MATHS IN FOCUS 12 MATHEMATICS ADVANCED

WORKED SOLUTIONS

Chapter 1: Sequences and series

Exercise 1.01 General sequences and series

Question 1

a 14, 17, 20 (add 3 to each term)

b 23, 28, 33 (add 5 to each term)

c 44, 55, 66 (add 11 to each term)

d 85, 80, 75 (subtract 5 from each term)

e 1, –1, –3 (subtract 2 from each term)

f –15, –24, –33 (subtract 9 from each term)

g $2,\ 2\frac{1}{2},\ 3$ (add $\frac{1}{2}$ to each term)

h 3.1, 3.7, 4.3 (add 0.6 to each term)

i 32, –64, 128 (multiply each term by –2)

j $\frac{27}{320}, \frac{81}{1280}, \frac{243}{5120}$ (multiply each term by $\frac{3}{4}$)

Question 2

a $4 + 12 + 36 + 108 + 324 + 972$ (multiply each term by 3)
Sum = 1456

b $1 + 2 + 4 + 8 + 16 + 32$ (multiply each term by 2)
Sum = 63

c $3 + 7 + 11 + 15 + 19 + 23$ (add 4 to each term)
Sum = 78

d $-6 + 12 - 24 + 48 - 96 + 192$ (multiply each term by -2)
Sum = 126

e $1 + 4 + 9 + 16 + 25 + 36$ (the nth term is n^2)
Sum = 91

f $1 + 8 + 27 + 64 + 125 + 216$ (the nth term is n^3)
Sum = 441

Question 3

$\frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ (multiply each term by $\frac{1}{2}$),

Question 4

38, 51, 66, 83

3	+	6	+	11	+	18	+	27	+	38	+	51		66		83
	+3		+5		+7		+9		+11		+13		+15		+17	

Question 5

21, 34, 55, 89, 144 (Each term is the sum of the two preceding terms)

Question 6

Each term is the sum of the two terms immediately above it in the preceding row)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Exercise 1.02 Arithmetic sequences

Question 1

a $d = 9 - 5 = 4$
$y = 9 + d = 13$

b $d = 2 - 8 = -6$
$x = 2 + d = -4$

c $x = \dfrac{45+99}{2} = 72$

d $b = \dfrac{16+6}{2} = 11$

e $x = 21 - 14 = 7$

f $x - 1 = \dfrac{32+50}{2} = 41$
$x = 42$

g $5k + 2 = \dfrac{21+3}{2} = 12$
$5k = 10$
$k = 2$

h $(x+3) - x = (2x+5) - (x+3)$
$3 = x + 2$
$x = 1$

i $3t - (t-5) = (3t+1) - 3t$
$2t + 5 = 1$
$t = -2$

j $(3t+1) - (2t-3) = (5t+2) - (3t+1)$
$t + 4 = 2t + 1$
$t = 3$

Question 2

a $d = 7 - 4 = 3, \quad a = 4$
$T_{15} = 4 + (15-1) \times 3$
$T_{15} = 46$

b $d = 13 - 8 = 5, \quad a = 8$
$T_{15} = 8 + (15-1) \times 5$
$T_{15} = 78$

c $d = 16 - 10 = 6, \quad a = 10$
$T_{15} = 10 + (15-1) \times 6$
$T_{15} = 94$

d $d = 111 - 120 = -9, \quad a = 120$
$T_{15} = 120 + (15-1) \times -9$
$T_{15} = -6$

e $d = 2 - (-3) = 5, \quad a = -3$
$T_{15} = -3 + (15-1) \times 5$
$T_{15} = 67$

Question 3

a $d = 2-(-4) = 6, \quad a = -4$

$T_{100} = -4 + (100-1) \times 6$

$T_{100} = 590$

b $d = 32 - 41 = -9, \quad a = 41$

$T_{100} = 41 + (100-1) \times -9$

$T_{100} = -850$

c $d = 22 - 18 = 4, \quad a = 18$

$T_{100} = 18 + (100-1) \times 4$

$T_{100} = 414$

d $d = 140 - 125 = 15, \quad a = 125$

$T_{100} = 125 + (100-1) \times 15$

$T_{100} = 1610$

e $d = -5-(-1) = -4, \quad a = -1$

$T_{100} = -1 + (100-1) \times -4$

$T_{100} = -397$

Question 4

a $d = -18-(-14) = -4, \quad a = -14$

$T_{25} = -14 + (25-1) \times -4$

$T_{25} = -110$

b $d = 0.9 - 0.4 = 0.5, \quad a = 0.4$

$T_{25} = 0.4 + (25-1) \times 0.5$

$T_{25} = 12.4$

c $d = 0.9 - 1.3 = -0.4, \quad a = 1.3$

$T_{25} = 1.3 + (25-1) \times -0.4$

$T_{25} = -8.3$

d $d = 2\frac{1}{2} - 1 = 1\frac{1}{2}, \quad a = 1$

$T_{25} = 1 + (25-1) \times 1\frac{1}{2}$

$T_{25} = 37$

e $d = 2 - 1\frac{2}{5} = \frac{3}{5}, \quad a = 1\frac{2}{5}$

$T_{25} = 1\frac{2}{5} + (25-1) \times \frac{3}{5}$

$T_{25} = 15\frac{4}{5}$

Question 5

$T_n = 2n + 1$

$d = 5 - 3 = 2, \quad a = 3$

$T_n = 3 + (n-1) \times 2$

$= 3 + 2n - 2$

$= 2n + 1$

Question 6

a $d = 17 - 9 = 8, \quad a = 9$

$$\begin{aligned} T_n &= 9 + (n-1) \times 8 \\ &= 9 + 8n - 8 \\ &= 8n + 1 \end{aligned}$$

b $d = 102 - 100 = 2, \quad a = 100$

$$\begin{aligned} T_n &= 100 + (n-1) \times 2 \\ &= 100 + 2n - 2 \\ &= 2n + 98 \end{aligned}$$

c $d = 9 - 6 = 3, \quad a = 6$

$$\begin{aligned} T_n &= 6 + (n-1) \times 3 \\ &= 6 + 3n - 3 \\ &= 3n + 3 \end{aligned}$$

d $d = 86 - 80 = 6, \quad a = 80$

$$\begin{aligned} T_n &= 80 + (n-1) \times 6 \\ &= 80 + 6n - 6 \\ &= 6n + 74 \end{aligned}$$

e $d = -17 - (-21) = 4, \quad a = -21$

$$\begin{aligned} T_n &= -21 + (n-1) \times 4 \\ &= -21 + 4n - 4 \\ &= 4n - 25 \end{aligned}$$

f $d = 10 - 15 = -5, \quad a = 15$

$$\begin{aligned} T_n &= 15 + (n-1) \times -5 \\ &= 15 - 5n + 5 \\ &= 20 - 5n \end{aligned}$$

g $d = 1 - \frac{7}{8} = \frac{1}{8}, \quad a = \frac{7}{8}$

$$\begin{aligned} T_n &= \frac{7}{8} + (n-1) \times \frac{1}{8} \\ &= \frac{7}{8} + \frac{1}{8}n - \frac{1}{8} \\ &= \frac{1}{8}n + \frac{6}{8} \end{aligned}$$

h $d = -32 - (-30) = -2, \quad a = -30$

$$\begin{aligned} T_n &= -30 + (n-1) \times -2 \\ &= -30 - 2n + 2 \\ &= -2n - 28 \end{aligned}$$

i $d = 4.4 - 3.2 = 1.2, \quad a = 3.2$

$$\begin{aligned} T_n &= 3.2 + (n-1) \times 1.2 \\ &= 3.2 + 1.2n - 1.2 \\ &= 1.2n + 2 \end{aligned}$$

j $d = 1\frac{1}{4} - \frac{1}{2} = \frac{3}{4}, \quad a = \frac{1}{2}$

$$\begin{aligned} T_n &= \frac{1}{2} + (n-1) \times \frac{3}{4} \\ &= \frac{1}{2} + \frac{3}{4}n - \frac{3}{4} \\ &= \frac{3}{4}n - \frac{1}{4} \end{aligned}$$

Question 7

$d = 7 - 3 = 4, \quad a = 3$

$$\begin{aligned} T_n &= 3 + (n-1) \times 4 \\ &= 3 + 4n - 4 \\ &= 4n - 1 \end{aligned}$$

Require n such that $T_n = 111$

$4n - 1 = 111$

$4n = 112$

$n = 28$

28th term

Question 8

$d = 5 - 1 = 4, \quad a = 1$

$$\begin{aligned} T_n &= 1 + (n-1) \times 4 \\ &= 4n - 3 \end{aligned}$$

Require n such that $T_n = 213$

$4n - 3 = 213$

$4n = 216$

$n = 54$

54th term

Question 9

$d = 24 - 15 = 9, \quad a = 15$

$$\begin{aligned} T_n &= 15 + (n-1) \times 9 \\ &= 9n + 6 \end{aligned}$$

Require n such that $T_n = 276$

$9n + 6 = 276$

$9n = 270$

$n = 30$

30th term

Question 10

$d = 18 - 25 = -7, \quad a = 25$

$$T_n = 25 + (n-1) \times -7$$
$$= 32 - 7n$$

Require n such that $T_n = -73$

$32 - 7n = -73$

$-7n = -105$

$n = 15$

15th term

Question 11

$d = 45 - 48 = -3, \quad a = 48$

$$T_n = 48 + (n-1) \times -3$$
$$= 51 - 3n$$

Check to see if there is an n such that $T_n = 0$

$51 - 3n = 0$

$n = 17$

Yes, n = 17, 17th term

Question 12

$d = 11 - 3 = 8, \quad a = 3$

$$T_n = 3 + (n-1) \times 8$$
$$= 8n - 5$$

Check to see if there is an n such that $T_n = 270$

$8n - 5 = 270$

$8n = 275$

$n = 34\frac{3}{8}$

No, since n is not a positive whole number

Question 13

$d = 3 - 0 = 3, \quad a = 0$

$T_n = 0 + (n-1) \times 3$

$\quad = 3n$

Check to see if there is an n such that $T_n = 405$.

$3n = 405$

$n = 135$

Yes, n = 135, 135th term

Question 14

$d = 93 - 100 = -7, \quad a = 100$

$T_n = 100 + (n-1) \times -7$

$\quad = 107 - 7n$

Want $107 - 7n < 20$

$-7n < -87$

$n > \frac{87}{7}$

$n > 12\frac{3}{7}$

$n = 13$

Question 15

$d = -83 - (-86) = 3, \quad a = -86$

$T_n = -86 + (n-1) \times 3$

$\quad = 3n - 89$

Want $3n - 89 > 0$

$3n > 89$

$n > 29\frac{2}{7}$

$n \geq 30$, n = 30, 31, 32 …

Question 16

$d = 50 - 54 = -4, \quad a = 54$

$$\begin{aligned} T_n &= 54 + (n-1) \times -4 \\ &= 58 - 4n \end{aligned}$$

Want $58 - 4n < 0$

$58 < 4n$

$n > 14\frac{1}{2}$

Take $n = 15$

$T_{15} = 58 - 4 \times 15 = -2$

Question 17

$d = 7 - 3 = 4, \quad a = 3$

$$\begin{aligned} T_n &= 3 + (n-1) \times 4 \\ &= 4n - 1 \end{aligned}$$

Want $4n - 1 > 100$

$4n > 101$

$n > 25\frac{1}{4}$

Take $n = 26$

$T_{26} = 4(26) - 1 = 103$

Question 18

$a = -7, \quad d = 8$

$$\begin{aligned} T_n &= -7 + (n-1) \times 8 \\ &= 8n - 15 \end{aligned}$$

$$\begin{aligned} T_{100} &= 8(100) - 15 \\ &= 785 \end{aligned}$$

Question 19

a

$a = 15$

$T_n = 15 + (n-1)d$

$T_3 = 31$

$31 = 15 + (3-1)d$

$16 = 2d$

$d = 8$

b

$T_n = 15 + (n-1) \times 8$

$= 8n + 7$

$T_{10} = 8(10) + 7$

$= 87$

Question 20

$a = 3$

$T_n = 3 + (n-1)d$

$T_5 = 39$

$39 = 3 + (5-1)d$

$36 = 4d$

$d = 9$

Question 21

$T_2 = 19 \Rightarrow a + d = 19 \quad [1]$

$T_7 = 54 \Rightarrow a + 6d = 54 \quad [2]$

Equation [2] – equation [1] gives

$5d = 35$

$d = 7$

Substitute for d in [1]

$a + 7 = 19$

$a = 12$

Question 22

$T_4 = 29 \Rightarrow a + 3d = 29$ [1]

$T_{10} = 83 \Rightarrow a + 9d = 83$ [2]

Equation [2] – equation [1] gives

$6d = 54$

$d = 9$

Substitute for d in [1]

$a + 3 \times 9 = 29$

$a = 2$

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 2 + (n-1) \times 9 \\ &= 9n - 7 \end{aligned}$$

$T_{20} = 9 \times 20 - 7 = 173$

Question 23

$d = 6$

$T_5 = 29 \Rightarrow a + 4d = 29$

so $a + 4(6) = 29$

$a = 5$

Question 24

$T_3 = 45 \Rightarrow a + 2d = 45$ [1]

$T_9 = 75 \Rightarrow a + 8d = 75$ [2]

Equation [2] – equation [1] gives

$6d = 30$

$d = 5$

Substitute for d in [1]

$a + 2(5) = 45$

$a = 35$

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 35 + (n-1) \times 5 \\ &= 5n + 30 \end{aligned}$$

$T_{50} = 5(50) + 30 = 280$

Question 25

$T_7 = 17 \Rightarrow a + 6d = 17 \qquad [1]$

$T_{10} = 53 \Rightarrow a + 9d = 53 \qquad [2]$

Equation [2] – equation [1] gives

$3d = 36$

$d = 12$

Substitute for d in [1]

$a + 6(12) = 17$

$a = -55$

$$\begin{aligned} T_n &= a + (n-1)d \\ &= -55 + (n-1) \times 12 \\ &= 12n - 67 \end{aligned}$$

$$\begin{aligned} T_{100} &= 12(100) - 67 \\ &= 1133 \end{aligned}$$

Question 26

a $T_2 - T_1 = \log_5 x^2 - \log_5 x = 2\log_5 x - \log_5 x = \log_5 x$

$T_3 - T_2 = \log_5 x^3 - \log_5 x^2 = 3\log_5 x - 2\log_5 x = \log_5 x$

The difference between each pair of consecutive terms is the same.

b $a = \log_5 x, \quad d = \log_5 x$

$$\begin{aligned} T_n &= \log_5 x + (n-1) \times \log_5 x \\ &= n\log_5 x \end{aligned}$$

$T_{80} = 80\log_5 x$

c $T_{10} = 10\log_5 4 = 8.6$

Question 27

a

$$T_2 - T_1 = \sqrt{12} - \sqrt{3}$$
$$= 2\sqrt{3} - \sqrt{3}$$
$$= \sqrt{3}$$
$$T_3 - T_2 = \sqrt{27} - \sqrt{12}$$
$$= 3\sqrt{3} - 2\sqrt{3}$$
$$= \sqrt{3}$$

The difference between each pair of consecutive terms is the same.

b

$$a = \sqrt{3}, \quad d = \sqrt{3}$$
$$T_n = \sqrt{3} + (n-1) \times \sqrt{3}$$
$$= n\sqrt{3}$$
$$T_{50} = 50\sqrt{3}$$

Question 28

$$a = \log_2 4 = 2,\ \log_2 8 = 3,\ \log_2 16 = 4$$
$$a = 2, d = 1$$
$$T_n = 2 + (n-1) \times 1$$
$$= n + 1$$
$$T_{25} = 25 + 1 = 26$$

Question 29

$$a = 5b, d = 3b$$
$$T_n = 5b + (n-1) \times 3b$$
$$= 3bn + 2b$$
$$T_{40} = 3b \times 40 + 2b = 122b$$

Question 30

$$a = 28y, d = 5y$$
$$T_n = 28y + (n-1) \times 5y$$
$$= 5yn + 23y$$

Want n such that $T_n = 213y$

$$5yn + 23y = 213y$$
$$5n + 23 = 213, (y \neq 0)$$
$$5n = 190$$
$$n = 38$$

38th term

Exercise 1.03 Arithmetic series

Question 1

a $a = 4, d = 3, n = 15$

$$S_{15} = \frac{15}{2}[2\times 4 + (15-1)\times 3]$$
$$= 375$$

b $a = 2, d = 5, n = 15$

$$S_{15} = \frac{15}{2}[2\times 2 + (15-1)\times 5]$$
$$= 555$$

c $a = 60, d = -4, n = 15$

$$S_{15} = \frac{15}{2}[2\times 60 + (15-1)\times -4]$$
$$= 480$$

Question 2

a $a = 1,\ d = 6,\ n = 30$

$$S_{30} = \frac{30}{2}[2\times 1 + (30-1)\times 6]$$
$$= 2640$$

b $a = 15,\ d = 9,\ n = 30$

$$S_{30} = \frac{30}{2}[2\times 15 + (30-1)\times 9]$$
$$= 4365$$

c $a = 95,\ d = -6,\ n = 30$

$$S_{30} = \frac{30}{2}[2\times 95 + (30-1)\times -6]$$
$$= 240$$

Question 3

a $a = -2, d = 7, n = 25$

$$S_{30} = \frac{25}{2}[2\times -2 + (25-1)\times 7]$$
$$= 2050$$

b $a = 5, d = -9, n = 25$

$$S_{30} = \frac{25}{2}[2\times 5 + (25-1)\times -9]$$
$$= -2575$$

Question 4

a

$$a = 50, d = -6, n = 50$$

$$S_{30} = \frac{50}{2}[2\times 50 + (50-1)\times -6]$$

$$= -4850$$

b

$$a = 11, d = 3, n = 50$$

$$S_{30} = \frac{50}{2}[2\times 11 + (50-1)\times 3]$$

$$= 4225$$

Question 5

a

$$a = 15, d = 5, l = 535$$

$$T_n = 15 + (n-1)\times 5$$

$$= 5n + 10$$

$$T_n = 535$$

$$5n + 10 = 535$$

$$n = 105$$

$$S_{105} = \frac{105}{2}\left(15 + 535\right)$$

$$= 28\ 875$$

b

$$a = 9,\ d = 8,\ l = 225$$

$$T_n = 9 + (n-1)\times 8$$

$$= 8n + 1$$

$$T_n = 225$$

$$8n + 1 = 225$$

$$n = 28$$

$$S_{28} = \frac{28}{2}\left(9 + 225\right)$$

$$= 3276$$

c

$$a = 5,\ d = -3,\ l = -91$$

$$T_n = 5 + (n-1)\times -3$$

$$= -3n + 8$$

$$T_n = -91$$

$$-3n + 8 = -91$$

$$n = 33$$

$$S_{33} = \frac{33}{2}\left(5 - 91\right)$$

$$= -1419$$

d

$$a = 81,\ d = 11,\ l = 378$$

$$T_n = 81 + (n-1)\times 11$$

$$= 11n + 70$$

$$T_n = 378$$

$$11n + 70 = 378$$

$$n = 28$$

$$S_{28} = \frac{28}{2}\left(81 + 378\right)$$

$$= 6426$$

e

$$a = 229,\ d = -4,\ l = 25$$

$$T_n = 229 + (n-1)\times -4$$

$$= -4n + 233$$

$$T_n = 25$$

$$-4n + 233 = 25$$

$$n = 52$$

$$S_{105} = \frac{52}{2}\left(229 + 25\right)$$

$$= 6604$$

f

$$a = -2,\ d = 8,\ l = 94$$

$$T_n = -2 + (n-1)\times 8$$

$$= 8n - 10$$

$$T_n = 94$$

$$8n - 10 = 94$$

$$n = 13$$

$$S_{13} = \frac{13}{2}\left(-2 + 94\right)$$

$$= 598$$

g

$a = 0,\ d = -9,\ l = -216$

$T_n = 0 + (n-1) \times -9$

$\quad = 9 - 9n$

$T_n = -216$

$9 - 9n = -216$

$n = 25$

$S_{25} = \frac{25}{2}\left(0 - 216\right)$

$\quad = -2700$

h

$a = 79,\ d = 2,\ l = 229$

$T_n = 79 + (n-1) \times 2$

$\quad = 2n + 77$

$T_n = 229$

$2n + 77 = 229$

$n = 76$

$S_{76} = \frac{76}{2}\left(79 + 229\right)$

$\quad = 11\ 704$

i

$a = 14,\ d = -3,\ l = -43$

$T_n = 14 + (n-1) \times -3$

$\quad = 17 - 3n$

$T_n = -43$

$17 - 3n = -43$

$n = 20$

$S_{20} = \frac{20}{2}\left(14 - 43\right)$

$\quad = -290$

j

$a = 1\frac{1}{2}, d = \frac{1}{4}, l = 25\frac{1}{4}$

$T_n = 1\frac{1}{2} + (n-1) \times \frac{1}{4}$

$\quad = \frac{1}{4}n + 1\frac{1}{4}$

$T_n = 25\frac{1}{4}$

$\frac{1}{4}n + 1\frac{1}{4} = 25\frac{1}{4}$

$n = 96$

$S_{96} = \frac{96}{2}\left(1\frac{1}{2} + 25\frac{1}{4}\right)$

$\quad = 1284$

Question 6

$a = 45, d = 2, S = 1365$

$1365 = \frac{n}{2}[2 \times 45 + (n-1) \times 2]$

$1365 = n[n + 44]$

$n^2 + 44n - 1365 = 0$

$(n-21)(n+65) = 0$

$n = 21, n = -65$ (or use quadratic formula)

Take $n = 21$

Question 7

$a = 5, d = 4, S = 152$

$152 = \frac{n}{2}[2 \times 5 + (n-1) \times 4]$

$152 = n[2n + 3]$

$2n^2 + 3n - 152 = 0$

$(n-8)(2n+19) = 0$

$n = 8, n = -9.5$ (or use quadratic formula)

Take $n = 8$

Question 8

$a = 80, d = -7, S = 495$

$495 = \frac{n}{2}[2 \times 80 + (n-1) \times -7]$

$495 = n[-7n + 167]$

$7n^2 - 167n + 990 = 0$

$(n-11)(7n+90) = 0$

$n = 11, n = -\frac{90}{7}$ (or use quadratic formula)

Take $n = 11$

Question 9

$a = 20, d = -2, S = 104$

$104 = \frac{n}{2}[2 \times 20 + (n-1) \times -2]$

$104 = n[-n + 21]$

$n^2 - 21n + 104 = 0$

$(n-8)(n-13) = 0$

$n = 8, n = 13$ (or use quadratic formula)

8 and 13 terms

Question 10

$S_5 = 110$

$\frac{5}{2}[2a+(5-1)d] = 110 \Rightarrow a+2d = 22$ [1]

$S_{10} = 320$

$\frac{10}{2}[2a+9d] = 320 \Rightarrow 2a+9d = 64$ [2]

$[2] - 2\times[1]$

$5d = 20 \Rightarrow d = 4$

Substitute for d in [1]

$a+8 = 22 \Rightarrow a = 14$

Question 11

$S_5 = 35$

$\frac{5}{2}[2a+(5-1)d] = 35 \Rightarrow a+2d = 7$ [1]

$S_{10} = 35+160 = 195$

$\frac{10}{2}[2a+9d] = 195 \Rightarrow 2a+9d = 39$ [2]

$[2] - 2\times[1]$

$5d = 25 \Rightarrow d = 5$

Substitute for d in [1]

$a+10 = 7 \Rightarrow a = -3$

Question 12

$T_8 = 16$

$a+(8-1)d = 16 \Rightarrow a+7d = 16$ [1]

$T_{13} = 81$

$a+(13-1)d = 81 \Rightarrow a+12d = 81$ [2]

$(n-8)(n-13) = 0$

$[2]-[1]$

$5d = 65 \Rightarrow d = 13$

Substitute for d in [1]

$a+7\times 13 = 16 \Rightarrow a = -75$

$S_{25} = \frac{25}{2}[2\times -75+(25-1)\times 13] = 2025$

Question 13

$S_{12} = 186$

$\frac{12}{2}[2a + (12 - 1)d] = 186$

$2a + 11d = 31 \quad [1]$

$T_{20} = 83$

$a + (20 - 1) \times d = 83$

$a + 19d = 83 \quad [2]$

$2 \times [2] - [1]$

$27d = 135 \Rightarrow d = 5$

Substitute for d in [2]

$a + 19 \times 5 = 83 \Rightarrow a = -12$

$S_{40} = \frac{40}{2}[2 \times -12 + (40 - 1) \times 5] = 3420$

Question 14

$S_4 = 42$

$\frac{4}{2}[2a + (4 - 1)d] = 42$

$2a + 3d = 21 \quad [1]$

$T_3 + T_7 = 46$

$[a + (3 - 1) \times d] + [a + (7 - 1) \times d] = 46$

$2a + 8d = 46 \quad [2]$

$[2] - [1]$

$5d = 25 \Rightarrow d = 5$

Substitute for d in [1]

$2a + 3 \times 5 = 21 \Rightarrow a = 3$

$$\begin{aligned} S_{20} &= \frac{20}{2}[2 \times 3 + (20 - 1) \times 5] \\ &= 1010 \end{aligned}$$

Question 15

a $T_2 - T_1 = (2x+4)-(x+1) = x+3$

$T_3 - T_2 = (3x+7)-(2x+4) = x+3$

The difference between consecutive terms is the same.

b $A = x + 1$, $d = x + 3$, $n = 50$.

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$\begin{aligned} S_{50} &= \frac{50}{2}\left[2(x+1)+(50-1)(x+3)\right] \\ &= 25\left[2x+2+49(x+3)\right] \\ &= 25(2x+2+49x+147) \\ &= 25(51x+149) \end{aligned}$$

Question 16

$T_{20} = 131$

$a+(20-1)d] = 131$

$a+19d = 131 \qquad [1]$

$S_{10} - S_5 = 235$

$\frac{10}{2}[2a+(10-1)d] - \frac{5}{2}[2a+(5-1)d] = 235$

$10(2a+9d)-5(2a+4d) = 470$

$a+7d = 47 \qquad [2]$

$[1]-[2]$

$12d = 84 \Rightarrow d = 7$

Substitute for d in [2]

$a+7\times7 = 47 \Rightarrow a = -2$

$S_{20} = \frac{20}{2}[2\times -2+(20-1)\times 7] = 1290$

Question 17

$$\begin{aligned} T_{50} &= S_{50} - S_{49} \\ &= 249-233 \\ &= 16 \end{aligned}$$

Question 18

$$S_n - S_{n-1} = \frac{n}{2}[2a + (n-1)d] - \frac{n-1}{2}[2a + ((n-1)-1)d]$$

$$\begin{aligned}
2(S_n - S_{n-1}) &= n[2a + (n-1)d] - n[2a + ((n-1)-1)d] + [2a + ((n-1)-1)d] \\
&= 2an + n(n-1)d - 2an - n((n-1)-1)d + 2a + ((n-1)-1)d \\
&= n(n-1)d - n(n-1)d + nd + 2a + (n-1)d - d \\
&= nd + 2a + (n-1)d - d \\
&= nd + 2a + nd - 2d \\
&= 2(a + nd - d)
\end{aligned}$$

$$2(S_n - S_{n-1}) = 2T_n$$
$$S_n - S_{n-1} = T_n$$

Question 19

a The arithmetic sequence is $6, 12, 18, ...,$

$a = 6, d = 6$

$T_n = 96$

$6 + (n-1) \times 6 = 96$

$6n = 96$

$n = 16$

So $l = 16 \times 6 = 96$

$S_{16} = \frac{16}{2}(6 + 96) = 816$

b Require the sum of all the integers from 1 to 100 less the sum of the multiples of 6 from 1 to 100.

The sum of all integers from 1 and 100.

$a = 1,\ l = 100,\ n = 100$

$S_{100} = \frac{100}{2}(1 + 100) = 5050$

Sum of multiples of 6 is 816 (from **a**).

Required sum is 5050 – 816 = 4234

Exercise 1.04 Geometric sequences

Question 1

a No

$\frac{T_2}{T_1}=\frac{20}{5}=4$

$\frac{T_3}{T_2}=\frac{60}{20}=3$

The ratios are not the same.

b Yes

$\frac{T_2}{T_1}=-\frac{3}{4}$

$\frac{T_3}{T_2}=-2\frac{1}{4}\div 3=-\frac{3}{4}$

The ratios are the same. $r=-\frac{3}{4}$

c Yes

$\frac{T_2}{T_1}=\frac{3}{14}\div\frac{3}{4}=\frac{2}{7}$

$\frac{T_3}{T_2}=\frac{3}{49}\div\frac{3}{14}=\frac{2}{7}$

The ratios are the same. $r=\frac{2}{7}$

d No

$\frac{T_2}{T_1}=5\frac{5}{6}\div 7=\frac{5}{6}$

$\frac{T_3}{T_2}=3\frac{1}{3}\div 5\frac{5}{6}=\frac{4}{7}$

The ratios are not the same.

e No

$\frac{T_2}{T_1}=\frac{42}{-14}=-3$

$\frac{T_3}{T_2}=\frac{-168}{42}=-4$

The ratios are not the same.

f No

$\frac{T_2}{T_1}=\frac{8}{9}\div 1\frac{1}{3}=\frac{2}{3}$

$\frac{T_3}{T_2}=\frac{8}{27}\div\frac{8}{9}=\frac{1}{3}$

The ratios are not the same.

g Yes

$\frac{T_2}{T_1}=\frac{1.71}{5.7}=0.3$

$\frac{T_3}{T_2}=\frac{0.513}{1.71}=0.3$

The ratios are the same. $r=0.3$

h Yes

$\frac{T_2}{T_1}=-1\frac{7}{20}\div 2\frac{1}{4}=-\frac{3}{5}$

$\frac{T_3}{T_2}=\frac{81}{100}\div -1\frac{7}{20}=-\frac{3}{5}$

The ratios are the same. $r=-\frac{3}{5}$

i No

$\frac{T_2}{T_1}=\frac{9}{63}=\frac{1}{7}$

$\frac{T_3}{T_2}=1\frac{7}{8}\div 9=\frac{5}{18}$

The ratios are not the same.

j Yes

$\frac{T_2}{T_1}=15\div -1\frac{7}{8}=-8$

$\frac{T_3}{T_2}=-120\div 15=-8$

The ratios are the same. $r=-8$

Question 2

a $\frac{x}{28}=\frac{28}{4}$

$x=196$

b $\frac{y}{12}=\frac{12}{-3}$

$y=-48$

c $a=\pm\sqrt{2\times 72}=\pm 12$

d $\frac{y}{2}=\frac{2}{6}$

$y=\frac{2}{3}$

e $\frac{x}{8}=\frac{8}{32}$

$x=2$

f $p=\pm\sqrt{5\times 20}=\pm 10$

g $y=\pm\sqrt{7\times 63}=\pm 21$

h $m=\pm\sqrt{-3\times -12}=\pm 6$

i $x-4=\pm\sqrt{3\times 15}$

$x=4\pm 3\sqrt{5}$

j $k-1=\pm\sqrt{3\times 21}$

$k=1\pm 3\sqrt{7}$

k $t=\pm\sqrt{\frac{1}{4}\times\frac{1}{9}}=\pm\frac{1}{6}$

l $t=\pm\sqrt{\frac{1}{3}\times\frac{4}{3}}=\pm\frac{2}{3}$

Question 3

a $a=1, r=\frac{5}{1}=5$

$T_n=1\times 5^{n-1}=5^{n-1}$

b $a=1, r=\frac{1.02}{1}=1.02$

$T_n=1\times 1.02^{n-1}=1.02^{n-1}$

c $a=1, r=\frac{9}{1}=9$

$T_n=1\times 9^{n-1}=9^{n-1}$

d $a=2, r=\frac{10}{2}=5$

$T_n=2\times 5^{n-1}$

e $a=6, r=\frac{18}{6}=3$

$T_n=6\times 3^{n-1}$

f $a=8, r=\frac{16}{8}=2$

$T_n=8\times 2^{n-1}=2^3\times 2^{n-1}=2^{n+2}$

g $a=\frac{1}{4}, r=1\div\frac{1}{4}=4$

$T_n=\frac{1}{4}\times 4^{n-1}=4^{n-2}$

h $a=1000, r=-\frac{100}{1000}=-0.1$

$T_n=1000\times(-0.1)^{n-1}$

i $a=-3, r=\frac{9}{-3}=-3$

$T_n=-3\times(-3)^{n-1}=(-3)^n$

j $a=\frac{1}{3}, r=\frac{2}{15}\div\frac{1}{3}=\frac{2}{5}$

$T_n=\frac{1}{3}\left(\frac{2}{5}\right)^{n-1}$

Question 4

a $a = 8, r = \frac{24}{8} = 3, n = 6$

$T_6 = 8 \times 3^{6-1} = 1944$

b $a = 9, r = \frac{36}{9} = 4, n = 6$

$T_6 = 9 \times 4^{6-1} = 9216$

c $a = 8, r = \frac{-32}{8} = -4, n = 6$

$T_6 = 8 \times (-4)^{6-1} = -8192$

d $a = -1, r = \frac{5}{-1} = -5, n = 6$

$T_6 = -1 \times (-5)^{6-1} = 3125$

e $a = \frac{2}{3}, r = \frac{4}{9} \div \frac{2}{3} = \frac{2}{3}, n = 6$

$T_6 = \frac{2}{3} \times \left(\frac{2}{3}\right)^{6-1} = \left(\frac{2}{3}\right)^6 = \frac{64}{729}$

Question 5

a $a = 1, r = 2, n = 9$

$T_9 = 1 \times 2^{9-1} = 256$

b $a = 4, r = 3, n = 9$

$T_9 = 4 \times 3^{9-1} = 26\ 244$

c $a = 1, r = \frac{1.04}{1} = 1.04, n = 9$

$T_9 = 1 \times 1.04^{9-1} \approx 1.369$

d $a = -3, r = \frac{6}{-3} = -2, n = 9$

$T_9 = -3 \times (-2)^{9-1} = -768$

e $a = \frac{3}{4}, r = -\frac{3}{8} \div \frac{3}{4} = \frac{1}{2}, n = 9$

$T_9 = \frac{3}{4} \times \left(\frac{1}{2}\right)^{9-1} = \frac{3}{1024}$

Question 6

a $a = 3,\ r = 5,\ n = 8$

$T_8 = 3 \times 5^{8-1} = 234375$

b $a = 2.1,\ r = \frac{4.2}{2.1} = 2,\ n = 8$

$T_8 = 2.1 \times 2^{8-1} = 268.8$

c $a = 5, r = \frac{-20}{5} = -4, n = 8$

$T_8 = 5 \times (-4)^{8-1} = -81\ 920$

d $a = -\frac{1}{2}, r = \frac{3}{10} \div -\frac{1}{2} = -\frac{3}{5}, n = 8$

$T_8 = -\frac{1}{2} \times \left(-\frac{3}{5}\right)^{8-1} = \frac{2187}{156\ 250}$

e $a = 1\frac{47}{81}, r = 2\frac{10}{27} \div 1\frac{47}{81} = \frac{3}{2}, n = 8$

$T_8 = 1\frac{47}{81} \times \left(\frac{3}{2}\right)^{8-1} = 27$

Question 7

a $a = 3,\ r = \dfrac{6}{3} = 2,\ n = 20$

$T_{20} = 3 \times 2^{20-1} = 3 \times 2^{19}$

b $a = 1,\ r = \dfrac{7}{1} = 7,\ n = 20$

$T_{20} = 7 \times 7^{20-1} = 7^{19}$

c $a = 1.04,\ r = 1.04,\ n = 20$

$T_{20} = 1.04 \times 1.04^{20-1} = 1.04 \times 1.04^{19} = 1.04^{20}$

d $a = \dfrac{1}{4},\ r = \dfrac{1}{8} \div \dfrac{1}{4} = \dfrac{1}{2},\ n = 20$

$T_{20} = \dfrac{1}{4} \times \left(\dfrac{1}{2}\right)^{20-1} = \left(\dfrac{1}{2}\right)^{2} \times \left(\dfrac{1}{2}\right)^{19} = \left(\dfrac{1}{2}\right)^{21} = \dfrac{1}{2^{21}}$

e $a = \dfrac{3}{4},\ r = \dfrac{9}{16} \div \dfrac{3}{4} = \dfrac{3}{4},\ n = 20$

$T_{20} = \dfrac{3}{4} \times \left(\dfrac{3}{4}\right)^{20-1} = \dfrac{3}{4} \times \left(\dfrac{3}{4}\right)^{19} = \left(\dfrac{3}{4}\right)^{20}$

Question 8

$a = 1, r = 11, n = 50$

$T_{50} = 1 \times 11^{50-1} = 11^{49}$

Question 9

$a = 4, r = 5, T_n = 12500$

$4 \times 5^{n-1} = 12500$

$5^{n-1} = 3125$

$\log_{10}\left(5^{n-1}\right) = \log_{10}\left(3125\right)$

$(n-1)\log_{10}\left(5\right) = \log_{10}\left(3125\right)$

$n = 1 + \dfrac{\log_{10}\left(3125\right)}{\log_{10}\left(5\right)}$

$n = 6$

6th term

Question 10

$a = 6, r = 6, T_n = 7776$

$6 \times 6^{n-1} = 7776$

$6^n = 7776$

$\log_{10}\left(6^n\right) = \log_{10}\left(7776\right)$

$n\log_{10}\left(6\right) = \log_{10}\left(7776\right)$

$n = \frac{\log_{10}\left(7776\right)}{\log_{10}\left(6\right)}$

$n = 5$

5th term

Question 11

$a = 2, r = 8$

Assume $T_n = 1200$ is a term of the sequence

$2 \times 8^{n-1} = 1200$

$8^{n-1} = 600$

$\log_{10}\left(8^{n-1}\right) = \log_{10}\left(600\right)$

$(n-1)\log_{10}\left(8\right) = \log_{10}\left(600\right)$

$n = 1 + \frac{\log_{10}\left(600\right)}{\log_{10}\left(8\right)}$

$n \approx 4.07$

n is not a positive whole value, so 1200 is not a term of the sequence.

Question 12

$a = 3, r = 7, T_n = 352\ 947$

$3 \times 7^{n-1} = 352\ 947$

$7^{n-1} = 117\ 649$

$\log_{10}\left(7^{n-1}\right) = \log_{10}\left(117\ 649\right)$

$(n-1)\log_{10}\left(7\right) = \log_{10}\left(117\ 649\right)$

$n = 1 + \frac{\log_{10}\left(117\ 649\right)}{\log_{10}\left(7\right)}$

$n = 7$

7th term

Question 13

$$a = 8, r = \frac{-4}{8} = -\frac{1}{2}, T_n = \frac{1}{128}$$

$$8 \times \left(-\frac{1}{2}\right)^{n-1} = \frac{1}{128}$$

$$\left(-\frac{1}{2}\right)^{n-1} = \frac{1}{1024}$$

$$\log_{10}\left(\left(\left|-\frac{1}{2}\right|\right)^{n-1}\right) = \log_{10}\left(\frac{1}{1024}\right)$$

$$(n-1)\log_{10}\left(\left|-\frac{1}{2}\right|\right) = \log_{10}\left(\frac{1}{1024}\right)$$

$$n = 1 + \frac{\log_{10}\left(\frac{1}{1024}\right)}{\log_{10}\left(\left|-\frac{1}{2}\right|\right)}$$

$$n = 11$$

11th term

Question 14

$$a = 54, r = \frac{18}{54} = \frac{1}{3}, T_n = \frac{2}{243}$$

$$54 \times \left(\frac{1}{3}\right)^{n-1} = \frac{2}{243}$$

$$\left(\frac{1}{3}\right)^{n-1} = \frac{1}{6561}$$

$$\log_{10}\left(\left(\frac{1}{3}\right)^{n-1}\right) = \log_{10}\left(\frac{1}{6561}\right)$$

$$(n-1)\log_{10}\left(\frac{1}{3}\right) = \log_{10}\left(\frac{1}{6561}\right)$$

$$n = 1 + \frac{\log_{10}\left(\frac{1}{6561}\right)}{\log_{10}\left(\frac{1}{3}\right)}$$

$$n = 9$$

9th term

Question 15

$$a=-2, r=-1\frac{1}{2}\div -2=-\frac{3}{4}, T_n=-\frac{81}{128}$$

$$-2\times\left(-\frac{3}{4}\right)^{n-1}=-\frac{81}{128}$$

$$\left(-\frac{3}{4}\right)^{n-1}=\frac{81}{256}$$

$$(n-1)\log_{10}\left(\left(\left|-\frac{3}{4}\right|\right)^{n-1}\right)=\log_{10}\left(\frac{81}{256}\right)$$

$$n=1+\frac{\log_{10}\left(\frac{81}{256}\right)}{\log_{10}\left(\left|-\frac{3}{4}\right|\right)}=5$$

Question 16

$$a=7$$

$$T_6=1701$$

$$7r^{6-1}=1701$$

$$r^5=243$$

$$r=243^{\frac{1}{5}}=3$$

Question 17

a

$$T_4=-648\Rightarrow ar^3=-648 \qquad [1]$$

$$T_5=3888\Rightarrow ar^4=3888 \qquad [2]$$

$$[2]\div[1]$$

$$\frac{ar^4}{ar^3}=\frac{3888}{-648}$$

$$r=-6$$

b

$$ar^3=-648$$

$$a(-6)^3=-648$$

$$a=3$$

$$T_2=3\times(-6)^{2-1}=-18$$

Question 18

$T_3 = \frac{2}{5} \Rightarrow ar^2 = \frac{2}{5}$ [1]

$T_5 = 1\frac{3}{5} \Rightarrow ar^4 = 1\frac{3}{5}$ [2]

$[2] \div [1]$

$\frac{ar^4}{ar^2} = 1\frac{3}{5} \div \frac{2}{5}$

$r^2 = 4$

$r = \pm 2$

Substitute for r in equation [1].

$a(2)^2 = \frac{2}{5}$

$a = \frac{1}{10}$

Question 19

$a = 5000, r = \frac{1000}{5000} = \frac{1}{5}$

$T_n = 5000 \times \left(\frac{1}{5}\right)^{n-1}$

Require $T_n < 1$

Use $5000 \times \left(\frac{1}{5}\right)^{n-1} > 1$, since $\left(\frac{1}{5}\right)^{n-1}$ is a decreasing function

$\left(\frac{1}{5}\right)^{n-1} > \frac{1}{5000}$

$(n-1)\log_{10}\left(\frac{1}{5}\right) > \log_{10}\left(\frac{1}{5000}\right)$

$n-1 > \frac{\log_{10}\left(\frac{1}{5000}\right)}{\log_{10}\left(\frac{1}{5}\right)}$

$n-1 > 5.292$

$n > 6.292$

Take $n = 7$

Question 20

$a = \frac{2}{7}, r = \frac{6}{7} \div \frac{2}{7} = 3$

$T_n = \frac{2}{7} \times 3^{n-1}$

Require $T_n > 100$

$\frac{2}{7} \times 3^{n-1} > 100$

$3^{n-1} > 350$

$(n-1)\log_{10} 3 > \log_{10} 350$

$n - 1 > \frac{\log_{10} 350}{\log_{10} 3}$

$n - 1 > 5.33$

$n > 6.33$

Take $n = 7$

$T_7 = \frac{2}{7} \times 3^{7-1} = 208\frac{2}{7}$

Exercise 1.05 Geometric series

Question 1

a

$$a = 6, r = \frac{24}{6} = 4, n = 10$$

$$S_{10} = \frac{6(4^{10} - 1)}{4 - 1} = 2\,097\,150$$

b

$$a = 3, r = \frac{15}{3} = 5, n = 10$$

$$S_{10} = \frac{3(5^{10} - 1)}{5 - 1} = 7\,324\,218$$

Question 2

a

$$a = -1, r = \frac{7}{-1} = -7, n = 8$$

$$S_8 = \frac{-1((-7)^8 - 1)}{-7 - 1} = 720\,600$$

b

$$a = 8, r = \frac{24}{8} = 3, n = 8$$

$$S_8 = \frac{8(3^8 - 1)}{3 - 1} = 26\,240$$

Question 3

a

$$a = 4, r = \frac{8}{4} = 2, n = 15$$

$$S_{15} = \frac{4(2^{15} - 1)}{2 - 1} = 131\,068$$

b

$$a = \frac{3}{4}, r = -\frac{3}{8} \div \frac{3}{4} = -\frac{1}{2}, n = 15$$

$$S_{15} = \frac{\frac{3}{4}\left(1 - \left(-\frac{1}{2}\right)^{15}\right)}{1 - (-0.5)} = 0.5$$

Question 4

a

$$a = 2, r = 5$$

$$T_n = 6250$$

$$2 \times 5^{n-1} = 6250$$

$$5^{n-1} = 3125$$

$$(n-1)\log_{10} 5 = \log_{10} 3125$$

$$n = 1 + \frac{\log_{10} 3125}{\log_{10} 5} = 6$$

$$S_6 = \frac{2(5^6 - 1)}{5 - 1} = 7812$$

b

$$a = 18, r = 0.5$$

$$T_n = \frac{9}{64}$$

$$18 \times 0.5^{n-1} = \frac{9}{64}$$

$$0.5^{n-1} = \frac{1}{128}$$

$$(n-1)\log_{10} 0.5 = \log_{10}\left(\frac{1}{128}\right)$$

$$n = 1 + \frac{\log_{10}\left(\frac{1}{128}\right)}{\log_{10} 0.5} = 8$$

$$S_8 = \frac{18(1 - 0.5^8)}{1 - 0.5} = 35\frac{55}{64}$$

c

$a = 3, r = 7$

$T_n = 7203$

$3 \times 7^{n-1} = 7203$

$7^{n-1} = 2401$

$(n-1)\log_{10} 7 = \log_{10} 2401$

$n = 1 + \dfrac{\log_{10} 2401}{\log_{10} 7} = 5$

$S_5 = \dfrac{3(7^5 - 1)}{7 - 1} = 8403$

d

$a = \dfrac{3}{4}, r = 2\dfrac{1}{4} \div \dfrac{3}{4} = 3$

$T_n = 182\dfrac{1}{4}$

$\dfrac{3}{4} \times 3^{n-1} = 182\dfrac{1}{4}$

$3^{n-1} = 243$

$(n-1)\log_{10} 3 = \log_{10} 243$

$n = 1 + \dfrac{\log_{10} 243}{\log_{10} 3} = 6$

$S_6 = \dfrac{\frac{3}{4}\left(3^6 - 1\right)}{3 - 1} = 273$

e

$a = -3, r = -2$

$T_n = 384$

$-3 \times (-2)^{n-1} = 384$

$(-2)^{n-1} = -128$

$(n-1)\log_{10}|-2| = \log_{10}|-128|$

$n = 1 + \dfrac{\log_{10}|-128|}{\log_{10}|-2|} = 8$

$S_8 = \dfrac{-3\left((-2)^8 - 1\right)}{-2 - 1} = 255$

Question 5

a

$a = 7, r = 2, n = 9$

$T_9 = 7 \times 2^{9-1} = 1792$

b

$S_9 = \dfrac{7(2^9 - 1)}{2 - 1} = 3577$

Question 6

$a = 1.09, r = 1.09, n = 30$

$S_{30} = \dfrac{1.09(1.09^{30} - 1)}{1.09 - 1} = 148.58$

Question 7

$a = 1, r = 1.12, n = 25$

$S_{25} = \dfrac{1(1.12^{25} - 1)}{1.12 - 1} = 133.33$

Question 8

$$a = 11, r = 3$$

$$S_n = 108\ 251$$

$$\frac{11(3^n - 1)}{3 - 1} = 108\ 251$$

$$3^n - 1 = 19\ 682$$

$$3^n = 19\ 683$$

$$n \log_{10} 3 = \log_{10} 19\ 682$$

$$n = \frac{\log_{10} 19\ 682}{\log_{10} 3} = 9$$

Question 9

$$a = \frac{1}{2}, r = \frac{1}{2}$$

$$S_n = \frac{1023}{1024}$$

$$\frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} = \frac{1023}{1024}$$

$$1 - \left(\frac{1}{2}\right)^n = \frac{1023}{1024}$$

$$\left(\frac{1}{2}\right)^n = \frac{1}{1024}$$

$$\frac{1}{2^n} = \frac{1}{1024}$$

$$2^n = 1024$$

$$n \log_{10} 2 = \log_{10} 1024$$

$$n = \frac{\log_{10} 1024}{\log_{10} 2} = 10$$

10 terms

Question 10

$r = 4, S_5 = 3069$

$$\frac{a\left(4^5 - 1\right)}{4 - 1} = 3069$$

$341a = 3069$

$a = 9$

Question 11

$a = 4, r = 4, S_n > 1\ 000\ 000$

$$\frac{4\left(4^n - 1\right)}{4 - 1} > 1\ 000\ 000$$

$4^n > 750\ 001$

$n \log_{10} 4 > \log_{10} 750\ 001$

$$n > \frac{\log_{10} 750\ 001}{\log_{10} 4}$$

$n > 9.758$

Take $n = 10$

10 terms

Question 12

a $a = 2, r = 2, n = 10$

$$S_{10} = \frac{2\left(2^{10} - 1\right)}{2 - 1} = 2046$$

b $a = 1, d = 2, n = 10$

$$S_{10} = \frac{10}{2}[2 \times 1 + (10 - 1) \times 2] = 100$$

c $3 + 7 + 13 + \ldots = (1 + 2) + (3 + 4) + (5 + 8) + ..$

This is the sum of the first 10 terms of $2 + 4 + 8 + \ldots$ and $1 + 3 + 5 + \ldots$

Therefore, the sum is $2046 + 100 = 2146$ (using the answers from **a** and **b**).

Question 1

Choice 1 is an arithmetic series.

a = \$1, d = \$1, n = 30

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$
$$S_{30} = \frac{30}{2}\left[2(1) + 29(1)\right] = \$465$$

Choice 2 is a geometric series.

a = 1c, r = 2, n = 30

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_{30} = \frac{1(2^{30} - 1)}{2 - 1} = 1\ 073\ 741\ 823\text{ c} = \$10\ 737\ 418.23$$

Choice 2 is better

Question 2

Working backwards, if after the 7th gate he had 1 apple left, then after the 6th gate, add another apple and double, so (1 + 1) × 2 = 4 apples. After the 5th gate, add another apple and double, so (4 + 1) × 2 = 10 apples. So following this rule:

The man originally had 382 apples.

(You can check this answer by halving and subtracting 1 each time).

After gate	**Apples left**
7	1
6	4
5	10
4	22
3	46
2	94
1	190
0	382

Exercise 1.06 Limiting sum of an infinite geometric series

Question 1

a $r=\frac{3}{9}=\frac{1}{3}$

$|r|<1,$ so there is a limiting sum.

$a=9$

$S=\frac{9}{1-\frac{1}{3}}=13\frac{1}{2}$

b No

$r=1\div\frac{1}{2}=2$

$|r|>1,$ so there is no limiting sum.

c $r=\frac{2}{-4}=-\frac{1}{2}$

$|r|<1,$ so there is a limiting sum.

$a=-\frac{1}{4}$

$S=\frac{16}{1-\left(-\frac{1}{4}\right)}=12\frac{4}{5}$

d No

$r=\frac{7}{9}\div\frac{2}{3}=1\frac{1}{6}$

$|r|>1,$ so there is no limiting sum.

e $r=\frac{2}{3}$

$|r|<1,$ so there is a limiting sum.

$a=1$

$S=\frac{1}{1-\frac{2}{3}}=3$

f $r=\frac{1}{8}\div\frac{5}{8}=\frac{1}{5}$

$|r|<1,$ so there is a limiting sum.

$a=\frac{5}{8}$

$S=\frac{\frac{5}{8}}{1-\frac{1}{5}}=\frac{25}{32}$

g No

$r=\frac{36}{-6}=-6$

$|r|>1,$ so there is no limiting sum.

h $r=1\frac{7}{8}\div-2\frac{1}{4}=-\frac{5}{6}$

$|r|<1,$ so there is a limiting sum.

$a=-2\frac{1}{4}$

$S=\frac{-2\frac{1}{4}}{1-\left(-\frac{5}{6}\right)}=-1\frac{5}{22}$

i No

$r=\frac{1}{6}\div\frac{1}{9}=1\frac{1}{2}$

$|r|>1,$ so there is no limiting sum.

j $r=-\frac{4}{5}\div 2=-\frac{2}{5}$

$|r|<1,$ so there is a limiting sum.

$a=2$

$S=\frac{2}{1-\left(-\frac{2}{5}\right)}=1\frac{3}{7}$

Question 2

a 40 + 20 + 10 + …

$r = \frac{20}{40} = \frac{1}{2}$

$|r| < 1,$ so there is a limiting sum.

$a = 40$

$S = \dfrac{40}{1 - \frac{1}{2}} = 80$

b 320 + 80 + 20 + …

$r = \frac{320}{80} = \frac{1}{4}$

$|r| < 1,$ so there is a limiting sum.

$a = 320$

$S = \dfrac{320}{1 - \frac{1}{4}} = 426\frac{2}{3}$

c 100 – 50 + 25 – …

$r = \frac{-50}{100} = -\frac{1}{2}$

$|r| < 1,$ so there is a limiting sum.

$a = 100$

$S = \dfrac{100}{1 - \left(-\frac{1}{2}\right)} = 66\frac{2}{3}$

d $6 + 3 + 1\frac{1}{2} + \ldots$

$r = \frac{3}{6} = \frac{1}{2}$

$|r| < 1,$ so there is a limiting sum.

$a = 6$

$S = \dfrac{6}{1 - \frac{1}{2}} = 12$

e $\frac{2}{5} + \frac{6}{35} + \frac{18}{245} + \ldots$

$r = \dfrac{\frac{6}{35}}{\frac{2}{5}} = \frac{3}{7}$

$|r| < 1,$ so there is a limiting sum.

$a = \frac{2}{5}$

$S = \dfrac{\frac{2}{5}}{1 - \frac{3}{7}} = \frac{7}{10} = 0.7$

f 72 – 24 + 8 – …

$r = \frac{-24}{72} = -\frac{1}{3}$

$|r| < 1,$ so there is a limiting sum.

$a = 72$

$S = \dfrac{72}{1 - \left(-\frac{1}{3}\right)} = 54$

g $-12 + 2 - \frac{1}{3} + \dots$

$r = \frac{2}{-12} = -\frac{1}{6}$

$|r| < 1,$ so there is a limiting sum.

$a = -12$

$S = \frac{-12}{1-\left(-\frac{1}{6}\right)} = -\frac{72}{7} = -10\frac{2}{7}$

i $12 + 9 + 6\frac{3}{4} + \dots$

$r = \frac{9}{12} = \frac{3}{4}$

$|r| < 1,$ so there is a limiting sum.

$a = 12$

$S = \frac{12}{1-\frac{3}{4}} = 48$

h $\frac{3}{4} - \frac{1}{2} + \frac{1}{3} - \dots$

$r = \frac{-\frac{1}{2}}{\frac{3}{4}} = -\frac{2}{3}$

$|r| < 1,$ so there is a limiting sum.

$a = \frac{3}{4}$

$S = \frac{\frac{3}{4}}{1-\left(-\frac{2}{3}\right)} = \frac{9}{20} = 0.45$

j $-\frac{2}{3} + \frac{5}{12} - \frac{25}{96} + \dots$

$r = \frac{\frac{5}{12}}{-\frac{2}{3}} = -\frac{5}{8}$

$|r| < 1,$ so there is a limiting sum.

$a = -\frac{2}{3}$

$S = \frac{-\frac{2}{3}}{1-\left(-\frac{5}{8}\right)} = -\frac{16}{39}$

Question 3

a

$$a = 56, r = -\frac{1}{2}$$

$$S = \frac{56}{1-\left(-\frac{1}{2}\right)} = 37\frac{1}{3}$$

$$S_6 = \frac{56\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\left(-\frac{1}{2}\right)} = 36.75$$

$$37\frac{1}{3} - 36.75 = 0.58$$

(to 2 significant figures)

b

$$a = 72, r = \frac{1}{3}$$

$$S = \frac{72}{1-\frac{1}{3}} = 108$$

$$S_6 = \frac{72\left(1-\left(\frac{1}{3}\right)^6\right)}{1-\frac{1}{3}} \approx 107.852$$

$$108 - 107.852 = 0.15$$

(to 2 significant figures)

c

$$a = 1, r = \frac{1}{5}$$

$$S = \frac{1}{1-\frac{1}{5}} = 1.25$$

$$S_6 = \frac{1\left(1-\left(\frac{1}{5}\right)^6\right)}{1-\frac{1}{5}} = 1.24992$$

$$1.25 - 1.24992 = 0.000080$$

(to 2 significant figures)

d

$$a = \frac{1}{2}, r = \frac{1}{2}$$

$$S = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$S_6 = \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = 0.984375$$

$$1 - 0.984375 = 0.016$$

(to 2 significant figures)

e

$$a = 1\frac{1}{4}, r = \frac{3}{4}$$

$$S = \frac{1\frac{1}{4}}{1-\frac{3}{4}} = 5$$

$$S_6 = \frac{1\frac{1}{4}\left(1-\left(\frac{3}{4}\right)^6\right)}{1-\frac{3}{4}} \approx 4.1101$$

$$5 - 4.1101 = 0.89$$

(to 2 significant figures)

Question 4

$$LS = 6, r = \frac{1}{3}$$

$$\frac{a}{1-\frac{1}{3}} = 6$$

$$a = 4$$

Question 5

$$LS = 5, a = 3$$

$$\frac{3}{1-r} = 5$$

$$5 - 5r = 3$$

$$r = \frac{2}{5}$$

Question 6

$$LS = 9\frac{1}{3}, r = \frac{2}{5}$$

$$\frac{a}{1-\frac{2}{5}} = 9\frac{1}{3}$$

$$a = 9\frac{1}{3}\left(1-\frac{2}{5}\right) = 5\frac{3}{5}$$

Question 7

$$LS = 40, a = 5$$

$$\frac{5}{1-r} = 40$$

$$40 - 40r = 5$$

$$r = \frac{7}{8}$$

Question 8

$$LS = -6\frac{2}{5}, a = -8$$
$$\frac{-8}{1-r} = -6\frac{2}{5}$$
$$-6\frac{2}{5} + 6\frac{2}{5}r = -8$$
$$r = -\frac{1}{4}$$

Question 9

$$LS = -\frac{3}{10}, a = -\frac{1}{2}$$
$$\frac{-\frac{1}{2}}{1-r} = -\frac{3}{10}$$
$$-\frac{3}{10} + \frac{3}{10}r = -\frac{1}{2}$$
$$r = -\frac{2}{3}$$

Question 10

$$ar^{2-1} = 2$$
$$r = \frac{2}{a}$$
$$LS = 9$$
$$\frac{a}{1-r} = 9$$
$$\frac{a}{1-\frac{2}{a}} = 9$$
$$\frac{a^2}{a-2} = 9$$
$$a^2 - 9a - 18 = 0$$
$$(a-3)(a-6) = 0$$
$$a = 3, r = \frac{2}{a} = \frac{2}{3}$$
$$a = 6, r = \frac{2}{6} = \frac{1}{3}$$

Question 11

$ar^2 = 12 \quad [1]$

$ar^3 = -3 \quad [2]$

equation [2] ÷ equation [1]

$r = \frac{-3}{12} = -\frac{1}{4}$

$ar^2 = 12$

$a\left(-\frac{1}{4}\right)^2 = 12$

$a = 192$

$S = \frac{192}{1-\left(-\frac{1}{4}\right)} = 153\frac{3}{5}$

Question 12

$ar = \frac{2}{3} \quad [1]$

$ar^3 = \frac{8}{27} \quad [2]$

Equation [2] ÷ equation [1]

$r^2 = \frac{8}{27} \div \frac{2}{3} = \frac{4}{9}$

$r = \pm\frac{2}{3}$

$ar = \frac{2}{3}$

$\pm\frac{2}{3}a = \frac{2}{3}$

$a = \pm 1$

$a = 1, r = \frac{2}{3}$

$S = \frac{a}{1-r} = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3$

$a = -1, r = -\frac{2}{3}$

$S = \frac{a}{1-r} = \frac{-1}{1-\left(-\frac{2}{3}\right)} = \frac{-1}{\frac{5}{3}} = -\frac{3}{5}$

Question 13

$ar^2 = 54 \qquad [1]$

$ar^5 = 11\frac{83}{125} \qquad [2]$

Equation [2] ÷ equation [1]

$r^3 = 11\frac{83}{125} \div 54 = \frac{27}{125}$

$r = \left(\frac{27}{125}\right)^{\frac{1}{3}} = \frac{3}{5}$

$ar^2 = 54$

$\left(\frac{3}{5}\right)^2 a = 54$

$a = 150$

$a = 150, r = \frac{3}{5}$

$S = \frac{a}{1-r} = \frac{150}{1-\frac{3}{5}} = \frac{150}{\frac{2}{5}} = 150 \times \frac{5}{2} = 375$

Question 14

$ar = \frac{4}{15}$ [1]

$ar^4 = \frac{32}{405}$ [2]

Equation [2] ÷ equation [1]

$r^3 = \frac{32}{405} \div \frac{4}{15} = \frac{8}{27}$

$r = \left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{2}{3}$

$ar = \frac{4}{15}$

$\frac{2}{3}a = \frac{4}{15}$

$a = \frac{2}{5}$

$S = \frac{\frac{2}{5}}{1-\frac{2}{3}} = 1\frac{1}{5}$

Question 15

$\frac{a}{1-r} = 5$ [1]

$ar = 1\frac{1}{5} \quad \Rightarrow r = \frac{6}{5a}$ [2]

Substitute equation [2] in equation [1] for r

$\frac{a}{1-\frac{6}{5a}} = 5$

$\frac{5a^2}{5a-6} = 5$

$a^2 - 5a + 6 = 0$

$(a-2)(a-3) = 0$

$a = 2, a = 3$

$a = 2,\ r = \frac{6}{5\times 2} = \frac{3}{5}$

$a = 3,\ r = \frac{6}{5\times 3} = \frac{2}{5}$

Question 16

$$x = \frac{21}{32}$$

$$a = x, r = \frac{1}{4}$$

$$LS = \frac{7}{8}$$

$$\frac{x}{1-\frac{1}{4}} = \frac{7}{8}$$

$$\frac{4x}{3} = \frac{7}{8}$$

$$x = \frac{21}{32}$$

Question 17

a Common ratio is k

For a limiting sum, require $|k| < 1$

b $a = k$

$$LS = \frac{k}{1-k} = \frac{-\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = -\frac{2}{5}$$

c $\frac{k}{1-k} = 3$

$k = 3 - 3k$

$4k = 3$

$k = \frac{3}{4}$

Question 18

$$S - S_n = \frac{a}{1-r} - \frac{a(1-r^n)}{1-r} = \frac{a - a(1-r^n)}{1-r} = \frac{a - a + ar^n}{1-r} = \frac{ar^n}{1-r}$$

Test yourself 1

Question 1

C

A and D are for arithmetic sequences, and B is for the limiting sum of a geometric sequence.

Question 2

C

The limiting sum only exists when the magnitude of the common ratio is less than 1.

Question 3

B

$$
\begin{aligned}
a &= 12, d = -3 \\
T_n &= a + (n-1)d \\
&= 12 + (n-1) \times -3 \\
&= 15 - 3n
\end{aligned}
$$

Question 4

a

$$
\begin{aligned}
a &= 9, d = 4 \\
T_n &= 9 + (n-1) \times 4 \\
&= 4n + 5
\end{aligned}
$$

b

$$
\begin{aligned}
a &= 7, d = -7 \\
T_n &= 7 + (n-1) \times -7 \\
&= 14 - 7n
\end{aligned}
$$

c

$$
\begin{aligned}
a &= 2, r = 3 \\
T_n &= 2 \times 3^{n-1}
\end{aligned}
$$

d

$$
a = 200, r = \frac{50}{200} = \frac{1}{4}
$$

$$
T_n = 200 \times \left(\frac{1}{4}\right)^{n-1}
$$

e

$$
\begin{aligned}
a &= -2, r = -2 \\
T_n &= -2 \times (-2)^{n-1} \\
&= (-2)^n
\end{aligned}
$$

Question 5

a $a = 156, d = 145 - 156 = -11, n = 15$

$T_{15} = 156 + (15 - 1) \times -11 = 2$

b $S_{15} = \frac{15}{2}[2 \times 156 + (15 - 1) \times -11] = 1185$

c $S_{14} = \frac{14}{2}[2 \times 156 + (14 - 1) \times -11] = 1183$

d $T_{15} = S_{15} - S_{14}$

e $156 + (n - 1) \times -11 < 0$

$167 - 11n < 0$

$n > 15.18$

Take $n = 16$

Question 6

a

i, $\left.\begin{aligned} T_2 - T_1 &= -4 \\ T_3 - T_2 &= -4 \end{aligned}\right\}$ common difference

$\left.\begin{aligned} \frac{T_2}{T_1} &\approx 0.9587 \\ \frac{T_3}{T_2} &\approx 0.9570 \end{aligned}\right\}$ no common ratio

b

ii, $\left.\begin{aligned} T_2 - T_1 &= -\frac{1}{6} \\ T_3 - T_2 &= -\frac{1}{8} \end{aligned}\right\}$ no common difference

$\left.\begin{aligned} \frac{T_2}{T_1} &= \frac{3}{4} \\ \frac{T_3}{T_2} &= \frac{3}{4} \end{aligned}\right\}$ common ratio

c

i, $\left.\begin{aligned} T_2 - T_1 &= \sqrt{20} - \sqrt{5} = 2\sqrt{5} - \sqrt{5} = \sqrt{5} \\ T_3 - T_2 &= \sqrt{45} - \sqrt{20} = 3\sqrt{5} - 2\sqrt{5} = \sqrt{5} \end{aligned}\right\}$ common difference

$\left.\begin{aligned} \frac{T_2}{T_1} &= \frac{\sqrt{20}}{\sqrt{5}} = 2 \\ \frac{T_3}{T_2} &= \frac{\sqrt{45}}{\sqrt{20}} = 1.5 \end{aligned}\right\}$ no common ratio

d

iii, $\left.\begin{aligned} T_2 - T_1 &= -0.4 - (-1.6) = 1.2 \\ T_3 - T_2 &= 0.6 - (-0.4) = 1 \end{aligned}\right\}$ no common difference

$\left.\begin{aligned} \frac{T_2}{T_1} &= \frac{-0.4}{-1.6} = \frac{1}{4} \\ \frac{T_3}{T_2} &= \frac{0.6}{-0.4} = -\frac{3}{2} \end{aligned}\right\}$ no common ratio

e

i, $\left.\begin{aligned} T_2 - T_1 &= 7.5 - 3.4 = 4.1 \\ T_3 - T_2 &= 11.6 - 7.5 = 4.1 \end{aligned}\right\}$ common difference

$\left.\begin{aligned} \frac{T_2}{T_1} &= \frac{7.5}{3.4} \approx 2.206 \\ \frac{T_3}{T_2} &= \frac{11.6}{7.5} \approx 1.547 \end{aligned}\right\}$ no common ratio

f

ii, $\left.\begin{aligned} T_2 - T_1 &= 24 - 48 = -24 \\ T_3 - T_2 &= 12 - 24 = -12 \end{aligned}\right\}$ no common difference

$\left.\begin{aligned} \frac{T_2}{T_1} &= \frac{24}{48} = \frac{1}{2} \\ \frac{T_3}{T_2} &= \frac{12}{24} = \frac{1}{2} \end{aligned}\right\}$ common ratio

g

ii, $\left.\begin{aligned} T_2 - T_1 &= 1 - \left(-\frac{1}{5}\right) = 1\frac{1}{5} \\ T_3 - T_2 &= -5 - 1 = -6 \end{aligned}\right\}$ no common difference

$\left.\begin{aligned} \frac{T_2}{T_1} &= 1 \div -\frac{1}{5} = -5 \\ \frac{T_3}{T_2} &= \frac{-5}{1} = -5 \end{aligned}\right\}$ common ratio

h

i, $\left.\begin{aligned} T_2 - T_1 &= 100 - 105 = -5 \\ T_3 - T_2 &= 95 - 100 = -5 \end{aligned}\right\}$ common difference

$\left.\begin{aligned} \frac{T_2}{T_1} &= \frac{100}{105} \approx 0.9524 \\ \frac{T_3}{T_2} &= \frac{95}{100} = 0.95 \end{aligned}\right\}$ common ratio

i

i, $\left.\begin{aligned} T_2 - T_1 &= 1\frac{1}{4} - 1\frac{1}{2} = -\frac{1}{4} \\ T_3 - T_2 &= 1 - 1\frac{1}{4} = -\frac{1}{4} \end{aligned}\right\}$ common difference

$\left.\begin{aligned} \frac{T_2}{T_1} &= 1\frac{1}{4} \div 1\frac{1}{2} = \frac{5}{6} \\ \frac{T_3}{T_2} &= 1 \div 1\frac{1}{4} = \frac{4}{5} \end{aligned}\right\}$ no common ratio

j

i, $\left.\begin{aligned} T_2 - T_1 &= \log x^2 - \log x = 2\log x - \log x = \log x \\ T_3 - T_2 &= \log x^3 - \log x^2 = 3\log x - 2\log x = \log x \end{aligned}\right\}$ common difference

$\left.\begin{aligned} \frac{T_2}{T_1} &= \frac{2\log x}{\log x} = 2 \\ \frac{T_3}{T_2} &= \frac{3\log x}{\log x} = 3 \end{aligned}\right\}$ no common ratio

Question 7

$a = 8, d = 5, T_n = 543$

$8 + (n-1) \times 5 = 543$

$5n + 3 = 543$

$n = 108$

Question 8

$T_{11} = 97$

$a + 10d = 97 \quad [1]$

$T_6 = 32$

$a + 5d = 32 \quad [2]$

$[1] - [2]$

$5d = 65 \quad \Rightarrow \quad d = 13$

Substitute for d in [2].

$a + 65 = 32 \quad \Rightarrow \quad a = -33$

Question 9

a $T_4 = 4^3 - 5 = 59$

b $T_1 = 1^3 - 5 = -4$

$S_1 = 1^3 - 5 = -4$

$S_2 = 2^3 - 5 = 3$

$S_3 = 3^3 - 5 = 22$

$S_4 = 4^3 - 5 = 59$ (or from **b**)

Sum is $-4 + 3 + 22 + 59 = 80$

c $n^3 - 5 = 5827$

$n^3 = 5832$

$n = 18$

18th term

Question 10

a If the sequence is arithmetic,

$d = x - 5 = 45 - x$

$2x = 50$

$x = 25$

b If the sequence is geometric,

$r = \dfrac{x}{5} = \dfrac{45}{x}$

$x^2 = 225$

$x = \pm 15$

Question 11

$T_2 - T_1 = T_3 - T_2$

$(2x+3) - x = 5x - (2x+3)$

$x + 3 = 3x - 3$

$2x = 6 \quad \Rightarrow \quad x = 3$

Question 12

a $a = 3, d = 7, n = 20$

$T_{20} = 3 + (20-1) \times 7 = 136$

b $a = 101, d = -3, n = 20$

$T_{20} = 101 + (20-1) \times -3 = 44$

c $a = 0.3, d = 0.3, n = 20$

$T_{20} = 0.3 + (20-1) \times 0.3 = 6$

Question 13

$a = 81, r = \frac{9}{27} = \frac{1}{3}$

$LS = \frac{81}{1 - \frac{1}{3}} = 121.5$

Question 14

a $a = 5, d = 4$

$S_n = \frac{n}{2}[2 \times 5 + (n-1) \times 4]$

$= \frac{n}{2}[4n + 6]$

$= 2n^2 + 3n$

b $a = 1, r = 1.07$

$S_n = \frac{1(1.07^n - 1)}{1.07 - 1}$

$= \frac{100(1.07^n - 1)}{7}$

Question 15

a Common ratio $r = x$

For limiting sum, $|r| < 1$.

Hence require $|x| < 1$.

b $a = 1, r = x = \frac{3}{5}$

$S = \frac{1}{1 - \frac{3}{5}} = 2.5$

c $\frac{1}{1-x} = 1.5$

$1.5(1-x) = 1$

$1.5 - 1.5x = 1$

$x = \frac{1-1.5}{-1.5} = \frac{1}{3}$

Question 16

$a = 4, S_{10} = 265$

$\frac{10}{2}[2 \times 4 + (10-1)d] = 265$

$5(8 + 9d) = 265$

$8 + 9d = 53$

$d = 5$

Question 17

$(7x-2)^2 = (x+2)(15x+6)$

$49x^2 - 28x + 4 = 15x^2 + 36x + 12$

$17x^2 - 32x - 4 = 0$

$(17x+2)(x-2) = 0$

$17x + 2 = 0 \Rightarrow x = -\frac{2}{17}$

$x - 2 = 0 \Rightarrow x = 2$

Question 18

$a = 8, d = 6$

$T_n = 122$

$8 + (n-1) \times 6 = 122$

$6n + 2 = 122$

$n = 20$

$S_{20} = \frac{20}{2}[2 \times 8 + (20-1) \times 6] = 1300$

or

$S_{20} = \frac{20}{2}[8 + 122] = 1300$

Question 19

a Sequence is $7, 14, 21, ...98$

$a = 7, d = 7$

$T_n = 98$

$7 + (n-1) \times 7 = 98$

$7n = 98$

$n = 14$

$S_{14} = \frac{14}{2}[2 \times 7 + (14-1) \times 7] = 735$

or

$S_{14} = \frac{14}{2}[7 + 98] = 735$

b First consider sum of all integers from 1 to 100.

$a = 1$, $d = 1$, $l = 100$, $n = 100$.

$S_n = \frac{n}{2}(a + l)$

$S_{100} = \frac{100}{2}(1 + 101) = 5050$

Sum of integers that are not multiples of 7 = 5050 – 735 (from **a**)

= 4315

Question 20

$a = 214, d = -8$

$T_n = 2760$

$\frac{n}{2}[2 \times 214 + (n-1) \times -8] = 2760$

$n(218 - 4n) = 2760$

$2n^2 - 109n + 1380 = 0$

$(n-20)(2n-69) = 0$

$n = 20, n = 34.5$ (not possible)

Question 21

$a = 4, r = 3$

$T_n = 236\,196$

$4 \times 3^{n-1} = 236\,196$

$3^n = 177\,147$

$n = \dfrac{\log_{10} 177\,147}{\log_{10} 3} = 11$

Challenge exercise 1

Question 1

a $n = 9$

$$T_9 = \frac{9^2}{9+1} = 8.1$$

b $\frac{n^2}{n+1} = 18.05$

$n^2 - 18.05n - 18.05 = 0$

$20n^2 - 361n - 361 = 0$

$(n-19)(20n+19) = 0$

$n - 19 = 0 \Rightarrow n = 19$

$20n + 19 = 0 \Rightarrow n = -\frac{20}{19}$ (not a valid solution)

So take $n = 19$

19th term

Question 2

a $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$

b $a = \frac{3\pi}{4}, d = \frac{\pi}{4}, n = 7$

$$T_7 = \frac{3\pi}{4} + (7-1) \times \frac{\pi}{4} = \frac{9\pi}{4}$$

c $a = \frac{3\pi}{4}, d = \frac{\pi}{4}, n = 6$

$$S_6 = \frac{6}{2}\left[2 \times \frac{3\pi}{4} + (6-1) \times \frac{\pi}{4}\right] = \frac{33\pi}{4}$$

Question 3

a $T_2 - T_1 = 2 = 2^1$

$T_3 - T_2 = 4 = 2^2$

$T_4 - T_3 = 8 = 2^3$

$T_5 - T_4 = 16 = 2^4$

$T_n - T_{n-1} = 2^{n-1}$

Hence

$$(T_2 + T_3 + T_4 + ... + T_n) - (T_1 + T_2 + T_3 + ... + T_{n-1}) = 2^1 + 2^2 + 2^3 + ... + 2^{n-1}$$

$$2^1 + 2^2 + 2^3 + ... + 2^{n-1} = \frac{2(2^{n-1} - 1)}{2 - 1} = 2^n - 2$$

$T_n - T_1 = 2^n - 2$

$T_n = 2^n + 1$

Let S be the sum of n terms

$$\begin{aligned} S &= T_1 + T_2 + T_3 + T_4 + ...T_n \\ &= (2^1 + 1) + (2^2 + 1) + ... + (2^n + 1) \\ &= (2^1 + 2^2 + ... + 2^n) + n \times 1 \\ &= 2^{n+1} - 2 + n \end{aligned}$$

$$\begin{aligned} S_{20} &= 2^{21} - 2 + 20 \\ &= 2\,097\,170 \end{aligned}$$

b $n = 10$ for each series below.

$$T_1 + T_3 + T_5 + ... = 5 + 10 + 15 + ... = \frac{5n(n+1)}{2} = 275 \ (n = 10)$$

$$\begin{aligned} T_2 + T_4 + T_6 + ... &= -2 - 8 - 32 - ... \\ &= \frac{-2(4^n - 1)}{4 - 1} \\ &= \frac{-2(4^n - 1)}{3} \\ &= -699\,050 \ (n = 10) \end{aligned}$$

Total sum is $275 + [-699\,050] = -698\,775$

Question 4

$$a=\frac{7}{9}, r=\frac{2}{5}, T_n=\frac{224}{28\,125}$$

$$\frac{7}{9}\times\left(\frac{2}{5}\right)^{n-1}=\frac{224}{28\,125}$$

$$\left(\frac{2}{5}\right)^{n-1}=\frac{32}{3125}$$

$$n=1+\frac{\log_{10}\left(\frac{32}{3125}\right)}{\log_{10}\left(\frac{2}{5}\right)}=6$$

6th term

Question 5

Find the difference between the sum of the first 200 terms of 1, 2, 3, . and the multiples of 9 between 1 and 200.

For multiples of 9, $T_n=9n$

$$9n>200$$
$$n>22.22$$

There are 22 terms between 1 and 200 that are multiples of 9.

$$a=9, d=9, n=22$$

$$S_n=\frac{22}{2}[2\times 9+(22-1)\times 9]=2277$$

The sum of the sequence 1, 2, 3, …, 200 is $\frac{200}{2}[2\times 1+(200-1)\times 1]=20\,100$

The required sum is $20\,100-2277=17\,823$.

Question 6

$a = 20, r = \frac{1}{5}$

Require $S_n > 24.99$

$$\frac{20\left(1-\left(\frac{1}{5}\right)^n\right)}{1-\frac{1}{5}} > 24.99$$

$$1-\left(\frac{1}{5}\right)^n > 0.9996$$

$$\left(\frac{1}{5}\right)^n < 0.0004$$

$$n > \frac{\log_{10} 0.0004}{\log_{10}\left(\frac{1}{5}\right)}$$

$n > 4.86$

So $n \geq 5$.

Question 7

a

$$S_5 = 77$$

$$\frac{a(r^5-1)}{r-1} = 77 \Rightarrow \frac{a}{r-1} = \frac{77}{r^5-1}$$

$$S_{10} = 77 + (-2464) = -2387$$

$$\frac{a(r^{10}-1)}{r-1} = -2387 \Rightarrow \frac{a}{r-1} = -\frac{2387}{r^{10}-1}$$

$$\frac{77}{r^5-1} = -\frac{2387}{r^{10}-1}$$

$$r^{10} + 31r^5 - 32 = 0$$

$$(r^5+32)(r^5-1) = 0$$

$$r^5 + 32 = 0 \Rightarrow r = -2$$

$$r^5 - 1 = 0 \Rightarrow r = 1 \text{ (not acceptable)}$$

$$\frac{a((-2)^5-1)}{-2-1} = 77$$

$$\frac{-33a}{-3} = 77$$

$$a = 7$$

b $T_4 = ar^3 = 7 \times (-2)^3 = -56$

Question 8

a

$$a = 1, r = \cos^2 x$$

$$LS = \frac{1}{1-\cos^2 x} = \frac{1}{\sin^2 x} = \text{cosec}^2 x$$

b For there to be a limiting sum, the magnitude of the common ratio must be less than 1.

So we require $\left|\cos^2 x\right| < 1$.

This is always true because for all x, $\left|\cos^2 x\right| = \cos^2 x$ and $\left|\cos x\right| < 1$.

MATHS IN FOCUS 12 MATHEMATICS ADVANCED

WORKED SOLUTIONS

Chapter 2: Transformations of functions

Exercise 2.01 Vertical translations of functions

Question 1

a Let $f(x) = x^2$

$y = f(x) + 3$

This is a vertical translation of $f(x)$ by 3 units up.

b Let $f(x) = x^2$

$y = f(x) - 7$

This is a vertical translation of $f(x)$ by 7 units down.

c Let $f(x) = x^2$

$y = f(x) - 1$

This is a vertical translation of $f(x)$ by 1 unit down.

d Let $f(x) = x^2$

$y = f(x) + 5$

This is a vertical translation of $f(x)$ by 5 units up.

Question 2

a Let $f(x) = x^3$

$y = f(x) + 1$

This is a vertical translation of $f(x)$ by 1 unit up.

b Let $f(x) = x^3$

$y = f(x) - 4$

This is a vertical translation of $f(x)$ by 4 units down.

c Let $f(x) = x^3$

$y = f(x) + 8$

This is a vertical translation of $f(x)$ by 8 units up.

Question 3

Let $f(x) = \frac{1}{x}$

$y = \frac{1}{x} + 9 = f(x) + 9$

This is a vertical translation of $f(x)$ by 9 units up.

Question 4

a Let $f(x) = x^2$

$y = f(x) - 3 = x^2 - 3$

b Let $f(x) = 2^x$

$y = f(x) + 8 = 2^x + 8$

c Let $f(x) = |x|$

$y = f(x) + 1 = |x| + 1$

d Let $f(x) = x^3$

$y = f(x) - 4 = x^3 - 4$

e $f(x) = \log x$

3 units upwards means

$f(x) = \log x + 3$

f Let $f(x) = \frac{2}{x}$

$y = f(x) - 7 = \frac{2}{x} - 7$

Question 5

a Let $y = x^4$

$f(x) = y - 1$

y is translated 1 unit down

b Let $y = x^4$

$f(x) = y + 6$

y is translated 6 units up

Question 6

a i Let $f(x) = 2x^3 + 3$

$y = f(x) - 5$

$= 2x^3 + 3 - 5$

$= 2x^3 - 2$

ii $y = f(x) + 3$

$= 2x^3 + 3 + 3$

$= 2x^3 + 6$

b i Let $f(x) = |x| - 4$

$y = f(x) + 1$

$= |x| - 4 + 1$

$= |x| - 3$

ii $y = f(x) - 2$

$= |x| - 4 - 2$

$= |x| - 6$

c i Let $f(x) = e^x + 2$

$y = f(x) - 1$

$= e^x + 2 - 1$

$= e^x + 1$

ii $y = f(x) + 3$

$= e^x + 2 + 3$

$= e^x + 5$

d i Let $f(x) = \log_e x - 1$

$y = f(x) + 11$

$= \log_e x - 1 + 11$

$= \log_e x + 10$

ii $y = f(x) - 7$

$= \log_e x - 1 - 7$

$= \log_e x - 8$

Question 7

a $y \to y + 2$

$(1, -3 + 2) = (1, -1)$

b $y \to y - 6$

$(1, -3 - 6) = (1, -9)$

c $y \to y + m$

$(1, -3 + m)$

Question 8

a

$$y_{\text{image}} = y_{\text{original}} + 1$$
$$2 = y_{\text{original}} + 1$$
$$y_{\text{original}} = 1$$

Original point is (–1, 1).

b

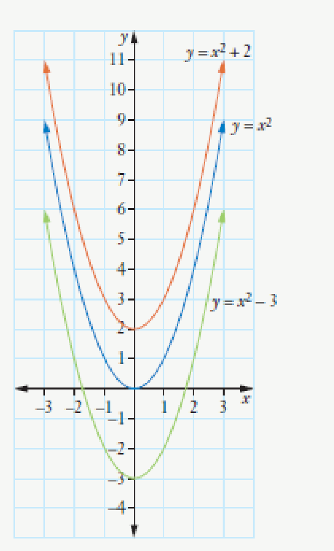
$$y_{\text{image}} = y_{\text{original}} - 3$$
$$2 = y_{\text{original}} - 3$$
$$y_{\text{original}} = 5$$

Original point is (–1, 5).

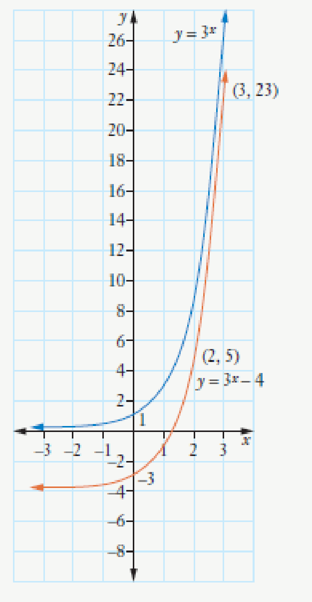
Question 9

a $y = x^2 + 2$ is the graph of $y = x^2$ translated 2 units up.

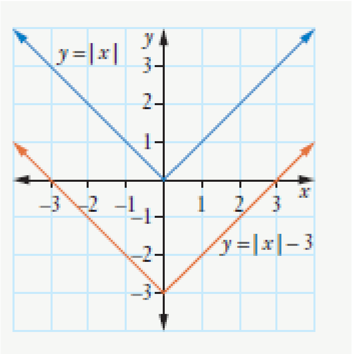
$y = x^2 - 3$ is the graph of $y = x^2$ translated 3 units down.



b $y = 3^x - 4$ is the graph of $y = 3^x$ translated 4 units down.



c $y = |x| - 3$ is the graph of $y = |x|$ translated 3 units down.



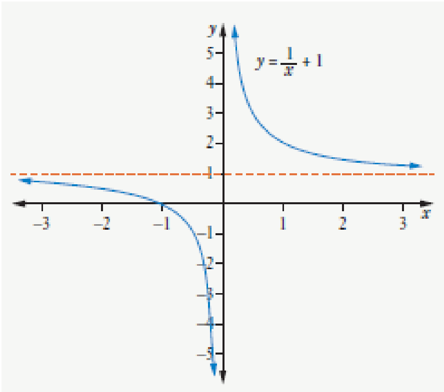
Question 10

a Let $f(x)=\frac{1}{x}$

$\frac{1}{x}+1=f(x)+1$

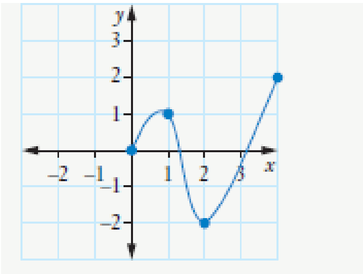
The graph of $y=\frac{1}{x}$ is vertically translated 1 unit up.

b $y=\frac{1}{x}+1$ is the graph of $y=\frac{1}{x}$ translated 1 unit up.

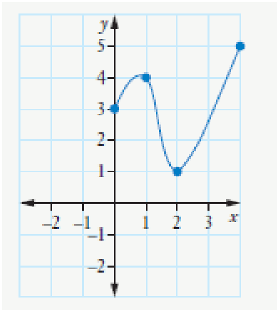


Question 11

a vertical translation of $f(x)$ 1 unit down



b vertical translation of $f(x)$ 2 units up

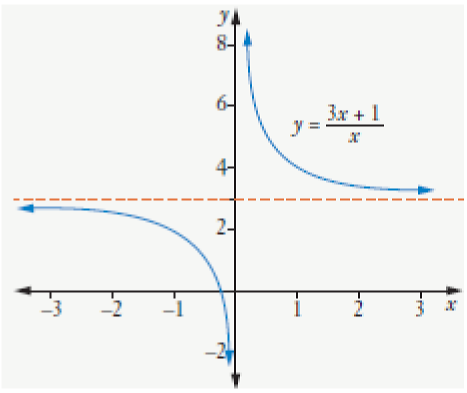


Question 12

a $\frac{3x+1}{x}=\frac{3x}{x}+\frac{1}{x}$

$=\frac{1}{x}+3$

b $y=\frac{3x+1}{x}$ translates the graph $y=\frac{1}{x}$ 3 units up.



Exercise 2.02 Horizontal translations of functions

Question 1

a Let $f(x) = x^2$

$y = f(x-4)$

The graph of $y = (x-4)^2$ is a horizontal translation 4 units right

b $y = f(x+2)$

The graph of $y = (x+2)^2$ is a horizontal translation 2 units left

Question 2

a Let $f(x) = x^3$

$y = f(x-5)$

The graph of $y = (x-5)^3$ is a horizontal translation 5 units right

b $y = f(x+3)$

The graph of $y = (x+3)^3$ is a horizontal translation 3 units left

Question 3

a Let $f(x) = x^2$

$y = f(x+3) = (x+3)^2$

b Let $f(x) = 2^x$

$y = f(x-8) = 2^{x-8}$

c Let $f(x) = |x|$

$y = f(x+1) = |x+1|$

d Let $f(x) = x^3$

$y = f(x-4) = (x-4)^3$

e Let $f(x) = \log x$

$y = f(x+3) = \log(x+3)$

Question 4

Let $f(x) = \dfrac{1}{x}$

$y = f(x-3) = \dfrac{1}{x-3}$

A translation of $y = \dfrac{1}{x}$, 3 units to the right.

Question 5

a Let $f(x) = x^4$

$f(x+2) = (x+2)^4$

A translation of $f(x) = \frac{1}{x}$, 2 units to the left.

b $f(x-5) = (x-5)^4$

A translation of $f(x) = \frac{1}{x}$, 5 units to the right.

Question 6

a **i** Let $f(x) = -x^2$

$f(x+4) = -(x+4)^2$

ii $f(x-8) = -(x-8)^2$

b **i** Let $f(x) = |x|$

$f(x-3) = |x-3|$

ii $f(x+4) = |x+4|$

c **i** Let $f(x) = e^{x+2}$

$f(x+4) = e^{x+2+4} = e^{x+6}$

ii $f(x-7) = e^{x+2-7}$

$= e^{x-5}$

d **i** Let $f(x) = \log_2(x-3)$

$f(x-2) = \log_2((x-2)-3)$

$= \log_2(x-5)$

ii $f(x+3) = \log_2((x+3)-3)$

$= \log_2 x$

Question 7

a $f(x-4)$ is a horizontal translation 4 units right.

Co-ordinates of image (1 + 4, –3) or (5, –3)

b $f(x+9)$ is a horizontal translation 9 units left.

Co-ordinates of image (1 – 9, –3) or (–8, –3)

c $f(x+t)$ is a horizontal translation t units.

Co-ordinates of image (1 + t, –3)

Question 8

a Image (–1, 2) after a translation 4 units left.

Original point is 4 units right: (–1 + 4, 2) ≡ (3, 2).

b Image (–1, 2) after a translation 8 units right

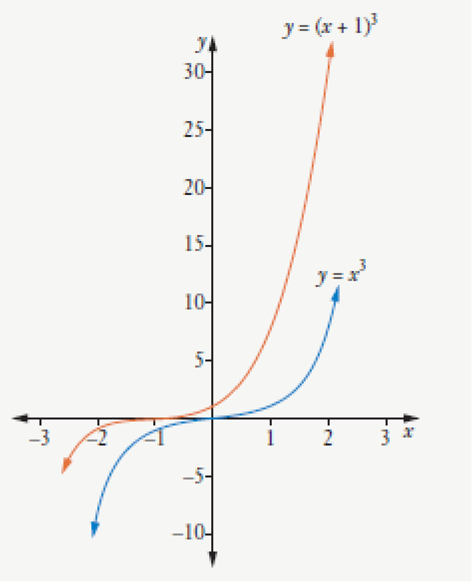
Original point is 8 units left: (–1 – 8, 2) ≡ (–9, 2).

Question 9

a Let $y = f(x) = x^3$

The graph of $f(x+1)$ is a horizontal translation of the graph of y 1 unit left.

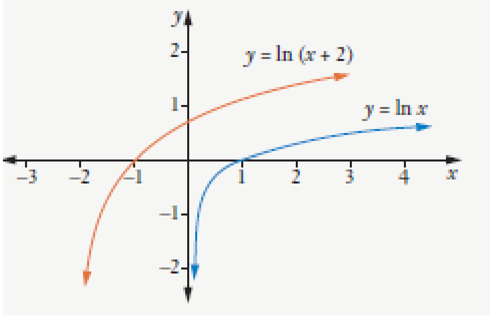
Each x-coordinate is translated 1 unit left.



b Let $y = f(x) = \ln x$

The graph of $f(x+2)$ is a horizontal translation of the graph of y 2 units left.

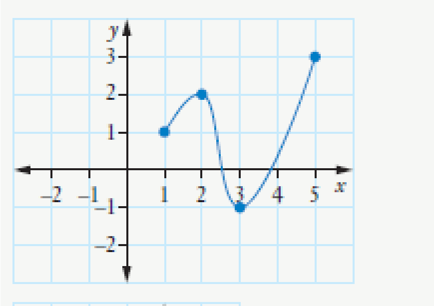
Each x-coordinate is translated 2 units left.



Question 10

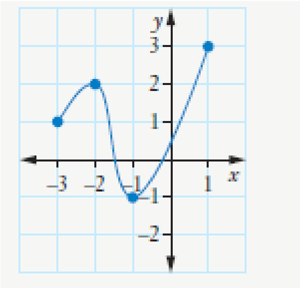
a The graph of $f(x-1)$ is a horizontal translation of the graph of y 1 unit right.

Each x-coordinate is translated 1 unit right.



b The graph of $f(x+3)$ is a horizontal translation of the graph of y 3 units left.

Each x-coordinate is translated 3 units left.



Question 11

Let $y = f(x)$.

a Transformed function is

$y = f(x) - 5 = x^5 - 5$

b Transformed function is

$y = f(x-3) = (x-3)^5$

c Transformed function is

$y = f(x) + 2 = x^5 + 2$

d Transformed function is

$y = f(x+7) = (x+7)^5$

Question 12

$x_{\text{image}} = x_{\text{original}} - 4$
$3 = x_{\text{original}} - 4$
$x_{\text{original}} = 7$

$y_{\text{image}} = y_{\text{original}}$

Original point is (7, –2).

Exercise 2.03 Vertical dilations of functions

Question 1

a **i** Vertical dilation scale factor 6 (stretched)

$y = x = 6 \times x$

Vertical scale factor $k = 6$

$k > 1$, so there is a vertical dilation (stretching) of $y = x$ by a factor of 6 from the *x*-axis.

ii Vertical dilation scale factor $\frac{1}{2}$ (compressed)

$y = \frac{1}{2} \times x$

Vertical scale factor $k = \frac{1}{2}$

$0 < k < 1$, so there is a vertical dilation (compression) of $y = x$ by a factor of $\frac{1}{2}$ from the *x*-axis.

iii Vertical dilation by a factor of –1 (reflection in the *x*-axis)

$y = -1 \times x$

Vertical scale factor $k = -1$, so there is a reflection of *y* in the *x*-axis.

b **i** Vertical dilation scale factor 2 (stretched)

$y = 2 \times x^2$

Vertical scale factor $k = 2$

$k > 1$, so there is a vertical dilation (stretch) by a factor of 2 from the *x*-axis.

ii Vertical dilation scale factor $\frac{1}{6}$ (compressed)

$y = \frac{1}{6} \times x^2$

Vertical scale factor $k = \frac{1}{6}$

$0 < k < 1$, so there is a vertical dilation (compression) by a factor of $\frac{1}{6}$ from the *x*-axis.

iii Vertical dilation by a factor of –1 (reflection in the *x*-axis)

$y = -1 \times x^2$

Vertical scale factor $k = -1$, so there is a reflection of *y* in the *x*-axis.

c **i** Vertical dilation scale factor 4 (stretched)

$y = 4 \times x^3$

Vertical scale factor $k = 4$

$k > 1$, so there is a vertical dilation (stretch) by a factor of 4 from the *x*-axis.

ii Vertical dilation scale factor $\frac{1}{7}$ (compressed)

$y = \frac{1}{7} \times x^3$

Vertical scale factor $k = \frac{1}{7}$

$0 < k < 1$, so there is a vertical dilation (compression) by a factor of $\frac{1}{7}$ from the *x*-axis.

iii Vertical dilation scale factor $\frac{4}{3}$ (stretched)

$y = \frac{4}{3} \times x^3$

Vertical scale factor $k = \frac{4}{3}$

$k > 1$, so there is a vertical dilation (stretching) by a factor of $\frac{4}{3}$ from the *x*-axis.

d **i** Vertical dilation scale factor 9 (stretched)

$y = 9 \times x^4$

Vertical scale factor $k = 9$

$k > 1$, so there is a vertical dilation (stretch) by a factor of 9 from the *x*-axis.

ii Vertical dilation scale factor $\frac{1}{3}$ (compressed)

$y = \frac{1}{3} \times x^4$

Vertical scale factor $k = \frac{1}{3}$

$0 < k < 1$, so there is a vertical dilation (compression) by a factor of $\frac{1}{3}$ from the *x*-axis.

iii Vertical dilation scale factor $\frac{3}{8}$ (compressed)

$y = \frac{3}{8} \times x^4$

Vertical scale factor $k = \frac{3}{8}$

$0 < k < 1$, so there is a vertical dilation (compression) by a factor of $\frac{3}{8}$ from the *x*-axis.

e **i** Vertical dilation scale factor 5 (stretched)

$y = 5 \times |x|$

Vertical scale factor $k = 5$

$k > 1$, so there is a vertical dilation (stretch) by a factor of 5 from the *x*-axis.

ii Vertical dilation scale factor $\frac{1}{8}$ (compressed)

$y = \frac{1}{8} \times |x|$

Vertical scale factor $k = \frac{1}{8}$

$0 < k < 1$, so there is a vertical dilation (compression) by a factor of $\frac{1}{8}$ from the *x*-axis.

iii Vertical dilation scale factor –1 (reflection in the *x*-axis)

$y = -1 \times |x|$

Vertical scale factor $k = -1$, $y = |x|$ is reflected in the *x*-axis.

f **i** Vertical dilation scale factor 9 (stretched)

$f(x) = 9 \times \log x$

Vertical scale factor $k = 9$

$k > 1$, so there is a vertical dilation (stretch) by a factor of 5 from the *x*-axis.

ii Vertical dilation scale factor –1 (reflection in the *x*-axis)

$f(x) = -1 \times \log x$

Vertical scale factor $k = -1$, so $f(x) = \log x$ is reflected in the *x*-axis.

iii Vertical dilation scale factor $\frac{2}{5}$ (compressed)

$y = \frac{2}{5} \times \log x$

Vertical scale factor $k = \frac{2}{5}$

$0 < k < 1$, so there is a vertical dilation (compression) by a factor of $\frac{2}{5}$ from the *x*-axis.

Question 2

a $y = 6x^2$, domain $(-\infty, \infty)$, range $[0, \infty)$

$k = 6$, the transformed function is $y = kx^2 = 6x^2$

$y = 6x^2$ is a parabola with a minimum turning point at (0, 0).

The domain is all real numbers, $(-\infty, \infty)$

The range is $[0, \infty)$ or $\{y : y \le 0\}$

b $y = \frac{1}{4}\ln x$, domain $(0, \infty)$, range $(-\infty, \infty)$

$k = \frac{1}{4}$, the transformed function is $y = k \ln x = \frac{1}{4}\ln x$

The domain and range of $y = \frac{1}{4}\ln x$ are the same as the domain and range of $y = \ln x$.

$x > 0$, so the domain is $(0, \infty)$

The range is all real numbers, $(-\infty, \infty)$

c $f(x) = -|x|$, domain $(-\infty, \infty)$, range $(-\infty, 0]$

$k = -1$, the transformed function is $f(x) = k|x| = -|x|$

There are no restrictions on *x*, so the domain of $f(x)$ is all real numbers, $(-\infty, \infty)$

The range of $|x|$ is $[0, \infty)$, so the range of $-|x|$ is $(-\infty, 0]$

d $f(x) = 4e^x$, domain $(-\infty, \infty)$, range $(0, \infty)$

$k = 4$, the transformed function is $f(x) = k \times e^x = 4e^x$

The domain and range of the transformed function is the same as the domain and range of the original function.

There are no restrictions on *x*, so the domain of $f(x)$ is all real numbers, $(-\infty, \infty)$

There is a horizontal asymptote at $y = 0$ and the function increases as *x* increases, so the range is $(0, \infty)$.

e $f(x) = \frac{7}{x}$, domain $(-\infty, 0) \cup (0, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

$k = 7$, the transformed function is $f(x) = k \times \frac{1}{x} = \frac{7}{x}$

The domain and range of the transformed function is the same as the domain and range of the original function.

$x \ne 0$ (vertical asymptote), so the domain is $(-\infty, 0) \cup (0, \infty)$

There is a horizontal asymptote at $y = 0$, so the range is $(-\infty, 0) \cup (0, \infty)$

Question 3

a $y = 5 \times 3^x$

Vertical scale factor is $k = 5$, so the transformed function is $f(x) = k \times 3^x = 5 \times 3^x$

b $f(x) = \frac{1}{3}x^2$

Vertical scale factor is $k = \frac{1}{3}$, so the transformed function is $f(x) = \frac{1}{3} \times x^2 = \frac{1}{3}x^2$

c $y = -x^3$

Vertical scale factor is $k = -1$, so the transformed function is $y = -1 \times x^3 = -x^3$

d $y = \frac{1}{2x}$

Vertical scale factor is $k = \frac{1}{2}$, so the transformed function is $y = \frac{1}{2} \times \frac{1}{x} = \frac{1}{2x}$

e $y = \frac{2}{3}|x|$

Vertical scale factor is $k = \frac{2}{3}$, so the transformed function is $y = \frac{2}{3} \times |x| = \frac{2}{3}|x|$

Question 4

a Vertical scale factor $k = 4$, the image is at $(3, 6k) = (3, 24)$

b Vertical scale factor $k = -1$, the image is at $(3, 6k) = (3, -6)$

c Vertical scale factor $k = 12$, the image is at $(3, 6k) = (3, 72)$

d Vertical scale factor $k = \frac{5}{6}$, the image is at $(3, 6 \times \frac{5}{6}) = (3, 5)$

Question 5

a $(4, 4)$

x-coordinate is unchanged.

$k = 3$

$ky = 12$

$3y = 12$

$y = 4$

b $(4, 6)$

x-coordinate is unchanged.

$k = 2$

$ky = 12$

$2y = 12$

$y = 6$

c $(4, 36)$

x-coordinate is unchanged.

$k = \frac{1}{3}$

$ky = 12$

$\frac{1}{3}y = 12$

$y = 36$

d $(4,16)$

x-coordinate is unchanged.

$k = \frac{3}{4}$

$ky = 12$

$\frac{3}{4}y = 12$

$y = 16$

e $(4, -12)$

x-coordinate is unchanged.

$k = -1$

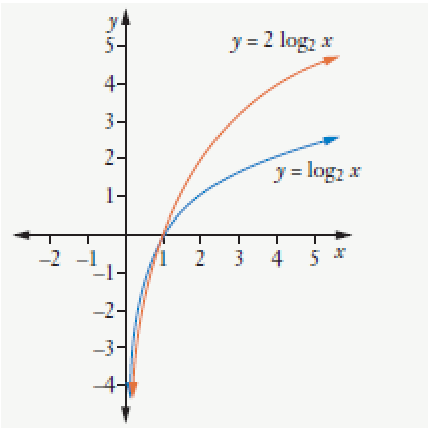
$ky = 12$

$-1y = 12$

$y = -12$

Question 6

a The graph of $f(x) = 2\log_2 x$ is a vertical dilation (stretching) by a scale factor of 2 of the graph of $f(x) = \log_2 x$. The x-intercepts $(x = 1)$ and the vertical asymptote $(x = 0)$ are the same for both functions.

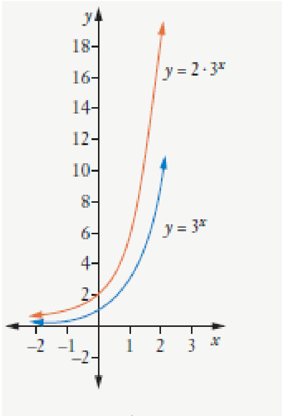


b The graph of $f(x)=2\times 3^x$ is a vertical dilation (stretching) by a scale factor of 2 of the graph of $f(x)=3^x$.

There are no x-intercepts. The y-intercept of $f(x)=3^x$ is at (0, 1) and the

y-intercept of $f(x)=2\times 3^x$ is at $(0,2\times 1)=(0,2)$

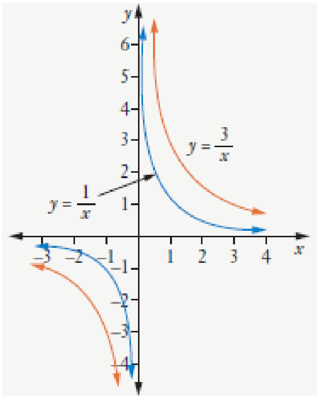
Both functions have a horizontal asymptote at $y=0$.



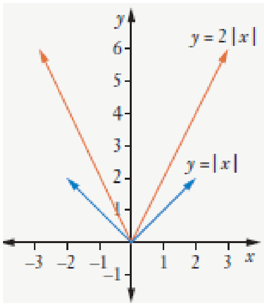
c The graph of $y=\frac{3}{x}=3\times\frac{1}{x}$ is a vertical dilation (stretching) by a scale factor of 3 of the graph $y=\frac{1}{x}$.

There are no axial intercepts.

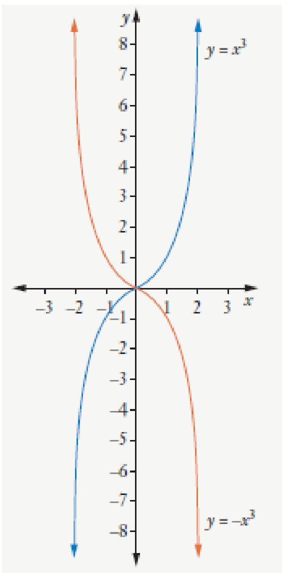
Both functions have a vertical asymptote at $x=0$ and a horizontal asymptote at $y=0$.



d The graph of $y=2|x|$ is a vertical dilation (stretching) by a scale factor of 2 of the graph of $y=|x|$. Axial intercepts are at 0.



e Scale factor is –1. The graph of $y=-x^3$ is a reflection in the x-axis of the graph of $y=x^3$. Axial intercepts are at 0.



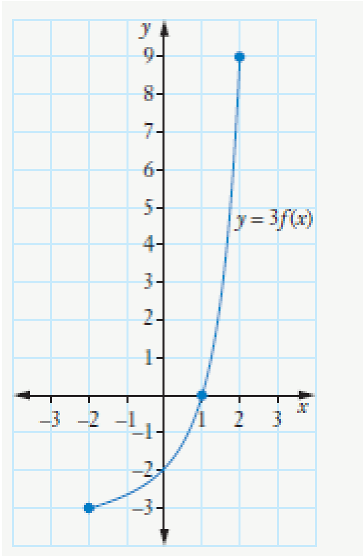
Question 7

a Vertical scale factor 3

$(-2,-1) \to (-2, 3\times -1) = (-2,-3)$

$(1,0) \to (1, 3\times 0) = (1,0)$

$(2,3) \to (2, 3\times 3) = (2,9)$

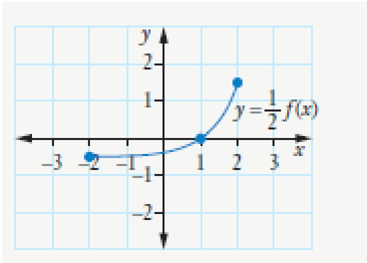


b Vertical scale factor $\frac{1}{2}$

$(-2,-1) \to (-2, -\frac{1}{2}\times -1) = (-2, \frac{1}{2})$

$(1,0) \to (1, \frac{1}{2}\times 0) = (1,0)$

$(2,3) \to (2, \frac{1}{2}\times 3) = (2, 1\frac{1}{2})$

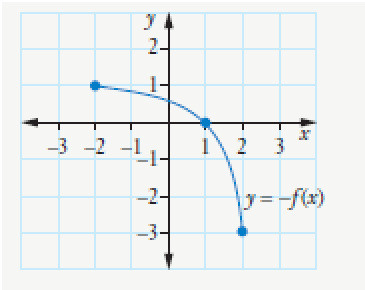


c Vertical scale factor -1

$(-2,-1) \to (-2, -1\times -1) = (-2,1)$

$(1,0) \to (1, -1\times 0) = (1,0)$

$(2,3) \to (2, -1\times 3) = (2,-3)$

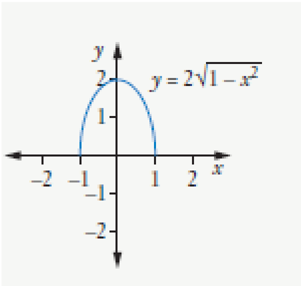


Question 8

The graph of $y = 2\sqrt{1-x^2}$ is a vertical dilation by a scale factor 2 (stretching) of the graph of the top half of the unit circle, $y = \sqrt{1-x^2}$.

The y-intercept is (0, 1) and the x-intercepts are at ± 1.

The dilation creates the ellipse with equation $x^2 + \frac{y^2}{4} = 1$.



The domain of $y = 2\sqrt{1-x^2}$ is the set of x values such that $1 - x^2 \geq 0$. That is, $[-1,1]$.

The range is $[0, 2\times 1] = [0,2]$.

Exercise 2.04 Horizontal dilations of functions

Question 1

a Horizontal dilation, scale factor $\frac{1}{8}$ (compressed)

Horizontal scale factor is $\frac{1}{8}$.

The scale factor is graph is dilated (compressed) horizontally using a scale factor $\frac{1}{8}$.

b Horizontal dilation, scale factor 5 (stretched)

Horizontal scale factor is $1 \div \frac{1}{5} = 5$.

The graph is dilated (stretched) horizontally using a scale factor 5.

c Horizontal dilation, scale factor $\frac{7}{3}$ (stretched)

Horizontal scale factor is $1 \div \frac{3}{7} = 2\frac{1}{3}$.

The graph is dilated (stretched) horizontally using a scale factor $2\frac{1}{3}$

d Horizontal dilation scale factor –1 (reflection in *y*-axis)

Horizontal scale factor is $1 \div -1 = -1$.

Since the magnitude of the scale factor is 1, there is no dilation, but there is reflection in the *y*-axis.

Question 2

a **i** $y=(2\times x)^2$, horizontal dilation (compression), scale factor is $\frac{1}{2}$.

ii $y=(5\times x)^2$, horizontal dilation (compression), scale factor is $\frac{1}{5}$.

iii $y=\left(\frac{1}{3}\times x\right)^2$, horizontal dilation (stretching), scale factor is $1\div\frac{1}{3}=3$.

b **i** $y=4\times x^3$, vertical dilation (stretching), scale factor is 4.

ii $y=\left(\frac{1}{2}\times x\right)^3$, horizontal dilation (stretching), scale factor is $1\div\frac{1}{2}=2$.

iii $y=(-1\times x)^3$, horizontal dilation, scale factor is -1.

There is no stretch or compression, but there is reflection in the *y*-axis.

c **i** $y=(7\times x)^4$, horizontal dilation (compression), scale factor is $\frac{1}{7}$.

ii $y=\frac{1}{8}\times x^4$, vertical dilation (compression), scale factor is $\frac{1}{8}$.

iii $y=\left(\frac{3}{4}\times x\right)^4$, horizontal dilation (stretching), scale factor is $1\div\frac{3}{4}=\frac{4}{3}$.

d **i** $y=\left|5\times x\right|$, horizontal dilation (compression), scale factor is $\frac{1}{5}$;

but function can also be written as $y=5\times\left|x\right|$, so vertical dilation (stretching), scale factor is 5

ii $y=\left|\frac{1}{2}\times x\right|$, horizontal dilation (stretching), scale factor is 2;

but function can also be written as $y=\frac{1}{2}\times\left|x\right|$, so vertical dilation (compression), scale factor is $\frac{1}{2}$

iii $y=\left|\frac{3}{5}\times x\right|$, horizontal dilation (stretching), scale factor is $\frac{5}{3}$;

but function can also be written as $y=\frac{3}{5}\times\left|x\right|$, so vertical dilation (compression), scale factor is $\frac{3}{5}$

e **i** $y = 5^{3\times x}$, horizontal dilation (compression), scale factor is $\frac{1}{3}$

ii $y = -1\times 5^{x}$, vertical dilation, scale factor is -1.

The magnitude of the scale factor is 1, so there is no compression or stretching. There is reflection in the *x*-axis.

iii $y = 5^{\frac{1}{2}\times x}$, horizontal dilation (stretching), scale factor is $1\div\frac{1}{2} = 2$.

f **i** $f(x) = 8\times\log x$, vertical dilation (stretching), scale factor is 8.

ii $f(x) = \log(-x)$, horizontal dilation (no stretching/compression), scale factor is $1\div -1 = -1$.

There is reflection in the *y*-axis.

iii $f(x) = \log\left(\frac{1}{7}\times x\right)$, horizontal dilation (stretching), scale factor is $1\div\frac{1}{7} = 7$.

Question 3

a $f(x) = |ax|$, $a = 1\div\frac{1}{5} = 5$

$f(x) = |5x|$

There is no restriction on the values *x* can take, so the domain is $(-\infty,\infty)$

$|x|$ is always positive, and the minimum *y*-value is 0. The range is $[0,\infty)$

b $y = (ax)^2$, $\frac{1}{a} = 3 \Rightarrow a = \frac{1}{3}$

$y = \left(\frac{1}{3}x\right)^2$

There is no restriction on the values *x* can take, so the domain is $(-\infty,\infty)$

$\left(\frac{1}{3}x\right)^2$ is always positive, and the minimum *y*-value is 0. The range is $[0,\infty)$

c Reflection in the *y*-axis means a horizontal dilation of –1.

$y = (ax)^3$, $a = -1$

$y = (-x)^3$

There is no restriction on the values *x* can take, so the domain is $(-\infty,\infty)$

There is no restriction on the values *y* can take, so the range is $(-\infty,\infty)$

d $y = ke^x$, where $k = \frac{1}{9}$

$y = \frac{1}{9}e^x$

There is no restriction on the values x can take, so the domain is $(-\infty, \infty)$

For all x, $\frac{1}{9}e^x > 0$, so the range is $(0, \infty)$

e Reflected in the x-axis means a vertical dilation with scale factor –1.

$y = k \times \log_4 x$, $k = -1$

$y = -\log_4 x$

$\log_4 x$ requires positive values for x. The domain is $(0, \infty)$

There is no restriction on the values y can take, so the range is $(-\infty, \infty)$

Question 4

a $a = 2 \Rightarrow y = f(2x)$ is a horizontal dilation (compression) with scale factor $\frac{1}{2}$

$X = (-2, 7)$ has image at $X = (-2 \times \frac{1}{2}, 7) = (-1, 7)$

b $a = -1 \Rightarrow y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis. The scale factor is -1

$X = (-2, 7)$ has image at $X = (-2 \times -1, 7) = (2, 7)$

c $a = \frac{1}{3} \Rightarrow y = f\left(\frac{1}{3}x\right)$ is a horizontal dilation (stretching) with scale factor 3

$X = (-2, 7)$ has image at $X = (-2 \times 3, 7) = (-6, 7)$

Question 5

a $a = 3 \Rightarrow y = f(3x)$ is a horizontal dilation (compression) with scale factor $\frac{1}{3}$

$(x \times \frac{1}{3}, y) = (-24, 1)$ so $(x, y) = (-24 \times 3, 1) = (-72, 1)$

b $a = 2 \Rightarrow y = f(2x)$ is a horizontal dilation (compression) with scale factor $\frac{1}{2}$

$(x \times \frac{1}{2}, y) = (-24, 1)$ so $(x, y) = (-24 \times 2, 1) = (-48, 1)$

c $a = \frac{1}{4} \Rightarrow y = f\left(\frac{1}{4}x\right)$ is a horizontal dilation (stretching) with scale factor 4

$(x \times 4, y) = (-24, 1)$ so $(x, y) = \left(\frac{-24}{4}, 1\right) = (-6, 1)$

Question 6

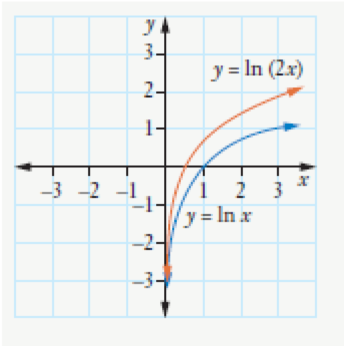
a The graph of $f(x)=\log_e(2x)$ is a horizontal dilation by a scale factor of $\frac{1}{2}$ (compression) of the graph of $f(x)=\log_e(x)$.

$f(x)=\log_e(x)$ has x-intercept at 1 with domain $(0,\infty)$ and range $(-\infty,\infty)$.

$f(x)=\log_e(2x)$ has x-intercept at $\frac{1}{2}$ with domain $(0,\infty)$ and range $(-\infty,\infty)$.

$x=0$ is the vertical asymptote for both functions.

As $x\to+\infty, y\to+\infty$ and as $x\to 0, y\to-\infty$.

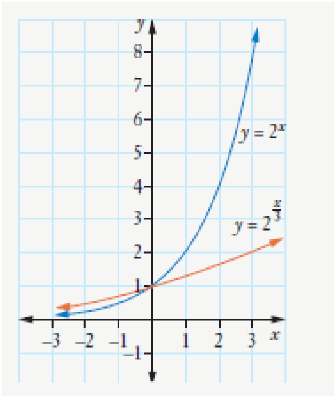


b The graph of $y=2^{\frac{x}{3}}=2^{\frac{1}{3}x}$ is a horizontal dilation by a scale factor of $1\div\frac{1}{3}=3$ (stretching) of the graph of $y=2^x$.

Both functions have y-intercept $2^0=1$, domain $(-\infty,\infty)$ and range $(0,\infty)$.

As $x\to+\infty, y\to+\infty$ and as $x\to-\infty, y\to 0$.

$y=0$ is the horizontal asymptote for both functions.



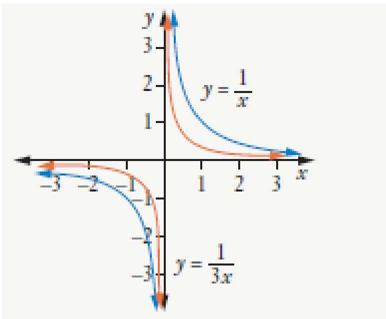
c The graph of $y=\frac{1}{3x}=\frac{1}{3}\times\frac{1}{x}$ is a vertical dilation by a scale factor of $\frac{1}{3}$ (compression) of the graph of $y=\frac{1}{x}=\frac{1}{x}$.

There are no axial intercepts.

Both functions have $y=0$ as the horizontal asymptote and $x=0$ as the vertical asymptote.

Domain of both functions is $(-\infty,0)\cup(0,\infty)$ and the range is $(-\infty,0)\cup(0,\infty)$.

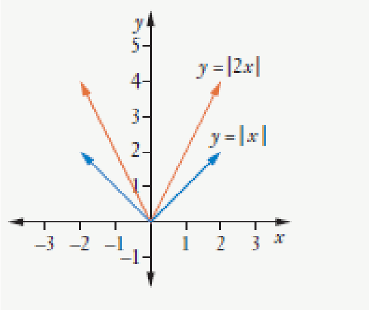
As $x\to+\infty, y\to 0$ and as $x\to-\infty, y\to 0$.



d The graph of $y=|2x|$ is a horizontal dilation by a scale factor of $\frac{1}{2}$ (compression) of the graph of $y=|x|$.

Both functions have an x-intercept and a y-intercept at $(0,0)$.

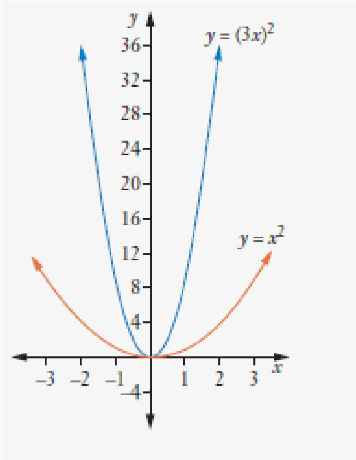
Domain of both functions is $(-\infty,\infty)$ and the range is $(0,\infty)$



e The graph of the parabola $f(x)=(3x)^2$ is a horizontal dilation by a scale factor of $\frac{1}{3}$ (compression) of the graph of the parabola $f(x)=x^2$.

Both functions have an x-intercept and a minimum turning point at $(0,0)$.

Domain of both functions is $(-\infty,\infty)$ and the range is $(0,\infty)$.

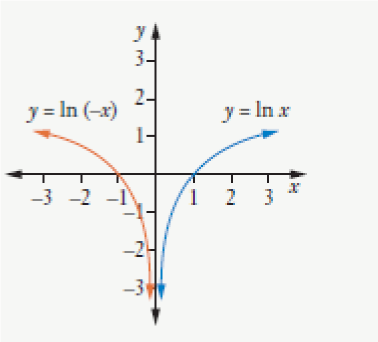


f The graph of the parabola $y=\ln(-x)$ is a horizontal dilation by a scale factor of -1 of the graph of the parabola $y=\ln(-x)$. It is a reflection in the y-axis.

$y=\ln(x)$ has an x-intercept at $(1,0)$ and $y=\ln(-x)$ has an x-intercept at $(-1,0)$.

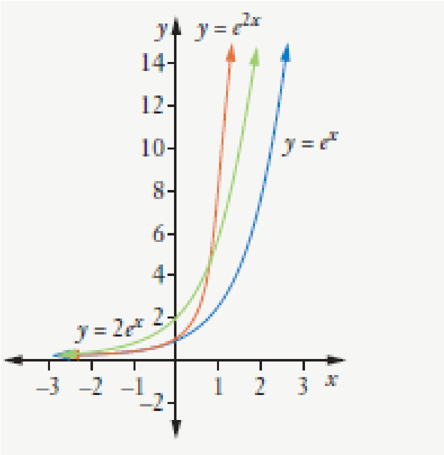
The domain of $y=\ln(x)$ is $(0,\infty)$ and the range is $(-\infty,\infty)$.

The domain of $y=\ln(-x)$ is $(0,-\infty)$ and the range is $(-\infty,\infty)$.



Question 7

The graph of the exponential $y = e^{2x}$ is a horizontal dilation by a scale factor of $\frac{1}{2}$ (compression) of the graph of the parabola $y = e^{x}$.



The graph of the exponential $y = 2e^{x}$ is a vertical dilation by a scale factor of 2(stretching) of the graph of the parabola $y = e^{x}$.

There are no x-intercepts. All three functions have $y = 0$ as a horizontal asymptote.

Their domain is $(-\infty, \infty)$ and their range is $(0, \infty)$

The y-intercept of $y = e^{x}$, $y = e^{2x}$ and $y = 2e^{x}$ is $(0,1)$, $(0,1)$ and $(0,2)$ respectively.

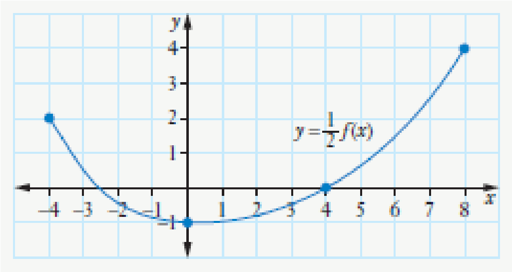
Question 8

a The graph of the parabola $y = x^2$ is an even function. That is, $y = f(-x) = f(x)$, so $y = (-x)^2 = x^2$. There is a reflection of $y = x^2$ in the y-axis.

b The graph of the parabola $y = |x|$ is an even function. That is, $y = f(-x) = f(x)$, so $y = |-x| = |x|$. There is a reflection of $y = |x|$ in the y-axis.

Question 9

a $y = f(ax) = f\left(\frac{1}{2}x\right)$ is a horizontal dilation (stretching) with scale factor $1 \div \frac{1}{2} = 2$



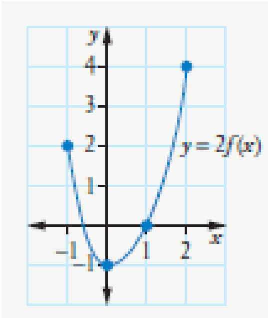
$(-2, 2) \to (-2\times 2, 2) = (-4, 2)$

$(0, -1) \to (0\times 2, -1) = (0, -1)$

$(2, 0) \to (2\times 2, 0) = (4, 0)$

$(4, 4) \to (4\times 2, 4) = (8, 4)$

b $y = f(ax) = f(2x)$ is a horizontal dilation (compression) with scale factor $\frac{1}{2}$



$(-2, 2) \to (-2\times \frac{1}{2}, 2) = (-1, 2)$

$(0, -1) \to (0\times \frac{1}{2}, -1) = (0, -1)$

$(2, 0) \to (2\times \frac{1}{2}, 0) = (1, 0)$

$(4, 4) \to (4\times \frac{1}{2}, 4) = (2, 4)$

Exercise 2.05 Combinations of transformations

Question 1

a translation 3 units right means $x = 2 \rightarrow x = 2 + 3 = 5$

translation 5 units down means $y = –6 \rightarrow y = –6 - 5 = -11$

image is at (5, –11)

b translation 3 units left means $x = 2 \rightarrow x = 2 - 3 = -1$

translation 4 units up means $y = –6 \rightarrow y = –6 + 4 = -2$

image is at (–1, –2)

c translation 7 units left means $x = 2 \rightarrow x = 2 + 7 = 9$

translation 9 units up means $y = –6 \rightarrow x = –6 + 9 = 3$

image is at (9, 3)

d translation 4 units left means $x = 2 \rightarrow x = 2 - 4 = -2$

translation 11 units down means $y = –6 \rightarrow y = –6 - 11 = -17$

image is at (–2, 17)

Question 2

a Reflection in the x-axis means $–f(x)$.

The transformed function is $f(x) = -x^5$.

$f(x)$ dilated vertically with scale factor 4 is $4f(x)$.

The transformed function is $f(x) = –4x^5$.

b $f(x)$ dilated horizontally with scale factor 3 is $f\left(\frac{1}{3}x\right) = \left(\frac{1}{3}x\right)^5$.

Reflection in the y-axis means $f(–x)$ = $= -\left(\frac{1}{3}x\right)^5 = -\frac{1}{243}x^5$.

The transformed function is $f(x) = -\frac{1}{243}x^5$.

Question 3

a Translated 3 units down means $f(x)-3$

The transformed function is $y=x^3-3$

Translated 4 units left means $f(x+4)$

The transformed function is $y=(x+4)^3-3$

b Translated 9 units up means $f(x)+9,$ so $f(x)=|x|+9$

The transformed function is $f(x)=|x|+9$

Translated 1 unit right means $f(x-1),$ so $f(x)=|x-1|+9$

The transformed function is $f(x)=|x-1|+9$

c $f(x)$ dilated vertically with scale factor 3 is $3f(x)$

The transformed function is $f(x)=3x$

Translated 6 units down means $f(x)-6$

The transformed function is $f(x)=3x-6$

d Reflected in the x-axis means $-f(x)$

The transformed function is $f(x)=-e^x$

Translated 2 units up means $f(x)+2$

The transformed function is $f(x)=-e^x+2$

e $f(x)$ dilated horizontally with scale factor $\frac{1}{2}$ is $f(2x)=(2x)^3=8x^3$

The transformed function is $y=8x^3$

Translated 5 units down means $f(x)-5$

The transformed function is $y=8x^3-5$

f $f(x)$ dilated vertically with scale factor 2 is $2f(x)=2\times\frac{1}{x}=\frac{2}{x}$

$f(x)$ dilated horizontally with scale factor 3 is $f(3x)=3\times\frac{2}{x}=\frac{6}{x}$

The transformed function is $f(x)=\frac{6}{x}$

g Reflected in the y-axis is $f(-x)$

The transformed function is $f(x)=\sqrt{-x}$

$f(x)$ dilated vertically with scale factor 3 is $3f(x)=3\sqrt{-x}$

The transformed function is $f(x)=-3\sqrt{x}$

$f(x)$ dilated horizontally with scale factor $\frac{1}{2}$ is $f(2x)=3\sqrt{-2x}$

The transformed function is $f(x)=3\sqrt{-2x}$

h Horizontal dilation by a scale factor of 3 $f\left(\frac{1}{3}x\right)=\ln\left(\frac{1}{3}x\right)$

Translated upward 2 units $f\left(\frac{1}{3}x\right)+2=\ln\left(\frac{1}{3}x\right)+2$

The transformed function is $f(x)=\ln\left(\frac{1}{3}x\right)+2$

i Horizontal dilation by a scale factor of $\frac{1}{4}$ $f(4x)=\log_2(4x)$

Vertical dilation by a scale factor of 3 $3f(4x)=3\log_2(4x)$

The transformed function is $3f(4x)=3\log_2(4x)$

j Horizontal dilation by a scale factor of 2 $f\left(\frac{1}{2}x\right)=\left(\frac{1}{2}x\right)^2=\frac{1}{4}x^2$

Translated down 3 units $f\left(\frac{1}{2}x\right)-3=\left(\frac{1}{2}x\right)^2-3=\frac{1}{4}x^2-3$

The transformed function is $f(x)=\left(\frac{1}{2}x\right)^2-3=\frac{1}{4}x^2-3$

Question 4

a Horizontal translation 1 unit to the right, vertical translation 7 units up

Let $y = f(x) = x^3$

horizontal translation 1 unit right $y = f(x-1) = (x-1)^3$

translation 7 units up $y = f(x-1) + 7 = (x-1)^3 + 7$

b Vertical dilation scale factor 4, vertical translation 1 unit down

Let $y = f(x) = x^3$

vertical dilation with scale factor 4 $y = 4f(x) = 4x^3$

translation 1 unit down $y = 4f(x) - 1 = 4x^3 - 1$

c Vertical dilation scale factor 5, reflection in *x*-axis, vertical translation 3 units down

Let $y = f(x) = x^3$

vertical dilation with scale factor 5 $y = 5f(x) = 5x^3$

reflection in the *x*-axis $y = -5f(x) = -5x^3$

translation 3 units down $y = -5f(x) - 3 = -5x^3 - 3$

d Horizontal translation 7 units to the left, vertical dilation scale factor 2

Let $y = f(x) = x^3$

horizontal translation 7 units left $y = f(x+7) = (x+7)^3$

vertical dilation with scale factor 2 $y = 2f(x+7) = 2(x+7)^3$

Note that in this case the transformations can also be performed in reverse order.

vertical dilation with scale factor 2 $y = 2f(x) = 2x^3$

horizontal translation 7 unit left $y = 2f(x+7) = 2(x+7)^3$

e Horizontal dilation scale factor $\frac{1}{2}$, horizontal translation 2 units to the right, vertical dilation scale factor 6, vertical translation 5 units up

Let $y = f(x) = x^3$

horizontal dilation by a scale factor of $\frac{1}{2}$ $y = f(2x) = (2x)^3$

horizontal translation 2 units right $y = f(2[x-2]) = (2x-4)^3$

vertical dilation by a scale factor of 6 $y = 6(2x-4)^3$

vertical translation 5 units up $y = 6(2x-4)^3 + 5$

f Horizontal dilation scale factor $\frac{1}{3}$, horizontal translation 3 units to the left, vertical dilation scale factor 2, vertical translation 10 units down

Let $y = f(x) = x^3$

horizontal dilation scale factor $\frac{1}{3}$ $y = f(3x) = (3x)^3$

horizontal translation 3 units to the left $y = f(3[x+3]) = (3x+9)^3$

vertical dilation by a scale factor of 2 $y = 2(3x+9)^3$

vertical translation 10 units down $y = 2(3x+9)^3 - 10$

Question 5

a Horizontal translation 3 units to the left, vertical dilation scale factor 2, vertical translation 1 unit down

Let $y = f(x) = \log(x)$

horizontal translation 3 unit left $y = f(x+3) = \log(x+3)$

vertical dilation with scale factor 2 $y = 2\log(x+3)$

vertical translation 1 unit down $y = 2\log(x+3) - 1$

b Horizontal dilation scale factor $\frac{1}{3}$, reflection in *x*-axis, vertical translation 9 units up

Let $y = f(x) = x^2$

Horizontal dilation by a scale factor of $y = f(3x) = (3x)^2$

Reflection in the *x*-axis (vertical dilation) $y = -f(3x) = -(3x)^2$

Vertical translation 9 units up $y = -f(3x) + 9 = -(3x)^2 + 9$

c Horizontal dilation scale factor $\frac{1}{5}$, vertical dilation scale factor 2, vertical translation 3 units down

Let $y = f(x) = e^x$

Horizontal dilation by a scale factor of $\frac{1}{5}$ $y = f(5x) = e^{5x}$

Vertical dilation by a scale factor of 2 $y = 2f(5x) = 2e^{5x}$

Vertical translation 3 units down $y = 2f(5x) - 3 = 2e^{5x} - 3$

d Horizontal translation 7 units to the right, vertical dilation scale factor 4, vertical translation 1 unit up

Let $y = f(x) = \sqrt{x}$

Horizontal translation 7 units right $y = f(x-7) = \sqrt{x-7}$

Vertical dilation by a scale factor of 4 $y = 4f(x-7) = 4\sqrt{x-7}$

Vertical translation 1 unit up $y = 4f(x-7) + 1 = 4\sqrt{x-7} + 1$

e Reflection in *y*-axis, horizontal dilation scale factor $\frac{1}{2}$, horizontal translation 1 unit to the left, vertical translation 1 unit down

Let $y = f(x) = |x|$

Reflection in the *y*-axis (horizontal dilation) $y = f(-x) = |-x|$

Horizontal dilation by a scale factor of $\frac{1}{2}$ $y = f(-2x) = |-2x|$

Horizontal translation 1 unit to the left $y = f(-2(x+1)) = |-2(x+1)|$

Vertical translation 1 unit down $y = f(-2(x+1)) - 1 = |-2(x+1)| - 1$

f Horizontal dilation scale factor $\frac{1}{2}$, reflection in *x*-axis, vertical translation 8 units up

Let $y = f(x) = \frac{1}{x}$

Horizontal dilation by a scale factor of $\frac{1}{2}$ $y = -f(2x) = \frac{1}{2x}$

Reflection in the *x*-axis (vertical dilation) $y = -f(2x) = -\frac{1}{2x}$

Vertical translation 8 units up $y = -f(2x) + 8 = -\frac{1}{2x} + 8$

Question 6

a $f(x-1)$ is a horizontal translation 1 unit to the right

$(8,-12) \to (8+1,-12) = (9,-12)$

The 3 in $3f(x-1)$ is a vertical dilation, scale factor 3.

$(9,-12) \to (9,-12\times 3) = (9,-36)$

The +5 in $3f(x-1)+5$ is a vertical translation 5 units up

$(9,-36) \to (9,-36+5) = (9,-31)$

b $f(2x)$ is a horizontal dilation scale factor $\frac{1}{2}$

$(8,-12) \to (8\times\frac{1}{2},-12) = (4,-12)$

$-f(2x)$ is a reflection of $f(2x)$ in the *x*-axis

$(4,-12) \to (4,-(-12)) = (4,12)$

The –7 in $-f(2x)-7$ is a vertical translation 7 units down

$(4,12) \to (4,12-7) = (4,5)$

c $f(x+3)$ is a horizontal translation 3 units to the left

$(8,-12) \to (8-3,-12) = (5,-12)$

$2f(x+3)$ is a vertical dilation of $f(x+3)$ with scale factor 2

$(5,-12) \to (5,-12\times 2) = (5,-24)$

The –1 in $2f(x+3)-1$ is a vertical translation 1 unit down

$(5,-24) \to (5,12-24-1) = (5,-25)$

d $f(-x)$ is a reflection of $f(x)$ in the *y*-axis

$(8,-12) \to (-8,-12)$

$6f(-x)$ is a vertical dilation of $f(-x)$ with scale factor 6

$(-8,-12) \to (-8,-12\times 6) = (-8,-72)$

The +5 in $6f(-x)+5$ is a vertical translation 5 units up

$(-8,-72) \to (-8,12-72+5) = (-8,-67)$

e $y = -2f(2x-4)-3 = -2f(2[x-2])-3$

The coefficient of *x* 2 is a horizontal dilation with scale factor $\frac{1}{2}$.

$(8,-12) \to (8\times\frac{1}{2},-12) = (4,-12)$

$f(2[x-2])$ is a horizontal translation 2 units to the right

$(4,-12) \to (4+2,-12) = (6,-12)$

The –2 in $-2f(2[x-2])$ is a vertical dilation of $f(2[x-2])$ with scale factor 2 followed by a reflection in the *x*-axis.

$(6,-12) \to (6,-12\times -2) = (6,24)$

The –3 in $-2f(2[x-2])-3$ is a vertical translation 3 units down

$(6,24) \to (6,24-3) = (6,21)$

Question 7

a Translated 6 units down

$(x, y) \to (x, y-6)$

Translated 3 units to the right

$(x, y-6) \to (x+3, y-6)$

b Reflected in the *y*-axis

$(x, y) \to (-x, y)$

Translated 6 units up

$(-x, y) \to (-x, y+6)$

c Vertically dilated with scale factor 3

$(x, y) \to (x, 2y)$

Translated 5 units left

$(x, 2y) \to (x-5, 2y)$

d Horizontal dilation with scale factor 3

$(x, y) \to (3x, y)$

Translated 5 units up

$(3x, y) \to (3x, y+5)$

e Reflection in the *x*-axis

$(x, y) \to (x, -y)$

Vertically dilated with scale factor 8

$(x, -y) \to (x, -8y)$

Translated 6 units left

$(x, -8y) \to (x-6, -8y)$

Horizontally dilated with scale factor 5

$(x-6, -8y) \to (5(x-6), -8y) = (5x-30, -8y)$

Translated 1 unit down

$(5x-30, -8y) \to (5x-30, -8y-1)$

Question 8

a Let $y = f(x)$

Translated 2 units down $\Rightarrow y \to y - 2$

$y = f(x) - 2$

Translated 1 unit to the left $\Rightarrow x \to x + 1$

$y = f(x+1) - 2$

b Let $y = f(x)$

Translated 5 units right $\Rightarrow x \to x - 5$

$y = f(x-5)$

Translated 3 units up $\Rightarrow y \to y + 3$

$y = f(x-5) + 3$

c Let $y = f(x)$

Reflected in the x-axis $\Rightarrow y \to -y$

$y = -f(x)$

Translated 4 units right $\Rightarrow x \to x - 4$

$y = -f(x-4)$

d Let $y = f(x)$

Reflected in the y-axis $\Rightarrow x \to -x$

$y = f(-x)$

Translated 2 units up $\Rightarrow y \to y + 2$

$y = f(-x) + 2$

e Let $y = f(x)$

Reflected in the x-axis $\Rightarrow y \to -y$

$y = -f(x)$

Horizontal dilation with scale factor 4 $\Rightarrow x \to \frac{1}{4}x$

$y = -f\left(\frac{1}{4}x\right)$

f Let $y = f(x)$

Vertically dilated by a scale factor of 2 $\Rightarrow y \to 2y$

$y = 2f(x)$

Translation 2 units down $\Rightarrow y \to y - 2$

$y = 2f(x) - 2$

Question 9

a Reflected in the *y*-axis $f(x)=-\frac{1}{x}$

Vertically dilated by a scale factor of 9 $f(x)=9\times\left(-\frac{1}{x}\right)=-\frac{9}{x}$

Translation 3 units up $f(x)=-\frac{9}{x}+3$

b Horizontal dilation by a scale factor of $\frac{1}{5}$ $f(x)=(5x)^2$

Translation 2 units left $f(x)=[5(x+2)]^2$

Translation 6 units down $f(x)=[5(x+2)]^2-6=25(x+2)^2-6$

c Horizontal dilation by a scale factor of 2 $f(x)=\ln\left(\frac{1}{2}x\right)$

Horizontal translation 5 units right $f(x)=\ln\left[\frac{1}{2}(x-5)\right]$

Vertical dilation by scale factor 8 $f(x)=8\ln\left[\frac{1}{2}(x-5)\right]$

Vertical translation 3 units down $f(x)=8\ln\left[\frac{1}{2}(x-5)\right]-3$

d Reflection in the *y*-axis (horizontal dilation) $y=\sqrt{-x}$

Horizontal translation 4 units left $y=\sqrt{-(x+4)}$

Vertical dilation by a scale factor of 9 $y=9\sqrt{-(x+4)}$

Vertical translation 4 units up $y=9\sqrt{-(x+4)}+4$

e Horizontal dilation by a scale factor of $\frac{1}{6}$ $y=|6x|$

Reflection in the *x*-axis $y=-|6x|$

Vertical translation 7 units up $y=-|6x|+7$

f Horizontal dilation by a scale factor of $\frac{1}{4}$ $y=(4x)^3$

Horizontal translation 4 units left $y=(4[x+4])^3=64(x+4)^3$

g Horizontal translation 2 units right $y=2^{x-2}$

Vertical translation 5 units up $y=2^{x-2}+5$

Vertical dilation by a scale factor of 6 $y=6(2^{x-2}+5)$

Question 10

a The function $f(x)=(x+3)^2+5$ is obtained by translating the parabola $f(x)=x^2$ left 3 units and 5 units up.

The domain of $f(x)=x^2$ is $(-\infty,\infty)$ and its range is $[0,\infty)$.

Hence, the domain of $f(x)=(x+3)^2+5$ is $(-\infty,\infty)$ and its range is $[5,\infty)$.

b The function $f(x)=5\left|-2x\right|-2$ is obtained by:

- reflecting the graph of $f(x)=\left|x\right|$ in the *y*-axis to get $f(x)=\left|-x\right|$, with domain $(-\infty,\infty)$, range $[0,\infty)$
- Then dilating horizontally by a scale factor of $\frac{1}{2}$ to get $f(x)=\left|-2x\right|$, with domain $(-\infty,\infty)$, range $[0,\infty)$
- Then dilating vertically by a scale factor of 5 to get $f(x)=5\left|-2x\right|$, with domain $(-\infty,\infty)$, range $[0,\infty)$
- Then translation 2 units down to get $f(x)=5\left|-2x\right|-2$ with domain $(-\infty,\infty)$, range $[-2,\infty)$

c The function $f(x)=\frac{1}{2x-4}+1=\frac{1}{2(x-2)}+1$ is obtained by:

- dilating $y=\frac{1}{x}$ horizontally by a scale factor of $\frac{1}{2}$ to get $f(x)=\frac{1}{2x}$, with domain $(-\infty,0)\cup(0,\infty)$, range $(-\infty,0)\cup(0,\infty)$
- Translating the graph of $f(x)=\frac{1}{2x}$ 2 units right to get $f(x)=\frac{1}{2\left(x-2\right)}$ with domain $(-\infty,-2)\cup(-2,\infty)$, range $(-\infty,0)\cup(0,\infty)$
- Then translate 1 unit up to get $f(x)=\frac{1}{2(x-2)}+1$, with domain $(-\infty,2)\cup(2,\infty)$, range $(-\infty,1)\cup(1,\infty)$

d The function $y=4^{3x}+2$ is obtained by:

- Dilating horizontally by a scale factor of 3 the graph of $y=4^x$ to get $y=4^{3x}$ with domain $(-\infty,\infty)$, range $[0,\infty)$
- Then translate 2 units up to get $y=4^{3x}+2$ with domain $(-\infty,\infty)$, range $(2,\infty)$

e The function $f(x) = 3\log(3x-6) - 5 = 3\log[3(x-2)] - 5$ is obtained by:

- Dilate $y = \log x$ horizontally with scale factor $\frac{1}{3}$ to get $f(x) = \log 3x$ with domain $(0, \infty)$, range $(-\infty, \infty)$
- Translating the graph of $f(x) = \log 3x$ 2 units right to get $f(x) = \log 3(x-2)$ with domain $(2, \infty)$, range $(-\infty, \infty)$
- Then dilate vertically with scale factor 3 to get $f(x) = 3\log[3(x-2)]$ with domain $(2, \infty)$, range $(-\infty, \infty)$
- Then translate vertically down 5 units to get $f(x) = 3\log[3(x-2)] - 5$ with domain $(2, \infty)$, range $(-\infty, \infty)$

Question 11

a

$$
\begin{aligned}
y &= x^2 + 2x - 7 \\
&= (x^2 + 2x + 1^2) - 1^2 - 7 \\
&= (x+1)^2 - 8
\end{aligned}
$$

b Horizontal translation 1 unit to the left, vertical translation 8 units down.

Translate the graph of $y = x^2$ 1 unit left to get $y = (x+1)^2$

Then translate the graph of $y = (x+1)^2$ 8 units down to get $y = (x+1)^2 - 8$

Question 12

Horizontal translation 5 units to the right, vertical translation 28 units down

$$
\begin{aligned}
y &= x^2 - 10x - 3 \\
&= (x-5)^2 - 28
\end{aligned}
$$

Translate the graph of $y = x^2$ 5 units right to get $y = (x-5)^2$

Then translate the graph of $y = (x-5)^2$ 28 units down to get $y = (x-5)^2 - 28$

Question 13

a $a=\frac{1}{2}$, $(x,y)\to(2x,y)$

$b=-3$, $(2x,y)\to(2x+3,y)$

$k=2$, $(2x+3,y)\to(2x+3,2y)$

$c=5$, $(2x+3,2y)\to(2x+3,2y+5)$

b $a=3$, $(x,y)\to\left(\frac{1}{3}x,y\right)$

$b=6$, $\left(\frac{1}{3}x,y\right)\to\left(\frac{1}{3}x-6,y\right)$

$k=-1$, $\left(\frac{1}{3}x-6,y\right)\to\left(\frac{1}{3}x-6,-y\right)$

$c=-2$, $\left(\frac{1}{3}x-6,-y\right)\to\left(\frac{1}{3}x-6,-y-2\right)$

Question 14

a Circle $(x-3)^2+(y-4)^2=9$ or $x^2-6x+y^2-8x+16=0$

3 units to the right, $(x-3)^2+y^2=9$

4 units up, $(x-3)^2+(y-4)^2=9$

This describes the translation of the circle $x^2+y^2=3^2$ of radius 3 units and centre at (0, 0) to its new centre at (3, 4).

b Translated 2 units to the right, 3 units down

$x^2-4x+y^2+6y+12=0$

$[(x-2)^2-2^2]+[(y+3)^2-3^2]+12=0$

$(x-2)^2+(y+3)^2-2^2-3^2+12=0$

$(x-2)^2+(y+3)^2=1^2$

This describes a circle of radius 1 unit with centre at (2, –3).

It is obtained by transforming the circle $x^2+y^2=1^2$ with centre at (0, 0) 2 units right and 3 units down.

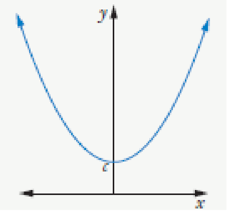
Exercise 2.06 Graphs of functions with combined transformations

Question 1

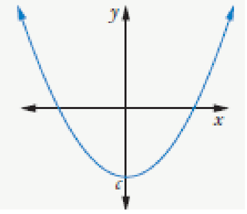
a The graph of $f(x) = x^2 + c$ is a parabola with a minimum turning point at (0, c).

The function is obtained by translating the graph of $f(x) = x^2$ up by c units.

i The turning point is above the x-axis.



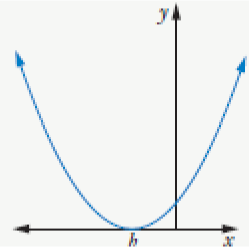
ii The turning point is below the x-axis.



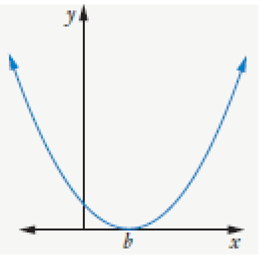
b The graph of $f(x) = (x + b)^2$ is a parabola with a minimum turning point at (0, 0).

The function is obtained by translating the graph of $f(x) = x^2$ horizontally by b units.

i The turning point is to the left of the origin.



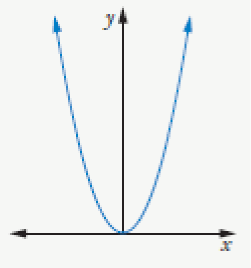
ii The turning point is to the right of the origin.



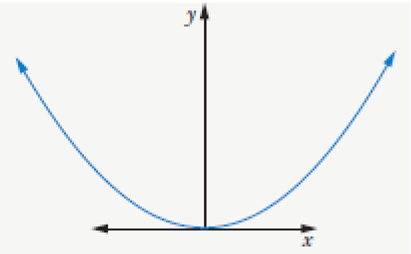
c The graph of $f(x) = kx^2$ is a parabola with a minimum turning point at (0, 0).

The function is obtained by vertically dilating the graph of $f(x) = x^2$ using scale factor k.

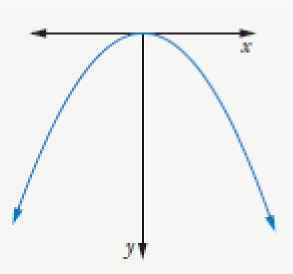
i The transformed function is the graph of $f(x) = x^2$ stretched vertically by factor k.



ii The transformed function is the graph of $f(x) = x^2$ compressed (stretched horizontally) by factor k.



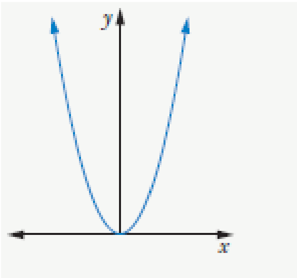
iii The transformed function is a reflection of the graph of $f(x) = x^2$ in the x-axis.



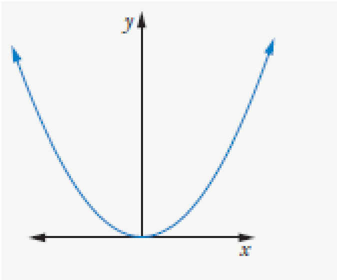
d The graph of $f(x)=(ax)^2$ is a parabola with a minimum turning point at (0, 0).

The function is obtained by horizontally dilating the graph of $f(x)=x^2$ using scale factor $\frac{1}{a}$.

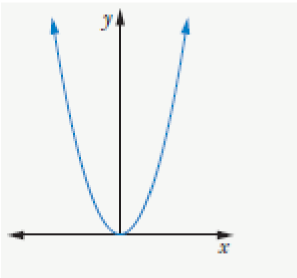
i The transformed function is the graph of $f(x)=x^2$ compressed horizontally by scale factor $\frac{1}{a}$.



ii The transformed function is the graph of $f(x)=x^2$ stretched horizontally by scale factor $\frac{1}{a}$.



iii The transformed function is a reflection of the graph of $f(x)=x^2$ in the y-axis.



Question 2

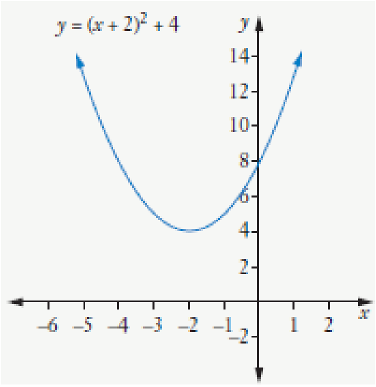
a The parabola $y = x^2$ is translated 2 units left and translated 4 units up.

Minimum turning point at (–2, 4)

There are no *x*-intercepts because the turning point is above the *x*-axis.

y-intercept

$y = (0+2)^2 + 4 = 8$ *y*-intercept at (0, 8).



b The parabola $y = x^2$ is translated 3 units right and translated 1 unit down.

Minimum turning point at (3, –1)

x-intercepts

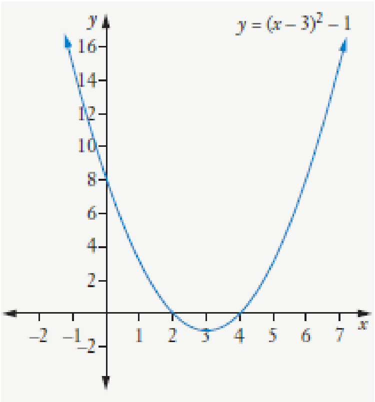
$(x-3)^2 - 1 = 0$

$x = 3 \pm 1$

x-intercepts at (2, 0), (4, 0)

y-intercept

$y = (0-3)^2 - 1 = 8$ *y*-intercept at (0, 8)



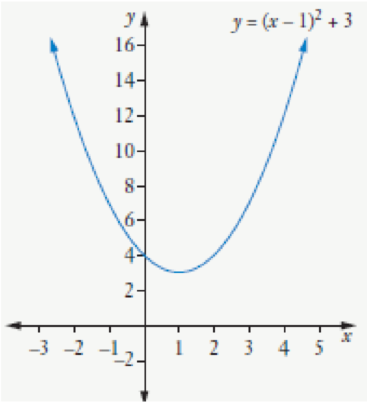
c The parabola $y = x^2$ is translated 1 unit right and translated 3 units up.

Minimum turning point at (1, 3)

There are no *x*-intercepts because the turning point is above the *x*-axis.

y-intercept

$y = (0-1)^2 + 3 = 4$ *y*-intercept at (0, 4)



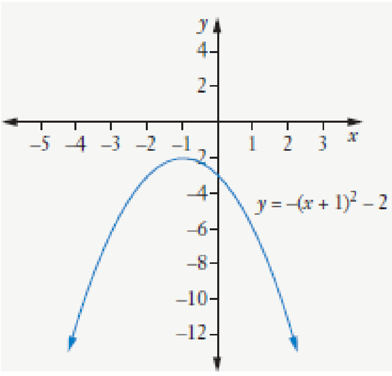
d The parabola $y = x^2$ is reflected in the *x*-axis, translated 1 unit left and translated 2 units down.

Maximum turning point at (–1, –2)

There are no *x*-intercepts because the turning point is below the *x*-axis.

y-intercept

$y = -(0+1)^2 - 2 = -3$ *y*-intercept at (0, –3)



e The parabola $y = x^2$ is dilated vertically by scale 2, translated 1 unit right and translated 4 units down.

Minimum turning point at (1, –4)

x-intercepts

$2(x-1)^2 - 4 = 0$

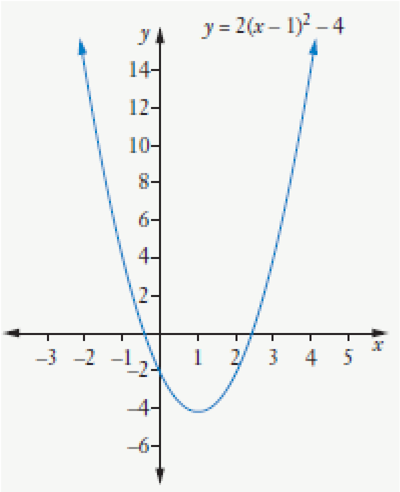
$x = 1 \pm \sqrt{2}$

$x \approx 2.41, -0.41$

x-intercepts at $(-0.41, 0), (2.41, 0)$

y-intercept

$y = 2(0-1)^2 - 4 = -2$ y-intercept at (0, –2)



Question 3

a The cubic $y = x^3$ has axial intercepts at (0, 0).

It is translated 1 unit right and translated 2 units up.

The image of (0, 0) is (1, 2)

x-intercept

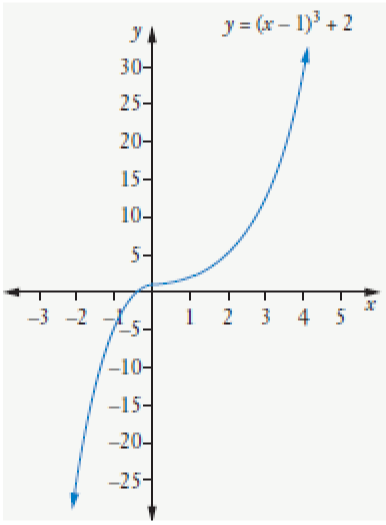
$(x-1)^3 + 2 = 0$

$x = 1 + \sqrt[3]{-2} \approx -0.26$

x-intercept at $(-0.26, 0)$

y-intercept

$y = (0-1)^3 + 2 = 1$ y-intercept at (0, 1)



b The graph of $y = x^3$ is translated 2 units right and translated 3 units down.

The image of (0, 0) is (2, –3)

x-intercept

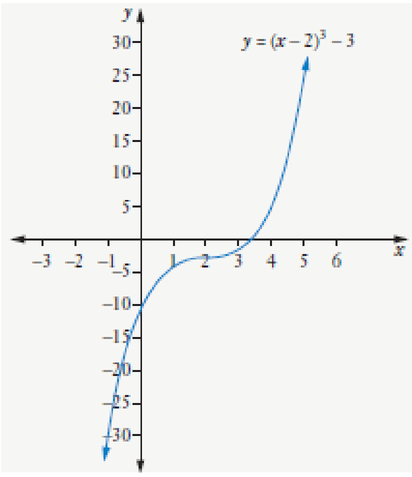
$(x-2)^3 - 3 = 0$

$x = 2 + \sqrt[3]{3} \approx 3.44$

x-intercept at $(3.44, 0)$

y-intercept

$y = (0-2)^3 - 3 = -11$ y-intercept at (0, –11)



c The graph of $y = x^3$ is reflected in the x-axis, translated 1 unit left and translated 4 units up.

The image of (0, 0) is (–1, 4).

x-intercept

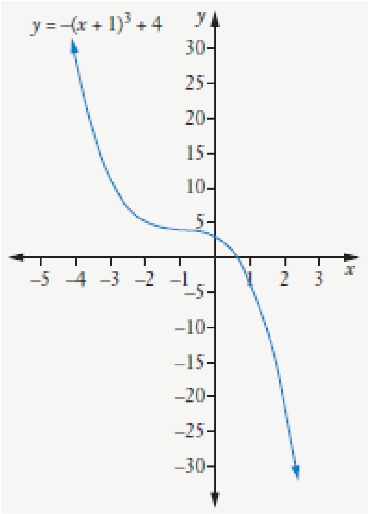
$-(x+1)^3 + 4 = 0$

$x = -1 + \sqrt[3]{4} \approx 0.59$

x-intercept at $(0.59, 0)$

y-intercept

$y = -(0+1)^3 + 4 = 3$ y-intercept at (0, 3)



d The graph of $y = x^3$ is vertically dilated by a scale factor of 2, translated 3 units left and translated 5 units down.

The image of (0, 0) is (–3, –5)

x-intercept

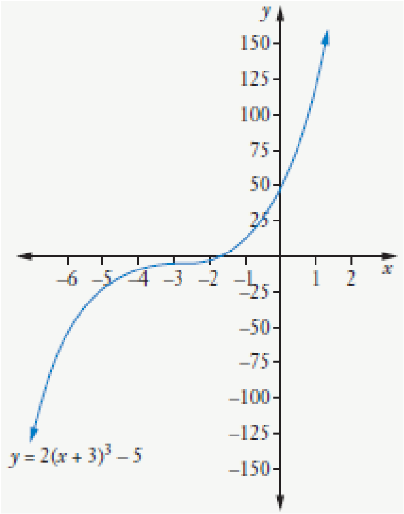
$2(x+3)^3 - 5 = 0$

$x = -3 + \sqrt[3]{2.5} \approx -1.64$

x-intercept at $(-1.64, 0)$

y-intercept

$y = 2(0+3)^3 - 5 = 49$ y-intercept at (0, 49)



e The graph of $y = x^3$ is vertically dilated by a scale factor of 3, translated 1 unit right and translated 2 units down.

The image of (0, 0) is (1, –2)

x-intercept

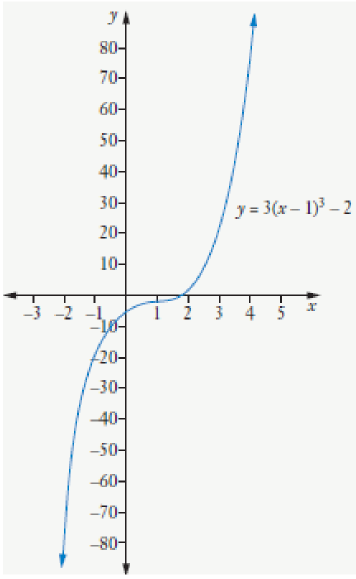
$3(x-1)^3 - 2 = 0$

$x = 1 + \sqrt[3]{\frac{2}{3}} \approx 1.87$

x-intercept at $(1.87, 0)$

y-intercept

$y = 3(0-1)^3 - 2 = -5$ y-intercept at (0, –5)



Question 4

a $y = -2f(3x) + 1$ transforms $f(x)$ by a horizontal dilation with scale factor $\frac{1}{3}$, then by a vertical dilation with scale factor 2, then reflection in the x-axis and a translation up of 1 unit.

$$(6,1) \to (6\times\frac{1}{3},1) \to (6\times\frac{1}{3},1\times 2) \to (6\times\frac{1}{3},-1\times 2) \to (6\times\frac{1}{3},-1\times 2+1) \text{ or } (2,-1)$$

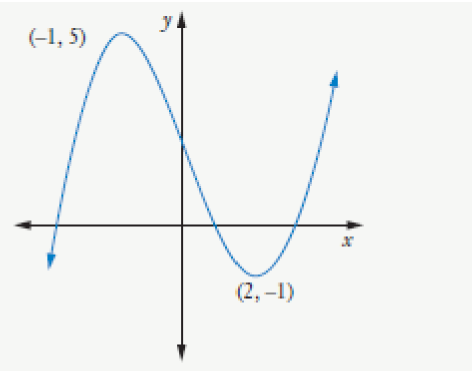
and

$$(-3,-2) \to (-2\times\frac{1}{3},-2) \to (-3\times\frac{1}{3},-2\times 2) \to (-3\times\frac{1}{3},-1\times -2\times 2) \to (-3\times\frac{1}{3},-1\times -2\times 2+1)$$

or $(-1,5)$

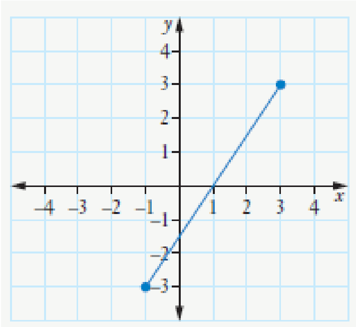
b (2, –1) is a local minimum stationary point and (–1, 5) is a local maximum stationary point of the transformed function.

The image of (x, y) is $\left(\frac{1}{3}x, -2y+1\right)$



Question 5

a **i** $y = 3f(x-1)$ transforms $y = f(x)$ by a horizontal translation 1 unit right followed by a vertical dilation with scale factor 3.

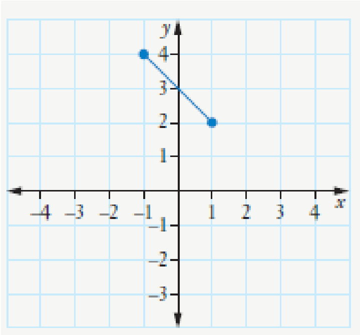


$(-2,-1) \to (-2+1,-1) \to (-2+1,-1\times 3)$ or $(-1,-3)$

and

$(2,1) \to (2+1,1) \to (2+1,1\times 3)$ or $(3,3)$

ii $y = -f(2x)+3$ transforms $y = f(x)$ by a reflection in the x-axis, then a horizontal dilation with scale factor $\frac{1}{2}$ followed by a vertical translation 3 units up.



$(-2,-1) \to (-2,-1\times -1) \to (-2\times\frac{1}{2},-1\times -1)$

$\to (-2\times\frac{1}{2},-1\times -1+3)$

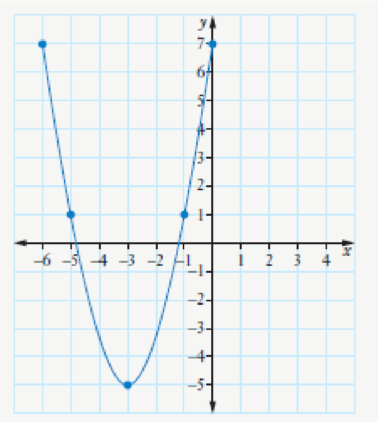
or $(-1,4)$

and

$(2,1) \to (2,1\times -1) \to (2\times\frac{1}{2},1\times -1) \to (2\times\frac{1}{2},1\times -1+3)$

or $(1,2)$

b **i** $y = 3f(x+3)-2$ transforms $y = f(x)$ by a vertical dilation with scale factor 3, then a horizontal translation 3 units left followed by a vertical translation 2 units down.



$(-3,3) \to (-3,3\times 3) \to (-3-3,3\times 3) \to (-3-3,3\times 3-2)$
or $(-6,7)$

$(-2,1) \to (-2,1\times 3) \to (-2-3,1\times 3) \to (-2-3,1\times 3-2)$
or $(-5,1)$

$(0,-1) \to (0,-1\times 3) \to (0-3,-1\times 3) \to (0-3,-1\times 3-2)$
or $(-3,-5)$

$(2,1) \to (2,1\times 3) \to (2-3,1\times 3) \to (2-3,1\times 3-2)$ or $(-1,1)$

$(3,3) \to (3,3\times 3) \to (3-3,3\times 3) \to (3-3,3\times 3-2)$ or $(0,7)$

ii $y=-2f\left(\dfrac{x}{4}\right)+3$ transforms $y=f(x)$ by a vertical dilation with scale factor 2, then reflection in the *x*-axis, then a horizontal dilation with scale factor 4 followed by a vertical translation 3 units up.

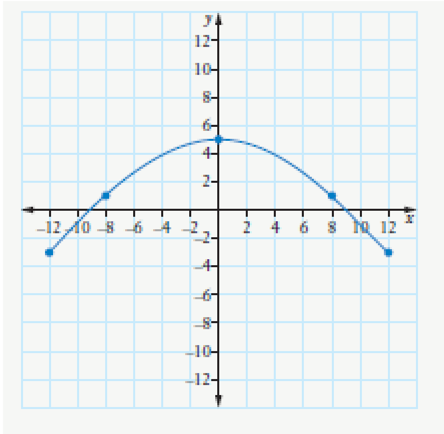
$(-3,3)\to(-3,3\times 2)\to(-3,-3\times 2)$
$\to(-3\times 4,-3\times 2)\to(-3\times 4,-3\times 2+3)$
or $(-12,-3)$

$(-2,1)\to(-2,1\times 2)\to(-2,-1\times 2)$
$\to(-2\times 4,-1\times 2)\to(-2\times 4,-1\times 2+3)$
or $(-8,1)$

$(0,-1)\to(0,-1\times 2)\to(0,-1\times -1\times 2)$
$\to(0\times 4,-1\times -1\times 2)\to(0\times 4,-1\times -1\times 2+3)$
or $(0,5)$

$(2,1)\to(2,1\times 2)\to(2,-1\times 1\times 2)$
$\to(2\times 4,-1\times 1\times 2)\to(2\times 4,-1\times 1\times 2+3)$
or $(8,1)$

$(3,3)\to(3,3\times 2)\to(3,-1\times 3\times 2)$
$\to(3\times 4,-1\times 3\times 2)\to(3\times 4,-1\times 3\times 2+3)$
or $(12,-3)$



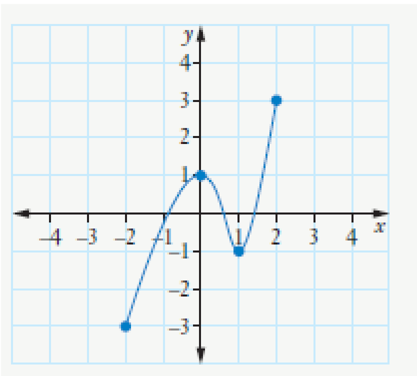
c **i** $y=2f(-x)-1$ transforms $y=f(x)$ by a vertical dilation with scale factor 2, then reflection in the *y*-axis followed by a vertical translation 1 unit down.

$(-2,2)\to(-2,2\times 2)\to(-2\times -1,2\times 2)\to(-2\times -1,2\times 2-1)$ or $(2,3)$

$(-1,0)\to(-1,2\times 0)\to(-1\times -1,2\times 0)\to(-1\times -1,2\times 0-1)$ or $(1,-1)$

$(0,1)\to(0,2\times 1)\to(0\times -1,2\times 1)\to(0\times -1,2\times 1-1)$ or $(0,1)$

$(2,-1)\to(2,2\times -1)\to(2\times -1,2\times -1)\to(2\times -1,2\times -1-1)$ or $(-2,-3)$



ii $y=-3f(2x+4)+2=-3f(2[x+2])+2$ transforms $y=f(x)$ by:

a vertical dilation with scale factor 3

reflection in the y-axis

a horizontal dilation with scale factor $\frac{1}{2}$

a horizontal translation 2 units left

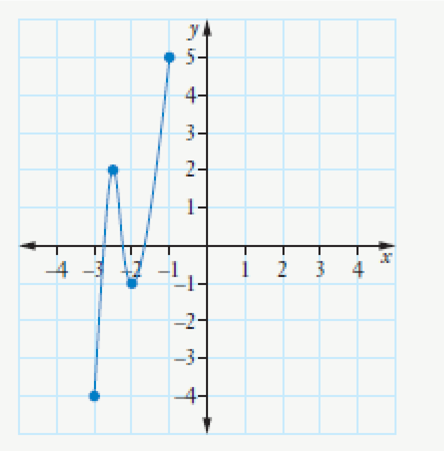
a vertical translation 2 units up

$(-2,2)\rightarrow(-2,2\times3)\rightarrow(-2,-2\times3)\rightarrow(-2\times\frac{1}{2},-2\times3)\rightarrow(-2\times\frac{1}{2}-2,-2\times3)\rightarrow(-2\times\frac{1}{2}-2,-2\times3+2)$ or $(-3,-4)$

$(-1,0)\rightarrow(-1,0\times3)\rightarrow(-1,-0\times3)\rightarrow(-1\times\frac{1}{2},-0\times3)\rightarrow(-1\times\frac{1}{2}-2,-0\times3)\rightarrow(-1\times\frac{1}{2}-2,-0\times3+2)$ or $(-2.5,2)$

$(0,1)\rightarrow(0,1\times3)\rightarrow(0,-1\times3)\rightarrow(0\times\frac{1}{2},-1\times3)\rightarrow(0\times\frac{1}{2}-2,-1\times3)\rightarrow(0\times\frac{1}{2}-2,-1\times3+2)$ or $(-2,-1)$

$(2,-1)\rightarrow(2,-1\times3)\rightarrow(2,1\times3)\rightarrow(2\times\frac{1}{2},1\times3)\rightarrow(2\times\frac{1}{2}-2,1\times3)\rightarrow(2\times\frac{1}{2}-2,1\times3+2)$ or $(-1,5)$



Question 6

Perform the vertical dilation first

$(-4,-7) \to (-4,-7\times 3) = (-4,-21)$, vertical dilation with scale factor 3

$(-4,-21) \to (-4,-21-4) = (-4,-25)$, vertical translation 4 units down

$(-4,-25) \to (-4-2,-25) = (-6,-25)$, horizontal translation 2 units left

The image of $(-4,-7)$ is $(-6,-25)$

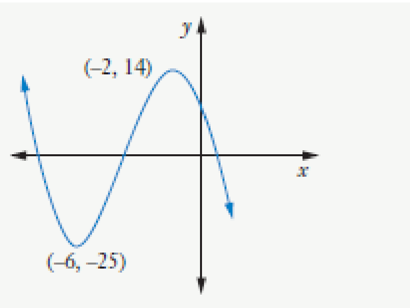
$(0,6) \to (0-2, 6\times 3-4) = (-2,14)$

The image of the *y*-intercept $(0,6)$ is $(-2,14)$

The most negative *x*-intercept is approximately at (–5.5, 0) so its image is $(-5.5-2,0) = (-7.5,0)$

The positive *x*-intercept is approximately at (1.5, 0) so its image is $(1.5-2,0) = (-0.5,0)$

The second *x*-intercept is approximately midway between the two stationary points.



Question 7

Order of transformations:

horizontal dilation of scale factor $\frac{1}{2}$

horizontal translation 1 unit to the right

reflection in the *x*-axis (vertical dilation)

vertical translation 5 units down

The images of the given points are:

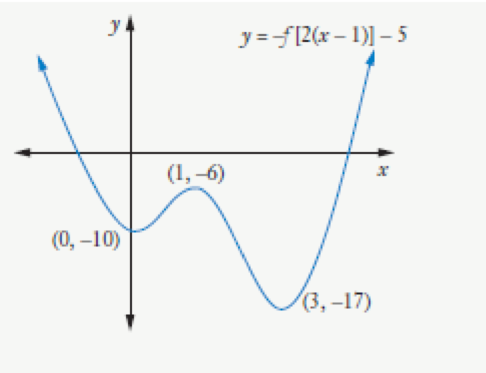
$(-2,5) \to (-1,5) \to (0,5) \to (0,-5) \to (0,-10)$

$(0,1) \to (0,1) \to (1,1) \to (1,-1) \to (1,-6)$

$(4,12) \to (2,12) \to (3,12) \to (3,-12) \to (3,-17)$

The negative *x*-intercept is approximately at (–3, 0) so its image is $(-3\times\frac{1}{2}+1,-0-5)=(-0.5,-5)$

The positive *x*-intercept is approximately at (6, 0) so its image is $(6\times\frac{1}{2}+1,-0-5)=(4,-5)$



Question 8

a Change $y = x^3$ to $y = -3(x-2)^3 + 1$ using the transformations below.

$y = (x-2)^3$, translation, 2 units to the right

$y = 3(x-2)^3$, vertical dilation with scale factor 3

$y = -3(x-2)^3$, reflection in the x-axis

$y = -3(x-2)^3 + 1$, vertical translation, 1 unit up

The image of (0, 0) is $(0+2, -0\times 3+1) = (2,1)$

x-intercept

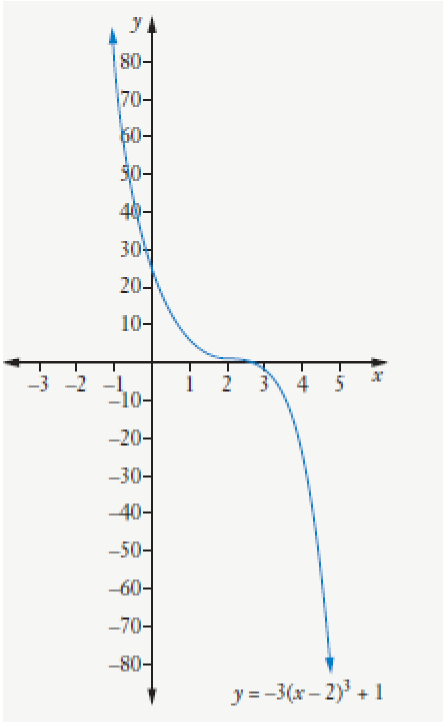
$-3(x-2)^3 + 1 = 0$, $x = 2 + \sqrt[3]{\frac{1}{3}} \approx 2.69$

x-intercept at $(2.69, 0)$

y-intercept

$y = -3(0-2)^3 + 1 = 25$

y-intercept at $(0, 25)$



b Change $y = e^x$ to $y = 2e^{x+1} - 4$ using the transformations below.

$y = e^{x+1}$, translation, 1 unit to the left

$y = 2e^{x+1}$, vertical dilation with scale factor 2

$y = 2e^{x+1} - 4$, vertical translation, 4 units down

The image of (0, 1) is $(0-1, 1\times 2-4) = (-1, -2)$

x-intercept

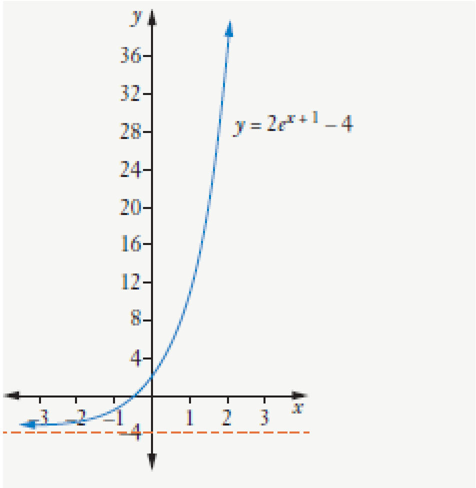
$2e^{x+1} - 4 = 0$, $x = -1 + \ln(2) \approx -0.31$

x-intercept at $(-0.31, 0)$

y-intercept

$y = 2e^{0+1} - 4 \approx 1.44$

y-intercept at $(0, 1.44)$



c Change $f(x)=\sqrt{x}$ to $f(x)=3\sqrt{x-2}-1$ using the transformations below.

$f(x)=\sqrt{x-2}$, translation, 2 units to the right

$f(x)=3\sqrt{x-2}$, vertical dilation with scale factor 3

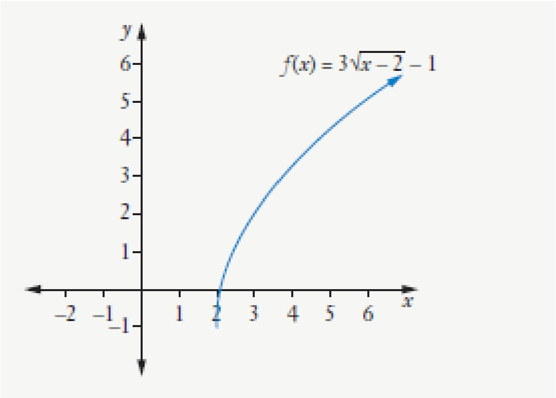
$f(x)=3\sqrt{x-2}-1$, vertical translation, 1 unit down

x-intercept

$3\sqrt{x-2}-1=0$, $x=2\frac{1}{9}\approx 2.11$

x-intercept at $(2.11,0)$

There is no y-intercept because the domain is $[2,\infty)$



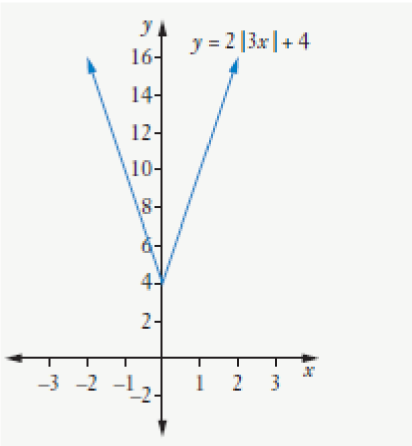
d Change $y=|x|$ to $y=2|3x|+4$ using the transformations below.

$y=|3x|$, horizontal dilation, scale factor $\frac{1}{3}$

$y=2|3x|$, vertical dilation with scale factor 2

$y=2|3x|+4$, vertical translation, 4 units up

The image of (0, 0) is $(0\times\frac{1}{3}, 0\times 2+4)=(0,4)$, which is the *y*-intercept.



e Change $y = x^2$ to $y = -(3x)^2 + 1$ using the transformations below.

$y = (3x)^2$, horizontal dilation, scale factor $\frac{1}{3}$

$y = -(3x)^2$, reflection in the x-axis

$y = -(3x)^2 + 1$, vertical translation, 1 unit up

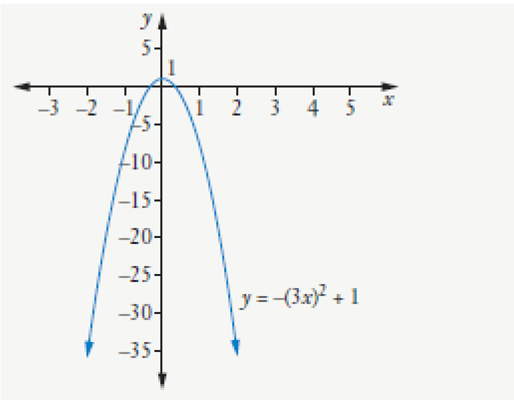
The image of (0, 0) is $(0 \times \frac{1}{3}, 1) = (0, 1)$

x-intercepts

$-(3x)^2 + 1 = 0$, $x = \pm\frac{1}{3}$

x-intercepts at $\left(-\frac{1}{3}, 0\right), \left(\frac{1}{3}, 0\right)$

The y-intercept is $-(3 \times 0)^2 + 1 = 1$



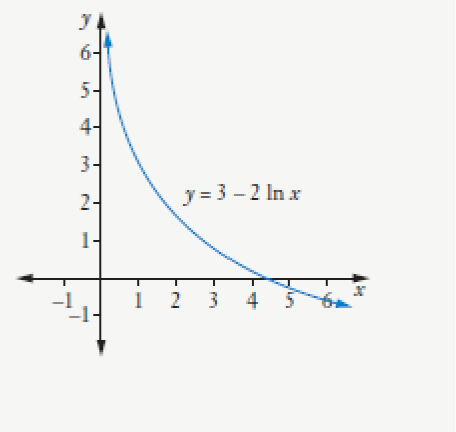
Question 9

a The log function $y = 3 - 2\ln x$ is obtained from the log function $y = \ln x$ by reflection in the x-axis, a vertical dilation with scale factor 2 and then a vertical translation 3 units up.

The image of (x, y) is $(x, -2y + 3)$

The x-intercept occurs when $3 - 2\ln x = 0 \Rightarrow x = e^{\frac{3}{2}} \approx 4.5$

There is no y-intercept because the domain is $(0, \infty)$

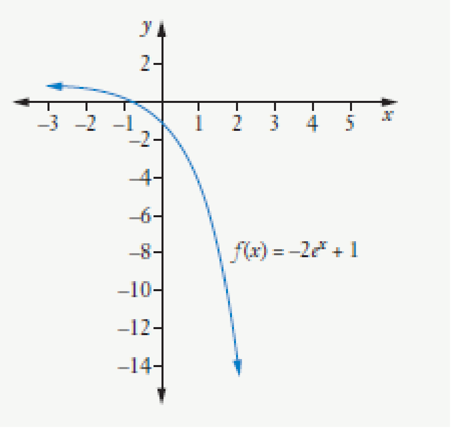


b The exponential $f(x) = -2e^x + 1$ is obtained from the exponential $f(x) = e^x$ by reflection in the x-axis, a vertical dilation with scale factor 2 and then a vertical translation 1 unit up.

The image of (x, y) is $(x, -2y + 1)$

The x-intercept occurs when $-2e^x + 1 = 0 \Rightarrow x = \ln\left(\frac{1}{2}\right) \approx -0.69$

The y-intercept is $-2e^0 + 1 = -1$

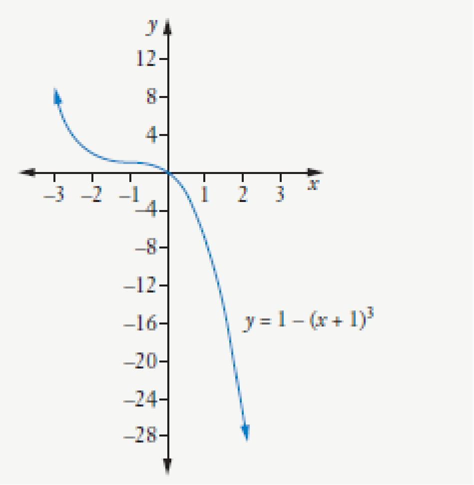


c The cubic $y=1-(x+1)^3$ is obtained from the cubic $y=x^3$ by reflection in the x-axis, a horizontal translation 1 unit left and then a vertical translation 1 unit up.

The image of (x, y) is $(x-1, -y+1)$

The x-intercept occurs when $1-(x+1)^3=0 \Rightarrow x=0$

The y-intercept is $1-(0+1)^3=0$



d The hyperbola $y=\dfrac{2}{x-1}+3$ is obtained from the hyperbola $y=\dfrac{1}{x}$ by a vertical dilation with scale factor 2, a horizontal translation 1 unit right and then a vertical translation 3 units up.

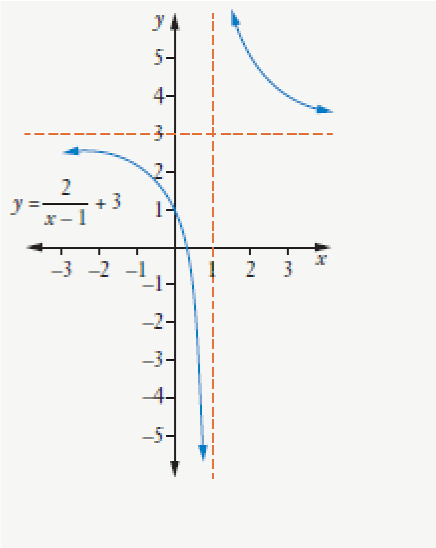
The image of (x, y) is $(x+1, 2y+3)$

The vertical asymptote at $x=0$ is now at $x=0+1=1$

The horizontal asymptote at $y=0$ is now at $y=2\times 0+3=3$

The x-intercept occurs when $\dfrac{2}{x-1}+3=0 \Rightarrow x=\dfrac{1}{3}$

The y-intercept is $\dfrac{2}{0-1}+3=1$

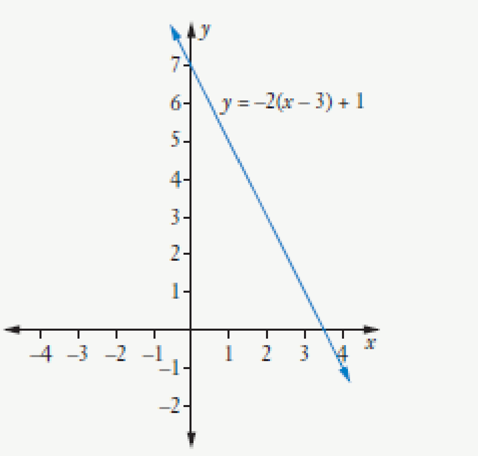


e The straight line $y = -2(x-3)+1$ is obtained from the straight line $y = x$ by a vertical dilation with scale factor 2, reflection in the x-axis, a horizontal translation 3 units right and then a vertical translation 1 unit up.

The image of (x, y) is $(x+3, -2y+1)$

The x-intercept occurs when $-2(x-3)+1=0 \Rightarrow x = 3\frac{1}{2}$

The y-intercept is $y = -2(0-3)+1 = 7$



Question 10

a The transformations on $y = f(x)$ are a horizontal translation 2 units right, vertical dilation with scale factor 3 and a vertical translation of 1 unit up.

Therefore, the image of (x, y) is $(x+2, 3y+1)$

So $(-3, 2) \to (x+2, 3y+1)$

$x+2=-3,\ 3y+1=2$

Hence

$x=-5,\ y=\frac{1}{3}$

b The image $(-3, 2)$ has original point $(-5,\ \frac{1}{3})$, from **a**

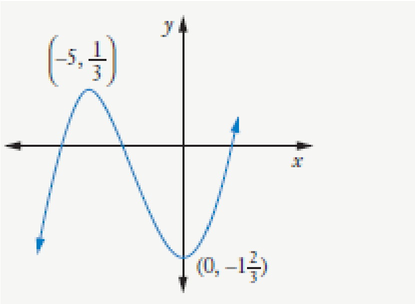
Also, $(2, -4) \to (x+2, 3y+1)$

$x+2=2,\ 3y+1=-4$

Hence

$x=0,\ y=-1\frac{2}{3}$

Hence the cubic has turning points at $(-5,\ \frac{1}{3})$, $(0,\ -1\frac{2}{3})$



Question 11

a $\left(-26, 7\frac{2}{3}\right)$

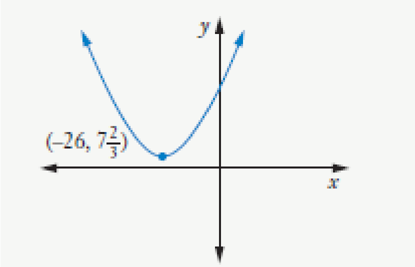
Let (x, y) be the vertex of $y = f(x)$, which is translated 2 units right and 5 units down to be the image $(-24, 18)$.

Then

$x + 2 = -24 \Rightarrow x = -26$, translated 2 units right

$3y - 5 = 18 \Rightarrow y = 7\frac{2}{3}$, translated 5 units down

The vertex of $y = f(x)$ is a minimum turning point.



b $\left(-69, -3\frac{3}{5}\right)$

Let (x, y) be the vertex of $y = f(x)$.

It is dilated horizontally with scale factor $\frac{1}{3}$, translated 1 unit left, dilated vertically with scale factor –5 to be the image $(-24, 18)$.

Then horizontally:

$$\frac{1}{3}x - 1 = -24$$
$$\frac{1}{3}x = -23$$
$$x = -69$$

Vertically:

$-5y = 18$

$y = -3\frac{3}{5}$

The $y = f(x)$ is reflected in the x-axis, so the vertex of $y = f(x)$ is a maximum turning point.

c $(-54, 10\frac{1}{2})$

Let (x, y) be the vertex of $y = f(x)$

The transformed function is $y = 2f(2x-6)-3 = 2f(2(x-3))-3$

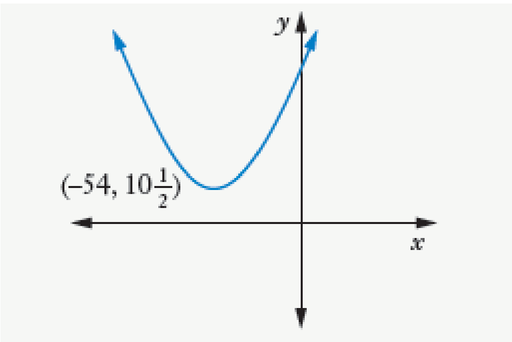
$y = f(x)$ is dilated horizontally with scale factor $\frac{1}{2}$, translated 3 unit right, dilated vertically with scale factor 2 and translated vertically 3 units down to be the image $(-24, 18)$.

Then

$\frac{1}{2}x + 3 = -24 \Rightarrow x = -54$, translated horizontally with scale factor $\frac{1}{2}$, translated 3 unit right

$2y - 3 = 18 \Rightarrow y = 10\frac{1}{2}$, dilated vertically with scale factor 2 and translated vertically 3 units

The vertex of $y = f(x)$ is a minimum turning point.



Exercise 2.07 Equations and inequalities

Question 1

a 2

There are 2 x-intercepts for the parabola.

b 0

The absolute value function does not intersect the x-axis, so there is no x-intercept.

c 1

The linear function intersects the x-axis at one point.

d 3

The cubic function intersects the x-axis at three points.

e 0

There are no x-intercepts. The function approaches the horizontal asymptote $y = 0$

f 1

The function intersects the x-axis at one point.

g 2

The cubic function touches the x-axis at one point and crosses over the x-axis at another point. Hence there are two intercepts.

h 0

There are no x-intercepts. As x increases on the positive side, the function approaches the horizontal asymptote $y = 0$

i 1

The function intersects the x-axis at one point.

j 0

The parabola has a maximum turning point below the x-axis, so there are no x-intercepts.

Question 2

a **i** $x = -2$, $x = 0$

From the graph, the *x*-values that correspond to $y = 1$ are $x = -2$, $x = 0$

Algebraically,

$$-2(x+1)^2 + 3 = 1$$
$$(x+1)^2 = 1$$
$$x = -1 \pm 1$$

ii $x = 0.6$, $x = -2.6$

Estimating from the graph, the *x*-values that correspond to $y = -2$ are $x = 0.6$, $x = -2.6$

Algebraically,

$$-2(x+1)^2 + 3 = -2$$
$$(x+1)^2 = 2.5$$
$$x = -1 \pm \sqrt{2.5}$$

iii $x = -2.2$, $x = 0.2$

We want the *x*-intercepts. Estimating from the graph, they are $x = -2.2$, $x = 0.2$

Algebraically,

$$-2(x+1)^2 + 3 = 0$$
$$(x+1)^2 = 1.5$$
$$x = -1 \pm \sqrt{1.5}$$

b Algebraically,

$$-2(x+1)^2 + 3 = 0$$
$$(x+1)^2 = 1.5$$
$$x = -1 \pm \sqrt{1.5}$$

$x = -2.2, 0.2$

Question 3

a $x = 1.4$

We want the *x*-intercept. Estimating from the graph, it is $x = 1.4$

Algebraically,

$$3(4x - 5) - 2 = 0$$
$$12x = 17$$
$$x = 1\frac{5}{12}$$

b $x = 1.9$

Estimating from the graph, the *x*-value corresponding to $y = 5$ is $x = 1.9$

Algebraically,

$$3(4x - 5) - 2 = 5$$
$$12x = 22$$
$$x = 1\frac{5}{6}$$

c $x = 0.2$

Estimating from the graph, the *x*-value corresponding to $y = -15$ is $x = 0.2$

Algebraically,

$$3(4x - 5) - 2 = -15$$
$$12x = 2$$
$$x = \frac{1}{6}$$

d $x > 2.2$

Estimating from the graph, the *x*-value corresponding to $y = 10$ is $x = 2.2$

Since $f(x)$ is linear with positive gradient, $y > 10$ requires $x > 2.2$

e $x \le 3.1$

Estimating from the graph, the *x*-value corresponding to $y = 20$ is $x = 3.1$

Since $f(x)$ is linear with positive gradient, $y \le 20$ requires $x \le 3.1$

Question 4

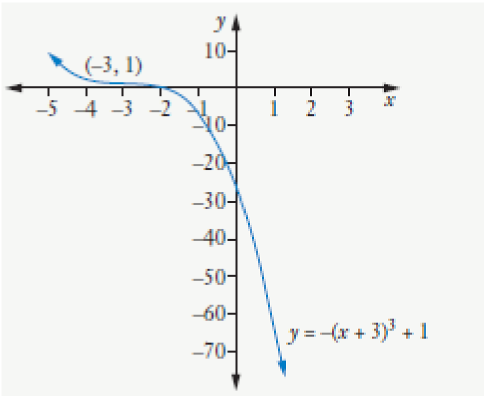
a Let $y = x^3$

The graph of $y = -(x+3)^3 + 1$ is obtained by performing the following operations on the graph of $y = x^3$.

$y = (x+3)^3$, horizontal translation, 3 units left

$y = -(x+3)^3$, reflection in the x-axis

$y = -(x+3)^3 + 1$, vertical translation, 1 unit up



b **i** $x = -2$

We want the x-intercept. From the graph it is $x = -2$

ii $x = -0.8$

Estimating from the graph, the x-value corresponding to $y = -10$ is $x = -0.8$

Algebraically,

$$-(x+3)^3 + 1 = -10$$
$$x = -3 + \sqrt[3]{11}$$

iii $x = -0.3$

Estimating from the graph, the x-value corresponding to $y = -20$ is $x = -0.3$

Algebraically,

c

$$-(x+3)^3 + 1 = 0$$
$$x + 3 = 1$$
$$x = -2$$

Question 5

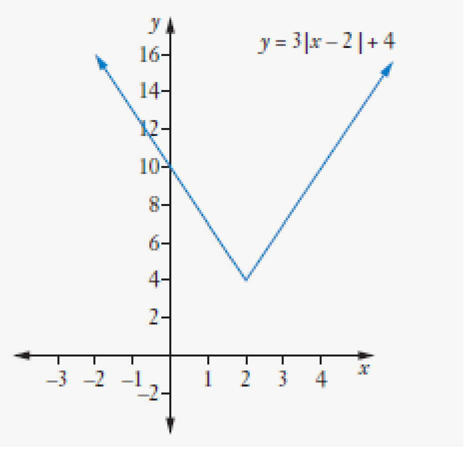
a Let $y = |x|$

The graph of $y = 3|x-2|+4$ is obtained by performing the following operations on the graph of $y = |x|$.

$y = |x-2|$, horizontal translation, 2 units right

$y = 3|x-2|$, vertical dilation, scale factor 2

$y = 3|x-2|+4$, vertical translation, 4 units up



b none

Let $y = |x|$

There are no values of x corresponding to $y = 1$ because from the vertical translation, the minimum value of y is 4.

c $x = 0, x = 4$

From the graph, the x-values corresponding to $y = 10$ are $x = 0, x = 4$

$$3|x-2|+4 = 10$$
$$|x-2| = 2$$

For $x \geq 2, x-2 = 2 \Rightarrow x = 4$

For $x < 2, -(x-2) = 2 \Rightarrow x = 0$

Question 6

a Let $f(x)=\frac{1}{x}$, which has a vertical asymptote at $x=0$ and a horizontal asymptote at $y=0$.

To transform $f(x)=\frac{1}{x}$ to $f(x)=\frac{2}{x-3}-4$, perform the transformations below.

$f(x)=\frac{1}{x-3}$, horizontal translation, 3 units right.

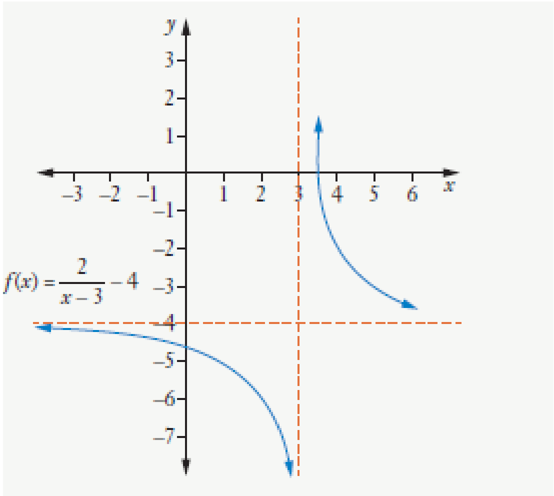
The vertical asymptote is at $x=3$

$f(x)=\frac{2}{x-3}$, vertical dilation, scale factor 2

$f(x)=\frac{2}{x-3}-4$, vertical translation, 4 units down

The horizontal asymptote is at $y=-4$

Hence the graph of $f(x)=\frac{2}{x-3}-4$ is the graph of $f(x)=\frac{1}{x}$ with the vertical asymptote at $x=3$ and the horizontal asymptote at $y=-4$.



b $x=1$

$$\frac{2}{x-3}-4=-5$$
$$\frac{2}{x-3}=-1$$
$$2=-x+3$$
$$x=1$$

c $x=4$

$$\frac{2}{x-3}-4=-2$$
$$\frac{2}{x-3}=2$$
$$2=2x-6$$
$$2x=8$$
$$x=4$$

Question 7

a The function $A=-3(x-2)^2+18$ is obtained from $A=x^2$ by:

horizontal translation, 2 units to the right

vertical dilation with scale factor 3

reflection in the x-axis

vertical translation 18 units up

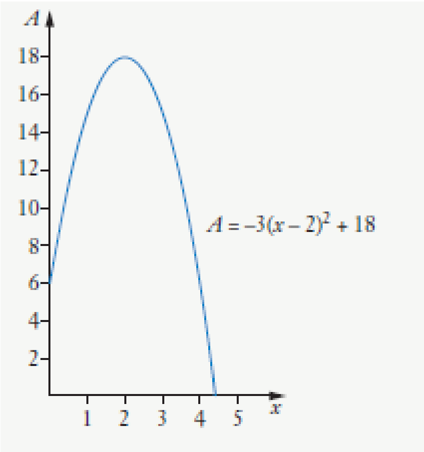
The minimum turning point $(0,0)$ of $A=x^2$ becomes the maximum turning point of $A=-3(x-2)^2+18$ at $(2,18)$.

The y-intercept is $-3(0-2)^2+18=6$

The x-intercepts occur when $-3(x-2)^2+18=0$

$$(x-2)^2=6$$
$$x=2\pm\sqrt{6}$$

The domain of A is $0\le x\le 2+\sqrt{6}$ or approximately $[0, 4.45]$



b $x=0.4, x=3.6$

From the graph, the estimated x values corresponding to $A=10$ are $x=0.4, x=3.6$

Question 8

a The function $C = 2(x+1)^2 + 3$ is obtained from $C = x^2$ by:

horizontal translation, 1 unit to the left

vertical dilation with scale factor 2

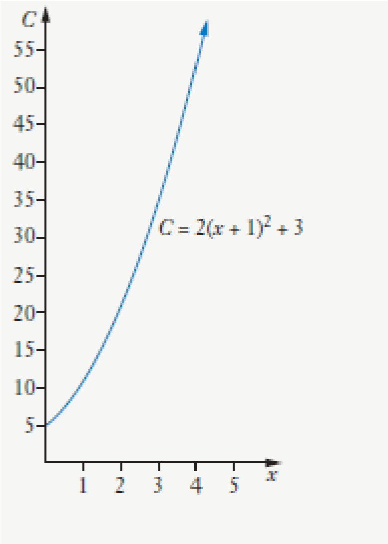
vertical translation 3 units up

The image of the minimum turning point $(0,0)$ of $C = x^2$ is $(-1,3)$.

The *y*-intercept is $2(0+1)^2 + 3 = 5$

There are no *x*-intercepts because the turning point is at $(-1,3)$.

The domain of C is $x \geq 0$



b \$5000

The factory overhead corresponds to the value of C for function $x = 0$.

This is $2(0+1)^2 + 3 = 5$

Hence, the overhead costs are $5 \times 1000 = \$5000$

c From the graph, the estimated x value corresponding to $C = 20$ is $x = 1.9$

Since x must be a positive integer, take $x = 2$

This means that to manufacture 2 products, the cost to the company is $20 \times 1000 = \$20\ 000$

Question 9

a The function $dB = 10\log\left(\frac{x}{I}\right)$ is obtained from $dB = \log(x)$ by:

horizontal dilation, scale factor I

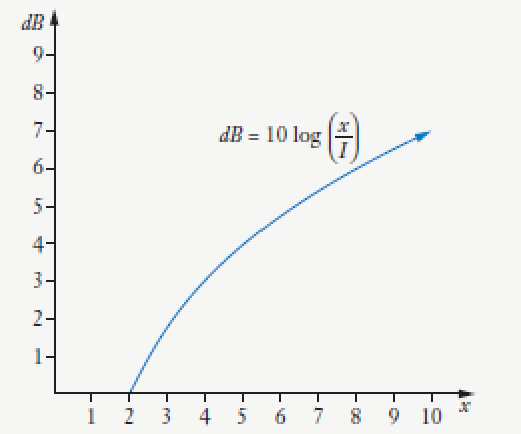
vertical dilation with scale factor 10

The domain is $\frac{x}{I} \geq 1 \Rightarrow x \geq I$

For $I = 2 \Rightarrow x \geq 2$

The *x*-intercept is at $x = I$.

Since $x \geq 2$ there is no *y*-intercept



b **i** From the graph, the estimated *x* value corresponding to $dB = 5$ is $x = 6$

ii From the graph, the estimated *x* value corresponding to $dB = 2$ is $x = 3$

Question 10

a **i** From the graph, the estimated t value corresponding to $T = 50$ is $t = 3.5$

The temperature of the cooling metal ball after 3.5 minutes is $50°\text{C}$

ii From the graph, the estimated t value corresponding to $T = 30$ is $t = 8$

b **i** 0.74 minutes

$$24 + 70e^{-0.3t} = 80$$
$$e^{-0.3t} = 0.8$$
$$-0.3t = \log_e 0.8$$
$$t = 0.74, \text{ to 2 decimal places}$$

ii 11.85 minutes

$$24 + 70e^{-0.3t} = 26$$
$$e^{-0.3t} = \frac{1}{35}$$
$$-0.3t = \log_e\left(\frac{1}{35}\right)$$
$$t = 11.85, \text{ to 2 decimal places}$$

c As the time t increases, the temperature $T = 24 + 70e^{-0.3t}$, $e^{-0.3t} \to 0$, hence $T \to 24 + 70 \times 0 = 24^0 C$.

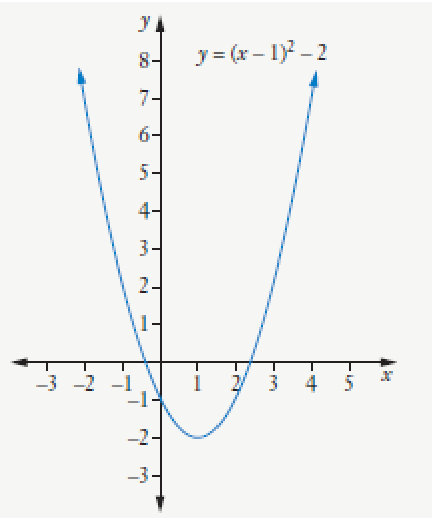
$24°\text{C}$ is the room temperature since the object's temperature can't drop below the temperature of its surroundings.

Question 11

a The function $y=(x-1)^2-2$ is obtained from the graph of $y=x^2$ by a horizontal translation 1 unit to the right followed by a vertical translation 2 units down.

The minimum turning point is at $(1,-2)$, the y-intercept is $(0-1)^2-2=-1$

At the x-intercepts, $(x-1)^2-2=0 \Rightarrow x=1\pm\sqrt{2}$



b **i** From the graph, the estimated x value corresponding to $y=2$ is $x=3, x=-1$

ii $y\geq 2$ for $x\geq 3$, and $y\geq 2$ for $x\leq -1$

iii $y<2$ for $-1<x<3$

Question 12

a The function $f(x) = -(2x+4)^2 + 1 = -(2[x+2])^2 + 1$ is obtained from the graph of $y = x^2$ by

horizontal translation 2 units to the left

horizontal dilation, scale factor $\frac{1}{2}$

reflection in the x-axis

vertical translation, 1 unit up

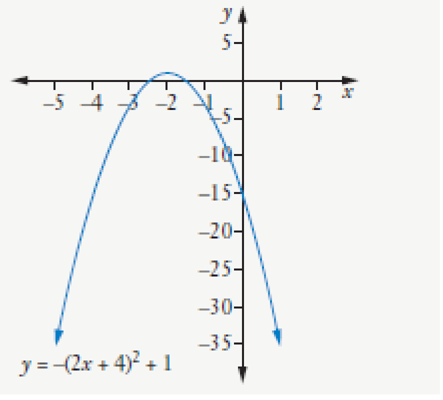
The maximum turning point is at $(-2,1)$, the y-intercept is $-(2\times 0+4)^2 + 1 = -15$

At the x-intercepts

$-(2x+4)^2 + 1 = 0$

$\Rightarrow x = \dfrac{-4\pm 1}{2}$

$x = -2.5, x = -1.5$



b **i** From the graph, the x values corresponding to $f(x) = -3$ are $x = -3, x = -1$

ii $f(x) > -3$ for $-3 < x < -1$

iii $f(x) \le -3$ for $x \le -3, x \ge -1$

Test yourself 2

Question 1

B

Question 2

D

The equation of the parabola can be written in the form $y = A(x-B)^2 + C$, where (B,C) are the co-ordinates of the turning point.

The maximum turning point is at $(3,0)$, so $y = A(x-3)^2$

The y-intercept is –9. Hence $-9 = A(0-3)^2 \Rightarrow A = -1$

The equation of the parabola is $y = -(x-3)^2$

Question 3

C

$(x, y) \to (x, -2y)$, vertical dilation with scale factor –2

$(x, -2y) \to (x-1, -2y)$, horizontal translation 1 unit to the left

$(x-1, -2y) \to (x, -2y+4)$, vertical translation 4 units up

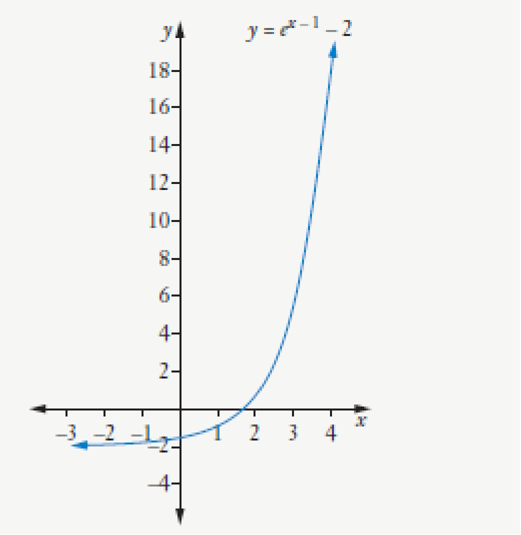
Question 4

a The graph of $y = e^{x-1} - 2$ is obtained from the graph of $y = e^x$ by a horizontal translation 1 unit right followed by a vertical translation of 2 units down.

Hence, the equation of the horizontal asymptote is $y = -2$

The y-intercept is $e^{0-1} - 2 \approx -1.6$

At the x-intercept, $e^{x-1} - 2 = 0 \Rightarrow x = 1 + \ln 2 \approx 1.7$



b From the graph, when $y = 8$ the corresponding x value is approximately 3.1

c

$e^{x-1} - 2 = 20$
$x = 1 + \ln 22 \approx 4.09$

Question 5

a $(24,36) \to (24\times\frac{1}{4},36) = (6,36)$, horizontal dilation with scale factor $\frac{1}{4}$

$(6,36) \to (6,36\times3) = (6,108)$, vertical dilation with scale factor 3

$(6,108) \to (6,108-1) = (6,107)$, vertical translation 1 unit down

b $(24,36) \to (24\times\frac{1}{3},36) = (8,36)$, horizontal dilation with scale factor $\frac{1}{3}$

$(8,36) \to (8-2,36) = (6,36)$, horizontal translation 2 units left

$(6,36) \to (6,36+4) = (6,40)$, vertical translation 4 units up

c $(24,36) \to (-24,36)$, reflection in the *y*-axis

$(-24,36) \to (-24,36\times5) = (-24,180)$, vertical dilation with scale factor 5

$(-24,180) \to (-24,180-3) = (-24,177)$, vertical translation 3 units down

d $(24,36) \to (24,36\times-2) = (24,-72)$, vertical dilation with scale factor –2

$(24,-72) \to (24-7,-72) = (17,-72)$, horizontal translation 7 units left

$(17,-72) \to (17,-72-3) = (17,-75)$, vertical translation 3 units down

e $y = -f(2x-8)+5 = -f(2(x-4))+5$

$(24,36) \to (24\times\frac{1}{2},36) = (12,36)$, horizontal dilation with scale factor $\frac{1}{2}$

$(12,36) \to (12+4,36) = (16,36)$, horizontal translation 4 units right

$(16,36) \to (16,-36)$, reflection in the *x*-axis

$(16,-36) \to (16,-36+5) = (16,-31)$, vertical translation 5 units up

Question 6

a **i** Let $y = f(x) = x^3$

Translation 3 units right, $y = f(x) + 3 = x^3 + 3$

ii Translation 3 units left, $y = f(x+7) = (x+7)^3$

b **i** Let $y = f(x) = |x|$

Dilated vertically with scale factor 3, $y = 3f(x) = 3|x|$

ii $y = \frac{1}{2}|x|$

Dilated horizontally with scale factor 2, $y = f\left(\frac{1}{2}x\right) = \left|\frac{1}{2}x\right| = \frac{1}{2}|x|$

c $5f(x) = 5\ln x$, dilated vertically with scale factor 5

$5f(-x) = 5\ln(-x)$, reflected in the *x*-axis

d $-f(x) = -\frac{1}{x}$, reflection in the *x*-axis

$-f(x-4) = -\frac{1}{x-4}$, horizontal translation, 4 units right

e $9f(x) = 9 \times 3^x$, vertical dilation with scale factor 9

$9f(3x) = 9 \times 3^{3x}$, horizontal dilation, scale factor $\frac{1}{3}$

$9f(3(x-2)) = 9 \times 3^{3(x-2)}$, horizontal translation, 2 units right

$9f(3(x-2)) - 6 = 9 \times 3^{3x-6} - 6$, vertical translation, 6 units down

Question 7

a c is a vertical translation c units up if $c > 0$ or down if $c < 0$

b is a horizontal translation b units to the right if $b < 0$ and to the left if $b > 0$

k is a vertical dilation with scale factor k, stretched if $k > 1$ and compressed if $0 < k < 1$

a is a horizontal dilation with scale factor $\frac{1}{a}$, compressed if $a > 1$ and stretched if $0 < a < 1$

b **i** Reflection in the x-axis

ii Reflection in the y-axis

Question 8

Let $y = f(x) = x^2$

$3f(x) = 3x^2$, vertical dilation with scale factor 3

$-3f(x) = -3x^2$, reflected in the x-axis

$-3f(x)+1 = -3x^2+1$, translation 1 unit up

Let $g(x) = -3x^2+1$

$$\begin{aligned} g(-x) &= -3(-x)^2+1 \\ &= -3x^2+1 \\ &= g(x) \end{aligned}$$

So even function.

Question 9

a The graph of $y = 2(x-3)+5$ can be obtained by transforming the graph of $y = x$ as follows.

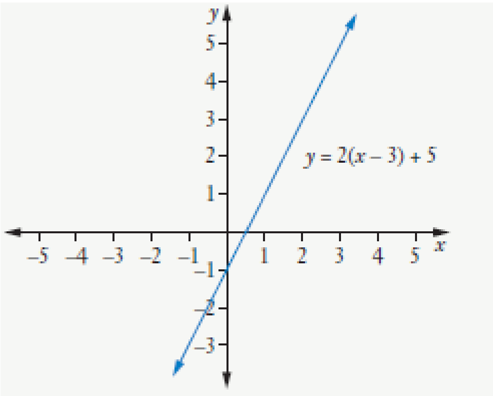
$y = x-3$, horizontal translation 3 units to the right

$y = 2(x-3)$, vertical dilation with scale factor 2

$y = 2(x-3)+5$, vertical translation, 5 units up

The y-intercept is $2(0-3)+5=-1$

The x-intercept satisfies $2(x-3)+5=0 \Rightarrow x=\frac{1}{2}$



b **i** From the graph, a y value of 7 corresponds to $x = 4$.

Hence, since the function is linear with a positive gradient, $y \le 7$ is true for $x \le 4$.

ii From the graph, a y value of 9 corresponds to $x = 5$.

Hence, since the function is linear with a positive gradient, $y > 9$ is true for $x > 5$.

Question 10

a The graph of $P = 2e^{0.4(t+1)}$ can be obtained by transforming the graph of $P = e^t$ as follows.

$P = e^{t+1}$, horizontal displacement, 1 unit left

$P = e^{0.4(t+1)}$, horizontal dilation, scale factor $\frac{1}{0.4} = 2.5$

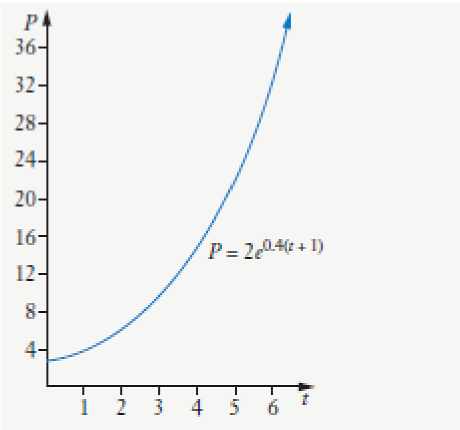
$P = 2e^{0.4(t+1)}$, vertical dilation, scale factor 2

There is no vertical translation, so the horizontal asymptote of the transformed function is the same as the horizontal asymptote of $P = e^t$. That is, $y = 0$

Because there is a horizontal asymptote at $y = 0$, here is no *x*-intercept.

The *y*-intercept is $2e^{0.4(0+1)} \approx 3$

Since *P* represents population, the domain of *P* is $[0, \infty)$ and the range is $[0, \infty)$.



b From the graph, when *P* = 5, $t \approx 1.3$, so it takes (*t*) 1.3 years for the population (*P*) to reach 5 × 10 000 = 50 000

Question 11

$f(x+a)$ is a horizontal translation to the left by *a* units.

Hence, $f(x+4) = (x+4)^4$

Question 12

$(8,2) \to (8\times\frac{1}{2}, 2) = (4,2)$, horizontal dilation, scale factor $\frac{1}{2}$

$(4,2) \to (4-1, 2) = (3,2)$, horizontal translation, 1 unit left

$(3,2) \to (3, 2\times 4) = (3,8)$, vertical dilation, scale factor 4

$(3,8) \to (3,-8)$, reflection in the *x*-axis

$(3,-8) \to (3,-8-3) = (3,-11)$, vertical translation, 3 units down

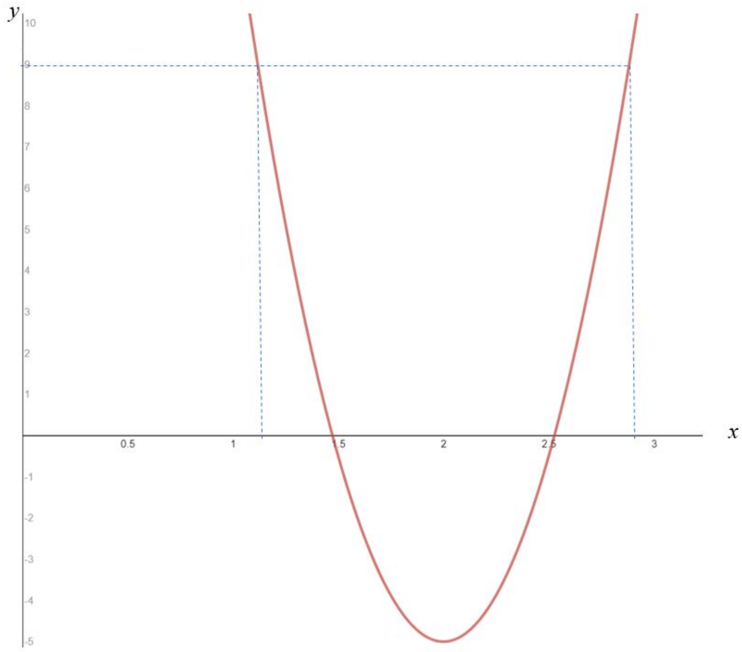
Question 13

a Let $y = 2(3x-6)^2 - 5$

When $y = 9$

$$2(3x-6)^2 - 5 = 9$$
$$(3x-6)^2 = 7$$
$$3x - 6 = \pm\sqrt{7}$$
$$x = \frac{6 \pm \sqrt{7}}{3}$$

$x \approx 1.12, x \approx 2.88$



b From the graph, $y > 9$ when $x < 1.12$ and when $x > 2.88$

c From the graph, $y \leq 9$ when $1.12 \leq x \leq 2.88$

Question 14

a $(x, y) \to (x+3, y)$, horizontal translation, 3 units to the right

$(x+3, y) \to (x+3, 7y)$, vertical dilation using a scale factor of 7

$(x+3, 7y) \to (x+3, -7y)$, reflection in the *x*-axis

$(x+3, -7y) \to (x+3, -7y-4)$, vertical translation, 4 units down

b $x = -6, y = -1$

The *x*-coordinate of the image is $x+3$

$x+3 = -3 \Rightarrow x = -6$

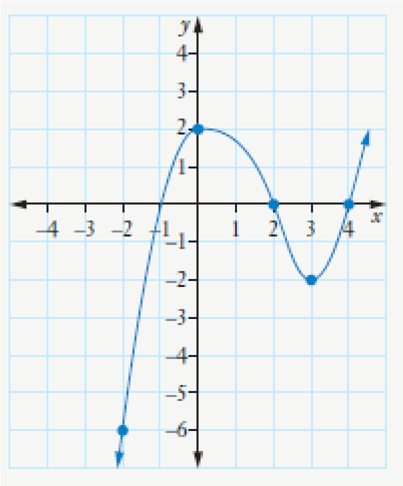
The *y*-coordinate of the image is $-7y-4$

$-7y-4 = 3 \Rightarrow y = -1$

Question 15

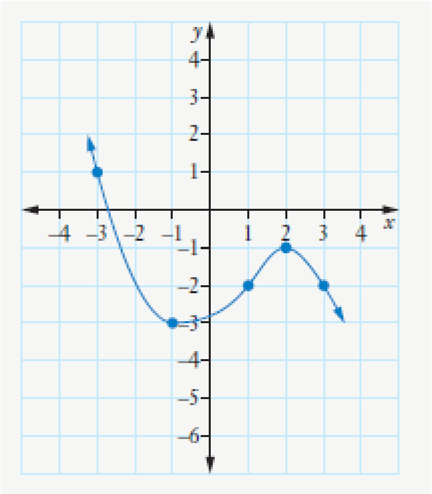
a The graph is translated 1 unit to the right and vertically dilated with scale factor 2.

$(-3,-3)\to(-2,-6)$, $(-1,1)\to(0,2)$, $(1,0)\to(2,0)$, $(2,-1)\to(3,-2)$, $(3,0)\to(4,0)$



b The graph is reflected in the *x*-axis and vertically translated 2 unit down.

$(-3,-3)\to(-3,1)$, $(-1,1)\to(-1,-3)$, $(1,0)\to(1,-2)$, $(2,-1)\to(2,-1)$,
$(3,0)\to(3,-2)$



Question 16

Let $y = 2(x+1)^2 - 8$

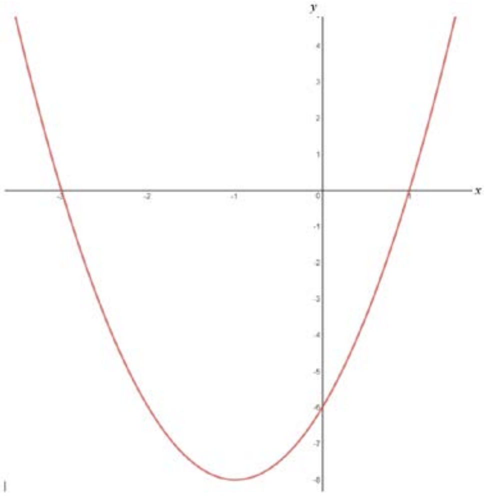
Its graph is the graph of $y = x^2$ vertically dilated with scale factor 2, translated 1 unit left and translated 8 units down.

Thus the maximum turning point is at $(-1, -8)$.

The y-intercept is $2(0+1)^2 - 8 = -6$

The x-intercept satisfies the equation $2(x+1)^2 - 8 = 0$.

Solving for x gives $x = -1 \pm 2$, so $x = -3, x = 1$



a $-3 \le x \le 1$

From the graph, $y \le 0$ when $-3 \le x \le 1$

b $x < -3$, $x > 1$

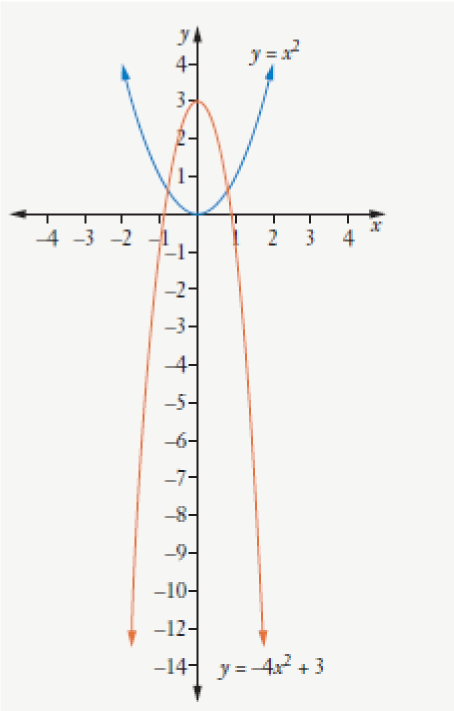
From the graph, $y > 0$ when $x < -3$ and when $x > 1$

Question 17

a The graph of $y = x^2$ is a parabola with minimum turning point, axial intercepts and axis of symmetry at (0, 0).

The graph of $y = -4x^2 + 3$ is the graph of $y = x^2$ reflected in the *x*-axis, dilated vertically with scale factor 4 and translated 3 units up. The maximum turning point is at (0, 3).

The *y*-intercept is 3 and the *x*-intercepts satisfy $-4x^2 + 3 = 0 \Rightarrow x = \pm\frac{\sqrt{3}}{2}$



b The graph of $y = -|x-1| + 2$ is obtained from the graph of $y = |x|$ by a translation 1 unit right,

reflection in the x-axis and a vertical translation 2 units up.

The axial intercepts and the axis of symmetry of $y = |x|$ are at (0, 0).

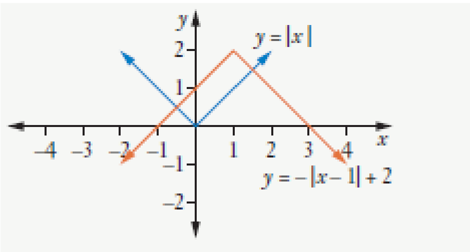
The axis of symmetry of $y = -|x-1| + 2$ is at (1, 2).

The y-intercept is $-|0-1| + 2 = 1$.

The x-intercepts satisfy $-|x-1| + 2 = 0$.

For $x \geq 1$, $-(x-1) + 2 = 0 \Rightarrow x = 3$

For $x < 1$, $(x-1) + 2 = 0 \Rightarrow x = -1$

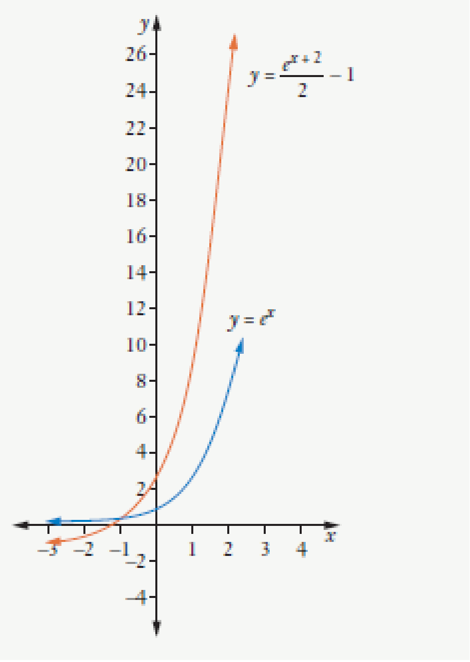


c The graph of $y = f(x) = \frac{1}{2}e^{x+2} - 1$ is obtained from the graph of $y = f(x) = e^x$ by a translation 2 units left, a vertical dilation with scale factor $\frac{1}{2}$ and a vertical translation 1 unit down.

The horizontal asymptote of $y = e^x$ is $y = 0$, which becomes $y = -1$ with the vertical translation.

The y-intercept is $\frac{1}{2}e^{0+2} - 1 \approx 2.7$.

Solving $\frac{1}{2}e^{x+2} - 1 = 0$ gives $x = -2 + \log_e 2 \approx -1.3$



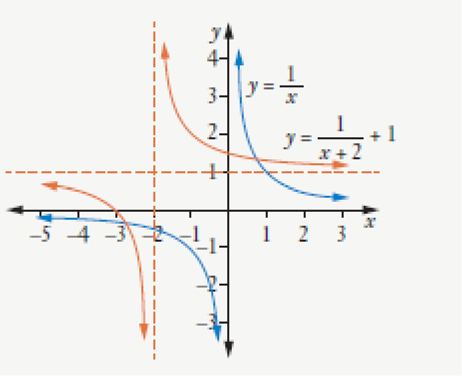
d The graph of $y=\dfrac{1}{x+2}+1$ is obtained from the graph of $y=\dfrac{1}{x}$ by a translation 2 units left and a vertical translation 1 unit up.

The horizontal asymptote of $y=\dfrac{1}{x}$ is $y=0$, which becomes $y=1$ with the vertical translation.

The vertical asymptote of $y=\dfrac{1}{x}$ is $x=0$, and the vertical asymptote of $y=\dfrac{1}{x+2}+1$ is $x=-2$ because of the horizontal translation.

The y-intercept is $\dfrac{1}{0+2}+1=1.5$.

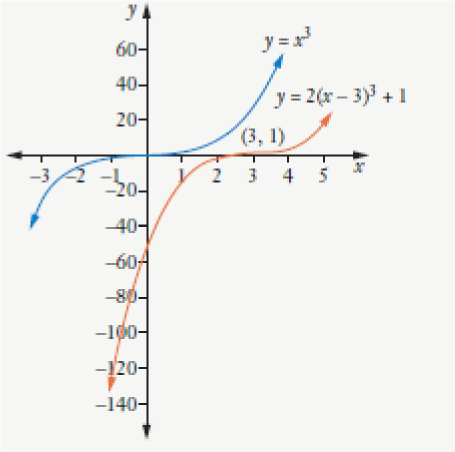
Solving $\dfrac{1}{x+2}+1=0$ gives $x=-3$ as the x-intercept.



e The graph of the cubic $y = 2(x-3)^3 + 1$ is obtained from the graph of $y = x^3$ by a translation 3 units right, a vertical dilation with scale factor 2 and a vertical translation 1 unit up.

The y-intercept is $2(0-3)^3 + 1 = -53$.

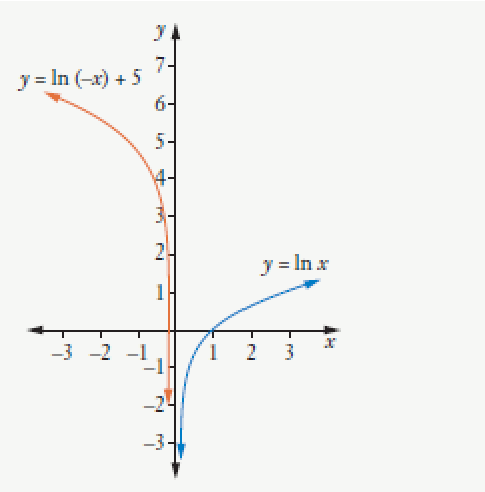
Solving $2(x-3)^3 + 1 = 0$ gives $x = 3 + \sqrt[3]{-\frac{1}{2}} \approx 2.2$ as the x-intercept.



f The graph of $y = f(x) = \log_e(-x) + 5$ is obtained from the graph of $y = f(x) = \log_e x$ by reflecting in the y-axis and by a vertical translation 5 units up.

The domain of $\log_e x$ is $(0, \infty)$ so the domain of $\log_e(-x)$ is $(-\infty, 0)$ because of the reflection in the y-axis. Hence there is no y-intercept.

Solving $\log_e(-x) + 5 = 0$ gives $x = -e^{-5} \approx -0.01$ as the x-intercept.

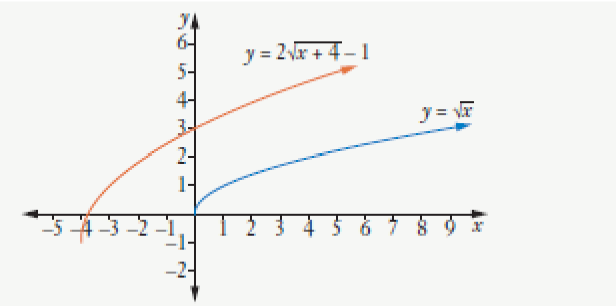


g The graph of $y = 2\sqrt{x+4} - 1$ is obtained from the graph of $y = \sqrt{x}$ a horizontal translation 4 units to the left, then a vertical dilation with scale factor 2 followed by a vertical translation 1 unit down.

The domain of $y = 2\sqrt{x+4} - 1$ is $[-4, \infty)$ and the range is $[-1, \infty)$.

The *y*-intercept is $2\sqrt{0+4} - 1 = 3$.

Solving $2\sqrt{x+4} - 1 = 0$ gives $x = -3\frac{3}{4}$ as the *x*-intercept.



Question 18

a 2

The parabola has two *x*-intercepts, so there are two solutions.

b 1

The graph has one *x*-intercept on the positive *x*-axis. There is a horizontal asymptote with equation $y = 0$, so there is no *x*-intercept on the negative *x*-axis.

c 0

The graph has a horizontal asymptote with equation $y = 0$, so there are no *x*-intercepts.

Hence there are no solutions for $f(x) = 0$

d 1

The curve intersects the *x*-axis at one point, so there is only 1 solution for $f(x) = 0$.

Hence there is solutions for $f(x) = 0$

e 1

The curve intersects the x-axis at one point, so there is only 1 solution for $f(x) = 0$.

f 0

$f(x) > 0$ for all values of x, so there are no solutions for $f(x) = 0$.

g 1

$y = f(x)$ is a linear function with non-zero gradient, so it has one x-intercept. Hence there is one solution for $f(x) = 0$.

h 0

The graph has a horizontal asymptote with equation $y = 0$, so there are no x-intercepts.

Hence there are no solutions for $f(x) = 0$

i 4

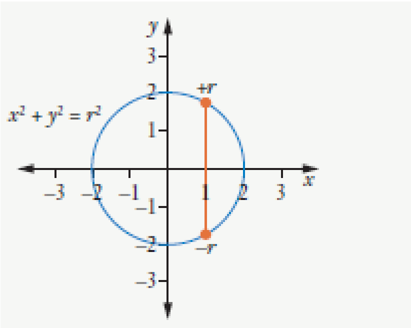
The quartic intersects the x-axis in four places, so there are four solutions for $f(x) = 0$.

j 3

The cubic intersects the x-axis in three places, so there are three solutions for $f(x) = 0$.

Question 19

a The equation describes a circle with radius r units and centre at the origin. It is not a function because there is a one-to-many relationship because there is more than one y-value for each x-value. This is observed by application of the vertical line test.

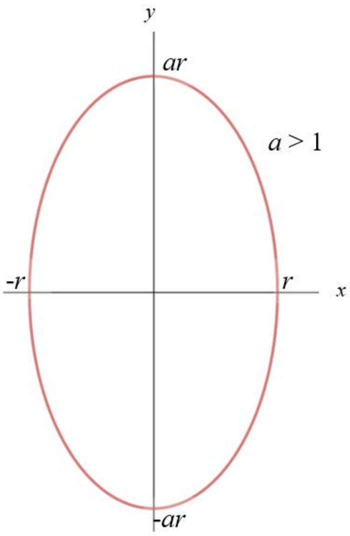


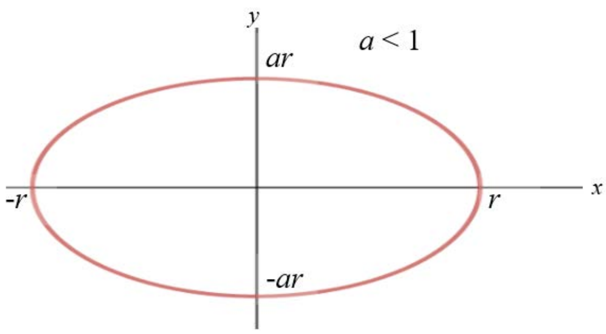
b The top half of the circle with equation $y=\sqrt{r^2-x^2}$, domain $[-r,r]$, range $[0,r]$ is a function.

Similarly, the lower half of the circle with equation $y=-\sqrt{r^2-x^2}$, domain $[-r,r]$, range $[-r,0]$ is a function.

Together, the graph of these two functions produce the entire circle.

c Applying a vertical dilation with scale factor a to $y=\sqrt{r^2-x^2}$ gives $y=a\sqrt{r^2-x^2}$, or, rearranging, $\frac{x^2}{r^2}+\frac{y^2}{(ra)^2}=1$, which is the equation of an ellipse.





Question 20

$y = -6f(2x+8) = -6f(2[x+4])$

Let (x, y) be the original point. Perform the transformations to obtain the function given.

$(\frac{1}{2}x, y)$, horizontal dilation, scale factor $\frac{1}{2}$

$(\frac{1}{2}x-4, y)$, horizontal translation, 4 units left

$(\frac{1}{2}x-4, 6y)$, vertical dilation, scale factor 6

$(\frac{1}{2}x-4, -6y)$, reflection in the *x*-axis

Hence $(\frac{1}{2}x-4, -6y) = (12, 6)$

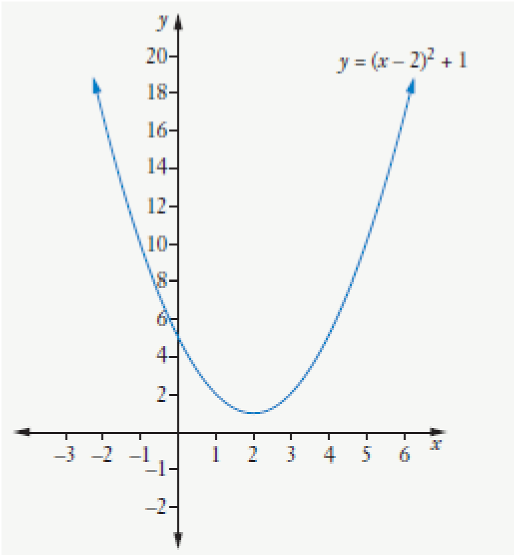
$\frac{1}{2}x-4=12 \Rightarrow x=32$ and $-6y=6 \Rightarrow y=-1$

(32, –1)

Question 21

a Sketch the parabola $y = (x-2)^2 + 1$ by applying to the graph of $y = x^2$ a horizontal translation 2 units to the right and a vertical translation 1 unit up.

The turning point is at (2, 1) and the y-intercept is 5. There are no x-intercepts because the vertical translation has moved the minimum turning point of $y = x^2$ above the y-axis.



b **i** $x = -1, x = 5$

$y = 10$ has corresponding x values $x = -1, x = 5$

ii $x < -1,\ x > 5$

When $x = 5, y = 10$ and for $x > 5$ as x increases, y increases.

When $x = -1, y = 10$ and for $x < -1$, as x decreases, y increases.

Hence $y > 10$ for $x < -1$ and for $x > 5$

iii $-1 \le x \le 5$

For $-1 \le x \le 5$, $y \le 10$

Question 22

a $(x, y) \to (x-1, y)$, horizontal translation, 1 unit left

$(x-1, y) \to (x-1, 3y)$, vertical dilation with scale factor 3

$(x-1, 3y) \to (x-1, 3y-5)$, vertical translation 5 units down

b $(x, y) \to (\frac{1}{2}x, y)$, horizontal dilation with scale factor $\frac{1}{2}$

$(\frac{1}{2}x, y) \to \left(\frac{1}{2}x+6, y\right)$, horizontal translation, 6 units right

$\left(\frac{1}{2}x+6, y\right) \to \left(\frac{1}{2}x+6, -2y\right)$, vertical dilation with scale factor –2

$\left(\frac{1}{2}x+6, -2y\right) \to \left(\frac{1}{2}x+6, -2y+4\right)$, vertical translation 4 units up

c $(x, y) \to (-x, y)$, reflection in the *y*-axis

$(-x, y) \to (-x, 5y)$, vertical dilation with scale factor 5

$(-x, 5y) \to (-x, 5y-3)$, vertical translation 3 units down

d $y = -3f(-3x+9)-1 = -3f(-3[x-3])-1$

$(x, y) \to (-\frac{1}{3}x, y)$, horizontal dilation with scale factor $-\frac{1}{3}$

$\left(-\frac{1}{3}x, y\right) \to \left(-\frac{1}{3}x+3, y\right)$, horizontal translation, 3 units right

$\left(-\frac{1}{3}x+3, y\right) \to \left(-\frac{1}{3}x+3, -3y\right)$, vertical dilation with scale factor –3

$\left(-\frac{1}{3}x+3, -3y\right) \to \left(-\frac{1}{3}x+3, -3y-1\right)$, vertical translation 1 unit down

Question 23

$af(x)$ is a vertical dilation with scale factor a.

If $a > 1$, the function is stretched, and if $a < 1$, the function is compressed.

$f(ax)$ is a horizontal dilation with scale factor $\frac{1}{a}$.

If $a > 1$, the function is compressed, and if $a < 1$, the function is stretched.

a stretched

vertical dilation with scale factor 7 means $a = 7$.

$a > 1$, so the function is stretched.

b compressed

horizontal dilation with scale factor $\frac{1}{7}$ means $a = 7$.

$a > 1$, so the function is compressed.

c stretched

horizontal dilation with scale factor 3 means $a = \frac{1}{3}$.

$a < 1$, so the function is stretched.

d compressed

vertical dilation with scale factor $\frac{1}{4}$ means $a = \frac{1}{4}$.

$a < 1$, so the function is compressed.

e stretched

horizontal dilation with scale factor $\frac{7}{6}$ means $a = \frac{6}{7}$.

$a < 1$, so the function is stretched.

Question 24

a domain $(-\infty, \infty)$, range $[-10, \infty)$

There is no restriction on the value of x in the parabola $y = 3(x-7)^2 - 10$, so its domain is $(-\infty, \infty)$.

It has a minimum turning point at $(7, -10)$, so the range is $[-10, \infty)$

b domain $(-\infty, \infty)$, range $[-\infty, 2)$

There is no restriction on the value of x in $y = -|x+1| + 2$, so its domain is $(-\infty, \infty)$.

The maximum value y can take is $y = -|-1+1| + 2 = 2$ when $x = -1$.

Hence the range is $(-\infty, 2]$

c $y = -\dfrac{2}{x-3} - 5$ is the graph of $y = \dfrac{1}{x}$ that is translated 3 units right, dilated vertically with scale factor 2, reflected in the x-axis and translated down 5 units.

$y = \dfrac{1}{x}$ has a vertical asymptote at $x = 0$, so its domain is $(-\infty, 0) \cup (0, -\infty)$.

The transformed function has a vertical asymptote at $x = 3$, so its domain is $(-\infty, 3) \cup (3, -\infty)$

$y = \dfrac{1}{x}$ has a horizontal asymptote at $y = 0$, so its range is $(-\infty, 0) \cup (0, -\infty)$.

There is a vertical translation of 5 units down, so the transformed function has a horizontal asymptote at $y = -5$. Hence the range is $(-\infty, -5) \cup (-5, -\infty)$.

Challenge exercise 2

Question 1

a The equation of the parabola is of the form $h=-A(t-B)^2+C$, where A, B and C are constants, with (B, C) the maximum height (turning point).

The maximum height of the ball is 3 m when $t=1$ second, so $B=1$ and $C=3$.

Hence, $h=-A(t-1)^2+3$

When $t=2, h=1$

$$1=-A(2-1)^2+3$$
$$A=2$$

So $h=-2(t-1)^2+3=-2t^2+4t+1$

b The ball will hit the ground when $h=0$.

$$-2(t-1)^2+3=0$$
$$t=1+\sqrt{1.5}\approx 2.2$$

The ball will hit the ground after 2.2 seconds

c $h=-2(t-1)^2+3$, from **a**.

$t-1$, horizontal translation, 1 unit to the right

–2 is a vertical dilation with scale factor 2 followed by reflection in the x-axis.

+3 is a vertical translation of 3 units up.

Question 2

a **i** $(4,-3)\to(4,-9)$, vertical dilation with scale factor 3

$(4,-9)\to(1,-9)$, horizontal translation 3 units left

$(1,-9)\to(1,-8)$, vertical translation 1 unit up

ii $(4,-3)\to(2,-3)$, horizontal dilation with scale factor $\frac{1}{2}$

$(2,-3)\to(2,3)$, reflection in the *x*-axis

$(2,3)\to(2,0)$, vertical translation 3 units down

iii $y=f(2x-2)+1=f(2[x-1])+1$

$(4,-3)\to(2,-3)$, horizontal dilation with scale factor $\frac{1}{2}$

$(2,-3)\to(3,-3)$, horizontal translation 1 unit to the right

$(3,-3)\to(3,-2)$, vertical translation 1 unit up

b From **a ii** and **a iii**, the co-ordinates of Q and R are $Q(2,0), R(3,-2)$.

The gradient, m_1, of the straight line that connects Q to R is $m_1=\frac{-2-0}{3-2}=-2$

The gradient, m_2, of the perpendicular is found using the property that $m_1\times m_2=-1$

$-2\times m_2=-1\Rightarrow m_2=\frac{1}{2}$

Let the equation of the perpendicular line be $y=m_2x+c=\frac{1}{2}x+c$

The co-ordinates of P are (1, –8), from **a i**

Hence $-8=\frac{1}{2}\times1+c\Rightarrow c=-8\frac{1}{2}$

The equation of the perpendicular is $y=\frac{1}{2}x-8\frac{1}{2}$ or $x-2y-17=0$

c Let $y = f(x) = x$

Then $y = \frac{1}{2}x - \frac{17}{2} = \frac{1}{2}(x-17)$

The transformations on $f(x)$ are:

$\frac{1}{2}x$, horizontal dilation with scale factor 2

–17, horizontal translation of 17 units to the right

OR $y = \frac{1}{2}x - \frac{17}{2}$

$\frac{1}{2}f(x)$, vertical dilation with scale factor $\frac{1}{2}$

$-\frac{17}{2}$, vertical translation of $\frac{17}{2}$ units down

Question 3

a

$$\begin{aligned}\text{RHS} &= -\frac{1}{x-3} + 2\\ &= \frac{-1+2(x-3)}{x-3}\\ &= \frac{2x-7}{x-3}\\ &= \text{LHS}\end{aligned}$$

b vertical asymptote when $x - 3 = 0 \Rightarrow x = 3$, so $x \neq 3$.

There are no other restrictions on the values *x* can take, so the domain is $(-\infty, 3) \cup (3, \infty)$

$y = -\frac{1}{x-3} + 2$, so the *y*-intercept is $y = -\frac{1}{0-3} + 2 = 2\frac{1}{3}$.

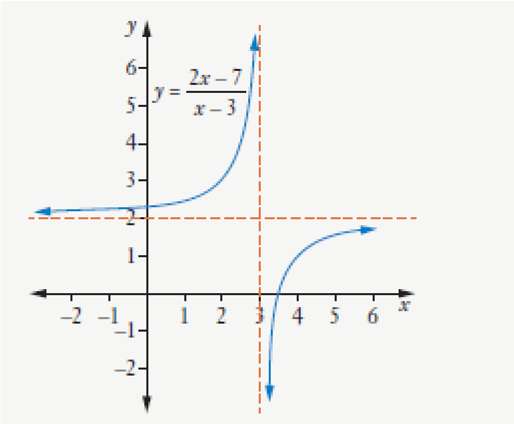
The *x*-intercept is found by solving $\frac{2x-7}{x-3} = 0$.

$\Rightarrow 2x - 7 = 0$

$x = 3.5$

As $x \to \pm\infty, y \to 2$, so $y = 2$ is a horizontal asymptote.

Hence the range is $(-\infty, 2) \cup (2, \infty)$



c **i** $x < 3$, $x \geq 3.5$

$y = \dfrac{2x-7}{x-3} \geq 0$ means values of x for which the y value is not negative.

From the graph, this is true for $x < 3$ and $x \geq 3.5$

ii $x > 3$

All y values are less than 2 (the horizontal asymptote) when x is greater than the vertical asymptote. Hence we want $x > 3$

Question 4

a Let $y = f(x) = \dfrac{2}{x}$. Horizontal dilation with scale factor 2 is $f\left(\dfrac{1}{2}x\right) = \dfrac{1}{\frac{1}{2}x} = \dfrac{2}{x}$

But $\dfrac{2}{x} = 2 \times \dfrac{1}{x} = 2f(x)$, which is a vertical dilation with scale factor 2.

b Let $y = f(x) = \dfrac{2}{x^2}$. Horizontal dilation with scale factor 2 is $f\left(\dfrac{1}{2}x\right) = \dfrac{1}{\left(\frac{1}{2}x\right)^2} = \dfrac{4}{x^2}$

But $\dfrac{4}{x^2} = 4 \times \dfrac{1}{x^2} = 4f(x)$, which is a vertical dilation with scale factor 4.

Therefore a horizontal dilation with scale factor 2 does not produce the same transformation as a vertical dilation with scale factor 2.

Question 5

a $f(x)=ax^2+bx+c$

$f(x)=a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right], a\neq 0$, take out common factor a

$=a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right]$, apply Complete The Square method

$=a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$, simplify

The axis of symmetry occurs at $x+\frac{b}{2a}=0\Rightarrow x=-\frac{b}{2a}$

b Only horizontal translations change the axis of symmetry by moving it left or right parallel to the *y*-axis.

c **i** $x+1=0\Rightarrow x=-1$

ii $x-3=0\Rightarrow x=3$

iii $x+b=0\Rightarrow x=-b$

iv $ax+b=0\Rightarrow x=-\frac{b}{a}$

Question 6

a $y = -\sin(x)$, reflection in the x-axis

$y = -3\sin(x)$, vertical dilation, scale factor 3

$y = -3\sin\left(\frac{1}{2}x\right)$, horizontal dilation, scale factor 2

$y = -3\sin\left(\frac{1}{2}x\right) - 1$, vertical translation, 1 unit down

b $y = |A|\sin(nx) + C$ has amplitude A, period $\frac{2\pi}{n}$ and the equation of its centre is $y = C$

For $y = -3\sin\left(\frac{1}{2}x\right) - 1$ the amplitude is 3 and the period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$.

The centre is $y = -1$

Question 7

The equation can be written as $(x+2)^2+(y-3)^2=1$.

Applying the transformations

$(x+2-5)^2+(y-3+3)^2=1$ or $(x-3)^2+y^2=1$

In expanded form, this is $x^2-6x+y^2+8=0$

Alternatively, apply the transformations directly to $x^2+4x+y^2-6y+12=0$.

$(x-5)^2+4(x-5)+(y+3)^2-6(y+3)+12=0$

Expanding and simplifying gives $x^2-6x+y^2+8=0$

Question 8

$y=3\times 2^{-3x-6}-5=3\times 2^{-3(x+2)}-5$.

The transformations are:

$y=2^{-3(x+2)}$, reflection in the *y*-axis, horizontal dilation, scale factor $\dfrac{1}{3}$

$y=2^{x+2}$, horizontal translation, 2 units left

$y=3\times 2^{-3(x+2)}$, vertical dilation, scale factor 3

$y=3\times 2^{-3(x+2)}-5$, vertical translation, 5 units down

Question 9

$P(x)+2=x^3-3x$, vertical translation, 2 units up

$P(-x)+2=(-x)^3-3(-x)=-x^3+3x$, reflected in the *y*-axis

MATHS IN FOCUS 12 MATHEMATICS ADVANCED

WORKED SOLUTIONS

Chapter 3: Trigonometric functions

Exercise 3.01 Transformations of trigonometric functions

Question 1

a phase shift

A horizontal translation is a phase shift. For a trigonometric function $y = f(x)$, a phase shift of b units is $y = f(x+b)$

b amplitude

A vertical dilation changes the amplitude of a trigonometric function. For $y = f(x)$, a vertical dilation with scale factor a gives $y = af(x)$

c period

A horizontal dilation changes the period of a trigonometric function. For $y = f(x)$, a horizontal dilation with scale factor k gives $y = f(kx)$.

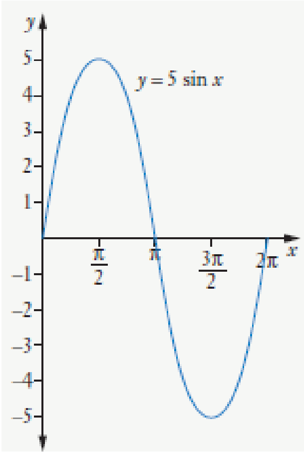
d centre

A vertical translation changes the centre of a trigonometric function. For $y = f(x)$, a vertical translation of c units is $y = f(x)+c$

Question 2

a The graph of $y = 5\sin x$ is a vertical dilation with scale factor 5 of $y = \sin x$.

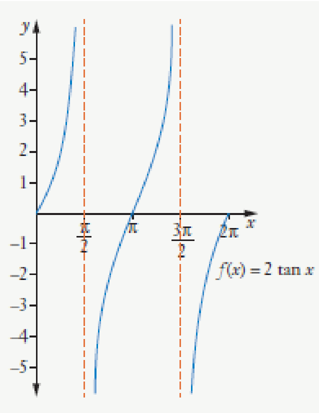
The amplitude is 5 and the period is 2π.
The y-intercept is $y = 5\sin 0 = 0$.
The x-intercepts are the same as the x-intercepts of $y = \sin x$. That is, $0, \pi, 2\pi$.



b The graph of $f(x) = 2\tan x$ is a vertical dilation with scale factor 2 of $f(x) = \tan x$.

The period is π and the x-intercepts are the same as $f(x) = \tan x$.
That is, $0, \pi, 2\pi$.

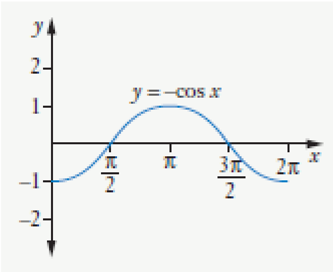
The y-intercept is $y = 2\tan 0 = 0$ and the centre is 0.



c The graph of $y = -\cos x$ is a reflection in the x-axis of $y = \cos x$.

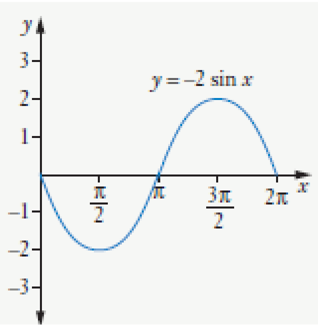
The amplitude, period and x-intercepts are the same as the graph of $y = \cos x$.

That is, $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$.
The y-intercept is $y = -\cos 0 = -1$.



d The graph of $y = -2\sin x$ is a vertical dilation with scale factor 2 of $y = \sin x$ followed by a reflection in the x-axis.

The amplitude is 2 and the period is 2π.
The x-intercepts are the same as the x-intercepts of $y = \sin x$. That is, $0, \pi, 2\pi$.
The y-intercept is $y = -2\sin 0 = 0$.

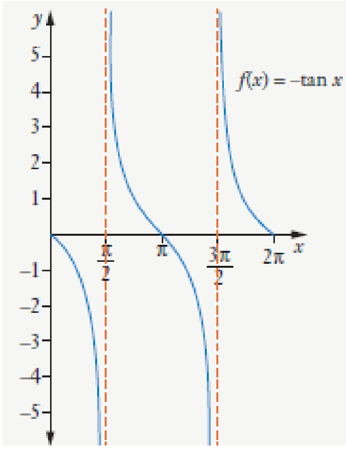


e The graph of $f(x) = -\tan x$ is a reflection in the x-axis of the graph of $f(x) = \tan x$.

The period is π and the x-intercepts are the same as $f(x) = \tan x$. That is, $0, \pi, 2\pi$.

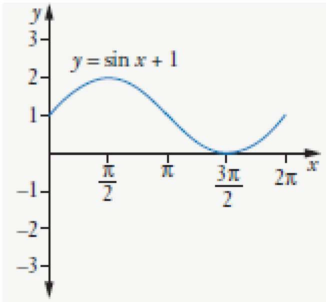
The y-intercept is $y = -\tan 0 = 0$. The centre is 0.

The asymptotes are the same as the asymptotes of $f(x) = \tan x$, $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$



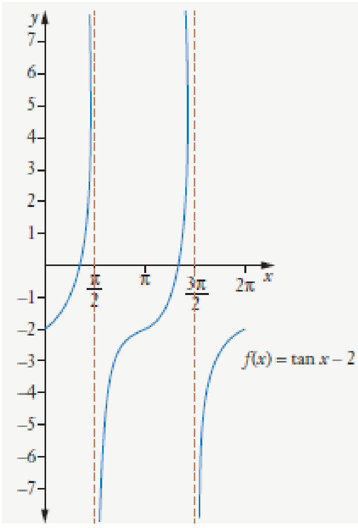
Question 3

a The graph of $y = \sin x + 1$ is a vertical translation of the graph of $y = \sin x$ 1 unit up. The centre is 1 and the amplitude and period are the same as for $y = \sin x$.
The y-intercept is $y = \sin 0 + 1 = 1$.
The centre is 1 and the amplitude is 1, so there is only 1 x-intercept in the given domain.

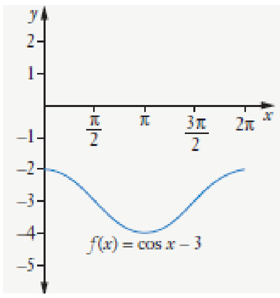


b The graph of $y = \tan x - 2$ is a vertical translation 2 units down of the graph of $y = \tan x$.
The period is π, the centre is –2 and the y-intercept is $f(0) = \tan 0 - 2 = -2$.

The asymptotes are the same as the asymptotes of $f(x) = \tan x$, namely $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$



c The graph of $f(x) = \cos x - 3$ is a vertical translation 3 units down of the graph of $f(x) = \cos x$, so its centre is at –3.
The y-intercept is $f(0) = \cos 0 - 3 = -2$.
The amplitude is 1, so the maximum value of $f(x)$ is $1 - 3 = -2$.
Hence there are no x-intercepts.



Question 4

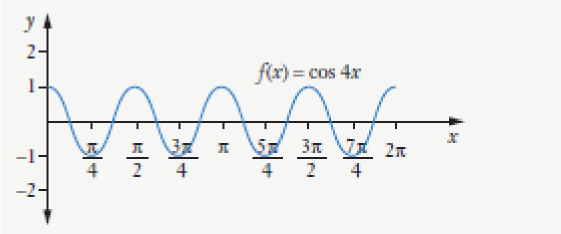
a The graph of $y = \cos 4x$ is a horizontal dilation with scale factor $\frac{1}{4}$ of the graph of $y = \cos x$. The period is $\frac{2\pi}{4} = \frac{\pi}{2}$. The amplitude is 1 and the centre is 0.

The y-intercept is $\cos 0 = 1$.

The x-intercepts are the solutions to $\cos 4x = 0$ in $[0, 2\pi]$.

They are the images of the solutions to $y = \cos x$ dilated (compressed) horizontally with scale factor 4.

$$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \frac{15\pi}{2} \ldots \to \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}, \ldots$$



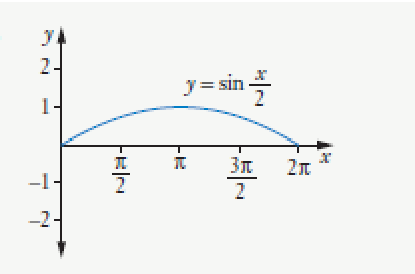
b The graph of $y = \sin\left(\frac{x}{2}\right)$ is a horizontal dilation with scale factor 2 of the graph of $y = \sin x$. The period is $\frac{2\pi}{\frac{1}{2}} = 4\pi$. The amplitude is 1 and the centre is 0.

The y-intercept is $\sin 0 = 0$.

The x-intercepts are the solutions to $\sin\left(\frac{x}{2}\right) = 0$ in $[0, 2\pi]$.

They are the images of the solutions to $y = \sin x$ dilated (stretched) horizontally with scale factor 2.

$0, \pi, 2\pi, \ldots \to 0, 2\pi, 4\pi..$



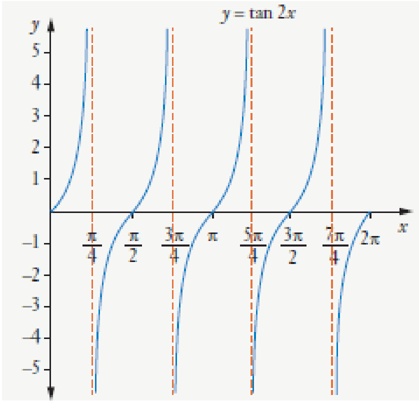
c The graph of $f(x) = \tan 2x$ is a horizontal dilation with scale factor $\frac{1}{2}$ of the graph of $f(x) = \tan x$.

The period is $\frac{\pi}{2}$ and the y-intercept is $\tan 0 = 0$.

The centre is 0.

The x-intercepts are the solutions to $f(x) = \tan 2x$ in $[0, 2\pi]$.

They are the images of the solutions to $\tan x = 0$ dilated (compressed) horizontally with scale factor 2.



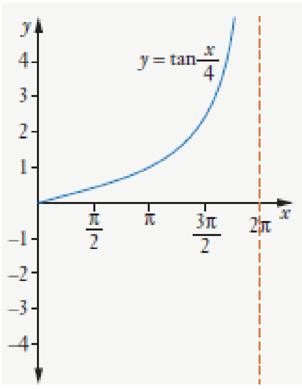
$$0, \pi, 2\pi, 3\pi, 4\pi, \ldots \to 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, ..$$

The asymptotes of $f(x) = \tan x$ are at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...,$ hence the asymptotes of $f(x) = \tan 2x$ are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, ..$

d The graph of $y = \tan\left(\frac{1}{4}x\right)$ is a horizontal dilation with scale factor 4 of the graph of $y = \tan x$. The period is 4π, the centre is 0 and the y-intercept is $\tan 0 = 0$.

The x-intercepts are the solutions to $y = \tan\left(\frac{1}{4}x\right)$ in $[0, 2\pi]$.

They are the images of the solutions to $\tan x = 0$ dilated (stretched) horizontally with scale factor 4.



$$0, \pi, 2\pi, 3\pi, \ldots \to 0, 4\pi, 8\pi, ..$$

The asymptotes of $y = \tan x$ are at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...,$ hence the asymptotes of $y = \tan\left(\frac{1}{4}x\right)$ are $x = 2\pi, 6\pi, 10\pi, ..$ which are $x = 2\pi$ in $[0, 2\pi]$

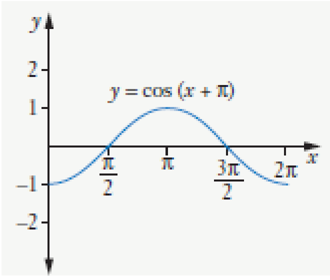
Question 5

a The graph is a horizontal translation π units to the left of the graph of $y = \cos x$. The period is 2π, the amplitude is 1 and the centre is 0.

The y-intercept is $\cos\pi = -1$.

The x-intercepts are solutions to $\cos x = 0$ which are then translated π units to the left.

$\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \to -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ or $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$ in $[0, 2\pi]$



b The graph is a horizontal translation π units to the right of the graph of $y = \tan x$. The period is $\frac{\pi}{2}$, the centre is 0 and there is no y-intercept because $\tan\left(0 - \frac{\pi}{2}\right)$ is an asymptote.

The x-intercepts are the x-intercepts of $y = \tan(x)$, $0, \pi, 2\pi, 3\pi, ..$ translated $\frac{\pi}{2}$ units to the right.

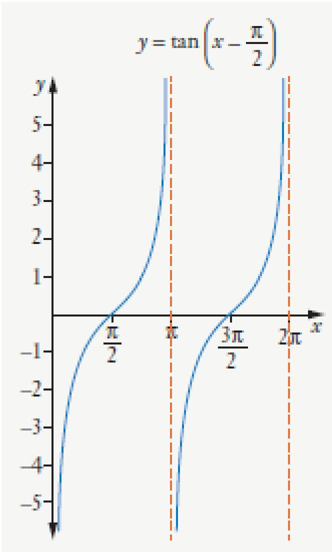
The solutions in $[0, 2\pi]$ of $y = \tan\left(x - \frac{\pi}{2}\right)$ are

$x = 0 + \frac{\pi}{2}, x = \pi + \frac{\pi}{2}$ or $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$

The asymptotes of $y = \tan x$ are at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots,$

hence the asymptotes of $y = \tan\left(x - \frac{\pi}{2}\right)$ in $[0, 2\pi]$ are

$x = \frac{\pi}{2} + \frac{\pi}{2}, x = \frac{3\pi}{2} + \frac{\pi}{2}$ or $x = \pi, x = 2\pi$

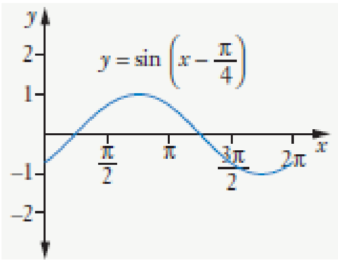


c The graph is a horizontal translation $\frac{\pi}{4}$ units to the right of the graph of $y = \sin x$. The period is 2π and the centre is 0. The y-intercept is $\sin\left(-\frac{\pi}{4}\right) \approx -0.7$.

The x-intercepts are the x-intercepts of $y = \sin x$ translated $\frac{\pi}{4}$ units to the right.

The solutions in $[0, 2\pi]$ of $y = \sin\left(x - \frac{\pi}{4}\right)$ are

$x = 0 + \frac{\pi}{4} = \frac{\pi}{4}, x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$



Question 6

a $y = 9\sin x$

$y = a \times f(x)$ has amplitude a and is a vertical dilation of scale factor a.

$y = 9\sin x$, amplitude 9

b $y = -\sin x$

$y = -f(x)$ is a reflection of $f(x)$ in the x-axis

$y = -\sin x$, reflection in the x-axis

c $y = \sin x - 4$

$y = f(x) + c$, y has centre at c

$y = \sin x - 4$

d $y = \sin 2x$

$y = f(nx)$, y has period $\dfrac{2\pi}{n}$

$\dfrac{2\pi}{n} = \pi \Rightarrow n = 2$

Hence $y = \sin 2x$

e $y = \sin(x - \pi)$

$y = f(x + a)$, phase shift a units left, $y = f(x - a)$, phase shift a units right

$y = \sin(x - \pi)$, phase shift π right

Question 7

a $y = 4\cos x$

$y = a \times f(x)$ has amplitude a and is a vertical dilation of scale factor a.

$y = 4\cos x$, amplitude 4

b $y = \cos\left(x + \frac{\pi}{3}\right)$

$y = f(x+a)$, phase shift a units left, $y = f(x-a)$, phase shift a units right

$y = \cos\left(x + \frac{\pi}{3}\right)$, phase shift $\frac{\pi}{3}$ units right

c $y = \cos x + 8$

$y = f(x) + c$, y has centre at c

$y = \cos x + 8$

d $y = \cos 4x$

$y = f(nx)$, y has period $\frac{2\pi}{n}$

$\frac{2\pi}{n} = \frac{\pi}{2} \Rightarrow n = 4$

Hence $y = \cos 4x$

e $y = 7\cos x$

$y = a \times f(x)$, vertical dilation with scale factor 7

$y = 7\cos x$

Question 8

a $y = \tan\left(\frac{1}{2}x\right)$

$y = \tan(nx)$, y has period $\frac{\pi}{n}$

$\frac{\pi}{n} = 2\pi \Rightarrow n = \frac{1}{2}$

Hence $y = \tan\left(\frac{1}{2}x\right)$

b $y = \tan\left(x - \frac{\pi}{6}\right)$

$y = f(x+a)$, phase shift a units left, $y = f(x-a)$, y phase shift a units right

$y = \tan\left(x - \frac{\pi}{6}\right)$, phase shift $\frac{\pi}{6}$ units right

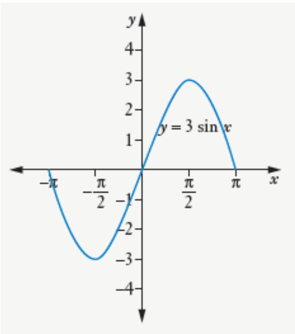
c $y = \tan\left(x - \frac{\pi}{6}\right)$

$y = f(-x)$, reflection in the y-axis

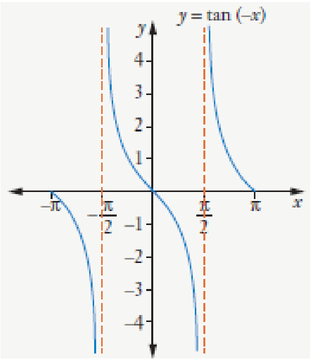
$y = \tan(-x)$

Question 9

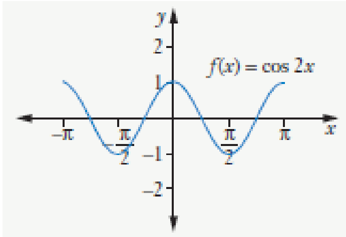
a The graph is a vertical dilation with scale factor 3 of the graph of $y = \sin x$.
The amplitude is 3 and the period is 2π.
The y-intercept is $y = \sin x = 0 = 0$.
The x-intercepts are the same as the x-intercepts of $y = \sin x$. In $[-\pi, \pi]$, they are $-\pi, 0, \pi$.



b The graph of $f(x) = \tan(-x)$ is a reflection in the y-axis of the graph of $f(x) = \tan x$.
The period is π, the centre is 0 and the x-intercepts are the same as $f(x) = \tan x$ in $[-\pi, \pi]$.
That is, $-\pi, 0, \pi$.
The y-intercept is $y = \tan 0 = 0$.
The asymptotes are the same as the asymptotes of $f(x) = \tan x$, namely $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$



c The graph is a horizontal dilation (compression) with scale factor $\frac{1}{2}$ of the graph of $y=\cos x$.

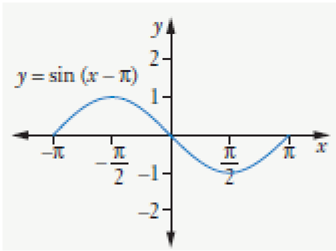


The period is $\frac{2\pi}{2}=\pi$. The amplitude is 1 and the centre is 0. The *y*-intercept is $\cos 0=1$.

The *x*-intercepts are the solutions to $\cos x=0$ in $[-\pi,\pi]$ dilated with scale factor $\frac{1}{2}$.

$$-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2}\to-\frac{3\pi}{4},-\frac{\pi}{4},\frac{\pi}{4},\frac{3\pi}{4}$$

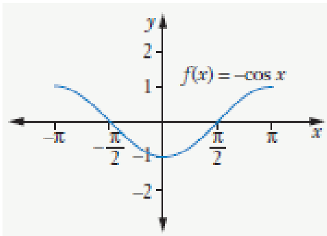
d The graph is a horizontal translation π units to the right of the graph of $y=\sin x$. The period is 2π, the amplitude is 1 and the centre is 0. The *y*-intercept is $\sin(-\pi)=0$.



The *x*-intercepts are solutions to $\sin x=0$ which are then translated π units to the right.

$-2\pi,-\pi,0\to-\pi,0,\pi$ in $[-\pi,\pi]$

e The graph is a reflection in the *x*-axis of $f(x)=\cos x$.



The amplitude, centre, period and *x*-intercepts are the same as the graph of $f(x)=\cos x$.

In $[-\pi,\pi]$, the *x*-intercepts are $-\frac{\pi}{2},\frac{\pi}{2}$.

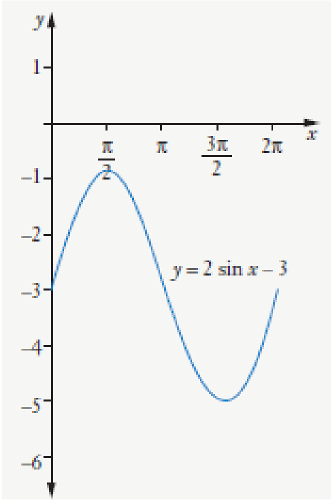
The *y*-intercept is $f(0)=-\cos 0=-1$.

Exercise 3.02 Combined transformations of trigonometric functions

Question 1

a $y = 2\sin x - 3$ has period 2π and is a vertical dilation of $y = \sin x$ with scale factor 2 followed by a vertical translation 3 units down.
Hence the amplitude is 2 and the centre is –3.

The domain is $[0, 2\pi]$ and the range is $[2-3, -2-3] = [-1, -5]$.

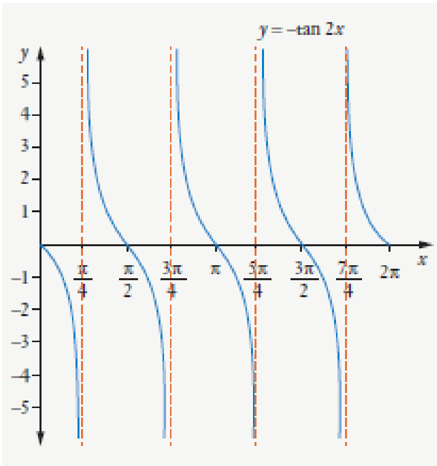


b $y = -\tan 2x$ has period $\frac{\pi}{2}$ and is a horizontal dilation of $y = \tan x$ with scale factor $\frac{1}{2}$ followed by reflection in the x-axis.

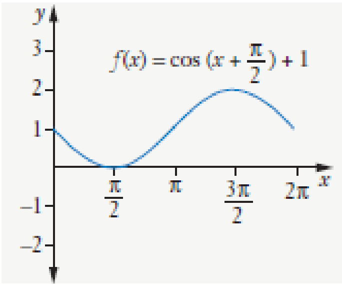
The x-intercepts of $y = \tan x$ are $0, \pi, 2\pi$.
The scale factor is $\frac{1}{2}$, so the x-intercepts of $y = -\tan 2x$ are $0, \frac{\pi}{2}, \frac{3\pi}{2}$.

The vertical asymptotes of $y = \tan x$ are at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$, so the asymptotes of $y = -\tan 2x$ in $[0, 2\pi]$ are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

The domain is $[0, 2\pi]$ and the range is $(-\infty, \infty)$.



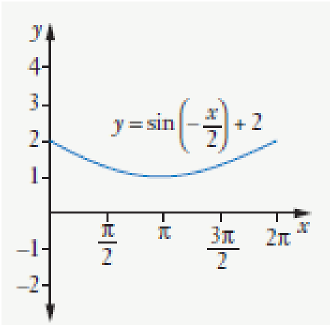
c $f(x)=\cos\left(x+\dfrac{\pi}{2}\right)+1$ has amplitude 1, period 2π, centre 1 and phase shift $\dfrac{\pi}{2}$.



It is a horizontal translation $\dfrac{\pi}{2}$ units to the left of the graph of $f(x)=\cos x$ followed by a vertical translation 1 unit up.

The domain is $[0,2\pi]$ and the range is $[-1+1,1+1]=[0,2]$.

d $y=\sin\left(-\dfrac{x}{2}\right)+2$ has period $\dfrac{2\pi}{\frac{1}{2}}=4\pi$ and amplitude 1 and centre 2.

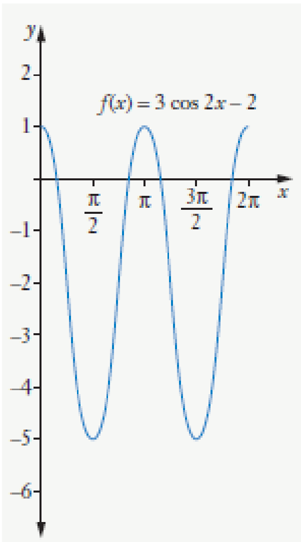


It is a horizontal dilation with scale factor 2 of $y=\sin x$ followed by reflection in the *y*-axis and then by a vertical translation 2 units up.

The *y*-intercept is $y=\sin(0)+2=2$.

The domain is $[0,2\pi]$ and the range is $[-1+2,1+2]=[1,3]$.

e $f(x)=3\cos 2x-2$ has period π, amplitude 3, and centre –2.



It is a horizontal dilation with scale factor $\dfrac{1}{2}$ of the graph of $f(x)=\cos x$, a vertical dilation with scale factor 3 units and a vertical translation 2 units down.

The domain is $[0,2\pi]$ and the range is $[-3-2,3-2]=[-5,1]$.

Question 2

a $y = 5f(3[x+5]) - 6$ or $y = 5\sin(3x+15) - 6$

Let $y = f(x) = \sin x$

$y = 5f(x) = 5\sin x$, vertical dilation with scale factor 5.

$y = 5f(3x) = 5\sin 3x$, horizontal dilation with scale factor $\frac{1}{3}$.

$y = 5f(3x) - 6 = 5\sin 3x - 6$, vertical translation 6 units down.

$y = 5f(3[x+5]) - 6 = 5\sin(3x+15) - 6$, horizontal translation 5 units left.

b A vertical dilation with scale factor 5 means the amplitude 5.

A horizontal dilation with scale factor $\frac{1}{3}$ has period $\frac{2\pi}{3}$.

A vertical translation 6 units down means he centre is –6.

A horizontal translation 5 units left is a phase shift 5 units to the left.

Question 3

a $y = 4\cos\left[6\left(x - \frac{\pi}{3}\right)\right] + 2$ or $y = 4\cos(6x - 2\pi) + 2$

Let $y = f(x) = \cos x$

$y = f(x) + 2 = \cos x + 2$, vertical translation 2 units up.

$y = f\left(x - \frac{\pi}{3}\right) + 2 = \cos\left(x - \frac{\pi}{3}\right) + 2$, horizontal translation $\frac{\pi}{3}$ units right.

$y = 4f(x) = 4\cos\left(x - \frac{\pi}{3}\right) + 2$, vertical dilation with scale factor 4.

$y = 4f(x) = 4\cos\left[6\left(x - \frac{\pi}{3}\right)\right] + 2$, horizontal dilation with scale factor 6.

b $y = f(x+\pi) - 5 = -\cos(x+\pi) - 5$

Let $y = f(x) = \cos x$

$y = -f(x) = -\cos x$, reflection in the x-axis.

$y = f(-x) = -\cos(-x) = -\cos x$, reflection in the y-axis.

Note that $\cos(-x) = \cos x$ because it is symmetrical about the y-axis.

$y = f(x) - 5 = -\cos x - 5$, vertical translation 5 units down.

$y = f(x+\pi) - 5 = -\cos(x+\pi) - 5$, horizontal translation π units left.

Question 4

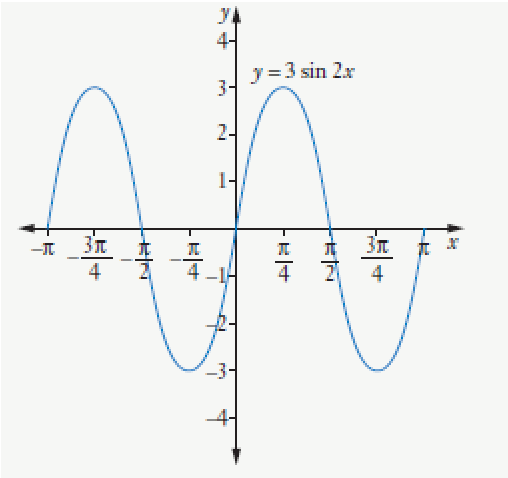
a $y = 3\sin 2x$ has period π, amplitude 3, and centre 0.

It is a horizontal dilation with scale factor $\frac{1}{2}$ of the graph of $y = \sin x$, followed by a vertical dilation with scale factor 3.

The domain is $[-\pi, \pi]$ and the range is $[-1\times 3, 1\times 3] = [-3, 3]$.

The y-intercept is $y = 3\sin 0 = 0$

The x-intercepts of $y = \sin x$ are $.., -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, ...$ so the x-intercepts of $y = 3\sin 2x$ in $[-\pi, \pi]$ are $-\pi, 0, \frac{\pi}{2}, \pi$



b $y = 2\tan\left(\frac{x}{2}\right) + 1$ has period $\pi \div \frac{1}{2} = 2\pi$.

It is a horizontal dilation with scale factor 2 of the graph of $y = \tan x$, followed by a vertical dilation with scale factor 2 and then a vertical translation of 1 unit up.

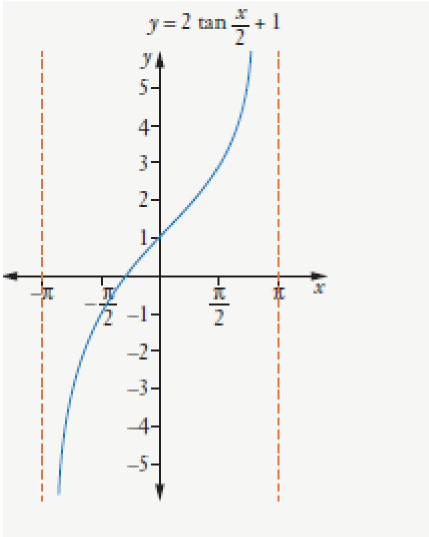
The y-intercept is $y = 2\tan 0 + 1 = 1$.

There is one x-intercept in $[-\pi, \pi]$ which is the solution to $2\tan\left(\frac{x}{2}\right) + 1 = 0$, or

$x = 2\tan^{-1}\left(-\frac{1}{2}\right) \approx -0.9$

The vertical asymptotes of $y = \tan x$ are at $., -\frac{\pi}{2}, \frac{\pi}{2}, ...$. Since the horizontal scale factor is 2, the asymptotes of $y = 2\tan\left(\frac{x}{2}\right) + 1$ in $[-\pi, \pi]$ are $2\times -\frac{\pi}{2}, 2\times \frac{\pi}{2} = [-\pi, \pi]$.

The domain is $[-\pi, \pi]$ and the range is $[-3\times\pi+1, 3\times\pi+1] = [-3\pi+1, 3\pi+1]$.



c $y = -2\cos 3x$ has period $\frac{2\pi}{3}$, amplitude 2, and centre 0.

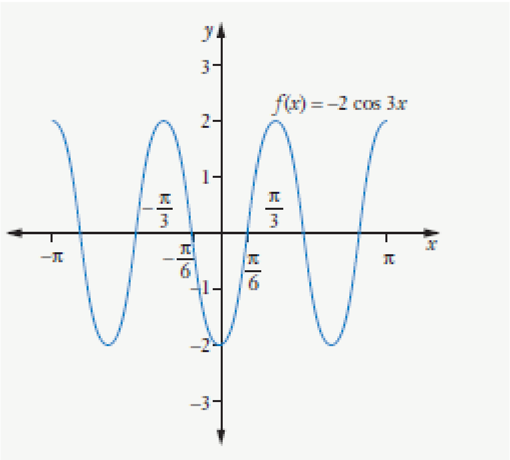
The graph is a horizontal dilation with scale factor $\frac{1}{3}$ of the graph of $y = \cos x$, which is followed by a vertical dilation with scale factor 2 and then a reflection in the *x*-axis.

The domain is $[-\pi, \pi]$ and the range is $[-1\times -2, 1\times -2] = [-2, 2]$.

The *y*-intercept is $y = -2\cos 0 = -2$.

The *x*-intercepts of $y = \cos x$ are $..-\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}...$ and the horizontal scale factor is $\frac{1}{3}$, so the *x*-intercepts of $y = -2\cos 3x$ in $[-\pi, \pi]$ are

$-\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$



d $y = 5\sin x - 3$ has period 2π, amplitude 5, and centre –3.

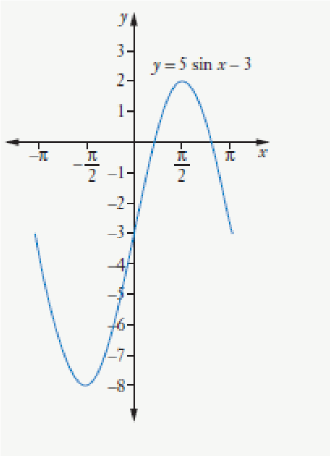
It is a vertical dilation with scale factor 5 of the graph of $y = \sin x$, followed by a vertical translation 3 units down.

The domain is $[-\pi, \pi]$ and the range is $[-1\times5-3, 1\times5-3] = [-8, 2]$.

The *y*-intercept is $y = 5\sin 0 - 3 = -3$.

The *x*-intercepts are solutions to $5\sin x - 3 = 0$ which in $[-\pi, \pi]$ are

$x = \sin^{-1}\left(\frac{3}{5}\right) \approx 0.64, x = \pi - \sin^{-1}\left(\frac{3}{5}\right) \approx 2.49$.



e $y = \cos(-2x) + 1$ has period π, amplitude 1, and centre 1.

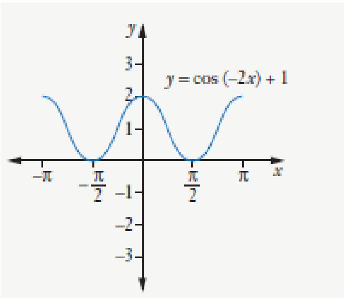
The graph is a horizontal dilation with scale factor $\frac{1}{2}$ of the graph of $y = \cos x$, which is followed by reflection in the *y*-axis and then a vertical translation 1 unit up.

The domain is $[-\pi, \pi]$ and the range is $[-1+1, 1+1] = [0, 2]$.

The *y*-intercept is $y = \cos 0 + 1 = 2$.

The *x*-intercepts are solutions to $y = \cos(-2x) + 1$ which in $[-\pi, \pi]$ are

$x = -\frac{1}{2}\cos^{-1}(-1) = -\frac{\pi}{2}, x = \pi - \frac{1}{2}\cos^{-1}(-1) = \frac{\pi}{2}$.



Question 5

a For $y = \tan x$ transformed to $y = k\tan[a(x+b)]+c$,

k is the vertical dilation factor, the period is $\frac{\pi}{a}$, the phase shift is b units and the centre is c.

$y = 3\tan 4x - 5$ has no amplitude, the period is $\frac{\pi}{4}$ and the centre is –5.

b For $y = \cos x$ transformed to $y = k\cos[a(x+b)]+c$,

k is the amplitude, the period is $\frac{2\pi}{a}$, the phase shift is b units and the centre is c.

For $y = 8\cos(x+\pi)-3$, the amplitude is 8, the period is 2π, the phase shift is π units to the left and the centre is –3.

c For $y = \sin x$ transformed to $y = k\sin[a(x+b)]+c$,

k is the amplitude, the period is $\frac{2\pi}{a}$, the phase shift is b units and the centre is c.

For $y = 5\sin[2(x-3)]+1$, the amplitude is 5, the period is $\frac{2\pi}{2} = \pi$, the phase shift is 3 units to the right and the centre is 1.

Question 6

a For $y = k\sin[a(x+b)]+c$:

The amplitude is 7 $\Rightarrow k = 7$.

The period is π so $\frac{2\pi}{a} = \pi \Rightarrow a = 2$.

The phase shift is 1 unit to the right means $b = -1$.

The centre is –3, so $c = -3$.

Hence the equation is $y = 7\sin[2(x-1)]-3$.

b For $y = k\cos[a(x+b)]+c$:

The amplitude is 1 $\Rightarrow k = 1$.

Reflection in the *x*-axis means $\Rightarrow k = -1$.

The period is $\frac{2\pi}{5}$ so $\frac{2\pi}{a} = \frac{2\pi}{5} \Rightarrow a = 5$.

No phase shift is stated, so $b = 0$.

The centre is 2, so $c = 2$.

Hence the equation is $y = -\cos 5x + 2$.

c For $y = k\tan[a(x+b)]+c$:

The period is 2π so $\frac{\pi}{a} = 2\pi \Rightarrow a = \frac{1}{2}$.

Reflection in the *x*-axis means $\Rightarrow k = -1$.

The phase shift is 2 units to the left, so $b = -2$.

The centre is not stated, so $c = 0$.

Hence the equation is $y = -\tan\left(\frac{1}{2}(x+2)\right)$.

d For $y = k\sin[a(x+b)]+c$:

Vertical dilation of scale factor 4, so $k = 4$.

Horizontal dilation of scale factor 3 with a reflection in the *y*-axis, so $a = -\frac{1}{3}$.

Vertical translation 2 units up, so $c = 2$.

Horizontal translation 5 units left, so $b = 5$.

Hence the equation is $y = 4\sin\left[-\frac{1}{3}(x+5)\right] + 2$.

Question 7

$y = k\text{cosec}[a(x+b)] + c = \dfrac{k}{\sin[a(x+b)]} + c$.

The properties of this reciprocal trigonometric function is defined in the same way as we define the properties of $y = k\sin[a(x+b)] + c$, but no amplitude.

No amplitude, the period is $\dfrac{2\pi}{a}$ and the centre is *c*.

The phase shift is *b* units to the left when $b > 0$ and *b* units to the right when $b < 0$.

Question 8

$y = \tan 4(x-3)$

$y = k\tan[a(x+b)] + c$,

There is a horizontal dilation with scale factor $\dfrac{1}{4}$ so $\dfrac{1}{a} = \dfrac{1}{4} \Rightarrow a = 4$.

A translation of 3 units to the right means a phase shift of 3 units, so $b = -3$.

The centre is not stated, so $c = 0$.

Hence $y = \tan 4(x-3)$

Question 9

a 15 m

The centre is at $\dfrac{5+25}{2} = 15$ metres.

b The water level, *h* m, of the tide at time *t* hours can be described by a sine or a cosine function.

$h = k\cos[a(t+b)] + c$

From **a**, the centre of the motion is 15 metres, so $c = 15$.

The amplitude is $25 - 15 = 15 - 5 = 10$ m, so $k = 10$

There is 12 hours between consecutive low tides, so the period is 12 hours.

c The period is 12 hours, so $\frac{2\pi}{a} = 12 \Rightarrow a = \frac{\pi}{6}$.

Assume there is no phase shift, so $b = 0$.

Hence $D = 10\cos\left(\frac{\pi}{6}t\right) + 15$.

Note that if a sine function were used, a phase shift of $\frac{\pi}{2}$ will produce the same results.

Question 10

$B = 20\sin\left(\frac{\pi t}{30}\right) + 100$

$B = k\sin[a(t+b)] + c$, where *BP* is the blood pressure at time *t* minutes.

Assume the phase shift is 0, so $b = 0$.

The period is 60 beats per minute, so $\frac{2\pi}{a} = 60 \Rightarrow a = \frac{\pi}{30}$.

The amplitude is $\frac{1}{2}(120 - 80) = 20$, so $k = 20$.

Hence, $B = 20\sin\left(\frac{\pi t}{30}\right) + c$.

Maximum blood pressure is 120, so $120 = 20 \times 1 + c$, since the maximum value of sine is 1.

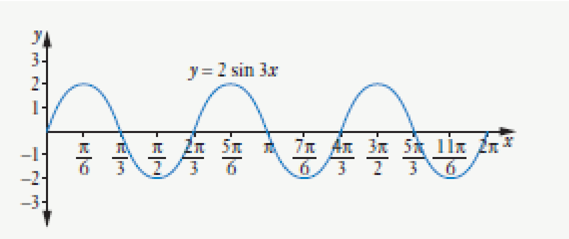
Hence $c = 100$.

Alternatively, the minimum blood pressure is 80, so $80 = 20 \times -1 + c \Rightarrow c = 100$, since the minimum value of sine is –1.

Hence $B = 20\sin\left(\frac{\pi t}{30}\right) + 100$.

Question 1

Sketch $y = 2\sin 3x$ in $[0, 2\pi]$ by horizontally dilating the graph of $y = \sin x$ with scale factor $\frac{1}{3}$ and then applying a vertical dilation with scale factor 2.



a The solutions to $2\sin 3x = 1$ are the points of intersection of the graph of $y = 2\sin 3x$ and the straight line $y = 1$.

The horizontal line $y = 1$ intersects the graph of $y = 2\sin 3x$ at 6 points. Hence there are 6 solutions.

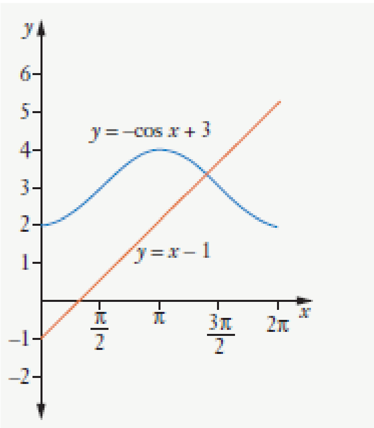
b From the graph, estimate the *x*-coordinates of the points of intersection.

$x = 0.2, 0.9, 2.3, 3, 4.4, 5.1$

Question 2

a Sketch $y = -\cos x + 3$ in $[0, 2\pi]$ by reflecting the graph of $y = \cos x$ in the *x*-axis and vertically translating 3 units up.

Sketch the linear graph $y = x - 1$ using $(1, 0)$ as the *x*-intercept and $(0, -1)$ as the *y*-intercept.



b **i** The solution is the point of intersection of the graph of $y = -\cos x + 3$

and the line $y = x - 1$.

From the graph, $x = 4.4$.

ii $x = 0$, $x = 2\pi$

The solutions are the points of intersection of the graph of $y = -\cos x + 3$ and the line $y = 2$.

The minimum value of $y = -\cos x + 3$ is 2 at $x = 0$ and $x = 2\pi$, so the horizontal line intersects the trigonometric function at $x = 0$ and $x = 2\pi$.

Question 3

a x = 15°, 75°, 195°, 255°

Change the domain $0° \leq x \leq 360°$, or $0° \leq 2x \leq 720°$.

$2\sin 2x = 1$

$\sin \theta°$ is positive in the 1st and 2nd quadrants.

$$2\sin 2x = 1$$
$$\sin 2x = \frac{1}{2}$$

1st quadrant

$$2x = 30° \text{ or } 390°$$
$$x = 15° \text{ or } 195°$$

2nd quadrant

$$2x = 150° \text{ or } 510°$$
$$x = 75° \text{ or } 255°$$

b x = 45°, 105°, 165°, 225°, 285°, 345°

$\tan 3x = -1$

$\tan\theta°$ is negative in the 2nd and 4th quadrants.

Change the domain $0° \le x \le 360°$, or $0° \le 3x \le 1080°$.

2nd quadrant

$3x = 180° - 45° = 135° \Rightarrow x = 45°$

$3x = 540° - 45° = 495° \Rightarrow x = 165°$

$3x = 900° - \tan^{-1} 45° = 855° \Rightarrow x = 285°$

4th quadrant

$3x = -45° \Rightarrow x = -15° = 345°$

$3x = 360° - 45° = 315° \Rightarrow x = 105°$

$3x = 720° - 45° = 315° \Rightarrow x = 225°$

c x = 240°, 300°

$\cos(x + 90°) = \dfrac{\sqrt{3}}{2}$

$\cos\theta°$ is positive in the 1st and 4th quadrants.

Change the domain $0° \le x \le 360°$, to $0° + 90° \le x + 90° \le 360° + 90°$,
or $90° \le x + 90° \le 450°$

1st quadrant

$x + 90° = 30° \Rightarrow x = -60° = 300°$

Adding multiples of 360° produces the same result.

4th quadrant

$x + 90° = -30° \Rightarrow x = -120° = 240°$

Adding multiples of 360° produces the same result.

d $x = 105°, 285°$

$\tan(x - 45°) = \sqrt{3}$

$\tan\theta°$ is positive in the 1st and 3rd quadrants.

Change the domain $0° \le x \le 360°$, to $0° - 45° \le x - 45° \le 360° - 45°$, or $-45° \le x - 45° \le 315°$

1st quadrant

$x - 45° = 60° \Rightarrow x = 105°$

Adding multiples of 360° will produce the same result.

3rd quadrant

$x - 45° = 180° + 60° = 240° \Rightarrow x = 285°$

Adding multiples of 360° will produce the same result.

e $x = 120°, x = 300°$

$\sin\theta°$ is 0 for multiples of 180°.

In [0°, 360°] we have $x + 60° = 0°, 180°, 360°$

$x = 120°, x = 300°$

Question 4

a $\tan 2x = \sqrt{3}$

$\tan\theta$ is positive in the 1st and 3rd quadrants.

Change the domain $0 \le x \le 2\pi$ to $0 \le 2x \le 4\pi$.

1st quadrant

$$2x = \tan^{-1}\sqrt{3} = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{6}$$

$$2x = 2\pi + \tan^{-1}\sqrt{3} = \frac{7\pi}{3} \Rightarrow x = \frac{7\pi}{6}$$

Adding more multiples of 2π will produce results outside the domain.

3rd quadrant

$$2x = \pi + \tan^{-1}\sqrt{3} = \frac{4\pi}{3} \Rightarrow x = \frac{2\pi}{3}$$

$$2x = 2\pi + \pi + \tan^{-1}\sqrt{3} = \frac{10\pi}{3} \Rightarrow x = \frac{5\pi}{3}$$

Adding more multiples of 2π will produce results outside the domain.

The solutions are $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$.

b $2\cos 3x + 1 = 0 \Rightarrow \cos 3x = -\frac{1}{2}$

$\cos\theta$ is negative in the 2nd and 3rd quadrants.

Change the domain $0 \le x \le 2\pi$ to $0 \le 3x \le 6\pi$.

2nd quadrant

$$3x = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} \Rightarrow x = \frac{2\pi}{9}$$

$$3x = 2\pi + \pi - \cos^{-1}\left(\frac{1}{2}\right) = \frac{8\pi}{3} \Rightarrow x = \frac{8\pi}{9}$$

$$3x = 4\pi + \pi - \cos^{-1}\left(\frac{1}{2}\right) = \frac{14\pi}{3} \Rightarrow x = \frac{14\pi}{9}$$

Adding more multiples of 2π will produce results outside the domain.

3rd quadrant

$$3x = \pi + \cos^{-1}\left(\frac{1}{2}\right) = \frac{4\pi}{3} \Rightarrow x = \frac{4\pi}{9}$$

$$3x = 2\pi + \pi + \cos^{-1}\left(\frac{1}{2}\right) = \frac{10\pi}{3} \Rightarrow x = \frac{10\pi}{9}$$

$$3x = 4\pi + \pi + \cos^{-1}\left(\frac{1}{2}\right) = \frac{16\pi}{3} \Rightarrow x = \frac{16\pi}{9}$$

Adding more multiples of 2π will produce results outside the domain.

The solutions are $x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$.

c $4\sin^2\left(x-\frac{\pi}{3}\right)=3 \Rightarrow \sin\left(x-\frac{\pi}{3}\right)=\pm\frac{\sqrt{3}}{2}$

Need to find solutions in all quadrants.

Change the domain $0 \le x \le 2\pi$ to $-\frac{\pi}{3} \le x-\frac{\pi}{3} \le \frac{5\pi}{3}$.

1st quadrant

$$x-\frac{\pi}{3}=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3} \Rightarrow x=\frac{2\pi}{3}$$

Adding more multiples of 2π will produce results outside the domain.

2nd quadrant

$$x-\frac{\pi}{3}=\pi-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{2\pi}{3} \Rightarrow x=\pi$$

Adding more multiples of 2π will produce results outside the domain.

3rd quadrant

$$x-\frac{\pi}{3}=\pi+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{4\pi}{3} \Rightarrow x=\frac{5\pi}{3}$$

Adding more multiples of 2π will produce results outside the domain.

4th quadrant

$$x-\frac{\pi}{3}=0-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3} \Rightarrow x=0$$

$$x-\frac{\pi}{3}=2\pi-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{3} \Rightarrow x=2\pi$$

Adding multiples of 2π will produce results outside the domain.

The solutions are $x=0,\frac{2\pi}{3},\pi,\frac{5\pi}{3},2\pi$.

d $$2\cos^2 2x - 1 = 0 \Rightarrow \cos 2x = \pm\frac{1}{\sqrt{2}}$$

Need to find solutions in all quadrants.

Change the domain $0 \le x \le 2\pi$ to $0 \le 2x \le 4\pi$.

1st quadrant

$$2x = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \Rightarrow x = \frac{\pi}{8}$$

$$2x = 2\pi + \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{9\pi}{4} \Rightarrow x = \frac{9\pi}{8}$$

Adding more multiples of 2π will produce results outside the domain.

2nd quadrant

$$2x = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{8}$$

$$2x = 2\pi + \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{11\pi}{4} \Rightarrow x = \frac{11\pi}{8}$$

Adding more multiples of 2π will produce results outside the domain.

3rd quadrant

$$2x = \pi + \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{5\pi}{4} \Rightarrow x = \frac{5\pi}{8}$$

$$2x = 2\pi + \pi + \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{13\pi}{4} \Rightarrow x = \frac{13\pi}{8}$$

Adding more multiples of 2π will produce results outside the domain.

4th quadrant

$$2x = 0 - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4} \Rightarrow x = \frac{7\pi}{8}$$

$$2x = 2\pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{7\pi}{4} \Rightarrow x = \frac{7\pi}{8}$$

$$2x = 4\pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{15\pi}{4} \Rightarrow x = \frac{15\pi}{8}$$

Adding more multiples of 2π will produce results outside the domain.

The solutions are $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

e

$$\cos(x+\pi) = 1$$
$$x + \pi = \cos^{-1} 1$$
$$= 0, 2\pi, 4\pi, 6\pi, \ldots$$

$$x = -\pi, \pi, 3\pi, 5\pi, \ldots$$

For the domain $[0, 2\pi]$, $x = \pi$.

Question 5

a $\tan 3x = 1$

$\tan\theta$ is positive in the 1st and 3rd quadrants.

Change the domain $-\pi \le x \le \pi$ to $-3\pi \le 3x \le 3\pi$.

1st quadrant

$$3x = \tan^{-1} 1 = \frac{\pi}{4} \Rightarrow x = \frac{\pi}{12}$$

$$3x = -2\pi + \tan^{-1} 1 = -\frac{7\pi}{4} \Rightarrow x = -\frac{7\pi}{12}$$

$$3x = 2\pi + \tan^{-1} 1 = \frac{9\pi}{4} \Rightarrow x = \frac{3\pi}{4}$$

Adding or subtracting more multiples of 2π will produce results outside the domain.

3rd quadrant

$$3x = \pi + \tan^{-1} 1 = \frac{5\pi}{4} \Rightarrow x = \frac{5\pi}{12}$$

$$3x = -2\pi + \pi + \tan^{-1} 1 = -\frac{3\pi}{4} \Rightarrow x = -\frac{\pi}{4}$$

$$3x = -4\pi + \pi + \tan^{-1} 1 = -\frac{11\pi}{4} \Rightarrow x = -\frac{11\pi}{12}$$

The solutions are $x = -\frac{11\pi}{12}, -\frac{7\pi}{12}, -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$.

b $\cos\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$\cos\theta$ is positive in the 1st and 4th quadrants.

Change the domain $-\pi \le x \le \pi$ to $-\pi+\frac{\pi}{4} \le x+\frac{\pi}{4} \le \pi+\frac{\pi}{4}$ or $-\frac{3\pi}{4} \le x+\frac{\pi}{4} \le \frac{5\pi}{4}$.

1st quadrant

$x+\frac{\pi}{4}=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4} \Rightarrow x=0$

Adding or subtracting multiples of 2π will produce results outside the domain.

4th quadrant

$x+\frac{\pi}{4}=0-\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=-\frac{\pi}{4} \Rightarrow x=-\frac{\pi}{2}$.

The solutions are $x=-\frac{\pi}{2},0$.

c $\sin 2x=-1$

$\sin\theta=-1$ for $\theta=\frac{3\pi}{2}\pm 2n\pi, n=0,1,2,..$

Hence $x=\frac{3\pi}{4}\pm n\pi$

For $-\pi \le x \le \pi$

$x=-\frac{\pi}{4},\frac{3\pi}{4}$.

d $\cos\left(x-\frac{\pi}{2}\right)=0$

$\cos\theta=0$ for $\theta=\frac{\pi}{2}\pm 2n\pi, n=0,1,2,..$ and for $\theta=\frac{3\pi}{2}\pm 2n\pi, n=0,1,2,..$

Hence $x=\pi\pm 2n\pi$ and $x=2\pi\pm 2n\pi$

For $-\pi\le x\le\pi$ the solutions are $x=-\pi,0,\pi$

e $\tan^2 4x=0\Rightarrow\tan 4x=0$

$\tan\theta=0$ for $\theta=0,\pm\pi,\pm 2\pi,$

Hence the solutions in $[-\pi,\pi]$ are $x=0,\pm\frac{\pi}{4},\pm\frac{2\pi}{4},\pm\frac{3\pi}{4},\pm\frac{4\pi}{4}$ or

$x=0,\pm\frac{\pi}{4},\pm\frac{\pi}{2},\pm\frac{3\pi}{4},\pm\pi$

Question 6

a $\cos 2\left(x-\frac{\pi}{2}\right)=\frac{1}{2}$

$\cos\theta$ is positive in the 1st and 4th quadrants.

Change the domain $0 \le x \le 2\pi$ to $-\frac{\pi}{2} \le x-\frac{\pi}{2} \le \frac{3\pi}{2}$

1st quadrant

$$2\left(x-\frac{\pi}{2}\right)=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}, x-\frac{\pi}{2}=\frac{\pi}{6} \Rightarrow x=\frac{2\pi}{3}$$

$$2\left(x-\frac{\pi}{2}\right)=2\pi+\frac{\pi}{3}=\frac{7\pi}{3}, x-\frac{\pi}{2}=\frac{7\pi}{6} \Rightarrow x=\frac{5\pi}{3}$$

Adding or subtracting more multiples of 2π will produce results outside the domain.

4th quadrant

$$2\left(x-\frac{\pi}{2}\right)=0-\frac{\pi}{3}=-\frac{\pi}{3}, x-\frac{\pi}{2}=-\frac{\pi}{6} \Rightarrow x=\frac{\pi}{3}$$

$$2\left(x-\frac{\pi}{2}\right)=2\pi-\frac{\pi}{3}=\frac{5\pi}{3}, x-\frac{\pi}{2}=\frac{5\pi}{6} \Rightarrow x=\frac{4\pi}{3}$$

Adding or subtracting more multiples of 2π will produce results outside the domain.

The solutions are $x=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$.

b $2\sin\left(3x+\frac{3\pi}{2}\right)=1 \Rightarrow \sin\left(3x+\frac{3\pi}{2}\right)=\frac{1}{2}$

$\sin\theta$ is positive in the 1st and 2nd quadrants.

Change the domain.

$0 \le x \le 2\pi$, $0 \le 3x \le 6\pi$, $\frac{3\pi}{2} \le 3x+\frac{3\pi}{2} \le 6\pi+\frac{3\pi}{2}$ or $\frac{3\pi}{2} \le 3x+\frac{3\pi}{2} \le \frac{15\pi}{2}$

1st quadrant

$3x+\frac{3\pi}{2}=\sin^{-1}\frac{1}{2}=\frac{\pi}{6} \Rightarrow x=-\frac{4\pi}{9}$.

The equivalent 1st quadrant angle is $2\pi-\frac{4\pi}{9}=\frac{14\pi}{9}$

$3x+\frac{3\pi}{2}=2\pi+\frac{\pi}{6} \Rightarrow x=\frac{2\pi}{9}$

$3x+\frac{3\pi}{2}=4\pi+\frac{\pi}{6} \Rightarrow x=\frac{8\pi}{9}$

Adding or subtracting more multiples of 2π will either produce results outside the domain or produce duplicates.

2nd quadrant

$3x+\frac{3\pi}{2}=\pi-\frac{\pi}{6} \Rightarrow x=-\frac{2\pi}{9}$. This angle is equivalent to $2\pi-\frac{2\pi}{9}=\frac{16\pi}{9}$

$3x+\frac{3\pi}{2}=2\pi+\pi-\frac{\pi}{6} \Rightarrow x=\frac{4\pi}{9}$

$3x+\frac{3\pi}{2}=4\pi+\pi-\frac{\pi}{6} \Rightarrow x=\frac{10\pi}{9}$

Adding or subtracting more multiples of 2π will either produce results outside the domain or produce duplicates.

The solutions are $x=\frac{2\pi}{9},\frac{4\pi}{9},\frac{8\pi}{9},\frac{10\pi}{9},\frac{14\pi}{9},\frac{16\pi}{9}$.

Question 7

a Amplitude 15, period 12, centre 20

The amplitude is the vertical dilation factor, 15

The period is $2\pi \div \frac{\pi}{6} = 12$

The centre is the vertical translation, 20

b Amplitude 15, period 12, centre 20

$$15\cos\left(\frac{\pi}{6}t\right) + 20 = 35$$

$$\cos\left(\frac{\pi}{6}t\right) = 1$$

$$\frac{\pi}{6}t = 0, 2\pi, 4\pi, \ldots$$
$$t = 0, 12, 24, \ldots$$

The maximum value of a cosine function is 1, so *t* specifies which month of each year (in this case, January) produces the maximum temperature of 35°C.

Question 8

a Let $h = A\sin(at+b)+c$, where h m is the height of the tidal wave at time t s.

Assume there is no phase shift, so $b=0$.

The centre, c, is $(20+6)/2=13$.

The period is 10, so $\frac{2\pi}{a}=10 \Rightarrow a=\frac{\pi}{5}$.

The minimum value is 6, which occurs when $\sin(ax+b)=-1$.

Hence

$$-A+c=6$$
$$-A+13=6 \Rightarrow A=7$$

So the function is $h=7\sin\left(\frac{\pi t}{5}\right)+13$.

b The maximum height occurs when $\sin\left(\frac{\pi t}{5}\right)=1$.

$$\frac{\pi t}{5}=\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2},\frac{13\pi}{2},\ldots$$

$$t=\frac{5}{2},\frac{25}{2},\frac{45}{2},\frac{65}{2},\ldots$$

The first four values of t for maximum height are t = 2.5, 12.5, 22.5. 32.5 seconds.

c The minimum height occurs when $\sin\left(\frac{\pi t}{5}\right)=-1$.

$$\frac{\pi t}{5}=\frac{3\pi}{2},\frac{7\pi}{2},\frac{11\pi}{2},\frac{15\pi}{2},\ldots$$

$$t=\frac{15}{2},\frac{35}{2},\frac{55}{2},\frac{75}{2},\ldots$$

The first value of t for minimum height is t = 7.5 seconds.

d Require $7\sin\left(\frac{\pi t}{5}\right)+13=13$.

$\sin\left(\frac{\pi t}{5}\right)=0$

$\frac{\pi t}{5}=0, \pi, 2\pi, 3\pi, \ldots$ seconds

T = 0, 5, 10, 15, … seconds

Question 9

a From the graph, the centre of the function is 0 and the maximum value is 1, hence the amplitude is 1.

Or from the equation, the amplitude is 1 and the period is $\frac{2\pi}{880\pi}=\frac{1}{440}$.

b **i** From the graph, a horizontal line with equation $y=0.5$ will intersect

the sine function 9 times, hence there are 9 solutions for *x*.

The solutions are *x* = 0.0002, 0.0010, 0.0025, 0.0032, 0.0047, 0.0055, 0.0070, 0.0078, 0.0093.

ii The solutions to $\sin(880\pi x)=0$ are the *x*-intercepts.

Reading these from the graph gives

x = 0, 0.001, 0.0021, 0.00035, 0.0045, 0.0056, 0.0068, 0.008, 0.009

c **i** $\sin(880\pi x)=0.5$ means the solutions are in the 1st and 2nd quadrants.

$x=\frac{1}{880\pi}\sin^{-1}0.5$

In the 1st quadrant

$\sin^{-1}0.5=\frac{\pi}{6}+2n\pi, n=0,1,2,\ldots$

$=\frac{\pi}{6},\frac{13\pi}{6},\frac{25\pi}{6},\frac{37\pi}{6},\frac{49\pi}{6}$ in $[0, 0.01]$

and in the 2nd quadrant

$$\sin^{-1} 0.5 = \pi - \frac{\pi}{6} + 2n\pi, n = 0, 1, 2, \ldots.$$

$$= \frac{5\pi}{6}, \frac{17\pi}{6}, \frac{29\pi}{6}, \frac{41\pi}{6} \text{ in } [0, 0.01]$$

Hence

$$x = \frac{1}{880\pi}\frac{\pi}{6}, \frac{1}{880\pi}\frac{5\pi}{6}, \frac{1}{880\pi}\frac{13\pi}{6}, \frac{1}{880\pi}\frac{17\pi}{6}, \frac{1}{880\pi}\frac{25\pi}{6}, \frac{1}{880\pi}\frac{29\pi}{6}, \frac{1}{880\pi}\frac{37\pi}{6}, \frac{1}{880\pi}\frac{41\pi}{6}, \frac{1}{880\pi}\frac{49\pi}{6}$$

$$= \frac{1}{5280}, \frac{5}{5280}, \frac{13}{5280}, \frac{17}{5280}, \frac{25}{5280}, \frac{29}{5280}, \frac{37}{5280}, \frac{41}{5280}, \frac{49}{5280}$$

$= 0.00019, 0.00095, 0.0025, 0.0032, 0.0047, 0.0055, 0.0070, 0.0078, 0.0093$, to 2 significant figures.

ii $880\pi x = \sin^{-1} 0 = 0, \pi, 2\pi, 3\pi, ..$

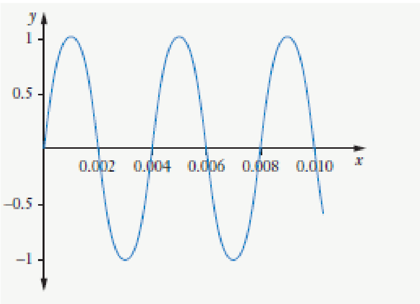
Hence $x = \frac{0}{880}, \frac{1}{880}, \frac{2}{880}, \frac{3}{880}, \ldots, \frac{9}{880}$

$= 0,\ 0.0011,\ 0.0023,\ 0.0034,\ 0.0045,\ 0.0057,\ 0.0068,\ 0.0080,\ 0.0091, 0.010$, to 2 significant figures.

d Three times louder means the amplitude is 3 times larger.

Hence the equation is $y = 3\sin(880\pi x)$

e



f The period is $\frac{2\pi}{880\pi} = \frac{1}{440}$ seconds.

Hence one wave cycle takes $\frac{2\pi}{880\pi} = \frac{1}{440}$ seconds, so there are 440 wave cycles

in 1 second.

Therefore note A has 440 hertz (Hz).

Question 10

$y = \sin\left(\frac{2\pi t}{23}\right)$, period is $y = 2\pi \div \frac{2\pi}{23} = 23$.

This means the physical cycle repeats after 23 days.

The amplitude is 1, so physically, during the month you are at your peak when $\frac{2\pi t}{23} = \sin^{-1} 1 = \frac{\pi}{2}$.

That is, on approximately the 6th day and 29th day ($t = \frac{23}{4} \approx 6$, $t = 6 + 23 = 29$).

Similarly, physically you are at your lowest level when $\frac{2\pi t}{23} = \sin^{-1}(-1) = \frac{3\pi}{2}$.

That is, on approximately the 17th day.

$y = \sin\left(\frac{2\pi t}{28}\right)$, period is $y = 2\pi \div \frac{2\pi}{28} = 28$.

This means the emotional cycle repeats after 28 days.

Emotionally, you will be at your best when $\frac{2\pi t}{28} = \frac{\pi}{2}$, the 7th day of the month, and at your lowest level when $\frac{2\pi t}{28} = \frac{3\pi}{2}$, which is the 21st day.

$y = \sin\left(\frac{2\pi t}{33}\right)$, period is $y = 2\pi \div \frac{2\pi}{33} = 33$

This means your intellectual cycle repeats after 33 days.

Your peak intellectual performance occurs when $\frac{2\pi t}{33} = \frac{\pi}{2}$, which is approximately the 8th day ($t = \frac{33}{4} = 8.25$)

Your lowest intellectual level occurs when $\frac{2\pi t}{33} = \frac{3\pi}{2}$, approximately the 25th day.

b From the graph, the points of intersection of $y = \sin\left(\frac{2\pi t}{23}\right)$ and $y = \sin\left(\frac{2\pi t}{33}\right)$ are $x \approx 7$ days and $x \approx 21$ days.

c From the graph, the points of intersection of $y = \sin\left(\frac{2\pi t}{23}\right)$ and $y = \sin\left(\frac{2\pi t}{28}\right)$ are $x \approx 6$ days and $x \approx 19$ days.

d The 3 functions are close to their maximum together at about the 7th day.

Question 11

a In the unstretched position (natural/equilibrium state), the spring is 15 cm in length.

It represents the centre of the motion, where the spring begins to stretch or contract.

b $h = 12\cos t + 15$

The maximum height is the maximum distance from the centre.
This is $12 + 15 = 27$ cm

The minimum height is the minimum distance from the centre.
This is $15 - 12 = 3$ cm

c After π seconds, the height of the spring is

$h = 12\cos\pi + 15 = 3$ cm.

d The minimum height is 3 cm.

$h = 12\cos t + 15 = 3$

$\cos t = -1$.

$t = \pi, 3\pi, 5\pi, ..$ seconds or $t = (2n-1)\pi, n = 1, 2, ...$ seconds

Test yourself 3

Question 1

D

The function $y = A\cos[a(x+b)]+c$ has amplitude *A*, period $\dfrac{2\pi}{a}$, phase shift *b* and centre *c*.

Hence, $y = 2\cos 3x - 7$ has amplitude 2, period $\dfrac{2\pi}{3}$, phase shift 0 and centre –7.

Question 2

A

The function $y = \tan[a(x+b)]+c$ has period $\dfrac{\pi}{a}$, phase shift *b* and centre *c*.

A phase shift π to the left means $b = \pi$.

Hence $y = \tan[a(x+\pi)]+c$. With $a = 1, c = 0$, $y = \tan(x+\pi)$.

Option B changes the period, option C changes the centre and option D is a phase shift to the right.

Question 3

B

$$\cos 2x = 1$$
$$2x = \cos^{-1} 1 = 0 \Rightarrow x = 0$$
$$2x = 2\pi + \cos^{-1} 1 \Rightarrow x = \pi$$
$$2x = 4\pi + \cos^{-1} 1 \Rightarrow x = 2\pi$$

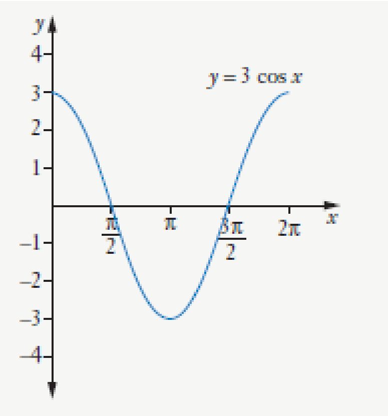
Question 4

a Amplitude is 3, period 2π, centre 0, phase shift 0.

The y-intercept is $3\cos 0 = 3$.

The x-intercepts satisfy $3\cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, x = \dfrac{3\pi}{2}$.

The function is the graph of $y = \cos x$ vertically dilated with scale factor 3.



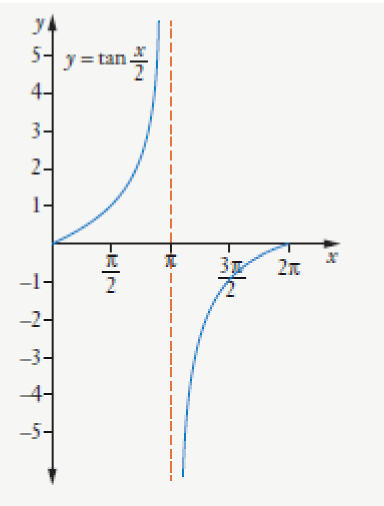
b Amplitude is 3, period $\pi \div \dfrac{1}{2} = 2\pi$, centre 0, phase shift 0.

The y-intercept is $\tan 0 = 0$.

The x-intercepts satisfy $\tan\left(\dfrac{1}{2}x\right) = 0$.

$\dfrac{1}{2}x = 0, \pi, 2\pi, .. \Rightarrow x = 0, x = 2\pi$ for given domain.

The function is the graph of $y = \tan x$ horizontally dilated with scale factor 2.



c Amplitude is 1, period 2π, centre –2, phase shift 0.

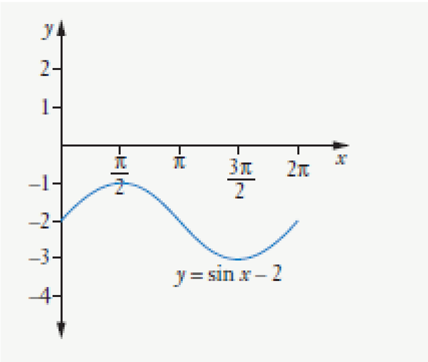
The y-intercept is $\sin 0 - 2 = -2$.

There are no x-intercepts as $\sin x - 2 = 0 \Rightarrow \sin x = 2$ which is not possible.

Also, the maximum value of y is $-2+1=-1$.

The function can be thought of as a vertical translation, 2 units down,

of the graph of $y = \sin x$.



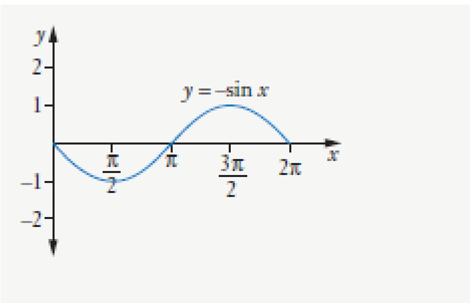
d Amplitude is 1, period 2π, centre 0, phase shift 0.

The y-intercept is $\sin 0 = 0$.

The x-intercepts satisfy $\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$.

$y = -1 \times \sin x$, so the function can also be thought of as a reflection in the x-axis

of the graph of $y = \sin x$.



e Amplitude is 3, period $\frac{2\pi}{2}=\pi$, centre –1, phase shift 0.

The y-intercept is $3\cos 0-1=2$.

The x-intercepts satisfy

$$3\cos 2x-1=0$$

$$2x=\cos^{-1}\left(\frac{1}{3}\right)\Rightarrow x\approx 0.6$$

$$2x=2\pi+\frac{1}{2}\cos^{-1}\left(\frac{1}{3}\right)\Rightarrow x\approx 3.8.$$

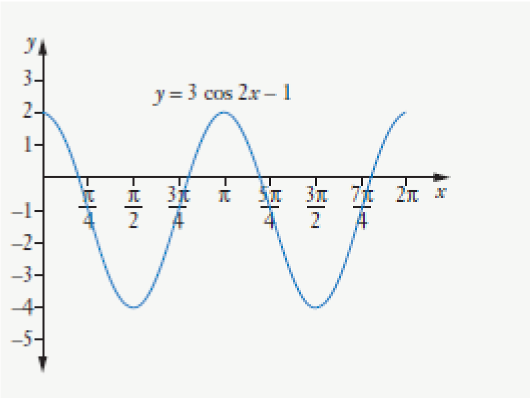
$$2x=2\pi-\frac{1}{2}\cos^{-1}\left(\frac{1}{3}\right)\Rightarrow x\approx 2.5$$

$$2x=4\pi-\frac{1}{2}\cos^{-1}\left(\frac{1}{3}\right)\Rightarrow x\approx 5.7$$

The function can be thought of as a horizontal dilation with scale factor $\frac{1}{2}$

of $y=\cos x$, then a vertical translation with scale factor 3

followed by a vertical translation 1 unit down.



Question 5

a $h = 3\cos\left(\frac{2\pi t}{3}\right) + 10$

The maximum level is the maximum value of the function from the centre.

The amplitude is 3, so the maximum is 10 + 3 = 13 m.

Maximum level occurs when $\cos\left(\frac{2\pi t}{3}\right) = 1$.

$$\frac{2\pi t}{3} = 0, 2\pi, 4\pi, 6\pi, \ldots$$
$$2\pi t = 0, 6\pi, 12\pi, 18\pi, \ldots$$
$$t = 0, 3, 6, 9, .. \text{ h}$$

The minimum level is the minimum value of the function from the centre,

10 – 3 = 7 m

The minimum level occurs when $\cos\left(\frac{2\pi t}{3}\right) = -1$.

$$\frac{2\pi t}{3} = \pi, 3\pi, 5\pi, 7\pi, \ldots$$
$$2\pi t = 3\pi, 9\pi, 15\pi, 21\pi, \ldots$$
$$t = 1.5, 4.5, 7.5, 10.5, .. \text{ h}$$

b

$$3\cos\left(\frac{2\pi t}{3}\right) + 10 = 11$$
$$3\cos\left(\frac{2\pi t}{3}\right) = 1$$
$$\cos\left(\frac{2\pi t}{3}\right) = \frac{1}{3}$$
$$\frac{2\pi t}{3} \approx 1.23, 2\pi - 1.23, 2\pi + 1.23, 4\pi - 1.23, 4\pi + 1.23, \ldots$$
$$2\pi t \approx 3.69, 6\pi - 3.69, 6\pi + 3.69, 12\pi - 3.69, 12\pi + 3.69, \ldots$$
$$t \approx 0.59, 2.41, 3.59, 5.41, 6.59, \ldots \text{h}$$

These are the times when the water level in the lock is 11 m.

Question 6

a

$$
\begin{aligned}
&2\cot^2 x + 2\\
&= 2(\cot^2 x + 1)\\
&= 2\left(\frac{\cos^2 x}{\sin^2 x} + 1\right)\\
&= 2\left(\frac{\cos^2 x + \sin^2 x}{\sin^2 x}\right), \cos^2 x + \sin^2 x = 1\\
&= 2 \times \frac{1}{\sin^2 x}\\
&= 2\operatorname{cosec}^2 x
\end{aligned}
$$

b

$$
\begin{aligned}
&\tan A \operatorname{cosec} A\\
&= \frac{\sin A}{\cos A} \times \frac{1}{\sin A}\\
&= \frac{1}{\cos A}\\
&= \sec A
\end{aligned}
$$

c

$$
\begin{aligned}
&(\sec A + \tan A)(\sec A - \tan A)\\
&= \sec^2 A - \tan^2 A, \text{ (expand brackets)}\\
&= 1 + \tan^2 A - \tan^2 A, \ (1 + \tan^2 A = \sec^2 A)\\
&= 1
\end{aligned}
$$

d

$$
\begin{aligned}
&\sin(180° - x) = \sin 180° \cos x - \cos 180° \sin x\\
&= 0 \times \cos x - (-1) \times \sin x\\
&= \sin x
\end{aligned}
$$

Question 7

a

$$4\cos^2 x = 3$$

$$\cos x = \pm\frac{\sqrt{3}}{2}$$

$$x = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}, \text{ quadrant 1}$$

$$x = \pi - \cos^{-1}\frac{\sqrt{3}}{2} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}, \text{ quadrant 2}$$

$$x = \pi + \cos^{-1}\frac{\sqrt{3}}{2} = \pi + \frac{\pi}{6} = \frac{7\pi}{6}, \text{ quadrant 3}$$

$$x = 2\pi - \cos^{-1}\frac{\sqrt{3}}{2} = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}, \text{ quadrant 4}$$

The solutions are $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

b $2\sin 2x = 1 \Rightarrow \sin 2x = \frac{1}{2}$

$$2x = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \Rightarrow x = \frac{\pi}{12}, \text{ quadrant 1}$$

and $2x = 2\pi + \sin^{-1}\left(\frac{1}{2}\right) = 2\pi + \frac{\pi}{6} \Rightarrow x = \frac{13\pi}{12}$

$$2x = \pi - \sin^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{6} \Rightarrow x = \frac{5\pi}{12}, \text{ quadrant 2}$$

$$2x = 2\pi + \pi - \sin^{-1}\left(\frac{1}{2}\right) = 3\pi - \frac{\pi}{6} \Rightarrow x = \frac{17\pi}{12}$$

The solutions are $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$.

c $\cos\left(x-\frac{\pi}{2}\right)=-1,\ x-\frac{\pi}{2}=\pi,3\pi,5\pi,...$ quadrant 1

$x-\frac{\pi}{2}=\pi,3\pi,5\pi,...\ \Rightarrow x=\frac{3\pi}{2}$ for the given domain.

$\cos\left(x-\frac{\pi}{2}\right)=-1$

$x-\frac{\pi}{2}=\pi+\frac{\pi}{2},2\pi+\pi+\frac{\pi}{2},4\pi+\pi+\frac{\pi}{2},...$ quadrant 3

$x=\frac{3\pi}{2},$

$x-\frac{\pi}{2}=\pi,3\pi,5\pi,...\ \Rightarrow x=\frac{3\pi}{2}$ for the given domain.

The solution is $x=\frac{3\pi}{2}$.

d

$\tan^2\left(x+\frac{\pi}{6}\right)=3$

$x+\frac{\pi}{6}=\tan^{-1}\sqrt{3}$

$x+\frac{\pi}{6}=\frac{\pi}{3}\Rightarrow x=\frac{\pi}{6}$, quadrant 1

$x+\frac{\pi}{6}=\pi-\frac{\pi}{3}\Rightarrow x=\frac{\pi}{2}$, quadrant 2

$x+\frac{\pi}{6}=\pi+\frac{\pi}{3}\Rightarrow x=\frac{7\pi}{6}$, quadrant 3

$x+\frac{\pi}{6}=2\pi-\frac{\pi}{3}\Rightarrow x=\frac{3\pi}{2}$, quadrant 3

The solutions are $x=\frac{\pi}{6},\frac{\pi}{2},\frac{7\pi}{6},\frac{3\pi}{2}$.

Question 8

a $y = k\sin[a(t+b)]+c$, where y is the blood pressure at time t minutes.

Assume the phase shift is 0, so $b = 0$.

The period is 70 beats per minute, so $\frac{2\pi}{a} = 70 \Rightarrow a = \frac{\pi}{35}$.

The amplitude is $\frac{1}{2}(135-85) = 25$, so $k = 25$.

Hence, $y = 25\sin\left(\frac{\pi}{35}t\right)+c$

Maximum blood pressure is 135, so $135 = 25\times 1 + c$,

since the maximum value of sine is 1.

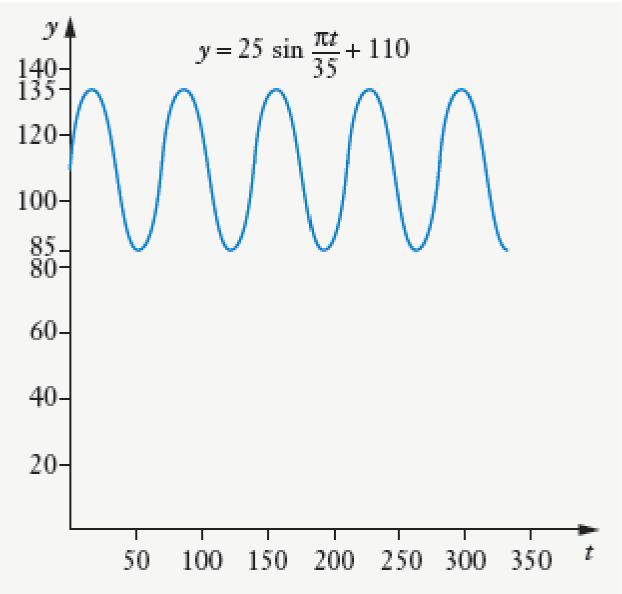
Hence $c = 110$.

Hence $y = 25\sin\left(\frac{\pi}{35}t\right)+110$.

b Sketch the function using amplitude 25, period 70 and centre 110.

The y-intercept is $25\sin 0 + 110 = 110$.

There are no x-intercepts because the minimum value of the function is 85.



Question 9

a

$$2\cos 2x = 1$$

$$\cos 2x = \frac{1}{2}$$

Quadrant 1

$$2x = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{6}$$

$$2x = -2\pi + \cos^{-1}\frac{1}{2} = -2\pi + \frac{\pi}{3} \Rightarrow x = -\frac{5\pi}{6}$$

Quadrant 4

$$2x = 2\pi - \cos^{-1}\frac{1}{2} = 2\pi - \frac{\pi}{3}$$

$x = \frac{5\pi}{6}$, (anti-clockwise), $x = -\frac{\pi}{6}$, (clockwise)

The solutions are $x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$.

b $\tan\left(x-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{3}}$

Quadrant 2

$$x-\frac{\pi}{4}=\pi-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$x=\frac{\pi}{4}+\pi-\frac{\pi}{6}$$

$$=\frac{13\pi}{12}\text{ (anti-clockwise, outside domain)}$$

$$=2\pi-\frac{13\pi}{12}=-\frac{11\pi}{12}\text{ (clockwise)}$$

Quadrant 4

$$x-\frac{\pi}{4}=2\pi-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$x=\frac{\pi}{4}+2\pi-\frac{\pi}{6}$$

$$=\frac{25\pi}{12}\text{ (anti-clockwise, outside domain)}$$

$$=2\pi-\frac{25\pi}{12}=-\frac{\pi}{12}\text{ (clockwise)}$$

The solutions are $x=\frac{\pi}{12}, x=-\frac{11\pi}{12}$.

c $\sin\left(x+\frac{\pi}{2}\right)=\frac{1}{\sqrt{2}}$

Quadrant 1

$$x+\frac{\pi}{2}=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$$

$$x=-\frac{\pi}{2}+\frac{\pi}{4}$$

$$=-\frac{\pi}{4}$$

Quadrant 2

$$x+\frac{\pi}{2}=\pi-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$x=-\frac{\pi}{2}+\pi-\frac{\pi}{4}$$

$$=\frac{\pi}{4}$$

The solutions are $x=\pm\frac{\pi}{4}$.

Question 10

a height $= k\sin[a(t+b)]+c$, where h is the height at time t hours.

Assume the phase shift is 0, so $b=0$.

The period is 13 hours, so $\frac{2\pi}{a}=13 \Rightarrow a=\frac{2\pi}{13}$.

The amplitude is $\frac{1}{2}(80-50)=15$, so $k=15$

Hence, $h=15\sin\left(\frac{2\pi}{13}t\right)+c$

Maximum height is 80 m, so $80=15\times 1+c$, since the maximum value of sine is 1.

Hence $c=65$ and the equation is $h=15\sin\left(\frac{2\pi}{13}t\right)+65$.

b Halfway between high tide and low tide means at the centre of the function.

$$15\sin\left(\frac{2\pi}{13}t\right)+65=65$$

$$\sin\left(\frac{2\pi}{13}t\right)=0$$

$$\frac{2\pi}{13}t=0,\pi,2\pi,3\pi,..$$

$$t=0,\frac{13}{2},13,\frac{39}{2},26,..\text{ hours}$$

Question 11

a $y = \cos ax$ has period $\dfrac{2\pi}{a}$

Period is π, hence $\dfrac{2\pi}{a} = \pi \Rightarrow a = 2$.

So $y = \cos 2x$

b $y = a\cos x$ has amplitude a

For $a = 5, y = 5\cos x$

c $y = -\cos x$ is a reflection of $y = \cos x$ in the x-axis.

d $y = \cos(x - a)$ is a phase shift of a units to the right, with $a > 0$

For $a = \dfrac{\pi}{6}, y = \cos\left(x - \dfrac{\pi}{6}\right)$

e $y = \cos x + c$ has centre at c.

For $c = 4, y = \cos x + 4$

Question 12

a $y = 2\sin\left(\frac{1}{2}x\right) - 1$

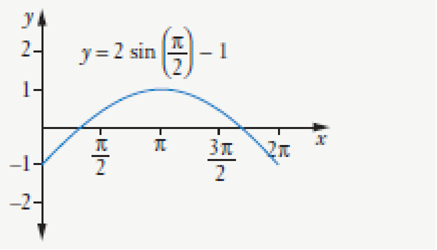
Amplitude is 2, period is $2\pi \div \frac{1}{2} = 4\pi$, phase shift is 0, centre at –1.

The function is the graph of $y = \sin x$ dilated horizontally with scale factor 2,

then dilated vertically with scale factor 2

followed by a vertical translation 1 unit down.

The *y*-intercept is $y = 2\sin 0 - 1 = -1$.



b Reading from the graph, the *x*-intercepts are approximately 1 and 5.2

c

$$2\sin\left(\frac{1}{2}x\right) - 1 = 0$$

$$\sin\left(\frac{1}{2}x\right) = \frac{1}{2}$$

$$\frac{1}{2}x = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

$$x = \frac{\pi}{3}$$

and

$$\frac{1}{2}x = \pi - \sin^{-1}\frac{1}{2} = \frac{5\pi}{6}$$

$$x = \frac{5\pi}{3}$$

Question 13

$y = k\sin(a[x+b])+c$, amplitude *k*, period $\frac{2\pi}{a}$, centre *c*, phase shift *b*

a amplitude 2, period $\frac{2\pi}{3}$, centre –1, no phase shift.

b $y = \cos\left(\frac{1}{2}(x+2\pi)\right)$

amplitude 1, period $2\pi \div \frac{1}{2} = 4\pi$, centre 0, phase shift 2π to the left.

c $y = -3\tan\left(5(x-\frac{\pi}{20})\right)$

No amplitude, reflection in *x*-axis, period $\frac{\pi}{5}$, centre 0, phase shift $\frac{\pi}{20}$ to the right.

Question 14

a x = 0°, 180°, 360°

1st quadrant

$$\tan\left(x+45°\right)=1$$
$$x+45° = \tan^{-1}1 = 45°$$

and

$$x+45° = 360° + \tan^{-1}1 = 405^0$$
$$x = 360^0$$

and in 3rd quadrant

$$x+45° = 180° + \tan^{-1}1 = 225°$$
$$x = 180°$$

b

$$\sqrt{2}\cos\left(x-20°\right)+1=0$$

$$\cos\left(x-20°\right)=-\frac{1}{\sqrt{2}}$$

2nd quadrant

$$x-20°=180°-\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$=200°-45°$$

$$x=155°$$

3rd quadrant

$$x-20°=180°+\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$=200°+45°$$

$$x=245°$$

c

$$3\sin[2(x+10°)]-2=0$$
$$\sin[2(x+10°)]=\frac{2}{3}$$

1st quadrant

$$2(x+10°)=\sin^{-1}\left(\frac{2}{3}\right)=41°49'$$
$$2x=21°49'$$
$$x=10°54'$$

and

$$2(x+10°)=360°+\sin^{-1}\left(\frac{2}{3}\right)$$
$$2x=381°49'$$
$$x=190°54'$$

2nd quadrant

$$2(x+10°)=180°-\sin^{-1}\left(\frac{2}{3}\right)$$
$$2x=118°21'$$
$$x=59°6'$$

and

$$2(x+10°)=360°+180°-\sin^{-1}\left(\frac{2}{3}\right)$$
$$2x=478°49'$$
$$x=239°6'$$

Question 15

a $5 \tan 2x = -5$

$\tan 2x = -1$

For x in [–180°, 180°], then $2x$ in [–360°, 360°].

tan is negative in 2nd and 4th quadrants

$2x = -180° - 45°, -45°, 180° - 45°, 360° - 45°$

$= -225°, -45°, 135°, 315°$

$x = -112.5°, -22.5°, 67.5°, 157.5°$

b $\cos [3(x - 30°)] + 1 = 0$

$\cos [3(x - 30°)] = -1$

For x in [–180°, 180°], then $[3(x - 30°)]$ in [–540 – 30°, 540° – 30°] = [–570°, 510°].

$3(x - 30°) = -540°, -180°, 180°$

$x - 30° = -180°, -60°, 60°$

$x = -150°, -30°, 90°$

Challenge exercise 3

Question 1

a $y = k\sin(a[x+b]) + c$, amplitude k, period $\frac{2\pi}{a}$, centre c, phase shift b.

$$y = 2\cos\left(2x - \frac{\pi}{2}\right) = 2\cos\left(2\left[x - \frac{\pi}{4}\right]\right)$$

amplitude 2, period $\frac{2\pi}{2} = \pi$, centre 0, phase shift $\frac{\pi}{4}$ units to the right.

b

$$2\cos(2x - \frac{\pi}{2}) = \sqrt{3}$$

$$\cos(2x - \frac{\pi}{2}) = \frac{\sqrt{3}}{2}$$

1st quadrant

$$\cos\left(2x - \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}$$

$$2x - \frac{\pi}{2} = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

$$x = \frac{\pi}{3}$$

and

$$2x - \frac{\pi}{2} = 2\pi + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$x = \frac{4\pi}{3}$$

4th quadrant

$$2x-\frac{\pi}{2}=2\pi-\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$x=\frac{7\pi}{6}$$

and $x = 2\pi - \frac{7\pi}{6}=\frac{\pi}{6}$

Question 2

a $y=k\cos(a[x+b])+c$, amplitude *k*, period $\frac{2\pi}{a}$, centre *c*, phase shift *b*.

Amplitude is 8, so $k=8$.

Centre at 4, so $c=4$.

Period $\frac{2\pi}{a}=2\pi \Rightarrow a=1$.

Assume phase shift is 0, so $b=0$.

$$y=8\cos(1[x+0])+4=8\cos x+4$$

b $y=k\sin(a[x+b])+c$, amplitude *k*, period $\frac{2\pi}{a}$, centre *c*, phase shift *b*.

Amplitude is 2, so $k=2$.

Centre at 3, so $c=3$.

Period $\frac{2\pi}{a}=\frac{\pi}{4}\Rightarrow a=8$.

Phase shift is $b=-\frac{\pi}{3}$.

$$y=2\sin\left(8\left[x-\frac{\pi}{3}\right]\right)+3$$

c $y = \tan(a[x+b]) + c$, period $\frac{\pi}{a}$, centre *c*, phase shift *b*.

Assume centre at 0, so $c = 0$.

Period $\frac{\pi}{a} = 2\pi \Rightarrow a = \frac{1}{2}$.

Phase shift is $b = \frac{\pi}{2}$.

$$y = \tan\left(\frac{1}{2}\left[x + \frac{\pi}{2}\right]\right) + 0 = \tan\left(\frac{1}{2}\left[x + \frac{\pi}{2}\right]\right)$$

Question 3

The graph of $y = 3 \sec 2x$ can be obtained by a vertical dilation with scale factor 2 of the graph of $y = \sec x$ followed by a horizontal dilation with scale factor $\frac{1}{2}$.

The period is $\frac{2\pi}{2} = \pi$.

The y-intercept is $y = 3 \sec 0 = 3$.

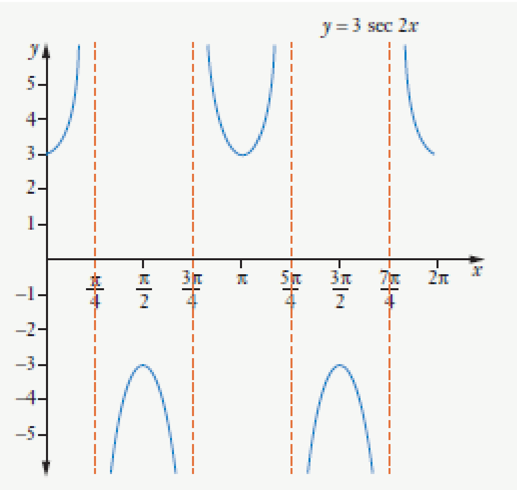
There are no x-intercepts because there are no x values that satisfy $\frac{3}{\cos 2x} = 0$.

$\cos 2x$ has maximum values at $x = 0, \pi, 2\pi, 3\pi, \ldots$ so $y = \frac{3}{\cos 2x}$ has local minimums at these values.

Similarly, $\cos 2x$ has minimum values at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ so $y = \frac{3}{\cos 2x}$ has local maximums at these values.

$y = 3\sec 2x = \frac{3}{\cos 2x}$ has vertical asymptotes where $\cos 2x = 0$.

This is when $2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$



Question 4

$$\operatorname{cosec} 2x = \sqrt{2}$$

$$\frac{1}{\sin 2x} = \sqrt{2} \Rightarrow \sin 2x = \frac{1}{\sqrt{2}}$$

1st quadrant

$$2x = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$x = \frac{\pi}{8}$$

and

$$2x = 2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$$

$$x = \frac{9\pi}{8}$$

2nd quadrant

$$2x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$x = \frac{3\pi}{8}$$

and

$$2x = 2\pi + \pi - \frac{\pi}{4} = \frac{9\pi}{4}$$

$$x = \frac{9\pi}{8}$$

$$\therefore x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$$

Question 5

a $y = \cos x - 5$, vertical translation

$y = \cos\left(x - \frac{\pi}{6}\right) - 5$, horizontal translation.

$y = 4\left[\cos\left(x - \frac{\pi}{6}\right) - 5\right]$, vertical dilation.

$y = 4\left[\cos\left(\frac{1}{3}\left\{x - \frac{\pi}{6}\right\}\right) - 5\right] = 4\cos\left[\frac{1}{3}\left(x - \frac{\pi}{6}\right)\right] - 20$, horizontal dilation.

b $y = k\cos(a[x+b]) + c$, amplitude k, period $\frac{2\pi}{a}$, centre c, phase shift b.

$y = 4\cos\left(\frac{1}{3}\left[x - \frac{\pi}{6}\right]\right) - 20$

Amplitude 4, period $2\pi \div \frac{1}{3} = 6\pi$, centre –20, phase shift $\frac{\pi}{6}$ to the right.

MATHS IN FOCUS 12 MATHEMATICS ADVANCED

WORKED SOLUTIONS

Chapter 4: Further differentiation

Exercise 4.01 Differentiation review

Question 1

a $\frac{d}{dx}\left(3x^4-2x^3+7x-4\right)=12x^3-6x^2+7$

b $\frac{d}{dx}(2x+5)=2$

c $\frac{d}{dx}\left(6x^2-3x-2\right)=12x-3$

Question 2

$f(x) = 4x^5 + 9x^2$

$f'(x) = 20x^4 + 18x$

Question 3

$x=2\pi t^3-3t^2+1$

$\frac{dx}{dt}=6\pi t^2-6t$

Question 4

$f(x) = 8x^3 + 5x - 2$

$f'(x) = 24x^2 + 5$

$f'(-2) = 24(-2)^2 + 5 = 101$

Question 5

a $\frac{d}{dx}\left(x^{-5}\right) = -5x^{-6}$

b $\frac{d}{dx}\left(x^{\frac{2}{3}}\right) = \frac{2}{3}x^{-\frac{1}{3}}$

c

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}\left(x^{-2}\right)$$
$$= -2x^{-3}$$
$$= \frac{-2}{x^3}$$

d

$$\frac{d}{dx}\left(\sqrt[4]{x}\right) = \frac{d}{dx}\left(x^{\frac{1}{4}}\right)$$
$$= \frac{1}{4}x^{-\frac{3}{4}}$$
$$= \frac{1}{4\sqrt[4]{x^3}}$$

e

$$\frac{d}{dx}\left(-\frac{5}{x^4}\right) = \frac{d}{dx}\left(-5^{-4}\right)$$
$$= 20x^{-5}$$
$$= \frac{20}{x^5}$$

Question 6

$y=\sqrt[3]{x}$

$y=x^{\frac{1}{3}}$

$y'=\frac{1}{3}x^{-\frac{2}{3}}$

$y'(8)=\frac{1}{3}(8)^{-\frac{2}{3}}=\frac{1}{12}$

Question 7

a

$$\frac{d}{dx}\left[(3x-1)^7\right]$$
$$=7(3x-1)^6\times 3$$
$$=21(3x-1)^6$$

b

$$\frac{d}{dx}\left[\left(x^2-x+2\right)^3\right]$$
$$=3\left(x^2-x+2\right)^2\times(2x-1)$$
$$=3(2x-1)\left(x^2-x+2\right)^2$$

c

$$\frac{d}{dx}\left(\sqrt{7x-2}\right)=\frac{d}{dx}\left[(7x-2)^{\frac{1}{2}}\right]$$
$$=\frac{1}{2}(7x-2)^{-\frac{1}{2}}\times 7$$
$$=\frac{7}{2(7x-2)^{\frac{1}{2}}}$$
$$=\frac{7}{2\sqrt{7x-2}}$$

d

$$\frac{d}{dx}\left(\frac{1}{3x-2}\right)=\frac{d}{dx}\left[\left(3x-2\right)^{-1}\right]$$
$$=-1\left(3x-2\right)^{-2}\times 3$$
$$=\frac{-3}{\left(3x-2\right)^{2}}$$

e

$$\frac{d}{dx}\left(\sqrt[3]{x^2-3}\right)=\frac{d}{dx}\left[\left(x^2-3\right)^{\frac{1}{3}}\right]$$
$$=\frac{1}{3}\left(x^2-3\right)^{-\frac{2}{3}}\times 2x$$
$$=\frac{2x}{3\left(x^2-3\right)^{\frac{2}{3}}}$$
$$=\frac{2x}{3\sqrt[3]{\left(x^2-3\right)^{2}}}$$

Question 8

a

$$\frac{d}{dx}\left[x^2\left(x+4\right)\right]$$
$$=x^2\times\left(1\right)+2x\left(x+4\right)$$
$$=x^2+2x^2+8x$$
$$=3x^2+8x$$

b

$$\frac{d}{dx}\left[\left(2x-1\right)\left(6x+5\right)\right]$$
$$=\left(2x-1\right)\times\left(6\right)+\left(2\right)\times\left(6x+5\right)$$
$$=12x-6+12x+10$$
$$=24x+4$$

c

$$\frac{d}{dx}\left[4x\left(x^2+1\right)\right]$$
$$=4x\times(2x)+(4)\times\left(x^2+1\right)$$
$$=8x^2+4x^2+4$$
$$=12x^2+4$$

d

$$\frac{d}{dx}\left[(4x+3)\left(x^2-1\right)^2\right]$$
$$=(4x+3)\times 2\left(x^2-1\right)\times 2x+(4)\times\left(x^2-1\right)^2$$
$$=4x(4x+3)\left(x^2-1\right)+4\left(x^2-1\right)^2$$
$$=4\left(x^2-1\right)\left[x(4x+3)+\left(x^2-1\right)\right]$$
$$=4\left(x^2-1\right)\left[4x^2+3x+x^2-1\right]$$
$$=4\left(x^2-1\right)\left(5x^2+3x-1\right)$$

e

$$\frac{d}{dx}\left(2x^3\sqrt{x+1}\right)=\frac{d}{dx}\left[2x^3(x+1)^{\frac{1}{2}}\right]$$
$$=2x^3\times\frac{1}{2}(x+1)^{-\frac{1}{2}}+6x^2(x+1)^{\frac{1}{2}}$$
$$=\frac{x^3}{(x+1)^{\frac{1}{2}}}+6x^2(x+1)^{\frac{1}{2}}$$
$$=\frac{x^3}{\sqrt{x+1}}+6x^2\sqrt{x+1}$$
$$=\frac{x^3}{\sqrt{x+1}}+\frac{6x^2(x+1)}{\sqrt{x+1}}$$
$$=\frac{x^3+6x^3+6x^2}{\sqrt{x+1}}$$
$$=\frac{7x^3+6x^2}{\sqrt{x+1}}$$
$$=\frac{x^2(7x+6)}{\sqrt{x+1}}$$

Question 9

a $$\frac{d}{dx}\left(\frac{2x+3}{x-5}\right)$$

Use the quotient rule. $u = 2x + 3$, $u' = 2$; $v = x - 5$, $v' = 1$

$$\frac{dy}{dx}=\frac{u'v-v'u}{v^2}=\frac{2(x-5)-1(2x+3)}{(x-5)^2}$$

$$=\frac{2x-10-2x-3}{(x-5)^2}$$

$$=\frac{-13}{(x-5)^2}$$

b

$$\frac{d}{dx}\left(\frac{x^3}{4x-7}\right)$$

$$=\frac{3x^2(4x-7)-4(x^3)}{(4x-7)^2}$$

$$=\frac{12x^3-21x^2-4x^3}{(4x-7)^2}$$

$$=\frac{8x^3-21x^2}{(4x-7)^2}$$

$$=\frac{x^2(8x-21)}{(4x-7)^2}$$

c

$$\frac{d}{dx}\left(\frac{x^2+3}{2x-3}\right)$$

$$=\frac{2x(2x-3)-2\left(x^2+3\right)}{(2x-3)^2}$$

$$=\frac{4x^2-6x-2x^2-6}{(2x-3)^2}$$

$$=\frac{2x^2-6x-6}{(2x-3)^2}$$

$$==\frac{2\left(x^2-3x-3\right)}{(2x-3)^2}$$

d

$$\frac{d}{dx}\left[\frac{3x+1}{(2x+9)^2}\right]$$

$$=\frac{3(2x+9)^2-2(2x+9)\times 2\times(3x+1)}{(2x+9)^4}$$

$$=\frac{3(2x+9)^2-4(3x+1)(2x+9)}{(2x+9)^4}$$

$$=\frac{3(2x+9)-4(3x+1)}{(2x+9)^3}$$

$$=\frac{6x+27-12x-4}{(2x+9)^3}$$

$$=\frac{-6x+23}{(2x+9)^3}$$

e

$$\frac{d}{dx}\left(\frac{3x+4}{\sqrt{2x-1}}\right)=\frac{d}{dx}\left[\frac{3x+4}{(2x-1)^{\frac{1}{2}}}\right]$$

$$=\frac{3(2x-1)^{\frac{1}{2}}-\left[\frac{1}{2}(2x-1)^{-\frac{1}{2}}\times 2\right](3x+4)}{(2x-1)}$$

$$=\frac{3\sqrt{2x-1}-\frac{3x+4}{\sqrt{2x-1}}}{2x-1}$$

$$=\frac{3(2x-1)-(3x+4)}{\sqrt{(2x-1)^3}}$$

$$=\frac{6x-3-3x-4}{\sqrt{(2x-1)^3}}$$

$$=\frac{3x-7}{\sqrt{(2x-1)^3}}$$

Question 10

a

$$y'=\frac{d}{dx}\left(x^2-2x+5\right)=2x-2$$

$$y'(-2)=2(-2)-2=-6$$

b

$$f'(x)=\frac{d}{dx}\left(x^3-3\right)=3x^2$$

$$f'(-1)=3(-1)^2=3$$

Question 11

a

$$f'(x)=\frac{d}{dx}\left(3x^4+x^2-2\right)=12x^3+2x$$

$$f'(-1)=12(-1)^3+2(-1)=-14$$

$$m_1 m_2=-1$$

$$-14\times m_2=-1$$

$$m_2=\frac{1}{14}$$

b

$$y'(x)=\frac{d}{dx}\left(x^2+x-3\right)=2x+1$$

$$y'(-3)=2(-3)+1=-5$$

$$m_1 m_2=-1$$

$$-5\times m_2=-1$$

$$m_2=\frac{1}{5}$$

Question 12

a

$$y'(x)=\frac{d}{dx}\left(2x^2-5x-6\right)=4x-5$$
$$y'(3)=4(3)-5=7$$
$$m=7$$
$$y-y_1=m\left(x-x_1\right)$$
$$y-(-3)=7(x-3)$$
$$y+3=7x-21$$
$$7x-y-24=0$$

b

$$y(2)=5(2)^3-2(2)^2-(2)=30$$
$$y'=\frac{d}{dx}\left(5x^3-2x^2-x\right)=15x^2-4x-1$$
$$y'(2)=15(2)^2-4(2)-1=51$$
$$m=51$$
$$y-y_1=m\left(x-x_1\right)$$
$$y-30=51(x-2)$$
$$y-30=51x-102$$
$$51x-y-72=0$$

Question 13

a

$$f'(x)=\frac{d}{dx}\left(x^3+2x^2-3x-5\right)=3x^2+4x-3$$
$$f'(-1)=3(-1)^2+4(-1)-3=-4$$
$$m_1=-4$$
$$m_1 m_2=-1$$
$$-4\times m_2=-1$$
$$m_2=\frac{1}{4}$$
$$y-y_1=m(x-x_1)$$
$$y-(-1)=\frac{1}{4}(x-(-1))$$
$$y+1=\frac{1}{4}x+\frac{1}{4}$$
$$4y+4=x+1$$
$$x-4y-3=0$$

b

$$y(3)=(3)^2-3(3)+1=1$$
$$y'=\frac{d}{dx}\left(x^2-3x+1\right)=2x-3$$
$$y'(3)=2(3)-3=3$$
$$m_1=3$$
$$m_1 m_2=-1$$
$$3\times m_2=-1$$
$$m_2=-\frac{1}{3}$$
$$y-y_1=m(x-x_1)$$
$$y-1=-\frac{1}{3}(x-3)$$
$$3y-3=-(x-3)$$
$$3y-3=-x+3$$
$$x+3y-6=0$$

Question 14

$y = x^2 - 8x + 15$

$y' = 2x - 8$

$0 = 2x - 8$

$2x = 8$

$x = 4$

Question 15.

$y = x^3 - 2$
$y' = 3x^2$
$12 = 3x^2$
$4 = x^2$
$x = \pm\sqrt{4}$
$x = \pm 2$
$y(2) = (2)^3 - 2 = 6,\ (2, 6)$
$y(-2) = (-2)^3 - 2 = -10,\ (-2, -10)$

Question 16

$3x + y - 4 = 0$

$y = -3x + 4$

$m = -3$

$f(x) = x^2 + x - 4$

$f'(x) = 2x + 1$

$-3 = 2x + 1$

$-4 = 2x$

$x = -2$

$f(-2) = (-2)^2 + (-2) - 4 = -2$, $(-2, -2)$

$y - y_1 = m(x - x_1)$

$y - (-2) = -3([x - (-2)]$

$y + 2 = -3x - 6$

$3x + y + 8 = 0$

Question 17

$y = \sqrt{x}$

$y = x^{\frac{1}{2}}$

$y' = \frac{1}{2}x^{-\frac{1}{2}} == \frac{1}{2\sqrt{x}}$

$\frac{1}{4} = \frac{1}{2\sqrt{x}}$

$4 = 2\sqrt{x}$

$2 = \sqrt{x}$

$4 = x$

$y = \sqrt{x}$

$y = \sqrt{4} = 2$

$(x, y) = (4, 2)$

Question 18

a

$$y=\frac{5x-3}{4x+1}$$
$$y(0)=\frac{5(0)-3}{4(0)+1}=-3$$
$$y'=\frac{5(4x+1)-4(5x-3)}{(4x+1)^2}$$
$$=\frac{20x+5-20x+12}{(4x+1)^2}$$
$$=\frac{17}{(4x+1)^2}$$
$$y'(0)=\frac{17}{\left[4(0)+1\right]^2}=17$$
$$m_1=17$$
$$(x,y)=(0,-3)$$
$$y-y_1=m(x-x_1)$$
$$y-(-3)=17(x-0)$$
$$y+3=17x$$
$$17x-y-3=0$$

b

$$m_1\,m_2=-1$$
$$17\times m_2=-1$$
$$m_2=-\frac{1}{17}$$
$$(x,\ y)=(0,-3)$$
$$y-y_1=m(x-x_1)$$
$$y-(-3)=-\frac{1}{17}(x-0)$$
$$y+3=-\frac{1}{17}x$$
$$17y+51=-x$$
$$x+17y+51=0$$

Question 19

a

$$Q = 3t^2 + 8$$

$$\frac{dQ}{dt} = 6t$$

b

$$Q = \frac{2}{t-3}$$

$$\frac{dQ}{dt} = \frac{0(t-3)-1(2)}{(t-3)^2} = \frac{-2}{(t-3)^2}$$

c

$$Q = \sqrt[3]{2x+3} = (x+3)^{\frac{1}{3}}$$

$$\frac{dQ}{dt} = \frac{1}{3}(2x+3)^{-\frac{2}{3}} \times 2$$

$$= \frac{2}{3\sqrt[3]{(2x+3)^2}}$$

Question 20

a **i**

$$M = t^2 + 3t + 4$$
$$M(2) = (2)^2 + 3(2) + 4 = 14$$
$$M(5) = (5)^2 + 3(5) + 4 = 44$$
$$\frac{44-14}{5-2} = 10 \text{ kg/s}$$

ii

$$M = t^2 + 3t + 4$$
$$M(6) = (6)^2 + 3(6) + 4 = 58$$
$$M(8) = (8)^2 + 3(8) + 4 = 92$$
$$\frac{92-58}{8-6} = 17 \text{ kg/s}$$

b **i**

$$M = t^2 + 3t + 4$$
$$M' = 2t + 3$$
$$M'(5) = 2(5) + 3 = 13 \text{ kg/s}$$

ii

$$M = t^2 + 3t + 4$$
$$M' = 2t + 3$$
$$M'(60) = 2(60) + 3 = 123 \text{ kg/s}$$

Question 21

$$P = \frac{k}{V}$$
$$P' = \frac{0-k}{V^2}$$
$$P' = \frac{-250}{V^2}$$
$$P'(10.7) = \frac{-250}{10.7^2}$$
$$P' = -2.18 \text{ Pa/m}^3$$

Question 22

a **i** $h = 4t - 2t^2$

$h\,(1) = 4\,(1) - 2\,(1)^2 = 2$ m

ii $h = 4t - 2t^2$

$h\,(1.5) = 4\,(1.5) - 2\,(12.5)^2 = 1.5$ m

b $h = 4t - 2t^2$

$0 = 4t - 2t^2$

$0 = 4 - 2t$

$2t = 4$

$t = 2$ s

c **i** $h = 4t - 2t^2$

$h' = 4 - 4t$

$h'\,(0.5) = 4 - 4\,(0.5) = 2$ m/s

ii $h = 4t - 2t^2$

$h' = 4 - 4t$

$h'\,(1) = 4 - 4\,(1) = 0$ m/s

iii $h = 4t - 2t^2$

$h' = 4 - 4t$

$h'\,(2) = 4 - 4\,(2) = -4$ m/s

Exercise 4.02 Derivative of exponential functions

Question 1

a $\frac{d}{dx}\left(e^{7x}\right)=7e^{7x}$

b $\frac{d}{dx}\left(e^{-x}\right)=-e^{-x}$

c $\frac{d}{dx}\left(e^{6x-2}\right)=6e^{6x-2}$

d $\frac{d}{dx}\left(e^{x^2+1}\right)=2xe^{x^2+1}$

e $\frac{d}{dx}\left(e^{x^3+5x+7}\right)=\left(3x^2+5\right)e^{x^3+5x+7}$

f $\frac{d}{dx}\left(e^{5x}\right)=5e^{5x}$

g $\frac{d}{dx}\left(e^{-2x}\right)=-2e^{-2x}$

h $\frac{d}{dx}\left(e^{10x}\right)=10e^{10x}$

i $\frac{d}{dx}\left(e^{2x}+x\right)=2e^{2x}+1$

j $\frac{d}{dx}\left(x^2+2x+e^{1-x}\right)=2x+2-e^{1-x}$

k

$$\frac{d}{dx}\left[\left(x+e^{4x}\right)^5\right]$$
$$=5\left(x+e^{4x}\right)^4\times\left(1+4e^{4x}\right)$$
$$=5\left(1+4e^{4x}\right)\left(x+e^{4x}\right)^4$$

l

$$\frac{d}{dx}\left(xe^{2x}\right)$$
$$=x\times 2e^{2x}+1\times e^{2x}$$
$$=2xe^{2x}+e^{2x}$$
$$=\left(2x+1\right)e^{2x}$$

m

$$\frac{d}{dx}\left(\frac{e^{3x}}{x^2}\right)$$
$$=\frac{3e^{3x}\times x^2-2x\times e^{3x}}{\left(x^2\right)^2}$$
$$=\frac{3x^2e^{3x}-2xe^{3x}}{x^4}$$
$$=\frac{3xe^{3x}-2e^{3x}}{x^3}$$
$$=\frac{\left(3x-2\right)e^{3x}}{x^3}$$

n

$$\frac{d}{dx}\left(x^3e^{5x}\right)$$
$$=3x^2\times e^{5x}+x^3\times 5e^{5x}$$
$$=3x^2e^{5x}+5x^3e^{5x}$$
$$=x^2\left(3e^{5x}+5xe^{5x}\right)$$
$$=x^2\left(3+5x\right)e^{5x}$$

o

$$\frac{d}{dx}\left(\frac{e^{2x+1}}{2x+5}\right)$$
$$=\frac{2e^{2x+1}(2x+5)-(2)e^{2x+1}}{(2x+5)^2}$$
$$=\frac{2e^{2x+1}(2x+5)-2e^{2x+1}}{(2x+5)^2}$$
$$=\frac{2e^{2x+1}(2x+5-1)}{(2x+5)^2}$$
$$=\frac{2e^{2x+1}(2x+4)}{(2x+5)^2}$$
$$=\frac{4e^{2x+1}(x+2)}{(2x+5)^2}$$

Question 2

$f(x) = e^{3x-2}$

$f'(x) = 3e^{3x-2}$

$f'(1) = 3e^{3(1)-2} = 3e$

Question 3

a $\frac{d}{dx}\left(3^x\right) = 3^x \ln 3$

b $\frac{d}{dx}\left(10^x\right) = 10^x \ln 10$

$2 = e^{\ln e}$

$2^{3x-4} = \left(e^{\ln 2}\right)^{3x-4}$

c $\frac{d}{dx}\left(e^{(x-4)\ln 2}\right)$

$= \left(3\ln 2\right)e^{(x-4)\ln 2}$

$= \left(3\ln 2\right)2^{3x-4}$

Question 4

$y = e^{5x}$

$y' = 5e^{5x}$

$y'(0) = 5e^{5(0)} = 5$

Question 5

$y = e^{2x} - 3x$

$y' = 2e^{2x} - 3$

$y'(0) = 2e^{2(0)} - 3 = -1$

$y - y_1 = m(x - \mathrm{x}_1)$

$y - 1 = -1(x - 0)$

$y - 1 = -x$

$x + y - 1 = 0$

Question 6

a $y = e^{3x}$

$y' = 3e^{3x}$

$y'(1) = 3e^{3(1)} = 3e^3$

b

$$m_1 m_2 = -1$$
$$3e^3 \times m_2 = -1$$
$$m_2 = -\frac{1}{3e^3}$$

Question 7

a

$$y = e^{x^2}$$
$$y' = 2x e^{x^2}$$
$$y'(1) = 2(1)e^{(1)^2} = 2e$$
$$m_1 = 2e$$
$$y - y_1 = m(x - x_1)$$
$$y - e = 2e(x - 1)$$
$$y - e = 2ex - 2e$$
$$2ex - y - e = 0$$

b

$$m_1 m_2 = -1$$
$$2e \times m_2 = -1$$
$$m_2 = -\frac{1}{2e}$$
$$y - y_1 = m(x - x_1)$$
$$y - e = -\frac{1}{2e}(x - 1)$$
$$2ey - 2e^2 = -x + 1$$
$$x + 2ey - 2e^2 - 1 = 0$$

Question 8

$$y = 4^{x+1}$$
$$y' = 4^{x+1}\ln 4$$
$$y'(0) = 4\ln 4$$
$$y - y_1 = m(x - x_1)$$
$$y - 4 = 4\ln 4(x - 0)$$
$$y - 4 = 4x\ln 4$$
$$4x\ln 4 - y + 4 = 0$$

Question 9

a **i** $P = 24\,500\, e^{0.038t}$

$P(5) = 24\,500\, e^{0.038(5)} = 29\,627$

ii $P = 24\,500\, e^{0.038t}$

$P(10) = 24\,500\, e^{0.038(10)} = 35\,826$

b **i** $P = 24\,500\, e^{0.038t}$

$P(1) = 24\,500\, e^{0.038(1)} = 25\,449$

$P(5) = 24\,500\, e^{0.038(5)} = 29\,627$

$$\frac{29\,627 - 25\,449}{5 - 1} = 1045 \text{ people/year}$$

ii $P = 24\,500\, e^{0.038t}$

$P(5) = 24\,500\, e^{0.038(5)} = 29\,627$

$P(10) = 24\,500\, e^{0.038(10)} = 35\,826$

$$\frac{35\,826 - 29\,627}{10 - 5} = 1240 \text{ people/year}$$

c **i** $P = 24\,500\,e^{0.038t}$

$P' = 931e^{0.038t}$

$P'(5) = 931e^{0.038(5)} = 1126$ people/year

ii $P = 24\,500\,e^{0.038t}$

$P' = 931e^{0.038t}$

$P'(10) = 931e^{0.038(10)} = 1361$ people/year

Question 10

a $s = 10e^{2t} - 5t$

$s(1) = 10e^{2(1)} - 5(1) = 68.9$

$s(5) = 10e^{2(5)} - 5(5) = 220\,239.7$

$$\frac{220\,239.7 - 69.9}{5 - 1} = 55\,042 \text{ cm/minute}$$

b **i** $s = 10e^{2t} - 5t$

$s' = 20e^{2t} - 5$

$s'(1) = 20e^{2(1)} - 5 = 142.8$ cm/minute

ii $s = 10e^{2t} - 5t$

$s' = 20e^{2t} - 5$

$s'(2) = 20e^{2(2)} - 5 = 1087$ cm/minute

iii $s = 10e^{2t} - 5t$

$s' = 20e^{2t} - 5$

$s'(8) = 20e^{2(8)} - 5 = 177\,722\,205.4$ cm/minute

Question 11

a $M = 20e^{-0.021t}$

$M(0) = 20e^{-0.021(0)} = 20$ g

b $M = 20e^{-0.021t}$

$M(50) = 20e^{-0.021(50)} = 7$ g

c $M = 20e^{-0.021t}$

$M(50) = 20e^{-0.021(50)} = 7$ g

$M(100) = 20e^{-0.021(100)} = 2.4$ g

$\frac{2.4-7}{100-50} = -0.091$ g/year

d **i** $M = 20e^{-0.021t}$

$M' = -0.42e^{-0.021t}$

$M'(50) = -0.42e^{-0.021(50)} = -0.147$ g/year

ii $M = 20e^{-0.021t}$

$M' = -0.42e^{-0.021t}$

$M'(100) = -0.42e^{-0.021(100)} = -0.051$ g/year

iii $M = 20e^{-0.021t}$

$M' = -0.42e^{-0.021t}$

$M'(200) = -0.42e^{-0.021(200)} = -0.0063$ g/year

Question 12

a $x = 3e^{2t}$

$x(5) = 3e^{2(5)} = 66\ 079.4$ cm

b $x = 3e^{2t}$

$x' = 6e^{2t}$

$x'(5) = 6e^{2(5)} = 132\ 159$ cm/s

Exercise 4.03 Derivative of logarithmic functions

Question 1.

a $\frac{d}{dx}\left(x+\log_e x\right)=1+\frac{1}{x}$

b $\frac{d}{dx}\left(1-\log_e 3x\right)=0-\frac{1}{3x}\times 3=-\frac{1}{x}$

c $\frac{d}{dx}\left[\ln\left(3x+1\right)\right]=\frac{1}{3x+1}\times 3=\frac{3}{3x+1}$

d $\frac{d}{dx}\left[\log_e\left(x^2-4\right)\right]=\frac{1}{x^2-4}\times 2x=\frac{2x}{x^2-4}$

e

$$\frac{d}{dx}\left[\ln\left(5x^3+3x-9\right)\right]$$
$$=\frac{1}{5x^3+3x-9}\times\left(15x^2+3\right)$$
$$=\frac{15x^2+3}{5x^3+3x-9}$$

f

$$\frac{d}{dx}\left[\log_e\left(5x+1\right)+x^2\right]$$
$$=\frac{1}{5x+1}\times 5+2x$$
$$=\frac{5}{5x+1}+2x$$
$$=\frac{5+10x^2+2x}{5x+1}$$

g $\dfrac{d}{dx}\left(3x^2+5x+5+\ln 4x\right)=6x+5+\dfrac{1}{x}$

h

$$\frac{d}{dx}\left[\log_e\left(8x-9\right)+2\right]$$
$$=\frac{1}{8x-9}\times 8$$
$$=\frac{8}{8x-9}$$

i

$$\frac{d}{dx}\left[\log_e\left(2x+4\right)\left(3x-1\right)\right]=\frac{d}{dx}\left[\log_e\left(2x+4\right)+\log_e\left(3x-1\right)\right]$$
$$=\frac{1}{2x+4}\times 2+\frac{1}{3x-1}\times 3$$
$$=\frac{1}{x+2}+\frac{3}{3x-1}$$
$$=\frac{1\left(3x-1\right)+3\left(x+2\right)}{\left(x+2\right)\left(3x-1\right)}$$
$$=\frac{3x-1+3x+6}{\left(x+2\right)\left(3x-1\right)}$$
$$=\frac{6x+5}{\left(x+2\right)\left(3x-1\right)}$$

j

$$\frac{d}{dx}\left(\log_e \frac{4x+1}{2x-7}\right)=\frac{d}{dx}\left[\log_e(4x+1)-\log_e(2x-7)\right]$$

$$=\frac{1}{4x+1}\times 4-\frac{1}{2x-7}\times 2$$

$$=\frac{4}{4x+1}-\frac{2}{2x-7}$$

$$=\frac{4(2x-7)-2(4x+1)}{(4x+1)(2x-7)}$$

$$=\frac{8x-28-8x-2}{(4x+1)(2x-7)}$$

$$=\frac{-30}{(4x+1)(2x-7)}$$

k

$$\frac{d}{dx}(1+\ln x)^5$$

$$=5(1+\ln x)^4\times\frac{1}{x}$$

$$=\frac{5}{x}(1+\ln x)^4$$

l

$$\frac{d}{dx}(\ln x-x)^9$$

$$=9(\ln x-x)^8\times\left(\frac{1}{x}-1\right)$$

$$=9\left(\frac{1}{x}-1\right)(\ln x-x)^8$$

m

$$\frac{d}{dx}(\ln x)^4$$

$$=4(\ln x)^3\times\frac{1}{x}$$

$$=\frac{4}{x}(\ln x)^3$$

n

$$\frac{d}{dx}\left(x^2+\ln x\right)^6$$
$$=6\left(x^2+\ln x\right)^5\times\left(2x+\frac{1}{x}\right)$$
$$=6\left(2x+\frac{1}{x}\right)\left(x^2+\ln x\right)^5$$

o

$$\frac{d}{dx}\left(x\ln x\right)$$
$$=1\times\ln x+x\times\frac{1}{x}$$
$$=1+\ln x$$

p

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right)$$
$$=\frac{\frac{1}{x}\times x-1\times\ln x}{x^2}$$
$$=\frac{1-\ln x}{x^2}$$

q

$$\frac{d}{dx}\left[\left(2x+1\right)\ln x\right]$$
$$=2\times\ln x+\left(2x+1\right)\times\frac{1}{x}$$
$$=\frac{2x\ln x+2x+1}{x}$$

r

$$\frac{d}{dx}\left[x^3\ln\left(x+1\right)\right]$$
$$=3x^2\times\ln\left(x+1\right)+x^3\times\frac{1}{x+1}$$
$$=3x^2\ln\left(x+1\right)+\frac{x^3}{x+1}$$

s

$$\frac{d}{dx}\ln\left(\ln x\right)$$
$$=\frac{1}{\ln x}\times\frac{1}{x}$$
$$=\frac{1}{x\ln x}$$

t

$$\frac{d}{dx}\left(\frac{\ln x}{x-2}\right)$$
$$=\frac{\frac{1}{x}\times\left(x-2\right)-1\times\ln x}{\left(x-2\right)^2}$$
$$=\frac{\frac{\left(x-2\right)}{x}-\frac{x\ln x}{x}}{\left(x-2\right)^2}$$
$$=\frac{x-2-x\ln x}{x\left(x-2\right)^2}$$

u

$$\frac{d}{dx}\left(\frac{e^{2x}}{\ln x}\right)$$
$$=\frac{2e^{2x}\times\ln x-e^{2x}\times\frac{1}{x}}{\left(\ln x\right)^2}$$
$$=\frac{2xe^{2x}\ \ln x-e^{2x}}{x\left(\ln x\right)^2}$$
$$=\frac{e^{2x}\left(2x\ln x-1\right)}{x\left(\ln x\right)^2}$$

v

$$\frac{d}{dx}\left(e^x \ln x\right)$$
$$= e^x \times \frac{1}{x} + e^x \times \ln x$$
$$= \frac{e^x}{x} + e^x \ln x$$
$$= e^x \left(\frac{1}{x} + \ln x\right)$$

w

$$\frac{d}{dx}\left[5\left(\ln x\right)^2\right]$$
$$= 5 \times 2 \times \left(\ln x\right) \times \frac{1}{x}$$
$$= \frac{10 \ln x}{x}$$

Question 2

$$f(x) = \log_e \sqrt{2-x}$$
$$f(x) = \log_e (2-x)^{\frac{1}{2}}$$
$$f(x) = \frac{\log_e (2-x)}{2}$$
$$f'(x) = \frac{1}{2} \times \frac{1}{2-x} \times -1$$
$$f'(x) = \frac{-1}{2(2-x)}$$
$$f'(x) = \frac{1}{2x-4}$$
$$f'(1) = \frac{1}{2(1)-4} = -\frac{1}{2}$$

Question 3

$$\frac{d}{dx}(\log_{10} x)=\frac{d}{dx}\left(\frac{\ln x}{\ln 10}\right)$$

$$=\frac{1}{\ln 10}\times\frac{1}{x}$$

$$=\frac{1}{x\ln 10}$$

Question 4

$$y=\ln x$$

$$y'=\frac{1}{x}$$

$$y'(2)=\frac{1}{2}$$

$$y-y_1=m(x-x_1)$$

$$y-\ln 2=\frac{1}{2}(x-2)$$

$$2y-2\ln 2=x-2$$

$$x-2y-2+2\ln 2=0$$

Question 5

$$y=\ln(x-1)$$

$$y(2)=\ln(2-1)=\ln 1=0$$

$$y'=\frac{1}{x-1}$$

$$y'(2)=\frac{1}{2-1}=1$$

$$y-y_1=m(x-x_1)$$

$$y-0=1(x-2)$$

$$y=x-2$$

$$x-y-2=0$$

Question 6

$y = \log_e\left(x^4 + x\right)$

$y' = \frac{1}{x^4 + x} \times \left(4x^3 + 1\right)$

$y' = \frac{4x^3 + 1}{x^4 + x}$

$y'(1) = \frac{4(1)^3 + 1}{(1)^4 + 1} = \frac{5}{2}$

$m_1 m_2 = -1$

$\frac{5}{2} \times m_2 = -1$

$m_2 = -\frac{2}{5}$

Question 7

$y = \ln x$

$y(5) = \ln 5$

$y' = \frac{1}{x}$

$y'(5) = \frac{1}{5}$

$m_1 m_2 = -1$

$\frac{1}{5} \times m_2 = -1$

$m_2 = -5$

$y - \ln 5 = -5(x - 5)$

$y - \ln 5 = -5x + 25$

$5x + y - \ln 5 - 25 = 0$

Question 8

$$y = \ln(5x+4)$$
$$y(3) = \ln[5(3)+4] = \ln(19)$$
$$y' = \frac{1}{5x+4} \times 5 = \frac{5}{5x+4}$$
$$y'(3) = \frac{5}{5(3)+4} = \frac{5}{19}$$
$$y - \ln 19 = \frac{5}{19}(x-3)$$
$$19y - 19\ln 19 = 5x - 15$$
$$5x - 19y + 19\ln 19 - 15 = 0$$

Question 9

$$y = \log_3(2x+5)$$
$$y = \frac{\ln(2x+5)}{\ln 3}$$
$$y' = \frac{1}{\ln 3} \times \frac{1}{2x+5} \times 2$$
$$y' = \frac{2}{(x+5)\ln 3}$$

Question 10

$y = \log_2 x$

$y(2) = \log_2 2 = 1$

$y = \dfrac{\ln x}{\ln 2}$

$y' = \dfrac{1}{\ln 2} \times \dfrac{1}{x}$

$y' = \dfrac{1}{x(\ln 2)}$

$y'(2) = \dfrac{1}{2(\ln 2)}$

$m_1 m_2 = -1$

$\dfrac{1}{2(\ln 2)} \times m_2 = -1$

$m_2 = -2(\ln 2)$

$y - y_1 = m(x - x_1)$

$y - 1 = -2(\ln 2)(x - 2)$

$y - 1 = -2x\ln 2 + 4\ln 2$

$(2\ln 2)x + y - 1 - 4\ln 2 = 0$

Question 11

a

$$t=\frac{\log_e\left(\frac{P}{20\ 000}\right)}{0.021}$$

$$t(0)=\frac{\log_e\left(\frac{0}{20\ 000}\right)}{0.021}=20\ 000$$

b **i**

$$t=\frac{\ln\left(\frac{P}{20\ 000}\right)}{0.021}$$

$$t(25\ 000)=\frac{\log_e\left(\frac{25\ 000}{20\ 000}\right)}{0.021}\approx 10.6\text{ years}$$

ii $$t(50\,000)=\frac{\ln\left(\frac{50\,000}{20\,000}\right)}{0.021}\approx 43.6\text{ years}$$

c

$$t=\frac{\log_e\left(\frac{P}{20\ 000}\right)}{0.021}$$

$$0.021t=\log_e\left(\frac{P}{20\ 000}\right)$$

$$e^{0.021t}=\frac{P}{20\ 000}$$

$$P=20\ 000e^{0.021t}$$

d $P = 20\,000\, e^{0.021t}$

$P(2) = 20\,000\, e^{0.021(2)} = 20\,858$

$P(5) = 20\,000\, e^{0.021(5)} = 22\,214$

$\frac{22\,214 - 20\,858}{5-2} = 452$ kangaroos/year

e **i** $P = 20\,000\, e^{0.021t}$

$P\,' = 420\, e^{0.021t}$

$P\,'(3) = e^{0.021\,(3)} = 447$ kangaroos/year

ii $P = 20\,000\, e^{0.021t}$

$P\,' = 420\, e^{0.021t}$

$P\,'(5) = e^{0.021\,(5)} = 466$ kangaroos/year

iii $P = 20\,000\, e^{0.021t}$

$P\,' = 420\, e^{0.021t}$

$P\,'(10) = e^{0.021\,(10)} = 518$ kangaroos/year

Exercise 4.04 Derivative of trigonometric functions

Question 1

a

$$\frac{d}{dx}(\sin 4x)$$
$$= \cos 4x \times 4$$
$$= 4\cos 4x$$

b

$$\frac{d}{dx}(\cos 3x)$$
$$= -\sin 3x \times 3$$
$$= -3\sin 3x$$

c

$$\frac{d}{dx}(\tan 5x)$$
$$= \sec^2 5x \times 5$$
$$= 5\sec^2 5x$$

d

$$\frac{d}{dx}\left[\tan(3x+1)\right]$$
$$= \sec^2(3x+1) \times 3$$
$$= 3\sec^2(3x+1)$$

e

$$\frac{d}{dx}\left[\cos(-x)\right]$$
$$= -\sin(-x) \times -1$$
$$= \sin(-x)$$

f

$$\frac{d}{dx}(3\sin x) = 3\cos x$$

g

$$\frac{d}{dx}\left[4\cos\left(5x-3\right)\right]$$
$$=4\times-\sin\left(5x-3\right)\times5$$
$$=-20\sin\left(5x-3\right)$$

h

$$\frac{d}{dx}\left[2\cos\left(x^3\right)\right]$$
$$=2\times-\sin\left(x^3\right)\times3x^2$$
$$=-6x^2\sin\left(x^3\right)$$

i

$$\frac{d}{dx}\left[7\tan\left(x^2+5\right)\right]$$
$$=7\times\sec^2\left(x^2+5\right)\times2x$$
$$=14x\sec^2\left(x^2+5\right)$$

j

$$\frac{d}{dx}\left(\sin 3x+\cos 8x\right)$$
$$=3\cos\left(3x\right)-8\sin\left(8x\right)$$

k

$$\frac{d}{dx}\left[\tan\left(\pi+x\right)+x^2\right]$$
$$=\sec^2\left(\pi+x\right)\times1+2x$$
$$=\sec^2\left(\pi+x\right)+2x$$

l

$$\frac{d}{dx}\left[x\tan\left(x\right)\right]$$
$$=x\times\sec^2\left(x\right)+1\times\tan\left(x\right)$$
$$=x\sec^2\left(x\right)+\tan\left(x\right)$$

m

$$\frac{d}{dx}(\sin 2x \tan 3x)$$
$$= \sin(2x)\sec^2(3x)\times 3 + \cos(2x)\tan(3x)\times 2$$
$$= 3\sin(2x)\sec^2(3x) + 2\cos(2x)\tan(3x)$$

n

$$\frac{d}{dx}\left(\frac{\sin x}{2x}\right)$$
$$= \frac{\cos x \times 2x - 2\times \sin x}{(2x)^2}$$
$$= \frac{2x\cos x - 2\sin x}{4x^2}$$
$$= \frac{x\cos x - \sin x}{2x^2}$$

o

$$\frac{d}{dx}\left(\frac{3x+4}{\sin 5x}\right)$$
$$= \frac{3\times \sin 5x - 5\cos 5x \times (3x+4)}{(\sin 5x)^2}$$
$$= \frac{3\sin 5x - 5(3x+4)\cos 5x}{\sin^2 5x}$$

p

$$\frac{d}{dx}\left[(2x+\tan 7x)^9\right]$$
$$= 9(2x+\tan 7x)^8 \times (2+7\sec^2 7x)$$
$$= 9(2+7\sec^2 7x)(2x+\tan 7x)^8$$

q

$$\frac{d}{dx}(\sin^2 x) = \frac{d}{dx}\left[(\sin x)^2\right]$$
$$= 2\sin x \times \cos x$$
$$= 2\sin x \cos x$$
$$= \sin 2x$$

r

$$\frac{d}{dx}\left(3\cos^3 5x\right)=\frac{d}{dx}\left[3\left(\cos\ 5x\right)^3\right]$$
$$=3\times3\left(\cos 5x\right)^2\times-5\sin 5x$$
$$=-45\cos^2 5x\sin 5x$$

s

$$\frac{d}{dx}\left(e^x-\cos 2x\right)$$
$$=e^x-\left(-\sin 2x\times 2\right)$$
$$=e^x+2\sin 2x$$

t

$$\frac{d}{dx}\left[\sin\left(1-\ln x\right)\right]$$
$$=\cos\left(1-\ln x\right)\times-\frac{1}{x}$$
$$=-\frac{1}{x}\cos\left(1-\ln x\right)$$

u

$$\frac{d}{dx}\left[\sin\left(e^x+x\right)\right]$$
$$=\cos\left(e^x+x\right)\times\left(e^x+1\right)$$
$$=\left(e^x+1\right)\cos\left(e^x+x\right)$$

v

$$\frac{d}{dx}\left[\ln\left(\sin x\right)\right]$$
$$=\frac{1}{\sin x}\times\cos x$$
$$=\frac{\cos x}{\sin x}$$
$$=\cot x$$

w

$$\begin{aligned}
&\frac{d}{dx}\left(e^{3x}\cos 2x\right)\\
&=e^{3x}\times\left[-\sin\left(2x\right)\times 2\right]+e^{3x}\cos 2x\\
&=-2\sin\left(2x\right)e^{3x}+3e^{3x}\cos 2x\\
&=e^{3x}\left(3\cos 2x-2\sin 2x\right)
\end{aligned}$$

x

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{e^{2x}}{\tan 7x}\right)\\
&=\frac{2e^{2x}\tan 7x-e^{2x}\sec^2 7x\times 7}{\left(\tan 7x\right)^2}\\
&=\frac{2e^{2x}\tan 7x-7e^{2x}\sec^2 7x}{\tan^2\left(7x\right)}\\
&=\frac{e^{2x}\left(2\tan 7x-7\sec^2 7x\right)}{\tan^2\left(7x\right)}
\end{aligned}$$

Question 2

$$\begin{aligned}
&y=\tan 3x\\
&y'=3\sec^2 3x\\
&y'\left(\frac{\pi}{9}\right)=3\sec^2\left[3\times\left(\frac{\pi}{9}\right)\right]\\
&y'\left(\frac{\pi}{9}\right)=3\sec^2\left(\frac{\pi}{3}\right)\\
&y'\left(\frac{\pi}{9}\right)=3\times 4\\
&y'\left(\frac{\pi}{9}\right)=12
\end{aligned}$$

Question 3

$$y = \sin(\pi - x)$$
$$y' = \cos(\pi - x) \times (-1) = -\cos(\pi - x)$$
$$y'\left(\frac{\pi}{6}\right) = -\cos\left(\pi - \frac{\pi}{6}\right) = -\cos\left(\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{2}$$
$$y - y_1 = m(x - x_1)$$
$$y - \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$$
$$2y - 1 = \sqrt{3}\,x - \frac{\sqrt{3}\,\pi}{6}$$
$$12y - 6 = 6\sqrt{3}\,x - \pi\sqrt{3}$$
$$6\sqrt{3}\,x - 12y + 6 - \pi\sqrt{3} = 0$$

Question 4

$$\frac{d}{dx}\left[\ln(\cos x)\right]$$
$$= \frac{1}{\cos x} \times -\sin x$$
$$= -\frac{\sin x}{\cos x}$$
$$= -\tan x$$

Question 5

$$y = \sin 3x$$
$$y' = \cos 3x \times 3 = 3\cos 3x$$
$$y'\left(\frac{\pi}{18}\right) = 3\cos\left[3 \times \left(\frac{\pi}{18}\right)\right] = 3\cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2}$$
$$m_1 m_2 = -1$$
$$\frac{3\sqrt{3}}{2} \times m_2 = -1$$
$$m_2 = -\frac{2}{3\sqrt{3}} = -\frac{2\sqrt{3}}{9}$$

Question 6

$$\frac{d}{dx}\left(e^{\tan x}\right)$$
$$=e^{\tan x}\times\sec^2 x$$
$$=e^{\tan x}\sec^2 x$$

Question 7

$$y=3\sin(2x)$$
$$y\left(\frac{\pi}{8}\right)=3\sin\left[2\left(\frac{\pi}{8}\right)\right]=3\sin\frac{\pi}{4}=3\times\frac{\sqrt{2}}{2}=\frac{3\sqrt{2}}{2}$$
$$y'=3\cos(2x)\times 2=6\cos(2x)$$
$$y'\left(\frac{\pi}{8}\right)=6\cos\left[2\left(\frac{\pi}{8}\right)\right]=6\cos\left(\frac{\pi}{4}\right)=6\times\frac{\sqrt{2}}{2}=\frac{6\sqrt{2}}{2}=3\sqrt{2}$$
$$m_1 m_2=-1$$
$$3\sqrt{2}\times m_2=-1$$
$$m_2=-\frac{1}{3\sqrt{2}}=-\frac{\sqrt{2}}{6}$$
$$y-y_1=m(x-x_1)$$
$$y-\frac{3\sqrt{2}}{2}=-\frac{\sqrt{2}}{6}\left(x-\frac{\pi}{8}\right)$$
$$6y-9\sqrt{2}=-\sqrt{2}\left(x-\frac{\pi}{8}\right)$$
$$6\sqrt{2}\,y-18=-2\left(x-\frac{\pi}{8}\right)$$
$$3\sqrt{2}\,y-9=-x+\frac{\pi}{8}$$
$$24\sqrt{2}\,y-72=-8x+\pi$$
$$8x+24\sqrt{2}\,y-72-\pi=0$$

Question 8

LHS = $\frac{d}{dx}$ [$\log_e$ (tan x)]

$$= \frac{\sec^2 x}{\tan x}$$

$$= \frac{\tan^2 x + 1}{\tan x}$$

$$= \frac{\tan^2 x}{\tan x} + \frac{1}{\tan x}$$

$$= \tan x + \cot x$$

= RHS

So $\frac{d}{dx}$ [$\log_e$ (tan x)] = tan x + cot x

Question 9

a

$$y = \tan x^\circ$$

$$y = \tan\left(\frac{\pi x^\circ}{180}\right)$$

$$y' = \sec^2\left(\frac{\pi x^\circ}{180}\right) \times \left(\frac{\pi}{180}\right)$$

$$y' = \frac{\pi}{180}\sec^2 x^\circ$$

b

$$y = 3\cos x^\circ$$

$$y = 3\cos\left(\frac{\pi x^\circ}{180}\right)$$

$$y' = -3\sin\left(\frac{\pi x^\circ}{180}\right) \times \left(\frac{\pi}{180}\right)$$

$$y' = -\frac{3\pi}{180}\sin\left(\frac{\pi x^\circ}{180}\right)$$

$$y' = -\frac{\pi}{60}\sin x^\circ$$

c

$$y = \frac{\sin x^\circ}{5}$$
$$y = \frac{1}{5}\sin\left(\frac{\pi x^\circ}{180}\right)$$
$$y' = \frac{1}{5}\cos\left(\frac{\pi x^\circ}{180}\right)\times\left(\frac{\pi}{180}\right)$$
$$y' = \frac{\pi}{900}\cos x^\circ$$

Question 10

$$\frac{d}{dx}\left(\cos x \sin^4 x\right) = \frac{d}{dx}\left[\cos x\left(\sin x\right)^4\right]$$
$$= \cos x \times 4\left(\sin x\right)^3 \cos x - \sin x\left(\sin x\right)^4$$
$$= 4\cos^2 x \sin^3 x - \sin^5 x$$
$$= \sin^3 x\left(4\cos^2 x - \sin^2 x\right)$$

Question 11

a where cos = 0

$P = 225 \cos\left(\frac{2\pi t}{9}\right) + 750$

$P = 225(0) + 750 = 750$

b where cos = –1

$P = 225 \cos\left(\frac{2\pi t}{9}\right) + 750$

$P = 225(-1) + 750 = 525$

c where cos = 1

$P = 225 \cos\left(\frac{2\pi t}{9}\right) + 750$

$P = 225(1) + 750 = 975$

d

$$P = 225\cos\left(\frac{2\pi t}{9}\right) + 750$$

$$700 = 225\cos\left(\frac{2\pi t}{9}\right) + 750$$

$$-50 = 225\cos\left(\frac{2\pi t}{9}\right)$$

$$-\frac{2}{9} = \cos\left(\frac{2\pi t}{9}\right)$$

$$\cos^{-1}\left(-\frac{2}{9}\right) = \frac{2\pi t}{9}$$

$$t = \frac{9}{2\pi}\cos^{-1}\left(-\frac{2}{9}\right)$$

$$t = 2.6,\ 6.4,\ 11.6,\ 15.4,\ \ldots \text{ days}$$

e **i**

$$P = 225\cos\left(\frac{2\pi t}{9}\right) + 750$$

$$P' = -225\sin\left(\frac{2\pi t}{9}\right) \times \left(\frac{2\pi}{9}\right)$$

$$P' = -50\pi\sin\left(\frac{2\pi t}{9}\right)$$

$$P'(3) = -50\pi\sin\left(\frac{2\pi(3)}{9}\right) = -136 \text{ fish/day}$$

ii

$$P = 225\cos\left(\frac{2\pi t}{9}\right) + 750$$

$$P' = -225\sin\left(\frac{2\pi t}{9}\right) \times \left(\frac{2\pi}{9}\right)$$

$$P' = -50\pi\sin\left(\frac{2\pi t}{9}\right)$$

$$P'(7) = -50\pi\sin\left(\frac{2\pi(7)}{9}\right) = 155 \text{ fish/day}$$

iii

$$P = 225\cos\left(\frac{2\pi t}{9}\right) + 750$$
$$P' = -225\sin\left(\frac{2\pi t}{9}\right) \times \left(\frac{2\pi}{9}\right)$$
$$P' = -50\pi\sin\left(\frac{2\pi t}{9}\right)$$
$$P'(10) = -50\pi\sin\left(\frac{2\pi(10)}{9}\right) = -101 \text{ fish/day}$$

iv

$$P = 225\cos\left(\frac{2\pi t}{9}\right) + 750$$
$$P' = -225\sin\left(\frac{2\pi t}{9}\right) \times \left(\frac{2\pi}{9}\right)$$
$$P' = -50\pi\sin\left(\frac{2\pi t}{9}\right)$$
$$P'(18) = -50\pi\sin\left(\frac{2\pi(18)}{9}\right) = 0 \text{ fish/day}$$

f

$$P = 225\cos\left(\frac{2\pi t}{9}\right) + 750$$
$$P' = -225\sin\left(\frac{2\pi t}{9}\right) \times \left(\frac{2\pi}{9}\right)$$
$$P' = -50\pi\sin\left(\frac{2\pi t}{9}\right)$$
$$25 = -50\pi\sin\left(\frac{2\pi t}{9}\right)$$
$$-\frac{1}{2\pi} = \sin\left(\frac{2\pi t}{9}\right)$$
$$\sin^{-1}\left(-\frac{1}{2\pi}\right) = \frac{2\pi t}{9}$$
$$t = \frac{9}{2\pi}\sin^{-1}\left(-\frac{1}{2\pi}\right)$$
$$t = 0.23,\ 4.27,\ 9.23,\ 13.27,\ \ldots \text{ days}$$

Question 12

a **i**

$$D=8\sin\left(\frac{\pi t}{6}\right)+9$$

$$D(0)=8\sin\left(\frac{\pi(0)}{6}\right)+9=9\,\text{m}$$

ii

$$D=8\sin\left(\frac{\pi t}{6}\right)+9$$

$$D(5)=8\sin\left(\frac{\pi(5)}{6}\right)+9=13\,\text{m}$$

b

$$D=8\sin\left(\frac{\pi t}{6}\right)+9$$

$$10=8\sin\left(\frac{\pi t}{6}\right)+9$$

$$1=8\sin\left(\frac{\pi t}{6}\right)$$

$$\frac{1}{8}=\sin\left(\frac{\pi t}{6}\right)$$

$$\sin^{-1}\left(\frac{1}{8}\right)=\frac{\pi t}{6}$$

$$t=\frac{6}{\pi}\sin^{-1}\left(\frac{1}{8}\right)$$

$$t=0.2,\ 5.8,\ 12.2,\ 17.8\ \ldots\ \text{hours}$$

c **i**

$$D=8\sin\left(\frac{\pi t}{6}\right)+9$$

$$D'=8\cos\left(\frac{\pi t}{6}\right)\times\frac{\pi}{6}$$

$$D'=\frac{4\pi}{3}\cos\left(\frac{\pi t}{6}\right)$$

$$D'(3)=\frac{4\pi}{3}\cos\left(\frac{\pi(3)}{6}\right)=0\ \text{m/hour}$$

ii

$$D = 8\sin\left(\frac{\pi t}{6}\right) + 9$$
$$D' = 8\cos\left(\frac{\pi t}{6}\right) \times \frac{\pi}{6}$$
$$D' = \frac{4\pi}{3}\cos\left(\frac{\pi t}{6}\right)$$
$$D'(11) = \frac{4\pi}{3}\cos\left(\frac{\pi(11)}{6}\right) = 3.6 \text{ m/hour}$$

iii

$$D = 8\sin\left(\frac{\pi t}{6}\right) + 9$$
$$D' = 8\cos\left(\frac{\pi t}{6}\right) \times \frac{\pi}{6}$$
$$D' = \frac{4\pi}{3}\cos\left(\frac{\pi t}{6}\right)$$
$$D'(12) = \frac{4\pi}{3}\cos\left(\frac{\pi(12)}{6}\right) = 4.2 \text{ m/hour}$$

d

$$D = 8\sin\left(\frac{\pi t}{6}\right) + 9$$
$$D' = 8\cos\left(\frac{\pi t}{6}\right) \times \frac{\pi}{6}$$
$$D' = \frac{4\pi}{3}\cos\left(\frac{\pi t}{6}\right)$$
$$3 = \frac{4\pi}{3}\cos\left(\frac{\pi t}{6}\right)$$
$$\frac{9}{4\pi} = \cos\left(\frac{\pi t}{6}\right)$$
$$\cos^{-1}\left(\frac{9}{4\pi}\right) = \frac{\pi t}{6}$$
$$t = \frac{6}{\pi}\cos^{-1}\left(\frac{9}{4\pi}\right)$$
$$t = 1.5,\ 10.5,\ 13.5,\ 22.5,\ \ldots \text{ hours}$$

Exercise 4.05 Second derivatives

Question 1

$$\frac{d}{dx}\left(x^7-2x^5+x^4-x-3\right)=7x^6-10x^4+4x^3-1$$
$$\frac{d}{dx}\left(7x^6-10x^4+4x^3-1\right)=42x^5-40x^3+12x^2$$
$$\frac{d}{dx}\left(42x^5-40x^3+12x^2\right)=210x^4-120x^2+24x$$
$$\frac{d}{dx}\left(210x^4-120x^2+24x\right)=840x^3-240x+24$$

Question 2

$f(x) = x^9 - 5$

$f'(x) = 9x^8$

$f''(x) = 72x^7$

Question 3

$f(x) = 2x^5 - x^3 + 1$

$f'(x) = 10x^4 - 3x^2$

$f''(x) = 40x^3 - 6x$

Question 4

$f(x) = 3t^4 - 2t^3 + 5t - 4$

$f'(x) = 12t^3 - 6t^2 + 5$

$f'(1) = 12(1)^3 - 6(1)^2 + 5 = 11$

$f''(x) = 36t^2 - 12t$

$f''(-2) = 36(-2)^2 - 12(-2) = 168$

Question 5

$$\frac{d}{dx}\left(x^7-2x^6+4x^4-7\right)=7x^6-12x^5+16x^3$$
$$\frac{d}{dx}\left(7x^6-12x^5+16x^3\right)=42x^5-60x^4+48x^2$$
$$\frac{d}{dx}\left(42x^5-60x^4+48x^2\right)=210x^4-240x^3+96x$$

Question 6

$y = 2x^2 - 3x + 3$

$y' = 4x - 3$

$y'' = 4$

Question 7

$f(x) = x^4 - x^3 + 2x^2 - 5x - 1$

$f'(x) = 4x^3 - 3x^2 + 4x - 5$

$f'(-1) = 4(-1)^3 - 3(-1)^2 + 4(-1) - 5 = -16$

$f''(x) = 12x^2 - 6x + 4$

$f''(2) = 12(2)^2 - 6(2) + 4 = 40$

Question 8

$$\frac{d}{dx}\left(x^{-4}\right)=-4x^{-5}$$
$$\frac{d}{dx}\left(-4x^{-5}\right)=20x^{-6}$$

Question 9

$g(x) = \sqrt{x}$

$g(x) = x^{\frac{1}{2}}$

$g'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$g''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

$g''(4) = -\frac{1}{4}(4)^{-\frac{3}{2}} = -\frac{1}{32}$

Question 10

$h = 5t^3 - 2t^2 + t + 5$

$\frac{dh}{dt} = 15t^2 - 4t + 1$

$\frac{d^2h}{dt^2} = 30t - 4$

$\frac{d^2h}{dt^2}(1) = 30(1) - 4 = 26$

Question 11

$y = 3x^3 - 2x^2 + 5x$

$y' = 9x^2 - 4x + 5$

$y'' = 18x - 4$

$3 = 18x - 4$

$18x = 7$

$x = \frac{7}{18}$

Question 12

$f(x)=x^3-x^2+x+9$

$f'(x)=3x^2-2x+1$

$f''(x)=6x-2$

$0<6x-2$

$2<6x$

$x>\frac{1}{3}$

Question 13

$\frac{d}{dx}\left[(4x-3)^5\right]$

$=5(4x-3)^4\times 4$

$=20(4x-3)^4$

$\frac{d}{dx}\left[20(4x-3)^4\right]$

$=80(4x-3)^3\times 4$

$=320(4x-3)^3$

Question 14

$f(x)=\sqrt{2-x}$

$f(x)=(2-x)^{\frac{1}{2}}$

$f'(x)=\frac{1}{2}(2-x)^{-\frac{1}{2}}\times -1$

$f'(x)=-\frac{1}{2}(2-x)^{-\frac{1}{2}}$

$f'(x)=-\frac{1}{2\sqrt{2-x}}$

$f''(x)=-\frac{1}{2}\times -\frac{1}{2}(2-x)^{-\frac{3}{2}}\times -1$

$f''(x)=-\frac{1}{4}(2-x)^{-\frac{3}{2}}$

$f''(x)=-\frac{1}{4\sqrt{(2-x)^3}}$

Question 15

$$f(x)=\frac{x+5}{3x-1}$$

$$f'(x)=\frac{1\times(3x-1)-(x+5)\times 3}{(3x-1)^2}$$

$$f'(x)=\frac{(3x-1)-(3x+15)}{(3x-1)^2}$$

$$f'(x)=\frac{-16}{(3x-1)^2}$$

$$f'(x)=-16(3x-1)^{-2}$$

$$f''(x)=-16\times-2(3x-1)^{-3}\times 3$$

$$f''(x)=96(3x-1)^{-3}$$

$$f''(x)=\frac{96}{(3x-1)^3}$$

Question 16

$v = (t + 3)(2t - 1)^2$

$v' = (t + 3) \times 2(2t - 1) \times 2 + 1 \times (2t - 1)^2$

$v' = 4(t + 3)(2t - 1) + (2t - 1)^2$

$v'' = 4[(t + 3) \times 2 + 1 \times (2t - 1)] + 2(2t - 1) \times 2$

$v'' = 4(2t + 6 + 2t - 1) + 4(2t - 1)$

$v'' = 4(4t + 5) + 8t - 4$

$v'' = 16t + 20 + 8t - 4$

$v'' = 24t + 16$

Question 17

$y = bx^3 - 2x^2 + 5x + 4$

$y' = 3bx^2 - 4x + 5$

$y'' = 6bx - 4$

$-2 = 6b\left(\frac{1}{2}\right) - 4$

$2 = 3b$

$b = \frac{2}{3}$

Question 18

$f\,(t) = t(2t \quad 1)^7$

$f'\,(t) = 1 \times (2t \quad 1)^7 + t \times 7(2t \quad 1)^6 \times 2$

$f'\,(t) = (2t \quad 1)^7 + 14t(2t \quad 1)^6$

$f''\,(t) = 7(2t \quad 1)^6 \times 2 + 14t \times 6(2t \quad 1)^5 \times 2 + 14(2t \quad 1)^6$

$f''\,(t) = 14(2t \quad 1)^6 + 168t(2t \quad 1)^5 + 14(2t \quad 1)^6$

$f''\,(1) = 14(2[1] - 1)^6 + 168[1](2[1] - 1)^5 + 14(2[1] - 1)^6 = 196$

Question 19

$f(x) = 5bx^2 - 4x^3$

$f'(x) = 10bx - 12x^2$

$f''(x) = 10b - 24x$

$-3 = 10b - 24(-1)$

$-3 = 10b + 24$

$-27 = 10b$

$b = \frac{-27}{10}$

$b = -2.7$

Question 20

$y = e^{4x} + e^{-4x}$

$y' = 4e^{4x} - 4e^{-4x}$

$y'' = 16e^{4x} + 16e^{-4x}$

$y'' = 16(e^{4x} + e^{-4x})$

$y'' = 16y$

Question 21

$y = 3e^{2x};\ \frac{dy}{dx} = 6e^{2x};\ \frac{d^2y}{dx^2} = 12e^{2x}$

$\text{LHS} = \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y$

$= 12e^{2x} - 3(6e^{2x}) + 2(3e^{2x})$

$= 0$

$= \text{RHS}$

So $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Question 22

$y = ae^{bx};\ \frac{dy}{dx} = bae^{bx};$

$\frac{d^2y}{dx^2} = b^2ae^{bx}$

$= b^2y$

Question 23

$y = e^{3x}$

$y' = 3e^{3x}$

$y'' = 9e^{3x}$

$9e^{3x} + 2 \times 3e^{3x} + ne^{3x} = 0$

$9e^{3x} + 6e^{3x} + ne^{3x} = 0$

$e^{3x}(9 + 6 + n) = 0$

$15 + n = 0$

$n = -15$

Question 24

$y = 2 \cos 5x$

$y' = -10 \sin 5x$

$y'' = -50 \cos 5x$

$y'' = -25(2 \cos 5x)$

$y'' = -25y$

Question 25

$f(x) = -2 \sin x$; $f'(x) = -2 \cos x$;

$f''(x) = 2 \sin x = -f(x)$

Question 26

$y = 2 \sin 3x - 5 \cos 3x$; $\frac{dy}{dx} = 6 \cos 3x + 15 \sin 3x$;

$\frac{d^2y}{dx^2} = -18 \sin 3x + 45 \cos 3x$

$= -9(2 \sin 3x - 5 \cos 3x)$

$= -9y$

Question 27

$y = e^{3x} \cos 4x$

$y' = 3e^{3x} \cos 4x - 4e^{3x} \sin 4x$

$y'' = 9e^{3x} \cos 4x - 12e^{3x} \sin 4x - 12e^{3x} \sin 4x - 16e^{3x} \cos 4x$

$y'' = -7e^{3x} \cos 4x - 24e^{3x} \sin 4x$

$\therefore\ b = -24,\ a = -7$

Question 28

$$f(x)=x\sqrt{3x-4}$$

$$f(x)=x(3x-4)^{\frac{1}{2}}$$

$$f'(x)=1\times(3x-4)^{\frac{1}{2}}+x\times\frac{1}{2}(3x-4)^{-\frac{1}{2}}\times 3$$

$$f'(x)=(3x-4)^{\frac{1}{2}}+\frac{3x}{2}(3x-4)^{-\frac{1}{2}}$$

$$f''(x)=3\times\frac{1}{2}(3x-4)^{-\frac{1}{2}}+\frac{3}{2}(3x-4)^{-\frac{1}{2}}+\frac{3x}{2}\times-\frac{1}{2}(3x-4)^{-\frac{3}{2}}\times 3$$

$$f''(x)=\frac{3}{2\sqrt{3x-4}}+\frac{3}{2\sqrt{3x-4}}-\frac{9x}{4\sqrt{(3x-4)^3}}$$

$$f''(x)=\frac{3}{\sqrt{3x-4}}-\frac{9x}{4\sqrt{(3x-4)^3}}$$

$$f''(2)=\frac{3}{\sqrt{3(2)-4}}-\frac{9(2)}{4\sqrt{(3(2)-4)^3}}$$

$$f''(2)=\frac{3}{\sqrt{2}}-\frac{18}{4\sqrt{(2)^3}}$$

$$f''(2)=\frac{3}{\sqrt{2}}-\frac{9}{2\sqrt{8}}$$

$$f''(2)=\frac{3}{\sqrt{2}}-\frac{9}{2\left(2\sqrt{2}\right)}$$

$$f''(2)=\frac{12-9}{4\sqrt{2}}$$

$$f''(2)=\frac{3}{4\sqrt{2}}$$

$$f''(2)=\frac{3\sqrt{2}}{4\times 2}$$

$$f''(2)=\frac{3\sqrt{2}}{8}$$

Question 29

a $x = 2t^3 - 5t^2 + 7t + 8$

$x\,(0) = 2(0)^3 - 5(0)^2 + 7(0) + 8 = 8$ m

b $x = 2t^3 - 5t^2 + 7t + 8$

$x\,(3) = 2(3)^3 - 5(3)^2 + 7(3) + 8 = 38$ m

c $x = 2t^3 - 5t^2 + 7t + 8$

$x' = 6t^2 - 10t + 7$

$x'\,(3) = 6(3)^2 - 10(3) + 7 = 31$ m/s

d $x = 2t^3 - 5t^2 + 7t + 8$

$x' = 6t^2 - 10t + 7$

$x'' = 12t - 10$

$x''\,(3) = 12(3) - 10 = 26$ m/s^2

Question 30

a $h = 8\cos(\pi t) + 12$

$h\,(3) = 8\cos[\pi\,(3)] + 12 = 4$ cm

b max when cos = 1

$h = 8\cos(\pi t) + 12$

$h = 8\,(1) + 12 = 20$ cm

min when cos = –1

$h = 8\cos(\pi t) + 12$

$h = 8\,(-1) + 12 = 4$ cm

c **i** $h = 8\cos(\pi t) + 12$

$h' = -8\sin(\pi t) \times \pi$

$h' = -8\pi\sin(\pi t)$

$h'(1) = -8\pi\sin[\pi(1)] = 0$ cm/s

ii $h = 8\cos(\pi t) + 12$

$h' = -8\sin(\pi t) \times \pi$

$h' = -8\pi\sin(\pi t)$

$h'(1.5) = -8\pi\sin[\pi(1.5)] = 25.1$ cm/s

d **i** $h = 8\cos(\pi t) + 12$

$h' = -8\sin(\pi t) \times \pi$

$h' = -8\pi\sin(\pi t)$

$h'' = -8\pi\cos(\pi t) \times \pi$

$h'' = -8\pi^2\cos(\pi t)$

$h''(0) = -8\pi^2\cos[\pi(0)] = -8\pi^2 = -79$ cm/s^2

ii $h = 8\cos(\pi t) + 12$

$h' = -8\sin(\pi t) \times \pi$

$h' = -8\pi\sin(\pi t)$

$h'' = -8\pi\cos(\pi t) \times \pi$

$h'' = -8\pi^2\cos(\pi t)$

$h''(1) = -8\pi^2\cos[\pi(1)] = 8\pi^2 = 79$ cm/s^2

iii $h = 8\cos(\pi t) + 12$

$h' = -8\sin(\pi t) \times \pi$

$h' = -8\pi\sin(\pi t)$

$h'' = -8\pi\cos(\pi t) \times \pi$

$h'' = -8\pi^2\cos(\pi t)$

$h''(1.5) = -8\pi^2\cos[\pi(1.5)] = 0$ cm/s^2

Exercise 4.06 Anti-derivative graphs

Question 1

a decreasing (negative gradient) $x < -1$

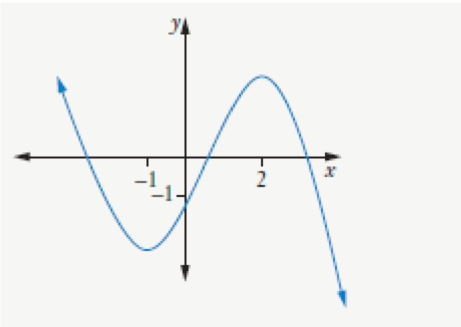
minimum at $x = -1$

increasing (positive gradient) $-1 < x < 2$

maximum at $x = 2$

decreasing (negative gradient) $x > 2$

y-intercept at $(0, -1)$

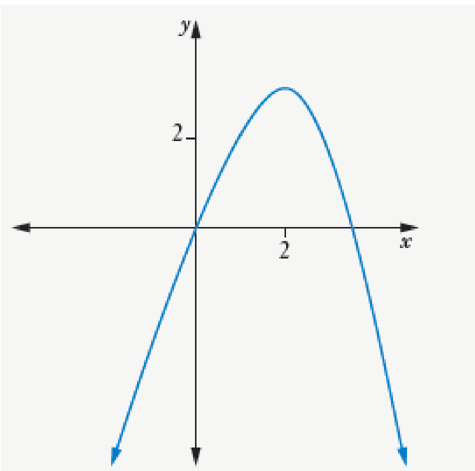


b decreasing (negative gradient) $x < 2$

Maximum $x = 2$

increasing (positive gradient) $x > 2$

passes through $(1, 2)$



c decreasing (negative gradient) $x < -4$

minimum at $x = -4$

increasing (positive gradient) $-4 < x < -1$

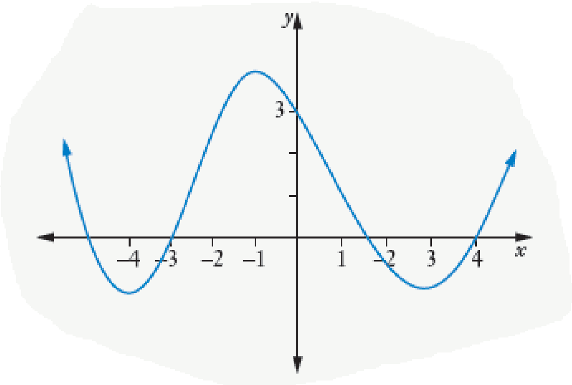
maximum at $x = -1$

decreasing (negative gradient) $-1 < x < 3$

minimum at $x = 3$

increasing (positive gradient) $x > 3$

passes through (0, 3)



d increasing (positive gradient) $x < -3$

maximum at $x = -3$

decreasing (negative gradient) $-3 < x < 0$

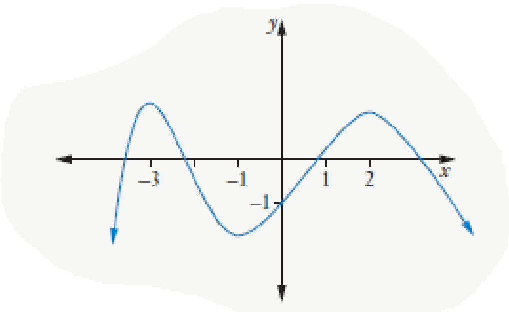
minimum at $x = 0$

increasing (positive gradient) $0 < x < 2$

maximum at $x = 2$

decreasing (negative gradient) $x > 2$

passes through (–1, –1)



e increasing (positive gradient) $x < 0$

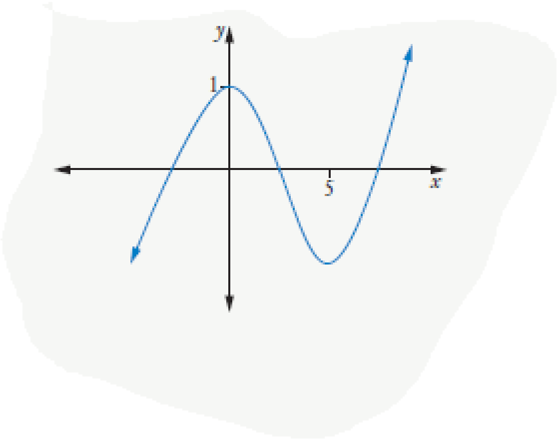
maximum at $x = 0$

decreasing (negative gradient) $0 < x < 5$

minimum at $x = 5$

increasing (positive gradient) $x > 5$

passes through (0, 1)

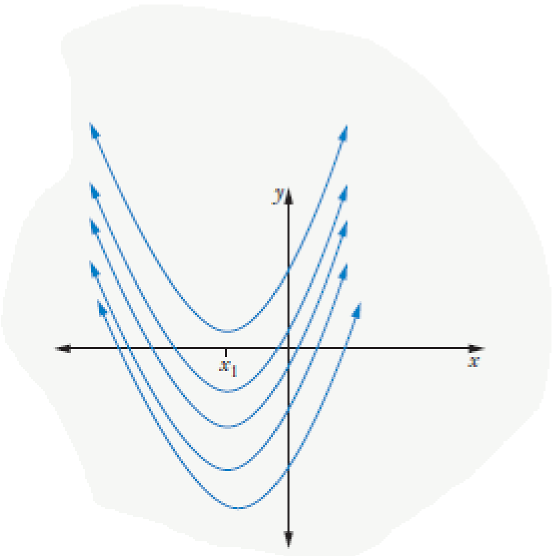


Question 2

a decreasing (negative gradient) $x < x_1$

minimum at $x = x_1$

increasing (positive gradient) $x > x_1$



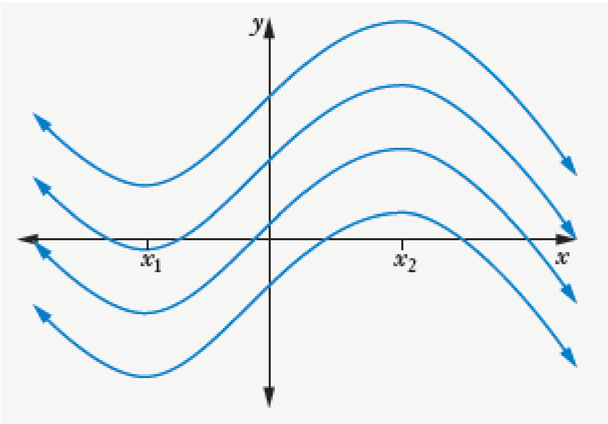
b decreasing (negative gradient) $x < x_1$

minimum at $x = x_1$

increasing (positive gradient) $x_1 < x < x_2$

maximum at $x = x_2$

decreasing (negative gradient) $x > x_2$



c increasing (positive gradient) $x < x_1$

maximum at $x = x_1$

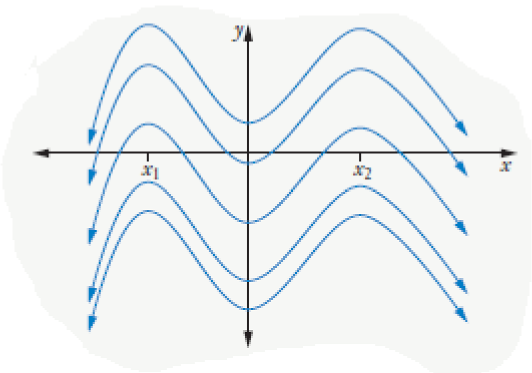
decreasing (negative gradient) $x_1 < x < 0$

minimum at $x = 0$

increasing (positive gradient) $0 < x < x_2$

maximum at $x = x_2$

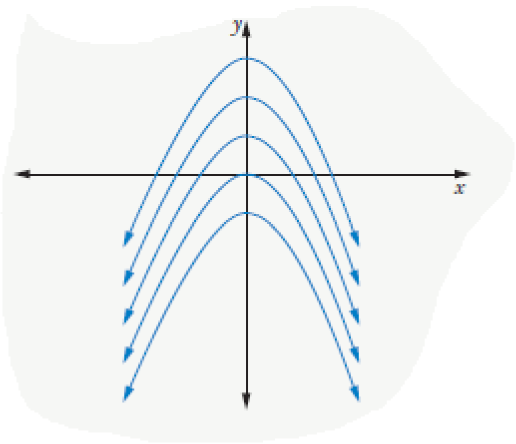
decreasing (negative gradient) $x > x_2$



d increasing (positive gradient) $x < 0$

maximum at $x = 0$

decreasing (negative gradient) $x > 0$



e decreasing (negative gradient) $x < x_1$

minimum at $x = x_1$

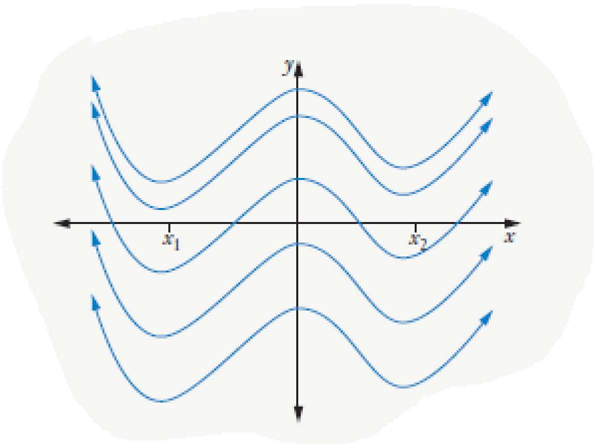
increasing (positive gradient) $x_1 < x < 0$

maximum at $x = 0$

decreasing (negative gradient) $0 < x < x_2$

minimum at $x = x_2$

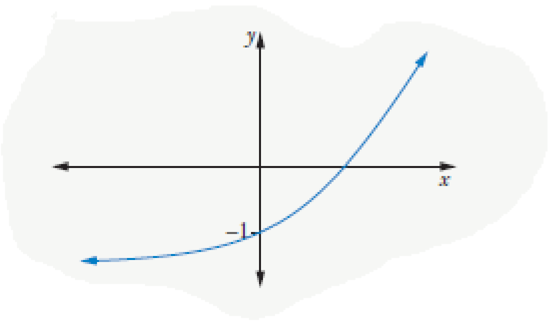
increasing (positive gradient) $x > x_2$



Question 3

increasing for all values of x

passes through (0, –1)



Question 4

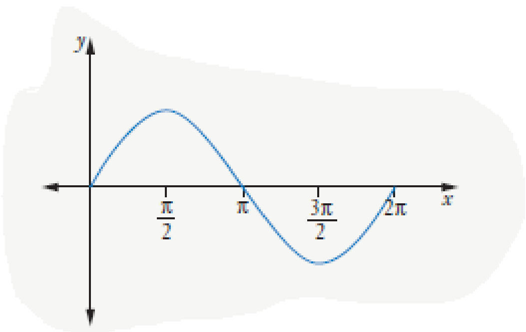
increasing (positive gradient) $x < \frac{\pi}{2}$

maximum at $x = \frac{\pi}{2}$

decreasing (negative gradient) $\frac{\pi}{2} < x < \frac{3\pi}{2}$

minimum at $x = \frac{3\pi}{2}$

increasing (positive gradient) $x > \frac{3\pi}{2}$

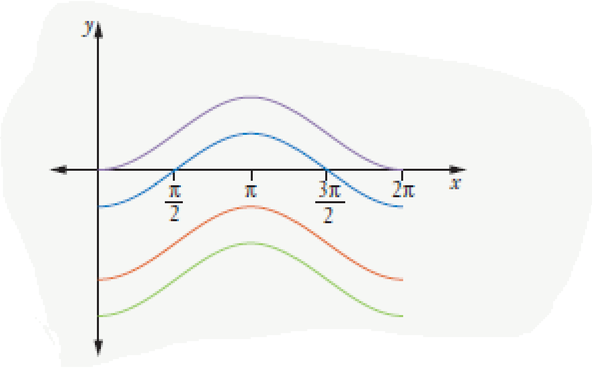


Question 5

increasing (positive gradient) $x < \pi$

maximum at $x = \pi$

decreasing (negative gradient) $x > \pi$



Exercise 4.07 Anti-derivatives

Question 1

a

$$\frac{dy}{dx}=2x-3$$
$$y=2\times\frac{1}{2}x^{1+1}-3x^{0+1}+C$$
$$y=x^2-3x+C$$

b

$$\frac{dy}{dx}=x^2+8x+1$$
$$y=\frac{1}{3}x^{2+1}+8\times\frac{1}{2}x^{1+1}+1x^{0+1}+C$$
$$y=\frac{x^3}{3}+4x^2+x+C$$

c

$$\frac{dy}{dx}=x^5-4x^3$$
$$y=\frac{1}{6}x^{5+1}-4\times\frac{1}{4}x^{3+1}+C$$
$$y=\frac{x^6}{6}-x^4+C$$

d

$$\frac{dy}{dx}=(x-1)^2$$
$$y=\frac{1}{3}(x-1)^{2+1}+C$$
$$y=\frac{(x-1)^3}{3}+C$$

e

$$\frac{dy}{dx}=6$$
$$y=6x^{0+1}+C$$
$$y=6x+C$$

f

$$\frac{dy}{dx}=(3x+2)^5$$
$$y=\frac{1}{6}(3x+2)^{5+1}\times\frac{1}{3}+C$$
$$y=\frac{1}{18}(3x+2)^6+C$$
$$y=\frac{(3x+2)^6}{18}+C$$

g

$$\frac{dy}{dx}=8(2x-7)^4$$
$$y=8\times\frac{1}{5}(2x-7)^{4+1}\times\frac{1}{2}+C$$
$$y=\frac{4}{5}(2x-7)^5+C$$
$$y=\frac{4(2x-7)^5}{5}+C$$

Question 2

a

$$f'(x)=6x^2-x$$
$$f(x)=6\times\frac{1}{3}x^{2+1}-\frac{1}{2}x^{1+1}+C$$
$$f(x)=2x^3-\frac{1}{2}x^2+C$$
$$f(x)=2x^3-\frac{x^2}{2}+C$$

b

$$f'(x)=x^4-3x^2+7$$

$$f(x)=\frac{1}{5}x^{4+1}-3\times\frac{1}{3}x^{2+1}+7x^{0+1}+C$$

$$f(x)=\frac{x^5}{5}-x^3+7x+C$$

c

$$f'(x)=x-2$$

$$f(x)=\frac{1}{2}x^2-2x^{0+1}+C$$

$$f(x)=\frac{x^2}{2}-2x+C$$

d

$$f'(x)=(x+1)(x-3)=x^2-3x+x-3=x^2-2x-3$$

$$f(x)=\frac{1}{2+1}x^{2+1}-\frac{2}{1+1}x^{1+1}-\frac{3}{0+1}x^{0+1}+C$$

$$f(x)=\frac{1}{3}x^3-x^2-3x+C$$

e

$$f'(x)=x^{\frac{1}{2}}$$

$$f(x)=\frac{1}{\frac{3}{2}}x^{\frac{1}{2}+1}+C$$

$$f(x)=\frac{2}{3}x^{\frac{3}{2}}+C$$

$$f(x)=\frac{2x^{\frac{3}{2}}}{3}+C$$

Question 3

a

$$\frac{dy}{dx}=5x^4-9$$
$$y=5\times\frac{1}{5}x^{4+1}-9x^{0+1}+C$$
$$y=x^5-9x+C$$

b

$$\frac{dy}{dx}=x^{-4}-2x^{-2}$$
$$y=\frac{1}{-3}x^{-4+1}-2\times\frac{1}{-1}x^{-2+1}+C$$
$$y=-\frac{1}{3}x^{-3}+2x^{-1}+C$$

c

$$\frac{dy}{dx}=\frac{x^3}{5}-x^2$$
$$y=\frac{1}{5}\times\frac{1}{4}x^{3+1}-\frac{1}{3}x^{2+1}+C$$
$$y=\frac{x^4}{20}-\frac{x^3}{3}+C$$

d

$$\frac{dy}{dx}=\frac{2}{x^2}=2x^{-2}$$
$$y=2\times\frac{1}{-1}x^{-2+1}+C$$
$$y=-2x^{-1}+C$$
$$y=\frac{-2}{x}+C$$

e

$$\frac{dy}{dx}=x^3-\frac{2x}{3}+1$$

$$y=\frac{1}{4}x^{3+1}-\frac{2}{3}\times\frac{1}{2}x^{1+1}+1x^{0+1}+C$$

$$y=\frac{x^4}{4}-\frac{x^2}{3}+x+C$$

Question 4

a

$$\frac{dy}{dx}=\sqrt{x}=x^{\frac{1}{2}}$$

$$y=\frac{1}{\frac{3}{2}}x^{\frac{1}{2}+1}+C$$

$$y=\frac{2}{3}x^{\frac{3}{2}}+C$$

$$y=\frac{2\sqrt{x^3}}{3}+C$$

b

$$\frac{dy}{dx}=x^{-3}$$

$$y=-\frac{1}{2}x^{-3+1}+C$$

$$y=-\frac{1}{2}x^{-2}+C$$

$$y=-\frac{1}{2x^2}+C$$

c

$$\frac{dy}{dx}=\frac{1}{x^8}=x^{-8}$$

$$y=-\frac{1}{7}x^{-8+1}+C$$

$$y=-\frac{1}{7x^7}+C$$

d

$$\frac{dy}{dx}=x^{-\frac{1}{2}}+2x^{-\frac{2}{3}}$$

$$y=\frac{1}{\frac{1}{2}}x^{-\frac{1}{2}+1}+2\times\frac{1}{\frac{1}{3}}x^{-\frac{2}{3}+1}+C$$

$$y=2x^{\frac{1}{2}}+6x^{\frac{1}{3}}+C$$

$$y=2\sqrt{x}+6\sqrt[3]{x}+C$$

e

$$\frac{dy}{dx}=x^{-7}-2x^{-2}$$

$$y=\frac{1}{-6}x^{-7+1}-2\times\frac{1}{-1}x^{-2+1}+C$$

$$y=-\frac{1}{6}x^{-6}+2x^{-1}+C$$

$$y=-\frac{1}{6x^{6}}+\frac{2}{x}+C$$

Question 5

a

$$f\left(x\right)=\left(x^2+5\right)^4$$

$$f'\left(x\right)=2x$$

$$\frac{dy}{dx}=f'\left(x\right)f\left(x\right)$$

$$y=\frac{1}{5}\left(x^2+5\right)^{4+1}+C$$

$$y=\frac{\left(x^2+5\right)^5}{5}+C$$

b

$$f(x)=\left(x^3-1\right)^9$$
$$f'(x)=3x^2$$
$$\frac{dy}{dx}=f'(x)\,f(x)$$
$$y=\frac{1}{10}\left(x^3-1\right)^{9+1}+C$$
$$y=\frac{\left(x^3-1\right)^{10}}{10}+C$$

c

$$f(x)=\left(2x^2+3\right)^3$$
$$f'(x)=4x$$
$$\frac{dy}{dx}=2f'(x)\,f(x)$$
$$y=2\times\frac{1}{4}\left(2x^2+3\right)^{3+1}+C$$
$$y=\frac{\left(2x^2+3\right)^4}{2}+C$$

d

$$f(x)=\left(x^5+1\right)^6$$
$$f'(x)=5x^4$$
$$\frac{dy}{dx}=3f'(x)\,f(x)$$
$$y=3\times\frac{1}{7}\left(x^5+1\right)^{6+1}+C$$
$$y=\frac{3\left(x^5+1\right)^7}{7}+C$$

e

$$f(x)=\left(x^2-4\right)^7$$
$$f'(x)=2x$$
$$\frac{dy}{dx}=\frac{1}{2}f'(x)\,f(x)$$
$$y=\frac{1}{2}\times\frac{1}{8}\left(x^2-4\right)^{7+1}+C$$
$$y=\frac{\left(x^2-4\right)^8}{16}+C$$

f

$$f(x)=\left(2x^6-7\right)^8$$
$$f'(x)=12x^5$$
$$\frac{dy}{dx}=\frac{1}{12}f'(x)\,f(x)$$
$$y=\frac{1}{12}\times\frac{1}{9}\left(2x^6-7\right)^{8+1}+C$$
$$y=\frac{\left(2x^6-7\right)^9}{108}+C$$

g

$$f(x)=\left(x^2-x+3\right)^4$$
$$f'(x)=2x-1$$
$$\frac{dy}{dx}=f'(x)\,f(x)$$
$$y=\frac{1}{5}\left(x^2-x+3\right)^{4+1}+C$$
$$y=\frac{\left(x^2-x+3\right)^5}{5}+C$$

h

$$f(x)=\left(x^3+2x^2-7x\right)^{10}$$
$$f'(x)=3x^2+4x-7$$
$$\frac{dy}{dx}=f'(x)\,f(x)$$
$$y=\frac{1}{11}\left(x^3+2x^2-7x\right)^{10+1}+C$$
$$y=\frac{\left(x^3+2x^2-7x\right)^{11}}{11}+C$$

i

$$f(x)=\left(x^2-6x-1\right)^5$$
$$f'(x)=2x-6$$
$$\frac{dy}{dx}=\frac{1}{2}f'(x)\,f(x)$$
$$y=\frac{1}{2}\times\frac{1}{6}\left(x^2-6x-1\right)^{5+1}+C$$
$$y=\frac{\left(x^2-6x-1\right)^6}{12}+C$$

Question 6

$$\frac{dy}{dx}=x^3-3x^2+5$$
$$y=\frac{1}{4}x^{3+1}-3\times\frac{1}{3}x^{2+1}+5x+C$$
$$y=\frac{x^4}{4}-x^3+5x+C$$
$$4=\frac{(1)^4}{4}-(1)^3+5(1)+C$$
$$4=4\frac{1}{4}+C$$
$$-\frac{1}{4}=C$$
$$y=\frac{x^4}{4}-x^3+5x-\frac{1}{4}$$

Question 7

$$f'(x)=4x-7$$

$$f(x)=4\times\frac{1}{2}x^{1+1}-7x+C$$

$$f(x)=2x^2-7x+C$$

$$5=2(2)^2-7(2)+C$$

$$5=8-14+C$$

$$5=-6+C$$

$$11=C$$

$$y=f(x)=2x^2-7x+11$$

Question 8

$$f'(x)=3x^2+4x-2$$

$$f(x)=3\times\frac{1}{3}x^{2+1}+4\times\frac{1}{2}x^{1+1}-2x+C$$

$$f(x)=x^3+2x^2-2x+C$$

$$4=(-3)^3+2(-3)^2-2(-3)+C$$

$$4=-27+18+6+C$$

$$4=-3+C$$

$$C=7$$

$$f(x)=x^3+2x^2-2x+7$$

$$f(1)=(1)^3+2(1)^2-2(1)+7=8$$

Question 9

$$\frac{dy}{dx}=2-6x$$

$$y=2x-6\times\frac{1}{2}x^{1+1}+C$$

$$y=2x-3x^2+C$$

$$3=2(-2)-3(-2)^2+C$$

$$3=-4-12+C$$

$$3=-16+C$$

$$C=19$$

$$y=2x-3x^2+19$$

Question 10

$$\frac{dx}{dt}=(t-3)^2$$

$$x=\frac{1}{3}(t-3)^{2+1}\times 1+C$$

$$x=\frac{1}{3}(t-3)^3+C$$

$$7=\frac{1}{3}(0-3)^3+C$$

$$7=-9+C$$

$$C=16$$

$$x=\frac{1}{3}(t-3)^3+16$$

$$x(4)=\frac{1}{3}(4-3)^3+16=\frac{1}{3}+16=16\frac{1}{3}$$

Question 11

$$\frac{d^2y}{dx^2}=8$$

$$\frac{dy}{dx}=8x+C$$

$$0=8(1)+C$$

$$C=-8$$

$$\frac{dy}{dx}=8x-8$$

$$y=8\times\frac{1}{2}x^2-8x+C$$

$$y=4x^2-8x+C$$

$$3=4(1)^2-8(1)+C$$

$$3=4-8+C$$

$$3=-4+C$$

$$C=7$$

$$y=4x^2-8x+7$$

Question 12

$$\frac{d^2y}{dx^2}=12x+6$$

$$\frac{dy}{dx}=12\times\frac{1}{2}x^{1+1}+6x+C$$

$$\frac{dy}{dx}=6x^2+6x+C$$

$$1=6(-1)^2+6(-1)+C$$

$$1=6-6+C$$

$$C=1$$

$$\frac{dy}{dx}=6x^2+6x+1$$

$$y=6\times\frac{1}{3}x^{2+1}+6\times\frac{1}{2}x^{1+1}+1x+C$$

$$y=2x^3+3x^2+x+C$$

$$-2=2(-1)^3+3(-1)^2+(-1)+C$$

$$-2=-2+3-1+C$$

$$C=-2$$

$$y=2x^3+3x^2+x-2$$

Question 13

$$f''(x)=6x-2$$

$$f'(x)=6\times\frac{1}{2}x^{1+1}-2x+C$$

$$f'(x)=3x^2-2x+C$$

$$7=3(2)^2-2(2)+C$$

$$7=12-4+C$$

$$C=-1$$

$$f'(x)=3x^2-2x-1$$

$$f(x)=3\times\frac{1}{3}x^{2+1}-2\times\frac{1}{2}x^{1+1}-1x+C$$

$$f(x)=x^3-x^2-x+C$$

$$7=(2)^3-(2)^2-(2)+C$$

$$7=8-4-2+C$$

$$C=5$$

$$f(x)=x^3-x^2-x+5$$

Question 14

$$f''(x)=5x^4$$

$$f'(x)=5\times\frac{1}{5}x^{4+1}+C$$

$$f'(x)=x^5+C$$

$$3=(0)^5+C$$

$$C=3$$

$$f'(x)=x^5+3$$

$$f(x)=\frac{1}{6}x^{5+1}+3x+C$$

$$f(x)=\frac{1}{6}x^6+3x+C$$

$$1=\frac{1}{6}(-1)^6+3(-1)+C$$

$$1=\frac{1}{6}-3+C$$

$$C=3\frac{5}{6}$$

$$f(x)=\frac{1}{6}x^6+3x+3\frac{5}{6}$$

$$f(2)=\frac{1}{6}(2)^6+3(2)+3\frac{5}{6}=10\frac{2}{3}+6+3\frac{5}{6}=20\frac{1}{2}$$

Question 15

$$\frac{d^2y}{dx^2}=8x$$
$$\frac{dy}{dx}=8\times\frac{1}{2}x^{1+1}+C$$
$$\frac{dy}{dx}=4x^2+C$$
$$\tan 45°=m$$
$$m=1$$
$$1=4(-2)^2+C$$
$$1=16+C$$
$$C=-15$$
$$\frac{dy}{dx}=4x^2-15$$
$$y=4\times\frac{1}{3}x^{2+1}-15x+C$$
$$y=\frac{4}{3}x^3-15x+C$$
$$5=\frac{4}{3}(-2)^3-15(-2)+C$$
$$5=-\frac{32}{3}+30+C$$
$$C=-\frac{43}{3}=-14\frac{1}{3}$$
$$y=\frac{4}{3}x^3-15x-14\frac{1}{3}$$

Question 16

$$\frac{d^2y}{dx^2}=2x-4$$

$$\frac{dy}{dx}=2\times\frac{1}{2}x^{1+1}-4x+C$$

$$\frac{dy}{dx}=x^2-4x+C$$

$$\tan 135°=m$$

$$m=-1$$

$$-1=(2)^2-4(2)+C$$

$$1=4-8+C$$

$$C=3$$

$$\frac{dy}{dx}=x^2-4x+3$$

$$y=\frac{1}{3}x^{2+1}-4\times\frac{1}{2}x^{1+1}+3x+C$$

$$y=\frac{1}{3}x^3-2x^2+3x+C$$

$$-4=\frac{1}{3}(2)^3-2(2)^2+3(2)+C$$

$$-4=\frac{8}{3}-8+6+C$$

$$C=-4\frac{2}{3}$$

$$y=\frac{1}{3}x^3-2x^2+3x-4\frac{2}{3}$$

Question 17

$f''(x)=12x^2-6x+4$

$f'(x)=12\times\frac{1}{3}x^{2+1}-6\times\frac{1}{2}x^{1+1}+4x+C$

$f'(x)=4x^3-3x^2+4x+C$

$4x-y-2=0$

$y=4x-2$

$m=4$

$4=4(0)^3-3(0)^2+4(0)+C$

$C=4$

$f'(x)=4x^3-3x^2+4x+4$

$f(x)=4\times\frac{1}{4}x^{3+1}-3\times\frac{1}{3}x^{2+1}+4\times\frac{1}{2}x^{1+1}+4x+C$

$f(x)=x^4-x^3+2x^2+4x+C$

$-2=(0)^4-(0)^3+2(0)^2+4(0)+C$

$C=-2$

$f(x)=x^4-x^3+2x^2+4x-2$

Question 18

$$\frac{d^2y}{dx^2}=6$$
$$\frac{dy}{dx}=6x+C$$
$$2x+4y-3=0$$
$$4y=3-2x$$
$$y=\frac{3}{4}-\frac{1}{2}x$$
$$m=-\frac{1}{2}$$
$$m_1 m_2=-1$$
$$-\frac{1}{2}m_2=-1$$
$$m_2=2$$
$$2=6(-1)+C$$
$$2=-6+C$$
$$C=8$$
$$\frac{dy}{dx}=6x+8$$
$$y=6\times\frac{1}{2}x^{1+1}+8x+C$$
$$y=3x^2+8x+C$$
$$3=3(-1)^2+8(-1)+C$$
$$3=3-8+C$$
$$C=8$$
$$y=3x^2+8x+8$$

Question 19

$$f''(x)=6x+18$$
$$f'(x)=6\times\frac{1}{2}x^{1+1}+18x+C$$
$$f'(x)=3x^2+18x+C$$
$$3=3(1)^2+18(1)+C$$
$$3=3+18+C$$
$$C=-18$$
$$f'(x)=3x^2+18x-18$$
$$f(x)=3\times\frac{1}{3}x^{2+1}+18\times\frac{1}{2}x^{1+1}-18x+C$$
$$f(x)=x^3+9x^2-18x+C$$
$$5=(1)^3+9(1)^2-18(1)+C$$
$$5=1+9-18+C$$
$$C=13$$
$$f(x)=x^3+9x^2-18x+13$$
$$f(-2)=(-2)^3+9(-2)^2-18(-2)+13=77$$

Question 20

$$\frac{dx}{dt}=6t-5$$
$$x=6\times\frac{1}{2}t^{1+1}-5t+C$$
$$x=3t^2-5t+C$$
$$(x,y)=(0,-2)$$
$$-2=3(0)^2-5(0)+C$$
$$C=-2$$
$$x=3t^2-5t-2$$

Question 21

$$\frac{d^2x}{dt^2}=24t^2-12t+6$$

$$\frac{dx}{dt}=24\times\frac{1}{3}t^{2+1}-12\times\frac{1}{2}t^{1+1}+6t+C$$

$$\frac{dx}{dt}=8t^3-6t^2+6t+C$$

$$0=8(1)^3-6(1)^2+6(1)+C$$

$$0=8-6+6+C$$

$$C=-8$$

$$\frac{dx}{dt}=8t^3-6t^2+6t-8$$

$$x=8\times\frac{1}{4}t^{3+1}-6\times\frac{1}{3}t^{2+1}+6\times\frac{1}{2}t^{1+1}-8t+C$$

$$x=2t^4-2t^3+3t^2-8t+C$$

$$-3=2(0)^4-2(0)^3+3(0)^2-8(0)+C$$

$$C=-3$$

$$x=2t^4-2t^3+3t^2-8t-3$$

Exercise 4.08 Further anti-derivatives

Question 1

a $y' = \sin x$

$y = -\cos x + C$

b $y' = \sec^2 x$

$y = \tan x + C$

c $y' = \cos x$

$y = \sin x + C$

d

$$y' = \sec^2 7x$$
$$y = \frac{1}{7} \times \tan 7x + C$$
$$y = \frac{\tan 7x}{7} + C$$

e

$$y' = \sin(2x - \pi)$$
$$y = \frac{1}{2} \times -\cos(2x - \pi) + C$$
$$y = -\frac{\cos(2x - \pi)}{2} + C$$

Question 2

a $y' = e^x$

$y = e^x + C$

b

$$y' = e^{6x}$$
$$y = \frac{1}{6} \times e^{6x} + C$$
$$y = \frac{e^{6x}}{6} + C$$

c

$$y' = \frac{1}{x}$$
$$y = \ln|x| + C$$

d

$$y' = \frac{3}{x-1}$$
$$f(x) = 3x - 1, \ f'(x) = 3$$
$$y' = \frac{f'(x)}{f(x)}$$
$$y = \ln|3x-1| + C$$

e

$$y' = \frac{x}{x^2+5}$$
$$f(x) = x^2 + 5, \ f'(x) = 2x$$
$$y' = \frac{1}{2} \times \frac{f'(x)}{f(x)}$$
$$y = \frac{1}{2} \times \ln|x^2+5| + C$$
$$y = \frac{1}{2} \ln|x^2+5| + C$$

Question 3

a $y' = e^x + 5$

$y = e^x + 5x + C$

b

$$y' = \cos x + 4x$$
$$y = \sin x + 4 \times \frac{1}{2} x^{1+1} + C$$
$$y = \sin x + 2x^2 + C$$

c

$$y' = x + \frac{1}{x}$$
$$y = \frac{1}{1+1} x^{1+1} + \ln|x| + C$$
$$y = \frac{x^2}{2} + \ln|x| + C$$

d

$$y' = 8x^3 - 3x^2 + 6x - 3 + x^{-1}$$
$$y = \frac{8}{3+1} x^{3+1} - \frac{3}{2+1} x^{2+1} + \frac{6}{1+1} x^{1+1} - \frac{3}{0+1} x^{0+1} + \ln|x| + C$$
$$y = 2x^4 - x^3 + 3x^2 - 3x + \ln|x| + C$$

e

$$y' = \sin 5x - \sec^2 9x$$
$$y = -\cos 5x \times \frac{1}{5} - \tan x \times \frac{1}{9} + C$$
$$y = -\frac{1}{5} \cos 5x - \frac{1}{9} \tan x + C$$

Question 4

$$\frac{dy}{dx}=\cos x$$
$$y=\sin x+C$$
$$-4=\sin\left(\frac{\pi}{2}\right)+C$$
$$-4=1+C$$
$$C=-5$$
$$y=\sin x-5$$

Question 5

$$f'(x)=\frac{5}{x}=5\times\frac{1}{x}$$
$$f(x)=5\times\ln|x|+C$$
$$3=5\ln(1)+C$$
$$3=0+C$$
$$C=3$$
$$f(x)=5\ln|x|+3$$

Question 6

$$\frac{dy}{dx}=4\cos 2x$$
$$y=4\times\sin 2x\times\frac{1}{2}+C$$
$$y=2\sin 2x+C$$
$$2\sqrt{3}=2\sin\left[2\left(\frac{\pi}{6}\right)\right]+C$$
$$2\sqrt{3}=2\times\frac{\sqrt{3}}{2}+C$$
$$C=\sqrt{3}$$
$$y=2\sin 2x+\sqrt{3}$$

Question 7

$$f''(x)=27e^{3x}$$
$$f'(x)=27\times\frac{1}{3}e^{3x}+C$$
$$f'(x)=9e^{3x}+C$$
$$e^{6}=9e^{3(2)}+C$$
$$e^{6}=9e^{6}+C$$
$$C=-8e^{6}$$
$$f'(x)=9e^{3x}-8e^{6}$$
$$f(x)=9\times\frac{1}{3}e^{3x}-8e^{6}x+C$$
$$f(x)=3e^{3x}-8e^{6}x+C$$
$$e^{6}=3e^{3(2)}-8e^{6}(2)+C$$
$$e^{6}=3e^{6}-16e^{6}+C$$
$$C=14e^{6}$$
$$f(x)=3e^{3x}-8e^{6}x+14e^{6}$$

Question 8

a

$$\frac{dP}{dt}=1350\,e^{0.054t}$$
$$P=1350\times\frac{1}{0.054}e^{0.054t}+C$$
$$P=25\,000\,e^{0.054t}+C$$
$$35\,000=25\,000\,e^{0.054(0)}+C$$
$$35\,000=25\,000+C$$
$$C=10\,000$$
$$P=25\,000\,e^{0.054t}+10\,000$$

b $P = 25\ 000\ e^{0.054t} + 10\ 000$

$P = 25\ 000\ e^{0.054(10)} + 10\ 000 = 52\ 900$

Question 9

$$\frac{dx}{dt}=3e^{t}$$

$$x=3\times\frac{1}{3}e^{3t}+C$$

$$x=e^{3t}+C$$

$$5=e^{3(0)}+C$$

$$5=1+C$$

$$C=4$$

$$x=e^{3t}+4$$

Question 10

a

$$\frac{d^2x}{dt^2} = -9\sin 3t$$
$$\frac{dx}{dt} = -9\times -\frac{1}{3}\cos 3t + C$$
$$\frac{dx}{dt} = 3\cos 3t + C$$
$$3 = 3\cos 3(0) + C$$
$$3 = 3 + C$$
$$C = 0$$
$$\frac{dx}{dt} = 3\cos 3t$$

b

$$\frac{dx}{dt} = 3\cos 3t$$
$$x = 3\times \frac{1}{3}\sin 3t + C$$
$$x = \sin 3t + C$$
$$0 = \sin 3(0) + C$$
$$0 = 0 + C$$
$$C = 0$$
$$x = \sin 3t$$
$$x = \sin 3(2) = -0.3\,\text{cm}$$

c

$$x = \sin 3t$$
$$0 = \sin 3t$$
$$\text{when } \sin 3t = 0$$
$$t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \ldots \text{ s}$$

Test yourself 4

Question 1

$$\frac{d}{dx}(\sin 6x)$$
$$=\frac{1}{6}\times -\cos 6x$$
$$=-\frac{1}{6}\cos 6x$$

D

Question 2

$y = e^{2x} + x$

$y' = 2e^{2x} + 1$

$y'(0) = 2e^{2(0)} + 1 = 3$

B

Question 3

$y = \cos 2x$

$y' = -2 \sin 2x$

$y'' = -2 \times 2 \cos 2x = -4 \cos 2x = -4y$

A

Question 4

decreasing to $x < x_1$

increasing to 0

decreasing to x_2

increasing $x > x_2$

C

Question 5

a

$$\frac{d}{dx}\left(e^{5x}\right)$$
$$=e^{5x}\times 5$$
$$=5e^{5x}$$

b

$$\frac{d}{dx}\left(2e^{1-x}\right)$$
$$=2e^{1-x}\times -1$$
$$=-2e^{1-x}$$

c

$$\frac{d}{dx}\left(\log_e 4x\right)$$
$$=\frac{1}{4x}\times 4$$
$$=\frac{1}{x}$$

d

$$\frac{d}{dx}\left[\ln\left(4x+5\right)\right]$$
$$=\frac{1}{4x+5}\times 4$$
$$=\frac{4}{4x+5}$$

e

$$\frac{d}{dx}\left(xe^{x}\right)$$
$$=xe^{x}+e^{x}$$
$$=e^{x}\left(x+1\right)$$

f

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right)$$
$$=\frac{\frac{1}{x}\times x-1\times\ln x}{x^2}$$
$$=\frac{1-\ln x}{x^2}$$

g

$$\frac{d}{dx}\left[\left(e^x+1\right)^{10}\right]$$
$$=10\left(e^x+1\right)^9\times e^x$$
$$=10e^x\left(e^x+1\right)^9$$

Question 6.

a $\frac{d}{dx}(\cos x)=-\sin x$

b $\frac{d}{dx}(2\sin x)=2\cos x$

c $\frac{d}{dx}(\tan x+1)=\sec^2 x$

d $\frac{d}{dx}(x\sin x)=x\cos x+\sin x$

e

$$\frac{d}{dx}\left(\frac{\tan x}{x}\right)$$
$$=\frac{\sec^2 x\times x-1\times\tan x}{x^2}$$
$$=\frac{x\sec^2 x-\tan x}{x^2}$$

f $\frac{d}{dx}(\cos 3x) = -3\sin 3x$

g $\frac{d}{dx}(\tan 5x) = 5\sec^2 5x$

Question 7

$$y = 2 + e^{3x}$$
$$y(0) = 2 + e^{3(0)} = 2 + e^0 = 2 + 1 = 3$$
$$y' = 3e^{3x}$$
$$y'(0) = 3e^{3(0)} = 3e^0 = 3 \times 1 = 3$$
$$y - 3 = 3(x - 0)$$
$$y - 3 = 3x$$
$$3x - y + 3 = 0$$

Question 8

$$y = \sin 3x$$
$$y' = 3\cos 3x$$
$$y'\left(\frac{\pi}{4}\right) = 3 \times \cos\left[3 \times \left(\frac{\pi}{4}\right)\right] = 3\cos\left(\frac{3\pi}{4}\right) = 3 \times -\frac{\sqrt{2}}{2} = -\frac{3\sqrt{2}}{2}$$
$$y - \frac{1}{\sqrt{2}} = -\frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$$
$$\sqrt{2}\,y - 1 = -3x + \frac{3\pi}{4}$$
$$4\sqrt{2}\,y - 4 = -12x + 3\pi$$
$$12x + 4\sqrt{2}\,y - 4 - 3\pi = 0$$

Question 9

$x = \cos 2t$

$\frac{dx}{dt} = -2 \sin 2t$

$\frac{d^2x}{dt^2} = -4 \cos 2t$

$= -4x$

Question 10

$$y = x - e^{-x}$$
$$y' = 1 + e^{-x}$$
$$y'(2) = 1 + e^{-(2)} = 1 + e^{-2}$$
$$m_1 m_2 = -1$$
$$\left(1 + e^{-2}\right) m_2 = -1$$
$$m_2 = \frac{-1}{1 + e^{-2}}$$
$$m_2 = -\frac{e^2}{e^2 + 1}$$

Question 11

a

$$\frac{dy}{dx} = 10x^4 - 4x^3 + 6x - 3$$
$$y = 10 \times \frac{1}{5} x^5 - 4 \times \frac{1}{4} x^4 + 6 \times \frac{1}{2} x^2 - 3x + C$$
$$y = 2x^5 - x^4 + 3x^2 - 3x + C$$

b

$$\frac{dy}{dx} = e^{5x}$$
$$y = \frac{1}{5} e^{5x} + C$$

c

$$\frac{dy}{dx}=\sec^2 9x$$
$$y=\frac{1}{9}\tan 9x+C$$
$$y=\frac{\tan 9x}{9}+C$$

d

$$\frac{dy}{dx}=\left(\frac{1}{x+5}\right)$$
$$y=\ln|x+5|+C$$

e

$$\frac{dy}{dx}=\cos 2x$$
$$y=\frac{1}{2}\times\sin 2x+C$$
$$y=\frac{1}{2}\sin 2x+C$$

f

$$\frac{dy}{dx}=\sin\left(\frac{x}{4}\right)$$
$$y=4\times-\cos\left(\frac{x}{4}\right)+C$$
$$y=-4\cos\left(\frac{x}{4}\right)+C$$

Question 12

$$y=3\cos 2x$$
$$y'=3\times-2\sin 2x=-6\sin 2x$$
$$y'\left(\frac{\pi}{6}\right)=-6\sin 2\left(\frac{\pi}{6}\right)=-6\times\frac{\sqrt{3}}{2}=-3\sqrt{3}$$

Question 13

$$\frac{dy}{dx} = 6x^2 + 12x - 5$$

$$y = 6 \times \frac{1}{3} x^3 + 12 \times \frac{1}{2} x^2 - 5x + C$$

$$y = 2x^3 + 6x^2 - 5x + C$$

$$-3 = 2(2)^3 + 6(2)^2 - 5(2) + C$$

$$-3 = 16 + 24 - 10 + C$$

$$C = -33$$

$$y = 2x^3 + 6x^2 - 5x - 33$$

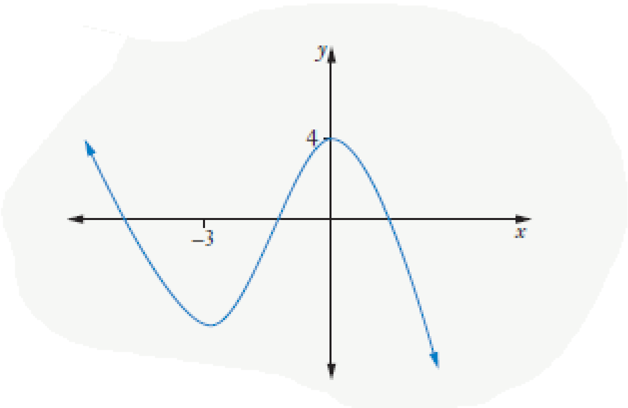
Question 14

decreasing $x < -3$

increasing $-3 < x < 0$

decreasing $x > 0$

point at (0, 4)



Question 15

$$y = \ln x$$
$$y' = \frac{1}{x}$$
$$y'(3) = \frac{1}{2}$$
$$m_1 m_2 = -1$$
$$\frac{1}{2} \times m_2 = -1$$
$$m_2 = -2$$
$$y - y_1 = m(x - x_1)$$
$$y - \ln 2 = -2(x - 2)$$
$$y - \ln 2 = -2x + 4$$
$$2x + y - \ln 2 - 4 = 0$$

Question 16

$$y = \tan x$$
$$y' = \sec^2 x$$
$$y'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^2 = 2$$
$$m_1 m_2 = -1$$
$$2 \times m_2 = -1$$
$$m_2 = -\frac{1}{2}$$
$$y - 1 = -\frac{1}{2}\left(x - \frac{\pi}{4}\right)$$
$$2y - 2 = -1\left(x - \frac{\pi}{4}\right)$$
$$2y - 2 = -x + \frac{\pi}{4}$$
$$8y - 8 = -4x + \pi$$
$$4x + 8y - 8 - \pi = 0$$

Question 17

a

$$\frac{d}{dx}\left[\left(5x^2+7\right)^4\right]$$
$$=4\left(5x^2+7\right)^3\times 10x$$
$$=40x\left(5x^2+7\right)^3$$

b

$$\frac{d}{dx}\left[4x\left(2x-3\right)^7\right]$$
$$=4x\times 7\left(2x-3\right)^6\times 2+4\left(2x-3\right)^7$$
$$=56x\left(2x-3\right)^6+4\left(2x-3\right)^7$$
$$=4\left(2x-3\right)^6\left[14x+\left(2x-3\right)\right]$$
$$=4\left(2x-3\right)^6\left(16x-3\right)$$

c

$$\frac{d}{dx}\left(\frac{5x-1}{3x+4}\right)$$
$$=\frac{5\left(3x+4\right)-3\left(5x-1\right)}{\left(3x+4\right)^2}$$
$$=\frac{15x+20-15x+3}{\left(3x+4\right)^2}$$
$$=\frac{23}{\left(3x+4\right)^2}$$

d

$$\frac{d}{dx}\left(2x^3e^x\right)$$
$$=6x^2e^x+2x^3e^x$$
$$=2x^2e^x\left(3+x\right)$$

e

$$\frac{d}{dx}\left(\frac{\tan 3x}{x+1}\right)$$

$$=\frac{3\sec^2 3x(x+1)-\tan 3x}{(x+1)^2}$$

$$=\frac{3(x+1)\sec^2 3x-\tan 3x}{(x+1)^2}$$

Question 18

$$f''(x)=15x+12$$

$$f'(x)=15\times\frac{1}{2}x^2+12x+C$$

$$f'(x)=\frac{15}{2}x^2+12x+C$$

$$5=\frac{15}{2}(2)^2+12(2)+C$$

$$5=30+24+C$$

$$C=-49$$

$$f'(x)=\frac{15}{2}x^2+12x-49$$

$$f(x)=\frac{15}{2}\times\frac{1}{3}x^3+12\times\frac{1}{2}x^2-49x+C$$

$$f(x)=\frac{5}{2}x^3+6x^2-49x+C$$

$$5=\frac{5}{2}(2)^3+6(2)^2-49(2)+C$$

$$5=20+24-98+C$$

$$C=59$$

$$f(x)=\frac{5}{2}x^3+6x^2-49x+59$$

Question 19

a $f(x) = 3x^5 - 2x^4 + x^3 - 2$

$f(-1) = 3(-1)^5 - 2(-1)^4 + (-1)^3 - 2 = -3 - 2 - 1 - 2 = -8$

b $f(x) = 3x^5 - 2x^4 + x^3 - 2$

$f'(x) = 3 \times 5x^4 - 2 \times 4x^3 + 3x^2 = 15x^4 - 8x^3 + 3x^2$

$f'(-1) = 15(-1)^4 - 8(-1)^3 + 3(-1)^2 = 15 + 8 + 3 = 26$

c $f'(x) = 15x^4 - 8x^3 + 3x^2$

$f''(x) = 15 \times 4x^3 - 8 \times 3x^2 + 3 \times 2x = 60x^3 - 24x^2 + 6x$

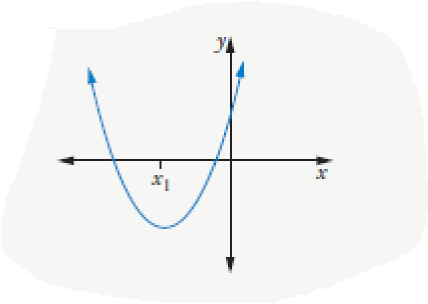
$f''(-1) = 60(-1)^3 - 24(-1)^2 + 6(-1) = -60 - 24 - 6 = -90$

Question 20

a decreasing $x < x_1$

increasing $x > x_1$

turning point at $x = x_1$

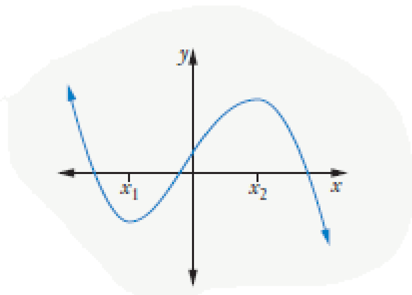


b decreasing $x < x_1$

increasing $x_1 < x < x_2$

decreasing $x > x_2$

turning points at x_1 and x_2



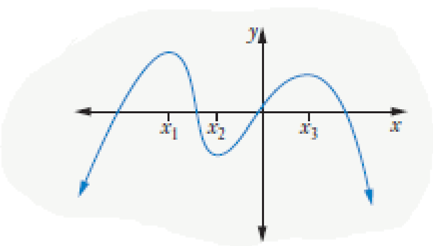
c increasing $x < x_1$

decreasing $x_1 < x < x_2$

increasing $x_2 < x < x_3$

decreasing $x > x_3$

turning points at x_1, x_2, x_3



Question 21

$$f''(x)=12x-6$$
$$f'(x)=12\times\frac{1}{2}x^2-6x+C$$
$$f'(x)=6x^2-6x+C$$
$$5=6(3)^2-6(3)+C$$
$$5=54-18+C$$
$$C=-31$$
$$f'(x)=6x^2-6x-31$$
$$f(x)=6\times\frac{1}{3}x^3-6\times\frac{1}{2}x^2-31x+C$$
$$f(x)=2x^3-3x^2-31x+C$$
$$2=2(3)^3-3(3)^2-31(3)+C$$
$$2=54-27-93+C$$
$$C=68$$
$$f(x)=2x^3-3x^2-31x+68$$

Question 22

a

$$\frac{dy}{dx} = x^3\left(3x^4 - 5\right)^6$$

$$u = 3x^4 - 5, du = 12x^3$$

$$\frac{dy}{dx} = \frac{1}{12} \times du \times u^6$$

$$y = \frac{1}{12} \times \frac{1}{7} u^7 + C$$

$$y = \frac{1}{84}\left(3x^4 - 5\right)^7 + C$$

$$y = \frac{\left(3x^4 - 5\right)^7}{84} + C$$

b

$$\frac{dy}{dx} = 3x\left(x^2 + 1\right)^9$$

$$u = x^2 + 1, du = 2x$$

$$\frac{dy}{dx} = \frac{3}{2} \times du \times u^9$$

$$y = \frac{3}{2} \times \frac{1}{10} u^{10} + C$$

$$y = \frac{3}{20}\left(x^2 + 1\right)^{10} + C$$

$$y = \frac{3\left(x^2 + 1\right)^{10}}{20} + C$$

Challenge exercise 4

Question 1.

$y = e^{x + \ln x}$

$y = e^x\, e^{\ln x}$

$y = x\, e^x$

$y' = 1 \times e^x + x\, e^x$

$y' = e^x + x\, e^x$

$y'(1) = e^1 + (1)e^1 = 2e$

Question 2

$$\frac{d}{dx}\left[\frac{5-x}{\left(4x^2+1\right)^3}\right]$$

$$=\frac{-1\times\left(4x^2+1\right)^3-3\times 8x\times\left(4x^2+1\right)^2\left(5-x\right)}{\left[\left(4x^2+1\right)^3\right]^2}$$

$$=\frac{-\left(4x^2+1\right)^3-24x\left(4x^2+1\right)^2\left(5-x\right)}{\left(4x^2+1\right)^6}$$

$$=\frac{-\left(4x^2+1\right)^3-\left(120x-24x^2\right)\left(4x^2+1\right)^2}{\left(4x^2+1\right)^6}$$

$$=\frac{-\left(4x^2+1\right)-\left(120x-24x^2\right)}{\left(4x^2+1\right)^4}$$

$$=\frac{-4x^2-1-120x+24x^2}{\left(4x^2+1\right)^4}$$

$$y'=\frac{20x^2-120x-1}{\left(4x^2+1\right)^4}$$

$$\frac{d}{dx}\left[\frac{20x^2-120x-1}{\left(4x^2+1\right)^4}\right]$$

$$=\frac{\left(40x-120\right)\left(4x^2+1\right)^4-4\times 8x\times\left(4x^2+1\right)^3\left(20x^2-120x-1\right)}{\left[\left(4x^2+1\right)^4\right]^2}$$

$$=\frac{\left(40x-120\right)\left(4x^2+1\right)^4-32x\left(4x^2+1\right)^3\left(20x^2-120x-1\right)}{\left(4x^2+1\right)^8}$$

$$=\frac{\left(40x-120\right)\left(4x^2+1\right)-32x\left(20x^2-120x-1\right)}{\left(4x^2+1\right)^5}$$

$$=\frac{160x^3+40x-480x^2-120-640x^3+3840x^2+32x}{\left(4x^2+1\right)^5}$$

$$=\frac{-480x^3+3360x^2+72x-120}{\left(4x^2+1\right)^5}$$

$$y''=\frac{-24\left(20x^3-140x^2-3x+5\right)}{\left(4x^2+1\right)^5}$$

Question 3

a

$$\frac{dy}{dx}\left(2x\,e^{x^2}\right)$$
$$u=x^2,\,du=2x$$
$$=du\times e^u$$
$$y=e^u+C$$
$$y=e^{x^2}+C$$

b

$$\frac{dy}{dx}\left[x^2\sin\left(x^3\right)\right]$$
$$u=x^3,\,du=3x^2$$
$$=\frac{1}{3}du\sin(u)$$
$$y=-\frac{1}{3}\cos(u)+C$$
$$y=-\frac{1}{3}\cos\left(x^3\right)+C$$

Question 4

$$\frac{dy}{dx}\left(e^{x\sin 2x}\right)$$
$$=e^{x\sin 2x}\times\left(\sin 2x+x\times 2\cos 2x\right)$$
$$=\left(\sin 2x+2x\cos 2x\right)e^{x\sin 2x}$$

Question 5

$$\frac{dy}{dx}=(x+3)(x-5)$$
$$=x^2+3x-5x-15$$
$$=x^2-2x-15$$
$$y=\frac{1}{3}x^3-2\times\frac{1}{2}x^2-15x+C$$
$$y=\frac{1}{3}x^3-x^2-15x+C$$
$$-1=\frac{1}{3}(0)^3-(0)^2-15(0)+C$$
$$C=-1$$
$$y=\frac{x^3}{3}-x^2-15x-1$$

Question 6

$$\frac{dV}{dt}=\left(2t-1\right)^2$$

$$u=2t-1,\frac{du}{dt}=2$$

$$dV=u^2\times\frac{1}{2}\times 2\,dt=\frac{1}{2}u^2\,\frac{du}{dt}\,dt=\frac{1}{2}u^2\,du$$

$$V=\frac{1}{2}\times\frac{1}{2+1}u^{2+1}+C$$

$$V=\frac{1}{6}\left(2t-1\right)^3+C$$

$$\text{Given: }\ 5=\frac{1}{6}\left[2\left(\frac{1}{2}\right)-1\right]^3+C$$

$$5=0+C$$

$$C=5$$

$$V\left(3\right)=\frac{1}{6}\left[2\left(3\right)-1\right]^3+5=\frac{1}{6}\left[5\right]^3+5=\frac{125}{6}+5=\frac{125+30}{6}=\frac{155}{6}=25\frac{5}{6}$$

Question 7

$$y=\frac{x\ln x}{e^x}$$

$$y'=\frac{\left(\ln x+1\right)e^x-\left(x\ln x\right)e^x}{\left(e^x\right)^2}$$

$$=\frac{\ln x+1-x\ln x}{e^x}$$

$$=\frac{1+\ln x-x\ln x}{e^x}$$

Question 8

a

$$\frac{d}{dx}\left[\ln\left(\tan x\right)\right]$$
$$=\frac{1}{\tan x}\times\sec^2 x$$
$$=\frac{\sec^2 x}{\tan x}$$
$$=\frac{1}{\sin x\cos x}$$

b

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$
$$u=\cos x,\ \frac{du}{dx}=-\sin x$$
$$\frac{d}{dx}\left(\frac{-du}{u}\right)=-\ln|u|+C$$
$$=-\ln|\cos x|+C$$

Question 9

a

$$u = x^3 - \pi, du = 3x^2$$

$$\begin{aligned}\int x^2 \sin\left(x^3 - \pi\right) dx &= \int \frac{1}{3} du \sin\left(u\right) \\ &= -\frac{1}{3}\cos u + C \\ &= -\frac{1}{3}\cos\left(x^3 - \pi\right) + C \\ &= -\frac{\cos\left(x^3 - \pi\right)}{3} + C\end{aligned}$$

b

$$u = x^2, du = 2x$$

$$\begin{aligned}\int x e^{x^2} dx &= \int \frac{1}{2} e^u du \\ &= \frac{1}{2} e^u + C \\ &= \frac{e^{x^2}}{2} + C\end{aligned}$$

MATHS IN FOCUS 12 MATHEMATICS ADVANCED

WORKED SOLUTIONS

Chapter 5: Geometrical applications of differentiation

Exercise 5.01 Increasing and decreasing curves

Question 1

$y = –2x^2 + 8x –1$

$y' = –4x + 8$

$–4x + 8 > 0$

$–4x > –8$

$x < 2$

Question 2

$y = 2x^2 – x$

$y' = –4x –1$

$4x – 1 < 0$

$4x > 1$

$x < \frac{1}{4}$

Question 3

$f(x) = 4 - x^2$

$f'(x) = -2x$

$-2x > 0$

$x < 0$

$(-\infty, 0)$

Question 4

a $y = x^2 - 3x - 4$

$y' = 2x - 3$

$2x - 3 < 0$

$2x < 3$

$x < \dfrac{3}{2}$

b $y = x^2 - 3x - 4$

$y' = 2x - 3$

$2x - 3 > 0$

$2x > 3$

$x > \dfrac{3}{2}$

c $y = x^2 - 3x - 4$

$y' = 2x - 3$

$2x - 3 = 0$

$2x = 3$

$x = \dfrac{3}{2}$

Question 5

$y = -2x - 7$

$y' = -2$

$y' = -2 < 0$ for all x

Question 6

$y = x^3$

$y' = 3x^2$

$3x^2 > 0$ for all x

Question 7

$y = x^3$

$y' = 3x^2$

$3x^2 = 0$

$x^2 = 0$

$x = 0$

$y(0) = (0)^3$

$y(0) = 0$

$(0, 0)$

Question 8

$y = 2x^3 + 3x^2 – 36x + 9$

$y' = 6x^2 + 6x – 36$

$0 = 6x^2 + 6x – 36$

$0 = x^2 + x – 6$

$0 = (x + 3)(x – 2)$

$0 = (x + 3)$

$x = –3$

$0 = (x – 2)$

$x = 2$

$x = –3, 2$

Question 9

a $y = x^2 – 2x –3$

$y' = 2x – 2$

$0 = 2x – 2$

$2 = 2x$

$x = 1$

$y(1) = (1)^2 – 2(1) – 3$

$y(1) = 1 – 2 – 3$

$y(1) = –4$

$(1, –4)$

b $f(x) = 9 - x^2$

$f'(x) = -2x$

$0 = -2x$

$x = 0$

$f(0) = 9 - (0)^2$

$f(0) = 9$

(0, 9)

c $y = 2x^3 - 9x^2 + 12x - 4$

$y' = 6x^2 - 18x + 12$

$0 = 6x^2 - 18x + 12$

$0 = x^2 - 3x + 2$

$0 = (x - 2)(x - 1)$

$0 = (x - 2)$

$x = 2$

$0 = (x - 1)$

$x = 1$

$y(2) = 2(2)^3 - 9(2)^2 + 12(2) - 4 = 0$

$y(1) = 2(1)^3 - 9(1)^2 + 12(1) - 4 = 1$

(1, 1) and (2, 0)

d $y = x^4 - 2x^2 + 1$

$y' = 4x^3 - 4x$

$0 = 4x^3 - 4x$

$0 = 4x(x^2 - 1)$

$0 = 4x(x + 1)(x - 1)$

$0 = 4x$

$x = 0$

$0 = (x + 1)$

$x = -1$

$0 = (x - 1)$

$x = 1$

$y(0) = (0)^4 - 2(0)^2 + 1 = 1$

$y(1) = (1)^4 - 2(1)^2 + 1 = 0$

$y(-1) = (-1)^4 - 2(-1)^2 + 1 = 1$

(0, 1), (1, 0) and (–1, 0)

Question 10

$y = (x - 2)^4$

$y' = 4(x - 2)^3 \times 1$

$y' = 4(x - 2)^3$

$0 = 4(x - 2)^3$

$(x - 2)^3 = 0$

$x - 2 = 0$

$x = 2$

$y(2) = ((2) - 2)^4 = 0$

(2, 0)

Question 11

$y = 2x^3 - 21x^2 + 60x - 3$

$y' = 6x^2 - 42x + 60$

$0 = 6x^2 - 42x + 60$

$0 = x^2 - 7x + 10$

$0 = (x - 5)(x - 2)$

$x - 2 = 0$

$x = 0$

$x - 5 = 0$

$x = 5$

$x = 2, 5$

Question 12

$f(x) = 2x^2 + px + 7$

$f'(x) = 4x + p$

$0 = 4(3) + p$

$0 = 12 + p$

$p = -12$

Question 13

$y = x^3 - ax^2 + bx - 3$

$y' = 3x^2 - 2ax + b$

$0 = 3(-1)^2 - 2a(-1) + b$

$0 = 3 + 2a + b$

$b = -2a - 3$

$0 = 3(2)^2 - 2a(2) + b$

$0 = 12 - 4a + b$

$0 = 12 - 4a + -2a - 3$

$0 = 9 - 6a$

$6a = 9$

$a = 1\frac{1}{2}$

$b = -2\left(1\frac{1}{2}\right) - 3 = -3 - 3$

$b = -6$

Question 14

a $y = x^3 - 3x^2 + 27x - 3$

$y' = 3x^2 - 6x + 27$

b $y = x^3 - 3x^2 + 27x - 3$

$y' = 3x^2 - 6x + 27$

$3x^2 - 6x + 27 = 0$

$x^2 - 2x + 3 = 0$

The quadratic function has $a > 0$.

$b^2 - 4ac = -288 < 0$

So $3x^2 - 6x + 27 > 0$ for all x.

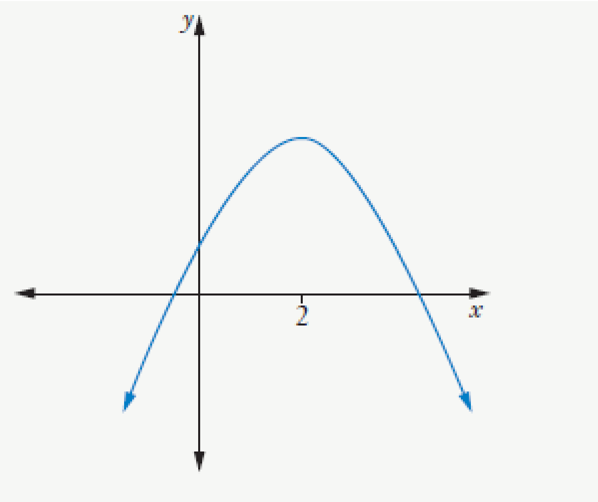
The function is monotonic increasing for all x.

Question 15

increasing for $x < 2$

stationary point at $x = 2$

decreasing for $x > 2$

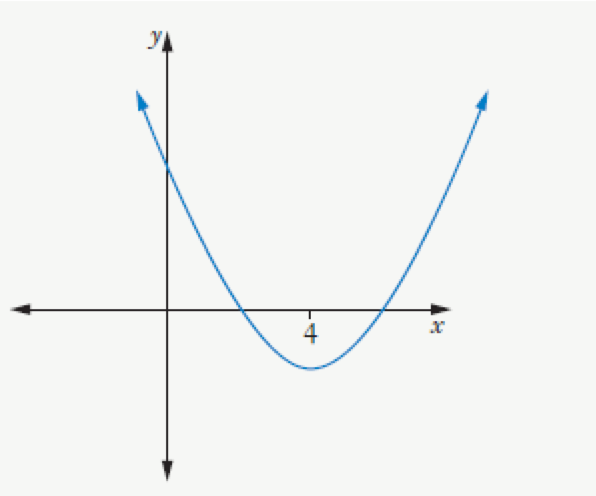


Question 16

decreasing for $x < 4$

stationary point at $x = 4$

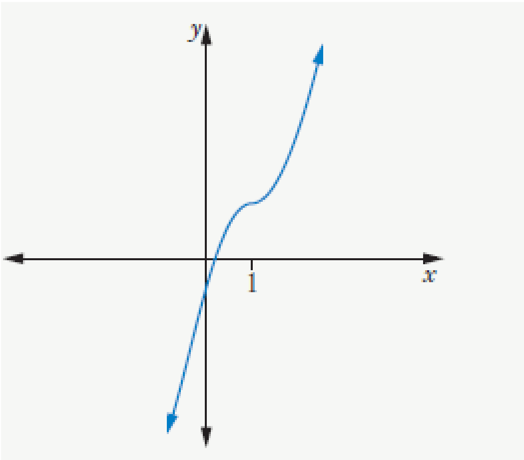
increasing for $x > 4$



Question 17

monotonic increasing $x \neq 1$

inflection point at $x = 1$



Question 18

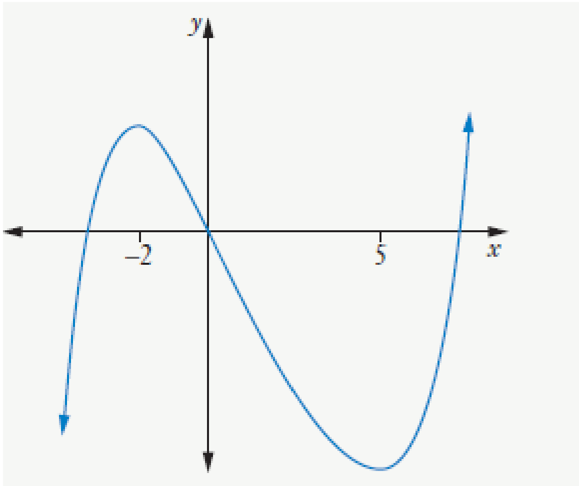
increasing $x < -2$

stationary point at $x = -2$

decreasing at $-2 < x < 5$

stationary point at $x = 5$

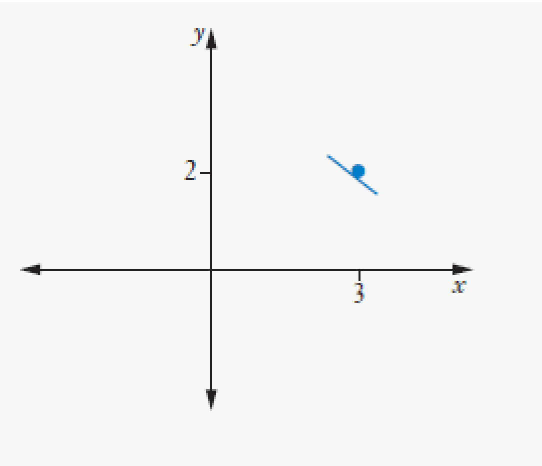
increasing $x > 5$



Question 19

point on graph at (3, 2)

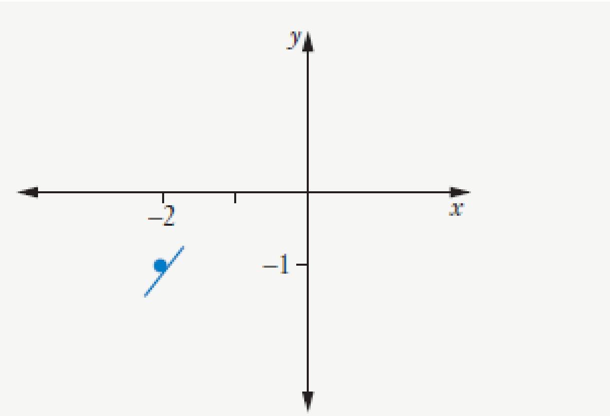
decreasing at $x = 3$



Question 20

point on graph at (–2, –1)

increasing at $x = –2$



Question 21

$y = (3x - 1)(x - 2)^4$

$y' = 3(x - 2)^4 + 4(x - 2)^3\,(3x - 1)$

$y' = (x - 2)^3\,(3(x - 2) + 4(3x - 1))$

$y' = (x - 2)^3\,(3x - 6 + 12x - 4)$

$y' = (x - 2)^3\,(15x - 10)$

$y' = 5(x - 2)^3\,(3x - 2)$

$0 = 5(x - 2)^3\,(3x - 2)$

$0 = (x - 2)^3\,(3x - 2)$

$(3x - 2) = 0$

$3x = 2$

$x = \frac{2}{3}$

$y\left(\frac{2}{3}\right) = \left[3\left(\frac{2}{3}\right) - 1\right]\left[\left(\frac{2}{3}\right) - 2\right]^4 = 3\frac{13}{81}$

$(x - 2)^3 = 0$

$x - 2 = 0$

$x = 2$

$y(2) = [3(2) - 1][(2) - 2]^4$

$y(2) = 0$

$(2, 0)$ and $(\frac{2}{3}, 3\frac{13}{81})$

Question 22

$$y = x\sqrt{x+1}$$

$$y = x(x+1)^{\frac{1}{2}}$$

$$y' = x \times \frac{1}{2}(x+1)^{-\frac{1}{2}} + 1 \times (x+1)^{\frac{1}{2}}$$

$$y' = \frac{x}{2\sqrt{x+1}} + \sqrt{x+1}$$

$$y' = \frac{x}{2\sqrt{x+1}} + \frac{2(x+1)}{2\sqrt{x+1}}$$

$$y' = \frac{x+2(x+1)}{2\sqrt{x+1}}$$

$$y' = \frac{x+2x+2}{2\sqrt{x+1}}$$

$$y' = \frac{3x+2}{2\sqrt{x+1}}$$

$$3x+2=0$$

$$3x=-2$$

$$x=-\frac{2}{3}$$

$$y\left(-\frac{2}{3}\right) = \left(-\frac{2}{3}\right)\sqrt{\left(-\frac{2}{3}\right)+1}$$

$$= \left(-\frac{2}{3}\right)\sqrt{\frac{1}{3}}$$

$$= \left(-\frac{2}{3}\right) \times \frac{\sqrt{3}}{3}$$

$$= -\frac{2\sqrt{3}}{9}$$

$$\left(-\frac{2}{3}, -\frac{2\sqrt{3}}{9}\right)$$

Question 23

$f(x) = ax^4 - 2x^3 + 7x^2 - x + 5$

$f'(x) = 4ax^3 - 6x^2 + 14x - 1$

$0 = 4a(1)^3 - 6(1)^2 + 14(1) - 1$

$0 = 4a - 6 + 14 - 1$

$-4a = 7$

$$a = -\frac{7}{4}$$

Question 24

$$f(x) = \sqrt{x}$$
$$f(x) = (x)^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2}(x)^{-\frac{1}{2}}$$
$$f'(x) = \frac{1}{2\sqrt{x}} \neq 0$$

Question 25

$$f(x) = \frac{1}{x^3}$$
$$f(x) = x^{-3}$$
$$f'(x) = -3x^{-4}$$
$$f'(x) = -\frac{3}{x^4} \neq 0$$

Exercise 5.02 Stationary points

Question 1

$y = x^2 - 1$

$y' = 2x$

$0 = 2x$

$x = 0$

$y(0) = (0)^2 - 1 = -1$

$(0, -1)$

x	-1	0	1
$\frac{dy}{dx}$	-2	0	2

Decreasing to increasing, so a minimum.

Question 2

$y = x^4$

$y' = 4x^3$

$0 = 4x^3$

$x^3 = 0$

$x = 0$

$y(0) = (0)^4 = 0$

$(0, 0)$

x	-1	0	1
$\frac{dy}{dx}$	-4	0	4

Decreasing to increasing, so a minimum.

Question 3

$f(x) = 7 - 4x - x^2$

$f'(x) = -4 - 2x$

$0 = -4 - 2x$

$4 = -2x$

$x = -2$

$f(-2) = 7 - 4(-2) - (-2)^2 = 11$

$(-2, 11)$

x	-4	-2	0
$\frac{dy}{dx}$	4	0	-6

Increasing to decreasing, so a maximum

Question 4

$y = 3x^2 + 6x + 1$

$y' = 6x + 6$

$0 = 6x + 6$

$6x = -6$

$x = -1$

$y(-1) = 3(-1)^2 + 6(-1) + 1 = -2$

$(-1, -2)$

x	-2	-1	0
$\frac{dy}{dx}$	-6	0	12

Decreasing to increasing, so a minimum.

Question 5

$y = (4 - x)^2$

$y' = 2(4 - x) \times -1$

$y' = -2(4 - x)$

$y' = -8 + 2x$

$0 = -8 + 2x$

$x = 4$

$y(4) = [4 - (4)]^2 = 0$

(4, 0)

x	2	4	6
$\frac{dy}{dx}$	–4	0	4

Decreasing to increasing, so a minimum.

Question 6

$y = x^3 - 6x^2 + 5$

$y' = 3x^2 - 12x$

$0 = 3x^2 - 12x$

$0 = x^2 - 4x$

$0 = x(x - 4)$

$x = 0$

$y(0) = (0)^3 + 6(0)^2 + 5 = 5$

(0, 5)

x	–1	0	1
$\frac{dy}{dx}$	5	0	–3

Increasing to decreasing, so a maximum

$x - 4 = 0$

$x = 4$

$y(4) = (4)^3 - 6(4)^2 + 5 = -27$

(4, –27)

x	1	4	5
$\frac{dy}{dx}$	–3	0	5

Decreasing to increasing, so a minimum.

Question 7

$y = x^3 - 3x^2 + 5$

$y' = 3x^2 - 6x$

$0 = 3x^2 - 6x$

$x^2 - 2x = 0$

$x(x - 2) = 0$

$x = 0$

$y(0) = (0)^3 - 3(0)^2 + 5 = 5$

(0, 5)

x	–1	0	1
$\frac{dy}{dx}$	9	0	–3

Increasing to decreasing, so a maximum.

$x - 2 = 0$

$x = 0$

$y(2) = (2)^3 - 3(2)^2 + 5 = 1$

(2, 1)

x	1	2	3
$\frac{dy}{dx}$	–3	0	9

Decreasing to increasing, so a minimum

Question 8

$f(x) = x^4 - 2x^2 - 3$

$f'(x) = 4x^3 - 4x$

$0 = 4x^3 - 4x$

$0 = 4x(x^2 - 1)$

$0 = 4x(x - 1)(x + 1)$

$4x = 0$

$x = 0$

$f(0) = (0)^4 - 2(0)^2 - 3 = -3$

$(0, -3)$

x	–0.5	0	0.5
$\frac{dy}{dx}$	1.5	0	–1.5

Increasing to decreasing, so a maximum

$(x - 1) = 0$

$x = 1$

$f(1) = (1)^4 - 2(1)^2 - 3 = -4$

$(1, -4)$

x	0.5	1	2
$\frac{dy}{dx}$	–1.5	0	24

Decreasing to increasing, so a minimum.

$(x + 1) = 0$

$x = -1$

$f(-1) = (-1)^4 - 2(-1)^2 - 3 \; f(-1) = -4$

$(-1, -4)$

x	-2	-1	0.5
$\frac{dy}{dx}$	-24	0	1.5

Decreasing to increasing, so a minimum.

Question 9

$y = x^3 - 3x + 2$

$y' = 3x^2 - 3$

$0 = 3x^2 - 3$

$0 = x^2 - 1$

$0 = (x - 1)(x + 1)$

$0 = (x - 1)$

$x = 1$

$y(1) = (1)^3 - 3(1) + 2 = 0$

(1, 0)

x	0	1	2
$\frac{dy}{dx}$	–3	0	9

Decreasing to increasing, so a minimum.

$0 = (x + 1)$

$x = -1$

$y(-1) = (-1)^3 - 3(-1) + 2 = 4$

(–1, 4)

x	–2	–1	0
$\frac{dy}{dx}$	9	0	–3

Increasing to decreasing, so a maximum.

Question 10

$y = x^5 + mx^3 - 2x^2 + 5$

$y' = 5x^4 + 3mx^2 - 4x$

$0 = 5(-1)^4 + 3m(-1)^2 - 4(-1)$

$0 = 5 + 3m + 4$

$-9 = 3m$

$m = -3$

Question 11

$f'(x) = 3 + x$

$0 = 3 + x$

$x = -3$

x	-4	-3	0
$\frac{dy}{dx}$	-1	0	3

Decreasing to increasing, so a minimum.

Question 12

$f'(x) = x(x + 1)$

$x = 0$

x	–0.5	0	1
$\frac{dy}{dx}$	–0.25	0	2

Decreasing to increasing, so a minimum

$x + 1 = 0$

$x = –1$

x	–2	–1	–0.5
$\frac{dy}{dx}$	2	0	–0.25

Increasing to decreasing, so a maximum.

Question 13

a

$$P = 2x + \frac{50}{x}$$

$$\frac{dP}{dx} = 2 - \frac{50}{x^2}$$

b

$$P = 2x + \frac{50}{x}$$

$$\frac{dP}{dx} = 2 - \frac{50}{x^2}$$

$$0 = 2 - \frac{50}{x^2}$$

$$2 = \frac{50}{x^2}$$

$$2x^2 = 50$$

$$x^2 = 50$$

$$x = \pm 5$$

$P(5) = 2(5) + \dfrac{50}{5} = 20$; $(5, 20)$

x	1	5	10
$\frac{dy}{dx}$	–48	0	1.5

Decreasing to increasing, so a minimum.

$P(–5) = 2(–5) + \dfrac{50}{(-5)} = –20$; $(–5, –20)$

x	–10	–5	–1
$\frac{dy}{dx}$	1.5	0	–48

Increasing to decreasing, so a maximum.

Question 14

$$A = \frac{h^2 - 2h + 5}{8}$$

$$A = \frac{1}{8}\left(h^2 - 2h + 5\right)$$

$$A' = \frac{1}{8}\left(2h - 2\right)$$

$$A' = \frac{h-1}{4}$$

$$0 = \frac{h-1}{4}$$

$0 = h - 1$

$h = 1$

$$A(1) = \frac{(1)^2 - 2(1) + 5}{8} = \frac{4}{8} = \frac{1}{2}$$

$(1, \frac{1}{2})$

x	0	1	2
$\frac{dy}{dx}$	–0.25	0	0.25

Decreasing to increasing, so a minimum.

Question 15

$V = 40r - \pi r^3$

$V' = 40 - 3\pi r^2$

$0 = 40 - 3\pi r^2$

$-40 = -3\pi r^2$

$\pi r^2 = \dfrac{40}{3}$

$r^2 = \dfrac{40}{3\pi}$

$r = \pm\sqrt{\dfrac{40}{3\pi}}$

$r = \pm 2.06$

$V(2.06) = 40(2.06) - \pi(2.06)^3 = 54.94$

(2.06, 54.94)

x	0	2.06	4
$\dfrac{dy}{dx}$	40	0	–110.8

Increasing to decreasing, so a maximum.

$V(-2.06) = 40(-2.06) - \pi(-2.06)^3 = -54.94$

(–2.06, –54.94)

x	–4	–2.06	0
$\dfrac{dy}{dx}$	–110.8	0	40

Decreasing to increasing, so a minimum

Question 16

$S = 2\pi r + \dfrac{120}{r}$

$S' = 2\pi - \dfrac{120}{r^2}$

$0 = 2\pi - \dfrac{120}{r^2}$

$\dfrac{120}{r^2} = 2\pi$

$120 = 2\pi r^2$

$r^2 = \dfrac{120}{2\pi}$

$r = \pm\sqrt{\dfrac{120}{2\pi}} = \pm 4.37$

$S(4.37) = 2\pi(4.37) + \dfrac{120}{4.37} = 54.92$

(4.37, 54.92)

x	1	4.37	5
$\dfrac{dy}{dx}$	–113.7	0	1.5

Decreasing to increasing, so a minimum.

$S(-4.37) = 2\pi(-4.37) + \dfrac{120}{(-4.37)} = -54.92$

(–4.37, –54.92)

X	–5	–4.37	1
$\dfrac{dy}{dx}$	1.5	0	–113.7

Increasing to decreasing, so a maximum.

Question 17

a

$$A = x\sqrt{3600 - x^2}$$

$$A = x\left(3600 - x^2\right)^{\frac{1}{2}}$$

$$A' = x\left(3600 - x^2\right)^{\frac{1}{2}} + x \times \frac{1}{2}\left(3600 - x^2\right)^{-\frac{1}{2}} \times -2x$$

$$A' = \left(3600 - x^2\right)^{\frac{1}{2}} - x^2\left(3600 - x^2\right)^{-\frac{1}{2}}$$

$$A' = \sqrt{3600 - x^2} - \frac{x^2}{\sqrt{3600 - x^2}}$$

$$A' = \frac{3600 - x^2}{\sqrt{3600 - x^2}} - \frac{x^2}{\sqrt{3600 - x^2}}$$

$$A' = \frac{3600 - x^2 - x^2}{\sqrt{3600 - x^2}}$$

$$A' = \frac{3600 - 2x^2}{\sqrt{3600 - x^2}}$$

b

$$A' = \frac{3600 - 2x^2}{\sqrt{3600 - x^2}}$$

$$0 = \frac{3600 - 2x^2}{\sqrt{3600 - x^2}}$$

$$0 = 3600 - 2x^2$$

$$2x^2 = 3600$$

$$x^2 = 1800$$

$$x = \pm 42.4$$

$A(42.4) = 42.4\sqrt{3600 - (42.4)^2} = 1800$

(42.4, 1800)

x	0	42.4	50
$\frac{dy}{dx}$	60	0	–42.2

Increasing to decreasing, so a maximum.

$A(–42.4) = -42.4\sqrt{3600 - (-42.4)^2} = -1800$

(–42.4, –1800)

x	–50	–42.4	0
$\frac{dy}{dx}$	–42.2	0	60

Decreasing to increasing, so a minimum

Exercise 5.03 Concavity and points of inflection

Question 1

$y = x^3 + x^2 - 2x - 1$

$y' = 3x^2 + 2x - 2$

$y'' = 6x + 2$

$0 > 6x + 2$

$6x > -2$

$x > -\frac{1}{3}$

Question 2

$y = (x-3)^3$

$y' = 3(x-3)^2$

$y'' = 6(x-3)$

$y'' = 6x - 18$

$0 < 6x - 18$

$6x < 18$

$x < 3$

Question 3

$y = 8 - 6x - 4x^2$

$y' = -6 - 8x$

$y'' = -8 < 0$

Question 4

$y = x^2$

$y' = 2x$

$y'' = 2 > 0$

Question 5

$y = x^3 - 7x^2 + 1$

$y' = 3x^2 - 14x$

$y'' = 6x - 14$

$0 > 6x - 14$

$6x < 14$

$x < \frac{14}{6}$

$(-\infty, 2\frac{1}{3})$

Question 6

$g(x) = x^3 - 3x^2 + 2x + 9$

$g'(x) = 3x^2 - 6x + 2$

$g'(x) = 6x - 6$

$0 = 6x - 6$

$6x = 6$

$x = 1$

$g(1) = (1)^3 - 3(1)^2 + 2(1) + 9 = 9$

(1, 9)

Question 7

$y = x^4 - 6x^2 + 12x - 24$

$y' = 4x^3 - 12x + 12$

$y'' = 12x^2 - 12$

$0 = 12x^2 - 12$

$12x^2 = 12$

$x^2 = 1$

$x = \pm 1$

$y(1) = (1)^4 - 6(1)^2 + 12(1) - 24 = -17$

(1, –17)

$y(-1) = (-1)^4 - 6(-1)^2 + 12(-1) - 24 = -41$

(–1, –41)

Question 8

$y = x^3 - 2$

$y' = 3x^2$

$y'' = 6x$

$0 = 6x$

$x = 0$

$y(0) = (0)^3 - 2 = -2$

(0, –2)

x	–1	0	1
y''	–6	0	6

$y'' < 0$ on LHS, $y'' > 0$ on RHS as required.

Question 9

a $y = x^6$

$y' = 6x^5$

$y'' = 30x^4 > 0$

No

b $y = x^7$

$y' = 7x^6$

$y'' = 42x^5$

$0 = 42x^5$

$x^5 = 0$

$x = 0$

$y(0) = (0)^7 = 0$

(0, 0)

Yes, point of inflection.

c $y = x^5$

$y' = 5x^4$

$y'' = 20x^3$

$0 = 20x^3$

$x^3 = 0$

$x = 0$

$y(0) = (0)^5 = 0$

(0, 0)

Yes, point of inflection.

d $y = x^9$

$y' = 9x^8$

$y'' = 72x^7$

$0 = 72x^7$

$x^7 = 0$

$x = 0$

$y(0) = (0)^9 = 0$

(0, 0)

Yes, point of inflection.

e $y = x^{12}$

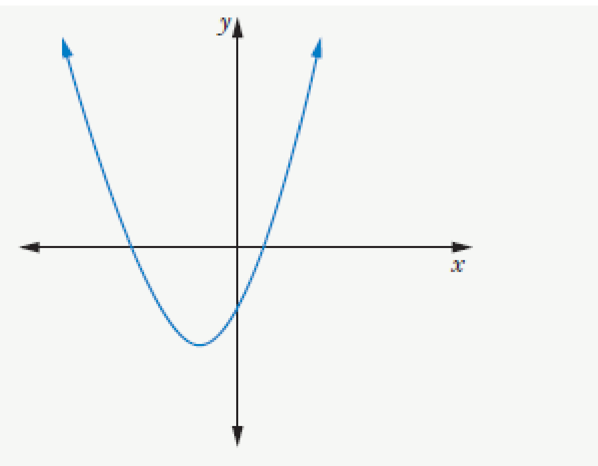
$y' = 12x^{11}$

$y'' = 121x^{10} > 0$

No

Question 10

Any ax^b where a is positive and b is even, etc

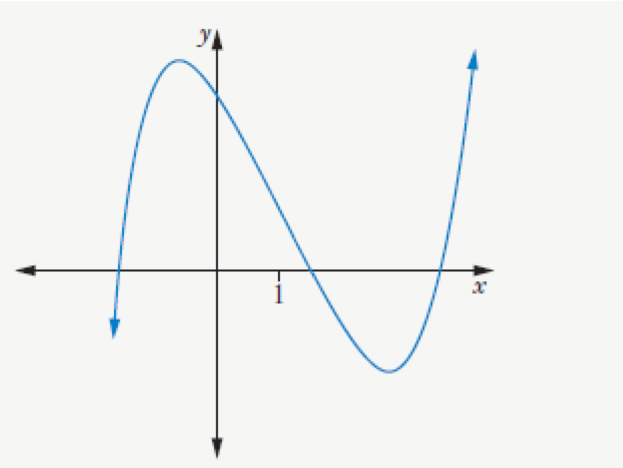


Question 11

concave down for $x > 1$

concave up for $x < 1$

point of inflection at 1



Question 12

$y = x^4 - 8x^3 + 24x^2 - 4x - 9$

$y' = 4x^3 - 24x^2 + 48x - 4$

$y'' = 12x^2 - 48x + 48$

$0 = 12x^2 - 48x + 48$

$0 = x^2 - 4x + 4$

$0 = (x-2)^2$

$x - 2 = 0$

$x = 2$

$y(2) = (2)^4 - 8(2)^3 + 24(2)^2 - 4(2) - 9 = 31$

(2, 31)

x	0	2	4
y''	48	0	48

Not an inflection point as concavity does not change

Question 13

$f(x) = \frac{2}{x^2}$

$f'(x) = \frac{-4}{x^3}$

$f''(x) = \frac{12}{x^4} > 0$

$x^4 > 0$ for all $x \neq 0$.

Question 14

a $f(x) = 3x^5 - 10x^3 + 7$

$f'(x) = 15x^4 - 30x^2$

$f''(x) = 60x^3 - 60x$

$0 = 60x^3 - 60x$

$0 = x^3 - x$

$0 = x(x^2 - 1)$

$0 = x(x-1)(x+1)$

$x = 0$

$f(0) = 3(0)^5 - 10(0)^3 + 7 = 7$

(0, 7)

$f(1) = 3(1)^5 - 10(1)^3 + 7 = 0$

(1, 0)

$x + 1 = 0$

$x = -1$

$f(-1) = 3(-1)^5 - 10(-1)^3 + 7 = 14$

(–1, 14)

b

x	–2	–1	–0.5	0	0.5	1	2
y''	–360	0	22.5	0	22.5	0	360

(0, 7) has no change in concavity, so it is a horizontal inflection point.

Question 15

a $y = x^4 + 12x^2 - 20x + 3$

$y' = 4x^3 + 24x - 20$

$y'' = 12x^2 + 24$

$\dfrac{d^2y}{dx^2} = 12x^2 + 24$

$x^2 \geq 0$ for all x

So $12x^2 \geq 0$ for all x

$12x^2 + 24 \geq 24$ for all x

$12x^2 + 24 \neq 0$ and there are no points of inflection.

b $y = x^4 + 12x^2 - 20x + 3$

$y' = 4x^3 + 24x - 20$

$y'' = 12x^2 + 24 > 0$

The curve is always concave upwards.

Question 16

$y = ax^3 - 12x^2 + 3x - 5$

$y' = 3ax^2 - 24x + 3$

$y'' = 6ax - 24$

$0 = 6a(2) - 24$

$24 = 12a$

$a = 2$

Question 17

$f(x) = x^4 - 6px^2 - 20x + 11$

$f'(x) = 4x^3 - 12px - 20$

$f''(x) = 12x^2 - 12p$

$0 = 12(-2)^2 - 12p$

$0 = 48 - 12p$

$12p = 48$

$p = 4$

Question 18

$y = 2ax^4 + 4bx^3 - 72x^2 + 4x - 3$

$y' = 8ax^3 + 12bx^2 - 144x + 4$

$y'' = 24ax^2 + 24bx - 144$

Using $x = 2$:

$0 = 24a(2)^2 + 24b(2) - 144$

$0 = 96a + 48b - 144$

$0 = 2a + b - 3$ [1]

Using $x = -1$:

$0 = 24a(-1)^2 + 24b(-1) - 144$

$0 = 24a - 24b - 144$

$0 = a - b - 6$ [2]

$0 = 3a - 9$ [1] + [2]

$3a = 9$

$a = 3$

$0 = 3 - b - 6$ Substitute for a in [2]

$b = 3 - 6$

$b = -3$

Question 19

a $y = x^6 - 3x^5 + 21x - 8$

$y' = 6x^5 - 15x^4 + 21$

$y'' = 30x^4 - 60x^3$

$0 = 30x^4 - 60x^3$

$0 = x^4 - 2x^3$

$0 = x^3(x-2)$

$x^3 = 0$

$x = 0$

$y(0) = (0)^6 - 3(0)^5 + 21(0) - 8 = -8$

(0, –8)

$x - 2 = 0$

$x = 2$

$y(2) = (2)^6 - 3(2)^5 + 21(2) - 8 = 2$

(2, 2)

b $\frac{dy}{dx} = 6x^5 - 15x^4 + 21$

At (0, –8):

$\frac{dy}{dx} = 6(0)^5 - 15(0)^4 + 21 = 21 \neq 0$

At (2, 2):

$\frac{dy}{dx} = 6(2)^5 - 15(2)^4 + 21 = -27 \neq 0$

So these points are not horizontal points of inflection.

Exercise 5.04 Interpreting rates of change graphically

Question 1

a gradient of curve is positive

$$\frac{dy}{dx} > 0$$

concave up

$$\frac{d^2y}{dx^2} > 0$$

b gradient of curve is negative

$$\frac{dy}{dx} < 0$$

concave down

$$\frac{d^2y}{dx^2} < 0$$

c gradient of curve is positive

$$\frac{dy}{dx} > 0$$

concave down

$$\frac{d^2y}{dx^2} < 0$$

d gradient of curve is negative

$$\frac{dy}{dx} < 0$$

concave up

$$\frac{d^2y}{dx^2} > 0$$

e gradient of curve is positive

$$\frac{dy}{dx} > 0$$

concave up

$$\frac{d^2P}{dt^2} > 0$$

Question 2

a gradient of curve is positive

$$\frac{dP}{dt} > 0$$

concave down

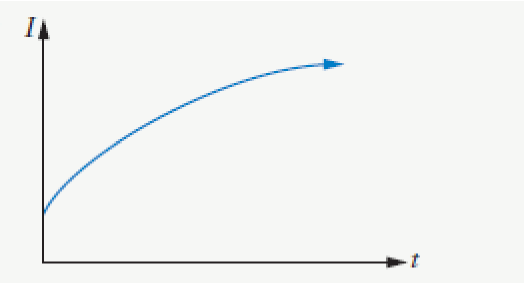
$$\frac{d^2P}{dt^2} < 0$$

b No, the rate is decreasing as the gradient is decreasing in magnitude.

Question 3

gradient of curve is positive

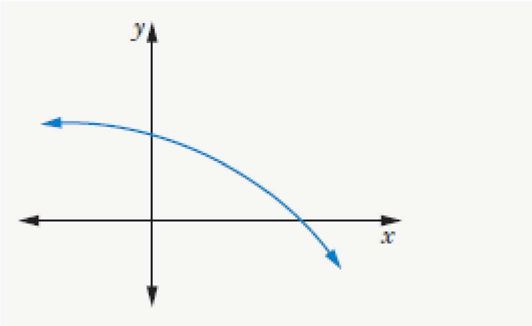
concave down



Question 4

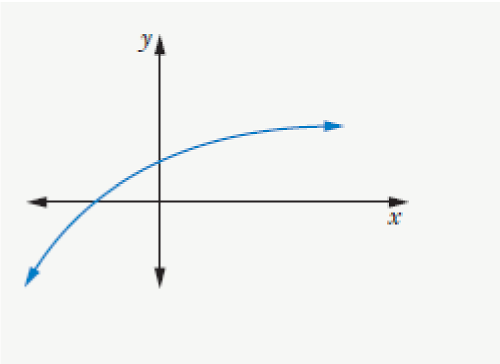
a gradient of curve is negative

concave down



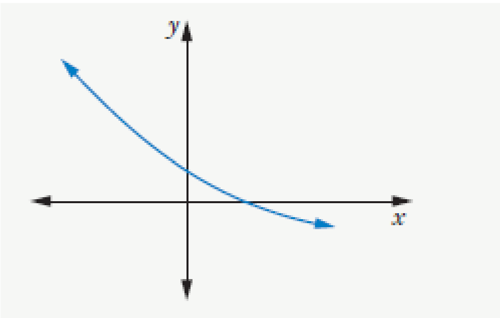
b gradient of curve is positive

concave down



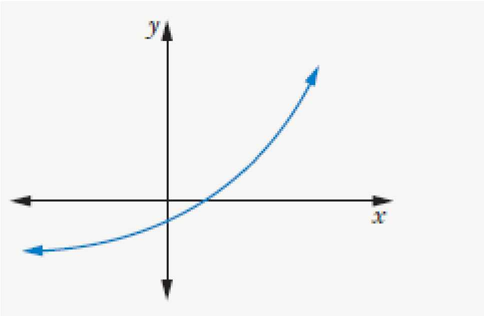
c gradient of curve is negative

concave up



d gradient of curve is positive

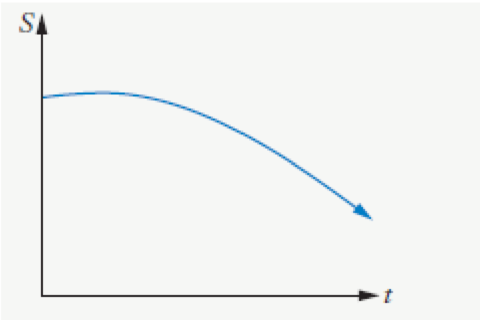
concave up



Question 5

gradient of curve is negative

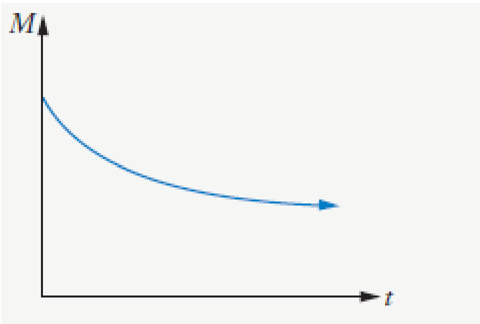
concave down



Question 6

gradient of curve is negative

concave up



Question 7

gradient of curve is negative

$$\frac{dM}{dt} < 0$$

concave up

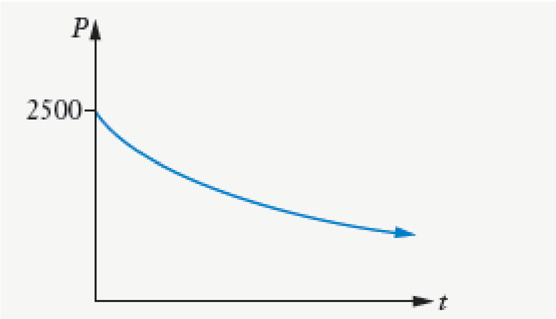
$$\frac{d^2M}{dt^2} > 0$$

Question 8

a Gradient is negative, so the number of fish is decreasing.

b Concave up, the population rate of change is increasing.

c



Question 9

Gradient is positive, so level of education is increasing.

Concave down, the rate of change is decreasing.

Question 10

Gradient is negative, school population is decreasing.

Concave down, the population rate of change is decreasing.

Exercise 5.05 Stationary points and the second derivative

Question 1

$y = x^2 - 2x + 1$

$y' = 2x - 2$

$0 = 2x - 2$

$2x = 2$

$x = 1$

$y(1) = (1)^2 - 2(1) + 1 = 0$

(1, 0)

$y'' = 2$

minimum

Question 2

$f(x) = 3x^4 + 1$

$f'(x) = 12x^3$

$0 = 12x^3$

$x^3 = 0$

$x = 0$

$f(0) = 3(0)^4 + 1 = 1$

(0, 1)

$f''(x) = 36x^2$

$f''(0) = 36(0)^2 = 0$

Minimum (flat).

Question 3

$f(x) = 3x^2 - 12x + 7$

$f'(x) = 6x - 12$

$0 = 6x - 12$

$12 = 6x$

$x = 2$

$f(2) = 3(2)^2 - 12(2) + 7 = -5$

(2, –5)

$f''(x) = 6 > 0$

So minimum.

Question 4

$y = x - x^2$

$y' = 1 - 2x$

$0 = 1 - 2x$

$2x = 1$

$x = \frac{1}{2}$

$y\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2$

$y\left(\frac{1}{2}\right) = \frac{1}{4}$

$(\frac{1}{2}, \frac{1}{4})$

$y'' = -2 < 0$

So maximum.

Question 5

$f(x) = 2x^3 - 5$

$f'(x) = 6x^2$

$0 = 6x^2$

$x^2 = 0$

$x = 0$

$f(0) = 2(0)^3 - 5 = -5$

(0, –5)

$f''(x) = 12x$

$\text{LHS}: f''(-1) = 12(-1) = -12 < 0$

$f''(0) = 12(0) = 0$

$\text{RHS}: f''(1) = 12(1) = 12 > 0$

Cavity changes.

So horizontal point of inflection.

Question 6

$f(x) = 2x^5 + 3$

$f'(x) = 10x^4$

$0 = 10x^4$

$x^4 = 0$

$x = 0$

$f(0) = 2(0)^5 + 3 = 3$

(0, 3)

$f''(x) = 40x^3$

$f''(0) = 40(0)^3 = 0$

$\text{LHS}: f''(-1) = 40(-1)^3 = -40 < 0$

$\text{RHS}: f''(1) = 40(1)^3 = 40 > 0$

Cavity changes.

So horizontal point of inflection.

Question 7

$f(x) = 2x^3 + 15x^2 + 36x - 50$

$f'(x) = 6x^2 + 30x + 36$

$0 = 6x^2 + 30x + 36$

$0 = x^2 + 5x + 6$

$0 = (x+2)(x+3)$

First factor: $x + 2 = 0$

$x = -2$

$f(-2) = 2(-2)^3 + 15(-2)^2 + 36(-2) - 50 = -78$

(–2, –78)

$f''(x) = 12x + 30$

$f''(-2) = 12(-2) + 30 = 6 > 0$

So minimum.

Second factor: $x + 3 = 0$

$x = -3$

$f(-3) = 2(-3)^3 + 15(-3)^2 + 36(-3) - 50 = -77$

(–3, –77)

$f''(x) = 12x + 30$

$f''(-3) = 12(-3) + 30 = -6 < 0$

So maximum.

Question 8

$f(x)=3x^4-4x^3-12x^2+1$

$f'(x)=12x^3-12x^2-24x$

$0=12x^3-12x^2-24x$

$0=x^3-x^2-2x$

$0=x\left(x^2-x-2\right)$

$0=x(x-2)(x+1)$

First factor: $x=0$

$f(0)=3(0)^4-4(0)^3-12(0)^2+1=1$

(0, 1)

$f''=36x^2-24x-24$

$f''(0)=36(0)^2-24(0)-24=-24<0$, so maximum.

Second factor: $x-2=0$

$x=2$

$f(2)=3(2)^4-4(2)^3-12(2)^2+1=-31$

(2, –31)

$f''(2)=36(2)^2-24(2)-24=72>0$, so minimum.

Third factor: $x+1=0$

$x=-1$

$f(-1)=3(-1)^4-4(-1)^3-12(-1)^2+1=-4$

(–1, –4)

$f''(-1)=36(-1)^2-24(-1)-24=36>0$, so minimum.

Question 9

$y=\left(4x^2-1\right)^4$

$y'=32x\left(4x^2-1\right)^3$

$0=32x\left(4x^2-1\right)^3$

$0=32x\left(4x^2-1\right)^3$

First factor: $32x=0$

$x=0$

$y\left(0\right)=\left(4\left[0\right]^2-1\right)^4=1$

(0, 1)

$y''=768x^2\left(4x^2-1\right)^2+32\left(4x^2-1\right)^3$

$y''\left(0\right)=768\left(0\right)^2\left(4\left[0\right]^2-1\right)^2+32\left(4\left[0\right]^2-1\right)^3=-32<0$

So maximum.

Second factor: $4x^2-1=0$

$4x^2=1$

$x^2=\frac{1}{4}$

$x=\pm\frac{1}{2}$

$y\left(\frac{1}{2}\right)=\left(4\left[\frac{1}{2}\right]^2-1\right)^4=0$

$\left(\frac{1}{2},0\right)$

x	0.25	0.5	1
$\frac{dy}{dx}$	–3.375	0	864

Gradient is decreasing to increasing, so a minimum.

$$y\left(-\frac{1}{2}\right)=\left(4\left[-\frac{1}{2}\right]^2-1\right)^4=0$$

$$\left(-\frac{1}{2},0\right)$$

x	–1	–0.5	–0.25
$\frac{dy}{dx}$	–864	0	3.375

Gradient is decreasing to increasing, so minimum.

Question 10

a $y = 2x^3 - 27x^2 + 120x$

$y' = 6x^2 - 54x + 120$

$y'' = 12x - 54$

$0 = 6x^2 - 54x + 120$

$0 = x^2 - 9x + 20$

$0 = (x-4)(x-5)$

First factor: $x - 4 = 0$

$x = 4$

$y(4) = 2(4)^3 - 27(4)^2 + 120(4) = 176$

(4, 176)

$y''(4) = 12(4) - 54 = -6 < 0$

So maximum.

Second factor: $x - 5 = 0$

$x = 5$

$y(5) = 2(5)^3 - 27(5)^2 + 120(5) = 176$

(5, 175)

$y''(5) = 12(5) - 54 = 6 > 0$

So minimum.

b $y'' = 12x - 54$

$0 = 12x - 54$

$12x = 54$

$x = 4.5$

$y(4.5) = 2(4.5)^3 - 27(4.5)^2 + 120(4.5) = 175.5$

(4.5, 175.5)

Question 11

$$y=(x-3)\sqrt{4-x}$$

$$y=(x-3)(4-x)^{\frac{1}{2}}$$

$$y'=\frac{(x-3)}{2(4-x)^{\frac{1}{2}}}+(4-x)^{\frac{1}{2}}$$

$$y'=\frac{2(4-x)}{2(4-x)^{\frac{1}{2}}}+\frac{(x-3)}{2(4-x)^{\frac{1}{2}}}$$

$$y'=\frac{8-2x-x+3}{2(4-x)^{\frac{1}{2}}}$$

$$y'=\frac{11-3x}{2(4-x)^{\frac{1}{2}}}$$

$$y''=\frac{-3(4-x)^{\frac{1}{2}}-(11-3x)\times-\frac{1}{2}(4-x)^{-\frac{1}{2}}}{2(4-x)}$$

$$y''=\frac{-3(4-x)^{\frac{1}{2}}+(11-3x)\frac{1}{2}(4-x)^{-\frac{1}{2}}}{2(4-x)}$$

$$y''=\frac{-3(4-x)+\frac{1}{2}(11-3x)}{2(4-x)^{\frac{3}{2}}}$$

$$y''=\frac{-12+3x+5.5-1.5x}{2(4-x)^{\frac{3}{2}}}$$

$$y''=\frac{1.5x-6.5}{2(4-x)^{\frac{3}{2}}}$$

$$0=\frac{11-3x}{2(4-x)^{\frac{1}{2}}}$$

$$0=11-3x$$

$3x = 11$

$x = 3.67$

$y(3.67) = (3.67 - 3)\sqrt{4 - 3.67} = 0.38$

(3.67, 0.38)

$y''(3.67) = \frac{1.5(3.67) - 6.5}{2(4 - 3.67))^{\frac{3}{2}}} = -0.09 < 0$

So maximum.

Question 12

$f(x) = x^4 + 8x^3 + 16x^2 - 1$

$f'(x) = 4x^3 + 24x^2 + 32x$

$0 = 4x^3 + 24x^2 + 32x$

$0 = x^3 + 6x^2 + 8x$

$0 = x(x^2 + 6x + 8)$

$0 = x(x+4)(x+2)$

First factor: $x = 0$

$f(0) = (0)^4 + 8(0)^3 + 16(0)^2 - 1 = -1$

(0, –1)

$f''(x) = 12x^2 + 48x + 32$

$f''(0) = 12(0)^2 + 48(0) + 32 = 32 > 0$, so minimum.

Second factor: $x + 4 = 0$

$x = -4$

$f(-4) = (-4)^4 + 8(-4)^3 + 16(-4)^2 - 1 = -1$

(–4, –1)

$f''(-4) = 12(-4)^2 + 48(-4) + 32 = 32 > 0$, so minimum.

Third factor: $x + 2 = 0$

$x = -2$

$f(-2) = (-2)^4 + 8(-2)^3 + 16(-2)^2 - 1 = 15$

(–2, 15)

$f''(-2) = 12(-2)^2 + 48(-2) + 32 = -40 < 0$, so maximum.

Question 13

a $y = ax^2 - 4x + 1$

$y' = 2ax - 4$

$0 = 2a\left(\frac{1}{2}\right) - 4$

$0 = a - 4$

$a = 4$

b $y = ax^2 - 4x + 1$

From part **a**, $y = 4x^2 - 4x + 1$

$y' = 8x - 4$

$0 = 8x - 4$

$8x = 4$

$x = \frac{1}{2}$

$y\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 1 = 0$

$(\frac{1}{2}, 0)$

$y'' = 8 > 0$

So minimum.

Question 14

$y = x^3 - mx^2 + 5x - 7$

$y' = 3x^2 - 2mx + 5$

$0 = 3(-1)^2 - 2m(-1) + 5$

$0 = 3 + 2m + 5$

$-2m = 8$

$m = -4$

Question 15

$y = ax^3 + bx^2 - x + 5$

$y' = 3ax^2 + 2bx - 1$

$y'' = 6ax + 2b$

At point of inflection: $y''(1) = 0 = 6a(1) + 2b$

$0 = 6a + 2b$

$0 = 3a + b$

$b = -3a$ [1]

$y(1) = -2 = a(1)^3 + b(1)^2 - (1) + 5$

$-2 = a + b + 4$

$-6 = a + b$

$-6 = a - 3a$ substituting in from [1]

$-6 = -2a$

$a = 3$

$b = -3(3)$ substituting onto [1]

$b = -9$

Exercise 5.06 Curve sketching

Question 1

Stationary point

$f(x) = x^2 - 3x - 4$

$f'(x) = 2x - 3$

$0 = 2x - 3$

$2x = 3$

$x = \frac{3}{2} = 1\frac{1}{2}$

$f''(x) = 2 > 0$, so minimum.

$f\left(1\frac{1}{2}\right) = \left(1\frac{1}{2}\right)^2 - 3\left(1\frac{1}{2}\right) - 4 = -6\frac{1}{4}$

$(1\frac{1}{2}, -6\frac{1}{4})$

x-intercepts

$0 = x^2 - 3x - 4$

$0 = (x-4)(x+1)$

$x - 4 = 0$

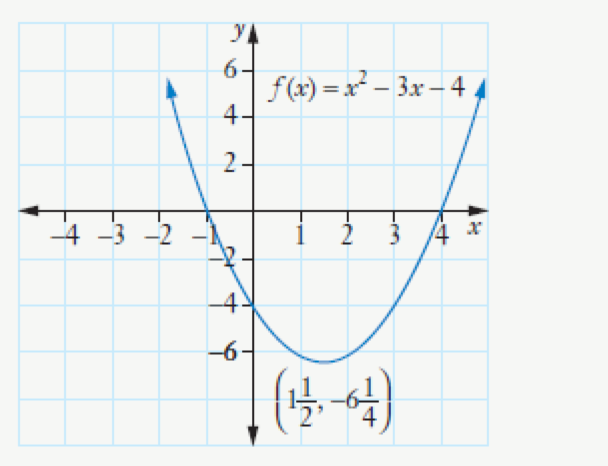
$x = 4,\ (4, 0)$

$x + 1 = 0$

$x = -1,\ (-1,\ 0)$

y-intercept

$y = (0)^2 - 3(0) - 4 = -4, (0,\ -4)$



Question 2

Stationary point

$y = 6 - 2x - x^2$

$y' = -2 - 2x$

$0 = -2 - 2x$

$-2x = 2$

$x = -1$

$y(-1) = 6 - 2(-1) - (-1)^2 = 7, \quad (-1, 7)$

$y'' = -2 < 0$, so maximum.

x-intercepts

$0 = 6 - 2x - x^2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(-1)(6)}}{2(-1)}$$

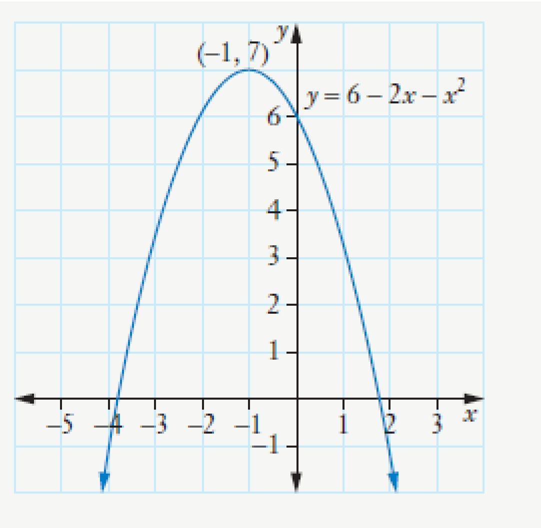
$$x = \frac{2 \pm \sqrt{28}}{-2}$$

$$x = \frac{1 \pm \sqrt{7}}{-1}$$

(–3.65, 0) and (1.65, 0)

y-intercept

$$y = 6 - 2(0) - (0)^2 = 6,\ (0,6)$$



Question 3

Stationary point

$$y = f(g(x))$$

$$y = (x-1)^3$$

$$y' = 3(x-1)^2$$

$$0 = 3(x-1)^2$$

$$0 = (x-1)^2$$

$$x - 1 = 0$$

$$x = 1$$

$$y(1) = ((1)-1)^3 = 0,\ (1,0)$$

$y'' = 6(x-1)$

$y''(1) = 6([1]-1) = 0$

$\text{LHS}: y''(0) = 6([0]-1) = -6 < 0$

$\text{RHS}: y''(2) = 6([2]-1) = 6 > 0$

Cavity changes.

So horizontal point of inflection.

x-intercepts

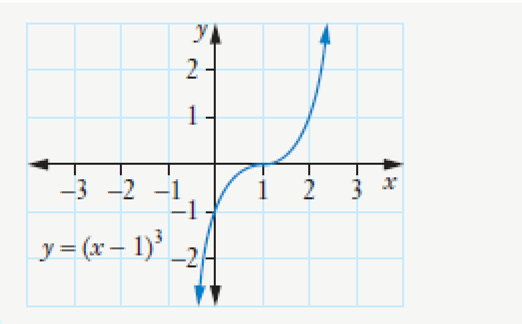
$0 = (x-1)^3$

$x - 1 = 0$

$x = 1, \quad (1, 0)$

y-intercept

$y = ([0]-1)^3 = -1, \quad (0, -1)$



Question 4

$y = x^4 + 3$

$y' = 4x^3$

$0 = 4x^3$

$x^3 = 0$

$x = 0$

$y(0) = (0)^4 + 3 = 3, \quad (0,3)$

x	–1	0	1
$\frac{dy}{dx}$	–4	3	4

Negative to positive, so a minimum.

x-intercepts

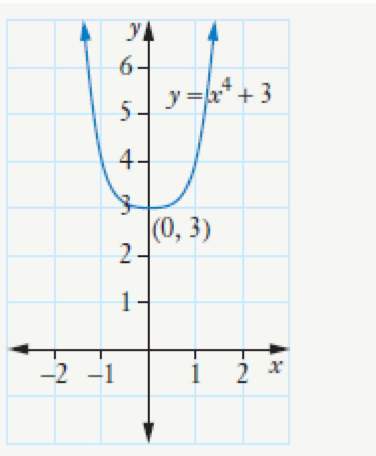
$0 = x^4 + 3$

$-3 = x^4$

none

y-intercepts

$y(0) = (0)^4 + 3 = 3, \quad (0,3)$



Question 5

$y = x^5$

$y' = 5x^4$

Stationary point

$0 = 5x^4$

$x^4 = 0$

$x = 0$

$y(0) = (0)^5 = 0, \ (0,0)$

$y'' = 20x^3$

$y''(0) = 20(0)^3 = 0$

$\text{LHS}: y''(-1) = 20(-1)^3 = -20 < 0$

$\text{RHS}: y''(1) = 20(1)^3 = 20 > 0$, concavity changes so a horizontal point of inflection.

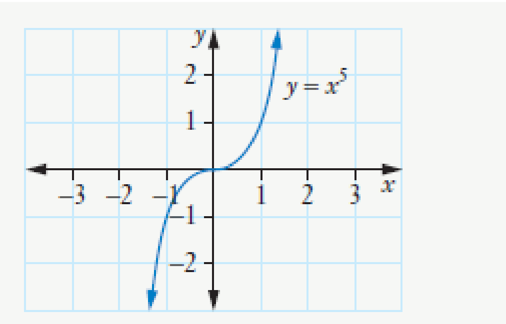
x-intercepts

$0 = x^5$

$x = 0, \ (0,0)$

y-intercepts

$y(0) = (0)^5 = 0, \ (0,0)$



Question 6

$f(x) = x^7$

$f'(x) = 7x^6$

Stationary points

$0 = 7x^6$

$x^6 = 0$

$x = 0$

$y(0) = (0)^7 = 0, \ (0,0)$

$f''(x) = 42x^5$

$y''(0) = 42(0)^5 = 0$

LHS: $y''(-1) = 42(-1)^5 = -42 < 0$

RHS: $y''(1) = 42(1)^5 = 42 > 0$, concavity changes so horizontal inflection point

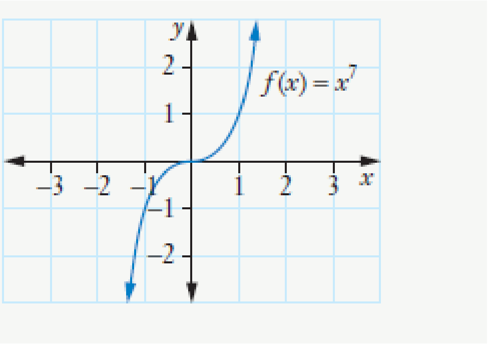
x-intercepts

$0 = x^7$

$x = 0, \ (0,0)$

y-intercepts

$y(0) = (0)^7 = 0, \ (0, 0)$



Question 7

$y = 2x^3 - 9x^2 - 24x + 30$

$y' = 6x^2 - 18x - 24$

Stationary points

$0 = 6x^2 - 18x - 24$

$0 = x^2 - 3x - 4$

$0 = (x-4)(x+1)$

First factor: $x - 4 = 0$, $x = 4$

$y(4) = 2(4)^3 - 9(4)^2 - 24(4) + 30 = -82$, $(4, -82)$

$y'' = 12x - 18$

$y''(4) = 12(4) - 18 = 30 > 0$, so a minimum

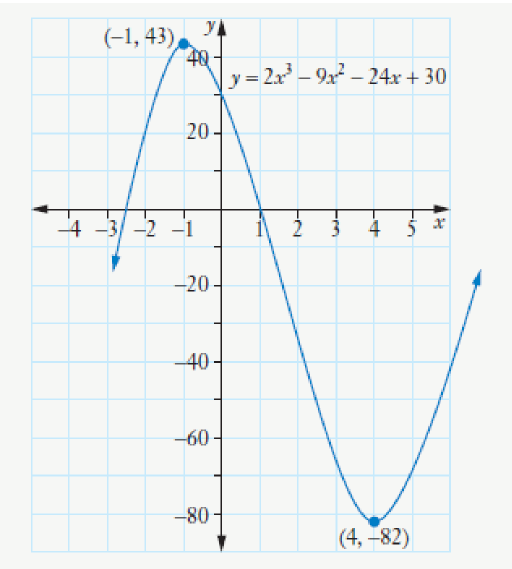
Second factor: $x + 1 = 0$, $x = -1$

$y(-1) = 2(-1)^3 - 9(-1)^2 - 24(-1) + 30 = 43$, $(-1, 43)$

$y''(-1) = 12(-1) - 18 = -30 < 0$, so a maximum

y-intercept

$y(0) = 2(0)^3 - 9(0)^2 - 24(0) + 30 = 30$, $(0, 30)$



Question 8

a $y = x^3 + 6x^2 - 7$

$y' = 3x^2 + 12x$

$y'' = 6x + 12$

$0 = 3x^2 + 12x$

$0 = 3x(x+4)$

First factor: $2x = 0$

$x = 0$

$y(0) = (0)^3 + 6(0)^2 - 7 = -7, \quad (0, -7)$

$y''(0) = 4(0) + 12 = 12 > 0$, so a minimum.

Second factor: $x + 4 = 0$

$x = -4$

$y(-4) = (-4)^3 + 6(-4)^2 - 7 = 25, \ (-4, 25)$

$y''(-4) = 4(-4) + 12 = -4 < 0$, so a maximum.

b $y = x^3 + 6x^2 - 7$

$y' = 3x^2 + 12x$

$y'' = 6x + 12$

$0 = 6x + 12$

$6x = -12$

$x = -2$

$y(-2) = (-2)^3 + 6(-2)^2 - 7 = 9, \quad (-2, 9)$

$\text{LHS}: y(-3) = (-3)^3 + 6(-3)^2 - 7 = 20 > 0$

$\text{RHS}: y(-1) = (-1)^3 + 6(-1)^2 - 7 = -2 < 0$

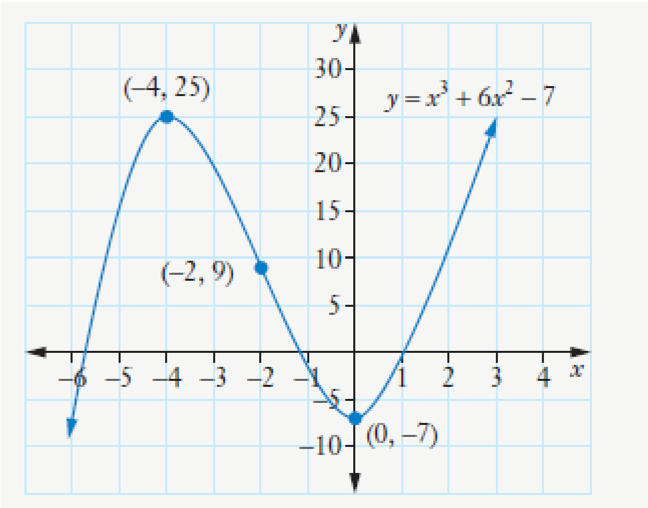
Concavity changes, so a point of inflection.

c y-intercept

$$y(0) = (0)^3 + 6(0)^2 - 7 = -7,\ \ (0, -7)$$

x-intercepts

$y(1) = (1)^3 + 6(1)^2 - 7 = 0$, so $x = 1$ is an x-intercept.



Question 9

$$y = f(x) + g(x)$$

$$y = (x^3 - 7x^2 - 1) + (x^2 + 4)$$

$$y = x^3 - 6x^2 + 3$$

$$y' = 3x^2 - 12x$$

Stationary points

$$0 = 3x^2 - 12x$$

$$0 = x^2 - 4x$$

$$0 = x(x - 4)$$

First factor: $x=0$

$y(0)=(0)^3-6(0)^2+3=3,\ (0,3)$

$y''=6x-12$

$y''(0)=6(0)-12=-12<0$, so a maximum.

Second factor: $x-4=0$, $x=4$

$y(4)=(4)^3-6(4)^2+3=-29,\ (4,-29)$

$y''(4)=6(4)-12=12>0$, so a minimum.

Point of inflection: $y''=6x-12$

$0=6x-12$

$6x=12$

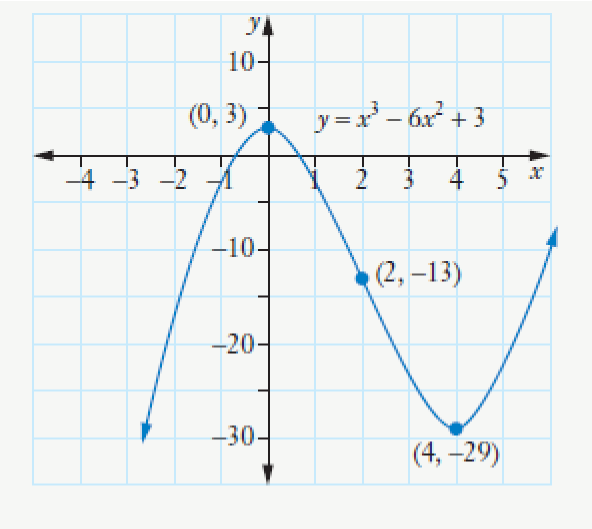
$x=2$

$y(2)=(2)^3-6(2)^2+3=-13,\ (2,-13)$

LHS: $y''(1)=6(1)-12=-12<0$
RHS: $y''(3)=6(3)-12=6>0$

Concavity changes, so a point of inflection.

y-intercept: $y(0)=(0)^3-6(0)^2+3=3,\ (0,3)$



Question 10

$y = 2 + 9x - 3x^2 - x^3$

$y' = 9 - 6x - 3x^2$

$y'' = -6 - 6x$

Stationary points

$0 = 9 - 6x - 3x^2$

$0 = 3 - 2x - x^2$

$0 = x^2 + 2x - 3$

$0 = (x+3)(x-1)$

First factor: $x + 3 = 0$

$x = -3$

$y(-3) = 2 + 9(-3) - 3(-3)^2 - (-3)^3 = -25,\ (-3, -25)$

$y''(-3) = -6 - 6(-3) = 12 > 0$, so a minimum.

Second factor: $x-1=0$

$x=1$

$y(1)=2+9(1)-3(1)^2-(1)^3=7,\ \ (1,7)$

$y''(1)=-6-6(1)=-12<0$, so a maximum.

Point of inflection

$0=-6-6x$

$6x=-6$

$x=-1$

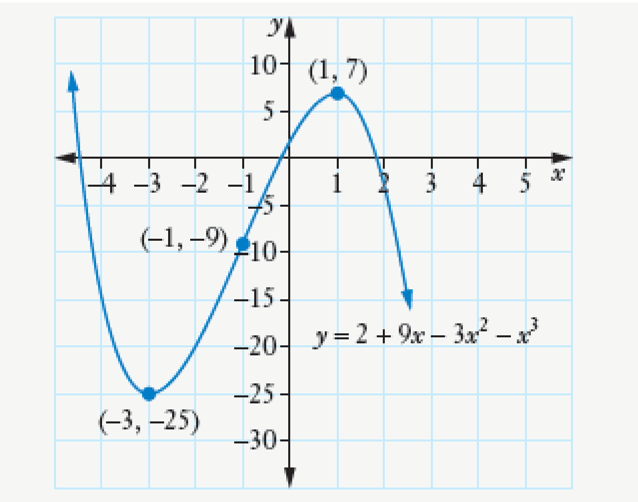
$y(-1)=2+9(-1)-3(-1)^2-(-1)^3=-9,\ \ (-1,-9)$

$\text{LHS}: y''(-2)=-6-6(-2)=6>0$
$\text{RHS}: y''(0)=-6-6(0)=-6<0$

Concavity changes, so a point of inflection.

y-intercept

$y(0)=2+9(0)-3(0)^2-(0)^3=2,\ \ (0,2)$



Question 11

$f(x)=3x^4+4x^3-12x^2-1$

$f'(x)=12x^3+12x^2-24x$

$f''(x)=36x^2+24x-24$

Stationary points

$0=12x^3+12x^2-24x$

$0=12x(x^2+x-2)$

$0=12x(x+2)(x-1)$

First factor: $12x=0$

$x=0$

$f(0)=3(0)^4+4(0)^3-12(0)^2-1=-1,\ (0,-1)$

$f''(0)=36(0)^2+24(0)-24=-24<0$, so a maximum.

Second factor: $x+2=0$

$x=-2$

$f(-2)=3(-2)^4+4(-2)^3-12(-2)^2-1=-33,\ (-2,-33)$

$f''(-2)=36(-2)^2+24(-2)-24=72>0$, so a minimum.

Third factor: $x-1=0$

$x=1$

$f(1)=3(1)^4+4(1)^3-12(1)^2-1=-6,\ (1,-6)$

$f''(1)=36(1)^2+24(1)-24=36>0$, so a minimum.

Points of inflection

$$f''(x) = 36x^2 + 24x - 24$$

$$0 = 36x^2 + 24x - 24$$
$$0 = 12(3x^2 + 2x - 2)$$

$x = -1.22$ and $x = 0.549$.

$x = -1.22$

$$f(-1.22) = 3(-1.22)^4 + 4(-1.22)^3 - 12(-1.22)^2 - 1 = -19.48,\ (-1.22, -19.48)$$

$$\text{LHS}: f''(-1.5) = 36(-1.5)^2 + 24(-1.5) - 24 = 21 > 0$$

$$\text{RHS}: f''(-1) = 36(-1)^2 + 24(-1) - 24 = -84 < 0$$

Concavity changes, so a point of inflection.

$x = 0.549$

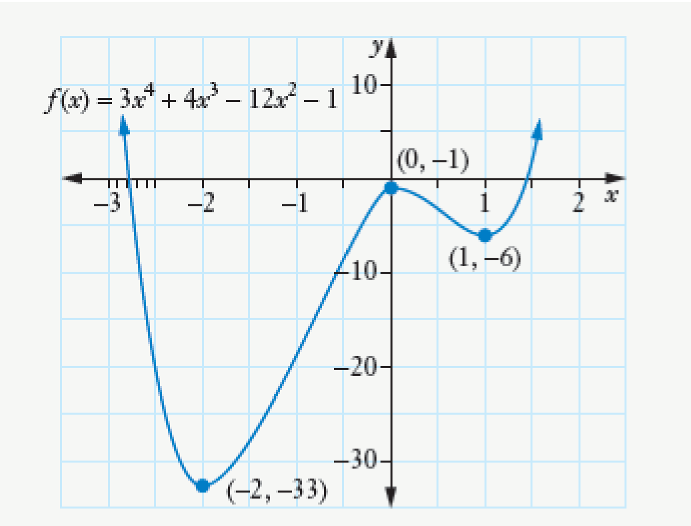
$$f(0.549) = 3(0.549)^4 + 4(0.549)^3 - 12(0.549)^2 - 1 = -3.68,\ (0.549, -3.68)$$

$$\text{LHS}: f''(0.5) = 36(0.5)^2 + 24(0.5) - 24 = -3 < 0$$

$$\text{RHS}: f''(0.6) = 36(0.6)^2 + 24(0.6) - 24 = 3.36 > 0$$

Concavity changes, so a point of inflection.

y-intercept: $f(0) = 3(0)^4 + 4(0)^3 - 12(0)^2 - 1 = -1,\ (0, -1)$



Question 12

$y=(2x+1)(x-2)^4$

$y'=4(2x+1)(x-2)^3+2(x-2)^4$

$y'=2(x-2)^3\left[2(2x+1)+(x-2)\right]$

$y'=2(x-2)^3(5x)$

$y'=10x(x-2)^3$

$y''=10(x-2)^3+30x(x-2)^2$

$y''=10(x-2)^2([x-2]+3x)$

$y''=10(x-2)^2(4x-2)$

$y''=20(x-2)^2(2x-1)$

Stationary points

$0=10x(x-2)^3$

First factor: $10x=0$

$x=0$

$y(0)=(2[0]+1)([0]-2)^4=16,\ (0,16)$

$y''(0)=20\left[(0)-2\right]^2\left[2(0)-1\right]=-80<0$, so a maximum.

Second factor: $(x-2)^3=0$

$x-2=0$

$x=2$

$y(2)=(2[2]+1)([2]-2)^4=0,\quad (2,0)$

x	1	2	3
$\frac{dy}{dx}$	–10	0	30

Gradient decreasing to increasing, so a minimum.

Points of inflection

$0 = 20(x-2)^2(2x-1)$

First factor: $(x-2)^2 = 0$

$x-2=0$

$x=2$

$y(2) = (2[2]+1)([2]-2)^4 = 0, \quad (2,0)$

Second factor: $2x-1=0$

$2x=1$

$x=0.5$

$y(0.5) = (2[0.5]+1)([0.5]-2)^4$

$y(0.5) = (2[0.5]+1)([0.5]-2)^4 = 10\frac{1}{8}, \quad (0.5, 10\frac{1}{8})$

$\text{LHS}: f''(0.4) = 20(0.4-2)^2(2[0.4]-1) = -10.24 < 0$

$\text{RHS}: f''(1) = 20(1-2)^2(2[1]-1) = 20 > 0$

Concavity changes, so a point of inflection.

y-intercept

$y(0) = (2[0]+1)([0]-2)^4 = 16, \quad (0,16)$

x-intercepts

$y=(2x+1)(x-2)^4$

First factor: $2x + 1 = 0$

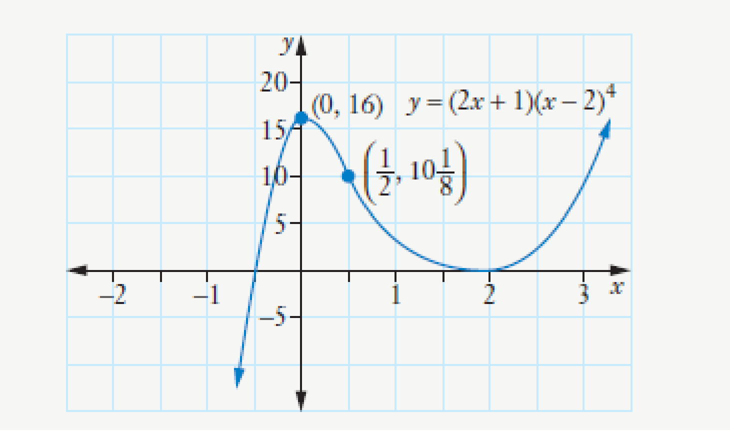
$2x = -1$

$x = -\frac{1}{2}$, $(-\frac{1}{2},0)$

Second factor: $x - 2 = 0$

$x = 2$

$y(2)=(2[2]+1)([2]-2)^4=0$, $(2,0)$



Question 13

$$y = \frac{2}{1+x}$$

$$y = 2(1+x)^{-1}$$

$$y' = -2(1+x)^{-2}$$

$$y' = \frac{-2}{(1+x)^2}$$

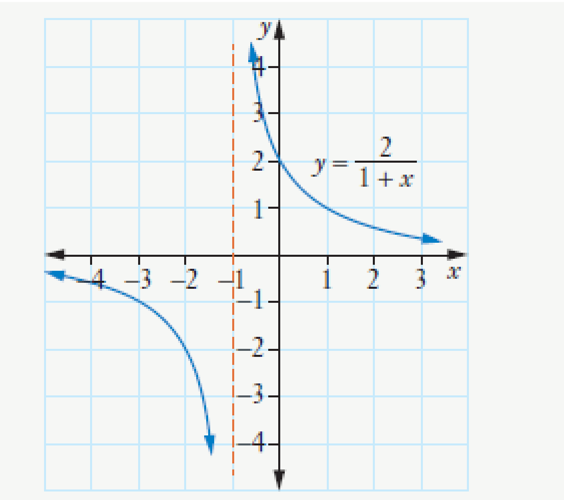
$$y' = \frac{-2}{(1+x)^2} \neq 0$$

$$y'' = 4(1+x)^{-3}$$

$$y'' = \frac{4}{(1+x)^3}$$

domain: all real x, $x \neq -1$

range: all real y, $y \neq 0$



Question 14

a $y=\cos 2x$

$y'=-2\sin 2x$

$y''=-4\cos 2x$

Stationary points

$0=-2\sin 2x$

$\sin 2x=0$

$2x=0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi,\frac{5\pi}{2},3\pi,\frac{7\pi}{2},4\pi$

$x=0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},\pi,\frac{5\pi}{4},\frac{3\pi}{2},\frac{7\pi}{4},2\pi$

$y(0)=\cos(2[0])=1,\ (0,1)$

$y''(0)=-4\cos[2(0)]=-4<0$, so a maximum.

$y\left(\frac{\pi}{4}\right)=\cos\left(2\left[\frac{\pi}{4}\right]\right)=0,\ (\frac{\pi}{4},0)$

$y''\left(\frac{\pi}{4}\right)=-4\cos\left[2\left(\frac{\pi}{4}\right)\right]=0$, possibly a point of inflection.

$\text{LHS}: y''\left(\frac{\pi}{8}\right)=-4\cos\left[2\left(\frac{\pi}{8}\right)\right]=-2\sqrt{2}<0$

$\text{RHS}: y''\left(\frac{3\pi}{8}\right)=-4\cos\left[2\left(\frac{3\pi}{8}\right)\right]=2\sqrt{2}>0$

Concavity changes, so it is a point of inflection.

$y\left(\frac{\pi}{2}\right)=\cos\left(2\left[\frac{\pi}{2}\right]\right)=-1,\ (\frac{\pi}{2},-1)$

$y''\left(\frac{\pi}{2}\right)=-4\cos\left[2\left(\frac{\pi}{2}\right)\right]=4>0$, so a minimum

$$y\left(\frac{3\pi}{4}\right)=\cos\left(2\left[\frac{3\pi}{4}\right]\right)=0,\ \left(\frac{3\pi}{4},0\right)$$

$$y''\left(\frac{3\pi}{4}\right)=-4\cos\left[2\left(\frac{3\pi}{4}\right)\right]=0$$, possibly a point of inflection

$$\text{LHS}: y''\left(\frac{5\pi}{8}\right)=-4\cos\left[2\left(\frac{5\pi}{8}\right)\right]=2\sqrt{2}>0$$

$$\text{RHS}: y''\left(\frac{7\pi}{8}\right)=-4\cos\left[2\left(\frac{7\pi}{8}\right)\right]=-2\sqrt{2}<0$$

Concavity changes, so it is a point of inflection.

$$y(\pi)=\cos(2[\pi])=1,\ (\pi, 1)$$

$$y''(\pi)=-4\cos[2(\pi)]=-4<0$$, so a maximum.

$$y\left(\frac{5\pi}{4}\right)=\cos\left(2\left[\frac{5\pi}{4}\right]\right)=0,\ \left(\frac{5\pi}{4},0\right)$$

$$y''\left(\frac{5\pi}{4}\right)=-4\cos\left[2\left(\frac{5\pi}{4}\right)\right]=0$$, possibly a point of inflection

$$\text{LHS}: y''\left(\frac{9\pi}{8}\right)=-4\cos\left[2\left(\frac{9\pi}{8}\right)\right]=-2\sqrt{2}<0$$

$$\text{RHS}: y''\left(\frac{11\pi}{8}\right)=-4\cos\left[2\left(\frac{11\pi}{8}\right)\right]=2\sqrt{2}>0$$

Concavity changes, so it is a point of inflection.

$$y\left(\frac{3\pi}{2}\right)=\cos\left(2\left[\frac{3\pi}{2}\right]\right)=-1,\ \left(\frac{3\pi}{2},-1\right)$$

$$y''\left(\frac{3\pi}{2}\right)=-4\cos\left[2\left(\frac{3\pi}{2}\right)\right]=4>0$$, so a minimum

$y\left(\frac{7\pi}{4}\right)=\cos\left(2\left[\frac{7\pi}{4}\right]\right)=0,\ (\frac{7\pi}{4},0)$

$y''\left(\frac{7\pi}{4}\right)=-4\cos\left[2\left(\frac{7\pi}{4}\right)\right]=0$, possibly a point of inflection

$\text{LHS}: y''\left(\frac{13\pi}{8}\right)=-4\cos\left[2\left(\frac{13\pi}{8}\right)\right]=2\sqrt{2}>0$

$\text{RHS}: y''\left(\frac{15\pi}{8}\right)=-4\cos\left[2\left(\frac{15\pi}{8}\right)\right]=-2\sqrt{2}<0$

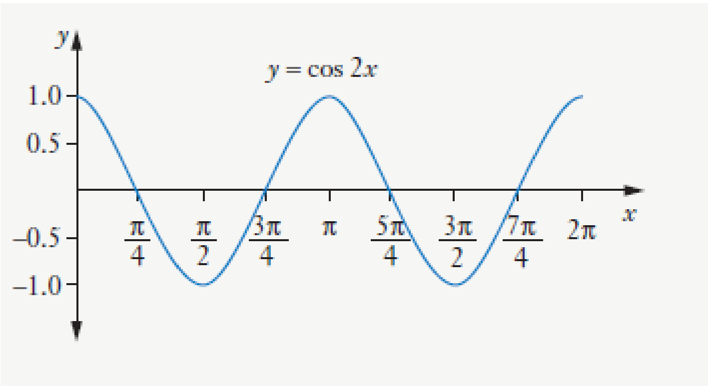
Concavity changes, so it is a point of inflection.

$y(2\pi)=\cos(2[2\pi])=1,\ (2\pi, 1)$

$y''(2\pi)=-4\cos[2(2\pi)]=-4<0$, so a maximum.

y-intercept already found (0, 1)

x-intercepts already found $(\frac{\pi}{4}, 0), (\frac{3\pi}{4}, 0), (\frac{5\pi}{4}, 0), (\frac{7\pi}{4}, 0)$



b $y = 5\sin 4x$

$y' = 20\cos 4x$

$y'' = -80\sin 4x$

Stationary points

$0 = 20\cos 4x$

$\cos 4x = 0$

$4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \frac{15\pi}{2}$

$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

$y\left(\frac{\pi}{8}\right) = 5\sin\left(4\left[\frac{\pi}{8}\right]\right) = 5,\ (\frac{\pi}{8}, 5)$

$y''\left(\frac{\pi}{8}\right) = -80\sin\left[4\left(\frac{\pi}{8}\right)\right] = -80 < 0$, so a maximum

$y\left(\frac{3\pi}{8}\right) = 5\sin\left(4\left[\frac{3\pi}{8}\right]\right) = -5,\ (\frac{3\pi}{8}, -5)$

$y''\left(\frac{3\pi}{8}\right) = -80\sin\left[4\left(\frac{3\pi}{8}\right)\right] = 80 > 0$, so a minimum

$y\left(\frac{5\pi}{8}\right) = 5\sin\left(4\left[\frac{5\pi}{8}\right]\right) = 5,\ (\frac{5\pi}{8}, 5)$

$y''\left(\frac{5\pi}{8}\right) = -80\sin\left[4\left(\frac{5\pi}{8}\right)\right] = -80 < 0$, so a maximum

$y\left(\frac{7\pi}{8}\right) = 5\sin\left(4\left[\frac{7\pi}{8}\right]\right) = -5,\ (\frac{7\pi}{8}, -5)$

$y''\left(\frac{7\pi}{8}\right) = -80\sin\left[4\left(\frac{7\pi}{8}\right)\right] = 80 > 0$, so a minimum

$$y\left(\frac{9\pi}{8}\right)=5\sin\left(4\left[\frac{9\pi}{8}\right]\right)=5,\ \left(\frac{9\pi}{8},5\right)$$

$$y''\left(\frac{9\pi}{8}\right)=-80\sin\left[4\left(\frac{9\pi}{8}\right)\right]=-80<0$$, so a maximum

$$y\left(\frac{11\pi}{8}\right)=5\sin\left(4\left[\frac{11\pi}{8}\right]\right)=-5,\ \left(\frac{11\pi}{8},-5\right)$$

$$y''\left(\frac{11\pi}{8}\right)=-80\sin\left[4\left(\frac{11\pi}{8}\right)\right]=80>0$$, so a minimum

$$y\left(\frac{13\pi}{8}\right)=5\sin\left(4\left[\frac{13\pi}{8}\right]\right)=5,\ \left(\frac{13\pi}{8},5\right)$$

$$y''\left(\frac{13\pi}{8}\right)=-80\sin\left[4\left(\frac{13\pi}{8}\right)\right]=-80<0$$, so a maximum

$$y\left(\frac{15\pi}{8}\right)=5\sin\left(4\left[\frac{15\pi}{8}\right]\right)=-5,\ \left(\frac{15\pi}{8},-5\right)$$

$$y''\left(\frac{15\pi}{8}\right)=-80\sin\left[4\left(\frac{15\pi}{8}\right)\right]=80>0$$, so a minimum

y-intercept

$$y(0)=5\sin(4[0])=0,\ (0,0)$$

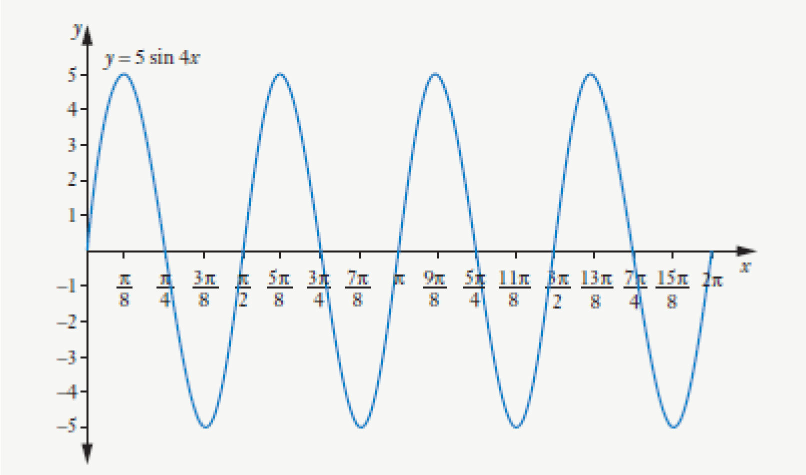
x-intercepts

$$0=5\sin 4x$$

$$\sin 4x=0$$

$$4x=0,\pi,2\pi,3\pi,4\pi,5\pi,6\pi,7\pi,8\pi$$

$$x=0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},\pi,\frac{5\pi}{4},\frac{3\pi}{2},\frac{7\pi}{4},2\pi$$



Question 15

a $y = x^2 \ln x$

$y' = 2x \ln x + x$

$y'' = 2\ln x + 2 + 1 = 2\ln x + 3$

Stationary points

$0 = 2x \ln x + x$

$0 = x(2\ln x + 1)$

First factor: $x = 0$

$y(0) = (0)^2 \ln(0) = 0, \ (0,0)$

Second factor: $2\ln x+1=0$

$\ln x=-\frac{1}{2}$

$x=\frac{1}{\sqrt{e}}$

$y\left(\frac{1}{\sqrt{e}}\right)=\left(\frac{1}{\sqrt{e}}\right)^2\ln\left(\frac{1}{\sqrt{e}}\right)=-\frac{1}{2e},\ (\frac{1}{\sqrt{e}},-\frac{1}{2e})\approx(0.607,-0.184)$

$y''\left(\frac{1}{\sqrt{e}}\right)=2\ln\left(\frac{1}{\sqrt{e}}\right)+3=1>0$, so a minimum

Points of inflection

$0=2\ln x+3$

$2\ln x=-3$

$\ln x=\frac{-3}{2}$

$x=\frac{1}{\sqrt{e^3}}$

$y\left(\frac{1}{\sqrt{e^3}}\right)=\left(\frac{1}{\sqrt{e^3}}\right)^2\ln\left(\frac{1}{\sqrt{e^3}}\right)=-\frac{3}{2e^3},\ \ (\frac{1}{\sqrt{e^3}},-\frac{3}{2e^3})\approx(0.223,-0.075)$

y-intercept already found (0, 0).

x-intercepts

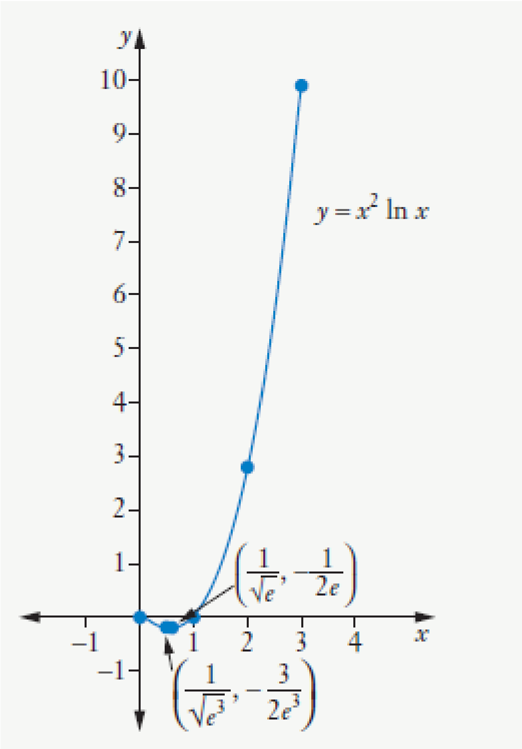
$0=x^2\ln x$

First factor: x = 0

$y(0)=(0)^2\ln(0)=0,\ (0,0)$

Second factor: ln x = 0

$x = e^0 = 1$, (1, 0)



b

$$y = \frac{x}{e^x}$$

$$y' = \frac{e^x - xe^x}{e^{2x}}$$

$$y' = \frac{1-x}{e^x}$$

$$y'' = \frac{-e^x - (1-x)e^x}{e^{2x}}$$

$$y'' = \frac{-1-(1-x)}{e^x}$$

$$y'' = \frac{x-2}{e^x}$$

Stationary points

$$0 = \frac{1-x}{e^x}$$

$$0 = 1-x$$

$$x = 1$$

$$y(1) = \frac{1}{e^1} = \frac{1}{e},\ \ (1, \frac{1}{e}) \approx (1, 0.368)$$

$y''(1) = \dfrac{(1)-2}{e^1} = -\dfrac{1}{e} < 0$, so a maximum

Point of inflection

$$y'' = \frac{x-2}{e^x}$$

$$0 = \frac{x-2}{e^x}$$

$$0 = x-2$$

$$x = 2$$

$$y(2) = \frac{2}{e^2},\ \ (2, \frac{2}{e^2}) \approx (2, 0.271)$$

$$\text{LHS}: y''(1.5) = \frac{(1.5)-2}{e^{1.5}} = -0.112 < 0$$

$$\text{RHS}: y''(2.5) = \frac{(2.5)-2}{e^{1.5}} = 0.112 > 0$$

Concavity changes, so point of inflection.

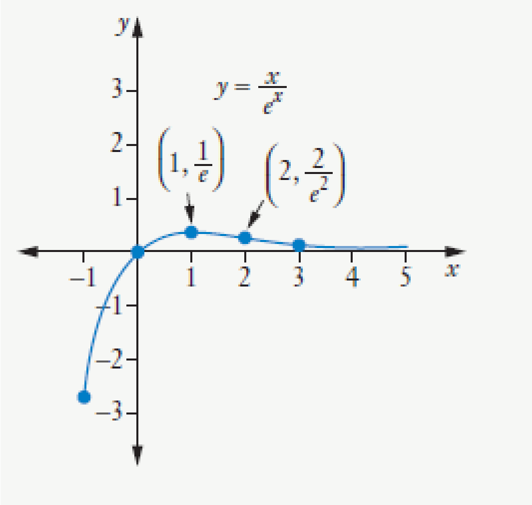
y-intercept

$$y(0) = \frac{0}{e^0} = \frac{0}{1} = 0,\ \ (0, 0)$$

x-intercepts

$$0 = \frac{x}{e^x}$$

$x = 0$, (0, 0)



c

$$y = \frac{1}{x^2 - 1}$$

$$y = \left(x^2 - 1\right)^{-1}$$

$$y' = -2x\left(x^2 - 1\right)^{-2}$$

$$y' = \frac{-2x}{\left(x^2 - 1\right)^2}$$

$$y'' = \frac{-2\left(x^2 - 1\right)^2 + 2x \times 4x\left(x^2 - 1\right)^1}{\left(x^2 - 1\right)^4}$$

$$y'' = \frac{-2\left(x^2 - 1\right) + 8x^2}{\left(x^2 - 1\right)^3}$$

$$y'' = \frac{6x^2 + 2}{\left(x^2 - 1\right)^3}$$

Stationary points

$$0=\frac{-2x}{\left(x^2-1\right)^2}$$

$$-2x=0$$

$$x=0$$

$$y(0)=\frac{1}{(0)-1}=-1,\ \ (0,1)$$

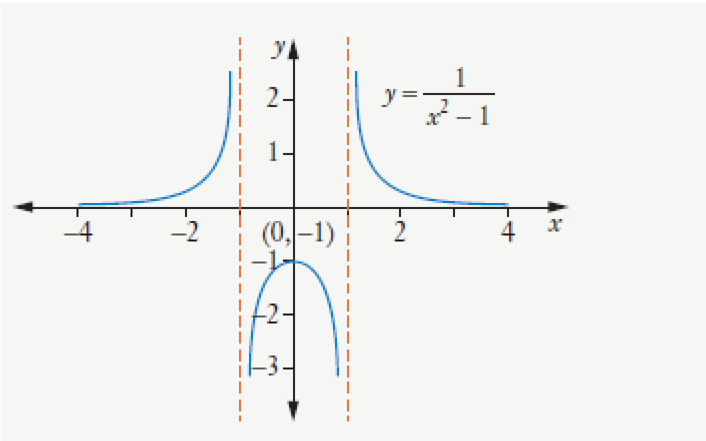
$$y''(0)=\frac{6(0)^2+2}{\left([0]^2-1\right)^3}=-2<0\text{, so a maximum}$$

Asymptotes

$$x^2-1=0$$

$$x^2=1$$

$$x=\pm1$$



Question 1

$y = x^2 + x - 2$

$y' = 2x + 1$

$y'' = 2$, so concave up for all x

Stationary points

$0 = 2x + 1$

$2x = -1$

$x = -\frac{1}{2}$

$y\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 2 = -2\frac{1}{4}$, $(-\frac{1}{2}, -2\frac{1}{4})$

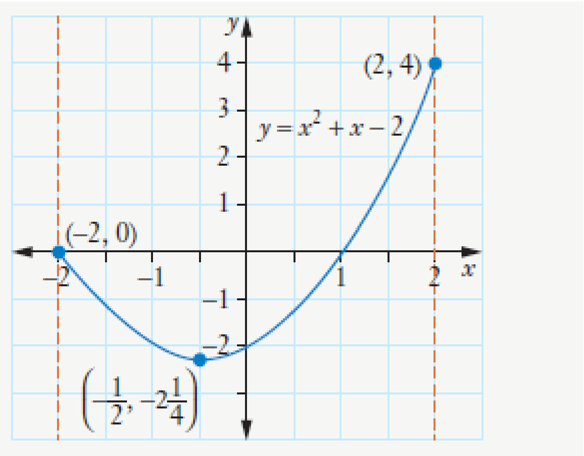
$y'' = 2 > 0$, so minimum.

Check values at domain ends.

$y(-2) = (-2)^2 + (-2) - 2 = 0$, $(-2, 0)$

$y(2) = (2)^2 + (2) - 2 = 4$, $(2, 4)$

Maximum value on the domain [–2, 2] is 4.



Question 2

$f(x) = 9 - x^2$

$f'(x) = -2x$

$f''(x) = -2$, concave down for all x

Stationary points

$0 = -2x$

$x = 0$

$f(0) = 9 - (0)^2 = 9$, $(0, 9)$

$f''(x) = -2 < 0$, so a maximum.

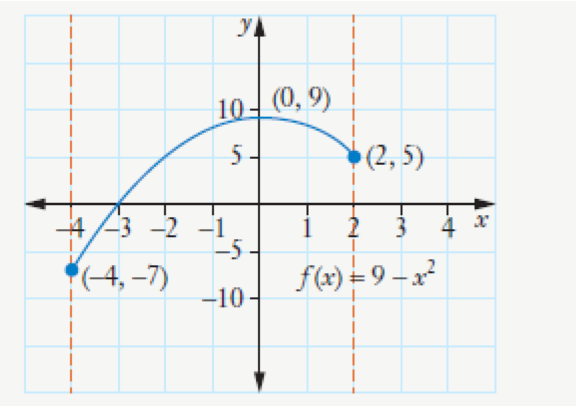
Check values at end points of domain

$f(-4) = 9 - (-4)^2 = -7$, $(-4, -7)$

$f(2) = 9 - (2)^2 = 5$, $(2, 5)$

$f(2) = 5$

Maximum value is 9, minimum value is –7.



Question 3

$y = x^2 - 4x + 4$

$y' = 2x - 4$

$y'' = 2$, so concave up for all *x*.

Stationary points.

$0 = 2x - 4$

$2x = 4$

$x = 2$

$y(2) = (2)^2 - 4(2) + 4 = 0$, $(2, 0)$

$y'' = 2 > 0$, so a minimum.

Check the values at the end of the domain.

$y(-3) = (-3)^2 - 4(-3) + 4 = 25$, $(-3, 25)$

$y(3) = (3)^2 - 4(3) + 4 = 1$, $(3, 1)$

The maximum value is 25.

Question 4

$y = 2x^3 + 3x^2 - 36x + 5$

$y' = 6x^2 + 6x - 36$

$y'' = 12x + 6$

Stationary points

$0 = 6x^2 + 6x - 36$

$0 = x^2 + x - 6$

$0 = (x+3)(x-2)$

First factor: $x + 3 = 0$

$x = -3$

$y(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) + 5 = 86$, $(-3, 86)$

$y''(-3) = 12(-3) + 6 = -30 < 0$, so a maximum.

Second factor: $x - 2 = 0$

$x = 2$

$y(2) = 2(2)^3 + 3(2)^2 - 36(2) + 5 = -39$, $(2, -39)$

$y''(2) = 12(2) + 6 = 30 > 0$, so a minimum.

y-intercept

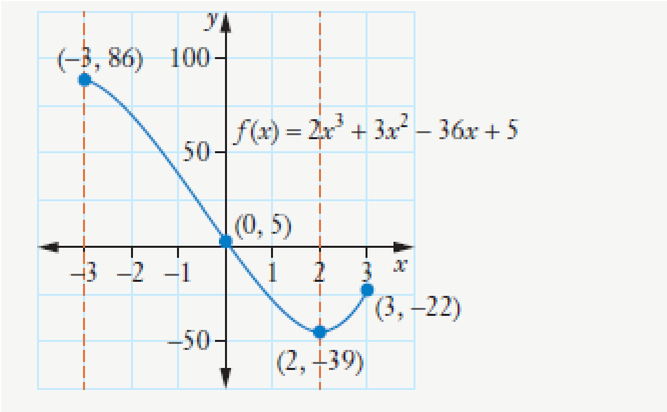
$y(0) = 2(0)^3 + 3(0)^2 - 36(0) + 5 = 5$, $(0, 5)$

Check values at ends of domain.

(–3, 86) already found.

$y(3) = 2(3)^3 + 3(3)^2 - 36(3) + 5 = -22$, $(3, -22)$

Maximum is 86, minimum is –39.



Question 5

$y = x^5 - 3$

$y' = 5x^4$

$y'' = 20x^3$

Stationary points

$0 = 5x^4$

$x^4 = 0$

$x = 0$

$y(0) = (0)^5 - 3 = -3, (0, -3)$

$y"(0) = 20(0)^3 = 0$, possible point of inflection

Check values at the ends of the domain.

$y(-2) = (-2)^5 - 3 = -35, (-2, -35)$

$y(1) = (1)^5 - 3 = -2, (1, -2)$

Maximum value is –2.

Question 6

$f(x) = 3x^2 - 16x + 5$

$f'(x) = 6x - 16$

$f''(x) = 6$, concave up for all x

Stationary points

$$\begin{aligned} 0 &= 6x - 16 \\ 6x &= 16 \\ x &= 2\frac{2}{3} \end{aligned}$$

$f\left(2\frac{2}{3}\right) = 3\left(2\frac{2}{3}\right)^2 - 16\left(2\frac{2}{3}\right) + 5 = -16\frac{1}{3}$, $\left(2\frac{2}{3}, -16\frac{1}{3}\right)$

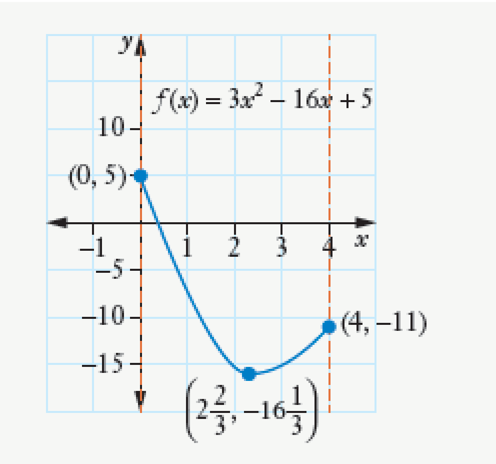
$f''(x) = 6 > 0$, so a minimum.

Check the values at the ends of the domain.

$f(0) = 3(0)^2 - 16(0) + 5 = 5$, $(0, 5)$

$f(4) = 3(4)^2 - 16(4) + 5 = -11$, $(4, -11)$

Maximum is 5 and minimum is $-16\frac{1}{3}$.



Question 7

$f(x)=3x^4+4x^3-12x^2-3$

$f'(x)=12x^3+12x^2-12x$

$f''(x)=36x^2+24x-24$

Stationary points

$0=12x^3+12x^2-12x$

$0=x^3+x^2-2x$

$0=x\left(x^2+x-2\right)$

$0=x(x+2)(x-1)$

First factor: $x=0$

$f(0)=3(0)^4+4(0)^3-12(0)^2-3=-3,\ (0,-3)$

$f''(0)=36(0)^2+24(0)-24=-24<0$, so a maximum

Second factor: $x+2=0$

$x=-2$

$f(-2)=3(-2)^4+4(-2)^3-12(-2)^2-3=-35,\quad (-2,-35)$

$f''(-2)=36(-2)^2+24(-2)-24=72>0$, so a minimum

Third factor: $x-1=0$

$x=1$

$f(1)=3(1)^4+4(1)^3-12(1)^2-3=-8,\ (1,-8)$

$f''(1)=36(1)^2+24(1)-24=36>0$, so a minimum

Check the values at the endpoints:

$f(-2)$ already found, $(-2, -35)$

$$f(2) = 3(2)^4 + 4(2)^3 - 12(2)^2 - 3 = 29, \quad (2, 29)$$

Absolute maximum 29, relative maximum –3,

absolute minimum –35, relative minimum –35, –8

Question 8

$y = x^3 + 2$

$y' = 3x^2$

$y'' = 6x$

Stationary points

$0 = 3x^2$

$x^2 = 0$

$x = 0$

$y(0) = (0)^3 + 2 = 2,\ (0, 2)$

$y''(0) = 6(0) = 0$, possible point of inflection

$\text{LHS}: y''(-1) = 6(-1) = -6 < 0$

$\text{RHS}: y''(1) = 6(1) = 6 > 0$

Change of concavity, so a point of inflection.

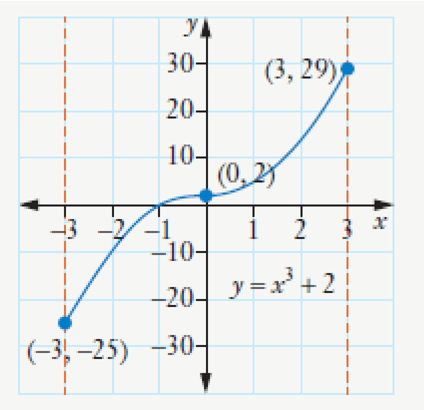
(0,0)

Check the values at the endpoints.

$y(-3) = (-3)^3 + 2 = -25,\ \ (-3, -25)$

$y(3) = (3)^3 + 2 = 29,\ \ (3, 29)$

Minimum –25, maximum 29.



Question 9

$$y=\sqrt{x+5}$$

$$y=(x+5)^{\frac{1}{2}}$$

$$y'=\frac{1}{2}(x+5)^{-\frac{1}{2}}$$

$$y'=\frac{1}{2\sqrt{x+5}}$$

$$y''=-\frac{1}{4}(x+5)^{-\frac{3}{2}}$$

$$y''=-\frac{1}{4\sqrt{x+5}^{3}}$$

Stationary points

$$0=\frac{1}{2\sqrt{x+5}}$$

$\frac{1}{2\sqrt{x+5}}\neq 0$, so no stationary points.

y-intercept

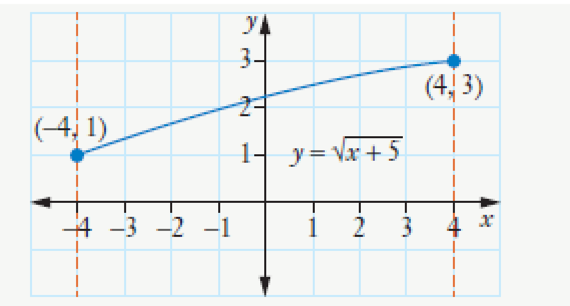
$$y(0)=\sqrt{(0)+5}=\sqrt{5}\approx 2.24,\quad (0,2.24)$$

Check the values at the endpoints.

$$y(-4)=\sqrt{(-4)+5}=1,\quad (-4,1)$$

$$y(4)=\sqrt{(4)+5}=3,\ (4,3)$$

Maximum 3, minimum 1



Question 10

$$y=\frac{1}{x-2}$$

$$y=(x-2)^{-1}$$
$$y'=-(x-2)^{-2}$$
$$y'=-\frac{1}{(x-2)^2}$$

For a stationary point $y'=0$, but $y'=-\frac{1}{(x-2)^2}$ has an asymptote y = 0, so it never equals 0.

$$x-2=0$$

$$x=2$$

asymptote at $x = 2$

from greater than 2, as $x\to 2^+$, $y\to\infty$, maximum.

from less than 2, as $x\to 2^-$, $y\to-\infty$, minimum.

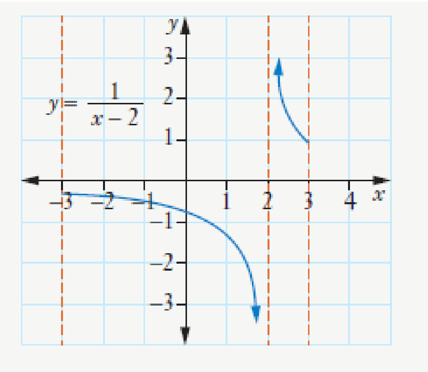
y-intercept

$$y(0)=\frac{1}{(0)-2}=-\frac{1}{2},\ (0,-\frac{1}{2})$$

Check the values at the endpoints.

$$y(-3)=\frac{1}{(-3)-2}=-\frac{1}{5},\ (-3,-\frac{1}{5})$$

$$y(3)=\frac{1}{(3)-2}=1,\ (3,1)$$



Exercise 5.08 Finding formulas for optimisation problems

Question 1

Area: $50 = xy$

$$\frac{50}{x} = y$$

Perimeter: $P = 2x + 2y$

Combine:

$$P = 2x + 2 \times \frac{50}{x} = 2x + \frac{100}{x}$$

Question 2

Perimeter: $2x + 2y = 120$

$2y = 120 - 2x$

$y = 60 - x$

Area: $A = xy$

Combine:

$A = x(60 - x)$

$A = 60x - x^2$

Question 3

Product: $20 = xy$

$$\frac{20}{x} = y$$

Sum: $S = x + y$

Combine:

$$S = x + \frac{20}{x}$$

Question 4

Volume: $400 = \pi r^2 h$

$$\frac{400}{\pi r^2} = h$$

Surface area: $S = 2\pi r^2 + 2\pi rh$

Combine:

$$S = 2\pi r^2 + 2\pi r\frac{(400)}{\pi r^2}$$

$$S = 2\pi r^2 + \frac{800}{r}$$

Question 5

a $x + y = 30$

$y = 30 - x$

b The perimeter of one square is x, so its side is $\frac{1}{4}x$. The other square has side $\frac{1}{4}y$.

$$A = \left(\frac{1}{4}x\right)^2 + \left(\frac{1}{4}y\right)^2$$

$$A = \frac{x^2}{16} + \frac{y^2}{16}$$

$$A = \frac{x^2}{16} + \frac{(30 - x)^2}{16}$$

$$A = \frac{x^2}{16} + \frac{900 - 60x + x^2}{16}$$

$$A = \frac{2x^2 - 60x + 900}{16}$$

$$A = \frac{x^2 - 30x + 450}{8}$$

Question 6

a $x^2 + y^2 = 280^2 = 78\ 400$

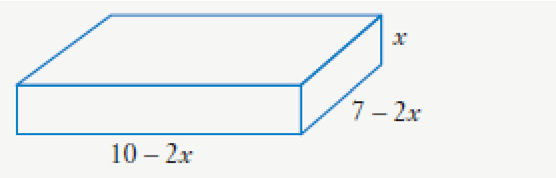
$y^2 = 78\ 400 - x^2$

$y = \sqrt{78400 - x^2}$

b Area: $A = xy$

$A = x\sqrt{78400 - x^2}$

Question 7



$V = x(10 - 2x)(7 - 2x)$

$= x(70 - 20x - 14x + 4x^2)$

$= x(70 - 34x + 4x^2)$

$= 70x - 34x^2 + 4x^3$

Question 8

Profit per person = Cost – Expenses

$P = (900 - 100x) - (200 + 400x)$

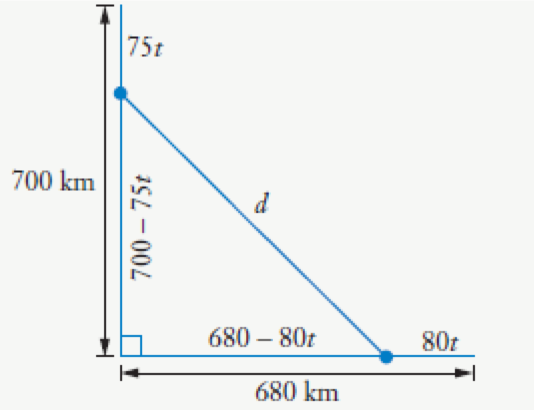
$= 900 - 100x - 200 - 400x$

$= 700 - 500x$

For x people $P = x(700 - 500x)$

$= 700x - 500x^2$

Question 9



After t hours, Joel has travelled $75t$ km. He is $700 – 75t$ km from the town.

After t hours, Nick has travelled $80t$ km. He is $680 – 80t$ km from the town.

$d^2 = (700 - 75t)^2 + (680 - 80t)^2$

$= 490\,000 - 105\,000t + 5625t^2 + 462\,400 - 108\,800t + 6400t^2$

$= 952\,400t^2 - 213\,800t + 12\,025t^2$

$$d = \sqrt{952\,400t^2 - 213800t + 12\,025t^2}$$

Question 10

The river is 500 m, or 0.5 km, wide.

Distance *AB*:

$$d = \sqrt{x^2 + 0.5^2}$$

$$d = \sqrt{x^2 + 0.25}$$

Speed = $\frac{\text{distance}}{\text{time}}$, so time = $\frac{\text{distance}}{\text{speed}}$

$$t = \frac{\sqrt{x^2 + 0.25}}{5}$$

Distance *BC*:

$d = 7 - x$

Time = $\frac{\text{distance}}{\text{speed}}$

$$t = \frac{7 - x}{4}$$

So total time taken is $t = \frac{\sqrt{x^2} + 0.25}{5} + \frac{7 - x}{4}$

Exercise 5.09 Optimisation problems

Question 1

$h = 16t - 4t^2$

$h' = 16 - 8t$

$0 = 16 - 8t$

$8t = 16$

$t = 2$ s

$h(2) = 16(2) - 4(2)^2 = 16$ m

$h'' = -8 < 0$, so a maximum.

Question 2

$C = x^2 - 15x + 40$

$C' = 2x - 15$

$0 = 2x - 15$

$2x = 15$

$x = 7.5$ km

$C'' = 2 > 0$, so a minimum.

Question 3

a Perimeter: $2x + 2y = 60$

$2y = 60 - 2x$

$y = 30 - x$

Area: $A = xy$

$A = x(30 - x)$

$A = 30x - x^2$

b $A = 30x - x^2$

$A' = 30 - 2x$

$0 = 30 - 2x$

$2x = 30$

$x = 15$

$A(15) = 30(15) - (15)^2 = 225 \text{ m}^2$

$A'' = -2 < 0$, so a maximum

Question 4

a Area: $4000 = xy$

$$\frac{4000}{x} = y$$

$$P = 2x + 2y$$

$$P = 2x + 2 \times \frac{4000}{x}$$

$$P = 2x + \frac{8000}{x}$$

b $P = 2x + \dfrac{8000}{x}$

$$P = 2x + 8000x^{-1}$$

$$P' = 2 - 8000x^{-2}$$

$$P' = 2 - \frac{8000}{x^2}$$

$$0 = 2 - \frac{8000}{x^2}$$

$$\frac{8000}{x^2} = 2$$

$$8000 = 2x^2$$

$$x^2 = 4000$$

$$x = 63.2$$

$$P'' = \frac{16\ 000}{x^3}$$

$P''(63.2) = \dfrac{16\ 000}{(63.2)^3} = 488.3 > 0$, so a minimum.

63.2 m by 63.2 m

c $63.2 \times 4 \times 48.75 \ = \$12\ 322.88$

Question 5

$A = xy$

$P = x + y$

$8 = x + y$

$y = 8 - x$

$A = x(8 - x)$

$A = 8x - x^2$

$A' = 8 - 2x$

$0 = 8 - 2x$

$2x = 8$

$x = 4$

$A'' = -2 < 0$, so a maximum.

4 m by 4 m

Question 6

$S = x + y$

$28 = x + y$

$y = 28 - x$

$P = xy$

$P = x(28 - x)$

$P = 28x - x^2$

$P' = 28 - 2x$

$0 = 28 - 2x$

$2x = 28$

$x = 14$

$y = 14$

$P'' = -2 < 0$, so a maximum

14 and 14

Question 7

$D = x - y$

$5 = x - y$

$y = x - 5$

$P = xy$

$P = x(x-5)$

$P = x^2 - 5x$

$P' = 2x - 5$

$0 = 2x - 5$

$2x = 5$

$x = 2.5$

$y = x - 5 = -2.5$

$P'' = 2 > 0$, so a minimum.

2.5 and –2.5

Question 8

$P = 2x + 2y + 4x$

$P = 6x + 2y$

$10 = 6x + 2y$

$5 = 3x + y$

$y = 5 - 3x$

$A = xy + x^2$

$A = x(5 - 3x) + x^2$

$A = 5x - 3x^2 + x^2$

$A = 5x - 2x^2$

$A' = 5 - 4x$

$0 = 5 - 4x$

$4x = 5$

$x = 1.25$ m

$A'' = -4 < 0$, so a maximum.

$y = 1.25$ m

Question 9

a $V = x(30 - 2x)(80 - 2x)$

$= x(2400 - 220x + 4x^2)$

$= 2400x - 220x^2 + 4x^3$

b $V = 4x^3 - 220x^2 + 2400x$

$V' = 12x^2 - 440x + 2400$

$0 = 12x^2 - 440x + 2400$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{440 \pm \sqrt{(-440)^2 - 4(12)(2400)}}{2(12)}$$

$$x = \frac{440 \pm \sqrt{(-440)^2 - 4(12)(2400)}}{24}$$

$$x = \frac{440 \pm 280}{24}$$

$$x = 6\frac{2}{3}, 30$$

30 cm is unrealistic you cannot cut out 2 × 30 cm squares from a 30 cm side.

$V'' = 24x - 440$

$V''\left(6\frac{2}{3}\right) = 24\left(6\frac{2}{3}\right) - 440 = -280 < 0$, so a maximum.

$x = 6\frac{2}{3}$ cm

c $V\left(6\frac{2}{3}\right) = 4\left(6\frac{2}{3}\right)^3 - 220\left(6\frac{2}{3}\right)^2 + 2400\left(6\frac{2}{3}\right) = 7407.4\text{ cm}^3$

Question 10

a Volume: $V = 54\pi = \pi r^2 h$

$$\frac{54}{r^2} = h$$

Surface area: $S = 2\pi r(r + h)$

Combining:

$$S = 2\pi r\left(r + \frac{54}{r^2}\right)$$

$$S = 2\pi r^2 + \frac{108\pi}{r}$$

b $$S = 2\pi r^2 + \frac{108\pi}{r}$$

$$S' = 4\pi r - \frac{108\pi}{r^2}$$

$$0 = 4\pi r - \frac{108\pi}{r^2}$$

$$0 = 4\pi r^3 - 108\pi$$

$$0 = r^3 - 27$$

$$r^3 = 27$$

$$r = 3 \text{ m}$$

$$S'' = 4\pi + \frac{216\pi}{r^3}$$

$S''(3) = 4\pi + \dfrac{216\pi}{(3)^3} = 12\pi > 0$, so a minimum.

Question 11

a Volume: $V = \pi r^2 h$

$8600 = \pi r^2 h$

$h = \dfrac{8600}{\pi r^2}$

Surface area: $S = 2\pi r^2 + 2\pi rh$

Combining: $S = 2\pi r^2 + 2\pi r\left(\dfrac{8600}{\pi r^2}\right)$

$S = 2\pi r^2 + \dfrac{17\ 200}{r}$

b $S' = 4\pi r - \dfrac{17\ 200}{r^2}$

$0 = 4\pi r - \dfrac{17\ 200}{r^2}$

$0 = 4\pi r^3 - 17\ 200$

$4\pi r^3 = 17\ 200$

$\pi r^3 = 4300$

$r^3 = \dfrac{4300}{\pi}$

$r = \sqrt[3]{\dfrac{4300}{\pi}}$

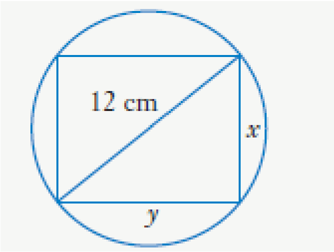
$S\left(\sqrt[3]{\dfrac{4300}{\pi}}\right) = 2\pi\left(\sqrt[3]{\dfrac{4300}{\pi}}\right)^2 + \dfrac{17200}{\left(\sqrt[3]{\dfrac{4300}{\pi}}\right)} = 2324\ \text{m}^2$

$S'' = 4\pi + \dfrac{34\ 400}{x^3}$

$S''\left(\sqrt[3]{\dfrac{4300}{\pi}}\right) = 4\pi + \dfrac{34\ 400}{\left(\sqrt[3]{\dfrac{4300}{\pi}}\right)^3} = 37.7 > 0$, so a minimum.

Question 12

a



$x^2 + y^2 = 12^2$

$y^2 = 144 - x^2$

$y = \sqrt{144 - x^2}$

$A = xy$

$A = x\sqrt{144 - x^2}$

b $A = x\sqrt{144 - x^2}$

$A' = \sqrt{144 - x^2} + x \times \frac{1}{2} \times -2x \times \left(144 - x^2\right)^{-\frac{1}{2}}$

$A' = \sqrt{144 - x^2} - \frac{x^2}{\sqrt{144 - x^2}}$

$0 = \sqrt{144 - x^2} - \frac{x^2}{\sqrt{144 - x^2}}$

$0 = 144 - x^2 - x^2$

$2x^2 = 144$

$x^2 = 72$

$x = \sqrt{72} \approx 8.5$

$$A\left(\sqrt{72}\right)=\left(\sqrt{72}\right)\sqrt{144-\left(\sqrt{72}\right)^2}=72\text{ cm}^2$$

$$\text{LHS}: A'(8)=\sqrt{144-8^2}-\frac{8^2}{\sqrt{144-8^2}}=1.789>0$$

$$\text{RHS}: A'(8)=\sqrt{144-9^2}-\frac{9^2}{\sqrt{144-9^2}}=-2.27<0$$

Gradient changes from increasing to decreasing, so a maximum.

Question 13

a Area of poster: $400 = xy$

$$\frac{400}{x} = y$$

Area of photograph: $A = (x - 10)(y - 10) = xy - 10x - 10y + 100$

Combine: $A = \ = 400 - 10x - 10 \times \dfrac{400}{x} + 100$

$$A = 500 - 10x - \frac{4000}{x}$$

b $A = 500 - 10x - \dfrac{4000}{x}$

$$A' = -10 + \frac{4000}{x^2}$$

$$0 = -10 + \frac{4000}{x^2}$$

$$0 = -10x^2 + 4000$$

$$10x^2 = 4000$$

$$x^2 = 400$$

$$x = 20$$

$$A(20) = 500 - 10(20) - \frac{4000}{(20)} = 100 \text{ cm}^2$$

$$A'' = -\frac{8000}{x^3}$$

$A''(20) = -\dfrac{8000}{(20)^3} = -1 < 0$, so a maximum.

Question 14

Perimeter: $P = \pi r + 2r + 2y$

$4 = \pi r + 2r + 2y$

$2y = 4 - \pi r - 2r$

$y = 2 - \frac{1}{2}\pi r - r$

Area: $A = \frac{1}{2}\pi r^2 + 2ry$

Combine: $A = \frac{1}{2}\pi r^2 + 2r\left(2 - \frac{1}{2}\pi r - r\right)$

$A = \frac{1}{2}\pi r^2 + 4r - \pi r^2 - 2r^2$

$A = -\frac{1}{2}\pi r^2 - 2r^2 + 4r$

$A = -\left(2 + \frac{\pi}{2}\right)r^2 + 4r$

$A' = -2\left(2 + \frac{\pi}{2}\right)r + 4$

$A' = (-4 - \pi)r + 4$

$0 = (-4 - \pi)r + 4$

$-4 = r(-4 - \pi)$

$r = \frac{4}{\pi + 4}$

$A\left(\frac{4}{\pi + 4}\right) = \frac{1}{2}\pi\left(\frac{4}{\pi + 4}\right)^2 + 4\left(\frac{4}{\pi + 4}\right) - \pi\left(\frac{4}{\pi + 4}\right)^2 - 2\left(\frac{4}{\pi + 4}\right)^2 = 1.12 \text{ m}^2$

$A'' = \pi - 2\pi - 4 = -7.14 < 0$, so a maximum.

Question 15

a Perimeter: $P = 2x + 2\pi r$

$30 = 2x + 2\pi r$

$15 = x + \pi r$

$x = 15 - \pi r$

Surface area: $S = \pi r x$

Combine: $S = \pi r\left(15 - \pi r\right)$

$S = 15\pi r - \pi^2 r^2$

$S' = 15\pi - 2\pi^2 r$

$0 = 15\pi - 2\pi^2 r$

$2\pi^2 r = 15\pi$

$2\pi r = 15$

$r = \dfrac{15}{2\pi}$ m

$S'' = -2\pi^2 < 0$, so a maximum.

$x = 15 - \pi \times \dfrac{15}{2\pi} = 7.5$ m

7.5 m by 7.5m

b $h = r = \dfrac{15}{2\pi} \approx 2.4$ m

Question 16

Area of picture: $A = xy - (x-6)(y-4)$

Area of border: $A = xy - xy + 4x + 6y - 24$

$100 = 4x + 6y - 24$

$124 = 4x + 6y$

$62 = 2x + 3y$

$y = \frac{62-2x}{3} \times x$

Area of frame: $A = xy$

$A = \frac{62x - 2x^2}{3}$

$A' = \frac{62-4x}{3}$

$0 = \frac{62-4x}{3}$

$62 - 4x = 0$

$4x = 62$

$x = 15.5$ cm

$A(15.5) = \frac{62(15.5) - 2(15.5)^2}{3} = 160.17 \text{ cm}^2$

$A'' = -\frac{4}{3} < 0$, so a maximum.

Question 17

Perimeter: $P = 4x + 2\pi r$

$3 = 4x + 2\pi r$

$r = \dfrac{3 - 4x}{2\pi}$

Area: $A = x^2 + \pi r^2$

Combine: $A = x^2 + \pi\left(\dfrac{3 - 4x}{2\pi}\right)^2$

$A = x^2 + \dfrac{(3 - 4x)^2}{4\pi}$

$A' = 2x + \dfrac{8x - 6}{\pi}$

$0 = 2x + \dfrac{8x - 6}{\pi}$

$0 = 2\pi x + 8x - 6$

$0 = \pi x + 4x - 3$

$3 = \pi x + 4x$

$x(\pi + 4) = 3$

$x = \dfrac{3}{\pi + 4}$

$A'' = 2 + \dfrac{8}{\pi} > 0$, so a minimum.

Square perimeter $= 4 \times \dfrac{3}{\pi + 4} \approx 1.68$ m

Circle perimeter = 3 – 1.68 = 1.32 m.

Question 18

a $d^2 = (200 - 80t)^2 + (120 - 60t)^2$

$= 40\ 000 - 32\ 000t + 6400t^2 + 14\ 400 - 14\ 400t + 3600t^2$

$= 10\ 000t^2 - 46\ 400t + 54\ 400$

$d = \sqrt{10\ 000t^2 - 46\ 400t + 54\ 400}$

b $d = \sqrt{10000t^2 - 46400t + 54400}$

$$d' = \frac{1}{2}\left(10\ 000t^2 - 46\ 400t + 54\ 400\right)^{-\frac{1}{2}}\left(20\ 000t - 46\ 400\right)$$

$$d' = \frac{20\ 000t - 46\ 400}{2\sqrt{10\ 000t^2 - 46\ 400t + 54\ 400}}$$

$$0 = \frac{20\ 000t - 46\ 400}{2\sqrt{10\ 000t^2 - 46\ 400t + 54\ 400}}$$

$0 = 20\ 000t - 46\ 400$

$20\ 000t = 46\ 400$

$t = 2.32$ h

$d(2.32) = \sqrt{10\ 000(2.32)^2 - 46\ 400(2.32) + 54400} = 24$ km

$$\text{LHS}: d'(2) = \frac{20\ 000(2) - 46\ 400}{2\sqrt{10\ 000(2)^2 - 46\ 400(2) + 54\ 400}} = -80$$

$$\text{LHS}: d'(3) = \frac{20\ 000(3) - 46\ 400}{2\sqrt{10\ 000(3)^2 - 46\ 400(3) + 54\ 400}} = 94.3$$

Gradient changes from decreasing to increasing, so a minimum.

Question 19

a $d = (x^2 - 2x + 5) - (4x - x^2)$

$= x^2 - 2x + 5 - 4x + x^2$

$= 2x^2 - 6x + 5$

b $d = 2x^2 - 6x + 5$

$d' = 4x - 6$

$0 = 4x - 6$

$4x = 6$

$x = 1.5$

$d(1.5) = 2(1.5)^2 - 6(1.5) + 5 = 0.5$ unit

$d'' = 4 > 0$, so a minimum.

Question 20

a $s = \frac{d}{t}$, so $t = \frac{d}{s} = \frac{1500}{s}$

Cost of trip taking t hours: $C = (s^2 + 9000)t$

$= \left(s^2 + 9000\right)\frac{1500}{s}$

$= \frac{1500}{s}\left(s^2 + 9000\right)$

b $C = 1500\left(s + \frac{9000}{s}\right)$

$C = 1500s + \frac{13\ 500\ 000}{s}$

$C' = 1500 - \frac{13\ 500\ 000}{s^2}$

$0 = 1500 - \frac{13\ 500\ 000}{s^2}$

$0 = 1500s^2 - 13\ 500\ 000$

$1500s^2 = 13\ 500\ 000$

$s^2 = 9000$

$s = 94.8683\ldots \approx 95$ km/h

$C'' = \frac{27\ 000\ 000}{s^3}$

$C''(95) = \frac{27\ 000\ 000}{(95)^3} > 0$, so a minimum.

c $C(95) = 1500\left([95] + \frac{9000}{[95]}\right) = 284605$ c = \$2846.05 $\approx$ \$2846

Test yourself 5

Question 1

A

Question 2

decreasing so $\frac{dy}{dx} < 0$

concave up so $\frac{d^2y}{dx^2} > 0$

C

Question 3

D

Question 4

gradient is positive, temperature increasing

concave down, rate is decreasing

C

Question 5

$y = x^3 + 6x^2 + 9x - 11$

$y' = 3x^2 + 12x + 9$

$y'' = 6x + 12$

Stationary points

$0 = 3x^2 + 12x + 9$

$0 = x^2 + 4x + 3$

$0 = (x+3)(x+1)$

First factor: $x + 3 = 0$

$x = -3$

$y(-3) = (-3)^3 + 6(-3)^2 + 9(-3) - 11 = -11, \quad (-3, -11)$

$y''(-3) = 6(-3) + 12 = -6 < 0$, so a maximum.

Second factor: $x + 1 = 0$

$x = -1$

$y(-1) = (-1)^3 + 6(-1)^2 + 9(-1) - 11 = -15, \ (-1, -15)$

$y''(-1) = 6(-1) + 12 = 6 > 0$, so a minimum.

Question 6

$y = 2x^3 - 7x^2 - 3x + 1$

$y' = 6x^2 - 14x - 3$

$y'' = 12x - 14$

$0 < 12x - 14$

$12x > 14$

$x > \frac{14}{12}$

$x > 1\frac{1}{6}$

Question 7

$h = 20t - 2t^2$

$h' = 20 - 4t$

$0 = 20 - 4t$

$4t = 20$

$t = 5$

$h(5) = 20(5) - 2(5)^2 = 50$ m

$h'' = -4 < 0$, so a maximum.

Question 8

$y = 5 - 6x - 3x^2$

$y' = -6 - 6x$

$-6 - 6x < 0$

$-6x < 6$

$x > -1$

Question 9

$y = 2x^3 - 3x^2 + 3x - 2$

$y' = 6x^2 - 6x + 3$

$y'' = 12x - 6$

$0 = 12x - 6$

$12x = 6$

$x = \frac{1}{2}$

$y\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) - 2 = -1$

$(\frac{1}{2}, -1)$

Question 10

a Volume: $375 = \pi r^2 h$

$$\frac{375}{\pi r^2} = h$$

Surface area: $S = 2\pi r^2 + 2\pi rh$

Combine: $S = 2\pi r^2 + 2\pi r\left(\frac{375}{\pi r^2}\right)$

$$S = 2\pi r^2 + \frac{750}{r}$$

b $S = 2\pi r^2 + \frac{750}{r}$

$$S' = 4\pi r - \frac{750}{r^2}$$

$$0 = 4\pi r - \frac{750}{r^2}$$

$$0 = 4\pi r^3 - 750$$

$$4\pi r^3 = 750$$

$$\pi r^3 = 187.5$$

$$r^3 = \frac{187.5}{\pi}$$

$$r = \sqrt[3]{\frac{187.5}{\pi}} \approx 3.9 \text{ cm}$$

$S''(3.9) = 4\pi + \frac{150}{(3.9)^3} > 0$, so a minimum.

Question 11

a $y = 3x^4 + 8x^3 + 6x^2$

$y' = 12x^3 + 24x^2 + 12x$

$y'' = 36x^2 + 48x + 12$

Stationary points

$0 = 12x^3 + 24x^2 + 12x$

$0 = x^3 + 2x^2 + x$

$0 = x\left(x^2 + 2x + 1\right)$

$0 = x(x+1)(x+1)$

First factor: $x = 0$

$y(0) = 3(0)^4 + 8(0)^3 + 6(0)^2 = 0,\ (0, 0)$

Second factor: $x + 1 = 0$

$x = -1$

$y(-1) = 3(-1)^4 + 8(-1)^3 + 6(-1)^2 = 1,\ \ (-1, 1)$

b (0, 0)

$y''(0) = 36(0)^2 + 48(0) + 12 = 12 > 0$, so a minimum.

(–1, 1)

$y''(-1) = 36(-1)^2 + 48(-1) + 12 = 0$, so possible point of inflection.

$\text{LHS}: y''(-2) = 36(-2)^2 + 48(-2) + 12 = 60 > 0$

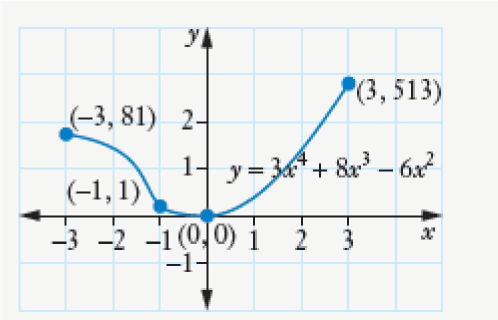
$\text{RHS}: y''(-0.5) = 36(-0.5)^2 + 48(-0.5) + 12 = -3 < 0$

Concavity changes, $y' = 0$, so horizontal point of inflection.

c Calculate values at endpoints.

$$y(3)=3(3)^4+8(3)^3+6(3)^2=513,\ (3, 513)$$

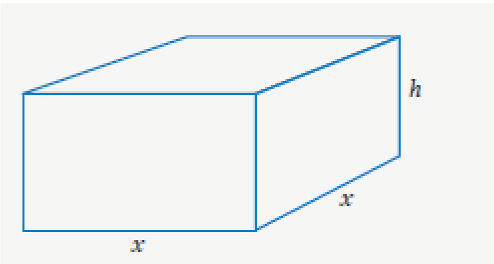
$$y(-3)=3(-3)^4+8(-3)^3+6(-3)^2=81, (-3, 81)$$



d Using points above, minimum = 0, maximum = 513.

Question 12

a



Volume: $250 = 2x^2 + 4xh$

$250 - 2x^2 = 4xh$

$$\frac{250-2x^2}{4x}=h$$

$$\frac{2(125-x^2)}{4x}=h$$

$$\frac{125-x^2}{2x}=h$$

$V = x^2h$

$$= x^2\left(\frac{125-x^2}{2x}\right)$$

$$= x\left(\frac{125-x^2}{2}\right)$$

$$= \frac{125x-x^3}{2}$$

b

$$V = \frac{125x-x^3}{2}$$

$V = 62.5x - 0.5x^3$

$V' = 62.5 - 1.5x^2$

$0 = 62.5 - 1.5x^2$

$1.5x^2 = 62.5$

$$x^2 = \frac{125}{3}$$

$$x = \sqrt{\frac{125}{3}} \approx 6.45\text{ cm}$$

$V'' = -3x$

$V''(6.45) = -3(6.45) < 0$, so a maximum.

$$h = \frac{125-x^2}{2x} = \frac{125-\frac{125}{3}}{2\sqrt{\frac{125}{3}}} = \frac{2\left(\frac{125}{3}\right)}{2\sqrt{\frac{125}{3}}} = \sqrt{\frac{125}{3}} \approx 6.45\text{ cm}$$

6.45cm by 6.45cm by 6.45cm

Question 13

$C = x^2 - 300x + 9000$

$C' = 2x - 300$

$0 = 2x - 300$

$2x = 300$

$x = 150$

$C'' = 2 > 0$, so a minimum.

150 products

Question 14

a $x^2 + y^2 = 5^2$

$y^2 = 25 - x^2$

$y = \sqrt{25 - x^2}$

$A = \frac{1}{2}xy$

$A = \frac{1}{2}x\sqrt{25 - x^2}$

b $A=\frac{1}{2}x\sqrt{25-x^2}$

$A=\frac{1}{2}x\left(25-x^2\right)^{\frac{1}{2}}$

$A'=\frac{1}{2}\left(25-x^2\right)^{\frac{1}{2}}+\frac{1}{2}x\times\frac{1}{2}\left(25-x^2\right)^{-\frac{1}{2}}\times-2x$

$A'=\frac{1}{2}\left(25-x^2\right)^{\frac{1}{2}}-\frac{1}{2}x^2\left(25-x^2\right)^{-\frac{1}{2}}$

$A'=\frac{\sqrt{25-x^2}}{2}-\frac{x^2}{2\sqrt{25-x^2}}$

$A'=\frac{25-x^2}{2\sqrt{25-x^2}}-\frac{x^2}{2\sqrt{25-x^2}}$

$A'=\frac{25-2x^2}{2\sqrt{25-x^2}}$

$0=\frac{25-2x^2}{2\sqrt{25-x^2}}$

$25-2x^2=0$

$2x^2=25$

$x^2=12.5$

$x=\sqrt{12.5}\approx 3.54\text{ m}$

$\text{LHS}: A'(3)=\frac{25-2(3)^2}{2\sqrt{25-(3)^2}}=0.875$

$\text{RHS}: A'(4)=\frac{25-2(4)^2}{2\sqrt{25-(4)^2}}=-1.17$

Gradient changes from increasing to decreasing, so maximum.

$A\left(\sqrt{12.5}\right)=\frac{1}{2}\left(\sqrt{12.5}\right)\sqrt{25-\left(\sqrt{12.5}\right)^2}=6.25\text{ m}^2$

Question 15

$y = x^4 - 6x^3 + 2x + 1$

$y' = 4x^3 - 18x^2 + 2$

$y'' = 12x^2 - 36x$

Points of inflection

$0 = 12x^2 - 36x$

$0 = x^2 - 3x$

$0 = x(x-3)$

First factor: $x = 0$

$y(0) = (0)^4 - 6(0)^3 + 2(0) + 1 = 1$, (0,1)

Second factor: $x - 3 = 0$

$x = 3$

$y(3) = (3)^4 - 6(3)^3 + 2(3) + 1 = -74$, $(3, -74)$

Question 16

$y = x^3 + 3x^2 - 24x - 1$

$y = 3x^2 + 6x - 24$

$y'' = 6x + 6$

Stationary points

$0 = 3x^2 + 6x - 24$

$0 = x^2 + 2x - 8$

$0 = (x+4)(x-2)$

First factor: $x + 4 = 0$

$x = -4$

$y(-4) = (-4)^3 + 3(-4)^2 - 24(-4) - 1 = 79$

$y''(-4) = 6(-4) + 6 = -18 < 0$, so a maximum

Second factor: $x - 2 = 0$

$x = 2$

$y(2) = (2)^3 + 3(2)^2 - 24(2) - 1 = -29,\ (2, -29)$

Check the values of the endpoints:

$y(-5) = (-5)^3 + 3(-5)x^2 - 24(-5) - 1 = 69$, $(-5, 69)$

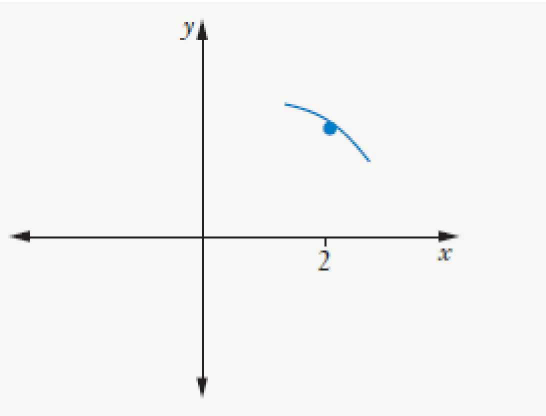
$y(6) = (6)^3 + 3(6)x^2 - 24(6) - 1 = 179$, $(6, 179)$

Maximum = 179.

Question 17

$f'(2) < 0$, so gradient is negative, decreasing

$f''(2) < 0$, so concave down



Question 18

$f(x) = xe^{2x}$

$f'(x) = e^{2x} + 2xe^{2x} = e^{2x}(1+2x)$

$f''(x) = 2e^{2x} + 2e^{2x} + 4xe^{2x} = 4e^{2x}(1+x)$

Stationary points

$0 = e^{2x}(1+2x)$

$1 + 2x = 0$

$2x = -1$

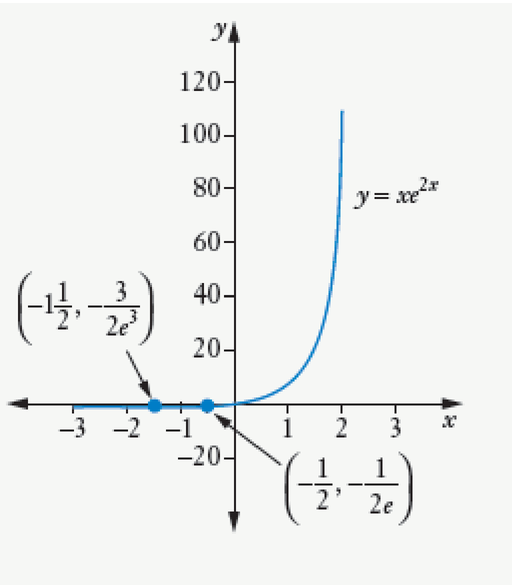
$x = -\frac{1}{2}$

$f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)e^{2\left(-\frac{1}{2}\right)} = -\frac{1}{2e}$, $\left(-\frac{1}{2}, -\frac{1}{2e}\right) \approx (-0.5, -0.184)$

$f''\left(-\frac{1}{2}\right) = 4e^{2\left(-\frac{1}{2}\right)}\left[1 + \left(-\frac{1}{2}\right)\right] = \frac{4}{e}\left(\frac{1}{2}\right) = \frac{2}{e} > 0$, so a minimum.

y-intercept

$f(0) = (0)e^{2(0)} = 0$, $(0,0)$



Question 19

$y = 2\cos 4x$

$y' = -8\sin 4x$

$y'' = -32\cos 4x$

Stationary points

$0 = -8\sin 4x$

$\sin 4x = 0$

$4x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

amplitude: 2

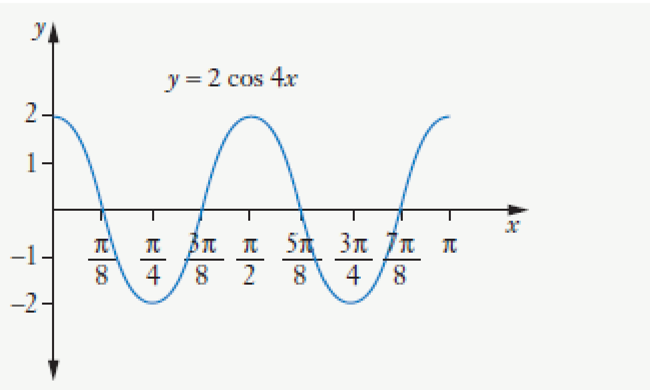
$\text{maximums}: (0, 2), (\frac{\pi}{2}, 2), (\pi, 2)$

$\text{minimums}: (\frac{\pi}{4}, -2), (\frac{3\pi}{4}, -2)$

y-intercept (0, 2)

x-intercepts

$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$



Challenge exercise 5

Question 1

$y = x(x-2)^3$

$y' = (x-2)^3 + 3x(x-2)^2$

$y'' = 3(x-2)^2 + 3(x-2)^2 + 6x(x-2)$

$y'' = 6(x-2)^2 + 6x(x-2)$

Stationary points

$0 = (x-2)^3 + 3x(x-2)^2$

$0 = (x-2) + 3x$

$0 = 4x - 2$

$4x = 2$

$x = \frac{1}{2}$

$y\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left[\left(\frac{1}{2}\right) - 2\right]^3 = -1\frac{11}{16},\ (\frac{1}{2}, -1\frac{11}{16})$

$y''\left(\frac{1}{2}\right) = 6\left[\left(\frac{1}{2}\right) - 2\right]^2 + 6\left(\frac{1}{2}\right)\left[\left(\frac{1}{2}\right) - 2\right] = 9 > 0$, so a minimum.

y-intercept

$y(0) = (0)\left[(0) - 2\right]^3 = 0,\ (0,0)$

x-intercepts

$0 = x(x-2)^3$

First factor: $x = 0$, (0, 0)

Second factor: $x - 2 = 0$

$x = 2$, (2, 0)

Possible points of inflection

$y'' = 6(x-2)^2 + 6x(x-2)$

$0 = 6(x-2)^2 + 6x(x-2)$

$0 = x - 2 + x$

$0 = 2x - 2$

$2x = 2$

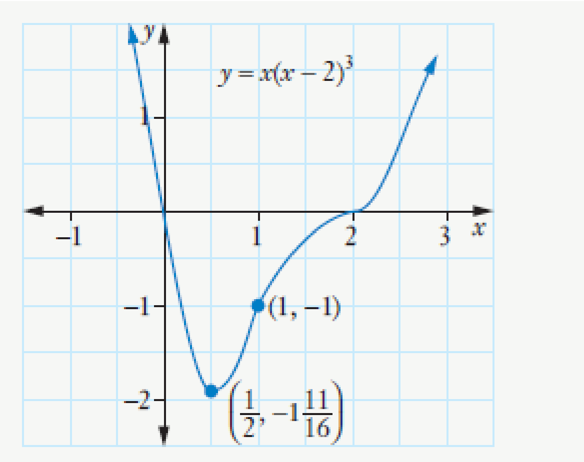
$x = 1$

$y(1) = [1]([1]-2)^3 = -1, \quad (1,-1)$

$\text{LHS}: y''(0.9) = 6(0.9-2)^2 + 6(0.9)(0.9-2) = 1.32$

$\text{RHS}: y''(1.1) = 6(1.1-2)^2 + 6(1.1)(1.1-2) = -1.08$

Concavity changes, so it is a point of inflection.



Question 2

$c^2 = a^2 + b^2$

$8^2 = a^2 + b^2$

$64 = a^2 + b^2$

$a^2 = 64 - b^2$

$a = \sqrt{64 - b^2}$

$A = \frac{1}{2}ab$

$A = \frac{1}{2}b\sqrt{64 - b^2}$

$A' = \frac{1}{2}\sqrt{64 - b^2} + \frac{1}{2}b \times \frac{1}{2}\left(64 - b^2\right)^{-\frac{1}{2}} \times -2b$

$A' = \frac{1}{2}\sqrt{64 - b^2} - \frac{1}{2}b^2\left(64 - b^2\right)^{-\frac{1}{2}}$

$A' = \frac{\sqrt{64 - b^2}}{2} - \frac{b^2}{2\sqrt{64 - b^2}}$

$A' = \frac{64 - b^2}{2\sqrt{64 - b^2}} - \frac{b^2}{2\sqrt{64 - b^2}}$

$A' = \frac{64 - 2b^2}{2\sqrt{64 - b^2}}$

$A' = \frac{32 - b^2}{\sqrt{64 - b^2}}$

$0 = \frac{32 - b^2}{\sqrt{64 - b^2}}$

$0 = 32 - b^2$

$b^2 = 32$

$b = \sqrt{32} \approx 5.66$

$$\text{LHS}: A'(5) = \frac{32-(5)^2}{\sqrt{64-(5)^2}} = 1.12$$

$$\text{LHS}: A'(6) = \frac{32-(6)^2}{\sqrt{64-(6)^2}} = -0.76$$

Gradient changes from increasing to decreasing, so a maximum.

$$A = \frac{1}{2} \times \sqrt{32} \times \sqrt{64-\sqrt{32}^2}$$

$$A = 16 \text{ m}^2$$

Question 3

$f(x)=4x^3-3x^2-18x$

$f'(x)=12x^2-6x-18$

$f''(x)=24x-6$

Stationary points

$0=12x^2-6x-18$

$0=2x^2-x-3$

$0=(2x-3)(x+1)$

First factor: $2x-3=0$

$2x=3$

$x=1.5$

$f(1.5)=4(1.5)^3-3(1.5)^2-18(1.5)=-20.25,\ \ (1.5,-20.25)$

$f''(1.5)=24(1.5)-6=30>0$, so a minimum.

Second factor: $x+1=0$

$x=-1$

$f(-1)=4(-1)^3-3(-1)^2-18(-1)=11,\ \ (-1,11)$

$f''(-1)=24(-1)-6=-30<0$, so a maximum.

Check the values at the endpoints:

$f(-2)=4(-2)^3-3(-2)^2-18(-2)=-8,\ \ (-2,-8)$

$f(3)=4(3)^3-3(3)^2-18(3)=27,\ \ (3,27)$

In the domain maximum = 27, minimum = –20.25.

Question 4

$f(x) = 2(5x-3)^3$

$f'(x) = 6(5x-3)^2 \times 5$

$f'(x) = 30(5x-3)^2$

$f''(x) = 60(5x-3) \times 5$

$f''(x) = 300(5x-3)$

$f(0.6) = 2(5[0.6]-3)^3 = 0, \quad (0.6,\ 0)$

$f'(0.6) = 30(5[0.6]-3)^2 = 0$

$f''(0.6) = 300(5[0.6]-3) = 0$

$\text{LHS}: f''(0.5) = 300(5[0.5]-3) = -150$

$\text{RHS}: f''(0.7) = 300(5[0.7]-3) = 150$, so concavity changes.

f' (0.6) = 0, f'' (0.6) = 0 and concavity changes at (0.6, 0) so it is a horizontal point of inflection.

Question 5

$r+s=25$

$s=25-r$

$A=\pi r^2+\pi s^2$

$A=\pi r^2+\pi\left(25-r\right)^2$

$A'=2\pi r+2\pi\left(25-r\right)\times -1$

$A'=2\pi r-2\pi\left(25-r\right)$

Stationary points

$0=2\pi r-2\pi\left(25-r\right)$

$0=r-\left(25-r\right)$

$0=r-25+r$

$25=2r$

$r=12.5$

$A''=2\pi+25\pi=27\pi\ >0$, so a minimum as required.

$s=25-r=25-12.5=12.5$

Question 6

$f''(x) = a$

$f'(x) = ax + C$

At the stationary point at (–1, 2):

$0 = a(-1) + C$

$0 = -a + C$

$C = a$

$f'(x) = ax + a$

$f(x) = \frac{ax^2}{2} + ax + D$

Ay the *y*-intercept (0, 3):

$3 = \frac{a(0)^2}{2} + a(0) + D$

$D = 3$

$f(x) = \frac{ax^2}{2} + ax + 3$

$2 = \frac{a(-1)^2}{2} + a(-1) + 3$

$-1 = \frac{a}{2} - a$

$-2 = a - 2a$

$-2 = -a$

$a = 2$

$f(x) = \frac{2x^2}{2} + 2x + 3$

$f(x) = x^2 + 2x + 3$

Question 7

a $y = x^n$

$y' = nx^{n-1}$

$y'' = n\left(n-1\right)x^{n-2}$

$y\left(0\right) = \left(0\right)^n,\ (0,0)$

If $y' = 0$ at (0, 0), then there is a stationary point there.

$y'\left(0\right) = n\left(0\right)^{n-1} = 0$

b $y'' = n\left(n-1\right)x^{n-2}$, n is a positive integer.

For n even, n – 2 is even, so $\text{LHS:}\, y''(-1) = n\left(n-1\right)\left(-1\right)^{n-2} = n\left(n-1\right) > 0$

For n even, n – 2 is even, so $\text{RHS:}\, y''(1) = n\left(n-1\right)\left(1\right)^{n-2} = n\left(n-1\right) > 0$

So a minimum.

c $y'' = n\left(n-1\right)x^{n-2}$, n is a positive integer.

For n odd, n – 2 is odd, so $\text{LHS:}\, y''(-1) = n\left(n-1\right)\left(-1\right)^{n-2} = -n\left(n-1\right) < 0$

For n odd, n – 2 is odd, so $\text{RHS:}\, y''(1) = n\left(n-1\right)\left(1\right)^{n-2} = n\left(n-1\right) > 0$

Concavity changes, so it is a horizontal point of inflection.

Question 8

$$y = \frac{x+3}{x^2-9}$$

$$y = \frac{x+3}{(x-3)(x+3)}$$

$$y = \frac{1}{x-3},\ x \neq 3$$

$$y' = \frac{-1}{(x-3)^2}$$

$y'' = \dfrac{2}{(x-3)^3} > 0$, so no stationary points in domain [–2, 2].

Check endpoint values.

$$y(-2) = \frac{-2+3}{(-2)^2-9} = -\frac{1}{5},\quad (-2, -\frac{1}{5})$$

$$y(2) = \frac{2+3}{(2)^2-9} = -1,\ (2, -1)$$

So maximum $-\dfrac{1}{5}$, minimum –1.

Question 9

$$t=\frac{d}{s}$$

$$t=\frac{1000\ \text{km}}{V\ \text{km h}^{-1}}=\frac{1000}{V}\ \text{h}$$

$$c=\left(100+\frac{V^2}{75}\right)\times\frac{1000}{V}\ \text{cents}$$

$$c=\frac{100\ 000}{V}+\frac{1000V}{75}$$

$$c=\frac{100\ 000}{V}+\frac{40V}{3}$$

$$c'=-\frac{100\ 000}{V^2}+\frac{40}{3}$$

Stationary points

$$0=-\frac{100\,000}{V^2}+\frac{40}{3}$$

$$0=-100\,000+\frac{40}{3}V^2$$

$$100\,000=\frac{40}{3}V^2$$

$$300\ 000=40V^2$$

$$7500=V^2$$

$$V=86.6025\ldots\approx 87\ \text{km/h}$$

$$c''=\frac{200\ 000}{V^3}$$

$c''(87)=\dfrac{200\ 000}{(87)^3}>0$, so a minimum as required.

MATHS IN FOCUS 12 MATHEMATICS ADVANCED

WORKED SOLUTIONS

Chapter 6: Integration

Exercise 6.01 Approximating areas under a curve

Question 1

a $y = x^2 + 2x$

$y(1) = (1)^2 + 2(1) = 3$

$y(1.5) = (1.5)^2 + 2(1.5) = 5.25$

$A = 3 \times 0.5 + 5.25 \times 0.5$

$A = 4.125 \text{ units}^2$

b $y = x^2 + 2x$

$y(1.5) = (1.5)^2 + 2(1.5) = 5.25$

$y(2) = (2)^2 + 2(2) = 8$

$A = 8 \times 0.5 + 5.25 \times 0.5$

$A = 6.625 \text{ units}^2$

Question 2

a $y=\dfrac{2}{x+1}$

$y(2)=\dfrac{2}{(2)+1}=\dfrac{2}{3}=0.67$

$y(3)=\dfrac{2}{(3)+1}=\dfrac{2}{4}=0.5$

$A=0.67\times 1+0.5\times 1$

$A=1.17\ \text{units}^2$

b $y=\dfrac{2}{x+1}$

$y(1)=\dfrac{2}{(1)+1}=\dfrac{2}{2}=1$

$y(2)=\dfrac{2}{(2)+1}=\dfrac{2}{3}=0.67$

$A=1\times 1+0.67\times 1$

$A=1.67\ \text{units}^2$

c $y=\dfrac{2}{x+1}$

$y(1.5)=\dfrac{2}{(1.5)+1}=\dfrac{2}{2.5}=\dfrac{4}{5}=0.8$

$y(2)=\dfrac{2}{(2)+1}=\dfrac{2}{3}=0.67$

$y(2.5)=\dfrac{2}{(2.5)+1}=\dfrac{2}{3.5}=\dfrac{4}{7}=0.57$

$y(3)=\dfrac{2}{(3)+1}=\dfrac{2}{4}=0.5$

$A=0.8\times 0.5+0.67\times 0.5+0.57\times 0.5+0.5\times 0.5$

$A=1.27\ \text{units}^2$

d $y=\frac{2}{x+1}$

$y(1)=\frac{2}{(1)+1}=\frac{2}{2}=1$

$y(1.5)=\frac{2}{(1.5)+1}=\frac{2}{2.5}=\frac{4}{5}=0.8$

$y(2)=\frac{2}{(2)+1}=\frac{2}{3}=0.67$

$y(2.5)=\frac{2}{(2.5)+1}=\frac{2}{3.5}=\frac{4}{7}=0.57$

$A=1\times 0.5+0.8\times 0.5+0.67\times 0.5+0.57\times 0.5$

$A=1.52\ \text{units}^2$

Question 3

a $f(x)=x^2$

$f(2)=(2^2)=4$

$f(3)=(3^2)=9$

$A=\frac{1}{2}(4+9)=\frac{1}{2}\times 13=6.5$

$A=6.5\ \text{units}^2$

b $f(x)=\ln x$

$f(4)=\ln(4)$

$f(7)=\ln(7)$

$A=\frac{3}{2}(\ln(4)+\ln(7))=1.5\times\ln(28)=4.99831\ldots$

$A=5\ \text{units}^2$

c $f\left(x\right)=x^3+1$

$f\left(0\right)=\left(0\right)^3+1=1$

$f\left(4\right)=\left(4\right)^3+1=65$

$A=\frac{4}{2}\left(1+65\right)=2\times66=132$

$A=132\ \text{units}^2$

d $f\left(x\right)=\sin x$

$f\left(\frac{\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$

$f\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)=1$

$A=\frac{1}{2}\times\frac{\pi}{4}\left(\frac{\sqrt{2}}{2}+1\right)$

$A=\frac{\pi}{8}\left(\frac{\sqrt{2+2}}{2}\right)$

$A=\frac{\pi}{16}\left(\sqrt{2}+2\right)\ \text{units}^2$

e $y=9-x^2$

$y\left(1\right)=9-\left(1\right)^2=8$

$y\left(2\right)=9-\left(2\right)^2=5$

$A=\frac{1}{2}\left(8+5\right)=\frac{1}{2}\times13=6.5$

$A=6.5\ \text{units}^2$

Question 4

a $y = x^3 + 3$

$y(0) = (0)^3 + 3 = 3$

$y(2) = (2)^3 + 3 = 11$

$A = 3 \times 2 + 11 \times 2 = 6 + 22 = 28$

$A = 28 \text{ units}^2$

b $y = x^3 + 3$

$y(2) = (2)^3 + 3 = 11$

$y(4) = (4)^3 + 3 = 67$

$A = 67 \times 2 + 11 \times 2 = 134 + 22 = 156$

$A = 156 \text{ units}^2$

c $y = x^3 + 3$

$y(0) = (0)^3 + 3 = 3$

$y(4) = (4)^3 + 3 = 67$

$A = \frac{4}{2}(67 + 3) = 2 \times 70 = 140$

$A = 140 \text{ units}^2$

Question 5

a $y=\frac{1}{x}$

$y(1)=\frac{1}{(1)}=1$

$y(7)=\frac{1}{(7)}=\frac{1}{7}$

$A=\frac{6}{2}\left(1+\frac{1}{7}\right)=3\times\frac{8}{7}=\frac{24}{7}=3\frac{3}{7}$

$A=3\frac{3}{7}\ \text{units}^2$

b $y=x^2+5$

$y(0)=(0)^2+5=5$

$y(1)=(1)^2+5=6$

$A=\frac{1}{2}(5+6)=\frac{1}{2}\times 11=5.5$

$A=5.5\ \text{units}^2$

c $f(x)=\cos x$

$f(0)=\cos(0)=1$

$f\left(\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=0.5$

$A=\frac{1}{2}\times\frac{\pi}{3}(1+0.5)=\frac{\pi}{6}\times\frac{3}{2}=\frac{\pi}{4}$

$A=\frac{\pi}{4}\ \text{units}^2$

d $y = e^x$

$y(1) = e^{(1)} = e$

$y(4) = e^{(4)} = e^4$

$A = \frac{3}{2}\left(e + e^4\right)$

$A = \frac{3}{2}\left(e + e^4\right)$ units^2

e $f(x) = x(x-4)(x-9)$

$f(2) = (2)((2)-4)((2)-9) = 2 \times (-2) \times (-7) = 28$

$f(3) = (3)((3)-4)((3)-9) = 3 \times (-1) \times (-6) = 18$

$A = \frac{1}{2}(28 + 18) = \frac{1}{2} \times 46 = 23$

$A = 23$ units^2

Question 6

a $y = 1 - x^2$

$y' = -2x$

$y'' = -2$

$0 = -2x$

$x = 0$

max

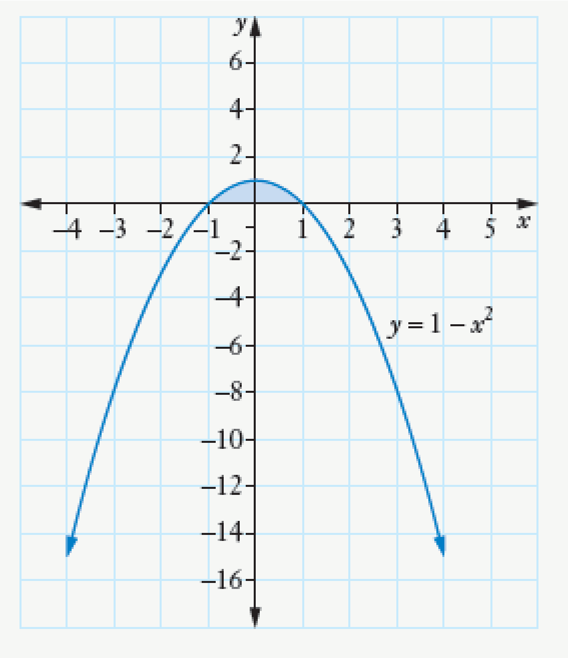
concave down

$1 - x^2 = 0$

$x^2 = 1$

$x = \pm 1$

x-intercepts



b Estimate using a triangle with base 2 and height 1.1.

h = 1.1 unit

b = 2 units

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2} \times 2 \times 1.1$$

$$A = 1.1 \text{units}$$

Question 7

a $\boldsymbol{y} = \sqrt{\boldsymbol{x} - 1}$

$$y(2) = \sqrt{(2) - 1} = 1$$

$$y(2.5) = \sqrt{(2.5) - 1} = \sqrt{1.5}$$

$$y(3) = \sqrt{(3) - 1} = \sqrt{2}$$

$$y(3.5) = \sqrt{(3.5) - 1} = \sqrt{2.5}$$

$$y(4) = \sqrt{(4) - 1} = \sqrt{3}$$

$$y(4.5) = \sqrt{(4.5) - 1} = \sqrt{3.5}$$

$$A = 0.5\left(\sqrt{3.5} + \sqrt{3} + \sqrt{2.5} + \sqrt{2} + \sqrt{1.5} + 1\right)$$

$$A = 4.4 \ \text{units}^2$$

b $y(2.5) = \sqrt{(2.5) - 1} = \sqrt{1.5}$

$y(3) = \sqrt{(3) - 1} = \sqrt{2}$

$y(3.5) = \sqrt{(3.5) - 1} = \sqrt{2.5}$

$y(4) = \sqrt{(4) - 1} = \sqrt{3}$

$y(4.5) = \sqrt{(4.5) - 1} = \sqrt{3.5}$

$y(5) = \sqrt{(5) - 1} = 2$

$A = 0.5\left(2 + \sqrt{3.5} + \sqrt{3} + \sqrt{2.5} + \sqrt{2} + \sqrt{1.5}\right)$

$A = 4.9 \text{ units}^2$

c $\boldsymbol{y} = \sqrt{\boldsymbol{x} - 1}$

$y(2) = \sqrt{(2) - 1} = 1$

$y(5) = \sqrt{(5) - 1} = 2$

$A = \frac{3}{2}(2+1) = \frac{3}{2} \times 3 = \frac{9}{2} = 4\frac{1}{2}$

$A = 4.5 \text{ units}^2$

d 4.5 units^2 by counting square units.

Question 8

$y = \sqrt{25 - x^2}$

$r = 5$

$A = \frac{1}{2}\pi r^2$

$A(5) = \frac{1}{2}\pi(5)^2$

$A(5) = \frac{25\pi}{2}\text{units}^2$

Question 9

a $y = \sqrt{9 - x^2}$

$r = 3$

$A = \frac{1}{2}\pi r^2$

$A(3) = \frac{1}{2}\pi(3)^2$

$A(3) = \frac{9\pi}{2}\ \text{units}^2$

b **i** $y = \sqrt{9 - x^2}$

$y(1) = \sqrt{9 - (1)^2}\ = \sqrt{8}$

$y(2) = \sqrt{9 - (2)}\ = \sqrt{5}$

$A = \frac{1}{2}\left(\sqrt{8} + \sqrt{5}\right)$

$A = 2.5\ \text{units}^2$

ii $y = \sqrt{9 - x^2}$

$y(0) = \sqrt{9 - (0)^2} = 3$

$y(1) = \sqrt{9 - (1)^2} = \sqrt{8}$

$y(2) = \sqrt{9 - (2)^2} = \sqrt{5}$

$A = \left(3 + \sqrt{8} + \sqrt{5}\right)$

$A = 8.1 \text{ units}^2$

Question 10

a $b = 4 - 1.2 = 2.8$

$h = f(4) = 4^2 = 16$

$A = \frac{1}{2} \times 2.8 \times 16 = 22.4 \text{ units}^2$

b $b = 3$

$h = 2.2$

$A = \frac{1}{2} \times 3 \times 2.2 = 3.3 \text{ units}^2$

c $b = 2$

$h = 1$

$A = \frac{1}{2} \times 2 \times 1 = 1 \text{ unit}^2$

Question 11

a **i** $y = -x^2 + 4x$

$y(0) = -(0)^2 + 4(0) = 0$

$y(1) = -(1)^2 + 4(1) = 3$

$y(3) = -(3)^2 + 4(3) = 3$

$y(4) = -(4)^2 + 4(4) = 0$

$A = (0+3+3+0) = 6$

$A = 6 \text{ units}^2$

ii $y = -x^2 + 4x$

$y(1) = -(1)^2 + 4(1) = 3$

$y(2) = -(2)^2 + 4(2) = 4$

$y(3) = -(3)^2 + 4(3) = 3$

$A = (4+3+3+4) = 14$

$A = 14 \text{ units}^2$

b **i** $y = \sin x$

$y\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$y(0) = \sin(0) = 0$

$A = \frac{\pi}{4}\left(0 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0\right)$

$A = \frac{\pi\sqrt{2}}{4} \text{ units}^2$

ii $y = \sin x$

$$y\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$y\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$A = \frac{\pi}{4}\left(1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 1\right)$$

$$A = \frac{\pi\left(\sqrt{2}+2\right)}{4} \text{ units}^2$$

Question 12

a $y = x^2 + 5$

$$y(0) = (0)^2 + 5 = 5$$

$$y(0.5) = (0.5)^2 + 5 = 5.25$$

$$y(1) = (1)^2 + 5 = 6$$

$$y(1.5) = (1.5)^2 + 5 = 7.25$$

$$y(2) = (2)^2 + 5 = 9$$

$$y(2.5) = (2.5)^2 + 5 = 11.25$$

$$y(3) = (3)^2 + 5 = 14$$

$$y(3.5) = (3.5)^2 + 5 = 17.25$$

$$y(4) = (4)^2 + 5 = 21$$

$$y(4.5) = (4.5)^2 + 5 = 25.25$$

$$A = 0.5(5 + 5.25 + 6 + 7.25 + 9 + 11.25 + 14 + 17.25 + 21 + 25.5)$$

$$A = 60.625 \text{ units}^2$$

b $y = x^2 + 5$

$y(0.5) = (0.5)^2 + 5 = 5.25$

$y(1) = (1)^2 + 5 = 6$

$y(1.5) = (1.5)^2 + 5 = 7.25$

$y(2) = (2)^2 + 5 = 9$

$y(2.5) = (2.5)^2 + 5 = 11.25$

$y(3) = (3)^2 + 5 = 14$

$y(3.5) = (3.5)^2 + 5 = 17.25$

$y(4) = (4)^2 + 5 = 21$

$y(4.5) = (4.5)^2 + 5 = 25.25$

$y(5) = (5)^2 + 5 = 30$

$A = 0.5(5.25 + 6 + 7.25 + 9 + 11.25 + 14 + 17.25 + 21 + 25.5 + 30)$

$A = 73.125 \text{ units}^2$

Exercise 6.02 Trapezoidal rule

Question 1

a $\int_1^2 x^2\,dx$

$=\frac{1}{2}(2-1)\left(f(2)+f(1)\right)$

$=\frac{1}{2}(2-1)(4+1)$

$=2.5$

b $\int_0^2 (x^3+1)\,dx$

$=\frac{1}{2}(2-0)\left(f(2)+f(0)\right)$

$=\frac{1}{2}(2-0)(9+1)$

$=10$

c $\int_1^5 \frac{dx}{x}$

$=\frac{1}{2}(5-1)\left(f(5)+f(1)\right)$

$=\frac{1}{2}(5-1)(0.2+1)$

$=2.4$

d $\int_1^2 \frac{dx}{x+3}$

$=\frac{1}{2}(2-1)\left(f(2)+f(1)\right)$

$=\frac{1}{2}(2-1)(0.2+0.25)$

$=0.225$

Question 2

a $\int_1^3 x^3 dx$

$= \frac{2}{2}\left(f(3) + f(1)\right)$

$= \frac{2}{2}(27 + 1)$

$= 28$

b $\int_1^3 x^3 dx$

$= \frac{1}{2}\left(f(3) + f(1) + 2f(2)\right)$

$= \frac{1}{2}(27 + 1 + 16)$

$= 22$

Question 3

a $\int_2^3 \log x\ dx = \int_2^{2.5} \log x\ dx + \int_{2.5}^3 \log x\ dx$

$= \frac{0.5}{2}\left(f(2) + f(2.5)\right) + \frac{0.5}{2}\left(f(3) + f(2.5)\right)$

$= 0.39$

b $\int_0^2 \frac{dx}{x+4} = \int_0^1 \frac{dx}{x+4} + \int_1^2 \frac{dx}{x+4}$

$= \frac{1}{2}\left(f(1) + f(0)\right) + \frac{1}{2}\left(f(2) + f(1)\right)$

$= 0.41$

Question 4

a $\int_1^4 \log x\, dx = \int_1^2 \log x\, dx + \int_2^3 \log x\, dx + \int_3^4 \log x\, dx$

$= \frac{1}{2}\left(f(2)+f(1)\right)+\frac{1}{2}\left(f(2)+f(3)\right)+\frac{1}{2}\left(f(4)+f(3)\right)$

$= 1.08$

b $\int_0^2 \left(x^2-x\right)dx$

$= \int_0^{0.5}\left(x^2-x\right)dx+\int_{0.5}^{1}\left(x^2-x\right)dx+\int_1^{1.5}\left(x^2-x\right)dx+\int_{1.5}^{2}\left(x^2-x\right)dx$

$= \frac{0.5}{2}\left(f(0)+f(0.5)\right)+\frac{0.5}{2}\left(f(0.5)+f(1)\right)$
$\quad +\frac{0.5}{2}\left(f(1)+f(1.5)\right)+\frac{0.5}{2}\left(f(2)+f(1.5)\right)$

$= 0.75$

c $\int_0^1 \sqrt{x\, dx}$

$= \int_0^{0.2}\sqrt{x\, dx}+\int_{0.2}^{0.4}\sqrt{x\, dx}+\int_{0.4}^{0.6}\sqrt{x\, dx}+\int_{0.6}^{0.8}\sqrt{x\, dx}+\int_{0.8}^{1}\sqrt{x\, dx}$

$= \frac{0.2}{2}\left(f(0)+f(0.2)\right)+\frac{0.2}{2}\left(f(0.2)+f(0.4)\right)+\frac{0.2}{2}\left(f(0.4)+f(0.6)\right)$
$\quad +\frac{0.2}{2}\left(f(0.6)+f(0.8)\right)+\frac{0.2}{2}\left(f(0.8)+f(1)\right)$

$= 0.65$

d $\int_1^5 \frac{dx}{x^2}$

$= \int_1^2 \frac{dx}{x^2}+\int_2^3 \frac{dx}{x^2}+\int_3^4 \frac{dx}{x^2}+\int_4^5 \frac{dx}{x^2}$

$= \frac{1}{2}\left(f(1)+f(2)\right)+\frac{1}{2}\left(f(3)+f(2)\right)+\frac{1}{2}\left(f(3)+f(4)\right)+\frac{1}{2}\left(f(5)+f(4)\right)$

$= 0.94$

e $\int_3^6 \frac{dx}{x-1}$

$= \int_3^{3.5} \frac{dx}{x-1} + \int_{3.5}^4 \frac{dx}{x-1} + \int_4^{4.5} \frac{dx}{x-1} + \int_{4.5}^5 \frac{dx}{x-1} + \int_5^{5.5} \frac{dx}{x-1} + \int_{5.5}^6 \frac{dx}{x-1}$

$= \frac{0.5}{2}\left(f(3) + 2f(3.5)\right) + 2f(4) + 2f(4.5) + 2f(5) + 2f(5.5) + f(6))$

$= 0.92$

Question 5

a $\int_1^9 f(x)$

$= \int_1^3 f(x) + \int_3^5 f(x) + \int_5^7 f(x) + \int_7^9 f(x)$

$= \frac{2}{2}(3.2 + 2\times 5.9 + 2\times 8.4 + 2\times 11.6 + 20.1)$

$= 75.1$

b $\int_1^4 f(t)$

$= \int_1^2 f(t) + \int_2^3 f(t) + \int_3^4 f(t)$

$= \frac{1}{2}(8.9 + 2\times 6.5 + 2\times 4.1 + 2.9)$

$= 16.5$

c $\int_2^{14} f(x)$

$= \int_2^4 f(x) + \int_4^6 f(x) + \int_6^8 f(x) + \int_8^{10} f(x) + \int_{10}^{12} f(x) + \int_{12}^{14} f(x)$

$= \frac{2}{2}(25.1 + 2\times 37.8 + 2\times 52.3 + 2\times 89.3 + 2\times 67.8 + 2\times 45.4 + 39.9)$

$= 650.2$

Question 6

a $= \frac{1}{2}(3.9 + 2\times 5.4 + 2\times 5.1 + 2\times 4.7 + 2\times 4.4 + 2\times 5.3 + 4.1)$

$= 28.9\ \text{m}^2$

b $= \frac{1}{2}(9.8 + 2\times 11.3 + 2\times 9.1 + 2\times 9.7 + 8.5)$

$= 39.25\ \text{m}^2$

c $= \frac{1}{2}(2.9 + 2\times 2.3 + 2\times 2.1 + 3.2)$

$= 7.45\ \text{km}^2$

d $= \frac{5}{2}(18.3 + 2\times 27.6 + 2\times 24.1 + 2\times 26.3 + 22.6)$

$= 492.25\ \text{m}^2$

Exercise 6.03 Definite integrals

Question 1

a $\int_0^2 4x\ dx$

$= \left[2x^2\right]_0^2$

$= 2(2)^2 - 2(0)^2 = 8$

b $\int_1^3 (2x+1)dx$

$= \left[x^2 + x\right]_1^3$

$= \left((3)^2 + (3)\right) - \left((1)^2 + (1)\right) = 10$

c $\int_{-1}^6 3x^2\ dx$

$= \left[x^3\right]_{-1}^6$

$= (6)^3 - (-1)^3 = 217$

d $\int_1^2 (4t-7)\ dt$

$= \left[2t^2 - 7t\right]_1^2$

$= \left(2(2)^2 - 7(2)\right) - \left(2(1)^2 - 7(1)\right)$

$= -1$

e $\int_{-1}^1 (6y-5)dy$

$= \left[3y^2 - 5y\right]_{-1}^1$

$= \left(3(1)^2 + 5(1)\right) - \left(3(-1)^2 + 5(-1)\right)$

$= 10$

f $\int_0^3 6x^2\,dx$

$=\left[2x^3\right]_0^3$

$=2(3)^3-2(0)^3$

$=54$

g $\int_1^2 \left(x^2+1\right)\,dx$

$=\left[\frac{x^3}{3}+x\right]_1^2$

$=\left(\frac{(2)^3}{3}+(2)\right)-\left(\frac{(1)^3}{3}+(1)\right)$

$=3\frac{1}{3}$

h $\int_0^2 4x^3\,dx$

$=\left[x^4\right]_0^2$

$=(2)^4-(0)^4$

$=16$

i $\int_{-1}^4 3x^2-2x\ dx$

$=\left[x^3-x^2\right]_{-1}^4$

$=\left((4)^3-(4)^2\right)-\left((-1)^3-(-1)^2\right)$

$=50$

Question 2

a $\int_{-1}^{1} x^2 \, dx$

$= \left[\frac{x^3}{3} \right]_{-1}^{1}$

$= \frac{(1)^3}{3} - \frac{(-1)^3}{3} = \frac{2}{3}$

b $\int_{-2}^{3} (x^3 + 1) \, dx$

$= \left[\frac{x^4}{3} + x \right]_{-2}^{3}$

$= \left(\frac{(3)^4}{4} + (3) \right) - \left(\frac{(-2)^4}{4} + (-2) \right)$

$= 21\frac{1}{4}$

c $\int_{-2}^{2} (x^5) \, dx$

$= \left[\frac{x^5}{5} \right]_{-2}^{2}$

$= \left(\frac{(2)^5}{5} \right) - \left(\frac{(2)^5}{5} \right)$

$= 0$

d $\int_{1}^{4} \left(\sqrt{x} \right) dx$

$= \left[\frac{2}{3} x^{\frac{3}{2}} \right]_{1}^{4}$

$= \left(\frac{2}{3} (4)^{\frac{3}{2}} \right) - \left(\frac{2}{3} (1)^{\frac{3}{2}} \right)$

$= 4\frac{2}{3}$

e $\int_0^1 (x^3 - 3x^2 + 4x)\ dx$

$$= \left[\frac{x^4}{4} - x^3 + 2x^2 \right]_0^1$$

$$= \left(\frac{(1)^4}{4} - (1)^3 + 2(1)^2 \right) - \left(\frac{(0)^4}{4} - (0)^3 + 2(0)^2 \right)$$

$$= 1\frac{1}{4}$$

f $\int_1^2 (2x-1)^2\ dx$

$$= \left[\frac{1}{6}(2x-1)^3 \right]_1^2$$

$$= \left(\frac{1}{6}(2(2)-1)^3 \right) - \left(\frac{1}{6}(2(1)-1)^3 \right)$$

$$= 4\frac{1}{3}$$

g $\int_{-1}^1 \left(y^3 + y \right)\ dx$

$$= \left[\frac{y^4}{4} + \frac{y^2}{2} \right]_{-1}^1$$

$$= \left(\frac{(1)^4}{4} + \frac{(1)^2}{2} \right) - \left(\frac{(-1)^4}{4} + \frac{(-1)^2}{2} \right) = 0$$

h $\int_3^4 (2-x)^2\ dx$

$$= \left[-\frac{1}{3}(2-x)^3 \right]_3^4$$

$$= \left(-\frac{1}{3}(2-(4))^3 \right) - \left(-\frac{1}{3}(2-(3))^3 \right) -$$

$$= 2\frac{1}{3}$$

i $\int_{-2}^{2} 4t^3 \; dt$

$=\left[t^4\right]_3^4$

$\left((2)^4\right)-\left((-2)^4\right)=0$

j $\int_2^4 \frac{x^2}{3} dx$

$=\left[\frac{x^3}{9}\right]_2^4$

$=\left(\frac{(4)^3}{9}\right)-\left(\frac{(2)^3}{9}\right)$

$=6\frac{2}{9}$

k $\int_1^3 \frac{5x^4}{x} \; dx$

$=\int_1^3 5x^3 dx$

$=\left[\frac{5x^4}{4}\right]_1^3$

$=\left(\frac{5(3)^4}{4}\right)-\left(\frac{5(1)^4}{4}\right)=100$

l $\int_2^4 \frac{x^4-3x}{x} \; dx=\int_2^4 \left(x^3-3\right) \; dx$

$=\left[\frac{x^4}{4}-3x\right]_2^4$

$=\left(\frac{(4)^4}{4}-3(4)\right)-\left(\frac{(2)^4}{4}-3(2)\right)$

$=54$

m

$$\int_1^2 \frac{4x^3+x^2+5x}{x}\,dx=\int_1^2\left(4x^2+x+5\right)dx$$

$$=\left[\frac{x^3}{3}+\frac{x^2}{2}+5x\right]_1^2$$

$$=\left(\frac{(2)^3}{3}+\frac{(2)^2}{2}+5(2)\right)-\left(\frac{(1)^3}{3}+\frac{(1)^2}{2}+5(1)\right)$$

$$=15\frac{5}{6}$$

n $$\int_3^5 \frac{x^3-2x^2+3x}{x}\,dx=\int_3^5\left(x^2-2x+3\right)dx$$

$$=\left[\frac{x^3}{3}-x^2+3x\right]_3^5$$

$$=\left(\frac{(5)^3}{3}-(5)^2+3(5)\right)-\left(\frac{(3)^3}{3}-(3)^2+3(3)\right)$$

$$=22\frac{2}{3}$$

o $$\int_3^4 \frac{x^2+x+3}{3x^5}\,dx=\int_3^4\left(\frac{1}{3x^3}+\frac{1}{3x^4}+\frac{1}{x^5}\right)dx$$

$$=\left[-\frac{1}{6x^2}-\frac{1}{9x^3}-\frac{1}{4x^4}\right]_3^4$$

$$=\left(-\frac{1}{6(4)^2}-\frac{1}{9(4)^3}-\frac{1}{4(4)^4}\right)-\left(-\frac{1}{6(3)^2}-\frac{1}{9(3)^3}-\frac{1}{4(3)^4}\right)$$

$$=0.0126$$

Question 3

a $\int_2^4 \left(3t^2+7\right)\,dt$

$=\left[t^3+7t\right]_2^4$

$=\left((4)^3+7(4)\right)-\left((2)^3+7(2)\right)=70\ \text{ m}$

b $\int_2^4 (8t-5)\,dt$

$=\left[4t^2-5t\right]_2^4$

$=\left(4(4)^2-5(4)\right)-\left(4(2)^2-5(2)\right)=38\ \text{ km}$

c $\int_2^4 \left(4t^3+2t+3\right)\,dt$

$=\left[t^4+t^2+3t\right]_2^4$

$=\left((4)^4+(4)^2\right)+3(4))-\left((2)^4+(2)^2\right)+3(2))=258\ \text{ cm}$

d $\int_2^4 (t+3)^2\,dt$

$=\left[\frac{1}{3}(t+3)^3\right]_2^4$

$=\left(\frac{1}{3}((4)+3)^3\right)-\left(\frac{1}{3}((2)+3)^3\right)$

$=72\frac{2}{3}\ \text{ m}$

e $\int_2^4 \left(5-6t+9t^2\right)\,dt$

$=\left[5t-3t^2+3t^3\right]_2^4$

$\left(5(4)-3(4)^2+3(4)^3\right)-\left(5(2)-3(2)^2+3(2)^3\right)$

$=142\ \text{ cm}$

Question 4

a $\int_0^5 \left(25+4t^3\right)\,dt$

$=\left[25t+t^4\right]_0^5$

$=\left(25(5)+(5)^4\right)-\left(25(0)+t(0)^4\right)$

$=750\ \text{L}$

b $\int_0^{15} \left(25+4t^3\right)\,dt$

$=\left[25t+t^4\right]_0^{15}$

$=\left(25(15)+(15)^4\right)-\left(25(0)+t(0)^4\right)$

$=51\,000\ \text{L}$

c $\int_0^{30} \left(25+4t^3\right)\,dt$

$=\left[25t+t^4\right]_0^{30}$

$=\left(25(30)+(30)^4\right)-\left(25(0)+t(0)^4\right)$

$=810\,750\ \text{L}$

Exercise 6.04 Indefinite integrals

Question 1

a $\int x^2 = \frac{x^3}{3} + C$

b $\int 3x^5 = \frac{3x^6}{6} + C = \frac{x^6}{2} + C$

c $\int 2x^4 = \frac{2x^5}{5} + C$

d $\int m+1 = \frac{m^2}{2} + m + C$

e $\int t^2 - 7 = \frac{t^3}{3} - 7t + C$

f $\int h^7 + 5 = \frac{h^8}{8} + 5h + C$

g $\int y - 3 = \frac{y^2}{2} - 3y + C$

h $\int 2x + 4 = x^2 + 4x + C$

i $\int b^2 + b = \frac{b^3}{3} + \frac{b^2}{2} + C$

Question 2

a $\int x^2+2x+5 \ =\dfrac{x^3}{3}+x^2+5x+C$

b $\int 4x^3-3x^2+8x-1 \ = x^4-x^3+4x^2-x+C$

c $\int 6x^5+x^4+2x^3 \ = x^6+\dfrac{x^5}{5}+\dfrac{x^4}{2}+C$

d $\int x^7-3x^6-9=\dfrac{x^8}{8}-\dfrac{3x^7}{7}-9x+C$

e $\int 2x^3+x^2-x-2 \ =\dfrac{x^4}{2}+\dfrac{x^3}{3}-\dfrac{x^2}{2}-2x+C$

f $\int x^5+x^3+4 \ =\dfrac{x^6}{6}+\dfrac{x^4}{4}+4x+C$

g $\int 4x^2-5x-8 \ =\dfrac{4x^3}{3}-\dfrac{5x^2}{2}-8x+C$

h $\int 3x^4-2x^3+x \ =\dfrac{3x^5}{5}-\dfrac{x^4}{2}+\dfrac{x^2}{2}+C$

i $\int 6x^3+5x^2-4=\dfrac{3x^4}{2}+\dfrac{5x^3}{3}-4x+C$

j $\int 3x^{-4}+x^{-3}+2x^{-2}$

$$=\frac{3x^{-3}}{-3}+\frac{x^{-2}}{-2}+\frac{2x^{-1}}{-1}+C$$

$$=x^{-3}-\frac{x^{-2}}{2}-2x^{-1}+C$$

Question 3

a $\int \frac{dx}{x^8} = \int x^{-8}dx$

$= \frac{x^{-7}}{-7} + C$

$= -\frac{1}{7x^7} + C$

b $\int x^{\frac{1}{3}}dx = \frac{3x^{\frac{4}{3}}}{4} + C$

c $\int \frac{x^6 - 3x^5 + 2x^4}{x^3}dx = \int (x^3 - 3x^2 + 2x)dx$

$= \frac{x^4}{4} - x^3 + x^2 + C$

d $\int (1-2x)^2\,dx = \int \left(1 - 4x + 4x^2\right)dx$

$= x - 2x^2 + \frac{4x^3}{3} + C$

e $\int (x-2)(x+5)dx = \int \left(x^2 + 3x - 10\right)dx$

$= \frac{x^3}{3} + \frac{3x^2}{2} - 10x + C$

f $\int \frac{3}{x^2}dx = \int 3x^{-2}dx$

$= -3x^{-1} + C$

$= -\frac{3}{x} + C$

g $\int \frac{dx}{x^3} = \int x^{-3}dx$

$= \frac{x^{-2}}{-2} + C$

$= -\frac{1}{2x^2} + C$

h $\int \frac{4x^3 - x^5 - 3x^2 + 7}{x^5} dx = \int \left(\frac{4}{x^2} - 1 - \frac{3}{x^3} + \frac{7}{x^5} \right) dx = \int \left(4x^{-2} - 1 - 3x^{-3} + 7x^{-5} \right) dx$

$= -4x - x + \frac{3x^{-2}}{2} - \frac{7x^{-4}}{4} + C$

$= -\frac{4}{x} - x + \frac{3}{2x^2} - \frac{7}{4x^4} + C$

i $\int \left(y^2 - y^{-7} + 5 \right) dy = \frac{y^3}{3} - \frac{y^{-6}}{6} + 5y + C$

j $\int \left(t^2 - 4 \right)\left(t - 1 \right) dt = \int \left(t^3 - t^2 - 4t + 4 \right) dt$

$= \frac{t^4}{4} - \frac{t^3}{3} - 2t^2 + 4t + C$

k $\int \left(\sqrt{x} \right) dx = \int x^{\frac{1}{2}} dx$

$= \frac{2}{3} x^{\frac{3}{2}} + C$

$= \frac{2\sqrt{x^3}}{3} + C$

l $\int \left(\frac{2}{t^5} \right) dt = \int 2t^{-5} dt$

$= \frac{2t^{-4}}{-4} + C$

$= -\frac{1}{2t^4} + C$

m $\int \left(\sqrt[3]{x} \right) dx = \int x^{\frac{1}{3}} dx$

$= \frac{3x^{\frac{4}{3}}}{4} + C$

$= \frac{3\sqrt[3]{x^4}}{4} + C$

n $\int\left(x\sqrt{x}\right)\,dx = \int x\times x^{\frac{1}{2}}\,dx = \int x^{\frac{3}{2}}\,dx$

$=\dfrac{2x^{\frac{5}{2}}}{5}+C$

$=\dfrac{2\sqrt{x^5}}{5}+C$

o $\int\left(\sqrt{x}\left(1+\dfrac{1}{\sqrt{x}}\right)\right)dx = \int\left(\sqrt{x}+1\right)dx = \int\left(x^{\frac{1}{2}}+1\right)dx$

$=\dfrac{2x^{\frac{3}{2}}}{3}+x+C$

$=\dfrac{2\sqrt{x^3}}{3}+x+C$

Question 4

$\dfrac{dS}{dn}=180$

$S=\int 180dn$

$S=180n+C$

$360=180(4)+C$

$360=720+C$

$C=-360$

$S=180n-360$

$S(7)=180(7)-360$

$S(7)=900°$

Question 5

$R = 3x^2 - 2x + 1$

$\int(3x^2 - 2x + 1)dx$

$= x^3 - x^2 + x + C$

$3 = (-1)^3 - (-1)^2 + (-1) + C$

$3 = -3 + C$

$C = 6$

$= x^3 - x^2 + x + 6$

Question 6

$\frac{dx}{dt} = 4t + t^2 - t^3$

$\int(4t + t^2 - t^3)dt$

$= 2t^2 + \frac{t^3}{3} - \frac{t^4}{4} + C$

$2 = 2(0)^2 + \frac{(0)^3}{3} - \frac{(0)^4}{4} + C$

$C = 2$

$x = 2t^2 + \frac{t^3}{3} - \frac{t^4}{4} + 2$

$x(15) = 2(15)^2 + \frac{(15)^3}{3} - \frac{(15)^4}{4} + 2$

$x(15) = -11\ 079.25\ \text{ cm s}^{-1}$

Question 7

$R = 500 + 20t$

$\int (500 + 20t)\,dt$

$= 500t + 10t^2 + C$

$15\ 000 = 500(0) + 10(0)^2 + C$

$C = 15\ 000$

$= 500t + 10t^2 + 15\ 000$

$= 500(10) + 10(10)^2 + 15\ 000$

$= 21\ 000\ \text{L}$

Exercise 6.05 Chain rule

Question 1

a $\int (3x-4)^2\,dx$

$=\frac{(3x-4)^3}{3(3)}+C$

$=\frac{(3x-4)^3}{9}+C$

b $\int (x+1)^4\,dx$

$=\frac{(x+1)^5}{5(1)}+C$

$=\frac{(x+1)^5}{5}+C$

c $\int (5x-1)^9\,dx$

$=\frac{(5x-1)^{10}}{10(5)}+C$

$=\frac{(5x-1)^{10}}{50}+C$

d $\int (3y-2)^7\,dy$

$=\frac{(3y-2)^8}{8(3)}+C$

$=\frac{(3y-2)^8}{24}+C$

e $\int (4+3x)^4\,dx$

$$= \frac{(4+3x)^5}{5(3)} + C$$

$$= \frac{(4+3x)^5}{15} + C$$

f $\int (7x+8)^{12}\,dx$

$$= \frac{(7x+8)^{13}}{13(7)} + C$$

$$= \frac{(7x+8)^{13}}{91} + C$$

g $\int (1-x)^6\,dx$

$$= \frac{(1-x)^7}{7(-1)} + C$$

$$= -\frac{(1-x)^7}{7} + C$$

h $\int \sqrt{2x-5}\,dx = \int (2x-5)^{\frac{1}{2}}\,dx$

$$= \frac{(2x-5)^{\frac{3}{2}}}{\frac{3}{2}(2)} + C$$

$$= \frac{(2x-5)^{\frac{3}{2}}}{3} + C$$

$$= \frac{\sqrt{(2x-5)^3}}{3} + C$$

i $\int 2(3x+1)^{-4}\,dx$

$=\dfrac{2(3x+1)^{-3}}{-3(3)}+C$

$=\dfrac{2(3x+1)^{-3}}{9}+C$

j $\int 3(x+7)^{-2}\,dx$

$=\dfrac{3(x+7)^{-1}}{(-1)}+C$

$=-3(x+7)^{-1}+C$

k $\int \dfrac{1}{2(4x-5)^{3}}\,dx = \int \dfrac{1}{2}(4x-5)^{-3}\,dx$

$=\dfrac{(4x-5)^{-2}}{2\times 4\times -2}+C$

$=-\dfrac{1}{16(4x-5)^{2}}+C$

l $\int \sqrt[3]{4x+3}\,dx = \int (4x+3)^{\frac{1}{3}}\,dx$

$=(4x+3)^{\frac{4}{3}}$

$\dfrac{4}{3}\times 4$

$=\dfrac{3\sqrt[3]{(4x+3)^{4}}}{16}+C$

m $\int (2-x)^{-\frac{1}{2}}\,dx$

$=\dfrac{2(2-x)^{\frac{1}{2}}}{-1}+C$

$=-2\sqrt{2-x}+C$

n $\int \sqrt{(t+3)^2}\,dt = = \int (t+3)^{\frac{3}{2}}\,dt$

$= \frac{2}{5}(t+3)^{\frac{5}{2}} + C$

$= \frac{2\sqrt{(t+3)^5}}{5} + C$

o $\int \sqrt{(5x+2)^5}\,dt = = \int (5x+2)^{\frac{5}{2}}\,dt$

$= \frac{\frac{2}{7}(5x+2)^{\frac{7}{2}}}{5} + C$

$= \frac{2\sqrt{(5x+2)^7}}{35} + C$

Question 2

a $\int_1^2 (2x+1)^4\,dx$

$= \left[\frac{(2x+1)^5}{5\times 2}\right]_1^2$

$= \left[\frac{(2x+1)^5}{10}\right]_1^2$

$= \frac{(2(2)+1)^5}{10} - \frac{(2(1)+1)^5}{10}$

$= 288.2$

b $\int_0^1 (3y-2)^3\,dy$

$$=\left[\frac{(3y-2)^4}{3\times 4}\right]_0^1$$

$$=\left[\frac{(3y-2)^4}{12}\right]_0^1$$

$$=\frac{(3(1)-2)^4}{12}-\frac{(3(0)-2)^4}{12}$$

$$=-1\frac{1}{4}$$

c $\int_1^2 (1-x)^7\,dx$

$$=\left[\frac{(1-x)^8}{8\times -1}\right]_1^2$$

$$=\left[-\frac{(1-x)^8}{8}\right]_1^2$$

$$=-\frac{(1-(2))^8}{8}+\frac{(1-(1))^8}{8}$$

$$=-\frac{1}{8}$$

d $\int_0^2 (3-2x)^5\,dx$

$$=\left[\frac{(3\times 2x)^6}{6\times -2}\right]_0^2$$

$$=\left[-\frac{(3\times 2x)^6}{12}\right]_0^2$$

$$=-\frac{(3-2(2))^6}{12}+\frac{(3-2(0))^6}{12}$$

$$=60\frac{2}{3}$$

e $\int_0^1 \frac{(3x-1)^2}{6}\,dx$

$$=\left[\frac{(3x-1)^3}{6\times 3\times 3}\right]_0^1$$

$$=\left[\frac{(3x-1)^3}{54}\right]_0^1$$

$$=\frac{(3(1)-1)^3}{54}-\frac{(3(0)-1)^3}{54}$$

$$=\frac{1}{6}$$

f $\int_4^5 (5-x)^6\,dx$

$$=\left[\frac{(5-x)^7}{-1\times 7}\right]_4^5$$

$$=\left[-\frac{(5-x)^7}{7}\right]_4^5$$

$$=-\frac{(5-(5))^7}{7}+\frac{(5-(4))^7}{7}$$

$$=-\frac{(5-(5))^7}{7}+\frac{(5-(4))^7}{7}$$

$$=\frac{1}{7}$$

g $\int_3^6 (x-2)^{\frac{1}{2}}\,dx$

$$=\left[\frac{(x-2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_3^6$$

$$=\left[\frac{2(x-2)^{\frac{3}{2}}}{3}\right]_3^6$$

$$=\frac{2((6)-2)^{\frac{3}{2}}}{3}-\frac{2((3)-2)^{\frac{3}{2}}}{3}$$

$$=4\frac{2}{3}$$

h $\int_0^2 \frac{5}{(2n+1)^3}\,dn = \int_0^2 5(2n+1)^{-3}\,dn$

$= \left[\frac{5(2n+1)^{-2}}{-2\times 2}\right]_0^2$

$= \left[-\frac{5(2n+1)^{-2}}{4}\right]_0^2$

$= -\frac{5(2(2)+1)^{-2}}{4} + \frac{5(2(0)+1)^{-2}}{4}$

$= 1\frac{1}{5}$

i $\int_1^4 \frac{2}{\sqrt{(5x-4)^3}}\,dx = \int_1^4 2(5x-4)^{-\frac{3}{2}}\,dx$

$= \left[\frac{2(5x-4)^{\frac{1}{2}}}{5\times -\frac{1}{2}}\right]_1^4$

$= \left[\frac{-4(5x-4)^{\frac{1}{2}}}{5}\right]_1^4$

$= -\frac{4(5(4)-4)^{-\frac{1}{2}}}{5} + \frac{4(5(1)-4)^{-\frac{1}{2}}}{5}$

$= \frac{3}{5}$

Question 3

a $\int 4x^3\left(x^4+5\right)^2 dx$

$u = x^4+5,\ du = 4x^3\,dx$

$\int u^2 du$

$= \frac{u^3}{3}+C$

$= \frac{\left(x^4+5\right)^3}{3}+C$

b $\int 2x\left(x^2-3\right)^5 dx$

$u = x^2-3,\ du = 2x\ dx$

$\int u^5 du$

$= \frac{u^6}{6}+C$

$= \frac{\left(x^2-3\right)^6}{6}+C$

c $\int 3x^2\left(x^3+1\right)^3 dx$

$u = x^3+1,\ du = 3x^2\,dx$

$\int u^3 du$

$= \frac{u^4}{4}+C$

$= \frac{\left(x^3+1\right)^4}{4}+C$

d $\int (2x+3)\left(x^2+3x-2\right)^4 dx$

$u = x^2+3x-2,\ du = 2x+3\ dx$

$\int u^4 du$

$= \frac{u^5}{5} + C$

$= \frac{\left(x^2+3x-2\right)^5}{5} + C$

e $\int x\left(3x^2-7\right)^6 dx$

$u = 3x^2-7,\ du = 6x\ dx$

$\int \frac{1}{6} u^6 du$

$= \frac{u^7}{6\times 7} + C$

$= \frac{\left(3x^2-7\right)^7}{42} + C$

f $\int x^2\left(4-5x^3\right)^2 dx$

$u = 4-5x^3,\ du = -15x^2\ dx$

$\int \frac{1}{-15} u^2 du$

$= -\frac{u^3}{15\times 3} + C$

$= -\frac{\left(4-5x^3\right)^3}{45} + C$

g $\int 4x^5\left(2x^6-3\right)^4 dx$

$u=2x^6-3,\ du=12x^5\ dx$

$\int \frac{1}{3}u^4 du$

$=\frac{u^5}{3\times 5}+C$

$=\frac{\left(2x^6-3\right)^5}{15}+C$

h $\int 3x\left(5x^2+3\right)^7$

$u=5x^2+3,\ du=10x\ dx$

$\int \frac{3}{10}u^7 du$

$=\frac{3u^8}{10\times 8}+C$

$=\frac{3\left(5x^2+3\right)^8}{80}+C$

i $\int \left(x+2\right)\left(x^2+4x\right)^5 dx$

$u=x^2+4x,\ du=2x+4\ dx$

$\int \frac{1}{2}u^5 du$

$=\frac{u^6}{2\times 6}+C$

$=\frac{\left(x^2+4x\right)^6}{12}+C$

j $\int\left(3x^2-2\right)\left(3x^3-6x-2\right)^3 dx$

$u=3x^3-6x-2,\ du=\left(9x^2-6\right) dx$

$\int\frac{1}{3}u^3du$

$=\frac{u^4}{3\times4}+C$

$=\frac{\left(3x^3-6x-2\right)^4}{12}+C$

Question 4

a $\int_0^2 x\left(2x^2+3\right)^2 dx$

$u=2x^2+3,\ du=4x\ dx$

$\int_0^2\frac{1}{4}u^2du$

$=\left[\frac{u^3}{4\times3}\right]_0^2$

$=\left[\frac{\left(2x^2+3\right)^3}{12}\right]_0^2$

$=\frac{\left(2(2)^2+3\right)^3}{12}-\frac{\left(2(0)^2+3\right)^3}{12}$

$=108\frac{2}{3}$

b $\int_0^1 x^2\left(x^3-1\right)^5 dx$

$u = x^3 - 1,\ du = 3x^2\ dx$

$\int_0^1 \frac{1}{3}u^5 du$

$= \left[\begin{matrix} u^6 \\ 3\times 6 \end{matrix}\right]_0^1$

$= \left[\frac{\left(x^3-1\right)^6}{18}\right]_0^1$

$= -\frac{1}{18}$

c $\int_1^2 x^4\left(x^5+2\right)^3 dx$

$u = x^5 + 2,\ du = 5x^4\ dx$

$\int_1^2 \frac{1}{5}u^3 du$

$= \left[\frac{u^4}{5\times 4}\right]_1^2$

$= \left[\frac{\left(x^5+2\right)^4}{20}\right]_1^2$

$= \frac{\left((2)^5+2\right)^4}{20} - \frac{\left((1)^5+2\right)^4}{20}$

$= 66\,812.75$

d $\int_0^1 x^3\left(5-x^4\right)^7\,dx$

$u=5-x^4,\ du=-4x^3\ dx$

$\int_0^1 \frac{1}{-4}u^7du$

$=\left[\frac{u^8}{-4\times 8}\right]_0^1$

$=\left[-\frac{\left(5-x^4\right)^8}{32}\right]_0^1$

$=-\frac{\left(5-(1)^4\right)^8}{32}+\frac{\left(5-(0)^4\right)^8}{32}$

$=10\,159\frac{1}{32}$

e $\int_2^4 3x\left(x^2+2\right)^4\,dx$

$u=x^2+2,\ du=2x\ dx$

$\int_2^4 \frac{3}{2}u^4du$

$=\left[\frac{3u^5}{2\times 5}\right]_2^4$

$=\left[\frac{3\left(x^2+2\right)^5}{10}\right]_2^4$

$=\frac{3\left((4)^2+2\right)^5}{10}-\frac{3\left((2)^2+2\right)^5}{10}$

$=564\,537.6$

f $\int_{-1}^{1} 5x^2\left(2x^3-7\right)^3 dx$

$u = 2x^3 - 7,\ du = 6x^2\ dx$

$\int_{-1}^{1} \frac{5}{6}u^3 du$

$= \left[\frac{5u^4}{6\times 4}\right]_{-1}^{1}$

$= \left[\frac{5\left(2x^3-7\right)^4}{24}\right]_{-1}^{1}$

$= \frac{5\left(2(1)^3-7\right)^4}{24} - \frac{5\left(2(-1)^3-7\right)^4}{24}$

$= -1236\frac{2}{3}$

g $\int_{-1}^{0} (x-1)\left(x^2-2x+3\right)^6 dx$

$u = x^2 - 2x + 3,\ du = (2x-2)\ dx$

$\int_{-1}^{0} \frac{1}{2}u^6 du$

$= \left[\frac{u^7}{2\times 7}\right]_{-1}^{0}$

$= \left[\frac{\left(x^2-2x+3\right)^7}{14}\right]_{-1}^{0}$

$= \frac{\left((0)^2-2(0)+3\right)^7}{14} - \frac{\left((-1)^2-2(-1)+3\right)^7}{14}$

$= -19\ 839\frac{3}{14}$

h $4\int_0^1 \left(x^2+2\right)\left(x^3+6x-1\right)^2\,dx$

$u = x^3+6x-1,\ du = \left(3x^2+6\right)\,dx$

$4\int_0^1 \frac{1}{3}u^2du$

$= 4\left[\frac{u^3}{3\times 3}\right]_0^1$

$= 4\left[\frac{\left(x^3+6x-1\right)^3}{9}\right]_0^1$

$= 4\left(\frac{\left(1(1)^3+6(1)-1\right)^3}{9} - \frac{\left((0)^3+6(0)-1\right)^3}{9}\right)$

$= 96\frac{4}{9}$

i

$5\int_{-2}^{2} x^2\left(x^3-1\right)\left(x^6-2x^3-1\right)^4\ dx$

$u = x^6-2x^3-1,\ du = \left(6x^5-6x^2\right)\,dx$

$5\int_{-2}^{2}\left(x^5-x^2\right)\left(x^6-2x^3-1\right)^4\ dx$

$= 5\int_{-2}^{2}\frac{1}{6}u^4\,du$

$= 5\left[\frac{u^5}{6\times 5}\right]_{-2}^{2}$

$= 5\left[\frac{\left(x^6-2x^3-1\right)^5}{30}\right]_{-2}^{2}$

$= 5\left(\frac{\left[(2)^6-2(2)^3-1\right]^5}{30} - \frac{\left[(-2)^6-2(-2)^3-1\right]^5}{30}\right)$

$= -474\ 618\ 565.3$

Question 5

$$\int x^2\left(x^3-2\right)^4\ dx$$

$$u=\left(x^3-2\right),\ du=3x^2\,dx$$

$$\int \frac{1}{3}u^4\,dx$$

$$=\frac{u^5}{3\times 5}+C$$

$$=\frac{\left(x^3-2\right)^5}{15}+C$$

$$4=\frac{\left((1)^3-2\right)^5}{15}+C$$

$$4=-\frac{1}{15}+C$$

$$C=4\frac{1}{15}$$

$$=\frac{\left(x^3-2\right)^5}{15}+4\frac{1}{15}$$

$$=\frac{1}{15}\left[\left(x^3-2\right)^5+61\right]$$

Question 6

a

$$\int x\left(x^2-3\right)^4\ dx$$

$$u=x^2-3,\ \ du\ =\ 2x\ dx$$

$$\int \frac{1}{2}u^4\ du$$

$$=\frac{u^5}{2\times 5}+C$$

$$=\frac{\left(x^2-3\right)^5}{10}+C$$

$$0=\frac{\left((2)^2-3\right)^5}{10}+C$$

$$0=\frac{1}{10}+C$$

$$C=-\frac{1}{10}$$

$$=\frac{\left(x^2-3\right)^5}{10}-\frac{1}{10}$$

$$=\frac{1}{10}\left[\left(x^2-3\right)^5-1\right]$$

b

$$x=\frac{1}{10}\left[\left(x^2-3\right)^5-1\right]$$

$$x(3)=\frac{1}{10}\left[\left([3]^2-3\right)^5-1\right]$$

$$x(3)=777.5\ \ \text{m}$$

Exercise 6.06 Integration involving exponential functions

Question 1

a

$$\int e^{4x}\ dx$$

$$u = 4x,\ du = 4\ dx$$

$$\int \frac{1}{4} \times 4e^{u}\ dx$$

$$= \int \frac{1}{4} \times e^{u}\ du$$

$$= \frac{1}{4} e^{4x} + C$$

b

$$\int e^{-x} dx$$

$$u = -x, du = -1dx$$

$$\int -1 \times e^{u} dx$$

$$= \int -1 \times e^{u} du$$

$$= -e^{-x} + C$$

c

$$\int e^{5x} dx$$

$$u = 5x, du = 5dx$$

$$\int \frac{1}{5} \times 5e^{u} dx$$

$$= \int \frac{1}{5} \times e^{u} du$$

$$= \frac{1}{5} e^{5x} + C$$

d $\int e^{-2x}dx$

$u = -2x, du = -2\,dx$

$\int \frac{1}{-2} \times -2e^{u}dx$

$= \int -\frac{1}{2} \times e^{u}du$

$= -\frac{1}{2}e^{-2x} + C$

e $\int e^{4x+1}dx$

$u = 4x + 1, du = 4\,dx$

$\int \frac{1}{4} \times 4e^{u}dx$

$= \int \frac{1}{4} \times e^{u}du$

$= \frac{1}{4}e^{4x+1} + C$

f $\int -3e^{5x}dx$

$u = 5x, du = 5\,dx$

$\int -3 \times \frac{1}{5} \times 5e^{u}dx$

$= \int \frac{-3}{5} \times e^{u}du$

$= -\frac{3}{5}e^{5x} + C$

g $\int e^{2t} dt$

$u = 2t, du = 2\,dt$

$\int \frac{1}{2} \times 2e^{u} dt$

$= \int \frac{1}{2} \times e^{u} du$

$= \frac{1}{2} e^{2t} + C$

h $\int e^{7x} - 2\ dx$

$u = 7x, du = 7\,dx$

$\int \frac{1}{7} \times 7e^{u} dx + \int -2\,dx$

$= \int \frac{1}{7} \times e^{u} du + \int -2\ dx$

$= \frac{1}{7} e^{7x} - 2x + C$

i $\int e^{x-3} + x\ dx$

$u = x - 3, du = 1dx$

$\int e^{u} dx + \int x\ dx$

$= \int e^{u} du + \int x\ dx$

$= e^{x-3} + \frac{x^2}{2} + C$

Question 2

a $\int_0^1 e^{5x}dx$

$u = 5x, du = 5dx$

$\int_0^1 \frac{1}{5}e^u du$

$= \left[\frac{1}{5}e^{5x}\right]_0^1$

$= \frac{1}{5}e^{5(1)} - \frac{1}{5}e^{5(0)}$

$= \frac{1}{5}\left(e^5 - 1\right)$

b $\int_0^2 -e^{-x}dx$

$u = -x, du = -1dx$

$\int_0^2 e^u du$

$= \left[e^{-x}\right]_0^2$

$= e^{-2} - e^{-0}$

$= e^{-2} - 1$

$= \frac{1}{e^2} - 1$

c $\int_1^4 2e^{3x+4}dx$

$u = 3x + 4, du = 3dx$

$\int_1^4 2\times\frac{1}{3}e^u du$

$=\left[\frac{2}{3}e^{3x+4}\right]_1^4$

$=\frac{2}{3}e^{3(4)+4}-\frac{2}{3}e^{3(1)+4}$

$=\frac{2}{3}\left(e^{16}-e^{7}\right)$

$=\frac{2e^7}{3}\left(e^{9}-1\right)$

d $\int_2^3 \left(3x^2 - e^{2x}\right)dx$

$u = 2x, du = 2dx$

$\int_2^3 3x^2 dx + \int_2^3 -\frac{1}{2}e^u du$

$=\left[x^3-\frac{1}{2}e^{2x}\right]_2^3$

$=\left(\left(3\right)^3-\frac{1}{2}e^{2(3)}\right)-\left(\left(2\right)^3-\frac{1}{2}e^{2(2)}\right)$

$=27-\frac{1}{2}e^6-8+\frac{1}{2}e^4$

$=19-\frac{e^4}{2}\left(e^2-1\right)$

e $\int_0^2 \left(e^{2x}+1\right)dx$

$u = 2x,\ du = 2\,dx$

$\int_0^2 \frac{1}{2}e^u du + \int_0^2 1\,dx$

$= \left[\frac{1}{2}e^{2x} + x\right]_0^2$

$= \left(\frac{1}{2}e^{2(2)} + 2\right) - \left(\frac{1}{2}e^{2(0)} + (0)\right)$

$= \frac{1}{2}e^4 + 2 - \frac{1}{2}$

$= 1\frac{1}{2} + \frac{1}{2}e^4$

f $\int_1^2 \left(e^x - x\right)dx$

$u = x, du = dx$

$\int_1^2 e^u du - \int_1^2 x\,dx$

$= \left[e^x - \frac{x^2}{2}\right]_1^2$

$= \left(e^{(2)} - \frac{(2)^2}{2}\right) - \left(e^{(1)} - \frac{(1)^2}{2}\right)$

$= e^2 - 2 - e + \frac{1}{2}$

$= e^2 - e - 1\frac{1}{2}$

g $\int_0^3 \left(e^{2x} - e^{-x}\right) dx$

$u = 2x, du = 2\,dx$

$v = -x, dv = -dx$

$\int_0^3 \frac{1}{2} e^u du + \int_0^3 e^v dv$

$= \left[\frac{1}{2}e^{2x} + e^{-x}\right]_0^3$

$= \left(\frac{1}{2}e^{2(3)} + e^{-(3)}\right) - (\frac{1}{2}e^{2(0)} + e^{-(0)}$

$= \frac{1}{2}e^6 + e^{-3} - \frac{1}{2} - 1$

$= \frac{1}{2}e^6 + e^{-3} - 1\frac{1}{2}$

Question 3

a $\int_1^3 \left(e^{-x}\right) dx$

$u = -x, du = -dx$

$\int_1^3 -e^u du$

$= \left[-e^{-x}\right]_1^3$

$= \left(-e^{-(3)}\right) - \left(-e^{-(1)}\right)$

$= 0.32$

b $\int_0^2 \left(2e^{3y}\right)dy$

$u = 3y,\ du = 3dy$

$\int_0^2 2\times\frac{1}{3}e^u du$

$=\left[\frac{2}{3}e^{3y}\right]_0^2$

$=\left(\frac{2}{3}e^{3(2)}\right)-\left(\frac{2}{3}e^{3(0)}\right)$

$=268.29$

c $\int_5^6 \left(e^{x+5}+2x-3\right)dx$

$u = x+5,\ du = dx$

$\int_5^6 e^u du+\int_5^6 2x\,dx+\int_5^6 -3\,dx$

$=\left[e^{x+5}+x^2-3x\right]_5^6$

$=\left(e^{(6)+5}+(6)^5-3(6)\right)-\left(e^{(5)+5}+(5)^5-3(5)\right)$

$=37855.68$

d $\int_0^1 \left(e^{3t+4}-t\right)dt$

$u = 3t+4,\ du = 3dt$

$\int_0^1 \frac{1}{3}e^u du+\int_0^1 -t\,dt$

$=\left[\frac{1}{3}e^{3t+4}-\frac{t^2}{2}\right]_0^1$

$=\left(\frac{1}{3}e^{3(1)+4}-\frac{(1)^2}{2}\right)-\left(\frac{1}{3}e^{3(0)+4}-\frac{(0)^2}{2}\right)$

$=346.85$

e $\int_1^2 \left(e^{4x}+e^{2x}\right)dx$

$u = 4x, du = 4\,dx$

$v = 2x, dv = 2dx$

$\int_1^2 \frac{1}{4}e^u du + \int_1^2 \frac{1}{2}e^v dv$

$= \left[\frac{1}{4}e^{4x}+\frac{1}{2}e^{2x}\right]_1^2$

$= \left(\frac{1}{4}e^{4(2)}+\frac{1}{2}e^{2(2)}\right)-\left(\frac{1}{4}e^{4(1)}+\frac{1}{2}e^{2(1)}\right)$

$= 755.19$

Question 4

a $\int 5^x dx$

$5 = e^{\ln 5}$

$5^x = e^{x\ln x}$

$\int e^{x\ln x}dx$

$= \frac{1}{\ln 5}e^{x\ln 5}+C$

$= \frac{1}{\ln 5}5^x + C$

b $\int 7^{3x}dx$

$7 = e^{\ln 7}$

$7^{3x} = e^{3x\ln 7}$

$\int e^{3x\ln 7}dx$

$= \frac{1}{3\ln 7}e^{3x\ln 7} + C$

$= \frac{1}{3\ln 7}7^{3x} + C$

c $\int 3^{2x-1}dx$

$3 = e^{\ln 3}$

$3^{2x-1} = e^{(2x-1)\ln 3}$

$\int e^{(3x-1)\ln 3}dx$

$= \frac{1}{2\ln 3}e^{(2x-1)\ln 3} + C$

$= \frac{1}{2\ln 3}3^{2x-1} + C$

Question 5

a $\frac{dy}{dx}\left(x^2e^x\right)$

$= 2xe^x + x^2e^x$

$= xe^x(2+x)$

b $\int x(2+x)e^x dx$

$= x^2e^x + C$

Question 6

$$f'(x) = x^2e^{2x^3}$$

$$\int x^2e^{2x^3}dx$$

$$u = 2x^3, du = 6x^2\,dx$$

$$= \int \frac{1}{6}e^u du$$

$$= \frac{1}{6}e^{2x^3} + C$$

$$0 = \frac{1}{6}e^{2(0)^3} + C$$

$$0 = \frac{1}{6} + C$$

$$C = -\frac{1}{6}$$

$$f(x) = \frac{1}{6}e^{2x^3} - \frac{1}{6}$$

$$f(x) = \frac{1}{6}\left(e^{2x^3} - 1\right)$$

Question 7

$v = 2e^t - 1$

$\int \left(2e^t - 1\right) dt$

$= 2\int e^t dt - \int dt$

$x = 2e^t - t + C$

$10 = 2e^0 - 0 + C$

$10 = 2 + C$

$C = 8$

$x = 2e^t - t + 8$

$x\left(3\right) = 2e^3 - 3 + 8$

$x(3) = \left(2e^3 + 5\right)$ m

Exercise 6.07 Integration involving logarithmic functions

Question 1

a $\int \frac{2}{2x+5} dx$

$u = 2x+5, du = 2\,dx$

$\int \frac{du}{u}$

$= \ln|u| + C$

$= \ln|2x+5| + C$

b $\int \frac{4x}{2x^2+1} dx$

$u = 2x^2+1, du = 4x\,dx$

$\int \frac{du}{u}$

$= \ln|u| + C$

$= \ln|2x^2+1| + C$

c $\int \frac{5x^4}{x^5-2} dx$

$u = x^5-2, du = 5x^4\,dx$

$\int \frac{du}{u}$

$= \ln|u| + C$

$= \ln|x^5-2| + C$

d $\int \frac{1}{2x}\,dx$

$u = 2x, du = 2\,dx$

$\int \frac{1}{2}\frac{du}{u}$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|2x| + C$

e $\int \frac{2}{x}\,dx$

$u = x, du = dx$

$2\int \frac{du}{u}$

$= 2\ln|u| + C$

$= 2\ln|x| + C$

f $\int \frac{5}{3x}\,dx$

$u = x,\ du = dx$

$\frac{5}{3}\int \frac{du}{u}$

$= \frac{5}{3}\ln|u| + C$

$= \frac{5}{3}\ln|x| + C$

g $\int \frac{2x-3}{x^2-3x}dx$

$u = x^2 - 3x, du = (2x-3)dx$

$\int \frac{du}{u}$

$= \ln|u| + C$

$= \ln|x^2 - 3x| + C$

h $\int \frac{x}{x^2+2}dx$

$u = x^2 + 2, du = 2x\ dx$

$\int \frac{1}{2}\frac{du}{u}$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2 + 2| + C$

i $\int \frac{3x}{x^2+7}dx$

$u = x^2 + 7,\ du = 2x\ dx$

$3\int \frac{1}{2}\frac{du}{u}$

$= \frac{3}{2}\ln|u| + C$

$= \frac{3}{2}\ln|x^2 + 7| + C$

j $\int \frac{x+1}{x^2+2x-5}\,dx$

$u = x^2+2x-5,\ du = (2x+2)\,dx$

$\int \frac{1}{2}\frac{du}{u}$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln\left|x^2+2x-5\right| + C$

Question 2

a $\int \frac{4}{4x-1}\,dx$

$u = 4x-1,\ du = 4\,dx$

$\int \frac{du}{u}$

$= \ln|u| + C$

$= \ln|4x-1| + C$

b $\int \frac{dx}{x+3}$

$u = x+3,\ du = dx$

$\int \frac{du}{u}$

$= \ln|u| + C$

$= \ln|x+3| + C$

c $\int \frac{x^2}{2x^3-7}dx$

$u = 2x^3 - 7,\ du = 6x^2\ dx$

$\int \frac{1}{6}\frac{du}{u}$

$= \frac{1}{6}\ln|u| + C$

$= \frac{1}{6}\ln|2x^3 - 7| + C$

d $\int \frac{x^5}{2x^6+5}dx$

$u = 2x^6 + 5,\ du = 12x^5\ dx$

$\int \frac{1}{12}\frac{du}{u}$

$= \frac{1}{12}\ln|u| + C$

$= \frac{1}{12}\ln|2x^6 + 5| + C$

e $\int \frac{x+3}{x^2+6x+2}dx$

$u = x^2 + 6x + 2,\ du = (2x+6)\ dx$

$\int \frac{1}{2}\frac{du}{u}$

$= \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2 + 6x + 2| + C$

Question 3

a $\int_1^3 \frac{2}{2x+5}\,dx$

$u = 2x+5,\ du = 2\ dx$

$\int_1^3 \frac{du}{u}$

$= \left[\ln|u|\right]_1^3$

$= \left[\ln|2x+5|\right]_1^3$

$= \ln|2(3)+5| - \ln|2(1)+5|$

$= 0.5$

b $\int_2^5 \frac{dx}{x+1}$

$u = x+1,\ du = dx$

$\int_2^5 \frac{du}{u}$

$= \left[\ln|u|\right]_2^5$

$= \left[\ln|x+1|\right]_2^5$

$= \ln|(5)+1| - \ln|(2)+1|$

$= 0.7$

c $\int_1^7 \frac{x^2}{x^3+2}$

$u = x^3 = 2,\ du = 3x^2\,dx$

$\int_1^7 \frac{1}{3}\frac{du}{u}$

$= \left[\frac{1}{3}\ln|u|\right]_1^7$

$= \left[\frac{1}{3}\ln\left|x^3+2\right|\right]_1^7$

$= \frac{1}{3}\ln\left|(7)^3+2\right| - \frac{1}{3}\ln\left|(1)^3+2\right|$

$= 1.6$

d $\int_0^3 \frac{4x+1}{2x^2+x+1}dx$

$u = 2x^2+x+1,\ du = (4x+1)\ dx$

$\int_0^3 \frac{du}{u}$

$= \left[\ln|u|\right]_0^3$

$= \left[\ln\left|2x^2+x+1\right|\right]_0^3$

$= \ln\left|2(3)^2+(3)+1\right| - \ln\left|2(0)^2+(0)+1\right|$

$= 3.1$

e $\int_3^4 \frac{x-1}{x^2-2x}dx$

$u = x^2 - 2x,\ du = (2x-2)dx$

$\int_3^4 \frac{1}{2}\frac{du}{u}$

$= \left[\frac{1}{2}\ln|u|\right]_3^4$

$= \left[\frac{1}{2}\ln\left|x^2-2x\right|\right]_3^4$

$= \frac{1}{2}\ln\left|(4)^2 - 2(4)\right| - \frac{1}{2}\ln\left|(3)^2 - 2(3)\right|$

$= 0.5$

Question 4

a $\text{RHS} = \frac{1}{x+3} + \frac{2}{x-3}$

$= \frac{1(x-3)}{(x+3)(x-3)} + \frac{2(x+3)}{(x-3)(x+3)}$

$= \frac{x-3}{x^2-9} + \frac{2x+6}{x^2-9}$

$= \frac{3x+3}{x^2-9}$

$= \text{LHS}$

$\therefore \frac{3x+3}{x^2-9} = \frac{1}{x+3} + \frac{3}{x-3}$

b $\int \frac{3x+3}{x^2-9}dx$

$= \int \frac{1}{x+3} + \frac{2}{x-3}\ dx$

$= \ln|x+3| + 2\ln|x-3| + C$

Question 5

a $\text{RHS} = 1 - \dfrac{5}{x-1}$

$= \dfrac{x-1}{x-1} - \dfrac{5}{x-1}$

$= \dfrac{x-6}{x-1}$

$= \text{LHS}$

$\therefore \dfrac{x-6}{x-1} = 1 - \dfrac{5}{x-1}$

b $\displaystyle\int \frac{x-6}{x-1}\,dx$

$\displaystyle\int 1 - \frac{5}{x-1}\,dx$

$= x - 5\ln|x-1| + C$

Question 6

$$\int \frac{x^2}{3x^3-1}dx$$

$u = 3x^3 - 1,\ du = 9x^2\ dx$

$$= \int \frac{1}{9}\frac{du}{u}$$

$$= \frac{1}{9}\ln|u| + C$$

$$= \frac{1}{9}\ln\left|3x^3-1\right| + C$$

$$0 = \frac{1}{9}\left|\ln 3(1)^3 - 1\right| + C$$

$$0 = \frac{\ln 2}{9} + C$$

$$C = -\frac{\ln 2}{9}$$

$$= \frac{1}{9}\ln\left|3x^3-1\right| - \frac{\ln 2}{9}$$

$$= \frac{1}{9}\left(\ln\left|3x^3-1\right| - \ln 2\right)$$

$$f(x) = \frac{1}{9}\ln\left|\frac{3x^3-1}{2}\right|$$

Question 7

$$\int \frac{5t}{t^2+4}\,dt$$

$$u = t^2 + 4,\ du = 2t\ dt$$

$$5\int \frac{1}{2}\frac{du}{u}$$

$$= \frac{5}{2}\ln|u| + C$$

$$= \frac{5}{2}\ln\left|t^2 + 4\right| + C$$

$$4 = \frac{5}{2}\ln\left|(0)^2 + 4\right| + C$$

$$4 = \frac{5}{2}\ln|4| + C$$

$$C = 4 - \frac{5}{2}\ln|4|$$

$$x = \frac{5}{2}\ln\left|t^2 + 4\right| + 4 - \frac{5}{2}\ln|4|$$

$$x(5) = \frac{5}{2}\ln\left|5^2 + 4\right| + 4 - \frac{5}{2}\ln|4|$$

= 8.95 m

Question 8

$$\int \frac{x^2}{3x^3+1}dx$$

$$u = 3x^3+1,\ du = 9x^2\ dx$$

$$= \int \frac{1}{9}\frac{du}{u}$$

$$= \frac{1}{9}\ln|u| + C$$

$$= \frac{1}{9}\ln\left|3x^3+1\right| + C$$

$$3 = \frac{1}{9}\ln\left|3(0)^3+1\right| + C$$

$$3 = \frac{1}{9}\ln|1| + C$$

$$C = 3 - \frac{1}{9}\ln|1|$$

$$= \frac{1}{9}\ln\left|3(8)^3+1\right| + 3 - \frac{1}{9}\ln|1|$$

$= 4$

Exercise 6.08 Integration involving trigonometric functions

Question 1

a $\int \cos x = \sin x + C$

b $\int \sin x = -\cos x + C$

c $\int \sec^2 x = \tan x + C$

d $\int \frac{\sin x}{4}$

$= -\frac{1}{4} \times \frac{180}{\pi} \cos x° + C$

$= -\frac{45 \cos x°}{\pi} + C$

e $\int \sin 3x \, dx$

$u = 3x, \; du = 3 \; dx$

$= \int \frac{1}{3} \sin u \, du$

$= -\frac{1}{3} \cos 3x + C$

f $\int -\sin 7x \; dx$

$u = 7x, \; du = 7 \, dx$

$\int -\frac{1}{7} \sin u \; du$

$= \frac{1}{7} \cos 7x + C$

g $\int \sec^2 5x\ dx$

$u = 5x,\ du = 5\ dx$

$\int \frac{1}{5}\sec^2 u\, du$

$= \frac{1}{5}\tan 5x + C$

h $\int \cos(x+1)\, dx$

$u = (x+1),\ du = dx$

$\int \cos u\ du$

$= \sin\ (x+1) + C$

i $\int \sin(2x-3)\, dx$

$u = (2x-3),\ du = 2\ dx$

$\int \frac{1}{2}\sin u\ du$

$= -\frac{1}{2}\cos\ (2x-3) + C$

j $\int \cos(2x-1)\, dx$

$u = (2x-1),\ du = 2\ dx$

$\int \frac{1}{2}\cos u\ du$

$= \frac{1}{2}\sin(2x-1) + C$

k $\int \sin\left(\pi - x\right)dx$

$u = (\pi - x),\ du = -1\ dx$

$\int -\sin u\ du\ .$

$= \cos\left(\pi - x\right) + C$

$= -\cos x + C$

l $\int \cos\left(\pi + x\right)dx$

$u = \left(\pi + x\right),\ du = dx$

$= \int \cos u\, du$

$= \sin\left(\pi + x\right) + C$

$-\sin x + C$

m $\int 2\sec^2 7x\ dx$

$u = 7x,\ du = 7\ dx$

$= 2\int \frac{1}{7}\sec^2 u\, du$

$= \frac{2}{7}\tan 7x + C$

n $\int 4\sin\left(\frac{x}{2}\right)dx$

$u = \left(\frac{x}{2}\right),\ du = \left(\frac{1}{2}\right)dx$

$= 4\int 2\sin u\ du$

$= -8\cos\left(\frac{x}{2}\right) + C$

o $\int 3\sec^2\left(\frac{x}{3}\right)dx$

$u=\left(\frac{x}{3}\right),\ du=\left(\frac{1}{3}\right)dx$

$=3\int 3\sec^2 u\,du$

$=9\tan\left(\frac{x}{3}\right)+C$

Question 2

a $\int_0^{\frac{\pi}{2}} \cos x\ dx$

$=\left[\sin x\right]_0^{\frac{\pi}{2}}$

$=\sin\left(\frac{\pi}{2}\right)-\sin(0)$

$=1$

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x\ dx$

$=\left[\tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$=\tan\left(\frac{\pi}{3}\right)-\tan\left(\frac{\pi}{6}\right)$

$=\sqrt{3}-\frac{1}{\sqrt{3}}$

$=\frac{2\sqrt{3}}{3}$

c $\int_{\frac{\pi}{2}}^{\pi} 2\sin u \; du$

$$=\left[-2\cos\left(\frac{x}{2}\right)\right]_{\frac{\pi}{2}}^{\pi}$$

$$=-2\cos\left(\frac{\pi}{2}\right)+2\cos\left(\frac{\pi}{4}\right)$$

$$=0+2\times\frac{\sqrt{2}}{2}$$

$$=\sqrt{2}$$

d $\int_{0}^{\frac{\pi}{2}} \cos 3x \; dx$

$u=3x,\; du=3\; dx$

$$\int_{0}^{\frac{\pi}{2}} \frac{1}{3}\cos u \; du$$

$$=\left[\frac{1}{3}\sin 3x\right]_{0}^{\frac{\pi}{2}}$$

$$=\frac{1}{3}\sin\left(3\times\frac{\pi}{2}\right)-\frac{1}{3}\sin\left(3\times 0\right)$$

$$=-\frac{1}{3}$$

e $\int_0^{\frac{1}{2}} \sin(\pi x)\,dx$

$u = \pi x,\ du = \pi\ dx$

$\int_0^{\frac{1}{2}} \frac{1}{\pi} \sin u\ du$

$= \left[-\frac{1}{\pi} \cos \pi x \right]_0^{\frac{1}{2}}$

$= -\frac{1}{\pi} \cos\left(\pi \times \frac{1}{2} \right) + \frac{1}{\pi} \cos(\pi \times 0)$

$= \frac{1}{\pi}$

f $\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$

$u = 2x,\ du = 2\ dx$

$\int_0^{\frac{\pi}{8}} \frac{1}{2} \sec^2 u\ du$

$= \left[\frac{1}{2} \tan 2x \right]_0^{\frac{\pi}{8}}$

$= \frac{1}{2} \tan\left(2 \times \frac{\pi}{8} \right) - \frac{1}{2} \tan(2 \times 0)$

$= \frac{1}{2}$

g $\int_0^{\frac{\pi}{12}} 3\cos(2x)\ dx$

$u = 2x,\ du = 2\ dx$

$3\int_0^{\frac{\pi}{12}} \frac{1}{2}\cos u\ du$

$= \left[\frac{3}{2}\sin 2x\right]_0^{\frac{\pi}{12}}$

$= \frac{3}{2}\sin\left(2\times\frac{\pi}{12}\right) - \frac{3}{2}\sin(2\times 0)$

$= \frac{3}{4}$

h $\int_0^{\frac{\pi}{10}} -\sin(5x) dx$

$u = 5x,\ du = 5\ dx$

$-\int_0^{\frac{\pi}{10}} \frac{1}{5}\sin u\ du$

$= \frac{1}{5}\cos\left(5\times\frac{\pi}{10}\right) - \frac{1}{5}\cos(2\times 0)$

$= -\frac{1}{5}$

Question 3

a $$\int\left(\cos x\cos\frac{\pi}{3}-\sin x\sin\frac{\pi}{3}\right)dx=\int\left(\frac{1}{2}\cos x-\left(\frac{\sqrt{3}}{2}\right)\sin x\right)dx$$

$$=\frac{1}{2}\sin x+\left(\frac{\sqrt{3}}{2}\right)\cos x+C$$

$$=\frac{\sin x+\sqrt{3}\cos x}{2}+C$$

or

$$\int\left(\cos x\cos\frac{\pi}{3}-\sin x\sin\frac{\pi}{3}\right)dx=\int\cos\left(x+\frac{\pi}{3}\right)dx$$ [EXT1]

$$=\sin\left(x+\frac{\pi}{3}\right)+C$$

$$=\sin x\cos\frac{\pi}{3}+\cos x\sin\frac{\pi}{3}+\mathrm{C}$$

$$=\sin x\left(\frac{1}{2}\right)+\cos x\left(\frac{\sqrt{3}}{2}\right)+C$$

$$=\frac{\sin x+\sqrt{3}\cos x}{2}+C$$

b $$\int(\sin\pi\cos x-\cos\pi\sin x)dx=\int(0\cos x-(-1)\sin x)dx$$

$$=\int\sin x\ dx$$

$$=-\cos x+C$$

or

$$\int(\sin\pi\cos x-\cos\pi\sin x)dx=\int\sin(\pi-x)dx$$ [EXT1]

$$=\int\sin x\ dx$$

$$=-\cos x+C$$

Question 4

$\int \cos\ 4x\ dx$

$u = 4x,\ du = 4\ dx$

$= \int \frac{1}{4} \cos 4x\ dx$

$= \frac{1}{4} \sin 4x + C$

$\frac{\pi}{4} = \frac{1}{4} \sin 4(\pi) + C$

$C = \frac{\pi}{4}$

$y = \frac{1}{4} \sin 4x + \frac{\pi}{4}$

Question 5

a $\int 12\pi\cos\left(\frac{2\pi t}{3}\right)dt$

$u=\frac{2\pi t}{3},\ du=\frac{2\pi}{3}\ dx$

$=12\pi\int\frac{3}{2\pi}\cos(u)du$

$=18\sin\left(\frac{2\pi t}{3}\right)+C$

$2=18\sin\left(\frac{2\pi(0)}{3}\right)+C$

$2=C$

$C=2$

$x=18\sin\left(\frac{2\pi t}{3}\right)+2\ \text{cm}$

b **i** $x=18\sin\left(\frac{2\pi t}{3}\right)+2$

$x(1)=18\sin\left(\frac{2\pi(1)}{3}\right)+2$

$x(1)=18\times\frac{\sqrt{3}}{2}+2$

$x(1)=9\sqrt{3}+2\ \text{cm}$

ii $x=18\sin\left(\frac{2\pi t}{3}\right)+2$

$x(1)=x=18\sin\left(\frac{2\pi(5)}{3}\right)+2$

$x(1)=18\times-\frac{\sqrt{3}}{2}+2$

$x(1)=-9\sqrt{3}+2\ \text{cm}$

Question 6

a $R = 4\pi\sin\left(\frac{\pi t}{6}\right)$

$\int 4\pi\sin\left(\frac{\pi t}{6}\right)dt$

$u = \frac{\pi t}{6},\ du = \frac{\pi}{6}\ dx$

$= 4\pi\int\frac{6}{\pi}\sin(u)\,du$

$= -24\cos\left(\frac{\pi t}{6}\right) + C$

$2 = -24\cos\left(\frac{\pi(0)}{6}\right) + C$

$2 = -24 + C$

$C = 26$

$= -24\cos\left(\frac{\pi t}{6}\right) + 26$

b $= -24\cos\left(\frac{\pi t}{6}\right) + 26$

$= -24\cos\left(\frac{\pi(2)}{6}\right) + 26$

$= -24\times\frac{1}{2} + 26 = 14\ \text{m}$

c $= -24\cos\left(\frac{\pi t}{6}\right) + 26$

amplitude is 24

$\text{minimum} = -24 + 26 = 2$ m

$\text{maximum} = 24 + 26 = 50$ m

$\text{centre} = 0 + 26 = 26$ m

d Period of cosine curve

$2\pi \div \frac{\pi}{6} = 12$ hours

Exercise 6.09 Areas enclosed by the *x*-axis

Question 1

$y = 1 - x^2$

$0 = 1 - x^2$

$x^2 = 1$

$x = \pm 1$

$\int_{-1}^{1} \left(1 - x^2\right) dx$

$= \left[x - \frac{x^3}{3} \right]_{-1}^{1}$

$= \left((1) - \frac{(1)^3}{3} \right) - \left((-1) - \frac{(-1)^3}{3} \right)$

$= \frac{2}{3} + \frac{2}{3}$

$= 1\frac{1}{3} \text{ units}^2$

Question 2

$y = x^2 - 9$

$0 = x^2 - 9$

$x^2 = 9$

$x = \pm 3$

$\int_{-3}^{3} \left(x^2 - 9 \right) dx$

$= \left[\frac{x^3}{3} - 9x \right]_{-3}^{3}$

$= \left(\frac{(3)^3}{3} - 9(3) \right) - \left(\frac{(-3)^3}{3} - 9(-3) \right)$

$= |-36|$

$= 36 \text{ units}^2$

Question 3

$y = x^2 + 5x + 4$

$0 = (x+4)(x+1)$

$x = -4, -1$

$\int_{-4}^{-1} \left(x^2 + 5x + 4 \right) dx$

$= \left[\frac{x^3}{3} + \frac{5x^2}{2} + 4x \right]_{-4}^{-1}$

$= \left(\frac{(-1)^3}{3} + \frac{5(-1)^2}{2} + 4(-1) \right) - \left(\frac{(-4)^3}{3} + \frac{5(-4)^2}{2} + 4(-4) \right)$

$= |-4.5|$

$= 4.5 \text{ units}^2$

Question 4

$y = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

$x = -1, 3$

$\int_{-1}^{3}\left(x^2 - 2x - 3\right)dx$

$= \left[\frac{x^3}{3} - x^2 - 3x\right]_{1}^{3}$

$= \left(\frac{(3)^3}{3} - 3(2)^2 - 3(3)\right) - \left(\frac{(-1)^3}{3} - (-1)^2 - 3(-1)\right)$

$= \left|-10\frac{2}{3}\right|$

$= 10\frac{2}{3}$ units^2

Question 5

$y = -x^2 + 9x - 20$

$0 = x^2 - 9x + 20$

$0 = (x-4)(x-5)$

$x = 4,\ 5$

$\int_4^5 \left(-x^2 + 9x - 20\right) dx$

$= \left[-\frac{x^3}{3} + \frac{9x^2}{2} - 20x\right]_4^5$

$= \left(-\frac{(5)^3}{3} + \frac{9(5)^2}{2} - 20(5)\right) - \left(-\frac{(4)^3}{3} + \frac{9(4)^2}{2} - 20(4)\right)$

$= \frac{1}{6}$ units^2

Question 6

$y=-2x^2-5x+3$

$0=2x^2+5x-3$

$0=(2x-1)(x+3)$

$x=-3,\ \frac{1}{2}$

$\int_{-3}^{\frac{1}{2}}\left(-2x^2-5x+3\right)dx$

$=\left[-\frac{2x^3}{3}-\frac{5x^2}{2}+3x\right]_{3}^{\frac{1}{2}}$

$=\left(-\frac{2\left(\frac{1}{2}\right)^3}{3}-\frac{5\left(\frac{1}{2}\right)^2}{2}+3\left(\frac{1}{2}\right)\right)-\left(-\frac{2(-3)^3}{3}-\frac{5(-3)^2}{2}+3(-3)\right)$

$=14.3\ \text{units}^2$

Question 7

$y=x^3$

$\int_0^2 x^3dx$

$=\left[\frac{x^4}{4}\right]_0^2$

$=\left(\frac{(2)^4}{4}\right)-\left(\frac{(0)^4}{4}\right)$

$=4\ \text{units}^2$

Question 8

$y = x^4$

$\int_{-1}^{1} x^4 dx$

$= \left[\frac{x^5}{5} \right]_{-1}^{1}$

$= \left(\frac{(1)^5}{5} \right) - \left(\frac{(-1)^5}{5} \right)$

$= 0.4 \text{ units}^2$

Question 9

$y = x^3$

$\int_{-2}^{2} x^3 dx$

$= \left[\frac{x^4}{4} \right]_{-2}^{0} + \left[\frac{x^4}{4} \right]_{-2}^{0}$

$= \left| \frac{(0)^4}{4} - \frac{(-2)^4}{4} \right| + \left| \frac{(2)^4}{4} - \frac{(0)^4}{4} \right|$

$= 8 \text{ units}^2$

Question 10

$y = x^3$

$\int_{-3}^{2} x^3 dx$

$= \left[\frac{x^4}{4}\right]_{-3}^{0} + \left[\frac{x^4}{4}\right]_{0}^{2}$

$= \left|\frac{(0)^4}{4} - \frac{(-3)^4}{4}\right| + \left|\frac{(2)^4}{4} - \frac{(0)^4}{4}\right|$

$= 24.25 \text{ units}^2$

Question 11

$y = 2e^{2x}$

$\int_{1}^{2} 2e^{2x} dx$

$= \left[e^{2x}\right]_{1}^{2}$

$= e^{2(2)} - e^{2(1)}$

$= e^2\left(e^2 - 1\right) \text{ units}^2$

Question 12

$y = e^{4x-3}$

$\int_0^1 e^{4x-3} dx$

$= \left[\frac{1}{4} e^{4x-3} \right]_0^1$

$= \frac{1}{4} e^{4(1)-3} - \frac{1}{4} e^{4(0)-3}$

$= \frac{1}{4}\left(e - e^{-3}\right) \text{ units}^2$

Question 13

$y = x + e^{-x}$

$\int_0^2 x + e^{-x}\, dx$

$= \left[\frac{x^2}{2} - e^{-x} \right]_0^2$

$= \left(\frac{(2)^2}{2} - e^{-(2)} \right) - \left(\frac{(0)^2}{2} - e^{-(0)} \right)$

$= 2.86 \text{ units}^2$

Question 14

$y = e^{5x}$

$\int_0^1 e^{5x} dx$

$= \left[\frac{e^{5x}}{5} \right]_0^1$

$= \left(\frac{e^{5(1)}}{5} \right) - \left(\frac{e^{5(10)}}{5} \right)$

$= 29.5 \text{ units}^2$

Question 15

$y = \sin x$

$\int_0^{2\pi} \sin x \ dx$

$= [-\cos x]_0^{\pi} + [-\cos x]_{\pi}^{2\pi}$

$= |-\cos(\pi) + \cos(0)| + |-\cos(2\pi) + \cos(\pi)|$

$= 4 \text{ units}^2$

Question 16

$y = \cos 3x$

$\int_0^{\frac{\pi}{12}} \cos 3x \; dx$

$= \left| \frac{1}{3}\sin 3\left(\frac{\pi}{12}\right) - \frac{1}{3}\sin 3(0) \right|$

$= \frac{1}{3} \times \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{6} \text{ units}^2$

Question 17

$y = \sec^2 \frac{x}{4}$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sec^2 \frac{x}{4} dx$

$= \left[4\tan \frac{x}{4} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \left| 4\tan \frac{\frac{\pi}{2}}{4} - 4\tan \frac{\frac{\pi}{4}}{4} \right|$

$= 0.86 \text{ units}^2$

Question 18

$y = 3x^2$

$\int_{-1}^{1} 3x^2 dx$

$= \left[x^3\right]_{-1}^{1}$

$= (1)^3 - (-1)^3$

$= 2 \text{ units}^2$

Question 19

$y = x^2 + 1$

$\int_{-2}^{2} \left(x^2 + 1\right) dx$

$= \left[\frac{x^3}{3} + x\right]_{-2}^{2}$

$= \left(\frac{(2)^3}{3} + (2)\right) - \left(\frac{(-2)^3}{3} + (-2)\right)$

$= 9\frac{1}{3} \text{units}^2$

Question 20

$y = x^2$

$\int_{-3}^{2} \left(x^2\right) dx$

$= \left[\frac{x^3}{3}\right]_{-3}^{2}$

$= 11\frac{2}{3} \text{ units}^2$

Question 21

$0 = x^2 + x$

$0 = x(x+1)$

$x = -1,\ 0$

$\int_{-1}^{0}\left(x^2 + x\right)dx$

$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{0}$

$= \left(\frac{(0)^3}{3} + \frac{(0)^2}{2}\right) - \left(\frac{(-1)^3}{3} + \frac{(-1)^2}{2}\right)$

$= \left|-\frac{1}{6}\right|$

$= \frac{1}{6}$ units^2

Question 22

$y = \frac{1}{x^2}$

$\int_{1}^{3}\left(\frac{1}{x^2}\right)dx$

$= \left[-\frac{1}{x}\right]_{1}^{3}$

$= \left(-\frac{1}{3}\right) - \left(-\frac{1}{1}\right)$

$= \frac{2}{3}$ units^2

Question 23

$$y=\frac{2}{(x-3)^2}$$

$$\int_0^1\left(\frac{2}{(x-3)^2}\right)dx$$

$$=\left[-\frac{2}{x-3}\right]_0^1$$

$$=\left(-\frac{2}{(1)-3}\right)-\left(\frac{2}{(0)-3}\right)$$

$$=\frac{1}{3}\ \text{units}^2$$

Question 24

$$y=\frac{1}{x}$$

$$\int_2^3\left(\frac{1}{x}\right)dx$$

$$=[\ln\ x]_2^3$$

$$=(\ln(3))-(\ln(2))$$

$$=\ln 1.5\ \text{units}^2$$

Question 25

$$y=\frac{1}{x-1}$$

$$\int_4^7\left(\frac{1}{x-1}\right)dx$$

$$=\left[\ln(x-1)\right]_4^7$$

$$=(\ln(7-1))-(\ln(4-1))$$

$$=\ln 6-\ln 3$$

$$=\ln 2\ \text{ units}^2.$$

Question 26

$$y=\frac{x}{x^2+1}$$

$$\int_2^4\left(\frac{x}{x^2+1}\right)dx$$

$$u=x^2+1,\ du=2x\ dx$$

$$\int_2^4\left(\frac{1}{2}\frac{du}{u}\right)$$

$$=\left[\frac{1}{2}\ln u\right]_2^4$$

$$=\left[\frac{1}{2}\ln\left(x^2+1\right)\right]_2^4$$

$$=\frac{1}{2}\ln\left((4)^2+1\right)-\frac{1}{2}\ln\left((2)^2+1\right)$$

$$=0.61\ \text{units}^2$$

Question 27

$y = \sqrt{x}$

$\int_0^4 x^{\frac{1}{2}}\,dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4$

$= \frac{2}{3}(4)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}$

$= 5\frac{1}{3}\ \text{units}^2$

Question 28

$y = \sqrt{x+2}$

$0 = \sqrt{x+2}$

$0 = x+2$

$x = -2$

$\int_{-2}^{7} (x+2)^{\frac{1}{2}}\,dx$

$= \left[\frac{2}{3}(x+2)^{\frac{3}{2}}\right]_{-2}^{7}$

$= \frac{2}{3}((7)+2)^{\frac{3}{2}} - \frac{2}{3}((-2)+2)^{\frac{3}{2}}$

$= 18\ \text{units}^2$

Question 29

$y = \ln x$

$0 = \ln x$

$e^0 = x$

$x = 1$

$\int_1^5 \ln x \, dx$

$\approx \frac{1}{2}\left[\ln(1) + 2\ln(2) + 2\ln(3) + 2\ln(4) + \ln(5)\right]$

$= 3.98 \text{ units}^2$

Question 30

$y = x^3$

$\int_{-a}^{a} x^3 \, dx$

$= \left[\frac{x^4}{4}\right]_{-a}^{0} + \left[\frac{x^4}{4}\right]_{0}^{a}$

$= \left|\frac{(0)^4}{4} - \frac{(a)^4}{4}\right| + \left|\frac{(a)^4}{4} - \frac{(0)^4}{4}\right|$

$= \frac{a^4}{4} + \frac{a^4}{4}$

$= \frac{a^4}{2} \text{ units}^2$

Exercise 6.10 Areas enclosed by the *y*-axis

Question 1

$x = y^2$

$\int_0^4 y^2 dy$

$= \left[\frac{y^3}{3}\right]_0^4$

$= \left|\frac{(4)^3}{3} - \frac{(0)^3}{3}\right|$

$= 21\frac{1}{3} \text{ units}^2$

Question 2

$x = y^3$

$\int_1^3 y^3 dy$

$= \left[\frac{y^4}{4}\right]_1^3$

$= \left|\frac{(3)^4}{4} - \frac{(1)^4}{4}\right|$

$= 20 \text{ units}^2$

Question 3

$y = x^2$

$x = \sqrt{y}$

$\int_1^4 \sqrt{y}\ dy$

$= \int_1^4 y^{\frac{1}{2}}\ dy$

$= \left[\frac{2}{3}y^{\frac{3}{2}}\right]_1^4$

$= \left|\frac{2}{3}(4)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}\right|$

$= 4\frac{2}{3}\text{units}^2$

Question 4

$y = x - 1$

$x = y + 1$

$\int_0^1 (y+1)\ dy$

$= \left[\frac{y^2}{2} + y\right]_0^1$

$= \left|\left(\frac{(1)^2}{2} + (1)\right) - \left(\frac{(0)^2}{2} + (0)\right)\right|$

$= 1\frac{1}{2}\ \text{units}^2$

Question 5

$y = 2x + 1$

$2x = y - 1$

$x = \frac{y-1}{2}$

$\int_3^4 \left(\frac{y-1}{2}\right) dy$

$= \left[\frac{y^2}{4} - \frac{1}{2}y\right]_3^4$

$= \left|\left(\frac{(4)^2}{4} - \frac{1}{2}(4)\right) - \left(\frac{(3)^2}{4} - \frac{1}{2}(3)\right)\right|$

$= 1\frac{1}{4}\,\text{units}^2$

Question 6

$y = \sqrt{x}$

$x = y^2$

$\int_1^2 \left(y^2\right)\, dy$

$= \left[\frac{y^3}{3}\right]_1^2$

$= \left|\left(\frac{(2)^3}{3}\right) - \left(\frac{(1)^3}{3}\right)\right|$

$= 2\frac{1}{3}\ \text{units}^2$

Question 7

$x = y^2 - 2y - 3$

$0 = y^2 - 2y - 3$

$0 = (y+1)(y-3)$

$y = -1,\ 3$

$\int_{-1}^{3} \left(y^2 - 2y - 3\right)\, dy$

$= \left[\dfrac{y^3}{3} - y^2 - 3y\right]_{-1}^{3}$

$= \left|\left(\dfrac{(3)^2}{3} - (3)^2 - 3(3)\right) - \left(\dfrac{(-1)^3}{3} - (-1)^2 - 3(-1)\right)\right|$

$= 10\dfrac{2}{3}\ \text{units}^2$

Question 8

$x=-y^2-5y-6$

$0=-y^2-5y-6$

$0=y^2+5y+6$

$y=(y+2)(y+3)$

$y=-2,-3$

$\int_{-3}^{-2}\left(-y^2-5y-6\right)dy$

$=\left[-\frac{y^3}{3}-\frac{5y^2}{2}-6y\right]_{-3}^{-2}$

$=\left|\left(\frac{(-2)^3}{3}-\frac{5(-2)^2}{2}-6(-2)\right)-\left(-\frac{(-3)^3}{3}-\frac{5(-3)^2}{2}-6(-3)\right)\right|$

$=\frac{1}{6}\text{ units}^2$

Question 9

$y = \sqrt{3x - 5}$

$3x - 5 = y^2$

$3x = \frac{y^2 + 5}{3}$

$\int_2^3 \left(\frac{y^2 + 5}{3} \right) dy$

$= \left[\frac{y^3}{9} + \frac{5y}{3} \right]_2^3$

$= \left| \left(\frac{(3)^3}{9} + \frac{5(3)}{3} \right) - \left(\frac{(2)^3}{9} + \frac{5(2)}{3} \right) \right|$

$= 3\frac{7}{9}\ \text{units}^2$

Question 10

$$y = \frac{1}{x^2}$$

$$x^2 = \frac{1}{y}$$

$$x = \frac{1}{\sqrt{y}}$$

$$x = y^{-\frac{1}{2}}$$

$$\int_1^4 \left(y^{-\frac{1}{2}} \right) dy$$

$$= \left[2y^{\frac{1}{2}} \right]_1^4$$

$$= \left| 2y^{\frac{1}{2}} \right|_1^4$$

$$= \left| \left(2(4)^{\frac{1}{2}} \right) - \left(2(1)^{\frac{1}{2}} \right) \right|$$

$$= 2 \text{ units}^2$$

Question 11

$y = x^3$

$x = \sqrt[3]{y}$

$x = y^{\frac{1}{3}}$

$\int_1^8 \left(y^{\frac{1}{3}} \right) dy$

$= \left[\frac{3}{4} y^{\frac{4}{3}} \right]_1^8$

$= \left| \left(\frac{3}{4}(8)^{\frac{4}{3}} \right) - \left(\frac{3}{4}(1)^{\frac{4}{3}} \right) \right|$

$= 11\frac{1}{4} \text{ units}^2$

Question 12

$y = x^3 - 2$

$x^3 = y + 2$

$x = \sqrt[3]{y+2}$

$x = (y+2)^{\frac{1}{3}}$

$\int_{-1}^{25} \left[(y+2)^{\frac{1}{3}} \right] dy$

$= \left[\frac{3}{4}(y+2)^{\frac{4}{3}} \right]_{-1}^{25}$

$= \left| \left(\frac{3}{4}(25+2)^{\frac{4}{3}} \right) - \left(\frac{3}{4}(-1+2)^{\frac{4}{3}} \right) \right|$

$= 60 \text{ units}^2$

Question 13

$y = 1 - x$

$x = 1 - y$

$\int_1^4 (1 - y)\ dy$

$= \left[y - \frac{y^2}{2} \right]_1^4$

$= \left| \left((4) - \frac{(4)^2}{2} \right) - \left((1) - \frac{(1)^2}{2} \right) \right|$

$= 4.5\ \text{units}^2$

Question 14

$x = y(y - 2)$

$x = 0,\ 2$

$\int_0^2 \left(y^2 - 2y \right)\ dy$

$= \left[\frac{y^3}{3} - y^2 \right]_0^2$

$= \left| \left(\frac{(2)^3}{3} - (2)^2 \right) - \left(\frac{(0)^3}{3} - (0)^2 \right) \right|$

$= 1\frac{1}{3}\ \text{units}^2$

Question 15

$y = x^4 + 1$

$x^4 = y - 1$

$x = \sqrt[4]{y-1}$

$x = (y-1)^{\frac{1}{4}}$

$\int_1^3 (y-1)^{\frac{1}{4}}\ dy$

$= \left[\frac{4}{5}(y-1)^{\frac{5}{4}}\right]_1^3$

$= \left|\left(\frac{4}{5}((3)-1)^{\frac{5}{4}}\right) - \left(\frac{4}{5}((1)-1)^{\frac{5}{4}}\right)\right|$

$= 1.9\ \text{units}^2$

Question 16

$y = \ln x$

$e^y = e^{\ln x}$

$x = e^y$

$\int_2^4 e^y\, dy$

$= \left[e^y\right]_2^4$

$= \left(e^4\right) - \left(e^2\right)$

$= 47.2\ \text{units}^2$

Exercise 6.11 Sums and differences of areas

Question 1

$1 = x^2$

$x = \pm 1$

$A_1 = 1 \times (1 - -1) = 2$

$A_2 = \int_{-1}^{1} x^2 dx$

$= \left[\frac{x^3}{3} \right]_{-1}^{1}$

$= \left| \frac{(1)^3}{3} - \frac{(-1)^3}{3} \right|$

$A_2 = \frac{2}{3}$

$A_1 - A_2 = 2 - \frac{2}{3} = 1\frac{1}{3}$ units^2

Question 2

$2 = x^2 + 1$

$x^2 = 1$

$x = \pm 1$

$A_1 = 2 \times (1 - -1) = 4$

$A_2 = \int_{-1}^{1} x^2 + 1 \, dx$

$= \left[\frac{x^3}{3} + x \right]_1^1$

$= \left| \left(\frac{(1)^3}{3} + (1) \right) - \left(\frac{(-1)^3}{3} + (-1) \right) \right|$

$A_2 = 2\frac{2}{3}$

$A_1 - A_2 \;= 4 - 2\frac{2}{3} \;= 1\frac{1}{3}$ units^2

Question 3

$x = x^2$

$x = 0, 1$

$\int_0^1 x \, dx - \int_0^1 x^2 \, dx$

$= \left[\frac{x^2}{2} \right]_0^1 - \left[\frac{x^3}{3} \right]_0^1$

$= \left| \left(\frac{(1)^2}{2} - \frac{(0)^2}{2} \right) - \left(\frac{(1)^3}{3} - \frac{(0)^3}{3} \right) \right|$

$= \frac{1}{6}$ units^2

Question 4

$5 = 9 - x^2$

$x^2 = 4$

$x = \pm 2$

$\int_{-2}^{2} 9 - x^2 \ dx - \int_{-2}^{2} 5\,dx$

$= \left[9x - \frac{x^3}{3}\right]_{-2}^{2} - \left[5x\right]_{-2}^{2}$

$= \left|\left(9(2) - \frac{(2)^3}{3}\right) - \left(9(-2) - \frac{(-2)^3}{3}\right)\right| - \left|\left(5(2) - 5(-2)\right)\right|$

$= 10\frac{2}{3} \text{ units}^2$

Question 5

$x + 6 = x^2$

$0 = x^2 - x - 6$

$0 = (x-3)(x+2)$

$x = -2, 3$

$\int_{-2}^{3} (x+6)\,dx - \int_{-2}^{3} x^2 dx$

$= \left[\frac{x^2}{2} + 6x\right]_{-2}^{3} - \left[\frac{x^3}{3}\right]_{-2}^{3}$

$= \left|\left(\frac{(3)^2}{2} + 6(3)\right) - \left(\frac{(-2)^2}{2} + 6(-2)\right)\right| - \left|\left(\frac{(3)^3}{3}\right) - \left(\frac{(-2)^3}{3}\right)\right|$

$= 20\frac{5}{6} \text{ units}^2$

Question 6

$4x = x^3$

$4 = x^2$

$x = \pm 2$

$\int_{-2}^{2}\left(x^3 - 4x\right)dx$

$=\left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} + \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$

$=\left|\left(\frac{(0)^4}{4} - 2(0)^2\right) - \left(\frac{(0)^4}{4} - 2(0)^2\right)\right|$

$= 8 \text{ units}^2$

Question 7

$(x+1)^2 = 0$

$x + 1 = 0$

$x = -1$

$(x-1)^2 = 0$

$x - 1 = 0$

$x = 1$

$\int_{-1}^{0}(x+1)^2\,dx + \int_{0}^{1}(x-1)^2\,dx$

$=\left[\frac{(x+1)^3}{3}\right]_{-1}^{0} + \left[\frac{(x-1)^3}{3}\right]_{0}^{1}$

$=\left|\left(\frac{((0)+1)^3}{3}\right) - \left(\frac{((-1)+1)^3}{3}\right)\right| + \left|\left(\frac{((1)-1)^3}{3}\right) - \left(\frac{((0)+1)^3}{3}\right)\right|$

$= \frac{2}{3} \text{ units}^2$

Question 8

$x^2 = -6x + 16$

$0 = x^2 + 6x - 16$

$0 = (x+8)(x-2)$

$x = -8,\ 2$

$\int_{-8}^{2} (-6x+16) - (x^2)\, dx$

$= \left[-3x^2 + 16x\right]_{-8}^{2} - \left[\frac{x^3}{3}\right]_{-8}^{2}$

$= \left|\left(-3(2)^2 + 16(2)\right) - \left(-3(-8)^2 + 16(-8)\right)\right|$

$- \left|\left(\frac{(2)^3}{3}\right) - \left(\frac{(-8)^3}{3}\right)\right|$

$= 166\frac{2}{3}\ \text{units}^2$

Question 9

$x^3 = -3x + 4$

$x = 1$

$x^3 = 0$

$x = 0$

$-3x + 4 = 0$

$x = \dfrac{4}{3}$

$\int_0^1 x^3 dx + \int_1^{\frac{4}{3}} (-3x + 4)\, dx$

$= \left[\dfrac{x^4}{4}\right]_0^1 + \left[\dfrac{-3x^2}{2} + 4x\right]_1^{\frac{4}{3}}$

$= \left|\left(\dfrac{(1)^4}{4}\right) - \left(\dfrac{(0)^4}{4}\right)\right| + \left|\left(\dfrac{-3\left(\frac{4}{3}\right)^2}{2} + 4\left(\dfrac{4}{3}\right)\right) - \left(\dfrac{-3(1)^2}{2} + 4(1)\right)\right|$

$= \dfrac{5}{12}$ units^2

Question 10

$$(x-2)^2=(x-4)^2$$

$$x=3$$

$$(x-2)^2=0$$

$$x=2$$

$$(x-4)^2=0$$

$$x=4$$

$$\int_2^3 (x-2)^2\,dx+\int_3^4 (x-4)^2\,dx$$

$$=\left[\frac{(x-2)^3}{3}\right]_2^3+\left[\frac{(x-4)^3}{3}\right]_3^4$$

$$=\left|\left(\frac{((3)-2)^3}{3}\right)-\left(\frac{((2)-2)^3}{3}\right)\right|+\left|(\frac{((4)-4)^3}{3}-\left(\frac{((3)-4)^3}{3}\right)\right|$$

$$=\frac{2}{3}\ \text{units}^2$$

Question 11

$$x^2=x^3$$

$$x=0,1$$

$$\int_0^1 \left(x^2-x^3\right)dx$$

$$=\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1$$

$$=\left|\left(\frac{(1)^3}{3}-\frac{(1)^4}{4}\right)-\left(\frac{(0)^3}{3}-\frac{(0)^4}{4}\right)\right|$$

$$=\frac{1}{12}\ \text{units}^2$$

Question 12

$y^2 = x$

$y = \sqrt{x}$

$\sqrt{x} = x^2$

$x = 0, 1$

$\int_0^1 \left(x^{\frac{1}{2}} - x^2 \right) dx$

$= \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1$

$= \left| \left[\frac{2}{3}(1)^{\frac{3}{2}} - \frac{(1)^3}{3} \right] - \left[\frac{2}{3}(0)^{\frac{3}{2}} - \frac{(0)^3}{3} \right] \right|$

$= \frac{1}{3}$ units^2

Question 13

$2x + 1 = x^2 + 2x - 8$

$0 = x^2 - 9$

$x^2 = 9$

$x = \pm 3$

$\int_{-3}^{3} \left(9 - x^2 \right) dx$

$= \left[9x - \frac{x^3}{3} \right]_{-3}^{3}$

$= \left| \left(9(3) - \frac{(3)^3}{9} \right) - \left(9(-3) - \frac{(-3)^3}{3} \right) \right|$

$= 36$ units^2

Question 14

$1-x^2=x^2-1$

$2x^2=2$

$x^2=1$

$x=\pm 1$

$\int_{-1}^{1}\left[\left(1-x^2\right)-\left(x^2+1\right)\right]dx=\int_{-1}^{1}\left(2-2x^2\right)\,dx$

$=\left[2x-\frac{2x^3}{3}\right]_{-1}^{1}$

$=\left|\left(2(1)-\frac{2(1)^3}{3}\right)-\left(2(-1)-\frac{2(-1)^3}{3}\right)\right|$

$=2\frac{2}{3}$ units^2

Question 15

$x-y+2-0$

$y=x+2$

$\sqrt{4-x^2}=x+2$

$4-x^2=(x+2)^2$

$4-x^2=x^2+4x+4$

$0=2x^2+4x$

$0=x^2+2x$

$0=x(x+2)$

$x=-2,\ 0$

$A_1=\int_{-2}^{0}\sqrt{4-x^2}\ dx=\frac{1}{4}\pi r^2=\frac{1}{4}\pi(2)^2=\pi$

$A_2=\int_{-2}^{0}(x-2)dx$

$=\left[\frac{x^2}{2}-2x\right]_{-2}^{0}$

$=\left|\left(\frac{(0)^2}{2}-2(0)\right)-\left(\frac{(-2)^2}{2}-2(-2)\right)\right|$

$=2$

$A_1-A_2=\pi-2\ \text{units}^2$

Question 16

$$\frac{1}{x} = x$$

$$x^2 = 1$$

$$x = \pm 1$$

$$\int_0^1 x\,dx + \int_1^2 \frac{1}{x}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[\ln x\right]_1^2$$

$$= \left|\left(\frac{(1)^2}{2}\right) - \left(\frac{(0)^2}{2}\right)\right| + \left|(\ln 2) - (\ln 2)\right|$$

$$= \frac{1}{2} + \ln 2 \ \text{units}^2$$

Question 17

$$\sin x = \cos x$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\int_{\frac{\pi}{4}}^{4} (\sin x - \cos x)\,dx$$

$$= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{5\pi}$$

$$= \left|(-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}) - (-\cos\frac{\pi}{4} - \sin\frac{\pi}{4})\right|$$

$$= 2\sqrt{2} \ \text{units}^2$$

Question 18

$y = e^{2x}$

$\int_0^2 \left(e^{2x} - 1\right) dx$

$= \left[\dfrac{e^{2x}}{2} - x\right]_0^2$

$= \left(\dfrac{e^{2(2)}}{2} - 2\right) - \left(\dfrac{e^{2(0)}}{2} - 0\right)$

$= \dfrac{e^4}{2} - 2 - \dfrac{1}{2}$

$= \dfrac{e^4}{2} - \dfrac{5}{2}$

$= \dfrac{1}{2}\left(e^4 - 5\right)$ units^2

Question 19

$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\sin x - \dfrac{1}{2}\right) dx$

$= [-\cos x - \dfrac{1}{2}x]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$

$= \left[\left[-\cos\left(\dfrac{5\pi}{6}\right) - \dfrac{1}{2}\left(\dfrac{5\pi}{6}\right)\right] - \left[-\cos\left(\dfrac{\pi}{6}\right) - \dfrac{1}{2}\left(\dfrac{\pi}{6}\right)\right]\right]$

$= \sqrt{3} - \dfrac{\pi}{3}$

$= \dfrac{3\sqrt{3} - \pi}{3}$ units^2

Test yourself 6

Question 1

$$\int \sin(6x)\,dx \;= -\frac{1}{6}\cos(6x) + C$$

D

Question 2

$$\int_{-3.5}^{1} \left(-x^2 - 3x + 4\right)dx - \int_{-3.5}^{1} \left(x^2 + 2x - 3\right)dx$$

B

Question 3

$$\int 4e^{3x}\,dx \;= \frac{4}{3}e^{3x} + C$$

A

Question 4

$$\int \frac{x}{x^2+3}\,dx$$

$$u = x^2 + 3,\ du = 2x\ dx$$

$$\int \frac{1}{2}\frac{du}{u}$$

$$= \frac{1}{2}\ln u + C$$

$$= \frac{1}{2}\ln\left(x^2+3\right) + C$$

D

Question 5

a $\int_1^2 \frac{dx}{x^2}$

$= \frac{0.5}{2}\left(\frac{1}{1^2} + 2\times\frac{1}{1.5^2} + \frac{1}{2^2}\right)$

$= 0.535$

b $\int_1^2 \frac{dx}{x^2}$

$= \left[-\frac{1}{x}\right]_1^2$

$= \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)$

$= 0.5$

Question 6

a $\int 3x+1\,dx = \frac{3x^2}{2} + x + C$

b $\int \frac{5x^2 - x}{x}\,dx = \int 5x-1\,dx$

$= \frac{5x^2}{2} - x + C$

c $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx$

$= \frac{2}{3}x^{\frac{3}{2}} + C$

$= \frac{2\sqrt{x^3}}{3} + C$

d $\int (2x+5)^7 \ dx$

$u = 2x+5,\ du = 2\ dx$

$= \int \frac{1}{2} u^7 du$

$= \frac{1}{2} \times \frac{1}{8} (2x+5)^8 + C$

$= \frac{(2x+5)^8}{16} + C$

e $\int x^3 \left(3x^4 - 2\right)^4 dx$

$u = 3x^4 - 2,\ du = 12x^3\ dx$

$= \int \frac{1}{12} u^4 du$

$= \frac{1}{12} \times \frac{1}{5} \left(3x^4 - 2\right)^5 + C$

$= \frac{\left(3x^4 - 2\right)^5}{60} + C$

Question 7

$\int 3^x \quad = \frac{3^x}{\ln 3} + C$

Question 8

a $\int_1^3 x^3$

$= \left(0.5 \times 1^3\right) + \left(0.5 \times 1.5^3\right) + \left(0.5 \times 2^3\right) + \left(0.5 \times 2.5^3\right)$

$= 14\ \text{units}^2$

b $\int_1^3 x^3$

$=\left(0.5\times1.5^3\right)+\left(0.5\times2^3\right)+\left(0.5\times2.5\right)^3+\left(0.5\times3^3\right)$

$=27\ \text{units}^2$

c $\int_1^3 x^3$

$\frac{2}{2}\left(1^3+3^3\right)$

$=28\ \text{units}^2$

Question 9

a $\int_0^2\left(x^3-1\right)\,dx$

$=\left[\frac{x^4}{4}-x\right]_0^2$

$=\left(\frac{2^4}{4}-2\right)-\left(\frac{0^4}{4}-0\right)$

$=2$

b $\int_{-1}^1\left(x^5\right)\,dx$

$=\left[\frac{x^6}{6}\right]_{-1}^1$

$=\left(\frac{(1)^6}{6}\right)-\left(\frac{(-1)^6}{6}\right)$

$=0$

c $\int_0^1 (3x-1)^4 \, dx$

$u = 3x - 1,\ du = 3\ dx$

$\int_0^1 \frac{1}{3} u^4 \ du$

$= \left[\frac{u^5}{15}\right]_0^1$

$= \left[\frac{(3x-1)^5}{15}\right]_0^1$

$= \left(\frac{(3(1)-1)^5}{15}\right) - \left(\frac{(3(0)-1)^5}{15}\right)$

$= 2\frac{1}{5}$

d $\int_0^1 x^2 \left(x^3 - 5\right) dx$

$u = x^3 - 5, du = 3x^2\, dx$

$\int_0^1 \frac{1}{3} u^2 \, du$

$= \left[\frac{u^3}{9}\right]_0^1$

$= \left[\frac{(x^3-5)^3}{9}\right]_0^1$

$= \left(\frac{((1)^3-5)^3}{9}\right) - \left(\frac{((0)^3-5)^3}{9}\right)$

$= 6\frac{7}{9}$

e $\int_{-1}^{2} 3x\left(x^2+1\right)^3 dx$

$u = x^2 + 1,\ du = 2x\ dx$

$3\int_{0}^{1} \frac{1}{2} u^3\, du$

$= \left[\frac{3u^4}{8}\right]_{-1}^{2}$

$= \left[\frac{3\left(x^2+1\right)^4}{8}\right]_{-1}^{2}$

$= \left(\frac{3\left((2)^2+1\right)^4}{8}\right) - \left(\frac{3\left((-1)^2+1\right)^4}{8}\right)$

$= 228\frac{3}{8}$

Question 10

$y = \ln x$

$x = e^y$

$\int_{1}^{3} e^y\, dy$

$= \left[e^y\right]_{1}^{3}$

$= \left(e^3\right) - \left(e^1\right)$

$= e\left(e^2 - 1\right)$ units^2

Question 11

$$\int_{-1}^{2} x^2\,dx$$

$$=\left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$=\left(\frac{(2)^3}{3}\right)-\left(\frac{(-1)^3}{3}\right)$$

$$=3\text{ units}^2$$

Question 12

$$\int \sin\,x^\circ\,dx$$
$$=-\frac{180}{\pi}\cos x^\circ+C$$

Question 13

$$x^2=2-x^2$$

$$2x^2=2$$

$$x^2=1$$

$$x\pm1$$

$$\int_{-1}^{1}\left(2-2x^2\right)dx$$

$$=\left[2x-\frac{2x^3}{3}\right]_{-1}^{1}$$

$$=\left(2(1)-\frac{2(1)^3}{3}\right)-\left(2(-1)-\frac{2(-1)^3}{3}\right)$$

$$=2\frac{2}{3}\text{ units}^2$$

Question 14

a $\int e^{4x} dx \ = \frac{1}{4}e^{4x} + C$

b $\int \frac{x}{x^2 - 9}\, dx$

$u = x^2 - 9,\ du = 2x\ dx$

$\int \frac{1}{2}\frac{du}{u}$

$= \frac{1}{2}\ln u + C$

$= \frac{1}{2}\ln\left|x^2 - 9\right| + C$

c $\int e^{-x}\, dx \ = -e^{-x} + C$

d $\int \frac{1}{x+4}\, dx = \ln\left|x+4\right| + C$

e $\int (x-3)\left(x^2 - 6x + 1\right)^8 dx$

$u = x^2 - 6x + 1,\ du = (2x - 6)\ dx$

$\int \frac{1}{2}u^8\, du$

$= \frac{1}{2}\frac{u^9}{9} + C$

$= \frac{\left(x^2 - 6x + 1\right)^9}{18} + C$

Question 15

$$\int_1^2 \frac{3x^4-2x^3+x^2-1}{x^2}\,dx = \int_1^2\left(3x^2-2x+1-\frac{1}{x^2}\right)dx$$

$$=\left[x^3-x^2=x+\frac{1}{x}\right]_1^2$$

$$=\left[(2)^3-(2)^2+(2)+\frac{1}{(2)}\right]-\left[(1)^3-(1)^2+(1)+\frac{1}{(1)}\right]$$

$$=4\frac{1}{2}\text{ units}^2$$

Question 16

$$x^2+y^2=9$$

$$r=3$$

$$A=\frac{1}{4}\times\pi r^2$$

$$=\frac{1}{4}\times\pi(3)^2$$

$$=\frac{9\pi}{4}\text{units}^2$$

Question 17

$y = x^3$

$x = \sqrt[3]{y}$

$x = y^{\frac{1}{3}}$

$\int_0^1 y^{\frac{1}{3}}\,dy$

$= \left[\frac{3}{4}y^{\frac{4}{3}}\right]_0^1$

$= \left[\frac{3}{4}(1)^{\frac{4}{3}}\right] - \left[\frac{3}{4}(0)^{\frac{4}{3}}\right]$

$= \frac{3}{4}\text{ units}^2$

Question 18

$\int (7x+3)^{11}\,dx$

$u = 7x+3,\ du = 7\ dx$

$\int \frac{1}{7}u^{11}du$

$= \frac{1}{7} \times \frac{u^{12}}{12} + C$

$= \frac{(7x+3)^{12}}{84}$

Question 19

Graph of $y = x^2 - x - 2 = (x - 2)(x + 1)$ is a parabola, concave up with x-intercepts at -1 and 2, so for between the limits of this integral $x = 1$ and $x = 3$, the function is negative between $x = 1$ and $x = 2$.

$$\text{Area} = \left|\int_1^2 x^2 - x - 2\ dx\right| + \int_2^3 x^2 - x - 2\ dx$$

$$= \left|\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_1^2\right| + \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_2^3$$

$$= \left|-\frac{10}{3} - \left[-\frac{13}{6}\right]\right| + \left[-\frac{3}{2}\right] - \left[-\frac{10}{3}\right]$$

$$= \left|-\frac{7}{6}\right| + \frac{11}{6}$$

$$= \frac{18}{6}$$

$$= 3 \text{ units}^2$$

Question 20

$$\int_2^5 e^{2x} dx$$

$$= \left[\frac{1}{2}e^{2x}\right]_2^5$$

$$= \left(\frac{1}{2}e^{2(5)}\right) - \left(\frac{1}{2}e^{2(2)}\right)$$

$$= \frac{1}{2}e^{10} - \frac{1}{2}e^4$$

$$\frac{e^4}{2}\left(e^6 - 1\right) \text{ units}^2$$

Question 21

$$\int_3^5 \ln\left(x^2-1\right)\,dx$$

$$=\frac{0.5}{2}\left[\ln\left(3^2-1\right)+\ln\left(5^2-1\right)+2\ln\left(3.5^2-1\right)+2\ln\left(4^2-1\right)+\ln\left(4.5^2-1\right)\right]$$

$$=5.36\ \text{units}^2$$

Question 22

$$\int_0^4 \left(3t^2-6t+5\right)dt$$

$$=\left[t^3-3t^2+5t\right]_0^4$$

$$=\left[(4)^3-3(4)^2+5(4)\right]-\left[(0)^3-3(0)^2+5(0)\right]$$

$$=36$$

Question 23

a $\int \sin\ 2x\ dx\ =-\frac{1}{2}\cos\ 2x+C$

b $\int 3\cos x\ dx\ =3\sin\ x+C$

c $\int \sec^2 5x\ dx\ =\frac{1}{5}\tan 5x+\mathrm{C}$

d $\int 1+\sin x\ dx\ =x-\cos x+C$

Question 24

$y = x^2 + 2x - 15$

$0 = x^2 + 2x - 15$

$0 = (x+5)(x-3)$

$x = -5, 3$

$\int_{-5}^{3} \left(x^2 + 2x - 15\right) dx$

$= \left[\frac{x^3}{3} + x^2 - 15x\right]_{-5}^{3}$

$= \left|\left(\frac{(3)^3}{3} + (3)^2 - 15(3)\right) - \left(\frac{(-5)^3}{3} + (-5)62 - 15(-5)\right)\right|$

$= 85\frac{1}{3} \text{ units}^2$

Question 25

a $R = -16e^{-0.4t}$

$\int -16e^{-0.4t} dt$

$= \frac{-16}{-0.4} e^{-0.4t} + C$

$= 40e^{-0.4t} + C$

$215 = 40e^{-0.4(0)} + C$

$215 = 40 + C$

$C = 175$

$T = 40e^{-0.4t} + 175$

b **i** $T\left(5\right)=40e^{-0.4(5)}+175$

$T\left(5\right)=180°$

ii $T\left(30\right)=40e^{-0.4(30)}+175$

$T\left(30\right)=175°$

Question 26

a $\int_0^{\frac{\pi}{4}} \cos x \; dx$

$=\left[\sin x\right]_0^{\frac{\pi}{4}}$

$=\left|(\sin\frac{\pi}{4})-(\sin 0)\right|$

$=\frac{1}{\sqrt{2}}$ units^2

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \; dx$

$=[\tan x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$=\left|(\tan\frac{\pi}{3})-(\tan\frac{\pi}{6})\right|$

$=\sqrt{3}-\frac{1}{\sqrt{3}}$

$=\frac{2\sqrt{3}}{\sqrt{3}}$ units^2

Question 27

a $\int 5(2x-1)^4\,dx$

$u = 2x-1,\ du = 2\ dx$

$5\int \frac{1}{2}u^4 du$

$= \frac{5}{2} \times \frac{u^5}{5} + C$

$= \frac{(2x-1)^5}{2} + C$

b $\int \frac{3x^5}{4}\ dx$

$= \frac{3}{4} \times \frac{x^6}{6} + C$

$= \frac{x^6}{8} + C$

Question 28

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\ \ dx$

$= [-\cos x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \left|(-\cos\frac{\pi}{2}) - (-\cos\frac{\pi}{4})\right|$

$= \frac{1}{\sqrt{2}}\ \text{units}^2$

Question 29

a $\int x^2\left(x^3-2\right)^5\ dx$

$u=x^3-2,\ du=3x^2\,dx$

$\int\frac{1}{3}u^5\,du$

$=\frac{1}{3}\frac{u^6}{6}+C$

$=\frac{\left(x^3-2\right)^6}{18}+C$

b $\int x\left(5x^2+2\right)^4 dx$

$u=5x^2+2,\ du=10x\ dx$

$\int\frac{1}{10}u^4\,du$

$=\frac{1}{10}\frac{u^5}{5}+C$

$=\frac{\left(5x^2+2\right)^5}{50}+C$

c $\int 5x^3\left(2x^4-1\right)^2 dx$

$u=2x^4-1,\ du=8x^3\,dx$

$5\int\frac{1}{8}u^2\,du$

$=\frac{5}{8}\times\frac{u^3}{3}+C$

$=\frac{5\left(2x^4-1\right)^3}{24}+C$

d $\int (x+2)\left(x^2+4x-3\right)^3 dx$

$u = x^2+4x-3,\ du = (2x+4)\ dx$

$\int \frac{1}{2}u^3 du$

$= \frac{1}{2}\times\frac{u^4}{4}+C$

$= \frac{\left(x^2+4x-3\right)^4}{8}+C$

Question 30

$$\int_0^{\pi} \cos 2x\ dx\ = \int_0^{\frac{\pi}{4}} \cos 2x\ dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos 2x\ dx + \int_{\frac{3\pi}{4}}^{\pi} x\cos 2x\ dx$$

$$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} + \left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \left[\frac{1}{2}\sin 2x\right]_{\frac{3\pi}{4}}^{\pi}$$

$$= \left|\left[\frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)\right] - \left[\frac{1}{2}\sin 2(0)\right]\right| + \left|\left[\frac{1}{2}\sin 2\left(\frac{3\pi}{4}\right)\right] - \left[\frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)\right]\right|$$

$$+\left[\frac{1}{2}\sin 2(\pi)\right] - \left[\frac{1}{2}\sin 2\left(\frac{3\pi}{4}\right)\right]$$

$= 2\ \text{units}^2$

Question 31

a $y=\sqrt{x-2}$

$0=\sqrt{x-2}$

$x-2=0$

$x=2$

$A=\frac{1}{2}bh$

$A=\frac{1}{2}\times 2\times\left(\sqrt{4-2}\right)$

$A=\sqrt{2}\ \text{units}^2$

b $y=\sqrt{x-2}$

$0=\sqrt{x-2}$

$x-2=0$

$x=2$

$A=\left(1\times\sqrt{2-2}\right)+\left(1\times\sqrt{3-2}\right)$

$A=1\ \text{unit}^2$

c $y=\sqrt{x-2}$

$0=\sqrt{x-2}$

$x-2=0$

$x=2$

$A=\left(1\times\sqrt{3-2}\right)+\left(1\times\sqrt{4-2}\right)$

$A=\left(1+\sqrt{2}\right)\ \text{units}^2$

Question 32

a $\int 3x\left(2x^2-1\right)^4 dx$

$u = 2x^2 - 1,\ du = 4x\ dx$

$3\int \frac{1}{4}u^4\, du$

$= \frac{3}{4} \times \frac{u^5}{5} + C$

$= \frac{3\left(2x^2-1\right)^5}{20} + C$

$3 = \frac{3\left[2(1)^2 - 1\right]^5}{20} + C$

$3 = \frac{3}{20} + C$

$C = \frac{57}{20}$

$f(x) = \frac{3\left(2x^2-1\right)^5}{20} + \frac{57}{20}$

$f(x) = \frac{1}{20}\left(3\left(2x^2-1\right)^5 + 57\right)$

b $\int \sec^2 2x\ dx$

$= \frac{1}{2}\tan 2x + C$

$\frac{\sqrt{3}}{2} = \frac{1}{2}\tan 2\left(\frac{\pi}{6}\right) + C$

$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} + C$

$C = 0$

$f(x) = \frac{1}{2}\tan 2x$

c $\int e^{5x} dx$

$= \frac{1}{5} e^{5x} + C$

$\frac{1}{5} = \frac{1}{5} + C$

$C = 0$

$f(x) = \frac{1}{5} e^{5x}$

d $\int x^3 \left(x^4 - 15\right)^3 dx$

$u = x^4 - 15,\ du = 4x^3\ dx$

$\int \frac{1}{4} u^3\, du$

$= \frac{1}{4} \times \frac{u^4}{4} + C$

$= \frac{\left(x^4 - 15\right)^4}{16} + C$

$0 = \frac{\left(2^4 - 15\right)^4}{16} + C$

$0 = \frac{1}{16} + C$

$C = -\frac{1}{16}$

$f(x) = \frac{\left(x^4 - 15\right)^4}{16} - \frac{1}{16}$

$f(x) = \frac{1}{16}\left(\left(x^4 - 15\right)^4 - 1\right)$

e $\int \frac{3x^3}{x^4+1}\,dx$

$u = x^4 + 1, du = 4x^3\,dx$

$3\int \frac{1}{4u}\,du$

$= \frac{3}{4}\ln u + C$

$= \frac{3}{4}\ln\left(x^4+1\right) + C$

$2 = \frac{3}{4}\ln\left((0)^4+1\right) + C$

$2 = C$

$f(x) = \frac{3}{4}\ln\left(x^4+1\right) + 2$

Question 33

a $\int \frac{t^2}{\sqrt{t^3+9}}\,dt$

$u = t^3 + 9,\ du = 3t^2\,dt$

$\int \frac{1}{3u^{\frac{1}{2}}}\,du$

$\int \frac{1}{3}u^{-\frac{1}{2}}\,du$

$= \frac{1}{3}\times 2u^{\frac{1}{2}} + C$

$x = \frac{2}{3}\sqrt{t^3+9} + C$

$-2 = \frac{2}{3}\sqrt{(0)^3+9} + C$

$-2 = 2 + C$

$C = -4$

$x = \frac{2}{3}\sqrt{t^3+9} - 4$

b $x = \frac{2}{3}\sqrt{t^3+9} - 4$

$x(5) = \frac{2}{3}\sqrt{(5)^3+9} - 4$

$x(5) = 3.7\text{ m}$

c $x = \frac{2}{3}\sqrt{t^3 + 9} - 4$

$10 = \frac{2}{3}\sqrt{t^3 + 9} - 4$

$14 = \frac{2}{3}\sqrt{t^3 + 9}$

$21 = \sqrt{t^3 + 9}$

$441 = t^3 + 9$

$t^3 = 432$

$t = 7.6\text{ s}$

Challenge exercise 6

Question 1

a $f(x) = x^3 + x$

Show $-f(x) = f(-x)$

$-f(x) = -x^3 - x$

$f(-x) = (-x)^3 + (-x) = -x^3 - x$

$\therefore -f(x) = f(-x)$

b $\int_{-2}^{2} (x^3 + x)\, dx$

$= \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_{-2}^{2}$

$= \left|\left(\frac{(2)^4}{4} + \frac{(2)^2}{2}\right) - \left(\frac{(-2)^4}{4} + \frac{(-2)^2}{2}\right)\right|$

$= 0$

c $f(x) = x^3 + x$

$\int_{-2}^{2} (x^3 + x)\, dx$

$= \int_{-2}^{0} (x^3 + x)\, dx + \int_{0}^{2} (x^3 + x)\, dx$

$= \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_{0}^{2}$

$= \left|\left(\frac{(0)^4}{4} + \frac{(0)^2}{2}\right) - \left(\frac{(-2)^4}{4} + \frac{(-2)^2}{2}\right)\right| + \left|\left(\frac{(2)^4}{4} + \frac{(2)^2}{2}\right) - \left(\frac{(0)^4}{4} + \frac{(0)^2}{2}\right)\right|$

$= 12 \text{ units}^2$

Question 2

a RHS = $\dfrac{\sec^2 x}{\tan x}$

$= \dfrac{\dfrac{1}{\cos^2 x}}{\dfrac{\sin x}{\cos x}}$

$= \dfrac{\dfrac{1}{\cos^2 x}}{\dfrac{\sin x}{\cos x}}$

$= \dfrac{1}{\cos^2 x} \times \dfrac{\cos x}{\sin x}$

$= \dfrac{1}{\cos x} \times \dfrac{1}{\sin x}$

= sec x cosec x

= LHS

b $= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec} x \sec x \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \dfrac{\sec^2 x}{\tan x} \, dx$

$u = \tan x, \; du = \sec^2 x \, dx$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \dfrac{du}{u}.$

$= [\ln u]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \left[\ln(\tan x)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \left|\left(\ln\left[\tan\left(\dfrac{\pi}{3}\right)\right]\right) - \left(\ln\left[\tan\left(\dfrac{\pi}{4}\right)\right]\right)\right|$

$= \ln\sqrt{3} - 0$

$= \dfrac{\ln 3}{2}$

Question 3

$$(x-1)^2 = 5 - x^2$$

$$x^2 - 2x + 1 = 5 - x^2$$

$$2x^2 - 2x - 4 = 0$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = -1,\ 2$$

$$\int_{-1}^{2} \left(5 - x^2 - x^2 + 2x - 1\right)\, dx$$

$$= \int_{-1}^{2} \left(4 - 2x^2 + 2x\right)\, dx$$

$$= \left[-\frac{2x^3}{3} + x^2 + 4x\right]_{-1}^{2}$$

$$= \left[-\frac{2(2)^3}{3} + (2)^2 + 4(2)\right] - \left[-\frac{2(-1)^3}{3} + (-1)^2 + 4(-1)\right]$$

$$= 9 \text{ units}^2$$

Question 4

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2 2x\, dx$$

$$= \left[\frac{1}{2}\tan 2x\right]_{\frac{\pi}{12}}^{\frac{\pi}{8}}$$

$$= \left[\frac{1}{2}\tan\left(2 \times \frac{\pi}{8}\right)\right] - \left[\frac{1}{2}\tan\left(2 \times \frac{\pi}{12}\right)\right]$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{6} \quad = \frac{3 - \sqrt{3}}{6} \text{ units}^2$$

Question 5

$$\int_0^1 \frac{x}{\left(3x^2-4\right)^2}\,dx$$

$$u = 3x^2 - 4,\ du = 6x\ dx$$

$$\int_0^1 \frac{1}{6}\frac{du}{u^2}$$

$$= \left[-\frac{1}{6u}\right]_0^1$$

$$= \left[-\frac{1}{6\left(3x^2-4\right)}\right]_0^1$$

$$= \left[-\frac{1}{6\left(3\left(1^2\right)-4\right)}\right] - \left[-\frac{1}{6\left(3(0)^2-4\right)}\right]$$

$$= \frac{1}{8}$$

Question 6

$$y = \frac{3}{x-2}$$

$$x - 2 = \frac{3}{y}$$

$$x = \frac{3}{y} + 2$$

$$\int_1^3 \left(\frac{3}{y} + 2\right) dy$$

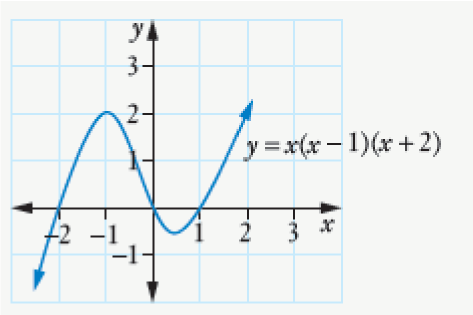
$$= \frac{0.5}{2}\left[\left(\frac{3}{1}+2\right) + \left(\frac{3}{3}+2\right) + 2\left(\frac{3}{1.5}+2\right) + 2\left(\frac{3}{2}+2\right) + 2\left(\frac{3}{2.5}+2\right)\right]$$

$$= 7.35\ \text{units}^2$$

Question 7

a $y = x(x - 1)(x + 2)$

$x = -2, 0, 1$



b $y = x(x-1)(x+2)$

$y = x^3 + x^2 - 2x$

$\int_{-2}^{0}\left(x^3 + x^2 - 2x\right)\,dx + \int_{0}^{1}\left(x^3 + x^2 - 2x\right)\,dx$

$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{-2}^{0} + \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{0}^{1}$

$= \left|\left[\frac{(0)^4}{4} + \frac{(0)^3}{3} - (0)^2\right] - \left[\frac{(-2)^4}{4} + \frac{(-2)^3}{3} - (-2)^2\right]\right|$

$+ \left|\left[\frac{(1)^4}{4} + \frac{(1)^3}{3} - (1)^2\right] - \left[\frac{(0)^4}{4} + \frac{(0)^3}{3} - (0)^2\right]\right|$

$= 3\frac{1}{12}$ units^2

Question 8

$x^2 = 4 - x$

$x^2 + x - 4 = 0$

$x = -2.56,\ 1.56$

$\int_{-2.56}^{1.56} \left(4 - x - x^2\right)\, dx$

$= \left[4x - \dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_{-2.56}^{1.56}$

$= \left|\left[4(1.56) - \dfrac{(1.56)^2}{2} - \dfrac{(1.56)^3}{3}\right] - \left[4(-2.56) - \dfrac{(-2.56)^2}{2} - \dfrac{(2.56)^3}{3}\right]\right|$

$= 11.68\ \text{units}^2$

Question 9

a $y = x\sqrt{x+3}$

$y' = \sqrt{x+3} + \dfrac{x}{2\sqrt{x+3}}$

$y' = \dfrac{3(x+2)}{2\sqrt{x+3}}$

b $\dfrac{2}{3}\int \dfrac{x+2}{\sqrt{x+3}}\, dx$

$= \dfrac{2x\sqrt{x+3}}{3} + C$

Question 10

a $\frac{d}{dx}\left(x^2 \ln x\right)$

$= x^2 \times \frac{1}{x} + 2x \ln x$

$= x + 2x \ln x$

$= x\left(1 + 2\ln[x]\right)$

b $2\int_1^3 \left(1 + 2\ln[x]\right)$

$= 2\left[x^2 \ln x\right]_1^3$

$= 2\left|(3)^2 \ln(3) - (1)^2 \ln(1)\right|$

$= 18\ln 3$

Question 11

$\sqrt{x} = x^3$

$x = x^6$

$0 = x\left(x^5 - 1\right)$

$x = 0, 1$

$\int_0^1 \left(x^{\frac{1}{2}} - x^3 \right) dx$

$= \left[\dfrac{2x^{\frac{3}{2}}}{3} - \dfrac{x^4}{4} \right]_0^1$

$= \left| \left(\dfrac{2(1)^{\frac{3}{2}}}{3} - \dfrac{(1)^4}{4} \right) - \left(\dfrac{2(0)^{\frac{3}{2}}}{3} - \dfrac{(0)^4}{4} \right) \right|$

$= \dfrac{5}{12}$ units^2

Question 12

a $\sum_{0}^{49} 2^{0.2n}$

$= 1023\left(1 + 2^{0.2} + 2^{0.4} + 2^{0.6} + 2^{0.8}\right)$

$= 6879.7$

b $A = \dfrac{10-0}{50} \times 6879.7$

$= 1375.94 \text{ units}^2$

c $\sum_{1}^{100} 2^{0.1n}$

$= 15\ 276.2$

$A = \dfrac{10-0}{100} \times 15\ 276.2$

$= 1527.6 \text{ units}^2$

MATHS IN FOCUS 12 MATHEMATICS ADVANCED

WORKED SOLUTIONS

Chapter 7: Statistics

Exercise 7.01 Types of data

Question 1

a N

b N

c N

d C

e N

f N

g N

h C

i N

j C

k N

l C

m N

n N

o N

p N

q N

r N

s C

t N

Question 2

a O

b C

c N

d N

e C

f D

g D

h N

i C

j N

k D

l N

m N

Question 3

a e.g. types of sport, hair colour, types of pizza

b e.g. heights, test scores, prices

c e.g. rankings, test scores, clothing sizes

d e.g. karate belt colour, size of take-away coffee cups, Olympic medals

e e.g. race times, length, rainfall

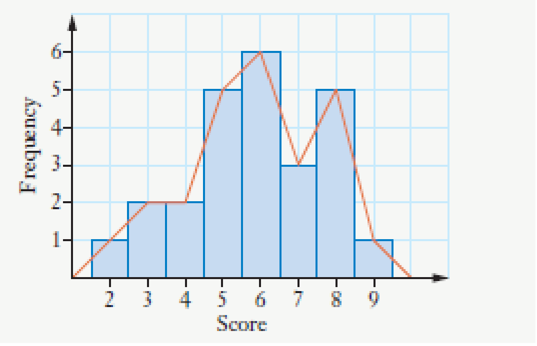
f e.g. types of trees, housing, cars

Question 1

a **i**

Score	Frequency
2	1
3	2
4	2
5	5
6	6
7	3
8	5
9	1

ii



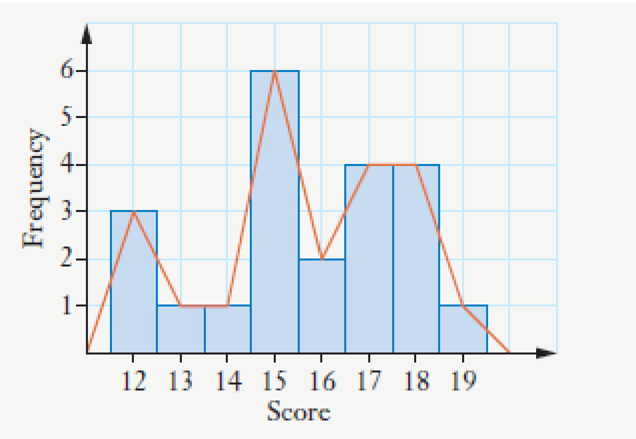
iii Highest 9, lowest 2

iv Most frequent score = 6

b **i**

Pizzas	Frequency
12	3
13	1
14	1
15	6
16	2
17	4
18	4
19	1

ii



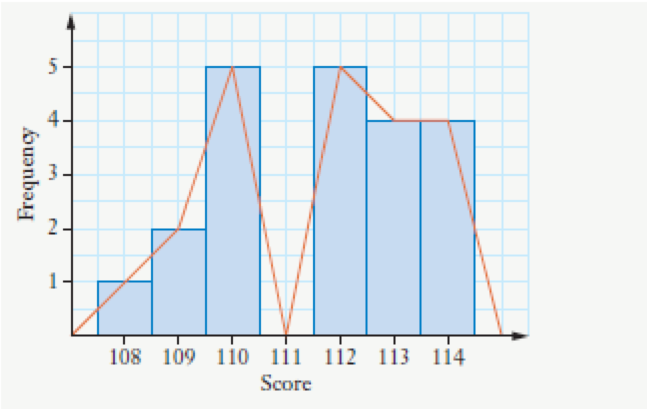
iii Highest 19, lowest 12

iv Most frequent score = 15

c **i**

Gym attendance	Frequency
108	1
109	2
110	5
111	0
112	5
113	4
114	4

ii



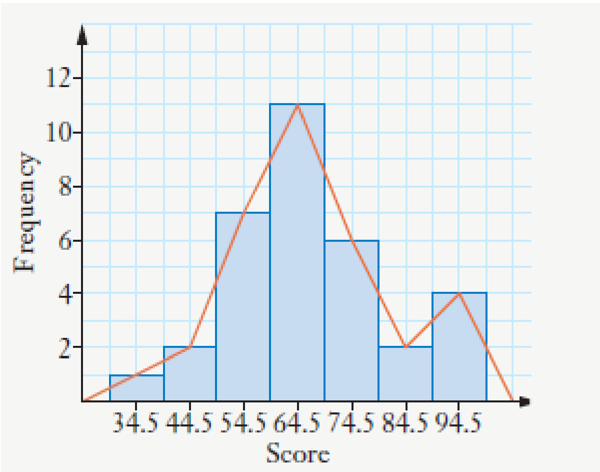
iii Highest 114, lowest 108

iv Most frequent scores = 110 and 112

d **i**

Results	Class centre	Frequency
30–39	34.5	1
40–49	44.5	2
50–59	54.5	7
60–69	64.5	11
70–79	74.5	6
80–89	84.5	2
90–99	94.5	4

ii



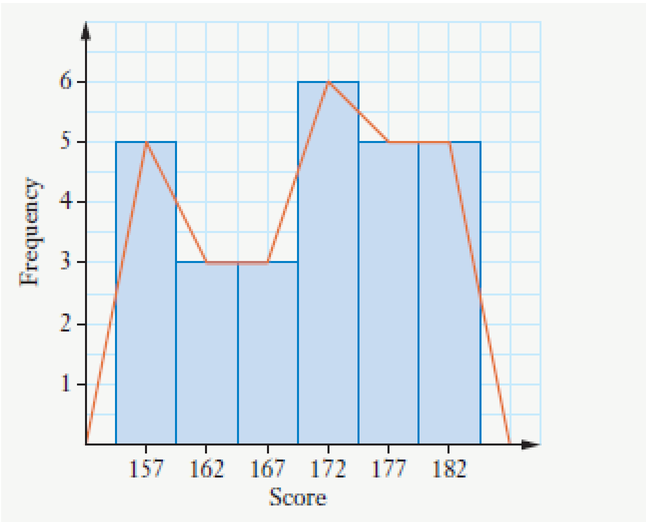
iii Highest group 90–99, lowest group 30–39

iv Most frequent group = 60–69

e **i**

Height (cm)	Class centre	Frequency
155–159	157	5
160–164	162	3
165–169	167	3
170–174	172	6
175–179	177	5
180–184	182	5

ii



iii Highest group 180–184, lowest group 155–159

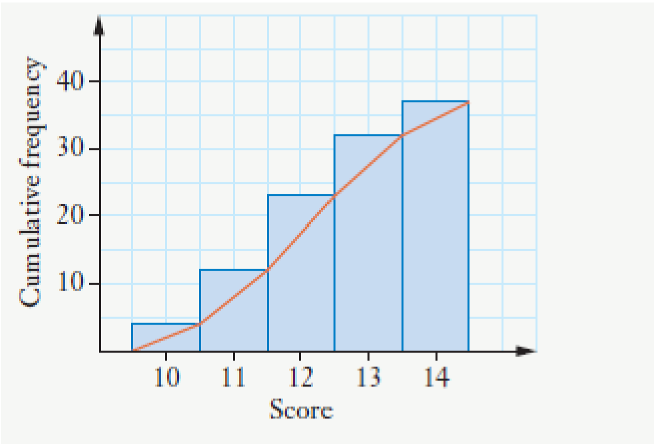
iv Most frequent group = 170–174

Question 2

a **i**

Number of cars	Frequency	Cumulative frequency
10	4	4
11	8	12
12	11	23
13	9	32
14	5	37

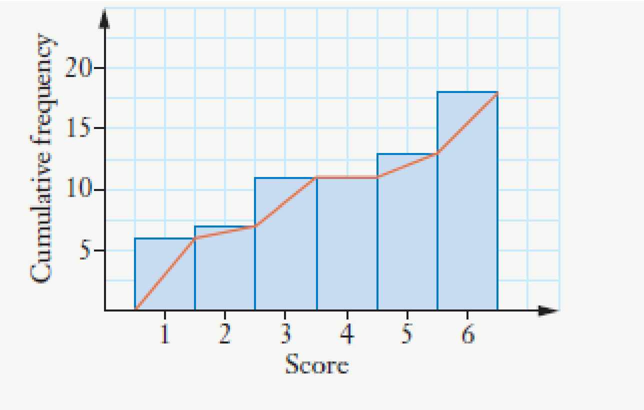
ii



b **i**

Score	Frequency	Cumulative frequency
1	7	7
2	1	8
3	3	11
4	0	11
5	2	13
6	5	18

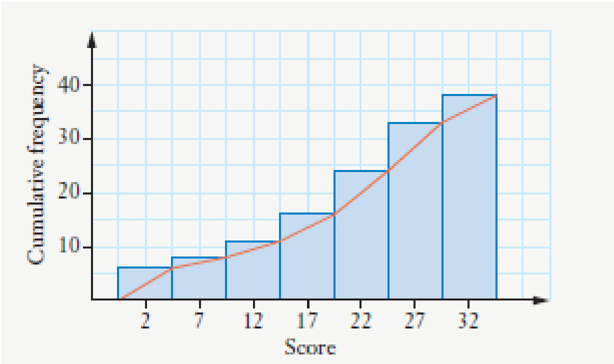
ii



c **i**

Sales	Class centre	Frequency	Cumulative frequency
0–4	2	6	6
5–9	7	2	8
10–14	12	3	11
15–19	17	5	16
20–24	22	8	24
25–29	27	9	33
30–34	32	5	38

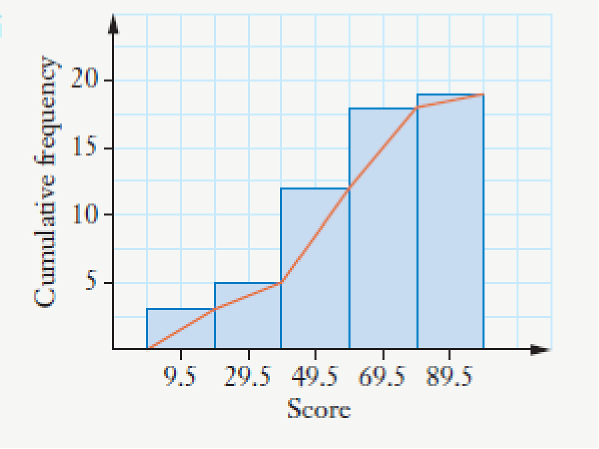
ii



d **i**

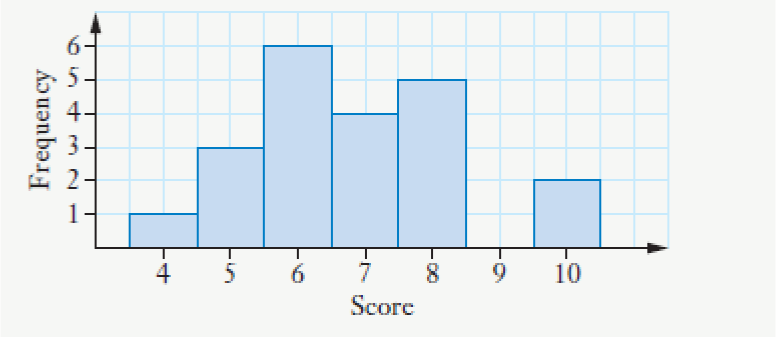
Scores	Class centre	Frequency	Cumulative frequency
0–19	9.5	3	3
20–39	29.5	2	5
40–59	49.5	7	12
60–79	69.5	6	18
80–99	89.5	1	19

ii



Question 3

a

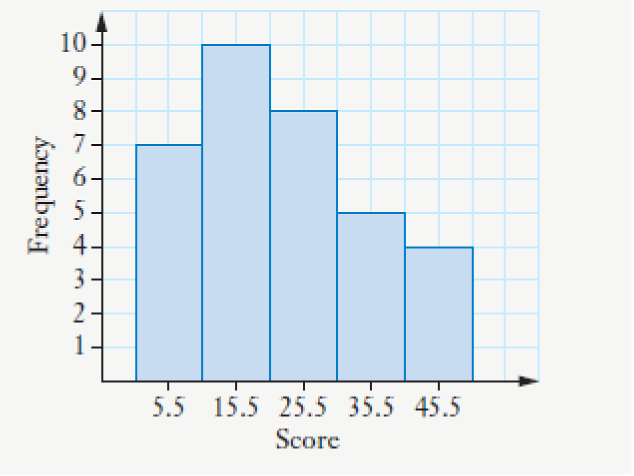


b More than 6 rescues = 4 + 5 + 0 + 2 = 11 times

c Most common number = 6 rescues (highest frequency)

Question 4

a



b Frequency between 21 and 40 min = 8 + 5 = 13

Total frequency = 7 + 10 + 8 + 5 + 4= 34

Percentage between 21 and 40 min = $\frac{13}{34} \times 100\% \approx 38.2\%$

Question 5

a

	Play soccer	Do not play soccer
Play tennis	12	35
Do not play tennis	27	28

b **i** Total = 27 + 35 + 28 + 12 = 102

People playing both sports = 12

Percentage of people playing both sports = $\frac{12}{102} \times 100\% \approx 11.8\%$

ii People playing neither sport = 28

Percentage of people playing neither sport = $\frac{28}{102} \times 100\% \approx 27.5\%$

c People playing at least one sport = 12 + 35 + 27 = 74

People playing soccer but not tennis = 27

Percentage = $\frac{27}{74} \times 100\% \approx 36.5\%$

d People playing soccer = 12 + 27 = 39

People playing soccer and tennis = 12

Fraction = $\frac{12}{39} = \frac{4}{13}$

e People playing tennis = 12 + 35 = 47

People not playing soccer = 35

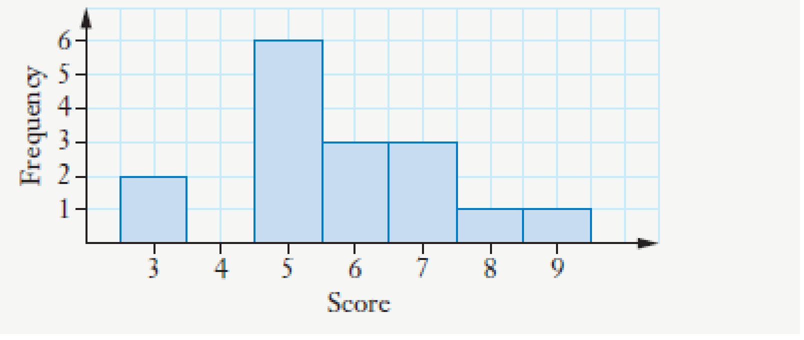
Percentage = $\frac{35}{47} \times 100\% \approx 74.5\%$

Question 6

a Total = 2 + 6 + 3 + 3 + 1 + 1 = 16

b 5 (most dots)

c



d Number above 4 = 6 + 3 + 3 + 1 + 1 = 14

Percentage above 4 = $\frac{14}{16} \times 100\% = 87.5\%$

e Number below 4 = 2

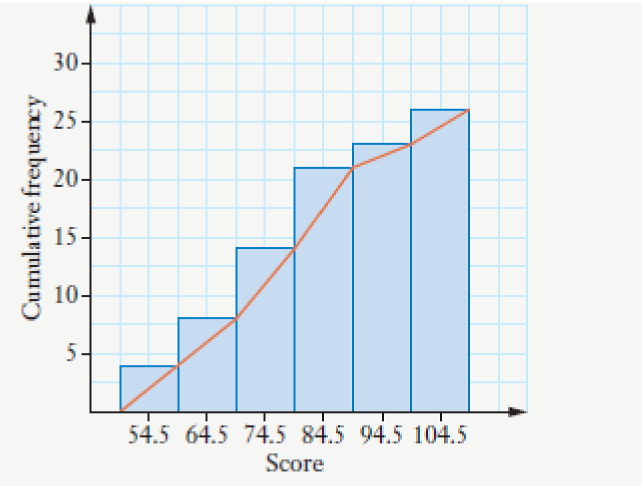
Fraction below 4 = $\frac{2}{16} = \frac{1}{8}$

Question 7

a

Weight (kg)	Class centre	Frequency	Cumulative frequency
50–59	54.5	4	4
60–69	64.5	4	8
70–79	74.5	6	14
80–89	84.5	7	21
90–99	94.5	2	23
100–109	104.5	3	26

b



c **i** People weighing 80 kg or more = 7 + 2 + 3 = 12

ii People weighing less than 80 kg = 4

d People surveyed = 4 + 4 + 6 + 7 + 2 + 3 = 26

People weighing from 70 to 89 kg = 6 + 7 = 13

Percentage of people weighing from 70 to 89 kg = $\frac{13}{26} \times 100\% = 50\%$

e People weighing between 50 and 80 kg = 4 + 4 + 6 = 14

Fraction of people weighing between 50 and 80 kg = $\frac{14}{26} = \frac{7}{13}$

Question 8

a 25%, as cricket takes up $\frac{1}{4}$ of the pie chart.

b

Sport	**Angle**	**Frequency**
Tennis	45°	$\frac{45}{360} \times 720 = 90$
Soccer	60°	$\frac{60}{360} \times 720 = 120$
Athletics	30°	$\frac{30}{360} \times 720 = 60$
Cricket	90°	$\frac{90}{360} \times 720 = 180$
Basketball	75°	$\frac{75}{360} \times 720 = 150$
Volleyball	60°	$\frac{60}{360} \times 720 = 120$

Question 9

a Total = 104 + 87 + 58 + 93 + 79 + 101 = 522

Percentage of students studying law = $\frac{101}{522} \times 100\% \approx 19.3\%$

b Students studying medicine or music = 104 + 58 = 162

Percentage of students studying law = $\frac{162}{522} \times 100\% \approx 31.0\%$

c Medicine: sector angle = $\frac{104}{522} \times 360° \approx 72°$

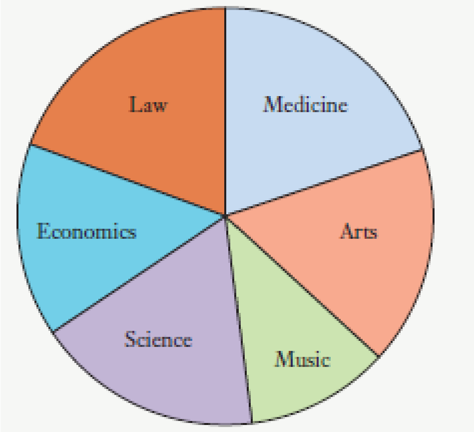
Arts: sector angle = $\frac{87}{522} \times 360° = 60°$

Music: sector angle = $\frac{58}{522} \times 360° = 40°$

Science: sector angle = $\frac{93}{522} \times 360° \approx 64°$

Economics: sector angle = $\frac{79}{522} \times 360° \approx 54°$

Law: sector angle = $\frac{101}{522} \times 360° \approx 70°$



Question 10

a People infected with virus = 11 + 76 = 87

b People vaccinated = 11 + 159 = 170

Total = 11 + 76 + 159 + 58 = 304

Percentage of people vaccinated = $\frac{170}{304} \times 100\% \approx 55.9\%$

c Vaccinated people who had virus = 11

Percentage of vaccinated people who had virus = $\frac{11}{170} \times 100\% \approx 6.5\%$

d People with virus who were not vaccinated = 76

Question 11

a Total = 104 + 105 + 112 + 409 = 430

b People who are asthmatic = 104 + 105 = 209

Percentage of people who are asthmatic = $\frac{209}{430} \times 100\% \approx 48.6\%$

c People who are asthmatic in control group = 105

Percentage of asthmatic people in control group = $\frac{105}{209} \times 100\% \approx 50.2\%$

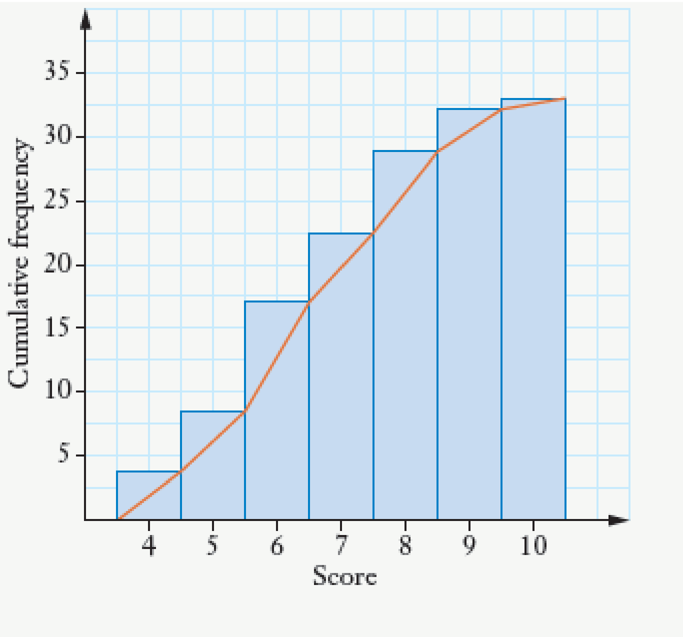
d People who are not asthmatic = 112 + 109 = 221

e Non-asthmatic people who took medication = 112

Fraction of non-asthmatic people who took medication = $\frac{112}{221}$

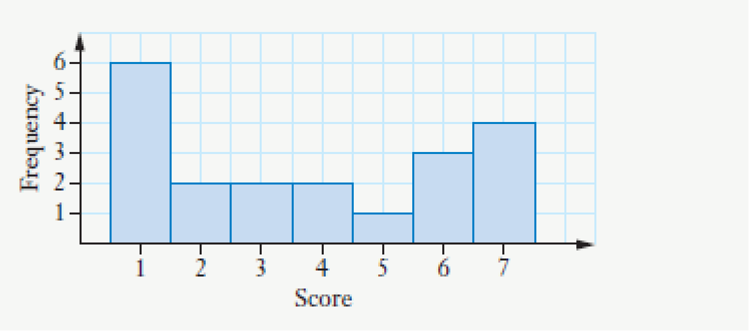
Question 12

Score	Frequency	Cumulative frequency
4	3	3
5	4	7
6	10	17
7	5	22
8	6	28
9	4	32
10	1	33



Question 13

a



b

Junk mail items	Cumulative frequency	Frequency
1	6	6
2	8	2
3	10	2
4	12	2
5	13	1
6	16	3
7	20	4

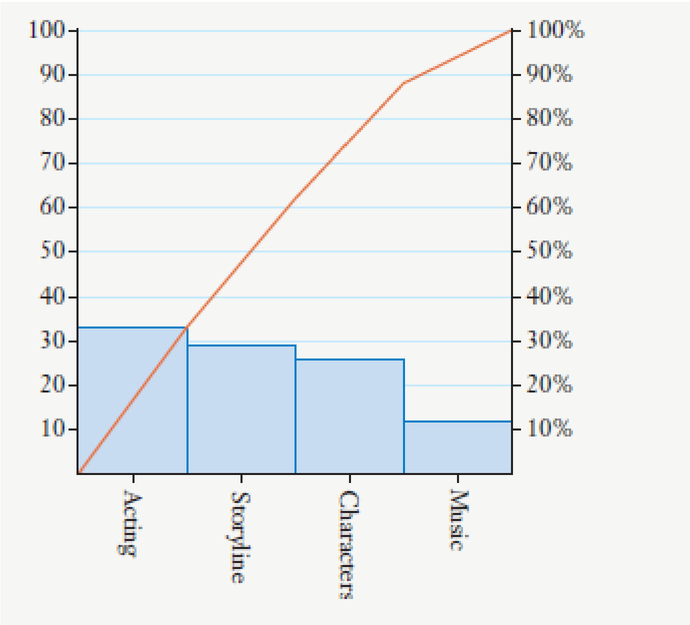
Question 14

Stem-and-leaf plots list individual scores so retain all details. Not easy to draw, and can be long. A grouped frequency distribution table groups scores so individual data is lost. Easy to draw and compact.

Question 15

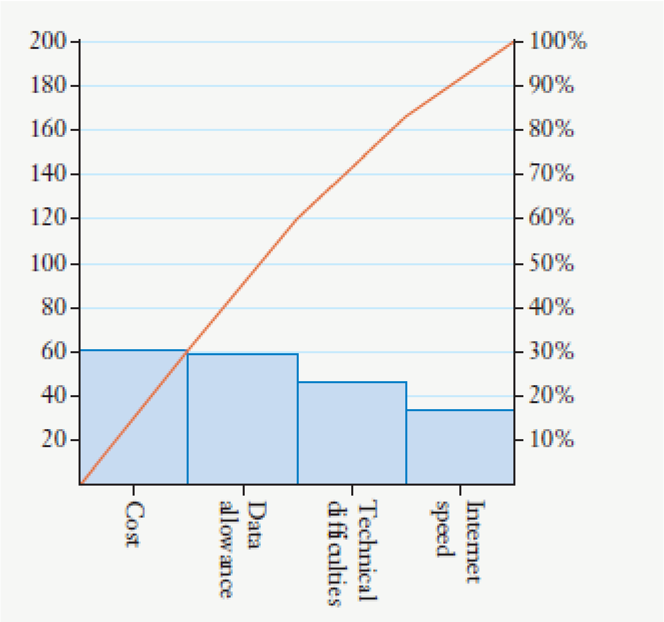
a

Reason	**Frequency**	**Percentage frequency**	**Cumulative percentage frequency**
Acting	33	33%	33%
Storyline	29	29%	62%
Characters	26	26%	88%
Music	12	12%	100%
Total	100		



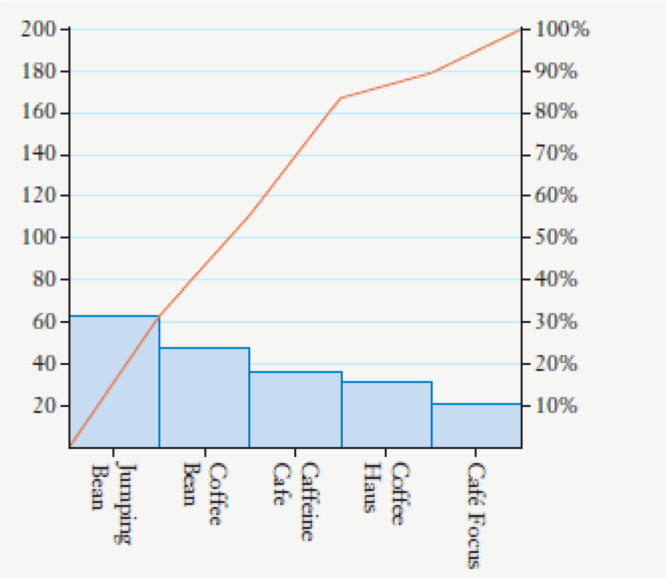
b

Complaint	Frequency	Percentage frequency	Cumulative percentage frequency
Cost	61	30.5%	30.5%
Data allowance	59	29.5%	60%
Technical difficulties	46	23%	83%
Internet speed	34	17%	100%
Total	200		



c

Café	Frequency	Percentage frequency	Cumulative percentage frequency
Jumping Bean	63	31.5%	31.5%
Coffee Bean	48	24%	55.5%
Caffeine Café	36	18%	73.5%
Coffee Haus	32	16%	89.5%
Café Focus	21	10.5%	100%
Total	200		



Question 1

a **i** Mean = $\dfrac{5+5+7+6+5+6+1}{7} = \dfrac{35}{7}$ = 5

ii Mode = 5 (highest frequency: 3)

iii 1, 5, 5, **5**, 6, 6, 7 Median = 5

b **i** Mean = $\dfrac{1+4+6+8+7+4+6+4+5}{9} = \dfrac{45}{9}$ = 5

ii Mode = 4 (highest frequency: 3)

iii 1, 4, 4, 4, **5**, 6, 6, 7, 8 Median = 5

c **i** Mean = $\dfrac{15+18+14+19+18+17+11}{7} = \dfrac{112}{7}$ = 16

ii Mode = 18 (highest frequency: 2)

iii 11, 14, 15, **17**, 18, 18, 19 Median = 17

d **i** Mean = $\dfrac{4+6+5+4+7+8}{6} = \dfrac{34}{9}$ ≈ 5.7

ii Mode = 4 (highest frequency: 2)

iii 4, 4, **5, 6,** 7, 8 Median = $\dfrac{5+6}{2}$ = 5.5

e **i** Mean = $\dfrac{1.43+1.66+1.55+1.49+1.27+1.81+1.49+1.38}{8} = \dfrac{12.08}{8}$ = 1.51

ii Mode = 1.49 (highest frequency: 2)

iii 127, 1.38, 1.43, **1.49, 1.49,** 1.55, 1.66, 1.81 Median = 1.49

Question 2

(Highest frequency)

a Brown

b Tabby

Question 3

a **i** Mean = $\frac{\sum fx}{\sum f} = \frac{162}{27} = 6$

ii

Score	Frequency	Cumulative frequency
3	3	3
4	4	7
5	2	9
6	7	16
7	6	22
8	2	24
9	3	27

Median is the $\frac{27+1}{2}$ = 14th score, so reading from the cumulative frequency column, median = 6.

iii Mode = 6 (highest frequency)

b **i** Mean = $\frac{\sum fx}{\sum f} = \frac{1101}{21} \approx 52.4$

ii

Score	Frequency	Cumulative frequency
50	1	1
51	6	7
52	5	12
53	3	15
54	4	19
55	2	21

Median is the $\frac{21+1}{2}$ = 11th score, so reading from the cumulative frequency column, median = 52.

iii Mode = 51 (highest frequency)

c **i** Mean = $\dfrac{\sum fx}{\sum f} = \dfrac{439}{25}$ = 17.56

ii

Score	Frequency	Cumulative frequency
14	4	4
15	2	6
16	1	7
17	4	11
18	3	14
19	5	19
20	6	25

Median is the $\dfrac{25+1}{2}$ = 13th score, so reading from the cumulative frequency column, median = 18.

iii Mode = 20 (highest frequency)

d **i** Mean = $\frac{\sum fx}{\sum f} = \frac{1756}{17} \approx 103.3$

ii

Score	Frequency	Cumulative frequency
100	3	3
101	0	3
102	2	5
103	1	6
104	6	12
105	5	17

Median is the $\frac{17+1}{2}$ = 9th score, so reading from the cumulative frequency column, median = 104.

iii Mode = 104 (highest frequency)

Question 4

a 24 scores, so reading from the 12 on the cumulative frequency axis, the median is 3 (on the Rankings axis).

b 27 scores, so reading from the 13.5 on the cumulative frequency axis, the median is 1 (on the Siblings axis).

c 100 scores, so reading from the 50 on the cumulative frequency axis, the median is $\frac{3+4}{2}$ = 3.5 (on the Hours axis).

d 10 scores, so reading from the 5 on the cumulative frequency axis, the median is 3 (on the Meetings axis).

Question 5

a **i**

Score	Class centre (x)	f	fx
2–4	3	5	15
5–7	6	4	24
8–10	9	7	63
11–13	12	4	48
14–16	15	3	45
17–19	18	2	36
		$\Sigma f = 25$	$\Sigma fx = 231$

$$\text{Mean} = \bar{x} = \frac{\sum fx}{\sum f} = \frac{231}{25} = 9.24$$

ii Modal class is 8–10 (highest frequency)

b **i**

Score	Class centre (x)	f	fx
0–4	2	3	6
5–9	7	2	14
10–14	12	6	72
15–19	17	8	136
20–24	22	9	198
25–29	27	5	135
		$\Sigma f = 33$	$\Sigma fx = 561$

$$\text{Mean} = \bar{x} = \frac{\sum fx}{\sum f} = \frac{561}{33} = 17$$

ii Modal class is 20–24 (highest frequency)

c **i**

Score	Class centre (x)	f	fx
10–24	17	4	68
25–39	32	0	0
40–54	47	1	47
55–69	62	5	310
70–84	77	9	693
85–99	92	8	736
		$\Sigma f = 27$	$\Sigma fx = 1854$

$$\text{Mean} = \bar{x} = \frac{\sum fx}{\sum f} = \frac{1854}{27} \approx 68.7$$

ii Modal class is 70–84 (highest frequency)

d **i**

Score	Class centre (x)	f	fx
20–24	22	12	264
25–29	27	8	216
30–34	32	9	288
35–39	37	7	259
40–44	42	8	336
45–49	47	11	517
50–54	52	12	624
55–59	57	6	342
		$\Sigma f = 73$	$\Sigma fx = 2846$

$$\text{Mean} = \bar{x} = \frac{\sum fx}{\sum f} = \frac{2846}{73} \approx 39.0$$

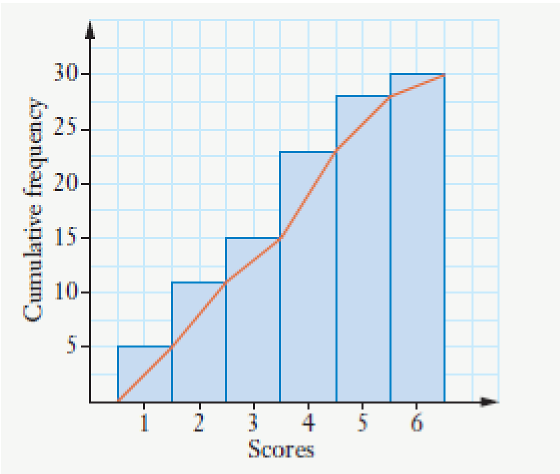
ii Modal classes are 20–24 and 50–54 (bimodal)

Question 6

a **i**

Athletes	Frequency	Cumulative frequency
1	5	5
2	6	11
3	4	15
4	8	23
5	5	28
6	2	30

ii

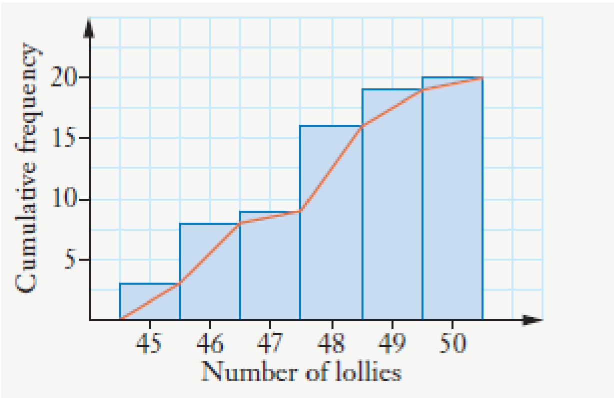


iii 30 scores, so reading from the 15 on the cumulative frequency axis, the median is $\dfrac{3+4}{2}$ = 3.5 (on the Scores axis).

b **i**

Lollies	Frequency	Cumulative frequency
45	3	3
46	5	8
47	1	9
48	7	16
49	3	19
50	1	20

ii

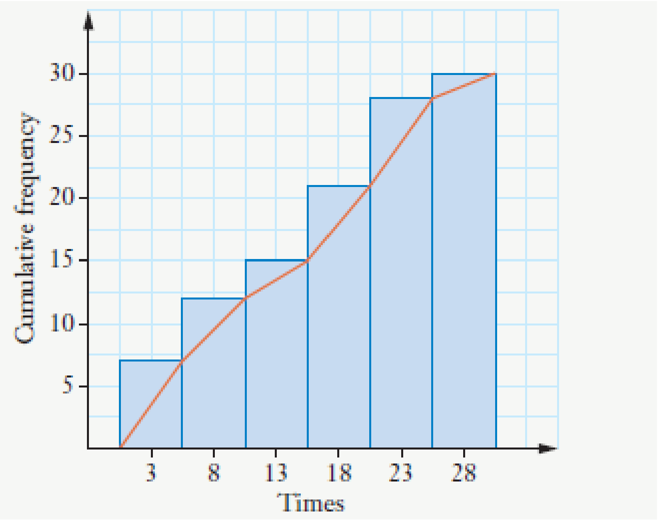


iii 20 scores, so reading from the 10 on the cumulative frequency axis, the median is 48 (on the Lollies axis).

c **i**

Time (h)	Class centre	Frequency	Cumulative frequency
1–5	3	7	7
6–10	8	5	12
11–15	13	3	15
16–20	18	6	21
21–25	23	7	28
26–30	28	2	30

ii

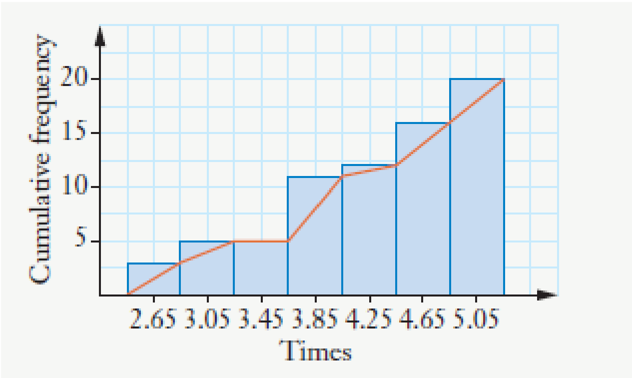


iii 30 scores, so reading from the 15 on the cumulative frequency axis, the median is $\frac{13+18}{2}$ = 15.5 (on the Times axis).

d **i**

Time (min)	Class centre	Frequency	Cumulative frequency
2.5–2.8	2.65	3	3
2.9–3.2	3.05	2	5
3.3–3.6	3.45	0	5
3.7–4.0	3.85	6	11
4.1–4.4	4.25	1	12
4.5–4.8	4.65	4	16
4.9–5.2	5.05	4	20

ii



iii 20 scores, so reading from the 10 on the cumulative frequency axis, the median is about 4.0 (on the Times axis).

Question 7

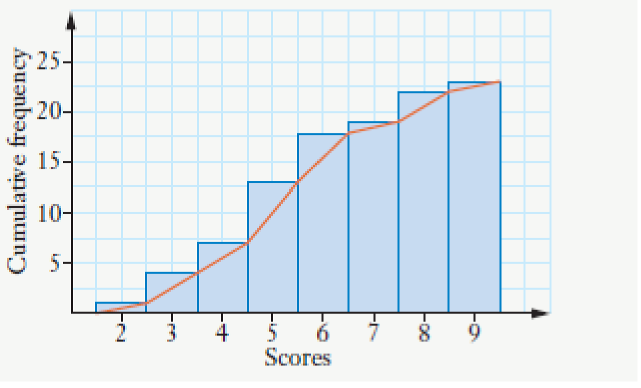
a **i**

Score	Frequency	Cumulative frequency
2	1	1
3	3	4
4	3	7
5	6	13
6	5	18
7	1	19
8	3	22
9	1	23

ii Mean = $\bar{x} = \frac{\sum fx}{\sum f} = \frac{2+9+12+30+30+7+24+9}{23} = \frac{123}{23} \approx 5.3$

iii Mode = 5 (highest frequency)

iv



v 23 scores, so reading from the 11.5 on the cumulative frequency axis, the median is 5 (on the Scores axis).

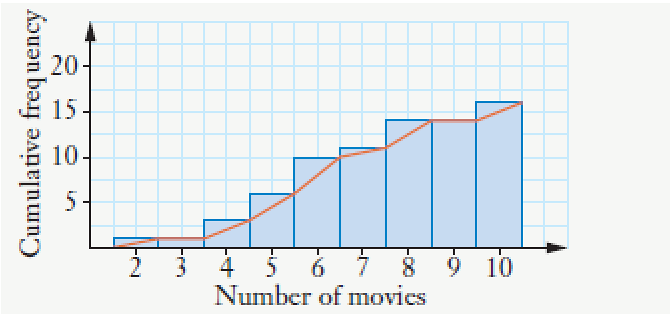
b **i**

Number of movies	**Frequency**	**Cumulative frequency**
2	1	1
3	0	1
4	2	3
5	3	6
6	4	10
7	1	11
8	3	14
9	0	14
10	2	16

ii Mean = $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{2+0+8+15+24+7+24+0+20}{16} = \dfrac{100}{16} = 6.25$

iii Mode = 6 (highest frequency)

iv



v 16 scores, so reading from the 8 on the cumulative frequency axis, the median is 6 (on the Movies axis).

c **i**

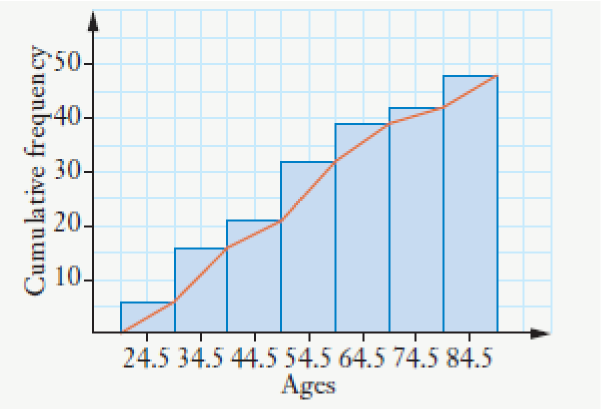
Ages	Class centre	Frequency	Cumulative frequency
20–29	24.5	6	6
30–39	34.5	10	16
40–49	44.5	5	21
50–59	54.5	11	32
60–69	64.5	7	39
70–79	74.5	3	42
80–89	84.5	6	48

ii Mean = $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{147+345+222.5+599.5+451.5+223.5+507}{48} = \dfrac{2496}{48}$

$= 52$

iii Mode = 50–59 (highest frequency)

iv



v 48 scores, so reading from the 24 on the cumulative frequency axis, the median about 54.5 (on the Ages axis).

Question 8

a In order: 32, 58, 59, 60, 64, 68, 69, 75, 77

i Outlier = 32

ii Mean = $\dfrac{32+58+59+60+64+68+69+75+77}{9}=\dfrac{562}{9}\approx 62.4$

No mode

Median = 64

iii Mean = $\dfrac{58+59+60+64+68+69+75+77}{8}=\dfrac{530}{8}=66.25$

No mode

58, 59, 60, 64, 68, 69, 75, 77; Median = $\dfrac{64+68}{2}$ = 66

b **i** Outlier = 1

ii Mean = $\dfrac{\Sigma fx}{\Sigma f}=\dfrac{1+0+0+4+15+36+35+24+36}{1+0+0+1+3+6+5+3+4}=\dfrac{151}{23}\approx 6.6$

Mode = 6 (highest frequency)

23 scores, so median is the 12th score, which is 7.

iii Mean = $\dfrac{\Sigma fx}{\Sigma f}=\dfrac{0+0+4+15+36+35+24+36}{0+0+1+3+6+5+3+4}=\dfrac{150}{22}\approx 6.8$

Mode = 6 (highest frequency)

22 scores, so median is the average of the 11th and 12th scores, which is 7.

c In order: 23, 25, 26, 32, 37, 41, 43, 50, 50, 53, 54, 63, 65, 97

i Outlier = 97

ii Mean =

$$\frac{23+25+26+32+37+41+43+50+50+53+54+63+65+97}{14}=\frac{659}{14}\approx 47.1$$

Mode = 50 (highest frequency)

Median = $\frac{43+50}{2}$ = 46.5

iii Mean = $\frac{23+25+26+32+37+41+43+50+50+53+54+63+65}{13}=\frac{562}{13}$

≈ 43.2

Mode = 50 (highest frequency)

23, 25, 26, 32, 37, 41, 43, 50, 50, 53, 54, 63, 65; Median = 43

d **i** Outlier = 3

ii Mean = $\frac{\Sigma fx}{\Sigma f}=\frac{3+16+30+18+28+16+36}{1+4+6+3+4+2+4}=\frac{147}{24}\approx 6.1$

Mode = 5 (highest frequency)

24 scores, so median is the average of the 12th and 13th scores, which is 6.

iii Mean = $\frac{\Sigma fx}{\Sigma f}=\frac{16+30+18+28+16+36}{4+6+3+4+2+4}=\frac{144}{23}\approx 6.3$

Mode = 5 (highest frequency)

23 scores, so median is the 12th score, which is 6.

Question 9

Outlier is 1. It changes the mean.

Question 10

a 17. One student had a low score compared to the other students in the class.

b

Class	Class centre	Frequency	Cumulative frequency
10–19	14.5	1	1
20–29	24.5	0	1
30–39	34.5	2	3
40–49	44.5	3	6
50–59	54.5	6	12
60–69	64.5	9	21
70–79	74.5	7	28
80–89	84.5	4	32
90–99	94.5	4	36

c **i** Mean = $\frac{\Sigma fx}{\Sigma f} = \frac{14.5 + 0 + 69 + 133.5 + 327 + 580.5 + 521.5 + 338 + 378}{36} = \frac{2362}{36}$

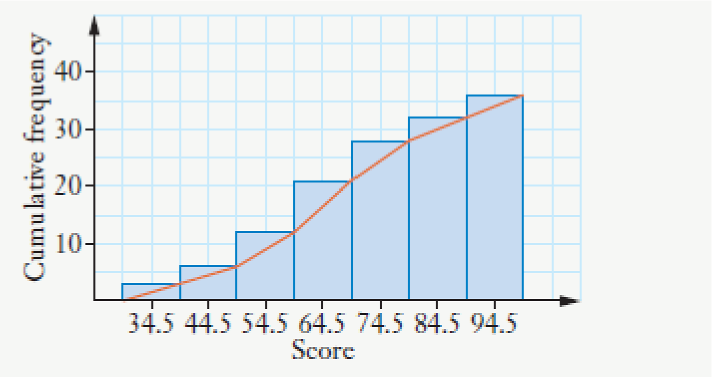
≈ 65.6

Modal class = 60–69

ii Mean = $\frac{\Sigma fx}{\Sigma f} = \frac{0 + 69 + 133.5 + 327 + 580.5 + 521.5 + 338 + 378}{35} = \frac{2347.5}{35} \approx 67.1$

Modal class = 60–69

d



35 scores, so reading from the 18 on the cumulative frequency axis, the median is about 66 (on the Score axis).

Exercise 7.04 Quantiles, deciles and percentiles

Question 1

a **i** 40 scores, so reading from the $\frac{1}{4} \times 40 = 10$ on the cumulative frequency axis,

the 1st quartile is 2 (on the Score axis).

ii Reading from the $\frac{1}{2} \times 40 = 20$ on the cumulative frequency axis,

the 2nd quartile is 3.

iii Reading from the $\frac{3}{4} \times 40 = 30$ on the cumulative frequency axis,

the 3rd quartile is $\frac{4+5}{2} = 4.5$.

b

Score	Frequency	Cumulative frequency
10	3	3
11	5	8
12	4	12
13	6	18
14	0	18
15	2	20

i 20 scores, so the 1st quartile is the average of the 5th and 6th scores,

which is 11.

ii The 2nd quartile is the average of the 10th and 11th scores, which is 12.

iii The 3rd quartile is the average of the 15th and 16th scores, which is 13.

c **i** 20 scores, so the 1st quartile is the average of the 5th and 6th scores, which is 7.

ii The 2nd quartile is the average of the 10th and 11th scores, which is 8.

iii The 3rd quartile is the average of the 15th and 16th scores, which is 9.

d **i** 25 scores, so the 1st quartile is the average of the 6th and 7th scores, which is 2.

ii The 2nd quartile is the 13th score, which is 3.

iii The 3rd quartile is the average of the 19th and 20th scores, which is 5.

Question 2

40 scores, so reading from the $\frac{1}{4} \times 20 = 5$ on the cumulative frequency axis,

the 1st quartile is 2 (on the Score axis).

Reading from the $\frac{3}{4} \times 20 = 15$ on the cumulative frequency axis,

the 3rd quartile is $\frac{4+5}{2} = 4.5$.

Question 3

Score	Frequency	Cumulative frequency
23	13	13
24	19	32
25	23	55
26	21	76
27	9	85
28	15	100

a 100 scores, so the 23rd percentile is the 23rd score, which is 24.

The 55th percentile is the 55th score, which is 25.

The 91st percentile is the 91st score, which is 28.

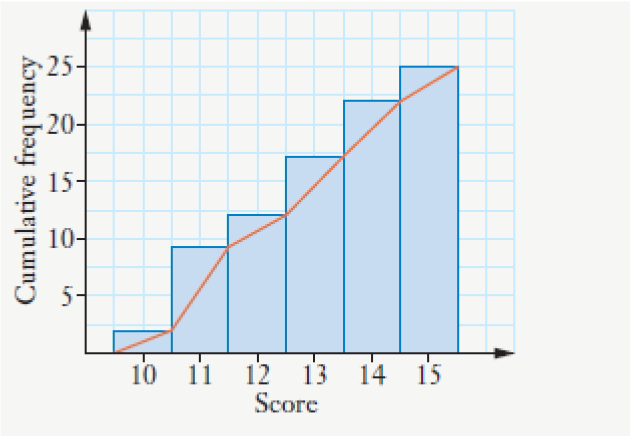
b 100 scores, so the 2nd decile is the $\frac{2}{10} \times 100 = 20$th score, which is 24.

The 8th decile is the $\frac{8}{10} \times 100 = 80$th score, score, which is 27.

Question 4

a

Score	Frequency	Cumulative frequency
10	2	2
11	7	9
12	3	12
13	5	17
14	4	21
15	4	25



b **i** 25 scores, so reading from the $\frac{1}{4} \times 25 = 6.25$ on the cumulative frequency axis, the 1st quartile is 11 (on the Score axis).

ii Reading from the $\frac{3}{4} \times 25 = 18.75$ on the cumulative frequency axis, the 3rd quartile is 14.

iii Reading from the $35\% \times 25 = 8.75$ on the cumulative frequency axis, the 35th percentile is 11.

iv Reading from the $\frac{7}{10} \times 25 = 17.5$ on the cumulative frequency axis, the 7th decile is 14.5.

v Reading from the $\frac{1}{10} \times 25 = 2.5$ on the cumulative frequency axis, the 1st decile is 11.

Question 5

Score	Class centre	Frequency	Cumulative frequency
30–34	32	1	1
35–39	37	9	10
40–44	42	8	18
45–49	47	5	23
50–54	52	2	27

a 27 scores, so the median is the 14th score, which is about 42.

b The 1st quartile is the $\frac{1}{4} \times 27 = 6.75$th score, which is about 37.

c The 3rd quartile is the $\frac{3}{4} \times 27 = 20.25$th score, which is about 47.

d The 60th percentile is the 60% × 27 = 16.2th score, which is about 44.

Question 6

Size	**Frequency**	**Cumulative frequency**
8	12	12
10	23	35
12	20	55
14	21	76
16	13	89
18	11	100

a 100

b $\frac{20}{100} \times 100\% = 20\%$

c 100 scores, so the median is the average of the 49th and 50th scores, which is 12.

d The 3rd quartile is the average of the 75th and 76th scores, which is 14.

e The first 14 is the 56th score, so it is the 56th percentile.

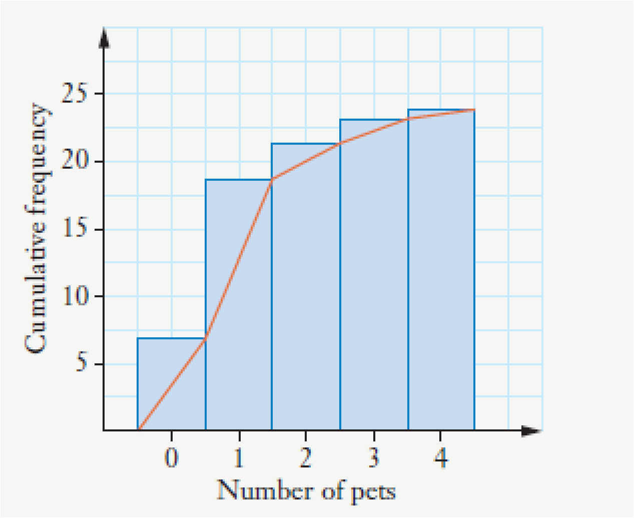
Question 7

Pets	Frequency	Cumulative frequency
0	7	7
1	11	18
2	3	21
3	2	23
4	1	24

a 3 people had 2 pets, total people surveyed = 24.

$$\frac{3}{24} \times 100\% = 12.5\%$$

b



c **i** 24 scores, so reading from the 12 on the cumulative frequency axis, the median is 1.

ii Reading from the $\frac{1}{4} \times 24 = 6$ on the cumulative frequency axis, the 1st quartile is 0.

iii Reading from the $\frac{3}{4} \times 24 = 18$ on the cumulative frequency axis, the 3rd quartile is $\frac{1+2}{2} = 1.5$.

Question 8

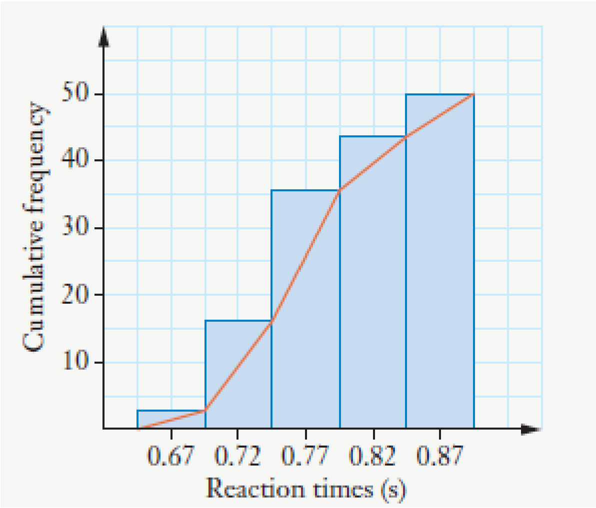
Time	Class centre (x)	Frequency (f)	fx	Cumulative frequency
0.65–0.69	0.67	2	1.34	2
0.70–0.74	0.72	14	10.08	16
0.75–0.79	0.77	19	14.63	35
0.80–0.84	0.82	8	6.56	43
0.85–0.89	0.87	7	6.09	50
		50	38.7	

a Mean = $\frac{38.7}{50} = 0.774$

b People with 0.75–0.79 = 19

Percentage of people with 0.75-0.79 = $\frac{19}{50} \times 100\% = 38\%$

c



d **i** 50 scores, so reading from the 30% × 50 = 15 on the cumulative frequency axis, the 30th percentile is 0.74.

ii Reading from the 25 on the cumulative frequency axis, the median is 0.77.

iii Reading from the $\frac{1}{4} \times 50 = 12.5$ on the cumulative frequency axis,

the 1st quartile is 0.72.

Reading from the $\frac{3}{4} \times 50 = 37.5$ on the cumulative frequency axis,

the 3rd quartile is 0.82.

So 0.72 to 0.82.

Question 9

a In order: 29, 36, 38, 45, 47, 51, 64, 72, 79, 83, 85

i Median = 51

ii Lower quartile = 38

iii Upper quartile = 79

b In order: 3, 4, 8, 9, 11, 12, 14, 15, 17

i Median = 11

ii Lower quartile = $\frac{4+8}{2}$ = 6

iii Upper quartile = $\frac{14+15}{2}$ = 14.5

c In order: 99.5, 103.7, 115.3, 125.4, 128.3, 137.5, 143.2, 154.6, 192.3, 203.4

i Median = $\frac{128.3+137.5}{2}$ = 132.9

ii Lower quartile = 115.3

iii Upper quartile = 154.6

d In order: 12, 14, 15, 15, 16, 17, 18, 19

i Median = $\frac{15+16}{2}$ = 15.5

ii Lower quartile = $\frac{14+15}{2}$ = 14.5

iii Upper quartile = $\frac{17+18}{2}$ = 17.5

Exercise 7.05 Range and interquartile range

Question 1

a Range = 19 – 3 = 16

b Range = 99 – 28 = 71

c Range = 126 – 87 = 39

d Range = 13 – 8 = 5

Question 2

a **i** 30 scores, so median is the 15th score. Reading from 15 on the cumulative frequency axis gives median = $\frac{6+7}{2}$ = 6.5

ii Range = 10 – 5 = 5

iii Q_1 is the $\frac{1}{4}$ × 30 = 7.5th score. Reading from 7.5 on the cumulative frequency axis gives Q_1 = 5.

Q_3 is the $\frac{3}{4}$ × 30 = 22.5th score. Reading from 22.5 on the cumulative frequency axis gives Q_3 = 9.

Interquartile range = 9 – 5 = 4

b **i** 25 scores, so median is the 12.5th score. Reading from 12.5 on the cumulative frequency axis gives median = 2

ii Range = 5 – 0 = 5

iii Q_1 is the $\frac{1}{4}$ × 25 = 6.25th score. Reading from 6.25 on the cumulative frequency axis gives Q_1 = 1.

Q_3 is the $\frac{3}{4}$ × 25 = 18.75th score. Reading from 18.75 on the cumulative frequency axis gives Q_3 = 2.

Interquartile range = 2 – 1 = 1

c **i** 8 scores, so median is the 4th score. Reading from 4 on the cumulative frequency axis gives median = $\frac{60+70}{2}$ = 65

ii Range = 90 – 50 = 40

iii Q_1 is the $\frac{1}{4}$ × 8 = 2nd score. Reading from 2 on the cumulative frequency axis gives Q_1 = 60.

Q_3 is the $\frac{3}{4}$ × 8 = 6th score. Reading from 6 on the cumulative frequency axis gives Q_3 = 80.

Interquartile range = 80 – 60 = 20

d **i** 14 scores, so median is the 7th score. Reading from 7 on the cumulative frequency axis gives median = 2

ii Range = 7 – 1 = 6

iii Q_1 is the $\frac{1}{4}$ × 14 = 3.5th score. Reading from 3.5 on the cumulative frequency axis gives Q_1 = 1.

Q_3 is the $\frac{3}{4}$ × 14 = 10.5th score. Reading from 10.5 on the cumulative frequency axis gives Q_3 = 3.

Interquartile range = 3 – 1 = 2

Question 3

a **i** Median = 7

ii Range = 9 – 4 = 5

iii IQR = 8 – 6 = 2

b **i** Median = 5

ii Range = 8 – 3 = 5

iii IQR = 6 – 4 = 2

c **i** Median = 17

ii Range = 20 – 11 = 9

iii IQR = 19 – 15 = 4

d **i** Median = 14

ii Range = 20 – 11 = 9

iii IQR = 16 – 11 = 5

e **i** Median = 6

ii Range = 8 – 1 = 7

iii IQR = 7 – 5 = 2

Question 4

Rain (mm) ***x***	**Frequency** ***f***	***fx***	**Cumulative frequency**
5	4	20	4
6	7	42	11
7	8	56	19
8	3	24	22
9	3	27	25
Total	25	169	

a Mean = $\frac{169}{25}$ = 6.76

b Mode = 7 (highest frequency)

c 25 scores, median is the 13th score, 7

d Range = 9 – 5 = 4

e Q_1 is the average of 6th and 7th scores, 6.

Q_3 is the average of 19th and 20th scores, $\frac{7+8}{2}$ = 7.5

IQR = 7.5 – 6 = 1.5

Question 5

a In order: 32, 51, 53, 54, 66, 76, 80, 93, 97, 100

$Q_1 = 53$, $Q_3 = 93$, IQR = 93 – 53 = 40.

Is 32 an outlier?

$Q_1 - 1.5\text{ IQR} = 53 - 1.5 \times 40 = 23$

32 is not less than 23 so it is not an outlier.

b In order: 1, 2, 3, 4, 5, 5, 5, 5, 6, 6, 7, 7, 7, 10, 11, 11, 19

$Q_1 = \dfrac{4+5}{2} = 4.5$, $Q_3 = \dfrac{7+10}{2} = 8.5$, IQR = 8.5 – 4.5 = 4.

Is 19 an outlier?

$Q_3 + 1.5\text{ IQR} = 8.5 + 1.5 \times 4 = 14.5$

19 is more than 14.5 so it is an outlier.

c In order: 1, 5, 5, 5, 5, 5, 6, 6, 6, 6

$Q_1 = 5$, $Q_3 = 6$, IQR = 6 – 5 = 1.

Is 1 an outlier?

$Q_1 - 1.5\text{ IQR} = 5 - 1.5 \times 1 = 3.5$

1 is less than 3.5 so it is an outlier.

Exercise 7.06 Variance and standard deviation

Question 1

a **i** Mean 5.4

ii Standard deviation 2.1

b **i** Mean 52.5

ii Standard deviation 14.6

c **i** Mean 123.3

ii Standard deviation 16.1

d **i** Mean 6.3

ii Standard deviation 1.8

e **i** Mean 17.6

ii Standard deviation 1.96

Question 2

a **i** Standard deviation 7.2

ii Variance 52.1

b **i** Standard deviation 1.96

ii Variance 3.8

c **i** Standard deviation 13.5

ii Variance 183.2

d **i** Standard deviation 14

ii Variance 196.6

e **i** Standard deviation 2.3

ii Variance 5.2

Question 3

a	**i**	Mean 8.4
	ii	Standard deviation 2.4
	iii	Variance 5.8
b	**i**	Mean 4.4
	ii	Standard deviation 1.7
	iii	Variance 2.9
c	**i**	Mean 34.1
	ii	Standard deviation 1.5
	iii	Variance 2.39
d	**i**	Mean 51.2
	ii	Standard deviation 14.9
	iii	Variance 222

Question 4

Rank	Frequency	Cumulative frequency
1	5	5
2	11	16
3	18	34
4	21	55
5	9	64

a Range = 5 – 1 = 4

b 64 scores, so $Q_1 = \frac{1}{4} \times 64$ = average of 16th and 17th score $\frac{2+3}{2}$ = 2.5.

$Q_3 = \frac{3}{4} \times 64$ = average of 48th and 49th score = 4.

Interquartile range = 4 – 2.5 = 1.5.

c Standard deviation = 1.14.

Question 5

a **i** Mean = 73.4

ii Standard deviation = 16.3

b Yes; 1 score in 20–29 class.

i Mean = 74.8

ii Standard deviation = 14.4

Question 6

a In order: 17, 18, 19, 20, 20, 23, 25, 25, 27, 29, 29, 30, 31, 31, 34, 53.

$Q_1 = 20$, $Q_3 = \frac{30+31}{2} = 30.5$, IQR = 30.5 – 20 = 10.5

Is 53 an outlier?

Q_3 + 1.5 IQR = 30.5 + 15.75 = 46.25

53 > 46.25, so 53 is an outlier.

b **i** Standard deviation = 8.4

ii Standard deviation without 53 = 5.3

Question 7

a Mean = 9.2

b Standard deviation = 3.5

c Variance = 12.6

Exercise 7.07 Shape and modality of data sets

Question 1

a Positively skewed, unimodal

b Symmetrical, unimodal

c Positively skewed, multimodal

d Negatively skewed, unimodal

e Positively skewed, unimodal

f Bimodal

g Negatively skewed, unimodal

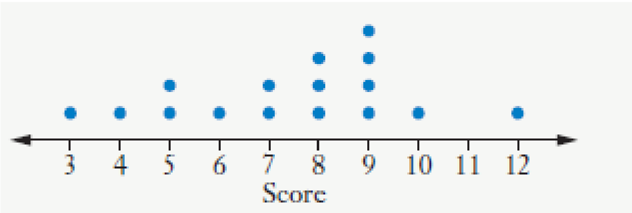
h Bimodal

i Positively skewed, unimodal

j Multimodal

Question 2

a



b Positively skewed

Question 3

a Positively skewed (tail at the higher scores), unimodal

b Symmetrical (middle scores have higher frequency), unimodal

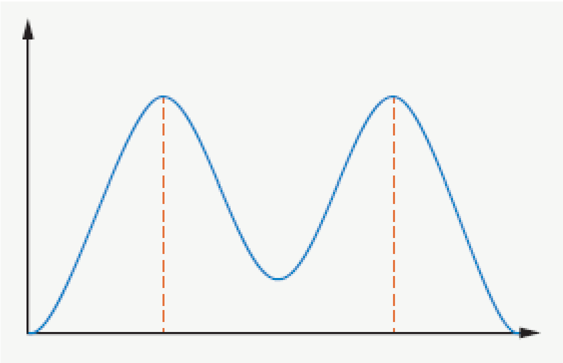
c Bimodal

d Negatively skewed (tail at the lower scores), unimodal

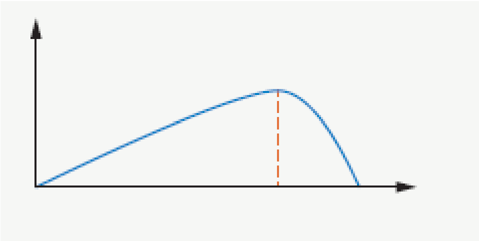
e Symmetrical (middle scores have higher frequency), unimodal

Question 4

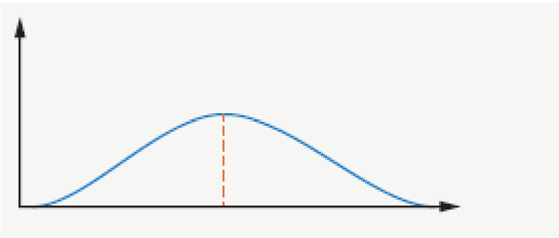
a



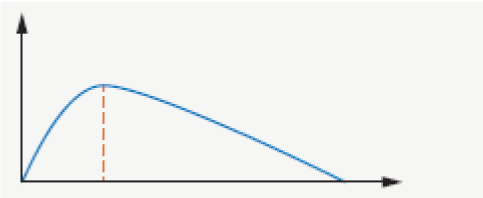
b



c



d



Question 5

Set 1: bimodal

Set 2: positively skewed, unimodal

Question 6

a Plot 1: positively skewed

Plot 2: negatively skewed

b 8 – 3 = 5

c Plot 1 IQR = 5 – 2 = 3

Plot 2 IQR = 10 – 6 = 4

Difference = 4 – 3 = 1

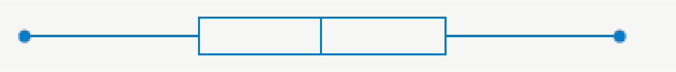
Question 7

a

Score	Class centre	Frequency
145–149	147	3
150–154	152	3
155–159	157	3
160–164	162	6
165–169	167	3
170–174	172	1
175–179	177	6
180–184	182	4
185–189	187	1

b Bimodal

Question 8



Question 9

x	*f*	*fx*	Cumulative frequency
4	3	12	3
5	5	25	8
6	6	36	14
7	9	63	23
8	6	48	29
9	5	45	34
10	3	30	37
Total	37	259	

a **i** Mean $= \frac{259}{37} = 7$

ii 37 scores, so the middle score is the 19th score, 7

iii Mode = 7 (highest frequency)

b Symmetrical (middle scores have higher frequency)

Question 10

Class discussion

Exercise 7.08 Analysing data sets

Question 1

a **i** Highest score = 7

ii Highest score = 4

b **i** Median = 2

ii Median = 2

c **i** Interquartile range = 3 – 0 = 3

ii Interquartile range = 3 – 1 = 2

d Class discussion

Question 2

a

Sample 1	Stem	Sample 2
9 8 7	14	
9 8 7 5 3 0	15	1 7 7 9
9 7 6 4 2 0	16	2 4 5 6 8
7 6 2 1 0	17	2 3 3 6 8 9
	18	0 1 1 1 2

b Sample 1: 20 scores, so median is the average of the 10th and 11th scores,

$\frac{160+162}{2} = 161.$

Sample 2: 20 scores, so median is the average of the 10th and 11th scores,

$\frac{172+173}{2} = 172.5$

c Sample 1 range = 177 – 147 = 30

Sample 2 range = 182 – 151 = 31

d Class discussion

Question 3

a

Bank 1

Score	Class centre	Bank 1	fx	Cumulative frequency
0–2	1	29	29	29
3–5	4	38	152	67
6–8	7	15	105	82
9–11	10	9	90	91
12–14	13	5	65	96
15–17	16	3	48	99
18–20	19	1	19	100
	Total	100	508	

i Mean = $\frac{508}{100}$ = 5.08

ii 100 scores, so the median score is the average of the 49th and 50th score, 4.

Bank 2

Score	**Class centre**	**Bank 2**	fx	**Cumulative frequency**
0–2	1	59	59	59
3–5	4	26	104	85
6–8	7	12	84	97
9–11	10	3	30	100
12–14	13	0	0	100
15–17	16	0	0	100
18–20	19	0	0	100
	Total	100	277	

i Mean = $\frac{277}{100}$ = 2.77

ii 100 scores, so the median score is the average of the 49th and 50th score, 1.

Bank 3

Score	Class centre	Bank 3	*fx*	Cumulative frequency
0–2	1	2	2	2
3–5	4	8	32	10
6–8	7	11	77	21
9–11	10	21	210	42
12–14	13	28	364	70
15–17	16	20	320	90
18–20	19	10	190	100
	Total	100	1195	

i Mean = $\frac{1195}{100}$ = 11.95

ii 100 scores, so the median score is the average of the 49th and 50th score, 13.

b Bank 1: 4.1, Bank 2: 2.4, Bank 3: 4.4

c Class discussion

Question 4

a **Class 1:** 54, 63, 71, 72, 74, 78, 80, 85, 85, 86, 89, 91, 91, 92

Median = $\frac{80+85}{2}$ = 82.5

Q_1 = 72

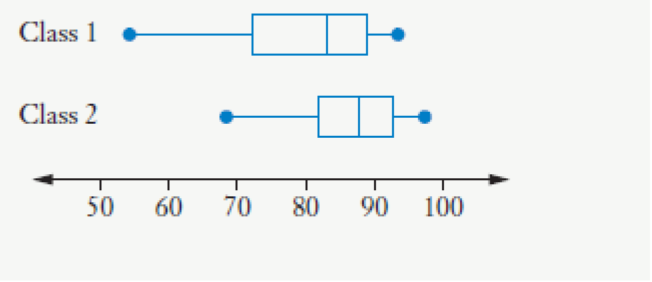
Q_3 = 89

Class 2: 69, 71, 79, 84, 85, 87, 88, 88, 89, 91, 93, 94, 97

Median = 88

$Q_1 = \frac{79+84}{2} = 81.5$

$Q_3 = \frac{91+93}{2} = 92$



b **i** Class 1 median = 82.5, Class 2 median = 88

ii Class 1 IQR = 89 – 72 = 17, Class 2 IQR = 92 – 81.5 = 10.5

iii Class 1 mean = $\frac{1111}{14} \approx 79.4$, Class 2 mean = $\frac{1115}{13} \approx 85.8$

iv Class 1 range = 92 – 54 = 38, Class 2 range = 97 – 69 = 28

Question 5

a

Camera 1		Camera 2
9 9 9 6 5	6	
7 6 5 4 4 3 3 2	7	
9 5 4 3 0	8	
0	9	
3	10	
	11	3 6 8 9 9
	12	0 0 2 2 3 4 5
	13	0 0 1 5 5 8
	14	0 2

b Camera 1 mean = $\frac{1546}{20}$ = 77.3 km/h, Camera 2 mean = $\frac{2522}{20}$ = 126.1 km/h

c Camera 1: 60 km/h, Camera 2: 110 km/h

Question 6

a Year 1 mean = $\frac{294}{6} = 49$

Year 2 mean = $\frac{399}{6} = 66.5$

Difference in means = 66.5 – 49 = 17.5

b Year 1 in order: 35, 41, 48, 53, 56, 61; median = $\frac{48+53}{2} = 50.5$

Year 2 in order: 58, 59, 67, 68, 73, 74; median = $\frac{67+68}{2} = 67.5$

Difference in medians = 67.5 – 50.5 = 17

c Year 1 range = 61 – 35 = 26

Year 2 range = 74 – 58 = 16

Difference in ranges = 26 – 16 = 10

d Year 1 standard deviation = 8.85

Year 2 standard deviation = 6.18

Difference in standard deviations = 8.85 – 6.18 = 2.67

Question 7

a **i** Mean = $\frac{39}{7} \approx 5.57$

ii Mean = $\frac{41}{7} \approx 5.86$

b **i** 3.16

ii 1.25

c The average test score is similar in both tests, but test 1 has a wider spread of scores.

Question 8

a Parents: 2, Children: 4

b **i** Parents: 4 – 0 = 4, Children 5 – 2 = 3

ii Parents: 5 – 0 = 5, Children: 6 – 0 = 6

c 6 (by children)

Question 9

a Science: 20 scores, median is the average of the 10th and 11th scores = $\frac{68+72}{2}$ = 70

English: 14 scores, median is the average of the 7th and 8th scores = $\frac{62+65}{2}$ = 63.5

b Science: mean = $\frac{1396}{20}$ = 69.8, English: mean = $\frac{963}{14}$ ≈ 68.8

c Science: standard deviation ≈ 12.8, English: standard deviation ≈ 13.4

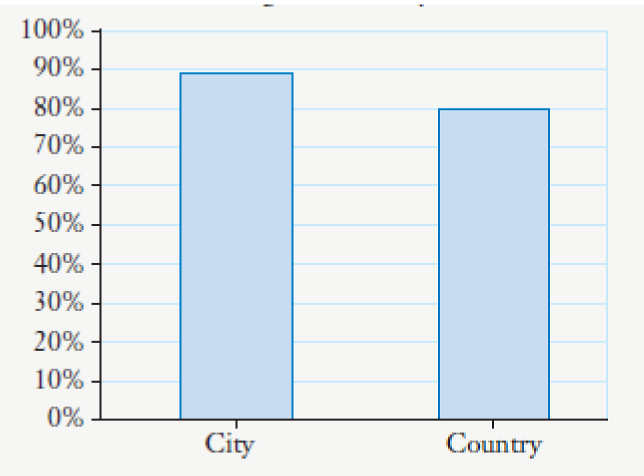
d Science: range = 94 – 50 = 44, English: range = 97 – 43 = 54

Difference in ranges = 54 – 44 = 10

Question 10

a The graph makes it look as if city access is about 3 times more than country access.

b



Question 11

Test yourself 7

Question 1

B

Temperature is a number and is measured on a smooth scale: numerical continuous.

Question 2

C

Negative skewed: tail points to the left.

Question 3

D

Colour is a category and is not ordered: categorical nominal.

Question 4

D

In order: 2, 3, 3, 4, 6, 7, 8, 9

Median = $\frac{4+6}{2}$ = 5

Question 5

D

The mean and range are most affected by outliers.

Question 6

A

The centres (means) are the same and test 2 has a larger spread (standard deviation) than test 1.

Question 7

In order: 4, 4, 5, 5, 6, 6, 6, 7, 7, 8, 8, 8, 8, 8, 9

Mode = 8 (most frequent)

Median = 7

Range = 9 – 4 = 5

Question 8

a Modal class = 5–9 (most frequent)

b

x	**Class centre**	f	fx	**Cumulative frequency**
0–4	2	6	12	6
5–9	7	8	56	14
10–14	12	4	48	18
15–19	17	7	119	25
20–24	22	3	66	28
	Totals	28	301	

Mean = $\frac{301}{28}$ = 10.75

c 28 scores, so the median is the average of the 14th and 15th scores, $\frac{7+12}{2}$ = 9.5

d Standard deviation = 6.6

Variance = 43.97

Question 9

Mean 55.7, standard deviation 23.5

Question 10

a Median = 10

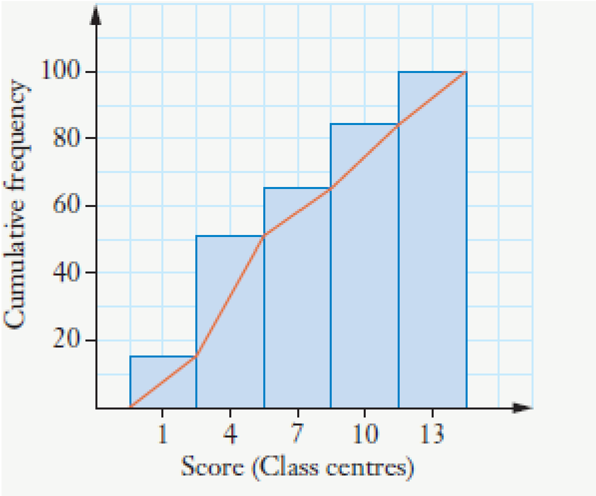
b Range = 16 – 5 = 11

c $Q_3 = 14$

d IQR = 14 – 8 = 6

Question 11

a



b 100 scores, so the median score is the 50th score. Reading from 50 on the cumulative frequency axis gives median = $\frac{4+7}{2}$ = 5.5

c **i** 100 scores, so the 20th percentile is the 20% × 100 = 20th score. Reading from 20 on the cumulative frequency axis gives a 20th percentile of about 3.

ii The 3rd quartile is the $\frac{3}{4}$ × 100 = 75th score. Reading from 75 on the cumulative frequency axis gives a 3rd quartile of about 10.

iii The 91st percentile is the 91% × 100 = 91st score. Reading from 91 on the cumulative frequency axis gives a 91st percentile of about 13.

iv The 3rd decile is the $\frac{3}{10}$ × 100 = 30th score. Reading from 30 on the cumulative frequency axis gives a 3rd decile of about 4.

v The 9th decile is the $\frac{9}{10}$ × 100 = 90th score. Reading from 90 on the cumulative frequency axis gives a 9th decile of about 12.

Question 12

x	f	fx	**Cumulative frequency**
0	9	0	9
1	8	8	17
2	5	10	22
3	1	3	23
4	3	12	26
5	2	10	28
	28	43	

a Mean = $\frac{43}{28} \approx 1.5$

b 28 scores, so the median is the average of the 14th and 15th scores, 1

c Range = 5 – 0 = 5

d 28 scores, so Q_1 is the $\frac{1}{4} \times 28 = 7$th score, 0, and Q_3 is the $\frac{3}{4} \times 28 = 21$st score, 1.

IQR = 1 – 0 = 1.

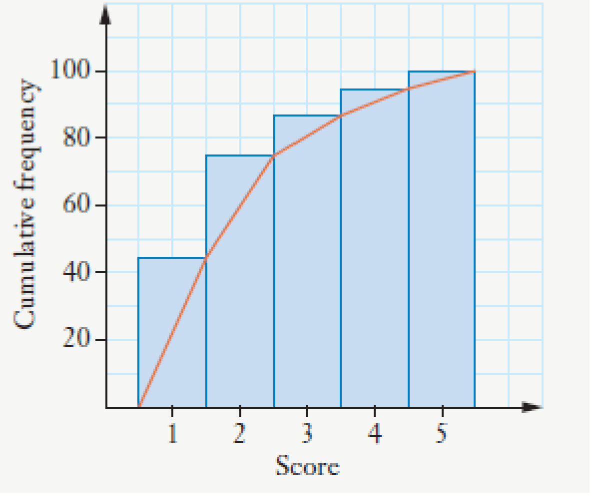
Question 13

Mode, because it's describing the most frequent or popular score.

Question 14

a

x	f	**Cumulative frequency**
1	43	43
2	32	75
3	12	87
4	8	95
5	5	100



b **i** 100 scores, so the median is the 50th score. Reading from 50 on the cumulative frequency axis, median = 2

ii 100 scores, so Q_1 is the $\frac{1}{4} \times 100 = 25$th score. Reading from 25 on the cumulative frequency axis, $Q_1 = 1$.

Q_3 is the $\frac{3}{4} \times 100 = 75$th score. Reading from 75 on the cumulative frequency axis, $Q_3 = \frac{2+3}{2} = 2.5$.

IQR = 2.5 – 1 = 1.5.

Question 15

a Mean $= \dfrac{182}{24} \approx 7.6$

b Standard deviation = 1.35

c Mode = 6 (most frequent score)

Question 16

In order: 40, 40, 41, 42, 43, 44, 45, 45, 48, 48, 49, 49, 49

Mean $= \dfrac{583}{13} \approx 44.8$

Mode = 49 (most frequent)

Median = 45.

Question 17

a 50 scores, so Q_1 is the $\dfrac{1}{4} \times 50 = 12.5$th score. Reading from 12.5 on the cumulative frequency axis, $Q_1 = 2$.

Q_3 is the $\dfrac{3}{4} \times 50 = 37.5$th score. Reading from 37.5 on the cumulative frequency axis, $Q_3 = 5$.

IQR = 5 – 2 = 3.

b 50 scores, so the median is the 25th score. Reading from 25 on the cumulative frequency axis, the median is 4.

Question 18

a In order: 2, 11, 11, 12, 12, 13, 14, 14, 14, 15, 15, 16, 16, 17, 18, 18, 18, 19, 19

i Range = 19 – 2 = 17

ii Mean = $\frac{274}{19} \approx 14.4$

iii Mode = 14, 18 (highest frequency)

iv Median = 15

b Is 2 an outlier?

$Q_1 = 12$, $Q_3 = 18$, IQR = 18 – 12 = 6.

$Q_1 - 1.5$ IQR $= 12 - 1.5 \times 6 = 3$

2 < 3, so 2 is an outlier

c In order: 11, 11, 12, 12, 13, 14, 14, 14, 15, 15, 16, 16, 17, 18, 18, 18, 19, 19

i Range = 19 – 11 = 8

ii Mean = $\frac{272}{18} \approx 15.11$

iii Mode = 14, 18 (highest frequency)

iv Median = $\frac{15+15}{2} = 15$

d The outlier affects the range and mean only.

Question 19

a Total microchipped animals = 2084

i Percentage female = $\frac{1137}{2084} \times 100\% \approx 54.6\%$

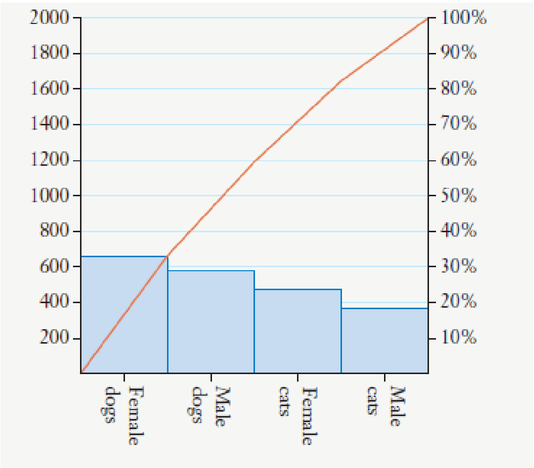
ii Percentage female cats = $\frac{473}{2084} \times 100\% \approx 22.7\%$

iii Percentage male dogs = $\frac{578}{2084} \times 100\% \approx 27.7\%$

iv Percentage dogs = $\frac{1242}{2084} \times 100\% \approx 59.6\%$

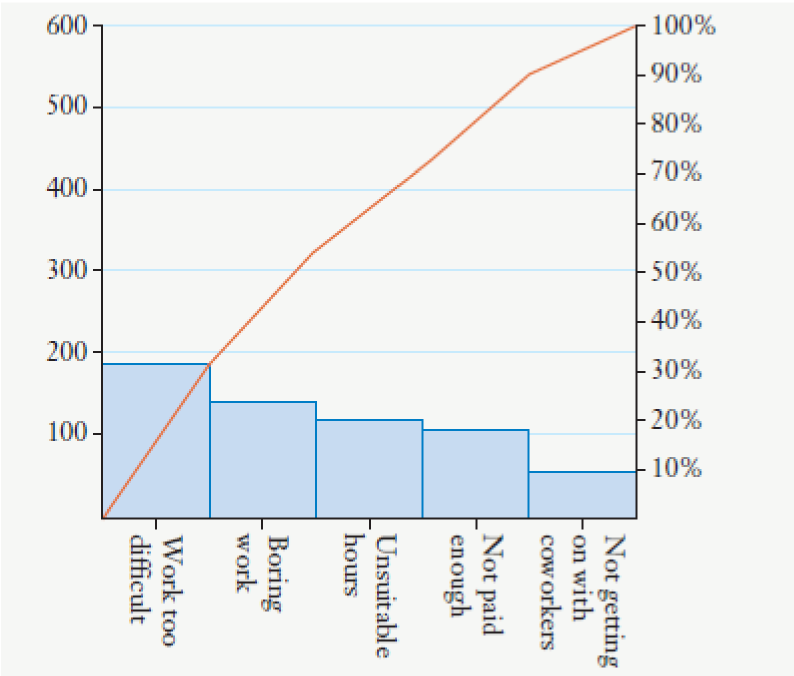
b

Category	Frequency	Percentage frequency	Cumulative percentage frequency
Female dogs	664	$\frac{664}{2084} \times 100\% \approx 32\%$	32%
Male dogs	578	$\frac{578}{2084} \times 100\% \approx 28\%$	60%
Female cats	473	$\frac{473}{2084} \times 100\% \approx 23\%$	83%
Male cats	369	$\frac{369}{2084} \times 100\% \approx 18\%$	100%
	2084		



Question 20

Reason	Frequency	Percentage frequency	Cumulative percentage frequency
Work too difficult	185	$\frac{185}{600} \times 100\% \approx 31\%$	31%
Boring work	139	$\frac{139}{600} \times 100\% \approx 23\%$	54%
Unsuitable hours	116	$\frac{116}{600} \times 100\% \approx 19\%$	73%
Not paid enough	104	$\frac{104}{600} \times 100\% \approx 17\%$	90%
Not getting on with co-workers	56	$\frac{56}{600} \times 100\% \approx 9\%$	100%
	600		



Question 21

a North has 14 scores, so the median is the average of the 7th and 8th scores.

$$\text{Median} = \frac{47+48}{2} = 47.5$$

West has 19 scores, so the median is the 10th score.

Median = 43

b North region: Mean 46.5, standard deviation 7.8

c West region: Mean 43.4, standard deviation 10.7

d Slightly more mushrooms found in the North region on average, spread higher in West region so results more variable.

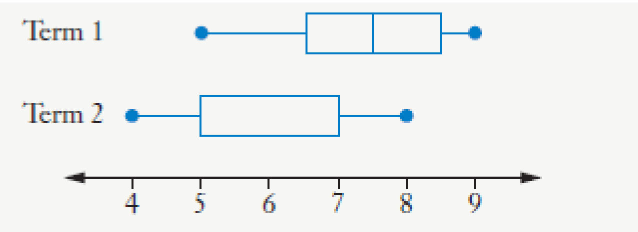
Question 22

a Term 1 in order: 5, 5, 6, 7, 7, 7, 8, 8, 8, 9, 9, 9

$$Q_1 = \frac{6+7}{2} = 6.5,\ Q_2 = \frac{7+8}{2} = 7.5,\ Q_3 = \frac{8+9}{2} = 8.5$$

Term 2 in order: 4, 4, 5, 5, 5, 5, 6, 6, 7, 7, 8

$Q_1 = 5$, $Q_2 = 5$, $Q_3 = 7$



b Median: Term 1: 7.5, Term 2: 5

c IQR: Term 1: 8.5 – 6.5 = 2, Term 2: 7 – 5 = 2

d Term 1: Mean 7.3, standard deviation 1.4

Term 2: Mean 5.6, standard deviation 1.2

e Students did better on average in Term 1.

Both assessments had a similar spread of results.

Challenge exercise 7

Question 1

a Bimodal. Female heights may have their own mode and male heights have their own (higher) mode

b Mean 168.3, variance 92.2

Question 2

Let the missing frequency be a.

x	f	fx
2	2	4
3	5	15
4	a	$4a$
5	3	15
6	2	12
7	4	28
Totals	$a + 16$	$4a + 74$

Mean = $\dfrac{4a+74}{a+16} = 4.5$

$4a + 74 = 4.5a + 72$

$2 = 0.5a$

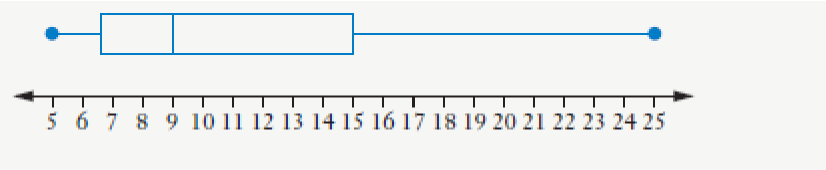
$a = 4$

The missing frequency is 4.

Question 3

a In order: 5, 6, 7, 8, 9, 11, 14, 16, 25

$Q_1 = \frac{6+7}{2} = 6.5$, $Q_2 = 9$, $Q_3 = \frac{14+16}{2} = 15$

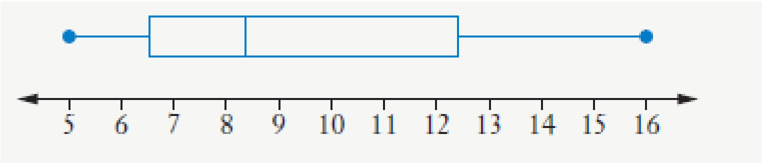


b Positively skewed (most scores on left, tail at right).

There is an outlier at 25.

Taking out 25: 5, 6, 7, 8, 9, 11, 14, 16

$Q_1 = \frac{6+7}{2} = 6.5$, $Q_2 = \frac{8+9}{2} = 8.5$, $Q_3 = \frac{11+14}{2} = 12.5$



Without the outlier it is still slightly positively skewed, but it is more symmetrical.

Question 4

a **i** Mean = 15

ii Standard deviation = 5.66

b

Score *x*	$x - \bar{x}$	$(x - \bar{x})^2$
7	–8	64
11	–4	16
15	0	0
19	4	16
23	8	64
	$\Sigma(x - \bar{x}) = 0$	$\Sigma(x - \bar{x})^2 = 160$

c Standard deviation = $\sqrt{\dfrac{160}{5}} = \sqrt{32} = 5.65685\ \ldots \approx 5.66$

Question 5

Mean of 7 scores = 25.

Sum of 7 scores = 25 × 7 = 175

Sum of 8 scores = 175 + 28 = 203

New mean = $\dfrac{203}{8}$ = 25.375

Question 6

a Set B has a higher average or centre, and more consistent results (less spread out than set A).

b Set B has a higher average and is less consistent (more spread out).

Question 7

a Mean 6.04, standard deviation 1.41

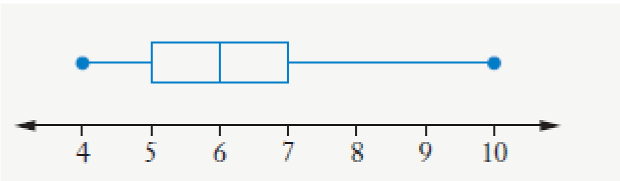
b

Score	Frequency	Cumulative frequency
4	7	7
5	11	18
6	15	33
7	9	42
8	4	46
9	2	48
10	1	49

49 scores, so the median is the 25th score, 6.

Q_1 is the average of the 12th and 13th scores, 5.

Q_3 is the average of the 37th and 38th scores, 7.



c Positively skewed

Question 8

a Looking at the differences between column heights, the differences are high for the low scores and low for the high scores. That means the frequencies are high on the left and low (tail) on the right, so the distribution is positively skewed.

b The differences are low for the low scores and high for the high scores. That means the frequencies are low (tail) on the left and high on the right, so the distribution is negatively skewed.

MATHS IN FOCUS 12 MATHEMATICS ADVANCED

WORKED SOLUTIONS

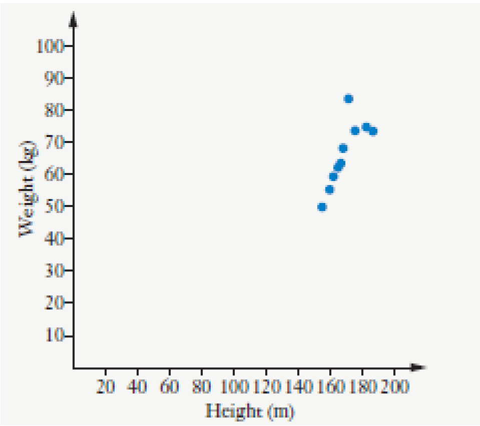
Chapter 8: Correlation and regression

Exercise 8.01 Bivariate data

Question 1

a The independent variable (height) is on the horizontal axis and the dependent variable (weight) on the vertical axis.

The graph can be drawn using graph paper, using a graphics calculator or a spreadsheet.



b The shape seems to be linear in a positive direction with perhaps one or two outliers.

The shape seems to follow a linear relationship (a straight line with positive gradient).

However, the data points (171.2, 83.6), (182.3, 74.8) and (186.4, 73.5) appear to be outliers.

c As height increases, weight increases.

Since there seems to be a linear relationship having a positive gradient, we can say that as height increases, weight also increases. That is, the rate of change of weight with respect to height is constant.

Question 2

a The scatterplot should have the independent variable (number of people in the family) on the horizontal axis and the dependent variable (number of bedrooms) on the vertical axis.

b The scatterplot may be non-linear, parabolic or linear (positive) with outliers.

There are not many data points, so a clear pattern is not evident.

One possibility is that the third to the sixth points follow a linear (positive) pattern and that the first, seventh and eight data points are outliers.

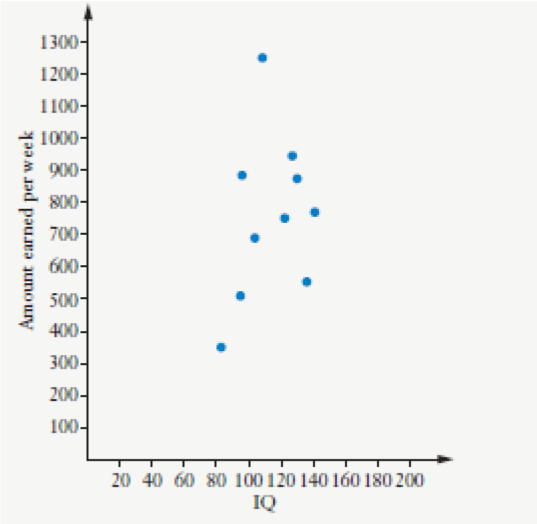
A second possibility is that the shape is parabolic, with the last two data points treated as outliers. This is plausible from the third point onwards if the first two points are thought of as outliers.

Finally, we can also consider the possibility that there is no pattern.

c If we assume a linear pattern with positive gradient, an increase in the number of family members requires a corresponding increase in the number of bedrooms.

Question 3

a On the scatterplot, display IQ (the independent variable) on the horizontal axis and the dependent variable (weekly earnings) on the vertical axis.



b The scatterplot is fairly unrelated, or linear (positive) with outliers.

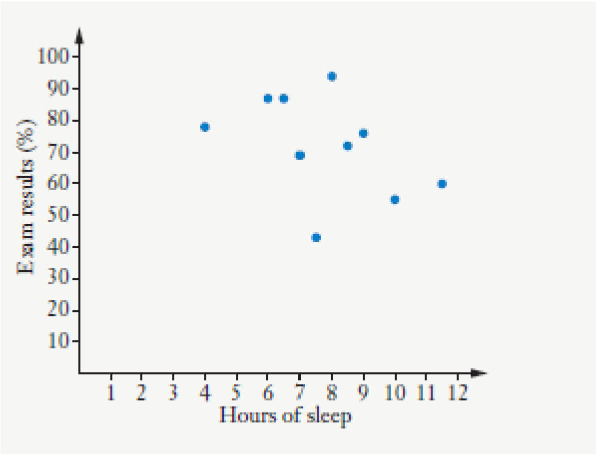
If we treat the points (96, 884), (109, 1250) and (136, 553) as outliers, the remaining points suggest a linear, positive relationship. If no outliers are included, then no obvious pattern is indicated.

c With some exceptions, the higher the IQ, the more money the person earns.

Allowing for some outliers, we conclude that a positive linear relationship means the higher the IQ, the higher will be their weekly income. Intuitively, this seems to be a reasonable claim. It suggests that smarter people make more money.

Question 4

a The scatterplot has the number of hours students sleep on the horizontal axis (the independent variable) and student exam results (the dependent variable) on the vertical axis.



b The scatterplot is unrelated.

There seems to be a slightly negative linear relationship, but if this is the case, it would be suggesting that the more hours students sleep, the lower will be their exam results. This seems to be counter-intuitive, unless we interpret it as meaning that students who spend more time sleeping instead of studying will perform worse in examinations.

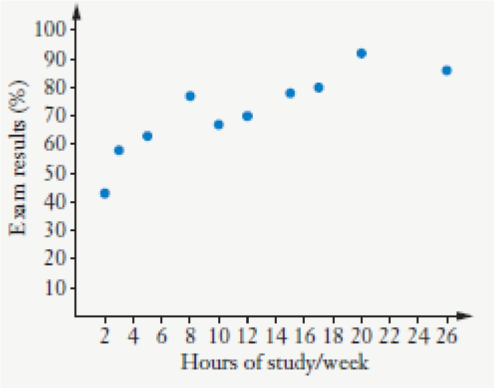
Otherwise, it appears that there is no pattern to the scatterplot.

c There is little or no correlation found between amount of sleep and exam results.

We would expect that more hours of sleep will enhance exam performance, but the scatterplot does not seem to bear this out. If we assume a positive linear model, then the relationship cannot be a strong one because of the wide spread of data.

Question 5

a On the scatterplot, the horizontal axis is the hours of weekly study (the independent variable) and the vertical axis (the dependent variable) represents the exam results.



b The shape is linear in a positive direction.

The shape shows a positive linear relationship with no clear outliers.

c As the number of hours of study increases, exam results increase.

A positive linear relationship indicates that as the number of hours of study time is increased, there is a corresponding increase in exam results. We may think of the gradient of the line as the improvement scale factor. For example, if the gradient is 2, each 1 hour increase in time spent studying will result in an extra 2% in exam results.

However, a linear relationship does not take into account that for some value of the independent variable, the dependent variable (exam result) will be greater than 100%, which is not possible.

Question 6

a Non-linear

If we consider the first, third last and second last data points as outliers, the relationship can be linear in the negative direction. However, for the number of data points (nine), treating three of them as outliers does not leave enough points to propose with confidence that there is a negative linear relationship.

Hence we can conclude that the relationship is non-linear.

b Linear in negative direction

Treating the fifth data point as an outlier, the relationship is seen to be linear in the negative direction.

c No pattern

The points appear randomly distributed, suggesting no relationship between the two variables.

d Linear in positive direction

The points seem to closely reflect that a straight line with positive gradient can be drawn through them. Hence the relationship between the variables is linear in the positive direction.

e No pattern or linear in negative direction with outliers

Without considering certain points to be outliers, the data are too randomly spread to represent a linear relationship. However, if we take the first and the last data points to be outliers, then the remaining values indicate a linear relationship in the negative direction.

f Non-linear

The pattern clearly shows a non-linear relationship, possibly quadratic, circular or trigonometric one.

g No pattern

The points appear randomly distributed, suggesting no relationship between the two variables, even if some points are considered to be outliers.

h Linear in positive direction

Even without omitting the two outliers, the scatterplot shows that a straight line with positive gradient can be drawn through them. The relationship between the variables is linear in the positive direction.

i Non-linear

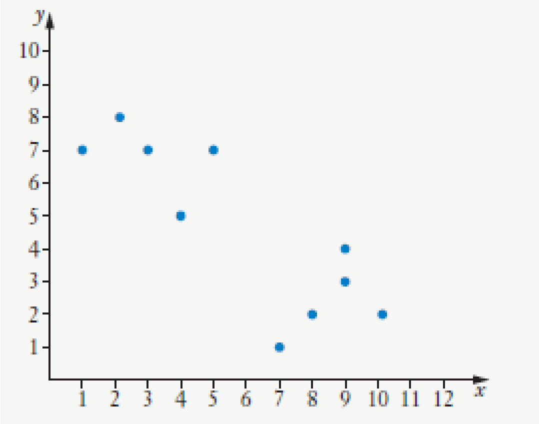
The scatterplot suggests three clustered groups, or two clustered groups if we treat the first three data points as outliers.

j Non-linear with outliers

No linear relationship is evident if all points are considered without outliers. Thus unrelated. If the bottom two data points are thought of as outliers, the remaining points suggest s non-linear relationship, possibly quadratic, circular or trigonometric one.

Question 7

a The independent variable is *x* and the dependent variable is *y*.



b The data is clustered.

There are two clusters. The first cluster has *x*-values 1 to 5 and *y*-values ranging from 5 to 8. The second cluster has *x*-values from 7 to 10 and *y*-values from 1 to 4.

Possible linear in a negative direction.

Question 8

a B

The scatterplot shows that as the independent variable increases, the dependent variable decreases at a near uniform rate. A linear relationship exists in the negative direction. Thus there is a negative linear correlation.

b C

A linear relationship does not exist for the data because the dependent variable values increase at a greater rate than the increase in the independent variable. The pattern appears to be exponential, quadratic or some other non-linear relationship.

c C

The first half of the data points follow a linear positive relationship and the second half follow a linear negative relationship. Hence the relationship spanning all the data is non-linear.

d D

The scatterplot displays points that appear randomly distributed, with no discernible pattern. Thus there does not seem to be any correlation between the independent and dependent variables.

e A

As the independent variable increases, the dependent variable also increases at an almost constant rate. A linear relationship exists in the positive direction, so there is a positive linear correlation.

f C

As the independent variable increases, the dependent variable decreases at a faster rate, so the relationship is not linear. Since some pattern is evident, there is a non-linear correlation.

g B

The scatterplot shows that the dependent variable decreases as the independent variable increases at close to a uniform rate. Hence a linear relationship exists in the negative direction, so there is a negative linear correlation.

h A

As the independent variable increases, the dependent variable also increases at an almost the same rate. A near perfect positive linear relationship exists, so there is a positive linear correlation.

i C

As the independent variable increases, the dependent variable decreases sharply at first and then reduces at a slower rate. This suggests an inverse relationship such as $y = \frac{1}{x}$, so there is a non-linear correlation.

j D

The points appear randomly distributed, with no discernible pattern. There is no correlation between the independent and dependent variables.

Exercise 8.02 Correlation

Question 1

a G

The pattern shows a linear negative relationship with most data points fairly close to where the intended straight line would be drawn. Hence there is a moderate negative correlation of about –0.7.

b B

The scatterplot shows the data points arranged in a straight line that would lie exactly on a line with positive gradient. Hence there is a perfect positive correlation of 1.

c F

The data points appear completely random, so there is no correlation. Hence the correlation coefficient is 0.

d A

The data indicates a negative relationship, with some points near to where the intended straight line would be drawn. We can describe it as a small (weak) negative correlation of about –0.5.

e C

The scatterplot displays data points that can lie close to a straight line with positive gradient. This can be described as a strong positive correlation of 0.8.

f D

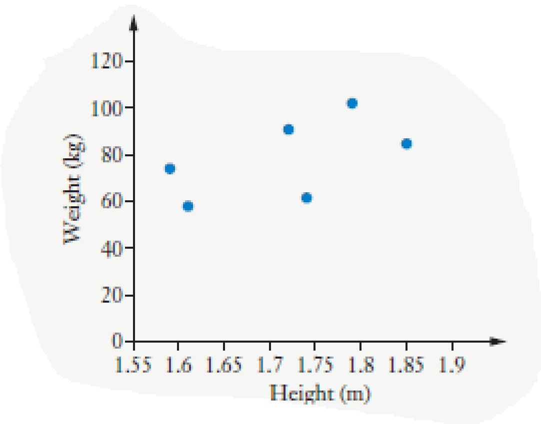
The data points lie exactly on a straight line with negative gradient. Hence there is a perfect negative correlation of –1.

Question 2

The correlation coefficient can be found using a variety of resources including a scientific calculator and graphing calculator.

Answers to this question are given using a spreadsheet because the scatterplot can be drawn at the same time. The function PEARSON(array1, array2) is used to calculate the correlation coefficient.

a i



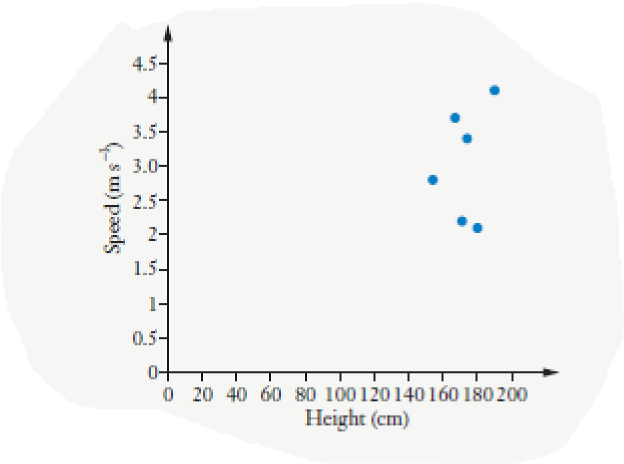
ii

	A	B	C	D	E	F	G
1	Height (m)	1.72	1.85	1.61	1.74	1.59	1.79
2	Weight (kg)	91.3	85.2	58.3	61.9	74.5	102.6
3							
4			Pearson correlation	0.57			

0.57

The correlation coefficient is 0.57. This is moderate positive, indicating a good linear relationship between height and weight. As the height increases, so does the weight.

b **i**



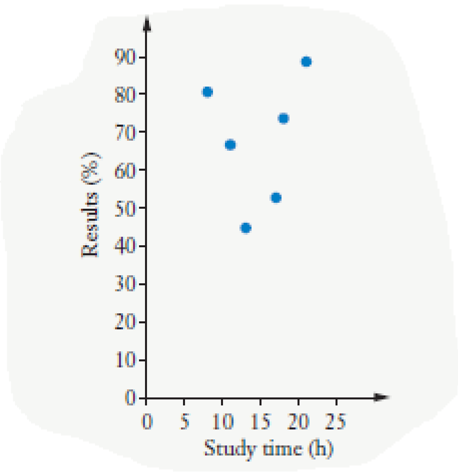
ii

	A	B	C	D	E	F	G
1	Height (cm)	167	180	174	171	154	190
2	Speed (m/s)	3.7	2.1	3.4	2.2	2.8	4.1
3							
4	Pearson correlation	0.28					

0.28

The correlation coefficient is 0.28. This is weak positive, indicating a small linear relationship between height and speed. There is little confidence in using the linear function for prediction.

c **i**



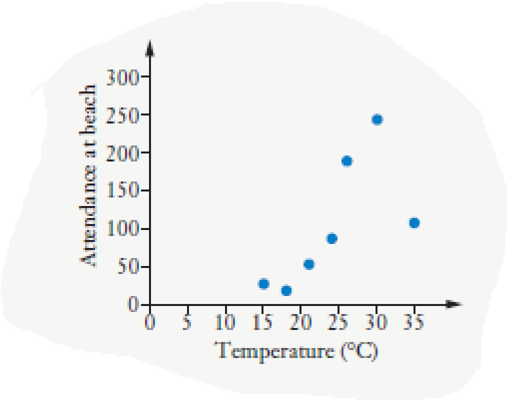
ii

	A	B	C	D	E	F	G
1	Study time (h)	13	21	8	11	18	17
2	Results (%)	45	89	81	67	74	53
3							
4			Pearson correlation	0.18			

0.18

The correlation coefficient is 0.18. This is weak positive, indicating a small linear relationship between study time and results. There is little confidence in using the linear function for prediction. However, in the scatterplot, if we treat the first two points as outliers then there will be a strong linear relationship. This suggests that study times of less than about 13 hours are counterproductive.

d i



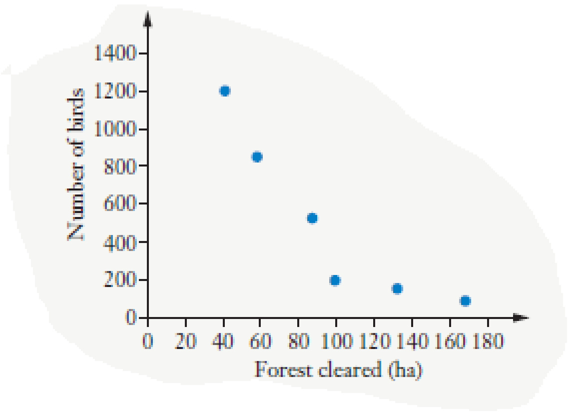
ii

	A	B	C	D	E	F	G	H
1	Temperature (^{0}C)	15	18	21	24	26	30	35
2	Attendance at beach	28	19	54	88	190	245	108
3								
4		Pearson correlation		0.68				

0.68

The correlation coefficient is 0.68. This is moderate positive, indicating a good linear relationship between temperature and beach attendance. There is confidence in using the linear function for prediction. We expect a positive correlation, since attendance generally increases with warmer weather.

e **i**



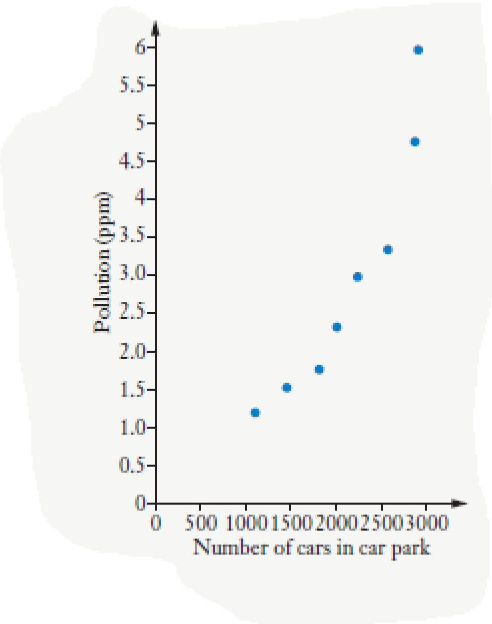
ii

	A	B	C	D	E	F	G
1	Forest cleared (ha)	41	58	87	99	132	168
2	Number of birds	1200	854	530	201	157	92
3							
4			Pearson correlation	-0.91			

–0.91

The correlation coefficient is –0.91. This is strong negative, indicating a good linear relationship between the two variables. That is, that a decrease in forest area produces a corresponding decrease in the bird population. We can confidently use the linear function to make predictions.

f **i**



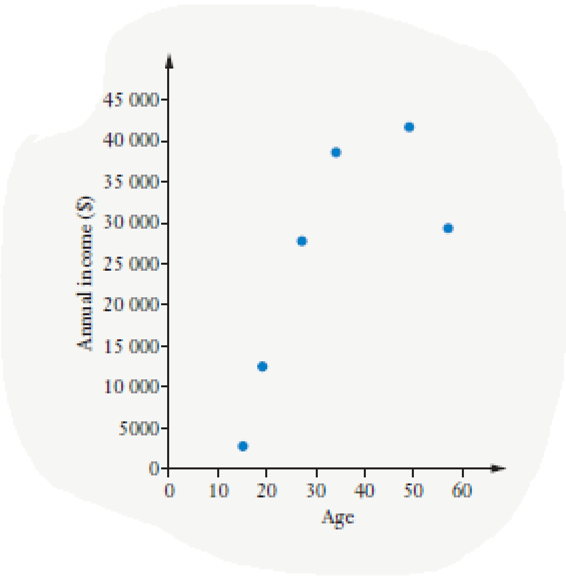
ii

	B	C	D	E	F	G	H	I
1	1100	1450	1809	2004	2234	2569	2871	2906
2	1.21	1.54	1.78	2.34	2.99	3.35	4.76	5.97
3								
4	Pearson correlation		0.93					

0.93

The correlation coefficient is 0.93. This is strong positive, indicating a strong linear relationship between the two variables. An increase in car numbers produces a corresponding increase in pollution.

g **i**



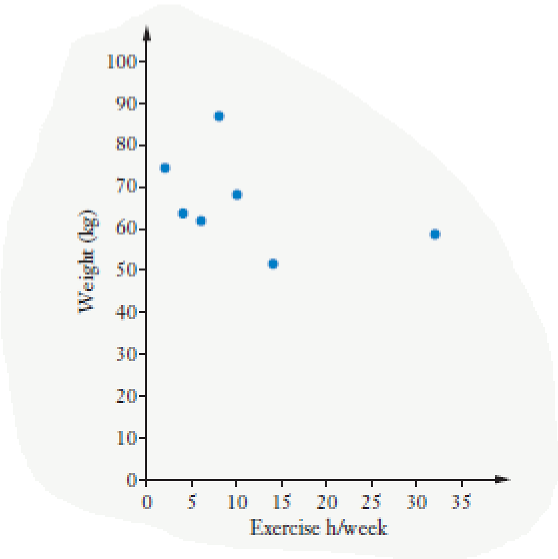
ii

	A	B	C	D	E	F	G
1	Age	15	19	27	34	49	57
2	Annual income	2.851	12600	27890	38740	41834	29450
3							
4	Pearson correlation			0.75			

0.75

The correlation coefficient is 0.75. This is moderate positive, indicating a good linear relationship between Age and Annual income. It suggests a good prediction can be made between a person's age and their income.

h **i**



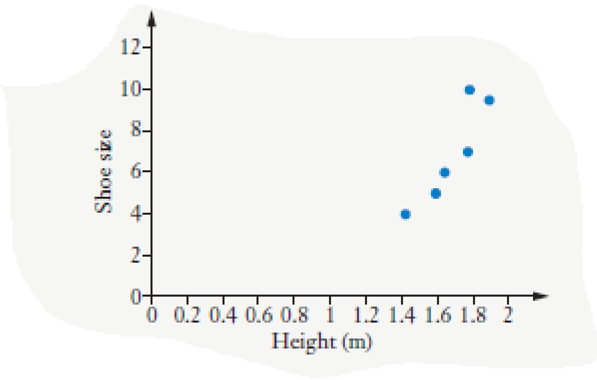
ii

	A	B	C	D	E	F	G	H
1	Exercise h/week	14	8	2	10	6	4	32
2	Weight (kg)	51.8	87.2	74.8	68.4	62.1	63.9	58.9
3								
4			Pearson correlation	-0.43				

–0.43

The correlation coefficient is –0.43. This is negative, indicating some degree of confidence in associating a linear relationship between Exercise and Weight. It suggests there is some degree of confidence in accepting that an increase in weekly exercise correspondingly reduces weight at the same rate.

i i



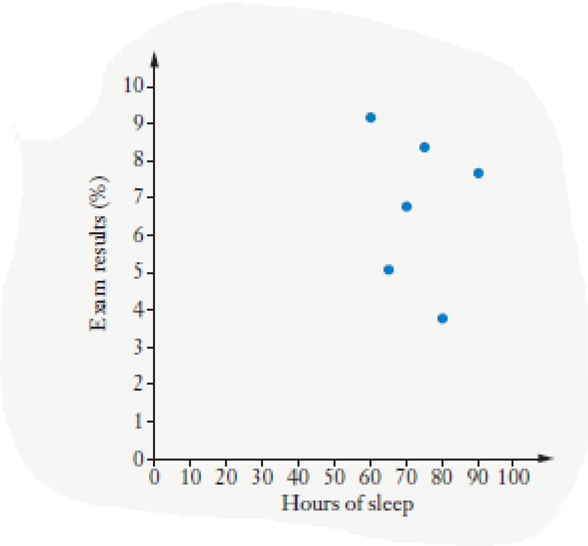
ii

	A	B	C	D	E	F	G
1	Height (m)	1.59	1.77	1.64	1.78	1.89	1.42
2	Shoe size	5	7	6	10	9.5	4
3							
4			Pearson correlation	0.90			

0.90

The correlation coefficient is 0.90. This is a high positive association, indicating a strong reliability in suggesting a linear relationship between Height and Shoe size, and for predicting the value of one variable given the value of the other variable.

j **i**



ii

	A	B	C	D	E	F	G
1	Exam results (%)	68	92	38	51	77	84
2	Hours of sleep	7	6	8	6.5	9	7.5
3							
4			Pearson correlation	-0.18			

–0.18

The correlation coefficient is –0.18. This is small negative, suggesting there is some linear relationship between the two variables. Specifically, that an increase in the percentage exam results is a consequence of a corresponding decrease in the number of hours of sleep.

Question 3

a No. A person may be tall and skinny, so their weight may be less than a shorter person.

Other factors such as muscle tone and bone structure come into play.

b No. It may be true that a taller person’s stride may give them some advantage, but other factors such as fitness, stamina and style are also considerations.

c No. Studying for longer periods of time may help, but it is not the only determining factor. For example, one may spend many hours attempting to understand the work without understanding it. Other issues such as the student’s performance throughout the year may also be significant.

d Yes. Generally, people go to the beach on hot days, although there are exceptions such as surfers and fishermen.

e Yes.

If the forest is the habitat for birdlife, then less forest area will restrict the food supply and nesting opportunities, as well as increase the likelihood of encounters with predators who also are restricted to search for food in a diminishing area.

f Yes. More cars in the car park will increase the amount of exhaust emissions, especially as many cars will be simultaneously looking for parking at low speeds.

g No. For the professional person, their career is usually mapped out in their twenties, after they have completed schooling. Income increment may then depend on the type of work (example, sales commission), type of employer (example, government or private) and how specialised/in demand the work is (example, surgeon). Granting salary increments simply on age is usually not practised by employers.

h No. The amount of exercise would have little impact on one’s weight. Overweight people may lose weight, but others aspiring for ultimate fitness, such as body builders, will gain weight by promoting muscle development through eating more protein-based food.

i Yes. Taller people tend to have proportionally longer feet, so their shoe size will be higher. This is illustrated by the strong correlation of 0.9.

j No.

Question 4

For each question, a spreadsheet was used to enter the independent variable in column A and the dependent variable in column B.

However, scientific or graphics calculators will also provide accurate results.

The function PEARSON(array1, array2) is used to calculate the correlation coefficient.

The range of the arrays changes for each question.

a 0.40

	A	B
1	x	y
2	3	7
3	5	9
4	4	3
5	11	7
6	15	12
7	8	4
8	9	1
9		
10	correlation	0.40

In cell B10 enter the formula = PEARSON(A2:A8, B2:B8)

b –0.39

	A	B
1	x	y
2	5	67
3	6	49
4	3	81
5	9	23
6	11	55
7	8	91
8	4	61
9		
10	correlation	-0.39

In cell B10 enter the formula = PEARSON(A2:A8, B2:B8)

c 0.99

	A	B
1	x	y
2	8	11
3	4	8
4	7	11
5	2	4
6	9	12
7	14	16
8	23	23
9		
10	correlation	0.99

In cell B10 enter the formula = PEARSON(A2:A8, B2:B8)

d –0.60

	A	B
1	x	y
2	5	21
3	3	28
4	6	19
5	5	17
6	9	21
7	4	26
8	11	15
9	15	*18*
10	9	12
11		
12	correlation	-0.60

In cell B12 enter the formula = PEARSON(A2:A10, B2:B10)

Question 5

a Yes

We can take population to be the independent variable and pollution to be the dependent variable.

It is reasonable to think that an increase in a city's population will result in more pollution because of the extra demand on resources and services. For example, more cars on the road will mean more exhaust fumes, more litter on the roads, an increase in sewerage running into rivers, more household garden waste and garbage, and more gases released by manufacturing industries as a result of keeping up with consumer demand.

b No

It is plausible that as a person becomes taller, their head circumference increases proportionately. However, it may not always be true that as a person gains weight their head circumference also increases. For instance, a person who has stopped growing may still be putting on weight by overeating.

c Yes

We can take the number of hours of training to be the independent variable and fitness level to be the dependent variable.

Dedicating more hours to a program of physical training will improve a person's fitness, assuming a proper diet and factors such as illnesses and disease are negligible.

d No

It is medically documented that being overweight increases the chance of poor health.

However, if a causal relationship is to exist, the independent variable would be 'weight', not 'overweight'. Hence there is no correlation between weight and health. For example, a 2 m tall physical trainer who weighs 90 kg is probably a lot healthier than a bus driver who weighs 60 kg.

e No

It is implied that the more pets there are in the household, the greater house size is. This is clearly not the case in real life. Most families do not upsize to a larger house on the basis that they now have another dog or cat to look after. Generally, pets such as cats and dogs are kept outside, so house size becomes irrelevant. For indoor pets such as fish, it is hard to imagine that having more of them will necessitate purchasing a larger home.

f Yes

Generally, the larger the house, the greater will be its value. This may be because of more land needed to build a larger house, a greater demand for larger homes by families or because investors see it as a better proposition. It's also true that to build a larger house will be more expensive than to build a smaller one.

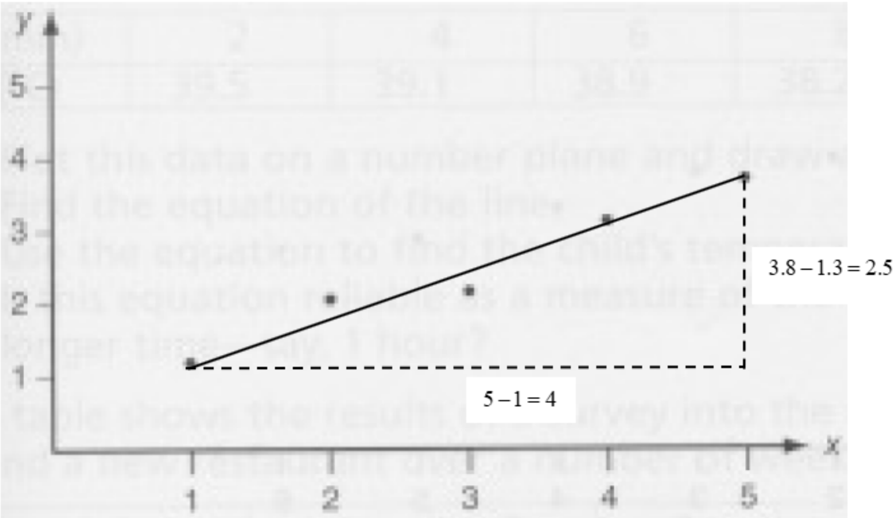
Exercise 8.03 Line of best fit

The answers below are based on the position and accuracy of the line of best fit.

Hence, slightly different answers are possible.

Question 1

a



Gradient of straight line $m = \dfrac{2.5}{4} = 0.625$

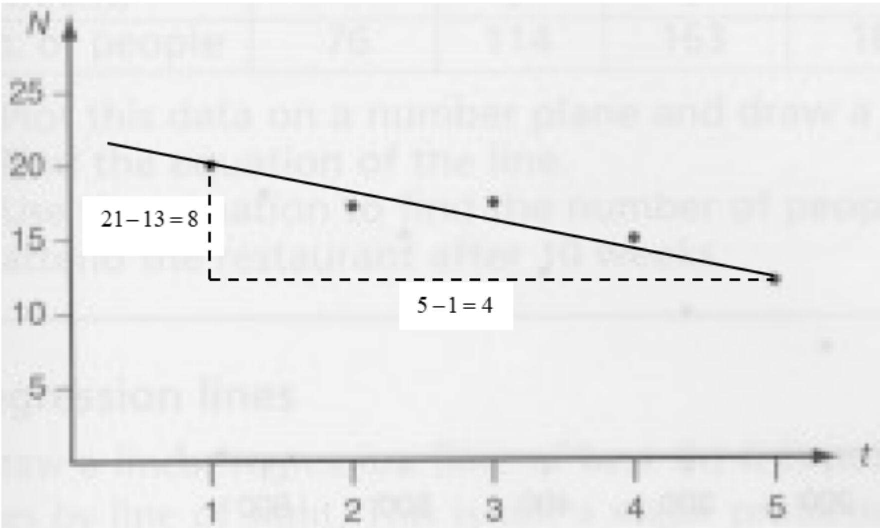
Equation of straight line $y = mx + c = 0.625x + c$

When $x = 1, y = 1.3$

$1.3 = 0.625 \times 1 + c \Rightarrow c = 0.675$

Regression line $y = 0.625x + 0.675$

b



Gradient $m = -\frac{8}{4} = -2$

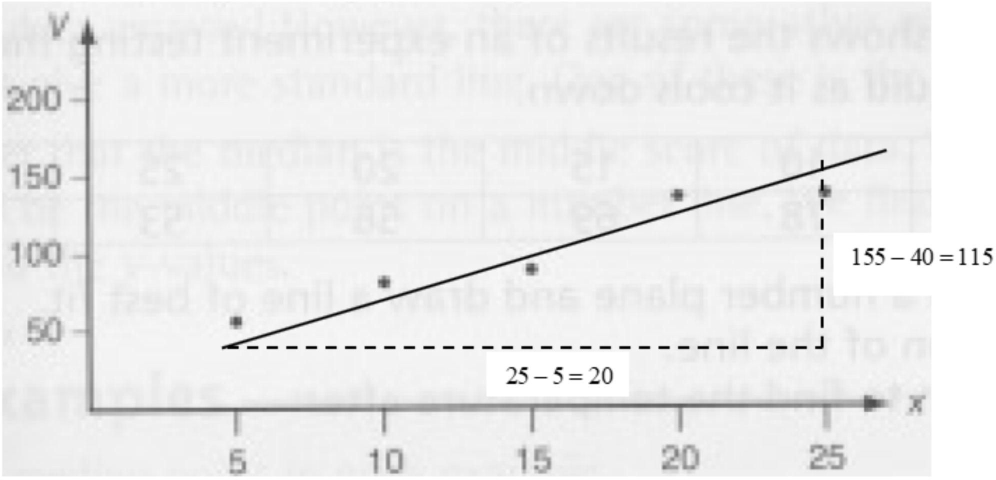
$N = mt + c = -2t + c$

When $t = 1, N = 21$

$21 = -2 \times 1 + c \Rightarrow c = 23$

Regression line $N = -2t + 23$

c



Gradient $m = \frac{115}{20} = 5.6$

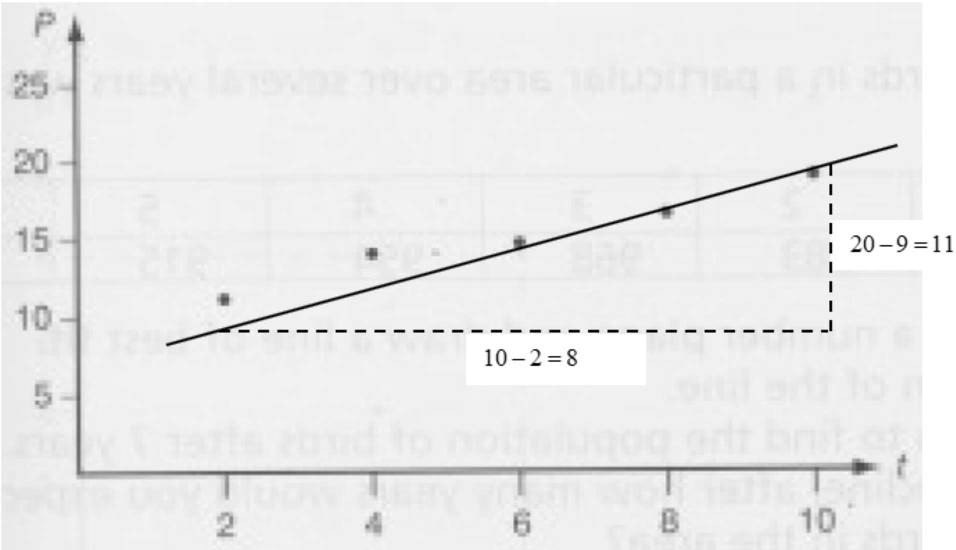
$V = mx + c = 5.6x + c$

When $x = 5, V = 40$

$40 = 1.6 \times 5 + c \Rightarrow c = 32$

Regression line $V = 5.6x + 32$

d



Gradient $m=\frac{11}{8}=1.375$

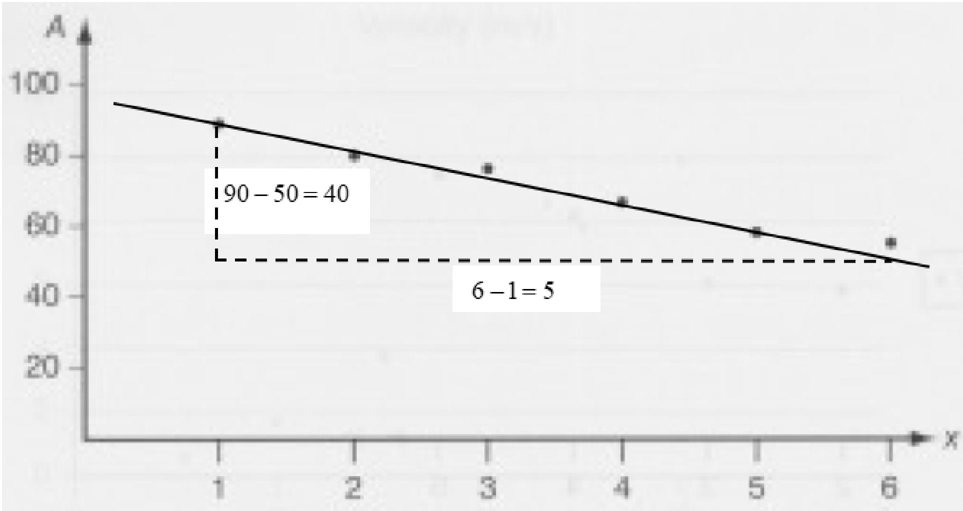
$P=mt+c=1.375t+c$

When $t=2, P=9$

$9=1.375\times 2+c \Rightarrow c=6.25$

Regression line $P=1.375t+6.25$

e



Gradient $m = -\dfrac{40}{5} = -8$

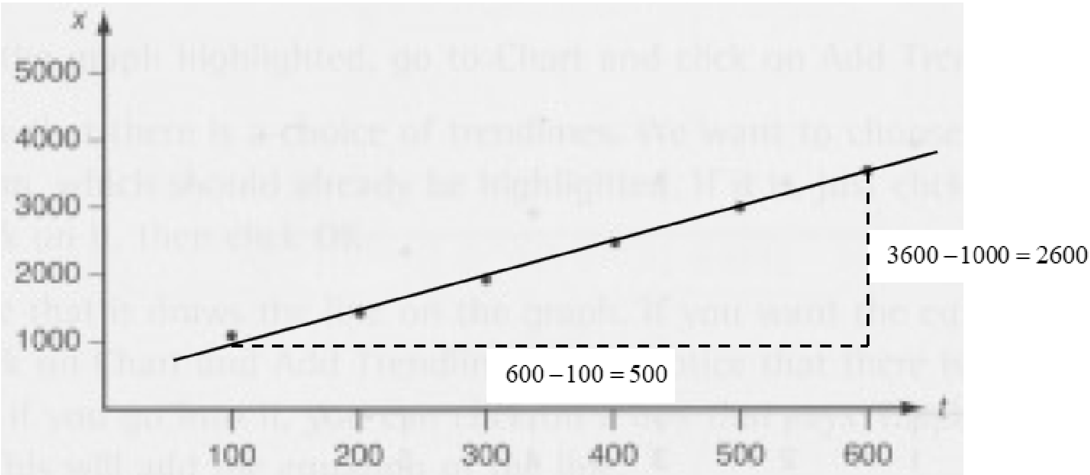
$A = mx + c = -8x + c$

When $x = 1, A = 90$

$90 = -8 \times 1 + c \Rightarrow c = 98$

Regression line $A = -8x + 98$

f



Gradient $m = \frac{2600}{500} = 5.2$

$x = mt + c = 5.2t + c$

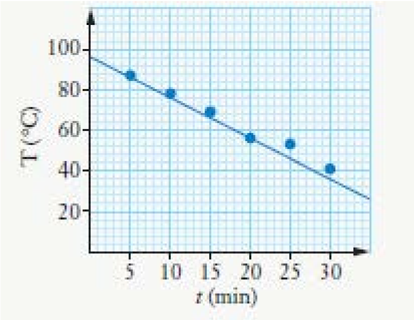
When $t = 600, x = 3600$

$3600 = 5.2 \times 600 + c \Rightarrow c = 480$

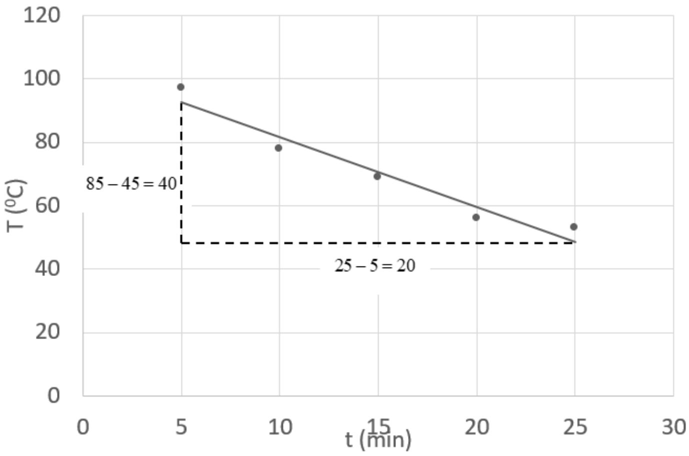
Regression line $x = 5.2t + 480$

Question 2

a



b



Gradient $m = -\frac{40}{20} = -2$

$T = mt + c = -2t + c$

When $t = 5, T = 85$

$85 = -2 \times 5 + c \Rightarrow c = 95$

Regression line $T = -2t + 95$

c **i** When $t = 17$, $T = -2 \times 17 + 95 = 61°\text{C}$

ii When $t = 35$, $T = -2 \times 35 + 95 = 25°\text{C}$

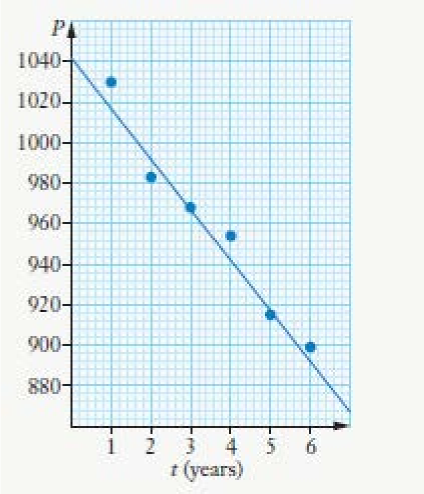
d The object's temperature cannot drop below room temperature, so the model is useful up to $23°\text{C}$.

When T = 23°C, $23 = -2t + 95 \Rightarrow t = 36$ minutes.

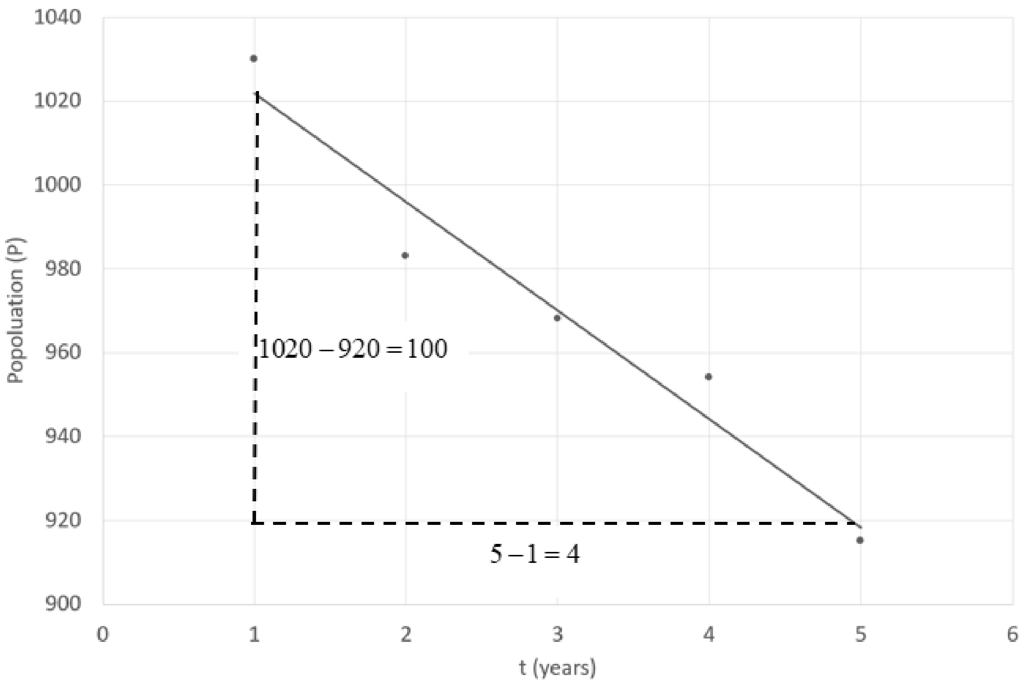
The model can be used for extrapolation during the first 5 minutes and then for interpolation for the next $36 - 5 = 31$ minutes.

Question 3

a



b



Gradient $m = -\frac{100}{4} = -25$

$P = mt + c = 25t + c$

When $t = 1, P = 1020$

$1020 = -25 \times 1 + c \Rightarrow c = 1045$

Regression line $P = -25t + 1045$

c When $t = 7$, $P = -25 \times 7 + 1045 = 870$

The bird population after 7 years will be 870.

d When $P = 0$,

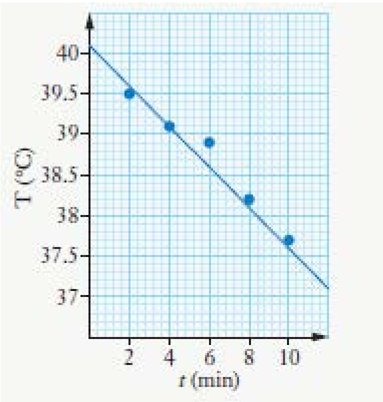
$0 = -25t + 1045$

$t = \dfrac{1045}{25} = 41.8 \approx 42$

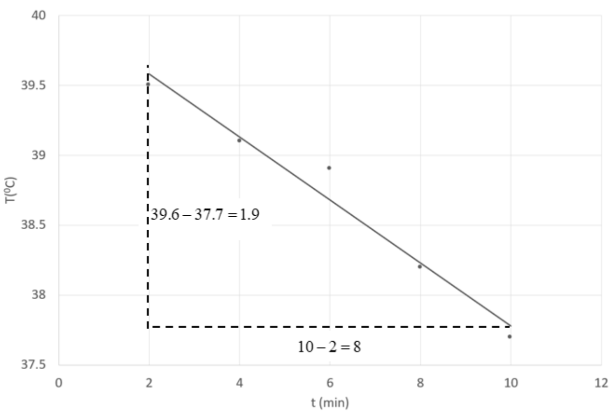
It is expected that the bird population will disappear after 42 years.

Question 4

a



b



Gradient $m = -\frac{1.9}{8} = -0.2375$

$T = mt + c = -0.2375t + c$

When $t = 2, T = 39.6$

$39.6 = -0.2375 \times 2 + c$
$c = 40.075$

Regression line

$T = -0.2375t + 40.075$
$T = -0.24t + 40.1$

C When $t = 15$, $T = -0.24 \times 15 + 40.1 = 36.5$

The temperature after 15 minutes is 36.5°C.

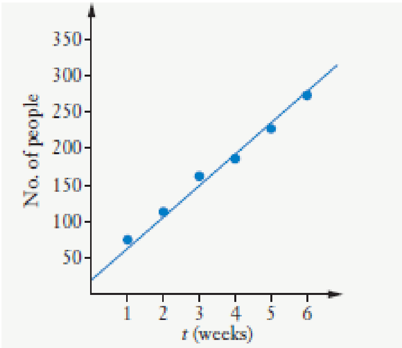
d No

When $t = 60$, $T = -0.24 \times 60 + 40.1 = 25.7°\text{C}$

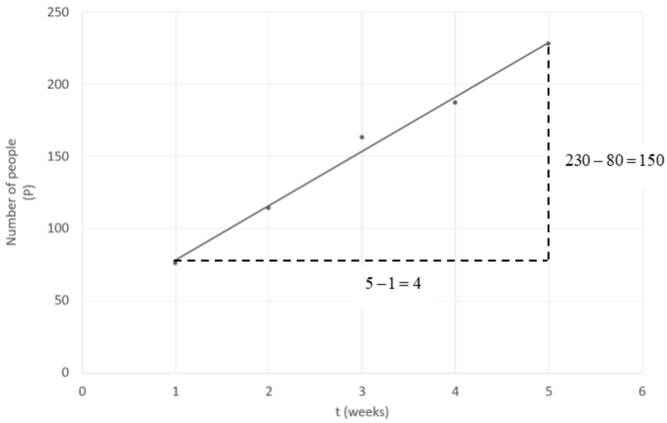
The human body experiences hypothermia begins when the body temperature drops to below 35°C. According to the model, this will occur 22 minutes after the medicine has been given, which is a short period of time. A non-linear relationship such as the power series $T = 40t^{0.03}$ give a better approximation. For example, when $t = 60$, $T = 40 \times 60^{0.03} = 45°\text{C}$, which is a reasonable result.

Question 5

a



b



Gradient $m = \dfrac{150}{4} = 37.5$

$N = mt + c = 37.5t + c$

When $t = 1, N = 80$

$80 = 37.5 \times 1 + c$
$c = 42.5$

Regression line $N = 37.5t + 42.5$

c When $t = 10$,

$P = 37.5 \times 10 + 42.5 = 417.5 \approx 418$

There will be 418 people attending the restaurant after 10 weeks.

d The restaurant can only accommodate a certain number of patrons, but the linear model allows for an infinite number of customers.

Exercise 8.04 Least-squares regression line

Question 1

The required calculations can be performed using technology.

Note: Solutions are given correct to two decimal places where appropriate.

a **i** $r = 0.99$ **ii** $m = 2.43$

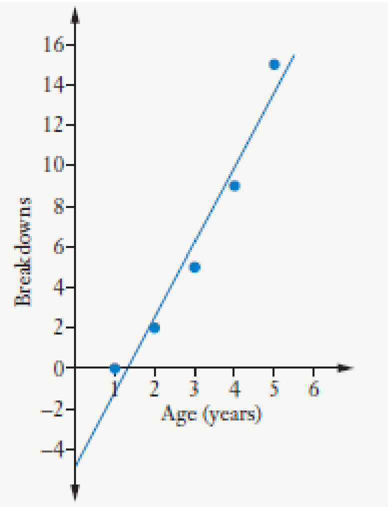
b **i** $r = 0.98$ **ii** $m = 3.89$

Question 2

We will establish the equation of the regression line using a spreadsheet, which was the method in question 2. To draw the scatterplot, highlight the data and choose insert scatterplot from the menu. Then right-click on any data point on the graph and select Add Trendline, then Linear.

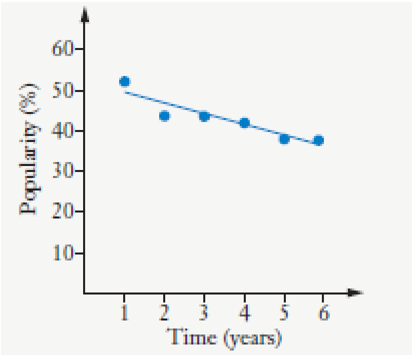
a **i** $\text{breakdowns} = 3.70 \times \text{age} - 4.90$

ii



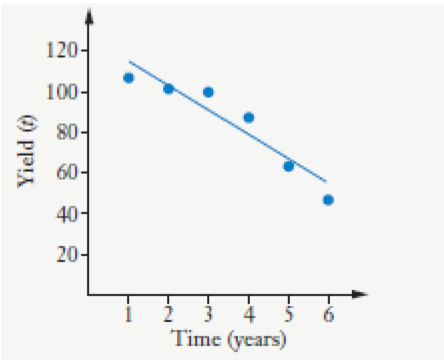
b **i** $\text{popularity} = -2.65 \times \text{time} + 52.18$

ii



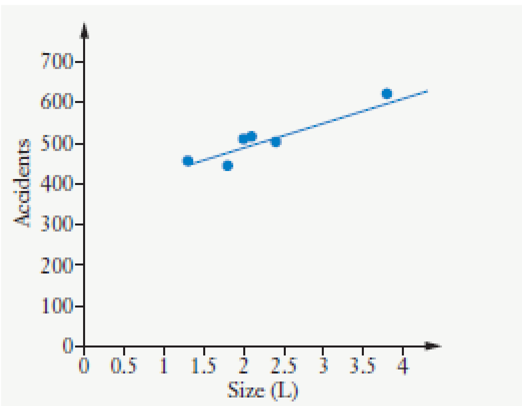
c **i** $\text{yield} = -12.24 \times \text{time} + 127.43$

ii



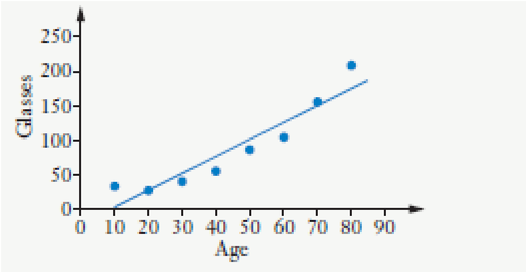
d **i** $\text{accidents} = 69.88 \times \text{size} + 355.43$

ii



e **i** glasses $= 2.49 \times$ age $- 22.36$

ii



Question 3

a $r = -0.95$

This is a high negative correlation. The gradient of the regression line is negative and there is a strong linear relationship between the mass of the ice as time passes. As time increases, the mass of ice reduces at a rate which is the gradient of the regression line.

b $m = -0.61t + 23.48$

$$m = r\frac{s_y}{s_x} = -0.95 \times \frac{7.88}{12.25} = -0.61, \quad c = \bar{y} - m\bar{x} = 9.75 - (-0.61) \times 22.50 = 23.48$$

Hence, $y = mx + c \Rightarrow m = -0.61t + 23.48$

c **i** 12.50 kg

When $t = 18$, $m = -0.61 \times 18 + 23.48 = 12.50$ kg

ii -13.12 kg

When $t = 60$, $m = -0.61 \times 60 + 23.48 = -13.12$ kg

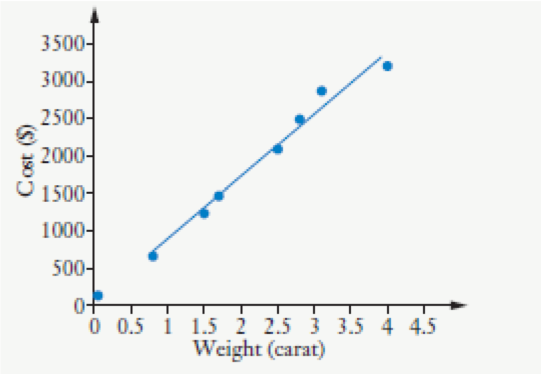
d When $m = 0$, $0 = -0.61t + 23.48 \Rightarrow t = 38.49$ minutes

This means the linear model predicts that the ice will completely melt after about 38 minutes.

Hence using any value for *t* greater than 38 will give a negative value for the mass. This is not plausible, so the model is not useful for interpolation between 38.49 minutes and 40 minutes (the last data point given). Also, the model breaks down for extrapolation for values of *t* greater than 40 minutes, since a negative value for the mass is obtained.

Question 4

a



b $r = 0.99$

This is a high positive value, which means the linear model is very effective in demonstrating that the price of gemstone increases linearly with an increase in its weight.

c $\text{price} = 831.7 \times \text{weight} + 69.2$

d When $\text{weight} = 2$, $\text{price} = 831.7 \times 2 + 69.2 = 1732.60$

The value of a 2 carat gemstone will be $1732.60.

e When $\text{price} = 10\,000$,

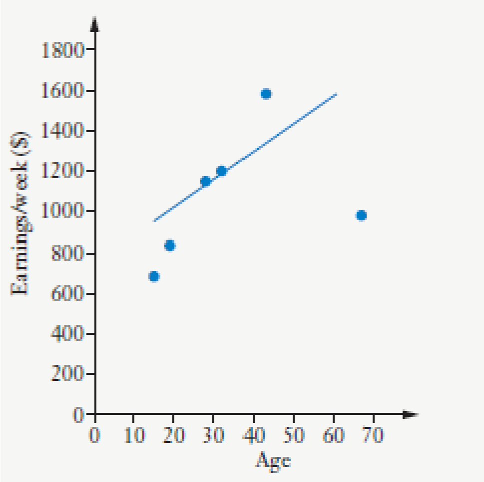
$$10\ 000 = 831.7 \times \text{weight} + 69.2$$

$$\text{weight} = \frac{10\ 000 - 69.2}{831.7} = 11.94$$

The weight of the gemstone will be 11.94 carats.

Question 5

a



b $r = 0.39$

This is a low positive value, which means the linear model is not very accurate in predicting the earnings for different age groups.

c $\text{earnings} = 6.49 \times \text{age} + 856.17$

$\bar{x} = 34.00, s_x = 18.95, \bar{y} = 1076.83, s_y = 315.41$

$m = r\dfrac{s_y}{s_x} = 0.39 \times \dfrac{315.41}{18.95} = 6.49$, $c = \bar{y} - m\bar{x} = 1076.83 - 6.49 \times 34.00 = 856.17$

Hence, $\text{earnings} = 6.49 \times \text{age} + 856.17$

d When $\text{age} = 50$, $\text{earnings} = 6.49 \times 50 + 856.17 = 1180.67$

The weekly earnings of a 50-year-old is $1180.67

e No

A non-linear model would be more suitable to make predictions because extrapolating to predict the age does not provide a realistic value.

For example, the age required for an income of $2000 is $\dfrac{2000 - 856.17}{6.49} = 176.24$ years.

Test yourself 8

Question 1

C

A. Yes

It is reasonable to expect that the taller you are, the longer your hand will be, especially since body proportions are maintained.

B. Yes

Spending more time preparing for exams will usually result in better exam performance.

C. No

It would be hard, if not impossible, to provide a valid relationship between a person's height and the number of hours they work.

D. Yes

We can accept that the faster a car travels, more/less fuel will be used. For example, accelerating requires more fuel, but moving at a constant speed requires less fuel.

Question 2

B

A line of best fit will have a positive gradient. The scatterplot displays points fairly close together, indicating a moderate linear relationship.

Question 3

D

The equation can be found from the graph by calculating the gradient and *y*-intercept.

The *y*-intercept is 10. Using the co-ordinates of the y-intercept (0, 10) and the last data point at approximately (3, 100), the gradient is $m = \frac{100-10}{3-0} = 30$.

Hence $y = mx + c$ is $y = 30x + 10$.

Question 4

A

A line of best fit will have a negative gradient. This eliminates B and D.

A value of –1 represents a perfect negative correlation where every point lies exactly on the line of best fit. This is not the case. The scatterplot shows that the points fairly close to the line of best fit, so the correlation coefficient would be around 0.5–0.7

Question 5

C

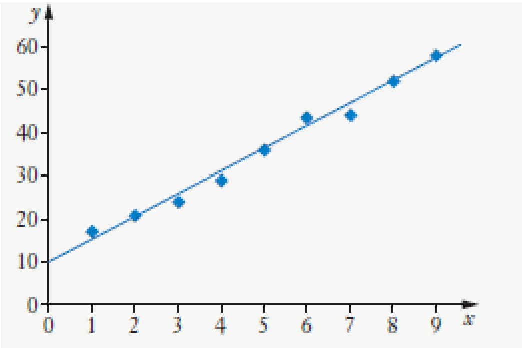
Extrapolation uses values outside the domain of the data set to make predictions.

Causality refers to the relationship between dependent and independent variables.

Correlation is a measure of the strength of the linear relationship between two variables.

Question 6

a Construct the scatterplot and draw the line of best fit. Then determine the gradient and y-intercept.



Gradient $m = \frac{25}{5} = 5$

$y = 5t + c$

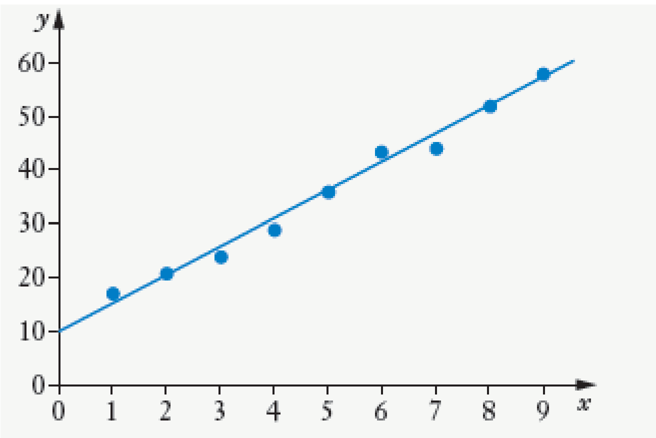
When $x = 5, y = 36$

$35 = 5 \times 5 + c \Rightarrow c = 10$

Line of best fit $y = 5x + 10$

b $y = 5.1833x + 10.083$

Draw the scatterplot and obtain the least-squares regression line using a graphing calculator or a spreadsheet.



Question 7

a C

If the last two data points are taken as outliers, the relationship might be negative linear. However, there are only 8 data points, so using two of them as outliers is significant. Taking all points without outliers, the scatterplot is non-linear.

b A

The data points lie closely on a straight line with positive gradient. Hence the correlation is positive linear.

c D

The points display no type of pattern, so there appears to be little or no correlation.

d C

The pattern is non-linear. It may possibly be parabolic, trigonometric or circular.

e B

The pattern of points shows a negative slope.

f A

The data points show a clear relationship to a straight line with positive gradient, so the correlation is positive linear.

g C

If the first two data points are taken as outliers, the relationship might be positive linear. However, there are only a few data points, so taking all points without outliers suggests the scatterplot is non-linear.

h D

The points are randomly distributed, so no linear relationship exists.

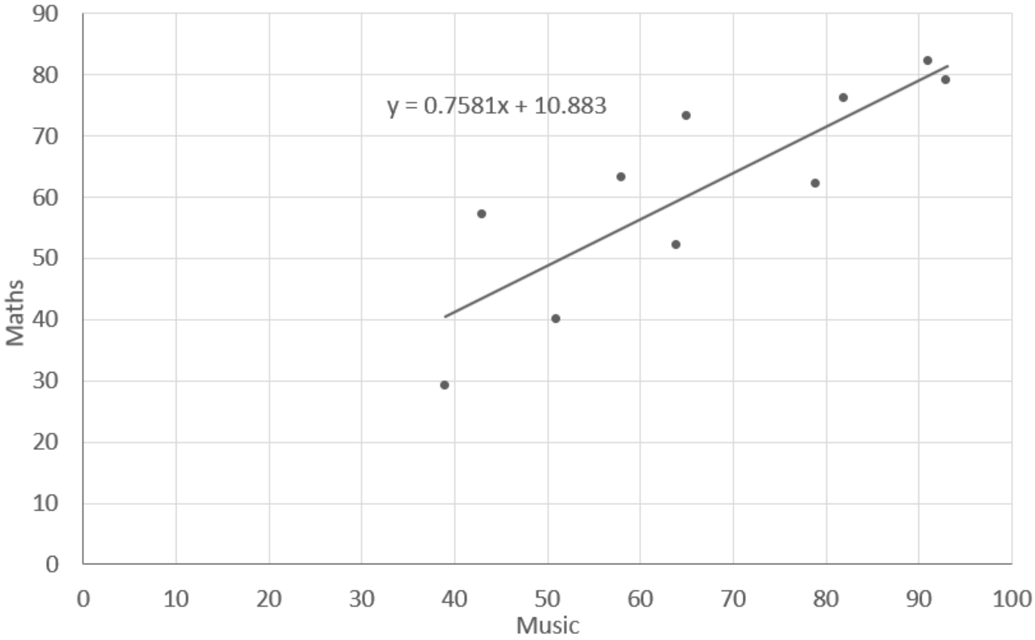
Question 8

a 0.84

	A	B	C	D	E	F	G	H	I	J	K
1	Music	79	58	91	93	65	43	39	64	82	51
2	Maths	62	63	82	79	73	57	29	52	76	40
3											
4	correlation	0.84176									

For the correlation coefficient, in cell B4 enter = PEARSON(B1:K1, B2:K2)

b $\text{Maths} = 0.7581 \times \text{Music} + 10.883$



c For $\text{Music} = 60$, $\text{Maths} = 0.7581 \times 60 + 10.883 = 56.369$

The Maths mark is 56.

d For $\text{Maths} = 70$,

$$70 = 0.7581 \times \text{Music} + 10.883$$

$$\text{Music} = \frac{70 - 10.883}{0.7581}$$

$$= 77.98$$

The Music mark is 78.

Question 9

a No

Provided one eats the normal amount of food with the required variety, then it cannot be said that the more food you eat the taller/shorter you will be. More food may lead to obesity, not to an increase in height.

b No

In some cases, such as jockeys riding horses, weight can decide how many years of the sport (horse riding) there will be. In general, however, participation in a sport can continue independent of weight if ability and achievement is not considered.

c Yes

The human body grows proportionately, so that arm length will increase as height increases.

d Yes

Assuming each chicken is capable of laying an egg, then the more chickens there are, the more eggs will be laid.

e No

The density (number of books/unit length) of a bookshelf can vary. Library bookshelves would have a greater density of books than the home library, where only a few books are displayed for decoration.

Challenge exercise 8

Question 1

The gradient of the regression line is $m = r\frac{s_y}{s_x}$.

$\bar{x} = 1.2, s_x = 1.8$

From $y = 2x + 4, m = 2$ and $c = 4$

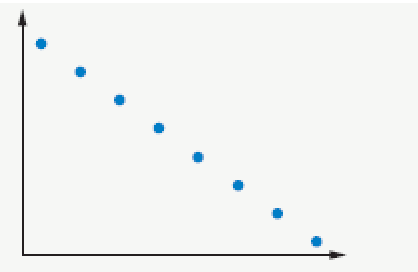
Hence,

$$2 = r \times \frac{4.5}{1.8}$$
$$r = 0.8$$

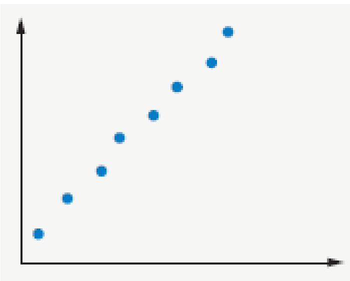
$$c = \bar{y} - m\bar{x}$$
$$4 = \bar{y} - 2 \times 1.2 = 0.21$$
$$\bar{y} = 6.4$$

Question 2

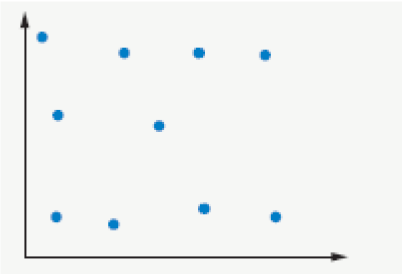
a The scatterplot will show a perfect negative correlation. This means the regression line will have negative gradient and pass through each data point.



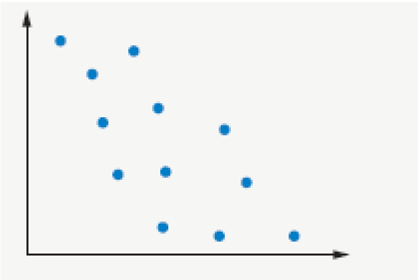
b A correlation of 0.8 indicates a strong positive relationship. This can be shown by having data points close to a regression line having positive gradient.



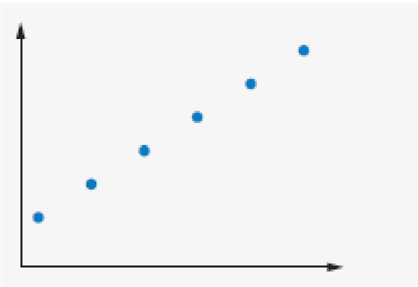
c A correlation of 0 means there is no observable linear relationship between the variables. This is shown by points that display no linear pattern.



d A correlation of –0.2 illustrates a small negative relationship. This means a regression line can be drawn, but most points will not be located near it.

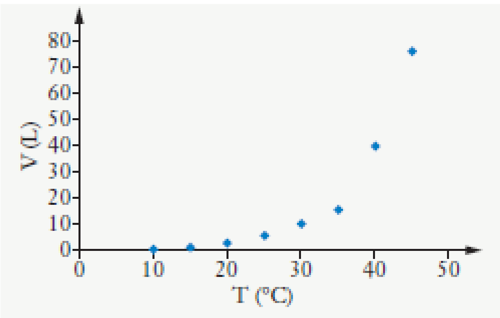


e The value of 1 represents a perfect positive correlation. All data points line up exactly to match a line of positive gradient.



Question 3

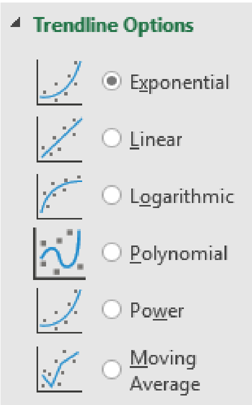
a

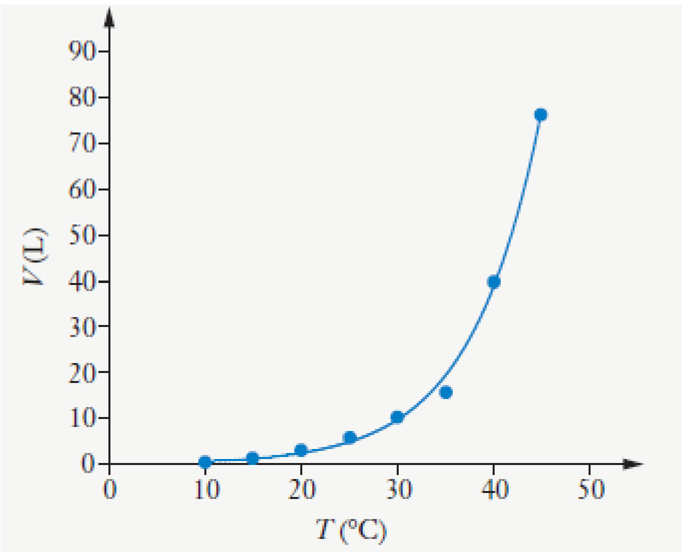


b For temperatures up to 35°C, a linear regression model might apply, but this will not be the case for temperatures above 35°C because the volume rises sharply at a non-linear rate.

c $V = 0.1563e^{0.1379T}$

The scatterplot suggests an exponential or power function might work better than a linear regression line. In the spreadsheet, Add an exponential trendline.



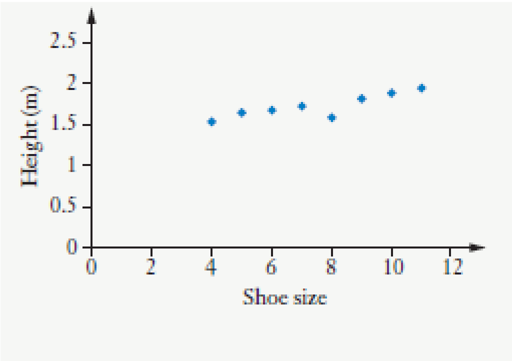


The equation of the exponential function is $V = 0.1563e^{0.1379T}$.

Question 4

	A	B	C	D	E	F	G	H	I
1	Shoe size	4	5	6	7	8	9	10	11
2	Height (m)	1.54	1.65	1.68	1.73	1.59	1.82	1.89	1.95
3									
4	correlation	0.87627							

a



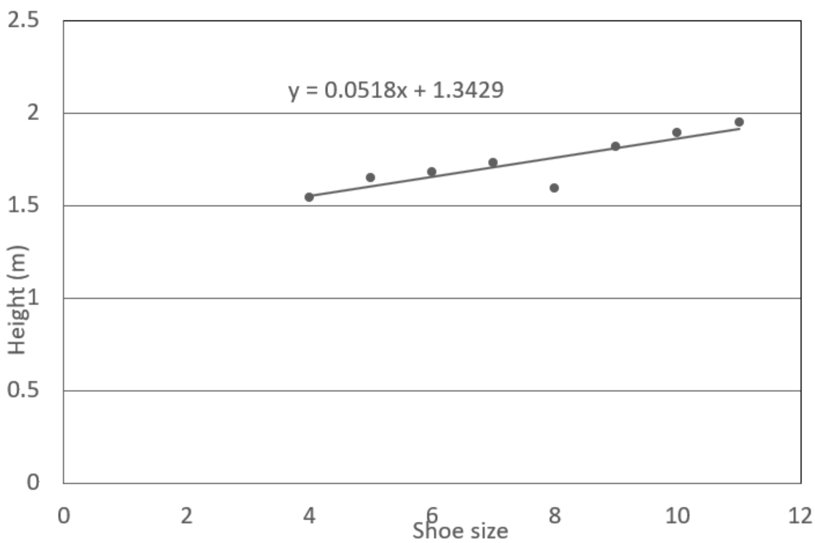
b 0.88

From the spreadsheet data, the correlation coefficient is 0.88

c $\text{height} = 0.052 \times \text{shoe size} + 1.34$

The equation of the regression line can be obtained from the Add trendline option in the spreadsheet.

The equation is $\text{height} = 0.052 \times \text{shoe size} + 1.34$



d

	A	B	C	D	E	F	G	H
1	Shoe size	4	5	6	7	9	10	11
2	Height (m)	1.54	1.65	1.68	1.73	1.82	1.89	1.95
3								
4	correlation	0.99227						

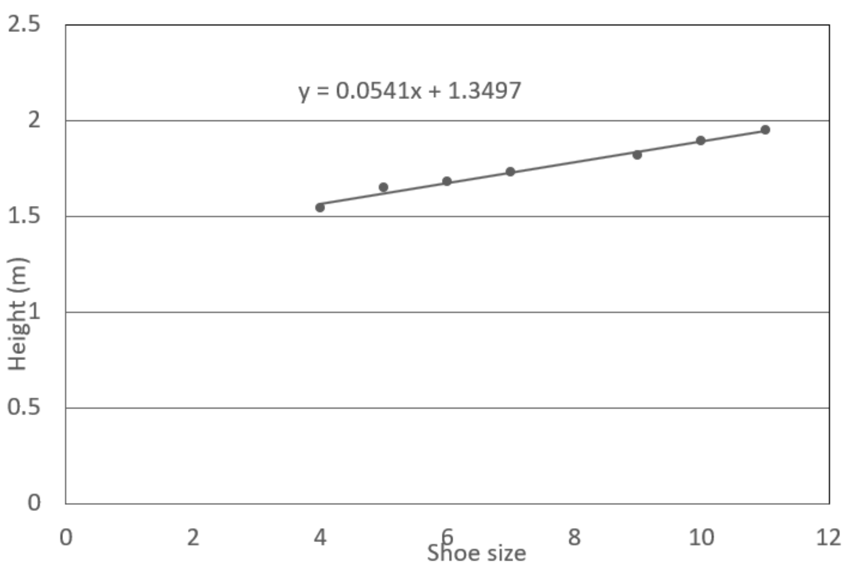
i 0.99

From the table, the correlation coefficient is 0.99227

ii $\text{height} = 0.054 \times \text{shoe size} + 1.35$

The equation of the regression line can be obtained from the Add trendline option in the spreadsheet.

The equation is $\text{height} = 0.054 \times \text{shoe size} + 1.35$



e No

A correlation value of 0.99 is very high, showing a very strong linear relationship between shoe size and height. However, the regression equation is not very useful for interpolation because shoe sizes are discrete, generally being 4, 5, 6, and sometimes $4, 4\frac{1}{2}, 5, 5\frac{1}{2}$.

This reduces the possible heights to a given number, which is not a large range. For extrapolation, the linear regression model does not take into account that shoes are not manufactured to an infinite size. That is, the model allows for all shoe sizes above 11.

MATHS IN FOCUS 12 MATHEMATICS ADVANCED

WORKED SOLUTIONS

Chapter 9: Investments, annuities and loans

Exercise 9.01 Arithmetic growth and decay

Question 1

$a = 45$, $d = 5$

a

$$t_n = a + (n-1)d$$
$$t_{34} = 45 + (34-1) \times 5$$
$$= 210$$

b

$$t_n = a + (n-1)d$$
$$100 < 45 + (n-1) \times 5$$
$$55 < (n-1) \times 5$$
$$11 < (n-1)$$
$$12 < n$$

The 13th now will have more than 100 balls

c

$$
\begin{aligned}
S_n &= \frac{n}{2}\left(2a+(n-1)d\right) \\
10545 &= \frac{n}{2}\left(90+(n-1)\times 5\right) \\
21090 &= n(85+5n) \\
5n^2+85n-21090 &= 0 \\
n &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
n &= \frac{-85 \pm \sqrt{85^2-4\times 5\times 21090}}{10} \\
n &= \frac{-85+655}{10} \\
n &= 57
\end{aligned}
$$

57 rows

Question 2

$a = 1$, $d = 2$

a

$$
\begin{aligned}
t_n &= a+(n-1)d \\
t_{34} &= 1+(20-1)\times 2 \\
&= 39
\end{aligned}
$$

b

$$
\begin{aligned}
t_n &= a+(n-1)d \\
57 &= 1+(n-1)\times 2 \\
56 &= (n-1)\times 2 \\
28 &= n-1 \\
n &= 29
\end{aligned}
$$

29th row

c

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$1024 = \frac{n}{2}\left(2 + (n-1)\times 2\right)$$
$$2048 = n(2n)$$
$$n^2 = 1024$$
$$n = 32$$

32 rows

Question 3

$a = 6$, $d = 3$

a

$$t_n = a + (n-1)d$$
$$t_n = 6 + 3(n-1)$$
$$= 6 + 3n - 3$$
$$= 3n + 3$$

b

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$S_n = \frac{n}{2}\left(12 + 3(n-1)\right)$$
$$S_n = \frac{n}{2}(9 + 3n)$$
$$S_n = \frac{3n}{2}(3 + n)$$

Question 4

a $d=\dfrac{0.6}{60}=0.01\text{ m}$

b a = 1.8, d = –0.01, n = 61

$$S_n=\frac{n}{2}\left(2a+\left(n-1\right)d\right)$$
$$S_{61}=\frac{61}{2}\left(2\times1.8+\left(61-1\right)\times\left(-0.01\right)\right)$$
$$S_{61}=\frac{61}{2}\left(3.6-0.6\right)$$
$$S_{61}=91.5\text{ m}$$

Question 5

a = 2.4, d = –0.3

a

$$t_n=a+\left(n-1\right)d$$
$$0.6=2.4+\left(n-1\right)\times\left(-0.3\right)$$
$$-1.8=\left(n-1\right)\times\left(-0.3\right)$$
$$6=n-1$$
$$n=7$$

7 poles

b

$$S_n=\frac{n}{2}\left(2a+\left(n-1\right)d\right)$$
$$S_7=\frac{7}{2}\left(2\times2.4+\left(7-1\right)\times\left(-0.3\right)\right)$$
$$S_7=\frac{7}{2}\left(3\right)$$
$$S_7=10.5\text{ m}$$

Question 6

$I = Prn$

a $I = 2000 \times 0.025 \times 1 = 50$

She has $2050.

b $I = 2000 \times 0.025 \times 2 = 100$

She has $2100.

c $I = 2000 \times 0.025 \times 3 = 150$

She has $2150.

d $I = 2000 \times 0.025 \times 10 = 500$

She has $2500.

e $I = 2000 \times 0.025 \times 30 = 1500$

She has $3500.

Question 7

$a = 1$, $d = 0.05$

a

$$\begin{aligned} t_n &= a + (n-1)d \\ t_6 &= 1 + (6-1) \times 0.05 \\ &= 1.25 \text{ m} \end{aligned}$$

b

$$\begin{aligned} t_n &= a + (n-1)d \\ 1.35 &= 1 + (n-1) \times 0.05 \\ 0.35 &= (n-1) \times 0.05 \\ \frac{0.35}{0.05} &= n-1 \\ n-1 &= 7 \\ n &= 8 \end{aligned}$$

There are 8 houses.

Question 8

$a = 0$, $d = 0.5$

a

$$t_n = a + (n-1)d$$
$$t_{10} = 0 + (10-1) \times 0.5$$
$$= 4.5$$

10th weight category 4.5–5.0 kg

b

$$t_n = a + (n-1)d$$
$$8.5 = 0 + (n-1) \times 0.5$$
$$8.5 = (n-1) \times 0.5$$
$$\frac{8.5}{0.5} = n - 1$$
$$n - 1 = 17$$
$$n = 18$$

The 18th category.

Question 9

a

$$S_n = \frac{n}{2}(a + l)$$
$$5929 = \frac{n}{2}(25 + 217)$$
$$\frac{n}{2} = \frac{5929}{242}$$
$$n = 49$$

b

$$t_n = a + (n-1)d$$
$$217 = 25 + (49-1) \times d$$
$$192 = 48 \times d$$
$$d = \frac{192}{48}$$
$$d = 4 \text{ mm}$$

Question 10

a kth apple

distance = $3k$ m

b $a = 6$, $d = 6$

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$S_k = \frac{k}{2}\left(2\times 6 + (k-1)\times 6\right)$$
$$S_k = \frac{k}{2}\left(6 + 6k\right)$$
$$S_k = k\left(3 + 3k\right)$$
$$S_k = 3k\left(1 + k\right) \text{ m}$$

c

$$S_k = 270$$
$$270 = 3k\left(1 + k\right)$$
$$3k^2 + 3k - 270 = 0$$
$$k^2 + k - 90 = 0$$
$$(k+10)(k-9) = 0$$
$$k = -10, 9$$

Only take the positive answer

$$k = 9$$

Exercise 9.02 Geometric growth and decay

Question 1

a **i** 93%

ii $(0.93)^2 \times 100 = 86.49\%$

iii $(0.93)^3 \times 100 = 80.44\%$

b $(0.93)^{15} \times 100 = 33.67\%$

c

$$0.25 = (0.93)^n$$
$$\ln 0.25 = \ln\left(0.93^n\right)$$
$$n \ln(0.93) = \ln 0.25$$
$$n = \frac{\ln 0.93}{\ln 0.25}$$
$$n = 19.1$$

19 weeks

Question 2

a $(0.98)^2 = 0.9604 = 96.04\%$

b

$$0.5 = (0.98)^n$$
$$\ln(0.98)^n = \ln 0.5$$
$$n \ln(0.98) = \ln 0.5$$
$$n = \frac{\ln 0.5}{\ln 0.98}$$
$$n = 34.3$$

34 days

c

$$0.1 = (0.98)^n$$
$$\ln(0.98)^n = \ln 0.1$$
$$n\ln(0.98) = \ln 0.1$$
$$n = \frac{\ln 0.1}{\ln 0.98}$$
$$n = 114$$

114 days

Question 3

a **i** 20 000 × 1.16 = \$23 200

ii 23 200 × 1.16 = \$26 912

iii 26 912 × 1.16 = \$31 217.92

b 20 000 × 1.16^{11} = \$102 345.29

c

$$50000 = 20000 \times 1.16^n$$
$$1.16^n = \frac{50000}{20000}$$
$$1.16^n = 2.5$$
$$\ln 1.16^n = \ln 2.5$$
$$n\ln 1.16 = \ln 2.5$$
$$n = \frac{\ln 2.5}{\ln 1.16}$$
$$n = 6.17$$

6.2 years

Question 4

a P = $(0.95)^5 = 0.774 = 77.4\%$

b

$$0.5 = (0.95)^n$$
$$\ln(0.95)^n = \ln 0.5$$
$$n\ln(0.95) = \ln 0.5$$
$$n = \frac{\ln 0.5}{\ln 0.95}$$
$$n = 13.5$$

After 13 years.

c

$$0.2 = (0.95)^n$$
$$\ln(0.95)^n = \ln 0.2$$
$$n\ln(0.95) = \ln 0.2$$
$$n = \frac{\ln 0.2}{\ln 0.95}$$
$$n = 31.4$$

After 31 years.

Question 5

a

$$0.\dot{4} = 0.4 + 0.04 + 0.004 + ...$$
$$a = 0.4 \qquad r = 0.1$$
$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{0.4}{1-0.1}$$
$$S_\infty = \frac{0.4}{0.9}$$
$$S_\infty = \frac{4}{9}$$

b

$$0.\dot{7} = 0.7 + 0.07 + 0.007 + ...$$
$$a = 0.7 \qquad r = 0.1$$
$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{0.7}{1-0.1}$$
$$S_\infty = \frac{0.7}{0.9}$$
$$S_\infty = \frac{7}{9}$$

c

$$1.\dot{2} = 1 + 0.2 + 0.02 + 0.002 + ...$$

$$a = 0.2 \qquad r = 0.1$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{0.2}{1-0.1}$$

$$S_\infty = \frac{0.2}{0.9}$$

$$S_\infty = \frac{2}{9}$$

$$1.\dot{2} = 1 + \frac{2}{9}$$

$$= 1\frac{2}{9}$$

d

$$0.\dot{2}\dot{5} = 0.25 + 0.0025 + 0.000025 + ...$$

$$a = 0.25 \qquad r = 0.01$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{0.25}{1-0.01}$$

$$S_\infty = \frac{0.25}{0.99}$$

$$S_\infty = \frac{25}{99}$$

e

$$2.\dot{8}\dot{1} = 2 + 0.81 + 0.0081 + 0.000081 + ...$$

$a = 0.81 \qquad r = 0.01$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{0.81}{1-0.01}$$

$$S_\infty = \frac{0.81}{0.99}$$

$$S_\infty = \frac{9}{11}$$

$$2.\dot{8}\dot{1} = 2 + \frac{9}{11}$$

$$= 2\frac{9}{11}$$

f

$$0.2\dot{3} = 0.2 + 0.03 + 0.003 + 0.0003 + ...$$

$a = 0.03 \qquad r = 0.1$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{0.03}{1-0.1}$$

$$S_\infty = \frac{0.03}{0.9}$$

$$S_\infty = \frac{1}{30}$$

$$0.2\dot{3} = \frac{1}{5} + \frac{1}{30}$$

$$= \frac{7}{30}$$

g

$$1.4\dot{7} = 1.4 + 0.07 + 0.007 + 0.0007 + ...$$

$a = 0.07 \qquad r = 0.1$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{0.07}{1-0.1}$$

$$S_\infty = \frac{0.07}{0.9}$$

$$S_\infty = \frac{7}{90}$$

$$1.4\dot{7} = 1\frac{2}{5} + \frac{7}{90}$$

$$= 1\frac{43}{90}$$

h

$$1.01\dot{5} = 1.01 + 0.005 + 0.0005 + 0.00005 + ...$$

$a = 0.005 \qquad r = 0.1$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{0.005}{1-0.1}$$

$$S_\infty = \frac{0.005}{0.9}$$

$$S_\infty = \frac{1}{180}$$

$$1.01\dot{5} = 1\frac{1}{100} + \frac{1}{180}$$

$$= 1\frac{7}{450}$$

i

$$0.1\dot{3}\dot{2} = 0.1 + 0.032 + 0.00032 + \ldots$$

$$a = 0.032 \qquad r = 0.01$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{0.032}{1-0.01}$$

$$S_\infty = \frac{0.032}{0.99}$$

$$S_\infty = \frac{32}{990} = \frac{16}{495}$$

$$0.1\dot{3}\dot{2} = \frac{1}{10} + \frac{16}{495} = \frac{131}{990}$$

j

$$2.\dot{3}6\dot{1} = 2 + 0.361 + 0.000361 + 0.000000361 + \ldots$$

$$a = 0.361 \qquad r = 0.001$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{0.361}{1-0.001}$$

$$S_\infty = \frac{0.361}{0.999}$$

$$S_\infty = \frac{361}{999}$$

$$2.\dot{3}6\dot{1} = 2 + \frac{361}{999}$$

$$= 2\frac{361}{999}$$

Question 6

$a = 0.5 \quad r = 0.2$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{0.5}{1-0.02}$

$S_\infty = \frac{0.5}{0.8}$

$S_\infty = \frac{5}{8} = 0.625 \text{ m}$

Question 7

$a = 3 \quad r = \frac{4}{5}$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{3}{1-\frac{4}{5}}$

$S_\infty = 15 \text{ m}$

Question 8

$a = 8 \quad r = 0.6$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{8}{1-0.6}$

$S_\infty = 20 \text{ cm}$

Question 9

$a = 0.5 \qquad r = \frac{5}{6}$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{0.5}{1-\frac{5}{6}}$$

$$S_\infty = 3 \text{ m}$$

Question 10

$a = 100 \qquad r = \frac{5}{7}$

a

$$t_n = ar^{n-1}$$

$$t_{10} = 100 \times \left(\frac{5}{7}\right)^9$$

$$t_{10} = 4.84 \text{ m}$$

b

$$t_n = ar^{n-1}$$

$$50 = 100 \times \left(\frac{5}{7}\right)^{n-1}$$

$$\left(\frac{5}{7}\right)^{n-1} = \frac{1}{2}$$

$$\ln\left(\frac{5}{7}\right)^{n-1} = \ln\frac{1}{2}$$

$$(n-1)\ln\left(\frac{5}{7}\right) = \ln\frac{1}{2}$$

$$(n-1) = \frac{\ln\frac{1}{2}}{\ln\left(\frac{5}{7}\right)}$$

$$n-1 = 2$$

$$n = 3$$

3 years

Question 11

$a = 45 \qquad r = \frac{2}{5}$

a

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$S_6 = \frac{45\left(\left(\frac{2}{5}\right)^6 - 1\right)}{\frac{2}{5} - 1}$$

$$S_6 = 74.69 \text{ cm}$$

b

$$S_\infty = \frac{a}{1 - r}$$

$$S_\infty = \frac{45}{1 - \frac{2}{5}}$$

$$S_\infty = 75 \text{ cm}$$

Question 12

$$60+2\times 60\times\left(\frac{2}{3}\right)+2\times 60\times\left(\frac{2}{3}\right)^2+2\times 60\times\left(\frac{2}{3}\right)^3+\ldots$$

Take out the first 60

$$2\times 60\times\left(\frac{2}{3}\right)+2\times 60\times\left(\frac{2}{3}\right)^2+2\times 60\times\left(\frac{2}{3}\right)^3+\ldots$$

$$a=120\times\left(\frac{2}{3}\right)\qquad r=\frac{2}{3}$$

$$S_\infty=\frac{a}{1-r}$$

$$S_\infty=\frac{120\times\left(\frac{2}{3}\right)}{1-\frac{2}{3}}$$

$$S_\infty=240$$

Total distance = 240 + 60 = 300 cm

Question 13

$$1.5+2\times 1.5\times\left(\frac{2}{5}\right)+2\times 1.5\times\left(\frac{2}{5}\right)^2+2\times 1.5\times\left(\frac{2}{5}\right)^3+\ldots$$

Take out the first 1.5

$$3\times\left(\frac{2}{5}\right)+3\times\left(\frac{2}{5}\right)^2+3\times\left(\frac{2}{5}\right)^3+\ldots$$

$$a=3\times\left(\frac{2}{5}\right)\qquad r=\frac{2}{5}$$

$$S_\infty=\frac{a}{1-r}$$

$$S_\infty=\frac{3\times\left(\frac{2}{5}\right)}{1-\frac{2}{5}}$$

$$S_\infty=2$$

Total distance = 2 + 1.5 = 3.5 m

Question 14

$a = 4 \qquad r = \frac{7}{8}$

$S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{4}{1-\frac{7}{8}}$

$S_\infty = 32\ \text{m}$

Question 15

a $1 + 8 + 8^2 + 8^3 + \ldots = 1, 8, 64, 512, \ldots$

b

$t_n = ar^{n-1}$

$t_9 = 1 \times 8^8$

$t_9 = 8^8$

$= 16\ 777\ 216$

c

$S_n = \frac{a\left(r^n - 1\right)}{r-1}$

$S_9 = \frac{1\left(8^9 - 1\right)}{8-1}$

$S_9 = 19\ 173\ 961$

Exercise 9.03 Compound interest

Question 1

a

$$FV = PV(1+r)^n$$
$$FV = 6500(1+0.03)^2$$
$$= \$6895.85$$

b

$$FV = PV(1+r)^n$$
$$FV = 6500(1+0.025)^3$$
$$= \$6999.79$$

c

$$FV = PV(1+r)^n$$
$$FV = 6500(1+0.041)^4$$
$$= \$7633.37$$

d

$$FV = PV(1+r)^n$$
$$FV = 6500(1+0.018)^3$$
$$= \$6857.36$$

e

$$FV = PV(1+r)^n$$
$$FV = 6500(1+0.053)^2$$
$$= \$7207.26$$

Question 2

a

$$FV = PV(1+r)^n$$
$$FV = 2500(1+0.045)^3$$
$$= \$2852.92$$

b

$$FV = PV(1+r)^n$$
$$FV = 10\ 000(1+0.062)^4$$
$$= \$12\ 720.32$$

c

$$FV = PV(1+r)^n$$
$$FV = 3400(1+0.035)^5$$
$$= \$4038.13$$

d

$$FV = PV(1+r)^n$$
$$FV = 5000(1+0.06)^3$$
$$= \$5955.08$$

e

$$FV = PV(1+r)^n$$
$$FV = 80\ 000(1+0.045)^2$$
$$= \$87\ 362$$

Question 3

$$FV = PV(1+r)^n$$
$$FV = 750\ 000(1+0.06)^3$$
$$= \$893\ 262$$

Question 4

$$FV = PV(1+r)^n$$
$$FV = 3000(1+0.058)^3$$
$$= \$3552.86$$

Question 5

$$FV = PV(1+r)^n$$
$$FV = 15\ 000(1+0.09)^4$$
$$= \$21\,173.72$$

Question 6

$$FV = PV(1+r)^n$$
$$FV = 950(1+0.03)^4$$
$$= \$1069.23$$

Question 7

$$FV = PV(1+r)^n$$
$$FV = 4500(1+0.029)^3$$
$$= \$4902.96$$

Question 8

a $800, 7 years, 5%

$800 \times 1.4071 = \$1125.68$

b $2000, 10 years, 1%

$2000 \times 1.1046 = \$2209.20$

c $5000, 6 years, 20%

$5000 \times 2.9860 = \$14\ 930$

d $60 000, 5 years, 10%

$60\ 000 \times 1.6105 = \$96\ 630$

e $100 000, 8 years, 15%

$100\ 000 \times 3.0590 = \$305\ 900$

f $673.25, 6 years, 5%

$673.25 \times 1.3401 = \$902.22$

g $1249.53, 4 years, 1%

$1249.53 \times 1.0406 = \$1300.26$

h $3000, 3 months, 12% p.a.

$3000, 3 months, 1%

$3000 \times 1.0303 = \$3090.90$

i $1000, 6 months, 12% p.a.

$1000, 3 months, 1%

$1000 \times 1.0615 = \$1061.50$

j $3500, 10 months, 12% p.a.

$3500, 10 months, 1%

$3500 \times 1.1046 = \$3866.10$

Question 9

FV = $10 000

a 7 years, 2%

10 000 ÷ 1.1487 = $8705.49

b 5 years, 15%

10 000 ÷ 2.0114 = $4971.66

c 10 years, 8%

10 000 ÷ 2.1589 = $4631.99

d 3 years, 1%

10 000 ÷ 1.0303 = $9705.91

e 4 years, 5%

10 000 ÷ 1.2155 = $8227.07

Exercise 9.04 Compound interest formula

Question 1

a

$$FV = PV\left(1+r\right)^n$$
$$FV = 500\left(1+0.04\right)^{10}$$
$$= \$740.12$$

b

$$FV = PV\left(1+r\right)^n$$
$$FV = 7500\left(1+0.07\right)^{10}$$
$$= \$14\ 753.64$$

c

$$FV = PV\left(1+r\right)^n$$
$$FV = 8000\left(1+0.08\right)^{10}$$
$$= \$17\ 271.40$$

d

$$FV = PV\left(1+r\right)^n$$
$$FV = 5000\left(1+0.065\right)^{10}$$
$$= \$9385.69$$

e

$$FV = PV\left(1+r\right)^n$$
$$FV = 2500\left(1+0.078\right)^{10}$$
$$= \$5298.19$$

Question 2

$PV = \$1500$

a

$$FV = PV(1+r)^n$$
$$FV = 1500(1+0.06)^5$$
$$= \$2007.34$$

b

$$FV = PV(1+r)^n$$
$$FV = 1500(1+0.03)^{10}$$
$$= \$2015.87$$

c

$$FV = PV(1+r)^n$$
$$FV = 1500(1+0.015)^{20}$$
$$= \$2020.28$$

Question 3

a

$$FV = PV(1+r)^n$$
$$FV = 3000\left(1+\frac{0.05}{4}\right)^{40}$$
$$= \$4930.86$$

b

$$FV = PV(1+r)^n$$
$$FV = 3000\left(1+\frac{0.05}{12}\right)^{120}$$
$$= \$4941.03$$

Question 4

a

$$FV = PV(1+r)^n$$
$$FV = 350(1+0.08)^2$$
$$= \$408.24$$

b

$$FV = PV(1+r)^n$$
$$FV = 350\left(1+\frac{0.08}{12}\right)^{24}$$
$$= \$410.51$$

Question 5

a

$$FV = PV(1+r)^n$$
$$FV = 850\left(1+\frac{0.045}{2}\right)^{6}$$
$$= \$971.40$$

b

$$FV = PV(1+r)^n$$
$$FV = 850\left(1+\frac{0.045}{4}\right)^{12}$$
$$= \$972.12$$

Question 6

a

$$FV = PV\left(1+r\right)^{n}$$
$$FV = 1000\left(1+\frac{0.07}{2}\right)^{16}$$
$$= \$1733.99$$

b

$$FV = PV\left(1+r\right)^{n}$$
$$FV = 1000\left(1+\frac{0.07}{4}\right)^{32}$$
$$= \$1742.21$$

c

$$FV = PV\left(1+r\right)^{n}$$
$$FV = 1000\left(1+\frac{0.07}{12}\right)^{96}$$
$$= \$1747.83$$

Question 7

a

$$FV = PV\left(1+r\right)^{n}$$
$$FV = 2500\left(1+0.055\right)^{4}$$
$$= \$3097.06$$

b

$$FV = PV\left(1+r\right)^{n}$$
$$FV = 2500\left(1+\frac{0.055}{4}\right)^{16}$$
$$= \$3110.53$$

Difference = \$3110.53 – \$3097.06 = \$13.47

Question 8

a

$$FV = PV\left(1+r\right)^n$$

$$FV = 6000\left(1+\frac{0.09}{4}\right)^{60}$$

$$= \$22\ 800.81$$

b

$$FV = PV\left(1+r\right)^n$$

$$FV = 6000\left(1+0.09\right)^{15}$$

$$= \$21\ 854.89$$

Difference = \$22 800.81 – \$21 854.89 = \$945.92

Question 9

$$FV = PV\left(1+r\right)^n$$

$$FV = 500\left(1+\frac{0.065}{12}\right)^{60}$$

$$= \$691.41$$

Question 10

$$FV = PV\left(1+r\right)^n$$

$$FV = 500\left(1+\frac{0.06}{4}\right)^{16}$$

$$= \$1776.58$$

Question 11

$$FV = PV\left(1+r\right)^n$$

$$FV = 8000\left(1+\frac{0.075}{12}\right)^{96}$$

$$= \$14\ 549.76$$

Question 12

$$FV = PV\left(1+r\right)^n$$

$$FV = 500\ 000\left(1+\frac{0.08}{12}\right)^{144}$$

$$= \$1\ 301\ 694.62$$

Question 13

$$FV = PV\left(1+r\right)^n$$

$$PV\left(1+r\right)^n = FV$$

$$PV = \frac{FV}{\left(1+r\right)^n}$$

$$PV = \frac{FV}{\left(1.05\right)^4}$$

a $FV = \$5000$

$$PV = \frac{FV}{\left(1+r\right)^n}$$

$$PV = \frac{5000}{\left(1+0.05\right)^4}$$

$$PV = \$4113.51$$

b $FV = \$675$

$$PV = \frac{FV}{\left(1+r\right)^n}$$

$$PV = \frac{675}{\left(1+0.05\right)^4}$$

$$PV = \$555.32$$

c $FV = \$12\ 000$

$$PV = \frac{FV}{\left(1+r\right)^n}$$

$$PV = \frac{12\ 000}{\left(1+0.05\right)^4}$$

$$PV = \$9872.43$$

d $FV = \$289.50$

$$PV = \frac{FV}{(1+r)^n}$$

$$PV = \frac{289.50}{(1+0.05)^4}$$

$$PV = \$238.17$$

e $FV = \$12\ 800$

$$PV = \frac{FV}{(1+r)^n}$$

$$PV = \frac{12\ 800}{(1+0.05)^4}$$

$$PV = \$10\ 530.59$$

Question 14

$$FV = PV(1+r)^n$$

$$5400 = PV\left(1+\frac{0.058}{4}\right)^{12}$$

$$PV = \frac{5400}{\left(1+\frac{0.058}{4}\right)^{12}}$$

$$PV = \$4543.28$$

Question 15

$$FV = PV(1+r)^n$$

$$6352.45 = 5000(1+0.005)^n$$

$$1.005^n = \frac{6352.45}{5000}$$

$$\ln\left(1.005^n\right) = \ln\left(\frac{6352.45}{5000}\right)$$

$$n\ln 1.005 = \ln\left(\frac{6352.45}{5000}\right)$$

$$n = \frac{\ln\left(\frac{6352.45}{5000}\right)}{\ln 1.005}$$

$$n = 48 \text{ months}$$

4 years

Question 16

$$FV = PV(1+r)^n$$

$$18\ 729.81 = 10\ 000(1+0.04)^n$$

$$1.04^n = \frac{18\ 729.81}{10\ 000}$$

$$\ln\left(1.04^n\right) = \ln\left(\frac{18\ 729.81}{10\ 000}\right)$$

$$n\ln 1.04 = \ln\left(\frac{18\ 729.81}{10\ 000}\right)$$

$$n = \frac{\ln\left(\frac{18\ 729.81}{10\ 000}\right)}{\ln 1.04}$$

$$n = 15.99$$

$$n = 16$$

8 years

Question 17

a

$$FV = PV(1+r)^n$$
$$6311.48 = 4500(1+r)^5$$
$$(1+r)^5 = \frac{6311.48}{4500}$$
$$1+r = \sqrt[5]{\frac{6311.48}{4500}}$$
$$1+r = 1.069$$
$$r = 0.069 \approx 7\%$$
$$x = 7$$

b

$$FV = PV(1+r)^n$$
$$5743.27 = 4500(1+r)^5$$
$$(1+r)^5 = \frac{5743.27}{4500}$$
$$1+r = \sqrt[5]{\frac{5743.27}{4500}}$$
$$1+r = 1.05$$
$$r = 0.05 = 5\%$$
$$x = 5$$

c

$$FV = PV(1+r)^n$$
$$6611.98 = 4500(1+r)^5$$
$$(1+r)^5 = \frac{6611.98}{4500}$$
$$1+r = \sqrt[5]{\frac{6611.98}{4500}}$$
$$1+r = 1.08$$
$$r = 0.08 = 8\%$$
$$x = 8$$

d

$$FV = PV(1+r)^n$$
$$6165.39 = 4500(1+r)^5$$
$$(1+r)^5 = \frac{6165.39}{4500}$$
$$1+r = \sqrt[5]{\frac{6165.39}{4500}}$$
$$1+r = 1.065$$
$$r = 0.065 = 6.5\%$$
$$x = 6.5$$

e

$$FV = PV(1+r)^n$$
$$6766.46 = 4500(1+r)^5$$
$$(1+r)^5 = \frac{6766.46}{4500}$$
$$1+r = \sqrt[5]{\frac{6766.46}{4500}}$$
$$1+r = 1.065$$
$$r = 0.085 = 8.5\%$$
$$x = 8.5$$

Question 18

$$FV = PV(1+r)^n$$
$$FV = 1200(1+0.07)^3$$
$$= \$1470.05$$

$$FV = PV(1+r)^n$$
$$FV = 1200\left(1+\frac{0.07}{4}\right)^{12}$$
$$= \$1477.73$$

Difference = \$1477.73 – \$1470.05 = \$7.68

Question 19

Kate

$$FV = PV(1+r)^n$$
$$FV = 4000(1+0.05)^5$$
$$= \$5105.13$$

Rachel

$$FV = PV(1+r)^n$$
$$FV = 4000(1+0.01)^{20}$$
$$= \$4880.76$$

Difference \$5105.13 – \$4880.76 = \$224.37

Kate earns \$224.37 more.

Question 20

A

$$FV = PV(1+r)^n$$
$$FV = 5000\left(1+\frac{0.08}{2}\right)^{12}$$
$$= \$8005.16$$

B

$$FV = PV(1+r)^n$$
$$FV = 5000\left(1+\frac{0.06}{12}\right)^{72}$$
$$= \$7160.22$$

Difference = \$8005.16 – \$7160.22 = \$844.94

Account A pays more by \$844.94.

Question 21

a **i** His 5th year of work means after 4 increases from $36 400.

$$\begin{aligned} FV &= PV(1+r)^n \\ &= 36\ 400(1+0.02)^4 \end{aligned}$$

= 39 400.53062

≈ $39 400.53

ii His 8th year of work means after 7 increases from $36 400.

$$\begin{aligned} FV &= PV(1+r)^n \\ &= 36\ 400(1+0.02)^7 \end{aligned}$$

= 41 812.1583

≈ $41 812.16

b

$$FV = PV(1+r)^n$$

$$60\ 000 = 36\ 400(1+0.02)^n$$

$$\frac{60\ 000}{36\ 400} = 1.02^n$$

$$1.02^n = 1.6483...$$

$$\ln(1.02^n) = \ln 1.6483...$$

$$n \ln 1.02 = \ln 1.6483...$$

$$\begin{aligned} n &= \frac{\ln 1.6483...}{\ln 1.02} \\ &= 25.2378... \\ &\approx 26 \quad \text{(rounding up)} \end{aligned}$$

26 increases from $36 400 means in his 27th year of work.

Question 22

a **i** Yuron's 3rd year means after 2 increases from $120 000.

$$\begin{aligned} FV &= PV\left(1+r\right)^n \\ &= 120\ 000\left(1+0.035\right)^2 \\ &= \$128\ 547 \end{aligned}$$

ii Yuron's 12th year means after 11 increases from $120 000.

$$\begin{aligned} FV &= PV\left(1+r\right)^n \\ &= 120\ 000\left(1+0.035\right)^{11} \\ &= \$175\,196.3661 \\ &\approx \$175\,196.37 \end{aligned}$$

iii Yuron's 20th year means after 19 increases from $120 000.

$$\begin{aligned} FV &= PV\left(1+r\right)^n \\ &= 120\ 000\left(1+0.035\right)^{19} \\ &= \$230\ 700.1581 \\ &\approx \$230\ 700.16 \end{aligned}$$

b

$$FV = PV\left(1+r\right)^n$$

$$300\ 000 = 120\ 000\left(1+0.035\right)^n$$

$$\frac{300\ 000}{120\ 000} = 1.035^n$$

$$1.035^n = 2.5$$

$$\ln\left(1.035^n\right) = \ln 2.5$$

$$n \ln 1.035 = \ln 2.5$$

$$\begin{aligned} n &= \frac{\ln 2.5}{\ln 1.035} \\ &= 26.6352... \\ &\approx 27 \quad \text{(rounding up)} \end{aligned}$$

27 increases from $120 000 means in his 28th year of work.

Question 23

a i

$$FV = PV(1+r)^n$$
$$5410 = 5000(1+0.02)^n$$
$$\frac{5410}{5000} = 1.02^n$$
$$1.02^n = 1.082$$
$$\ln(1.02^n) = \ln 1.082$$
$$n \ln 1.02 = \ln 1.082$$
$$\begin{aligned} n &= \frac{\ln 1.082}{\ln 1.02} \\ &= 3.9798... \\ &\approx 4 \text{ years} \quad \text{(rounding up)} \end{aligned}$$

ii

$$FV = PV(1+r)^n$$
$$7000 = 5000(1+0.02)^n$$
$$\frac{7000}{5000} = 1.02^n$$
$$1.02^n = 1.4$$
$$\ln(1.02^n) = \ln 1.4$$
$$n \ln 1.02 = \ln 1.4$$
$$\begin{aligned} n &= \frac{\ln 1.4}{\ln 1.02} \\ &= 16.9912... \\ &\approx 17 \text{ years} \quad \text{(rounding up)} \end{aligned}$$

b **i**

$$FV = PV\left(1+r\right)^n$$
$$6000 = 5000\left(1+r\right)^6$$
$$\frac{6000}{5000} = \left(1+r\right)^6$$
$$\left(1+r\right)^6 = 1.2$$
$$\begin{aligned}1+r &= \sqrt[6]{1.2}\\ &= 1.03085...\end{aligned}$$
$$\begin{aligned}r &= 1.03085...-1\\ &= 0.03085...\\ &\approx 0.031\\ &= 3.1\%\end{aligned}$$

ii

$$FV = PV\left(1+r\right)^n$$
$$6000 = 5000\left(1+r\right)^{10}$$
$$\frac{6000}{5000} = \left(1+r\right)^{10}$$
$$\left(1+r\right)^{10} = 1.2$$
$$\begin{aligned}1+r &= \sqrt[10]{1.2}\\ &= 1.01839...\end{aligned}$$
$$\begin{aligned}r &= 1.01839...-1\\ &= 0.01839...\\ &\approx 0.018\\ &= 1.8\%\end{aligned}$$

Question 24

a From the table

Interest factor = 1.9990

b

$$FV = PV(1+r)^n$$
$$FV = 1(1+0.08)^9$$
$$= 1.9990$$

Question 25

a From the table

Interest factor = 1.5209

b

$$FV = PV(1+r)^n$$
$$FV = 1(1+0.15)^3$$
$$= 1.5209$$

Therefore it is correct to 4 decimal places.

Exercise 9.05 Annuities

Question 1

a $5000 \times 1.025 + 5000 = \$10\ 125$

b $1200 \times 1.04^2 + 1200 \times 1.04 + 1200 = \$3745.92$

c $875 \times 1.036^2 + 875 \times 1.036 + 875 = \$2720.63$

d $10000 \times 1.041 + 10\ 000 = \$20\ 410$

e $2000 \times 1.032^2 + 2000 \times 1.032 + 2000 = \$6194.05$

Question 2

a $6300 \times 7.8983 = \$49\ 759.29$

b $980 \times 5.6371 = \$5524.36$

c $7500 \times 12.5779 = \$94\ 334.25$

d $495.75 \times 4.1836 = \$2074.02$

e $20\ 500 \times 13.4121 = \$274\ 948.05$

f $647.12 \times 6.1520 = \$3981.08$

g $800 \times 25.129 = \$20\ 103.20$

h $598 \times 26.0192 = \$15\ 559.48$

i $15\ 000 \times 21.8143 = \$327\ 214.50$

j $160\ 000\times 9.2142 = \$1\ 474\ 272$

Question 3

a $n = 24$, $r = 1\%$

$400 \times 26.9735 = \$10\ 789.40$

b $n = 20$, $r = 2\%$

$940 \times 24.2974 = \$22\ 839.56$

c $n = 16$, $r = 7\%$

$2500 \times 27.8881 = \$69\ 720.25$

d $n = 15$, $r = 2\%$

$550 \times 17.2934 = \$9511.37$

e $n = 18$, $r = 1\%$

$587 \times 19.6147 = \$11\ 513.83$

Question 4

$3500 \times 136.3075 = \$477\ 076.25$

Question 5

$$35\ 000 = x \times 16.699$$
$$x = \frac{35\ 000}{16.699}$$
$$x = \$2074.70$$

Question 6

a

$$8450 = x \times 5.7507$$
$$x = \frac{8450}{5.7507}$$
$$x = \$1469.39$$

b

$$25000 = x \times 8.8923$$
$$x = \frac{25000}{8.8923}$$
$$x = \$2811.42$$

c

$$10\ 000 = x \times 7.8983$$
$$x = \frac{10\ 000}{7.8983}$$
$$x = \$1266.10$$

d

$$3200 = x \times 5.2040$$
$$x = \frac{3200}{5.2040}$$
$$x = \$614.91$$

e

$$1000\ 000 = x \times 33.066$$
$$x = \frac{1000\ 000}{33.066}$$
$$x = \$30\ 242.55$$

Question 7

$$8000 = x \times 26.9735$$
$$x = \frac{8000}{26.9735}$$
$$x = \$296.59$$

Question 8

a **i** $Y_1 = 50\ 000 \times 1.04 - 5000 = \$47\ 000$

ii

$$Y_2 = \left(50\ 000 \times 1.04 - 5000\right) \times 1.04 - 5000$$
$$= 50\ 000 \times 1.04^2 - 5000\left(1 + 1.04\right)$$
$$= \$43\ 880$$

iii

$$Y_3 = \left(50\ 000 \times 1.04^2 - 5000\left(1 + 1.04\right)\right) \times 1.04 - 5000$$
$$= 50\ 000 \times 1.04^3 - 5000\left(1 + 1.04 + 1.04^2\right)$$
$$= \$40\ 635.20$$

b

$$Y_3 = \left(50\ 000 \times 1.04^2 - 4000\left(1 + 1.04\right)\right) \times 1.04 - 4000$$
$$= 50\ 000 \times 1.04^3 - 4000\left(1 + 1.04 + 1.04^2\right)$$
$$= \$43\ 756.80$$

c

$$Y_3 = \left(50\ 000 \times 1.027^2 - 4000\left(1 + 1.027\right)\right) \times 1.027 - 4000$$
$$= 50\ 000 \times 1.027^3 - 4000\left(1 + 1.027 + 1.027^2\right)$$
$$= \$41\ 833.42$$

Question 9

a

$$M_1 = 125\ 000 \times 1.01 - 500 = \$125\ 750$$

b

$$M_2 = \left(125\ 000 \times 1.01 - 500\right) \times 1.01 - 500$$
$$= 125\ 000 \times 1.01^2 - 500\left(1 + 1.01\right)$$
$$= \$126\ 507.50$$

c

$$M_3 = \left(125\ 000 \times 1.01^2 - 500\left(1 + 1.01\right)\right) \times 1.01 - 500$$
$$= 125\ 000 \times 1.01^3 - 500\left(1 + 1.01 + 1.01^2\right)$$
$$= \$127\ 272.58$$

d

$$M_3 = 125\ 000 \times 1.01^3 - 1000\left(1 + 1.01 + 1.01^2\right)$$
$$= 125\ 757.53$$

e

$$M_3 = 125\ 000 \times 1.005^3 - 500\left(1 + 1.005 + 1.005^2\right)$$
$$= \$125\ 376.88$$

Question 10

a **i** 18 months

ii Approximately \$10 400

iii 36 months

b **i** 27 months

ii Approximately \$12 200

iii 46 months

Question 11

a

$$Y_1 = 2000$$
$$Y_2 = 2000 + 2000 \times 1.06$$
$$= 2000(1 + 1.06)$$
$$Y_3 = 2000(1 + 1.06 + 1.06^2)$$
$$Y_n = 2000(1 + 1.06 + 1.06^2 + .. + 1.06^{n-1})$$

$$1 + 1.06 + 1.06^2 + .. + 1.06^{n-1}$$
$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$S_n = \frac{1(1.06^n - 1)}{0.06}$$

$$Y_n = 2000\frac{(1.06^n - 1)}{0.06}$$
$$50\ 000 = 2000\frac{(1.06^n - 1)}{0.06}$$
$$25 = \frac{(1.06^n - 1)}{0.06}$$
$$1.5 = 1.06^n - 1$$
$$1.06^n = 2.5$$
$$\ln 1.06^n = \ln 2.5$$
$$n \ln 1.06 = \ln 2.5$$
$$n = \frac{\ln 2.5}{\ln 1.06}$$
$$n = 17.73$$

18 payments

b from part **a**.

$$80\ 000 = 2000\frac{\left(1.06^n - 1\right)}{0.06}$$

$$40 = \frac{\left(1.06^n - 1\right)}{0.06}$$

$$2.4 = 1.06^n - 1$$

$$1.06^n = 3.4$$

$$\ln 1.06^n = \ln 3.4$$

$$n \ln 1.06 = \ln 3.4$$

$$n = \frac{\ln 3.4}{\ln 1.06}$$

$$n = 21$$

21 payments

Question 12

a Contribution of $1500 grows to $11 284.95 after 6 years.

Dividing both figures by 1500:

Contribution of $1 grows to $7.5233 after 6 years.

Reading from the future value of an annuity table:

In the row for 6 periods (years), the cell with the value $7.5233 is in the 9% column.

So the interest rate is 9% p.a.

b Contribution of $1500 grows to $17 195.85 after 10 years.

Dividing both figures by 1500:

Contribution of $1 grows to $11.4639 after 10 years.

Reading from the future value of an annuity table:

In the row for 10 periods (years), the cell with the value $11.4639 is in the 3% column.

So the interest rate is 3% p.a.

Exercise 9.06 Annuities and geometric series

Question 1

$Y_1 = 1500$

$Y_2 = 1500 \times 1.08 + 1500$

$= 1500(1 + 1.08)$

$Y_{15} = 1500(1 + 1.08 + 1.08^2 + ... + 1.08^{14})$

$1 + 1.08 + 1.08^2 + ... + 1.08^{14}$

$a = 1 \quad r = 1.08 \quad n = 15$

$$S_{15} = \frac{a(r^n - 1)}{r - 1}$$

$$S_{15} = \frac{1(1.08^{15} - 1)}{1.08 - 1}$$

$S_{15} = 27.15$

$Y_{15} = 1500 \times 27.15$

$Y_{15} = \$40\ 728.17$

Question 2

$Y_1 = 2000$
$Y_2 = 2000 \times 1.075 + 2000$
$= 2000(1 + 1.075)$
$Y_5 = 2000\left(1 + 1.075 + 1.075^2 + ... + 1.075^4\right)$

$1 + 1.075 + 1.075^2 + ... + 1.075^4$
$a = 1 \quad r = 1.075 \quad n = 5$

$$S_5 = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$S_5 = \frac{1\left(1.075^5 - 1\right)}{1.075 - 1}$$

$S_5 = 5.8$

$Y_5 = 2000 \times 5.8$
$Y_5 = 11\ 616.78$
$15\ 000 - 11\ 616.78 = 3383.22$

He will have to pay \$3383.22 more.

Question 3

$Y_1 = 5000$
$Y_2 = 5000 \times 1.06 + 5000$
$= 5000(1 + 1.06)$
$Y_{10} = 5000\left(1 + 1.06 + 1.06^2 + ... + 1.06^9\right)$

$1 + 1.06 + 1.06^2 + ... + 1.06^9$
$a = 1 \quad r = 1.06 \quad n = 10$

$$S_{10} = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$S_{10} = \frac{1\left(1.06^{10} - 1\right)}{1.06 - 1}$$

$S_{10} = 13.18$

$Y_{10} = 5000 \times 13.18$
$Y_{10} = \$65\ 903.97$

Question 4

$$Y_1 = 500$$
$$Y_2 = 500 \times 1.065 + 5000$$
$$= 500(1 + 1.065)$$
$$Y_5 = 500\left(1 + 1.065 + 1.065^2 + ... + 1.065^4\right)$$

$$1 + 1.065 + 1.065^2 + ... + 1.065^4$$
$$a = 1 \quad r = 1.065 \quad n = 5$$
$$S_5 = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_5 = \frac{1\left(1.065^5 - 1\right)}{1.065 - 1}$$
$$S_5 = 4.084$$

$$Y_5 = 500 \times 4.084$$
$$Y_5 = \$2846.82$$

Question 5

$$Y_1 = 200$$
$$Y_2 = (200 \times 1.06) + 200 = 200(1 + 1.06)$$
$$Y_{18} = 200\left(1 + 1.06 + 1.06^2 + ... + 1.06^{17}\right)$$

$$1.06 + 1.06^2 + ... + 1.06^{17}$$
$$a = 1 \quad r = 1.06 \quad n = 18$$
$$S_{17} = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{17} = \frac{1\left(1.06^{18} - 1\right)}{1.06 - 1}$$
$$S_{17} = 30.90565\ldots$$

$$Y_{18} = 200(30.90565\ldots)$$
$$Y_{18} = 6181.13$$

There would be \$6181.13 in the account on his 18th birthday.

Question 6

$Y_1 = 800$
$Y_2 = 800 \times 1.075 + 800$
$= 800(1 + 1.075)$
$Y_5 = 800\left(1 + 1.075 + 1.075^2 + \ldots + 1.075^4\right)$
$Y_5 = \$4646.71$

Question 7

$A_1 = 3000 \times 1.05$
$A_2 = (3000 \times 1.05 + 3000) \times 1.05$
$= 3000(1.05)^2 + 3000 \times 1.05$
$= 3000(1.05) + 3000(1.05)^2$
$= 3000\left(1.05 + 1.05^2\right)$
$A_6 = 3000\left(1.05 + 1.05^2 + \ldots + 1.05^6\right)$

$1.05 + 1.05^2 + \ldots + 1.05^6$
$a = 1.05 \quad r = 1.05 \quad n = 6$
$$S_6 = \frac{a\left(r^{20} - 1\right)}{r - 1}$$
$$S_6 = \frac{1.05\left(1.05^6 - 1\right)}{1.05 - 1}$$
$S_6 = 7.1420\ldots$

$A_6 = 3000 \times 7.1420\ldots$
$A_6 = \$21\,426.03$

Question 8

a

$$Y_1 = 2000$$
$$Y_2 = 2000 \times 1.06 + 2000$$
$$= 2000(1 + 1.06)$$
$$Y_{10} = 2000(1 + 1.06 + 1.06^2 + \ldots + 1.06^9)$$

$$1 + 1.06 + 1.06^2 + \ldots + 1.06^9$$
$$a = 1 \quad r = 1.06 \quad n = 10$$
$$S_{10} = \frac{a(r^n - 1)}{r - 1}$$
$$S_{10} = \frac{1(1.06^{10} - 1)}{1.06 - 1}$$
$$S_{10} = 13.18$$

$$Y_{10} = 2000 \times 13.18$$
$$Y_{10} = \$26\ 361.59$$

b

$$Y_1 = 2000$$
$$Y_2 = 2000 \times 1.06 + 2000$$
$$= 2000(1 + 1.06)$$
$$Y_{15} = 2000(1 + 1.06 + 1.06^2 + \ldots + 1.06^{14})$$

$$1 + 1.06 + 1.06^2 + \ldots + 1.06^{14}$$
$$a = 1 \quad r = 1.06 \quad n = 10$$
$$S_{15} = \frac{a(r^n - 1)}{r - 1}$$
$$S_{15} = \frac{1(1.06^{15} - 1)}{1.06 - 1}$$
$$S_{15} = 23.28$$

$$Y_{15} = 2000 \times 23.28$$
$$Y_{15} = \$46\ 551.94$$

Question 9

$Y_1 = 1000$

$Y_2 = 1000 \times 1.1 + 1000$

$= 1000(1 + 1.1)$

$Y_{18} = 1000\left(1 + 1.1 + 1.1^2 + ... + 1.1^{17}\right)$

$1 + 1.1 + 1.1^2 + ... + 1.1^{17}$

$a = 1 \quad r = 1.1 \quad n = 18$

$S_{18} = \dfrac{a\left(r^n - 1\right)}{r - 1}$

$S_{18} = \dfrac{1\left(1.1^{18} - 1\right)}{1.1 - 1}$

$S_{18} = 45.6$

$Y_{18} = 1000 \times 45.6$

$Y_{18} = \$45\ 599.17$

Question 10

a

$$A_1 = 1000 \times 1.08$$
$$A_2 = (1000 \times 1.08 + 1000) \times 1.08$$
$$= 1000(1.08)^2 + 1000 \times 1.08$$
$$= 1000(1.08) + 1000(1.08)^2$$
$$= 1000(1.08 + 1.08^2)$$
$$A_6 = 1000(1.08 + 1.08^2 + ... + 1.08^6)$$

$$1.08 + 1.08^2 + ... + 1.08^6$$
$$a = 1.08 \quad r = 1.08 \quad n = 6$$
$$S_6 = \frac{a(r^6 - 1)}{r - 1}$$
$$S_6 = \frac{1.08(1.08^6 - 1)}{1.08 - 1}$$
$$S_6 = 7.9228...$$

$$A_6 = 1000 \times 7.9228...$$
$$A_6 = \$7922.80$$

b

$$A_1 = 1200 \times 1.08$$
$$A_2 = (1200 \times 1.08 + 1200) \times 1.08$$
$$= 1200(1.08)^2 + 1200 \times 1.08$$
$$= 1200(1.08) + 1200(1.08)^2$$
$$= 1200(1.08 + 1.08^2)$$
$$A_6 = 1200(1.08 + 1.08^2 + ... + 1.08^6)$$
$$A_6 = 1200 \times 7.9228...$$
$$A_6 = \$9507.36$$

\$9507.36 – \$7922.80 = \$1584.56

Question 11

With no calculation Jack will clearly get more money by investing 500 each year for 30 years as the same amount of money is invested but some of it is (by comparison) left in the account for longer and so will accrue more interest.

$Y_1 = 1000$

$Y_2 = 1000 \times 1.05 + 1000$

$= 1000(1 + 1.05)$

$Y_{15} = 1000\left(1 + 1.05 + 1.05^2 + ... + 1.05^{14}\right)$

$1 + 1.05 + 1.05^2 + ... + 1.05^{14}$

$a = 1 \quad r = 1.05 \quad n = 15$

$$S_{15} = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$S_{15} = \frac{1\left(1.05^{15} - 1\right)}{1.05 - 1}$$

$S_{15} = 21.58$

$Y_{15} = 1000 \times 21.58$

$Y_{15} = \$21\ 578.56$

$Y_1 = 500$

$Y_2 = 500 \times 1.05 + 500$

$= 500(1 + 1.05)$

$Y_{30} = 500\left(1 + 1.05 + 1.05^2 + ... + 1.05^{29}\right)$

$1 + 1.05 + 1.05^2 + ... + 1.05^{29}$

$a = 1 \quad r = 1.05 \quad n = 30$

$$S_{30} = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$S_{30} = \frac{1\left(1.05^{30} - 1\right)}{1.05 - 1}$$

$S_{30} = 66.44$

$Y_{30} = 500 \times 66.44$

$Y_{30} = \$33\ 219.42$

Difference: \$33 219.42 – \$21 578.56 = \$11 640.86

Question 12

$Y_1 = 1000$

$Y_2 = 1000 \times 1.07 + 1000$

$= 1000(1 + 1.07)$

$Y_8 = 1000\left(1 + 1.07 + 1.07^2 + ... + 1.07^7\right)$

$1 + 1.07 + 1.07^2 + ... + 1.07^7$

$a = 1 \quad r = 1.07 \quad n = 8$

$S_8 = \dfrac{a\left(r^n - 1\right)}{r - 1}$

$S_8 = \dfrac{1\left(1.07^8 - 1\right)}{1.05 - 1}$

$S_8 = 10.26$

$Y_8 = 1000 \times 10.26$

$Y_8 = \$10\ 259.80$

She will have enough with $259.80 left over.

Question 13

$A_1 = 20 \times 1.0068\dot{3}$

$A_2 = \left(20 \times 1.068\dot{3} + 20\right) \times 1.0068\dot{3}$

$= 20\left(1.0068\dot{3}\right)^2 + 20 \times 1.00068\dot{3}$

$= 20\left(1.0068\dot{3}\right) + 20\left(1.0068\dot{3}\right)^2$

$= 20\left(1.0068\dot{3} + 1.0068\dot{3}^2\right)$

$A_{36} = 20\left(1.0068\dot{3} + 1.0068\dot{3}^2 + \ldots + 1.0068\dot{3}^{36}\right)$

$1.0068\dot{3} + 1.0068\dot{3}^2 + \ldots + 1.0068\dot{3}^{36}$

$a = 1.0068\dot{3} \quad r = 1.0068\dot{3} \quad n = 36$

$S_{36} = \dfrac{a\left(r^{36} - 1\right)}{r - 1}$

$S_{36} = \dfrac{1.0068\dot{3}\left(1.0068\dot{3}^{36} - 1\right)}{1.0068\dot{3} - 1}$

$S_{36} = 40.9358\ldots$

$A_{36} = 20 \times 40.9358\ldots$

$A_{36} = \$818.72$

Question 14

a $\$37.3790 \approx \$37.38$

b

$$Y_1 = 1$$
$$Y_2 = 1 \times 1.07 + 1$$
$$= 1(1 + 1.07)$$
$$Y_{19} = 1\left(1 + 1.07 + 1.07^2 + \ldots + 1.07^{18}\right)$$

$$1 + 1.07 + 1.07^2 + \ldots + 1.07^{18}$$
$$a = 1 \quad r = 1.07 \quad n = 19$$
$$S_{19} = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{19} = \frac{1\left(1.07^{19} - 1\right)}{1.05 - 1}$$
$$S_{19} = 37.3790$$

$$Y_{19} = 1 \times 37.3790$$
$$Y_{19} = 37.3790$$

which is the same as in part **a**.

Exercise 9.07 Reducing balance loans

Question 1

a

$$\begin{aligned}M_1 &= 20\ 000\times1.009-432.87\\ M_2 &= (20\ 000\times1.009-432.87)\times1.009-432.87\\ &= 20\ 000\times1.009^2-432.87(1+1.009)\\ M_3 &= 20\ 000\times1.009^3-432.87\left(1+1.009+1.009^2\right)\\ &= \$19\ 234.54\end{aligned}$$

b

$$\begin{aligned}M_1 &= 3500\times1.013-151.53\\ M_2 &= (3500\times1.013-151.53)\times1.013-151.53\\ &= 3500\times1.013^2-151.53(1+1.013)\\ M_3 &= 3500\times1.013^3-151.53\left(1+1.013+1.013^2\right)\\ &= \$3177.64\end{aligned}$$

c

$$\begin{aligned}M_1 &= 100\ 000\times1.022-2203.22\\ M_2 &= (100\ 000\times1.022-2203.22)\times1.022-2203.22\\ &= 100\ 000\times1.022^2-2203.22(1+1.022)\\ M_3 &= 100\ 000\times1.022^3-2203.22\left(1+1.022+1.022^2\right)\\ &= \$99\ 990.13\end{aligned}$$

d

$$\begin{aligned}M_1 &= 2000\times1.02-105.74\\ M_2 &= (2000\times1.02-105.74)\times1.02-105.74\\ &= 2000\times1.02^2-105.74(1+1.02)\\ M_3 &= 2000\times1.02^3-105.74\left(1+1.02+1.02^2\right)\\ &= \$1798.81\end{aligned}$$

e

$$\begin{aligned}M_1 &= 45\ 800\times1.01-504.30\\ M_2 &= \left(45\ 800\times1.01-504.30\right)\times1.01-504.30\\ &= 45\ 800\times1.01^2-504.30\left(1+1.01\right)\\ M_3 &= 45\ 800\times1.01^3-504.30\left(1+1.01+1.01^2\right)\\ &= \$45\ 659.71\end{aligned}$$

Question 2

a **i** $3 \times 12 \times 166.07 = \$5978.52$

ii $I = 5978.52 - 5000 = \$978.52$

iii

$$\begin{aligned}I &= \Pr n\\ 978.52 &= 5000\times r\times3\\ r &= \frac{978.52}{15000}\\ r &= 0.065\\ r &= 6.5\%\end{aligned}$$

b **i** $5 \times 12 \times 403.76 = \$24\ 225.60$

ii $I = 24\ 225.60 - 15\ 900 = \$8325.60$

iii

$$\begin{aligned}I &= \Pr n\\ 8325.60 &= 15900\times r\times5\\ r &= \frac{8325.60}{15900\times5}\\ r &= 0.105\\ r &= 10.5\%\end{aligned}$$

c **i** $12 \times 12 \times 1109.62 = \$159\ 785.28$

ii $I = 159\ 785.28 - 80\ 000 = \$79\ 785.28$

iii

$$I = Prn$$
$$79785.28 = 80000 \times r \times 12$$
$$r = \frac{79785.28}{80000 \times 12}$$
$$r = 0.0831$$
$$r = 8.3\%$$

d **i** $25 \times 12 \times 907.09 = \$272\ 127$

ii $I = 272\ 127 - 235\ 000 = \$37\ 127$

iii

$$I = \Pr n$$
$$37127 = 235000 \times r \times 25$$
$$r = \frac{37127}{235000 \times 25}$$
$$r = 0.0063$$
$$r = 0.63\%$$

e **i** $2 \times 12 \times 71.27 = \$1710.48$

ii $I = 1710.48 - 1348 = \$362.48$

iii

$$I = \Pr n$$
$$362.48 = 1348 \times r \times 2$$
$$r = \frac{362.48}{1348 \times 2}$$
$$r = 0.134$$
$$r = 13.4\%$$

Question 3

a $8 \times 19.33 = \$154.64$

b $15 \times 20.28 = \$304.20$

c $72 \times 11.87 = \$854.64$

d $430 \times 6.06 = \$2605.80$

e $312 \times 8.17 = \$2549.04$

f $137 \times 5.01 = \$686.37$

g $49 \times 11.61 = \$568.89$

h $765 \times 3.95 = \$3021.75$

i $925 \times 4.24 = \$3922$

j $1000 \times 5.68 = \$5680$

Question 4

a $680.50 \times 5.80 = \$3946.90$

b $3946.90 \times 20 \times 12 = \$947\ 256$

c $947\ 256 - 680\ 500 = \$266\ 756$

d

$$\begin{aligned}
I &= Prn \\
266\ 756 &= 680\ 500 \times r \times 20 \\
r &= \frac{266\ 756}{680\ 500 \times 20} \\
r &= 0.0196 \\
r &= 1.96\%
\end{aligned}$$

Question 5

a 81.12 ÷ 4 = 20.28

5 years

b 777 ÷ 75 = 10.36

10 years

c 937.95 ÷ 169 = 5.55

20 years

d 1560.25 ÷ 395 = 3.95

30 years

e 232.20 ÷ 20 = 11.61

10 years

f 3131.25 ÷ 625 = 5.01

25 years

g 1302 ÷ 120 = 10.85

10 years

h 1809.64 ÷ 281 = 6.44

15 years

i 72.15 ÷ 6.5 = 11.1

10 years

j 474.15 ÷ 81.75 = 5.8

20 years

Question 6

a $57.99 \div 3 = 13.99$

$r = 6\%$

b $619.65 \div 81 = 7.65$

$r = 4.5\%$

c $2307.36 \div 456 = 5.06$

$r = 2\%$

d $2571.56 \div 212 = 12.13$

$r = 8\%$

e $6515.36 \div 947 = 6.88$

$r = 5.5\%$

f $178.20 \div 9 = 19.8$

$r = 7\%$

g $2709.70 \div 686 = 3.95$

$r = 2.5\%$

h $1422 \div 300 = 4.74$

$r = 3\%$

i $6814.73 \div 845.5 = 8.06$

$r = 7.5\%$

j $3127.24 \div 422.6 = 7.4$

$r = 4\%$

Question 1

$$Y_1 = 3000 \times 1.22 - M$$
$$Y_2 = (3000 \times 1.22 - M) \times 1.22 - M$$
$$= 3000 \times 1.22^2 - M(1 + 1.22)$$
$$Y_5 = 3000 \times 1.22^5 - M(1 + 1.22 + .. + 1.22^4)$$
$$Y_5 = 3000 \times 1.22^5 - M(7.74)$$
$$0 = 3000 \times 1.22^5 - M(7.74)$$
$$7.74M = 3000 \times 1.22^5$$
$$M = \frac{3000 \times 1.22^5}{7.74}$$
$$M = \$1047.62$$

Question 2

$$A_1 = 20\ 000 \times 1.015 - M$$
$$A_2 = (20\ 000 \times 1.015 - M) \times 1.015 - M$$
$$= 20\ 000 \times 1.015^2 - M(1 + 1.015)$$
$$A_8 = 20\ 000 \times 1.015^8 - M(1 + 1.015 + .. + 1.015^7)$$

$$1 + 1.015 + 1.015^2 + ... + 1.015^7$$
$$a = 1 \quad r = 1.015 \quad n = 96$$
$$S_8 = \frac{a(r^n - 1)}{r - 1}$$
$$S_8 = \frac{1(1.015^{96} - 1)}{1.015 - 1}$$
$$S_8 = 211.7202...$$

$$A_8 = 20\ 000 \times 1.015^{96} - M(211.7202...)$$
$$0 = 20\ 000 \times 1.015^{96} - M(211.7202...)$$
$$211.7202......M = 20\ 000 \times 1.015^{96}$$
$$M = \frac{20\ 000 \times 1.015^{96}}{211.7202......}$$
$$M = \$394.46$$

Question 3

$$Y_1 = 5000 \times 1.0125 - M$$
$$Y_2 = \left(5000 \times 1.0125 - M\right) \times 1.0125 - M$$
$$= 5000 \times 1.0125^2 - M1.0125$$
$$Y_{48} = 5000 \times 1.0125^{48} - M\left(1 + 1.0125 + .. + 1.0125^{47}\right)$$
$$Y_{48} = 5000 \times 1.0125^{48} - M\left(65.2283...\right)$$
$$0 = 5000 \times 1.0125^{48} - M\left(65.2283...\right)$$
$$65.2283...M = 5000 \times 1.0125^{48}$$
$$M = \frac{5000 \times 1.0125^{48}}{65.2283...}$$
$$M = \$139.15$$

Question 4

a

$$M_1 = 150\ 000 \times 1.005 - M$$
$$M_2 = \left(150\ 000 \times 1.005 - M\right) \times 1.005 - M$$
$$= 150\ 000 \times 1.005^2 - M\left(1 + 1.005\right)$$
$$M_{300} = 150\ 000 \times 1.005^{300} - M\left(1 + 1.005 + .. + 1.005^{299}\right)$$

$$1 + 1.005 + 1.005^2 + ... + 1.005^{299}$$
$$a = 1 \quad r = 1.005 \quad n = 300$$
$$S_{300} = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{300} = \frac{1\left(1.005^{300} - 1\right)}{1.005 - 1}$$
$$S_{300} = 692.99$$

$$M_{300} = 150\ 000 \times 1.005^{300} - M\left(692.99\right)$$
$$0 = 150\ 000 \times 1.005^{300} - M\left(692.99\right)$$
$$692.99M = 150\ 000 \times 1.005^{300}$$
$$M = \frac{150\ 000 \times 1.005^{300}}{692.99}$$
$$M = 966.45$$

b

$$M_1 = 150\ 000 \times 1.005 - M$$
$$M_2 = \left(150\ 000 \times 1.005 - M\right) \times 1.005 - M$$
$$= 150\ 000 \times 1.005^2 - M\left(1 + 1.005\right)$$
$$M_{180} = 150\ 000 \times 1.005^{180} - M\left(1 + 1.005 + .. + 1.005^{179}\right)$$

$$1 + 1.005 + 1.005^2 + ... + 1.005^{179}$$
$$a = 1 \quad r = 1.005 \quad n = 180$$
$$S_{180} = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{180} = \frac{1\left(1.005^{180} - 1\right)}{1.005 - 1}$$
$$S_{180} = 290.8187...$$

$$M_{180} = 150\ 000 \times 1.005^{180} - M\left(290.82\right)$$
$$0 = 150\ 000 \times 1.005^{180} - M\left(290.82\right)$$
$$290.82M = 150\ 000 \times 1.005^{180}$$
$$M = \frac{150\ 000 \times 1.005^{180}}{290.8187...}$$
$$M = 1265.79$$

Question 5

$$A_1 = 6000 \times 1.125 - M$$
$$A_2 = \left(6000 \times 1.125 - M\right) \times 1.125 - M$$
$$= 6000 \times 1.125^2 - M\left(1 + 1.125\right)$$
$$A_3 = 6000 \times 1.125^3 - M\left(1 + 1.125 + 1.125^2\right)$$
$$A_3 = 6000 \times 1.125^3 - M\left(3.390625\right)$$
$$0 = 6000 \times 1.125^3 - M\left(3.390625\right)$$
$$3.390625M = 6000 \times 1.125^3$$
$$M = \frac{6000 \times 1.125^3}{3.390625}$$
$$M = \$2519.59$$

Question 6

a Loan = \$38 000 × 0.9 = \$34 200

$$A_1 = 34\ 200 \times 1.015 - M$$
$$A_2 = (34\ 200 \times 1.015 - M) \times 1.015 - M$$
$$= 34\ 200 \times 1.015^2 - M(1 + 1.015)$$
$$A_{60} = 34\ 200 \times 1.015^{60} - M(1 + 1.015 + .. + 1.015^{59})$$
$$A_{60} = 34\ 200 \times 1.015^{60} - M(96.2146...)$$
$$0 = 34\ 200 \times 1.015^{60} - M(96.2146...)$$
$$96.2146...M = 34\,200 \times 1.015^{60}$$
$$M = \frac{34\ 200 \times 1.015^{60}}{96.2146...}$$
$$M = \$868.46$$

b \$3800 (deposit) + \$868.46 × 60 = \$55 907.60

Question 7

a

$$M_1 = 2000 \times 1.015 - M$$
$$M_2 = \left(2000 \times 1.015 - M\right) \times 1.015 - M$$
$$= 2000 \times 1.015^2 - M\left(1 + 1.015\right)$$
$$M_{34} = 2000 \times 1.015^{34} - M\left(1 + 1.015 + .. + 1.015^{33}\right)$$

$$1 + 1.015 + 1.015^2 + ... + 1.015^{33}$$
$$a = 1 \quad r = 1.015 \quad n = 34$$
$$S_{34} = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{34} = \frac{1\left(1.015^{34} - 1\right)}{1.015 - 1}$$
$$S_{34} = 43.93$$

$$M_{34} = 2000 \times 1.015^{34} - M\left(43.93\right)$$
$$0 = 2000 \times 1.015^{34} - M\left(43.93\right)$$
$$43.93M = 2000 \times 1.015^{34}$$
$$M = \frac{2000 \times 1.015^{34}}{43.93}$$
$$M = \$77.81$$

b $77.81 \times 34 = \$2645.42$

Question 8

$M_1 = A\times 1.00958\dot{3} - 800$

$M_2 = \left(A\times 1.00958\dot{3} - 800\right)\times 1.00958\dot{3} - 800$

$= A\times 1.00958\dot{3}^2 - 800\left(1 + 1.00958\dot{3}\right)$

$M_{300} = A\times 1.00958\dot{3}^{300} - 800\left(1 + 1.00958\dot{3} + .. + 1.00958\dot{3}^{299}\right)$

$1 + 1.00958\dot{3} + 1.00958\dot{3}^2 + ... + 1.00958\dot{3}^{299}$

$a = 1 \quad r = 1.00958\dot{3} \quad n = 300$

$S_{300} = \dfrac{a\left(r^n - 1\right)}{r - 1}$

$S_{300} = \dfrac{1\left(1.00958\dot{3}^{300} - 1\right)}{1.00958\dot{3} - 1}$

$S_{300} = 1720.12$

$M_{300} = A\times 1.00958\dot{3}^{300} - 800\left(1720.12\right)$

$0 = A\times 1.00958\dot{3}^{300} - 800\left(1720.12\right)$

$1720.12\times 800 = A\times 1.00958\dot{3}^{300}$

$A = \dfrac{1720.12\times 800}{1.00958\dot{3}^{300}}$

$A = 78\ 703.83$

He can borrow $78 700.

Question 9

a **Get Rich**

$$M_1 = 80\ 000 \times 1.00625 - M$$
$$M_2 = (80\ 000 \times 1.00625 - M) \times 1.00625 - M$$
$$= 80\ 000 \times 1.00625^2 - M(1 + 1.00625)$$
$$M_{120} = 80\ 000 \times 1.00625^{120} - M(1 + 1.00625 + .. + 1.00625^{119})$$

$$1 + 1.00625 + 1.00625^2 + ... + 1.00625^{119}$$
$$a = 1 \quad r = 1.00625 \quad n = 120$$
$$S_{120} = \frac{a(r^n - 1)}{r - 1}$$
$$S_{120} = \frac{1(1.00625^{120} - 1)}{1.00625 - 1}$$
$$S_{120} = 177.9303...$$

$$A_{120} = 80\ 000 \times 1.00625^{120} - M(177.9303...)$$
$$0 = 80\ 000 \times 1.00625^{120} - M(177.9303...)$$
$$177.9303...M = 80\ 000 \times 1.00625^{120}$$
$$M = \frac{80\ 000 \times 1.00625^{120}}{177.9303...}$$
$$M = \$949.61$$

Capital

$$A_1 = 80\ 000 \times 1.00458\dot{3} - M$$

$$A_2 = \left(80\ 000 \times 1.00458\dot{3} - M\right) \times 1.0048\dot{3} - M$$

$$= 80\ 000 \times 1.00458\dot{3}^2 - M\left(1 + 1.00458\dot{3}\right)$$

$$A_{300} = 80\ 000 \times 1.00458\dot{3}^{300} - M\left(1 + 1.00458\dot{3} + .. + 1.00458\dot{3}^{299}\right)$$

$$1 + 1.00458\dot{3} + 1.00458\dot{3}^2 + \ldots + 1.00458\dot{3}^{299}$$

$$a = 1 \quad r = 1.00458\dot{3} \quad n = 300$$

$$S_{300} = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$S_{300} = \frac{1\left(1.00458\dot{3}^{300} - 1\right)}{1.00458\dot{3} - 1}$$

$$S_{300} = 642.0374...$$

$$A_{300} = 80\ 000 \times 1.00458\dot{3}^{300} - M\left(642.0374...\right)$$

$$0 = 80\ 000 \times 1.00458\dot{3}^{300} - M\left(642.0374...\right)$$

$$642.0374...M = 80\ 000 \times 1.00458\dot{3}^{300}$$

$$M = \frac{80\ 000 \times 1.00458\dot{3}^{300}}{642.0374...}$$

$$M = \$491.27$$

b **Get Rich**

949.61 × 120 = $113 953.20

Capital

$491.27 × 300 = $147 381

Difference:

$147 381 – $112 993.20 = $33 427.80 more through Capital.

Question 10

35 000 × 0.95 = 33 250

a

$$M_1 = 33\ 250 \times 1.01 - M$$
$$M_2 = \left(33\ 250 \times 1.01M\right) \times 1.01 - M$$
$$= 33\ 250 \times 1.01^2 - M\left(1 + 1.01\right)$$
$$M_{48} = 33\ 250 \times 1.01^{48} - M\left(1 + 1.01 + .. + 1.01^{47}\right)$$

$$1 + 1.01 + 1.01^2 + \ldots + 1.01^{47}$$
$$a = 1 \quad r = 1.01 \quad n = 48$$
$$S_{48} = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{48} = \frac{1\left(1.01^{48} - 1\right)}{1.01 - 1}$$
$$S_{48} = 61.22$$

$$M_{48} = 33\ 250 \times 1.01^{48} - M\left(61.22\right)$$
$$0 = 33\ 250 \times 1.01^{48} - M\left(61.22\right)$$
$$61.22M = 33\ 250 \times 1.01^{48}$$
$$M = \frac{33\ 250 \times 1.01^{48}}{61.22}$$
$$M = \$875.60$$

b 875.60 × 48 = 42 028.80

Total payments = 1750 + 42 028.80 = $43 778.80

Question 11

$A_1 = P\times 1.01291\dot{6} - 1200$

$A_2 = \left(P\times 1.01291\dot{6} - 1200\right)\times 1.01291\dot{6} - 1200$

$= P\times 1.01291\dot{6}^2 - 1200\left(1 + 1.01291\dot{6}\right)$

$A_{84} = P\times 1.01291\dot{6}^{84} - 1200\left(1 + 1.01291\dot{6} + .. + 1.01291\dot{6}^{83}\right)$

$1 + 1.0129 + 1.0129^2 + ... + 1.0129^{83}$

$a = 1 \quad r = 1.01291\dot{6} \quad n = 84$

$$S_{84} = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$S_{84} = \frac{1\left(1.01291\dot{6}^{84} - 1\right)}{1.01291\dot{6} - 1}$$

$S_{84} = 150.1122...$

$A_{84} = P\times 1.01291\dot{6}^{84} - 1200(150.1122...)$

$0 = P\times 1.01291\dot{6}^{84} - 1200(150.1122...)$

$150.1122...\times 1200 = P\times 1.01291\dot{6}^{84}$

$$P = \frac{150.1122...\times 1200}{1.01291\dot{6}^{84}}$$

$P = \$61\ 292.20$

Amy borrowed $61 292.20.

Question 12

NSW Bank

Let M stand for the monthly repayment.

Number of months is $3 \times 12 = 36$.

Monthly interest is $0.09 \div 12 = 0.0075$

$A_1 = 5\,000(1 + 0.0075)^1$

$A_2 = 5\,000(1.0075)^2$

$A_3 = 5\,000(1.0075)^3$

$A_4 = 5\,000(1.0075)^3(1.0075)^1 - M$

$= 5\,000(1.0075)^4 - M$

$A_5 = A_4(1.0075)^1 - M$

$= [5\,000(1.0075)^4 - M](1.0075)^1 - M$

$= 5\,000(1.0075)^5 - M(1.0075^1) - M$

$= 5\,000(1.0075)^5 - M(1.0075^1 + 1)$

Similarly:

$A_6 = 5\,000(1.0075)^6 - M(1.0075^2 + 1.0075^1 + 1)$

Continuing this pattern the last payment is:

$A_{36} = 5\,000(1.0075)^{36} - M(1.0075^{32} + 1.0075^{31} + \ldots + 1)$

But the loan is paid out after 36 months.

So $A_{36} = 0$

$0 = 5\,000(1.0075)^{36} - M(1.0075^{32} + 1.0075^{31} + 1.0075^{30} + \ldots + 1.0075^1 + 1)$

$M(1.0075)^{36} - M(1.0075^{32} + 1.0075^{31} + \ldots + 1.0075^1 + 1) = 5\,000(1.0075)^{36}$

$$M = \frac{5000(1.0075)^{36}}{1.0075^{32} + 1.0075^{31} + 1.0075^{30} + \ldots + 1.0075^1 + 1}$$

$$= \frac{5000(1.0075)^{36}}{1 + 1.0075^1 + \ldots + 1.0075^{30} + 1.0075^{31} + 1.0075^{32}}$$

$1 + 1.0075^1 + \ldots + 1.0075^{30} + 1.0075^{31} + 1.0075^{32}$ is a geometric series with

$a = 1$, $r = 1.0075$ and $n = 33$.

$S_n = \dfrac{a(r^n - 1)}{r - 1}$

$S_{33} = \dfrac{1(1.0075^{33} - 1)}{1.0075 - 1}$

$= \dfrac{(1.0075^{33} - 1)}{0.0075}$

$= 37.28$

$M = \dfrac{5\,000(1.0075)^{36}}{37.28} = 175.49$

So the monthly repayment is \$175.49.

Total amount repaid = \$175.49 $\times$ 33 = \$5791.25.

Sydney Bank

Let M stand for the monthly repayment.

Number of months is $3 \times 12 = 36$.

Monthly interest is $0.07 \div 12 = 0.0058$

$A_1 = 5\,000(1 + 0.0058)^1 - M$

$= 5\,000(1.0058)^1 - M$

$A_2 = A_1(1.0058)^1 - M$

$= [5\,000(1.0058)^1 - M](1.0058)^1 - M$

$= 5\,000(1.0075)^2 - M(1.0058^1) - M$

$= 5\,000(1.0075)^2 - M(1.0058^1 + 1)$

Similarly:

$A_3 = 5\,000(1.0075)^3 - M(1.0058^2 + 1.0058^1 + 1)$

Continuing this pattern the last payment is:

$A_{36} = 5\,000(1.0058)^{36} - M(1.0058^{35} + 1.0058^{34} + \ldots + 1)$

But the loan is paid out after 36 months.

So $A_{36} = 0$

$0 = 5\,000(1.0058)^{36} - M(1.0058^{35} + 1.0058^{34} + \ldots + 1)$

$M(1.0058)^{36} - M(1.0058^{35} + 1.0058^{34} + \ldots + 1.0058^{1} + 1) = 5\,000(1.0058)^{36}$

$$M = \frac{5000(1.0058)^{36}}{1.0058^{35} + 1.0058^{34} + \ldots + 1.0058^{1} + 1}$$

$$= \frac{5000(1.0058)^{36}}{1 + 1.0058^{1} + \ldots + 1.0058^{34} + 1.0058^{35}}$$

$1 + 1.0058^{1} + \ldots + 1.0058^{34} + 1.0058^{35}$ is a geometric series with

$a = 1$, $r = 1.0058$ and $n = 36$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{36} = \frac{1(1.0058^{36} - 1)}{1.0058 - 1}$$

$$= \frac{(1.0058^{36} - 1)}{0.0058}$$

$= 39.93$

$$M = \frac{5\,000(1.0058)^{36}}{39.93} = 154.39$$

So the monthly repayment is \$154.39.

Total amount repaid = \$154.39 $\times$ 36 = \$5557.88.

So Sydney Bank is better than NSW Bank because the total amount paid is lower.

Question 13

10 000 – 1500 = 8500

a

$$M_1 = 8500 \times 1.015 - M$$
$$M_2 = \left(8500 \times 1.015 M\right) \times 1.015 - M$$
$$= 8500 \times 1.015^2 - M\left(1 + 1.015\right)$$
$$M_{48} = 8500 \times 1.015^{48} - M\left(1 + 1.015 + .. + 1.015^{47}\right)$$

$$1 + 1.015 + 1.015^2 + ... + 1.015^{47}$$
$$a = 1 \quad r = 1.015 \quad n = 48$$
$$S_{48} = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{48} = \frac{1\left(1.015^{48} - 1\right)}{1.015 - 1}$$
$$S_{48} = 69.57$$

$$M_{48} = 8500 \times 1.015^{48} - M\left(69.57\right)$$
$$0 = 8500 \times 1.015^{48} - M\left(69.57\right)$$
$$69.57M = 8500 \times 1.015^{48}$$
$$M = \frac{8500 \times 1.015^{48}}{69.57}$$
$$M = \$249.69$$

b Total payments = 249.69 × 48 + 1500 = $13 485.12

Question 14

a

$$FV = PV\left(1+r\right)^n$$

$$FV = 12\ 000\left(1+\frac{0.02}{12}\right)^6$$

$$= \$13\ 251.13$$

b

$$M_7 = 13\ 251.13 \times 1.01\dot{6} - M$$

$$M_8 = \left(13\ 251.13 \times 1.01\dot{6}M\right) \times 1.01\dot{6} - M$$

$$= 13\ 251.13 \times 1.01\dot{6}^2 - M\left(1+1.01\dot{6}\right)$$

$$M_{60} = 13\ 251.13 \times 1.01\dot{6}^{54} - M\left(1+1.01\dot{6}+..+1.01\dot{6}^{53}\right)$$

$$1+1.01\dot{6}+1.01\dot{6}^2+...+1.01\dot{6}^{53}$$

$$a = 1 \quad r = 1.01\dot{6} \quad n = 54$$

$$S_{48} = \frac{a\left(r^n - 1\right)}{r-1}$$

$$S_{48} = \frac{1\left(1.01\dot{6}^{48} - 1\right)}{1.015-1}$$

$$S_{48} = 86.49$$

$$M_{48} = 13\ 251.13 \times 1.01\dot{6}^{48} - M\left(86.49\right)$$

$$0 = 13\ 251.13 \times 1.01\dot{6}^{48} - M\left(86.49\right)$$

$$86.49M = 13\ 251.13 \times 1.01\dot{6}^{48}$$

$$M = \frac{13\ 251.13 \times 1.01\dot{6}^{48}}{86.49}$$

$$M = \$374.07$$

c 374.07 × 54 = \$20 199.78

Question 15

a

$$Y_1 = 6000 \times 1.0125^{12} - M$$
$$Y_2 = \left(6000 \times 1.0125^{12} - M\right) \times 1.0125^{12} - M$$
$$= 6000 \times 1.0125^{24} - M\left(1 + 1.0125^{12}\right)$$
$$Y_5 = 6000 \times 1.0125^{60} - M\left(1 + 1.0125^{12} + .. + 1.0125^{48}\right)$$

$$1 + 1.0125^{12} + 1.0125^{24} + ... + 1.0125^{48}$$
$$a = 1 \quad r = 1.0125^{12} \quad n = 5$$
$$S_5 = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_5 = \frac{1\left(1.0125^{60} - 1\right)}{1.0125^{12} - 1}$$
$$S_8 = 6.89$$

$$Y_5 = 6000 \times 1.0125^{60} - M\left(6.89\right)$$
$$0 = 6000 \times 1.0125^{60} - M\left(6.89\right)$$
$$6.89M = 6000 \times 1.0125^{60}$$
$$M = \frac{6000 \times 1.0125^{60}}{6.89}$$
$$M = \$1835.68$$

b Payments = 1835.68 × 5 = \$9178.41

Question 16

a 10.36

b

$$M_1 = 1000 \times 1.00375 - M$$
$$M_2 = \left(1000 \times 1.00375 - M\right) \times 1.00375 - M$$
$$= 1000 \times 1.00375^2 - M\left(1 + 1.00375\right)$$
$$M_{120} = 1000 \times 1.00375^{120} - M\left(1 + 1.00375 + .. + 1.00375^{119}\right)$$

$$1 + 1.00375 + 1.00375^2 + ... + 1.00375^{119}$$
$$a = 1 \quad r = 1.00375 \quad n = 120$$
$$S_{120} = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{120} = \frac{1\left(1.00375^{120} - 1\right)}{1.00375 - 1}$$
$$S_{120} = 151.20$$

$$M_{120} = 1000 \times 1.00375^{120} - M\left(151.20\right)$$
$$0 = 1000 \times 1.00375^{120} - M\left(151.20\right)$$
$$151.20M = 1000 \times 1.00375^{120}$$
$$M = \frac{1000 \times 1.00375^{120}}{151.20}$$
$$M = 10.36$$

$10.36 \times 12 = 124.37$ (rounding difference)

Test yourself 9

Question 1

$$FV = PV(1+r)^n$$
$$FV = 2500(1+0.03)^{10}$$

C

Question 2

I = 79 500 – 68 000 = 11 500

$$I = \Pr n$$
$$11\,500 = 68\ 000 \times r$$
$$r = \frac{11\,500}{68\ 000} \times 100$$
$$r = 16.9\%$$

B

Question 3

$$a = 1.2 \qquad r = \frac{3}{8}$$
$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{1.2}{1-\frac{3}{8}}$$
$$S_\infty = 1.92$$

A

Question 4

$$\begin{aligned}M_1 &= 22\ 000\times1.01-226.29\\ M_2 &= \left(22\ 000\times1.01-226.29\right)\times1.01-226.29\\ &= 22\ 000\times1.01^2-226.29\left(1+1.01\right)\\ M_3 &= 22\ 000\times1.01^3-226.29\left(1+1.01+1.01^2\right)\\ &= \$21\,980.94\end{aligned}$$

Question 5

$$\begin{aligned}FV &= PV\left(1+r\right)^n\\ FV &= 1500\left(1+0.037\right)^3\\ FV &= \$1672.74\end{aligned}$$

Question 6

a 30 slats with a 3 mm gap

$30 \times 3 = 90$ mm

b $90 - 3 = 87$ mm

c Each slat rises 3mm less than the previous one hence $d = -3$.

d

$$\begin{aligned}t_n &= a+\left(n-1\right)d\\ t_{17} &= 90+\left(17-1\right)\times\left(-3\right)\\ &= 42\text{ mm}\end{aligned}$$

e

$$\begin{aligned}S_n &= \frac{n}{2}\left(a+l\right)\\ S_{30} &= \frac{30}{2}\left(90+3\right)\\ S_{30} &= 1395\text{ mm}\end{aligned}$$

Question 7

a 2022 × 5 × 12 = $121 320

b I = 121 320 – 62 500 = $58 820

c

$$\begin{aligned}&I = Prn\\&58820 = 62500 \times r \times 5\\&r = \frac{58820}{62500 \times 5} \times 100\\&r = 18.8\%\end{aligned}$$

Question 8

a 595 × 1.2155 = $723.22

b 5000 × 1.9990 = $9995

c 1651.20 × 1.7716 = $2925.27

d 13500 × 1.2434 = $16 785.90

e 9485 × 1.1961 = $11 345.01

Question 9

a

$$\begin{aligned}&t_n = a + (n-1)d\\&t_{10} = 20\ 000 + (10-1) \times 450\\&= \$24\ 050\end{aligned}$$

b

$$\begin{aligned}&S_n = \frac{n}{2}(a + l)\\&S_{10} = \frac{10}{2}(20\ 000 + 24\ 050)\\&S_{10} = \$220\ 250\end{aligned}$$

Question 10

a

$$M_1 = 186\ 900 \times 1.0025 - 2500$$
$$M_1 = \$184\ 867.25$$

b

$$M_2 = (186\ 900 \times 1.0025 - 2500) \times 1.0025 - 2500$$
$$= 186\ 900 \times 1.0025^2 - 2500(1 + 1.0025)$$
$$M_2 = \$182\ 829.42$$

c

$$M_3 = 186\ 900 \times 1.0025^3 - 2500\left(1 + 1.0025 + 1.0025^2\right)$$
$$M_3 = \$180\ 786.49$$

Question 11

$$Y_1 = M$$
$$Y_2 = M \times 1.13 + M$$
$$= M(1. + 1.13)$$
$$Y_{25} = 200\ 000 = M\left(1 + 1.13 + 1.13^2 + ... + 1.13^{24}\right)$$

$$1 + 1.13 + 1.13^2 + ... + 1.13^{24}$$
$$a = 1 \quad r = 1.13 \quad n = 25$$
$$S_{25} = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{25} = \frac{1\left(1.13^{25} - 1\right)}{1.13 - 1}$$
$$S_{25} = 155.62$$

$$200\ 000 = M \times 155.62$$
$$M = \frac{200\ 000}{155.62}$$
$$M = \$1285.19$$

Question 12

a

$$t_n = a + (n-1)d$$
$$0 = 20 + (n-1) \times (-2)$$
$$-20 = -2(n-1)$$
$$n - 1 = 10$$
$$n = 11$$

The 11th row has 0 boxes, so there are 10 rows.

b

$$S_n = \frac{n}{2}(a + l)$$
$$S_{10} = \frac{10}{2}(20 + 2)$$
$$S_{10} = 110$$

There are 110 boxes.

Question 13

a

$$0.\dot{4} = 0.4 + 0.04 + 0.004 + ...$$
$$a = 0.4 \qquad r = 0.1$$
$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{0.4}{1-0.1}$$
$$S_\infty = \frac{4}{9}$$

b

$$0.7\dot{2} = 0.7 + 0.02 + 0.002 + ...$$
$$= \frac{7}{10} + 0.02 + 0.002 + ...$$
$$a = 0.02 \qquad r = 0.1$$
$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{0.02}{0.9}$$
$$S_\infty = \frac{1}{45}$$
$$0.7\dot{2} = \frac{7}{10} + \frac{1}{45}$$
$$= \frac{13}{18}$$

c

$$1.\dot{5}\dot{7} = 1 + 0.57 + 0.0057 + 0.000057 + ...$$
$$0.57 + 0.0057 + 0.000057 + ...$$
$$a = 0.57 \qquad r = 0.01$$
$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{0.57}{0.99}$$
$$S_\infty = \frac{57}{99}$$
$$1.\dot{5}\dot{7} = 1\frac{57}{99} = 1\frac{19}{33}$$

Question 14

$$FV = PV(1+r)^n$$
$$5860.91 = PV(1+0.095)^6$$
$$PV = \frac{5860.91}{(1+0.095)^6}$$
$$PV = \$3400.01$$

Question 15

a 50 000 × 1.18 = $59 000

b

$$\begin{aligned} Y_1 &= 50\ 000 \times 1.18 - M \\ Y_2 &= (50\ 000 \times 1.18 - M) \times 1.18 - M \\ &= 50\ 000 \times 1.18^2 - M(1+1.18) \\ Y_5 &= 50\ 000 \times 1.18^5 - M(1+1.18+..+1.18^4) \\ Y_5 &= 50\ 000 \times 1.18^5 - M(7.15) \\ 0 &= 50\ 000 \times 1.18^5 - M(7.15) \\ 7.15M &= 50\ 000 \times 1.18^5 \\ M &= \frac{50\ 000 \times 1.18^5}{7.15} \\ M &= \$15\ 988.89 \end{aligned}$$

Question 16

a

$$FV = PV(1+r)^n$$
$$FV = 2000(1+0.045)^4$$
$$FV = \$2385.04$$

b

$$FV = PV(1+r)^n$$
$$FV = 2000\left(1+\frac{0.045}{4}\right)^{16}$$
$$FV = \$2392.03$$

Question 17

a

$$M_1 = 200\ 000 \times 1.005 - M$$
$$M_2 = \left(200\ 000 \times 1.005 - M\right) \times 1.005 - M$$
$$= 200\ 000 \times 1.005^2 - M\left(1 + 1.005\right)$$
$$M_{240} = 200\ 000 \times 1.005^{240} - M\left(1 + 1.005 + .. + 1.005^{239}\right)$$

$$1 + 1.005 + 1.005^2 + ... + 1.005^{239}$$
$$a = 1 \quad r = 1.005 \quad n = 240$$
$$S_{240} = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_{240} = \frac{1\left(1.005^{240} - 1\right)}{1.005 - 1}$$
$$S_{240} = 462.04$$

$$M_{240} = 200\ 000 \times 1.005^{300} - M\left(462.04\right)$$
$$0 = 200\ 000 \times 1.005^{240} - M\left(462.04\right)$$
$$462.04M = 200\ 000 \times 1.005^{240}$$
$$M = \frac{200\ 000 \times 1.005^{240}}{462.04}$$
$$M = \$1432.86$$

b 1432.86 × 240 = \$343 886.91

Question 18

a

$$A_1 = M \times 1.05$$
$$A_2 = \left(M \times 1.05 + M\right) \times 1.05$$
$$= M\left(1.05 + 1.05^2\right)$$
$$A_4 = M\left(1.05 + 1.05^2 + 1.05^4 + 1.05^4\right)$$
$$A_4 = M\left(4.5256...\right)$$
$$12\ 000 = M\left(4.5256...\right)$$
$$M = \frac{12\ 000}{4.5256...}$$
$$M = \$2651.56$$

b

$$FV = PV\left(1+r\right)^n$$
$$12\ 000 = PV\left(1+0.05\right)^4$$
$$PV = \frac{12\ 000}{\left(1+0.05\right)^4}$$
$$PV = \$9872.43$$

Question 19

a

$$a = 18 \quad d = 3$$
$$t_n = a + (n-1)d$$
$$t_8 = 18 + (8-1) \times 3$$
$$t_8 = 18 + 21$$
$$t_8 = 39 \text{ words per minute}$$

b

$$a = 18 \quad d = 3$$
$$t_n = a + (n-1)d$$
$$60 = 18 + (n-1) \times 3$$
$$42 = 3(n-1)$$
$$14 = n - 1$$
$$n = 15$$

15 weeks

Question 20

$$a = 1.2 \quad r = \frac{3}{5}$$
$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{1.2}{1-\frac{3}{5}}$$
$$S_\infty = 3m$$

Double this result and subtract 1.2 for the first bounce.

Distance = 2 × 3 – 1.2 = 4.8 m

Question 21

a **i** Approximately $33 000

ii Approximately $23 000

b **i** 14 years

ii 28 years

c 32 years

Challenge exercise 9

Question 1

$$Y_1 = 300 \times 1.015^4$$
$$Y_2 = \left(300 \times 1.015^4 + 300\right) \times 1.015^4$$
$$= 300\left(1.015^4 + 1.015^8\right)$$
$$Y_5 = 300\left(1.015^4.+1.015^8 + 1.015^{12} + 1.015^{16} + 1.015^{20}\right)$$
$$Y_5 = \$1799.79$$

Question 2

a

$$n = 10 \quad a = 58 \quad d = -2$$
$$t_n = a + (n-1)d$$
$$t_{10} = 58 + (10-1) \times (-2)$$
$$t_{10} = 40$$

Each pair of shoes will cost $40.

b

$$n = 6 \quad a = 58 \quad d = -2$$
$$t_n = a + (n-1)d$$
$$t_6 = 58 + (6-1) \times (-2)$$
$$t_6 = 48$$

The shoes cost $48 per pair, so the total cost is 60 × $48 = $2880.

Question 3

For 4 years.

$$FV = PV\left(1+r\right)^n$$
$$FV = 5000\left(1+\frac{0.085}{12}\right)^{48}$$
$$FV = 7016.32$$

Following 3 years

$$FV = PV\left(1+r\right)^n$$
$$FV = 7016.32\left(1+\frac{0.065}{12}\right)^{36}$$
$$FV = \$8522.53$$

Question 4

$r = \frac{425}{500} = 0.85$, $a = 500$

a **i**

$$t_n = ar^{n-1}$$
$$t_{10} = 500 \times 0.85^9$$
$$t_{10} = 115.81°$$

ii

$$t_n = ar^{n-1}$$
$$t_{15} = 500 \times 0.85^{14}$$
$$t_{10} = 51.38°$$

b **i**

$$t_n = ar^{n-1}$$
$$200 = 500 \times 0.85^{n-1}$$
$$0.85^{n-1} = 0.4$$
$$\ln 0.85^{n-1} = \ln 0.4$$
$$(n-1)\ln 0.85 = \ln 0.4$$
$$(n-1) = \frac{\ln 0.4}{\ln 0.85}$$
$$n-1 = 5.64$$
$$n = 6.64$$

6.64 minutes

ii

$$t_n = ar^{n-1}$$
$$100 = 500 \times 0.85^{n-1}$$
$$0.85^{n-1} = 0.2$$
$$\ln 0.85^{n-1} = \ln 0.2$$
$$(n-1)\ln 0.85 = \ln 0.2$$
$$(n-1) = \frac{\ln 0.2}{\ln 0.85}$$
$$n-1 = 9.9$$
$$n = 10.9$$

10.9 minutes

Question 5

a

$$FV = PV\left(1+r\right)^n$$
$$FV = \$1000\left(1.06\right)^{25}$$

b

$$FV = PV\left(1+r\right)^n$$
$$FV = \$1000\left(1.06\right)^{24}$$

c

$$FV = PV\left(1+r\right)^n$$
$$FV = \$1000\left(1.06\right)^{23}$$

d

$$FV = PV\left(1+r\right)^n$$
$$FV = \$1000\left(1.06\right)$$

e

$$Y_{25} = 1000\left(1.06+1.06^2+\ldots+1.06^{25}\right)$$

$$1.06+1.06^2+\ldots+1.06^{25}$$
$$a = 1.06 \quad r = 1.06 \quad n = 25$$
$$S_{25} = \frac{a\left(r^n-1\right)}{r-1}$$
$$S_{25} = \frac{1.06\left(1.06^{25}-1\right)}{1.06-1}$$

$$Y_{25} = 1000\times\frac{1.06\left(1.06^{25}-1\right)}{0.06}$$

f

$$Y_{25} = 1000\times\frac{1.06\left(1.06^{25}-1\right)}{0.06}$$
$$Y_{25} = \$58\,156.38$$

Question 6

a $M_1 = 10\ 000 \times 1.01$ = \$10 100

b $M_{12} = 10\ 000 \times 1.01^{12}$ = \$11 268.25

c

$$Y_1 = 10\ 000 \times 1.01^{12} - M$$
$$Y_2 = \left(10\ 000 \times 1.01^{12} - M\right) \times 1.01^{12} - M$$
$$Y_2 = 10\ 000 \times 1.01^{24} - M\left(1 + 1.01^{12}\right)$$
$$Y_3 = 10\ 000 \times 1.01^{36} - M\left(1 + 1.01^{12} + 1.01^{24}\right)$$

$$1 + 1.01^{12} + 1.01^{24}$$
$$a = 1 \quad r = 1.01^{12} \quad n = 3$$
$$S_3 = \frac{a\left(r^n - 1\right)}{r - 1}$$
$$S_3 = \frac{1\left(1.01^{36} - 1\right)}{1.01^{12} - 1}$$
$$S_3 = 3.40$$

$$Y_3 = 0 = 10\ 000 \times 1.01^{36} - M\left(3.40\right)$$
$$M\left(3.40\right) = 10000 \times 1.01^{36}$$
$$M = \frac{10\ 000 \times 1.01^{36}}{3.40}$$
$$M = \$4212.41$$

d Total payments = \$4212.41 × 3 = \$12 637.23

Interest = \$12 637.23 – \$10 000 = \$2637.23

Question 7

$Y_1 = 5000 \times 1.06$

$Y_2 = \left(5000 \times 1.06 + 5000\right) \times 1.06$

$= 5000\left(1.06 + 1.06^2\right)$

$Y_{10} = 5000\left(1.06 + 1.06^2 + ... + 1.06^{10}\right)$

$1.06 + 1.06^2 + ... + 1.06^{10}$

$a = 1.06 \quad r = 1.06 \quad n = 10$

$$S_{10} = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$S_{10} = \frac{1.06\left(1.06^{10} - 1\right)}{1.06 - 1}$$

$S_{10} = 13.97$

$Y_{10} = 5000 \times 13.97$

$Y_{10} = 69\ 858.21$

This amount left to earn compound interest for 15 years gives

$FV = PV\left(1 + r\right)^n$

$FV = 69\ 858.21\left(1.1\right)^{15}$

$FV = 291\ 815.09$

Starting fresh in year 11

$Y_{11}=5000\times 1.1$

$Y_{12}=\left(5000\times 1.1+5000\right)\times 1.1$

$=5000\left(1.1+1.1^2\right)$

$Y_{25}=5000\left(1.1+1.1^2+...+1.1^{15}\right)$

$1.1+1.1^2+...+1.1^{10}$

$a=1.1\quad r=1.1\quad n=15$

$$S_{15}=\frac{a\left(r^n-1\right)}{r-1}$$

$$S_{15}=\frac{1.1\left(1.1^{10}-1\right)}{1.1-1}$$

$S_{15}=34.95$

$Y_{25}=5000\times 34.5$

$Y_{25}=174\ 748.65$

Total = 291 815.09 + 174 748.65 = $466 563.74

MATHS IN FOCUS 12 MATHEMATICS ADVANCED

WORKED SOLUTIONS

Chapter 10: Continuous probability distributions

Exercise 10.01 Probability density functions

Question 1

a Discrete

b Continuous

c Continuous

d Discrete

e Continuous

Question 2

a $5 \times 0.2 = 1$; Yes, it does.

b $\frac{0}{12}+\frac{1}{12}+\frac{2}{12}+\frac{3}{12}+\frac{4}{12}+\frac{5}{12}+\frac{6}{12} \geq 1$; It does not.

c

$$f(x)=\frac{x^3}{324} \qquad 0\le x\le 3$$

$$\int_0^3 \frac{x^3}{324}dx$$

$$=\left[\frac{x^4}{1296}\right]_0^3$$; It does not.

$$=\frac{3^4}{1296}$$

$$=\frac{81}{1296}\ne 1$$

d

$$f(x)=\frac{x^2}{21} \qquad 1\le x\le 4$$

$$\int_1^4 \frac{x^2}{21}dx$$

$$=\left[\frac{x^3}{63}\right]_1^4$$; Yes, it does.

$$=\frac{4^3}{63}-\frac{1^3}{63}$$

$$=\frac{63}{63}=1$$

e

$$f(x)=\frac{x}{8} \qquad 1\le x\le 8$$

$$\int_1^8 \frac{x}{8}dx$$

$$=\left[\frac{x^2}{16}\right]_1^8$$; It does not.

$$=\frac{8^2}{16}-\frac{1^2}{16}$$

$$=\frac{63}{16}\ne 1$$

Question 3

a $6\times\frac{1}{6}=1$: Yes, it does.

b $12\times\frac{1}{10}=\frac{6}{5}\neq 1$; It does not.

c $\frac{1}{2}\times 8\times\frac{1}{8}=\frac{1}{2}\neq 1$; It does not.

d $\frac{1}{2}\times 20\times\frac{1}{10}=1$; Yes, it does.

e $\frac{1}{2}\times 9\times\frac{1}{9}=\frac{1}{2}\neq 1$; No, it does not.

Question 4

$$f(x)=\frac{x^4}{3355} \qquad [2,b]$$

$$\int_2^b \frac{x^4}{3355}\,dx=1$$

$$\left[\frac{x^5}{16775}\right]_2^b=1$$

$$\frac{b^5}{16775}-\frac{2^5}{16775}=1$$

$$b^5-32=16775$$

$$b^5=16807$$

$$b=7$$

Question 5

$$f(x) = kx^3 \qquad [0,5]$$

$$\int_0^5 kx^3 dx = 1$$

$$\left[\frac{kx^4}{4}\right]_0^5 = 1$$

$$\frac{k \times 5^4}{4} - \frac{k \times 0^4}{4} = 1$$

$$\frac{625k}{4} = 1$$

$$k = \frac{4}{625}$$

Question 6

a

$$f(x) = ae^x \qquad [1,3]$$

$$\int_1^3 ae^x dx = 1$$

$$\left[ae^x\right]_1^3 = 1$$

$$ae^3 - ae^1 = 1$$

$$a\left(e^3 - e\right) = 1$$

$$a = \frac{1}{e^3 - e}$$

b

$$f(x) = ae^x \qquad [1,7]$$

$$\int_1^7 ae^x dx = 1$$

$$\left[ae^x\right]_1^7 = 1$$

$$ae^7 - ae^1 = 1$$

$$a\left(e^7 - e\right) = 1$$

$$a = \frac{1}{e^7 - e}$$

c

$$f(x) = ae^x \qquad [0,4]$$
$$\int_0^4 ae^x dx = 1$$
$$\left[ae^x\right]_0^4 = 1$$
$$ae^4 - ae^0 = 1$$
$$a\left(e^4 - 1\right) = 1$$
$$a = \frac{1}{e^4 - 1}$$

Question 7

$$f(x) = \frac{x^2}{72} \qquad [0,a]$$
$$\int_0^a \frac{x^2}{72} dx = 1$$
$$\left[\frac{x^3}{216}\right]_0^a = 1$$
$$\frac{a^3}{216} - \frac{0^3}{216} = 1$$
$$a^3 = 216$$
$$a = 6$$

Domain is [0, 6]

Question 8

$$f\left(x\right)=\frac{2x^5}{87\ 381} \qquad 1\le x\le b$$

$$\int_1^b \frac{2x^5}{87\ 381}\,dx=1$$

$$\left[\frac{x^6}{262\ 143}\right]_1^b=1$$

$$\frac{b^6}{262\ 143}-\frac{1^6}{262\ 143}=1$$

$$b^6-1=262\ 143$$

$$b^6=262\ 144$$

$$b=8$$

Exercise 10.02 Calculating probabilities

Question 1

a

$$P(X \le 3)$$
$$= 3 \times \frac{1}{6} = \frac{1}{2}$$

b

$$P(1 \le X \le 2)$$
$$= 1 \times \frac{1}{6} = \frac{1}{6}$$

c

$$P(1 \le X \le 4)$$
$$= 3 \times \frac{1}{6} = \frac{1}{2}$$

d

$$P(X < 4)$$
$$= 4 \times \frac{1}{6} = \frac{2}{3}$$

e

$$P(X > 4)$$
$$= 1 - P(X \le 4)$$
$$= 1 - \frac{2}{3}$$
$$= \frac{1}{3}$$

Question 2

a

$$m=\frac{\text{rise}}{\text{run}}=\frac{\frac{1}{5}}{10}=\frac{1}{50}$$

$$y=\frac{x}{50}$$

b **i**

$$P\left(X<9\right)$$

$$y=\frac{9}{50}$$

$$A=\frac{1}{2}bh$$

$$=\frac{1}{2}\times\frac{9}{50}\times 9=\frac{81}{100}$$

ii

$$P\left(X\le 3\right)$$

$$y=\frac{3}{50}$$

$$A=\frac{1}{2}bh$$

$$=\frac{1}{2}\times\frac{3}{50}\times 3=\frac{9}{100}$$

iii

$$P(4 \le X \le 7)$$
$$= P(X \le 7) - P(X \le 4)$$
$$P(X \le 4)$$
$$y = \frac{4}{50}$$
$$A = \frac{1}{2}bh$$
$$= \frac{1}{2} \times \frac{4}{50} \times 4 = \frac{16}{100}$$
$$P(X \le 7)$$
$$y = \frac{7}{50}$$
$$A = \frac{1}{2}bh$$
$$= \frac{1}{2} \times \frac{7}{50} \times 7 = \frac{49}{100}$$
$$P(X \le 7) - P(X \le 4)$$
$$= \frac{49}{100} - \frac{16}{100} = \frac{33}{100}$$

iv

$$P(2 \le X \le 6)$$
$$= P(X \le 6) - P(X \le 2)$$
$$P(X \le 2)$$
$$y = \frac{2}{50}$$
$$A = \frac{1}{2}bh$$
$$= \frac{1}{2} \times \frac{2}{50} \times 2 = \frac{4}{100}$$
$$P(X \le 6)$$
$$y = \frac{6}{50}$$
$$A = \frac{1}{2}bh$$
$$= \frac{1}{2} \times \frac{6}{50} \times 6 = \frac{36}{100}$$
$$P(X \le 6) - P(X \le 2)$$
$$= \frac{36}{100} - \frac{4}{100} = \frac{32}{100} = \frac{8}{25}$$

v

$$P(X > 5)$$
$$1 - P(X \le 5)$$
$$y = \frac{5}{50}$$
$$A = \frac{1}{2}bh$$
$$= \frac{1}{2} \times \frac{5}{50} \times 5 = \frac{25}{100} = \frac{1}{4}$$
$$1 - P(X \le 5)$$
$$= 1 - \frac{1}{4}$$
$$= \frac{3}{4}$$

Question 3

a

$$f(x) = ax^2 \qquad 0 \le x \le 5$$

$$\int_0^5 ax^2 dx = 1$$

$$\left[\frac{ax^3}{3}\right]_0^5 = 1$$

$$\frac{a\times 5^3}{3} - \frac{a\times 0^3}{3} = 1$$

$$\frac{125a}{3} = 1$$

$$a = \frac{3}{125}$$

$$f(x) = \frac{3x^2}{125}$$

b **i**

$$P(X \le 3)$$

$$\int_0^3 \frac{3x^2}{125}\, dx$$

$$= \left[\frac{x^3}{125}\right]_0^3$$

$$= \frac{3^3}{125} - \frac{0^3}{125} = \frac{27}{125}$$

ii

$$P(1 < X < 4)$$

$$\int_1^4 \frac{3x^2}{125} dx$$

$$= \left[\frac{x^3}{125}\right]_1^4$$

$$= \frac{4^3}{125} - \frac{1^3}{125}$$

$$= \frac{64-1}{125}$$

$$= \frac{63}{125}$$

iii

$$P(X > 2)$$

$$\int_2^5 \frac{3x^2}{125}dx$$

$$= \left[\frac{x^3}{125}\right]_2^5$$

$$= \frac{5^3}{125} - \frac{2^3}{125}$$

$$= \frac{125-8}{125}$$

$$= \frac{117}{125}$$

iv

$$P(X < 1)$$

$$\int_0^1 \frac{3x^2}{125}dx$$

$$= \left[\frac{x^3}{125}\right]_0^1$$

$$= \frac{1^3}{125} - \frac{0^3}{125}$$

$$= \frac{1}{125}$$

v

$$P(3 \le X < 4)$$

$$\int_3^4 \frac{3x^2}{125}dx$$

$$= \left[\frac{x^3}{125}\right]_3^4$$

$$= \frac{4^3}{125} - \frac{3^3}{125}$$

$$= \frac{64-27}{125}$$

$$= \frac{37}{125}$$

Question 4

a

$$f(x) = ax^3 \qquad 0 \le x \le 3$$

$$\int_0^3 ax^3 dx = 1$$

$$\left[\frac{ax^4}{4}\right]_0^3 = 1$$

$$\frac{a \times 3^4}{4} - \frac{a \times 0^4}{4} = 1$$

$$\frac{81a}{4} = 1$$

$$a = \frac{4}{81}$$

$$f(x) = \frac{4x^3}{81}$$

b **i**

$$P(1 \le X \le 3)$$

$$\int_1^3 \frac{4x^3}{81} dx$$

$$= \left[\frac{x^4}{81}\right]_1^3$$

$$= \frac{3^4}{81} - \frac{1^4}{81}$$

$$= \frac{81-1}{81}$$

$$= \frac{80}{81}$$

ii

$$P(X < 2)$$

$$\int_0^2 \frac{4x^3}{81} dx$$

$$= \left[\frac{x^4}{81}\right]_0^2$$

$$= \frac{2^4}{81} - \frac{0^4}{81} = \frac{16}{81}$$

iii

$P(1 \le X \le 2)$

$\int_1^2 \frac{4x^3}{81}\,dx$

$= \left[\frac{x^4}{81}\right]_1^2$

$= \frac{2^4}{81} - \frac{1^4}{81}$

$= \frac{16-1}{81}$

$= \frac{15}{81}$

$= \frac{5}{27}$

iv

$P(X \le 1)$

$\int_0^1 \frac{4x^3}{81}\,dx$

$= \left[\frac{x^4}{81}\right]_0^1$

$= \frac{1^4}{81} - \frac{0^4}{81}$

$= \frac{1}{81}$

Question 5

a

$$f(x) = ke^x \qquad [1, 6]$$

$$\int_1^6 ke^x dx = 1$$

$$\left[ke^x\right]_1^6 = 1$$

$$ke^6 - ke^1 = 1$$

$$k\left(e^6 - e\right) = 1$$

$$k = \frac{1}{e^6 - e}$$

b **i**

$$P\left(2 \le X \le 5\right)$$

$$= \frac{1}{e^6 - e}\int_2^5 e^x\, dx$$

$$= \frac{1}{e^6 - e}\left[e^x\right]_2^5$$

$$= \frac{e^5 - e^2}{e^6 - e}$$

$$= \frac{e^4 - e}{e^5 - 1}$$

$$= \frac{e\left(e^3 - 1\right)}{e^5 - 1}$$

ii

$$P\left(X < 4\right)$$

$$= \frac{1}{e^6 - e}\int_1^4 e^x\, dx$$

$$= \frac{1}{e^6 - e}\left[e^x\right]_1^4$$

$$= \frac{e^4 - e^1}{e^6 - e}$$

$$= \frac{e^3 - 1}{e^5 - 1}$$

iii

$$P(X \geq 3)$$
$$= \frac{1}{e^6 - e}\int_3^6 e^x \, dx$$
$$= \frac{1}{e^6 - e}\left[e^x\right]_3^6$$
$$= \frac{e^6 - e^3}{e^6 - e}$$
$$= \frac{e^5 - e^2}{e^5 - 1}$$
$$= \frac{e^2\left(e^3 - 1\right)}{e^5 - 1}$$

Question 6

a

$$f(x) = \sin x \qquad \left[0, \frac{\pi}{2}\right]$$
$$\int_0^{\frac{\pi}{2}} \sin x \, dx$$
$$= \left[-\cos x\right]_0^{\frac{\pi}{2}}$$
$$= -\cos\left(\frac{\pi}{2}\right) - \left(-\cos[0]\right)$$
$$= 0 - (-1)$$
$$= 1$$

It is a probability density function.

b **i**

$$P\left(X \le \frac{\pi}{3}\right)$$
$$= \int_0^{\frac{\pi}{3}} \sin x \, dx$$
$$= \left[-\cos x\right]_0^{\frac{\pi}{3}}$$
$$= -\cos\left(\frac{\pi}{3}\right) - \left(-\cos[0]\right)$$
$$= -\frac{1}{2} - (-1)$$
$$= \frac{1}{2}$$

ii

$$P\left(0 \le X \le \frac{\pi}{4}\right)$$
$$\int_0^{\frac{\pi}{4}} \sin x \, dx$$
$$= \left[-\cos x\right]_0^{\frac{\pi}{4}}$$
$$= -\cos\left(\frac{\pi}{4}\right) - \left(-\cos[0]\right)$$
$$= -\frac{\sqrt{2}}{2} - (-1)$$
$$= \frac{2-\sqrt{2}}{2}$$

iii

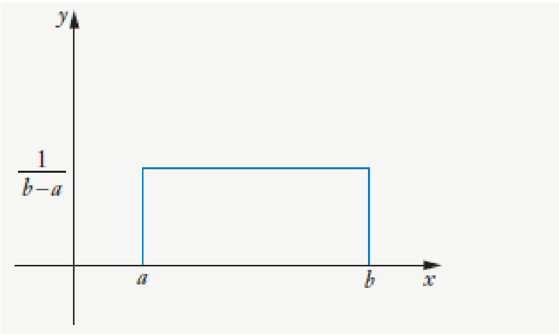
$$P\left(X > \frac{\pi}{6}\right)$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x \, dx$$
$$= [-\cos x]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= -\cos\left(\frac{\pi}{2}\right) - \left(-\cos\left[\frac{\pi}{6}\right]\right)$$
$$= \cos\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{2}$$

Question 7

a

$$f(x) = \frac{1}{b-a} \qquad [a,b]$$
$$\int_a^b \frac{1}{b-a} \, dx$$
$$= \left[\frac{x}{b-a}\right]_a^b$$
$$= \frac{b}{b-a} - \frac{a}{b-a}$$
$$= \frac{b-a}{b-a}$$
$$= 1$$

It is a probability density function.



b $a = 3, b = 7$

$$f(x) = \frac{1}{7-3}$$
$$f(x) = \frac{1}{4}$$

i

$$P(X \le 6)$$
$$= 3 \times \frac{1}{4} = \frac{3}{4}$$

ii

$$P(X \ge 5)$$
$$= 2 \times \frac{1}{4} = \frac{1}{2}$$

iii

$$P(5 \le X \le 6)$$
$$= 1 \times \frac{1}{4} = \frac{1}{4}$$

Exercise 10.03 Cumulative distribution function

Question 1

a

$$f(x)=\frac{x^2}{9} \qquad 0\le x\le 3$$

$$F(x)=\int_0^x \frac{x^2}{9}\,dx$$

$$=\left[\frac{x^3}{27}\right]_0^x$$

$$=\frac{x^3}{27}$$

b

$$f(x)=\frac{4x^3}{1296} \qquad [0,6]$$

$$F(x)=\int_0^x \frac{4x^3}{1296}\,dx$$

$$=\left[\frac{x^4}{1296}\right]_0^x$$

$$=\frac{x^4}{1296}-\frac{0^4}{1296}$$

$$=\frac{x^4}{1296}$$

c

$$f(x)=\frac{e^x}{e^4-1} \qquad 0\le x\le 4$$

$$F(x)=\int_0^x \frac{e^x}{e^4-1}\,dx$$

$$=\left[\frac{e^x}{e^4-1}\right]_0^x$$

$$=\frac{e^x}{e^4-1}-\frac{e^0}{e^4-1}$$

$$=\frac{e^x-1}{e^4-1}$$

d

$$f(x)=\frac{4(x-2)^3}{625} \qquad [2,7]$$

$$F(x)=\int_2^x \frac{4(x-2)^3}{625}\,dx$$

$$=\left[\frac{(x-2)^4}{625}\right]_2^x$$

$$=\frac{(x-2)^4}{625}-\frac{(2-2)^4}{625}$$

$$=\frac{(x-2)^4}{625}$$

e

$$f(x)=\frac{3x(8-x)}{135} \qquad [2,5]$$

$$f(x)=\frac{24x-3x^2}{135}$$

$$F(x)=\int_2^x \frac{24x-3x^2}{135}\,dx$$

$$=\left[\frac{12x^2-x^3}{135}\right]_2^x$$

$$=\frac{12x^2-x^3}{135}-\left(\frac{12\times 2^2-2^3}{135}\right)$$

$$=\frac{12x^2-x^3}{135}-\frac{40}{135}$$

$$=\frac{12x^2-x^3-40}{135}$$

Question 2

a

$$f(x) = \frac{5x^4}{7776} \qquad 1 \le x \le 6$$

$$F(x) = \int_1^x \frac{5x^4}{7776}\,dx$$

$$= \left[\frac{x^5}{7776}\right]_1^x$$

$$= \frac{x^5}{7776} - \frac{1^5}{7776}$$

$$= \frac{x^5 - 1}{7776}$$

b **i**

$$P(X \le 3)$$

$$= F(3)$$

$$= \frac{3^5 - 1}{7776}$$

$$= \frac{242}{7776}$$

$$= \frac{121}{3888}$$

ii

$$P(X \le 2)$$

$$= F(2)$$

$$= \frac{2^5 - 1}{7776}$$

$$= \frac{31}{7776}$$

iii

$$P(X<5)$$
$$=F(5)$$
$$=\frac{5^5-1}{7776}$$
$$=\frac{3124}{7776}$$
$$=\frac{781}{1944}$$

iv

$$P(X>4)$$
$$=1-P(X\le 4)$$
$$=1-F(4)$$
$$=1-\frac{4^5-1}{7776}$$
$$=\frac{6753}{7776}$$
$$=\frac{2251}{2592}$$

v

$$P(2\le X\le 4)$$
$$=F(4)-F(2)$$
$$=\frac{4^5-1}{7776}-\frac{2^5-1}{7776}$$
$$=\frac{1023-31}{7776}$$
$$=\frac{992}{7776}$$
$$=\frac{31}{243}$$

Question 3

a

$$f(x)=\frac{4x^3}{2320} \qquad [3,7]$$

$$F(x)=\int_3^x \frac{4x^3}{2320}\,dx$$

$$=\left[\frac{x^4}{2320}\right]_3^x$$

$$=\frac{x^4}{2320}-\frac{3^4}{2320}$$

$$=\frac{x^4-81}{2320}$$

b **i**

$$P(X\le 4)$$

$$=F(4)$$

$$=\frac{4^4-81}{2320}$$

$$=\frac{175}{2320}$$

$$=\frac{35}{464}$$

ii

$$P(X\le 6)$$

$$=F(6)$$

$$=\frac{6^4-81}{2320}$$

$$=\frac{1215}{2320}$$

$$=\frac{243}{464}$$

iii

$$
\begin{aligned}
&P(X \geq 5)\\
&= 1 - P(X \leq 5)\\
&= 1 - F(5)\\
&= 1 - \frac{5^4 - 81}{2320}\\
&= \frac{1776}{2320}\\
&= \frac{111}{145}
\end{aligned}
$$

iv

$$
\begin{aligned}
&P(X \geq 4)\\
&= 1 - P(X \leq 4)\\
&= 1 - F(4)\\
&= 1 - \frac{35}{464}\\
&= \frac{429}{464}
\end{aligned}
$$

v

$$
\begin{aligned}
&P(4 \leq X \leq 6)\\
&= P(X \leq 6) - P(X \leq 4)\\
&= F(6) - F(4)\\
&= \frac{243}{464} - \frac{35}{464}\\
&= \frac{208}{464}\\
&= \frac{13}{29}
\end{aligned}
$$

Question 4

a

$$f\left(x\right)=\frac{2e^{2x}}{e^{10}-1} \qquad \left[0,5\right]$$

$$F\left(x\right)=\int_0^x \frac{2e^{2x}}{e^{10}-1}dx$$

$$=\left[\frac{e^{2x}}{e^{10}-1}\right]_0^x$$

$$=\frac{e^{2x}}{e^{10}-1}-\frac{e^{0}}{e^{10}-1}$$

$$=\frac{e^{2x}-1}{e^{10}-1}$$

b **i**

$$P\left(X\le 2\right)$$

$$=F\left(2\right)$$

$$=\frac{e^{4}-1}{e^{10}-1}$$

$$=0.0024$$

ii

$$P\left(X\le 4\right)$$

$$=F\left(4\right)$$

$$=\frac{e^{8}-1}{e^{10}-1}$$

$$=0.14$$

iii

$$P\left(X>3\right)$$

$$=1-P\left(X\le 3\right)$$

$$=1-F\left(3\right)$$

$$=1-\frac{e^{6}-1}{e^{10}-1}$$

$$=0.98$$

iv

$$
\begin{aligned}
&P(X > 2.8)\\
&= 1 - P(X \le 2.8)\\
&= 1 - F(2.8)\\
&= 1 - \frac{e^{5.6} - 1}{e^{10} - 1}\\
&= 0.99
\end{aligned}
$$

v

$$
\begin{aligned}
&P(2 \le X \le 4)\\
&= F(4) - F(2)\\
&= \frac{e^{8} - 1}{e^{10} - 1} - \frac{e^{4} - 1}{e^{10} - 1}\\
&= 0.13
\end{aligned}
$$

Question 5

a

$$
\begin{aligned}
&f(x) = ax^3 \qquad [0,9]\\
&\int_0^9 ax^3 dx = 1\\
&\left[\frac{ax^4}{4}\right]_0^9 = 1\\
&\frac{a \times 9^4}{4} - \frac{a \times 0^4}{4} = 1\\
&\frac{6561a}{4} = 1\\
&a = \frac{4}{6561}\\
&f(x) = \frac{4x^3}{6561}
\end{aligned}
$$

b

$$f(x)=\frac{4x^3}{6561} \qquad [0,9]$$

$$F(x)=\int_0^x \frac{4x^3}{6561}\,dx$$

$$=\left[\frac{x^4}{6561}\right]_0^x$$

$$=\frac{x^4}{6561}-\frac{0^4}{6561}$$

$$=\frac{x^4}{6561}$$

c **i**

$$P(X\le 5)$$

$$=F(5)$$

$$=\frac{5^4}{6561}$$

$$=\frac{625}{6561}$$

ii

$$P(X\le 4)$$

$$=F(4)$$

$$=\frac{4^4}{6561}$$

$$=\frac{256}{6561}$$

iii

$$P(X>8)$$

$$=1-(X\le 8)$$

$$=1-F(8)$$

$$=1-\frac{8^4}{6561}$$

$$=\frac{2465}{6561}$$

iv

$$\begin{aligned}
&P(X \geq 3) \\
&= 1-(X \leq 3) \\
&= 1-F(3) \\
&= 1-\frac{3^4}{6561} \\
&= \frac{6480}{6561} \\
&= \frac{80}{81}
\end{aligned}$$

v

$$\begin{aligned}
&P(2 \leq X \leq 6) \\
&= F(6)-F(2) \\
&= \frac{6^4}{6561}-\frac{2^4}{6561} \\
&= \frac{1280}{6561}
\end{aligned}$$

Question 6

a

$$f(x)=\frac{a}{x} \qquad [1,6]$$

$$\int_1^6 \frac{a}{x}dx \quad =1$$

$$a[\ln x]_1^6 =1$$

$$a(\ln 6-\ln 1)=1$$

$$a=\frac{1}{\ln 6}$$

b

$$f(x)=\frac{1}{x\ln 6}\qquad [1,6]$$

$$F(x)=\int_1^x \frac{1}{x\ln 6}\,dx$$

$$=\frac{1}{\ln 6}\left[\ln x\right]_1^x$$

$$=\frac{\ln x}{\ln 6}$$

c **i**

$$P(X\le 3)$$

$$=F(3)$$

$$=\frac{\ln 3}{\ln 6}$$

$$=0.61$$

ii

$$P(X\le 2)$$

$$=F(2)$$

$$=\frac{\ln 2}{\ln 6}$$

$$=0.39$$

iii

$$P(X>5)$$

$$1-P(X\le 5)$$

$$=1-F(5)$$

$$=1-\frac{\ln 5}{\ln 6}$$

$$=0.10$$

iv

$$P(X\ge 4)$$

$$1-P(X\le 4)$$

$$=1-F(4)$$

$$=1-\frac{\ln 4}{\ln 6}$$

$$=0.23$$

v

$$
\begin{aligned}
&P(2 \le X \le 5) \\
&= F(5) - F(2) \\
&= \frac{\ln 5}{\ln 6} - \frac{\ln 2}{\ln 6} \\
&= 0.51
\end{aligned}
$$

Question 7

a

$$
\begin{aligned}
&f(x) = \cos x \qquad \left[\frac{3\pi}{2}, 2\pi\right] \\
&\int_{\frac{3\pi}{2}}^{2\pi} \cos x \; dx \\
&= [\sin x]_{\frac{3\pi}{2}}^{2\pi} \\
&= \sin(2\pi) - \left(\sin\left[\frac{3\pi}{2}\right]\right) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

Yes, it is a probability distribution function.

b

$$
\begin{aligned}
&f(x) = \cos x \qquad \left[\frac{3\pi}{2}, 2\pi\right] \\
&F(x) = \int_{\frac{3\pi}{2}}^{x} \cos x \; dx \\
&= [\sin x]_{\frac{3\pi}{2}}^{x} \\
&= \sin x - \sin\left[\frac{3\pi}{2}\right] \\
&= \sin x - (-1) \\
&= \sin x + 1
\end{aligned}
$$

c **i**

$$P\left(X \le \frac{5\pi}{3}\right)$$
$$= F\left(\frac{5\pi}{3}\right)$$
$$= \sin\left(\frac{5\pi}{3}\right) + 1$$
$$= -\frac{\sqrt{3}}{2} + 1$$
$$= \frac{2-\sqrt{3}}{2}$$

ii

$$P\left(X \ge \frac{7\pi}{4}\right)$$
$$= 1 - F\left(\frac{7\pi}{4}\right)$$
$$= 1 - \left[\sin\left(\frac{7\pi}{4}\right) + 1\right]$$
$$= 1 - \left(-\frac{1}{\sqrt{2}}\right) - 1$$
$$= \frac{1}{\sqrt{2}}$$

iii

$$P\left(\frac{5\pi}{3} \le X \le \frac{11\pi}{6}\right)$$
$$= F\left(\frac{11\pi}{6}\right) - F\left(\frac{5\pi}{3}\right)$$
$$= \sin\left(\frac{11\pi}{6}\right) + 1 - \left(\sin\left(\frac{5\pi}{3}\right) + 1\right)$$
$$= -\frac{1}{2} - \left(-\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{3}-1}{2}$$

Question 8

Mode is the value with the highest frequency, the value that has the highest point on the graph.

a 2

b 3

c 3

d 7

e 4

f

$$f(x) = -\frac{3}{434}\left(x^2 - 8x - 9\right) \qquad [0,7]$$

$$f'(x) = -\frac{3}{434}(2x - 8)$$

Let $f'(x) = 0$

$$0 = -\frac{3}{434}(2x - 8)$$

$$2x - 8 = 0$$

$$2x = 8$$

$$x = 4$$

g

$$f(x) = \frac{4e^{4x}}{e^8\left(e^{16} - 1\right)} \qquad 2 \le x \le 6$$

$$f'(x) = \frac{16e^{4x}}{e^8\left(e^{16} - 1\right)}$$

Let $f'(x) = 0$

$$0 = \frac{16e^{4x}}{e^8\left(e^{16} - 1\right)}$$

$$16e^{4x} = 0$$

This function is monotonic increasing over the domain, therefore the mode is maximum x or $x = 6$.

h

$$f(x) = -\frac{3}{1100}\left(x^2 - 16x + 15\right) \qquad [1, 11]$$
$$f'(x) = -\frac{3}{1100}(2x - 16)$$
Let $f'(x) = 0$
$$0 = -\frac{3}{1100}(2x - 16)$$
$$2x - 16 = 0$$
$$2x = 16$$
$$x = 8$$

i

$$f(x) = \frac{2}{2105}\left(2x^3 - 33x^2 + 168x + 3\right) \qquad 0 \le x \le 5$$
$$f'(x) = \frac{2}{2105}\left(6x^2 - 66x + 168\right)$$
Let $f'(x) = 0$
$$0 = \frac{12}{2105}\left(x^2 - 11x + 28\right)$$
$$x^2 - 11x + 28 = 0$$
$$(x - 7)(x - 4) = 0$$
$$x = 4, 7$$
$$f''(x) = \frac{2}{2105}(12x - 66)$$
$$f''(4) < 0$$
It is a maximum, so 4 is the mode.

j

$$f(x) = \frac{3x^2}{342} \qquad 1 \le x \le 7$$

Is monotonic increasing over the domain so the mode is maximum $x = 7$.

Question 9

a

$$f(x) = -\frac{3}{22}\left(x^2 - 6x + 5\right) \qquad [2, 4]$$

$$f'(x) = -\frac{3}{22}(2x - 6)$$

Let $f'(x) = 0$

$$0 = -\frac{3}{22}(2x - 6)$$

$$2x - 6 = 0$$

$$2x = 6$$

$$x = 3$$

b

$$f(x) = -\frac{3}{22}\left(x^2 - 6x + 5\right) \qquad [2, 4]$$

$$F(x) = \int_2^x -\frac{3}{22}\left(x^2 - 6x + 5\right)\, dx$$

$$= \left[-\frac{3}{22}\left(\frac{x^3}{3} - 3x^2 + 5x\right)\right]_2^x$$

$$= -\frac{3}{22}\left(\left(\frac{x^3}{3} - 3x^2 + 5x\right) - \left(\frac{2^3}{3} - 3\times 2^2 + 5\times 2\right)\right)$$

$$= -\frac{3}{22}\left(\frac{x^3}{3} - 3x^2 + 5x - \frac{2}{3}\right)$$

$$= -\frac{3}{66}\left(x^3 - 9x^2 + 15x - 2\right)$$

$$= -\frac{x^3 - 9x^2 + 15x - 2}{22}$$

c

$$
\begin{aligned}
&P(X \le a)\\
&= P(X \le 3)\\
&= F(3)\\
&= -\frac{3^3 - 9\times 3^2 + 15\times 3 - 2}{22}\\
&= -\frac{-11}{22}\\
&= \frac{1}{2}
\end{aligned}
$$

Question 10

a

$$
\begin{aligned}
&f(x) = \frac{1}{116}\left(x^3 - 9x^2 + 24x + 1\right) \qquad [3,7]\\
&F(x) = \int_3^x \frac{1}{116}\left(x^3 - 9x^2 + 24x + 1\right)\,dx\\
&= \left[\frac{1}{116}\left(\frac{x^4}{4} - 3x^3 + 12x^2 + x\right)\right]_3^x\\
&= \frac{1}{116}\left(\left(\frac{x^4}{4} - 3x^3 + 12x^2 + x\right) - \left(\frac{3^4}{4} - 3\times 3^3 + 12\times 3^2 + 3\right)\right)\\
&= \frac{1}{116}\left(\frac{x^4}{4} - 3x^3 + 12x^2 + x - \frac{201}{4}\right)\\
&= \frac{1}{464}\left(x^4 - 12x^3 + 48x^2 + 4x - 201\right)
\end{aligned}
$$

b **i**

$$\begin{aligned}&P(X \le 5)\\&= P(X \le 5)\\&= F(5)\\&= \frac{1}{464}\left(5^4 - 12\times 5^3 + 48\times 5^2 + 4\times 5 - 201\right)\\&= \frac{144}{464}\\&= \frac{9}{29}\end{aligned}$$

ii

$$\begin{aligned}&P(X > 4)\\&= 1 - P(X \le 4)\\&= 1 - F(4)\\&= 1 - \frac{1}{464}\left(4^4 - 12\times 4^3 + 48\times 4^2 + 4\times 4 - 201\right)\\&= 1 - \frac{71}{464}\\&= \frac{393}{464}\end{aligned}$$

iii

$$\begin{aligned}&P(4 \le X \le 5)\\&= P(X \le 5) - P(X \le 4)\\&= F(5) - F(4)\\&= \frac{1}{464}\left[\left(5^4 - 12\times 5^3 + 48\times 5^2 + 4\times 5 - 201\right) - \left(4^4 - 12\times 4^3 + 48\times 4^2 + 4\times 4 - 201\right)\right]\\&= \frac{9}{29} - \frac{71}{464}\\&= \frac{73}{464}\end{aligned}$$

c

$f(x)=\frac{1}{116}\left(x^3-9x^2+24x+1\right) \qquad [3,7]$

$f'(x)=\frac{1}{116}\left(3x^2-18x+24\right)$

Let $f'(x)=0$

$0=\frac{1}{116}\left(3x^2-18x+24\right)$

$3x^2-18x+24=0$

$3(x-4)(x-2)=0$

$x=2,4$

x = 2 is outside the domain

x = 4 but need to test the endpoints and maximum

$f(3)=\frac{1}{116}\left(3^3-9\times3^2+24\times3+1\right)$

$=\frac{19}{116}$

$f(4)=\frac{1}{116}\left(4^3-9\times4^2+24\times4+1\right)$

$=\frac{17}{116}$

$f(7)=\frac{1}{116}\left(7^3-9\times7^2+24\times7+1\right)$

$=\frac{71}{116}$

7 minutes is the mode.

Exercise 10.04 Quantiles

Question 1

a

$$f(x)=\frac{3x^2}{511} \qquad 1\le x\le 8$$

$$\text{Median}=F(x)=\int_1^x \frac{3x^2}{511}dx=0.5$$

$$\left[\frac{x^3}{511}\right]_1^x=0.5$$

$$\frac{x^3}{511}-\frac{1^3}{511}=0.5$$

$$x^3-1=\frac{511}{2}$$

$$x^3=\frac{513}{2}$$

$$x=\sqrt[3]{\frac{513}{2}}$$

$$x=6.35$$

b

$$f(x)=\frac{4x^3}{2401} \qquad [0,7]$$

$$\text{Median}=F(x)=\int_0^x \frac{4x^3}{2401}\ dx=0.5$$

$$\left[\frac{x^4}{2401}\right]_0^x=0.5$$

$$\frac{x^4}{2401}-\frac{0^4}{2401}=0.5$$

$$x^4=\frac{2401}{2}$$

$$x=\sqrt[4]{\frac{2401}{2}}$$

$$x=5.89$$

c

$$f(x)=\frac{5x^4}{16\ 807} \qquad 0\le x\le 7$$

$$\text{Median}=F(x)=\int_0^x \frac{5x^4}{16\ 807}\,dx = 0.5$$

$$\left[\frac{x^5}{16\ 807}\right]_0^x = 0.5$$

$$\frac{x^5}{16\ 807}-\frac{0^5}{16\ 807}=0.5$$

$$x^5=\frac{16\ 807}{2}$$

$$x=\sqrt[5]{\frac{16\ 807}{2}}$$

$$x=6.09$$

d

$$f(x)=\frac{3(x-3)^2}{16} \qquad [1,5]$$

$$\text{Median}=F(x)=\int_1^x \frac{3(x-3)^2}{16}\,dx=0.5$$

$$\left[\frac{(x-3)^3}{16}\right]_1^x=0.5$$

$$\frac{(x-3)^3}{16}-\frac{(1-3)^3}{16}=0.5$$

$$\frac{(x-3)^3+8}{16}=0.5$$

$$(x-3)^3+8=8$$

$$(x-3)^3=0$$

$$x-3=0$$

$$x=3$$

e

$$f(x)=\frac{(3x+1)^2}{244} \qquad 0\le x\le 4$$

$$\text{Median} = F(x)=\int_0^x \frac{(3x+1)^2}{244}\,dx = 0.5$$

$$\left[\frac{(3x+1)^3}{2196}\right]_0^x = 0.5$$

$$\frac{(3x+1)^3}{2196}-\frac{(3x+1)^3}{2196}=0.5$$

$$(3x+1)^3-1=1098$$

$$(3x+1)^3=1099$$

$$3x+1=\sqrt[3]{1099}$$

$$3x+1=10.32$$

$$3x=9.32$$

$$x=3.11$$

f

$$f(x)=\frac{4x^3}{6560} \qquad [1,9]$$

$$\text{Median} = F(x)=\int_1^x \frac{4x^3}{6560}\,dx = 0.5$$

$$\left[\frac{x^4}{6560}\right]_1^x = 0.5$$

$$\frac{x^4}{6560}-\frac{1^4}{6560}=0.5$$

$$x^4-1=3280$$

$$x^4=3281$$

$$x=\sqrt[4]{3281}$$

$$x=7.57$$

g

$$f(x) = \frac{3x^2}{1304} \qquad [3,11]$$

$$\text{Median} = F(x) = \int_3^x \frac{3x^2}{1304}\,dx = 0.5$$

$$\left[\frac{x^3}{1304}\right]_3^x = 0.5$$

$$\frac{x^3}{1304} - \frac{3^3}{1304} = 0.5$$

$$x^3 - 27 = 652$$

$$x^3 = 679$$

$$x = \sqrt[3]{679}$$

$$x = 8.79$$

h

$$f(x) = \frac{6x^5}{15\ 625} \qquad 0 \le x \le 5$$

$$\text{Median=}F(x) = \int_0^x \frac{6x^5}{15\ 625}\,dx = 0.5$$

$$\left[\frac{x^6}{15\ 625}\right]_0^x = 0.5$$

$$\frac{x^6}{15\ 625} - \frac{0^6}{15\ 625} = 0.5$$

$$x^6 = \frac{15\ 625}{2}$$

$$x = \sqrt[6]{\frac{15\ 625}{2}}$$

$$x = 4.45$$

i

$$f(x)=\frac{(2x-1)^4}{16\ 105} \qquad [1,6]$$

$$\text{Median}=F(x)=\int_1^x \frac{(2x-1)^4}{16\ 105}\,dx=0.5$$

$$\left[\frac{(2x-1)^5}{161\ 050}\right]_1^x=0.5$$

$$\frac{(2x-1)^5}{161\ 050}-\frac{(2\times 0-1)^5}{161\ 050}=0.5$$

$$\frac{(2x-1)^5+1}{161\ 050}=0.5$$

$$(2x-1)^5+1=80\ 525$$

$$(2x-1)^5=80\ 524$$

$$2x-1=\sqrt[5]{80\ 524}$$

$$2x-1=9.58$$

$$2x=10.58$$

$$x=5.29$$

j

$$f(x)=\frac{x\left(x^2-3\right)^3}{3570} \qquad [2,4]$$

$$f(x)=\frac{2x\left(x^2-3\right)^3}{7140}$$

$$\text{Median} = F(x)=\int_2^x \frac{2x\left(x^2-3\right)^3}{7140}\,dx=0.5$$

$$\left[\frac{\left(x^2-3\right)^4}{28\,560}\right]_2^x=0.5$$

$$\frac{\left(x^2-3\right)^4}{28\,560}-\frac{\left(2^2-3\right)^4}{28\,560}=0.5$$

$$\frac{\left(x^2-3\right)^4-1}{28\,560}=0.5$$

$$\left(x^2-3\right)^4-1=14\,280$$

$$\left(x^2-3\right)^4=14\,281$$

$$x^2-3=\sqrt[4]{14\,281}$$

$$x^2-3=10.93$$

$$x^2=13.93$$

$$x=3.73$$

Question 2

a

$$f(x) = \frac{3x^2}{973} \qquad [3,10]$$

$$F(x) = \int_3^x \frac{3x^2}{973}\,dx$$

$$= \left[\frac{x^3}{973}\right]_3^x$$

$$= \frac{x^3}{973} - \frac{3^3}{973}$$

$$= \frac{x^3 - 27}{973}$$

i

$$F(x) = 0.25$$

$$\frac{x^3 - 27}{973} = 0.25$$

$$x^3 - 27 = \frac{973}{4}$$

$$x^3 = 27 + \frac{973}{4}$$

$$x = \sqrt[3]{27 + \frac{973}{4}}$$

$$x = 6.47$$

ii

$$F(x) = 0.2$$

$$\frac{x^3 - 27}{973} = 0.2$$

$$x^3 - 27 = \frac{973}{5}$$

$$x^3 = 27 + \frac{973}{5}$$

$$x = \sqrt[3]{27 + \frac{973}{5}}$$

$$x = 6.05$$

iii

$$F(x) = 0.77$$
$$\frac{x^3 - 27}{973} = 0.77$$
$$x^3 - 27 = 749.21$$
$$x^3 = 27 + 749.21$$
$$x = \sqrt[3]{27 + 749.21}$$
$$x = 9.19$$

b

$$f(x) = \frac{x^3}{324} \qquad 0 \le x \le 6$$
$$F(x) = \int_0^x \frac{x^3}{324}\,dx$$
$$= \left[\frac{x^4}{1296}\right]_0^x$$
$$= \frac{x^4}{1296} - \frac{0^4}{1296}$$
$$= \frac{x^4}{1296}$$

i

$$F(x) = 0.25$$
$$\frac{x^4}{1296} = 0.25$$
$$x^4 = \frac{1296}{4}$$
$$x = \sqrt[4]{\frac{1296}{4}}$$
$$x = 4.24$$

ii

$$F(x) = 0.2$$
$$\frac{x^4}{1296} = 0.2$$
$$x^4 = \frac{1296}{5}$$
$$x = \sqrt[4]{\frac{1296}{5}}$$
$$x = 4.01$$

iii

$$F(x) = 0.77$$
$$\frac{x^4}{1296} = 0.77$$
$$x^4 = 997.92$$
$$x = \sqrt[4]{997.92}$$
$$x = 5.62$$

c

$$f(x) = \frac{5x^4}{3124} \qquad 1 \le x \le 5$$
$$F(x) = \int_1^x \frac{5x^4}{3124}\,dx$$
$$= \left[\frac{x^5}{3124}\right]_1^x$$
$$= \frac{x^5}{3124} - \frac{1^5}{3124}$$
$$= \frac{x^5 - 1}{3124}$$

i

$$F(x) = 0.25$$
$$\frac{x^5 - 1}{3124} = 0.25$$
$$x^5 - 1 = 781$$
$$x^3 = 1 + 781$$
$$x = \sqrt[3]{782}$$
$$x = 3.79$$

ii

$$F(x) = 0.2$$
$$\frac{x^5 - 1}{3124} = 0.2$$
$$x^5 - 1 = 624.8$$
$$x^3 = 1 + 624.8$$
$$x = \sqrt[3]{625.8}$$
$$x = 3.62$$

iii

$$F(x) = 0.77$$
$$\frac{x^5 - 1}{3124} = 0.77$$
$$x^5 - 1 = 2405.48$$
$$x^3 = 1 + 2405.48$$
$$x = \sqrt[3]{2406.48}$$
$$x = 4.75$$

Question 3

$$f\left(x\right)=\frac{3x^2}{512} \qquad \left[0,8\right]$$

$$F\left(x\right)=\int_0^x \frac{3x^2}{512}\ dx$$

$$=\left[\frac{x^3}{512}\right]_0^x$$

$$=\frac{x^3}{512}-\frac{0^3}{512}$$

$$=\frac{x^3}{512}$$

a

$$F\left(x\right)=0.5$$

$$\frac{x^3}{512}=0.5$$

$$x^3=256$$

$$x=\sqrt[3]{256}$$

$$x=6.35$$

b

$$F\left(x\right)=0.35$$

$$\frac{x^3}{512}=0.35$$

$$x^3=179.2$$

$$x=\sqrt[3]{179.2}$$

$$x=5.64$$

Question 4

$$f(x)=\frac{x^2}{168} \qquad 2\le x\le 8$$

$$F(x)=\int_2^x \frac{x^2}{168}dx$$

$$=\left[\frac{x^3}{504}\right]_2^x$$

$$=\frac{x^3}{504}-\frac{2^3}{504}$$

$$=\frac{x^3-8}{504}$$

a

$$F(x)=0.5$$

$$\frac{x^3-8}{504}=0.5$$

$$x^3-8=252$$

$$x^3=260$$

$$x=\sqrt[3]{260}$$

$$x=6.38$$

b

$$F(x)=0.25$$

$$\frac{x^3-8}{504}=0.25$$

$$x^3-8=126$$

$$x^3=134$$

$$x=\sqrt[3]{134}$$

$$x=5.12$$

c

$$F(x) = 0.75$$
$$\frac{x^3 - 8}{504} = 0.75$$
$$x^3 - 8 = 378$$
$$x^3 = 386$$
$$x = \sqrt[3]{386}$$
$$x = 7.28$$

d

$$F(x) = 0.67$$
$$\frac{x^3 - 8}{504} = 0.67$$
$$x^3 - 8 = 337.68$$
$$x^3 = 345.68$$
$$x = \sqrt[3]{345.68}$$
$$x = 7.02$$

e

$$F(x) = 0.14$$
$$\frac{x^3 - 8}{504} = 0.14$$
$$x^3 - 8 = 70.56$$
$$x^3 = 78.56$$
$$x = \sqrt[3]{78.56}$$
$$x = 4.28$$

f

$$F(x) = 0.8$$
$$\frac{x^3 - 8}{504} = 0.8$$
$$x^3 - 8 = 403.2$$
$$x^3 = 411.2$$
$$x = \sqrt[3]{411.2}$$
$$x = 7.44$$

Question 5

$$f(x)=\frac{x^2}{576} \qquad 0\le x\le 12$$

$$F(x)=\int_0^x \frac{x^2}{576}\,dx$$

$$=\left[\frac{x^3}{1728}\right]_0^x$$

$$=\frac{x^3}{1728}-\frac{0^3}{1728}$$

$$=\frac{x^3}{1728}$$

a

$$F(x)=0.2$$

$$\frac{x^3}{1728}=0.2$$

$$x^3=345.6$$

$$x=\sqrt[3]{345.6}$$

$$x=7.02$$

b

$$F(x)=0.5$$

$$\frac{x^3}{1728}=0.5$$

$$x^3=864$$

$$x=\sqrt[3]{864}$$

$$x=9.52$$

c

$$F(x)=0.75$$

$$\frac{x^3}{1728}=0.75$$

$$x^3=1296$$

$$x=\sqrt[3]{1296}$$

$$x=10.90$$

Question 6

a

$$f(x) = \frac{x^3}{1020} \qquad 2 \le x \le 8$$

$$F(x) = \int_2^x \frac{x^3}{1020}\,dx$$

$$= \left[\frac{x^4}{4080}\right]_2^x$$

$$= \frac{x^4}{4080} - \frac{2^4}{4080}$$

$$= \frac{x^4 - 16}{4080}$$

b

$$P(X \le 5)$$

$$= F(5)$$

$$= \frac{5^4 - 16}{4080}$$

$$= \frac{203}{1360}$$

c

$$P(X > 4)$$

$$1 - P(X \le 4)$$

$$= 1 - F(4)$$

$$= 1 - \frac{4^4 - 16}{4080}$$

$$= \frac{16}{17}$$

d

$$P(3 \le X \le 7)$$

$$= F(7) - F(3)$$

$$= \frac{7^4 - 16}{4080} - \frac{7^4 - 16}{4080}$$

$$= \frac{29}{51}$$

e

$$F(x)=0.5$$
$$\frac{x^4-16}{4080}=0.5$$
$$x^4-16=2040$$
$$x^4=2056$$
$$x=\sqrt[4]{2056}$$
$$x=6.73$$

f

$$F(x)=0.75$$
$$\frac{x^4-16}{4080}=0.75$$
$$x^4-16=3060$$
$$x^4=3076$$
$$x=\sqrt[4]{3076}$$
$$x=7.45$$

g

$$F(x)=0.9$$
$$\frac{x^4-16}{4080}=0.9$$
$$x^4-16=3672$$
$$x^4=3688$$
$$x=\sqrt[4]{3688}$$
$$x=7.79$$

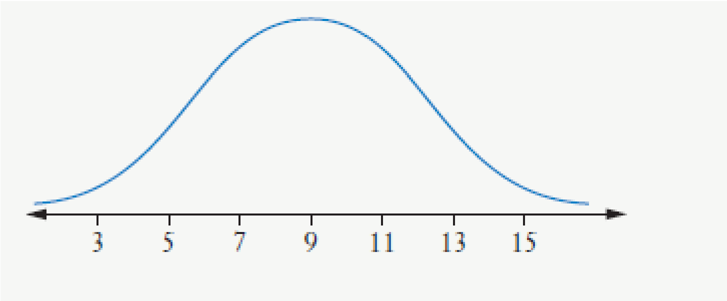
h

$$F(x)=0.23$$
$$\frac{x^4-16}{4080}=0.23$$
$$x^4-16=938.4$$
$$x^4=954.4$$
$$x=\sqrt[4]{954.4}$$
$$x=5.56$$

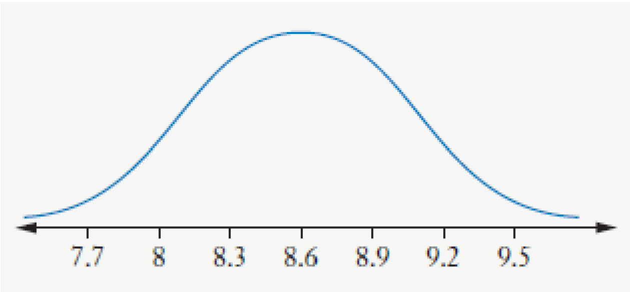
Exercise 10.05 Normal distribution

Question 1

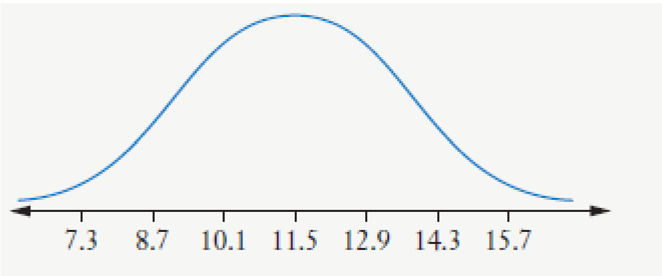
a



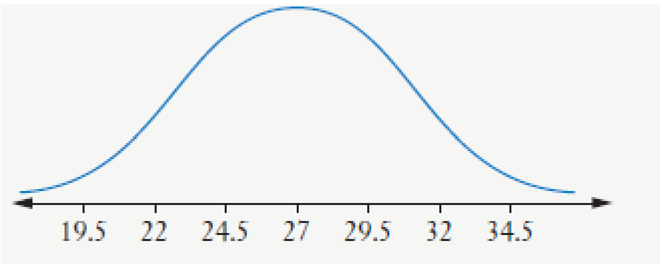
b



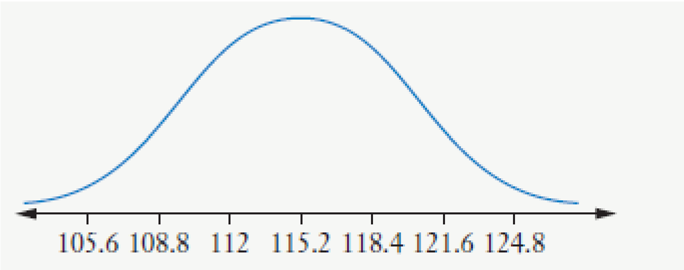
c



d



e



Question 2

a $\mu = 23.7$, $\sigma = 4.2$

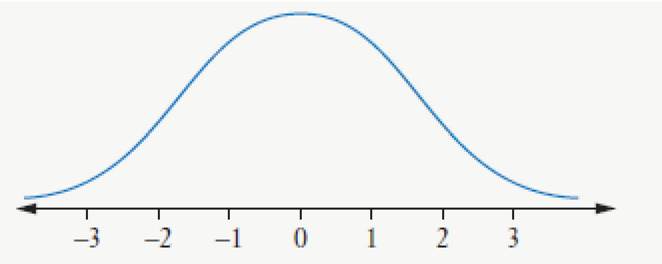
b $\mu = 5.4$, $\sigma = 0.9$

c $\mu = 59.7$, $\sigma = 5.4$

d $\mu = 209$, $\sigma = 10.6$

e $\mu = 11.3$, $\sigma = 2.2$

Question 3



Question 4

a $P(Z \le 0) = 0.5$

b $P(Z \le 1) = 0.8413$

c $P(Z \le 2) = 0.9772$

d $P(Z \le 3) = 0.9987$

e $P(Z \le -1) = 0.1587$

f $P(Z \le -2) = 0.0228$

g $P(Z \le -3) = 0.0013$

h $P(Z \le 1.5) = 0.9332$

i $P(Z < -2.67) = 0.0038$

j $P(Z \le 3.09) = 0.9990$

Question 5

a

$$\begin{aligned} &P(Z \ge -0.46) \\ &= P(Z \le 0.46) \quad \text{by symmetry} \\ &= 0.6772 \end{aligned}$$

b

$$\begin{aligned} &P(Z > 2.11) \\ &= P(Z \le -2.11) \quad \text{by symmetry} \\ &= 0.0174 \end{aligned}$$

c

$$\begin{aligned} &P(Z \ge -2.01) \\ &= P(Z \le 2.01) \quad \text{by symmetry} \\ &= 0.9778 \end{aligned}$$

d

$$\begin{aligned}
&P(-2.4 \le Z \le -1.76)\\
&= P(Z \le -1.76) - P(Z \le -2.4)\\
&= 0.0392 - 0.0082\\
&= 0.031
\end{aligned}$$

e

$$\begin{aligned}
&P(-2.2 \le Z \le 2.2)\\
&= P(Z \le 2.2) - P(Z \le -2.2)\\
&= 0.9861 - 0.0139\\
&= 0.9722
\end{aligned}$$

f

$$\begin{aligned}
&P(1.21 < Z < 1.89)\\
&= P(Z < 1.89) - P(Z < 1.21)\\
&= 0.9706 - 0.8869\\
&= 0.0837
\end{aligned}$$

g

$$\begin{aligned}
&P(-1.45 \le Z \le 3.1)\\
&= P(Z \le 3.1) - P(Z \le -1.45)\\
&= 0.9990 - 0.0735\\
&= 0.9255
\end{aligned}$$

h

$$\begin{aligned}
&P(-1 \le Z \le 1)\\
&= P(Z \le 1) - P(Z \le -1)\\
&= 0.8413 - 0.1587\\
&= 0.6826
\end{aligned}$$

i

$$\begin{aligned}
&P(-2 \le Z \le 2)\\
&= P(Z \le 2) - P(Z \le -2)\\
&= 0.9772 - 0.0228\\
&= 0.9544
\end{aligned}$$

j

$$P(-3 \le Z \le 3)$$
$$= P(Z \le 3) - P(Z \le -3)$$
$$= 0.9987 - 0.0013$$
$$= 0.9974$$

Question 6

a 0.84

b 0.67

c –0.53

d –0.84

e 1.23

f –1.17

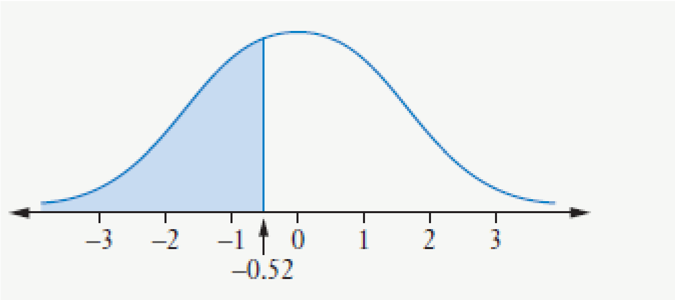
g –0.52

h –0.67

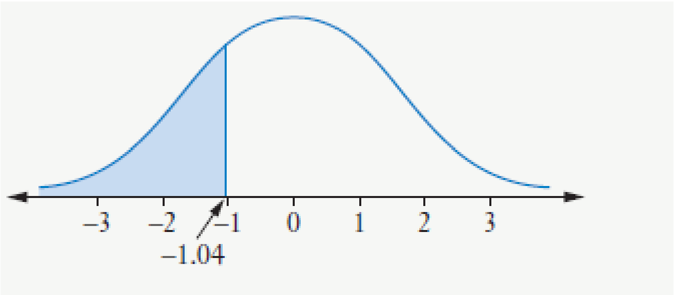
i 0.33

Question 7

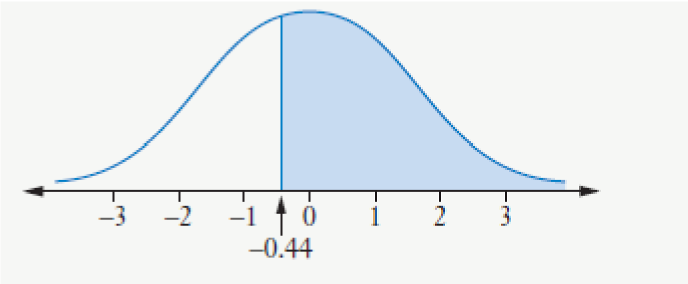
a



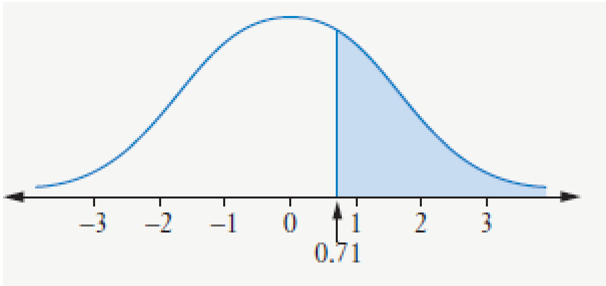
b



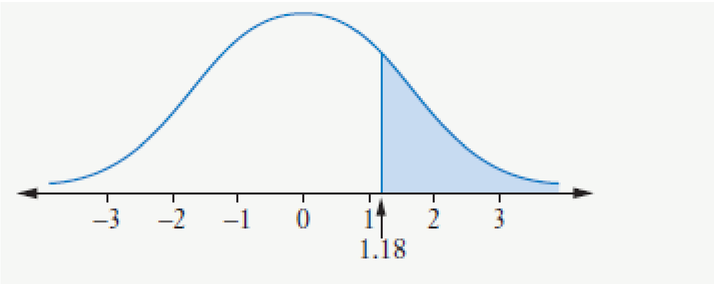
c



d



e



Exercise 10.06 Empirical rule

Question 1

a 68%

b 95%

c 99.7%

Question 2

$\mu = 15, \sigma = 1.5$

a 13.5, 16.5

$\mu - \sigma, \mu + \sigma$

68%

b 12, 18

$\mu - 2\sigma, \mu + 2\sigma$

95%

c 10.5 , 19.5

$\mu - 3\sigma, \mu + 3\sigma$

99.7%

Question 3

$\mu = 8.4, \sigma = 0.9$

a 7.5, 9.3

$\mu - \sigma, \mu + \sigma$

68%

b 6.6, 10.2

$\mu - 2\sigma, \mu + 2\sigma$

95%

c 5.7, 11.1

$\mu - 3\sigma, \mu + 3\sigma$

99.7%

Question 4

$\mu = 18, \sigma = 2$

a 16, 20

$\mu - \sigma, \mu + \sigma$

68%

b 14, 22

$\mu - 2\sigma, \mu + 2\sigma$

95%

c 12, 24

$\mu - 3\sigma, \mu + 3\sigma$

99.7%

d 16, 18

$\mu - \sigma, \mu$

$0.5 \times 68\% = 34\%$

e 18, 24

$\mu, \mu + 3\sigma$

$0.5 \times 99.7\% = 49.85\%$

f 12, 22

$\mu - 3\sigma, \mu + 2\sigma$

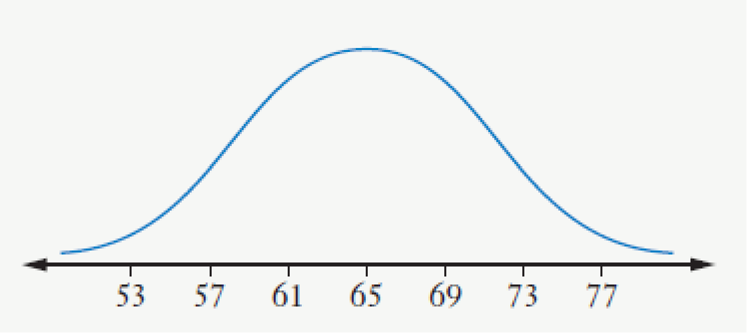
$0.5 \times 99.7\% + 0.5 \times 95\%$

$= 49.85\% + 47.5\%$

$= 97.35\%$

Question 5

a



b $\mu = 65, \sigma = 4$

i 57, 73

$\mu - 2\sigma$, $\mu + 2\sigma$

95%

ii 61, 65

$\mu - \sigma, \mu$

$0.5 \times 68\% = 34\%$

iii 65, 77

$\mu, \mu + 3\sigma$

$0.5 \times 99.7\% = 49.85\%$

iv 57, 69

$\mu - 2\sigma, \mu + \sigma$

$0.5 \times 95\% + 0.5 \times 68\%$

$= 47.5\% + 34\%$

$= 81.5\%$

v 61, 73

$\mu - \sigma$, $\mu + 2\sigma$

$0.5 \times 68\% + 0.5 \times 95\%$

$= 34\% + 47.5\%$

$= 81.5\%$

Question 6

$\mu = 9.7$, $\sigma = 2.1$

a 7.6, 11.8

$\mu - \sigma$, $\mu + \sigma$

68%

b 9.7, 11.8

μ , $\mu + \sigma$

$0.5 \times 68\% = 34\%$

c 9.7, 13.9

μ , $\mu + 2\sigma$

$0.5 \times 95\% = 47.5\%$

d 5.5, 9.7

$\mu - 2\sigma$, μ

$0.5 \times 95\% = 47.5\%$

e 3.4, 11.8

$\mu - 3\sigma$, $\mu + \sigma$

$0.5 \times 99.7\% + 0.5 \times 68\%$

$= 49.85\% + 34\%$

$= 83.85\%$

Exercise 10.07 *z*-scores

Question 1

$\mu = 18$, $\sigma = 1.3$

a **i**

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{18 - 18}{1.3}$$
$$z = 0$$

ii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{19.3 - 18}{1.3}$$
$$z = 1$$

iii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{20.6 - 18}{1.3}$$
$$z = 2$$

iv

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{21.9 - 18}{1.3}$$
$$z = 3$$

v

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{16.7 - 18}{1.3}$$
$$z = -1$$

vi

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{15.4-18}{1.3}$$
$$z = -2$$

vii

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{14.1-18}{1.3}$$
$$z = -3$$

b **i**

$$z = \frac{x-\mu}{\sigma}$$
$$1.5 = \frac{x-18}{1.3}$$
$$x-18 = 1.5\times1.3$$
$$x = 18+1.5\times1.3$$
$$x = 19.95$$

ii

$$z = \frac{x-\mu}{\sigma}$$
$$-2.1 = \frac{x-18}{1.3}$$
$$x-18 = -2.1\times1.3$$
$$x = 18-2.1\times1.3$$
$$x = 15.27$$

Question 2

$\mu = 53.1$, $\sigma = 8.7$

a $\mu - 3\sigma$, $\mu + 3\sigma$

$53.1 - 3 \times 8.7$, $53.1 + 3 \times 8.7$

27 cm, 79.2 cm

b **i**

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{53.1-53.1}{8.7}$$
$$z = 0$$

ii

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{61.8-53.1}{8.7}$$
$$z = 1$$

iii

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{44.4-53.1}{8.7}$$
$$z = -1$$

iv

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{70.5-53.1}{8.7}$$
$$z = 2$$

v

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{35.7 - 53.1}{8.7}$$
$$z = -2$$

vi

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{79.2 - 53.1}{8.7}$$
$$z = 3$$

vii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{27 - 53.1}{8.7}$$
$$z = -3$$

viii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{65 - 53.1}{8.7}$$
$$z = 1.37$$

Question 3

$\mu = 6.8$, $\sigma = 1.1$

a $\mu - 3\sigma$, $\mu + 3\sigma$

$6.8 - 3 \times 1.1$, $6.8 + 3 \times 1.1$

3.5°C, 10.1°C

b **i**

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{6.8 - 6.8}{1.1}$$
$$z = 0$$

ii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{7.9 - 6.8}{1.1}$$
$$z = 1$$

iii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{9 - 6.8}{1.1}$$
$$z = 2$$

iv

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{10.1 - 6.8}{1.1}$$
$$z = 3$$

v

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{5.7 - 6.8}{1.1}$$
$$z = -1$$

vi

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{4.6 - 6.8}{1.1}$$
$$z = -2$$

vii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{3.5 - 6.8}{1.1}$$
$$z = -3$$

viii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{6 - 6.8}{1.1}$$
$$z = -0.7$$

Question 4

$\mu = 66.4$, $\sigma = 5.8$

a $\mu - 2\sigma$, $\mu + 2\sigma$

$66.4 - 2 \times 5.8$, $66.4 + 2 \times 5.8$

54.8 mL, 78 mL

b **i**

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{66.4 - 66.4}{5.8}$$
$$z = 0$$

ii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{72.2 - 66.4}{5.8}$$
$$z = 1$$

iii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{78 - 66.4}{5.8}$$
$$z = 2$$

iv

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{83.8 - 66.4}{5.8}$$
$$z = 3$$

v

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{60.6 - 66.4}{5.8}$$
$$z = -1$$

vi

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{54.8 - 66.4}{5.8}$$
$$z = -2$$

vii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{49 - 66.4}{5.8}$$
$$z = -3$$

viii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{90 - 66.4}{5.8}$$
$$z = 4.1$$

c **i**

$$z=\frac{x-\mu}{\sigma}$$
$$1.2=\frac{x-66.4}{5.8}$$
$$x-66.4=1.2\times 5.8$$
$$x=66.4+1.2\times 5.8$$
$$x=73.36\text{ mL}$$

ii

$$z=\frac{x-\mu}{\sigma}$$
$$2.9=\frac{x-66.4}{5.8}$$
$$x-66.4=2.9\times 5.8$$
$$x=66.4+2.9\times 5.8$$
$$x=83.22\text{ mL}$$

iii

$$z=\frac{x-\mu}{\sigma}$$
$$-0.6=\frac{x-66.4}{5.8}$$
$$x-66.4=-0.6\times 5.8$$
$$x=66.4-0.6\times 5.8$$
$$x=62.92\text{ mL}$$

iv

$$z=\frac{x-\mu}{\sigma}$$
$$-2.3=\frac{x-66.4}{5.8}$$
$$x-66.4=-2.3\times 5.8$$
$$x=66.4-2.3\times 5.8$$
$$x=53.06\text{ mL}$$

Question 5

$\mu = 68$, $\sigma = 4.5$

a **i**

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{80-68}{4.5}$$
$$z = 2.7$$

ii

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{53.2-68}{4.5}$$
$$z = -3.3$$

iii

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{78.6-68}{4.5}$$
$$z = 2.4$$

iv

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{62.1-68}{4.5}$$
$$z = -1.3$$

v

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{90-68}{4.5}$$
$$z = 4.9$$

vi

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{59.7 - 68}{4.5}$$
$$z = -1.8$$

vii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{82.7 - 68}{4.5}$$
$$z = 3.3$$

viii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{56.4 - 68}{4.5}$$
$$z = -2.6$$

b 53.2, 90, 82.7

c 62.1, 59.7

d 80, 78.6, 62.1, 59.7, 56.4

Question 6

$\mu = 14.2$, $\sigma = 1.4$

a **i**

$$z=\frac{x-\mu}{\sigma}$$
$$z=\frac{16-14.2}{1.4}$$
$$z=1.29$$

ii

$$z=\frac{x-\mu}{\sigma}$$
$$z=\frac{12-14.2}{1.4}$$
$$z=-1.57$$

b **i**

$$z=\frac{x-\mu}{\sigma}$$
$$-2.1=\frac{x-14.2}{1.4}$$
$$x-14.2=-2.1\times1.4$$
$$x=14.2-2.1\times1.4$$
$$x=11.26\text{ mm}$$

ii

$$z=\frac{x-\mu}{\sigma}$$
$$1.3=\frac{x-14.2}{1.4}$$
$$x-14.2=1.3\times1.4$$
$$x=14.2+1.3\times1.4$$
$$x=16.02\text{ mm}$$

iii

$$z = \frac{x-\mu}{\sigma}$$
$$3.2 = \frac{x-14.2}{1.4}$$
$$x-14.2 = 3.2\times1.4$$
$$x = 14.2+3.2\times1.4$$
$$x = 18.68\text{ mm}$$

iv

$$z = \frac{x-\mu}{\sigma}$$
$$-0.76 = \frac{x-14.2}{1.4}$$
$$x-14.2 = -0.76\times1.4$$
$$x = 14.2-0.76\times1.4$$
$$x = 13.136\text{ mm}$$

v

$$z = \frac{x-\mu}{\sigma}$$
$$1.95 = \frac{x-14.2}{1.4}$$
$$x-14.2 = 1.95\times1.4$$
$$x = 14.2+1.95\times1.4$$
$$x = 16.93\text{ mm}$$

Question 7

$\mu = 23 \qquad \sigma = 2$

$$z = \frac{x-\mu}{\sigma}$$

$$2.5 = \frac{x-23}{2}$$

$$x - 23 = 2.5 \times 2$$

$$x = 23 + 2.5 \times 2$$

$$x = 28$$

Question 8

$$z = \frac{x-\mu}{\sigma}$$

$$2.7 = \frac{39-\mu}{2}$$

$$39 - \mu = 2.7 \times 4.5$$

$$\mu = 39 - 2.7 \times 4.5$$

$$x = 26.85$$

Question 9

$$z = \frac{x-\mu}{\sigma}$$

$$-0.6 = \frac{59-89}{\sigma}$$

$$59 - 89 = -0.6 \times \sigma$$

$$\sigma = \frac{-30}{-0.6}$$

$$\sigma = 50$$

Question 10

$\mu = 53.4$, $\sigma = 5.6$

a

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ 0 &= \frac{x-53.4}{5.6} \\ x-53.4 &= 0\times 5.6 \\ x &= 53.4 \end{aligned}$$

b

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ -2 &= \frac{x-53.4}{5.6} \\ x-53.4 &= -2\times 5.6 \\ x &= 53.4-2\times 5.6 \\ x &= 42.2 \end{aligned}$$

c

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ 1 &= \frac{x-53.4}{5.6} \\ x-53.4 &= 1\times 5.6 \\ x &= 53.4+1\times 5.6 \\ x &= 59 \end{aligned}$$

d

$$\begin{aligned} z &= \frac{x-\mu}{\sigma} \\ 2.8 &= \frac{x-53.4}{5.6} \\ x-53.4 &= 2.8\times 5.6 \\ x &= 53.4+2.8\times 5.6 \\ x &= 69.08 \end{aligned}$$

e

$$z=\frac{x-\mu}{\sigma}$$
$$-1.7=\frac{x-53.4}{5.6}$$
$$x-53.4=-1.7\times5.6$$
$$x=53.4-1.7\times5.6$$
$$x=43.88$$

Question 11

$$z=\frac{x-\mu}{\sigma}$$
$$-1=\frac{45-\mu}{3.3}$$
$$45-\mu=-1\times3.3$$
$$\mu=45+3.3$$
$$x=48.3$$

Question 12

$\mu = 16$, $\sigma = 1.9$

a **i** 95% = $\mu \pm 2\sigma$

$\mu - 2\sigma$, $\mu + 2\sigma$

$16 - 2 \times 1.9$, $16 + 2 \times 1.9$

12.2, 19.8

ii 68% = $\mu \pm \sigma$

$\mu - \sigma$, $\mu + \sigma$

$16 - 1.9$, $16 + 1.9$

14.1, 17.9

iii $99.7\% = \mu \pm 3\sigma$

$\mu - 3\sigma, \mu + 3\sigma$

$16 - 3 \times 1.9, 16 + 3 \times 1.9$

$10.3, 21.7$

b **i**

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{20 - 16}{1.9}$$
$$z = 2.1$$

ii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{13.5 - 16}{1.9}$$
$$z = -1.3$$

c **i**

$$z = \frac{x - \mu}{\sigma}$$
$$-3 = \frac{x - 16}{1.9}$$
$$x - 16 = -3 \times 1.9$$
$$x = 16 - 3 \times 1.9$$
$$x = 10.3$$

ii

$$z = \frac{x - \mu}{\sigma}$$
$$1.1 = \frac{x - 16}{1.9}$$
$$x - 16 = 1.1 \times 1.9$$
$$x = 16 + 1.1 \times 1.9$$
$$x = 18.09$$

Question 13

$\mu = 104.7$, $\sigma = 5.1$

a $\mu - \sigma$, $\mu + \sigma$

$104.7 - 5.1$, $104.7 + 5.1$

99.6, 109.8

b **i**

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{80-104.7}{5.1}$$
$$z = -4.84$$

ii

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{103-104.7}{5.1}$$
$$z = -0.3$$

c **i**

$$z = \frac{x-\mu}{\sigma}$$
$$2 = \frac{x-104.7}{5.1}$$
$$x - 104.7 = 2 \times 5.1$$
$$x = 104.7 + 2 \times 5.1$$
$$x = 114.9$$

ii

$$z = \frac{x-\mu}{\sigma}$$
$$-1.3 = \frac{x-104.7}{5.1}$$
$$x - 104.7 = -1.3 \times 5.1$$
$$x = 104.7 - 1.3 \times 5.1$$
$$x = 98.07$$

Exercise 10.08 Applications of the normal distribution

Question 1

$\mu = 24$, $\sigma = 0.2$

a **i** 23.4, 24.6

$\mu - 3\sigma$, $\mu + 3\sigma$

99.7%

ii 24, 24.4

μ , $\mu + 2\sigma$

$0.5 \times 95\%$

$= 47.5\%$

iii 23.8, 24.2

$\mu - \sigma$, $\mu + \sigma$

68%

iv 23.4, 24

$\mu - 3\sigma$, μ

$0.5 \times 99.7\% = 49.85\%$

v 23.4, 24.2

$\mu - 3\sigma$, $\mu + \sigma$

$0.5 \times 99.7\% + 0.5 \times 68\%$

$= 49.85\% + 34\%$

$= 83.85\%$

b **i**

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{24.8 - 24}{0.2}$$
$$z = 4$$

ii Yes, it is unusual as it is greater than 3 standard deviations from the mean.

c **i**

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{23.7 - 24}{0.2}$$
$$z = -1.5$$

ii $P(X \le 23.7) = P(Z \le -1.5)$

$= 0.0668$

iii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{23.7 - 24}{0.2}$$
$$z = -1.5$$
$$z = \frac{23.9 - 24}{0.2}$$
$$z = -0.5$$
$$P\left(23.7 \le X \le 23.9\right)$$
$$= P\left(-1.5 \le Z \le -0.5\right)$$
$$= P\left(Z \le -0.5\right) - P\left(Z \le -1.5\right)$$
$$= 0.3085 - 0.0668$$
$$= 0.2417$$

Question 2

$\mu = 25.3$, $\sigma = 3.4$

a **i** 21.9, 28.7

$\mu - \sigma$, $\mu + \sigma$

68%

ii 18.5, 32.1

$\mu - 2\sigma$, $\mu + 2\sigma$

95%

iii 15.1, 35.5

$\mu - 3\sigma$, $\mu + 3\sigma$

99.7%

iv 25.3, 28.7

μ , $\mu + \sigma$

$0.5 \times 68 = 34\%$

v 21.9, 32.1

$\mu - \sigma$, $\mu + 2\sigma$

$0.5 \times 68\% + 0.5 \times 95\%$

$= 34\% + 47.5\%$

$= 81.5\%$

b **i**

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{14 - 25.3}{3.4}$$

$$z = -3.3$$

ii Yes, it is unusual as it is more than 3 standard deviations away from the mean.

c **i**

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{38 - 25.3}{3.4}$$
$$z = 3.7$$

ii Yes, it is unusual as it is more than 3 standard deviations away from the mean.

Question 3

$\mu = 8$, $\sigma = 1.7$

a

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{15 - 8}{1.7}$$
$$z = 4.1$$

It is unacceptable as it is outside of 3 standard deviations from the mean.

b 95% = $\mu \pm 2\sigma$

$\mu - 2\sigma$, $\mu + 2\sigma$

$8 - 2 \times 1.7$, $8 + 2 \times 1.7$

4.6 minutes, 11.4 minutes

c 4.6, 13.1

$\mu - 2\sigma$, $\mu + 3\sigma$

$0.5 \times 95\% + 0.5 \times 99.7\%$

$= 47.5\% + 49.85\%$

$= 97.35\%$

d **i**

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{7.5 - 8}{1.7}$$
$$z = -0.29$$

ii

$$P(Z \le -0.29) = 0.3859$$
$$P(7.5 \le X \le 8)$$
$$= P(-0.29 \le Z \le 0)$$
$$= P(Z \le 0) - P(Z \le -0.29)$$
$$= 0.5 - 0.3859$$
$$= 0.1141$$
$$= 11.41\%$$

e **i**

$$Z = \frac{x - \mu}{\sigma}$$
$$Z = \frac{6 - 8}{1.7}$$
$$Z = -1.18$$
$$P(Z \le -1.18)$$
$$= 0.1190$$
$$P(6 \le X \le 8)$$
$$= P(-1.18 \le Z \le 0)$$
$$= P(Z \le 0) - P(Z \le -1.18)$$
$$= 0.5 - 0.1190$$
$$= 0.381$$

ii 6, 9.7

$P(6 \le X \le 8) = 0.381$ from part **i**;

$P(8 < X \le 9.7) = 0.34$ (using empirical rule of $0.5 \times \mu \pm \sigma$)

$= 0.381 + 0.34 = 0.721$

iii

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{5-8}{1.7} = -1.76$$
$$z = \frac{10-8}{1.7} = 1.18$$
$$P(5 \le X \le 10)$$
$$= P(-1.76 \le Z \le 1.18)$$
$$= P(Z \le 1.18) - P(Z \le -1.76)$$
$$= 0.8810 - 0.0392$$
$$= 0.8418$$

f **i**

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{3-8}{1.7} = -2.94$$
$$z = \frac{12-8}{1.7} = 2.35$$
$$P(3 \le X \le 12)$$
$$= P(-2.94 \le Z \le 2.35)$$
$$= P(Z \le 2.35) - P(Z \le -2.94)$$
$$= 0.9906 - 0.0016$$
$$= 0.989$$
$$= 98.9\%$$

ii

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{7 - 8}{1.7} = -0.59$$

$$z = \frac{11 - 8}{1.7} = 1.76$$

$$P(7 \le X \le 11)$$
$$= P(-0.59 \le Z \le 1.76)$$
$$= P(Z \le 1.76) - P(Z \le -0.59)$$
$$= 0.9608 - 0.2776$$
$$= 0.6832$$
$$= 68.32\%$$

iii

$$z = \frac{x - \mu}{\sigma}$$

$$Z = \frac{3.5 - 8}{1.7} = -2.65$$

$$z = \frac{11.9 - 8}{1.7} = 2.29$$

$$P(3.5 \le X \le 11.9)$$
$$= P(-2.65 \le Z \le 2.29)$$
$$= P(Z \le 2.29) - P(Z \le -2.65)$$
$$= 0.9890 - 0.0040$$
$$= 0.985$$
$$= 98.5\%$$

Question 4

$\mu = 19.9$, $\sigma = 0.4$

a **i** 19.1, 20.7

$\mu - 2\sigma$, $\mu + 2\sigma$

95%

ii 19.9, 21.7

μ , $\mu + 3\sigma$

$0.5 \times 99.7\% = 49.85\%$

iii 20, 21

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{20-19.9}{0.4} = 0.25$$
$$z = \frac{21-19.9}{0.4} = 2.75$$
$$\begin{aligned} &P(20 \le X \le 21) \\ &= P(0.25 \le Z \le 2.75) \\ &= P(Z \le 2.75) - P(Z \le 0.25) \\ &= 0.9970 - 0.5987 \\ &= 0.3983 \\ &= 39.83\% \end{aligned}$$

b $99.7\% = \mu \pm 3\sigma$

$\mu - 3\sigma$, $\mu + 3\sigma$

$19.9 - 3 \times 0.4$, $19.9 + 3 \times 0.4$

18.7 mL, 21.1 mL

c

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{23-19.9}{0.4} = 7.75$$

A bottle whose volume is 23 mL is greater than 3 standard deviations from the mean which is unusual and outside the expected range.

d

$$z = \frac{x - \mu}{\sigma}$$

$$z = \frac{18.9 - 19.9}{0.4} = -2.5$$

$$z = \frac{19.3 - 19.9}{0.4} = -1.5$$

$$P(18.9 \le X \le 19.3)$$

$$= P(-2.5 \le Z \le -1.5)$$

$$= P(Z \le -1.5) - P(Z \le -2.5)$$

$$= 0.0668 - 0.0062$$

$$= 0.0606$$

Question 5

$\mu = 7.5$, $\sigma = 0.3$

a $\mu \pm 2\sigma = 95\%$

b Largest diameter: $\mu + 2\sigma$

$= 7.5 + 2 \times 0.3$

$= 8.1$ cm

c Smallest diameter: $\mu - 2\sigma$

$= 7.5 - 2 \times 0.3$

$= 6.9$ cm

Question 6

$\mu = 3.1$, $\sigma = 0.3$

a $\mu - 3\sigma$

$= 3.1 - 3 \times 0.3$

$= 2.2$ years

b **i** 2.8, 3.4

$\mu - \sigma$, $\mu + \sigma$

68%

ii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{2.5 - 3.1}{0.3} = -2$$
$$z = \frac{3.5 - 3.1}{0.3} = 1.33$$
$$P(2.5 \le X \le 3.5)$$
$$= P(-2 \le Z \le 1.33)$$
$$= P(Z \le 1.33) - P(Z \le -2)$$
$$= 0.9082 - 0.0228$$
$$= 0.8854$$
$$= 88.54\%$$

c

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{4 - 3.1}{0.3} = 3$$

Yes, as 4 years is 3 standard deviations above the mean so any occurrence greater than this is unusual.

Question 7

$\mu = 28$, $\sigma = 0.833$

a $\mu - 3\sigma$, $\mu + 3\sigma$

$28 - 3 \times 0.833$, $28 + 3 \times 0.833$

25.5 cm, 30.5 cm

b

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{27.2 - 28}{0.833}$$
$$z = -0.96$$

c

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{30 - 28}{0.833} = 2.4$$
$$P(27.2 \le X \le 30)$$
$$= P(-0.96 \le Z \le 2.4)$$
$$= P(Z \le 2.4) - P(Z \le -0.96)$$
$$= 0.9918 - 0.1685$$
$$= 0.8233$$
$$= 82.33\%$$

d From part **a**, 24 cm tall is outside the range, so it would be unusual.

Question 8

$\mu = 4.95$, $\sigma = 0.15$

a $4.65 = \mu - 2\sigma$

$5.25 = \mu + 2\sigma$

95% are accepted

So 100% – 95% = 5% are rejected

b $0.5 \times 5\% = 2.5\%$

2.5% are too small.

c **i** $\mu \pm 3\sigma = 99.7\%$

0.3% are rejected

ii $\mu - 3\sigma$, $\mu + 3\sigma$

$4.95 - 3 \times 0.15$, $4.95 + 3 \times 0.15$

4.5, 5.4

If the mass is less than 4.5 kg or greater than 5.4 kg.

Question 9

$\mu = 10.6$, $\sigma = 0.5$

a **i** $100 - 0.3 = 99.7\%$

ii $\mu + 3\sigma = 10.6 + 3 \times 0.5 = 12.1$ cm

iii $\mu - 3\sigma = 10.6 - 3 \times 0.5 = 9.1$ cm

b 8.3 and 12.6 are rejected.

Question 10

$\mu = 252.5$, $\sigma = 0.4$

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{250 - 252.5}{0.4}$$
$$z = -6.25$$

Yes, the claim is realistic as 250 is 6.25 standard deviations below the mean.

Question 11

$\mu = 1$, $\sigma = 0.01$

$$Z = \frac{x - \mu}{\sigma}$$
$$z = \frac{0.98 - 1}{0.01}$$
$$z = -2$$

2.5% of all minces are below 0.98 kg

Question 12

a $826 - 814 = 12 = 6\sigma$

$\sigma = 12 \div 6 = 2$ mm

b

$$\mu = \frac{826 + 814}{2}$$
$$\mu = 820 \text{ mm}$$

c $\mu - 2\sigma$, $\mu + 2\sigma$

$820 - 2 \times 2$, $820 + 2 \times 2$

816 mm, 824 mm

Question 13

a

$$\mu = 3.1$$
$$\mu + 2\sigma = 3.5$$
$$3.1 + 2\sigma = 3.5$$
$$2\sigma = 0.4$$
$$\sigma = 0.2 \text{ h}$$

b $\mu - 3\sigma$

$3.1 - 3 \times 0.2$

$= 2.5$ h

Question 14

a Epping

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{78 - 75.3}{2.6}$$
$$z = 1.04$$

City

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{65 - 62.7}{1.7}$$
$$z = 1.35$$

b The city is the longest trip in comparison to the relative means and standard deviations.

Question 15

a Kieran

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{78 - 69.5}{8.5}$$
$$z = 1$$

b Cameron

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{71 - 61.2}{4.8}$$
$$z = 2.04$$

c Cameron did better as he has a higher z-score.

Test yourself 10

Question 1

$$f(x)=\frac{x^2}{21} \qquad 1\le x\le 4$$

$$\int_1^4 \frac{x^2}{21}\,dx$$

$$=\left[\frac{x^3}{63}\right]_1^4$$

$$=\frac{4^3}{63}-\frac{1^3}{63}$$

$$=\frac{63}{63}=1$$

$$f(x)=\frac{e^x}{e^3-1} \qquad [0,3]$$

$$\int_0^3 \frac{e^x}{e^3-1}\,dx$$

$$=\left[\frac{e^x}{e^3-1}\right]_0^3$$

$$=\frac{e^3}{e^3-1}-\frac{e^x}{e^3-1}$$

$$=\frac{e^3-1}{e^3-1}$$

$$=1$$

$$f(x)=\frac{x^4}{625} \qquad [1,5]$$

$$\int_1^5 \frac{x^4}{625}\,dx$$

$$=\left[\frac{x^5}{3125}\right]_1^5$$

$$=\frac{5^5}{3125}-\frac{1^5}{3125}$$

$$\frac{5^5-1}{3125}\ne 1$$

$$f(x)=\frac{4x^3}{625} \qquad [0,5]$$

$$\int_0^5 \frac{4x^3}{625}dx$$

$$=\left[\frac{x^4}{625}\right]_0^5$$

$$=\frac{x^4}{625}-\frac{0^4}{625}$$

$$=\frac{625}{625}=1$$

C

Question 2

A

Question 3

D

Question 4

a $P(X \le 15)=0.75$

b $P(X \le 15)=\frac{8}{20}=\frac{2}{5}=0.4$

c

$$P(7 \le X \le 18)$$

$$=\frac{18}{20}-\frac{7}{20}=\frac{11}{20}=0.55$$

d

$$P(4 < X < 13)$$

$$=\frac{13}{20}-\frac{4}{20}=\frac{9}{20}=0.45$$

e $P(X \geq 6) = \frac{14}{20} = \frac{7}{10} = 0.7$

f Median = 10

g 18th percentile

$\frac{18}{100} \times 20 = 3.6$

h 89th percentile

$\frac{89}{100} \times 20 = 17.8$

i 6th decile

$\frac{6}{10} \times 20 = 12$

j 3rd quartile

$\frac{3}{4} \times 20 = 15$

Question 5

a

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2} \times 0.5 \times 4$$
$$= 1$$

Yes, it is.

b

$$A_1 = \frac{h}{2}(a+b)$$
$$= \frac{2}{2}(0.5+0.2)$$
$$= 0.7$$
$$A_2 = \frac{1}{2}bh$$
$$= \frac{1}{2}\times 0.2\times 3$$
$$= 0.3$$
$$\text{Total Area} = A_1 + A_2$$
$$= 0.7+0.3$$
$$= 1$$

Yes, it is.

c

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}\times 0.6\times 3$$
$$= 0.9$$

No, it is not.

Question 6

$$f(x)=\frac{3x^2}{124} \qquad 1\le x\le 5$$

$$F(x)=\int_1^x \frac{3x^2}{124}\,dx$$

$$=\left[\frac{x^3}{124}\right]_1^x$$

$$=\frac{x^3}{124}-\frac{1^3}{124}$$

$$=\frac{x^3-1}{124}$$

$$F(x)=0.5$$

$$0.5=\frac{x^3-1}{124}$$

$$x^3-1=62$$

$$x^3=63$$

$$x=\sqrt[3]{63}$$

$$x=3.98$$

Question 7

a

$$f(x)=\frac{3x^2}{511} \qquad 1\le x\le 8$$

$$F(x)=\int_1^x \frac{3x^2}{511}\,dx$$

$$=\left[\frac{x^3}{511}\right]_1^x$$

$$=\frac{x^3}{511}-\frac{1^3}{511}$$

$$=\frac{x^3-1}{511}$$

b **i**

$$P(X \le 3)$$
$$F(3) = \frac{3^3 - 1}{511} = \frac{26}{511}$$

ii

$$P(X \le 5)$$
$$F(5) = \frac{5^3 - 1}{511} = \frac{124}{511}$$

iii

$$P(X > 6)$$
$$1 - P(X \le 6)$$
$$1 - F(6)$$
$$= 1 - \frac{6^3 - 1}{511}$$
$$= \frac{296}{511}$$

iv

$$P(X \ge 4)$$
$$1 - P(X \le 4)$$
$$1 - F(4)$$
$$= 1 - \frac{4^3 - 1}{511}$$
$$= \frac{448}{511}$$
$$= \frac{64}{73}$$

v

$$
\begin{aligned}
&P(2 \le X \le 7) \\
&= F(7) - F(2) \\
&= \frac{7^3 - 1}{511} - \frac{2^3 - 1}{511} \\
&= \frac{7^3 - 2^3}{511} \\
&= \frac{335}{511}
\end{aligned}
$$

Question 8

a i

$$
\begin{aligned}
&f(x) = \frac{2x}{15} \qquad [1, 4] \\
&f'(x) = \frac{2}{15}
\end{aligned}
$$

Mode is end point 4.

ii

$$
\begin{aligned}
&f(x) = \frac{2x}{15} \qquad [1, 4] \\
&F(x) = \int_1^x \frac{2x}{15}\,dx \\
&= \left[\frac{x^2}{15}\right]_1^x \\
&= \frac{x^2}{15} - \frac{1^1}{15} \\
&= \frac{x^2 - 1}{15}
\end{aligned}
$$

Let $F(x) = 0.5$

$$
\begin{aligned}
&0.5 = \frac{x^2 - 1}{15} \\
&x^2 - 1 = 7.5 \\
&x^2 = 8.5 \\
&x = \sqrt{8.5} \\
&x = 2.9
\end{aligned}
$$

b **i**

$$f\left(x\right)=\frac{x^2}{243} \qquad 0 \le x \le 9$$
$$f'\left(x\right)=\frac{2x}{243}$$

This function is monotonic increasing over the domain, so the mode is 9.

ii

$$f\left(x\right)=\frac{x^2}{243} \qquad 0 \le x \le 9$$
$$F\left(x\right)=\int_0^x \frac{x^2}{243}dx$$
$$=\left[\frac{x^3}{729}\right]_0^x$$
$$=\frac{x^3}{729}-\frac{0^3}{729}$$
$$=\frac{x^3}{729}$$
Let $F\left(x\right)=0.5$
$$0.5=\frac{x^3}{729}$$
$$x^3=\frac{729}{2}$$
$$x=\sqrt[3]{\frac{729}{2}}$$
$$x=7.14$$

Question 9

$\mu = 3.2$, $\sigma = 0.31$

a $\mu - 3\sigma$, $\mu + 3\sigma$

$3.2 - 3 \times 0.31$, $3.2 + 3 \times 0.31$

2.27 kg, 4.13 kg

b **i**

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{3.9-3.2}{0.31}$$
$$z = 2.26$$

ii

$$z = \frac{x-\mu}{\sigma}$$
$$z = \frac{3.5-3.2}{0.31}$$
$$z = 0.97$$

c

$$P(3.5 \le X \le 3.9)$$
$$= P(0.97 \le Z \le 2.26)$$
$$= P(Z \le 2.26) - P(Z \le 0.97)$$
$$= 0.9881 - 0.8340$$
$$= 0.1541$$

Question 10

$$f(x)=\frac{3x^5}{2048} \qquad 0\le x\le a$$

$$\int_0^a \frac{3x^5}{2048}\,dx=1$$

$$\left[\frac{x^6}{4096}\right]_0^a=1$$

$$\frac{a^6}{4096}-\frac{0^6}{4096}=1$$

$$a^6=4096$$

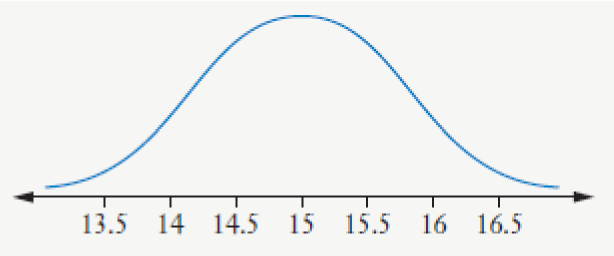
$$a=\sqrt[6]{4096}$$

$$a=4$$

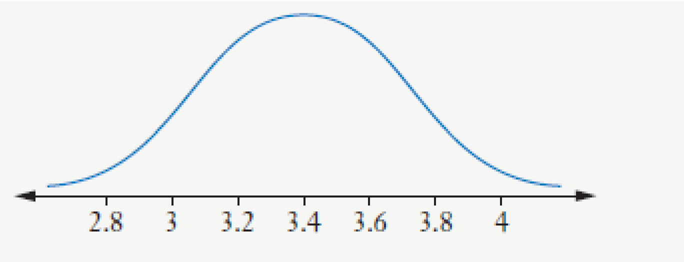
Domain [0, 4]

Question 11

a



b



Question 12

$\mu = 1.95$, $\sigma = 0.08$

a 1.71 kg to 2.19 kg = $\mu - 3\sigma$ to $\mu + 3\sigma$

So 99.7% accepted.

100% – 99.7% = 0.3% rejected

b $2.11 = \mu + 2\sigma$

$1.87 = \mu - \sigma$

Accepted = 0.5 × 95% + 0.5 × 68%

= 47.5% + 34%

= 81.5%

Rejected = 100% – 81.5% = 18.5%

Question 13

$$f(x) = \frac{3x^2}{316} \qquad 3 \le x \le 7$$

$$F(x) = \int_3^x \frac{3x^2}{316}\, dx$$

$$= \left[\frac{x^3}{316}\right]_3^x$$

$$= \frac{x^3}{316} - \frac{3^3}{316}$$

$$= \frac{x^3 - 27}{316}$$

a

$$F(x) = 0.5$$

$$\frac{x^3 - 27}{316} = 0.5$$

$$x^3 - 27 = 158$$

$$x^3 = 185$$

$$x = \sqrt[3]{185}$$

$$x = 5.7$$

b

$$F(x) = 0.75$$
$$\frac{x^3 - 27}{316} = 0.75$$
$$x^3 - 27 = 237$$
$$x^3 = 264$$
$$x = \sqrt[3]{264}$$
$$x = 6.4$$

c

$$F(x) = 0.4$$
$$\frac{x^3 - 27}{316} = 0.4$$
$$x^3 - 27 = 126.4$$
$$x^3 = 153.4$$
$$x = \sqrt[3]{153.4}$$
$$x = 5.35$$

d

$$F(x) = 0.63$$
$$\frac{x^3 - 27}{316} = 0.63$$
$$x^3 - 27 = 199.08$$
$$x^3 = 226.08$$
$$x = \sqrt[3]{226.08}$$
$$x = 6.09$$

e

$$F(x) = 0.28$$
$$\frac{x^3 - 27}{316} = 0.28$$
$$x^3 - 27 = 88.48$$
$$x^3 = 115.48$$
$$x = \sqrt[3]{115.48}$$
$$x = 4.87$$

Question 14

$\mu = 2.9$, $\sigma = 3.6 - 2.9 = 0.7$

Question 15

a

$$f(x) = \frac{x^4}{625} \qquad 0 \le x \le 5$$
$$F(x) = \int_0^x \frac{x^4}{625}\,dx$$
$$= \left[\frac{x^5}{3125}\right]_0^x$$
$$= \frac{x^5}{3125} - \frac{0^5}{3125}$$
$$= \frac{x^5}{3125}$$

b

$$f(x) = \frac{x^6}{117\ 649} \qquad [0,7]$$
$$F(x) = \int_0^x \frac{x^6}{117\ 649}\,dx$$
$$= \left[\frac{x^7}{823\ 543}\right]_0^x$$
$$= \frac{x^7}{823\ 543} - \frac{0^7}{823\ 543}$$
$$= \frac{x^7}{823\ 543}$$

c

$$f(x) = \frac{e^x}{e^6 - 1} \qquad 0 \le x \le 6$$

$$F(x) = \int_0^x \frac{e^x}{e^6 - 1} dx$$

$$= \left[\frac{e^x}{e^6 - 1} \right]_0^x$$

$$= \frac{e^x}{e^4 - 1} - \frac{e^0}{e^4 - 1}$$

$$= \frac{e^x - 1}{e^6 - 1}$$

d

$$f(x) = \frac{x}{40} \qquad [1, 9]$$

$$F(x) = \int_1^x \frac{x}{40} dx$$

$$= \left[\frac{x^2}{80} \right]_1^x$$

$$= \frac{x^2}{80} - \frac{1^2}{80}$$

$$= \frac{x^2 - 1}{80}$$

Question 16

a $P(Z \le 0.54) = 0.7054$

b $P(Z \le 1.32) = 0.9066$

c $P(Z \le -3) = 0.0013$

d $P(Z \le -0.71) = 0.2389$

e

$$\begin{aligned}&P(Z>-1)\\&=1-P(Z\le -1)\\&=1-0.1587\\&=0.8413\end{aligned}$$

f

$$\begin{aligned}&P(Z\ge 2.5)\\&=1-P(Z\le 2.5)\\&=1-0.9938\\&=0.0062\end{aligned}$$

g

$$\begin{aligned}&P(Z\ge -1.08)\\&=1-P(Z\le -1.08)\\&=1-0.1401\\&=0.8599\end{aligned}$$

h

$$\begin{aligned}&P(-2.3\le Z\le -1.09)\\&=P(Z\le -1.09)-P(Z\le -2.3)\\&=0.1379-0.0107\\&=0.1272\end{aligned}$$

i

$$\begin{aligned}&P(1.1\le Z\le 3.11)\\&=P(Z\le 3.11)-P(Z\le 1.1)\\&=0.9991-0.8643\\&=0.1348\end{aligned}$$

Question 17

$\mu = 12.5$, $\sigma = 1.5$

a 9.5, 15.5

$\mu - 2\sigma$, $\mu + 2\sigma$

95%

b 12.5, 14

μ , $\mu + \sigma$

$0.5 \times 68\% = 34\%$

c 11, 17

$\mu - \sigma$, $\mu + 3\sigma$

$0.5 \times 68\% + 0.5 \times 99.7\%$

$= 34\% + 49.85\%$

$= 83.85\%$

d 10, 15

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{10 - 12.5}{1.5} = -1.67$$
$$Z = \frac{15 - 12.5}{1.5} = 1.67$$
$$P(10 \le X \le 15)$$
$$= P(-1.67 \le Z \le 1.67)$$
$$= 1 - 2P(Z \le -1.67) \quad \text{By symmetry of bell curve}$$
$$= 1 - 2 \times 0.0475$$
$$= 0.905$$
$$= 90.5\%$$

e 12, 13

$$z=\frac{x-\mu}{\sigma}$$
$$z=\frac{12-12.5}{1.5}=-0.33$$
$$z=\frac{13-12.5}{1.5}=0.33$$
$$P(12\le X\le 13)$$
$$=P(-0.33\le Z\le 0.33)$$
$$=1-2P(Z\le -0.33)\quad \text{By symmetry of bell curve}$$
$$=1-2\times 0.3707$$
$$=0.2586$$
$$=25.86\%$$

Question 18

a

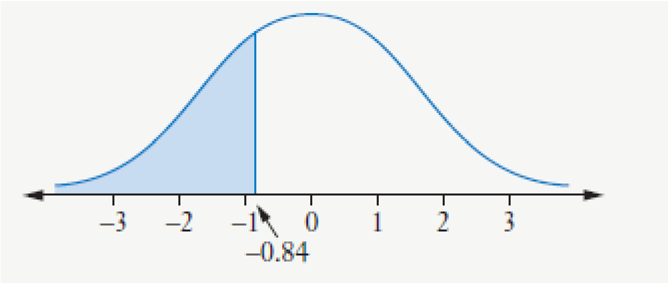
$$f(x)=ae^x \qquad [0,5]$$
$$\int_0^5 ae^x dx=1$$
$$\left[ae^x\right]_0^5=1$$
$$ae^5-ae^0=1$$
$$a\left(e^5-1\right)=1$$
$$a=\frac{1}{e^5-1}$$

b

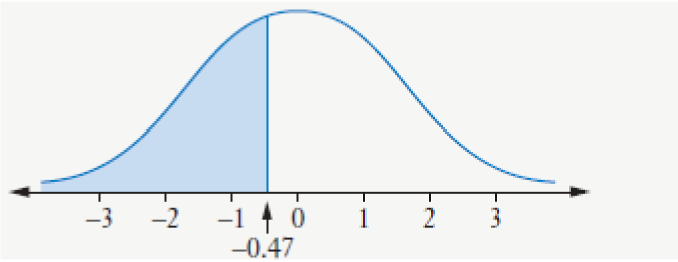
$$f(x)=ae^x \qquad [1,4]$$
$$\int_1^4 ae^x dx=1$$
$$\left[ae^x\right]_1^4=1$$
$$ae^4-ae^1=1$$
$$a\left(e^4-e\right)=1$$
$$a=\frac{1}{e^4-e}$$

Question 19

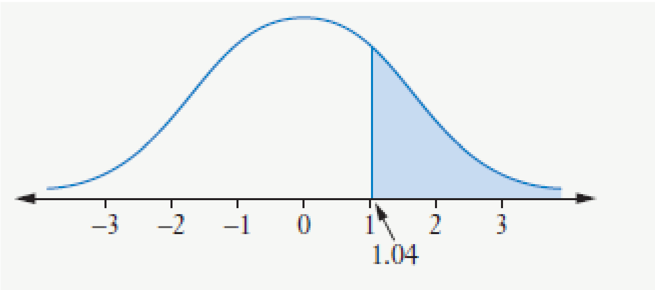
a



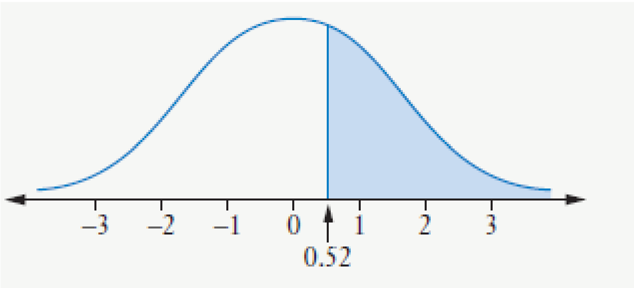
b



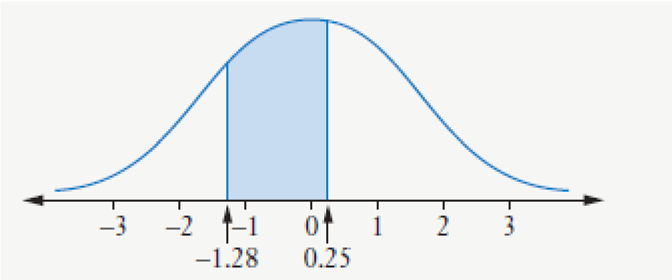
c



d



e



Question 20

a $f(x)$ is monotonic increasing, so mode = maximum $x = 8$.

b Mode = 1.5

c

$$f(x)=\frac{4}{189}\left(x^3-9x^2+24x\right) \qquad [0,3]$$

$$f'(x)=\frac{4}{189}\left(3x^2-18x+24\right)$$

Let $f'(x)=0$

$$0=\frac{4}{189}\left(3x^2-18x+24\right)$$

$$3x^2-18x+24=0$$

$$3(x-4)(x-2)=0$$

$$x=2,4$$

$x = 4$ is outside the domain

$x = 2$

but need to test for maximum

$$f''(x)=\frac{4}{189}(6x-18)$$

$$f''(2)=\frac{4}{189}(6\times 2-18)<0$$

$x = 2$ is a maximum

Question 21

a Klare

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{9.3 - 8.3}{1.2} \approx 0.83$$

b Simon

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{8.9 - 8.1}{0.8} = 1$$

c Klare had the lower *z*-score so she had the better time.

Question 22

$$f(x) = \frac{x^3}{600} \qquad 1 \le x \le 7$$

$$\int_1^7 \frac{x^3}{600}\,dx$$
$$= \left[\frac{x^4}{2400}\right]_1^7$$
$$= \frac{7^4}{2400} - \frac{1^4}{2400}$$
$$= \frac{2401 - 1}{2400}$$
$$= \frac{2400}{2400}$$
$$= 1$$

Yes, it is a probability density function.

Question 23

$\mu = 1.1$, $\sigma = 0.02$

a **i**

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{1.15 - 1.1}{0.02}$$
$$z = 2.5$$

ii

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{1.07 - 1.1}{0.02}$$
$$z = -1.5$$

b **i** $z = 1 \Rightarrow \mu + \sigma = 1.1 + 0.02 = 1.12$ m

ii $z = -2 \Rightarrow \mu - 2\sigma = 1.1 - 0.04 = 1.06$ m

iii

$$z = \frac{x - \mu}{\sigma}$$
$$3.1 = \frac{x - 1.1}{0.02}$$
$$x - 1.1 = 3.1 \times 0.02$$
$$x = 1.1 + 3.1 \times 0.02$$
$$x = 1.162 \text{ m}$$

iv

$$z = \frac{x - \mu}{\sigma}$$
$$-0.63 = \frac{x - 1.1}{0.02}$$
$$x - 1.1 = -0.63 \times 0.02$$
$$x = 1.1 - 0.63 \times 0.02$$
$$x = 1.087 \text{ m}$$

v

$$
\begin{aligned}
z &= \frac{x-\mu}{\sigma}\\
1.27 &= \frac{x-1.1}{0.02}\\
x-1.1 &= 1.27\times 0.02\\
x &= 1.1+1.27\times 0.02\\
x &= 1.125 \text{ m}
\end{aligned}
$$

Question 24

$$
\begin{aligned}
z &= \frac{x-\mu}{\sigma}\\
-1.3 &= \frac{87.9-\mu}{1.6}\\
87.9-\mu &= -1.3\times 1.6\\
\mu &= 87.9+1.3\times 1.6\\
\mu &= 89.98
\end{aligned}
$$

Question 25

$$
\begin{aligned}
z &= \frac{x-\mu}{\sigma}\\
1.7 &= \frac{52.4-\mu}{1.9}\\
52.4-\mu &= 1.7\times 1.9\\
\mu &= 52.4-1.7\times 1.9\\
\mu &= 49.17
\end{aligned}
$$

Challenge exercise 10

Question 1

$$f(x)=\frac{5x^4}{3124} \qquad a\le x\le b$$

$$F(x)=\int_a^x \frac{5x^4}{3124}dx$$

$$=\left[\frac{x^5}{3124}\right]_a^x$$

$$=\frac{x^5}{3124}-\frac{a^5}{3124}$$

$$=\frac{x^5-a^5}{3124}$$

$$F(b)=1$$

$$\frac{b^5-a^5}{3124}=1$$

$$b^5-a^5=3124$$

$$F(4.353031)=0.5$$

$$\frac{(4.353031)^5-a^5}{3124}=0.5$$

$$(4.353031)^5-a^5=1562$$

$$a^5=(4.353031)^5-1562$$

$$a^5=1$$

$$a=1$$

$$b^5-1=3124$$

$$b^5=3125$$

$$b=\sqrt[5]{3125}$$

$$b=5$$

Question 2

$\mu - 2\sigma = 12.4$

$\mu + 2\sigma = 14$

$2\mu = 26.4$

$\mu = 13.2$

$13.2 + 2\sigma = 14$

$2\sigma = 0.8$

$\sigma = 0.4$

Question 3

$$f(x) = \frac{4}{249}(x+2)(x-4)^2 \qquad [0,3]$$

$$f'(x) = \frac{4}{249}\left[(x-4)^2 + 2(x+2)(x-4)\right]$$

Let $f'(x) = 0$

$$0 = \frac{4}{249}\left[(x-4)^2 + 2(x+2)(x-4)\right]$$

$$0 = (x-4)^2 + 2(x+2)(x-4)$$

$$0 = (3x+4)(x-4)$$

$$x = -\frac{4}{3}, 4$$

Both of these solutions are outside of the range, so the end points need to be tested.

$$f(0) = \frac{4}{249}(0+2)(0-4)^2$$

$$= \frac{128}{249}$$

$$f(3) = \frac{4}{249}(3+2)(3-4)^2$$

$$= \frac{20}{249}$$

$$f(0) > f(3)$$

0 is the mode

Question 4

$$f(x)=\frac{3x\left(x^2+1\right)^2}{62\ 000} \qquad [3,7]$$

$$F(x)=\int_3^x \frac{3x\left(x^2+1\right)^2}{62\ 000}\ dx$$

$$=\frac{3}{124\ 000}\int_3^x 2x\left(x^2+1\right)^2\ dx$$

$$=\frac{1}{124\ 000}\left[\left(x^2+1\right)^3\right]_3^x$$

$$=\frac{\left(x^2+1\right)^3}{124\ 000}-\frac{\left(3^2+1\right)^3}{124\ 000}$$

$$=\frac{\left(x^2+1\right)^3-1000}{124\ 000}$$

Question 5

$P(X \le 23.8) = 0.9192$

$z = 1.4$

$z = \dfrac{x - \mu}{\sigma}$

$1.4 = \dfrac{23.8 - \mu}{\sigma}$

$23.8 - \mu = 1.4\sigma$

$P(X \le 17.15) = 0.3085$

$z = -0.5$

$-0.5 = \dfrac{17.15 - \mu}{\sigma}$

$17.15 - \mu = -0.5\sigma$ [1]

$23.8 - \mu = 1.4\sigma$ [2]

$[2] - [1]$:

$6.65 = 1.9\sigma$

$\sigma = 3.5$

Sub into [2]:

$23.8 - \mu = 1.4 \times 3.5$

$= 4.9$

$\mu = 23.8 - 4.9$

$\mu = 18.9$

Question 6

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Given $\mu = 6$ and $\sigma = 1$.

$$f(x)=\frac{1}{1\times\sqrt{2\pi}}e^{-\frac{(x-6)^2}{2(1)^2}}$$

$$=\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-6)^2}{2}}$$

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-6)^2}{2}}\,dx$$

$$=1$$