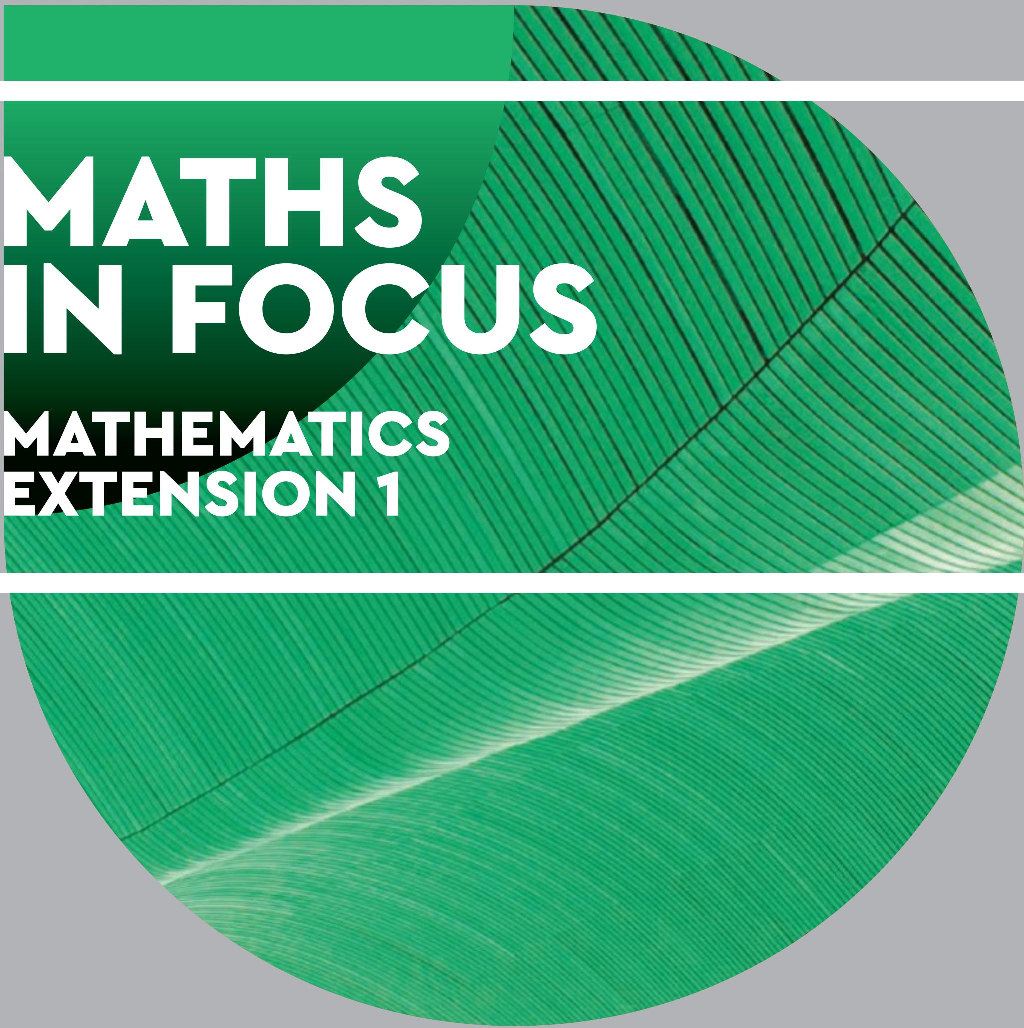
MATHS IN FOCUS



MATHEMATICS EXTENSION 1

YEAR 11

Margaret Grove
Series editor: Robert Yen

3RD EDITION

NELSON
A Cengage Company

Maths In Focus 11 Mathematics Extension 1
3rd Edition
Margaret Grove

Publisher: Robert Yen and Alan Stewart
Project editor: Anna Pang
Editor: Elaine Cochrane
Cover design: Chris Starr (MakeWork)
Text design: Sarah Anderson
Art direction: Danielle Maccarone and Aisling Gallagher
Cover image: iStock.com/markgoddard
Permissions researcher: Helen Mammides
Production controller: Christine Fotis
Typeset by: Cenveo Publisher Services

National Library of Australia Cataloguing-in-Publication Data
Grove, Margaret, author.

Maths in focus : year 11 mathematics extension 1 / Margaret Grove.

9780170413299 (paperback)
For secondary school age

Mathematics--Problems, exercises, etc.
Mathematics--Textbooks.

Cengage Learning Australia
Level 7, 80 Dorcas Street
South Melbourne, Victoria Australia 3205

Cengage Learning New Zealand
Unit 4B Rosedale Office Park
331 Rosedale Road, Albany, North Shore 0632, NZ

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Printed in China by China Translation & Printing Services.
1 2 3 4 5 6 7 22 21 20 19 18

PREFACE

Maths in Focus 11 Mathematics Extension 1 has been rewritten for the new Mathematics Extension 1 syllabus (2017). In this 3rd edition of the book, teachers will find those familiar features that have made Maths in Focus a leading senior mathematics series, such as clear and abundant worked examples in plain English, comprehensive sets of graded exercises, chapter *Test Yourself* and *Challenge* exercises, Investigations, and practice sets of mixed revision and exam-style questions.

The Mathematics Extension 1 course is designed for students who intend to study mathematics at university, possibly majoring in the subject.

This book covers the Year 11 content of the course, which includes the Year 11 Mathematics Advanced course. The specific Mathematics Extension 1 content is labelled EXT1. The theory follows a logical order, although some topics may be learned in any order. We have endeavoured to produce a practical text that captures the spirit of the course, providing relevant and meaningful applications of mathematics.

The *NelsonNet* student and teacher websites contain additional resources such as worksheets, video tutorials and topic tests. We wish all teachers and students using this book every success in embracing the new senior mathematics course.

ABOUT THE AUTHOR

Margaret Grove has spent over 30 years teaching HSC Mathematics, most recently at Bankstown TAFE College. She has written numerous senior mathematics texts and study guides over the past 25 years, including the bestselling *Maths in Focus* series for Mathematics and Mathematics Extension 1.

Margaret thanks her family, especially her husband Geoff, for their support in writing this book.

CONTRIBUTING AUTHORS

Gaspare Carrozza and **Haroon Ha** from Homebush Boys High School wrote many of the *NelsonNet* worksheets.

Scott Smith and **Cherylanne Saywell** created the video tutorials.

Tania Eastcott and **Elizabeth Nabhan** wrote the topic tests.

Roger Walter wrote the *ExamView* questions.

Shane Scott, **Brandon Pettis** and **George Dimitriadis** wrote the worked solutions to all exercise sets.

CONTENTS

EXT1 = Mathematics Extension 1 content additional to Mathematics Advanced

* = Revision

ISBN 97807043299

ISBN 97807043299

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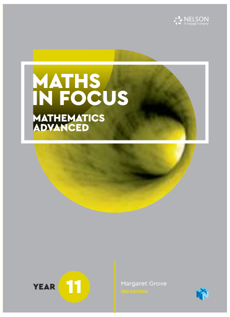
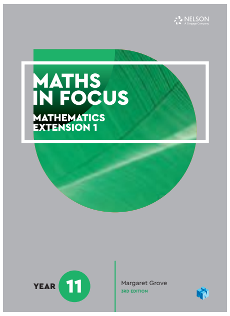
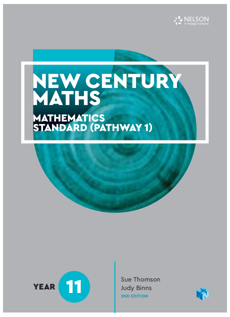
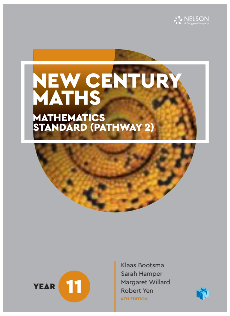
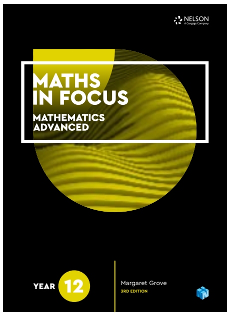
MATHS IN FOCUS AND NEW CENTURY MATHS 11–12

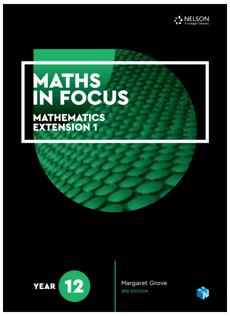


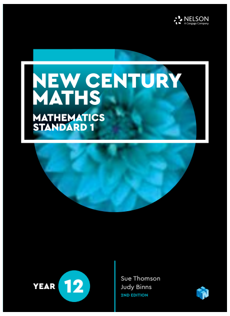


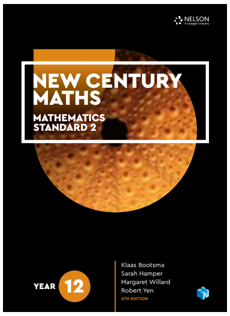


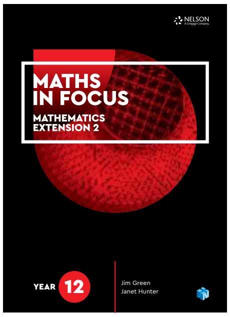












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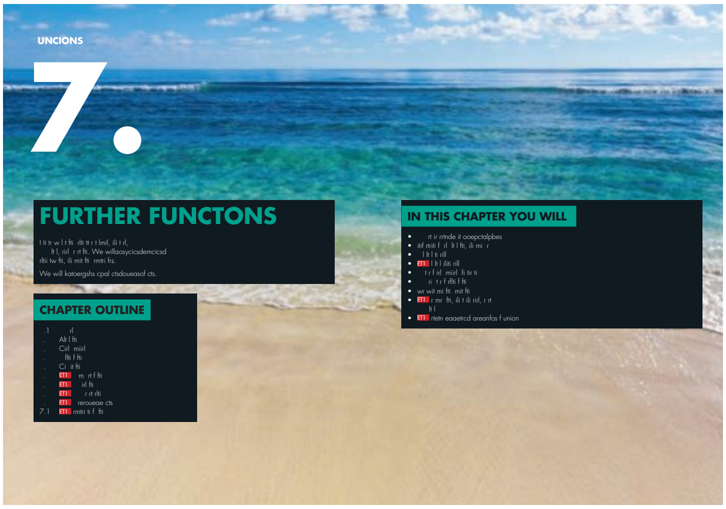
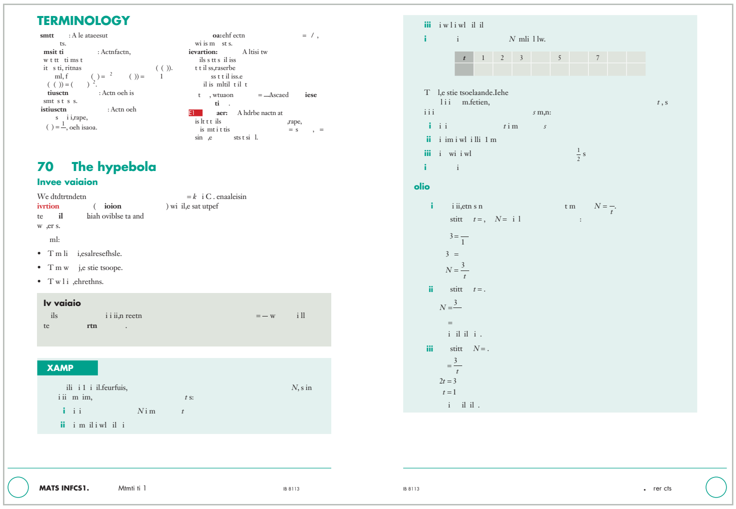
ABOUT THIS BOOK

AT THE BEGINNING OF EACH CHAPTER

- Each chapter begins on a double-page spread showing the **Chapter contents** and a list of chapter outcomes



- **Terminology** is a chapter glossary that previews the key words and phrases from within the chapter



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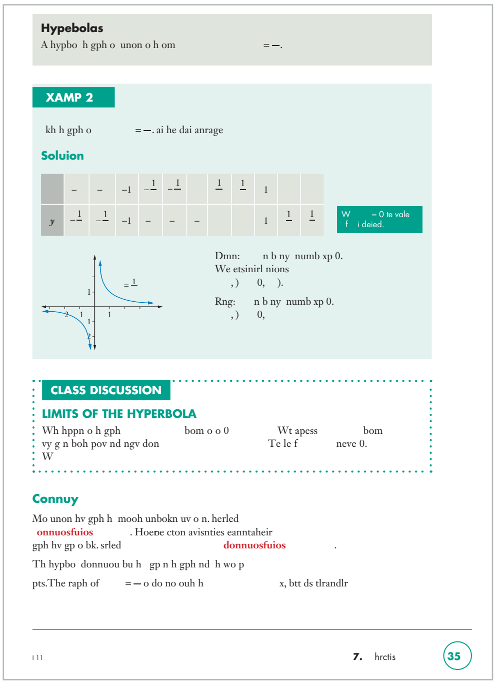
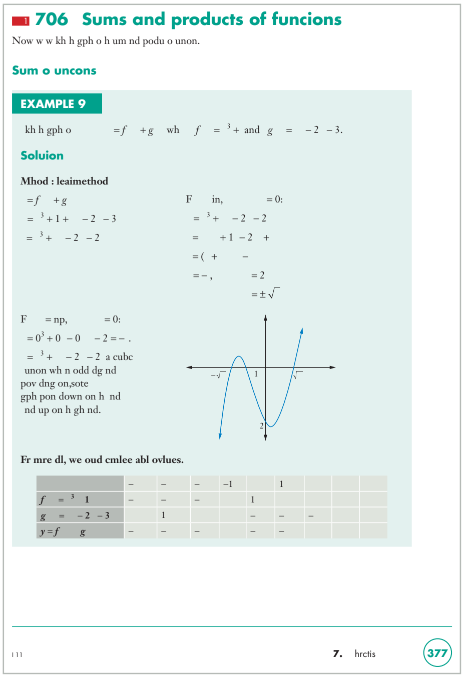
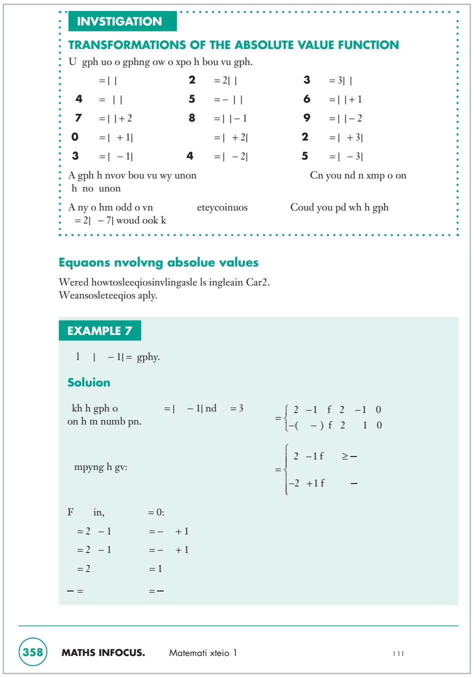
IN EACH CHAPTER

- Important facts and formulas are highlighted in a shaded box.
- Important words and phrases are printed in red and listed in the Terminology chapter glossary.
- The specific Mathematics Extension 1 content is labelled EXT1.
- Graded exercises include exam-style problems and realistic applications.
- Worked solutions to all exercise questions are provided on the *NelsonNet* teacher website
- **Investigations** explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.
- **Did you know?** contains interesting facts and applications of the mathematics learned in the chapter.







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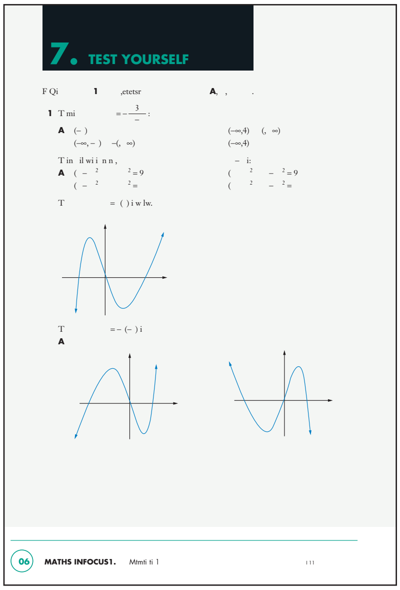
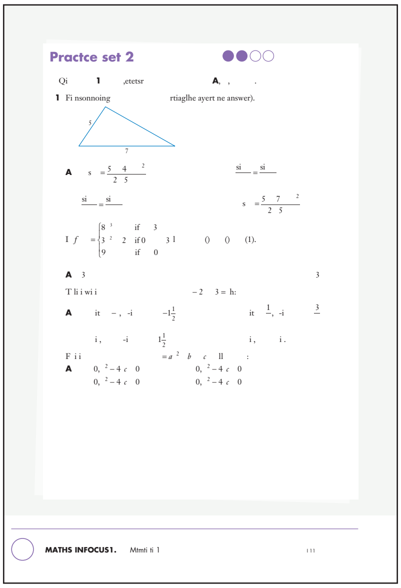
AT THE END OF EACH CHAPTER

- **Test Yourself** contains chapter revision exercises.
- If you have trouble completing the *Test Yourself* exercises, you need to go back and revise the chapter before trying the exercises again.
- **Challenge Exercise** contains chapter extension questions. Attempt these only after you are confident with the *Test Yourself* exercises, because these are more difficult and are designed for students who understand the topic really well.
- **Practice sets** (after several chapters) provide a comprehensive variety of mixed exam-style questions from various chapters, including short-answer, free-response and multiple-choice questions.



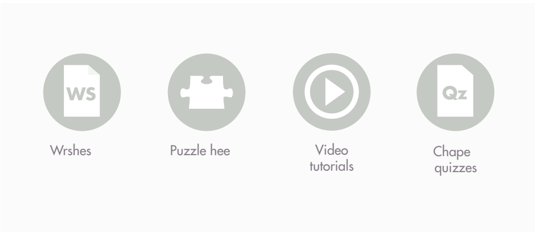
AT THE END OF THE BOOK

- **Answers** and **Index** (worked solutions on the teacher website).



NELSONNET STUDENT WEBSITE

Margin icons link to print (PDF) and multimedia resources found on the *NelsonNet* student website, **www.nelsonnet.com.au**. These include:



- **Worksheets** and **puzzle sheets** that are write-in enabled PDFs
- **Video tutorials:** worked examples explained by ‘flipped classroom’ teachers
- ***ExamView* quizzes:** interactive and self-marking

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NELSONNET TEACHER WEBSITE

The *NelsonNet* teacher website, also at **www.nelsonnet.com.au**, contains:

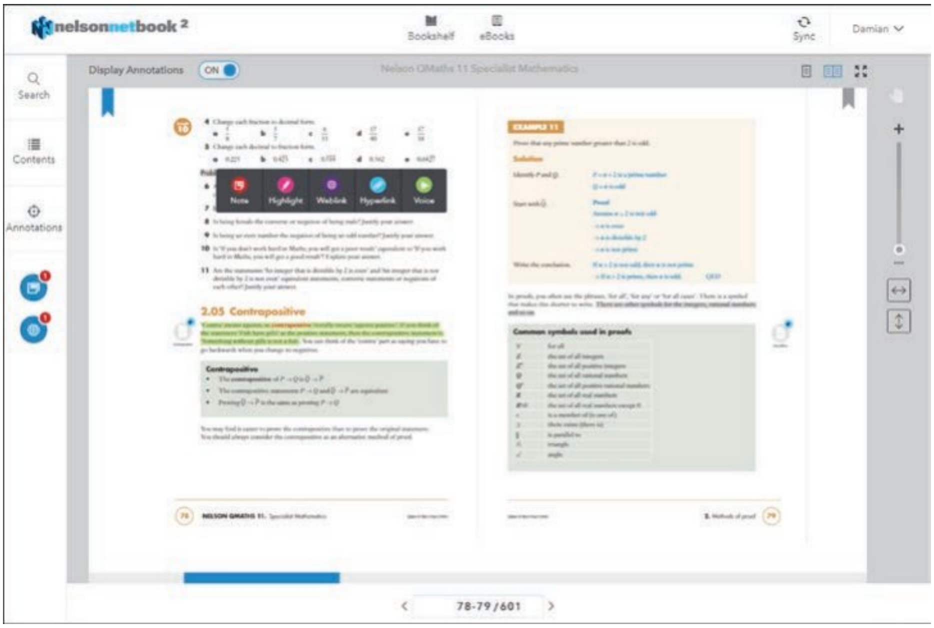
- A **teaching program,** in Microsoft Word and PDF formats
- **Topic tests,** in Microsoft Word and PDF formats
- **Worked solutions** to each exercise set
- **Chapter PDFs** of the textbook
- ***ExamView*** exam-writing software and questionbanks
- **Resource Finder:** search engine for *NelsonNet* resources

Note: Complimentary access to these resources is only available to teachers who use this book as a core educational resource in their classroom. Contact your Cengage Education Consultant for information about access codes and conditions.

NELSONNETBOOK

NelsonNetBook is the web-based interactive version of this book found on *NelsonNet*.

- To each page of NelsonNetBook you can add notes, voice and sound bites, highlighting, weblinks and bookmarks
- **Zoom** and **Search** functions
- Chapters can be customised for different groups of students



STUDY SKILLS

The Year 11 course introduces the basics of topics such as calculus that are then applied in the Year 12 course. You will struggle in the HSC if you don't set yourself up to revise the Year 11 topics as you learn new Year 12 topics. Your teachers will be able to help you build up and manage good study habits. Here are a few hints to get you started. There is no right or wrong way to learn. Different styles of learning suit different people. There is also no magical number of hours a week that you should study, because this will be different for every student. But just listening in class and taking notes is not enough, especially when you are learning material that is totally new.

If a skill is not practised within the first 24 hours, up to 50% can be forgotten. If it is not practised within 72 hours, up to 85–90% can be forgotten! So it is really important that, whatever your study timetable, new work must be looked at soon after it is presented to you.

With a continual succession of new work to learn and retain, this is a challenge. But the good news is that you don't have to study for hours on end!

IN THE CLASSROOM

In order to remember, first you need to focus on what is being said and done.

According to an ancient proverb:

I hear and I forget
I see and I remember
I do and I understand.

If you chat to friends and just take notes without really paying attention, you aren't giving yourself a chance to remember anything and will have to study harder at home.

If you are unsure of something that the teacher has said, the chances are that others are also not sure. Asking questions and clarifying things will ultimately help you gain better results, especially in a subject like mathematics where much of the knowledge and skills depend on being able to understand the basics.

Learning is all about knowing what you know and what you don't know. Many students feel like they don't know anything, but it's surprising just how much they know already. Picking up the main concepts in class and not worrying too much about other less important parts can really help. The teacher can guide you on this.

Here are some pointers to get the best out of classroom learning:

- Take control and be responsible for your own learning
- Clear your head of other issues in the classroom
- Active, not passive, learning is more memorable
- Ask questions if you don't understand something

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- Listen for cues from the teacher
- Look out for what are the main concepts.

Note-taking varies from class to class, but here are some general guidelines:

- Write legibly
- Use different colours to highlight important points or formulas
- Make notes in textbooks (using pencil if you don't own the textbook)
- Use highlighter pens to point out important points
- Summarise the main points
- If notes are scribbled, rewrite them at home.

AT HOME

You are responsible for your own learning and nobody else can tell you how best to study. Some people need more revision time than others, some study better in the mornings while others do better at night, and some can work at home while others prefer a library.

- Revise both new and older topics regularly
- Have a realistic timetable and be flexible
- Summarise the main points
- Revise when you are fresh and energetic
- Divide study time into smaller rather than longer chunks
- Study in a quiet environment
- Have a balanced life and don't forget to have fun!

If you are given exercises out of a textbook to do for homework, consider asking the teacher if you can leave some of them till later and use these for revision. It is not necessary to do every exercise at one sitting, and you learn better if you can spread these over time.

People use different learning styles to help them study. The more variety the better, and you will find some that help you more than others. Some people (around 35%) learn best visually, some (25%) learn best by hearing and others (40%) learn by doing.

- Summarise on cue cards or in a small notebook
- Use colourful posters
- Use mindmaps and diagrams
- Discuss work with a group of friends
- Read notes out aloud
- Make up songs and rhymes
- Exercise regularly
- Role-play teaching someone else

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ASSESSMENT TASKS AND EXAMS

You will cope better in exams if you have practised doing sample exams under exam conditions. Regular revision will give you confidence, and if you feel well prepared this will help get rid of nerves in the exam. You will also cope better if you have had a reasonable night's sleep before the exam.

One of the biggest problems students have with exams is in timing. Make sure you don't spend too much time on questions you're unsure about, but work through and find questions you can do first.

Divide the time up into smaller chunks for each question and allow some extra time to go back to questions you couldn't do or finish. For example, in a 2-hour exam with 6 questions, allow around 15 minutes for each question. This will give an extra half hour at the end to tidy up and finish off questions.

- Read through and ensure you know how many questions there are
- Divide your time between questions with extra time at the end
- Don't spend too much time on one question
- Read each question carefully, underlining key words
- Show all working out, including diagrams and formulas
- Cross out mistakes with a single line so it can still be read
- Write legibly

AND FINALLY...

Study involves knowing what you don't know, and putting in a lot of time into concentrating on these areas. This is a positive way to learn. Rather than just saying, 'I can't do this', say instead, 'I can't do this yet', and use your teachers, friends, textbooks and other ways of finding out.

With the parts of the course that you do know, make sure you can remember these easily under exam pressure by putting in lots of practice.

Remember to look at new work:

today, tomorrow, in a week, in a month.

Some people hardly ever find time to study while others give up their outside lives to devote their time to study. The ideal situation is to balance study with other aspects of your life, including going out with friends, working, and keeping up with sport and other activities that you enjoy.

Good luck with your studies!

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MATHEMATICAL VERBS

A glossary of 'doing words' commonly found in mathematics problems

analyse: study in detail the parts of a situation

apply: use knowledge or a procedure in a given situation

classify, identify: state the type, name or feature of an item or situation

comment: express an observation or opinion about a result

compare: show how two or more things are similar or different

construct: draw an accurate diagram

describe: state the features of a situation

estimate: make an educated guess for a number, measurement or solution, to find roughly or approximately

evaluate, calculate: find the value of a numerical expression, for example 3×8^2 or $4x + 1$ when $x = 5$

expand: remove brackets in an algebraic expression for example expanding $3(2y + 1)$ gives $6y + 3$

explain: describe why or how

factorise: opposite to **expand,** to insert brackets by taking out a common factor, for example factorising $6y + 3$ gives $3(2y + 1)$

give reasons: show the rules or thinking used when solving a problem. *See also* **justify.**

hence find/prove: find an answer or prove a result using previous answers or information supplied

interpret: find meaning in a mathematical result

justify: give reasons or evidence to support your argument or conclusion. *See also* **give reasons**

rationalise: make rational, remove surds

show that, prove: (in questions where the answer is given) use calculation, procedure or reasoning to prove that an answer or result is true

simplify: give a result in its most basic, shortest, neatest form, for example simplifying a ratio or algebraic expression

sketch: draw a rough diagram that shows the general shape or ideas, less accurate than **construct**

solve: find the value(s) of an unknown pronumeral in an equation or inequality

substitute: replace a variable by a number and evaluate

verify: check that a solution or result is correct, usually by substituting back into the equation or referring back to the problem

write, state: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

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FUNCTIONS

1.

ALGEBRAIC TECHNIQUES

This chapter revises and extends the algebraic techniques that you will need for this course. These include indices, algebraic expressions, expansion, factorisation, algebraic fractions and surds.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- identify and use index rules including fractional and negative indices
- simplify algebraic expressions
- remove grouping symbols including perfect squares and the difference of 2 squares
- factorise expressions including binomials and special factors
- simplify algebraic fractions
- use algebra to substitute into formulas
- simplify and use surds including rationalising the denominator

TERMINOLOGY

binomial: A mathematical expression consisting of 2 terms; for example, $x + 3$ and $3x - 1$

binomial product: The product of binomial expressions for example $(x + 3)(2x - 1)$

expression: A mathematical statement involving numbers, pronumerals and symbols; for example, $2x - 3$

factor: A whole number that divides exactly into another number. For example, 4 is a factor of 28

factorise: To write an expression as a product of its factors; that is, take out the highest common factor in an expression and place the rest in brackets For exampl, $2y - 8 = 2(y - 4)$

index: The power or exponent of a number. For example, 2^3 has a base number of 2 and an index of 3. The plural of index is **indices**

power: The index or exponent of a number. For example, 2^3 has a base number of 2 and a power of 3

root: A number that when multiplied by itself a given number of times equals another number. For example, $\sqrt{25} = 5$ because $5^2 = 25$

surd A root that ca't be simplifid; for example, $\sqrt{3}$

term: A part of an expression containing pronumerals and/or numbers separated by an operation such as +, −, × or ÷. For example, in $2x - 3$ the terms are $2x$ and 3

trinomial: An expression with 3 terms; for example, $3x^2 - 2x + 1$

1.01 Index laws

An **index** (or **power** or **exponent**) of a number shows how many times a number is multiplied by itself. A **root** of a number is the inverse of the power.

For example:

- $4^3 = 4 \times 4 \times 4 = 64$
- $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$
- $\sqrt{36} = 6$ since $6^2 = 36$
- $\sqrt[3]{8} = 2$ since $2^3 = 8$
- $\sqrt[6]{64} = 2$ since $2^6 = 64$

Note: In 4^3 the 4 is called the base number and the 3 s called the ndex or power.

There are some general laws that simplify calculations with indices. These laws work for any m and n, including fractions and negative numbers.

Index laws

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

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EXAMPLE 1

Simplify:

a $m^9 \times m^7 \div m^2$ **b** $(2y^4)^3$ **c** $\dfrac{(y^6)^3 \times y^{-4}}{y^5}$

Solution

a $m^9 \times m^7 \div m^2 = m^{9+7-2}$
$= m^{14}$

b $(2y^4)^3 = 2^3(y^4)^3$
$= 2^3 y^{4\times 3}$
$= 8y^{12}$

c $\dfrac{(y^6)^3 \times y^{-4}}{y^5} = \dfrac{y^{18} \times y^{-4}}{y^5}$
$= \dfrac{y^{18+(-4}}{y^5}$
$= \dfrac{y^{14}}{y^5}$
$= y^{14-5}$
$= y^9$

Exercise 1.01 Index laws

1 Evaluate without using a calculator:

a $5^3 \times 2^2$ **b** $3^4 + 8^2$ **c** $\left(\dfrac{1}{4}\right)^3$

d $\sqrt[3]{27}$ **e** $\sqrt[4]{16}$

2 Evaluate correct to 1 decimal place:

a 3.7^2 **b** $1.06^{1.5}$ **c** $2.3^{-0.2}$

d $\sqrt[3]{19}$ **e** $\sqrt[3]{34.8 - 12 \times 43.1}$ **f** $\dfrac{1}{\sqrt[3]{0.99 + 5.61}}$

3 Simplify:

a $a^6 \times a^9 \times a^2$ **b** $y^3 \times y^{-8} \times y^5$ **c** $a^{-1} \times a^{-3}$

d $w^{\frac{1}{2}} \times w^{\frac{1}{2}}$ **e** $x^6 \div x$ **f** $p^3 \div p^{-7}$

g $\dfrac{y^{11}}{y^5}$ **h** $(x^7)^3$ **i** $(2x^5)^2$

j $(3y^{-2})^4$ **k** $a^3 \times a^5 \div a^7$ **l** $\left(\dfrac{x^2}{y^9}\right)^5$

m $\dfrac{w^6 \times w^7}{w^3}$ **n** $\dfrac{p^2 \times (p^3)^4}{p^9}$ **o** $\dfrac{x^6 \div x^7}{x^2}$

p $\dfrac{a^2 \times (b^2)^6}{a^4 \times b^9}$ **q** $\dfrac{(x^2)^{-3} \times (y^3)^2}{x^{-1} \times y^4}$

4 Simplify:

a $x^5 \times x^9$

b $a^{-1} \times a^{-6}$

c $\frac{m^7}{m^3}$

d $k^{13} \times k^6 \div k^9$

e $a^{-5} \times a^4 \times a^{-7}$

f $x^{\frac{2}{5}} \times x^{\frac{3}{5}}$

g $\frac{m^5 \times n^4}{m^4 \times n^2}$

h $\frac{p^{\frac{1}{2}} \times p^{\frac{1}{2}}}{p^2}$

i $(3x^{11})^2$

j $\frac{(x^4)^6}{x^3}$

5 Expand each expression and simplify where possible:

a $(pq^3)^5$

b $\left(\frac{a}{b}\right)^8$

c $\left(\frac{4a}{b^4}\right)^3$

d $(7a^5b)^2$

e $\frac{(2m^7)^3}{m^4}$

f $\frac{xy^3 \times (xy^2)^4}{xy}$

g $\frac{(2k^8)^4}{(6k^3)^3}$

h $(2y^5)^7 \times \frac{y^{12}}{8}$

i $\left(\frac{a^6 \times a^4}{a^{11}}\right)^{-3}$

j $\left(\frac{5xy^9}{x^8 \times y^3}\right)^3$

6 Evaluate a^3b^2 when $a = 2$ and $b = \frac{3}{4}$.

7 If $x = \frac{2}{3}$ and $y = \frac{1}{9}$, find the value of $\frac{x^3y^2}{xy^5}$.

8 If $a = \frac{1}{2}$, $b = \frac{1}{3}$ and $c = \frac{1}{4}$, evaluate $\frac{a^2b^3}{c^4}$ as a fraction.

9 **a** Simplify $\frac{a^{11}b^8}{a^8b^7}$.

b Hence evaluate $\frac{a^{11}b^8}{a^8b^7}$ as a fraction when $a = \frac{2}{5}$ and $b = \frac{5}{8}$.

10 **a** Simplify $\frac{p^5q^8r^4}{p^4q^6r^2}$.

b Hence evaluate $\frac{p^5q^8r^4}{p^4q^6r^2}$ as a fraction when $p = \frac{7}{8}$, $q = \frac{2}{3}$ and $r = \frac{3}{4}$.

11 Evaluate $(a^4)^3$ when $a = \left(\frac{2}{3}\right)^{\frac{1}{6}}$.

12 Evaluate $\frac{a^3b^6}{b^4}$ when $a = \frac{1}{2}$ and $b = \frac{2}{3}$.

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13 Evaluate $\dfrac{x^4 y^7}{x^5 y^5}$ when $x = \dfrac{1}{3}$ and $y = \dfrac{2}{9}$.

14 Evaluate $\dfrac{k^{-5}}{k^{-9}}$ when $k = \dfrac{1}{3}$.

15 Evaluate $\dfrac{a^4 b^6}{a^3 (b^2)^2}$ when $a = \dfrac{3}{4}$ and $b = \dfrac{1}{9}$.

16 Evaluate $\dfrac{a^6 \times b^3}{a^5 \times b^2}$ as a fraction when $a = \dfrac{1}{9}$ and $b = \dfrac{3}{4}$.

1.02 Zero and negative indices

Zero and negative indices

$$x^0 = 1$$

$$x^{-n} = \frac{1}{x^n}$$

WS
Review o
index laws

EXAMPLE 2

a Simplify $\left(\dfrac{ab^5c}{abc^4}\right)^0$.

b Evaluate 2^{-3}.

c Write in index form:

i $\dfrac{1}{x^2}$ **ii** $\dfrac{3}{x^5}$ **iii** $\dfrac{1}{5x}$ **v** $\dfrac{1}{x+1}$

d Write a^{-3} without the negative index.

Solution

a $\left(\dfrac{ab^5c}{abc^4}\right)^0 = 1$

b $2^{-3} = \dfrac{1}{2^3}$

$= \dfrac{1}{8}$

c **i** $\dfrac{1}{x^2} = x^{-2}$

ii $\dfrac{3}{x^5} = 3 \times \dfrac{1}{x^5}$

$= 3x^{-5}$

iii $\dfrac{1}{5x} = \dfrac{1}{5} \times \dfrac{1}{x}$

$= \dfrac{1}{5}x^{-}$

iv $\dfrac{1}{x+1} = \dfrac{1}{(x+1)}$

$= (x+1)^{-1}$

d $a^{-3} = \dfrac{1}{a^3}$

Exercise 1.02 Zero and negative indices

1 Evaluate as a fraction or whole number:

a 3^{-3} **b** 4^{-1} **c** 7^{-3} **d** 10^{-4} **e** 2^{-8}

f 6^0 **g** 2^{-5} **h** 3^{-4} **i** 7^{-1} **j** 9^{-2}

k 2^{-6} **l** 3^{-2} **m** 4^0 **n** 6^{-2} **o** 5^{-3}

p 10^{-5} **q** 2^{-7} **r** 2^0 **s** 8^{-2} **t** 4^{-3}

2 Evaluate:

a 2^0 **b** $\left(\dfrac{1}{2}\right)^{-4}$ **c** $\left(\dfrac{2}{3}\right)^{-1}$ **d** $\left(\dfrac{5}{6}\right)^{-2}$ **e** $\left(\dfrac{x+2y}{3x-y}\right)^{0}$

f $\left(\dfrac{1}{5}\right)^{-3}$ **g** $\left(\dfrac{3}{4}\right)^{-1}$ **h** $\left(\dfrac{1}{7}\right)^{-2}$ **i** $\left(\dfrac{2}{3}\right)^{-3}$ **j** $\left(\dfrac{1}{2}\right)^{-5}$

k $\left(\dfrac{3}{7}\right)^{-1}$ **l** $\left(\dfrac{8}{9}\right)^{0}$ **m** $\left(\dfrac{6}{7}\right)^{-2}$ **n** $\left(\dfrac{9}{10}\right)^{-2}$ **o** $\left(\dfrac{6}{11}\right)^{0}$

p $\left(-\dfrac{1}{4}\right)^{-2}$ **q** $\left(-\dfrac{2}{5}\right)^{-3}$ **r** $\left(-3\tfrac{2}{7}\right)^{-1}$ **s** $\left(-\dfrac{3}{8}\right)^{0}$ **t** $\left(-1\tfrac{1}{4}\right)^{-2}$

3 Change into index form:

a $\dfrac{1}{m^3}$ **b** $\dfrac{1}{x}$ **c** $\dfrac{1}{p^7}$ **d** $\dfrac{1}{d^9}$ **e** $\dfrac{1}{k^5}$

f $\dfrac{1}{x^2}$ **g** $\dfrac{2}{x^4}$ **h** $\dfrac{3}{y^2}$ **i** $\dfrac{1}{2z^6}$ **j** $\dfrac{3}{5t^8}$

k $\dfrac{2}{7x}$ **l** $\dfrac{5}{2m^6}$ **m** $\dfrac{2}{3y^7}$ **n** $\dfrac{1}{(3x+4)^2}$ **o** $\dfrac{1}{(a+b)^8}$

p $\dfrac{1}{x-2}$ **q** $\dfrac{1}{(5p+1)^3}$ **r** $\dfrac{2}{(4t-9)^5}$ **s** $\dfrac{1}{4(x+1)^{11}}$ **t** $\dfrac{5}{9(a+3b)^7}$

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4 Write without negative indices:

a	t^{-5}	**b**	x^{-6}	**c**	y^{-3}	**d**	n^{-8}	**e**	w^{-10}
f	$2x^{-1}$	**g**	$3m^{-4}$	**h**	$5x^{-7}$	**i**	$(2x)^{-3}$	**j**	$(4n)^{-1}$
k	$(x+1)^{-6}$	**l**	$(8y+z)^{-1}$	**m**	$(k-3)^{-2}$	**n**	$(3x+2y)^{-9}$	**o**	$\left(\frac{1}{x}\right)^{-5}$
p	$\left(\frac{1}{y}\right)^{-10}$	**q**	$\left(\frac{2}{p}\right)^{-1}$	**r**	$\left(\frac{1}{a+b}\right)^{-2}$	**s**	$\left(\frac{x+y}{x-y}\right)^{-1}$	**t**	$\left(\frac{2w-z}{3x+y}\right)^{-7}$

1.03 Fractional indices

WS Indice

WS Facional indices and radicals

INVESTIGATION

FRACTIONAL INDICES

Consider the following examples.

$$\left(x^{\frac{1}{2}}\right)^2 = x^1 \text{ (by index laws)}$$
$$= x$$

$$\left(\sqrt{x}\right)^2 = x$$

$$\text{So } \left(x^{\frac{1}{2}}\right)^2 = \left(\sqrt{x}\right)^2$$
$$= x$$
$$\therefore \quad x^{\frac{1}{2}} = \sqrt{x}$$

Now simplify these expressions.

1	$\left(x^2\right)^{\frac{1}{2}}$	**2**	$\sqrt{x^2}$	**3**	$\left(x^{\frac{1}{3}}\right)^3$	**4**	$\left(x^3\right)^{\frac{1}{3}}$	**5**	$\left(\sqrt[3]{x}\right)^3$
6	$\sqrt[3]{x^3}$	**7**	$\left(x^{\frac{1}{4}}\right)^4$	**8**	$\left(x^4\right)^{\frac{1}{4}}$	**9**	$\left(\sqrt[4]{x}\right)^4$	**10**	$\sqrt[4]{x^4}$

Use your results to complete:

$x^{\frac{1}{n}} =$

Power of $\frac{1}{n}$

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Proof

$$\left(a^{\frac{1}{n}}\right)^n = a \quad \text{(by index laws)}$$

$$(\sqrt[n]{a})^n = a$$

$$\therefore a^{\frac{1}{n}} = \sqrt[n]{a}$$

EXAMPLE 3

a Evaluate:

i $49^{\frac{1}{2}}$ **ii** $27^{\frac{1}{3}}$

b Write $\sqrt{3x-2}$ in index form.

c Write $(a+b)^{\frac{1}{7}}$ without fractional indices.

Solution

a **i** $49^{\frac{1}{2}} = \sqrt{49} = 7$ **ii** $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

b $\sqrt{3x-2} = (3x-2)^{\frac{1}{2}}$ **c** $(a+b)^{\frac{1}{7}} = \sqrt[7]{a+b}$

Further fractional indices

$$a^{-\frac{1}{n}} = \frac{1}{\sqrt[n]{a}}$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ or } \left(\sqrt[n]{a}\right)^m$$

Proof

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$
$$= \left(\sqrt[n]{a}\right)^m$$

$$a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}}$$
$$= \sqrt[n]{a^m}$$

ISBN 97807043299

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

Proof

$$\begin{aligned}
\left(\frac{a}{b}\right)^{-n} &= \frac{1}{\left(\frac{a}{b}\right)^{n}} \\
&= \frac{1}{\frac{a^n}{b^n}} \\
&= 1 \div \frac{a^n}{b^n} \\
&= 1 \times \frac{b^n}{a^n} \\
&= \frac{b^n}{a^n} \\
&= \left(\frac{b}{a}\right)^{n}
\end{aligned}$$

EXAMPLE 4

a Evaluate:

i $8^{\frac{4}{3}}$ **ii** $125^{-\frac{\ }{3}}$ **iii** $\left(\frac{2}{3}\right)^{-3}$

b Write in index form:

i $\sqrt{x^5}$ **ii** $\frac{1}{\sqrt[3]{\left(4x^2-1\right)^2}}$

c Write $r^{-\frac{3}{5}}$ without the negative and fractional indices.

Solution

a **i** $8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4 \text{ (or } \sqrt[3]{8^4}\text{)}$
$= 2^4$
$= 16$

ii $125^{-\frac{1}{3}} = \dfrac{1}{125^{\frac{1}{3}}}$
$= \dfrac{1}{\sqrt[3]{125}}$
$= \dfrac{1}{5}$

iii $\left(\dfrac{2}{3}\right)^{-3} = \left(\dfrac{3}{2}\right)^3$
$= \dfrac{27}{8}$
$= 3\frac{3}{8}$

b **i** $\sqrt{x^5} = x^{\frac{5}{2}}$

ii $\dfrac{1}{\sqrt[3]{(4x^2-1)^2}} = \dfrac{1}{(4x^2-1)^{\frac{2}{3}}}$
$= (4x^2-1)^{-\frac{2}{3}}$

c $r^{-\frac{3}{5}} = \dfrac{1}{r^{\frac{3}{5}}} = \dfrac{1}{\sqrt[5]{r^3}}$

DID YOU KNOW?

Fractional indices

Nicole Oresme (1323–82) was the first mathematician to use fractional indices.

John Wallis (1616–1703) was the first person to explain the significance of zero, negative and fractional indices. He also introduced the symbol ∞ for infinity.

Research these mathematicians and find out more about their work and backgrounds. You could use keywords such as indices and infinity as well as their names to find this information.

Exercise 1.03 Fractional indices

1 Evaluate:

a $81^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $16^{\frac{1}{2}}$ **d** $8^{\frac{1}{3}}$ **e** $49^{\frac{1}{2}}$

f $1000^{\frac{1}{3}}$ **g** $16^{\frac{1}{4}}$ **h** $64^{\frac{1}{2}}$ **i** $64^{\frac{1}{3}}$ **j** $1^{\frac{1}{7}}$

k $81^{\frac{1}{4}}$ **l** $32^{\frac{1}{5}}$ **m** $0^{\frac{1}{8}}$ **n** $125^{\frac{1}{3}}$ **o** $343^{\frac{1}{3}}$

p $128^{\frac{1}{7}}$ **q** $256^{\frac{1}{4}}$ **r** $125^{\frac{2}{3}}$ **s** $4^{\frac{5}{2}}$ **t** $8^{\frac{2}{3}}$

u $9^{\frac{3}{2}}$ **v** $8^{-\frac{1}{3}}$ **w** $9^{-\frac{1}{2}}$ **x** $16^{-\frac{1}{4}}$ **y** $64^{-\frac{2}{3}}$

 ISBN 97807043299

2 Evaluate correct to 2 decimal places:

a $23^{\frac{1}{4}}$ **b** $\sqrt[4]{45.8}$ **c** $\sqrt[7]{1.24+4.3^2}$

d $\frac{1}{\sqrt[5]{12.9}}$ **e** $\sqrt[8]{\frac{3.6-1.4}{1.5+3.7}}$ **f** $\frac{\sqrt[4]{5.9\times 3.7}}{8.79-1.4}$

3 Write without fractional or negative indices:

a $y^{\frac{1}{3}}$ **b** $x^{\frac{1}{6}}$ **c** $a^{\frac{1}{2}}$ **d** $t^{\frac{1}{9}}$ **e** $y^{\frac{2}{3}}$

f $x^{\frac{3}{4}}$ **g** $b^{\frac{2}{5}}$ **h** $a^{\frac{4}{7}}$ **i** $x^{-\frac{1}{2}}$ **j** $d^{-\frac{1}{3}}$

k $x^{-\frac{1}{8}}$ **l** $y^{-\frac{1}{3}}$ **m** $a^{-\frac{1}{4}}$ **n** $z^{-\frac{3}{4}}$ **o** $y^{-\frac{3}{5}}$

p $(2x+5)^{\frac{1}{2}}$ **q** $(6q+r)^{\frac{1}{3}}$ **r** $(a+b)^{\frac{1}{9}}$ **s** $(3x-1)^{-\frac{1}{2}}$ **t** $(x+7)^{-\frac{2}{5}}$

4 Write in index form:

a $\sqrt{t}$ **b** $\sqrt[5]{y}$ **c** $\sqrt{x^3}$ **d** $\sqrt[3]{9-x}$ **e** $\sqrt{4s+1}$

f $\sqrt{(3x+1)^5}$ **g** $\frac{1}{\sqrt{2t+3}}$ **h** $\frac{1}{\sqrt{(5x-y)^3}}$ **i** $\frac{1}{\sqrt[3]{(x-2)^2}}$ **j** $\frac{1}{2\sqrt{y+7}}$

k $\frac{5}{\sqrt[3]{x+4}}$ **l** $\frac{1}{3\sqrt{y^2-1}}$ **m** $\frac{3}{5\sqrt[4]{(x^2+2)^3}}$

5 Write in index form and simplify:

a $x\sqrt{x}$ **b** $\frac{\sqrt{x}}{x}$ **c** $\frac{x}{\sqrt[3]{x}}$ **d** $\frac{x^2}{\sqrt[3]{x}}$ **e** $x\sqrt[4]{x}$

6 Write without fractional or negative indices:

a $(a-2b)^{-\frac{1}{3}}$ **b** $(y-3)^{-\frac{2}{3}}$ **c** $4(6a+1)^{-\frac{4}{7}}$ **d** $\frac{(x+y)^{-\frac{5}{4}}}{3}$ **e** $\frac{6(3x+8)^{-\frac{2}{9}}}{7}$

DID YOU KNOW?

The beginnings of algebra

One of the earliest mathematicians to use algebra was **Diophantus of Alexandria** in Greece. It is not known when he lived, but it is thought this may have been around 250 CE.

In Persia around 700–800 CE a mathematician named **Muhammad ibn Musa al-Khwarizmi** wrote books on algebra and Hindu numerals. One of his books was named *Al-Jabr wa'l Muqabala*, and the word **algebra** comes from the first word in this title.

ISBN 97807043299

1.04 Simplifying algebraic expressions

EXAMPLE 5

Simplify:

a $4x^2 - 3x^2 + 6x^2$ **b** $x^3 - 3x - 5x + 4$ **c** $3a - 4b - 5a - b$

Solution

a $4x^2 - 3x^2 + 6x^2 = x^2 + 6x^2$

$= 7x^2$

b $x^3 - 3x - 5x + 4 = x^3 - 8x + 4$

Only 'like' terms can be added or subtracted.

c $3a - 4b - 5a - b = 3a - 5a - 4b - b$

$= -2a - 5b$

EXAMPLE 6

Simplify:

a $-5x \times 3y \times 2x$ **b** $\dfrac{5a^3b}{15ab^2}$

Solution

a $-5x \times 3y \times 2x = -30xyx$

$= -30x^2y$

b $\dfrac{5a^3b}{15ab^2} = \dfrac{1}{3}a^{3-1}b^{1-2}$

$= \dfrac{1}{3}a^2b^{-1}$

$= \dfrac{a^2}{3b}$

ISBN 97807043299

Exercise 1.04 Simplifying algebraic expressions

1 Simplify:

a $9a - 6a$ **b** $5z - 4z$ **c** $4b - b$

d $2r - 5r$ **e** $-4y + 3y$ **f** $-2x - 3x$

g $2a - 2a$ **h** $-4k + 7k$ **i** $3t + 4t + 2t$

j $8w - w + 3w$ **k** $4m - 3m - 2m$ **l** $x + 3x - 5x$

m $8h - h - 7h$ **n** $3b - 5b + 4b + 9b$ **o** $-5x + 3x - x - 7x$

p $6x - 5y - y$ **q** $8a + b - 4b - 7a$ **r** $xy + 2y + 3xy$

s $2ab^2 - 5ab^2 - 3ab^2$ **t** $m^2 - 5m - m + 12$ **u** $p^2 - 7p + 5p - 6$

v $ab + 2b - 3ab + 8b$ **w** $ab + bc - ab - ac + bc$

x $a^5 - 7x^3 + a^5 - 2x^3 + 1$ **y** $x^3 - 3xy^2 + 4x^2y - x^2y + xy^2 + 2y^3$

2 Simplify:

a $5 \times 2b$ **b** $2x \times 4y$ **c** $5p \times 2p$

d $-3z \times 2w$ **e** $-5a \times -3b$ **f** $x \times 2y \times 7z$

g $8ab \times 6c$ **h** $4d \times 3d$ **i** $3a \times 4a \times a$

j $(-3y)^3$ **k** $(2x^2)^5$ **l** $2ab^3 \times 3a$

m $5a^2b \times -2ab$ **n** $7pq^2 \times 3p^2q^2$ **o** $5ab \times a^2b^2$

p $4h^3 \times -2h^7$ **q** $k^3p \times p^2$ **r** $(-3t^3)^4$

s $7m^6 \times -2m^5$ **t** $-2x^2 \times 3x^3y \times -4xy^2$

3 Simplify:

a $30x \div 5$ **b** $2y \div y$ **c** $\frac{8a^2}{2}$

d $\frac{8a^2}{a}$ **e** $\frac{8a^2}{2a}$ **f** $\frac{xy}{2x}$

g $12p^3 \div 4p^2$ **h** $\frac{3a^2b^2}{6ab}$ **i** $\frac{20x}{15xy}$

j $\frac{-9x^7}{3x^4}$ **k** $-15ab \div -5b$ **l** $\frac{2ab}{6a^2b^3}$

m $\frac{-8p}{4pqs}$ **n** $14cd^2 \div 21c^3d^3$ **o** $\frac{2xy^2z^3}{4x^3y^2z}$

p $\frac{42p^5q^4}{7pq^3}$ **q** $5a^9b^4c^{-2} \div 20a^5b^{-3}c^{-1}$ **r** $\frac{2(a^{-5})^2b^4}{4a^{-9}(b^2)^{-1}}$

s $-5x^4y^7z \div 15xy^8z^{-2}$ **t** $-9(a^4b^{-1})^3 \div -18a^{-1}b^3$

ISBN 97807043299

WS
Expanding algebraic expressions

1.05 Expansion

When we remove grouping symbols we say that we are **expanding** an expression.

Expanding expressions

To expand an expression, use the distributive law:

$$a(b+c) = ab + ac$$

EXAMPLE 7

Expand and simplify:

a $5a^2(4+3ab-c)$ **b** $5-2(y+3)$ **c** $2(b-5)-(b+1)$

Solution

a $5a^2(4+3ab-c) = 5a^2 \times 4 + 5a^2 \times 3ab - 5a^2 \times c$

$= 20a^2 + 15a^3b - 5a^2c$

b $5-2(y+3) = 5 - 2 \times y - 2 \times 3$

$= 5 - 2y - 6$

$= -2y - 1$

c $2(b-5)-(b+1) = 2 \times b + 2 \times -5 - 1 \times b - 1 \times 1$

$= 2b - 10 - b - 1$

$= b - 11$

Exercise 1.05 Expansion

Expand and simplify each expression.

1 $2(x-4)$ **2** $3(2h+3)$ **3** $-5(a-2)$

4 $x(2y+3)$ **5** $x(x-2)$ **6** $2a(3a-8b)$

7 $ab(2a+b)$ **8** $5n(n-4)$ **9** $3x^2y(xy+2y^2)$

10 $3+4(k+1)$ **11** $2(t-7)-3$ **12** $y(4y+3)+8y$

13 $9-5(b+3)$ **14** $3-(2x-5)$ **15** $5(3-2m)+7(m-2)$

16 $2(h+4)+3(2h-9)$ **17** $3(2d-3)-(5d-3)$ **18** $a(2a+1)-(a^2+3a-4)$

19 $x(3x-4)-5(x+1)$ **20** $2ab(3-a)-b(4a-1)$ **21** $5x-(x-2)-3$

22 $8-4(2y+1)+y$ **23** $(a+b)-(a-b)$ **24** $2(3t-4)-(t+1)+3$

ISBN 97807043299

1.06 Binomial products

Expanding binomial products

A **binomial expression** consists of 2 **terms**; for example, $x + 3$.

A set of 2 binomial expressions multiplied together is called a **binomial product**; for example $(x + 3)(x - 2)$.

Each term in the first bracket is multiplied by each term in the second bracket.

Binomial product

$$(x + a)(x + b) = x^2 + bx + ax + ab$$

EXAMPLE 8

Expand and simplify:

a $(p + 3)(q - 4)$ **b** $(a + 5)^2$ **c** $(x + 4)(2x - 3y - 1)$

Solution

a $(p + 3)(q - 4) = pq - 4p + 3q - 12$

b $(a + 5)^2 = (a + 5)(a + 5)$
$= a^2 + 5a + 5a + 25$
$= a^2 + 10a + 25$

c $(x + 4)(2x - 3y - 1) = 2x^2 - 3xy - x + 8x - 12y - 4$
$= 2x^2 - 3xy + 7x - 12y - 4$

Exercise 1.06 Binomial products

Expand and simplify:

1 $(a + 5)(a + 2)$ **2** $(x + 3)(x - 1)$ **3** $(2y - 3)(y + 5)$ **4** $(m - 4)(m - 2)$

5 $(x + 4)(x + 3)$ **6** $(y + 2)(y - 5)$ **7** $(2x - 3)(x + 2)$ **8** $(h - 7)(h - 3)$

9 $(x + 5)(x - 5)$ **10** $(5a - 4)(3a - 1)$ **11** $(2y + 3)(4y - 3)$ **12** $(x - 4)(y + 7)$

13 $(x^2 + 3)(x - 2)$ **14** $(n + 2)(n - 2)$ **15** $(2x + 3)(2x - 3)$ **16** $(4 - 7y)(4 + 7y)$

17 $(a + 2b)(a - 2b)$ **18** $(3x - 4y)(3x + 4y)$ **19** $(x + 3)(x - 3)$ **20** $(y - 6)(y + 6)$

21 $(3a + 1)(3a - 1)$ **22** $(2z - 7)(2z + 7)$ **23** $(x + 9)(x - 2y + 2)$

24 $(b - 3)(2a + 2b - 1)$ **25** $(x + 2)(x^2 - 2x + 4)$ **26** $(a - 3)(a^2 + 3a + 9)$

27 $(a + 9)^2$ **28** $(k - 4)^2$ **29** $(x + 2)^2$ **30** $(y - 7)^2$

31 $(2x + 3)^2$ **32** $(2t - 1)^2$ **33** $(3a + 4b)^2$ **34** $(x - 5y)^2$

ISBN 97807043299

35 $(2a + b)^2$ **36** $(a - b)(a + b)$ **37** $(a + b)^2$ **38** $(a - b)^2$

39 $(a + b)(a^2 - ab + b^2)$ **40** $(a - b)(a^2 + ab + b^2)$

Expanding expressions

Special binomial products

1.07 Special products

Some binomial products have special results and can be simplified quickly using their special properties. Did you notice some of these in Exercise 1.06?

Difference of two squares

$$(a + b)(a - b) = a^2 - b^2$$

Perfect squares

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

EXAMPLE 9

Expand and simplify:

a $(2x - 3)^2$

b $(3y - 4)(3y + 4)$

Solution

a $(2x - 3)^2 = (2x)^2 - 2(2x)3 + 3^2$
$= 4x^2 - 12x + 9$

b $(3y - 4)(3y + 4) = (3y)^2 - 4^2$
$= 9y^2 - 16$

Exercise 1.07 Special products

Expand and simplify:

1 $(t + 4)^2$ **2** $(z - 6)^2$ **3** $(x - 1)^2$

4 $(y + 8)^2$ **5** $(q + 3)^2$ **6** $(k - 7)^2$

7 $(n + 1)^2$ **8** $(2b + 5)^2$ **9** $(3 - x)^2$

10 $(3y - 1)^2$ **11** $(x + y)^2$ **12** $(3a - b)^2$

13 $(4d + 5e)^2$ **14** $(t + 4)(t - 4)$ **15** $(x - 3)(x + 3)$

16 $(p + 1)(p - 1)$ **17** $(r + 6)(r - 6)$ **18** $(x - 10)(x + 10)$

19 $(2a + 3)(2a - 3)$ **20** $(x - 5y)(x + 5y)$ **21** $(4a + 1)(4a - 1)$

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22 $(7-3x)(7+3x)$ **23** $(x^2+2)(x^2-2)$ **24** $(x^2+5)^2$

25 $(3ab-4c)(3ab+4c)$ **26** $\left(x+\frac{2}{x}\right)^2$ **27** $\left(a-\frac{1}{a}\right)\left(a+\frac{1}{a}\right)$

28 $[x+(y-2)][x-(y-2)]$ **29** $[(a+b)+c]^2$ **30** $[(x+1)-y]^2$

31 $(a+3)^2-(a-3)^2$ **32** $16-(z-4)(z+4)$ **33** $2x+(3x+1)^2-4$

34 $(x+y)^2-x(2-y)$ **35** $(4n-3)(4n+3)-2n^2+5$ **36** $(x-4)^3$

37 $\left(x-\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)^2+2$ **38** $(x^2+y^2)^2-4x^2y^2$ **39** $(2a+5)^3$

1.08 Factorisation

WS Facoiing algebraic expressions

Factors divide exactly into an equal or larger number or term, without leaving a remainder.

Factorising

To **factorise** an expression, we use the distributive law in the opposite way from when we expand brackets.

$$ax+bx=x(a+b)$$

EXAMPLE 10

Factorise:

a $3x+12$ **b** y^2-2y **c** x^3-2x^2

d $5(x+3)+2y(x+3)$ **e** $8a^3b^2-2ab^3$

Solution

a The highest common factor is 3. $3x+12=3(x+4)$

b The highest common factor is y. $y^2-2y=y(y-2)$

c x and x^2 are both common factors. Take out the highest common factor, which is x^2. $x^3-2x^2=x^2(x-2)$

d The highest common factor is $x+3$. $5(x+3)+2y(x+3)=(x+3)(5+2y)$

e The highest common factor is $2ab^2$. $8a^3b^2-2ab^3=2ab^2(4a^2-b)$

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Exercise 1.08 Factorisation

Factorise:

1 $2y + 6$ **2** $5x - 10$ **3** $3m - 9$

4 $8x + 2$ **5** $24 - 18y$ **6** $x^2 + 2x$

7 $m^2 - 3m$ **8** $2y^2 + 4y$ **9** $15a - 3a^2$

10 $ab^2 + ab$ **11** $4x^2y - 2xy$ **12** $3mn^3 + 9mn$

13 $8x^2z - 2xz^2$ **14** $6ab + 3a - 2a^2$ **15** $5x^2 - 2x + xy$

16 $3q^5 - 2q^2$ **17** $5b^3 + 15b^2$ **18** $6a^2b^3 - 3a^3b^2$

19 $x(m + 5) + 7(m + 5)$ **20** $2(y - 1) - y(y - 1)$ **21** $4(7 + y) - 3x(7 + y)$

22 $6x(a - 2) + 5(a - 2)$ **23** $x(2t + 1) - y(2t + 1)$

24 $a(3x - 2) + 2b(3x - 2) - 3c(3x - 2)$ **25** $6x^3 + 9x^2$

26 $3pq^5 - 6q^3$ **27** $15a^4b^3 + 3ab$ **28** $4x^3 - 24x^2$

29 $35m^3n^4 - 25m^2n$ **30** $24a^2b^5 + 16ab^2$ **31** $2\pi r^2 + 2\pi rh$

32 $(x - 3)^2 + 5(x - 3)$ **33** $y^2(x + 4) + 2(x + 4)$ **34** $a(a + 1) - (a + 1)^2$

1.09 Factorisation by grouping in pairs

Factorising by grouping in pairs

If an expression has 4 terms, it can sometimes be factorised in pairs.

$$\begin{aligned} ax + bx + ay + by &= x(a + b) + y(a + b) \\ &= (a + b)(x + y) \end{aligned}$$

EXAMPLE 11

Factorise:

a $x^2 - 2x + 3x - 6$ **b** $2x - 4 + 6y - 3xy$

Solution

a
$$\begin{aligned} x^2 - 2x + 3x - 6 &= x(x - 2) + 3(x - 2) \\ &= (x - 2)(x + 3) \end{aligned}$$

b
$$\begin{aligned} 2x - 4 + 6y - 3xy &= 2(x - 2) + 3y(2 - x) \\ &= 2(x - 2) - 3y(x - 2) \\ &= (x - 2)(2 - 3y) \end{aligned}$$

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Exercise 1.09 Factorisation by grouping in pairs

Factorise:

1 $2x + 8 + bx + 4b$	**2** $ay - 3a + by - 3b$	**3** $x^2 + 5x + 2x + 10$
4 $m^2 - 2m + 3m - 6$	**5** $ad - ac + bd - bc$	**6** $x^3 + x^2 + 3x + 3$
7 $5ab - 3b + 10a - 6$	**8** $2xy - x^2 + 2y^2 - xy$	**9** $ay + a + y + 1$
10 $x^2 + 5x - x - 5$	**11** $y + 3 + ay + 3a$	**12** $m - 2 + 4y - 2my$
13 $2x^2 + 10xy - 3xy - 15y^2$	**14** $a^2b + ab^3 - 4a - 4b^2$	**15** $5x - x^2 - 3x + 15$
16 $x^4 + 7x^3 - 4x - 28$	**17** $7x - 21 - xy + 3y$	**18** $4d + 12 - de - 3e$
19 $3x - 12 + xy - 4y$	**20** $2a + 6 - ab - 3b$	**21** $x^3 - 3x^2 + 6x - 18$
22 $pq - 3p + q^2 - 3q$	**23** $3x^3 - 6x^2 - 5x + 10$	**24** $4a - 12b + ac - 3bc$
25 $xy + 7x - 4y - 28$	**26** $x^4 - 4x^3 - 5x + 20$	**27** $4x^3 - 6x^2 + 8x - 12$
28 $3a^2 + 9a + 6ab + 18b$	**29** $5y - 15 + 10xy - 30x$	**30** $\pi r^2 + 2\pi r - 3r - 6$

1.10 Factorising trinomials

Factorising quadratic expressions

A **trinomial** is an expression with 3 terms; for example, $x^2 - 4x + 3$. Factorising a trinomial usually gives a **binomial product**.

We know tha: $(x + a)(x + b) = x^2 + bx + ax + ab$

$$= x^2 + (a + b)x + ab$$

Factorising trinomials

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$

Find values for a and b so that the sum $a + b$ is the middle term and the product ab is the last term.

EXAMPLE 12

Factorise:

a $m^2 - 5m + 6$

b $y^2 + y - 2$

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Solution

a $a + b = -5$ and $ab = 6$

To have $a + b = -5$, at least one number must be negative.

To have $ab = 6$, both numbers have the same sign. So both are negative.

For $ab = 6$: we could have $-6 \times (-1)$ or $-3 \times (-2)$

$-3 + (-2) = -5$ so $a = -3$ and $b = -2$.

So $m^2 - 5m + 6 = (m - 3)(m - 2)$

Check: $(m - 3)(m - 2) = m^2 - 2m - 3m + 6$

$= m^2 - 5m + 6$

b $a + b = 1$ and $ab = -2$

To have $ab = -2$, the numbers must have opposite signs. So one is positive and one is negative.

For $ab = -2$: we could have -2×1 or -1×2

$-1 + 2 = 1$ so $a = -1$ and $b = 2$.

So $y^2 + y - 2 = (y - 1)(y + 2)$

Check: $(y - 1)(y + 2) = y^2 + 2y - y - 2$

$= y^2 + y - 2$

Exercise 1.10 Factorising trinomials

Factorise:

1 $x^2 + 4x + 3$

2 $y^2 + 7y + 12$

3 $m^2 + 2m + 1$

4 $t^2 + 8t + 16$

5 $z^2 + z - 6$

6 $x^2 - 5x - 6$

7 $v^2 - 8v + 15$

8 $t^2 - 6t + 9$

9 $x^2 + 9x - 10$

10 $y^2 - 10y + 21$

11 $m^2 - 9m + 18$

12 $y^2 + 9y - 36$

13 $x^2 - 5x - 24$

14 $a^2 - 4a + 4$

15 $x^2 + 14x - 32$

16 $y^2 - 5y - 36$

17 $n^2 - 10n + 24$

18 $x^2 - 10x + 25$

19 $p^2 + 8p - 9$

20 $k^2 - 7k + 10$

21 $x^2 + x - 12$

22 $m^2 - 6m - 7$

23 $q^2 + 12q + 20$

24 $d^2 - 4d - 5$

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1.11 Further trinomials

When the coefficient of the first term is not 1, for example $5x^2 - 13x + 6$, we need to use a different method to factorise the trinomial.

The coefficient of the first term is the number in front of the x^2.

This method still involves finding 2 numbers that give a required sum and product but it also involves grouping in pairs.

Factorising quadratic expressions (Advanced)

WS

Excel worksheet: Factorising trinomials

Excel spreadsheet: Factorising trinomials

EXAMPLE 13

Factorise:

a $5x^2 - 13x + 6$

b $4y^2 + 4y - 3$

Solution

a First, multiply the coefficient of the first term by the last term: $5 \times 6 = 30$.

Now $a + b = -13$ and $ab = 30$.

Since the sum is negative and the product is positive, a and b must be both negative.

2 numbers with product 30 and sum −13 are −10 and −3.

Now write the trinomial with the middle term split into 2 terms $-10x$ and $-3x$, and then factorise by grouping in pairs.

$$\begin{aligned} 5x^2 - 13x + 6 &= 5x^2 - 10x - 3x + 6 \\ &= 5x(x - 2) - 3(x - 2) \end{aligned}$$

If you factorise correctly, you should always find a common factor remaining, such as $(x - 2)$ here.

$$= (x - 2)(5x - 3)$$

b First, multiply the coefficient of the first term by the last term: $4(-3) = -12$

Now $a + b = 4$ and $ab = -12$.

Since the product is negative, a and b have opposite signs (one positive and one negative).

2 numbers with product −12 and sum 4 are 6 and −2.

Now write the trinomial with the middle term split into 2 terms $6y$ and $-2y$, and then factorise by grouping in pairs.

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$$4y^2 + 4y - 3 = 4y^2 + 6y - 2y - 3$$
$$= 2y(2y + 3) - 1(2y + 3)$$
$$= (2y + 3)(2y - 1)$$

There are other ways of factorising these trinomials. Your teacher may show you some of these.

Exercise 1.11 Further trinomials

Factorise:

1 $2a^2 + 11a + 5$

2 $5y^2 + 7y + 2$

3 $3x^2 + 10x + 7$

4 $3x^2 + 8x + 4$

5 $2b^2 - 5b + 3$

6 $7x^2 - 9x + 2$

7 $3y^2 + 5y - 2$

8 $2x^2 + 11x + 12$

9 $5p^2 + 13p - 6$

10 $6x^2 + 13x + 5$

11 $2y^2 - 11y - 6$

12 $10x^2 + 3x - 1$

13 $8t^2 - 14t + 3$

14 $6x^2 - x - 12$

15 $6y^2 + 47y - 8$

16 $4n^2 - 11n + 6$

17 $8t^2 + 18t - 5$

18 $12q^2 + 23q + 10$

19 $4r^2 + 11r - 3$

20 $4x^2 - 4x - 15$

21 $6y^2 - 13y + 2$

22 $6p^2 - 5p - 6$

23 $8x^2 + 31x + 21$

24 $12b^2 - 43b + 36$

25 $6x^2 - 53x - 9$

26 $9x^2 + 30x + 25$

27 $16y^2 + 24y + 9$

28 $25k^2 - 20k + 4$

29 $36a^2 - 12a + 1$

30 $49m^2 + 84m + 36$

1.12 Perfect squares

You have looked at expanding $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$. These are called **perfect squares**.

When factorising, use these results the other way around.

EXAMPLE 14

Factorise:

a $x^2 - 8x + 16$

b $4a^2 + 20a + 25$

Solution

a $x^2 - 8x + 16 = x^2 - 2(4)x + 4^2$
$= (x - 4)^2$

b $4a^2 + 20a + 25 = (2a)^2 + 2(2a)(5) + 5^2$
$= (2a + 5)^2$

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Exercise 1.12 Perfect squares

Factorise:

1 $y^2 - 2y + 1$

2 $x^2 + 6x + 9$

3 $m^2 + 10m + 25$

4 $t^2 - 4t + 4$

5 $x^2 - 12x + 36$

6 $4x^2 + 12x + 9$

7 $16b^2 - 8b + 1$

8 $9a^2 + 12a + 4$

9 $25x^2 - 40x + 16$

10 $49y^2 + 14y + 1$

11 $9y^2 - 30y + 25$

12 $16k^2 - 24k + 9$

13 $25x^2 + 10x + 1$

14 $81a^2 - 36a + 4$

15 $49m^2 + 84m + 36$

16 $t^2 + t + \frac{1}{4}$

17 $x^2 - \frac{4x}{3} + \frac{4}{9}$

18 $9y^2 + \frac{6y}{5} + \frac{1}{25}$

19 $x^2 + 2 + \frac{1}{x^2}$

20 $25k^2 - 20 + \frac{4}{k^2}$

1.13 Difference of two squares

Difference of two squares

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLE 15

Factorise:

a $d^2 - 36$

b $1 - 9b^2$

c $(a + 3)^2 - (b - 1)^2$

Solution

a $d^2 - 36 = d^2 - 6^2$

$= (d + 6)(d - 6)$

b $1 - 9b^2 = 1^2 - (3b)^2$

$= (1 + 3b)(1 - 3b)$

c $(a + 3)^2 - (b - 1)^2 = [(a + 3) + (b - 1)][(a + 3) - (b - 1)]$

$= (a + 3 + b - 1)(a + 3 - b + 1)$

$= (a + b + 2)(a - b + 4)$

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Exercise 1.13 Difference of two squares

Factorise:

1 $a^2 - 4$
2 $x^2 - 9$
3 $y^2 - 1$
4 $x^2 - 25$
5 $4x^2 - 49$
6 $16y^2 - 9$
7 $1 - 4z^2$
8 $25t^2 - 1$
9 $9t^2 - 4$
10 $9 - 16x^2$
11 $x^2 - 4y^2$
12 $36x^2 - y^2$
13 $4a^2 - 9b^2$
14 $x^2 - 100y^2$
15 $4a^2 - 81b^2$
16 $(x + 2)^2 - y^2$
17 $(a - 1)^2 - (b - 2)^2$
18 $z^2 - (1 + w)^2$
19 $x^2 - \frac{1}{4}$
20 $\frac{y^2}{9} - 1$
21 $(x + 2)^2 - (2y + 1)^2$
22 $x^4 - 1$
23 $9x^6 - 4y^2$
24 $x^4 - 16y^4$

1.14 Mixed factorisation

WS Facoiing expressions

EXAMPLE 16

Factorise $5x^2 - 45$.

Solution

Using simple factors: $5x^2 - 45 = 5(x^2 - 9)$

The difference of 2 squares: $= 5(x + 3)(x - 3)$

Exercise 1.14 Mixed factorisation

Factorise:

1 $4a^3 - 36a$
2 $2x^2 - 18$
3 $3p^2 - 3p - 36$
4 $5y^2 - 5$
5 $5a^2 - 10a + 5$
6 $3z^3 + 27z^2 + 60z$
7 $9ab - 4a^3b^3$
8 $x^3 - x$
9 $6x^2 + 8x - 8$
10 $y^2(y + 5) - 16(y + 5)$
11 $x^4 + 8x^3 - x^2 - 8x$
12 $y^6 - 4$
13 $x^3 - 3x^2 - 10x$
14 $x^3 - 3x^2 - 9x + 27$
15 $4x^2y^3 - y$
16 $24 - 6b^2$
17 $18x^2 + 33x - 30$
18 $3x^2 - 6x + 3$

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19 $x^3+2x^2-25x-50$ **20** z^3+6z^2+9z **21** $3y^2+30y+75$

22 ab^2-9a **23** $4k^3+40k^2+100k$ **24** $3x^3+9x^2-3x-9$

25 $4a^3b+8a^2b^2-4ab^2-2a^2b$

1.15 Simplifying algebraic fractions

EXAMPLE 17

Simplify:

a $\dfrac{4x+2}{2}$ **b** $\dfrac{2x^2-3x-2}{x^2-4}$

Solution

a $\dfrac{4x+2}{2}=\dfrac{2(2x+1)}{2}$
$=2x+1$

b Factorise both top and bottom.

$\dfrac{2x^2-3x-2}{x^2-4}=\dfrac{(2x+1)(x-2)}{(x-2)(x+2)}$
$=\dfrac{2x+1}{x+2}$

Exercise 1.15 Simplifying algebraic fractions

Simplify:

1 $\dfrac{5a+10}{5}$ **2** $\dfrac{6t-3}{3}$ **3** $\dfrac{8y+2}{6}$

4 $\dfrac{8}{4d-2}$ **5** $\dfrac{x^2}{5x^2-2x}$ **6** $\dfrac{y-4}{y^2-8y+16}$

7 $\dfrac{2ab-4a^2}{a^2-3a}$ **8** $\dfrac{s^2+s-2}{s^2+5s+6}$ **9** $\dfrac{b^4-1}{b^2-1}$

10 $\dfrac{2p^2+7p-15}{6p-9}$ **11** $\dfrac{a^2-1}{a^2+2a-3}$ **12** $\dfrac{3(x-2)+y(x-2)}{x^2-4}$

13 $\dfrac{x^3+3x^2-9x-27}{x^2+6x+9}$ **14** $\dfrac{2p^2-3p-2}{2p^2+p}$ **15** $\dfrac{ay-ax+by-bx}{2ay-by-2ax+bx}$

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WS

Algebraic fractions

1.16 Operations with algebraic fractions

EXAMPLE 18

Simplify:

a $\dfrac{x-1}{5}-\dfrac{x+3}{4}$ **b** $\dfrac{2a^2b+10ab}{b^2-9}\div\dfrac{a^2-25}{4b+12}$ **c** $\dfrac{2}{x-5}+\dfrac{1}{x+2}$ **d** $\dfrac{2}{x+1}-\dfrac{1}{x^2-1}$

Solution

a
$$\begin{aligned}\frac{x-1}{5}-\frac{x+3}{4}&=\frac{4(x-1)-5(x+3)}{20}\\&=\frac{4x-4-5x-15}{20}\\&=\frac{-x-19}{20}\end{aligned}$$

b
$$\begin{aligned}\frac{2a^2b+10ab}{b^2-9}\div\frac{a^2-25}{4b+12}&=\frac{2a^2b+10ab}{b^2-9}\times\frac{4b+12}{a^2-25}\\&=\frac{2ab(a+5)}{(b+3)(b-3)}\times\frac{4(b+3)}{(a+5)(a-5)}\\&=\frac{8ab}{(a-5)(b-3)}\end{aligned}$$

c
$$\begin{aligned}\frac{2}{x-5}+\frac{1}{x+2}&=\frac{2(x+2)+1(x-5)}{(x-5)(x+2)}\\&=\frac{2x+4+x-5}{(x-5)(x+2)}\\&=\frac{3x-1}{(x-5)(x+2)}\end{aligned}$$

d
$$\begin{aligned}\frac{2}{x+1}-\frac{1}{x^2-1}&=\frac{2}{x+1}-\frac{1}{(x+1)(x-1)}\\&=\frac{2(x-1)}{(x+1)(x-1)}-\frac{1}{(x+1)(x-1)}\\&=\frac{2x-2}{(x+1)(x-1)}-\frac{1}{(x+1)(x-1)}\\&=\frac{2x-2-1}{(x+1)(x-1)}\\&=\frac{2x-3}{(x+1)(x-1)}\end{aligned}$$

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Exercise 1.16 Operations with algebraic fractions

1 Simplify:

a $\frac{x}{2}+\frac{3x}{4}$ **b** $\frac{y+1}{5}+\frac{2y}{3}$ **c** $\frac{a+2}{3}-\frac{a}{4}$

d $\frac{p-3}{6}+\frac{p+2}{2}$ **e** $\frac{x-5}{2}-\frac{x-1}{3}$

2 Simplify:

a $\frac{3x+6}{5}\times\frac{10}{x+2}$ **b** $\frac{a^2-4}{3}\times\frac{5b}{a+2}$

c $\frac{t^2+3t-10}{xy^2}\div\frac{5t-10}{2xy}$ **d** $\frac{2a-6}{2x+4}\times\frac{5x+10}{4}$

e $\frac{5x+10-xy-2y}{15}\div\frac{7x+14}{3}$ **f** $\frac{3}{b+2}\times\frac{b^2+2b}{6a-3}$

g $\frac{3ab^2}{5xy}\div\frac{12ab-6a}{x^2y+2xy^2}$ **h** $\frac{ax-ay+bx-by}{x^2-y^2}\times\frac{x^2y+xy^2}{ab^2+a^2b}$

i $\frac{x^2-6x+9}{x^2-25}\div\frac{x^2-5x+6}{x^2+4x-5}$ **j** $\frac{p^2-4}{q^2+2q+1}\times\frac{5q+5}{3p+6}$

3 Simplify:

a $\frac{2}{x}+\frac{3}{x}$ **b** $\frac{1}{x-1}-\frac{2}{x}$ **c** $1+\frac{3}{a+b}$

d $x-\frac{x^2}{x+2}$ **e** $p-q+\frac{1}{p+q}$ **f** $\frac{1}{x+1}+\frac{1}{x-3}$

g $\frac{2}{x^2-4}-\frac{3}{x+2}$ **h** $\frac{1}{a^2+2a+1}+\frac{1}{a+1}$

4 Simplify:

a $\frac{a^2-5a}{y^2-4y+4}\div\frac{3a-15}{y^2-4}\times\frac{y^2-y-2}{5ay}$ **b** $\frac{3}{x-3}+\frac{2x+8}{x^2-9}\times\frac{x^2+3x}{4x-16}$

c $\frac{5b}{2b+6}\div\frac{b^2}{b^2+b-6}-\frac{b}{b+1}$ **d** $\frac{x^2-8x+15}{5x^2+10x}\div\frac{x^2-9}{10x^2}\times\frac{x^2+5x+6}{2x-10}$

5 Simplify:

a $\frac{5}{x^2-4}-\frac{3}{x-2}-\frac{2}{x+2}$ **b** $\frac{2}{p^2+pq}+\frac{3}{pq-q^2}$ **c** $\frac{a}{a+b}-\frac{b}{a-b}+\frac{1}{a^2-b^2}$

1.17 Substitution

Algebra is used for writing general formulas or rules, and we substitute numbers into these formulas to solve a problem.

EXAMPLE 19

a $V = \pi r^2 h$ is the formula for finding the volume of a cylinder with radius r and height h. Find V (correct to 1 decimal place) when $r = 2.1$ and $h = 8.7$.

b If $F = \frac{9C}{5} + 32$ is the formula for converting degrees Celsius (°C) into degrees Fahrenheit (°F), find F when $C = 25$.

Solution

a When $r = 2.1$, $h = 8.7$,

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(2.1)^2(8.7) \\ &= 120.533\ldots \\ &\approx 120.5 \end{aligned}$$

b When $C = 25$,

$$\begin{aligned} F &= \frac{9C}{5} + 32 \\ &= \frac{9(25)}{5} + 32 \\ &= 77 \end{aligned}$$

This means that 25°C is the same as 77°F.

Exercise 1.17 Substitution

1 Given $a = 3.1$ and $b = -2.3$ find, correct to 1 decimal place:

a ab **b** $3b$ **c** $5a^2$ **d** ab^3

e $(a+b)^2$ **f** $\sqrt{a-b}$ **g** $-b^2$

2 For the formula $T = a + (n-1)d$, find T when $a = -4$, $n = 18$ and $d = 3$.

3 Given $y = mx + c$, the equation of a straight line, find y if $m = 3$, $x = -2$ and $c = -1$.

4 If $h = 100t - 5t^2$ is the height of a particle at time t, find h when $t = 5$.

5 Given vertical velocity $v = -gt$, find v when $g = 9.8$ and $t = 20$.

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6 If $y = 2^x + 3$ is the equation of a function, find y when $x = 1.3$, correct to 1 decimal place.

7 $S = 2\pi r(r + h)$ is the formula for the surface area of a cylinder. Find S when $r = 5$ and $h = 7$, correct to the nearest whole number.

8 $A = \pi r^2$ is the area of a circle with radius r. Find A when $r = 9.5$, correct to 3 significant figures.

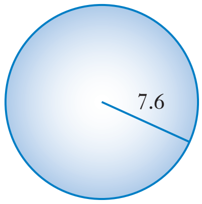
9 For the formula $u = ar^{n-1}$, find u if $a = 5$, $r = -2$ and $n = 4$.

10 Given $V = \frac{1}{3}lbh$ is the volume formula for a rectangular pyramid, find V if $l = 4.7$, $b = 5.1$ and $h = 6.5$.

11 The gradient of a straight line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$. Find m if $x_1 = 3$, $x_2 = -1$, $y_1 = -2$ and $y_2 = 5$.

12 If $A = \frac{1}{2}h(a + b)$ gives the area of a trapezium, find A when $h = 7$, $a = 2.5$ and $b = 3.9$.

13 $V = \frac{4}{3}\pi r^3$ is the volume formula for a sphere with radius r.
Find V to 1 decimal place for a sphere with radius $r = 7.6$.



14 The velocity of an object at time t is given by the formula $v = u + at$.
Find v when $u = \frac{1}{4}$, $a = \frac{3}{5}$ and $t = \frac{5}{6}$.

15 Given $S = \frac{a}{1-r}$, find S if $a = 5$ and $r = \frac{2}{3}$. S is the sum to infinity of a geometric series.

16 $c = \sqrt{a^2 + b^2}$, according to Pythagoras' theorem. Find the value of c if $a = 6$ and $b = 8$.

17 Given $y = \sqrt{16 - x^2}$ is the equation of a semicircle, find the exact value of y when $x = 2$.

18 Find the value of E in the energy equation $E = mc^2$ if $m = 8.3$ and $c = 1.7$.

19 $A = P\left(1 + \frac{r}{100}\right)^n$ is the formula for finding compound interest. Find A correct to 2 decimal places when $P = 200$, $r = 12$ and $n = 5$.

20 If $S = \frac{a(r^n - 1)}{r - 1}$ is the sum of a geometric series, find S if $a = 3$, $r = 2$ and $n = 5$.

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1.18 Simplifying surds

An **irrational number** is a number that cannot be written as a ratio or fraction.

Surds such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ are special types of irrational numbers.

If a question involving surds asks for an exact answer, then leave it as a surd.

Properties of surds

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

$$\left(\sqrt{x}\right)^2 = \sqrt{x^2} = x \text{ for } x \geq 0$$

EXAMPLE 20

a Express $\sqrt{45}$ in simplest surd form.

b Simplify $3\sqrt{40}$.

c Write $5\sqrt{2}$ as a single surd.

Solution

a
$$\begin{aligned}\sqrt{45} &= \sqrt{9 \times 5}\\ &= \sqrt{9} \times \sqrt{5}\\ &= 3 \times \sqrt{5}\\ &= 3\sqrt{5}\end{aligned}$$

b
$$\begin{aligned}3\sqrt{40} &= 3 \times \sqrt{4} \times \sqrt{10}\\ &= 3 \times 2 \times \sqrt{10}\\ &= 6\sqrt{10}\end{aligned}$$

c
$$\begin{aligned}5\sqrt{2} &= \sqrt{25} \times \sqrt{2}\\ &= \sqrt{50}\end{aligned}$$

ISBN 97807043299

Exercise 1.18 Simplifying surds

1 Express these surds in simplest surd form:

a	$\sqrt{12}$	**b**	$\sqrt{63}$	**c**	$\sqrt{24}$	**d**	$\sqrt{50}$	**e**	$\sqrt{72}$
f	$\sqrt{200}$	**g**	$\sqrt{48}$	**h**	$\sqrt{75}$	**i**	$\sqrt{32}$	**j**	$\sqrt{54}$
k	$\sqrt{112}$	**l**	$\sqrt{300}$	**m**	$\sqrt{128}$	**n**	$\sqrt{243}$	**o**	$\sqrt{245}$
p	$\sqrt{108}$	**q**	$\sqrt{99}$	**r**	$\sqrt{125}$				

2 Simplify:

a	$2\sqrt{27}$	**b**	$5\sqrt{80}$	**c**	$4\sqrt{98}$	**d**	$2\sqrt{28}$	**e**	$8\sqrt{20}$
f	$4\sqrt{56}$	**g**	$8\sqrt{405}$	**h**	$15\sqrt{8}$	**i**	$7\sqrt{40}$	**j**	$8\sqrt{45}$

3 Write as a single surd:

a	$3\sqrt{2}$	**b**	$2\sqrt{5}$	**c**	$4\sqrt{11}$	**d**	$8\sqrt{2}$	**e**	$5\sqrt{3}$
f	$4\sqrt{10}$	**g**	$3\sqrt{13}$	**h**	$7\sqrt{2}$	**i**	$11\sqrt{3}$	**j**	$12\sqrt{7}$

4 Evaluate x if:

a	$\sqrt{x}=3\sqrt{5}$	**b**	$2\sqrt{3}=\sqrt{x}$	**c**	$3\sqrt{7}=\sqrt{x}$	**d**	$5\sqrt{2}=\sqrt{x}$	**e**	$2\sqrt{11}=\sqrt{x}$
f	$\sqrt{x}=7\sqrt{3}$	**g**	$4\sqrt{19}=\sqrt{x}$	**h**	$\sqrt{x}=6\sqrt{23}$	**i**	$5\sqrt{31}=\sqrt{x}$	**j**	$\sqrt{x}=8\sqrt{15}$

1.19 Operations with surds

EXAMPLE 21

Simplify $\sqrt{3}-\sqrt{12}$.

Solution

First, change into like surds.

$$\begin{aligned}\sqrt{3}-\sqrt{12} &= \sqrt{3}-\sqrt{4}\times\sqrt{3}\\ &= \sqrt{3}-2\sqrt{3}\\ &= -\sqrt{3}\end{aligned}$$

Multiplication and division, as in algebra, are easier to do than adding and subtracting.

EXAMPLE 22

Simplify:

a $4\sqrt{2}\times 5\sqrt{18}$ **b** $\dfrac{2\sqrt{14}}{4\sqrt{2}}$ **c** $\left(\sqrt{\dfrac{10}{3}}\right)^2$

Solution

a $4\sqrt{2}\times 5\sqrt{18}=20\sqrt{36}$
$=20\times 6$
$=120$

b $\dfrac{2\sqrt{14}}{4\sqrt{2}}=\dfrac{2\times\sqrt{7}}{4}$
$=\dfrac{\sqrt{7}}{2}$

c $\left(\sqrt{\dfrac{10}{3}}\right)^2=\dfrac{10}{3}$
$=3\dfrac{1}{3}$

EXAMPLE 23

Expand and simplify:

a $3\sqrt{7}(2\sqrt{3}-3\sqrt{2})$ **b** $(\sqrt{2}+3\sqrt{5})(\sqrt{3}-\sqrt{2})$ **c** $(\sqrt{5}+2\sqrt{3})(\sqrt{5}-2\sqrt{3})$

Solution

a $3\sqrt{7}(2\sqrt{3}-3\sqrt{2})=3\sqrt{7}\times 2\sqrt{3}-3\sqrt{7}\times 3\sqrt{2}$
$=6\sqrt{21}-9\sqrt{14}$

b $(\sqrt{2}+3\sqrt{5})(\sqrt{3}-\sqrt{2})=\sqrt{2}\times\sqrt{3}-\sqrt{2}\times\sqrt{2}+3\sqrt{5}\times\sqrt{3}-3\sqrt{5}\times\sqrt{2}$
$=\sqrt{6}-2+3\sqrt{15}-3\sqrt{10}$

c Using the difference of 2 squares: $(\sqrt{5}+2\sqrt{3})(\sqrt{5}-2\sqrt{3})=\left(\sqrt{5}\right)^2-\left(2\sqrt{3}\right)^2$
$=5-4\times 3$
$=-7$

Exercise 1.19 Operations with surds

1 Simplify:

a $\sqrt{5}+2\sqrt{5}$ **b** $3\sqrt{2}-2\sqrt{2}$ **c** $\sqrt{3}+5\sqrt{3}$

d $7\sqrt{3}-4\sqrt{3}$ **e** $\sqrt{5}-4\sqrt{5}$ **f** $4\sqrt{6}-\sqrt{6}$

g $\sqrt{2}-8\sqrt{2}$ **h** $\sqrt{5}+4\sqrt{5}+3\sqrt{5}$ **i** $\sqrt{2}-2\sqrt{2}-3\sqrt{2}$

ISBN 97807043299

j $\sqrt{5}+\sqrt{45}$ **k** $\sqrt{8}-\sqrt{2}$ **l** $\sqrt{3}+\sqrt{48}$

m $\sqrt{12}-\sqrt{27}$ **n** $\sqrt{50}-\sqrt{32}$ **o** $\sqrt{28}+\sqrt{63}$

p $2\sqrt{8}-\sqrt{18}$ **q** $3\sqrt{54}+2\sqrt{24}$ **r** $\sqrt{90}-5\sqrt{40}-2\sqrt{10}$

s $4\sqrt{48}+3\sqrt{147}+5\sqrt{12}$ **t** $3\sqrt{2}+\sqrt{8}-\sqrt{12}$ **u** $\sqrt{63}-\sqrt{28}-\sqrt{50}$

v $\sqrt{12}-\sqrt{45}-\sqrt{48}-\sqrt{5}$

2 Simplify:

a $\sqrt{7}\times\sqrt{3}$ **b** $\sqrt{3}\times\sqrt{5}$ **c** $\sqrt{2}\times 3\sqrt{3}$

d $5\sqrt{7}\times 2\sqrt{2}$ **e** $-3\sqrt{3}\times 2\sqrt{2}$ **f** $5\sqrt{3}\times 2\sqrt{3}$

g $-4\sqrt{5}\times 3\sqrt{11}$ **h** $2\sqrt{7}\times\sqrt{7}$ **i** $2\sqrt{3}\times 5\sqrt{12}$

j $\sqrt{6}\times\sqrt{2}$ **k** $\left(\sqrt{2}\right)^2$ **l** $\left(2\sqrt{7}\right)^2$

m $\sqrt{3}\times\sqrt{5}\times\sqrt{2}$ **n** $2\sqrt{3}\times\sqrt{7}\times-\sqrt{5}$ **o** $\sqrt{2}\times\sqrt{6}\times 3\sqrt{3}$

3 Simplify:

a $\dfrac{4\sqrt{12}}{2\sqrt{2}}$ **b** $\dfrac{12\sqrt{18}}{3\sqrt{6}}$ **c** $\dfrac{5\sqrt{8}}{10\sqrt{2}}$ **d** $\dfrac{16\sqrt{2}}{2\sqrt{12}}$

e $\dfrac{10\sqrt{30}}{5\sqrt{10}}$ **f** $\dfrac{2\sqrt{2}}{6\sqrt{20}}$ **g** $\dfrac{4\sqrt{2}}{8\sqrt{10}}$ **h** $\dfrac{\sqrt{3}}{3\sqrt{15}}$

i $\dfrac{\sqrt{2}}{\sqrt{8}}$ **j** $\dfrac{3\sqrt{15}}{6\sqrt{10}}$ **k** $\dfrac{5\sqrt{12}}{5\sqrt{8}}$ **l** $\dfrac{15\sqrt{18}}{10\sqrt{10}}$

m $\dfrac{\sqrt{15}}{2\sqrt{6}}$ **n** $\left(\sqrt{\dfrac{2}{3}}\right)^2$ **o** $\left(\sqrt{\dfrac{5}{7}}\right)^2$

4 Expand and simplify:

a $\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)$ **b** $\sqrt{3}\left(2\sqrt{2}-\sqrt{5}\right)$ **c** $4\sqrt{3}\left(\sqrt{3}+2\sqrt{5}\right)$

d $\sqrt{7}\left(5\sqrt{2}-2\sqrt{3}\right)$ **e** $-\sqrt{3}\left(\sqrt{2}-4\sqrt{6}\right)$ **f** $\sqrt{3}\left(5\sqrt{11}+3\sqrt{7}\right)$

g $-3\sqrt{2}\left(\sqrt{2}+4\sqrt{3}\right)$ **h** $\sqrt{5}\left(\sqrt{5}-5\sqrt{3}\right)$ **i** $\sqrt{3}\left(\sqrt{12}+\sqrt{10}\right)$

j $2\sqrt{3}\left(\sqrt{18}+\sqrt{3}\right)$ **k** $-4\sqrt{2}\left(\sqrt{2}-3\sqrt{6}\right)$ **l** $-7\sqrt{5}\left(-3\sqrt{20}+2\sqrt{3}\right)$

m $10\sqrt{3}\left(\sqrt{2}-2\sqrt{12}\right)$ **n** $-\sqrt{2}\left(\sqrt{5}+2\right)$ **o** $2\sqrt{3}\left(2-\sqrt{12}\right)$

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5 Expand and simplify:

a $\left(\sqrt{2}+3\right)\left(\sqrt{5}+3\sqrt{3}\right)$ **b** $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{2}-\sqrt{7}\right)$ **c** $\left(\sqrt{2}+5\sqrt{3}\right)\left(2\sqrt{5}-3\sqrt{2}\right)$

d $\left(3\sqrt{10}-2\sqrt{5}\right)\left(4\sqrt{2}+6\sqrt{6}\right)$ **e** $\left(2\sqrt{5}-7\sqrt{2}\right)\left(\sqrt{5}-3\sqrt{2}\right)$ **f** $\left(\sqrt{5}+6\sqrt{2}\right)\left(3\sqrt{5}-\sqrt{3}\right)$

g $\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)$ **h** $\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)$ **i** $\left(\sqrt{6}+3\sqrt{2}\right)\left(\sqrt{6}-3\sqrt{2}\right)$

j $\left(3\sqrt{5}+\sqrt{2}\right)\left(3\sqrt{5}-\sqrt{2}\right)$ **k** $\left(\sqrt{8}-\sqrt{5}\right)\left(\sqrt{8}+\sqrt{5}\right)$ **l** $\left(\sqrt{2}+9\sqrt{3}\right)\left(\sqrt{2}-9\sqrt{3}\right)$

m $\left(2\sqrt{11}+5\sqrt{2}\right)\left(2\sqrt{11}-5\sqrt{2}\right)$ **n** $\left(\sqrt{5}+\sqrt{2}\right)^2$

o $\left(2\sqrt{2}-\sqrt{3}\right)^2$ **p** $\left(3\sqrt{2}+\sqrt{7}\right)^2$ **q** $\left(2\sqrt{3}+3\sqrt{5}\right)^2$

r $\left(\sqrt{7}-2\sqrt{5}\right)^2$ **s** $\left(2\sqrt{8}-3\sqrt{5}\right)^2$ **t** $\left(3\sqrt{5}+2\sqrt{2}\right)^2$

6 If $a=3\sqrt{2}$, simplify:

a a^2 **b** $2a^3$ **c** $(2a)^3$

d $(a+1)^2$ **e** $(a+3)(a-3)$

7 Evaluate a and b if:

a $\left(2\sqrt{5}+1\right)^2=a+\sqrt{b}$ **b** $\left(2\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}-3\sqrt{5}\right)=a+b\sqrt{10}$

8 Expand and simplify:

a $\left(\sqrt{a+3}-2\right)\left(\sqrt{a+3}+2\right)$ **b** $\left(\sqrt{p-1}-\sqrt{p}\right)^2$

9 Evaluate $\left(2\sqrt{7}-\sqrt{3}\right)\left(2\sqrt{7}+\sqrt{3}\right)$.

10 Simplify $\left(2\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-3\sqrt{y}\right)$.

11 If $\left(2\sqrt{3}-\sqrt{5}\right)^2=a-\sqrt{b}$, evaluate a and b.

12 Evaluate a and b if $\left(7\sqrt{2}-3\right)^2=a+b\sqrt{2}$.

1.20 Rationalising the denominator

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Rationalising the denominator of a fractional surd means writing it with a rational number (not a surd) in the denominator. For example, after rationalising the denominator $\frac{3}{\sqrt{5}}$ becomes $\frac{3\sqrt{5}}{5}$.

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To rationalise the denominator, multiply top and bottom by the same surd as in the denominator:

Rationalising the denominator

$$\frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$$

EXAMPLE 24

Rationalise the denominator of $\frac{2}{5\sqrt{3}}$.

Solution

$$\begin{aligned}\frac{2}{5\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} &= \frac{2\sqrt{3}}{5\sqrt{9}} \\ &= \frac{2\sqrt{3}}{5\times 3} \\ &= \frac{2\sqrt{3}}{15}\end{aligned}$$

When there is a binomial denominator, we use the difference of 2 squares to rationalise it.

Rationalising a binomial denominator

To rationalise the denominator of $\frac{b}{\sqrt{c}+\sqrt{d}}$, multiply by $\frac{\sqrt{c}-\sqrt{d}}{\sqrt{c}-\sqrt{d}}$.

To rationalise the denominator of $\frac{b}{\sqrt{c}-\sqrt{d}}$, multiply by $\frac{\sqrt{c}+\sqrt{d}}{\sqrt{c}+\sqrt{d}}$.

EXAMPLE 25

a Write with a rational denominator:

i $\frac{\sqrt{5}}{\sqrt{2}-3}$ **ii** $\frac{2\sqrt{3}+\sqrt{5}}{\sqrt{3}+4\sqrt{2}}$

b Evaluate a and b if $\frac{3\sqrt{3}}{\sqrt{3}-\sqrt{2}} = a + \sqrt{b}$.

c Evaluate $\frac{2}{\sqrt{3}+2} + \frac{\sqrt{5}}{\sqrt{3}-2}$ as a fraction with rational denominator.

Solution

a **i** $$\frac{\sqrt{5}}{\sqrt{2}-3}\times\frac{\sqrt{2}+3}{\sqrt{2}+3}=\frac{\sqrt{5}\left(\sqrt{2}+3\right)}{\left(\sqrt{2}\right)^2-3^2}$$

$$=\frac{\sqrt{10}+3\sqrt{5}}{2-9}$$

$$=-\frac{\sqrt{10}+3\sqrt{5}}{7}$$

ii $$\frac{2\sqrt{3}+\sqrt{5}}{\sqrt{3}+4\sqrt{2}}\times\frac{\sqrt{3}-4\sqrt{2}}{\sqrt{3}-4\sqrt{2}}=\frac{\left(2\sqrt{3}+\sqrt{5}\right)\left(\sqrt{3}-4\sqrt{2}\right)}{\left(\sqrt{3}\right)^2-\left(4\sqrt{2}\right)^2}$$

$$=\frac{2\times3-8\sqrt{6}+\sqrt{15}-4\sqrt{10}}{3-16\times2}$$

$$=\frac{6-8\sqrt{6}+\sqrt{15}-4\sqrt{10}}{-29}$$

$$=\frac{-6+8\sqrt{6}-\sqrt{15}+4\sqrt{10}}{29}$$

b $$\frac{3\sqrt{3}}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{3\sqrt{3}\left(\sqrt{3}+\sqrt{2}\right)}{\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)}$$

$$=\frac{3\sqrt{9}+3\sqrt{6}}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2}$$

$$=\frac{3\times3+3\sqrt{6}}{3-2}$$

$$=\frac{9+3\sqrt{6}}{1}$$

$$=9+3\sqrt{6}$$

$$=9+\sqrt{9}\times\sqrt{6}$$

$$=9+\sqrt{54}$$

So $a = 9$ and $b = 54$.

ISBN 97807043299

c $\dfrac{2}{\sqrt{3}+2}+\dfrac{\sqrt{5}}{\sqrt{3}-2}=\dfrac{2\left(\sqrt{3}-2\right)+\sqrt{5}\left(\sqrt{3}+2\right)}{\left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right)}$

$$=\frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{\left(\sqrt{3}\right)^2-2^2}$$

$$=\frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{3-4}$$

$$=\frac{2\sqrt{3}-4+\sqrt{15}+2\sqrt{5}}{-1}$$

$$=-2\sqrt{3}+4-\sqrt{15}-2\sqrt{5}$$

Exercise 1.20 Rationalising the denominator

1 Express with a rational denominator:

a $\dfrac{1}{\sqrt{7}}$ **b** $\dfrac{\sqrt{3}}{2\sqrt{2}}$ **c** $\dfrac{2\sqrt{3}}{\sqrt{5}}$ **d** $\dfrac{6\sqrt{7}}{5\sqrt{2}}$

e $\dfrac{1+\sqrt{2}}{\sqrt{3}}$ **f** $\dfrac{\sqrt{6}-5}{\sqrt{2}}$ **g** $\dfrac{\sqrt{5}+2\sqrt{2}}{\sqrt{5}}$ **h** $\dfrac{3\sqrt{2}-4}{2\sqrt{7}}$

i $\dfrac{8+3\sqrt{2}}{4\sqrt{5}}$ **j** $\dfrac{4\sqrt{3}-2\sqrt{2}}{7\sqrt{5}}$

2 Express with a rational denominator:

a $\dfrac{4}{\sqrt{3}+\sqrt{2}}$ **b** $\dfrac{\sqrt{3}}{\sqrt{2}-7}$ **c** $\dfrac{2\sqrt{3}}{\sqrt{5}+2\sqrt{6}}$

d $\dfrac{\sqrt{3}-4}{\sqrt{3}+4}$ **e** $\dfrac{\sqrt{2}+5}{\sqrt{3}-\sqrt{2}}$ **f** $\dfrac{3\sqrt{3}+\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}$

3 Express as a single fraction with a rational denominator:

a $\dfrac{1}{\sqrt{2}+1}+\dfrac{1}{\sqrt{2}-1}$ **b** $\dfrac{\sqrt{2}}{\sqrt{2}-\sqrt{3}}-\dfrac{3}{\sqrt{2}+\sqrt{3}}$

c $t+\dfrac{1}{t}$ where $t=\sqrt{3}-2$ **d** $z^2-\dfrac{1}{z^2}$ where $z=1+\sqrt{2}$

e $\dfrac{\sqrt{2}+3}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}$ **f** $\dfrac{\sqrt{3}}{\sqrt{2}+3}+\dfrac{\sqrt{2}}{\sqrt{3}}$

g $\dfrac{\sqrt{5}}{\sqrt{6}+2}-\dfrac{2}{5\sqrt{3}}$ **h** $\dfrac{\sqrt{2}+7}{4+\sqrt{3}}-\dfrac{\sqrt{2}}{4-\sqrt{3}}$

i $\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\dfrac{2+\sqrt{3}}{\sqrt{3}+1}$

4 Find a and b if:

a $\frac{3}{2\sqrt{5}}=\frac{\sqrt{a}}{b}$

b $\frac{\sqrt{3}}{4\sqrt{2}}=\frac{a\sqrt{6}}{b}$

c $\frac{2}{\sqrt{5}+1}=a+b\sqrt{5}$

d $\frac{2\sqrt{7}}{\sqrt{7}-4}=a+b\sqrt{7}$

e $\frac{\sqrt{2}+3}{\sqrt{2}-1}=a+\sqrt{b}$

5 Show that $\frac{\sqrt{2}-1}{\sqrt{2}+1}+\frac{4}{\sqrt{2}}$ is rational.

6 If $x=\sqrt{3}+2$, simplify:

a $x+\frac{1}{x}$

b $x^2+\frac{1}{x^2}$

c $\left(x+\frac{1}{x}\right)^2$

ISBN 97807043299

1. TEST YOURSELF

For Questions **1** to **8**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

1 Rationalise the denominator of $\dfrac{\sqrt{3}}{2\sqrt{7}}$ (there may be more than one answer).

A $\dfrac{\sqrt{21}}{28}$ **B** $\dfrac{2\sqrt{21}}{28}$ **C** $\dfrac{\sqrt{21}}{14}$ **D** $\dfrac{\sqrt{21}}{7}$

2 Simplify $\dfrac{x-3}{5}-\dfrac{x+1}{4}$.

A $\dfrac{-(x+7)}{20}$ **B** $\dfrac{x+7}{20}$ **C** $\dfrac{x+17}{20}$ **D** $\dfrac{-(x+17)}{20}$

3 Factorise $x^3 - 4x^2 - x + 4$ (there may be more than one answer).

A $(x^2 - 1)(x - 4)$ **B** $(x^2 + 1)(x - 4)$
C $x^2(x - 4)$ **D** $(x - 4)(x + 1)(x - 1)$

4 Simplify $3\sqrt{2}+2\sqrt{98}$.

A $5\sqrt{2}$ **B** $5\sqrt{10}$ **C** $17\sqrt{2}$ **D** $10\sqrt{2}$

5 Simplify $\dfrac{3}{x^2-4}+\dfrac{2}{x-2}-\dfrac{1}{x+2}$.

A $\dfrac{x+5}{(x+2)(x-2)}$ **B** $\dfrac{x+1}{(x+2)(x-2)}$ **C** $\dfrac{x+9}{(x+2)(x-2)}$ **D** $\dfrac{x-3}{(x+2)(x-2)}$

6 Simplify $5ab - 2a^2 - 7ab - 3a^2$.

A $2ab + a^2$ **B** $-2ab - 5a^2$ **C** $-13a^3b$ **D** $-2ab + 5a^2$

7 Simplify $\sqrt{\dfrac{80}{27}}$.

A $\dfrac{4\sqrt{5}}{3\sqrt{3}}$ **B** $\dfrac{4\sqrt{5}}{9\sqrt{3}}$ **C** $\dfrac{8\sqrt{5}}{9\sqrt{3}}$ **D** $\dfrac{8\sqrt{5}}{3\sqrt{3}}$

8 Expand and simplify $(3x - 2y)^2$.

A $3x^2 - 12xy - 2y^2$ **B** $9x^2 - 12xy - 4y^2$
C $3x^2 - 6xy + 2y^2$ **D** $9x^2 - 12xy + 4y^2$

9 Evaluate as a fraction:

a 7^{-2} **b** 5^{-1} **c** $9^{-\frac{}{2}}$

10 Simplify:

a $x^5 \times x^7 \div x^3$ **b** $(5y^3)^2$ **c** $\dfrac{(a^5)^4 b^7}{a^9 b}$ **d** $\left(\dfrac{2x^6}{3}\right)^3$ **e** $\left(\dfrac{ab^4}{a^5 b^6}\right)^0$

11 Evaluate:

a $36^{\frac{}{2}}$ **b** 4^{-3} as fraction **c** $8^{\frac{2}{3}}$

d $49^{-\frac{}{2}}$ as a fraction **e** $16^{\frac{}{4}}$ **f** $(-3)^0$

12 Simplify:

a $a^{14} \div a^9$ **b** $(x^5 y^3)^6$ **c** $p^6 \times p^5 \div p^2$

d $(2b^9)^4$ **e** $\dfrac{(2x^7)^3 y^2}{x^{10} y}$

13 Write in index form:

a $\sqrt{n}$ **b** $\dfrac{1}{x^5}$ **c** $\dfrac{1}{x+y}$ **d** $\sqrt[4]{x+1}$ **e** $\sqrt[7]{a+b}$

f $\dfrac{2}{x}$ **g** $\dfrac{1}{2x^3}$ **h** $\sqrt[3]{x^4}$ **i** $\sqrt[7]{(5x+3)^9}$ **j** $\dfrac{1}{\sqrt[4]{m^3}}$

14 Write without fractional or negative indices:

a a^{-5} **b** $n^{\frac{}{4}}$ **c** $(x+1)^{\frac{}{2}}$ **d** $(x-y)^{-1}$ **e** $(4t-7)^{-4}$

f $(a+b)^{\frac{}{5}}$ **g** $x^{-\frac{}{3}}$ **h** $b^{\frac{3}{4}}$ **i** $(2x+3)^{\frac{4}{3}}$ **j** $x^{-\frac{3}{2}}$

15 Evaluate $a^2 b^4$ when $a = \dfrac{9}{25}$ and $b = 1\dfrac{2}{3}$.

16 If $a = \left(\dfrac{1}{3}\right)^4$ and $b = \dfrac{3}{4}$, evaluate ab^3 as a fraction.

17 Write in index form:

a $\sqrt{x}$ **b** $\dfrac{1}{y}$ **c** $\sqrt[6]{x+3}$ **d** $\dfrac{1}{(2x-3)^{11}}$ **e** $\sqrt[3]{y^7}$

18 Write without the negative index:

a x^{-3} **b** $(2a+5)^{-1}$ **c** $\left(\dfrac{a}{b}\right)^{-5}$

19 Simplify:

a $5y - 7y$ **b** $\dfrac{3a+12}{3}$ **c** $-2k^3 \times 3k^2$ **d** $\dfrac{x}{3} + \dfrac{y}{5}$

e $4a - 3b - a - 5b$ **f** $\sqrt{8} + \sqrt{32}$ **g** $3\sqrt{5} - \sqrt{20} + \sqrt{45}$

ISBN 97807043299

20 Factorise:

a $x^2 - 36$ **b** $a^2 + 2a - 3$ **c** $4ab^2 - 8ab$

d $5y - 15 + xy - 3x$ **e** $4n - 2p + 6$

21 Expand and simplify:

a $b + 3(b - 2)$ **b** $(2x - 1)(x + 3)$ **c** $5(m + 3) - (m - 2)$

d $(4x - 3)^2$ **e** $(p - 5)(p + 5)$ **f** $7 - 2(a + 4) - 5a$

g $\sqrt{3}\left(2\sqrt{2}-5\right)$ **h** $\left(3+\sqrt{7}\right)\left(\sqrt{3}-2\right)$

22 Simplify:

a $\dfrac{4a-12}{5b^3}\times\dfrac{10b}{a^2-9}$ **b** $\dfrac{5m+10}{m^2-m-2}\div\dfrac{m^2-4}{3m+3}$

23 The volume of a cube is $V = s^3$. Evaluate V when $s = 5.4$.

24 **a** Expand and simplify $\left(2\sqrt{5}+\sqrt{3}\right)\left(2\sqrt{5}-\sqrt{3}\right)$.

b Rationalise the denominator of $\dfrac{3\sqrt{3}}{2\sqrt{5}+\sqrt{3}}$.

25 Simplify $\dfrac{3}{x-2}+\dfrac{1}{x+3}-\dfrac{2}{x^2+x-6}$.

26 If $a = 4$, $b = -3$ and $c = -2$, find the value of:

a ab^2 **b** $a - bc$ **c** $\sqrt{a}$ **d** $(\text{bc})^3$ **e** $c(2a + 3b)$

27 Simplify:

a $\dfrac{3\sqrt{12}}{6\sqrt{15}}$ **b** $\dfrac{4\sqrt{32}}{2\sqrt{2}}$

28 The formula for the distance an object falls is given by $d = 5t^2$. Find d when $t = 1.5$.

29 Rationalise the denominator of:

a $\dfrac{2}{5\sqrt{3}}$ **b** $\dfrac{1+\sqrt{3}}{\sqrt{2}}$

30 Expand and simplify:

a $\left(3\sqrt{2}-4\right)\left(\sqrt{3}-\sqrt{2}\right)$ **b** $\left(\sqrt{7}+2\right)^2$

31 Factorise fully:

a $3x^2 - 27$ **b** $6x^2 - 12x - 18$ **c** $5y^2 - 30y + 45$

32 Simplify:

a $\dfrac{3x^4y}{9xy^5}$ **b** $\dfrac{5}{15x-5}$

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33 Simplify:

a $\left(3\sqrt{11}\right)^2$ **b** $\left(2\sqrt{3}\right)^3$

34 Expand and simplify:

a $(a+b)(a-b)$ **b** $(a+b)^2$

35 Factorise:

a $a^2-2ab+b^2$ **b** a^2-b^2

36 If $x=\sqrt{3}+1$, simplify $x+\dfrac{1}{x}$ and give your answer with a rational denominator.

37 Simplify:

a $\dfrac{4}{a}+\dfrac{3}{b}$ **b** $\dfrac{x-3}{2}-\dfrac{x-2}{5}$

38 Simplify $\dfrac{3}{\sqrt{5}+2}-\dfrac{\sqrt{2}}{2\sqrt{2}-1}$, writing your answer with a rational denominator.

39 Simplify:

a $3\sqrt{8}$ **b** $-2\sqrt{2}\times 4\sqrt{3}$ **c** $\sqrt{108}-\sqrt{48}$ **d** $\dfrac{8\sqrt{6}}{2\sqrt{18}}$

e $5a\times -3b\times -2a$ **f** $\dfrac{2m^3n}{6m^2n^5}$ **g** $3x-2y-x-y$

40 Expand and simplify:

a $2\sqrt{2}\left(\sqrt{3}+\sqrt{2}\right)$ **b** $\left(5\sqrt{7}-3\sqrt{5}\right)\left(2\sqrt{2}-\sqrt{3}\right)$ **c** $\left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right)$

d $\left(4\sqrt{3}-\sqrt{5}\right)\left(4\sqrt{3}+\sqrt{5}\right)$ **e** $\left(3\sqrt{7}-\sqrt{2}\right)^2$

41 Rationalise the denominator of:

a $\dfrac{3}{\sqrt{7}}$ **b** $\dfrac{\sqrt{2}}{5\sqrt{3}}$ **c** $\dfrac{2}{\sqrt{5}-1}$ **d** $\dfrac{2\sqrt{2}}{3\sqrt{2}+\sqrt{3}}$ **e** $\dfrac{\sqrt{5}+\sqrt{2}}{4\sqrt{5}-3\sqrt{3}}$

42 Simplify:

a $\dfrac{3x}{5}-\dfrac{x-2}{2}$ **b** $\dfrac{a+2}{7}+\dfrac{2a-3}{3}$ **c** $\dfrac{1}{x^2-1}-\dfrac{2}{x+1}$

d $\dfrac{4}{k^2+2k-3}+\dfrac{1}{k+3}$ **e** $\dfrac{\sqrt{3}}{\sqrt{2}+\sqrt{5}}-\dfrac{5}{\sqrt{3}-\sqrt{2}}$

43 Evaluate n if:

a $\sqrt{108}-\sqrt{12}=\sqrt{n}$ **b** $\sqrt{112}+\sqrt{7}=\sqrt{n}$ **c** $2\sqrt{8}+\sqrt{200}=\sqrt{n}$

d $4\sqrt{147}+3\sqrt{75}=\sqrt{n}$ **e** $2\sqrt{245}+\dfrac{\sqrt{180}}{2}=\sqrt{n}$

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1. CHALLENGE EXERCISE

1 Write $64^{-\frac{2}{3}}$ as a rational number.

2 Show that $2(2^k - 1) + 2^{k+1} = 2(2^{k+1} - 1)$.

3 Find the value of $\frac{a}{b^3c^2}$ in index form if $a = \left(\frac{2}{5}\right)^4$, $b = \left(-\frac{1}{3}\right)^3$ and $c = \left(\frac{3}{5}\right)^2$.

4 Expand and simplify:

a $4ab(a - 2b) - 2a^2(b - 3a)$ **b** $(y^2 - 2)(y^2 + 2)$ **c** $(2x - 5)^3$

5 Find the value of $x + y$ with rational denominator if $x = \sqrt{3} + 1$ and $y = \frac{1}{2\sqrt{5} - 3}$.

6 Simplify $\frac{2\sqrt{3}}{7\sqrt{6} - \sqrt{54}}$.

7 Factorise:

a $(x + 4)^2 + 5(x + 4)$ **b** $x^4 - x^2y - 6y^2$ **c** $a^2b - 2a^2 - 4b + 8$

8 Simplify $\frac{2xy + 2x - 6 - 6y}{4x^2 - 16x + 12}$.

9 Simplify $\frac{(a+1)^3}{a^2 - 1}$.

10 Factorise $\frac{4}{x^2} - \frac{a^2}{b^2}$.

11 **a** Expand $(2x - 1)^3$.

b Hence, or otherwise, simplify $\frac{6x^2 + 5x - 4}{8x^3 - 12x^2 + 6x - 1}$.

12 If $V = \pi r^2 h$ is the volume of a cylinder, find the exact value of r when $V = 9$ and $h = 16$.

13 If $s = u + \frac{1}{2}at^2$, find the exact value of s when $u = 2$, $a = \sqrt{3}$ and $t = 2\sqrt{3}$.

14 Expand and simplify, and write in index form:

a $\left(\sqrt{x} + x\right)^2$ **b** $\left(\sqrt[3]{a} + \sqrt[3]{b}\right)\left(\sqrt[3]{a} - \sqrt[3]{b}\right)$

c $\left(p + \frac{1}{\sqrt{p}}\right)^2$ **d** $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

15 Find the value of $\frac{a^3b^2}{c^2}$ if $a = \left(\frac{3}{4}\right)^2$, $b = \left(\frac{2}{3}\right)^3$ and $c = \left(\frac{1}{2}\right)^4$.

FUNCTIONS

2.

EQUATIONS AND INEQUALITIES

Equations are found in most branches of mathematics. They are also important in many other fields, such as science, economics, statistics and engineering. In this chapter you will revise basic equations and inequalities, including those involving absolute values, exponential equations, quadratic equations and simultaneous equations.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- solve equations and inequalities
- understand and use absolute values in equations
- solve simple exponential equations
- solve quadratic equations using 3 different methods
- understand how to substitute into and rearrange formulas
- solve linear and non-linear simultaneous equations
- EXT1 solve quadratic and absolute value inequalities
- EXT1 solve inequalities involving algebraic fractions, including those with an unknown in the denominator

TERMINOLOGY

absolute value $|x|$ is the absolute value of x, its size without sign or direction. Also the distance of x from 0 on the number line in either direction

equation A mathematical statement that has a pronumeral or unknown number and an equal sign. An equation can be solved to find the value of the unknown number, for example, $3x + 1 = 7$

exponential equation An equation where the unknown pronumeral is the power or index, for example, $2^x = 8$

inequality A mathematical statement involving an inequality sign with an unknown pronumeral, for example, $x - 7 \le 12$

quadratic equation An equation involving x^2 in which the highest power of x is 2

quadratic inequality An inequality involving x^2 in which the highest power of x is 2

simultaneous equations 2 or more equations that can be solved together to produce a solution that makes each equation true at the same time

PROBLEM

The age of Diophantus at his death can be calculated from his epitaph:

Diophantus passed one-sixth of his life in childhood one-twelft h in youth and one-seventh more as a bachelor; five years after his marriage a son was born who died four years before his father at half his father's final age. How old was Diophantus?

2.01 Equations

WS
Equaion

EXAMPLE 1

Solve each equation.

a $4y - 3 = 8y + 21$

b $2(3x + 7) = 6 - (x - 1)$

Solution

a

$$\begin{aligned} 4y - 3 &= 8y + 21 \\ 4y - 4y - 3 &= 8y - 4y + 21 \\ -3 &= 4y + 21 \\ -3 - 21 &= 4y + 21 - 21 \\ -24 &= 4y \\ \frac{-24}{4} &= \frac{4y}{4} \\ -6 &= y \\ y &= -6 \end{aligned}$$

b

$$\begin{aligned} 2(3x + 7) &= 6 - (x - 1) \\ 6x + 14 &= 6 - x + 1 \\ &= 7 - x \\ 6x + x + 14 &= 7 - x + x \\ 7x + 14 &= 7 \\ 7x + 14 - 14 &= 7 - 14 \\ 7x &= -7 \\ \frac{7x}{7} &= \frac{-7}{7} \\ x &= -1 \end{aligned}$$

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When an equation involves fractions, multiply both sides of the equation by the common denominator of the fractions.

EXAMPLE 2

Solve:

a $\frac{m}{3}-4=\frac{1}{2}$

b $\frac{x+1}{3}+\frac{x}{4}=5$

Solution

a

$$\begin{aligned}\frac{m}{3}-4&=\frac{1}{2}\\ 6\left(\frac{m}{3}\right)-6(4)&=6\left(\frac{1}{2}\right)\\ 2m-24&=3\\ 2m-24+24&=3+24\\ 2m&=27\\ \frac{2m}{2}&=\frac{27}{2}\\ m&=\frac{27}{2}\\ &=13\tfrac{1}{2}\end{aligned}$$

b

$$\begin{aligned}\frac{x+1}{3}+\frac{x}{4}&=5\\ 12\left(\frac{x+1}{3}\right)+12\left(\frac{x}{4}\right)&=12(5)\\ 4(x+1)+3x&=60\\ 4x+4+3x&=60\\ 7x+4&=60\\ 7x+4-4&=60-4\\ 7x&=56\\ \frac{7x}{7}&=\frac{56}{7}\\ x&=8\end{aligned}$$

DID YOU KNOW?

History of algebra

Algebra was known in ancient civilisations. Many equations were known in Babylon, although general solutions were difficult because symbols were not used in those times.

Diophantus, around 250 CE, first used algebraic notation and symbols (e.g. the minus sign). He wrote a treatise on algebra in his *Arithmetica*, comprising 13 books. Only six of these books survived. About 400 CE, Hypatia of Alexandria wrote a commentary on them.

Hypatia was the first female mathematician on record, and was a philosopher and teacher. She was the daughter of Theon, who was also a mathematician and who ensured that she had the best education.

In 1799 **Carl Friedrich Gauss** proved the Fundamental Theorem of Algebra: that every algebraic equation involving a power of x has at least one solution, which may be a real number or a non-real number.

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Exercise 2.01 Equations

Solve each equation.

1 $t + 4 = -1$

2 $z + 1.7 = -3.9$

3 $y - 3 = -2$

4 $w - 2.6 = 4.1$

5 $5 = x - 7$

6 $1.5x = 6$

7 $5y = \frac{1}{3}$

8 $\frac{b}{7} = 5$

9 $-2 = \frac{n}{8}$

10 $\frac{r}{6} = \frac{2}{3}$

11 $2y + 1 = 19$

12 $33 = 4k + 9$

13 $7d - 2 = 12$

14 $-2 = 5x - 27$

15 $\frac{y}{3} + 4 = 9$

16 $\frac{x}{2} - 3 = 7$

17 $\frac{m}{5} + 7 = 11$

18 $3x + 5 = 17$

19 $4a + 7 = -21$

20 $7y - 1 = 20$

21 $3(x + 2) = 15$

22 $-2(3a + 1) = 8$

23 $7t + 4 = 3t - 12$

24 $x - 3 = 6x - 9$

25 $2(a - 2) = 4 - 3a$

26 $5b + 2 = -3(b - 1)$

27 $3(t + 7) = 2(2t - 9)$

28 $2 + 5(p - 1) = 5p - (p - 2)$

29 $3.7x + 1.2 = 54x - 6.3$

30 $\frac{b}{5} = \frac{2}{3}$

31 $\frac{5x}{4} = \frac{11}{7}$

32 $\frac{x}{3} - 4 = 8$

33 $\frac{5 + x}{7} = \frac{2}{7}$

34 $\frac{y}{2} = -\frac{3}{5}$

35 $\frac{x}{9} - \frac{2}{3} = 7$

36 $\frac{w - 3}{2} = 5$

37 $\frac{2t}{5} - \frac{t}{3} = 2$

38 $\frac{x}{4} + \frac{1}{2} = 4$

39 $\frac{x}{5} - \frac{x}{2} = \frac{3}{10}$

40 $\frac{x + 4}{3} + \frac{x}{2} = 1$

41 $\frac{p - 3}{2} + \frac{2p}{3} = 2$

42 $\frac{t + 3}{7} + \frac{t - 1}{3} = 4$

43 $\frac{x + 5}{9} - \frac{x + 2}{5} = 1$

44 $\frac{q - 1}{3} - \frac{q - 2}{4} = 2$

45 $\frac{x + 3}{5} + 2 = \frac{x + 7}{2}$

COULD THIS BE TRUE?

Half full = half empty

$\therefore$ full = empty

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2.02 Inequalities

$>$ means greater than. $\geq$ means greater than or equal to.

$<$ means less than. $\leq$ means less than or equal to.

Solving inequalities

The **inequality sign reverses** when:

- multiplying by a negative
- dividing by a negative
- taking the reciprocal of both sides.

On the number plane, we graph inequalities using arrows and circles (open for greater than and less than and closed in for greater than or equal to and less than or equal to).

Inequalities on a number line

WS Inequaliies on a number line

EXAMPLE 3

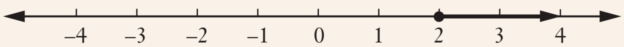
Solve each inequality and show its solution on a number line.

a $5x + 7 \geq 17$ **b** $3t - 2 > 5t + 4$ **c** $1 < 2z + 7 \leq 11$

Solution

a

$$5x + 7 \geq 17$$
$$5x + 7 - 7 \geq 17 - 7$$
$$5x \geq 10$$
$$\frac{5x}{5} \geq \frac{10}{5}$$
$$x \geq 2$$



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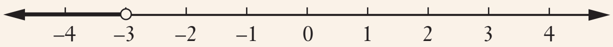
b

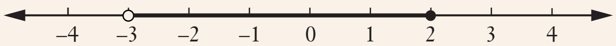
$$3t - 2 > 5t + 4$$
$$3t - 5t - 2 > 5t - 5t + 4$$
$$-2t - 2 > 4$$
$$-2t - 2 + 2 > 4 + 2$$
$$-2t > 6$$
$$\frac{-2t}{-2} < \frac{6}{-2}$$
$$t < -3$$

Remember to change the inequality sign when dividing by -2.



c

$$1 < 2z + 7 \le 11$$
$$1 - 7 < 2z + 7 - 7 \le 11 - 7$$
$$-6 < 2z \le 4$$
$$-3 < z \le 2$$



Exercise 2.02 Inequalities

1 Solve each equation and plot the solution on a number line.

a $x + 4 > 7$ **b** $y - 3 \le 1$

2 Solve:

a $5t > 35$ **b** $3x - 7 \ge 2$ **c** $2(p + 5) > 8$

d $4 - (x - 1) \le 7$ **e** $3y + 5 > 2y - 4$ **f** $2a - 6 \le 5a - 3$

g $3 + 4y \ge -2(1 - y)$ **h** $2x + 9 < 1 - 4(x + 1)$ **i** $\frac{a}{2} \le -3$

j $8 > \frac{2y}{3}$ **k** $\frac{b}{2} + 5 < -4$ **l** $\frac{x}{3} - 4 > 6$

m EXT1 $\frac{1}{4} + \frac{x}{5} \le 1$ **n** EXT1 $\frac{m}{4} - 3 > \frac{2}{3}$ **o** EXT1 $\frac{2b}{5} - \frac{1}{2} \ge 6$

p EXT1 $\frac{r-3}{2} \le -6$ **q** EXT1 $\frac{z+1}{9} + 2 > 3$ **r** EXT1 $\frac{w}{6} + \frac{2w+5}{3} < 4$

s EXT1 $\frac{x+1}{2} - \frac{x-2}{3} \ge 7$ **t** EXT1 $\frac{t+2}{7} - \frac{t+3}{2} \le 2$ **u** EXT1 $\frac{q-2}{3} < 2 + \frac{3q}{4}$

v EXT1 $\frac{2x}{3} - \frac{x-1}{2} > \frac{2}{9}$ **w** EXT1 $\frac{2b-5}{8} + 3 \le \frac{b+6}{12}$

3 Solve and plot each solution on a number line.

a $3 < x + 2 < 9$ **b** $-4 \le 2p < 10$ **c** $2 < 3x - 1 < 11$

d $-6 \le 5y + 9 \le 34$ **e** $-2 < 3(2y - 1) < 7$

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2.03 Absolute value

The **absolute value** of a number is the size of the number without the sign or direction. So absolute value is always positive or zero.

We write the absolute value of x as $|x|$.

For example, $|4| = 4$ and $|-3| = 3$.

We can also define $|x|$ as the distance of x from 0 on the number line.

If x is positive, then its absolute value is itself.

If $x = 0$, then its absolute value is 0.

If x is negative, then its absolute value is its opposite, $-x$. Because x is already negative, the effect of the negative sign in front of it is to make it positive; for example, $-(-5) = 5$.

Absolute value

$$|x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

$|4| = 4$ since $4 \geq 0$.

$|-3| = -(-3)$ since $-3 < 0$

$= 3$

Properties of absolute value

Property	Example
$\|ab\| = \|a\| \times \|b\|$	$\|2 \times -3\| = \|2\| \times \|-3\| = 6$
$\|a\|^2 = a^2$	$\|-3\|^2 = (-3)^2 = 9$
$\sqrt{a^2} = \|a\|$	$\sqrt{(-5)^2} = \|-5\| = 5$
$\|-a\| = \|a\|$	$\|-7\| = \|7\| = 7$
$\|a - b\| = \|b - a\|$	$\|2 - 3\| = \|3 - 2\| = 1$
$\|a + b\| \leq \|a\| + \|b\|$	$\|2 + 3\| = \|2\| + \|3\|$ but $\|-3 + 4\| < \|-3\| + \|4\|$

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EXAMPLE 4

a Evaluate $|2| - |-1| + |-3|^2$.

b Show that $|a + b| \le |a| + |b|$ when $a = -2$ and $b = 3$.

c Write expressions for $|2x - 4|$ without the absolute value signs.

Solution

a
$$\begin{aligned} |2| - |-1| + |-3|^2 &= 2 - 1 + 3^2 \\ &= 10 \end{aligned}$$

b
$$\begin{aligned} \text{LHS} &= |a + b| \\ &= |-2 + 3| \\ &= |1| \\ &= 1 \end{aligned} \qquad \begin{aligned} \text{RHS} &= |a| + |b| \\ &= |-2| + |3| \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

Note: LHS means left-hand side and RHS means right-hand side.

Since $1 < 5$,

$|a + b| \le |a| + |b|$

c $|2x - 4| = 2x - 4$ when $2x - 4 \ge 0$ i.e. when $2x \ge 4$

i.e. when $x \ge 2$

$|2x - 4| = -(2x - 4)$ when $2x - 4 < 0$ i.e. when $2x < 4$

$= -2x + 4$ i.e. when $x < 2$

CLASS DISCUSSION

ABSOLUTE VALUE

Are these statements true? If so, are there some values for which the expression is undefined (values of x or y that the expression cannot have)?

1 $\dfrac{x}{|x|} = 1$

2 $|2x| = 2x$

3 $|2x| = 2|x|$

4 $|x| + |y| = |x + y|$

5 $|x|^2 = x^2$

6 $|x|^3 = x^3$

7 $|x + 1| = |x| + 1$

8 $\dfrac{|3x - 2|}{3x - 2} = 1$

9 $\dfrac{|x|}{x^2} = 1$

10 $|x| \ge 0$

Discuss absolute value and its definition in relation to these statements.

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Exercise 2.03 Absolute value

1 Evaluate:

a $|7|$ **b** $|-5|$ **c** $|-6|$ **d** $|0|$

e $|2|$ **f** $|-11|$ **g** $|-2||3|$ **h** $3|-8|$

i $|-5|^2$ **j** $|-5|^3$

2 Evaluate:

a $|3| + |-2|$ **b** $|-3| - |4|$ **c** $|-5 + 3|$

d $|2 \times -7|$ **e** $|-3| + |-1|$ **f** $5 - |-2| \times |6|^2$

g $|-2 + 5 \times -1|$ **h** $3|-4|$ **i** $2|-3| - 3|-4|$

j $|5 - 7| + 4|-2|$

3 Evaluate $|a - b|$ if:

a $a = 5$ and $b = 2$ **b** $a = -1$ and $b = 2$ **c** $a = -2$ and $b = -3$

d $a = 4$ and $b = 7$ **e** $a = -1$ and $b = -2$

4 Write an expression for:

a $|a|$ when $a > 0$ **b** $|a|$ when $a < 0$ **c** $|a|$ when $a = 0$

d $|3a|$ when $a > 0$ **e** $|3a|$ when $a < 0$ **f** $|3a|$ when $a = 0$

g $|a + 1|$ when $a > -1$ **h** $|a + 1|$ when $a < -1$ **i** $|x - 2|$ when $x > 2$

5 Show that $|a + b| \le |a| + |b|$ when:

a $a = 2$ and $b = 4$ **b** $a = -1$ and $b = -2$ **c** $a = -2$ and $b = 3$

d $a = -4$ and $b = 5$ **e** $a = -7$ and $b = -3$

6 Show that $\sqrt{x^2} = |x|$ when:

a $x = 5$ **b** $x = -2$ **c** $x = -3$

d $x = 4$ **e** $x = -9$

7 Use the definition of absolute value to write each expression without the absolute value signs.

a $|x + 5|$ **b** $|b - 3|$ **c** $|a + 4|$

d $|2y - 6|$ **e** $|3x + 9|$ **f** $|4 - x|$

g $|2k + 1|$ **h** $|5x - 2|$ **i** $|a + b|$

8 Find values of x for which $|x| = 3$.

9 Simplify $\frac{|n|}{n}$ where $n \ne 0$.

10 Simplify $\frac{x-2}{|x-2|}$ and state which value x cannot be.

2.04 Equations involving absolute values

WS
Abolue value equations and inequalities

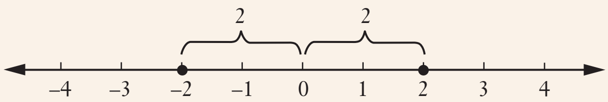
On a number line, $|x|$ means the distance of x from 0 in either direction.

EXAMPLE 5

Solve $|x| = 2$.

Solution

$|x| = 2$ means the distance of x from zero is 2 (in either direction).



$x = \pm 2$

CLASS DISCUSSION

ABSOLUTE VALUE AND THE NUMBER LINE

What does $|a - b|$ mean as a distance along the number line?

Select different values of a and b to help with this discussion.

EXAMPLE 6

Solve:

a $|x + 4| = 7$

b $|2x - 3| = 9$

Solution

a This means that the distance from $x + 4$ to 0 is 7 in either direction.

So $x + 4 = \pm 7$

$$x + 4 = 7 \qquad \text{or} \qquad x + 4 = -7$$

$$x + 4 - 4 = 7 - 4 \qquad\qquad x + 4 - 4 = -7 - 4$$

$$x = 3 \qquad\qquad x = -11$$

So $x = 3$ or -11.

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Checking your answer:

$\text{LHS} = |3 + 4|$
$= |7|$
$= 7$
$= \text{RHS}$

$\text{LHS} = |-11 + 4|$
$= |-7|$
$= 7$
$= \text{RHS}$

b $|2x - 3| = 9$

$2x - 3 = 9$
$2x = 12$
$x = 6$

or $2x - 3 = -9$
$2x = -6$
$x = -3$

So $x = 6$ or -3.

Checking your answer:

$\text{LHS} = |2 \times 6 - 3|$
$= |9|$
$= 9$
$= \text{RHS}$

$\text{LHS} = |2 \times (-3) - 3|$
$= |-9|$
$= 9$
$= \text{RHS}$

Exercise 2.04 Equations involving absolute values

1 Solve:

a $|x| = 5$ **b** $|y| = 8$ **c** $|x| = 0$

2 Solve:

a $|x + 2| = 7$ **b** $|n - 1| = 3$ **c** $9 = |2x + 3|$

d $|7x - 1| = 34$ **e** $\left|\frac{x}{3}\right| = 4$

3 Solve:

a $|8x - 5| = 11$ **b** $|5 - 3n| = 1$ **c** $16 = |5t + 4|$

d $21 = |9 - 2y|$ **e** $|3x + 2| - 7 = 0$

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WS

Exponenial equations

2.05 Exponential equations

The word **exponent** means the power or index of a number.

So an **exponential equation** involves an unknown index or power; for example, $2^x = 8$.

EXAMPLE 7

Solve:

a $3^x = 81$ **b** $5^{2k-1} = 25$ **c** $8^n = 4$

Solution

a

$$3^x = 81$$
$$3^x = 3^4$$
$$\therefore x = 4$$

b

$$5^{2k-1} = 25$$
$$5^{2k-1} = 5^2$$
$$\therefore 2k - 1 = 2$$
$$2k = 3$$
$$\frac{2k}{2} = \frac{3}{2}$$
$$k = 1\tfrac{1}{2}$$

c It is hard to write 8 as a power of 4 or 4 as a power of 8, but both can be written as powers of 2.

$$8^n = 4$$
$$(2^3)^n = 2^2$$
$$2^{3n} = 2^2$$
$$\therefore 3n = 2$$
$$\frac{3n}{3} = \frac{2}{3}$$
$$n = \frac{2}{3}$$

To solve other equations involving indices, we do the opposite or inverse operation. For example, squares and square roots are inverse operations, and cubes and cube roots are inverse operations.

EXAMPLE 8

Solve:

a $x^2 = 9$ **b** $5n^3 = 40$

Solution

a There are two possible numbers whose square is 9.

$$x^2 = 9$$
$$x = \pm\sqrt{9}$$
$$\therefore x = \pm 3$$

b

$$5n^3 = 40$$
$$\frac{5n^3}{5} = \frac{40}{5}$$
$$n^3 = 8$$
$$n = \sqrt[3]{8}$$
$$n = 2$$

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INVESTIGATION

SOLUTIONS FOR EQUATIONS INVOLVING x^n

Investigate equations of the type $x^n = k$ where k is a constant; for example, $x^n = 9$.

Look at these questions.

1 What is the solution when $n = 0$?

2 What is the solution when $n = 1$?

3 How many solutions are there when $n = 2$?

4 How many solutions are there when $n = 3$?

5 How many solutions are there when n is even?

6 How many solutions are there when n is odd?

Exercise 2.05 Exponential equations

1 Solve:

a $2^n = 16$ **b** $3^y = 243$ **c** $2^m = 512$ **d** $10^x = 100\,000$

e $6^m = 1$ **f** $4^x = 64$ **g** $4^x + 3 = 19$ **h** $5(3^x) = 45$

i $4^x = 4$ **j** $\dfrac{6^k}{2} = 18$

2 Solve:

a $3^{2x} = 81$ **b** $2^{5x-1} = 16$ **c** $4^{x+3} = 4$ **d** $3^{n-2} = 1$

e $7^{2x+1} = 7$ **f** $3^{x-3} = 27$ **g** $5^{3y+2} = 125$ **h** $7^{3x-4} = 49$

i $2^{4x} = 256$ **j** $9^{3a+1} = 9$

3 Solve:

a $4^m = 2$ **b** $27^x = 3$ **c** $125^x = 5$ **d** $\left(\dfrac{1}{49}\right)^k = 7$

e $\left(\dfrac{1}{1000}\right)^k = 100$ **f** $16^n = 8$ **g** $25^x = 125$ **h** $64^n = 16$

i $\left(\dfrac{1}{4}\right)^{3k} = 2$ **j** $8^{x-1} = 4$

4 Solve:

a $2^{4x+1} = 8^x$ **b** $3^{5x} = 9^{x-2}$ **c** $7^{2k+3} = 7^{k-1}$

d $4^{3n} = 8^{n+3}$ **e** $6^{x-5} = 216^x$ **f** $16^{2x-1} = 4^{x-4}$

g $27^{x+3} = 3^x$ **h** $\left(\dfrac{1}{2}\right)^x = \left(\dfrac{1}{64}\right)^{2x+3}$ **i** $\left(\dfrac{3}{4}\right)^x = \left(\dfrac{27}{64}\right)^{2x-3}$

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5 Solve:

a $4^m = \sqrt{2}$ **b** $\left(\frac{9}{25}\right)^{k+3} = \sqrt{\frac{3}{5}}$ **c** $\frac{1}{\sqrt{2}} = 4^{2x-5}$

d $3^k = 3\sqrt{3}$ **e** $\left(\frac{1}{27}\right)^{3n+1} = \frac{\sqrt{3}}{81}$ **f** $\left(\frac{2}{5}\right)^{3n+1} = \left(\frac{5}{2}\right)^{-n}$

g $32^{-x} = \frac{1}{16}$ **h** $9^{2b+5} = 3^b\sqrt{3}$ **i** $81^{x+1} = \sqrt{3^x}$

6 Solve, giving exact answers:

a $x^3 = 27$ **b** $y^2 = 64$ **c** $n^4 = 16$

d $x^2 = 20$ **e** $p^3 = 1000$ **f** $2x^2 = 50$

g $6y^4 = 486$ **h** $w^3 + 7 = 15$ **i** $6n^2 - 4 = 92$

7 Solve and give the answer correct to 2 decimal places:

a $p^2 = 45$ **b** $x^3 = 100$ **c** $n^5 = 240$

d $2x^2 = 70$ **e** $4y^3 + 7 = 34$ **f** $\frac{d^4}{3} = 14$

g $\frac{k^2}{2} - 3 = 7$ **h** $\frac{x^3 - 1}{5} = 2$ **i** $2y^2 - 9 = 20$

8 Solve:

a $x^{-1} = 5$ **b** $a^{-3} = 8$ **c** $y^{-5} = 32$ **d** $x^{-2} + 1 = 50$

e $2n^{-1} = 3$ **f** $a^{-3} = \frac{1}{8}$ **g** $x^{-2} = \frac{1}{4}$ **h** $b^{-1} = \frac{1}{9}$

i $x^{-2} = 2\frac{1}{4}$ **j** $b^{-4} = \frac{16}{81}$

PUZZLE

Test your logical thinking and that of your friends.

1 How many months have 28 days?

2 If I have 128 sheep and take away all but 10, how many do I have left?

3 A bottle and its cork cost \$1.10 to make. If the bottle costs \$1 more than the cork, how much does each cost?

4 What do you get if you add 1 to 15 four times?

5 On what day of the week does Good Friday fall in 2030?

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2.06 Solving quadratic equations by factorisation

Quadratic equations by factorising

A **quadratic equation** is an equation involving a square. For example, $x^2 - 4 = 0$.

When solving quadratic equations by factorising, we use a property of zero.

For any real numbers a and b, if $ab = 0$ then $a = 0$ or $b = 0$.

EXAMPLE 9

Solve:

a $x^2 + x - 6 = 0$ **b** $y^2 - 7y = 0$ **c** $3a^2 - 14a = -8$

Solution

a

$x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$\therefore x + 3 = 0 \text{ or } x - 2 = 0$

$x = -3 \text{ or } x = 2$

So the solution is $x = -3$ or 2.

b

$y^2 - 7y = 0$

$y(y - 7) = 0$

$\therefore y = 0 \text{ or } y - 7 = 0$

$y = 7$

So the solution is $y = 0$ or 7.

c First we make the equation equal to zero so we can factorise and use the rule for zero.

$3a^2 - 14a = -8$

$3a^2 - 14a + 8 = -8 + 8$

$3a^2 - 14a + 8 = 0$

$(3a - 2)(a - 4) = 0$

$\therefore 3a - 2 = 0 \text{ or } a - 4 = 0$

$3a = 2 \text{ or } a = 4$

$\frac{3a}{3} = \frac{2}{3}$

$a = \frac{2}{3}$

So the solution is $a = \frac{2}{3}$ or 4.

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Exercise 2.06 Solving quadratic equations by factorisation

Solve each quadratic equation.

1 $y^2 + y = 0$

2 $b^2 - b - 2 = 0$

3 $p^2 + 2p - 15 = 0$

4 $t^2 - 5t = 0$

5 $x^2 + 9x + 14 = 0$

6 $q^2 - 9 = 0$

7 $x^2 - 1 = 0$

8 $a^2 + 3a = 0$

9 $2x^2 + 8x = 0$

10 $4x^2 - 1 = 0$

11 $3x^2 + 7x + 4 = 0$

12 $2y^2 + y - 3 = 0$

13 $8b^2 - 10b + 3 = 0$

14 $x^2 - 3x = 10$

15 $3x^2 = 2x$

16 $2x^2 = 7x - 5$

17 $5x - x^2 = 0$

18 $y^2 = y + 2$

19 $8n = n^2 + 15$

20 $12 = 7x - x^2$

21 $m^2 = 6 - 5m$

22 $x(x + 1)(x + 2) = 0$

23 $(y - 1)(y + 5)(y + 2) = 0$

24 $(x + 3)(x - 1) = 32$

25 $(m - 3)(m - 4) = 20$

WS

Compleing the square

2.07 Solving quadratic equations by completing the square

Not all trinomials will factorise, so other methods need to be used to solve quadratic equations.

EXAMPLE 10

Solve:

a $(x + 3)^2 = 11$

b $(y - 2)^2 = 7$

Solution

a
$$\begin{aligned} (x+3)^2 &= 11 \\ x+3 &= \pm\sqrt{11} \\ x+3-3 &= \pm\sqrt{11}-3 \\ x &= \pm\sqrt{11}-3 \end{aligned}$$

b
$$\begin{aligned} (y-2)^2 &= 7 \\ y-2 &= \pm\sqrt{7} \\ y-2+2 &= \pm\sqrt{7}+2 \\ y &= \pm\sqrt{7}+2 \end{aligned}$$

To solve a quadratic equation such as $x^2 - 6x + 3 = 0$, which will not factorise, we can use the method of **completing the square**.

We use the perfect square:

$$a^2 + 2ab + b^2 = (a + b)^2$$

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EXAMPLE 11

Complete the square on $a^2 + 6a$.

Solution

Compare with $a^2 + 2ab + b^2$: $2ab = 6a$

$$b = 3$$

To complete the square: $a^2 + 2ab + b^2 = (a + b)^2$

$$a^2 + 2a(3) + 3^2 = (a + 3)^2$$

$$a^2 + 6x + 9 = (a + 3)^2$$

Completing the square

To complete the square on $a^2 \pm pa$, divide p by 2 and square it.

$$a^2 \pm pa + \left(\frac{p}{2}\right)^2 = \left(a \pm \frac{p}{2}\right)^2$$

EXAMPLE 12

Solve by completing the square:

a $x^2 - 6x + 3 = 0$

b $y^2 + 2y - 7 = 0$ (correct to 3 significant figures)

Solution

a

$$x^2 - 6x + 3 = 0$$

$$x^2 - 6x = -3$$

$$x^2 - 6x + 9 = -3 + 9 \qquad \left(\frac{6}{2}\right)^2 = 3^2 = 9$$

$$(x - 3)^2 = 6$$

$$\therefore x - 3 = \pm\sqrt{6}$$

$$x = \pm\sqrt{6} + 3$$

The 3rd line shows the 'completing the square' step in both solutions.

b

$$y^2 + 2y - 7 = 0$$

$$y^2 + 2y = 7$$

$$y^2 + 2y + 1 = 7 + 1 \qquad \left(\frac{2}{2}\right)^2 = 1^2 = 1$$

$$(y + 1)^2 = 8$$

$$\therefore y + 1 = \pm\sqrt{8}$$

$$y = \pm\sqrt{8} - 1$$

$$y \approx 1.83 \text{ or } -3.83$$

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Exercise 2.07 Solving quadratic equations by completing the square

1 Solve and give exact solutions:

a $(x+1)^2 = 7$ **b** $(y+5)^2 = 5$ **c** $(a-3)^2 = 6$

d $(x-2)^2 = 13$ **e** $(2y+3)^2 = 2$

2 Solve and give solutions correct to one decimal place:

a $(h+2)^2 = 15$ **b** $(a-1)^2 = 8$ **c** $(x-4)^2 = 17$

d $(y+7)^2 = 21$ **e** $(3x-1)^2 = 12$

3 Solve by completing the square, giving exact solutions in simplest surd form:

a $x^2 + 4x - 1 = 0$ **b** $a^2 - 6a + 2 = 0$ **c** $y^2 - 8y - 7 = 0$

d $x^2 + 2x - 12 = 0$ **e** $p^2 + 14p + 5 = 0$ **f** $x^2 - 10x - 3 = 0$

g $y^2 + 20y + 12 = 0$ **h** $x^2 - 2x - 1 = 0$ **i** $n^2 + 24n + 7 = 0$

4 Solve by completing the square and writing answers correct to 3 significant figures:

a $x^2 - 2x - 5 = 0$ **b** $x^2 + 12x + 34 = 0$ **c** $q^2 + 18q - 1 = 0$

d $x^2 - 4x - 2 = 0$ **e** $b^2 + 16b + 50 = 0$ **f** $x^2 - 24x + 112 = 0$

g $r^2 - 22r - 7 = 0$ **h** $x^2 + 8x + 5 = 0$ **i** $a^2 + 6a - 1 = 0$

2.08 Solving quadratic equations by quadratic formula

WS Quadaic formula

WS Quadaic equations

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The quadratic formula

Completing the square is difficult with harder quadratic equations such as $2x^2 - x - 5 = 0$. Completing the square on a general quadratic equation gives the following formula.

The quadratic formula

For the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof

Solve $ax^2 + bx + c = 0$ by completing the square.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

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Completing the square:

$$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$=\frac{-4ac+b^2}{4a^2}$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{-4ac+b^2}{4a^2}}$$

$$=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Excel worksheet: The quadratic formula

Excel spreadsheet: The quadratic formula

Solving algebraic equations

EXAMPLE 13

a Solve $x^2 - x - 2 = 0$ by using the quadratic formula

b Solve $2y^2 - 9y + 3 = 0$ by formula and give your answer correct to 2 decimal places

Solution

a $a = 1, b = -1, c = -2$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-2)}}{2(1)}$$

$$=\frac{1\pm\sqrt{1+8}}{2}$$

$$=\frac{1\pm\sqrt{9}}{2}$$

$$=\frac{1\pm 3}{2}$$

$$=2 \text{ or } -1$$

b $a = 2, b = -9, c = 3$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-9)\pm\sqrt{(-9)^2-4(2)(3)}}{2(2)}$$

$$=\frac{9\pm\sqrt{81-24}}{4}$$

$$=\frac{9\pm\sqrt{57}}{4}$$

$$\approx 414 \text{ or } 036$$

Exercise 2.08 Solving quadratic equations by quadratic formula

1 Solve by formula, correct to 3 significant figures where necessary:

a $y^2 + 6y + 2 = 0$ **b** $2x^2 - 5x + 3 = 0$ **c** $b^2 - b - 9 = 0$

d $2x^2 - x - 1 = 0$ **e** $-8x^2 + x + 3 = 0$ **f** $n^2 + 8n - 2 = 0$

g $m^2 + 7m + 10 = 0$ **h** $x^2 - 7x = 0$ **i** $x^2 + 5x = 6$

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2 Solve by formula, leaving the answer in simplest surd form:

a	$x^2 + x - 4 = 0$	**b**	$3x^2 - 5x + 1 = 0$	**c**	$q^2 - 4q - 3 = 0$
d	$4h^2 + 12h + 1 = 0$	**e**	$3s^2 - 8s + 2 = 0$	**f**	$x^2 + 11x - 3 = 0$
g	$6d^2 + 5d - 2 = 0$	**h**	$x^2 - 2x = 7$	**i**	$t^2 = t + 1$

CLASS INVESTIGATION

FAULTY PROOF

Here is a proof that 1 = 2 Can you see the fault in the proof ?

$$x^2 - x^2 = x^2 - x^2$$

$$x(x - x) = (x + x)(x - x)$$

$$x = x + x$$

$$x = 2x$$

$$\therefore \quad 1 = 2$$

2.09 Formulas and equations

Sometimes substituting values into a formula involves solving an equation.

EXAMPLE 14

a The formula for the surface area of a rectangular prism is given by $S = 2(lb + bh + lh)$. Find the value of b when $S = 180$, $l = 9$ and $h = 6$.

b The volume of a cylinder is given by $V = \pi r^2 h$. Evaluate the radius r, correct to 2 decimal places, when $V = 350$ and $h = 6.5$.

Solution

a

$$S = 2(lb + bh + lh)$$

$$180 = 2(9b + 6b + 9 \times 6)$$

$$= 2\,(15b + 54)$$

$$= 30b + 108$$

$$72 = 30b$$

$$\frac{72}{30} = \frac{30b}{30}$$

$$2.4 = b$$

b

$$V = \pi r^2 h$$

$$350 = \pi r^2(6.5)$$

$$\frac{350}{6\ 5\pi} = \frac{\pi r^2(6\ 5)}{6\ 5\pi}$$

$$\frac{350}{6\ 5\pi} = r^2$$

$$\sqrt{\frac{350}{6\ 5\pi}} = r$$

$$414 = r$$

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Exercise 2.09 Formulas and equations

1 Given that $v = u + at$ is the formula for the velocity of a particle at time t, find the value of t when $u = 17.3$, $v = 100.6$ and $a = 9.8$.

2 The sum of an arithmetic series is given by $S = \frac{n}{2}(a + l)$. Find l if $a = 3$, $n = 26$ and $S = 1625$.

3 The formula for finding the area of a triangle is $A = \frac{1}{2}bh$. Find b when $A = 36$ and $h = 9$.

4 The area of a trapezium is given by $A = \frac{1}{2}h(a + b)$. Find the value of a when $A = 120$, $h = 5$ and $b = 7$.

5 Find the value of y when $x = 3$, given the straight line equation $5x - 2y - 7 = 0$.

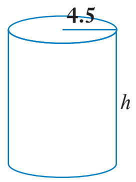
6 The area of a circle is given by $A = \pi r^2$. Find r correct to 3 significant figures if $A = 140$.

7 The area of a rhombus is given by the formula $A = \frac{1}{2}xy$ where x and y are its diagonals. Find the value of x correct to 2 decimal places when $y = 7.8$ and $A = 25.1$.

8 The simple interest formula is $I = Prn$. Find n if $r = 0.145$, $P = 150$ and $I = 326.25$.

9 The gradient of a straight line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$. Find y_1 when $m = -\frac{5}{6}$, $y_2 = 7$, $x_2 = -3$ and $x_1 = 1$.

10 The surface area of a cylinder is given by the formula $S = 2\pi r(r + h)$. Evaluate h correct to 1 decimal place if $S = 232$ and $r = 4.5$.



11 The formula for body mass index is $\text{BMI} = \frac{w}{h^2}$. Evaluate:

- **a** the BMI when $w = 65$ and $h = 1.6$
- **b** w when BMI = 21.5 and $h = 1.8$
- **c** h when BMI = 19.7 and $w = 73.8$.

12 A formula for depreciation is $D = P(1 - r)^n$. Find r if $D = 12\ 000$, $P = 15\ 000$ and $n = 3$.

13 The x value of the midpoint is given by $x = \frac{x_1 + x_2}{2}$. Find x_1 when $x = -2$ and $x_2 = 5$.

14 Given the height of a particle at time t is $h = 5t^2$, evaluate t when $h = 23$.

15 If $y = x^2 + 1$, evaluate x when $y = 5$.

16 If the surface area of a sphere is $S = 4\pi r^2$, evaluate r to 3 significant figures when $S = 56.3$.

17 The area of a sector of a circle is $A = \frac{1}{2}r^2\theta$. Evaluate r when $A = 24.6$ and $\theta = 0.45$.

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18 If $y = \frac{2}{x^3 - 1}$, find the value of x when $y = 3$.

19 Given $y = \sqrt{2x + 5}$ evaluate x when $y = 4$.

20 The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Evaluate r to 1 decimal place when $V = 150$.

INVESTIGATION

BODY MASS INDEX

Body mass index (BMI) is a formula that is used by health professionals to screen for weight categories that may lead to health problems.

The formula for BMI is $BMI = \frac{m}{h^2}$ where m is the mass of a person in kg and h is the height in metres.

For adults over 20, a BMI under 18.5 means that the person is underweight and over 25 is overweight. Over 30 is considered obese.

The BMI may not always be a reliable measurement of body fat. Can you think of some reasons?

Is it important where the body fat is stored? Does it make a difference if it is on the hips or the stomach?

Research more about BMI generally.

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2.10 Linear simultaneous equations

You can solve two equations together to find one solution that satisfies both equations. Such equations are called **simultaneous equations** and there are two ways of solving them. The **elimination method** adds or subtracts the equations. The **substitution method** substitutes one equation into the other.

EXAMPLE 15

Solve simultaneously using the elimination method:

a $3a + 2b = 5$ and $2a - b = -6$

b $5x - 3y = 19$ and $2x - 4y = 16$

Solution

a

$3a + 2b = 5$ [1]

$2a - b = -6$ [2]

[2] × 2: $4a - 2b = -12$ [3]

[1] + [3]: $3a + 2b = 5$ [1]

$7a = -7$

$a = -1$

Substitute $a = -1$ in [1]: $3(-1) + 2b = 5$

$-3 + 2b = 5$

$2b = 8$

$b = 4$

Check that the solution is correct by substituting back into *both* equatons

$\therefore$ Solution is $a = -1, b = 4$

b

$5x - 3y = 19$ [1]

$2x - 4y = 16$ [2]

[1] × 4: $20x - 12y = 76$ [3]

[2] × 3: $6x - 12y = 48$ [4]

[3] − [4]: $14x = 28$

$x = 2$

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Substitute $x = 2$ in [2]:

$$2(2) - 4y = 16$$
$$4 - 4y = 16$$
$$-4y = 12$$
$$y = -3$$

$\therefore$ Solution is $x = 2, y = -3$

Exercise 2.10 Linear simultaneous equations

Solve each pair of simultaneous equations.

1 $a - b = -2$ and $a + b = 4$

2 $5x + 2y = 12$ and $3x - 2y = 4$

3 $4p - 3q = 11$ and $5p + 3q = 7$

4 $y = 3x - 1$ and $y = 2x + 5$

5 $2x + 3y = -14$ and $x + 3y = -4$

6 $7t + v = 22$ and $4t + v = 13$

7 $4x + 5y + 2 = 0$ and $4x + y + 10 = 0$

8 $2x - 4y = 28$ and $2x - 3y = -11$

9 $5x - y = 19$ and $2x + 5y = -14$

10 $5m + 4n = 22$ and $m - 5n = -13$

11 $4w_1 + 3w_2 = 11$ and $3w_1 + w_2 = 2$

12 $3a - 4b = -16$ and $2a + 3b = 12$

13 $5p + 2q + 18 = 0$ and $2p - 3q + 11 = 0$

14 $7x_1 + 3x_2 = 4$ and $3x_1 + 5x_2 = -2$

15 $9x - 2y = -1$ and $7x - 4y = 9$

16 $5s - 3t - 13 = 0$ and $3s - 7t - 13 = 0$

17 $3a - 2b = -6$ and $a - 3b = -2$

18 $3k - 2h = -14$ and $2k - 5h = -13$

PROBLEM

A group of 39 people went to see a play. There were both adults and children in the group. The total cost of the tickets was $939, with children paying $17 each and adults paying $29 each. How many in the group were adults and how many were children? (Hint: let x be the number of adults and y the number of children.)

2.11 Non-linear simultaneous equations

In simultaneous equations involving **non-linear equations** there may be more than one set of solutions. When solving these, you need to use the substitution method.

Non-linear simultaneous equations

EXAMPLE 16

Solve each pair of equations simultaneously using the substitution method:

a $xy = 6$ and $x + y = 5$

b $x^2 + y^2 = 16$ and $3x - 4y - 20 = 0$

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Solution

a

$xy = 6$ [1]

$x + y = 5$ [2]

From [2]: $y = 5 - x$ [3]

Substitute [3] in [1]:

$$x(5 - x) = 6$$
$$5x - x^2 = 6$$
$$0 = x^2 - 5x + 6$$
$$0 = (x - 2)(x - 3)$$
$$\therefore\ x = 2 \text{ or } x = 3$$

Substitute $x = 2$ in [3]: $y = 5 - 2 = 3$

Substitute $x = 3$ in [3]: $y = 5 - 3 = 2$

Solutions are $x = 2, y = 3$ and $x = 3, y = 2$

b

$x^2 + y^2 = 16$ [1]

$3x - 4y - 20 = 0$ [2]

From [2]:

$$3x - 20 = 4y$$

$\dfrac{3x - 20}{4} = y$ [3]

Substitute [3] into [1]:

$$x^2 + \left(\frac{3x - 20}{4}\right)^2 = 16$$
$$x^2 + \left(\frac{9x^2 - 120x + 400}{16}\right) = 16$$
$$16x^2 + 9x^2 - 120x + 400 = 256$$
$$25x^2 - 120x + 144 = 0$$
$$(5x - 12)^2 = 0$$
$$\therefore\ 5x - 12 = 0$$
$$x = 2.4$$

Substitute $x = 2.4$ into [3]:

$$y = \frac{3(2\ 4) - 20}{4}$$
$$= -3\ 2$$

So the solution is $x = 2.4, y = -3.2$

Exercise 2.11 Non-linear simultaneous equations

Solve each pair of simultaneous equations.

1 $y = x^2$ and $y = x$

2 $y = x^2$ and $2x + y = 0$

3 $x^2 + y^2 = 9$ and $x + y = 3$

4 $x - y = 7$ and $xy = -12$

5 $y = x^2 + 4x$ and $2x - y - 1 = 0$

6 $y = x^2$ and $6x - y - 9 = 0$

7 $x = t^2$ and $x + t - 2 = 0$

8 $m^2 + n^2 = 16$ and $m + n + 4 = 0$

9 $xy = 2$ and $y = 2x$

10 $y = x^3$ and $y = x^2$

11 $y = x - 1$ and $y = x^2 - 3$

12 $y = x^2 + 1$ and $y = 1 - x^2$

13 $y = x^2 - 3x + 7$ and $y = 2x + 3$

14 $xy = 1$ and $4x - y + 3 = 0$

15 $h = t^2$ and $h = (t + 1)^2$

16 $x + y = 2$ and $2x^2 + xy - y^2 = 8$

17 $y = x^3$ and $y = x^2 + 6x$

18 $y = |x|$ and $y = x^2$

19 $y = x^2 - 7x + 6$ and $24x + 4y - 23 = 0$

20 $x^2 + y^2 = 1$ and $5x + 12y + 13 = 0$

WS

Simulaneou equations

2.12 Simultaneous equations with three unknown variables

Three equations can be solved simultaneously to find 3 unknown pronumerals.

EXAMPLE 17

Solve simultaneously: $a - b + c = 7$, $a + 2b - c = -4$ and $3a - b - c = 3$.

Solution

$$a - b + c = 7 \quad [1]$$

$$a + 2b - c = -4 \quad [2]$$

$$3a - b - c = 3 \quad [3]$$

[1] + [2]:

$$\begin{aligned} a - b + c &= 7 \\ a + 2b - c &= -4 \\ \hline 2a + b &= 3 \quad [4] \end{aligned}$$

[1] + [3]:

$$\begin{aligned} a - b + c &= 7 \\ 3a - b - c &= 3 \\ \hline 4a - 2b &= 10 \end{aligned}$$

or

$$2a - b = 5 \quad [5]$$

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$$
\begin{aligned}
&[4] + [5]: && 2a + b = 3\\
& && 2a - b = 5\\
& && \overline{4a \quad\; = 8}\\
& && a = 2\\
&\text{Substitute } a = 2 \text{ in } [4]: && 2(2) + b = 3\\
& && 4 + b = 3\\
& && b = -1\\
&\text{Substitute } a = 2 \text{ and } b = -1 \text{ in } [1]: && 2 - (-1) + c = 7\\
& && 2 + 1 + c = 7\\
& && 3 + c = 7\\
& && c = 4
\end{aligned}
$$

$\therefore$ solution is $a = 2, b = -1, c = 4$

Exercise 2.12 Simultaneous equations with three unknown variables

Solve each set of simultaneous equations.

1 $x = -2$, $2x - y = 4$ and $x - y + 6z = 0$

2 $a = -2$, $2a - 3b = -1$ and $a - b + 5c = 9$

3 $2a + b + c = 1$, $a + b = -2$ and $c = 7$

4 $a + b + c = 0$, $a - b + c = -4$ and $2a - 3b - c = -1$

5 $x + y - z = 7$, $x + y + 2z = 1$ and $3x + y - 2z = 19$

6 $2p + 5q - r = 25$, $2p - 2q - r = -24$ and $3p - q + 5r = 4$

7 $2x - y + 3z = 9$, $3x + y - 2z = -2$ and $3x - y + 5z = 14$

8 $x - y - z = 1$, $2x + y - z = -9$ and $2x - 3y - 2z = 7$

9 $3h + j - k = -3$, $h + 2j + k = -3$ and $5h - 3j - 2k = -13$

10 $2a - 7b + 3c = 7$, $a + 3b + 2c = -4$ and $4a + 5b - c = 9$

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EXT1 2.13 Quadratic inequalities

Solving **quadratic inequalities** is similar to solving quadratic equations, but you need to check the inequality on a number line.

EXAMPLE 18

Solve:

a $x^2 + x - 6 > 0$ **b** $9 - x^2 \geq 0$

Solution

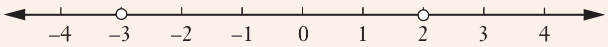
a First solve $x^2 + x - 6 = 0$

$(x - 2)(x + 3) = 0$

$\therefore x = 2 \text{ or } -3$

We can also solve quadratic inequalities by graphing a parabola, which is shown in Chapter 4 on page 185.

Now look at the number line.



Choose a number between –3 and 2, say $x = 0$.

Substitute $x = 0$ into the inequality.

$x^2 + x - 6 > 0$

$0 + 0 - 6 > 0$

$-6 > 0$ (false)

So the solution is not between –3 and 2.

$\therefore$ the solution lies either side of –3 and 2.

Check by choosing a number on either side of the two numbers.

Choose a number on the RHS of 2, say $x = 3$.

Substitute $x = 3$ into the inequality.

$3^2 + 3 - 6 > 0$

$6 > 0$ (true)

So the solution is on the RHS of 2.

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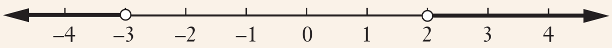
Choose a number on the LHS of -3, say $x = -4$.

Substitute $x = -4$ into the inequality.

$$(-4)^2 + (-4) - 6 > 0$$
$$6 > 0 \quad \text{(true)}$$

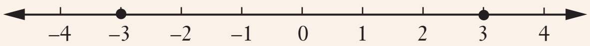
So the solution is on the LHS of -3.



This gives the solution $x < -3$, $x > 2$.

b First solve $9 - x^2 = 0$

$$9 = x^2$$
$$x = \pm\sqrt{9}$$
$$x = \pm 3$$

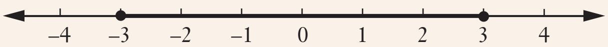


Choose a number between -3 and 3, say $x = 0$.

Substitute $x = 0$ into the inequality.

$$9 - x^2 \geq 0$$
$$9 - 0^2 \geq 0$$
$$9 \geq 0 \quad \text{(true)}$$

So the solution is between -3 and 3, that is $-3 \leq x \leq 3$ on the number line:



EXT1 Exercise 2.13 Quadratic inequalities

Solve each quadratic inequality.

1 $x^2 + 3x < 0$
2 $y^2 - 4y < 0$
3 $n^2 - n \geq 0$
4 $x^2 - 4 \geq 0$
5 $1 - n^2 < 0$
6 $n^2 + 2n - 15 \leq 0$
7 $c^2 - c - 2 > 0$
8 $x^2 + 6x + 8 \leq 0$
9 $x^2 - 9x + 20 < 0$
10 $2b^2 + 5b + 2 \geq 0$
11 $1 - 2a - 3a^2 < 0$
12 $2y^2 - y - 6 > 0$
13 $3x^2 - 5x + 2 \geq 0$
14 $6 - 13b - 5b^2 < 0$
15 $6x^2 + 11x + 3 \leq 0$
16 $y^2 + y \leq 12$
17 $x^2 > 16$
18 $a^2 \leq 1$
19 $x^2 < x + 6$
20 $x^2 \geq 2x + 3$
21 $x^2 < 2x$
22 $2a^2 \leq 5a - 3$
23 $5y^2 + 6y \geq 8$
24 $6m^2 > 15 - m$

EXT1 2.14 Inequalities involving the unknown in the denominator

Inequalities with the unknown in the denominator

EXAMPLE 19

Solve:

a $\dfrac{1}{x} < 3$ **b** $\dfrac{6}{x+3} \geq 1$ **c** $\dfrac{y^2 - 6}{y} \leq 1$

Solution

a $x \neq 0$

Method 1: Critical values

Solve $\dfrac{1}{x} = 3$.

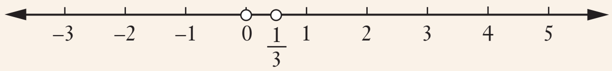
$$\frac{1}{x} \times x = 3 \times x$$

$$1 = 3x$$

$$\frac{1}{3} = x$$

$x = \dfrac{1}{3}$ is not a solution of the inequality $\dfrac{1}{x} < 3$.

Place $x = 0$ and $x = \dfrac{1}{3}$ on a number line and test x values on either side of these values in the inequality.



Test for $x < 0$, say $x = -1$.

Substitute into the inequality:

$$\frac{1}{x} < 3$$

$$\frac{1}{-1} < 3$$

$$-1 < 3 \qquad \text{(true)}$$

So $x < 0$ is part of the solution.

Test for $0 < x < \dfrac{1}{3}$, say $x = \dfrac{1}{10}$.

$$\frac{1}{\frac{1}{10}} < 3$$

$$10 < 3 \qquad \text{(false)}$$

So $0 < x < \dfrac{1}{3}$ is not part of the solution.

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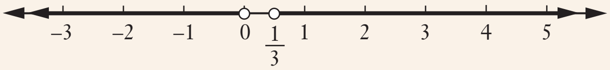
Test for $x > \frac{1}{3}$, say $x = 1$.

$\frac{1}{1} < 3$

$1 < 3$ (true)

So $x > \frac{1}{3}$ is part of the solution.

Solution is $x < 0$, $x > \frac{1}{3}$



Method 2: Multiplying by the square of the denominator

$\frac{1}{x} < 3$.

First multiply both sides by x^2. This will not change the inequality since $x^2 > 0$ for $x \neq 0$.

$$\frac{1}{x} \times x^2 < 3 \times x^2$$

$$x < 3x^2$$

$$0 < 3x^2 - x$$

$$< x(3x - 1)$$

$x(3x - 1) > 0$

Solving this quadratic inequality gives the solution:

$x < 0$, $x > \frac{1}{3}$.

b $x + 3 \neq 0$

$x \neq -3$

Method 1: Critical values

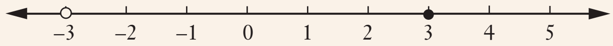
Solve $\frac{6}{x+3} = 1$.

$6 = x + 3$

$3 = x$

$x = 3$ is a solution of the inequality $\frac{6}{x+3} \geq 1$.

Place $x = -3$ and $x = 3$ on a number line and test values on either side in the inequality.



Test for $x < -3$, say $x = -4$.

Substitute into the inequality:

$$\frac{6}{x+3} \geq 1$$

$$\frac{6}{-4+3} \geq 1$$

$$-6 \geq 1 \qquad \text{(false)}$$

So $x < -3$ is not part of the solution.

Test for $-3 < x \leq 3$, say $x = 0$.

$$\frac{6}{0+3} \geq 1$$

$$2 \geq 1 \qquad \text{(true)}$$

So $-3 < x \leq 3$ is part of the solution.

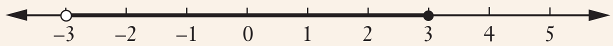
Test for $x \geq 3$, say $x = 4$.

$$\frac{6}{4+3} \geq 1$$

$$\frac{6}{7} \geq 1 \qquad \text{(false)}$$

So $x \geq 3$ is not part of the solution.

Solution is $-3 < x \leq 3$.



Method 2: Multiplying by the square of the denominator

$$\frac{6}{x+3} \geq 1.$$

First multiply both sides by $(x + 3)^2$.

$$\frac{6}{x+3} \times (x+3)^2 \geq 1 \times (x+3)^2$$

$$6(x + 3) \geq (x + 3)^2$$

$$0 \geq (x + 3)^2 - 6(x + 3)$$

$$\geq (x + 3)(x + 3 - 6) \qquad \text{factorising}$$

$$\geq (x + 3)(x - 3)$$

$$\geq x^2 - 9$$

$$x^2 - 9 \leq 0$$

Solving this quadratic inequality gives the solution:

$-3 \leq x \leq 3$

But $x \neq -3$, so the solution is $-3 < x \leq 3$.

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c $y \neq 0$

Method 1: Critical values

Solve $\frac{y^2 - 6}{y} = 1$.

$$\frac{y^2 - 6}{y} \times y = 1 \times y$$

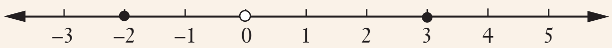
$$y^2 - 6 = y$$

$$y^2 - y - 6 = 0$$

$$(y - 3)(y + 2) = 0$$

$y = 3 \qquad y = -2$

Sketch these on a number line and test values on either side.



Test for $y \leq -2$, say $y = -3$.

Substitute into the inequality:

$$\frac{y^2 - 6}{y} \leq 1$$

$$\frac{(-3)^2 - 6}{-3} \leq 1$$

$$-1 \leq 1 \qquad \text{(true)}$$

So $y \leq -2$ is part of the solution.

Test for $-2 \leq y < 0$, say $y = -1$.

$$\frac{(-1)^2 - 6}{-1} \leq 1$$

$$5 \leq 1 \qquad \text{(false)}$$

So $2 \leq y < 0$ is not part of the solution.

Test $0 < y \leq 3$, say $y = 1$.

$$\frac{1^2 - 6}{1} \leq 1$$

$$-5 \leq 1 \qquad \text{(true)}$$

So $0 < y \leq 3$ is part of the solution.

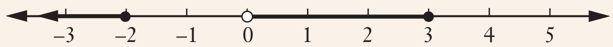
Test $y \geq 3$, say $y = 4$.

$$\frac{4^2 - 6}{4} \leq 1$$

$$2\frac{1}{2} \leq 1 \quad \text{(false)}$$

So $y \geq 3$ is not part of the solution.

The solution is $y \leq -2$, $0 < y \leq 3$.



Method 2: Multiplying by the square of the denominator

$$\frac{y^2 - 6}{y} \leq 1$$

$$\frac{y^2 - 6}{y} \times y^2 \leq 1 \times y^2$$

$$y(y^2 - 6) \leq y^2$$

$$y(y^2 - 6) - y^2 \leq 0$$

$$y(y^2 - 6 - y) \leq 0$$

$$y(y - 3)(y + 2) \leq 0$$

Solving $y(y - 3)(y + 2) = 0$ gives $y = 0, 3, -2$.

Testing points on the number line gives the solution $y \leq -2$, $0 < y \leq 3$.

EXT1 Exercise 2.14 Inequalities involving the unknown in the denominator

Solve:

1 $\frac{1}{y} < 1$ **2** $\frac{1}{x} > 2$ **3** $\frac{3}{x} < 2$ **4** $\frac{2}{m} \geq 7$

5 $\frac{3}{x} > -5$ **6** $\frac{2}{b} \leq -1$ **7** $\frac{1}{x-1} > 4$ **8** $\frac{1}{z+3} < -5$

9 $\frac{3}{x-2} \geq 4$ **10** $\frac{-1}{2-x} < 6$ **11** $\frac{5}{x+4} \leq -9$ **12** $\frac{2}{3x-4} > 5$

13 $\frac{-3}{2a+5} < 2$ **14** $\frac{x}{2x-1} > 5$ **15** $\frac{y}{y+1} < 2$ **16** $\frac{3x+1}{x-4} \geq \frac{1}{3}$

17 $\frac{8p+7}{2p-9} > 5$ **18** $\frac{x-2}{5x+1} \leq \frac{3}{4}$ **19** $\frac{x^2-5}{x} < -4$ **20** $\frac{2x^2}{3x-2} \leq -1$

21 $\frac{3x^2}{7x+4} < -2$ **22** $\frac{2x(x-4)}{x-1} \leq 7$

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EXT1 2.15 Inequalities involving absolute values

WS Futhr inequalities

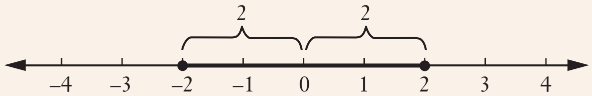
EXAMPLE 20

Solve:

a $|x| \le 2$ **b** $|x| > 2$

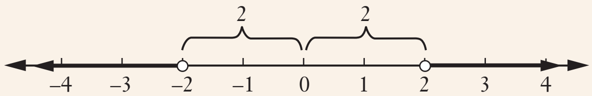
Solution

a $|x| \le 2$ means the distance of x from zero is less than or equal to 2 (in both directions).



Notice that there is one region on the number line. We can write this as the single statement $-2 \le x \le 2$.

b $|x| > 2$ means the distance of x from zero is greater than 2 (in both directions).



There are two regions on the number line, so we write two separate inequalities.
$x < -2, x > 2$.

Absolute value inequalities

$|x| = a$ means $x = \pm a$.

$|x| < a$ means $-a < x < a$.

$|x| > a$ means $x < -a, x > a$.

EXAMPLE 21

Solve:

a $|2y - 1| < 5$ **b** $|5b - 7| \geq 3$

Solution

a This means that the distance from $2y - 1$ to 0 is less than 5 in both directions. So it means $-5 < 2y - 1 < 5$.

$$-5 < 2y - 1 < 5$$
$$-4 < 2y < 6$$
$$-2 < y < 3$$

b $|5b - 7| \geq 3$ means that the distance from $5b - 7$ to 0 is greater than or equal to 3 in both directions.

$$5b - 7 \leq -3 \quad \text{or} \quad 5b - 7 \geq 3$$
$$5b \leq 4 \quad\quad 5b \geq 10$$
$$b \leq \frac{4}{5} \quad\quad b \geq 2$$

So $b \leq \frac{4}{5}, b \geq 2$.

EXT1 Exercise 2.15 Inequalities involving absolute values

1 Solve:

a $|a| < 4$ **b** $|k| \geq 1$ **c** $|x| > 6$
d $|p| \leq 10$ **e** $|a| > 14$ **f** $|y| < 12$
g $|b| \geq 20$

2 Solve:

a $|2a| > 4$ **b** $|x - 5| \leq 1$ **c** $|4y + 3| < 11$
d $|2x - 3| \geq 15$ **e** $\left|\frac{a}{2} - 3\right| \leq 2$

3 Solve:

a $|5y - 3| \geq 7$ **b** $|7 + 6a| < 5$ **c** $|10t - 3| \leq 17$
d $14 > |2x - 8|$ **e** $11 \leq |6 - 5n|$

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2. TEST YOURSELF

For Questions **1** to **4**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

1 Find the exact solution of $x^2 - 5x - 1 = 0$.

A $\dfrac{-5 \pm \sqrt{29}}{2}$ **B** $\dfrac{5 \pm \sqrt{21}}{2}$ **C** $\dfrac{5 \pm \sqrt{29}}{2}$ **D** $\dfrac{-5 \pm \sqrt{21}}{2}$

2 If $S = 4\pi r^2$, find the value of r when $S = 200$ (there may be more than one answer).

A $5\sqrt{\dfrac{2}{\pi}}$ **B** $\sqrt{\dfrac{200}{\pi}}$ **C** $10\sqrt{\dfrac{2}{\pi}}$ **D** $\sqrt{\dfrac{50}{\pi}}$

3 Solve the simultaneous equations $x - y = 7$ and $x + 2y = 1$.

A $x = 5, y = 2$ **B** $x = 5, y = -2$ **C** $x = -5, y = -2$ **D** $x = -5, y = 2$

4 EXT1 Solve $\dfrac{2x}{x-2} \geq 1$.

A $-2 < x < 2$ **B** $-2 \leq x \leq 2$ **C** $x \leq -2, x > 2$ **D** $x < -2, x > 2$

5 Solve:

a $8 = 3b - 22$ **b** $\dfrac{a}{4} - \dfrac{a+2}{3} = 9$ **c** $4(3x + 1) = 11x - 3$

d EXT1 $\dfrac{-4}{x+3} \leq 3$ **e** $3p + 1 \leq p + 9$

6 The compound interest formula is $A = P(1 + r)^n$. Find, correct to 2 decimal places:

a A when $P = 1000$, $r = 0.06$ and $n = 4$

b P when $A = 12\,450$, $r = 0.055$ and $n = 7$.

7 Solve each pair of simultaneous equations.

a $x - y + 7 = 0$ and $3x - 4y + 26 = 0$ **b** $xy = 4$ and $2x - y - 7 = 0$

8 Solve:

a $3^{x+2} = 81$ **b** $16^y = 2$

9 Solve:

a $|3b - 1| = 5$ **b** EXT1 $|2x - 7| \geq 1$

10 The area of a trapezium is given by $A = \dfrac{1}{2}h(a + b)$. Find:

a A when $h = 6$, $a = 5$ and $b = 7$

b b when $A = 40$, $h = 5$ and $a = 4$.

11 Solve $2x^2 - 3x + 1 = 0$.

12 Solve $-2 < 3y + 1 \leq 10$ and plot the solution on a number line.

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13 Solve, correct to 3 significant figures:

a $x^2 + 7x + 2 = 0$ **b** $y^2 - 2y - 9 = 0$ **c** $3n^2 + 2n - 4 = 0$

14 The surface area of a sphere is given by $A = 4\pi r^2$. Evaluate to 1 decimal place:

a A when $r = 7.8$ **b** r when $A = 102.9$

15 EXT1 Solve $\frac{x-3}{7} - \frac{3}{4} > 9$.

16 EXT1 Solve $x^2 - 11x + 18 > 0$.

17 Solve the simultaneous equations $x^2 + y^2 = 16$ and $3x + 4y - 20 = 0$.

18 The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Evaluate to 2 significant figures:

a V when $r = 8$ **b** r when $V = 250$

19 For each equation, decide if it has:

A 2 solutions **B** 1 solution **C** no solutions.

a $x^2 - 6x + 9 = 0$ **b** $|2x - 3| = 7$ **c** $x^2 - x - 5 = 0$

d $2x^2 - x + 4 = 0$ **e** $3x + 2 = 7$

20 Solve simultaneously $a + b = 5$, $2a + b + c = 4$, $a - b - c = 5$.

21 EXT1 Solve $|3n + 5| > 5$ and plot the solution on a number line.

22 EXT1 Solve $\frac{7t+4}{3t-8} \geq -1$.

23 Solve $9^{2x+1} = 27^x$.

24 Solve:

a $2(3y - 5) > y + 5$ **b** EXT1 $n^2 + 3n \leq 0$ **c** $3^{2x-1} = 27$

d $5x^3 - 1 = 39$ **e** $|5x - 4| = 11$ **f** EXT1 $|2t + 1| \geq 3$

g EXT1 $x^2 + 2x - 8 \leq 0$ **h** $8^{x+1} = 4^x$ **i** EXT1 $y^2 - 4 > 0$

j EXT1 $1 - x^2 \leq 0$ **k** $27^{2x-1} = 9$ **l** EXT1 $|4b - 3| \leq 5$

m EXT1 $x^2 < 2x + 3$ **n** EXT1 $m^2 + m \geq 6$ **o** EXT1 $\frac{2t-3}{t} < 5$

p EXT1 $\frac{y+1}{y-1} > 2$ **q** EXT1 $\frac{n}{2n-4} \geq 3$ **r** EXT1 $\frac{3x-2}{2x+1} \leq -1$

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2. CHALLENGE EXERCISE

1 Find the value of y if $a^{3y-5} = \frac{1}{a^2}$.

2 EXT1 Solve $x^2 > a^2$.

3 The solutions of $x^2 - 6x - 3 = 0$ are in the form $a + b\sqrt{3}$. Find the values of a and b.

4 EXT1 Solve $\frac{2}{x-1} - \frac{1}{x+1} = 1$ correct to 3 significant figures.

5 EXT1 Solve $\frac{6-2y}{y} \geq y - 3$.

6 **a** Factorise $x^5 - 9x^3 - 8x^2 + 72$.

b Hence or otherwise solve $x^5 - 9x^3 - 8x^2 + 72 = 0$.

7 Solve the simultaneous equations $y = x^3 + x^2$ and $y = x + 1$.

8 Find the value of b if $x^2 - 8x + b$ is a perfect square. Hence solve $x^2 - 8x - 1 = 0$ by completing the square.

9 Considering the definition of absolute value, solve $\frac{|x-3|}{3-x} = x$ where $x \neq 3$.

10 EXT1 Solve $(x - 4)(x - 1) \leq 28$.

11 Solve $x^{\frac{3}{2}} = \frac{1}{8}$.

12 Find the solutions of $x^2 - 2ax - b = 0$ by completing the square.

13 EXT1 Solve $\frac{y^2 - 5y + 2}{3y - 2} \geq y$.

14 Solve $3x^2 = 8(2x - 1)$ and write the solution in the simplest surd form.

15 Solve $|2x - 1| = 5 - x$ and check solutions.

COMBINATORICS

3. PERMUTATIONS AND COMBINATIONS

Probability is the study of how likely it is that something will happen. In order to calculate this, we need to find the total number of possible outcomes. In this Mathematics Extension 1 chapter, we will look at counting techniques using permutations and combinations to find the number of possible outcomes. You will learn about the pigeonhole principle and determining the number of possible arrangements or selections in a probability situation. You will also explore Pascal's triangle and its relevance to combinations and binomial products.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- EXT1 use factorials and other counting techniques to find numbers of arrangements
- EXT1 use the pigeonhole principle to solve problems
- EXT1 distinguish between permutations and combinations and use them to find numbers of arrangements and selections to calculate simple probabilities
- EXT1 identify the relationship between Pascal's triangle and binomial coefficients

EXT1 TERMINOLOGY

arrangements: Different ways of organising objects

binomial expansion: The algebraic expansion of powers of a binomial expression; for example, $(3x - 5)^7$

combinations: Arrangements of objects when order is not important

factorial: The product of n consecutive positive integers from n down to 1. For example, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

fundamental counting principle: If one event can occur in p ways and a second independent event can occur in q ways, then the two successive events can occur in $p \times q$ different ways

ordered selections: Selections that are taken in a particular position or order

permutations: Arrangements of objects when order is important

unordered selections: Selections that are made when the order of arrangements is not important or relevant

EXT1 3.01 Counting techniques

To find the probability of an event happening, we compare the number of ways the event can occur with the total number of possible outcomes (the sample space):

$$\text{Probability of an event} = \frac{\text{Number of ways the event can occur}}{\text{Total number of possible outcomes}}$$

The hardest part of calculating probabilities is finding all the possible outcomes. This can become quite difficult when the numbers of outcomes are large. There are some counting techniques that help in these cases.

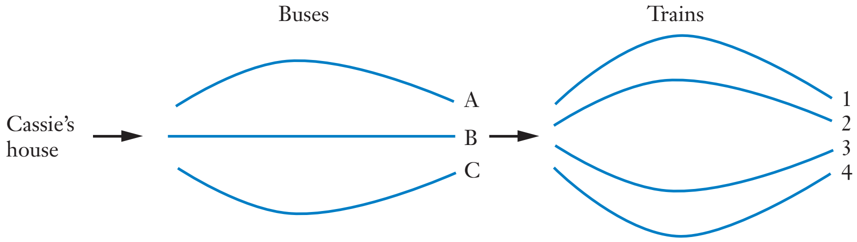
ISBN 97807043299

INVESTIGATION

COUNTING

1 Cassie needs to catch a bus and a train to work. There are 3 different buses she could catch into town. When she arrives in town, she needs to catch one of 4 trains to work.

If there are 3 buses and 4 trains possible for Cassie to catch, in how many ways is it possible for her to travel to work?



2 A restaurant offers 3 entrees, 4 main meals and 2 desserts. Every time Rick eats at the restaurant he chooses to eat a different combination of courses. How many times would he need to go to the restaurant to cover all possible combinations?

The **fundamental counting principle** comes from the product rule of probability, which you will study in detail in Chapter 9, *Probability*. The investigation above shows how it works.

For example, Cassie could travel by 3×4 or 12 different routes:

A1	A2	A3	A4
B1	B2	B3	B4
C1	C2	C3	C4

If one event can happen in p different ways and another event can happen in q different ways, then the 2 successive events can happen in pq different ways.

We can generalise even further to many events:

Fundamental counting principle

If one event can happen in a different ways, a second event in b different ways, a third event in c different ways and so on, then the successive events can happen in abc … different ways.

EXAMPLE 1

a The number plate on a car has 2 letters, followed by 4 numbers. How many different number plates of this type are possible?

b I have 12 pairs of earrings, 3 necklaces, 8 rings and 2 watches in my jewellery box.

i If I can wear any combination of earrings, necklaces, rings and watches, how many different sets of jewellery can I wear?

ii If my friend makes a guess at the combination of jewellery that I will wear, what is the probability that she will guess correctly?

c A restaurant serves 5 different types of entree, 12 main courses and 6 desserts.

i If I order any combination of entree, main course and dessert at random, how many different combinations are possible?

ii If my friend makes 3 guesses at which combination I will order, what is the probability that she will guess correctly?

Solution

a There are 26 letters and 10 numbers (0 to 9) possible for each position in the number plate. Using the fundamental counting principle:

$$\begin{aligned}\text{Total number} &= 26 \times 26 \times 10 \times 10 \times 10 \times 10\\ &= 26^2 \times 10^4\\ &= 6\ 760\ 000\end{aligned}$$

So 6 760 000 number plates are possible.

b **i** $$\begin{aligned}\text{Total number} &= 12 \times 3 \times 8 \times 2\\ &= 576\end{aligned}$$

ii The friend makes 1 guess and there are 576 possible outcomes.

$$P(\text{correct guess}) = \frac{1}{576}$$

c **i** $$\begin{aligned}\text{Total number of combinations} &= 5 \times 12 \times 6\\ &= 360\end{aligned}$$

ii The friend makes 3 guesses and there are 360 possible outcomes.

$$\begin{aligned}P(\text{correct guess}) &= \frac{3}{360}\\ &= \frac{1}{120}\end{aligned}$$

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Sometimes an outcome depends on what happens previously.

EXAMPLE 2

a To win a trifecta bet in a race, a person has to pick the horses that come first, second and third in the race, in the correct order. If a race has 9 horses, how many different trifecta bets are possible?

b A group of 15 people attend a concert and 3 of them are randomly chosen to receive a free backstage pass. The first person receives a gold pass, the second one a silver pass and the third one a bronze pass. In how many different ways can the passes be given out?

c In Lotto Strike, a machine contains 45 balls, each with a different number from 1 to 45. Players must guess the first 4 numbers to be drawn, in the correct order, to win first prize.

- **i** In how many ways can 4 balls be randomly drawn in order?
- **ii** Lisa has 3 entries in the same draw of Lotto Strike. What is the probability that she will win first prize?

Solution

a Any of the 9 horses could come first.

Any of the remaining 8 could come second.

Any of the remaining 7 horses could come third.

$$\text{Total ways} = 9 \times 8 \times 7 = 504$$

b Any of the 15 people can receive the first pass.

There are 14 people left who could receive the second pass.

Similarly there are 13 people who could receive the third pass.

$$\text{Total number of possibilities} = 15 \times 14 \times 13 = 2730$$

c **i** The first ball could be any of the 45 balls.

The second could be any of the remaining 44 balls and so on.

$$\text{The number of ways} = 45 \times 44 \times 43 \times 42 = 3\,575\,880$$

ii $$P(\text{first prize}) = \frac{3}{3\,575\,880} = \frac{1}{1\,191\,960}$$

EXT1 Exercise 3.01 Counting techniques

1 A password has 4 letters. How many passwords are possible?

2 A motorcycle number plate is made up of 2 letters followed by 2 numbers. How many number plates of this type are available?

3 A password can have up to 5 letters followed by 4 numbers. If I could use any letter of the alphabet or number, how many different passwords could be formed? Leave your answer in index form.

4 A witness saw most of the number plate on a getaway car except for the first letter and the last number. How many different cars do the police need to check in order to find this car?

5 A certain brand of computer has a serial number made up of 10 letters then 15 numbers. How many computers with this type of serial number can be made? Leave your answer in index form.

6 Victoria has postcodes starting with 3. How many different postcodes are available in Victoria?

7 A country town has telephone numbers starting with 63 followed by any 6 other numbers from 0 to 9. How many telephone numbers are possible in this town?

8 Jarred has 12 tops, 5 pairs of jeans and 5 pairs of shoes in his wardrobe. If he chooses a top, pair of jeans and shoes at random, how many combinations are possible?

9 A car manufacturer produces cars in 8 different colours, with either manual or automatic gear transmission, and 4 different types of wheels. How many different combinations can it produce?

10 A PIN has 4 numbers. If I forget my PIN I am allowed 3 tries to get it right. Find the probability that I get it within the 3 tries.

11 A restaurant offers 7 main courses and 4 desserts, as well as 3 different types of coffee.

- **a** How many different combinations of main course, dessert and coffee are possible?
- **b** Find the probability that I randomly pick the combination most often voted favourite.

12 A telephone number in a capital city can start with a 9 and has 8 digits altogether.

- **a** How many telephone numbers are possible?
- **b** If I forget the last 3 digits of my friend's telephone number, how many numbers would I have to try for the correct number?

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13 A company manufactures 20 000 000 computer chips. If it uses a serial number on each one consisting of 10 letters, will there be enough serial numbers for all these chips?

14 A password consists of 2 letters followed by 5 numbers. What is the probability that Izak randomly guesses the correct password?

15 A city has a population of 3 500 000. How many digits should its telephone numbers have so that every person can have one?

16 A manufacturer of computer parts puts a serial number on each part, consisting of 3 letters, 4 numbers then 4 letters. The number of parts sold is estimated as 5 million. Will there be enough combinations on this serial number to cope with these sales?

17 A bridal shop carries 12 different types of bridal dresses, 18 types of veils and 24 different types of shoes. If Kate chooses a combination of dress, veil and shoes at random, what is the probability that she chooses the same combination as her friend Yasmin?

18 Kate chooses a different coloured dress for each of her 3 bridesmaids. If the colours are randomly given to each bridesmaid, how many different possibilities are there?

19 In a computer car race game, the cars that come first, second and third are awarded at random. If there are 20 cars, how many possible combinations of first, second and third are there?

20 Jordan only has 4 different chocolates left and decides to randomly choose which of his 6 friends will receive one each. How many possible ways are there in which he can give the chocolates away?

21 Three different prizes are given away at a concert by taping them underneath random seats. If there are 200 people in the audience, in how many ways can these prizes be won?

22 There are 7 clients at a barber shop. If there are 3 barbers working, in how many ways could 3 clients be selected to have their haircut first?

23 A family of 5 people each choose a flavour of ice cream from vanilla, strawberry and chocolate. In how many ways can this happen?

24 A set of cards is numbered 1 to 100 and 2 chosen at random.

- **a** How many different arrangements of ordered pairs are possible?
- **b** What is the probability that a particular ordered pair is chosen?

25 Each of 10 cards has a letter written on it from A to J. If 3 cards are selected in order at random, find the probability that they spell out CAB.

EXT1 3.02 The pigeonhole principle

The **pigeonhole principle** is another useful counting technique.

Pigeonhole principle

If $n + 1$ or more pigeons are placed into n pigeonholes, then at least one pigeonhole must contain 2 or more pigeons.

Proof

Suppose there are n pigeonholes and only one pigeon in each hole.

Then the maximum number of pigeons is n.

But there are $n + 1$ pigeons, so the assumption that there is only one pigeon per hole is wrong.

Therefore at least one pigeonhole must contain 2 or more pigeons.

EXAMPLE 3

a A bag contains green, black, yellow, white, red and blue jellybeans. How many jellybeans must Keira take out of the bag so that she is sure to take 2 of the same colour?

b Is it certain that at least 2 students will have the same birthday in a school with 750 students?

c A paragraph contains 33 words. Show that there must be at least 2 words that start with the same letter.

d A square with side length 2 cm has 9 points drawn at random inside the square. Show that it is possible for 3 of these points to form a triangle with an area less than 1 cm^2.

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Solution

a There are 6 different colours. Using the pigeonhole principle, Keira must take out 7 jellybeans to be sure of taking 2 of the same colour (since the first 6 could be different colours).

b There are 366 possible birthdays, including leap years.

So there only need to be 367 people for at least 2 to have the same birthday.

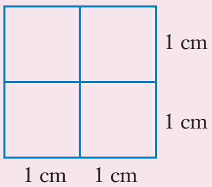
Since there are 750 students, which is more than 366, at least 2 students must share the same birthday.

c There are 26 letters in the alphabet.

So there only need to be 27 words so that at least 2 will start with the same letter.

Since the paragraph contains 33 words, which is more than 26, at least 2 must start with the same letter.

d Divide the square into 4 smaller squares with area 1 cm^2 as shown.



For 3 points to form a triangle with area less than 1 cm^2, they must lie within the same smaller square (since the triangle inside will have a smaller area than the square's area).

When placing 4 points inside the square, it is possible that each could lie in a different smaller square. Placing the next 4 points could also result in each being in a different smaller square. This means that now the smaller squares must have at least 2 points inside.

The next (9th) point must go into one of the 4 smaller squares, so even if there were only 2 points in each smaller square previously, now there must be 3 points in at least one of the smaller squares.

So it is possible to form a triangle from these 3 points (out of the 9 points) with an area less than 1 cm^2.

Generalised pigeonhole principle

If n pigeons are placed into k pigeonholes, where $n > k$, then at least one pigeonhole must contain at least $\frac{n}{k}$ pigeons.

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Proof

Suppose there is only one pigeon in each of the k pigeonholes.

Then the maximum number of pigeons is k.

But $n > k$, so the assumption of one pigeon per hole is wrong.

Therefore, there must be more than one pigeon in at least one hole.

The average number of pigeons per pigeonhole must be $\frac{\text{Number of pigeons}}{\text{Number of pigeonholes}} = \frac{n}{k}$.

Each pigeonhole will be below or above the average, or on the average.

So at least one pigeonhole must contain at least $\frac{n}{k}$ pigeons.

(If $\frac{n}{k}$ is not a whole number, then because some pigeonholes will be above the average we can round *up* to the next whole number.)

EXAMPLE 4

a A group of 75 people in a singing contest are placed into different audition rooms according to their category:

- males
- females
- children
- groups

If there are at least x people in one of the rooms, find the value of x.

b A group of 117 people rated a TV show from 1 to 5. Find r if there were at least r people who gave the same rating.

Solution

a There are 75 people and 4 rooms.

So $n = 75$ and $k = 4$.

$\frac{n}{k} = \frac{75}{4} = 18\frac{3}{4}$

$x \geq 18\frac{3}{4} = 19$ (as x is a whole number).

There are at least 19 people in at least one of the rooms.

> If $\frac{n}{k}$ is not a whole number, always round *up*.

b $n = 117$ and $k = 5$

$\frac{n}{k} = \frac{117}{5} = 23\frac{2}{5}$

$r \geq 23\frac{2}{5} = 24$ (as r is a whole number).

There were at least 24 people who gave the same rating.

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DID YOU KNOW?

The Dirichlet principle

The pigeonhole principle is also called the **Dirichlet principle**. The German mathematician Johann Dirichlet (1805–1859) was the first person to come up with this principle, in 1834. He was also involved in other branches of mathematics.

Research his other contributions and his place in the mathematics of the times.

EXT1 Exercise 3.02 The pigeonhole principle

1 A set of blocks contains red, blue, yellow and green blocks. How many blocks must Stevie choose at random to ensure that there are at least 2 blocks with the same colour?

2 A national committee is made up of members from NSW, Victoria, Queensland, South Australia and Tasmania. How many committee members are needed so that at least 2 of them must be from the same state?

3 A school has 9 different sports for its weekly sports afternoon. How many students would you need to survey to ensure that at least 2 of them are from the same sporting group?

4 A farm has 20 sheep, 20 cows and 20 pigs in a paddock. Show that if 4 animals escape from the paddock, at least 2 must be the same type of animal.

5 Show that if a wardrobe contains 8 pairs of black socks and 8 pairs of white socks, only 3 need to be chosen to find a pair of socks with the same colour.

6 Show that if you choose 5 cards from a deck of playing cards, then at least 2 must be the same suit.

7 Show that if eye colour can be described as blue, green, hazel or brown, then only 5 people need to be chosen for at least 2 to have the same eye colour.

8 A farmer picks 83 oranges and places them in barrels according to their size: small, medium, large and extra large. Find the value of x if at least one barrel has at least x oranges.

9 The long-term car park at the airport has 1024 cars, in sections labeled A, B, C, … M. Find the minimum number of cars parked in at least one of the sections.

10 A herd of 129 dairy cows are put into 3 pens: those too young to milk, those ready to be milked and those already milked. Find the minimum number of cows in at least one of the pens.

11 On New Year's Eve there were 9 different parks that were best for watching the fireworks. If there were 2495 people in these parks, find the minimum number of people in at least one of these parks.

12 The numbers 1 to 30 are divided by 7 and the remainder recorded. Find the value of x if at least x of the numbers have the same remainder.

13 There are n people placed in 8 levels of karate. If there are at least 29 people in at least one level, find the value of n.

14 In a survey of 450 people, there were at least 35 who preferred the same type of take-away food. How many different types of take-away foods were surveyed?

15 A group of friends split up into different groups, with some going to the cinema, some going to a concert and others going out to dinner. How many friends must you select so that at least 3 of them went to the same event?

WS
Facoial notation

EXT1 3.03 Factorial notation

Counting outcomes when repetition or replacement is allowed is straightforward, even when the numbers become very large.

EXAMPLE 5

A card is drawn at random from a set of 25 cards numbered 1 to 25. The card is then replaced before the next is selected. How many possible outcomes are there if 25 cards are chosen this way? Answer in scientific notation, correct to 3 significant figures.

Solution

Each time there is a card drawn, there are 25 possibilities.

$$\begin{aligned}\text{Total number} &= 25 \times 25 \times 25 \times \ldots \times 25 \text{ (25 times)}\\ &= 25^{25}\\ &\approx 8.88 \times 10^{34}\end{aligned}$$

When there is no repetition or replacement, the calculations can be long.

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EXAMPLE 6

A card is drawn at random from a set of 25 cards numbered 1 to 25. The card is not replaced before the next is drawn. How many possible outcomes are there if all 25 cards are drawn? Answer in scientific notation, correct to 3 significant figures.

Solution

For the first card, there are 25 possibilities.

For the second card, there are only 24 possibilities because one card has already been drawn.

For the third card, there are 23 possibilities, and so on.

$$\begin{aligned}\text{Total number of possible outcomes} &= 25 \times 24 \times 23 \times \ldots\ 3 \times 2 \times 1 \\ &\approx 1.55 \times 10^{25}\end{aligned}$$

The product of consecutive whole numbers $25 \times 24 \times 23 \times \ldots \times 3 \times 2 \times 1$ is called '25 factorial' and is written as '25!'.
Factorial notation allows us to write the number of possible outcomes when selecting all objects in order with no replacement or repetition.

Factorial notation

The number of ways of selecting n objects in order with no replacement or repetition is $n!$ (n factorial).

$$n! = n(n-1)(n-2)(n-3)(n-4)\ \ldots\ 3 \times 2 \times 1$$

Mathematicians find it convenient to define zero factorial as being equal to 1.

$$0! = 1$$

EXAMPLE 7

a Evaluate:

i 4! **ii** 7!

iii 25! (in scientific notation correct to 3 significant figures.)

b A group of 9 teenagers is waiting to be served in a café. They are each randomly assigned a number from 1 to 9.

i In how many ways is it possible for the numbers to be assigned?

ii One of the group needs to be served quickly because he has to leave. If he is given the first number, in how many ways is it possible for the numbers to be assigned?

Solution

a **i** $4! = 4 \times 3 \times 2 \times 1$

$= 24$

ii $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 5040$

iii

Operation	Casio scientific	Sharp scientific
Enter data	25 [x!] [=]	25 [2ndF] [x!] [=]

$25! \approx 1.55 \times 10^{25}$

b **i** The first number could be assigned 9 ways.

The second number could be assigned 8 ways and so on.

Total ways = 9!

Operation	Casio scientific	Sharp scientific
Enter data:	9 [x!] [=]	9 [2ndF] [x!] [=]

9! = 362 880

So there are 362 880 ways for the numbers to be assigned.

ii One of the group is given the first ticket. This can only happen in one way.

The second number could be assigned 8 ways, and so on.

Total ways $= 1 \times 8!$

$= 40\ 320$

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EXT1 Exercise 3.03 Factorial notation

1 Evaluate:

a $6!$ **b** $10!$ **c** $0!$ **d** $8! - 7!$ **e** $5 \times 4!$

f $\frac{7!}{4!}$ **g** $\frac{12!}{5!}$ **h** $\frac{13!}{4!9!}$ **i** $\frac{8!}{3!5!}$ **j** $\frac{11!}{4!7!}$

2 A group of 9 jockeys are each given a set of riding colours to wear. If these are given out randomly, how many different arrangements are possible?

3 Each of 6 people at a restaurant is given a different-coloured glass. How many possible combinations are there?

4 A mountain trail has room for only one person at a time. If 12 people are waiting at the bottom of the trail and are picked at random to start out, in how many ways can this happen?

5 A dog walker has 5 dogs and 5 leashes. In how many different ways is it possible to put a leash on each dog?

6 There are 11 actors in a play and each receives a script highlighting different parts.

a In how many different ways could the scripts be handed out?

b Anthony, the director, also needs a script. In how many ways could the scripts be handed out for the actors and the director?

7 A row of seats in a theatre seats 8 people. In how many ways could a group of 8 friends be seated at random in this row?

8 A group of 7 people line up to do karaoke. If they are each given a song at random to sing, how many possible outcomes are there?

9 A kindergarten class has a rabbit, a mouse and a parrot. Three children are selected to take these pets home for the holidays. If the pets are given out at random to these children, how many different ways are possible?

10 A group of 6 students are each given a different topic for a speech. In how many ways can the 6 topics be given to the 6 students?

11 In a chorus for a school musical, 7 students each wear a different mask. In how many different ways can the masks be worn by these students?

12 If 15 people play a game of Kelly pool, each person in turn chooses a number at random between 1 and 15. In how many different ways can this occur? Answer in scientific notation, correct to one decimal place.

13 **a** A school talent quest has 11 performers and each one is randomly given the order in which to perform. In how many ways can the order of performances be arranged?

b If one performer is chosen to perform first, in how many ways can the others be arranged?

14 A group of 6 friends sit in the same row at a concert.

a In how many different ways can they arrange themselves?

b If one friend must sit on the centre aisle, in how many ways can they be arranged?

15 A group of 8 friends go to a restaurant and sit at a round table. If the first person can sit anywhere, in how many ways can the others be arranged around the table?

16 In a pack of cards, the 4 aces are taken out and shuffled.

a What is the probability of picking out the ace of hearts at random?

b If all the aces are arranged in order, what is the probability of guessing the correct order?

17 At a wedding, each of the 12 tables is to have a centrepiece with a different coloured rose.

a In how many different ways can the roses be arranged at random?

b What is the probability that the bride will have a pink rose at her table?

18 In a maths exam, a student has to arrange 5 decimals in the correct order. If he has no idea how to do this and arranges them randomly, what is the probability that he makes the right guess for all the decimals?

19 In a car race, the fastest car is given pole position and the other cars are given their starting positions at random. If there are 14 cars altogether, in how many ways can this be arranged?

20 Show that:

a $\frac{8!}{4!} = 8 \times 7 \times 6 \times 5$

b $\frac{11!}{6!} = 11 \times 10 \times 9 \times 8 \times 7$

c $\frac{n!}{r!} = n(n-1)(n-2)(n-3) \ldots (r+1)$ where $n > r$

d $\frac{n!}{(n-r)!} = n(n-1)(n-2)(n-3) \ldots (n-r+1)$ where $n > r$

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EXT1 3.04 Permutations

Factorial notation is useful for writing the number of possible outcomes when arranging **all** objects in order without replacement. It becomes slightly more complex when we arrange only **some** of the objects in order without replacement.

WS Counting techniques

WS Permutation calculations

WS Permutations

EXAMPLE 8

13 cards are chosen at random from 20 cards without replacement. Find the possible number of ways the cards can be chosen.

Solution

The first card can be any of the 20 numbers.

The second card can be any of the remaining 19 numbers.

The third can be any of the remaining 18 numbers, and so on.

Then the number of ways the cards can be chosen $= 20 \times 19 \times 18 \times 17 \times \ldots \times 8$

$$\approx 4.8 \times 10^{14}$$

A **permutation** describes an **arrangement** or **ordered selection** of r objects from a total of n objects without replacement or repetition.

Permutations

The permutation ${}^nP_r$ is the number of ways of making ordered selections of r objects from a total of n objects.

$$\begin{aligned} {}^nP_r &= n \times (n-1) \times (n-2) \times (n-3) \times \ldots \qquad (r \text{ times}) \\ &= n(n-1)(n-2)(n-3) \times \ldots \times (n-r+1) \end{aligned}$$

$${}^nP_r = \frac{n}{(n-r)}$$

Proof

$$\begin{aligned} {}^nP_r &= n(n-1)(n-2)(n-3)\times\ldots\times(n-r+1) \\ &= n(n-1)(n-2)(n-3)\times\ldots\times(n-r+1)\times\frac{(n-r)(n-r-1)(n-r-2)\times\ldots\times3\times2\times1}{(n-r)(n-r-1)(n-r-2)\times\ldots\times3\times2\times1} \\ &= \frac{n(n-1)(n-2)(n-3)\times\ldots\times(n-r+1)(n-r)(n-r-1)(n-r-2)\times\ldots\times3\times2\times1}{(n-r)(n-r-1)(n-r-2)\times\ldots\times3\times2\times1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

A special case of this result is:

$$^nP_n = n!$$

Proof

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\therefore {}^nP_n = \frac{n!}{(n-n)!}$$

$$= \frac{n!}{0!}$$

$$= \frac{n!}{1}$$

$$= n!$$

EXAMPLE 9

a Evaluate $^9P_4$.

b **i** Find the number of arrangements of 3 digits that can be formed using the digits 0 to 9 if each digit can be used only once.

ii How many 3-digit numbers greater than 700 can be formed?

Solution

a

$$^9P_4 = \frac{9}{(9-4)}$$

$$= \frac{9}{5}$$

$$= \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1}$$

$$= 9\times8\times7\times6$$

$$= 3024$$

Operation	Casio scientific	Sharp scientific
Enter data	9 SHIFT ^nP_r 4 =	9 2ndF ^nP_r 4 =

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b **i** There are 10 digits from 0 to 9.

The first digit can be any of the 10 digits.

The second digit can be any of the remaining 9 digits.

The third digit can be any of the remaining 8 digits.

$$\text{Total permutations} = 10 \times 9 \times 8$$
$$= 720$$

or

$$^{10}P_3 = \frac{10}{(10-3)}$$
$$= \frac{10}{7}$$
$$= 720$$

ii The first digit must be 7 or 8 or 9 (3 possible digits).

The second digit can be any of the remaining 9 digits.

The third digit can be any of the remaining 8 digits.

$$\text{Total arrangements} = 3 \times 9 \times 8$$
$$= 216$$

Using permutations:

There are 3 ways to get the first digit.

The possible arrangements of the remaining 2 digits is $^9P_2$.

$$\text{Total arrangements} = 3 \times {}^9P_2$$
$$= 3 \times 72$$
$$= 216$$

WS
Cicula permutations

WS
Pemttios with repetitions

Permutations with restrictions

Some examples need very careful counting. As you saw in the above example, sometimes we can use permutations and sometimes factorials.

EXAMPLE 10

a **i** In how many ways can 6 people sit around a circular table?

ii If seating is random, find the probability that 3 particular people will sit together.

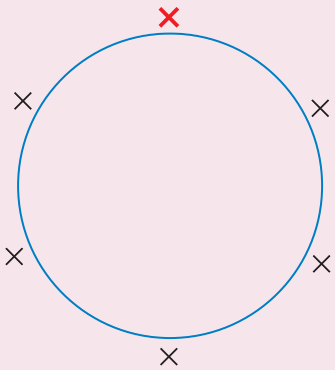
b In how many ways can the letters of the word EXCEPTIONAL be arranged?

Solution

a **i** The first person can sit anywhere around the table so we only need to arrange the other 5 people.

The second person can sit in any of the 5 remaining seats.

The third person can sit in any of the remaining 4 seats and so on.



$$\begin{aligned}\text{Total arrangements} &= 5! \\ &= 120\end{aligned}$$

ii The 3 people can sit anywhere around the table together in $3 \times 2 \times 1$ or $3!$ ways.

The remaining 3 people can sit together in $3!$ ways.

$$\begin{aligned}\text{Total arrangements} &= 3! \times 3! \\ &= 36\end{aligned}$$

$$\begin{aligned}P(\text{3 sit together}) &= \frac{36}{120} \\ &= \frac{3}{10}\end{aligned}$$

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b EXCEPTIONAL has 11 letters with the letter E repeated.

If each E was different, i.e. E_1 and E_2, then there would be 11! arrangements.

However, we cannot tell the difference between the 2 Es.

Since there are 2! ways of arranging the Es, then there are 2! arrangements of the word EXCEPTIONAL that are identical. We need to divide by 2! to eliminate these identical arrangements.

$$\text{Total arrangements} = \frac{11}{2}$$
$$= 19\,958\,400$$

Permutations involving repeated objects

The number of different ways of arranging n objects in which a of the objects are of one kind, b objects are of another kind, c of another kind and so on, is given by $\frac{n}{a!b!c}$ where $a + b + c + \ldots \leq n$.

EXAMPLE 11

Find the number of ways that the letters of the place name ULLADULLA can be arranged.

Solution

There are 9 letters, including 4 Ls, 2 As and 2 Us. There are 9! ways of arranging the letters, with 4! ways of arranging the Ls, 2! of arranging the As and 2! ways of arranging the Us.

$$\text{Total arrangements} = \frac{9}{4\ 2\ 2}$$
$$= 3780$$

There are different ways of working out the number of arrangements. Sometimes it is just a matter of drawing a diagram or counting carefully.

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Permutations

EXAMPLE 12

A bag contains 5 balls of different colours – red, yellow, blue, green and white. In how many ways can these 5 balls be arranged:

a with no restrictions

b if the yellow ball must be first

c if the first ball must not be red or white

d if blue and green must be together

e if red, blue and green must be together?

Solution

a The first can be any of the 5 balls.

The second can be any of the remaining 4 balls and so on.

$$\begin{aligned}\text{Total arrangements} &= 5! \\ &= 120\end{aligned}$$

b The first ball must be yellow, so there is only 1 way of arranging this.

The second ball can be any of the remaining 4 balls.

The third ball can be any of the remaining 3 balls and so on.

$$\begin{aligned}\text{Total arrangements} &= 4! \\ &= 24\end{aligned}$$

c The first ball could be yellow, blue or green so there are 3 possible arrangements.

The second ball could be any of the remaining 4 balls and so on.

$$\begin{aligned}\text{Total arrangements} &= 3 \times 4! \\ &= 72\end{aligned}$$

d When two objects must be together, we treat them as a single object with 2! possible arrangements.

So we arrange 4 balls in 4! ways: R, Y, BG and W.

But there are 2! ways in which to arrange the blue and green balls.

$$\begin{aligned}\text{Total arrangements} &= 4! \times 2! \\ &= 48\end{aligned}$$

e When three objects are together, we treat them as a single object with 3! possible arrangements.

We are then arranging 3 balls in 3! ways: RBG, Y, W.

But there are 3! ways in which to arrange the red, blue and green balls.

$$\begin{aligned}\text{Total arrangements} &= 3! \times 3! \\ &= 36\end{aligned}$$

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EXT1 Exercise 3.04 Permutations

1 Evaluate each permutation.

a ${}^6P_3$ **b** ${}^5P_2$ **c** ${}^8P_3$ **d** ${}^{10}P_7$ **e** ${}^9P_6$

f ${}^7P_5$ **g** ${}^8P_6$ **h** ${}^{11}P_8$ **i** ${}^9P_1$ **j** ${}^6P_6$

2 A set of 26 cards, each with a different letter of the alphabet, is placed into a hat and cards drawn out at random without replacement. Find the number of 'words' possible if selecting:

a 2 cards **b** 3 cards

c 4 cards **d** 5 cards.

3 A random 3-digit number is made from cards containing the numbers 0 to 9.

a In how many ways can this be done if the cards cannot be used more than once and 0 cannot be the first number?

b How many numbers over 400 can be made?

c How many numbers less than 300 can be made?

4 A set of 5 cards, each with a number from 1 to 5 on it, is placed in a box and 2 drawn out at random. Find the possible number of combinations:

a altogether **b** of numbers greater than 50

c of odd numbers **d** of even numbers.

5 a How many arrangements of the letters A, B, C and D are possible if no letter can be used twice?

b How many arrangements of any 3 of these letters are possible?

6 A 4 digit number is to be selected at random from the numbers 0 to 9 with a non-zero first digit and no repetition.

a How many arrangements can there be?

b How many arrangements of numbers over 6000 are there?

c How many arrangements of numbers less than 8000 are there?

7 The numbers 1, 2, 3, 4 and 5 are arranged in a line. How many arrangements are possible if:

a there is no restriction **b** the number is less than 30 000

c the number is greater than 20 000 **d** the number is odd

e any 3 numbers are selected at random?

8 There are 12 swimmers in a race.

a In how many ways could they finish?

b In how many ways could they come in first, second and third?

9 How many different ordered arrangements can be made from the word COMPUTER with:

a 2 letters? **b** 3 letters? **c** 4 letters?

10 How many different ordered arrangements can be made from these words?

a CENTIPEDE **b** ALGEBRA **c** TELEVISION
d ANTARCTICA **e** DONOR **f** BASKETBALL
g GREEDY **h** DUTIFUL **i** MANUFACTURER
j AEROPLANE

11 A group of friends queue in a straight line outside a night club. Find how many ways the friends can be arranged if there are:

a 4 friends **b** 7 friends **c** 8 friends
d 10 friends **e** 11 friends.

12 A group of friends go into a restaurant and are seated around a circular table. Find how many arrangements are possible if there are:

a 4 friends **b** 7 friends **c** 8 friends
d 10 friends **e** 11 friends.

13 A string necklace contains a circle of beads, but each possible arrangement of beads can also be worn back-to-front (flipped). Find the number of different arrangements possible with:

a 10 beads **b** 12 beads **c** 9 beads
d 11 beads **e** 13 beads.

14 In how many ways can a group of 6 people be arranged:

a in a line? **b** in a circle?

15 Find how many different ways a group of 9 people can be arranged in:

a a line **b** a circle.

16 In how many ways can a set of 10 beads be arranged:

a in a line?
b in a circle around the edge of a poster?
c on a bracelet?

17 **a** How many different arrangements can be made from the jack, queen, king and ace of hearts?

b If I choose 2 of these cards at random, how many different arrangements could I make?

c If I choose 3 of these cards at random, how many different arrangements could I make?

18 A group of 7 people sit around a table. In how many ways can they be arranged:

a with no restrictions? **b** if 2 people want to sit together?
c if 2 people cannot sit together? **d** if 3 people sit together?

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19 A group of 5 boys and 5 girls line up outside a cinema. In how many ways can they be arranged:

a with no restriction?

b if a particular girl stands in line first?

c if boys and girls alternate (with either a girl or boy in first place)?

20 Find the probability that if 10 people sit around a table at random, 2 particular people will be seated together.

21 A bookshelf is to hold 5 mathematics books, 8 novels and 7 cookbooks.

a In how many different ways could they be arranged? (Leave your answer in factorial notation.)

b If the books are grouped in categories, in how many ways can they be arranged? (Answer in factorial notation.)

c If one book is chosen at random, find the probability that it is a cookbook.

22 a How many different arrangements can be made from the numbers 3, 4, 4, 5 and 6?

b How many arrangements form numbers greater than 40 000?

c How many form numbers less than 50 000?

d If an arrangement is made at random, find the probability that it is less than 40 000.

23 Find the probability that an arrangement of the word LAPTOP will start with T.

24 What is the probability that, if a 3-letter 'word' is formed from the letters of PHYSICAL at random, it will be CAL?

25 A minibus has 6 forward-facing and 2 backward-facing seats. If 8 people use the bus, in how many ways can they be seated:

a with no restrictions

b if one person must sit in a forward-facing seat

c if 2 people must sit in a forward-facing seat?

26 If 3 letters of the word VALUED are selected at random, find the number of possible arrangements if:

a the first letter is D

b the first letter is a vowel.

27 The letters of the word THEORY are arranged randomly. Find the number of arrangements:

a with no restrictions

b if the E is at the beginning

c if the first letter is a consonant and the last letter is a vowel.

28 Find the number of arrangements possible if x people are:

- **a** in a straight line
- **b** in a circle
- **c** in a circle with 2 people together
- **d** in a straight line with 3 people together
- **e** in a circle with 2 people not together.

29 **a** Use factorial notation to show that $\frac{{}^8P_3}{3!} = \frac{{}^8P_5}{5!}$.

b Prove that $\frac{{}^nP_r}{r!} = \frac{{}^nP_{n-r}}{(n-r)!}$.

30 Prove that ${}^{n+1}P_r = {}^nP_r + r\,{}^nP_{r-1}$.

EXT1 3.05 Combinations

WS Combinaion calculations

WS Combinaion

WS Pemtiios and combinations

The permutation ${}^nP_r$ is the number of arrangements possible for an ordered selection of r objects from a total of n objects.

When the order is not important, for example when AB is the same as BA, the number of arrangements is called a **combination**.

A combination describes an **unordered selection** of r objects from a total of n objects without replacement or repetition.

EXAMPLE 13

a A committee of 2 is chosen from Scott, Rachel and Frankie. In how many ways can this be done?

b There are 3 vacancies on a school council and 8 people who are available. If the vacancies are filled at random, in how many ways can this happen?

Solution

a Number of ordered arrangements $= {}^3P_2$

$= 6$

However, a committee of Scott and Rachel is the same as a committee of Rachel and Scott. This is the same for all other arrangements of the committee. There are 2! ways of arranging each committee of 2 people.

To get the number of unordered arrangements, we divide the number of ordered arrangements by 2!

Total arrangements $= \frac{{}^3P_2}{2}$

$= 3$

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b Number of ordered arrangements = ${}^{8}P_{3}$

However, order is not necessary here, since the 3 vacancies filled by, say, Henry, Amie and Wade, would be the same in any order.

There are 3! different ways of arranging Henry, Amie and Wade.

$$\text{So, total arrangements} = \frac{{}^{8}P_{3}}{3} = 56$$

Combinations

The combination ${}^{n}C_{r}$ is the number of ways of making unordered selections of r objects from a total of n objects.

$$
\begin{aligned}
{}^{n}C_{r} &= \frac{{}^{n}P_{r}}{r} \\
&= \frac{n}{(n-r)!r!}
\end{aligned}
$$

Proof

${}^{n}P_{r}$ is the ordered selection of r objects from n objects.

There are $r!$ ways of arranging r objects.

If order is unimportant, the unordered selection of r objects from n is given by $\frac{{}^{n}P_{r}}{r!}$.

$$
\begin{aligned}
\frac{{}^{n}P_{r}}{r!} &= \frac{\frac{n!}{(n-r)!}}{r!} \\
&= \frac{n!}{(n-r)!} \times \frac{1}{r!} \\
&= \frac{n!}{(n-r)!r!}
\end{aligned}
$$

${}^{n}C_{r}$ can also be written as $\binom{n}{r}$.

EXAMPLE 14

a A bag contains 3 white and 2 black counters labelled W_1 W_2 W_3 and B_1, B_2. If 2 counters are drawn out of the bag in how many ways can this happen if order is not important ?

b If 12 coins are tossed, find the number of ways of tossing 7 tails.

c **i** A committee of 5 people is formed at random from a group of 15 students. In how many different ways can the committee be formed?

ii If the group consists of 9 senior and 6 junior students, in how many ways can the committee be formed if it is to have 3 senior and 2 junior students in it?

d A team of 6 men and 5 women is chosen at random from a group of 10 men and 9 women. If Kaye and Greg both hope to be chosen in the team, find the probability that:

i both will be chosen **ii** neither will be chosen.

Solution

a Possible arrangements (unordered) are:

W_1W_2 W_2W_3 W_3B_1 B_1B_2

W_1W_3 W_2B_1 W_3B_2

W_1B_1 W_2B_2

W_1B_2

There are 10 different combinations.

Using combinations, the number of different arrangements of choosing 2 counters from 5 is ${}^5C_2$.

$$\begin{aligned} {}^5C_2 &= \frac{5}{(5-2)\,2} \\ &= \frac{5}{3\ 2} \\ &= 10 \end{aligned}$$

b The order is not important.

There are ${}^{12}C_7$ ways of tossing 7 tails from 12 coins.

$$\begin{aligned} {}^{12}C_7 &= \frac{12}{(12-7)\ 7} \\ &= \frac{12}{5\ 7} \\ &= 792 \end{aligned}$$

Operation	Casio scientific	Sharp scientific
Enter data	12 SHIFT ^nC_r 7 =	12 2ndF ^nC_r 7 =

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c **i** The order of the committee is not important.

$$\text{Number of arrangements} = \binom{15}{5}$$

$$= 3003$$

Operation	Casio scientific	Sharp scientific
Enter data	15 SHIFT ^nC_r 5 =	15 2ndF ^nC_r 5 =

ii 3 senior students can be chosen in $\binom{9}{3}$ or 84 ways.

2 junior students can be chosen in $\binom{6}{2}$ or 15 ways.

$$\text{Total number of arrangements} = \binom{9}{3} \times \binom{6}{2}$$

$$= 84 \times 15$$

$$= 1260$$

d **i** The number of possible teams $= {}^{10}C_6 \times {}^9C_5$

$$= 210 \times 126$$

$$= 26\ 460$$

For Kaye to be chosen, then 4 out of the other 8 women will be chosen i.e. ${}^8C_4$.

For Greg to be chosen, 5 out of the other 9 men will be chosen i.e. ${}^9C_5$.

Number of combinations $= {}^8C_4 \times {}^9C_5$

$$= 70 \times 126$$

$$= 8820$$

$$\text{Probability} = \frac{8820}{26\ 460}$$

$$= \frac{1}{3}$$

ii For Kaye and Greg not to be included, then 5 out of the other 8 women and 6 out of the other 9 men will be chosen.

Number of combinations $= {}^8C_5 \times {}^9C_6$

$$= 56 \times 84$$

$$= 4704$$

$$\text{Probability} = \frac{4704}{26\ 460}$$

$$= \frac{8}{45}$$

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EXT1 Exercise 3.05 Combinations

1 Evaluate:

a $\binom{9}{5}$ **b** $\binom{12}{7}$ **c** $\binom{8}{3}$ **d** ${}^{10}C_4$ **e** ${}^{11}C_5$

2 **a** Evaluate:

i ${}^{10}C_0$ **ii** ${}^{7}C_0$ **iii** $\binom{14}{0}$ **v** ${}^{9}C_9$ **v** $\binom{11}{11}$

b Hence evaluate:

i ${}^{n}C_0$ **ii** ${}^{n}C_n$

3 Find the number of different ways that a random committee of 6 people can be made from a group of:

a 8 people **b** 9 people **c** 11 people

d 15 people **e** 20 people.

4 **a** A set of 3 red cards and 3 blue cards is placed in a box. By naming the red cards R_1, R_2 and R_3 and the blue cards B_1, B_2 and B_3, list the number of different arrangements possible when 2 cards are drawn out at random, with order not important. How many arrangements are possible?

b If there are 10 red and 10 blue cards and 7 are drawn out at random, how many different combinations are possible?

5 A coin is tossed 20 times. How many different arrangements are there for tossing 5 heads?

6 A set of 10 marbles are placed in a bag and 6 selected at random. In how many different ways can this happen?

7 In poker, 5 cards are dealt from a pack of 52 playing cards. How many different arrangements are possible?

8 Three cards are drawn at random from a set of 10 cards with the numbers 0 to 9 on them. How many different arrangements are possible if order is:

a important **b** unimportant?

9 A debating team of 3 is chosen from a class of 14 students. In how many ways can the team be selected if order is:

a important **b** unimportant?

10 A bag contains 23 lollies. If I take 6 lollies out of the bag, how many different combinations are possible?

11 A team of 4 players is chosen at random from a group of 20 tennis players to play an exhibition match. In how many ways could the team be chosen?

12 A group of 3 students is chosen at random from a class of 27 to go on a student representative council. In how many different ways could this be done?

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13 A board of 8 people is chosen from a membership of 35. How many different combinations are possible?

14 A basketball team of 5 players is selected at random from a group of 12 PE students.

a In how many ways can the team be selected?

b Find the probability that Erik is selected as one of the team members.

c Find the probability that Erik and Jens are both selected.

15 A committee of 6 people is to be selected at random from a group of 11 men and 12 women. Find the number of possible committees if:

a there is no restriction on who is on the committee

b all committee members are to be male

c all committee members are to be female

d there are to be 3 men and 3 women

e Anna is included

f Bruce is not included

g there are to be 4 women and 2 men.

16 A horse race has 15 horses competing. At the TAB, a quinella pays out on the horses that come in first and second, in either order. Ryan decides to bet on all possible combinations of quinellas. If it costs him $1 a bet, how much does he pay?

17 A group of 25 students consist of 11 who play a musical instrument and 14 who do not. Find the number of different arrangements possible if a group of 9 students is selected at random:

a with no restriction

b who all play musical instruments

c where 5 play musical instruments

d where 2 do not play musical instruments.

18 A set of cards consists of 8 yellow and 7 red cards, each showing a different picture.

a If 10 cards are selected at random, find the number of different arrangements possible.

b If 8 cards are selected, find the number of arrangements of selecting:

i 4 yellow cards

ii 6 yellow cards

iii 7 yellow cards

v 5 red cards.

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19 Ten cards are selected at random from a set of 52 playing cards. Find the number of combinations selected if:

a there are no restrictions (answer in scientific notation correct to 3 significant figures)

b they are all hearts

c there are 7 hearts

d they are all red cards

e there are 4 aces.

20 An animal refuge has 17 dogs and 21 cats. If a nursing home orders 12 animals at random to be companion animals, find the number of ways that the order would have:

a 7 dogs **b** 9 dogs **c** 10 dogs

d 4 cats **e** 6 cats.

21 There are 8 white, 9 red and 5 blue marbles in a bag and 7 are drawn out at random. Find the number of arrangements possible:

a with no restriction

b if all marbles are red

c if there are 3 white and 2 red marbles

d if there are 4 red and 1 blue marbles

e if there are 4 white and 2 blue marbles.

22 Out of a group of 25 students, 7 walk to school, 12 catch a train and 6 catch a bus. If 6 students are selected, find the number of combinations if:

a all walk to school

b no-one catches a bus

c 3 walk to school and 1 catches a bus

d 1 walks to school and 4 catch a train

e 3 catch a train and 1 catches a bus.

23 At a karaoke night, a group of 14 friends decide that 4 of them will sing a song together. Of the friends, 5 have previously sung this song before. In how many ways can they do this if they select:

a friends who have all sung the song previously

b 2 of the friends who sang the song previously

c none of the friends who sang the song previously?

24 **a** Evaluate ${}^{12}C_5$.

b Evaluate ${}^{12}C_7$.

c By using factorial notation, show why ${}^{12}C_5 = {}^{12}C_7$.

25 By evaluating both sides, show that ${}^{9}C_6 = {}^{8}C_6 + {}^{8}C_5$.

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26 Show that $\binom{13}{7} = \binom{13}{6}$.

27 Show that $\binom{10}{4} = \binom{9}{4} + \binom{9}{3}$.

28 Prove that $\binom{n}{r} = \binom{n}{n-r}$.

29 Prove that ${}^nP_r = r!\,{}^nC_r$.

30 Prove that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

EXT1 3.06 Pascal's triangle and binomial coefficients

WS Pacals triangle

WS Pacals triangle

The binomial expansion

Pascal's triangle

Here is Pascal's triangle. Can you see the patterns in each row and between rows? Could you add the next row using these patterns?

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

INVESTIGATION

COMBINATIONS AND PASCAL'S TRIANGLE

There is a relationship between Pascal's triangle and combinations ${}^nC_r$.

Find ${}^1C_0$, ${}^1C_1$, ${}^2C_0$, ${}^2C_1$, ${}^2C_2$, ${}^3C_0$, ${}^3C_1$, ${}^3C_2$, ${}^3C_3$ and so on. Compare these with Pascal's triangle. Can you see any relationships or patterns?

The numbers in Pascal's triangle are the same as combinations as shown below.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$\begin{array}{ccccccccc} & & & & {}^0C_0 & & & & \\ & & & {}^1C_0 & & {}^1C_1 & & & \\ & & {}^2C_0 & & {}^2C_1 & & {}^2C_2 & & \\ & {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 & \\ {}^4C_0 & & {}^4C_1 & & {}^4C_2 & & {}^4C_3 & & {}^4C_4 \end{array}$$

You can use combinations to complete further rows in Pascal's triangle.

EXAMPLE 15

Add the next 2 lines to Pascal's triangle.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Solution

The 1st row after this is ${}^5C_0 = 1, {}^5C_1 = 5, {}^5C_2 = 10, {}^5C_3 = 10, {}^5C_4 = 5, {}^5C_5 = 1$

The 2nd row after this is ${}^6C_0 = 1, {}^6C_1 = 6, {}^6C_2 = 15, {}^6C_3 = 20, {}^6C_4 = 15, {}^6C_5 = 6, {}^6C_6 = 1$

So Pascal's triangle becomes:

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

ISBN 97807043299

There is a relationship between Pascal's triangle and the expansions of binomial products $(x+y)^n$.

EXAMPLE 16

Expand each binomial product:

a $(x+y)^0$ **b** $(x+y)^1$ **c** $(x+y)^2$ **d** $(x+y)^3$ **e** $(x+y)^4$

Solution

a $(x+y)^0 = 1$ since $a^0 = 1$

b $(x+y)^1 = x+y$ since $a^1 = a$

c $(x+y)^2 = x^2 + 2xy + y^2$ perfect square

d
$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)^2\\ &= (x+y)(x^2+2xy+y^2)\\ &= x^3+2x^2y+xy^2+x^2y+2xy^2+y^3\\ &= x^3+3x^2y+3xy^2+y^3\end{aligned}$$

e
$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)^3\\ &= (x+y)(x^3+3x^2y+3xy^2+y^3)\\ &= x^4+3x^3y+3x^2y^2+xy^3+x^3y+3x^2y^2+3xy^3+y^4\\ &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\end{aligned}$$

Look at the coefficients of each of the **binomial expansions** in the example.
The coefficients of each term in these binomial expansions form Pascal's triangle.

$(x+y)^0$					1				
$(x+y)^1$				1		1			
$(x+y)^2$			1		2		1		
$(x+y)^3$		1		3		3		1	
$(x+y)^4$	1		4		6		4		1

ISBN 97807043299

Since the numbers in Pascal's triangle are also combinations, this means that the coefficients in the binomial expansion of $(x + y)^n$ can be written as combinations.

Binomial expansion

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + {}^nC_3x^{n-3}y^3 + \ldots + {}^nC_kx^{n-k}y^k + \ldots + {}^nC_ny^n$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \quad + \binom{n}{k}x^{n-k}y^k + \quad + \binom{n}{n}y^n$$

EXAMPLE 17

Use ${}^nC_r$ to expand each binomial product.

a $(p + q)^2$ **b** $(x + y)^3$ **c** $(a + b)^5$

Solution

a $(p+q)^2 = {}^2C_0p^2q^0 + {}^2C_1p^1q^1 + {}^2C_2p^0q^2$

$= 1p^2 + 2pq + 1q^2$

$= p^2 + 2pq + q^2$

This agrees with the formula for a perfect square

b $(x+y)^3 = {}^3C_0x^3y^0 + {}^3C_1x^2y^1 + {}^3C_2x^1y^2 + {}^3C_3x^0y^3$

$= 1x^3 + 3x^2y + 3xy^2 + 1y^3$

$= x^3 + 3x^2y + 3xy^2 + y^3$

c $(a+b)^5 = {}^5C_0a^5 + {}^5C_1a^4b + {}^5C_2a^3b^2 + {}^5C_3a^2b^3 + {}^5C_4ab^4 + {}^5C_5b^5$

$= 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

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We can use this formula to expand other binomial products.

EXAMPLE 18

a Expand $(4 + x)^5$.

b Expand $(3x - 2)^4$.

c If $(1 + \sqrt{3})^4 = a + b\sqrt{3}$, evaluate a and b.

Solution

a
$$\begin{aligned}(4 + x)^5 &= {}^5C_0 4^5 + {}^5C_1 4^4 x + {}^5C_2 4^3 x^2 + {}^5C_3 4^2 x^3 + {}^5C_4 4x^4 + {}^5C_5 x^5 \\ &= 1(1024) + 5(256)x + 10(64)x^2 + 10(16)x^3 + 5(4)x^4 + (1)x^5 \\ &= 1024 + 1280x + 640x^2 + 160x^3 + 20x^4 + x^5\end{aligned}$$

b
$$\begin{aligned}(3x - 2)^4 &= {}^4C_0(3x)^4 + {}^4C_1(3x)^3(-2) + {}^4C_2(3x)^2(-2)^2 + {}^4C_3(3x)(-2)^3 + {}^4C_4(-2)^4 \\ &= 1(81x^4) + 4(27x^3)(-2) + 6(9x^2)(4) + 4(3x)(-8) + (1)16 \\ &= 81x^4 - 216x^3 + 216x^2 - 96x + 16\end{aligned}$$

c Expand $(1 + \sqrt{3})^4$.

$$\begin{aligned}(1 + \sqrt{3})^4 &= {}^4C_0 1^4 + {}^4C_1 1^3(\sqrt{3}) + {}^4C_2 1^2(\sqrt{3})^2 + {}^4C_3 1^1(\sqrt{3})^3 + {}^4C_4(\sqrt{3})^4 \\ &= 1 + 4\sqrt{3} + 6(\sqrt{3})^2 + 4(\sqrt{3})^3 + (\sqrt{3})^4 \\ &= 1 + 4\sqrt{3} + 6\sqrt{9} + 4\sqrt{27} + \sqrt{81} \\ &= 1 + 4\sqrt{3} + 18 + 4 \times 3\sqrt{3} + 9 \\ &= 28 + 16\sqrt{3}\end{aligned}$$

So $a = 28$ and $b = 16$.

Properties of coefficients

The patterns and symmetry of Pascal's triangle show some properties of combinations and binomial expansions. We can prove these by using the definition ${}^nC_r = \frac{n!}{r!(n-r)!}$ that you saw in the previous secton.

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Since the first and last values in each row of Pascal's triangle are 1, we have the property:

$$^nC_0 = {}^nC_n = 1$$

Proof

$$^nC_r = \frac{n!}{r!(n-r)!}$$

$$\begin{aligned} ^nC_n &= \frac{n!}{n!(n-n)!} \\ &= \frac{n!}{n!0!} \\ &= \frac{n!}{n!} \\ &= 1 \end{aligned}$$

$$\begin{aligned} ^nC_0 &= \frac{n!}{0!(n-0)!} \\ &= \frac{n!}{0!n!} \\ &= 1 \end{aligned}$$

By symmetry of Pascal's triangle:

$$^nC_k = {}^nC_{n-k}$$

Proof

$$^nC_k = \frac{n!}{k!(n-k)!}$$

$$\begin{aligned} ^nC_{n-k} &= \frac{n!}{(n-k)!(n-[n-k])!} \\ &= \frac{n!}{(n-k)!k!} \end{aligned}$$

$$\therefore {}^nC_k = {}^nC_{n-k}$$

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Since each number in Pascal's triangle is the sum of the 2 numbers in the row above it:

Pascal's triangle identity

$$^{n}C_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$$

Proof

$$^{n}C_k = \frac{n!}{k!(n-k)!}$$

$$\begin{aligned}
{}^{n-1}C_{k-1} + {}^{n-1}C_k &= \frac{(n-1)!}{(k-1)!([n-1]-[k-1])!} + \frac{(n-1)!}{k!([n-1]-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{k(n-1)!}{k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{(n-k)k!(n-k-1)!} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{(n-1)!(k+n-k)}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!}
\end{aligned}$$

$$\therefore\ ^{n}C_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$$

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EXAMPLE 19

Show that:

a $^8C_3 = {^8C_5}$

b $\binom{7}{4} = \binom{6}{3} + \binom{6}{4}$

Solution

a $^8C_3 = \frac{8}{(8-3)\ 3} = \frac{8}{5\ 3}$

$^8C_5 = \frac{8}{(8-5)\ 5} = \frac{8}{3\ 5}$

So $^8C_3 = {^8C_5}$.

b LHS $= \binom{7}{4} = \frac{7}{(7-4)\ 4} = \frac{7}{3\ 4}$

RHS $= \binom{6}{3} + \binom{6}{4}$

$= \frac{6}{(6-3)\ 3} + \frac{6}{(6-4)\ 4}$

$= \frac{6}{3\ 3} + \frac{6}{2\ 4}$

$= \frac{6\ \times 4}{3\ 3 \times 4} + \frac{6\ \times 3}{2\ 4\ \times 3}$

$= \frac{6\ \times 4}{4\ 3} + \frac{6\ \times 3}{4\ 3}$

$= \frac{4(6!) + 3(6\ )}{4\ 3}$

$= \frac{7(6\ )}{4\ 3}$

$= \frac{7}{4\ 3}$

$=$ LHS

So $\binom{7}{4} = \binom{6}{3} + \binom{6}{4}$.

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INVESTIGATION

FIBONACCI AND OTHER SEQUENCES

Pascal's triangle is not the only interesting pattern of numbers that has practical uses. The Fibonacci sequence is also very interesting and can be seen in nature. Research Fibonacci and the golden ratio.

Discover how the number phi (ϕ) is related to Fibonacci, trigonometry and the number π.

Can you find any other interesting number patterns or sequences?

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EXT1 Exercise 3.06 Pascal's triangle and binomial coefficients

1 Show that:

a $\binom{9}{5}=\binom{9}{4}$ **b** ${}^7C_2={}^7C_5$ **c** $\binom{12}{5}=\binom{12}{7}$

2 Prove that:

a ${}^7C_5={}^6C_4+{}^6C_5$ **b** $\binom{10}{6}=\binom{9}{5}+\binom{9}{6}$ **c** $\binom{7}{3}=\binom{6}{2}+\binom{6}{3}$

3 Show that $\binom{n}{1}=\binom{n}{n-1}$.

4 Use the symmetry of Pascal's triangle to find x if ${}^7C_x={}^7C_2$.

5 If $\binom{12}{3}=\binom{12}{y}$, use the symmetry of Pascal's triangle to find y.

6 Find the value of a if ${}^{11}C_a={}^{11}C_8$.

7 Use Pascal's triangle identity to find n if $\binom{n}{6}=\binom{10}{5}+\binom{10}{6}$.

8 Use Pascal's triangle identity to find k if ${}^{20}C_7={}^{19}C_k+{}^{19}C_7$.

9 Expand each binomial product.

a $(a+x)^4$ **b** $(a+x)^6$ **c** $(a+x)^5$

d $(2a+1)^3$ **e** $(x-2)^7$ **f** $(4x^2+3)^4$

g $(3-2x)^6$ **h** $(4a-5b)^3$ **i** $(2+3m)^5$

j $(1-2x)^8$

10 Expand:

a $\left(\sqrt{2}+1\right)^5$ **b** $\left(\sqrt{3}-1\right)^6$ **c** $\left(\sqrt{3}+\sqrt{5}\right)^4$

d $\left(3+\frac{x}{2}\right)^4$ **e** $\left(x+\frac{1}{x}\right)^5$ **f** $\left(1-\frac{x}{2}\right)^3$

g $\left(\frac{a}{3}-\frac{b}{2}\right)^3$

11 Evaluate a and b if $\left(\sqrt{2}+3\right)^3=a+b\sqrt{2}$.

12 If $\left(2-\sqrt{5}\right)^4=a+b\sqrt{5}$, evaluate a and b.

13 Evaluate x and y if $\left(\sqrt{3}-1\right)^5=x+\sqrt{y}$.

14 If $\left(\sqrt{2}+\sqrt{3}\right)^3=a\sqrt{2}+b\sqrt{3}$, evaluate a and b.

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3. TEST YOURSELF

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

1 A group of 4 people sit together in a bus with 2 seats facing forwards and 2 facing backwards. If one person cannot sit facing backwards, in how many ways can the 4 people be arranged?

A 36 **B** 24 **C** 12 **D** 48

2 Which one of these formulas is correct?

A ${}^{n}C_k = {}^{n}C_{k-1}$

B ${}^{n}C_k = {}^{n+1}C_{k+1} + {}^{n+1}C_k$

C ${}^{n-1}C_k = {}^{n}C_{k-1} + {}^{n}C_k$

D ${}^{n}C_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$

3 Find the smallest number of balls chosen from a bag containing yellow, white, blue, black and green balls so that 2 must be the same colour.

A 5 **B** 6 **C** 7 **D** 4

4 Expand $(x - 3)^5$.

5 12 people are to be seated around a table.

a In how many ways can they be seated?

b In how many ways can they be seated if 2 particular people are not to sit together?

c Find the probability that 2 friends will be seated together.

6 Show that:

a ${}^{11}C_3 = {}^{11}C_8$

b ${}^{10}C_1 = {}^{10}C_9$

c $\binom{9}{7} = \binom{8}{6} + \binom{8}{7}$

d $\binom{11}{6} = \binom{10}{5} + \binom{10}{6}$

7 A committee of 5 people is chosen at random from a group of 10 women and 12 men. Find the number of ways in which the committee could be formed with:

a no restriction on how many men or women are on the committee

b a committee of 2 men and 3 women.

8 Expand $(2x + 3y)^4$.

9 Any one person has up to 150 000 hairs on their head. A city has a population of 256 840. Show that there are at least 2 people in this city that have exactly the same number of hairs on their head.

10 In how many ways can the letters of the word AUSTRALIA be arranged?

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11 Find the number of ways of arranging 7 people:

a in a straight line **b** in a circle.

12 Expand:

a $(\sqrt{5}-2)^6$ **b** $\left(x-\frac{3}{x^2}\right)^4$ **c** $(2-3\sqrt{2})^7$

13 A set of 10 cards, numbered 1 to 10, is placed into a box and 3 drawn out at random. Find the number of arrangements possible if order is:

a important **b** unimportant.

14 A group of 10 boys and 16 girls are on a school excursion. Five of them are chosen at random to help the teacher check the rolls. Find the number of ways these students could be selected if:

a there is no restriction

b 3 are to be girls

c all are to be boys.

15 If $(\sqrt{5}-\sqrt{3})^5 = a\sqrt{5}+b\sqrt{3}$, evaluate a and b.

16 In how many ways can 3-letter combinations be selected at random from the word RANDOM?

17 How many different arrangements are there of the letters in the word:

a PERMUTATION **b** COMBINATION **c** FACTORIAL

d PROBABILITY **e** SELECTION?

18 A batch of 750 bricks is sorted into 7 different categories according to their colour, density and other properties. Find the smallest number of bricks that will be in at least one of the categories.

19 Evaluate:

a $\binom{6}{4}$ **b** ${}^9P_7$

20 A serial number is made up of 4 letters and 2 numbers. If 0 is not allowed, find how many serial numbers are possible.

21 **a** Evaluate 0!

b Show that $\binom{n}{0}=\binom{n}{n}$.

22 Show that if 33 points are drawn inside a square with sides 4 cm, at least 3 of these points can join to give a triangle with an area less than 1 cm^2.

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3. CHALLENGE EXERCISE

1 A bag contains 8 cards, each with a different number from 1 to 8. If you select 5 numbers at random, show that at least 2 of the numbers add up to 9.

2 If a computer randomly generates 4-letter 'words' from the letters in MATHEMATICS, find the probability that the word made is CAME.

3 Simplify $\frac{{}^{n}C_{k}}{{}^{n}C_{k-}}$.

4 Numbers are formed from the digits 1, 2, 3, 3 and 7 at random.

a In how many ways can they be arranged with no restrictions?

b In how many ways can they be arranged to form a number greater than 30 000?

5 A charm bracelet has 6 different charms on it. In how many ways can the charms be arranged if the bracelet:

a has a clasp? **b** has no clasp?

6 A management committee is made up of 5 athletes and 3 managers. If the committee is formed from a group of 20 athletes and 10 managers at random, find:

a the number of different ways in which the committee could be formed

b the probability that Patrick, an athlete, is included

c the probability that both Patrick and his sister Alexis, who is a manager, are included

d the probability that Patrick and Alexis are excluded from the committee.

7 A group of n people sit around a circular table.

a In how many ways can they be arranged?

b How many arrangements are possible if k people sit together?

8 By writing 0.99 as 1 – 0.01 and expanding $(1 - 0.01)^3$, evaluate 0.99^3 to 4 decimal places.

9 An equilateral triangle has sides 3 cm. If 10 points are randomly drawn inside the triangle, show that there are at least 2 points whose distance apart is less than or equal to 1 cm.

Practice set 1

In Questions **1** to **12**, select the correct answer **A**, **B**, **C** or **D**.

1 Write $\dfrac{1}{3\sqrt{(x-2)^5}}$ in index form.

A $(x-2)^{-\frac{5}{3}}$ **B** $\dfrac{(x-2)^{-\frac{5}{2}}}{3}$ **C** $3(x-2)^{-\frac{5}{2}}$ **D** $\dfrac{1}{(x-2)^{\frac{5}{3}}}$

2 Simplify $\dfrac{(2a^3b)^3}{(ab)^2}$.

A $8a^7b$ **B** $8a^8b$ **C** $2a^7b$ **D** $2a^8b$

3 Evaluate $4^{-\frac{3}{2}}$.

A -8 **B** $\dfrac{1}{8}$ **C** $\dfrac{1}{6}$ **D** -6

4 Simplify $\dfrac{a^2-6a+9}{a^2-9}$.

A $\dfrac{1}{a+3}$ **B** $\dfrac{a-3}{a+3}$

C $\dfrac{a+3}{a-3}$ **D** $\dfrac{-6a+9}{a-9}$

5 Factorise $a^2-\dfrac{b^2}{4}$.

A $\left(a-\dfrac{b}{2}\right)^2$ **B** $\left(a+\dfrac{b}{4}\right)\left(a-\dfrac{b}{4}\right)$

C $\left(a+\dfrac{b}{2}\right)^2$ **D** $\left(a+\dfrac{b}{2}\right)\left(a-\dfrac{b}{2}\right)$

6 The solution to $x^2+2x-6=0$ is:

A $x=-1\pm 2\sqrt{7}$ **B** $x=\dfrac{2\pm\sqrt{28}}{2}$

C $x=\dfrac{-2\pm\sqrt{-20}}{2}$ **D** $x=-1\pm\sqrt{7}$

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7 EXT1 The solution to the equation $\frac{3}{x+5} \leq 2$ is:

A $x < -5, x > -3\frac{1}{2}$ **B** $-5 < x \leq -3\frac{1}{2}$

C $x < -5, x \geq -3\frac{1}{2}$ **D** $3\frac{1}{2} \leq x < 5$

8 EXT1 What is the number of possible outcomes when arranging the letters of the word LITERATURE?

A $\frac{10!}{2!2!}$ **B** $\frac{10!}{2!2!2!}$ **C** $10!$ **D** $\frac{10!}{3!}$

9 EXT1 The number of possible different PINs with a combination of 4 numbers and 2 letters is:

A 4 435 236 **B** 6 760 000 **C** 1 000 000 **D** 10 676

10 EXT1 The number of possible seating positions for 12 people sitting at a round table is:

A ${}^{12}C_{11}$ **B** $11!$ **C** $12!$ **D** ${}^{12}P_{11}$

11 EXT1 Combination ${}^{n}C_{r}$ is equal to:

A $(n-r)!{}^{n}P_{r}$ **B** $\frac{{}^{n}P_{r}}{(n-r)!}$ **C** $r!{}^{n}P_{r}$ **D** $\frac{{}^{n}P_{r}}{r!}$

12 EXT1 The binomial expansion of $(x+3)^4$ is:

A $x^4 + 12x^3 + 54x^2 + 108x + 81$ **B** $x^4 + 4x^3 + 18x^2 + 12x + 3$

C $x^4 + 3x^3 + 9x^2 + 27x + 81$ **D** $x^4 + 4x^3 + 6x^2 + 4x + 1$

13 Solve:

a $3x - 7 = 23$ **b** $5(b-3) = 15$ **c** $\frac{x}{3} + 4 = 5$

d $4y - 7 = 3y + 9$ **e** $8z + 1 = 11z - 17$ **f** $2^x = 32$

g $9^{y-1} = 3$ **h** $x^2 - 3x = 0$ **i** $|x+2| = 5$

j $|5a - 2| = 8$

14 Solve for p: $\frac{p-3}{2} - \frac{p+1}{5} = 1$.

15 Simplify $2\sqrt{12}$.

16 EXT1 Find the number of ways of seating 10 people around a table at random:

a if 3 people are to sit together

b if 2 people must not sit together.

17 EXT1 A batch of 2300 spare parts for cars was placed on 11 different shelves. Find the smallest number of spare parts that were placed on at least one shelf.

18 Factorise fully: $10x + 2xy - 10y - 2y^2$.

19 Write in index form:

a $\frac{1}{x}$ **b** $\sqrt[3]{x^4}$

20 Simplify the expression $8y - 2(y + 5)$.

21 Rationalise the denominator of $\frac{5}{5-\sqrt{2}}$.

22 EXT1 In how many different ways can a committee of 4 people be selected from a group of 9 people?

23 EXT1 A team of 3 boys and 5 girls is chosen at random from a class of 12 boys and 18 girls. In how many ways can this be done?

24 Solve $2x^2 - 3x - 1 = 0$ correct to 3 significant figures.

25 EXT1 How many committees of 5 people could be formed randomly from a meeting of 20 people?

26 Simplify $\frac{x+1}{5} \div \frac{x^2-2x-3}{10}$.

27 Evaluate $(3.9)^4$ correct to 1 decimal place.

28 Simplify $2\sqrt{3} - \sqrt{27}$.

29 EXT1 Find the probability that if 12 people sit around a table at random, 3 particular friends will be seated together.

30 EXT1 In how many ways can 4 different letters be selected from the word TRIGONOMETRY?

31 Expand and simplify $(x - 3)(x^2 + 5x - 1)$.

32 EXT1 Show that:

a ${}^{10}C_4 = {}^{10}C_6$ **b** $\binom{5}{2} = \binom{4}{1} + \binom{4}{2}$

33 Expand and simplify $\sqrt{2}(3\sqrt{5} - 2\sqrt{2})$.

34 Simplify $\frac{2x+6}{2}$.

35 Solve $4a - 5 < 7a + 4$.

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36 EXT1 Evaluate:

a $5!$ **b** ${}^8C_6$

37 The radius r of a circle with area A is given by $r = \sqrt{\frac{A}{\pi}}$. Find r, correct to 2 decimal places, if $A = 7.59$.

38 EXT1 A store sells T-shirts in 7 different sizes. How many T-shirts need to be selected so that 2 must be the same size?

39 Solve each set of simultaneous equations.

a $3a - b = 7$ and $2a + b = 8$

b $a + b - c = 8$, $b + c = 5$ and $a + 2c = 3$

40 Solve $5 - 2x < 3$ and show the solution on a number line.

41 Solve the equation $x^2 - 4x + 1 = 0$, giving exact solutions in simplest surd form.

42 Write 7^{-2} as a rational number.

43 Solve the simultaneous equations $y = 3x - 1$ and $y = x^2 - 5$.

44 EXT1 Expand:

a $(3x + y)^5$ **b** $(\sqrt{2} - 3)^4$

45 EXT1 Evaluate a and b if $(2\sqrt{3} + \sqrt{2})^3 = a\sqrt{3} + b\sqrt{2}$.

46 Find integers x and y such that $\frac{\sqrt{3}}{2\sqrt{3} + 3} = x + y\sqrt{3}$.

47 Evaluate $|-2|^2 - |-1| + |4|$.

48 Factorise $8x^2 - 32$.

49 Rationalise the denominator of $\frac{2\sqrt{3}}{3\sqrt{5} - \sqrt{2}}$.

50 Simplify $2|-4| - |3| + |-2|$.

51 Rationalise the denominator of $\frac{\sqrt{5} + 1}{2\sqrt{2} + 3}$.

52 Simplify $\frac{(a^{-4})^3 \times b^6}{a^9 \times (b^{-1})^4}$.

53 Evaluate $4^{-\frac{3}{2}}$ as a rational number.

54 EXT1 A committee of 5 people is to be formed from a group of 12 women and 9 men. Find the number of ways of forming the committee if:

a there are no restrictions

b there are to be 3 women and 2 men on the committee:

  i with no restrictions

  ii if Sue will be on the committee.

55 Simplify $2(x - 5) - 3(x - 1)$.

56 Solve $4^{2x+1} = 8$.

57 Write $\frac{1}{x+3}$ in index form.

58 Find the value of a^3b^{-2} in index form if $a = \left(\frac{1}{2}\right)^3$ and $b = \left(\frac{4}{5}\right)^2$.

59 Write $(3x+2)^{-\frac{1}{2}}$ without an index.

60 Simplify:

a $8x - 7y - y + 4x$

b $\sqrt{124}$

c $\frac{x^2 - 9}{2x^2 + 5x - 3}$

d $\frac{1}{\sqrt{2}+1} + \frac{2}{\sqrt{2}-1}$

e $\frac{3}{x+1} + \frac{2}{x^2-1} - \frac{4}{x-1}$

f $x - \frac{1}{x}$ when $x = 2\sqrt{3}$

g $\frac{(x^{-2})^5 y^4 z^{-3}}{x^4 (y^3)^{-1} (z^{-4})^{-2}}$

h $\frac{a+b}{5a - 20ab^2} \div \frac{a^2 + 2ab + b^2}{3 - 6b}$

i $8\sqrt{5} - 3\sqrt{20} + 2\sqrt{45}$

j $\frac{a^3 b^2 (c^4)^2}{(a^2)^2 bc^5}$ if $a = \left(\frac{1}{2}\right)^2$, $b = \left(\frac{2}{3}\right)^3$ and $c = \left(\frac{4}{9}\right)^{-1}$.

61 The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$. Find the exact radius r if the volume V is $10\frac{2}{3}$ cm^3.

62 Find the value of k if $(2x + 5)^2 = 4x^2 + kx + 25$.

63 Simplify $\sqrt{81x^2y^3}$.

64 Factorise:

a $5(a - 2)^2 + 40(a - 2)$

b $(2a - b + c)^2 - (a + 5b - c)^2$

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65 Solve $-2 \le \frac{8x-1}{5} < 9$.

66 Simplify $\frac{x+1}{5} - \frac{x+2}{3}$.

67 Solve $x^2 - 5x = 0$.

68 Solve $x^2 - 5x - 1 = 0$ and write the solutions correct to 2 decimal places.

69 Simplify $\sqrt{8} + \sqrt{98}$.

70 Write $\frac{3}{x^2+5x} - \frac{4}{x} + \frac{2}{x+5}$ as a single fraction.

71 Solve for x: $4^{2x-1} = \frac{1}{8}$.

72 Factorise:

a $x^2 - 2x - 8$ **b** $a^2 - 9$ **c** $y^2 + 6y + 9$

d $t^2 + 8t + 16$ **e** $3x^2 - 11x + 6$

73 EXT1 Solve each inequality:

a $a^2 - 1 < 0$ **b** $y^2 + 3y \ge 0$ **c** $y^2 - y - 2 \le 0$

d $x^2 > 9$ **e** $2d - d^2 \ge 0$

74 EXT1 Solve each inequality:

a $|5x - 9| > 21$ **b** $|3x - 7| < 2$

75 EXT1 Solve each inequality:

a $\frac{5}{x} > 1$ **b** $\frac{x}{x-2} \le 3$ **c** $\frac{3}{x-4} < 5$

d $\frac{-1}{2x-1} \le 4$ **e** $\frac{x}{x-2} > 3$

76 EXT1 Expand $(2x - 3)^4$.

77 EXT1 How many different 11-letter 'words' can be made at random from the word MISSISSIPPI?

78 EXT1 Solve each inequality:

a $m^2 - 5m + 6 \ge 0$ **b** $x^2 - 4 > 0$ **c** $p^2 - p < 0$

79 Solve:

a $5x - 4 = 2x + 11$ **b** $y^2 - 2y - 13 = 0$ (correct to 2 decimal places)

c $4^{2x} = 8$ **d** $|2b + 3| = 7$

e EXT1 $m^2 \le 9$ **f** EXT1 $|5n - 1| > 9$

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4.

FUNCTIONS

Functions and their graphs are used in many areas such as mathematics, science and economics. In this chapter you will explore what functions are and how to sketch some types of graphs, including straight lines, parabolas and cubics.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- understand the definition of a function and use function notation
- test a function using the vertical line test
- identify a one-to-one function using the horizontal line test
- find the domain and range of functions including composite functions using interval notation
- identify even and odd functions
- understand a linear function, its graph and properties, including the gradient and axes intercepts
- graph situations involving direct linear variation
- find the equation of a line, including parallel and perpendicular lines
- identify a quadratic function, its graph and properties, including its axis of symmetry, turning point and axes intercepts
- solve quadratic equations and use the discriminant to identify the numbers and types of solutions
- find the quadratic equation of a parabola
- EXT1 solve quadratic inequalities
- identify a cubic function, its graph and properties, including the shape, horizontal point of inflection and axes intercepts
- find a cubic equation
- identify a polynomial and its characteristics
- draw the graph of a polynomial showing intercepts
- solve simultaneous equations involving linear and quadratic equations, both algebraically and graphically, and solve problems involving intersection of graphs of functions (for example, break-even points)

TERMINOLOGY

angle of inclination The angle a straight line makes with the positive x-axis measured anticlockwise

axis of symmetry A line that divides a shape into halves that are mirror-images of each other

break-even point The point at which a business' income equals its costs, making neither a profit nor a loss

coefficient A constant multiplied by a pronumeral in an algebraic term. For example, in ax^3 the a is the coefficient

constant term The term in a polynomial function that is independent of x

cubic function A function with x^3 as its highest power or degree

degree The highest power of x in a polynomial

dependent variable A variable whose value depends on another (independent) variable, such as y (depending on x)

direct variation A relationship between two variables such that as one variable increases so does the other, or as one variable decreases so does the other. One variable is a multiple of the other, with equation $y = kx$. Also called **direct proportion**

discriminant The expression $b^2 - 4ac$ that shows how many roots the quadratic equation $ax^2 + bx + c = 0$ has

domain The set of all possible values of x for a function or relation; the set of 'input' values

even function A function $f(x)$ that has the property $f(-x) = f(x)$; its graph is symmetrical about the y-axis

function A relation where every x value in the domain has a unique y value in the range

gradient The steepness of a graph at a point on the graph, measured by the ratio $\frac{\text{rise}}{\text{run}}$; or the change in y values as x values change

horizontal line test A test that checks if a function is one-to-one, whereby any horizontal line drawn on the graph of a function should cut the graph at most once. If the horizontal line cuts the graph more than once, it is not one-to-one

independent variable A variable whose value does not depend on another variable; for example, x in $y = f(x)$

intercepts The values where a graph cuts the x- and y- axes

interval notation A notation that represents an interval by writing its endpoints in square brackets [ ] when they are included and in parentheses ( ) when they are not included

leading coefficient The coefficient of the highest power of x. For example, $2x^4 - x^3 + 3x + 1$ has a leading coefficient of 2

leading term The term with the highest power of x. For example, $2x^4 - x^3 + 3x + 1$ has a leading term of $2x^4$

linear function A function with x as its highest power or degree

monic polynomial A polynomial whose leading coefficient is 1

odd function A function $f(x)$ that has the property $f(-x) = -f(x)$; its graph has point symmetry about the origin (0, 0)

one-to-one function A function in which every y value in the range corresponds to exactly one x value in the domain

parabola The graph of a quadratic function

piecewise function A function that has different functions defined on different intervals

point of inflection A point on a curve where the concavity changes, such as the turning point on the graph of a cubic function

polynomial An expression in the form $P(x) = a_nx^n + \ldots + a_2x^2 + a_1x + a_0$ where n is a positive integer or zero

quadratic function A function with x^2 as the highest power of x

range The set of all possible y values of a function or relation; the set of 'output' values

root A solution of an equation

turning point Where a graph changes from increasing to decreasing or vice versa; sometimes a turning point (horizontal inflection) where concavity changes

vertex A turning point

vertical line test A test that checks if a relation is a function, whereby any vertical line drawn on the graph of a relation should cut the graph at most once. If the vertical line cuts the graph more than once, it is not a function

zero An x value of a function or polynomial for which the y value is zero, that is, $f(x) = 0$

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4.01 Functions

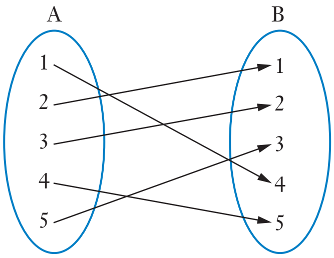
WS
Funcions and relations

A **relation** is a set of **ordered pairs** (x, y) where the **variables** x and y are related according to some pattern or rule. The x is called the **independent variable** and the y is called the **dependent variable** because the value of y depends on the value of x. We usually choose a value of x and use it to find the corresponding value of y.

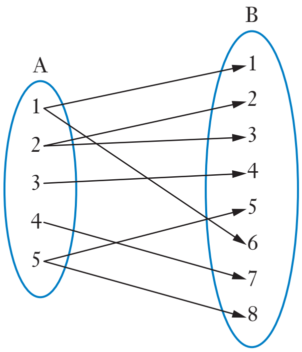
A relation can also be described as a mapping between 2 sets of numbers, with the set of x values, A, on the left and the set of y values B, on the right.

Types of relations

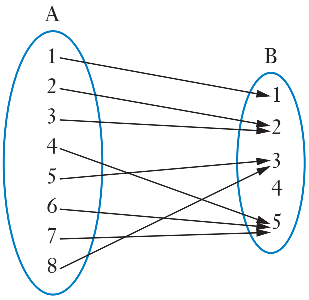
A **one-to-one** relation is a mapping where every element of A corresponds with exactly one element of B and every element of B corresponds with exactly one element of A. Each element has its own unique match.



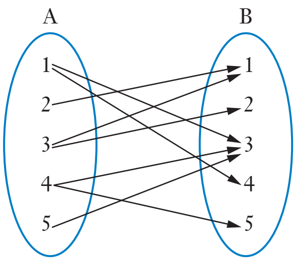
A **one-to-many** relation is a mapping where an element of A corresponds with 2 or more elements of B. For example, 5 in set A matches with 5 and 8 in set B.



A **many-to-one** relation is a mapping where 2 or more elements of A correspond with the same one element of B. For example, 4, 6 and 7 in set A match with 5 in set B.



A **many-to-many** relation is a mapping where 2 or more elements of A correspond with 2 or more elements of B. This is a combination of the one-to-many and many-to-one relations.



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Function

A **function** is a special type of relation where for every value of x there is a unique value of y.

The **domain** is the set of all values of x for which a function is defined.

The **range** is the set of all values of y as x varies.

A function could be a one-to-one or many-to-one relation.

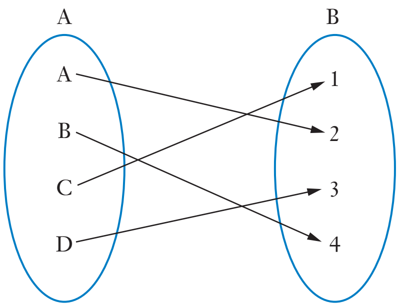
For example, this table matches a group of people with their eye colours.

Person	Anne	Jacquie	Donna	Hien	Marco	Russell	Trang
Colour	Blue	Brown	Grey	Brown	Green	Brown	Brown

The ordered pairs are (Anne, Blue), (Jacquie, Brown), (Donna, Grey), (Hien, Brown), (Marco, Green), (Russell, Brown) and (Trang, Brown).

This table represents a function, since for every person there is a unique eye colour. The domain is the set of people; the range is the set of eye colours. It is a many-to-one function since more than one person can correspond to one eye colour.

Here is a different function:



Set A is the domain, set B is the range.

The ordered pairs are (A, 2), (B, 4), (C, 1) and (D, 3).

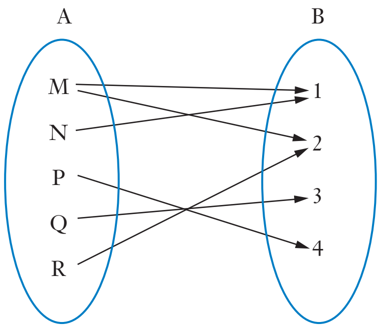
It is a function because every x value in A corresponds to exactly one x value in B.

It is a one-to-one function because every y value in B corresponds to exactly one x value in A.

Here is an example of a relation that is **not** a function. Can you see why?

In this example the ordered pairs are (M, 1), (M, 2), (N, 1), (P, 4), (Q, 3) and (R, 2).

Notice that M corresponds to 2 values in set B: 1 and 2. This means that it is **not** a function. Notice also that M and R both correspond with the same value 2. This is a many-to-many relation.



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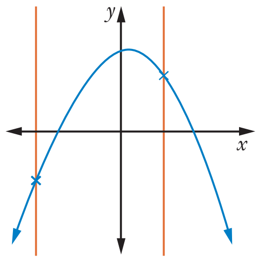
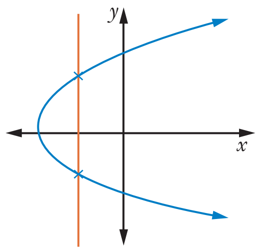
The vertical line test

Relations can also be described by algebraic rules or equations such as $y = x^2 + 1$ and $x^2 + y^2 = 4$, and hence graphed on a number plane. There is a very simple test called the **vertical line test** to test if a graph represents a function.

If any vertical line crosses a graph at only one point, the graph represents a function. This shows that, for every value of x, there is only one value of y.



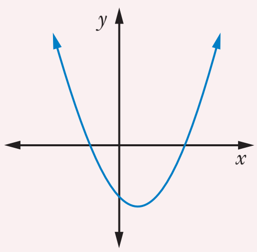
If any vertical line crosses a graph at more than one point, the graph does not represent a function. This shows that, for some value of x, there is more than one value of y.



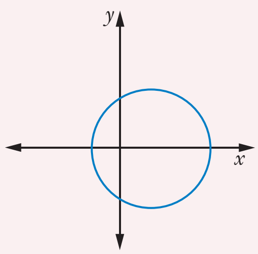
EXAMPLE 1

Does each graph or set of ordered pairs represent a function?

a

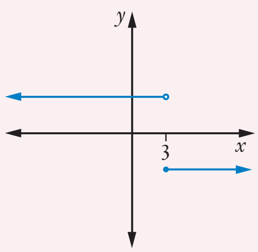


b



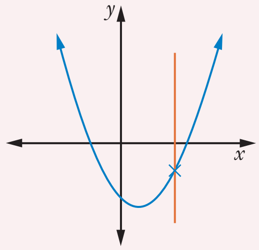
c (–2, 3), (–1, 4), (0, 5), (1, 3), (2, 4)

d



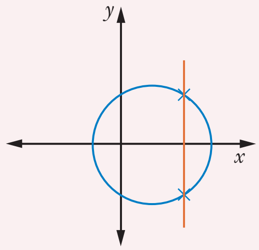
Solution

a



A vertical line only cuts the graph once. So the graph represents a function.

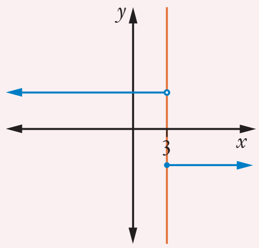
b



A vertical line can cut the curve in more than one place. So the circle does not represent a function.

c For each x value there is only one y value, so this set of ordered pairs is a function.

d



The open circle at $x = 3$ on the top line means that $x = 3$ is not included, while the closed circle on the bottom line means that $x = 3$ is included on this line.

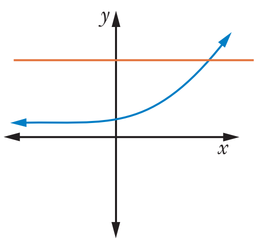
So a vertical line only touches the graph once at $x = 3$.

The graph represents a function.

The horizontal line test

The **horizontal line test** is used on the graph of a function to test whether the function is **one-to-one**.

If any horizontal line crosses a graph at only one point, there is only one x value for every y value. The graph represents a one-to-one function.



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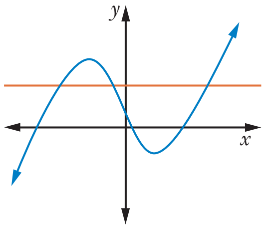
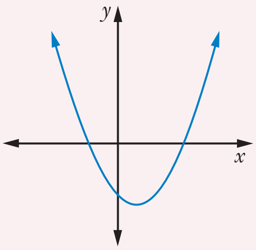
If any horizontal line crosses a graph at more than one point, this means that there are 2 or more x values that have the same y value. The graph does not represent a one-to-one function.



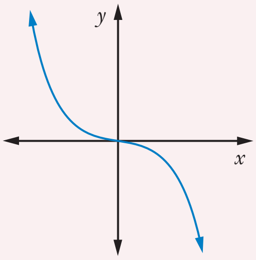
EXAMPLE 2

Does each graph represent a one-to-one function?

a

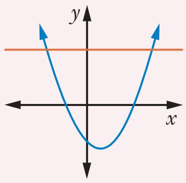


b

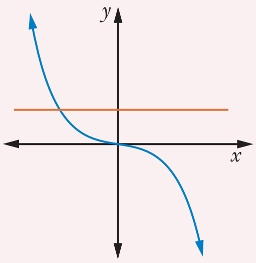


Solution

a A horizontal line cuts the curve in more than one place. The function is not one-to-one.



b A horizontal line cuts the curve in only one place. The function is one-to-one.



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DID YOU KNOW?

René Descartes

The number plane is called the **Cartesian plane** after René Descartes (1596–1650). Descartes used the number plane to develop analytical geometry. He discovered that any equation with two unknown variables can be represented by a line. The points in the number plane can be called Cartesian coordinates.

Descartes used letters at the beginning of the alphabet to stand for numbers that are known, and letters near the end of the alphabet for unknown numbers. This is why we still use x and y so often!

Research Descartes to find out more about his life and work.

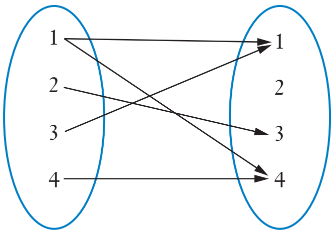
Exercise 4.01 Functions

1 List the ordered pairs for each relation, then state whether the relation is a one-to-one, one-to-many, many-to-one or many-to-many.

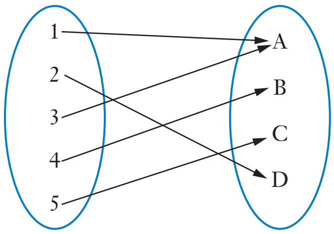
a

Name	Wade	Scott	Geoff	Deng	Mila	Stevie
Hair colour	Black	Blond	Grey	Black	Brown	Blond

b



c



d

x	3	5	8	9	5	8
y	5	±2	−7	3	6	0

e

x	y
1	9
2	15
3	27
4	33
5	45

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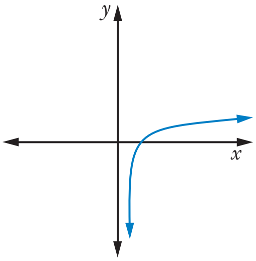
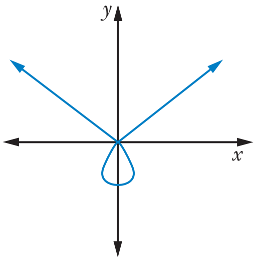
2 Does each graph or set of ordered pairs represent a function? If it does, state whether it is one-to-one.

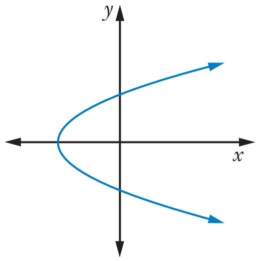
a



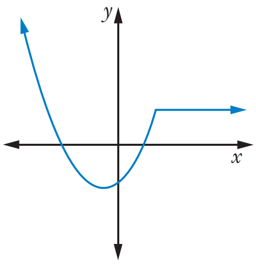
b



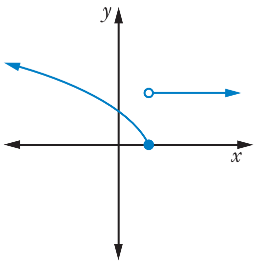
c



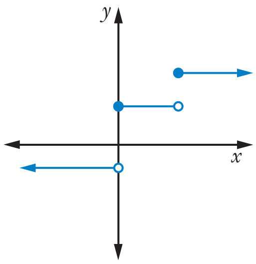
d



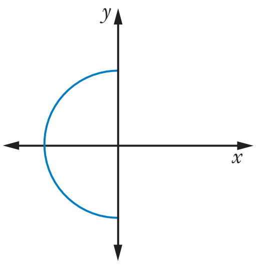
e



f



g



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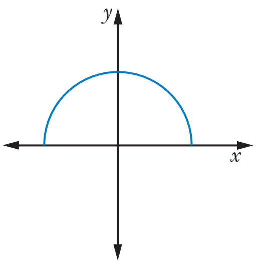
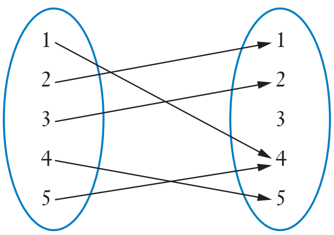


i (1, 3), (2, −1), (3, 3), (4, 0)

j (1, 3), (2, −1), (2, 7), (4, 0)

k



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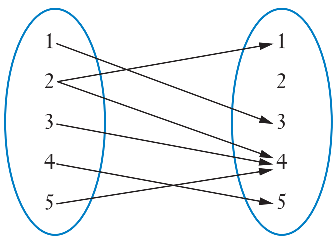

m (2, 5), (3, −1), (4, 0), (−1, 3), (−2, 7)

n

Person	Ben	Paula	Pierre	Hamish	Jacob	Leanne	Pierre	Lien
Sport	Tennis	Football	Tennis	Football	Football	Badminton	Football	Badminton

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o

A	3
B	4
C	7
D	3
E	5
F	7
G	4

3 A relation consists of the ordered pairs (–3, 4), (–1, 5), (0, –2), (1, 4) and (6, 8).

a Write the set of independent variables, x.

b Write the set of dependent variables, y.

c Describe the relation as one-to-one, one-to-many, many-to-one or many-to-many.

d Is the relation a function?

WS

Function notation

4.02 Function notation

Since the value of y depends on the value of x, we say that y is a function of x. We write this using **function notation** as $y = f(x)$.

EXAMPLE 3

a Find the value of y when $x = 3$ in the equation $y = 2x - 1$.

b Evaluate $f(3)$, given $f(x) = 2x - 1$.

Solution

a When $x = 3$:

$$y = 2(3) - 1$$
$$= 6 - 1$$
$$= 5$$

b $f(x) = 2x - 1$

$$f(3) = 2(3) - 1$$
$$= 6 - 1$$
$$= 5$$

Both questions in Example 3 are the same, but the second one looks different because it uses function notation.

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EXAMPLE 4

a If $f(x) = x^2 + 3x + 1$, find $f(-2)$.

b If $f(x) = x^3 - x^2$, find the value of $f(-1)$.

c Find the values of x for which $f(x) = 0$ given that $f(x) = x^2 + 3x - 10$.

Solution

a
$$\begin{aligned} f(x) &= x^2 + 3x + 1 \\ f(-2) &= (-2)^2 + 3(-2) + 1 \\ &= 4 - 6 + 1 \\ &= -1 \end{aligned}$$

b
$$\begin{aligned} f(x) &= x^3 - x^2 \\ f(-1) &= (-1)^3 - (-1)^2 \\ &= -1 - 1 \\ &= -2 \end{aligned}$$

c
$$\begin{aligned} f(x) &= 0 \\ x^2 + 3x - 10 &= 0 \\ (x + 5)(x - 2) &= 0 \\ x = -5&, x = 2 \end{aligned}$$

A **piecewise function** is a function made up of 2 or more functions defined on different intervals.

EXAMPLE 5

a
$$f(x) = \begin{cases} 3x + 4 & \text{when } x \geq 2 \\ -2x & \text{when } x < 2 \end{cases}$$

Find $f(3)$, $f(2)$, $f(0)$ and $f(-4)$.

b
$$g(x) = \begin{cases} x^2 & \text{when } x > 2 \\ 2x - 1 & \text{when } -1 \leq x \leq 2 \\ 5 & \text{when } x < -1 \end{cases}$$

Find $g(1) + g(-2) - g(3)$.

Solution

a
$$\begin{aligned} f(3) &= 3(3) + 4 && \text{since } 3 \geq 2 \\ &= 13 \\ f(2) &= 3(2) + 4 && \text{since } 2 \geq 2 \\ &= 10 \\ f(0) &= -2(0) && \text{since } 0 < 2 \\ &= 0 \\ f(-4) &= -2(-4) && \text{since } -4 < 2 \\ &= 8 \end{aligned}$$

b
$$\begin{aligned} g(1) &= 2(1) - 1 && \text{since } -1 \leq 1 \leq 2 \\ &= 1 \\ g(-2) &= 5 && \text{since } -2 < -1 \\ g(3) &= 3^2 && \text{since } 3 > 2 \\ &= 9 \end{aligned}$$

So $g(1) + g(-2) - g(3) = 1 + 5 - 9$
$= -3$

You can also substitute pronumerals instead of numbers into functions.

EXAMPLE 6

Find $f(h + 1)$ given $f(x) = 5x + 4$.

Solution

Substitute $h + 1$ for x:

$$\begin{aligned} f(h+1) &= 5(h+1) + 4 \\ &= 5h + 5 + 4 \\ &= 5h + 9 \end{aligned}$$

DID YOU KNOW?

Leonhard Euler

Leonhard Euler (1707–83), from Switzerland, studied functions and invented the function notation $f(x)$. He studied theology, astronomy, medicine, physics and oriental languages as well as mathematics, and wrote more than 500 books and articles on mathematics. He found time between books to marry and have 13 children, and even when he went blind he kept on having books published.

Exercise 4.02 Function notation

1 Given $f(x) = x + 3$, find $f(1)$ and $f(-3)$.

2 If $h(x) = x^2 - 2$, find $h(0)$, $h(2)$ and $h(-4)$.

3 If $f(x) = -x^2$, find $f(5)$, $f(-1)$, $f(3)$ and $f(-2)$.

4 Find the value of $f(0) + f(-2)$ if $f(x) = x^4 - x^2 + 1$.

5 Find $f(-3)$ if $f(x) = 2x^3 - 5x + 4$.

6 If $f(x) = 2x - 5$, find x when $f(x) = 13$.

7 Given $f(x) = x^2 + 3$, find any values of x for which $f(x) = 28$.

8 If $f(x) = 3^x$, find x when $f(x) = \frac{1}{27}$.

9 Find values of z for which $f(z) = 5$ given $f(z) = |2z + 3|$.

10 If $f(x) = 2x - 9$, find $f(p)$ and $f(x + h)$.

11 Find $g(x - 1)$ when $g(x) = x^2 + 2x + 3$.

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12 If $f(x) = x^2 - 1$, find $f(k)$ as a product of factors.

13 Given $f(t) = t^2 - 2t + 1$, find:

a t when $f(t) = 0$ **b** any values of t for which $f(t) = 9$.

14 Given $f(t) = t^4 + t^2 - 5$, find the value of $f(b) - f(-b)$.

15 $f(x) = \begin{cases} x^3 & \text{for } x > 1 \\ x & \text{for } x \le 1 \end{cases}$

Find $f(5)$, $f(1)$ and $f(-1)$.

16 $f(x) = \begin{cases} 2x - 4 & \text{if } x > 1 \\ x + 3 & \text{if } -1 \le x \le 1 \\ x^2 & \text{if } x < -1 \end{cases}$

Find the value of $f(2) - f(-2) + f(-1)$.

17 Find $g(3) + g(0) + g(-2)$ if $g(x) = \begin{cases} x + 1 & \text{when } x \ge 0 \\ -2x + 1 & \text{when } x < 0 \end{cases}$

18 Find the value of $f(3) - f(2) + 2f(-3)$ when $f(x) = \begin{cases} x & \text{for } x > 2 \\ x^2 & \text{for } -2 \le x \le 2 \\ 4 & \text{for } x < -2 \end{cases}$

19 Find the value of $f(-1) - f(3)$ if $f(x) = \begin{cases} x^3 - 1 & \text{for } x \ge 2 \\ 2x^2 + 3x - 1 & \text{for } x < 2 \end{cases}$

20 If $f(x) = x^2 - 5x + 4$, find $f(x + h) - f(x)$ in its simplest form.

21 Simplify $\dfrac{f(x+h) - f(h)}{h}$ where $f(x) = 2x^2 + x$.

22 If $f(x) = 5x - 4$, find $f(x) - f(c)$ in its simplest form.

23 Find the value of $f(k^2)$ if $f(x) = \begin{cases} 3x + 5 & \text{for } x \ge 0 \\ x^2 & \text{for } x < 0 \end{cases}$

24 If $f(x) = \begin{cases} x^3 & \text{when } x \ge 3 \\ 5 & \text{when } 0 < x < 3 \\ x^2 - x + 2 & \text{when } x \le 0 \end{cases}$

evaluate:

a $f(0)$ **b** $f(2) - f(1)$ **c** $f(-n^2)$

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25 If $f(x) = \dfrac{x^2 - 2x - 3}{x - 3}$:

a evaluate $f(2)$.

b explain why the function does not exist for $x = 3$.

c by taking several x values close to 3, find the value of y that the function is moving towards as x moves towards 3.

WS

Function notation

4.03 Properties of functions

We can use the properties of functions, such as their **intercepts**, to draw their graphs.

Intercepts

The ***x*-intercept** of a graph is the value of x where the graph crosses the x-axis.

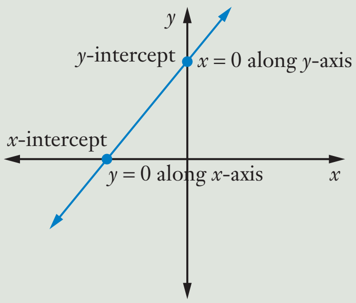
The ***y*-intercept** of a graph is the value of y where the graph crosses the y-axis.

Intercepts of the graph of a function

For x-intercept(s), substitute $y = 0$.

For y-intercept, substitute $x = 0$.

For the graph of $y = f(x)$, solving $f(x) = 0$ gives the x-intercepts and evaluating $f(0)$ gives the y-intercept.



EXAMPLE 7

Find the x- and y-intercepts of the function $f(x) = x^2 + 7x - 8$.

Solution

For x-intercepts, $y = f(x) = 0$:

$0 = x^2 + 7x - 8$

$= (x + 8)(x - 1)$

$x = -8, x = 1$

So x-intercepts are -8 and 1.

For y-intercept, $x = 0$:

$f(0) = 0^2 + 7(0) - 8$

$= -8$

So the y-intercept is -8.

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Domain and range

The **domain** of a function $y = f(x)$ is the set of all x values for which $f(x)$ is defined.

The **range** of a function $y = f(x)$ is the set of all y values for which $f(x)$ is defined.

Interval notation

- $[a, b]$ means the interval is between a and b, including a and b
- (a, b) means the interval is between a and b, excluding a and b
- $[a, b)$ means the interval is between a and b, including a but excluding b
- $(a, b]$ means the interval is between a and b, excluding a but including b
- $(-\infty, \infty)$ means that the interval includes the set of all real numbers R

EXAMPLE 8

Domain and range

Find the domain and range of each function.

a $f(x) = x^2$ **b** $y = \sqrt{x-1}$

Solution

a You can find the domain and range from the equation or the graph.

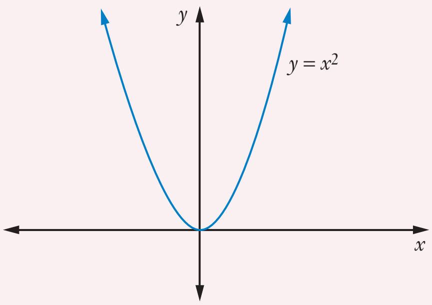
For $f(x) = x^2$, you can substitute any value for x. The y values will be 0 or positive.

So the domain is all real values of x and the range is all $y \geq 0$.

We can write this using interval notation:

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$



b The function $y = \sqrt{x-1}$ is only defined if $x - 1 \geq 0$ because we can only evaluate the square root of a positive number or 0.

For example, $x = 0$ gives $y = \sqrt{-1}$, which is undefined for real numbers.

So $x - 1 \geq 0$

$x \geq 1$

Domain: $[1, \infty)$

The value of $\sqrt{x-1}$ is always positive or zero. So $y \geq 0$.

Range: $[0, \infty)$

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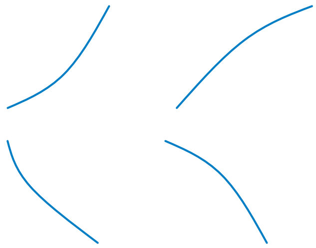
Increasing and decreasing graphs

When you draw a graph, it helps to know whether the function is increasing or decreasing on an interval.

If a graph is **increasing**, y increases as x increases, and the graph is moving upwards.

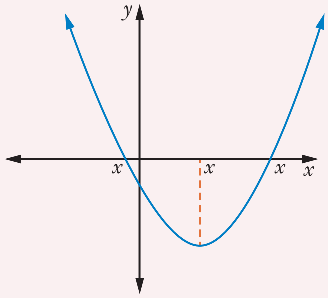
If a graph is **decreasing**, then y decreases as x increases, and the curve moves downwards.



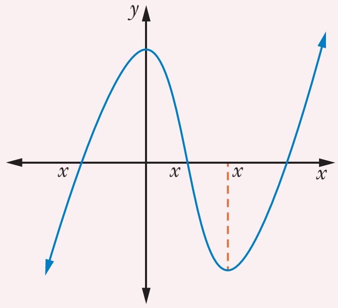
EXAMPLE 9

State the domain over which each curve is increasing.

a



b



Solution

a The curve is decreasing to the left of x_2 and increasing to the right of x_2, that is, when $x > x_2$.

So the domain over which the graph is increasing is (x_2, ∞).

b The curve is increasing on the left of the y-axis ($x = 0$), decreasing from $x = 0$ to $x = x_3$, then increasing again from $x = x_3$.

So the curve is increasing for $x < 0$, $x > x_3$.

So the domain over which the graph is increasing is $(-\infty, 0) \cup (x_3, \infty)$.

The symbol $\cup$ is for 'union' and means 'and'. It stands for the union or joining of 2 separate parts. You will meet this symbol again in probability.

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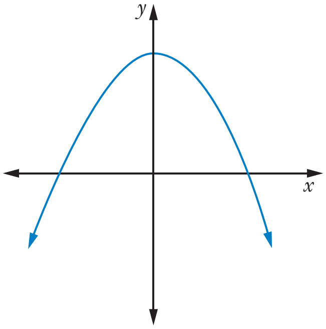
Even and odd functions

Even functions have graphs that are symmetrical about the y-axis. The graph has line symmetry about the y-axis. The left and right halves are mirror-images of each other.



Even functions

A function is even if $f(x) = f(-x)$ for all values of x in the domain.

EXAMPLE 10

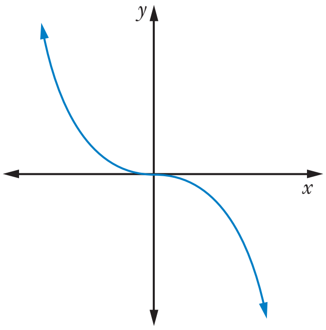
Show that $f(x) = x^2 + 3$ is an even function.

Solution

$$
\begin{aligned}
f(-x) &= (-x)^2 + 3 \\
&= x^2 + 3 \\
&= f(x)
\end{aligned}
$$

So $f(x) = x^2 + 3$ is an even function.

Odd functions have graphs that have point symmetry about the origin. A graph rotated 180° about the origin gives the original graph.



Odd functions

A function is odd if $f(-x) = -f(x)$ for all values of x in the domain.

Odd and even functions

EXAMPLE 11

Show that $f(x) = x^3 - x$ is an odd function.

Solution

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) \\ &= -f(x) \end{aligned}$$

So $f(x) = x^3 - x$ is an odd function.

INVESTIGATION

EVEN AND ODD FUNCTIONS

Explore the family of graphs of $f(x) = kx^n$, the **power functions**.

For what values of n is the function even?

For what values of n is the function odd?

Does the value of k change this?

Are these families of functions below even or odd? Does the value of k change this?

1 $f(x) = x^n + k$ **2** $f(x) = (x + k)^n$

Exercise 4.03 Properties of functions

1 Find the x- and y-intercepts of each function.

a $y = 3x - 2$ **b** $2x - 5y + 20 = 0$ **c** $x + 3y - 12 = 0$
d $f(x) = x^2 + 3x$ **e** $f(x) = x^2 - 4$ **f** $p(x) = x^2 + 5x + 6$
g $y = x^2 - 8x + 15$ **h** $p(x) = x^3 + 5$ **i** $y = \dfrac{x+3}{x}$
j $g(x) = 9 - x^2$

2 $f(x) = 3x - 6$

a Solve $f(x) = 0$.
b Find the x- and y-intercepts.

3 Show that $f(x) = f(-x)$ where $f(x) = x^2 - 2$. What type of function is it?

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4 $f(x) = x^3 + 1$

a Find $f(x^2)$. **b** Find $[f(x)]^2$. **c** Find $f(-x)$.

d Is $f(x) = x^3 + 1$ an even or odd function?

e Solve $f(x) = 0$.

f Find the intercepts of the function.

5 Show that $g(x) = x^8 + 3x^4 - 2x^2$ is an even function.

6 Show that $f(x)$ is odd, given $f(x) = x$.

7 Show that $f(x) = x^2 - 1$ is an even function.

8 Show that $f(x) = 4x - x^3$ is an odd function.

9 **a** Prove that $f(x) = x^4 + x^2$ is an even function.

b Find $f(x) - f(-x)$.

10 Are these functions even, odd or neither?

a $y = \dfrac{x^3}{x^4 - x^2}$ **b** $f(x) = \dfrac{1}{x^3 - 1}$ **c** $f(x) = \dfrac{3}{x^2 - 4}$

d $y = \dfrac{x-3}{x+3}$ **e** $f(x) = \dfrac{x^3}{x^5 - x^2}$

11 If n is a positive integer, for what values of n is the power function $f(x) = kx^n$:

a even? **b** odd?

12 Can the function $f(x) = x^n + x$ ever be:

a even? **b** odd?

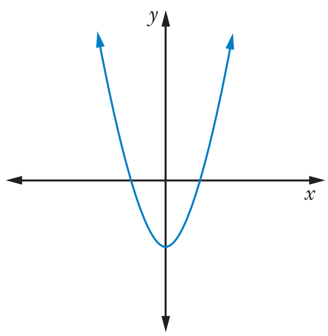
13 For the functions below, state:

i the domain over which the graph is increasing

ii the domain over which the graph is decreasing

iii whether the graph is odd, even or neither.

a



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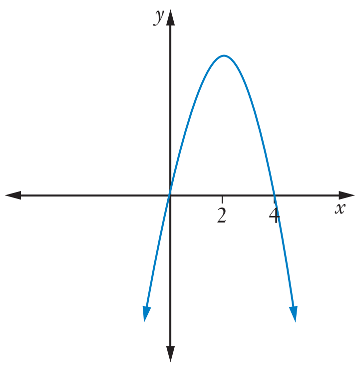

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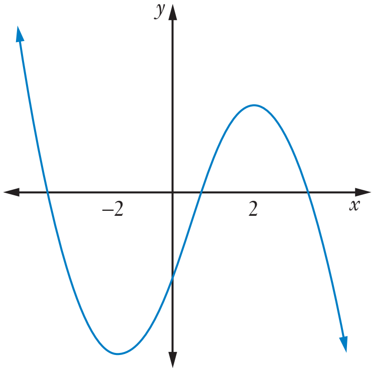
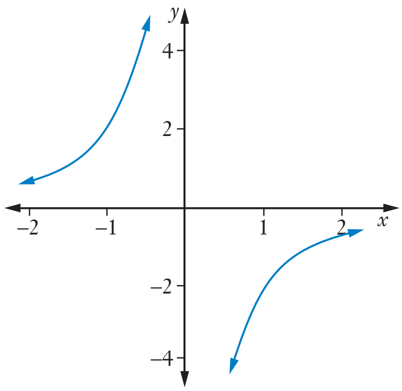
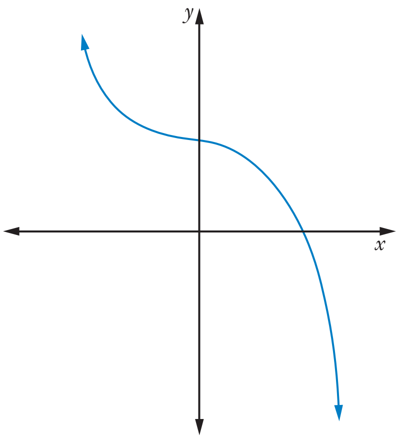
c



d



e



14 State the domain and range for each function.

a $f(x) = x^2 + 1$ **b** $y = x^3$ **c** $y = \sqrt{x}$

d $f(x) = \sqrt{x + 5}$ **e** $y = -\sqrt{2x - 6}$

15 $f(x) = (x - 2)^2$

a Find $f(3)$.

b Find $f(-5)$.

c Solve $f(x) = 0$.

d Find the x- and y-intercepts.

e State the domain and range of $f(x)$.

f Find $f(-x)$.

g Is $f(x)$ even, odd or neither?

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4.04 Linear functions

Linear functions

A **linear function** has an equation of the form $y = mx + c$ or $ax + by + c = 0$.

Its graph is a straight line with one x-intercept and one y-intercept.

WS A page o number planes

WS Gaphing linear functions

WS x and y-intercepts

Direct variation

When one variable is in **direct variation** (or **direct proportion**) with another variable, one is a constant multiple of the other. This means that as one increases, so does the other.

Direct variation

If variables x and y are in direct proportion we can write the equation $y = kx$, where k is called the **proportionality constant**.

EXAMPLE 12

Huang earns \$20 an hour. Find an equation for Huang's income (I) for working x hours and draw its graph.

Solution

Income for 1 hour is \$20.

Income for 2 hours is $\$20 \times 2$ or \$40.

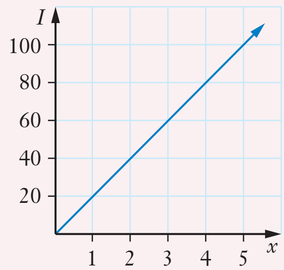
Income for 3 hours is $\$20 \times 3$ or \$60.

Income for x hours is $\$20 \times x$ or $\$20x$.

We can write the equation as $I = 20x$.

We can graph the equation using a table of values.

x	1	2	3
I	20	40	60



$I = 20x$ is an example of direct variation. Direct variation graphs are always straight lines passing through the origin.

Graphing linear functions

EXAMPLE 13

a Find the x- and y-intercepts of the graph of $y = 2x - 4$ and draw its graph on the number plane.

b Find the x- and y-intercepts of the line with equation $x + 2y + 6 = 0$ and draw its graph.

Solution

a For x-intercept, $y = 0$:

$0 = 2x - 4$

$4 = 2x$

$2 = x$

So the x-intercept is 2.

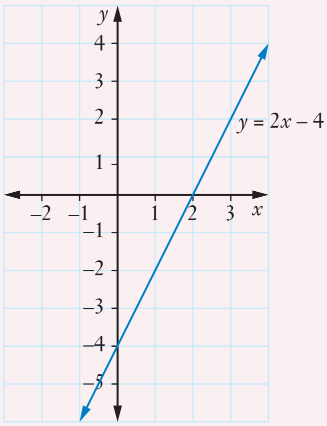
For y-intercept, $x = 0$:

$y = 2(0) - 4$

$= -4$

So the y-intercept is -4.

Use the intercepts to graph the line.



b For x-intercept, $y = 0$:

$x + 2(0) + 6 = 0$

$x + 6 = 0$

$x = -6$

So the x-intercept is -6.

For y-intercept, $x = 0$

$0 + 2y + 6 = 0$

$2y = -6$

$y = -3$

So the y-intercept is -3.

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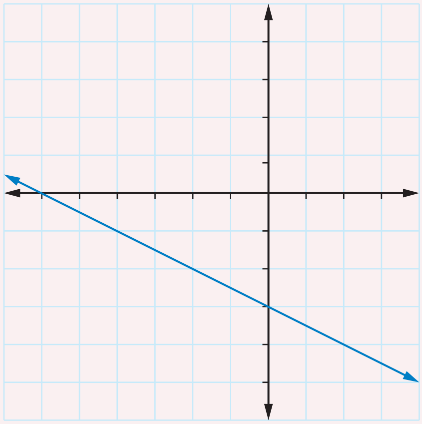

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Horizontal and vertical ines

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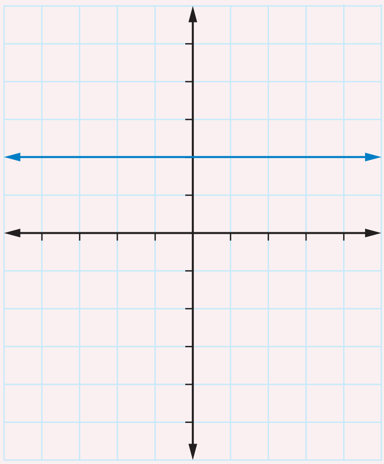

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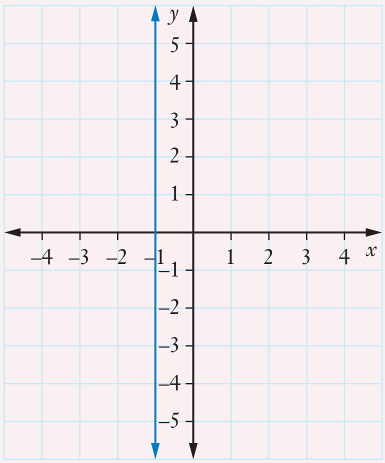
b y can have any value and x is always -1.

Some of the points on the line will be $(-1, 0)$, $(-1, 1)$ and $(-1, 2)$.

This gives a vertical line with x-intercept -1.

Domain $[-1]$, Range $(-\infty, \infty)$.



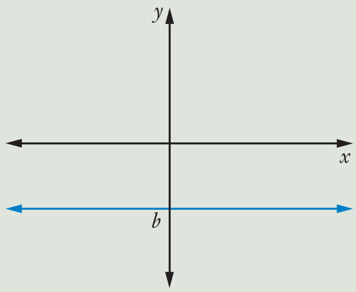
Horizontal lines

$y = b$ is a horizontal line with y-intercept b.

$y = b$ is a many-to-one function.

Domain $(-\infty, \infty)$

Range $[b]$



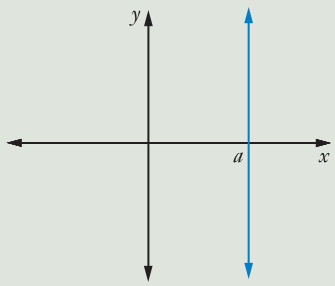
Vertical lines

$x = a$ is a vertical line with x-intercept a.

$x = a$ is not a function.

Domain $[a]$

Range $(-\infty, \infty)$



Exercise 4.04 Linear functions

1 Write an equation for:

- **a** the number of months (N) in x years
- **b** the amount of juice (A) in n lots of 2 litre bottles
- **c** the cost (c) of x litres of petrol at \$1.50 per litre
- **d** the number (y) of people in x debating teams if there are 4 people in each team
- **e** the weight (w) of x lots of 400 g cans of peaches.

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2 Find the equation and draw the graph of the cost (c) of x refrigerators if each refrigerator costs \$850.

3 Find the x- and y-intercepts of the graph of each function.

a $y = x - 2$ **b** $y = 3x + 9$ **c** $y = 4 - 2x$

d $f(x) = 2x + 3$ **e** $f(x) = 5x - 4$ **f** $f(x) = 10x + 5$

g $x + y - 2 = 0$ **h** $2x - y + 4 = 0$ **i** $x - y + 3 = 0$

j $3x - 6y - 2 = 0$

4 Draw the graph of each linear function.

a $y = x + 4$ **b** $f(x) = 2x - 1$ **c** $f(x) = 3x + 2$

d $x + y = 3$ **e** $x - y - 1 = 0$

5 Find the domain and range of each equation.

a $3x - 2y + 7 = 0$ **b** $y = 2$ **c** $x = -4$

d $x - 2 = 0$ **e** $3 - y = 0$

6 Sketch each equation's graph and state its domain and range.

a $x = 4$ **b** $x - 3 = 0$

c $y = 5$ **d** $y + 1 = 0$

7 A supermarket has boxes containing cans of dog food. The number of cans of dog food is directly proportional to the number of boxes.

a If there are 144 cans in 4 boxes find an equation for the number of cans (N) in x boxes.

b How many cans are in 28 boxes?

c How many boxes would be needed for 612 cans of dog food?

8 By sketching the graphs of $x - y - 4 = 0$ and $2x + 3y - 3 = 0$ on the same set of axes, find the point where they cross.

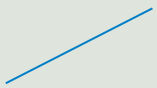
4.05 The gradient of a straight line

Gradient and y-intercept of a line

The **gradient** of a line measures its slope. It compares the vertical rise with the horizontal run.

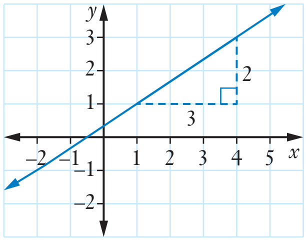
The gradient of a line

$$\text{Gradient} = \frac{\text{rise}}{\text{run}}$$



Positive gradient leans to the right.

Negative gradient leans to the left.



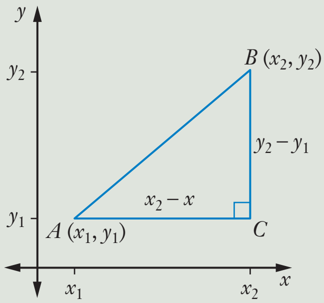
$$\text{Gradient} = \frac{\text{rise}}{\text{run}} = \frac{2}{3}$$

On the number plane, gradient is a measure of the rate of change of y with respect to x.

Gradient formula

The gradient of the line joining points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



EXAMPLE 15

Find the gradient of the line joining points (2, 3) and (–3, 4).

Solution

$$\begin{aligned}\text{Gradient } m &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{4-3}{-3-2}\\ &= \frac{1}{-5}\\ &= -\frac{1}{5}\end{aligned}$$

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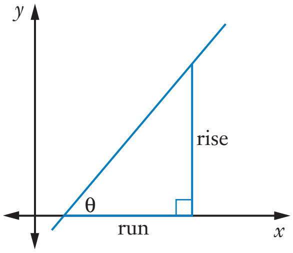
The angle of inclination of a line

The **angle of inclination**, θ, is the angle a straight line makes with the positive x-axis, measured anticlockwise.

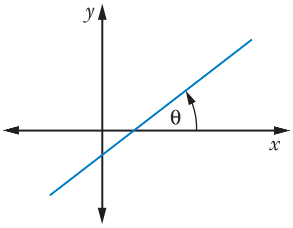
$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{\text{opposite}}{\text{adjacent}}$$
$$= \tan \theta$$



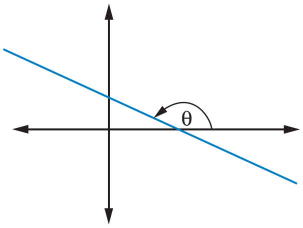
Gradient and angle of inclination of a line

$$m = \tan \theta$$

where m is the gradient and θ is the **angle of inclination**.



For an acute angle, $\tan \theta > 0$.



For an obtuse angle, $\tan \theta < 0$.

DISCUSSION

ANGLES AND GRADIENTS

1 What type of angles give a positive gradient?

2 What type of angles give a negative gradient? Why?

3 What is the gradient of a horizontal line? What angle does it make with the x-axis?

4 What angle does a vertical line make with the x-axis? Can you find its gradient?

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EXAMPLE 16

a Find the gradient of the line that makes an angle of inclination of 135°.

b Find correct to the nearest minute the angle of inclination of a straight line whose gradient is:

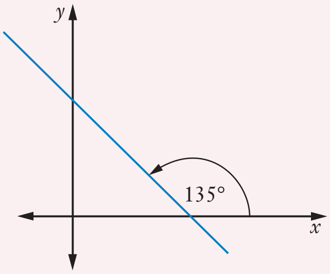
i 0.5 **ii** –3

Solution

a $m = \tan \theta$

$= \tan 135°$

$= -1$



b **i** $m = \tan \theta$

$\therefore \tan \theta = 0.5$

$\theta = \tan^{-1}(0.5)$

$= 26°33'54.18''$

$\approx 26°34'$

See page 228 of Chapter 5 if you need to revise rounding an angle to the nearest minute.

Operation	Casio scientific	Sharp scientific
Enter data.	0.5	0.5
Change to degrees and minutes.	° ' "	'

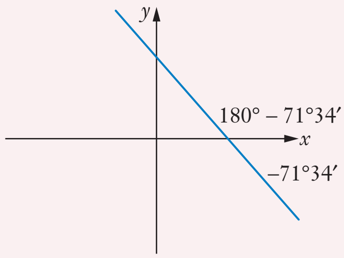
ii $m = \tan \theta$

$\therefore \tan \theta = -3$

$\theta = \tan^{-1}(-3)$

$= -71°33'54.18''$

$\approx -71°34'$



A negative gradient means that the angle of inclination is obtuse.

To find this angle, subtract the acute angle from 180°.

$\theta = 180° - 71°34'$

$= 108°26'$

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INVESTIGATION

GRAPHING $y = mx + c$

Graph each linear function using a graphics calculator or graphing software. Find the gradient of each function. What do you notice?

1 $y = x$ **2** $y = 2x$ **3** $y = 3x$ **4** $y = 4x$

5 $y = -x$ **6** $y = -2x$ **7** $y = -3x$ **8** $y = -4x$

Graph each function and find the y-intercept.

9 $y = x$ **10** $y = x + 1$ **11** $y = x + 2$ **12** $y = x + 3$

13 $y = x - 1$ **14** $y = x - 2$ **15** $y = x - 3$

The gradient–intercept equation of a straight line

The linear function with equation $y = mx + c$ has gradient m and y-intercept c.

WS

$y = mx + c$

EXAMPLE 17

a Find the gradient and y-intercept of the linear function $y = 7x - 5$.

b Find the gradient of the straight line with equation $2x + 3y - 6 = 0$.

Solution

a Gradient = 7, y-intercept = −5.

b First, change the equation into the form $y = mx + c$.

$$\begin{aligned} 2x + 3y - 6 &= 0 \\ 2x + 3y &= 6 \\ 3y &= 6 - 2x \\ &= -2x + 6 \end{aligned}$$

$$\begin{aligned} y &= \frac{-2x}{3} + \frac{6}{3} \\ &= -\frac{2}{3}x + 2 \end{aligned}$$

So the gradient is $-\frac{2}{3}$.

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Exercise 4.05 The gradient of a straight line

1 Find the gradient of the line joining the points:

a (3, 2) and (1, –2) **b** (0, 2) and (3, 6) **c** (–2, 3) and (4, –5)
d (2, –5) and (–3, 7) **e** (2, 3) and (–1, 1) **f** (–5, 1) and (3, 0)
g (–2, –3) and (–4, 6) **h** (–1, 3) and (–7, 7) **i** (1, –4) and (5, 5)

2 Find the gradient of the straight line, correct to 1 decimal place, whose angle of inclination is:

a 25° **b** 82° **c** 68°
d 100° **e** 130° **f** 164°

3 For each linear function, find:

i the gradient **ii** the y-intercept.

a $y = 3x + 5$ **b** $f(x) = 2x + 1$ **c** $y = 6x - 7$
d $y = -x$ **e** $y = -4x + 3$ **f** $y = x - 2$
g $f(x) = 6 - 2x$ **h** $y = 1 - x$ **i** $y = 9x$

4 Find the gradient of the linear function:

a with x-intercept 3 and y-intercept –1
b passing through (2, 4) and x-intercept 5
c passing through (1, 1) and (–2, 7)
d with x-intercept –3 and passing through (2, 3)
e passing through the origin and (–3, –1).

5 Find the angle of inclination, to the nearest minute, of a line with gradient:

a 2 **b** 1.7 **c** 6
d –5 **e** –0.85 **f** –1.2

6 For each linear function, find:

i the gradient **ii** the y-intercept.

a $2x + y - 3 = 0$ **b** $5x + y + 6 = 0$ **c** $6x - y - 1 = 0$
d $x - y + 4 = 0$ **e** $4x + 2y - 1 = 0$ **f** $6x - 2y + 3 = 0$
g $x + 3y + 6 = 0$ **h** $4x + 5y - 10 = 0$ **i** $7x - 2y - 1 = 0$

7 Find the gradient of each linear function.

a $y = -2x - 1$ **b** $y = 2$ **c** $x + y + 1 = 0$
d $3x + y = 8$ **e** $2x - y + 5 = 0$ **f** $x + 4y - 12 = 0$
g $3x - 2y + 4 = 0$ **h** $5x - 4y = 15$ **i** $y = \frac{2}{3}x + 3$

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j $y = \frac{x}{5} - 1$ **k** $y = \frac{2x}{7} + 5$ **l** $y = -\frac{3x}{5} - 2$

m $2y = -\frac{x}{7} + \frac{1}{3}$ **n** $3x - \frac{y}{5} = 8$ **o** $\frac{x}{2} + \frac{y}{3} = 1$

8 If the gradient of the line joining $(8, y_1)$ and $(-1, 3)$ is 2, find the value of y_1.

9 The gradient of the line through $(2, -1)$ and $(x, 0)$ is -5. Find the value of x.

10 The gradient of a line is -1 and the line passes through the points (4 2) and $(x, -3)$. Find the value of x.

11 The number of frequent flyer points that Mario earns on his credit card is directly proportional to the amount of money he spends on his card.

a If Mario earns 150 points when he spends \$450, find an equation for the number of points (P) he earns when spending d dollars.

b Find the number of points Mario earns when he spends \$840.

c If Mario earns 57 points, how much did he spend?

12 The points $A(-1, 2)$, $B(1, 5)$, $C(6, 5)$ and $D(4, 2)$ form a parallelogram. Find the gradients of all 4 sides of the parallelogram. What do you notice?

4.06 Finding a linear equation

WS Linea uncions code puzzle

Linear modelling

WS Finding he equation of a line

WS Equaion o lines

EXAMPLE 18

Find the equation of the line with gradient 3 and y-intercept -1.

Solution

The equation is $y = mx + c$ where m = gradient and c = y-intercept.

$m = 3$ and $c = -1$.

Equation is $y = 3x - 1$.

There is a formula you can use if you know the gradient and the coordinates of a point on the line.

The point–gradient equation of a straight line

The linear function with equation $y - y_1 = m(x - x_1)$ has gradient m and the point (x_1, y_1) lies on the line.

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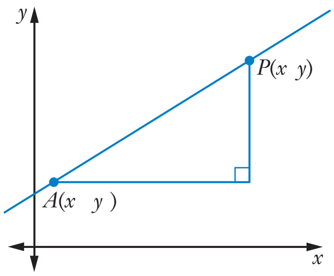
Proof

Let $P(x, y)$ be a general point on the line with gradient m that passes through $A(x_1, y_1)$.

Then line AP has gradient

$$m = \frac{y - y}{x - x}$$

$$m(x - x_1) = y - y_1$$



EXAMPLE 19

Find the equation of the line:

a with gradient −4 and x-intercept 1

b passing through (2, 3) and (−1, 4).

Solution

a The x-intercept of 1 means the line passes through the point (1, 0).

Substituting $m = -4$, $x_1 = 1$ and $y_1 = 0$ into the formula:

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -4(x - 1)$$

$$y = -4x + 4$$

b First find the gradient.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{4 - 3}{-1 - 2}$$

$$= -\frac{1}{3}$$

Substitute the gradient and one of the points, say (2, 3), into the formula.

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -\frac{1}{3}(x - 2)$$

$$3 \times (y - 3) = 3 \times -\frac{1}{3}(x - 2)$$

$$3y - 9 = -(x - 2)$$

$$= -x + 2$$

$$x + 3y - 9 = 2$$

$$x + 3y - 11 = 0$$

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Applications of linear functions

EXAMPLE 20

A solar panel company has fixed overhead costs of \$3000 per day and earns \$150 for each solar cell sold.

a Write the amount (\$$A$) that the company earns on selling x solar cells each day.

b Find the amount the company earns on a day when it sells 54 solar cells.

c If the company earns \$2850 on another day, how many solar cells did it sell that day?

d What is the **break-even point** for this company (where income and costs of production are equal)?

Solution

a The company earns \$150 per cell, so it earns \$$150x$ for x cells.

Daily amount earned = value of solar cells sold – overhead costs.

So $A = 150x - 3000$

b Substitute $x = 54$:

$A = 150(54) - 3000 = 5100$

The company earns \$5100 when it sells 54 solar cells.

c Substitute $A = 2850$:

$2850 = 150x - 3000$

$5850 = 150x$

$39 = x$

The company sold 39 solar cells that day.

d At the break-even point:

Income = overhead costs

$150x = 3000$ (or $A = 0$)

$x = 20$

So the break-even point is where the company sells 20 solar cells.

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Exercise 4.06 Finding a linear equation

1 Find the equation of the straight line:

- **a** with gradient 4 and y-intercept -1
- **b** with gradient -3 and passing through $(0, 4)$
- **c** passing through the origin with gradient 5
- **d** with gradient 4 and x-intercept -5
- **e** with x-intercept 1 and y-intercept 3
- **f** with x-intercept 3, y-intercept -4.

2 Find the equation of the straight line passing through the points:

a $(2, 5)$ and $(-1, 1)$ **b** $(0, 1)$ and $(-4, -2)$ **c** $(-2, 1)$ and $(3, 5)$

d $(3, 4)$ and $(-1, 7)$ **e** $(-4, -1)$ and $(-2, 0)$.

3 What is **a** the gradient and **b** the equation of the line with x-intercept 2 that passes through $(3, -4)$?

4 Find the equation of the line:

- **a** parallel to the x-axis and passing through $(2, 3)$
- **b** parallel to the y-axis and passing through $(-1, 2)$.

5 A straight line passing through the origin has a gradient of -2. Find:

a the y-intercept **b** its equation.

6 In a game, each person starts with 20 points, then earns 15 points for every level completed.

- **a** Write an equation for the number of points earned (P) for x levels completed.
- **b** Find the number of points earned for completing:
  i 24 levels **ii** 55 levels **iii** 247 levels
- **c** Find the number of levels completed if the number of points earned is:
  i 2195 **ii** 7700 **iii** 12 665

7 A TV manufacturing business has fixed costs of \$1500 rental, \$3000 wages and other costs of \$2500 each week. It costs \$250 to produce each TV.

- **a** Write an equation for the cost (c) of producing n TVs each week.
- **b** From the equation, find the cost of producing:
  i 100 TVs **ii** 270 TVs **iii** 1200 TVs
- **c** From the equation find the number of TVs produced if the cost is:
  i \$52 000 **ii** \$78 250 **iii** \$367 000
- **d** If each TV sells for \$950, find the number of TVs needed to sell to break even.

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8 There are 450 litres of water in a pond, and 8 litres of water evaporate out of the pond every hour.

a Write an equation for the amount of water in the pond (A) after h hours.

b Find the amount of water in the pond after:

i 3 hours **ii** a day.

c After how many hours will the pond be empty?

9 Geordie has a \$20 iTunes credit. He uses the credit to buy singles at \$1.69 each.

a Write an equation for the amount of credit (C) left if Geordie buys x singles.

b How many songs can Geordie buy before his credit runs out?

10 Emily-Rose owes \$20 000 and she pays back \$320 a month.

a Write an equation for the amount of money she owes (A) after x months.

b How much does Emily-Rose owe after:

i 5 months? **ii** 1 year? **iii** 5 years?

c How long will it take for Emily-Rose to pay all the money back?

11 Acme Party Supplies earns \$5 for every helium balloon it sells.

a If overhead costs are \$100 each day, find an equation for the profit (P) of selling x balloons.

b How much profit does Acme make if it sells 300 balloons?

c How many balloons does it sell if it makes a profit of \$1055?

d What is the break-even point for this business?

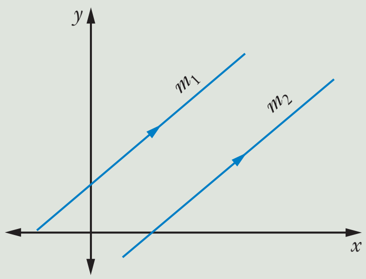
4.07 Parallel and perpendicular lines

Paallel and perpendicular lines

Linea uncions

Gradients of parallel lines

If 2 lines are parallel, then they have the same gradient. That is, $m_1 = m_2$.



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EXAMPLE 21

a Prove that the straight lines with equations $5x - 2y - 1 = 0$ and $5x - 2y + 7 = 0$ are parallel.

b Find the equation of a straight line parallel to the line $2x - y - 3 = 0$ and passing through $(1, -5)$.

Solution

a First, change the equation into the form $y = mx + c$.

$$5x - 2y - 1 = 0$$
$$5x - 1 = 2y$$
$$\frac{5}{2}x - \frac{1}{2} = y$$
$$\therefore \quad m = \frac{5}{2}$$

$$5x - 2y + 7 = 0$$
$$5x + 7 = 2y$$
$$\frac{5}{2}x + \frac{7}{2} = y$$
$$\therefore \quad m_2 = \frac{5}{2}$$

$$m_1 = m_2 = \frac{5}{2}$$

$\therefore$ the lines are parallel.

b
$$2x - y - 3 = 0$$
$$2x - 3 = y$$
$$\therefore m_1 = 2$$

For parallel lines $m_1 = m_2$.

$$\therefore m_2 = 2$$

Substitute this and $(1, -5)$ into $y - y_1 = m(x - x_1)$:

$$y - (-5) = 2(x - 1)$$
$$y + 5 = 2x - 2$$
$$y = 2x - 7$$

CLASS INVESTIGATION

PERPENDICULAR LINES

Sketch each pair of straight lines on the same number plane.

1 $3x - 4y + 12 = 0$ and $4x + 3y - 8 = 0$

2 $2x + y + 4 = 0$ and $x - 2y + 2 = 0$

What do you notice about each pair of lines?

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Gradients of perpendicular lines

If 2 lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$,

that is, $m_2 = -\frac{1}{m}$.

EXAMPLE 22

a Show that the lines with equations $3x + y - 11 = 0$ and $x - 3y + 1 = 0$ are perpendicular.

b Find the equation of the straight line through (2, 3) that is perpendicular to the line passing through (–1, 7) and (3, 3).

Solution

a

$$3x + y - 11 = 0$$
$$y = -3x + 11$$
$$\therefore m_1 = -3$$
$$x - 3y + 1 = 0$$
$$x + 1 = 3y$$
$$\frac{1}{3}x + \frac{1}{3} = y$$
$$\therefore m_2 = \frac{1}{3}$$
$$m_1 m_2 = -3 \times \frac{1}{3} = -1$$

$\therefore$ the lines are perpendicular.

b Line through (–1, 7) and (3, 3):

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{7-3}{-1-3}$$
$$= \frac{4}{-4}$$
$$= -1$$

For perpendicular lines, $m_1 m_2 = -1$:

$$-1m_2 = -1$$
$$m_2 = 1$$

Substitute $m = 1$ and the point (2, 3) into $y - y_1 = m(x - x_1)$:

$$y - 3 = 1(x - 2)$$
$$= x - 2$$
$$y = x + 1$$

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Exercise 4.07 Parallel and perpendicular lines

1 Find the gradient of the straight line:

a parallel to the line $3x + y - 4 = 0$

b perpendicular to the line $3x + y - 4 = 0$

c parallel to the line joining (3, 5) and (–1, 2)

d perpendicular to the line with x-intercept 3 and y-intercept 2

e perpendicular to the line that has an angle of inclination of 135°

f perpendicular to the line $6x - 5y - 4 = 0$

g parallel to the line $x - 3y - 7 = 0$

h perpendicular to the line passing through (4, –2) and (3, 3).

2 Find the equation of the straight line:

a passing through (2, 3) and parallel to the line $y = x + 6$

b through (–1, 5) and parallel to the line $x - 3y - 7 = 0$

c with x-intercept 5 and parallel to the line $y = 4 - x$

d through (3, –4) and perpendicular to the line $y = 2x$

e through (–2, 1) and perpendicular to the line $2x + y + 3 = 0$

f through (7, –2) and perpendicular to the line $3x - y - 5 = 0$

g through (–3, –1) and perpendicular to the line $4x - 3y + 2 = 0$

h passing through the origin and parallel to the line $x + y + 3 = 0$

i through (3, 7) and parallel to the line $5x - y - 2 = 0$

j through (0, –2) and perpendicular to the line $x - 2y = 9$

k perpendicular to the line $3x + 2y - 1 = 0$ and passing through the point (–2, 4).

3 Show that the lines with equations $y = 3x - 2$ and $6x - 2y - 9 = 0$ are parallel.

4 Show that lines $x + 5y = 0$ and $y = 5x + 3$ are perpendicular.

5 Show that lines $6x - 5y + 1 = 0$ and $6x - 5y - 3 = 0$ are parallel.

6 Show that lines $7x + 3y + 2 = 0$ and $3x - 7y = 0$ are perpendicular.

7 If the lines $3x - 2y + 5 = 0$ and $y = kx - 1$ are perpendicular, find the value of k.

8 Show that the line joining (3, –1) and (2, –5) is parallel to the line $8x - 2y - 3 = 0$.

9 Show that the points A(–3, –2), B(–1, 4), C(7, –1) and D(5, –7) are the vertices of a parallelogram.

10 The points A(–2, 0), B(1, 4), C(6, 4) and D(3, 0) form a rhombus. Show that the diagonals are perpendicular.

11 Find the equation of the straight line passing through (6, –3) that is perpendicular to the line joining (2, –1) and (–5, –7).

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4.08 Quadratic functions

WS Graphing quadratic functions

WS Graphing quadratics

Quadratic functions

A **quadratic function** has an equation in the form $y = ax^2 + bx + c$, where the highest power of x is 2. The graph of a quadratic function is a **parabola**.

EXAMPLE 23

Graph the quadratic function $y = x^2 - x$.

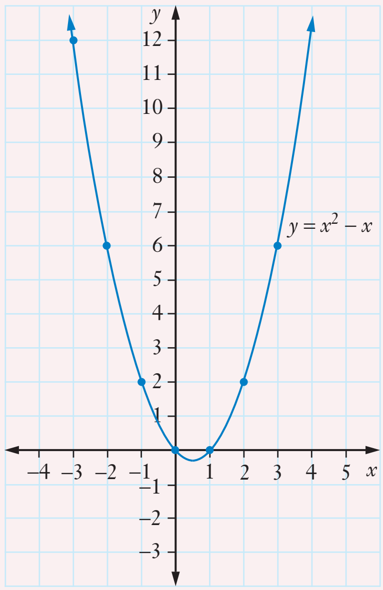
Solution

Draw up a table of values for $y = x^2 - x$.

x	–3	–2	–1	0	1	2	3
y	12	6	2	0	0	2	6

Plot (–3, 12), (–2, 6), (–1, 2), (0, 0), (1, 0), (2, 2) and (3, 6) and draw a parabola through them.

Label the graph with its equation.



TECHNOLOGY

Transforming quadratic functions

Use a graphics calculator or graphing software to graph these quadratic functions. Look for any patterns.

$y = x^2$	$y = x^2 + 1$	$y = x^2 + 2$	$y = x^2 + 3$
$y = x^2 - 1$	$y = x^2 - 2$	$y = x^2 - 3$	$y = 2x^2$
$y = 3x^2$	$y = x^2 + x$	$y = x^2 + 2x$	$y = x^2 + 3x$
$y = x^2 - x$	$y = x^2 - 2x$	$y = x^2 - 3x$	$y = -x^2$
$y = -x^2 + 1$	$y = -x^2 + 2$	$y = -x^2 + 3$	$y = -x^2 - 1$
$y = -x^2 - 2$	$y = -x^2 - 3$	$y = -2x^2$	$y = -3x^2$
$y = -x^2 + x$	$y = -x^2 + 2x$	$y = -x^2 - x$	$y = -x^2 - 2x$

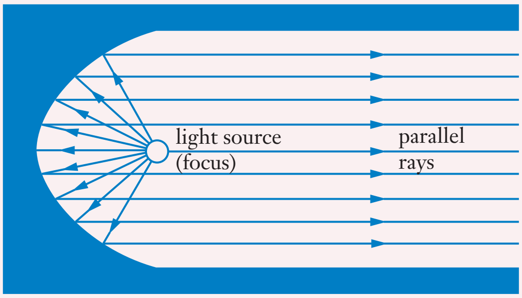
Could you predict where the graphs $y = x^2 + 9$, $y = 5x^2$ or $y = x^2 + 6x$ would lie?

Is the parabola always a function? Can you find an example of a parabola that is not a function?

DID YOU KNOW?

The parabola

The parabola shape has special properties that are very useful. For example, if a light is placed inside a parabolic mirror at a special place called the focus, then all light rays coming from this point and reflecting off the parabola shape will radiate out parallel to each other, giving a strong light. This is how car headlights work. The dishes of radio telescopes also use this property of the parabola, because radio signals coming in to the dish will reflect back to the focus.



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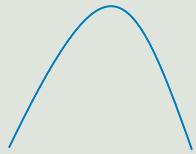
Concavity and turning points

For the parabola $y = ax^2 + bx + c$:

- if $a > 0$ the parabola is **concave upwards** and has a **minimum turning point**.

- if $a < 0$ the parabola is **concave downwards** and has a **maximum turning point**.



The **turning point** is also called the **vertex** or **stationary point** of the parabola.

Notice also that the parabola is always symmetrical.

EXAMPLE 24

a **i** Sketch the graph of $y = x^2 - 1$, showing intercepts.

ii State the domain and range.

b **i** Find the x- and y-intercepts of the quadratic function $f(x) = -x^2 + 4x + 5$.

ii Sketch a graph of the function.

iii Find the maximum value of the function.

v State the domain and range.

Solution

a **i** Since $a > 0$, the graph is concave upwards.

For x-intercepts, $y = 0$:

$0 = x^2 - 1$

$1 = x^2$

$x = \pm 1$

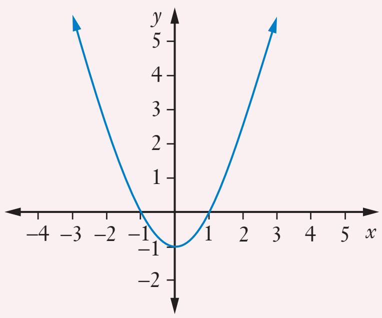
For y-intercept, $x = 0$:

$y = 0^2 - 1$

$= -1$

Since the parabola is symmetrical, the turning point is at $x = 0$, halfway between the x-intercepts -1 and 1.

When $x = 0$, $y = -1$: Vertex is $(0, -1)$.



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ii From the equation and the graph, x can have any value.

Domain $(-\infty, \infty)$

The values of y are greater than or equal to -1.

Range $[-1, \infty)$

b **i** For x-intercepts, $f(x) = 0$.

$0 = -x^2 + 4x + 5$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

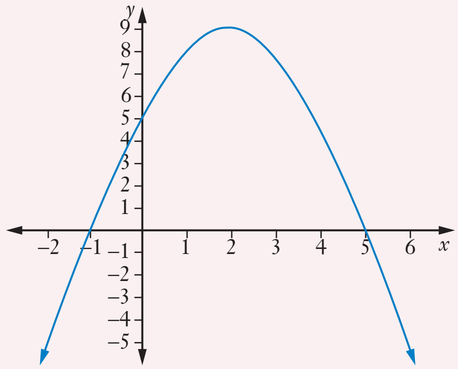
$x = 5, x = -1$

For y-intercept, $x = 0$.

$f(0) = -(0)^2 + 4(0) + 5$

$= 5$

ii Since $a < 0$, the quadratic function is concave downwards.



iii The turning point is halfway between $x = -1$ and $x = 5$.

$$x = \frac{-1+5}{2}$$

$$= 2$$

$$f(2) = -(2)^2 + 4(2) + 5$$

$$= 9$$

The maximum value of $f(x)$ is 9.

iv For the domain, the function can take on all real numbers for x.

Domain $(-\infty, \infty)$

For the range, $y \leq 9$.

Range $(-\infty, 9]$

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Exercise 4.08 Quadratic functions

1 Find the x- and y-intercepts of the graph of each quadratic function:

a $y = x^2 + 2x$ **b** $y = -x^2 + 3x$ **c** $f(x) = x^2 - 1$

d $y = x^2 - x - 2$ **e** $y = x^2 - 9x + 8$

2 Sketch each parabola and find its maximum or minimum value:

a $y = x^2 + 2$ **b** $y = -x^2 + 1$ **c** $f(x) = x^2 - 4$

d $y = x^2 + 2x$ **e** $y = -x^2 - x$ **f** $f(x) = (x - 3)^2$

g $f(x) = (x + 1)^2$ **h** $y = x^2 + 3x - 4$ **i** $y = 2x^2 - 5x + 3$

j $f(x) = -x^2 + 3x - 2$

3 For each parabola, find:

i the x- and y-intercepts **ii** the domain and range.

a $y = x^2 - 7x + 12$ **b** $f(x) = x^2 + 4x$ **c** $y = x^2 - 2x - 8$

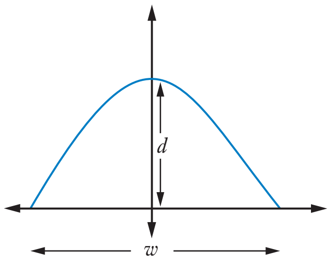
d $y = x^2 - 6x + 9$ **e** $f(x) = 4 - x^2$

4 Find the domain and range of:

a $y = x^2 - 5$ **b** $f(x) = x^2 - 6x$ **c** $f(x) = x^2 - x - 2$

d $y = -x^2$ **e** $f(x) = (x - 7)^2$

5 A satellite dish is in the shape of a parabola with equation $y = -3x^2 + 6$, and all dimensions are in metres.



a Find d, the depth of the dish.

b Find w, the width of the dish, to 1 decimal place.

WS

Quadaic functions

WS

Skeching quadratic functions

WS

Feaue o a parabola

4.09 Axis of symmetry

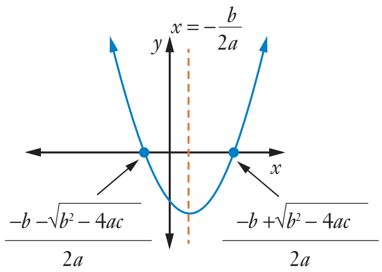
Axis of symmetry of a parabola

The **axis of symmetry** of a parabola with the equation $y = ax^2 + bx + c$ is the vertical line with equation:

$$x = -\frac{b}{2a}.$$

Proof

The axis of symmetry of a parabola lies halfway between the x-intercepts.



For the x-intercepts, $y = 0$

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The x-coordinate of the axis of symmetry is the average of the x-intercepts.

$$x = \frac{\frac{-b - \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a}}{2} = \frac{\frac{-2b}{2a}}{2} = \frac{-2b}{4a} = -\frac{b}{2a}$$

Turning point of a parabola

The quadratic function $y = ax^2 + bx + c$ has a minimum value if $a > 0$ and a maximum value if $a < 0$.

The minimum or maximum value of the quadratic function is $f\left(-\frac{b}{2a}\right)$.

The turning point or vertex of a parabola is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

EXAMPLE 25

a Find the equation of the axis of symmetry and the minimum value of the quadratic function $y = x^2 - 5x + 1$.

b Find the equation of the axis of symmetry, the maximum value and the turning point of the quadratic function $y = -3x^2 + x - 5$.

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Solution

a Axis of symmetry:

$$x = -\frac{b}{2a}$$
$$= -\frac{(-5)}{2(1)}$$
$$= \frac{5}{2}$$
$$= 2\frac{1}{2}$$

∴ Axis of symmetry is the line $x = 2\frac{1}{2}$.

Minimum value $y = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 1$
$$= \frac{25}{4} - \frac{25}{2} + 1$$
$$= -5\frac{1}{4}$$

b $x = -\frac{b}{2a}$
$$= -\frac{1}{2(-3)}$$
$$= \frac{1}{6}$$

∴ Axis of symmetry is the line $x = \frac{1}{6}$.

Maximum value $y = -3\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right) - 5$
$$= -\frac{1}{12} + \frac{1}{6} - 5$$
$$= -4\frac{11}{12}$$

The turning point is $\left(\frac{1}{6}, -4\frac{11}{12}\right)$.

EXAMPLE 26

Determine whether each function is even.

a $f(x) = x^2 + 3$ **b** $y = -x^2 + 3x$

Solution

a $f(x) = x^2 + 3$
$$f(-x) = (-x)^2 + 3$$
$$= x^2 + 3$$
$$= f(x)$$
So $f(x) = x^2 + 3$ is an even function.

b Let $f(x) = -x^2 + 3x$
$$f(-x) = -(-x)^2 + 3(-x)$$
$$= -x^2 - 3x$$
$$\neq f(x)$$
So $y = -x^2 + 3x$ is not an even function.

Exercise 4.09 Axis of symmetry

1 For the parabola $y = x^2 + 2x$, find the equation of its axis of symmetry and the minimum value.

2 Find the equation of the axis of symmetry and the minimum value of the parabola $y = x^2 - 4$.

3 Find the equation of the axis of symmetry and the minimum turning point of the parabola $y = 4x^2 - 3x + 1$.

4 Find the equation of the axis of symmetry and the maximum value of the parabola $y = -x^2 + 2x - 7$.

5 Find the equation of the axis of symmetry and the vertex of the parabola $y = -2x^2 - 4x + 5$.

6 Find the equation of the axis of symmetry and the minimum value of the parabola $y = x^2 + 3x + 2$.

7 Find the equation of the axis of symmetry and the coordinates of the vertex for each parabola:

a $y = x^2 + 6x - 3$ **b** $y = -x^2 - 8x + 1$ **c** $y = 3x^2 + 18x + 4$
d $y = -2x^2 + 5x$ **e** $y = 4x^2 + 10x - 7$

8 For each parabola, find:

i the equation of the axis of symmetry
ii the minimum or maximum value
iii the vertex.

a $y = x^2 + 2x - 2$ **b** $y = -2x^2 + 4x - 1$

9 Find the turning point of each function and state whether it is a maximum or minimum.

a $y = x^2 + 2x + 1$ **b** $y = x^2 - 8x - 7$ **c** $f(x) = x^2 + 4x - 3$
d $y = x^2 - 2x$ **e** $f(x) = x^2 - 4x - 7$ **f** $f(x) = 2x^2 + x - 3$
g $y = -x^2 - 2x + 5$ **h** $y = -2x^2 + 8x + 3$ **i** $f(x) = -3x^2 + 3x + 7$

10 For each quadratic function:

i find x-intercepts using the quadratic formula
ii state whether the function has a maximum or minimum value and find this value
iii sketch the graph of the function on a number plane
v solve the quadratic equation $f(x) = 0$ graphically

a $f(x) = x^2 + 4x + 4$ **b** $f(x) = x^2 - 2x - 3$ **c** $y = x^2 - 6x + 1$
d $f(x) = -x^2 - 2x + 6$ **e** $f(x) = -x^2 - x + 3$

11 **a** Find the minimum value of the parabola with equation $y = x^2 - 2x + 5$.
b How many solutions does the quadratic equation $x^2 - 2x + 5 = 0$ have?
c Sketch the parabola.

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12 **a** Find the maximum value of the quadratic function $f(x) = -2x^2 + x - 4$.

b How many solutions are there to the quadratic equation $-2x^2 + x - 4 = 0$?

c Sketch the graph of the quadratic function.

13 Show that $f(x) = -x^2$ is an even function.

14 Determine which of these functions are even.

a $y = x^2 + 1$ **b** $f(x) = x^2 - 3$ **c** $y = -2x^2$

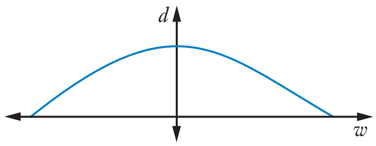
d $f(x) = x^2 - 3x$ **e** $f(x) = x^2 + x$ **f** $y = x^2 - 4$

g $y = x^2 - 2x - 3$ **h** $y = x^2 - 5x + 4$ **i** $p(x) = (x + 1)^2$

15 A bridge has a parabolic span as shown,

with equation $d = -\dfrac{w^2}{800} + 200$

where d is the depth of the arch in metres.



a Show that the quadratic function is even.

b Find the depth of the arch from the top of the span.

c Find the total width of the span.

d Find the depth of the arch at a point 10 m from its widest span.

e Find the width across the span at a depth of 100 m.

EXT1 4.10 Quadratic inequalities

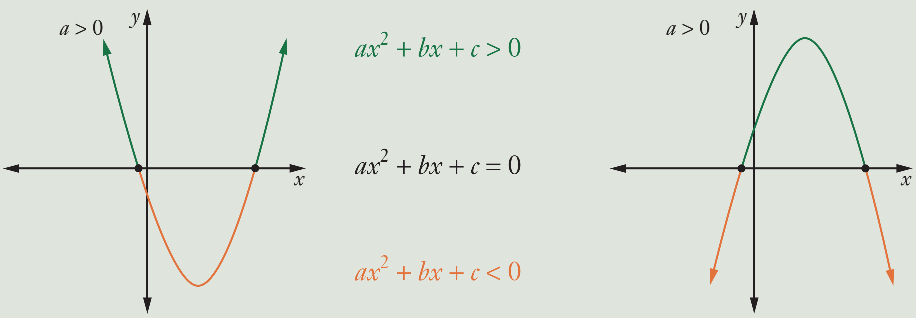
WS Quadaic inequalities

In Chapter 2, *Equations and inequalities*, you solved quadratic inequalities using the number line. You can also solve quadratic inequalities using the graph of a parabola.

The parabola and quadratic inequalities

For the graph of the quadratic function $y = ax^2 + bx + c$:

- $ax^2 + bx + c = 0$ on the x-axis
- $ax^2 + bx + c > 0$ above the x-axis
- $ax^2 + bx + c < 0$ below the x-axis



Quadratic inequalities

EXAMPLE 27

Solve:

a $x^2 - 3x + 2 \geq 0$

b $4x - x^2 > 0$

Solution

a Sketch the graph of $y = x^2 - 3x + 2$ showing x-intercepts.

$a > 0$ so it is concave upwards.

For x-intercepts, $y = 0$.

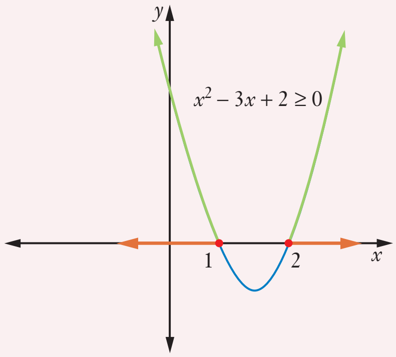
$0 = x^2 - 3x + 2$

$= (x - 2)(x - 1)$

$x = 2, x = 1$

$x^2 - 3x + 2 \geq 0$ on and above the x-axis.

$\therefore\ x \leq 1, x \geq 2$



b For $y = 4x - x^2$, $a < 0$ so its graph is concave downwards.

For x-intercepts, $y = 0$:

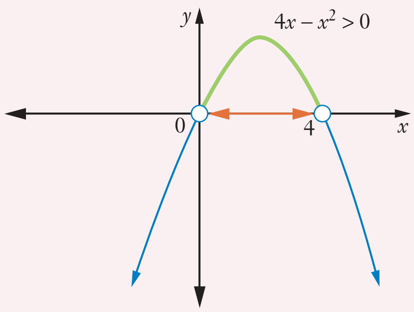
$0 = 4x - x^2$

$= x(4 - x)$

$x = 0, x = 4$

$4x - x^2 > 0$ above the x-axis.

$\therefore\ 0 < x < 4$



EXT1 Exercise 4.10 Quadratic inequalities

Solve each quadratic inequality.

1 $x^2 - 9 > 0$

2 $n^2 + n \leq 0$

3 $a^2 - 2a \geq 0$

4 $4 - x^2 < 0$

5 $y^2 - 6y \leq 0$

6 $2t - t^2 > 0$

7 $x^2 + 2x - 8 > 0$

8 $p^2 + 4p + 3 \geq 0$

9 $m^2 - 6m + 8 > 0$

10 $6 - x - x^2 \leq 0$

11 $2h^2 - 7h + 6 < 0$

12 $x^2 - x - 20 \leq 0$

13 $35 + 9k - 2k^2 \geq 0$

14 $q^2 - 9q + 18 > 0$

15 $(x + 2)^2 \geq 0$

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16 $12 - n - n^2 \le 0$ **17** $x^2 - 2x < 15$ **18** $-t^2 \ge 4t - 12$

19 $3y^2 > 14y + 5$ **20** $(x - 3)(x + 1) \ge 5$

4.11 The discriminant

WS The discriminant

The solutions of an equation are also called the **roots** of the equation.

In the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression $b^2 - 4ac$ is called the **discriminant**.

It gives us information about the roots of the quadratic equation $ax^2 + bx + c = 0$.

EXAMPLE 28

Use the quadratic formula to find how many real roots each quadratic equation has.

a $x^2 + 5x - 3 = 0$ **b** $x^2 - x + 4 = 0$ **c** $x^2 - 2x + 1 = 0$

Solution

a
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times (-3)}}{2 \times 1}$$
$$= \frac{-5 \pm \sqrt{25 + 12}}{2}$$
$$= \frac{-5 \pm \sqrt{37}}{2}$$

There are 2 real roots:

$$x = \frac{-5 + \sqrt{37}}{2}, \frac{-5 - \sqrt{37}}{2}$$

b
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times 4}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{-15}}{2}$$

There are no real roots since $\sqrt{-15}$ has no real value.

c
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 1}}{2 \times 1}$$
$$= \frac{2 \pm \sqrt{0}}{2}$$
$$= 1$$

There are 2 real roots:

$x = 1, 1$

However, these are equal roots.

The discriminant

The value of the **discriminant** $\Delta = b^2 - 4ac$ tells us information about the roots of the quadratic equation $ax^2 + bx + c = 0$.

When $\Delta \geq 0$, there are 2 real roots.

- If Δ is a perfect square, the roots are rational.
- If Δ is not a perfect square, the roots are irrational.

When $\Delta = 0$, there are 2 equal rational roots (or 1 rational root).

When $\Delta < 0$, there are no real roots.

EXAMPLE 29

a Show that the equation $2x^2 + x + 4 = 0$ has no real roots.

b Describe the roots of the equation:

i $2x^2 - 7x - 1 = 0$ **ii** $x^2 + 6x + 9 = 0$

c Find the values of k for which the quadratic equation $5x^2 - 2x + k = 0$ has real roots.

Solution

a $\Delta = b^2 - 4ac$

$= 1^2 - 4(2)(4)$

$= -31$

< 0

$\Delta < 0$, so the equation has no real roots.

b **i** $\Delta = b^2 - 4ac$

$= (-7)^2 - 4(2)(-1)$

$= 57$

> 0

$\Delta > 0$, so there are 2 real irrational roots.

Roots are irrational because 57 is not a perfect square.

ii $\Delta = b^2 - 4ac$

$= (6)^2 - 4(1)(9)$

$= 0$

$\Delta = 0$ so there are 2 real equal rational roots.

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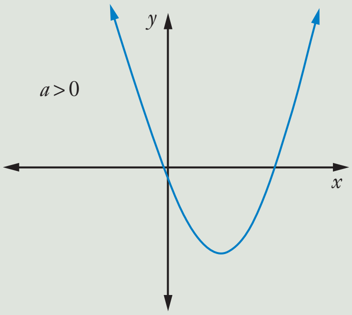
c For real roots, $\Delta \geq 0$.

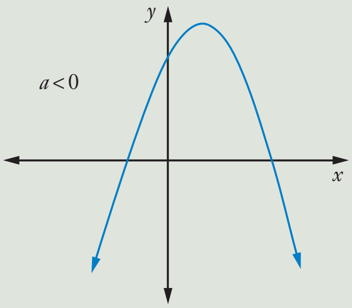
$$b^2 - 4ac \geq 0$$
$$(-2)^2 - 4(5)(k) \geq 0$$
$$4 - 20k \geq 0$$
$$4 \geq 20k$$
$$k \leq \frac{1}{5}$$

The discriminant and the parabola

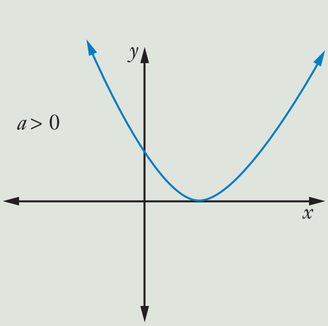
The roots of the quadratic equation $ax^2 + bx + c = 0$ give the x-intercepts of the parabola $y = ax^2 + bx + c$.

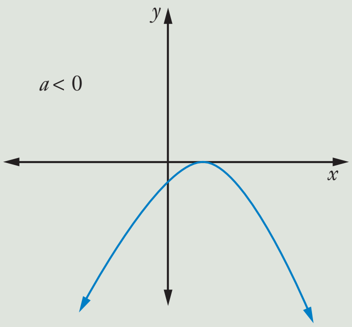
If $\Delta > 0$, then the quadratic equation has 2 real roots and the parabola has 2 x-intercepts.





If $\Delta = 0$, then the quadratic equation has 1 real root or 2 equal roots and the parabola has one x-intercept.





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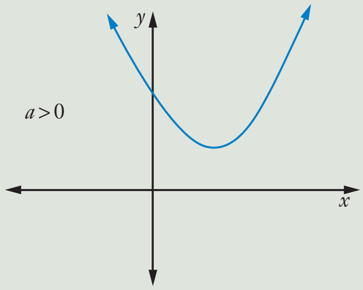
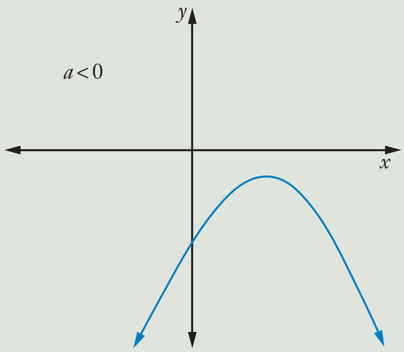
If $\Delta < 0$, then the quadratic equation has no real roots and the parabola has no x-intercepts.



If $\Delta < 0$ and $a > 0$, then $ax^2 + bx + c > 0$ for all x.



If $\Delta < 0$ and $a < 0$, then $ax^2 + bx + c < 0$ for all x.

EXAMPLE 30

a Show that the parabola $f(x) = x^2 - x - 2$ has 2 x-intercepts.

b Show that $x^2 - 2x + 4 > 0$ for all x.

Solution

a $\Delta = b^2 - 4ac$

$= (-1)^2 - 4(1)(-2)$

$= 9$

> 0

So there are 2 real roots and the parabola has 2 x-intercepts.

b If $a > 0$ and $\Delta < 0$, then $ax^2 + bx + c > 0$ for all x.

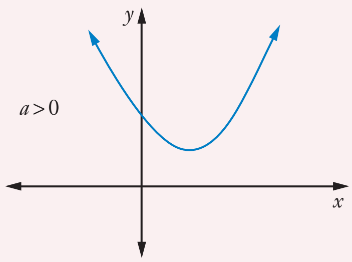
$a = 1 > 0$

$\Delta = b^2 - 4ac$

$= (-2)^2 - 4(1)(4)$

$= -12$

< 0



Since $a > 0$ and $\Delta < 0$, $x^2 - 2x + 4 > 0$ for all x.

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Exercise 4.11 The discriminant

1 Find the discriminant of each quadratic equation.

a $x^2 - 4x - 1 = 0$ **b** $2x^2 + 3x + 7 = 0$ **c** $-4x^2 + 2x - 1 = 0$

d $6x^2 - x - 2 = 0$ **e** $-x^2 - 3x = 0$ **f** $x^2 + 4 = 0$

g $x^2 - 2x + 1 = 0$ **h** $-3x^2 - 2x + 5 = 0$ **i** $-2x^2 + x + 2 = 0$

2 Find the discriminant and state whether the roots of the quadratic equation are real or not real. If the roots are real, state whether they are equal or unequal, rational or irrational.

a $x^2 - x - 4 = 0$ **b** $2x^2 + 3x + 6 = 0$ **c** $x^2 - 9x + 20 = 0$

d $x^2 + 6x + 9 = 0$ **e** $2x^2 - 5x - 1 = 0$ **f** $-x^2 + 2x - 5 = 0$

g $-2x^2 - 5x + 3 = 0$ **h** $-5x^2 + 2x - 6 = 0$ **i** $-x^2 + x = 0$

3 Find the value of p for which the quadratic equation $x^2 + 2x + p = 0$ has equal roots.

4 Find any values of k for which the quadratic equation $x^2 + kx + 1 = 0$ has equal roots.

5 Find all the values of b for which $2x^2 + x + b + 1 = 0$ has real roots.

6 Evaluate p if $px^2 + 4x + 2 = 0$ has no real roots.

7 Find all values of k for which $(k + 2)x^2 + x - 3 = 0$ has 2 real unequal roots.

8 Prove that $3x^2 - x + 7 > 0$ for all real x.

9 Show that the line $y = 2x + 6$ cuts the parabola $y = x^2 + 3$ in 2 points.

10 Show that the line $3x + y - 4 = 0$ cuts the parabola $y = x^2 + 5x + 3$ in 2 points.

11 Show that the line $y = -x - 4$ does not touch the parabola $y = x^2$.

12 Show that the line $y = 5x - 2$ is a tangent to the parabola $y = x^2 + 3x - 1$.

13 EXT1 Find the values of k for which $x^2 + (k + 1)x + 4 = 0$ has real roots.

14 EXT1 Find values of k for which the expression $kx^2 + 3kx + 9 > 0$ for all real x.

15 EXT1 Find the values of m for which the quadratic equation $x^2 - 2mx + 9 = 0$ has real and different roots.

WS
A page o parabolas

4.12 Finding a quadratic equation

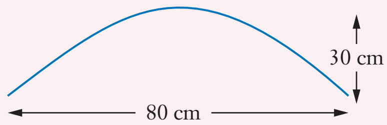
EXAMPLE 31

a Find the equation of the parabola that passes through the points (–1, –3), (0, 3) and (2, 21).

b A parabolic satellite dish is built so it is 30 cm deep and 80 cm wide, as shown.

i Find an equation for the parabola.

ii Find the depth of the dish 10 cm out from the vertex.



Solution

a The parabola has equation in the form $y = ax^2 + bx + c$.

Substitute the points into the equation.

(–1, –3):

$$-3 = a(-1)^2 + b(-1) + c$$
$$= a - b + c$$
$$\therefore a - b + c = -3 \qquad [1]$$

(0, 3):

$$3 = a(0)^2 + b(0) + c$$
$$= c$$
$$\therefore c = 3 \qquad [2]$$

(2, 21):

$$21 = a(2)^2 + b(2) + c$$
$$= 4a + 2b + c$$
$$\therefore 4a + 2b + c = 21 \qquad [3]$$

Solve simultaneous equations to find a, b and c.

Substitute [2] into [1]:

$$a - b + 3 = -3$$
$$a - b = -6 \qquad [4]$$

Substitute [2] into [3]:

$$4a + 2b + 3 = 21$$
$$4a + 2b = 18 \qquad [5]$$

[4] × 2:

$$2a - 2b = -12 \qquad [6]$$

[5] + [6]:

$$6a = 6$$
$$a = 1$$

Substitute $a = 1$ into [5]:

$$4(1) + 2b = 18$$
$$4 + 2b = 18$$
$$2b = 14$$
$$b = 7$$

$$\therefore a = 1, b = 7, c = 3$$

Thus the parabola has equation $y = x^2 + 7x + 3$.

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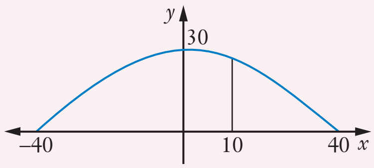
b **i** We can put the dish onto a number plane as shown. Since the parabola is symmetrical, the width of 80 cm means 40 cm either side of the y-axis.



The parabola passes through points (0, 30), (40, 0) and (–40, 0).

Substitute these points into $y = ax^2 + bx + c$.

(0, 30): $30 = a(0)^2 + b(0) + c = c$

So $y = ax^2 + bx + 30$

Substitute (40, 0) into $y = ax^2 + bx + 30$:

$0 = a(40)^2 + b(40) + 30$

$0 = 1600a + 40b + 30$ [1]

Substitute (–40, 0) into $y = ax^2 + bx + 30$:

$0 = a(-40)^2 + b(-40) + 30$

$0 = 1600a - 40b + 30$ [2]

[1] + [2]:

$$0 = 3200a + 60$$

$$a = \frac{-60}{3200}$$

$$= -\frac{3}{160}$$

Substitute a into [1]:

$$0 = 1600\left(-\frac{3}{160}\right) + 40b + 30$$

$$= -30 + 40b + 30$$

$$= 40b$$

$$0 = b$$

So $y = -\frac{3}{160}x^2 + 30$

ii Substitute $x = 10$:

$$y = -\frac{3}{160}(10)^2 + 30$$

$$= 28.125$$

So the depth of the dish at 10 cm is 28.125 cm.

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Exercise 4.12 Finding a quadratic equation

1 The braking distance of a car travelling at 100 km/h is 40 metres. The formula for braking distance (d) in metres is $d = kx^2$ where k is a constant and x is speed in km/h.

- **a** Find the value of k.
- **b** Find the braking distance at 80 km/h.
- **c** A dog runs out onto the road 15 m in front of a car travelling at 50 km/h. Will the car be able to stop in time without hitting the dog?
- **d** If the dog was 40 m in front of a car travelling at 110 km/h, would the car stop in time?

2 The area (A) of a figure is directly proportional to the square of its length (x). When $x = 5$ cm, its area is 125 cm^2.

- **a** Find the equation for the area.
- **b** Find the area when the length is 4.2 cm.
- **c** Find the length correct to 1 decimal place when the area is 250 cm^2.

3 The volume of a cylinder is given by $V = \pi r^2 h$ where r is the radius and h is height.

- **a** Find the equation for volume if the height is fixed at 8 cm.
- **b** Find the volume of a cylinder with radius 5 cm.
- **c** Find the radius if the volume is 100 cm^3.

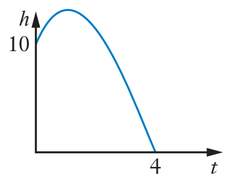
4 A rectangle has sides x and $3 - x$.

- **a** Write an equation for its area.
- **b** Draw the graph of the area.
- **c** Find the value of x that gives the maximum area.
- **d** Find the maximum area of the rectangle.

5 Find the equation of the parabola that passes through the points:

- **a** (0, –5), (2, –3) and (–3, 7)
- **b** (1, –2), (3, 0) and (–2, 10)
- **c** (–2, 21), (1, 6) and (–1, 12)
- **d** (2, 3), (1, –4) and (–1, –12)
- **e** (0, 1), (–2, 1) and (2, –7)

6 Grania throws a ball off a 10 m high cliff. After 1 s it is 22.5 m above ground and it reaches the ground after 4 s.



- **a** Find the equation for the height (h metres) of the ball after time t seconds.
- **b** Find the height of the ball after 2 seconds.
- **c** Find when the ball is in line with the cliff.

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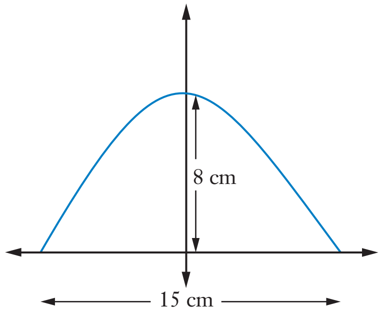
7 A parabolic shaped headlight is 15 cm wide and 8 cm deep as shown.



a Find an equation for the parabola.

b Find the depth of the headlight at a point 3 cm out from its axis of symmetry.

c At what width from the axis of symmetry does the headlight have a depth of 5 cm?

8 a Find the equation of the parabola passing through (0, 0), (3, –3) and (–1, 5).

b Find the value of y when:

i $x = 5$ **ii** $x = -4$

c Find values of x when $y = -4$.

d Find exact values of x when $y = 2$.

9 a Find the equation of the quadratic function $f(x)$ that passes through points (1, 10), (0, 7) and (–1, 6).

b Evaluate $f(-5)$.

c Show that $f(x) > 0$ for all x.

10 Find the equation of a parabola with axis of symmetry $x = 1$, minimum value –2 and passing through (0, 0).

11 Find the equation of the quadratic function with axis $x = 3$, maximum value 13 and passing through (0, 4).

WS Cubic uncions

WS Graphing cubics

WS Graphing cubics 2

4.13 Cubic functions

A **cubic function** has an equation where the highest power of x is 3, such as $f(x) = kx^3$, $f(x) = k(x - b)^3 + c$ and $f(x) = k(x - a)(x - b)(x - c)$ where a, b, c and k are constants.

EXAMPLE 32

a Sketch the graph of the cubic function $f(x) = x^3 + 2$.

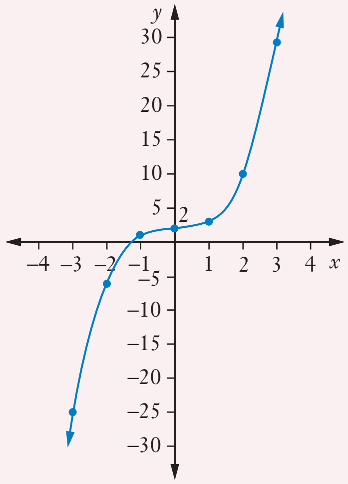
b State its domain and range.

c Solve the equation $x^3 + 2 = 0$ graphically.

Solution

a Draw up a table of values.

x	−3	−2	−1	0	1	2	3
y	−25	−6	1	2	3	10	29



b The function can have any real x or y value.

Domain $(-\infty, \infty)$

Range $(-\infty, \infty)$

c From the graph, the x-intercept is approximately −1.3.

So the root of $x^3 + 2 = 0$ is approximately $x = -1.3$.

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INVESTIGATION

TRANSFORMING CUBIC FUNCTIONS

Use a graphics calculator or graphing software to sketch the graphs of some cubic functions, such as:

$y = x^3$	$y = x^3 + 1$	$y = x^3 + 3$	$y = x^3 - 1$
$y = x^3 - 2$	$y = 2x^3$	$y = 3x^3$	$y = -x^3$
$y = -2x^3$	$y = -3x^3$	$y = 2x^3 + 1$	$y = (x + 1)^3$
$y = (x + 2)^3$	$y = (x - 1)^3$	$y = 2(x - 2)^3$	$y = 3(x + 2)^3 + 1$

$y = (x - 1)(x - 2)(x - 3)$ $\quad y = x(x + 1)(x + 4)$ $\quad y = 2(x + 1)(x - 2)(x + 5)$

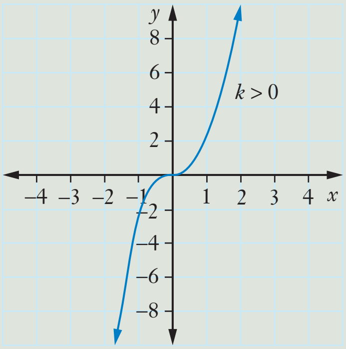
Can you see any patterns? Could you describe the shape of the cubic function? Could you predict where the graphs of different cubic functions would lie?

Is the cubic graph always a function? Can you find an example of a cubic that is not a function?

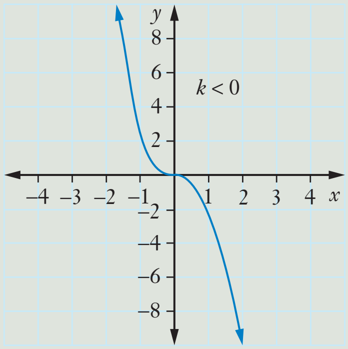
Point of inflection

The flat turning point of the cubic function $y = kx^3$ is called a **point of inflection**, which is where the concavity of the curve changes.

The graph of $y = kx^3$



This cubic curve is increasing and has a point of inflection at (0, 0) where the curve changes from concave downwards to concave upwards.



This cubic curve is decreasing and has a point of inflection at (0, 0) where the curve changes from concave upwards to concave downwards.

The graph of $y = k(x - b)^3 + c$

The graph of $y = k(x - b)^3 + c$ is the graph of $y = kx^3$ shifted so that its point of inflection is at (b, c).

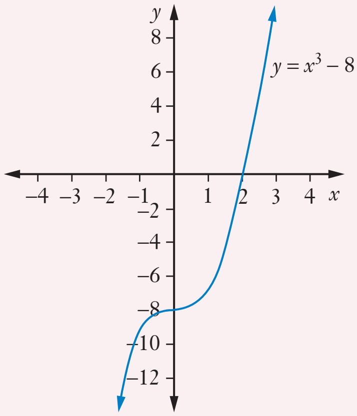
EXAMPLE 33

a Sketch the graph of $y = x^3 - 8$, showing intercepts.

b Sketch the graph of $f(x) = -2(x - 3)^3 + 2$.

Solution

a This is the graph of $y = x^3$ shifted downwards 8 units so that its point of inflection is at $(0, -8)$. Since $k > 0$, the function is increasing.



For x-intercepts, $y = 0$:

$0 = x^3 - 8$

$8 = x^3$

$x = 2$

For y-intercept, $x = 0$:

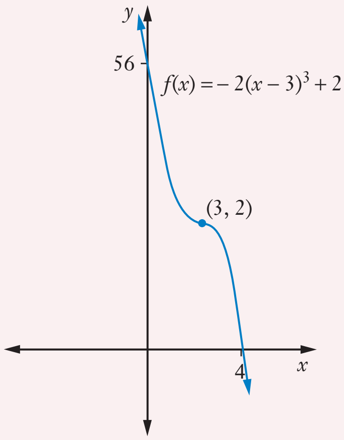
$y = 0^3 - 8$

$= -8$

The point of inflection is at $(0, -8)$, where the curve changes from concave downwards to concave upwards.

b Since $k < 0$, $f(x)$ is decreasing.

This is the graph of $y = -2x^3$ shifted upwards and to the right so that its point of inflection is at $(3, 2)$.



For x-intercepts, $f(x) = 0$:

$0 = -2(x - 3)^3 + 2$

$-2 = -2(x - 3)^3$

$1 = (x - 3)^3$

$1 = x - 3$

$x = 4$

For y-intercept, $x = 0$:

$y = -2(0 - 3)^3 + 2$

$= -2(-27) + 2$

$= 56$

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EXAMPLE 34

Show that $y = 2x^3$ is an odd function.

Solution

Let $f(x) = 2x^3$.

$$f(-x) = 2(-x)^3$$
$$= -2x^3$$
$$= -f(x)$$

So $y = 2x^3$ is an odd function.

A cubic function has one y-intercept and up to 3 x-intercepts. We can sketch the graph of a more general cubic function using intercepts. This will not give a very accurate graph but it will show the shape and important features.

The graph of $y = k(x - a)(x - b)(x - c)$

The graph of $y = k(x - a)(x - b)(x - c)$ has x-intercepts at a, b and c.

EXAMPLE 35

a **i** Sketch the graph of the cubic function $f(x) = x(x + 3)(x - 2)$.

ii Describe the shape of the graph and state its domain and range.

b Sketch the graph of the cubic function $f(x) = (x - 3)(x + 1)^2$ and describe its shape.

Solution

a **i** For x-intercepts, $f(x) = 0$:

$0 = x(x + 3)(x - 2)$

$x = 0, -3, 2$

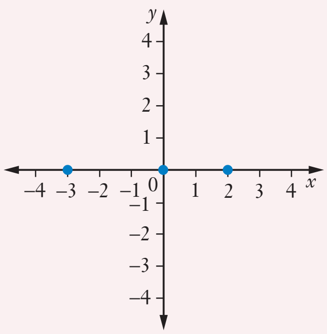
Plot x-intercepts on graph.

For y-intercept, $x = 0$:

$f(0) = 0(0 + 3)(0 - 2)$

$= 0$

So y-intercept is 0.



We look at which parts of the graph are above and which are below the x-axis between the x-intercepts.

Test $x < -3$, say $x = -4$:

$f(-4) = -4(-4 + 3)(-4 - 2) = -24 < 0$

So here the curve is below the x-axis.

Test $-3 < x < 0$, say $x = -1$:

$f(-1) = -1(-1 + 3)(-1 - 2) = 6 > 0$

So here the curve is above the x-axis.

Test $0 < x < 2$, say $x = 1$:

$f(1) = 1(1 + 3)(1 - 2) = -4 < 0$

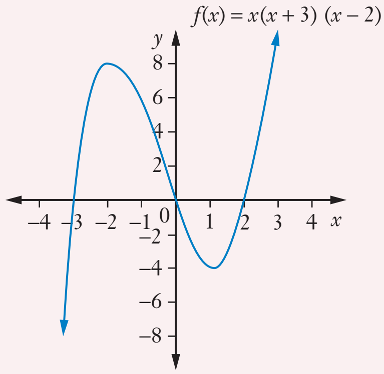
So here the curve is below the x-axis.

Test $x > 2$, say $x = 3$:

$f(3) = 3(3 + 3)(3 - 2) = 18 > 0$

So here the curve is above the x-axis.

We can sketch the cubic curve as shown.



ii The graph increases to a maximum turning point, then decreases to a minimum turning point. Then it increases again.

Domain $(-\infty, \infty)$

Range $(-\infty, \infty)$

b For x-intercepts, $f(x) = 0$:

$0 = (x - 3)(x + 1)^2$

$x = 3, x = -1$

So x-intercepts are -1 and 3.

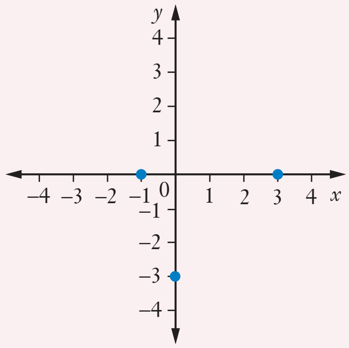
For y-intercept, $x = 0$:

$f(0) = (0 - 3)(0 + 1)^2$

$= (-3)(1)$

$= -3$

So y-intercept is -3.



We look at which parts of the graph are above and below the x-axis.

Test $x < -1$, say $x = -2$:

$f(-2) = (-2 - 3)(-2 + 1)^2 = -5 < 0$

So here the curve is below the x-axis.

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Test $-1 < x < 3$, say $x = 0$:

$f(0) = (0 - 3)(0 + 1)^2 = -3 < 0$

So here the curve is below the x-axis.

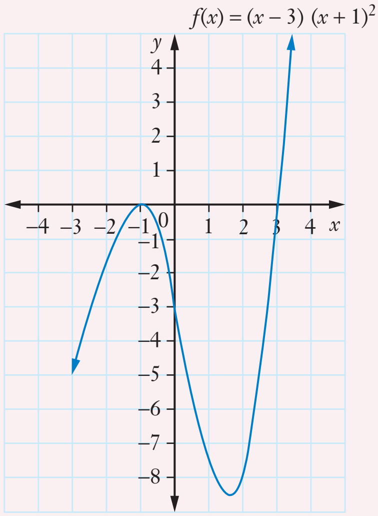
Test $x > 3$, say $x = 4$:

$f(4) = (4 - 3)(4 + 1)^2 = 25 > 0$

So here the curve is above the x-axis.

We can sketch the cubic curve as shown.

The graph increases to a maximum turning point, then decreases to a minimum turning point, then increases.



Finding a cubic equation

EXAMPLE 36

a Find the equation of the cubic function $y = kx^3 + c$ if it passes through (0, 16) and (4, 0).

b Find the equation of the cubic function $f(x) = k(x - a)(x - b)(x - c)$ if it has x-intercepts -1, 3 and 4 and passes through (1, 12).

Solution

a Substitute (0, 16) into $y = kx^3 + c$.

$16 = k(0)^3 + c$

$= c$

So $y = kx^3 + 16$.

Substitute (4, 0) into $y = kx^3 + 16$.

$0 = k(4)^3 + 16$

$= 64k + 16$

$-16 = 64k$

$k = -\frac{16}{64}$

$= -\frac{1}{4}$

So the equation is $y = -\frac{1}{4}x^3 + 16$.

b $f(x) = k(x - a)(x - b)(x - c)$ has x-intercepts when $f(x) = 0$.

$0 = k(x - a)(x - b)(x - c)$

$x = a, b, c$

But we know x-intercepts are at -1, 3 and 4.

So $a = -1$, $b = 3$ and $c = 4$ (in any order).

So $f(x) = k(x - (-1))(x - 3)(x - 4)$

$= k(x + 1)(x - 3)(x - 4)$

To find k, substitute (1, 12):

$12 = k(1 + 1)(1 - 3)(1 - 4)$

$= k(2)(-2)(-3)$

$= 12k$

$1 = k$

So the cubic function is $f(x) = (x + 1)(x - 3)(x - 4)$.

Exercise 4.13 Cubic functions

1 Find the x- and y-intercept(s) of the graph of each cubic function.

a $y = x^3 - 1$ **b** $f(x) = -x^3 + 8$ **c** $y = (x + 5)^3$

d $f(x) = -(x - 4)^3$ **e** $f(x) = 3(x + 7)^3 - 3$ **f** $y = (x - 2)(x - 1)(x + 5)$

2 Draw each graph on a number plane.

a $y = -x^3$ **b** $p(x) = 2x^3$ **c** $g(x) = x^3 + 1$

d $y = (x + 2)^3$ **e** $y = -(x - 3)^3 + 1$ **f** $f(x) = -x(x + 2)(x - 4)$

g $y = (x + 2)(x - 3)(x + 6)$ **h** $y = x^2(x - 2)$ **i** $f(x) = (x - 1)(x + 3)^2$

3 Find the point of inflection of the graph of each cubic function by sketching each graph.

a $y = 8x^3 + 1$ **b** $y = -x^3 + 27$ **c** $f(x) = (x + 2)^3$

d $y = 2(x - 1)^3 - 16$ **e** $f(x) = -(x + 1)^3 + 1$

4 Find the x-intercept of the graph of each cubic function correct to one decimal place.

a $y = 2x^3 - 5$ **b** $f(x) = (x - 1)^3 + 2$ **c** $f(x) = -3x^3 + 1$

d $y = 2(x + 3)^3 - 3$ **e** $y = -3(2x - 1)^3 + 2$

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5 Describe the shape of each cubic function.

a $y = x^3 - 64$ **b** $f(x) = -(x - 3)^3$ **c** $y = x(x + 2)(x + 4)$

d $f(x) = -2(x + 3)(x + 1)(x - 4)$ **e** $y = x(x + 5)^2$

6 Solve graphically:

a $x^3 - 5 = 0$ **b** $x^3 + 2 = 0$ **c** $2x^3 - 9 = 0$

d $3x^3 + 4 = 0$ **e** $(x - 1)^3 + 6 = 0$ **f** $x(x + 2)(x - 1) = 0$

7 The volume of a certain solid has equation $V = kx^3$ where x is the length of its side in cm.

a Find the equation if $V = 120$ when $x = 3.5$.

b Find the volume when $x = 6$.

c Find x when $V = 250$.

8 The volume of a solid is directly proportional to the cube of its radius.

a If radius $r = 12$ mm when the volume V is 7238 mm^3, find an equation for the volume.

b Find the volume if the radius is 2.5 mm.

c Find the radius if the volume is 7000 mm^3.

9 Show that $f(x) = -x^3$ is an odd function.

10 Determine whether each function is odd.

a $y = 3x^3$ **b** $y = (x + 1)^3$ **c** $f(x) = -2x^3 - 1$

d $y = -5x^3$ **e** $y = (x - 2)^3 + 3$

11 A cubic function is in the form $y = kx^3 + c$. Find its equation if it passes through:

a (0, 0) and (1, 2) **b** (0, 5) and (2, –3) **c** (1, –4) and (–2, 23)

d (1, –2) and (2, 33) **e** (2, –29) and (–3, 111)

12 A cubic function is in the form $y = k(x - a)(x - b)(x - c)$. Find its equation if:

a it has x-intercepts 2, 3 and –5 and passes through the point (–2, –120)

b it has x-intercepts –1, 4 and 6 and passes through the point (3, 96)

c it has x-intercepts 1 and 3, y-intercept –27 and $k = -3$.

4.14 Polynomial functions

Graphing power functions

A **polynomial** is a function defined for all real x involving powers of x in the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

where n is a positive integer or zero and $a_0, a_1, a_2, \ldots, a_n$ are real numbers.

We generally write polynomials from the highest power down to the lowest, for example $P(x) = x^2 - 5x + 4$. We have already studied some polynomial functions, as linear, quadratic and cubic functions are all polynomials.

Polynomial terminology

$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ is called a **polynomial expression**.

$P(x)$ has **degree** n (where n is the highest power of x).

$a_n, a_{n-1}, a_{n-2}, \ldots a_2, a_1$ and a_0 are called **coefficients**.

$a_n x^n$ is called the **leading term**.

a_n is the **leading coefficient**.

a_0 is called the **constant term**.

If $a_n = 1$, $P(x)$ is called a **monic polynomial**.

EXAMPLE 37

a Which of the following are polynomial expressions?

A $4 - x + 3x^2$ **B** $3x^4 - x^2 + 5x - 1$ **C** $x^2 - 3x + x^{-1}$

b $P(x) = x^6 - 2x^4 + 3x^3 + x^2 - 7x - 3$.

i Find the degree of $P(x)$.

ii Is the polynomial monic?

iii State the leading term.

v What is the constant term?

v Find the coefficient of x^4.

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Solution

a **A** and **B** are polynomials but **C** is not, because it has a term of x^{-1} that is not a positive integer power of x.

b **i** Degree is 6 since x^6 is the highest power.

ii Yes, the polynomial is monic because the coefficient of x^6 is 1.

iii The leading term is x^6.

v The constant term is –3.

v The coefficient of x^4 is –2.

Polynomial equations

$P(x) = 0$ is a **polynomial equation** of degree n.

The values of x that satisfy the equation are called the **roots** of the equation or the **zeros** of the polynomial $P(x)$.

EXAMPLE 38

a Find the zeros of the polynomial $P(x) = x^2 - 5x$.

b Show that the polynomial $p(x) = x^2 - x + 4$ has no real zeros.

Solution

a To find the zeros of the polynomial, solve $P(x) = 0$.

$$x^2 - 5x = 0$$

$$x(x - 5) = 0$$

$x = 0, 5$

So the zeros are 0, 5.

b Solve $p(x) = 0$.

$x^2 - x + 4 = 0$

The discriminant will show whether the polynomial has real zeros.

$$b^2 - 4ac = (-1)^2 - 4(1)(4)$$
$$= -15$$
$$< 0$$

So the polynomial has no real zeros.

Graphing polynomials

EXAMPLE 39

a Write the polynomial $P(x) = x^4 + 2x^3 - 3x^2$ as a product of its factors.

b Sketch the graph of the polynomial.

Solution

a $P(x) = x^4 + 2x^3 - 3x^2$

$= x^2(x^2 + 2x - 3)$

$= x^2(x + 3)(x - 1)$

b For x-intercepts, $P(x) = 0$:

$0 = x^4 + 2x^3 - 3x^2$

$= x^2(x + 3)(x - 1)$

$x = 0, -3, 1$

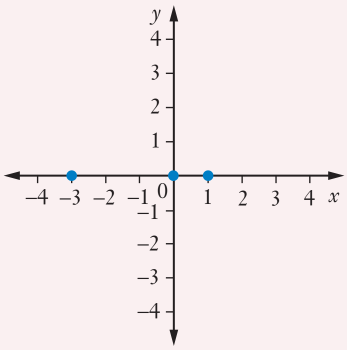
So the x-intercepts are $-3, 0, 1$.

For y-intercepts, $x = 0$:

$P(0) = 0^4 + 2(0)^3 - 3(0)^2$

$= 0$

So y-intercept is 0.



Test $x < -3$, say $x = -4$:

$P(-4) = (-4)^4 + 2(-4)^3 - 3(-4)^2 = 80 > 0$

So here the curve is above the x-axis.

Test $-3 < x < 0$, say $x = -1$:

$P(-1) = (-1)^4 + 2(-1)^3 - 3(-1)^2 = -4 < 0$

So here the curve is below the x-axis.

Test $0 < x < 1$, say $x = 0.5$:

$P(0.5) = (0.5)^4 + 2(0.5)^3 - 3(0.5)^2$

$= -0.4375 < 0$

So here the curve is below the x-axis.

Test $x > 1$, say $x = 3$:

$P(3) = 3^4 + 2(3)^3 - 3(3)^2 = 108 > 0$

So here the curve is above the x-axis.

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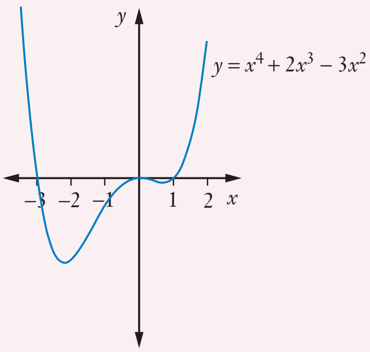
We can sketch the graph of the polynomial as shown.



DID YOU KNOW?

'Poly' means many

The word 'polynomial' means an expression with many terms. (A binomial has 2 terms and a trinomial has 3 terms.) 'Poly' means 'many', and is used in many words, for example polygamy, polyglot, polygon, polyhedron, polymer, polyphonic, polypod and polytechnic. Do you know what all these words mean? Do you know any others with 'poly-'?

Exercise 4.14 Polynomial functions

1 Write down the degree of each polynomial:

a $5x^7 - 3x^5 + 2x^3 - 3x + 1$ **b** $3 + x + x^2 - x^3 + 2x^4$ **c** $3x + 5$

d $x^{11} - 5x^8 + 4$ **e** $2 - x - 5x^2 + 3x^3$ **f** 3

2 For the polynomial $P(x) = x^3 - 7x^2 + x - 1$, find:

a $P(2)$ **b** $P(-1)$ **c** $P(0)$

3 Given $P(x) = x + 5$ and $Q(x) = 2x - 1$, find:

a $P(-11)$ **b** $Q(3)$ **c** $P(2) + Q(-2)$

d the degree of $P(x) + Q(x)$ **e** the degree of $P(x)Q(x)$

4 For the polynomial $P(x) = x^5 - 3x^4 - 5x + 4$, find:

a the degree of $P(x)$ **b** the constant term

c the coefficient of x^4 **d** the coefficient of x^2.

5 Find the zeros of each polynomial.

a $P(x) = x^2 - 9$ **b** $p(x) = x + 5$ **c** $f(x) = x^2 + x - 2$

d $P(x) = x^2 - 8x + 16$ **e** $g(x) = x^3 - 2x^2 + 5x$

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6 Which of the following are not polynomials?

a $5x^4 - 3x^2 + x + \frac{1}{x}$ **b** $x^2 + 3^x$ **c** $x^2 + 3x - 7$

d $3x + 5$ **e** 0 **f** $4x^3 + 7x^{-2} + 5$

7 For the polynomial $P(x) = (a + 1)x^3 + (b - 7)x^2 + c + 5$, find values for a, b or c if:

a $P(x)$ is monic **b** the coefficient of x^2 is 3

c the constant term is -1 **d** $P(x)$ has degree 2

e the leading term has a coefficient of 5.

8 Given $P(x) = 2x + 5$, $Q(x) = x^2 - x - 2$ and $R(x) = x^3 + 9x$, find:

a any zeros of $P(x)$ **b** the roots of $Q(x) = 0$

c the degree of $P(x) + R(x)$ **d** the degree of $P(x)Q(x)$

e the leading term of $Q(x)R(x)$.

9 Given $f(x) = 3x^2 - 2x + 1$ and $g(x) = 3x - 3$:

a show $f(x)$ has no zeros **b** find the leading term of $f(x)g(x)$

c find the constant term of $f(x) + g(x)$ **d** find the coefficient of x in $f(x)g(x)$

e find the roots of $f(x) + g(x) = 0$.

10 State how many real roots there are for each polynomial equation $P(x) = 0$.

a $P(x) = x^2 - 9$ **b** $P(x) = x^2 + 4$

c $P(x) = x^2 - 3x - 7$ **d** $P(x) = 2x^2 + x + 3$

e $P(x) = 3x^2 - 5x - 2$ **f** $P(x) = x(x - 1)(x + 4)(x + 6)$

11 Sketch the graph of each polynomial by finding its zeros and showing the x- and y-intercepts.

a $f(x) = (x + 1)(x - 2)(x - 3)$ **b** $P(x) = x\,(x + 4)(x - 2)$

c $p(x) = -x(x - 1)(x - 3)$ **d** $f(x) = x(x + 2)^2$

e $g(x) = (5 - x)(x + 2)(x + 5)$

12 **i** Write each polynomial as a product of its factors.

ii Sketch the graph of the polynomial and describe its shape.

a $P(x) = x^3 - 2x^2 - 8x$ **b** $f(x) = -x^3 - 4x^2 + 5x$

c $P(x) = x^4 + 3x^3 + 2x^2$ **d** $A(x) = 2x^3 + x^2 - 15x$

e $P(x) = -x^4 + 2x^3 + 3x^2$

13 **a** Find the x-intercepts of the polynomial $P(x) = x(x - 1)(x + 2)^2$.

b Sketch the graph of the polynomial.

14 **a** Show that $(x - 3)(x - 2)(x + 2) = x^3 - 3x^2 - 4x + 12$.

b Sketch the graph of the polynomial $P(x) = x^3 - 3x^2 - 4x + 12$.

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4.15 Intersection of graphs

Solving equations graphically

EXAMPLE 40

a Sketch $y = x^2$ and $y = 1$ on the same set of axes, and hence solve $x^2 = 1$ graphically.

b Sketch $y = x^2 - x$ and $y = 2$ on the same set of axes, and hence solve $x^2 - x = 2$ graphically.

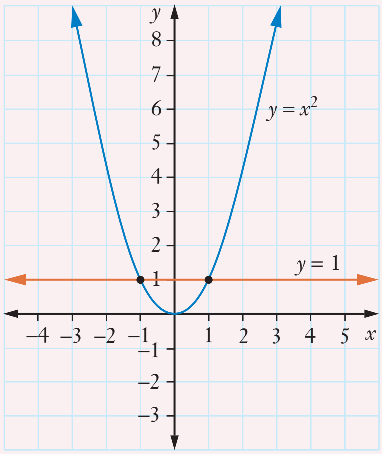
c EXT1 Solve $x^2 - x \le 2$ graphically.

Solution

a $y = x^2$ is a parabola and $y = 1$ is a horizontal line, as shown.

To solve $x^2 = 1$ graphically, find the x values where the 2 graphs $y = x^2$ and $y = 1$ intersect.

The solution is $x = \pm 1$.

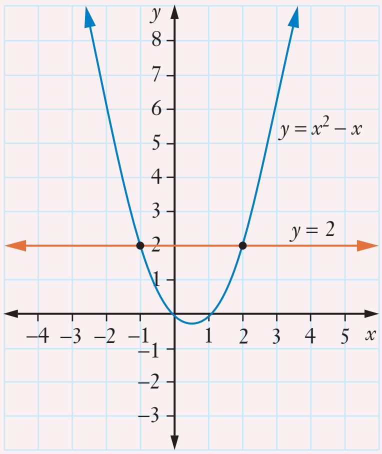


b $y = x^2 - x$ is a parabola with x-intercepts 0, 1 and y-intercept 0. Since $a > 0$, it is concave upwards.

$y = 2$ is a horizontal line.

The solutions of $x^2 - x = 2$ are the x values at the intersection of the 2 graphs.

$x = -1, 2.$



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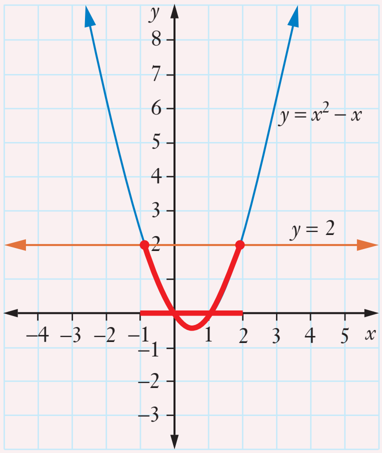
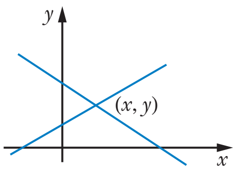
c The solutions of $x^2 - x \le 2$ are the x values at and below the intersection of the 2 graphs.

$-1 \le x \le 2.$



Intersecting lines

Two straight lines intersect at a single point (x, y).



The point of intersection can be found graphically or algebraically using simultaneous equations.

EXAMPLE 41

Find the point of intersection between lines $2x - 3y - 3 = 0$ and $5x - 2y - 13 = 0$.

Solution

Solve simultaneous equations.

$2x - 3y - 3 = 0$ [1]

$5x - 2y - 13 = 0$ [2]

[1] × 2:

$4x - 6y - 6 = 0$ [3]

[2] × 3:

$15x - 6y - 39 = 0$ [4]

[3] – [4]:

$$-11x + 33 = 0$$
$$33 = 11x$$
$$3 = x$$

Substitute $x = 3$ into [1]:

$$2(3) - 3y - 3 = 0$$
$$-3y + 3 = 0$$
$$3 = 3y$$
$$1 = y$$

So the point of intersection is (3, 1).

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Break-even points

Break-even points

EXAMPLE 42

A company that manufactures cables sells them for \$2 each. It costs 50 cents to produce each cable and the company has fixed costs of \$1500 per week.

a Find the equation for the income, $\$I$, on x cables per week.

b Find the equation for the costs, $\$C$, of manufacturing x cables per week.

c Find the break-even point (where income = costs).

d Find the profit on 1450 cables.

Solution

a $I = 2x$

b $C = 0.5x + 1500$

c Solving simultaneous equations:

$I = 2x$ [1]

$C = 0.5x + 1500$ [2]

Substitute [1] into [2]:

$2x = 0.5x + 1500$

$1.5x = 1500$

$x = 1000$

1000 cables is where income = costs.
Substitute $x = 1000$ into [1] (or [2]):

$I = 2(1000)$

$= 2000$

So the break-even point is (1000, 2000). 1000 cables gives an income and cost of \$2000.

d Profit = income – costs = $I - C$

Substitute $x = 1450$ into both equations.

$I = 2x$

$= 2(1450)$

$= 2900$

So income is \$2900.

$C = 0.5x + 1500$

$= 0.5(1450) + 1500$

$= 2225$

So costs are \$2225.

Profit = \$2900 – \$2225

= \$675

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Break-even point

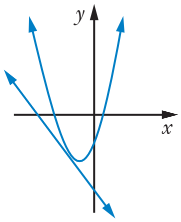
In business, the **break-even point** is the point where the **income** (or **revenue**) **equals costs.**

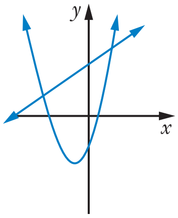
If income > costs, the business makes a **profit**.

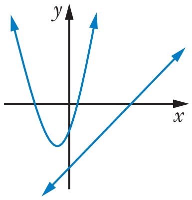
If income < costs, the business makes a **loss**.

Intersecting lines and parabolas

A line and a parabola can intersect at 1 or 2 points, or they may not intersect at all.







EXAMPLE 43

Find the points of intersection of the line $y = x - 1$ with the parabola $y = x^2 + 4x + 1$.

Solution

Solve simultaneous equations.

$y = x - 1$ [1]

$y = x^2 + 4x + 1$ [2]

Substitute [1] into [2]:

$$x - 1 = x^2 + 4x + 1$$
$$0 = x^2 + 3x + 2$$
$$= (x + 2)(x + 1)$$
$$x = -2, -1$$

Substitute $x = -2$ into [1]:

$$y = -2 - 1$$
$$= -3$$

Substitute $x = -1$ into [1]:

$$y = -1 - 1$$
$$= -2$$

So the 2 points of intersection are $(-2, -3)$ and $(-1, -2)$.

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Exercise 4.15 Intersection of graphs

1 **a** Given $f(x) = 2x - 4$, solve graphically:

i $f(x) = 0$

ii $f(x) = -2$

iii $f(x) = 4$

b By sketching the graph of $f(x) = x^2 - 2x$, solve graphically:

i $f(x) = 0$

ii $f(x) = 3$

c Use the sketch of $f(x) = x^3 - 1$ to solve graphically:

i $f(x) = 0$

ii $f(x) = 7$

iii $f(x) = -2$

d EXT1 Solve graphically:

i $x^2 - 3x > 0$

ii $x^2 - 4 \leq 0$

iii $x^2 - 7x > -12$

v $m^2 < 4m$

2 Find the point of intersection between:

a $y = x + 3$ and $y = 2x + 2$

b $y = 3x - 1$ and $y = 5x + 1$

c $x + 2y - 4 = 0$ and $2x - y + 2 = 0$

d $3x + y - 2 = 0$ and $2x - 3y - 5 = 0$

e $4x - 3y - 5 = 0$ and $7x - 2y - 12 = 0$

3 Find points of intersection between:

a $y = x^2$ and $y = x$

b $y = x^2$ and $y = 4$

c $y = x^2$ and $y = x + 2$

d $y = x^2$ and $y = -2x + 3$

e $y = x^2 - 5$ and $y = 4x$

4 **a** Draw the graphs of $f(x) = x^2$ and $f(x) = (x - 2)^2$ on the same number plane.

b From the graph, find the number of points of intersection of the functions.

c From the graph or by using algebra, find any points of intersection.

5 Find any points of intersection between the functions $f(x) = x^2$ and $f(x) = (x + 2)^2$.

6 Find any points of intersection between the curves $y = x^2 - 5$ and $y = 2x^2 + 5x + 1$.

7 Find any points of intersection between $y = 3x^2 - 4x - 4$ and $y = 5x^2 - 2$.

8 **a** If Paula's Posies' income on x roses is given by $y = 10x$ and the costs are $y = 3x + 980$, find the break-even point.

b Find the profit on 189 roses.

c Find the loss on 45 roses.

9 Find the number of calculators that a company needs to sell to break even each week if it costs \$3 to make each calculator and they are sold for \$15 each. Fixed overheads are \$852 a week.

10 Cupcakes Online sells cupcakes at \$5 each. The cost of making each cupcake is \$1 and the company has fixed overheads of \$264 a day.

a Find the equations for daily income and costs.

b Find how many cupcakes the company needs to sell daily to break even.

c What is the profit on 250 cupcakes?

d What is the loss on 50 cupcakes?

11 **a** The perimeter of a figure is in direct proportion to its side x. Find an equation for perimeter if the perimeter $y = 90$ cm when side $x = 5$ cm.

b The area of the figure is in direct proportion to the square of its side x. If the area of the figure is $y = 108$ cm^2 when $x = 3$ cm, find its equation.

c Find any x values for the side for which the perimeter and area will have the same y value.

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4. TEST YOURSELF

Practice quiz

For Questions **1** to **5**, select the correct answer **A**, **B**, **C** or **D**.

1 Which polynomial below is a monic polynomial with constant term 5 and degree 6?

A $P(x) = -x^6 + 5$ **B** $P(x) = 6x^5 - 3x^4 + 5$

C $P(x) = x^6 - 3x^4 + 5$ **D** $P(x) = 5x^6 - 3x^4 + 1$

2 The axis of symmetry and turning point of the quadratic function $f(x) = 1 + 2x - x^2$ are, respectively:

A $x = 1$, $(1, 2)$ **B** $x = -1$, $(-1, 4)$

C $x = 2$, $(2, 5)$ **D** $x = -2$, $(-2, 5)$

3 The linear function $2x - 3y - 6 = 0$ has x- and y-intercepts, respectively:

A -3 and 2 **B** 3 and -2 **C** -3 and -2 **D** 3 and 2

4 The domain and range of the straight line with equation $x = -2$ are:

A Domain $(-\infty, \infty)$, range $[-2]$ **B** Domain $[-2]$, range $(-\infty, \infty)$

C Domain $(-\infty, \infty)$, range $(-\infty, \infty)$ **D** Domain $[-2]$, range $[-2]$

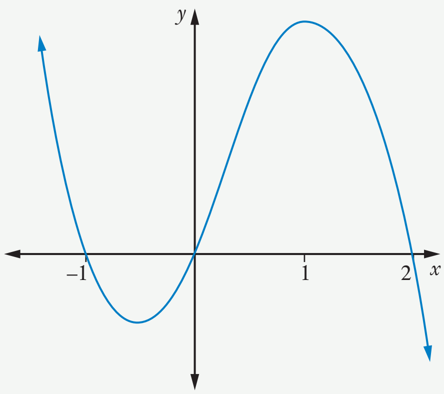
5 Which cubic function has this graph?

A $y = x(x + 1)(x - 2)$

B $y = -x(x - 1)(x + 2)$

C $y = x(x - 1)(x + 2)$

D $y = -x(x + 1)(x - 2)$



6 If $f(x) = x^2 - 3x - 4$, find:

a $f(-2)$ **b** $f(a)$ **c** x when $f(x) = 0$

7 Sketch each graph and find its domain and range.

a $y = x^2 - 3x - 4$ **b** $f(x) = x^3$ **c** $2x - 5y + 10 = 0$

d $x = 3$ **e** $y = (x + 1)^3$ **f** $y = -2$

g $f(x) = -x^2 + x$ **h** $f(x) = x^2 + 4x + 4$

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8 If $f(x) = 3x - 4$, find:

a $f(2)$ **b** x when $f(x) = 7$ **c** x when $f(x) = 0$

9 Sketch the graph of $P(x) = 2x^3 - 2x^2 - 4x$.

10 Find the gradient of the straight line:

a passing through $(3, -1)$ and $(-2, 5)$ **b** with equation $2x - y + 1 = 0$

c perpendicular to the line $5x + 3y - 8 = 0$ **d** making an angle of inclination of 45°.

11 For the parabola $y = x^2 - 4x + 1$, find:

a the equation of the axis of symmetry

b the minimum value.

12 Sketch the graph of $f(x) = (x - 2)(x + 3)^2$, showing the intercepts.

13 For the polynomial $P(x) = x^3 + 2x^2 - 3x$, find:

a the degree **b** the coefficient of x

c the zeros **d** the leading term.

14 Find the x- and y-intercepts of:

a $2x - 5y + 20 = 0$ **b** $y = x^2 - 5x - 14$

c $y = (x + 2)^3$ **d** $2x - 5y - 10 = 0$

15 Find the point of intersection between lines $y = 2x + 3$ and $x - 5y + 6 = 0$.

16 For the quadratic function $y = -2x^2 - x + 6$, find:

a the equation of the axis of symmetry

b the maximum value.

17 Find the domain and range of $y = -2x^2 - x + 6$.

18 For each quadratic equation, select the correct property of its roots **A**, **B**, **C** or **D**.

A real, different and rational **B** real, different and irrational

C equal **D** unreal.

a $2x^2 - x + 3 = 0$ **b** $x^2 - 10x - 25 = 0$ **c** $x^2 - 10x + 25 = 0$

d $3x^2 + 7x - 2 = 0$ **e** $6x^2 - x - 2 = 0$

19 Find the equation of the line:

a passing through $(2, 3)$ and with gradient 7

b parallel to the line $5x + y - 3 = 0$ and passing through $(1, 1)$

c through the origin, and perpendicular to the line $2x - 3y + 6 = 0$

d through $(3, 1)$ and $(-2, 4)$

e with x-intercept 3 and y-intercept -1.

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20 The polynomial $f(x) = ax^2 + bx + c$ has zeros 4 and 5, and $f(-1) = 60$. Evaluate a, b and c.

21 Determine whether each function is even, odd or neither.

a $y = x^2 - 1$ **b** $y = x + 1$ **c** $y = x^3$

d $y = (x + 1)^2$ **e** $y = -5x^3$

22 Show that $f(x) = x^3 - x$ is odd.

23 Prove that the line between (–1 4) and (, 3) is perpendicular to the line $4 \quad x - y - 6 = 0$.

24 Show that $-4 + 3x - x^2 < 0$ for all x.

25 For each pair of equations, state whether their graphs have 0, 1 or 2 points of intersection.

a $xy = 7$ and $3x - 5y - 1 = 0$ **b** $x^2 + y^2 = 9$ and $y = 3x - 3$

c $x^2 + y^2 = 1$ and $x - 2y - 3 = 0$ **d** $y = x^2$ and $y = 4x - 4$

e EXT1 $y = \frac{2}{x}$ and $y = 3x + 1$

26 Prove that the lines with equations $y = 5x - 7$ and $10x - 2y + 1 = 0$ are parallel.

27 Find the zeros of $g(x) = -x^2 + 9x - 20$.

28 Sketch the graph of $P(x) = 2x(x - 3)(x + 5)$, showing intercepts.

29 Solve $P(x) = 0$ when $P(x) = x^3 - 4x^2 + 4x$.

30 Find x if the gradient of the line through (3 –4) and (x, 2) is –5.

31 If $f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x^2 - 3 & \text{if } x < 1 \end{cases}$, find $f(5) - f(0) + f(1)$.

32 Given $f(x) = \begin{cases} 3 & \text{if } x > 3 \\ x^2 & \text{if } 1 \leq x \leq 3 \\ 2 - x & \text{if } x < 1 \end{cases}$

find:

a $f(2)$ **b** $f(-3)$ **c** $f(3)$

d $f(5)$ **e** $f(0)$

33 Find the equation of the parabola:

a that passes through the points (–2, 18), (3, –2) and (1, 0)

b with x-intercepts 3 and –2 and y-intercept 12.

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34 The area (A) of a certain shape is in direct proportion to the square of its length x. If the area is 448 cm^2 when $x = 8$, find:

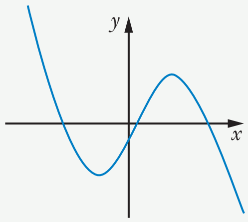
a the equation for area

b the area when $x = 10$

c x when the area is 1093.75 cm^2.

35 For each graph and set of ordered pairs, state whether it represents a function, and for those that do, whether it represents a one-to-one function.

a



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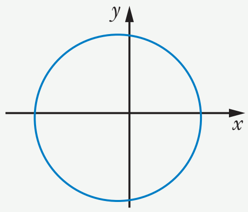

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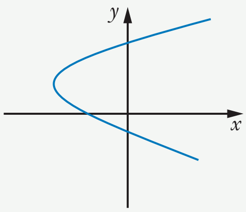

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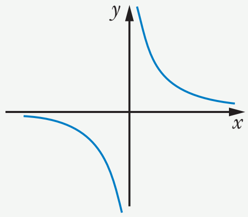

e (1, 2), (2, 5), (–1, 4), (1, 3), (3, 4)

36 Find the equation of a cubic function $f(x) = kx^3 + c$ if it passes through the point (1, 2) and has y-intercept 5.

37 A company has costs given by $y = 7x + 15$ and income $y = 12x$. Find the break-even point.

38 **a** Find the equation of the straight line that is perpendicular to the line $y = \frac{1}{2}x - 3$ and passes through (1, –1).

b Find the x-intercept of this line.

39 Find values of m such that $mx^2 + 3x - 4 < 0$ for all x.

40 Find any points of intersection of the graphs of:

a $y = 3x - 4$ and $y = 1 - 2x$

b $y = x^2 - x$ and $y = 2x - 2$

c $y = x^2$ and $y = 2x^2 - 9$

41 Find the equation of the straight line passing through the origin and parallel to the line with equation $3x - 4y + 5 = 0$.

42 Find the equation of the line with y-intercept –2 and perpendicular to the line passing through (3, –2) and (0, 5).

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43 The amount of petrol used in a car is directly proportional to the distance travelled.

a If the car uses 10.8 litres of petrol for an 87 km trip, find the equation for the amount of petrol used (A) over a distance of d km.

b Find the amount of petrol used for a 250 km trip.

c Find how far the car travelled if it used 35.5 L of petrol.

44 EXT1 Solve each inequality.

a $x^2 - 3x \le 0$ **b** $n^2 - 9 > 0$ **c** $4 - y^2 \ge 0$

45 A function has equation $f(x) = x^3 - x^2 - 4x + 4$.

a Solve $f(x) = 0$.

b Find its x- and y-intercepts.

c Sketch the graph of the function.

d From the graph, state how many solutions there are for:

i $f(x) = 1$

ii $f(x) = -2$

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4. CHALLENGE EXERCISE

1 Find the values of b if $f(x) = 3x^2 - 7x + 1$ and $f(b) = 7$.

2 Sketch the graph of $y = (x + 2)^2 - 1$ in the domain $[-3, 0]$.

3 If points $(-3k, 1)$, $(k - 1, k - 3)$ and $(k - 4, k - 5)$ are collinear (lie on a straight line), find the value of k.

4 Find the equation of the line that passes through the point of intersection of the lines $2x + 5y + 19 = 0$ and $4x - 3y - 1 = 0$ and is perpendicular to the line $3x - 2y + 1 = 0$.

5 If $ax - y - 2 = 0$ and $bx - 5y + 11 = 0$ intersect at the point $(3, 4)$, find the values of a and b.

6 By writing each as a quadratic equation, solve:

a $(3x - 2)^2 - 2(3x - 2) - 3 = 0$

b $5^{2x} - 26(5^x) + 25 = 0$

c $2^{2x} - 10(2^x) + 16 = 0$

d $2^{2x+1} - 5(2^x) + 2 = 0$

e EXT1 $\left(x+\frac{1}{x}\right)^2 - 5\left(x+\frac{1}{x}\right)+6=0$

7 Find the equation of the straight line through $(1, 3)$ that passes through the intersection of the lines $2x - y + 5 = 0$ and $x + 2y - 5 = 0$.

8 $f(x) = \begin{cases} 2x+3 & \text{when } x > 2 \\ 1 & \text{when } -2 \le x \le 2 \\ x^2 & \text{when } x < -2 \end{cases}$

Find $f(3)$, $f(-4)$, $f(0)$ and sketch the graph of the function.

9 If $h(t) = \begin{cases} 1-t^2 & \text{if } t > 1 \\ t^2-1 & \text{if } t \le 1 \end{cases}$, find the value of $h(2) + h(-1) - h(0)$ and sketch the curve.

10 If $f(x) = 2x^3 - 2x^2 - 12x$, find x when $f(x) = 0$.

11 Show that the quadratic equation $2x^2 - kx + k - 2 = 0$ has real rational roots.

12 Find the values of p for which $x^2 - x + 3p - 2 > 0$ for all x.

13 If $f(x) = 2x - 1$ show that $f(a^2) = f[(-a)^2]$ for all real a.

14 Find the equation of the straight line through $(3, -4)$ that is perpendicular to the line with x-intercept -2 and y-intercept 5.

15 Find any points of intersection between $y = x^2$ and $y = x^3$.

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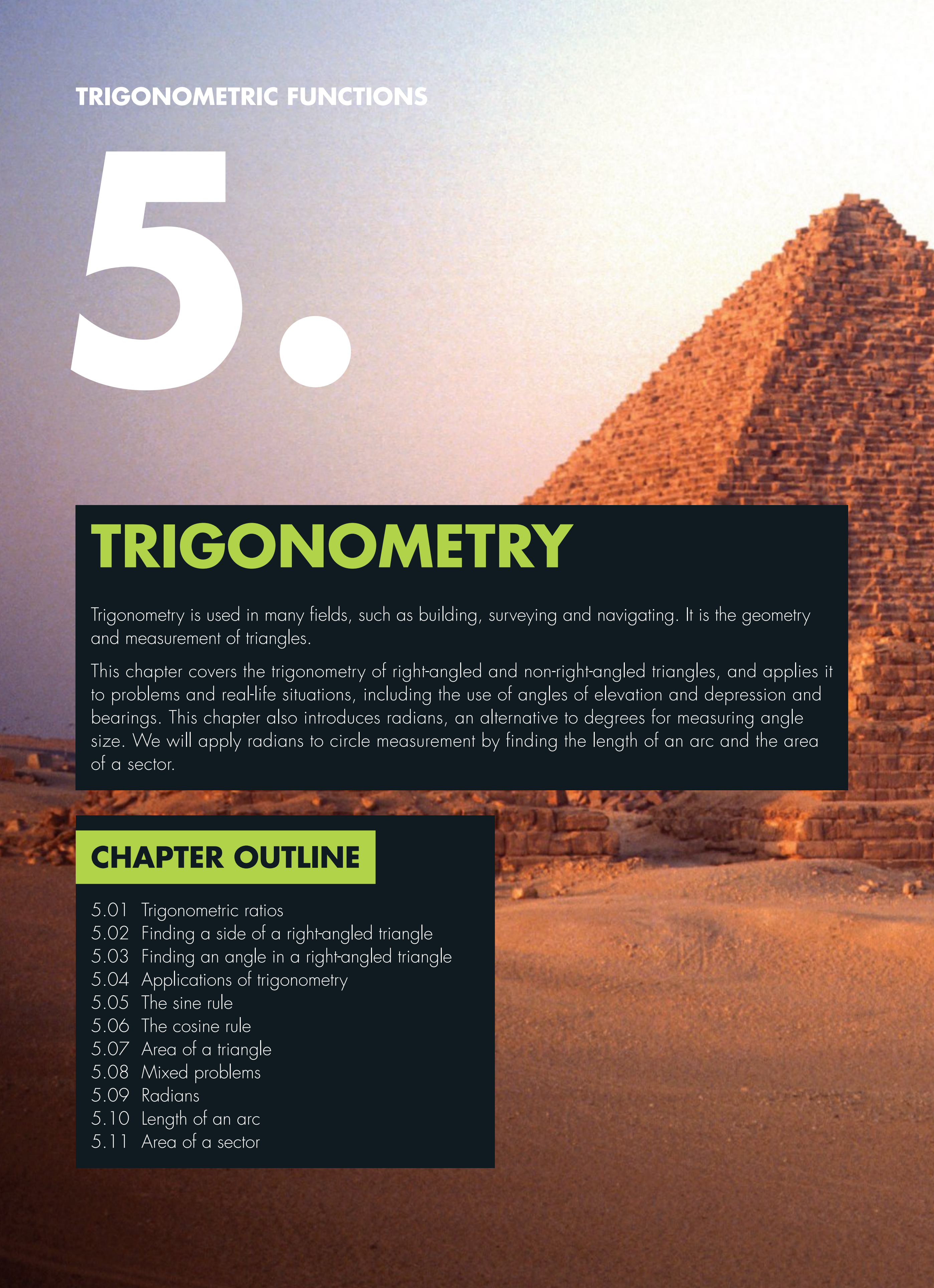
16 Find the equation of a cubic function $y = ax^3 + bx^2 + cx + d$ if it passes through (0, 1), (1, 3), (–1, 3) and (2, 15).

17 Show that the quadratic equation $x^2 - 2px + p^2 = 0$ has equal roots.

18 EXT1 Solve $x^2 + 1 + \frac{25}{x^2 + 1} = 10$.

19 EXT1 Find exact values of k for which $x^2 + 2kx + k + 5 = 0$ has real roots.

20 A monic polynomial $P(x)$ of degree 3 has zeros –2, 1 and 6. Write down the equation of the polynomial.



5. TRIGONOMETRY

Trigonometry is used in many fields, such as building, surveying and navigating. It is the geometry and measurement of triangles.

This chapter covers the trigonometry of right-angled and non-right-angled triangles, and applies it to problems and real-life situations, including the use of angles of elevation and depression and bearings. This chapter also introduces radians, an alternative to degrees for measuring angle size. We will apply radians to circle measurement by finding the length of an arc and the area of a sector.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- identify the trigonometric ratios
- solve right-angled triangle problems
- apply trigonometry to angles of elevation and depression and bearings
- understand and apply the sine and cosine rules
- find the area of a triangle given the length of two sides and the size of their included angle
- understand radians and convert between degrees and radians
- find the length of an arc and area of a sector of a circle

TERMINOLOGY

ambiguous case: When using the sine rule to find an angle, there may be 2 possible angles – one acute and one obtuse

angle of depression: The angle between the horizontal and the line of sight when looking down to an object below

angle of elevation: The angle between the horizontal and the line of sight when looking up to an object above

bearing: A direction from one point on Earth's surface to another, measured in degrees. Bearings may be written as true bearings (clockwise from north) or as compass bearings (using N, S, E and W)

compass bearing: Angles specified as either side of north or south, for example N 20° W or S 67° E.

cosine rule: In any triangle $c^2 = a^2 + b^2 - 2ab \cos C$

radian: A unit of angle measurement equal to the size of the angle subtended at the centre of a unit circle by an arc of length 1 unit

sine rule: In any triangle $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

true bearing: True or three-figure bearings are measured from north and turning clockwise

DID YOU KNOW?

Ptolemy

Ptolemy (Claudius Ptolemaeus), in the second century, wrote *Hē mathē matikē syntaxis* (or *Almagest* as it is now known) on astronomy. This is considered to be the first treatise on trigonometry, but it was based on circles and spheres rather than on triangles. The notation 'chord of an angle' was used rather than sin, cos or tan.

Ptolemy constructed a table of sines from 0° to 90° in steps of a quarter of a degree. He also calculated a value of π to 5 decimal places, and established the relationship for $\sin(x \pm y)$ and $\cos(x \pm y)$.

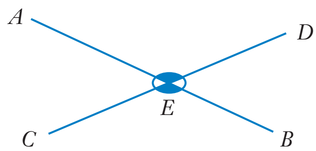
Geometry results

You will need to use some geometry when solving trigonometry problems. Here is a summary of the rules you may need.

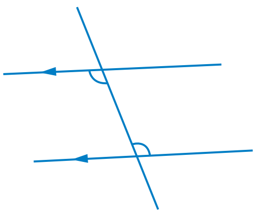
$\angle AEC$ and $\angle DEB$ are **vertically opposite angles**.

$\angle AED$ and $\angle CEB$ are also vertically opposite.

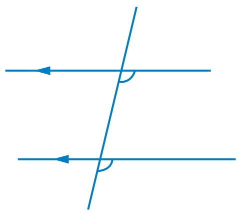
Vertically opposite angles are equal.



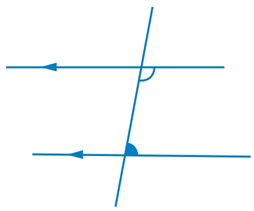
If lines are parallel, then:



alternate angles are equal.



corresponding angles are equal.



cointerior angles are **supplementary** (their sum is 180°).

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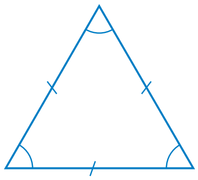
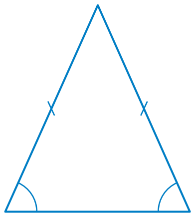
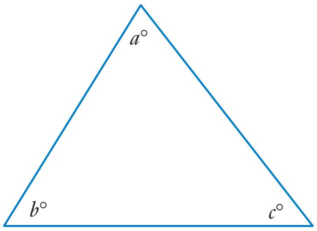


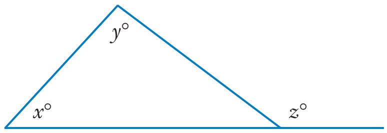
An **equilateral triangle** has 3 equal sides and 3 equal angles of size 60°.



An **isosceles triangle** has 2 equal sides and 2 equal angles.



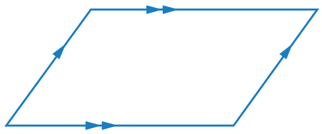
The **sum of the interior angles** in any triangle is 180°, that is, $a + b + c = 180$.



The **exterior angle** in any triangle is equal to the sum of the two opposite interior angles. That is, $x + y = z$.

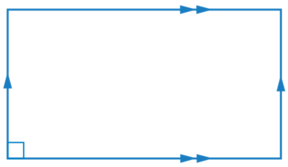
A **parallelogram** is a quadrilateral with opposite sides parallel.

- Opposite sides are equal.
- Opposite angles are equal.
- Diagonals bisect each other.



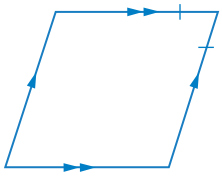
A **rectangle** is a parallelogram with one angle a right angle.

- Opposite sides are equal.
- All angles are right angles.
- Diagonals are equal and bisect each other.



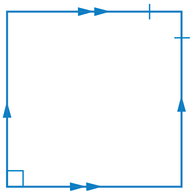
A **rhombus** is a parallelogram with a pair of adjacent sides equal.

- All sides are equal.
- Opposite angles are equal.
- Diagonals bisect each other at right angles.



A **square** is a rectangle with a pair of adjacent sides equal.

- All sides are equal.
- All angles are right angles.
- Diagonals are equal and bisect each other at right angles.
- Diagonals make angles of 45° with the sides.



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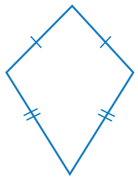
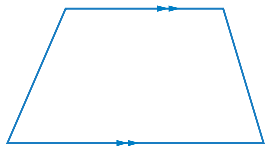
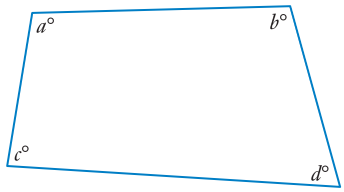
A **kite** is a quadrilateral with two pairs of adjacent sides equal.



A **trapezium** is a quadrilateral with one pair of sides parallel.



The **sum of the interior angles** in any quadrilateral is 360°, that is, $a + b + c + d = 360$.

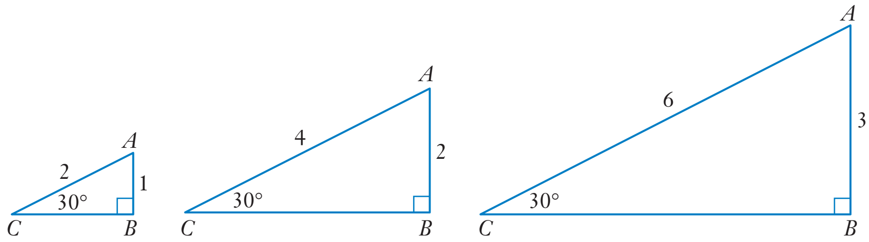


Trigonometric ratios

Trigonometry calculations

5.01 Trigonometric ratios

In similar triangles, pairs of corresponding angles are equal and sides are in proportion. For example:

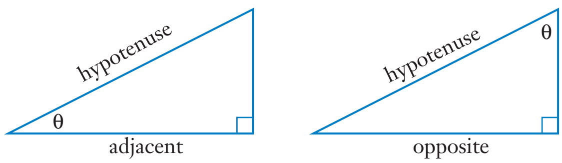


In any triangle containing an angle of 30°, the ratio $AB : AC = 1 : 2$. Similarly, the ratios of other corresponding sides will be equal. These ratios of sides form the basis of the trigonometric ratios.

The sides of a right-angled triangle

- The **hypotenuse** is the longest side, and is always opposite the right angle.
- The **opposite** side is opposite the angle marked in the triangle.
- The **adjacent** side is next to the angle marked.

The opposite and adjacent sides vary according to where the angle is marked. For example:



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The trigonometric ratios

Sine $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

Cosine $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

Tangent $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

DID YOU KNOW?

The origins of trigonometry

Trigonometry, or **triangle measurement**, progressed from the study of geometry in ancient Greece. Trigonometry was seen as applied mathematics. It gave a tool for the measurement of planets and their motion. It was also used extensively in navigation, surveying and mapping, and it is still used in these fields today.

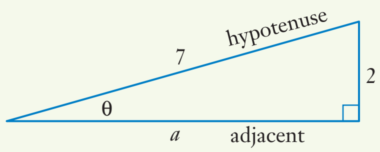
Trigonometry was crucial in setting up an accurate calendar, since this involved measuring the distances between the Earth, Sun and Moon.

EXAMPLE 1

If $\sin\theta = \frac{2}{7}$, find the exact ratios of $\cos\theta$ and $\tan\theta$.

Solution

$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{2}{7}$.



First draw a triangle with opposite side 2 and hypotenuse 7, then use Pythagoras' theorem to find the adjacent side.

$$c^2 = a^2 + b^2$$
$$7^2 = a^2 + 2^2$$
$$49 = a^2 + 4$$
$$45 = a^2$$
$$a = \sqrt{45}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{45}}{7}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{\sqrt{45}}$$

Degrees, minutes, seconds

Angles are measured in degrees, minutes and seconds.

> 60 minutes = 1 degree (60′ = 1°)
>
> 60 seconds = 1 minute (60″ = 1′)

When rounding numbers, you round up if the digit to the right is 5 or more. However, with angles, you round up to the next degree if there are 30 minutes or more.

Similarly, round angles up to the nearest minute if there are 30 seconds or more.

EXAMPLE 2

a Round to the nearest degree:

i 54°17′45″ **ii** 29°32′52″

b Round to the nearest minute:

i 23°12′22″ **ii** 57°34′41″ **iii** 84°19′30″

Solution

a **i** 17′ is less than 30′ so rounding gives 54°.

ii 32′ is more than 30′ so rounding gives 30°.

b **i** 22″ is less than 30″ so rounding gives 23°12′.

ii 41″ is more than 30″ so rounding gives 57°35′.

iii 30″ is exactly halfway so round up to 84°20′.

Decimal degrees and degrees-minutes-seconds

Scientific calculators ha ° ’ ” D°M’S key for converting between decimal degrees and degrees, minutes, seconds.

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EXAMPLE 3

a Change 58°19′ into a decimal.

b Change 45.236° into degrees and minutes.

Solution

a

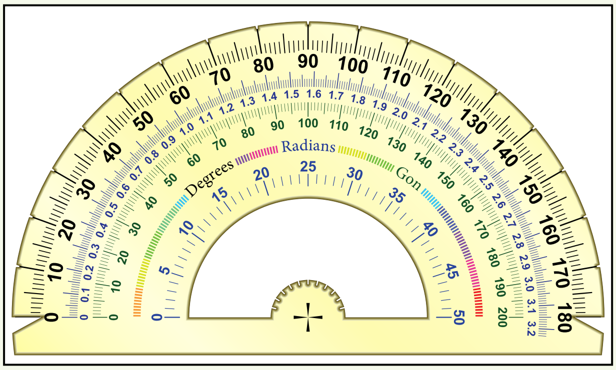
Operation	Casio scientific	Sharp scientific
Make sure the calculator is in degrees.	SHIFT SET UP deg	Press DRG until deg is on the screen
Enter data.	58 °′″ 19 °′″ =	58 D°M'S 19 D°M'S
Change to a decimal.	°′″	2ndF D°M'S

So 58°19′ = 58.31666 … ≈ 58.32

b

Operation	Casio scientific	Sharp scientific
Enter data.	45.236 =	45.236 =
Change to degrees and minutes.	°′″	2ndF D°M'S

So 45.236° = 45°14′9.6″ ≈ 45°14′



EXAMPLE 4

a Find cos 58°19′ correct to 3 decimal places.

b If tan θ = 0.348, find θ in degrees and minutes.

Solution

a

Operation	Casio scientific	Sharp scientific
Enter data.	cos 58 °’’’ 19 °’’’ =	cos 58 D°M′S 19 D°M′S =

So cos 58°19′ = 0.52522 … ≈ 0.525.

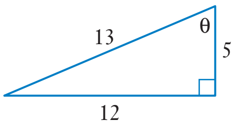
b To find the angle given the ratio, use the inverse key ($\tan^{-1}$).

Operation	Casio scientific	Sharp scientific
Enter data.	SHIFT $\tan^{-1}$ 0.348 =	2ndF $\tan^{-1}$ 0.348 =
Change to degrees and minutes.	°’’’	2ndF D°M′S

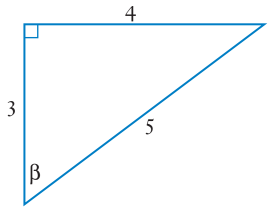
So θ = 19°11′16.43″ ≈ 19°11′.

Exercise 5.01 Trigonometric ratios

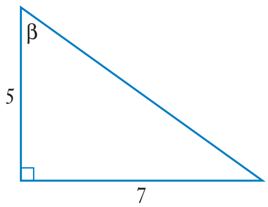
1 Write down the ratios of cos θ, sin θ and tan θ.



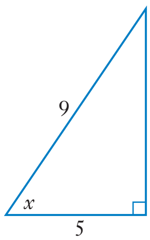
2 Find sin β, tan β and cos β.



3 Find the exact ratios of sin β, tan β and cos β.



4 Find exact values for cos x, tan x and sin x.



5 If $\tan\theta = \frac{4}{3}$, find cos θ and sin θ.

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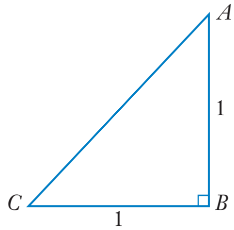
6 If $\cos\theta = \frac{2}{3}$, find exact values for $\tan\theta$ and $\sin\theta$.

7 If $\sin\theta = \frac{1}{6}$, find the exact ratios of $\cos\theta$ and $\tan\theta$.

8 If $\cos\theta = 0.7$, find exact values for $\tan\theta$ and $\sin\theta$.

9 $\triangle ABC$ is a right-angled isosceles triangle with $\angle ABC = 90°$ and $AB = BC = 1$.



a Find the exact length of AC.

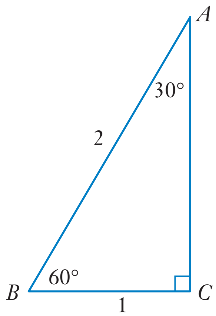
b Find $\angle BAC$.

c From the triangle, write down the exact ratios of sin 45°, cos 45° and tan 45°.

10 **a** Using Pythagoras' theorem, find the exact length of AC.

b Write down the exact ratios of sin 30°, cos 30° and tan 30°.

c Write down the exact ratios of sin 60°, cos 60° and tan 60°.



11 Round each angle to the nearest degree.

a 47°13′12″ **b** 81°45′43″ **c** 19°25′34″ **d** 76°37′19″ **e** 52°29′54″

12 Round each angle to the nearest minute.

a 47°13′12″ **b** 81°45′43″ **c** 19°25′34″ **d** 76°37′19″ **e** 52°29′54″

13 Change to a decimal:

a 77°45′ **b** 65°30′ **c** 24°51′ **d** 68°21′ **e** 82°31′

14 Change into degrees and minutes:

a 59.53° **b** 72.231° **c** 85.887° **d** 46.9° **e** 73.213°

15 Find correct to 3 decimal places:

a sin 39°25′ **b** cos 45°51′ **c** tan 18°43′ **d** sin 68°06′ **e** tan 54°20′

16 Find θ in degrees and minutes if:

a $\sin\theta = 0.298$ **b** $\tan\theta = 0.683$ **c** $\cos\theta = 0.827$

d $\tan\theta = 1.056$ **e** $\cos\theta = 0.188$

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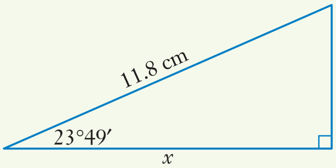
Finding an unknown side

5.02 Finding a side of a right-angled triangle

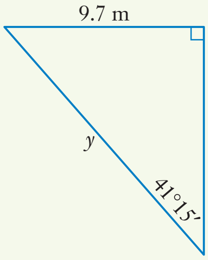
We can use trigonometry to find an unknown side of a triangle.

EXAMPLE 5

a Find the value of x, correct to 1 decimal place.



b Find the value of y, correct to 3 significant figures.



Solution

a

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos\ 23°49' = \frac{x}{11\ 8}$$

$$11\ 8\cos\ 23°49' = x$$

$$x \approx 10\ 8\text{ cm}$$

b

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 41°15' = \frac{9\ 7}{y}$$

$$y\sin 41°15' = 9\ 7$$

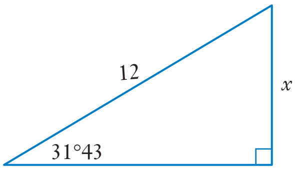
$$y = \frac{9\ 7}{\sin 41°15'}$$

$$\approx 14.7\text{ m}$$

Exercise 5.02 Finding a side of a right-angled triangle

1 Find the values of all pronumerals, correct to 1 decimal place:

a



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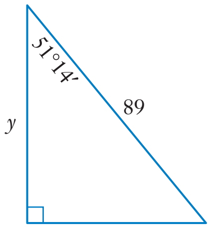

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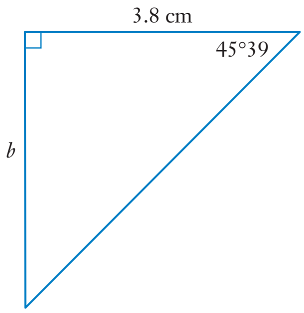

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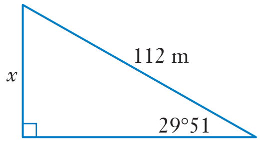

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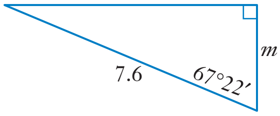

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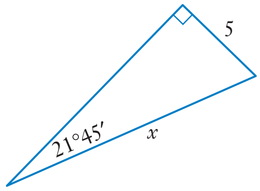

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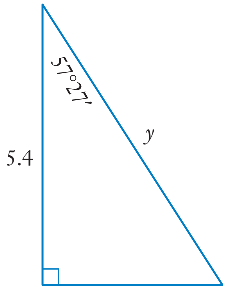

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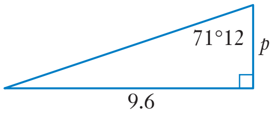

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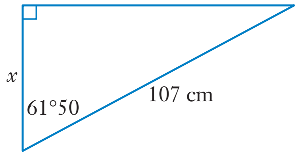

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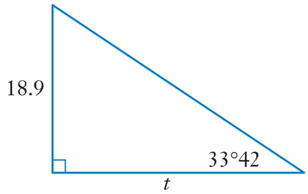

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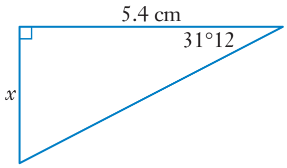

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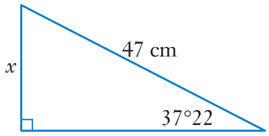

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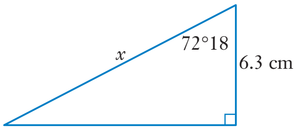

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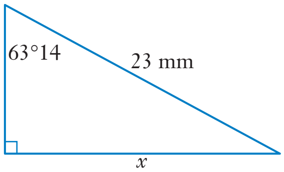

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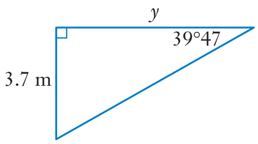

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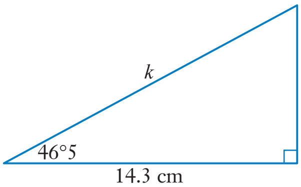

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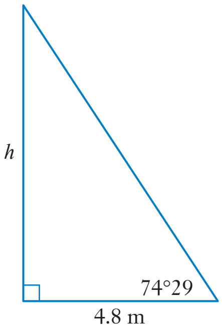

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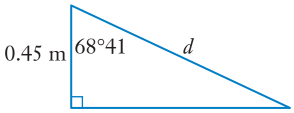

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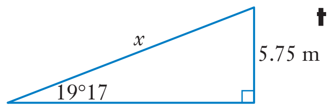

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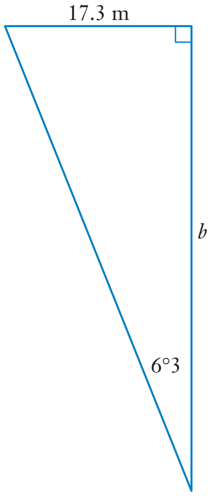
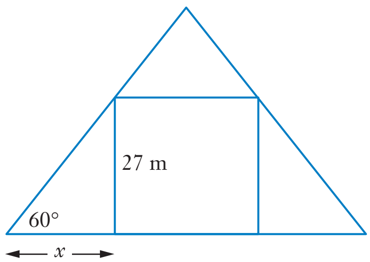
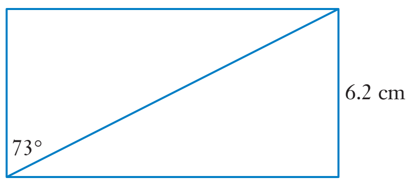


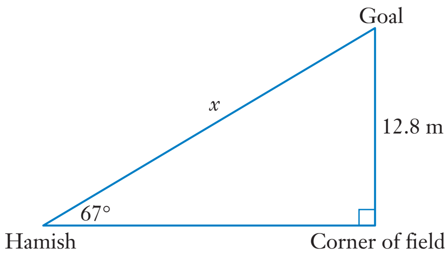
2 A roof is pitched at 60°. A room built inside the roof space is to have a 2.7 m high ceiling. How far in from the side of the roof will the wall for the room go?



3 A diagonal in a rectangle with width 6.2 cm makes an angle of 73° with the vertex as shown. Find the length of the rectangle correct to 1 decimal place.



4 Hamish is standing on the sideline of a soccer field, and the goal is at an angle of 67° from his position as shown. The goal is 12.8 m from the corner of the field. How far does he need to kick a ball for it to reach the goal?



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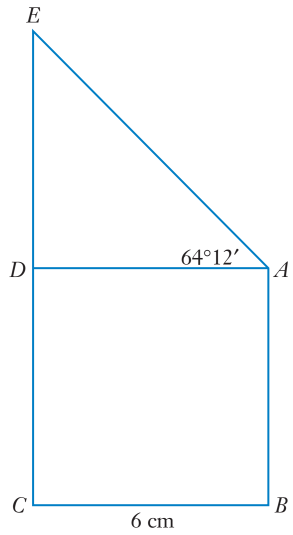
5 Square $ABCD$ with side 6 cm has line CD produced to E as shown so that $\angle EAD = 64°12'$. Evaluate the length, correct to 1 decimal place, of:

a CE **b** AE



6 A right-angled triangle with hypotenuse 14.5 cm long has one interior angle of $43°36'$. Find the lengths of the other two sides of the triangle.

7 A right-angled triangle ABC with the right angle at A has $\angle B = 56°44'$ and $AB = 26$ mm. Find the length of the hypotenuse.

8 A triangular fence is made for a garden inside a park. Three holes A, B and C for fence posts are made at the corners so that A and B are 10.2 m apart, AB and CB are perpendicular, and angle CAB is $59°54'$. How far apart are A and C?

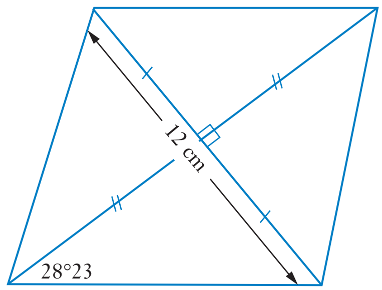
9 Triangle ABC has $\angle BAC = 46°$and $\angle ABC = 54°$. An altitude (perpendicular line) is drawn from C to meet AB at point D. If the altitude is 5.3 cm long, find, correct to 1 decimal place, the length of:

a AC **b** BC **c** AB

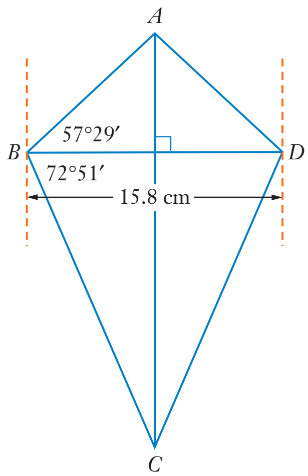
10 A rhombus has one diagonal 12 cm long and the other diagonal makes an angle of $28°23'$ with the side of the rhombus.

a Find the length of the side of the rhombus.

b Find the length of the other diagonal.



11 Kite $ABCD$ has diagonal $BD = 15.8$ cm as shown. If $\angle ABD = 57°29'$ and $\angle DBC = 72°51'$, find the length of the other diagonal AC.



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Finding an unknown angle

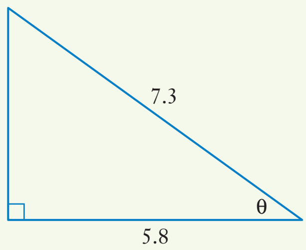
5.03 Finding an angle in a right-angled triangle

We can use trigonometry to find an unknown angle in a triangle.

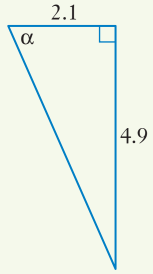
EXAMPLE 6

Find the value of the pronumeral, in degrees and minutes.

a



b



Solution

a $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

$= \frac{5\ 8}{7\ 3}$

$\therefore \theta = \cos^{-}\left(\frac{5\ 8}{7\ 3}\right)$

$\approx 37°23$

b $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

$= \frac{4\ 9}{2\ 1}$

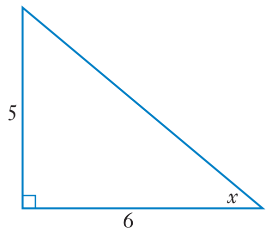
$\therefore \alpha = \tan^{-}\left(\frac{4\ 9}{2\ 1}\right)$

$= 66°48'$

Exercise 5.03 Finding an angle in a right-angled triangle

1 Find the value of each pronumeral, in degrees and minutes:

a



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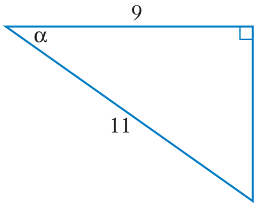

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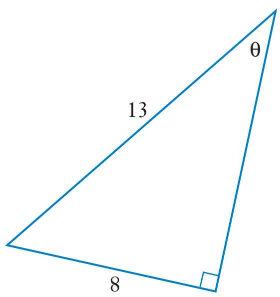

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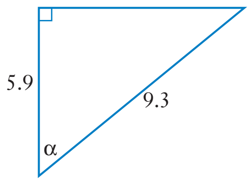
d



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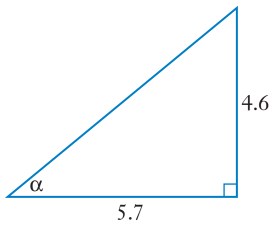

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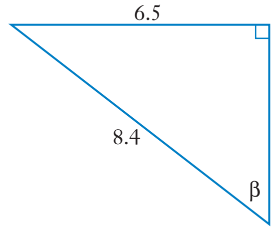

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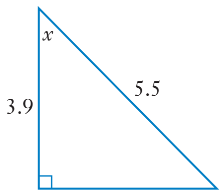

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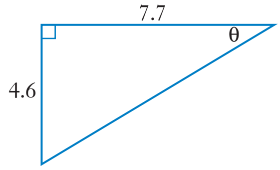

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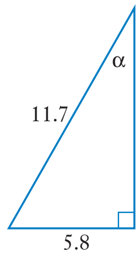

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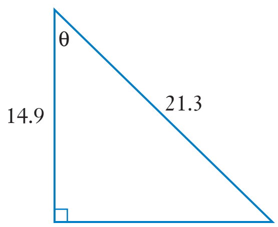

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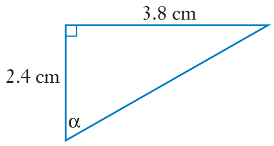

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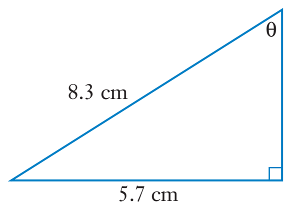
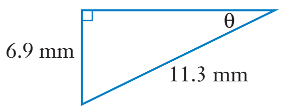
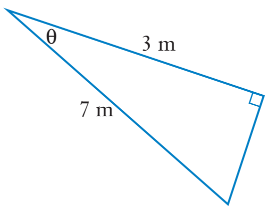


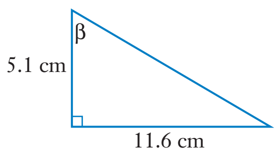
m



n



o



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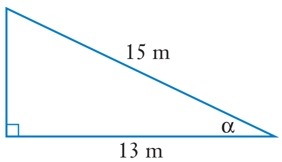

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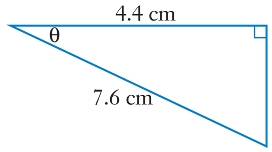

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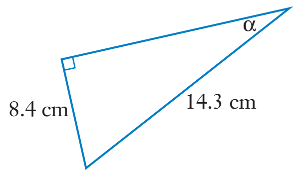

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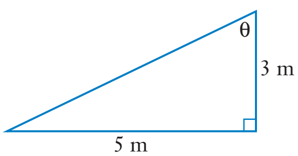

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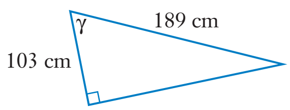
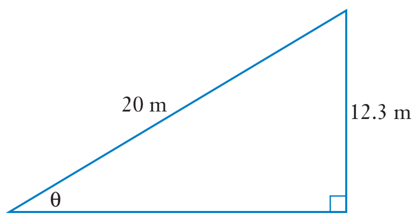
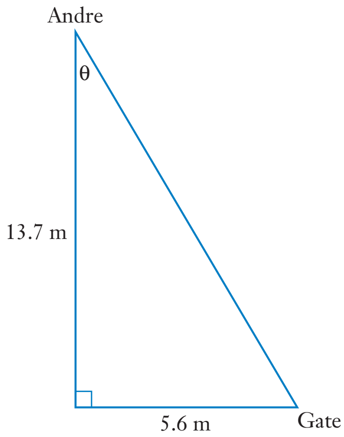


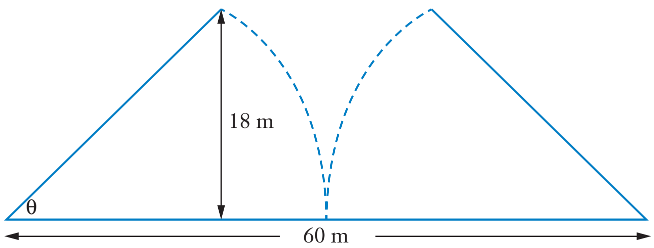
2 A kite is flying at an angle of θ above the ground as shown. If the kite is 12.3 m above the ground and has 20 m of string, find angle θ.



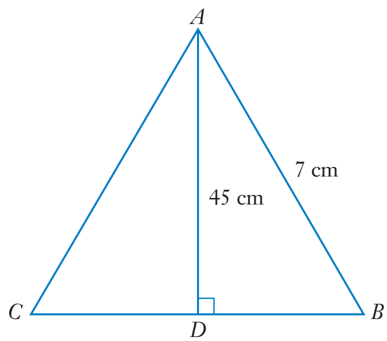
3 A field is 13.7 m wide and Andre is on one side. There is a gate on the opposite side and 5.6 m along from where Andre is. At what angle will he walk to get to the gate?



4 A 60 m long bridge has an opening in the middle and both sides open up to let boats pass underneath. The two parts of the bridge floor rise up to a height of 18 m. Through what angle do they move?



5 An equilateral triangle *ABC* with side 7 cm has an altitude *AD* 4.5 cm long. Evaluate the angle the altitude makes with vertex *A* (∠*DAB*).



6 Rectangle *ABCD* has dimensions 18 cm × 7 cm. A line *AE* is drawn so that *E* bisects *DC*.

a How long is line *AE*? (Answer to 1 decimal place.)

b Evaluate ∠*DEA*.

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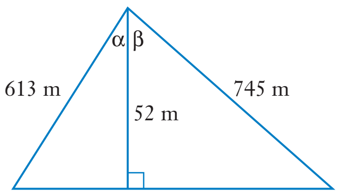
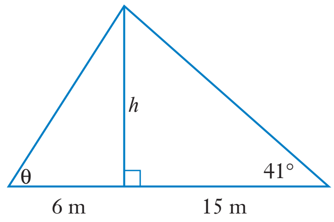
7 A 52 m tall tower has wire stays on either side to minimise wind movement. One stay is 61.3 m long and the other is 74.5 m long, as shown. Find the angles that the tower makes with each stay.



8 The angle up from the ground to the top of a pole is 41° from a position 15 m to one side.

a Find the height h of the pole, to the nearest metre.

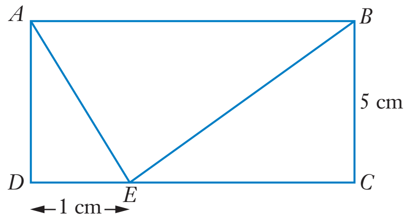
b If Sarah stands 6 m away on the other side, find the angle of elevation θ from Sarah to the top of the pole.



9 Rectangle $ABCD$ has a line BE drawn so that $\angle AEB = 90°$ and $DE = 1$ cm. The width of the rectangle is 5 cm.

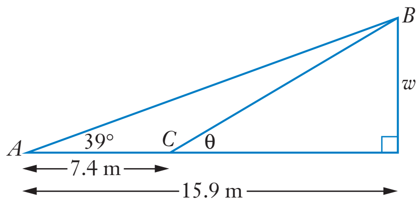
a Find $\angle BEC$.

b Find the length of the rectangle.



10 **a** Frankie is standing at the side of a road at point A, 15.9 m away from an intersection. She is at an angle of 39° from point B on the other side of the road. What is the width w of the road?

b Frankie walks 7.4 m to point C. At what angle is she from point B?



INVESTIGATION

LEANING TOWER OF PISA

The Tower of Pisa was built as a belltower for the cathedral nearby. Work started in 1174, but when it was half-completed the soil underneath one side of it sank. This made the tower lean to one side. Work stopped, and it wasn't until 100 years later that architects found a way of completing the tower. The third and fifth storeys were built close to the vertical to compensate for the lean. Later a vertical top storey was added. The tower is about 55 m tall and 16 m in diameter. It is tilted about 5 m from the vertical at the top, and tilts by an extra 6 mm each year.

Discuss some of the problems with the Leaning Tower of Pisa.

1 Find the angle at which it is tilted from the vertical.

2 Work out how far it will be tilted in 10 years.

3 Use research to find out if the tower will fall over, and if so, when.

5.04 Applications of trigonometry

WS

Right-angled trigonometry

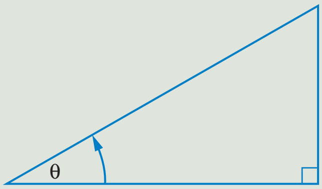
Angles of elevation and depression

Angle of elevation

The **angle of elevation** can be used to measure the height of tall objects that cannot be measured directly; for example, a tree, cliff, tower or building. Stand outside a tall building and look up to the top of the building. Think about what angle your eyes pass through to look up to the top of the building.

Angle of elevation

The angle of elevation, θ, is the angle measured when looking from the ground up to the top of the object. We assume that the ground is horizontal.



EXAMPLE 7

The angle of elevation of a tree from a point 50 m out from its base is 38°14′.

Find the height of the tree to the nearest metre.

Solution

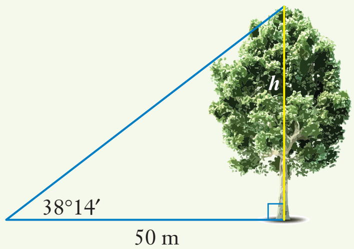
We assume that the tree is vertical.

$$\tan 38°14' = \frac{h}{50}$$

$$50 \tan 38°14' = h$$

$$39 \approx h$$

So the tree is 39 m tall.



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Angle of depression

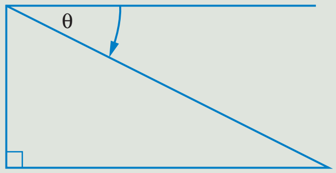
Angle of depression

The **angle of depression** is the angle formed when looking down from a high place to an object below. Find a tall building, hill or other high place, and look down to something below. Through what angle do your eyes pass as you look down?

Angle of depression

Angle of depression

The angle of depression, θ, is the angle measured when looking down from the horizontal to an object below.



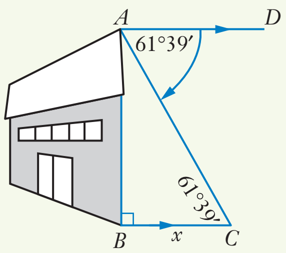
EXAMPLE 8

a The angle of depression from the top of a 20 m building to Gina below is 61°39′. How far is Gina from the building, to 1 decimal place?

b A bird sitting on top of an 8 m tall tree looks down at a possum 3.5 m out from the base of the tree. Find the angle of depression to the nearest minute.

Solution

a



$\angle DAC = \angle ACB = 61°39'$ (alternate angles, $AD \| BC$)

$$\tan 61°39' = \frac{20}{x}$$

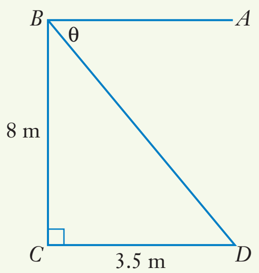
$$x \tan 61°39' = 20$$

$$x = \frac{20}{\tan 61°39'}$$

$$\approx 10.8$$

So Gina is 10.8 m from the building.

b



The angle of depression is θ.

$\angle ABD = \angle BDC = \theta$ (alternate angles, $AB \| DC$).

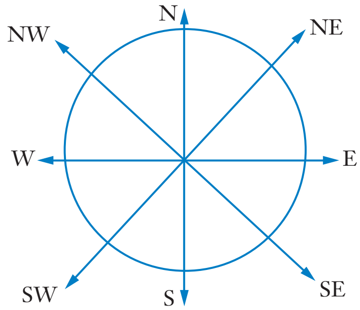
$$\tan \theta = \frac{8}{3\ 5}$$

$$\therefore \theta \approx 66°22$$

Compass bearings

Compass bearings

A **bearing** is a direction according to a compass. The main points on a compass are north (N), south (S), east (E) and west (W). Halfway between these are NE, NW, SE, SW. We write **compass bearings** with north or south first, followed by an angle and then east or west.



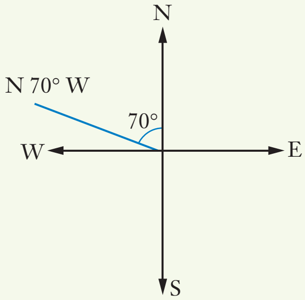
EXAMPLE 9

a Draw a compass bearing of N 70° W.

b Eli walks from his house on a bearing of S 25° E. If he walks 5.7 km, how far south is he from his house?

Solution

a Start at north and turn 70° towards west.



b Start at south and turn 25° towards east.

The hypotenuse is 5.7 and we want to measure the adjacent side (x).

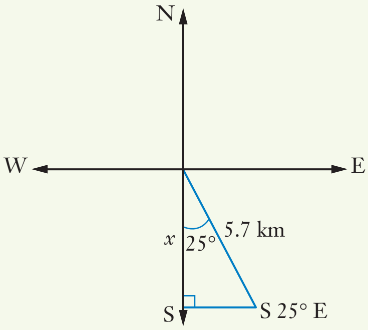
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 25° = \frac{x}{5\ 7}$$

$$5\ 7\cos 25° = x$$

$$x \approx 5\ 2$$

So Eli is 5.2 km south of his house.



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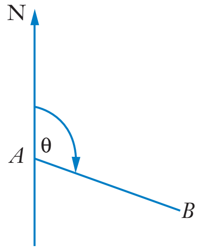
True bearings

True bearings measure angles clockwise from north.

We say B is on a bearing of θ from A.

A true bearing uses 3 digits from 000° to 360°.



Bearings

True bearings

A page of bearings

Bearings match-up

Elevations and bearings

EXAMPLE 10

a X is on a bearing of 030° from Y. Sketch this diagram.

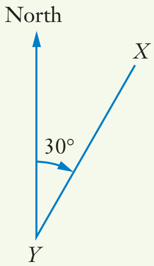
b A house is on a bearing of 305° from a school. What is the bearing of the school from the house?

c A plane leaves Sydney and flies 100 km due east, then 125 km due north. Find the bearing of the plane from Sydney, to the nearest degree.

d A ship sails on a bearing of 140° from Sydney for 250 km. How far east of Sydney is the ship now, to the nearest km?

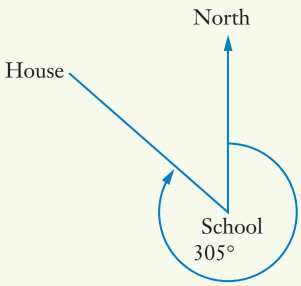
Solution

a



Note: Bearings use 3 digits so a bearing of 030° is a 30° angle.

b



The diagram below shows the bearing of the house from the school.

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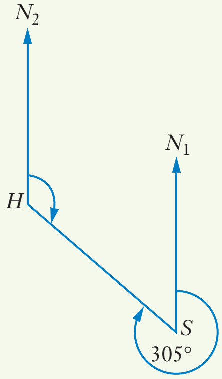
To find the bearing of the school from the house, draw in north from the house and use geometry to find the bearing as follows:



The bearing of the school from the house is $\angle N_2HS$.

$\angle N_1SH = 360° - 305°$ (angles in a revolution)

$= 55°$

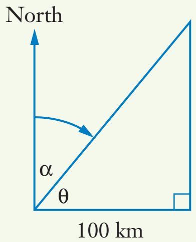
With parallel lines, the sum of cointerior angles is 180°.

$\angle N_2HS = 180° - 55°$

$= 125°$

So the bearing of the school from the house is 125°.

c



$$\tan\theta = \frac{125}{100}$$

$$\theta \approx 51°$$

$$\alpha = 90° - 51° = 39°$$

So the bearing of the plane from Sydney is 039°.

d

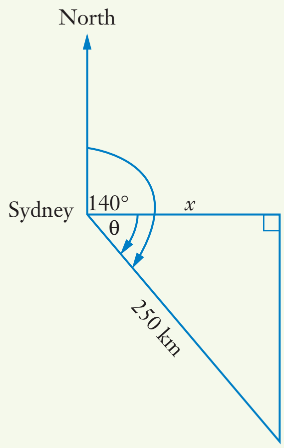
$$\theta = 140° - 90° = 50°$$

$$\cos 50° = \frac{x}{250}$$

$$250\cos 50° = x$$

$$x \approx 161$$

So the ship is 161 km east of Sydney.



Note: A navigator on a ship uses a sextant to measure angles. A clinometer measures angles of elevation and depression.

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Exercise 5.04 Applications of trigonometry

1 Draw a diagram to show the bearing in each question.

a N 50° E **b** S 60° W **c** S 80° E **d** N 40° W

e A boat is on a bearing of 100° from a beach house.

f Jamie is on a bearing of 320° from a campsite.

g A seagull is on a bearing of 200° from a jetty.

h Alistair is on a bearing of 050° from the bus stop.

i A plane is on a bearing of 285° from Broken Hill.

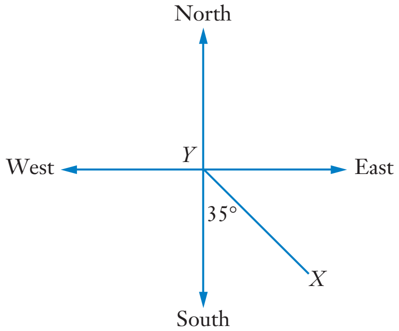
j A farmhouse is on a bearing of 012° from a dam.

k Mohammed is on a bearing of 160° from his house.

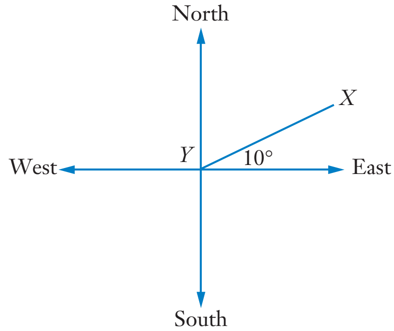
2 Find the bearing of X from Y in each question using:

i compass bearings **ii** true bearings.

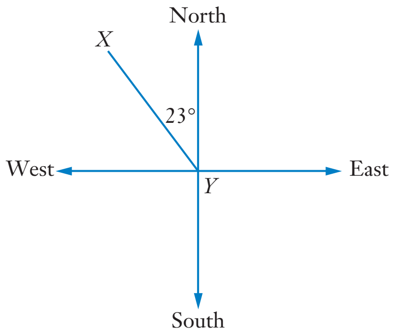
a



b



c



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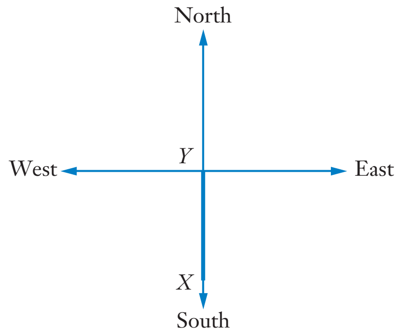


3 Jack is on a bearing of 260° from Jill What is Jil's bearing from Jack ?

4 A tower is on a bearing of 030° from a house. What is the bearing of the house from the tower?

5 Tamworth is on a bearing of 340° from Newcastle. What is the bearing of Newcastle from Tamworth?

6 The angle of elevation from a point 11.5 m away from the base of a tree up to the top of the tree is 42°12′. Find the height of the tree to one decimal place.

7 Geoff stands 25.8 m away from the base of a tower and measures the angle of elevation as 39°20′. Find the height of the tower to the nearest metre.

8 A wire is suspended from the top of a 100 m tall bridge tower down to the bridge at an angle of elevation of 52°. How long is the wire, to 1 decimal place?

9 A cat crouches at the top of a 4.2 m high cliff and looks down at a mouse 1.3 m out from the foot (base) of the cliff. What is the angle of depression, to the nearest minute?

10 A plane leaves Melbourne and flies on a bearing of 065° for 2500 km.

- **a** How far north of Melbourne is the plane?
- **b** How far east of Melbourne is it?
- **c** What is the bearing of Melbourne from the plane?

11 The angle of elevation of a tower is 39°44′ when measured at a point 100 m from its base. Find the height of the tower, to 1 decimal place.

12 Kim leaves her house and walks for 2 km on a bearing of 155°. How far south is Kim from her house now, to 1 decimal place?

13 The angle of depression from the top of an 8 m tree down to a rabbit is 43°52′. If an eagle is perched in the top of the tree, how far does it need to fly to reach the rabbit, to the nearest metre?

14 Sanjay rides a motorbike through his property, starting at his house. If he rides south for 1.3 km, then rides west for 2.4 km, what is his bearing from the house, to the nearest degree?

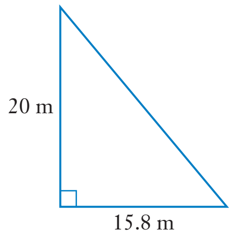
15 A plane flies north from Sydney for 560 km, then turns and flies east for 390 km. What is its bearing from Sydney, to the nearest degree?

16 Find the height of a pole, correct to 1 decimal place, if a 10 m rope tied to it at the top and stretched out straight to reach the ground makes an angle of elevation of 67°13′.

17 The angle of depression from the top of a cliff down to a boat 100 m out from the foot of the cliff is 59°42′. How high is the cliff, to the nearest metre?

18 A group of students are bushwalking. They walk north from their camp for 7.5 km, then walk west until their bearing from camp is 320°. How far are they from camp, to 1 decimal place?

19 A 20 m tall tower casts a shadow 15.8 m long at a certain time of day. What is the angle of elevation from the edge of the shadow up to the top of the tower at this time?



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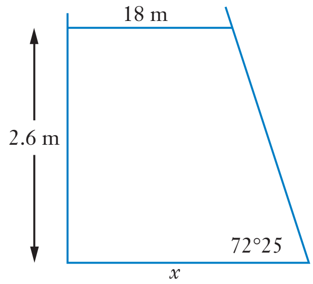
20 A flat verandah roof 1.8 m deep is 2.6 m up from the ground. At a certain time of day, the sun makes an angle of elevation of 72°25′. How much shade is provided on the ground by the verandah roof at that time, to 1 decimal place?



21 Find the angle of elevation of a 15.9 m cliff from a point 100 m out from its base.

22 A plane leaves Sydney and flies for 2000 km on a bearing of 195°. How far due south of Sydney is it?

23 The angle of depression from the top of a 15 m tree down to a pond is 25°41′. If a bird is perched in the top of the tree, how far does it need to fly to reach the pond, to the nearest metre?

24 Robin starts at her house, walks south for 2.7 km then walks east for 1.6 km. What is her bearing from the house, to the nearest degree?

25 The angle of depression from the top of a tower down to a car 250 m out from the foot of the tower is 38°19′. How high is the tower, to the nearest metre?

26 A blimp flies south for 3.6 km then turns and flies east until it is on a bearing of 127° from where it started. How far east does it fly?

27 A 24 m wire is attached to the top of a pole and runs down to the ground where the angle of elevation is 22°32′. Find the height of the pole.

28 A train depot has train tracks running north for 7.8 km where they meet another set of tracks going east for 5.8 km into a station. What is the bearing of the depot from the station, to the nearest degree?

29 Jessica leaves home and walks for 4.7 km on a bearing of 075°. She then turns and walks for 2.9 km on a bearing of 115° and she is then due east of her home.

- **a** What is the furthest north that Jessica walks?
- **b** How far is she from home?

30 Builder Jo stands 4.5 m out from the foot of a building and looks up to the top of the building where the angle of elevation is 71°. Builder Ben stands at the top of the building looking down at his wheelbarrow that is 10.8 m out from the foot of the building on the opposite side from where Jo is standing.

- **a** Find the height of the building.
- **b** Find the angle of depression from Ben down to his wheelbarrow.

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WS

Sine rule problems

5.05 The sine rule

The sin, cos and tan of angles greater than 90° give some interesting results. You will explore these in Chapter 11, *Trigonometric functions*. For now, we just need to know about **obtuse angles** (between 90° and 180°) so we can solve problems involving obtuse-angled triangles.

INVESTIGATION

LARGER ANGLES

1 Use your calculator to find the sin, cos and tan of some angles greater than 90°. What do you notice?

2 Can you see a pattern for angles between 90° and 180° for:

i sin? **ii** cos? **iii** tan?

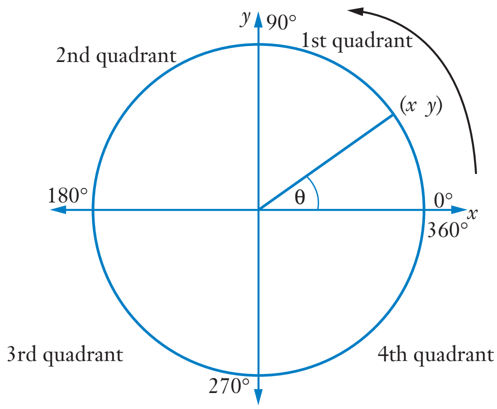
We can use a circle to show angles, starting with 0° at the *x*-axis and turning anticlockwise to show other angles. We divide the number plane into 4 **quadrants** as shown:

1st quadrant: 0° to 90°

2nd quadrant: 90° to 180°

3rd quadrant: 180° to 270°

4th quadrant: 270° to 360°



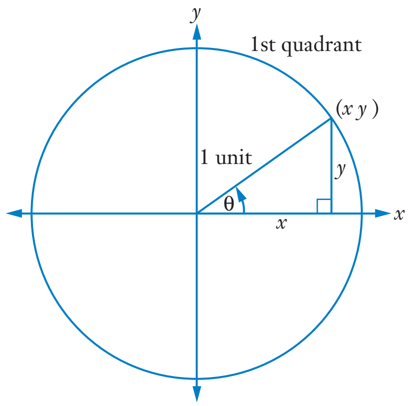
To make it easier to explore these results, we use a **unit circle** with radius 1.

We can find the trigonometric ratios for angle θ.

$\sin\theta = \dfrac{y}{1} = y$

$\cos\theta = \dfrac{x}{1} = x$

$\tan\theta = \dfrac{y}{x}$



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In the 2nd quadrant, notice that x values are negative and y values are positive.

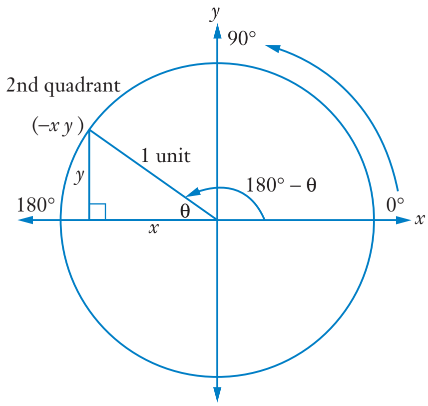
So the point in the 2nd quadrant will be $(-x, y)$.

Since $\sin \theta = y$, sin will be **positive** in the 2nd quadrant.

Since $\cos \theta = -x$, cos will **negative** in the 2nd quadrant.

Since $\tan \theta = \frac{y}{-x}$, tan will be negative in the 2nd quadrant (positive divided by negative).

To have an angle of θ in the triangle, the obtuse angle in the 2nd quadrant is 180° – θ.



Trigonometric ratios of obtuse angles

$$\sin (180° - \theta) = \sin \theta$$

$$\cos (180° - \theta) = -\cos \theta$$

$$\tan (180° - \theta) = -\tan \theta$$

EXAMPLE 11

a If cos 80° = 0.174, evaluate cos 100°.

b If sin 55° = 0.819, find the value of sin 125°.

Solution

a

$$\cos (180° - \theta) = -\cos \theta$$

So $\cos (180° - 80°) = -\cos 80°$

$$\cos 100° = -\cos 80°$$

$$= -0.174$$

b

$$\sin (180° - \theta) = \sin \theta$$

So $\sin (180° - 55°) = \sin 55°$

$$\sin 125° = \sin 55°$$

$$= 0.819$$

You can check that this is true by finding both ratios on the calculator.

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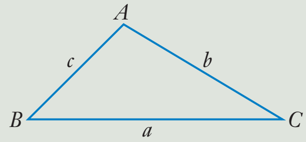
Naming the sides and angles of a triangle

Side a is opposite angle A, side b is opposite angle B and side c is opposite angle C.

The shortest side is opposite the smallest angle.

The longest side is opposite the largest angle.



The sine rule

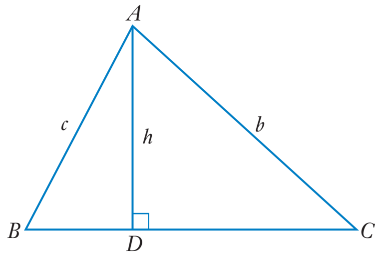
The **sine rule** is used to find unknown sides and angles in non-right-angled triangles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof

In $\triangle ABC$ draw perpendicular AD and call it h.



From $\triangle ABD$,

$\sin B = \dfrac{h}{c}$

$\therefore h = c \sin B$ [1]

From $\triangle ACD$,

$\sin C = \dfrac{h}{b}$

$\therefore h = b \sin C$ [2]

From [1] and [2],

$c \sin B = b \sin C$

$\dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Similarly, by drawing a perpendicular from C it can be proved that

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

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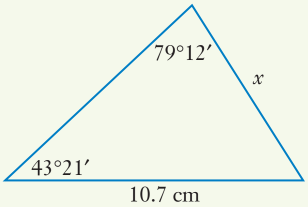
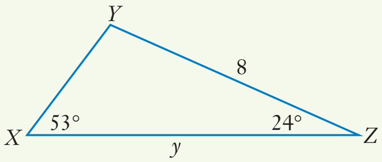
EXAMPLE 12

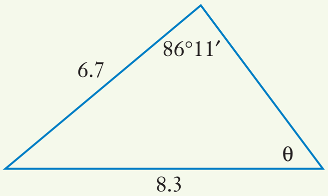
a Find the value of x, correct to 1 decimal place.



b Find the value of y, to the nearest whole number.



c Find the value of θ, in degrees and minutes, given θ is acute.



Solution

a Name the sides a and b, and opposite angles A and B.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{x}{\sin 43°21'} = \frac{10\ 7}{\sin 79°12'}$$

$$x = \frac{10\ 7 \sin 43°21'}{\sin 79°12'}$$

$$\approx 7.5 \text{ cm}$$

b First we need to find angle Y since it is opposite side y.

$$\angle Y = 180^\circ - (53^\circ + 24^\circ) = 103^\circ$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{y}{\sin 103^\circ} = \frac{8}{\sin 53^\circ}$$

$$y = \frac{8 \sin 103^\circ}{\sin 53^\circ}$$

$$\approx 10$$

c

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{6\ 7} = \frac{\sin 86^\circ 11'}{8\ 3}$$

$$\sin \theta = \frac{6\ 7 \sin 86^\circ 11'}{8\ 3}$$

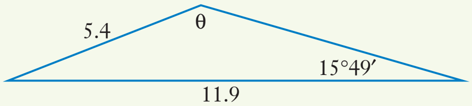
$$= 08054$$

$$\theta = \sin^{-}(08054\ \ )$$ SHIFT sin ANS

$$\approx 53^\circ 39'$$

EXAMPLE 13

Find the value of θ, in degrees and minutes, given θ is obtuse.



Solution

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{11\ 9} = \frac{\sin 15^\circ 49'}{5\ 4}$$

$$\sin \theta = \frac{11\ 9 \sin 15^\circ 49'}{5\ 4}$$

$$= 06006$$

$$\theta = \sin^{-}(06006\ \ )$$

$$\approx 36^\circ 55'$$

But θ is obtuse.

$$\therefore\ \theta = 180^\circ - 36^\circ 55'$$

$$= 143^\circ 05'$$

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Ambiguous case

When using the sine rule to find an unknown angle, there are 2 possible solutions: one acute and one obtuse. This is called the **ambiguous case** of the sine rule.

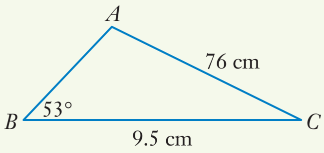
EXAMPLE 14

a Triangle ABC has $\angle B = 53°$, $AC = 7.6$ cm and $BC = 9.5$ cm. Find $\angle A$ to the nearest degree.

b In triangle XYZ, $\angle Y = 118°35'$, $YZ = 12.5$ mm and $XZ = 14.3$ mm. Find $\angle X$ in degrees and minutes.

Solution

a Draw a diagram.



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{9\ 5} = \frac{\sin 53°}{7\ 6}$$

$$\sin A = \frac{9\ 5 \sin 53°}{7\ 6}$$

$$= 0.998$$

$$A = \sin^{-1}(0.998)$$

$$\approx 87°$$

But $\angle A$ could be obtuse.

So $\angle A = 180° - 87°$

$= 93°$

Checking angle sum of a triangle:

$53° + 87° = 140° < 180°$, so 87° is a possible answer.

$53° + 93° = 146° < 180°$, so 93° is a possible answer.

So $\angle A = 87°$ or $93°$.

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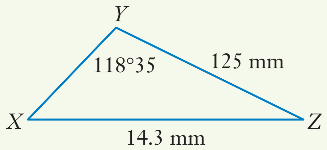
b



$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin X}{12\ 5} = \frac{\sin 118°35'}{14\ 3}$$

$$\sin X = \frac{12\ 5 \sin 118°35'}{14\ 3}$$

$$= 0.768$$

$$X = \sin^{-1}(0.768)$$

$$\approx 50°8'$$

But $\angle X$ could be obtuse.

So $\angle X = 180° - 50°8'$

$$= 129°52'$$

Checking angle sum of a triangle:

$118°35' + 50°8' = 168°43' < 180°$, so a possible answer.

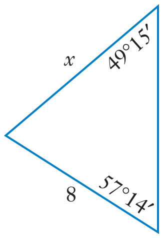
$118°35' + 129°52' = 248°27' > 180°$, so an impossible answer.

So $\angle X = 50°8'$.

Exercise 5.05 The sine rule

1 Evaluate each pronumeral, correct to 1 decimal place:

a



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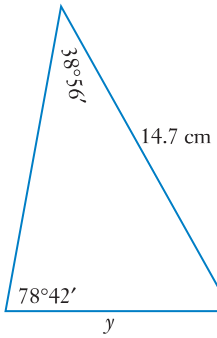
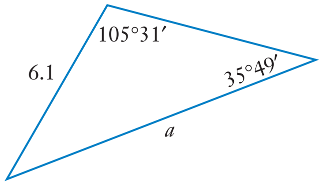
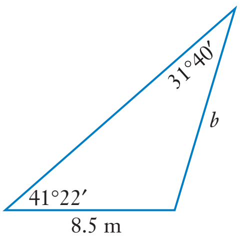


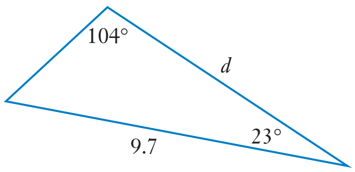
c



d



e



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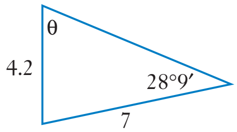
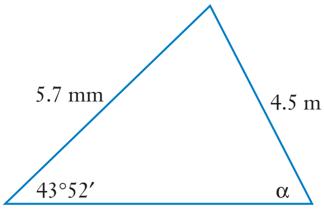
2 Find the value of all pronumerals, in degrees and minutes (triangles not to scale):

a



b



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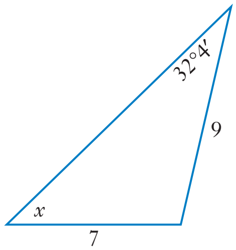

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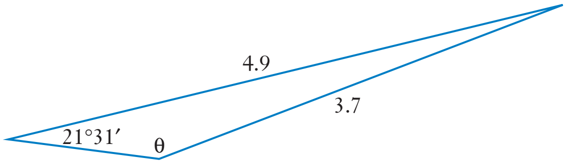

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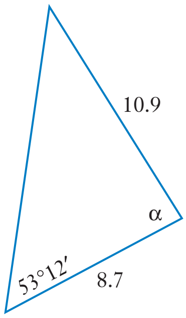


3 Triangle ABC has an obtuse angle at A. Evaluate this angle to the nearest minute if $AB = 3.2$ cm, $BC = 4.6$ cm and $\angle ACB = 33°47'$.

4 Triangle EFG has $\angle FEG = 48°$, $\angle EGF = 32°$ and $FG = 18.9$ mm. Find the length of:

a the shortest side **b** the longest side.

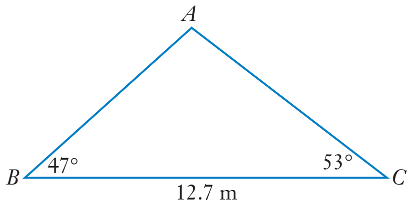
5 Triangle XYZ has $\angle XYZ = 51°$, $\angle YXZ = 86°$ and $XZ = 2.1$ m. Find the length of:

a the shortest side **b** the longest side.

6 Triangle XYZ has $XY = 5.4$ cm, $\angle ZXY = 48°$ and $\angle XZY = 63°$. Find the length of XZ.

7 Triangle ABC has $BC = 12.7$ m, $\angle ABC = 47°$and $\angle ACB = 53°$ as shown. Find the length of:

a AB **b** AC



8 Triangle PQR has sides $PQ = 15$ mm, $QR = 14.7$ mm and $\angle PRQ = 62°29'$. Find to the nearest minute:

a $\angle QPR$ **b** $\angle PQR$

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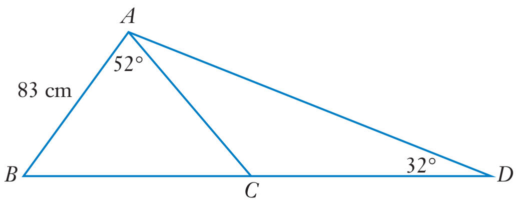
9 Triangle ABC is isosceles with $AB = AC$. BC is produced to D as shown.

If $AB = 8.3$ cm, $\angle BAC = 52°$ and $\angle ADC = 32°$, find the length of:

a AD **b** BD



10 Triangle ABC is equilateral with side 63 mm. A line is drawn from A to BC where it meets BC at D and $\angle DAB = 26°15'$. Find the length of:

a AD **b** DC

11 In triangle ABC, find $\angle B$ to the nearest degree given:

a $\angle C = 67°, AB = 7.2, AC = 7.5$

b $\angle A = 92°, BC = 10.7, AC = 8.4$

c $\angle A = 29°, BC = 4.9, AC = 8.3$

WS Cosine rule problems

WS The sine and cosine rules

WS Finding an unknown side

WS Finding an unknown angle

5.06 The cosine rule

The cosine rule

The **cosine rule** is also used to find unknown sides and angles in non-right-angled triangles.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Proof

In triangle ABC, draw perpendicular AD with length p and let $CD = x$.

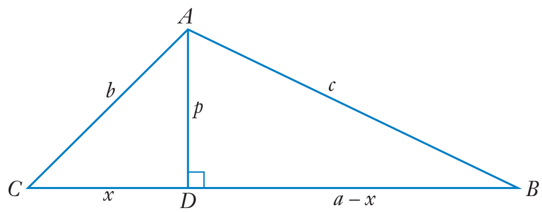
Since $BC = a$, $BD = a - x$

From triangle ACD:

$$b^2 = x^2 + p^2 \qquad [1]$$

$$\cos C = \frac{x}{b}$$

$$\therefore b \cos C = x \qquad [2]$$



From triangle DAB:

$$c^2 = p^2 + (a - x)^2$$

$$= p^2 + a^2 - 2ax + x^2$$

$$= p^2 + x^2 + a^2 - 2ax \qquad [3]$$

Substitute [1] into [3]:

$$c^2 = b^2 + a^2 - 2ax \qquad [4]$$

Substituting [2] into [4]:

$$c^2 = b^2 + a^2 - 2ab \cos C$$

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DID YOU KNOW?

The cosine rule for right-angled triangles

Pythagoras' theorem is a special case of the cosine rule when the triangle is right-angled.

$c^2 = a^2 + b^2 - 2ab \cos C$

When $C = 90°$

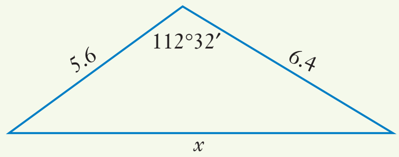
$c^2 = a^2 + b^2 - 2ab \cos 90°$

$\quad = a^2 + b^2 - 2ab \times 0$

$\quad = a^2 + b^2$

EXAMPLE 15

Find the value of x, correct to the nearest whole number.



Solution

$c^2 = a^2 + b^2 - 2ab \cos C$

$x^2 = 5.6^2 + 6.4^2 - 2(5.6)(6.4) \cos 112°32'$

$\quad = 99.7892 \ldots$

$x = \sqrt{997892 \ldots}$

$\quad = 9.9894 \ldots$

$\quad \approx 10$

When using the cosine rule to find an unknown angle, it may be more convenient to change the subject of this formula to cos C.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$2ab \cos C = a^2 + b^2 - c^2$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

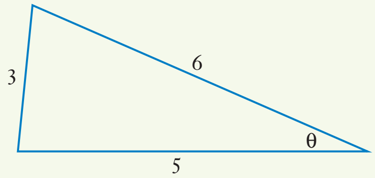
The cosine rule for angles

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

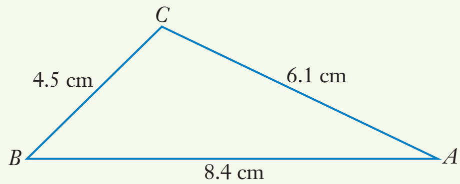
The cosine rule for angles

EXAMPLE 16

a Find θ, in degrees and minutes.



b Evaluate $\angle BCA$ in degrees and minutes.



Solution

Naming sides and opposite angles, side c is opposite the unknown angle C.

a

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\theta = \frac{5^2 + 6^2 - 3^2}{2(5)(6)}$$

$$= \frac{52}{60}$$

$$\theta = \cos^{-}\left(\frac{52}{60}\right)$$

$$\approx 29°56'$$

b

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\angle BCA = \frac{4\ 5^2 + 6\ 1^2 - 8\ 4^2}{2(4\ 5)(6\ 1)}$$

$$= -0.2386\ \ldots$$

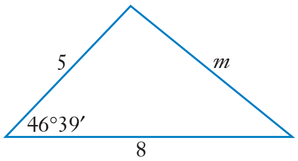
$$\angle BCA = \cos^{-1}(-0.2386\ldots)$$

$$\approx 103°48'$$

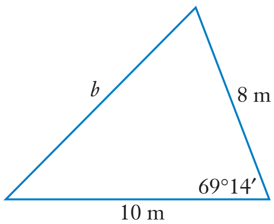
Exercise 5.06 The cosine rule

1 Find the value of each pronumeral, correct to 1 decimal place:

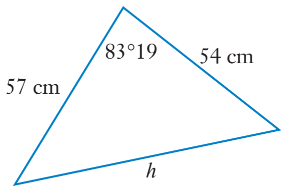
a



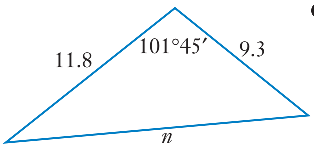
b



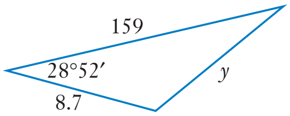
c



d



e



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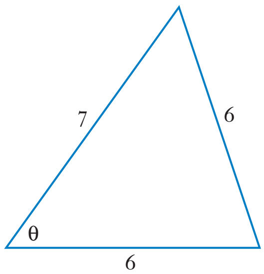
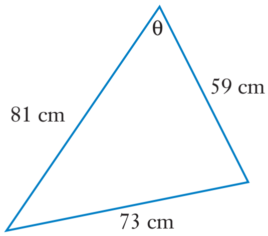
2 Evaluate each pronumeral, correct to the nearest minute:

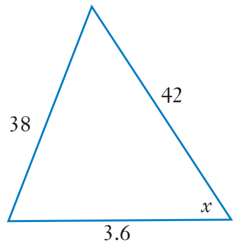
a



b



c



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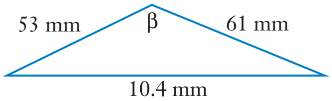
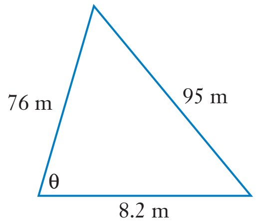
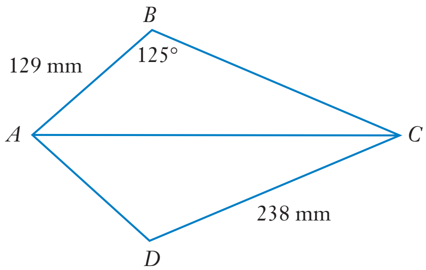


e



3 Kite *ABCD* has *AB* = 12.9 mm, *CD* = 23.8 mm and ∠*ABC* = 125° as shown. Find the length of diagonal *AC*.



4 Parallelogram *ABCD* has sides 11 cm and 5 cm, and one interior angle 79°25′. Find the length of the diagonals.

5 Quadrilateral *ABCD* has sides *AB* = 12 cm, *BC* = 10.4 cm, *CD* = 8.4 cm and *AD* = 9.7 cm with ∠*ABC* = 63°57′. Find:

a the length of diagonal *AC* **b** ∠*DAC* **c** ∠*ADC*

6 Triangle *XYZ* is isosceles with *XY* = *XZ* = 7.3 cm and *YZ* = 5.9 cm. Find the value of all angles, to the nearest minute.

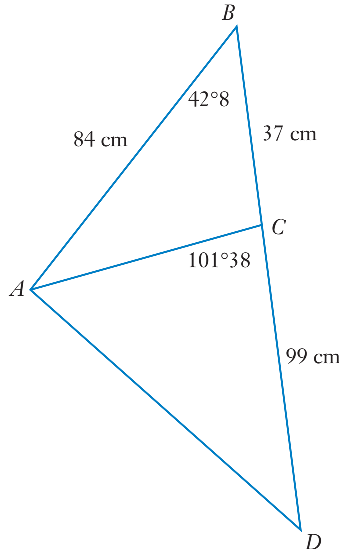
7 Quadrilateral *MNOP* has *MP* = 12 mm, *NO* = 12.7 mm, *MN* = 8.9 mm, *OP* = 15.6 mm and ∠*NMP* = 119°15′. Find:

a the length of diagonal *NP* **b** ∠*NOP*

8 Given the figure, find the length of:

a AC

b AD



9 In a regular pentagon $ABCDE$ with sides 8 cm, find the length of diagonal AD.

10 A regular hexagon $ABCDEF$ has sides 5.5 cm. Find:

a the length of AD **b** $\angle ADF$

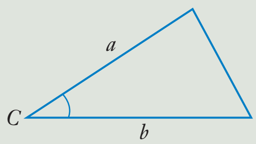
WS Areas of triangles

5.07 Area of a triangle

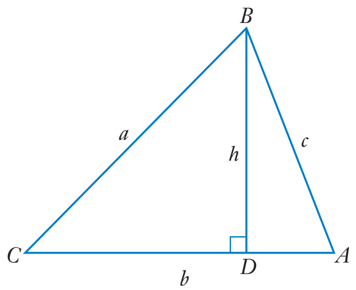
Trigonometry allows us to find the area of a triangle if we know 2 sides and their included angle.

Sine formula for the area of a triangle

$$A = \frac{1}{2}ab \sin C$$



Proof



From $\triangle BCD$,

$$\sin C = \frac{h}{a}$$

$$\therefore\ h = a \sin C$$

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}ba \sin C$$

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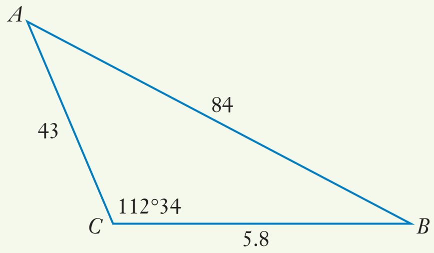
EXAMPLE 17

Find the area of $\triangle ABC$ correct to 2 decimal places.



Solution

$$A = \frac{1}{2}ab \sin C$$

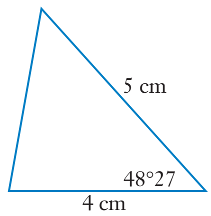
$$= \frac{1}{2}(4.3)(5.8) \sin 112°34'$$

$$\approx 11.52 \text{ units}^2$$

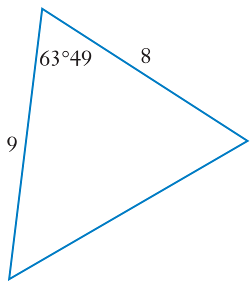
Exercise 5.07 Area of a triangle

1 Find the area of each triangle correct to 1 decimal place:

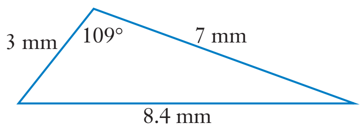
a



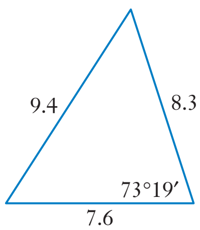
b



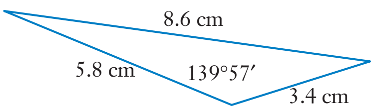
c



d



e



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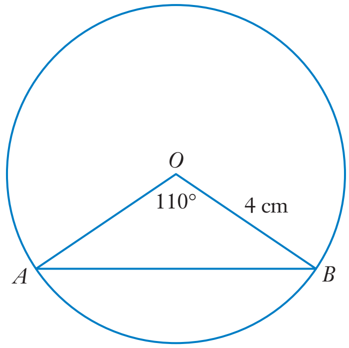
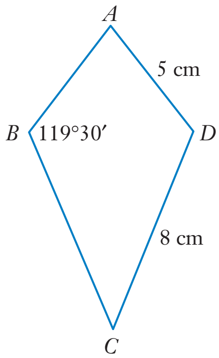
2 Find the area of $\triangle OAB$ correct to 1 decimal place (O is the centre of the circle):

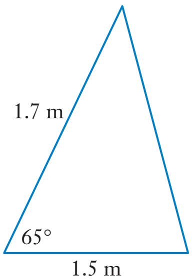


3 Find the area of a parallelogram with sides 3.5 cm and 4.8 cm and with one of its internal angles $67°13'$, correct to 1 decimal place.

4 Find the area of kite $ABCD$, correct to 3 significant figures:



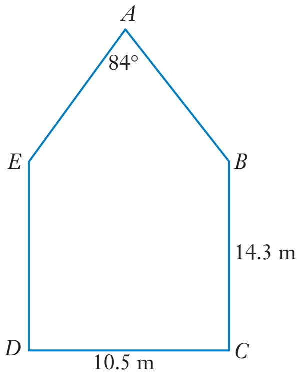
5 Find the area of this sail, correct to 1 decimal place:



6 This pentagon is made from a rectangle and isosceles triangle with $AE = AB$, as shown. Find:

a the length of AE

b the area of the figure.



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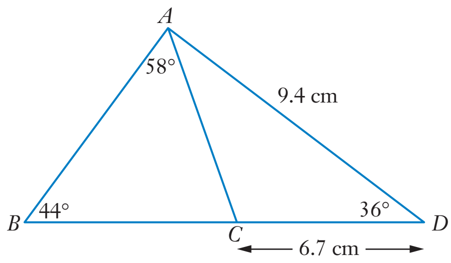
7 For this figure, find:

- **a** the length of AC
- **b** the area of triangle ACD
- **c** the area of triangle ABC.



8 Find the exact area of an equilateral triangle with sides 5 cm.

5.08 Mixed problems

The sine and cosine rules

Use the **sine rule** to find:

- a side, given one side and 2 angles
- an angle, given 2 sides and one angle

Use the **cosine rule** to find:

- a side, given 2 sides and one angle
- an angle, given 3 sides

EXAMPLE 18

a The angle of elevation of a tower from point A is 72°. From point B, 50 m further away from the tower than A, the angle of elevation is 47°.

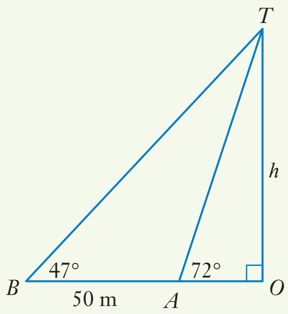
- **i** Find the exact length of AT, the distance from A to the top of the tower.
- **ii** Hence, or otherwise, find the height h of the tower to 1 decimal place.

b A ship sails from Sydney for 200 km on a bearing of 040° then sails on a bearing of 157° for 345 km.

- **i** How far from Sydney is the ship, to the nearest km?
- **ii** What is the bearing of the ship from Sydney, to the nearest degree?

Solution

a



i $\angle BAT = 180° - 72° = 108°$ (straight angle)

$\angle BTA = 180° - (47° + 108°)$ (angle sum of $\triangle BTA$)

$= 25°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{AT}{\sin 47°} = \frac{50}{\sin 25°}$$

$$\therefore AT = \frac{50 \sin 47°}{\sin 25°}$$

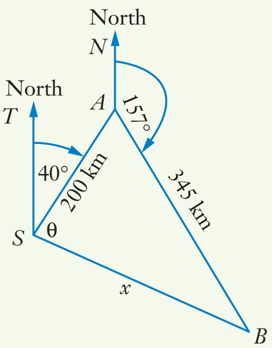
ii $\sin 72° = \dfrac{h}{AT}$

$\therefore h = AT \sin 72°$

$$= \frac{50 \sin 47°}{\sin 25°} \times \sin 72°$$

≈ 82.3 m

b



i $\angle SAN = 180° - 40° = 140°$ (cointerior angles)

$\therefore \angle SAB = 360° - (140° + 157°)$ (angle of revolution)

$= 63°$

$c^2 = a^2 + b^2 - 2ab \cos C$

$x^2 = 200^2 + 345^2 - 2(200)(345) \cos 63°$

$= 96\ 374.3110\ldots$

$x = \sqrt{963743110}$

$= 310.4421\ldots$

≈ 310

So the ship is 310 km from Sydney.

ii $$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{345} = \frac{\sin 63°}{3104421}$$

$$\therefore \sin \theta = \frac{345 \sin 63°}{3104421}$$

$= 0.9901\ldots$

$\theta \approx 82°$

The bearing from Sydney $= 40° + 82° = 122°$.

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We can also use trigonometry to solve 3-dimensional problems.

WS
3D trigonometry

EXAMPLE 19

a From point X, 25 m due south of the base of a tower, the angle of elevation is 47°. Point Y is 15 m due east of the tower. Find:

i the height, h, of the tower, correct to 1 decimal place

ii the angle of elevation, θ, of the tower from point Y.

b A cone has a base diameter of 18 cm and a slant height of 15 cm. Find the vertical angle at the top of the cone.

Solution

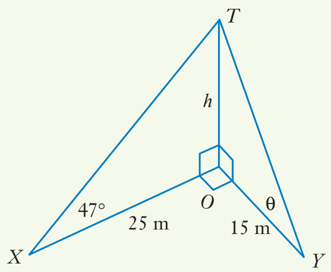
a **i** From $\triangle XTO$

$$\tan 47° = \frac{h}{25}$$

$$25 \tan 47° = h$$

$$26.8 = h$$

So the tower is 26.8 m high.



ii From $\triangle YTO$

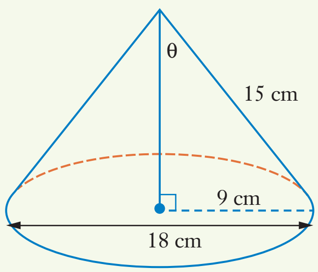
$$\tan \theta = \frac{26\ 8}{15}$$

$$\therefore \theta = \tan^{-1} \frac{26\ 8}{15}$$

$$= 60°46'$$

So the angle of elevation from Y is 60°46′.

b The radius of the base is 9 cm.



$$\sin \theta = \frac{9}{15}$$

$$\therefore \theta = \sin^{-1} \frac{9}{15}$$

$$= 36°52'$$

$$\text{Vertical angle} = 2\theta$$

$$= 73°44'$$

The vertical angle is the angle at the vertex of the cone.

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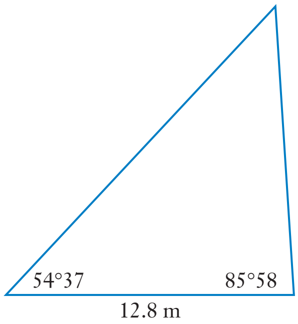
Exercise 5.08 Mixed problems

1 A car is broken down to the north of 2 towns. The car is 39 km from town A and 52 km from town B. If A is due west of B and the 2 towns are 68 km apart, what is the bearing, to the nearest degree, of the car from:

a town A **b** town B?

2 The angle of elevation to the top of a tower is 54°37′ from a point 12.8 m out from its base. The tower is leaning at an angle of 85°58′ as shown. Find the height of the tower.



3 Rugby league goal posts are 5.5 m apart. If a footballer is standing 8 m from one post and 11 m from the other, find the angle within which the ball must be kicked to score a goal, to the nearest degree.

4 A boat is sinking 1.3 km out to sea from a marina. Its bearing is 041° from the marina and 324° from a rescue boat. The rescue boat is due east of the marina.

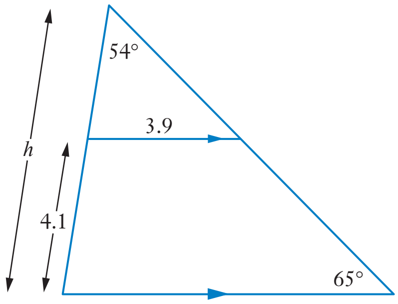
a How far, correct to 2 decimal places, is the rescue boat from the sinking boat?

b How long will it take the rescue boat, to the nearest minute, to reach the other boat if it travels at 80 km/h?

5 The angle of elevation of the top of a flagpole is 20° from where Thuy stands a certain distance away from its base. After walking 80 m towards the flagpole, Thuy finds the angle of elevation is 75°. Find the height of the flagpole, to the nearest metre.

6 A triangular field ABC has sides $AB = 85$ m and $AC = 50$ m. If B is on a bearing of 065° from A and C is on a bearing of 166° from A, find the length of BC, correct to the nearest metre.

7 Find the value of h, correct to 1 decimal place.



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8 A motorbike and a car leave a service station at the same time. The motorbike travels on a bearing of 080° and the car travels for 15.7 km on a bearing of 108° until the bearing of the motorbike from the car is 310°. How far, correct to 1 decimal place, has the motorbike travelled?

9 A submarine is being followed by two ships, A and B, 3.8 km apart, with A due east of B. If A is on a bearing of 165° from the submarine and B is on a bearing of 205° from the submarine, find the distance from the submarine to both ships.

10 A plane flies from Dubbo on a bearing of 139° for 852 km, then turns and flies on a bearing of 285° until it is due west of Dubbo. How far from Dubbo is the plane, to the nearest km?

11 Rhombus $ABCD$ with side 8 cm has diagonal BD 11.3 cm long. Find $\angle DAB$.

12 Zeke leaves school and runs for 8.7 km on a bearing of 338°, then turns and runs on a bearing of 061° until he is due north of school. How far north of school is he?

13 A car drives due east for 83.7 km then turns and travels for 105.6 km on a bearing of 029°. How far is the car from its starting point?

14 A plane leaves Sydney and flies for 1280 km on a bearing of 050°. It then turns and flies for 3215 km on a bearing of 149°. How far is the plane from Sydney, to the nearest km?

15 Trapezium $ABCD$ has $AD \parallel BC$, with $AB = 4.6$ cm, $BC = 11.3$ cm, $CD = 6.4$ cm, $\angle DAC = 23°30'$ and $\angle ABC = 78°$. Find:

a the length of AC **b** $\angle ADC$ to the nearest minute

16 A plane leaves Adelaide and flies for 875 km on a bearing of 056°. It then turns and flies on a bearing of θ for 630 km until it is due east of Adelaide. Evaluate θ to the nearest degree.

17 Quadrilateral $ABCD$ has $AB = AD = 7.2$ cm, $BC = 8.9$ cm and $CD = 10.4$ cm, with $\angle DAB = 107°$. Find:

a the length of diagonal BD **b** $\angle BCD$

18 A wall leans inwards and makes an angle of 88° with the floor.

a A 4 m long ladder leans against the wall with its base 2.3 m out from the wall. Find the angle that the top of the ladder makes with the wall.

b A longer ladder is placed the same distance out from the wall and its top makes an angle of 31° with the wall.

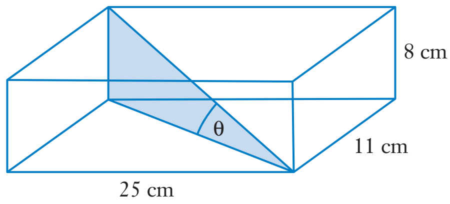
i How long is this ladder?

ii How much further does it reach up the wall than the first ladder?

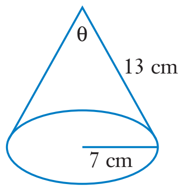
19 A 25 cm × 11 cm × 8 cm cardboard box contains an insert (the shaded area) made of foam.

a Find the area of foam in the insert, to the nearest cm^2.

b Find θ, the angle that the insert makes at the corner of the box.



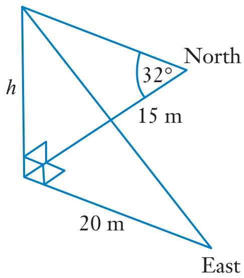
20 A cone has radius 7 cm and a slant height of 13 cm. Find the vertical angle, θ, at the top of the cone, in degrees and minutes.



21 From a point 15 m due north of a tower, the angle of elevation of the tower is 32°.

a Find the height of the tower, correct to 2 decimal places.

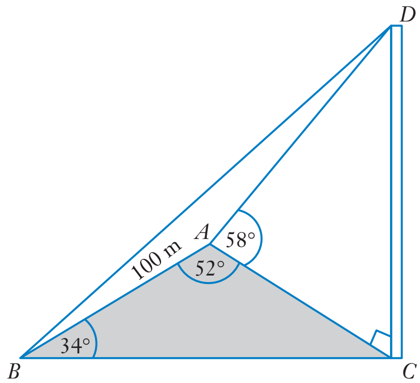
b Find correct to the nearest degree the angle of elevation of the tower at a point 20 m due east of the tower.



22 A pole DC is seen from two points A and B. The angle of elevation from A is 58°. If $\angle CAB = 52°$, $\angle ABC = 34°$, and A and B are 100 m apart, find:

a how far A is from the foot of the pole, to the nearest metre.

b the height of the pole, to 1 decimal place.



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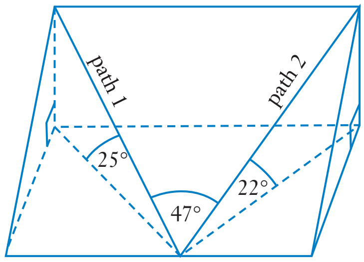
23 Two straight paths to the top of a cliff are inclined at angles of 25° and 22° to the horizontal.

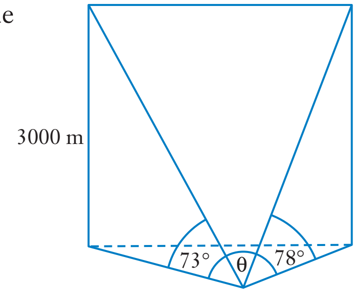
a If path 1 is 114 m long, find the height of the cliff, to the nearest metre.

b Find the length of path 2, to 1 decimal place.

c If the paths meet at 47° at the base of the cliff, find their distance apart at the top of the cliff, correct to 1 decimal place.



24 A hot-air balloon floating at 950 m/h at a constant altitude of 3000 m is observed to have an angle of elevation of 78°. After 20 minutes, the angle of elevation is 73°. Calculate the angle through which the observer has turned during those 20 minutes.



5.09 Radians

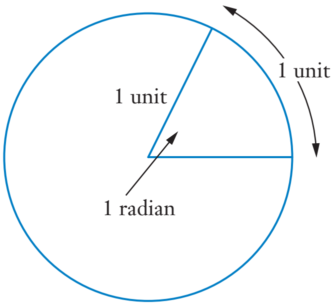
WS Radians

WS Converting degrees and radians

We use degrees to measure angles in geometry and trigonometry, but there are other units for measuring angles.

A **radian** is a unit for measuring angles based on the length of an arc in a circle.

One radian is the angle subtended by an arc with length 1 unit in a unit circle (of radius 1).



Conversions

We can change between radians and degrees using this equation:

Radians and degrees

$$\pi \text{ radians} = 180°$$

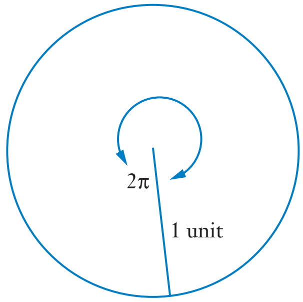
Proof

The circumference of a circle with radius 1 unit is:

$C = 2\pi r$

$= 2\pi(1)$

$= 2\pi$



The arc length of the whole circle is 2π.

$\therefore$ there are 2π radians in a whole circle.

But there are 360° in a whole circle (angle of revolution).

So $2\pi = 360°$

$\pi = 180°$

Converting between radians and degrees

To change from radians to degrees: multiply by $\frac{180}{\pi}$.

To change from degrees to radians: multiply by $\frac{\pi}{180}$.

Notice that $1° = \frac{\pi}{180} \approx 0.017$ radians

Also 1 radian $= \frac{180}{\pi} \approx 57°18'$

Degrees and radians

EXAMPLE 20

a Convert $\frac{3\pi}{2}$ into degrees.

b Change 60° to radians, leaving your answer in terms of π.

c Convert 50° into radians, correct to 2 decimal places.

d Change 1.145 radians into degrees, to the nearest minute.

e Convert 38°41′ into radians, correct to 3 decimal places.

f Evaluate cos 1.145 correct to 2 decimal places.

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Solution

a Since $\pi = 180°$,

$$\frac{3\pi}{2} = \frac{3(180°)}{2} = 270°$$

b $180° = \pi$ radians

So $1° = \frac{\pi}{180}$ radians

$$60° = \frac{\pi}{180} \times 60 = \frac{60\pi}{180} = \frac{\pi}{3}$$

c $180° = \pi$ radians

So $1° = \frac{\pi}{180}$ radians

$$50° = \frac{\pi}{180} \times 50 = \frac{50\pi}{180} \approx 0.87$$

d π radians $= 180°$

$\therefore$ 1 radian $= \frac{180°}{\pi}$

$$1.145 \text{ radians} = \frac{180°}{\pi} \times 1.145 \approx 65.6° = 65°36'$$

e $180° = \pi$ radians

$$1° = \frac{\pi}{180°}$$

$$38°41' = \frac{\pi}{180°} \times 38°41' = 0.675$$

f

Operation	Casio scientific	Sharp scientific
Make sure the calculator is in radians.	SHIFT SET UP Rad	Press DRG until rad is on the screen.
Enter data.	cos 1.145 =	cos 1.145 =

$\cos 1.145 = 0.4130 \ldots$

≈ 0.41

Special angles

$$30° = \frac{\pi}{6} \quad 45° = \frac{\pi}{4} \quad 60° = \frac{\pi}{3} \quad 90° = \frac{\pi}{2}$$

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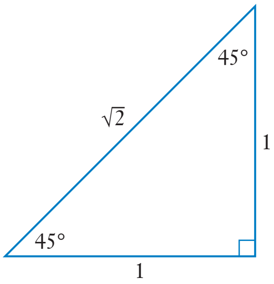
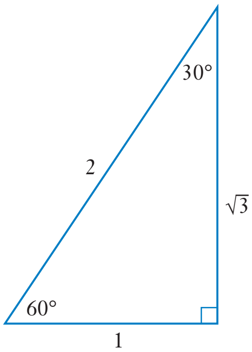
The angles 30°, 45° and 60° give exact results in trigonometry using 2 special triangles. You looked at these in Exercise 5.01, Questions 9 and 10, on page 231.





From these triangles we have the exact trigonometric ratios:

Exact values

Exact values 2

Exact trigonometric values

The exact ratios

$\sin 30° = \frac{1}{2}$ $\quad$ $\sin 45° = \frac{1}{\sqrt{2}}$ $\quad$ $\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\cos 45° = \frac{1}{\sqrt{2}}$ $\quad$ $\cos 60° = \frac{1}{2}$

$\tan 30° = \frac{1}{\sqrt{3}}$ $\quad$ $\tan 45° = 1$ $\quad$ $\tan 60° = \sqrt{3}$

We can write these same results in radians:

The exact ratios in radians

$\sin\frac{\pi}{6} = \frac{1}{2}$ $\quad$ $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad$ $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\quad$ $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad$ $\cos\frac{\pi}{3} = \frac{1}{2}$

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ $\quad$ $\tan\frac{\pi}{4} = 1$ $\quad$ $\tan\frac{\pi}{3} = \sqrt{3}$

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EXAMPLE 21

a **i** Convert $\frac{\pi}{3}$ to degrees.

ii Find the exact value of $\tan\frac{\pi}{3}$.

b Find the exact value of $\cos\frac{\pi}{4}$.

Solution

a **i** $\frac{\pi}{3}=\frac{180°}{3}$

$=60°$

ii $\tan\frac{\pi}{3}=\tan 60°$

$=\sqrt{3}$

b $\cos\frac{\pi}{4}=\cos 45°$

$=\frac{1}{\sqrt{2}}$

Exercise 5.09 Radians

1 Convert to degrees:

a $\frac{\pi}{5}$ **b** $\frac{2\pi}{3}$ **c** $\frac{5\pi}{4}$ **d** $\frac{7\pi}{6}$ **e** 3π

f $\frac{7\pi}{9}$ **g** $\frac{4\pi}{3}$ **h** $\frac{7\pi}{3}$ **i** $\frac{\pi}{9}$ **j** $\frac{5\pi}{18}$

2 Convert to radians in terms of π:

a 135° **b** 30° **c** 150° **d** 240° **e** 300°

f 63° **g** 15° **h** 450° **i** 225° **j** 120°

3 Change to radians, correct to 2 decimal places:

a 56° **b** 68° **c** 127° **d** 289° **e** 312°

4 Change to radians, correct to 2 decimal places:

a 18°34′ **b** 35°12′ **c** 101°56′ **d** 88°29′ **e** 50°39′

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5 Convert each radian measure into degrees and minutes, to the nearest minute:

a	1.09	**b**	0.768	**c**	1.16	**d**	0.99	**e**	0.32
f	3.2	**g**	2.7	**h**	4.31	**i**	5.6	**j**	0.11

6 Find correct to 2 decimal places:

a	sin 0.342	**b**	cos 1.5	**c**	tan 0.056	**d**	cos 0.589	**e**	tan 2.29
f	sin 2.8	**g**	tan 5.3	**h**	cos 4.77	**i**	cos 3.9	**j**	sin 2.98

7 Find the exact value of:

a	$\sin\frac{\pi}{4}$	**b**	$\cos\frac{\pi}{3}$	**c**	$\tan\frac{\pi}{6}$	**d**	$\sin\frac{\pi}{3}$	**e**	$\tan\frac{\pi}{4}$
f	$\sin\frac{\pi}{6}$	**g**	$\cos\frac{\pi}{4}$	**h**	$\cos\frac{\pi}{6}$	**i**	$\tan\frac{\pi}{3}$		

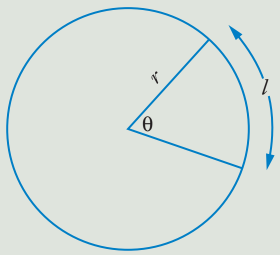
5.10 Length of an arc

Since radians are defined from the length of an arc of a circle, we can use radians to find the arc length of a circle.

You can find formulas for these using degrees, but they are not as simple. All the work on circles in this chapter uses radians.

Length of an arc

$l = r\theta$



Proof

$$\frac{\text{arc length } l}{\text{circumference}} = \frac{\text{angle } \theta}{\text{whole revolution}}$$

$$\frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

$$\therefore l = \frac{\theta 2\pi r}{2\pi}$$

$$= r\theta$$

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EXAMPLE 22

a Find the length of the arc formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m.

b Find the length of the arc formed given the angle subtended is 30° and the radius is 9 cm.

c The area of a circle is 450 cm^2. Find, in degrees and minutes, the angle subtended at the centre of the circle by a 2.7 cm arc.

Solution

a

$$l = r\theta$$
$$= 5\left(\frac{\pi}{4}\right)$$
$$= \frac{5\pi}{4} \text{ m}$$

b First change 30° into radians.

$$\theta = \frac{\pi}{6}$$
$$l = r\theta$$
$$= 9\left(\frac{\pi}{6}\right)$$
$$= \frac{3\pi}{2} \text{ cm}$$

c

$$A = \pi r^2$$
$$450 = \pi r^2$$
$$\frac{450}{\pi} = r^2$$
$$\sqrt{\frac{450}{\pi}} = r$$
$$11.9682\ldots = r$$

Now $l = r\theta$

$$2.7 = 11.9682\ldots\theta$$
$$\frac{2\ 7}{119682} = \theta$$
$$0.2255\ldots = \theta$$

$$\pi \text{ radians} = 180°$$
$$1 \text{ radian} = \frac{180°}{\pi}$$
$$0.2255\ldots \text{ radians} = \frac{180°}{\pi} \times 0.2255\ldots$$
$$= 12.9257°$$
$$\approx 12°56'$$

So $\theta = 12°56'$

Exercise 5.10 Length of an arc

1 Find the exact arc length of a circle with:

- **a** radius 4 cm and angle subtended π
- **b** radius 3 m and angle subtended $\frac{\pi}{3}$
- **c** radius 10 cm and angle subtended $\frac{5\pi}{6}$
- **d** radius 3 cm and angle subtended $30°$
- **e** radius 7 mm and angle subtended $45°$.

2 Find the arc length, correct to 2 decimal places, given:

- **a** radius 1.5 m and angle subtended 0.43
- **b** radius 3.21 cm and angle subtended 1.22
- **c** radius 7.2 mm and angle subtended $55°$
- **d** radius 5.9 cm and angle subtended $23°12'$
- **e** radius 2.1 m and angle subtended $82°35'$.

3 The angle subtended at the centre of a circle of radius 3.4 m is $29°51'$. Find the length of the arc cut off by this angle, correct to 1 decimal place.

4 The arc length when a sector of a circle is subtended by an angle of $\frac{\pi}{5}$ at the centre is $\frac{3\pi}{2}$ m. Find the radius of the circle.

5 The radius of a circle is 3 cm and an arc is $\frac{2\pi}{7}$ cm long. Find the angle subtended at the centre of the circle by the arc.

6 The circumference of a circle is 300 mm. Find the length of the arc that is formed by an angle of $\frac{\pi}{6}$ subtended at the centre of the circle.

7 A circle with area 60 cm^2 has an arc 8 cm long. Find the angle that is subtended at the centre of the circle by the arc.

8 A circle with circumference 124 mm has a chord cut off it that subtends an angle of $40°$ at the centre. Find the length of the arc cut off by the chord.

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9 A circle has a chord of 25 mm with an angle of $\frac{\pi}{6}$ subtended at the centre. Find, to 1 decimal place:

a the radius

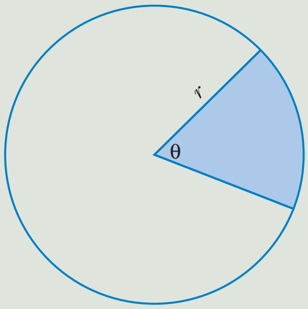
b the length of the arc cut off by the chord.

10 A sector of a circle with radius 5 cm and an angle of $\frac{\pi}{3}$ subtended at the centre is cut out of cardboard. It is then curved around to form an open cone. Find its exact volume.

5.11 Area of a sector

Area of a sector

$$A = \frac{1}{2}r^2\theta$$



Proof

$$\frac{\text{area of sector } A}{\text{area of circle}} = \frac{\text{angle } \theta}{\text{whole revolution}}$$

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$\therefore A = \frac{\theta \pi r^2}{2\pi}$$

$$= \frac{1}{2}r^2\theta$$

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EXAMPLE 23

a Find the area of the sector formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m.

b The area of the sector of a circle with radius 4 cm is $\frac{6\pi}{5}$ cm^2. Find the angle, in degrees, that is subtended at the centre of the circle.

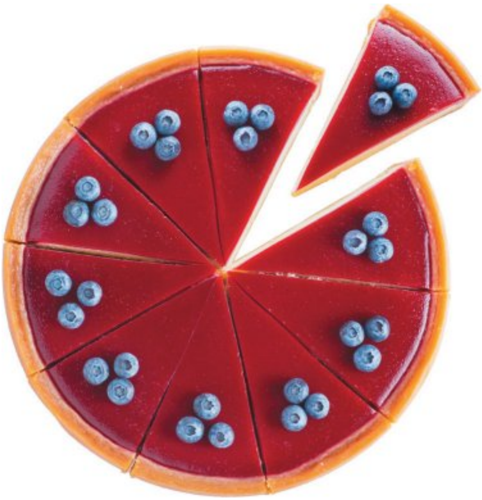
Solution

a

$$A = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2}(5)^2\left(\frac{\pi}{4}\right)$$
$$= \frac{25\pi}{8} \text{ m}^2$$

b

$$A = \frac{1}{2}r^2\theta$$
$$\frac{6\pi}{5} = \frac{1}{2}(4)^2\theta$$
$$= 8\theta$$
$$\frac{6\pi}{40} = \theta$$
$$\theta = \frac{3\pi}{20}$$
$$= \frac{3(180°)}{20}$$
$$= 27°$$



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Exercise 5.11 Area of a sector

1 Find the exact area of the sector of a circle whose radius is:

a 4 cm and the subtended angle is π

b 3 m and the subtended angle is $\frac{\pi}{3}$

c 10 cm and the subtended angle is $\frac{5\pi}{6}$

d 3 cm and the subtended angle is 30°

e 7 mm and the subtended angle is 45°.

2 Find the area of the sector, correct to 2 decimal places, given the radius is:

a 1.5 m and the subtended angle is 0.43

b 3.21 cm and the subtended angle is 1.22

c 7.2 mm and the subtended angle is 55°

d 5.9 cm and the subtended angle is 23°12′

e 2.1 m and the subtended angle is 82°35′.

3 Find the area, correct to 3 significant figures, of the sector of a circle with radius 4.3 m and an angle of 1.8 subtended at the centre.

4 The area of a sector of a circle is 20 cm^2. If the radius of the circle is 3 cm, find the angle subtended at the centre of the circle by the sector.

5 The area of the sector of a circle that is subtended by an angle of $\frac{\pi}{3}$ at the centre is 6π m^2. Find the radius of the circle.

6 A circle with radius 7 cm has a sector cut off by an angle of 30° subtended at the centre of the circle. Find:

a the arc length

b the area of the sector.

7 A circle has a circumference of 185 mm. Find the area of the sector cut off by an angle of $\frac{\pi}{5}$ subtended at the centre.

8 If the area of a circle is 200 cm^2 and a sector is cut off by an angle of $\frac{3\pi}{4}$ at the centre, find the area of the sector.

9 Find the area of the sector of a circle with radius 5.7 cm if the length of the arc formed by this sector is 4.2 cm.

10 The area of a sector is $\frac{3\pi}{10}$ cm^2 and the arc length cut off by the sector is $\frac{\pi}{5}$ cm. Find the angle subtended at the centre of the circle and the radius of the circle.

11 If an angle of $\frac{\pi}{7}$ is subtended at the centre of a circle with radius 3 cm, find:

a the exact arc length

b the exact area of the sector.

12 An angle of $\frac{\pi}{6}$ is subtended at the centre of a circle with radius 5 cm. Find:

a the length of the arc

b the area of the sector

c the length of the chord.

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13 A chord 8 mm long is formed by an angle of 45° subtended at the centre of a circle. Find, correct to 1 decimal place:

a the radius of the circle

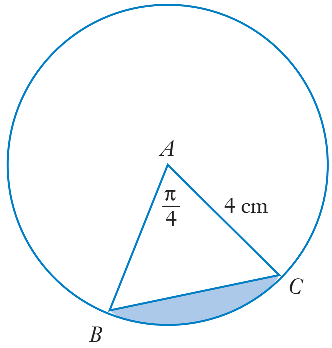
b the area of the sector cut off by the angle.

14 **a** Find the area of the sector of a circle with radius 4 cm if the angle subtended at the centre is $\frac{\pi}{4}$.

b Find the length of *BC* to 1 decimal place.

c Find the exact area of triangle *ABC*.

d Hence find the exact area of the shaded minor segment of the circle.



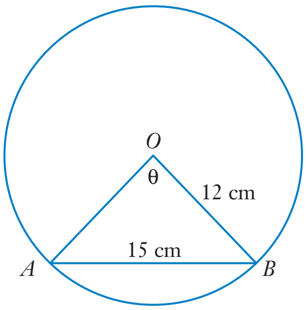
15 A triangle *OAB* is formed where *O* is the centre of a circle of radius 12 cm and *A* and *B* are endpoints of a 15 cm chord.

a Find the angle subtended at the centre of the circle, in degrees and minutes.

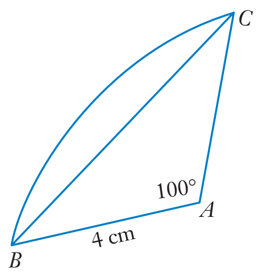
b Find the area of ΔOAB correct to 1 decimal place.

c Find the area of the minor segment cut off by the chord, correct to 2 decimal places.

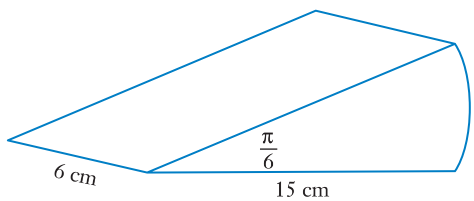
d Find the area of the major segment cut off by the chord, correct to 2 decimal places.



16 Arc *BC* subtends an angle of 100° at the centre *A* of a circle with radius 4 cm. Find the perimeter of sector *ABC*.



17 A wedge is cut so that its cross-sectional area is a sector of a circle with radius 15 cm and subtending an angle of $\frac{\pi}{6}$ at the centre. Find the exact volume of the wedge.



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5. TEST YOURSELF

Qz
Practice quiz

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

1 Find the exact length of the radius of a circle if the arc length cut off by an angle of $\frac{5\pi}{4}$ is $\frac{25\pi}{8}$ cm.

A 5π cm **B** 5 cm **C** 2.5 cm **D** $\frac{5\pi}{2}$ cm

2 The cosine rule is (there is more than one answer):

A $c^2 = a^2 + b^2 - 2ab \cos C$

B $c^2 = a^2 + c^2 - 2ac \cos C$

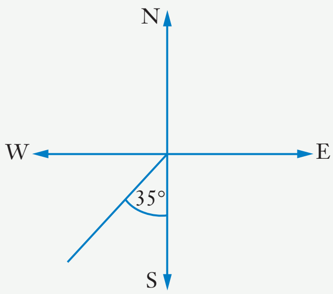
C $a^2 = b^2 + c^2 - 2bc \cos A$

D $a^2 = b^2 + c^2 - 2ab \cos A$

3 What bearing is shown on the diagram (there may be more than one answer)?

A 035° **B** W 35° S

C S 35° W **D** 215°



4 Find the exact value of $\cos \theta$ and $\sin \theta$ if $\tan \theta = \frac{3}{5}$.

5 Evaluate to 2 decimal places:

a $\sin 39°54'$ **b** $\tan 61°30'$ **c** $\cos 19°2'$ **d** $\sin 0.14$ **e** $\tan 3.5$

6 Find θ to the nearest minute if:

a $\sin \theta = 0.72$ **b** $\cos \theta = 0.286$ **c** $\tan \theta = \frac{5}{7}$

7 A ship sails on a bearing of 215° from port until it is 100 km due south of port. How far does it sail, to the nearest km?

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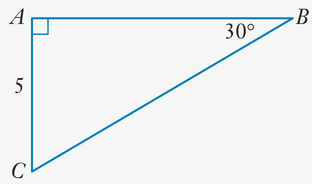
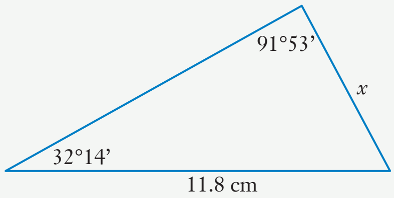
8 Find the length of AB as a surd.

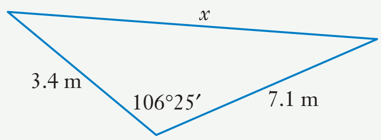


9 Evaluate x, correct to 2 significant figures.

a



b

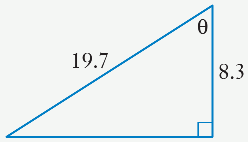


10 Convert each radian measure to degrees and minutes.

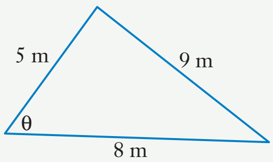
a 0.75 **b** 1.3 **c** 3.95 **d** 4.2 **e** 5.66

11 Evaluate θ to the nearest minute.

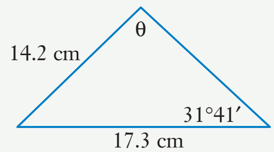
a



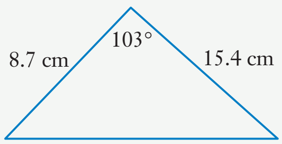
b



c



12 Find the area of this triangle.



13 Jacquie walks south from home for 3.2 km, then turns and walks west for 1.8 km. What is the bearing, to the nearest degree, of:

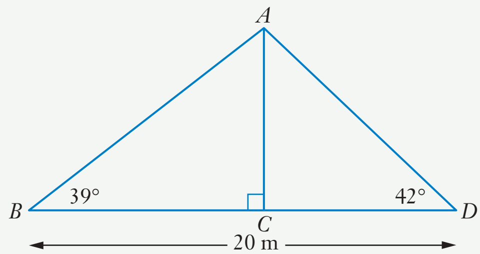
a Jacquie from her home?

b her home from where Jacquie is now?

14 The angle of elevation from point B to the top of a pole AC is 39° and the angle of elevation from D, on the other side of the pole, is 42°. B and D are 20 m apart.

a Find an expression for the length of AD.

b Find the height of the pole, to 1 decimal place.



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15 A plane flies from Orange for 1800 km on a bearing of 300°. It then turns and flies for 2500 km on a bearing of 205°. How far is the plane from Orange, to the nearest km?

16 Convert to radians, leaving in terms of π:

a 60° **b** 45° **c** 150° **d** 180° **e** 20°

17 A circle with radius 5 cm has an angle of $\frac{\pi}{6}$ subtended at the centre. Find:

a the exact arc length

b the exact area of the sector.

18 Find the exact value of:

a $\tan\frac{\pi}{3}$ **b** $\cos\frac{\pi}{6}$ **c** $\sin\frac{\pi}{4}$ **d** $\tan\frac{\pi}{6}$ **e** $\cos\frac{\pi}{4}$

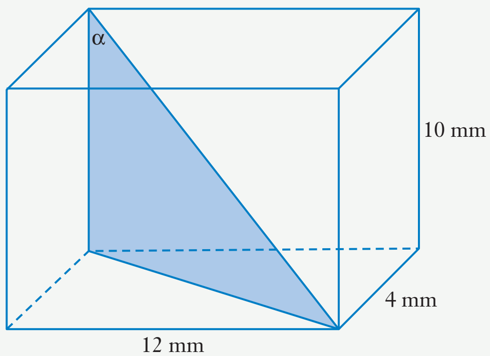
f $\sin\frac{\pi}{6}$ **g** $\tan\frac{\pi}{4}$ **h** $\cos\frac{\pi}{3}$ **i** $\sin\frac{\pi}{3}$

19 A circle has a circumference of 8π cm. If an angle of $\frac{\pi}{7}$ is subtended at the centre of the circle, find:

a the exact area of the sector

b the area of the minor segment, to 2 decimal places.

20 Evaluate α in this figure.

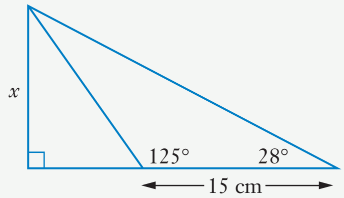


21 In triangle MNP, $NP = 14.9$ cm, $MP = 12.7$ cm and $\angle N = 43°49'$. Find $\angle M$ in degrees and minutes.

5. CHALLENGE EXERCISE

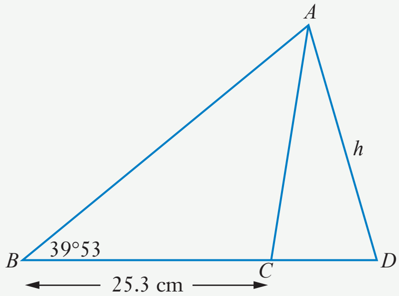
1 Two cars leave an intersection at the same time, one travelling at 70 km/h along one straight road and the other car travelling at 80 km/h along another straight road. After 2 hours they are 218 km apart. At what angle, to the nearest minute, do the roads meet at the intersection?

2 Evaluate x correct to 3 significant figures.



3 **a** Find an exact expression for the length of AC.

b Hence, or otherwise, find the value of h correct to 1 decimal place.



4 From the top of a vertical pole the angle of depression to Ian standing at the foot of the pole is 43°. Liam is on the other side of the pole, and the angle of depression from the top of the pole to Liam is 52°. The boys are standing 58 m apart. Find the height of the pole, to the nearest metre.

5 From point A, 93 m due south of the base of a tower, the angle of elevation is 35°. Point B is 124 m due east of the tower. Find:

a the height of the tower, to the nearest metre

b the angle of elevation of the tower from point B.

6 A cable car 100 m above the ground is seen to have an angle of elevation of 65° when it is on a bearing of 345°. After a minute, it has an angle of elevation of 69° and is on a bearing of 025°. Find:

a how far it travels in that minute

b its speed in m s^{-1}.

7 Find the area of a regular hexagon with sides 4 cm, to the nearest cm^2.

8 Calculate correct to one decimal place the area of a regular pentagon with sides 12 mm.

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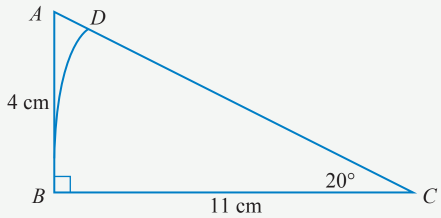
9 The length of an arc is 8.9 cm and the area of the sector is 24.3 cm^2 when an angle of θ is subtended at the centre of a circle. Find the area of the minor segment cut off by θ, correct to 1 decimal place.

10 *BD* is the arc of a circle with centre *C*. Find, correct to 2 decimal places:

a the length of arc *BD*

b the area of region *ABD*

c the perimeter of sector *BDC*.



11 David walks along a straight road. At one point he notices a tower on a bearing of 053° with an angle of elevation of 21°. After David walks 230 m, the tower is on a bearing of 342° with an angle of elevation of 26°. Find the height of the tower correct to the nearest metre.

12 The hour hand of a clock is 12 cm long. Find:

a the length of the arc through which the hand would turn in 5 hours

b the area through which the hand would pass in 2 hours.

FUNCTIONS

6. POLYNOMIALS AND INVERSE FUNCTIONS

You were introduced to functions and polynomials in Chapter 4, *Functions*. In this Mathematics Extension 1 chapter, you will study polynomials in more detail and look at inverse functions.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- EXT1 divide polynomials and write them as products of their factors
- EXT1 understand and apply the remainder and factor theorems
- EXT1 solve polynomial equations
- EXT1 draw polynomial graphs using intercepts and limiting behaviour
- EXT1 understand multiplicity of roots and their effect on graphs
- EXT1 find and graph inverses of functions and identify whether the inverse is also a function
- EXT1 understand how to restrict the domain of a function so that its inverse is a function
- EXT1 understand properties of inverse functions

EXT1 TERMINOLOGY

dividend: In division, the dividend is the polynomial or number being divided

divisor: In division, the divisor is the number or polynomial that divides another of the same type

factor theorem: The theorem that states that a polynomial $P(x)$ has a factor $x - k$ if and only if $P(k) = 0$

horizontal line test: A test that determines whether the inverse of a function is a function: any horizontal line drawn on the graph of the original function should cut the graph at most once

inverse function: An inverse function undoes the original function and can be shown by exchanging the x and y values of the original function

monotonic decreasing: Always decreasing

monotonic increasing: Always increasing

multiplicity: If $P(x) = (x - k)^r Q(x)$ where $Q(x) \neq 0$ and r is a positive integer, then the root $x = k$ has multiplicity r

quotient: The result when dividing two numbers or polynomials

remainder: A number or polynomial that is left over after dividing two numbers or polynomials

remainder theorem: The theorem that states that if a polynomial $P(x)$ is divided by $x - k$, then the remainder is given by $P(k)$

restricted domain: Domain restricted to the x values that will make the inverse relation a function

EXT1 6.01 Division of polynomials

Long division is a way to divide by a two-digit number without using a calculator. We can also use this method to divide polynomials. This allows us to factorise polynomials.

INVESTIGATION

LONG DIVISION

Study this example of long division: 5715 ÷ 48.

$$\begin{array}{r} 119\text{ r}3 \\ 48\overline{)5715} \\ \underline{48} \\ 91 \\ \underline{48} \\ 435 \\ \underline{432} \\ 3 \end{array}$$

$$\frac{5715}{48} = 119 + \frac{3}{48}$$

This means $\frac{5715}{48} \times 48 = 119 \times 48 + \frac{3}{48} \times 48$.

So 5715 = 48 × 119 + 3. (Check this on your calculator.)

The number 5715 is called the dividend, the 48 is the divisor, 119 is the quotient and 3 is the remainder.

 ISBN 97807043299

In Chapter 4, *Functions*, we learned that a polynomial is an expression in the form $P(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$, where n is a positive integer or zero. If we divide a polynomial $P(x)$ by $A(x)$, we can write $P(x)$ in the form $\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)}$ where $Q(x)$ is the quotient and $R(x)$ is the remainder.

$$\frac{P(x)}{A(x)} \times A(x) = Q(x) \times A(x) + \frac{R(x)}{A(x)} \times A(x)$$

$$P(x) = A(x)Q(x) + R(x)$$

Dividing polynomials

A polynomial $P(x)$ can be written as $P(x) = A(x)Q(x) + R(x)$

where $P(x)$ is the **dividend**, $A(x)$ is the **divisor**, $Q(x)$ is the **quotient** and $R(x)$ is the **remainder**.

The degree of the remainder $R(x)$ is always less than the degree of the divisor $A(x)$.

EXAMPLE 1

a **i** Divide $P(x) = 3x^4 - x^3 + 7x^2 - 2x + 3$ by $x - 2$.

ii Hence write $P(x)$ in the form $P(x) = A(x)Q(x) + R(x)$.

iii Show that $P(2)$ is equal to the remainder.

b For each pair of polynomials, divide $P(x)$ by $A(x)$ and then write $P(x)$ in the form $P(x) = A(x)Q(x) + R(x)$.

i $P(x) = x^3 - 3x^2 + x + 4,\ A(x) = x^2 - x$

ii $P(x) = x^5 + x^3 + 5x^2 - 6x + 15,\ A(x) = x^2 + 3$

Solution

a **i** Step 1: Dividing $3x^4$ by x gives $3x^3$.

$$\begin{array}{r|l} & 3x^3 \\ \hline x-2 & 3x^4 - x^3 + 7x^2 - 2x + 3 \end{array}$$

Step 2: Multiply $3x^3$ by $(x - 2)$ and find the remainder by subtraction.

$3x^3(x - 2) = 3x^4 - 6x^3$

$$\begin{array}{r|l} & 3x^3 \\ \hline x-2 & 3x^4 - x^3 + 7x^2 - 2x + 3 \\ & \underline{3x^4 - 6x^3} \\ & \qquad 5x^3 \end{array}$$

Step 3: Bring down the $7x^2$ and next divide $5x^3$ by x to give $5x^2$.

$$\begin{array}{r|l} & 3x^3 + 5x^2 \\ \hline x-2 & 3x^4 - x^3 + 7x^2 - 2x + 3 \\ & \underline{3x^4 - 6x^3} \\ & \qquad 5x^3 + 7x^2 \end{array}$$

Step 4: Multiply $5x^2$ by $(x - 2)$ and find the remainder by subtraction.

$5x^2(x - 2) = 5x^3 - 10x^2$

$$
\begin{array}{r}
3x^3 + 5x^2 \qquad\qquad\quad \\
x-2\,\overline{)\,3x^4 - x^3 + 7x^2 - 2x + 3} \\
\underline{3x^4 - 6x^3} \qquad\qquad\qquad\quad \\
5x^3 + 7x^2 \qquad\qquad\quad \\
\underline{5x^3 - 10x^2} \qquad\qquad\quad \\
17x^2 \qquad\qquad\quad
\end{array}
$$

Continue this way until we have a number (67) as the remainder.

$$
\begin{array}{r}
3x^3 + 5x^2 + 17x + 32 \\
x-2\,\overline{)\,3x^4 - x^3 + 7x^2 - 2x + 3} \\
\underline{3x^4 - 6x^3} \qquad\qquad\qquad\quad \\
5x^3 + 7x^2 \qquad\qquad\quad \\
\underline{5x^3 - 10x^2} \qquad\qquad\quad \\
17x^2 - 2x \qquad \\
\underline{17x^2 - 34x} \qquad \\
32x + 3 \\
\underline{32x - 64} \\
67
\end{array}
$$

ii $P(x) = 3x^4 - x^3 + 7x^2 - 2x + 3$ is the dividend.

$A(x) = x - 2$ is the divisor.

$Q(x) = 3x^3 + 5x^2 + 17x + 32$ is the quotient.

$R(x) = 67$ is the remainder.

$P(x) = A(x)Q(x) + R(x)$.

So $3x^4 - x^3 + 7x^2 - 2x + 3 = (x - 2)(3x^3 + 5x^2 + 17x + 32) + 67$.

iii $P(2) = 3(2)^4 - (2)^3 + 7(2)^2 - 2(2) + 3$

$= 48 - 8 + 28 - 4 + 3$

$= 67$

$\therefore$ $P(2)$ is equal to the remainder.

ISBN 97807043299

b **i**

$$\begin{array}{r l}
 & \quad\; x - 2 \\
x^2 - x \big) & x^3 - 3x^2 + x + 4 \\
 & \underline{x^3 - x^2} \\
 & \quad -2x^2 + x \\
 & \quad \underline{-2x^2 + 2x} \\
 & \qquad\quad -x + 4
\end{array}$$

$(x^3 - 3x^2 + x + 4) \div (x^2 - x) = x - 2$, remainder $-x + 4$

So $x^3 - 3x^2 + x + 4 = (x - 2)(x^2 - x) + (-x + 4)$

ii

$$\begin{array}{r l}
 & \quad x^3 \qquad\quad - 2x + 5 \\
x^2 + 3 \big) & x^5 + x^3 + 5x^2 - 6x + 15 \\
 & \underline{x^5 + 3x^3} \\
 & \quad -2x^3 + 5x^2 - 6x \\
 & \quad \underline{-2x^3 \qquad\quad - 6x} \\
 & \qquad\qquad 5x^2 \qquad + 15 \\
 & \qquad\qquad \underline{5x^2 \qquad + 15} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}$$

So $x^5 + x^3 + 5x^2 - 6x + 15 = (x^3 - 2x + 5)(x^2 + 3)$

EXT1 Exercise 6.01 Division of polynomials

Divide each pair of polynomials and write the dividend in the form $P(x) = A(x)Q(x) + R(x)$.

1 $(3x^2 + 2x + 5) \div (x + 4)$
2 $(x^2 + 5x - 2) \div (x + 1)$
3 $(x^2 - 7x + 4) \div (x - 1)$
4 $(x^3 + x^2 + 2x - 1) \div (x - 3)$
5 $(4x^2 + 2x - 3) \div (2x + 3)$
6 $(x^3 + x^2 - x - 3) \div (x - 2)$
7 $(x^4 - x^3 - 2x^2 + x - 3) \div (x + 4)$
8 $(4x^3 - 2x^2 + 6x - 1) \div (2x + 1)$
9 $(3x^5 - 2x^4 - 3x^3 + x^2 - x - 1) \div (x + 2)$
10 $(x^4 - 2x^2 + 5x + 4) \div (x - 3)$
11 $(2x^3 + 4x^2 - x + 8) \div (x^2 + 3x + 2)$
12 $(x^4 - 2x^3 + 4x^2 + 2x + 5) \div (x^2 + 2x - 1)$
13 $(3x^5 - 2x^3 + x - 1) \div (x + 1)$
14 $(x^3 - 3x^2 + 3x - 1) \div (x^2 + 5)$
15 $(2x^4 - 5x^3 + 2x^2 + 2x - 5) \div (x^2 - 2x)$

ISBN 97807043299

WS The remainder theorem

WS Factorising polynomials

EXT1 6.02 Remainder and factor theorems

Remainder theorem

If a polynomial $P(x)$ is divided by $x - k$, then the remainder is $P(k)$.

Proof

$P(x) = A(x)Q(x) + R(x)$ where $A(x) = x - k$

$P(x) = (x - k)Q(x) + R(x)$

The degree of $A(x)$ is 1, so the degree of $R(x)$ must be 0.

So $R(x) = c$ where c is a constant.

$\therefore P(x) = (x - k)Q(x) + c$

Substituting $x = k$:

$P(k) = (k - k)Q(k) + c$

$\quad = 0 \cdot Q(k) + c$

$\quad = c$

So $P(k)$ is the remainder.

EXAMPLE 2

a Find the remainder when $3x^4 - 2x^2 + 5x + 1$ is divided by $x - 2$.

b Evaluate m if the remainder is 4 when $2x^4 + mx + 5$ is divided by $x + 3$.

Solution

a When $P(x)$ is divided by $x - 2$ the remainder is $P(2)$.

$P(2) = 3(2)^4 - 2(2)^2 + 5(2) + 1$

$\quad = 51$

So the remainder is 51.

ISBN 97807043299

b The remainder when $P(x)$ is divided by $x + 3$ is $P(-3)$ since $x + 3 = x - (-3)$.

$$\begin{aligned} \text{So} \quad P(-3) &= 4 \\ 2(-3)^4 + m(-3) + 5 &= 4 \\ 162 - 3m + 5 &= 4 \\ 167 - 3m &= 4 \\ 167 &= 3m + 4 \\ 163 &= 3m \\ 54\tfrac{1}{3} &= m \end{aligned}$$

The **factor theorem** is a direct result of the **remainder theorem**.

Factor theorem

For a polynomial $P(x)$, if $P(k) = 0$ then $x - k$ is a factor of the polynomial.

Proof

$P(x) = (x - k)\,Q(x) + R(x)$.

The remainder theorem states that when $P(x)$ is divided by $x - k$, the remainder is $P(k)$.

So $P(x) = (x - k)\,Q(x) + P(k)$.

But if $P(k) = 0$:

$$\begin{aligned} P(x) &= (x - k)\,Q(x) + 0 \\ &= (x - k)\,Q(x) \end{aligned}$$

So $x - k$ is a factor of $P(x)$.

The converse is also true:

Converse of the factor theorem

For a polynomial $P(x)$, if $x - k$ is a factor of the polynomial, then $P(k) = 0$.

EXAMPLE 3

a Show that $x - 1$ is a factor of $P(x) = x^3 - 7x^2 + 8x - 2$.

b Divide $P(x)$ by $x - 1$ and write $P(x)$ in the form $P(x) = (x - 1)Q(x)$.

Solution

a The remainder when dividing the polynomial by $x - 1$ is $P(1)$.

$$P(1) = 1^3 - 7(1)^2 + 8(1) - 2$$
$$= 0$$

So $x - 1$ is a factor of $P(x)$.

b

$$\begin{array}{r}
x^2 - 6x + 2 \\
x - 1 \overline{\big) x^3 - 7x^2 + 8x - 2} \\
\underline{x^3 - x^2 } \\
-6x^2 + 8x \\
\underline{-6x^2 + 6x } \\
2x - 2 \\
\underline{2x - 2} \\
0
\end{array}$$

So $x^3 - 7x^2 + 8x - 2 = (x - 1)(x^2 - 6x + 2)$.

Some properties of polynomials come from the remainder and factor theorems.

The **zeros** of the polynomial $P(x)$ are those values of x for which $P(x) = 0$.

Properties of polynomials

- If polynomial $P(x)$ has n distinct zeros $k_1, k_2, k_3, \ldots, k_n$, then $(x - k_1)(x - k_2)(x - k_3) \ldots (x - k_n)$ is a factor of $P(x)$.
- If polynomial $P(x)$ has degree n and n distinct zeros $k_1, k_2, k_3, \ldots, k_n$, then $P(x) = a_n(x - k_1)(x - k_2)(x - k_3) \ldots (x - k_n)$.
- A polynomial of degree n cannot have more than n distinct real zeros.
- A polynomial of degree n with more than n distinct real zeros is the **zero polynomial** $P(x) = 0x^n + 0x^{n-1} + \ldots + 0x^2 + 0x + 0$.
- If 2 polynomials of degree n are equal for more than n distinct values of x, then the coefficients of like powers of x are equal:

  if $a_nx^n + \ldots + a_2x^2 + a_1x + a_0 \equiv b_nx^n + \ldots + b_2x^2 + b_1x + b_0$, then $a_n = b_n, \ldots, a_2 = b_2$, $a_1 = b_1$, $a_0 = b_0$.

ISBN 97807043299

EXAMPLE 4

If a polynomial has degree 2, show that it cannot have 3 zeros.

Solution

Let $P(x) = a_2x^2 + a_1x + a_0$ where $a_2 \neq 0$.

Assume $P(x)$ has 3 zeros, k_1, k_2 and k_3.

Then $(x - k_1)(x - k_2)(x - k_3)$ is a factor of the polynomial.

$\therefore P(x) = (x - k_1)(x - k_2)(x - k_3)Q(x)$

But this polynomial has degree 3 and $P(x)$ only has degree 2.

So $P(x)$ cannot have 3 zeros.

EXAMPLE 5

Write $x^3 - 2x^2 + 5$ in the form $ax^3 + b(x + 3)^2 + c(x + 3) + d$.

Solution

$$\begin{aligned} ax^3 + b(x+3)^2 + c(x+3) + d &= ax^3 + b(x^2 + 6x + 9) + c(x + 3) + d \\ &= ax^3 + bx^2 + 6bx + 9b + cx + 3c + d \\ &= ax^3 + bx^2 + (6b + c)x + 9b + 3c + d \end{aligned}$$

For $x^3 - 2x^2 + 5 \equiv ax^3 + bx^2 + (6b + c)x + 9b + 3c + d$:

by equating coefficients

$$\begin{aligned} a &= 1 \qquad [1] \\ b &= -2 \qquad [2] \\ 6b + c &= 0 \qquad [3] \\ 9b + 3c + d &= 5 \qquad [4] \end{aligned}$$

Substitute [2] into [3]:

$$\begin{aligned} 6(-2) + c &= 0 \\ -12 + c &= 0 \\ c &= 12 \end{aligned}$$

Substitute $b = -2$ and $c = 12$ into [4]:

$$\begin{aligned} 9(-2) + 3(12) + d &= 5 \\ -18 + 36 + d &= 5 \\ d &= -13 \end{aligned}$$

$\therefore x^3 - 2x^2 + 5 \equiv x^3 - 2(x + 3)^2 + 12(x + 3) - 13.$

ISBN 97807043299

Factorising polynomials

Factorising polynomials

If $x - k$ is a factor of polynomial $P(x)$, then k is a factor of the constant term of the polynomial.

You already use this property to factorise quadratic trinomials of the form $ax^2 + bx + c$.

Proof

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$ where $a_n \neq 0$.

If $x - k$ is a factor of $P(x)$, then:

$P(x) = (x - k)Q(x)$ where $Q(x)$ has degree $n - 1$.

$$\begin{aligned}
P(x) &= (x - k)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_2x^2 + b_1x + b_0) \text{ where } b_{n-1} \neq 0 \\
&= xb_{n-1}x^{n-1} + xb_{n-2}x^{n-2} + \ldots + xb_1x + xb_0 - kb_{n-1}x^{n-1} - kb_{n-2}x^{n-2} - \ldots \\
&\quad - kb_2x^2 - kb_1x - kb_0 \\
&= b_{n-1}x^n + b_{n-2}x^{n-1} + \ldots + b_1x^2 + b_0x - kb_{n-1}x^{n-1} - kb_{n-2}x^{n-2} - \ldots - kb_2x^2 - kb_1x - kb_0 \\
&= b_{n-1}x^n + (b_{n-2} - kb_{n-1})x^{n-1} + \ldots + (b_1 - k)x^2 + (b_0 - k)x - kb_0
\end{aligned}$$

$\therefore a_0 = -kb_0$

So k is a factor of a_0.

ISBN 97807043299

EXAMPLE 6

Factorise each polynomial.

a $P(x) = x^3 + 3x^2 - 4x - 12$

b $P(x) = x^3 + 3x^2 + 5x + 15$

Solution

a Try factors of the constant term, −12 (that is, ±1, ±2, ±3, ±4, ±6, ±12).

Substitute these into $P(x)$ until you find one where $P(k) = 0$.

$P(1) = 1^3 + 3(1)^2 - 4(1) - 12 = -12 \neq 0$

$\therefore x - 1$ is not a factor of $P(x)$.

$P(2) = 2^3 + 3(2)^2 - 4(2) - 12 = 0$

$\therefore x - 2$ is a factor of $P(x)$.

Divide $P(x)$ by $x - 2$ to find other factors:

$$\begin{array}{r|l} & x^2 + 5x + 6 \\ x-2 & x^3 + 3x^2 - 4x - 12 \\ & \underline{x^3 - 2x^2} \\ & \quad 5x^2 - 4x \\ & \quad \underline{5x^2 - 10x} \\ & \qquad 6x - 12 \\ & \qquad \underline{6x - 12} \\ & \qquad\quad 0 \end{array}$$

$\therefore P(x) = (x - 2)(x^2 + 5x + 6)$

$= (x - 2)(x + 2)(x + 3)$

b Try factors of 15 (that is, ±1, ±3, ±5, ±15).

$P(-3) = (-3)^3 + 3(-3)^2 + 5(-3) + 15 = 0$

$\therefore x + 3$ is a factor of $f(x)$.

Divide $P(x)$ by $x + 3$ to find other factors:

$$\begin{array}{r|l} & x^2 \qquad + 5 \\ x+3 & x^3 + 3x^2 + 5x + 15 \\ & \underline{x^3 + 3x^2} \\ & \qquad 0 + 5x + 15 \\ & \qquad\quad \underline{5x + 15} \\ & \qquad\qquad 0 \end{array}$$

$\therefore P(x) = (x + 3)(x^2 + 5)$

ISBN 97807043299

EXT1 Exercise 6.02 Remainder and factor theorems

1 Use the remainder theorem to find the remainder in each division.

a $(x^3 - 2x^2 + x + 5) \div (x - 4)$

b $(x^2 + 5x + 3) \div (x + 2)$

c $(2x^3 - 4x - 1) \div (x + 3)$

d $(3x^5 + 2x^2 - x + 4) \div (x - 5)$

e $(5x^3 + 2x^2 + 2x - 9) \div (x - 1)$

f $(x^4 - x^3 + 3x^2 - x - 1) \div (x + 2)$

g $(2x^2 + 7x - 2) \div (x + 7)$

h $(x^7 + 5x^3 - 1) \div (x - 3)$

i $(2x^6 - 3x^2 + x + 4) \div (x + 5)$

j $(3x^4 - x^3 - x^2 - x - 7) \div (x + 1)$

2 Find the value of k if:

a the remainder is 3 when $5x^2 - 10x + k$ is divided by $x - 1$

b the remainder is -14 when $x^3 - (k - 1)x^2 + 5kx + 4$ is divided by $x + 2$

c the remainder is 0 when $2x^5 + 7x^2 + 1 + k$ is divided by $x + 6$

d $2x^4 - kx^3 + 3x^2 + x - 3$ is divisible by $x - 3$

e the remainder is 25 when $2x^4 - 3x^2 + 5$ is divided by $x - k$.

3 **a** Find the remainder when $f(x) = x^3 - 4x^2 + x + 6$ is divided by $x - 2$.

b Is $x - 2$ a factor of $f(x)$?

c Divide $x^3 - 4x^2 + x + 6$ by $x - 2$.

d Factorise $f(x)$ fully and write $f(x)$ as a product of its factors.

4 **a** Show that $x + 3$ is a factor of $P(x) = x^4 + 3x^3 - 9x^2 - 27x$.

b Divide $P(x)$ by $x + 3$ and write $P(x)$ as a product of its factors.

5 The remainder is 89 when $P(x) = ax^3 - 4bx^2 + x - 4$ is divided by $x - 3$, and the remainder is -3 when $P(x)$ is divided by $x + 1$. Find the values of a and b.

6 When $f(x) = ax^2 - 3x + 1$ and $g(x) = x^3 - 3x^2 + 2$ are divided by $x + 1$ they leave the same remainder. Find the value of a.

7 **a** Show that $x - 3$ is not a factor of $P(x) = x^5 - 2x^4 + 7x^2 - 3x + 5$.

b Find a value of k such that $x - 3$ is a factor of $Q(x) = 2x^3 - 5x + k$.

8 The polynomial $P(x) = x^3 + ax^2 + bx + 2$ has factors $x + 1$ and $x - 2$.

a Find the values of a and b.

b Write $P(x)$ as a product of its factors.

9 **a** The remainder when $f(x) = ax^4 + bx^3 + 15x^2 + 9x + 2$ is divided by $x - 2$ is 216, and $x + 1$ is a factor of $f(x)$. Find a and b.

b Divide $f(x)$ by $x + 1$ and write the polynomial in the form $f(x) = (x + 1)g(x)$.

c Show that $x + 1$ is a factor of $g(x)$.

d Write $f(x)$ as a product of its factors.

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10 Write each polynomial as a product of its factors.

a $P(x) = x^2 - 2x - 8$

b $P(x) = x^3 + x^2 - 2x$

c $f(x) = x^3 + x^2 - 10x + 8$

d $g(x) = x^3 + 4x^2 - 11x - 30$

e $G(x) = x^3 - 11x^2 + 31x - 21$

f $P(x) = x^3 - 12x^2 + 17x + 90$

g $Q(x) = x^3 - 7x^2 + 16x - 12$

h $R(x) = x^4 + 6x^3 + 9x^2 + 4x$

11 **a** Write $P(x) = x^3 - 7x + 6$ as a product of its factors.

b What are the zeros of $P(x)$?

c Is $(x - 2)(x + 3)$ a factor of $P(x)$?

12 If $f(x) = x^4 + 10x^3 + 23x^2 - 34x - 120$ has zeros -5 and 2:

a show that $(x + 5)(x - 2)$ is a factor of $f(x)$

b write $f(x)$ as a product of its linear factors.

13 If $P(x) = x^4 + 3x^3 - 13x^2 - 51x - 36$ has zeros -3 and 4, write $P(x)$ as a product of its linear factors.

14 **a** Show that $P(x) = x^3 - 3x^2 - 34x + 120$ has zeros -6 and 5.

b Write $P(x)$ as a product of its linear factors.

15 Evaluate a, b, c and d if:

a $x^2 + 4x - 3 \equiv a(x + 1)^2 + b(x + 1) + c$

b $2x^2 - 3x + 1 \equiv a(x + 2)^2 + b(x + 2) + c$

c $x^2 - x - 2 \equiv a(x - 1)^2 + b(x - 1) + c$

d $x^2 + x + 6 \equiv a(x - 3)^2 + b(x - 3) + c$

e $3x^2 - 5x - 2 \equiv a(x + 1)^2 + b(x - 1) + c$

f $x^3 + 3x^2 - 2x + 1 \equiv ax^3 + b(x - 1)^2 + cx + d$

The congruency symbol ≡ means 'is identical to' when applied to algebra.

16 A monic polynomial of degree 3 has zeros -3, 0 and 4. Find the polynomial.

17 Polynomial $P(x) = ax^3 - bx^2 + cx - 8$ has zeros 2 and -1, and $P(3) = 28$. Evaluate a, b and c.

18 A polynomial with leading term $2x^4$ has zeros -2, 0, 1 and 3. Find the polynomial.

19 Show that a polynomial of degree 2 cannot have 3 zeros.

20 Show that a polynomial of degree 3 cannot have 4 zeros.

EXT1 6.03 Polynomial equations

$P(x)$ is a **polynomial** while $P(x) = 0$ is a **polynomial equation**.

The solutions to $P(x) = 0$ are called the **roots** of the equation or the **zeros** of the polynomial $P(x)$.

EXAMPLE 7

a Find all zeros of $P(x) = x^3 - 7x + 6$.

b Find the roots of $x^4 + 4x^3 - 7x^2 - 10x = 0$.

Solution

a Factorise $P(x)$ by trying factors of the constant term, 6 (that is, ±1, ±2, ±3, ±6).

$P(1) = 1^3 - 7(1) + 6 = 0$

So $x - 1$ is a factor of $P(x)$.

$$\begin{array}{r|l} & x^2 + x - 6 \\ x-1 & x^3 \quad - 7x + 6 \\ & \underline{x^3 - x^2} \\ & x^2 - 7x \\ & \underline{x^2 - x + 6} \\ & -6x + 6 \\ & \underline{-6x + 6} \\ & 0 \end{array}$$

$P(x) = (x - 1)(x^2 + x - 6)$

$= (x - 1)(x + 3)(x - 2)$

For zeros, $P(x) = 0$:

$(x - 1)(x + 3)(x - 2) = 0$

$x = 1, -3, 2$

b Factorising: $x^4 + 4x^3 - 7x^2 - 10x = x(x^3 + 4x^2 - 7x - 10)$

To factorise $x^3 + 4x^2 - 7x - 10$, try factors of -10:

$P(1) = 1^3 + 4(1)^2 - 7(1) - 10 = -12 \neq 0$

$P(2) = 2^3 + 4(2)^2 - 7(2) - 10 = 0$

So $x - 2$ is a factor of $P(x)$.

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$$
\begin{array}{r}
x^2 + \ 6x + 5 \\
x-2 \overline{\smash{)}\, x^3 + 4x^2 - \ 7x - 10} \\
\underline{x^3 - 2x^2} \qquad\qquad \\
6x^2 - \ 7x \qquad \\
\underline{6x^2 - 12x} \qquad \\
5x - 10 \\
\underline{5x - 10} \\
0
\end{array}
$$

So $x^4 + 4x^3 - 7x^2 - 10x = x(x - 2)(x^2 + 6x + 5)$

$= x(x - 2)(x + 5)(x + 1)$

Solving $x^4 + 4x^3 - 7x^2 - 10x = 0$

Roots are $x = 0, 2, -5, -1$.

EXT1 Exercise 6.03 Polynomial equations

1 Find all the zeros of each polynomial.

- **a** $P(x) = x^3 - 4x^2 + x + 6$
- **b** $R(x) = x^3 - 3x^2 - x + 3$
- **c** $P(x) = x^3 - 3x^2 - 6x + 8$
- **d** $f(x) = x^3 + x^2 - 16x + 20$
- **e** $P(x) = x^3 - 11x^2 + 23x + 35$
- **f** $P(x) = x^3 + 7x^2 - 17x + 9$
- **g** $f(x) = x^4 - 7x^2 - 6x$
- **h** $Q(x) = x^4 - x^3 - 7x^2 + x + 6$
- **i** $f(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$
- **j** $P(x) = x^4 + 3x^3 - 15x^2 - 19x + 30$

2 Find the roots of each polynomial equation.

- **a** $x^3 + x^2 - 5x + 3 = 0$
- **b** $x^3 - 3x^2 - x + 3 = 0$
- **c** $x^3 - 9x^2 + 26x - 24 = 0$
- **d** $x^3 - 2x^2 - 13x - 10 = 0$
- **e** $x^3 - 10x^2 + 23x - 14 = 0$
- **f** $x^3 - 13x - 12 = 0$
- **g** $x^4 - 9x^3 + 11x^2 + 21x = 0$
- **h** $x^4 + x^3 - 16x^2 - 4x + 48 = 0$
- **i** $x^4 - 5x^2 + 4 = 0$
- **j** $x^4 - x^3 - 13x^2 + x + 12 = 0$

3 Solve:

- **a** $2x^3 - 3x^2 - 3x + 2 = 0$
- **b** $2x^3 - 3x^2 - 2x + 3 = 0$
- **c** $5x^3 - 4x^2 - 11x - 2 = 0$
- **d** $4x^3 - 25x^2 + 49x - 30 = 0$
- **e** $6x^3 - 13x^2 + 9x - 2 = 0$

4 Find the zeros of $P(x) = x^4 - 6x^3 - 19x^2 + 84x + 180$.

5 Find the roots of $2x^4 - 5x^3 + 5x - 2 = 0$.

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WS

Roots and coefficients

EXT1 6.04 Roots and coefficients of polynomial equations

Quadratic equations

If a quadratic equation $ax^2 + bx + c = 0$ has roots α and β, then the equation can be written as:

$$(x - \alpha)(x - \beta) = 0$$
$$x^2 - \beta x - \alpha x + \alpha\beta = 0$$
$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

But $ax^2 + bx + c = 0$ can be written in monic form as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

$$\therefore\ x^2 - (\alpha + \beta)x + \alpha\beta \equiv x^2 + \frac{b}{a}x + \frac{c}{a}$$

$$\therefore\ -(\alpha + \beta) = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

giving us formulas for the sum and product of the roots in terms of the coefficients a, b and c:

Sum and product of the roots of a quadratic equation

For the quadratic equation $ax^2 + bx + c = 0$:

Sum of roots:

$$\alpha + \beta = -\frac{b}{a}$$

Product of roots:

$$\alpha\beta = \frac{c}{a}$$

EXAMPLE 8

a Find the quadratic equation that has roots $3 + \sqrt{2}$ and $3 - \sqrt{2}$.

b If α and β are the roots of $2x^2 - 6x + 1 = 0$, find:

- **i** $\alpha + \beta$
- **ii** $\alpha\beta$
- **iii** $\alpha^2 + \beta^2$

c Find the value of k if one root of $kx^2 - 7x + k + 1 = 0$ is -2.

d Evaluate p if one root of $x^2 + 2x - 5p = 0$ is double the other root.

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Solution

a $\alpha + \beta = 3 + \sqrt{2} + 3 - \sqrt{2}$

$= 6$

$\alpha\beta = (3 + \sqrt{2}) \times (3 - \sqrt{2})$

$= 3^2 - \left(\sqrt{2}\right)^2$

$= 9 - 2$

$= 7$

Substituting into $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ gives $x^2 - 6x + 7 = 0$.

b **i** $\alpha + \beta = -\frac{b}{a}$

$= -\frac{(-6)}{2}$

$= 3$

ii $\alpha\beta = \frac{c}{a}$

$= \frac{1}{2}$

iii Use $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

So $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (3)^2 - 2\left(\frac{1}{2}\right)$ from **i** and **ii**

$= 9 - 1$

$= 8$

c If -2 is a root of the equation then $x = -2$ satisfies the equation.

$$k(-2)^2 - 7(-2) + k + 1 = 0$$

$$4k + 14 + k + 1 = 0$$

$$5k + 15 = 0$$

$$5k = -15$$

$$k = -3$$

d If one root is α then the other root is 2α.

Sum of roots:

$$\alpha + \beta = -\frac{b}{a}$$

$$\alpha + 2\alpha = -\frac{2}{1}$$

$$3\alpha = -2$$

$$\alpha = -\frac{2}{3}$$

Product of roots:

$$\alpha\beta = \frac{c}{a}$$

$$\alpha \times 2\alpha = \frac{-5p}{1}$$

$$2\alpha^2 = -5p$$

Substituting $\alpha = -\frac{2}{3}$:

$$2\left(-\frac{2}{3}\right)^2 = -5p$$

$$2\left(\frac{4}{9}\right) = -5p$$

$$\frac{8}{9} = -5p$$

$$p = -\frac{8}{45}$$

Cubic equations

If a cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ then:

$$(x - \alpha)(x - \beta)(x - \gamma) = 0$$

$$(x^2 - \beta x - \alpha x + \alpha\beta)(x - \gamma) = 0$$

$$x^3 - \gamma x^2 - \beta x^2 + \beta\gamma x - \alpha x^2 + \alpha\gamma x + \alpha\beta x - \alpha\beta\gamma = 0$$

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$$

The cubic equation $ax^3 + bx^2 + cx + d = 0$ can be written in monic form as $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$.

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma \equiv x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a}$$

Equating coefficients gives the formulas below.

Sum and product of the roots of a cubic equation

For the cubic equation $ax^3 + bx^2 + cx + d = 0$:

Sum of roots 1 at a time:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

Sum of roots 2 at a time:

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

Do you notice a pattern in these formulas?

Product of roots:

$$\alpha\beta\gamma = -\frac{d}{a}$$

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EXAMPLE 9

a If α, β, γ are the roots of $2x^3 - 5x^2 + x - 1 = 0$, find:

i $(\alpha + \beta + \gamma)^2$ **ii** $(\alpha + 1)(\beta + 1)(\gamma + 1)$ **iii** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

b If one root of $x^3 - x^2 + 2x - 3 = 0$ is 4, find the sum and product of the other two roots.

c Solve $12x^3 + 32x^2 + 15x - 9 = 0$ given that 2 roots are equal.

Solution

a **i**

$$\alpha + \beta + \gamma = -\frac{b}{a}$$
$$= -\frac{(-5)}{2}$$
$$= \frac{5}{2}$$

$$(\alpha + \beta + \gamma)^2 = \left(\frac{5}{2}\right)^2$$
$$= 6\frac{1}{4}$$

ii

$$(\alpha + 1)(\beta + 1)(\gamma + 1)$$
$$= (\alpha + 1)(\beta\gamma + \beta + \gamma + 1)$$
$$= \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \alpha + \beta\gamma + \beta + \gamma + 1$$
$$= \alpha\beta\gamma + (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) + 1$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$
$$= \frac{1}{2}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$
$$= -\frac{(-1)}{2}$$
$$= \frac{1}{2}$$

$$\therefore (\alpha + 1)(\beta + 1)(\gamma + 1) = \frac{1}{2} + \frac{1}{2} + \frac{5}{2} + 1$$
$$= 4\frac{1}{2}$$

iii

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$$
$$= \frac{\frac{1}{2}}{\frac{1}{2}}$$
$$= 1$$

b Roots are α, β, γ where, say, γ = 4.

$$\alpha + \beta + \gamma = -\frac{b}{a}$$
$$\therefore \alpha + \beta + 4 = 1$$
$$\alpha + \beta = -3$$

$$\alpha\beta\gamma = -\frac{d}{a}$$
$$\alpha\beta(4) = 3$$
$$\therefore \alpha\beta = \frac{3}{4}$$

c Let the roots be α, α and β.

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha + \alpha + \beta = -\frac{32}{12}$$

$$\therefore 2\alpha + \beta = -\frac{8}{3} \qquad [1]$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\alpha + \alpha\beta + \alpha\beta = \frac{15}{12}$$

$$\therefore \alpha^2 + 2\alpha\beta = \frac{5}{4} \qquad [2]$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

$$\alpha\alpha\beta = -\frac{-9}{12}$$

$$\therefore \alpha^2\beta = \frac{3}{4} \qquad [3]$$

From [1]:

$$\beta = -\frac{8}{3} - 2\alpha \qquad [4]$$

Substitute in [2]:

$$\alpha^2 + 2\alpha\left(-\frac{8}{3} - 2\alpha\right) = \frac{5}{4}$$

$$12\alpha^2 + 24\alpha\left(-\frac{8}{3} - 2\alpha\right) = 15$$

$$12\alpha^2 - 64\alpha - 48\alpha^2 = 15$$

$$36\alpha^2 + 64\alpha + 15 = 0$$

$$(2\alpha + 3)(18\alpha + 5) = 0$$

$$2\alpha = -3 \qquad 18\alpha = -5$$

$$\alpha = -1\tfrac{1}{2} \qquad \alpha = -\frac{5}{18}$$

To find β, substitute each value in [4].

$\alpha = -1\frac{1}{2}$:

$$\beta = -\frac{8}{3} - 2\left(-1\tfrac{1}{2}\right)$$

$$= \frac{1}{3}$$

$\alpha = -\frac{5}{18}$:

$$\beta = -\frac{8}{3} - 2\left(-\frac{5}{18}\right)$$

$$= -2\tfrac{1}{9}$$

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Only one of these values for α can be correct. Test by substituting each in the LHS of [3]:

$\alpha = -1\frac{1}{2}, \beta = \frac{1}{3}$:

$$\left(-1\frac{1}{2}\right)^2\left(\frac{1}{3}\right) = \frac{3}{4}$$

$= \text{RHS}$

$\alpha = -\frac{5}{18}, \beta = -2\frac{1}{9}$:

$$\left(-\frac{5}{18}\right)^2\left(-2\frac{1}{9}\right) = -\frac{475}{2916}$$

$\neq \text{RHS}$

$\therefore$ The roots are $-1\frac{1}{2}$ and $\frac{1}{3}$.

Quartic equations

If a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots α, β, γ and δ then:

$$(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) = 0$$

When expanded fully, this is:

$$x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta = 0$$

The quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ can be written in monic form as:

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0.$$

$$\therefore x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta$$

$$\equiv x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a}$$

Equating coefficients gives the formulas below:

Sum and product of the roots of a quartic equation

For the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$:

Sum of roots: $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$

Sum of roots 2 at a time: $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$

Sum of roots 3 at a time: $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$

Product of roots: $\alpha\beta\gamma\delta = \frac{e}{a}$

Do you notice a pattern in these formulas?

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INVESTIGATION

HIGHER DEGREE POLYNOMIALS

This pattern of roots and coefficients extends to polynomials of any degree.

Can you find results for sums and products of roots for polynomial equations of degree 5, 6 and so on?

EXAMPLE 10

If α, β, γ and δ are the roots of $x^4 - 2x^3 + 7x - 3 = 0$, find:

a $\alpha\beta\gamma\delta$ **b** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **c** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

Solution

a
$$\begin{aligned}\alpha\beta\gamma\delta &= \frac{e}{a} \\ &= \frac{-3}{1} \\ &= -3\end{aligned}$$

b
$$\begin{aligned}\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= -\frac{d}{a} \\ &= -\frac{7}{1} \\ &= -7\end{aligned}$$

c
$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} &= \frac{\beta\gamma\delta}{\alpha\beta\gamma\delta} + \frac{\alpha\gamma\delta}{\alpha\beta\gamma\delta} + \frac{\alpha\beta\delta}{\alpha\beta\gamma\delta} + \frac{\alpha\beta\gamma}{\alpha\beta\gamma\delta} \\ &= \frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta} \\ &= \frac{-7}{-3} \\ &= 2\tfrac{1}{3}\end{aligned}$$

EXT1 Exercise 6.04 Roots and coefficients of polynomial equations

1 Given that α and β are the roots of the equation, for each quadratic equation find:

i $\alpha + \beta$ **ii** $\alpha\beta$

a $x^2 - 2x + 8 = 0$ **b** $3x^2 + 6x - 2 = 0$ **c** $x^2 + 7x + 1 = 0$

d $4x^2 - 9x - 12 = 0$ **e** $5x^2 + 15x = 0$

2 Where α, β, and γ are the roots of the equation, for each cubic equation find:

i $\alpha + \beta + \gamma$ **ii** $\alpha\beta + \alpha\gamma + \beta\gamma$ **iii** $\alpha\beta\gamma$

a $x^3 + x^2 - 2x + 8 = 0$ **b** $x^3 - 3x^2 + 5x - 2 = 0$ **c** $2x^3 - x^2 + 6x + 2 = 0$

d $-x^3 - 3x^2 - 11 = 0$ **e** $x^3 + 7x - 3 = 0$

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3 For each quartic equation, where α, β, γ and δ are the roots of the equation, find:

i $\alpha + \beta + \gamma + \delta$ **ii** $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

iii $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **v** $\alpha\beta\gamma\delta$

a $x^4 + 2x^3 - x^2 - x + 5 = 0$ **b** $x^4 - x^3 - 3x^2 + 2x - 7 = 0$

c $-x^4 + x^3 + 3x^2 - 2x + 4 = 0$ **d** $2x^4 - 2x^3 - 4x^2 + 3x - 2 = 0$

e $2x^4 - 12x^3 + 7 = 0$

4 If α and β are the roots of $x^2 - 5x - 5 = 0$, find:

a $\alpha + \beta$ **b** $\alpha\beta$ **c** $\frac{1}{\alpha} + \frac{1}{\beta}$ **d** $\alpha^2 + \beta^2$

5 If α, β and γ are the roots of $2x^3 + 5x^2 - x - 3 = 0$, find:

a $\alpha\beta\gamma$ **b** $\alpha\beta + \alpha\gamma + \beta\gamma$ **c** $\alpha + \beta + \gamma$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **e** $(\alpha + 1)(\beta + 1)(\gamma + 1)$

6 If α, β, γ and δ are the roots of $x^4 - 2x^3 + 5x - 3 = 0$, find:

a $\alpha\beta\gamma\delta$ **b** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **c** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

7 One root of $x^2 - 3x + k - 2 = 0$ is -4. Find the value of k.

8 One root of $x^3 - 5x^2 - x + 21 = 0$ is 3. Find the sum $\alpha + \beta$ and the product $\alpha\beta$ of the other 2 roots.

9 Given $P(x) = 2x^3 - 7x^2 + 4x + 1$, if the equation $P(x) = 0$ has zero $x = 1$, find the sum and product of its other roots.

10 Find the value(s) of k if the quadratic equation $x^2 - (k + 2)x + k + 1 = 0$ has:

a equal roots **b** one root equal to 5

c consecutive roots **d** one root double the other

e reciprocal roots.

11 Two roots of $x^3 + ax^2 + bx + 24 = 0$ are equal to 4 and -2. Find the values of a and b.

12 **a** Show that 1 is a zero of the polynomial $P(x) = x^4 - 2x^3 + 7x - 6$.

b If α, β and γ are the other 3 zeros, find the value of $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$.

13 If $x = 2$ is a double root of $ax^4 - 2x^3 - 8x + 16 = 0$, find the value of a and the sum of the other 2 roots.

14 Two of the roots of $x^3 - px^2 - qx + 30 = 0$ are 3 and 5.

a Find the other root. **b** Find p and q.

15 The product of two of the roots of $x^4 + 2x^3 - 18x - 5 = 0$ is -5 Find the product of the other 2 roots.

16 The sum of 2 of the roots of $x^4 + x^3 + 7x^2 + 14x - 1 = 0$ is 4. Find the sum of the other 2 roots.

17 Find the roots of $x^3 - 3x^2 + 4 = 0$ given that 2 of the roots are equal.

18 Solve $12x^3 - 4x^2 - 3x + 1 = 0$ if the sum of 2 of its roots is 0.

19 Solve $6x^4 + 5x^3 - 24x^2 - 15x + 18 = 0$ if the sum of 2 of its roots is zero.

20 Two roots of $x^3 + mx^2 - 3x - 18 = 0$ are equal and rational. Find m.

EXT1 6.05 Graphing polynomial functions

WS Polynomial graphs

Graphing polynomials

WS Sketching curves

To graph polynomial functions, factorise polynomials to find their zeros first.

EXAMPLE 11

a Factorise the polynomial $P(x) = x^3 - x^2 - 5x - 3$.

b Sketch the graph of the polynomial.

Solution

a Factors of −3 are ±1 and ±3.

$P(-1) = (-1)^3 - (-1)^2 - 5(-1) - 3$

$= 0$

So $x + 1$ is a factor of the polynomial.

By long division,

$$\begin{array}{r} x^2 - 2x - 3 \\ x+1 \overline{)\, x^3 - x^2 - 5x - 3} \\ \underline{x^3 + x^2} \qquad\qquad \\ -2x^2 - 5x \qquad \\ \underline{-2x^2 - 2x} \qquad \\ -3x - 3 \\ \underline{-3x - 3} \\ 0 \end{array}$$

$P(x) = (x + 1)(x^2 - 2x - 3)$

$= (x + 1)(x - 3)(x + 1)$

$= (x + 1)^2(x - 3)$

b For the graph of $P(x) = x^3 - x^2 - 5x - 3$,

For x-intercepts, $P(x) = 0$:

$0 = x^3 - x^2 - 5x - 3$

$= (x + 1)^2(x - 3)$

$x = -1, 3$

So x-intercepts are −1 and 3.

For y-intercept, $x = 0$:

$P(0) = 0^3 - (0)^2 - 5(0) - 3$

$= -3$

So y-intercept is −3.

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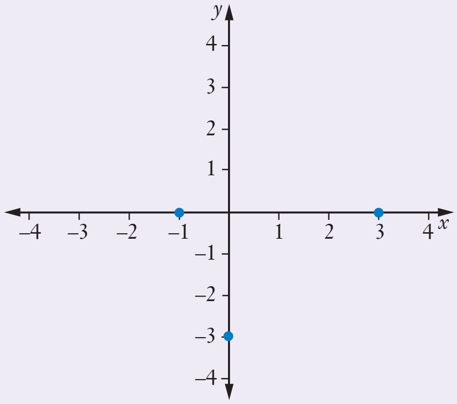


Test $x < -1$, say $x = -2$:

$P(-2) = (-2 + 1)^2(-2 - 3)$

$= (-1)^2(-5)$

$= -5$

< 0

So the curve is below the x-axis for $x < -1$.

Test $-1 < x < 3$, say $x = 0$:

$P(0) = (0 + 1)^2(0 - 3)$

$= (1)^2(-3)$

$= -3$

< 0

So the curve is below the x-axis for $-1 < x < 3$.

Test $x > 3$, say $x = 4$:

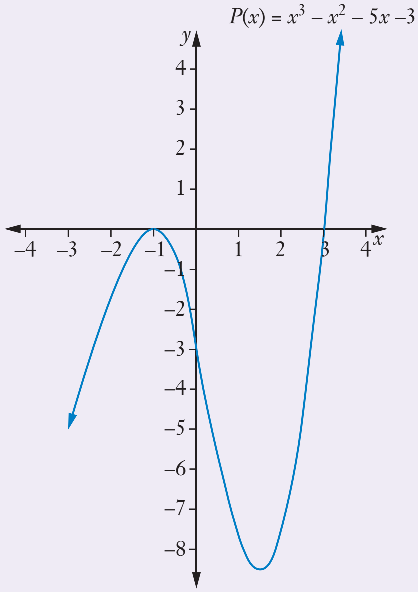
$P(4) = (4 + 1)^2(4 - 3)$

$= (5)^2(1)$

$= 25$

> 0

So the curve is above the x-axis for $x > 3$.



Limiting behaviour of polynomials

What does the graph of a polynomial look like for large positive and negative values of x as $x \to \pm\infty$?

INVESTIGATION

LIMITING BEHAVIOUR OF POLYNOMIALS

Use a graphics calculator or graphing software to explore the behaviour of polynomials as x becomes large (both negative and positive values).

For example, sketch $f(x) = 2x^5 + 3x^2 - 7x - 1$ and $f(x) = 2x^5$ together. What do you notice at both ends of the graphs where x is large? Zoom out on these graphs and watch the graph of the polynomial and the graph of the leading term come together.

Try sketching the graphs of other polynomials along with graphs of their leading terms. Do you find the same results?

The leading term, $a_n x^n$ of a polynomial function shows us what the limiting behaviour of the function will be.

For very large $|x|$, $P(x) \approx a_n x^n$.

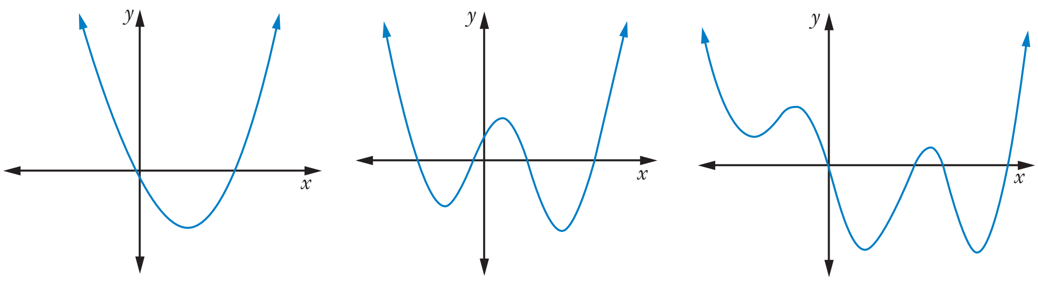
As x becomes large, the leading term $a_n x^n$ becomes very large compared with the other terms because it has the highest power of x and the other powers of x are relatively small.

Consider a polynomial of even degree; for example, a polynomial whose leading term $a_n x^n$ is $3x^2$ or x^4 or $-5x^6$.

If n is even, x^n is always positive.

So if n is even and $a_n > 0$, then $a_n x^n > 0$.

As $x \to \pm\infty$, $P(x) \to \infty$, as shown by these 3 graphs of polynomials.



Degree 2 (quadratic) Degree 4 (quartic) Degree 6

If n is even and $a_n < 0$, then $a_n x^n < 0$.

As $x \to \pm\infty$, $P(x) \to -\infty$, as shown by the 3 graphs of polynomials on the next page.

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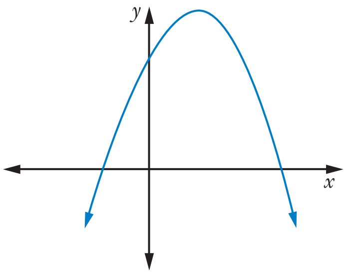
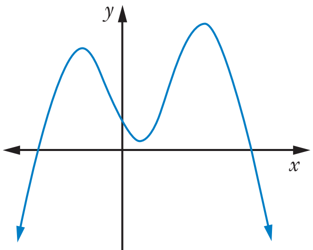
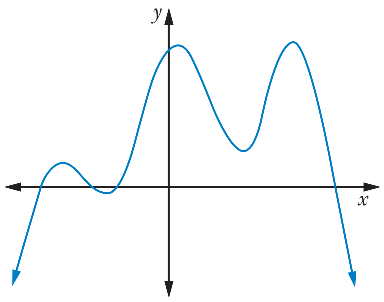


Degree 2 (quadratic)



Degree 4 (quartic)



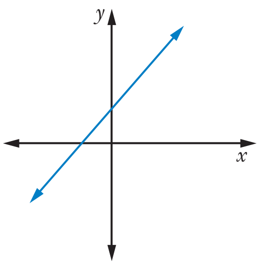
Degree 6

Now consider a polynomial of odd degree; for example, a polynomial whose leading term $a_n x^n$ is x^3 or $-4x^5$ or $2x^7$.

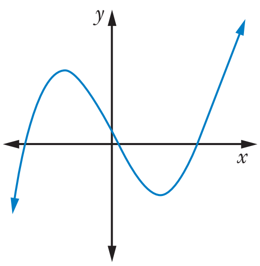
If n is odd, $x^n > 0$ for $x > 0$ and $x^n < 0$ for $x < 0$.

So if n is odd and a_n is positive, then $a_n x^n > 0$ for $x > 0$ and $a_n x^n < 0$ for $x < 0$.

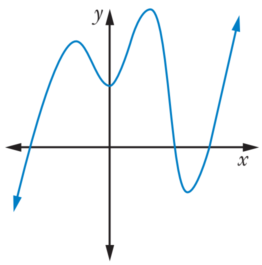
As $x \to -\infty$, $P(x) \to -\infty$ and as $x \to \infty$, $P(x) \to \infty$, as shown by these 3 graphs of polynomials.



Degree 1 (linear)



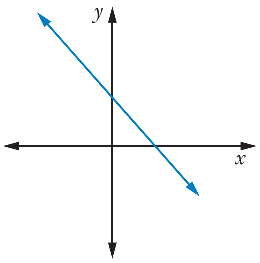
Degree 3 (cubic)



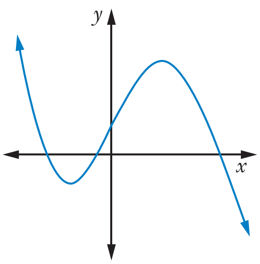
Degree 5

If n is odd and a_n is negative, then $a_n x^n < 0$ for $x > 0$ and $a_n x^n > 0$ for $x < 0$.

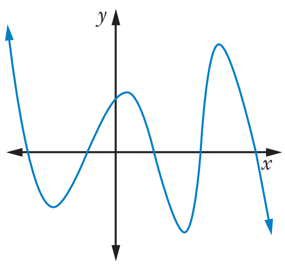
As $x \to -\infty$, $P(x) \to \infty$ and as $x \to \infty$, $P(x) \to -\infty$, as shown by these 3 graphs of polynomials.



Degree 1 (linear)



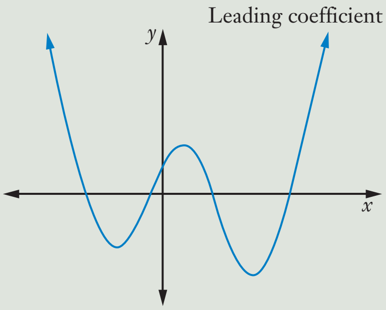
Degree 3 (cubic)

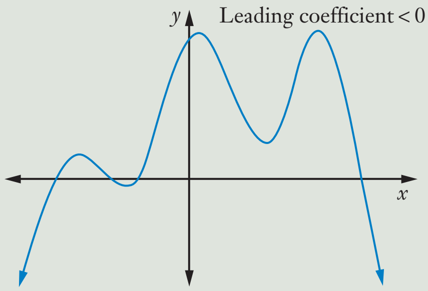


Degree 5

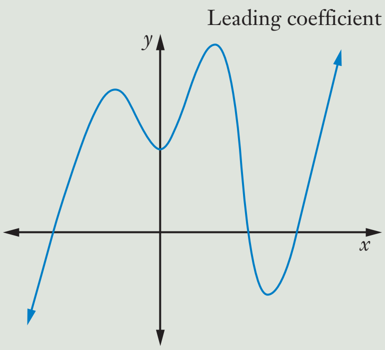
The graph of a polynomial

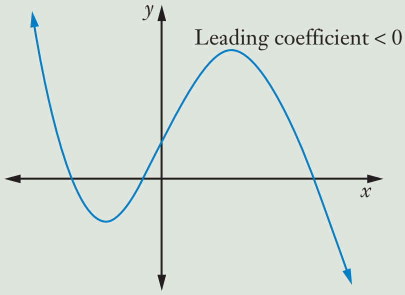
If $P(x)$ has **even degree**, the ends of the graph both point in the same direction.





If $P(x)$ has **odd degree**, the ends of the graph point in opposite directions.





EXT1 Exercise 6.05 Graphing polynomial functions

1 **a** Show that $x - 2$ is a factor of $P(x) = x^3 - 3x^2 - 4x + 12$.

b Write $P(x)$ as a product of its factors.

c Sketch the graph of the polynomial.

2 Sketch the graph of each polynomial, showing all x- and y-intercepts.

a $P(x) = x^3 + 3x^2 - 10x - 24$

b $P(x) = x^3 + x^2 - 9x - 9$

c $P(x) = 12 - 19x + 8x^2 - x^3$

d $P(x) = x^3 - 13x + 12$

e $P(x) = -x^3 + 2x^2 + 9x - 18$

f $P(x) = x^3 + 2x^2 - 4x - 8$

g $P(x) = x^3 - 5x^2 + 8x - 4$

h $P(x) = x^3 + x^2 - 5x + 3$

i $P(x) = 16x + 12x^2 - x^4$

j $P(x) = x^4 - 2x^2 + 1$

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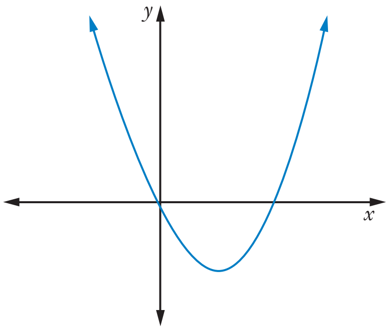
3 For each graph, state if:

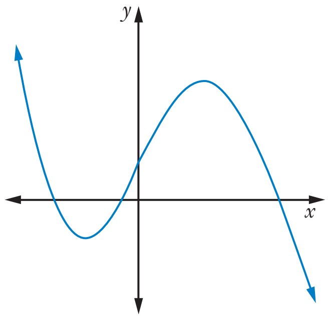
i the leading coefficient is positive or negative

ii the degree of the polynomial is even or odd.

a



b



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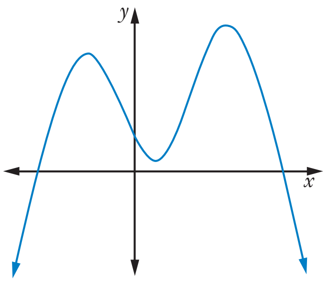

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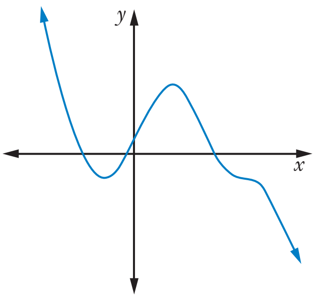

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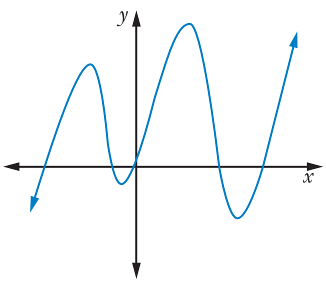

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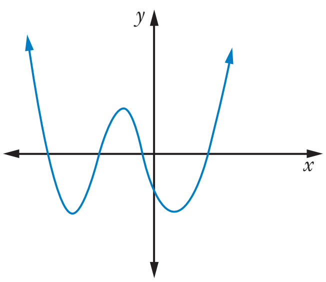

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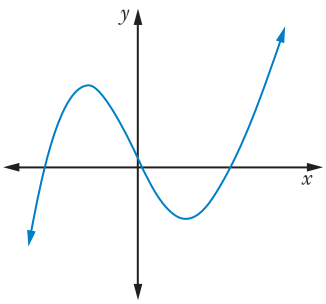

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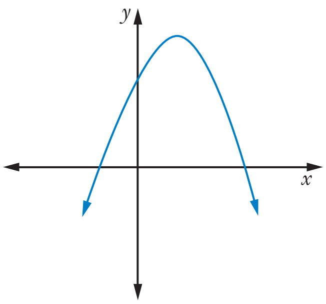
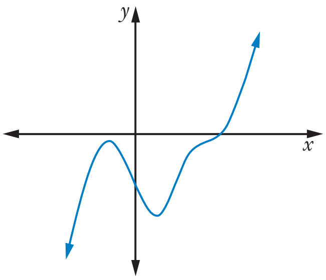


i



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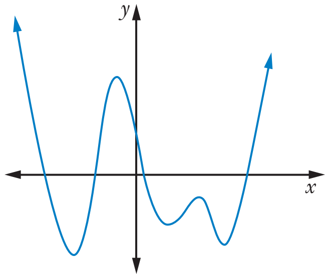


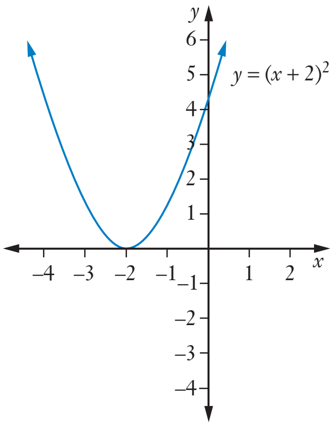
4 Draw an example of a polynomial with leading term:

a x^3 **b** $-2x^5$ **c** $3x^2$

d $-x^4$ **e** $-2x^3$

EXT1 6.06 Multiple roots

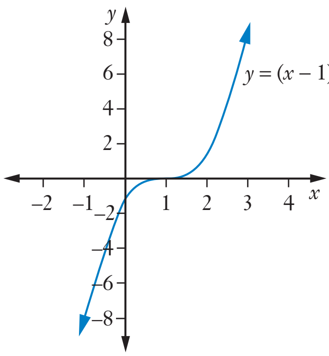
If $f(x) = (x + 2)^2$, we say that the quadratic equation $f(x) = 0$ has a double root at $x = -2$ since there are 2 equal roots.



Double root at $x = -2$
Turning point at $x = -2$

Similarly, if $f(x) = (x - 1)^3$, the cubic equation $f(x) = 0$ has a triple root at $x = 1$.

Notice that there is always a turning point or point of inflection where there is a multiple root.



Triple root at $x = 1$
Point of inflection at $x = 1$

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INVESTIGATION

MULTIPLE ROOTS

Use a graphics calculator or graphing software to graph polynomials with multiple roots.

a Examine values close to the roots.

b Look at the relationship between the degree of the polynomial, the leading coefficient and its graph.

1 $P(x) = (x + 1)(x - 3)$

2 $P(x) = (x + 1)^2(x - 3)$

3 $P(x) = -(x + 1)^3(x - 3)$

4 $P(x) = -(x + 1)^4(x - 3)$

5 $P(x) = (x + 1)(x - 3)^2$

6 $P(x) = (x + 1)(x - 3)^3$

7 $P(x) = -(x + 1)(x - 3)^4$

8 $P(x) = -(x + 1)^2(x - 3)^2$

9 $P(x) = -(x + 1)^2(x - 3)^3$

10 $P(x) = (x + 1)^3(x - 3)^2$

Multiple roots

If $P(x) = (x - k)^2Q(x)$ then $P(x) = 0$ has a **double root** at $x = k$ (2 equal roots)

If $P(x) = (x - k)^3Q(x)$ then $P(x) = 0$ has a **triple root** at $x = k$ (3 equal roots)

If $P(x) = (x - k)^rQ(x)$ then $P(x) = 0$ has a **multiple root** at $x = k$ (r equal roots).

We can also say that $P(x)$ has a root with **multiplicity** r at $x = k$.

EXAMPLE 12

a Examine the behaviour of the polynomial $P(x) = (x + 2)^2(x - 1)$ close to its multiple root and describe how this affects its graph at this root.

b Describe the limiting behaviour of the polynomial.

c Sketch the graph of the polynomial.

Solution

a $P(x) = (x + 2)^2(x - 1)$ has a double root at $x = -2$.

Look at the sign of $P(x)$ close to $x = -2$:

On LHS:

$$P(-2.1) = (-2.1 + 2)^2(-2.1 - 1) = -0.031 < 0$$

So the curve is below the x-axis on the LHS.

On RHS:

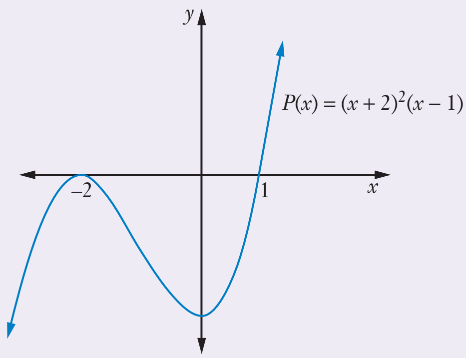
$$P(-1.9) = (-1.9 + 2)^2(-1.9 - 1) = -0.029 < 0$$

So the curve is also below the x-axis on the RHS.

As $P(x)$ is negative on both sides of the double root at $x = -2$, its graph is below the x-axis. But $x = -2$ is an x-intercept, so there is a maximum turning point at that point.

b $P(x) = (x + 2)^2(x - 1)$ has the leading term x^3 so it has an odd degree (3) and a positive leading coefficient (1). As $x \to \infty$, $P(x) \to \infty$. Since $P(x)$ is odd, as $x \to -\infty$, $P(x) \to -\infty$.

c



Turning points at multiple roots on polynomial graphs

If the multiplicity r of a root is even, there is a maximum or minimum turning point at the multiple root.

If the multiplicity r of a root is odd, there is a point of inflection at the multiple root.

EXAMPLE 13

Sketch the graph of $P(x) = -x(x - 3)^3$.

Solution

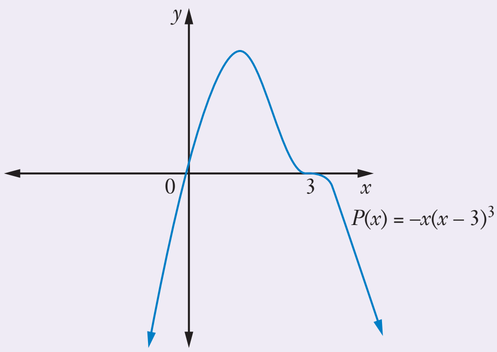
$P(x) = -x(x - 3)^3 = 0$ has roots at $x = 0$, $x = 3$.

$x = 0$ is a single root so the curve crosses the x-axis at this point.

$x = 3$ is a triple root. Since r is odd, there is a point of inflection at $x = 3$.

$P(x) = -x(x - 3)^3$ has leading term $-x^4$, so $P(x)$ has an even degree and a negative leading coefficient.

As $x \to \pm\infty$, $P(x) \to -\infty$.



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EXT1 Exercise 6.06 Multiple roots

1 Find the roots of each polynomial equation $P(x) = 0$ and state if they are multiple roots.

a $P(x) = x^2 - 6x + 9$

b $P(x) = x^3 - 9x^2 + 14x$

c $P(x) = x^3 - 3x^2$

d $f(x) = x^3 - 2x^2 - 4x + 8$

e $P(x) = x^3 - 6x^2 + 12x - 8$

f $A(x) = x^4 - 4x^3 + 5x^2 - 2x$

g $P(x) = x^4 - 4x^3 - 2x^2 + 12x + 9$

h $Q(x) = x^5 - 8x^4 + 16x^3$

i $P(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

j $f(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$

2 A monic polynomial of degree 2 has a double root at $x = -4$. Write down an expression for the polynomial $P(x)$. Is this a unique expression?

3 A polynomial of degree 3 has a triple root at $x = 1$.

a Write down an expression for the polynomial. Is this unique?

b If $P(2) = 5$, write the expression for the polynomial.

4 Sketch the graph of a polynomial with a double root at $x = 2$ and leading term $2x^3$.

5 Sketch the graph of a polynomial with a double root at $x = -1$ and leading term $-x^3$.

6 Sketch the graph of a polynomial with a double root at $x = 2$ and a leading term x^4.

7 Sketch the graph of a polynomial with a double root at $x = -3$ and leading term x^6.

8 A polynomial has a triple root at $x = 1$ and leading term x^3. Sketch a graph showing this information.

9 Given a polynomial with a triple root at $x = 0$ and leading term $-x^4$, sketch the graph of a polynomial that fits this information.

10 If a polynomial has a triple root at $x = -2$ and a leading term of x^8, sketch the graph of a polynomial fitting this information.

11 A polynomial has a triple root at $x = 4$ and its leading term is $-4x^3$. Sketch its graph.

12 A monic polynomial has degree 3 and a double root at $x = -1$. Show on a sketch that the polynomial has another root.

13 A polynomial with leading term $-x^8$ has a triple root at $x = -2$. Show by a sketch that the polynomial has at least one other root.

14 A polynomial has a double root at $x = 2$ and a double root at $x = -3$. Its leading term is $2x^5$. By sketching a graph, show that the polynomial has another root.

EXT1 6.07 The inverse of a function

The inverse of a function is an operation that 'undoes' the original function.
For example:

- The inverse relation of $y = 2x$ is $y = \frac{x}{2}$.
- The inverse relation of $y = \sqrt{x}$ is $y = x^2$.

EXAMPLE 14

Change the subject of each function to x, and then find the inverse relation of the function.

a $y = 2x + 1$ **b** $y = x^3 - 2$

Solution

a Make x the subject of the function:

$$y = 2x + 1$$
$$y - 1 = 2x$$
$$\frac{y-1}{2} = x$$

The inverse operations of 'multiplying by 2 then adding 1' are 'subtracting 1 then dividing by 2'.

So the inverse relation of $y = 2x + 1$ is $y = \frac{x-1}{2}$.

b
$$y = x^3 - 2$$
$$y + 2 = x^3$$
$$\sqrt[3]{y+2} = x$$

The inverse operations of 'cubing then subtracting 2' are 'adding 2 then finding the cube root'.

So the inverse relation of $y = x^3 - 2$ is $y = \sqrt[3]{x+2}$.

Notice in the example that for the inverse relation, we can swap x and y.

Finding the inverse relation of a function

The inverse relation of $y = f(x)$ can be found by interchanging the x and y of the function, then making y the subject.

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EXAMPLE 15

Find the inverse relation of:

a $y = 3x - 8$ **b** $f(x) = 2x^5 + 7$ **c** $y = x^2 + 4x - 7$

Solution

a

$$x = 3y - 8$$
$$x + 8 = 3y$$
$$\frac{x+8}{3} = y$$

b

$$x = 2y^5 + 7$$
$$x - 7 = 2y^5$$
$$\frac{x-7}{2} = y^5$$
$$\sqrt[5]{\frac{x-7}{2}} = y$$

c

$$x = y^2 + 4y - 7$$
$$x + 7 = y^2 + 4y$$
$$x + 7 + 4 = y^2 + 4y + 4$$
$$x + 11 = (y + 2)^2$$
$$\pm\sqrt{x+11} = y + 2$$
$$\pm\sqrt{x+11} - 2 = y$$

Completing the square

EXT1 Exercise 6.07 The inverse of a function

1 Find the inverse relation of each function.

a $y = 3x$ **b** $y = -x$ **c** $f(x) = \frac{x}{5}$
d $y = \sqrt[3]{x}$ **e** $y = 7x$ **f** $f(x) = x + 1$
g $y = x - 5$ **h** $f(x) = x + 3$ **i** $y = x^3$
j $y = x^5$ **k** $f(x) = x - 9$ **l** $f(x) = 5 - x$
m $y = -3x$ **n** $y = x^2$ **o** $y = \sqrt[7]{x}$
p $y = \frac{x}{9}$ **q** $y = x^8$

2 Find the inverse relation of each function.

a $y = x^3 + 5$ **b** $f(x) = x^7 - 1$ **c** $y = \sqrt[3]{x-2}$
d $y = \frac{2}{x}$ **e** $y = \frac{3}{x+5}$ **f** $y = \frac{x+1}{2}$
g $f(x) = \sqrt{x+2}$ **h** $y = \sqrt[3]{x-7}$ **i** $y = \frac{3}{\sqrt{x}}$
j $y = 3x^5 - 2$ **k** $f(x) = 2\sqrt{x} + 5$ **l** $y = 3\sqrt[3]{2x+1}$
m $y = 2x^4$ **n** $y = x^2 + 5$ **o** $y = x^6 - 3$
p $y = x^2 + 8x$ **q** $y = 4x - x^2$ **r** $y = x^2 - 2x + 3$
s $y = x^2 + 10x - 1$ **t** $y = x^2 - 6x - 3$ **u** $y = x^2 + 12x - 11$

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EXT1 6.08 Graphing the inverse of a function

Graph of the inverse of a function

On the number plane, the graph of the inverse relation is a reflection of the graph of the original function in the line $y = x$.

If a point (x, y) on the number plane has its x- and y-coordinates swapped, then the point (y, x) is the reflection of (x, y) in the line $y = x$.

EXAMPLE 16

Sketch the graph of the original function, its inverse and the line $y = x$ on the same set of axes.

a $y = x + 3$ **b** $y = x^3$ **c** $y = x^2$

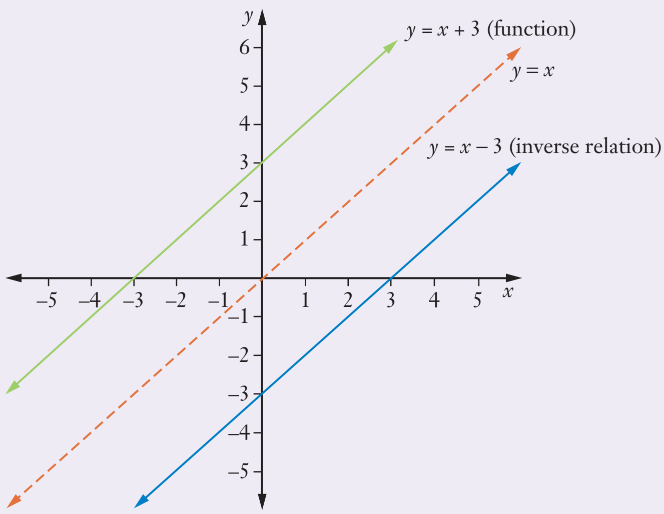
Solution

a $y = x + 3$ is a line with gradient 1 and y-intercept 3.

For x-intercept, $y = 0$:
$0 = x + 3$
$x = -3$

Inverse of $y = x + 3$:
$x = y + 3$
$y = x - 3$
This is a line with gradient 1 and y-intercept -3.

For x-intercept, $y = 0$:
$0 = x - 3$
$x = 3$



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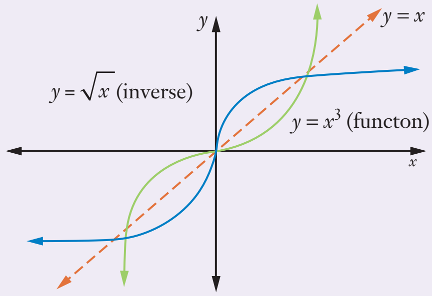
b $y = x^3$ is a cubic function with a point of inflection at (0, 0).

Inverse of $y = x^3$:

$x = y^3$

$y = \sqrt[3]{x}$

x	−2	−1	0	1	2
y	−1.26	−1	0	1	1.26



c $y = x^2$ is a quadratic function with a turning point at (0, 0).

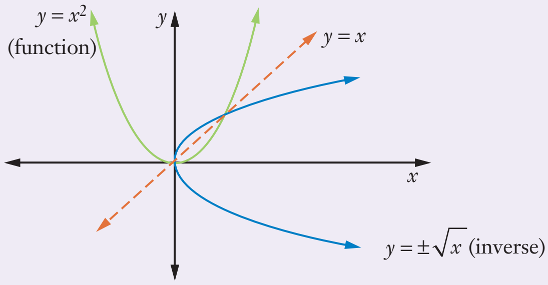
Inverse of $y = x^2$:

$x = y^2$

$y = \pm\sqrt{x}$

x	0	1	4	9
y	0	±1	±2	±9

Notice that $y = \pm\sqrt{x}$ is not a function.



The inverse relations of $y = x + 3$ and $y = x^3$ are also functions, while the inverse relation of $y = x^2$ is not. How could you test the original function to see if its inverse is a function?

Horizontal line test

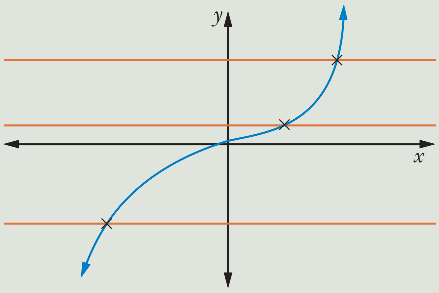
A function has a unique y value for every x value. This can be determined by a vertical line test.

Since the inverse is an exchange of the x and y values, the **inverse function** exists if there is a unique value of x for every y value in the original function, that is, if the original function is **one-to-one**. As we saw in Chapter 4, this can be determined by a **horizontal line test**.

Horizontal line test

If any horizontal line crosses the graph of a function at only one point, then the inverse relation is a function.

This also means that the original function is a **one-to-one function**.



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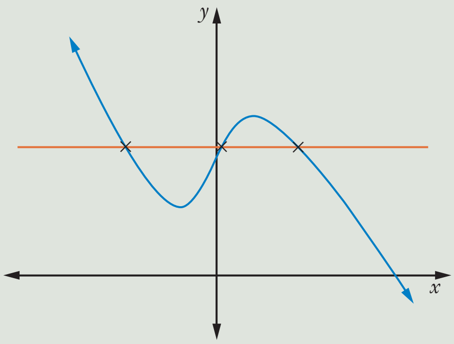
If a horizontal line crosses the graph at more than one point, then the inverse relation is not a function.



Notice that functions that pass the horizontal line test are either always increasing or always decreasing. They do not have turning points. We call these functions **monotonic increasing** or **monotonic decreasing**.

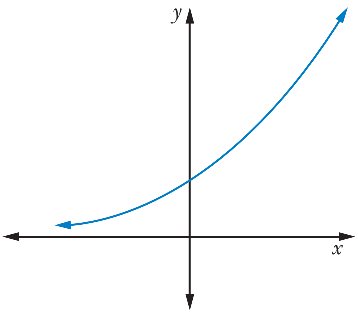
EXT1 Exercise 6.08 Graphing the inverse of a function

1 Sketch the graph of each function, its inverse and the line $y = x$ on the same set of axes.

a $f(x) = 2x + 1$ **b** $y = x^3 - 1$ **c** $f(x) = \dfrac{x}{4}$ **d** $y = \sqrt{x+1}$

2 Does the function represented by each graph have an inverse function?

a



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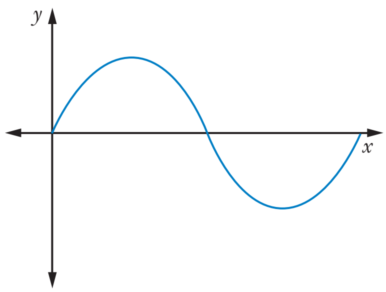

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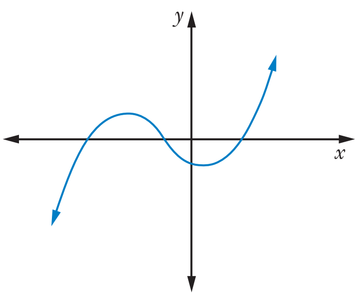

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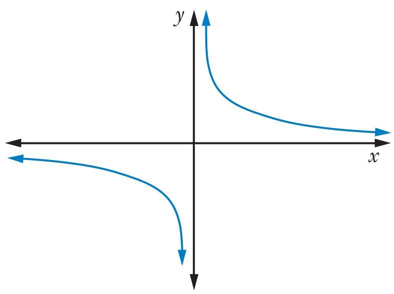

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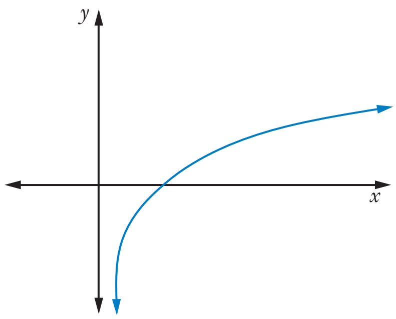

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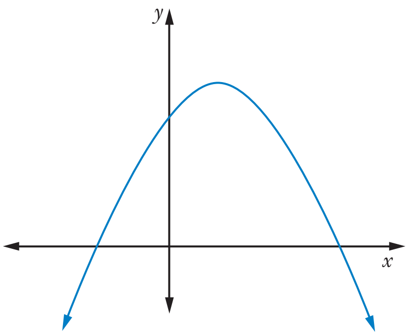

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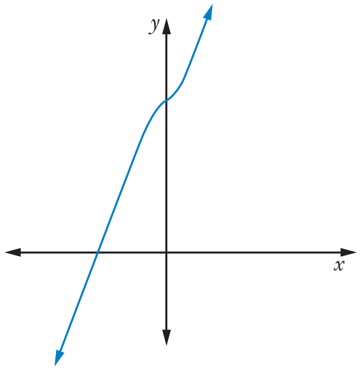

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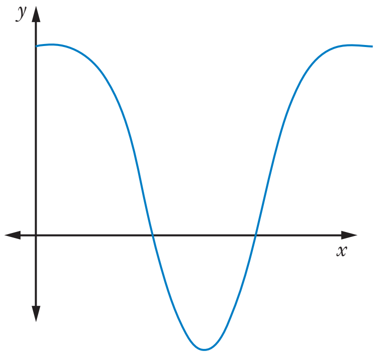

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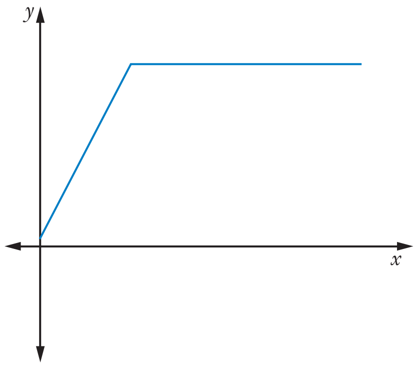

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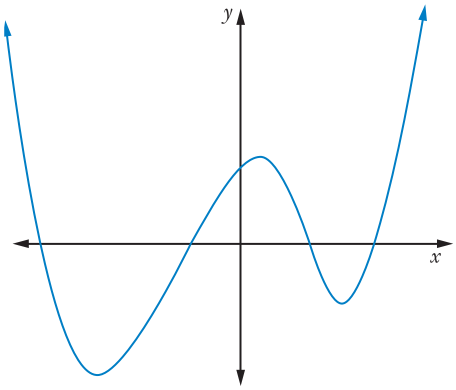

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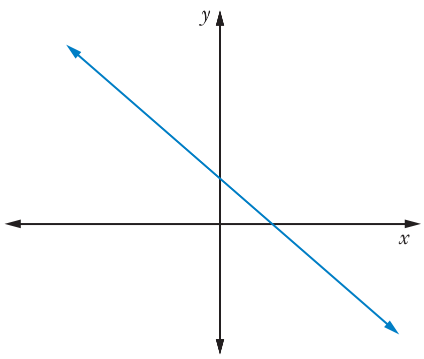

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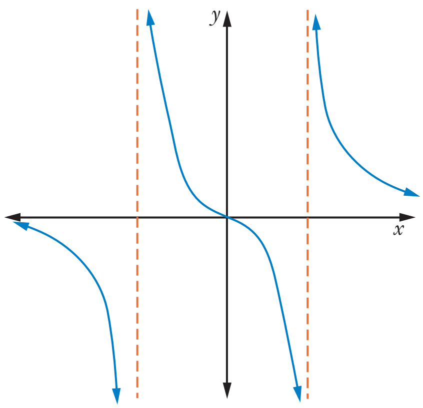

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WS

Inverse functions

WS

Inverse functions code puzzle

EXT1 6.09 Inverse functions

Inverse function notation

If the original function is $y = f(x)$, then we write the inverse function as $y = f^{-1}(x)$.

Note: $f^{-1}(x)$ is not the same as the reciprocal function $[f(x)]^{-1} = \dfrac{1}{f(x)}$.

Because x and y are interchanged in inverse functions, the domain of the inverse function is the range of the original function, and the range of the inverse function is the domain of the original function.

Domain and range of inverse functions

If $y = f(x)$ is a one-to-one function with domain $[a, b]$ and range $[f(a), f(b)]$, the inverse function $y = f^{-1}(x)$ has domain $[f(a), f(b)]$ and range $[a, b]$.

Inverse functions

EXAMPLE 17

a Find the domain and range of the function $y = \dfrac{1}{x-2}$.

b Find the inverse function.

c Find the domain and range of the inverse function.

Solution

a The denominator can't be 0.

$x - 2 \neq 0$

$x \neq 2$

Domain: $(-\infty, 2) \cup (2, \infty)$

$\dfrac{1}{x-2} \neq 0$

So $y \neq 0$

Range: $(-\infty, 0) \cup (0, \infty)$

b

$$x = \frac{1}{y-2}$$

$$y - 2 = \frac{1}{x}$$

$$y = \frac{1}{x} + 2$$

c Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 2) \cup (2, \infty)$

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Restricting the domain

If a function fails the horizontal line test and is not one-to-one, we can still create an inverse function if we restrict its domain to where it is monotonic increasing or decreasing only (no turning points). Then it will have an inverse function over that **restricted domain**.

EXAMPLE 18

Restrict the domain of each function to find an inverse function and its domain and range.

a $y = x^2$

b $f(x) = x^2 - 4x$

Solution

a The inverse relation is

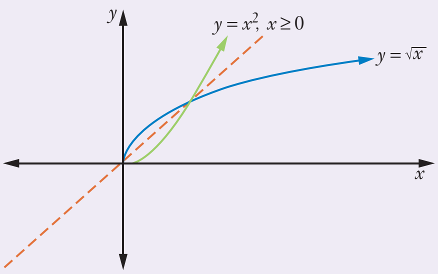
$$x = y^2$$
$$y = \pm\sqrt{x}$$

$y = x^2$ is a parabola with a minimum turning point at (0, 0).

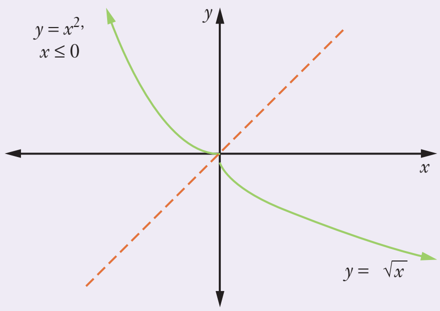
We can restrict its domain to where it is monotonic increasing in the interval $x \geq 0$.

So $f(x)$ has domain $[0, \infty)$ and range $[0, \infty)$, and $f^{-1}(x)$ must have domain $[0, \infty)$ and range $[0, \infty)$.

$\therefore$ The inverse function is $y = \sqrt{x}$.



Alternatively, if the domain of $y = x^2$ is restricted to $x \leq 0$ where it is monotonic decreasing, then the inverse function is $y = -\sqrt{x}$ with domain $[0, \infty)$ and range $(-\infty, 0]$.



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b Inverse relation:

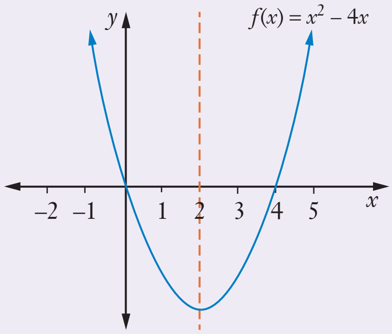
$$x = y^2 - 4y$$

$$x + 4 = y^2 - 4y + 4 \quad \leftarrow \text{Completing the square}$$

$$= (y - 2)^2$$

$$\pm\sqrt{x+4} = y - 2$$

$$\therefore y = \pm\sqrt{x+4} + 2$$



$f(x) = x^2 - 4x$ is a concave upwards parabola with x-intercepts 0, 4 and axis of symmetry at $x = 2$.

$$f(2) = 2^2 - 4(2)$$

$$= -4$$

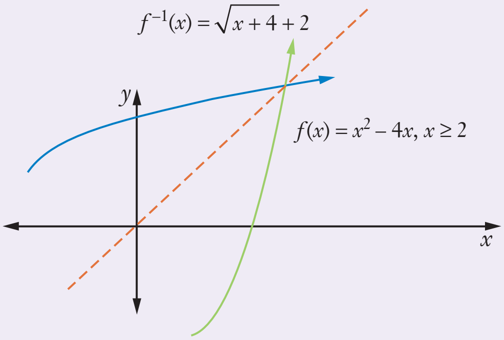
Minimum turning point at (2, –4).

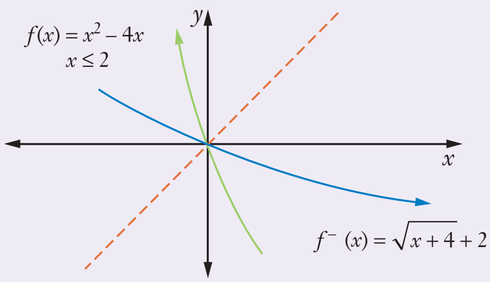
$f(x)$ is monotonic increasing for $x \geq 2$. If the domain is restricted to $[2, \infty)$, the range will be $[-4, \infty)$.

$f^{-1}(x)$ will have domain $[-4, \infty)$ and range $[2, \infty)$.

$$\therefore f^{-1}(x) = \sqrt{x+4} + 2$$

Similarly, if the domain of $f(x)$ is restricted to $(-\infty, 2]$ the inverse function will be $f^{-1}(x) = -\sqrt{x+4} + 2$ with domain $[-4, \infty)$ and range $(-\infty, 2]$.





A function and its inverse 'undo each other'. That is, $f^{-1}[f(x)] = f[f^{-1}(x)] = x$.

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EXAMPLE 19

If $y = 2x - 5$, find the inverse function and show that $f^{-1}[f(x)] = f[f^{-1}(x)] = x$.

Solution

$$x = 2y - 5$$
$$x + 5 = 2y$$
$$\frac{x+5}{2} = y$$
$$f^{-}(x) = \frac{x+5}{2}$$
$$\begin{aligned} f^{-1}[f(x)] &= f^{-1}(2x - 5) \\ &= \frac{(2x-5)+5}{2} \\ &= \frac{2x}{2} \\ &= x \end{aligned}$$

$$\begin{aligned} f[f^{-1}(x)] &= f\left(\frac{x+5}{2}\right) \\ &= 2\left(\frac{x+5}{2}\right) - 5 \\ &= x + 5 - 5 \\ &= x \end{aligned}$$
$$\therefore f^{-1}[f(x)] = f[f^{-1}(x)] = x$$

EXT1 Exercise 6.09 Inverse functions

1 Which of these functions has an inverse function? There is more than one answer.

A $f(x) = 5x - 7$ **B** $y = \frac{4}{x}$ **C** $y = x^2 + 1$ **D** $y = x^3$

2 Find the inverse function of each function, and state its domain and range.

a $y = x^3$ **b** $y = 3x - 2$ **c** $f(x) = \frac{2}{x}$ **d** $y = \frac{1}{x+1}$

3 If the domain of each function is restricted to a monotonic increasing curve, find the inverse function and its domain and range.

a $y = 2x^2$ **b** $y = x^2 + 2$ **c** $y = (x - 3)^2$
d $y = x^2 - 2x$ **e** $y = x^6$ **f** $y = 1 - x^2$
g $y = x^4 - 1$ **h** $y = \frac{1}{x^2}$

4 **a** Find the domain over which the function $y = x^2 + 6x$ is monotonic increasing.

b Find the inverse function over this restricted domain, and state its domain and range.

c Find the domain over which $y = x^2 + 6x$ is monotonic decreasing.

d Find the inverse function over this restricted domain, and state its domain and range.

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5 Restrict the domain of each function to a monotonic decreasing curve and find the inverse function over this domain.

a $y = x^2$ **b** $y = 3x^2 - 1$ **c** $f(x) = (x - 2)^4$

d $y = \frac{3}{x^2}$ **e** $f(x) = \frac{2}{x^4}$

6 For each function and its inverse, show that $f^{-1}[f(x)] = f[f^{-1}(x)] = x$.

a $f(x) = x + 7$ **b** $y = 3x$ **c** $y = \sqrt{x}$

d $y = 3x + 1$

7 **a** Find the domain and range of $y = \frac{2}{x-1}$.

b Find the inverse function.

c State the domain and range of the inverse function.

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6. TEST YOURSELF

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

Polynomials review

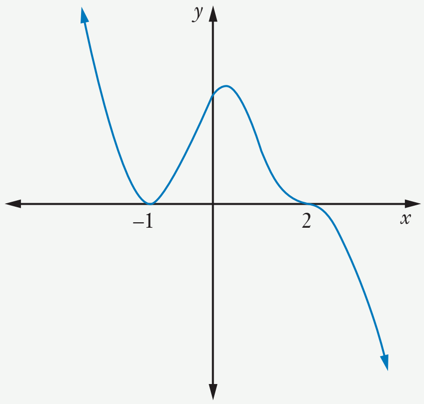
1 Which is a possible equation for this graph?

A $P(x) = (x + 1)^2(x - 2)$

B $P(x) = (x - 1)(x + 2)^2$

C $P(x) = -(x + 1)^2(x - 2)^3$

D $P(x) = -(x - 1)^2(x + 2)^3$



2 If $f(x) = \dfrac{1}{x - 3}$, find $f^{-1}(x)$:

A $f^{-1}(x) = \dfrac{3}{x}$ **B** $f^{-1}(x) = \dfrac{1}{x} + 3$ **C** $f^{-1}(x) = x - 3$ **D** $f^{-1}(x) = \dfrac{1}{x + 3}$

3 If the roots of the quadratic equation $x^2 + 3x + k - 1 = 0$ are consecutive, evaluate k.

A $k = -1$ **B** $k = 1$ **C** $k = 2$ **D** $k = 3$

4 Write $p(x) = x^4 + 4x^3 - 14x^2 - 36x + 45$ as a product of its factors.

5 If α, β and γ are the roots of $x^3 - 3x^2 + x - 9 = 0$, find:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha\beta + \alpha\gamma + \beta\gamma$ **d** $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

6 A monic polynomial $P(x)$ of degree 3 has zeros –2, 1 and 6. Write down the polynomial.

7 **a** Divide $P(x) = x^4 + x^3 - 19x^2 - 49x - 30$ by $x^2 - 2x - 15$.

b Hence, write $P(x)$ as a product of its factors.

8 Find the inverse function of $f(x) = 3 - 2x$.

9 For the polynomial $P(x) = x^3 + 2x^2 - 3x$, find:

a the degree

b the coefficient of x

c the zeros

d the leading term.

10 Sketch the graph of $f(x) = (x - 2)(x + 3)^2$, showing the intercepts.

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11 If $ax^4 + 3x^3 - 48x^2 + 60x = 0$ has a double root at $x = 2$, find:

a the value of a **b** the sum of the other 2 roots.

12 **a** State the domain and range of $y = \sqrt{x-1}$.

b Find the inverse of this function and state its domain and range.

13 **a** Find the domain and range of $y = \frac{1}{x+2}$.

b Find the inverse function.

c Find the domain and range of the inverse function.

14 Show that $x + 7$ is not a factor of $x^3 - 7x^2 + 5x - 4$.

15 If the sum of 2 roots of $x^4 + 2x^3 - 8x^2 - 18x - 9 = 0$ is 0, find the roots of the equation.

16 **a** Find the domain over which the curve $y = x^2 - 4x$ is monotonic increasing.

b Find the inverse function over this domain.

17 If $p(x) = x^3 - 1$ and $q(x) = 2x + 5$, evaluate:

a $p^{-1}(7)$ **b** $q^{-1}(p(3))$

18 The polynomial $f(x) = ax^2 + bx + c$ has zeros 4 and 5, and $f(-1) = 60$. Evaluate a, b and c.

19 Find the x- and y-intercepts of the curve $y = x^3 - 3x^2 - 10x + 24$.

20 Divide $p(x) = 3x^5 - 7x^3 + 8x^2 - 5$ by $x - 2$ and write $p(x)$ in the form $p(x) = (x - 2)\,a(x) + b(x)$.

21 When $8x^3 - 5kx + 9$ is divided by $x - 2$ the remainder is 3. Evaluate k.

22 Write $P(x) = x^5 + 2x^4 + x^3 - x^2 - 2x - 1$ as a product of its factors.

23 By restricting the domain of $f(x) = x^2 - 4$ to monotonic decreasing, find its inverse function.

24 Find the zeros of $g(x) = -x^2 + 9x - 20$.

25 Sketch the graph of $P(x) = 2x(x - 3)(x + 5)$, showing intercepts.

26 Find the value of k if the remainder is -4 when $x^3 + 2x^2 - 3x + k$ is divided by $x - 2$.

27 The sum of 2 roots of $x^4 - 7x^3 + 5x^2 - x + 3 = 0$ is 3. Find the sum of the other 2 roots.

28 The leading term of a polynomial is $3x^3$ and there is a double root at $x = 3$. Sketch a graph of the polynomial.

29 A polynomial $P(x)$ has a triple root at $x = -6$.

a Write an expression for $P(x)$.

b If $P(x)$ has leading coefficient 3 and degree 4, sketch a graph showing this information.

30 Draw an example of a polynomial with leading term $3x^5$.

31 If $f(x) = x^3$, show that $f[f^{-1}(x)] = f^{-1}[f(x)] = x$.

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6. CHALLENGE EXERCISE

1 **a** Write the polynomial $P(u) = u^3 - 4u^2 + 5u - 2$ as a product of its factors.

b Hence or otherwise solve $(x - 1)^3 - 4(x - 1)^2 + 5(x - 1) - 2 = 0$.

2 **a** Write $f(u) = u^3 - 13u^2 + 39u - 27$ as a product of its factors.

b Hence or otherwise, solve $3^{3x} - 13(3^{2x}) + 39(3^x) - 27 = 0$.

3 Find the points of intersection between the polynomial $P(x) = x^3 + 5x^2 + 4x - 1$ and the line $3x + y + 4 = 0$.

4 Divide $6x^2 - 3x + 1$ by $3x - 2$.

5 If $P(x) = ax^3 + bx^2 + cx + d$ has a remainder of 8 when divided by $x - 1$, $P(2) = 17$, $P(-1) = -4$ and $P(0) = 5$, evaluate a, b, c and d.

6 If α, β and γ are roots of the cubic equation $2x^3 + 8x^2 - x + 6 = 0$, find:

a $\alpha\beta\gamma$

b $\alpha^2 + \beta^2 + \gamma^2$

7 Find the value of a if $(x + 1)(x - 2)$ is a factor of $2x^3 - x^2 + ax - 2$.

8 Prove that if $x - k$ is a factor of polynomial $P(x)$, then $P(k) = 0$.

9 Sketch a graph of a polynomial with a double root at $x = k_1$ and a double root at $x = k_2$, if the polynomial is monic, has even degree, and $k_2 > k_1$.

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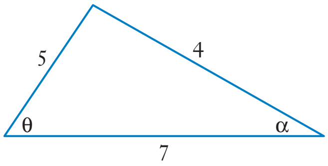
Practice set 2

In Questions **1** to **12**, select the correct answer **A**, **B**, **C** or **D**.

1 Find an expression involving θ for this triangle (there may be more than one answer).



A $\cos\theta = \frac{5^2 + 4^2 - 7^2}{2\times5\times4}$

B $\frac{\sin\theta}{4} = \frac{\sin\alpha}{5}$

C $\frac{\sin\theta}{5} = \frac{\sin\alpha}{4}$

D $\cos\theta = \frac{5^2 + 7^2 - 4^2}{2\times5\times7}$

2 If $f(x) = \begin{cases} 8x^3 & \text{if } x > 3 \\ 3x^2 - 2 & \text{if } 0 \le x \le 3 \\ 9 & \text{if } x < 0 \end{cases}$ evaluate $f(3) + f(1) + f(-1)$.

A 35 **B** 226 **C** 233 **D** 53

3 The linear function with equation $4x - 2y + 3 = 0$ has:

A gradient -2, y-intercept $-1\frac{1}{2}$

B gradient $\frac{1}{2}$, y-intercept $\frac{3}{4}$

C gradient 2, y-intercept $1\frac{1}{2}$

D gradient 4, y-intercept 3.

4 For the quadratic function $y = ax^2 + bx + c > 0$ for all x:

A $a > 0, b^2 - 4ac > 0$

B $a < 0, b^2 - 4ac > 0$

C $a > 0, b^2 - 4ac < 0$

D $a < 0, b^2 - 4ac < 0$

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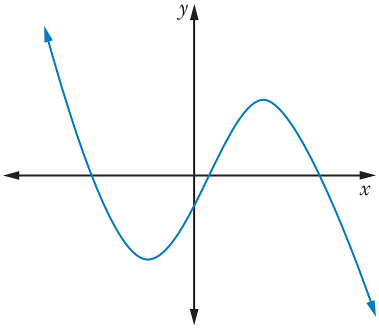
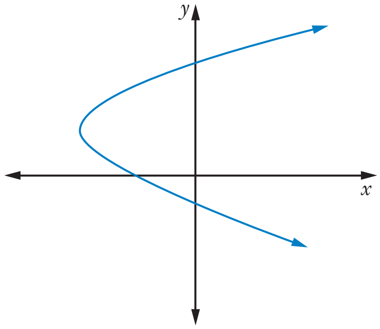
5 Which of the following is not the graph of a function?

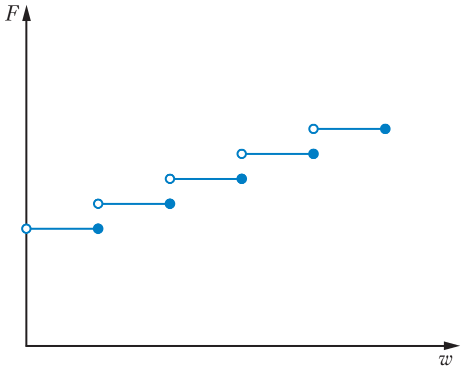
A



B



C



D (0, 3), (1, 3), (2, 5), (3, 1)

6 EXT1 The quadratic equation $x^2 + (k - 3)x + k = 0$ has real roots. Evaluate k.

A $k \leq 1, k \geq 9$

B $k = 1, 9$

C $1 \leq k \leq 9$

D $k < 1, k > 9$

7 The polynomial $P(x) = x^3 - 5x^2 + 3x - 8$ (there is more than one answer):

A is monic

B has degree 3

C has leading coefficient –8

D has constant term –8.

8 EXT1 Which of these functions has an inverse function? There is more than one answer.

A $f(x) = x^2$

B $f(x) = x^3$

C $f(x) = \frac{5}{x^2}$

D $f(x) = -\frac{3}{x}$

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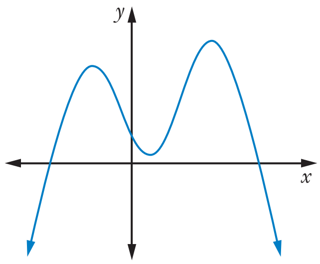
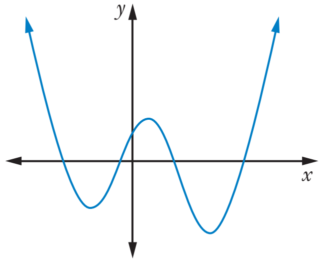
9 EXT1 Which of these is the graph of a polynomial with leading coefficient –3 and degree 4?

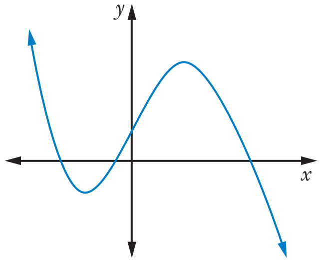
A



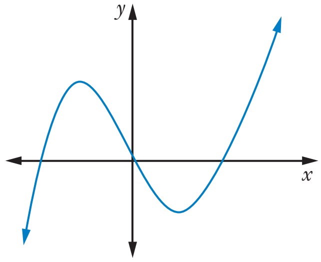
B



C

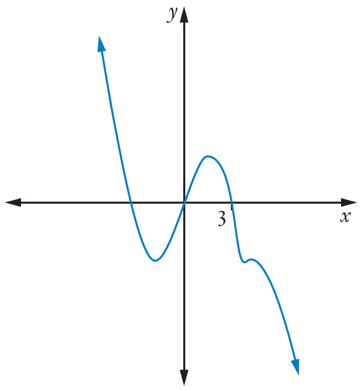


D

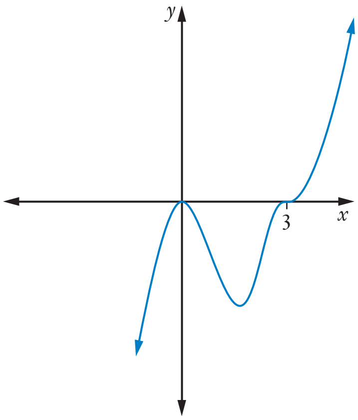


10 EXT1 Which of these is the graph of the polynomial $P(x) = x^3(x + 3)^2$?

A



B



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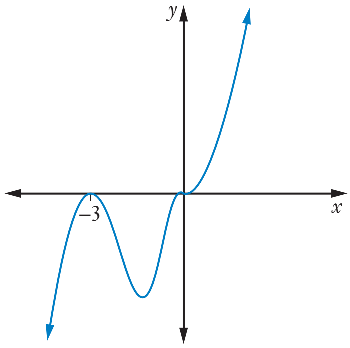

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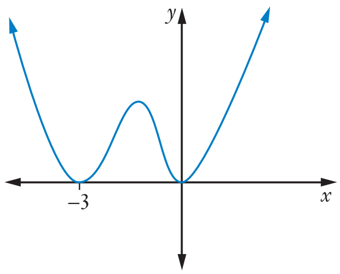

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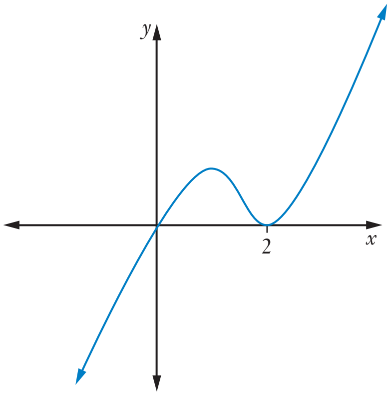
11 EXT1 Which of these polynomials has the graph shown below?

A $P(x) = x(x - 2)^2$

B $P(x) = x(x + 2)^2$

C $P(x) = x^2(x + 2)$

D $P(x) = x^2(x - 2)$



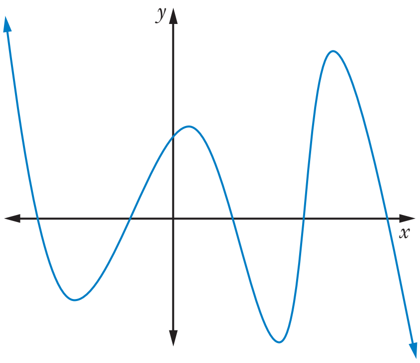
12 EXT1 What could be the leading term of the polynomial whose graph is shown below?

A x^6

B x^5

C $-x^4$

D $-x^5$



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13 A triangle has sides of length 5.1 m, 6.5 m and 8.2 m.

a Find the size of the angle opposite the 6.5 m side, correct to the nearest minute.

b Find the area of the triangle correct to one decimal place.

14 Find the equation of the straight line:

a with gradient –2 and y-intercept 3

b with x-intercept 5 and y-intercept –1

c passing through (2, 0) and (–3, –4)

d through (5, –4) and parallel to the line through (7, 4) and (3, –1)

e through (3, –1) perpendicular to the line $3x - 2y - 7 = 0$

f through (1, 2) parallel to the line through (–3, 4) and (5, 5)

g through (1, 3) and an angle of inclination of 135°.

15 Simplify:

a $\frac{6x}{2x-8}$ **b** $\frac{5y+10}{xy^2} \div \frac{y^2-4}{x^2y}$ **c** $\frac{4a-3}{5} - \frac{a+1}{4}$

16 Convert these angles into radians in terms of π:

a 60° **b** 150° **c** 90°

d 10° **e** 315°

17 Sketch the graph of:

a $5x - 2y - 10 = 0$ **b** $x = 2$ **c** $f(x) = (x-3)^2$

d $y = x^2 - 5x + 4$ **e** $y = (x-1)^3 + 2$

18 EXT1 Find the remainder when dividing $x^3 + 7x^2 - 3x - 4$ by $x - 2$.

19 Convert each value in radians into degrees and minutes:

a 1.7 **b** 0.36 **c** 2.54

20 The lines AB and AC have equations $3x - 4y + 9 = 0$ and $8x + 6y - 1 = 0$ respectively.

a Show that the lines are perpendicular.

b Find the coordinates of A.

21 EXT1 Find the inverse function of $f(x) = \frac{1}{x-1}$ and state its domain and range.

22 Find the gradient of the line through the origin and (–3, 5).

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23 If $g(x) = \begin{cases} 3 - x & \text{if } x > 1 \\ 2x & \text{if } x \le 1 \end{cases}$:

a find $g(2)$ and $g(-3)$

b sketch the graph of $y = g(x)$.

24 Find the value of x if $f(x) = 7$ where $f(x) = 2^x - 1$.

25 If $f(x) = 9 - 2x^2$, find the value of $f(-1)$.

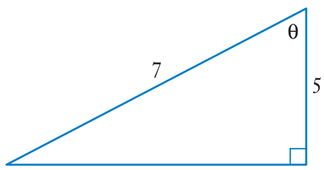
26 Show that $3x - 4y + 10 = 0$ is a tangent to the circle $x^2 + y^2 = 4$.

27 Change each value in radians into degrees:

a $\frac{\pi}{4}$ **b** $\frac{3\pi}{2}$ **c** $\frac{\pi}{5}$ **d** $\frac{7\pi}{8}$ **e** 6π

28 EXT1 Find the zeros of $f(x) = (2x - 1)^5$.

29 Given the triangle ABC, find exact values of $\cos\theta$, $\sin\theta$ and $\tan\theta$.



30 Show that:

a $-x^2 + x - 9 < 0$ for all x

b $x^2 - x + 3 > 0$ for all x.

31 EXT1 Write $P(x) = x^3 + 4x^2 - x - 4$ as a product of its factors.

32 The distance travelled by a runner is directly proportional to the time she takes. If Vesna runs 12 km in 2 hours 30 minutes, find:

a an equation for distance d in terms of time t

b how far Vesna runs in:

i 2 hours **ii** 5 hours

c how long it takes Vesna to run:

i 30 km **ii** 19 km

d her average speed.

33 EXT1 For the polynomial $f(x) = 3x^4 - 2x^3 - x + 8$, what graph does the polynomial approach as x becomes very large?

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34 EXT1 For each function:

i find the inverse

ii state whether or not the inverse is a function

iii write down the domain and range of the inverse.

a $f(x) = 3x + 5$ **b** $f(x) = 2x^3$

c $f(x) = (x - 1)^2$ **d** $f(x) = \frac{2}{x+1}$

35 Find the equation of the parabola with x-intercepts 3 and −1 and y-intercept −3.

36 Show that the quadratic equation $6x^2 + x - 15 = 0$ has 2 real, rational roots.

37 The area of a circle is 5π and an arc 3 cm long cuts off a sector with an angle of θ subtended at the centre. Find θ in degrees and minutes.

38 A soccer goal is 8 m wide. Tim shoots for goal when he is 9 m from one post and 11 m from the other. Within what angle must a shot be made in order to score a goal?

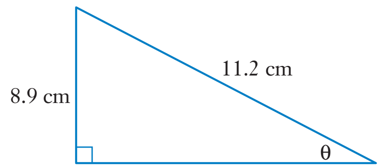
39 EXT1 Find the zeros of the polynomial $f(x) = x^4 - x^3 + x^2 - 3x - 6$.

40 EXT1 If α, β and γ are the roots of $x^3 + 2x^2 - 3x + 4 = 0$, find:

a $\alpha\beta\gamma$ **b** $\alpha + \beta + \gamma$

c $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **d** $\alpha^2 + \beta^2 + \gamma^2$

41 Evaluate θ in degrees and minutes, to the nearest minute:



42 **a** Find the equation of the straight line l through (−1, 2) that is perpendicular to the line $3x + 6y - 7 = 0$.

b Line l cuts the x-axis at P and the y-axis at Q. Find the coordinates of P and Q.

43 Show that $f(x) = x^6 - x^2 - 3$ is an even function.

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44 Find the angle of depression from the top of a 5.6 m tall cliff down to a boat that is 150 m out from the base of the cliff.

45 EXT1 In the quadratic equation $(k - 1)x^2 - 5x + 3k + 4 = 0$, the roots are reciprocals of each other. Find the value of k.

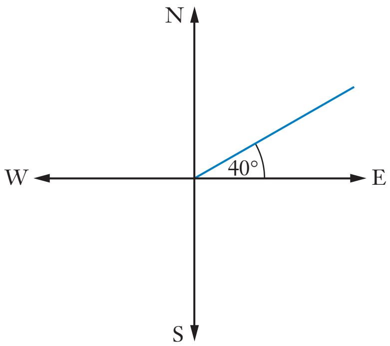
46 EXT1 If $f(x) = x^5 - 3$, show that $f^{-1}[f(x)] = f[f^{-1}(x) = x$.

47 EXT1 By restricting the domain of $f(x) = x^2 - 4x + 7$ to monotonic increasing, find the inverse function and state its domain and range.

48 Write each direction shown as:

i a compass bearing **ii** a true bearing

a



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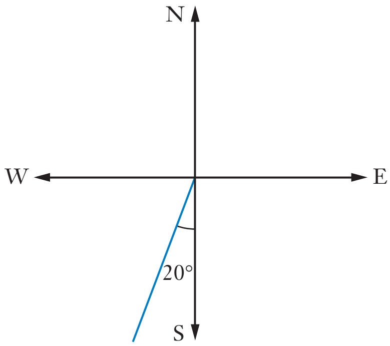

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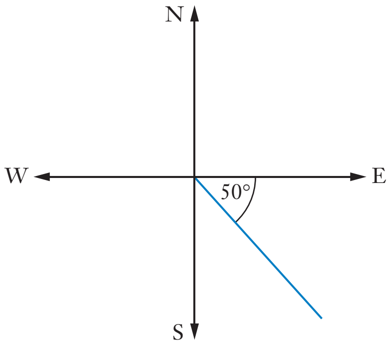

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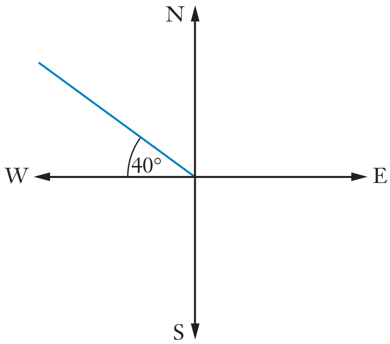


49 EXT1 If $f(x) = x^3 - 1$, find the inverse function and state its domain and range.

50 EXT1 Expand $(3a - b)^3$.

51 An angle of 30° is subtended at the centre of a circle with radius 5 cm. Find the exact:

a arc length **b** area of the sector.

52 EXT1 If α, β, γ and δ are the roots of the equation $x^4 - 3x^3 + 2x + 1 = 0$, find:

a $\alpha\beta\gamma\delta$ **b** $\alpha + \beta + \gamma + \delta$

c $\alpha\beta$ and $\alpha + \beta$ if $\gamma = -2$ and $\delta = 5$

53 EXT1 Find the remainder if $P(x) = 3x^5 - 4x^3 + 2x^2 - 7x - 3$ is divided by $x + 1$.

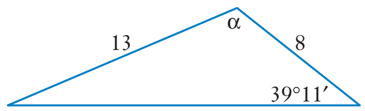
54 Factorise:

a $x^2 - 4x + 4$ **b** $9x^2 - 1$

55 EXT1 **a** Write $P(x) = x^3 - 7x^2 + 15x - 9$ as a product of its factors.

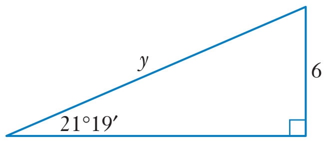
EXT1 **b** Sketch the graph of the polynomial on a number plane.

56 Find α in degrees and minutes.



57 EXT1 Show that the remainder on dividing $P(x) = 2x^3 - 7x^2 + x - 9$ by $x + 2$ is $P(-2)$.

58 Find the value of y correct to 3 significant figures:



59 Find the intersection of the graphs:

a $x + 3y - 1 = 0$ and $x - 2y - 6 = 0$

b $y = x^2$ and $x - 2y + 15 = 0$

60 For each quadratic function:

i find the equation of the axis of symmetry

ii state whether it has a maximum or minimum turning point and find its coordinates.

a $y = x^2 - 6x + 1$ **b** $y = -2x^2 - 4x - 3$

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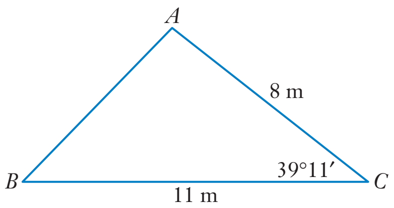
61 A hawk at the top of a 10 m tree sees a mouse on the ground. If the angle of depression is 34°51′, how far, to 1 decimal place, does the bird need to fly to reach the mouse?

62



a Find AB, to the nearest metre.

b Find the area of ΔABC, to 3 significant figures.

63 EXT1 If α, β, γ and δ are the roots of $x^4 - x^3 + x^2 - 1 = 0$:

a show that $\alpha\beta\gamma + \alpha\gamma\delta + \beta\gamma\delta + \alpha\beta\delta = 0$

b find $(\alpha + \beta + \gamma + \delta)^2$

c find $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$.

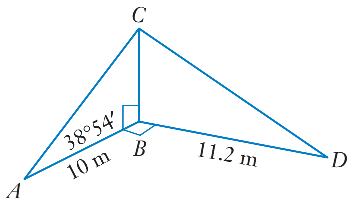
64 Two points A and B are 100 m apart on the same side of a tower. The angle of elevation of A to the top of the tower is 20° and the angle of elevation from B is 27°. Find the height of the tower, to the nearest metre.

65 The length of an arc in a circle of radius 6 cm is 7π cm. Find the area of the sector cut off by this arc.

66 EXT1 Find values of a, b and c for which $3x^2 - 2x - 7 = a(x + 2)^2 + b(x + 2) + c$.

67 Jordan walks for 3.1 km due west, then turns and walks for 2.7 km on a bearing of 205°. How far is he from his starting point?

68 The angle of elevation from a point A to the top of a tower BC is 38°54′. A is 10 m due south of the tower.



a Find the height of the tower, to 1 decimal place.

b If point D is 11.2 m due east of the tower, find the angle of elevation from D to the tower.

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69 Find the domain and range of:

a $f(x) = \dfrac{3}{x+4}$ **b** $y = |x| + 2$ EXT1 **c** $y = -\sqrt{4 - x^2}$

d $y = 4$ **e** $y = x^2 - 3$

70 EXT1 Show that $x - 3$ is a factor of $f(x) = 3x^3 - 7x^2 - 5x - 3$.

71 Nalini leaves home and cycles west for 12.5 km then turns and rides south for 11.3 km.

a How far is Nalini from home?

b Find the bearing of Nalini from home.

72 EXT1 What graph does the graph of the polynomial $f(x) = -x^4 + x^3 - 2x^2 + x - 2$ approach as x becomes very large?

73 Show that $f(x) = x^3 - 5x$ is an odd function.

74 Sketch the graph of:

a $3x - 2y + 6 = 0$ **b** $y = x^2 - x - 2$

c $y = x^3 - 1$ **d** $y = x(x + 2)(x - 3)$

75 EXT1 If α, β and γ are the roots of the cubic equation $x^3 - 4x^2 - 3x + 2 = 0$, evaluate:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha\beta + \beta\gamma + \alpha\gamma$

d $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **e** $\alpha^2 + \beta^2 + \gamma^2$

76 EXT1 By dividing the polynomial $P(x) = x^3 - 2x^2 + x + 3$ by $x - 2$, write $P(x)$ in the form $P(x) = (x - 2)Q(x) + R(x)$.

77 The length of an arc in a circle of radius 2 cm is 1.6 cm. Find the area of the sector.

78 EXT1 Write the polynomial $P(x) = -x^3 + 3x^2 + 9x + 5$ as a product of its factors.

79 Change these angles into degrees.

a 2π **b** $\dfrac{\pi}{6}$ **c** $\dfrac{9\pi}{4}$

80 EXT1 A type of car number plate has 3 letters followed by 3 numbers. How many different number plates of this type are possible?

81 EXT1 By restricting the domain of $y = x^2 - 4$ to monotonic decreasing, find the inverse function and state its domain and range.

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82 A plane flies on a bearing of 034° from Sydney for 875 km. How far due east of Sydney is the plane?

83 Solve:

a $5b - 3 \geq 7$ **b** $x^2 - 3x = 0$ **c** $|2n + 5| = 9$

d EXT1 $n^2 - 9 < 0$ **e** EXT1 $|3x - 1| \geq 2$

84 EXT1 A group of 12 people is offered 5 free backstage tickets. Find the number of different ways that the tickets can be given out if the order is:

a important **b** unimportant.

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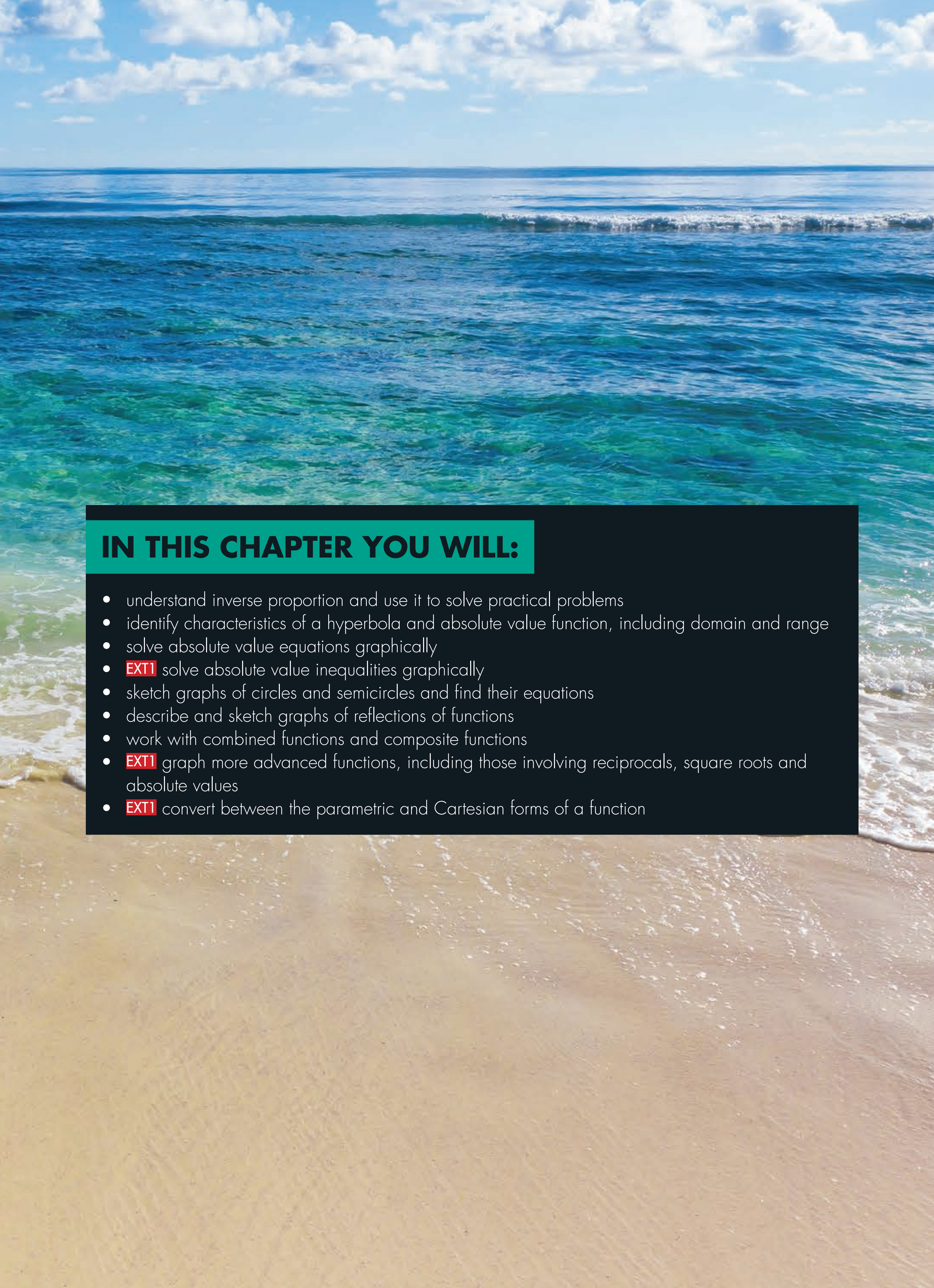
7.

FURTHER FUNCTIONS

In this chapter, we look at functions and relations that are not polynomials, including the hyperbola, absolute value, circles and semicircles. We will also study reflections and relationships between functions, including combined functions, composite functions, reciprocal functions, square root relations and parametric forms of a function.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- understand inverse proportion and use it to solve practical problems
- identify characteristics of a hyperbola and absolute value function, including domain and range
- solve absolute value equations graphically
- EXT1 solve absolute value inequalities graphically
- sketch graphs of circles and semicircles and find their equations
- describe and sketch graphs of reflections of functions
- work with combined functions and composite functions
- EXT1 graph more advanced functions, including those involving reciprocals, square roots and absolute values
- EXT1 convert between the parametric and Cartesian forms of a function

TERMINOLOGY

asymptote: A line that a curve approaches but doesn't touch

composite function: A function of a function, where the output of one function becomes the input of a second function, written as $f(g(x))$. For example, if $f(x) = x^2$ and $g(x) = 3x + 1$ then $f(g(x)) = (3x + 1)^2$

continuous function: A function whose graph is smooth and does not have gaps or breaks

discontinuous function: A function whose graph has a gap or break in it; for example, $f(x) = \frac{1}{x}$, whose graph is a hyperbola

hyperbola: The graph of the function $y = \frac{k}{x}$, which is made up of 2 separate curves

inverse variation: A relationship between 2 variables such that as one variable increases the other variable decreases, or as one variable decreases the other variable increases. One variable is a multiple of the reciprocal of the other, with equation $y = \frac{k}{x}$. Also called **inverse proportion**

EXT1 **parameter:** A third variable in a function that is related to the 2 variables x and y; for example, θ is a parameter in the equations $x = 4 \cos \theta$, $y = 4 \sin \theta$, where θ is the size of an angle

7.01 The hyperbola

WS The hyperbola

Graphing hyperbolas

WS Graphing $y = \frac{10}{x}$

Inverse variation

We looked at direct variation and the equation $y = kx$ in Chapter 4. When one variable is in inverse variation (or **inverse proportion**) with another variable, one is a constant multiple of the **reciprocal** of the other. This means that as one variable increases, the other decreases and when one decreases, the other increases.

For example:

- The more slices you cut a pizza into, the smaller the size of each slice
- The more workers there are on a project, the less time it takes to complete
- The fewer people sharing a house, the higher the rent each person pays.

Inverse variation

If variables x and y are in inverse variation, we can write the equation $y = \frac{k}{x}$ where k is called the **constant of variation**.

EXAMPLE 1

a Building a shed in 12 hours requires 3 builders. If the number of builders, N, is in inverse variation to the amount of time, t hours:

i find the equation for N in terms of t

ii find the number of builders it would take to build the shed in 9 hours

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iii find how long it would take 2 builders to build the shed

v graph the equation for N after completing the table below.

t	1	2	3	4	5	6	7	8	9
N									

b The faster a car travels, the less time it takes to travel a certain distance. It takes the car 2 hours to travel this distance at a speed of 80 km/h. If the time taken, t hours, is in inverse proportion to the speed s km/h, then:

i find the equation for t in terms of s

ii find the time it would take if travelling at 100 km/h

iii find the speed at which the trip would take $2\frac{1}{2}$ hours

v graph the equation.

Solution

a **i** For inverse variation, the equation is in the form $N = \frac{k}{t}$.

Substitute $t = 12$, $N = 3$ to find the value of k:

$$3 = \frac{k}{12}$$

$$36 = k$$

$$\therefore N = \frac{36}{t}$$

ii Substitute $t = 9$.

$$N = \frac{36}{9}$$

$$= 4$$

So it takes 4 builders to build the shed in 9 hours.

iii Substitute $N = 2$.

$$2 = \frac{36}{t}$$

$$2t = 36$$

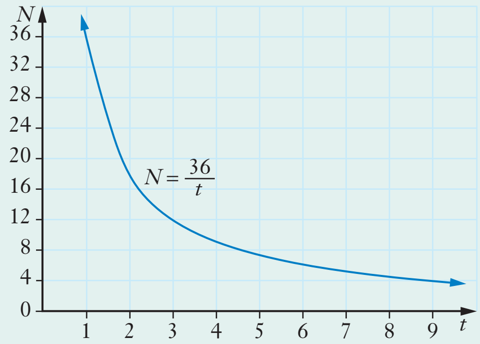
$$t = 18$$

So it takes 18 hours for 2 builders to build the shed.

iv $N = \frac{36}{t}$

Completing a table of values:

t	1	2	3	4	5	6	7	8	9
N	36	18	12	9	7.2	6	5.1	4.5	4



b **i** For inverse proportion, the equation is in the form $t = \frac{k}{s}$.

Substitute $s = 80$, $t = 2$ to find k.

$$2 = \frac{k}{80}$$

$$k = 160$$

$$\therefore \quad t = \frac{160}{s}$$

ii Substitute $s = 100$.

$$t = \frac{160}{100}$$

$$= 1.6 \text{ hours}$$

$$= 1 \text{ h } 36 \text{ min}$$

So the car takes 1 h 36 min to travel the distance if travelling at 100 km/h.

iii Substitute $t = 2.5$.

$$2.5 = \frac{160}{s}$$

$$2.5s = 160$$

$$s = \frac{160}{2\ 5}$$

$$= 64$$

So the car travels at 64 km/h if the trip takes $2\frac{1}{2}$ hours.

iv To graph this function, complete a table of values for $t = \frac{160}{s}$.

s	20	40	60	80	100	120
t	8	4	2.667	2	1.6	1.333

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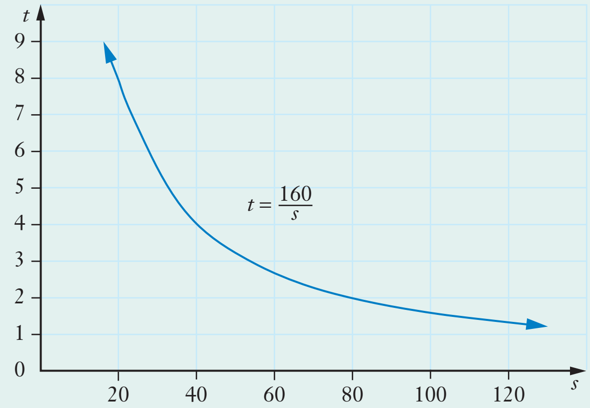


The graph of the function $y = \frac{k}{x}$ is a **hyperbola**.

Hyperbolas

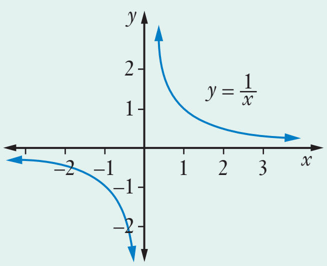
A hyperbola is the graph of a function of the form $y = \frac{k}{x}$.

EXAMPLE 2

Sketch the graph of $y = \frac{1}{x}$. What is the domain and range?

Solution

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	–	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$



When $x = 0$ the value of y is undefined.

Domain: x can be any real number except 0.
$(-\infty, 0) \cup (0, \infty)$

Range: y can be any real number except 0.
$(-\infty, 0) \cup (0, \infty)$

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CLASS DISCUSSION

LIMITS OF THE HYPERBOLA

What happens to the graph as x becomes closer to 0? What happens as x becomes very large in both positive and negative directions? The value of y is never 0. Why?

Continuity

Most functions have graphs that are smooth unbroken curves (or lines). They are called **continuous functions**. However, some functions have discontinuities, meaning that their graphs have gaps or breaks. These are called **discontinuous functions**.

The hyperbola is discontinuous because there is a gap in the graph and it has two separate parts. The graph of $y = \frac{1}{x}$ also does not touch the x- or y-axes, but it gets closer and closer to them. We call the x- and y-axes **asymptotes**: lines that the curve approaches but never touches.

To find the shape of the graph close to the asymptotes or as $x \to \pm\infty$, we can check points nearby.

EXAMPLE 3

Find the domain and range of $f(x) = \frac{3}{x-3}$ and sketch the graph of the function.

Solution

To find the domain, we notice that $x - 3 \neq 0$. So $x \neq 3$.

Domain $(-\infty, 3) \cup (3, \infty)$

Also y cannot be zero: $y \neq 0$.

Range $(-\infty, 0) \cup (0, \infty)$

The lines $x = 3$ and $y = 0$ (the x-axis) are the asymptotes of the hyperbola.

To find the limiting behaviour of the graph, look at what is happening as $x \to \pm\infty$.

As x increases and approaches ∞, $\frac{3}{x-3}$ becomes closer to 0 and is positive.
Substitute large values of x into the function, for example, $x = 1000$.

As $x \to \infty$, $y \to 0^+$ (as x approaches infinity, y approaches 0 from above, the positive side).

Similarly, as x decreases and approaches $-\infty$, y becomes closer to 0 and is negative.
Substitute $x = -1000$, for example.

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As $x \to -\infty, y \to 0^-$ (as x approaches negative infinity, y approaches 0 from below, the negative side).

To see the behaviour of the function near the asymptote $x = 3$ we can test values either side.

LHS: When $x = 2.999$,

$$\frac{3}{2999 - 3} = -3000$$
$$< 0$$

As $x \to 3^-, y \to -\infty$

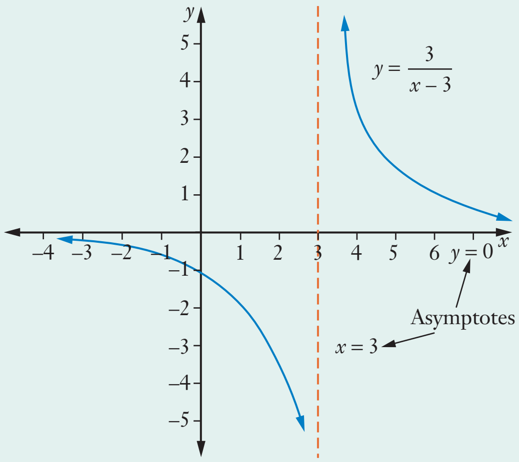
RHS: When $x = 3.001$,

$$\frac{3}{3001 - 3} = 3000$$
$$> 0$$

As $x \to 3^+, y \to \infty$

For y-intercept, $x = 0$:

$$y = \frac{3}{0-3}$$
$$= -1$$



EXAMPLE 4

Sketch the graph of $y = -\dfrac{1}{2x+4}$.

Solution

To find the domain, notice that:

$$2x + 4 \neq 0$$
$$2x \neq -4$$
$$x \neq -2$$

Domain $(-\infty, -2) \cup (-2, \infty)$

For the range, $y \neq 0$.

Range $(-\infty, 0) \cup (0, \infty)$

So there are asymptotes at $x = -2$ and $y = 0$.

Limiting behaviour:

As $x \to \infty, y \to 0^-$.

As $x \to -\infty, y \to 0^+$.

Substitute, say, $x = 5000$ and $x = -5000$.

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To see the shape of the graph near the asymptote $x = -2$, we can test values either side.

LHS: When $x = -2.0001$,

$$y = -\frac{1}{2(-20001) + 4} = 5000$$
$$> 0$$

As $x \to 2^-, y \to \infty$

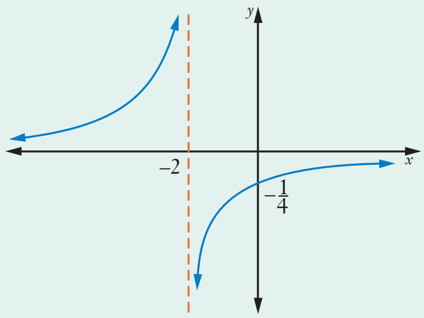
RHS: When $x = -1.9999$,

$$y = -\frac{1}{2(-19999) + 4} = -5000$$
$$< 0$$

As $x \to 2^+, y \to -\infty$

For y-intercept, $x = 0$

$$y = -\frac{1}{2(0) + 4}$$
$$= -\frac{1}{4}$$



The hyperbola

The hyperbola $y = \frac{k}{bx + c}$ is a discontinuous function with 2 parts, separated by vertical and horizontal asymptotes.

Exercise 7.01 The hyperbola

1 The diameter of a balloon varies inversely with the thickness of the rubber. The diameter of the balloon is 80 mm when the rubber is 2 mm thick.

- **a** Find an equation for the diameter D in terms of the thickness x.
- **b** Find the diameter when the thickness is 0.8 mm.
- **c** Find the thickness correct to one decimal place when the diameter is 115.3 mm.
- **d** Sketch the graph showing this information.

2 The more boxes a factory produces, the less it costs to produce each box. When 128 boxes are produced, it costs \$2 per box.

- **a** Write an equation for the cost c to produce each box when manufacturing n boxes.
- **b** Find the cost of each box when 100 boxes are produced.
- **c** Find how many boxes must be produced for the cost for each box to be 50 cents.
- **d** Sketch the graph of this information.

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3 For each function:

i state the domain and range

ii find the y-intercept if it exists

iii sketch the graph.

a $y = \frac{2}{x}$ **b** $y = -\frac{1}{x}$ **c** $f(x) = \frac{1}{x+1}$

d $f(x) = \frac{3}{x-2}$ **e** $y = \frac{1}{3x+6}$ **f** $f(x) = -\frac{2}{x-3}$

g $f(x) = \frac{4}{x-1}$ **h** $y = -\frac{2}{x+1}$ **i** $f(x) = \frac{2}{6x-3}$

4 Show that $f(x) = \frac{2}{x}$ is an odd function.

5 **a** Is the hyperbola $y = -\frac{2}{x+1}$:

i a function? **ii** even, odd or neither? **iii** continuous?

b What are the equations of the asymptotes?

c State its domain and range.

7.02 Absolute value functions

WS Absolute value functions

An absolute value function is an example of a piecewise function with 2 sections. We were introduced to absolute value in Chapter 4, *Functions*.

The absolute value function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

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EXAMPLE 5

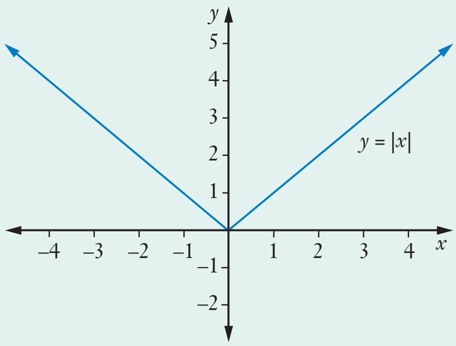
Sketch the graph of $y = |x|$ and state its domain and range.

Solution

$y = |x|$ gives the piecewise function:

$$y = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

We can draw $y = x$ for $x \geq 0$ and $y = -x$ for $x < 0$ on the same set of axes.



From the graph, notice that x can be any real number while $y \geq 0$.

Domain $(-\infty, \infty)$

Range $[0, \infty)$

Absolute value graphs

EXAMPLE 6

a Sketch the graph of $f(x) = |x| - 1$ and state its domain and range.

b Sketch the graph of $y = |x + 2|$.

Solution

a Using the definition of absolute value:

$$y = \begin{cases} x - 1 & \text{for } x \geq 0 \\ -x - 1 & \text{for } x < 0 \end{cases}$$

Draw $y = x - 1$ for $x \geq 0$ and $y = -x - 1$ for $x < 0$.

For x-intercepts, $y = 0$:

$y = x - 1$	$y = -x - 1$
$0 = x - 1$	$0 = -x - 1$
$1 = x$	$x = -1$

For y-intercept, $x = 0$:

$$\begin{aligned} y &= x - 1 \text{ for } x = 0 \\ &= 0 - 1 \\ &= -1 \end{aligned}$$

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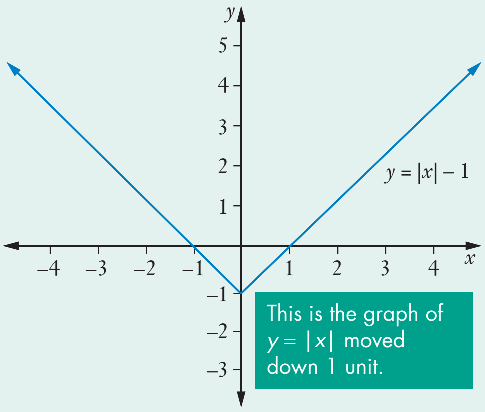
From the graph, notice that x can be any real number while $y \geq -1$.

Domain: $(-\infty, \infty)$

Range: $[-1, \infty)$



b Using the definition of absolute value:

$$y = \begin{cases} x+2 & \text{for } x+2 \geq 0 \\ -(x+2) & \text{for } x+2 < 0 \end{cases}$$

Simplifying this gives:

$$y = \begin{cases} x+2 & \text{for } x \geq -2 \\ -x-2 & \text{for } x < -2 \end{cases}$$

For x-intercepts, $y = 0$:

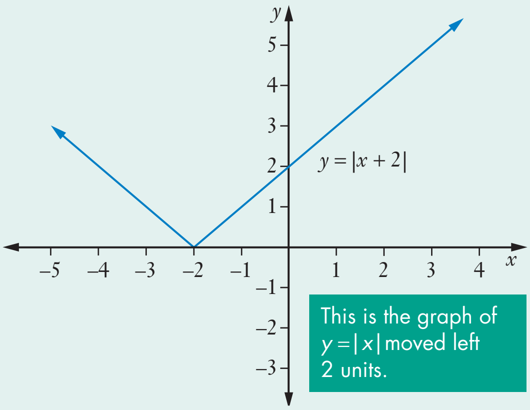
$y = x + 2$	$y = -x - 2$
$0 = x + 2$	$0 = -x - 2$
$-2 = x$	$x = -2$

For y-intercept, $x = 0$:

$y = x + 2$ for $x = 0$

$= 0 + 2$

$= 2$



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INVESTIGATION

TRANSFORMATIONS OF THE ABSOLUTE VALUE FUNCTION

Use a graphics calculator or graphing software to explore each absolute value graph.

1 $y = \|x\|$	**2** $y = 2\|x\|$	**3** $y = 3\|x\|$
4 $y = -\|x\|$	**5** $y = -2\|x\|$	**6** $y = \|x\| + 1$
7 $y = \|x\| + 2$	**8** $y = \|x\| - 1$	**9** $y = \|x\| - 2$
10 $y = \|x + 1\|$	**11** $y = \|x + 2\|$	**12** $y = \|x + 3\|$
13 $y = \|x - 1\|$	**14** $y = \|x - 2\|$	**15** $y = \|x - 3\|$

Are graphs that involve absolute value always functions? Can you find an example of one that is not a function?

Are any of them odd or even? Are they continuous? Could you predict what the graph $y = 2|x - 7|$ would look like?

Equations involving absolute values

We learned how to solve equations involving absolute values using algebra in Chapter 2. We can also solve these equations graphically.

EXAMPLE 7

Solve $|2x - 1| = 3$ graphically.

Solution

Sketch the graphs of $y = |2x - 1|$ and $y = 3$ on the same number plane.

$$y = \begin{cases} 2x - 1 & \text{for } 2x - 1 \geq 0 \\ -(2x - 1) & \text{for } 2x - 1 < 0 \end{cases}$$

Simplifying this gives:

$$y = \begin{cases} 2x - 1 & \text{for } x \geq \frac{1}{2} \\ -2x + 1 & \text{for } x < \frac{1}{2} \end{cases}$$

For x-intercepts, $y = 0$:

$$\begin{aligned} y &= 2x - 1 \\ 0 &= 2x - 1 \\ 1 &= 2x \\ \frac{1}{2} &= x \end{aligned} \qquad \begin{aligned} y &= -2x + 1 \\ 0 &= -2x + 1 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

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For y-intercept, $x = 0$:

$y = -2x + 1$ for $x = 0$

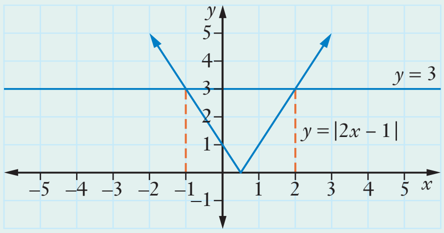
$= -2(0) + 1$

$= 1$

The graph of $y = 3$ is a horizontal line through 3 on the y-axis.

The solutions of $|2x - 1| = 3$ are the values of x at the point of intersection of the graphs.

$x = -1, 2$.



We can check that our solutions are correct by substituting them back into the equation.

EXT1 Inequalities involving absolute values

We learned how to solve inequalities involving absolute values using algebra in Chapter 2, *Equations and inequalities*. We can also solve these inequalities graphically.

WS Absolute value inequalities

Absolute value inequalities

$|x| \le a$ means $-a \le x \le a$

$|x| \ge a$ means $x \le -a$, $x \ge a$

EXAMPLE 8

Solve $|x + 1| < 2$ graphically.

Solution

Sketch $y = |x + 1|$ and $y = 2$ on the same number plane. Graph $y = |x + 1|$ first.

$$y = \begin{cases} x+1 & \text{for } x+1 \ge 0 \\ -(x+1) & \text{for } x+1 < 0 \end{cases}$$

$$y = \begin{cases} x+1 & \text{for } x \ge -1 \\ -x-1 & \text{for } x < -1 \end{cases}$$

For x-intercepts, $y = 0$:

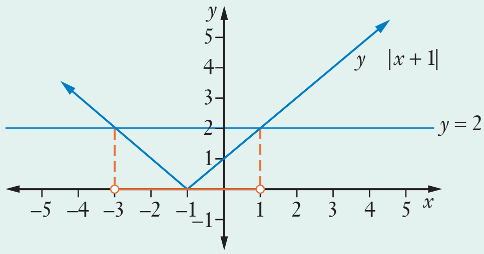
$y = x + 1$	$y = -x - 1$
$0 = x + 1$	$0 = -x - 1$
$-1 = x$	$x = -1$

For y-intercept, $x = 0$:

$$y = x + 1$$
$$= 0 + 1$$
$$= 1$$

The graph of $y = 2$ is the horizontal line with y-intercept 2. The solutions of $|x + 1| < 2$ are the values of x where the graph of $y = |x + 1|$ is below the graph of $y = 2$.

$-3 < x < 1$



Exercise 7.02 Absolute value functions

1 Find the x- and y-intercepts of the graph of each function.

a $f(x) = |x| + 7$ **b** $f(x) = |x| - 2$ **c** $y = 5|x|$

d $f(x) = -|x| + 3$ **e** $y = |x + 6|$ **f** $f(x) = |3x - 2|$

g $y = |5x + 4|$ **h** $y = |7x - 1|$ **i** $f(x) = |2x| + 9$

2 Sketch the graph of each function.

a $y = |x|$ **b** $f(x) = |x| + 1$ **c** $f(x) = |x| - 3$

d $y = 2|x|$ **e** $f(x) = -|x|$ **f** $y = |x + 1|$

g $f(x) = -|x - 1|$ **h** $y = |2x - 3|$ **i** $f(x) = |3x| + 1$

3 Find the domain and range of each function.

a $y = |x - 1|$ **b** $f(x) = |x| - 8$ **c** $f(x) = |2x + 5|$

d $y = 2|x| - 3$ **e** $f(x) = -|x - 3|$

4 Solve each equation graphically.

a $|x| = 3$ **b** $|x + 2| = 1$ **c** $|x - 3| = 0$

d $|2x - 3| = 1$ **e** $|2x + 3| = 11$ **f** $|5b - 2| = 8$

g $|3x + 1| = 2$ **h** $5 = |2x + 1|$ **i** $0 = |6t - 3|$

5 EXT1 Solve each inequality graphically.

a $|x| > 1$ **b** $|x| \leq 2$ **c** $|x - 1| < 4$

d $|x + 1| \leq 3$ **e** $|x - 2| > 2$ **f** $|x - 3| \geq 1$

g $|2x + 3| \leq 5$ **h** $|2x - 1| \geq 1$

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7.03 Circles and semicircles

The circle is not a function. It does not pass the vertical line test.

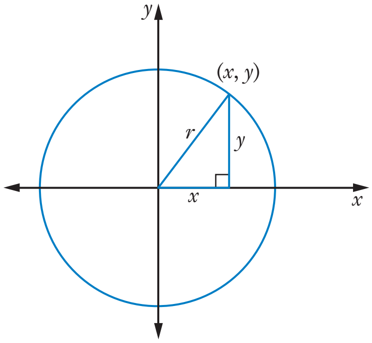
Equations of circles

Circle with centre (0, 0)

We can use Pythagoras' theorem to find the equation of a circle using a general point (x, y) on a circle with centre (0, 0) and radius r.

$$c^2 = a^2 + b^2$$

$$\therefore \quad r^2 = x^2 + y^2$$



Equation of a circle with centre (0, 0)

The equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

EXAMPLE 9

a Sketch the graph of $x^2 + y^2 = 4$.

b Why is it not a function?

c State its domain and range.

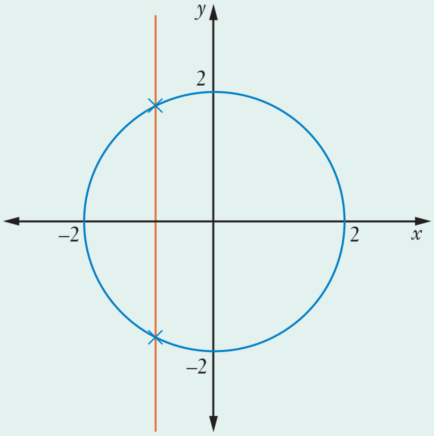
Solution

a The equation is in the form $x^2 + y^2 = r^2$ where $r^2 = 4$.

Radius $r = \sqrt{4} = 2$

This is a circle with radius 2 and centre (0, 0).

b The circle is not a function because a vertical line will cut the graph in more than one place.



c The x values for this graph lie between –2 and 2 and the y values also lie between –2 and 2.

Domain: [–2, 2]

Range: [–2, 2]

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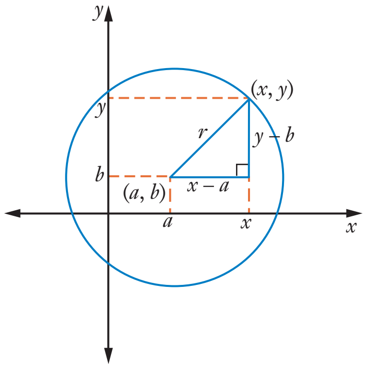
Circle with centre (*a*, *b*)

We can use Pythagoras' theorem to find the equation of a circle using a general point (x, y) on a circle with centre (a, b) and radius r.

The smaller sides of the triangle are $x - a$ and $y - b$ and the hypotenuse is r, the radius.

$c^2 = a^2 + b^2$

$r^2 = (x - a)^2 + (y - b)^2$



Equation of a circle with centre (*a*, *b*)

The equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$.

EXAMPLE 10

a **i** Sketch the graph of the circle $(x - 1)^2 + (y + 2)^2 = 4$.

ii State its domain and range.

b Find the equation of a circle with radius 3 and centre $(-2, 1)$ in expanded form.

c Find the centre and radius of the circle with equation $x^2 + 2x + y^2 - 6y - 6 = 0$.

Solution

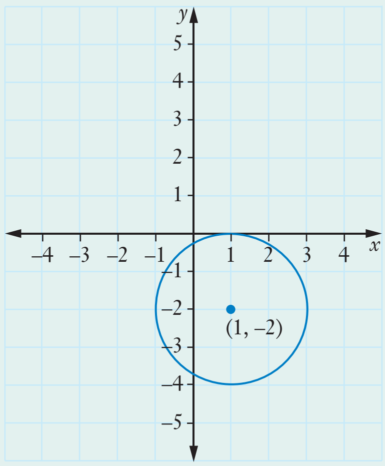
a **i** The equation is in the form $(x - a)^2 + (y - b)^2 = r^2$.

$(x - 1)^2 + (y + 2)^2 = 4$

$(x - 1)^2 + (y - (-2))^2 = 2^2$

So $a = 1$, $b = -2$ and $r = 2$.

This is a circle with centre $(1, -2)$ and radius 2.



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ii From the graph, we can see that all x values lie between –1 and 3 and all y values lie between –4 and 0.

Domain: [–1, 3]

Range: [–4, 0]

b Centre is (–2, 1) so $a = -2$ and $b = 1$.

Radius is 3 so $r = 3$.

$$(x - a)^2 + (y - b)^2 = r^2$$
$$(x - (-2))^2 + (y - 1)^2 = 3^2$$
$$(x + 2)^2 + (y - 1)^2 = 9$$

Expanding:

$$x^2 + 4x + 4 + y^2 - 2y + 1 = 9$$
$$x^2 + 4x + y^2 - 2y - 4 = 0$$

c The equation of a circle is $(x - a)^2 + (y - b)^2 = r^2$.

We need to complete the square to put the equation into this form.

To complete the square on $x^2 + 2x$, we add $\left(\frac{2}{2}\right)^2 = 1$.

To complete the square on $y^2 - 6y$, we add $\left(\frac{6}{2}\right)^2 = 9$.

$$x^2 + 2x + y^2 - 6y - 6 = 0$$
$$x^2 + 2x + y^2 - 6y = 6$$
$$x^2 + 2x + 1 + y^2 - 6y + 9 = 6 + 1 + 9$$
$$(x + 1)^2 + (y - 3)^2 = 16$$
$$(x - (-1))^2 + (y - 3)^2 = 4^2$$

This is in the form $(x - a)^2 + (y - b)^2 = r^2$ where $a = -1$, $b = 3$ and $r = 4$.

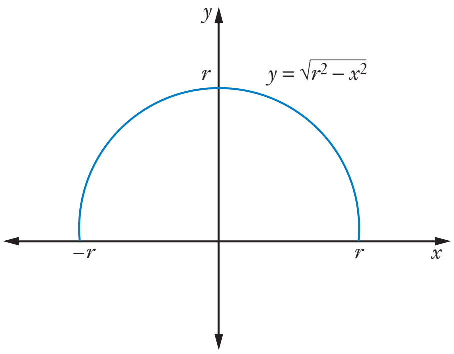
So it is a circle with centre (–1, 3) and radius 4 units.

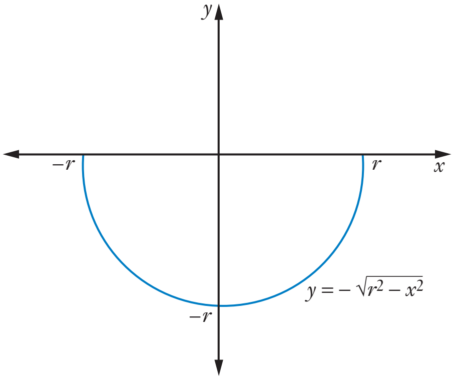
Semicircles

By rearranging the equation of a circle, we can find the equations of 2 semicircles.

$$x^2 + y^2 = r^2$$
$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

This gives 2 separate functions:





$y = \sqrt{r^2 - x^2}$ is the semicircle above the x-axis since $y \geq 0$.

$y = -\sqrt{r^2 - x^2}$ is the semicircle below the x-axis since $y \leq 0$.

Equations of a semicircle with centre (0, 0)

The equation of a semicircle above the x-axis with centre (0, 0) and radius r is

$$y = \sqrt{r^2 - x^2}.$$

The equation of a semicircle below the x-axis with centre (0, 0) and radius r is

$$y = -\sqrt{r^2 - x^2}.$$

EXAMPLE 11

Sketch the graph of each function and state the domain and range.

a $f(x) = \sqrt{9 - x^2}$ **b** $y = -\sqrt{4 - x^2}$

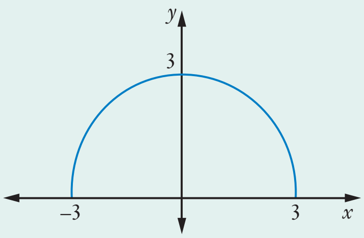
Solution

a This is in the form $f(x) = \sqrt{r^2 - x^2}$ where $r^2 = 9$, so $r = 3$.

It is a semicircle above the x-axis with centre (0, 0) and radius 3.

Domain: [–3, 3]

Range: [0, 3]



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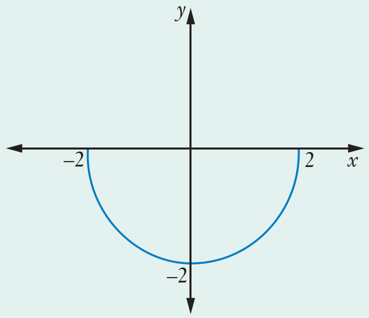
b This is in the form $y = -\sqrt{r^2 - x^2}$ where $r^2 = 4$, so $r = 2$.

It is a semicircle below the x-axis with centre (0 0) and radius .

Domain: [–2, 2]

Range: [–2, 0]



Exercise 7.03 Circles and semicircles

1 For each equation:

i sketch the graph **ii** state the domain and range.

a $x^2 + y^2 = 9$ **b** $x^2 + y^2 - 16 = 0$

c $(x - 2)^2 + (y - 1)^2 = 4$ **d** $(x + 1)^2 + y^2 = 9$

e $(x + 2)^2 + (y - 1)^2 = 1$

2 For each semicircle:

i state whether it is above or below the x-axis

ii sketch the graph

iii state the domain and range.

a $y = -\sqrt{25 - x^2}$ **b** $y = \sqrt{1 - x^2}$

c $y = \sqrt{36 - x^2}$ **d** $y = -\sqrt{64 - x^2}$

e $y = -\sqrt{7 - x^2}$

3 Find the radius and the centre of each circle.

a $x^2 + y^2 = 100$ **b** $x^2 + y^2 = 5$

c $(x - 4)^2 + (y - 5)^2 = 16$ **d** $(x - 5)^2 + (y + 6)^2 = 49$

e $x^2 + (y - 3)^2 = 81$

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4 Find the equation of each circle in expanded form.

- **a** centre (0, 0) and radius 4
- **b** centre (3, 2) and radius 5
- **c** centre (–1, 5) and radius 3
- **d** centre (2, 3) and radius 6
- **e** centre (–4, 2) and radius 5
- **f** centre (0, –2) and radius 1
- **g** centre (4, 2) and radius 7
- **h** centre (–3, –4) and radius 9
- **i** centre (–2, 0) and radius $\sqrt{5}$
- **j** centre (–4, –7) and radius $\sqrt{3}$

5 Find the radius and the centre of each circle.

- **a** $x^2 - 4x + y^2 - 2y - 4 = 0$
- **b** $x^2 + 8x + y^2 - 4y - 5 = 0$
- **c** $x^2 + y^2 - 2y = 0$
- **d** $x^2 - 10x + y^2 + 6y - 2 = 0$
- **e** $x^2 + 2x + y^2 - 2y + 1 = 0$
- **f** $x^2 - 12x + y^2 = 0$
- **g** $x^2 + 6x + y^2 - 8y = 0$
- **h** $x^2 + 20x + y^2 - 4y + 40 = 0$
- **i** $x^2 - 14x + y^2 + 2y + 25 = 0$
- **j** $x^2 + 2x + y^2 + 4y - 5 = 0$

6 Find the centre and radius of the circle with equation:

- **a** $x^2 - 6x + y^2 + 2y - 6 = 0$
- **b** $x^2 - 4x + y^2 - 10y + 4 = 0$
- **c** $x^2 + 2x + y^2 + 12y - 12 = 0$
- **d** $x^2 - 8x + y^2 - 14y + 1 = 0$

7 Sketch the circle whose equation is given by $x^2 + 4x + y^2 - 2y + 1 = 0$.

WS Advanced graphs

WS Matching graphs (Advanced)

7.04 Reflections of functions

The graph of $y = -f(x)$

EXAMPLE 12

For each function, sketch the graph of $y = f(x)$ and $y = -f(x)$ on the same number plane.

a $f(x) = x^2 - 2x$ **b** $f(x) = x^3$

Solution

a $f(x) = x^2 - 2x$ is a concave upwards parabola.

For x-intercepts: $f(x) = 0$

$$x^2 - 2x = 0$$

$$x(x - 2) = 0$$

$$x = 0, 2$$

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For y-intercept, $x = 0$:

$f(0) = 0^2 - 2(0) = 0$

Axis of symmetry at $x = 1$ (halfway between 0 and 2):

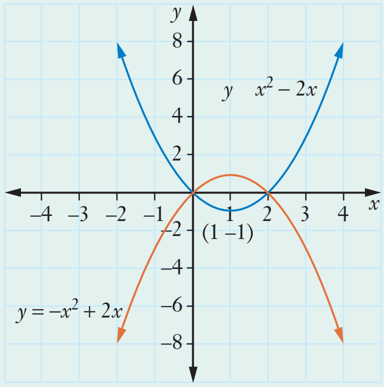
$f(1) = 1^2 - 2(1) = -1$

Minimum turning point at (1, –1).

$y = -f(x)$

$= -(x^2 - 2x)$

$= -x^2 + 2x$



$y = -x^2 + 2x$ is a concave downwards parabola also with x-intercepts 0, 2 with y-intercept 0 and axis of symmetry at $x = 1$.

$f(1) = -1^2 + 2(1) = 1$

Maximum turning point at (1, 1).

Draw both graphs on the same set of axes.

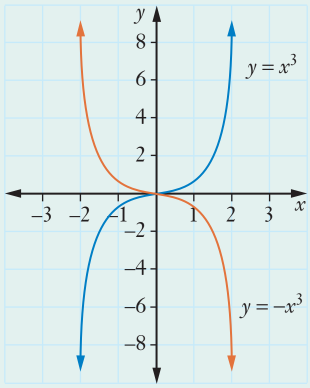
b $f(x) = x^3$ is a cubic curve with a point of inflection at (0, 0).

x	–3	–2	–1	0	1	2	3
y	–27	–8	–1	0	1	8	27

$y = -f(x)$

$= -x^3$

x	–3	–2	–1	0	1	2	3
y	27	8	1	0	–1	–8	–27



$y = -f(x)$ changes the sign of the y values of the original function: from positive to negative, or negative to positive. On the number plane, this means reflecting the graph in the x-axis.

The graph of $y = -f(x)$

The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

The graph of $y = f(-x)$

We have already seen that some functions are even or odd by finding $f(-x)$ We can see the relationship between $f(x)$ and $f(-x)$ by drawing their graphs

hehf

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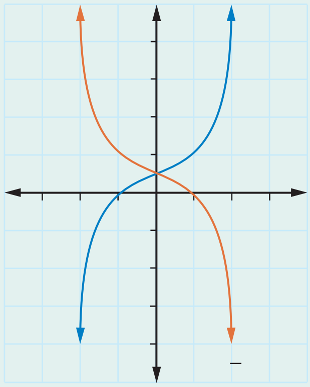

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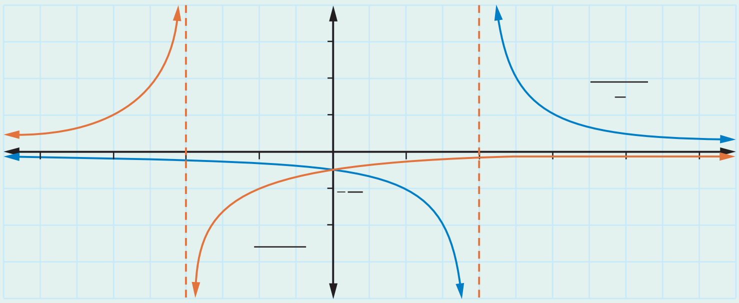

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$y = f(-x)$ changes the sign of the x value of the original function: from positive to negative, or negative to positive. On the number plane, this means reflecting the graph in the y-axis.

The graph of $y = f(-x)$

The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

The graph of $y = -f(-x)$

EXAMPLE 14

For each function, sketch the graph of $y = f(x)$ and $y = -f(-x)$ on the same number plane.

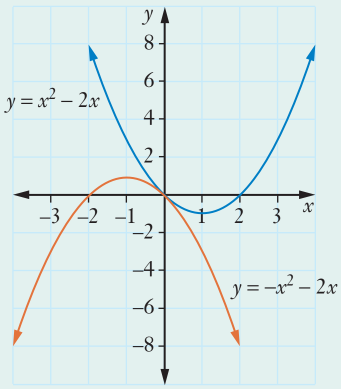
a $f(x) = x^2 - 2x$ **b** $f(x) = \dfrac{2}{x+1}$

Solution

a From Example **12a**, $f(x) = x^2 - 2x$ is a concave upwards parabola with x-intercepts 0, 2.

$$\begin{aligned} y &= -f(-x) \\ &= -\left([-x]^2 - 2[-x]\right) \\ &= -\left(x^2 + 2x\right) \\ &= -x^2 - 2x \end{aligned}$$

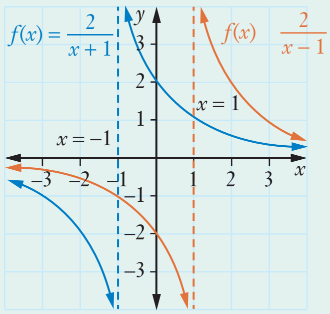
A concave downwards parabola with x-intercepts 0, −2.



b $f(x) = \dfrac{2}{x+1}$

This is a hyperbola with asymptotes at $x = -1$, $y = 0$ and y-intercept $f(0) = 2$.

$$\begin{aligned} y &= -f(-x) \\ &= -\frac{2}{-x+1} \\ &= \frac{2}{x-1} \end{aligned}$$



This is a hyperbola with asymptotes at $x = 1$, $y = 0$ and y-intercept $f(0) = -2$.

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The graph of $y = -f(-x)$

$y = -f(-x)$ is a reflection of the graph of $y = f(x)$ in both the x- and y-axes.

CLASS DISCUSSION

REFLECTIONS OF FUNCTIONS

Use a graphics calculator or graphing software to draw the graphs of different functions $y = f(x)$ together with:

1 $y = -f(x)$

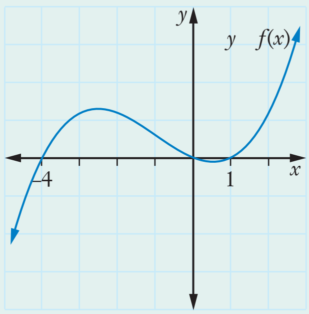
2 $y = f(-x)$

3 $y = -f(-x)$.

Are any of these functions the same as $y = f(x)$ if $f(x)$ is an even or odd function? Why?

EXAMPLE 15

The graph of $y = f(x)$ is shown below.

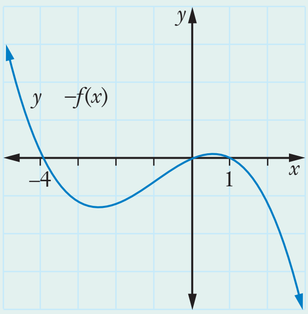


Sketch the graph of:

a $y = -f(x)$ **b** $y = f(-x)$ **c** $y = -f(-x)$

Solution

a $y = -f(x)$ is a reflection in the x-axis



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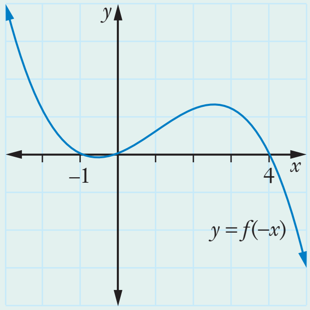
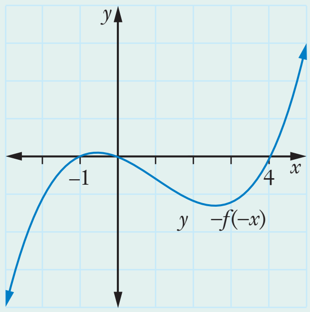
b $y = f(-x)$ is a reflection in the y-axis



c $y = -f(-x)$ is a reflection in both the x-and y-axes Using the graph from **a** that has been reflected in the x-axis and reflecting it in the y-axis gives the graph below.



Exercise 7.04 Reflections of functions

1 For each function, find the equation of:

i $y = -f(x)$ **ii** $y = f(-x)$ **iii** $y = -f(-x)$

a $f(x) = x^2 - 2$ **b** $f(x) = (x + 1)^3$ **c** $y = 5x - 3$

d $y = |2x + 5|$ **e** $f(x) = \dfrac{1}{x - 1}$

2 Describe the type of reflection that each function has on $y = f(x)$.

a $y = -f(x)$ **b** $y = f(-x)$ **c** $y = -f(-x)$

3 Sketch the graphs of the function $f(x) = (x - 1)^2$ and $y = -f(-x)$ on the same number plane.

4 Sketch the graphs of the function $f(x) = 1 - x^3$ and $y = -f(x)$ on the same number plane.

5 For the function $f(x) = x^2 + 2x$, sketch the graph of:

a $y = f(x)$ **b** $y = -f(x)$ **c** $y = f(-x)$ **d** $y = -f(-x)$

6 **a** Show that $f(x) = 2x^2$ is an even function.

b Find the equation of:

i $y = f(-x)$ **ii** $y = -f(x)$

c Sketch the graph of $y = -f(-x)$.

7 **a** Show that $f(x) = -x^3$ is an odd function.

b Find the equation of:

i $y = -f(x)$ **ii** $y = -f(-x)$

c Sketch the graph of $y = f(-x)$.

8 **a** Find the x- and y-intercepts of the graph of $f(x) = x^3 - 7x^2 + 12x$ and sketch the graph.

b Sketch the graph of:

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = -f(-x)$

EXT1 **9** **a** Write $P(x) = x^3 - 3x^2 + 4$ as a product of its factors.

b Sketch its graph.

c Sketch the graph of:

i $y = P(-x)$ **ii** $y = -P(x)$ **iii** $y = -P(-x)$

WS Circles and composite functions

7.05 Combined and composite functions

Sometimes we use different operations to combine 2 different functions.

EXAMPLE 16

For $f(x) = 2x^2 - x + 1$ and $g(x) = x^3 - 2$, write each combined function below as a polynomial and find its degree and constant term.

a $y = f(x) + g(x)$ **b** $y = f(x) - g(x)$ **c** $y = f(x)g(x)$

Solution

a $y = f(x) + g(x)$

$= 2x^2 - x + 1 + x^3 - 2$

$= x^3 + 2x^2 - x - 1$

This polynomial has degree 3 and constant term -1.

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b $y = f(x) - g(x)$

$= 2x^2 - x + 1 - (x^3 - 2)$

$= 2x^2 - x + 1 - x^3 + 2$

$= -x^3 + 2x^2 - x + 3$

This polynomial has degree 3 and constant term 3.

c $y = f(x)g(x)$

$= (2x^2 - x + 1)(x^3 - 2)$

$= 2x^5 - 4x^2 - x^4 + 2x + x^3 - 2$

$= 2x^5 - x^4 + x^3 - 4x^2 + 2x - 2$

This polynomial has degree 5 and constant term −2.

We could also find the degree by multiplying just the 2 leading terms: $2x^2 \times x^3 = 2x^5$ and find the constant term by multiplying just the 2 constant terms: $1 \times (-2) = (-2)$.

EXAMPLE 17

a Find the domain and range of each function below given $f(x) = x^2 - x - 2$ and $g(x) = x - 2$.

i $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$ **iii** $y = f(x)g(x)$

b Find the domain of $y = \dfrac{f(x)}{g(x)}$ if $f(x) = x^3 + 1$ and $g(x) = x^2 - x - 6$.

Solution

a **i** $y = f(x) + g(x)$

$= x^2 - x - 2 + x - 2$

$= x^2 - 4$

This is a quadratic function with a minimum turning point at (0, −4).

Domain $(-\infty, \infty)$, range $[-4, \infty)$

ii $y = f(x) - g(x)$

$= x^2 - x - 2 - (x - 2)$

$= x^2 - x - 2 - x + 2$

$= x^2 - 2x$

This is a quadratic function with x-intercepts 0, 2.

Axis of symmetry: $x = 1$

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Minimum value:

$f(1) = 1^2 - 2(1)$

$= -1$

Domain $(-\infty, \infty)$, range $[-1, \infty)$

iii $y = f(x)g(x)$

$= (x^2 - x - 2)(x - 2)$

$= x^3 - 2x^2 - x^2 + 2x - 2x + 4$

$= x^3 - 3x^2 + 4$

This is a cubic function.

Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

b $y = \dfrac{f(x)}{g(x)}$

$= \dfrac{x^3 + 1}{x^2 - x - 6}$

For domain: $x^2 - x - 6 \neq 0$

$(x - 3)(x + 2) \neq 0$

$x \neq 3, -2$

So domain is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

Composite functions

A **composite function** $f(g(x))$ is a relationship between functions where the output of one function $g(x)$ becomes the input of a second function $f(x)$.

EXAMPLE 18

a Find the composite function $f(g(x))$ given:

i $f(x) = x^2$ and $g(x) = 2x - 5$

ii $f(x) = x^3$ and $g(x) = x^2 + 3$

iii $f(x) = 5x - 3$ and $g(x) = x^3 + 2$

b Given $f(x) = 5x + 2$ and $g(x) = \dfrac{1}{x}$, find:

i $f(g(x))$ **ii** $g(f(x))$

c Find the domain and range of $f(g(x))$ given $f(x) = \sqrt{x}$ and $g(x) = 9 - x^2$.

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Solution

a **i** $f(g(x)) = (2x - 5)^2$
$= 4x^2 - 20x + 25$

ii $f(g(x)) = (x^2 + 3)^3$
$= (x^2 + 3)(x^2 + 3)^2$
$= (x^2 + 3)(x^4 + 6x^2 + 9)$
$= x^6 + 9x^4 + 27x^2 + 27$

iii $f(g(x)) = 5(x^3 + 2) - 3$
$= 5x^3 + 10 - 3$
$= 5x^3 + 7$

b **i** $f(g(x)) = 5\left(\dfrac{1}{x}\right) + 2$
$= \dfrac{5}{x} + 2$

ii $g(f(x)) = \dfrac{1}{5x + 2}$

c $f(g(x)) = \sqrt{9 - x^2}$

This is a semicircle above the x-axis with centre (0, 0) and radius 3.

Domain [–3, 3], range [0, 3]

Exercise 7.05 Combined and composite functions

1 For each pair of functions, find the combined function:

i $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$

iii $y = f(x)g(x)$ **v** $y = \dfrac{f(x)}{g(x)}$

a $f(x) = 4x + 1$ and $g(x) = 2x^2 + x$

b $f(x) = x^4 + 5x - 4$ and $g(x) = x^3 + 5$

c $f(x) = x^2 + 3$ and $g(x) = 5x^2 - 7x - 2$

d $f(x) = 3x^2 + 2x - 1$ and $g(x) = x^2 - x + 5$

e $f(x) = 4x^5 + 7$ and $g(x) = 3x - 4$

2 For each pair of functions, find the degree of:

i $f(x) + g(x)$ **ii** $f(x) - g(x)$ **iii** $f(x)g(x)$ without expanding

a $f(x) = 2x + 1$ and $g(x) = 5x - 7$

b $f(x) = x^2$ and $g(x) = 3x + 4$

c $f(x) = (x - 3)^2$ and $g(x) = x^2 - 6x + 1$

d $f(x) = 2x^3$ and $g(x) = x - 2$

3 For each pair of functions, find the constant term of:

i $f(x) + g(x)$ **ii** $f(x) - g(x)$ **iii** $f(x)g(x)$ without expanding

a $f(x) = 5x^2 + 4$ and $g(x) = x - 7$

b $f(x) = 3x^2 + 1$ and $g(x) = 2x - 5$

c $f(x) = (2x - 5)^2$ and $g(x) = 4x - 3$

d $f(x) = x^3 + 7$ and $g(x) = 2x^2$

4 Find the domain and range of $y = f(x) + g(x)$ given:

a $f(x) = x + 2$ and $g(x) = x - 4$

b $f(x) = 2x^2 + x - 1$ and $g(x) = -x - 1$

c $f(x) = x^3$ and $g(x) = x + 2$

d $f(x) = x^2 - 1$ and $g(x) = x - 1$

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5 Find the domain and range of $y = f(x) - g(x)$ given:

a $f(x) = 3x + 2$ and $g(x) = x - 1$

b $f(x) = x^2 - 1$ and $g(x) = x - 1$

c $f(x) = x^3 + x$ and $g(x) = x + 2$

d $f(x) = 3x^2 - x - 1$ and $g(x) = x^2 + x + 3$

6 Find the domain and range of $y = f(x)g(x)$ given:

a $f(x) = x + 2$ and $g(x) = x - 4$

b $f(x) = x - 5$ and $g(x) = x + 5$

c $f(x) = x^2$ and $g(x) = x$

7 Find the domain of $y = \dfrac{f(x)}{g(x)}$ given:

a $f(x) = 5$ and $g(x) = x - 4$

b $f(x) = x - 1$ and $g(x) = x + 1$

c $f(x) = 2x$ and $g(x) = x - 3$

d $f(x) = x + 3$ and $g(x) = x^3$

8 Find the composite function $f(g(x))$ given:

a $f(x) = x^2$ and $g(x) = x^2 + 1$

b $f(x) = x^3$ and $g(x) = 5x - 3$

c $f(x) = x^7$ and $g(x) = x^2 - 3x + 2$

d $f(x) = \sqrt{x}$ and $g(x) = 2x - 1$

e $f(x) = \sqrt[3]{x}$ and $g(x) = x^4 + 7x^2 - 4$

f $f(x) = 3x$ and $g(x) = 2x + 1$

g $f(x) = 2x - 7$ and $g(x) = x^3$

h $f(x) = 6x - 5$ and $g(x) = x^2$

i $f(x) = 2x^2$ and $g(x) = 3x$

j $f(x) = 4x^2 + 1$ and $g(x) = x^2 + 3$

9 Find the domain and range of the composite function $f(g(x))$ given that:

a $f(x) = x^2$ and $g(x) = x - 1$

b $f(x) = x^3$ and $g(x) = x + 5$

c $f(x) = \sqrt{x}$ and $g(x) = x - 2$

d $f(x) = -\sqrt{x}$ and $g(x) = 3x + 9$

e $f(x) = \sqrt{x}$ and $g(x) = 4 - x^2$

f $f(x) = -\sqrt{x}$ and $g(x) = 1 - x^2$

10 If $f(x) = \sqrt{x}$ and $g(x) = x^3$, find:

a $f(g(x))$

b $g(f(x))$

11 If $f(x) = \dfrac{1}{x}$ and $g(x) = x^2 + 3$, find:

a $y = f(x)g(x)$

b $y = f(g(x))$

c $y = \dfrac{f(x)}{g(x)}$

d $y = \dfrac{g(x)}{f(x)}$

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EXT1 # 7.06 Sums and products of functions

Now we will sketch the graph of the sums and products of functions.

WS Sums and products of functions

Sum of functions

EXAMPLE 19

Sketch the graph of $y = f(x) + g(x)$ where $f(x) = x^3 + 1$ and $g(x) = x^2 - 2x - 3$.

Solution

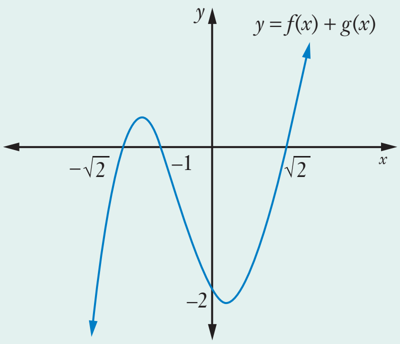
Method 1: Algebraic method

$y = f(x) + g(x)$
$= x^3 + 1 + x^2 - 2x - 3$
$= x^3 + x^2 - 2x - 2$

For x-intercepts, $y = 0$:
$0 = x^3 + x^2 - 2x - 2$
$= x^2(x + 1) - 2(x + 1)$
$= (x + 1)(x^2 - 2)$
$x = -1, \quad x^2 = 2$
$x = \pm\sqrt{2}$

For y = intercept, $x = 0$:
$y = 0^3 + 0^2 - 2(0) - 2$
$= -2$

$y = x^3 + x^2 - 2x - 2$ is a cubic function with an odd degree and a positive leading coefficient, so the graph points down on the left end and up on the right end.



For more detail, we could complete a table of values.

x	−4	−3	−2	−1	0	1	2	3	4
$f(x) = x^3 + 1$	−63	−26	−7	0	1	2	9	28	65
$g(x) = x^2 - 2x - 3$	21	12	5	0	−3	−4	−3	0	5
$y = f(x) + g(x)$	−42	−14	−2	0	−2	−2	6	28	60

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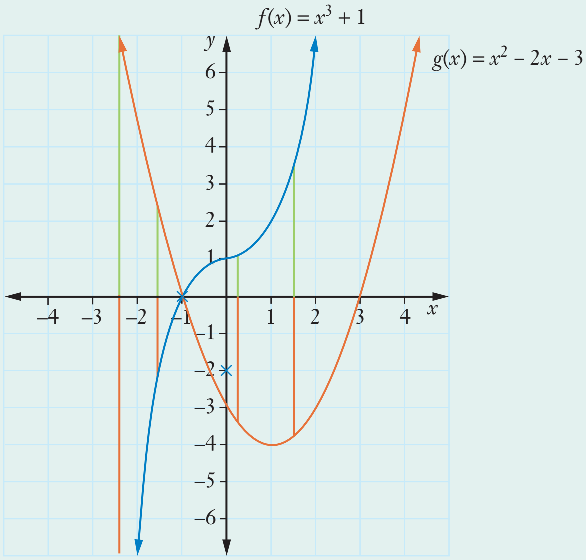
Method 2: Adding graphs

Sketch the graphs of $f(x) = x^3 + 1$ and $g(x) = x^2 - 2x - 3$ on the same number plane.

$f(x) = x^3 + 1$ is a cubic function with a point of inflection at (0, 1) and x-intercept -1.

$g(x) = x^2 - 2x - 3$ is a quadratic function with x-intercepts 3, -1 and y-intercept -3.

We add the y-values (heights) of the 2 graphs. Where the graph is below the y-axis, these values are negative (shown by orange lines).



On the LHS, $f(x) \to -\infty$ (leading term x^3) and $g(x) \to \infty$ (leading term x^2). However, $x^3 > x^2$ (orange lines are longer than the green lines) so $f(x) + g(x) \to -\infty$. Around $x = -1.5$, the green lines become longer than the orange lines so $f(x) + g(x) > 0$.

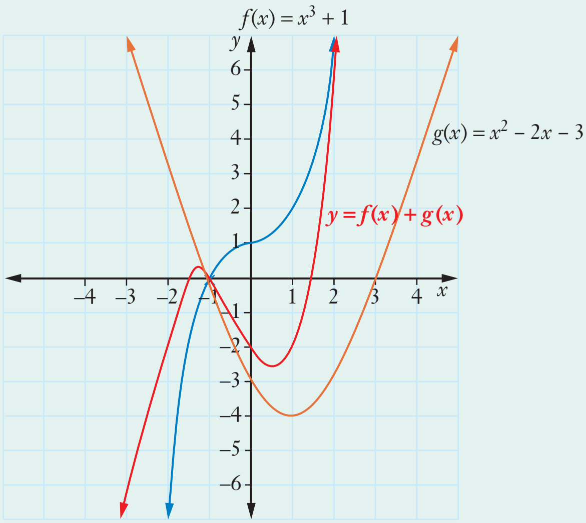
At $x = -1$, $f(x) + g(x) = 0 + 0 = 0$.

For $x > -1$, $f(x) + g(x) < 0$ (orange line longer again) until around $x = 1.5$ when $f(x) + g(x) > 0$ (green line longer again).

At $x = 0$, $f(x) + g(x) = 1 + (-3) = -2$

At $x = 3$, $f(x) + g(x) = f(x) + 0 = f(x)$

On the RHS, both $f(x)$ and $g(x)$ increase to ∞ so $f(x) + g(x)$ increases even faster to ∞. Drawing this information gives the general shape of the function $y = f(x) + g(x)$.



Check your answer using graphing software or website.

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Products of functions

EXAMPLE 20

Sketch the graph of $y = f(x)g(x)$ where $f(x) = x - 2$ and $g(x) = x^2 + 2x + 4$.

Solution

Method 1: Algebraic method

$y = f(x)g(x)$

$= (x - 2)(x^2 + 2x + 4)$

$= x^3 + 2x^2 + 4x - 2x^2 - 4x - 8$

$= x^3 - 8$

This is a cubic function with a point of inflection at (0, –8).

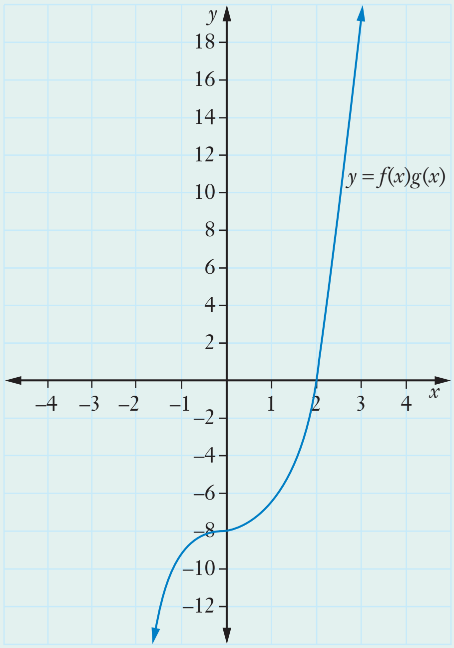
For x-intercepts, $y = 0$:

$0 = x^3 - 8$

$2 = x$

For y-intercept, $x = 0$:

$y = -8$



For more detail, we could complete a table of values.

x	–4	–3	–2	–1	0	1	2	3	4
$f(x) = x - 2$	–6	–5	–4	–3	–2	–1	0	1	2
$g(x) = x^2 + 2x + 4$	12	7	4	3	4	7	16	19	28
$y = f(x)g(x)$	–72	–35	–16	–9	–8	–7	0	19	56

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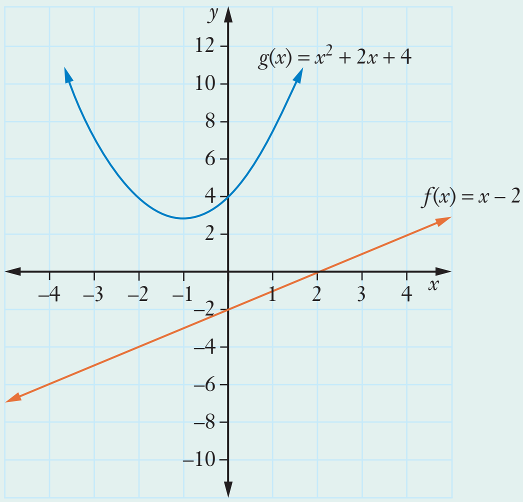
Method 2: Multiplying graphs

Sketch the graphs of $f(x) = x - 2$ and $g(x) = x^2 - 2x + 4$ on the same set of axes. $f(x) = x - 2$ is a linear function with gradient 1 and y-intercept -2.



$g(x) = x^2 + 2x + 4$ is a quadratic function with y-intercept 4 and no x-intercepts.

Axis of symmetry:

$$x = -\frac{b}{2a}$$
$$= -\frac{2}{2(1)}$$
$$= -1$$

$g(-1) = (-1)^2 + 2(-1) + 4 = 3$

Vertex at $(-1, 3)$

We multiply the y values (heights) of the 2 graphs, noting their signs. Where the graph is below the y-axis, these values are negative.

On the LHS, $f(x) \to -\infty$ (leading term x) and $g(x) \to \infty$ (leading term x^2). So $f(x)g(x) \to -\infty$.

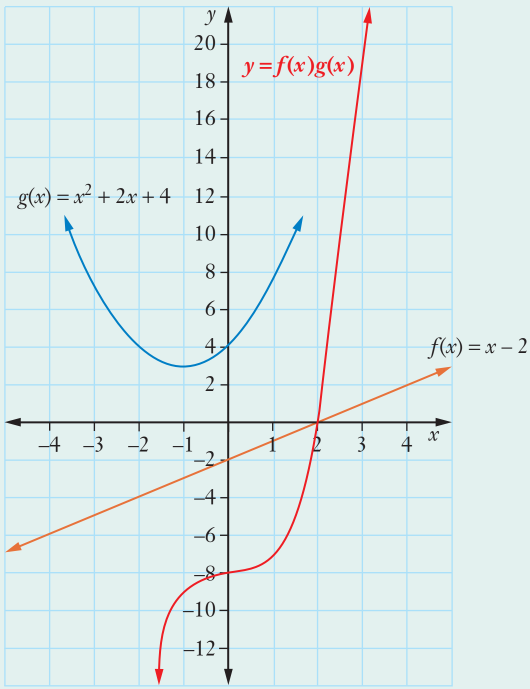
At $x = -1$, $f(x)g(x) = (-3) \times 3 = (-9)$.

At $x = 0$, $f(x)g(x) = (-2) \times 4 = (-8)$.

At $x = 2$, $f(x)g(x) = 0 \times 12 = 0$.

For $x > 2$, $f(x) > 0$ and $g(x) > 0$ so $f(x)g(x) > 0$.

On the RHS, both $f(x)$ and $g(x) \to \infty$ so $f(x)g(x) \to \infty$. Drawing this information gives the general shape of the function.



Check your answer using graphing software or website.

EXT1 Exercise 7.06 Sums and products of functions

1 Sketch the graph of $y = f(x) + g(x)$ given:

a $f(x) = 3x + 5$ and $g(x) = x - 1$ **b** $f(x) = x - 2$ and $g(x) = x^2$

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c $f(x) = 3x$ and $g(x) = x^2 + x$

d $f(x) = -x - 3$ and $g(x) = -x^2 - 3x - 1$

e $f(x) = 3x - 4$ and $g(x) = x^2 + 2x - 2$

f $f(x) = -x^2 - 5$ and $g(x) = 2x + 1$

g $f(x) = -x + 1$ and $g(x) = x^3 + x + 7$

h $f(x) = x^3 - 4x - 1$ and $g(x) = x^2 - 3$

2 Sketch the graph of $y = f(x)g(x)$ if:

a $f(x) = x + 1$ and $g(x) = x + 5$

b $f(x) = x - 3$ and $g(x) = 2x + 4$

c $f(x) = x^2 - 1$ and $g(x) = x + 3$

d $f(x) = -x - 2$ and $g(x) = x^2 + 2x - 3$

e $f(x) = x^2$ and $g(x) = x - 2$

f $f(x) = x + 4$ and $g(x) = x^2 - 6x + 5$

g $f(x) = -x^2$ and $g(x) = x^2 - 2x - 8$

h $f(x) = x^3$ and $g(x) = x^2 + 2x$

3 For the functions $f(x) = x + 1$ and $g(x) = x^2 - 3$, sketch the graphs of:

a $y = f(x) + g(x)$

b $y = f(x)g(x)$

EXT1 7.07 Reciprocal functions

WS Reciprocal functions

The reciprocal function of $y = f(x)$ is $y = \dfrac{1}{f(x)}$.

The hyperbola $f(x) = \dfrac{k}{bx + c}$ is a reciprocal function. As with a hyperbola, the graph of $y = \dfrac{1}{f(x)}$ has a vertical asymptote where $f(x) = 0$. It also has a horizontal asymptote at where $y = 0$ because $\dfrac{1}{f(x)} \neq 0$.

EXAMPLE 21

Reciprocal functions

Sketch the graph of $y = \dfrac{1}{f(x)}$ if $f(x) = x^2 + 2x - 3$, and find its domain and range.

Solution

$$y = \frac{1}{f(x)} = \frac{1}{x^2 + 2x - 3}$$

First sketch the graph of $f(x) = x^2 + 2x - 3$.

For x-intercepts, $y = 0$:

$0 = x^2 + 2x - 3$

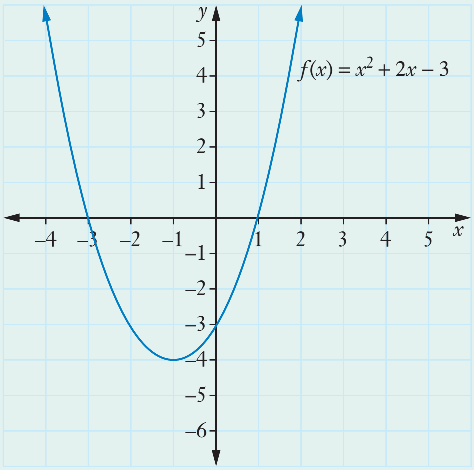
$= (x + 3)(x - 1)$

$x = -3, 1$

For y-intercept, $x = 0$:

$y = 0^2 + 2(0) - 3$

$= -3$



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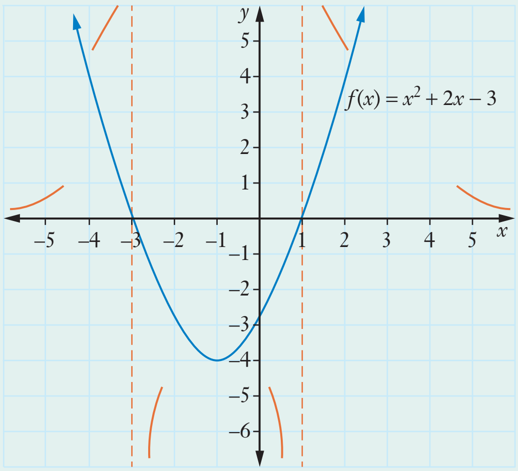
Axis of symmetry: $x = -1$ (halfway between x-intercepts)

$f(-1) = (-1)^2 + 2(-1) - 3 = -4$

Vertex is $(-1, -4)$

Now use this graph to sketch the graph of $y = \dfrac{1}{f(x)}$.



$f(x) = 0$ at the x-intercepts -3 and 1, so the reciprocal function $y = \dfrac{1}{f(x)}$ is undefined there. Draw vertical asymptotes at $x = -3, 1$.

Looking at the limiting behaviour:

As $x \to -\infty, f(x) \to \infty$, so $\dfrac{1}{f(x)} \to 0^+$ (0 from the positive direction).

As $x \to \infty, f(x) \to \infty$, so $\dfrac{1}{f(x)} \to 0^+$.

Looking at the behaviour near the asymptotes:

As x approaches -3 from the negative (left) side, $f(x) > 0$ and small, so $\dfrac{1}{f(x)} > 0$ and large.

As $x \to -3^-, \dfrac{1}{f(x)} \to \infty$.

You can check this by substituting values of x very close to -3.

As x approaches -3 from the positive (right) side, $f(x) < 0$ and small, so $\dfrac{1}{f(x)} < 0$ and large.

As $x \to -3^+, \dfrac{1}{f(x)} \to -\infty$.

Similarly:

As $x \to 1^-, \dfrac{1}{f(x)} \to -\infty$.

As $x \to 1^+, \dfrac{1}{f(x)} \to \infty$.

The vertex of the parabola is $(-1, -4)$, so the corresponding point on the reciprocal function $y = \dfrac{1}{f(x)}$ is $\left(-1\ -\dfrac{1}{4}\right)$.

The y-intercept of the parabola is -3, so the y-intercept of $y = \dfrac{1}{f(x)}$ is $-\dfrac{1}{3}$.

As values of $f(x)$ increase, the values of $\dfrac{1}{f(x)}$ decrease, and vice versa. This means that where $y = f(x)$ has a minimum turning point, $y = \dfrac{1}{f(x)}$ will have a maximum turning point.

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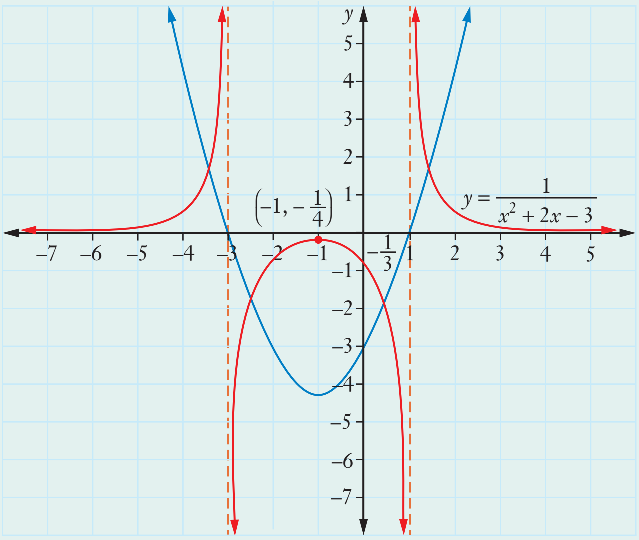
Putting all this information together gives the graph.



Domain $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$

Range $(-\infty, -\frac{1}{4}) \cup (0, \infty)$

Properties of reciprocal functions

- As $f(x)$ increases, $\frac{1}{f(x)}$ decreases.
- As $f(x)$ decreases, $\frac{1}{f(x)}$ increases.
- As $f(x) \to \pm\infty$, then $\frac{1}{f(x)} \to 0$.
- Where the graph of $y = f(x)$ has x-intercepts, the graph of $y = \frac{1}{f(x)}$ will have vertical asymptotes at these x values.
- If $y = f(x)$ has vertex (a, b), then $y = \frac{1}{f(x)}$ will have vertex $\left(a \;\; \frac{1}{b}\right)$.
- If $y = f(x)$ has a maximum or minimum turning point, then $y = \frac{1}{f(x)}$ will have a minimum or maximum turning point respectively at the same x value.

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EXAMPLE 22

Sketch the graph of $y = \frac{1}{f(x)}$ if $f(x) = -(x - 2)^2$, and find its domain and range.

Solution

$$y = \frac{1}{f(x)} = \frac{1}{-(x-2)^2} = -\frac{1}{(x-2)^2}$$

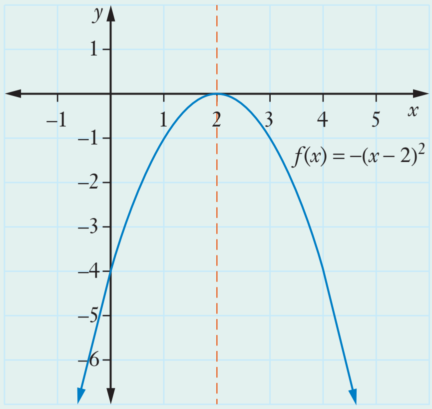
First sketch $f(x) = -(x - 2)^2$, a concave downward parabola.

For x-intercepts, $f(x) = 0$:

$$0 = -(x - 2)^2$$

$$x - 2 = 0$$

$$x = 2$$



For y-intercepts, $x = 0$:

$f(0) = -(0 - 2)^2 = -4$

Axis of symmetry: $x = 2$

Vertex: $(2, 0)$

$y = \frac{1}{f(x)}$ will have an asymptote at $x = 2$.

Looking at the limiting behaviour:

As $x \to \pm\infty$, $f(x) \to -\infty$

So $\frac{1}{f(x)} \to 0^-$.

As x approaches 2 from the negative side, $f(x) < 0$ so $\frac{1}{f(x)} < 0$.

As $x \to 2^-$, $\frac{1}{f(x)} \to -\infty$.

As x approaches 2 from the positive side, $f(x) < 0$ (still) so $\frac{1}{f(x)} < 0$.

As $x \to 2^+$, $\frac{1}{f(x)} \to -\infty$.

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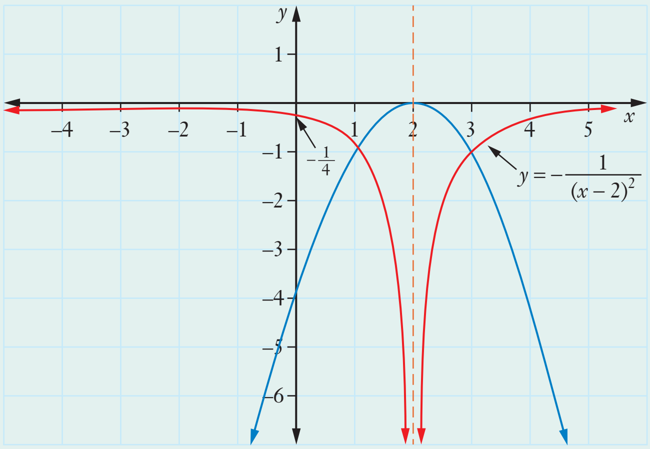
The y-intercept of $y = f(x)$ is -4 so the y-intercept of $y = \dfrac{1}{f(x)}$ is $-\dfrac{1}{4}$.

When $x = 0$, $y = -\dfrac{1}{4}$, so $\left(0 \quad -\dfrac{1}{4}\right)$.



Domain $(-\infty, 2) \cup (2, \infty)$ $x \neq 2$

Range $(-\infty, 0)$ $y < 0$

EXAMPLE 23

Sketch the graph of $y = \dfrac{1}{f(x)}$ if $f(x) = x(x + 1)(x - 2)$ and find its domain and range.

Solution

$$y = \frac{1}{f(x)} = \frac{1}{x(x+1)(x-2)}$$

First sketch the graph of $f(x) = x(x + 1)(x - 2)$.

This is a cubic function.

For x-intercept, $y = 0$:

$0 = x(x + 1)(x - 2)$

$x = 0, -1, 2$

For y-intercept, $x = 0$:

$y = 0(0 + 1)(0 - 2)$

$= 0$

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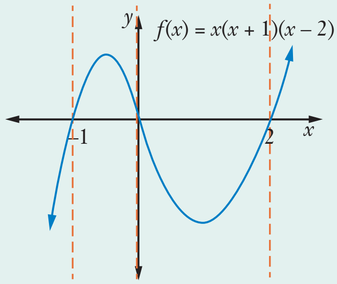
$f(x) = x(x + 1)(x - 2)$ has an odd degree and a positive leading term, so its graph points down on the left end and points up on the right end.

$y = \dfrac{1}{x(x+1)(x-2)}$ has asymptotes at $x = -1$, 0 and 2.



Looking at the limiting behaviour:

As $x \to -\infty, f(x) \to -\infty$.

So $\dfrac{1}{f(x)} \to 0^-$

As $x \to \infty, f(x) \to \infty$

So $\dfrac{1}{f(x)} \to 0^+$.

Looking at the behaviour near the asymptotes:

As $x \to -1^-, f(x) < 0$, so $\dfrac{1}{f(x)} \to -\infty$.

As $x \to -1^+, f(x) > 0$, so $\dfrac{1}{f(x)} \to \infty$.

As $x \to 0^-, f(x) > 0$, so $\dfrac{1}{f(x)} \to \infty$.

As $x \to 0^+, f(x) < 0$, so $\dfrac{1}{f(x)} \to -\infty$.

As $x \to 2^-, f(x) < 0$, so $\dfrac{1}{f(x)} \to -\infty$.

As $x \to 2^+, f(x) > 0$, so $\dfrac{1}{f(x)} \to \infty$.

We don't have enough information to find the values of the turning points in a cubic function, so we can only draw a rough sketch.

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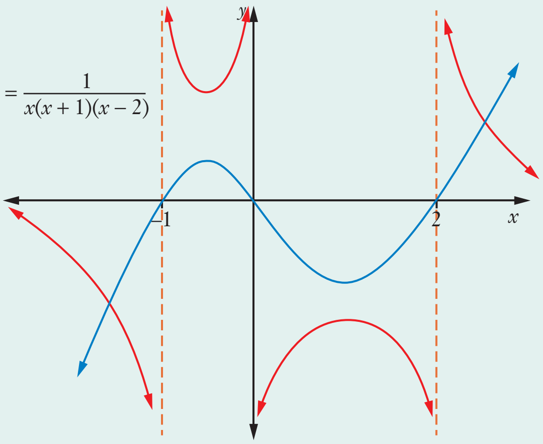

Domain $(-\infty, -1) \cup (-1, 0) \cup (0, 2) \cup (2, \infty)$ $x \neq -1, 0, 2$

Range $(-\infty, 0) \cup (0, \infty)$ $y \neq 0$

EXT1 Exercise 7.07 Reciprocal functions

1 For each function, sketch the graph of the reciprocal function $y = \dfrac{1}{f(x)}$ and find its domain and range.

a $f(x) = x - 2$
b $f(x) = 2x + 6$
c $f(x) = x^2 - 4$
d $f(x) = x^2 - 2x - 8$
e $f(x) = (x + 1)^2$
f $f(x) = -x^2 - 9$
g $f(x) = x^3$
h $f(x) = x(x - 2)(x + 2)$
i $f(x) = (x + 1)(x + 3)(x - 2)$
j $f(x) = -x(x - 1)^2$

2 Sketch the graph of the reciprocal function of:

a $f(x) = -5x - 2$
b $f(x) = x^2$
c $f(x) = x^2 + 4x + 3$
d $f(x) = 1 - x^2$
e $f(x) = x^2 + 3$
f $f(x) = -x^3$
g $f(x) = x^3 - 1$
h $f(x) = -x^3 - 8$
i $f(x) = -(x - 2)(x + 4)(x - 1)$
j $f(x) = (x - 3)(x + 1)^2$

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Square root functions

EXT1 7.08 Square root relations

The relation $y^2 = f(x)$ is not a function. It can be broken into 2 separate functions:

$$y = \pm\sqrt{f(x)}.$$

EXAMPLE 24

For the function $f(x) = x - 2$, sketch the graph of $y^2 = f(x)$.

Solution

$y^2 = f(x)$

$y^2 = x - 2$

$y = \pm\sqrt{x-2}$

We can only take square roots of positive numbers or 0, so the domain is $x - 2 \geq 0$, or $x \geq 2$.

Domain is $[2, \infty)$

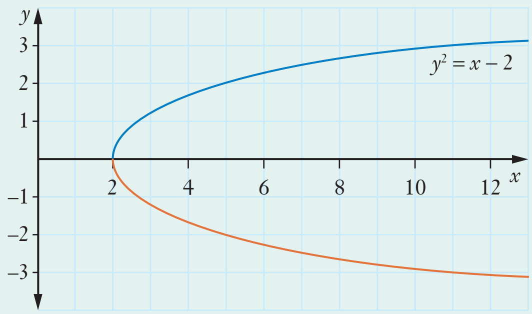
We can use a table to find values on the graph:

x	2	6	11
$y = \sqrt{x-2}$	0	2	3

x	2	6	11
$y = -\sqrt{x-2}$	0	–2	–3

This is the reflection of $y = \sqrt{x-2}$ in the x-axis.

Putting both graphs together gives us the graph of $y^2 = x - 2$:



Notice that this is a sideways parabola, but it is not a function. (It is actually the inverse relation of $y = x^2 + 2$.)

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As you can see from the example, the 2 separate functions are symmetrical about the x-axis. This is always the case for graphs of this type.

EXAMPLE 25

Sketch the graph of $y = \sqrt{f(x)}$ given $f(x) = x^2 + x - 2$, and state its domain and range.

Solution

For $y = \sqrt{x^2 + x - 2}$:

$x^2 + x - 2 \geq 0$

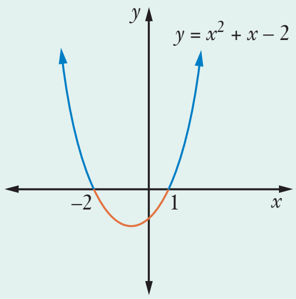
$(x + 2)(x - 1) \geq 0$

$x \leq -2, x \geq 1$

So for $y = \sqrt{f(x)}$

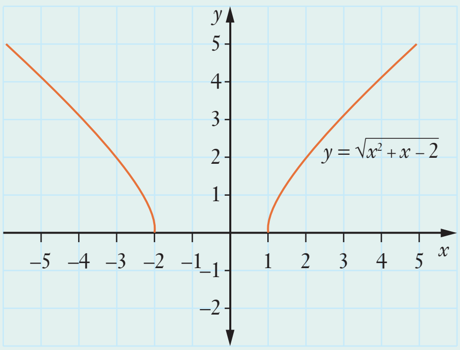
Domain $(-\infty, -2] \cup [1, \infty)$

Range $[0, \infty)$



We can use a table to find values on the graph:

x	-5	-4	-3	-2	1	2	3	4
$y = \sqrt{x^2 + x - 2}$	$\sqrt{18} = 4.24$	$\sqrt{10} = 3.16$	2	0	0	2	$\sqrt{10} = 3.16$	$\sqrt{18} = 4.24$



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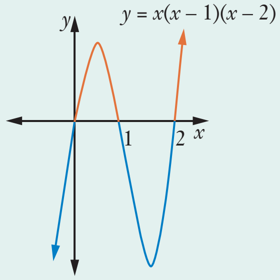
EXAMPLE 26

Sketch the graph of $y^2 = f(x)$ if $f(x) = x(x - 1)(x - 2)$.

Solution

$$y^2 = x(x - 1)(x - 2)$$
$$y = \pm\sqrt{x(x-1)(x-2)}$$



For $y = \sqrt{x(x-1)(x-2)}$, $x(x - 1)(x - 2) \geq 0$.

$y = x(x - 1)(x - 2)$ is a cubic function (odd degree) with a positive leading term, with x-intercepts at 0, 1, 2.

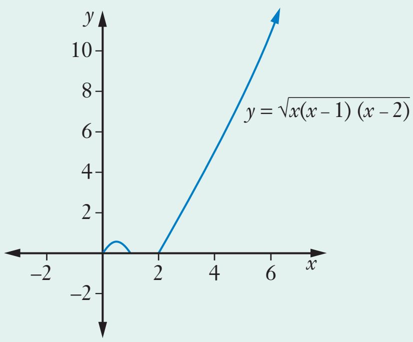
$0 \leq x \leq 1, x \geq 2$

Domain: $[0, 1] \cup [2, \infty)$

Range is $[0, \infty)$

We can use a table to find values on the square root graph:

x	0	0.25	0.5	0.75	1	2	3	4	5
$y = \sqrt{x(x-1)(x-2)}$	0	$\sqrt{033}$ ≈ 0.57	$\sqrt{0375}$ ≈ 0.61	$\sqrt{023}$ ≈ 0.48	0	0	$\sqrt{6}$ ≈ 2.45	$\sqrt{24}$ ≈ 4.9	$\sqrt{60}$ ≈ 7.75



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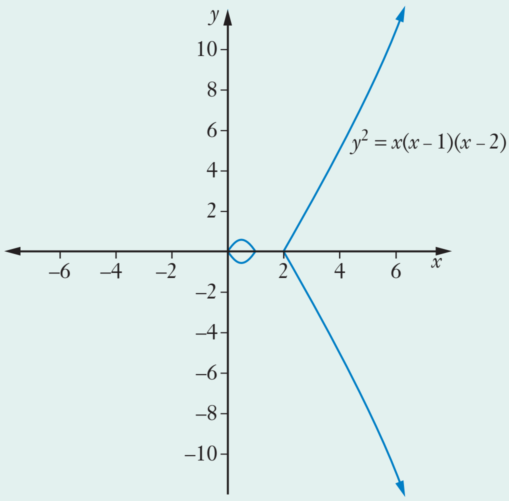
The graph of $y = -\sqrt{x(x-1)(x-2)}$ is a reflection of the above graph in the x-axis.

Putting both graphs together gives us the graph of $y^2 = x(x - 1)(x - 2)$.



EXT1 Exercise 7.08 Square root relations

1 Sketch the graph of $y = \sqrt{f(x)}$ and state its domain and range, given:

a $f(x) = x + 1$

b $f(x) = x - 3$

c $f(x) = 2x + 4$

d $f(x) = x^2 - 1$

e $f(x) = x^2 - 5x + 6$

f $f(x) = x^2 - 2x - 3$

g $f(x) = 4 - x^2$

h $f(x) = x^3 - 8$

i $f(x) = (x - 2)(x + 1)(x + 3)$

j $f(x) = x(x + 1)^2$

2 Sketch the graph of $y^2 = f(x)$, given:

a $f(x) = x - 4$

b $f(x) = x + 2$

c $f(x) = x^2 - 9$

d $f(x) = 9 - x^2$

e $f(x) = x^2 - 7x + 10$

f $f(x) = 2x^2 - 5x - 3$

g $f(x) = x^3 + 27$

h $f(x) = x(x + 3)(x - 1)$

i $f(x) = (x + 4)(x - 2)(x - 1)$

j $f(x) = (x - 2)(x - 3)^2$

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EXT1 7.09 Further absolute value functions

EXAMPLE 27

Given $f(x) = x^2 - 4x + 3$, sketch the graph of:

a $y = |f(x)|$ **b** $y = f(|x|)$

Solution

a $y = |f(x)|$

So $y = |x^2 - 4x + 3|$

Using the definition of absolute value:

$$y = \begin{cases} f(x) & \text{for } f(x) \geq 0 \\ -f(x) & \text{for } f(x) < 0 \end{cases}$$

Graph $y = x^2 - 4x + 3$ first.

For x-intercepts, $f(x) = 0$:

$0 = x^2 - 4x + 3$

$= (x - 3)(x - 1)$

$x = 1, 3$

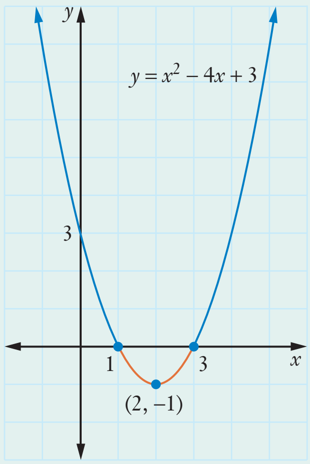
For y-intercept, $x = 0$:

$f(0) = 0^2 - 4(0) + 3 = 3$

Axis of symmetry at $x = 2$ (halfway between x-intercepts)

$f(2) = 2^2 - 4(2) + 3 = -1$

Vertex at $(2, -1)$.



When $f(x) \geq 0$ (above or on the x-axis), $|f(x)| = f(x)$, so the graph of $y = |f(x)|$ is the blue part of the graph.

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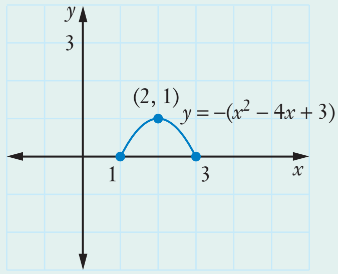
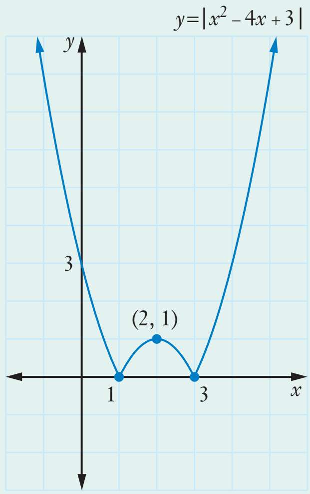
When $f(x) < 0$ (the orange part of the graph below the x-axis), $|f(x)| = -f(x)$, which is the reflection of the graph of $f(x)$ in the x-axis, as shown in this diagram. So the vertex $(2, -1)$ becomes $(2, 1)$.



Putting these 2 graphs together gives the graph of $y = |x^2 - 4x + 3|$.



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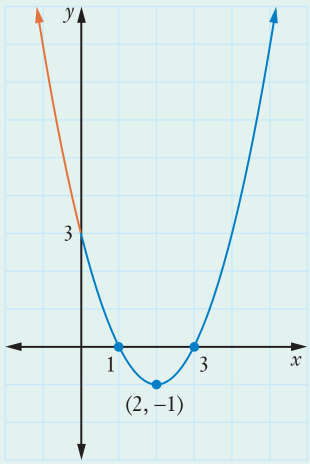
b $y = f(|x|)$

So $y = |x|^2 - 4|x| + 3$

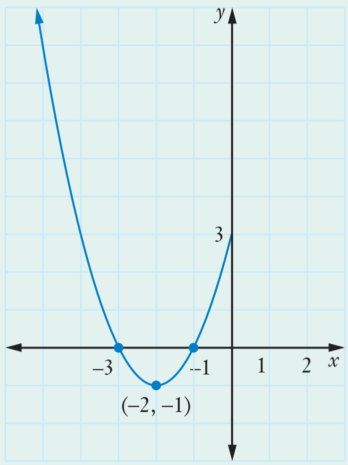
Using the definition of absolute value:

$$y = \begin{cases} f(x) & \text{for } x \geq 0 \\ f(-x) & \text{for } x < 0 \end{cases}$$

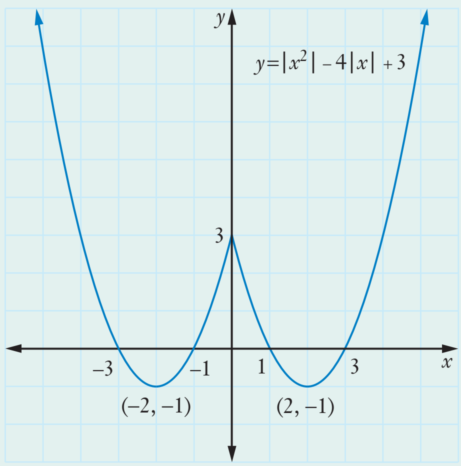
First graph $y = f(x) = x^2 - 4x + 3$ for $x \geq 0$, shown in blue here.



When $x < 0$, $f(|x|) = f(-x)$. This is a reflection of the graph of $f(x)$ in the y-axis as shown in this diagram.



Putting these 2 graphs together gives the graph of $y = |x|^2 - 4|x| + 3$.



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Absolute value functions

The graph of $y = |f(x)|$ is the graph of $y = f(x)$ when $f(x) \geq 0$ and the graph of $y = f(x)$ reflected in the x-axis when $f(x) < 0$. The graph of $y = f(|x|)$ is always above or on the x-axis.

The graph of $y = f(|x|)$ is the graph of $y = f(x)$ for $x \geq 0$ (right of the y-axis), and its reflection in the y-axis for $x < 0$ (left of the y-axis). The graph of $y = f(|x|)$ is symmetrical about the y-axis and it is an even function.

EXAMPLE 28

Given $f(x) = x(x - 1)(x + 3)$, sketch the graph of:

a $y = |f(x)|$ **b** $y = f(|x|)$

Solution

a $y = |x(x - 1)(x + 3)|$

First graph $f(x) = x(x - 1)(x + 3)$.

For x-intercepts, $f(x) = 0$:

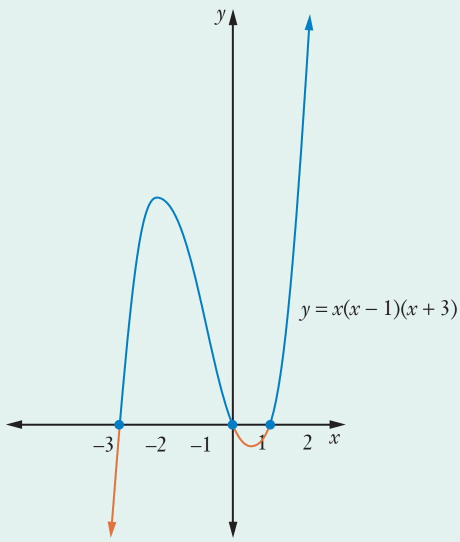
$0 = x(x - 1)(x + 3)$

$x = 0, 1, -3$

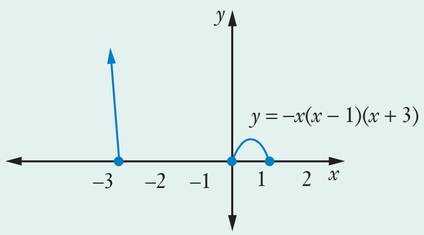
For y-intercept, $x = 0$:

$f(0) = 0(0 - 1)(0 + 3) = 0$.

When $f(x) \geq 0$ (above or on the x-axis, the blue part of the graph), $|f(x)| = f(x)$.



When $f(x) < 0$ (the orange part of the graph), $|f(x)| = -f(x)$, so the graph of $y = |f(x)|$ is the graph of $y = f(x)$ reflected in the x-axis.



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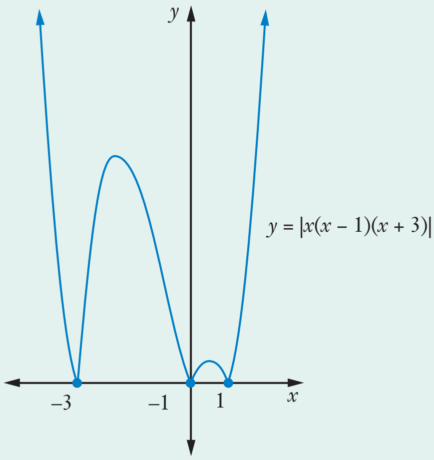
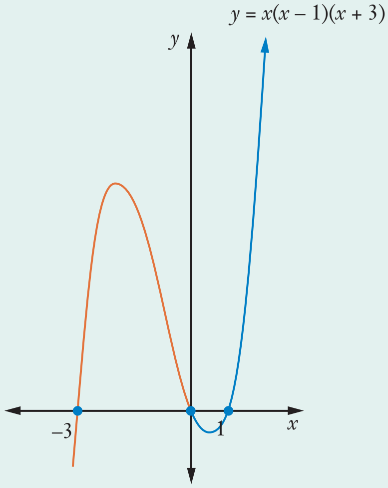
Combining the 2 graphs gives the graph of $y = |x(x-1)(x+3)|$:



b $y = |x|(|x|-1)(|x|+3)$

$$y = \begin{cases} f(x) & \text{for } x \geq 0 \\ f(-x) & \text{for } x < 0 \end{cases}$$

Drawing $y = x(x-1)(x+3)$ for $x \geq 0$ gives the blue part of this graph.



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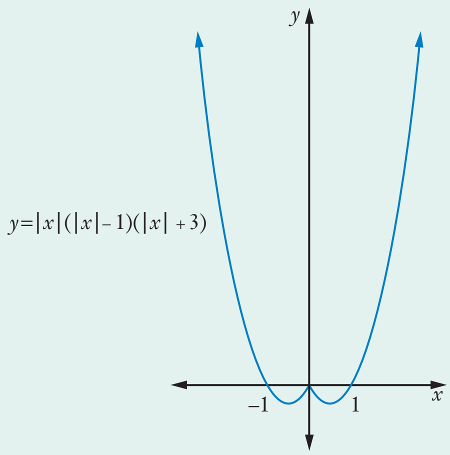
Now reflect this graph in the y-axis.

Putting these 2 graphs together gives the graph of $y = |x|(|x| - 1)(|x| + 3)$:



EXT1 Exercise 7.09 Further absolute value functions

1 For each function, sketch the graph of:

i $y = |f(x)|$ **ii** $y = f(|x|)$

a $f(x) = x + 5$ **b** $f(x) = 3x - 6$

c $f(x) = x^2 - 4$ **d** $f(x) = x^2 + 4x + 4$

e $f(x) = x^2 + x - 12$ **f** $f(x) = (x - 3)(x - 5)$

g $f(x) = x^3 + 1$ **h** $f(x) = (x - 2)(x + 4)(x - 1)$

i $f(x) = x(x - 3)^2$ **j** $f(x) = x^2(x + 1)$

2 Sketch the graph of $y = |f(x)|$ if $f(x) = x^2 + x - 6$, and state its domain and range.

3 For the function $f(x) = x + 4$, sketch the graph of:

a $y = \dfrac{1}{f(x)}$ **b** $y = f(|x|)$ **c** $y^2 = f(x)$ **d** $y = |f(x)|$

4 If $f(x) = x^2 - 1$ and $g(x) = 3x - 3$, sketch the graph of:

a $y = f(|x|)$ **b** $y = |g(x)|$ **c** $y = f(x) + g(x)$

d $y = \dfrac{1}{g(x)}$ **e** $y^2 = g(x)$ **f** $y = f(x)g(x)$

g $y = g(|x|)$ **h** $y^2 = f(x)$ **i** $y = |f(x)|$

EXT1 7.10 Parametric equations of a function

In Chapter 5, *Trigonometry*, we used points on a unit circle to find the trigonometric ratios for obtuse angles. The formulas were:

$$\sin\theta = y, \cos\theta = x \text{ and } \tan\theta = \frac{y}{x}.$$

When x and y are described by another variable such as θ in a relation, then θ is called a **parameter**. This variable gives the characteristics of x and y in a different form.

Until now, we have written the equation of a function in **Cartesian form**. This is one equation with 2 variables, such as x and y.

When using a parameter such as θ, we have 3 variables and 2 equations, one with x and the parameter, and one with y and the parameter. This is the **parametric form** of a function.

In the trigonometry example above, the parametric equations are $x = \cos\theta$, $y = \sin\theta$.

Another example is $x = 2p + 4$, $y = p - 2$.

Instead of substituting x into a function to find y, we substitute the parameter, p, into the parametric equations to find x and y. This table of values show the values of x and y for different values of p for this example.

p	-2	-1	0	1	2
x	0	2	4	6	8
y	-4	-3	-2	-1	0

Any Cartesian equation can be written in parametric form and any pair of parametric equations can be written in Cartesian form.

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Linear functions

EXAMPLE 29

a Find the Cartesian equation of the straight line defined by the parametric equations $x = 3t, y = 2t - 3$.

b Sketch the graph of the linear function $x = 2p + 4, y = p - 2$.

Solution

a Solve simultaneous equations to eliminate the parameter.

$x = 3t + 1$ [1]

$y = 2t - 3$ [2]

Make t the subject of [1]:

$$x - 1 = 3t$$
$$\frac{x-1}{3} = t$$

Substitute in [2]:

$$y = 2t - 3$$
$$= 2\left(\frac{x-1}{3}\right) - 3$$
$$3y = 2(x-1) - 9$$
$$= 2x - 2 - 9$$
$$= 2x - 11$$
$$0 = 2x - 3y - 11$$

b $x = 2p + 4$ [1]

$y = p - 2$ [2]

Make p the subject of [2]:

$$y + 2 = p$$

Substitute into [1]:

$$x = 2(y + 2) + 4$$
$$= 2y + 4 + 4$$
$$= 2y + 8$$
$$x - 2y - 8 = 0$$

Now graph $x - 2y - 8 = 0$.

For x-intercept, $y = 0$:

$x - 2(0) - 8 = 0$

$x = 8$

For y-intercept, $x = 0$:

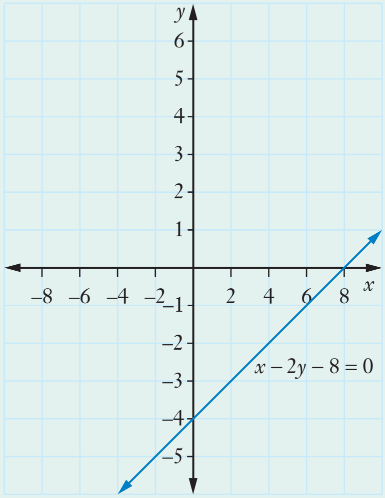
$0 - 2y - 8 = 0$

$-2y = 8$

$y = -4$

Note that the points on this line agree with the table of values on page 398.

p	−2	−1	0	1	2
x	0	2	4	6	8
y	−4	−3	−2	−1	0



We can convert Cartesian equations into parametric form. There are many different ways to do this, depending on what we choose for the parameter.

EXAMPLE 30

Convert each linear function to parametric form using the given equation for x.

a $y = 3x + 1, x = p$

b $5x - 2y + 1 = 0, x = 4t$

Solution

a Use the given parametric equation $x = p$.

Substitute into the Cartesian equation:

$y = 3x + 1$

$= 3p + 1$

So the function in parametric form is $x = p, y = 3p + 1$

b Use the given parametric equation $x = 4t$.

Substitute into the Cartesian equation:

$5(4t) - 2y + 10 = 0$

$20t - 2y + 10 = 0$

$20t + 10 = 2y$

$10t + 5 = y$

So the equation in parametric form is $x = 4t, y = 10t + 5$

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Quadratic functions

EXAMPLE 31

a The parametric equations $x = 2p, y = p^2 - 3$ describe a quadratic function. Find its equation in Cartesian form.

b Sketch the graph of $x = t + 1, y = t^2 - 2t - 8$.

c Write the quadratic function $y = x^2 + 3x - 1$ in parametric form using $x = 2m - 2$.

Solution

a $x = 2p$ [1]

$y = p^2 - 3$ [2]

Make p the subject of [1]:

$x = 2p$

$\frac{x}{2} = p$

Substitute in [2]:

$$\begin{aligned} y &= p^2 - 3 \\ &= \left(\frac{x}{2}\right)^2 - 3 \\ &= \frac{x^2}{4} - 3 \end{aligned}$$

b We can draw up a table of values using different values for t or change the equation into Cartesian form.

$x = t + 1$ [1]

$y = t^2 - 2t - 8$ [2]

Make t the subject of [1]:

$$\begin{aligned} x &= t + 1 \\ x - 1 &= t \end{aligned}$$

Substitute in [2]:

$$\begin{aligned} y &= t^2 - 2t - 8 \\ &= (x - 1)^2 - 2(x - 1) - 8 \\ &= x^2 - 2x + 1 - 2x + 2 - 8 \\ &= x^2 - 4x - 5 \end{aligned}$$

For x-intercepts, $y = 0$:

$0 = x^2 - 4x - 5$

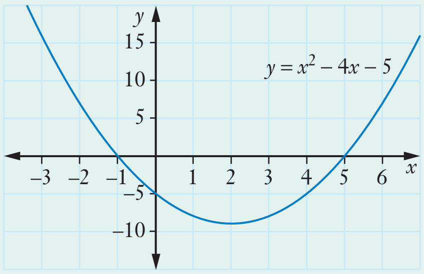
$= (x - 5)(x + 1)$

$x = -1, 5$

For y-intercepts, $x = 0$:

$y = 0^2 - 4(0) - 5$

$= -5$



c $x = 2m - 2$

Substitute into the Cartesian equation:

$y = (2m - 2)^2 + 3(2m - 2) - 1$

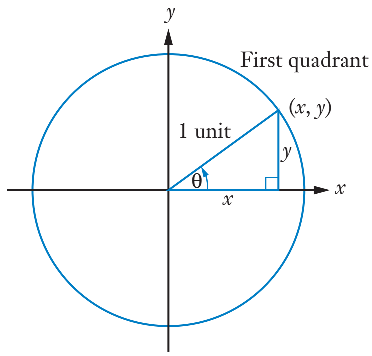
$= 4m^2 - 8m + 4 + 6m - 6 - 1$

$= 4m^2 - 2m - 3$

So the equation in parametric form is $x = 2m - 2, y = 4m^2 - 2m - 3$.

Circles

While the circle is not a function, we can still write it in parametric form. To do this, we can look at the unit circle from Chapter 5 *Trigonometry*.



From the circle:

$\sin\theta = y$ and $\cos\theta = x$

Using Pythagoras' theorem:

$x^2 + y^2 = 1$ (the equation of the unit circle)

We can substitute $\sin\theta = y$ and $\cos\theta = x$ into this equation:

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

We can write this as $\cos^2\theta + \sin^2\theta = 1$. This formula is called an **identity** because it is true for any value of θ.

Pythagorean identity

$\cos^2\theta + \sin^2\theta = 1$

We use this identity to convert between the parametric and Cartesian equations of a circle.

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EXAMPLE 32

a Convert the parametric equations $x = 5\sin\theta$, $y = 5\cos\theta$ to Cartesian form and sketch its graph.

b Find the Cartesian form of the equation for $x = 4 + 3\cos\theta$, $y = 3 + 3\sin\theta$ and describe its shape.

c Write $x^2 + 2x + y^2 - 4y - 4 = 0$ in parametric form.

Solution

a $x = 5\sin\theta$ [1]

$y = 5\cos\theta$ [2]

Make sin θ the subject of [1]:

$$\sin\theta = \frac{x}{5}$$

Make cos θ the subject of [2]:

$$\cos\theta = \frac{y}{5}$$

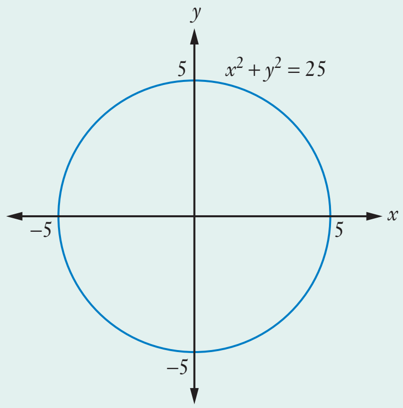
Substitute into the identity $\cos^2\theta + \sin^2\theta = 1$:

$$\left(\frac{y}{5}\right)^2 + \left(\frac{x}{5}\right)^2 = 1$$

$$\frac{y^2}{25} + \frac{x^2}{25} = 1$$

$$y^2 + x^2 = 25$$

$$x^2 + y^2 = 25$$



This is the equation of a circle, radius 5 and centre (0, 0).

b $x = 4 + 3\cos\theta$ [1]

$y = 3 + 3\sin\theta$ [2]

Make cos θ the subject of [1]:

$$x - 4 = 3\cos\theta$$

$$\frac{x-4}{3} = \cos\theta$$

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Make $\sin\theta$ the subject of [2]:

$y - 3 = 3\sin\theta$

$\frac{y-3}{3} = \sin\theta$

Since $\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{x-4}{3}\right)^2 + \left(\frac{y-3}{3}\right)^2 = 1$

$\frac{(x-4)^2}{9} + \frac{(y-3)^2}{9} = 1$

$(x-4)^2 + (y-3)^2 = 9$

This is the equation of a circle, radius 3 and centre (4, 3).

c This is the equation of a circle.

Complete the square to put the equation in the form $(x-a)^2 + (y-b)^2 = r^2$.

$x^2 + 2x + y^2 - 4y - 4 = 0$

$x^2 + 2x + y^2 - 4y = 4$

$x^2 + 2x + 1 + y^2 - 4y + 4 = 4 + 1 + 4$

$(x+1)^2 + (y-2)^2 = 9$

$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{9} = 1$

$\left(\frac{x+1}{3}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$

We know $\cos^2\theta + \sin^2\theta = 1$, so let

$\frac{x+1}{3} = \cos\theta$ and $\frac{y-2}{3} = \sin\theta$

$x + 1 = 3\cos\theta$ $\quad$ $y - 2 = 3\sin\theta$

$x = 3\cos\theta - 1$ $\quad$ $y = 2 + 3\sin\theta$

The parametric equations are $x = 3\cos\theta - 1$, $y = 2 + 3\sin\theta$.

Parametric equations of a circle

The parametric equations of a circle with centre (0, 0) and radius r are:

$$x = r\cos\theta,\ y = r\sin\theta$$

The parametric equations of a circle with centre (a, b) and radius r are:

$$x = a + r\cos\theta,\ y = b + r\sin\theta$$

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EXT1 Exercise 7.10 Parametric equations of a function

1 Write each set of parametric equations in Cartesian form.

a $x = 2t, y = 3t - 4$ **b** $x = 5q + 1, y = 2q$

c $x = 3n - 2, y = 2n - 5$ **d** $x = 7p + 3, y = 2p + 1$

e $x = -6t, y = 3t + 2$

2 Write as a Cartesian equation:

a $x = t, y = t^2 - 3t + 6$ **b** $x = r - 3, y = r^2 + 1$ **c** $x = 2p + 3, y = p^2$

d $x = 3s + 1, y = s^2 + 2s$ **e** $x = 4k - 7, y = 2k^2 + k$

3 Sketch the graph of each function given in parametric form.

a $x = 2t, y = 3t - 1$ **b** $x = q + 4, y = 3q + 6$ **c** $x = p + 1, y = p^2 - 3p$

d $x = p - 2, y = p^2 - 1$ **e** $x = p + 4, y = p^2 - 1$

4 Convert each pair of parametric equations to a Cartesian equation.

a $x = 6 \cos\theta, y = 6 \sin\theta$ **b** $x = -2 \cos\theta, y = -2 \sin\theta$

c $x = \sin\theta, y = \cos\theta + 1$ **d** $x = 5 + 4 \cos\theta, y = 2 + 4 \sin\theta$

e $x = 3 + \cos\theta, y = \sin\theta - 2$

5 Use the parameter $x = p$ to write $x - y + 1 = 0$ in parametric form.

6 Write each function in parametric form, using the given equation for x.

a $3x - 4y + 24 = 0, x = 4t$ **b** $x + 7y - 21 = 0, x = t + 5$

c $y = x^2, x = 2t - 1$ **d** $y = 2x^2 - 3x + 4, x = 1 - t$

e $x^2 + y^2 = 9, x = 3 \cos t$

7 Find the radius and centre of each circle given in parametric form.

a $x = 1 + \cos\alpha, y = \sin\alpha - 2$ **b** $x = -3 + 5 \cos\beta, y = 2 + 5 \sin\beta$

c $x = 4 - 2 \cos\theta, y = -2 + 2 \sin\theta$ **d** $x = 6 + 7 \sin\theta, y = -5 + 7 \sin\theta$

e $x = -2 \cos\theta - 8, y = 2 \sin\theta + 9$

8 Sketch the graph of each circle.

a $x = 4 \cos\theta, y = -4 \sin\theta$ **b** $x = 2 + \cos\alpha, y = 3 + \sin\alpha$

9 Use the parameter θ to write each equation of a circle in parametric form.

a $x^2 + y^2 = 4$ **b** $x^2 + 4x + y^2 - 5 = 0$

c $x^2 - 6x + y^2 + 2y - 26 = 0$ **d** $x^2 - 8x + y^2 - 10y - 8 = 0$

e $x^2 + 2x + y^2 - 4y + 1 = 0$

10 a Write $x = 4p, y = 2p^2 - 3$ as a Cartesian equation.

b Find the x- and y-intercepts of its graph.

c State the domain and range of the function.

d Sketch the graph of the function.

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7. TEST YOURSELF

Practice quiz

For Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

1 The domain of $y = -\dfrac{3}{x-4}$ is:

A (-4) **B** $(-\infty, 4) \cup (4, \infty)$

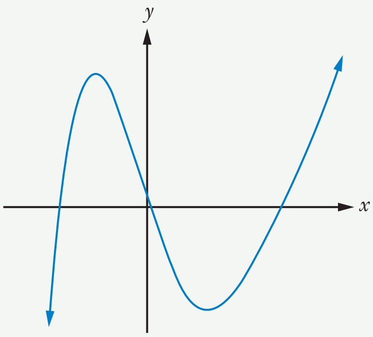
C $(-\infty, -4) \cup (-4, \infty)$ **D** $(-\infty, 4)$

2 The equation of a circle with radius 3 and centre $(1, -2)$ is:

A $(x-1)^2 + (y+2)^2 = 9$ **B** $(x+1)^2 + (y-2)^2 = 9$

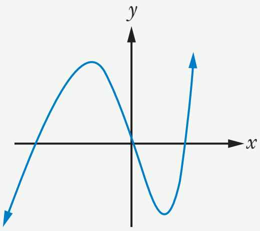
C $(x-1)^2 + (y+2)^2 = 3$ **D** $(x+1)^2 + (y-2)^2 = 3$

3 The graph of $y = f(x)$ is shown below.

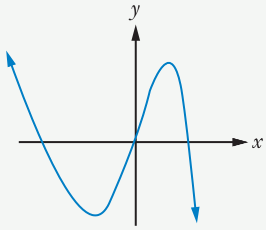


Which one of these is the graph of $y = -f(-x)$?

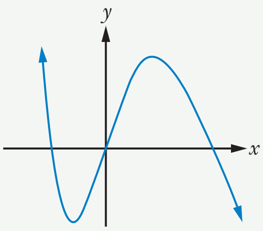
A



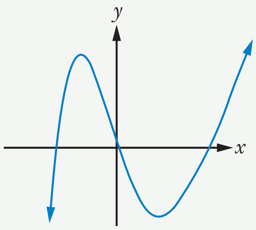
B



C



D



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4 The area of a pizza slice decreases as the number of people sharing it evenly increases. When 5 people share the pizza, the area of each slice is 30 cm^2.

a Find the equation of the area, A, of a pizza slice in terms of the number of people sharing, n.

b What is the area of one pizza slice when:

i 10 people share? **ii** 8 people share?

c How many people are sharing the pizza when each slice has an area of:

i 16. 67 cm^2? **ii** 25 cm^2?

5 Sketch the graph of each function or relation.

a $x^2 + y^2 = 1$ **b** $y = \frac{2}{x}$ **c** $y = |x + 2|$

d $y = -\sqrt{4 - x^2}$ **e** $y = f(-x)$ given $f(x) = \frac{2}{x - 1}$

f $y = -f(x)$ if $f(x) = 3x - 6$ **g** $y = -f(-x)$ given $f(x) = x^2 + x$

6 Find the radius and centre of the circle $x^2 - 6x + y^2 - 2y - 6 = 0$.

7 If $f(x) = x^3$ and $g(x) = 3x - 1$, find the equation of:

a $y = f(x) + g(x)$ **b** $y = f(x)g(x)$

c $y = f(g(x))$ **d** $y = g(f(x))$

8 EXT1 Convert each equation from Cartesian to parametric form, using the given equation for x.

a $x - 3y + 6 = 0, x = 3p + 5$

b $y = x^2 - 2x + 5, x = -2p$

c $x^2 + y^2 = 81, x = 9 \cos p$

9 **a** Is the circle $x^2 + y^2 = 1$ a function?

b Change the subject of the equation to y in terms of x.

c Sketch the graphs of 2 separate functions that together make up the circle $x^2 + y^2 = 1$.

10 Find the domain and range of each relation.

a $x^2 + y^2 = 16$ **b** $y = \frac{1}{x + 2}$

c $f(x) = |x| + 3$ **d** $y = \sqrt{9 - x^2}$

11 EXT1 Given $f(x) = x^2 + 3x$ and $g(x) = x + 4$, sketch:

a $y = \frac{1}{f(x)}$ **b** $y = |g(x)|$ **c** $y^2 = f(x)$

d $y = f(|x|)$ **e** $y^2 = g(x)$ **f** $y = f(x)g(x)$

12 Find the domain and range of $y = f(x) + g(x)$ given $f(x) = x^2 - 4x$ and $g(x) = 2x - 3$.

13 EXT1 Sketch the graph of each pair of parametric equations.

a $x = \frac{t-8}{3}, y = t - 2$ **b** $x = 3\cos\theta, y = 3\sin\theta$ **c** $x = 2p, y = 4p^2 + 8p$

14 **a** Write down the domain and range of the curve $y = \frac{2}{x-3}$.

b Sketch the graph of $y = \frac{2}{x-3}$.

15 EXT1 The parametric equations of a circle are $x = 4 + 5\cos\theta, y = -3 + 5\sin\theta$.

a Write the equation of the circle in Cartesian form.

b Find the radius and centre of the circle.

16 **a** Sketch the graph of $y = |x + 1|$.

b From the graph, solve:

i $|x + 1| = 3$ EXT1 **ii** $|x + 1| < 3$ EXT1 **iii** $|x + 1| > 3$

17 Solve graphically: $|x - 3| = 2$.

18 Find the centre and radius of the circle with equation:

a $x^2 + y^2 = 100$

b $(x - 3)^2 + (y - 2)^2 = 121$

c $x^2 + 6x + y^2 + 2y + 1 = 0$

19 Find the x- and y-intercepts (where they exist) of:

a $P(x) = x^3 - 4x$ **b** $y = -\frac{2}{x+1}$ **c** $x^2 + y^2 = 9$

d $y = \sqrt{25 - x^2}$ **e** $f(x) = |x - 2| + 3$

20 If $f(x) = 2x^2 + x - 6$ and $g(x) = 5x^3 + 1$, find:

a the degree of $y = f(x) + g(x)$

b the leading term of $y = f(x)g(x)$

c the constant term of $y = f(x) - g(x)$

d EXT1 the equation of:

i $y = [g(x)]^2$ **ii** $y = g(x^2)$ **iii** $y = \frac{1}{f(x)}$

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7. CHALLENGE EXERCISE

1 EXT1 Sketch the graph of $y^2 = f(x)$ and state the domain and range, given:

a $f(x) = x(x - 2)(x - 4)$ **b** $f(x) = (x - 1)^3$

2 Solve $|2x + 1| = 3x - 2$ graphically.

3 Given $f(x) = |x| + 3x - 4$, sketch the graph of:

a $y = f(x)$ **b** $y = -f(x)$ EXT1 **c** $y = \dfrac{1}{f(x)}$

4 A variable a is inversely proportional to the square of b. When $b = 3$, $a = 2$.

a Find the equation of a in terms of b.

b Evaluate a when $b = 2$.

c Evaluate b when $a = 10$, correct to 2 decimal places, if $b > 0$.

5 EXT1 Find the domain and range of $y = \dfrac{1}{x^2 - 1}$.

6 EXT1 Given $f(x) = x^2 - 1$, sketch the graph of $y^2 = \dfrac{1}{f(x)}$.

7 Find the centre and radius of the circle with equation given by $x^2 + 3x + y^2 - 2y - 3 = 0$.

8 Find the equation of the straight line through the centres of the circles with equations $x^2 + 4x + y^2 - 8y - 5 = 0$ and $x^2 - 2x + y^2 + 10y + 10 = 0$.

9 Sketch the graph of $y = \dfrac{|x|}{x^2}$.

10 a Show that $\dfrac{2x+7}{x+3} = 2 + \dfrac{1}{x+3}$.

b Find the domain and range of $y = \dfrac{2x+7}{x+3}$.

c Hence sketch the graph of $y = \dfrac{2x+7}{x+3}$.

11 Show that $x^2 - 2x + y^2 + 4y + 1 = 0$ and $x^2 - 2x + y^2 + 4y - 4 = 0$ are concentric.

12 Sketch the graph of $f(x) = 1 - \dfrac{1}{x^2}$.

13 a Sketch the graph of $f(x) = \begin{cases} x & \text{for } x < -2 \\ x^2 & \text{for } -2 \le x \le 0 \\ 2 & \text{for } x > 0 \end{cases}$.

b Find any x values for which the function is discontinuous.

c Find the domain and range of the function.

14 EXT1 Solve graphically $|x| - 3 \ge x$.

15 EXT1 **a** Find the Cartesian equation of $x = 2\cos\theta$, $y = 3\sin\theta$.

b Sketch the graph and describe its shape.

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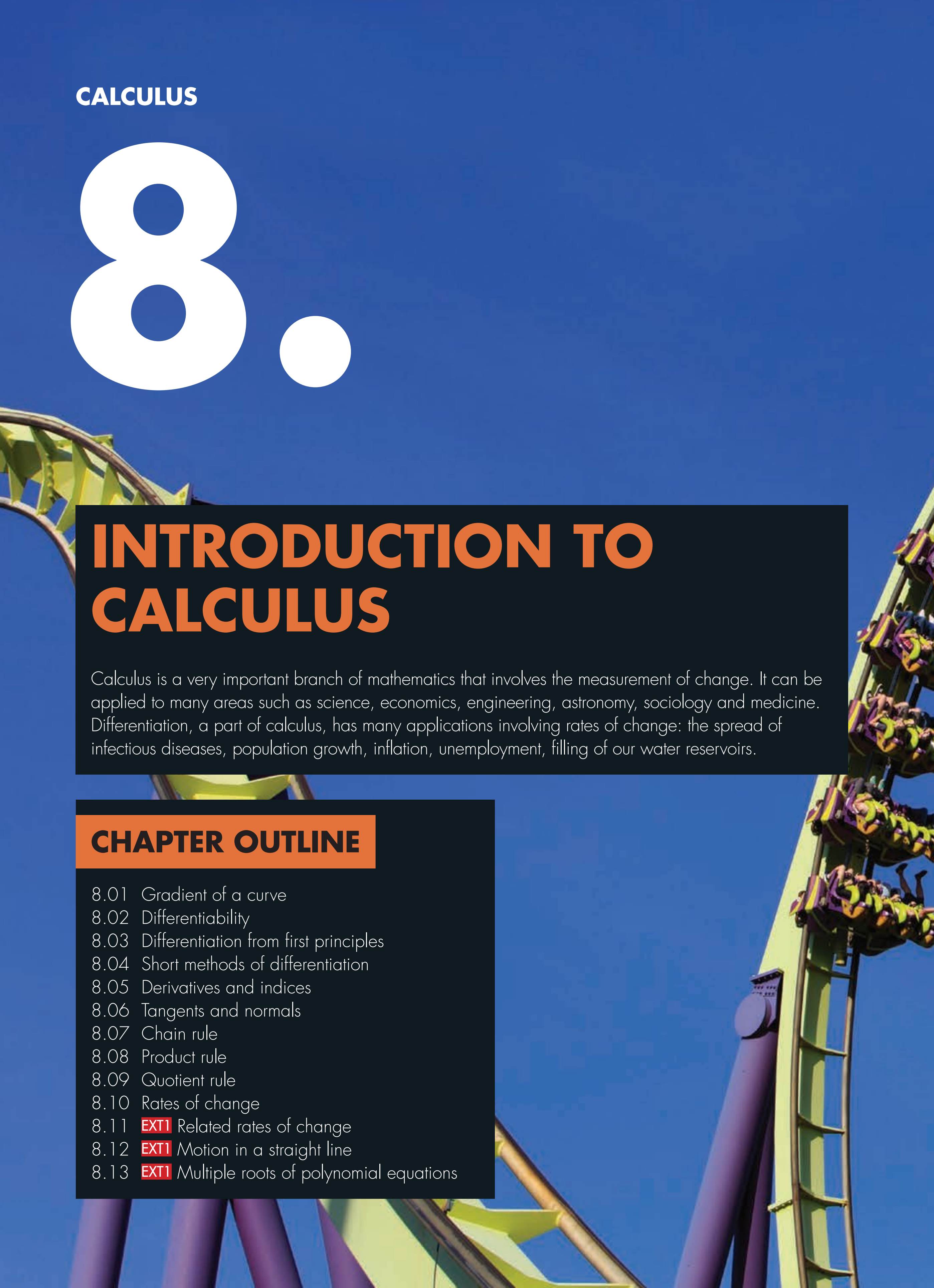
CALCULUS

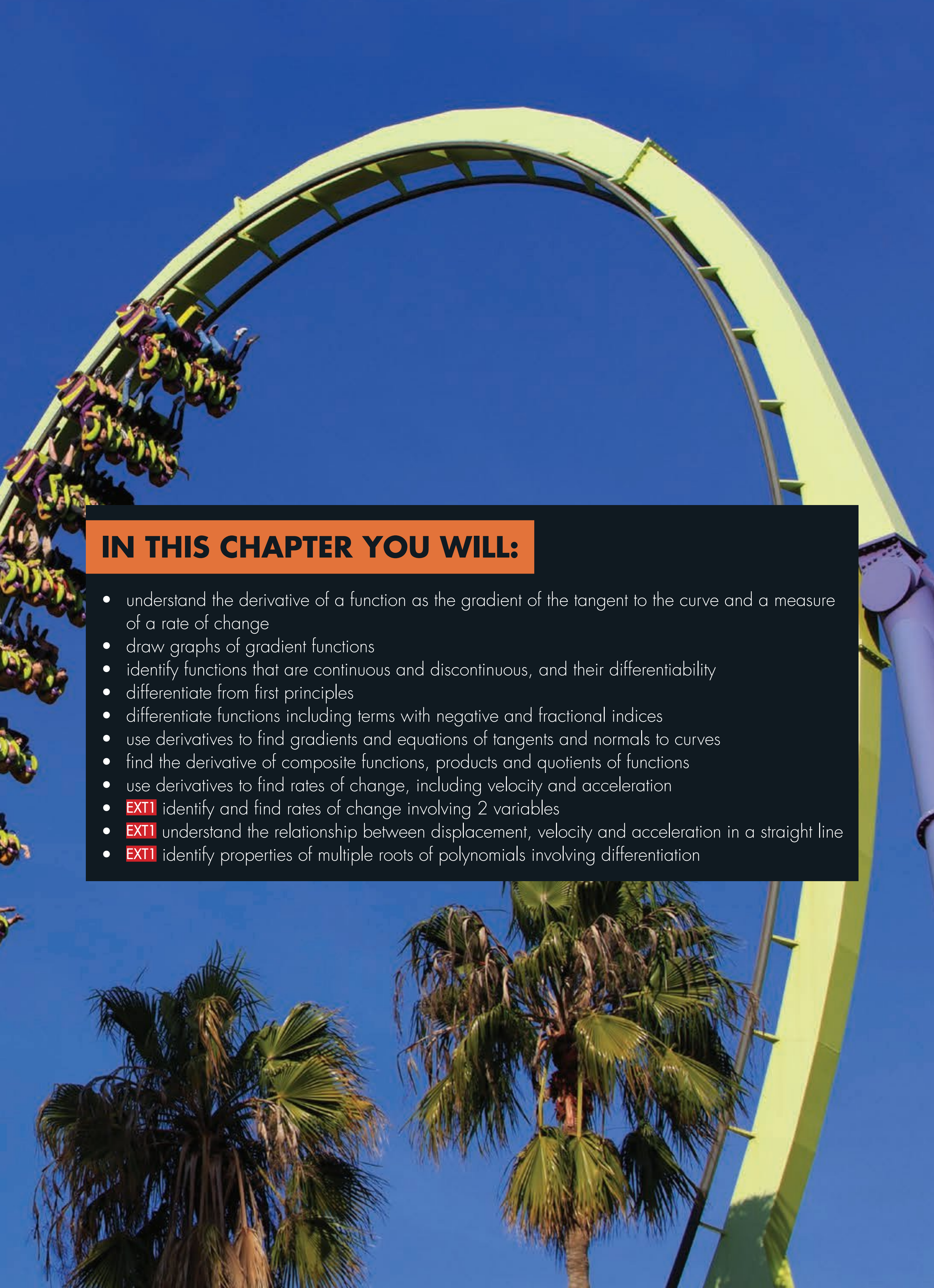
8.



INTRODUCTION TO CALCULUS

Calculus is a very important branch of mathematics that involves the measurement of change. It can be applied to many areas such as science, economics, engineering, astronomy, sociology and medicine. Differentiation, a part of calculus, has many applications involving rates of change: the spread of infectious diseases, population growth, inflation, unemployment, filling of our water reservoirs.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- understand the derivative of a function as the gradient of the tangent to the curve and a measure of a rate of change
- draw graphs of gradient functions
- identify functions that are continuous and discontinuous, and their differentiability
- differentiate from first principles
- differentiate functions including terms with negative and fractional indices
- use derivatives to find gradients and equations of tangents and normals to curves
- find the derivative of composite functions, products and quotients of functions
- use derivatives to find rates of change, including velocity and acceleration
- EXT1 identify and find rates of change involving 2 variables
- EXT1 understand the relationship between displacement, velocity and acceleration in a straight line
- EXT1 identify properties of multiple roots of polynomials involving differentiation

TERMINOLOGY

acceleration: The rate of change of velocity with respect to time

average rate of change: The rate of change between 2 points on a function; the gradient of the line (secant) passing through those points

chain rule: A method for differentiating composite functions

derivative function: The gradient function $y = f'(x)$ of a function $y = f(x)$ obtained through differentiation

differentiability: A function is differentiable wherever its gradient is defined

differentiation: The process of finding the gradient function

differentiation from first principles: The process of finding the gradient of a tangent to a curve by finding the gradient of the secant between 2 points and finding the limit as the secant becomes a tangent

displacement: The distance and direction of an object in relation to the origin

gradient of a secant: The gradient (slope) of the line between 2 points on a function; measures the average rate of change between the 2 points

gradient of a tangent: The gradient of a line that is a tangent to the curve at a point on a function; measures the instantaneous rate of change of the function at that point

instantaneous rate of change: The rate of change at a particular point on a function; the gradient of the tangent at this point

limit: The value that a function approaches as the independent variable approaches some value

normal: A line that is perpendicular to the tangent at a given point on a curve

product rule: A method for differentiating the product of 2 functions

quotient rule: A method for differentiating the quotient of 2 functions

secant: A straight line passing through 2 points on the graph of a function

stationary point: A point on the graph of $y = f(x)$ where the tangent is horizontal and its gradient $f'(x) = 0$. It could be a maximum point, minimum point or a horizontal point of inflection

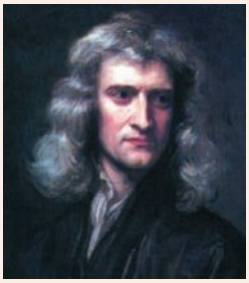
tangent: A straight line that just touches a curve at one point. The curve has the same gradient or direction as the tangent at that point

turning point: A maximum or minimum point on a curve, where the curve turns around

velocity: The rate of change of displacement of an object with respect to time; involves speed and direction

DID YOU KNOW?

Newton and Leibniz


Isaac Newton

'Calculus' comes from the Latin meaning 'pebble' or 'small stone'. In many ancient civilisations stones were used for counting, but the mathematics they practised was quite sophisticated.

It was not until the 17th century that there was a breakthrough in calculus when scientists were searching for ways of measuring motion of objects such as planets, pendulums and projectiles.

Isaac Newton (1642–1727), an Englishman, discovered the main principles of calculus when he was 23 years old. At this time an epidemic of bubonic plague had closed Cambridge University where he was studying, so many of his discoveries were made at home. He first wrote about his calculus methods, which he called fluxions, in 1671, but his *Method of fluxions* was not published until 1704.

Gottfried Leibniz (1646–1716), in Germany, was studying the same methods and there was intense rivalry between the two countries over who was first to discover calculus!

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8.01 Gradient of a curve

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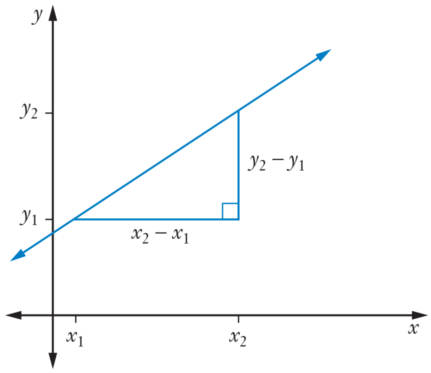
Gadien functions

WS
Gadien functions

The **gradient** of a straight line measures the **rate of change** of y (the dependent variable) with respect to the change in x (the independent variable).



Gradient

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$

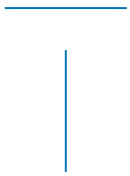
Notice that when the gradient of a straight line is positive the line is increasing, and when the gradient is negative the line is decreasing. Straight lines increase or decrease at a constant rate and the gradient is the same everywhere along the line.

CLASS DISCUSSION

Remember that an **increasing** line has a positive gradient and a **decreasing** line has a negative gradient.

What is the gradient of a horizontal line?

Can you find the gradient of a vertical line? Why?



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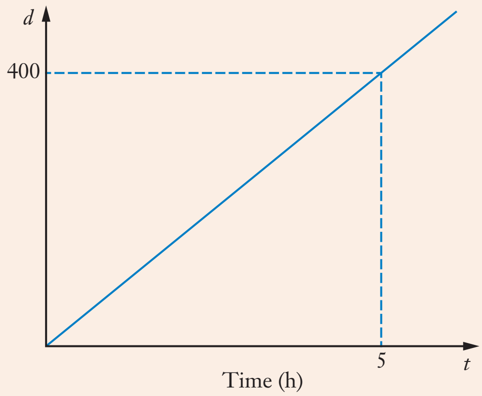
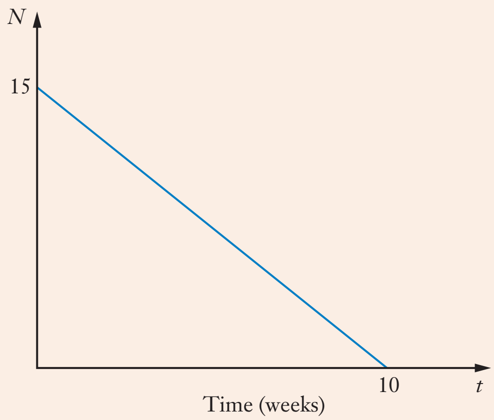
EXAMPLE 1

a The graph shows the distance travelled by a car over time. Find the gradient and describe it as a rate.



b The graph shows the number of cases of flu reported in a town over several weeks. Find the gradient and describe it as a rate.



Solution

a

$$m = \frac{\text{rise}}{\text{run}}$$

$$= \frac{400}{5}$$

$$= 80$$

The line is increasing, so it has a positive gradient.

This means that the car is travelling at a constant rate (speed) of 80 km/h.

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b $m = \dfrac{\text{rise}}{\text{run}}$

$= \dfrac{-1500}{10}$

$= -150$

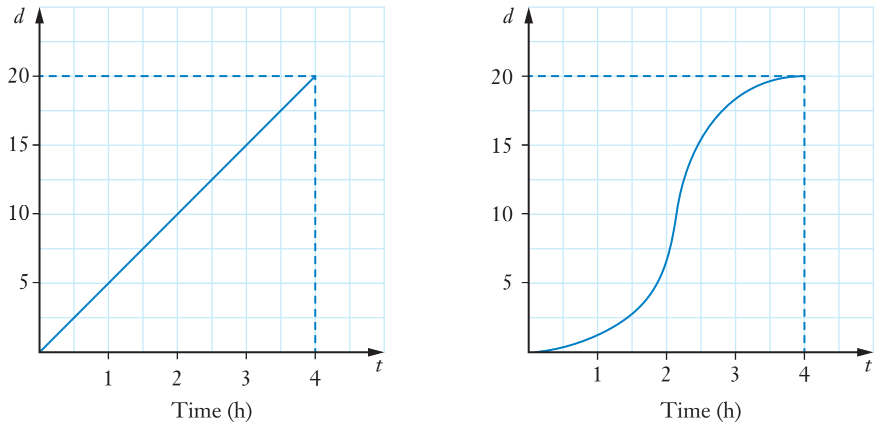
The line is decreasing, so it will have a negative gradient.

The 'rise' is a drop so it's negative.

This means that the rate is –150 cases/week, or the number of cases reported is decreasing by 150 cases/week.

CLASS DISCUSSION

The 2 graphs below show the distance that a bicycle travels over time. One is a straight line and the other is a curve.



Is the average speed of the bicycle the same in both cases? What is different about the speed in the 2 graphs?

How could you measure the speed in the second graph at any one time? Does it change? If so, how does it change?

We can start finding rates of change along a curve by looking at its shape and how it behaves. We started looking at this in Chapter 4, *Functions*.

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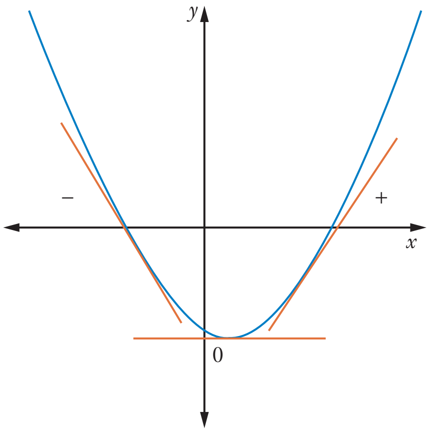
The gradient of a curve shows the **rate of change of *y*** as *x* changes. A **tangent** to a curve is a straight line that just touches the curve at one point. We can see where the gradient of a curve is positive, negative or zero by drawing **tangents to the curve** at different places around the curve and finding the gradients of the tangents.

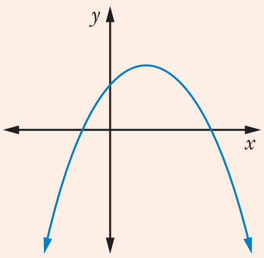
Notice that when the curve increases it has a positive gradient, when it decreases it has a negative gradient, and when it is a **turning point** the gradient is zero.



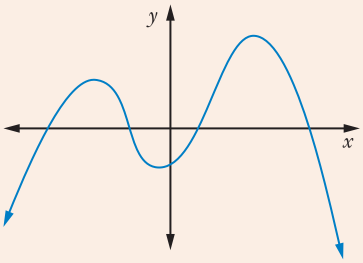
EXAMPLE 2

Copy each curve and write the sign of its gradient along the curve.

a



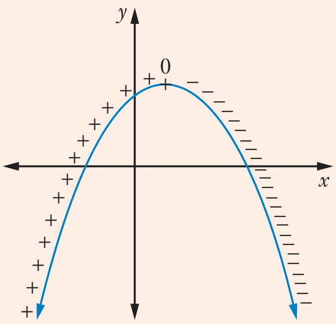
b



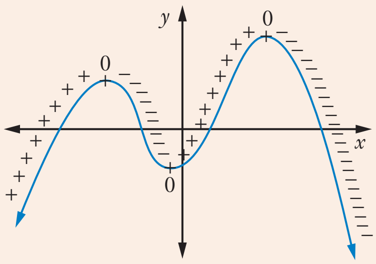
Solution

Where the curve increases, the gradient is positive. Where it decreases, it is negative. Where it is a turning point it has a zero gradien.

a



b



We find the gradient of a curve by measuring the **gradient of a tangent** to the curve at different points around the curve.

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We can then sketch the graph of these gradient values, which we call $y = f'(x)$, the **gradient function** or the **derivative function**.

EXAMPLE 3

a Make an accurate sketch of $f(x) = x^2$ on graph paper, or use graphing software.

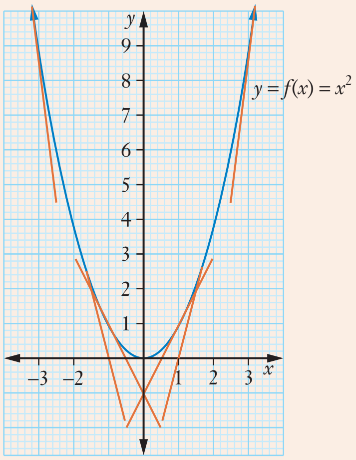
b Draw tangents to this curve at the points where $x = -3$, $x = -2$, $x = -1$, $x = 0$, $x = 1$, $x = 2$ and $x = 3$.

c Find the gradient of each of these tangents.

d Draw the graph of $y = f'(x)$ (the derivative or gradient function).

Solution

a and **b**



c At $x = -3$, $m = -6$

At $x = -2$, $m = -4$

At $x = -1$, $m = -2$

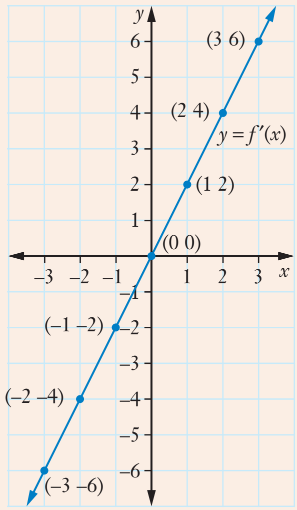
At $x = 0$, $m = 0$

At $x = 1$, $m = 2$

At $x = 2$, $m = 4$

At $x = 3$, $m = 6$

d Using the values from part **c**, $y = f'(x)$ is a linear function.



Notice in Example 3 that where $m > 0$, the gradient function is above the x-axis; where $m = 0$, the gradient function is on the x-axis; and where $m < 0$, the gradient function is below the x-axis. Since $m = f'(x)$, we can write the following:

Sketching gradient (derivative) functions

$f'(x) > 0$: gradient function is above the x-axis

$f'(x) < 0$: gradient function is below the x-axis

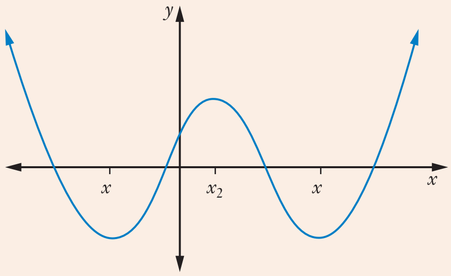
$f'(x) = 0$: gradient function is on the x-axis

Sketching gradient functions

EXAMPLE 4

Sketch a gradient function for each curve.

a



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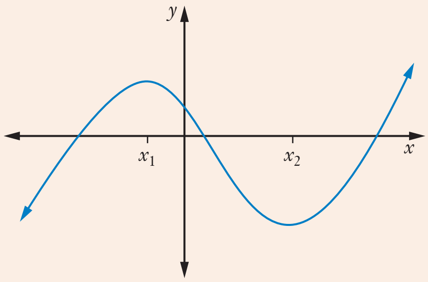
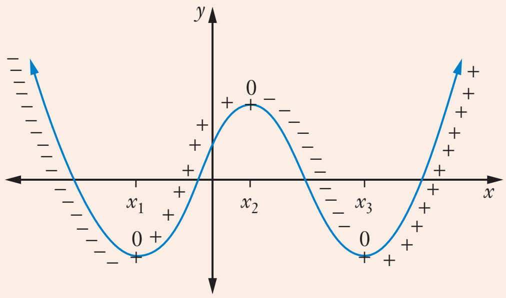
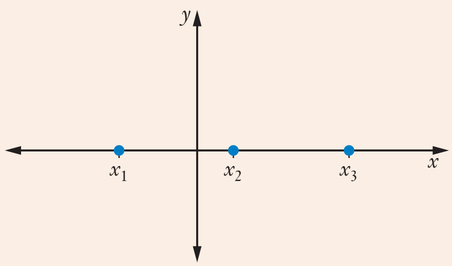


Solution

a First we mark in where the gradient is positive, negative and zero.



$f'(x) = 0$ at x_1, x_2 and x_3, so on the gradient graph these points will be on the x-axis (the x-intercepts of the gradient graph).



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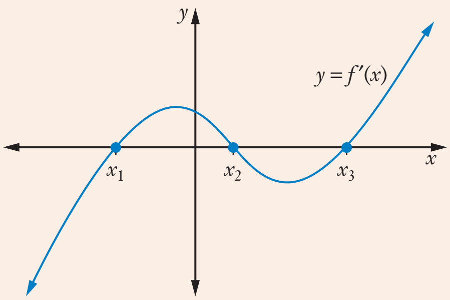
$f'(x) < 0$ to the left of x_1, so this part of the gradient graph will be below the x-axis.

$f'(x) > 0$ between x_1 and x_2, so the graph will be above the x-axis here.

$f'(x) < 0$ between x_2 and x_3, so the graph will be below the x-axis here.



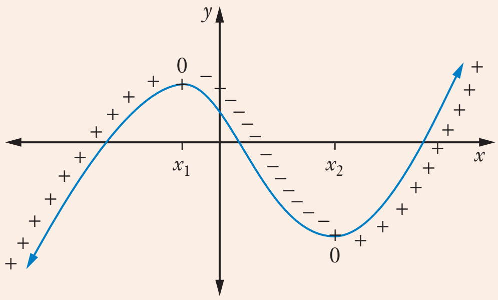
$f'(x) > 0$ to the right of x_3, so this part of the graph will be above the x-axis.

Sketching this information gives the graph of the gradient function $y = f'(x)$. Note that this is only a rough graph that shows the shape and sign rather than precise values.

b First mark in where the gradient is positive, negative and zero.

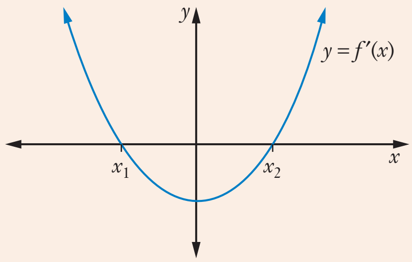
$f'(x) = 0$ at x_1 and x_2. These points will be the x-intercepts of the gradient function graph.

$f'(x) > 0$ to the left of x_1, so the graph will be above the x-axis here.



$f'(x) < 0$ between x_1 and x_2, so the graph will be below the x-axis here.

$f'(x) > 0$ to the right of x_2, so the graph will be above the x-axis here.



TECHNOLOGY

TANGENTS TO A CURVE

There are some excellent graphing software, online apps and websites that will draw tangents to a curve and sketch the gradient function.

Explore how to sketch gradient functions from the previous examples.

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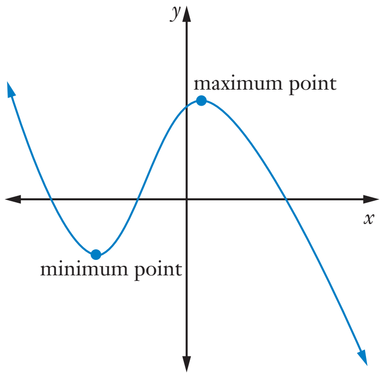
Stationary points

The points on a curve where the gradient $f'(x) = 0$ are called **stationary points** because the gradient there is neither increasing nor decreasing.

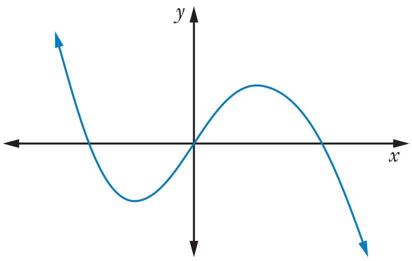
For example, the curve shown decreases to a **minimum turning point**, which is a type of **stationary point**. It then increases to a **maximum turning point** (also a stationary point) and then decreases again.



Exercise 8.01 Gradient of a curve

Sketch a gradient function for each curve.

1



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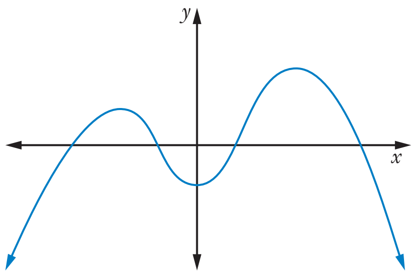

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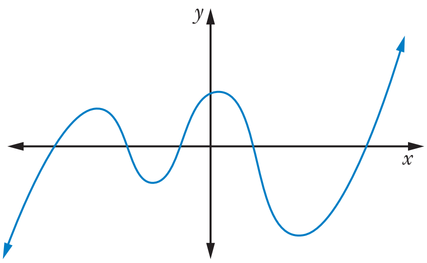

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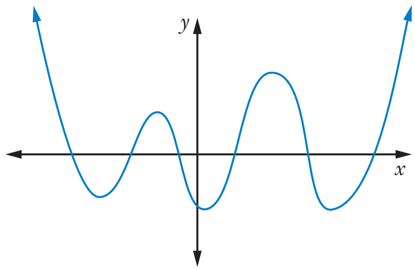

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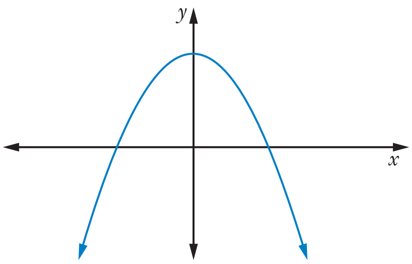

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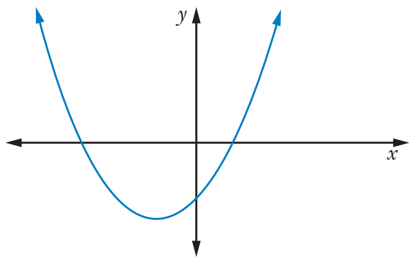

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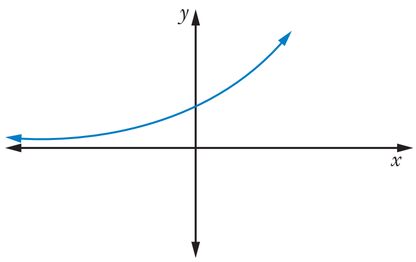

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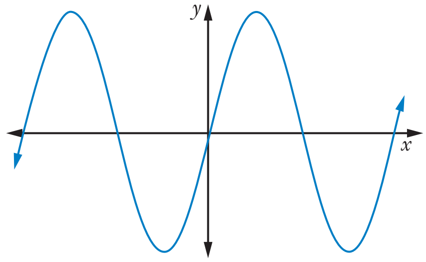

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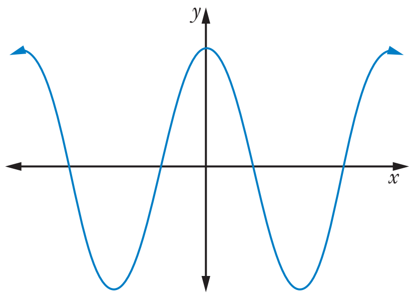

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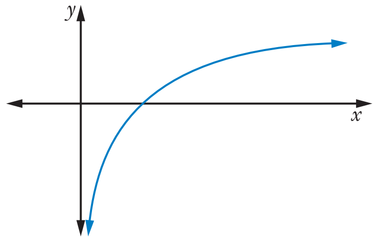


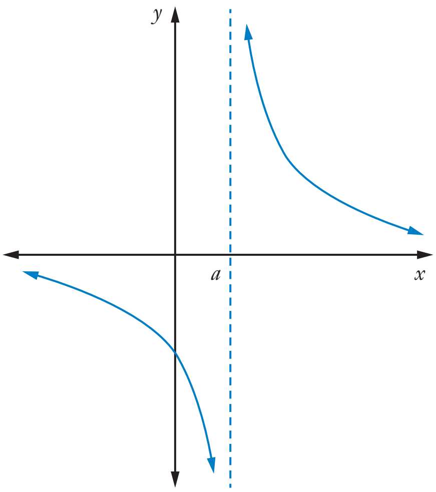
8.02 Differentiability

The process of finding the gradient function $y = f'(x)$ is called **differentiation**.

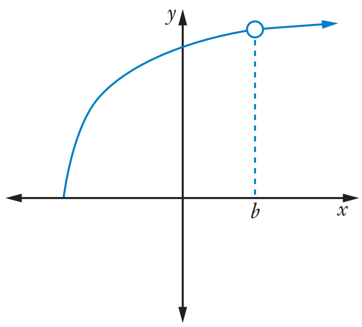
$y = f'(x)$ is called the **derivative function**, or just the **derivative**.

A function is **differentiable** at any point where it is continuous because we can find its gradient at that point. Linear, quadratic, cubic and other polynomial functions are differentiable at all points because their graphs are smooth and unbroken. A function is **not differentiable** at any point where it is **discontinuous**, where there is a gap or break in its graph.

This hyperbola is not differentiable at $x = a$ because the curve is discontinuous at this point.



This function is not differentiable at $x = b$ because the curve is discontinuous at this point.



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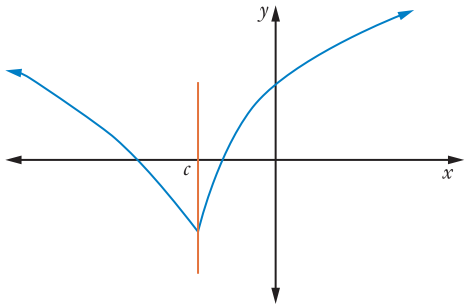
A function is also **not differentiable** where it is not smooth.

This function is not **differentiable** at $x = c$ since it is not smooth at that point. We cannot draw a unique tangent there so we cannot find the gradient of the function at that point.

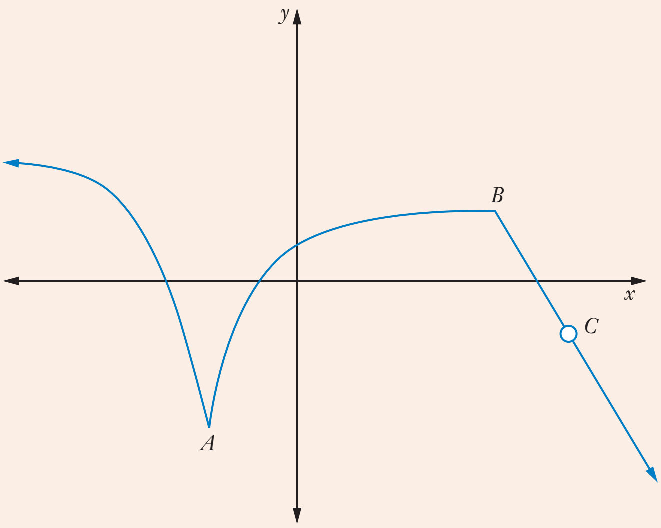


Differentiability at a point

A function $y = f(x)$ is **differentiable** at the point $x = a$ if its graph is **continuous** and **smooth** at $x = a$.

EXAMPLE 5

a Find all points where the function below is not differentiable.



b Is the function $f(x) = \begin{cases} x^2 & \text{for } x \geq 1 \\ 3x - 2 & \text{for } x < 1 \end{cases}$ differentiable at all points?

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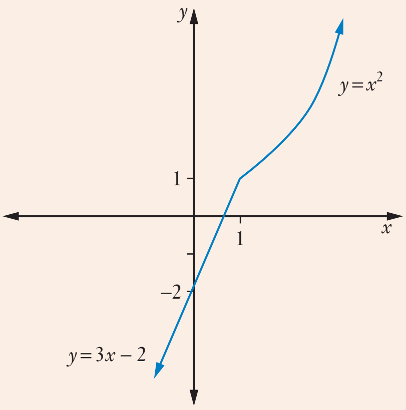
Solution

a The function is not differentiable at points A and B because the curve is not smooth at these points.

It is not differentiable at point C because the function is discontinuous at this point.

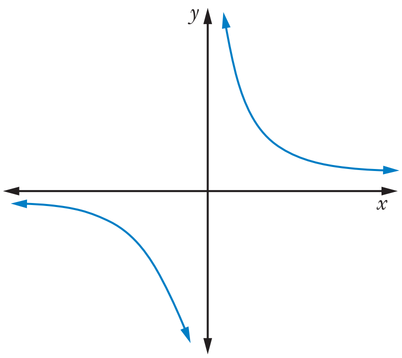
b Sketching this piecewise function shows that it is not smooth where the 2 parts meet, so it is not differentiable at $x = 1$.



Exercise 8.02 Differentiability

For each graph of a function, state any x values where the function is not differentiable.

1



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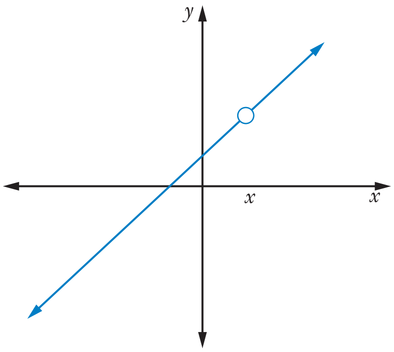

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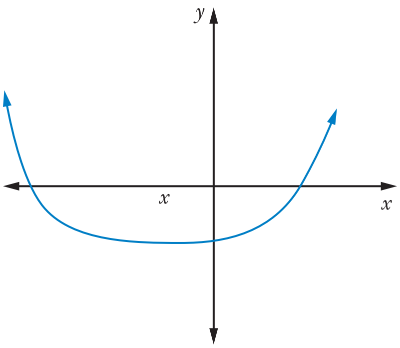

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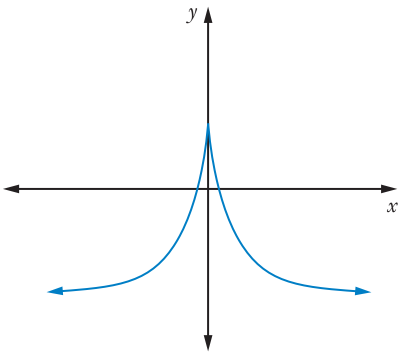

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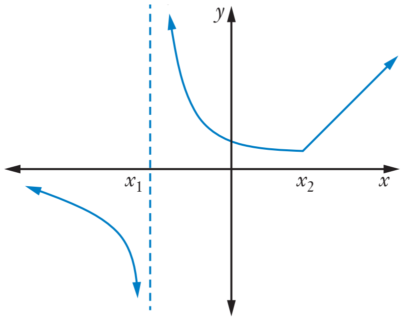


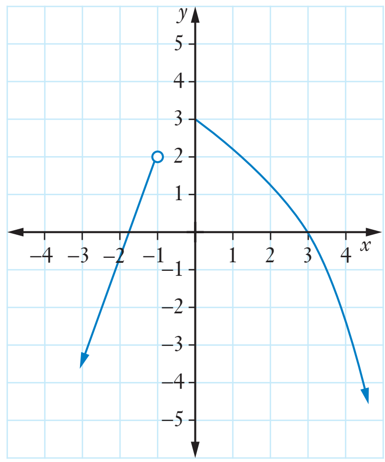
6 $f(x) = \dfrac{4}{x}$

7 $y = -\dfrac{1}{x+3}$

8 $f(x) = \begin{cases} x^3 & \text{for } x > 2 \\ x+1 & \text{for } x \leq 2 \end{cases}$

9 $f(x) = \begin{cases} 2x & \text{for } x > 3 \\ 3 & \text{for } -2 \leq x \leq 3 \\ 1 - x^2 & \text{for } x < -2 \end{cases}$

10



11 $f(x) = \dfrac{|x|}{x}$

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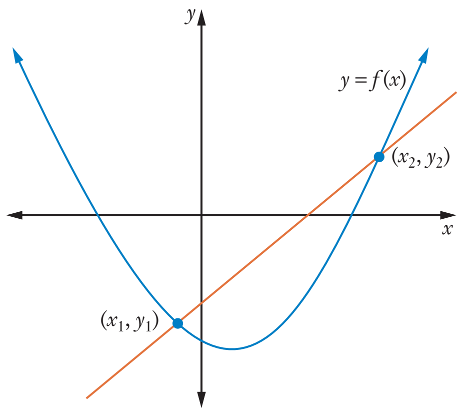
WS Finding derivatives from first principles

WS Rae o change - Gradients of secants

8.03 Differentiation from first principles

Gradient of a secant



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The line passing through the 2 points (x_1, y_1) and (x_2, y_2) on the graph of a function $y = f(x)$ is called a **secant**.

Differentiation from first principles

WS

imi

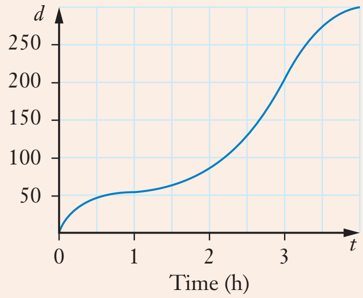
Gradient of the secant

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The **gradient of a secant** gives the **average rate of change** between the 2 points.

EXAMPLE 6

a This graph shows the distance d in km that a car travels over time t in hours. After 1 hour the car has travelled 55 km and after 3 hours the car has travelled 205 km. Find the average speed of the car.



b Given the function $f(x) = x^2$, find the average rate of change between $x = 1$ and $x = 1.1$.

Solution

a Speed is the change in distance over time.

The gradient of the secant will give the average speed.

Average rate of change:

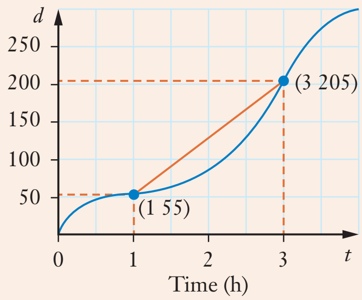
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{205 - 55}{3 - 1}$$

$$= \frac{150}{2}$$

$$= 75$$

So the average speed is 75 km/h.



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b When $x = 1$:

$f(1) = 1^2$

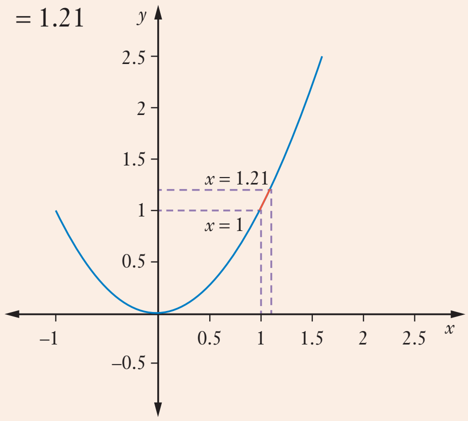
$= 1$

When $x = 1.1$:

$f(1.1) = 1.1^2$

$= 1.21$

So points are (1, 1) and (1.1, 1.21).



Average rate of change:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{121 - 1}{1\ 1 - 1}$$

$$= 2\ 1$$

So the average rate of change is 2.1.

Notice that the secant (orange interval) is very close to the shape of the curve itself. This is because the 2 points chosen are close together.

Estimating the gradient of a tangent

By taking 2 points close together, the **average rate of change** is quite close to the gradient of the tangent to the curve at one of those points, which is called the **instantaneous rate of change** at that point.

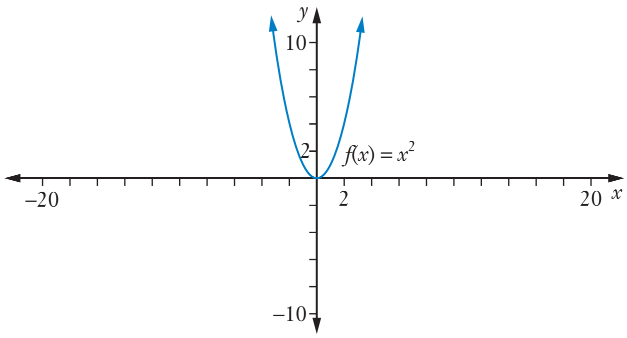
If you look at a close-up of a graph, you can get some idea of this concept. When the curve is magnified, any 2 points close together appear to be joined by a straight line. We say the curve is **locally straight**.

TECHNOLOGY

LOCALLY STRAIGHT CURVES

Use a graphics calculator or graphing software to sketch a curve and then zoom in on a section of the curve to see that it is locally straight.

For example, here is the parabola $y = x^2$.



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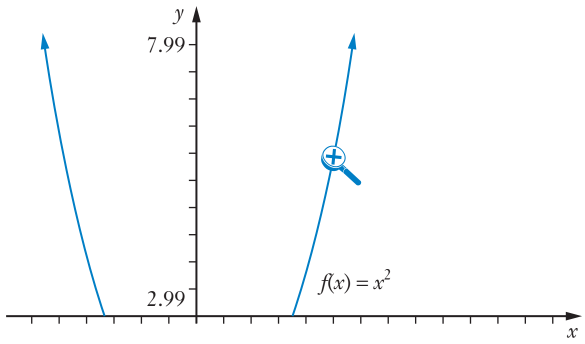
Notice how it looks straight when we zoom in on a point on the parabola.

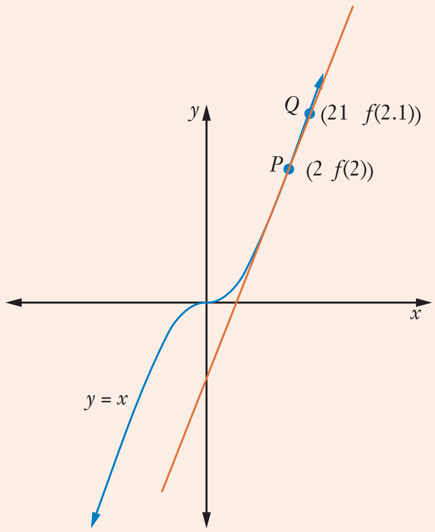


Use technology to sketch other curves and zoom in to show that they are locally straight.

We can calculate an approximate value for the gradient of the tangent at a point on a curve by taking another point close by, then calculating the gradient of the secant joining those 2 points.

EXAMPLE 7

a For $f(x) = x^3$, find the gradient of the secant PQ where P is the point on the curve where $x = 2$ and Q is another point on the curve where $x = 2.1$. Then choose different values for Q and use these results to estimate $f'(2)$, the gradient of the tangent to the curve at P.



b For the curve $y = x^2$, find the gradient of the secant AB where A is the point on the curve where $x = 5$ and point B is close to A. Find an estimate of the gradient of the tangent to the curve at A by using 3 different values for B.

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Solution

a P is $(2, f(2))$. Take different values of x for point Q, starting with $x = 2.1$, and find the gradient of the secant using $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Point Q	Gradient of secant PQ	Point Q	Gradient of secant PQ
$(2.1, f(2.1))$	$m = \frac{f(2\ 1) - f(2)}{2\ 1 - 2}$ $= \frac{2\ 1^3 - 2^3}{0\ 1}$ $= 1261$	$(1.9, f(1.9))$	$m = \frac{f(1\ 9) - f(2)}{1\ 9 - 2}$ $= \frac{1\ 9^3 - 2^3}{-0\ 1}$ $= 1141$
$(2.01, f(2.01))$	$m = \frac{f(2\ 01) - f(2)}{201 - 2}$ $= \frac{201^3 - 2^3}{001}$ $= 120601$	$(1.99, f(1.99))$	$m = \frac{f(1\ 99) - f(2)}{199 - 2}$ $= \frac{199^3 - 2^3}{-001}$ $= 119401$
$(2.001, f(2.001))$	$m = \frac{f(2\ 001) - f(2)}{2001 - 2}$ $= \frac{2001^3 - 2^3}{0001}$ $= 12006\ 001$	$(1.999, f(1.999))$	$m = \frac{f(1\ 999) - f(2)}{1999 - 2}$ $= \frac{1999^3 - 2^3}{-0001}$ $= 11994\ 001$

From these results, we can see that a good estimate for $f'(2)$, the gradient at P, is 12.

As $x \to 2, f'(2) \to 12$.

We use a special notation for **limits** to show this.

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$
$$= 12$$

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b $A = (5, f(5))$

Take 3 different values of x for point B; for example, $x = 4.9$, $x = 5.1$ and $x = 5.01$.

$B = (4.9, f(4.9))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(4\ 9) - f(5)}{4\ 9 - 5}$$
$$= \frac{4\ 9^2 - 5^2}{-0\ 1}$$
$$= 9\ 9$$

$B = (5.1, f(5.1))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(5\ 1) - f(5)}{5\ 1 - 5}$$
$$= \frac{5\ 1^2 - 5^2}{0\ 1}$$
$$= 10\ 1$$

$B = (5.01, f(5.01))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{f(5\ 0\ 1) - f(5)}{501 - 5}$$
$$= \frac{501^2 - 5^2}{001}$$
$$= 1001$$

As $x \to 5, f'(5) \to 10$.

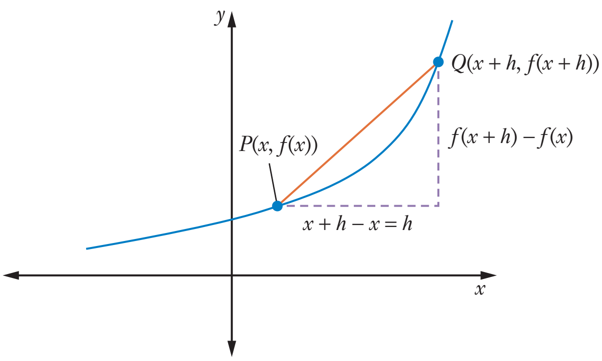
$$f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5}$$
$$= 10$$

The difference quotient

We measure the instantaneous rate of change of any point on the graph of a function by using limits to find the gradient of the tangent to the curve at that point. This is called **differentiation from first principles**. Using the method from the examples above, we can find a general formula for the derivative function $y = f'(x)$.

We want to find the instantaneous rate of change or gradient of the tangent to a curve $y = f(x)$ at point $P(x, f(x))$.

We choose a second point Q close to P with coordinates $(x + h, f(x + h))$ where h is small.



Now find the gradient of the secant PQ.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x+h) - f(x)}{x+h-x}$$

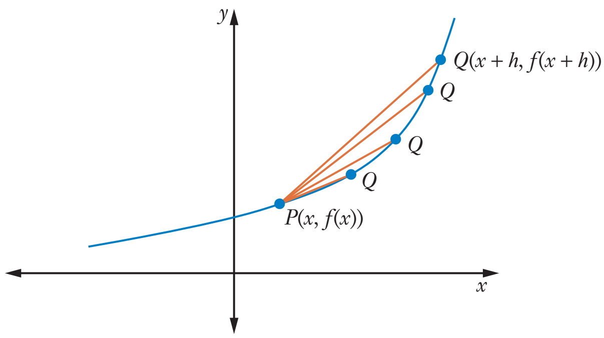
$$= \frac{f(x+h) - f(x)}{h}$$

$\dfrac{f(x+h) - f(x)}{h}$ is called the **difference quotient** and it gives an **average rate of change**.

To find the gradient of the tangent at P, we make h smaller as shown, so that Q becomes closer and closer to P.

As h approaches 0, the gradient of the tangent becomes $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

We call this $f'(x)$ or $\dfrac{dy}{dx}$ or y'.



Differentiation from first principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

INVESTIGATION

CALCULUS NOTATION

On p.412, we learned about the mathematicians Isaac Newton and Gottfried Leibniz. Newton used the notation $f'(x)$ for the derivative function while Leibniz used the notation $\dfrac{dy}{dx}$ where d stood for 'difference'. Can you see why he would have used this?

Use the Internet to explore the different notations used in calculus and where they came from.

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EXAMPLE 8

a Differentiate from first principles to find the gradient of the tangent to the curve $y = x^2 + 3$ at the point where $x = 1$.

b Differentiate $f(x) = 2x^2 + 7x - 3$ from first principles.

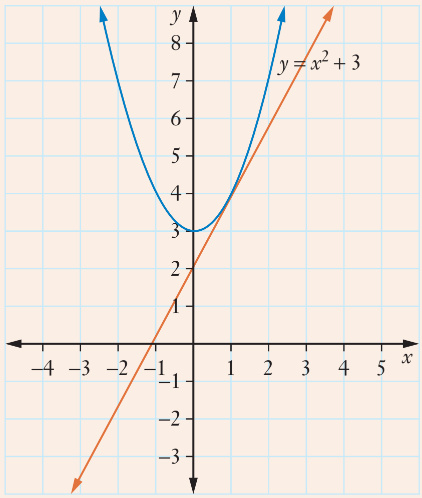
Solution

a

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = x^2 + 3$$

$$f(x+h) = (x+h)^2 + 3$$
$$= x^2 + 2xh + h^2 + 3$$



Substitute $x = 1$:

$$f(1) = 1^2 + 3$$
$$= 4$$

$$f(1+h) = 1^2 + 2(1)h + h^2 + 3$$
$$= 4 + 2h + h^2$$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{4 + 2h + h^2 - 4}{h}$$
$$= \lim_{h \to 0} \frac{2h + h^2}{h}$$
$$= \lim_{h \to 0} \frac{h(2+h)}{h}$$
$$= \lim_{h \to 0} (2+h)$$
$$= 2 + 0$$
$$= 2$$

So the gradient of the tangent to the curve $y = x^2 + 3$ at the point (1, 4) is 2.

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b

$$f(x) = 2x^2 + 7x - 3$$
$$f(x+h) = 2(x+h)^2 + 7(x+h) - 3$$
$$= 2(x^2 + 2xh + h^2) + 7x + 7h - 3$$
$$= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3$$
$$f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - (2x^2 + 7x - 3)$$
$$= 2x^2 + 4xh + 2h^2 + 7x + 7h - 3 - 2x^2 - 7x + 3$$
$$= 4xh + 2h^2 + 7h$$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{4xh + 2h^2 + 7h}{h}$$
$$= \lim_{h \to 0} \frac{h(4x + 2h + 7)}{h}$$
$$= \lim_{h \to 0} (4x + 2h + 7)$$
$$= 4x + 0 + 7$$
$$= 4x + 7$$

So the gradient function (derivative) of $f(x) = 2x^2 + 7x - 3$ is $f'(x) = 4x + 7$.

Exercise 8.03 Differentiation from first principles

1 **a** For the curve $y = x^4 + 1$, find the gradient of the secant between the point (1, 2) and the point where $x = 1.01$.

b Find the gradient of the secant between (1, 2) and the point where $x = 0.999$ on the curve.

c Use these results to find an approximation to the gradient of the tangent to the curve $y = x^4 + 1$ at the point (1, 2).

2 For the function $f(x) = x^3 + x$, find the average rate of change between the point (2, 10) and the point on the curve where:

a $x = 2.1$ **b** $x = 2.01$ **c** $x = 1.99$

d Hence find an approximation to the gradient of the tangent at the point (2, 10).

3 For the function $f(x) = x^2 - 4$, find the gradient of the tangent at point P where $x = 3$ by selecting points near P and finding the gradient of the secant.

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4 A function is given by $f(x) = x^2 + x + 5$.

a Find $f(2)$

b Find $f(2 + h)$

c Find $f(2 + h) - f(2)$

d Show that $\dfrac{f(2+h)-f(2)}{h} = 5 + h$

e Find $f'(2)$

5 Given the curve $f(x) = 4x^2 - 3$ fin:

a $f(-1)$ **b** $f(-1 + h) - f(-1)$

c the gradient of the tangent to the curve at the point where $x = -1$

6 For the parabola $y = x^2 - 1$, fin:

a $f(3)$ **b** $f(3 + h) - f(3)$ **c** $f'(3)$

7 For the function $f(x) = 4 - 3x - 5x^2$ fin:

a $f'(1)$ **b** the gradient of the tangent at the point (–2 –10.

8 If $f(x) = x^2$

a find $f(x + h)$ **b** show that $f(x + h) - f(x) = 2xh + h^2$

c show that $\dfrac{f(x+h)-f(x)}{h} = 2x + h$ **d** show that $f'(x) = 2x$

9 A function is given by $f(x) = 2x^2 - 7x + 3$.

a Show that $f(x + h) = 2x^2 + 4xh + 2h^2 - 7x - 7h + 3$.

b Show that $f(x + h) - f(x) = 4xh + 2h^2 - 7h$

c Show that $\dfrac{f(x+h)-f(x)}{h} = 4x + 2h - 7$.

d Find $f'(x)$

10 Differentiate from first principles to find the gradient of the tangent to the curve

a $f(x) = x^2$ at the point where $x = 1$

b $y = x^2 + x$ at the point (2 6)

c $f(x) = 2x^2 - 5$ at the point where $x = -3$

d $y = 3x^2 + 3x + 1$ at the point where $x = 2$

e $f(x) = x^2 - 7x - 4$ at the point (–1 4.

11 Find the derivative function for each function from first principles

a $f(x) = x^2$ **b** $y = x^2 + 5x$

c $f(x) = 4x^2 - 4x - 3$ **d** $y = 5x^2 - x - 1$

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Deivaive o linear products

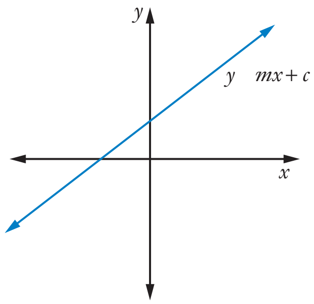
WS
Deivaive o polynomials

8.04 Short methods of differentiation

Derivative of x^n

Remember that the gradient of a straight line $y = mx + c$ is m. The tangent to the line is the line itself, so the gradient of the tangent is m everywhere along the line.

So if $y = mx$, $\frac{dy}{dx} = m$.

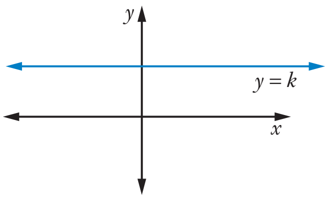


Derivative of kx

$$\frac{d}{dx}(kx) = k$$

A horizontal line $y = k$ has a gradient of zero.

So if $y = k$, $\frac{dy}{dx} = 0$.



Derivative of k

$$\frac{d}{dx}(k) = 0$$

TECHNOLOGY

DIFFERENTIATION OF POWERS OF X

Find an approximation to the derivative of power functions such as $y = x^2, y = x^3, y = x^4$, $y = x^5$ by drawing the graph of $y = \frac{f(x+001) - f(x)}{001}$. You could use a graphics calculator or graphing software/website to sketch the derivative for these functions and find its equation. Can you find a pattern? Could you predict what the result would be for x^n?

When differentiating $y = x^n$ from first principles, a simple pattern appears:

- For $y = x$, $f'(x) = 1x^0 = 1$
- For $y = x^2$, $f'(x) = 2x^1 = 2x$
- For $y = x^3$, $f'(x) = 3x^2$
- For $y = x^4$, $f'(x) = 4x^3$
- For $y = x^5$, $f'(x) = 5x^4$

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Derivative of x^n

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$.

There are some more properties of differentiation.

Derivative of kx^n

$$\frac{d}{dx}(kx^n) = knx^{n-1}$$

More generally:

Derivative of a constant multiple of a function

$$\frac{d}{dx}\left(kf(x)\right) = kf'(x)$$

EXAMPLE 9

a Find the derivative of $3x^8$.

b Differentiate $f(x) = 7x^3$.

Solution

a $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\frac{d}{dx}(3x^8) = 3 \times 8x^{8-}$

$= 24x^7$

b $f'(x) = knx^{n-}$

$f'(x) = 7 \times 3x^{3-1}$

$= 21x^2$

If there are several terms in an expression, we differentiate each one separately.

Derivative of a sum of functions

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

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EXAMPLE 10

a Differentiate $x^3 + x^4$.

b Find the derivative of $7x$.

c Differentiate $f(x) = x^4 - x^3 + 5$.

d Find the derivative of $y = 4x^7$.

e If $f(x) = 2x^5 - 7x^3 + 5x - 4$, evaluate $f'(-1)$.

f Find the derivative of $f(x) = 2x^2(3x - 7)$.

g Find the derivative of $\frac{3x^2 + 5x}{2x}$.

h Differentiate $S = 6r^2 - 12r$ with respect to r.

Solution

a $\frac{d}{dx}(x^3 + x^4) = 3x^2 + 4x^3$

b $\frac{d}{dx}(7x) = 7$

c $f'(x) = 4x^3 - 3x^2 + 0$

$= 4x^3 - 3x^2$

d $\frac{dy}{dx} = 4 \times 7x^6$

$= 28x^6$

e $f'(x) = 10x^4 - 21x^2 + 5$

$f'(-1) = 10(-1)^4 - 21(-1)^2 + 5$

$= -6$

f Expand first.

$f(x) = 2x^2(3x - 7)$

$= 6x^3 - 14x^2$

$f'(x) = 18x^2 - 28x$

g Simplify first.

$$\frac{3x^2 + 5x}{2x} = \frac{3x^2}{2x} + \frac{5x}{2x}$$

$$= \frac{3x}{2} + \frac{5}{2}$$

$$\frac{d}{dx}\left(\frac{3x^2 + 5x}{2x}\right) = \frac{3}{2}$$

$$= 1\frac{1}{2}$$

h Differentiating with respect to r rather than x:

$S = 6r^2 - 12r$

$\frac{dS}{dr} = 12r - 12$

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INVESTIGATION

FAMILIES OF CURVES

1 Differentiate:

a $x^2 + 1$ **b** $x^2 - 3$ **c** $x^2 + 7$

d x^2 **e** $x^2 + 20$ **f** $x^2 - 100$

What do you notice?

2 Differentiate:

a $x^3 + 5$ **b** $x^3 + 11$ **c** $x^3 - 1$

d $x^3 - 6$ **e** x^3 **f** $x^3 + 15$

What do you notice?

These groups of functions are families because they have the same derivatives. Can you find others?

Exercise 8.04 Short methods of differentiation

1 Differentiate:

a $x + 2$ **b** $5x - 9$ **c** $x^2 + 3x + 4$

d $5x^2 - x - 8$ **e** $x^3 + 2x^2 - 7x - 3$ **f** $2x^3 - 7x^2 + 7x - 1$

g $3x^4 - 2x^2 + 5x$ **h** $x^6 - 5x^5 - 2x^4$ **i** $2x^5 - 4x^3 + x^2 - 2x + 4$

j $4x^{10} - 7x^9$

2 Find the derivative of:

a $x(2x + 1)$ **b** $(2x - 3)^2$ **c** $(x + 4)(x - 4)$

d $(2x^2 - 3)^2$ **e** $(2x + 5)(x^2 - x + 1)$

3 Find the derivative of:

a $\frac{x^2}{6} - x$ **b** $\frac{x^4}{2} - \frac{x^3}{3} + 4$ **c** $\frac{1}{3}x^6(x^2 - 3)$

d $\frac{2x^3 + 5x}{x}$ **e** $\frac{x^2 + 2x}{4x}$ **f** $\frac{2x^5 - 3x^4 + 6x^3 - 2x^2}{3x^2}$

4 Find $f'(x)$ when $f(x) = 8x^2 - 7x + 4$.

5 If $y = x^4 - 2x^3 + 5$, find $\frac{dy}{dx}$ when $x = -2$.

6 Find $\frac{dy}{dx}$ if $y = 6x^{10} - 5x^8 + 7x^5 - 3x + 8$.

7 If $s = 5t^2 - 20t$, find $\frac{ds}{dt}$.

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8 Find $g'(x)$ given $g(x) = 5x^4$.

9 Find $\frac{dv}{dt}$ when $v = 15t^2 - 9$.

10 If $h = 40t - 2t^2$, find $\frac{dh}{dt}$.

11 Given $V = \frac{4}{3}\pi r^3$, find $\frac{dV}{dr}$.

12 If $f(x) = 2x^3 - 3x + 4$, evaluate $f'(1)$.

13 Given $f(x) = x^2 - x + 5$, evaluate:

a $f'(3)$ **b** $f'(-2)$ **c** x when $f'(x) = 7$

14 If $y = x^3 - 7$, evaluate:

a the derivative when $x = 2$ **b** x when $\frac{dy}{dx} = 12$

15 Evaluate $g'(2)$ when $g(t) = 3t^3 - 4t^2 - 2t + 1$.

DID YOU KNOW?

Motion and calculus

Galileo

Galileo (1564–1642) was very interested in the behaviour of bodies in motion. He dropped stones from the Leaning Tower of Pisa to try to prove that they would fall with equal speed. He rolled balls down slopes to prove that they move with uniform speed until friction slows them down. He showed that a body moving through the air follows a curved path at a fairly constant speed.

John Wallis (1616–1703) continued this study with his publication *Mechanica, sive Tractatus de Motu Geometricus*. He applied mathematical principles to the laws of motion and stimulated interest in the subject of mechanics.

Soon after Wallis' publication, **Christiaan Huygens** (1629–1695) wrote *Horologium Oscillatorium sive de Motu Pendulorum*, in which he described various mechanical principles. He invented the pendulum clock, improved the telescope and investigated circular motion and the descent of heavy bodies.

These three mathematicians provided the foundations of mechanics. **Sir Isaac Newton** (1642–1727) used calculus to increase the understanding of the laws of motion. He also used these concepts as a basis for his theories on gravity and inertia.

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8.05 Derivatives and indices

INVESTIGATION

DERIVATIVES AND INDICES

1 **a** Show that $\frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)}$.

b Hence differentiate $y = \frac{1}{x}$ from first principles.

c Differentiate $y = x^{-1}$ using the formula. Do you get the same answer as in part **b**?

2 **a** Show that $\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right) = h$.

b Hence differentiate $y = \sqrt{x}$ from first principles.

c Differentiate $y = x^{\frac{1}{2}}$ and show that this gives the same answer as in part **b**.

EXAMPLE 11

a Differentiate $f(x) = 7\sqrt[3]{x}$.

b Find the derivative of $y = \frac{4}{x^2}$ at the point where $x = 2$.

Solution

a $f(x) = 7\sqrt[3]{x} = 7x^{\frac{1}{3}}$ ← Convert the function to a power of *x* first.

$f'(x) = 7 \times \frac{1}{3}x^{\frac{1}{3}-1}$

$= \frac{7}{3}x^{-\frac{2}{3}}$

$= \frac{7}{3} \times \frac{1}{x^{\frac{2}{3}}}$

$= \frac{7}{3} \times \frac{1}{\sqrt[3]{x^2}}$

$= \frac{7}{3\sqrt[3]{x^2}}$

b $y = \frac{4}{x^2}$

$= 4x^{-2}$

$\frac{dy}{dx} = -8x^{-3}$

$= -\frac{8}{x^3}$

When $x = 2$:

$\frac{dy}{dx} = -\frac{8}{2^3}$

$= -1$

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Exercise 8.05 Derivatives and indices

1 Differentiate:

a x^{-3} **b** $x^{1.4}$ **c** $6x^{0.2}$ **d** $x^{\frac{}{2}}$

e $2x^{\frac{}{2}} - 3x^{-}$ **f** $3x^{\frac{}{3}}$ **g** $8x^{\frac{3}{4}}$ **h** $-2x^{-\frac{}{2}}$

2 Find the derivative function.

a $\frac{1}{x}$ **b** $5\sqrt{x}$ **c** $\sqrt[6]{x}$ **d** $\frac{2}{x^5}$

e $-\frac{5}{x^3}$ **f** $\frac{1}{\sqrt{x}}$ **g** $\frac{1}{2x^6}$ **h** $x\sqrt{x}$

i $\frac{2}{3x}$ **j** $\frac{1}{4x^2} + \frac{3}{x^4}$

3 Find the derivative of $y = \sqrt[3]{x}$ at the point where $x = 27$.

4 If $x = \frac{12}{t}$, find $\frac{dx}{dt}$ when $t = 2$.

5 A function is given by $f(x) = \sqrt[4]{x}$. Evaluate $f'(16)$.

6 Find the derivative of $y = \frac{3}{2x^2}$ at the point $\left(1\ 1\frac{1}{2}\right)$.

7 Find $\frac{dy}{dx}$ if $y = \left(x + \sqrt{x}\right)^2$.

8 A function $f(x) = \frac{\sqrt{x}}{2}$ has a tangent at (4, 1). Find its gradient.

9 **a** Differentiate $\frac{\sqrt{x}}{x}$.

b Hence find the derivative of $y = \frac{\sqrt{x}}{x}$ at the point where $x = 4$.

10 The function $f(x) = 3\sqrt{x}$ has $f'(x) = \frac{3}{4}$ at $x = a$. Find a.

11 The hyperbola $y = \frac{2}{x}$ has 2 tangents with gradient $-\frac{2}{25}$. Find the points where these tangents touch the hyperbola.

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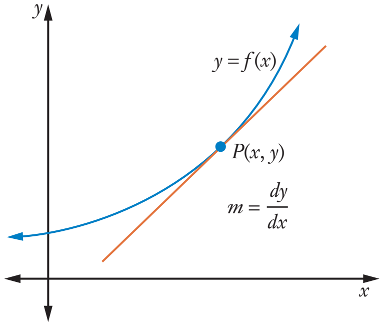
8.06 Tangents and normals

Tangents to a curve

Remember that the derivative is a function that gives the instantaneous rate of change or gradient of the tangent to the curve.

A tangent is a line so we can use the formula $y = mx + c$ or $y - y_1 = m(x - x_1)$ to find its equation.



WS Tanges and normals

Equation of a tangent

WS Slope o cuvs

WS Tangesto a curve

EXAMPLE 12

a Find the gradient of the tangent to the parabola $y = x^2 + 1$ at the point (1, 2).

b Find values of x for which the gradient of the tangent to the curve $y = 2x^3 - 6x^2 + 1$ is equal to 18.

c Find the equation of the tangent to the curve $y = x^4 - 3x^3 + 7x - 2$ at the point (2, 4).

Solution

a The gradient of a tangent to a curve is $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 2x + 0$$
$$= 2x$$

Substitute $x = 1$ from the point (1, 2):

$$\frac{dy}{dx} = 2(1)$$
$$= 2$$

So the gradient of the tangent at (1, 2) is 2.

b $\frac{dy}{dx} = 6x^2 - 12x$

Gradient is 18 so $\frac{dy}{dx} = 18$.

$$18 = 6x^2 - 12x$$
$$0 = 6x^2 - 12x - 18$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$\therefore x = 3, -1$$

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c $$\frac{dy}{dx} = 4x^3 - 9x^2 + 7$$

At (2, 4), $\frac{dy}{dx} = 4(2)^3 - 9(2)^2 + 7$

$= 3$

So the gradient of the tangent at (2, 4) is 3.

Equation of the tangent:

$y - y_1 = m(x - x_1)$

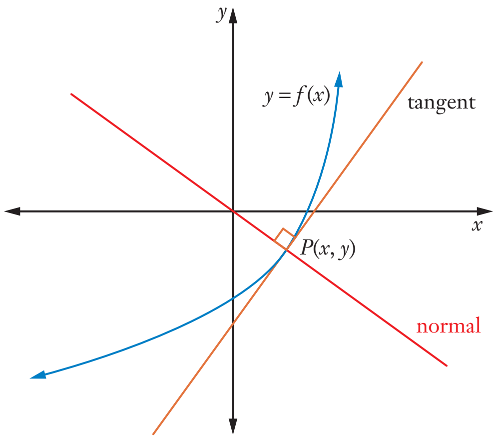
$y - 4 = 3(x - 2)$

$= 3x - 6$

$y = 3x - 2$ or $3x - y - 2 = 0$

Normals to a curve

The **normal** is a straight line **perpendicular** to the tangent at the same point of contact with the curve.



Remember the rule for perpendicular lines from Chapter 4, *Functions*:

Gradients of perpendicular lines

If 2 lines with gradients m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$ or $m_2 = -\frac{1}{m}$.

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EXAMPLE 13

a Find the gradient of the normal to the curve $y = 2x^2 - 3x + 5$ at the point where $x = 4$.

b Find the equation of the normal to the curve $y = x^3 + 3x^2 - 2x - 1$ at $(-1, 3)$.

Solution

a $\frac{dy}{dx} = 4x - 3$

When $x = 4$:

$\frac{dy}{dx} = 4 \times 4 - 3$

$= 13$

So $m_1 = 13$

The normal is perpendicular to the tangent, so $m_1 m_2 = -1$.

$13m_2 = -1$

$m_2 = -\frac{1}{13}$

So the gradient of the normal is $-\frac{1}{13}$.

b $\frac{dy}{dx} = 3x^2 + 6x - 2$

When $x = -1$:

$\frac{dy}{dx} = 3(-1)^2 + 6(-1) - 2$

$= -5$

So $m_1 = -5$

The normal is perpendicular to the tangent, so $m_1 m_2 = -1$.

$-5m_2 = -1$

$m_2 = \frac{1}{5}$

So the gradient of the normal is $\frac{1}{5}$.

Equation of the normal: $y - y_1 = m(x - x_1)$

$y - 3 = \frac{1}{5}(x - (-1))$

$5y - 15 = x + 1$

$x - 5y + 16 = 0$

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Exercise 8.06 Tangents and normals

1 Find the gradient of the tangent to the curve:

a $y = x^3 - 3x$ at the point where $x = 5$

b $f(x) = x^2 + x - 4$ at the point $(-7, 38)$

c $f(x) = 5x^3 - 4x - 1$ at the point where $x = -1$

d $y = 5x^2 + 2x + 3$ at $(-2, 19)$

e $y = 2x^9$ at the point where $x = 1$

f $f(x) = x^3 - 7$ at the point where $x = 3$

g $v = 2t^2 + 3t - 5$ at the point where $t = 2$

h $Q = 3r^3 - 2r^2 + 8r - 4$ at the point where $r = 4$

i $h = t^4 - 4t$ where $t = 0$

j $f(t) = 3t^5 - 8t^3 + 5t$ at the point where $t = 2$.

2 Find the gradient of the normal to the curve:

a $f(x) = 2x^3 + 2x - 1$ at the point where $x = -2$

b $y = 3x^2 + 5x - 2$ at $(-5, 48)$

c $f(x) = x^2 - 2x - 7$ at the point where $x = -9$

d $y = x^3 + x^2 + 3x - 2$ at $(-4, -62)$

e $f(x) = x^{10}$ at the point where $x = -1$

f $y = x^2 + 7x - 5$ at $(-7, -5)$

g $A = 2x^3 + 3x^2 - x + 1$ at the point where $x = 3$

h $f(a) = 3a^2 - 2a - 6$ at the point where $a = -3$.

i $V = h^3 - 4h + 9$ at $(2, 9)$

j $g(x) = x^4 - 2x^2 + 5x - 3$ at the point where $x = -1$.

3 Find the gradient of **i** the tangent and **ii** the normal to the curve:

a $y = x^2 + 1$ at $(3, 10)$

b $f(x) = 5 - x^2$ where $x = -4$

c $y = 2x^5 - 7x^2 + 4$ where $x = -1$

d $p(x) = x^6 - 3x^4 - 2x + 8$ where $x = 1$

e $f(x) = 4 - x - x^2$ at $(-6, 26)$

4 Find the equation of the tangent to the curve:

a $y = x^4 - 5x + 1$ at $(2, 7)$

b $f(x) = 5x^3 - 3x^2 - 2x + 6$ at $(1, 6)$

c $y = x^2 + 2x - 8$ at $(-3, -5)$

d $y = 3x^3 + 1$ where $x = 2$

e $v = 4t^4 - 7t^3 - 2$ where $t = 2$

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5 Find the equation of the normal to the curve:

a $f(x) = x^3 - 3x + 5$ at (3, 23)

b $y = x^2 - 4x - 5$ at (–2, 7)

c $f(x) = 7x - 2x^2$ where $x = 6$

d $y = 7x^2 - 3x - 3$ at (–3, 69)

e $y = x^4 - 2x^3 + 4x + 1$ where $x = 1$

6 Find the equation of **i** the tangent and **ii** the normal to the curve:

a $f(x) = 4x^2 - x + 8$ at (1, 11)

b $y = x^3 - 2x^2 - 5x$ at (–3, –30)

c $F(x) = x^5 - 5x^3$ where $x = 1$

d $y = x^2 - 8x + 7$ at (3, –8).

7 For the curve $y = x^3 - 27x - 5$, find values of x for which $\frac{dy}{dx} = 0$.

8 Find the coordinates of the points at which the curve $y = x^3 + 1$ has a tangent with a gradient of 3.

9 A function $f(x) = x^2 + 4x - 12$ has a tangent with a gradient of –6 at point P on the curve. Find the coordinates of P.

10 The tangent at point P on the curve $y = 4x^2 + 1$ is parallel to the x-axis. Find the coordinates of P.

11 Find the coordinates of point Q where the tangent to the curve $y = 5x^2 - 3x$ is parallel to the line $7x - y + 3 = 0$.

12 Find the coordinates of point S where the tangent to the curve $y = x^2 + 4x - 1$ is perpendicular to the line $4x + 2y + 7 = 0$.

13 The curve $y = 3x^2 - 4$ has a gradient of 6 at point A.

a Find the coordinates of A.

b Find the equation of the tangent to the curve at A.

14 A function $h = 3t^2 - 2t + 5$ has a tangent at the point where $t = 2$. Find the equation of the tangent.

15 A function $f(x) = 2x^2 - 8x + 3$ has a tangent parallel to the line $4x - 2y + 1 = 0$ at point P. Find the equation of the tangent at P.

16 Find the equation of the tangent to the curve $y = \frac{1}{x^3}$ at $\left(2\ \frac{1}{8}\right)$.

17 Find the equation of the tangent to $f(x) = 6\sqrt{x}$ at the point where $x = 9$.

18 Find the equation of the tangent to the curve $y = \frac{4}{x}$ at $\left(8\ \frac{1}{2}\right)$.

19 If the gradient of the tangent to $y = \sqrt{x}$ is $\frac{1}{6}$ at point A, find the coordinates of A.

8.07 Chain rule

Chain ule

We looked at composite functions in Chapter 7, *Further functions*.

The **chain rule** is a method for differentiating composite functions. It is also called the **composite function** rule or the '**function of a function**' **rule**.

The chain rule

If a function y can be written as a composite function where $y = f(u(x))$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The chain rule

EXAMPLE 14

Differentiate:

a $y = (5x + 4)^7$ **b** $y = (3x^2 + 2x - 1)^9$ **c** $y = \sqrt{3 - x}$

Solution

a Let $u = 5x + 4$

Then $\frac{du}{dx} = 5$

$y = u^7$

$\therefore \frac{dy}{du} = 7u^6$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 7u^6 \times 5 \\ &= 35u^6 \\ &= 35(5x + 4)^6\end{aligned}$$

b Let $u = 3x^2 + 2x - 1$

Then $\frac{du}{dx} = 6x + 2$

$y = u^9$

$\therefore \frac{dy}{du} = 9u^8$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 9u^8 \times (6x + 2) \\ &= 9(3x^2 + 2x - 1)^8(6x + 2) \\ &= 9(6x + 2)(3x^2 + 2x - 1)^8\end{aligned}$$

c $y = \sqrt{3 - x} = (3 - x)^{\frac{1}{2}}$

Let $u = 3 - x$

Then $\frac{du}{dx} = -1$

$y = u^{\frac{1}{2}}$

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{2}u^{-\frac{1}{2}} \times (-1) \\ &= -\frac{1}{2}(3 - x)^{-\frac{1}{2}} \\ &= -\frac{1}{2\sqrt{3 - x}}\end{aligned}$$

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You might see a pattern when using the chain rule. The derivative of a composite function is the product of the derivatives of 2 functions.

The derivative of $[f(x)]^n$

$$\frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-1}$$

EXAMPLE 15

Differentiate:

a $y = (8x^3 - 1)^5$ **b** $y = (3x + 8)^{11}$ **c** $y = \frac{1}{(6x+1)^2}$

Solution

a $\frac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$

$= 24x^2 \times 5(8x^3 - 1)^4$

$= 120x^2(8x^3 - 1)^4$

b $\frac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$

$= 3 \times 11(3x + 8)^{10}$

$= 33(3x + 8)^{10}$

c $y = \frac{1}{(6x+1)^2} = (6x+1)^{-2}$

$\frac{dy}{dx} = f'(x) \times n[f(x)]^{n-1}$

$= 6 \times (-2)(6x + 1)^{-3}$

$= -12(6x + 1)^{-3}$

$= -\frac{12}{(6x+1)^3}$

Exercise 8.07 Chain rule

1 Differentiate:

a $y = (x + 3)^4$ **b** $y = (2x - 1)^3$ **c** $y = (5x^2 - 4)^7$

d $y = (8x + 3)^6$ **e** $y = (1 - x)^5$ **f** $y = 3(5x + 9)^9$

g $y = 2(x - 4)^2$ **h** $y = (2x^3 + 3x)^4$ **i** $y = (x^2 + 5x - 1)^8$

j $y = (x^6 - 2x^2 + 3)^6$ **k** $y = (3x - 1)^{\frac{}{2}}$ **l** $y = (4 - x)^{-2}$

m $y = (x^2 - 9)^{-3}$ **n** $y = (5x + 4)^{\frac{}{3}}$ **o** $y = (x^3 - 7x^2 + x)^{\frac{3}{4}}$

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p $y = \sqrt{3x+4}$

q $y = \dfrac{1}{5x-2}$

r $y = \dfrac{1}{(x^2+1)^4}$

s $y = \sqrt[3]{(7-3x)^2}$

t $y = \dfrac{5}{\sqrt{4+x}}$

u $y = \dfrac{1}{2\sqrt{3x-1}}$

v $y = \dfrac{3}{4(2x+7)^9}$

w $y = \dfrac{1}{x^4-3x^3+3x}$

x $y = \sqrt[3]{(4x+1)^4}$

y $y = \dfrac{1}{\sqrt[4]{(7-x)^5}}$

2 Find the gradient of the tangent to the curve $y = (3x - 2)^3$ at the point (1, 1).

3 If $f(x) = 2(x^2 - 3)^5$, evaluate $f'(2)$.

4 The curve $y = \sqrt{x-3}$ has a tangent with gradient $\dfrac{1}{2}$ at point N. Find the coordinates of N.

5 For what values of x does the function $f(x) = \dfrac{1}{4x-1}$ have $f'(x) = -\dfrac{4}{49}$?

6 Find the equation of the tangent to $y = (2x + 1)^4$ at the point where $x = -1$.

7 Find the equation of the tangent to the curve $y = (2x - 1)^8$ at the point where $x = 1$.

8 Find the equation of the normal to the curve $y = (3x - 4)^3$ at (1, –1).

9 Find the equation of the normal to the curve $y = (x^2 + 1)^4$ at (1, 16).

10 Find the equation of **a** the tangent and **b** the normal to the curve $f(x) = \dfrac{1}{2x+3}$ at the point where $x = -1$.

8.08 Product rule

WS Poduc ule

The **product rule** is a method for differentiating the product of 2 functions.

The product rule

If $y = uv$ where u and v are functions, then:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

or $\quad y' = u'v + v'u.$

We can also write the product rule the other way round (differentiating v first), but the above formulas will also help us to remember the quotient rule in the next section.

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EXAMPLE 16

Differentiate:

a $y = (3x + 1)(x - 5)$ **b** $y = 9x^3(2x - 7)$

Solution

a You could expand the brackets and then differentiate:

$$y = (3x + 1)(x - 5)$$
$$= 3x^2 - 15x + x - 5$$
$$= 3x^2 - 14x - 5$$
$$\frac{dy}{dx} = 6x - 14$$

Using the product rule:

$y = uv$ where $u = 3x + 1$ and $v = x - 5$

$u' = 3$ $v' = 1$

$$y' = u'v + v'u$$
$$= 3(x - 5) + 1(3x + 1)$$
$$= 3x - 15 + 3x + 1$$
$$= 6x - 14$$

b $y = uv$ where $u = 9x^3$ and $v = 2x - 7$

$u' = 27x^2$ $v' = 2$

$$y' = u'v + v'u$$
$$= 27x^2(2x - 7) + 2(9x^3)$$
$$= 54x^3 - 189x^2 + 18x^3$$
$$= 72x^3 - 189x^2$$

We can use the product rule together with the chain rule.

EXAMPLE 17

Differentiate:

a $y = 2x^5(5x+3)^3$ **b** $y = (3x-4)\sqrt{5-2x}$

Solution

a $y = uv$ where $u = 2x^5$ and $v = (5x+3)^3$

$u' = 10x^4$ $\quad v' = 5 \times 3(5x+3)^2$ using chain rule

$= 15(5x+3)^2$

$$\begin{aligned}
y' &= u'v + v'u \\
&= 10x^4\,(5x+3)^3 + 15(5x+3)^2\,2x^5 \\
&= 10x^4(5x+3)^3 + 30x^5(5x+3)^2 \\
&= 10x^4(5x+3)^2[(5x+3) + 3x] \\
&= 10x^4(5x+3)^2(8x+3)
\end{aligned}$$

b $y = uv$ where $u = 3x-4$ and $v = \sqrt{5-2x} = (5-2x)^{\frac{1}{2}}$

$u' = 3$ $\quad v' = -2 \times \frac{1}{2}(5-2x)^{-\frac{1}{2}}$ using chain rule

$$\begin{aligned}
&= -(5-2x)^{-\frac{1}{2}} \\
&= -\frac{1}{(5-2x)^{\frac{1}{2}}} \\
&= -\frac{1}{\sqrt{5-2x}}
\end{aligned}$$

$$\begin{aligned}
y' &= u'v + v'u \\
&= 3\sqrt{5-2x} + -\frac{1}{\sqrt{5-2x}}(3x-4) \\
&= 3\sqrt{5-2x} - \frac{3x-4}{\sqrt{5-2x}} \\
&= \frac{3\sqrt{5-2x} \times \sqrt{5-2x}}{\sqrt{5-2x}} - \frac{3x-4}{\sqrt{5-2x}} \\
&= \frac{3(5-2x)}{\sqrt{5-2x}} - \frac{3x-4}{\sqrt{5-2x}}
\end{aligned}$$

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$$= \frac{3(5-2x)-(3x-4)}{\sqrt{5-2x}}$$

$$= \frac{15-6x-3x+4}{\sqrt{5-2x}}$$

$$= \frac{19-9x}{\sqrt{5-2x}}$$

Exercise 8.08 Product rule

1 Differentiate:

a $y = x^3(2x+3)$ **b** $y = (3x-2)(2x+1)$ **c** $y = 3x(5x+7)$

d $y = 4x^4(3x^2-1)$ **e** $y = 2x(3x^4-x)$ **f** $y = x^2(x+1)^3$

g $y = 4x(3x-2)^5$ **h** $y = 3x^4(4-x)^3$ **i** $y = (x+1)(2x+5)^4$

2 Find the gradient of the tangent to the curve $y = 2x(3x-2)^4$ at (1, 2).

3 If $f(x) = (2x+3)(3x-1)^5$, evaluate $f'(1)$.

4 Find the exact gradient of the tangent to the curve $y = x\sqrt{2x+5}$ at the point where $x = 1$.

5 Find the gradient of the tangent where $t = 3$ given $x = (2t-5)(t+1)^3$.

6 Find the equation of the tangent to the curve $y = x^2(2x-1)^4$ at (1, 1).

7 Find the equation of the tangent to $h = (t+1)^2(t-1)^7$ at (2, 9).

8 Find exact values of x for which the gradient of the tangent to the curve $y = 2x(x+3)^2$ is 14.

9 Given $f(x) = (4x-1)(3x+2)^2$, find the equation of the tangent at the point where $x = -1$.

8.09 Quotient rule

WS Quoien ule

WS Rule of differentiation

WS Mixed differentiation problems

The **quotient rule** is a method for differentiating the ratio of 2 functions.

The quotient rule

If $y = \frac{u}{v}$ where u and v are functions, then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\text{or } y' = \frac{u'v - v'u}{v^2}.$$

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EXAMPLE 18

Differentiate:

a $y = \dfrac{3x-5}{5x+2}$ **b** $y = \dfrac{4x^3-5x+2}{x^3-1}$

Solution

a $y = \dfrac{u}{v}$ where $u = 3x - 5$ and $v = 5x + 2$

$u' = 3$ $v' = 5$

$$\begin{aligned} y' &= \frac{u'v - v'u}{v^2} \\ &= \frac{3(5x+2)-5(3x-5)}{(5x+2)^2} \\ &= \frac{15x+6-15x+25}{(5x+2)^2} \\ &= \frac{31}{(5x+2)^2} \end{aligned}$$

b $y = \dfrac{u}{v}$ where $u = 4x^3 - 5x + 2$ and $v = x^3 - 1$

$u' = 12x^2 - 5$ $v' = 3x^2$

$$\begin{aligned} y' &= \frac{u'v - v'u}{v^2} \\ &= \frac{(12x^2-5)(x^3-1)-3x^2(4x^3-5x+2)}{(x^3-1)^2} \\ &= \frac{12x^5-12x^2-5x^3+5-12x^5+15x^3-6x^2}{(x^3-1)^2} \\ &= \frac{10x^3-18x^2+5}{(x^3-1)^2} \end{aligned}$$

Exercise 8.09 Quotient rule

1 Differentiate:

a $y = \dfrac{1}{2x-1}$ **b** $y = \dfrac{3x}{x+5}$ **c** $y = \dfrac{x^3}{x^2-4}$ **d** $y = \dfrac{x-3}{5x+1}$

e $y = \dfrac{x-7}{x^2}$ **f** $y = \dfrac{5x+4}{x+3}$ **g** $y = \dfrac{x}{2x^2-1}$ **h** $y = \dfrac{x+4}{x-2}$

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i $y = \dfrac{2x+7}{4x-3}$ **j** $y = \dfrac{x+5}{3x+1}$ **k** $y = \dfrac{x+1}{3x^2-7}$ **l** $y = \dfrac{2x^2}{2x-3}$

m $y = \dfrac{x^2+4}{x^2-5}$ **n** $y = \dfrac{x^3}{x+4}$ **o** $y = \dfrac{x^3+2x-1}{x+3}$ **p** $y = \dfrac{x^2-2x-1}{3x+4}$

q $y = \dfrac{2x}{(x+5)^{2}}$ **r** $y = \dfrac{x-1}{(7x+2)^4}$ **s** $y = \dfrac{3x+1}{\sqrt{x+1}}$ **t** $y = \dfrac{\sqrt{x-1}}{2x-3}$

2 Find the gradient of the tangent to the curve $y = \dfrac{2x}{3x+1}$ at $\left(1, \dfrac{1}{2}\right)$.

3 If $f(x) = \dfrac{4x+5}{2x-1}$, evaluate $f'(2)$.

4 Find values of x for which the gradient of the tangent to $y = \dfrac{4x-1}{2x-1}$ is –2.

5 Given $f(x) = \dfrac{2x}{x+3}$, find x if $f'(x) = \dfrac{1}{6}$.

6 Find the equation of the tangent to the curve $y = \dfrac{x}{x+2}$ at $\left(4, \dfrac{2}{3}\right)$.

7 Find the equation of the tangent to the curve $y = \dfrac{x^2-1}{x+3}$ at $x = 2$.

8.10 Rates of change

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WS Graph o ae of change

We know that the gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ of the secant passing through 2 points on the graph of a function gives the **average rate of change** between those 2 points.

Now consider a quantity Q that changes with time, giving the function $Q(t)$.

Average rate of change

The average rate of change of a quantity Q with respect to time t is $\dfrac{Q_2 - Q_1}{t_2 - t_1}$.

We know that the gradient $\dfrac{dy}{dx}$ of the tangent at a point on the graph of a function gives the **instantaneous rate of change** at that point.

Instantaneous rate of change

The instantaneous rate of change of a quantity Q with respect to time t is $\dfrac{dQ}{dt}$.

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EXAMPLE 19

a The number of bacteria in a culture increases according to the function $B = 2t^4 - t^2 + 2000$, where t is time in hours. Find:

i the number of bacteria initially

ii the average rate of change in number of bacteria between 2 and 3 hours

iii the number of bacteria after 5 hours

v the rate at which the number of bacteria is increasing after 5 hours.

b An object travels a distance according to the function $D = t^2 + t + 5$, where D is in metres and t is in seconds. Find the speed at which it is travelling at:

i 4 s ii 10 s

Solution

a i $B = 2t^4 - t^2 + 2000$

Initially, $t = 0$:

$B = 2(0)^4 - (0)^2 + 2000$

$= 2000$

So there are 2000 bacteria initially.

ii When $t = 2$, $B = 2(2)^4 - (2)^2 + 2000$

$= 2028$

When $t = 3$, $B = 2(3)^4 - (3)^2 + 2000$

$= 2153$

$$\text{Average rate of change} = \frac{B_2 - B_1}{t_2 - t_1}$$

$$= \frac{2153 - 2028}{3 - 2}$$

$= 125$ bacteria/hour

So the average rate of change is 125 bacteria per hour.

iii When $t = 5$, $B = 2(5)^4 - (5)^2 + 2000$

$= 3225$

So there will be 3225 bacteria after 5 hours.

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v The instantaneous rate of change is given by the derivative $\frac{dB}{dt} = 8t^3 - 2t$.

When $t = 5$, $\frac{dB}{dt} = 8(5)^3 - 2(5)$

$= 990$

So the rate of increase after 5 hours will be 990 bacteria per hour.

b Speed is the rate of change of distance over time: $\frac{dD}{dt} = 2t + 1$.

i When $t = 4$, $\frac{dD}{dt} = 2(4) + 1$

$= 9$

So speed after 4 s is 9 m/s.

ii When $t = 10$, $\frac{dD}{dt} = 2(10) + 1$

$= 21$

So speed after 10 s is 21 m/s.

Displacement, velocity and acceleration

Displacement (x) measures the distance of an object from a fixed point (origin). It can be positive or negative or 0, according to where the object is.

Velocity (v) is the rate of change of displacement with respect to time, and involves speed and direction.

Velocity

Velocity $v = \frac{dx}{dt}$ is the instantaneous rate of change of displacement x over time t.

Acceleration (a) is the rate of change of velocity with respect to time.

Acceleration

Acceleration $a = \frac{dv}{dt}$ is the instantaneous rate of change of velocity v over time t.

We usually write velocity units as km/h or m/s, but we can also use index notation and write km h^{-1} or m s^{-1}.

With acceleration units, we write km/h/h as km/h^2, or in index notation we write km h^{-2}.

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EXAMPLE 20

A ball rolls down a ramp so that its displacement x cm in t seconds is $x = 16 - t^2$.

a Find its initial displacement.

b Find its displacement at 3 s.

c Find its velocity at 2 s.

d Show that the ball has a constant acceleration of -2 cm s^{-2}.

Solution

a $x = 16 - t^2$

Initially, $t = 0$:

$x = 16 - 0^2$

$= 16$

So the ball's initial displacement is 16 cm.

b When $t = 3$:

$x = 16 - 3^2$

$= 7$

So the ball's displacement at 3 s is 7 cm.

c $v = \frac{dx}{dt}$

$= -2t$

When $t = 2$:

$v = -2(2)$

$= -4$

So the ball's velocity at 2 s is -4 cm s^{-1}.

> x is measured in cm, is measured in s, so v is measured in cm/s or cm s^{-1}.

d $a = \frac{dv}{dt}$

$= -2$

So acceleration is constant at -2 cm s^{-2}.

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Exercise 8.10 Rates of change

1 Find the formula for the rate of change for each function.

a $h = 20t - 4t^2$ **b** $D = 5t^3 + 2t^2 + 1$ **c** $A = 16x - 2x^2$

d $x = 3t^5 - t^4 + 2t - 3$ **e** $V = \frac{4}{3}\pi r^3$ **f** $S = 2\pi r + \frac{50}{r^2}$

g $D = \sqrt{x^2 - 4}$ **h** $S = 800r + \frac{400}{r}$

2 If $h = t^3 - 7t + 5$, find:

a the average rate of change of h between $t = 3$ and $t = 4$

b the instantaneous rate of change of h when $t = 3$.

3 The volume of water V in litres flowing through a pipe after t seconds is given by $V = t^2 + 3t$. Find the rate at which the water is flowing when $t = 5$.

4 The mass in grams of a melting ice block is given by the formula $M = t - 2t^2 + 100$, where t is time in minutes.

a Find the average rate of change at which the ice block is melting between:

i 1 and 3 minutes **ii** 2 and 5 minutes.

b Find the rate at which it will be melting at 5 minutes.

5 The surface area in cm^2 of a balloon being inflated is given by $S = t^3 - 2t^2 + 5t + 2$, where t is time in seconds. Find the rate of increase in the balloon's surface area at 8 s.

6 A circular disc expands as it is heated. The area, in cm^2, of the disc increases according to the formula $A = 4t^2 + t$, where t is time in minutes. Find the rate of increase in the area after 5 minutes.

7 A car is d km from home after t hours according to the formula $d = 10t^2 + 5t + 11$.

a How far is the car from home:

i initially? **ii** after 3 hours? **iii** after 5 hours?

b At what speed is the car travelling after:

i 3 hours? **ii** 5 hours?

8 According to Boyle's Law, the pressure of a gas is given by the formula $P = \frac{k}{V}$ where k is a constant and V is the volume of the gas. If $k = 100$ for a certain gas, find the rate of change in the pressure when $V = 20$.

9 The displacement of a particle is $x = t^3 - 9t$ cm, where t is time in seconds.

a Find the velocity of the particle at 3 s.

b Find the acceleration at 2 s.

c Show that the particle is initially at the origin, and find any other times that the particle will be at the origin.

d At what time will the acceleration be 30 cm s^{-2}?

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10 A particle is moving with displacement $s = 2t^2 - 8t + 3$, where s is in metres and t is in seconds.

- **a** Find its initial velocity.
- **b** Show that its acceleration is constant and find its value.
- **c** Find its displacement at 5 s.
- **d** Find when the particle's velocity is zero.
- **e** What will the particle's displacement be at that time?

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EXT1 8.11 Related rates of change

Rates of change with respect to time are harder to calculate when there are **2 or more related variables**. For example, when inflating a balloon, both its radius and its volume increase, but at different rates.

Related rates of change

If y is related to x and x is related to time t, then the instantaneous rate of change of y with respect to t uses the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

EXAMPLE 21

Related rates of change

- **a** Given $y = 2x^2 - 3x + 1$ and $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = -2$.
- **b** A spherical metal ball is heated so that its radius is expanding at the rate of 0.04 mm per second. At what rate will its volume be increasing when the radius is 3.4 mm?
- **c** A pool holds a volume of water given by $V = 2x + 3x^2$, where x is the depth of water. If the pool is filled with water at the rate of 1.3 m^3/h, at what rate will the level of water be increasing when the depth is 0.78 m?
- **d** Car A is north of an intersection and travelling towards it, while car B is moving away from the intersection eastwards at a constant speed of 60 km h^{-1}. The distance between the cars at any one time is 10 km. Find the rate at which car A will be moving when car B is 8 km from the intersection.

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Solution

a $y = 2x^2 - 3x + 1,$

so $\frac{dy}{dx} = 4x - 3.$

Also, $\frac{dx}{dt} = 5.$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= (4x - 3) \times 5$$

$$= 20x - 15$$

When $x = -2$,

$$\frac{dy}{dt} = 20(-2) - 15$$

$$= -55$$

b $V = \frac{4}{3}\pi r^3$

$\therefore \frac{dV}{dr} = 4\pi r^2$

Also, $\frac{dr}{dt} = 0.04$ mm/s

Now $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$$= 4\pi r^2 \times 0.04$$

$$= 0.16\pi r^2$$

When $r = 3.4$, $\frac{dV}{dt} = 0.16\pi(3.4)^2$

$$\approx 5.81$$

So the volume is increasing at a rate of 5.81 mm^3/s.

c $V = 2x + 3x^2$

$\therefore \frac{dV}{dx} = 2 + 6x$

Also, $\frac{dV}{dt} = 1.3$ m^3/h

Now $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$$1.3 = (2 + 6x) \times \frac{dx}{dt}$$

When $x = 0.78$,

$$1.3 = [2 + 6(0.78)] \times \frac{dx}{dt}$$

$$= 6.68\frac{dx}{dt}$$

$$0.195 \approx \frac{dx}{dt}$$

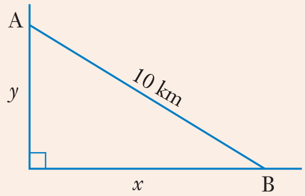
So the level of water is increasing at a rate of 0.195 m/h.

d $x^2 + y^2 = 100$ (by Pythagoras' theorem)

$y^2 = 100 - x^2$

$\therefore y = \sqrt{100 - x^2}$

$= (100 - x^2)^{\frac{1}{2}}$



$$\frac{dy}{dx} = \frac{1}{2}(100 - x^2)^{-\frac{1}{2}}(-2x)$$

$$= -x(100 - x^2)^{-\frac{1}{2}}$$

$$= \frac{-x}{\sqrt{100 - x^2}}$$

$\frac{dx}{dt} = -60$ (the speed of car B)

Note: The negative sign means the car is moving **away** from the intersection.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= \frac{-x}{\sqrt{100 - x^2}} \times (-60)$$

$$= \frac{60x}{\sqrt{100 - x^2}}$$

When $x = 8$,

$$\frac{dy}{dt} = \frac{60(8)}{\sqrt{100 - 8^2}}$$

$$= \frac{480}{6}$$

$$= 80$$

So car A will be travelling at a speed of 80 km h^{-1} when car B is 8 km from the intersection.

EXT1 Exercise 8.11 Rates involving two variables

1 Find an expression for $\frac{dy}{dt}$ given:

a $y = x^4$ and $\frac{dx}{dt} = 2$

b $y = 3x^3 + 7$ and $\frac{dx}{dt} = 6$

c $y = x^2 - x - 2$ and $\frac{dx}{dt} = -3$

2 Evaluate $\frac{dy}{dt}$ when $x = 4$, given:

a $y = 2x^3 + 3x - 7$ and $\frac{dx}{dt} = 3$

b $y = (3x + 1)^3$ and $\frac{dx}{dt} = -4$

c $y = (5 - x)^5$ and $\frac{dx}{dt} = 4$

3 If $y = x^3 + 5x - 4$ and $\frac{dy}{dt} = 6$, evaluate $\frac{dx}{dt}$ when $x = 2$.

4 If $y = x^2 + x$ and $\frac{dy}{dt} = -5$, evaluate $\frac{dx}{dt}$ when $x = 4$.

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5 If $x = 3t^2 - t$ and $\frac{dt}{du} = 6$, evaluate $\frac{dx}{du}$ when $t = 12$.

6 If $V = 5\pi r^2$ and $\frac{dr}{dt} = 2$, evaluate $\frac{dV}{dt}$ correct to the nearest unit when $r = 4.6$.

7 If $A = 16x - 2x^2$ and $\frac{dx}{dt} = 11$, evaluate $\frac{dA}{dt}$ when $x = 3$.

8 If $V = x^3 + 4x^2 - 3x + 4$ and $\frac{dV}{dt} = 10$, evaluate $\frac{dx}{dt}$ when $x = 1$.

9 A cube is expanding so that its side length is increasing at the constant rate of 0.12 mm s^{-1}. Find the rate of increase in its volume when its side is 150 mm.

10 The radius of a cylindrical pipe 2 m long expands with heat at a constant rate of 1.2×10^{-3} mm s^{-1}. Find the rate at which the volume of the pipe will be increasing when its radius is 19 mm.

11 Find the rate of change of the surface area of a balloon when its radius is 6.3 cm, if the radius is expanding at a constant rate of 1.3 cm s^{-1}.

12 A cone contains liquid with volume given by $V = \frac{6\pi h^2}{7}$, where h is the height of the liquid in the cone. If the height of the liquid is increasing at a rate of 2.3 cm s^{-1}, find the rate of increase in the volume of the liquid when its height is 12.9 cm.

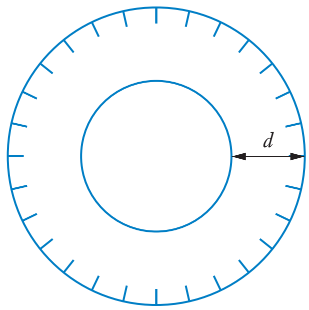
13 A point, P, moves along the curve $y = 2x^2 - 7x + 9$. What will be the rate of change in the y-coordinate of P when the x-coordinate is increasing at a rate of 8 units per second and the value of x is 3?

14 An ice cube with sides x mm is melting so that the length of its sides is decreasing at 0.8 mm s^{-1}. What will the rate of decrease in volume be when the sides are 120 mm long?

15 A factory produces a quantity of radios according to the formula $N = x^2 + 7x$, where x is the number of workers. If the number of workers decreases by a constant rate of 2 per week, find the rate at which the quantity of radios made will decrease when there are 150 workers.

16 A particle is moving so that its velocity is given by the formula $v = 8x^3 - 5x^2 - 3x - 1$, where x is its displacement. If the rate of change in displacement is a constant 4.2 cm s^{-1}, find the rate of change in velocity when the displacement is –4.7 cm.

17 A car tyre has a volume given by the formula $V = 0.53\pi d^2$, where d is the diameter of the wall of the tyre. If the diameter decreases at the constant rate of 0.02 mm s^{-1}, find the rate at which the volume of the tyre will be decreasing when the diameter is 167 mm.



18 The number of burrows for a colony of rabbits is decreasing owing to the clearing of land, at a constant rate of 5 burrows per day. If the number of rabbits is given by the formula $N = 5x^2 + 3x$, where x is the number of burrows, find the rate of decrease in the rabbit population when there are 55 burrows.

19 The volume of a balloon being inflated is increasing at a constant rate of 115 $cm^3\ s^{-1}$. Find the rate of increase in its radius when the radius is 3 cm.

20 A population increases at a constant rate of 15 000 people per year. If the population has the formula $P = x^2 - 3000x + 100$, where x is the number of houses available, find the rate at which the number of houses will be increasing when there are 5000 houses.

21 A cone-shaped candle whose height is 3 times its radius is melting at the constant rate of 1.4 $cm^3\ s^{-1}$. If the proportion of radius to height is preserved, find the rate at which the radius will be decreasing when it is 3.7 cm.

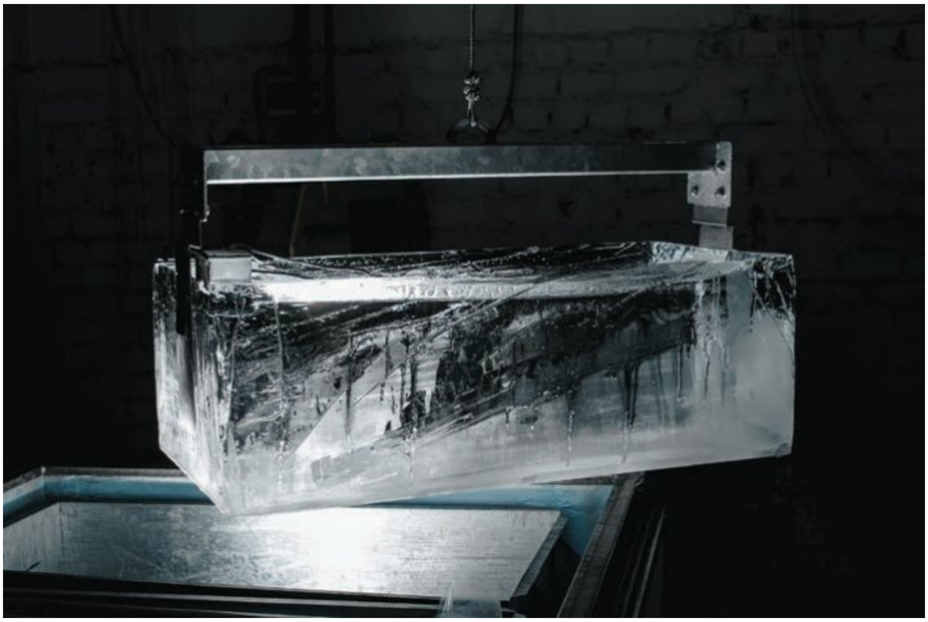
22 The rate of change in the radius of a sphere is 0.3 mm s^{-1}. Given that the radius is 88 mm at a certain time, find the rate of change at that time in:

a surface area　　**b** volume

23 If the volume of a cube is increasing at the rate of 23 $mm^3\ s^{-1}$, find the increase in its surface area when its side length is 140 mm.

24 The surface area of a spherical bubble is increasing at a constant rate of 1.9 $mm^2\ s^{-1}$. Find the rate of increase in its volume when its radius is 0.6 mm.

25 A rectangular block of ice with a square base has a height half the side of the base. As it melts, the volume of the block of ice is decreasing at the rate of 12 $cm^3\ s^{-1}$. Find the rate at which its surface area will be decreasing when the side of its base is 2.1 cm.



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EXT1 8.12 Motion in a straight line

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Now we will look more closely at the rate of change of motion along a straight line.

When studying the motion of a particle, the term 'particle' describes any moving object, such as a golf ball or a car. We will also ignore friction, gravity and other influences on the motion.

DID YOU KNOW?

The origins of calculus

Calculus was developed in the 17th century as a solution to problems about the **study of motion**. Some problems of the time included finding the speed and acceleration of planets, calculating the lengths of their orbits, finding maximum and minimum values of functions, finding the direction in which an object is moving at any one time and calculating the areas and volumes of certain figures.

Displacement

Displacement (x) measures the distance of a particle from a fixed point (origin). Displacement can be positive or negative, according to which side of the origin, O, it is on. Usually, it is **positive** to the **right** of O and **negative** to the **left** of O.

Zero displacement

When the particle is at the **origin**, its **displacement** is **zero**. That is, $x = 0$.

Velocity

Velocity (v) is the rate of change of displacement:

Velocity

$$v = x = \frac{dx}{dt}$$

$\dot{x}$ is another way of writing $\frac{dx}{dt}$.

If the particle is moving to the **right**, velocity is **positive**. If it is moving to the **left**, velocity is **negative**.

Zero velocity

When the particle is not moving, we say that it is **at rest**. That is, $x = 0$.

Acceleration

Acceleration is the rate of change of velocity:

Acceleration

$$a = x = \frac{dv}{dt}$$

x means the derivative of the derivative of x, or the second derivative of x. If the acceleration (and the force on the particle) is to the **right**, it is **positive**. If the acceleration is to the **left**, it is **negative**.

If the acceleration is in the **same direction** as the velocity, the particle is **speeding up** (accelerating). If the acceleration is in the **opposite direction** from the velocity, the particle is **slowing down** (decelerating). If the particle is not speeding up or slowing down, then its speed is not changing and we say it has constant velocity.

Zero acceleration

When the particle is not accelerating, we say that it has **constant velocity**. That is, $x = 0$.

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Motion graphs

We can describe the velocity of a particle by looking at the derivative function of its displacement graph.

We can describe acceleration by looking at the derivative function of its velocity graph.

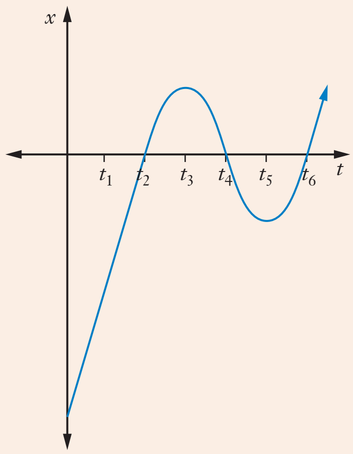
EXAMPLE 22

a This graph shows the displacement of a particle from the origin as it moves in a straight line.

i When is the particle at rest?

ii When is it at the origin?

iii When is it moving at its greatest speed?



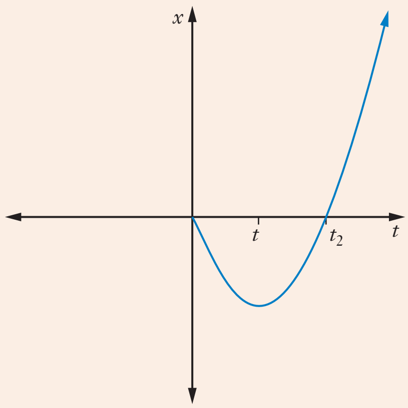
b This graph shows the displacement x of a particle over time t.

i Sketch a graph of its velocity.

ii Sketch its acceleration graph.

iii Find when the particle is at the origin.

v Find when the particle is at rest.



Solution

a **i** The particle is at rest when $v = \frac{dx}{dt} = 0$.

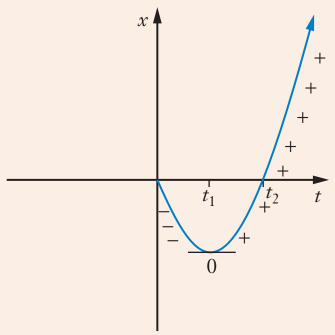
This is where the displacement is neither increasing nor decreasing.

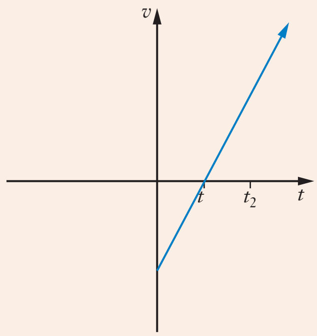
From the graph, the particle is at rest at times t_3 and t_5.

ii The particle is at the origin when $x = 0$, that is, on the t-axis.
So the particle is at the origin at times t_2, t_4 and t_6.

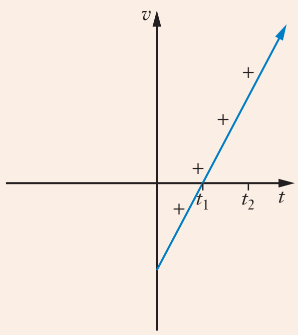
iii The speed is greatest when the graph is at its steepest. From the graph, this occurs at t_4. The particle is moving at its greatest speed at time t_4.

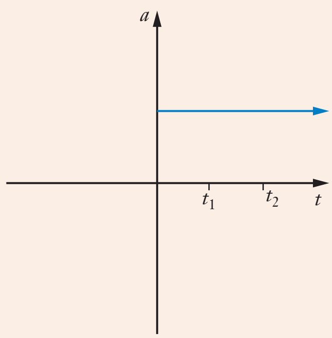
b **i** Velocity $v = \frac{dx}{dt}$. By noting where the gradient of the displacement graph is positive, negative and zero, we draw its gradient function for the velocity graph.





ii Acceleration $a = \frac{dv}{dt}$. By noting where the gradient of the velocity graph is positive, negative and zero, we draw its gradient function for the acceleration graph.





iii At the origin, $x = 0$.

From the displacement graph, this is at 0 and t_2.

v At rest, $v = 0$.

From the velocity graph, this is at t_1.

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Motion functions

EXAMPLE 23

a The displacement x cm of a particle at t seconds is given by $x = 4t - t^2$.

i Find when the particle is at rest.

ii How far does the particle move in the first 3 seconds?

b The displacement of a particle is given by $x = -t^2 + 2t + 3$ cm where t is in seconds.

i Find the initial velocity of the particle.

ii Show that the particle has constant acceleration.

iii Find when the particle is at the origin.

v Find the particle's greatest displacement.

v Sketch the graph showing the particle's motion.

Solution

a **i** At rest, $v = 0$.

$$v = \frac{dx}{dt} = 4 - 2t$$

$$4 - 2t = 0$$

$$4 = 2t$$

$$2 = t$$

So the particle is at rest at 2 seconds.

ii When $t = 0$:

$$x = 4(0) - 0^2$$
$$= 0$$

So the particle is initially at the origin.

When $t = 3$:

$$x = 4(3) - 3^2$$
$$= 3$$

So the particle is 3 cm from the origin at 3 seconds.

When $t = 2$:

$$x = 4(2) - 2^2$$
$$= 4$$

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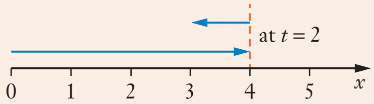
The particle moves from $x = 0$ to 4 in the first 2 seconds, then it turns and moves to $x = 3$ in the 3rd second, going back 1 cm (see diagram).



$$\begin{aligned}\text{Total distance travelled} &= 4 + 1 \\ &= 5 \text{ cm}\end{aligned}$$

b **i** $v = \dfrac{dx}{dt} = -2t + 2$

Initially, $t = 0$:

$$\begin{aligned}v &= -2(0) + 2 \\ &= 2\end{aligned}$$

So the initial velocity is 2 cm s^{-1}.

ii $v = -2t + 2$

Acceleration $a = \dfrac{dv}{dt} = -2$

$\therefore$ the particle has a constant acceleration of -2 cm s^{-2}.

iii At the origin $\quad x = 0$

$$\begin{aligned}-t^2 + 2t + 3 &= 0 \\ -(t + 1)(t - 3) &= 0 \\ t &= -1 \text{ or } 3\end{aligned}$$

Since time cannot be negative, the particle will be at the origin at 3 s.

v Greatest displacement will be at the turning point of the displacement graph (or when $v = 0$):

$$\begin{aligned}\frac{dx}{dt} &= 0 \\ -2t + 2 &= 0 \\ 2 &= 2t \\ 1 &= t\end{aligned}$$

So the greatest displacement occurs when $t = 1$.

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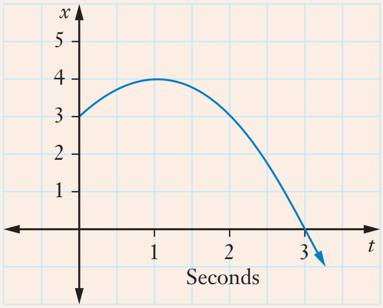
When $t = 1$:

$x = -1^2 + 2(1) + 3$

$= 4$

So the greatest displacement is 4 cm.

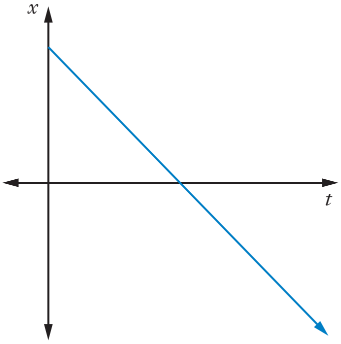
v The graph of $x = -t^2 + 2t + 3$ is a concave downward parabola. From above, we know that it has x-intercepts at −1 and 3 and goes through (0, 3) and (1, 4). We draw the graph for $t \geq 0$ only.



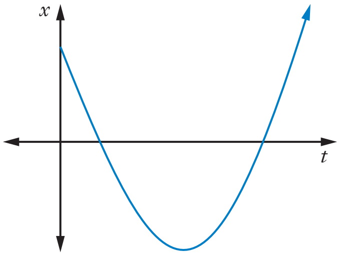
EXT1 Exercise 8.12 Motion in a straight line

1 For each displacement graph, sketch the graphs for velocity and acceleration.

a



b



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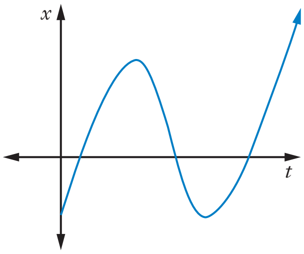

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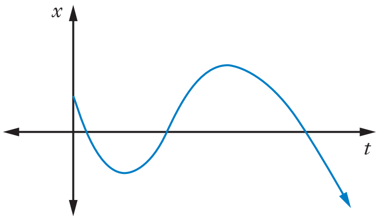

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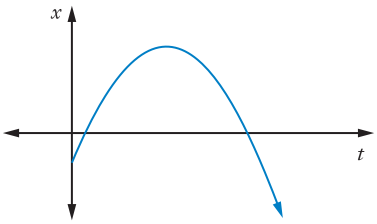


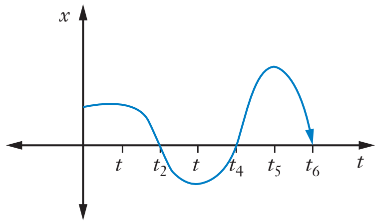
2 The graph shows the displacement of a particle as it moves along a straight line.

When is the particle:

a at the origin?

b at rest?

c furthest from the origin?



3 This graph shows the velocity of a particle.

a When is the particle at rest?

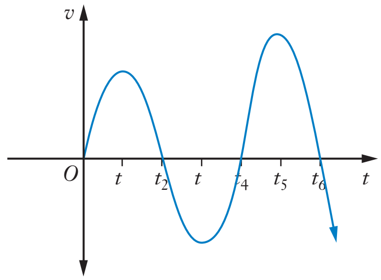
b When is the acceleration zero?

c When is the speed the greatest?

d Describe the motion of the particle at:

i t_2

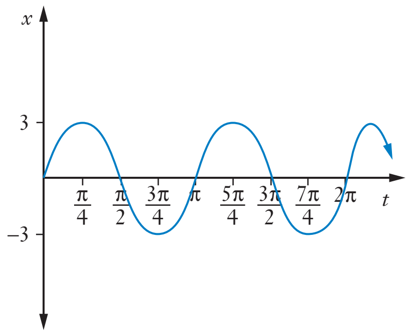
ii t_3



4 This graph shows the displacement of a pendulum.

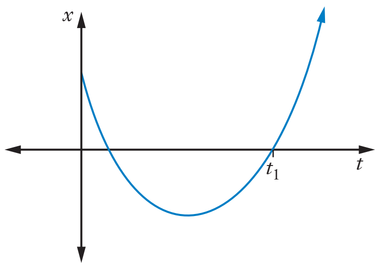
a When is the pendulum at rest?

b When is the pendulum in its equilibrium position (at the origin)?



5 Describe the displacement and velocity of the particle at t_1 for each displacement graph.

a



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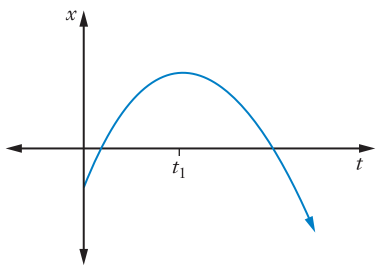

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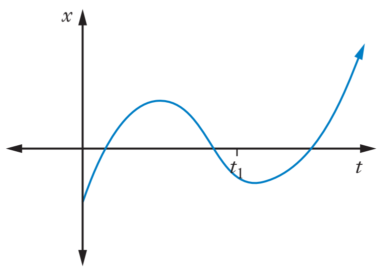

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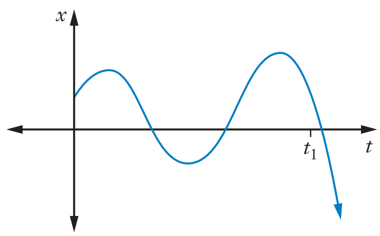

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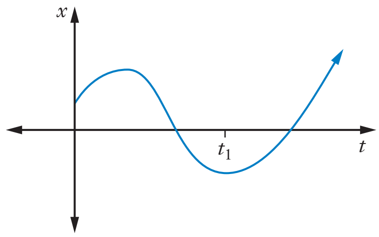

6 A projectile is fired into the air and its height in metres is given by $h = 40t - 5t^2 + 4$, where t is in seconds.

- **a** Find the initial height.
- **b** Find the initial velocity.
- **c** Find the height at 1 s.
- **d** What is the maximum height of the projectile?
- **e** Sketch the graph of the height against time.

7 The displacement in cm after time t s of a particle moving in a straight line is given by $x = 2 - t - t^2$.

- **a** Find the initial displacement.
- **b** Find when the particle will be at the origin.
- **c** Find the displacement at 2 s.
- **d** How far will the particle move in the first 2 seconds?
- **e** Find its velocity at 3 s.

8 An object is travelling along a straight line over time t seconds, with displacement $x = t^3 + 6t^2 - 2t + 1$ m.

- **a** Find the equations of its velocity and acceleration.
- **b** What will its displacement be after 5 s?
- **c** What will its velocity be after 5 s?
- **d** Find its acceleration after 5 s.

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9 The displacement, in centimetres, of a body is given by $x = (4t - 3)^5$ where t is time in seconds.

- **a** Find the equations for velocity $\dot{x}$ and acceleration $\ddot{x}$.
- **b** Find the values of x, $\dot{x}$ and $\ddot{x}$ after 1 s.
- **c** Describe the motion of the body after 1 s.

10 The displacement of a particle, in metres, over time t seconds is $s = ut + \frac{1}{2}gt^2$ where $u = 5$ and $g = -10$.

- **a** Find the equation of the velocity of the particle.
- **b** Find the velocity at 10 s.
- **c** Show that the acceleration is equal to g.

11 The displacement in metres after t seconds is given by $s = \frac{2t-5}{3t+1}$. Find the equations for velocity and acceleration.

12 The displacement of a particle is given by $x = t^3 - 4t^2 + 3t$ where x is in metres and t is in seconds.

- **a** Find the initial velocity.
- **b** Find the times when the particle will be at the origin.
- **c** Find the acceleration after 3 s.

13 The height of a projectile is given by $h = 7 + 6t - t^2$ where height is in metres and time is in seconds.

- **a** Find the initial height.
- **b** Find the maximum height reached.
- **c** When will the projectile reach the ground?
- **d** Sketch the graph showing the height of the projectile over time t.
- **e** How far will the projectile travel in the first 4 s?

14 A ball is rolled up a slope at a distance from the base of the slope, after time t seconds, given by $x = 15t - 3t^2$ metres.

- **a** How far up the slope will the ball roll before it starts to roll back down?
- **b** What will its velocity be when it reaches the base of the slope?
- **c** How long will the motion of the ball take altogether?

15 The displacement of a particle is given by $x = 2t^3 - 3t^2 + 42t$.

- **a** Show that the particle is initially at the origin but never returns to the origin.
- **b** Show that the particle is never at rest.

16 A particle is moving in a straight line so that its displacement x cm over time t seconds is given by $x = t\sqrt{49 - t^2}$.

- **a** For how many seconds does the particle travel?
- **b** Find the exact time at which the particle comes to rest.
- **c** How far does the particle move altogether?

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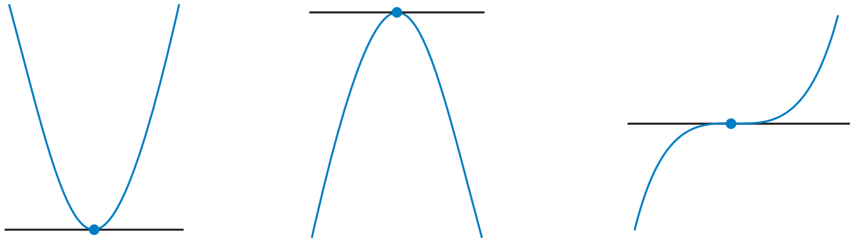
EXT1 8.13 Multiple roots of polynomial equations

WS Muliple oo

We saw in Chapter 6, *Polynomials and inverse functions*, that there is always a stationary point at a multiple root of a polynomial equation $P(x) = 0$



Minimum turning point   Maximum turning point   Horizontal point of inflection

As stationary points have a horizontal tangent, $P'(x) = 0$ for multiple roots.

Multiple roots of polynomial equations

If $P(x) = 0$ has a multiple root at $x = k$, then $P(k) = P'(k) = 0$.

Proof

$$\begin{aligned}
P(x) &= (x-k)^r Q(x) \text{ where } Q(x) \text{ is another polynomial} \\
P(k) &= (k-k)^r Q(k) \\
&= 0^r Q(0) \\
&= 0 \\
P'(x) &= u'v + v'u \\
&= r(x-k)^{r-1} Q(x) + Q'(x)(x-k)^r \\
P'(k) &= r(k-k)^{r-1} Q(k) + Q'(k)(k-k)^r \\
&= r(0)^{r-1} Q(k) + Q'(k)(0)^r \\
&= 0
\end{aligned}$$

So $P(k) = P'(k) = 0$

Stationary points on polynomial graphs

If the multiplicity r of a root is even, there is a maximum or minimum turning point at the multiple root.

If the multiplicity r of a root is odd, there is a horizontal point of inflection at the multiple root.

EXAMPLE 24

A polynomial has a double root at $x = 5$.

a Write an expression for the polynomial.

b Prove that $P(5) = P'(5) = 0$.

Solution

a If $P(x)$ has a double root at $x = 5$, then $(x - 5)^2$ is a factor.

So $P(x) = (x - 5)^2 Q(x)$

b $P(5) = (5 - 5)^2 Q(5)$

$= 0^2 Q(5)$

$= 0$

$P'(x) = u'v + v'u$ where $u = (x - 5)^2$ and $v = Q(x)$

$= 2(x - 5)\,Q(x) + Q'(x)(x - 5)^2$

$P'(5) = 2(5 - 5)\,Q(5) + Q'(5)(5 - 5)^2$

$= 2(0)\,Q(5) + Q'(5)\,0^2$

$= 0$

So $P(5) = P'(5) = 0$

Multiplicity of roots of *P(x)* and *P′(x)*

If $P(x) = 0$ has a root at $x = k$ of multiplicity $r > 1$, then $P'(x) = 0$ has a root of multiplicity $r - 1$.

Proof

If $P(x) = 0$ has a root of multiplicity r then we can write:

$$P(x) = (x - k)^r Q(x)$$

$$\begin{aligned}P'(x) &= u'v + v'u\\ &= r(x - k)^{r-1} Q(x) + Q'(x)(x - k)^r\\ &= (x - k)^{r-1}[rQ(x) + Q'(x)(x - k)]\\ &= (x - k)^{r-1} R(x)\end{aligned}$$

So $P'(x) = 0$ has a root of multiplicity $r - 1$.

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EXAMPLE 25

If a polynomial $P(x) = 0$ has a root of multiplicity 4, show that $P'(x) = 0$ has a root of multiplicity 3.

Solution

If $P(x) = 0$ has a root of multiplicity 4 then we can write:

$P(x) = (x - k)^4 Q(x)$

$P'(x) = u'v + v'u$

where $u = (x - k)^4$ and $v = Q(x)$

$= 4(x - k)^3 Q(x) + Q'(x)(x - k)^4$

$= (x - k)^3 [4Q(x) + Q'(x)(x - k)]$

$= (x - k)^3 R(x)$

So $P'(x) = 0$ has a root of multiplicity 3.

EXT1 Exercise 8.13 Multiple roots of polynomial equations

1 $P(x) = x^3 - 7x^2 + 8x + 16$ has a double root at $x = 4$.

- **a** Show that $(x - 4)^2$ is a factor of $P(x)$.
- **b** Write $P(x)$ as a product of its factors.
- **c** Prove $P(4) = P'(4) = 0$.

2 $f(x) = x^4 + 7x^3 + 9x^2 - 27x - 54$ has a triple root at $x = -3$.

- **a** Show that $(x + 3)^3$ is a factor of $f(x)$.
- **b** Write $f(x)$ as a product of its factors.
- **c** Prove $f(-3) = f'(-3) = 0$.

3 A polynomial has a triple root at $x = k$.

- **a** Write an expression for the polynomial.
- **b** Prove that $P(k) = P'(k) = 0$.

4
- **a** Write $P(x) = x^3 + x^2 - 8x - 12$ as a product of its factors.
- **b** Find the roots of $P(x) = 0$ and state the multiplicity of each root.
- **c** For each multiple root a, prove that $P(a) = P'(a) = 0$.

5
- **a** Write $P(x) = x^5 - 2x^4 + x^3$ as a product of its factors.
- **b** Find the roots of $P(x) = 0$ and state the multiplicity of each root.
- **c** For each multiple root a, prove that $P(a) = P'(a) = 0$.

6 A polynomial equation $P(x) = 0$ has a triple root at $x = 5$. Show that $P'(x) = 0$ has a double root at $x = 5$.

7 $P(x) = 0$ has a root of multiplicity 6 at $x = -3$. Show that $P'(x) = 0$ has a root of multiplicity 5 at $x = -3$.

8 $P(x) = 0$ has a root of multiplicity n at $x = p$. Show that $P'(x) = 0$ has a root of multiplicity $n - 1$ at $x = p$.

9 **a** Divide $P(x) = x^3 + 5x^2 + 3x - 9$ by $x^2 + 6x + 9$.

b What root of $P(x) = 0$ has multiplicity 2?

c What is the multiplicity of this root for the polynomial equation $P'(x) = 0$?

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8. TEST YOURSELF

For Questions **1** to **4**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

Derivatives find-a-word

1 Find the derivative of $\frac{2}{3x^4}$.

A $\frac{8}{3x^5}$ **B** $-\frac{8}{3x^3}$ **C** $-\frac{8}{3x^5}$ **D** $\frac{8}{3x^3}$

2 Differentiate $3x(x^3 - 5)$.

A $4x^3$ **B** $12x^3 - 15$ **C** $9x^2$ **D** $3x^4 - 15x$

3 The derivative of $y = f(x)$ is given by:

A $\lim_{h\to 0} \frac{f(x+h) - f(x)}{x - h}$ **B** $\lim_{x\to 0} \frac{f(x+h) - f(x)}{x}$

C $\lim_{h\to 0} \frac{f(x) - f(x+h)}{h}$ **D** $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

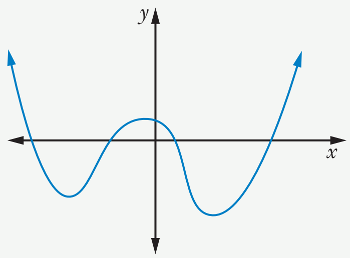
4 Which of the following is the chain rule (there is more than one answer)?

A $\frac{dy}{dx} = \frac{dy}{du} \times \frac{dx}{du}$ **B** $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

C $\frac{dy}{dx} = nf'(x)f(x)^{n-1}$ **D** $\frac{dy}{dx} = nf(x)^{n-1}$

5 Sketch the derivative function of each graph.

a



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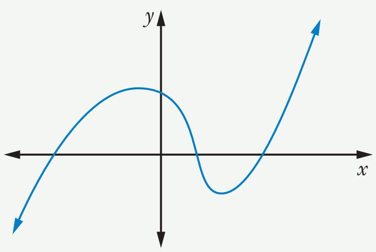

6 Differentiate $y = 5x^2 - 3x + 2$ from first principles.

7 Differentiate:

a $y = 7x^6 - 3x^3 + x^2 - 8x - 4$ **b** $y = 3x^{-4}$ **c** $y = \frac{2}{(x+1)^4}$

d $y = x^2\sqrt{x}$ **e** $y = (x^2 + 4x - 2)^9$ **f** $y = \frac{3x - 2}{2x + 1}$

g $y = x^3(3x + 1)^6$

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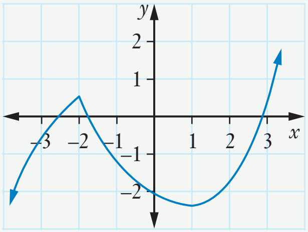
8 Find $\dfrac{dv}{dt}$ if $v = 2t^2 - 3t - 4$.

9 Find the gradient of the tangent to the curve $y = x^3 + 3x^2 + x - 5$ at (1, 0).

10 If $h = 60t - 3t^2$, find $\dfrac{dh}{dt}$ when $t = 3$.

11 For each graph of a function, find all values of x where it is not differentiable.

a



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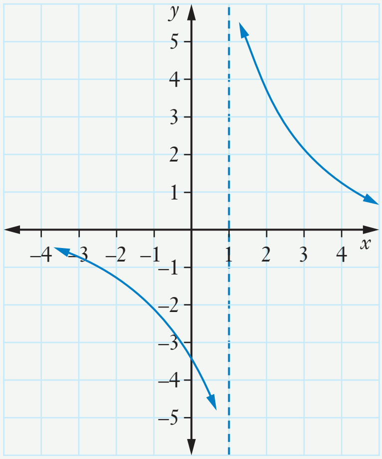

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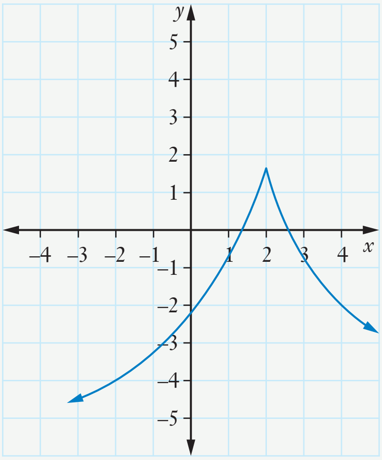

12 Differentiate:

a $y = \dfrac{4}{x}$

b $f(x) = \sqrt[5]{x}$

c $f(x) = 2(4x + 9)^4$

d $y = (3x + 2)(x - 1)^3$

e $f(x) = \dfrac{x^3 - 3}{2x + 5}$

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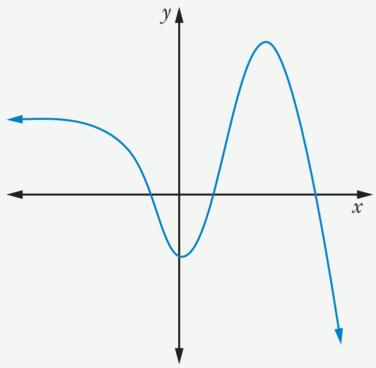
13 Sketch the derivative function of this curve.



14 Find the equation of the tangent to the curve $y = x^2 + 5x - 3$ at (2, 11).

15 Find the point on the curve $y = x^2 - x + 1$ at which the tangent has a gradient of 3.

16 Find $\frac{dS}{dr}$ if $S = 4\pi r^2$.

17 Find the gradient of the secant on the curve $f(x) = x^2 - 3x + 1$ between the points where $x = 1$ and $x = 1.1$.

18 At which points on the curve $y = 2x^3 - 9x^2 - 60x + 3$ are the tangents horizontal?

19 Find the equation of the tangent to the curve $y = x^2 + 2x - 5$ that is parallel to the line $y = 4x - 1$.

20 **a** Differentiate $s = ut + \frac{1}{2}at^2$ with respect to t.

b Find the value of t for which $\frac{ds}{dt} = 5$, $u = 7$ and $a = -10$.

21 Find the equation of the tangent to the curve $y = \frac{1}{3x}$ at the point where $x = \frac{1}{6}$.

22 A ball is thrown into the air and its height h metres over t seconds is given by $h = 4t - t^2$.

a Find the height of the ball:

i initially **ii** at 2 s

iii at 3 s **v** at 3.5 s

b Find the average rate of change of the height between:

i 1 and 2 seconds **ii** 2 and 3 seconds

c Find the rate at which the ball is moving:

i initially **ii** at 2 s **iii** at 3 s

23 If $f(x) = x^2 - 3x + 5$, find:

a $f(x + h)$ **b** $f(x + h) - f(x)$ **c** $f'(x)$

24 Given $f(x) = (4x - 3)^5$, find the value of:

a $f(1)$ **b** $f'(1)$

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25 Find $f'(4)$ when $f(x) = (x - 3)^9$.

26 Differentiate:

a $y = 3(x^2 - 6x + 1)^4$ **b** $y = \dfrac{2}{\sqrt{3x - 1}}$

27 EXT1 The volume of a sphere increases at a constant rate of 35 mm^3 s^{-1}. When the radius is 12 mm, at what rate is the radius increasing?

28 The displacement x in cm of a particle over time t seconds is given by $x = 5 + 6t - 3t^2$.

a Find the initial:
 i displacement **ii** velocity

b What is the velocity at 1 s?

c When is the particle at rest ($v = 0$)?

d What is the maximum displacement?

e Show that the particle is moving with constant acceleration.

29 A particle moves so that its displacement after t seconds is $x = 4t^2 - 5t^3$ metres. Find:

a its initial displacement, velocity and acceleration

b when $x = 0$

c its velocity and acceleration at 2 s.

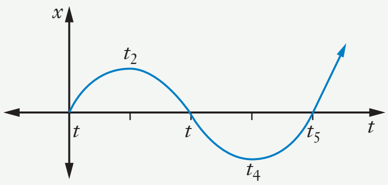
30 EXT1 A particle has displacement $x = t^3 - 12t^2 + 36t - 9$ cm at time t seconds.

a When is the particle at rest?

b At 1 s, what is:
 i the displacement? **ii** the velocity? **iii** the acceleration?

c Describe the motion of the particle after 1 s.

31 EXT1 This graph shows the displacement of a particle.



a When is the particle at the origin?

b When is it at rest?

c When is it travelling at its greatest speed?

d Sketch the graph of:
 i its velocity **ii** its acceleration.

32 EXT1 The height of a ball is $h = 20t - 5t^2$ metres after t seconds.

a Find the height at 1 s.

b What is the maximum height of the ball?

c What is the time of flight of the ball?

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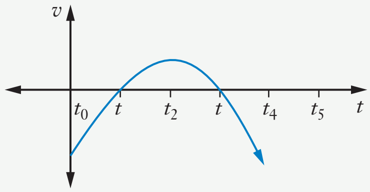
33 EXT1 This graph shows the velocity of a particle.

a Sketch a graph that shows:

i displacement **ii** acceleration.

b When is the particle at rest?



34 EXT1 $P(x) = (x - b)^7$.

a Show that $P(b) = P'(b) = 0$.

b Hence find a and b if $(x - 1)^7$ is a factor of $P(x) = x^7 + 3x^6 + ax^5 + x^4 + 3x^3 + bx^2 - x + 1$.

35 EXT1 **a** Show that $x - 5$ is a factor of $f(x) = x^3 - 7x^2 - 5x + 75$.

b Show that $f(5) = f'(5) = 0$.

c What can you say about the root at $x = 5$?

d Write $f(x)$ as a product of its factors.

8. CHALLENGE EXERCISE

1 Find the equations of the tangents to the curve $y = x(x - 1)(x + 2)$ at the points where the curve cuts the x-axis.

2 **a** Find the points on the curve $y = x^3 - 6$ where the tangents are parallel to the line $y = 12x - 1$.

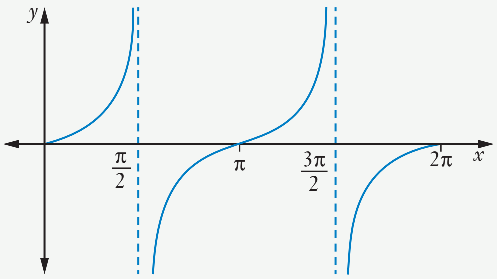
b Hence find the equations of the normals to the curve at those points.

3 The normal to the curve $y = x^2 + 1$ at the point where $x = 2$ cuts the curve again at point P. Find the coordinates of P.

4 The equation of the tangent to the curve $y = x^4 - nx^2 + 3x - 2$ at the point where $x = -2$ is given by $3x - y - 2 = 0$. Evaluate n.

5 **a** Find any points at which the graphed function is not differentiable.

b Sketch the derivative function for the graph.



6 EXT1 A sunflower grows so that the diameter of its disc increases at a constant rate of 0.1 mm per hour until it reaches 200 mm.

a Find the rate of increase in the surface area of the flower when its diameter is 50 mm.

b What is the maximum surface area of the sunflower, to the nearest mm?

7 Find the exact gradient of the tangent to the curve $y = \sqrt{x^2 - 3}$ at the point where $x = 5$.

8 Find the equation of the normal to the curve $y = 3\sqrt{x+1}$ at the point where $x = 8$.

9 **a** Find the equations of the tangents to the parabola $y = 2x^2$ at the points where the line $6x - 8y + 1 = 0$ intersects with the parabola.

b Show that the tangents are perpendicular.

10 Find any x values of the function $f(x) = \dfrac{2}{x^3 - 8x^2 + 12x}$ where it is not differentiable.

11 EXT1 A spherical balloon is being inflated so that the surface area is increasing at the rate of 0.3 cm^2 s^{-1}. When the balloon's radius is 5 cm, find the rate of increase in:

a the radius

b the volume.

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12 Find the equation of the chord joining the points of contact of the tangents to the curve $y = x^2 - x - 4$ with gradients 3 and –1.

13 For the function $f(x) = ax^2 + bx + c$, $f(2) = 4$, $f'(1) = 0$ and $f'(-3) = 8$. Evaluate a, b and c.

14 For the function $f(x) = x^3$:

a Show that $f(x + h) = x^3 + 3x^2h + 3xh^2 + h^3$.

b Show that $f'(x) = 3x^2$ by differentiating from first principles.

15 Consider the function $f(x) = \frac{1}{x}$.

a Find the gradient of the secant between:

i $f(1)$ and $f(1.1)$ **ii** $f(1)$ and $f(1.01)$ **iii** $f(1)$ and $f(0.99)$

b Estimate the gradient of the tangent to the curve at the point where $x = 1$.

c Show that $\frac{1}{x+h} - \frac{1}{x} = \frac{-h}{x(x+h)}$.

d Hence show that $f'(x) = -\frac{1}{x^2}$ by differentiating from first principles.

16 EXT1 A container is being filled with sand such that its volume at any one time is given by $V = 21x + x^2$ cm^3, where x is the depth of sand in the container. If sand is poured into the container at a constant rate of 15 cm^3 per second, at what rate, correct to 2 decimal places, will the level of sand be rising when the depth is 20 cm?

17 The displacement of a particle is given by $x = (t^3 + 1)^6$, where x is in metres and t is in seconds.

a Find the initial displacement and velocity of the particle.

b Find its acceleration after 2 s in scientific notation, correct to 3 significant figures.

c Show that the particle is never at the origin.

18 EXT1 For each function:

i find the inverse function $f^{-1}(x)$ with x in terms of y

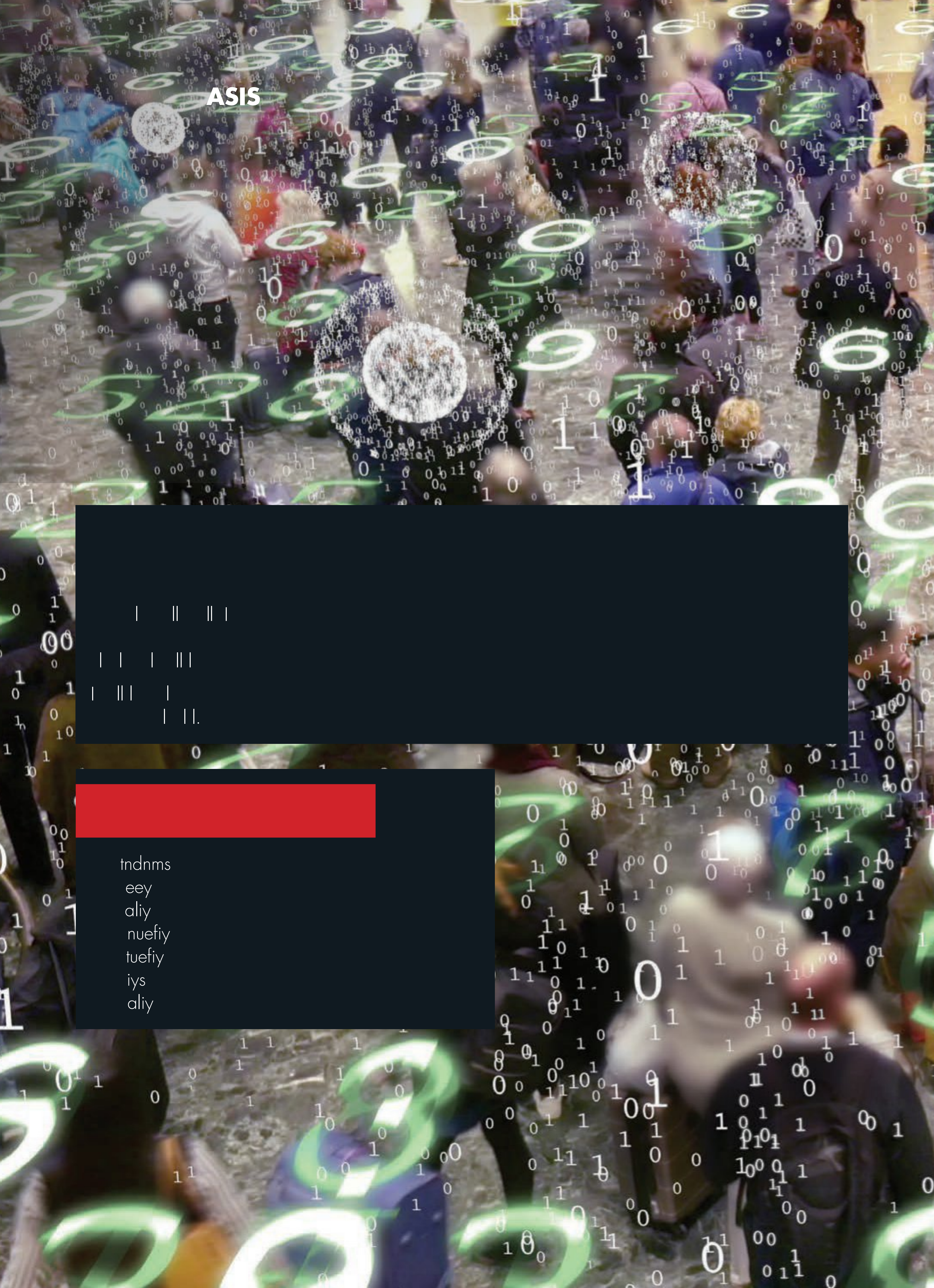
ii write $f^{-1}(x)$ with y in terms of x

iii find $\frac{dy}{dx}$ of the inverse function

v find $\frac{dx}{dy}$ of the inverse function in terms of x

v show $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.

a $y = 3x + 1$ **b** $f(x) = 3x^5$



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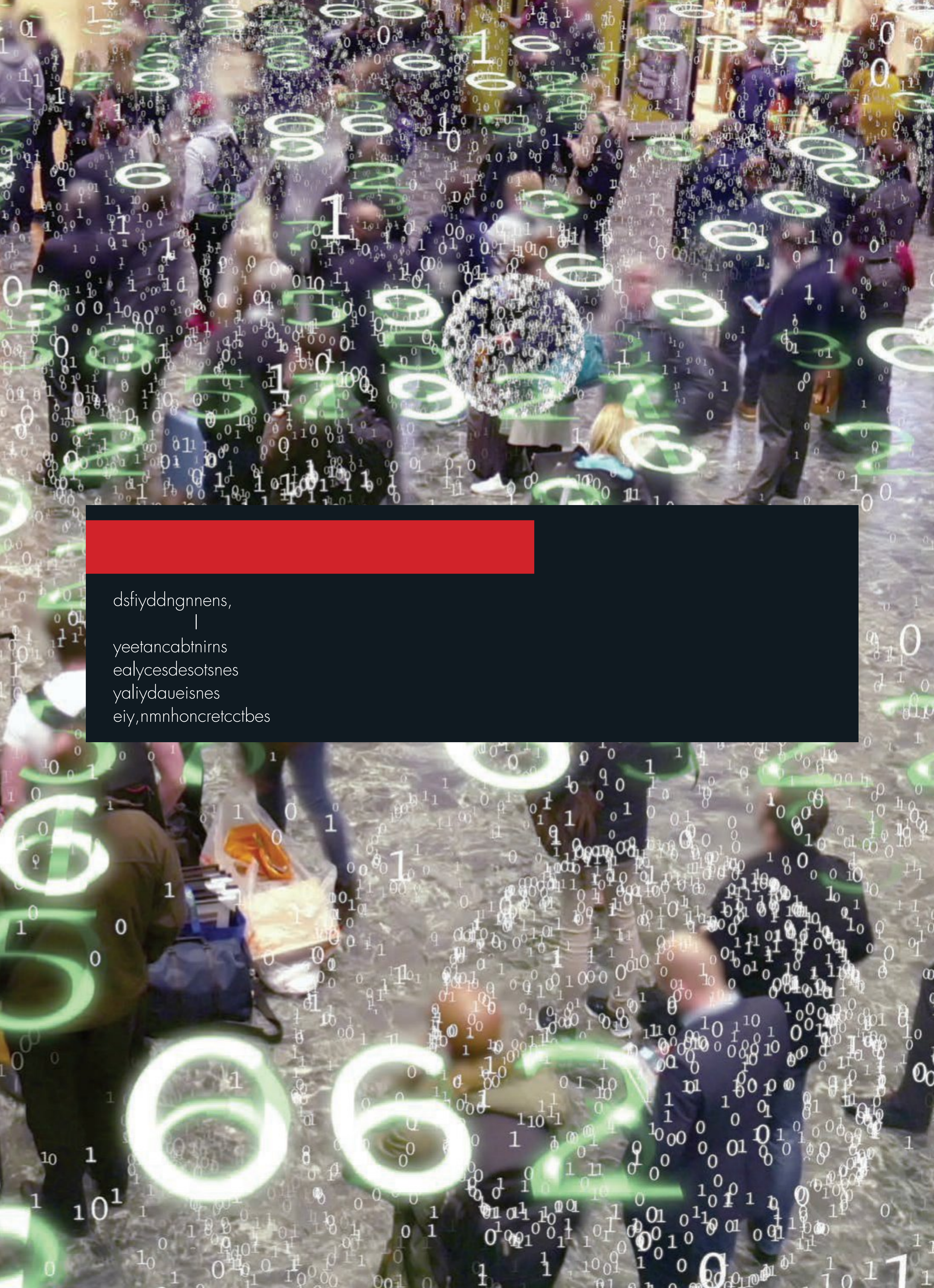

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TERMINOLOGY

complement The complement of an event is when the event does not occur

conditional probability The probability that an event A occurs when it is known that another event B has occurred

equally likely outcoes: Outcomes that have the same chance of occurring

independent events Events where the occurrence of one event does not affect the probability of another event

mutually exclusive events Events within the same sample space that cannot both occur at the same time for example rolling an even number on a die and rolling a 5 on the same die

non-mutually exclusive events Events within the same sample space that can occur at the same time for example rolling a prime number on a die and rolling an odd number

probability tree A diagram that uses branches to show multi-stage events and sets out the probability on each branch

relative frequeny: The frequency of an event relative to the total frequency

sample space The set of all possible outcomes in an event

set A collection of distinct objects called elements or members
For example set $A = \{1, 2, 3, 4, 5, 6\}$

tree diagram A diagram that uses branches to show multi-stage events

Venn diagram A diagram that shows the relationship between 2 or more sets using circles (usually overlapping) drawn inside a rectangle

9.01 Set notation and Venn diagrams

Venn diaras

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Venn diagam

Probability and statistics do not provide exact answers in real lifebut they can help in making decisions Here are some examples of where statistics and probability are use.

- An actuary is a mathematician who looks at statistics and makes decisions for insurance companies Life expectancy statistics help to decide the cost of life insurance for people of different ages Statistics about car accidents will help set car insurance premium.
- Stockbrokers use a chart or formula to predict when to buy and sell share. This chart is usually based on statistics of past trends
- A business does a feasibility study in a local area to decide whether to open up a new leisure centre It uses this data to make a decision based on the likelihood that local people will want to join the centre

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To find the probability of an event happenin, we compare the number of ways the event can occur with the total number of possible outcomes (the sample space)

$$\text{Probability of an event} = \frac{\text{Number of ways the event can occur}}{\text{Total number of possible outcomes}}$$

If we call the event E and the sample space S we can write this as

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We usually write a probability as a fractio, but we could also write it as a decimal or percentae.

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Set notation

When working with probabilities we often use **set notation**

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Venn diagrams

A **Venn diagram** is a special way to show the relationship between two or more sets

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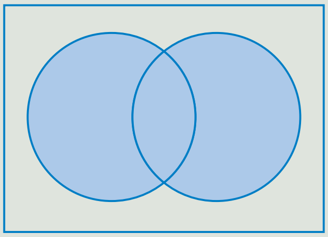

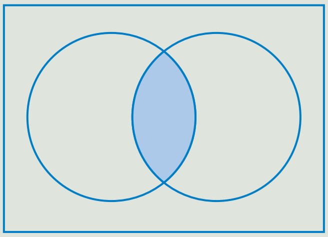

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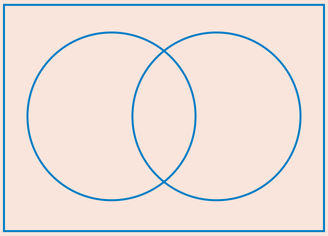

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Exercise 9.01 Set notation and Venn diagrams

1 Write the sample space in set notation for each chance situatin.

- **a** Tossing a coin
- **b** Rating a radio station between 1 and 5
- **c** Rolling a die
- **d** Selecting a jelly bean from a packet containing re, gren, yellow and blue jelly beans
- **e** Rolling an 8-sided die with a different number from 1 to 8 on each fac.

2 For each pair of set, fid:

i $X \cap Y$ **ii** $X \cup Y$

- **a** $X = \{1, 2, 3, 4, 5\}$ and $Y = \{2, 4, 6\}$
- **b** X = {red yello, white} and Y = {red white}
- **c** $X = \{4, 5, 7, 11, 15\}$ and $Y = \{6, 8, 9, 10, 12\}$
- **d** X = {blue gree, bron, hazel} and Y = {brown gre, blue}
- **e** $X = \{1, 3, 5, 7, 9\}$ and $Y = \{2, 4, 6, 8, 10\}$

3 Draw a Venn diagram for each pair of sts.

a $A = \{10, 12, 13, 14, 15\}$ and $B = \{12, 14, 15, 16\}$

b P = {red yello, white} and Q = {red gree, white}

c $X = \{2, 3, 5, 7, 8\}$ and $Y = \{1, 2, 5, 7, 9, 10\}$

d A = {Toyot, Maza, MW, Nssan, Porsche} and B = {Mazda Nissa, Holdn, Ford Porsche}

e X = {rectangle squar, trapezium} and Y = {square parallelogra, trapezim, kite}

4 Discuss whether each probability statement is tru.

a The probability of one particular horse winning the Melbourne Cup is — if there are 20 horses in the race

b The probability of a player winning a masters golf tournament is — if there are 15 players in the tournament

c A coin came up tails 8 times in a ro So the next toss must be a had.

d A family has 3 sons and is expecting a fourth chil. There is more chance of the new baby being a daughter.

e The probability of a Ducati winning a MotoGP this year is — because there are 6 Ducatis and 47 motobikes altogether.

5 To start playing a board gae, Simone must roll an even number on a ie.

a Write down the sample space for rolling a de.

b What is the set of even numbers on a die?

6 Draw a Venn diagram for each pair of sts.

a Event K = {Monday, Thursay, Friday} and Event L = {Tuesda, Thurday, Saturday} out of days of the week

b Event $A = \{3, 5, 6, 8\}$ and Event $B = \{4, 7, 9\}$ with cards each with a number from 1 to 10 drawn out of a hat

n

nsedr **nnn** nhtd

9.02 Relative frequency

Expeimenal pobabiliy

We can use frequency distribution tables to find the probability of an event using **relative frequency** the frequency of the event relative to the total frequenc

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Exercise 9.02 Relative frequency

1 The table shows the scores that a class earned on a maths test

a Find the relative frequency for each score in the table in fraction for.

b If a student is chosen at random from this clas, find the probability that this student

i scored 8

ii scored less than 7

iii pased, if the pass markis 5.

c What score i:

i most likely?

ii least likely?

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2 The table shows the results of a survey into the number of days students study each week

s	

a Find the relative frequency as a percentage for each number of days

b If a student was selected at rando, find the most likely number of days this student studies

c Find the probability that this student would study for

i 1 day

ii 5 days

iii 3 or 4 days

iv at least 4 days

v fewer than 3 day.

3 The table shows the results of a trial HSC exa.

9	
7	
	4

a Calculate the relative frequency as a decimal for each class

b Find the probability that a student chosen at random from these students scored

i between 20 and 39

ii between 60 and 99

iii less than 4.

4 This table shows the results of a science experiment to find the velocity of an object when it is rolled down a ramp

yncy	

a Write the relative frequency of each velocity as a fraction

b Find the probability that an object selected at random rolls down the ramp with a velocity between

i 5 and 7 m/s **ii** 11 and 13 m/s **iii** 8 and 10 m/s

v 11 and 16 m/s **v** 2 and 10 m/.

c Find the probability that the object has a velocit:

i less than 8 m/s **ii** 5 m/s or more **iii** more than 7 m/.

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5 A telemarketing company records the number of sales it makes per minute over a half-hour period The results are in the tabe.

4	
6	
	3
	0
	5

a What percentage of the time were there 3 sales per minute?

b Write the relative frequencies as percentages

c What is the most likely number of sales/minute?

d Find the probability of makin:

i 2 sales/minute **ii** 5 sales/minute

iii more than 2 sales per minut.

6 **a** Organise the scores below in a frequency distribution tabl.

9, 5, 4, 7, 7, 9, 4, 6, 5, 8, 9, 6, 7, 4, 4, 3, 8, 5, 6, 9

b Find the probability of an outcome chosen at random having a score o:

i 7 **ii** at least 8 **iii** less than 5 **v** 7 or less

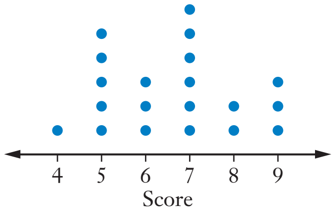
7 **a** From the dot plo, draw up a frequency distribution table

b Find the probability (as a decimal) that an outcome chosen at random has a score of

i 8 **ii** at least 6

iii less than 7 **v** 5 or more

v 8 or less



8 The stem-and-leaf plot shows the ages of people attending a meeting

Stem	Leaf
1	8 9 9
2	0 3 5 6
3	0 1 2 2 3 5 7 9
4	2 4 6 7 8
5	1 2 4 4 6

a Organise this data into a frequency distribution tabl, using groups of 10–19 20–29 and so o.

b What percentage of people at the meeting wer:

i in their 30s?

ii younger than 20?

iii in their 40s or 50s?

c Find the probability that a person selected at random from this meeting i:

i younger than 40

ii 50 or over

iii between 20 and 49

v over 29

v between 10 and 4.

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9 The table shows the quantity of food that a pet shop uses each day for a month

dncy	

a In which month was this survey done?

b For what fraction of the month was 45–59 kg of food used?

c For what percentage of the month did the pet shop need more than 29 kg of pet food?

d Find the relative frequency for all groups as a fractio.

e If this survey is typical of the quantities of food that the pet shop use, find the probability that on any day it will use between

i 30 and 44 kg **ii** 45 and 74 kg **iii** 0 and 29 kg

v 15 and 59 kg **v** 30 and 74 kg

Theoeical pobabiliy

Maching pobabiliie

9.03 Theoretical probability

While experiments and surveys can give a good prediction of the probability of future events they are not very accurate The larger the number of trias, the closer the results can become to the theoretical probability. Howevr, this is not guaraneed.

For example it is reasonable to assume that if you toss a coin many times you would get similar numbers of heads and tails Yet in an experiment a coin may come up heads every tme.

TOSSING A COIN

Toss a coin 20 times and count the number of heads and tail. What would you expect to happen when tossing a coin this many times? Did your results surprise you?

Combine your results with others in your classroom into a table with relative frequencies

1 Do the combined results differ from your own?

2 From the tabl, find the probability of tossing

a heads **b** tails.

If a coin came up tails every time it was tossed 20 timesdo you think it would it be more likely to come up heads the next time? Why?

Even though we might think that theoretical probability should be more accurate than experiments in real life these probabilities will not happen exactly as in theor!

Mutually exclusive events are events that cannot occur at the same timeFor exampl, when throwing a die you cannot throw a number that is both a 5 and a 6

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We can use probability to make predictions or decision.

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Exercise 9.03 Theoretical probability

1 Alannah is in a class of 30 student. If one student is chosen at random to make a speeh, find the probability that the student chosen

a will be Alannah **b** will not be Alannh.

2 A pack of cards contains 52 different card, one of which is the ace of diamons. If one card is chosen at random find the probability that i:

a will be the ace of diamonds **b** will not be the ace of diamond.

3 There are 6 different newspapers sold at the local newsagent each da Wendy sends her little brother Rupert to buy her a newspaper one morning but forgets to tell him which one What is the probability that Rupert will buy the correct newspaper ?

4 A raffle is held in which 200 tickets are sol. If I buy 5 tickes, what is the probabilityof:

a my winning **b** my not winning the prize in the raffle?

5 In a lotter, 200 000 tickets are sld. If Lucia buys 10 ticets, what is the probability of her winning first prize?

6 A bag contains 6 red balls and 8 white ball. If Peter draws one ball out of the bag at random find the probability that it will b:

a white **b** red.

7 A shoe shop orders in 20 pairs of blac, 14 pairs of navy and 3 pairs of brown school shoes If the boxes are all mixed u, find the probability that one box selected at random will contain brown shoes

8 The probability of a bus arriving on time is estimated at $\frac{18}{33}$

a What is the probability that the bus will not arrive on time?

b If there are 352 buses each da, how many would be expected to arrive on time ?

9 A bag contains 5 black marble, 4 yellow marbles and 11 green marbls. Find the probability of drawing 1 marble out at random and getting

a a green marble **b** a yellow or a green marbl.

10 The probability of a certain seed producing a plant with a pink flower is $\frac{7}{9}$

a Find the probability of the seed producing a flower of a different colou

b If 189 of these plants are grow, how many of them would be expected to have a pink flower?

11 If a baby has a .2% chance of being born with a disabilit find the probability of the baby being born without a disability.

12 A die is throw. Calculate the probability of throwig:

a a 6 **b** an even number **c** a number less than .

13 A book has 124 page. If any page is selected at randm, find the probability of the page number being

a either 80 or 90 **b** a multiple of 10

c an odd number **d** less than 10.

14 A machine has a .5% chance of breaking down at any given tim.

a What is the probability of the machine not breaking down?

b If 2600 of these machines are manufacture, how many of them would be expected o:

i break down? **ii** not break down?

15 The probabilities when 3 coins are tossed are as follow:

$P(3 \text{ heads}) = \frac{1}{8}$ $P(2 \text{ heads}) = \frac{3}{8}$

$P(1 \text{ head}) = \frac{3}{8}$ $P(3 \text{ tails}) = \frac{1}{8}$

Find the probability of tossing at least one head

16 In the game of poo, there are 15 bals, each with the number 1 to 15 onit. In Kelly pool each person chooses a number at random to determine which ball to sin. If Tracey chooses a number, find the probability that her ball will e:

a an odd number **b** a number less than 8 **c** the 8 ball.

17 **a** Find the probability of a coin coming up heads when tosse.

b If the coin is double-heade, find the probability of tossing a hed.

18 A student is chosen at random to write about his or her favourite spor. If 12 students like tennis best 7 prefer socce, 3 prefer sqush, 5 prefer basketball and 4 prefer swimming find the probability that the student chosen will write abou:

a soccer **b** squash or swimming **c** tennis.

19 There are 29 re, 17 ble, 21 yellow and 19 green chocolate beans in a pacet. If Kate chooses one at random find the probability that it will be red or yello

20 The probability of breeding a white budgerigar is $\frac{2}{9}$ If Mr Seed breeds 153 budgerigars over the year, how many would be expected to be white?

21 A biased coin is weighted so that heads comes up twice as often as tail. Find the probability of tossing a tail

22 A die has the centre dot painted white on the 5 so that it appears as a . Find the probability of throwing

a a 2 **b** a 4 **c** a number less than .

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23 The probabilities of a certain number of seeds germinating when 4 seeds are planted ae:

	—	—	—	—	—

Find the probability of at least one seed germinating

24 The probabilities of 4 friends being chosen for a soccer team ar:

$P(4 \text{ chosen}) = \frac{1}{15}$ $P(3 \text{ chosen}) = \frac{4}{15}$

$P(2 \text{ chosen}) = \frac{6}{15}$ $P(1 \text{ chosen}) = \frac{2}{15}$

Find the probability of

a none of the friends being chosen

b at least 1 of the friends being chose.

25 If 2 events are mutually exclusiv, what could you say about $A \cap B$?

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9.04 Addition rule of probability

Sometimes there is an overlap where more than one event can occur at the same tim. We call these **non-mutually exclusive events** It is important to count the possible outcomes carefully when this happens We need to be careful not to count the overlapping outcomes $A \cap B$ twice

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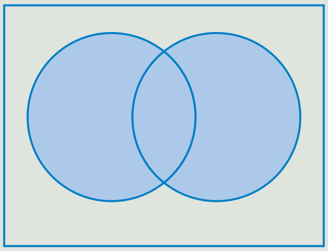

If A and B are mutually exclusive then $P(A \cap B) = 0$ so $P(A \cup B) = P(A) + P(B)$

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Sometimes for more complex problems a Venn diagram is useul.

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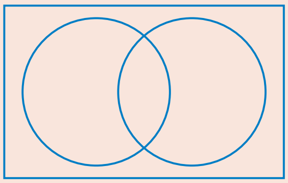

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Exercise 9.04 Addition rule of probability

1 A number is chosen at random from the numbers 1 to 2. Find the probability that the number chosen will be

a divisible by 3
b less than 10 or divisible by 3
c a composite number
d a composite number or a number greater than 1.

2 A set of 50 cards is labelled from 1 to 5. One card is drawn out at randm. Find the probability that the card will be

a a multiple of 5
b an odd number
c a multiple of 5 or an odd number
d a number greater than 40 or an even number
e less than 2.

3 A set of 26 card, each with a different letter of the alphabet on t, is placed in a box and one card is drawn out at random Find the probability that the letter on the card i:

a a vowel
b a vowel or one of the letters in the word 'randm'
c a consonant or one of the letters in the word 'movi'.

4 A set of discs is numbered 1 to 100 and one is chosen at rando. Find the probability that the number on the disc will be

a less than 30
b an odd number or a number greater than 70
c divisible by 5 or less than 2.

5 In Lott, a machine holds 45 bals, each with a different number between 1 and 45 onit. The machine draws out one ball at a time at random Find the probability that the first ball drawn out will be

a less than 10 or an even number
b between 1 and 15 inclusiv, or divisible by 6
c greater than 30 or an odd numbe.

6 A class of 28 students puts on a concert with all class members performin. If 15 dance and 19 sing in the performance find the probability that any one student chosen at random from the class will

a both sing and dance
b only sing
c only danc.

7 A survey of 80 people with dark hair or brown eyes showed that 63 had dark hair and 59 had brown eyes Find the probability that one of the people surveyed chosen at random has

- **a** dark hair but not brown eyes
- **b** brown eyes but not dark hair
- **c** both brown eyes and dark hai.

8 A list is made up of 30 people with experience in coding or graphical desig. On the list 13 have coding experience while 9 have graphical design experienc. Find the probability that a person chosen at random from the list will have experience in

- **a** both coding and graphical design
- **b** coding only
- **c** graphical design onl.

9 Of a group of 75 student, all study either history or geograpy. Altogether 54 take history and 31 take geography. Find the probability that a student selected at random studies

- **a** only geography
- **b** both history and geography
- **c** history but not geograph.

10 In a group of 20 dogs at obedience schoo, 14 dogs will walk to heel and 12 will stay when told All dogs will do one or the othr, or oth. If one dog is chosen at rndom, find the probability that it will

- **a** both walk to heel and stay
- **b** walk to heel but not stay
- **c** stay but not walk to hee.

9.05 Product rule of probability

Muliage poblem

TWO-STAGE EVENTS

Work in pairs and try these experiments with one person doing the activity and one recording the results Toss two coins as many times as you can in a 5-minute period and record the results in a table

	eddds		

Compare your results with others in the class What do you notice ? Is this surprising?

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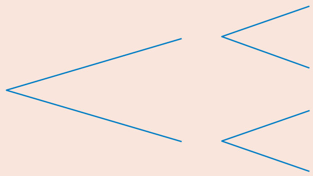
Roll 2 dice as many times as you can in a 5-minute periodfind the total of the 2 numbers rolled and record the results in a table

Compare your results with others in the classWhat do you notice ? Is this surprising?

Tossing 2 coins and rolling 2 dice are examples of **multi-stage experiments** where two outcomes happen together. The sample space becomes more complicaed, so to list all possible outcomes we use tables and **tree diagrams**



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					7 8		
					8 9		
					9		

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If A and B are **independent events** then A occurring does not affect the probability of B occurring The probability of both occurring is the product of their probabilitis.

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The sample space changes when events are not independentThe second event is **conditional** on the first event

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Exercise 9.05 Product rule of probability

1 Find the probability of getting 2 heads if a coin is tossed twic.

2 A coin is tossed 3 time. Find the probability of tossing 3 tais.

3 A family has 2 childre. What is the probability that they are both girls ?

4 A box contains 2 black ball, 5 red balls and 4 green bals. If I draw out 2 balls at ranom, replacing the first before drawing out the secondfind the probability that they will both be red

5 The probability of a conveyor belt in a factory breaking down at any one time is .2. If the factory has 2 conveyor beltsfind the probability that at any one tim:

a both conveyor belts will break down

b neither conveyor belt will break dow.

6 The probability of a certain plant flowering is 93. If a nursery has 3 of these plans, find the probability that they will all flower.

7 An archery student has a 69% chance of hitting a targe. If she fires 3 arrows at a targt, find the probability that she will hit the target each time

8 The probability of a pair of small parrots breeding an albino bird is $\frac{2}{33}$ If they lay 3 eggs find the probability of the pai:

a not breeding any albinos **b** having all 3 albinos

c breeding at least one albin.

9 A photocopier has a paper jam on average around once every 2400 sheets of pape

a What is the probability that a particular sheet of paper will jam?

b What is the probability that 2 particular sheets of paper will jam?

c What is the probability that 2 particular sheets of paper will both not jam?

10 In the game Yahtee, 5 dice are roled. Find the probability of roling:

a five 6s **b** no 6s **c** at least one 6.

11 The probability of a faulty computer part being manufactured at Omikron Computer Factory is $\frac{3}{5000}$ If 2 computer parts are examine, find the probability tht:

a both are faulty **b** neither is faulty

c at least one is fault.

12 A set of 10 cards is numbered 1 to 10 and 2 cards are drawn out at random with replacement Find the probability that the numbers on both cards ar:

a odd numbers **b** divisible by 3

c less than 4.

13 The probability of an arrow hitting a target is 85. If 3 arrows are sht, find the probability as a percentage correct to 2 decimal place, f:

a all arrows hitting the target

b no arrows hitting the target

c at least one arrow hitting the targe.

14 A coin is tossed n times Find the probability in terms of n of tossing

a no tails **b** at least one tai.

15 A bag contains 8 yellow and 6 green lollie. If I choose 2 lollies at randm, find the probability that they will both be green

a if I replace the first lolly before selecting the second

b if I do't replace the first loly.

16 Mala buys 10 tickets in a raffle in which 250 tickets are sol. Find the probability that she wins both first and second prizes

17 Two cards are drawn from a deck of 20 red and 25 blue cards (without replacemen). Find the probability that they will both be red

18 A bag contains 100 cards numbered 1 to 10. Scott draws 2 cards out of the bg. Find the probability that

a both cards are less than 10

b both cards are even

c neither card is a multiple of .

19 A box of pegs contains 23 green pegs and 19 red peg. If 2 pegs are taken out of the box at random find the probability that both will b:

a green **b** red

20 Find the probability of selecting 2 apples at random from a fruit bowl that contains 8 apples 9 oranges and 3 peache.

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9.06 Probability trees

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Tree diagam

A **probability tree** is a tree diagram that shows the probabilities on the branches

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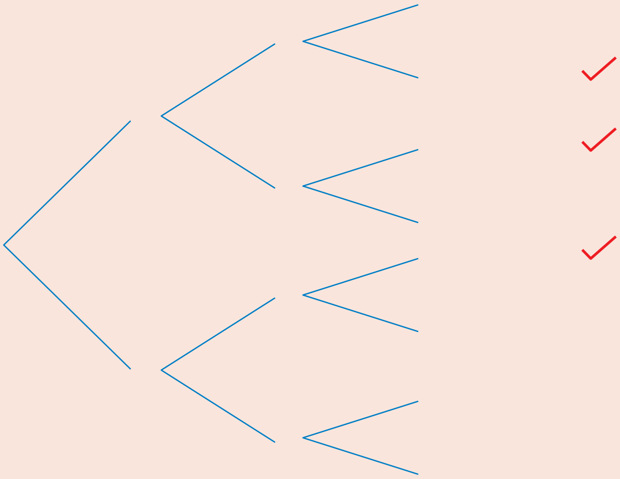

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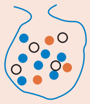

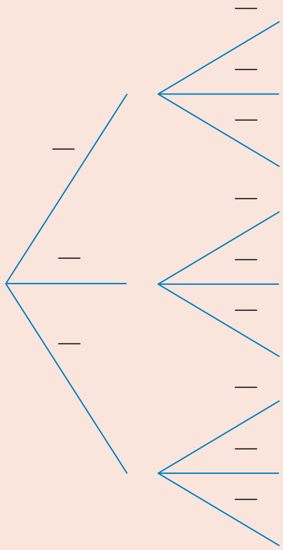

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Exercise 9.06 Probability trees

1 Three coins are tosse. Find the probability of gettig:

a 3 tails **b** 2 heads and 1 tail **c** at least 1 head.

2 In a set of 30 card, each one has a number on it from 1 to 0. If 1 card is drawn ut, then replaced and another drawn outfind the probability of gettin:

a two 8s

b a 3 on the first card and an 18 on the second card

c a 3 on one card and an 18 on the other car.

3 A bag contains 5 red marbles and 8 blue marble. If 2 marbles are chosen at randm, with the first replaced before the second is drawn outfind the probability of gettin:

a 2 red marbles **b** a red and a blue marbl.

4 A certain breed of cat has a 35% probability of producing a white kitte. If a cat has 3 kittens find the probability that she will produc:

a no white kittens **b** 2 white kittens **c** at least 1 white kitte.

5 The probability of rain on any day in May each year is given by $\frac{3}{10}$ A school holds a fete on a Sunday in May for 3 years running Find the probability that it will rai:

a during 2 of the fetes **b** during 1 fete **c** during least 1 fet.

6 A certain type of plant has a probability of .85 of producing a variegated lea. If I grow 3 of these plants find the probability of getting a variegated leaf i:

a 2 of the plants **b** none of the plants **c** at least 1 plant.

7 A bag contains 3 yellow ball, 4 pink balls and 2 black bals. If 2 balls are chosen at random find the probability of getting a yellow and a black bal:

a with replacement **b** without replacemen.

8 Anh buys 4 tickets in a raffle in which 100 tickets are sold altogethe There are 2 prizes in the raffle Find the probability that Anh will wn:

a first prize **b** both prizes **c** 1 prize

d no prizes **e** at least 1 prize.

9 Two singers are selected at random to compete against each other in a TV singing contest One person is chosen from Tea A, which has 8 females and 7 ales, and the other is chosen from TeamB, which has 6 females and 9 maes. Find the probability of choosing

a 2 females **b** 1 female and 1 mal.

10 Two tennis players are said to have a probability of $\frac{2}{5}$ and $\frac{3}{4}$ respectively of winning a tournament Find the probability tha:

a 1 of them will win **b** neither one will wi.

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11 In a batch of 100 car, past experience would suggest that 3 could be fauly. If 3 cars are selected at random find the probability tha:

a 1 is faulty **b** none is faulty **c** all 3 cars are fault.

12 In a certain pol, 46% of people surveyed liked the current governmet, 42% liked the Opposition and 12% had no preferenceIf 2 people from the survey are selected at random find the probability tha:

a both will prefer the Opposition

b one will prefer the government and the other will have no preference

c both will prefer the governmen.

13 A manufacturer of X energy drink surveyed a group of people and found that 31 people liked X drinks best 19 liked another brand better and 5 did not drink energy drink. If any 2 people are selected at random from that groupfind the probability tha:

a one person likes the X brand of energy drink

b both people do not drink energy drink.

14 In a group of peopl, 32 are Australian-brn, 12 were born in Asia and 7 were born in Europe If 2 of the people are selected at rando, find the probability tht:

a they were both born in Asia

b at least 1 of them will be Australian-born

c both were born in Europ.

15 There are 34 men and 32 women at a part Of thse, 13 men and 19 women are married If 2 people are chosen at rando, find the probability tht:

a both will be men

b one will be a married woman and the other an unmarried man

c both will be marrie.

16 Frankie rolls 3 dic. Find the probability she rols:

a 3 sixes **b** 2 sixes **c** at least 1 six.

17 A set of 5 card, each labelled with one of the lettersA, C, Dand E, is placed in a hat and 2 cards are selected at random without replacementFind the probability of gettin:

a D and E

b neither D nor E on either card

c at least one .

18 The ratio of girls to boys at a school is 4: . Two students are surveyed at random from the school Find the probability that the students ar:

a both boys **b** a girl and a boy **c** at least one gir.

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9.07 Conditional probability

Conditional probability is the probability that an event A occurs when it is known that another event B has already occurred You have already used conditional probability in multi-stage events when the outcome of the second event was dependent on the outcome of the first event Examples include selections **without replacement**

We write the probability of event A happening given that event B has happened as $P(A|B)$

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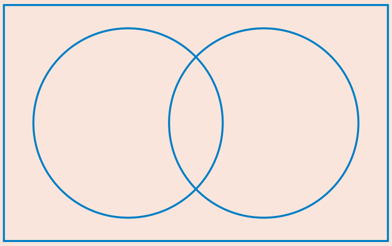

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With conditional probabilit, knowing that an event has already occurred reduces the sample space In the example abov, the sample space changed from 30 to 2.

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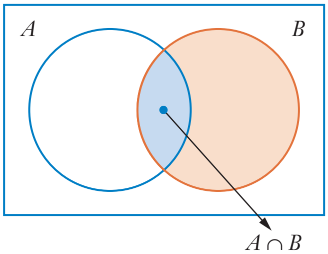
The $P(B)$ in the denominator is a result of the sample space being reduced to B (the orange circle in the Venn diagram)



Proof

For conditional probability, the product rule becomes $P(A \cap B) = P(A|B)P(B)$

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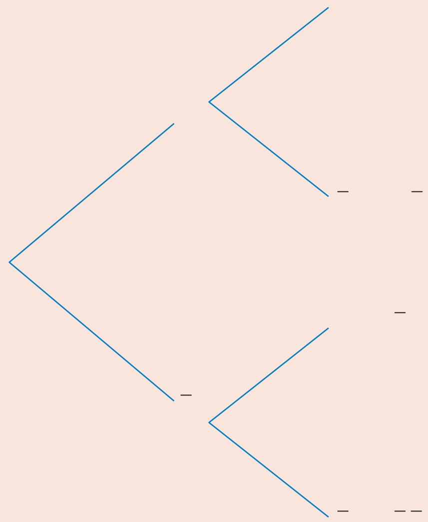

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Conditional probability and independent events

We saw earlier tha:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Rearranging this gives $P(A \cap B) = P(A|B)\ P(B)$

But if A and B are **independent events** $P(A \cap B) = P(A)P(B)$ (the product rule) which means

$P(A|B) = P(A)$

Similarly, $P(B|A) = P(B)$

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Exercise 9.07 Conditionl probablity

1 A bag contains 9 black and 8 white ball. I draw out two at randm. If the first ball is white find the probability that the next ball i:

a black **b** white

2 A class has 13 boys and 15 girl. Two students are chosen at random to carry a box of equipment Find the probability that the second person chosen is a boy given that the first student chosen was a girl

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3 Two dice are rolld. Find the probability of rollng:

a a double six if the first die was a si.

b a total of 8 or more if the first die was a .

4 A team has a probability of 52% of winning its first season and a 39% chance of winning both seasons 1 and 2 What is the probability of the team winning the second season given that it wins the first season?

5 A missile has a probability of .75 of hitting a targe. It has a probability of0.65 of hitting two targets in a row. What is the probability that the missile will hit the second target given that it has hit the first target?

6 Danuta has an 80% probability of passing her first English assessment and she has a 45% probability of passing both the first and second assessmentsFind the probability that Danuta will pass the second assessment given that she passes the first one

7 A group of 10 friends all prepared to go out in the sun by putting on either sunscreen or a hat If 5 put on only sunscreen and 3 put on both sunscreen and a ha, find the probability that a friend who

a put on sunscreen also put on a hat

b put on a hat did't put on sunscren.

8 A container holds 20 cards numbered 1 to 2. Two cards are selected at radom. Find the probability that the second card is

a an odd number given that the first card was a 7

b a number less than 5 given that the first card was a 12

c a number divisible by 3 if the first number was .

9 A group of 12 people met at a café for lunc. If 9 people had a pie and 7 had chis, find the probability that one of the people

a had chip, given that this person had a pie

b did not have a pie given that the person had chip.

10 All except for 3 people out of 25 on a European tour had studied either French or Spanish Nine people studied only French and 5 studied both French and Spanis. Find the probability that one of these people

a studied Spanish if that person studied French

b did not study French given that the person studied Spanis.

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11 The two-way table shows the numbers of students who own smartphones and tablet.

	oals		
	31		
	68		

Find the probability that a person selected at random

a owns a smartphone given that the perso:

i owns a tablet **ii** doesn't own a tabet.

b owns a tablet given that the perso:

i owns a smartphone **ii** doen't own a smartphne.

12 The table below shows the number of local people with casual and permanent job.

Find the probability that a person chosen at random

a has a permanent job given that she is a woman

b has a casual job given that he is a man

c is a man given that the person has a casual job

d is a man if the person has a permanent jo.

13 In a group of 35 friend, all either play sport or a musical instrumet. If 14 play both and 8 only play sport find the probability that a friend chosen at random wil:

a play a musical instrument given that the friend plays sport

b not play sport given that the friend plays a musical instrumen.

14 The two-way table shows the results of a survey into attendance at a local TAFE collge.

	e2nvotals			
E	8	155		
tFE	5	278		

Find the probability that a person

a attends TAFE given that this person is over 50

b is between 25 and 50 if that person does not attend TAFE

c is not at TAFE given that this person is under 25

d is over 50 if the person is at TAFE

e is at TAFE given the person is aged 25 or oer.

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15 A tennis team has a probability of 76% of winning a match when they are at home and 45% of winning a match when they are away. If the team plays 58% of their matches away, find the probability that the tem:

a wins their match given that they are away

b are at home given that they win a match

c are away given that they lose a matc.

16 A factory produces solar batterie. The probability of a new battery being defective is 3% Howeve, if the manager is on dty, the probability of a new battery being defective changes to 2% The manager is on duty 39% of the tie. Find the probability that the manager is on duty if a new battery is defective

17 The chance of a bushfire is 85% after a period of no rain and 21% after rai. The chance of rain is 46% Find the probability tha:

a there is not a bushfire given that it has rained

b it has rained given that there is a bushfire

c it has not rained given there is a bushfire

d it has rained given there is not a bushfir.

18 If $P(A|B) = 067$ and $P(B) = 031$ find the value of $P(A \cap B)$

19 If $P(L) = 017$, $P(L \cap M) = 00204$ and $P(M) = 012$ show that L and M are independent

20 Given $P(X) = 0.3$, $P(Y) = 042$ and $P(X \cup Y) = 0594$ show that X and Y are independent eyfgtte1ngoe:

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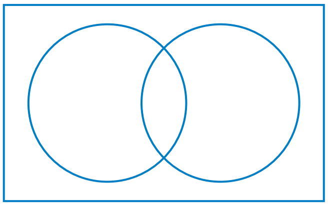

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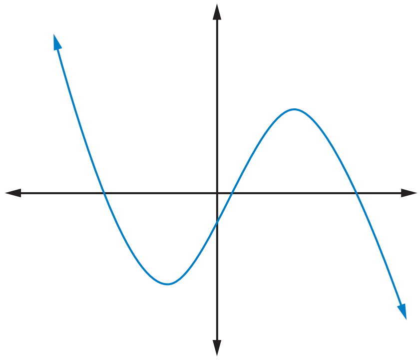

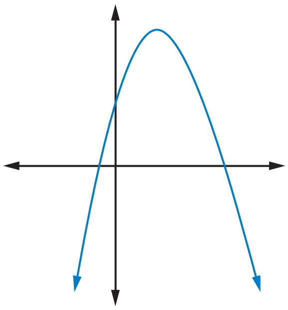

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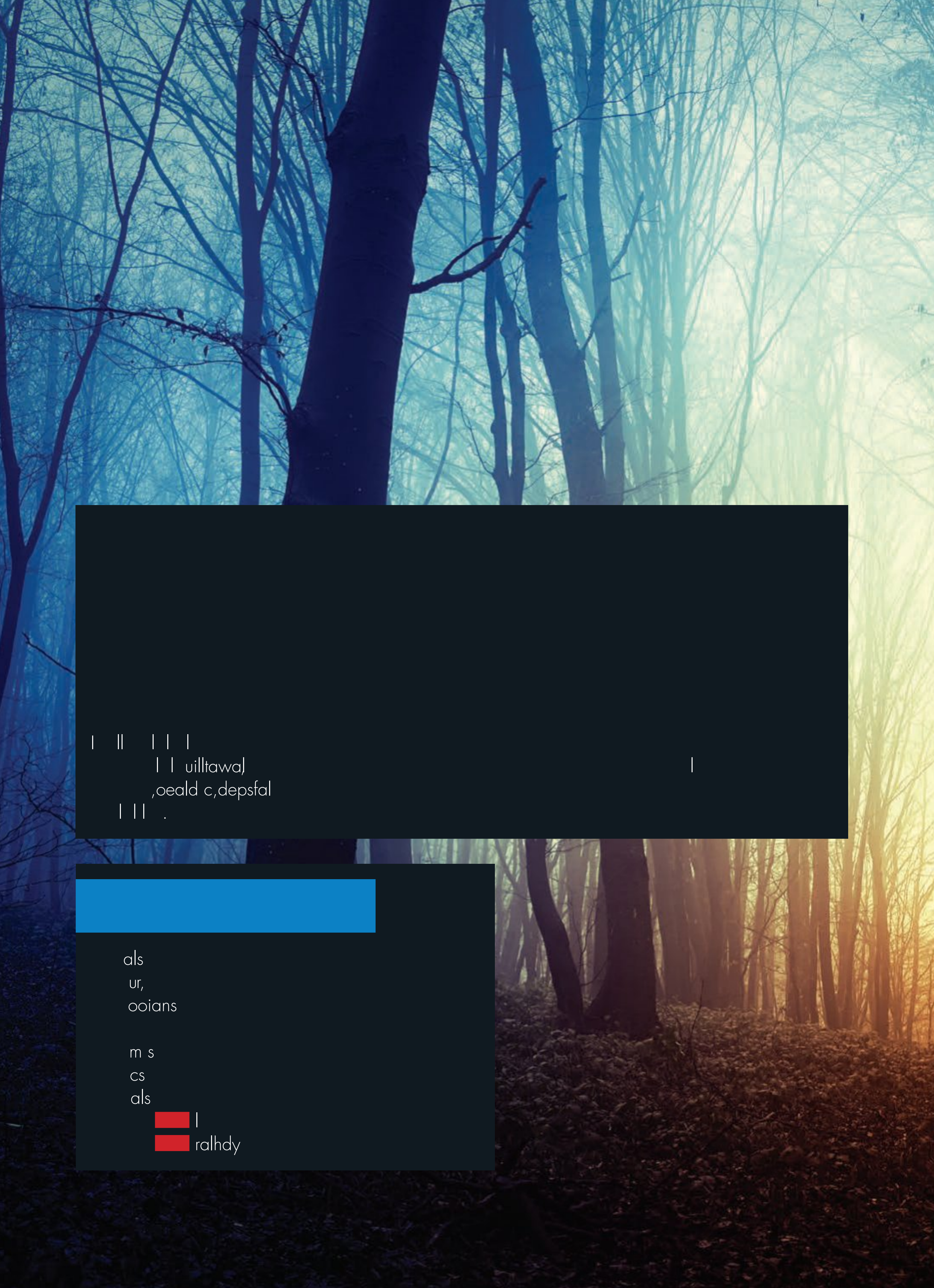

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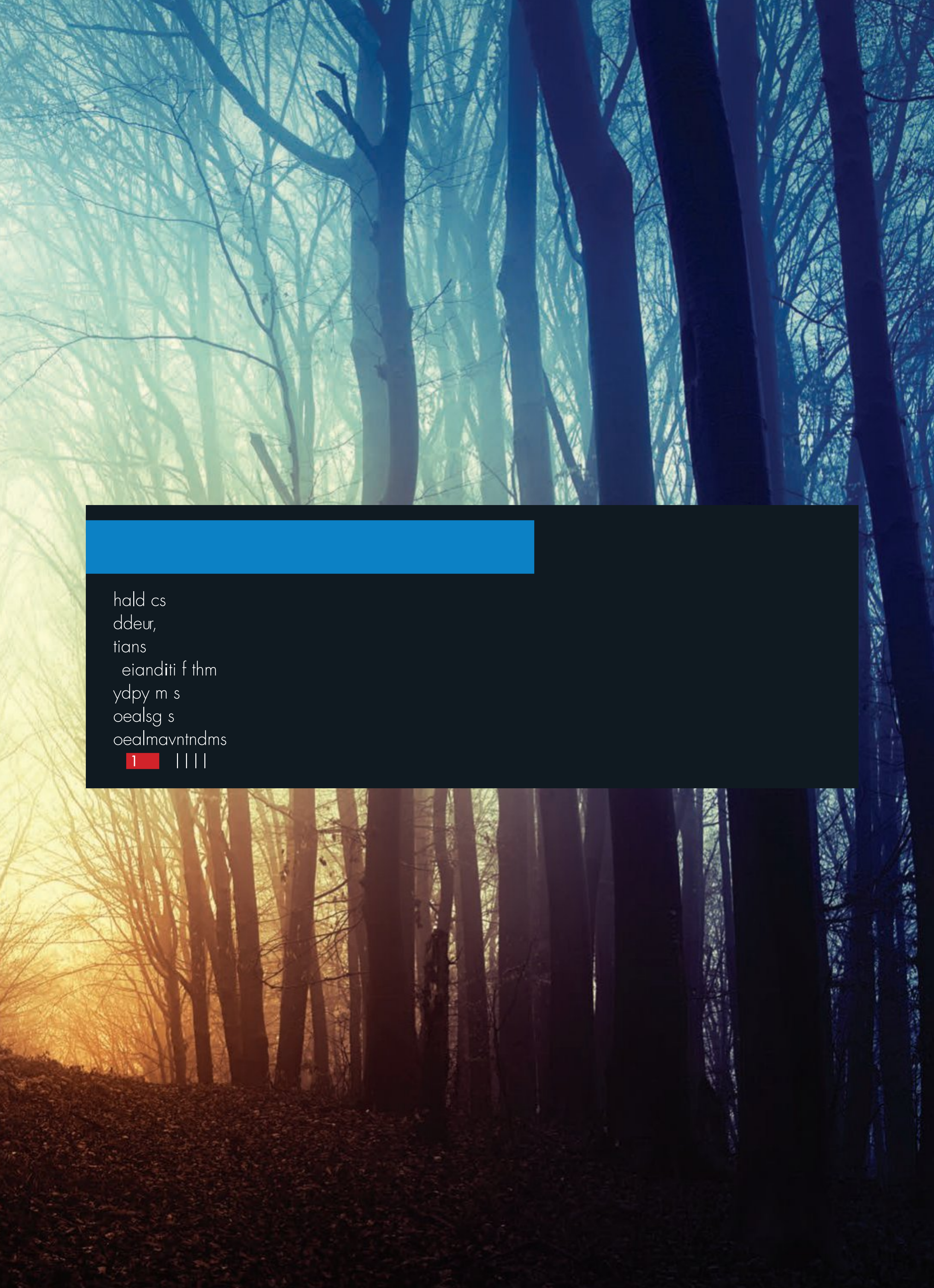

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TERMINOLOGY

Eulers number This numbe, e approximately 2718 28 is an important constant that is the base of natural logarithms

exponential decay When a quantity decreases according to the exponential function $N = Ae^{kt}$ where k is negative

exponential growth When a quantity increases according to the exponential function $N = Ae^{kt}$ where k is positive

exponential function A function in the form $y = a^x$

logarithm The logarithm of a positive number y is the power to which a given number a called the base must be raised in order to produce the number y so $\log_a y = x$ means $y = a^x$

logarithmic function A function in the form $y = \log_a x$

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10.01 Exponential functions

An exponential function is in the form $y = a^x$ where $a > 0$

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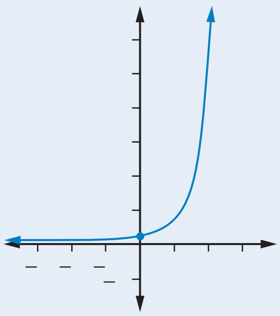

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THE VALUE OF *a* IN $y = a^x$

Notice that the exponential function $y = a^x$ is only defined for $a > 0$.

1 Suppose $a = 0$ What would the function $y = 0^x$ look like? Try completing a table of values or use technology to sketch the graphIs the function defined for positive values of x negative values of x or when $x = 0$? What if x is a fraction?

2 Suppose $a < 0$ What would the function $y = (-2)^x$ look like?

3 For $y = 0^x$ and $y = (-2)^x$

a is it possible to graph these functions at all?

b are there any discontinuities on the graphs?

c do they have a domain and range?

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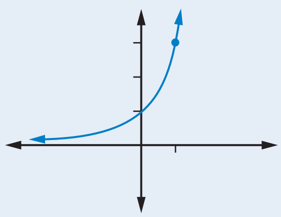

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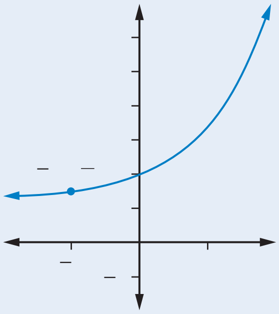

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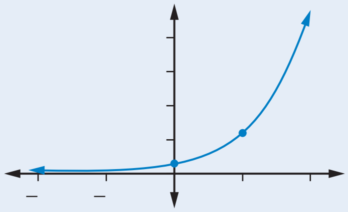

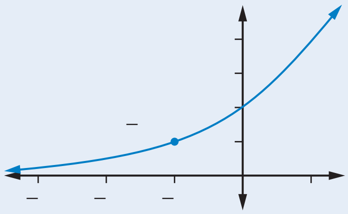

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Reflections of exponential functions

We can reflect the graph of $y = a^x$ using what we learned in Chapter 7 *Further functions*

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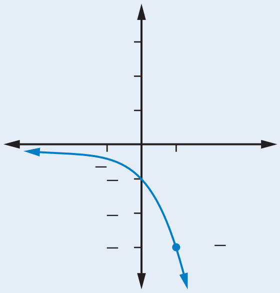

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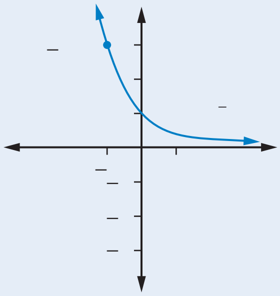

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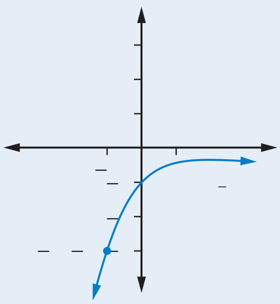

GRAPHS OF EXPONENTIAL FUNCTIONS

Use a graphics calculator or graphing software to sketch the graphs of the exponential functions below. Look for similarities and differences within each st.

a $y = 2^x \ \ y = 2^x + 1, y = 2^x + 3, y = 2^x - 5$

b $y = 3(2^x), y = 4(2^x), y = -2^x \ \ y = -3(2^x)$

c $y = 3(2^x) + 1, y = 4(2^x) + 3, y = -2^x + 1, y = -3(2^x) - 3$

d $y = 2^{x+} \quad y = 2^{x+2} \ \ y = 2^{x-} \quad y = 2^{x-3} \ \ y = 2^{-x}$

e $y = 2^{-x} \ \ y = 2\left(2^{-x}\right) \ \ y = -2^{-x} \ \ y = -3\left(2^{-x}\right) \ \ y = 2^{-x-}$

Exercise 10.01 Exponential functions

1 Sketch each exponential functio.

a $y = 2^x$ **b** $y = 4^x$ **c** $f(x) = 3^x + 2$ **d** $y = 2^x - 1$

e $f(x) = 3(2^x)$ **f** $y = 4^{x+1}$ **g** $y = 3(4^{2x}) - 1$ **h** $f(x) = -2^x$

i $y = 2(4^{-x})$ **j** $f(x) = -3(5^{-x}) + 4$

2 State the domain and range of each functio.

a $f(x) = 2^x$ **b** $y = 3^x + 5$ **c** $f(x) = 10^{-x}$ **d** $f(x) = -5^x + 1$

3 Given $f(x) = 2^x$ and $g(x) = 3x - 4$ fin:

a $f(g(x))$ **b** $g(f(x))$

4 **a** Sketch the graph of $f(x) = 4(3^x) + 1$.

b Sketch the graph o:

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = -f(-x)$

5 Sales number, N of a new solar battery are growing over t years according to the formula $N = 450(3^{0.9t})$

a Draw a graph of this functio.

b Find the initial number of sales when $t = 0$

c Find the number of sales afte:

i 3 years

ii 5 years

iii 10 years

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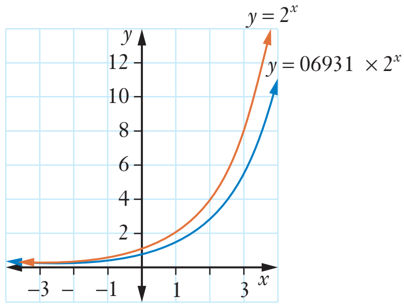
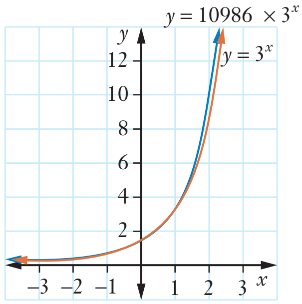
10.02 Euler's number, e

The gradient function of exponential functions is interestingNotice that the gradient of an exponential function is always increasingand increases at an increasing rat.

If you sketch the derivative function of an exponential functionthen it too is an exponential function Here are the graphs of the derivative functions (in blue) of $y = 2^x$ and $y = 3^x$ (in red) together with their equations





Notice that the graph of the derivative function of $y = 3^x$ is very close to the graph of the original function

We can find a number close to 3 that gives exactly the same derivative function as the original graph This number is approximately2.718 8, and is called **Eulers number** *e* Like π the number *e* is irrational

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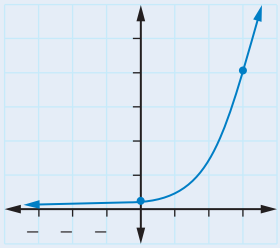

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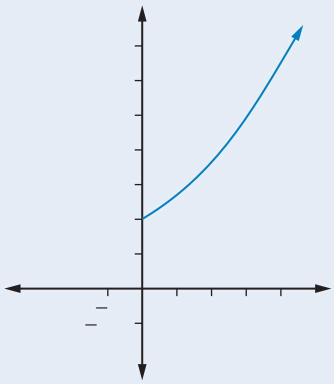

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Exercise 10.02 Euler's number, *e*

1 Sketch the curve $f(x) = 2e^{x-2}$

2 Evaluat, correct to 2 decimal placs:

a e^{15} **b** e^{-2} **c** $2e^{0.3}$ **d** $\frac{1}{e^3}$ **e** $-3e^{-31}$

3 Sketch each exponential functio.

a $y = 2e^x$ **b** $f(x) = e^x + 1$ **c** $y = -e^x$ **d** $y = e^{-x}$ **e** $y = -e^{-x}$

4 State the domain and range of $f(x) = e^x - 2$.

5 If $f(x) = e^x$ and $g(x) = x^3 + 3$, fin:

a $f(g(x))$ **b** $g(f(x))$

6 The volume V of a metal in mm^3 expands as it is heated over time according to the formula $V = 25e^{07\,t}$ where t is in minutes

a Sketch the graph of $V = 25e^{07\,t}$

b Find the volume of the metal a:

i 3 minutes **ii** 8 minutes

c Is this formula a good model for the rise in volume? Why?

7 The mass of a radioactive substance in g is given by $M = 150e^{-0014\,t}$ where t is in years Find the mass after

a 10 years **b** 50 years **c** 250 years

8 The number of koalas in a forest is declining according to the formula $N = 873e^{-0078\,t}$ where t is the time in years

a Sketch a graph showing this decline in numbers of koalas for the first 6 year.

b Find the number of koala:

i initially **ii** after 5 years **iii** after 10 years

9 An object is cooling down according to the exponential function $T = 23 + 125e^{-006\,t}$ where T is the temperature in $°C$ and t is time in minutes

a Find the initial temperatur.

b Find the temperature a:

i 2 minutes **ii** 5 minutes **iii** 10 minutes **v** 2 hours

c What temperature is the object tending towards? Can you explain why?

10 A population is growing exponentiall If the initial population is 20 000 and after 5 years the population is 80 000 draw a graph showing this informatio.

11 The temperature of a piece of iron in a smelter is 1000°C and it is cooling down exponentially. After 10 minutes the temperature is 650°C Draw a graph showing this information

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10.03 Differentiation of exponential functions

Difretiting exponenial uncion

Eulers numbe, e is the special number such that the derivative function of $y = e^x$ is itself
The derivative of e^x is e^x

The rule for differentiating $kf(x)$ works with the rule for e^x as well

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We can also use other differentiation rules such as the chain rul, product rule and quotient rule with the exponential function

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Proof

Let $u = ax$

Then $\frac{du}{dx} = a$

$y = e^u$

$\frac{dy}{du} = e^u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^u \times a$

$= ae^u$

$= ae^{ax}$

b

Exercise 10.03 Differentiation of exponential functions

1 Differentiate

a $y = 9e^x$ **b** $y = -e^x$ **c** $y = e^x + x^2$

d $y = 2x^3 - 3x^2 + 5x - e^x$ **e** $y = (e^x + 1)^3$ $y = (e^x + 5)^7$

g $y = (2e^x - 3)^2$ **h** $y = xe^x$ **i** $y = \dfrac{e^x}{x}$

j $y = x^2e^x$ **k** $y = e^x(2x + 1)$ $y = \dfrac{e^x}{7x - 3}$

m $y = \dfrac{5x}{e^x}$

2 Find the derivative o:

a $y = e^{2x}$ **b** $y = e^{-x}$ **c** $y = 2e^{3x}$

d $y = -e^{7x}$ **e** $y = -3e^{2x} + x^2$ $y = e^{2x} - e^{-2x}$

g $y = 5e^{-x} - 3x + 2$ **h** $y = xe^{4x}$ **i** $y = \dfrac{2e^{3x} - 3}{x + 1}$

j $y = (9e^{3x} + 2)^5$

3 If $f(x) = x^3 + 3x - e^x$ find $f'(1)$ in terms of e

4 Find the exact gradient of the tangent to the curve $y = e^x$ at the point $(1 \ \ e)$

5 Find the exact gradient of the normal to the curve $y = e^{2x}$ at the point where $x = 5$.

6 Find the gradient of the tangent to the curve $y = 4e^x$ at the point where $x = 16$ correct to 2 decimal places

7 Find the equation of the tangent to the curve $y = -e^x$ at the point $(1 \ \ -e)$

8 Find the equation of the normal to the curve $y = e^{-x}$ at the point where $x = 3$ in exact form

9 A population P of insects over time t weeks is given by $P = 3e^{14\,t} + 12\ 569$

a What is the initial population?

b Find the rate of change in the number of insects afte:

i 3 weeks **ii** 7 weeks

10 The displacement of a particle over time t seconds is given by $x = 2e^{4t}$ m

a What is the initial displacement?

b What is the exact velocity after 10 s?

c Find the acceleration after 2 s correct to 1 decimal plac.

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11 The displacement of an object in cm over time t seconds is given by $x = 6e^{-034\,t} - 5$ Fin:

a the initial displacement

b the initial velocity

c the displacement after 4 s

d the velocity after 9 s

e the acceleration after 2 s

12 The volume V of a balloon in mm^3 as it expands over time t seconds is given by $V = 3e^{08\,t}$

a Find the volume of the balloon a:

i 3 s **ii** 5 s

b Find the rate at which the volume is increasing a:

i 3 s **ii** 5 s

13 The population of a city is changing over t years according to the formula $P = 34\,500e^{0025\,t}$

a Find (to the nearest whole number) the population afte:

i 5 years **ii** 10 years

b Find the rate at which the population is changing afte:

i 5 years **ii** 10 years

14 The depth of water (in metres) in a dam is decreasing over t months according to the formula $D = 3e^{-0017\,t}$

a Find correct to 2 decimal places the depth afte:

i 1 month **ii** 2 months **iii** 3 months

b Find correct to 3 decimal places the rate at which the depth is changing afte:

i 1 month **ii** 2 months **iii** 3 months

10.04 Logarithms

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The **logarithm** of a positive number, y is the **power** to which a **base** a must be raised in order to produce the number y For exampl, $\log_2 8 = 3$ because $2^3 = 8$.

If $y = a^x$ then x is called the **logarithm of y to the base a**

Just as the exponential function $y = a^x$ is defined for positive bases only ($a > 0$) logarithms are also defined for $a > 0$ Furthermor, $a \neq 1$ because $1^x = 1$ for all values of x

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Logarithms are related to exponential functions and allow us to solve equations like $2^x = 5$.

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Notice that logarithms and exponentials are inverse operations

eoms

Common logarithms and natural logarithms

There are 2 types of logarithms that you can find on your calculator.

Common logarithms (base 10) $\log_{10} x$ or $\log x$

Natural (Naperian) logarithms (base *e*) $\log_e x$ or $\ln x$

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Exercise 10.04 Logarithms

1 Evaluate

a $\log_2 16$ **b** $\log_4 16$ **c** $\log_5 125$
d $\log_3 3$ **e** $\log_7 49$ $\log_7 7$
g $\log_5 1$ **h** $\log_2 128$ **i** $\log_8 8$

2 Evaluate

a $2^{\text{og } 3}$ **b** $7^{\text{og } 4}$ **c** $3^{\text{og } 29}$

3 Evaluate

a $3 \log_2 8$ **b** $\log_5 25 + 1$ **c** $3 - \log_3 81$
d $4 \log_3 27$ **e** $2 \log_{10} 10\,000$ $1 + \log_4 64$
g $3 \log_4 64 + 5$ **h** $\dfrac{\log_3 9}{2}$ **i** $\dfrac{\log_8 64 + 4}{\log_2 8}$

4 Evaluate

a $\log_2 \dfrac{1}{2}$ **b** $\log_3 \sqrt{3}$ **c** $\log_4 2$ **d** $\log_5 \dfrac{1}{25}$
e $\log_7 \sqrt[4]{7}$ **f** $\log_3 \dfrac{1}{\sqrt[3]{3}}$ **g** $\log_4 \dfrac{1}{2}$ **h** $\log_8 2$
i $\log_6 6\sqrt{6}$ **j** $\log_2 \dfrac{\sqrt{2}}{4}$

5 Evaluate correct to 2 decimal place:

a $\log_{10} 1200$ **b** $\log_{10} 875$ **c** $\log_e 25$

d $\ln 140$ **e** $5 \ln 8$ $\log_{10} 350 + 45$

g $\frac{\log_{10} 15}{2}$ **h** $\ln 9.8 + \log_{10} 17$ **i** $\frac{\log_{10} 30}{\log_e 30}$

6 Write in logarithmic fom:

a $3^x = y$ **b** $5^x = z$ **c** $x^2 = y$ **d** $2^b = a$

e $b^3 = d$ **f** $y = 8^x$ **g** $y = 6^x$ **h** $y = e^x$

i $y = a^x$ **j** $Q = e^x$

7 Write in index fom:

a $\log_3 5 = x$ **b** $\log_a 7 = x$ **c** $\log_3 a = b$

d $\log_x y = 9$ **e** $\log_a b = y$ $y = \log_2 6$

g $y = \log_3 x$ **h** $y = \log_{10} 9$ **i** $y = \ln 4$

8 Solve for x correct to 1 decimal place where necessar:

a $\log_{10} x = 6$ **b** $\log_3 x = 5$ **c** $\log_x 343 = 3$

d $\log_x 64 = 6$ **e** $\log_5 \frac{1}{5} = x$ $\log_x \sqrt{3} = \frac{1}{2}$

g $\ln x = 3.8$ **h** $3 \log_{10} x - 2 = 10$ **i** $\log_4 x = \frac{3}{2}$

9 Evaluate y given that $\log_y 125 = 3$.

10 If $\log_{10} x = 165$ evaluate x correct to 1 decimal place

11 Evaluate b to 3 significant figures if $\log_e b = 0894$

12 Find the value of $\log_2 1$ What is the value of $\log_a 1$?

13 Evaluate $\log_5 5$ What is the value of $\log_a a$?

14 **a** Evaluate $\ln e$ without a calculator.

b Using a calculato, evalute:

i $\log_e e^3$ **ii** $\log_e e^2$ **iii** $\ln_e e^5$ **v** $\log_e \sqrt{e}$

v $\ln_e \frac{1}{e}$ **v** $e^{\ln 2}$ **vii** $e^{\ln 3}$ **viii** $e^{\ln 5}$

x $e^{\ln 7}$ **x** $e^{\ln 1}$ **x** $e^{n\, e}$

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15 A class was given musical facts to lear. The students were then tested on these facts and each week they were given similar tests to find out how much they were able to remember. The formula $A = 85 - 55 \log_{10}(t + 2)$ seemed to model the average score after t weeks

a What was the initial average score?

b What was the average score afte:

i 1 week? **ii** 3 weeks?

c After how many weeks was the average score 30?

16 The pH of a solution is defined as pH = –log [H$^+$] where [H$^+$] is the hydrogen ion concentration A solution is acidic if its pH is less than7, alkaline if pH is greater than 7 and neutral if pH is 7 For each question find its pH and state whether it is acidi, alkaline or neutral

a Fruit juice whose hydrogen ion concentration is .0035

b Water with [H$^+$] = 10^{-7}

c Baking soda with [H$^+$] = 10^{-9}

d Coca Cola whose hydrogen ion concentration is .01

e Bleach with [H$^+$] = 1.2×10^{-12}

Coffee with [H$^+$] = 0000 01

17 If $f(x) = \log x$ and $g(x) = 2x - 7$ fin:

a $f(g(x))$ **b** $g(f(x))$

HISTORY OF BASES AND NUMBER SYSTEMS

Common logarithms use base 10 like our decimal number systemWe might have developed a different system if we had a different number of fingers The Mayan, in ancient times used base 20 for their number system since they counted with both their fingers and toes

1 Research the history and types of other number system, including those of Aboriginal and Torres Strait Islander peopls. Did any cultures use systems other than base 10? Why?

2 Explore computer-based systes. Computers have used both binary (base 2) and octal (base 8) Find out why these bases are use.

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ogaihm law

10.05 Logarithm laws

Because logarithms are just another way of writing indices (powers)there are logarithm laws that correspond to the index laws

Proof

Let $x = a^m$ and $y = a^n$

Then $m = \log_a x$ and $n = \log_a y$

$$xy = a^m \times a^n$$
$$= a^{m+n}$$

$\therefore \quad \log_a (xy) = m + n$ (by definition)

$$= \log_a x + \log_a y$$

| − | −

Proof

Let $x = a^m$ and $y = a^n$

Then $m = \log_a x$ and $n = \log_a y$

$$\frac{x}{y} = a^m \div a^n$$
$$= a^{m-n}$$

$\therefore \log_a \left(\frac{x}{y}\right) = m - n$ (by definition)

$$= \log_a x - \log_a y$$

Proof

Let $x = a^m$

Then $m = \log_a x$

$$x^n = (a^m)^n$$
$$= a^{mn}$$

$\therefore \quad \log_a x^n = mn$ (by definition)

$$= n \log_a x$$

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Proof

$$\log_a\left(\frac{1}{x}\right) = \log_a 1 - \log_a x$$
$$= 0 - \log_a x$$
$$= -\log_a x$$

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Change of base

If we need to evaluate logarithms such as $\log_5 2$ we use the change of base formul.

$$\frac{\quad}{\quad}$$

Proof

Let　$y = \log_a x$

Then　$x = a^y$

Take logarithms to the base b of both sides of the equation

$$\log_b x = \log_b a^y$$
$$= y \log_b a$$
$$\therefore \quad \frac{\log_b x}{\log_b a} = y$$
$$= \log_a x$$

To find the logarithm of any numbe, such as $\log_5 2$ you can change it to either $\log_{10} x$ or $\log_e x$

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Exercise 10.05 Logarithm laws

1 Simplify

a $\log_a 4 + \log_a y$

b $\log_a 4 + \log_a 5$

c $\log_a 12 - \log_a 3$

d $\log_a b - \log_a 5$

e $3 \log_x y + \log_x z$

f $2 \log_k 3 + 3 \log_k y$

g $5 \log_a x - 2 \log_a y$

h $\log_a x + \log_a y - \log_a z$

i $\log_{10} a + 4 \log_{10} b + 3 \log_{10} c$

j $3 \log_3 p + \log_3 q - 2 \log_3 r$

k $\log_4 \frac{1}{n}$

$\log_x \frac{1}{6}$

2 Evaluate

a $\log_5 5^2$

b $\log_7 7^6$

3 Given $\log_7 2 = 036$ and $\log_7 5 = 083$ fin:

a $\log_7 10$

b $\log_7 04$

c $\log_7 20$

d $\log_7 25$

e $\log_7 8$

$\log_7 14$

g $\log_7 50$

h $\log_7 35$

i $\log_7 98$

4 Use the logarithm laws to evaluat:

a $\log_5 50 - \log_5 2$

b $\log_2 16 + \log_2 4$

c $\log_4 2 + \log_4 8$

d $\log_5 500 - \log_5 4$

e $\log_9 117 - \log_9 13$

f $\log_8 32 + \log_8 16$

g $3 \log_2 2 + 2 \log_2 4$

h $2 \log_4 6 - (2 \log_4 3 + \log_4 2)$

i $\log_6 4 - 2 \log_6 12$

j $2 \log_3 6 + \log_3 18 - 3 \log_3 2$

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5 If $\log_a 3 = x$ and $\log_a 5 = y$ find an expression in terms of x and y for

a $\log_a 15$ **b** $\log_a 06$ **c** $\log_a 27$

d $\log_a 25$ **e** $\log_a 9$ $\log_a 75$

g $\log_a 3a$ **h** $\log_a \frac{a}{5}$ **i** $\log_a 9a$

6 If $\log_a x = p$ and $\log_a y = q$ fin, in terms of p and q

a $\log_a xy$ **b** $\log_a y^3$ **c** $\log_a \frac{y}{x}$ **d** $\log_a x^2$

e $\log_a xy^5$ **f** $\log_a \frac{x^2}{y}$ **g** $\log_a ax$ **h** $\log_a \frac{a}{y^2}$

i $\log_a a^3y$ **j** $\log_a \frac{x}{ay}$

7 If $\log_a b = 34$ and $\log_a c = 47$ evaluat:

a $\log_a \frac{c}{b}$ **b** $\log_a bc^2$ **c** $\log_a (bc)^2$

d $\log_a abc$ **e** $\log_a a^2c$ $\log_a b^7$

g $\log_a \frac{a}{c}$ **h** $\log_a a^3$ **i** $\log_a bc^4$

8 Solve

a $\log_4 12 = \log_4 x + \log_4 3$ **b** $\log_3 4 = \log_3 y - \log_3 7$

c $\log_a 6 = \log_a x - 3 \log_a 2$ **d** $\log_2 81 = 4 \log_2 x$

e $\log_x 54 = \log_x k + 2 \log_x 3$

9 **a** Change the subject of $\text{dB} = 10 \log\left(\frac{I}{I_0}\right)$ to I

b Find the value of I in terms of I_0 when dB = 45

10 **a** Show that the formula $A = 100 - 50 \log (t + 1)$ can be written as

i $\log (t + 1) = \frac{100 - A}{50}$ **ii** $t = 10^{\frac{100-A}{50}} - 1$

b Hence fin:

i A when $t = 3$ **ii** t when $A = 75$

11 Evaluate to 2 decimal place:

a $\log_4 9$ **b** $\log_6 25$ **c** $\log_9 200$ **d** $\log_2 12$

e $\log_3 23$ **f** $\log_8 250$ **g** $\log_5 95$ **h** $2 \log_4 234$

i $7 - \log_7 108$ **j** $3 \log_{11} 340$

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10.06 Logarithmic functions

A **logarithmic function** is a function of the form $y = \log_a x$

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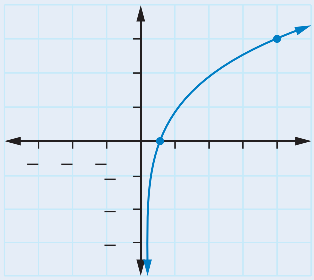

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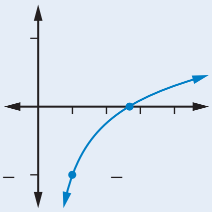

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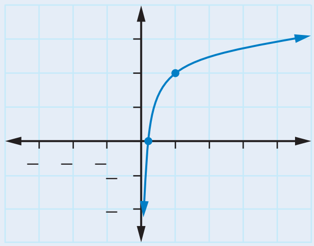

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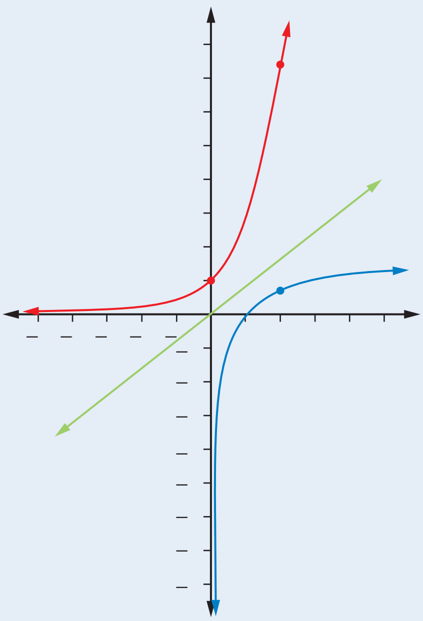

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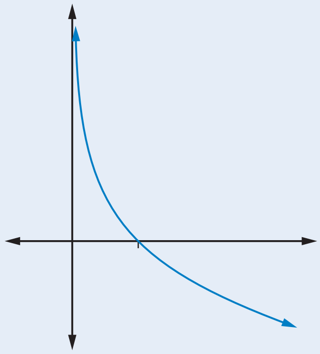

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GRAPHS OF LOGARTHMC FUNCTIONS

1 Substitute different values of x into the logarithmic function $y = \log x$ positive negative and zero What do you notice ?

2 Use a graphics calculator or graphing software to sketch the graphs of different logarithmic functions such as

a $y = \log_2 x \;\; y = \log_3 x \;\; y = \log_4 x \;\; y = \log_5 x \;\; y = \log_6 x$

b $y = \log_2 x + 1, y = \log_2 x + 2, y = \log_2 x + 3, y = \log_2 x - 1, y = \log_2 x - 2$

c $y = 2 \log_2 x \;\; y = 3 \log_2 x \;\; y = -\log_2 x \;\; y = -2 \log_2 x \;\; y = -3 \log_2 x$

d $y = 2 \log_2 x + 1, y = 2 \log_2 x + 2, y = 2 \log_2 x + 3, y = 2 \log_2 x - 1, y = 2 \log_2 x - 2$

e $y = 3 \log_4 x + 1, y = 5 \log_3 x + 2, y = -\log_5 x + 3, y = -2 \log_2 x - 1, y = 4 \log_7 x - 2$

3 Try sketching the graph of $y = \log_{-2} x$ What does the table of values look like? Are there any discontinuities on the graph? Why? Could you find the domain and range? Use a graphics calculator or graphing software to sketch this graphWhat do you find ?

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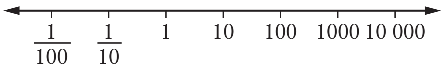
Logarithmic scales

It is difficult to describe and graph exponential functions because their y values increase so quickly. We use logarithms and **logarithmic scales** to solve this problem

On a base 10 logarithmic scale an axis or number line has units that don't increase by1, but by powers of 10



Examples of base 10 logarithmic scales are

- the Richter scale for measuring earthquake magnitude
- the pH scale for measuring acidity in chemistry
- the decibel scale for measuring loudness
- the octave (frequency) scale in music

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Exercise 10.06 Logarithmic functions

1 Sketch the graph of each logarithmic function and state its domain and rang.

a $y = \log_3 x$ **b** $f(x) = 2\log_4 x$ **c** $y = \log_2 x + 1$

d $y = \log_5 x - 1$ **e** $f(x) = \log_4 x - 2$ $y = 5\ln x + 3$

g $f(x) = -3\log_{10} x + 2$

2 Sketch the graphs of $y = 10^x$ $y = \log_{10} x$ and $y = x$ on the same number plane
What do you notice about the relationship of the curves to the line?

3 Sketch the graph of $f(x) = \log_2 x$ and $y = \log_2 (-x)$ on the same set of axes and describe their relationship

4 **a** Sketch the graphs of $y = \log_2 x$ $y = 2^x$ and $y = x$ on the same set of axes

b Find the inverse function of $y = \log_2 x$

5 Find the inverse function of each functio.

a $y = \log_7 x$ **b** $y = \log_9 x$ **c** $y = \log_e x$

d $y = 2^x$ **e** $y = 6^x$ $y = e^x$

6 This table lists some of the earthquakes experienced in Australa.

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The Richter scale for earthquakes is logarithmicUse the table to find the difference in magnitude (correct to the nearest whole number) between the earthquakes in

a Newcastle and Swan Hill

b Collier Bay and Orange

c Newcastle and Orange

d Coral Sea and Kalgoorlie

e Collier Bay and Coffs Harbour

7 The decibel (dB) scale for loudness is logarithmi. Find (correct to the nearest whole number) the difference in loudness between

a 20 and 23 dB **b** 40 and 41 dB **c** 65.2 and 66.5 dB

d 8.4 and 8.9 dB **e** 52.3 and 58.6 dB

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10.07 Exponential equations

Exponential equations can be solved using logarithms or the change of base formula

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Exercise 10.07 Exponential equations

1 Solve each equation correct to 2 significant figure:

a $4^x = 9$	**b** $3^x = 5$	**c** $7^x = 14$	**d** $2^x = 15$
e $5^x = 34$	**f** $6^x = 60$	**g** $2^x = 76$	**h** $4^x = 50$
i $3^x = 23$	**j** $9^x = 210$		

2 Solv, correct to 2 decimal placs:

a $2^x = 6$	**b** $5^y = 15$	**c** $3^x = 20$	**d** $7^m = 32$
e $4^k = 50$	**f** $3^t = 4$	**g** $8^x = 11$	**h** $2^p = 57$
i $4^x = 81.3$	**j** $6^n = 1026$		

3 Solv, to 1 decimal plae:

a $3^{x+1} = 8$	**b** $5^{3n} = 71$	**c** $2^{x-3} = 12$
d $4^{2n-1} = 7$	**e** $7^{5x+2} = 11$	$8^{3-n} = 57$
g $2^{x+2} = 18.3$	**h** $3^{7k-3} = 329$	**i** $9^{\frac{x}{2}} = 50$

4 Solve each equation correct to 3 significant figures

a $e^x = 200$	**b** $e^{3t} = 5$	**c** $2e^t = 75$
d $45 = e^x$	**e** $3000 = 100e^n$	$100 = 20e^{3t}$
g $2000 = 50e^{015\,t}$	**h** $15\,000 = 2000e^{003\,k}$	**i** $3Q = Qe^{0.02t}$

5 The amount A of money in a bank account after n years grows with compound interest according to the formula $A = 850(1025)^n$

a Find

i the initial amount in the bank **ii** the amount after 7 year.

b Find how many years it will take for the amount in the bank to be $100.

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6 The population of a city is given by $P = 35\ 000e^{0.024t}$ where t is time in years

a Find the populatio:

i initially **ii** after 10 years **iii** after 50 year.

b Find when the population will reac:

i 80 000 **ii** 200 000

7 A species of wattle is gradually dying out in a Blue Mountains region The number of wattle trees over time t years is given by $N = 8900e^{-0.048t}$

a Find the number of wattle trees

i initially

ii after 5 years

iii after 70 year.

b After how many years will there be

i 5000 wattle trees?

ii 200 wattle trees?

8 A formula for the mass M g of plutonium after t years is given by $M = 100e^{-0000\ 03\ t}$ Fin:

a initial mass **b** mass after 50 years **c** mass after 500 years

d its half-life (the time taken to decay to half of its initial mass)

9 The temperature of an electronic sensor is given by the formula $T = 18 + 12e^{0.002t}$ where t is in hours

a What is the temperature of the sensor after 5 hours?

b When the temperature reaches 50°C the sensor needs to be shut down to coo. After how many hours does this happen?

10 A particle is moving along a straight line with displacement x cm over time t s according to the formula $x = 5e^t + 23$.

a Find

i the initial displacement

ii the exact velocity after 20 s

iii the displacement after 6 s

iv the time when displacement is 85 cm

v the time when the velocity is 1000 cm s^{-}

b Show that acceleration $a = x - 23$.

c Find the acceleration when displacement is 85 c.

11 Find the inverse function of

a $y = e^{2x}$ **b** $y = \ln(x + 1)$ **c** $f(x) = e^{3x} + 1$

12 **a** Find the domain and range of $f(x) = 3^x$

b What is its inverse function?

c Write down the domain and range of the inverse functon.

Exponenial gowh and deay

Exponenial deay

10.08 Exponential growth and decay

Exponential growth and **exponential decay** are terms that describe a **special rate of change** that occurs in many situations They describe a quantity that is increasing or decreasing according to an exponential function Population growth and growth of bacteria in a culture are examples of exponential growth The decay of radioactive substances and the cooling of a substance are examples of exponential decay.

When a quantity grows or decays exponentially, its rate of change is directly proportional to the current amount of the quantity itself The more of the quantity there s, the faster it grows or decays For exampl, in a population of rabbis, there is fairly slow growth in the numbers at first but the more rabbits there ar, the higher the rate of growth in rabbit numbers

For exponential growth and decay, the rate of change of a quantity over time is directly proportional to the quantity itself If we call the quantity N and time t this gives

$$\frac{dN}{dt} = kN$$

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Proof

$$N = Ae^{kt}$$

$$\frac{dN}{dt} = kAe^{kt}$$

$$= kN$$

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Initially $t = 0$.

$N = Ae^{k \times 0}$

$= A$

$\therefore A$ is the initial quantity.

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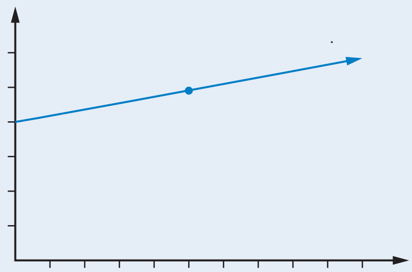

Sometimes you need to find the constants A and k before you can answer questionsWhen calculating with the exponential functiondo't round the decimal value of k So that your answers are accurate store the value of k in your calculators memory or write it down with many decimal places

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1 Exercise 10.08 Exponential growth and decay

1 The number of migratory birds in a colony is given by $N = 80e^{0.002t}$ where t is in days

a How many birds are there in the colony initially?

b How many birds will there be after 30 days?

c After how many days will there be 500 birds?

d Sketch the curve of the population over the first 100 days

2 The number of bacteria in a culture is given by $N = N_0e^{032\,t}$ where t is time in hours

a If there are initially 20 000 bacteri, how many will there be after 5 hours ?

b How many hour, to the nearest hor, would it take for the number of bacteria to reach 200 000?

3 The rate of decay of radium is proportional to its mas, and 100 kg of radium takes 5 years to decay to 95 kg

a Show that the mass of radium is given by $M = 100e^{-001\,t}$

b Find its mass after 10 year.

c Find its half-life (the time taken for the radium to halve its mass.

d Sketch the graph of the deca.

4 A chemical reaction causes the amount of chlorine to be reduced at a rate proportional to the amount of chlorine present at any one timeIf the amount of chlorine is given by the formula $A = A_0e^{-kt}$ and 100 L reduces to 65 L after 5 minutesfin:

a the amount of chlorine after 12 minutes

b how long it will take for the chlorine to reduce to 10 .

5 The production output in a factory increases according to the equation $P = P_0e^{kt}$ where t is in years

a Find P_0 if the initial output is 5000 units

b The factory produces 8000 units after 3 year. Find the value of k to 3 decimal place.

c How many units will the factory produce after 6 years?

d The factory needs to produce 20 000 units to make a maximum profi. After how many years correct to 1 decimal plac, will this happen ?

6 The rate of depletion of rainforests can be estimated as proportional to the area of rainforest If 3 million m ^{2} of rainforest is reduced to 27 million m ^{2} after 20 years find how much rainforest there will be after 50 years

7 The population of a country is increasing at a yearly rate of .9; that i, $\frac{dP}{dt} = 0069\,P$ If the population was 50 000 in 2015 fin:

a a formula for the population growth

b the population in the year 2020

c the rate at which the population will be growing in the year 2020

d in which year the population will reach 300 00.

8 An object is cooling according to the formula $T = T_0 e^{-kt}$ where T is temperature in degrees Celsius and t is time in minutes If the temperature is initially 90°C and the object cools down to 81°C after 10 minutes fin:

a its temperature after half an hour

b how long (in hours and minutes) it will take to cool down to 30°.

9 In the process of the inversion of suga, the amount of sugar present is given by the formula $S = Ae^{kt}$ If 150 kg of sugar is reduced to 125 kg after 3 hour, fid:

a the amount of sugar after 8 hour, to the nearest kilogram

b the rate at which the sugar will be reducing after 8 hours

c how long it will take to reduce to 50 k.

10 The mas, in gras, of a radioactive substance is given by $M = M_0 e^{-kt}$ where t is time in years Fin:

a M_0 and k if a mass of 200 kg decays to 195 kg after 10 years

b the mass after 15 years

c the rate of decay after 15 years

d the half-life of the substance (time taken to decay to half its mass.

11 The number of bacteria in a culture increases from 15 000 to 25 000 in 7 hour. If the rate of bacterial growth is proportional to the number of bacteria at that timefin:

a a formula for the number of bacteria

b the number of bacteria after 12 hours

c how long it will take for the culture to produce 5 000 000 bacteri.

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12 A population in a certain city is growing at a rate proportional to the population itself After 3 years the population increases by 20% How long will it take for the population to double?

13 The half-life of radium is 1600 year.

a Find the percentage of radium that will be decayed after 500 year.

b Find the number of years that it will take for 75% of the radium to deca

14 The population of a city is $P(t)$ at any one time The rate of decline in population is proportional to the population $P(t)$ that i, $\frac{dP}{dt} = -kP$

a Show that $P = P_0 e^{-kt}$ is a solution of the differential equation $\frac{dP}{dt} = -kP$

b What percentage decline in population will there be after 10 year, given a 10% decline in 4 years? Answer to the nearest percenage.

c What will the percentage rate of decline in population be after 10 years? Answer to the nearest percenage.

d When will the population fall by 20%? Answer to the neares 0.1 year.

15 The rate of leakage of water out of a container is proportional to the amount of water in the container at any one time If the container is 60% empty after 5 minute, find how long it will take for the container to be 90% empty.

16 Numbers of sheep in a certain district are dropping exponentially due to drought A survey found that numbers had declined by 15% after 3 years If the drought continues how long would it take to halve the number of sheep in that district?

17 Anthony has a blood alcohol level of 150 mg/d. The amount of alcohol in the bloodstream decays exponentially. If it decreases by 20% in the first hor, ind:

a the level of alcohol in Anthoy's blood after 3 hours

b when the blood alcohol level reaches 20 mg/d.

18 The current C flowing in a conductor dissipates according to the formula $\frac{dC}{dt} = -kC$

If it dissipates by 40% in 5 seconds how long will it take to dissipate to 20% of the original current?

19 Pollution levels in a city have been rising exponentially with a 10% increase in pollution levels in the past two years At this rae, how long will it take for pollution levels to increase by 50%?

20 If $\frac{dQ}{dt} = kQ$ prove that $Q = Ae^{kt}$ satisfies this equation

EXPONENTIAL DECAY AND THE ENVIRONMENT

Exponential decay is often related to environmental problems such as extinction and lack of sustainability. Choose one or more of the following issues to research

It is estimated that some animals such as pandas and koalas will be extinct soon How soon will pandas be extinct? Can we do anything to stop this extinction?

How does climate change affect the Earth? Is it affecting us now? If not how soon will it have a noticeable effect on us?

How long do radioactive substances such as radium and plutonium take to decay? What are some of the issues concerning the storage of radioactive waste?

The erosion and salination of Australian soils are problems that affect our farmin. Find out about this issue and some possible solution.

The effect of blue-green algae in some of our rivers is becoming a major problem What steps have been taken to remedy this situation?

Look at the mathematical aspects of these issues For exampl, what formulas are used to make predictions? What kinds of time scales are involved in these issues?

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10.09 Further exponential growth and decay

Futhr exponenial gowh and deay

The formulas $\frac{dN}{dt} = kN$ and $N = Ae^{kt}$ are based on the work of **Thomas Malthus** (1766–1834). It is accurate in many cases but it is a simple formula that does't take in the various factors that may influence the rate of population growthA more realistic formula for exponential growth and decay has the rate of change of a quantity not being directly proportional to the current quantity, but directly proportional to the **difference between the quantity and a constant *P***

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Proof

$N = P + Ae^{kt}$

$\frac{dN}{dt} = kAe^{kt}$

$= k(P + Ae^{kt} - P)$

$= k(N - P)$

The next example shows that some population growths and **Newto's Law of Cooling** both use this formula

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For modified exponential decay, $N = P + Ae^{kt}$ and $k < 0$, so as $t \to \infty$ $N \to P$ This means N has a limiting value whether it be room temperatur, carrying capacity or a certain population value

1 Exercise 10.09 Further exponential growth and decay

1 a Show that $x = 100 + Ae^{2t}$ is a solution of $\frac{dx}{dt} = 2(x - 100)$

b If $x = 180$ when $t = 3$ find A to 3 significant figures

c Find t to 3 significant figures when $x = 150$.

2 a Show that $N = 45 + Ae^{014\,t}$ is a solution of $\frac{dN}{dt} = 014(N - 45)$

b Given $N = 82$ when $t = 2$ find A to 2 decimal places

c What is N when $t = 5$?

d Find t when $N = 120$

e Sketch the graph of this function for values of t from 0 to 25

3 The rate of change in the volume of water in a dam is given by $\frac{dV}{dt} = k(V - 5000)$ where k is a constant

a Show that a solution of this differential equation is $V = 5000 + Ae^{kt}$

b If the initial volume is 87 000 kL and after 10 hours the volume is 129 000 k, find the values of A and k

c What volume of water will be in the dam after 3 days?

d Calculate how long it will take for the volume to reach .2 million k. Answer in days and hours to the nearest hou.

4 a Show that $N = P + Ae^{kt}$ is a solution of $\frac{dN}{dt} = k(N - P)$ where k P and A are constants

b $\frac{dN}{dt} = k(N - 1000)$ and initially $N = 150$. When $t = 5, N = 2200$ Find t when $N = 2500$

5 According to Newto's Law of Coolig, the rate of change in the temperature of an object is proportional to the difference between its temperature and the temperature of the air (or surrounding matter)The air temperature is assumed to be constat.
A piece of metal is heated to 80°C and placed in a room where the temperature is a constant 18°C

a If the metal cools to 68°C after 15 minute, show that $T = 18 + 62e^{-00143\,t}$

b When will the temperature reach 30°C?

c Show that as t approaches infinity, the temperature of the metal approaches room temperature

6 The population of an ant colony is given by $P = 950 + Ae^{kt}$ If there are initially 14 000 ants and after 6 weeks there are 20 000 fin:

a the ant population after 10 weeks

b when the population will reach 1 millio.

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7 A piece of mea, initially at 14C, is placed in a freezer whose temperature is a constant −10°C After 25 seconds the temperature of the meat is 11°CFin:

a the mea's temperature after 5 minutes

b when the temperature will reach −8°C (to the nearest minute)

8 When a body fall, the rate of change in velocity is given by $\frac{dv}{dt} = -k(v - P)$ where k and P are constants

a Show that $v = P + Ae^{-kt}$ is a solution of this differential equation

b When $P = 500$ initial velocity is 0 and velocity v after 5 seconds is 21 m s^{-} Find values of A and k

c Find the velocity after 20 second.

d Find the maximum possible velocity as t tends to infinity.

9 An ice-block with temperature −14°C is left out in the sun The air temperature is a constant 25°C and after 40 seconds the temperature of the ice-block has reached −5°C Fin:

a its predicted temperature after 2 minutes

b when the ice-block will start to melt (.. when its temperature will reach 0°).

10 The population of sheep on a farm is given by $\frac{dN}{dt} = k(N - 1800)$ where N is the number of sheep If there are initially 3000 sheep and after 3 years there are 3400 sheep fin:

a the number of sheep on the farm after 5 years

b when the sheep population will reach 8000

11 Wilhey's Law states that the rate of transformation of a substance in a chemical reaction is proportional to its concentration That s, $\frac{dx}{dt} = k(x - c)$ where x is the amount of substance transformed and c is the initial concentration of the substance Initially none of the substance is transformed If the initial concentration is .9 and the amount transformed after 2 minutes is 27 find how much of the substance will be transformed after 5 minute.

12 The rate of growth of a certain tow's population is proportional to the excess of the population over 10 000 If the town initially has 18 000 people and after 4 years the population grows to 25 000 find how long it would take t:

a reach 40 000

b be double the initial populatio.

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13 A formula for the rate of change in population of a certain species of animal is given by $P = 200 + 1600e^{-kt}$ If the population reduces to half after 56 year, find how long it would take to reduce to a quarter of the original population

14 A saucepan of water is brought nearly to the boil then removed from the hea. The temperature reaches 95°C and room temperature is a constant 25°CIf the water has lost 25% of its excess heat after 25 minutesfind how long it will take for it to cool down to 30°C

15 The velocity of a particle is given by $v = 100 + 280e^{-kt}$ If the velocity decreases by 20% after 50 seconds find the percentage decrease in velocity after 3 minute.

16 A population of herons is increasing according to the formula $P = 800 + 2000e^{kt}$ where t is measured in years

a What is the initial heron population?

b If the population has increased by 20% in 5 years how long will it take to double the population?

c If the carrying capacity of this region is 7500 herons how long it will take to reach this capacity?

17 The ozone layer over a certain region is decreasing according to the equation $Q = 50 + 80e^{-kt}$ where t is measured in years If it decreases by 4% over 10 year, fid:

a by what percentage it decreases over 50 years

b how long it takes to decrease by 40, to the nearest yer.

18 The production of a mine is decreasing exponentiall and in the past 5 years there has been a decline of 18% If production declines by 90, the mine will cloe. The equation of production P after t years is given by $P = 500 + 6500e^{-kt}$ Fin:

a the percentage of production decline after 10 years

b how long it will take for the mine to clos.

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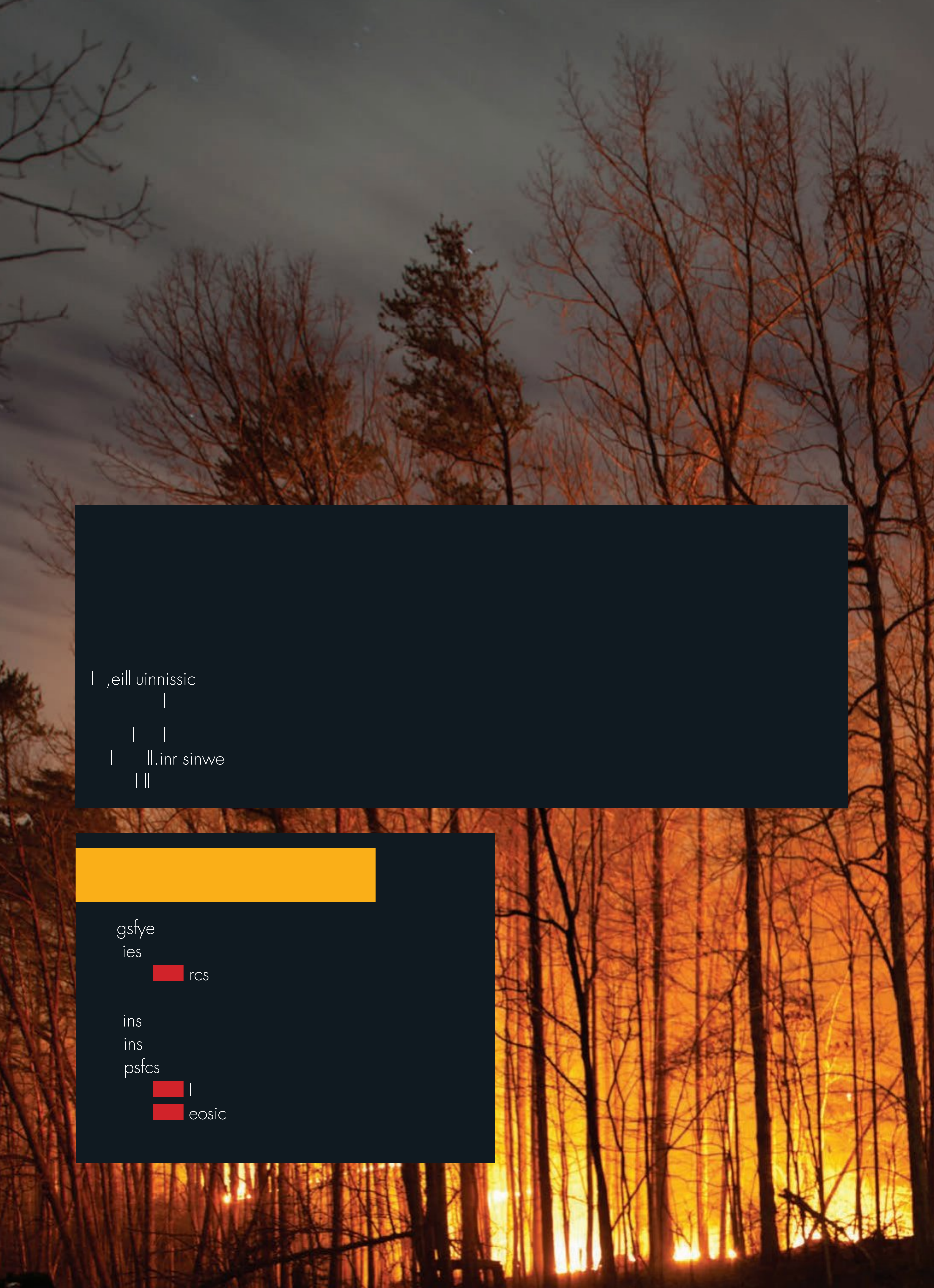

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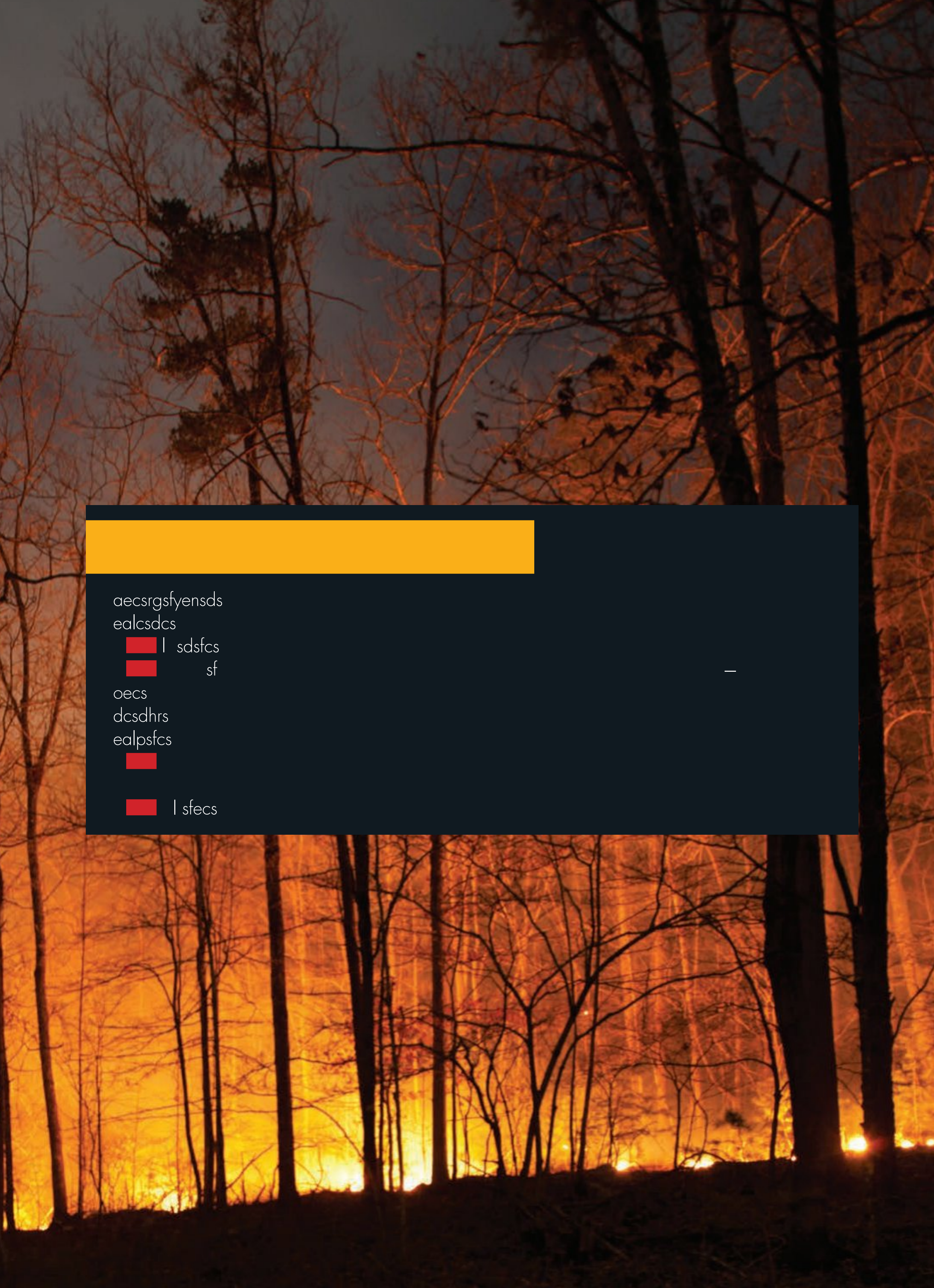

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TERMINOLOGY

amplitude The height from the centre of a periodic function to the maximum or minimum values (peaks and troughs of its graph respectively) For $y = k \sin ax$ the amplitude is k

centre The mean value of a periodic function that is equidistant from the maximum and minimum values For $y = k \sin ax + c$ the centre is c

identity An equation that shows the equivalence of 2 algebraic expressions for all values of the variables

inverse trigonometric functins: The $\sin^{-} x$ $\cos^{-} x$ and $\tan^{-} x$ functions which are the inverse functions of the sine cosine and tangent functions respectively

period The length of one cycle of a periodic function on the x-axis before the function repeats itself For $y = k \sin ax$ the period is $\frac{2\pi}{a}$

periodic function A function that repeats itself regularly

phase A horizontal shift (translation. For $y = k \sin [a(x + b)]$ the phase is b that i, the graph of $y = k \sin ax$ shifted b units to the left

reciprocal trigonometric ratios The cosecan, secant and cotangent ratios which are the reciprocals of sine cosine and tangent respectively

***t*-formulas** Formulas for $\sin A$ $\cos A$ and $\tan A$ in terms of $t = \tan$ —

Angles o any magniude

11.01 Angles of any magnitude

In Chapter 5 *Trigonometry* we examined acute and obtuse obtuse angles by looking at angles turning around a unit circle We can find angles of *any* size by continuing around the circle

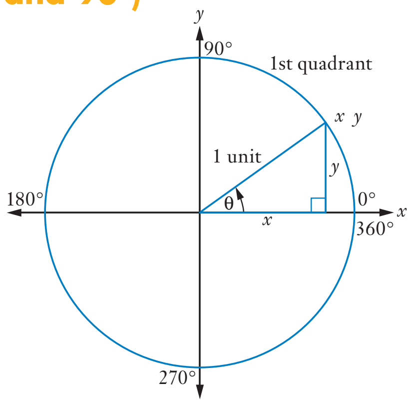
1st quadrant: acute angles (between 0° and 90°)

You can see from the triangle in the unit circle with angle θ that

$$\sin \theta = y$$

$$\cos \theta = x$$

$$\tan \theta = \frac{y}{x}$$



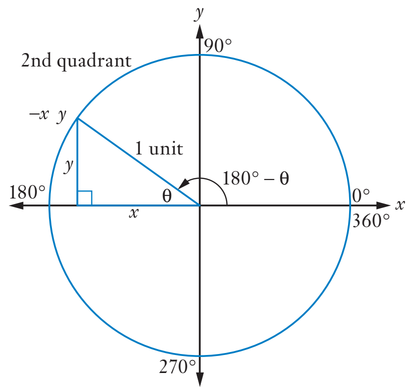
In the 1st quadrant x and y are both positive so all ratios are positive in the 1st quadrant

2nd quadrant: obtuse angles (between 90° and 180°)

$\sin \theta = y$ (positive)

$\cos \theta = -x$ (negative)

$\tan \theta = \frac{y}{-x}$ (negative)



The angle that gives θ in the triangle is 180° − θ

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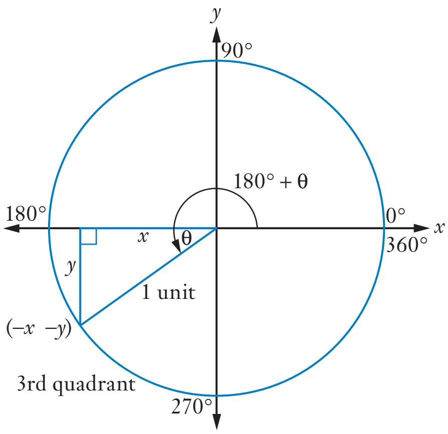
3rd quadrant: angles between 180° and 270°

$\sin \theta = -y$ (negative)

$\cos \theta = -x$ (negative)

$\tan \theta = \dfrac{-y}{-x} = \dfrac{y}{x}$ (positive)

The angle that gives θ in the triangle is 180° + θ



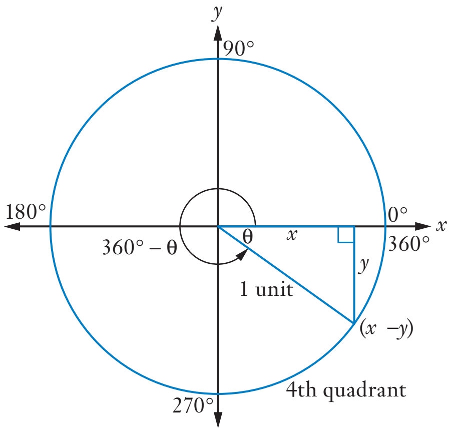
4th quadrant: angles between 270° and 360°

$\sin \theta = -y$ (negative)

$\cos \theta = x$ (positive)

$\tan \theta = \dfrac{-y}{x}$ (negative)

The angle that gives θ in the triangle is 360° – θ



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Putting all of these results together gives a rule for all 4 quadrants that we usually call the **ASTC rule**

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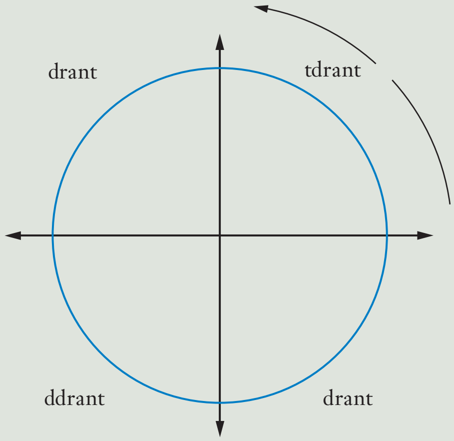

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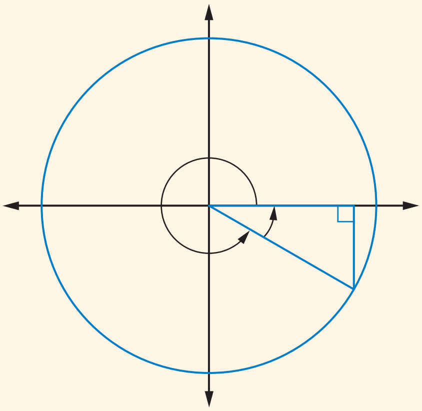

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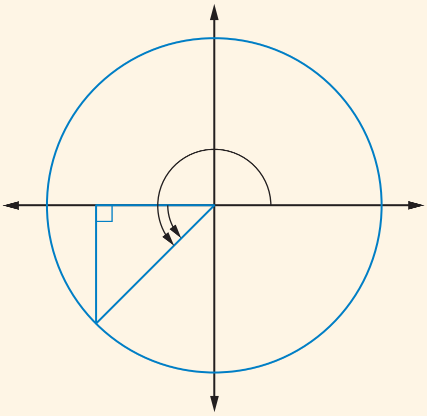

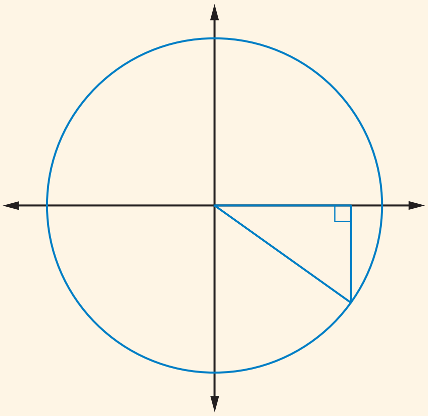

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We can find trigonometric ratios of angles greater than 360° by turning around the circle more than once

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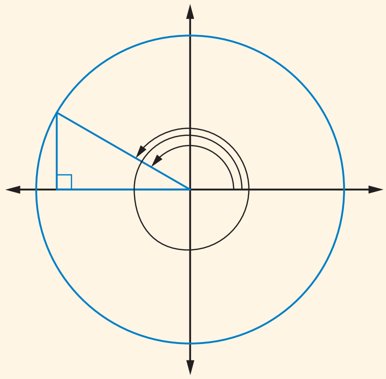
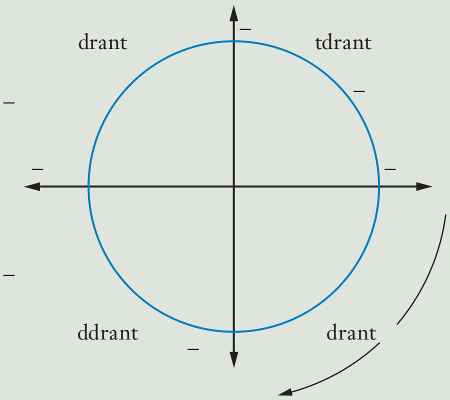


Negative angles

The ASTC rule also works for negative angle. These are measured in the opposite direction (clockwise) from positive angles as shown



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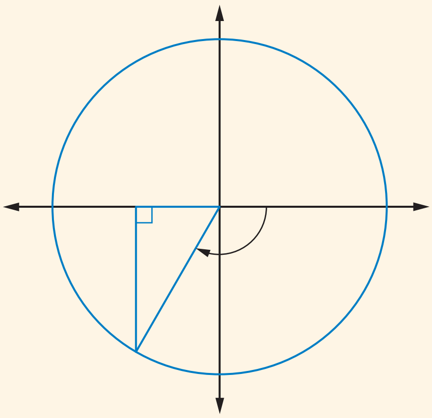

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Exercise 11.01 Angles of any magnitude

1 Find all quadrants wher:

a $\cos\theta > 0$ **b** $\tan\theta > 0$ **c** $\sin\theta > 0$
d $\tan\theta < 0$ **e** $\sin\theta < 0$ $\cos\theta < 0$
g $\sin\theta < 0$ and $\tan\theta > 0$ **h** $\cos\theta < 0$ and $\tan\theta < 0$
i $\cos\theta > 0$ and $\tan\theta < 0$ **j** $\sin\theta < 0$ and $\tan\theta < 0$

2 **a** Which quadrant is the angle 240° in?
b Find the exact value of cos 240°

3 **a** Which quadrant is the angle 315° in?
b Find the exact value of sin 315°

4 **a** Which quadrant is the angle 120° in?
b Find the exact value of tan 120°

5 **a** Which quadrant is the angle –225° in?
b Find the exact value of sin (–225°)

6 **a** Which quadrant is the angle –330° in?
b Find the exact value of cos (–330°)

7 Find the exact value o:

a tan 225° **b** cos 315° **c** tan 300° **d** sin 150°
e cos 120° **f** sin 210° **g** cos 330° **h** tan 150°
i sin 300° **j** cos 135°

8 Find the exact value o:

a cos (–225°) **b** cos (–210°) **c** tan (–300°) **d** cos (–150°)
e sin (–60°) **f** tan (–240°) **g** cos (–300°) **h** tan (–30°)
i cos (–45°) **j** sin (–135°)

9 Find the exact value o:

a cos 570° **b** tan 420° **c** sin 480° **d** cos 660°
e sin 690° **f** tan 600° **g** sin 495° **h** cos 405°
i tan 675° **j** sin 390°

10 If $\tan\theta = \frac{3}{4}$ and $\cos\theta < 0$ find $\sin\theta$ and $\cos\theta$ as fractions

11 Given $\sin\theta = \frac{4}{7}$ and $\tan\theta < 0$ find the exact value of $\cos\theta$ and $\tan\theta$

12 If $\sin x < 0$ and $\tan x = -\frac{5}{8}$ find the exact value of $\cos x$

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13 Given $\cos x = \frac{2}{5}$ and $\tan x < 0$ find the exact value of $\sin x$ and $\tan x$

14 If $\cos x < 0$ and $\sin x > 0$ find $\cos x$ and $\sin x$ in surd form if $\tan x = \frac{5}{7}$

15 If $\sin \theta = -\frac{4}{9}$ and $270° < \theta < 360°$ find the exact value of $\tan \theta$ and $\cos \theta$

16 If $\cos x = -\frac{3}{8}$ and $180° < x < 270°$ find the exact value of $\tan x$ and $\sin x$

17 Given $\sin x = 03$ and $\tan x < 0$

a express $\sin x$ as a fraction

b find the exact value of $\cos x$ and $\tan x$

18 If $\tan \alpha = -12$ and $270° < \alpha < 360°$ find the exact values of $\cos \alpha$ and $\sin \alpha$

19 Given that $\cos \theta = -07$ and $90° < \theta < 180°$ find the exact value of $\sin \theta$ and $\tan \theta$

20 Simplify

a $\sin (180° - \theta)$ **b** $\cos (360° - x)$ **c** $\tan (180° + \beta)$

d $\sin (180° + \alpha)$ **e** $\tan (360° - \theta)$ $\sin (-\theta)$

g $\cos (-\alpha)$ **h** $\tan (-x)$

11.02 Trigonometric identities

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The reciprocal trigonometric ratios

The **reciprocal trigonometric ratios** are the reciprocals of the sine cosine and tangent ratios

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The reciprocal ratios have the same signs as their related ratios in the different quadrants For example in the 3rd and 4th quadrant, $\sin \theta < 0$ so $\operatorname{cosec} \theta < 0$

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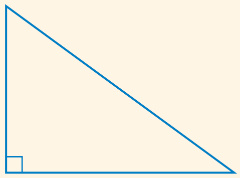

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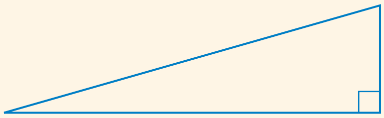

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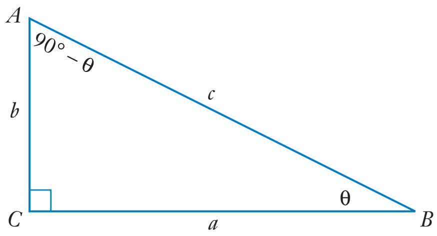
Compementary angles

In $\triangle ABC$ if $\angle B = \theta$ then $\angle A = 90° - \theta$ (by the angle sum of a triangle) $\angle B$ and $\angle A$ are **complementary angles** because they add up to 90°



$\sin\theta = \frac{b}{c}$	$\sin(90° - \theta) = \frac{a}{c}$
$\cos\theta = \frac{a}{c}$	$\cos(90° - \theta) = \frac{b}{c}$
$\tan\theta = \frac{b}{a}$	$\tan(90° - \theta) = \frac{a}{b}$
$\sec\theta = \frac{c}{a}$	$\sec(90° - \theta) = \frac{c}{b}$
$\operatorname{cosec}\theta = \frac{c}{b}$	$\operatorname{cosec}(90° - \theta) = \frac{c}{a}$
$\cot\theta = \frac{a}{b}$	$\cot(90° - \theta) = \frac{b}{a}$

Notice the pairs of trigonometric ratios that are equal

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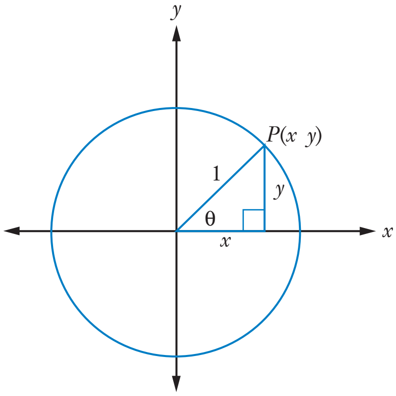
The tangent identity

In the work on angles of any magnitudewe saw that

$\sin\theta = y \;\; \cos\;\theta = x$ and $\tan\theta = \frac{y}{x}$

From this we get the following trigonometric identities



An **identity** is an equation that shows the equivalence of 2 algebraic expressions for all values of the variables for exampl, $a^2 - b^2 = (a + b)(a - b)$ is an identity.

The Pythagorean identities

The unit circle above has equation $x^2 + y^2 = 1$ because of Pythagora' theore.

But $\sin\theta = y$ and $\cos\theta = x$ so

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

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A shorter way of writing this is

$\cos^2\theta + \sin^2\theta = 1$

This formula is called a Pythagorean identity because it is based on Pythagoras theorem in the unit circle

There are 2 other identities that can be derived from this identity.

Dividing each term by $\cos^2\theta$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

Dividing each term by $\sin^2\theta$

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 = \text{cosec}^2\theta$$

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$\cos^2\theta + \sin^2\theta = 1$ can also be rearranged to give

$$\cos^2\theta = 1 - \sin^2\theta \text{ or}$$

$$\sin^2\theta = 1 - \cos^2\theta$$

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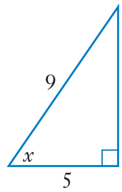
Exercise 11.02 Trigonometric identities



1 For this triangl, find the exact ratios of sec x cot x and cosec x

2 If $\sin\theta = \frac{5}{13}$ find cosec θ sec θ and cot θ

3 If $\cos\theta = \frac{4}{7}$ find exact values of cosec θ sec θ and cot θ

4 If $\sec\theta = -\frac{6}{5}$ and $\sin\theta > 0$ find exact values of tan θ cosec θ and cot θ

5 If $\cot\theta = 06$ and $\operatorname{cosec}\theta < 0$ find the exact values of sin θ cosec θ tan θ and sec θ

6 Show $\sin 67° = \cos 23°$

7 Show $\sec 82° = \operatorname{cosec} 8°$

8 Show $\tan 48° = \cot 42°$

9 Simplify

a $\cos 61° + \sin 29°$

b $\sec\theta - \operatorname{cosec}(90° - \theta)$

c $\tan 70° + \cot 20° - 2\tan 70°$

d $\frac{\sin 55°}{\cos 35°}$

e $\frac{\cot 25° + \tan 65°}{\cot 25°}$

10 Find the value of x if $\sin 80° = \cos(90 - x)°$

11 Find the value of y if $\tan 22° = \cot(90 - y)°$

12 Find the value of p if $\cos 49° = \sin(p + 10)°$

13 Find the value of b if $\sin 35° = \cos(b + 30)°$

14 Find the value of t if $\cot(2t + 5)° = \tan(3t - 15)°$

15 Find the value of k if $\tan(15 - k)° = \cot(2k + 60)°$

16 Simplify

a $\tan\theta\cos\theta$

b $\tan\theta\operatorname{cosec}\theta$

c $\sec x \cot x$

d $1 - \sin^2 x$

e $\sqrt{1 - \cos^2\alpha}$

f $\cot^2 x + 1$

g $1 + \tan^2 x$

h $\sec^2\theta - 1$

i $5\cot^2\theta + 5$

j $\frac{1}{\operatorname{cosec}^2 x}$

k $\sin^2\alpha\operatorname{cosec}^2\alpha$

$\cot\theta - \cot\theta\cos^2\theta$

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17 Prove tha:

a $\cos^2 x - 1 = -\sin^2 x$

b $\sec\theta + \tan\theta = \dfrac{1+\sin\theta}{\cos\theta}$

c $3 + 3\tan^2\alpha = \dfrac{3}{1-\sin^2\alpha}$

d $\sec^2 x - \tan^2 x = \text{cosec}^2 x - \cot^2 x$

e $(\sin x - \cos x)^3 = \sin x - \cos x - 2\sin^2 x\cos x + 2\sin x\cos^2 x$

$\cot\theta + 2\sec\theta = \dfrac{1-\sin^2\theta + 2\sin\theta}{\sin\theta\cos\theta}$

g $\cos^2(90° - \theta)\cot\theta = \sin\theta\cos\theta$

h $(\text{cosec}\, x + \cot x)(\text{cosec}\, x - \cot x) = 1$

i $\dfrac{1-\sin^2\theta\cos^2\theta}{\cos^2\theta} = \tan^2\theta + \cos^2\theta$

11.03 Further trigonometric identities

Sums and differences of angles

There are special formulas or identities for the trigonometric ratios of sums and differences of angles

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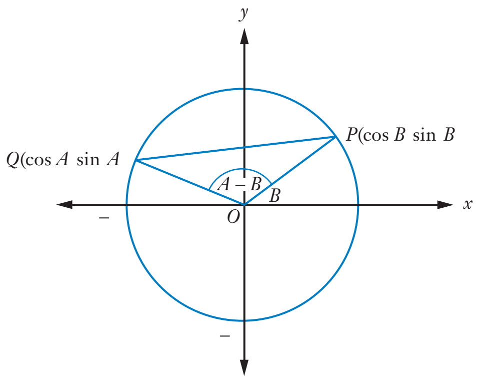
Proof

Since cos θ = x and sin θ = y in the unit circle we can write the coordinates of points (x y) on the unit circle as (cos θ sin θ). Let point P have coordinates (cos B sin B) and Q have coordinates (cos A sin A) where B and A are the angles of inclination of OP and OQ respectively.



Lets now find the length of PQ^2

By the distance formula

$$d^2 = (x_2 - x\ )^2 + (y_2 - y\ )^2$$

$$PQ^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$

$$= \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B$$

$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2\cos A\cos B - 2\sin A\sin B$$

$$= 1 + 1 - 2(\cos A\cos B + \sin A\sin B)$$

$$= 2 - 2(\cos A\cos B + \sin A\sin B) \qquad [1]$$

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By the cosine rule

$c^2 = a^2 + b^2 - 2ab \cos C$

$PQ^2 = 1^2 + 1^2 - 2(1)(1) \cos (A - B)$

$= 2 - 2 \cos (A - B)$ [2]

From [1] and [2]

$2 - 2 \cos (A - B) = 2 - 2(\cos A \cos B + \sin A \sin B)$

So $\cos (A - B) = \cos A \cos B + \sin A \sin B$

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Proof

Substitute $-B$ for B in the formula $\cos (A - B) = \cos A \cos B + \sin A \sin B$

$\cos (A - (-B)) = \cos A \cos (-B) + \sin A \sin (-B)$

$\cos (A + B) = \cos A \cos B + \sin A (-\sin B)$ since $\cos (-B) = \cos B$ and $\sin (-B) = -\sin B$

$= \cos A \cos B - \sin A \sin B$

Proof

Substitute $90° - A$ for A in the formula $\cos (A - B) = \cos A \cos B + \sin A \sin B$

$\cos (90° - A - B) = \cos (90° - A) \cos B + \sin (90° - A) \sin B$

$\cos (90° - (A + B)) = \sin A \cos B + \cos A \sin B$ using complementary angle results

$\sin (A + B) = \sin A \cos B + \cos A \sin B$

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Proof

Substitute $-B$ for B in the formula $\sin (A + B) = \sin A \cos B + \cos A \sin B$

$\sin (A + (-B)) = \sin A \cos (-B) + \cos A \sin (-B)$

$= \sin A \cos B + \cos A (-\sin B)$ since $\cos (-B) = \cos B$ and $\sin (-B) = -\sin B$

$= \sin A \cos B - \cos A \sin B$

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Proof

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \cos B\sin A}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B - \sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Proof

Substitute $-B$ for B in the formula $\frac{\tan A + \tan B}{1 - \tan A\tan B}$

$$\tan(A-B) = \frac{\tan A + \tan(-B)}{1 - \tan A\tan(-B)}$$

$$= \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Summarising all these results gives

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Double angles

By using the sum of angles we can find the trigonometric ratios for double angle.

Proof

$\sin 2A = \sin (A + A)$

$= \sin A \cos A + \cos A \sin A$

$= 2 \sin A \cos A$

Proof

$\cos 2A = \cos (A + A)$

$= \cos A \cos A - \sin A \sin A$

$= \cos^2 A - \sin^2 A$

Proof

$\tan 2A = \tan (A + A)$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2 \tan A}{1 - \tan^2 A}$$

Summarising all these results gives

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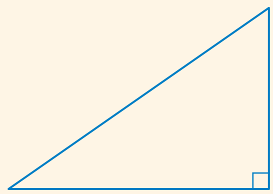

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Ulug Beg (1393–1449) used the relation $\sin^3 \theta = \frac{1}{4}(3 \sin \theta - \sin 3\theta)$ to draw up a table of sine ratios Can you prove this relation ?

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Products to sums and differences

We can rearrange the sum and difference identities to find the product of trigonometric ratio.

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Proof

$$\cos (A + B) + \cos (A - B) = \cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B$$

$$= 2 \cos A \cos B$$

$$\frac{1}{2} [\cos(A + B) + \cos (A - B)] = \cos A \cos B$$

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Proof

$$\cos (A - B) - \cos (A + B) = \cos A \cos B + \sin A \sin B - (\cos A \cos B - \sin A \sin B)$$

$$= 2 \sin A \sin B$$

$$\frac{1}{2} [\cos(A - B) - \cos (A + B)] = \sin A \sin B$$

– –

Proof

$$\sin (A + B) + \sin (A - B) = \sin A \cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$$

$$= 2 \sin A \cos B$$

$$\frac{1}{2}[\sin(A + B) + \sin (A - B)] = \sin A \cos B$$

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Proof

$$\sin(A + B) - \sin(A - B) = \sin A \cos B + \cos A \sin B - (\sin A \cos B - \cos A \sin B)$$
$$= 2\cos A \sin B$$
$$\frac{1}{2}[\sin(A + B) - \sin(A - B)] = \cos A \sin B$$

We can summarise these rule:

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The ***t*-formulas** express sin A cos A and tan A in terms of $t = \tan \frac{A}{2}$

Proof for tan A

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$$\tan A = \frac{2\tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$$

$$= \frac{2t}{1 - t^2}$$

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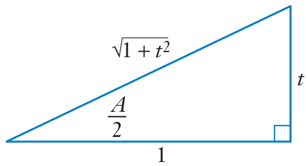
Proof for sin A

Drawing a right–angled triangle for $\tan \frac{A}{2}$



$$\tan \frac{A}{2} = t$$

$$= \frac{t}{1}$$

By Pythagoras theorem the hypotenuse is $\sqrt{1+t^2}$

$$\sin 2A = 2 \sin A \cos A$$

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right)$$

$$= \frac{2t}{1+t^2}$$

Proof for cos A

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$$

$$= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2 \quad \text{(from the triangle)}$$

$$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$$

$$= \frac{1-t^2}{1+t^2}$$

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Exercise 11.03 Further trigonometric identities

1 Expand

a $\sin(a-b)$ **b** $\cos(p+q)$ **c** $\tan(\alpha+\beta)$

d $\sin(x+20°)$ **e** $\tan(48°+x)$ $\cos(2\theta-\alpha)$

g $\cos(x+75°)$ **h** $\tan(5x-7y)$ **i** $\sin(4\alpha-\beta)$

2 Simplify

a $\sin a\cos b+\cos a\sin b$ **b** $\dfrac{\tan 36°+\tan 29°}{1-\tan 36°\tan 29°}$

c $\cos 28°\cos 27°-\sin 28°\sin 27°$ **d** $\sin 2x\cos 3y+\cos 2x\sin 3y$

e $\dfrac{\tan 3\theta-\tan\theta}{1+\tan 3\theta\tan\theta}$ **f** $\sin 74°\cos 42°-\cos 74°\sin 42°$

g $\sin(45°+30°)+\sin(45°-30°)$ **h** $\sin(x+y)-\sin(x-y)$

i $\cos(x-y)-\cos(x+y)$ **j** $\cos(m+n)+\cos(m-n)$

3 Simplify each expressio, given $t=\tan\dfrac{A}{2}$

a $\dfrac{2t}{1-t^2}$ **b** $\dfrac{1-t^2}{1+t^2}$ **c** $\dfrac{2\tan 10°}{1-\tan^2 10°}$

d $\dfrac{1-\tan^2 25°}{1+\tan^2 25°}$ **e** $\dfrac{2\tan A}{1+\tan^2 A}$ $\dfrac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

4 Find the exact value o:

a $\sin 75°$ **b** $\cos 15°$ **c** $\tan 75°$

d $\tan 105°$ **e** $\cos 105°$ $\sin 15°$

g $\sin 105°$ **h** $\tan 285°$ **i** $\sin(x+30°)+\cos(x+30°)$

j $\cos(45°-y)+\cos(45°+y)$

5 Simplify $\dfrac{\tan(x+y)+\tan(x-y)}{1-\tan(x+y)\tan(x-y)}$

6 If $\sin x=\dfrac{2}{3}$ and $\cos y=\dfrac{3}{4}$ find the exact value o:

a $\sin(x+y)$ **b** $\cos(x-y)$ **c** $\tan(x+y)$

7 By taking $2\theta=\theta+\theta$ find an expression fo:

a $\sin 2\theta$ **b** $\cos 2\theta$ **c** $\tan 2\theta$

8 By writing 3θ as $2\theta+\theta$ find an expression in terms of θ for

a $\sin 3\theta$ **b** $\cos 3\theta$ **c** $\tan 3\theta$

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9 **a** Simplify $\dfrac{\tan 7\theta - \tan 3\theta}{1 + \tan 7\theta \tan 3\theta}$

b Find an expression for sin 4θ in terms of 7θ and 3θ

10 Find an expression for cos $9x$ in terms of $2x$ and $7x$

11 Find the exact value of each expression

a ________ **b** ________

c ________ **d** ________

12 Write each expression as a sum or difference of trigonometric ratis.

a sin $3a$ sin $2b$ **b** cos $5y$ sin $3z$ **c** cos $2p$ cos $3q$

d sin $4x$ cos $9y$ **e** cos $7x$ cos $2x$ sin $4y$ sin y

g sin $6a$ cos $5a$ **h** cos $2x$ sin $5x$

13 Find the exact value o:

a cos 23° cos 22° – sin 23° sin 22° **b** $\dfrac{\tan 85° - \tan 25°}{1 + \tan 85° \tan 25°}$

c sin 180° cos 60° + cos 180° sin 60° **d** cos 290° cos 80° + sin 290° sin 80°

e $\dfrac{\tan 11° + \tan 19°}{1 - \tan 11° \tan 19°}$ **f** cos 165° cos 15°

g sin 105° cos 75°

14 If $\sin x = \dfrac{3}{5}$ and $\cos y = \dfrac{5}{13}$ find the value o:

a cos x **b** sin y **c** sin $(x - y)$ **d** tan y **e** tan $(x + y)$

15 **a** Write an expression for cos $(x + y)$ + cos $(x - y)$

b Hence write an expression for cos 50° cos 65°

16 Write each expression in terms of t where t = tan —

a cosec A **b** sec A

c cot A **d** sin A + cos A

e 1 + tan A **f** 1 + tan A tan —

g 3 cos A + 4 sin A **h** ________

i tan A + sec A **j** sin $2A$

17 Find an expression fo:

a $\sin(x + y) + \sin(x - y)$

b $\cos(x + y) - \cos(x - y)$

c $\sin(x - y) - \sin(x + y)$

d $\tan(x + y) + \tan(x - y)$

18 Simplify

a $2\cos 3x \sin 3x$

b $\cos^2 7y - \sin^2 7y$

c $\dfrac{2\tan 5\theta}{1 - \tan^2 5\theta}$

d $1 - 2\sin^2 y$

e $\sin 6\theta \cos 6\theta$

f $(\sin x + \cos x)^2$

g $2\cos^2 3\alpha - 1$

h $1 - 2\sin^2 40°$

i $\dfrac{2\tan\beta}{1 - \tan^2\beta}$

j $(\sin 3x - \cos 3x)^2$

19 **a** Simplify $\dfrac{\sin 2x}{1 + \cos 2x}$

b Henc, find the exact value of tan 15 °

20 Find the exact value of $\tan 22\frac{1}{2}°$ by using the expression for $\tan 2x$

21 Prove tha:

a $\sin^2\theta = \dfrac{1}{2}\sin 2\theta \tan\theta$

b $\tan\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{\sin\theta}$

c ―――――――― = tan ―

22 Show that $\sin^2 7\theta - \sin^2 4\theta = \sin 11\theta \sin 3\theta$

23 Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

24 Find an expression for $\sin 2A - \cos 2A$ in terms of $t = \tan$ ―

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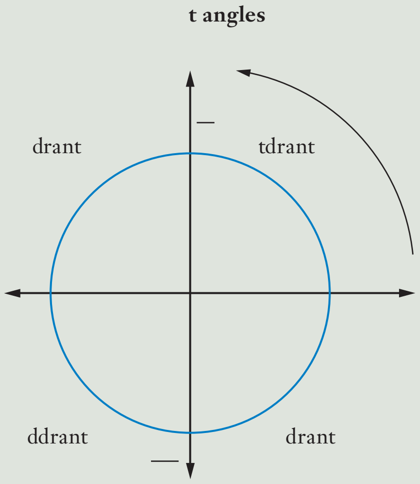
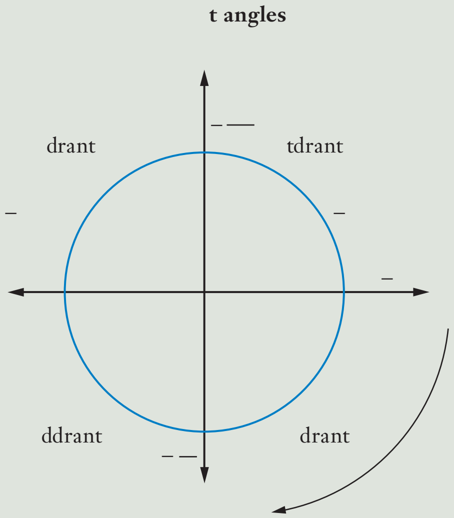
11.04 Radians

The rules and formulas learned in this chapter can also be expressed in radianswhich we learned about in Chapter 5 *Trigonometry*

Radians o any magniude





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Exercise 1104 Radians

1 Find the exact value of each expressio.

a cosec — **b** sec — **c** cot —

d — **e** $1 - \cos^2$ — tan — cos —

g — **h** cosec^2 — $- 1$ **i** —

2 **a** Show that $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$

b In which quadrant is the angle $\frac{3\pi}{4}$?

c Find the exact value of $\cos \frac{3\pi}{4}$

3 **a** Show that $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$

b In which quadrant is the angle $\frac{5\pi}{6}$?

c Find the exact value of $\sin \frac{5\pi}{6}$

4 **a** Show that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$

b In which quadrant is the angle $\frac{7\pi}{4}$?

c Find the exact value of $\tan \frac{7\pi}{4}$

5 **a** Show that $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$

b In which quadrant is the angle $\frac{4\pi}{3}$?

c Find the exact value of $\cos \frac{4\pi}{3}$

6 **a** Show that $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$

b In which quadrant is the angle $\frac{5\pi}{3}$?

c Find the exact value of $\sin \frac{5\pi}{3}$

7 **a** **i** Show that $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$

ii In which quadrant is the angle $\frac{13\pi}{6}$?

iii Find the exact value of $\cos \frac{13\pi}{6}$

b Find the exact value o:

i $\sin \frac{9\pi}{4}$ **ii** $\tan \frac{7\pi}{3}$ **iii** $\cos \frac{11\pi}{4}$

v $\tan \frac{19\pi}{6}$ **v** $\sin \frac{10\pi}{3}$

8 Copy and complete each table with exact value.

a

	—	—	—	—	—	—	—	—

b

	—	—	—	—	—	—	—	—

c

	—	—	—	—	—	—	—	—

9 Copy and complete the table where possibl.

		—		—		—		—	

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10 Find the exact value of

a $\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$

b $\cos\left(\frac{\pi}{6}-\frac{\pi}{3}\right)$

c $\sin\frac{2\pi}{3}\cos\frac{\pi}{4}-\cos\frac{2\pi}{3}\sin\frac{\pi}{4}$

d $2\sin\frac{\pi}{8}\cos\frac{\pi}{8}$

e $\dfrac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}}$

11 Simplify

a $\cos\frac{\pi}{9}\cos\frac{\pi}{5}-\sin\frac{\pi}{9}\sin\frac{\pi}{5}$

b $\sin\frac{5\pi}{7}\cos\frac{\pi}{8}-\cos\frac{5\pi}{7}\sin\frac{\pi}{8}$

c $\dfrac{\tan\pi-\tan\frac{\pi}{5}}{1+\tan\pi\tan\frac{\pi}{5}}$

d $\sin\frac{\pi}{11}\cos\frac{\pi}{9}$

e $\dfrac{2\tan\frac{\pi}{7}}{1+\tan^2\frac{\pi}{7}}$

12 **a** **i** Show that $\frac{\pi}{6}+\frac{\pi}{4}=\frac{5\pi}{12}$

ii Find the exact value of $\tan\frac{5\pi}{12}$

b **i** Show that $\frac{2\pi}{3}+\frac{\pi}{4}=\frac{11\pi}{12}$

ii Find the exact value of $\cos\frac{11\pi}{12}$

c **i** Show that $\frac{9\pi}{4}-\frac{5\pi}{3}=\frac{7\pi}{12}$

ii Find the exact value of $\sin\frac{7\pi}{12}$

d Find the exact value o:

i $\sin\frac{\pi}{12}$

ii $\cos\frac{13\pi}{12}$

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11.05 Trigonometric functions

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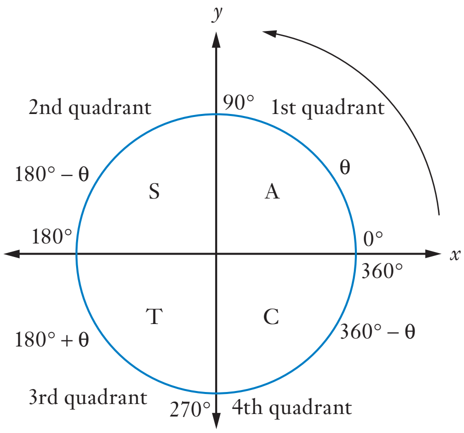
TRGONOMETRIC RATIOS OF 0°, 90°, 180°, 270° AND 360°

Remember the results from the unit circle

$\sin\theta = y$

$\cos\theta = x$

$\tan\theta = \frac{y}{x}$



1 Angle 0° is at the point (1 0) on the unit circl. Use the circle results to find sin °, cos 0° and tan 0°

2 Angle 90° is at the point (0 1. Use the circle results to find sin 90 ° cos 90° and tan 90° Discuss the result for tan 90° and why this happens

3 Angle 180° is at the point (–1 0. Find sin 180 °, cos 180° and tan 180°

4 Angle 270° is at the point (0 –1) Find sin 270 ° cos 270° and tan 270° Discuss the result for tan 270° and why this happens

5 What are the results for sin 360° cos 360° and tan 360°? Why?

6 Check these results on your calculato

The sine function

Using all the results from the investigationwe can draw up a table of values for $y = \sin x$

				–	

We could add in all the exact value results we know for a more accurate grap. Remember that sin x is positive in the 1st and 2nd quadrants and negative in the 3rd and 4th quadrants

		–	$\frac{}{\sqrt{}}$	$\frac{\sqrt{}}{}$		$\frac{\sqrt{}}{}$	$\frac{}{\sqrt{}}$	–		– –	$-\frac{}{\sqrt{}}$	$-\frac{\sqrt{}}{}$	–	$-\frac{\sqrt{}}{}$	$-\frac{}{\sqrt{}}$	– –	

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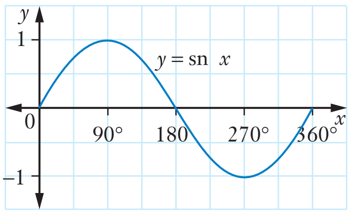
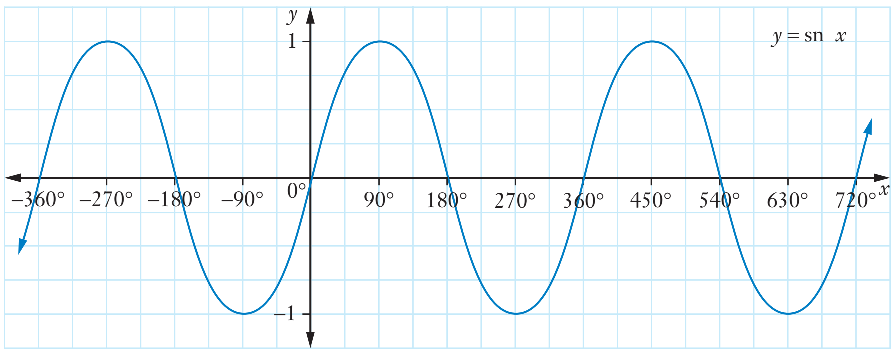
Drawing the graph gives a smooth ‘wav’ curv.



As we go around the unit circle and graph the y values of the points on the circle the graph should repeat itself every 360°

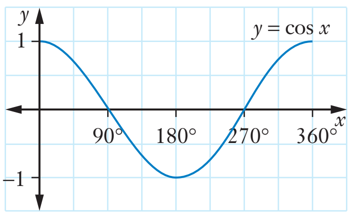
$y = \sin x$ has domain (–∞ ∞) and range [–1 1. It is an odd functin.



The cosine function

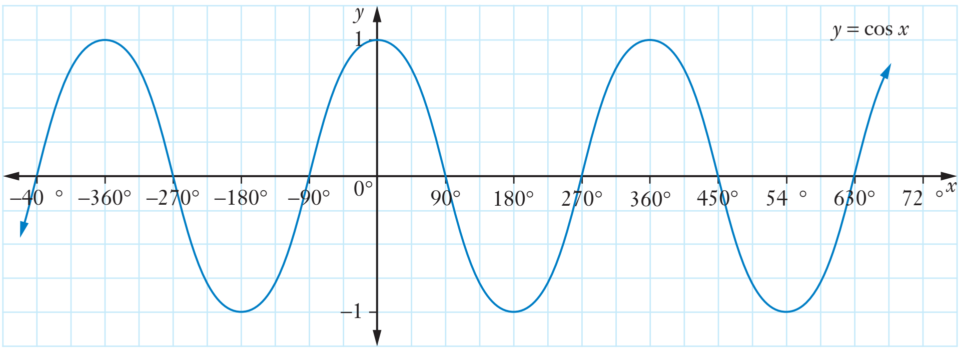
Similarly for $y = \cos x$ which is positive in the 1st and 4th quadrants and negative in the 2nd and 3rd quadrants Its graph has the same shape as the graph of the sine functio.

		$\frac{\sqrt{\ }}{\ }$	$\frac{\ }{\sqrt{\ }}$	–		– –	$-\frac{\ }{\sqrt{\ }}$	$-\frac{\sqrt{\ }}{\ }$		$-\frac{\sqrt{\ }}{\ }$	$-\frac{\ }{\sqrt{\ }}$	– –	–	–	$\frac{\ }{\sqrt{\ }}$	$\frac{\sqrt{\ }}{\ }$	



As we go around the unit circle and graph the x values of the points on the circle the graph should repeat itself every 360°

$y = \cos x$ has domain (–∞ ∞) and range [–1 1. It is an even functin.



The tangent function

$y = \tan x$ is positive in the 1st and 3rd quadrants and negative in the 2nd and 4th quadrants

It is also undefined for 90° and 270° so there are vertical asymptotes at those x values where the function is discontinuous

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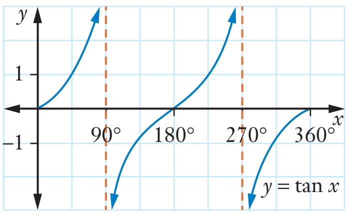
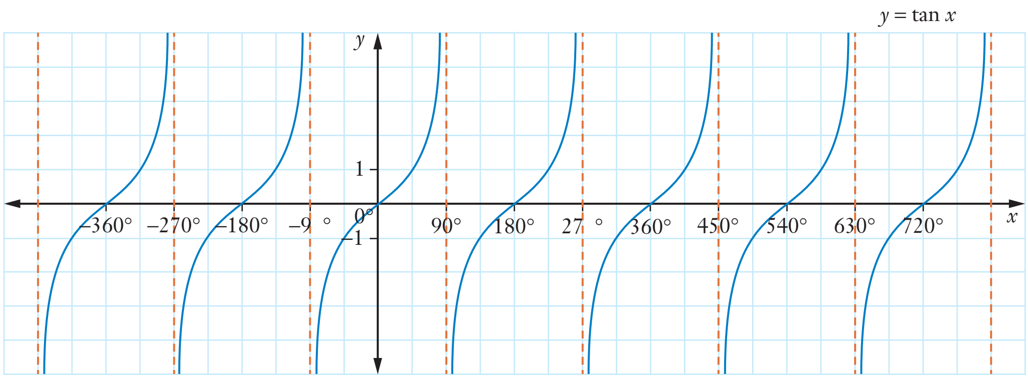


As we go around the unit circle and graph the values of $\frac{y}{x}$ of the points on the circle the graph repeats itself every 180°

$y = \tan x$ has domain (–∞ ∞) except for 90° 270, 54°, … (odd multiples of 90°) and range (–∞ ∞) It is an odd functio.



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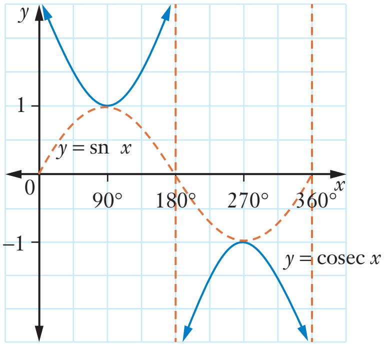
The cosecant function

$$\text{cosec } x = \frac{1}{\sin x}$$

We learned about reciprocal functions in Chapter , *Further functions* Each y value of $y = \text{cosec } x$ will be the reciprocal of $y = \sin x$ Because $\sin x = 0$ at $x = 0°, 180°, 360°, \ldots,$ $y = \text{cosec } x$ will have vertical asymptotes at those values

We can use a table of values and explore the limits as x approaches any asymptotes

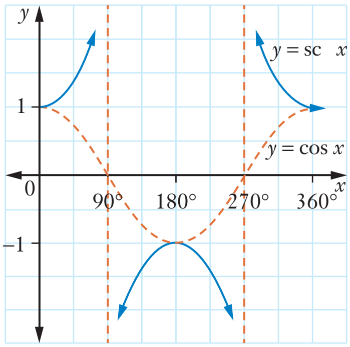
			√	√		√	√			–	–√	√	–	√	√	–	–



The secant function

$\sec x = \dfrac{1}{\cos x}$ so each y value of $y = \sec x$ will be the reciprocal of $y = \cos x$ Because $\cos x = 0$ at $x = 90°, 270°, 450°, \ldots, y = \sec x$ will have vertical asymptotes at those values

		√	√			–	√	√	–	√	–√	–			√	√	



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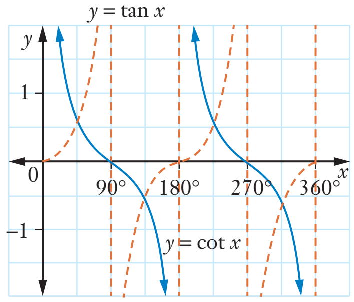
The cotangent function

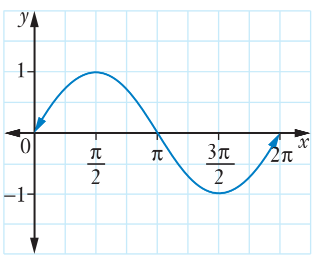
$\cot x = \dfrac{1}{\tan x}$ so each y value of $y = \cot x$ will be the reciprocal of $y = \tan x$ Because $\tan x = 0$ at $x = 0°, 180°, 360°, \ldots, y = \cot x$ will have vertical asymptotes at those valuesAlo, because $\tan x$ has asymptotes at $x = 90°, 270°, 450°, \ldots, y = \cot x = 0$ and there are x-intercepts at those values

	−	$\sqrt{\ }$		$\overline{\sqrt{\ }}$		$\overline{\sqrt{\ }}$	−	$\sqrt{\ }$	−	$\sqrt{\ }$		$\overline{\sqrt{\ }}$		$\overline{\sqrt{\ }}$	−	$\sqrt{\ }$	−



It is more practical to express the trigonometric functions in terms of radians (not degrees) so here are the graphs in radians

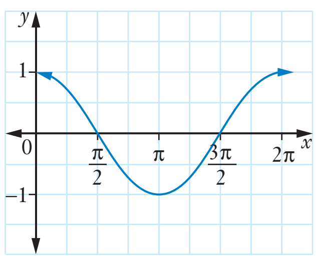
$y = \sin x$



Domain $(-\infty\ \ \infty)$ range $[-1, 1]$

Odd function

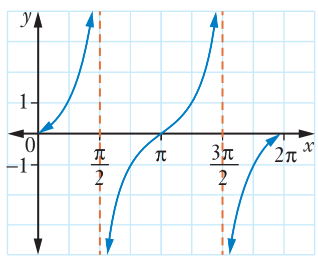
$y = \cos x$



Domain $(-\infty\ \ \infty)$ range $[-1, 1]$

Even function

$y = \tan x$



Domain $(-\infty\ \ \infty)$

except for $\frac{\pi}{2}\ \frac{3\pi}{2}\ \frac{5\pi}{2}$

(odd multiples of $\frac{\pi}{2}$)
range $(-\infty\ \ \infty)$
Odd function

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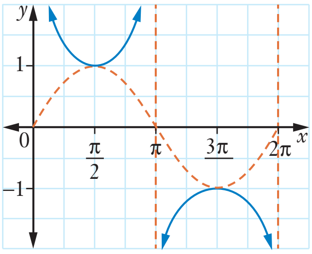
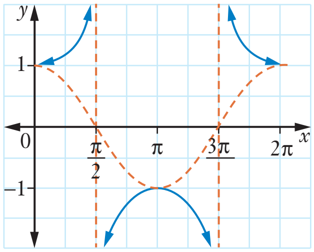
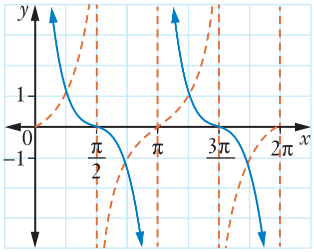
$y = \text{cosec } x$



$y = \sec x$



$y = \cot x$



Properties of the trigonometric functions

All the trigonometric functions have graphs that repeat at regular intervalsso they are called **periodic functions** The **period** is the length of one cycle of a periodic function on the x-axis before the function repeats itsel.

The **centre** of a periodic function is its mean value and is equidistant from the maximum and minimum values The mean value of $y = \sin x$ $y = \cos x$ and $y = \tan x$ is 0 represented by the x-axis

The **amplitude** is the height from the centre of a periodic function to the maximum or minimum values (peaks and troughs of its graph respectively)The range of $y = \sin x$ and $y = \cos x$ is $[-1, 1]$.

$y = \sin x$ has period 2π and amplitude 1

$y = \cos x$ has period 2π and amplitude 1

$y = \tan x$ has period π and no amplitude

TRANSFORMING TRIGONOMETRC GRAPHS

Use a graphics calculator or graphing software to draw the graphs of trigonometric functions with different values

1 Graphs in the form $y = k \sin x$ $y = k \cos x$ and $y = k \tan x$ where $k = \ldots, -3, -2, -1, 2, 3, \ldots$

2 Graphs in the form $y = \sin ax$ $y = \cos ax$ and $y = \tan ax$ where $a = \ldots, -3, -2, -1, 2, 3, \ldots$

3 Graphs in the form $y = \sin x + c$ $y = \cos x + c$ and $y = \tan x + c$ where $c = \ldots, -3, -2, -1, 2, 3, \ldots$

4 Graphs in the form $y = \sin(x + b)$, $y = \cos(x + b)$ and $y = \tan(x + b)$ where $b = \ldots, \pm\frac{\pi}{2}$ $\pm\pi$ $\pm\frac{\pi}{4}$ $\ldots$

Can you see patterns? Could you predict what different graphs look like?

Now we shall examine more general trigonometric functions of the form $y = k \sin ax$ $y = k \cos ax$ and $y = k \tan ax$ where k and a are constants

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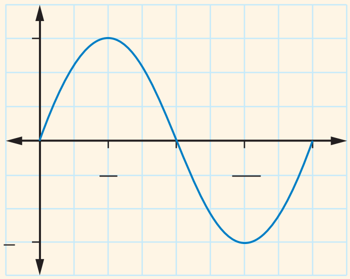

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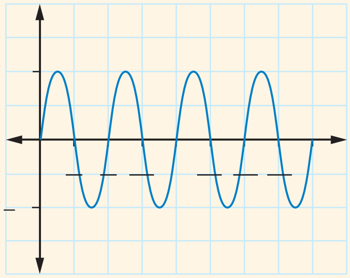

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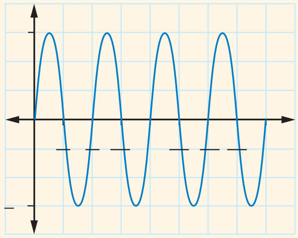

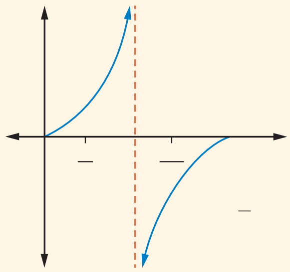

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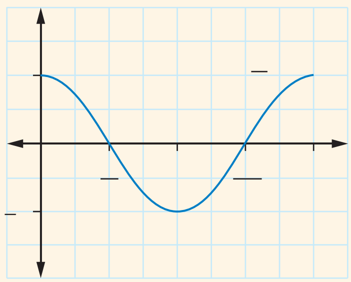

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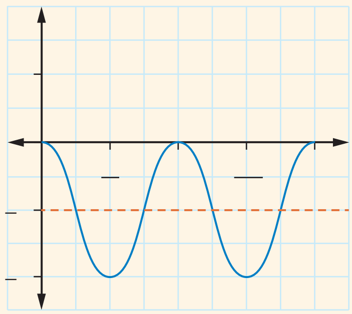

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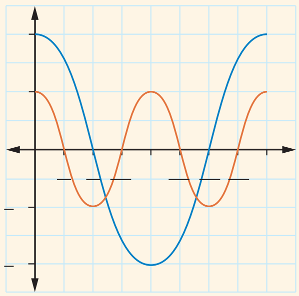

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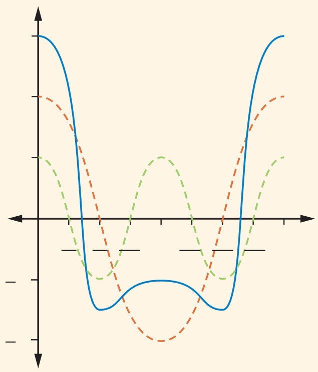

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Exercise 11.05 Trigonometric functions

1 **a** Sketch the graph of $f(x) = \cos x$ in the domain [0 2π.

b Sketch the graph of $y = -f(x)$ in the same domain

2 Sketch the graph of each function in the domain [, 2 π]

a $f(x) = 2 \sin x$ **b** $y = 1 + \sin x$ **c** $y = 2 - \sin x$

d $f(x) = -3 \cos x$ **e** $y = 4 \sin x$ $f(x) = \cos x + 3$

g $y = 5 \tan x$ **h** $f(x) = \tan x + 3$ $y = 1 - 2 \tan x$

3 Sketch the graph of each function in the domain [, 2 π]

a $y = \cos 2x$ **b** $y = \tan 2x$ **c** $y = \sin 3x$

d $f(x) = 3 \cos 4x$ **e** $y = 6 \cos 3x$ $y = \tan \frac{x}{2}$

g $f(x) = 2 \tan 3x$ **h** $y = 3 \cos \frac{x}{2}$ **i** $y = 2 \sin \frac{x}{2}$

4 Sketch the graph of each function in the domain [−π π]

a $y = -\sin 2x$ **b** $y = 7 \cos 4x$ **c** $f(x) = -\tan 4x$

d $y = 5 \sin 4x$ **e** $f(x) = 2 \cos 2x$ $f(x) = 3 \tan x - 1$

5 Sketch the graph of $y = 8 \sin \frac{x}{2}$ in the domain [0 4 π]

6 Sketch over the interval [, 2 π] the graph of

a $y = \sin (x + \pi)$ **b** $y = \tan \left(x + \frac{\pi}{2}\right)$ **c** $f(x) = \cos (x - \pi)$

d $y = 3 \sin \left(x - \frac{\pi}{2}\right)$ **e** $f(x) = 2 \cos \left(x + \frac{\pi}{2}\right)$ $y = 4 \sin \left(2x + \frac{\pi}{2}\right)$

g $y = \cos \left(x - \frac{\pi}{4}\right)$ **h** $y = \tan \left(x + \frac{\pi}{4}\right)$

7 Sketch over the interval [−2 2] the graph o:

a $y = \sin \pi x$ **b** $y = 3 \cos 2\pi x$

8 For each function fin:

i the amplitude **ii** the period **iii** the centre **v** the phase

a $y = 5 \sin 2x$ **b** $f(x) = -\cos (x - \pi)$ **c** $y = 2 \tan (4x) - 2$

d $y = 3 \sin \quad - + 1$ **e** $y = 8 \cos (\pi x - 2) - 3$ $f(x) = 3 \tan \quad - + 2$

9 Find the domain and range of each function

a $y = 4 \sin x - 1$ **b** $f(x) = -3 \cos 5x + 7$

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10 Sketch in the domain [, 2 π] the graphs of

a $y = \sin x$ and $y = \sin 2x$ on the same set of axes

b $y = \sin x + \sin 2x$

11 Sketch for the interval [, 2 π] the graphs of

a $y = 2 \cos x$ and $y = 3 \sin x$ on the same set of axes

b $y = 2 \cos x + 3 \sin x$

12 By sketching the graphs of $y = \cos x$ and $y = \cos 2x$ on the same set of axes for [0 2 π], sketch the graph of $y = \cos 2x - \cos x$

13 Sketch the graph of $y = \cos x + \sin x$

11.06 Trigonometric equations

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If we are solving an equation involving $2x$ or $3x$ for exampl, we need to change the domain to find all possible solutions

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You can solve trigonometric equations involving **radians** You can recognise these because the domain is in radians

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Exercise 11.06 Trigonometric equations

1 Solve each equation for [0°, 360°]

a $\sin\theta = 035$ **b** $\cos\theta = -\frac{1}{2}$ **c** $\tan\theta = -1$

d $\sin\theta = \frac{\sqrt{3}}{2}$ **e** $\tan\theta = -\frac{1}{\sqrt{3}}$ $2\cos\theta = \sqrt{3}$

g $\tan 2\theta = \sqrt{3}$ **h** $2\cos 2\theta - 1 = 0$ **i** $2\sin 3\theta = -1$

j $\tan^2 3\theta = 1$ **k** $\sin^2 x = 1$ $2\cos^2 x - \cos x = 0$

2 Solve for $0° \le x \le 360°$

a $\cos x = 1$ **b** $\sin x + 1 = 0$ **c** $\cos^2 x = 1$

d $\sin x = 1$ **e** $\tan x = 0$ $\sin^2 x + \sin x = 0$

g $\cos^2 x - \cos x = 0$ **h** $\tan^2 x = \tan x$ **i** $\tan^2 x = 3$

3 Solve for $[0, 2\pi]$

a $\sin x = 0$ **b** $\tan 2x = 0$ **c** $\sin x = -1$

d $\cos x - 1 = 0$ **e** $\cos x = -1$

4 Solve for [–180°, 180°]

a $\cos\theta = 0187$ **b** $\sin\theta = \frac{1}{2}$ **c** $\tan\theta = 1$

d $\sin\theta = -\frac{\sqrt{3}}{2}$ **e** $\tan\theta = -\frac{1}{\sqrt{3}}$ $3\tan^2\theta = 1$

g $\tan\theta + 1 = 0$ **h** $\tan 2\theta = 1$

5 Solve for $0 \le x \le 2\pi$

a $\cos x = \frac{1}{2}$ **b** $\sin x = -\frac{1}{\sqrt{2}}$ **c** $\tan x = 1$

d $\tan x = \sqrt{3}$ **e** $\cos x = -\frac{\sqrt{3}}{2}$

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6 Solve for $-\pi \le x \le \pi$

a $2 \sin x = \sqrt{3}$ **b** $2 \cos x = 0$ **c** $3 \tan^2 x = 1$

7 Solve $2 \cos x = -1$ in the domain $[-2\pi, 2\pi]$

8 Solve for $[0, 2\pi]$

a $\tan^2 x + \tan x = 0$ **b** $\sin^2 x - \sin x = 0$ **c** $2 \cos^2 x - \cos x - 1 = 0$

d $4 \sin^2 x = 1$ **e** $\tan x \cos x + \tan x = 0$ $\sin^2 x + 2 \cos x - 2 = 0$

11.07 Applications of trigonometric functions

Trigonometric graphs can model real-life situation.

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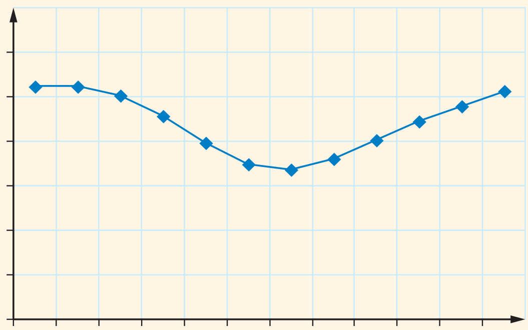

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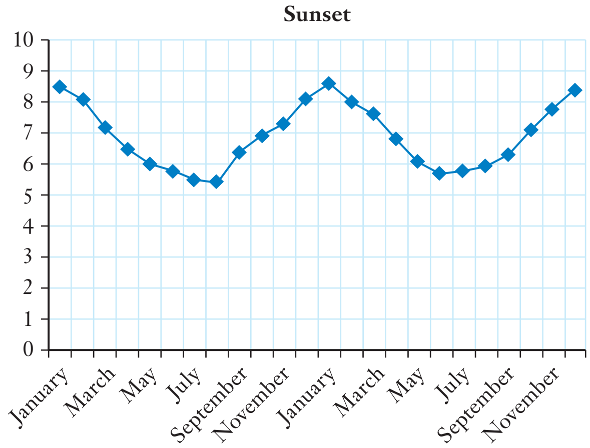
Exercise 11.07 Applications of trigonometric functions

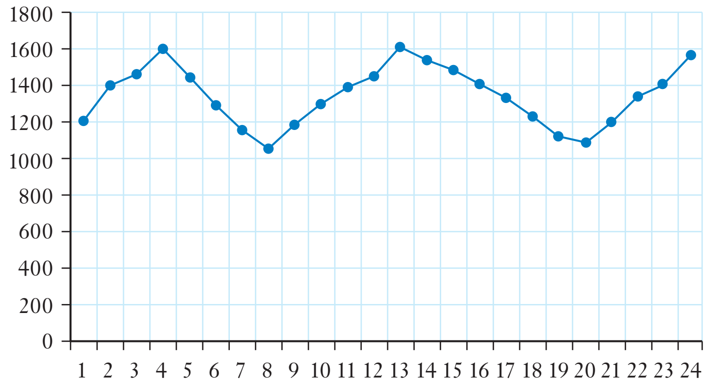
1 This graph shows the time of sunset in a city over a period of 2 years



a Find the approximate period and amplitude of the graph

b At approximately what time would you expect the Sun to set in July?

2 The graph shows the incidence of crimes committed over 24 years in Gotham Cit



a Approximately how many crimes were committed in the 10th year?

b What wa:

i the highest number of crimes? **ii** the lowest number of crimes?

c Find the approximate amplitude and the period of the grap.

3 This table shows the tides (in metres) at a jetty measured 4 times each day for 3 day.

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a Draw a graph showing the tide.

b Find the period and amplitud.

c Estimate the height of the tide at around 8 .. on Fridy.

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11.08 Inverse trigonometric functions

We looked at inverse functions in Chapter , *Polynomials and inverse functions*

The sine cosine and tangent functions all have inverse function, called the **inverse trigonometric functions**

The inverse sine function $y = \sin^{-}\ x = \arcsin x$

The inverse cosine function $y = \cos^{-}\ x = \arccos x$

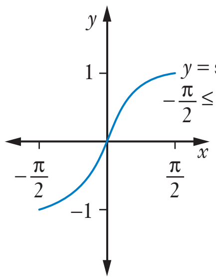
The inverse tangent function $y = \tan^{-}\ x = \arctan x$

We already use the [i] [ ] and [ ] keys on our calculator to find angles in trigonometry problems Because the inverse function notation $\sin^{-1} x$ $\cos^{-1} x$ and $\tan^{-1} x$ could be confused with the reciprocal notation (power of −1) we can use **arcsin *x*** **arcos *x*** and **arctan *x*** instead

The trigonometric functions fail the horizontal line testso their inverses are not functions and we must restrict the domain of each function so that their inverse is a function

The inverse sine function

For $y = \sin^{-}\ x$ we restrict $y = \sin x$ to a monotonic increasing domain $\left[-\frac{\pi}{2}\ \frac{\pi}{2}\right]$ and its range is $[-1, 1]$.



Remember that the domain of the inverse function is the range of the original function and the range of the inverse function is the domain of the original functionThe graph of the inverse function is a reflection of the original function in the line $y = x$

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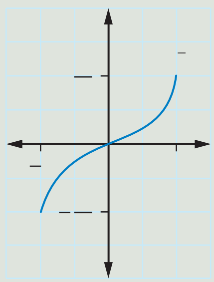

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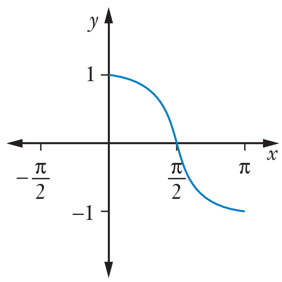
Inverse cosine function

For $y = \cos^{-}\ x$ we restrict $y = \cos x$ to a monotonic decreasing domain $[0\ \ \pi]$ and its range is $[-1, 1]$.



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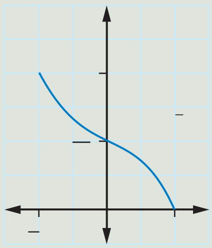
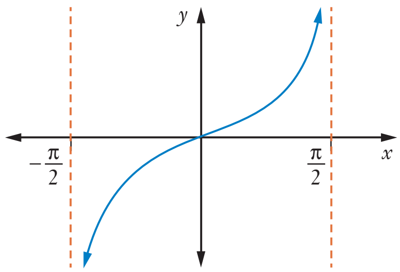


Inverse tangent function

For $y = \tan^{-}\ x$ we restrict $y = \tan x$ to a monotonic increasing domain $\left[-\frac{\pi}{2}\ \frac{\pi}{2}\right]$ and its range is $(-\infty\ \infty$.



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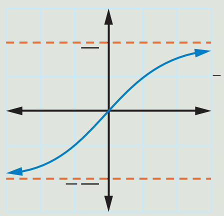

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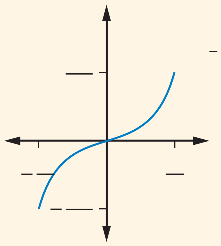

TRANSFORMNG NVERSE TRIGONOMETRIC FUNCTIONS

1 Investigate how the values of the constants a and b affect the domain range and graph of the functions

a $y = a \sin^{-} bx$ **b** $y = a \cos^{-} bx$ **c** $y = a \tan^{-} bx$

Use different values of a and b

2 What is the domain and range of $y = \sin^{-} (\sin x)$? Can you sketch its graph?

3 Does $y = \sin (\sin^{-} x)$ have the same domain range and graph as the above ?

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Exercise 11.08 Inverse trigonometric functions

1 State the domain and range o:

a $y = \arcsin x$ **b** $y = \tan^{-} x$ **c** $f(x) = \cos^{-} x$

2 Sketch the graph o:

a $y = \cos^{-} x$ **b** $y = \arctan x$ **c** $y = \sin^{-} x$

d $y = \cos^{-} 2x$ **e** $y = \arcsin 3x$ $y = 2 \arccos x$

g $y = 5 \cos^{-} 3x$ **h** $y = 3 \sin^{-} \frac{x}{2}$ **i** $y = 2 \sin^{-} 4x$

j $y = 2 \cos^{-} 7x$

3 State the domain and range o:

a $y = \arcsin (x^2)$ **b** $y = \sin^{-} (\tan x)$

11.09 Properties of inverse trigonometric functions

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When evaluating an inverse trigonometric functionwe must ensure that we stay within the range for that function

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We will now prove some properties of the inverse trigonometric function.

Proof

Let $y = \sin^{-1}(-x)$

Then $\sin y = -x$ in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore x = -\sin y$

$= \sin(-y)$ (4th quadrant)

So $-y = \sin^{-1} x$

$y = -\sin^{-1} x$

But $y = \sin^{-1}(-x)$

$\therefore \sin^{-1}(-x) = -\sin^{-1} x$

$y = \sin^{-1}(-x)$ is an odd function

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Proof

Let $y = \cos^{-}\ (-x)$

Then $\cos y = -x$ in the range $[0\ \ \pi]$

$\therefore x = -\cos y$

$= \cos(\pi - y)$ (2nd quadrant)

So $\pi - y = \cos^{-}\ x$

$y = \pi - \cos^{-}\ x$

But $y = \cos^{-}\ (-x)$

$\therefore \cos^{-}\ (-x) = \pi - \cos^{-}\ x$

Proof

Let $y = \tan^{-}\ (-x)$

Then $\tan y = -x$ in the range $\left(-\frac{\pi}{2}\ \ \frac{\pi}{2}\right)$

$\therefore x = -\tan y$

$= \tan(-y)$ (4th quadrant)

So $-y = \tan^{-}\ x$

$y = -\tan^{-}\ x$

But $y = \tan^{-}\ (-x)$

$\therefore \tan^{-}\ (-x) = -\tan^{-}\ x$

$y = \tan^{-}\ (-x)$ is an odd function

Proof

Let $y = \sin^{-}\ x$ [1]

Then $\sin y = x$ in the range $\left[-\frac{\pi}{2}\ \ \frac{\pi}{2}\right]$

$\therefore x = \sin y$

$= \cos\left(\frac{\pi}{2} - y\right)$ (using complementary angle results)

$\therefore \frac{\pi}{2} - y = \cos^{-}\ x$ [2]

From [1] and [2]

$$\sin^{-}\ x + \cos^{-}\ x = y + \frac{\pi}{2} - y$$

$$= \frac{\pi}{2}$$

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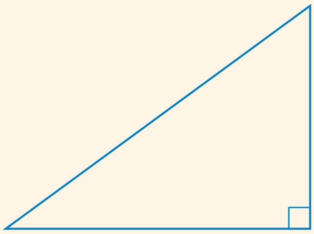

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Inverse of an inverse trigonometric function

You saw in Chapter , *Polynomials and inverse functions* that $f^{-}[f(x)] = f[f^{-}(x)] = x$

However, because the domain of trigonometric inverse functions is restrictd, this limits the x values for which these statements are true

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$\sin(\sin^{-} x) = x$ only if $\sin^{-} x$ is defined that i, for $-1 \le x \le 1$.

$\cos(\cos^{-} x) = x$ only if $\cos^{-} x$ is defined that i, for $-1 \le x \le 1$.

$\tan(\tan^{-} x) = x$ only if $\tan^{-} x$ is defined that i, for all real values of x

$\sin^{-} (\sin x) = x$ only if x is within the range of $\sin^{-}$ that i, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\cos^{-} (\cos x) = x$ only if x is within the range of $\cos^{-}$ that i, $0 \le x \le \pi$

$\tan^{-} (\tan x) = x$ only if x is within the range of $\tan^{-}$ that i, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

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Exercise 11.09 Using inverse trigonometric functions

1 Evaluat, giving exact answes:

a $\sin^{-}\ 1$ **b** $\tan^{-}\ 0$ **c** $\cos^{-}\ 1$

d $\sin^{-}\left(\frac{1}{2}\right)$ **e** $\tan^{-}\ (-1)$ $\sin^{-}\ (-1)$

g $\cos^{-}\ 0$ **h** $\cos^{-}\left(\frac{1}{\sqrt{2}}\right)$ **i** $\tan^{-}\left(\frac{1}{\sqrt{3}}\right)$

j $\tan^{-}\left(-\sqrt{3}\right)$ **k** $\sin^{-}\left(\frac{1}{\sqrt{2}}\right)$ $\cos^{-}\left(-\frac{\sqrt{3}}{2}\right)$

2 Evaluate

a $\tan\ [\cos^{-}\ 1]$ **b** $\cos\ [\cos^{-}\ (-1)]$ **c** $\cos^{-}\ (\sin \pi)$

d $\cos\left[\cos^{-}\left(\frac{1}{\sqrt{2}}\right)\right]$ **e** $\sin\ [\tan^{-}\ 1]$ $\tan\left[\cos^{-}\left(\frac{1}{2}\right)\right]$

g $\sin^{-}\ (\tan 0)$ **h** $\tan\left[\sin^{-}\left(-\frac{\sqrt{3}}{2}\right)\right]$ **i** $\cos\left[\tan^{-}\left(-\frac{1}{\sqrt{3}}\right)\right]$

j $\tan\ [\tan^{-}\ (-1)]$

3 Evaluate correct to 2 decimal place:

a $\sin^{-}\ 04$ **b** $\tan^{-}\ 1.72$ **c** $\cos^{-}\ 0569$

d $\sin^{-}\ -06$ **e** $\tan^{-}\ (-37)$

4 Evaluate correct to 2 decimal place:

a $\sin^{-}\ (\sin 067)$ **b** $\tan^{-}\ [\tan (-014)]$ **c** $\cos^{-}\ (\cos 164)$

d $\sin\ [\cos^{-}\ 026]$ **e** $\tan\ [\sin^{-}\ (-067)]$

5 Find exact values fo:

a $\cos^{-}\ (-1) + \cos^{-}\ 1$ **b** $\sin^{-}\ 1 + \sin^{-}\ (-1)$

c $\tan^{-}\ 1 + \tan^{-}\ (-1)$ **d** $\sin^{-}\left(\frac{1}{2}\right) + \cos^{-}\left(\frac{1}{2}\right)$

e $\cos^{-}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-}\left(\frac{\sqrt{3}}{2}\right)$ **f** $\sin^{-}\left(\frac{1}{\sqrt{2}}\right) + \cos^{-}\left(\frac{1}{\sqrt{2}}\right)$

6 Find exact values fo:

a $\sin\left[\sin^{-}\left(\frac{4}{5}\right)\right]$ **b** $\cos\left[\sin^{-}\left(\frac{4}{5}\right)\right]$

c $\tan\left[\cos^{-}\left(\frac{12}{13}\right)\right]$ **d** $\sin\left[\tan^{-}\left(\frac{3}{7}\right)\right]$

e $\cos^{-}\left[\sin\left(\frac{\pi}{4}\right)\right]$ **f** $\tan^{-}\ (\cos \pi)$

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7 Show tha:

a $\tan^{-}(-1) = -\tan^{-} 1$

b $\sin^{-}(-1) = -\sin^{-} 1$

c $\tan^{-}(-3) = -\tan^{-} 3$

d $\cos^{-}\left(-\frac{1}{2}\right) = \pi - \cos^{-}\left(\frac{1}{2}\right)$

e $\sin^{-}\left(-\frac{1}{\sqrt{2}}\right) = -\sin^{-}\left(\frac{1}{\sqrt{2}}\right)$

8 a Sketch the graphs of $y = \sin^{-} x$ and $y = \cos^{-} x$ on the same set of axes

b On the same set of axe, sketch the graph of $y = \sin^{-} x + \cos^{-} x$ to show graphically that $\sin^{-} x + \cos^{-} x = \frac{\pi}{2}$

9 Prove tha:

a $\sin^{-}\left(\frac{3}{7}\right) + \cos^{-}\left(\frac{3}{7}\right) = \frac{\pi}{2}$

b $\sin^{-}\left(-\frac{5}{9}\right) = -\sin^{-}\left(\frac{5}{9}\right)$

c $\cos^{-}\left(-\frac{2}{5}\right) = \pi - \cos^{-}\left(\frac{2}{5}\right)$

d $\tan^{-}\left(-\frac{7}{10}\right) = -\tan^{-}\left(\frac{7}{10}\right)$

10 a Show that $f(x) = \sin^{-} x$ and $f(x) = \tan^{-} x$ are odd functions

b Show that $f(x) = \cos^{-} x$ is neither even nor odd

11 Test whether each statement is tre.

a $\cos^{-}\left[\cos\left(\frac{\pi}{2}\right)\right] = \frac{\pi}{2}$

b $\sin[\sin^{-}(-1)] = -1$

c $\tan^{-}(\tan \pi) = \pi$

d $\cos\left[\arccos\left(\frac{1}{2}\right)\right] = \left(\frac{1}{2}\right)$

e $\tan[\tan^{-1} 12] = 12$

f arctan — $= \frac{2\pi}{3}$

12 a Show that $y = -2\tan^{-1} 7x$ is an odd function

b Sketch the graph of $y = -2\tan^{-1} 7x$

13 Show that $\cos(\tan^{-1} x) = \sqrt{}$

14 Find the conditions for whic:

a $\sin^{-}(\sin a)$ is defined

b $\sin(\sin^{-} b) = b$

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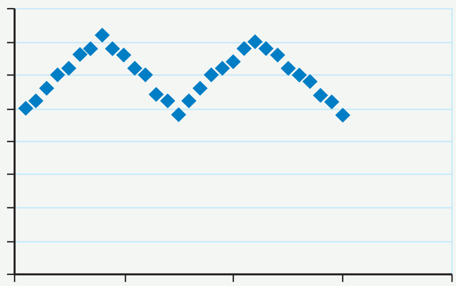

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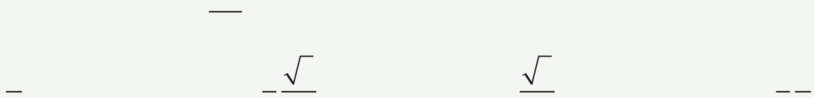

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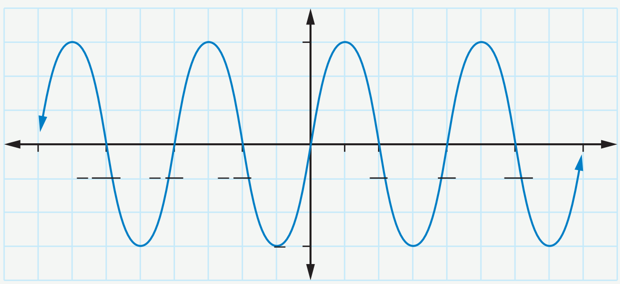

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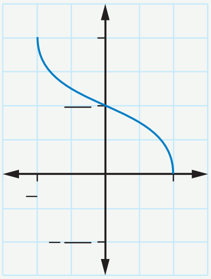

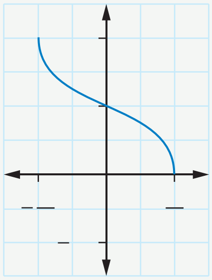

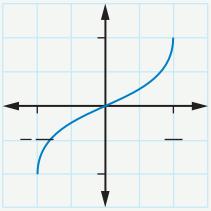

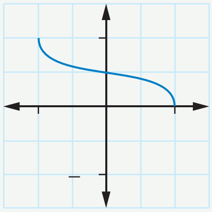

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TERMINOLOGY

discrete random variable A random variable that can take on a number of discrete values for example the number of children in a family

expected value Average or mean value of a probability distribution

population The whole data set from which a sample can be taken

probability distribution A function that sets out all possible values of a random variable together with their probabilities

random variable A variable whose values are based on a chance experiment for example the number of road accidents in an hour

standard deviatio: A measure of the spread of values from the mean of a distribution the square root of the variance

uniform probability distribution A probability distribution in which every outcome has the same probability

variance A measure of the spread of values from the mean of a distribution the square of the standard deviation

12.01 Random variables

We studied probability in Chapter . Now we will look at probability distributions which use random variables to predict and model random situations in areas such as science economics and medicine

A random variable is a variable that can take on different values depending on the outcome of a random process such as an experimen. Random variables can be discrete or continuous Discrete variables such as goals scored or number of children take on specific finite values while continuous variables such as length or temperature are measured along a continuous scale

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In this chapter we will look at discrete random variables We will look at continuous random variables in Year 2.

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We use a capital letter such as X for a random variable and a lower-case letter such as x for the values of X

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Exercise 12.01 Random variables

1 For each random variabl, state whether it is discrete or continuos:

a A film criti’s rating of a fim, from 0 to 4 stars

b The speed of a car

c The sum rolled on a pair of dice

d The winning ticket number drawn from a raffle

e The weight of parcels at a post office

The size of jeans in a shop

g The temperature of a metal as it cools

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h The amount of water in different types of fruit drink

i The number of cars passing the school over a 10-minute period

j The number of cities in each country in Europe

k The number of heads when tossing a coin 50 times

The number of correct answers in a 10-question tes.

2 Write the set of possible values for each discrete random variabe:

a Number of daughters in a one-child family

b Number of 6s on 10 rolls of a die

c Number of people aged over 50 in a group of 20 people

d The number of days it rains in March

e The sum of the 2 numbers rolled on a pair of dic.

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12.02 Discrete probability distributions

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A discrete probability distribution can be displayed in a table or graphor represented by an equation or set of ordered pairsIt is also called a **discrete probability function**

We can write a probability function that uses X as the random variable as $P(X = x)$ or $p(x)$.

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Remember that all probabilities lie between 0 and 1 and their total is 1These same rules apply to a probability distribution

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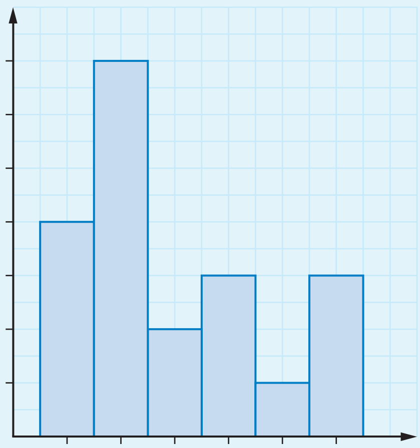

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In a **uniform probability distribution** the probability for each value is the sam.

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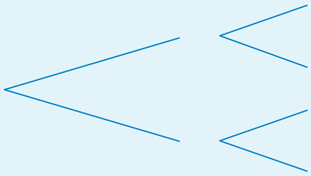

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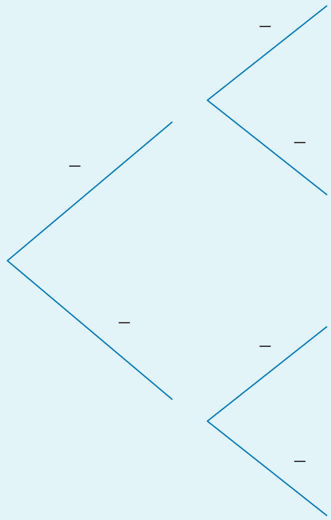

Exercise 12.02 Discrete probability distributions

1 Draw a probability distribution table for the sum of the numbers rolled on 2 dic.

2 Write the probability distribution as a set of ordered pairs ($x \;\; P(X = x)$) for the number of heads when tossing

a 1 coin **b** 2 coins **c** 3 coins.

3 A survey of a sample of bags of 50 jelly beans found that they did't all hold exactly 0. The table shows the results of the study.

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a Draw a probability distribution table for the result.

b If a bag of jelly beans is chosen at rando, find the probability that the bag contais:

i at least 50 jelly beans

ii fewer than 51 jelly beans

4 A function is given by $p(x) = \frac{x-2}{6}$ for $x = 3, 4, 5$.

a Show that the function is a probability distributio.

b Draw up a probability distribution tabl.

c Find:

i $P(X > 3)$ **ii** $P(X = \text{odd})$ **iii** $P(3 \le X < 5)$

5 Draw a histogram to show this discrete probability distributio.

6 **a** Draw a probability distribution table for rolling a di.

b Is this a uniform distribution?

c Find:

i $P(X \ge 4)$ **ii** $P(X < 3)$ **iii** $P(1 < X \le 4)$

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7 For each functio, state whether it is a probability distributin:

a $\left(0\ \frac{1}{5}\right)$ $\left(1\ \frac{2}{5}\right)$, (2, 0), $\left(3\ \frac{2}{5}\right)$ $\left(4\ \frac{1}{5}\right)$

b

	—	–	–	—

c $p(x) = \frac{x+2}{4}$ for $x = 0, 1, 2$

8 Find k if each function is a probability distribution

a $p(x) = k(x + 1)$ for $x = 1, 2, 3, 4$

b

c $(1, k)$, $\left(\ \frac{1}{10}\right)$, $(3, 0)$, $\left(4\ \frac{1}{5}\right)$ — $\left(6\ \frac{2}{5}\right)$

9 The probability function for the random variable X is given by $p(x) = \frac{kx^2}{x+5}$ for $x = 1, 2, 3, 4$.

a Construct a probability distribution table for the functio.

b Find the value of k

10 In a gam, each player rolls 2 die. The game pays \$1 if one of the numbers isa 6, \$3 for double 6 and \$2 for any other doubleThere is no payout for other resuls.

a Draw a probability distribution table for the game payout Y

b Find the probability of winnin:

i \$3 **ii** at least \$2 **iii** less than \$.

11 Given the probability function belo, evaluate p

12 Simon plays a game where he selects a card at random from 100 cards numbered 1 to 100 He wins \$1 for selecting a number less than 2, \$2 for a number greater than 90 \$3 for any number from 61 and 69 (inclusive) and \$5 for any number from 41 to 50 (inclusive)

a Create a probability distribution table for the random variable X for the prize values

b Find the probability of winning more than \$.

c Find the probability of winning less than \$.

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13 The table below shows the probability function for random variable X

	—	–	—	—	—	–

Find

a $P(X = 6)$ **b** $P(X = \text{even})$ **c** $P(X > 8)$ **d** $P(X \leq 7)$

e $P(6 < X < 9)$ **f** $P(7 \leq X < 10)$ **g** $P(6 \leq X \leq 9)$

14 The table below shows the probability function for random variable Q

	—	—	–	–	—	—

Find

a $p(8)$ **b** $P(Q \geq 4)$ **c** $P(2 < Q \leq 6)$

d $P(4 \leq Q \leq 10)$ **e** $P(0 \leq Q < 4)$ $P(2 \leq Q \leq 8)$

15 A company makes washing machine. On averae, there are 3 faulty machines made for every 1000 machines Two washing machines are selected at random for a quality control inspection

a Draw a probability distribution table for the number of these machines that could be faulty.

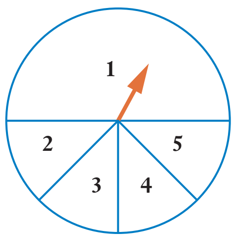
b Find the probability tha:

i one will be faulty **ii** at least one will be fault.

16 A bag contains 7 re, 6 white and 8 blue bals. Create a probability function table for the number of white balls selected when drawing 2 balls at random from the bag

a with replacement **b** without replacemen.

17 The spinner below has the numbers 1–5 distributed as shown



a What is the probability that the arrow points to the number 3 when it is spun?

b Is the probability distribution of the spun numbers uniform?

c Draw a table showing the probability distribution for spinning the number.

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18 The probability of a traffic light showing green as a car approaches it is 12. Draw a probability distribution table for the number of green traffic lights on approach when a car passes through 3 traffic lights

19 There is a 51% chance of giving birth to a bo If a family has 4 childen, construct a probability function to show the number of boys in the family.

20 A raffle has 2 prizes with 100 tickets sold altogethe Iris buys 5 tickts.

a Draw a probability distribution table to show the number of prizes Iris could win in the raffle

b Find the probability that Iris wins at least one priz.

12.03 Mean or expected value

Expeced value

The **expected value** $E(X)$ of a probability distribution measures the centre of the distribution It is the same as finding the **mean** or averag, which has symbol μ It is the expected value of the random variable

We use ¯ for the mean of a **sample** and μ for the mean of a **population** For probability distributions we use the population mea, μ The sample men, ¯ is an estimate of μ and as the sample size increases the sample represents the population better and the value of ¯ approaches μ

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MEAN

Find the mean in Example 9 using the formula $\bar{x} = \frac{\sum fx}{\sum f}$ Can you see why the sum of scores multiplied by relative frequencies also gives this mean?

The symbol Σ means ‘the sum o’. It is the Greek capital lettr ‘sma’.

$\Sigma xp(x)$ is the sum of the products of x times $p(x)$

Proof

$$\mu = \frac{\sum fx}{\sum f}$$

$$= \frac{\sum fx}{n} \text{ where } n \text{ is the sum of frequencies}$$

$$= \sum x \frac{f}{n}$$

$$= \sum xp(x)$$

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You can find the expected value on your calculator using the statistics mod, in the same way you would find the mean of data presented in a frequency table

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You can solve problems using expected value.

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Exercise 12.03 Mean or expected value

1 Find the expected value of each probability distributio.

a $\begin{pmatrix} 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & \frac{1}{4} \end{pmatrix}$

b

c

	–	—	–	–	—

d $p(x) = \dfrac{x+1}{8}$ for $x = 0, 1, 4$

e

$$p(x) = \begin{cases} \dfrac{x}{2} & \text{for } x = 1 \\ \dfrac{x}{8} & \text{for } x = 2 \\ \dfrac{x-3}{4} & \text{for } x = 4 \\ 0 & \text{for all other values of } x \end{cases}$$

2 For each questio:

i evaluate k **ii** find the mean of the probability distributio.

a $(1, k)$, – — –

b $p(x) = k(x + 3)$ for $x = 0, 1, 2$

c

3 Find the expected value of each probability distributio:

a The number of heads when tossing 2 coins

b The sum of the 2 numbers rolled on a pair of dice

c The number of girls in a 3-child family

d The number of faulty cars when testing 3 cars if 1 in every 1000 cars is faulty

e The number of red counters when 2 counters are selected at random from a bag containing 7 red and 12 white counters

i with replacement

ii without replacemen.

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4 The expected value $E(X) = 635$ for this probability function Find p and q

5 The mean of the probability distribution below is $3\frac{3}{4}$ Evaluate a and b

$(1, a)$, — , $(3, b)$, — —

6 A uniform discrete random variable X has values $x = 1, 2, 3, 4$.

a Draw up a probability distribution table for X

b Find $E(X)$

7 Find the expected number of heads when tossing 3 coin.

8 A bag contains 8 white and 3 yellow marble. If 3 marbles are selected at randm, find the mean number of white marbles

a with replacement

b without replacemen.

9 In a gam, 2 dice are rolled and the difference between the 2 numbers is calculatd. A player wins \$1 if the difference is 3 \$2 if it is 4 and \$3 if it is .

a Draw a probability distribution table for the winning value.

b Find the expected valu.

c It costs \$1 for a player to roll the dic. How much would the player be expected to win or lose?

10 Staff at a call centre must make at least 1 phone sale every hou The probability that Yasmin will make a sale on a phone call is $\frac{2}{5}$ She makes 4 phone calls in an hou.

a Draw a probability distribution for the number of sales Yasmin maes.

b Find the expected valu.

c Will Yasmin make at least one phone sale in an hour?

11 A game uses a spinner with the numbers 1 to 12 equally spread around i. A player wins \$3 for spinning a number greater than 10\$2 for a number less than 4 and loses \$1 for any other number. How much money would a player be expected to win or lose?

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12.04 Variance and standard deviation

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Variance and **standard deviation** measure the spread of data in a distribution by finding the difference of each value from the meanVariance is the square of the standard deviation

The variance σ^2 involves the average of the squared differences of each value from the mea.

σ is the Greek lower-case letter'sig'.

, σ

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Proof

$$\sigma^2 = \frac{\Sigma f(x-\mu)^2}{\Sigma f}$$

$$= \frac{\Sigma f(x-\mu)^2}{n} \quad \text{where } n = \Sigma f \text{ is the sum of frequencies}$$

$$= \Sigma(x-\mu)^2 \times \frac{f}{n}$$

$$= \Sigma(x-\mu)^2 p(x)$$

$$= E[(X-\mu)^2]$$

Standard deviation is the square root of variance

σ

$\sqrt{}$

$\sqrt{}$

We use s for the standard deviation of a **sample** and σ (the lower-case Greek letter sigma) for the standard deviation of a **population** For probability distribution, we use the population standard deviation σ The sample standard deviatin, s is an estimate of σ and as the sample size increases the sample represents the population better and the value of s approaches σ

The formula for variance is a little tedious since we subtract the mean from every value
There is a simpler formula for variance

dn

Proof

$$\sigma^2 = \sum (x - \mu)^2 p(x)$$

$= \sum [x^2 p(x) - 2\mu x p(x) + \mu^2 p(x)]$ expanding $(x - \mu)^2$

$= \sum x^2 p(x) - \sum 2\mu x p(x) + \sum \mu^2 p(x)$ taking separate sums of each part

$= \sum x^2 p(x) - 2\mu \sum x p(x) + \mu^2 \sum p(x)$ since μ is a constant

$= \sum x^2 p(x) - 2\mu \times \mu + \mu^2 \times 1$ since $\sum x p(x) = \mu$ and $\sum p(x) = 1$ is a constant

$= \sum [x^2 p(x)] - \mu^2$

$= E(X^2) - \mu^2$

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If we use the same probability distribution as Example 13we can see that this formula gives us the same result

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You can use a calculator to work out the variance and standard deviatio.

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Exercise 12.04 Variance and standard deviation

In this exercise round answers to 2 decimal places where necessar

1 For each probability distributio, fid:

i the standard deviation **ii** the varianc.

a

b

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c $\left(1\ \frac{3}{8}\right)\left(2\ \frac{1}{4}\right)\left(3\ \frac{1}{8}\right)\left(4\ \frac{1}{16}\right)\left(5\ \frac{3}{16}\right)$

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2 Find the mea, variance and standard deviation of each probability functin.

a

b $P(x) = \frac{x+1}{9}$ for $x = 0, 2, 4$

3 Evaluate n and find the expected value and variance for this probability distribution

	–	—	—	—	

4 For the probability distribution below with $E(X) = 332$, fin:

a the values of a and b

b the variance

c the standard deviatio.

5 For each probability distribution fin:

i the expected value

ii the standard deviation

iii the varianc.

a The number of tails when tossing 3 coin.

b The number of blue marbles when 2 marbles are selected randomly from a bag containing 10 blue and 12 white marbles

6 A uniform discrete random variable X has values $x = 1, 2, 3, 4, 5$.Find:

a the mean

b the standard deviation

c the varianc.

7 **a** Create a probability distribution table for the number of 6s rolled on a pair of dic.

b Find the mea, variance and standard deviation of this functin.

8 The probability of selecting a black jelly bean at random from a packet is 4.
If 2 jelly beans are selected at random fin:

a the expected number of black jelly beans

b the standard deviation

c the varianc.

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9 A set of cards contains 5 blue and 7 white card. If 3 are drawn out at randm, the discrete random variable X is the number of blue cards drawn outFind the mean and variance of X if the cards are drawn out

a with replacement

b without replacement

10 In a gam, 2 cards are drawn from a deck of 52 standard playing cars. A player wins 5 points if one of the cards is an ace and 10 points for double aces

a If random variable X is the number of aces drawn

i create a probability distribution for X

ii find the mea, variance and standard deviation for this distributin.

b If random variable Y is the number of points won

i create a probability distribution for Y

ii find the mea, variance and standard deviation for this distributin.

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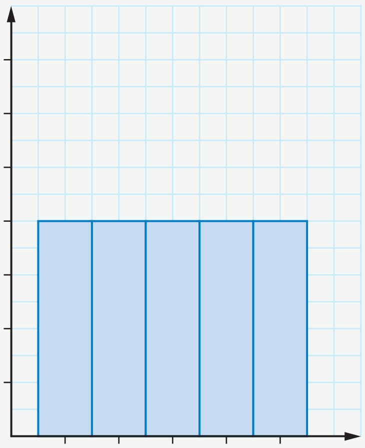

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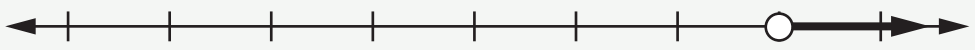

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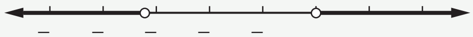

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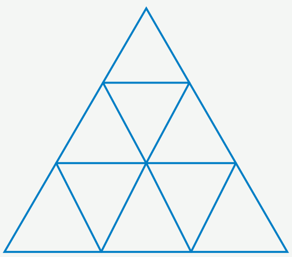

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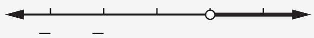

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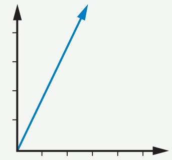

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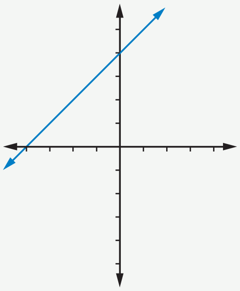

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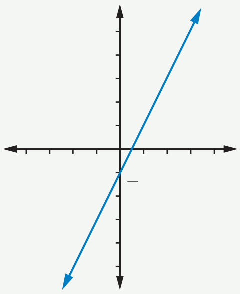

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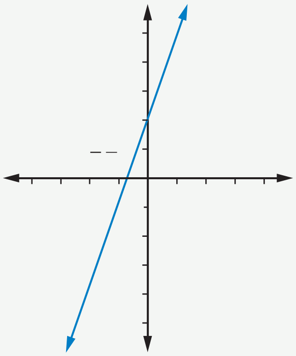

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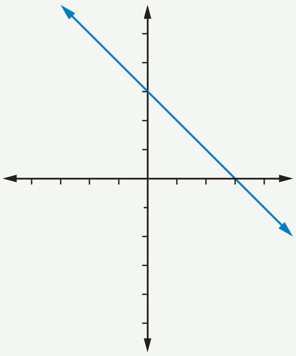

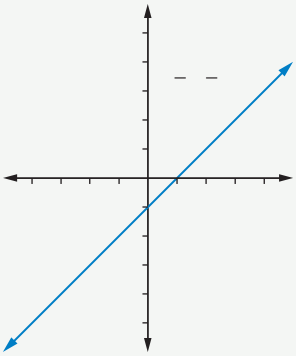

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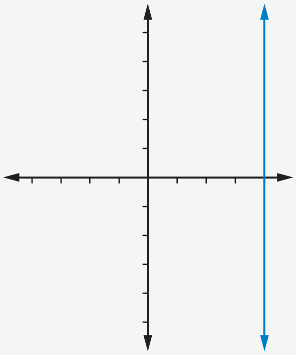

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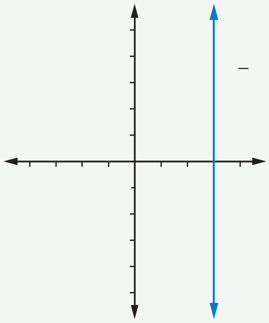

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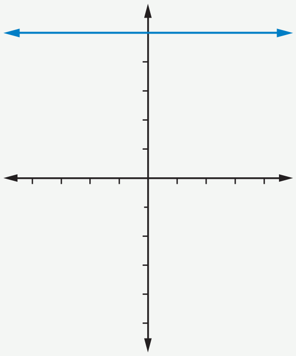

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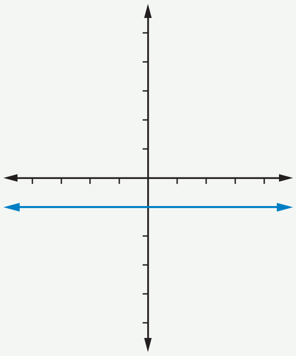

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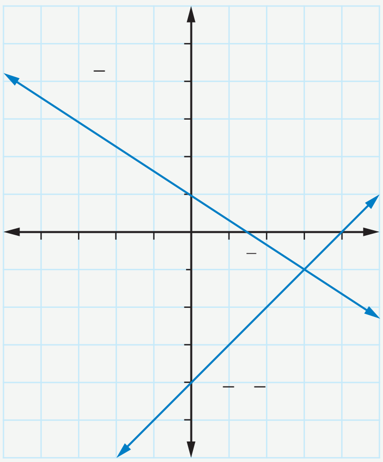

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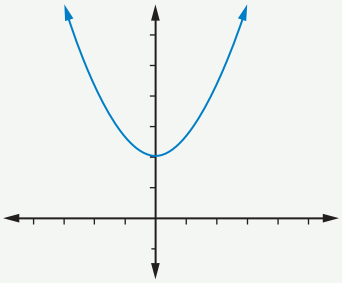

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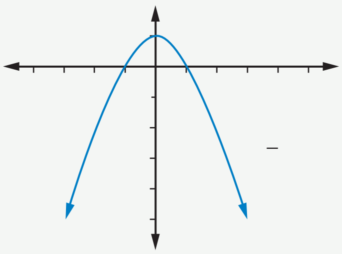

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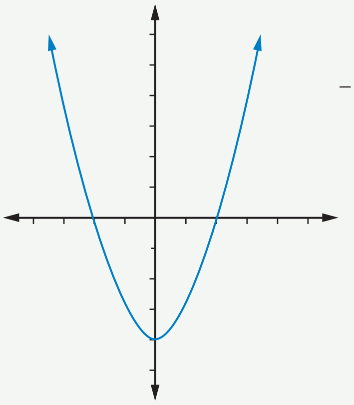

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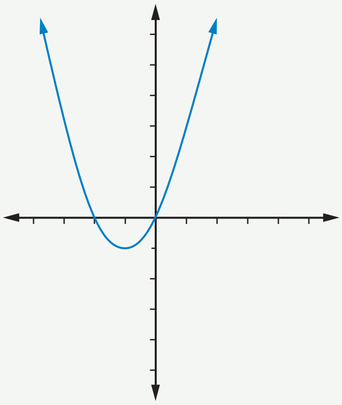

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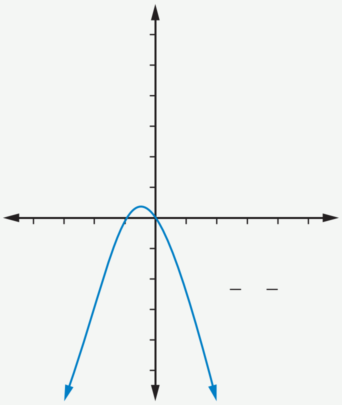

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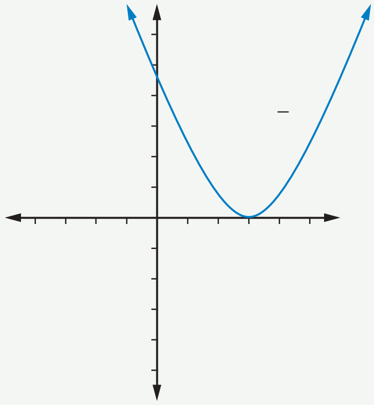

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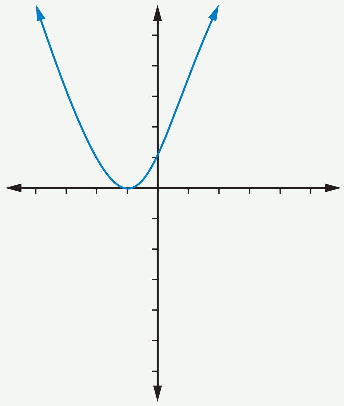

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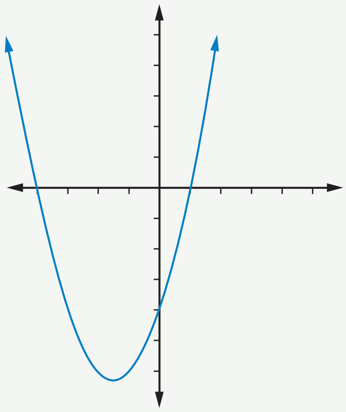

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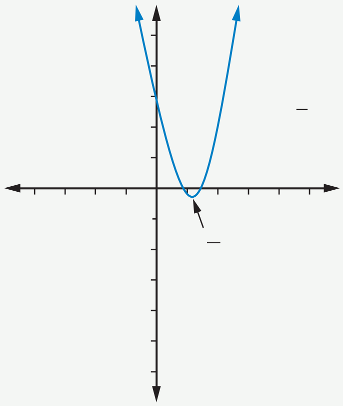

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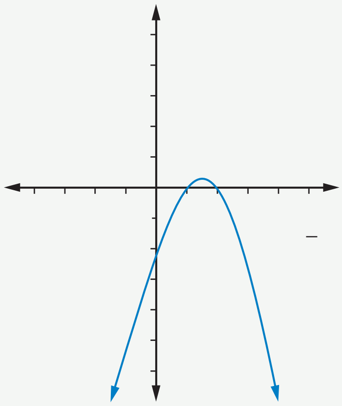

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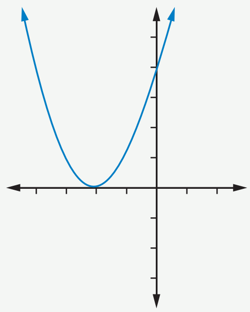

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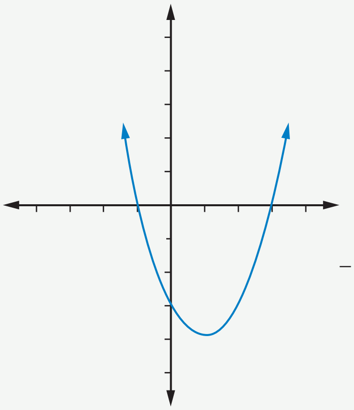

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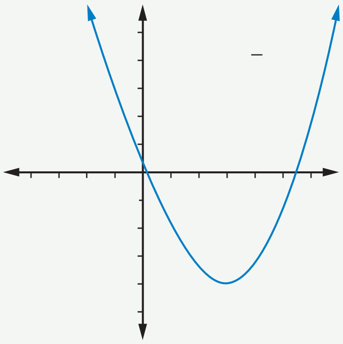

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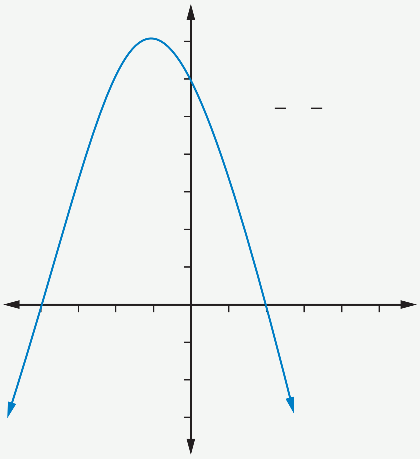

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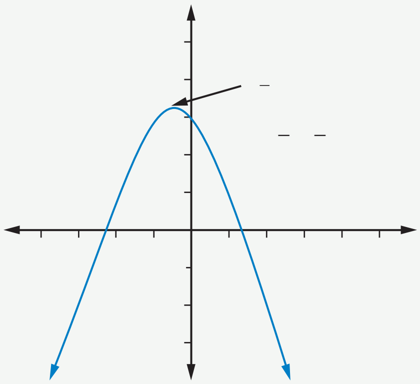

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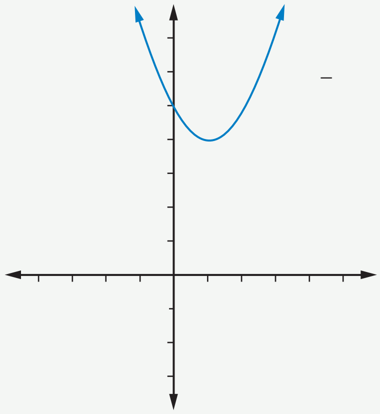

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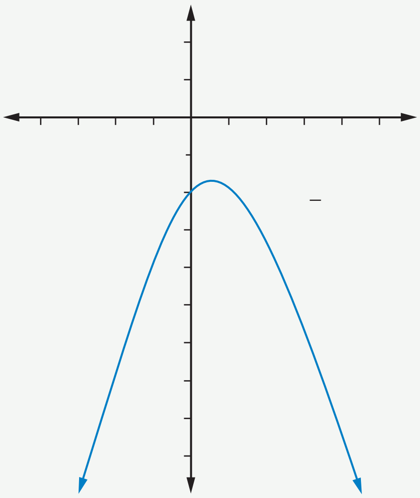

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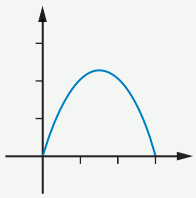

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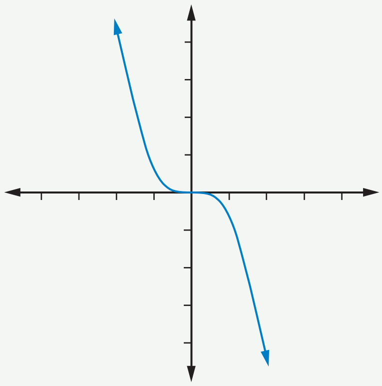

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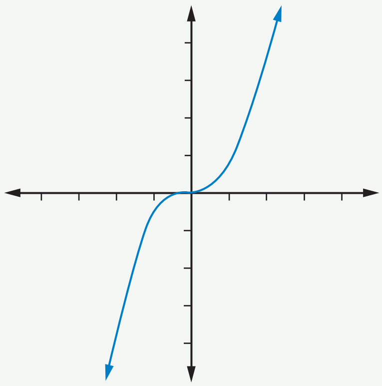

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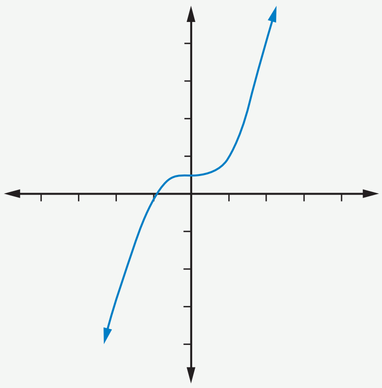

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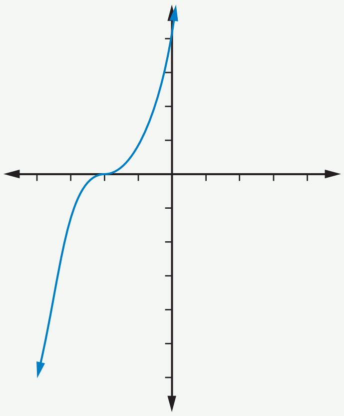

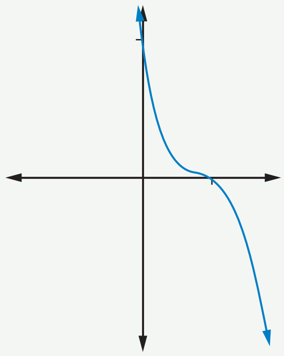

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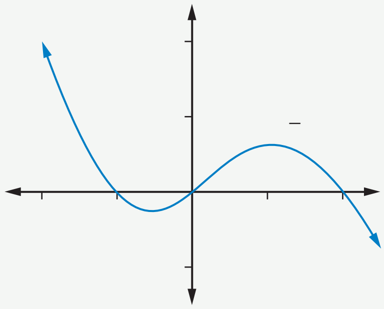

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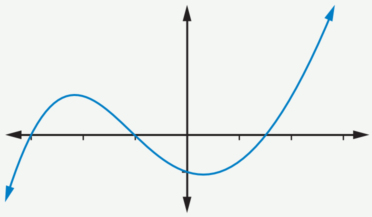

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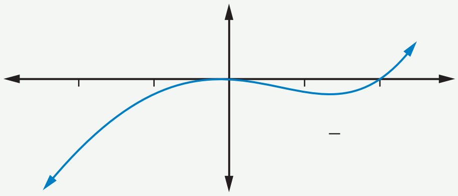

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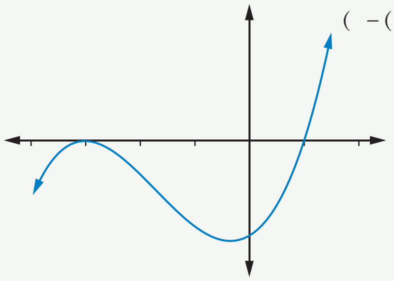

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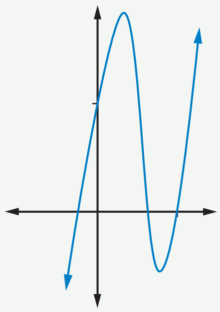

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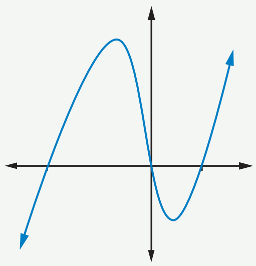

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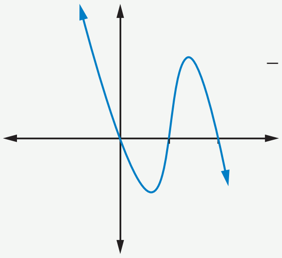

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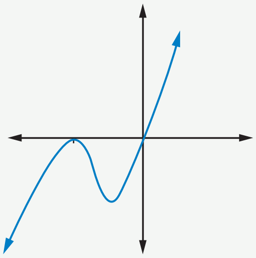

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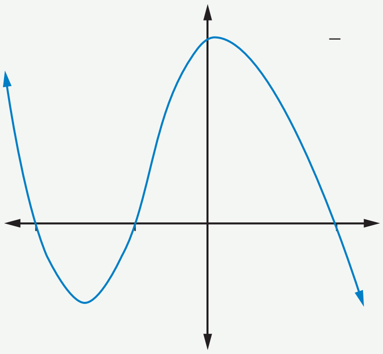

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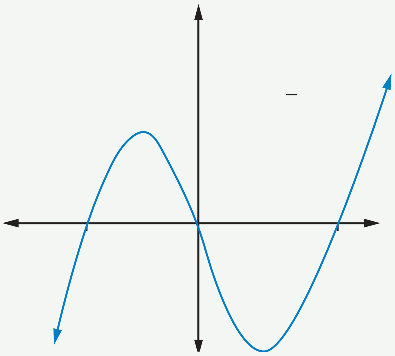

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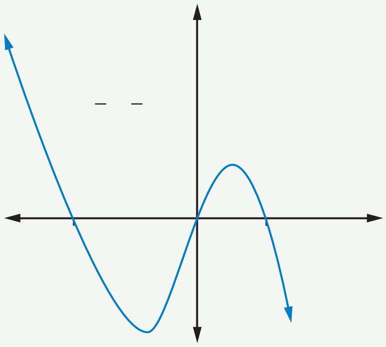

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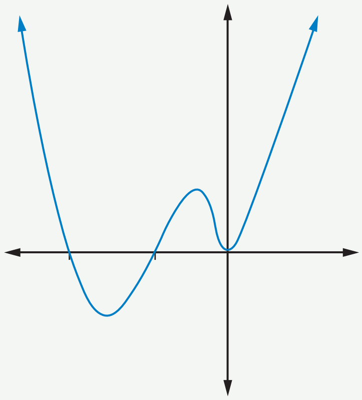

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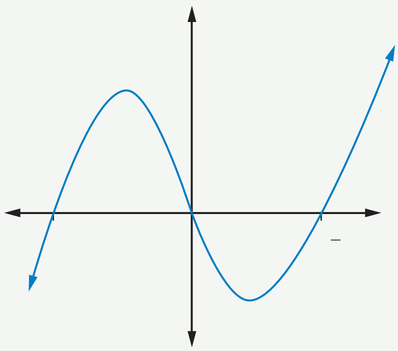

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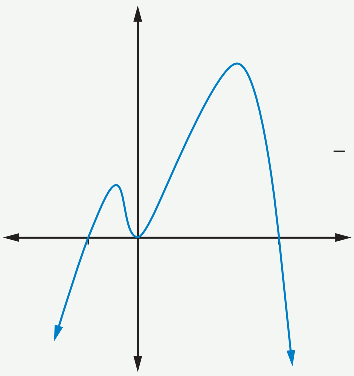

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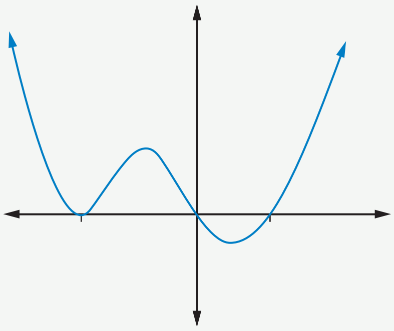

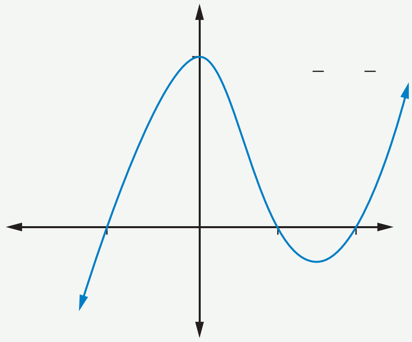

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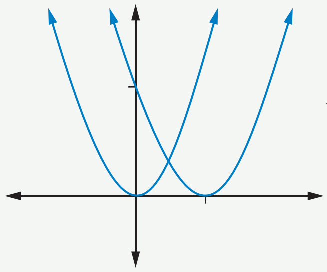

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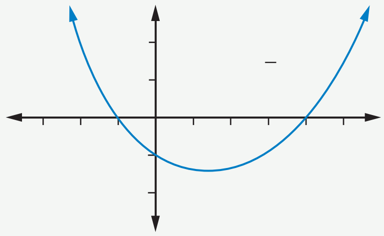

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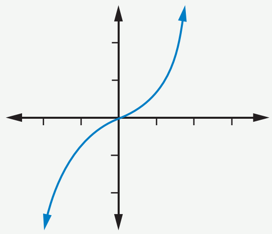

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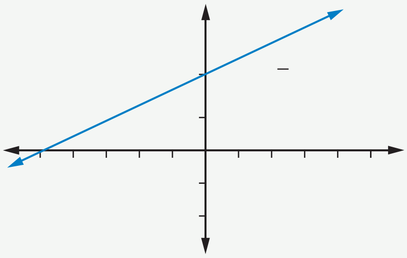

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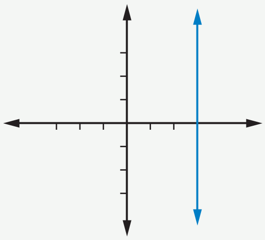

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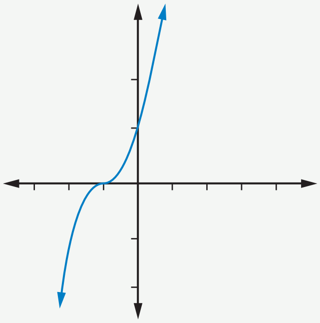

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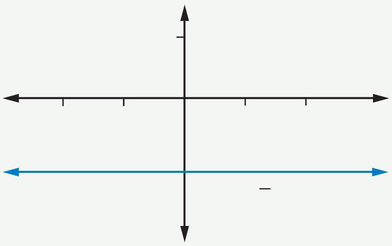

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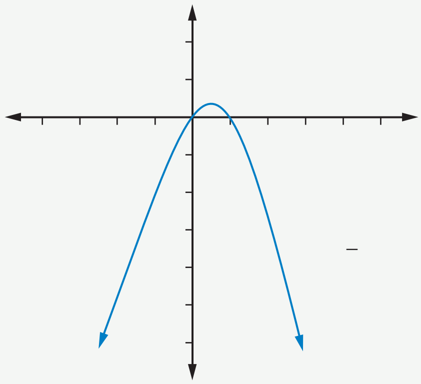

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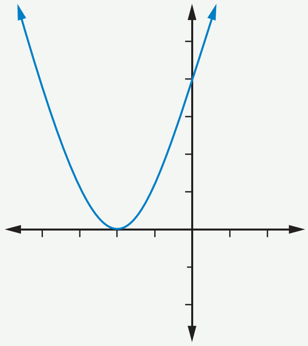

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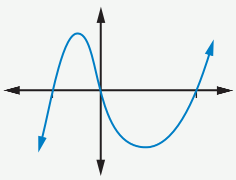

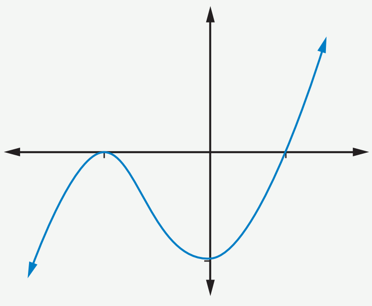

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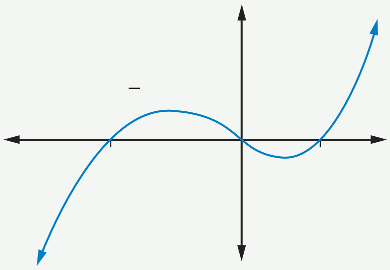

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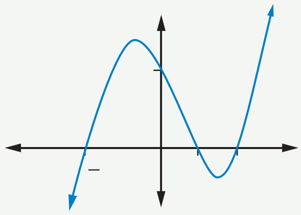

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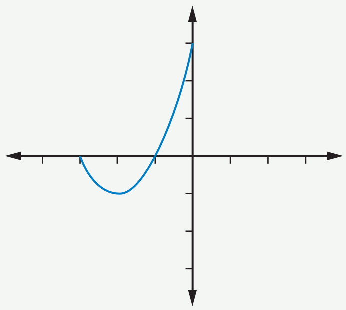

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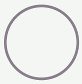

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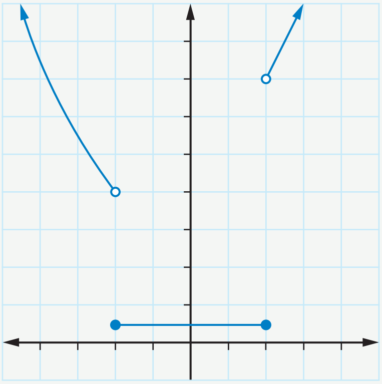

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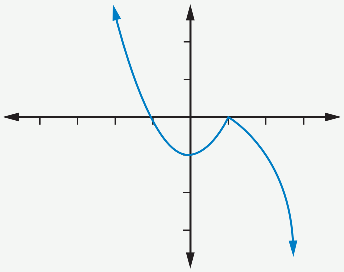

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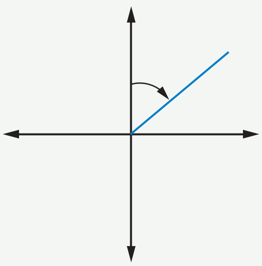

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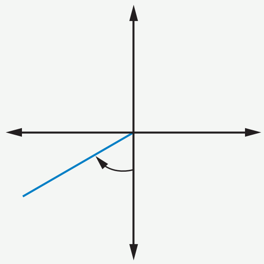

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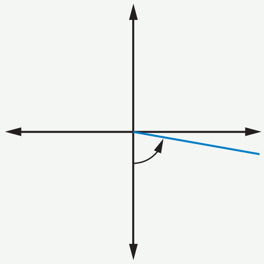

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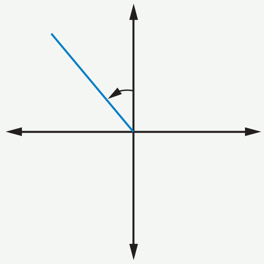

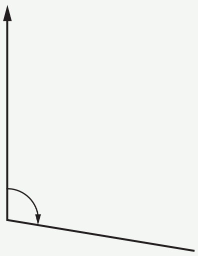

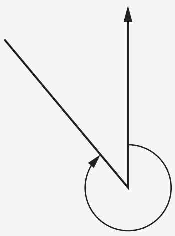

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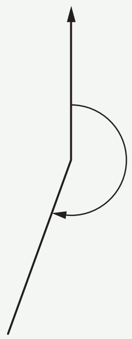

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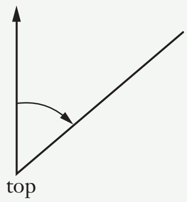

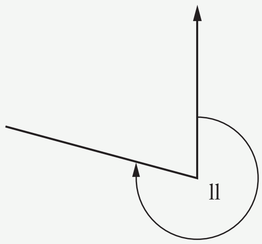

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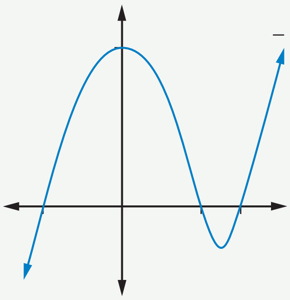

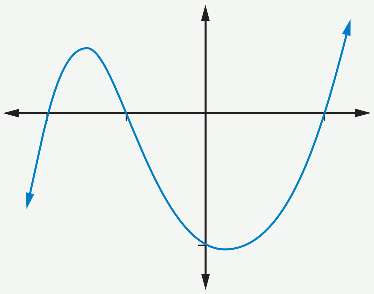

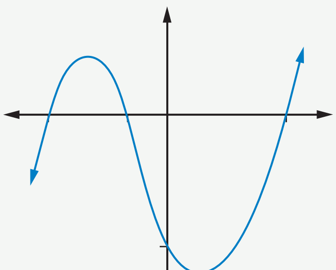

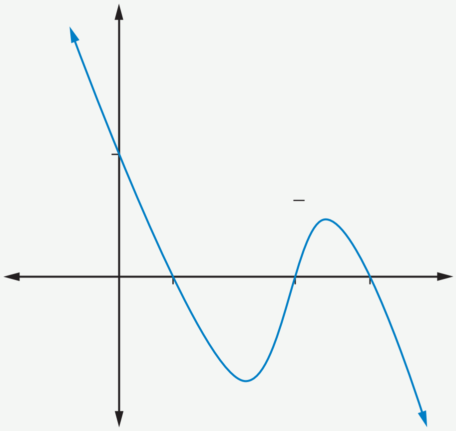

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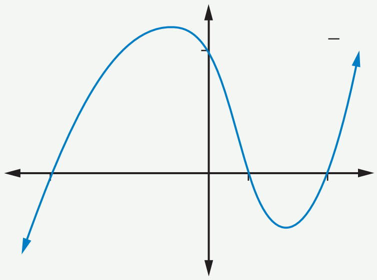

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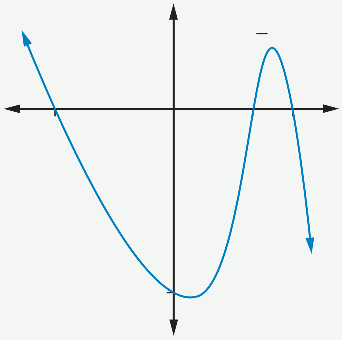

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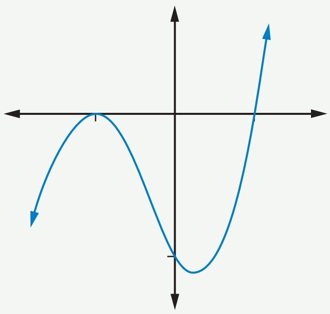

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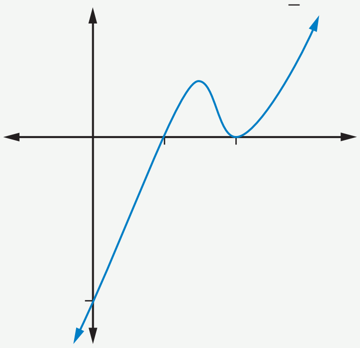

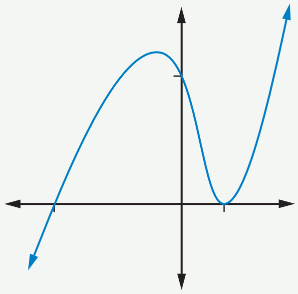

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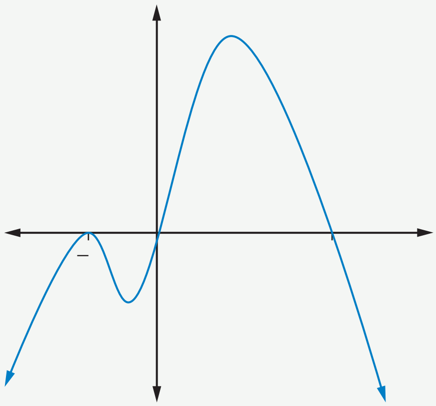

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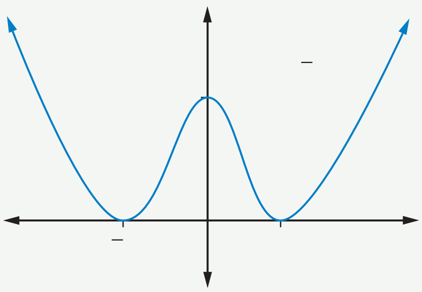

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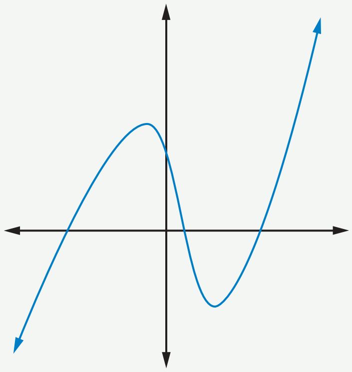

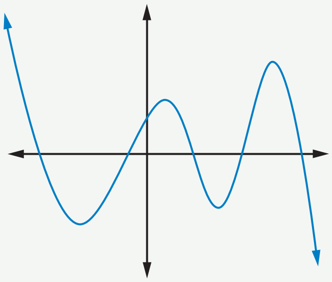

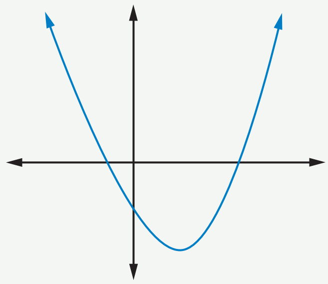

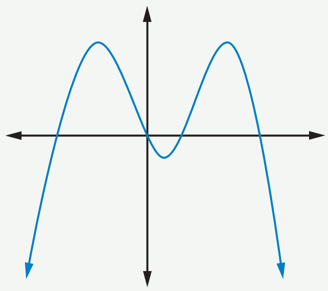

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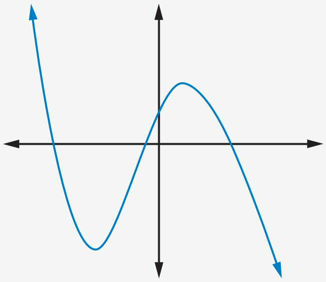

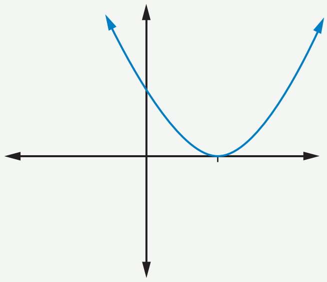

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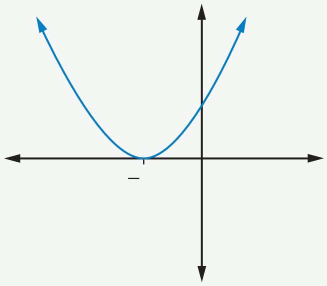

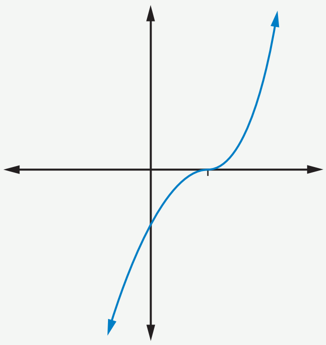

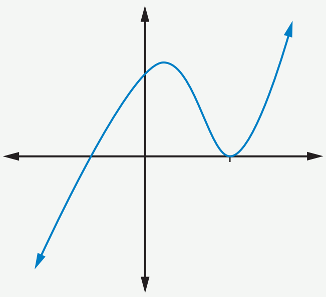

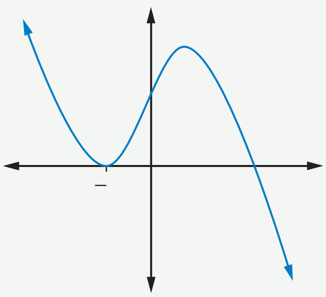

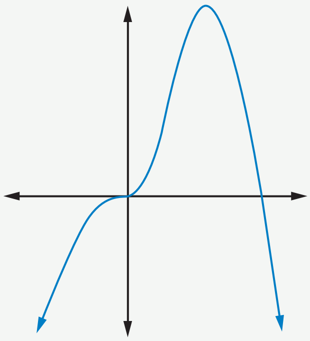

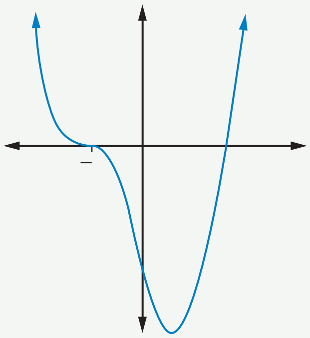

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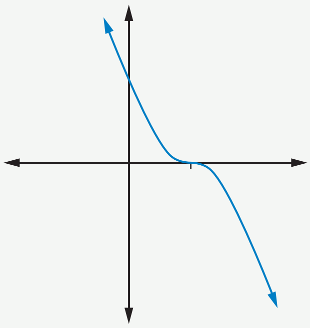

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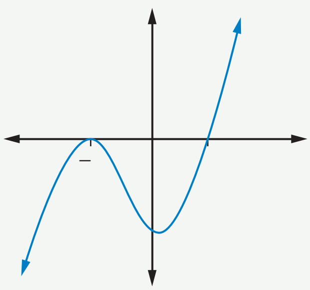

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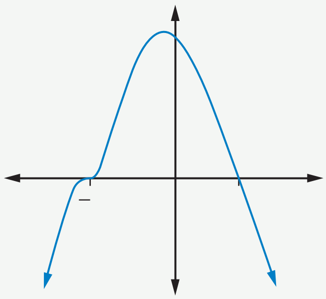

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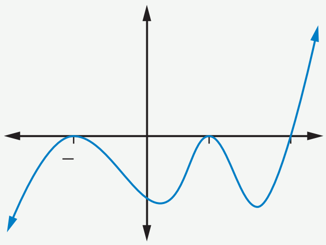

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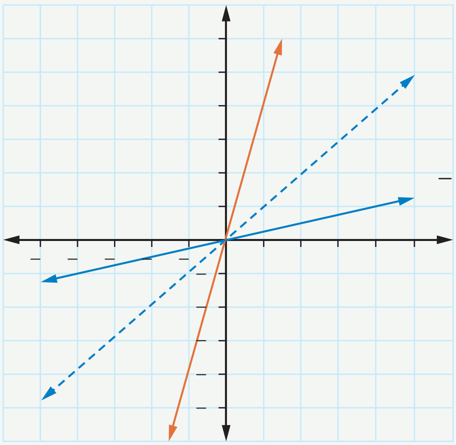

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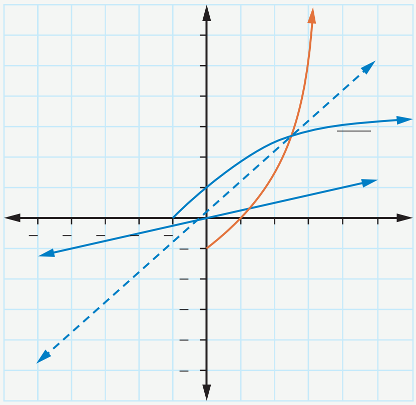

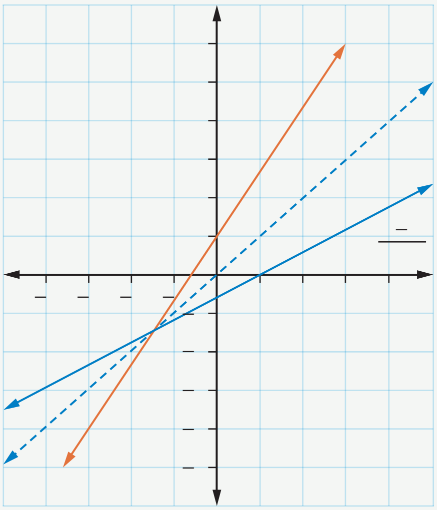

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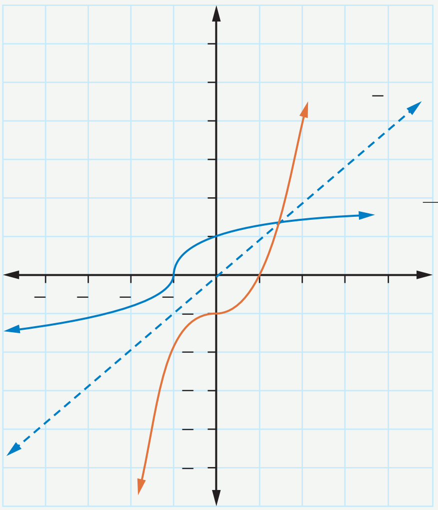

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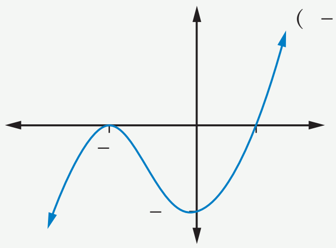

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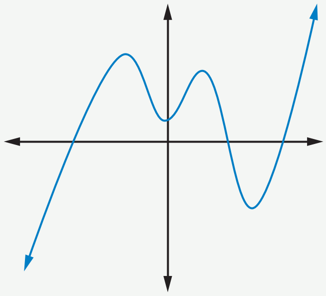

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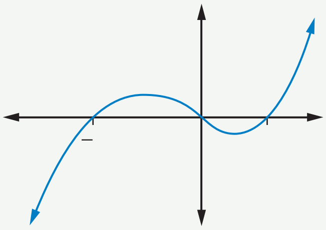

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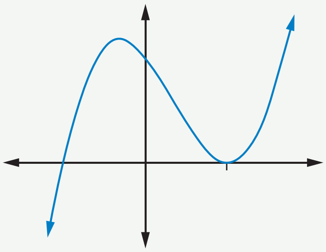

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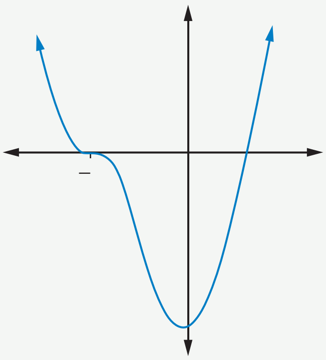

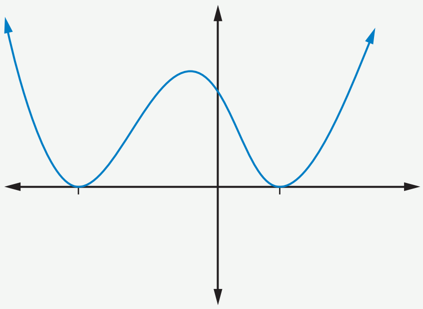

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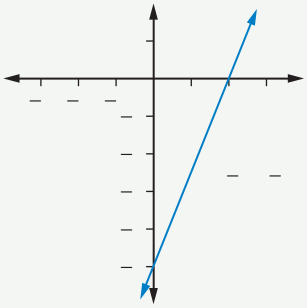

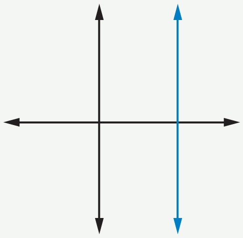

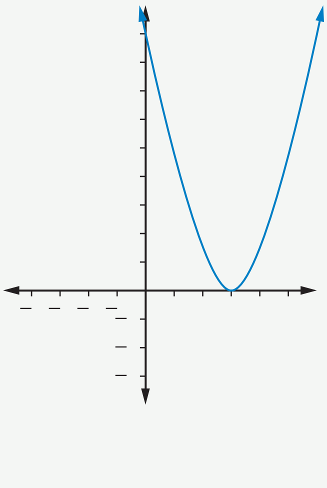

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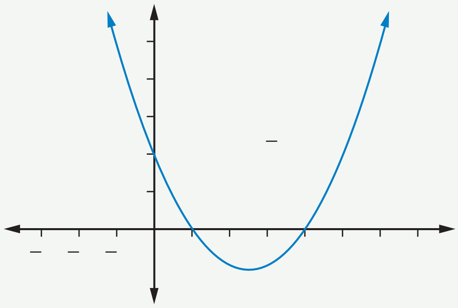

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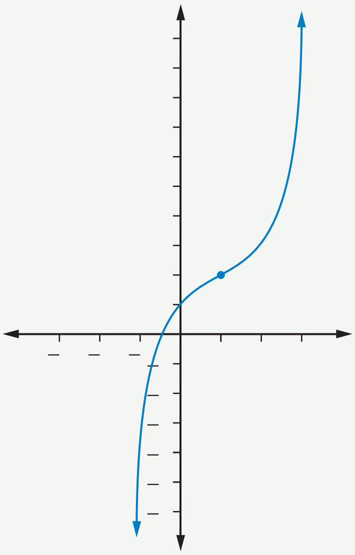

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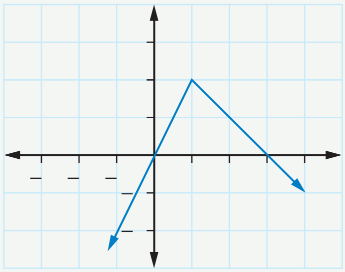

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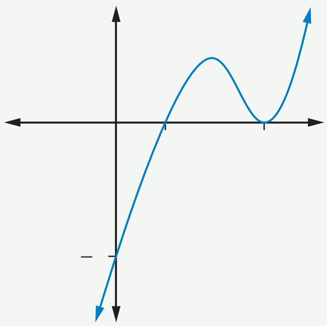

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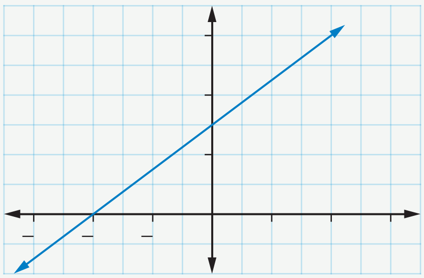

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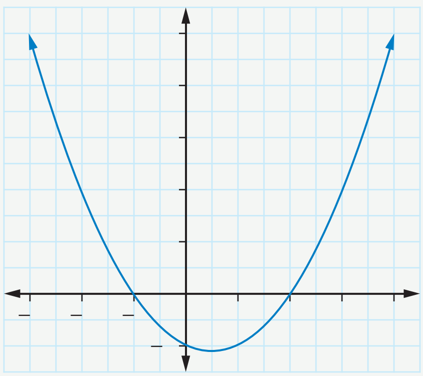

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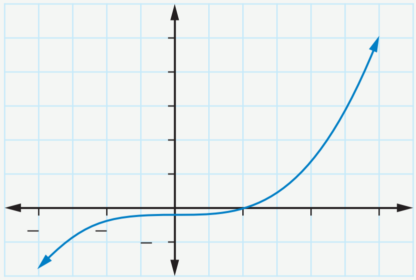

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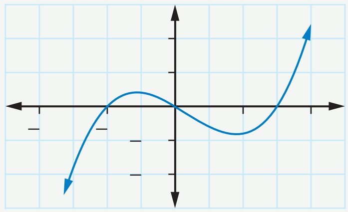

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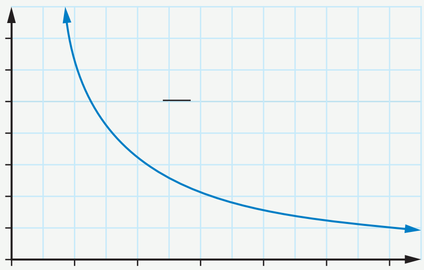

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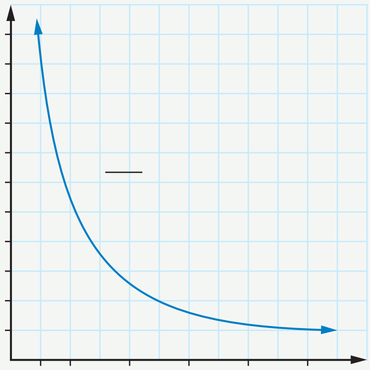

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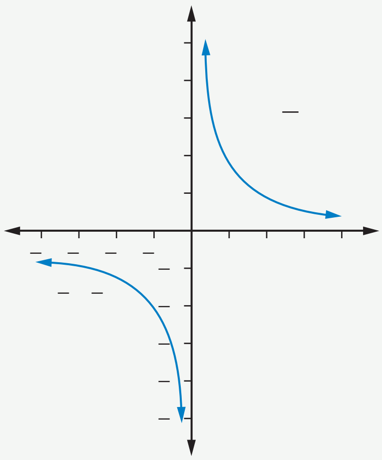

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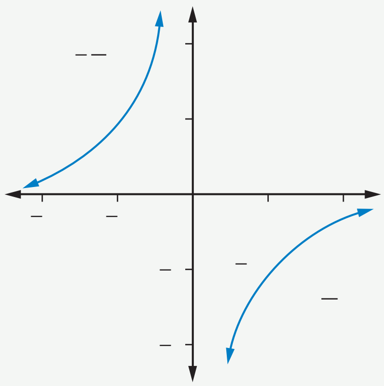

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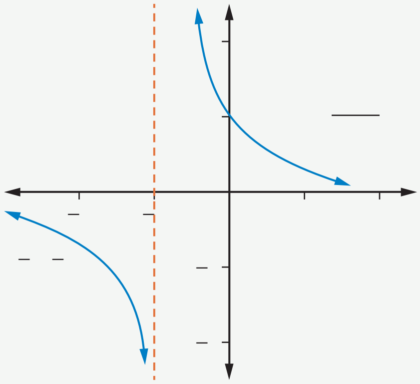

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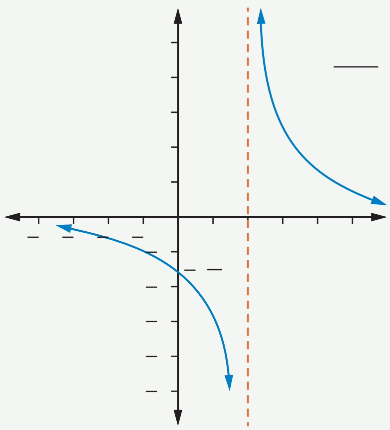

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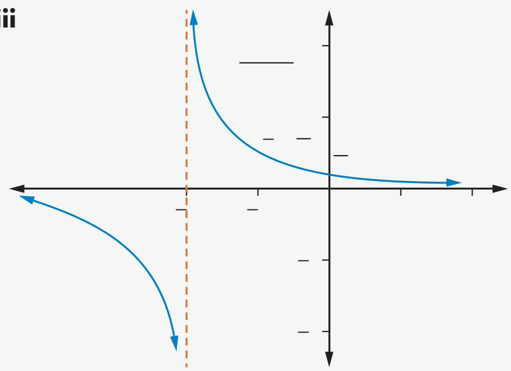

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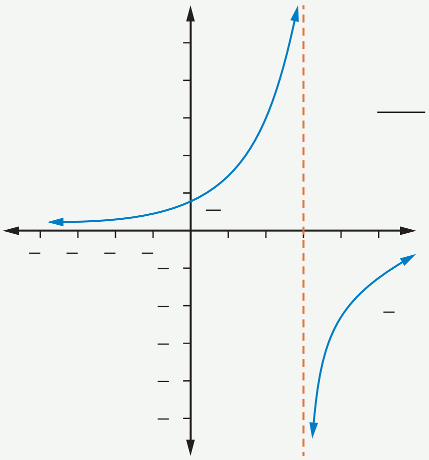

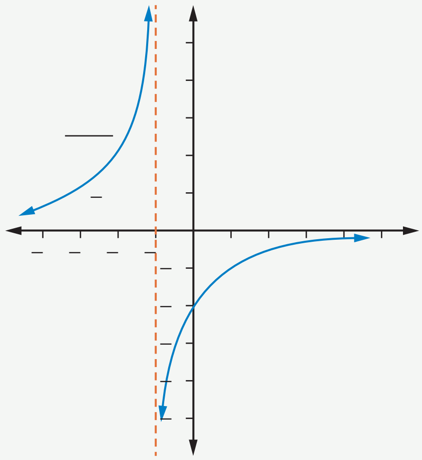

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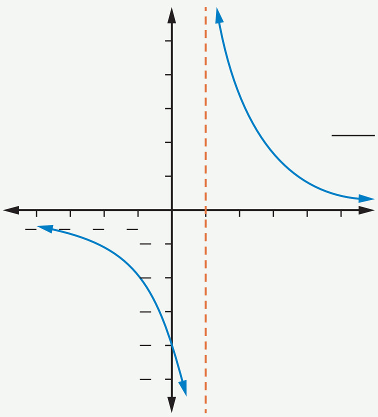

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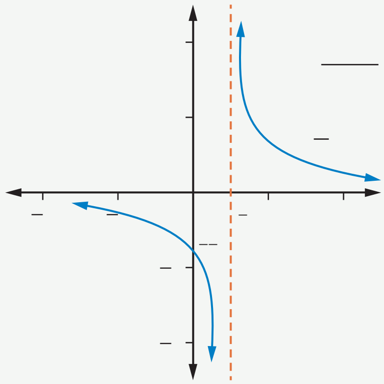

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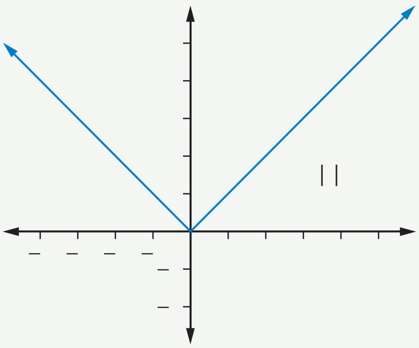

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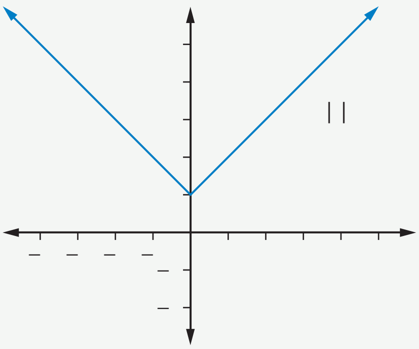

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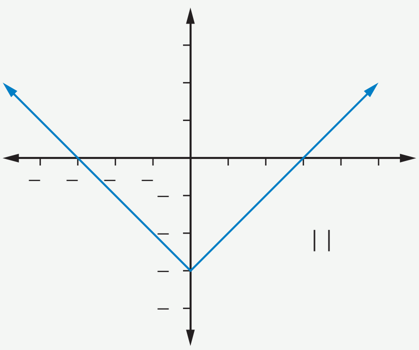

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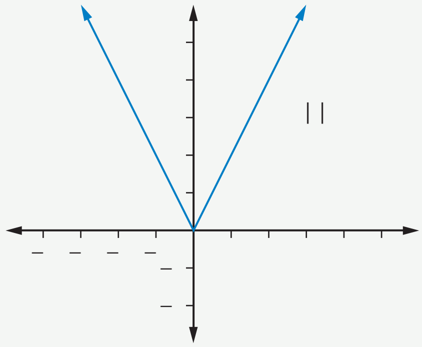

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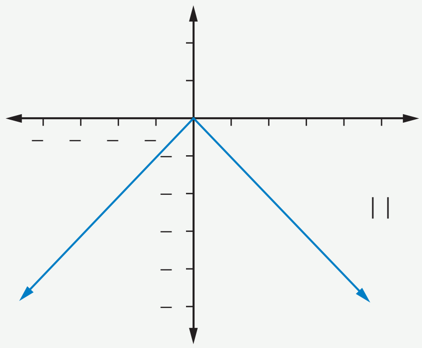

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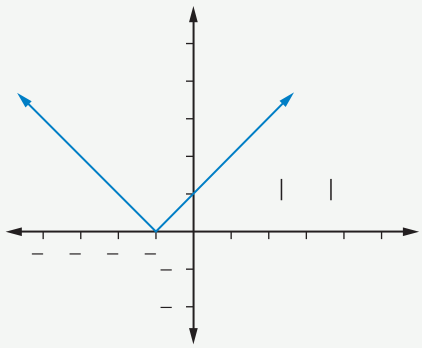

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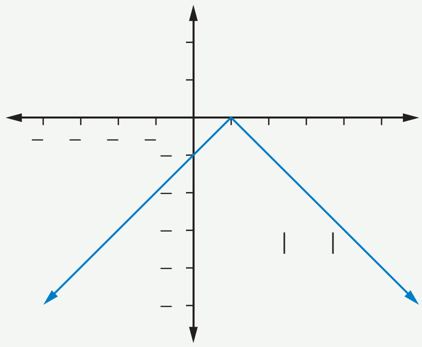

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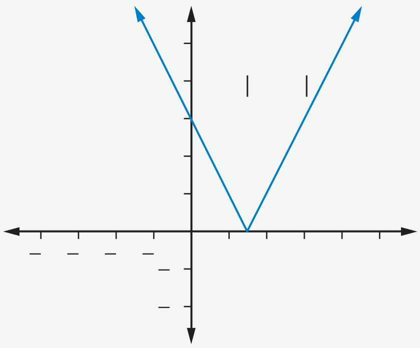

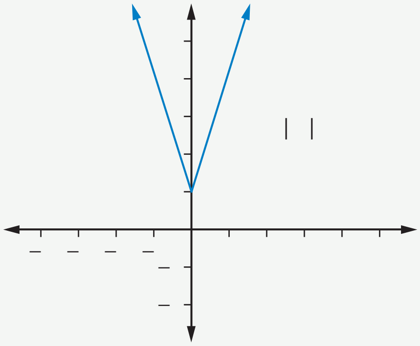

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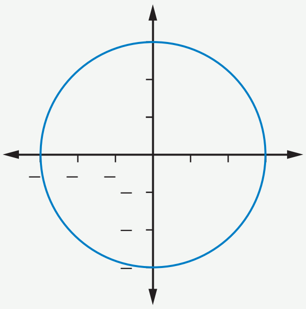

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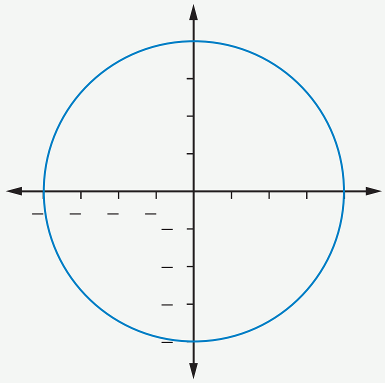

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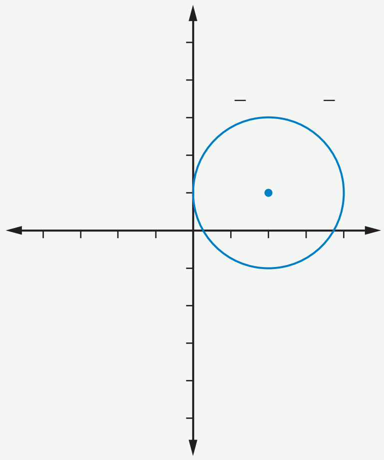

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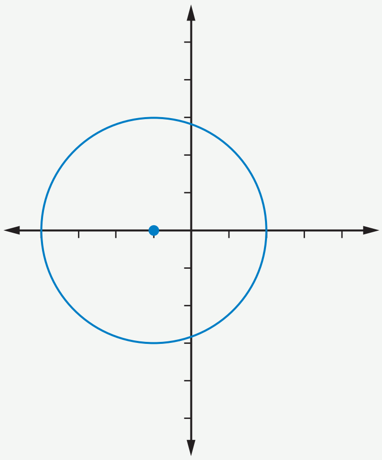

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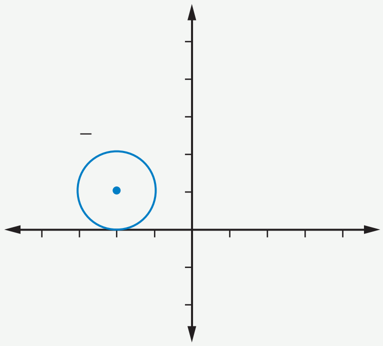

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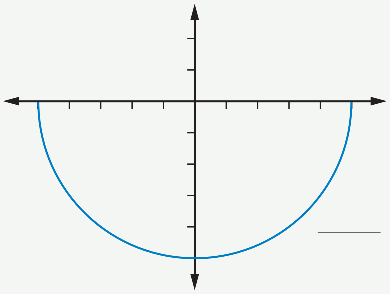

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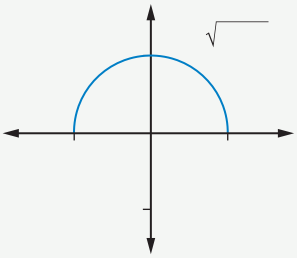

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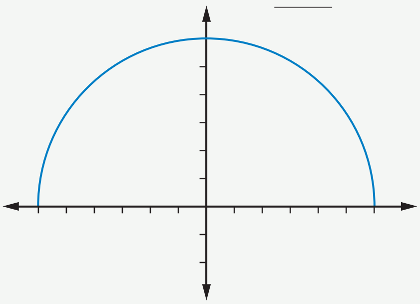

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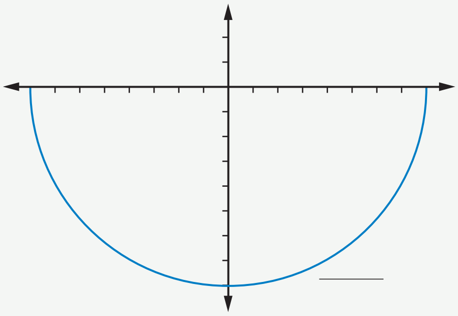

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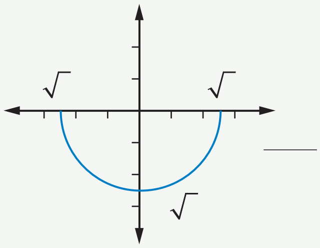

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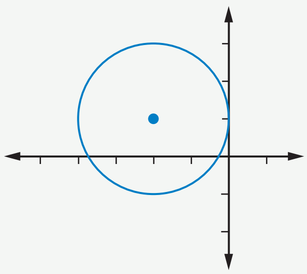

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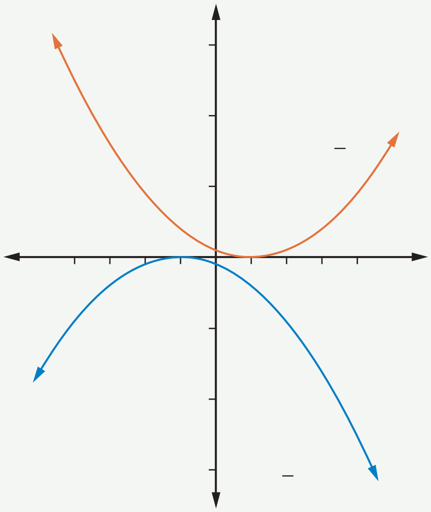

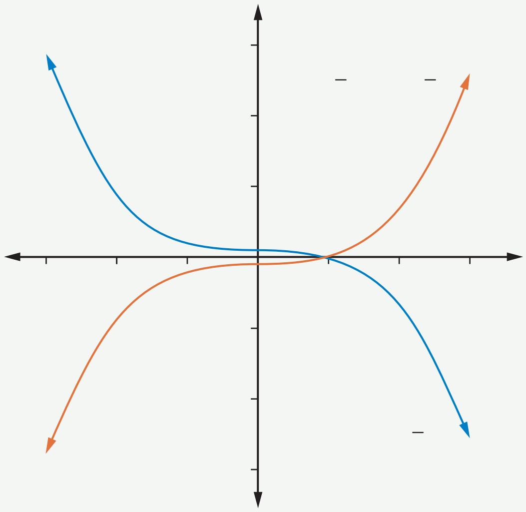

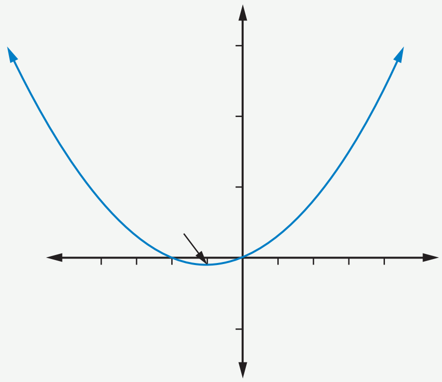

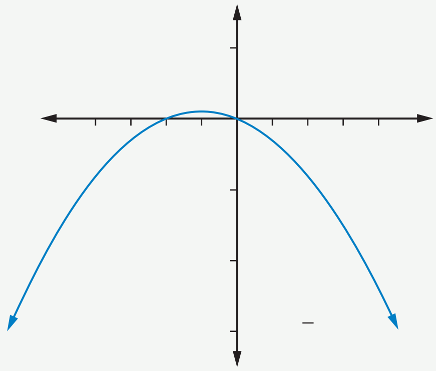

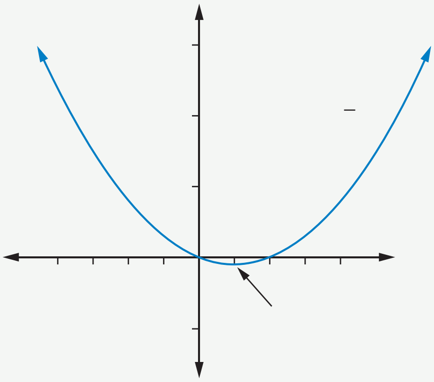

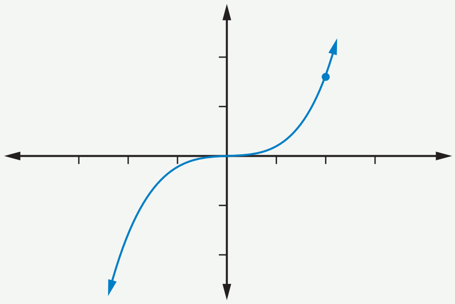

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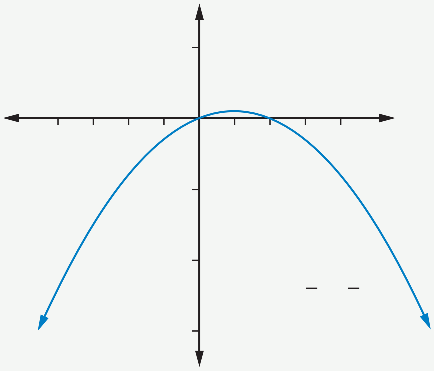

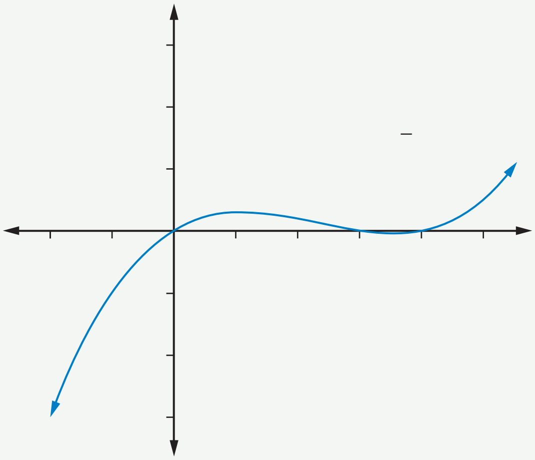

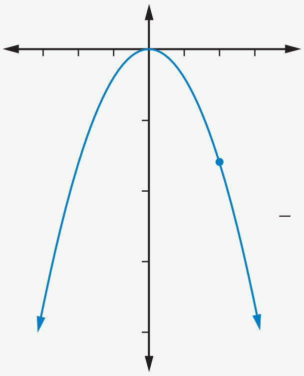

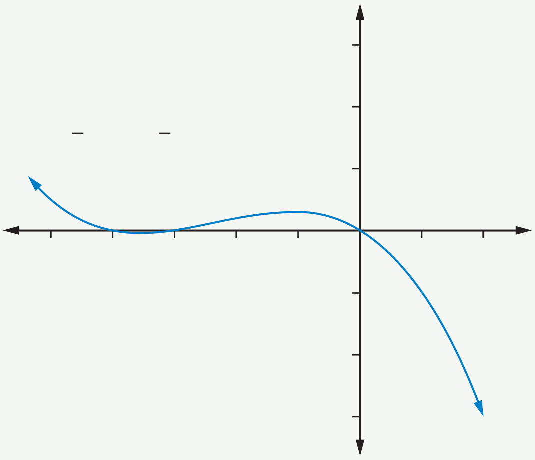

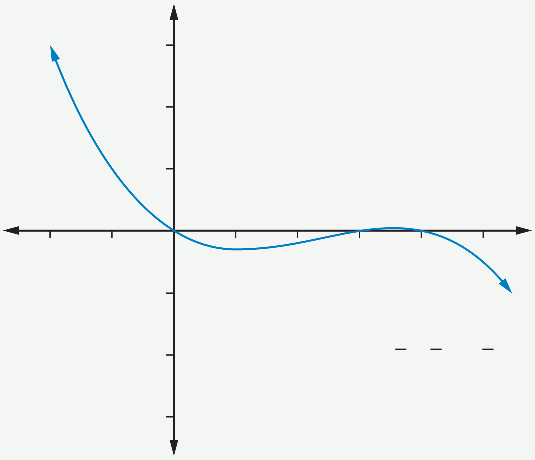

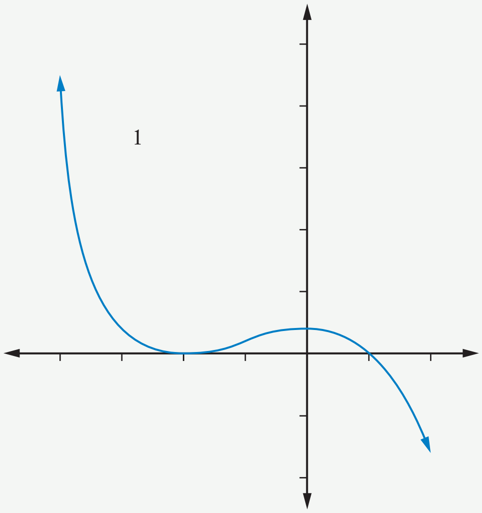

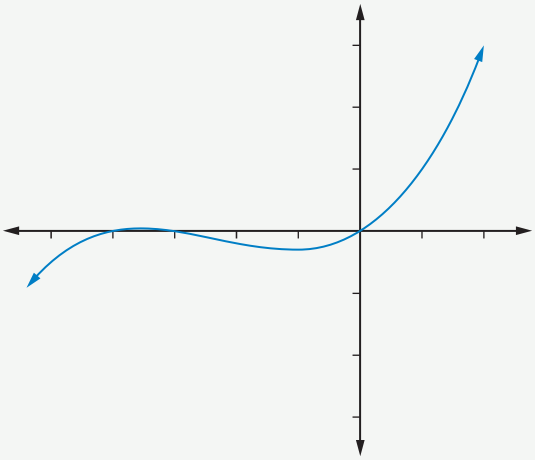

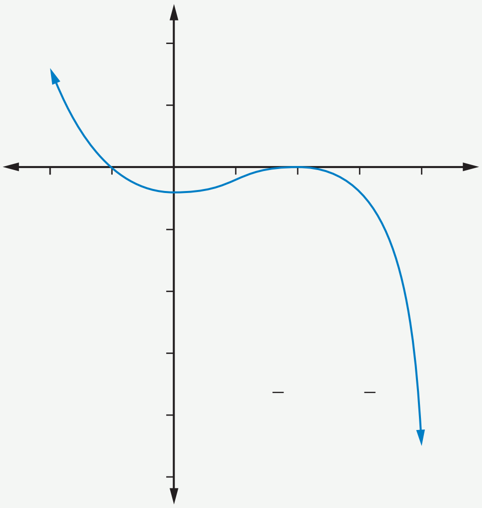

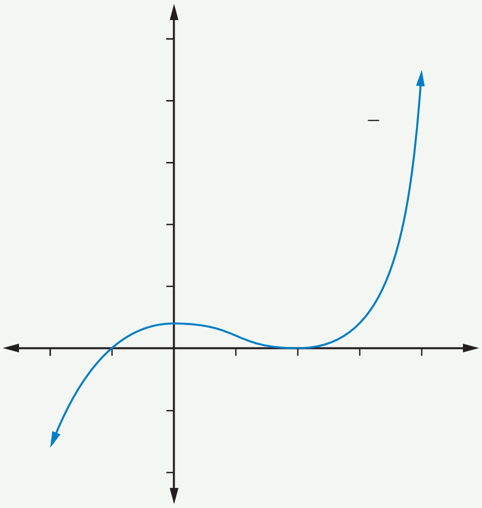

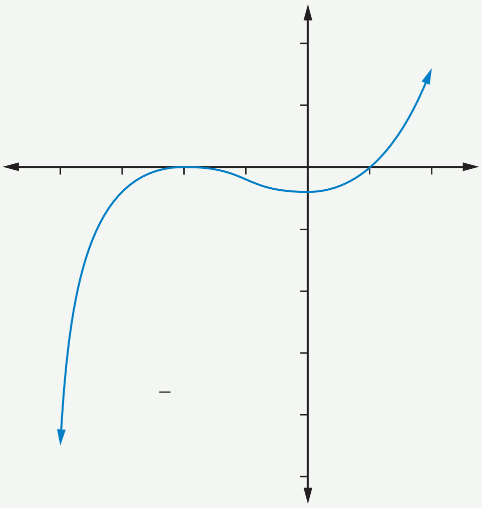

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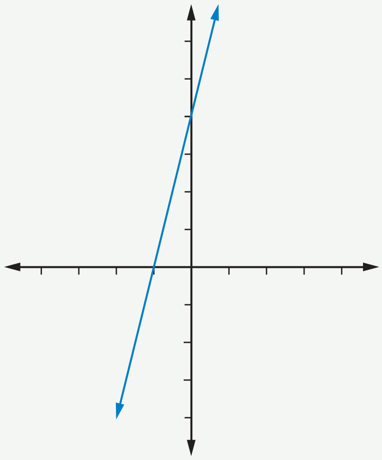

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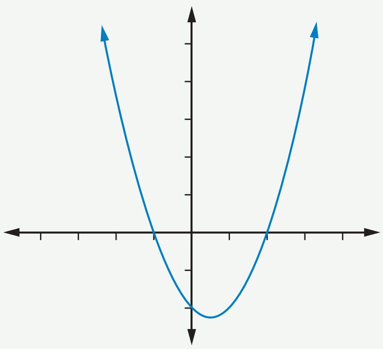

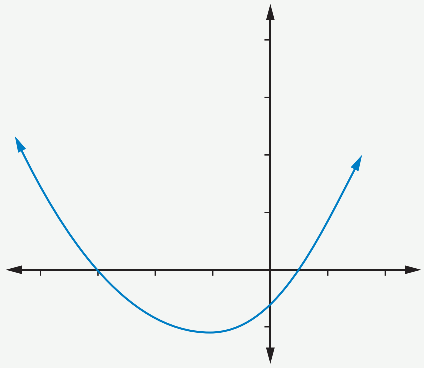

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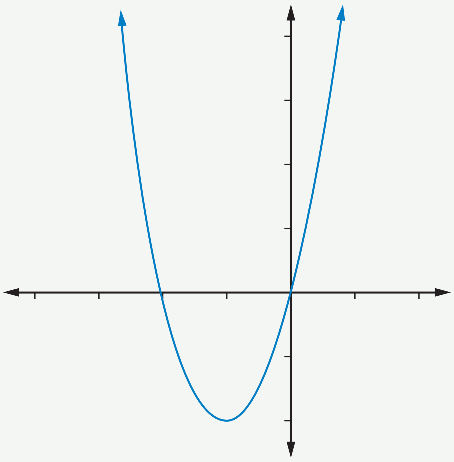

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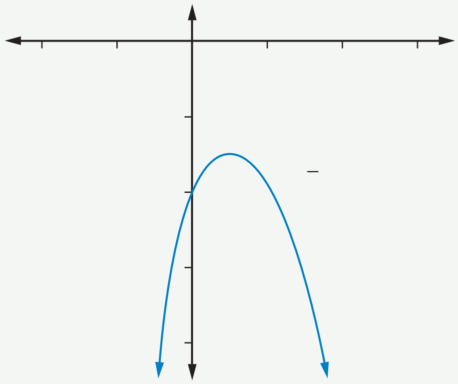

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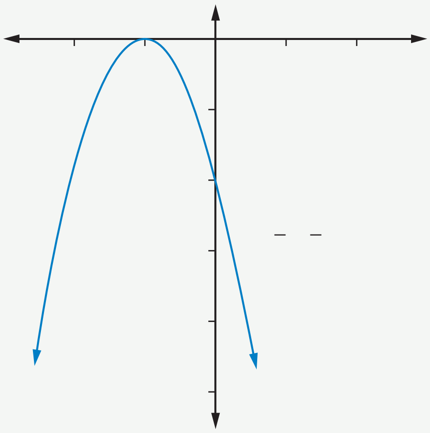

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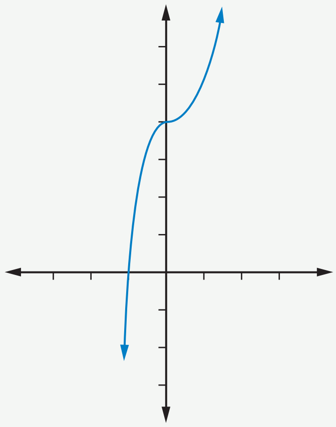

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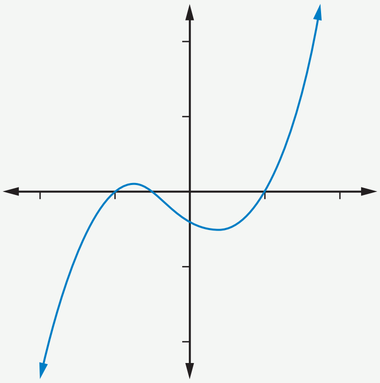

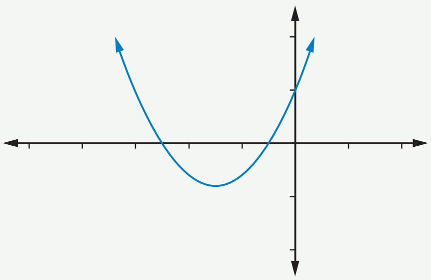

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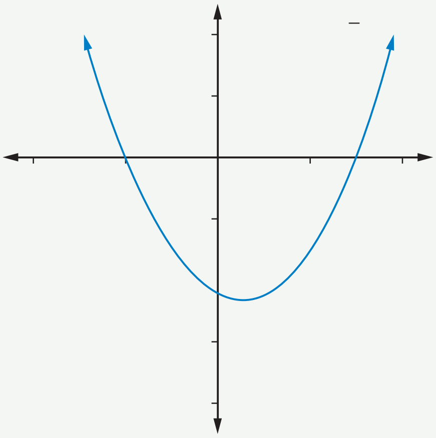

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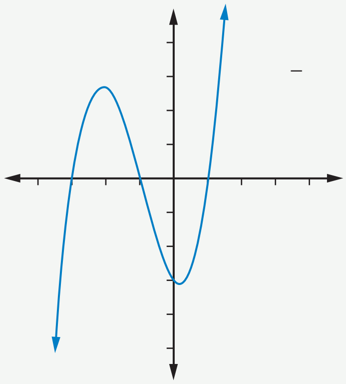

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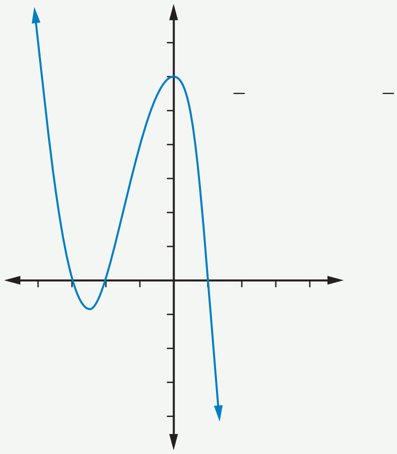

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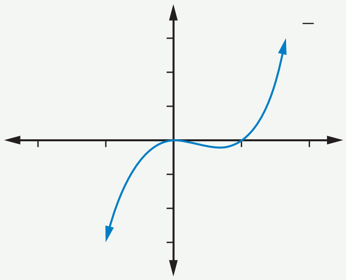

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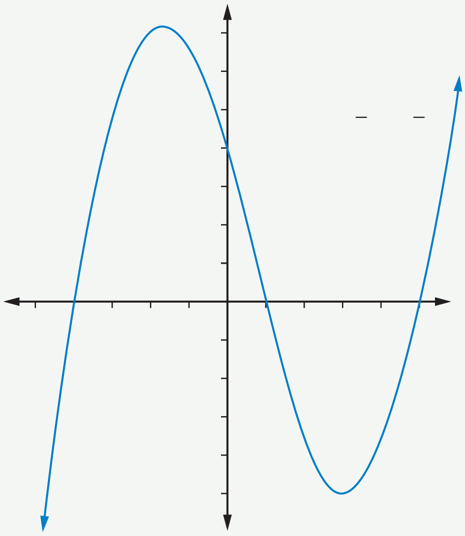

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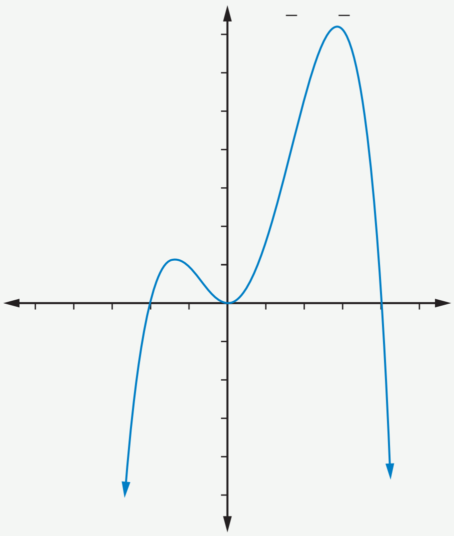

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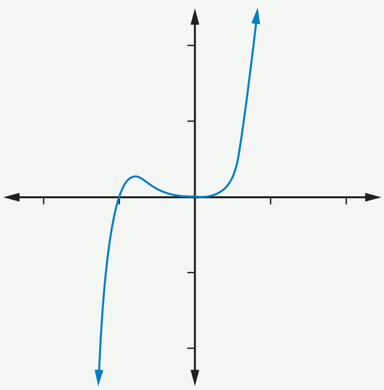

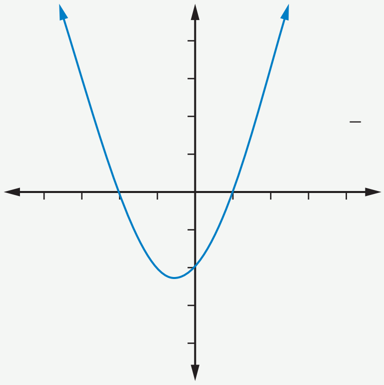

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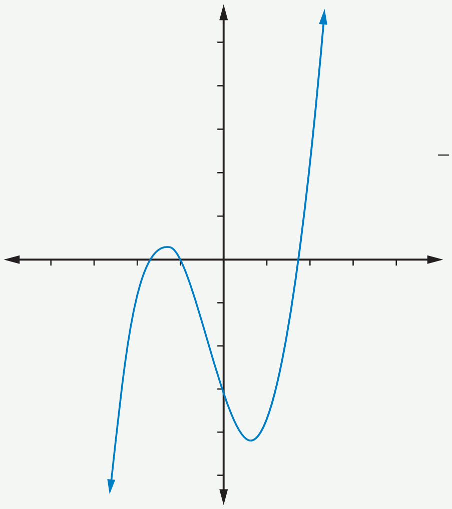

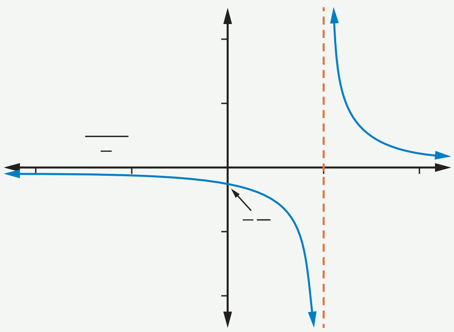

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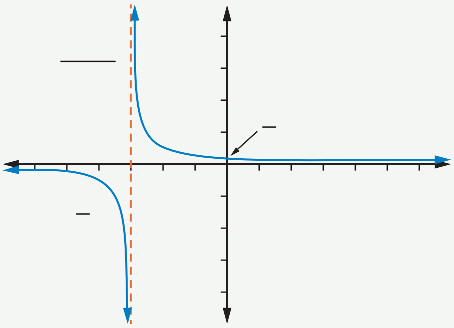

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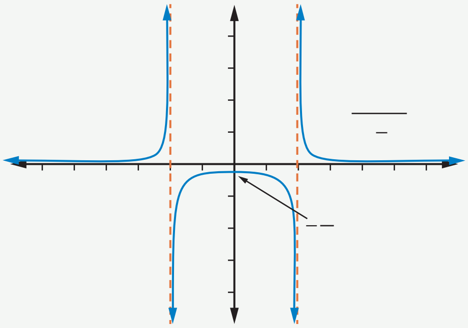

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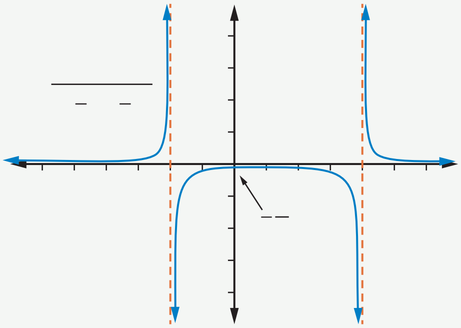

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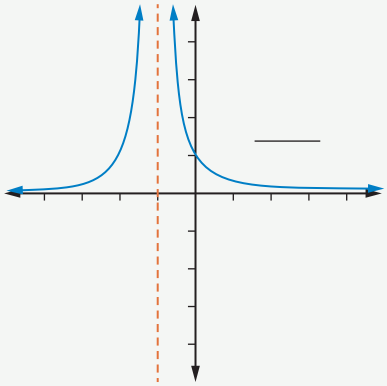

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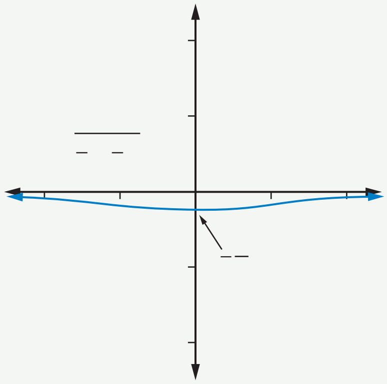

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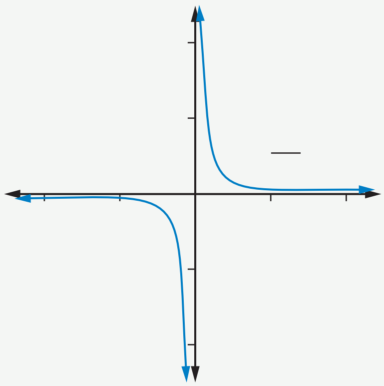

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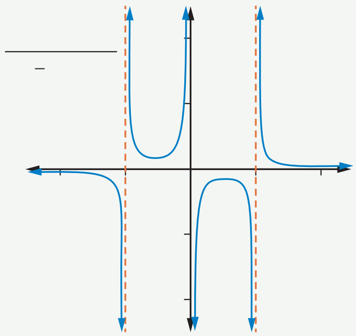

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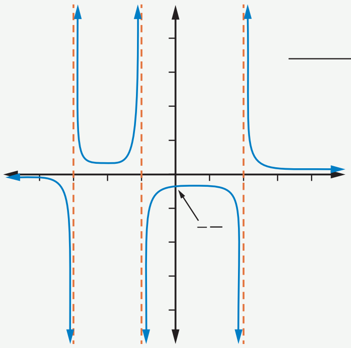

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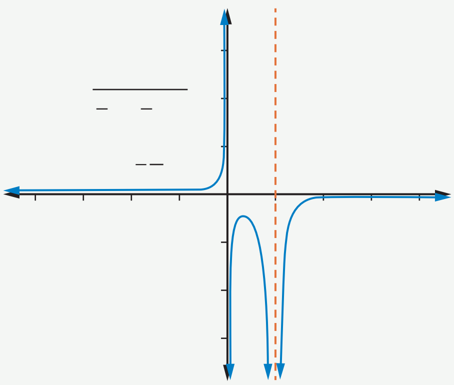

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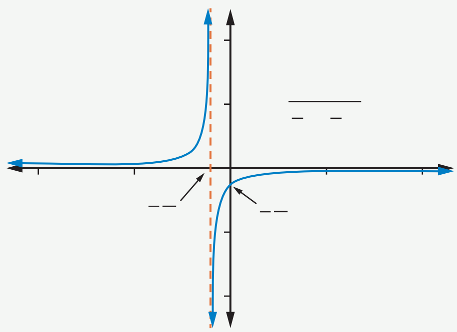

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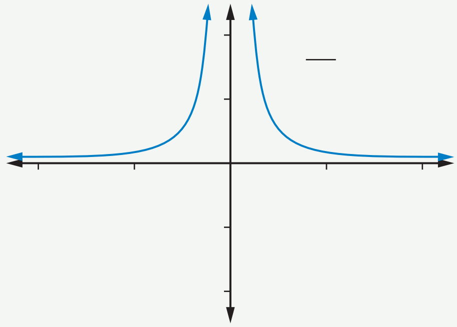

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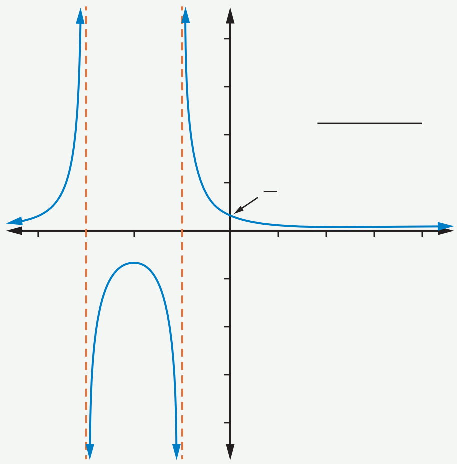

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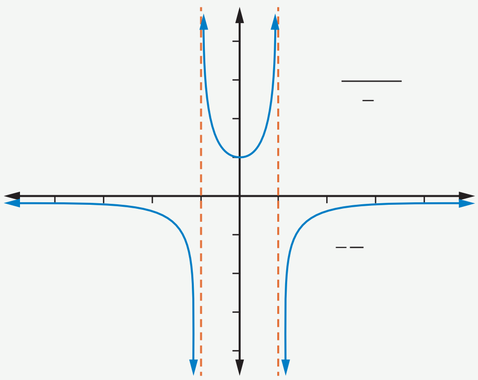

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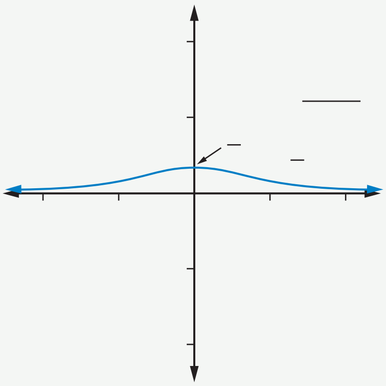

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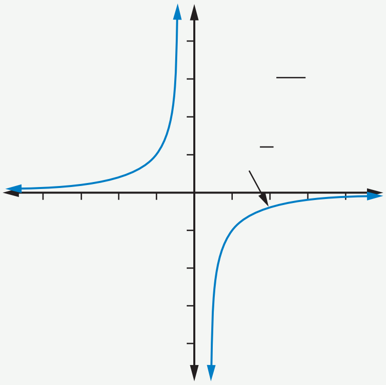

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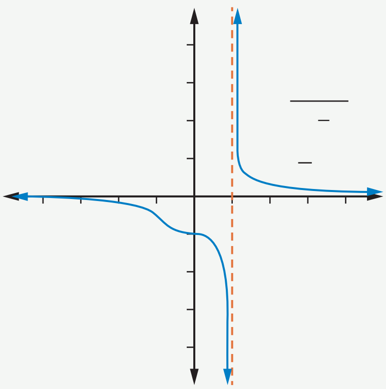

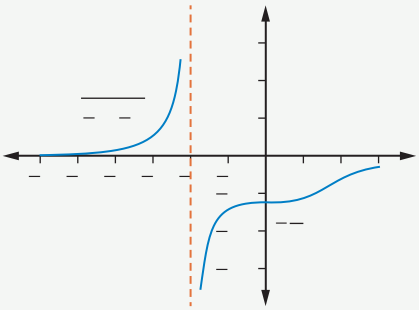

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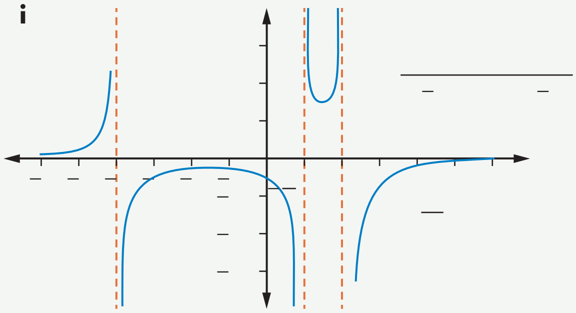

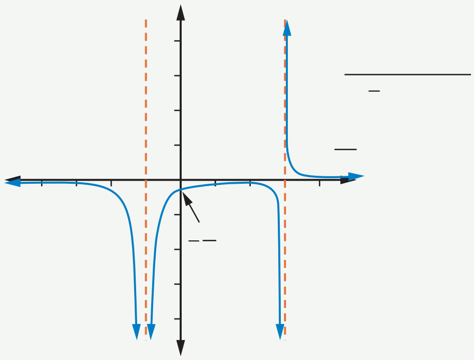

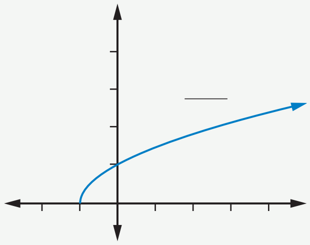

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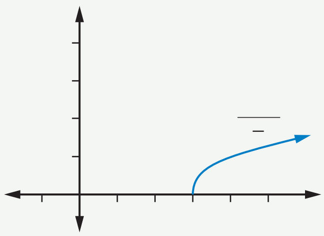

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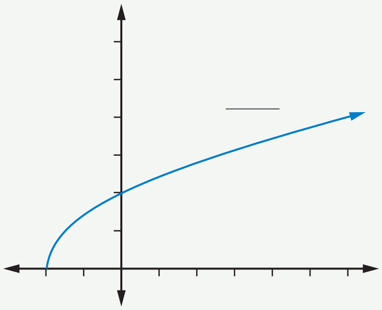

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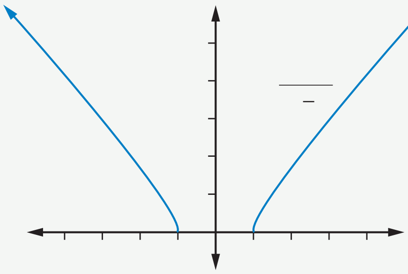

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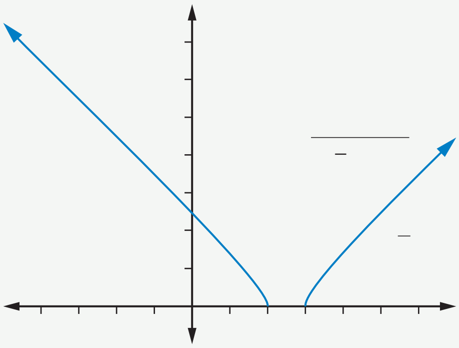

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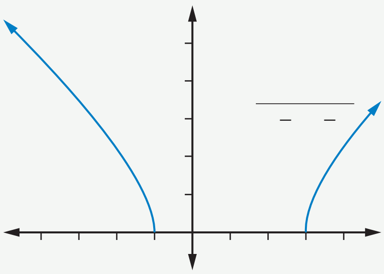

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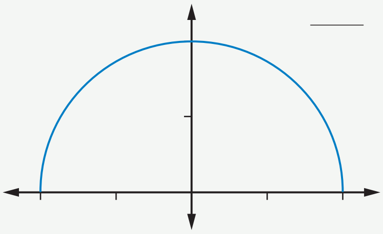

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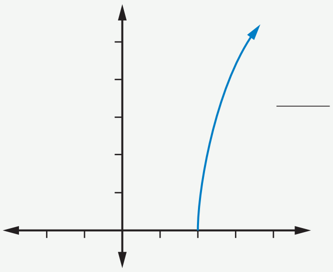

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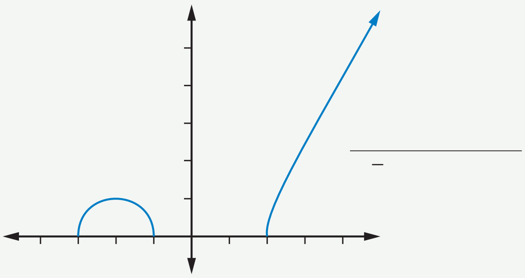

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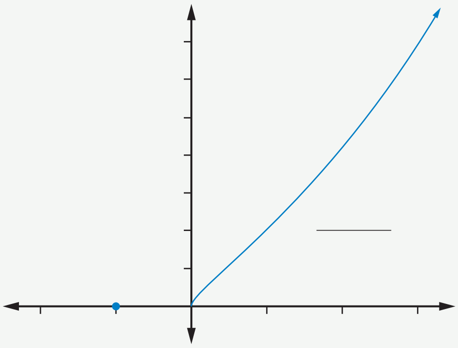

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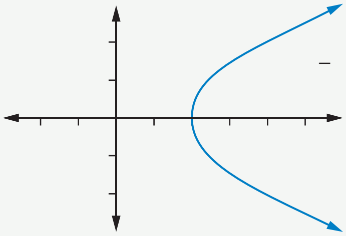

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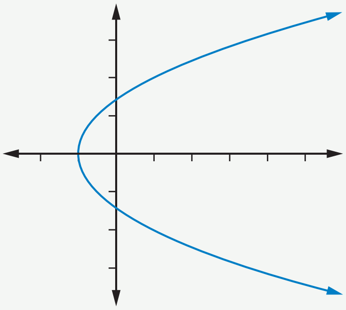

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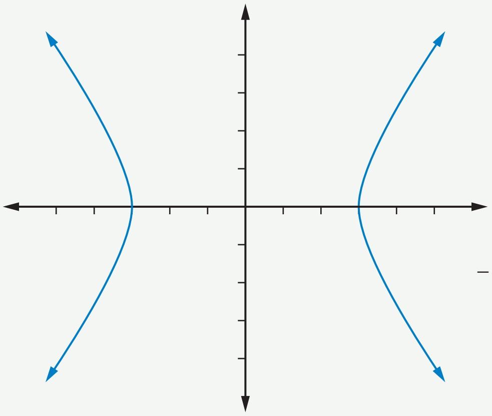

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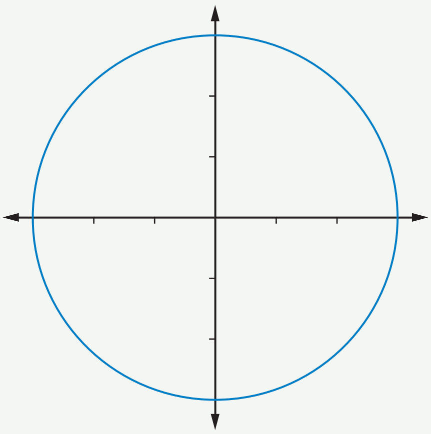

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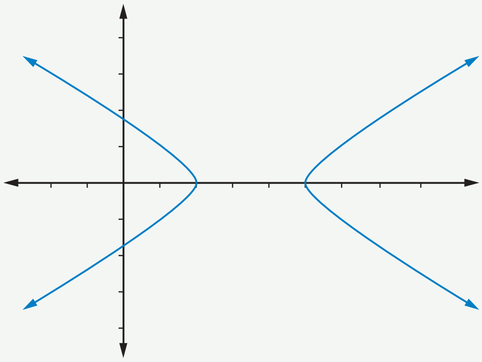

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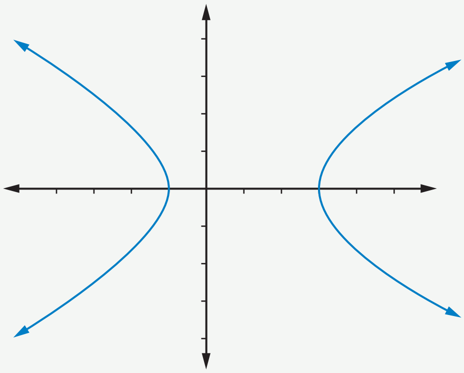

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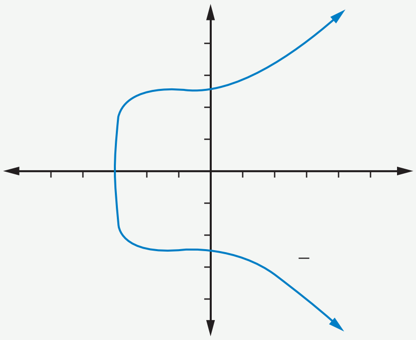

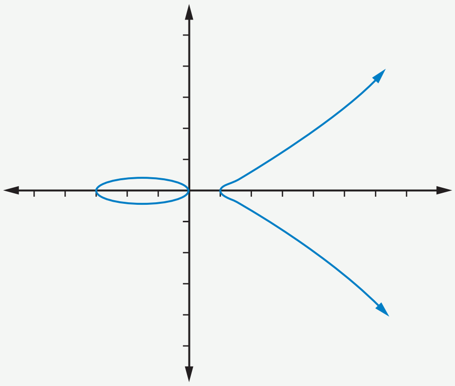

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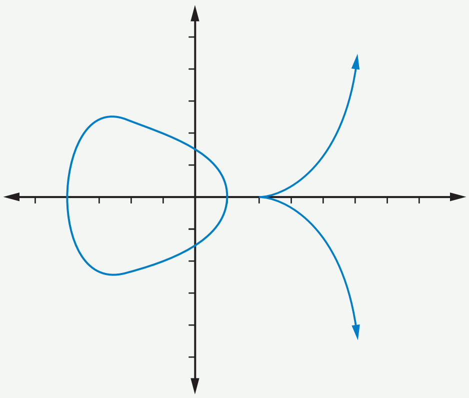

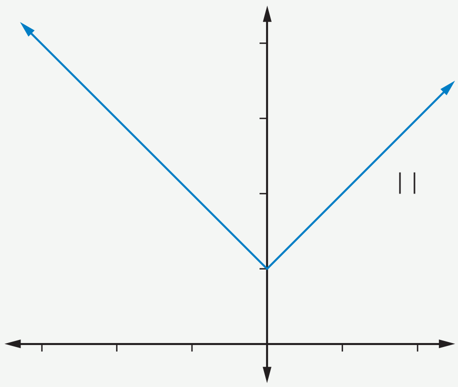

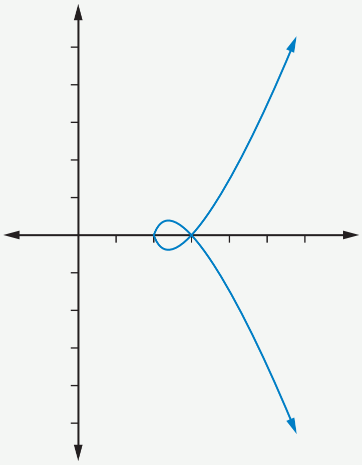

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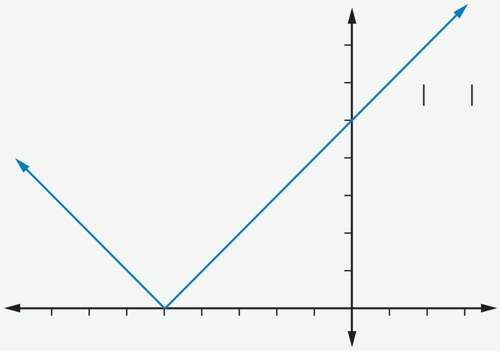

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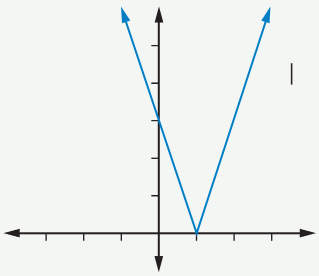

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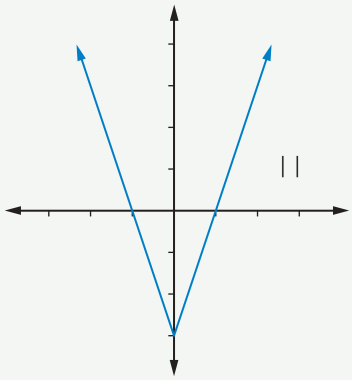

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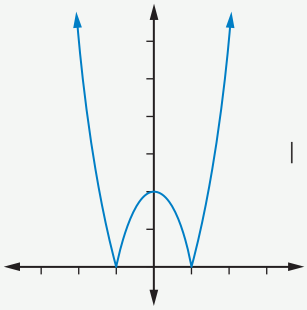

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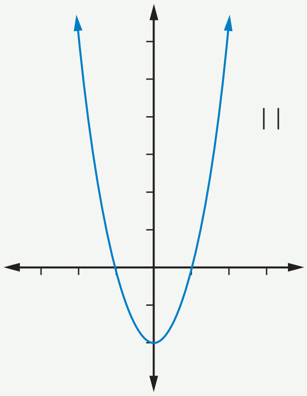

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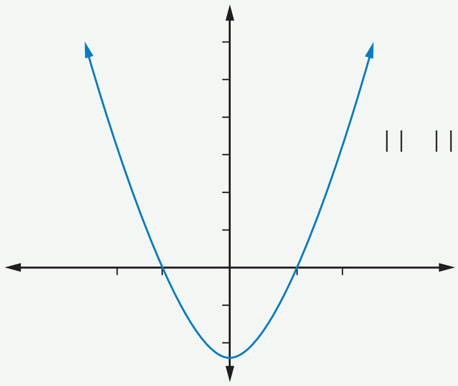

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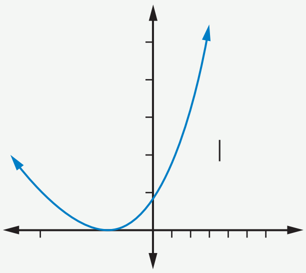

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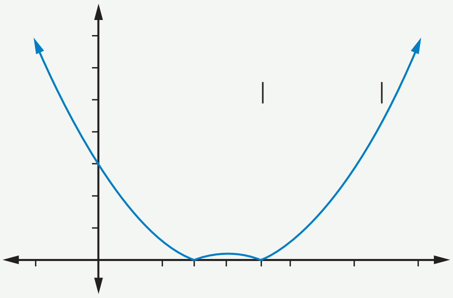

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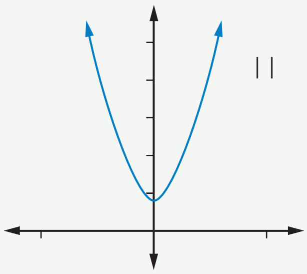

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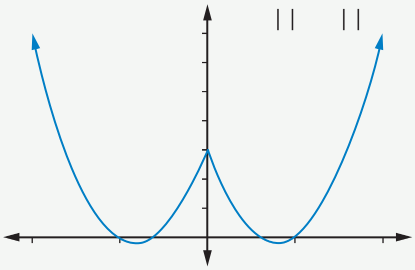

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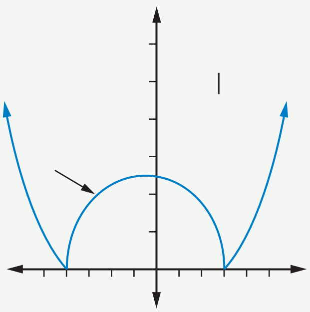

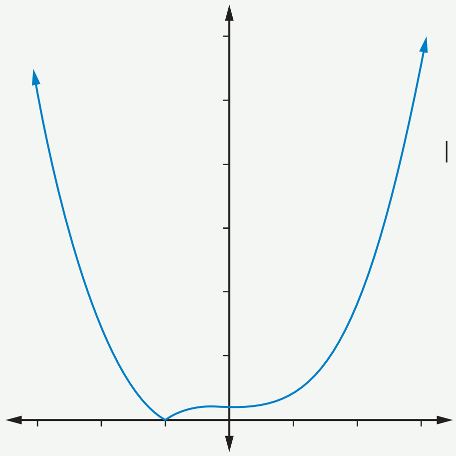

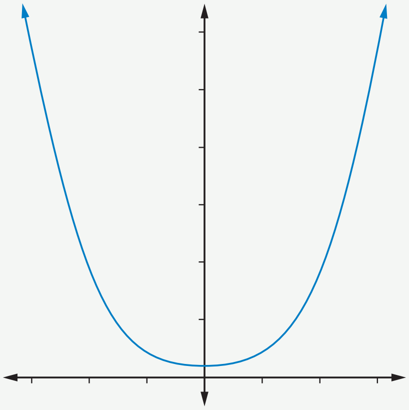

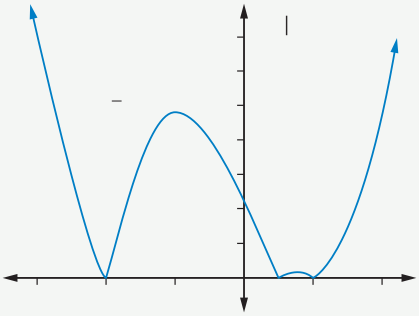

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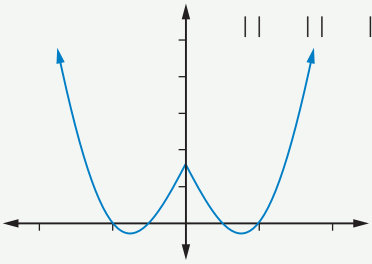

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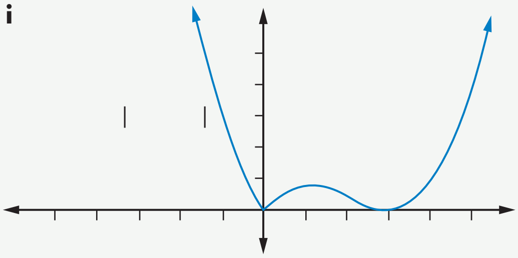

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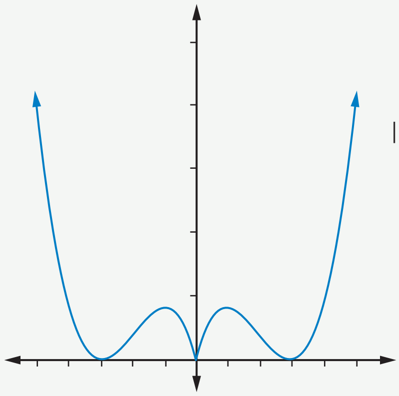

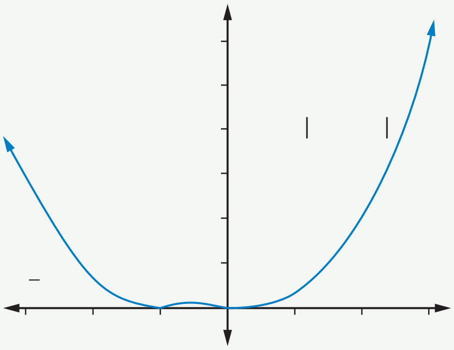

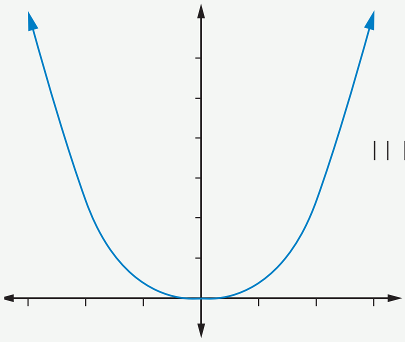

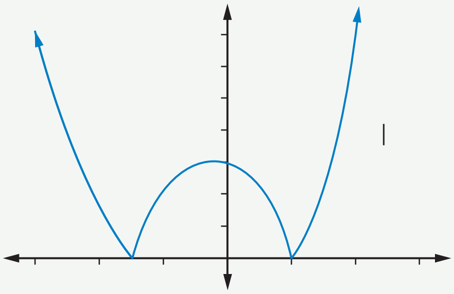

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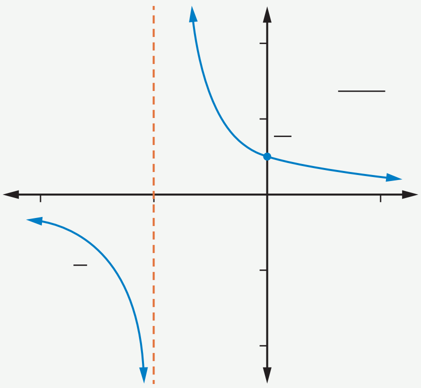

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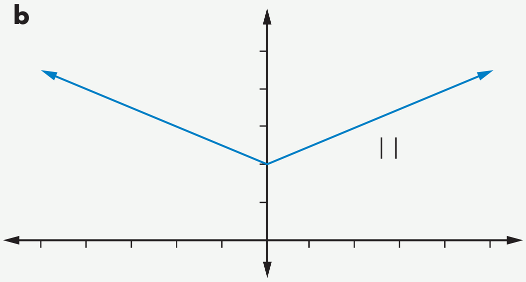

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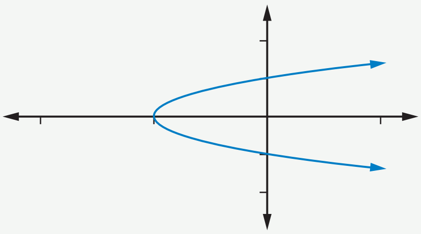

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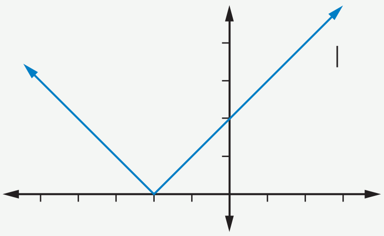

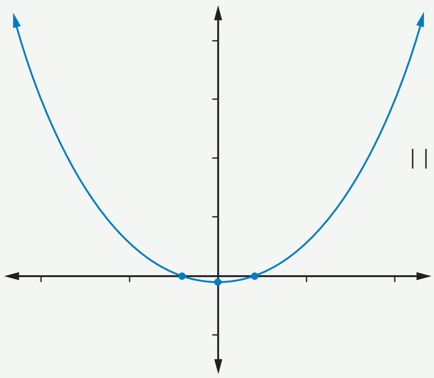

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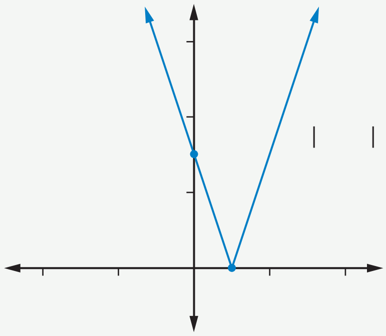

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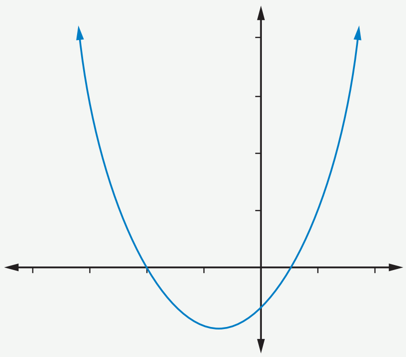

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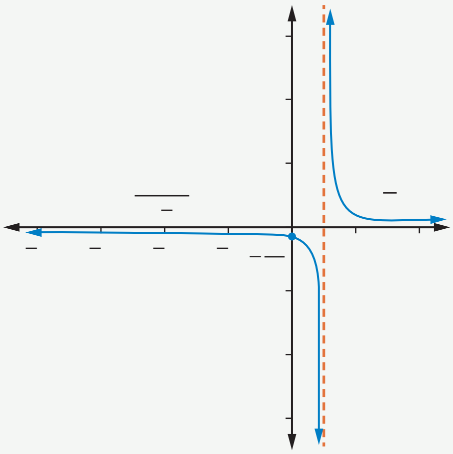

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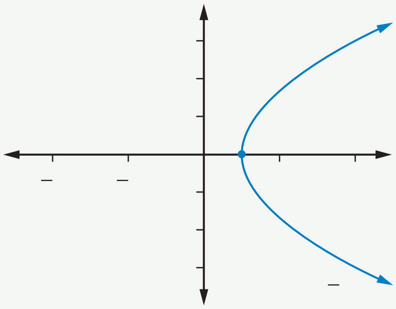

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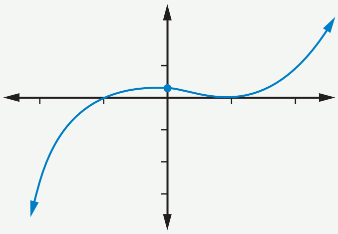

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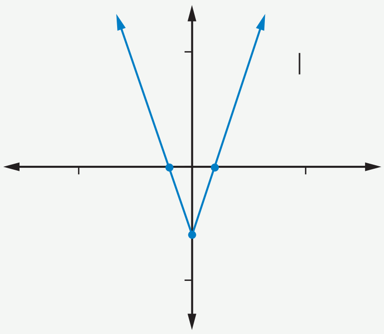

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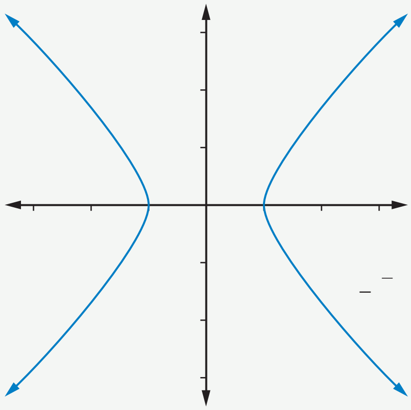

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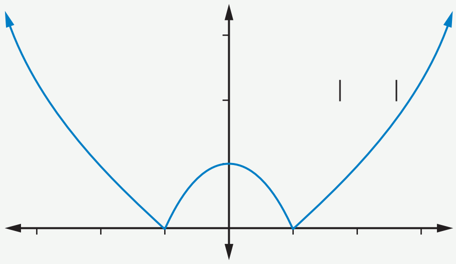

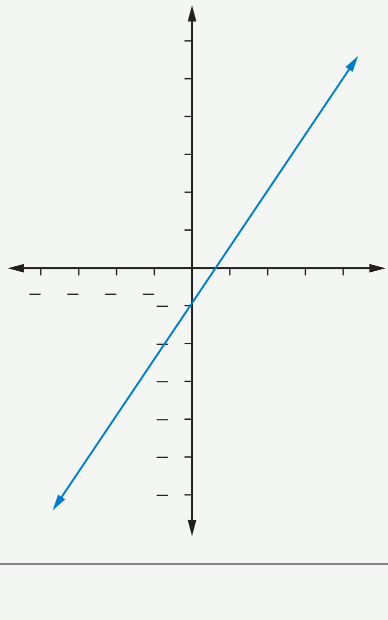

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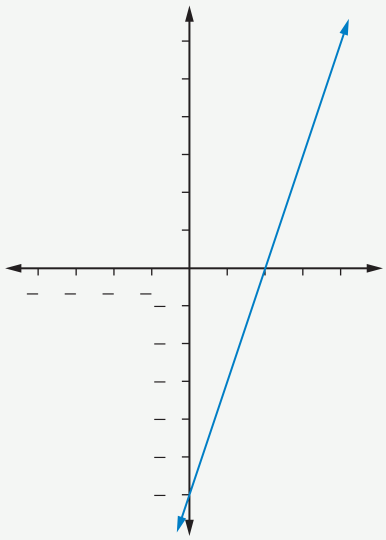

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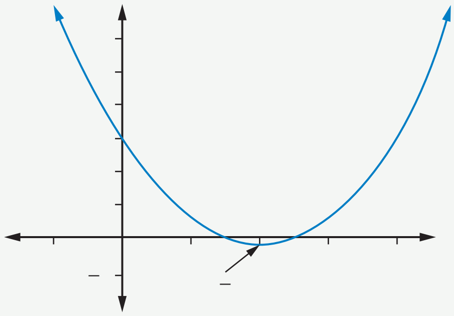

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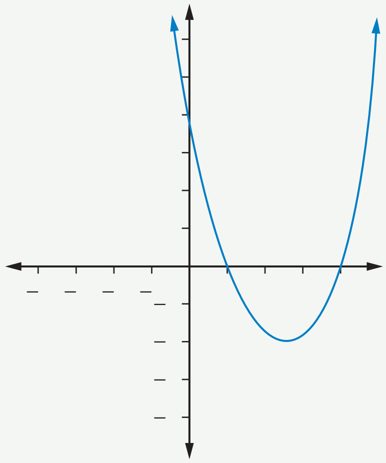

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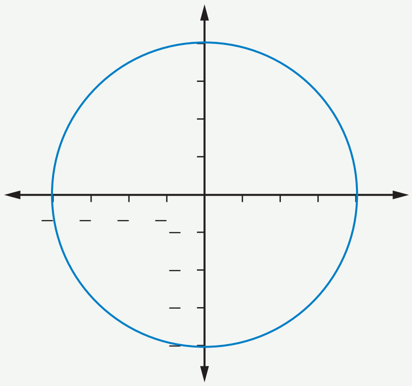

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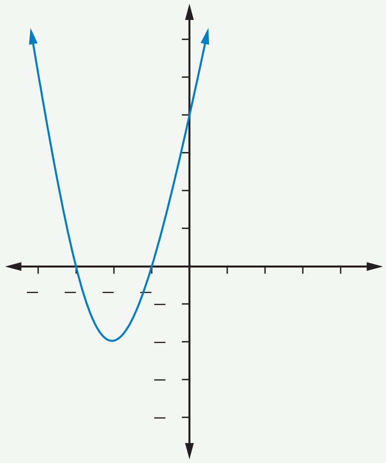

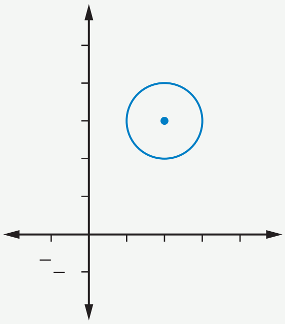

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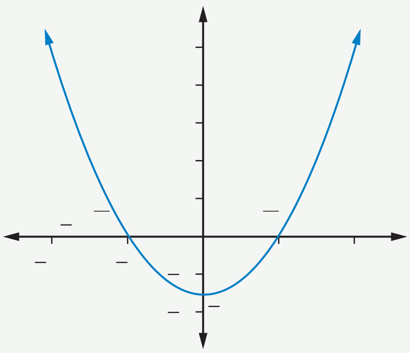

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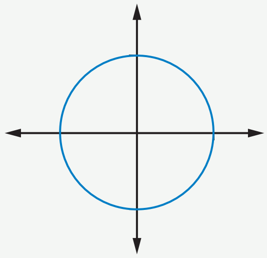

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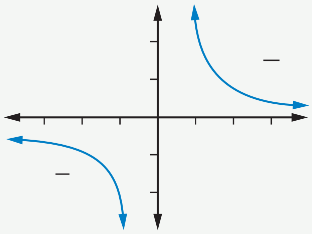

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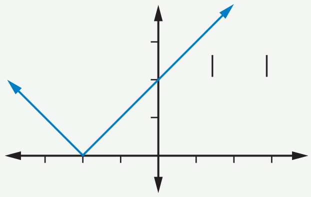

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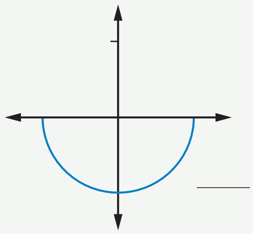

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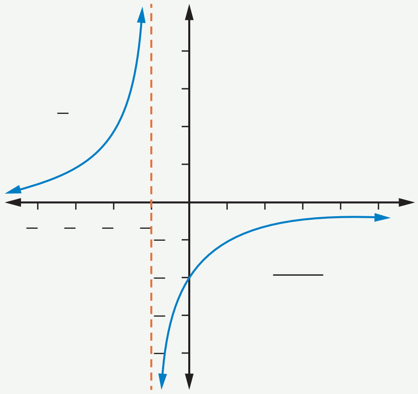

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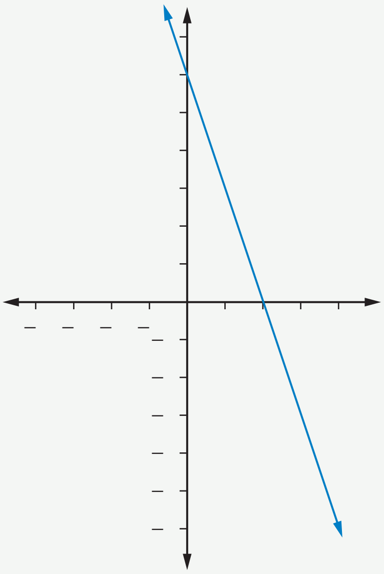

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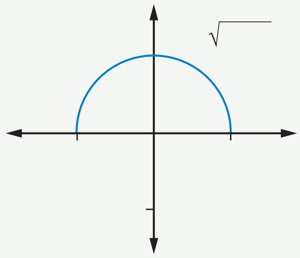

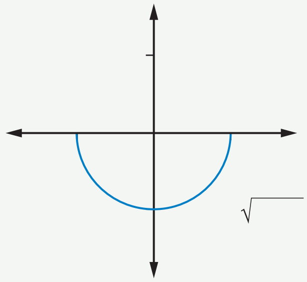

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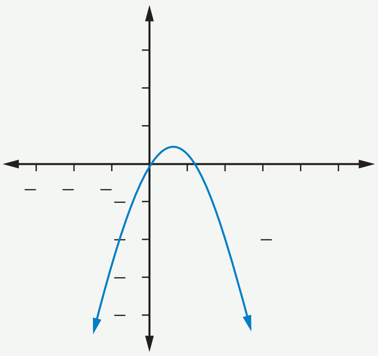

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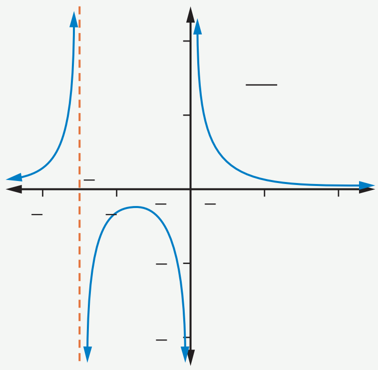

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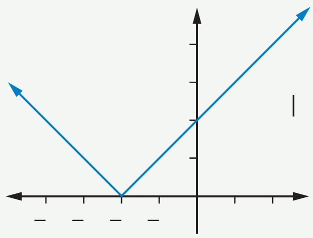

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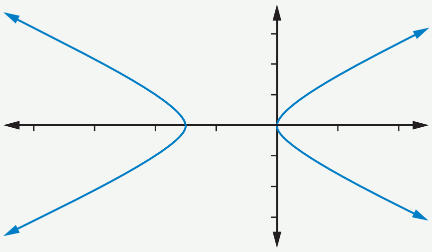

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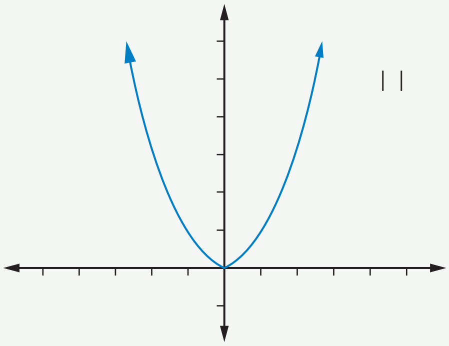

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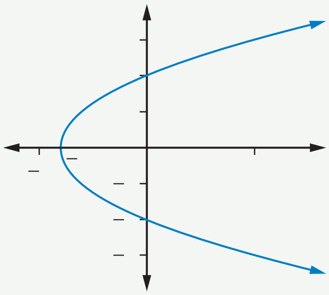

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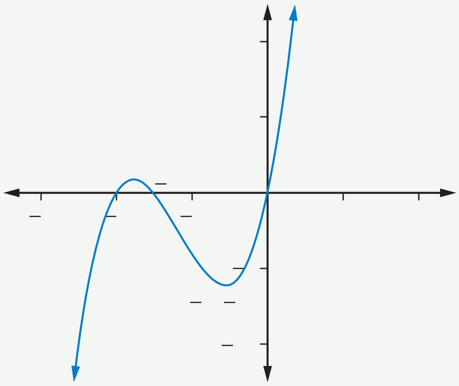

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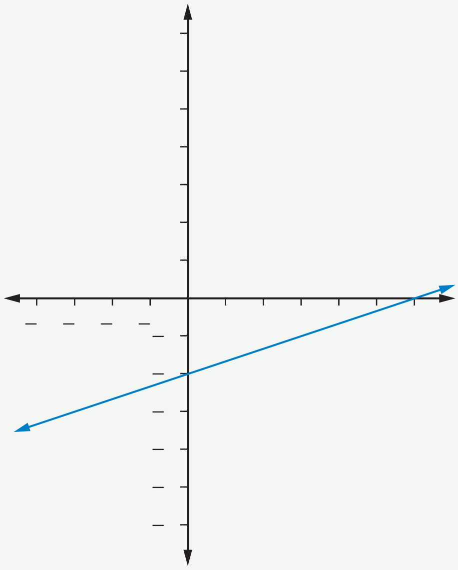

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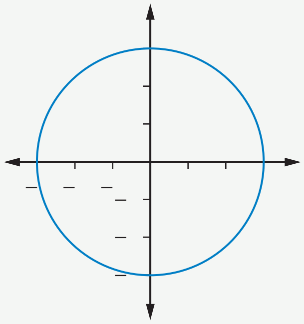

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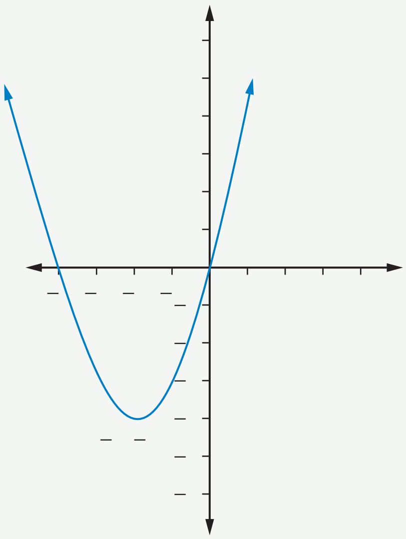

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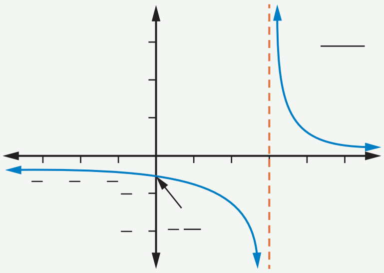

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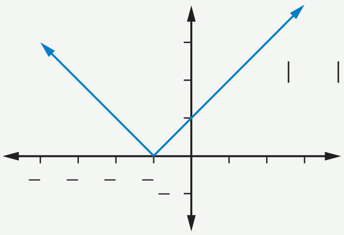

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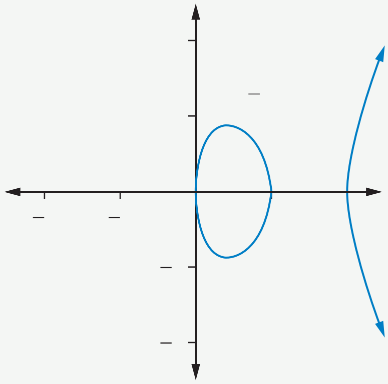

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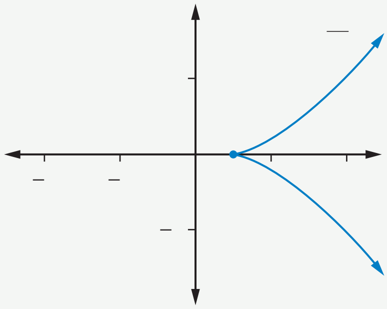

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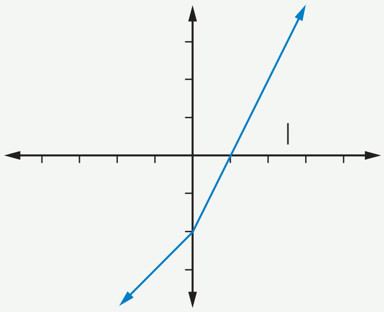

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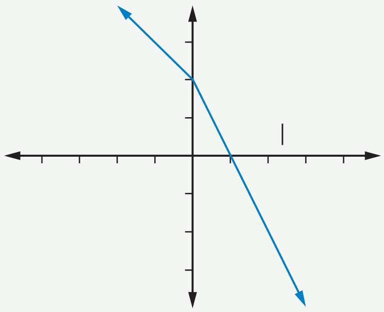

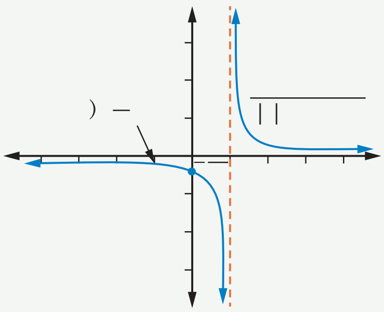

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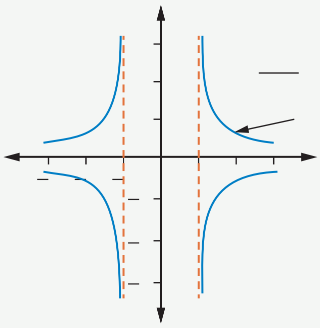

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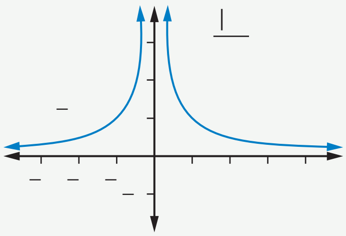

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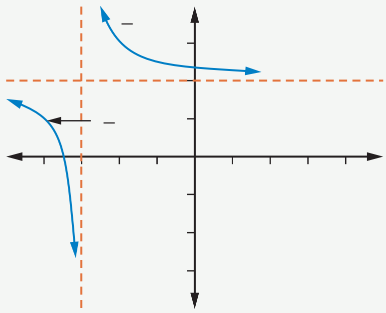

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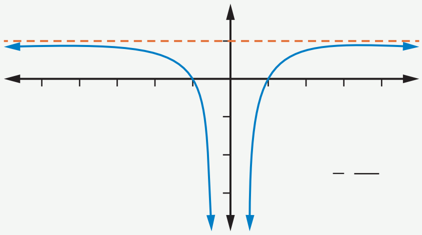

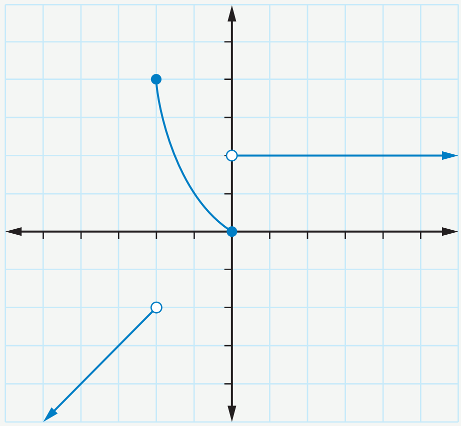

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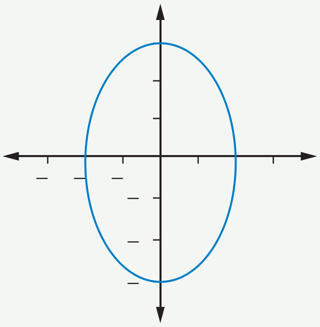

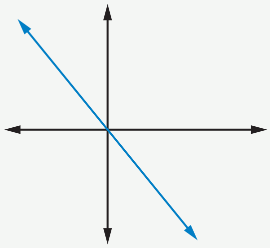

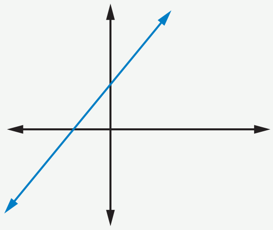

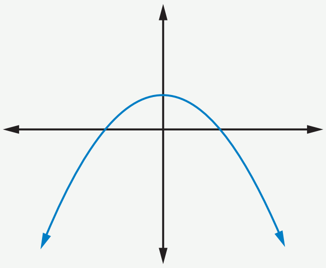

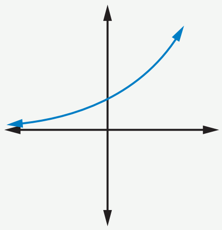

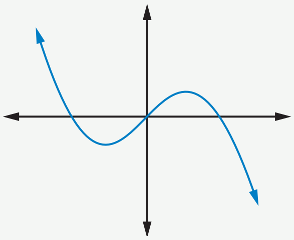

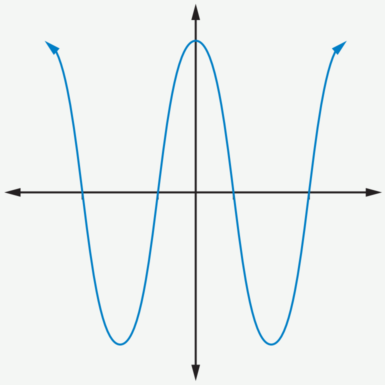

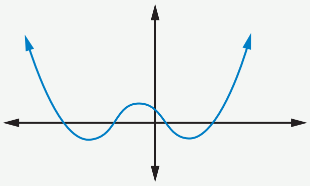

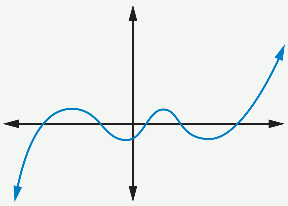

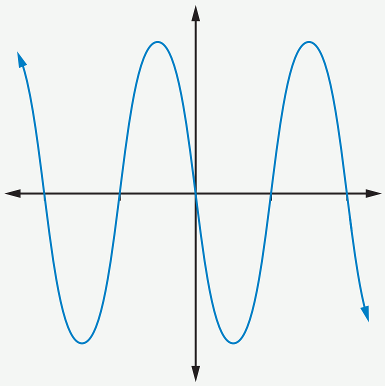

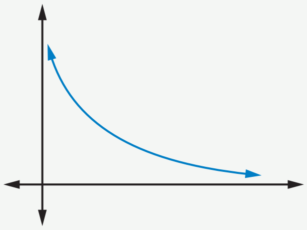

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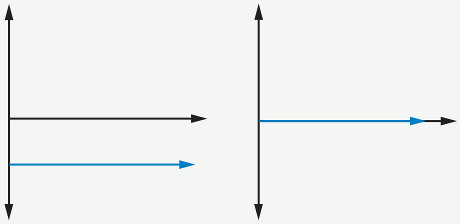

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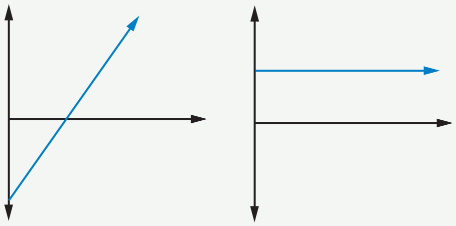

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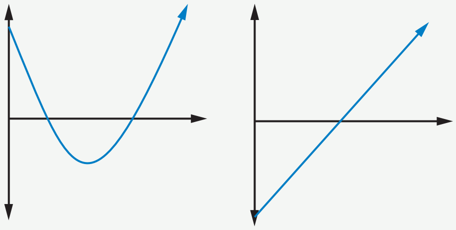

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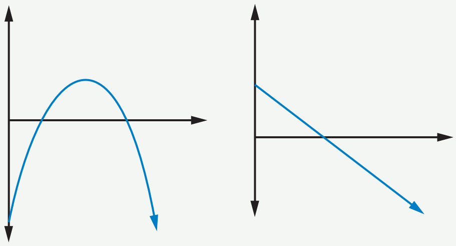

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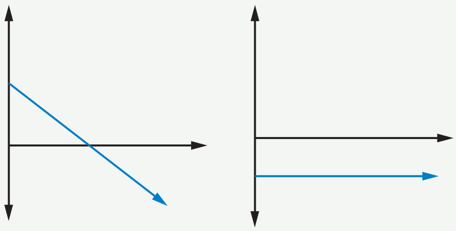

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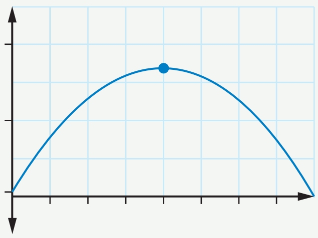

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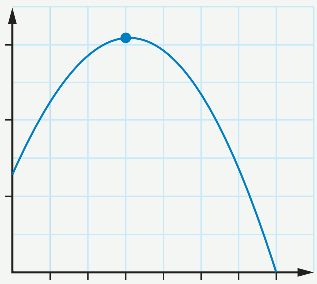

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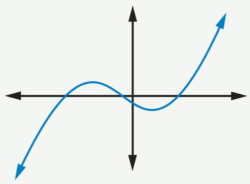

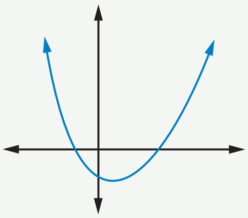

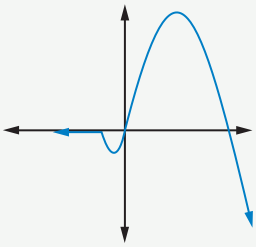

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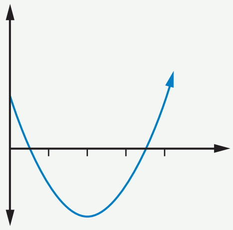

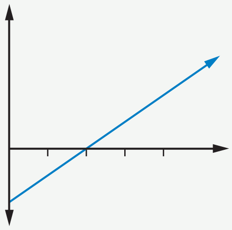

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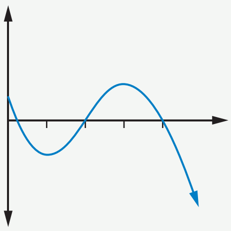

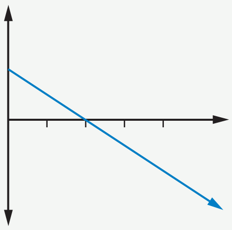

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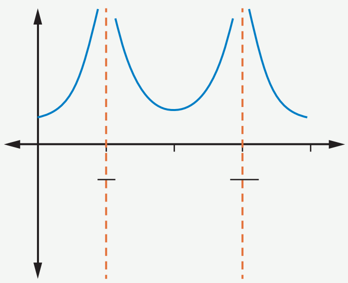

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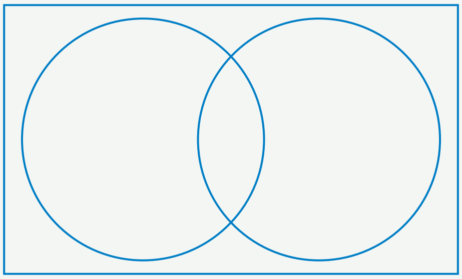

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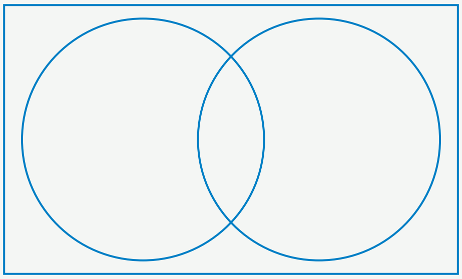

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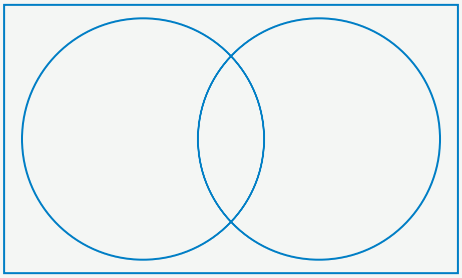

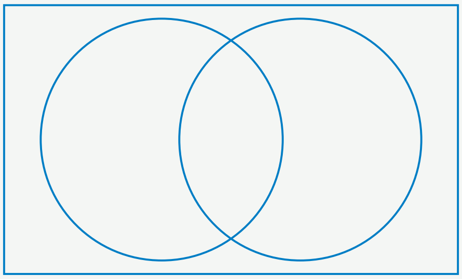

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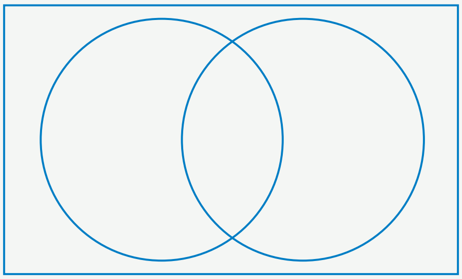

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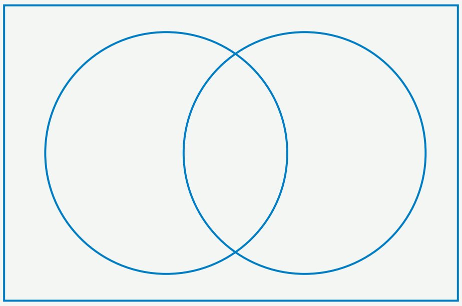

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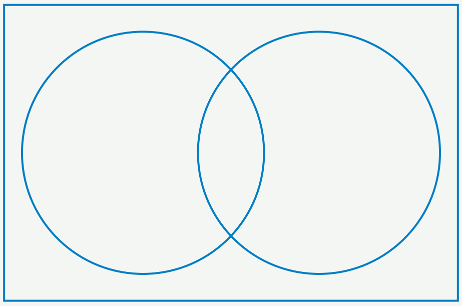

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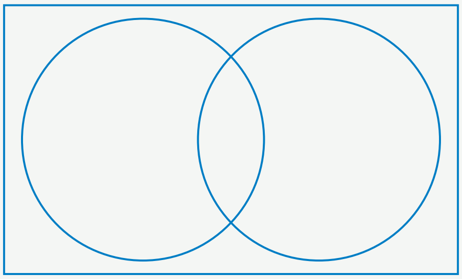

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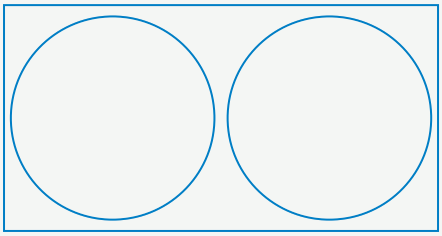

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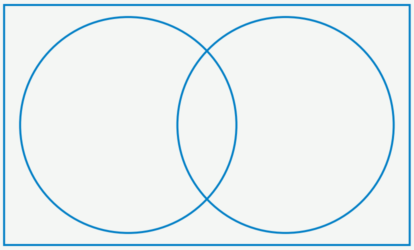

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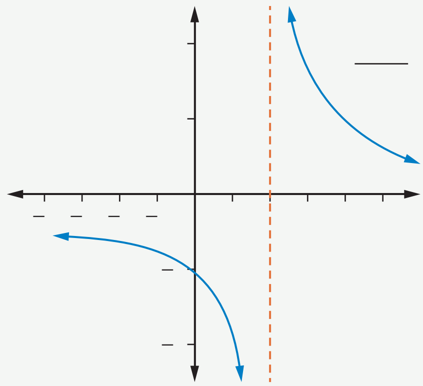

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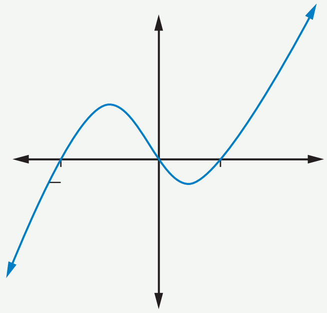

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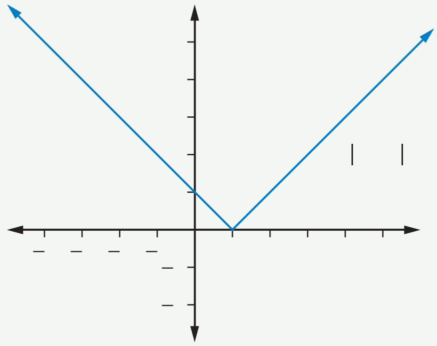

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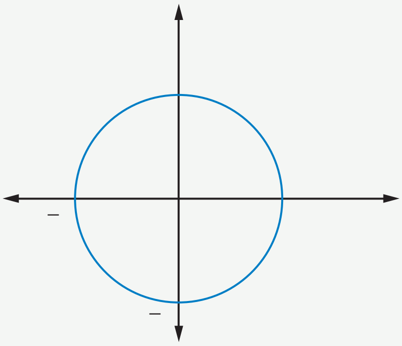

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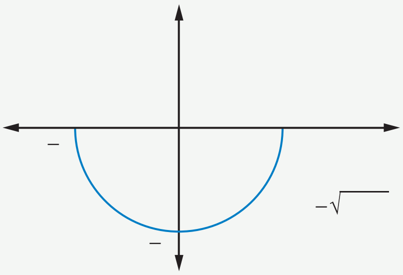

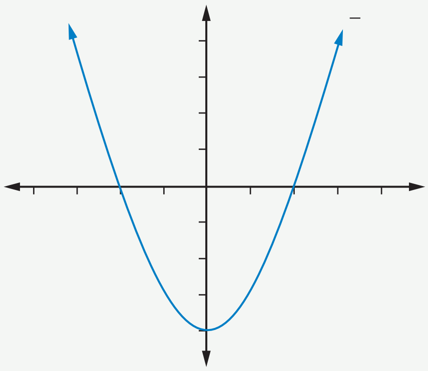

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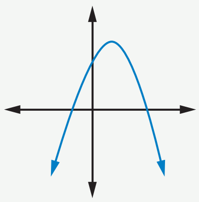

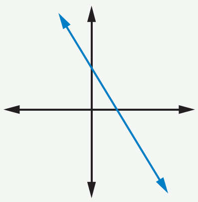

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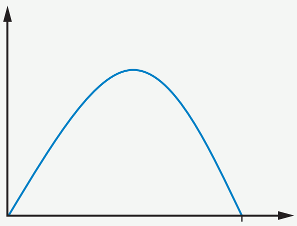

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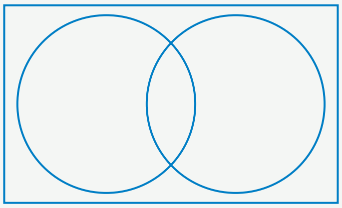

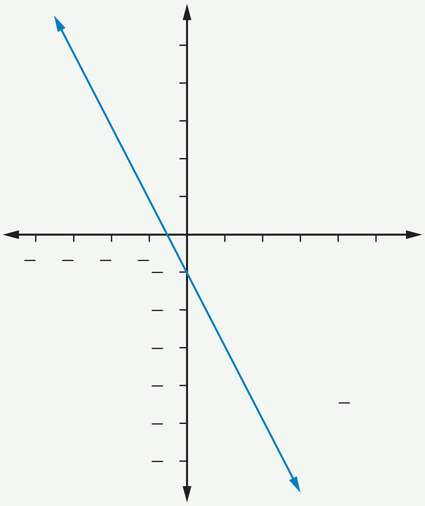

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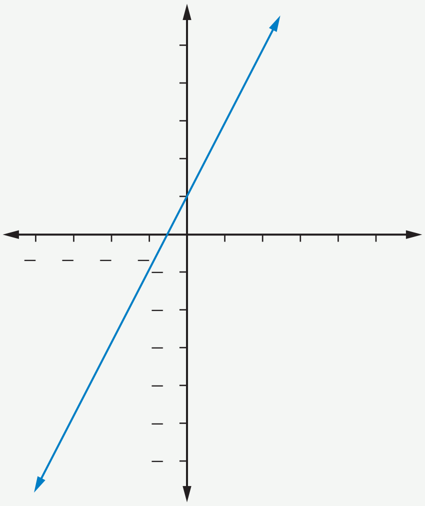

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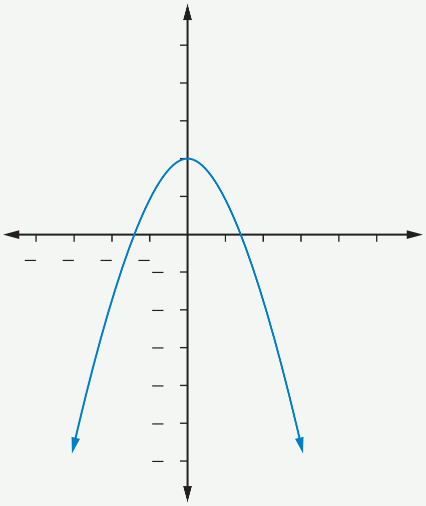

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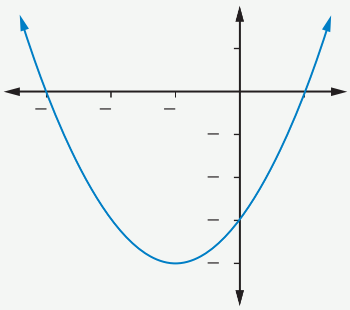

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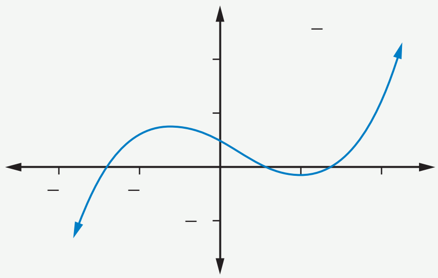

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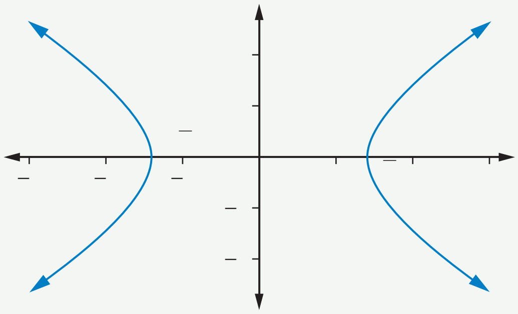

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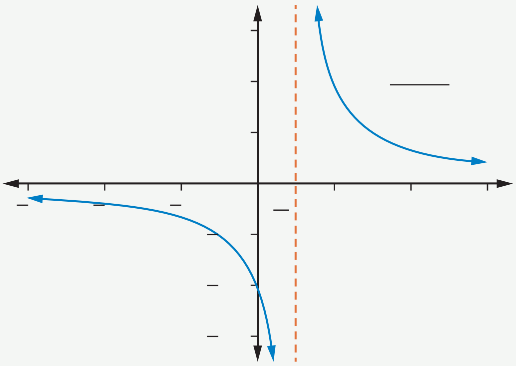

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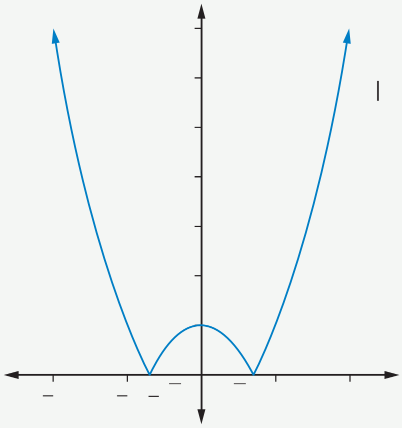

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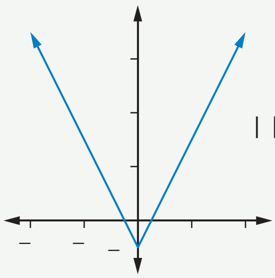

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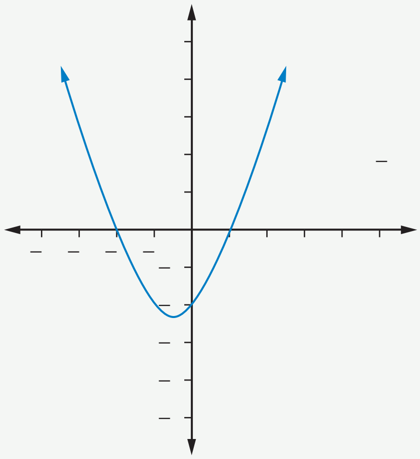

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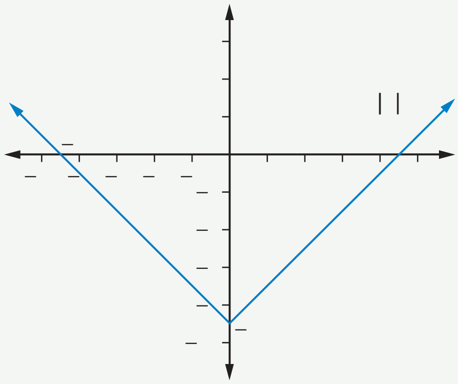

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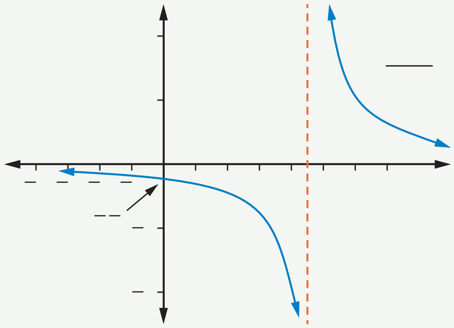

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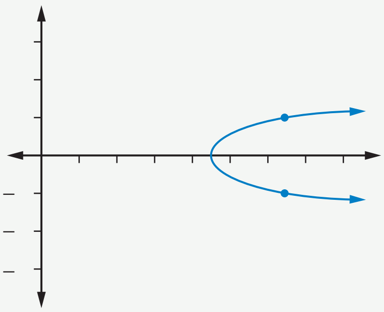

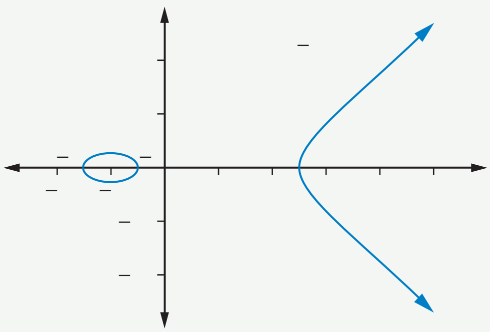

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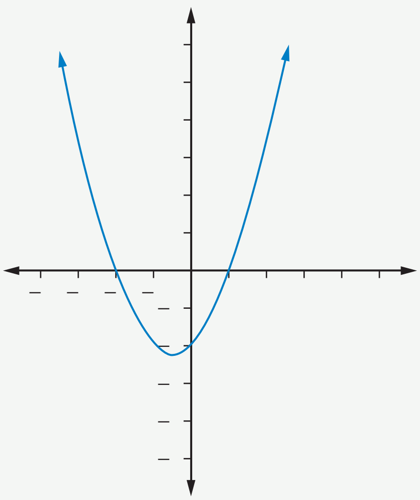

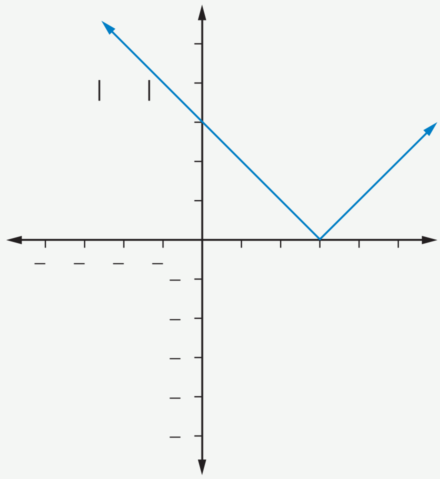

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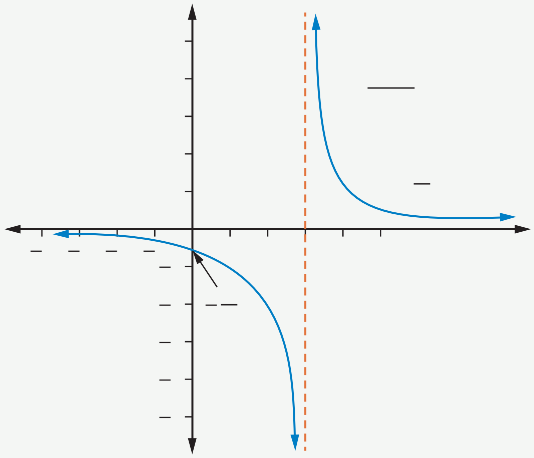

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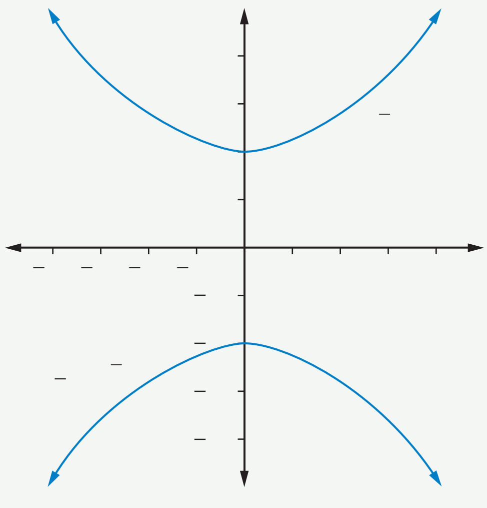

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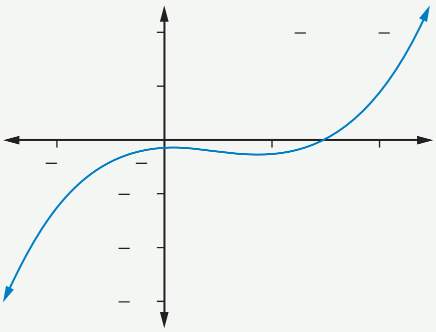

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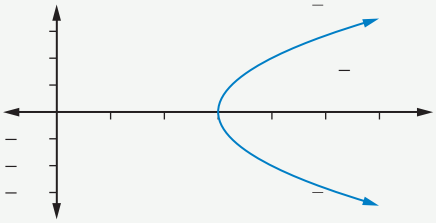

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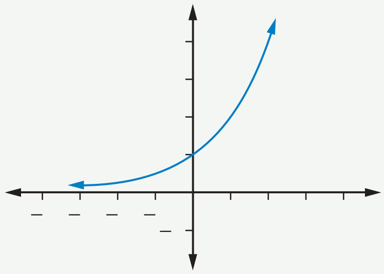

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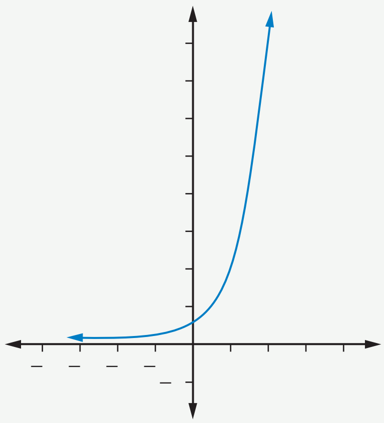

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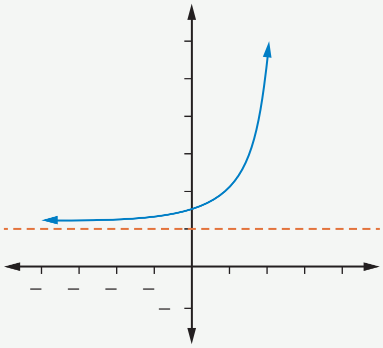

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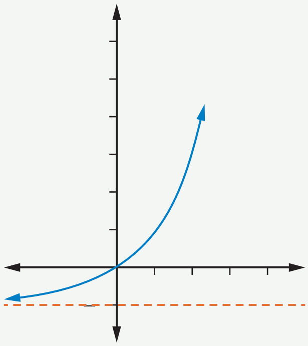

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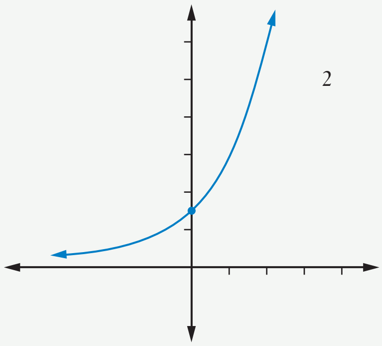

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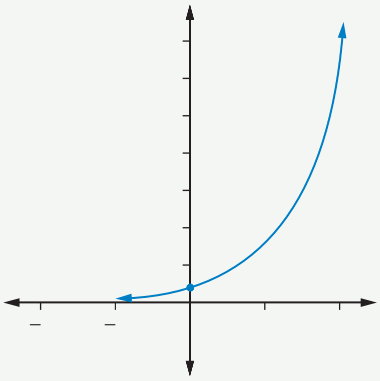

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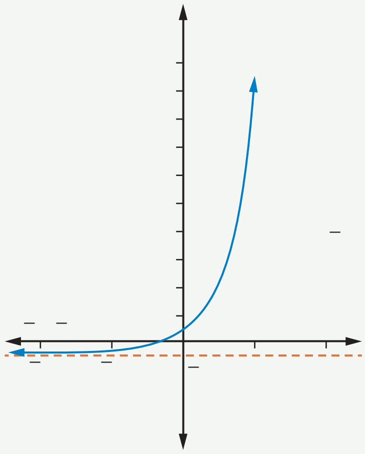

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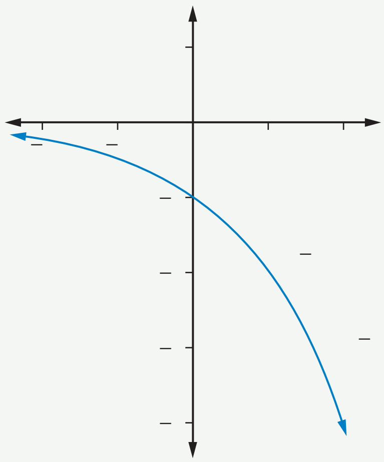

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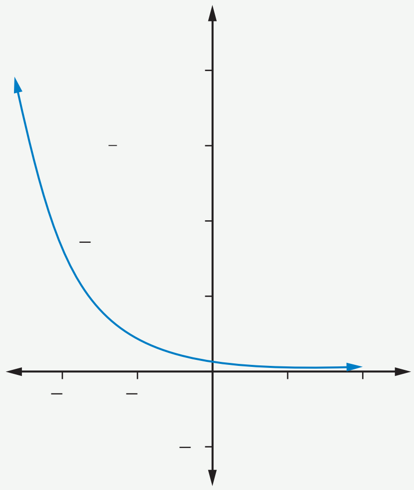

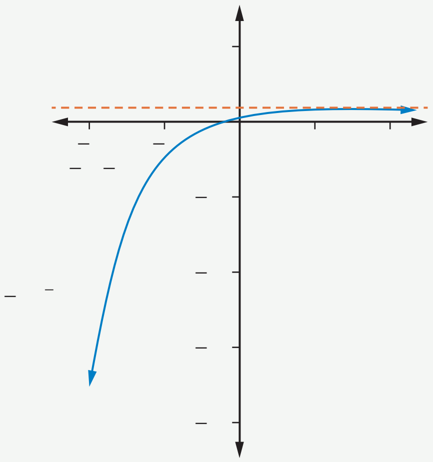

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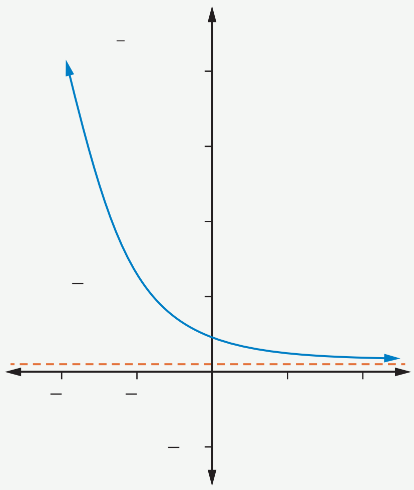

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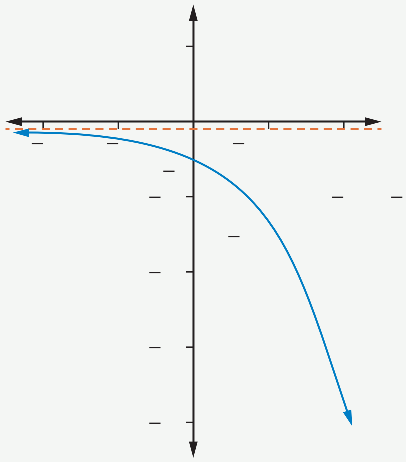

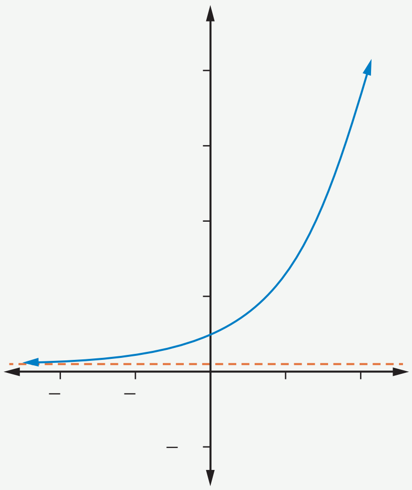

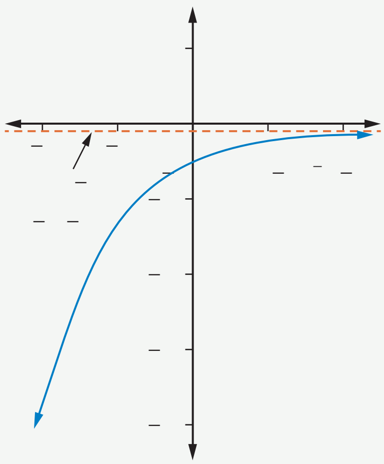

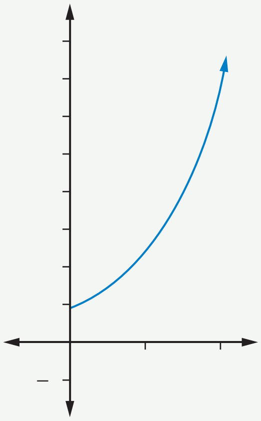

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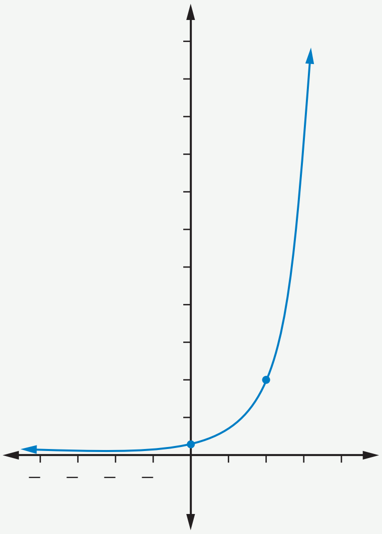

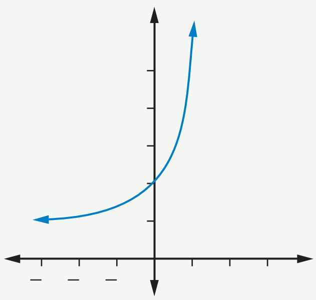

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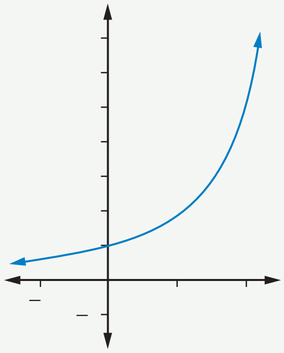

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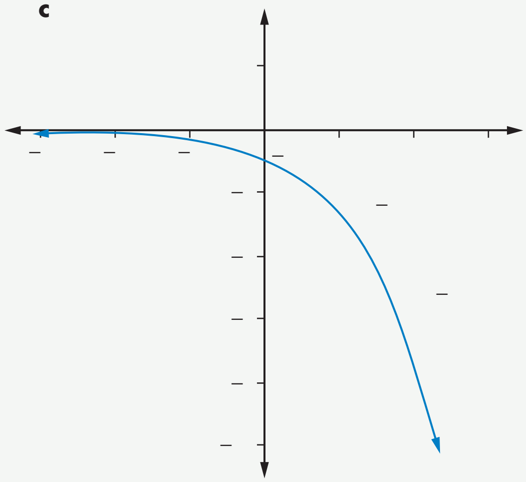

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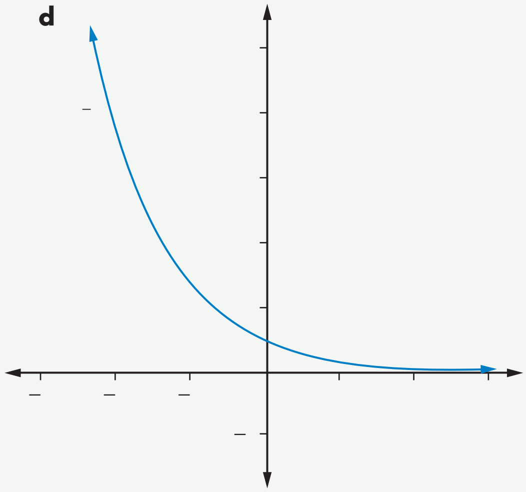

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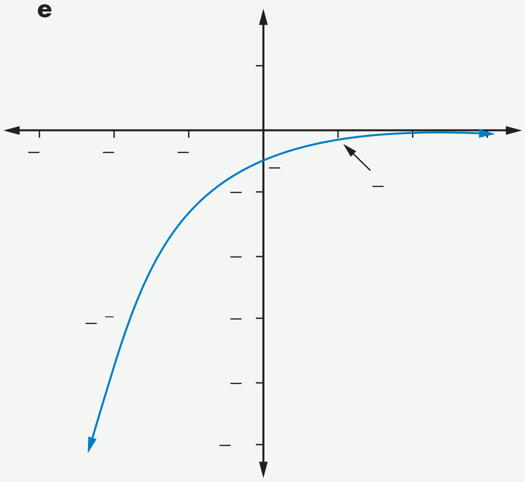

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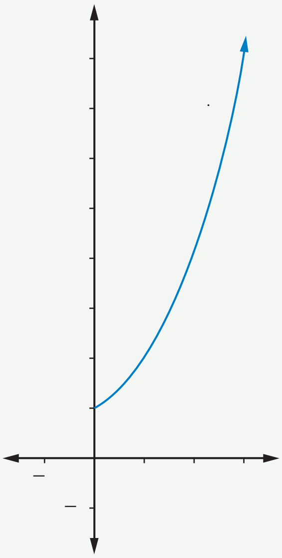

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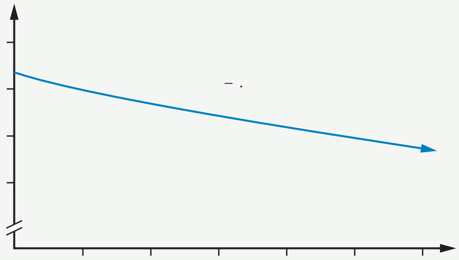

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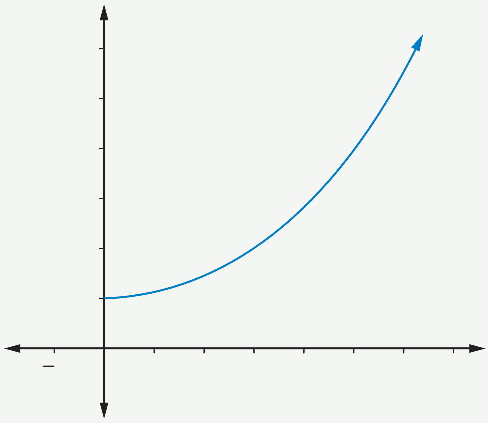

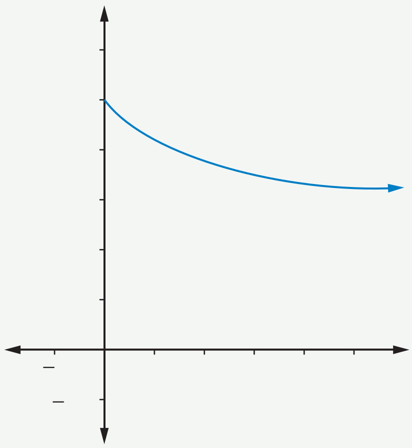

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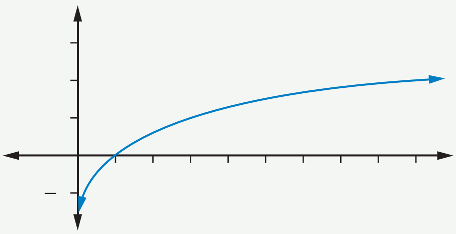

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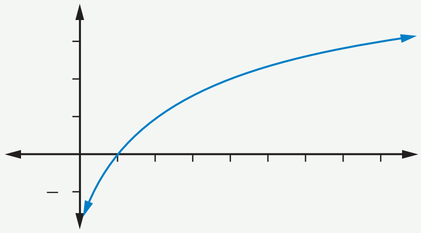

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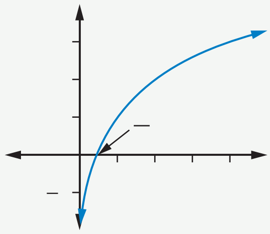

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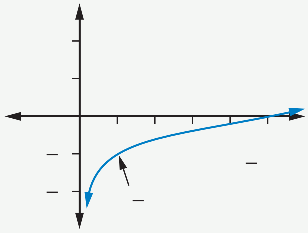

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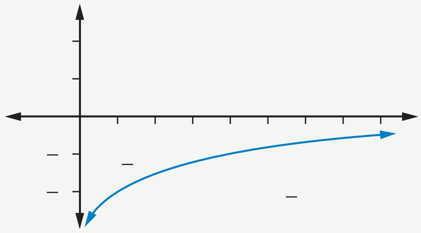

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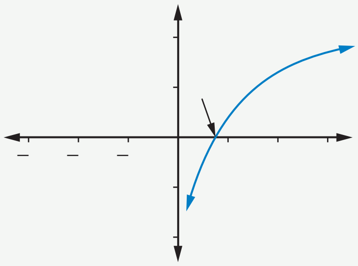

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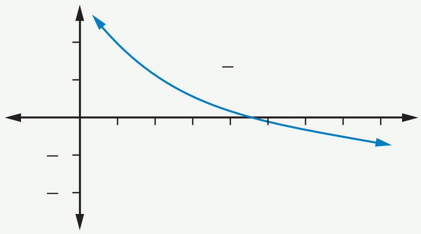

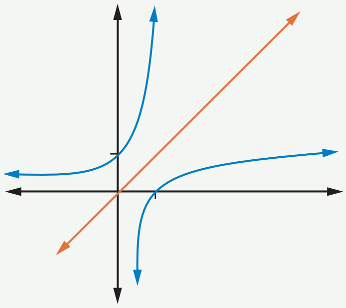

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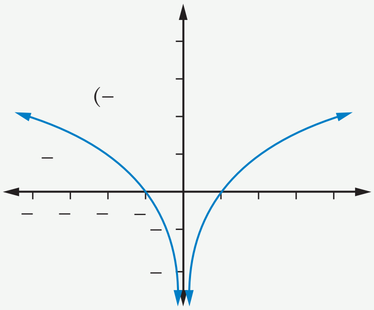

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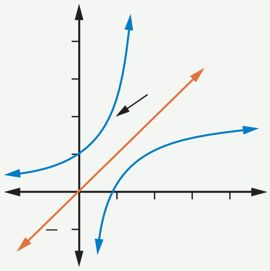

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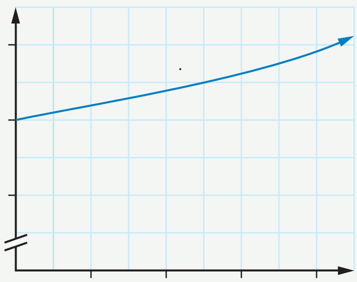

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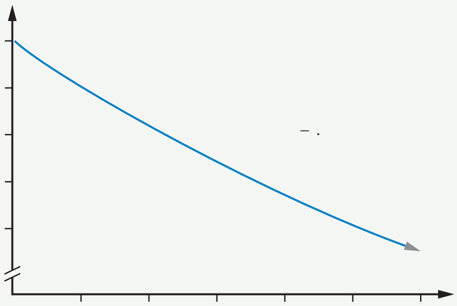

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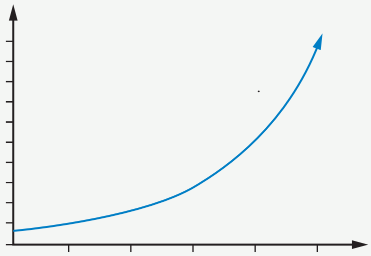

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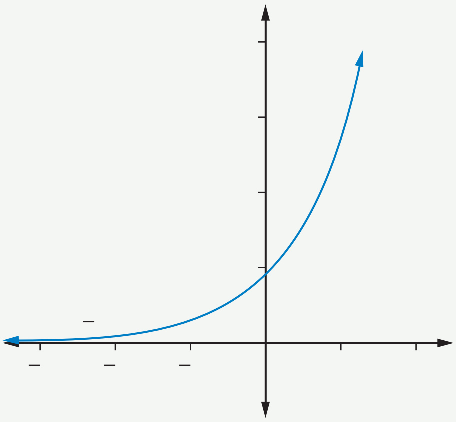

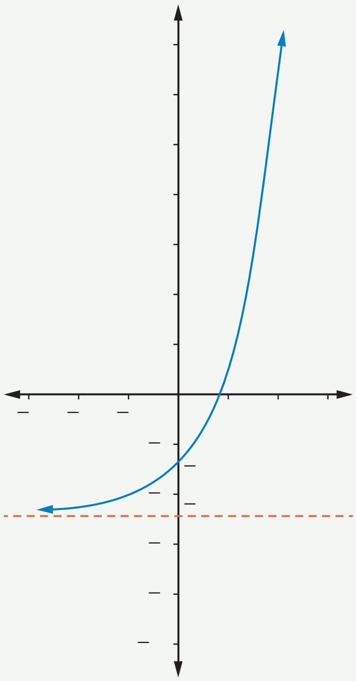

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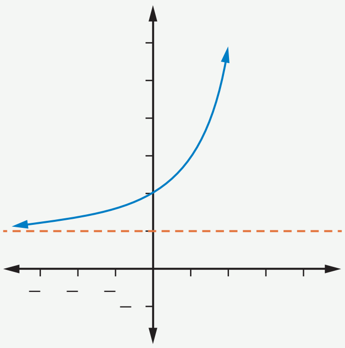

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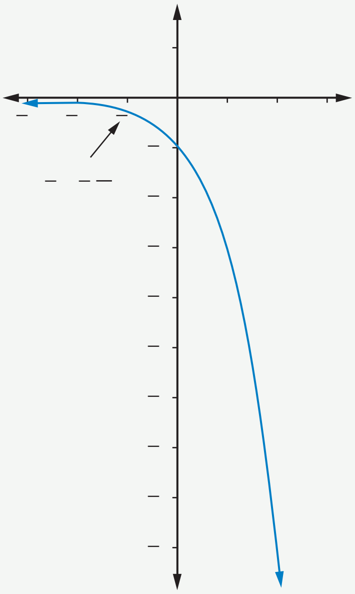

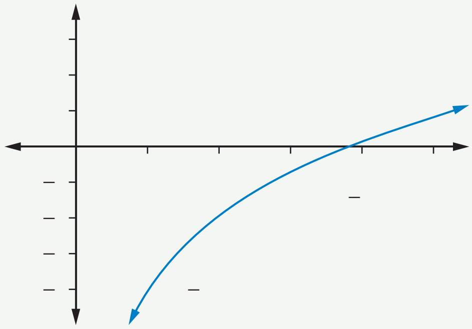

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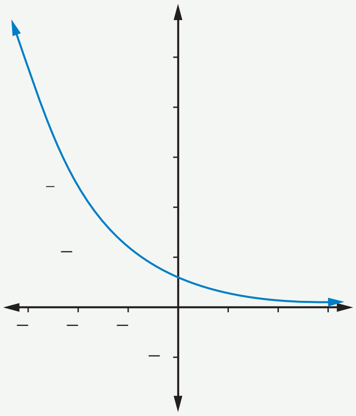

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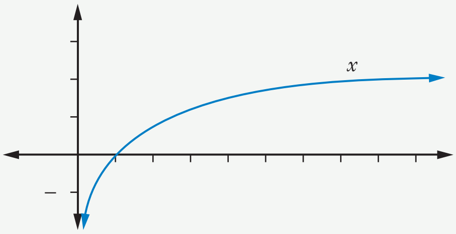

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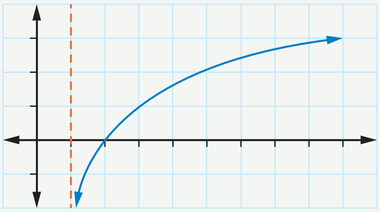

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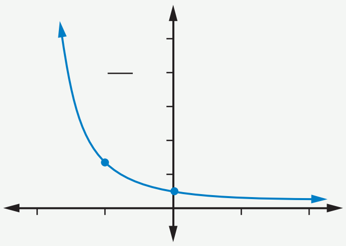

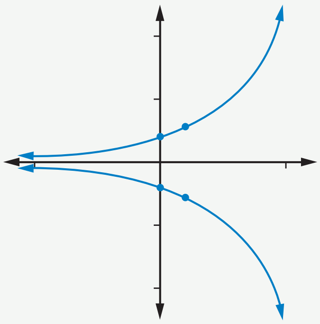

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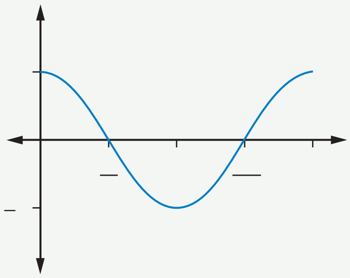

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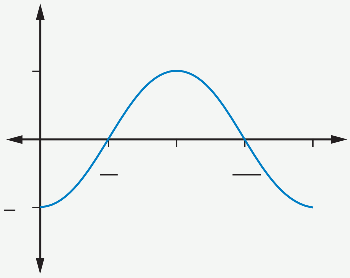

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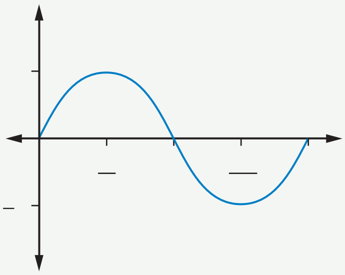

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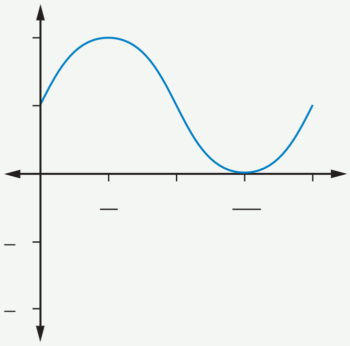

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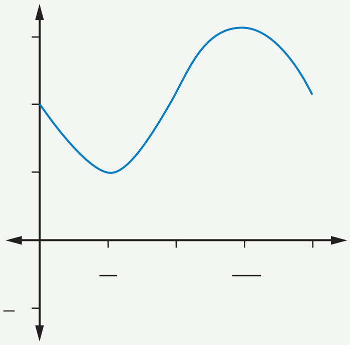

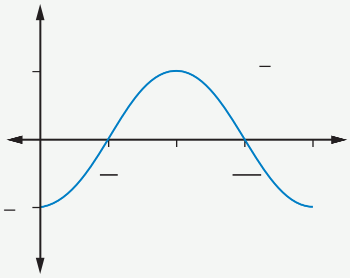

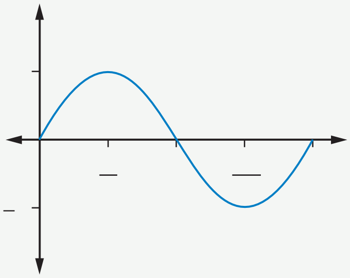

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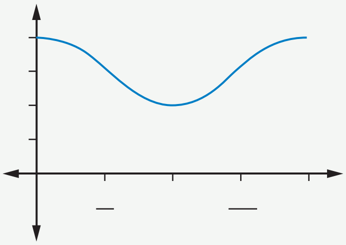

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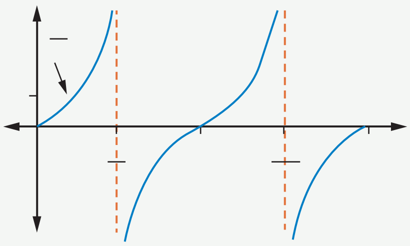

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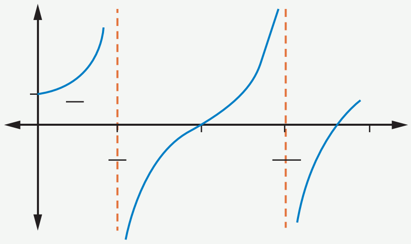

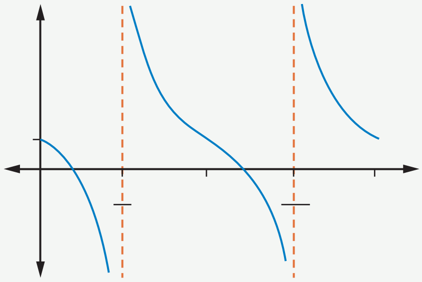

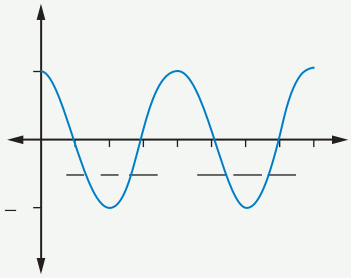

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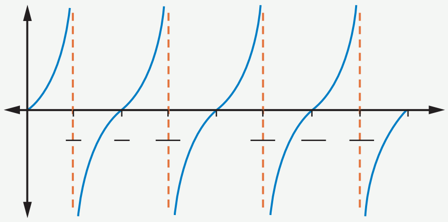

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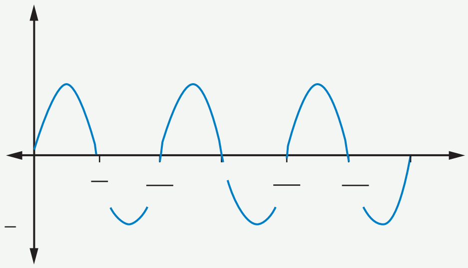

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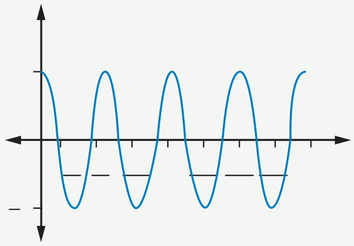

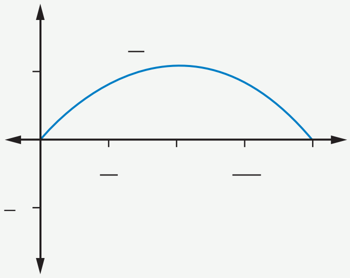

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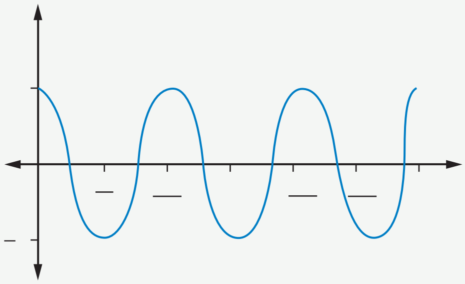

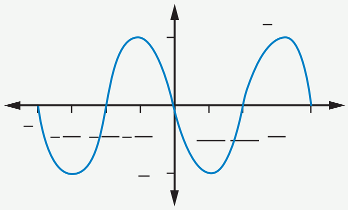

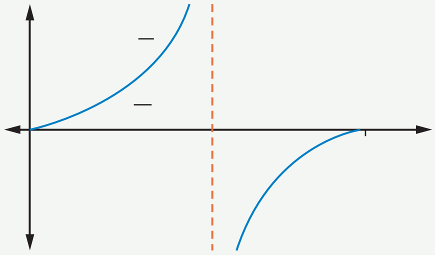

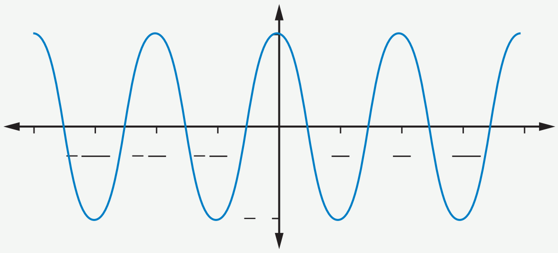

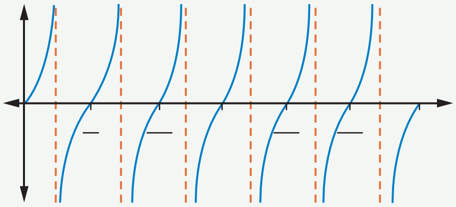

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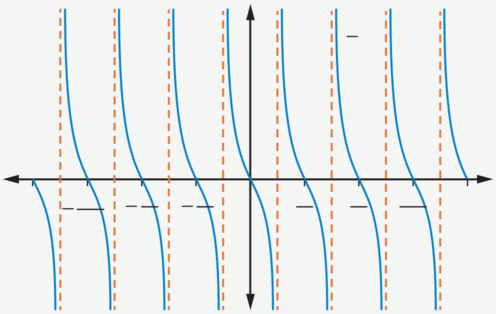

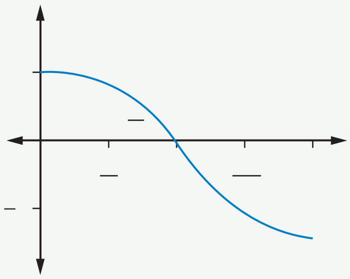

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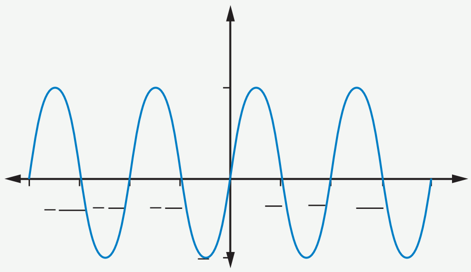

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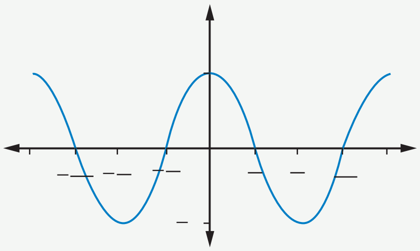

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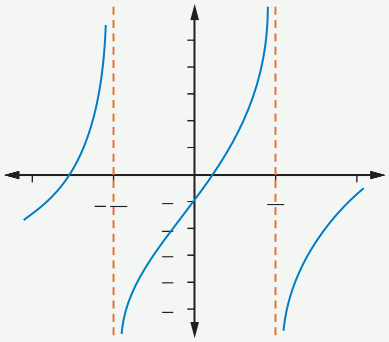

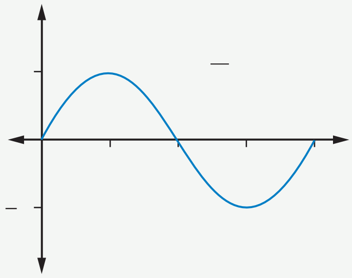

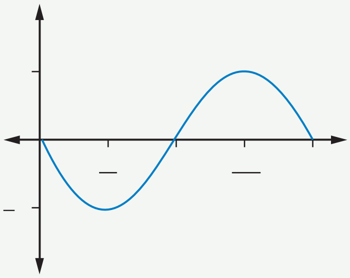

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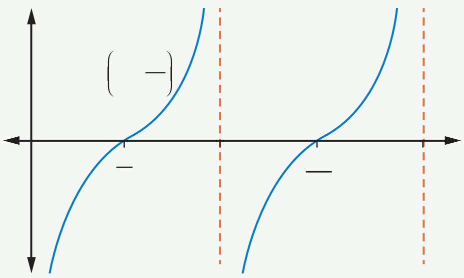

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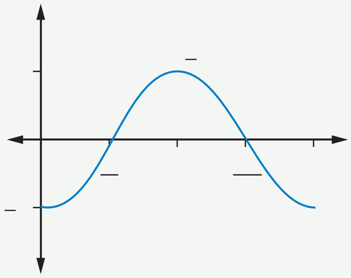

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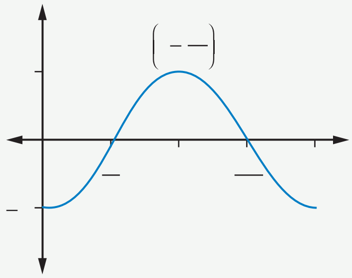

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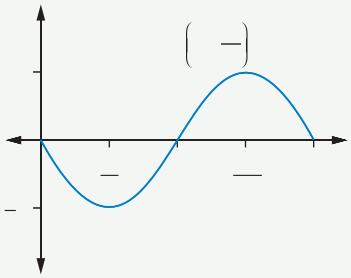

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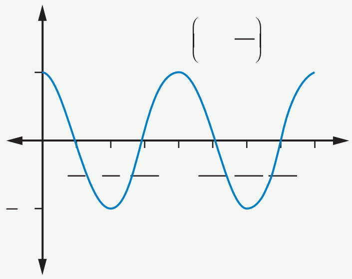

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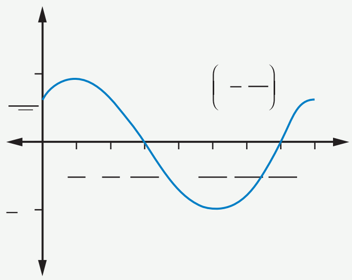

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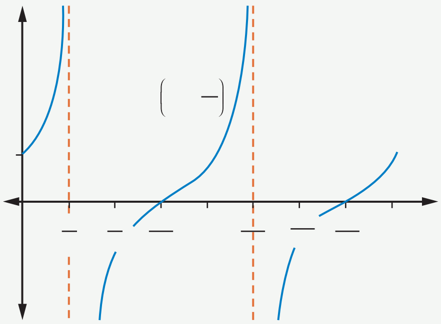

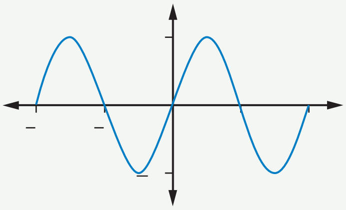

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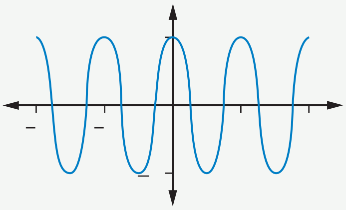

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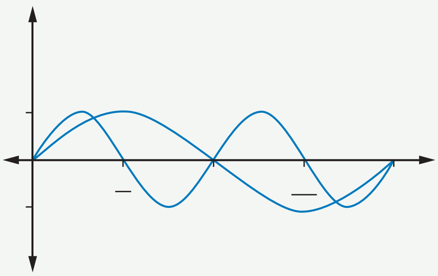

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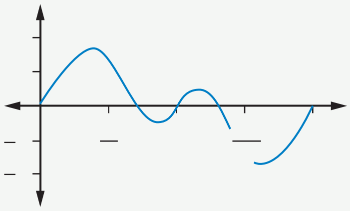

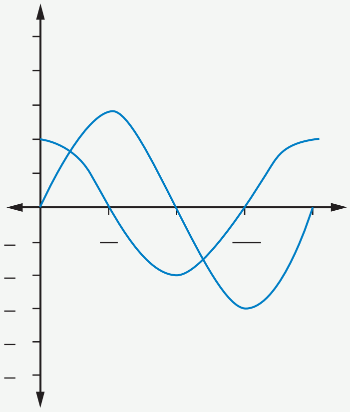

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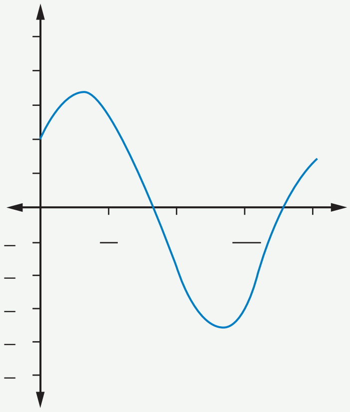

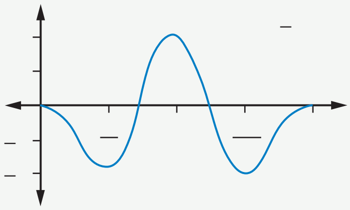

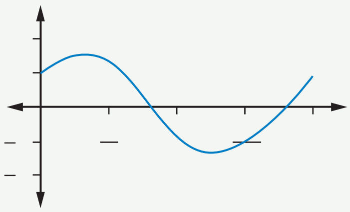

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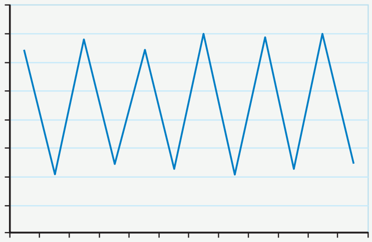

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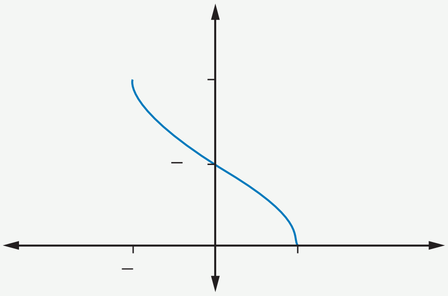

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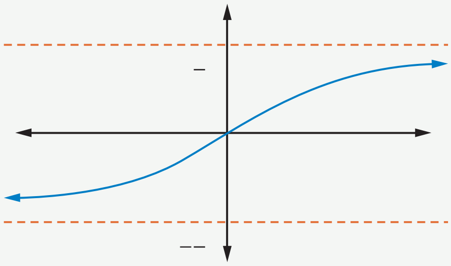

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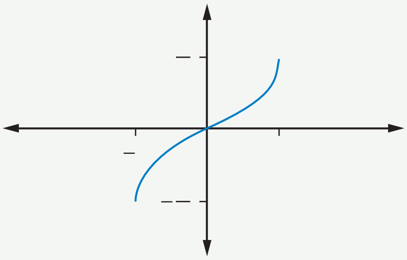

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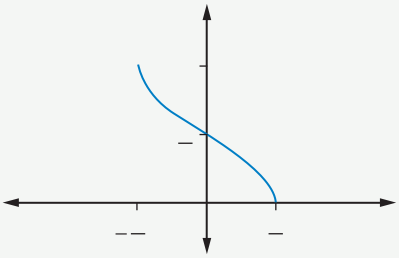

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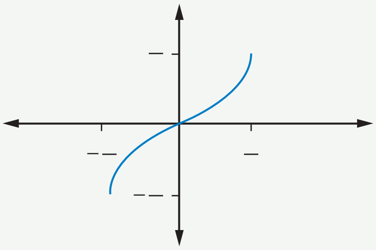

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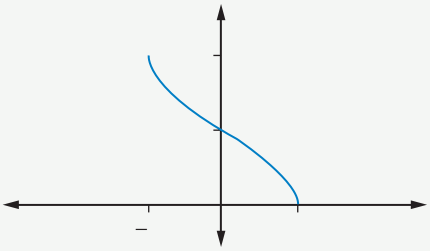

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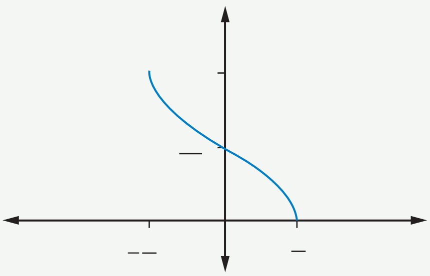

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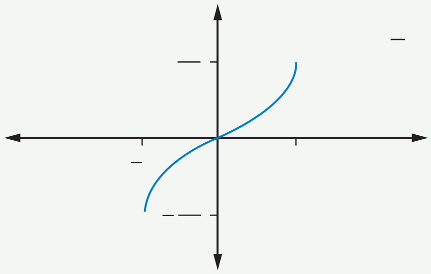

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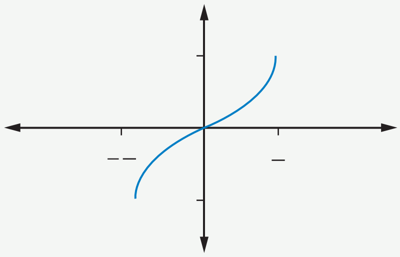

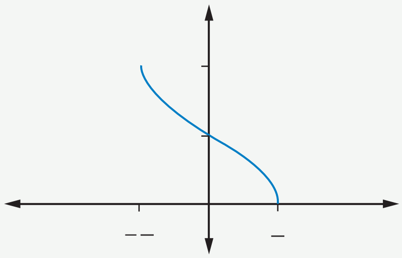

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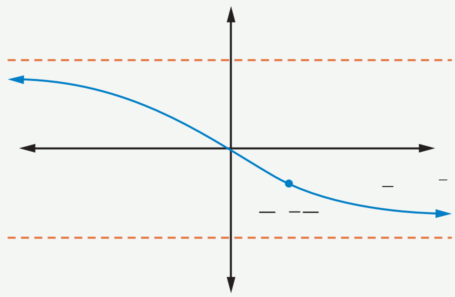

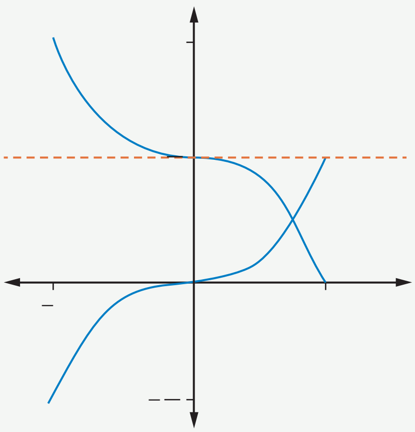

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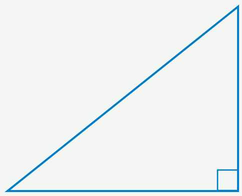

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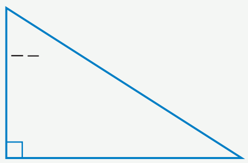

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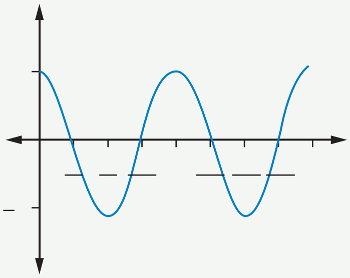

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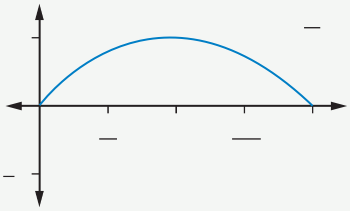

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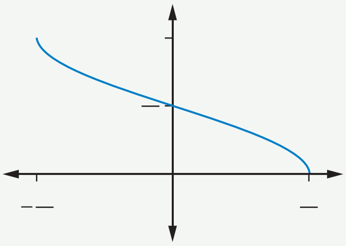

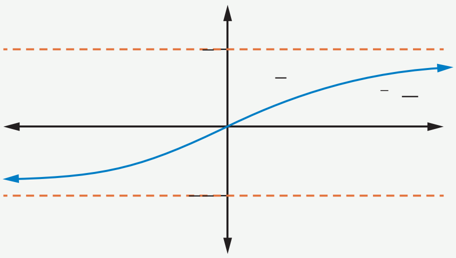

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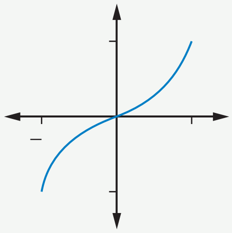

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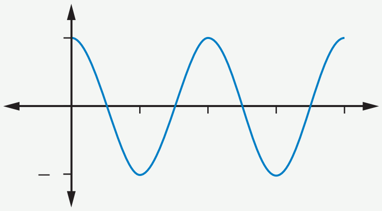

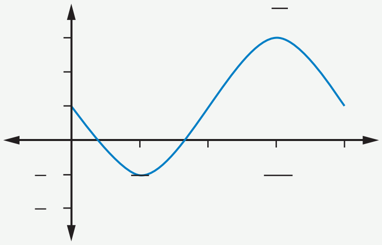

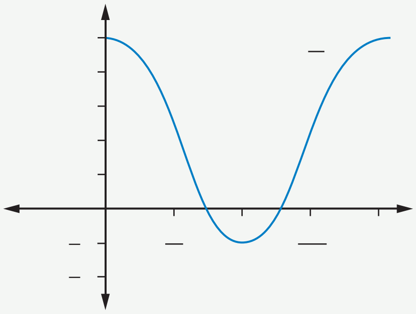

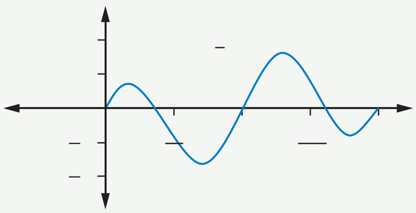

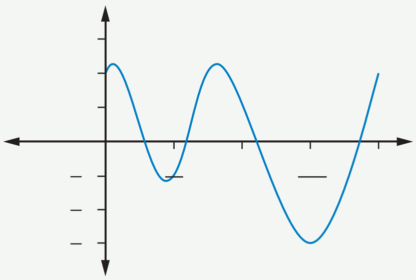

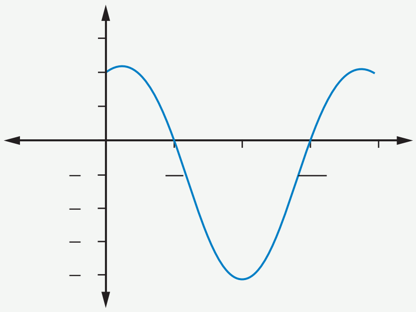

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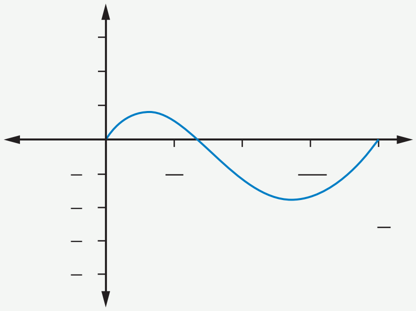

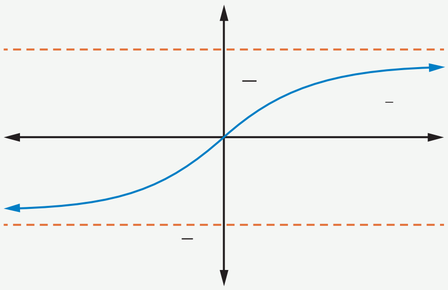

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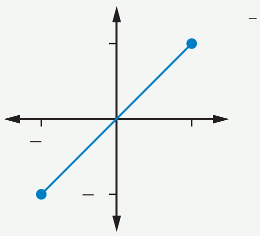

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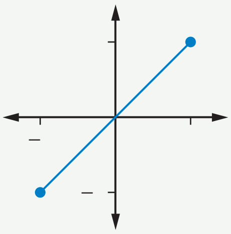

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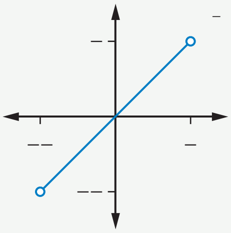

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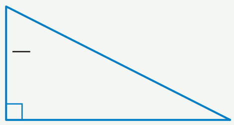

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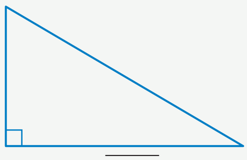

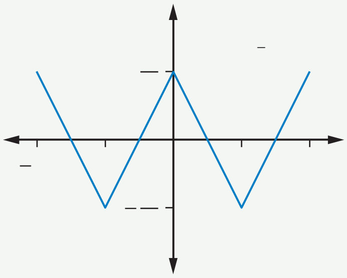

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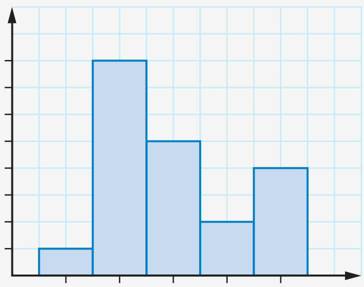

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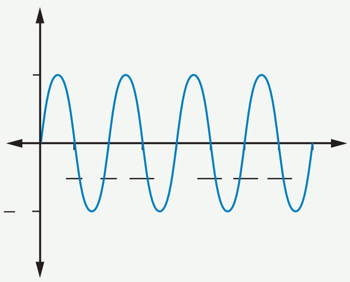

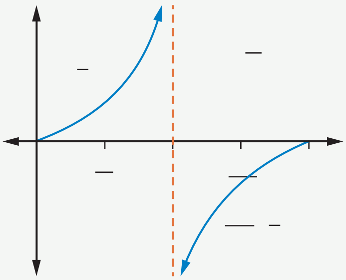

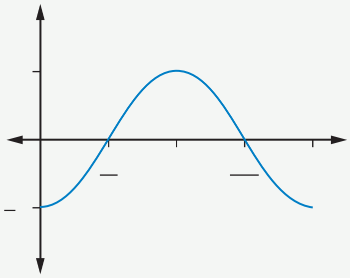

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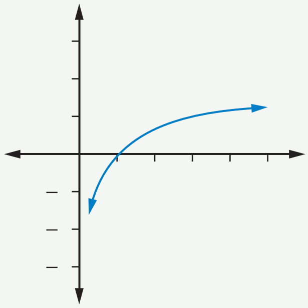

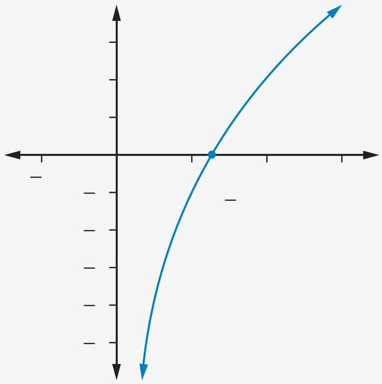

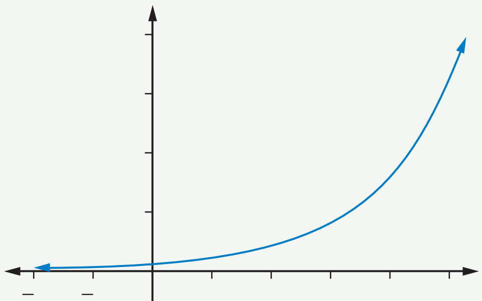

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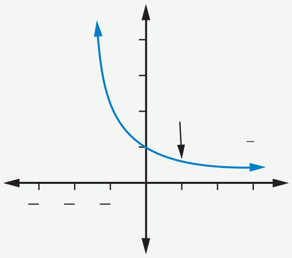

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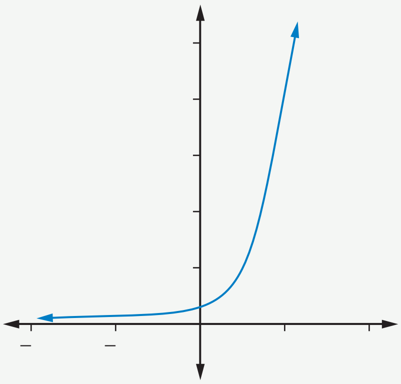

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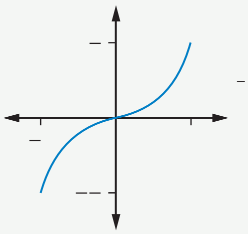

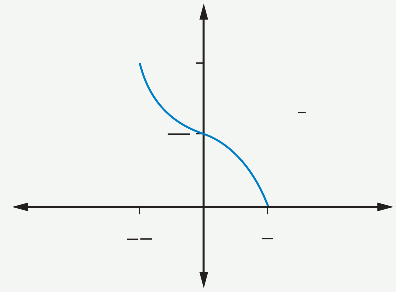

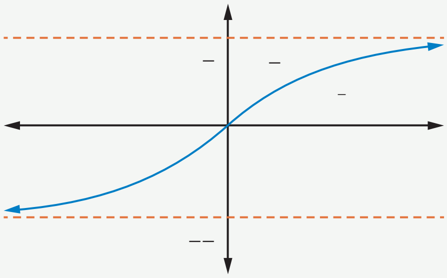

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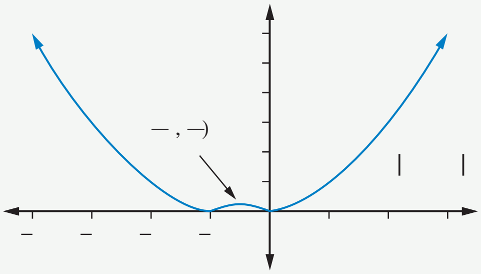

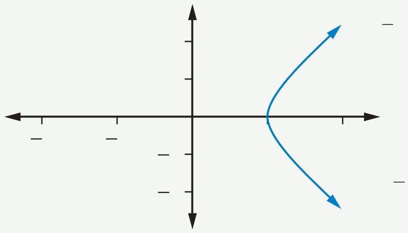

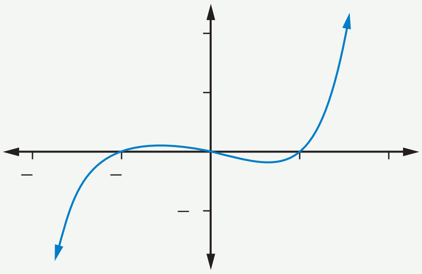

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