 3t^5 - 8t^3 + 5t$ at the point where $t = 2$.

2 Find the gradient of the normal to the curve

a $f(x) = 2x^3 + 2x - 1$ at the point where $x = -2$

b $y = 3x^2 + 5x - 2$ at (–5, 48)

c $f(x) = x^2 - 2x - 7$ at the point where $x = -9$

d $y = x^3 + x^2 + 3x - 2$ at (–4 –62)

e $f(x) = x^{10}$ at the point where $x = -1$

f $y = x^2 + 7x - 5$ at (–7, –5)

g $A = 2x^3 + 3x^2 - x + 1$ at the point where $x = 3$

h $f(a) = 3a^2 - 2a - 6$ at the point where $a = -3$

i $V = h^3 - 4h + 9$ at (2, 9)

j $g(x) = x^4 - 2x^2 + 5x - 3$ at the point where $x = -1$

3 Find the gradient of **i** the tangent and **ii** the normal to the curve

a $y = x^2 + 1$ at (3, 10)

b $f(x) = 5 - x^2$ where $x = -4$

c $y = 2x^5 - 7x^2 + 4$ where $x = -1$

d $p(x) = x^6 - 3x^4 - 2x + 8$ where $x = 1$

e $f(x) = 4 - x - x^2$ at (–6 26)

4 Find the equation of the tangent to the curve

a $y = x^4 - 5x + 1$ at (2, 7)

b $f(x) = 5x^3 - 3x^2 - 2x + 6$ at (1, 6)

c $y = x^2 + 2x - 8$ at (–3, –5)

d $y = 3x^3 + 1$ where $x = 2$

e $v = 4t^4 - 7t^3 - 2$ where $t = 2$

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5 Find the equation of the normal to the curve

a $f(x) = x^3 - 3x + 5$ at $(3, 23)$

b $y = x^2 - 4x - 5$ at $(-2, 7)$

c $f(x) = 7x - 2x^2$ where $x = 6$

d $y = 7x^2 - 3x - 3$ at $(-3, 69)$

e $y = x^4 - 2x^3 + 4x + 1$ where $x = 1$

6 Find the equation of **i** the tangent and **ii** the normal to the curve

a $f(x) = 4x^2 - x + 8$ at $(1, 11)$

b $y = x^3 - 2x^2 - 5x$ at $(-3, -30)$

c $F(x) = x^5 - 5x^3$ where $x = 1$

d $y = x^2 - 8x + 7$ at $(3, -8.$

7 For the curve $y = x^3 - 27x - 5$ find values of x for which $\frac{dy}{dx} = 0$

8 Find the coordinates of the points at which the curve $y = x^3 + 1$ has a tangent with a gradient of 3

9 A function $f(x) = x^2 + 4x - 12$ has a tangent with a gradient of -6 at point P on the curve Find the coordinates of P

10 The tangent at point P on the curve $y = 4x^2 + 1$ is parallel to the x-axis Find the coordinates of P

11 Find the coordinates of point Q where the tangent to the curve $y = 5x^2 - 3x$ is parallel to the line $7x - y + 3 = 0$

12 Find the coordinates of point S where the tangent to the curve $y = x^2 + 4x - 1$ is perpendicular to the line $4x + 2y + 7 = 0$

13 The curve $y = 3x^2 - 4$ has a gradient of 6 at point A

a Find the coordinates of A

b Find the equation of the tangent to the curve at A

14 A function $h = 3t^2 - 2t + 5$ has a tangent at the point where $t = 2$ Find the equation of the tangent

15 A function $f(x) = 2x^2 - 8x + 3$ has a tangent parallel to the line $4x - 2y + 1 = 0$ at point P Find the equation of the tangent at P

16 Find the equation of the tangent to the curve $y = \frac{1}{x^3}$ at $\left(2 \; \frac{1}{8}\right)$

17 Find the equation of the tangent to $f(x) = 6\sqrt{x}$ at the point where $x = 9$

18 Find the equation of the tangent to the curve $y = \frac{4}{x}$ at $\left(8 \; \frac{1}{2}\right)$

19 If the gradient of the tangent to $y = \sqrt{x}$ is $\frac{1}{6}$ at point A find the coordinates of A

6.07 Chain rule

WS Chain ule

We looked at composite functions in Chapter , *Further functions*

The **chain rule** is a method for differentiating composite functions It is also called the **composite function** rule or the '**function of a function rule**

The chain rule

If a function y can be written as a composite function where $y = f(u(x))$ the:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The chain ule

EXAMPLE 14

Differentiate

a $y = (5x + 4)^7$ **b** $y = (3x^2 + 2x - 1)^9$ **c** $y = \sqrt{3 - x}$

Solution

a Let $u = 5x + 4$

Then $\frac{du}{dx} = 5$

$y = u^7$

$\therefore \frac{dy}{du} = 7u^6$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 7u^6 \times 5$$
$$= 35u^6$$
$$= 35(5x + 4)^6$$

b Let $u = 3x^2 + 2x - 1$

Then $\frac{du}{dx} = 6x + 2$

$y = u^9$

$\therefore \frac{dy}{du} = 9u^8$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 9u^8 \times (6x + 2)$$
$$= 9(3x^2 + 2x - 1)^8(6x + 2)$$
$$= 9(6x + 2)(3x^2 + 2x - 1)^8$$

c $y = \sqrt{3 - x} = (3 - x)^{\frac{}{2}}$

Let $u = 3 - x$

Then $\frac{du}{dx} = -1$

$y = u^{\frac{}{2}}$

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{}{2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{2}u^{-\frac{}{2}} \times (-1)$$
$$= -\frac{1}{2}(3 - x)^{-\frac{}{2}}$$
$$= -\frac{1}{2\sqrt{3 - x}}$$

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You might see a pattern when using the chain rul. The derivative of a composite function is the product of the derivatives of 2 functions

The derivative of $[f(x)]^n$

$$\frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-1}$$

EXAMPLE 15

Differentiate

a $y = (8x^3 - 1)^5$ **b** $y = (3x + 8)^{11}$ **c** $y = \dfrac{1}{(6x+1)^2}$

Solution

a $\dfrac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$

$= 24x^2 \times 5(8x^3 - 1)^4$

$= 120x^2(8x^3 - 1)^4$

b $\dfrac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$

$= 3 \times 11(3x + 8)^{10}$

$= 33(3x + 8)^{10}$

c $y = \dfrac{1}{(6x+1)^2} = (6x+1)^{-2}$

$\dfrac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$

$= 6 \times (-2)(6x + 1)^{-3}$

$= -12(6x + 1)^{-3}$

$= -\dfrac{12}{(6x+1)^3}$

Exercise 6.07 Chain rule

1 Differentiate

a $y = (x + 3)^4$ **b** $y = (2x - 1)^3$ **c** $y = (5x^2 - 4)^7$

d $y = (8x + 3)^6$ **e** $y = (1 - x)^5$ **f** $y = 3(5x + 9)^9$

g $y = 2(x - 4)^2$ **h** $y = (2x^3 + 3x)^4$ **i** $y = (x^2 + 5x - 1)^8$

j $y = (x^6 - 2x^2 + 3)^6$ **k** $y = (3x - 1)^{\overline{2}}$ $y = (4 - x)^{-2}$

m $y = (x^2 - 9)^{-3}$ **n** $y = (5x + 4)^{\overline{3}}$ **o** $y = (x^3 - 7x^2 + x)^{\frac{3}{4}}$

p $y = \sqrt{3x+4}$

q $y = \dfrac{1}{5x-2}$

r $y = \dfrac{1}{(x^2+1)^4}$

s $y = \sqrt[3]{(7-3x)^2}$

t $y = \dfrac{5}{\sqrt{4+x}}$

u $y = \dfrac{1}{2\sqrt{3x-1}}$

v $y = \dfrac{3}{4(2x+7)^9}$

w $y = \dfrac{1}{x^4 - 3x^3 + 3x}$

x $y = \sqrt[3]{(4x+1)^4}$

y $y = \dfrac{1}{\sqrt[4]{(7-x)^5}}$

2 Find the gradient of the tangent to the curve $y = (3x-2)^3$ at the point (1, 1.

3 If $f(x) = 2(x^2 - 3)^5$ evaluate $f'(2)$

4 The curve $y = \sqrt{x-3}$ has a tangent with gradient $\dfrac{1}{2}$ at point N
Find the coordinates of N

5 For what values of x does the function $f(x) = \dfrac{1}{4x-1}$ have $f'(x) = -\dfrac{4}{49}$?

6 Find the equation of the tangent to $y = (2x+1)^4$ at the point where $x = -1$

7 Find the equation of the tangent to the curve $y = (2x-1)^8$ at the point where $x = 1$.

8 Find the equation of the normal to the curve $y = (3x-4)^3$ at (1, –1.

9 Find the equation of the normal to the curve $y = (x^2+1)^4$ at (1, 16).

10 Find the equation of **a** the tangent and **b** the normal to the curve $f(x) = \dfrac{1}{2x+3}$ at the point where $x = -1$

WS
Poduc ule

6.08 Product rule

The **product rule** is a method for differentiating the product of 2 functions

The product rule

If $y = uv$ where u and v are functions the:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\text{or} \quad y' = u'v + v'u$$

We can also wite the product rule the other way round(ifferenaiing v frst but the above formulas ill also help us to remember the quoient rulein the next secio.

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EXAMPLE 16

Differentiate

a $y = (3x + 1)(x - 5)$ **b** $y = 9x^3(2x - 7)$

Solution

a You could expand the brackets and then differentiat:

$$\begin{aligned} y &= (3x + 1)(x - 5) \\ &= 3x^2 - 15x + x - 5 \\ &= 3x^2 - 14x - 5 \\ \frac{dy}{dx} &= 6x - 14 \end{aligned}$$

Using the product rule

$y = uv$ where $u = 3x + 1$ and $v = x - 5$

$u' = 3$ $v' = 1$

$$\begin{aligned} y' &= u'v + v'u \\ &= 3(x - 5) + 1(3x + 1) \\ &= 3x - 15 + 3x + 1 \\ &= 6x - 14 \end{aligned}$$

b $y = uv$ where $u = 9x^3$ and $v = 2x - 7$

$u' = 27x^2$ $v' = 2$

$$\begin{aligned} y' &= u'v + v'u \\ &= 27x^2(2x - 7) + 2(9x^3) \\ &= 54x^3 - 189x^2 + 18x^3 \\ &= 72x^3 - 189x^2 \end{aligned}$$

We can use the product rule together with the chain rul.

EXAMPLE 17

Differentiate

a $y = 2x^5(5x+3)^3$ **b** $y = (3x-4)\sqrt{5-2x}$

Solution

a $y = uv$ where $u = 2x^5$ and $v = (5x+3)^3$

$u' = 10x^4$ $\quad v' = 5 \times 3(5x+3)^2$ using chain rule

$= 15(5x+3)^2$

$$\begin{aligned} y' &= u'v + v'u \\ &= 10x^4\,(5x+3)^3 + 15(5x+3)^2\,2x^5 \\ &= 10x^4(5x+3)^3 + 30x^5(5x+3)^2 \\ &= 10x^4(5x+3)^2[(5x+3)+3x] \\ &= 10x^4(5x+3)^2(8x+3) \end{aligned}$$

b $y = uv$ where $u = 3x-4$ and $v = \sqrt{5-2x} = (5-2x)^{\frac{1}{2}}$

$u' = 3$ $\quad v' = -2 \times \frac{1}{2}(5-2x)^{-\frac{1}{2}}$ using chain rule

$$\begin{aligned} &= -(5-2x)^{-\frac{1}{2}} \\ &= -\frac{1}{(5-2x)^{\frac{1}{2}}} \\ &= -\frac{1}{\sqrt{5-2x}} \end{aligned}$$

$$\begin{aligned} y' &= u'v + v'u \\ &= 3\ \sqrt{5-2x} + -\frac{1}{\sqrt{5-2x}}(3x-4) \\ &= 3\sqrt{5-2x} - \frac{3x-4}{\sqrt{5-2x}} \\ &= \frac{3\sqrt{5-2x} \times \sqrt{5-2x}}{\sqrt{5-2x}} - \frac{3x-4}{\sqrt{5-2x}} \\ &= \frac{3(5-2x)}{\sqrt{5-2x}} - \frac{3x-4}{\sqrt{5-2x}} \end{aligned}$$

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$$= \frac{3(5-2x)-(3x-4)}{\sqrt{5-2x}}$$

$$= \frac{15-6x-3x+4}{\sqrt{5-2x}}$$

$$= \frac{19-9x}{\sqrt{5-2x}}$$

Exercise 6.08 Product rule

1 Differentiate

a $y = x^3(2x+3)$ **b** $y = (3x-2)(2x+1)$ **c** $y = 3x(5x+7)$

d $y = 4x^4(3x^2-1)$ **e** $y = 2x(3x^4-x)$ **f** $y = x^2(x+1)^3$

g $y = 4x(3x-2)^5$ **h** $y = 3x^4(4-x)^3$ **i** $y = (x+1)(2x+5)^4$

2 Find the gradient of the tangent to the curve $y = 2x(3x-2)^4$ at (1, 2).

3 If $f(x) = (2x+3)(3x-1)^5$ evaluate $f'(1)$

4 Find the exact gradient of the tangent to the curve $y = x\sqrt{2x+5}$ at the point where $x = 1$.

5 Find the gradient of the tangent where $t = 3$ given $x = (2t-5)(t+1)^3$

6 Find the equation of the tangent to the curve $y = x^2(2x-1)^4$ at (1, 1).

7 Find the equation of the tangent to $h = (t+1)^2(t-1)^7$ at (2, 9.

8 Find exact values of x for which the gradient of the tangent to the curve $y = 2x(x+3)^2$ is 14

9 Given $f(x) = (4x-1)(3x+2)^2$ find the equation of the tangent at the point where $x = -1$

WS Quoien ule

WS Rule o difretition

WS Mixed difretition poblem

6.09 Quotient rule

The **quotient rule** is a method for differentiating the ratio of 2 functions

The quotient rule

If $y = \frac{u}{v}$ where u and v are functions the:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\text{or } y' = \frac{u'v - v'u}{v^2}$$

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EXAMPLE 18

Differentiate

a $y = \dfrac{3x-5}{5x+2}$ **b** $y = \dfrac{4x^3-5x+2}{x^3-1}$

Solution

a $y = \dfrac{u}{v}$ where $u = 3x - 5$ and $v = 5x + 2$

$u' = 3$ $v' = 5$

$$\begin{aligned} y' &= \frac{u'v - v'u}{v^2} \\ &= \frac{3(5x+2) - 5(3x-5)}{(5x+2)^2} \\ &= \frac{15x + 6 - 15x + 25}{(5x+2)^2} \\ &= \frac{31}{(5x+2)^2} \end{aligned}$$

b $y = \dfrac{u}{v}$ where $u = 4x^3 - 5x + 2$ and $v = x^3 - 1$

$u' = 12x^2 - 5$ $v' = 3x^2$

$$\begin{aligned} y' &= \frac{u'v - v'u}{v^2} \\ &= \frac{(12x^2-5)(x^3-1) - 3x^2(4x^3-5x+2)}{(x^3-1)^2} \\ &= \frac{12x^5 - 12x^2 - 5x^3 + 5 - 12x^5 + 15x^3 - 6x^2}{(x^3-1)^2} \\ &= \frac{10x^3 - 18x^2 + 5}{(x^3-1)^2} \end{aligned}$$

Exercise 6.09 Quotient rule

1 Differentiate

a $y = \dfrac{1}{2x-1}$ **b** $y = \dfrac{3x}{x+5}$ **c** $y = \dfrac{x^3}{x^2-4}$ **d** $y = \dfrac{x-3}{5x+1}$

e $y = \dfrac{x-7}{x^2}$ **f** $y = \dfrac{5x+4}{x+3}$ **g** $y = \dfrac{x}{2x^2-1}$ **h** $y = \dfrac{x+4}{x-2}$

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i $y = \dfrac{2x+7}{4x-3}$ **j** $y = \dfrac{x+5}{3x+1}$ **k** $y = \dfrac{x+1}{3x^2-7}$ $y = \dfrac{2x^2}{2x-3}$

m $y = \dfrac{x^2+4}{x^2-5}$ **n** $y = \dfrac{x^3}{x+4}$ **o** $y = \dfrac{x^3+2x-1}{x+3}$ **p** $y = \dfrac{x^2-2x-1}{3x+4}$

q $y = \dfrac{2x}{(x+5)^{\frac{1}{2}}}$ **r** $y = \dfrac{x-1}{(7x+2)^4}$ **s** $y = \dfrac{3x+1}{\sqrt{x+1}}$ **t** $y = \dfrac{\sqrt{x-1}}{2x-3}$

2 Find the gradient of the tangent to the curve $y = \dfrac{2x}{3x+1}$ at $\left(1, \dfrac{1}{2}\right)$

3 If $f(x) = \dfrac{4x+5}{2x-1}$ evaluate $f'(2)$

4 Find values of x for which the gradient of the tangent to $y = \dfrac{4x-1}{2x-1}$ is -2

5 Given $f(x) = \dfrac{2x}{x+3}$ find x if $f'(x) = \dfrac{1}{6}$

6 Find the equation of the tangent to the curve $y = \dfrac{x}{x+2}$ at $\left(4, \dfrac{2}{3}\right)$

7 Find the equation of the tangent to the curve $y = \dfrac{x^2-1}{x+3}$ at $x = 2$.

6.10 Rates of change

WS Rae o change

WS Inananeou ae o change

WS Graph o ae o change

We know that the gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ of the secant passing through 2 points on the graph of a function gives the **average rate of change** between those 2 points

Now consider a quantity Q that changes with time giving the function $Q(t)$

Average rate of change

The average rate of change of a quantity Q with respect to time t is $\dfrac{Q_2 - Q_1}{t_2 - t_1}$

We know that the gradient $\dfrac{dy}{dx}$ of the tangent at a point on the graph of a function gives the **instantaneous rate of change** at that point

Instantaneous rate of change

The instantaneous rate of change of a quantity Q with respect to time t is $\dfrac{dQ}{dt}$

EXAMPLE 19

a The number of bacteria in a culture increases according to the function $B = 2t^4 - t^2 + 2000$ where t is time in hours Fin:

- **i** the number of bacteria initially
- **ii** the average rate of change in number of bacteria between 2 and 3 hours
- **iii** the number of bacteria after 5 hours
- **v** the rate at which the number of bacteria is increasing after 5 hours

b An object travels a distance according to the function $D = t^2 + t + 5$ where D is in metres and t is in seconds Find the speed at which it is travelling a:

i 4 s **ii** 10 s

Solution

a **i** $B = 2t^4 - t^2 + 2000$

Initially, $t = 0$

$$B = 2(0)^4 - (0)^2 + 2000$$
$$= 2000$$

So there are 2000 bacteria initially.

ii When $t = 2$, $B = 2(2)^4 - (2)^2 + 2000$
$$= 2028$$

When $t = 3$, $B = 2(3)^4 - (3)^2 + 2000$
$$= 2153$$

$$\text{Average rate of change} = \frac{B_2 - B_1}{t_2 - t_1}$$
$$= \frac{2153 - 2028}{3 - 2}$$
$$= 125 \text{ bacteria/hour}$$

So the average rate of change is 125 bacteria per hour.

iii When $t = 5$, $B = 2(5)^4 - (5)^2 + 2000$
$$= 3225$$

So there will be 3225 bacteria after 5 hours

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v The instantaneous rate of change is given by the derivative $\frac{dB}{dt} = 8t^3 - 2t$

When $t = 5$, $\frac{dB}{dt} = 8(5)^3 - 2(5)$

$= 990$

So the rate of increase after 5 hours will be 990 bacteria per hour.

b Speed is the rate of change of distance over time $\frac{dD}{dt} = 2t + 1$

i When $t = 4$, $\frac{dD}{dt} = 2(4) + 1$

$= 9$

So speed after 4 s is 9 m/s

ii When $t = 10$, $\frac{dD}{dt} = 2(10) + 1$

$= 21$

So speed after 10 s is 21 m/s

Displacement, velocity and acceleration

Displacement (x) measures the distance of an object from a fixed point (origin) It can be positive or negative or 0 according to where the object i.

Velocity (v) is the rate of change of displacement with respect to time and involves speed and direction

Velocity

Velocity $v = \frac{dx}{dt}$ is the instantaneous rate of change of displacement x over time t

Acceleration (a) is the rate of change of velocity with respect to time

Acceleration

Acceleration $a = \frac{dv}{dt}$ is the instantaneous rate of change of velocity v over time t

We usually write velocity units as km/h or m/, but we can also use index notation and write km h^{-} or m s^{-}

With acceleration unit, we write km/h/h as km/h ^{2} or in index notation we write km h^{-2}

EXAMPLE 20

A ball rolls down a ramp so that its displacement x cm in t seconds is $x = 16 - t^2$

a Find its initial displacement

b Find its displacement at 3 s

c Find its velocity at 2 s

d Show that the ball has a constant acceleration of –2 cm s^{-2}

Solution

a $x = 16 - t^2$

Initially, $t = 0$

$x = 16 - 0^2$

$\quad = 16$

So the balls initial displacement is 16 c.

b When $t = 3$:

$x = 16 - 3^2$

$\quad = 7$

So the balls displacement at 3 s is 7 c.

c $v = \dfrac{dx}{dt}$

$\quad = -2t$

When $t = 2$:

$v = -2(2)$

$\quad = -4$

So the balls velocity at 2 s is –4 cm s^{-}

x s measured n cm, s measured n s,
so v s measured n cm/s or cm s $^{-}$.

d $a = \dfrac{dv}{dt}$

$\quad = -2$

So acceleration is constant at –2 cm s^{-2}

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Exercise 6.10 Rates of change

1 Find the formula for the rate of change for each function

a $h = 20t - 4t^2$ **b** $D = 5t^3 + 2t^2 + 1$ **c** $A = 16x - 2x^2$

d $x = 3t^5 - t^4 + 2t - 3$ **e** $V = \frac{4}{3}\circ r^3$ **f** $S = 2\pi r + \frac{50}{r^2}$

g $D = \sqrt{x^2 - 4}$ **h** $S = 800r + \frac{400}{r}$

2 If $h = t^3 - 7t + 5$ fin:

a the average rate of change of h between $t = 3$ and $t = 4$

b the instantaneous rate of change of h when $t = 3$.

3 The volume of water V in litres flowing through a pipe after t seconds is given by $V = t^2 + 3t$ Find the rate at which the water is flowing when $t = 5$.

4 The mass in grams of a melting ice block is given by the formula $M = t - 2t^2 + 100$, where t is time in minutes

a Find the average rate of change at which the ice block is melting between

i 1 and 3 minutes **ii** 2 and 5 minutes

b Find the rate at which it will be melting at 5 minutes

5 The surface area in cm^2 of a balloon being inflated is given by $S = t^3 - 2t^2 + 5t + 2$, where t is time in seconds Find the rate of increase in the balloo's surface area at 8s.

6 A circular disc expands as it is heated The ara, in cm^2 of the disc increases according to the formula $A = 4t^2 + t$ where t is time in minutes Find the rate of increase in the area after 5 minutes

7 A car is d km from home after t hours according to the formula $d = 10t^2 + 5t + 11$.

a How far is the car from home

i initially? **ii** after 3 hours? **iii** after 5 hours?

b At what speed is the car travelling after

i 3 hours? **ii** 5 hours?

8 According to Boyles La, the pressure of a gas is given by the formula $P = \frac{k}{V}$ where k is a constant and V is the volume of the gas If $k = 100$ for a certain gas find the rate of change in the pressure when $V = 20$

9 The displacement of a particle is $x = t^3 - 9t$ cm where t is time in seconds

a Find the velocity of the particle at 3 s

b Find the acceleration at 2 s

c Show that the particle is initially at the origin and find any other times that the particle will be at the origin

d At what time will the acceleration be 30 cm s^{-2}?

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10 A particle is moving with displacement $s = 2t^2 - 8t + 3$ where s is in metres and t is in seconds

a Find its initial velocity.

b Show that its acceleration is constant and find its value

c Find its displacement at 5 s

d Find when the particles velocity is zer.

e What will the particles displacement be at that time ?

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6. TEST YOURSELF

Pacice quiz

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For Questions **1** to **4** select the correct answer **A B C** or **D**

1 Find the derivative of $\frac{2}{3x^4}$

A $\frac{8}{3x^5}$ **B** $-\frac{8}{3x^3}$ **C** $-\frac{8}{3x^5}$ **D** $\frac{8}{3x^3}$

2 Differentiate $3x(x^3 - 5)$

A $4x^3$ **B** $12x^3 - 15$ **C** $9x^2$ **D** $3x^4 - 15x$

3 The derivative of $y = f(x)$ is given by

A $\lim_{h \to 0} \frac{f(x+h) - f(x)}{x-h}$ **B** $\lim_{x \to 0} \frac{f(x+h) - f(x)}{x}$

C $\lim_{h \to 0} \frac{f(x) - f(x+h)}{h}$ **D** $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

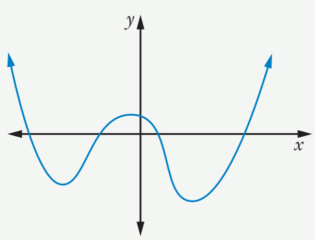
4 Which of the following is the chain rule (there is more than one answer)?

A $\frac{dy}{dx} = \frac{dy}{du} \times \frac{dx}{du}$ **B** $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

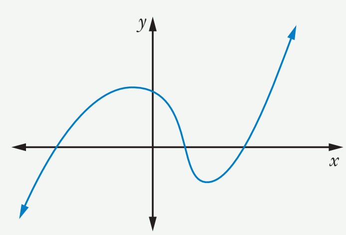
C $\frac{dy}{dx} = nf'(x)f(x)^{n-1}$ **D** $\frac{dy}{dx} = nf(x)^{n-1}$

5 Sketch the derivative function of each graph

a



b



6 Differentiate $y = 5x^2 - 3x + 2$ from first principles

7 Differentiate

a $y = 7x^6 - 3x^3 + x^2 - 8x - 4$ **b** $y = 3x^{-4}$ **c** $y = \frac{2}{(x+1)^4}$

d $y = x^2\sqrt{x}$ **e** $y = (x^2 + 4x - 2)^9$ **f** $y = \frac{3x - 2}{2x + 1}$

g $y = x^3(3x + 1)^6$

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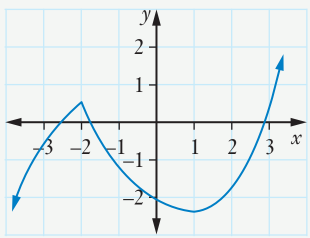
8 Find $\dfrac{dv}{dt}$ if $v = 2t^2 - 3t - 4$

9 Find the gradient of the tangent to the curve $y = x^3 + 3x^2 + x - 5$ at (1, 0).

10 If $h = 60t - 3t^2$ find $\dfrac{dh}{dt}$ when $t = 3$.

11 For each graph of a function find all values of x where it is not differentiable

a



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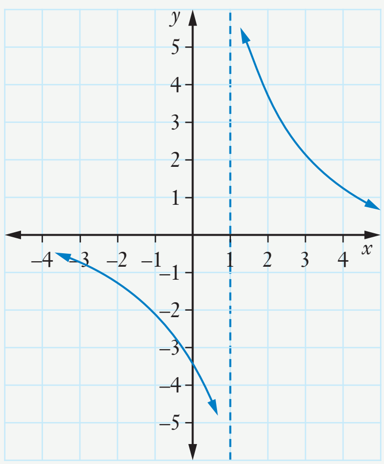
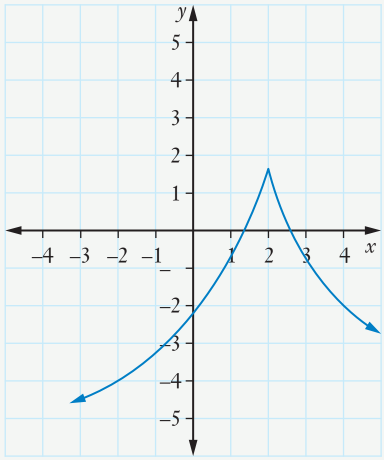


c



12 Differentiate

a $y = \dfrac{4}{x}$

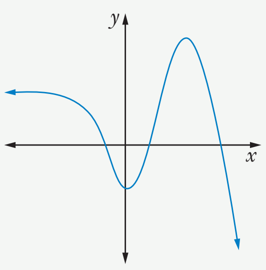
b $f(x) = \sqrt[5]{x}$

c $f(x) = 2(4x + 9)^4$

d $y = (3x + 2)(x - 1)^3$

e $f(x) = \dfrac{x^3 - 3}{2x + 5}$

13 Sketch the derivative function of this curve



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14 Find the equation of the tangent to the curve $y = x^2 + 5x - 3$ at (2, 11.

15 Find the point on the curve $y = x^2 - x + 1$ at which the tangent has a gradient of 3

16 Find $\frac{dS}{dr}$ if $S = 4\pi r^2$

17 Find the gradient of the secant on the curve $f(x) = x^2 - 3x + 1$ between the points where $x = 1$ and $x = 1.1$.

18 At which points on the curve $y = 2x^3 - 9x^2 - 60x + 3$ are the tangents horizontal?

19 Find the equation of the tangent to the curve $y = x^2 + 2x - 5$ that is parallel to the line $y = 4x - 1$.

20 **a** Differentiate $s = ut + \frac{1}{2}at^2$ with respect to t

b Find the value of t for which $\frac{ds}{dt} = 5$, $u = 7$ and $a = -10$

21 Find the equation of the tangent to the curve $y = \frac{1}{3x}$ at the point where $x = \frac{1}{6}$

22 A ball is thrown into the air and its height h metres over t seconds is given by $h = 4t - t^2$

a Find the height of the ball

i initially **ii** at 2 s **iii** at 3 s **v** at 35 s

b Find the average rate of change of the height between

i 1 and 2 seconds **ii** 2 and 3 seconds

c Find the rate at which the ball is moving

i initially **ii** at 2 s **iii** at 3 s

23 If $f(x) = x^2 - 3x + 5$ fin:

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $f'(x)$

24 Given $f(x) = (4x - 3)^5$ find the value o:

a $f(1)$ **b** $f'(1)$

25 Find $f'(4)$ when $f(x) = (x - 3)^9$

26 Differentiate

a $y = 3(x^2 - 6x + 1)^4$ **b** $y = \frac{2}{\sqrt{3x - 1}}$

27 A particle moves so that its displacement after t seconds is $x = 4t^2 - 5t^3$ metres Fin:

a its initial displacement velocity and acceleration

b when $x = 0$

c its velocity and acceleration at 2 s

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6. CHALLENGE EXERCISE

1 Find the equations of the tangents to the curve $y = x(x - 1)(x + 2)$ at the points where the curve cuts the x-axis

2 **a** Find the points on the curve $y = x^3 - 6$ where the tangents are parallel to the line $y = 12x - 1$.

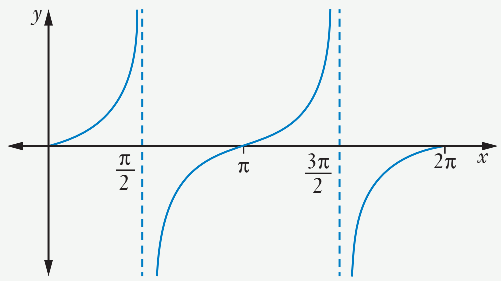
b Hence find the equations of the normals to the curve at those points

3 The normal to the curve $y = x^2 + 1$ at the point where $x = 2$ cuts the curve again at point P Find the coordinates of P

4 The equation of the tangent to the curve $y = x^4 - nx^2 + 3x - 2$ at the point where $x = -2$ is given by $3x - y - 2 = 0$ Evaluate n

5 **a** Find any points at which the graphed function is not differentiable

b Sketch the derivative function for the graph



6 Find the exact gradient of the tangent to the curve $y = \sqrt{x^2 - 3}$ at the point where $x = 5$.

7 Find the equation of the normal to the curve $y = 3\sqrt{x + 1}$ at the point where $x = 8$.

8 **a** Find the equations of the tangents to the parabola $y = 2x^2$ at the points where the line $6x - 8y + 1 = 0$ intersects with the parabola

b Show that the tangents are perpendicular.

9 Find any x values of the function $f(x) = \dfrac{2}{x^3 - 8x^2 + 12x}$ where it is not differentiable

10 Find the equation of the chord joining the points of contact of the tangents to the curve $y = x^2 - x - 4$ with gradients 3 and -1

11 For the function $f(x) = ax^2 + bx + c$ $f(2) = 4, f'(1) = 0$ and $f'(-3) = 8$ Evaluate a b and c

12 For the function $f(x) = x^3$

a Show that $f(x + h) = x^3 + 3x^2h + 3xh^2 + h^3$

b Show that $f'(x) = 3x^2$ by differentiating from first principles

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13 Consider the function $f(x) = \dfrac{1}{x}$

a Find the gradient of the secant between

i $f(1)$ and $f(11)$ **ii** $f(1)$ and $f(101)$ **iii** $f(1)$ and $f(099)$

b Estimate the gradient of the tangent to the curve at the point where $x = 1$.

c Show that $\dfrac{1}{x+h} - \dfrac{1}{x} = \dfrac{-h}{x(x+h)}$

d Hence show that $f'(x) = -\dfrac{1}{x^2}$ by differentiating from first principles

14 The displacement of a particle is given by $x = (t^3 + 1)^6$ where x is in metres and t is in seconds

a Find the initial displacement and velocity of the particle

b Find its acceleration after 2 s in scientific notation correct to 3 significant figure.

c Show that the particle is never at the origin

STATISTICAL ANALYSIS

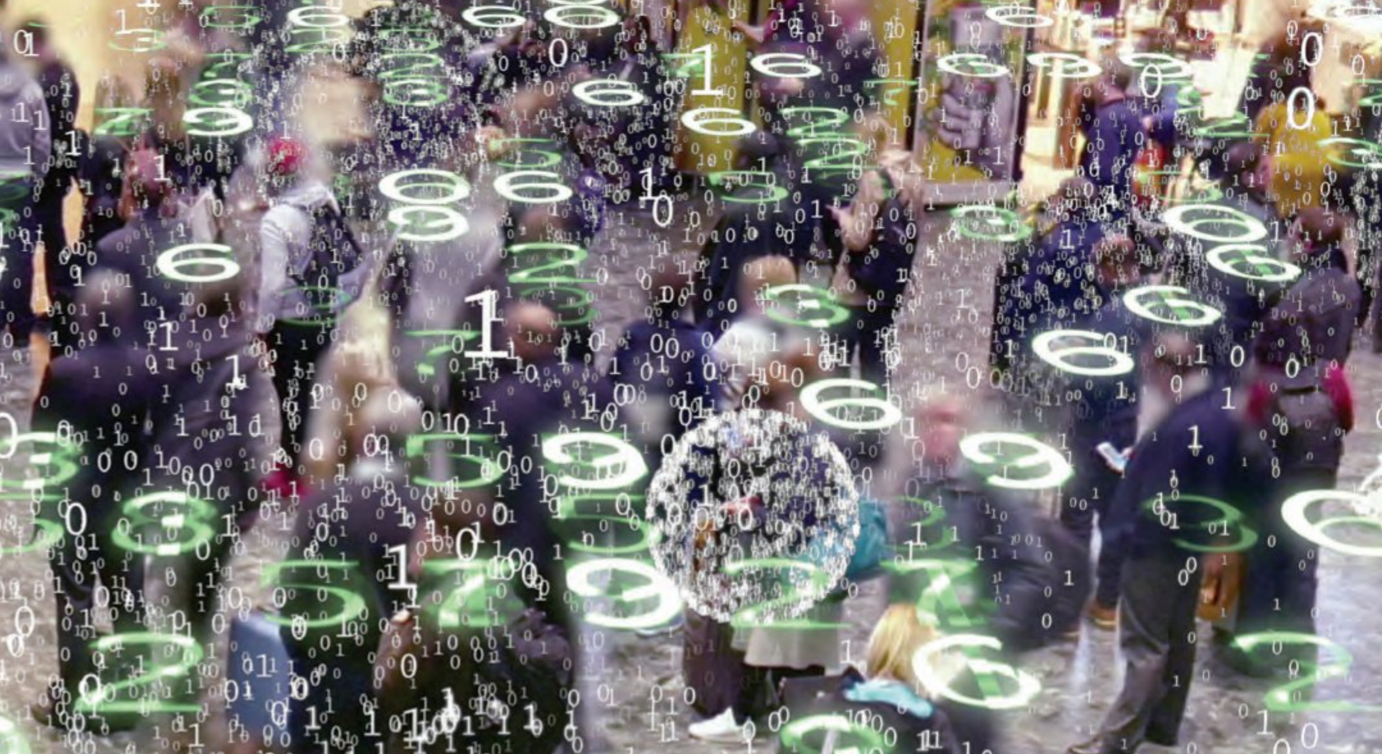
7.

PROBABILITY

Probablty s the study of how lkely t s that somethng wll happen It s used to make predctons and decsons n areas such as busness nvestment weather forecastng and nsurance Statstcs also s about analysng data to make decsons Probablty and statstcs are closely related

In ths chapter we wll look at how to fnd the probablty of somethng happenng usng both expermental data and theoretcal probablty.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- understand defntons of probablty and assocated terinologyincluing set notaio, event, outcomes and sample space
- dentfy dfferences between expeimental and theoreical probaiity and thir iitaions
- recognse non-mutually exclusve events and use technques to count outcomes n these cases
- dentfy condtonal probablty and calculate probabltes n these cases
- use probablty trees Venn iagrams and the adiion and product rules to calculate probaiiies

TERMINOLOGY

complement The complement of an event is when the event does not occur
conditional probability The probability that an event A occurs when it is known that another event B has occurred
equally likely outcoes: Outcomes that have the same chance of occurring
independent events Events where the occurrence of one event does not affect the probability of another event
mutually exclusive events Events within the same sample space that cannot both occur at the same time for example rolling an even number on a die and rolling a 5 on the same die
non-mutually exclusive events Events within the same sample space that can occur at the same time for example rolling a prime number on a die and rolling an odd number
probability tree A diagram that uses branches to show multi-stage events and sets out the probability on each branch
relative frequeny: The frequency of an event relative to the total frequency
sample space The set of all possible outcomes in an event
set A collection of distinct objects called elements or members
For example set $A = \{1, 2, 3, 4, 5, 6\}$
tree diagram A diagram that uses branches to show multi-stage events
Venn diagram A diagram that shows the relationship between 2 or more sets using circles (usually overlapping) drawn inside a rectangle

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Venn diagam

7.01 Set notation and Venn diagrams

Probability and statistics do not provide exact answers in real life but they can help in making decisions Here are some examples of where statistics and probability are use.

- An actuary is a mathematician who looks at statistics and makes decisions for insurance companies Life expectancy statistics help to decide the cost of life insurance for people of different ages Statistics about car accidents will help set car insurance premium.
- Stockbrokers use a chart or formula to predict when to buy and sell shares This chart is usually based on statistics of past trends
- A business does a feasibility study in a local area to decide whether to open up a new leisure centre It uses this data to make a decision based on the likelihood that local people will want to join the centre

Sample space

An **outcome** is a possible result of a random experiment

The **sample space** is the **set** of all possible outcomes for an experiment

An **event** is a set of one or more outcomes

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EXAMPLE 1

In a survey, a TV show is given a rating from 1 to10.

a Write down the sample spac.

b Give an example of

i an outcome **ii** an event

Solution

a The sample space is the set of all possible ratings from 1 to 10

Sample space = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

b **i** There are 10 different possible outcomes One outcome is a rating of .

ii One example of an event is ‘a rating higher than .

To find the probability of an event happenin, we compare the number of ways the event can occur with the total number of possible outcomes (the sample space)

$$\text{Probability of an event} = \frac{\text{Number of ways the event can occur}}{\text{Total number of possible outcomes}}$$

If we call the event E and the sample space S we can write this as

Probability formula

$$P(E) = \frac{n(E)}{n(S)}$$

where $P(E)$ means the probability of E $n(E)$ means the number of outcomes in E and $n(S)$ means the number of outcomes in the sample space S, where each **outcome** is **equally likely**

We usually write a probability as a fractio, but we could also write it as a decimal or percentae.

EXAMPLE 2

30 people were surveyed on their favourite sport 11 liked football 4 liked basketbal, 7 liked tennis 2 liked golf and 6 liked swimmin.

Find the probability that any one of these people selected at random will like

a swimming **b** football **c** golf

Solution

The size of the sample space $n(S) = 30$ since 30 people were surveyed

a $P(\text{swimming}) = \frac{6}{30}$
$= \frac{1}{5}$

b $P(\text{football}) = \frac{11}{30}$

c $P(\text{golf}) = \frac{2}{30}$
$= \frac{1}{15}$

Set notation

When working with probabilities we often use **set notation**

Union and intersection

$A \cup B$ means A **union** B and is the set of all elements in set A or set B

$A \cap B$ means A **intersection** B and is the set of all elements that are in **both** sets A and B

EXAMPLE 3

Set A contains the numbers 3 , 12 and 5.
Set B contains the numbers 2 , 2, 13 and17.

a Write sets A and B in set notation

b Find $A \cup B$ and $A \cap B$

Solution

a Set $A = \{3, 7, 12, 15\}$.

Set $B = \{2, 9, 12, 13, 17\}$.

b $A \cup B = \{2, 3, 7, 9, 12, 13, 1, 17\}$. It includes all the numbers in either set A or set B

$A \cap B = \{12\}$ It includes any numbers that are in both set A and set B

Venn diagrams

A **Venn diagram** is a special way to show the relationship between two or more sets

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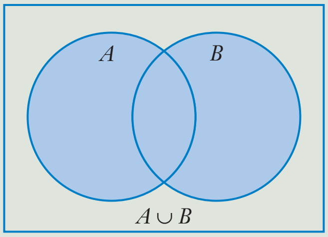
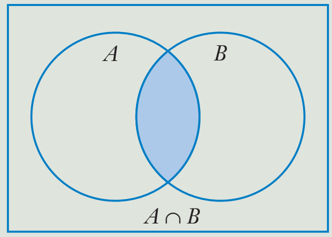
Venn diagram





EXAMPLE 4

Draw a Venn diagram for the integers from 1 to 0, given $A = \{1, 3, 4, 5, 8\}$ and $B = \{3, 6, 8, 9, 10\}$.

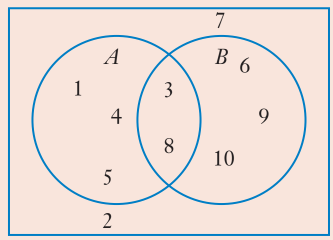
Solution

Draw two overlapping circles and name them A and B

$A \cap B = \{3$ 8, so place these numbers in the overlapping part

Place the remaining elements of A in the other part of circle A and the remaining elements of B in the other part of circle B

The numbers are 2 and 7 are not in A or B so place them outside the circles



Exercise 7.01 Set notation and Venn diagrams

1 Write the sample space in set notation for each chance situatio.

- **a** Tossing a coin
- **b** Rating a radio station between 1 and 5
- **c** Rolling a die
- **d** Selecting a jelly bean from a packet containing red gree, yellow and blue jelly beans
- **e** Rolling an 8-sided die with a different number from 1 to 8 on each face

2 For each pair of sets fin:

i $X \cap Y$ **ii** $X \cup Y$

- **a** $X = \{1, 2, 3, 4, 5\}$ and $Y = \{2, 4, 6\}$
- **b** X = {red yello, white} and Y = {red white}
- **c** $X = \{4, 5, 7, 11, 15\}$ and $Y = \{6, 8, 9, 10, 12\}$
- **d** X = {blue gree, bron, hazel} and Y = {brown gre, blue}
- **e** $X = \{1, 3, 5, 7, 9\}$ and $Y = \{2, 4, 6, 8, 10\}$

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3 Draw a Venn diagram for each pair of ses.

- **a** $A = \{10, 12, 13, 14, 15\}$ and $B = \{12, 14, 15, 16\}$
- **b** P = {red yello, white} and Q = {red gree, white}
- **c** $X = \{2, 3, 5, 7, 8\}$ and $Y = \{1, 2, 5, 7, 9, 10\}$
- **d** A = {Toyot, Maza, MW, Nssan, Porsche} and B = {Mazda Nissa, Holdn, Ford Porsche}
- **e** X = {rectangle squar, trapezium} and Y = {square parallelogra, trapezim, kite}

4 Discuss whether each probability statement is true

- **a** The probability of one particular horse winning the Melbourne Cup is $\frac{1}{20}$ if there are 20 horses in the race
- **b** The probability of a player winning a masters golf tournament is $\frac{1}{15}$ if there are 15 players in the tournament
- **c** A coin came up tails 8 times in a row. So the next toss must be a hed.
- **d** A family has 3 sons and is expecting a fourth child There is more chance of the new baby being a daughter.
- **e** The probability of a Ducati winning a MotoGP this year is $\frac{6}{47}$ because there are 6 Ducatis and 47 motobikes altogether.

5 To start playing a board gam, Simone must roll an even number on a de.

- **a** Write down the sample space for rolling a di.
- **b** What is the set of even numbers on a die?

6 Draw a Venn diagram for each pair of ses.

- **a** Event K = {Monday, Thursay, Friday} and Event L = {Tuesda, Thurday, Saturday} out of days of the week
- **b** Event $A = \{3, 5, 6, 8\}$ and Event $B = \{4 ,7, 9\}$ with cards each with a number from 1 to 10 drawn out of a hat

DID YOU KNOW?

John Venn

Venn diagrams are named after **John Venn** (1834–1923) an English probabilist and logician

WS
Expeimenal pobabiliy

7.02 Relative frequency

We can use frequency distribution tables to find the probability of an event using **relative frequency** the frequency of the event relative to the total frequenc.

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EXAMPLE 5

This table shows the number of items bought by a group of people surveyed in a shopping centre

Number of items	Frequency
0	6
1	4
2	5
3	3
4	7

a Find the relative frequency for each number of items as a fraction

b Find the probability that a person surveyed at random would buy

i no items

ii at least 3 items

Solution

a The sum of the frequencies is 25 This means that 25 people were surveyd.

From the table 0 has a frequency of . The relative frequency is 6 out of 25= $\frac{6}{25}$
Similarly, other relative frequencies ae:

1 item $\frac{4}{25}$ 2 item: $\frac{5}{25} = \frac{1}{5}$ 3 item: $\frac{3}{25}$ 4 item: $\frac{7}{25}$

b **i** $P(0) = \frac{6}{25}$

ii At least 3 items means 3 or 4 items Relative frequency of 3 or 4 is 3 + 7 = 10.

$P(\geq 3) = \frac{10}{25} = \frac{2}{5}$

Exercise 7.02 Relative frequency

1 The table shows the scores that a class earned on a maths test

Score	Frequency
4	6
5	4
6	1
7	7
8	2
9	3

a Find the relative frequency for each score in the table in fraction for.

b If a student is chosen at random from this class find the probability that this student

i scored 8

ii scored less than 7

iii passed if the pass mark is .

c What score is

i most likely?

ii least likely?

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2 The table shows the results of a survey into the number of days students study each week

Number of days	Frequency
1	3
2	6
3	1
4	7
5	2
6	1

a Find the relative frequency as a percentage for each number of days

b If a student was selected at random find the most likely number of days this student studies

c Find the probability that this student would study for

i 1 day

ii 5 days

iii 3 or 4 days

v at least 4 days

v fewer than 3 days

3 The table shows the results of a trial HSC exam

Class	Frequency
0–19	9
20–39	12
40–59	18
60–79	7
80–99	4

a Calculate the relative frequency as a decimal for each class

b Find the probability that a student chosen at random from these students scored

i between 20 and 39

ii between 60 and 99

iii less than 40

4 This table shows the results of a science experiment to find the velocity of an object when it is rolled down a ramp

Velocity (m/s)	Frequency
2–4	2
5–7	7
8–10	4
11–13	1
14–16	6

a Write the relative frequency of each velocity as a fraction

b Find the probability that an object selected at random rolls down the ramp with a velocity between

i 5 and 7 m/s **ii** 11 and 13 m/s **iii** 8 and 10 m/s

v 11 and 16 m/s **v** 2 and 10 m/s

c Find the probability that the object has a velocity

i less than 8 m/s **ii** 5 m/s or more **iii** more than 7 m/s

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5 A telemarketing company records the number of sales it makes per minute over a half-hour period The results are in the tabe.

Sales/min	Frequency
0	4
1	12
2	6
3	3
4	0
5	5

a What percentage of the time were there 3 sales per minute?

b Write the relative frequencies as percentages

c What is the most likely number of sales/minute?

d Find the probability of making

i 2 sales/minute **ii** 5 sales/minute

iii more than 2 sales per minute

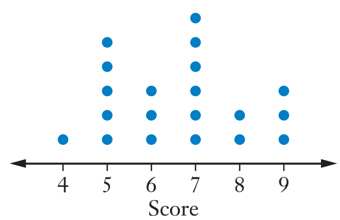
6 **a** Organise the scores below in a frequency distribution table

9, 5, 4, 7, 7, 9, 4, 6, 5, 8, 9, 6, 7, 4, 4, 3, 8, 5, 6, 9

b Find the probability of an outcome chosen at random having a score of

i 7 **ii** at least 8 **iii** less than 5 **v** 7 or less

7 **a** From the dot plot draw up a frequency distribution table



b Find the probability (as a decimal) that an outcome chosen at random has a score of

i 8 **ii** at least 6

iii less than 7 **v** 5 or more

v 8 or less

8 The stem-and-leaf plot shows the ages of people attending a meeting

Stem	Leaf
1	8 9 9
2	0 3 5 6
3	0 1 2 2 3 5 7 9
4	2 4 6 7 8
5	1 2 4 4 6

a Organise this data into a frequency distribution table using groups of 10–19 20–29 and so o.

b What percentage of people at the meeting were

i in their 30s?

ii younger than 20?

iii in their 40s or 50s?

c Find the probability that a person selected at random from this meeting is

i younger than 40

ii 50 or over

iii between 20 and 49

v over 29

v between 10 and 49

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9 The table shows the quantity of food that a pet shop uses each day for a month

Food (kg)	Frequency
0–14	3
15–29	11
30–44	8
45–59	4
60–74	2

a In which month was this survey done?

b For what fraction of the month was 45–59 kg of food used?

c For what percentage of the month did the pet shop need more than 29 kg of pet food?

d Find the relative frequency for all groups as a fraction

e If this survey is typical of the quantities of food that the pet shop uses find the probability that on any day it will use between

i 30 and 44 kg **ii** 45 and 74 kg **iii** 0 and 29 kg

v 15 and 59 kg **v** 30 and 74 kg

7.03 Theoretical probability

WS Theoeical pobabiliy

Maching pobabiliie

While experiments and surveys can give a good prediction of the probability of future events they are not very accurate The larger the number of trias, the closer the results can become to the theoretical probability. Howevr, this is not guaraneed.

For example it is reasonable to assume that if you toss a coin many times you would get similar numbers of heads and tails Yet in an experiment a coin may come up heads every tme.

INVESTIGATION

TOSSING A COIN

Toss a coin 20 times and count the number of heads and tail. What would you expect to happen when tossing a coin this many times? Did your results surprise you?

Combine your results with others in your classroom into a table with relative frequencies

1 Do the combined results differ from your own?

2 From the table find the probability of tossing

a heads **b** tails

If a coin came up tails every time it was tossed 20 times do you think it would it be more likely to come up heads the next time? Why?

Even though we might think that theoretical probability should be more accurate than experiments in real life these probabilities will not happen exactly as in theor!

Mutually exclusive events are events that cannot occur at the same time For exampl, when throwing a die you cannot throw a number that is both a 5 and a 6

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The addition rule for mutually exclusive events

When events A and B are mutually exclusive

$$P(A \cup B) = P(A) + P(B)$$

EXAMPLE 6

A container holds 5 blue 3 white and 7 yellow marble. If one marble is selected at random find the probability of selectin:

a a white marble

b a white or blue marble

c a yellow, white or blue marble

d a red marble

Solution

Blue whte and yellow are mutually exclusve events

$n(S) = 5 + 3 + 7 = 15$

a $P(\text{W}) = \frac{3}{15} = \frac{1}{5}$

b $P(\text{W} \cup \text{B}) = P(\text{W}) + P(\text{B}) = \frac{3}{15} + \frac{5}{15} = \frac{8}{15}$

c $P(\text{Y} \cup \text{W} \cup \text{B}) = P(\text{Y}) + P(\text{W}) + P(\text{B}) = \frac{7}{15} + \frac{3}{15} + \frac{5}{15} = \frac{15}{15} = 1$

d $P(\text{R}) = \frac{0}{15} = 0$

The range of probabilities

If $P(E) = 0$ the event is impossible

If $P(E) = 1$ the event is certain (it has to happen)

$0 \le P(E) \le 1$

The sum of all (mutually exclusive) probabilities is 1

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Complementary events

The **complement** of set E is the set of all elements that are not in E We write $\overline{E}$ or E^c

The **complement** $\overline{E}$ of an event E happening is the event **not** happening

$P(\overline{E}) = 1 - P(E)$

$P(E) + P(\overline{E}) = 1$

EXAMPLE 7

a The probability of winning a raffle is $\frac{1}{350}$ What is the probability of not winning?

b The probability of a tree surviving a fire is 72% Find the probability of the tree failing to survive a fire

Solution

a

$$P(\text{not win}) = 1 - P(\text{win})$$
$$= 1 - \frac{1}{350}$$
$$= \frac{349}{350}$$

b

$$P(\text{failing to survive}) = 1 - P(\text{surviving})$$
$$= 100\% - 72\%$$
$$= 28\%$$

We can use probability to make predictions or decision.

EXAMPLE 8

The probability that a traffic light will turn green as a car approaches it is $\frac{5}{12}$ A taxi goes through 192 intersections where there are traffic lights How many of these would be expected to turn green as the taxi approached?

Solution

It is expected that $\frac{5}{12}$ of the traffic lights would turn green

$$\frac{5}{12} \times 192 = 80$$

So it would be expected that 80 traffic lights would turn green as the taxi approached

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Exercise 7.03 Theoretical probability

1 Alannah is in a class of 30 students If one student is chosen at random to make a speec, find the probability that the student chosen

a will be Alannah **b** will not be Alanna.

2 A pack of cards contains 52 different cards one of which is the ace of diamond. If one card is chosen at random find the probability that i:

a will be the ace of diamonds **b** will not be the ace of diamonds

3 There are 6 different newspapers sold at the local newsagent each day. Wendy sends her little brother Rupert to buy her a newspaper one morning but forgets to tell him which one What is the probability that Rupert will buy the correct newspaper ?

4 A raffle is held in which 200 tickets are sold If I buy 5 ticket, what is the probability f:

a my winning **b** my not winning the prize in the raffle?

5 In a lottery, 200 000 tickets are sod. If Lucia buys 10 tickts, what is the probability of her winning first prize?

6 A bag contains 6 red balls and 8 white balls If Peter draws one ball out of the bag at random find the probability that it will b:

a white **b** red

7 A shoe shop orders in 20 pairs of black 14 pairs of navy and 3 pairs of brown school shoes If the boxes are all mixed u, find the probability that one box selected at random will contain brown shoes

8 The probability of a bus arriving on time is estimated at $\frac{18}{33}$

a What is the probability that the bus will not arrive on time?

b If there are 352 buses each day, how many would be expected to arrive on time?

9 A bag contains 5 black marbles 4 yellow marbles and 11 green marble. Find the probability of drawing 1 marble out at random and getting

a a green marble **b** a yellow or a green marble

10 The probability of a certain seed producing a plant with a pink flower is $\frac{7}{9}$

a Find the probability of the seed producing a flower of a different colour.

b If 189 of these plants are grown how many of them would be expected to have a pink flower?

11 If a baby has a 02% chance of being born with a disability, find the probability of the baby being born without a disability.

12 A die is thrown Calculate the probability of throwin:

a a 6 **b** an even number **c** a number less than 3

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13 A book has 124 pages If any page is selected at rando, find the probability of the page number being

a either 80 or 90 **b** a multiple of 10

c an odd number **d** less than 100

14 A machine has a 15% chance of breaking down at any given time

a What is the probability of the machine not breaking down?

b If 2600 of these machines are manufactured how many of them would be expected t:

i break down? **ii** not break down?

15 The probabilities when 3 coins are tossed are as follows

$P(3\text{ heads}) = \frac{1}{8}$ $P(2\text{ heads}) = \frac{3}{8}$

$P(1\text{ head}) = \frac{3}{8}$ $P(3\text{ tails}) = \frac{1}{8}$

Find the probability of tossing at least one head

16 In the game of pool there are 15 ball, each with the number 1 to 15 on t. In Kelly pool each person chooses a number at random to determine which ball to sin. If Tracey chooses a number, find the probability that her ball will e:

a an odd number **b** a number less than 8 **c** the 8 ball

17 **a** Find the probability of a coin coming up heads when tossed

b If the coin is double-headed find the probability of tossing a hea.

18 A student is chosen at random to write about his or her favourite sport If 12 students like tennis best 7 prefer socce, 3 prefer sqush, 5 prefer basketball and 4 prefer swimming find the probability that the student chosen will write abou:

a soccer **b** squash or swimming **c** tennis

19 There are 29 red 17 blu, 21 yellow and 19 green chocolate beans in a packt. If Kate chooses one at random find the probability that it will be red or yello.

20 The probability of breeding a white budgerigar is $\frac{2}{9}$ If Mr Seed breeds 153 budgerigars over the year, how many would be expected to be white?

21 A biased coin is weighted so that heads comes up twice as often as tails Find the probability of tossing a tail

22 A die has the centre dot painted white on the 5 so that it appears as a 4 Find the probability of throwing

a a 2 **b** a 4 **c** a number less than 5

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23 The probabilities of a certain number of seeds germinating when 4 seeds are planted ar:

Number of seeds	0	1	2	3	4
Probability	$\frac{3}{49}$	$\frac{18}{49}$	$\frac{16}{49}$	$\frac{8}{49}$	$\frac{4}{49}$

Find the probability of at least one seed germinating

24 The probabilities of 4 friends being chosen for a soccer team are

$P(4 \text{ chosen}) = \frac{1}{15}$ $P(3 \text{ chosen}) = \frac{4}{15}$

$P(2 \text{ chosen}) = \frac{6}{15}$ $P(1 \text{ chosen}) = \frac{2}{15}$

Find the probability of

a none of the friends being chosen

b at least 1 of the friends being chosen

25 If 2 events are mutually exclusive what could you say about $A \cap B$?

DID YOU KNOW?

The origins of probability

Girolamo Cardano (1501–76) was a doctor and mathematician who developed the first theory of probability. He was a great gamblr, and he wrote *De Ludo Aleae* (On Games of Chance) This work was largely ignord, and it is said that the first book on probability was written by **Christiaan Huygens** (1629–95)

The main study of probability was done by **Blaise Pascal** (1623–62) and **Pierre de Fermat** (1601–65) Pascal developed the'arithmetical triange' called Pascl's triangle that has properties that are applicable to probability as well

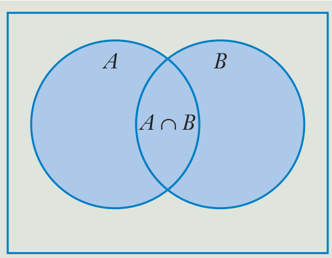
7.04 Addition rule of probability

Sometimes there is an overlap where more than one event can occur at the same tim. We call these **non-mutually exclusive events** It is important to count the possible outcomes carefully when this happens We need to be careful not to count the overlapping outcomes $A \cap B$ twice

Addition rule of probability

For events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



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If A and B are mutually exclusive then $P(A \cap B) = 0$ so $P(A \cup B) = P(A) + P(B)$

EXAMPLE 9

One card is selected at random from a pack of 100 cards numbered from 1 to 100
Find the probability that the number on this card is even or less than 20

Solution

Even $A = \{2, 4, 6, \ldots, 100\}$

There are 50 even numbers between 1 and 100

Less than 20 $B = \{1, 2, 3, \ldots, 19\}$

There are 19 numbers less than 20

Even and less than 20 $A \cap B = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$

There are 9 numbers that are both even and less than 20

P(Even or less than 20)

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{50}{100} + \frac{19}{100} - \frac{9}{100} \\ &= \frac{60}{100} \\ &= \frac{3}{5} \end{aligned}$$

Ths s to avod countng the 9 overlappng numbers twce

Sometimes for more complex problems a Venn diagram is useul.

EXAMPLE 10

In Year 7 at Mt Random High Schol, every student must do art or muic. In a group of 100 students surveyed 47 do music and 59 do ar. If one student is chosen at random from Year , find the probability that this student dos:

a both art and music **b** only art **c** only music

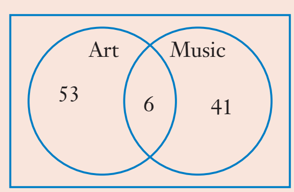
Solution

Number of students 47 + 59 = 106

But there are only 100 students

This means 6 students have been counted twice

That is 6 students do both art and musi.



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Students doing art only $59 - 6 = 53$

Students doing music only $47 - 6 = 41$

a $P(\text{both}) = \dfrac{6}{100} = \dfrac{3}{50}$

b $P(\text{art only}) = \dfrac{53}{100}$

c $P(\text{music only}) = \dfrac{41}{100}$

Exercise 7.04 Addition rule of probability

1 A number is chosen at random from the numbers 1 to 20 Find the probability that the number chosen will be

a divisible by 3

b less than 10 or divisible by 3

c a composite number

d a composite number or a number greater than 12

2 A set of 50 cards is labelled from 1 to 50 One card is drawn out at rando. Find the probability that the card will be

a a multiple of 5

b an odd number

c a multiple of 5 or an odd number

d a number greater than 40 or an even number

e less than 20

3 A set of 26 cards each with a different letter of the alphabet on i, is placed in a box and one card is drawn out at random Find the probability that the letter on the card i:

a a vowel

b a vowel or one of the letters in the word 'rando'

c a consonant or one of the letters in the word 'movie.

4 A set of discs is numbered 1 to 100 and one is chosen at random Find the probability that the number on the disc will be

a less than 30

b an odd number or a number greater than 70

c divisible by 5 or less than 20

5 In Lotto a machine holds 45 ball, each with a different number between 1 and 45 on t. The machine draws out one ball at a time at random Find the probability that the first ball drawn out will be

a less than 10 or an even number

b between 1 and 15 inclusive or divisible by 6

c greater than 30 or an odd number.

6 A class of 28 students puts on a concert with all class members performing If 15 dance and 19 sing in the performance find the probability that any one student chosen at random from the class will

a both sing and dance

b only sing

c only dance

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7 A survey of 80 people with dark hair or brown eyes showed that 63 had dark hair and 59 had brown eyes Find the probability that one of the people surveyed chosen at random has

a dark hair but not brown eyes

b brown eyes but not dark hair

c both brown eyes and dark hair.

8 A list is made up of 30 people with experience in coding or graphical design On the list 13 have coding experience while 9 have graphical design experienc. Find the probability that a person chosen at random from the list will have experience in

a both coding and graphical design

b coding only

c graphical design only.

9 Of a group of 75 students all study either history or geograph. Altogether 54 take history and 31 take geography. Find the probability that a student selected at random studies

a only geography

b both history and geography

c history but not geography.

10 In a group of 20 dogs at obedience school 14 dogs will walk to heel and 12 will stay when told All dogs will do one or the othr, or oth. If one dog is chosen at rndom, find the probability that it will

a both walk to heel and stay

b walk to heel but not stay

c stay but not walk to heel

WS Muliage poblem

7.05 Product rule of probability

CLASS DISCUSSION

TWO-STAGE EVENTS

Work in pairs and try these experiments with one person doing the activity and one recording the results Toss two coins as many times as you can in a 5-minute period and record the results in a table

Result	2 heads	One head and one tail	2 tails
Tally			

Compare your results with others in the class What do you notice ? Is this surprising?

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Roll 2 dice as many times as you can in a 5-minute period find the total of the 2 numbers rolled and record the results in a table

Total	2	3	4	5	6	7	8	9	10	11	12
Tally											

Compare your results with others in the class What do you notice ? Is this surprising?

Tossing 2 coins and rolling 2 dice are examples of **multi-stage experiments** where two outcomes happen together. The sample space becomes more complicaed, so to list all possible outcomes we use tables and **tree diagrams**

EXAMPLE 11

Find the sample space and the probability of each outcome for

a tossing 2 coins

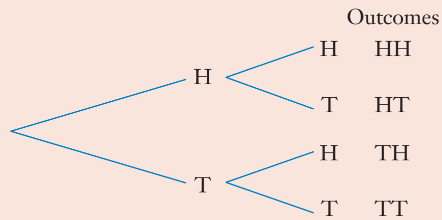
b rolling 2 dice and calculating their sum

Solution

a Using a table gives

		2nd coin	
		H	**T**
1st coin	**H**	HH	HT
	T	TH	TT

Using a tree diagram gives



Since there are four possible outcomes (HH HT,TH, TT) each outcome has a probability of $\frac{1}{4}$

Remember that each outcome when tossing 1 coin is $\frac{1}{2}$

Notice that $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

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b A tree diagram would be too big to draw for this question

Using a table

		2nd die					
		1	**2**	**3**	**4**	**5**	**6**
1st die	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Since there are 36 outcomes each has a probability of $\frac{1}{36}$

Remember that each outcome when rolling 1 die is $\frac{1}{6}$

Notice that $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

If A and B are **independent events** then A occurring does not affect the probability of B occurring The probability of both occurring is the product of their probabilitis.

The product rule for independent events

$$P(A \cap B) = P(A)P(B)$$

EXAMPLE 12

a Find the probability of rolling a double 6 on 2 dice

b The probability that an archer will hit a target is $\frac{7}{8}$ Find the probability that the archer will

i hit the target twice

ii miss the target twice

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Solution

a $P(A \cap B) = P(A)P(B)$

$$P(6 \cap 6) = P(6)P(6)$$
$$= \frac{1}{6} \times \frac{1}{6}$$
$$= \frac{1}{36}$$

b **i** $P(A \cap B) = P(A)P(B)$

Let H = hit, M = miss

$$P(\text{H} \cap \text{H}) = P(\text{H})P(\text{H})$$
$$= \frac{7}{8} \times \frac{7}{8}$$
$$= \frac{49}{64}$$

ii $P(\text{M}) = P(\bar{\text{H}})$

$$= 1 - \frac{7}{8}$$
$$= \frac{1}{8}$$

$$P(\text{M} \cap \text{M}) = P(\text{M})P(\text{M})$$
$$= \frac{1}{8} \times \frac{1}{8}$$
$$= \frac{1}{64}$$

The sample space changes when events are not independent The second event is **conditional** on the first event

EXAMPLE 13

Maryam buys 5 tickets in a raffle in which 95 tickets are sold altogether. There are 2 prizes in the raffle What is the probability of her winnig:

a both first and second prizes?

b neither prize?

c at least one of the prizes?

Solution

a Probability of winning first prize $P(\text{W} \ ) = \frac{5}{95}$

After winning first prize she has 4 tickets left in the raffle out of a total of 94 tickets left

Probability of winning second prize $P(\text{W}_2) = \frac{4}{94}$

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Probability of winning both prizes

$$P(W_1 \cap W_2) = \frac{5}{95} \times \frac{4}{94}$$
$$= \frac{20}{8930}$$
$$= \frac{2}{893}$$

b Probability of not winning first prize

$$P(\overline{W}\ ) = 1 - \frac{5}{95}$$
$$= \frac{90}{95}$$
$$= \frac{18}{19}$$

After not winning first prize Marya’s 5 tickets are all left in the drw, but the winning ticket is taken out leaving 94 tickets in the raffl.

Probability of winning second prize $P(W_2) = \frac{5}{94}$

Probability of not winning second prize

$$P(\overline{W_2}) = 1 - \frac{5}{94}$$
$$= \frac{89}{94}$$

Probability of winning neither prize

$$P(\overline{W}_1 \cap \overline{W}_2) = P(\overline{W}_1)P(\overline{W}_2)$$
$$= \frac{90}{95} \times \frac{89}{94}$$
$$= \frac{8010}{8930}$$
$$= \frac{801}{893}$$

c Probability of at least one win ← 'at least one' s the same as not none

$$P(\geq 1 \text{ win}) = 1 - P(0 \text{ wins})$$
$$= 1 - \frac{801}{893} \quad \text{from } \mathbf{b}$$
$$= \frac{92}{893}$$

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Exercise 7.05 Product rule of probability

1 Find the probability of getting 2 heads if a coin is tossed twice

2 A coin is tossed 3 times Find the probability of tossing 3 tail.

3 A family has 2 children What is the probability that they are both girls ?

4 A box contains 2 black balls 5 red balls and 4 green ball. If I draw out 2 balls at randm, replacing the first before drawing out the second find the probability that they will both be red

5 The probability of a conveyor belt in a factory breaking down at any one time is 021 If the factory has 2 conveyor belts find the probability that at any one tim:

a both conveyor belts will break down

b neither conveyor belt will break down

6 The probability of a certain plant flowering is 93% If a nursery has 3 of these plant, find the probability that they will all flower.

7 An archery student has a 69% chance of hitting a target If she fires 3 arrows at a targe, find the probability that she will hit the target each time

8 The probability of a pair of small parrots breeding an albino bird is $\frac{2}{33}$ If they lay 3 eggs find the probability of the pai:

a not breeding any albinos **b** having all 3 albinos

c breeding at least one albino

9 A photocopier has a paper jam on average around once every 2400 sheets of paper.

a What is the probability that a particular sheet of paper will jam?

b What is the probability that 2 particular sheets of paper will jam?

c What is the probability that 2 particular sheets of paper will both not jam?

10 In the game Yahtze, 5 dice are roled. Find the probability of roling:

a five 6s **b** no 6s **c** at least one 6

11 The probability of a faulty computer part being manufactured at Omikron Computer Factory is $\frac{3}{5000}$ If 2 computer parts are examine, find the probability tht:

a both are faulty **b** neither is faulty

c at least one is faulty.

12 A set of 10 cards is numbered 1 to 10 and 2 cards are drawn out at random with replacement Find the probability that the numbers on both cards ar:

a odd numbers **b** divisible by 3

c less than 4

13 The probability of an arrow hitting a target is 85% If 3 arrows are sho, find the probability as a percentage correct to 2 decimal place, f:

a all arrows hitting the target

b no arrows hitting the target

c at least one arrow hitting the target

14 A coin is tossed n times Find the probability in terms of n of tossing

a no tails

b at least one tail

15 A bag contains 8 yellow and 6 green lollies If I choose 2 lollies at rando, find the probability that they will both be green

a if I replace the first lolly before selecting the second

b if I dont replace the first loll.

16 Mala buys 10 tickets in a raffle in which 250 tickets are sold Find the probability that she wins both first and second prizes

17 Two cards are drawn from a deck of 20 red and 25 blue cards (without replacement. Find the probability that they will both be red

18 A bag contains 100 cards numbered 1 to 100 Scott draws 2 cards out of the ba. Find the probability that

a both cards are less than 10

b both cards are even

c neither card is a multiple of 5

19 A box of pegs contains 23 green pegs and 19 red pegs If 2 pegs are taken out of the box at random find the probability that both will b:

a green

b red

20 Find the probability of selecting 2 apples at random from a fruit bowl that contains 8 apples 9 oranges and 3 peache.

WS

Tree diagam

Tree diagam

Tree diagam

7.06 Probability trees

A **probability tree** is a tree diagram that shows the probabilities on the branches

Probability trees

Use the product rule along the branches to find $P(A \cap B)$ the probability of' A and B

Use the addition rule for different branches to find $P(A \cup B)$ the probability of' A or B

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EXAMPLE 14

WS
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ee

a Robert has a chance of 02 of winning a prize in a Taekwondo competitin. If he enters 3 competitions find the probability of his winnin:

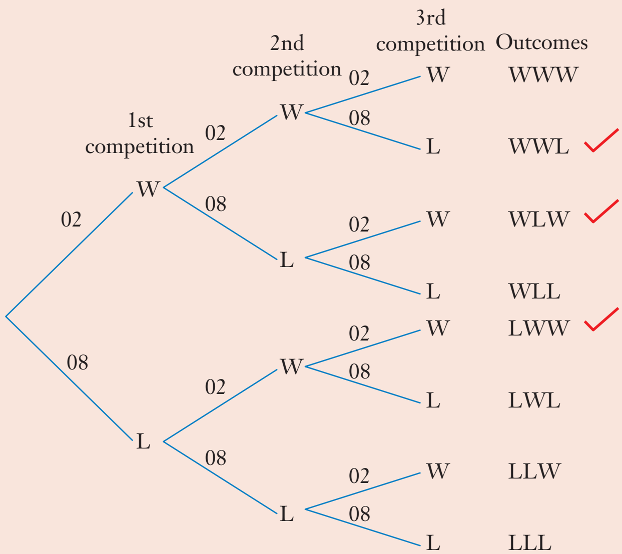
i 2 competitions **ii** at least 1 competition

b A bag contains 3 red 4 white and 7 blue marble. Two marbles are drawn at random from the bag without replacement Find the probability that the marbles are red and whit.

Solution

a $P(\text{W}) = 02,\ P(\text{L}) = 1 - 02\ = 08$ W = win L = lose

Draw a probability tree with 3 levels of branches as shown



i There are 3 different ways of winning 2 competitions (WWL WLW and WW, shown by the red ticks)

Using the product rule along the branches

$P(\text{WWL}) = 02\ \times 02\ \times 08\ = 0032$ P(win and win and lose)

$P(\text{WLW}) = 02\ \times 08\ \times 02\ = 0032$

$P(\text{LWW}) = 08\ \times 02\ \times 02\ = 0032$

Using the addition rule for the different results

$P(2\text{ wins}) = P(\text{WWL}) + P(\text{WLW}) + P(\text{LWW})$ P(WWL or WLW or LWW)

$= 0032\ + 0032\ + 0032$

$= 0096$

ii $P(\geq 1W) = 1 - P(LLL)$

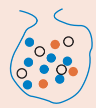
$= 1 - 0.8 \times 08 \times 08$

$= 0488$

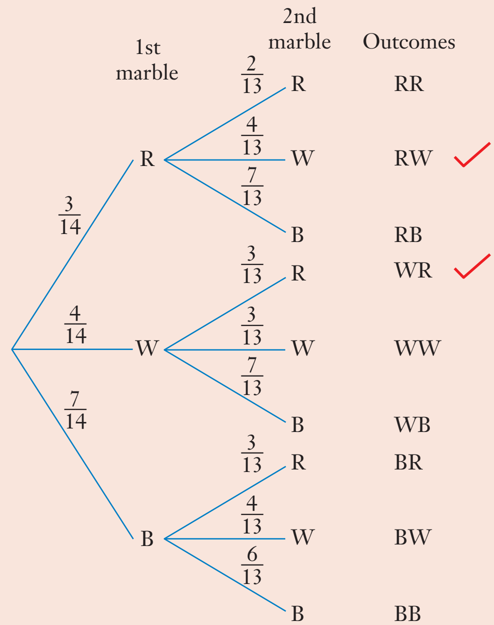
b First marble

$P(R) = \frac{3}{14}$ $P(W) = \frac{4}{14}$

$P(B) = \frac{7}{14}$



The probabilities for the second marble are dependent on the outcome of the first draw.



There are 2 different ways of drawing out a red and a white marble as shown by the red ticks RWR.

Using the product rule along the branches

$$P(RW) = \frac{3}{14} \times \frac{4}{13} = \frac{12}{182} = \frac{6}{91}$$

$$P(WR) = \frac{4}{14} \times \frac{3}{13} = \frac{12}{182} = \frac{6}{91}$$

Using the addition rule for the different results

$$P(RW \text{ or } WR) = \frac{6}{91} + \frac{6}{91} = \frac{12}{91}$$

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Exercise 7.06 Probability trees

1 Three coins are tossed Find the probability of gettin:

a 3 tails **b** 2 heads and 1 tail **c** at least 1 head

2 In a set of 30 cards each one has a number on it from 1 to 3. If 1 card is drawn ot, then replaced and another drawn out find the probability of gettin:

a two 8s

b a 3 on the first card and an 18 on the second card

c a 3 on one card and an 18 on the other card

3 A bag contains 5 red marbles and 8 blue marbles If 2 marbles are chosen at rando, with the first replaced before the second is drawn out find the probability of gettin:

a 2 red marbles **b** a red and a blue marble

4 A certain breed of cat has a 35% probability of producing a white kitten If a cat has 3 kittens find the probability that she will produc:

a no white kittens **b** 2 white kittens **c** at least 1 white kitten

5 The probability of rain on any day in May each year is given by $\frac{3}{10}$ A school holds a fete on a Sunday in May for 3 years running Find the probability that it will rai:

a during 2 of the fetes **b** during 1 fete **c** during least 1 fete

6 A certain type of plant has a probability of 085 of producing a variegated leaf If I grow 3 of these plants find the probability of getting a variegated leaf i:

a 2 of the plants **b** none of the plants **c** at least 1 plant

7 A bag contains 3 yellow balls 4 pink balls and 2 black ball. If 2 balls are chosen at random find the probability of getting a yellow and a black bal:

a with replacement **b** without replacement

8 Anh buys 4 tickets in a raffle in which 100 tickets are sold altogether. There are 2 prizes in the raffle Find the probability that Anh will wn:

a first prize **b** both prizes **c** 1 prize

d no prizes **e** at least 1 prize

9 Two singers are selected at random to compete against each other in a TV singing contest One person is chosen from Tea A, which has 8 females and 7 ales, and the other is chosen from TeamB, which has 6 females and 9 maes. Find the probability of choosing

a 2 females **b** 1 female and 1 male

10 Two tennis players are said to have a probability of $\frac{2}{5}$ and $\frac{3}{4}$ respectively of winning a tournament Find the probability tha:

a 1 of them will win **b** neither one will win

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11 In a batch of 100 cars past experience would suggest that 3 could be fault. If 3 cars are selected at random find the probability tha:

a 1 is faulty **b** none is faulty **c** all 3 cars are faulty.

12 In a certain poll 46% of people surveyed liked the current governmen, 42% liked the Opposition and 12% had no preference If 2 people from the survey are selected at random find the probability tha:

a both will prefer the Opposition

b one will prefer the government and the other will have no preference

c both will prefer the government

13 A manufacturer of X energy drink surveyed a group of people and found that 31 people liked X drinks best 19 liked another brand better and 5 did not drink energy drink. If any 2 people are selected at random from that group find the probability tha:

a one person likes the X brand of energy drink

b both people do not drink energy drinks

14 In a group of people 32 are Australian-bon, 12 were born in Asia and 7 were born in Europe If 2 of the people are selected at rando, find the probability tht:

a they were both born in Asia

b at least 1 of them will be Australian-born

c both were born in Europe

15 There are 34 men and 32 women at a party. Of thee, 13 men and 19 women are married If 2 people are chosen at rando, find the probability tht:

a both will be men

b one will be a married woman and the other an unmarried man

c both will be married

16 Frankie rolls 3 dice Find the probability she roll:

a 3 sixes **b** 2 sixes **c** at least 1 six

17 A set of 5 cards each labelled with one of the letters A, B C, D nd E, is placed in a hat and 2 cards are selected at random without replacement Find the probability of gettin:

a D and E

b neither D nor E on either card

c at least one D.

18 The ratio of girls to boys at a school is 4 5 Two students are surveyed at random from the school Find the probability that the students ar:

a both boys **b** a girl and a boy **c** at least one girl

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7.07 Conditional probability

Conditional probability is the probability that an event A occurs when it is known that another event B has already occurred You have already used conditional probability in multi-stage events when the outcome of the second event was dependent on the outcome of the first event Examples include selections **without replacement**

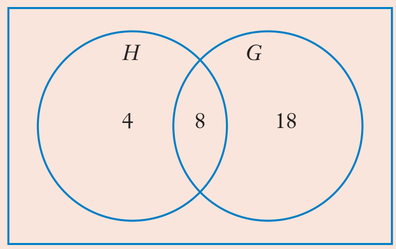
We write the probability of event A happening given that event B has happened as $P(A|B)$

EXAMPLE 15

All 30 students in a class study either history or geography. If 18 only do geography and 8 do both subjects find the probability that a student does geograph, given that the student does history.

Solution

Draw a Venn diagram using H = history and G = geography.



8 students do both history and geography.

18 students only do geography.

So $n(G) = 18 + 8 = 26$

So $n(H \text{ only}) = 30 - 26 = 4$

There are $8 + 4 = 12$ students doing histor, of whom 8 also do geograhy.

$$\text{So } P(G|H) = \frac{8}{12} = \frac{2}{3}$$

With conditional probabilit, knowing that an event has already occurred reduces the sample space In the example abov, the sample space changed from 30 to 2.

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EXAMPLE 16

The table shows the results of a survey into vaccinations against a new virus

	Vaccinated	Not vaccinated	Totals
Infected	13	159	172
Not infected	227	38	265
Totals	240	197	437

Find the probability that a person selected at random is

a not vaccinated

b infected given that the person is vaccinated

c not infected given that the person is not vaccinated

d vaccinated given that the person is infected

Solution

a $n(S) = 437$, $n(\text{not vaccinated}) = 197$

$P(\text{not vaccinated}) = \dfrac{197}{437}$

b $n(\text{vaccinated}) = 240$

$n(\text{infected} \quad \text{vaccinated}) = 13$

$P(\text{infected} \quad \text{vaccinated}) = \dfrac{13}{240}$

c $n(\text{not vaccinated}) = 197$

$n(\text{not infected} \quad \text{not vaccinated}) = 38$

$P(\text{not infected} \quad \text{not vaccinated}) = \dfrac{38}{197}$

d $n(\text{infected}) = 172$

$n(\text{vaccinated} \quad \text{infected}) = 13$

$P(\text{vaccinated} \quad \text{infected}) = \dfrac{13}{172}$

Notice that $P(\text{infected} \quad \text{vaccinated}) \neq P(\text{vaccinated} \mid \text{infected})$

Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

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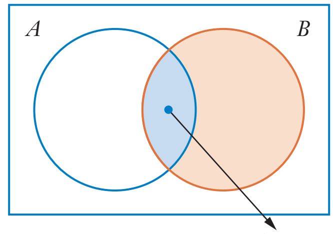
The $P(B)$ in the denominator is a result of the sample space being reduced to B (the orange circle in the Venn diagram)



Proof

$$P(A|B) = \frac{n(A \cap B)}{n(B)}$$

$$= \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}}$$

$$= \frac{P(A \cap B)}{P(B)}$$

For conditional probability, the product rule becomes $P(A \cap B) = P(A|B)P(B)$

EXAMPLE 17

Lara is an athlete who enters a swimming and running race She has a 44% chance of winning the swimming race and a 37% chance of winning both races Find to the nearest whole percentage the probability that she wins the running race if she has won the swimming race

Solution

If Lara has already won the swimming race (S) then the probability of her winning the running race (R) is conditional

$$P(S) = 44\%$$

$$= 044$$

$$P(R \text{ and } S) = P(R \cap S)$$

$$= 37\%$$

$$= 037$$

$$P(R|S) = \frac{P(R \cap S)}{P(S)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{037}{044}$$

$$\approx 084\ 09$$

$$\approx 84\%$$

So the probability of Lara winning the running race given that she has won the swimming race is 84%

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Condiional pobabiliy

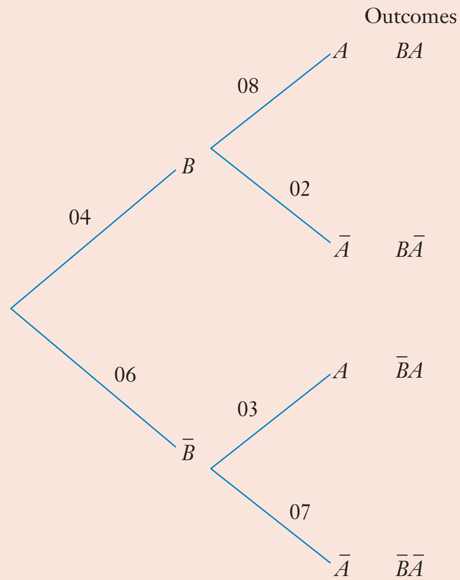
EXAMPLE 18

A zoo has a probability of 80% of having an article published in the newspaper when there is a birth of a baby animal When there is no birh, the zoo has a probability of only 30% of having an article published The probability of the zoo having an animal born at any one time is 40%

Find the percentage probability that a baby animal was born given that an article was published

Solution

We can draw up a probability tree showing the probabilities of having an article published (A) and a baby animal being born (B)



We want $P(B|A)$ According to the formua:

$$P(B|A)=\frac{P(B^\circ\ A)}{P(A)}$$

From the probability tree

$P(B\cap A)=P(BA)$ The event (numerator)

$=04\ \times 08$

$=032$

$P(A)=04\ \times 08\ +06\ \times .3$ The sample space (denominator)

$=05$

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$$P(B|A) = \frac{P(B^{\circ}\quad A)}{P(A)}$$
$$= \frac{032}{0\ 5}$$
$$= 064$$
$$= 64\%$$

So the probability that an animal was born given that an article was published is 64%

Conditional probability and independent events

We saw earlier tha:

$$P(A|B) = \frac{P(A^{\circ}\quad B)}{P(B)}$$

Rearranging this gives $P(A \cap B) = P(A|B)\ P(B)$

But if A and B are **independent events** $P(A \cap B) = P(A)P(B)$ (the product rule) which means

$P(A|B) = P(A)$

Similarly, $P(B|A) = P(B)$

Conditional probability and independent events

For independent events A and B

$$P(A|B) = P(A)$$
$$P(B|A) = P(B)$$
$$P(A \cap B) = P(A)P(B)$$

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EXAMPLE 19

a $P(X) = 02$ and $P(X \cap Y) = 006$ Determine whether X and Y are independent if

i $P(Y) = 06$ **ii** $P(Y) = 03$

b Show that A and B are independent given that $P(A) = 06$, $P(B) = 045$ $P(A \cup B) = 078$

Solution

a For independent events the product rule is $P(X \cap Y) = P(X)P(Y)$

i

$$\begin{aligned} P(X \cap Y) &= 006 \\ P(X)P(Y) &= 02 \times 06 \\ &= 012 \\ &\neq P(X \cap Y) \end{aligned}$$

$\therefore$ X and Y are not independent

ii

$$\begin{aligned} P(X \cap Y) &= 006 \\ P(X)P(Y) &= 02 \times 03 \\ &= 006 \\ &= P(X \cap Y) \end{aligned}$$

$\therefore$ X and Y are independent

b Using the addition rule

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ 078 &= 06 + 045 - P(A \cap B) \\ 078 &= 105 - P(A \cap B) \\ P(A \cap B) &= 105 - 078 \\ &= 027 \end{aligned}$$

Using the product rule for independent events

$$\begin{aligned} P(A \cap B) &= P(A)\,P(B) \\ &= 06 \times 045 \\ &= 027 \text{ as above} \end{aligned}$$

$\therefore$ A and B are independent

Exercise 7.07 Conditional probability

1 A bag contains 9 black and 8 white balls I draw out two at rando. If the first ball is white find the probability that the next ball i:

a black **b** white

2 A class has 13 boys and 15 girls Two students are chosen at random to carry a box of equipment Find the probability that the second person chosen is a boy given that the first student chosen was a girl

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3 Two dice are rolle. Find the probability of rollig:

a a double six if the first die was a six

b a total of 8 or more if the first die was a 3

4 A team has a probability of 52% of winning its first season and a 39% chance of winning both seasons 1 and 2 What is the probability of the team winning the second season given that it wins the first season?

5 A missile has a probability of 075 of hitting a target It has a probability of .65 of hitting two targets in a row. What is the probability that the missile will hit the second target given that it has hit the first target?

6 Danuta has an 80% probability of passing her first English assessment and she has a 45% probability of passing both the first and second assessments Find the probability that Danuta will pass the second assessment given that she passes the first one

7 A group of 10 friends all prepared to go out in the sun by putting on either sunscreen or a hat If 5 put on only sunscreen and 3 put on both sunscreen and a ha, find the probability that a friend who

a put on sunscreen also put on a hat

b put on a hat didnt put on sunscree.

8 A container holds 20 cards numbered 1 to 20 Two cards are selected at ranom. Find the probability that the second card is

a an odd number given that the first card was a 7

b a number less than 5 given that the first card was a 12

c a number divisible by 3 if the first number was 6

9 A group of 12 people met at a café for lunch If 9 people had a pie and 7 had chip, find the probability that one of the people

a had chips given that this person had a pie

b did not have a pie given that the person had chips

10 All except for 3 people out of 25 on a European tour had studied either French or Spanish Nine people studied only French and 5 studied both French and Spanis. Find the probability that one of these people

a studied Spanish if that person studied French

b did not study French given that the person studied Spanish

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11 The two-way table shows the numbers of students who own smartphones and tablets

	Smartphone	No smartphone	Totals
Tablet	23	8	31
No tablet	65	3	68
Totals	88	11	99

Find the probability that a person selected at random

a owns a smartphone given that the person

i owns a tablet **ii** doesnt own a table.

b owns a tablet given that the person

i owns a smartphone **ii** doesnt own a smartphon.

12 The table below shows the number of local people with casual and permanent jobs

	Women	Men
Permanent	23	38
Casual	79	64

Find the probability that a person chosen at random

a has a permanent job given that she is a woman

b has a casual job given that he is a man

c is a man given that the person has a casual job

d is a man if the person has a permanent job

13 In a group of 35 friends all either play sport or a musical instrumen. If 14 play both and 8 only play sport find the probability that a friend chosen at random wil:

a play a musical instrument given that the friend plays sport

b not play sport given that the friend plays a musical instrument

14 The two-way table shows the results of a survey into attendance at a local TAFE collee.

	Under 25	Between 25 and 50	Over 50	Totals
At TAFE	53	68	34	155
Not at TAFE	85	105	88	278
Totals	138	173	122	433

Find the probability that a person

a attends TAFE given that this person is over 50

b is between 25 and 50 if that person does not attend TAFE

c is not at TAFE given that this person is under 25

d is over 50 if the person is at TAFE

e is at TAFE given the person is aged 25 or ovr.

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15 A tennis team has a probability of 76% of winning a match when they are at home and 45% of winning a match when they are away. If the team plays 58% of their matches away, find the probability that the tem:

a wins their match given that they are away

b are at home given that they win a match

c are away given that they lose a match

16 A factory produces solar batteries The probability of a new battery being defective is 3% Howeve, if the manager is on dty, the probability of a new battery being defective changes to 2% The manager is on duty 39% of the tie. Find the probability that the manager is on duty if a new battery is defective

17 The chance of a bushfire is 85% after a period of no rain and 21% after rain The chance of rain is 46% Find the probability tha:

a there is not a bushfire given that it has rained

b it has rained given that there is a bushfire

c it has not rained given there is a bushfire

d it has rained given there is not a bushfire

18 If $P(A|B) = 067$ and $P(B) = 031$ find the value of $P(A \cap B)$

19 If $P(L) = 017$, $P(L \cap M) = 00204$ and $P(M) = 012$ show that L and M are independent

20 Given $P(X) = 0.3$, $P(Y) = 042$ and $P(X \cup Y) = 0594$ show that X and Y are independent

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7. TEST YOURSELF

Pacice quiz

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Pobabiliy indawod

For Questions **1–4** select the correct answer **A B C** or **D**

1 The probability of getting at least one 1 when rolling two dice is

A $\frac{1}{3}$ **B** $\frac{1}{6}$ **C** $\frac{11}{36}$ **D** $\frac{5}{18}$

2 A bag contains 7 white and 5 blue balls Two balls are selected at random without replacement The probability of selecting a white and a blue ball s:

A $\frac{35}{132}$ **B** $\frac{35}{72}$ **C** $\frac{35}{144}$ **D** $\frac{35}{66}$

3 If $A = \{5, 7, 8\}$ and $B = \{3 , 9\}$ then the set $\{7\}$ represens:

A $A - B$ **B** $A \cap B$ **C** $A + B$ **D** $A \cup B$

4 For the table the relative frequency of a score of 11 is (there may be more than one answer)

A 32% **B** $\frac{8}{11}$

C 0032 **D** $\frac{8}{25}$

Score	Frequency
8	5
9	2
10	9
11	8
12	1

5 **a** Given event $A = \{3, 5, 6, 8, 10\}$ and event $B = \{5, 7, 8, 9, 11, 12\}$, find:

i $A \cup B$ **ii** $A \cap B$

b Draw a Venn diagram showing this informatin.

6 Find the sample space for each situation

a Tossing two coins

b Choosing a colour from the Australian fla.

7 The table shows the results of an experiment when throwing a die

a Add a column for relative frequencies (as fractions)

b From the table find the probability of throwin:

i 3 **ii** more than 4

iii 6 **v** 1 or 2

v less than 4

Face	Frequency
1	17
2	21
3	14
4	20
5	18
6	10

8 The probability that a certain type of seed will germinate is 93%
If 3 of this type of seeds are planted find the probability tha:

a all will germinate **b** just 1 will germinate

c at least 1 will germinate

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9 A game is played where the differences of the numbers rolled on 2 dice are taken

a Draw a table showing the sample space (all possibilities)

b Find the probability of rolling a difference of

i 3 **ii** 0 **iii** 1 or 2

10 Mark buys 5 tickets in a raffle in which 200 are sold altogether.

a What is the probability that he will

i win the raffle? **ii** not win the raffle?

b If the raffle has 2 prizes find the probability that Mark will win just 1 priz.

11 In a class of 30 students 17 study histor, 11 study geography and 5 study neiter.

Find the probability that a student chosen at random studies

a geography but not history

b both history and geography

c geography, given that the student studies history

d history, given that the student studies geograpy.

12 In the casino when tossing 2 coin, 2 tails came up 10 times in a rw. So there is less chance that 2 tails will come up next time' Is this statement true ? Why?

13 A set of 100 cards numbered 1 to 100 is placed in a box and one is drawn at random Find the probability that the card chosen is

a odd **b** less than 30 **c** a multiple of 5

d less than 30 or a multiple of 5 **e** odd or less than 30

14 Jenny has a probability of $\frac{3}{5}$ of winning a game of chess and a probability of $\frac{2}{3}$ of winning a card game If she plays one of each gam, find the probability that she wis:

a both games **b** one game **c** neither game

15 A bag contains 5 black and 7 white marbles Two are chosen at random from the bag without replacement Find the probability of getting a black and a white marble

16 There are 7 different colours and 8 different sizes of leather jackets in a shop If Brady selects a jacket at random find the probability that he will select one the same size and colour as his friend does

17 Each machine in a factory has a probability of 45% of breaking down at any time If the factory has 3 of these machines find the probability tha:

a all will be broken down

b at least one will be broken down

18 A bag contains 4 yellow, 3 red and 6 blue bals. Two are chosen at rndom.

a Find the probability of choosing
  i 2 yellow balls **ii** a red and a blue ball **iii** 2 blue balls

b Find the probability that the second ball is
  i yellow, given that the first ball is blue
  ii red given the first ball is yello.

19 In a group of 12 friends 8 have seen the movie *Star Wars 20* and 9 have seen the movie *Mission Impossible 9* Everyone in the group has seen at least one of these movie. If one of the friends is chosen at random find the probability that this person has see:

a both movies **b** only *Mission Impossible 9*

20 A game of chance offers a $\frac{2}{5}$ probability of a win or a $\frac{3}{8}$ probability of a draw.

a If Billal plays one of these games find the probability that he lose.

b If Sonya plays 2 of these games find the probability o:
  i a win and a draw **ii** a loss and a draw **iii** 2 wins

21 A card is chosen at random from a set of 10 cards numbered 1 to 10 A second card is chosen from a set of 20 cards numbered 1 to 20 The 2 cards are placed together in order to make a number, for example 75. Find the probability that the combination number these cards make is

a 911 **b** less than 100 **c** between 300 and 500

22 A loaded die has a $\frac{2}{3}$ probability of coming up 6 The other numbers have an equal probability of coming up If the die is rolle, find the probability that it comes p:

a 2 **b** even

23 Amie buys 3 raffle tickets If 150 tickets are sold altogethe, find the probability that Amie wins

a 1st prize **b** only 2nd prize

c 1st and 2nd prizes **d** neither prize

24 A bag contains 6 white 8 red and 5 blue ball. If 2 balls are selected at randm, find the probability of choosing a red and a blue ball

a with replacement **b** without replacement

25 A group of 9 friends go to the movies All buy popcorn or an ice-crem. If 5 buy popcorn and 7 buy ice-creams find the probability that one friend chosen at random will hav:

a popcorn but not ice-cream **b** both popcorn and ice-cream

c popcorn given that the friend has an ice-cream

26 Eds probability of winning at tennis is $\frac{3}{5}$ and his probability of winning at squash is $\frac{7}{10}$
Find the probability of Ed winning

a both games **b** neither game **c** one game

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7. CHALLENGE EXERCISE

1 In a group of 35 students 25 go to the movies and 15 go to the league gam. If all the students like at least one of these activities and two students are chosen from this group at random find the probability tha:

a both only go to the movies

b one only goes to the league game and the other goes to both the game and movies

2 A certain soccer team has a probability of 05 of winning a match and a probability of 02 of a draw. If the team plays 2 matchs, find the probability that it wll:

a draw both matches **b** win at least 1 match **c** not win either match

3 A game of poker uses a deck of 52 cards with 4 suits (hearts diamond, spades and clubs) Each suit has 13 card, consisting of an ae, cards numbered from 2 to10, a ack, queen and king If a person is dealt 5 card, find the probability of getting four acs.

4 If a card is drawn out at random from a set of playing cards find the probability that it will be

a an ace or a heart

b a diamond or an odd number not including aces

c a jack or a spade

5 Bill does not select the numbers 1 ,3, 4, 5 and 6 for Lotto because he says this combination would never win Is he correct ?

6 Out of a class of 30 students 19 play a musical instrument and 7 play both a musical instrument and a sport Two students play neiter.

a One student is selected from the class at random Find the probability that this person plays a sport but not a musical instrument

b Two people are selected at random from the clas. Find the probability that both these people only play a sport

7 A game involves tossing 2 coins and rolling 2 dice The scoring is shown in the table.

Result	Score (points)
2 heads and double 6	5
2 heads and double (not 6)	3
2 tails and double 6	4
2 tails and double (not 6)	2

a Find the probability of getting 2 heads and a double 6

b Find the probability of getting 2 tails and a double that is not 6

c What is the probability that Andre will score 13 in three moves?

8 Silvana has a 38% probability of passing on a defective gene to a daughter and a 06% probability of passing a defective gene on to a son The probability of Silvana having a son is 52% Find the probability that she has a so, given that Silvana passes on a defective gee.

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Practice set 3

For Questions **1** to **11** select the correct answer **A B C** or **D**

1 The quotient rule for differentiating $y = \frac{u}{v}$ is

A $y' = \frac{uv' - vu'}{v^2}$ **B** $\frac{u'v - v'u}{v^2}$

C $y' = u'v + v'u$ **D** $y' = uv' + vu'$

2 If $f(x) = x^2$ and $g(x) = 2x + 1$ the composite function $g(f(x))$ is given by

A $(2x + 1)^2$ **B** $(2x)^2 + 1$

C $2x + 1^2$ **D** $2x^2 + 1$

3 The number of employees N is inversely proportional to the tim, t it takes to do a stocktake What is the equation showing this information ?

A $N = kt$ **B** $N = t + k$ **C** $N = \frac{k}{t}$ **D** $N = \frac{t}{k}$

4 Find the derivative of $(3x - 2)^8$

A $(3x - 2)^7$ **B** $8(3x - 2)^7$

C $8x^7(3x - 2)$ **D** $24(3x - 2)^7$

5 Find the probability of drawing out a blue and a white ball from a bag containing 7 blue and 5 white balls if the first ball is not replaced before taking out the second

A $\frac{70}{121}$ **B** $\frac{70}{144}$ **C** $\frac{1225}{17\,424}$ **D** $\frac{35}{66}$

6 The equation of a circle with radius 3 and centre $(-1, 4)$ i:

A $(x - 1)^2 + (y + 4)^2 = 3$ **B** $(x - 1)^2 + (y + 4)^2 = 9$

C $(x + 1)^2 + (y - 4)^2 = 9$ **D** $(x + 1)^2 + (y - 4)^2 = 3$

7 If $f(x) = 2x^2 - 3x + 1$ and $g(x) = (x + 3)^2$ find the degree of $y = f(x)g(x)$

A 2 **B** 4 **C** 3 **D** 5

8 Find the domain of $f(x) = \frac{2}{x + 7}$

A $(-\infty, 7) \cup (7, \infty)$ **B** $(-\infty, -7) \cup (-7\ \infty)$

C $(-\infty\ 7)\ \cap (7, \infty)$ **D** $(-\infty, -7) \cap (-7\ \infty)$

9 If the displacement of a particle is given by $x = 2t^3 + 6t^2 - 4t + 10$ the initial velocity is

A -4 **B** 10 **C** 12 **D** 14

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10 In a group of 25 students 19 catch a train to school and 21 catch a bu. If one of these students is chosen at random find the probability that the student only catches a bus to school

A $\frac{6}{25}$ **B** $\frac{21}{25}$ **C** $\frac{3}{5}$ **D** $\frac{3}{20}$

11 Conditional probability $P(A|B)$ is given by

A $\frac{P(A \cup B)}{P(B)}$ **B** $\frac{P(A \cap B)}{P(A)}$

C $\frac{P(A \cup B)}{P(A)}$ **D** $\frac{P(A \cap B)}{P(B)}$

12 Differentiate

a $y = x^9 - 4x^2 + 7x + 3$ **b** $y = 2x(x^2 - 1)$ **c** $y = 3x^{-4}$

d $y = \frac{5}{2x^5}$ **e** $y = \sqrt{x}$ **f** $y = (2x + 3)^7$

g $y = \frac{1}{(x^2 - 7)^4}$ **h** $y = \sqrt[3]{5x + 1}$ **i** $y = \frac{5x^2 - 1}{2x + 3}$

13 Sketch the graph of

a $y = \frac{4}{2x - 4}$ **b** $P(x) = x^3 + x^2 - 2x$ **c** $y = |x - 1|$

d $x^2 + y^2 = 25$ **e** $f(x) = -\sqrt{1 - x^2}$

14 In a class of 25 students 11 play guita, 9 play the pino, while 8don't play either instrument If one student is selected at random from the clas, find the probability that this student will play

a both guitar and piano

b neither guitar or piano

c only guitar.

15 The volume in litres of a rectangular container that is leaking over time t minutes is given by $V = -t^2 + 4t + 100$ Fin:

a the initial volume

b the volume after 10 minutes

c the rate of change in volume after 10 minutes

d how long it will take to 1 decimal plac, until the container is empy.

16 **a** Find the equation of the tangent to the curve $y = x^3 - 3x$ at the point $P = (-2, -2)$

b Find the equation of the normal to $y = x^3 - 3x$ at P

c Find the point Q where this normal cuts the x-axis

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17 Two dice are throw. Find the probability of throwig:

a double 1 **b** any double **c** at least one 3
d a total of 6 **e** a total of at least 8

18 The function $f(x) = ax^2 + bx + c$ has a tangent at (1 –3) with a gradient of –1. It also passes through (4 3. Find the values of a b and c

19 Find the equation of the circle with centre (–2, –3) and radius 5 units

20 Find the centre and radius of the circle with equation

a $x^2 + 6x + y^2 - 10y - 15 = 0$ **b** $x^2 + 10x + y^2 - 6y + 30 = 0$

21 $f(x) = 3x^2 - 4x + 9$

a Find $f(x + h) - f(x)$

b Show by differentiating from first principles that $f'(x) = 6x - 4$

22 **a** Find the equation of the tangent to the curve $y = x^3 - 2$ at the point $P(1, -1)$

b The curve $y = x^3 - 2$ meets the y-axis at Q Find the equation of PQ

c Find the equation of the normal to $y = x^3 - 2$ at the point (–1, –3)

d Find the point R where this normal cuts the x-axis

23 100 cards are numbered from 1 to 100 If one card is chosen at rando, find the probability of selecting

a an even number less than 30

b an odd number or a number divisible by 9

24 A bag contains 5 white 6 yellow and 3 blue ball. Two balls are chosen at random from the bag without replacement Find the probability of choosin:

a 2 blue balls **b** a white ball and a yellow ball

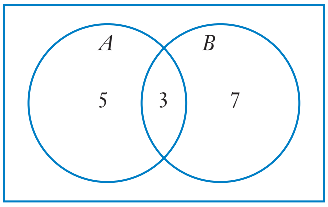
25 If Scott buys 10 tickets find the probability that he wins both first and second prizes in a raffle in which 100 tickets are sold

26 Two dice are rolle. Find the probability of rolling a totl:

a of 8 **b** less than 7 **c** greater than 9
d of 4 or 5 **e** that is an odd number.

27 For the Venn diagrm, fnd:

a $P(A|B)$ **b** $P(B\ A)$



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28 A bag contains 5 red 7 blue and 9 yellow ball. Cherylanne chooses 2 balls at random from the bag Find the probability of that she choose:

a blue given the first ball was yellow

b red given the first ball was blue

29 If $f(x) = 2x^3 - 5x^2 + 4x - 1$, find $f(-2)$ and $f'(-2)$

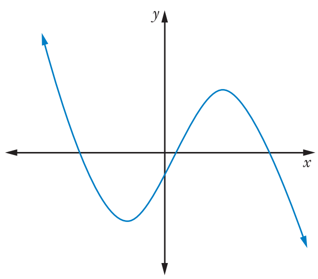
30 **a** Find the gradient of the secant to the curve $f(x) = 2x^3 - 7$ between the point (2 9) and the point wher:

i $x = 201$ **ii** $x = 199$

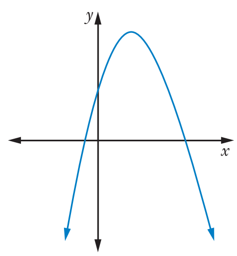
b Hence estimate the gradient of the tangent to the curve at (2 9.

31 Sketch the gradient function for each curve

a



b



32 The area of a community garden in m^2 is given by $A = 7x - x^2$ where x is the length of the garden

a Find the area when the length is

i 3 m **ii** 45 m

b Find the length when the area is 8 m^2 to 1 decimal plac.

c Sketch the graph of the area function

d Find the maximum possible area

33 Solve graphically

$$|x + 2| = 3.$$

34 If $f(x) = x^2 - 1$ and $g(x) = x^3 + 3$, fin:

a the degree of $y = f(x)g(x)$

b the leading coefficient of $y = f(x)g(x)$

c the constant term of $y = f(x)g(x)$

35 The displacement x cm of an object moving along a straight line over time t seconds is given by $x = 2t^3 - 13t^2 + 17t + 12$.

a Find the initial displacement velocity and acceleratio.

b Find the displacement velocity and acceleration after 2 second.

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36 If $A = \{1, 3, 4, 5\}$ and $B = \{2, 3, 5,6\}$:

a find $A \cup B$

b find $A \cap B$

c draw a Venn diagram showing this informatin.

37 Find the equation of the tangent to the curve $y = 3x^2 - 6x + 7$ at the point (2, 7.

38 Find the derivative of

a $y = x^{-3}$ **b** $y = \sqrt{x^3}$ **c** $y = \dfrac{1}{x}$ **d** $y = \dfrac{(7x+4)^2}{3x-1}$

e $y = (5x^2 + 1)(2x - 3)^4$ **f** $y = (3x + 1)^5$ **g** $y = \sqrt{2x-1}$

39 $f(x) = x^2 - 2$ and $g(x) = 2x - 1$.

a Find the equation of

i $y = f(x) + g(x)$ **ii** $y = f(x)g(x)$

iii $y = g(x) - f(x)$ **v** $y = \dfrac{g(x)}{f(x)}$

b Sketch the graph of

i $y = -f(x)$ **ii** $y = g(-x)$

iii $y = -g(-x)$

40 **a** Find the centre and radius of the circle $x^2 + 2x + y^2 - 6y - 6 = 0$

b Find its domain and range

41 Find the equation of the normal to the curve $y = x^2 - 4x + 1$ at the point (3, –2)

42 Differentiate

a $y = 2x^4 - 5x^3 + 3x^2 - x - 4$ **b** $y = \dfrac{1}{2x^5}$ **c** $y = \sqrt{x}$

d $y = (2x - 3)^7$ **e** $y = 3x^4(2x - 5)^7$ **f** $y = \dfrac{5x+7}{3x-2}$

43 If $f(x) = x^2 + 1$ and $g(x) = x - 3$ find the degree o:

a $f(x) + g(x)$ **b** $f(x)g(x)$

44 A coin is tossed and a die thrown Find the probability of gettin:

a a head and a 6 **b** a tail and an odd number.

45 Find the domain and range of

a $y = x^3 + 1$ **b** $y = 1 - x^2$

c $x^2 + 4x + y^2 - 2y - 20 = 0$ **d** $y = \dfrac{4}{x+2}$

46 If $f(x) = x^3$ and $g(x) = 2x + 5$ fin:

a $f(g(x))$ **b** $g(f(x))$

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47 The table below shows the results of an experiment in tossing 2 coins

Result	Frequency
HH	24
HT	15
TH	38
TT	23

a Add a column for relative frequencies as fractions

b From the table find the probability of tossin:

i 2 tails **ii** a head and a tail in any order

c What is the theoretical probability of tossing

i 2 tails? **ii** a head and a tail in any order?

48 Find the equation of the tangent to the curve $y = x^3 - 7x + 3$ at the point where $x = 2$.

49 Find in exact form

a the length of the arc

b the area of the sector

cut off by an angle of 40° at the centre of a circle with radius 4 cm

50 If $f(x) = |x| - 2$ find

a $f(-2)$ **b** $f(0)$ **c** $f(m+1)$

51 The probability that Despina passes her first maths test is 64% and the probability that she will pass both the first and second tests is 48% Find the probability that Despina passes the second test given that she passes the first test

52 If $P(L) = 45\%$ $P(L \cap M) = 54\%$ and $P(M) = 12\%$ show that L and M are independent

53 Given $P(X) = 026$, $P(Y) = 015$ and $P(X \cup Y) = 0371$ show that X and Y are independent

54 State whether events A and B are mutually exclusive if $P(A) = 0.18$, $P(A \cup B) = 05$ and $P(B) = 032$

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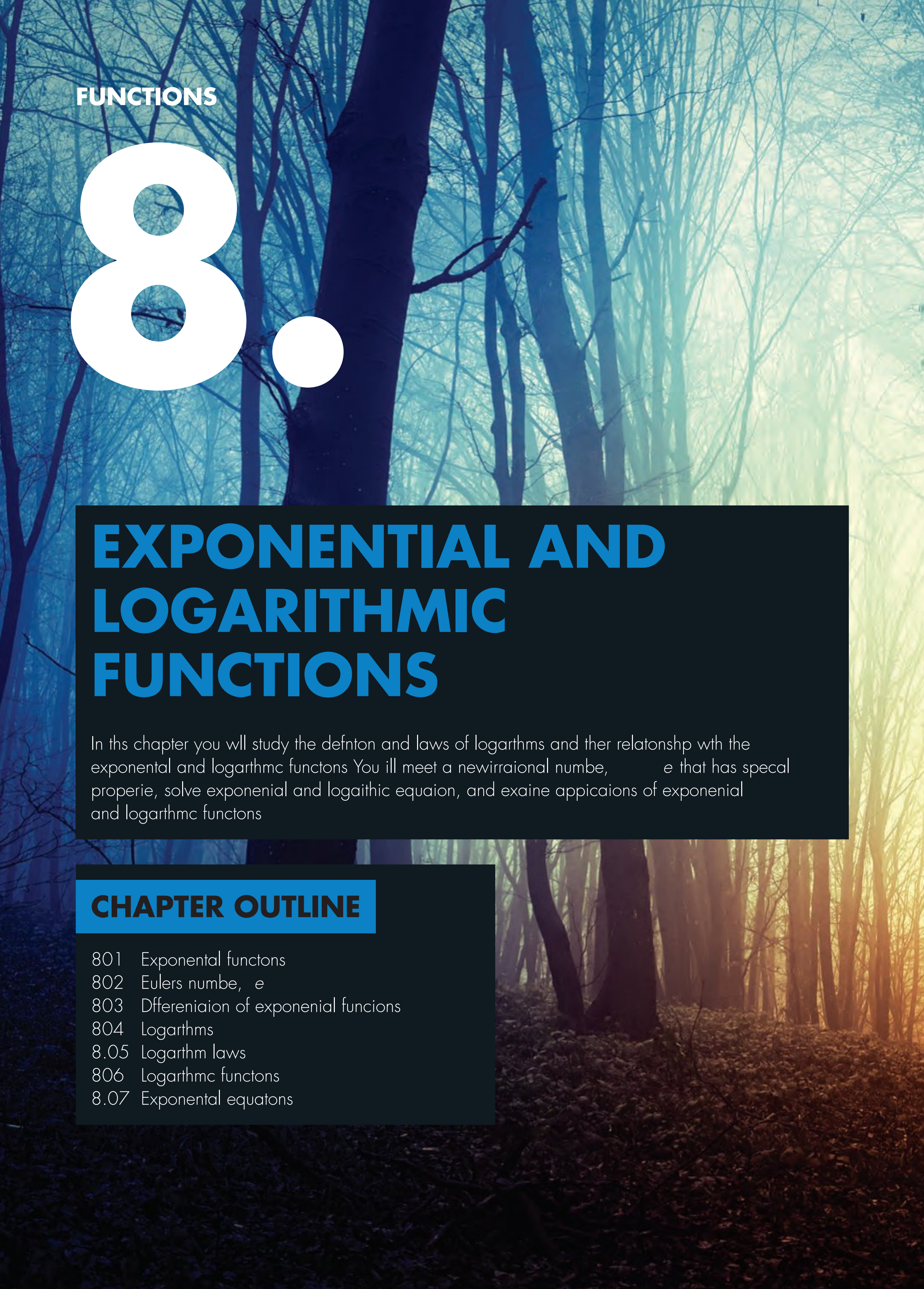


FUNCTIONS

8.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In ths chapter you wll study the defnton and laws of logarthms and ther relatonshp wth the exponental and logarthmc functons You ill meet a newirraional numbe, *e* that has specal properie, solve exponenial and logaithic equaion, and exaine appicaions of exponenial and logarthmc functons

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- graph exponental and logarthmc functons
- understand and use Eulers numbe, *e*
- dfferenlate exponenial funcions
- convert between exponenial and logaithic forms sing the dfntion of a logrithm
- dentfy and apply logarthm laws
- solve exponental equatons usng logarthms
- solve practcal formulasinvoling exponents and logaithms

TERMINOLOGY

Eulers number This numbe, e approximately 2718 28 is an important constant that is the base of natural logarithms

exponential function A function in the form $y = a^x$

logarithm The logarithm of a positive number y is the power to which a given number a called the base must be raised in order to produce the number y so $\log_a y = x$ means $y = a^x$

logarithmic function A function in the form $y = \log_a x$

Gaphing exponenial

Exponenial uncion

Traslting exponenial gaph

8.01 Exponential functions

An **exponential function** is in the form $y = a^x$ where $a > 0$

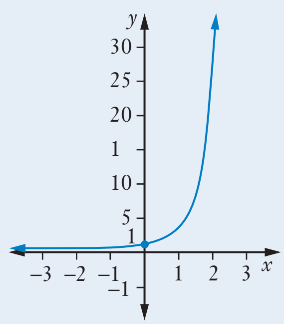
EXAMPLE 1

Sketch the graph of the function $y = 5^x$ and state its domain and range

Solution

Complete a table of values for $y = 5^x$

x	−3	−2	−1	0	1	2	3
y	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	125



Notice that a^x is always positive So there is no x-intercept and $y > 0$

For the y-intercept when $x = 0$, $y = 5^0 = 1$.

The y-intercept is 1

From the graph the domain is $(-\infty \ \infty)$ and the range is $(0 \ \infty)$

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INVESTIGATION

THE VALUE OF a IN $y = a^x$

Notice that the exponential function $y = a^x$ is only defined for $a > 0$.

1 Suppose $a = 0$ What would the function $y = 0^x$ look like? Try completing a table of values or use technology to sketch the graph Is the function defined for positive values of x negative values of x or when $x = 0$? What if x is a fraction?

2 Suppose $a < 0$ What would the function $y = (-2)^x$ look like?

3 For $y = 0^x$ and $y = (-2)^x$

a is it possible to graph these functions at all?

b are there any discontinuities on the graphs?

c do they have a domain and range?

The exponential function $y = a^x$

- Domain $(-\infty \ \infty)$ range $(, \ \infty)$
- The y-intercept ($x = 0$) is always 1 because $a^0 = 1$.
- The graph is always above the x-axis and there is no x-intercept ($y = 0$) because $a^x > 0$ for all values of x
- The x-axis is an **asymptote**

EXAMPLE 2

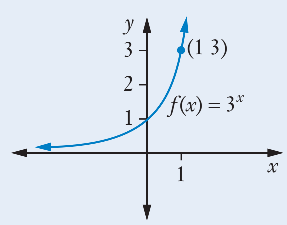
Sketch the graph of

a $f(x) = 3^x$ **b** $y = 2^x + 1$

Solution

a The curve is above the x-axis with y-intercept 1 We must show another point on the curve

$f(1) = 3 \ = 3.$



b For y-intercept $x = 0$

$y = 2^0 + 1$

$= 1 + 1$

$= 2$

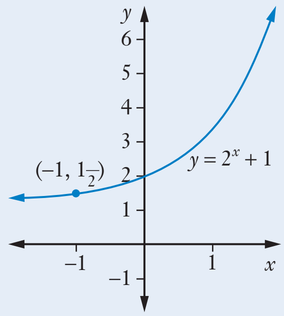
Find another point when $x = -1$

$y = 2^{-} + 1$

$= \frac{1}{2} + 1$

$= 1\frac{1}{2}$

Notce that ths s the graph o $y = 2^x$ moved up 1 unt



EXAMPLE 3

Sketch the graph of

a $f(x) = 3(4^x)$ **b** $y = 2^{x+1}$

Solution

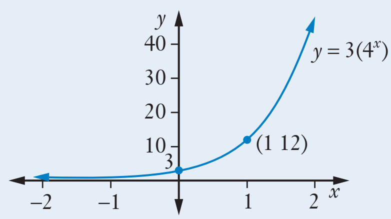
a The values of $f(x) = 3(4^x)$ will be 3 times greater than 4^x so its curve will be steeper.

$f(-1) = 3(4^{-}\ ) = 075$

$f(0) = 3(4^0) = 3$

$f(1) = 3(4\ ) = 12$

$f(2) = 3(4^2) = 48$

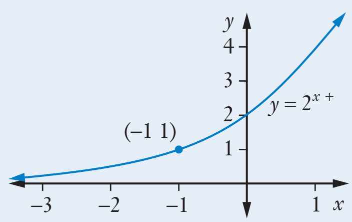


b $f(-1) = 2^{-1+1} = 1$

$f(0) = 2^{0+1} = 2$

$f(1) = 2^{1+1} = 4$

$f(2) = 2^{2+1} = 8$



Ths s the graph of $y = 2^x$ shfted 1 unt eft

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Reflections of exponential functions

We can reflect the graph of $y = a^x$ using what we learned in Chapter 5 *Further functions*

EXAMPLE 4

Given $f(x) = 3^x$ sketch the graph o:

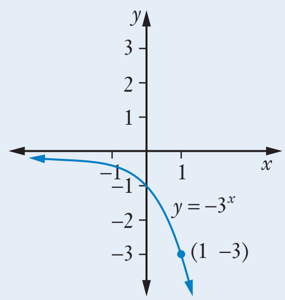
a $y = -3^x$ **b** $y = 3^{-x}$ **c** $y = -3^{-x}$

Solution

a Given $f(x) = 3^x$ then $y = -f(x) = -3^x$

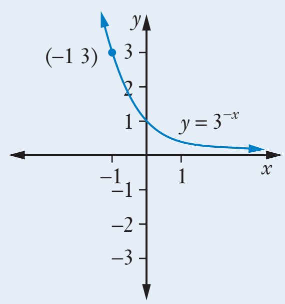
This is a reflection of $f(x)$ in the x-axis

Note -3^x means -3^x, *no* -3^x.



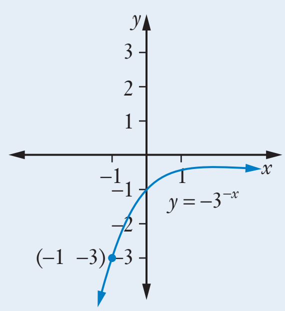
b Given $f(x) = 3^x$ then $y = f(-x) = 3^{-x}$

This is a reflection of $f(x)$ in the y-axis



c Given $f(x) = 3^x$ then $y = -f(-x) = -3^{-x}$

This is a reflection of $f(x)$ in both the x- and y-axes



INVESTIGATION

GRAPHS OF EXPONENTIAL FUNCTIONS

Use a graphics calculator or graphing software to sketch the graphs of the exponential functions below. Look for similarities and differences within each st.

a $y = 2^x \; y = 2^x + 1, y = 2^x + 3, y = 2^x - 5$

b $y = 3(2^x), y = 4(2^x), y = -2^x \; y = -3(2^x)$

c $y = 3(2^x) + 1, y = 4(2^x) + 3, y = -2^x + 1, y = -3(2^x) - 3$

d $y = 2^{x+} \quad y = 2^{x+2} \; y = 2^{x-} \quad y = 2^{x-3} \; y = 2^{-x}$

e $y = 2^{-x} \; y = 2\left(2^{-x}\right) \; y = -2^{-x} \; y = -3\left(2^{-x}\right) \; y = 2^{-x-}$

Exercise 8.01 Exponential functions

1 Sketch each exponential function

a $y = 2^x$ **b** $y = 4^x$ **c** $f(x) = 3^x + 2$ **d** $y = 2^x - 1$

e $f(x) = 3(2^x)$ **f** $y = 4^{x+1}$ **g** $y = 3(4^{2x}) - 1$ **h** $f(x) = -2^x$

i $y = 2(4^{-x})$ **j** $f(x) = -3(5^{-x}) + 4$

2 State the domain and range of each function

a $f(x) = 2^x$ **b** $y = 3^x + 5$ **c** $f(x) = 10^{-x}$ **d** $f(x) = -5^x + 1$

3 Given $f(x) = 2^x$ and $g(x) = 3x - 4$ fin:

a $f(g(x))$ **b** $g(f(x))$

4 **a** Sketch the graph of $f(x) = 4(3^x) + 1$.

b Sketch the graph of

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = -f(-x)$

5 Sales numbers N of a new solar battery are growing over t years according to the formula $N = 450(3^{0.9t})$

a Draw a graph of this function

b Find the initial number of sales when $t = 0$

c Find the number of sales after

i 3 years

ii 5 years

iii 10 years

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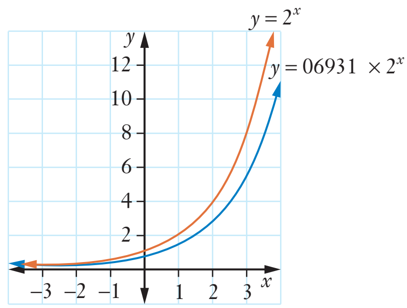
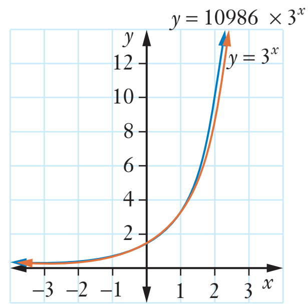
8.02 Euler's number, *e*

The gradient function of exponential functions is interesting Notice that the gradient of an exponential function is always increasing and increases at an increasing rat.

If you sketch the derivative function of an exponential function then it too is an exponential function Here are the graphs of the derivative functions (in blue) of $y = 2^x$ and $y = 3^x$ (in red) together with their equations





Notice that the graph of the derivative function of $y = 3^x$ is very close to the graph of the original function

We can find a number close to 3 that gives exactly the same derivative function as the original graph This number is approximately2.718 8, and is called **Eulers number** *e* Like π the number *e* is irrational

Euler's number

$$e \approx 2718\ 28$$

DID YOU KNOW?

Leonhard Euler

Like π Eule's numbr, *e*, is a **transcendental** number, which is an irrational number that is not a surd This was proven by a French mathematicin, **Charles Hermite**, in 187. The Swiss mathematician **Leonhard Euler** (1707–83) gave *e* its symbol and he gave an approximation of *e* to 23 decimal places Now *e* has been calculated to over a trillion decimal places

Euler gave mathematics much of its important notation He caused π to become standard notation for pi and used *i* for the square root of –1 He also introduced the symbol Σ for sums and $f(x)$ notation for functions

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EXAMPLE 5

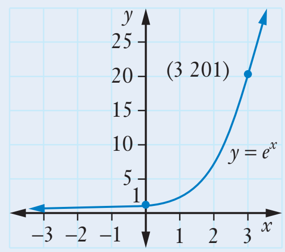
Sketch the graph of the exponential function $y = e^x$

Solution

Use e^x on your calculator to draw up a table of values For exampl, to calculate e^{-3}

Casio scientific	Sharp scientific
SHIFT e^x (−) 3 =	2ndF e^x +/− 3 =

$e^{-3} = 0049\ 78$



x	−3	−2	−1	0	1	2	3
y	0.05	0.1	0.4	1	2.7	7.4	20.1

(rounded figures)

EXAMPLE 6

The salmon population in a river over time can be described by the exponential function $P = 200e^{0.3t}$ where t is time in years

a Find the population after 3 years

b Draw the graph of the population

Solution

a $P = 200e^{0.3t}$

When $t = 3$:

$P = 200e^{0.3 \times 3}$

$= 4919206$

≈ 492

So after 3 years there are 492 salmon

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b The graph is an exponential curve Finding some points will help us graph it accurately.

When $t = 0$: $P = 200e^{0.3\times 0}$

$= 200$ This is also the P-intercept

When $t = 1$: $P = 200e^{0.3\times 1}$

$= 2699717$

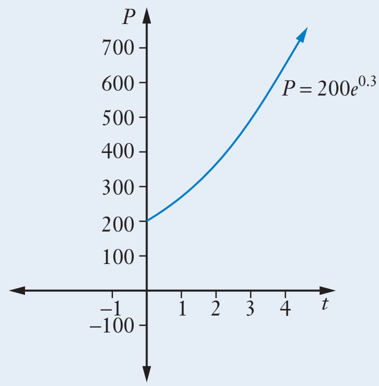
≈ 270

When $t = 2$: $P = 200e^{0.3\times 2}$

$= 3644237$

≈ 364

We already know $P \approx 492$ when $t = 3$.



Tim, ≥ 0 so dont sketch the curve for negatve values of .

Exercise 8.02 Euler's number, *e*

1 Sketch the curve $f(x) = 2e^{x-2}$

2 Evaluate correct to 2 decimal place:

a e^{15} **b** e^{-2} **c** $2e^{0.3}$ **d** $\frac{1}{e^3}$ **e** $-3e^{-31}$

3 Sketch each exponential function

a $y = 2e^x$ **b** $f(x) = e^x + 1$ **c** $y = -e^x$ **d** $y = e^{-x}$ **e** $y = -e^{-x}$

4 State the domain and range of $f(x) = e^x - 2$.

5 If $f(x) = e^x$ and $g(x) = x^3 + 3$, fin:

a $f(g(x))$ **b** $g(f(x))$

6 The volume V of a metal in mm^3 expands as it is heated over time according to the formula $V = 25e^{07\,t}$ where t is in minutes

a Sketch the graph of $V = 25e^{07\,t}$

b Find the volume of the metal at

i 3 minutes **ii** 8 minutes

c Is this formula a good model for the rise in volume? Why?

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7 The mass of a radioactive substance in g is given by $M = 150e^{-0014\,t}$ where t is in years
Find the mass after

a 10 years **b** 50 years **c** 250 years

8 The number of koalas in a forest is declining according to the formula $N = 873e^{-0078\,t}$ where t is the time in years

a Sketch a graph showing this decline in numbers of koalas for the first 6 years

b Find the number of koalas

i initially **ii** after 5 years **iii** after 10 years

9 An object is cooling down according to the exponential function $T = 23 + 125e^{-006\,t}$ where T is the temperature in °C and t is time in minutes

a Find the initial temperature

b Find the temperature at

i 2 minutes **ii** 5 minutes **iii** 10 minutes **v** 2 hours

c What temperature is the object tending towards? Can you explain why?

10 A population is growing exponentially. If the initial population is 20 000 and after 5 years the population is 80 000 draw a graph showing this informatio.

11 The temperature of a piece of iron in a smelter is 1000°C and it is cooling down exponentially. After 10 minutes the temperature is 650°C Draw a graph showing this information

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8.03 Differentiation of exponential functions

WS
Difretiting exponenial uncion

Eulers numbe, e is the special number such that the derivative function of $y = e^x$ is itself
The derivative of e^x is e^x

Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 7

a Differentiate $y = e^x - 5x^2$

b Find the equation of the tangent to the curve $y = e^x$ at the point $(1 \; e)$

Solution

a $\frac{dy}{dx} = e^x - 10x$

b Gradient of the tangent

$\frac{dy}{dx} = e^x$

At $(1 \; e)$

$\frac{dy}{dx} = e$

$= e$

So $m = e$

Equation

$y - y \; = m(x - x \;)$

$y - e = e(x - 1)$

$= ex - e$

$y = ex$

The rule for differentiating $kf(x)$ works with the rule for e^x as well

Derivative of ke^x

$$\frac{d}{dx}(ke^x) = ke^x$$

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EXAMPLE 8

a Differentiate $y = 5e^x$

b Find the gradient of the normal to the curve $y = 3e^x$ at the point (0 3.

Solution

a $\frac{dy}{dx} = 5e^x$

b Gradient of tangent

$\frac{dy}{dx} = 3e^x$

At (0 3:

$\frac{dy}{dx} = 3e^0$

$= 3$ since $e^0 = 1$

So $m = 3$

For normal

$m\ m_2 = -1$

$3m_2 = -1$

$m_2 = -\frac{1}{3}$

So the gradient of the normal at (0 3) is $-\frac{1}{3}$

We can also use other differentiation rules such as the chain rul, product rule and quotient rule with the exponential function

EXAMPLE 9

Differentiate

a $y = e^{9x}$

b $y = e^{-5x}$

Solution

a Let $u = 9x$

Then $\frac{du}{dx} = 9$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^u \times 9$

$= 9e^u$

$= 9e^{9x}$

b Let $u = -5x$

Then $\frac{du}{dx} = -5$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^u \times (-5)$

$= -5e^u$

$= -5e^{-5x}$

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The derivative of e^{ax}

$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$

Proof

Let $u = ax$

Then $\frac{du}{dx} = a$

$$y = e^u$$

$$\frac{dy}{du} = e^u$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= e^u \times a\\ &= ae^u\\ &= ae^{ax}\end{aligned}$$

EXAMPLE 10

Differentiate

a $y = (1 + e^x)^3$

b $y = \frac{2x + 3}{e^x}$

Solution

a $$\begin{aligned}\frac{dy}{dx} &= 3(1 + e^x)^2 \times e^x\\ &= 3e^x(1 + e^x)^2\end{aligned}$$

b $$\begin{aligned}\frac{dy}{dx} &= \frac{u'v - v'u}{v^2}\\ &= \frac{2e^x - e^x(2x + 3)}{(e^x)^2}\\ &= \frac{2e^x - 2xe^x - 3e^x}{e^{2x}}\\ &= \frac{-e^x - 2xe^x}{e^{2x}}\\ &= \frac{-e^x(1 + 2x)}{e^{2x}}\\ &= \frac{-(1 + 2x)}{e^x}\end{aligned}$$

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Exercise 8.03 Differentiation of exponential functions

1 Differentiate

a $y = 9e^x$ **b** $y = -e^x$ **c** $y = e^x + x^2$

d $y = 2x^3 - 3x^2 + 5x - e^x$ **e** $y = (e^x + 1)^3$ **f** $y = (e^x + 5)^7$

g $y = (2e^x - 3)^2$ **h** $y = xe^x$ **i** $y = \frac{e^x}{x}$

j $y = x^2e^x$ **k** $y = e^x(2x + 1)$ $y = \frac{e^x}{7x - 3}$

m $y = \frac{5x}{e^x}$

2 Find the derivative of

a $y = e^{2x}$ **b** $y = e^{-x}$ **c** $y = 2e^{3x}$

d $y = -e^{7x}$ **e** $y = -3e^{2x} + x^2$ **f** $y = e^{2x} - e^{-2x}$

g $y = 5e^{-x} - 3x + 2$ **h** $y = xe^{4x}$ **i** $y = \frac{2e^{3x} - 3}{x + 1}$

j $y = (9e^{3x} + 2)^5$

3 If $f(x) = x^3 + 3x - e^x$ find $f'(1)$ in terms of e

4 Find the exact gradient of the tangent to the curve $y = e^x$ at the point $(1 \;\; e)$

5 Find the exact gradient of the normal to the curve $y = e^{2x}$ at the point where $x = 5$.

6 Find the gradient of the tangent to the curve $y = 4e^x$ at the point where $x = 16$ correct to 2 decimal places

7 Find the equation of the tangent to the curve $y = -e^x$ at the point $(1 \;\; -e)$

8 Find the equation of the normal to the curve $y = e^{-x}$ at the point where $x = 3$ in exact form

9 A population P of insects over time t weeks is given by $P = 3e^{14\,t} + 12\,569$

a What is the initial population?

b Find the rate of change in the number of insects after

i 3 weeks **ii** 7 weeks

10 The displacement of a particle over time t seconds is given by $x = 2e^{4t}$ m

a What is the initial displacement?

b What is the exact velocity after 10 s?

c Find the acceleration after 2 s correct to 1 decimal place

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11 The displacement of an object in cm over time t seconds is given by $x = 6e^{-034\,t} - 5$ Fin:

a the initial displacement

b the initial velocity

c the displacement after 4 s

d the velocity after 9 s

e the acceleration after 2 s

12 The volume V of a balloon in mm^3 as it expands over time t seconds is given by $V = 3e^{08\,t}$

a Find the volume of the balloon at

i 3 s **ii** 5 s

b Find the rate at which the volume is increasing at

i 3 s **ii** 5 s

13 The population of a city is changing over t years according to the formula $P = 34\,500e^{0025\,t}$

a Find (to the nearest whole number) the population after

i 5 years **ii** 10 years

b Find the rate at which the population is changing after

i 5 years **ii** 10 years

14 The depth of water (in metres) in a dam is decreasing over t months according to the formula $D = 3e^{-0017\,t}$

a Find correct to 2 decimal places the depth after

i 1 month **ii** 2 months **iii** 3 months

b Find correct to 3 decimal places the rate at which the depth is changing after

i 1 month **ii** 2 months **iii** 3 months

8.04 Logarithms

The **logarithm** of a positive number, y is the **power** to which a **base** a must be raised in order to produce the number y For exampl, $\log_2 8 = 3$ because $2^3 = 8$.

If $y = a^x$ then x is called the **logarithm of y to the base a**

Just as the exponential function $y = a^x$ is defined for positive bases only ($a > 0$) logarithms are also defined for $a > 0$ Furthermor, $a \neq 1$ because $1^x = 1$ for all values of x

ogaihm

ogaihm

Logarithms

If $y = a^x$ then $x = \log_a y$ $\quad (a > 0, a \neq 1, y > 0)$

Logarithms are related to exponential functions and allow us to solve equations like $2^x = 5$.

EXAMPLE 11

a Write $\log_4 x = 3$ in index form and solve for x

b Write $5^2 = 25$ in logarithm form

c Solve $\log_x 36 = 2$.

d Evaluate $\log_3 81$.

e Find the value of $\log_2 \frac{1}{4}$

Solution

a $\log_a y = x$ means $y = a^x$

$\log_4 x = 3$ means $x = 4^3$

So $x = 64$

b $y = a^x$ means $\log_a y = x$

So $25 = 5^2$ means $\log_5 25 = 2$

c $\log_x 36 = 2$ means $36 = x^2$

$$x = \sqrt{36}$$
$$= 6$$

Note x is the base so $x > 0$

d $\log_3 81 = x$ means $81 = 3^x$

Solving $3^x = 81$

$3^x = 3^4$

So $x = 4$

$\log_3 81 = 4$

e Let $\log_2 \frac{1}{4} = x$

Then $2^x = \frac{1}{4}$

$= \frac{1}{2^2}$

$= 2^{-2}$

$\therefore x = -2$

So $\log_2 \frac{1}{4} = -2$

EXAMPLE 12

Simplify

a $\log_8 1$ **b** $\log_8 8$ **c** $\log_8 8^3$

d $\log_a a^x$ **e** $3^{\text{og } 7}$ **f** $a^{\text{og}_a x}$

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Solution

a $\log_8 1 = 0$ because $8^0 = 1$

b $\log_8 8 = 1$ because $8 \; = 8$

c $\log_8 8^3 = 3$ because $8^3 = 8^3$

d $\log_a a^x = x$ because $a^x = a^x$

e Let $\log_3 7 = y$

Then $3^y = 7$

So substituting for y

$3^{\text{og } 7} = 7$

f Let $\log_a x = y$

Then $a^y = x$

So substituting for y

$a^{\text{og}_a x} = x$

Notice that logarithms and exponentials are inverse operations

Properties of logarithms

$$\log_a a = 1$$

$$\log_a 1 = 0$$

$$\log_a a^x = x$$

$$a^{\text{og}_a x} = x$$

Common logarithms and natural logarithms

There are 2 types of logarithms that you can find on your calculator.

- **Common logarithms (base 10)** $\log_{10} x$ or $\log x$
- **Natural (Naperian) logarithms (base *e*)** $\log_e x$ or $\ln x$

EXAMPLE 13

a Find $\log_{10} 53$ correct to 1 decimal place

b Evaluate $\log_e 80$ correct to 3 significant figures

c Loudness in decibels is given by the formula $L = 10 \log_{10} \left(\frac{I}{I_0}\right)$ where I_0 is threshold sound or sound that can barely be hear. Sound louder than 85 decibels can cause hearing damage

  i The loudness of a vacuum cleaner is 10 000 000 times the threshold level or 10 000 000I_0 How many decibels is this?

  ii If the loudness of the sound of rustling leaves is 20 dB find its loudness in terms of I_0

Solution

a $\log_{10} 5.3 = 0724\ 2$ [log] 5.3 [=]

≈ 07

b $\log_e 80 = 4382\ 0$ [ln] 80 [=]

≈ 438

c **i** $L = 10 \log_{10}\left(\frac{10\,000\,000 I_0}{I_0}\right)$

$= 10 \log_{10} (10\ 000\ 000)$

$= 10 \times 7$

$= 70$

So the loudness of the vacuum cleaner is 70 dB

ii $L = 10 \log_{10}\left(\frac{I}{I_0}\right)$

$20 = 10 \log_{10}\left(\frac{I}{I_0}\right)$

$2 = \log_{10}\left(\frac{I}{I_0}\right)$

Using the definition of a logarithm

$10^2 = \frac{I}{I_0}$

$100 = \frac{I}{I_0}$

$100 I_0 = I$

So the loudness of rustling leaves is 100 times threshold sound

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DID YOU KNOW?

The origins of logarithms

John Napier (1550–1617) a Scottish theologian and an amateur mathematicia, was the first to invent logarithms Thes 'natul',or 'Napian', logarithms were based on *e* Napier originally used the compound interest formula to find the value of *e*

Napier was also one of the first mathematicians to use decimals rather than fractions He invented decimal notation using either a comma or a poin. The point was used in England but some European countries use a comm.

Henry Briggs (1561–1630) an Englishman who was a professor at Oxfor, decided that logarithms would be more useful if they were based on 10 (our decimal system) Briggs painstakingly produced a table of common logarithms correct to 14 decimal places

The work on logarithms was greatly appreciated by **Kepler** **Galileo** and other astronomers at the time since they allowed the computation of very large number.

Exercise 8.04 Logarithms

1 Evaluate

a $\log_2 16$ **b** $\log_4 16$ **c** $\log_5 125$

d $\log_3 3$ **e** $\log_7 49$ **f** $\log_7 7$

g $\log_5 1$ **h** $\log_2 128$ **i** $\log_8 8$

2 Evaluate

a $2^{og\ \ 3}$ **b** $7^{og\ \ 4}$ **c** $3^{og\ \ 29}$

3 Evaluate

a $3 \log_2 8$ **b** $\log_5 25 + 1$ **c** $3 - \log_3 81$

d $4 \log_3 27$ **e** $2 \log_{10} 10\,000$ **f** $1 + \log_4 64$

g $3 \log_4 64 + 5$ **h** $\dfrac{\log_3 9}{2}$ **i** $\dfrac{\log_8 64 + 4}{\log_2 8}$

4 Evaluate

a $\log_2 \dfrac{1}{2}$ **b** $\log_3 \sqrt{3}$ **c** $\log_4 2$ **d** $\log_5 \dfrac{1}{25}$

e $\log_7 \sqrt[4]{7}$ **f** $\log_3 \dfrac{1}{\sqrt[3]{3}}$ **g** $\log_4 \dfrac{1}{2}$ **h** $\log_8 2$

i $\log_6 6\sqrt{6}$ **j** $\log_2 \dfrac{\sqrt{2}}{4}$

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5 Evaluate correct to 2 decimal places

a $\log_{10} 1200$ **b** $\log_{10} 875$ **c** $\log_e 25$

d $\ln 140$ **e** $5 \ln 8$ **f** $\log_{10} 350 + 45$

g $\frac{\log_{10} 15}{2}$ **h** $\ln 98 + \log_{10} 17$ **i** $\frac{\log_{10} 30}{\log_e 30}$

6 Write in logarithmic for:

a $3^x = y$ **b** $5^x = z$ **c** $x^2 = y$ **d** $2^b = a$

e $b^3 = d$ **f** $y = 8^x$ **g** $y = 6^x$ **h** $y = e^x$

i $y = a^x$ **j** $Q = e^x$

7 Write in index for:

a $\log_3 5 = x$ **b** $\log_a 7 = x$ **c** $\log_3 a = b$

d $\log_x y = 9$ **e** $\log_a b = y$ **f** $y = \log_2 6$

g $y = \log_3 x$ **h** $y = \log_{10} 9$ **i** $y = \ln 4$

8 Solve for x correct to 1 decimal place where necessar:

a $\log_{10} x = 6$ **b** $\log_3 x = 5$ **c** $\log_x 343 = 3$

d $\log_x 64 = 6$ **e** $\log_5 \frac{1}{5} = x$ **f** $\log_x \sqrt{3} = \frac{1}{2}$

g $\ln x = 38$ **h** $3 \log_{10} x - 2 = 10$ **i** $\log_4 x = \frac{3}{2}$

9 Evaluate y given that $\log_y 125 = 3$.

10 If $\log_{10} x = 165$ evaluate x correct to 1 decimal place

11 Evaluate b to 3 significant figures if $\log_e b = 0894$

12 Find the value of $\log_2 1$ What is the value of $\log_a 1$?

13 Evaluate $\log_5 5$ What is the value of $\log_a a$?

14 **a** Evaluate $\ln e$ without a calculator.

b Using a calculator, evaluae:

i $\log_e e^3$ **ii** $\log_e e^2$ **iii** $\ln_e e^5$ **v** $\log_e \sqrt{e}$

v $\ln_e \frac{1}{e}$ **v** $e^{\ln 2}$ **vii** $e^{\ln 3}$ **viii** $e^{\ln 5}$

x $e^{\ln 7}$ **x** $e^{\ln 1}$ **x** $e^{n\, e}$

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15 A class was given musical facts to learn The students were then tested on these facts and each week they were given similar tests to find out how much they were able to remember. The formula $A = 85 - 55 \log_{10} (t + 2)$ seemed to model the average score after t weeks

a What was the initial average score?

b What was the average score after

i 1 week? **ii** 3 weeks?

c After how many weeks was the average score 30?

16 The pH of a solution is defined as pH = $-\log [H^+]$ where $[H^+]$ is the hydrogen ion concentration A solution is acidic if its pH is less than7, alkaline if pH is greater than 7 and neutral if pH is 7 For each question find its pH and state whether it is acidi, alkaline or neutral

a Fruit juice whose hydrogen ion concentration is 00035

b Water with $[H^+] = 10^{-7}$

c Baking soda with $[H^+] = 10^{-9}$

d Coca Cola whose hydrogen ion concentration is 001

e Bleach with $[H^+] = 1.2 \times 10^{-12}$

f Coffee with $[H^+]$ = 0000 01

17 If $f(x) = \log x$ and $g(x) = 2x - 7$ fin:

a $f(g(x))$ **b** $g(f(x))$

INVESTIGATION

HISTORY OF BASES AND NUMBER SYSTEMS

Common logarithms use base 10 like our decimal number system We might have developed a different system if we had a different number of fingers The Mayan, in ancient times used base 20 for their number system since they counted with both their fingers and toes

1 Research the history and types of other number systems including those of Aboriginal and Torres Strait Islander peopls. Did any cultures use systems other than base 10? Why?

2 Explore computer-based system. Computers have used both binary (base 2) and octal (base 8) Find out why these bases are use.

8.05 Logarithm laws

ogaihm law

ogaihm law

Because logarithms are just another way of writing indices (powers) there are logarithm laws that correspond to the index laws

$$\log_a (xy) = \log_a x + \log_a y$$

Proof

Let $\quad x = a^m \quad$ and $\quad y = a^n$

Then $\quad m = \log_a x$ and $n = \log_a y$

$$xy = a^m \times a^n$$
$$= a^{m+n}$$

$\therefore \quad \log_a (xy) = m + n \qquad$ (by definition)

$$= \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

Proof

Let $\quad x = a^m \quad$ and $\quad y = a^n$

Then $\quad m = \log_a x \quad$ and $\quad n = \log_a y$

$$\frac{x}{y} = a^m \div a^n$$
$$= a^{m-n}$$

$\therefore \log_a \left(\frac{x}{y}\right) = m - n \qquad$ (by definition)

$$= \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

Proof

Let $\quad x = a^m$

Then $\quad m = \log_a x$

$$x^n = (a^m)^n$$
$$= a^{mn}$$

$\therefore \quad \log_a x^n = mn \qquad$ (by definition)

$$= n \log_a x$$

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$$\log_a\left(\frac{1}{x}\right) = -\log_a x$$

Proof

$$\begin{aligned}\log_a\left(\frac{1}{x}\right) &= \log_a 1 - \log_a x\\ &= 0 - \log_a x\\ &= -\log_a x\end{aligned}$$

EXAMPLE 14

ogaihm law

a Given $\log_5 3 = 068$ and $\log_5 4 = 086$ fin:

i $\log_5 12$ **ii** $\log_5 075$ **iii** $\log_5 9$ **v** $\log_5 20$

b Solve $\log_2 12 = \log_2 3 + \log_2 x$

c Simplify $\log_a 21$ if $\log_a 3 = p$ and $\log_a 7 = q$

d The formula for measuring R the strength of an earthquake on the Richter scal, is $R = \log\left(\frac{I}{S}\right)$ where I is the maximum seismograph signal of the earthquake being measured and S is the signal of a standard earthquake

Show that

i $\log I = R + \log S$ **ii** $I = S(10^R)$

Solution

a **i**
$$\begin{aligned}\log_5 12 &= \log_5 (3 \times 4)\\ &= \log_5 3 + \log_5 4\\ &= 068 + 086\\ &= 154\end{aligned}$$

ii
$$\begin{aligned}\log_5 075 &= \log_5 \frac{3}{4}\\ &= \log_5 3 - \log_5 4\\ &= 068 - 086\\ &= -018\end{aligned}$$

iii
$$\begin{aligned}\log_5 9 &= \log_5 3^2\\ &= 2\log_5 3\\ &= 2 \times 068\\ &= 1.36\end{aligned}$$

v
$$\begin{aligned}\log_5 20 &= \log_5 (5 \times 4)\\ &= \log_5 5 + \log_5 4\\ &= 1 + 086\\ &= 186\end{aligned}$$

b $\log_2 12 = \log_2 3 + \log_2 x$

$= \log_2 3x$

So $12 = 3x$

$4 = x$

c $\log_a 21 = \log_a (3 \times 7)$

$= \log_a 3 + \log_a 7$

$= p + q$

d **i** $R = \log\left(\frac{I}{S}\right)$

$= \log I - \log S$

$R + \log S = \log I$

ii $R = \log\left(\frac{I}{S}\right)$

$\frac{I}{S} = 10^R$

$I = S(10^R)$

Change of base

If we need to evaluate logarithms such as $\log_5 2$ we use the change of base formul.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof

Let $y = \log_a x$

Then $x = a^y$

Take logarithms to the base b of both sides of the equation

$\log_b x = \log_b a^y$

$= y \log_b a$

$\therefore \frac{\log_b x}{\log_b a} = y$

$= \log_a x$

To find the logarithm of any numbe, such as $\log_5 2$ you can change it to either $\log_{10} x$ or $\log_e x$

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EXAMPLE 15

a Evaluate $\log_5 2$ correct to 2 decimal places

b Find the value of $\log_2 3$ to 1 decimal place

Solution

a $\log_5 2 = \frac{\log 2}{\log 5}$

≈ 043

b $\log_2 3 = \frac{\log 3}{\log 2}$

≈ 1.6

Exercise 8.05 Logarithm laws

1 Simplify

a $\log_a 4 + \log_a y$
b $\log_a 4 + \log_a 5$
c $\log_a 12 - \log_a 3$
d $\log_a b - \log_a 5$
e $3 \log_x y + \log_x z$
f $2 \log_k 3 + 3 \log_k y$
g $5 \log_a x - 2 \log_a y$
h $\log_a x + \log_a y - \log_a z$
i $\log_{10} a + 4 \log_{10} b + 3 \log_{10} c$
j $3 \log_3 p + \log_3 q - 2 \log_3 r$
k $\log_4 \frac{1}{n}$
$\log_x \frac{1}{6}$

2 Evaluate

a $\log_5 5^2$
b $\log_7 7^6$

3 Given $\log_7 2 = 036$ and $\log_7 5 = 083$ fin:

a $\log_7 10$
b $\log_7 04$
c $\log_7 20$
d $\log_7 25$
e $\log_7 8$
f $\log_7 14$
g $\log_7 50$
h $\log_7 35$
i $\log_7 98$

4 Use the logarithm laws to evaluate

a $\log_5 50 - \log_5 2$
b $\log_2 16 + \log_2 4$
c $\log_4 2 + \log_4 8$
d $\log_5 500 - \log_5 4$
e $\log_9 117 - \log_9 13$
f $\log_8 32 + \log_8 16$
g $3 \log_2 2 + 2 \log_2 4$
h $2 \log_4 6 - (2 \log_4 3 + \log_4 2)$
i $\log_6 4 - 2 \log_6 12$
j $2 \log_3 6 + \log_3 18 - 3 \log_3 2$

5 If $\log_a 3 = x$ and $\log_a 5 = y$ find an expression in terms of x and y for

a $\log_a 15$ **b** $\log_a 06$ **c** $\log_a 27$

d $\log_a 25$ **e** $\log_a 9$ **f** $\log_a 75$

g $\log_a 3a$ **h** $\log_a \frac{a}{5}$ **i** $\log_a 9a$

6 If $\log_a x = p$ and $\log_a y = q$ fin, in terms of p and q

a $\log_a xy$ **b** $\log_a y^3$ **c** $\log_a \frac{y}{x}$ **d** $\log_a x^2$

e $\log_a xy^5$ **f** $\log_a \frac{x^2}{y}$ **g** $\log_a ax$ **h** $\log_a \frac{a}{y^2}$

i $\log_a a^3y$ **j** $\log_a \frac{x}{ay}$

7 If $\log_a b = 34$ and $\log_a c = 47$ evaluat:

a $\log_a \frac{c}{b}$ **b** $\log_a bc^2$ **c** $\log_a (bc)^2$

d $\log_a abc$ **e** $\log_a a^2c$ **f** $\log_a b^7$

g $\log_a \frac{a}{c}$ **h** $\log_a a^3$ **i** $\log_a bc^4$

8 Solve

a $\log_4 12 = \log_4 x + \log_4 3$ **b** $\log_3 4 = \log_3 y - \log_3 7$

c $\log_a 6 = \log_a x - 3\log_a 2$ **d** $\log_2 81 = 4\log_2 x$

e $\log_x 54 = \log_x k + 2\log_x 3$

9 **a** Change the subject of $\text{dB} = 10\log\left(\frac{I}{I_0}\right)$ to I

b Find the value of I in terms of I_0 when dB = 45

10 **a** Show that the formula $A = 100 - 50\log(t+1)$ can be written as

i $\log(t+1) = \frac{100-A}{50}$ **ii** $t = 10^{\frac{100-A}{50}} - 1$

b Hence find

i A when $t = 3$ **ii** t when $A = 75$

11 Evaluate to 2 decimal places

a $\log_4 9$ **b** $\log_6 25$ **c** $\log_9 200$ **d** $\log_2 12$

e $\log_3 23$ **f** $\log_8 250$ **g** $\log_5 95$ **h** $2\log_4 234$

i $7 - \log_7 108$ **j** $3\log_{11} 340$

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8.06 Logarithmic functions

A **logarithmic function** is a function of the form $y = \log_a x$

Exponenial and log gaphs machup

EXAMPLE 16

Sketch the graph of $y = \log_2 x$

Solution

y-intercept ($x = 0$) No y-intercept because $x > 0$

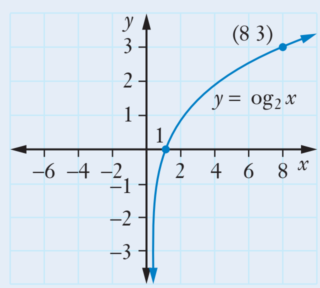
x-intercept ($y = 0$) $0 = \log_2 x$

$x = 2^0 = 1$, so x-intercept is 1 ($y = 0$)

Complete a table of values

$y = \log_2 x$ means $x = 2^y$ For $x = 6$ in the table use the change of base formul, $\log\ _2 x = \dfrac{\log x}{\log 2}$

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	6	8
y	–2	–1	0	1	2	2.58	3



Logarithmic functions

- The logarithmic function $y = \log_a x$ is the inverse function of an exponential function $y = a^x$
- Domain (0 ∞) range (–∞ ∞)
- $x > 0$ so the curve is always to the right of the y-axis (no y-intercept)
- The y-axis is an **asymptote**
- The x-intercept is always 1 because $\log_a 1 = 0$

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EXAMPLE 17

Sketch the graph of

a $y = \log_e x - 1$ **b** $y = 3\log_{10} x + 4$

Solution

a No y-intercept ($x = 0$) because $\log_e 0$ is undefined The y-axis is an asymptote

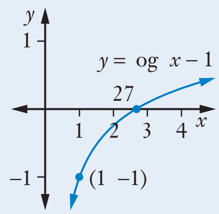
x-intercept ($y = 0$)

$0 = \log_e x - 1$

$1 = \log_e x$

$x = e$

≈ 27



Notce that ths s the graph of $y = \log_e x$ moved down 1 unt

Complete a table of values for this graph using the ln key on the calculator.

x	1	2	3	4
y	–1	–03	0.1	0.4

b Complete a table of values using the log key on the calculator.

$y = 3\log_{10} x + 4$

x	1	2	3	4
y	4	4.9	5.4	5.8

No y-intercept

For x-intercept $y = 0$

$$0 = 3\log_{10} x + 4$$

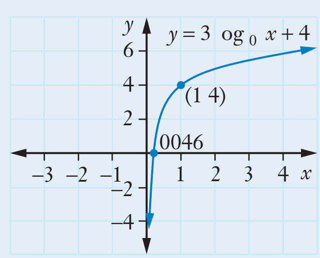
$$-4 = 3\log_{10} x$$

$$-\frac{4}{3} = \log_{10} x$$

$$10^{-\frac{4}{3}} = x$$

$$x = 004641\ldots$$

$$\approx 0046$$



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EXAMPLE 18

a Sketch the graphs of $y = e^x$ $y = \log_e x$ and $y = x$ on the same set of axes

b What relationship do these graphs have?

c If $f(x) = \log_a x$ sketch the graph of $y = -f(x)$ and state its domain and range

Solution

a Drawing $y = e^x$ gives an exponential curve with y-intercept 1

Find another point say $x = 2$:

$y = e^2$

$= 73890$

≈ 74

Drawing $y = \log_e x$ gives a logarithmic curve with x-intercept 1

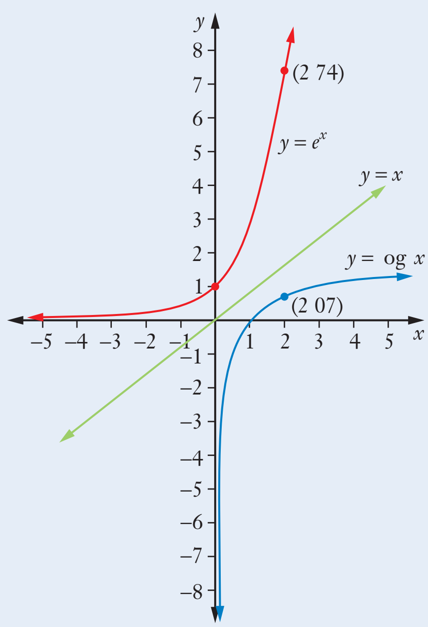
Find another point say $x = 2$:

$y = \ln 2$

$= 06931$

≈ 07

$y = x$ is a linear function with gradient 1 and y-intercept 0



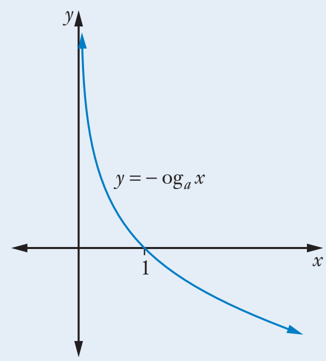
b The graphs of $y = e^x$ and $y = \log_e x$ are reflections of each other in the line $y = x$
They are **inverse functions**

c Given $f(x) = \log_a x$

$y = -f(x)$

$= -\log_a x$

This is a reflection of $f(x)$ in the x-axis



Domain (0 ∞) range (−∞ ∞)

The exponential and logarithmic functions

$f(x) = a^x$ and $f(x) = \log_a x$ are inverse functions Their graphs are reflections of each other in the line $y = x$

INVESTIGATION

GRAPHS OF LOGARITHMIC FUNCTIONS

1 Substitute different values of x into the logarithmic function $y = \log x$ positive negative and zero What do you notice ?

2 Use a graphics calculator or graphing software to sketch the graphs of different logarithmic functions such as

a $y = \log_2 x$ $y = \log_3 x$ $y = \log_4 x$ $y = \log_5 x$ $y = \log_6 x$

b $y = \log_2 x + 1, y = \log_2 x + 2, y = \log_2 x + 3, y = \log_2 x - 1, y = \log_2 x - 2$

c $y = 2 \log_2 x$ $y = 3 \log_2 x$ $y = -\log_2 x$ $y = -2 \log_2 x$ $y = -3 \log_2 x$

d $y = 2 \log_2 x + 1, y = 2 \log_2 x + 2, y = 2 \log_2 x + 3, y = 2 \log_2 x - 1, y = 2 \log_2 x - 2$

e $y = 3 \log_4 x + 1, y = 5 \log_3 x + 2, y = -\log_5 x + 3, y = -2 \log_2 x - 1, y = 4 \log_7 x - 2$

3 Try sketching the graph of $y = \log_{-2} x$ What does the table of values look like? Are there any discontinuities on the graph? Why? Could you find the domain and range? Use a graphics calculator or graphing software to sketch this graph What do you find ?

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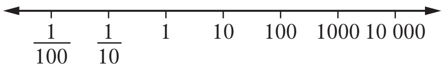
Logarithmic scales

It is difficult to describe and graph exponential functions because their y values increase so quickly. We use logarithms and **logarithmic scales** to solve this problem

On a base 10 logarithmic scale an axis or number line has units that don't increase by1, but by powers of 10



Examples of base 10 logarithmic scales are

- the Richter scale for measuring earthquake magnitude
- the pH scale for measuring acidity in chemistry
- the decibel scale for measuring loudness
- the octave (frequency) scale in music

EXAMPLE 19

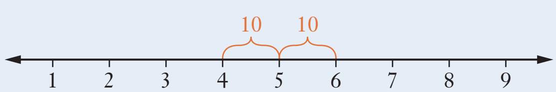
a Ged finds that the pH of soil is 4 in the eastern area of his garden and 6 in the western area The pH formula is logarithmic and pH < 7 is acidic What is the difference in acidity in these 2 areas of the garden?

b If Ged finds another area with a pH of 36 how much more acidic is this area than the eastern area?

Solution

a The difference in pH between 4 and 6 is 2 But this is a logarithmic scal.

Each interval on a logarithmic scale is a multiple of 10



So the difference is $10 \times 10 = 10^2 = 100$

The lower the pH the more acidi. So the soil in the eastern area is 100 times more acidic than the soil in the western area

b The difference in pH between 4 and 36 is 04

So the difference is $10^{0.4} = 2.5118 \approx 2.5.$

The soil in this area is about 25 times more acidic than the soil in the eastern area

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Exercise 8.06 Logarithmic functions

1 Sketch the graph of each logarithmic function and state its domain and range

a $y = \log_3 x$ **b** $f(x) = 2\log_4 x$ **c** $y = \log_2 x + 1$

d $y = \log_5 x - 1$ **e** $f(x) = \log_4 x - 2$ **f** $y = 5\ln x + 3$

g $f(x) = -3\log_{10} x + 2$

2 Sketch the graphs of $y = 10^x$ $y = \log_{10} x$ and $y = x$ on the same number plane
What do you notice about the relationship of the curves to the line?

3 Sketch the graph of $f(x) = \log_2 x$ and $y = \log_2(-x)$ on the same set of axes and describe their relationship

4 **a** Sketch the graphs of $y = \log_2 x$ $y = 2^x$ and $y = x$ on the same set of axes

b Find the inverse function of $y = \log_2 x$

5 This table lists some of the earthquakes experienced in Australi.

Year	Location	Strength on Richter scale
1989	Newcastle NSW	56
1997	Collier Bay WA	63
2001	Swan Hill Vic	48
2010	Kalgoorlie WA	52
2015	Coral Sea Qld	55
2017	Orange NSW	43
2018	Coffs Harbour NSW	42

The Richter scale for earthquakes is logarithmic Use the table to find the difference in magnitude (correct to the nearest whole number) between the earthquakes in

a Newcastle and Swan Hill

b Collier Bay and Orange

c Newcastle and Orange

d Coral Sea and Kalgoorlie

e Collier Bay and Coffs Harbour

6 The decibel (dB) scale for loudness is logarithmic Find (correct to the nearest whole number) the difference in loudness between

a 20 and 23 dB **b** 40 and 41 dB **c** 652 and 665 dB

d 854 and 889 dB **e** 523 and 586 dB

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8.07 Exponential equations

Exponential equations can be solved using logarithms or the change of base formula

Exponenial equaion

ogaihmic and exponenial equaion

Solving exponenial equaion

Uing exponenial model

EXAMPLE 20

Solve $5^x = 7$ correct to 1 decimal place

Solution

Method 1 Logarithms

Take logarithms of both side:

$\log 5^x = \log 7$

$x \log 5 = \log 7$

$x = \dfrac{\log 7}{\log 5}$

$= 12090$

≈ 1.2

Method 2 Change of base formula

Convert to logarithm form

$5^x = 7$ means $\log_5 7 = x$

Using the change of base to evaluate x

$x = \log_5 7$

$= \dfrac{\log 7}{\log 5}$

$= 12090$

≈ 1.2

EXAMPLE 21

a Solve $e^{34\,x} = 100$ correct to 2 decimal places

b The temperature T in °C of a metal as it cools down over t minutes is given by $T = 27 + 219e^{-0032\,t}$ Fin, correct to 1 decimal plae, the time it takes to cool down to 100°C

Exponenial uncion

Solution

a With an equation involving e we use $\ln x$ which is $\log_e x$

Take natural logs of both side:

$\ln e^{34\,x} = \ln 100$

$34\,x = \ln 100$ $\quad$ $\ln x$ and e^x are inverses

$x = \dfrac{\ln 100}{3\,4}$

$= 13544$

≈ 135

b When $T = 100$

$$100 = 27 + 219e^{-0032\,t}$$

$$73 = 219e^{-0032\,t}$$

$$\frac{73}{219} = e^{-0.032t}$$

$$\log_e\left(\frac{73}{219}\right) = \log_e(e^{-0.032t})$$

$$= -0032\,t$$

$$t = \frac{\log_e\left(\frac{73}{219}\right)}{-0032}$$

$$= 343316$$

$$= 343 \text{ to 1 dp}$$

So it takes 343 minutes to cool down to 100°C

Exercise 8.07 Exponential equations

1 Solve each equation correct to 2 significant figures

a $4^x = 9$ **b** $3^x = 5$ **c** $7^x = 14$ **d** $2^x = 15$

e $5^x = 34$ **f** $6^x = 60$ **g** $2^x = 76$ **h** $4^x = 50$

i $3^x = 23$ **j** $9^x = 210$

2 Solve correct to 2 decimal place:

a $2^x = 6$ **b** $5^y = 15$ **c** $3^x = 20$ **d** $7^m = 32$

e $4^k = 50$ **f** $3^t = 4$ **g** $8^x = 11$ **h** $2^p = 57$

i $4^x = 81.3$ **j** $6^n = 1026$

3 Solve to 1 decimal plac:

a $3^{x+1} = 8$ **b** $5^{3n} = 71$ **c** $2^{x-3} = 12$

d $4^{2n-1} = 7$ **e** $7^{5x+2} = 11$ **f** $8^{3-n} = 57$

g $2^{x+2} = 18.3$ **h** $3^{7k-3} = 329$ **i** $9^{\frac{x}{2}} = 50$

4 Solve each equation correct to 3 significant figures

a $e^x = 200$ **b** $e^{3t} = 5$ **c** $2e^t = 75$

d $45 = e^x$ **e** $3000 = 100e^n$ **f** $100 = 20e^{3t}$

g $2000 = 50e^{015\,t}$ **h** $15\,000 = 2000e^{003\,k}$ **i** $3Q = Qe^{0.02t}$

5 The amount A of money in a bank account after n years grows with compound interest according to the formula $A = 850(1025)^n$

a Find

i the initial amount in the bank **ii** the amount after 7 years

b Find how many years it will take for the amount in the bank to be $1000

6 The population of a city is given by $P = 35\,000e^{0.024t}$ where t is time in years

a Find the population

i initially **ii** after 10 years **iii** after 50 years

b Find when the population will reach

i 80 000 **ii** 200 000

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7 A species of wattle is gradually dying out in a Blue Mountains region The number of wattle trees over time t years is given by $N = 8900e^{-0.048t}$

a Find the number of wattle trees

i initially

ii after 5 years

iii after 70 years

b After how many years will there be

i 5000 wattle trees?

ii 200 wattle trees?

8 A formula for the mass M g of plutonium after t years is given by $M = 100e^{-0000\ 03\ t}$ Fin:

a initial mass **b** mass after 50 years **c** mass after 500 years

d its half-life (the time taken to decay to half of its initial mass)

9 The temperature of an electronic sensor is given by the formula $T = 18 + 12e^{0.002t}$ where t is in hours

a What is the temperature of the sensor after 5 hours?

b When the temperature reaches 50°C the sensor needs to be shut down to cool After how many hours does this happen?

10 A particle is moving along a straight line with displacement x cm over time t s according to the formula $x = 5e^{t} + 23$.

a Find

i the initial displacement

ii the exact velocity after 20 s

iii the displacement after 6 s

v the time when displacement is 85 cm

v the time when the velocity is 1000 cm s^{-}

b Show that acceleration $a = x - 23$.

c Find the acceleration when displacement is 85 cm

8. TEST YOURSELF

Pacice quiz

For Questions **1** to **3** select the correct answer **A B C** or **D**

1 Simplify $\log_a 15 - \log_a 3$:

A $\log_a 45$ **B** $\frac{\log_a 15}{\log_a 3}$ **C** $\log_a 15 \times \log_a 3$ **D** $\log_a 5$

2 Write $a^x = y$ as a logarithm

A $\log_y x = a$ **B** $\log_a y = x$ **C** $\log_a x = y$ **D** $\log_x a = y$

3 Solve $5^x = 4$ (there is more than one answer)

A $x = \frac{\log 4}{\log 5}$ **B** $x = \frac{\log 5}{\log 4}$ **C** $x = \frac{\ln 4}{\ln 5}$ **D** $x = \frac{\ln 5}{\ln 4}$

4 Evaluate

a $\log_2 8$ **b** $\log_7 7$ **c** $\log_{10} 1000$ **d** $\log_9 81$

e $\log_e e$ **f** $\log_4 64$ **g** $\log_9 3$ **h** $\log_2 \frac{1}{2}$

i $\log_5 \frac{1}{25}$ **j** $\ln e^3$

5 Evaluate to 3 significant figures

a $e^2 - 1$ **b** $\log_{10} 95$ **c** $\log_e 26$ **d** $\log_4 7$

e $\log_4 3$ **f** $\ln 50$ **g** $e + 3$ **h** $\frac{5e^3}{\ln 4}$

6 Evaluate

a $e^{n\,6}$ **b** $e^{\ln 2}$

7 Write in index for:

a $\log_3 a = x$ **b** $\ln b = y$ **c** $\log c = z$

8 If $\log_7 2 = 036$ and $\log_7 3 = 056$ find the value o:

a $\log_7 6$ **b** $\log_7 8$ **c** $\log_7 1.5$

d $\log_7 14$ **e** $\log_7 3.5$

9 Solve

a $3^x = 8$ **b** $2^{3x-4} = 3$ **c** $\log_x 81 = 4$ **d** $\log_6 x = 2$

10 Solve $12 = 10e^{001\,t}$

11 Evaluate $\log_9 8$ to 1 decimal place

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12 Simplify

a $5 \log_a x + 3 \log_a y$ **b** $2 \log_x k - \log_x 3 + \log_x p$

13 Evaluate to 2 significant figures

a $\log_{10} 45$ **b** $\ln 37$

14 Sketch the graph of $y = 2^x + 1$ and state its domain and range

15 Solve

a $2^x = 9$ **b** $3^x = 7$ **c** $5^{x+1} = 6$ **d** $4^{2y} = 11$

e $8^{3n-2} = 5$ **f** $\log_x 16 = 4$ **g** $\log_3 y = 3$ **h** $\log_7 n = 2$

i $\log_x 64 = \frac{1}{2}$ **j** $\log_8 m = \frac{1}{3}$

16 Write as a logarith:

a $2^x = y$ **b** $5^a = b$ **c** $10^x = y$

d $e^x = z$ **e** $3^{x+1} = y$

17 Sketch the graph of

a $y = 5(3^{x+2})$ **b** $y = 2(3^x) - 5$ **c** $f(x) = -3^x$ **d** $y = 3(2^{-x})$

18 Sketch the graph of

a $f(x) = \log_3 x$ **b** $y = 3 \ln x - 4$

19 If $\log_x 2 = a$ and $\log_x 3 = b$ find in terms of a and b

a $\log_x 6$ **b** $\log_x 1.5$ **c** $\log_x 8$

d $\log_x 18$ **e** $\log_x 27$

20 The formula for loudness is $L = 10 \log\left(\frac{I}{I_0}\right)$ where I_0 is threshold sound and L is measured in decibels (dB) Fin:

a the dB level of a $5500I_0$ sound

b the sound in terms of I_0 if its dB level is 32

21 Simplify

a $\log_a \frac{1}{x}$ **b** $\log_e \frac{1}{y}$

22 Evaluate

a $\log_6 12 + \log_6 3$ **b** $\log 25 + \log 4$ **c** $2 \log_4 8$

d $\log_8 72 - \log_8 9$ **e** $\log 53\ 000 - \log 53$

23 Solve correct to 1 decimal place

a $e^x = 15$ **b** $27^{\,x} = 21.8$ **c** $10^x = 1287$

24 The amount of money in the bank after n years is given by $A = 5280(1019)^{\,n}$

a Find the amount in the bank

i initially **ii** after 3 years **iii** after 4 years

b Find how long it will take for the amount of money in the bank to reach

i $6000 **ii** $10 000

25 Differentiate each function

a $y = e^{3x}$ **b** $y = e^{-2x}$ **c** $y = 5e^{4x}$

d $y = -2e^{8x} + 5x^3 - 1$ **e** $y = x^2e^{2x}$ **f** $y = (4e^{3x} - 1)^9$

g $y = \dfrac{x}{e^{2x}}$

26 The formula for the number of wombats in a region of New South Wales after t years is $N = 1118 - 37e^{0032\,t}$

a Find the initial number of wombats in this region

b How many wombats are there after 5 years?

c How long will it take until the number of wombats in the region is

i 500? **ii** 100?

27 Differentiate

a $y = e^x + x$ **b** $y = -4e^x$ **c** $y = 3e^{-x}$

d $y = (3 + e^x)^9$ **e** $y = 3x^5e^x$ **f** $y = \dfrac{e^x}{7x - 2}$

28 An earthquake has magnitude 67 and its aftershock has magnitude 47 on the base 10 logarithmic Richter scale How much larger is the first earthquake ?

29 Shampoo A has pH 72 and shampoo B has pH 85 The pH scale is base 10 logarithmc. How much more alkaline is shampoo B?

30 If $f(x) = \log_e x$ $g(x) = e^x$ and $h(x) = 6x^2 - 1$, fin:

a $f(h(x))$ **b** $g(h(x))$ **c** $h(g(x))$

d $f(g(x))$ **e** $g(f(x))$

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8. CHALLENGE EXERCISE

1 If $\log_b 2 = 06$ and $\log_b 3 = 11$, fin:

a $\log_b 6b$ **b** $\log_b 8b$ **c** $\log_b 1.5b^2$

2 Find the point of intersection of the curves $y = \log_e x$ and $y = \log_{10} x$

3 Sketch the graph of $y = \log_2 (x - 1)$ and state its domain and range

4 By substituting $u = 3^x$ solve $3^{2x} - 3^x - 2 = 0$ correct to 2 decimal places

5 The pH of a solution is given by $\text{pH} = -\log [H^+]$ where $[H^+]$ is the hydrogen ion concentration

a Show that pH could be given by $\text{pH} = \log \dfrac{1}{[H^+]}$

b Show that $[H^+] = \dfrac{1}{10^{\text{pH}}}$

c Find the hydrogen ion concentration to 1 significant figur, of a substance with a pH of

i 63

ii 77

6 If $y = 8 + \log_2 (x + 2)$

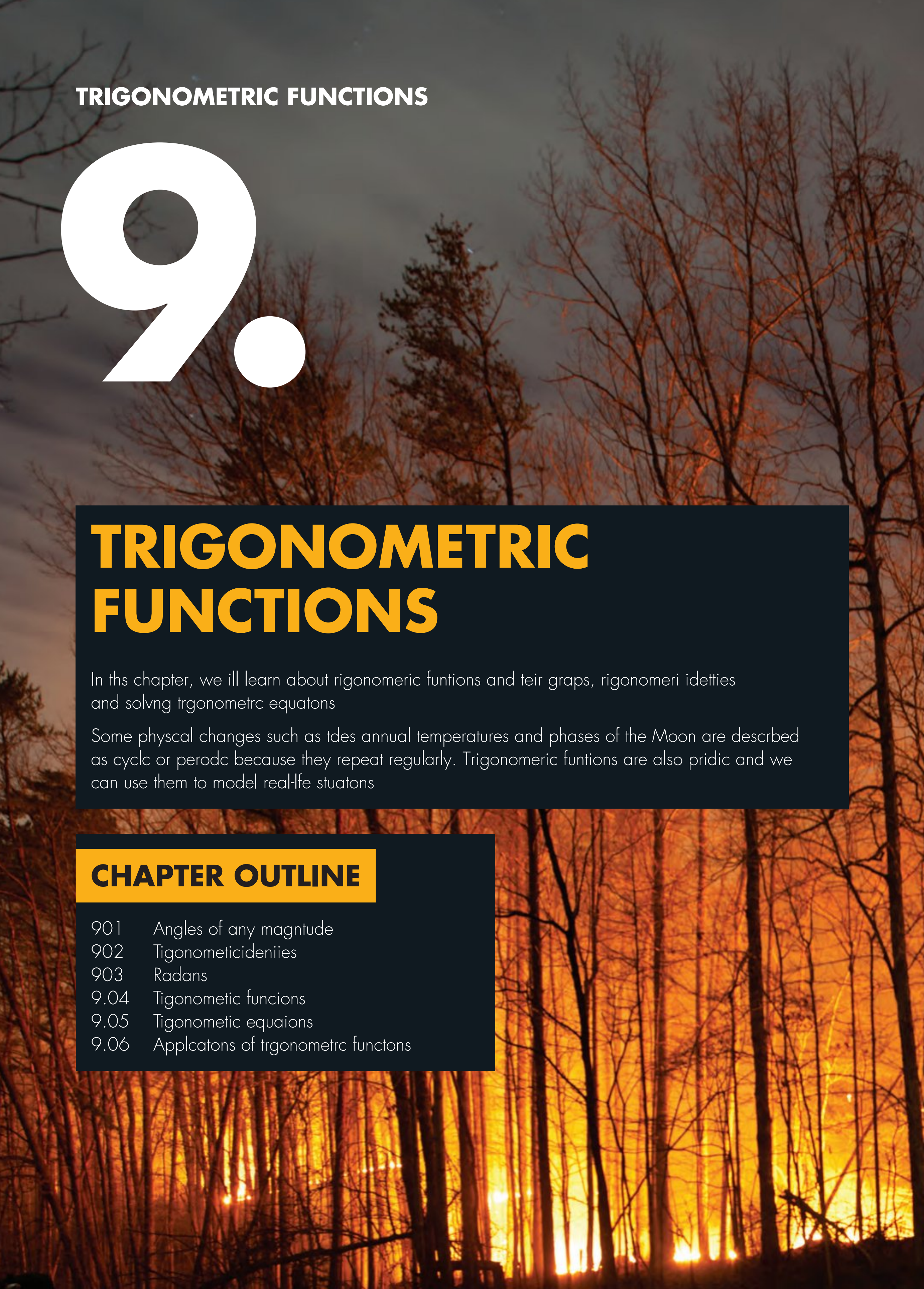
a show that $x = 2(2^{y-9} - 1)$

b find correct to 2 decimal place:

i y when $x = 5$

ii x when $y = 1$

7 Find the equation of **a** the tangent and **b** the normal to the curve $y = 3e^x - 5$ at the point $(2, 3e^2 - 5)$



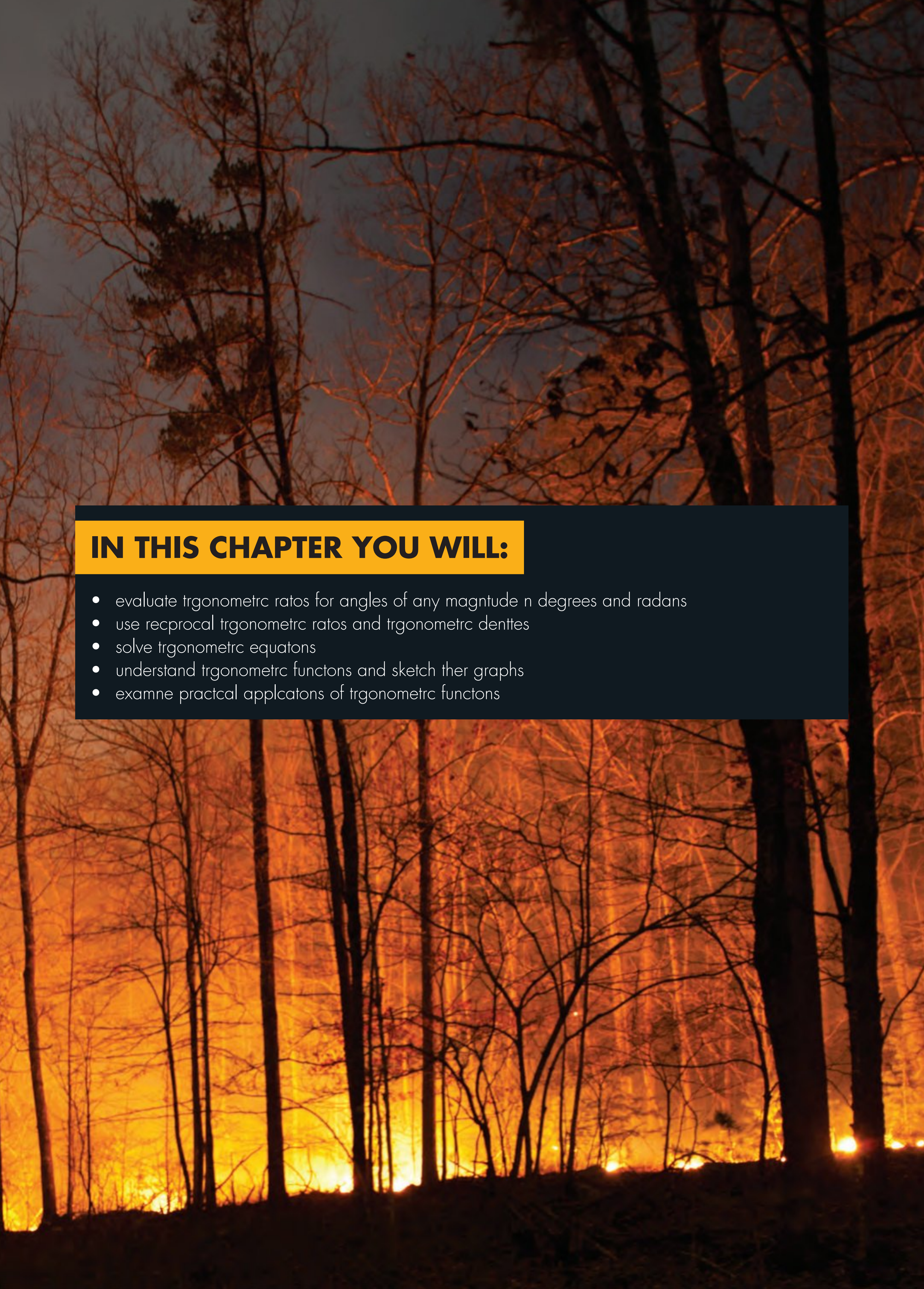
9.

TRIGONOMETRIC FUNCTIONS

In ths chapter, we ill learn about rigonomeric funtions and teir graps, rigonomeri idetties and solvng trgonometrc equatons

Some physcal changes such as tdes annual temperatures and phases of the Moon are descrbed as cyclc or perodc because they repeat regularly. Trigonomeric funtions are also pridic and we can use them to model real-lfe stuatons

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- evaluate trgonometrc ratos for angles of any magntude n degrees and radans
- use recprocal trgonometrc ratos and trgonometrc denttes
- solve trgonometrc equatons
- understand trgonometrc functons and sketch ther graphs
- examne practcal applcatons of trgonometrc functons

TERMINOLOGY

amplitude The height from the centre of a periodic function to the maximum or minimum values (peaks and troughs of its graph respectively) For $y = k \sin ax$ the amplitude is k

centre The mean value of a periodic function that is equidistant from the maximum and minimum values For $y = k \sin ax + c$ the centre is c

identity An equation that shows the equivalence of 2 algebraic expressions for all values of the variables

period The length of one cycle of a periodic function on the x-axis before the function repeats itself For $y = k \sin ax$ the period is $\frac{2\pi}{a}$

periodic function A function that repeats itself regularly

phase A horizontal shift (translation. For $y = k \sin [a(x + b)]$ the phase is b that i, the graph of $y = k \sin ax$ shifted b units to the left

reciprocal trigonometric ratios The cosecan, secant and cotangent ratios which are the reciprocals of sine cosine and tangent respectively

WS

Angles o any magniude

9.01 Angles of any magnitude

In Chapter 4 *Trigonometry* we examined acute and obtuse obtuse angles by looking at angles turning around a unit circle We can find angles of *any* size by continuing around the circle

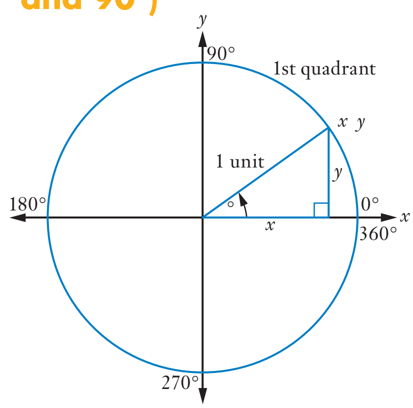
1st quadrant: acute angles (between 0° and 90°)

You can see from the triangle in the unit circle with angle θ that

$$\sin \theta = y$$

$$\cos \theta = x$$

$$\tan \theta = \frac{y}{x}$$



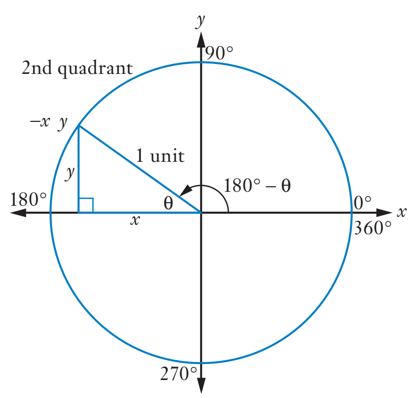
In the 1st quadrant x and y are both positive so all ratios are positive in the 1st quadrant

2nd quadrant: obtuse angles (between 90° and 180°)

$\sin \theta = y$ (positive)

$\cos \theta = -x$ (negative)

$\tan \theta = \frac{y}{-x}$ (negative)



The angle that gives θ in the triangle is $180° - \theta$

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2nd quadrant

$\sin(180° - \theta) = \sin\theta$

$\cos(180° - \theta) = -\cos\theta$

$\tan(180° - \theta) = -\tan\theta$

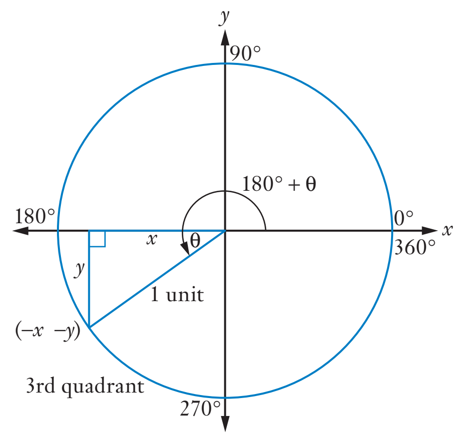
3rd quadrant: angles between 180° and 270°

$\sin\theta = -y$ (negative)

$\cos\theta = -x$ (negative)

$\tan\theta = \frac{-y}{-x} = \frac{y}{x}$ (positive)

The angle that gives θ in the triangle is 180° + θ



3rd quadrant

$\sin(180° + \theta) = -\sin\theta$

$\cos(180° + \theta) = -\cos\theta$

$\tan(180° + \theta) = \tan\theta$

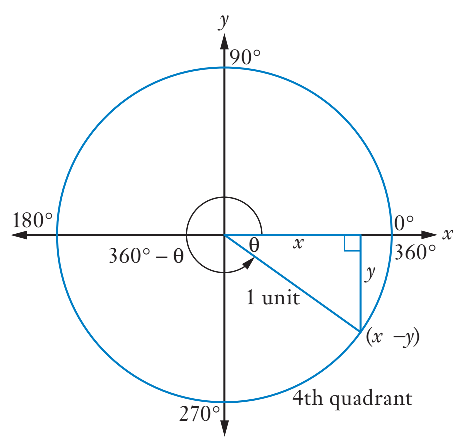
4th quadrant: angles between 270° and 360°

$\sin\theta = -y$ (negative)

$\cos\theta = x$ (positive)

$\tan\theta = \frac{-y}{x}$ (negative)

The angle that gives θ in the triangle is 360° – θ



4th quadrant

$\sin(360° - \theta) = -\sin\theta$

$\cos(360° - \theta) = \cos\theta$

$\tan(360° - \theta) = -\tan\theta$

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Putting all of these results together gives a rule for all 4 quadrants that we usually call the **ASTC rule**

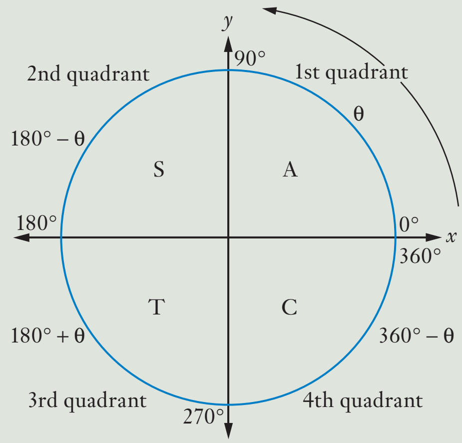
ASTC rule

A All ratios are positive in the 1st quadrant

S Sin is positive in the 2nd quadrant (cos and tan are negative)

T Tan is positive in the 3rd quadrant (sin and cos are negative)

C Cos is positive in the 4th quadrant (sin and tan are negative)



ASTC can be remembered usng the phrase **A** **S**tatons **T**o **C**entral

EXAMPLE 1

a Find all quadrants where

i $\sin\theta > 0$ **ii** $\cos\theta < 0$ **iii** $\tan\theta < 0$ and $\cos\theta > 0$

b Find the exact value of

i $\tan 330°$ **ii** $\sin 225°$

c Simplify $\cos(180° + x)$

d If $\sin x = -\frac{3}{5}$ and $\cos x > 0$ find the value of $\tan x$

Solution

a **i** Using the ASTC rul, $\sin\theta > 0$ in the 1st and 2nd quadrants

ii $\cos\theta > 0$ in the 1st and 4th quadrants so $\cos\theta < 0$ in the 2nd and 3rd quadrants

iii $\tan\theta > 0$ in the 1st and 3rd quadrants so $\tan\theta < 0$ in the 2nd and 4th quadrants
Also $\cos\theta > 0$ in the 1st and 4th quadrants

So $\tan\theta < 0$ and $\cos\theta > 0$ in the 4th quadrant

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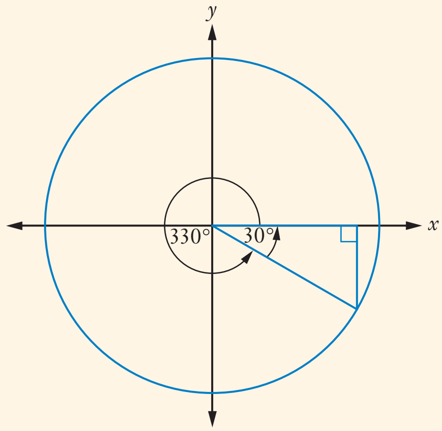
b **i** 330° lies in the 4th quadrant

The angle inside the triangle in the 4th quadrant is 360° – 330° = 30° and tan is negative in the 4th quadrant

$\tan 330° = -\tan 30°$

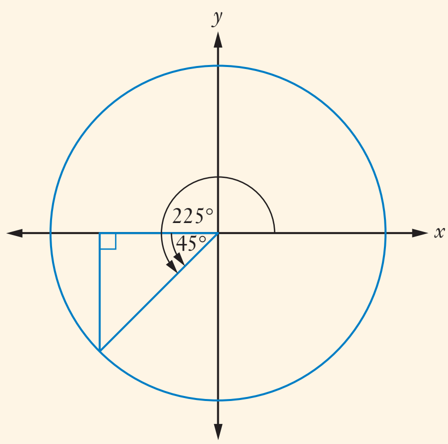
$= -\frac{1}{\sqrt{3}}$



ii 225° is in the 3rd quadrant
The angle in the triangle in the 3rd quadrant is 225° – 180° = 45° and sin is negative in the 3rd quadrant

$\sin 225° = -\sin 45°$

$= -\frac{1}{\sqrt{2}}$



c $180° + x$ is in the 3rd quadrant where $\cos x$ is negative

So $\cos(180° + x) = -\cos x$

d $\sin x < 0$ and $\cos x > 0$ so x is in the 4th quadrant

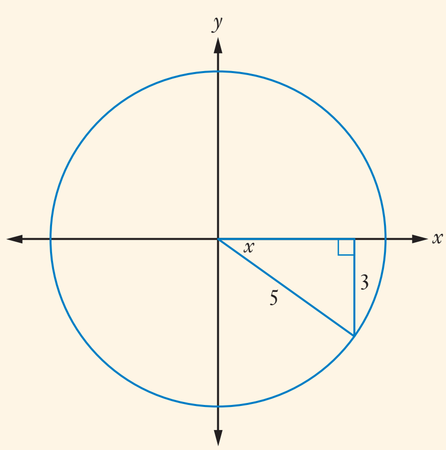
$\sin x = -\frac{3}{5}$

So the opposite side is 3 and the hypotenuse is 5

By Pythagoras theorem the adjacent side is 4 (,4, 5 triange).

$\tan x < 0$ in the 4th quadrant

So $\tan x = -\frac{3}{4}$



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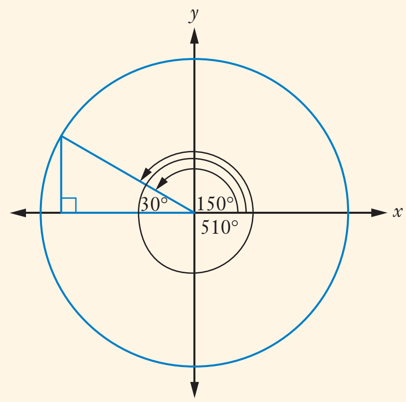
We can find trigonometric ratios of angles greater than 360° by turning around the circle more than once

EXAMPLE 2

Find the exact value of cos 510°

Solution

To find cos 510° we move around the circle more than once



$\cos(510° - 360°) = \cos(150°)$

The angle is in the 2nd quadrant where cos is negative The angle inside the triangle is $180° - 150° = 30°$

So $\cos 510° = \cos 150°$

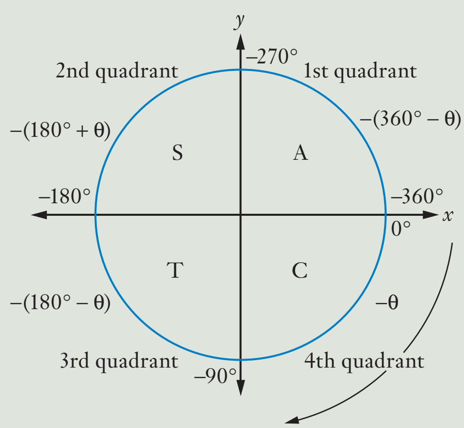
$= -\cos 30°$

$= -\dfrac{\sqrt{3}}{2}$

Negative angles

The ASTC rule also works for negative angle. These are measured in the opposite direction (clockwise) from positive angles as shown

Negative angles



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In the 4th quadrant	In the 3rd quadrant
$\sin(-\theta) = -\sin\theta$	$\sin(-(180° - \theta)) = -\sin\theta$
$\cos(-\theta) = \cos\theta$	$\cos(-(180° - \theta)) = -\cos\theta$
$\tan(-\theta) = -\tan\theta$	$\tan(-(180° - \theta)) = \tan\theta$
In the 2nd quadrant	In the 1st quadrant
$\sin(-(180° + \theta)) = \sin\theta$	$\sin(-(360° - \theta)) = \sin\theta$
$\cos(-(180° + \theta)) = -\cos\theta$	$\cos(-(360° - \theta)) = \cos\theta$
$\tan(-(180° + \theta)) = -\tan\theta$	$\tan(-(360° - \theta)) = \tan\theta$

EXAMPLE 3

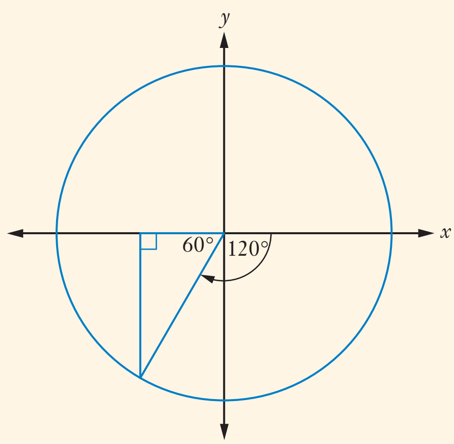
Find the exact value of tan (–120°)

Solution

Moving clockwise around the circle the angle is in the 3rd quadrant with $180° - 120° = 60°$ in the triangle

tan is positive in the 3rd quadrant

$$\tan(-120°) = \tan 60°$$
$$= \sqrt{3}$$



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Exercise 9.01 Angles of any magnitude

1 Find all quadrants where

a $\cos\theta > 0$ **b** $\tan\theta > 0$ **c** $\sin\theta > 0$

d $\tan\theta < 0$ **e** $\sin\theta < 0$ **f** $\cos\theta < 0$

g $\sin\theta < 0$ and $\tan\theta > 0$ **h** $\cos\theta < 0$ and $\tan\theta < 0$

i $\cos\theta > 0$ and $\tan\theta < 0$ **j** $\sin\theta < 0$ and $\tan\theta < 0$

2 a Which quadrant is the angle 240° in?

b Find the exact value of cos 240°

3 a Which quadrant is the angle 315° in?

b Find the exact value of sin 315°

4 a Which quadrant is the angle 120° in?

b Find the exact value of tan 120°

5 a Which quadrant is the angle –225° in?

b Find the exact value of sin (–225°)

6 a Which quadrant is the angle –330° in?

b Find the exact value of cos (–330°)

7 Find the exact value of

a tan 225° **b** cos 315° **c** tan 300° **d** sin 150°

e cos 120° **f** sin 210° **g** cos 330° **h** tan 150°

i sin 300° **j** cos 135°

8 Find the exact value of

a cos (–225°) **b** cos (–210°) **c** tan (–300°) **d** cos (–150°)

e sin (–60°) **f** tan (–240°) **g** cos (–300°) **h** tan (–30°)

i cos (–45°) **j** sin (–135°)

9 Find the exact value of

a cos 570° **b** tan 420° **c** sin 480° **d** cos 660°

e sin 690° **f** tan 600° **g** sin 495° **h** cos 405°

i tan 675° **j** sin 390°

10 If $\tan\theta = \frac{3}{4}$ and $\cos\theta < 0$ find $\sin\theta$ and $\cos\theta$ as fractions

11 Given $\sin\theta = \frac{4}{7}$ and $\tan\theta < 0$ find the exact value of $\cos\theta$ and $\tan\theta$

12 If $\sin x < 0$ and $\tan x = -\frac{5}{8}$ find the exact value of $\cos x$

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13 Given $\cos x = \frac{2}{5}$ and $\tan x < 0$ find the exact value of $\sin x$ and $\tan x$

14 If $\cos x < 0$ and $\sin x > 0$ find $\cos x$ and $\sin x$ in surd form if $\tan x = \frac{5}{7}$

15 If $\sin \theta = -\frac{4}{9}$ and $270° < \theta < 360°$ find the exact value of $\tan \theta$ and $\cos \theta$

16 If $\cos x = -\frac{3}{8}$ and $180° < x < 270°$ find the exact value of $\tan x$ and $\sin x$

17 Given $\sin x = 03$ and $\tan x < 0$

- **a** express $\sin x$ as a fraction
- **b** find the exact value of $\cos x$ and $\tan x$

18 If $\tan \alpha = -12$ and $270° < \alpha < 360°$ find the exact values of $\cos \alpha$ and $\sin \alpha$

19 Given that $\cos \theta = -07$ and $90° < \theta < 180°$ find the exact value of $\sin \theta$ and $\tan \theta$

20 Simplify

- **a** $\sin (180° - \theta)$
- **b** $\cos (360° - x)$
- **c** $\tan (180° + \beta)$
- **d** $\sin (180° + \alpha)$
- **e** $\tan (360° - \theta)$
- **f** $\sin (-\theta)$
- **g** $\cos (-\alpha)$
- **h** $\tan (-x)$

9.02 Trigonometric identities

WS Trigonomric ideniie

Simpliying igonomeic uncion

The reciprocal trigonometric ratios

The **reciprocal trigonometric ratios** are the reciprocals of the sine cosine and tangent ratios

The reciprocal trigonometric ratios

Cosecant $\quad \operatorname{cosec} \theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}}$

cosec θ can also be wrtten as *csc* θ.

Secant $\quad \sec \theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$

Cotangent $\quad \cot \theta = \frac{1}{\tan\theta} = \frac{\text{adjacent}}{\text{opposite}}$

The reciprocal ratios have the same signs as their related ratios in the different quadrants
For example in the 3rd and 4th quadrant, $\sin \theta < 0$ so $\operatorname{cosec} \theta < 0$

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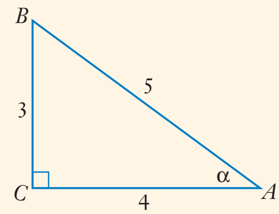
EXAMPLE 4

a Find cosec α sec α and cot α for this triangle

b If $\sin\theta = -\frac{2}{7}$ and tan > 0 find the exact ratios of cot θ sec θ and cosec θ

c State the quadrants where cosec θ is negative



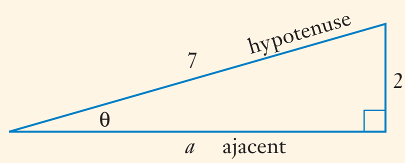
Solution

a
$$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3}$$
$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4}$$
$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}$$

b sin θ < 0 and tan θ > 0 in the 3rd quadrant So cos θ < 0

By Pythagoras theorem

$$7^2 = a^2 + 2^2$$
$$a^2 + 4 = 49$$
$$a^2 = 45$$
$$a = \sqrt{45}$$
$$= 3\sqrt{5}$$



$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent}}{\text{opposite}} = \frac{3\sqrt{5}}{2}$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}} = -\frac{7}{3\sqrt{5}} = -\frac{7\sqrt{5}}{15}$$
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}} = -\frac{7}{2}$$

c sin θ < 0 in the 3rd and 4th quadrants

So cosec θ < 0 in the 3rd and 4th quadrants

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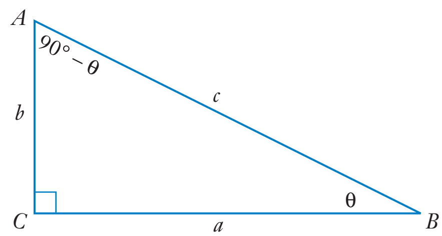
Complementary angles

In $\triangle ABC$ if $\angle B = \theta$ then $\angle A = 90° - \theta$ (by the angle sum of a triangle) $\angle B$ and $\angle A$ are **complementary angles** because they add up to 90°



$\sin\theta = \frac{b}{c}$ $\quad$ $\sin(90° - \theta) = \frac{a}{c}$

$\cos\theta = \frac{a}{c}$ $\quad$ $\cos(90° - \theta) = \frac{b}{c}$

$\tan\theta = \frac{b}{a}$ $\quad$ $\tan(90° - \theta) = \frac{a}{b}$

$\sec\theta = \frac{c}{a}$ $\quad$ $\sec(90° - \theta) = \frac{c}{b}$

$\operatorname{cosec}\theta = \frac{c}{b}$ $\quad$ $\operatorname{cosec}(90° - \theta) = \frac{c}{a}$

$\cot\theta = \frac{a}{b}$ $\quad$ $\cot(90° - \theta) = \frac{b}{a}$

Notice the pairs of trigonometric ratios that are equal

Complementary angle results

$\sin\theta = \cos(90° - \theta)$ $\quad$ $\tan\theta = \cot(90° - \theta)$ $\quad$ $\sec\theta = \operatorname{cosec}(90° - \theta)$

$\cos\theta = \sin(90° - \theta)$ $\quad$ $\cot\theta = \tan(90° - \theta)$ $\quad$ $\operatorname{cosec}\theta = \sec(90° - \theta)$

EXAMPLE 5

a Simplify $\tan 50° - \cot 40°$

b Find the value of m if $\sec 55° = \operatorname{cosec}(2m - 15)°$

Solution

a $\tan 50° - \cot 40° = \tan 50° - \cot(90° - 50°)$

$= \tan 50° - \tan 50°$

$= 0$

b $\sec 55° = \operatorname{cosec}(90° - 55°)$

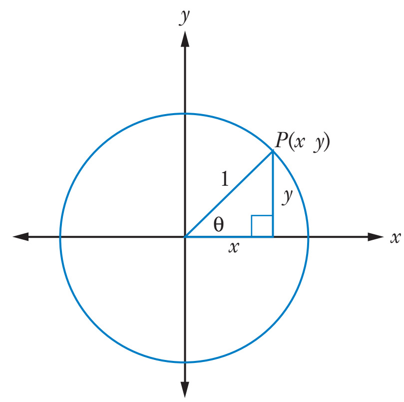
$= \operatorname{cosec} 35°$

So $2m - 15 = 35$

$2m = 50$

$m = 25$

The tangent identity



In the work on angles of any magnitude we saw that

$\sin \theta = y$ $\cos \theta = x$ and $\tan \theta = \frac{y}{x}$

From this we get the following trigonometric identities

> **The tangent identity**
>
> For any value of θ
>
> $$\tan \theta = \frac{\sin\theta}{\cos\theta} \qquad \cot \theta = \frac{\cos\theta}{\sin\theta}$$

An **identity** is an equation that shows the equivalence of 2 algebraic expressions for all values of the variables for exampl, $a^2 - b^2 = (a + b)(a - b)$ is an identity.

EXAMPLE 6

Simplify $\sin \theta \cot \theta$

Solution

$$\begin{aligned}\sin \theta \cot \theta &= \sin \theta \times \frac{\cos\theta}{\sin\theta}\\ &= \cos \theta\end{aligned}$$

The Pythagorean identities

The unit circle above has equation $x^2 + y^2 = 1$ because of Pythagora' theore.

But $\sin \theta = y$ and $\cos \theta = x$ so

$(\cos \theta)^2 + (\sin \theta)^2 = 1$

A shorter way of writing this is

$\cos^2 \theta + \sin^2 \theta = 1$

This formula is called a Pythagorean identity because it is based on Pythagoras theorem in the unit circle

There are 2 other identities that can be derived from this identity.

Dividing each term by $\cos^2 \theta$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

Dividing each term by $\sin^2 \theta$

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\cot^2 \theta + 1 = \text{cosec}^2 \theta$$

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Pythagorean identities

For any value of θ

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

$\cos^2 \theta + \sin^2 \theta = 1$ can also be rearranged to give

$$\cos^2 \theta = 1 - \sin^2 \theta \text{ or}$$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

EXAMPLE 7

Prove that

a $\cot x + \tan x = \operatorname{cosec} x \sec x$

b $\dfrac{1-\cos x}{\sin^2 x} = \dfrac{1}{1+\cos x}$

Solution

a
$$\begin{aligned}
\text{LHS} &= \cot x + \tan x \\
&= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \\
&= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} \\
&= \frac{1}{\sin x \cos x} \\
&= \frac{1}{\sin x} \times \frac{1}{\cos x} \\
&= \operatorname{cosec} x \sec x \\
&= \text{RHS}
\end{aligned}$$

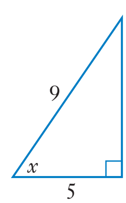
$\therefore \cot x + \tan x = \operatorname{cosec} x \sec x$

b
$$\begin{aligned}
\text{LHS} &= \frac{1-\cos x}{\sin^2 x} \\
&= \frac{1-\cos x}{1-\cos^2 x} \\
&= \frac{1-\cos x}{(1+\cos x)(1-\cos x)} \\
&= \frac{1}{1+\cos x} \\
&= \text{RHS}
\end{aligned}$$

$$\therefore \frac{1-\cos x}{\sin^2 x} = \frac{1}{1+\cos x}$$

Exercise 9.02 Trigonometric identities

1 For this triangle find the exact ratios of sec x cot x and cosec x



2 If $\sin \theta = \dfrac{5}{13}$ find cosec θ sec θ and cot θ

3 If $\cos \theta = \dfrac{4}{7}$ find exact values of cosec θ sec θ and cot θ

4 If $\sec \theta = -\dfrac{6}{5}$ and $\sin \theta > 0$ find exact values of tan θ cosec θ and cot θ

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5 If cot θ = 06 and cosec θ < 0 find the exact values of sin θ cosec θ tan θ and sec θ

6 Show sin 67° = cos 23°

7 Show sec 82° = cosec 8°

8 Show tan 48° = cot 42°

9 Simplify

a $\cos 61° + \sin 29°$

b $\sec \theta - \text{cosec}\,(90° - \theta)$

c $\tan 70° + \cot 20° - 2\tan 70°$

d $\dfrac{\sin 55°}{\cos 35°}$

e $\dfrac{\cot 25° + \tan 65°}{\cot 25°}$

10 Find the value of x if $\sin 80° = \cos\,(90 - x)°$

11 Find the value of y if $\tan 22° = \cot\,(90 - y)°$

12 Find the value of p if $\cos 49° = \sin\,(p + 10)°$

13 Find the value of b if $\sin 35° = \cos\,(b + 30)°$

14 Find the value of t if $\cot\,(2t + 5)° = \tan\,(3t - 15)°$

15 Find the value of k if $\tan\,(15 - k)° = \cot\,(2k + 60)°$

16 Simplify

a $\tan \theta \cos \theta$

b $\tan \theta\, \text{cosec}\, \theta$

c $\sec x \cot x$

d $1 - \sin^2 x$

e $\sqrt{1 - \cos^2 °}$

f $\cot^2 x + 1$

g $1 + \tan^2 x$

h $\sec^2 \theta - 1$

i $5\cot^2 \theta + 5$

j $\dfrac{1}{\text{cosec}^2 x}$

k $\sin^2 \alpha\, \text{cosec}^2 \alpha$

$\cot \theta - \cot \theta \cos^2 \theta$

17 Prove that

a $\cos^2 x - 1 = -\sin^2 x$

b $\sec \theta + \tan \theta = \dfrac{1 + \sin °}{\cos °}$

c $3 + 3\tan^2 \alpha = \dfrac{3}{1 - \sin^2 °}$

d $\sec^2 x - \tan^2 x = \text{cosec}^2 x - \cot^2 x$

e $(\sin x - \cos x)^3 = \sin x - \cos x - 2\sin^2 x \cos x + 2\sin x \cos^2 x$

f $\cot \theta + 2\sec \theta = \dfrac{1 - \sin^2 ° + 2\sin °}{\sin ° \cos °}$

g $\cos^2 (90° - \theta) \cot \theta = \sin \theta \cos \theta$

h $(\text{cosec}\, x + \cot x)(\text{cosec}\, x - \cot x) = 1$

i $\dfrac{1 - \sin^2 ° \cos^2 °}{\cos^2 °} = \tan^2 \theta + \cos^2 \theta$

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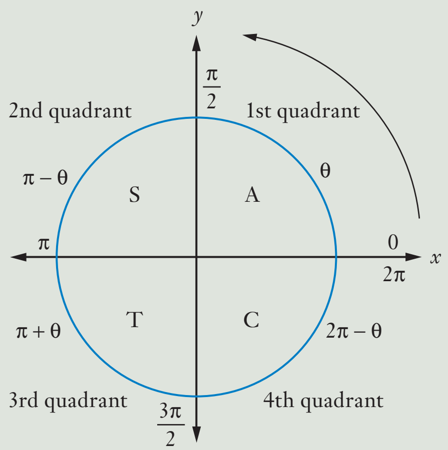
9.03 Radians

The rules and formulas learned in this chapter can also be expressed in radians which we learned about in Chapter 4 *Trigonometry*

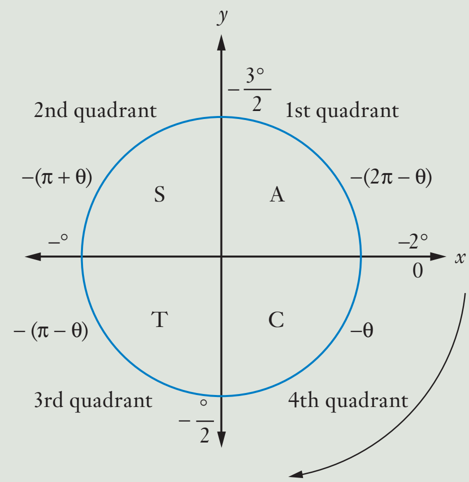
WS
Radians o any magniude

ASTC rule

Positive angles



Negative angles



In the 2nd quadran:

$\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

In the 3rd quadran:

$\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi + \theta) = -\cos\theta$

$\tan(\pi + \theta) = \tan\theta$

In the 4th quadran:

$\sin(2\pi - \theta) = -\sin\theta$

$\cos(2\pi - \theta) = \cos\theta$

$\tan(2\pi - \theta) = -\tan\theta$

In the 4th quadran:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

In the 3rd quadran:

$\sin(-(\pi - \theta)) = -\sin\theta$

$\cos(-(\pi - \theta)) = -\cos\theta$

$\tan(-(\pi - \theta)) = \tan\theta$

In the 2nd quadran:

$\sin(-(\pi + \theta)) = \sin\theta$

$\cos(-(\pi + \theta)) = -\cos\theta$

$\tan(-(\pi + \theta)) = -\tan\theta$

In the 1st quadran:

$\sin(-(2\pi - \theta)) = \sin\theta$

$\cos(-(2\pi - \theta)) = \cos\theta$

$\tan(-(2\pi - \theta)) = \tan\theta$

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EXAMPLE 8

Find the exact value of

a $\sin \frac{5\pi}{4}$

b $\cos \frac{11\pi}{6}$

Solution

a $\frac{5\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4}$

$= \pi + \frac{\pi}{4}$

in the 3rd quadrant so $\sin \theta < 0$

$$\sin\left(\frac{5\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right)$$
$$= -\sin\frac{\pi}{4}$$
$$= -\frac{1}{\sqrt{2}}$$

b $\frac{11\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6}$

$= 2\pi - \frac{\pi}{6}$

in the 4th quadrant so $\cos \theta > 0$

$$\cos\left(\frac{11\pi}{6}\right) = \cos\left(2\pi - \frac{\pi}{6}\right)$$
$$= \cos\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2}$$

Exercise 9.03 Radians

1 Find the exact value of each expression

a $\operatorname{cosec} \frac{\pi}{4}$

b $\sec \frac{\pi}{6}$

c $\cot \frac{\pi}{3}$

d $\dfrac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}$

e $1 - \cos^2 \frac{\pi}{4}$

f $\tan \frac{\pi}{3} \cos \frac{\pi}{3}$

g $\sqrt{1 + \tan^2 \frac{\pi}{4}}$

h $\operatorname{cosec}^2 \frac{\pi}{6} - 1$

i $\dfrac{\cot\frac{\pi}{5} + \tan\frac{3\pi}{10}}{\cot\frac{\pi}{5}}$

2 **a** Show that $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$

b In which quadrant is the angle $\frac{3\pi}{4}$?

c Find the exact value of $\cos \frac{3\pi}{4}$

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3 a Show that $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$

b In which quadrant is the angle $\frac{5\pi}{6}$?

c Find the exact value of $\sin \frac{5\pi}{6}$

4 a Show that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$

b In which quadrant is the angle $\frac{7\pi}{4}$?

c Find the exact value of $\tan \frac{7\pi}{4}$

5 a Show that $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$

b In which quadrant is the angle $\frac{4\pi}{3}$?

c Find the exact value of $\cos \frac{4\pi}{3}$

6 a Show that $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$

b In which quadrant is the angle $\frac{5\pi}{3}$?

c Find the exact value of $\sin \frac{5\pi}{3}$

7 a i Show that $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$

ii In which quadrant is the angle $\frac{13\pi}{6}$?

iii Find the exact value of $\cos \frac{13\pi}{6}$

b Find the exact value of

i $\sin \frac{9\pi}{4}$ **ii** $\tan \frac{7\pi}{3}$ **iii** $\cos \frac{11\pi}{4}$

v $\tan \frac{19\pi}{6}$ **v** $\sin \frac{10\pi}{3}$

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8 Copy and complete each table with exact values

a

	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{3}$	$\frac{8\pi}{3}$	$\frac{10\pi}{3}$	$\frac{11\pi}{3}$
sin								
cos								
tan								

b

	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$	$\frac{11\pi}{4}$	$\frac{13\pi}{4}$	$\frac{15\pi}{4}$
sin								
cos								
tan								

c

	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{13\pi}{6}$	$\frac{17\pi}{6}$	$\frac{19\pi}{6}$	$\frac{23\pi}{6}$
sin								
cos								
tan								

9 Copy and complete the table where possible

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
sin									
cos									
tan									

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9.04 Trigonometric functions

INVESTIGATION

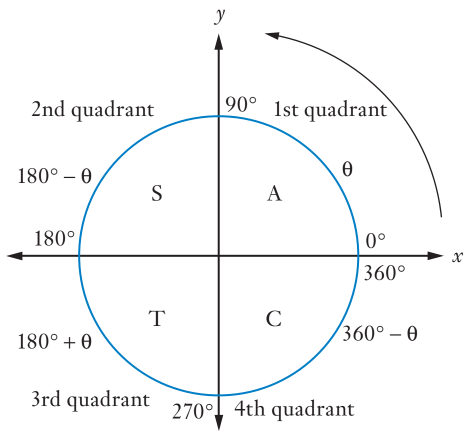
TRIGONOMETRIC RATIOS OF 0°, 90°, 180°, 270° AND 360°

Remember the results from the unit circle

$\sin \theta = y$

$\cos \theta = x$

$\tan \theta = \frac{y}{x}$



1 Angle 0° is at the point (1 0) on the unit circl. Use the circle results to find sin °, cos 0° and tan 0°

2 Angle 90° is at the point (0 1. Use the circle results to find sin 90 ° cos 90° and tan 90° Discuss the result for tan 90° and why this happens

3 Angle 180° is at the point (–1 0. Find sin 180 °, cos 180° and tan 180°

4 Angle 270° is at the point (0 –1) Find sin 270 ° cos 270° and tan 270° Discuss the result for tan 270° and why this happens

5 What are the results for sin 360° cos 360° and tan 360°? Why?

6 Check these results on your calculator.

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The sine function

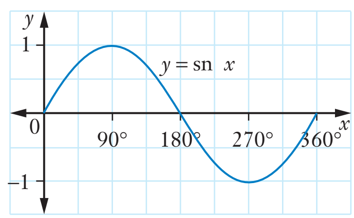
Using all the results from the investigation we can draw up a table of values for $y = \sin x$

x	0°	90°	180°	270°	360°
y	0	1	0	–1	0

We could add in all the exact value results we know for a more accurate grap. Remember that sin x is positive in the 1st and 2nd quadrants and negative in the 3rd and 4th quadrants

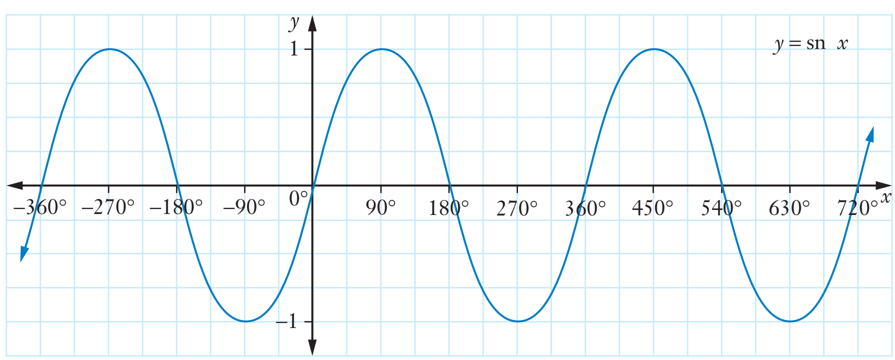
x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	–1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0

Drawing the graph gives a smooth 'wav' curv.



As we go around the unit circle and graph the y values of the points on the circle the graph should repeat itself every 360°

$y = \sin x$ has domain $(-\infty\ \infty)$ and range $[\,-1\ 1$. It is an odd functin.



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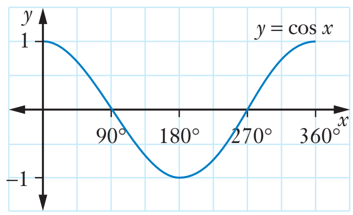
The cosine function

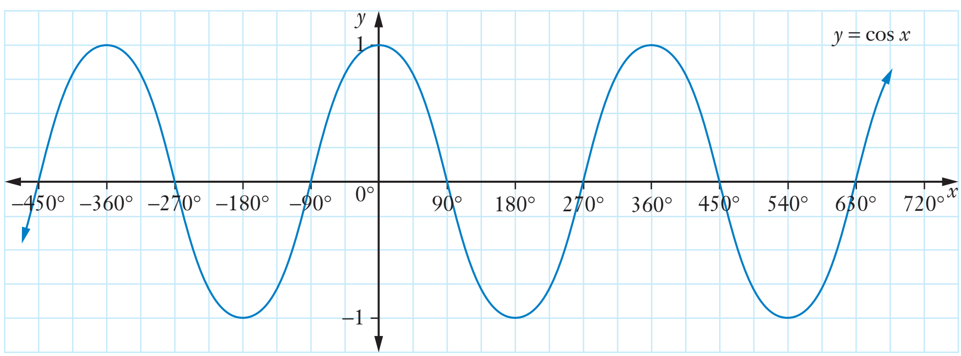
Similarly for $y = \cos x$ which is positive in the 1st and 4th quadrants and negative in the 2nd and 3rd quadrants Its graph has the same shape as the graph of the sine functio.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	0



As we go around the unit circle and graph the x values of the points on the circle the graph should repeat itself every 360°

$y = \cos x$ has domain $(-\infty \;\; \infty)$ and range $[-1 \; 1$. It is an even functin.



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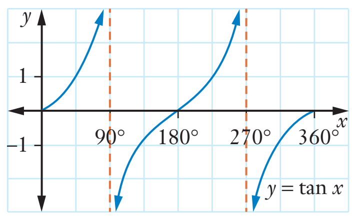
The tangent function

$y = \tan x$ is positive in the 1st and 3rd quadrants and negative in the 2nd and 4th quadrants

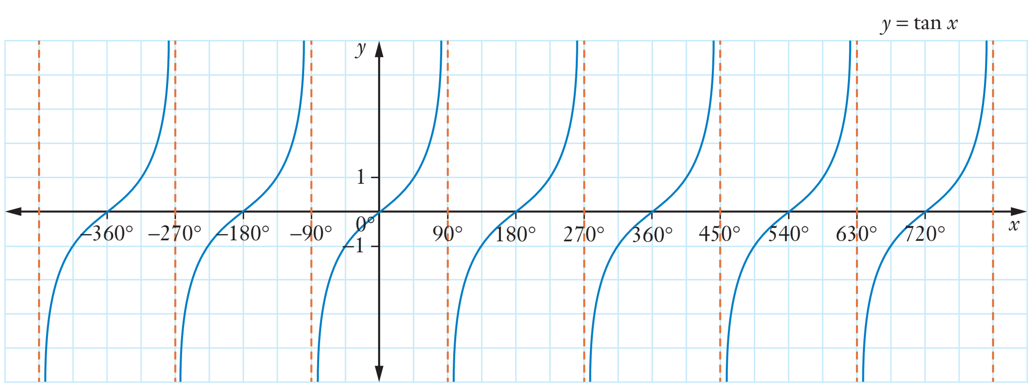
It is also undefined for 90° and 270° so there are vertical asymptotes at those x values where the function is discontinuous

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	–1	$-\frac{1}{\sqrt{3}}$	0



As we go around the unit circle and graph the values of $\frac{y}{x}$ of the points on the circle the graph repeats itself every 180°

$y = \tan x$ has domain (–∞ ∞) except for 90° 270, 54°, … (odd multiples of 90°) and range (–∞ ∞) It is an odd functio.



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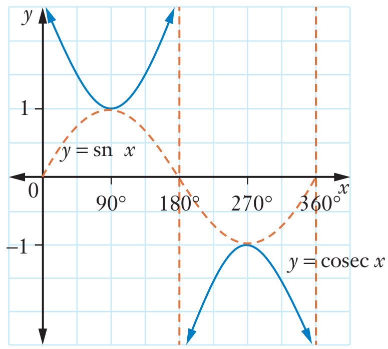
The cosecant function

$\text{cosec } x = \dfrac{1}{\sin x}$

Each y value of $y = \text{cosec } x$ will be the reciprocal of $y = \sin x$ Because $\sin x = 0$ at $x = 0°, 180°, 360°$, $y = \text{cosec } x$ will have vertical asymptotes at those values

We can use a table of values and explore the limits as x approaches any asymptotes

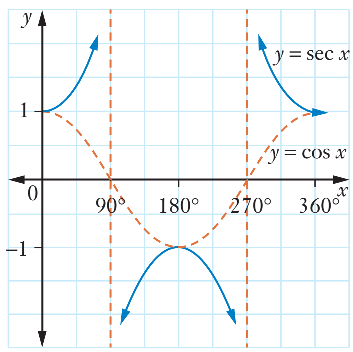
x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	–	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	–	–2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	–1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	–2	–



The secant function

$\sec x = \dfrac{1}{\cos x}$ so each y value of $y = \sec x$ will be the reciprocal of $y = \cos x$ Because $\cos x = 0$ at $x = 90°, 270°, 450°, \ldots, y = \sec x$ will have vertical asymptotes at those values

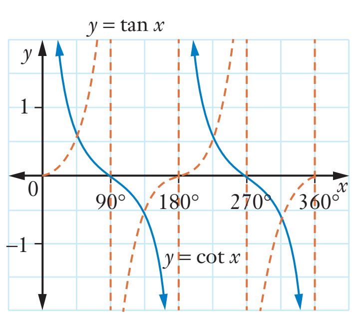
x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	–	−2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	−1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	−2	–	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1



The cotangent function

$\cot x = \dfrac{1}{\tan x}$ so each y value of $y = \cot x$ will be the reciprocal of $y = \tan x$ Because $\tan x = 0$ at $x = 0°, 180°, 360°, \ldots, y = \cot x$ will have vertical asymptotes at those values Alo, because $\tan x$ has asymptotes at $x = 90°, 270°, 450°, \ldots, y = \cot x = 0$ and there are x-intercepts at those values

x	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°	225°	240°	270°	300°	315°	330°	360°
y	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	−1	$-\sqrt{3}$	–



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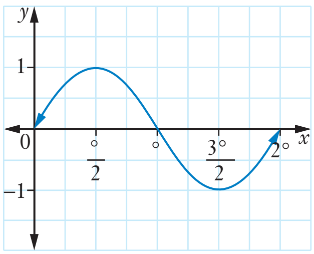
It is more practical to express the trigonometric functions in terms of radians (not degrees) so here are the graphs in radians

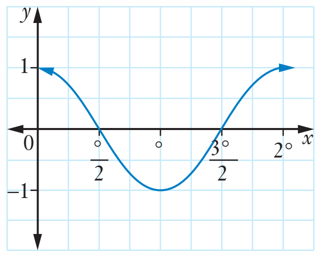
$y = \sin x$



Domain (–∞ ∞) range [–1, 1]

Odd function

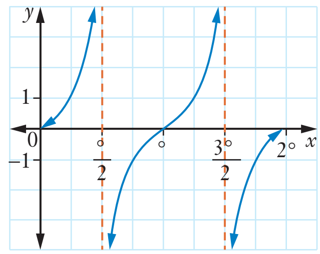
$y = \cos x$



Domain (–∞ ∞) range [–1, 1]

Even function

$y = \tan x$

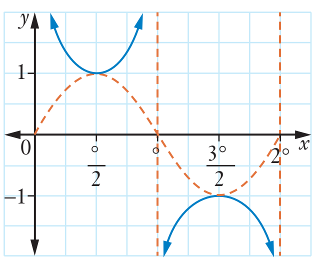


Domain (–∞ ∞)

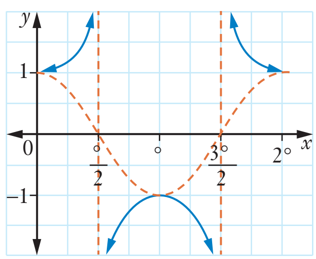
except for $\frac{\pi}{2}$ $\frac{3\pi}{2}$ $\frac{5\pi}{2}$

(odd multiples of $\frac{\pi}{2}$)
range (–∞ ∞)
Odd function

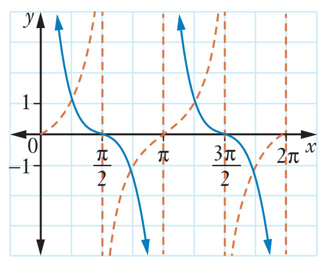
$y = \text{cosec}\, x$



$y = \sec x$



$y = \cot x$



Properties of the trigonometric functions

All the trigonometric functions have graphs that repeat at regular intervals so they are called **periodic functions** The **period** is the length of one cycle of a periodic function on the x-axis before the function repeats itsel.

The **centre** of a periodic function is its mean value and is equidistant from the maximum and minimum values The mean value of $y = \sin x$ $y = \cos x$ and $y = \tan x$ is 0 represented by the x-axis

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The **amplitude** is the height from the centre of a periodic function to the maximum or minimum values (peaks and troughs of its graph respectively) The range of $y = \sin x$ and $y = \cos x$ is $[-1, 1]$.

$$y = \sin x \text{ has period } 2\pi \text{ and amplitude } 1$$

$$y = \cos x \text{ has period } 2\pi \text{ and amplitude } 1$$

$$y = \tan x \text{ has period } \pi \text{ and no amplitude}$$

INVESTIGATION

TRANSFORMING TRIGONOMETRIC GRAPHS

Use a graphics calculator or graphing software to draw the graphs of trigonometric functions with different values

1. Graphs in the form $y = k \sin x$ $y = k \cos x$ and $y = k \tan x$ where $k = \ldots, -3, -2, -1, 2, 3, \ldots$
2. Graphs in the form $y = \sin ax$ $y = \cos ax$ and $y = \tan ax$ where $a = \ldots, -3, -2, -1, 2, 3, \ldots$
3. Graphs in the form $y = \sin x + c$ $y = \cos x + c$ and $y = \tan x + c$ where $c = \ldots, -3, -2, -1, 2, 3, \ldots$
4. Graphs in the form $y = \sin (x + b)$, $y = \cos (x + b)$ and $y = \tan (x + b)$ where $b = \ldots, \pm\frac{\circ}{2}$ $\pm\pi$ $\pm\frac{\circ}{4}$ $\ldots$

Can you see patterns? Could you predict what different graphs look like?

Now we shall examine more general trigonometric functions of the form $y = k \sin ax$ $y = k \cos ax$ and $y = k \tan ax$ where k and a are constants

Period and amplitude of trigonometric functions

$$y = k \sin ax \text{ has amplitude } k \text{ and period } \frac{2\pi}{a}$$

$$y = k \cos ax \text{ has amplitude } k \text{ and period } \frac{2\pi}{a}$$

$$y = k \tan ax \text{ has no amplitude and has period } \frac{\pi}{a}$$

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EXAMPLE 9

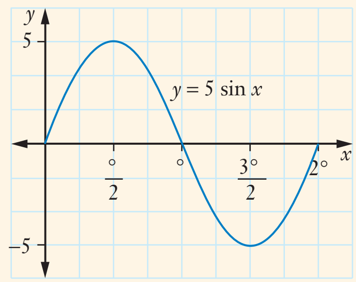
a Sketch each function in the domain [0 2 π]

i $y = 5 \sin x$ **ii** $y = \sin 4x$ **iii** $y = 5 \sin 4x$

b Sketch the graph of $y = 2 \tan \frac{x}{2}$ for [0, 2π]

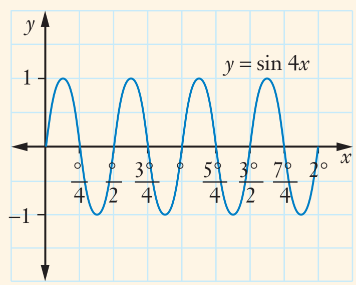
Solution

a **i** The graph of $y = 5 \sin x$ has y values that are 5 times as much as $y = \sin x$ so this function has amplitude 5 and period 2π We draw one period of the sine shape between ±5

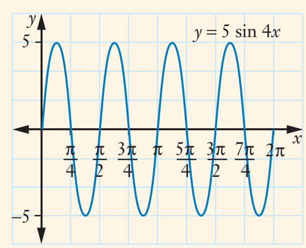


ii The graph $y = \sin 4x$ has amplitude 1 and period $\frac{2\pi}{4} = \frac{\pi}{2}$

The curve repeats every $\frac{\pi}{2}$, so in the domain [0 2 π] there will be 4 repetitions Th ‘4’ in sin 4 x compresses the graph of $y = \sin x$ horizontally.



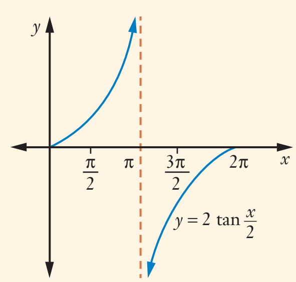
iii The graph $y = 5 \sin 4x$ has amplitude 5 and period $\frac{\pi}{2}$ It is a combination of graphs **i** and **ii**



b $y = 2 \tan \frac{x}{2}$ has no amplitude

$$\text{Period} = \frac{\pi}{\frac{1}{2}} = 2\pi$$

So there will be one period in the domain [0 2 π]



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The graphs of trigonometric functions can change their **phase** a shift to the left or righ.

Phase shift of trigonometric functions

$y = \sin(x + b)$, $y = \cos(x + b)$ and $y = \tan(x + b)$ have phase b which is a shift b units from $y = \sin x$ $y = \cos x$ and $y = \tan x$ respectively, to the left if $b > 0$ and to the right if $b < 0$

EXAMPLE 10

Sketch the graph of $f(x) = \sin\left(x + \frac{\circ}{2}\right)$ for $[0, 2\pi]$

Solution

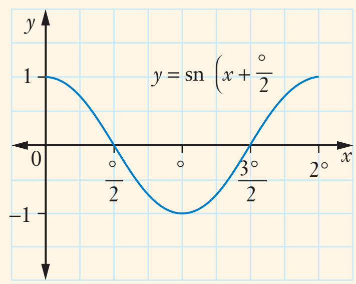
Amplitude = 1

Period $= \frac{2\pi}{1} = 2\pi$

Phase $b = \frac{\pi}{2}$

This is the graph of $y = \sin x$ moved $\frac{\pi}{2}$ units to the left If yo're unsure how the phase affects the graph draw a table of value.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	0	–1	0	1



The graphs of trigonometric functions can change their centre a shift up or dow.

Centre of trigonometric functions

$y = \sin x + c$ $y = \cos x + c$ and $y = \tan x + c$ have centre c which is a shift up from $y = \sin x$ $y = \cos x$ and $y = \tan x$ respectively if $c > 0$ and a shift down if $c < 0$

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EXAMPLE 11

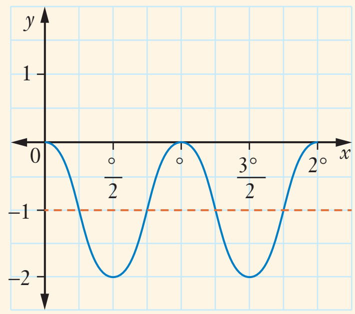
Sketch the graph of $y = \cos 2x - 1$ in the domain [0 2 π]

Solution

Amplitude = 1, period $\frac{2\pi}{2} = \pi$

$c = -1$ so the centre of the graph moves down 1 unit to −1

Instead of moving between −1 and 1 the graph moves between −2 and 0



Gaphing he coine uncion

General trigonometric functions

	Amplitude	Period	Phase	Centre
$y = k \sin [a(x + b)] + c$	k	$\frac{2\pi}{a}$	b Shift left if $b > 0$ Shift right if $b < 0$	$y = c$ Shift up if $c > 0$ Shift down if $c < 0$
$y = k \cos [a(x + b)] + c$	k	$\frac{2\pi}{a}$		
$y = k \tan [a(x + b)] + c$	No amplitude	$\frac{\pi}{a}$		

EXAMPLE 12

For the function $y = 3 \cos (2x - \pi)$ find

a the amplitude **b** the period **c** the phase

Solution

$y = 3 \cos (2x - \pi)$

$= 3 \cos \left[2\left(x - \frac{\pi}{2}\right)\right]$

a Amplitude = 3

b Period $= \frac{2\pi}{2}$
$= \pi$

c Phase $= \frac{\pi}{2}$ units

EXAMPLE 13

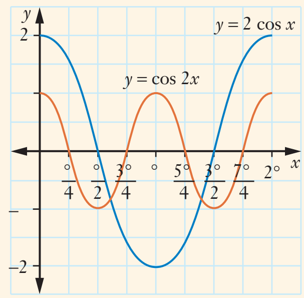
a Sketch the graph of $y = 2 \cos x$ and $y = \cos 2x$ on the same set of axes for [0 2 π]

b Hence sketch the graph of $y = \cos 2x + 2 \cos x$ for $[0, 2\pi]$

Solution

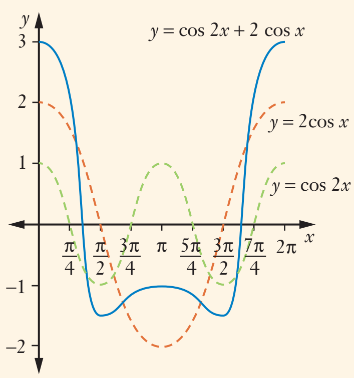
a $y = 2 \cos x$ has amplitude 2 and period 2π

$y = \cos 2x$ has amplitude 1 and period $\frac{2\pi}{2}$ or π



b Add y values on the graph using a table of values if more accuracy is neede.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos 2x$	1	0	–1	0	1	0	–1	0	1
$2 \cos x$	2	$\sqrt{2}$	0	$-\sqrt{2}$	–2	$-\sqrt{2}$	0	$\sqrt{2}$	2
$\cos 2x + 2 \cos x$	3	$\sqrt{2}$	–1	$-\sqrt{2}$	–1	$-\sqrt{2}$	–1	$\sqrt{2}$	3



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Exercise 9.04 Trigonometric functions

1 **a** Sketch the graph of $f(x) = \cos x$ in the domain $[0\ 2\pi$.

b Sketch the graph of $y = -f(x)$ in the same domain

2 Sketch the graph of each function in the domain $[0\ 2\ \pi]$

a $f(x) = 2 \sin x$	**b** $y = 1 + \sin x$	**c** $y = 2 - \sin x$
d $f(x) = -3 \cos x$	**e** $y = 4 \sin x$	**f** $f(x) = \cos x + 3$
g $y = 5 \tan x$	**h** $f(x) = \tan x + 3$	$y = 1 - 2 \tan x$

3 Sketch the graph of each function in the domain $[0\ 2\ \pi]$

a $y = \cos 2x$	**b** $y = \tan 2x$	**c** $y = \sin 3x$
d $f(x) = 3 \cos 4x$	**e** $y = 6 \cos 3x$	**f** $y = \tan \frac{x}{2}$
g $f(x) = 2 \tan 3x$	**h** $y = 3 \cos \frac{x}{2}$	**i** $y = 2 \sin \frac{x}{2}$

4 Sketch the graph of each function in the domain $[-\pi\ \ \pi]$

a $y = -\sin 2x$	**b** $y = 7 \cos 4x$	**c** $f(x) = -\tan 4x$
d $y = 5 \sin 4x$	**e** $f(x) = 2 \cos 2x$	**f** $f(x) = 3 \tan x - 1$

5 Sketch the graph of $y = 8 \sin \frac{x}{2}$ in the domain $[0\ 4\ \pi]$

6 Sketch over the interval $[0\ 2\ \pi]$ the graph of

a $y = \sin (x + \pi)$	**b** $y = \tan \left(x + \frac{\circ}{2} \right)$	**c** $f(x) = \cos (x - \pi)$
d $y = 3 \sin \left(x - \frac{\circ}{2} \right)$	**e** $f(x) = 2 \cos \left(x + \frac{\circ}{2} \right)$	**f** $y = 4 \sin \left(2x + \frac{\circ}{2} \right)$
g $y = \cos \left(x - \frac{\circ}{4} \right)$	**h** $y = \tan \left(x + \frac{\circ}{4} \right)$	

7 Sketch over the interval $[-2\ 2]$ the graph o:

a $y = \sin \pi x$ **b** $y = 3 \cos 2\pi x$

8 For each function fin:

i the amplitude **ii** the period **iii** the centre **v** the phase

a $y = 5 \sin 2x$	**b** $f(x) = -\cos (x - \pi)$	**c** $y = 2 \tan (4x) - 2$
d $y = 3 \sin \left(x + \frac{\pi}{4} \right) + 1$	**e** $y = 8 \cos (\pi x - 2) - 3$	**f** $f(x) = 3 \tan \left(5x + \frac{\pi}{2} \right) + 2$

9 Find the domain and range of each function

a $y = 4 \sin x - 1$ **b** $f(x) = -3 \cos 5x + 7$

10 Sketch in the domain [0 2 π] the graphs of

a $y = \sin x$ and $y = \sin 2x$ on the same set of axes

b $y = \sin x + \sin 2x$

11 Sketch for the interval [0 2 π] the graphs of

a $y = 2 \cos x$ and $y = 3 \sin x$ on the same set of axes

b $y = 2 \cos x + 3 \sin x$

12 By sketching the graphs of $y = \cos x$ and $y = \cos 2x$ on the same set of axes for [0 2 π], sketch the graph of $y = \cos 2x - \cos x$

13 Sketch the graph of $y = \cos x + \sin x$

9.05 Trigonometric equations

WS Trigonomric equaion

EXAMPLE 14

Solve each equation for [0°, 360°]

a $\sin x = 034$ **b** $\cos x = \frac{\sqrt{3}}{2}$ **c** $\tan \theta = -1$

Solution

> Make sure that your calculator s n **degrees mode** Check your soluton by substtutng back nto the equaton

a 034 is positive and $\sin x > 0$ in 1st and 2nd quadrants

$\sin x = 034$

$x \approx 19°53', 180° - 19°53'$

$= 19°53', 160°7'$

19°53′ is the **principal solution** but there is another solution in the 2nd quadrant

b $\cos x > 0$ in the 1st and 4th quadrants

$\cos x = \frac{\sqrt{3}}{2}$

$x = 30°, 360° - 30°$

$= 30°, 330°$

c $\tan \theta < 0$ in the 2nd and 4th quadrants

For $\tan \theta = -1$

$\theta = 180° - 45°, 360° - 45°$

$= 135°, 315°$

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EXAMPLE 15

Solve $\tan x = \sqrt{3}$ for [–180°, 180°]

Solution

In the domain [–180°, 180°] we use positive angles for $0° \le x \le 180°$ and negative angles for $-180° \le x \le 0°$

tan > 0 in the 1st and 3rd quadrants

$$\tan x = \sqrt{3}$$
$$x = 60° \quad -(180° - 60°)$$
$$= 60° \quad -120°$$

EXAMPLE 16

Solve $2\sin^2 x - 1 = 0$ for $0° \le x \le 360°$

Solution

$$2\sin^2 x - 1 = 0$$
$$2\sin^2 x = 1$$
$$\sin^2 x = \frac{1}{2}$$

$$\sin x = \pm\frac{\sqrt{1}}{\sqrt{2}}$$
$$= \pm\frac{1}{\sqrt{2}}$$

Since the ratio could be positive or negative there are solutions in all 4 quadrant.

$x = 45°, 180° - 45°, 180° + 45°, 360° - 45°$

$= 45°, 135°, 225°, 315°$

If we are solving an equation involving $2x$ or $3x$ for exampl, we need to change the domain to find all possible solutions

EXAMPLE 17

Solve $2\sin 2x - 1 = 0$ for [0°, 360°]

Solution

Notice that the angle is $2x$ but the domain is for x

If $\quad 0° \le x \le 360°$

then $0° \le 2x \le 720°$

This means that we can find the solutions by going around the circle twice

$$2 \sin 2x - 1 = 0$$

$$2 \sin 2x = 1$$

$$\sin 2x = \frac{1}{2}$$

Sin is positive in the 1st and 2nd quadrants

First time around the circle 1st quadrant is θ and the 2nd quadrant is 180° – θ

Second time around the circle add 360 ° to θ and 180° – θ

$$2x = 30°, 180° - 30°, 360° + 30°\ \ 360° + 180° - 30°$$

$$= 30°, 150°\ \ 390°, 510°$$

$$\therefore x = 15°, 75°, 195°, 255°$$

You can solve trigonometric equations involving **radians** You can recognise these because the domain is in radians

Trigonomric equaion

EXAMPLE 18

Solve each equation for [0 2 π]

a $\cos x = 034$

b $\sin \alpha = -\frac{1}{\sqrt{2}}$

c $\sin^2 x - \sin x = 2$

The doman [0 2 π] tells us that the solutons w be n **radans** Make sure that your calculator s n radans mode here

Solution

a $\cos x > 0$ in the 1st and 4th quadrants

$$\cos x = 034$$

$$x \approx 1224, 2\ \pi - 1224$$

$$= 1224, .059$$

b $\sin \alpha$ is negative in the 3rd and 4th quadrants

$$\sin \alpha = -\frac{1}{\sqrt{2}}$$

$$\alpha = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$

$$= \frac{5\pi}{4}\ \ \frac{7\pi}{4}$$

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c $\sin^2 x - \sin x = 2$

$\sin^2 x - \sin x - 2 = 0$ This is a quadratic equation

$(\sin x - 2)(\sin x + 1) = 0$

$\sin x = 2 \qquad \sin x = -1$

$\sin x = 2$ has no solutions since $-1 \le \sin x \le 1$

$\sin x = -1$ has solution $x = \frac{3\pi}{2}$

$x = \frac{3\pi}{2}$

Exercise 9.05 Trigonometric equations

1 Solve each equation for [0°, 360°]

a $\sin\theta = 035$ **b** $\cos\theta = -\frac{1}{2}$ **c** $\tan\theta = -1$

d $\sin\theta = \frac{\sqrt{3}}{2}$ **e** $\tan\theta = -\frac{1}{\sqrt{3}}$ **f** $2\cos\theta = \sqrt{3}$

g $\tan 2\theta = \sqrt{3}$ **h** $2\cos 2\theta - 1 = 0$ **i** $2\sin 3\theta = -1$

j $\tan^2 3\theta = 1$ **k** $\sin^2 x = 1$ $2\cos^2 x - \cos x = 0$

2 Solve for $0° \le x \le 360°$

a $\cos x = 1$ **b** $\sin x + 1 = 0$ **c** $\cos^2 x = 1$

d $\sin x = 1$ **e** $\tan x = 0$ **f** $\sin^2 x + \sin x = 0$

g $\cos^2 x - \cos x = 0$ **h** $\tan^2 x = \tan x$ **i** $\tan^2 x = 3$

3 Solve for [0 2 π]

a $\sin x = 0$ **b** $\tan 2x = 0$ **c** $\sin x = -1$

d $\cos x - 1 = 0$ **e** $\cos x = -1$

4 Solve for [–180°, 180°]

a $\cos\theta = 0187$ **b** $\sin\theta = \frac{1}{2}$ **c** $\tan\theta = 1$

d $\sin\theta = -\frac{\sqrt{3}}{2}$ **e** $\tan\theta = -\frac{1}{\sqrt{3}}$ **f** $3\tan^2\theta = 1$

g $\tan\theta + 1 = 0$ **h** $\tan 2\theta = 1$

5 Solve for $0 \le x \le 2\pi$

a $\cos x = \frac{1}{2}$ **b** $\sin x = -\frac{1}{\sqrt{2}}$ **c** $\tan x = 1$

d $\tan x = \sqrt{3}$ **e** $\cos x = -\frac{\sqrt{3}}{2}$

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6 Solve for $-\pi \le x \le \pi$

a $2 \sin x = \sqrt{3}$ **b** $2 \cos x = 0$ **c** $3 \tan^2 x = 1$

7 Solve $2 \cos x = -1$ in the domain $[-2\pi, 2\pi]$

8 Solve for $[0\ 2\ \pi]$

a $\tan^2 x + \tan x = 0$ **b** $\sin^2 x - \sin x = 0$ **c** $2 \cos^2 x - \cos x - 1 = 0$

d $4 \sin^2 x = 1$ **e** $\tan x \cos x + \tan x = 0$ **f** $\sin^2 x + 2 \cos x - 2 = 0$

9.06 Applications of trigonometric functions

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WS Applying igonomeic uncion

Trigonometric graphs can model real-life situation.

EXAMPLE 19

This table shows the average maximum monthly temperatures in Sydney.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
°C	26.1	26.1	25.1	22.8	19.8	17.4	16.8	18.0	20.1	22.2	23.9	25.6

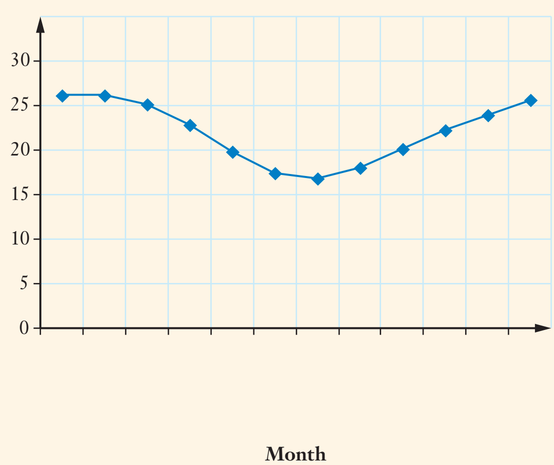
a Draw a graph of this data

b Is it periodic? Would you expect it to be periodic?

c What is the period and amplitude?

Solution

a



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b The graph looks like it is periodic and we would expect it to b, since the temperature varies with the seasons and these repeat every 12 months It goes up and down and reaches a highest value in summer and a lowest value in winte.

c This curve is approximately a cosine curve with a period of 12 months

The highest maximum temperature is around 26° and the lowest maximum temperature is around 18° so the centre of the graph is $\frac{26°+18°}{2} = 22°$ So the amplitude is 26 – 22 (or 22 – 18) = 4

DID YOU KNOW?

Waves

The sine and cosine curves are used in many applications including the study of waves There are many different types of waves including wate, light and sound waes. Oscilloscopes display patterns of electrical waves on the screen of a cathode-ray tube

Simple harmonic motion (such as the movement of a pendulum) is a wave-like or oscillatory motion when graphed against time In 158, when he was 17 years od, the Italian scientist Galileo noticed a lamp swinging backwards and forwards in Pisa cathedral He found that the lamp took the same time to swing to and fr, no matter how much weight it had on it This led him to discover the pendulm.

Galileo also invented the telescoe. Find out more about his life and work.

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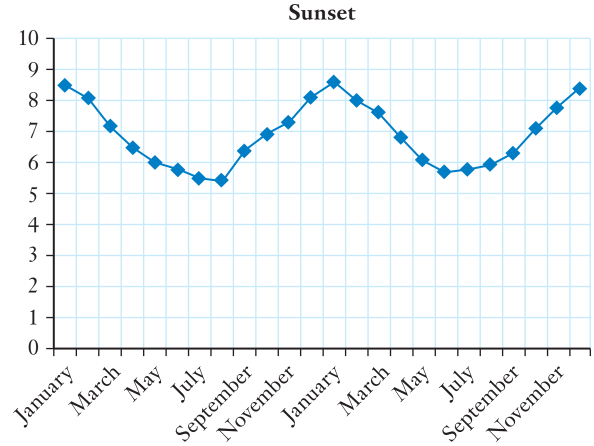
Exercise 9.06 Applications of trigonometric functions

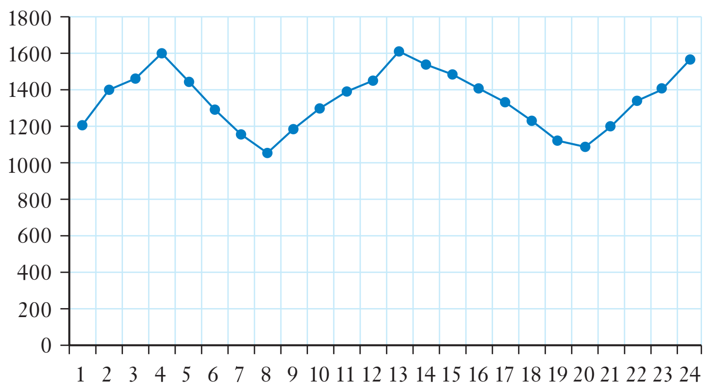
1 This graph shows the time of sunset in a city over a period of 2 years



a Find the approximate period and amplitude of the graph

b At approximately what time would you expect the Sun to set in July?

2 The graph shows the incidence of crimes committed over 24 years in Gotham City.



a Approximately how many crimes were committed in the 10th year?

b What was

i the highest number of crimes? **ii** the lowest number of crimes?

c Find the approximate amplitude and the period of the graph

3 This table shows the tides (in metres) at a jetty measured 4 times each day for 3 days

Day	Friday				Saturday				Sunday			
Time	620 am	1155 am	615 pm	1148 pm	620 am	1155 am	615 pm	1148 pm	620 am	1155 am	615 pm	1148 pm
Tide (m)	3.2	1.1	3.4	1.3	3.2	1.2	3.5	1.1	3.4	1.2	3.5	1.3

a Draw a graph showing the tides

b Find the period and amplitude

c Estimate the height of the tide at around 8 am on Frida.

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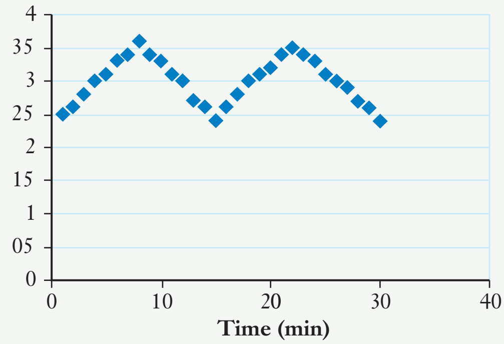
9. TEST YOURSELF

For Questions **1** to **4** select the correct answer **A B C** or **D**

Pacice quiz

1 This graph shows the water depth in metres as a lock opens and closes over time



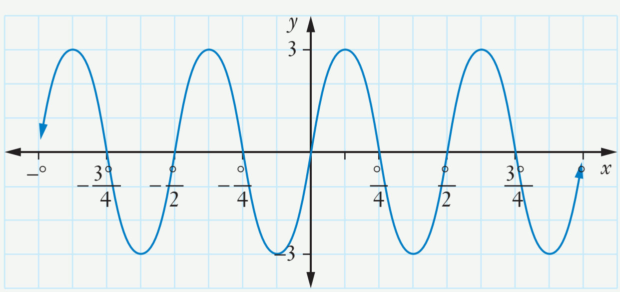
The approximate period and amplitude of the graph are

A Amplitude 1 period 15 min
B Amplitude 05 period .5 min
C Amplitude 1 period .5 min
D Amplitude 05 period 15 min

2 The exact value of $\cos \frac{2\pi}{3}$ is

A $\frac{1}{2}$ **B** $-\frac{\sqrt{3}}{2}$ **C** $\frac{\sqrt{3}}{2}$ **D** $-\frac{1}{2}$

3 The equation of the graph below is



A $y = 3 \cos 4x$ **B** $y = 3 \sin 4x$ **C** $y = 4 \sin 3x$ **D** $y = 4 \cos 3x$

4 $\text{cosec}^2\, x$ is equal to

A $\cot^2 x - 1$ **B** $1 - \cos^2 x$ **C** $1 + \cot^2 x$ **D** $\tan^2 x + 1$

5 Find the exact value of

a $\cos 315°$ **b** $\sin(-60°)$ **c** $\tan 120°$

6 Solve for $0° \le x \le 360°$

a $\sin x = \frac{\sqrt{3}}{2}$ **b** $\tan x = 1$ **c** $2\cos x + 1 = 0$

d $\sin^2 x = \frac{3}{4}$ **e** $\tan 2x = \frac{1}{\sqrt{3}}$

7 Solve for $0 \le x \le 2\pi$

a $\tan x = -1$ **b** $2\sin x = 1$ **c** $\tan^2 x = 3$

d $\cos x = 1$ **e** $\sin x = -1$

8 For $0 \le x \le 2\pi$ sketch the graph o:

a $y = 3\cos 2x$ **b** $y = 7\sin\frac{x}{2}$

9 If $\sin x = -\frac{12}{13}$ and $\cos x > 0$ evaluate $\cos x$ and $\tan x$

10 Simplify

a $\cos(180° + \theta)$ **b** $\tan(-\theta)$

c $\sin(\pi - \theta)$ **d** $\tan x \cos x$

e $\sqrt{4 - 4\sin^2 A}$ **f** $\cos(90 - x)°$

g $\cot\beta \tan\beta$

11 Find the exact value of

a $\sin\frac{5\pi}{4}$ **b** $\cos\frac{5\pi}{6}$ **c** $\tan\frac{4\pi}{3}$

12 Prove that $\frac{2\cos^2\theta}{1-\sin\theta} = 2 + 2\sin\theta$

13 Find the value of b if $\sin b = \cos(2b - 30)°$

14 Find the period amplitud, centre and phase of $y = -2\cos\left(3x + \frac{\circ}{12}\right) + 5$.

15 Find the exact value of

a $\sec\frac{\pi}{4}$ **b** $\cot\frac{\pi}{6}$

c $\operatorname{cosec}\frac{\pi}{3}$ **d** $\frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}$

16 Find the domain and range of each function

a $y = -6\sin(2x) + 5$ **b** $f(x) = 4\cos x - 3$

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9. CHALLENGE EXERCISE

1 Find the exact value of

a $\sin 600°$ **b** $\tan(-405°)$

2 Solve $2\cos(\theta + 10°) = -1$ for $0° \le \theta \le 360°$

3 If $f(x) = 3\cos \pi x$

a find the period and amplitude of the function

b sketch the graph of $f(x)$ for $0 \le x \le 4$

4 For $0 \le x \le 2\pi$ sketch the graph o:

a $f(x) = 2\cos\left(x + \frac{\circ}{2}\right) + 1$ **b** $y = 2 - 3\sin\left(x - \frac{\circ}{2}\right)$

c $y = \sin 2x - \sin x$ **d** $y = \sin x + 2\cos 2x$

e $y = 3\cos x - \cos 2x$ **f** $y = \sin x - \sin\frac{x}{2}$

5 Solve $\cos^2 x - \cos x = 0$ for $0 \le x \le 2\pi$

6 Find the exact value of $\sin 120° + \cos 135°$ as a surd with rational denominator.

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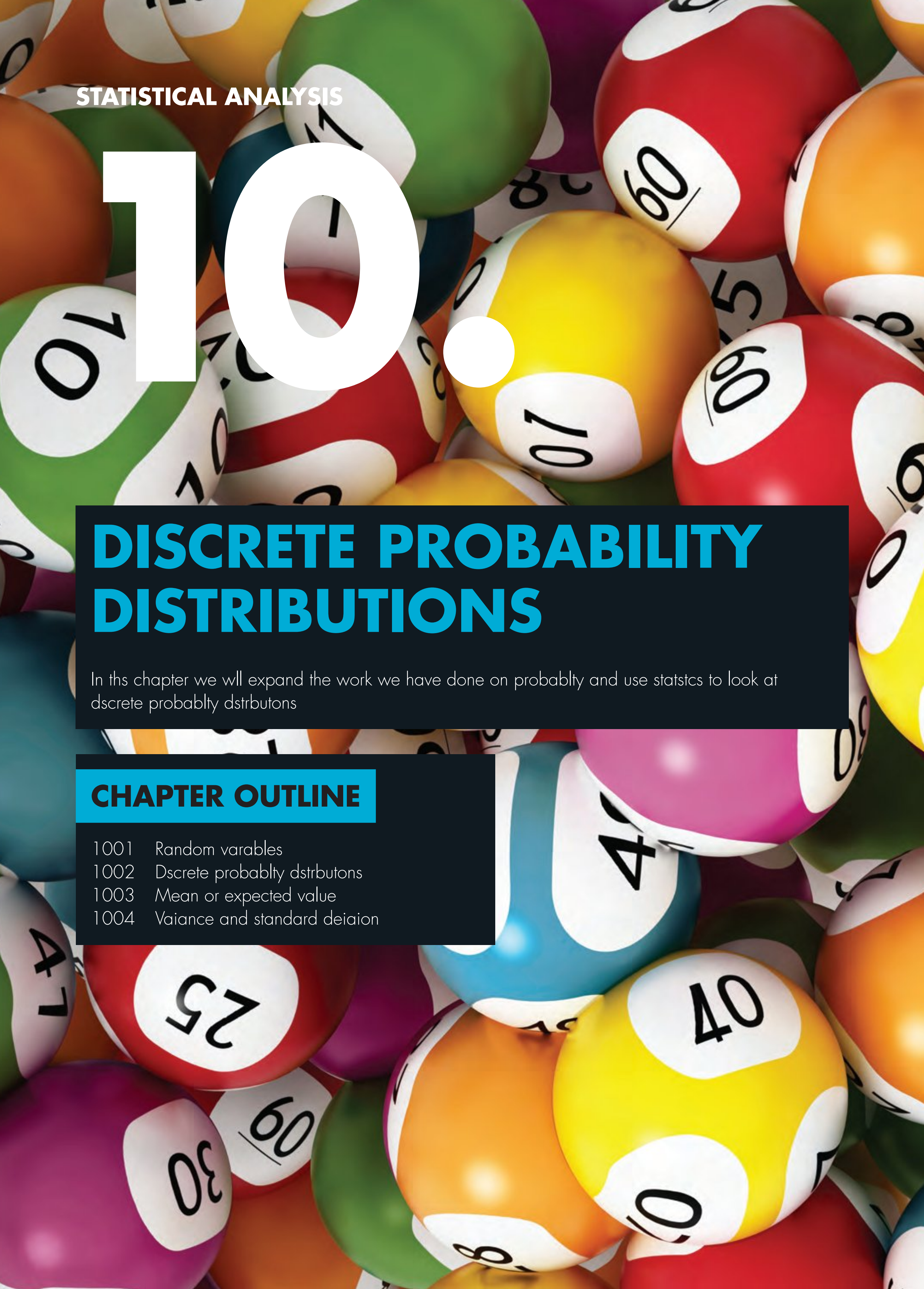

10.

DISCRETE PROBABILITY DISTRIBUTIONS

In ths chapter we wll expand the work we have done on probablty and use statstcs to look at dscrete probablty dstrbutons

CHAPTER OUTLINE

1001 Random varables
1002 Dscrete probablty dstrbutons
1003 Mean or expected value
1004 Vaiance and standard deiaion

IN THIS CHAPTER YOU WILL:

- understand random varables and defntons of dscrete contnuous unfor, iite andiniite varables
- recognse dscrete probablty dstrbutons and ther properies
- use probablty dstrbutons to solve practcal problems
- fnd expected values varance and standard devatons of probablty dstrbutons

TERMINOLOGY

discrete random variable A random variable that can take on a number of discrete values for example the number of children in a family

expected value Average or mean value of a probability distribution

population The whole data set from which a sample can be taken

probability distribution A function that sets out all possible values of a random variable together with their probabilities

random variable A variable whose values are based on a chance experiment for example the number of road accidents in an hour

standard deviation A measure of the spread of values from the mean of a distribution the square root of the variance

uniform probability distribution A probability distribution in which every outcome has the same probability

variance A measure of the spread of values from the mean of a distribution the square of the standard deviation

10.01 Random variables

We studied probability in Chapter . Now we will look at **probability distributions** which use random variables to predict and model random situations in areas such as science economics and medicine

A **random variable** is a variable that can take on different values depending on the outcome of a random process such as an experimen. Random variables can be **discrete** or **continuous** Discrete variables such as goals scored or number of children take on specific finite values while continuous variables such as length or temperature are measured along a continuous scale

Discrete random variables

A **discrete random variable** is a variable whose values are specific and can be listed

In this chapter we will look at **discrete random variables** We will look at continuous random variables in Year 2.

EXAMPLE 1

Is each random variable discrete or continuous?

a The number of goals scored by a netball team

b The height of a student

c The shoe size of a Year 11 studet.

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Solution

a The number of goals scored is a specific whole number so it is a discrete random variable

b Height is measured on a continuous scale so the height of a student is a continuous random variable

c Shoe sizes are specific values that can be listed so it is a discrete random variable

We use a capital letter such as X for a random variable and a lower-case letter such as x for the values of X

EXAMPLE 2

Find the set of possible values for each discrete random variable

a The number rolled on a die

b The number of girls in a family of 3 children

c The number of heads when tossing a coin 8 times

Solution

a Any number from 1 to 6 can be rolled on a die

So $X = \{1, 2, 3, 4, 5, 6\}$

b It is possible to have no girls 1 gir, 2 girls or 3 girs.

So $X = \{0, 1, 2, 3\}$

c The coin could come up heads 0 ,2, … 8 ties.

So $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

Exercise 10.01 Random variables

1 For each random variable state whether it is discrete or continuou:

a A film critics rating of a fil, from 0 to 4 stars

b The speed of a car

c The sum rolled on a pair of dice

d The winning ticket number drawn from a raffle

e The weight of parcels at a post office

f The size of jeans in a shop

g The temperature of a metal as it cools

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h The amount of water in different types of fruit drink

i The number of cars passing the school over a 10-minute period

j The number of cities in each country in Europe

k The number of heads when tossing a coin 50 times

The number of correct answers in a 10-question test

2 Write the set of possible values for each discrete random variabl:

a Number of daughters in a one-child family

b Number of 6s on 10 rolls of a die

c Number of people aged over 50 in a group of 20 people

d The number of days it rains in March

e The sum of the 2 numbers rolled on a pair of dice

10.02 Discrete probability distributions

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Discrete probability distribution

A **discrete probability distribution** lists the probability for each value of a discrete random variable

A discrete probability distribution can be displayed in a table or graph or represented by an equation or set of ordered pairs It is also called a **discrete probability function**

We can write a probability function that uses X as the random variable as $P(X = x)$ or $p(x)$.

EXAMPLE 3

a In a random experiment a die was rolled and the results recorded in the table belo.

Number	Frequency
1	18
2	23
3	17
4	28
5	12
6	22

i How many times was the die rolled?

ii Draw up a probability distribution for this experiment

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b Write the probability function of rolling a die as a set of ordered pairs $(x\ \ P(X=x))$

c A probability function for discrete random variable X is given by

$$P(X=x)=\begin{cases}\frac{1}{16}(4-x) & \text{for } x = 0\ \ 2\\ \frac{1}{8}(x-1) & \text{for } x = 3\ \ 4\\ 0 & \text{for any other } x \text{ value}\end{cases}$$

i Complete a discrete probability distribution table

ii Find the sum of all probabilities

iii Evaluate $P(X = \text{odd})$

v Evaluate $P(X \le 3)$.

Solution

a **i** Adding the frequencies the die was rolled 120 time.

ii For the probability of each outcome we use **relative frequencies** as we did in Chapter 7

x	1	2	3	4	5	6
$P(X=x)$	$\frac{18}{120}=\frac{3}{20}$	$\frac{23}{120}$	$\frac{17}{120}$	$\frac{28}{120}=\frac{7}{30}$	$\frac{12}{120}=\frac{1}{10}$	$\frac{22}{120}=\frac{11}{60}$

b The probability of each number being rolled on a die is $\frac{1}{6}$ So the probability function $P(X = x)$ can be written as $\left(1\ \frac{1}{6}\right)\ \left(2\ \frac{1}{6}\right)\ \left(3\ \frac{1}{6}\right)\ \left(4\ \frac{1}{6}\right)\ \left(5\ \frac{1}{6}\right)\ \left(6\ \frac{1}{6}\right)$

c **i**

$p(0) = \frac{1}{16}(4-0)$
$= \frac{1}{4}$

$p(2) = \frac{1}{16}(4-2)$
$= \frac{1}{8}$

$p(3) = \frac{1}{8}(3-1)$
$= \frac{1}{4}$

$p(4) = \frac{1}{8}(4-1)$
$= \frac{3}{8}$

All other values of x give $p(x) = 0$ We cannot put all of these in a tale. They will not make any difference to calculations anyway.

x	0	1	2	3	4
$p(x)$	$\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$

ii $p(0) + p(1) + p(2) + p(3) + p(4) = \frac{1}{4} + 0 + \frac{1}{8} + \frac{1}{4} + \frac{3}{8}$
$= 1$

iii $P(X = \text{odd}) = p(1) + p(3)$
$= 0 + \frac{1}{4}$
$= \frac{1}{4}$

v $P(X \le 3) = p(0) + p(1) + p(2) + p(3)$
$= \frac{1}{4} + 0 + \frac{1}{8} + \frac{1}{4}$
$= \frac{5}{8}$

Remember that all probabilities lie between 0 and 1 and their total is 1 These same rules apply to a probability distribution

Properties of discrete probability distributions

For a discrete probability distribution

- all possible values of X must be mutually exclusive
- the sum of all probabilities must be 1
- for each value of x, $0 \le P(X = x) \le 1$.

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EXAMPLE 4

Consider this discrete probability distribution

x	1	2	3	4	5	6
$P(X = x)$	0.2	0.35	0.1	0.15	0.05	0.15

a Find

i $P(X = 2)$ **ii** $P(X < 3)$

iii $P(X \geq 4)$ **v** $P(2 \leq X < 5)$

b Show that the sum of probabilities is 1

c Draw a histogram of the function

Solution

a **i** From the table $p(2) = 035$

ii $P(X < 3) = p(1) + p(2)$

$= 02 + 035$

$= 055$

iii $P(X \geq 4) = p(4) + p(5) + p(6)$

$= 0.15 + 005 + 015$

$= 035$

v $P(2 \leq X < 5) = p(2) + p(3) + p(4)$

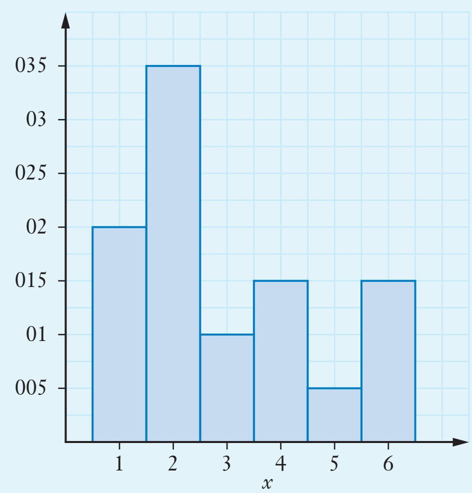
$= 035 + 0.1 + 015$

$= 06$

b $p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 02 + 035 + 0.1 + 0.15 + 005 + 015$

$= 1$

c A histogram is the best type of graph to draw a discrete probability distribution



EXAMPLE 5

a A function is given by $p(x) = \dfrac{x-1}{3}$ where $x = 1, 3$. Is the function a probability distribution?

b Find the value of n for which the table below is a discrete probability distribution

x	0	1	2	3	4
$p(x)$	$\dfrac{1}{12}$	$\dfrac{1}{6}$	$\dfrac{5}{12}$	$\dfrac{1}{12}$	n

Solution

a

$$p(1) = \frac{1-1}{3} = 0$$

$$p(2) = \frac{2-1}{3} = \frac{1}{3}$$

$$p(3) = \frac{3-1}{3} = \frac{2}{3}$$

$$p(1) + p(2) + p(3) = 0 + \frac{1}{3} + \frac{2}{3} = 1$$

So the function is a probability distribution

b The sum of the probabilities must be 1

$$\frac{1}{12} + \frac{1}{6} + \frac{5}{12} + \frac{1}{12} + n = 1$$

$$\frac{9}{12} + n = 1$$

$$n = 1 - \frac{9}{12}$$

$$n = \frac{1}{4}$$

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EXAMPLE 6

A game involves scoring points for selecting a card at random from a deck of 52 playing cards The table shows the scores awarded for different selections

Type of card	Score
Number 2–10	2
Picture card (jack queen or king)	5
Ace	8

a Draw a probability distribution table for random variable Y for the different scores

b Find

i $P(Y < 8)$

ii $P(Y > 2)$

Solution

a There are 9 cards numbered 2–10 in each of the 4 suits (hearts diamond, spades and clubs) So there are $9 \times 4 = 36$ cards that will score 2 points

There are 3 picture cards (jack queen and king) in each sui, so there are $3 \times 4 = 12$ cards that will score 5 points

There are 4 aces that will score 8 points

y	2	5	8
$P(Y = y)$	$\frac{36}{52} = \frac{9}{13}$	$\frac{12}{52} = \frac{3}{13}$	$\frac{4}{52} = \frac{1}{13}$

b **i** $P(X < 8) = p(2) + p(5)$

$$= \frac{9}{13} + \frac{3}{13}$$

$$= \frac{12}{13}$$

ii $P(X > 2) = p(5) + p(8)$

$$= \frac{3}{13} + \frac{1}{13}$$

$$= \frac{4}{13}$$

Uniform distribution

In a **uniform probability distribution** the probability for each value is the sam.

Uniform probability distribution

A uniform probability distribution occurs if random variable X has n values where

$P(X = x) = \frac{1}{n}$ for $x = 1, 2, 3, \ldots, n$

EXAMPLE 7

Which probability distribution is uniform?

a The number of heads when tossing a coin

b The number of heads when tossing 2 coins

Solution

a When tossing a coin we could get either 0 or 1 hea.

$X = \{0, 1\}$

$p(0) = \frac{1}{2}$

$p(1) = \frac{1}{2}$

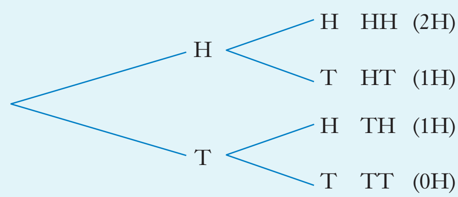
Since both values have the same probability, it is a uniform distributin.

b When tossing 2 coins the number of heads could be , 1 or2.

$X = \{0, 1, 2\}$

Using a probability tree or table we can find the probability for each outcom.

$p(0) = P(\text{TT})$
$= \frac{1}{4}$



$p(1) = P(\text{HT}) + P(\text{TH})$
$= \frac{2}{4}$
$= \frac{1}{2}$

$p(2) = P(\text{HH})$
$= \frac{1}{4}$

The probabilities are not all the same so it is not a uniform distribution

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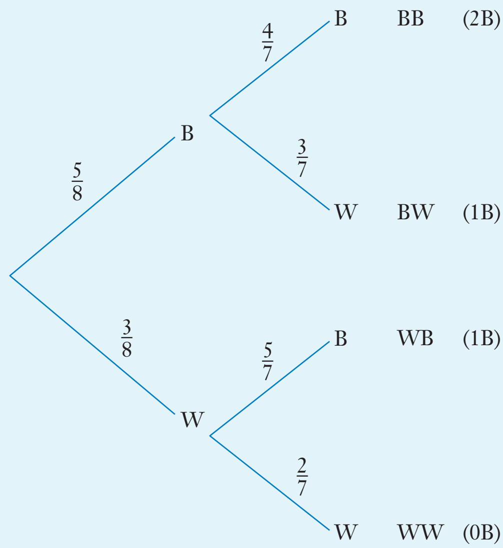
EXAMPLE 8

Draw a probability distribution table for the number of black balls that could be drawn out of a bag containing 5 black and 3 white balls when 2 balls are selected randomly without replacement

Solution

First draw a probability tree



$$P(0\text{B}) = P(\text{WW})$$
$$= \frac{3}{8} \times \frac{2}{7}$$
$$= \frac{6}{56}$$
$$= \frac{3}{28}$$

$$P(1\text{B}) = P(\text{BW}) + P(\text{WB})$$
$$= \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{5}{7}$$
$$= \frac{30}{56}$$
$$= \frac{15}{28}$$

$$P(2\text{B}) = P(\text{BB})$$
$$= \frac{5}{8} \times \frac{4}{7}$$
$$= \frac{20}{56}$$
$$= \frac{5}{14}$$

x	0	1	2
$P(X = x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{5}{14}$

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Exercise 10.02 Discrete probability distributions

1 Draw a probability distribution table for the sum of the numbers rolled on 2 dice

2 Write the probability distribution as a set of ordered pairs (x $P(X = x)$) for the number of heads when tossing

a 1 coin **b** 2 coins **c** 3 coins

3 A survey of a sample of bags of 50 jelly beans found that they didnt all hold exactly 5. The table shows the results of the study.

Number of jelly beans	Frequency
48	8
49	9
50	21
51	9
52	6

a Draw a probability distribution table for the results

b If a bag of jelly beans is chosen at random find the probability that the bag contain:

i at least 50 jelly beans

ii fewer than 51 jelly beans

4 A function is given by $p(x) = \frac{x-2}{6}$ for $x = 3, 4, 5$.

a Show that the function is a probability distribution

b Draw up a probability distribution table

c Find

i $P(X > 3)$ **ii** $P(X = \text{odd})$ **iii** $P(3 \le X < 5)$

5 Draw a histogram to show this discrete probability distribution

x	0	1	2	3	4
$P(X = x)$	0.05	0.4	0.25	0.1	0.2

6 **a** Draw a probability distribution table for rolling a die

b Is this a uniform distribution?

c Find

i $P(X \ge 4)$ **ii** $P(X < 3)$ **iii** $P(1 < X \le 4)$

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7 For each function state whether it is a probability distributio:

a $\left(0\ \frac{1}{5}\right)$ $\left(1\ \frac{2}{5}\right)$ (2 0, $\left(3\ \frac{2}{5}\right)$ $\left(4\ \frac{1}{5}\right)$

b

x	1	2	3	4
$P(X = x)$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$

c $p(x) = \frac{x+2}{4}$ for $x = 0, 1, 2$

8 Find k if each function is a probability distribution

a $p(x) = k(x + 1)$ for $x = 1, 2, 3, 4$

b

x	0	1	2	3	4
$P(X = x)$	02	k	0.15	0.34	0.12

c $(1, k), \left(2\ \frac{1}{10}\right), (3, 0), \left(4\ \frac{1}{5}\right)\ \left(5\ \frac{3}{10}\right)\ \left(6\ \frac{2}{5}\right)$

9 The probability function for the random variable X is given by $p(x) = \frac{kx^2}{x+5}$ for $x = 1, 2, 3, 4$.

a Construct a probability distribution table for the function

b Find the value of k

10 In a game each player rolls 2 dic. The game pays \$1 if one of the numbers is 6, \$3 for double 6 and \$2 for any other double There is no payout for other resuls.

a Draw a probability distribution table for the game payout Y

b Find the probability of winning

i \$3 **ii** at least \$2 **iii** less than \$3

11 Given the probability function below, evaluate p

x	3	4	5	6
$P(X = x)$	$2p$	$3p$	$5p$	p

12 Simon plays a game where he selects a card at random from 100 cards numbered 1 to 100 He wins \$1 for selecting a number less than 2, \$2 for a number greater than 90 \$3 for any number from 61 and 69 (inclusive) and \$5 for any number from 41 to 50 (inclusive)

a Create a probability distribution table for the random variable X for the prize values

b Find the probability of winning more than \$2

c Find the probability of winning less than \$5

13 The table below shows the probability function for random variable X

x	5	6	7	8	9	10
$P(X = x)$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{1}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{1}{7}$

Find

a $P(X = 6)$ **b** $P(X = \text{even})$ **c** $P(X > 8)$ **d** $P(X \le 7)$

e $P(6 < X < 9)$ **f** $P(7 \le X < 10)$ **g** $P(6 \le X \le 9)$

14 The table below shows the probability function for random variable Q

q	0	2	4	6	8	10
$P(Q = q)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{3}{16}$

Find

a $p(8)$ **b** $P(Q \ge 4)$ **c** $P(2 < Q \le 6)$

d $P(4 \le Q \le 10)$ **e** $P(0 \le Q < 4)$ **f** $P(2 \le Q \le 8)$

15 A company makes washing machines On averag, there are 3 faulty machines made for every 1000 machines Two washing machines are selected at random for a quality control inspection

a Draw a probability distribution table for the number of these machines that could be faulty.

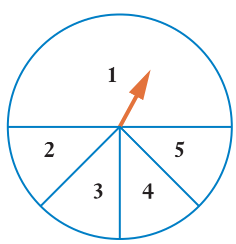
b Find the probability that

i one will be faulty **ii** at least one will be faulty.

16 A bag contains 7 red 6 white and 8 blue ball. Create a probability function table for the number of white balls selected when drawing 2 balls at random from the bag

a with replacement **b** without replacement

17 The spinner below has the numbers 1–5 distributed as shown



a What is the probability that the arrow points to the number 3 when it is spun?

b Is the probability distribution of the spun numbers uniform?

c Draw a table showing the probability distribution for spinning the numbers

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18 The probability of a traffic light showing green as a car approaches it is 12% Draw a probability distribution table for the number of green traffic lights on approach when a car passes through 3 traffic lights

19 There is a 51% chance of giving birth to a boy. If a family has 4 childrn, construct a probability function to show the number of boys in the family.

20 A raffle has 2 prizes with 100 tickets sold altogether. Iris buys 5 tickes.

a Draw a probability distribution table to show the number of prizes Iris could win in the raffle

b Find the probability that Iris wins at least one prize

10.03 Mean or expected value

WS

Expeced value

The **expected value** $E(X)$ of a probability distribution measures the centre of the distribution It is the same as finding the **mean** or averag, which has symbol μ It is the expected value of the random variable

We use $\bar{x}$ for the mean of a **sample** and μ for the mean of a **population** For probability distributions we use the population mea, μ The sample men, $\bar{x}$ is an estimate of μ and as the sample size increases the sample represents the population better and the value of $\bar{x}$ approaches μ

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EXAMPLE 9

This table shows Harrisons diving scores (out of 10) over one yea.

Score	Frequency
5	7
6	9
7	8
8	3
9	2
10	1

a Copy the table and add 2 columns to calculate the relative frequency for each score and the product of each score and its relative frequency.

b Calculate correct to 2 decimal place, the sum of the last column (products of scores and their relative frequencies) to find Harrisons expected value (average score) for the yea.

Solution

a Adding frequencies gives a total of 30

Score	Frequency	Relative frequency	Score × relative frequency
5	7	$\frac{7}{30}$	$5 \times \frac{7}{30} = \frac{35}{30}$
6	9	$\frac{9}{30}$	$6 \times \frac{9}{30} = \frac{54}{30}$
7	8	$\frac{8}{30}$	$7 \times \frac{8}{30} = \frac{56}{30}$
8	3	$\frac{3}{30}$	$8 \times \frac{3}{30} = \frac{24}{30}$
9	2	$\frac{2}{30}$	$9 \times \frac{2}{30} = \frac{18}{30}$
10	1	$\frac{1}{30}$	$10 \times \frac{1}{30} = \frac{10}{30}$

b

$$\begin{aligned}\text{Expected value} &= \frac{35}{30} + \frac{54}{30} + \frac{56}{30} + \frac{24}{30} + \frac{18}{30} + \frac{10}{30} \\ &= \frac{197}{30} \\ &= 65666\ldots \\ &= 657\end{aligned}$$

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INVESTIGATION

MEAN

Find the mean in Example 9 using the formula $\bar{x} = \frac{\circ\ fx}{\circ\ f}$ Can you see why the sum of scores multiplied by relative frequencies also gives this mean?

Expected value

$$E(X) = \mu = \Sigma xp(x)$$

The symbol Σ means 'the sum o'. It is the Greek capital lettr 'sma'.

$\Sigma xp(x)$ is the sum of the products of x times $p(x)$

Proof

$$\mu = \frac{\Sigma fx}{\Sigma f}$$

$$= \frac{\Sigma fx}{n} \text{ where } n \text{ is the sum of frequencies}$$

$$= \sum x \frac{f}{n}$$

$$= \Sigma xp(x)$$

EXAMPLE 10

a Find the expected value of this discrete probability distribution

x	1	2	3	4	5
$P(X = x)$	0.1	0.3	0.2	0.1	0.3

b In a game of chance Bethany tosses 2 coin. She wins \$10 for 2 heas, \$5 for 2 tails and nothing for a head and a tail

i Find the expected value of this game

ii If the game costs \$5 to play, would Bethany expect to win or lose money in the long term?

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Solution

a $E(X) = \sum xp(x)$

$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 02 \ + 4 \times 0.1 + 5 \times 0.3$

$= 3.2$

b **i** We can make $X = \{0 , 2\}$ where0, 1 and 2 is the number of heds.

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

The game pays different amounts of money for different outcomes

We can use Y as the random variable for the payout of money in dollars

$Y = \{0, 5, 10\}$

1 head earns \$0 0 heads earn \$, 2 heads earn \$0.

y	\$0	\$5	\$10
$P(Y = y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

$E(Y) = \sum yp(y)$

$= \$0 \times \frac{1}{2} + \$5 \times \frac{1}{4} + \$10 \times \frac{1}{4}$

$= \$375$

ii Since the game costs \$5 to play and the expected outcome is \$375 Bethany would expect to lose money in the long term

You can find the expected value on your calculator using the statistics mod, in the same way you would find the mean of data presented in a frequency table

EXAMPLE 11

Find the expected value of this discrete probability distribution

x	0	2	4	6	8
$P(X = x)$	0.14	0.13	0.25	0.36	0.12

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Solution

Operation	Casio scientific	Sharp scientific
Place your calculator in statistical mode	MODE STAT 1-VAR SHIFT MODE scroll down to STAT Frequency? ON	MODE STAT =
Clear the statistical memory	SHIFT 1 Edit Del-A	2ndF DEL
Enter data	SHIFT 1 Data to get table 0 = 2 = etc to enter in x column 014 = 0.13 = etc to enter in FREQ column AC to leave table	0 2ndF STO 014 M+ 2 2ndF STO 013 M+ etc
Calculate mean ($\bar{x} = 438$)	SHIFT VAR $\bar{x}$ =	RCL $\bar{x}$
Check the number of scores ($n = 1$)	SHIFT VAR n =	RCL n
Change back to normal mode	MODE COMP	MODE 0

Expected value $E(X) = 438$

You can solve problems using expected value.

EXAMPLE 12

a The probability of selling a red car, based on previous experiene, is 5%. Find the expected number of red cars sold in one week if a dealer sells 2 cars

b For the probability function below, evaluate a and b given that $E(X) = 2$.

x	1	2	3	4
$P(X = x)$	a	b	0.2	0.1

Solution

a $P(\text{R}) = 35\%$

$= 035$

So $P(\text{not R}) = P\left(\bar{\text{R}}\right)$

$= 1 - 035$

$= 065$

$P(X = 0) = P(\bar{\text{R}}\bar{\text{R}})$

$= 065 \times 065$

$= 04225$

$P(X = 1) = P(\bar{\text{R}}\text{R}) + P(\text{R}\bar{\text{R}})$

$= 065 \times 035 + 035 \times 065$

$= 0455$

$P(X = 2) = P(\text{RR})$

$= 035 \times 035$

$= 01225$

x	0	1	2
$P(X = x)$	04225	.4550.1225	

$E(X) = \sum xp(x)$

$= 0 \times 04225 + 1 \times 0455 + 2 \times 01225$

$= 07$

So it is expected that when 2 cars are sold .7 of them will be re.

Rounded to the nearest whole number, we could expect around 1 car to be rd.

Note This answer would be more meaningful for a much larger number of sale!

b $E(X) = \sum xp(x)$

$2 = 1 \times a + 2 \times b + 3 \times 02 + 4 \times 0.1$

$2 = a + 2b + 1$

$a + 2b = 1$ [1]

Since the function is a probability distribution

$a + b + 02 + 0.1 = 1$

$a + b + 0.3 = 1$

$a + b = 07$ [2]

[1] – [2] $b = 03$

Substitute into [2]

$a + 0.3 = 07$

$a = 04$

So $a = 04$, $b = 0.3$.

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Exercise 10.03 Mean or expected value

1 Find the expected value of each probability distribution

a $\left(0, \frac{1}{4}\right) \left(1, \frac{1}{2}\right) \left(2, \frac{1}{4}\right)$

b

x	1	2	3	4	5
$P(X = x)$	0.31	0.16	0.15	0.2	0.18

c

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{16}$

d $p(x) = \frac{x+1}{8}$ for $x = 0, 1, 4$

e

$$p(x) = \begin{cases} \frac{x}{2} & \text{for } x = 1 \\ \frac{x}{8} & \text{for } x = 2 \\ \frac{x-3}{4} & \text{for } x = 4 \\ 0 & \text{for all other values of } x \end{cases}$$

2 For each question

i evaluate k **ii** find the mean of the probability distribution

a $(1, k), \left(2, \frac{1}{5}\right) \left(3, \frac{3}{10}\right) \left(4, \frac{2}{5}\right)$

b $p(x) = k(x + 3)$ for $x = 0, 1, 2$

c

x	1	2	3	4	5	6
$P(X = x)$	0.1	0.02	0.17	0.24	k	032

3 Find the expected value of each probability distribution

a The number of heads when tossing 2 coins

b The sum of the 2 numbers rolled on a pair of dice

c The number of girls in a 3-child family

d The number of faulty cars when testing 3 cars if 1 in every 1000 cars is faulty

e The number of red counters when 2 counters are selected at random from a bag containing 7 red and 12 white counters

i with replacement

ii without replacement

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4 The expected value $E(X) = 635$ for this probability function Find p and q

x	3	7	8	9
$p(x)$	p	025	.35	q

5 The mean of the probability distribution below is $3\frac{3}{4}$ Evaluate a and b

$(1, a), \left(2\ \frac{1}{8}\right), (3, b), \left(4\ \frac{1}{4}\right)\ \left(5\ \frac{3}{8}\right)$

6 A uniform discrete random variable X has values $x = 1, 2, 3, 4$.

- **a** Draw up a probability distribution table for X
- **b** Find $E(X)$

7 Find the expected number of heads when tossing 3 coins

8 A bag contains 8 white and 3 yellow marbles If 3 marbles are selected at rando, find the mean number of white marbles

- **a** with replacement
- **b** without replacement

9 In a game 2 dice are rolled and the difference between the 2 numbers is calculate. A player wins \$1 if the difference is 3 \$2 if it is 4 and \$3 if it is .

- **a** Draw a probability distribution table for the winning values
- **b** Find the expected value
- **c** It costs \$1 for a player to roll the dice How much would the player be expected to win or lose?

10 Staff at a call centre must make at least 1 phone sale every hour. The probability that Yasmin will make a sale on a phone call is $\frac{2}{5}$ She makes 4 phone calls in an hou.

- **a** Draw a probability distribution for the number of sales Yasmin maks.
- **b** Find the expected value
- **c** Will Yasmin make at least one phone sale in an hour?

11 A game uses a spinner with the numbers 1 to 12 equally spread around it A player wins \$3 for spinning a number greater than 10 \$2 for a number less than 4 and loses \$1 for any other number. How much money would a player be expected to win or lose?

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10.04 Variance and standard deviation

WS

Vriance and andad deviaion

Vriance and andad deviaion o a dicee andom vaiable

Variance and **standard deviation** measure the spread of data in a distribution by finding the difference of each value from the mean Variance is the square of the standard deviation

The variance σ^2 involves the average of the squared differences of each value from the mea.

σ is the Greek lower-case letter‘sig’.

Variance, σ^2

$$Var(X) = \Sigma(x - \mu)^2 p(x)$$
$$= E[(X - \mu)^2]$$

Proof

$$\sigma^2 = \frac{\Sigma f(x-\mu)^2}{\Sigma f}$$

$$= \frac{\Sigma f(x-\mu)^2}{n}$$ where $n = \Sigma f$ is the sum of frequencies

$$= \Sigma(x-\mu)^2 \times \frac{f}{n}$$

$$= \Sigma(x-\mu)^2 p(x)$$

$$= E[(X-\mu)^2]$$

Standard deviation is the square root of variance

Standard deviation, σ

$$\sigma = \sqrt{\Sigma(x-\mu)^2 p(x)}$$
$$= \sqrt{E[(X-\mu)^2]}$$

We use s for the standard deviation of a **sample** and σ (the lower-case Greek letter sigma) for the standard deviation of a **population** For probability distribution, we use the population standard deviation σ The sample standard deviatin, s is an estimate of σ and as the sample size increases the sample represents the population better and the value of s approaches σ

EXAMPLE 13

Find the variance and standard deviation of this probability distribution

x	1	2	3	4
$P(X = x)$	0.16	0.32	0.42	0.1

Solution

$$E(X) = \sum xp(x)$$

$$= 1 \times 016 \; + 2 \times 032 \; + 3 \times 042 \; + 4 \times 0.1$$

$$= 246$$

$$Var(X) = \sum(x - \mu)^2 p(x)$$

$$= (1 - 246)^2 016 \; + (2 - 246)^2 032 \; + (3 - 246)^2 042 \; + (4 - 246)^2 01$$

$$= 07684$$

Standard deviation

$$\sigma = \sqrt{07684}$$

$$= 08765\ldots$$

$$\approx 08766$$

The formula for variance is a little tedious since we subtract the mean from every value
There is a simpler formula for variance

Calculation formulas for variance and standard deviation

$$Var(X) = \sum[x^2 p(x)] - \mu^2$$

$$= E(X^2) - \mu^2$$

$$\sigma = \sqrt{Var(X)}$$

Proof

$$\sigma^2 = \sum(x - \mu)^2 p(x)$$

$= \sum[x^2 p(x) - 2\mu x p(x) + \mu^2 p(x)]$ — expanding $(x - \mu)^2$

$= \sum x^2 p(x) - \sum 2\mu x p(x) + \sum \mu^2 p(x)$ — taking separate sums of each part

$= \sum x^2 p(x) - 2\mu \sum x p(x) + \mu^2 \sum p(x)$ — since μ is a constant

$= \sum x^2 p(x) - 2\mu \times \mu + \mu^2 \times 1$ — since $\sum xp(x) = \mu$ and $\sum p(x) = 1$ is a constant

$= \sum[x^2 p(x)] - \mu^2$

$= E(X^2) - \mu^2$

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If we use the same probability distribution as Example 13 we can see that this formula gives us the same result

EXAMPLE 14

Use the simpler calculation formulas to find the variance and standard deviation of this probability distribution

x	1	2	3	4
$P(X = x)$	0.16	0.32	0.42	0.1

Solution

$$\mu = E(X) = \sum xp(x)$$

$$= 1 \times 016 + 2 \times 032 + 3 \times 042 + 4 \times 0.1$$

$$= 246$$

$$Var(X) = \sum[x^2 p(x)] - \mu^2$$

$$= (1)^2\ 016 + (2)^2\ 032 + (3)^2\ 042 + (4)^2\ 0.1 - 246^{\ 2}$$

$$= 07684$$

Standard deviation

$$\sigma = \sqrt{07684}$$

$$= 08765\ldots$$

$$\approx 08766$$

You can use a calculator to work out the variance and standard deviatio.

EXAMPLE 15

Find the expected value standard deviation and variance of this probability distributio.

x	1	2	3	4	5
$P(X = x)$	0.1	0.25	0.2	0.35	0.1

Solution

Operation	Casio scientific	Sharp scientific
Place your calculator in statistical mode	MODE STAT 1-VAR SHIFT MODE scroll down to STAT Frequency? ON	MODE STAT =
Clear the statistical memory	SHIFT 1 Edit Del-A	2ndF DEL
Enter data	SHIFT 1 Data to get table 1 = 2 = etc to enter in x column 01 = 025 = etc to enter in FREQ column AC to leave table	1 2ndF STO 0.1 M+ 2 2ndF STO 025 M+ etc
Calculate mean ($\overline{x} = 31$)	SHIFT Var $\overline{x}$ =	RCL $\overline{X}$
Calculate the standard deviation ($\sigma_x = 11789\ldots$)	SHIFT Var σ_x =	RCL σ_x
Change back to normal mode	MODE COMP	MODE 0

Mean $\mu = 3.1$

Standard deviation $\sigma \approx 1.18$

Variance $\sigma^2 = 11789\ldots^{\,2}$

≈ 1.39

Exercise 10.04 Variance and standard deviation

In this exercise round answers to 2 decimal places where necessar.

1 For each probability distribution fin:

i the standard deviation **ii** the variance

a

x	1	2	3	4	5	6
$P(X = x)$	0.17	0.24	0.12	0.13	0.23	0.11

b

x	0	1	2	3
$P(X = x)$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{1}{14}$	$\frac{5}{14}$

c $\left(1 \quad \frac{3}{8}\right) \left(2 \quad \frac{1}{4}\right) \left(3 \quad \frac{1}{8}\right) \left(4 \quad \frac{1}{16}\right) \left(5 \quad \frac{3}{16}\right)$

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2 Find the mean variance and standard deviation of each probability functio.

a

x	1	4	7	9	10
$p(x)$	0.09	0.18	0.26	0.32	0.15

b $P(x) = \dfrac{x+1}{9}$ for $x = 0, 2, 4$

3 Evaluate n and find the expected value and variance for this probability distribution

x	1	2	3	4	5
$p(x)$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{3}{20}$	$\frac{1}{20}$	n

4 For the probability distribution below with $E(X) = 332$, fin:

x	1	2	3	4	5
$p(x)$	a	b	0.17	0.2	0.3

a the values of a and b

b the variance

c the standard deviation

5 For each probability distribution find

i the expected value

ii the standard deviation

iii the variance

a The number of tails when tossing 3 coins

b The number of blue marbles when 2 marbles are selected randomly from a bag containing 10 blue and 12 white marbles

6 A uniform discrete random variable X has values $x = 1, 2, 3, 4, 5$.Find:

a the mean

b the standard deviation

c the variance

7 a Create a probability distribution table for the number of 6s rolled on a pair of dice

b Find the mean variance and standard deviation of this functio.

8 The probability of selecting a black jelly bean at random from a packet is 4%
If 2 jelly beans are selected at random fin:

a the expected number of black jelly beans

b the standard deviation

c the variance

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9 A set of cards contains 5 blue and 7 white cards If 3 are drawn out at rando, the discrete random variable X is the number of blue cards drawn out Find the mean and variance of X if the cards are drawn out

a with replacement

b without replacement

10 In a game 2 cards are drawn from a deck of 52 standard playing card. A player wins 5 points if one of the cards is an ace and 10 points for double aces

a If random variable X is the number of aces drawn

i create a probability distribution for X

ii find the mean variance and standard deviation for this distributio.

b If random variable Y is the number of points won

i create a probability distribution for Y

ii find the mean variance and standard deviation for this distributio.

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10. TEST YOURSELF

For Questions **1** to **4** select the correct answer **A B C** or **D**

Pacice quiz

1 The table shows a discrete probability distribution

x	1	2	3	4	5	6
$P(X=x)$	0.24	0.16	0.08	0.14	0.21	0.17

Find $P(X \geq 2)$

A 04 **B** 052 **C** 076 **D** 048

2 The expected value of the probability distribution below is

x	1	2	3	4
$p(x)$	0.6	0.1	0.2	0.1

A $06 + 0.1 + 02 + 0.1$

B $\frac{1}{4}(1 \times 06 + 2 \times 0.1 + 3 \times 02 + 4 \times 01)$

C $\frac{1+2+3+4}{4}$

D $1 \times 06 + 2 \times 0.1 + 3 \times 02 + 4 \times 0.1$

3 The value of t in the probability distribution is

x	1	2	3	4	5	6
$p(x)$	0.28	0.16	0.04	0.1	t	025

A 013 **B** 017 **C** 027 **D** 007

4 Which table represents a probability function?

A

x	$f(x)$
1	0.6
2	0.2
3	0.1
4	0.2

B

x	$f(x)$
1	0.3
2	0.25
3	0.1
4	0.25

C

x	$f(x)$
1	0.15
2	0.25
3	0.3
4	0.4

D

x	$f(x)$
1	0.25
2	0.4
3	0.15
4	0.2

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5 For each random variable write the set of possible value.

a The number of 6s when rolling a die 5 times

b The number of heads when tossing a coin 10 times

c The first day the temperature rises above 28° in November

d The number of doubles when rolling 2 dice twice

e The number of red cards selected in 9 trials when pulling a card from a hat that contains 20 red and 20 blue cards

6 This table shows a discrete probability distribution Evaluate k

x	0	1	2	3	4
$P(X = x)$	$5k$	$3k$	$4k - 1$	$2k - 3$	$6k$

7 A probability function is given by $p(x) = \frac{x}{15}$ for $x = 1$,3, 4 5. Find itsmean, variance and standard deviation

8 The table represents a probability distribution

x	4	7	8	9
$P(X = x)$	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{1}{3}$	$\frac{1}{12}$

Find

a $P(X = 9)$ **b** $P(X < 8)$ **c** $P(X \geq 7)$

d $P(4 \leq X \leq 8)$ **e** $P(7 < X \leq 9)$

9 Draw a discrete probability distribution table for the number of tails when tossing 2 coins

10 State whether each probability distribution is uniform

a Number of tails when tossing a coin

b Number of heads when tossing 2 coins

c The number rolled on a die

d Number of 6s when rolling a die

11 State whether each random variable is discrete or continuous

a The number of heads when tossing 5 coins

b The distances between cars parked in the street

c The number of correct answers in an exam

d The masses of babies

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12 Find the expected value variance and standard deviation for this probability distribution

x	0	1	2	3	4
$P(X = x)$	31%	22%	18%	24%	5%

13 A spinner has the numbers 1 to 7 evenly spaced around it

a Draw a probability distribution table for the spinner.

b Is it a uniform distribution?

c Find the probability of spinning a number

i greater than 5 **ii** 3 or less **iii** at least 4

d Find the expected value of the spinner.

14 A function is given by

$$f(x) = \begin{cases} \dfrac{x-1}{10} & \text{for } x = 3 \\ \dfrac{x-4}{5} & \text{for } x = 5 \\ \dfrac{x}{15} & \text{for } x = 9 \end{cases}$$

a Find

i $f(3)$ **ii** $f(5)$ **iii** $f(9)$

b Show that $f(x)$ is a probability function

15 a Construct a probability distribution table for the number of tails when tossing 2 coins

b Is it a uniform distribution?

c Find the probability of tossing

i one tail **ii** at least one tail

16 State whether each function is a probability function

a

x	1	2	3	4	5	6
$f(x)$	0.2	0.07	0.15	0.2	0.3	0.08

b $f(x) = \dfrac{x+1}{6}$ for $x = 0, 1, 2, 3$

c $\left(0, \dfrac{1}{8}\right), \left(1, \dfrac{1}{4}\right), \left(2, \dfrac{1}{2}\right), \left(3, \dfrac{1}{16}\right), \left(4, \dfrac{3}{16}\right)$

17 In a game Jonas pays \$1 to toss 3 coins togethe. He wins\$1.50 for 3 heads and \$2 for 3 tails

a Find the expected value for this game

b How much would you expect Jonas to win or lose in the long term?

18 **a** Show that the points (3 21%, 5, 1%), (6, 47%) ad (9, 18%) represent a discrete probability function

b Find $E(X)$ and $Var(X)$

19 Each table represents a probability distribution Evaluate n

a

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{8}$	n	$\frac{1}{16}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{1}{16}$

b

x	0	1	2	3
$P(X = x)$	0.27	0.51	0.14	n

c $P(x) = n(2x - 1)$ for $x = 1, 2, 3$

20 The table represents a probability distribution

x	2	3	4	5	6
$P(X = x)$	0.2	0.2	0.3	a	b

If $E(X) = 38$ evaluate a and b

21 A game involves tossing 2 coins Tannika wins \$2 for 2 heads or 2 tails and loses \$1 for a head and a tail

a Draw up a probability distribution table for random variable Y showing the winning amounts

b If it costs Tannika \$1 to ply, would you expect her to win or lose the game ?

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10. CHALLENGE EXERCISE

1 For the probability distribution below, $E(X) = 294$ and $Var(X) = 22564$
Evaluate a b and c

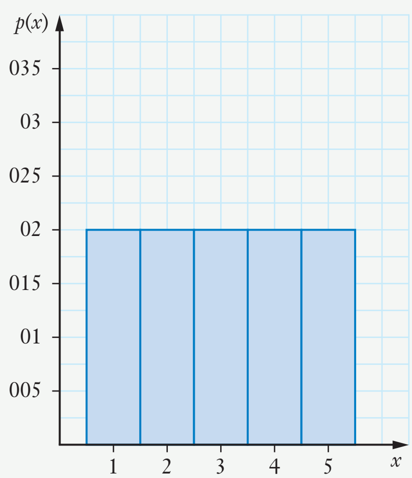
x	1	2	3	4	5
$p(x)$	a	b	c	0.16	0.24

2 The variance of the probability distribution (1 a), (2,0.3), (k, .4), (5, 0.1) i 1.89 and the mean is 29 Evaluate a and k

3 The probability distribution below has $E(X) = 334$ and $Var(X) = 43044$ Find the value of k and l

x	1	3	k	6
$P(X = x)$	0.4	0.12	0.3	l

4



a Show that the graph above represents a discrete probability distribution

b Is it a uniform distribution?

c Find

i $P(X \leq 3)$ **ii** $P(X > 2)$ **iii** $P(1 \leq X < 5)$

d Find $E(X)$ and $Var(X)$

e If this distribution changes so that $P(X = 1) = 035$ find $P(X = 2)$ if all the other probabilities remain the same

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5 A sample of people were surveyed to rate a TV show on a scale of 1 to 5

Rating	Frequency
1	4
2	15
3	23
4	59
5	19

a How many people were surveyed?

b Draw a probability distribution table for the survey results

c Is the sample mean from the survey a good estimate of the population mean of 25 ?

d Find the standard deviation Is this a good estimate of the population standard deviation of 1?

e Can you explain these results from **c** and **d**?

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Practice set 4

For Questions **1** to **5** select the correct answer **A B C** or **D**

1 Find the amplitude and period of $y = 5 \sin 3x$

A Amplitude 3 period 5

B Amplitude 5 period 3

C Amplitude 5 period $\frac{2\pi}{3}$

D Amplitude 3 period $\frac{2\pi}{5}$

2 The table is a discrete probability distribution

x	1	2	3	4	5	6
$P(X = x)$	0.14	0.16	0.08	0.14	0.31	0.17

Find $P(X \leq 4)$

A 038

B 052

C 014

D 062

3 Find the exact value of sin 135° + cos 120°

A $\frac{\sqrt{2}-\sqrt{3}}{2}$

B $\frac{\sqrt{2}+1}{2}$

C $\frac{\sqrt{2}+\sqrt{3}}{2}$

D $\frac{\sqrt{2}-1}{2}$

4 Which statement is the same as $3^x = 7$? There is more than one answe.

A $x = \log \frac{7}{3}$

B $\log_3 x = 7$

C $\log_3 7 = x$

D $x = \frac{\log 7}{\log 3}$

5 The derivative of $x^2(2x + 9)^2$ is

A $4x(2x + 9)$

B $2x(2x + 9)^2 + 2x^2(2x + 9)$

C $2x(2x + 9)$

D $2x(2x + 9)^2 + 4x^2(2x + 9)$

6 Differentiate

a $y = e^x - x$

b $y = 3e^x + 1$

c $y = (e^x - 2)^4$

d $y = e^x(4x + 1)^3$

e $y = \frac{e^x}{5x - 2}$

f $y = 5e^{7x}$

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7 A function is given by

$$f(x)=\begin{cases}\dfrac{x+1}{8} & \text{for } x=0,1,2\\[2mm] \dfrac{x-2}{4} & \text{for } x=3\end{cases}$$

a Find

i $f(0)$ **ii** $f(3)$

b Show that $f(x)$ is a probability function

8 Find $\log_5 \dfrac{1}{25}$

9 The table represents a probability distribution

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$

Find

a $P(X = 2)$ **b** $P(X < 4)$ **c** $P(X \geq 2)$

d $P(4 \leq X \leq 6)$ **e** $P(1 \leq X < 5)$

10 Simplify

a $\tan(180° - \theta)$ **b** $\sin(-\theta)$ **c** $\cos(2\pi - \theta)$

11 For $0 \leq x \leq 2\pi$ sketch the graph of

a $y = 2\sin 4x$ **b** $y = \tan\dfrac{x}{2}$ **c** $y = -\cos x$

12 For each random variable X write the set of possible values

a The number of rolls of a die until a 6 turns up

b The number of red cards selected when choosing 12 cards from a bag containing 15 red and 15 black cards

c The first rainy day in January.

13 Solve $\log_x \dfrac{1}{16} = 4$

14 The population of a city over t years is given by the formula $P = 100\ 000e^{071\ t}$ After how many years to 1 decimal plac, will the population become 1 million ?

15 A bag contains 7 white and 6 blue cards Create a probability distribution table for the number of blue cards selected when randomly selecting 3 cards

a with replacement **b** without replacement

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16 If $\tan x = -\frac{4}{3}$ and $\cos x > 0$ evaluate $\sin x$ and $\cos x$

17 Solve for $0 \leq x \leq 2\pi$

a $2\cos x + 1 = 0$ **b** $\tan^2 x = 1$

c $\cos x = 0$ **d** $\sin 2x = \frac{1}{2}$

18 This table represents a probability distribution

x	1	2	3	4	5
$P(X = x)$	0.16	0.23	0.22	a	b

If $E(X) = 304$ evaluate a and b

19 Find the expected value variance and standard deviation for the probability distribution below.

x	0	1	2	3	4
$P(X = x)$	0.2	0.1	0.3	0.1	0.3

20 Find the exact value of

a $\cos\frac{7\pi}{4}$ **b** $\sin\frac{4\pi}{3}$ **c** $\tan\frac{5\pi}{6}$

21 Draw a discrete probability distribution table for the number of tails when tossing 3 coins

22 Sketch the graph of

a $y = \log_3 x$ **b** $y = 3\log_2 x - 1$

23 a Write $\log_e x$ as an equation with x in terms of y

b Hence find the value of x to 3 significant figure, when $y = 1.23$.

24 Solve $7^{2x} = 3$.

25 This table shows a discrete probability distribution Evaluate k

x	0	1	2	3	4
$P(X = x)$	$2k$	$3k$	$4k - 2$	$5k - 1$	$6k$

26 State whether each probability distribution is uniform

a Number of heads when tossing 2 coins

b Number of heads when tossing a coin

c Number of even numbers when rolling one die

d Number of 1s when rolling one die

27 State whether each function is a probability function

a $f(x) = \frac{x+1}{10}$ for $x = 0, 1, 2, 3$

b $f(x) = \begin{cases} \frac{x}{11} & \text{for } x = 1, 2 \\ \frac{x-1}{22} & \text{for } x = 3, 4, 5 \end{cases}$

28 Solve for $0° \le x \le 360°$

a $\tan x = -1$ **b** $2 \sin x = 1$

c $2 \cos^2 x = 1$ **d** $\tan 2x = \sqrt{3}$

29 Evaluate to 2 decimal places where appropriat.

a $\log_2 16$ **b** $\log_3 3$ **c** $\log_4 2$

d $\log_{10} 1097$ **e** $\ln 431$ **f** $\log_3 11$

30 Sketch the graph of

a $y = e^{-x}$

b $y = 2e^{3x} + 1$

31 The probability of winning a game is 65% and the probability of losing the game is 12%

a Draw a probability distribution table showing 0 for a loss 1 for a draw and 2 for a win

b Find the expected value and variance

32 Find the equation of the tangent to the curve $y = 5e^x$ at the point $(2\ 5\ e^2)$

33 In a game Faizal pays \$1 to toss 2 coin. He wins \$2 for 2 heads or 2 tails and loses \$1 for a head and a tail

a Find the expected value for this game

b How much would you expect Faizal to win or lose in the long term?

34 A spinner has the numbers 1 to 8 equally placed around it

a Draw a probability distribution table for the spinner.

b Is it a uniform distribution?

c Find the probability of spinning a number

i greater than 4 **ii** 3 or less **iii** at least 4

d Find the expected value of the spinner.

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35 **a** Show that the points (1 27%, 2, 3%), (3, 28%) ad (4, 14%) represent a discrete probability function

b Find $E(X)$ and $Var(X)$

36 For the following probability distribution evaluate k

x	1	2	3	4	5	6
$p(x)$	$\frac{5}{16}$	k	$\frac{1}{16}$	$\frac{3}{8}$	$\frac{1}{16}$	$\frac{1}{8}$

37 Simplify

a $5 + 5\tan^2 x$

b $\dfrac{(1+\sin x)(1-\sin x)}{\sin x \cos x}$

38 Find the exact value of

a tan 150°

b cos (–45°)

c sin 240°

39 Find the value of x

a $x^2 - 2x - 3 = 0$

b $1 < 2x - 3 \le 7$

c $|3x + 1| = 4$

40 Find the centre and radius of the circle $x^2 - 4x + y^2 + 6y - 3 = 0$

41 Amanda leaves home and cycles south for 36 km She then turns and cycles for 54 km on a bearing of 243 °

a How far is Amanda from her hous, to 1 decimal place ?

b What is Amand's bearing from her houe, to the nearest degree ?

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ANSWERS

Answers are based on full calculator values and only rounded at the en, even when different parts of a question require rounding This gives more accurate answes. Answers based on reading graphs may not be accuate.

Chapter 1

Exercse 101

1 **a** 500 **b** 145 **c** $\frac{1}{64}$
d 3 **e** 2

2 **a** 137 **b** 11 **c** 08
d 27 **e** –26 **f** 05

3 **a** a^{17} **b** $y^0 = 1$ **c** a^{-4}
d w **e** x^5 **f** p^0
g y^6 **h** x^{21} $4x^0$
j $81y^{-8}$ **k** a $\frac{x^{10}}{y^{45}}$
m w^0 **n** p^5 **o** x^{-3}
p $a^{-2}b^3$ or $\frac{b^3}{a^2}$ **q** $x^{-5}y^2$ or $\frac{y^2}{x^5}$

4 **a** x^4 **b** a^{-7} **c** m^4
d k^0 **e** a^{-8} **f** x
g mn^2 **h** p^{-} $9x^{22}$
j x^{21}

5 **a** p^5q^{15} **b** $\frac{a^8}{b^8}$ **c** $\frac{64a^3}{b^{12}}$
d $49a^0b^2$ **e** $8m^{17}$ **f** x^4y^0
g $\frac{2k^{23}}{27}$ **h** $16y^{47}$ a^3
j $125x^{-21}y^{18}$

6 $4\frac{}{2}$ **7** 324 **8** $2\frac{10}{27}$

9 **a** a^3b **b** $\frac{1}{25}$

10 **a** pq^2r^2 **b** $\frac{7}{32}$

11 $\frac{4}{9}$ **12** $\frac{1}{18}$ **13** $\frac{4}{27}$

14 $\frac{1}{81}$ **15** $\frac{1}{108}$ **16** $\frac{1}{12}$

Exercse 102

1 **a** $\frac{1}{27}$ **b** $\frac{1}{4}$ **c** $\frac{1}{343}$
d $\frac{1}{10\,000}$ **e** $\frac{1}{256}$ **f** 1
g $\frac{1}{32}$ **h** $\frac{1}{81}$ $\frac{1}{7}$
j $\frac{1}{81}$ **k** $\frac{1}{64}$ $\frac{1}{9}$
m 1 **n** $\frac{1}{36}$ **o** $\frac{1}{125}$
p $\frac{1}{100\,000}$ **q** $\frac{1}{128}$ **r** 1
s $\frac{1}{64}$ **t** $\frac{1}{64}$

2 **a** 1 **b** 16 **c** $1\frac{}{2}$ **d** $1\frac{11}{25}$
e 1 **f** 125 **g** $1\frac{}{3}$ **h** 49
$3\frac{3}{8}$ **j** 32 **k** $2\frac{}{3}$ 1
m $1\frac{13}{36}$ **n** $1\frac{19}{81}$ **o** 1 **p** 16
q $-15\frac{5}{8}$ **r** $-\frac{7}{23}$ **s** 1 **t** $\frac{16}{25}$

3 **a** m^{-3} **b** x^{-}
c p^{-7} **d** d^{-9}
e k^{-5} **f** x^{-2}
g $2x^{-4}$ **h** $3y^{-2}$
$\frac{1}{2}z^{-6}$ or $\frac{z^{-6}}{2}$ **j** $\frac{3t^{-8}}{5}$

k $\frac{2x^{-}}{7}$ $\frac{5m^{-6}}{2}$

m $\frac{2y^{-7}}{3}$ **n** $(3x+4)^{-2}$

o $(a+b)^{-8}$ **p** $(x-2)^{-}$

q $(5p+1)^{-3}$ **r** $2(4t-9)^{-5}$

s $\frac{(x+1)^{-11}}{4}$ **t** $\frac{5(a+3b)^{-7}}{9}$

4 **a** $\frac{1}{t^5}$ **b** $\frac{1}{x^6}$ **c** $\frac{1}{y^3}$

d $\frac{1}{n^8}$ **e** $\frac{1}{w^{10}}$ **f** $\frac{2}{x}$

g $\frac{3}{m^4}$ **h** $\frac{5}{x^7}$ $\frac{1}{8x^3}$

j $\frac{1}{4n}$ **k** $\frac{1}{(x+1)^6}$ $\frac{1}{8y+z}$

m $\frac{1}{(k-3)^2}$ **n** $\frac{1}{(3x+2y)^9}$ **o** x^5

p y^0 **q** $\frac{p}{2}$ **r** $(a+b)^2$

s $\frac{x-y}{x+y}$ **t** $\left(\frac{3x+y}{2w-z}\right)^7$

Exercse 103

1 **a** 9 **b** 3 **c** 4 **d** 2

e 7 **f** 10 **g** 2 **h** 8

4 **j** 1 **k** 3 2

m 0 **n** 5 **o** 7 **p** 2

q 4 **r** 25 **s** 32 **t** 4

u 27 **v** $\frac{1}{2}$ **w** $\frac{1}{3}$ **x** $\frac{1}{2}$

y $\frac{1}{16}$

2 **a** 219 **b** 260 **c** 153 **d** 060

e 090 **f** 029

3 **a** $\sqrt[3]{y}$ **b** $\sqrt[6]{x}$ **c** $\sqrt{a}$

d $\sqrt[9]{t}$ **e** $\sqrt[3]{y^2}$ or $\left(\sqrt[3]{y}\right)^2$ **f** $\sqrt[4]{x^3}$

g $\sqrt[5]{b^2}$ **h** $\sqrt[7]{a^4}$ $\frac{1}{\sqrt{x}}$

j $\frac{1}{\sqrt[3]{d}}$ **k** $\frac{1}{\sqrt[8]{x}}$ $\frac{1}{\sqrt[3]{y}}$

m $\frac{1}{\sqrt[4]{a}}$ **n** $\frac{1}{\sqrt[4]{z^3}}$ **o** $\frac{1}{\sqrt[5]{y^3}}$

p $\sqrt{2x+5}$ **q** $\sqrt[3]{6q+r}$ **r** $\sqrt[9]{a+b}$

s $\frac{1}{\sqrt{3x-1}}$ **t** $\frac{1}{\sqrt[5]{(x+7)^2}}$ or $\frac{1}{\left(\sqrt[5]{x+7}\right)^2}$

4 **a** $t^{\frac{}{2}}$ **b** $y^{\frac{}{5}}$ **c** $x^{\frac{3}{2}}$

d $(9-x)^{\frac{}{3}}$ **e** $(4s+1)^{\frac{}{2}}$ **f** $(3x+1)^{\frac{5}{2}}$

g $(2t+3)^{-\frac{}{2}}$ **h** $(5x-y)^{-\frac{3}{2}}$ $(x-2)^{-\frac{2}{3}}$

j $\frac{1}{2}(y+7)^{-\frac{}{2}}$ **k** $5(x+4)^{-\frac{}{3}}$ $\frac{2}{3}(y^2-1)^{-\frac{}{2}}$

m $\frac{3}{5}(x^2+2)^{-\frac{3}{4}}$

5 **a** $x^{\frac{3}{2}}$ **b** $x^{-\frac{}{2}}$ **c** $x^{\frac{2}{3}}$

d $x^{\frac{5}{3}}$ **e** $x^{\frac{5}{4}}$

6 **a** $\frac{1}{\sqrt[3]{a-2b}}$ **b** $\frac{1}{\sqrt[3]{(y-3)^2}}$ **c** $\frac{4}{\sqrt[7]{(6a+1)^4}}$

d $\frac{1}{3\sqrt[4]{(x+y)^5}}$ **e** $\frac{6}{7\sqrt[9]{(3x+8)^2}}$

Exercse 104

1 **a** $3a$ **b** z **c** $3b$

d $-3r$ **e** $-y$ **f** $-5x$

g 0 **h** $3k$ $9t$

j $10w$ **k** $-m$ $-x$

m 0 **n** $11b$ **o** $-10x$

p $6x-6y$ **q** $a-3b$ **r** $4xy+2y$

s $-6ab^2$ **t** $m^2-6m+12$

u p^2-2p-6 **v** $-2ab+10b$

w $2bc-ac$ **x** $2a^5-9x^3+1$

y $x^3-2xy^2+3x^2y+2y^3$

2	**a** $10b$	**b** $8xy$	**c** $10p^2$
	d $-6wz$	**e** $15ab$	**f** $14xyz$
	g $48abc$	**h** $12d^2$	$12a^3$
	j $-27y^3$	**k** $32x^0$	$6a^2b^3$
	m $-10a^3b^2$	**n** $21p^3q^4$	**o** $5a^3b^3$
	p $-8h^0$	**q** k^3p^3	**r** $81t^{12}$
	s $-14m^{11}$	**t** $24x^6y^3$	
3	**a** $6x$	**b** 2	**c** $4a^2$
	d $8a$	**e** $4a$	**f** $\frac{y}{2}$
	g $3p$	**h** $\frac{ab}{2}$	$\frac{4}{3y}$
	j $-3x^3$	**k** $3a$	$\frac{1}{3ab^2}$
	m $\frac{-2}{qs}$	**n** $\frac{2}{3c^2d}$	**o** $\frac{z^2}{2x^2}$
	p $6p^4q$	**q** $\frac{a^4b^7}{4c}$	**r** $\frac{b^6}{2a}$
	s $-\frac{x^3z^3}{3y}$	**t** $\frac{a^{13}}{2b^6}$	

Exercse 105

1 $2x-8$
2 $6h+9$
3 $-5a+10$
4 $2xy+3x$
5 x^2-2x
6 $6a^2-16ab$
7 $2a^2b+ab^2$
8 $5n^2-20n$
9 $3x^3y^2+6x^2y^3$
10 $4k+7$
11 $2t-17$
12 $4y^2+11y$
13 $-5b-6$
14 $8-2x$
15 $-3m+1$
16 $8h-19$
17 $d-6$
18 a^2-2a+4
19 $3x^2-9x-5$
20 $2ab-2a^2b+b$
21 $4x-1$
22 $-7y+4$
23 $2b$
24 $5t-6$

Exercse 106

1 $a^2+7a+10$
2 x^2+2x-3
3 $2y^2+7y-15$
4 m^2-6m+8
5 $x^2+7x+12$
6 $y^2-3y-10$
7 $2x^2+x-6$
8 $h^2-10h+21$
9 x^2-25
10 $15a^2-17a+4$
11 $8y^2+6y-9$
12 $xy+7x-4y-28$
13 x^3-2x^2+3x-6
14 n^2-4
15 $4x^2-9$
16 $16-49y^2$
17 a^2-4b^2
18 $9x^2-16y^2$
19 x^2-9
20 y^2-36
21 $9a^2-1$
22 $4z^2-49$
23 $x^2-2xy+11x-18y+18$
24 $2ab+2b^2-7b-6a+3$
25 x^3+8
26 a^3-27
27 $a^2+18a+81$
28 $k^2-8k+16$
29 x^2+4x+4
30 $y^2-14y+49$
31 $4x^2+12x+9$
32 $4t^2-4t+1$
33 $9a^2+24ab+16b^2$
34 $x^2-10xy+25y^2$
35 $4a^2+4ab+b^2$
36 a^2-b^2
37 $a^2+2ab+b^2$
38 $a^2-2ab+b^2$
39 a^3+b^3
40 a^3-b^3

Exercse 107

1 $t^2+8t+16$
2 $z^2-12z+36$
3 x^2-2x+1
4 $y^2+16y+64$
5 q^2+6q+9
6 $k^2-14k+49$
7 n^2+2n+1
8 $4b^2+20b+25$
9 $9-6x+x^2$
10 $9y^2-6y+1$
11 $x^2+2xy+y^2$
12 $9a^2-6ab+b^2$
13 $16d^2+40de+25e^2$
14 t^2-16
15 x^2-9
16 p^2-1
17 r^2-36
18 x^2-100
19 $4a^2-9$
20 x^2-25y^2
21 $16a^2-1$
22 $49-9x^2$
23 x^4-4
24 x^4+10x^2+25
25 $9a^2b^2-16c^2$
26 $x^2+4+\frac{4}{x^2}$
27 $a^2-\frac{1}{a^2}$
28 x^2-y^2+4y-4
29 $a^2+2ab+b^2+2ac+2bc+c^2$
30 $x^2+2x+1-2xy-2y+y^2$
31 $12a$
32 $32-z^2$
33 $9x^2+8x-3$
34 $x^2+3xy+y^2-2x$
35 $14n^2-4$
36 $x^3-12x^2+48x-64$
37 x^2
38 $x^4-2x^2y^2+y^4$
39 $8a^3+60a^2+150a+125$

Exercse 108

1 $2(y+3)$ **2** $5(x-2)$
3 $3(m-3)$ **4** $2(4x+1)$
5 $6(4-3y)$ **6** $x(x+2)$
7 $m(m-3)$ **8** $2y(y+2)$
9 $3a(5-a)$ **10** $ab(b+1)$
11 $2xy(2x-1)$ **12** $3mn(n^2+3)$
13 $2xz(4x-z)$ **14** $a(6b+3-2a)$
15 $x(5x-2+y)$ **16** $q^2(3q^3-2)$
17 $5b^2(b+3)$ **18** $3a^2b^2(2b-a)$
19 $(m+5)(x+7)$ **20** $(y-1)(2-y)$
21 $(7+y)(4-3x)$ **22** $(a-2)(6x+5)$
23 $(2t+1)(x-y)$ **24** $(3x-2)(a+2b-3c)$
25 $3x^2(2x+3)$ **26** $3q^3(pq^2-2)$
27 $3ab(5a^3b^2+1)$ **28** $4x^2(x-6)$
29 $5m^2n(7mn^3-5)$ **30** $8ab^2(3ab^3+2)$
31 $2\pi r(r+h)$ **32** $(x-3)(x+2)$
33 $(x+4)(y^2+2)$ **34** $-(a+1)$

Exercse 109

1 $(x+4)(2+b)$ **2** $(y-3)(a+b)$
3 $(x+5)(x+2)$ **4** $(m-2)(m+3)$
5 $(d-c)(a+b)$ **6** $(x+1)(x^2+3)$
7 $(5a-3)(b+2)$ **8** $(2y-x)(x+y)$
9 $(y+1)(a+1)$ **10** $(x+5)(x-1)$
11 $(y+3)(1+a)$ **12** $(m-2)(1-2y)$
13 $(x+5y)(2x-3y)$ **14** $(a+b^2)(ab-4)$
15 $(5-x)(x+3)$ **16** $(x+7)(x^3-4)$
17 $(x-3)(7-y)$ **18** $(d+3)(4-e)$
19 $(x-4)(3+y)$ **20** $(a+3)(2-b)$
21 $(x-3)(x^2+6)$ **22** $(q-3)(p+q)$
23 $(x-2)(3x^2-5)$ **24** $(a-3b)(4+c)$
25 $(y+7)(x-4)$ **26** $(x-4)(x^3-5)$
27 $(2x-3)(2x^2+4)=2(2x-3)(x^2+2)$
28 $3(a+2b)(a+3)$ **29** $5(y-3)(1+2x)$
30 $(r+2)(\pi r-3)$

Exercse 110

1 $(x+3)(x+1)$ **2** $(y+4)(y+3)$
3 $(m+1)^2$ **4** $(t+4)^2$
5 $(z+3)(z-2)$ **6** $(x+1)(x-6)$
7 $(v-3)(v-5)$ **8** $(t-3)^2$
9 $(x+10)(x-1)$ **10** $(y-7)(y-3)$
11 $(m-6)(m-3)$ **12** $(y+12)(y-3)$
13 $(x-8)(x+3)$ **14** $(a-2)^2$
15 $(x-2)(x+16)$ **16** $(y+4)(y-9)$
17 $(n-6)(n-4)$ **18** $(x-5)^2$
19 $(p+9)(p-1)$ **20** $(k-2)(k-5)$
21 $(x+4)(x-3)$ **22** $(m-7)(m+1)$
23 $(q+10)(q+2)$ **24** $(d-5)(d+1)$

Exercse 111

1 $(2a+1)(a+5)$ **2** $(5y+2)(y+1)$
3 $(3x+7)(x+1)$ **4** $(3x+2)(x+2)$
5 $(2b-3)(b-1)$ **6** $(7x-2)(x-1)$
7 $(3y-1)(y+2)$ **8** $(2x+3)(x+4)$
9 $(5p-2)(p+3)$ **10** $(3x+5)(2x+1)$
11 $(2y+1)(y-6)$ **12** $(5x-1)(2x+1)$
13 $(4t-1)(2t-3)$ **14** $(3x+4)(2x-3)$
15 $(6y-1)(y+8)$ **16** $(4n-3)(n-2)$
17 $(4t-1)(2t+5)$ **18** $(3q+2)(4q+5)$
19 $(4r-1)(r+3)$ **20** $(2x-5)(2x+3)$
21 $(6y-1)(y-2)$ **22** $(2p-3)(3p+2)$
23 $(8x+7)(x+3)$ **24** $(3b-4)(4b-9)$
25 $(6x+1)(x-9)$ **26** $(3x+5)^2$
27 $(4y+3)^2$ **28** $(5k-2)^2$
29 $(6a-1)^2$ **30** $(7m+6)^2$

Exercse 112

1 $(y-1)^2$ **2** $(x+3)^2$ **3** $(m+5)^2$
4 $(t-2)^2$ **5** $(x-6)^2$ **6** $(2x+3)^2$
7 $(4b-1)^2$ **8** $(3a+2)^2$ **9** $(5x-4)^2$
10 $(7y+1)^2$ **11** $(3y-5)^2$ **12** $(4k-3)^2$
13 $(5x+1)^2$ **14** $(9a-2)^2$ **15** $(7m+6)^2$
16 $\left(t+\frac{1}{2}\right)^2$ **17** $\left(x-\frac{2}{3}\right)^2$ **18** $\left(3y+\frac{1}{5}\right)^2$
19 $\left(x+\frac{1}{x}\right)^2$ **20** $\left(5k-\frac{2}{k}\right)^2$

Exercse 113

1 $(a+2)(a-2)$ **2** $(x+3)(x-3)$
3 $(y+1)(y-1)$ **4** $(x+5)(x-5)$
5 $(2x+7)(2x-7)$ **6** $(4y+3)(4y-3)$
7 $(1+2z)(1-2z)$ **8** $(5t+1)(5t-1)$

9 $(3t+2)(3t-2)$
10 $(3+4x)(3-4x)$
11 $(x+2y)(x-2y)$
12 $(6x+y)(6x-y)$
13 $(2a+3b)(2-3b)$
14 $(x+10y)(x-10y)$
15 $(2a+9b)(2a-9b)$
16 $(x+2+y)(x+2-y)$
17 $(a+b-3)(a-b+1)$
18 $(z+w+1)(z-w-1)$
19 $\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$
20 $\left(\frac{y}{3}+1\right)\left(\frac{y}{3}-1\right)$
21 $(x+2y+3)(x-2y+1)$
22 $(x^2+1)(x-1)(x+1)$
23 $(3x^3+2y)(3x^3-2y)$
24 $(x^2+4y^2)(x+2y)(x-2y)$

Exercse 114

1 $4a(a+3)(a-3)$
2 $2(x+3)(x-3)$
3 $3(p+3)(p-4)$
4 $5(y+1)(y-1)$
5 $5(a-1)^2$
6 $3z(z+5)(z+4)$
7 $ab(3+2ab)(3-2ab)$
8 $x(x+1)(x-1)$
9 $2(3x-2)(x+2)$
10 $(y+5)(y+4)(y-4)$
11 $x(x+1)(x-1)(x+8)$
12 $(y^3+2)(y^3-2)$
13 $x(x+2)(x-5)$
14 $(x+3)(x-3)^2$
15 $y(2xy+1)(2xy-1)$
16 $6(2+b)(2-b)$
17 $3(3x-2)(2x+5)$
18 $3(x-1)^2$
19 $(x+2)(x+5)(x-5)$
20 $z(z+3)^2$
21 $3(y+5)^2$
22 $a(b+3)(b-3)$
23 $4k(k+5)^2$
24 $3(x+1)(x-1)(x+3)$
25 $2ab(a+2b)(2a-1)$

Exercse 115

1 $a+2$
2 $2t-1$
3 $\frac{4y+1}{3}$
4 $\frac{4}{2d-1}$
5 $\frac{x}{5x-2}$
6 $\frac{1}{y-4}$
7 $\frac{2(b-2a)}{a-3}$
8 $\frac{s-1}{s+3}$
9 b^2+1
10 $\frac{p+5}{3}$
11 $\frac{a+1}{a+3}$
12 $\frac{3+y}{x+2}$
13 $x-3$
14 $\frac{p-2}{p}$
15 $\frac{a+b}{2a-b}$

Exercse 116

1 **a** $\frac{5x}{4}$ **b** $\frac{13y+3}{15}$ **c** $\frac{a+8}{12}$
d $\frac{4p+3}{6}$ **e** $\frac{x-13}{6}$
2 **a** 6 **b** $\frac{5b(a-2)}{3}$ **c** $\frac{2(t+5)}{5y}$
d $\frac{5(a-3)}{4}$ **e** $\frac{5-y}{35}$ **f** $\frac{b}{2a-1}$
g $\frac{b^2(x+2y)}{10(2b-1)}$ **h** $\frac{xy}{ab}$
$\frac{(x-3)(x-1)}{(x-5)(x-2)}$ **j** $\frac{5(p-2)}{3(q+1)}$
3 **a** $\frac{5}{x}$ **b** $\frac{-x+2}{x(x-1)}$
c $\frac{a+b+3}{a+b}$ **d** $\frac{2x}{x+2}$
e $\frac{(p+q)(p-q)+1}{p+q}$ **f** $\frac{2(x-1)}{(x+1)(x-3)}$
g $\frac{-3x+8}{(x+2)(x-2)}$ **h** $\frac{a+2}{(a+1)^2}$
4 **a** $\frac{(y+2)(y+1)}{15y}$ **b** $\frac{x^2+10x-24}{2(x-3)(x-4)}$
c $\frac{3b^2-5b-10}{2b(b+1)}$ **d** x
5 **a** $\frac{3-5x}{(x+2)(x-2)}$ **b** $\frac{3p^2+5pq-2q^2}{pq(p+q)(p-q)}$
c $\frac{a^2-2ab-b^2+1}{(a+b)(a-b)}$

Exercse 117

1 **a** -71 **b** -69 **c** 481 **d** -377
e 06 **f** 23 **g** -53
2 $T=47$
3 $y=-7$
4 $h=375$
5 $v=-196$
6 $y=55$
7 $S=377$
8 $A=284$
9 $u=-40$
10 $V=51935$
11 $m=-1\frac{3}{4}$
12 $A=224$
13 $V=18388$
14 $v=\frac{3}{4}$
15 $S=15$
16 $c=10$
17 $y=\sqrt{12}=2\sqrt{3}$
18 $E=23987$
19 $A=35247$
20 $S=93$

Exercse 118

1 **a** $2\sqrt{3}$ **b** $3\sqrt{7}$ **c** $2\sqrt{6}$ **d** $5\sqrt{2}$
e $6\sqrt{2}$ **f** $10\sqrt{2}$ **g** $4\sqrt{3}$ **h** $5\sqrt{3}$
$4\sqrt{2}$ **j** $3\sqrt{6}$ **k** $4\sqrt{7}$ $10\sqrt{3}$

m $8\sqrt{2}$ **n** $9\sqrt{3}$ **o** $7\sqrt{5}$ **p** $6\sqrt{3}$
q $3\sqrt{11}$ **r** $5\sqrt{5}$

2 a $6\sqrt{3}$ **b** $20\sqrt{5}$ **c** $28\sqrt{2}$ **d** $4\sqrt{7}$
e $16\sqrt{5}$ **f** $8\sqrt{14}$ **g** $72\sqrt{5}$ **h** $30\sqrt{2}$
$14\sqrt{10}$ **j** $24\sqrt{5}$

3 a $\sqrt{18}$ **b** $\sqrt{20}$ **c** $\sqrt{176}$ **d** $\sqrt{128}$
e $\sqrt{75}$ **f** $\sqrt{160}$ **g** $\sqrt{117}$ **h** $\sqrt{98}$
$\sqrt{363}$ **j** $\sqrt{1008}$

4 a 45 **b** 12 **c** 63 **d** 50
e 44 **f** 147 **g** 304 **h** 828
775 **j** 960

Exercse 119

1 a $3\sqrt{5}$ **b** $\sqrt{2}$ **c** $6\sqrt{3}$
d $3\sqrt{3}$ **e** $-3\sqrt{5}$ **f** $3\sqrt{6}$
g $-7\sqrt{2}$ **h** $8\sqrt{5}$ $-4\sqrt{2}$
j $4\sqrt{5}$ **k** $\sqrt{2}$ $5\sqrt{3}$
m $-\sqrt{3}$ **n** $\sqrt{2}$ **o** $5\sqrt{7}$
p $\sqrt{2}$ **q** $13\sqrt{6}$ **r** $-9\sqrt{10}$
s $47\sqrt{3}$ **t** $5\sqrt{2}-2\sqrt{3}$ **u** $\sqrt{7}-5\sqrt{2}$
v $-2\sqrt{3}-4\sqrt{5}$

2 a $\sqrt{21}$ **b** $\sqrt{15}$ **c** $3\sqrt{6}$
d $10\sqrt{14}$ **e** $-6\sqrt{6}$ **f** 30
g $-12\sqrt{55}$ **h** 14 60
j $\sqrt{12}=2\sqrt{3}$ **k** 2 28
m $\sqrt{30}$ **n** $-2\sqrt{105}$ **o** 18

3 a $2\sqrt{6}$ **b** $4\sqrt{3}$ **c** 1
d $\frac{8}{\sqrt{6}}$ **e** $2\sqrt{3}$ **f** $\frac{1}{3\sqrt{10}}$
g $\frac{1}{2\sqrt{5}}$ **h** $\frac{1}{3\sqrt{5}}$ $\frac{1}{2}$
j $\frac{\sqrt{3}}{2\sqrt{2}}$ **k** $\frac{\sqrt{3}}{\sqrt{2}}$ $\frac{9}{2\sqrt{5}}$
m $\frac{\sqrt{5}}{2\sqrt{2}}$ **n** $\frac{2}{3}$ **o** $\frac{5}{7}$

4 a $\sqrt{10}+\sqrt{6}$ **b** $2\sqrt{6}-\sqrt{15}$
c $12+8\sqrt{15}$ **d** $5\sqrt{14}-2\sqrt{21}$
e $-\sqrt{6}+12\sqrt{2}$ **f** $5\sqrt{33}+3\sqrt{21}$
g $-6-12\sqrt{6}$ **h** $5-5\sqrt{15}$

$6+\sqrt{30}$ **j** $6\sqrt{6}+6$
k $-8+24\sqrt{3}$ $210-14\sqrt{15}$
m $10\sqrt{6}-120$ **n** $-\sqrt{10}-2\sqrt{2}$
o $4\sqrt{3}-12$

5 a $\sqrt{10}+3\sqrt{6}+3\sqrt{5}+9\sqrt{3}$
b $\sqrt{10}-\sqrt{35}-2+\sqrt{14}$
c $2\sqrt{10}-6+10\sqrt{15}-15\sqrt{6}$
d $24\sqrt{5}+36\sqrt{15}-8\sqrt{10}-12\sqrt{30}$
e $52-13\sqrt{10}$
f $15-\sqrt{15}+18\sqrt{10}-6\sqrt{6}$
g 4 **h** -1
-12 **j** 43
k 3 -241
m -6 **n** $7+2\sqrt{10}$
o $11-4\sqrt{6}$ **p** $25+6\sqrt{14}$
q $57+12\sqrt{15}$ **r** $27-4\sqrt{35}$
s $77-12\sqrt{40}=77-24\sqrt{10}$
t $53+12\sqrt{10}$

6 a 18 **b** $108\sqrt{2}$ **c** $432\sqrt{2}$
d $19+6\sqrt{2}$ **e** 9

7 a $a=21, b=80$ **b** $a=19, b=-7$

8 a $a-1$ **b** $2p-1-2\sqrt{p(p-1)}$

9 25 **10** $2x-3y-5\sqrt{xy}$

11 $a=17, b=240$ **12** $a=107, b=-42$

Exercse 120

1 a $\frac{\sqrt{7}}{7}$ **b** $\frac{\sqrt{6}}{4}$
c $\frac{2\sqrt{15}}{5}$ **d** $\frac{6\sqrt{14}}{10}=\frac{3\sqrt{14}}{5}$
e $\frac{\sqrt{3}+\sqrt{6}}{3}$ **f** $\frac{2\sqrt{3}-5\sqrt{2}}{2}$
g $\frac{5+2\sqrt{10}}{5}$ **h** $\frac{3\sqrt{14}-4\sqrt{7}}{14}$
$\frac{8\sqrt{5}+3\sqrt{10}}{20}$ **j** $\frac{4\sqrt{15}-2\sqrt{10}}{35}$

2 **a** $4\sqrt{3}-4\sqrt{2}=4\left(\sqrt{3}-\sqrt{2}\right)$

b $\frac{-\left(\sqrt{6}+7\sqrt{3}\right)}{47}$

c $\frac{-\left(2\sqrt{15}-4\sqrt{18}\right)}{19}=\frac{-2\left(\sqrt{15}-6\sqrt{2}\right)}{19}$

d $\frac{-\left(19-8\sqrt{3}\right)}{13}=\frac{8\sqrt{3}-19}{13}$

e $\sqrt{6}+2+5\sqrt{3}+5\sqrt{2}$

f $\frac{6\sqrt{15}-9\sqrt{6}+2\sqrt{10}-6}{2}$

3 **a** $2\sqrt{2}$

b $-2-\sqrt{6}+3\sqrt{2}-3\sqrt{3}$

c -4

d $4\sqrt{2}$

e $\frac{6+9\sqrt{2}+2\sqrt{3}}{6}$

f $\frac{4\sqrt{6}+9\sqrt{3}}{21}$

g $\frac{15\sqrt{30}-30\sqrt{5}-4\sqrt{3}}{30}$

h $\frac{28-2\sqrt{6}-7\sqrt{3}}{13}$

$\frac{2\sqrt{15}+2\sqrt{10}-2\sqrt{6}-\sqrt{3}-5}{2}$

4 **a** $a=45 \quad b=10$ **b** $a=1, b=8$

c $a=-\frac{1}{2} \quad b=\frac{1}{2}$ **d** $a=-1\frac{5}{9} \quad b=-\frac{8}{9}$

e $a=5, b=32$

5 3 so rationa.

6 **a** 4 **b** 14 **c** 16

Test yourslf 1

1 B C **2** D **3** A, D

4 C **5** C **6** B

7 A **8** D

9 **a** $\frac{1}{49}$ **b** $\frac{1}{5}$ **c** $\frac{1}{3}$

10 **a** x^9 **b** $25y^6$ **c** $a^{11}b^6$

d $\frac{8x^{18}}{27}$ **e** 1

11 **a** 6 **b** $\frac{1}{64}$ **c** 4

d $\frac{1}{7}$ **e** 2 **f** 1

12 **a** a^5 **b** $x^{30}y^{18}$ **c** p^9

d $16b^{36}$ **e** $8x^{11}y$

13 **a** $n^{\frac{}{2}}$ **b** x^{-5} **c** $(x+y)^{-}$

d $(x+1)^{\frac{}{4}}$ **e** $(a+b)^{\frac{}{7}}$ **f** $2x^{-}$

g $\frac{1}{2}x^{-3}$ **h** $x^{\frac{4}{3}}$ $(5x+3)^{\frac{9}{7}}$

j $m^{-\frac{3}{4}}$

14 **a** $\frac{1}{a^5}$ **b** $\sqrt[4]{n}$ **c** $\sqrt{x+1}$

d $\frac{1}{x-y}$ **e** $\frac{1}{(4t-7)^4}$ **f** $\sqrt[5]{a+b}$

g $\frac{1}{\sqrt[3]{x}}$ **h** $\sqrt[4]{b^3}$ $\sqrt[3]{(2x+3)^4}$

j $\frac{1}{\sqrt{x^3}}$

15 1 **16** $\frac{1}{192}$

17 **a** $x^{\frac{}{2}}$ **b** y^{-} **c** $(x+3)^{\frac{}{6}}$

d $(2x-3)^{-11}$ **e** $y^{\frac{7}{3}}$

18 **a** $\frac{1}{x^3}$ **b** $\frac{1}{2a+5}$ **c** $\left(\frac{b}{a}\right)^5$

19 **a** $-2y$ **b** $a+4$ **c** $-6k^5$

d $\frac{5x+3y}{15}$ **e** $3a-8b$ **f** $6\sqrt{2}$

g $4\sqrt{5}$

20 **a** $(x+6)(x-6)$ **b** $(a+3)(a-1)$

c $4ab(b-2)$ **d** $(y-3)(5+x)$

e $2(2n-p+3)$

21 **a** $4b-6$ **b** $2x^2+5x-3$

c $4m+17$ **d** $16x^2-24x+9$

e p^2-25 **f** $-1-7a$

g $2\sqrt{6}-5\sqrt{3}$ **h** $3\sqrt{3}-6+\sqrt{21}-2\sqrt{7}$

22 **a** $\frac{8}{b^2(a+3)}$ **b** $\frac{15}{(m-2)^2}$

23 $V = 157464$

24 **a** 17 **b** $\frac{6\sqrt{15}-9}{17}$

25 $\frac{4x+5}{(x+3)(x-2)}$

26 **a** 36 **b** -2 **c** 2
d 216 **e** 2

27 **a** $\frac{1}{\sqrt{5}}$ **b** 8

28 $d = 1125$

29 **a** $\frac{2\sqrt{3}}{15}$ **b** $\frac{\sqrt{2}+\sqrt{6}}{2}$

30 **a** $3\sqrt{6}-6-4\sqrt{3}+4\sqrt{2}$ **b** $11+4\sqrt{7}$

31 **a** $3(x-3)(x+3)$ **b** $6(x-3)(x+1)$
c $5(y-3)^2$

32 **a** $\frac{x^3}{3y^4}$ **b** $\frac{1}{3x-1}$

33 **a** 99 **b** $24\sqrt{3}$

34 **a** a^2-b^2 **b** $a^2+2ab+b^2$

35 **a** $(a-b)^2$ **b** $(a+b)(a-b)$

36 $\frac{3\sqrt{3}+1}{2}$

37 **a** $\frac{4b+3a}{ab}$ **b** $\frac{3x-11}{10}$

38 $\frac{21\sqrt{5}-46-\sqrt{2}}{7}$

39 **a** $6\sqrt{2}$ **b** $-8\sqrt{6}$ **c** $2\sqrt{3}$
d $\frac{4}{\sqrt{3}}$ **e** $30a^2b$ **f** $\frac{m}{3n^4}$
g $2x-3y$

40 **a** $2\sqrt{6}+4$
b $10\sqrt{14}-5\sqrt{21}-6\sqrt{10}+3\sqrt{15}$
c 7 **d** 43 **e** $65-6\sqrt{14}$

41 **a** $\frac{3\sqrt{7}}{7}$ **b** $\frac{\sqrt{6}}{15}$
c $\frac{\sqrt{5}+1}{2}$ **d** $\frac{12-2\sqrt{6}}{15}$
e $\frac{20+3\sqrt{15}+4\sqrt{10}+3\sqrt{6}}{53}$

42 **a** $\frac{x+10}{10}$ **b** $\frac{17a-15}{21}$
c $\frac{3-2x}{(x+1)(x-1)}$ **d** $\frac{1}{k-1}$
e $\frac{\sqrt{15}-\sqrt{6}-15\sqrt{3}-15\sqrt{2}}{3}$

43 **a** $n = 48$ **b** $n = 175$ **c** $n = 392$
d $n = 5547$ **e** $n = 1445$

Chaenge exercse 1

1 $\frac{1}{16}$

2 Proof involves $2 \times 2^{k+1} - 2$ (see worked solutions)

3 $-2^4 \times 3^5$

4 **a** $2a^2b - 8ab^2 + 6a^3$ **b** $y^4 - 4$
c $8x^3 - 60x^2 + 150x - 125$

5 $\frac{11\sqrt{3}+2\sqrt{5}+14}{11}$ **6** $\frac{1}{2\sqrt{2}}$ or $\frac{\sqrt{2}}{4}$

7 **a** $(x+4)(x+9)$ **b** $(x^2-3y)(x^2+2y)$
c $(b-2)(a+2)(a-2)$

8 $\frac{y+1}{2(x-1)}$ **9** $\frac{(a+1)^2}{a-1}$

10 $\left(\frac{2}{x}+\frac{a}{b}\right)\left(\frac{2}{x}-\frac{a}{b}\right)$

11 **a** $8x^3 - 12x^2 + 6x - 1$ **b** $\frac{3x+4}{(2x-1)^2}$

12 $r = \frac{3\sqrt{\pi}}{4\pi}$ **13** $s = 2 + 6\sqrt{3}$

14 **a** $x + x^2 + 2x^{\frac{3}{2}}$ **b** $a^{\frac{2}{3}} - b^{\frac{2}{3}}$
c $p^2 + p^{-} + 2p^{\overline{2}}$ **d** $x + x^{-} + 2$

15 4

Chapter 2

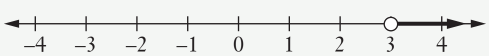
Exercse 201

1 $t = -5$ **2** $z = -56$ **3** $y = 1$
4 $w = 67$ **5** $x = 12$ **6** $x = 4$
7 $y = \frac{1}{15}$ **8** $b = 35$ **9** $n = -16$

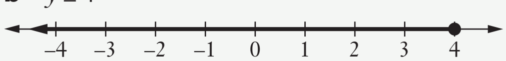
10	$r = 4$	**11**	$y = 9$	**12**	$k = 6$
13	$d = 2$	**14**	$x = 5$	**15**	$y = 15$
16	$x = 20$	**17**	$m = 20$	**18**	$x = 4$
19	$a = -7$	**20**	$y = 3$	**21**	$x = 3$
22	$a = -1\frac{2}{3}$	**23**	$t = -4$	**24**	$x = 1.2$
25	$a = 16$	**26**	$b = \frac{1}{8}$	**27**	$t = 39$
28	$p = 5$	**29**	$x \approx 441$	**30**	$b = 3\frac{}{3}$
31	$x = 1\frac{9}{35}$	**32**	$x = 36$	**33**	$x = -3$
34	$y = -12$	**35**	$x = 69$	**36**	$w = 13$
37	$t = 30$	**38**	$x = 14$	**39**	$x = -1$
40	$x = -04$	**41**	$p = 3$	**42**	$t = 82$
43	$x = -95$	**44**	$q = 22$	**45**	$x = -3$

Exercse 202

1 a $x > 3$

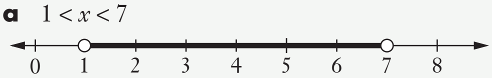


b $y \le 4$

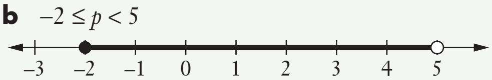


2 a $t > 7$ **b** $x \ge 3$ **c** $p > -1$
d $x \ge -2$ **e** $y > -9$ **f** $a \ge -1$
g $y \ge -2\frac{}{2}$ **h** $x < -2$ $a \le -6$
j $y < 12$ **k** $b < -18$ $x > 30$

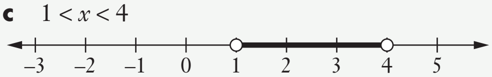
3 a $1 < x < 7$



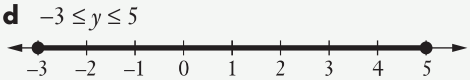
b $-2 \le p < 5$



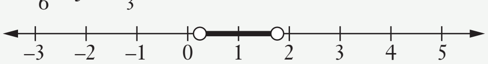
c $1 < x < 4$



d $-3 \le y \le 5$



e $\frac{1}{6} < y < 1\frac{2}{3}$



Exercse 203

1 a 7 **b** 5 **c** 6 **d** 0
e 2 **f** 11 **g** 6 **h** 24
25 **j** 125

2 a 5 **b** –1 **c** 2 **d** 14
e 4 **f** –67 **g** 7 **h** 12
–6 **j** 10

3 a 3 **b** 3 **c** 1
d 3 **e** 1

4 a a **b** $-a$ **c** 0
d $3a$ **e** $-3a$ **f** 0
g $a + 1$ **h** $-a - 1$ $x - 2$

5 a $6 \le 6$ **b** $3 \le 3$ **c** $1 \le 5$
d $1 \le 9$ **e** $10 \le 10$

6 See worked solutions

7 a $x + 5$ for $x \ge -5$ and $-x - 5$ for $x < -5$
b $b - 3$ for $b \ge 3$ and $3 - b$ for $b < 3$
c $a + 4$ for $a \ge -4$ and $-a - 4$ for $a < -4$
d $2y - 6$ for $y \ge 3$ and $6 - 2y$ for $y < 3$
e $3x + 9$ for $x \ge -3$ and $-3x - 9$ for $x < -3$
f $4 - x$ for $x \le 4$ and $x - 4$ for $x > 4$
g $2k + 1$ for $k \ge -\frac{1}{2}$ and $-2k - 1$ for $k < -\frac{1}{2}$
h $5x - 2$ for $x \ge \frac{2}{5}$ and $-5x + 2$ for $x < \frac{2}{5}$
$a + b$ for $a \ge -b$ and $-a - b$ for $a < -b$

8 ±3 **9** ±1 **10** $\pm 1, x \ne 2$

Exercse 204

1 a $x = \pm 5$ **b** $y = \pm 8$ **c** $x = 0$

2 a $x = 5, -9$ **b** $n = 4, -2$ **c** $x = 3, -6$
d $x = 5, -4\frac{5}{7}$ **e** $x = \pm 12$

3 a $x = 2, -075$ **b** $n = 1\frac{}{3}$ 2 **c** $t = 24$ –4
d $y = -6$ 15 **e** $x = -3, 1\frac{2}{3}$

Exercse 205

1 a $n = 4$ **b** $y = 5$ **c** $m = 9$ **d** $x = 5$
e $m = 0$ **f** $x = 3$ **g** $x = 2$ **h** $x = 2$
$x = 1$ **j** $k = 2$

2 **a** $x=2$ **b** $x=1$ **c** $x=-2$ **d** $n=2$
e $x=0$ **f** $x=6$ **g** $y=\frac{1}{3}$ **h** $x=2$
$x=2$ **j** $a=0$

3 **a** $m=\frac{1}{2}$ **b** $x=\frac{1}{3}$ **c** $x=\frac{1}{3}$
d $k=-\frac{1}{2}$ **e** $k=-\frac{2}{3}$ **f** $n=\frac{3}{4}$
g $x=1\frac{}{2}$ **h** $n=\frac{2}{3}$ $k=-\frac{1}{6}$
j $x=1\frac{2}{3}$

4 **a** $x=-1$ **b** $x=-1\frac{}{3}$ **c** $k=-4$
d $n=3$ **e** $x=-2\frac{}{2}$ **f** $x=-\frac{2}{3}$
g $x=-4\frac{}{2}$ **h** $x=-1\frac{7}{11}$ $x=1\frac{4}{5}$

5 **a** $m=\frac{1}{4}$ **b** $k=-2\frac{3}{4}$ **c** $x=2\frac{3}{8}$
d $k=1\frac{}{2}$ **e** $n=\frac{1}{18}$ **f** $n=-\frac{1}{2}$
g $x=\frac{4}{5}$ **h** $b=-3\frac{}{6}$ $x=-1\frac{}{7}$

6 **a** $x=3$ **b** $y=\pm 8$ **c** $n=\pm 2$
d $x=\pm 2\sqrt{5}$ **e** $p=10$ **f** $x=\pm 5$
g $y=\pm 3$ **h** $w=2$ $n=\pm 4$

7 **a** $p=\pm 671$ **b** $x=464$ **c** $n=299$
d $x=\pm 592$ **e** $y=189$ **f** $d=\pm 255$
g $k=\pm 447$ **h** $x=222$ $y=\pm 381$

8 **a** $x=\frac{1}{5}$ **b** $a=\frac{1}{2}$ **c** $y=\frac{1}{2}$
d $x=\pm\frac{1}{7}$ **e** $n=\frac{2}{3}$ **f** $a=2$
g $x=\pm 2$ **h** $b=9$ $x=\pm\frac{2}{3}$
j $b=\pm 1\frac{}{2}$

Puzze

1 All months have 28 days Some months have more days as well
2 10
3 Bottle \$105 cork 5 cents
4 16 each time
5 Friday

Exercse 206

1 $y=0,-1$
2 $b=2,-1$
3 $p=3,-5$
4 $t=0,5$
5 $x=-2,-7$
6 $q=\pm 3$
7 $x=\pm 1$
8 $a=0,-3$
9 $x=0,-4$
10 $x=\pm\frac{1}{2}$
11 $x=-1,-1\frac{}{3}$
12 $y=1,-1\frac{}{2}$
13 $b=\frac{3}{4}\ \frac{1}{2}$
14 $x=5,-2$
15 $x=0,\frac{2}{3}$
16 $x=1,2\frac{}{2}$
17 $x=0,5$
18 $y=-1,2$
19 $n=3,5$
20 $x=3,4$
21 $m=-6\ 1$
22 $x=0,-1,-2$
23 $y=1,-5,-2$
24 $x=5,-7$
25 $m=8,-1$

Exercse 207

1 **a** $x=\pm\sqrt{7}-1$ **b** $y=\pm\sqrt{5}-5$
c $a=\pm\sqrt{6}+3$ **d** $x=\pm\sqrt{13}+2$
e $y=\frac{\pm\sqrt{2}-3}{2}$

2 **a** $h=19,\ -59$ **b** $a=38,\ -18$
c $x=8.1,-01$ **d** $y=-24\ \ -116$
e $x=1.5,-08$

3 **a** $x=\pm\sqrt{5}-2$ **b** $a=\pm\sqrt{7}+3$
c $y=\pm\sqrt{23}+4$ **d** $x=\pm\sqrt{13}-1$
e $p=\pm\sqrt{44}-7=\pm 2\sqrt{11}-7$
f $x=\pm\sqrt{28}+5=\pm 2\sqrt{7}+5$
g $x=\pm\sqrt{88}-10=\pm 2\sqrt{22}-10=2\left(\pm\sqrt{22}-5\right)$
h $x=\pm\sqrt{2}+1$
$n=\pm\sqrt{137}-12$

4 **a** $x=345\ \ -145$ **b** $x=-459\ \ -741$
c $q=00554\ \ -181$ **d** $x=445\ \ -0449$
e $b=-426\ \ -117$ **f** $x=177,.34$
g $r=223,\ -0314$ **h** $x=-0683\ \ -732$
$a=0162\ \ -616$

Exercse 208

1 **a** $y = -0354 \quad -565$ **b** $x = 1, 1.5$
c $b = 354 \quad -254$ **d** $x = 1, -05$
e $x = -0553\ .678$ **f** $n = 0243 \quad -824$
g $m = -2, -5$ **h** $x = 0, 7$
$x = 1, -6$

2 **a** $x = \frac{-1 \pm \sqrt{17}}{2}$ **b** $x = \frac{5 \pm \sqrt{13}}{6}$
c $q = 2 \pm \sqrt{7}$ **d** $h = \frac{-3 \pm 2\sqrt{2}}{2}$
e $s = \frac{4 \pm \sqrt{10}}{3}$ **f** $x = \frac{-11 \pm \sqrt{133}}{2}$
g $d = \frac{-5 \pm \sqrt{73}}{12}$ **h** $x = 1 \pm 2\sqrt{2}$
$t = \frac{1 \pm \sqrt{5}}{2}$

Exercse 209

1 $t = 85$ **2** $l = 122$
3 $b = 8$ **4** $a = 41$
5 $y = 4$ **6** $r = 668$
7 $x = 644$ **8** $n = 15$
9 $y \ = 3\frac{2}{3}$ **10** $h = 37$
11 **a** BMI $= 2539$ **b** $w = 6966$
c $h = 194$
12 $r = 0072$ **13** $x \ = -9$
14 $t = 214$ **15** $x = \pm 2$
16 $r = 212$ **17** $r = 1046$
18 $x = 119$ **19** $x = 55$
20 $r = 33$

Exercse 210

1 $a = 1, b = 3$ **2** $x = 2, y = 1$
3 $p = 2, q = -1$ **4** $x = 6, y = 17$
5 $x = -10 \ y = 2$ **6** $t = 3, v = 1$
7 $x = -3, y = 2$ **8** $x = -64 \ y = -39$
9 $x = 3, y = -4$ **10** $m = 2, n = 3$
11 $w \ = -1, w_2 = 5$ **12** $a = 0, b = 4$
13 $p = -4, q = 1$ **14** $x \ = 1, x_2 = -1$
15 $x = -1, y = -4$ **16** $s = 2, t = -1$
17 $a = -2, b = 0$ **18** $k = -4, h = 1$

Probem

23 adults and 16 children

Exercse 211

1 $x = 0, y = 0$ and $x = 1, y = 1$
2 $x = 0, y = 0$ and $x = -2, y = 4$
3 $x = 0, y = 3$ and $x = 3, y = 0$
4 $x = 4, y = -3$ and $x = 3, y = -4$
5 $x = -1 \ y = -3$ **6** $x = 3, y = 9$
7 $t = -2, x = 4$ and $t = 1, x = 1$
8 $m = -4, n = 0$ and $m = 0, n = -4$
9 $x = 1, y = 2$ and $x = -1, y = -2$
10 $x = 0, y = 0$ and $x = 1, y = 1$
11 $x = 2, y = 1$ and $x = -1, y = -2$
12 $x = 0, y = 1$
13 $x = 1, y = 5$ and $x = 4, y = 11$
14 $x = \frac{1}{4} \ y = 4$ and $x = -1, y = -1$
15 $t = -\frac{1}{2} \ h = \frac{1}{4}$ **16** $x = 2, y = 0$
17 $x = 0, y = 0$ and $x = -2, y = -8$ and $x = 3, y = 27$
18 $x = 0, y = 0$ and $x = 1, y = 1$ and $x = -1, y = 1$
19 $x = \frac{1}{2} \ y = 2\frac{3}{4}$ **20** $x = -\frac{5}{13} \ y = -\frac{12}{13}$

Exercse 212

1 $x = -2, y = -8, z = -1$ **2** $a = -2, b = -1, c = 2$
3 $a = -4, b = 2, c = 7$ **4** $a = 1, b = 2, c = -3$
5 $x = 5, y = 0, z = -2$ **6** $p = -3, q = 7, r = 4$
7 $x = 1, y = -1, z = 2$ **8** $x = 0, y = -5, z = 4$
9 $h = -3, j = 2, k = -4$ **10** $a = 3, b = -1, c = -2$

Test yoursl f 2

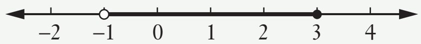
1 C **2** A, D
3 B
4 **a** $b = 10$ **b** $a = -116$
c $x = -7$ **d** $p \le 4$
5 **a** $A = 126248$ **b** $P = 855859$
6 **a** $x = -2, y = 5$
b $x = 4, y = 1$ and $x = -\frac{1}{2} \ y = -8$
7 **a** $x = 2$ **b** $y = \frac{1}{4}$

8 $b = 2, -1\frac{}{3}$

9 **a** $A = 36$ **b** $b = 12$

10 $x = \frac{1}{2}, 1$

11 $-1 < y \leq 3$



12 **a** $x = -0298 \quad -670$ **b** $y = 416 \quad -216$

c $n = 0869 \quad -154$

13 **a** $A = 7645$ **b** $r = 29$

14 $x = 24 \quad y = 32$

15 **a** $V = 2100$ **b** $r = 39$

16 **a** B **b** A **c** A

d C **e** B

17 $a = 3, b = 2, c = -4$

18 $x = -2$

19 **a** $y > 3$ **b** $x = 2$

c $x = 2$ **d** $x = 3, -1\frac{2}{5}$

e $x = -3$ **f** $x = \frac{5}{6}$

Chaenge exercse 2

1 $y = 1$ **2** $a = 3, b = \pm 2$

3 **a** $(x + 3)(x - 3)(x^3 - 8)$

b $x = \pm 3, 2$

4 $x = 1, y = 2$ and $x = -1, y = 0$

5 $b = 16, x = 4 \pm \sqrt{17}$

6 $x = 1$ **7** $x = \frac{1}{4}$

8 $x = \pm\sqrt{b + a^2} + a$

9 $x = \frac{2\left(4 \pm \sqrt{10}\right)}{3}$ **10** $x = 2, -4$

Practice set 1

1 B **2** A **3** B **4** B

5 D **6** D

7 **a** $x = 10$ **b** $b = 6$ **c** $x = 3$

d $y = 16$ **e** $z = 6$ **f** $x = 5$

g $y = 1.5$ **h** $x = 0, 3$ $x = 3, -7$

j $a = 2, -12$

8 $p = 9$ **9** $4\sqrt{3}$

10 $2(5 + y)(x - y)$

11 **a** x^{-} **b** $x^{\frac{4}{3}}$

12 $6y - 10$ **13** $\frac{25 + 5\sqrt{2}}{23}$

14 $x = 178, \ -0281$

15 $\frac{2}{x - 3}$ **16** 2313

17 $-\sqrt{3}$

18 $x^3 + 2x^2 - 16x + 3$

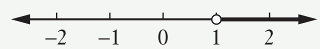
19 $3\sqrt{10} - 4$

20 $x + 3$ **21** $a > -3$

22 $r = 155$

23 **a** $a = 3, b = 2$ **b** $a = 3, b = 5, c = 0$

24 $x > 1$



25 $x = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$ **26** $\frac{1}{49}$

27 $x = 4, y = 11$ or $x = -1, y = -4$

28 $x = 2, y = -1$

29 7 **30** $8(x + 2)(x - 2)$

31 $\frac{6\sqrt{15} + 2\sqrt{6}}{43}$ **32** 7

33 $-2\sqrt{10} + 3\sqrt{5} - 2\sqrt{2} + 3$ **34** $a^{-21}b^{0} = \frac{b^{0}}{a^{21}}$

35 $\frac{1}{8}$

36 $-x - 7$ **37** $x = \frac{1}{4}$

38 $(x + 3)^{-}$ **39** $\frac{5^4}{2^{17}}$

40 $\frac{1}{\sqrt{3x + 2}}$

41 **a** $12x - 8y$ **b** $2\sqrt{31}$

c $\frac{x - 3}{2x - 1}$ **d** $3\sqrt{2} + 1$

e $\frac{-(x + 5)}{(x + 1)(x - 1)}$ **f** $\frac{11\sqrt{3}}{6}$

g $x^{-4}y^7z^{-11}$ or $\frac{y^7}{x^{14}z^{11}}$ **h** $\frac{3}{5a(a + b)(1 + 2b)}$

$8\sqrt{5}$ **j** $13\frac{}{2}$

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42 $r = \frac{2}{\sqrt[3]{\pi}}$ cm

43 $k = 20$

44 $9xy\sqrt{y}$

45 **a** $5(a-2)(a+6)$ **b** $(3a+4b)(a-6b+2c)$

46 $-1\frac{}{8} \le x < 5\frac{3}{4}$

47 $\frac{-2x-7}{15}$

48 $x = 0, 5$

49 $x = 519, \ -019$

50 $9\sqrt{2}$

51 $\frac{-2x-17}{x(x+5)}$

52 $x = -\frac{1}{4}$

53 **a** $(x-4)(x+2)$ **b** $(a+3)(a-3)$
c $(y+3)^2$ **d** $(t+4)^2$
e $(3x-2)(x-3)$

54 **a** $x = 5$ **b** $y = 474 \quad -274$
c $x = 075$ **d** $b = 2, -5$

Chapter 3

Exercse 301

1 **a** (Wad, blac), (Sctt, blnd), (eoff,grey), (Deng black, (Mia, bron), (Stvie, bond); many-to-one
b (1, 1), (1, 4), (2, 3), (3, 1), (4, 4); many-to-many
c (1),(2, D) (, A), (4 B, (5, C); many-to-one
d (3 5, 5, −2) (,),(8, −7) (,),(5, 6) (, 0); one-to-many
e (1, 9), (2, 15), (3, 27), (4, 33), (5, 45); one-to-one

2 **a** Yes (one-to-one) **b** No
c No **d** Yes
e Yes **f** Yes
g No **h** Yes
Yes **j** No
k Yes No
m Yes (one-to-one) **n** No
o Yes

3 **a** $\{-3, -1, 0, 1, 6\}$ **b** $\{-2, 4, 5, 8\}$
c many-to-one **d** Yes

Exercse 302

1 $f(1) = 4, f(-3) = 0$

2 $h(0) = -2, h(2) = 2, h(-4) = 14$

3 $f(5) = -25 \ \ f(-1) = -1, f(3) = -9, f(-2) = -4$

4 14 **5** −35

6 $x = 9$ **7** $x = \pm 5$

8 $x = -3$ **9** $z = 1, -4$

10 $f(p) = 2p - 9, f(x+h) = 2x + 2h - 9$

11 $g(x-1) = x^2 + 2$ **12** $f(k) = (k-1)(k+1)$

13 **a** $t = 1$ **b** $t = 4, -2$

14 0

15 $f(5) = 125, f(1) = 1, f(-1) = -1$

16 −2 **17** 10

18 7 **19** −28

20 $f(x+h) - f(x) = 2xh + h^2 - 5h$

21 $4x + 2h + 1$

22 $5(x-c)$ **23** $3k^2 + 5$

24 **a** 2 **b** 0 **c** $n^4 + n^2 + 2$

25 **a** 3
b Denominator cannot be 0 so $x - 3 \ne 0, \therefore x \ne 3$.
c 4

Exercse 303

1 **a** x-intercept $\frac{2}{3}$ y-intercept −2
b x-intercept −10 y-intercept 4
c x-intercept 12 y-intercept 4
d x-intercepts 0 −3, y-intercept 0
e x-intercepts ±2, y-intercept −4
f x-intercepts −2, −3, y-intercept 6
g x-intercepts 3 , y-intercept 15
h x-intercept $-\sqrt[3]{5}$ y-intercept 5
x-intercept −3 no y-intercept
j x-intercepts ±3, y-intercept 9

2 **a** $x = 2$
b x-intercept 2 y-intercept −6

3 $f(-x) = (-x)^2 - 2 = x^2 - 2 = f(x)$ so even

4 **a** $f(x^2) = x^6 + 1$
b $f(x)^{\ 2} = x^6 + 2x^3 + 1$
c $f(-x) = -x^3 + 1$
d Neither odd nor even
e $x = -1$
f x-intercept −1, y-intercept 1

5 $g(-x) = (-x)^8 + 3(-x)^4 - 2(-x)^2 = x^8 + 3x^4 - 2x^2$
$= g(x)$ so even

6 $f(-x) = -x = -f(x)$ so odd

7 $f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$ so even

8 $f(-x) = 4(-x) - (-x)^3 = -4x + x^3 = -(4x - x^3)$
$= -f(x)$ so odd

9 **a** $f(-x) = (-x)^4 + (-x)^2 = x^4 + x^2 = f(x)$ so even
b 0

10 **a** Odd **b** Neither
c Even **d** Neither
e Neither

11 **a** Even values .. $n = 2, 4, 6, \ldots$
b Odd values .. $n = 1, 3, 5, \ldots$

12 **a** No value of n
b Ye, when n is odd (1, 3, 5, …)

13 **a** $(0\ \infty)$ **ii** $(-\infty\ 0)$ **iii** Even
b $(-\infty\ 2)$ **ii** $(2\ \infty)$ **iii** Neither
c $(-2\ 2)$ **ii** $(-\infty\ -2) \cup (2\ \infty)$
iii Neither
d $(-\infty\ 0) \cup (0\ \infty)$
ii None **iii** Odd
e None **ii** $(-\infty\ \infty)$ **iii** Neither

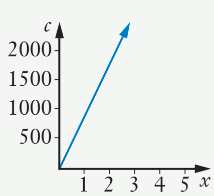
14 **a** Domain $(-\infty\ \infty)$ range $[,\ \ \infty)$
b Domain $(-\infty\ \infty)$ range $(-\infty\ \infty)$
c Domain $[0\ \infty)$ range $[,\ \ \infty)$
d Domain $[-5, \infty)$ range $[,\ \ \infty)$
e Domain $[3\ \infty)$ range $(-\infty\ 0]$

15 **a** 1 **b** 49 **c** $x = 2$
d x-intercept 2 y-intercept 4
e Domain $(-\infty\ \infty$, range $0, \infty)$
f $(-x - 2)^2$ **g** Neither

Exercse 304

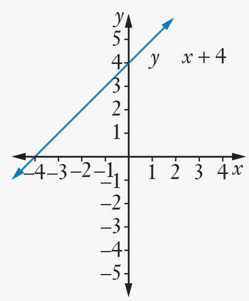
1 **a** $N = 12x$ **b** $A = 2n$ **c** $c = 1.5x$
d $y = 4x$ **e** $w = 400x$

2 $c = 850x$

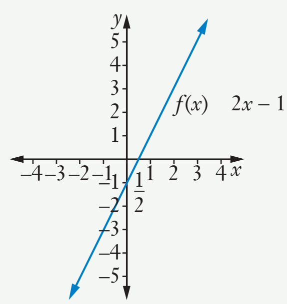


3 **a** x-intercept 2 y-intercept -2
b x-intercept -3, y-intercept 9
c x-intercept 2 y-intercept 4
d x-intercept $-1\frac{}{2}$ y-intercept 3
e x-intercept $\frac{4}{5}$ y-intercept -4
f x-intercept $-\frac{1}{2}$ y-intercept 5
g x-intercept 2 y-intercept 2
h x-intercept -2, y-intercept 4
x-intercept -3, y-intercept 3
j x-intercept $\frac{2}{3}$ y-intercept $-\frac{1}{3}$

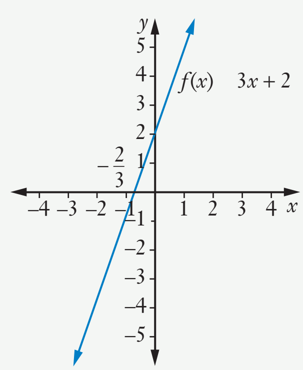
4 **a**



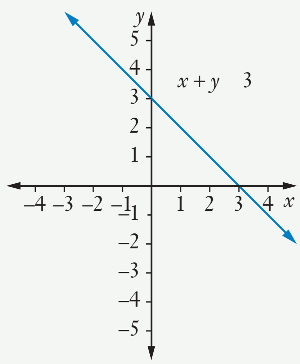
b



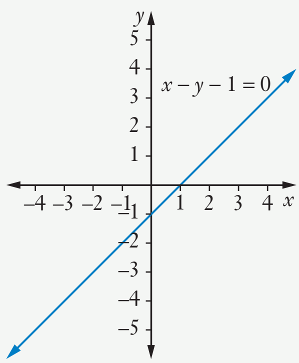
c



d



e



5 **a** Domain $(-\infty \quad \infty)$ range $(-\infty \quad \infty)$

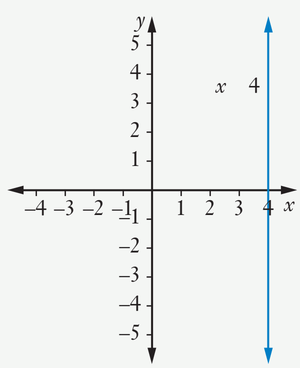
b Domain $(-\infty \quad \infty)$ range $[2]$

c Domain $[-4]$ range $(-\infty \quad \infty)$

d Domain $[2]$ range $(-\infty \quad \infty)$

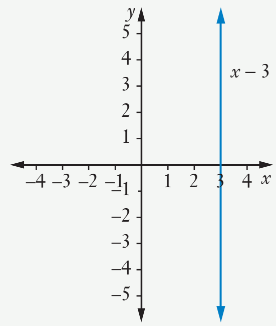
e Domain $(-\infty \quad \infty)$ range $[3]$

6 **a**



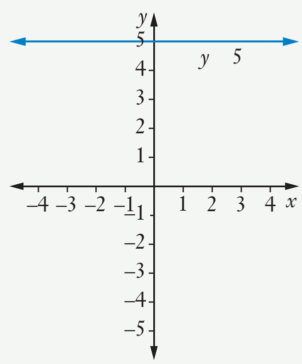
Domain $[4]$ range $(-\infty \quad \infty)$

b



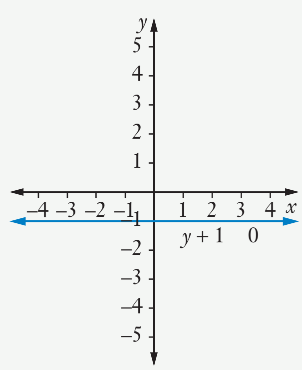
Domain $[3]$ range $(-\infty \quad \infty)$

c



Domain $(-\infty \quad \infty)$ range $[5]$

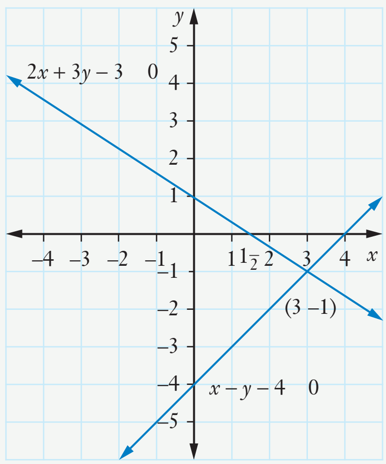
d



Domain $(-\infty \quad \infty)$ range $[-1]$

7 **a** $N = 36x$ **b** 1008 **c** 17

8



Exercse 305

1 **a** 2 **b** $1\frac{}{3}$ **c** $-1\frac{}{3}$

d $-2\frac{2}{5}$ **e** $\frac{2}{3}$ **f** $-\frac{1}{8}$

g $-4\frac{}{2}$ **h** $-\frac{2}{3}$ $2\frac{}{4}$

2 **a** 05 **b** 71 **c** 25

d –57 **e** –12 **f** –03

3 **a** 3 **ii** 5

b 2 **ii** 1

c 6 **ii** –7

d –1 **ii** 0

e –4 **ii** 3

f 1 **ii** –2

g –2 **ii** 6

h –1 **ii** 1

9 **ii** 0

4 **a** $\frac{1}{3}$ **b** $-1\frac{}{3}$ **c** –2

d $\frac{3}{5}$ **e** $\frac{1}{3}$

5 **a** 63°26 **b** 59°32 **c** 80°32

d 101°19 **e** 139°38 **f** 129°48

6 **a** –2 **ii** 3

b –5 **ii** –6

c 6 **ii** –1

d 1 **ii** 4

e –2 **ii** $\frac{1}{2}$

f 3 **ii** $1\frac{}{2}$

g $-\frac{1}{3}$ **ii** –2

h $-\frac{4}{5}$ **ii** 2

$3\frac{}{2}$ **ii** $-\frac{1}{2}$

7 **a** –2 **b** 0 **c** –1

d –3 **e** 2 **f** $-\frac{1}{4}$

g $1\frac{}{2}$ **h** $1\frac{}{4}$ $\frac{2}{3}$

j $\frac{1}{5}$ **k** $\frac{2}{7}$ $-\frac{3}{5}$

m $-\frac{1}{14}$ **n** 15 **o** $-1\frac{}{2}$

8 y = 21 **9** $x = 1.8$ **10** $x = 9$

11 **a** $P = \frac{d}{3}$ **b** 280 **c** $171

12 $m_{AB} = m_{CD} = 1.5, m_{AD} = m_{BC} = 0$ Opposite sides are parallel so $ABCD$ is a parallelogram

Exercse 306

1 **a** $y = 4x - 1$ **b** $y = -3x + 4$ **c** $y = 5x$

d $y = 4x + 20$ **e** $3x + y - 3 = 0$

f $4x - 3y - 12 = 0$

2 **a** $4x - 3y + 7 = 0$ **b** $3x - 4y + 4 = 0$

c $4x - 5y + 13 = 0$ **d** $3x + 4y - 25 = 0$

e $x - 2y + 2 = 0$

3 **a** –4 **b** $y = -4x + 8$

4 **a** $y = 3$ **b** $x = -1$

5 **a** 0 **b** $y = -2x$

6 **a** $P = 15x + 20$

b 380 **ii** 845 **iii** 3725

c 145 **ii** 512 **iii** 843

7 **a** $c = 250n + 7000$

b $32 000 **ii** $74 500 **iii** $307 000

c 180 **ii** 285 **iii** 1440

d 10

8 **a** $A = 450 - 8h$

b 426 L **ii** 258 L

c 5625 h or 2 days 8 hours 15 minutes

9 **a** $C = 20 - 169\ x$ **b** 11 songs

10 **a** $A = 20\ 000 - 320x$

b \$18 400 **ii** \$16 160 **iii** \$800

c 63 months or 5 years 3 months

11 **a** $P = 5x - 100$ **b** \$1400

c 231 **d** 20 balloons

Exercse 307

1 **a** -3 **b** $\frac{1}{3}$ **c** $\frac{3}{4}$

d $1\frac{}{2}$ **e** 1 **f** $-\frac{5}{6}$

g $\frac{1}{3}$ **h** $\frac{1}{5}$

2 **a** $x - y + 1 = 0$ **b** $x - 3y + 16 = 0$

c $x + y - 5 = 0$ **d** $x + 2y + 5 = 0$

e $x - 2y + 4 = 0$ **f** $x + 3y - 1 = 0$

g $3x + 4y + 13 = 0$ **h** $x + y = 0$

$5x - y - 8 = 0$ **j** $2x + y + 2 = 0$

k $2x - 3y + 16 = 0$

3 $m\ = m_2 = 3$ so parallel

4 $m\ m_2 = -\frac{1}{5} \times 5 = -1$ so perpendicular

5 $m\ = m_2 = 1\frac{}{5}$ so parallel

6 $m\ m_2 = -\frac{7}{3} \times \frac{3}{7} = -1$ so perpendicular

7 $k = -\frac{2}{3}$

8 $m\ = m_2 = 4$ so parallel

9 $AB\ \ CD$ ($m\ = m_2 = 3$) and $BC\ \ AD$

$\left(m_1 = m_2 = -\frac{5}{8}\right)$ So $ABCD$ is a parallelogram

10 Gradient $AC = m\ = \frac{1}{2}$ gradient $BD = m_2 = -2$.

$m\ m_2 = \frac{1}{2} \times -2 = -1$ so diagonals are perpendicular.

11 $7x + 6y - 24 = 0$

Exercse 308

1 **a** x-intercepts 0 -2, y-intercept 0

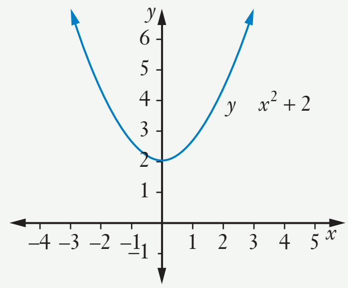
b x-intercepts 0 , y-intercept 0

c x-intercepts ± 1, y-intercept -1

d x-intercepts -1, 2, y-intercept -2

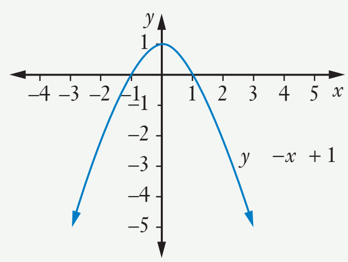
e x-intercepts 1 , y-intercept 8

2 **a**



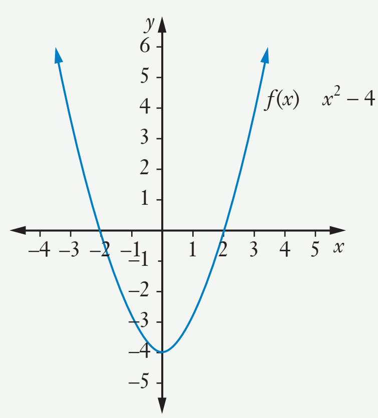
Minimum 2

b



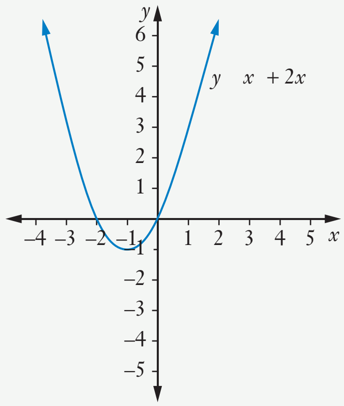
Maximum 1

c



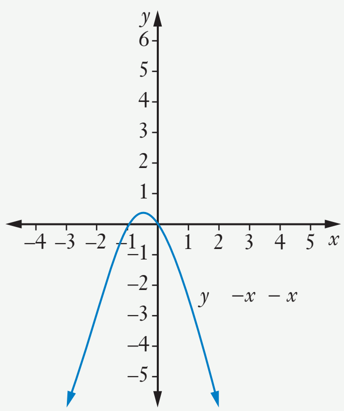
Minimum -4

d



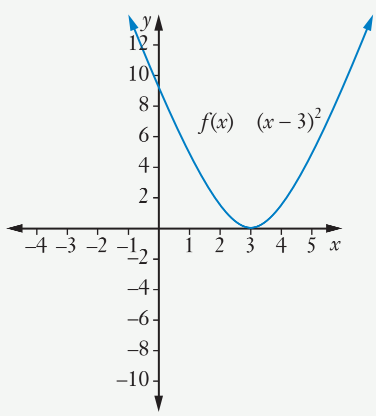
Minimum –1

e



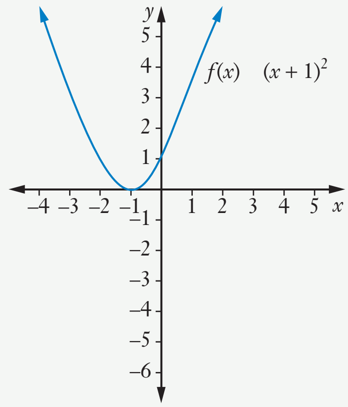
Maximum 025

f



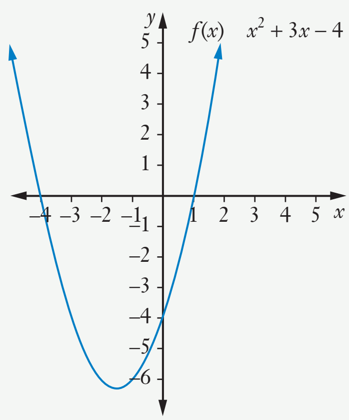
Minimum 0

g

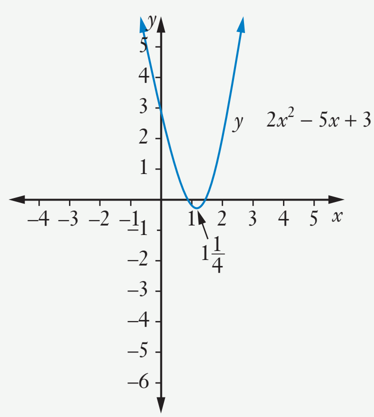


Minimum 0

h

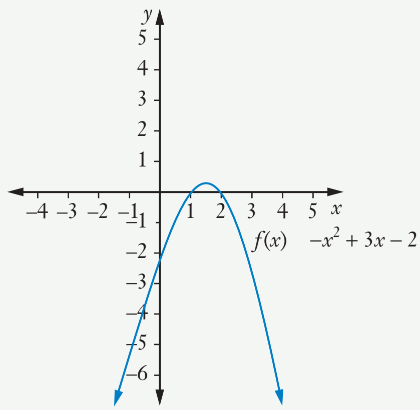


Minimum –625



Minimum –0125

j



Maximum 025

3 **a** x-intercepts 3 , y-intercept 12

ii Domain (−∞ ∞) range $[-\frac{1}{4}$ ∞)

b x-intercepts 0 −4, y-intercept 0

ii Domain (−∞ ∞) range [−4, ∞)

c x-intercepts −2, , y-intercept −8

ii Domain (−∞ ∞) range [−9, ∞)

d x-intercept 3 y-intercept 9

ii Domain (−∞ ∞) range [, ∞)

e x-intercepts ±2, y-intercept 4

ii Domain (−∞ ∞) range (−∞ 4]

4 **a** Domain (−∞ ∞) range [−5, ∞)

b Domain (−∞ ∞) range [−9, ∞)

c Domain (−∞ ∞) range $\left[-2\frac{}{4}, \infty\right)$

d Domain (−∞ ∞) range (−∞ 0]

e Domain (−∞ ∞) range [0 ∞)

5 **a** $d = 6$ m **b** $w = 28$ m

Exercse 309

1 Axis of symmetry $x = -1$ minimum value −1

2 Axis of symmetry $x = 0$ minimum value −4

3 Axis of symmetry $x = \frac{3}{8}$ minimum turning point $\left(\frac{3}{8}\ \frac{7}{16}\right)$

4 Axis of symmetry $x = 1$ maximum value −6

5 Axis of symmetry $x = -1$ vertex (−1 7)

6 Axis of symmetry $x = -15$ minimum value −025

7 **a** $x = -3$, (−3, −12) **b** $x = -4$, (−4 17)

c $x = -3$, (−3, −23) **d** $x = 1\frac{}{4}$ $\left(1\frac{}{4}, 3\frac{}{8}\right)$

e $x = -1\frac{}{4}$ $\left(-1\frac{}{4}, -13\frac{}{4}\right)$

8 **a** $x = -1$ **ii** −3 **iii** (−1, −3)

b $x = 1$ **ii** 1 **iii** (1, 1)

9 **a** (−1 0, min **b** (4 −23) min

c (−2, −7) min **d** (1 −1) min

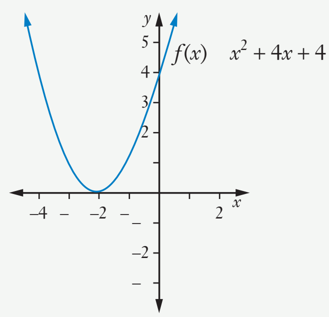
e (2 −11) min **f** $\left(-\frac{1}{4}, -3\frac{1}{8}\right)$ min

g (−1 6, max **h** (2 11, max

$\left(\frac{1}{2}\ 7\ \frac{3}{4}\right)$ max

10 **a** −2 **ii** Minimum 0

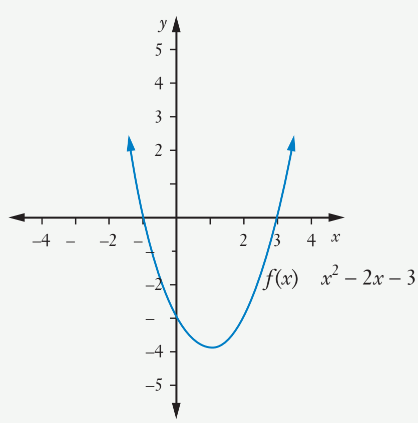
iii



v $x = -2$

b −1, 3 **ii** Minimum −4

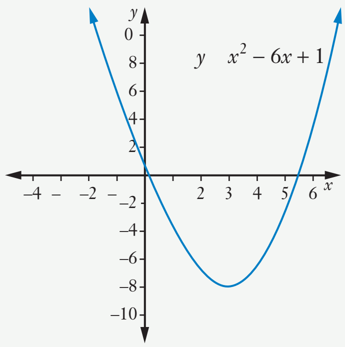
iii



v $x = 3, -1$

c 583 .17 **ii** Minimum −8

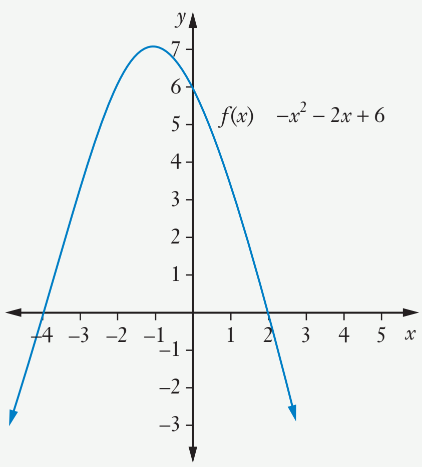
iii



v $x = 58, .2$

d 165 −365 **ii** Maximum 7

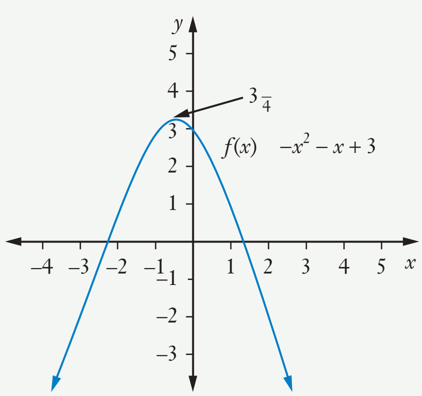
iii



v $x = -365 .65$

e 13 −23 **ii** Maximum $3\frac{\ }{4}$

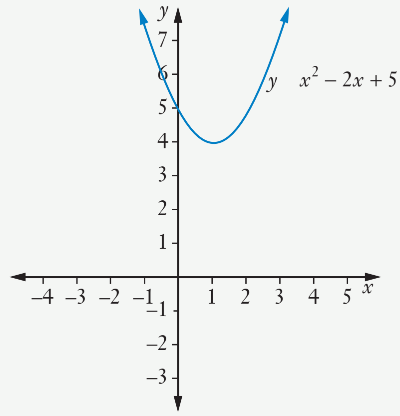
iii



v $x = -23, .3$

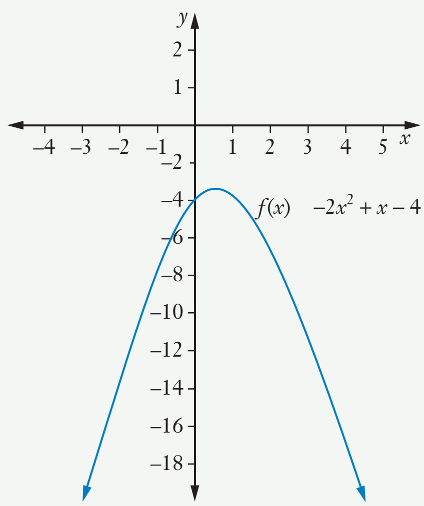
11 **a** 4 **b** None

c



12 **a** $-3\frac{7}{8}$ **b** None

c



13 $f(-x) = -(-x)^2 = -x^2 = f(x)$ ∴ even

14 a, ,c, f

15 **a** $f(-w) = -\frac{(-w)^2}{800} + 200 = -\frac{w^2}{800} + 200 = f(w)$

b 200 m **c** 800 m

d 9875 m **e** 5657 m

Exercse 310

1 **a** 20 **b** −47 **c** −12

d 49 **e** 9 **f** −16

g 0 **h** 64 17

2 **a** 17 unequal real irrational roots
b −39 no real roots
c 1 unequal real rational roots
d 0 equal real rational roots
e 33 unequal real irrational roots
f −16 no real roots
g 49 unequal real rational roots
h −116 no real roots
1 unequal real rational roots

3 $p = 1$ **4** $k = \pm 2$

5 $b \le -\frac{7}{8}$ **6** $p > 2$

7 $k > -2\frac{}{12}$

8 $a = 3 > 0$
$\Delta = -83 < 0$

9 Solving simultaneously
$x^2 - 2x - 3 = 0$
$\Delta = 16 > 0$
So there are 2 points of intersection

10 $\Delta = 68 > 0$
So there are 2 points of intersection

11 $\Delta = -15 < 0$
So there are no points of intersection

12 $\Delta = 0$
So there is 1 point of intersection
$\therefore$ the line is a tangent to the parabola

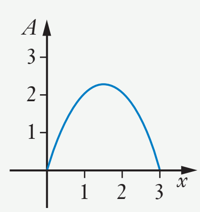
Exercse 311

1 **a** $k = 0004$ **b** $d = 256$ m
c Yes ($d = 10$ m) **d** No ($d = 484$ m)

2 **a** $A = 5x^2$ **b** 882 cm ^{2} **c** 71 cm

3 **a** $V = 8\pi r^2$ **b** 6283 cm ^{3} **c** 2 cm

4 **a** $A = -x^2 + 3x$
b



c $x = 1.5$ **d** $A = 225$ units ^{2}

5 **a** $y = x^2 - x - 5$ **b** $y = x^2 - 3x$
c $y = 2x^2 - 3x + 7$ **d** $y = x^2 + 4x - 9$
e $y = -x^2 - 2x + 1$

6 **a** $h = -5t^2 + 175\,t + 10$
b $h = 25$ m **c** At $t = 0, .5$ s

7 **a** $225y = -32x^2 + 1800$ **b** 672 cm
c 46 cm

8 **a** $y = x^2 - 4x$
b $y = 5$ **ii** $y = 32$
c $x = 2$ **d** $x = 2 \pm \sqrt{6}$

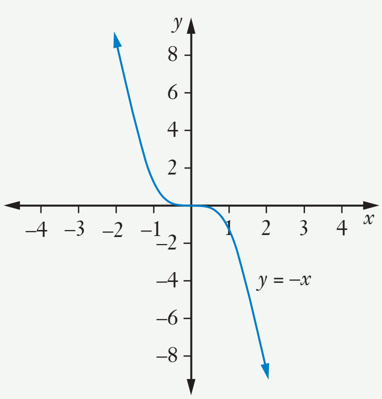
9 **a** $f(x) = x^2 + 2x + 7$ **b** $y = 22$
c $a = 1 > 0, \Delta = -24 < 0$

10 $y = 2x^2 - 4x$ **11** $y = -x^2 + 6x + 4$

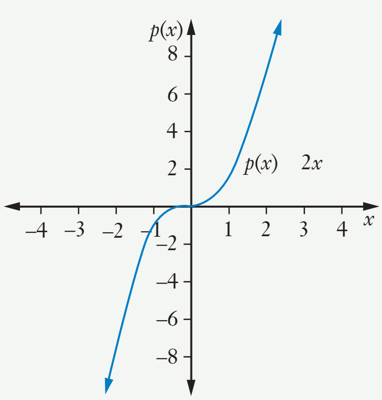
Exercse 312

1 **a** x-intercept 1 y-intercept −1
b x-intercept 2 y-intercept 8
c x-intercept −5, y-intercept 125
d x-intercept 4 y-intercept 64
e x-intercept −6, y-intercept 1026
f x-intercepts −5, 1, 2, y-intercept 10

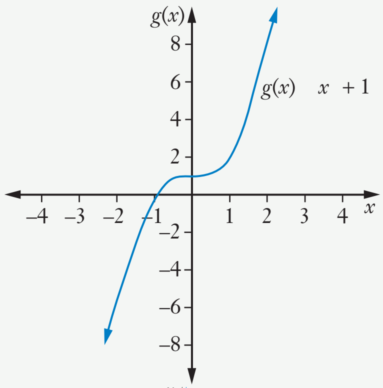
2 **a**



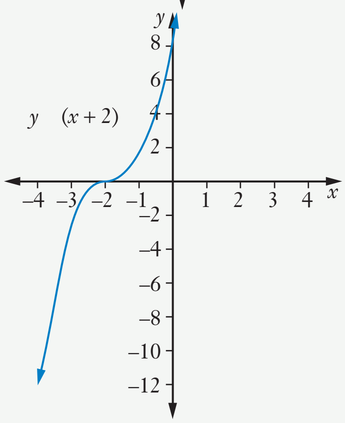
b



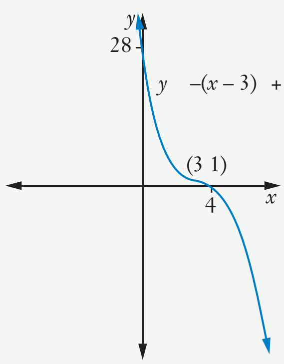
c



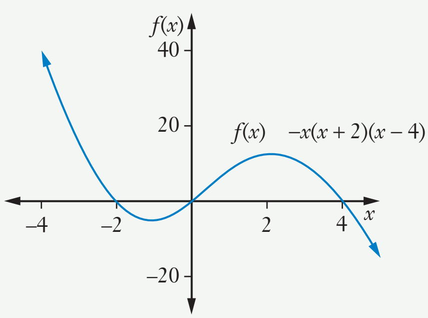
d



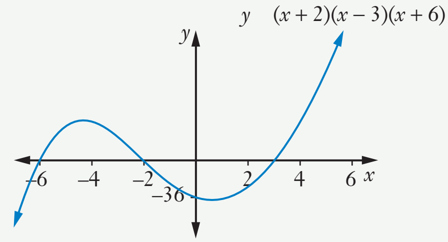
e



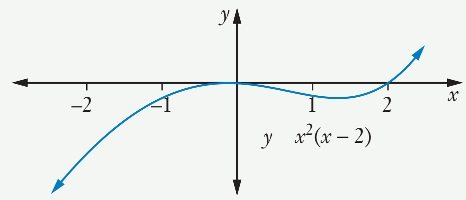
f

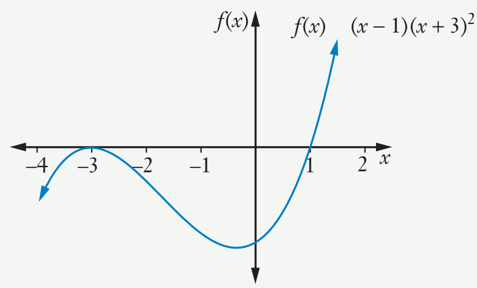


g



h





3 **a** (0 1) **b** (0 27) **c** (−2 0)
d (1 −16) **e** (−1, 1)

4 **a** 14 **b** −03 **c** 07
d −19 **e** 09

5 **a** Increasing curve with x-intercept 4 point of inflection (0 −64) and y-intercept −64

b Decreasing curve with x-intercept 3 point of inflection (3 0) and y-intercept 27

c LHS increasing to a maximum turning point between $x = -4$ and $x = -2$ then decreasing to a minimum turning point between $x = -2$ and $x = 0$ then increasing again on the RH, x-intercepts −4, −2, , y-intercept 0

d LHS decreasing to a minimum turning point between $x = -3$ and $x = -1$ then increasing to a maximum turning point between $x = -1$ and $x = 4$ then decreasing again on the RH, x-intercepts −3, −1, , y-intercept 24

e LHS increasing to a maximum turning point at $x = -5$ then decreasing to a minimum turning point between $x = -5$ and $x = 0$ then increasing again on the RHS x-intercepts −5, 0, y-intercept 0

6 **a** $x = 1.7$ **b** $x = -13$ **c** $x = 165$
d $x = -11$ **e** $x = -08$ **f** $x = -2, 0, 1$

7 **a** $V = 28\,x^3$ **b** 6045 cm ^{3} **c** 45 cm

8 **a** $V = 42\,r^3$ **b** 654 mm ^{3} **c** 119 mm

9 $f(-x) = -(-x)^3 = x^3 = -f(-x)$ So an odd function

10 a, d

11 **a** $y = 2x^3$ **b** $y = -x^3 + 5$ **c** $y = -3x^3 - 1$

d $y = 5x^3 - 7$ **e** $y = -4x^3 + 3$

12 **a** $y = -2(x - 2)(x - 3)(x + 5)$

b $y = 8(x + 1)(x - 4)(x - 6)$

c $y = -3(x - 1)(x - 3)(x + 3)$

Exercse 313

1 **a** 7 **b** 4 **c** 1 **d** 11

e 3 **f** 0

2 **a** −19 **b** −10 **c** −1

3 **a** −6 **b** 5 **c** 2 **d** 1 **e** 2

4 **a** 5 **b** 4 **c** −3 **d** 0

5 **a** ±3 **b** −5 **c** −2, 1

d 4 **e** 0

6 a, , f

7 **a** $a = 0$ **b** $b = 10$ **c** $c = -6$

d $a = -1$ **e** $a = 4$

8 **a** $x = -2\frac{1}{2}$ **b** $x = 2, -1$ **c** 3

d 3 **e** x^5

9 **a** $\Delta = -8 < 0$

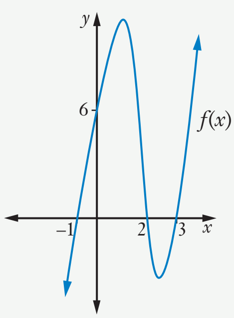
b $9x^3$ **c** −2 **d** 9

e $x = \frac{2}{3}\ \ -1$

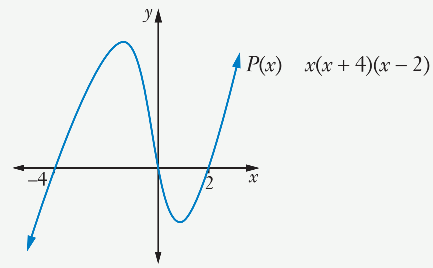
10 **a** 2 **b** 0 **c** 2 **d** 0

e 2 **f** 4

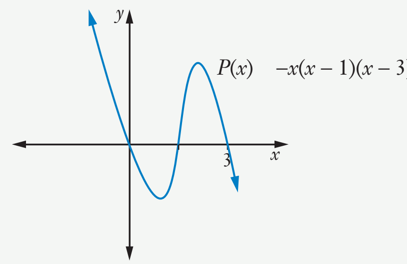
11 **a**



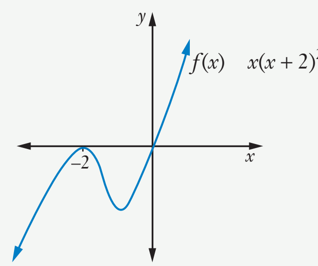
b



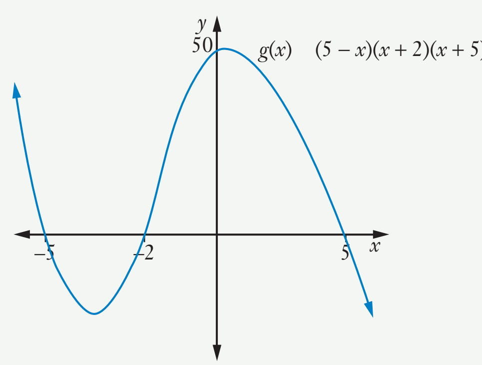
c



d



e



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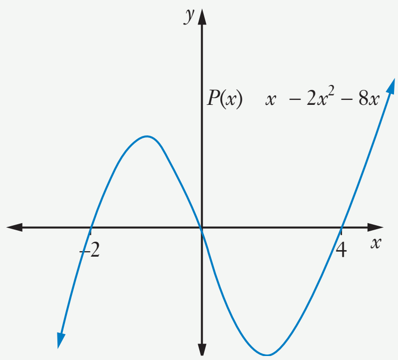
12 **a** $P(x) = x(x - 4)(x + 2)$

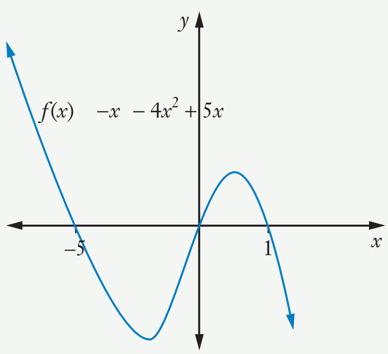
ii



Increases to a maximum turning point then decreases to a minimum turning point then increases

b $f(x) = -x(x - 1)(x + 5)$

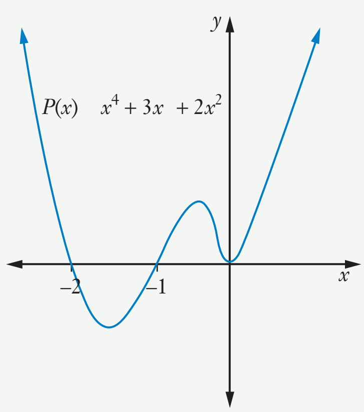
ii



Decreases to a minimum turning point then increases to a maximum turning point then decrease.

c $P(x) = x^2(x + 1)(x + 2)$

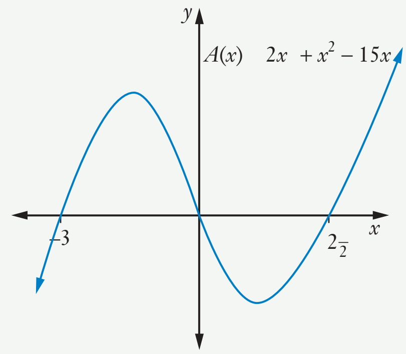
ii



Decreases to a minimum turning point then increases to a maximum turning point then decreases to a minimum turning point then increase.

d $A(x) = x(2x - 5)(x + 3)$

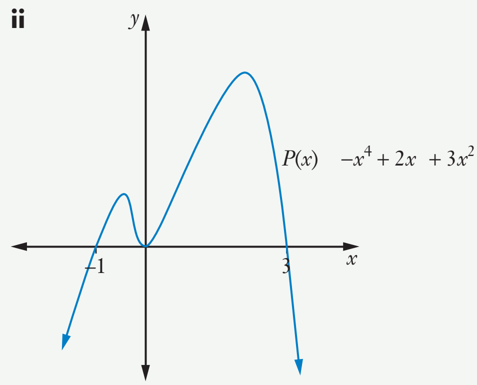
ii



Increases to a maximum turning point then decreases to a minimum turning point then increase.

e $P(x) = -x^2(x - 3)(x + 1)$

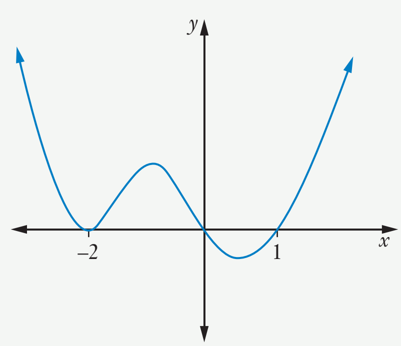
ii



Increases to a maximum turning point then decreases to a minimum turning point then increases to a maximum point then decreases

13 **a** $x = 0, 1, -2$

b



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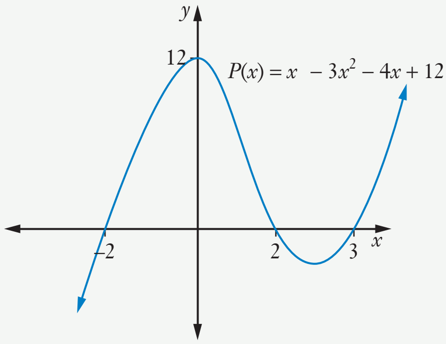
14 b



Exercse 314

1 a $x = 2$ **ii** $x = 1$ **iii** $x = 4$

b $x = 0, 2$ **ii** $x = 3, -1$

c $x = 1$ **ii** $x = 2$ **iii** $x = -1$

2 a (1 4) **b** (−1, −4)

c (0 2) **d** (1 −1)

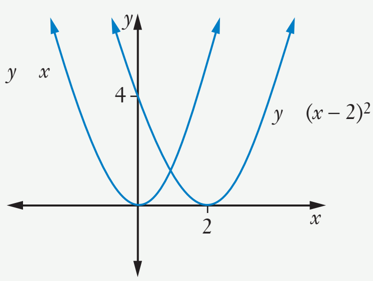
e (2 1)

3 a (0, 0), (1, 1) **b** (−2 4, 2, 4)

c (2 4, (−1, 1) **d** (−3, 9), (1, 1)

e (5 20, (−1, −4)

4 a



b 1 **c** (1, 1)

5 (−1, 1)

6 (−3 4, (−2, −1)

7 (−1, 3)

8 a (140 1400) .. 140 roses at \$1400

b \$343 **c** \$665 loss

9 71 calculators

10 a Income $y = 5x$ Cost: $y = x + 264$

b 66 cupcakes **c** \$736 **d** \$64 loss

11 a $y = 18x$ **b** $y = 12x^2$ **c** $x = 1.5$

Test yourslf 3

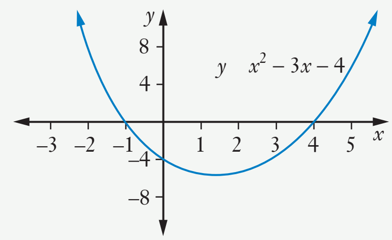
1 C **2** A **3** B

4 B **5** D

6 a $f(-2) = 6$ **b** $f(a) = a^2 - 3a - 4$

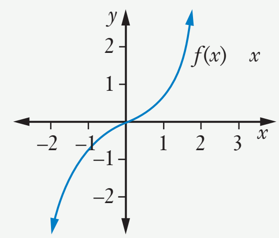
c $x = 4, -1$

7 a



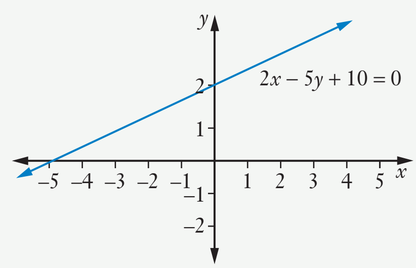
Domain (−∞ ∞) range [−6 4 ∞)

b



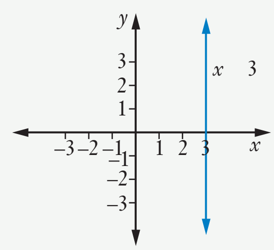
Domain (−∞ ∞) range (−∞ ∞)

c



Domain (−∞ ∞) range (−∞ ∞)

d



Domain [3] range (−∞ ∞)

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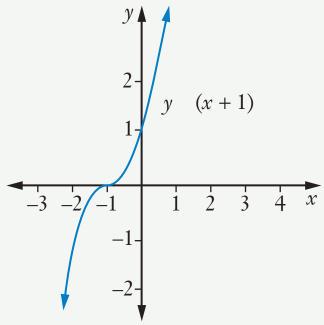
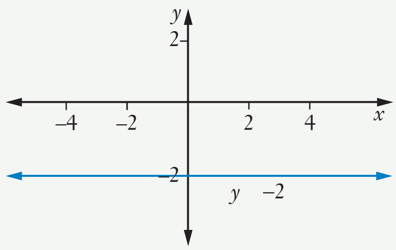
e



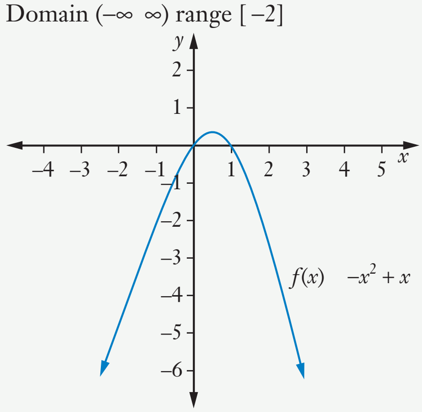
Domain (–∞ ∞) rang: (–∞ ∞)

f



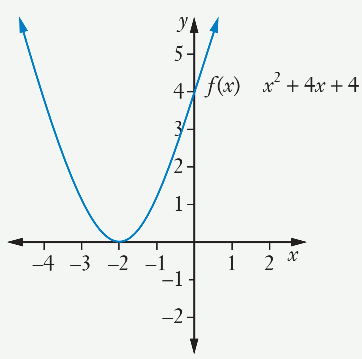
Domain (–∞ ∞) range [–2]

g



Domain (–∞ ∞) range (–∞ .25]

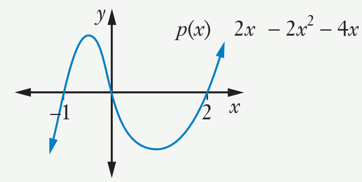
h



Domain (–∞ ∞) range [0 ∞)

8 **a** 2 **b** $x = 3\frac{2}{3}$ **c** $x = 1\frac{}{3}$

9

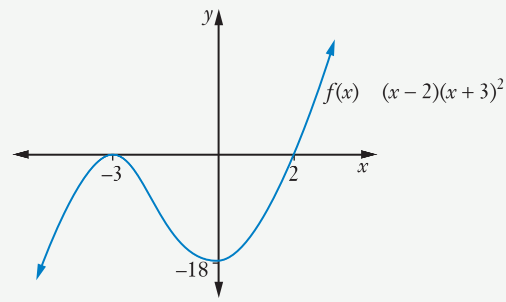


10 **a** $-1\frac{}{5}$ **b** 2

c $\frac{3}{5}$ **d** 1

11 **a** $x = 2$ **b** -3

12



13 **a** 3 **b** -3

c $x = 0, -3, 1$ **d** x^3

14 **a** x-intercept -10 y-intercept 4

b x-intercepts -2, , y-intercept -14

c x-intercept -2, y-intercept 8

d x-intercept 5 y-intercept -2

15 $(-1, 1)$

16 **a** $x = -\frac{1}{4}$ **b** $6\frac{}{8}$

17 Domain (–∞ ∞) range (–∞ $6\frac{}{8}$

18 **a** D **b** B **c** C

d B **e** A

19 **a** $7x - y - 11 = 0$ **b** $5x + y - 6 = 0$

c $3x + 2y = 0$ **d** $3x + 5y - 14 = 0$

e $x - 3y - 3 = 0$

20 $a = 2, b = -18$ $c = 40$

21 **a** Even **b** Neither **c** Odd

d Neither **e** Odd

22 $f(-x) = -(x^3 - x) = -f(x)$

23 $m\ \ = -\frac{1}{4}$ $m_2 = 4$ so $m\ m_2 = -1$

24 $a = -1 < 0$

$\Delta = -7 < 0$

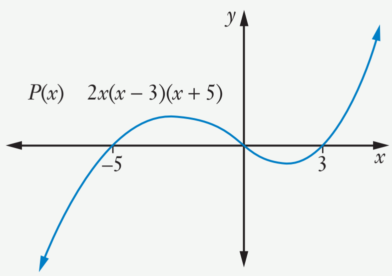
$\therefore -4 + 3x - x^2 < 0$ for all x

25 **a** 2 **b** 2 **c** 0
d 1

26 $m \ = m_2 = 5$ so lines are parallel

27 $x = 4, 5$

28



29 $x = 0, 2$

30 $x = 1\frac{4}{5}$

31 15

32 **a** 4 **b** 5 **c** 9
d 3 **e** 2

33 **a** $y = x^2 - 5x + 4$
b $y = -2x^2 + 2x + 12$

34 **a** $A = 7x^2$ **b** 700 cm^2
c 125 cm

35 **a** function not one-to-one
b not a function
c not a function
d function one-to-one
e not a function

36 $f(x) = -3x^3 + 5$

37 (3 36)

38 **a** $2x + y - 1 = 0$ **b** $\frac{1}{2}$

39 $m < -\frac{9}{16}$

40 **a** (1 −1) **b** (1 0, 2, 2)
c (−3 9, 3, 9)

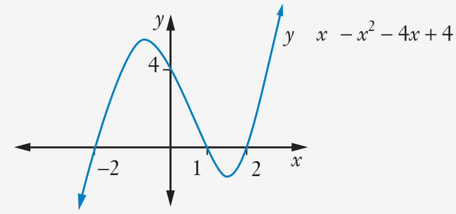
41 $3x - 4y = 0$

42 $3x - 7y - 14 = 0$

43 **a** $A = 012\ d$ **b** 31 L **c** 296 km

44 **a** $x = 1, \pm 2$
b x-intercepts 1 ±2 y-intercept 4

c

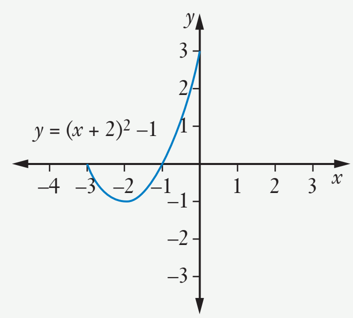


d 3 **ii** 1

Chaenge exercse 3

1 $b = -\frac{2}{3}, 3$

2



3 $k = -2$

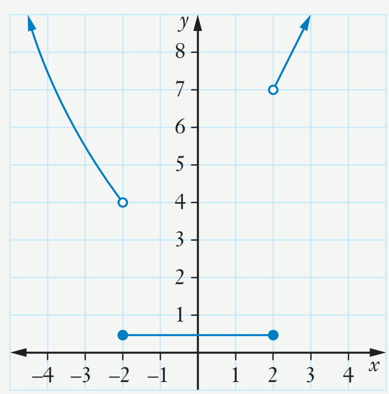
4 $2x + 3y + 13 = 0$

5 $a = 2, b = 3$

6 **a** $x = 1\frac{2}{3}, \frac{\ }{3}$ **b** $x = 0, 2$ **c** $x = 1, 3$
d $x = \pm 1$

7 $y = 3$

8 $f(3) = 9, f(-4) = 16, f(0) = 1$



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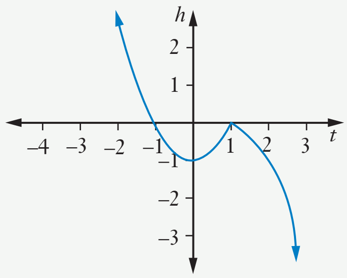
9 $h(2) + h(-1) - h(0) = -3 + 0 - (-1) = -2$



10 $x = 0, 3, -2$

11 $\Delta = (k - 4)^2 \geq 0$ and a perfect square
$\therefore$ real rational roots

12 $p > 075$

13 $f((-a)^2) = 2(-a^2) - 1 = 2a^2 - 1 = f(a^2)$

14 $2x + 5y + 14 = 0$

15 (0, 0), (1, 1)

16 $y = x^3 + 2x^2 - x + 1$

17 $b^2 - 4ac = 0$ So equal root.

18 $P(x) = (x + 2)(x - 1)(x - 6)$

Chapter 4

Exercse 401

1 $\cos\theta = \frac{5}{13}$ $\sin\theta = \frac{12}{13}$ $\tan\theta = \frac{12}{5}$

2 $\sin\beta = \frac{4}{5}$ $\tan\beta = \frac{4}{3}$ $\cos\beta = \frac{3}{5}$

3 $\sin\beta = \frac{7}{\sqrt{74}}$ $\tan\beta = \frac{7}{5}$ $\cos\beta = \frac{5}{\sqrt{74}}$

4 $\cos x = \frac{5}{9}$ $\tan x = \frac{\sqrt{56}}{5}$ $\sin x = \frac{\sqrt{56}}{9}$

5 $\cos\theta = \frac{3}{5}$ $\sin\theta = \frac{4}{5}$

6 $\tan\theta = \frac{\sqrt{5}}{2}$ $\sin\theta = \frac{\sqrt{5}}{3}$

7 $\cos\theta = \frac{\sqrt{35}}{6}$ $\tan\theta = \frac{1}{\sqrt{35}}$

8 $\tan\theta = \frac{\sqrt{51}}{7}$ $\sin\theta = \frac{\sqrt{51}}{10}$

9 **a** $\sqrt{2}$ **b** 45°
c $\sin 45° = \frac{1}{\sqrt{2}}$ $\cos 45° = \frac{1}{\sqrt{2}}$ $\tan 45° = 1$

10 **a** $\sqrt{3}$
b $\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}}$
c $\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \sqrt{3}$

11 **a** 47° **b** 82° **c** 19°
d 77° **e** 52°

12 **a** 47°13 **b** 81°46 **c** 19°26
d 76°37 **e** 52°30

13 **a** 7775 ° **b** 655 ° **c** 2485 °
d 6835 ° **e** 82517 °

14 **a** 59°32 **b** 72°14 **c** 85°53
d 46°54 **e** 73°13

15 **a** 0635 **b** 0697 **c** 0339
d 0928 **e** 1393

16 **a** 17°20 **b** 34°20 **c** 34°12
d 46°34 **e** 79°10

Exercse 402

1 **a** $x = 63$ **b** $y = 56$ **c** $b = 39$ cm
d $x = 56$ m **e** $m = 29$ **f** $x = 135$
g $y = 100$ **h** $p = 33$ $x = 51$ cm
j $t = 283$ **k** $x = 33$ cm $x = 29$ cm
m $x = 207$ cm **n** $x = 205$ mm **o** $y = 44$ m
p $k = 206$ cm **q** $h = 173$ m **r** $d = 12$ m
s $x = 174$ cm **t** $b = 1632$ m

2 16 m **3** 203 cm **4** 139 m

5 **a** 184 cm **b** 138 cm

6 10 cm and 105 cm

7 474 mm **8** 203 m

9 **a** 74 cm **b** 66 cm **c** 90 cm

10 **a** 126 cm **b** 222 cm **11** 38 cm

Exercse 403

1 **a** $x = 39°48$ **b** $\alpha = 35°06$ **c** $\theta = 37°59$
d $\alpha = 50°37$ **e** $\alpha = 38°54$ **f** $\beta = 50°42$
g $x = 44°50$ **h** $\theta = 30°51$ $\alpha = 29°43$
j $\theta = 45°37$ **k** $\alpha = 57°43$ $\theta = 43°22$
m $\theta = 37°38$ **n** $\theta = 64°37$ **o** $\beta = 66°16$
p $\alpha = 29°56$ **q** $\theta = 54°37$ **r** $\alpha = 35°58$
s $\theta = 59°2$ **t** $\gamma = 56°59$

2 37°57 **3** 22°14 **4** 36°52 **5** 50°

6 **a** 114 cm **b** 37°52

7 α = 31°58 β = 45°44

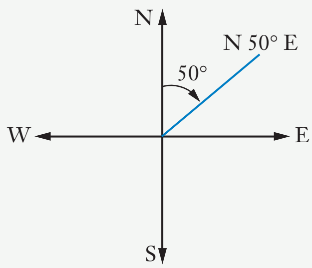
8 **a** 13 m **b** 65°13

9 **a** 11°19 **b** 26 cm

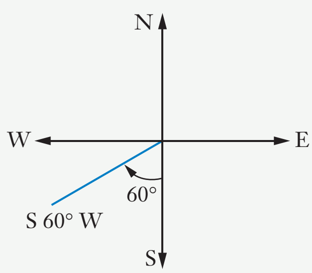
10 **a** 129 m **b** 56°37

Exercse 404

1 **a**



b



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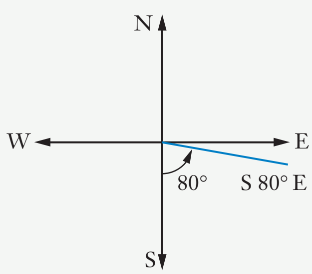

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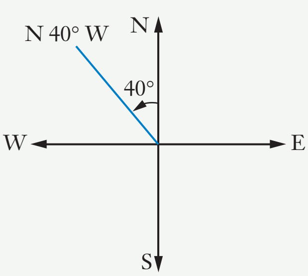

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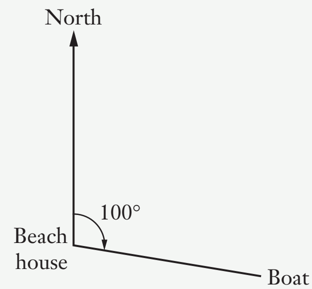

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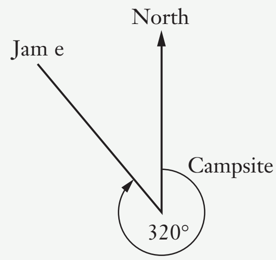
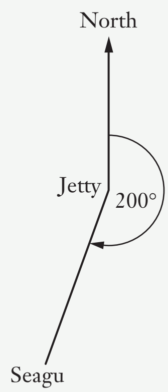
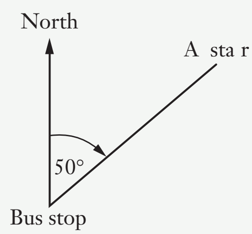
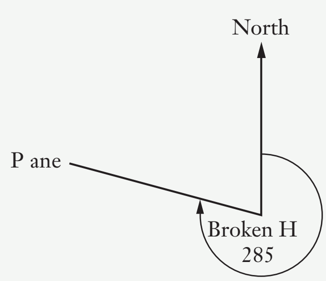


g



h





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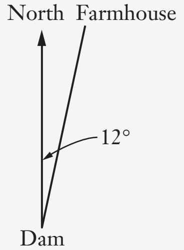
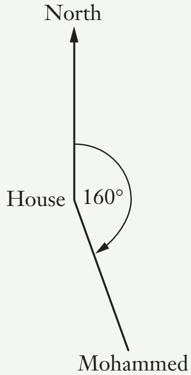
j



k



2 **a** S 35° E **ii** 145°
b N 80° E **ii** 080°
c N 23° W **ii** 337°
d S **ii** 180°

3 080° **4** 210° **5** 160° **6** 104 m
7 21 m **8** 1269 m **9** 72°48
10 **a** 10565 km **b** 22658 km **c** 245°
11 831 m **12** 18 km **13** 12 m **14** 242°
15 035° **16** 92 m **17** 171 m
18 98 km **19** 51°41 **20** 26 m
21 9°2 **22** 19319 km **23** 35 m
24 149° **25** 198 m **26** 48 km
27 92 m **28** 217°
29 **a** 12 km **b** 72 km
30 **a** 131 m **b** 50°26

Exercse 405

1 **a** $x = 89$ **b** $y = 94$ cm **c** $a = 100$
d $b = 107$ m **e** $d = 80$
2 **a** $\theta = 51°50$ or $128°10$
b $\alpha = 61°23$ or $118°37$
c $x = 43°03'$ or $136°57'$
d $\theta = 29°4$ or $150°56$
e $\alpha = 87°4$
3 126°56
4 **a** 135 mm **b** 25 mm
5 **a** 18 m **b** 27 m **6** 57 cm
7 **a** 103 m **b** 94 m
8 **a** 60°22 **b** 57°9
9 **a** 141 cm **b** 156 cm
10 **a** 547 mm **b** 351 mm
11 **a** 74° or 106° **b** 52° **c** 55° or 125°

Exercse 406

1 **a** $m = 58$ **b** $b = 104$ m **c** $h = 74$ cm
d $n = 164$ **e** $y = 93$
2 **a** $\theta = 54°19$ **b** $\theta = 60°27$ **c** $x = 57°42$
d $\beta = 131°31$ **e** $\theta = 73°49$
3 3294 mm **4** 112 cm and 129 cm
5 **a** 119 cm **b** 44°11 **c** 82°12
6 $\angle XYZ = \angle XZY = 66°10$ $\angle YXZ = 47°40$
7 **a** 181 mm **b** 78°47
8 **a** 62 cm **b** 127 cm
9 129 cm
10 **a** 11 cm **b** 30°

Exercse 407

1 **a** 75 cm^2 **b** 323 $units^2$ **c** 99 mm^2
d 302 $units^2$ **e** 63 cm^2
2 75 cm^2 **3** 155 cm^2 **4** 348 cm^2
5 12 m^2
6 **a** 78 m **b** 1807 m^2
7 **a** 56 cm **b** 185 cm^2 **c** 189 cm^2
8 $\frac{25\sqrt{3}}{4}$ cm^2

Exercise 408

1 **a** 040° **b** 305°
2 164 m **3** 28°
4 **a** 121 km **b** 1 minute
5 32 m **6** 107 m
7 $h = 85$ **8** 77 km
9 54 km from A and .7 km from B
10 1841 km **11** 89°52
12 99 km **13** 1635 km
14 3269 km
15 **a** 113 cm **b** 44°45 or 135°15

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16 141°

17 **a** 116 cm **b** 73°14

18 **a** 35°5 **b** 45 m **ii** 055 m

19 **a** 109 cm^2 **b** 16°20

20 65°9

21 **a** 937 m **b** 25°

22 **a** 56 m **b** 897 m

23 **a** 48 m **b** 1286 m **c** 977 m

24 11°10

Exercse 409

1 **a** 36° **b** 120° **c** 225°
d 210° **e** 540° **f** 140°
g 240° **h** 420° 20°
j 50°

2 **a** $\frac{3\pi}{4}$ **b** $\frac{\pi}{6}$ **c** $\frac{5\pi}{6}$
d $\frac{4\pi}{3}$ **e** $\frac{5\pi}{3}$ **f** $\frac{7\pi}{20}$
g $\frac{\pi}{12}$ **h** $\frac{5\pi}{2}$ $\frac{5\pi}{4}$
j $\frac{2\pi}{3}$

3 **a** 098 **b** 119 **c** 222
d 504 **e** 545

4 **a** 032 **b** 061 **c** 178
d 154 **e** 088

5 **a** 62°27 **b** 44°0 **c** 66°28
d 56°43 **e** 18°20 **f** 183°21
g 154°42 **h** 246°57 320°51
j 6°18

6 **a** 034 **b** 007 **c** 006
d 083 **e** –114 **f** 033
g –150 **h** 006 –073
j 016

7 **a** $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{2}$ **c** $\frac{1}{\sqrt{3}}$
d $\frac{\sqrt{3}}{2}$ **e** 1 **f** $\frac{1}{2}$
g $\frac{1}{\sqrt{2}}$ **h** $\frac{\sqrt{3}}{2}$ $\sqrt{3}$

Exercse 410

1 **a** 4π cm **b** π m **c** $\frac{25\pi}{3}$ cm
d $\frac{\pi}{2}$ cm **e** $\frac{7\pi}{4}$ mm

2 **a** 065 m **b** 392 cm **c** 691 mm
d 239 cm **e** 303 m

3 18 m **4** 75 m **5** $\frac{2\pi}{21}$

6 25 mm **7** 183 **8** $13\frac{7}{9}$ mm

9 **a** 483 mm **b** 253 mm

10 $\frac{125\sqrt{35}\,\pi}{648}$ cm^3

Exercse 411

1 **a** 8π cm^2 **b** $\frac{3\pi}{2}$ m^2 **c** $\frac{125\pi}{3}$ cm^2
d $\frac{3\pi}{4}$ cm^2 **e** $\frac{49\pi}{8}$ mm^2

2 **a** 048 m ^{2} **b** 629 cm ^{2} **c** 2488 mm ^{2}
d 705 cm ^{2} **e** 318 m ^{2}

3 166 m ^{2} **4** 44 **5** 6 m

6 **a** $\frac{7\pi}{6}$ cm **b** $\frac{49\pi}{12}$ cm^2

7 $\frac{6845}{8\pi}$ mm^2 **8** 75 cm^2 **9** 1197 cm ^{2}

10 $\frac{\pi}{15}$ 3 cm

11 **a** $\frac{3\pi}{7}$ cm **b** $\frac{9\pi}{14}$ cm^2

12 **a** $\frac{5\pi}{6}$ cm **b** $\frac{25\pi}{12}$ cm^2 **c** 26 cm

13 **a** 105 mm **b** 429 mm ^{2}

14 **a** 2π cm^2 **b** 31 cm **c** $4\sqrt{2}$ cm^2
d $2\pi - 4\sqrt{2}$ cm^2

15 **a** 77°22 **b** 703 cm ^{2} **c** 2696 cm ^{2}
d 42543 cm ^{2}

16 1498 cm **17** $\frac{225\pi}{2}$ cm^3

Test yourslf 4

1 C **2** A C **3** C, D

4 $\cos\theta = \frac{5}{\sqrt{34}}$ $\sin\theta = \frac{3}{\sqrt{34}}$

5 **a** 064 **b** 184 **c** 095
d 014 **e** 037

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6 **a** $\theta = 46°3$ **b** $\theta = 73°23$ **c** $\theta = 35°32$

7 122 km **8** $5\sqrt{3}$

9 **a** 63 cm **b** 87 m

10 **a** 42°58 **b** 74°29 **c** 226°19

d 240°39 **e** 324°18

11 **a** $\theta = 65°5$ **b** $\theta = 84°16$

c $\theta = 39°47'$ 140°13

12 653 cm ^{2}

13 **a** 209° **b** 029°

14 **a** $AD = \dfrac{20\sin 39°}{\sin 99°}$ **b** 85 m

15 2951 km

16 **a** $\dfrac{\pi}{3}$ **b** $\dfrac{\pi}{4}$ **c** $\dfrac{5\pi}{6}$

d π **e** $\dfrac{\pi}{9}$

17 **a** $\dfrac{5\pi}{6}$ cm **b** $\dfrac{25\pi}{12}$ cm^2

18 **a** $\sqrt{3}$ **b** $\dfrac{\sqrt{3}}{2}$ **c** $\dfrac{1}{\sqrt{2}}$

d $\dfrac{1}{\sqrt{3}}$ **e** $\dfrac{1}{\sqrt{2}}$ **f** $\dfrac{1}{2}$

g 1 **h** $\dfrac{1}{2}$ $\dfrac{\sqrt{3}}{2}$

19 **a** $\dfrac{8\pi}{7}$ cm^2 **b** 012 cm ^{2}

20 $\alpha = 51°40$ **21** 54°19 or 125°41

Chaenge exercse 4

1 92°58 **2** $x = 127$ cm

3 **a** $AC = \dfrac{25\ 3\sin 39°53}{\sin 41°21}$ **b** $h = 252$ cm

4 31 m

5 **a** 65 m **b** 27°42

6 **a** 301 m **b** 05 m s $^-$

7 42 cm^2 **8** 2477 mm ^{2} **9** 94 cm ^{2}

10 **a** 384 cm **b** 088 cm ^{2} **c** 2584 cm

11 84 m

12 **a** 314 cm **b** 754 cm ^{2}

Practice set 2

1 B D **2** A **3** C

4 C **5** B **6** A, B, D

7 **a** 52°26 **b** 166 m ^{2}

8 **a** $y = -2x + 3$ **b** $x - 5y - 5 = 0$

c $4x - 5y - 8 = 0$ **d** $5x - 4y - 41 = 0$

e $2x + 3y - 3 = 0$ **f** $x - 8y + 15 = 0$

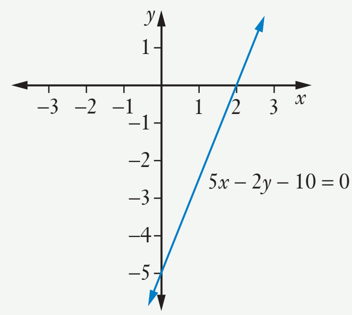
g $x + y - 4 = 0$

9 **a** $\dfrac{3x}{x-4}$ **b** $\dfrac{5x}{y(y-2)}$ **c** $\dfrac{11a-17}{20}$

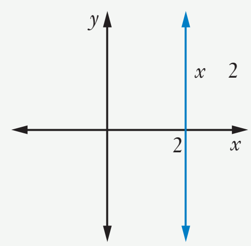
10 **a** $\dfrac{\pi}{3}$ **b** $\dfrac{5\pi}{6}$ **c** $\dfrac{\pi}{2}$

d $\dfrac{\pi}{18}$ **e** $\dfrac{7\pi}{4}$

11 **a**



b



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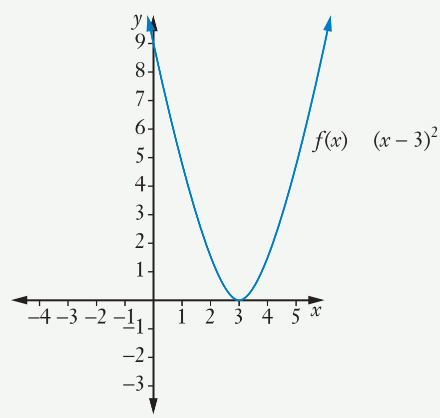

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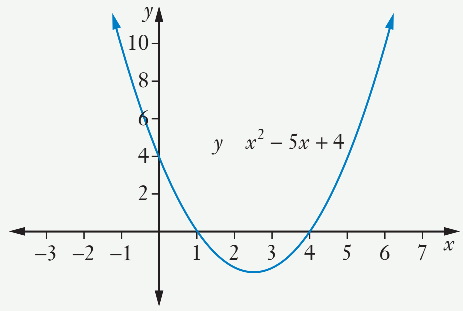
d



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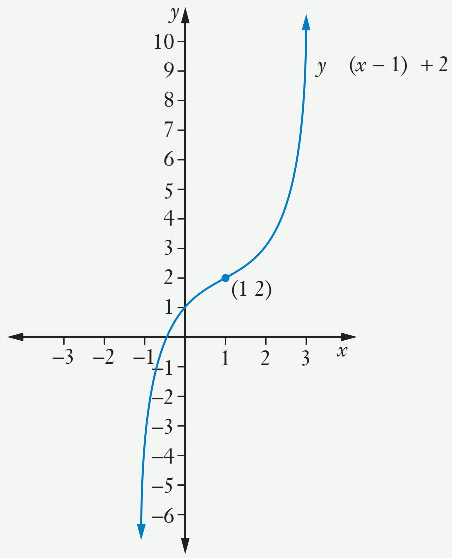


12 **a** 97°24 **b** 20°38 **c** 145°32

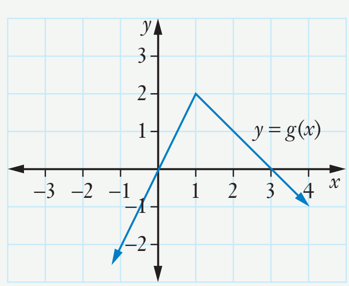
13 **a** $m\ m_2 = \frac{3}{4} \times \left(-\frac{8}{6}\right) = -1$ So perpendicula.

b $A = \left(-1\ 1\frac{}{2}\right)$

14 $-1\frac{2}{3}$

15 **a** $g(2) = 1, g(-3) = -6$

b



16 $x = 3$

17 $f(-1) = 7$

18 Solve simultaneous equations

$25x^2 + 60x + 36 = 0$

$\Delta = 0$

One solution so the line is a tangent to the circle

19 **a** 45° **b** 270° **c** 36°

d 157°30 **e** 1080°

20 $\cos\theta = \frac{5}{7}$ $\sin\theta = \frac{\sqrt{24}}{7}$ $\tan\theta = \frac{\sqrt{24}}{5}$

21 **a** $a < 0, \Delta < 0$

b $a > 0, \Delta < 0$

22 **a** $d = 48\ t$

b 96 km **ii** 24 km

c 625 h **ii** 396 h **d** 48 km/h

23 $y = x^2 - 2x - 3$

24 $\Delta = 361$ (> 0 and a perfect square)

25 76°52 **26** 45°49

27 52°37

28 **a** $2x - y + 4 = 0$ **b** $P(-2\ 0,\ \ Q(0\ 4)$

29 $f(-x) = x^6 - x^2 - 3$

30 2°8

31 **a** N 50° E **ii** 050°

b S 20° W **ii** 200°

c S 40° E **ii** 140°

d N 50° W **ii** 310°

32 **a** $\frac{5\pi}{6}$ cm **b** $\frac{25\pi}{12}$ cm^2

33 **a** $(x-2)^2$ **b** $(3x+1)(3x-1)$

34 117°56

35 $y = 165$

36 **a** (4 −1) **b** (3 9, (−25 .25)

37 **a** $x = 3$ **ii** Minimum (3 −8)

b $x = -1$ **ii** Maximum (−1, −1)

38 175 m

39 **a** 7 m **b** 278 m^2

40 127 m

41 21π cm^2

42 49 km

43 **a** 81 m **b** 35°46

44 **a** Domain (−∞ −4) ∪ (−4, ∞), range (−∞ 0) ∪ (0 ∞)

b Domain (−∞ ∞) range [, ∞)

c Domain (−∞ ∞) range [4]

d Domain (−∞ ∞) range [−3, ∞)

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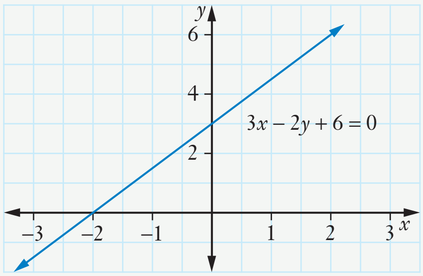
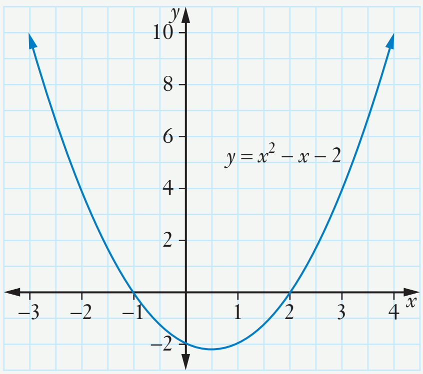
45 **a** 1685 km **b** 228°

46 $f(-x) = -x^3 + 5x = -(x^3 - 5x) = -f(x)$

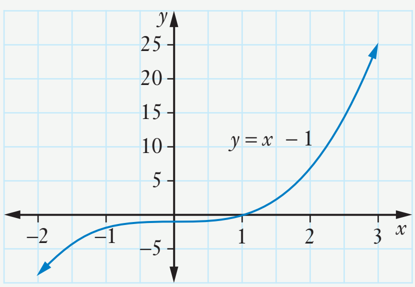
47 **a**



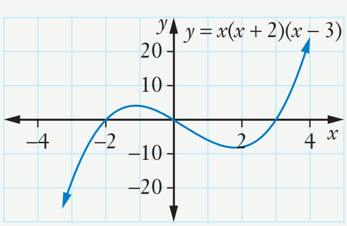
b



c



d



48 16 cm^2

49 **a** 360° **b** 30° **c** 405°

50 4893 km

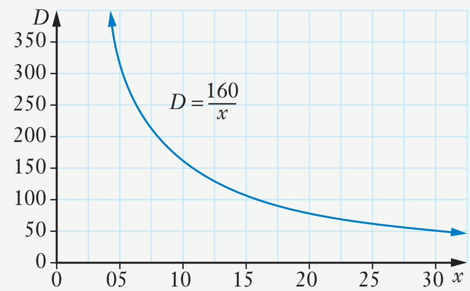
51 **a** $b \geq 2$ **b** $x = 0, 3$ **c** $n = 2, -7$

Chapter 5

Exercse 501

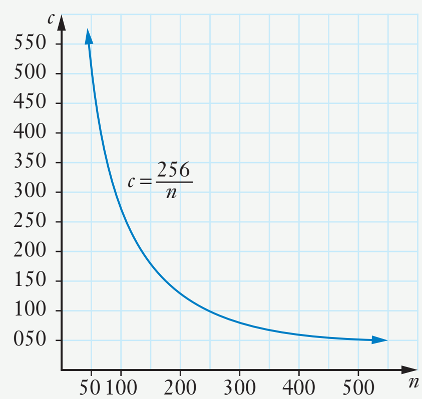
1 **a** $D = \frac{160}{x}$ **b** 200 mm **c** 14 mm

d



2 **a** $c = \frac{256}{n}$ **b** $256 **c** 512 boxes

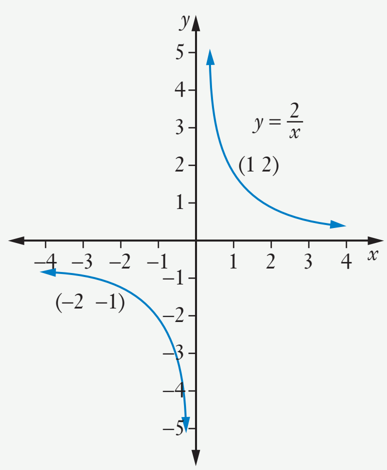
d



3 **a** Domain (–∞ 0) ∪ (0 ∞), range (–∞ 0) ∪ (0 ∞)

ii no y-intercept

iii



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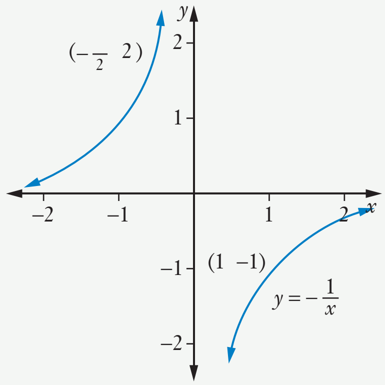
b Domain (−∞ 0) ∪ (0 ∞),
range (−∞ 0) ∪ (0 ∞)

ii no y-intercept

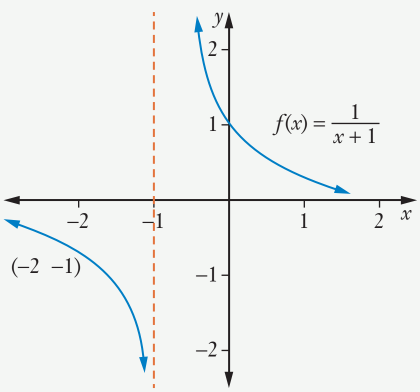
iii



c Domain (−∞ −1) ∪ (−1 ∞),
range (−∞ 0) ∪ (0 ∞)

ii 1

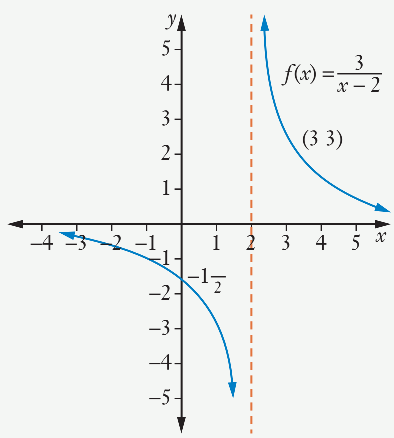
iii



d Domain (−∞ 2) ∪ (2 ∞),
range (−∞ 0) ∪ (0 ∞)

ii $-1\frac{}{2}$

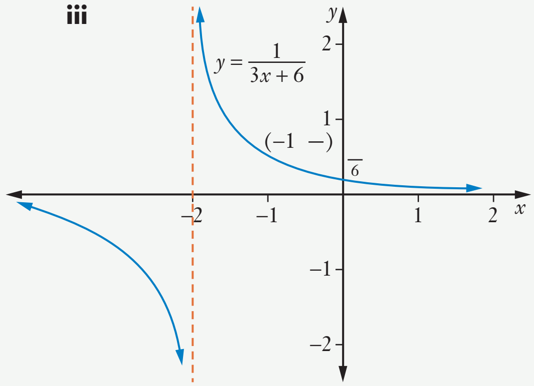
iii



e Domain (−∞ −2) ∪ (−2 ∞),
range (−∞ 0) ∪ (0 ∞)

ii $\frac{1}{6}$

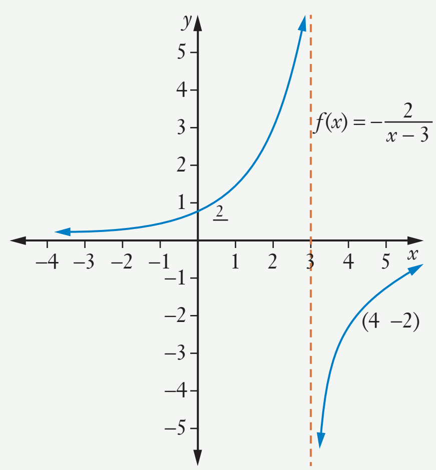
iii



f Domain (−∞, 3) ∪ (3, ∞),
range (−∞ 0) ∪ (0 ∞)

ii $\frac{2}{3}$

iii



g Domain (−∞, 1) ∪ (1, ∞),
range (−∞ 0) ∪ (0 ∞)

ii −4

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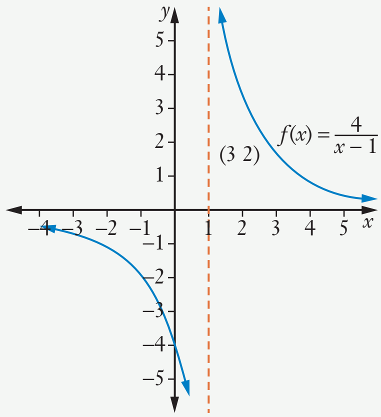
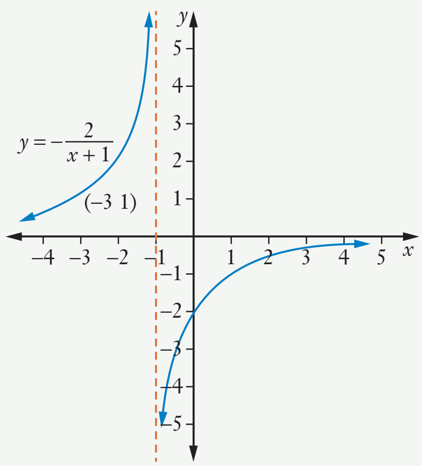
iii



h Domain (–∞ –1) ∪ (–1 ∞),
range (–∞ 0) ∪ (0 ∞)

ii –2

iii

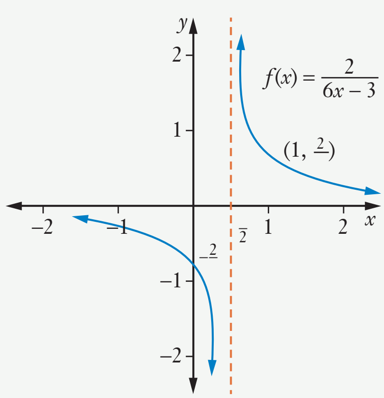


Domain $\left(-\infty \;\; \frac{1}{2}\right) \cup \left(\frac{1}{2} \;\; \infty\right)$

range (–∞ 0) ∪ (0 ∞)

ii $-\frac{2}{3}$

iii



4 $f(-x) = \frac{2}{-x} = -\frac{2}{x} = -f(x)$

∴ odd function

5 **a** Yes **ii** Neither **iii** No

b $x = -1, y = 0$

c Domain (–∞ –1) ∪ (–1 ∞),
range (–∞ 0) ∪ (0 ∞)

Exercse 502

1 **a** No x-intercepts y-intercept 7

b x-intercepts ±2 y-intercept –2

c x-intercept 0 y-intercept 0

d x-intercepts ±3 y-intercept 3

e x-intercept –6 y-intercept 6

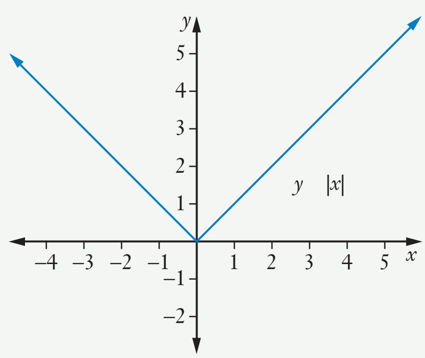
f x-intercept $\frac{2}{3}$ y-intercept 2

g x-intercept $-\frac{4}{5}$, y-intercept 4

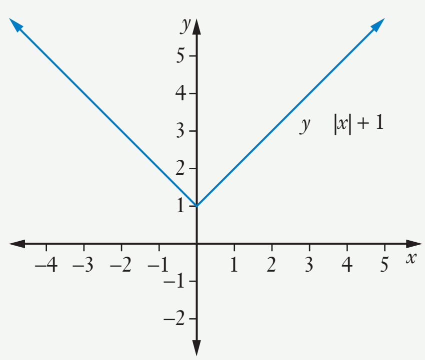
h x-intercept $\frac{1}{7}$ y-intercept 1

No x-intercepts y-intercept 9

2 **a**



b



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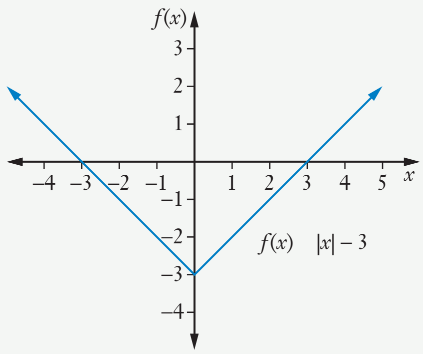
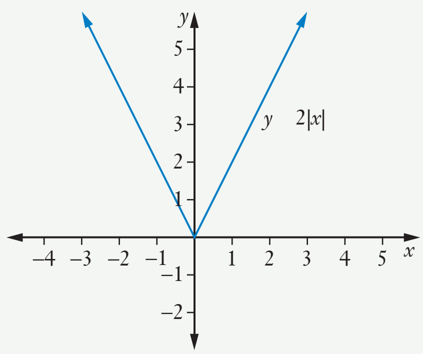
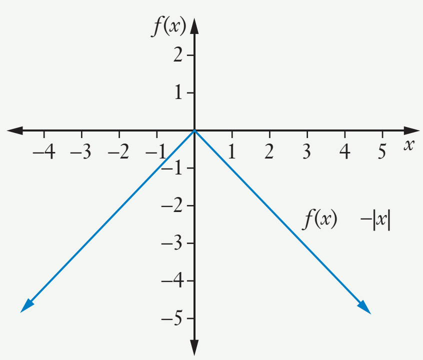
c



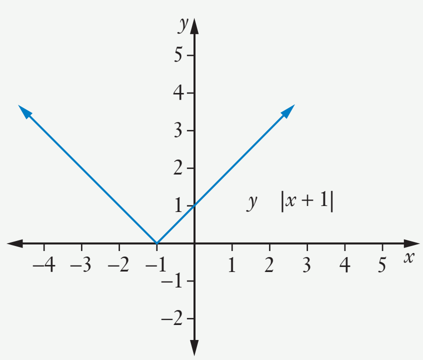
d



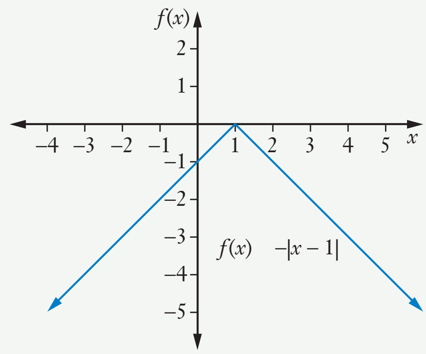
e



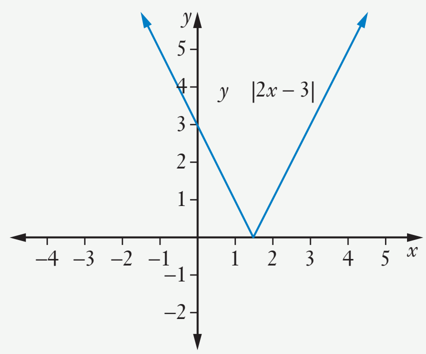
f

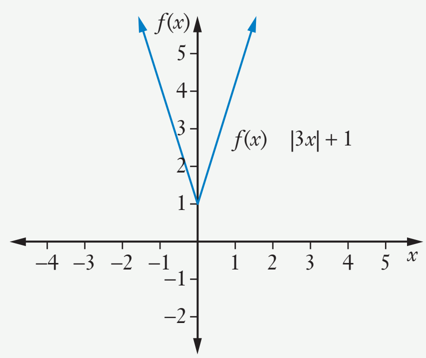


g



h





3 **a** Domain $(-\infty\ \infty)$ range $[,\ \ \infty)$
b Domain $(-\infty\ \infty)$ range $[-,\ \ \infty)$
c Domain $(-\infty\ \infty)$ range $[,\ \ \infty)$
d Domain $(-\infty\ \infty)$ range $[-,\ \ \infty)$
e Domain $(-\infty\ \infty)$ range $(-\infty\ 0]$

4 **a** $x = \pm 3$ **b** $x = -1\ -3$ **c** $x = 3$
d $x = 1, 2$ **e** $x = 4\ -7$ **f** $b = 2, -.2$
g $x = \frac{1}{3}\ -1$ **h** $x = 2\ -3$ $t = \frac{1}{2}$

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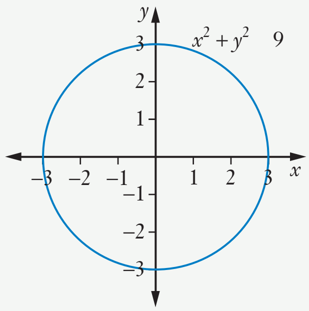
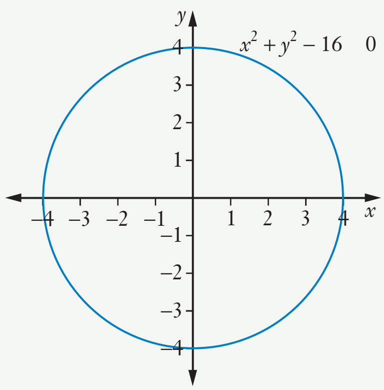
Exercse 503

1 **a**



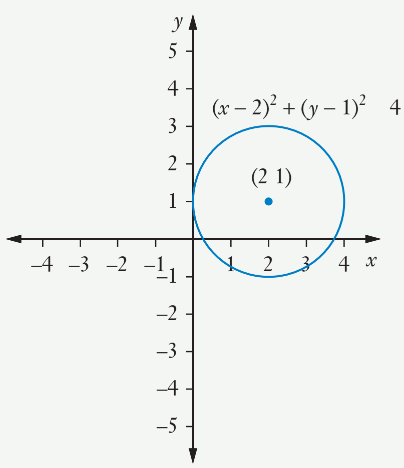
ii Domain [–3 3, range [3, 3]

b



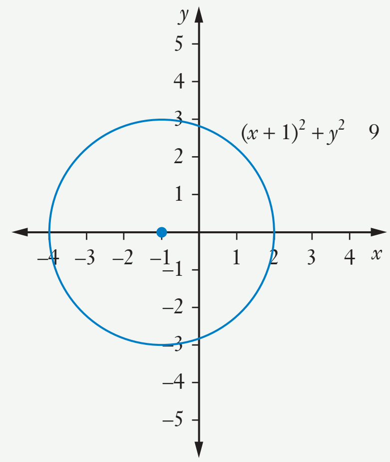
ii Domain [–4 4, range [4, 4]

c



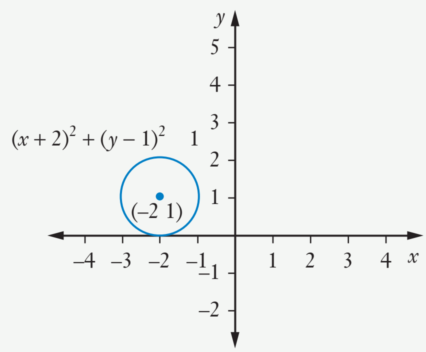
ii Domain [0 4, range [1, 3]

d



ii Domain [–4 2, range [3, 3]

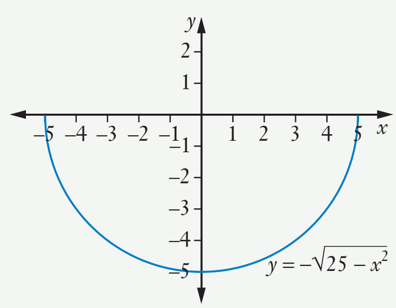
e



ii Domain [–3 –1, range 0, 2]

2 **a** Below x-axis

ii



iii Domain [–5 5, range [5, 0]

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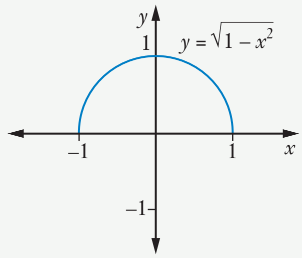
b Above x-axis

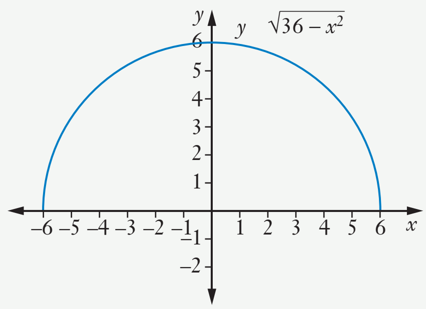
ii



iii Domain [–1 1, range 0, 1]

c Above x-axis

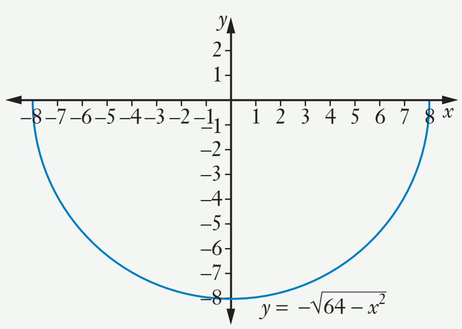
ii



iii Domain [–6 6, range 0, 6]

d Below x-axis

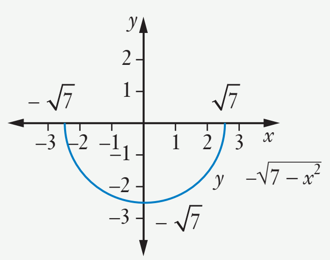
ii



iii Domain [–8 8, range [8, 0]

e Below x-axis

ii



iii Domain $[-\sqrt{7}\ \ \sqrt{7}\,]$ range $[-\sqrt{7}\ \ 0]$

3 **a** 10 (, 0) **b** $\sqrt{5}$, (0, 0)

c 4 (, 5) **d** 7 (, –6)

e 9, (0, 3)

4 **a** $x^2 + y^2 = 16$

b $x^2 - 6x + y^2 - 4y - 12 = 0$

c $x^2 + 2x + y^2 - 10y + 17 = 0$

d $x^2 - 4x + y^2 - 6y - 23 = 0$

e $x^2 + 8x + y^2 - 4y - 5 = 0$

f $x^2 + y^2 + 4y + 3 = 0$

g $x^2 - 8x + y^2 - 4y - 29 = 0$

h $x^2 + 6x + y^2 + 8y - 56 = 0$

$x^2 + 4x + y^2 - 1 = 0$

j $x^2 + 8x + y^2 + 14y + 62 = 0$

5 **a** 3, (2, 1) **b** 5 (–, 2)

c 1, (0, 1) **d** 6 (, –3)

e 1, (–1, 1) **f** 6 (, 0)

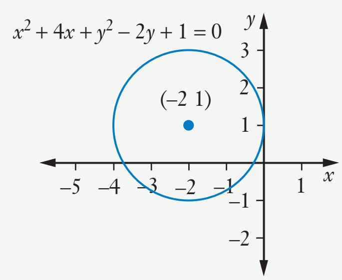
g 5 (–, 4) **h** 8 (–1, 2)

5 (, –1) **j** $\sqrt{10}$ (–, –2)

6 **a** (3 –1, 4 **b** (2 5, 5

c (–1 –6, 7 **d** (4 7, 8

7



Exercse 504

1 **a** $y = -x^2 + 2$ **ii** $y = x^2 - 2$

iii $y = -x^2 + 2$

b $y = -(x + 1)^3$ **ii** $y = (-x + 1)^3$

iii $y = -(-x + 1)^3$

c $y = -5x + 3$ **ii** $y = -5x - 3$

iii $y = 5x + 3$

d $y = -|2x + 5|$ **ii** $y = |-2x + 5|$

iii $y = -|-2x + 5|$

e $y = -\dfrac{1}{x - 1}$ **ii** $y = -\dfrac{1}{x + 1}$

iii $y = \dfrac{1}{x + 1}$

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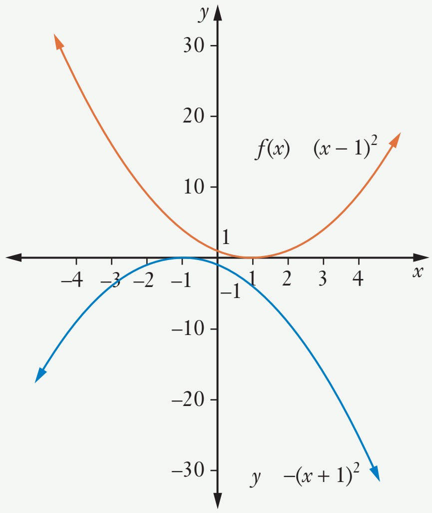
2 **a** Reflection in x-axis

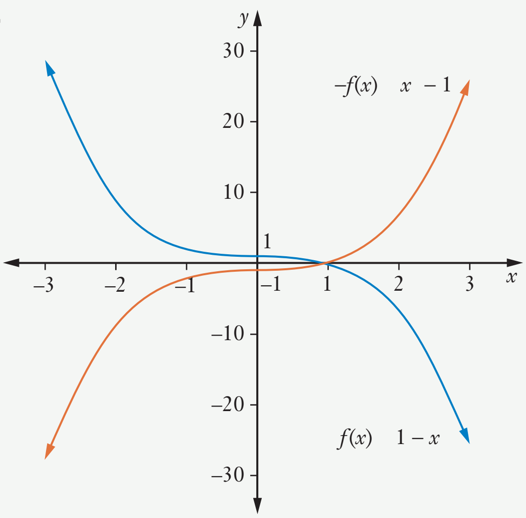
b Reflection in y-axis

c Reflection in both x-axis and y-axis

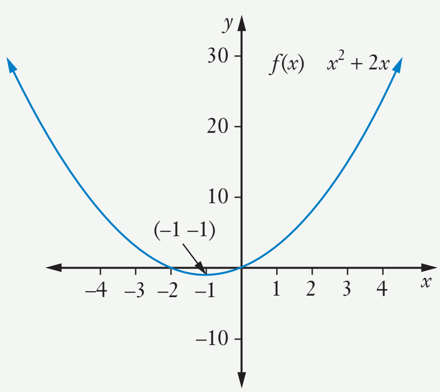
3



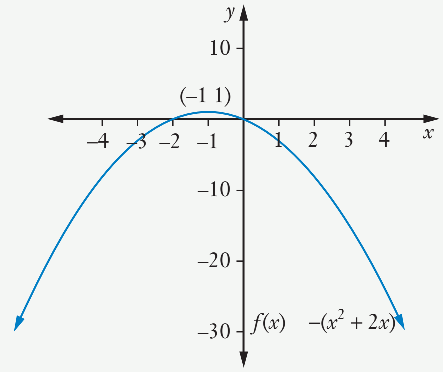
4



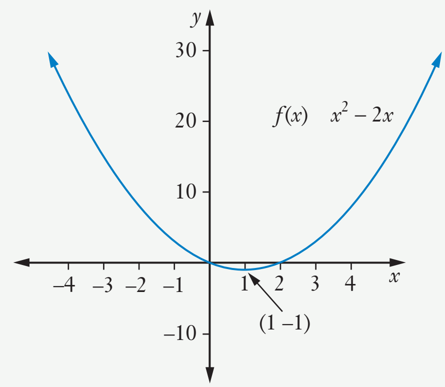
5 **a**



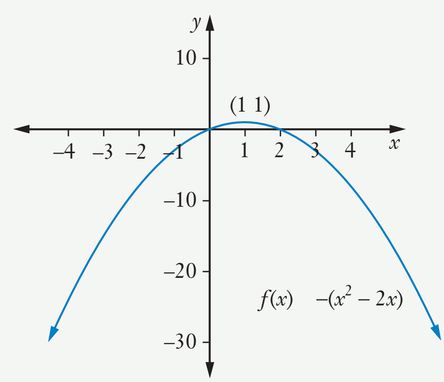
b



c



d



6 **a** $f(-x)=2(-x)^2=2x^2=f(x)$ so even

b $y=2x^2$ **ii** $y=-2x^2$

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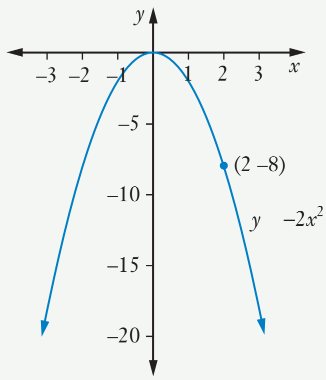
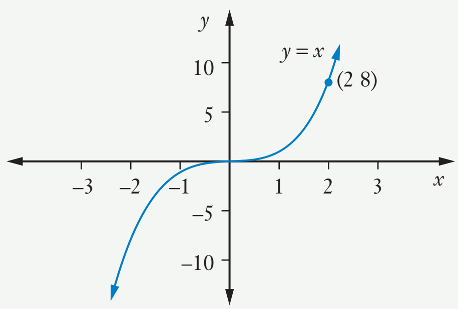
c



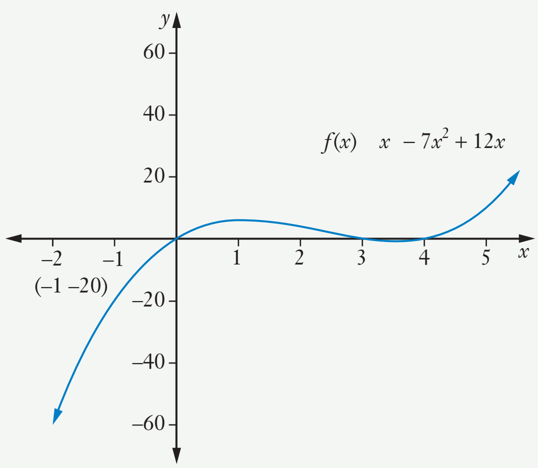
7 **a** $f(-x) = -(-x)^3 = x^3 = -f(x)$ so odd

b $y = x^3$ **ii** $y = -x^3$

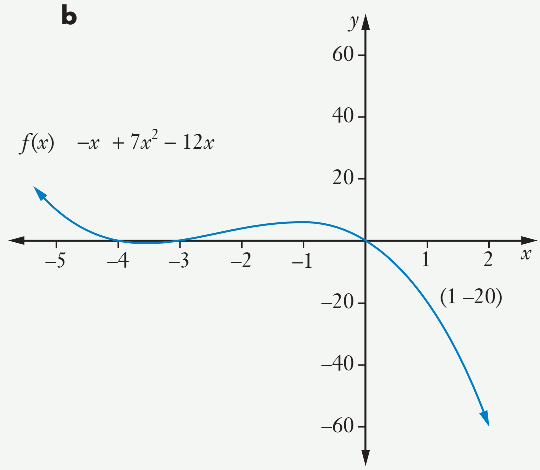
c



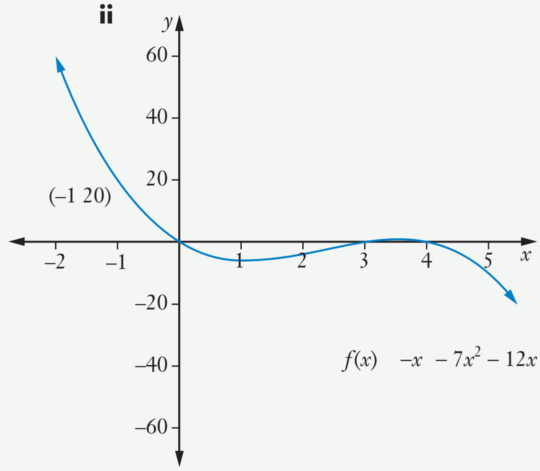
8 **a** x-intercepts 0 ,4; y-intercept 0



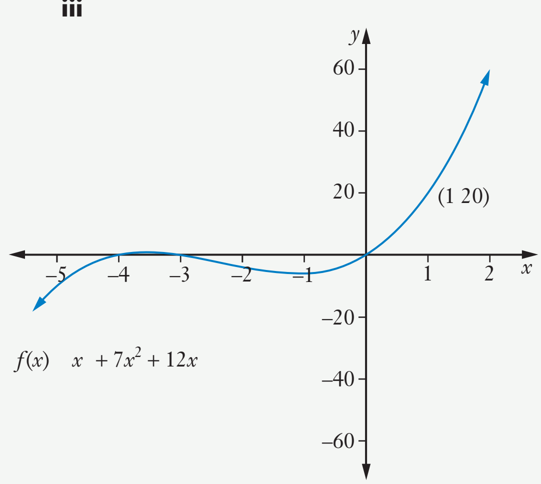
b



ii



iii



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Exercse 505

1 **a** $y = 2x^2 + 5x + 1$ **ii** $y = -2x^2 + 3x + 1$

iii $y = 8x^3 + 6x^2 + x$ **v** $y = \dfrac{4x+1}{2x^2+x}$

b $y = x^4 + x^3 + 5x + 1$ **ii** $y = x^4 - x^3 + 5x - 9$

iii $y = x^7 + 10x^4 - 4x^3 + 25x - 20$

v $y = \dfrac{x^4+5x-4}{x^3+5}$

c $y = 6x^2 - 7x + 1$ **ii** $y = -4x^2 + 7x + 5$

iii $y = 5x^4 - 7x^3 + 13x^2 - 21x - 6$

v $y = \dfrac{x^2+3}{5x^2-7x-2}$

d $y = 4x^2 + x + 4$ **ii** $y = 2x^2 + 3x - 6$

iii $y = 3x^4 - x^3 + 12x^2 + 11x - 5$

v $y = \dfrac{3x^2+2x-1}{x^2-x+5}$

e $y = 4x^5 + 3x + 3$ **ii** $y = 4x^5 - 3x + 11$

iii $y = 12x^6 - 16x^5 + 21x - 28$

iv $y = \dfrac{4x^5+7}{3x-4}$

2	**a**	1	**ii**	1	**iii**	2
	b	2	**ii**	2	**iii**	3
	c	2	**ii**	1	**iii**	4
	d	3	**ii**	3	**iii**	4
3	**a**	–3	**ii**	11	**iii**	–28
	b	–4	**ii**	6	**iii**	–5
	c	22	**ii**	28	**iii**	–75
	d	7	**ii**	7		

iii no constant term

4 **a** Domain (–∞ ∞) range (– ∞ ∞)
b Domain (–∞ ∞) range [–, ∞)
c Domain (–∞ ∞) range (– ∞ ∞)
d Domain (–∞ ∞) range [–.2, ∞)

5 **a** Domain (–∞ ∞) range (– ∞ ∞)
b Domain (–∞ ∞) range $\left[-\frac{\ }{4}\ \infty\right)$
c Domain (–∞ ∞) range (– ∞ ∞)
d Domain (–∞ ∞) range $[-45, \infty)$

6 **a** Domain (–∞ ∞) range [–, ∞)
b Domain (–∞ ∞) range [–2, ∞)
c Domain (–∞ ∞) range (– ∞ ∞)

7 **a** Domain (–∞ 4) ∪ (4 ∞)
b Domain (–∞ –1) ∪ (–1 ∞)
c Domain (–∞, 3) ∪ (3, ∞)
d Domain (–∞ 0) ∪ (0 ∞)

8 **a** $y = (x^2 + 1)^2 = x^4 + 2x^2 + 1$
b $y = (5x - 3)^3 = 125x^3 - 225x^2 + 135x - 27$
c $y = (x^2 - 3x + 2)^7$ **d** $y = \sqrt{2x-1}$
e $y = \sqrt[3]{x^4+7x^2-4}$ **f** $y = 6x + 3$
g $y = 2x^3 - 7$ **h** $y = 6x^2 - 5$
$y = 18x^2$ **j** $y = 4x^4 + 24x^2 + 37$

9 **a** Domain (–∞ ∞) range [, ∞)
b Domain (–∞ ∞) range (– ∞ ∞)
c Domain [2 ∞) range [, ∞)
d Domain [–3 ∞) range (– ∞ 0]
e Domain [–2 2, range 0, 2]
f Domain [–1 1, range [1, 0]

10 **a** $y = \sqrt{x^3}$ **b** $y = \left(\sqrt{x}\right)^3$

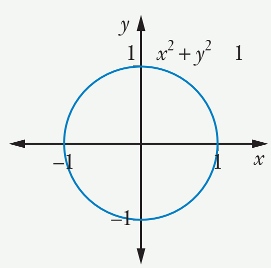
11 **a** $y = \dfrac{x^2+3}{x}$ **b** $y = \dfrac{1}{x^2+3}$
c $y = \dfrac{1}{x^3+3x}$ **d** $y = x^3 + 3x$

Test yourslf 5

1 B **2** A **3** A

4 **a** $A = \dfrac{150}{n}$
b 15 cm^2 **ii** 1875 cm^2
c 9 **ii** 6

5 **a**



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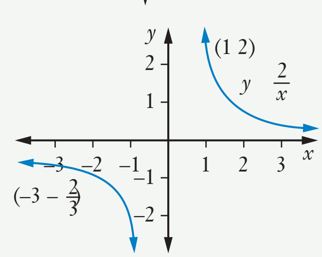

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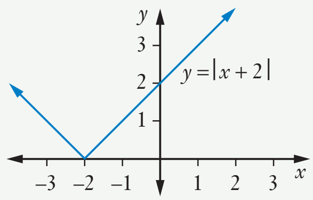

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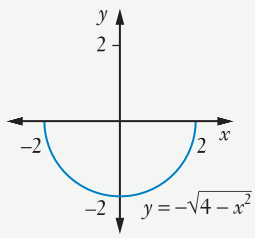

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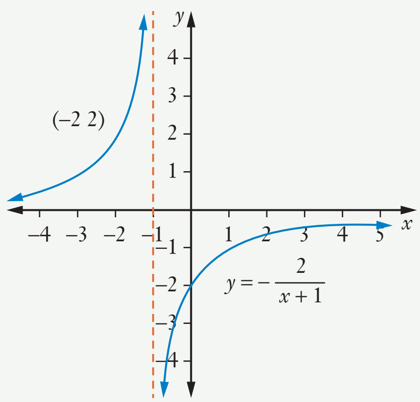

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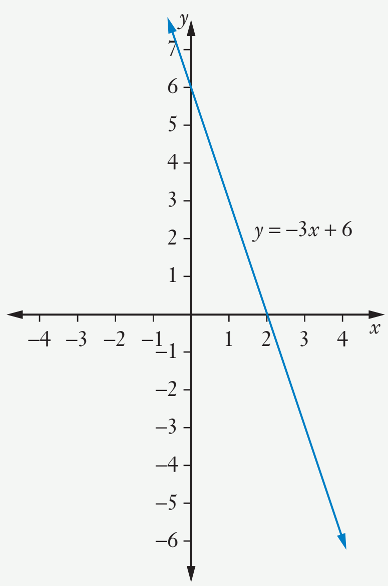

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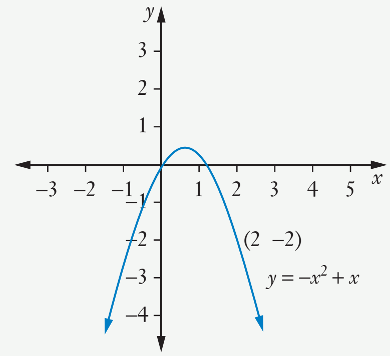


6 Radius 4 centre (, 1)

7 **a** $y = x^3 + 3x - 1$

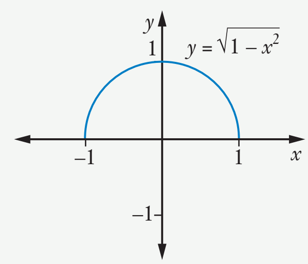
b $y = 3x^4 - x^3$

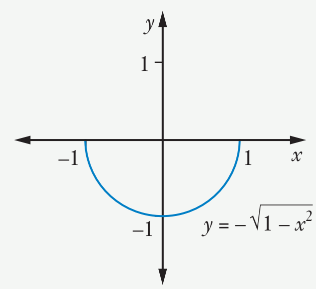
c $y = (3x - 1)^3 = 27x^3 - 27x^2 + 9x - 1$

d $y = 3x^3 - 1$

8 **a** No **b** $y = \pm\sqrt{1 - x^2}$

c





9 **a** Domain [−4 4, range [−4 4]

b Domain (−∞ −2) ∪ (−2, ∞), range (−∞ 0) ∪ (0 ∞)

c Domain (−∞ ∞) range [, ∞)

d Domain [−3 3, range 0, 3]

10 Domain (−∞ ∞) range [−4, ∞)

11 **a** Domain (−∞ 3) ∪ (3, ∞) range (−∞ 0) ∪ (0 ∞)

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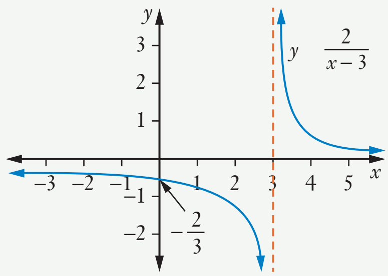
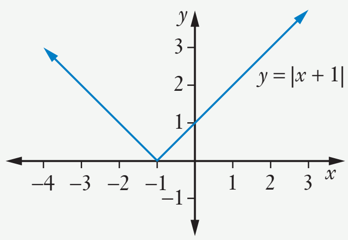
b



12 a



b $x = 2, -4$

13 $x = 1, 5$

14 a Radius 10 centre (, 0)

b Radius 11 centre (, 2)

c Radius 3 centre $(-3, -1)$

15 a x-intercepts 0 ±, y-intercept 0

b No x-intercepts y-intercept –2

c x-intercepts ±3 y-intercepts ±3

d x-intercepts ±5 y-intercept 5

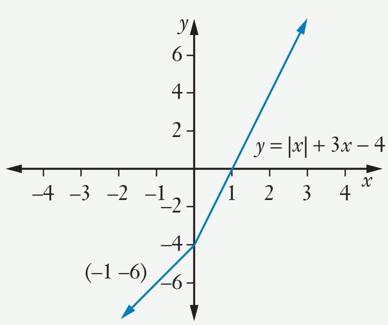
e No x-intercepts y-intercept 5

16 a 3 **b** $10x^5$ **c** –7

Chaenge exercse 5

1 $x = 3$

2 a



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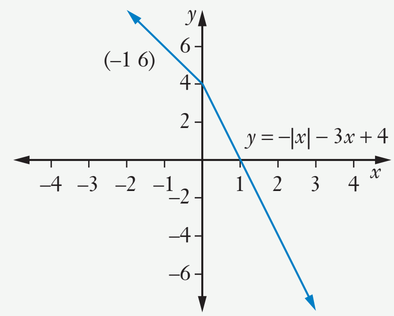
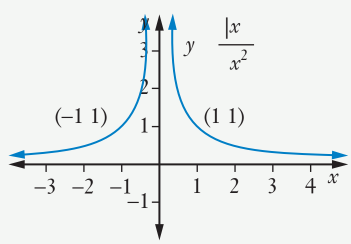


3 a $a = \frac{18}{b^2}$ **b** $a = 45$ **c** $b = 134$

4 Centre $\left(-1\frac{}{2}, 1\right)$ radius $2\frac{}{2}$

5 $3x + y + 2 = 0$

6

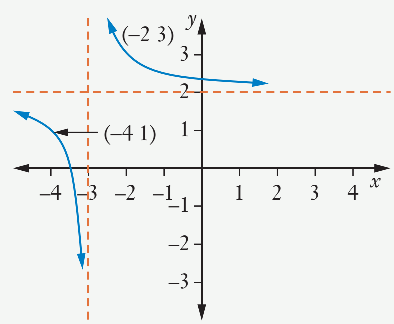


7 a $\frac{2(x+3)}{x+3} + \frac{1}{x+3} = \frac{2x+6+1}{x+3}$

$\therefore \frac{2x+7}{x+3} = 2 + \frac{1}{x+3}$

b Domain (–∞ –3) ∪ (–3, ∞), range (–∞ 2) ∪ (2 ∞)

c



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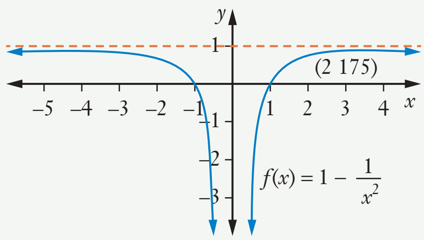
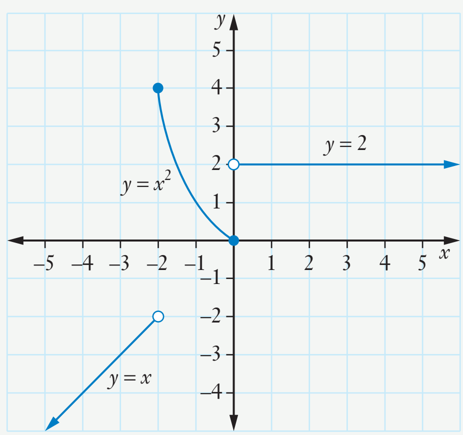
8 Both circles have same centre (1 −2) so concentric

9



10 **a**



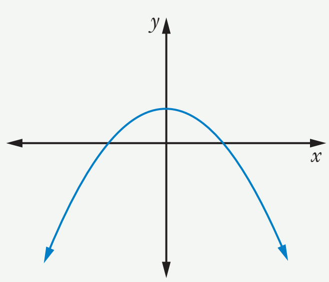
b $x = -2, 0$

c Domain $(-\infty \ \infty)$ range $(-\infty \ -2) \cup [0 \ 4]$

Chapter 6

Exercse 601

1



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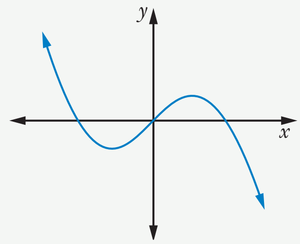

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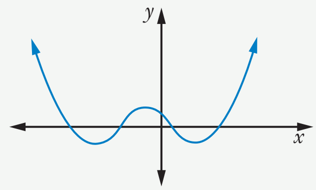

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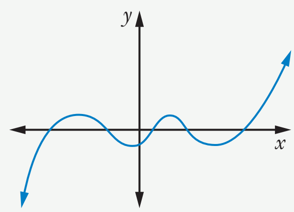

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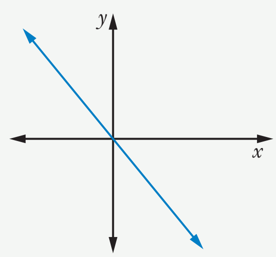

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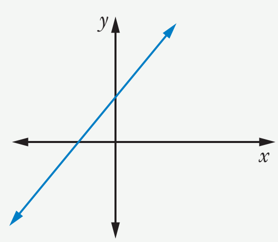

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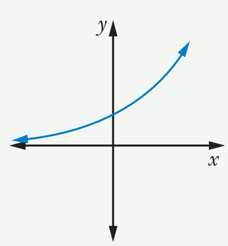

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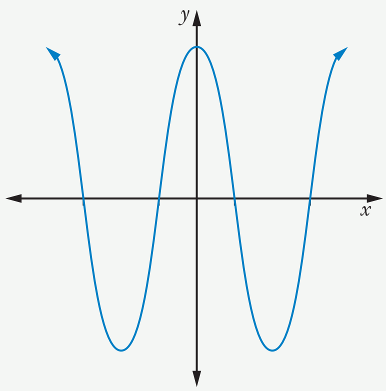

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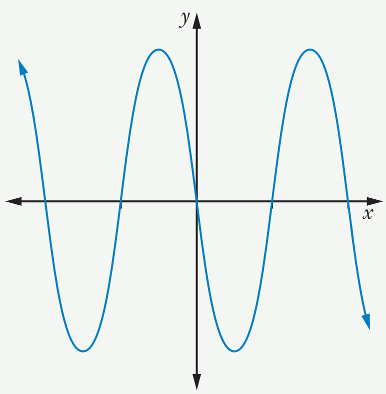

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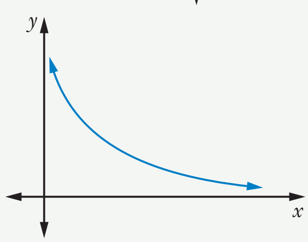

Exercse 602

1 $x = 0$ **2** $x = x$ **3** None
4 $x = 0$ **5** $x = x \quad x_2$
6 $x = 0$ **7** $x = -3$ **8** $x = 2$
9 $x = -2\ 3$ **10** $-1 \le x \le 0$ **11** $x = 0$

Exercse 603

1 a 406 **b** 3994 **c** 4
2 a 1361 **b** 130601 **c** 129401 13
3 6
4 a 11 **b** $h^2 + 5h + 11$ **c** $h^2 + 5h$
d $\frac{h^2+5h}{h} = \frac{h(h+5)}{h} = h+5$ **e** 5
5 a 1 **b** $4h^2 - 8h$ **c** -8

6 a 8 **b** $6h + h^2$ **c** 6
7 a -13 **b** 17
8 a $x^2 + 2xh + h^2$
b $x^2 + 2xh + h^2 - x^2 = 2xh + h^2$
c $\frac{2xh+h^2}{h} = \frac{h(2x+h)}{h} = 2x + h$
d $\lim_{h\to 0}(2x+h) = 2x$
9 a $2(x^2 + 2xh + h^2) - 7x - 7h + 3$
b $2x^2 + 4xh + 2h^2 - 7x - 7h + 3 - (2x^2 - 7x + 3)$
c $\frac{h(4x+2h-7)}{h}$ **d** $4x - 7$
10 a 2 **b** 5 **c** -12 **d** 15
e -9
11 a $2x$ **b** $2x + 5$ **c** $8x - 4$ **d** $10x - 1$

Exercse 604

1 a 1 **b** 5 **c** $2x + 3$ **d** $10x - 1$
e $3x^2 + 4x - 7$ **f** $6x^2 - 14x + 7$
g $12x^3 - 4x + 5$ **h** $6x^5 - 25x^4 - 8x^3$
$10x^4 - 12x^2 + 2x - 2$ **j** $40x^9 - 63x^8$
2 a $4x + 1$ **b** $8x - 12$ **c** $2x$
d $16x^3 - 24x$ **e** $6x^2 + 6x - 3$
3 a $\frac{x}{3} - 1$ **b** $2x^3 - x^2$ **c** $\frac{8x^7}{3} - 6x^5$
d $4x$ **e** $\frac{1}{4}$ **f** $2x^2 - 2x + 2$
4 $16x - 7$ **5** -56
6 $60x^9 - 40x^7 + 35x^4 - 3$ **7** $10t - 20$ **8** $20x^3$
9 $30t$ **10** $40 - 4t$ **11** $4\pi r^2$ **12** 3
13 a 5 **b** -5 **c** 4
14 a 12 **b** ± 2 **15** 18

Exercse 605

1 a $-3x^{-4}$ **b** $14\,x^{0.4}$ **c** $12\,x^{-0.8}$ **d** $\frac{1}{2}x^{-\frac{\ }{2}}$
e $x^{-\frac{\ }{2}} + 3x^{-2}$ **f** $x^{-\frac{2}{3}}$ **g** $6x^{-\frac{\ }{4}}$
h $x^{-\frac{3}{2}}$
2 a $-\frac{1}{x^2}$ **b** $\frac{5}{2\sqrt{x}}$ **c** $\frac{1}{6\sqrt[6]{x^5}}$

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d $-\frac{10}{x^6}$ e $\frac{15}{x^4}$ f $-\frac{1}{2\sqrt{x^3}}$

g $-\frac{3}{x^7}$ h $\frac{3\sqrt{x}}{2}$ $-\frac{2}{3x^2}$

j $-\frac{1}{2x^3}-\frac{12}{x^5}$

3 $\frac{1}{27}$ 4 -3 5 $\frac{1}{32}$ 6 -3

7 $2x+3\sqrt{x}+1$ 8 $\frac{1}{8}$

9 a $-\frac{1}{2\sqrt{x^3}}$ b $-\frac{1}{16}$

10 $a=4$ 11 $\left(5\ \frac{2}{5}\right)\left(-5\ -\frac{2}{5}\right)$

Exercse 606

1 a 72 b -13 c 11 d -18
e 18 f 27 g 11 h 136
-4 j 149

2 a $-\frac{1}{26}$ b $\frac{1}{25}$ c $\frac{1}{20}$ d $-\frac{1}{43}$
e $\frac{1}{10}$ f $\frac{1}{7}$ g $-\frac{1}{71}$ h $\frac{1}{20}$
$-\frac{1}{8}$ j $-\frac{1}{5}$

3 a 6 ii $-\frac{1}{6}$
b 8 ii $-\frac{1}{8}$
c 24 ii $-\frac{1}{24}$
d -8 ii $\frac{1}{8}$
e 11 ii $-\frac{1}{11}$

4 a $27x-y-47=0$ b $7x-y-1=0$
c $4x+y+17=0$ d $36x-y-47=0$
e $44t-v-82=0$

5 a $x+24y-555=0$ b $x-8y+58=0$
c $x-17y-516=0$ d $x-45y+3108=0$
e $x+2y-9=0$

6 a $7x-y+4=0$ ii $x+7y-78=0$
b $34x-y+72=0$ ii $x+34y+1023=0$
c $10x+y-6=0$ ii $x-10y-41=0$
d $2x+y+2=0$ ii $x-2y-19=0$

7 $x=\pm 3$ 8 (1 2) and $(-1\ 0)$
9 $(-5, -7)$ 10 (0 1)
11 (1 2) 12 $\left(-1\frac{3}{4}, -4\frac{15}{16}\right)$
13 a $(1\ -1)$ b $6x-y-7=0$
14 $10t-h-7=0$ 15 $4x-2y-19=0$
16 $3x+16y-8=0$ 17 $x-y+9=0$
18 $x+16y-16=0$ 19 (9 3)

Exercse 607

1 a $4(x+3)^3$ b $6(2x-1)^2$
c $70x(5x^2-4)^6$ d $48(8x+3)^5$
e $-5(1-x)^4$ f $135(5x+9)^8$
g $4(x-4)=4x-16$ h $4(6x^2+3)(2x^3+3x)^3$
$8(2x+5)(x^2+5x-1)^7$
j $6(6x^5-4x)(x^6-2x^2+3)^5$
$=12x(3x^4-2)\ (x^6-2x^2+3)^5$

k $\frac{3}{2}(3x-1)^{-\frac{\ }{2}}$ $2(4-x)^{-3}$

m $-6x(x^2-9)^{-4}$ n $\frac{5}{3}(5x+4)^{-\frac{2}{3}}$

o $\frac{3}{4}(3x^2-14x+1)(x^3-7x^2+x)^{-\frac{\ }{4}}$

p $\frac{3}{2\sqrt{3x+4}}$ q $-\frac{5}{(5x-2)^2}$

r $-\frac{8x}{(x^2+1)^5}$ s $-\frac{2}{\sqrt[3]{7-3x}}$

t $-\frac{5}{2\sqrt{(4+x)^3}}$ u $-\frac{3}{4\sqrt{(3x-1)^3}}$

v $-\frac{27}{2(2x+7)^{10}}$ w $-\frac{4x^3-9x^2+3}{(x^4-3x^3+3x)^2}$

x $\frac{16\sqrt[3]{4x+1}}{3}$ y $\frac{5}{4\sqrt[4]{(7-x)^9}}$

2 9 3 40 4 (4 1)

5 $x=2, -1\frac{\ }{2}$ 6 $8x+y+7=0$

7 $16x-y-15=0$ 8 $x+9y+8=0$

9 $x+64y-1025=0$

10 a $2x+y+1=0$ b $x-2y+3=0$

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Exercse 608

1 **a** $8x^3 + 9x^2$ **b** $12x - 1$
c $30x + 21$ **d** $72x^5 - 16x^3$
e $30x^4 - 4x$ **f** $x(5x + 2)(x + 1)^2$
g $8(9x - 1)(3x - 2)^4$ **h** $3x^3(16 - 7x)(4 - x)^2$
$(10x + 13)(2x + 5)^3$

2 26 **3** 1264

4 $\frac{8\sqrt{7}}{7}$ **5** 176

6 $10x - y - 9 = 0$ **7** $69t - h - 129 = 0$

8 $\frac{-6 \pm \sqrt{30}}{3}$ **9** $34x - y + 29 = 0$

Exercse 609

1 **a** $\frac{-2}{(2x-1)^2}$ **b** $\frac{15}{(x+5)^2}$

c $\frac{x^4 - 12x^2}{(x^2-4)^2} = \frac{x^2(x^2-12)}{(x^2-4)^2}$

d $\frac{16}{(5x+1)^2}$ **e** $\frac{-x^2+14x}{x^4} = \frac{14-x}{x^3}$

f $\frac{11}{(x+3)^2}$ **g** $\frac{-2x^2-1}{(2x^2-1)^2}$

h $\frac{-6}{(x-2)^2}$ $\frac{-34}{(4x-3)^2}$

j $\frac{-14}{(3x+1)^2}$ **k** $\frac{-3x^2-6x-7}{(3x^2-7)^2}$

$\frac{4x^2-12x}{(2x-3)^2} = \frac{4x(x-3)}{(2x-3)^2}$

m $\frac{-18x}{(x^2-5)^2}$

n $\frac{2x^3+12x^2}{(x+4)^2} = \frac{2x^2(x+6)}{(x+4)^2}$

o $\frac{2x^3+9x^2+7}{(x+3)^2}$ **p** $\frac{3x^2+8x-5}{(3x+4)^2}$

q $\frac{2(x+5)^{\frac{}{2}} - x(x+5)^{-\frac{}{2}}}{x+5}$

r $\frac{(7x+2)^4 - 28(x-1)(7x+2)^3}{(7x+2)^8} = \frac{30-21x}{(7x+2)^5}$

s $\frac{3\sqrt{x+1} - \frac{3x+1}{2\sqrt{x+1}}}{x+1} = \frac{3x+5}{2\sqrt{(x+1)^3}}$

t $\frac{\frac{2x-3}{2\sqrt{x-1}} - 2\sqrt{x-1}}{(2x-3)^2} = \frac{-2x+1}{2\sqrt{x-1}(2x-3)^2}$

2 $\frac{1}{8}$ **3** $-1\frac{5}{9}$ **4** $x = 0, 1$

5 $x = -9\ 3$ **6** $x - 18y + 8 = 0$

7 $17x - 25y - 19 = 0$

Exercse 610

1 **a** $\frac{dh}{dt} = 20 - 8t$ **b** $\frac{dD}{dt} = 15t^2 + 4t$

c $\frac{dA}{dx} = 16 - 4x$ **d** $\frac{dx}{dt} = 15t^4 - 4t^3 + 2$

e $\frac{dV}{dr} = 4\pi r^2$ **f** $\frac{dS}{dr} = 2\pi - \frac{100}{r^3}$

g $\frac{dD}{dx} = \frac{x}{\sqrt{x^2-4}}$ **h** $\frac{dS}{dr} = 800 - \frac{400}{r^2}$

2 **a** 30 **b** 20

3 13 L/s

4 **a** −7 g/min **ii** −13 g/min
b 19 g/min

5 181 cm^2/s **6** 41 cm^2/min

7 **a** 11 km **ii** 116 km **iii** 286 km
b 65 km/h **ii** 105 km/h

8 −025

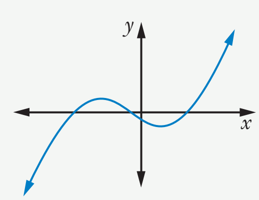
9 **a** 18 cm s^{-} **b** 12 cm s^{-2}
c When $t = 0$, $x = 0$; at 3 s
d 5 s

10 **a** −8 m s^{-}
b $a = 4$ constant acceleration of 4 m s^{-2}
c 13 m **d** 2 s **e** −5 m

Test yourslf 6

1 C **2** B **3** D **4** B, C

5 **a**



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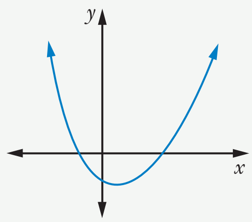


6 $\frac{dy}{dx} = 10x - 3$

7 **a** $42x^5 - 9x^2 + 2x - 8$ **b** $-12x^{-5}$

c $-\frac{8}{(x+1)^5}$ **d** $\frac{dy}{dx} = \frac{5\sqrt{x^3}}{2}$

e $18(x + 2)(x^2 + 4x - 2)^8$

f $\frac{7}{(2x+1)^2}$

g $3x^2(3x + 1)^5(9x + 1)$

8 $4t - 3$ **9** 10 **10** 42

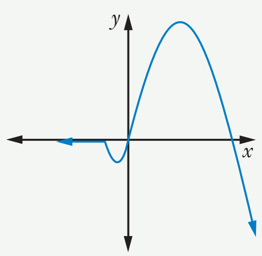
11 **a** $x = -2$ **b** $x = 1$ **c** $x = 2$

12 **a** $-\frac{4}{x^2}$ **b** $\frac{1}{5\sqrt[5]{x^4}}$ **c** $32(4x + 9)^3$

d $3(x - 1)^2(4x + 1)$

e $\frac{4x^3+15x^2+6}{(2x+5)^2}$

13



14 $9x - y - 7 = 0$ **15** (2 3)

16 $8\pi r$ **17** –09

18 (–2 71, 5, –272) **19** $4x - y - 6 = 0$

20 **a** $u + at$ **b** $\frac{1}{5}$ **21** $12x + y - 4 = 0$

22 **a** 0 m **ii** 4 m **iii** 3 m **v** 175 m

b 1 m s⁻ **ii** –1 m s⁻

c 4 m s⁻ **ii** 0 m s⁻ **iii** –2 m s⁻

23 **a** $x^2 + 2xh + h^2 - 3x - 3h + 5$

b $2xh + h^2 - 3h$ **c** $2x - 3$

24 **a** 1 **b** 20 **25** 9

26 **a** $24(x - 3)(x^2 - 6x + 1)^3$

b $-\frac{3}{(\sqrt{3x-1})^3}$

27 **a** 0 m 0 m s⁻ 8 m s^{-2}

b 0 .8 s **c** –44 m s⁻ –52 m s^{-2}

Chaenge exercse 6

1 $2x + y = 0$, $3x - y - 3 = 0$, $6x - y + 12 = 0$

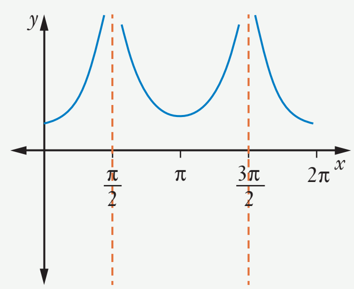
2 **a** (2 2, (2, –14)

b $x + 12y - 26 = 0$, $x + 12y + 170 = 0$

3 $\left(-2\frac{}{4}, 6\frac{}{16}\right)$ **4** $n = 8$

5 **a** $\frac{\pi}{2}$ $\frac{3\pi}{2}$

b



6 $\frac{5\sqrt{22}}{22}$ **7** $2x + y - 25 = 0$

8 **a** $16x + 32y + 1 = 0$, $4x - 2y - 1 = 0$

b $m_1m_2 = -\frac{1}{2} \times 2 = -1$ so perpendicular

9 $x = 0, 2, 6$

10 $x - y - 4 = 0$

11 $a = -1, b = 2, c = 4$

12 **a** Proof includes $x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3$

b Proof includes $\lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h}$

13 **a** –0909 **ii** –099 **iii** –101

b –1

c Proof includes $\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}$

d Proof includes $\lim_{h\to 0} \frac{-h}{x(x+h)} \times \frac{1}{h}$

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14 a $1 \text{ m } 0 \text{ m s}^{-}$ **b** $326 \times 10^{7} \text{ m s}^{-2}$
c $(t^3 + 1)^6 = 0$ has solution $t = -1$ but time $t \geq 0$

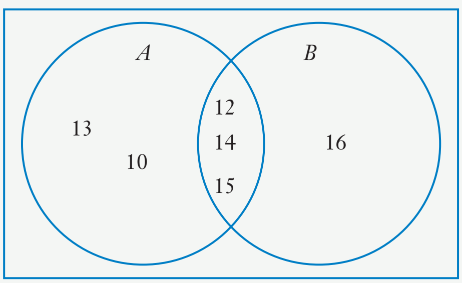
Chapter 7

Exercse 701

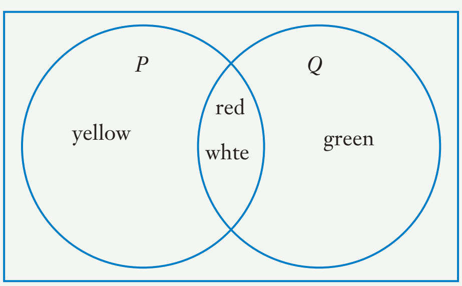
1 a {H T} **b** {1, 2, 3, 4, 5}
c {1, 2, 3, 4, 5, 6}
d {red gree, yellw, blue}
e {1, 2, 3, 4, 5, 6, 7, 8}

2 a {2 4} **ii** {1, 2, 3, 4, 5, 6}
b {red white} **ii** {red yello, white}
c { } **ii** {4, 5, 6, 7, 8, 9, 10, 11, 12, 15}
d {brown blue}
ii {blue gree, bron, hael, grey}
e { } **ii** {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

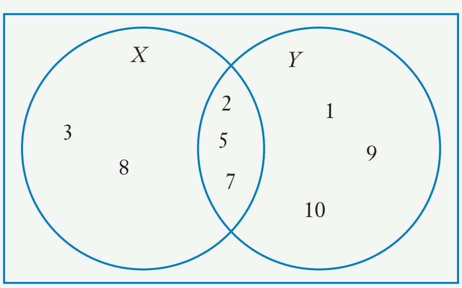
3 a



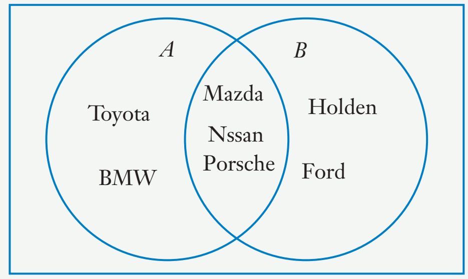
b



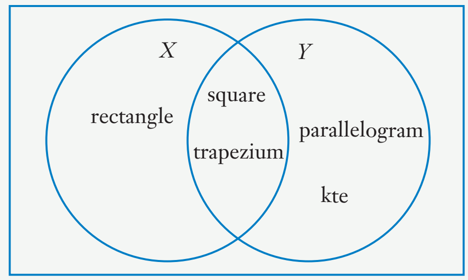
c



d



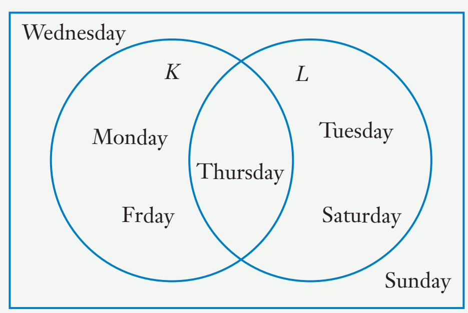
e



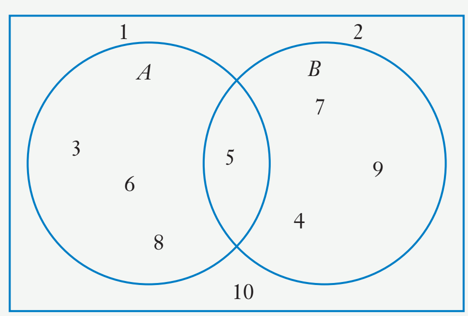
4 a False Each horse has different abilities and chances
b False Each player has different abilities and chances
c False Previous outcomes have no influenc.
d False Previous outcomes have no influenc.
e False Each bike rider has different abilities and chances

5 a {1, 2, 3, 4, 5, 6} **b** {2 , 6} **c** $\frac{1}{2}$

6 a



b



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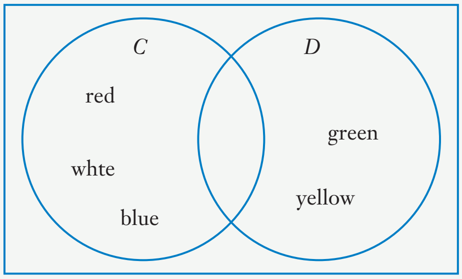
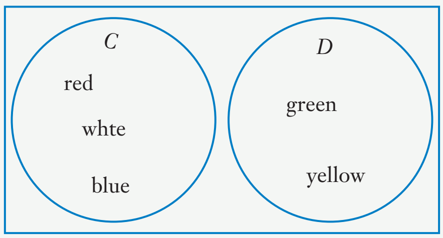
c 

or 

Exercse 702

1 **a**

Score	Frequency	Relative frequency
4	6	$\frac{6}{23}$
5	4	$\frac{4}{23}$
6	1	$\frac{1}{23}$
7	7	$\frac{7}{23}$
8	2	$\frac{2}{23}$
9	3	$\frac{3}{23}$

b $\frac{2}{23}$ **ii** $\frac{11}{23}$ **iii** $\frac{17}{23}$

c 7 **ii** 6

2 **a**

Number of days	Frequency	Relative frequency
1	3	15%
2	6	30%
3	1	5%
4	7	35%
5	2	10%
6	1	5%

b 4

c 15% **ii** 10% **iii** 40% **v** 50%

v 45%

3 **a**

Class	Frequency	Relative frequency
0–19	9	018
20–39	12	024
40–59	18	036
60–79	7	014
80–99	4	008

b 024 **ii** 022 **iii** 042

4 **a**

		$\frac{1}{10}$
		$\frac{7}{20}$
		$\frac{1}{5}$
		$\frac{1}{20}$
		$\frac{3}{10}$

$\frac{7}{20}$ $\frac{1}{20}$ $\frac{1}{5}$ $\frac{7}{20}$

$\frac{13}{20}$

$\frac{9}{20}$ $\frac{9}{10}$ $\frac{11}{20}$

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6 **a**

Score	Frequency
3	1
4	4
5	3
6	3
7	3
8	2
9	4

b $\frac{3}{20}$ **ii** $\frac{3}{10}$ **iii** $\frac{1}{4}$ **v** $\frac{7}{10}$

7 **a**

Score	Frequency
4	1
5	5
6	3
7	6
8	2
9	3

b 01 **ii** 07 **iii** 045 **v** 095
v 085

8 **a**

Ages	Frequency
10–19	3
20–29	4
30–39	8
40–49	5
50–59	5

b 32% **ii** 12% **iii** 40%
c 60% **ii** 20% **iii** 68% **v** 72%
v 80%

9 **a** February (28 days) **b** $\frac{1}{7}$ **c** 50%

d

Food (kg)	Frequency	Relative frequency
0–14	3	$\frac{3}{28}$
15–29	11	$\frac{11}{28}$
30–44	8	$\frac{2}{7}$
45–59	4	$\frac{1}{7}$
60–74	2	$\frac{1}{14}$

e $\frac{2}{7}$ **ii** $\frac{3}{14}$ **iii** $\frac{1}{2}$ **v** $\frac{23}{28}$
v $\frac{1}{2}$

Exercse 703

1 **a** $\frac{1}{30}$ **b** $\frac{29}{30}$

2 **a** $\frac{1}{52}$ **b** $\frac{51}{52}$ **3** $\frac{1}{6}$

4 **a** $\frac{1}{40}$ **b** $\frac{39}{40}$ **5** $\frac{1}{20\,000}$

6 **a** $\frac{4}{7}$ **b** $\frac{3}{7}$ **7** $\frac{3}{37}$

8 **a** $\frac{5}{11}$ **b** 192 **9** **a** $\frac{11}{20}$ **b** $\frac{3}{4}$

10 **a** $\frac{2}{9}$ **b** 147 **11** 998%

12 **a** $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** $\frac{1}{3}$

13 **a** $\frac{1}{62}$ **b** $\frac{3}{31}$ **c** $\frac{1}{2}$ **d** $\frac{99}{124}$

14 **a** 985% **b** 39 **ii** 2561

15 $\frac{7}{8}$ **16** **a** $\frac{8}{15}$ **b** $\frac{7}{15}$ **c** $\frac{1}{15}$

17 **a** $\frac{1}{2}$ **b** 1

18 **a** $\frac{7}{31}$ **b** $\frac{7}{31}$ **c** $\frac{12}{31}$

19 $\frac{25}{43}$ **20** 34 **21** $\frac{1}{3}$

22 **a** $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{5}{6}$

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23 $\frac{46}{49}$ **24** **a** $\frac{2}{15}$ **b** $\frac{13}{15}$

25 $A \cap B = 0$

Exercse 704

1 **a** $\frac{3}{10}$ **b** $\frac{3}{5}$ **c** $\frac{11}{20}$ **d** $\frac{7}{10}$

2 **a** $\frac{1}{5}$ **b** $\frac{1}{2}$ **c** $\frac{3}{5}$ **d** $\frac{3}{5}$

e $\frac{19}{50}$

3 **a** $\frac{5}{26}$ **b** $\frac{9}{26}$ **c** $\frac{12}{13}$

4 **a** $\frac{29}{100}$ **b** $\frac{13}{20}$ **c** $\frac{9}{25}$

5 **a** $\frac{3}{5}$ **b** $\frac{4}{9}$ **c** $\frac{2}{3}$

6 **a** $\frac{3}{14}$ **b** $\frac{13}{28}$ **c** $\frac{9}{28}$

7 **a** $\frac{21}{80}$ **b** $\frac{17}{80}$ **c** $\frac{21}{40}$

8 **a** $\frac{1}{10}$ **b** $\frac{11}{20}$ **c** $\frac{7}{20}$

9 **a** $\frac{7}{25}$ **b** $\frac{2}{15}$ **c** $\frac{44}{75}$

10 **a** $\frac{3}{10}$ **b** $\frac{2}{5}$ **c** $\frac{3}{10}$

Exercse 705

1 $\frac{1}{4}$ **2** $\frac{1}{8}$

3 $\frac{1}{4}$ **4** $\frac{25}{121}$

5 **a** 00441 **b** 06241

6 804% **7** 329%

8 **a** $\frac{29\ 791}{35\ 937}$ **b** $\frac{8}{35\ 937}$ **c** $\frac{6146}{35\ 937}$

9 **a** $\frac{1}{2400}$ **b** $\frac{1}{5\ 760\ 000}$ **c** $\frac{5\ 755\ 201}{5\ 760\ 000}$

10 **a** $\frac{1}{7776}$ **b** $\frac{3125}{7776}$ **c** $\frac{4651}{7776}$

11 **a** $\frac{9}{25\,000\,000}$ **b** $\frac{24\,970\,009}{25\,000\,000}$ **c** $\frac{29\,991}{25\,000\,000}$

12 **a** $\frac{1}{4}$ **b** $\frac{9}{100}$ **c** $\frac{9}{100}$

13 **a** 6141% **b** 034% **c** 9966%

14 **a** $\frac{1}{2^n}$ **b** $1-\frac{1}{2^n}=\frac{2^n-1}{2^n}$

15 **a** $\frac{9}{49}$ **b** $\frac{15}{91}$ **16** $\frac{3}{2075}$

17 $\frac{19}{99}$

18 **a** $\frac{2}{275}$ **b** $\frac{49}{198}$ **c** $\frac{316}{495}$

19 **a** $\frac{253}{861}$ **b** $\frac{57}{287}$ **20** $\frac{14}{95}$

Exercse 706

1 **a** $\frac{1}{8}$ **b** $\frac{3}{8}$ **c** $\frac{7}{8}$

2 **a** $\frac{1}{900}$ **b** $\frac{1}{900}$ **c** $\frac{1}{450}$

3 **a** $\frac{25}{169}$ **b** $\frac{80}{169}$

4 **a** 275% **b** 239% **c** 725%

5 **a** $\frac{189}{1000}$ **b** $\frac{441}{1000}$ **c** $\frac{657}{1000}$

6 **a** 0325 **b** 00034 **c** 0997

7 **a** $\frac{4}{27}$ **b** $\frac{1}{6}$

8 **a** $\frac{1}{25}$ **b** $\frac{1}{825}$ **c** $\frac{64}{825}$ **d** $\frac{152}{165}$

e $\frac{13}{165}$

9 **a** $\frac{16}{75}$ **b** $\frac{38}{75}$

10 **a** $\frac{11}{20}$ **b** $\frac{3}{20}$

11 **a** $\frac{84\,681}{1000\,000}$ **b** $\frac{912\,673}{1000\,000}$ **c** $\frac{27}{1000\,000}$

12 **a** 176% **b** 11% **c** 212%

13 **a** $\frac{1488}{3025}$ **b** $\frac{1}{121}$

14 **a** $\frac{22}{425}$ **b** $\frac{368}{425}$ **c** $\frac{7}{425}$

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15 **a** $\frac{17}{65}$ **b** $\frac{133}{715}$ **c** $\frac{496}{2145}$

16 **a** $\frac{1}{216}$ **b** $\frac{5}{72}$ **c** $\frac{91}{216}$

17 **a** $\frac{1}{10}$ **b** $\frac{3}{10}$ **c** $\frac{2}{5}$

18 **a** $\frac{25}{81}$ **b** $\frac{40}{81}$ **c** $\frac{56}{81}$

Exercse 707

1 **a** $\frac{9}{16}$ **b** $\frac{7}{16}$

2 $\frac{13}{27}$ **3** **a** $\frac{1}{6}$ **b** $\frac{1}{3}$ **4** 75%

5 867% **6** 5625% **7** **a** $\frac{3}{8}$ **b** $\frac{2}{5}$

8 **a** $\frac{9}{19}$ **b** $\frac{4}{19}$ **c** $\frac{5}{19}$

9 **a** $\frac{4}{9}$ **b** $\frac{3}{7}$ **10** **a** $\frac{5}{14}$ **b** $\frac{8}{13}$

11 **a** $\frac{23}{31}$ **ii** $\frac{65}{68}$ **b** $\frac{23}{88}$ **ii** $\frac{8}{11}$

12 **a** $\frac{23}{102}$ **b** $\frac{32}{51}$ **c** $\frac{64}{143}$ **d** $\frac{38}{61}$

13 **a** $\frac{7}{11}$ **b** $\frac{13}{27}$

14 **a** $\frac{17}{61}$ **b** $\frac{105}{278}$ **c** $\frac{85}{138}$ **d** $\frac{34}{155}$

e $\frac{102}{295}$

15 **a** 45% **b** 55% **c** 76% **16** 299%

17 **a** 79% **b** 174% **c** 826% **d** 818%

18 02077

19 $P(L \cap M) = 00204$

$P(L)P(M) = 017 \times 012 = 00204$

Since $P(L \cap M) = P(L)P(M)$, L and M are independent

20 $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

$0594 = 03 + 042 - P(X \cap Y)$

$P(X \cap Y) = 03 + 042 - 0594$

$= 0126$

$P(X \cap Y) = P(X)\,P(Y|X)$

$0.126 = 03 \times P(Y|X)$

$042 = P(Y|X)$

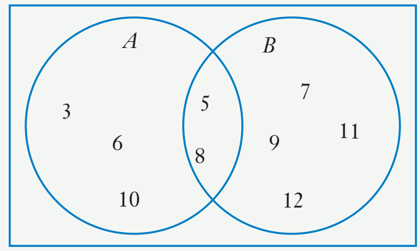
Since $P(Y|X) = P(Y) = 042$ X and Y are independent

Test yourslf 7

1 C **2** D **3** B **4** A, D

5 **a** {3, 5, 6, 7, 8, 9, 10, 11, 12} **ii** {5 8}

b



6 **a** {HH T, TH, TT} **b** {red whit, blue}

7 **a**

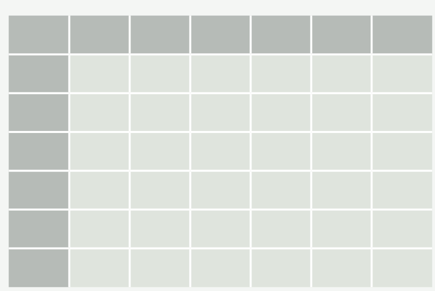
Face	Frequency	Relative frequency
1	17	$\frac{17}{100}$
2	21	$\frac{21}{100}$
3	14	$\frac{7}{50}$
4	20	$\frac{1}{5}$
5	18	$\frac{9}{50}$
6	10	$\frac{1}{10}$

b $\frac{7}{50}$ **ii** $\frac{7}{25}$ **iii** $\frac{1}{10}$ **v** $\frac{19}{50}$

v $\frac{13}{25}$

8 **a** 804% **b** 14% **c** 9997%

9 **a**



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b $\frac{1}{6}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{2}$

10 **a** $\frac{1}{40}$ **ii** $\frac{39}{40}$ **b** $\frac{39}{796}$

11 **a** $\frac{4}{15}$ **b** $\frac{1}{10}$ **c** $\frac{3}{17}$ **d** $\frac{3}{11}$

12 False the events are independent

13 **a** $\frac{1}{2}$ **b** $\frac{29}{100}$ **c** $\frac{1}{5}$ **d** $\frac{11}{25}$

e $\frac{16}{25}$

14 **a** $\frac{2}{5}$ **b** $\frac{7}{15}$ **c** $\frac{2}{15}$

15 $\frac{35}{66}$ **16** $\frac{1}{56}$

17 **a** 0009% **b** 129%

18 **a** $\frac{1}{13}$ **ii** $\frac{3}{13}$ **iii** $\frac{5}{26}$

b $\frac{1}{3}$ **ii** $\frac{1}{4}$

19 **a** $\frac{5}{12}$ **b** $\frac{1}{3}$

20 **a** $\frac{9}{40}$ **b** $\frac{3}{10}$ **ii** $\frac{27}{160}$ **iii** $\frac{4}{25}$

21 **a** $\frac{1}{200}$ **b** $\frac{81}{200}$ **c** $\frac{11}{100}$

22 **a** $\frac{1}{15}$ **b** $\frac{4}{5}$

23 **a** $\frac{1}{50}$ **b** $\frac{147}{7450}$ **c** $\frac{1}{3725}$ **d** $\frac{3577}{3725}$

24 **a** $\frac{80}{361}$ **b** $\frac{40}{171}$

25 **a** $\frac{2}{9}$ **b** $\frac{1}{3}$ **c** $\frac{3}{7}$

26 **a** $\frac{21}{50}$ **b** $\frac{3}{25}$ **c** $\frac{23}{50}$

Chaenge exercse 7

1 **a** $\frac{38}{119}$ **b** $\frac{10}{119}$

2 **a** 004 **b** 075 **c** 025 **3** $\frac{1}{54145}$

4 **a** $\frac{4}{13}$ **b** $\frac{25}{52}$ **c** $\frac{4}{13}$

5 No – all combinations are equally likely to win

6 **a** $\frac{3}{10}$ **b** $\frac{12}{145}$

7 **a** $\frac{1}{144}$ **b** $\frac{5}{144}$ **c** $\frac{1}{165\,888}$

8 146%

Practice set 3

1 B **2** D **3** C **4** D

5 D **6** C **7** B **8** B

9 A **10** A **11** D

12 **a** $9x^8 - 8x + 7$ **b** $6x^2 - 2$ **c** $-12x^{-5}$

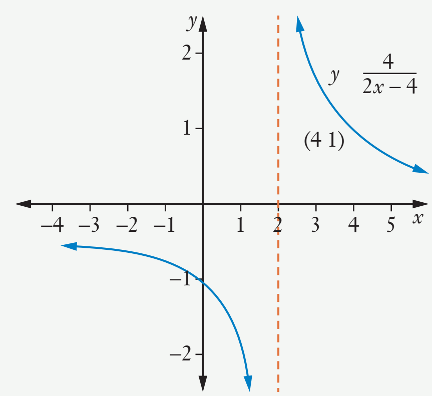
d $-\frac{25}{2x^6}$ **e** $\frac{3\sqrt{x}}{2}$

f $14(2x+3)^6$ **g** $-\frac{8x}{(x^2-7)^5}$

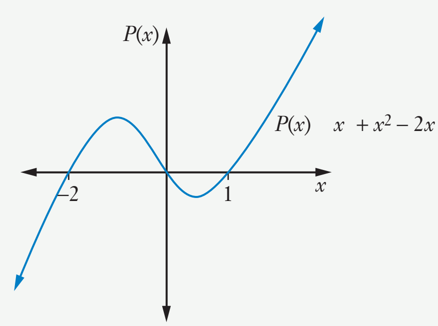
h $\frac{5}{3\sqrt[3]{(5x+1)^2}}$

$\frac{2(5x^2+15x+1)}{(2x+3)^2}$

13 **a**



b



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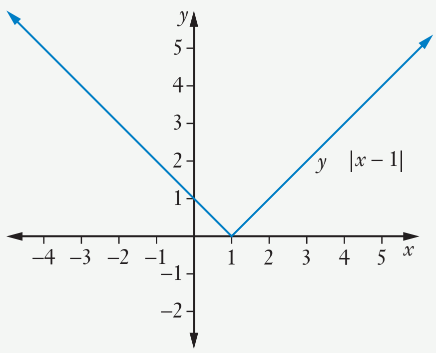
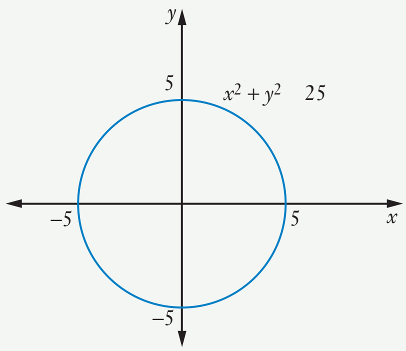
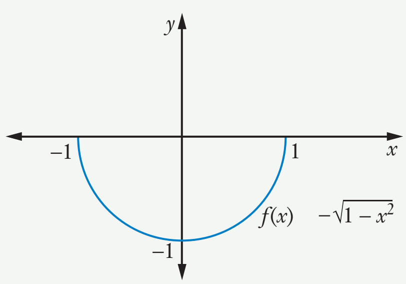
c



d



e



14 **a** $\frac{3}{25}$ **b** $\frac{8}{25}$ **c** $\frac{8}{25}$

15 **a** 100 L **b** 40 L

c –16 (leaking at the rate of 16 L/min)

d 122 min

16 **a** $9x - y + 16 = 0$ **b** $x + 9y + 20 = 0$

c $Q = (-20\ 0)$

17 **a** $\frac{1}{36}$ **b** $\frac{1}{6}$ **c** $\frac{11}{36}$

d $\frac{5}{36}$ **e** $\frac{5}{12}$

18 $a = 1, b = -3\ \ c = -1$

19 $x^2 + 4x + y^2 + 6y - 12 = 0$

20 **a** Centre (–3 5, radius 7

b Centre (–5 3, radius 2

21 **a** $3h^2 + 6xh - 4h$

b Proof involves $\lim_{h \to 0} \frac{3h^2 + 6xh - 4h}{h}$

22 **a** $3x - y - 4 = 0$ **b** $x - y - 2 = 0$

c $x + 3y + 10 = 0$ **d** $R = (-10\ 0)$

23 **a** $\frac{7}{50}$ **b** $\frac{11}{20}$

24 **a** $\frac{3}{91}$ **b** $\frac{30}{91}$

25 $\frac{1}{110}$

26 **a** $\frac{5}{36}$ **b** $\frac{5}{12}$ **c** $\frac{1}{6}$ **d** $\frac{7}{36}$

e $\frac{1}{2}$

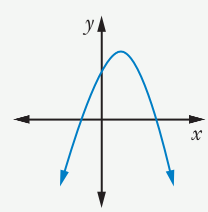
27 **a** $\frac{3}{10}$ **b** $\frac{3}{8}$

28 **a** $\frac{7}{20}$ **b** $\frac{1}{4}$

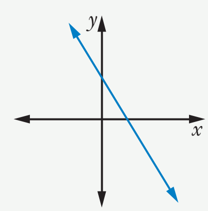
29 $f(-2) = -45\ \ f\ (-2) = 48$

30 **a** 241202 **ii** 238802 **b** 24

31 **a**



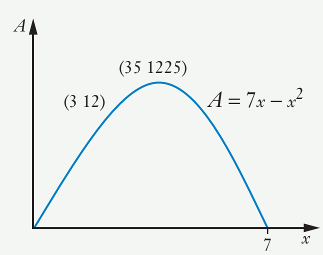
b



32 **a** 12 m^2 **ii** 1125 m^2

b 56 m or 14 m

c



d 1225 m^2

33 $x = 1, -5$

34 **a** 5 **b** 1 **c** –3

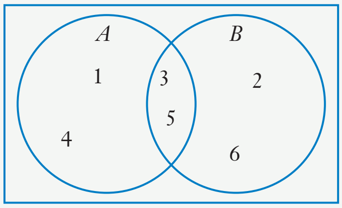
35 **a** 12 cm 17 cm s^{-} –26 cm s^{-2}

b 10 cm –11 cm s^{-} –2 cm s^{-2}

ISBN 9780704352

36 **a** $\{1, 2, 3, 4, 5, 6\}$ **b** $\{3\ 5\}$

c



37 $y = 6x - 5$

38 **a** $-3x^{-4}$ **b** $\dfrac{3\sqrt{x}}{2}$ **c** $-\dfrac{2}{x^3}$

d $\dfrac{(7x+4)(21x-26)}{(3x-1)^2}$

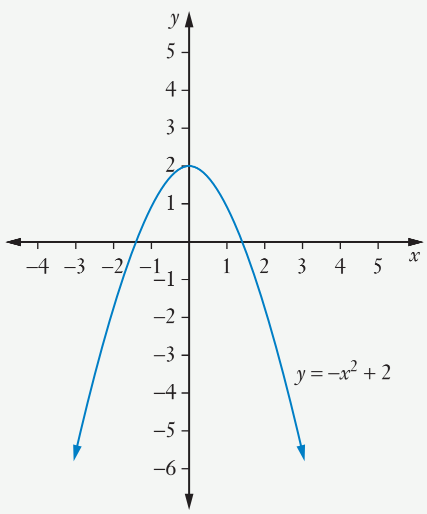
e $2(2x-3)^3(30x^2-15x+4)$

f $15(3x+1)^4$ **g** $\dfrac{1}{\sqrt{2x-1}}$

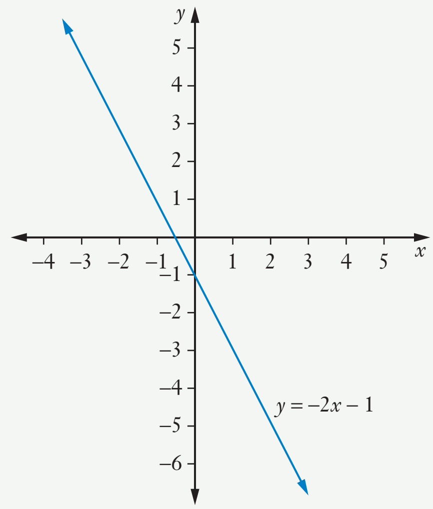
39 **a** $y = x^2 + 2x - 3$ **ii** $y = 2x^3 - x^2 - 4x + 2$

iii $y = -x^2 + 2x + 1$ **v** $y = \dfrac{2x-1}{x^2-2}$

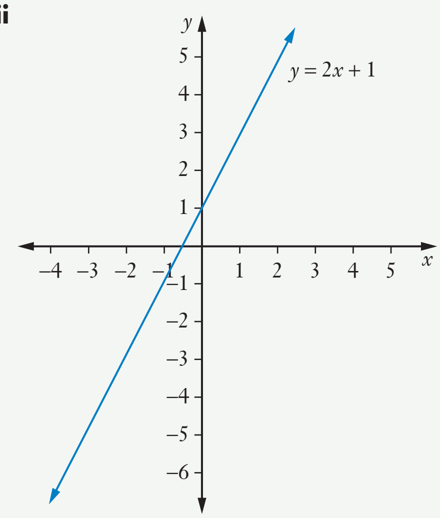
b



ii



iii



40 **a** Centre (–1 3, radius 4

b Domain [–5 3, range [1, 7]

41 $x + 2y + 1 = 0$

42 **a** $8x^3 - 15x^2 + 6x - 1$ **b** $-\dfrac{5}{2x^6}$

c $\dfrac{1}{2\sqrt{x}}$ **d** $14(2x-3)^6$

e $6x^3(2x-5)^6(11x-10)$ **f** $-\dfrac{31}{(3x-2)^2}$

43 **a** 2 **b** 3

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44 **a** $\frac{1}{12}$ **b** $\frac{1}{4}$

45 **a** Domain (−∞ ∞) range (− ∞ ∞)

b Domain (−∞ ∞) range (− ∞, 1]

c Domain [−7 3, range [4, 6]

d Domain (−∞ −2) ∪ (−2 ∞), range (−∞ 0) ∪ (0 ∞)

46 **a** $y = (2x + 5)^3$ **b** $y = 2x^3 + 5$

47 **a**

Result	Frequency	Relative frequency
HH	24	$\frac{6}{25}$
HT	15	$\frac{3}{20}$
TH	38	$\frac{19}{50}$
TT	23	$\frac{23}{100}$

b $\frac{23}{100}$ **ii** $\frac{53}{100}$

c $\frac{1}{4}$ **ii** $\frac{1}{2}$

48 $5x - y - 13 = 0$

49 **a** $\frac{8\pi}{9}$ cm **b** $\frac{16\pi}{9}$ cm^2

50 **a** 0 **b** −2 **c** $|m + 1| - 2$

51 75%

52 $P(L)P(M) = 045 \times 012 = 0054 = P(L \cap M)$

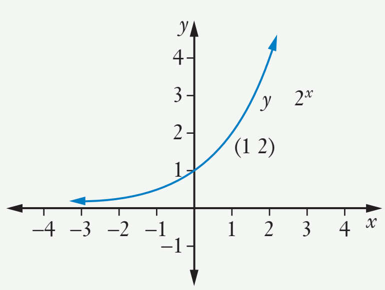
53 Show that $P(Y|X) = P(Y) = 015$

54 Mutually exclusive

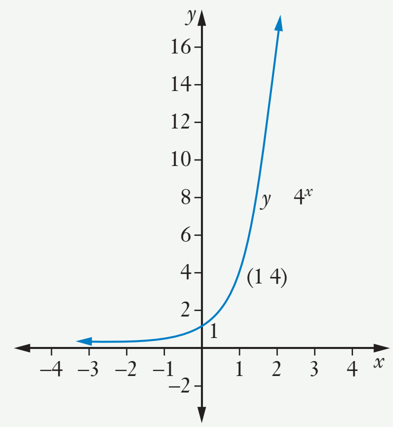
Chapter 8

Exercse 801

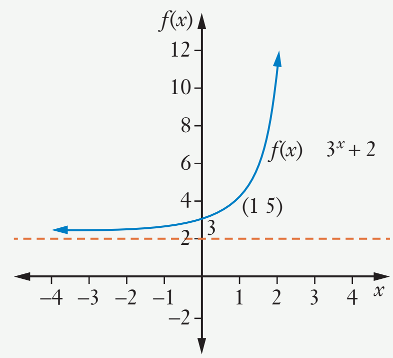
1 **a**



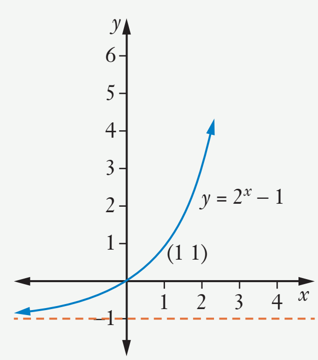
b



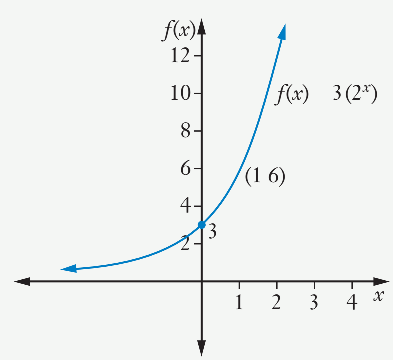
c



d



e



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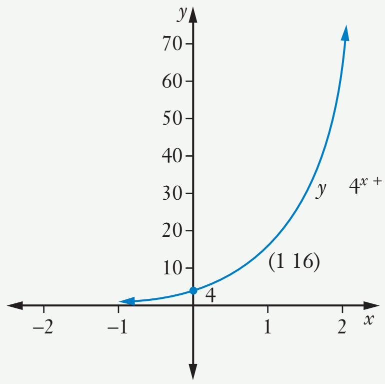
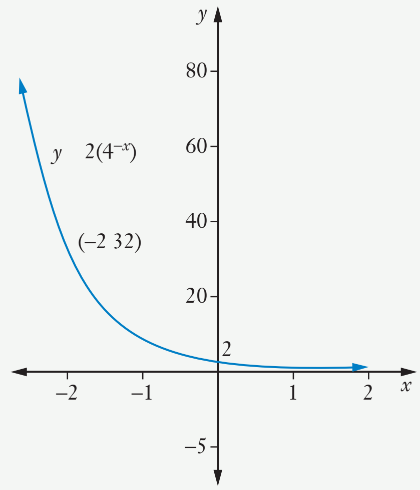
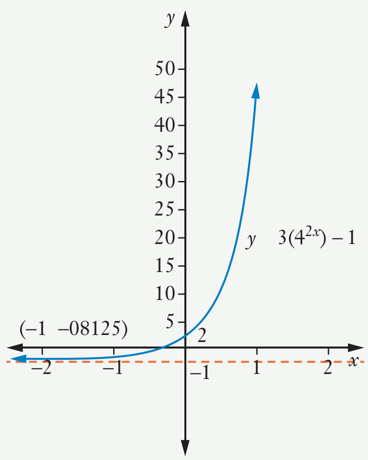
f

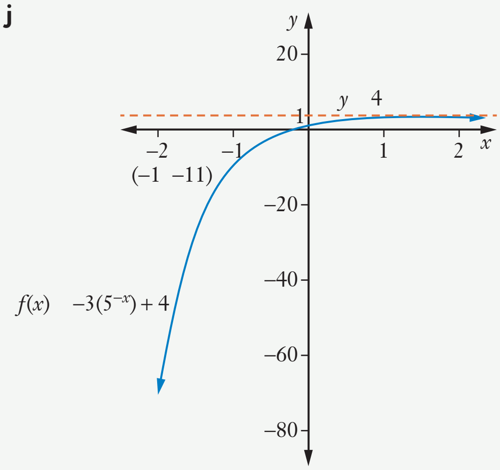




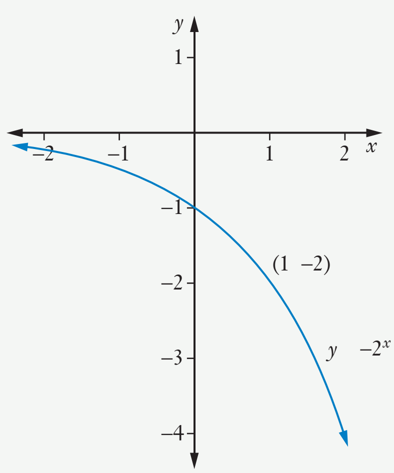
g



j



h



2 **a** Domain (–∞ ∞) range (, ∞)

b Domain (–∞ ∞) range (, ∞)

c Domain (–∞ ∞) range (, ∞)

d Domain (–∞ ∞) range (– ∞ 1)

3 **a** $y = 2^{3x-4}$ **b** $y = 3(2^x) - 4$

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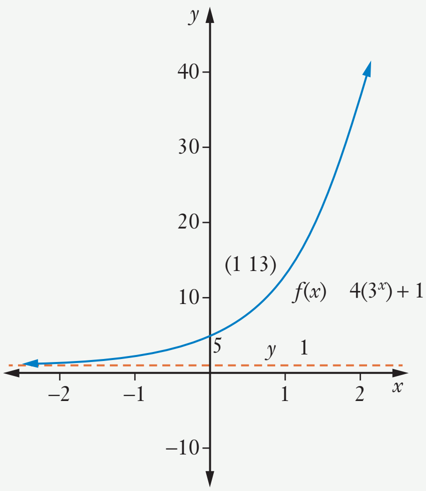
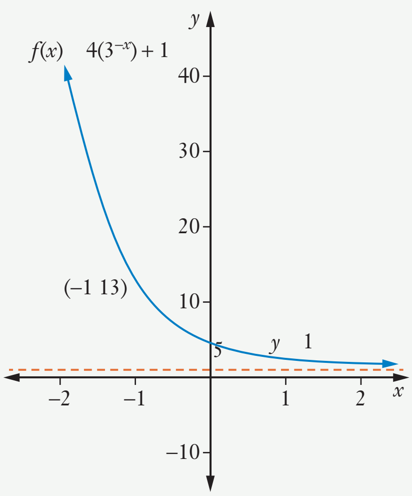
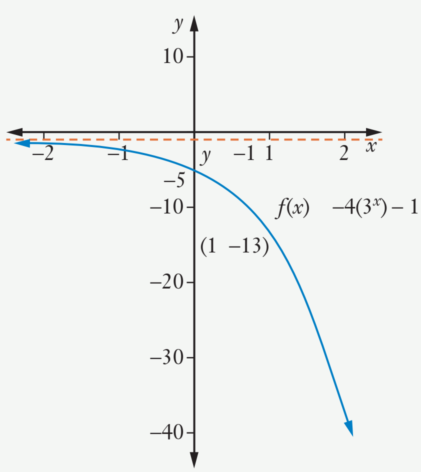
4 a



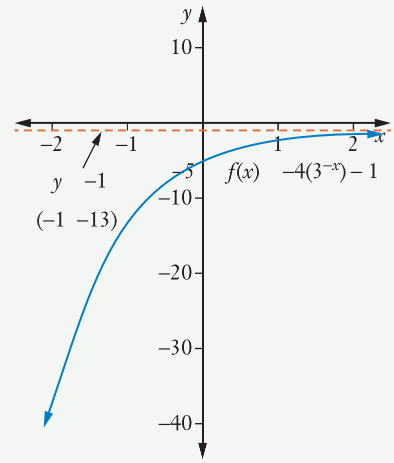
b



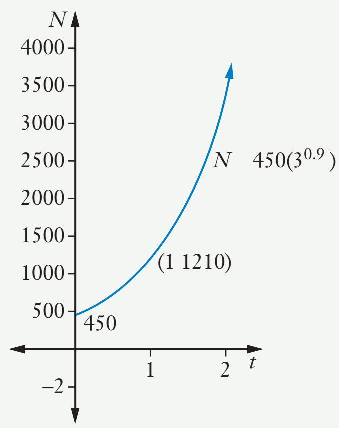
ii



iii



5 a

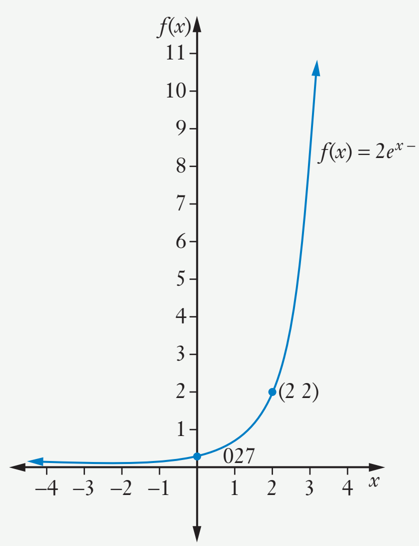


b 450 sales

c 8739 **ii** 63 133 **iii** 8 857 350

Exercse 802

1



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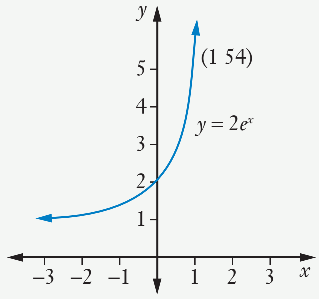
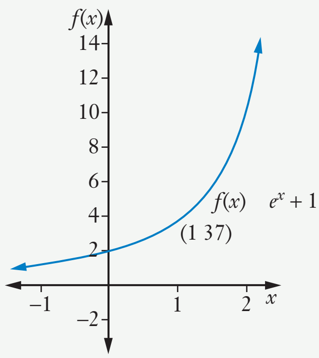
2 **a** 448 **b** 014 **c** 270 **d** 005
e –014

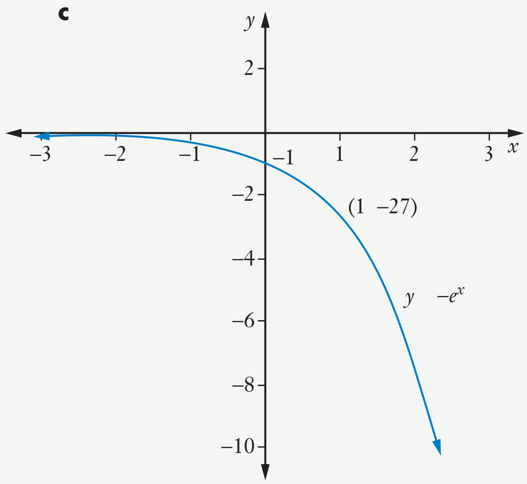
3 **a**



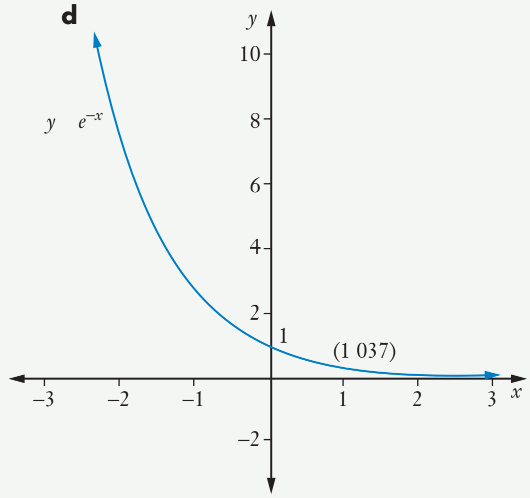
b



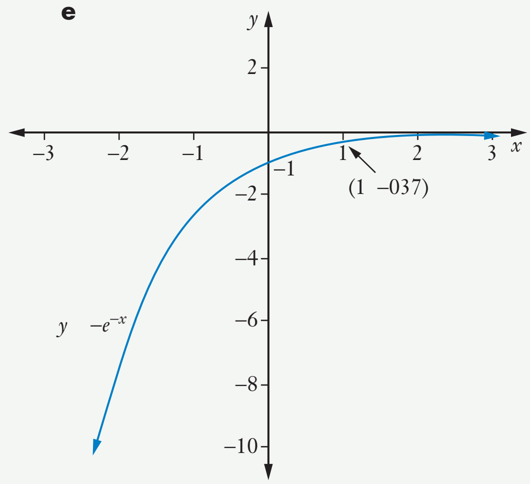
c



d



e



4 Domain (–∞ ∞) range (–, ∞)

5 **a** $y = e^{x} + 3$ **b** $y = e^{3x} + 3$

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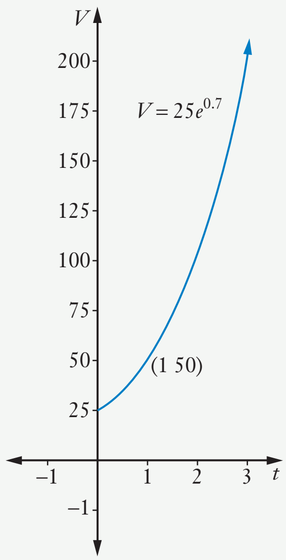
6 **a**

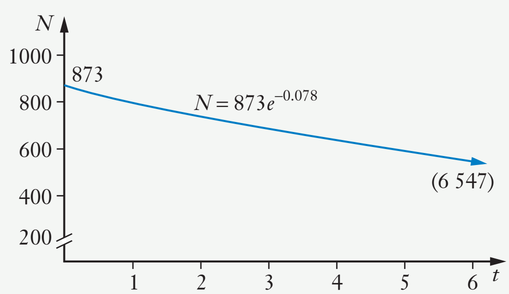


b 2042 mm ^{3} **ii** 67607 mm ^{3}

c No because it predicts the volume never stops increasing

7 **a** 1304 g **b** 745 g **c** 45 g

8 **a**



b 873 **ii** 591 **iii** 400

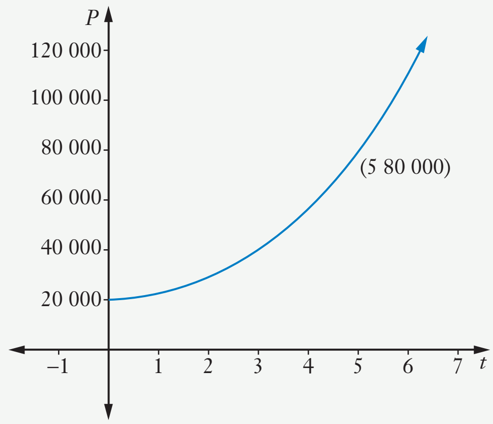
9 **a** 148°C

b 1339°C **ii** 1156°C

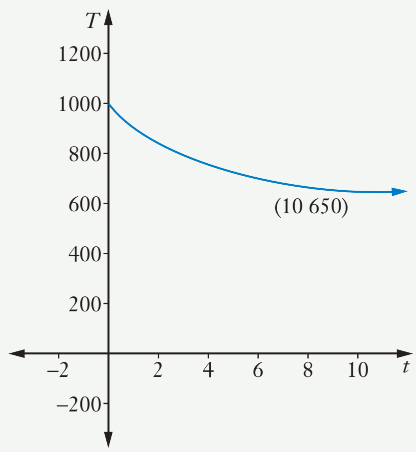
iii 916°C **v** 231°C

c 23°C (room temperature)

10



11



Exercse 803

1 **a** $9e^x$ **b** $-e^x$ **c** $e^x + 2x$

d $6x^2 - 6x + 5 - e^x$ **e** $3e^x(e^x + 1)^2$

f $7e^x(e^x + 5)^6$ **g** $4e^x(2e^x - 3)$ **h** $e^x(x + 1)$

$\dfrac{e^x(x-1)}{x^2}$ **j** $xe^x(x + 2)$

k $e^x(2x + 3)$

$\dfrac{e^x(7x-10)}{(7x-3)^2}$ **m** $\dfrac{5e^x - 5xe^x}{e^{2x}} = \dfrac{5(1-x)}{e^x}$

2 **a** $2e^{2x}$ **b** $-e^{-x}$ **c** $6e^{3x}$

d $-7e^{7x}$ **e** $-6e^{2x} + 2x$

f $2e^{2x} + 2e^{-2x} = 2(e^{2x} + e^{-2x})$ **g** $-5e^{-x} - 3$

h $e^{4x}(4x + 1)$ $\dfrac{6xe^{3x} + 4e^{3x} + 3}{(x+1)^2}$

j $135e^{3x}(9e^{3x} + 2)^4$

3 $6 - e$ **4** e **5** $-\dfrac{1}{2e^{10}}$

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6 1981 **7** $y = -ex$
8 $x + e^3 y - 3 - e^6 = 0$
9 a 12 572
b 280 insects/week **ii** 75 742 insects/week
10 a 2 m **b** $8e^{40}$ m s^{-}
c 95 3907 m s^{-2}
11 a 1 cm **b** –204 cm s^{-}
c –346 cm **d** –0096 cm s^{-}
e 035 cm s^{-2}
12 a 331 mm^3 **ii** 1638 mm^3
b 265 mm^3 s^{-} **ii** 1310 mm^3 s^{-}
13 a 39 094 **ii** 44 299
b 977 people/year **ii** 1107 people/year
14 a 295 m **ii** 290 m **iii** 285 m
b –0050 m/month
ii –0049 m/month **iii** –0048 m/month

Exercse 804

1 a 4 **b** 2 **c** 3 **d** 1
e 2 **f** 1 **g** 0 **h** 7
1
2 a 3 **b** 4 **c** 29
3 a 9 **b** 3 **c** –1 **d** 12
e 8 **f** 4 **g** 14 **h** 1
2
4 a –1 **b** $\frac{1}{2}$ **c** $\frac{1}{2}$ **d** –2
e $\frac{1}{4}$ **f** $-\frac{1}{3}$ **g** $-\frac{1}{2}$ **h** $\frac{1}{3}$
$1\frac{}{2}$ **j** $-1\frac{}{2}$
5 a 308 **b** 294 **c** 322 **d** 494
e 1040 **f** 704 **g** 059 **h** 351
043
6 a $\log_3 y = x$ **b** $\log_5 z = x$ **c** $\log_x y = 2$
d $\log_2 a = b$ **e** $\log_b d = 3$ **f** $\log_8 y = x$
g $\log_6 y = x$ **h** $\log\ y = x$ $\log_a y = x$
j $\log\ Q = x$
7 a $3^x = 5$ **b** $a^x = 7$ **c** $3^b = a$ **d** $x^9 = y$
e $a^y = b$ **f** $2^y = 6$ **g** $3^y = x$ **h** $10^y = 9$
$e^y = 4$
8 a $x = 1\ 000\ 000$ **b** $x = 243$ **c** $x = 7$
d $x = 2$ **e** $x = -1$ **f** $x = 3$
g $x = 447$ **h** $x = 10\ 000$ $x = 8$
9 $y = 5$ **10** 447 **11** 244 **12** 0, 0
13 1, 1
14 a 1
b 3 **ii** 2 **iii** 5 **v** $\frac{1}{2}$
v –1 **v** 2 **v** 3 **viii** 5
x 7 **x** 1 **x** e
15 a 684 **b** 588 **ii** 466
c 8 weeks
16 a 25 acidic **b** 7 neutral
c 9 alkaline **d** 2 acidic
e 119 alkaline **f** 5 acidic
17 a $y = \log(2x - 7)$ **b** $y = 2\log x - 7$

Exercse 805

1 a $\log_a 4y$ **b** $\log_a 20$ **c** $\log_a 4$ **d** $\log_a \frac{b}{5}$
e $\log_x y^3 z$ **f** $\log_k 9y^3$ **g** $\log_a \frac{x^5}{y^2}$
h $\log_a \frac{xy}{z}$ $\log_0 ab^4c^3$
j $\log_3 \frac{p^3 q}{r^2}$ **k** $-\log_4 n$ $-\log_x 6$
2 a 2 **b** 6
3 a 119 **b** –047 **c** 155 **d** 166
e 108 **f** 136 **g** 202 **h** 183
236
4 a 2 **b** 6 **c** 2 **d** 3
e 1 **f** 3 **g** 7 **h** $\frac{1}{2}$
–2 **j** 4
5 a $x + y$ **b** $x - y$ **c** $3x$ **d** $2y$
e $2x$ **f** $x + 2y$ **g** $x + 1$ **h** $1 - y$
$2x + 1$
6 a $p + q$ **b** $3q$ **c** $q - p$ **d** $2p$
e $p + 5q$ **f** $2p - q$ **g** $p + 1$ **h** $1 - 2q$
$3 + q$ **j** $p - 1 - q$
7 a 13 **b** 128 **c** 162 **d** 91
e 67 **f** 238 **g** –37 **h** 3
222

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8 **a** $x = 4$ **b** $y = 28$ **c** $x = 48$ **d** $x = 3$

e $k = 6$

9 **a** $I = 10^{\frac{L}{10}} I_0$ **b** $31\ 6228\ I_0$

10 **a** Proof see worked solutions

b 699 **ii** 22

11 **a** 158 **b** 180 **c** 241

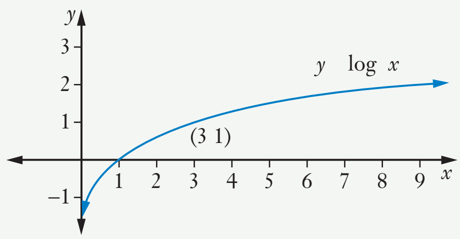
d 358 **e** 285 **f** 266

g 140 **h** 455 459

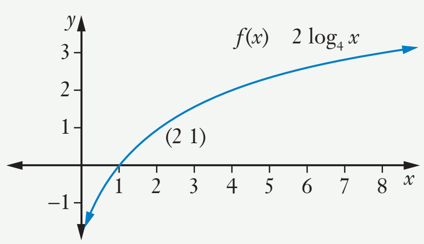
j 729

Exercse 806

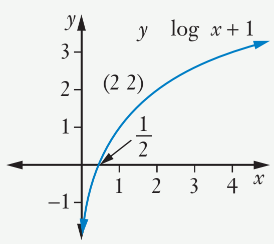
1 **a**



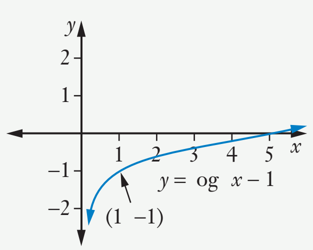
b



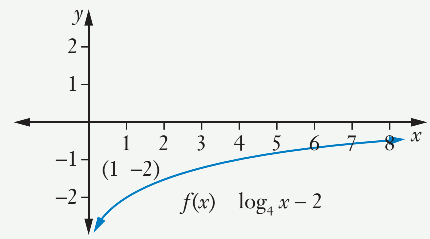
c



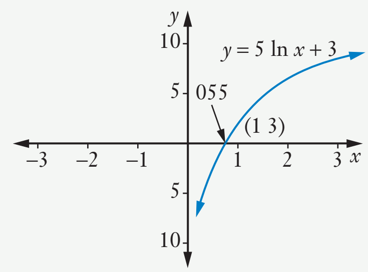
d



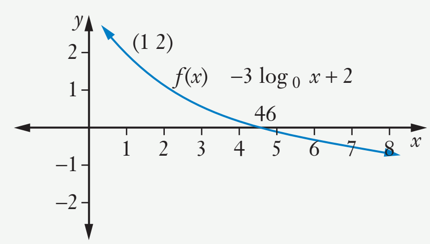
e



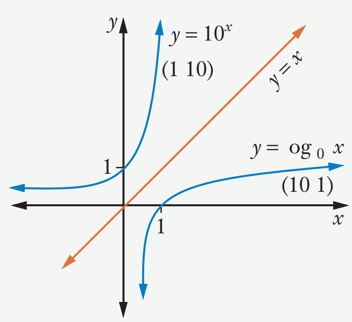
f



g

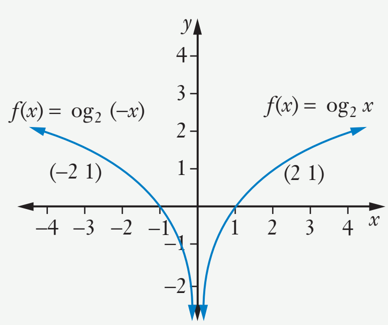


2



Curves are reflections of each other in the line $y = x$

3



$y = \log_2(-x)$ is a reflection of $f(x) = \log_2 x$ in the y-axis

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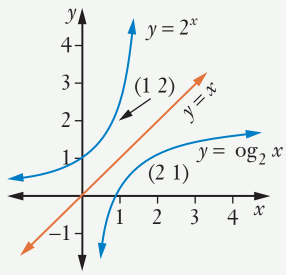
4 **a**



b $y = 2^x$

5 **a** 6 **b** 100 **c** 20 **d** 2
e 126

6 **a** 1000 **b** 10 **c** 20 **d** 3162
e 1 995 262

Exercse 807

1 **a** $x = 1.6$ **b** $x = 1.5$ **c** $x = 14$ **d** $x = 39$
e $x = 22$ **f** $x = 2.3$ **g** $x = 62$ **h** $x = 28$
$x = 29$ **j** $x = 24$

2 **a** $x = 258$ **b** $y = 168$ **c** $x = 273$
d $m = 178$ **e** $k = 282$ **f** $t = 126$
g $x = 115$ **h** $p = 583$ $x = 317$
j $n = 258$

3 **a** $x = 09$ **b** $n = 09$ **c** $x = 66$
d $n = 1.2$ **e** $x = –02$ **f** $n = 22$
g $x = 22$ **h** $k = 09$ $x = 36$

4 **a** $x = 530$ **b** $t = 0536$ **c** $t = 362$
d $x = 381$ **e** $n = 340$ **f** $t = 0536$
g $t = 246$ **h** $k = 672$ $t = 549$

5 **a** $850 **ii** $101038 **b** 66 years

6 **a** 35 000 **ii** 44 494 **iii** 116 204
b after 344 years **ii** after 726 years

7 **a** 8900 **ii** 7001 **iii** 309
b 12 years **ii** 79 years

8 **a** 100 g **b** 9985 g
c 985 g **d** 23 105 years

9 **a** 301 °C **b** 4904 hours

10 **a** 28 cm **ii** $5e^{20}$ cm s$^{–}$ **iii** 20401 cm
v 252 s **v** 530 s
b $a = 5e$
$= 5e \ + 23 – 23$
$= x – 23$
c 62 cm s$^{–2}$

Test yourslf 8

1 D **2** B **3** A, C

4 **a** 3 **b** 1 **c** 3 **d** 2
e 1 **f** 3 **g** $\frac{1}{2}$ **h** –1
–2 **j** 3

5 **a** 639 **b** 198 **c** 326
d 140 **e** 0792 **f** 391
g 572 **h** 724

6 **a** 6 **b** 2

7 **a** $3^x = a$ **b** $e^y = b$ **c** $10^z = c$

8 **a** 092 **b** 108 **c** 02
d 136 **e** 064

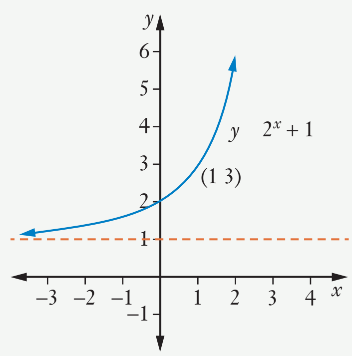
9 **a** $x = 1.9$ **b** $x = 1.9$ **c** $x = 3$ **d** $x = 36$

10 $t = 182$ **11** 09

12 **a** $\log_a x^5 y^3$ **b** $\log_x \frac{k^2 p}{3}$

13 **a** 065 **b** 13

14



Domain (–∞ ∞) range (, ∞)

15 **a** $x = 317$ **b** $x = 177$ **c** $x = 011$
d $y = 086$ **e** $n = 092$ **f** $x = 2$
g $y = 27$ **h** $n = 49$ $x = 4096$
j $m = 2$

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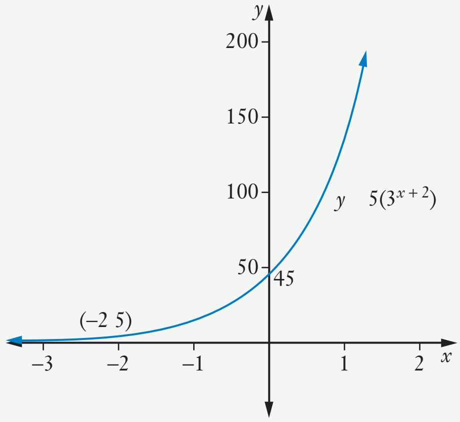
16 **a** $\log_2 y = x$ **b** $\log_5 b = a$ **c** $\log y = x$

d $\ln z = x$ **e** $\log_3 y = x + 1$

17 **a**



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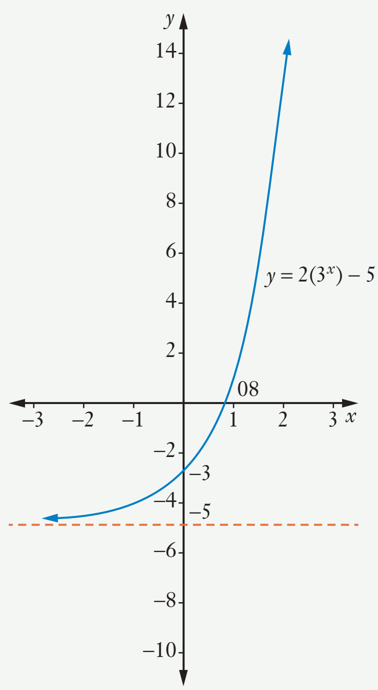

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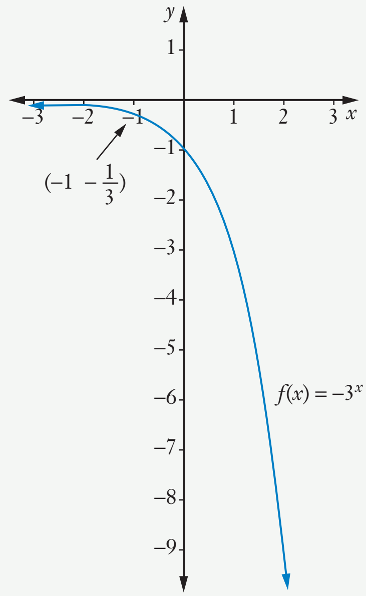

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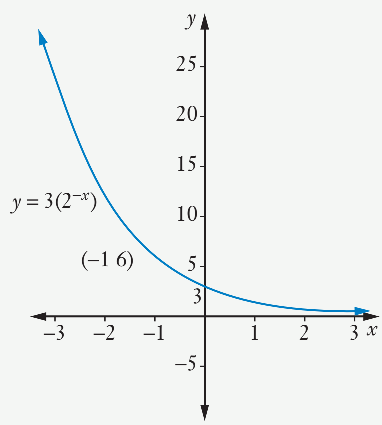
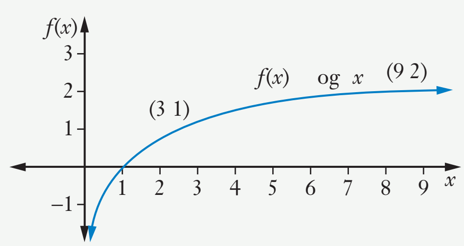


18 **a**



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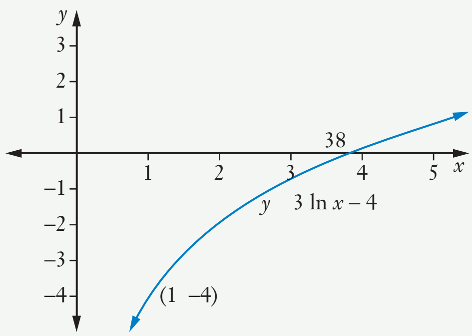
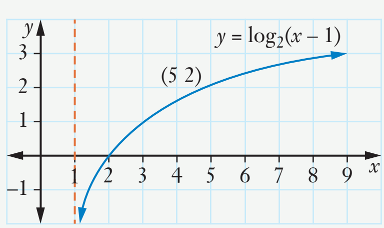


19 **a** $a + b$ **b** $b - a$ **c** $3a$ **d** $a + 2b$
e $3b$
20 **a** 374 dB **b** $1585I_0$
21 **a** $-\log_a x$ **b** $-\log y$
22 **a** 2 **b** 2 **c** 3 **d** 1 **e** 3
23 **a** $x = 27$ **b** $x = 3.1$ **c** $x = 21$
24 **a** $5280 **ii** $558671 **iii** $569286
b 68 years **ii** 339 years
25 **a** $3e^{3x}$ **b** $-2e^{-2x}$ **c** $20e^{4x}$
d $-16e^{8x} + 15x^2$ **e** $2xe^{2x}(x + 1)$
f $108e^{3x}(4e^{3x} - 1)^8$ **g** $\dfrac{1 - 2x}{e^{2x}}$
26 **a** 1081 **b** 1075
c 88 years **ii** 1036 years
27 **a** $e^x + 1$ **b** $-4e^x$ **c** $-3e^{-x}$
d $9e^x(3 + e^x)^8$ **e** $3x^4e^x(x + 5)$ **f** $\dfrac{e^x(7x - 9)}{(7x - 2)^2}$
28 100 times
29 20 times
30 **a** $y = \log(6x^2 - 1)$ **b** $y = e^{6x} - 1$
c $y = 6e^{2x} - 1$ **d** $y = x$
e $y = x$

Chaenge exercse 8

1 **a** 27 **b** 28 **c** 25
2 (1 0)
3



Domain (1 ∞) range (– ∞ ∞)
4 $x = 063$

5 **a** $\text{pH} = -\log [\text{H}^+$
$= \log 1 - \log [\text{H}^+$
$= \log \dfrac{1}{\text{H}^+}$
b $\text{pH} = -\log [\text{H}^+$
$-\text{pH} = \log [\text{H}^+$
$10^{-\text{pH}} = [\text{H}^+$
$\dfrac{1}{10^{\text{pH}}} = [\text{H}^+$
c 0000 000 5 **ii** 0000 000 02
6 **a** Proof includes $x + 2 = 2^{y-8}$ (see worked solutions)
b $y = 1081$ **ii** $x = -199$
7 **a** $3e^2x - y - 3e^2 - 5 = 0$
b $x + 3e^2y - 9e^4 + 15e^2 - 2 = 0$

Chapter 9

Exercise 901

1 **a** 1st 4th **b** 1st 3rd **c** 1st 2nd
d 2nd 4th **e** 3rd 4th **f** 2nd 3rd
g 3rd **h** 2nd 4th
j 4th
2 **a** 3rd **b** $-\dfrac{1}{2}$
3 **a** 4th **b** $-\dfrac{1}{\sqrt{2}}$
4 **a** 2nd **b** $-\sqrt{3}$
5 **a** 2nd **b** $\dfrac{1}{\sqrt{2}}$
6 **a** 1st **b** $\dfrac{\sqrt{3}}{2}$
7 **a** 1 **b** $\dfrac{1}{\sqrt{2}}$ **c** $-\sqrt{3}$ **d** $\dfrac{1}{2}$
e $-\dfrac{1}{2}$ **f** $-\dfrac{1}{2}$ **g** $\dfrac{\sqrt{3}}{2}$ **h** $-\dfrac{1}{\sqrt{3}}$
$-\dfrac{\sqrt{3}}{2}$ **j** $-\dfrac{1}{\sqrt{2}}$
8 **a** $-\dfrac{1}{\sqrt{2}}$ **b** $-\dfrac{\sqrt{3}}{2}$ **c** $\sqrt{3}$ **d** $-\dfrac{\sqrt{3}}{2}$
e $-\dfrac{\sqrt{3}}{2}$ **f** $-\sqrt{3}$ **g** $\dfrac{1}{2}$ **h** $-\dfrac{1}{\sqrt{3}}$
$\dfrac{1}{\sqrt{2}}$ **j** $-\dfrac{1}{\sqrt{2}}$

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9 **a** $-\frac{\sqrt{3}}{2}$ **b** $\sqrt{3}$ **c** $\frac{\sqrt{3}}{2}$ **d** $\frac{1}{2}$

e $-\frac{1}{2}$ **f** $\sqrt{3}$ **g** $\frac{1}{\sqrt{2}}$ **h** $\frac{1}{\sqrt{2}}$

-1 **j** $\frac{1}{2}$

10 $\sin\theta = -\frac{3}{5}$ $\cos\theta = -\frac{4}{5}$

11 $\cos\theta = -\frac{\sqrt{33}}{7}$ $\tan\theta = -\frac{4}{\sqrt{33}}$

12 $\cos x = \frac{8}{\sqrt{89}}$

13 $\sin x = -\frac{\sqrt{21}}{5}$ $\tan x = -\frac{\sqrt{21}}{2}$

14 $\sin x = \frac{5}{\sqrt{74}}$ $\cos x = -\frac{7}{\sqrt{74}}$

15 $\tan\theta = -\frac{4}{\sqrt{65}}$ $\cos\theta = \frac{\sqrt{65}}{9}$

16 $\tan x = \frac{\sqrt{55}}{3}$ $\sin x = -\frac{\sqrt{55}}{8}$

17 **a** $\sin x = \frac{3}{10}$ **b** $\cos x = -\frac{\sqrt{91}}{10}$ $\tan x = -\frac{3}{\sqrt{91}}$

18 $\cos\alpha = \frac{5}{\sqrt{61}}$ $\sin\alpha = -\frac{6}{\sqrt{61}}$

19 $\sin\theta = \frac{\sqrt{51}}{10}$ $\tan\theta = -\frac{\sqrt{51}}{7}$

20 **a** $\sin\theta$ **b** $\cos x$ **c** $\tan\beta$

d $-\sin\alpha$ **e** $-\tan\theta$ **f** $-\sin\theta$

g $\cos\alpha$ **h** $-\tan x$

Exercse 902

1 $\sec x = \frac{9}{5}$ $\cot x = \frac{5}{\sqrt{56}}$ $\operatorname{cosec} x = \frac{9}{\sqrt{56}}$

2 $\operatorname{cosec}\theta = \frac{13}{5}$ $\sec\theta = \frac{13}{12}$ $\cot\theta = \frac{12}{5}$

3 $\operatorname{cosec}\theta = \frac{7}{\sqrt{33}}$ $\sec\theta = \frac{7}{4}$ $\cot\theta = \frac{4}{\sqrt{33}}$

4 $\tan\theta = -\frac{\sqrt{11}}{5}$ $\operatorname{cosec}\theta = \frac{6}{\sqrt{11}}$ $\cot\theta = -\frac{5}{\sqrt{11}}$

5 $\sin\theta = -\frac{5}{\sqrt{34}}$ $\operatorname{cosec}\theta = -\frac{\sqrt{34}}{5}$ $\tan\theta = \frac{5}{3}$

$\sec\theta = -\frac{\sqrt{34}}{3}$

6 $\sin 67° = \cos(90° - 67°) = \cos 23°$

7 $\sec 82° = \operatorname{cosec}(90° - 82°) = \operatorname{cosec} 8°$

8 $\tan 48° = \cot(90° - 48°) = \cot 42°$

9 **a** $2\cos 61°$ or $2\sin 29°$

b 0 **c** 0 **d** 1 **e** 2

10 $x = 80$ **11** $y = 22$ **12** $p = 31$

13 $b = 25$ **14** $t = 20$ **15** $k = 15$

16 **a** $\sin\theta$ **b** $\sec\theta$ **c** $\operatorname{cosec} x$

d $\cos^2 x$ **e** $\sin\alpha$ **f** $\operatorname{cosec}^2 x$

g $\sec^2 x$ **h** $\tan^2\theta$ $5\operatorname{cosec}^2\theta$

j $\sin^2 x$ **k** 1 $\sin\theta\cos\theta$

17 See worked solutions

Exercse 903

1 **a** $\sqrt{2}$ **b** $\frac{2}{\sqrt{3}}$ **c** $\frac{1}{\sqrt{3}}$

d $\sqrt{3}$ **e** $\frac{1}{2}$ **f** $\frac{\sqrt{3}}{2}$

g $\sqrt{2}$ **h** 3 2

2 **b** 2nd **c** $-\frac{1}{\sqrt{2}}$

3 **b** 2nd **c** $\frac{1}{2}$

4 **b** 4th **c** -1

5 **b** 3rd **c** $-\frac{1}{2}$

6 **b** 4th **c** $-\frac{\sqrt{3}}{2}$

7 **a** **ii** 1st **iii** $\frac{\sqrt{3}}{2}$

b $\frac{1}{\sqrt{2}}$ **ii** $\sqrt{3}$ **iii** $-\frac{1}{\sqrt{2}}$ **v** $\frac{1}{\sqrt{3}}$

v $-\frac{\sqrt{3}}{2}$

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8 **a**

	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{3}$	$\frac{8\pi}{3}$	$\frac{10\pi}{3}$	$\frac{11\pi}{3}$
sin	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
cos	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
tan	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$

b

	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$	$\frac{11\pi}{4}$	$\frac{13\pi}{4}$	$\frac{15\pi}{4}$
sin	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
cos	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{\ }{\sqrt{2}}$
tan	1	–1	1	–1	1	–1	1	–1

c

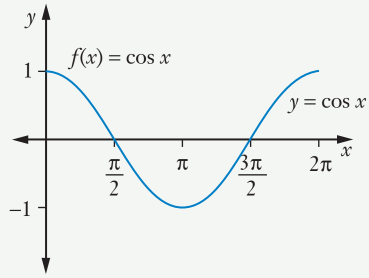
	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{13\pi}{6}$	$\frac{17\pi}{6}$	$\frac{19\pi}{6}$	$\frac{23\pi}{6}$
sin	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
cos	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
tan	$\frac{\ }{\sqrt{3}}$	$-\frac{\ }{\sqrt{3}}$	$\frac{\ }{\sqrt{3}}$	$-\frac{\ }{\sqrt{3}}$	$\frac{\ }{\sqrt{3}}$	$-\frac{\ }{\sqrt{3}}$	$\frac{\ }{\sqrt{3}}$	$-\frac{\ }{\sqrt{3}}$

9

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
sin	0	1	0	–1	0	1	0	–1	0
cos	1	0	–1	0	1	0	–1	0	1
tan	0	Not defned	0	Not defned	0	Not defned	0	Not defned	0

Exercse 904

1 **a**



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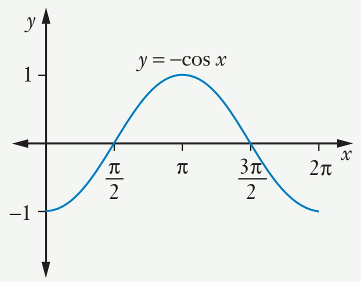
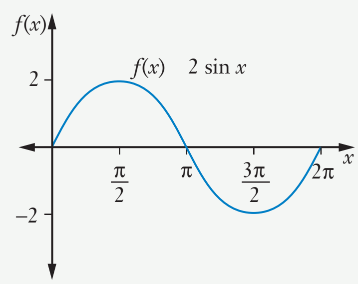


2 **a**



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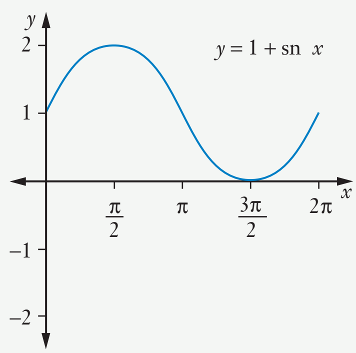

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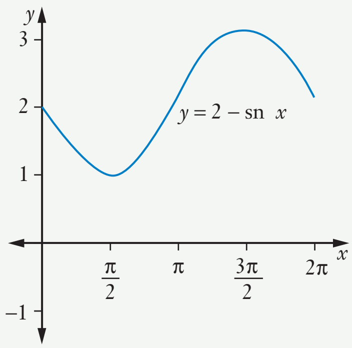

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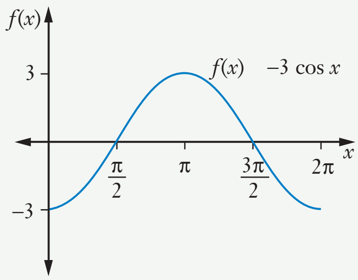

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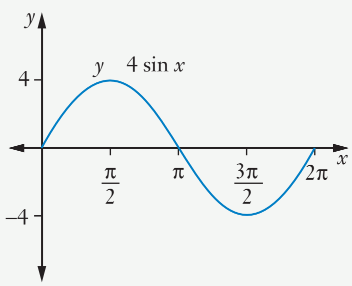

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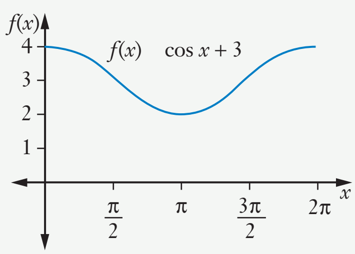

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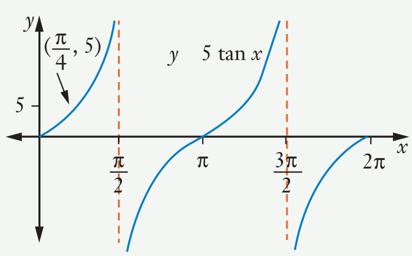

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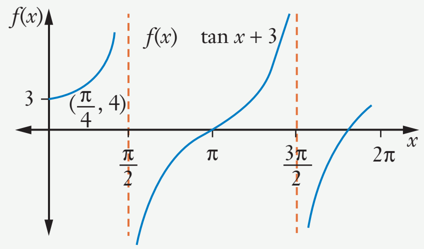

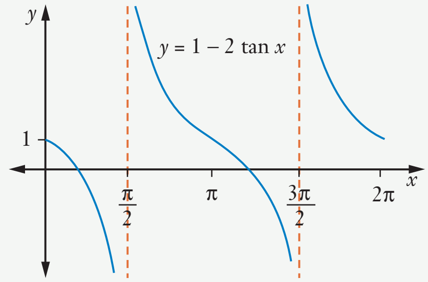

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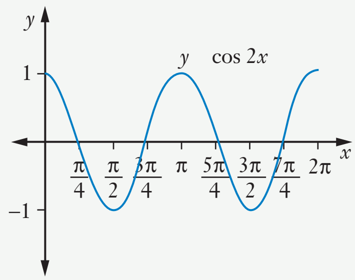

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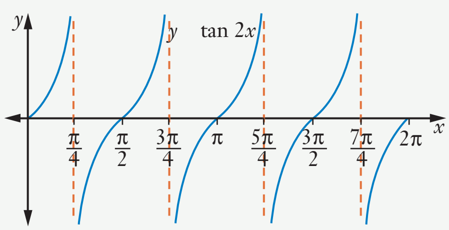

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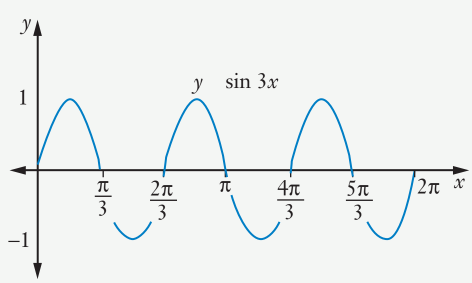

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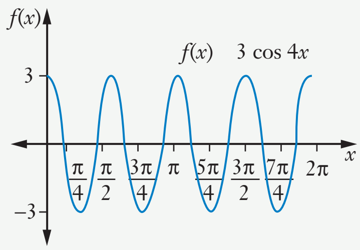

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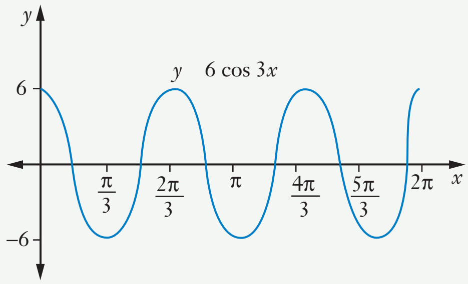

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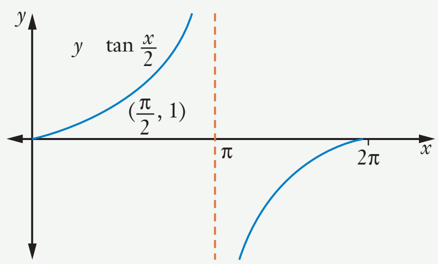

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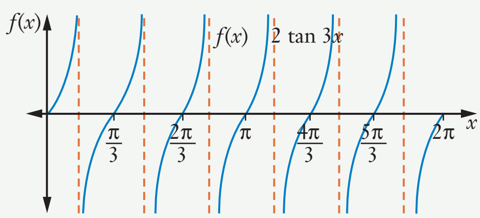

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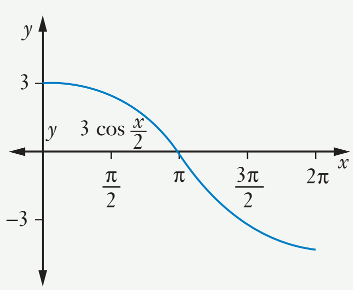

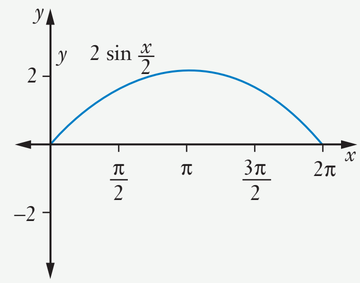

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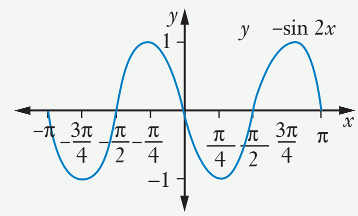

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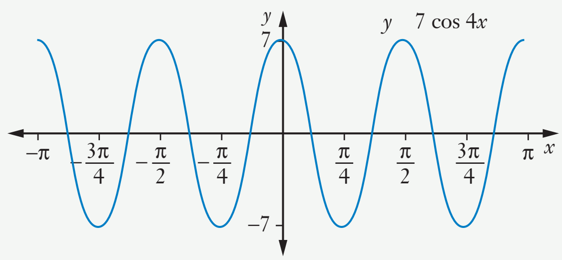

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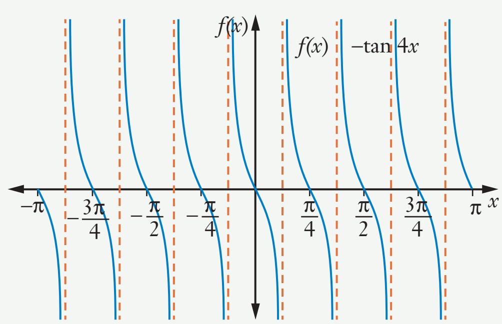

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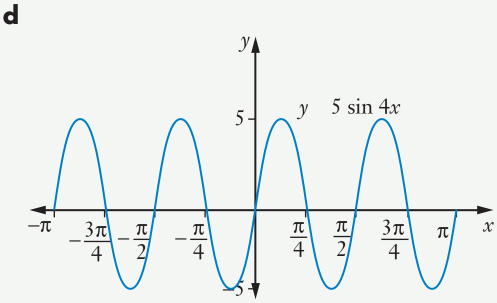

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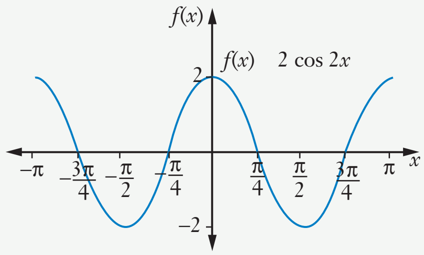

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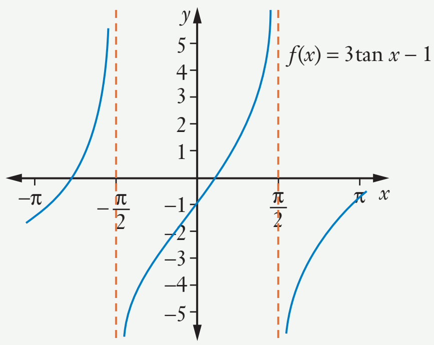

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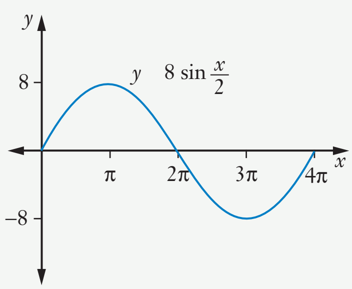

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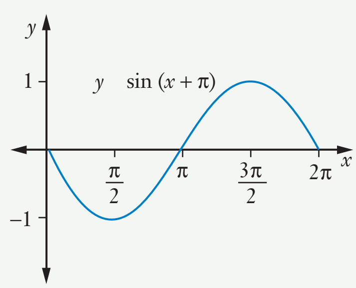

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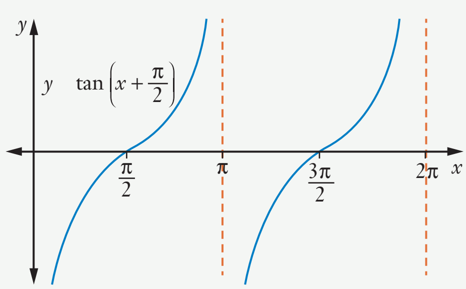

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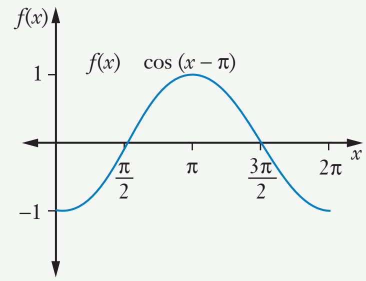

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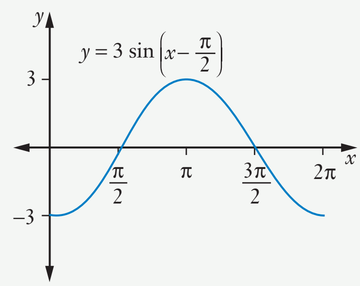

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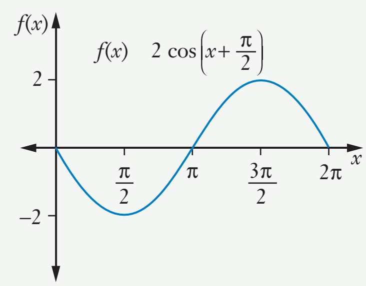

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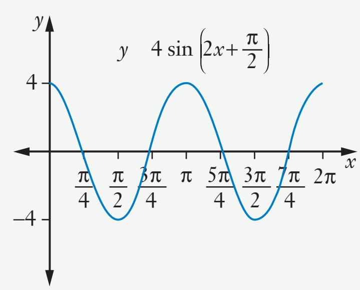

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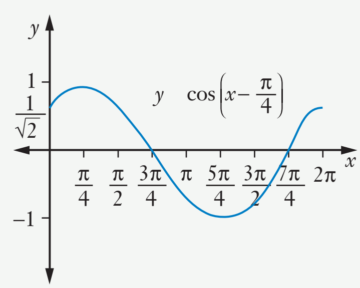

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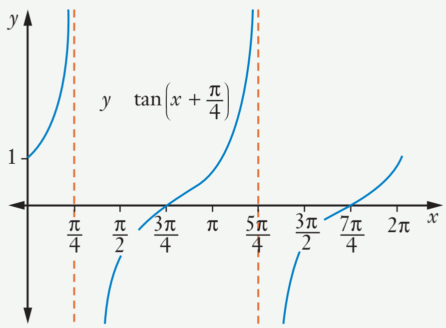
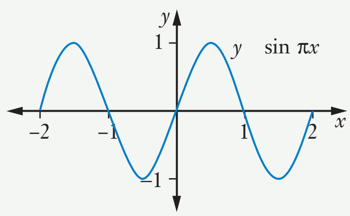
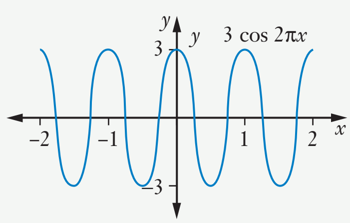


7 a



b

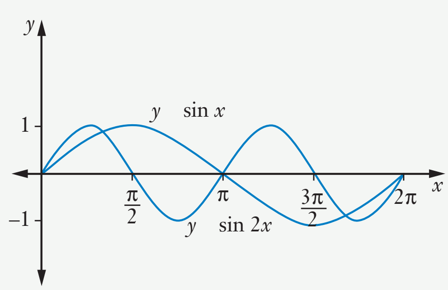


8	**a**	5	**ii**	π	**iii**	0	**v**	0
	b	1	**ii**	2π	**iii**	0	**v**	–π
	c	None	**ii**	$\frac{\pi}{4}$	**iii**	–2	**v**	0
	d	3	**ii**	2π	**iii**	1	**v**	$\frac{\pi}{4}$
	e	8	**ii**	2	**iii**	–3	**v**	$-\frac{2}{\pi}$
	f	None	**ii**	$\frac{\pi}{5}$	**iii**	2	**v**	$\frac{\pi}{10}$

9 a Domain (–∞ ∞) range [–, 3]

b Domain (–∞ ∞) range [, 10]

10 a



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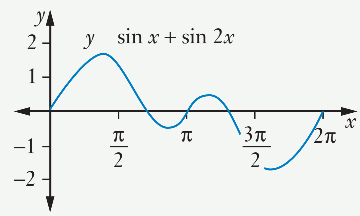
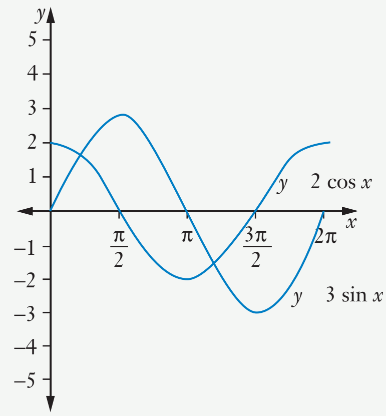


11 a



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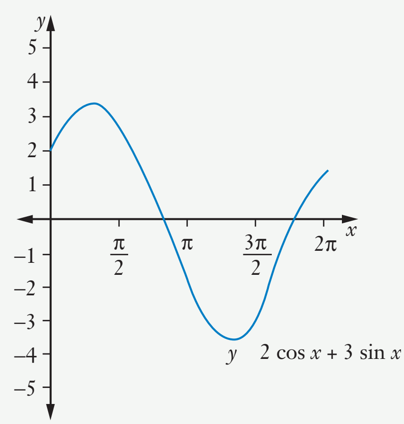

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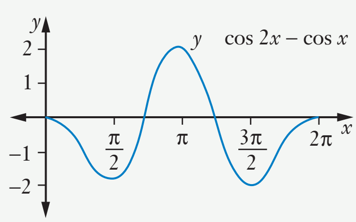
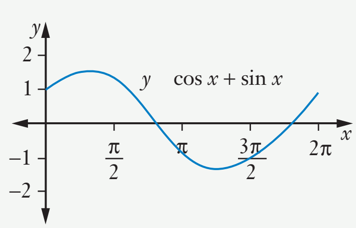


13



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Exercse 905

1 **a** $\theta = 20°29 \quad 159°31$
b $\theta = 120° \ 240°$
c $\theta = 135°, 315°$
d $\theta = 60° \ 120°$
e $\theta = 150° \ 330°$
f $\theta = 30° \ 330°$
g $\theta = 30° \ 120° \ 210° \ 300°$
h $\theta = 30° \ 150° \ 210° \ 330°$
$\theta = 70°, 110° \ 190° \ 230°, 310° \ 350°$
j $\theta = 15° \ 45° \ 75° \ 105°, 135° \ 165° \ 195° \ 225° \ 255° \ 285°, 315° \ 345°$
k $x = 90° \ 270°$
$x = 60° \ 90° \ 270° \ 300°$

2 **a** $x = 0° \ 360°$ **b** $x = 270°$
c $x = 0° \ 180° \ 360°$ **d** $x = 90°$
e $x = 0° \ 180° \ 360°$
f $x = 0° \ 180° \ 270° \ 360°$
g $x = 0° \ 90° \ 270° \ 360°$
h $x = 0° \ 45° \ 180° \ 225° \ 360°$
$x = 60° \ 120° \ 240° \ 300°$

3 **a** $x = 0, \pi \ 2\pi$ **b** $x = 0, \frac{\pi}{2} \ \pi \ \frac{3\pi}{2} \ 2\pi$
c $x = \frac{3\pi}{2}$ **d** $x = 0, 2\pi$ **e** $x = \pi$

4 **a** $\theta = \pm 79°13$ **b** $\theta = 30° \ 150°$
c $\theta = 45° \ -135°$ **d** $\theta = -60° \ -120°$
e $\theta = 150° \ -30°$ **f** $\theta = \pm 30° \ \pm 150°$
g $\theta = 135° \ -45°$
h $\theta = 225°, 11.5° \ -6.5° \ -15.5°$

5 **a** $x = \frac{\pi}{3} \ \frac{5\pi}{3}$ **b** $x = \frac{5\pi}{4} \ \frac{7\pi}{4}$
c $x = \frac{\pi}{4} \ \frac{5\pi}{4}$ **d** $x = \frac{\pi}{3} \ \frac{4\pi}{3}$
e $x = \frac{5\pi}{6} \ \frac{7\pi}{6}$

6 **a** $x = \frac{\pi}{3} \ \frac{2\pi}{3}$ **b** $x = \pm\frac{\pi}{2}$
c $x = \pm\frac{\pi}{6} \ \pm\frac{5\pi}{6}$

7 $x = \pm\frac{2\pi}{3} \ \pm\frac{4\pi}{3}$

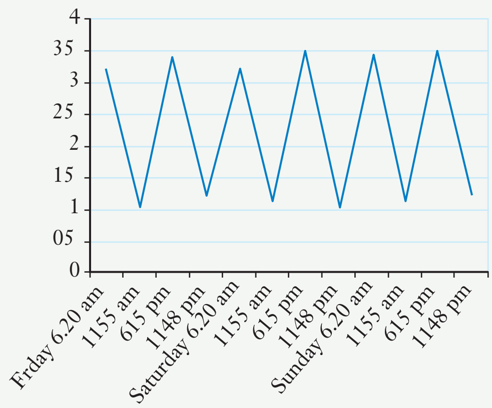
8 **a** $x = 0, \frac{3\pi}{4} \ \pi \ \frac{7\pi}{4} \ 2\pi$ **b** $x = 0, \frac{\pi}{2} \ \pi \ 2\pi$
c $x = 0, \frac{2\pi}{3} \ \frac{4\pi}{3} \ 2\pi$ **d** $x = \frac{\pi}{6} \ \frac{5\pi}{6} \ \frac{7\pi}{6} \ \frac{11\pi}{6}$
e $x = 0, \pi \ 2\pi$ **f** $x = 0, 2\pi$

Exercse 906

1 **a** Period 12 months amplitude .5
b 530 pm

2 **a** 1300 **b** 1600 **ii** 1100
c Amplitude 250 period 8 or 9 years

3 **a**



b Period 12 hours amplitude .25 m
c 25 m

Test yourslf 9

1 D **2** D **3** B **4** C

5 **a** $\frac{1}{\sqrt{2}}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $-\sqrt{3}$

6 **a** $x = 60° \ 120°$ **b** $x = 45° \ 225°$
c $x = 120° \ 240°$
d $x = 60° \ 120° \ 240° \ 300°$
e $x = 15° \ 105° \ 195° \ 285°$

7 **a** $x = \frac{3\pi}{4} \ \frac{7\pi}{4}$ **b** $x = \frac{\pi}{6} \ \frac{5\pi}{6}$
c $x = \frac{\pi}{3} \ \frac{2\pi}{3} \ \frac{4\pi}{3} \ \frac{5\pi}{3}$ **d** $x = 0, 2\pi$
e $x = \frac{3\pi}{2}$

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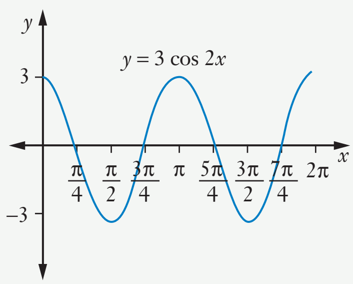
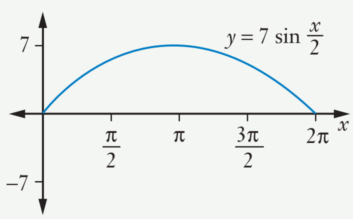
8 **a**



b



9 $\cos x = \frac{5}{13}$ $\tan x = -\frac{12}{5}$

10 **a** –cos θ **b** –tan θ **c** sin θ
d $\sin x$ **e** $2 \cos A$ **f** $\sin x$
g 1

11 **a** $-\frac{1}{\sqrt{2}}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $\sqrt{3}$

12 See worked solutions

13 $b = 40$

14 Period $\frac{2\pi}{3}$ amplitude , centre5, phase $\frac{\pi}{36}$

15 **a** $\sqrt{2}$ **b** $\sqrt{3}$ **c** $\frac{2}{\sqrt{3}}$
d $\sqrt{3}$

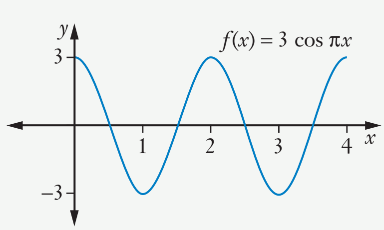
16 **a** Domain (– ∞ ∞) range [–, 11]
b Domain (– ∞ ∞) range [–, 1]

Chaenge exercse 9

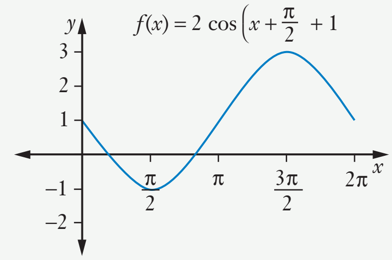
1 **a** $-\frac{\sqrt{3}}{2}$ **b** –1

2 θ = 110° 230°

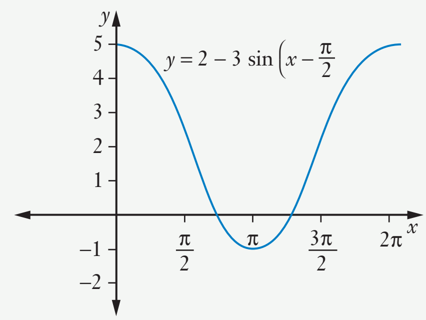
3 **a** Period = 2 amplitude = 3
b



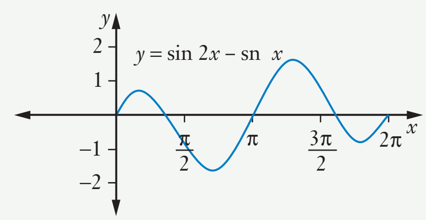
4 **a**



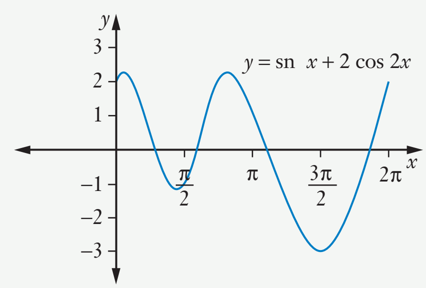
b



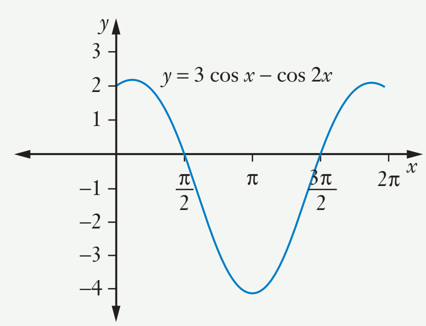
c



d



e



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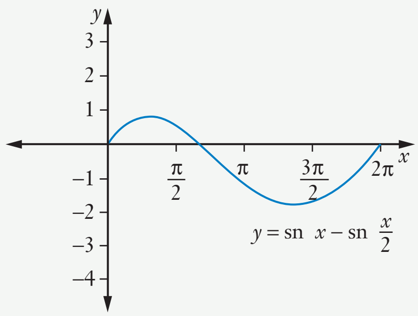
f



5 $x = 0, \frac{\pi}{2}\ \frac{3\pi}{2}\ 2\pi$

6 $\frac{\sqrt{3}-\sqrt{2}}{2}$

Chapter 10

Exercse 1001

1 **a** Discrete **b** Continuous **c** Discrete
d Discrete **e** Continuous **f** Discrete
g Continuous **h** Continuous Discrete
j Discrete **k** Discrete Discrete

2 **a** $X = \{0, 1\}$
b $X = \{0, 1, 2, \ldots, 10\}$
c $X = \{0, 1, 2, 3, \ldots, 20\}$
d $X = \{0, 1, 2, \ldots 31\}$
e $X = \{2, 3, 4, \ldots, 12\}$

Exercse 1002

1

x	2	3	4	5	6	7	8	9	10	11	12
$P(X\ \ x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

2 **a** $\left(0\ \frac{1}{2}\right)\left(1\ \frac{1}{2}\right)$ **b** $\left(0\ \frac{1}{4}\right)\left(1\ \frac{1}{2}\right)\left(2\ \frac{1}{4}\right)$

c $\left(0\ \frac{1}{8}\right)\left(1\ \frac{3}{8}\right)\left(2\ \frac{3}{8}\right)\left(3\ \frac{1}{8}\right)$

3 **a**

x	48	49	50	51	52
$P(X\ \ x)$	$\frac{8}{53}$	$\frac{9}{53}$	$\frac{21}{53}$	$\frac{9}{53}$	$\frac{6}{53}$

b $\frac{36}{53}$ **ii** $\frac{38}{53}$

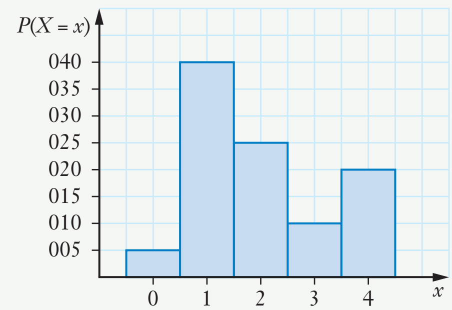
4 **a** $p(3) + p(4) + p(5) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6}$
$= 1$

b

x	3	4	5
$P(X\ \ x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

c $\frac{5}{6}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{2}$

5



6

x	1	2	3	4	5	6
$P(X\ \ x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b Yes

c $\frac{1}{2}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{2}$

7 **a** No **b** Yes **c** No

8 **a** $k = \frac{1}{14}$ **b** $k = 019$ **c** $k = 0$

9 **a**

x	1	2	3	4
$P(X\ \ x)$	$\frac{k}{6}$	$\frac{4k}{7}$	$\frac{9k}{8}$	$\frac{16k}{9}$

b $k = \frac{504}{1835}$

10 **a**

y	0	1	2	3
$P(Y\ \ y)$	$\frac{5}{9}$	$\frac{5}{18}$	$\frac{5}{36}$	$\frac{1}{36}$

b $\frac{1}{36}$ **ii** $\frac{1}{6}$ **iii** $\frac{35}{36}$

11 $p = \frac{1}{11}$

12 **a**

x	0	1	2	3	5
$P(X\ \ x)$	$\frac{13}{25}$	$\frac{19}{100}$	$\frac{1}{10}$	$\frac{9}{100}$	$\frac{1}{10}$

b $\frac{19}{100}$ **c** $\frac{9}{10}$

13 **a** $\frac{2}{7}$ **b** $\frac{1}{2}$ **c** $\frac{5}{14}$ **d** $\frac{4}{7}$

e $\frac{1}{7}$ **f** $\frac{5}{14}$ **g** $\frac{9}{14}$

14 **a** $\frac{3}{16}$ **b** $\frac{3}{4}$ **c** $\frac{3}{8}$ **d** $\frac{3}{4}$

e $\frac{1}{4}$ **f** $\frac{3}{4}$

15 **a**

x	0	1	2
$P(X \quad x)$	$\frac{994\,009}{1\,000\,000}$	$\frac{5982}{1\,000\,000}$	$\frac{9}{1\,000\,000}$

b $\frac{5982}{1\,000\,000}$ **ii** $\frac{5991}{1\,000\,000}$

16 **a**

x	0	1	2
$P(X \quad x)$	$\frac{25}{49}$	$\frac{20}{49}$	$\frac{4}{49}$

b

x	0	1	2
$P(X \quad x)$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{1}{14}$

17 **a** $\frac{1}{8}$ **b** No

c

x	1	2	3	4	5
$P(X \quad x)$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

18

x	0	1	2	3
$P(X \quad x)$	681%	279%	38%	02%

19

x	0	1	2	3	4
$P(X \quad x)$	576%	24%	3747%	26%	677%

20 **a**

x	0	1	2
$P(X \quad x)$	$\frac{893}{990}$	$\frac{19}{198}$	$\frac{1}{495}$

b $\frac{97}{990}$

Exercse 1003

1 **a** 1 **b** 278 **c** $3\frac{7}{16}$

d $2\frac{3}{4}$ **e** 2

2 **a** $k = \frac{1}{10}$ **ii** 3

b $k = \frac{1}{12}$ **ii** $1\frac{}{6}$

c $k = 015$ **ii** 428

3 **a** 1 **b** 7 **c** $1\frac{}{2}$ **d** $\frac{3}{1000}$

e $\frac{14}{19}$ **ii** $\frac{14}{19}$

4 $p = 03, \; q = 01$

5 $a = \frac{1}{16} \; b = \frac{3}{16}$

6 **a**

x	1	2	3	4
$P(X \quad x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

b $2\frac{}{2}$

7 $1\frac{}{2}$

8 **a** $2\frac{2}{11}$ **b** $2\frac{2}{11}$

9 **a**

x	0	1	2	3
$P(X \quad x)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

b 56 cents **c** Lose 44 cents

10 **a**

x	0	1	2	3	4
$P(X \quad x)$	$\frac{81}{625}$	$\frac{216}{625}$	$\frac{216}{625}$	$\frac{96}{625}$	$\frac{16}{625}$

b 16

c Ye, Yasmin will make at least one phone sale in an hour.

11 Win 42 cents

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Exercse 1004

1 **a** 169 **ii** 285
b 111 **ii** 123
c 150 **ii** 225

2 **a** Mean 701 variance .4, σ 274
b Mean 289 variance .8, σ 137

3 $n = \frac{3}{10}$ $E(X) = 275$ variance $= 289$

4 **a** $a = 015,\ b = 018$ **b** 208 **c** 144

5 **a** 15 **ii** 087 **iii** 075
b 091 **ii** 069 **iii** 047

6 **a** 3 **b** 141 **c** 2

7 **a**

x	0	1	2
$P(X\ \ x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

b Mean 033 variance .2, standard deviation 053

8 **a** 008 **b** 028 **c** 008

9 **a** Mean 125 variance .73
b Mean 125 variance .60

10 **a**

x	0	1	2
$P(X\ \ x)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

ii Mean 015 variance .1, standard deviation 037

b

y	0	5	10
$P(Y\ \ y)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

ii Mean 077 variance .4, standard deviation 187

Test yourslf 10

1 C **2** D **3** B **4** D

5 **a** $X = \{0, 1, 2, 3, 4, 5\}$
b $X = \{0, 1, 2, 3, \ldots, 10\}$
c $X = \{0, 1, 2, 3, \ldots, 30\}$
d $X = \{0, 1, 2\}$
e $X = \{0, 1, 2, 3, \ldots, 9\}$

6 $k = \frac{1}{4}$

7 Mean 367 variance .5, standard deviation1.25

8 **a** $\frac{1}{12}$ **b** $\frac{7}{12}$ **c** $\frac{5}{6}$
d $\frac{11}{12}$ **e** $\frac{5}{12}$

9

x	0	1	2
$P(X\ \ x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

10 **a** uniform **b** not uniform
c uniform **d** not uniform

11 **a** Discrete **b** Continuous
c Discrete **d** Continuous

12 $E(X) = 1.5$, $Var(X) = 165$, standard deviation $= 128$

13 **a**

x	1	2	3	4	5	6	7
$P(X\ \ x)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

b Yes **c** $\frac{2}{7}$ **ii** $\frac{3}{7}$ **iii** $\frac{4}{7}$
d $E(X) = 4$

14 **a** $\frac{1}{5}$ **ii** $\frac{1}{5}$ **iii** $\frac{3}{5}$
b $f(3) + f(5) + f(9) = \frac{1}{5} + \frac{1}{5} + \frac{3}{5} = 1.$
So it is a probability function

15 **a**

x	0	1	2
$P(X\ \ x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b No **c** $\frac{1}{2}$ **ii** $\frac{3}{4}$

16 **a** Yes **b** No **c** No

17 **a** $E(X) = 044$ **b** Jonas loses 56 cents

18 **a** 21% + 14% + 47% + 18% = 100% = 1
b $E(X) = 577$ $Var(X) = 35971$

19 **a** $n = \frac{1}{16}$ **b** $n = 008$ **c** $n = \frac{1}{9}$

20 $a = 02,\ b = 01$

21 **a**

y	$2	–$1	$2
$P(Y\ \ y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

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b $E(X) = 50$ cents so the player is expected to lose 50 cents

Chaenge exercse 10

1 $a = 022 \quad b = 026 \quad c = 012$

2 $a = 02, \; k = 4$ **3** $k = 5, l = 018$

4 **a** $5 \times 02 \; = 1$ **b** Yes

c 06 **ii** 06 **iii** 08

d $E(X) = 3, Var(X) = 2$ **e** $P(X = 2) = 005$

5 **a** 120

b

x	1	2	3	4	5
$P(X \quad x)$	$\frac{1}{30}$	$\frac{1}{8}$	$\frac{23}{120}$	$\frac{59}{120}$	$\frac{19}{120}$

c No mean 362

d Ye, standard deviation 1

e Class discussion

Practice set 4

1 C **2** B **3** D

4 C D **5** D

6 **a** $e^x - 1$ **b** $3e^x$ **c** $4e^x(e^x - 2)^3$

d $e^x(4x + 1)^2(4x + 13)$ **e** $\frac{e^x(5x-7)}{(5x-2)^2}$

f $35e^{7x}$

7 **a** $\frac{1}{8}$ **ii** $\frac{1}{4}$

b $\frac{1}{8} + \frac{1}{4} + \frac{3}{8} + \frac{1}{4} = 1$

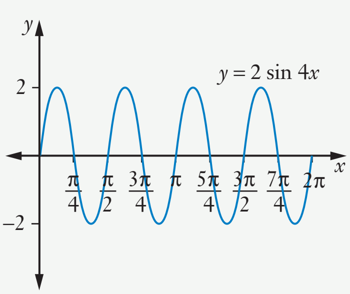
8 -2

9 **a** $\frac{3}{10}$ **b** $\frac{3}{5}$ **c** $\frac{9}{10}$

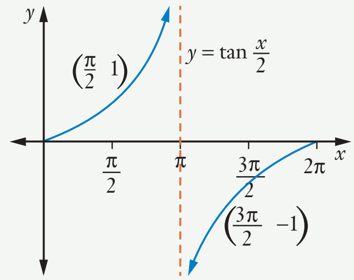
d $\frac{2}{5}$ **e** $\frac{7}{10}$

10 **a** $-\tan\theta$ **b** $-\sin\theta$ **c** $\cos\theta$

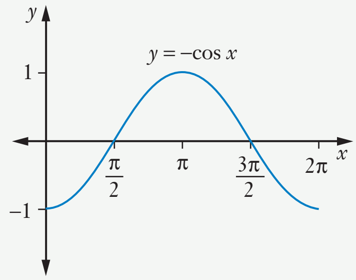
11 **a**



b



c



12 **a** $X = \{0, 1, 2, 3, \ldots\}$

b $X = \{0, 1, 2, \ldots, 12\}$

c $X = \{0, 1, 2, \ldots, 31\}$

13 $x = \frac{1}{2}$ **14** 32 years

15 **a**

x	0	1	2	3
$P(X \quad x)$	$\frac{343}{2197}$	$\frac{882}{2197}$	$\frac{756}{2197}$	$\frac{216}{2197}$

b

x	0	1	2	3
$P(X \quad x)$	$\frac{35}{286}$	$\frac{63}{143}$	$\frac{105}{286}$	$\frac{10}{143}$

16 $\sin x = -\frac{4}{5} \; \cos x = \frac{3}{5}$

17 **a** $x = \frac{2\pi}{3} \; \frac{4\pi}{3}$ **b** $x = \frac{\pi}{4} \; \frac{3\pi}{4} \; \frac{5\pi}{4} \; \frac{7\pi}{4}$

c $x = \frac{\pi}{2} \; \frac{3\pi}{2}$ **d** $x = \frac{\pi}{12} \; \frac{5\pi}{12} \; \frac{13\pi}{12} \; \frac{17\pi}{12}$

18 $a = 019 \quad b = 02$

19 $E(X) = 22, \; Var(X) = 216 \quad \sigma = 147$

20 **a** $\frac{1}{\sqrt{2}}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $-\frac{1}{\sqrt{3}}$

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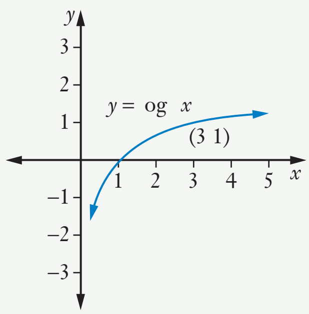
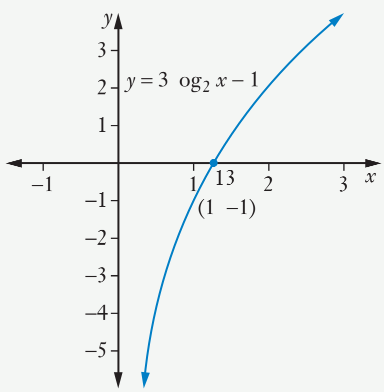
21

x	0	1	2	3
$P(X \quad x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

22 **a**



b



23 **a** $x = e^y$ **b** $x \approx 342$

24 $x \approx 028$

25 $k = \frac{1}{5}$

26 **a** No **b** Yes **c** Yes **d** No

27 **a** Yes **b** No

28 **a** $x = 135°, 315°$ **b** $x = 30° \ 150°$

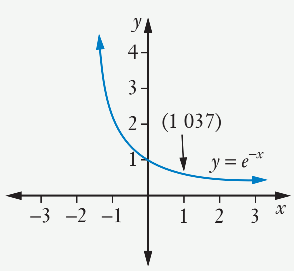
c $x = 45°, 135° \ 225°, 315°$

d $x = 30° \ 120° \ 210° \ 300°$

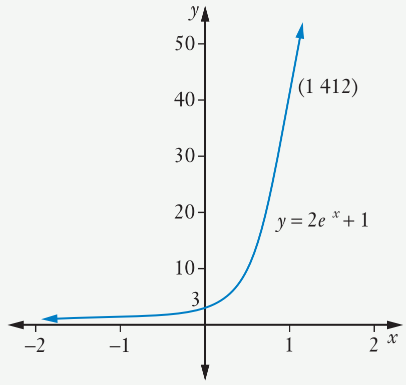
29 **a** 4 **b** 1 **c** $\frac{1}{2}$

d 204 **e** 376 **f** 218

30 **a**



b



31 **a**

x	0	1	2
$P(X \quad x)$	12%	23%	65%

b $E(X) = 153$, $Var(X) = 04891$

32 $y = 5e^2x - 5e^2$

33 **a** 50 cents **b** Lose 50 cents

34 **a**

x	1	2	3	4	5	6	7	8
$P(X \quad x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

b Yes **c** $\frac{1}{2}$ **ii** $\frac{3}{8}$ **iii** $\frac{5}{8}$

d $4\frac{\ }{2}$

35 **a** $027 + 031 + 028 + 014 = 1$

b $E(X) = 229 \quad Var(X) = 10259$

36 $k = \frac{1}{16}$

37 **a** $5\sec^2 x$ **b** $\cot x$

38 **a** $-\frac{1}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{2}}$ **c** $-\frac{\sqrt{3}}{2}$

39 **a** $x = -1, 3$ **b** $2 < x \le 5$ **c** $x = -1\frac{2}{3}, 1$

40 Centre (2 −3, radius 4

41 **a** 77 km **b** 218°

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