MATHS IN FOCUS

MATHEMATICS ADVANCED

YEAR 12

Margaret Grove

3RD EIION

NELSON
A Cengage Company

Maths n Focus 12 Advanced
3rd Edton
Margaret Grove
ISBN 9780170413367

Publisher: Robert Yen
Proect editor: Alan Stewart
Editor: Elaine Cochrane
Cover design: Chris Starr MakeWork
Text design: Sarah Anderson
Cover image: iStock.com/ChrisHepburn
Permissions researcher: Wendy Duncan
Production manager: em Wolfenden
Production manager: Alice Kane

For product information and technology assistance
in Australia call **1300 790 853**;
in New Zealand call **0800 449 725**

For permission to use material from this text or product please email
austpermissions@cengagecom

ISBN 978 0 17 041336 7

Cengage Learning Australia
Level 7 80 Dorcas Street
South Melbourne Victoria Australia 3205

Cengage Learning New Zealand
Unit 4B Rosedale Office Park
331 Rosedale Road Albany North Shore 0632 NZ

For learning solutions visit **cengagecomau**

Printed in Singapore by 1010 Printing International Limited
1 2 3 4 5 6 7 23 22 21 20 19

PREFACE

Maths in Focus 12 Mathematics Advanced has been rewritten for the new Mathematics Advanced syllabus (2017) In this 3rd edition of the book teachers will find those familiar features that have made Maths in Focus a leading senior mathematics series such as clear and abundant worked examples in plain English comprehensive sets of graded exercise, chapter *Test Yourself* and *Challenge* exercises Investigations and practice sets of mixed revision and exam-style questions

The Mathematics Advanced course is designed for students who intend to study at university in a field that requires mathematics especially calculus and statistics This book covers the content of the Year 12 Mathematics Advanced course The theory follows a logical ordr, although some topics may be learned in any order. We have endeavoured to produce a practical text that captures the spirit of the course providing relevant and meaningful applications of mathematics

The *NelsonNet* student and teacher websites contain additional resources such as worksheets video tutorials and topic tests We wish all teachers and students using this book every success in embracing the new senior mathematics course

ABOUT THE AUTHOR

Margaret Grove has spent over 30 years teaching HSC Mathematics most recently at Bankstown TAFE Collee. She has written numerous senior mathematics texts and study guides over the past 25 years including the bestselling *Maths in Focus* series for Mathematics and Mathematics Extension 1

Margaret thanks her family, especially her husband Geoff for their support in writing this book

CONTRIBUTING AUTHORS

Gaspare Carrozza and **Haroon Ha** from Homebush Boys High School wrote many of the *NelsonNet* worksheets

Scott Smith created the video tutorials

Tania Eastcott and **Elizabeth Nabhan** wrote the topic tests

Roger Walter wrote the *ExamView* questions

Shane Scott Brandon Pettis and **George Dimitriadis** wrote the worked solutions to all exercise sets

CONTENTS

ISBN 97807043220

5

GEOMETRICAL APPLICATIONS OF DIFFERENTIATION 164

6

INTEGRATION 210

7

STATISTICS 268

8

CORRELATION AND REGRESSION 332

ISBN 97807043220

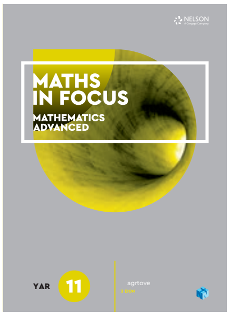
ISBN 97807043220

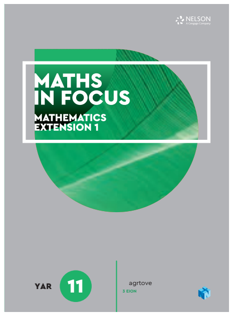
SYLLABUS REFERENCE GRID

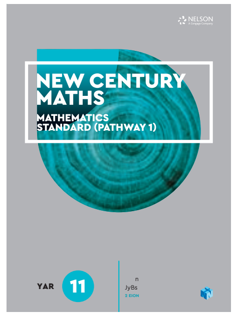
Topic and subtopic	***Maths in Focus 12 Mathematics Advanced chapter***
FUNCTIONS	
MA-F2 Graphng technques	2 Transformations of functions
TRIGONOMETRIC FUNCTIONS	
MA-T3 Tigonometic funcions and graphs **MA-T2 Tigonometic funcions andideniies**	3 Trigonometric functions 3 Trigonometric functions
CALCULUS	
MA-C2 Dfferenil clclus	
C21 Dfferenaion of tigonometi, exponenial and logarthmc functons C22 Rules of dfferenaion	4 Further ifferetition
MA-C3 Appcatons of df ferentaton	
C31 The frst and second dervatves C32 Applcatons of the dervatve	4 Further ifferetition 5 Geometrcal applcatons of dfferenaion
MA-C4 ntegra cacuus	
C41 The ant-dervatve C42 Areas and the defnte ntegral	4 Further if ferentiation 6 Integraton
FNANCAL MATHEMATICS	
MA-M1 Modeng fnanca stuatons	
M11 Modellng nvestments and loans M12 Arthmetc sequences and seres M13 Geometrc sequences and seres M14 Fnancal applcatons of sequences and seres	1 Sequences and seres 9 Investments annutes and loans
STATITICAL ANALYSIS	
MA-S2 Descrptve statstcs and bvarate data anayss	
S21 Data grouped and ungrouped and summary staisics S22 Bvarate data analyss	7 Statistics 8 Correlaton and regresson
MA-S3 Random varabes	
S31 Contnuous random varables S32 The normal istibuion	10 Contnuous probablty dstrbutons

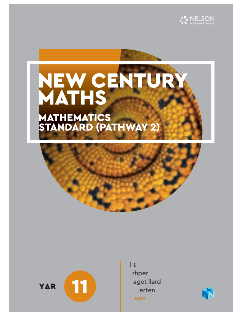
ISBN 97807043220

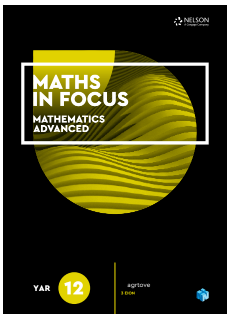
MATHS IN FOCUS AND NEW CENTURY MATHS 11–12

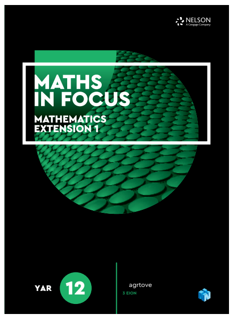


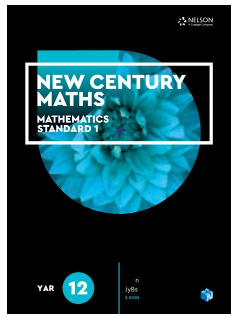


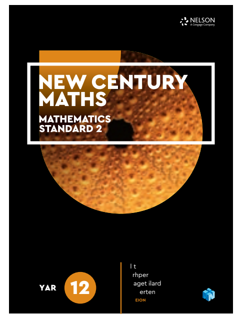


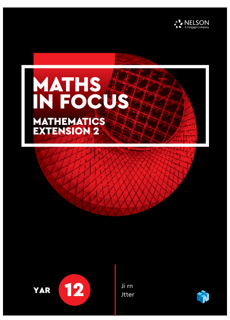








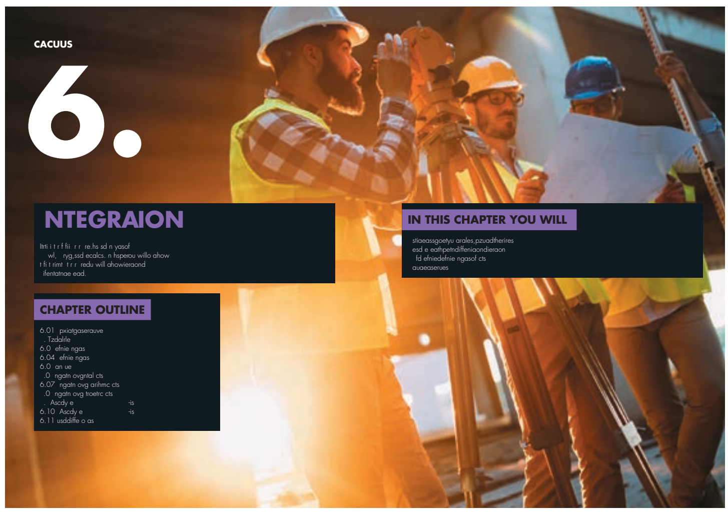




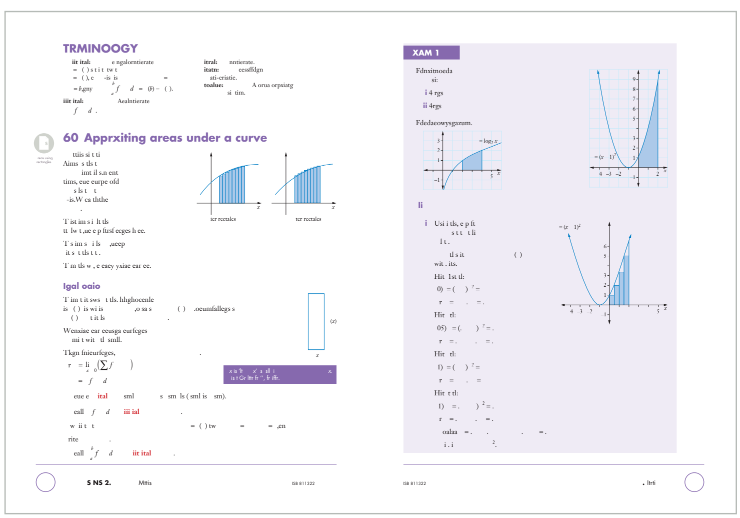
ABOUT THIS BOOK

AT THE BEGINNING OF EACH CHAPTER

- Each chapter begins on a double-page spread showing the **Chapter contents** and a list of chapter outcomes



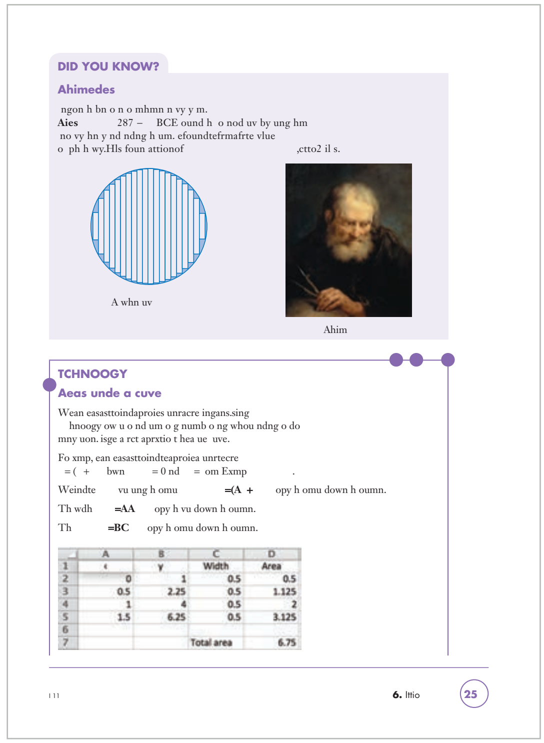
- **Terminology** is a chapter glossary that previews the key words and phrases from within the chapter

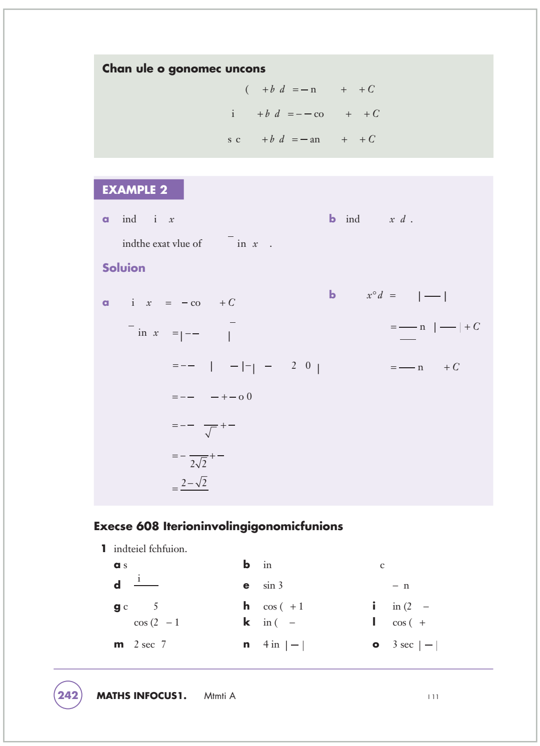


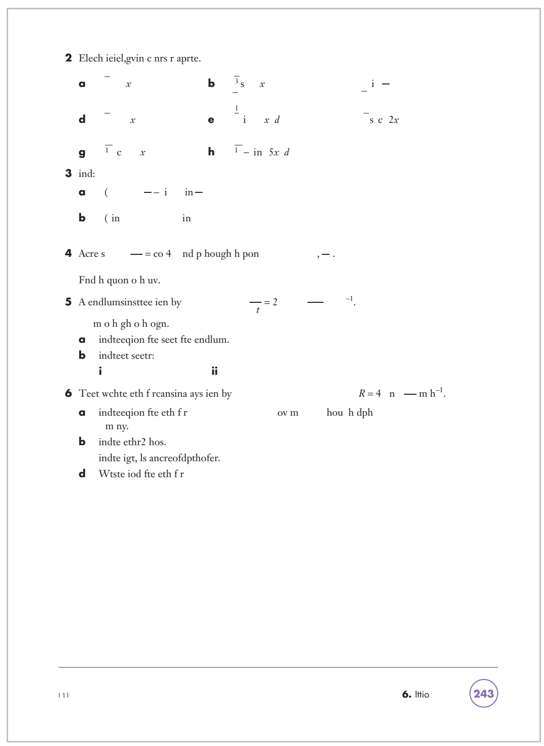
ISBN 97807043220

IN EACH CHAPTER

- Important facts and formulas are highlighted in a shaded box
- Important words and phrases are printed in red and listed in the Terminology chapter glossary.
- Graded exercises include exam-style problems and realistic applications
- Worked solutions to all exercise questions are provided on the *NelsonNet* teacher website
- **Investigations** explore the syllabus in more detail providing ideas for modelling activities and assessment tasks
- **Did you know?** contains interesting facts and applications of the mathematics learned in the chapter



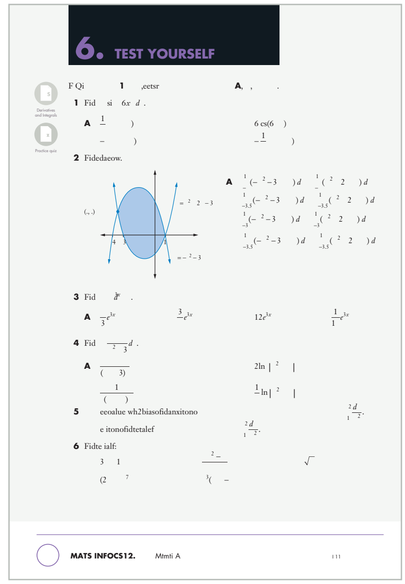




ISBN 97807043220

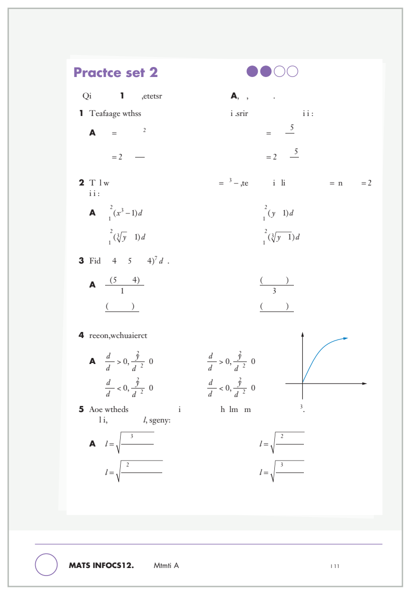
AT THE END OF EACH CHAPTER

- **Test Yourself** contains chapter revision exercises
- If you have trouble completing the *Test Yourself* exercises you need to go back and revise the chapter before trying the exercises again
- **Challenge Exercise** contains chapter extension questions Attempt these only after you are confident with the *Test Yourself* exercises because these are more difficult and are designed for students who understand the topic really well
- **Practice sets** (after several chapters) provide a comprehensive variety of mixed exam-style questions from various chapters including short-answe, free-response and multiple-choice questions



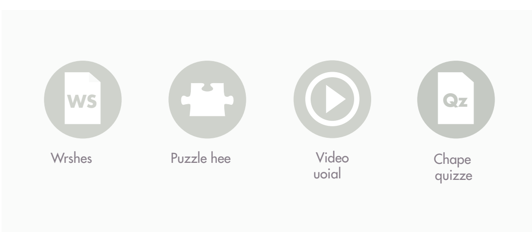
AT THE END OF THE BOOK

- **Answers** and **Index** (worked solutions on the teacher website)



NELSONNET STUDENT WEBSITE

Margin icons link to print (PDF) and multimedia resources found on the *NelsonNet* student website **www.nelsonne.co.au** These inclue:



- **Worksheets** and **puzzle sheets** that are write-in enabled PDFs
- **Video tutorial:** worked examples explained by flipped classroom teachers
- ***ExamView* quizzes** interactive and self-marking

NELSONNET TEACHER WEBSITE

The *NelsonNet* teacher website also at **www.nelsonne.co.au** contain:

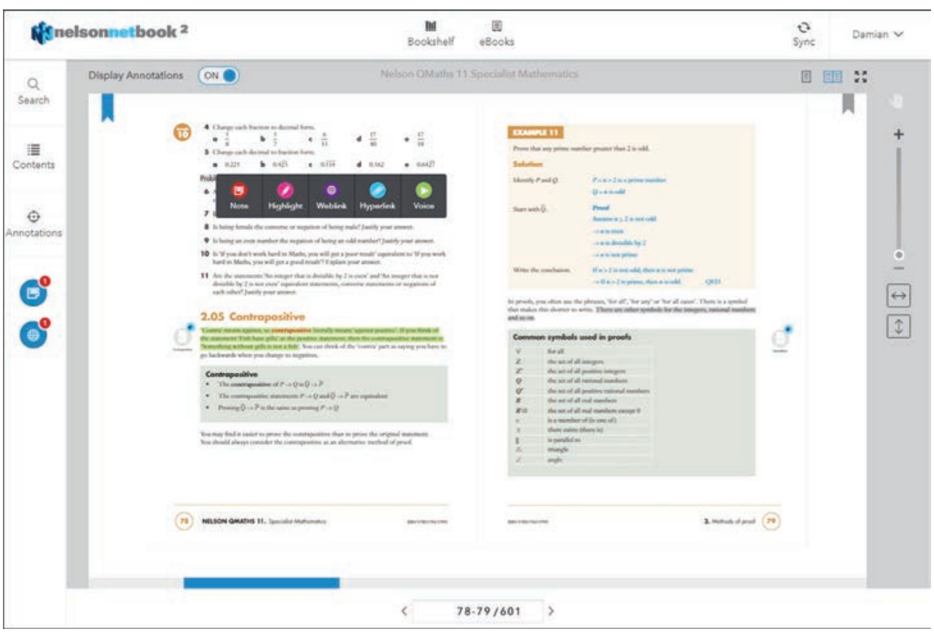
- A **teaching program** in Microsoft Word and PDF formats
- **Topic tests** in Microsoft Word and PDF formats
- **Worked solutions** to each exercise set
- **Chapter PDFs** of the textbook
- ***ExamView*** exam-writing software and questionbanks
- **Resource finder** search engine for *NelsonNet* resources

Note Complimentary access to these resources is only available to teachers who use this book as a core educational resource in their classroom Contact your Cengage Education Consultant for information about access codes and conditions

NELSONNETBOOK

NelsonNetBook is the web-based interactive version of this book found on *NelsonNet*

- To each page of NelsonNetBook you can add note, voice and sound bits, highlightng, weblinks and bookmarks
- **Zoom** and **Search** functions
- Chapters can be customised for different groups of students



ISBN 97807043220

STUDY SKILLS

The Year 11 course introduces the basics of topics such as calculus that are then applied in the Year 12 cours. You will struggle in the HSC if youdon't set yourself up to revise the Year 11 topics as you learn new Year 12 topis. Your teachers will be able to help you build up and manage good study habits Here are a few hints to get you starte. There is no right or wrong way to lern. Different styles of learning suit different people There is also no magical number of hours a week that you should study, as this will be different for every studet. But just listening in class and taking notes is not enough especially when learning material that is totally ne.

If a skill is not practised within the first 24 hours up to 50% can be forgotte. If it is not practised within 72 hours up to 85–90% can be forgotte! So it is really important tha, whatever your study timetable new work must be looked at soon after it is presented to yo.

With a continual succession of new work to learn and retai, this is a challene. But the good news is that you dont have to study for hours on en!

IN THE CLASSROOM

In order to remember, first you need to focus on what is being said and doe.

According to an ancient proverb

I hear and I forget
I see and I remember
I do and I understan.

If you chat to friends and just take notes without really paying attention you are't giving yourself a chance to remember anything and will have to study harder at home

If you are unsure of something that the teacher has said the chances are that others are also not sur. Asking questions and clarifying things will ultimately help you gain better results especially in a subject like mathematics where much of the knowledge and skills depends on being able to understand the basics

Learning is all about knowing what you know and what you dont kno. Many students feel like they dont know anythin, but t's surprising just how much they know alredy. Picking up the main concepts in class and not worrying too much about other less important parts can really help The teacher can guide you on this

Here are some pointers to get the best out of classroom learning

- Take control and be responsible for your own learning
- Clear your head of other issues in the classroom
- Active not passiv, learning is more memorable
- Ask questions if you dont understand something

ISBN 97807043220

- Listen for cues from the teacher
- Look out for what are the main concepts

Note-taking varies from class to class but here are some general guideline:

- Write legibly
- Use different colours to highlight important points or formulas
- Make notes in textbooks (using pencil if you dont own the textbook)
- Use highlighter pens to point out important points
- Summarise the main points
- If notes are scribbled rewrite them at hom.

AT HOME

You are responsible for your own learning and nobody else can tell you how best to stud. Some people need more revision time than others some study better in the mornings while others do better at night and some can work at home while others prefer a librar.

- Revise both new and older topics regularly
- Have a realistic timetable and be flexible
- Summarise the main points
- Revise when you are fresh and energetic
- Divide study time into smaller rather than longer chunks
- Study in a quiet environment
- Have a balanced life and dont forget to have fu!

If you are given exercises out of a textbook to do for homework consider asking the teacher if you can leave some of them till later and use these for revision It is not necessary to do every exercise at one sitting and you learn better if you can spread these over tim.

People use different learning styles to help them study The more variety the bettr, and you will find some that help you more than others Some people (around 35%) learn best visuall, some (25%) learn best by hearing and others (40%) learn by doing

- Summarise on cue cards or in a small notebook
- Use colourful posters
- Use mind maps and diagrams
- Discuss work with a group of friends
- Read notes out aloud
- Make up songs and rhymes
- Do exercises regularly
- Role-play teaching someone else

ISBN 97807043220

ASSESSMENT TASKS AND EXAMS

You will cope better in exams if you have practised doing sample exams under exam condition. Regular revision will give you confidence and if you feel well prepared this will help get rid of nerves in the exam You will also cope better if you have had a reasonable niht's sleep before the xam.

One of the biggest problems students have with exams is in timing Make sure you do't spend too much time on questions youre unsure about but work through and find questions you can do firs.

Divide the time up into smaller chunks for each question and allow some extra time to go back to questions you couldnt do or finis. For exampe, in a 3-hour exam with 50 questins, allow around 3 minutes for each question This will give an extra half hour at the end to tidy up and finish off questions Alternativey, in a 3-hour exam with questions worth a total of 100 mrks, allow around 15 minutes per mark

- Read through and ensure you know how many questions there are
- Divide your time between questions with extra time at the end
- Dont spend too much time on one question
- Read each question carefully, underlining key words
- Show all working out including diagrams and formulas
- Cross out mistakes with a single line so it can still be read
- Write legibly

AND FINALLY...

Study involves knowing what you dont kno, and putting in a lot of time into concentrating on these areas This is a positive way to lean. Rather than just sayig 'Ican't dohis', say intead 'I can't do this *yet* and use your teacher, friens, textbooks and other ways of finding ut.

With the parts of the course that you do kno, make sure you can remember these easily under exam pressure by putting in lots of practice

Remember to look at new work

todą tomorow, in week, ina month.

Some people hardly ever find time to study while others give up their outside lives to devote their time to study. The ideal situation is to balance study with other aspects of your lfe, including going out with friends working and keeping up with sport and other activities that you enjo.

Good luck with your studies!

ISBN 97807043220

MATHEMATICAL VERBS

A glossary of 'doing words' commonly found in mathematics problems

analys: study in detail the parts of a situation

appl: use knowledge or a procedure in a given situation

classify, identiy: state the type name or feature of an item or situation

comment express an observation or opinion about a result

compare show how two or more things are similar or different

construct draw an accurate diagram

describe state the features of a situation

estimate make an educated guess for a number, measurement or solution to find roughly or approximately

evaluae, calculte: find the value of a numerical expression for exampl, 3×8^2 or $4x + 1$ when $x = 5$

expan: remove brackets in an algebraic expression for exampl, expanding $3(2y + 1)$ gives $6y + 3$

explai: describe why or how

factoris: opposite to **expand** to insert brackets by taking out a common factor, for exampe, factorising $6y + 3$ gives $3(2y + 1)$

give reasos: show the rules or thinking used when solving a problem See also **justify**

increase make larger

interpret find meaning in a mathematical result

justify give reasons or evidence to support your argument or conclusion See also **give reasons**

rationalis: make rational remove surds

show tha, prove: (in questions where the answer is given) use calculation procedure or reasoning to prove that an answer or result is true

simplify give a result in its most basic shortes, neatest form for exampl, simplifying a ratio or algebraic expression

sketch: draw a rough diagram that shows the general shape or ideas less accurate than **construct**

solv: find the value(s) of an unknown pronumeral in an equation or inequality

substitute replace a variable by a number and evaluate

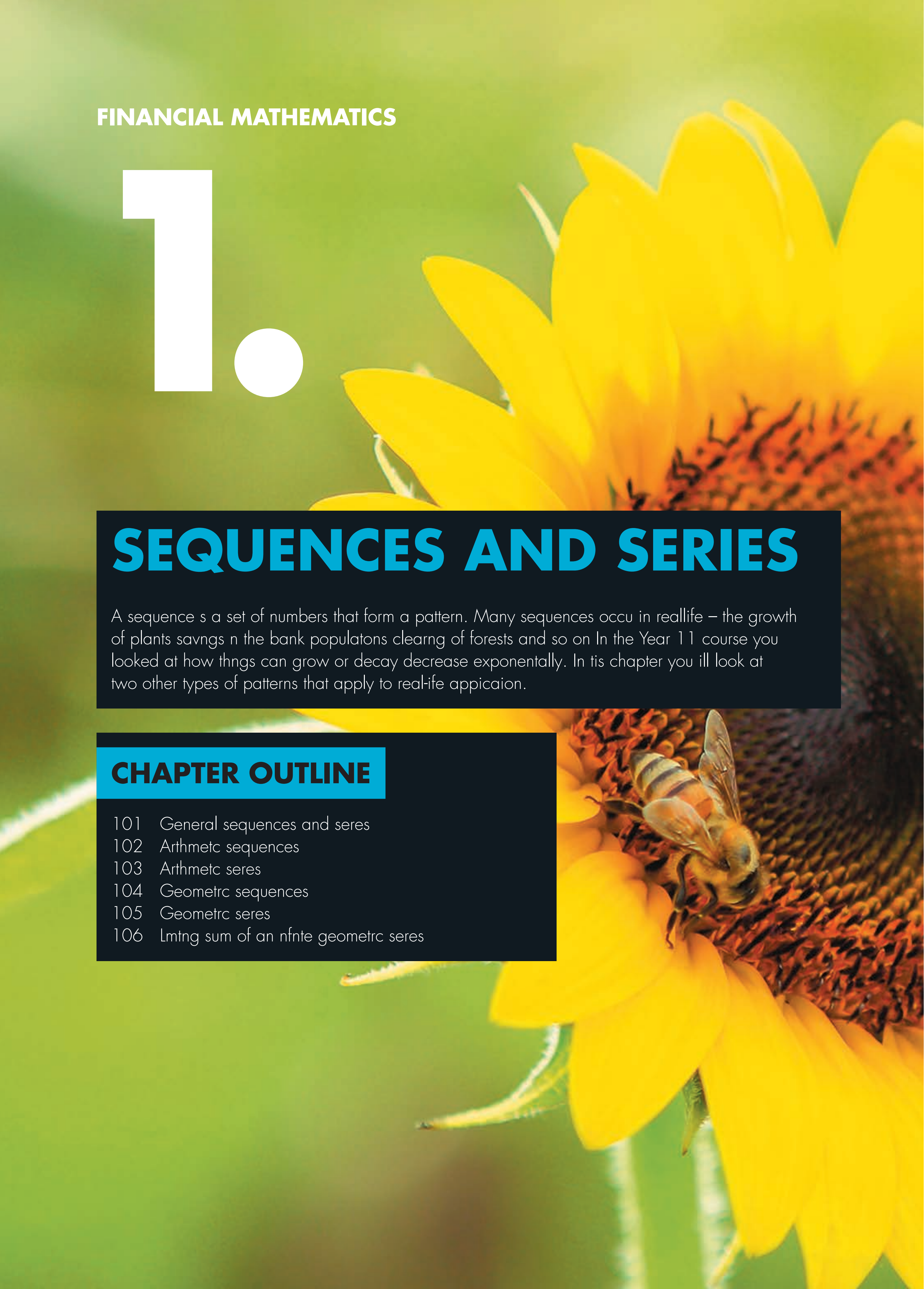
verif: check that a solution or result is correct usually by substituting back into the equation or referring back to the problem

write stat: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

ISBN 97807043220

FINANCIAL MATHEMATICS

1.



SEQUENCES AND SERIES

A sequence s a set of numbers that form a pattern. Many sequences occu in reallife – the growth of plants savngs n the bank populatons clearng of forests and so on In the Year 11 course you looked at how thngs can grow or decay decrease exponentally. In tis chapter you ill look at two other types of patterns that apply to real-ife appicaion.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- dentfy the dfference between a sequence and a seies
- dentfy the dfference between aithmeic and geometic sequences and seies
- fnd the *n*th term of aithmeic and geometic sequences
- fnd the sum to *n* terms of aithmeic and geometic seies
- understand and apply the lmtng sum formula foriniite geometic seies

TERMINOLOGY

arithmetic sequence A list of numbers where the difference between successive terms is a constant (called the common difference)
arithmetic series A sum of the terms forming an arithmetic sequence
common differenc: The constant difference between successive terms of an arithmetic sequence
common rati: The constant multiplier of successive terms in a geometric sequence
geometric sequence A list of numbers where the ratio of successive terms is a constant (called the common ratio)
geometric series A sum of the terms forming a geometric sequence
limiting sum The limi, where it exiss, of a geometric series as $n \to \infty$
recurrence relation An equation that defines a term of a sequence or series by referring to its previous term(s)
sequence A list of numbers where each term of the sequence is related to the previous term by a particular pattern
series The sum of terms of a sequence of numbers
term A value of a sequence

WS Sequence and seies

WS Claiying equence

1.01 General sequences and series

Sequences

A **sequence** is an ordered list of numbers called **terms** of the sequence which follow a pattern Some patterns are easy to see and some are more difficult to fin.

EXAMPLE 1

Find the next 3 terms in the sequence

a 14, 17, 20, **b** 5, 10, 20, 40, **c** 5, 1, –3,

Solution

a For the sequence 14 1, 0, we add 3 to each term for the next term

$14 + 3 = 17$ and $17 + 3 = 20$

Following this pattern the next 3 terms are 2, 26 and 9.

b For 5, 10, 20, 40, we multiply each term by 2 for the next term

$5 \times 2 = 10$, $10 \times 2 = 20$, $20 \times 2 = 40$

So the next 3 terms are 80 160 and 32.

c For the sequence 5 , –3, we subtract 4 (or add –4) to each term for the next term

$5 - 4 = 1$ and $1 - 4 = -3$

So the next 3 terms are –7, –11 and –15

ISBN 97807043220

Series

A **series** is a sum of terms that form a sequence

EXAMPLE 2

Find the sum of the series with 5 terms

a $8 + 15 + 22 +$

b $4 + 8 + 16 +$

Solution

a We add 7 to each term in the series $8 + 15 + 22 +$ to find the next term

So the series with 5 terms is $8 + 15 + 22 + 29 + 36$

$$\begin{aligned}\text{Sum} &= 8 + 15 + 22 + 29 + 36 \\ &= 110\end{aligned}$$

b We multiply each term in the series $4 + 8 + 16 +$ by 2 to find the next term

So the series with 5 terms is $4 + 8 + 16 + 32 + 64$

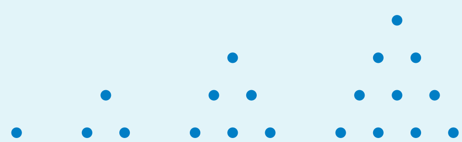
$$\begin{aligned}\text{Sum} &= 4 + 8 + 16 + 32 + 64 \\ &= 124\end{aligned}$$

DID YOU KNOW?

Polygonal numbers

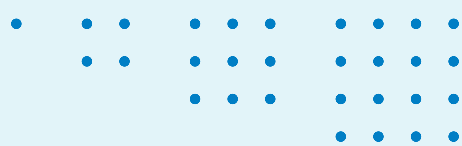
Around 500 BC the Pythagoreans explored different polygonal numbers

Triangular number: $1 + 2 + 3 + 4 +$



1 $\quad 1 + 2$ $\quad 1 + 2 + 3$ $\quad 1 + 2 + 3 + 4$

Square numbers $1 + 3 + 5 + 7 +$



1 $\quad 1 + 3$ $\quad 1 + 3 + 5$ $\quad 1 + 3 + 5 + 7$

ISBN 97807043220

Exercise 1.01 General sequences and series

1 Find the next 3 terms in each sequence

a 5, 8, 11, ...

b 8, 13, 18, ...

c 11, 22, 33,

d 100, 95, 90...

e 7, 5, 3,

f 12, 3, –6, ...

g $\frac{1}{2}$, 1, $1\frac{1}{2}$

h 1.3, 1.9, 2.5,

i 2, – 4, 8, –16, ...

j $\frac{1}{5}$ $\frac{3}{20}$ $\frac{9}{80}$

2 Find the sum of each series if it has 6 terms

a 4 + 12 + 36 +

b 1 + 2 + 4 +

c 3 + 7 + 11 +

d –6 + 12 – 24 +

e 1 + 4 + 9 + 16 +

f 1 + 8 + 27 + 64 +

3 Find the next 3 terms of the sequence $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$

4 Find the next 4 terms in the series 3 + 6 + 11 + 18 + 27 +

5 What are the next 5 terms in the sequence 1, 1, 2, 3, 5, 8,3, ... ?

6 Complete the next 3 rows in Pascals triangl:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

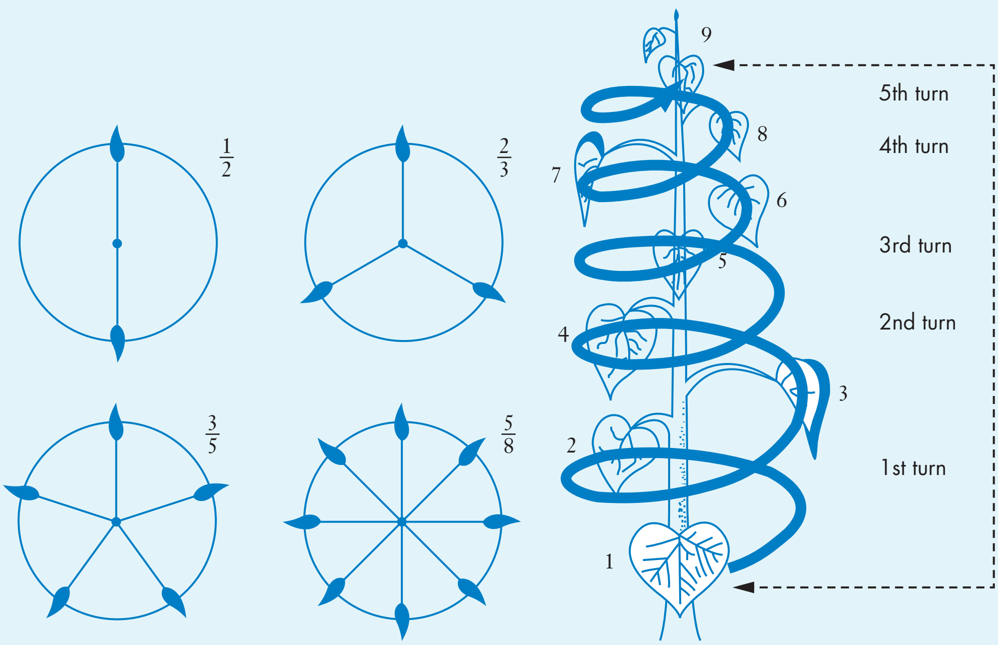
ISBN 97807043220

DID YOU KNOW?

Fibonacci numbers

The numbers 1, 1, 2, 3, 5, 8, are called **Fibonacci** numbers after **Leonardo Fibonacci** (1170–1250) These numbers occur in many natural situatios.

For example when new leaves grow on a plan's stm, they spiral around the sem.The ratio of the number of turns to the number of spaces between successive leaves gives the sequence of fractions $\frac{1}{1}\ \frac{1}{2}\ \frac{2}{3}\ \frac{3}{5}\ \frac{5}{8}\ \frac{8}{13}\ \frac{13}{21}\ \frac{21}{34}\ \frac{34}{55}$



The Fibonacci ratio is the number of turns divided by the number of spaces

Research Fibonacci numbers and find out where else they appear in naure.

ISBN 97807043220

Aihmeic equence

Aihmeic pogeion

1.02 Arithmetic sequences

In an **arithmetic sequence** each term is a constant amount more than the previous ter. The constant is called the **common difference** d

EXAMPLE 3

Find the common difference of the arithmetic sequence

a 5, 9, 13, 17,

b 85, 80, 75,

Solution

a For this sequence $9 - 5 = 4$, $13 - 9 = 4$ and $17 - 13 = 4$

So common difference $d = 4$

b For this sequence $80 - 85 = -5$ and $75 - 80 = -5$

So common difference $d = -5$

A **recurrence relation** is an equation that defines a term of a sequence by referring to its previous term In any arithmetic sequenc, a term is d more than the previous term We can write this as a recurrence relation

$$T_n = T_{n-1} + d \text{ where } T_n \text{ is the } n\text{th term of the sequence}$$

or $T_n - T_{n-1} = d$

EXAMPLE 4

a If 5 x, 31, is an arithmetic sequence find x

b **i** Evaluate k if $k + 2, 3k + 2, 6k - 1,$ is an arithmetic sequence

ii Write down the first 3 terms of the sequenc.

iii Find the common difference d

Solution

a For an arithmetic sequence

$T_2 - T \ = d$ and $T_3 - T_2 = d$

So $T_2 - T \ = T_3 - T_2$

$$\begin{aligned} x - 5 &= 31 - x \\ 2x - 5 &= 31 \\ 2x &= 36 \\ x &= 18 \end{aligned}$$

Note x is called the **arithmetic mean** because $x = \dfrac{5 + 31}{2}$

ISBN 97807043220

b **i** For an arithmetic sequence

$$T_2 - T \;\; = T_3 - T_2$$
$$(3k + 2) - (k + 2) = (6k - 1) - (3k + 2)$$
$$3k + 2 - k - 2 = 6k - 1 - 3k - 2$$
$$2k = 3k - 3$$
$$2k + 3 = 3k$$
$$3 = k$$

ii Substituting $k = 3$ into the terms of the sequence

$$\begin{aligned} T \;\; &= k + 2 \\ &= 3 + 2 \\ &= 5 \end{aligned} \qquad \begin{aligned} T_2 &= 3k + 2 \\ &= 3(3) + 2 \\ &= 11 \end{aligned} \qquad \begin{aligned} T_3 &= 6k - 1 \\ &= 6(3) - 1 \\ &= 17 \end{aligned}$$

iii The sequence is 5, 11, 17,

$$\begin{aligned} d &= 11 - 5 \text{ or } 17 - 11 \\ &= 6 \end{aligned}$$

So $\quad d = 6$

The general term of an arithmetic sequence

Given an arithmetic sequence with 1st term $T \;\; = a$ and common difference d

$$T \;\; = a \qquad \begin{aligned} T_2 &= T \;\; + d \\ &= a + d \end{aligned} \qquad \begin{aligned} T_3 &= T_2 + d \\ &= (a + d) + d \\ &= a + 2d \end{aligned} \qquad \begin{aligned} T_4 &= T_3 + d \\ &= (a + 2d) + d \\ &= a + 3d \end{aligned}$$

Notice that the multiple of d is one less than the number of the term So the multiple of d for the nth term T_n is $n - 1$.

*n*th term of an arithmetic sequence

$$T_n = a + (n - 1)d$$

ISBN 97807043220

EXAMPLE 5

a Find the 20th term of the sequence 3 1, 7,

b Find a formula for the nth term of the sequence 2 , 4,

c Find the first positive term of the sequence −50 −47 −44

Solution

a $a = 3, d = 7, n = 20$

$T_n = a + (n - 1)d$

$T_{20} = 3 + (20 - 1) \times 7$

$= 3 + 19 \times 7$

$= 136$

b $a = 2, d = 6$

$T_n = a + (n - 1)d$

$= 2 + (n - 1) \times 6$

$= 2 + 6n - 6$

$= 6n - 4$

c $a = -50 \quad d = 3$

For the first positive term

$T_n > 0$

$a + (n - 1)d > 0$

$-50 + (n - 1) \times 3 > 0$

$-50 + 3n - 3 > 0$

$3n - 53 > 0$

$3n > 53$

$n > 1766$

So $n = 18$ gives the first positive term

$T_{18} = -50 + (18 - 1) \times 3$

$= 1$

So the first positive term is 1

EXAMPLE 6

The 5th term of an arithmetic sequence is 37 and the 8th term is 55 Find the common difference and the first term of the sequence

Solution

$T_n = a + (n - 1)d$

Given $T_5 = 37$

$a + (5 - 1)d = 37$

$a + 4d = 37$ [1]

Given $T_8 = 55$

$a + (8 - 1)d = 55$

$a + 7d = 55$ [2]

Solve [1] and [2] simultaneously

$3d = 18$ [2] − [1]

$d = 6$

Substitute $d = 6$ into [1]

$a + 4(6) = 37$

$a + 24 = 37$

$a = 13$

So the common difference is 6 and the first term is 13

ISBN 97807043220

Exercise 1.02 Arithmetic sequences

1 Find the value of the pronumeral in each arithmetic sequence

a $5, 9, y$ **b** $8, 2, x$ **c** $45\ x\ 9$,
d $16\ b, 6,$ **e** $x, 14, 21,$ **f** $32\ x - 1, 50,$
g $3, 5k + 2, 21,$ **h** $x\ x + 3, 2x + 5,$ **i** $t - 5, 3t, 3t + 1,$
j $2t - 3, 3t + 1, 5t + 2,$

2 Find the 15th term of each sequence

a 4, 7, 10, **b** 8, 13, 18, **c** 10, 16, 22,
d 120, 111, 102, **e** –3, 2, 7,

3 Find the 100th term of each sequence

a –4, 2, 8, **b** 41, 32, 23, **c** 18, 22, 26,
d 125, 140, 155, **e** –1, –5, –9,

4 What is the 25th term of each sequence?

a –14 –18 –22 **b** 04, .9,1.4, **c** 13, .9,0.5,
d $1, 2\frac{1}{2}, 4,$ **e** $1\frac{2}{5}, 2, 2\frac{3}{5}$

5 Find the formula for the nth term of the sequence 3 ,7,

6 Find the formula for the nth term of each sequence

a 9, 17, 25, **b** 100, 102, 104, **c** 6, 9, 12,
d 80, 86, 92, **e** –21, –17 –13, **f** 15, 10, 5,
g $\frac{7}{8}, 1, 1\frac{1}{8}$ **h** –30 –32 –34 **i** 32, .4,5.6,
j $\frac{1}{2}, 1\frac{1}{4}, 2,$

7 Find which term of 3 , 1, is equal to 111

8 Which term of the sequence 1 ,9, is 213?

9 Which term of the sequence 15 2, 3, is 276?

10 Which term of the sequence 25 1, 1, is equal to –73?

11 Is 0 a term of the sequence 48 4, 2, ?

12 Is 270 a term of the sequence 3 1, 9, ?

13 Is 405 a term of the sequence 0 ,6, ?

14 Find the first value of n for which the terms of the sequence 100 9, 6, is less than 20

15 Find the values of n for which the terms of the sequence –86 –83 –80 are positive

16 Find the first negative term of the sequence 54 5, 6,

17 Find the first term that is greater than 100 in the sequence 3 , 1,

18 The first term of an arithmetic sequence is –7 and the common difference is 8
Find the 100th term

19 The first term of an arithmetic sequence is 15 and the 3rd term is 31

a Find the common difference

b Find the 10th term of the sequence

20 The first term of an arithmetic sequence is 3 and the 5th term is 39
Find the common difference

21 The 2nd term of an arithmetic sequence is 19 and the 7th term is 54
Find the first term and common difference

22 Find the 20th term in an arithmetic sequence with 4th term 29 and 10th term 83

23 The common difference of an arithmetic sequence is 6 and the 5th term is 29
Find the first term of the sequence

24 If the 3rd term of an arithmetic sequence is 45 and the 9th term is 75
find the 50th term of the sequence

25 The 7th term of an arithmetic sequence is 17 and the 10th term is 53
Find the 100th term of the sequence

26 **a** Show that $\log_5 x \ \log_5 x^2 \ \log_5 x^3$ is an arithmetic sequence

b Find the 80th term

c If $x = 4$ evaluate the 10th term correct to 1 decimal plac.

27 **a** Show that $\sqrt{3}\ \sqrt{12}, \sqrt{27}$ is an arithmetic sequence

b Find the 50th term in simplest form

28 Find the 25th term of $\log_2 4 \ \log_2 8, \log_2 16,$

29 Find the 40th term of $5b$, $8b$, $11b$

30 Which term is $213y$ of the sequence $28y$, $33y$, $38y$?

ISBN 97807043220

1.03 Arithmetic series

The sum of an **arithmetic series** with n terms is given by the formula

Sum of an arithmetic series with *n* terms

$S_n = \frac{n}{2}(a + l)$ where a = 1st term and l = last (nth) term

Proof

Let the last or nth term be l

$S_n = a + (a + d) + (a + 2d) + \quad + l$ [1]

Writing this around the other wa:

$S_n = l + (l - d) + (l - 2d) + \quad + a$ [2]

[1] + [2]

$$2S_n = (a + l) + (a + l) + (a + l) + \quad + (a + l) \; n \text{ times}$$

$$= n(a + l)$$

$$S_n = \frac{n}{2}(a + l)$$

We can find a more general formula if we substitute $T_n = a + (n - 1)d$ for l

Sum of an arithmetic series with *n* terms

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Proof

$$S_n = \frac{n}{2}(a + l)$$

$$= \frac{n}{2}[a + a + (n - 1)d]$$

$$= \frac{n}{2}[2a + (n - 1)d]$$

We can also use these formulas to find the sum of the first n terms of an arithmetic sequence (also called the **nth partial sum**)

WS Aihmeic eie

WS Aihmeic eie

Aihmeic eie

WS Aihmeic eie aignmen

WS Aihmeic eie exenion

ISBN 97807043220

EXAMPLE 7

a Evaluate $9 + 14 + 19 + \quad + 224$

b For what value of n is the sum of the series $2 + 11 + 20 + \quad$ equal to 618?

c The 6th term of an arithmetic sequence is 23 and the sum of the first 10 terms is 210 Find the sum of the first 20 terms of the sequence

Solution

a $a = 9, d = 5, T_n = 224$

$$\begin{aligned}
T_n &= a + (n-1)d \\
224 &= 9 + (n-1) \times 5 \\
&= 9 + 5n - 5 \\
&= 5n + 4 \\
220 &= 5n \\
44 &= n
\end{aligned}$$

$$\begin{aligned}
S_n &= \frac{n}{2}(a + l) \\
&= \frac{44}{2}(9 + 224) \\
&= 5126
\end{aligned}$$

b $a = 2, d = 9, S_n = 618$

$$\begin{aligned}
S_n &= \frac{n}{2}[2a + (n-1)d] \\
618 &= \frac{n}{2}[2 \times 2 + (n-1) \times 9] \\
1236 &= n(4 + 9n - 9) \\
&= n(9n - 5) \\
&= 9n^2 - 5n
\end{aligned}$$

$$\begin{aligned}
0 &= 9n^2 - 5n - 1236 \\
&= (n - 12)(9n + 103)
\end{aligned}$$

(or use the quadratic formula)

$n - 12 = 0, 9n + 103 = 0$

$n = 12$

($9n + 103 = 0$ gives a negative value of n)

c

$$\begin{aligned}
T_n &= a + (n-1)d \\
T_6 &= a + (6-1)d = 23 \\
a + 5d &= 23 \qquad [1] \\
S_n &= \frac{n}{2}[2a + (n-1)d] \\
S_{10} &= \frac{10}{2}[2a + (10-1)d] = 210 \\
5(2a + 9d) &= 210 \\
2a + 9d &= 42 \qquad [2]
\end{aligned}$$

$[1] \times 2: \quad 2a + 10d = 46 \qquad [3]$

$[3] - [2] \quad d = 4$

ISBN 97807043220

Substitute $d = 4$ in [1]

$$a + 5(4) = 23$$
$$a + 20 = 23$$
$$a = 3$$

Substitute $a = 3, d = 2, n = 20$ into the formula for S_n

$$S_{20} = \frac{20}{2}[2(3) + (20 - 1)4]$$
$$= 10(6 + 19 \times 4)$$
$$= 820$$

Exercise 1.03 Arithmetic series

1 Find the sum of 15 terms of each series

a $4 + 7 + 10 + \ldots$ **b** $2 + 7 + 12 + \ldots$ **c** $60 + 56 + 52 + \ldots$

2 Find the sum of 30 terms of each series

a $1 + 7 + 13 + \ldots$ **b** $15 + 24 + 33 + \ldots$ **c** $95 + 89 + 83 + \ldots$

3 Find the sum of 25 terms of each series

a $-2 + 5 + 12 + \ldots$ **b** $5 - 4 - 13 - \ldots$

4 Find the sum of 50 terms of each series

a $50 + 44 + 38 + \ldots$ **b** $11 + 14 + 17 + \ldots$

5 Evaluate each arithmetic series

a $15 + 20 + 25 + \ldots + 535$ **b** $9 + 17 + 25 + \ldots + 225$

c $5 + 2 - 1 - \ldots - 91$ **d** $81 + 92 + 103 + \ldots + 378$

e $229 + 225 + 221 + \ldots + 25$ **f** $-2 + 6 + 14 + \ldots + 94$

g $0 - 9 - 18 - \ldots - 216$ **h** $79 + 81 + 83 + \ldots + 229$

i $14 + 11 + 8 + \ldots - 43$ **j** $1\frac{1}{2} + 1\frac{3}{4} + 2 + \ldots + 25\frac{1}{4}$

6 How many terms of the series $45 + 47 + 49 + \ldots$ are needed to give a sum of 1365?

7 For what value of n is the sum of the arithmetic series $5 + 9 + 13 + \ldots$ equal to 152?

8 How many terms of the series $80 + 73 + 66 + \ldots$ are needed to give a sum of 495?

9 How many terms of the series $20 + 18 + 16 + \ldots$ are needed to give a sum of 104?

10 The sum of the first 5 terms of an arithmetic sequence is 110 and the sum of the first 10 terms is 320 Find the first term and the common differenc.

11 The sum of the first 5 terms of an arithmetic sequence is 35 and the sum of the next 5 terms is 160 Find the first term and the common differenc.

ISBN 97807043220

12 Find S_{25} given an arithmetic series with 8th term 16 and 13th term 8.

13 The sum of 12 terms of an arithmetic series is 186 and the 20th term is 83 Find the sum of 40 terms of the series

14 The sum of the first 4 terms of an arithmetic series is 42 and the sum of the 3rd and 7th term is 46 Find the sum of the first 20 term.

15 **a** Show that $x + 1, 2x + 4, 3x + 7,$ are the first 3 terms in an arithmetic sequence

b Find the sum of the first 50 terms of the sequence

16 The 20th term of an arithmetic series is 131 and the sum of the 6th to 10th terms inclusive is 235 Find the sum of the first 20 term.

17 The sum of 50 terms of an arithmetic series is 249 and the sum of 49 terms of the series is 233 Find the 50th term of the serie.

18 Prove that $T_n = S_n - S_{n-1}$ for any arithmetic sequence

19 **a** Find the sum of all integers from 1 to 100 that are multiples of 6

b Find the sum of all integers from 1 to 100 that are not multiples of 6

WS Geomeic equence

WS Geomeic pogeion

1.04 Geometric sequences

In a **geometric sequence** each term is formed by multiplying the previous term by a constant The constant is called the **common ratio** r

EXAMPLE 8

Find the common ratio of the geometric sequence

a 3, 6, 12,

b −2, 10, −50

c $\frac{1}{2}\ \frac{1}{5}\ \frac{2}{25}$

Solution

a For this sequence $6 \div 3 = 2, 12 \div 6 = 2$

So common ratio $r = 2$.

b For this sequence $10 \div -2 = -5, -50 \div 10 = -5$

So common ratio $r = -5$

c $\frac{1}{5} \div \frac{1}{2} = \frac{2}{5}\quad \frac{2}{25} \div \frac{1}{5} = \frac{2}{5}$

So common ratio $r = \frac{2}{5}$

ISBN 97807043220

In any geometric sequence a term is r times more than the previous term We can write this as a recurrence relation

$$T_n = rT_{n-1}$$

or $$\frac{T_n}{T_{n-}} = r$$

EXAMPLE 9

a **i** Find x if $5, x\ 4\ ,$ is a geometric sequence

ii Find the sequence

b Is $\frac{1}{4}\ \frac{1}{6}\ \frac{1}{18}$ a geometric sequence?

Solution

a **i** For a geometric sequence

$$\frac{T_2}{T} = r \text{ and } \frac{T_3}{T_2} = r$$

So $$\frac{T_2}{T} = \frac{T_3}{T_2}$$

$$\frac{x}{5} = \frac{45}{x}$$

$$x^2 = 225$$

$$x = \pm\sqrt{225}$$

$$= \pm 15$$

Note x is called the **geometric mean** because $x = \sqrt{5 \times 45}$

ii If $x = 15$ the sequence is $5, 15, 45,$ $(r = 3)$

If $x = -15$ the sequence is $5\ -15, 45,$ $(r = -3)$

b $$\frac{T_2}{T} = \frac{1}{6} \div \frac{1}{4} = \frac{2}{3}$$ $$\frac{T_3}{T_2} = \frac{1}{18} \div \frac{1}{6} = \frac{1}{3}$$

$\frac{T_2}{T} \neq \frac{T_3}{T_2}$ so the series is not geometric

ISBN 97807043220

General term of a geometric sequence

Given a geometric sequence with 1st term $T \ = a$ and common ratio r

$T \ = a$	$T_3 = T_2 \times r$	$T_4 = T_3 \times r$
$T_2 = T \ \times r$	$= (ar) \times r$	$= (ar^2) \times r$
$= ar$	$= ar^2$	$= ar^3$

Notice that the power of r is one less than the number of the term So the power of r for T_n is $n - 1$.

*n*th term of a geometric sequence

$$T_n = ar^{n-1}$$

EXAMPLE 10

a **i** Find the 10th term of the sequence 3 , 2,

ii Find the formula for the nth term of the sequence

b Find the 10th term of the sequence $-5, 10, -20$

c Which term of the sequence 4 1, 6,   is equal to 78 732?

d The 3rd term of a geometric sequence is 18 and the 7th term is 1458
Find the first term and the common ratio

Geomeic
equence

Solution

a **i** This is a geometric sequence with $a = 3$, $r = 2$ and $n = 10$.

$$T_n = ar^{n-1}$$
$$T_{10} = 3(2)^{10-1}$$
$$= 3 \times 2^9$$
$$= 1536$$

ii $T_n = ar^{n-1}$

$$= 3(2)^{n-1}$$

b $a = -5$, $r = -2$, $n = 10$

$$T_n = ar^{n-1}$$
$$T_{10} = -5(-2)^{10-1}$$
$$= -5 \times (-2)^9$$
$$= 2560$$

c This is a geometric sequence with $a = 4$, $r = 3$ and $T_n = 78\,732$

$$T_n = ar^{n-1}$$
$$78\,732 = 4(3)^{n-1}$$
$$19\,683 = 3^{n-1}$$
$$\log 19\,683 = \log 3^{n-1}$$
$$= (n-1)\log 3$$
$$\frac{\log 19\,683}{\log 3} = n - 1$$
$$\frac{\log 19\,683}{\log 3} + 1 = n$$
$$10 = n$$

So the 10th term is 78 732

ISBN 97807043220

d Given $T_3 = 18$

$ar^{3-1} = 18$

$ar^2 = 18$ [1]

Given $T_7 = 1458$

$ar^{7-1} = 1458$

$ar^6 = 1458$ [2]

[2] ÷ [1]:

$\frac{ar^6}{ar^2} = \frac{1458}{18}$

$r^4 = 81$

$r = \pm\sqrt[4]{81}$

$= \pm 3$

Substitute $r = 3$ into [1]

$a(3)^2 = 18$

$9a = 18$

$a = 2$

Substitute $r = -3$ into [1]

$a(-3)^2 = 18$

$9a = 18$

$a = 2$

The first term is 2 and the common ratio is ±3

Here is an example of a geometric sequence involving fractions

EXAMPLE 11

a Find the 8th term of $\frac{2}{3}$ $\frac{4}{15}$ $\frac{8}{75}$ in index form

b Find the first value of n for which the terms of the sequence $\frac{1}{5}$, 1, 5, exceed 3000

Solution

a $\frac{T_2}{T} = \frac{4}{15} \div \frac{2}{3}$

$= \frac{2}{5}$

$\frac{T_3}{T_2} = \frac{8}{75} \div \frac{4}{15}$

$= \frac{2}{5}$

So $r = \frac{2}{5}$

$T_n = ar^{n-1}$

$= \frac{2}{3}\left(\frac{2}{5}\right)^{8-1}$

$= \frac{2}{3}\left(\frac{2}{5}\right)^{7}$

$= \frac{2^8}{3(5^7)}$

b $a = \frac{1}{5}$ $r = 5$

$T_n > 3000$

$ar^{n-1} > 3000$

$\frac{1}{5}(5)^{n-1} > 3000$

$5^{n-1} > 15\,000$

$\log 5^{n-1} > \log 15\,000$

$(n-1)\log 5 > \log 15\,000$

$n - 1 > \frac{\log 15\,000}{\log 5}$

$n > \frac{\log 15\,000}{\log 5} + 1$

> 6974

So $n = 7$

The 7th term is the first term to exceed 3000

ISBN 97807043220

Exercise 1.04 Geometric sequences

1 Is each sequence geometric? If so find the common rati.

a 5, 20, 60,   **b** $-4, 3, -2\frac{1}{4}$ **c** $\frac{3}{4}\ \frac{3}{14}\ \frac{3}{49}$

d $7, 5\frac{5}{6}, 3\frac{1}{3}$ **e** $-14\ 4,\ \ -168$ **f** $1\frac{1}{3}, \frac{8}{9}\ \frac{8}{27}$

g 5.7, 1.71, 0.513,   **h** $2\frac{1}{4}, -1\frac{7}{20}, \frac{81}{100}$ **i** $63, 9, 1\frac{7}{8}$

j $-1\frac{7}{8}, 15, -120$

2 Find the pronumeral in each geometric sequence

a 4 2, x **b** $-3, 12, y$ **c** $2, a, 72,$

d $y, 2, 6,$ **e** $x, 8, 32,$ **f** $5, p, 20,$

g $7, y, 63,$ **h** $-3, m\ \ -12$ **i** $3, x-4, 15,$

j $3, k-1, 21,$ **k** $\frac{1}{4}\ t\ \frac{1}{9}$ $\frac{1}{3}\ t\ \frac{4}{3}$

3 Find the formula for the nth term of each sequence

a 1, 5, 25,   **b** 1, .02,1.0404,   **c** 1, 9, 81,

d 2, 10, 50,   **e** 6, 18, 54,   **f** 8, 16, 32,

g $\frac{1}{4}, 1, 4,$ **h** 1000 –100, 10,   **i** –3, 9, –27

j $\frac{1}{3}\ \frac{2}{15}\ \frac{4}{75}$

4 Find the 6th term of each sequence

a 8, 24, 72,   **b** 9, 36, 144,   **c** 8, –32, 128,

d –1, 5, –25   **e** $\frac{2}{3}\ \frac{4}{9}\ \frac{8}{27}$

5 What is the 9th term of each sequence?

a 1, 2, 4,   **b** 4, 12, 36,   **c** 1, .04,1.0816,

d –3, 6, –12   **e** $\frac{3}{4}\ -\frac{3}{8}\ \frac{3}{16}$

6 Find the 8th term of each sequence

a 3, 15, 75,   **b** 21, .2,8.4,   **c** 5, –20 8,

d $-\frac{1}{2}\ \frac{3}{10}\ -\frac{9}{50}$ **e** $1\frac{47}{81}, 2\frac{10}{27}, 3\frac{5}{9},$

7 Find the 20th term of each sequence leaving the answer in index for.

a 3, 6, 12,   **b** 1, 7, 49,   **c** $104\ .04\ ^{2}, .04\ ^{3}$

d $\frac{1}{4}\ \frac{1}{8}\ \frac{1}{16}$ **e** $\frac{3}{4}\ \frac{9}{16}\ \frac{27}{64}$

8 Find the 50th term of 1, 11, 121,   in index form

9 Which term of the sequence 4 2, 10,   is equal to 12 500?

ISBN 97807043220

10 Which term of 6 3, 26, is equal to 7776?

11 Is 1200 a term of the sequence 2 1, 18, ?

12 Which term of 3 2, 17, is equal to 352 947?

13 Which term of the sequence 8 −4, 2, is $\frac{1}{128}$?

14 Which term of 54 1,6, is $\frac{2}{243}$?

15 Find the value of n if the nth term of the sequence $-2, 1\frac{1}{2}\ -1\frac{1}{8}$ is $-\frac{81}{128}$

16 The first term of a geometric sequence is 7 and the 6th term is 1701
Find the common ratio

17 The 4th term of a geometric sequence is –648 and the 5th term is 3888

a Find the common ratio

b Find the 2nd term

18 The 3rd term of a geometric sequence is $\frac{2}{5}$ and the 5th term is $1\frac{3}{5}$
Find the first term and common ratio

19 Find the value of n for the first term of the sequence 5000 100, 20, that is less than 1

20 Find the first term of the sequence $\frac{2}{7}\ \frac{6}{7}, 2\frac{4}{7}$ that is greater than 100

1.05 Geometric series

WS Geomeic eie

WS Geomeic eie

WS Geomeic eie aignmen

WS Geomeic eie poblem

The sum of a **geometric series** with n terms is given by the formulas

Sum of a geometric series with *n* terms

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

This formula can also be written as

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

to be used if r is a fraction that i, $-1 < r < 1$ also written as $|r| < 1$.

Proof

The sum of a geometric series can be written

$$S_n = a + ar + ar^2 + \quad + ar^{n-1} \qquad [1]$$

Multiplying both sides by r

$$rS_n = r(a + ar + ar^2 + \quad + ar^{n-1})$$
$$= ar + ar^2 + ar^3 + \quad + ar^n \qquad [2]$$

[2] – [1]

$rS_n - S_n = ar^n - a$

$S_n(r - 1) = a(r^n - 1)$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

[1] – [2] gives the formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

We can also use these formulas to find the sum of the first n terms of a geometric sequence (also called the ***n*th partial sum**)

EXAMPLE 12

a Find the sum of the first 10 terms of the series 3 + 12 + 48 +

b Evaluate $60 + 20 + 6\frac{2}{3} + \quad + \frac{20}{81}$

c The sum of n terms of $1 + 4 + 16 + \quad$ is 21 845 Find the value of n

Solution

a This is a geometric series with $a = 3, r = 4, n = 10.$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{3(4^{10} - 1)}{4 - 1}$$

$$= \frac{3(4^{10} - 1)}{3}$$

$$= 4^{10} - 1$$

$$= 1\ 048\ 575$$

b $a = 60, \ r = \frac{1}{3} \ \ T_n = \frac{20}{81}$

$$T_n = ar^{n-1}$$

$$ar^{n-} = \frac{20}{81}$$

$$60\left(\frac{1}{3}\right)^{n-} = \frac{20}{81}$$

$$\left(\frac{1}{3}\right)^{n-} = \frac{1}{243}$$

$$\frac{1}{3^{n-}} = \frac{1}{243}$$

ISBN 97807043220

So $3^{n-1} = 243$

$$= 3^5$$

$$n - 1 = 5$$

$$n = 6$$

Since $|r| < 1$ we use the second formula

$$S_n = \frac{a(1-r^n)}{r-1}$$

$$S_6 = \frac{60\left(1-\left(\frac{1}{3}\right)^6\right)}{1-\frac{1}{3}}$$

$$= \frac{60\left(1-\frac{1}{729}\right)}{\frac{2}{3}}$$

$$= 60 \times \frac{728}{729} \times \frac{3}{2}$$

$$= 89\frac{71}{81}$$

c $a = 1, r = 4, S_n = 21\ 845$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$21\ 845 = \frac{1(4^n - 1)}{4-1}$$

$$= \frac{4^n - 1}{3}$$

$$65\ 535 = 4^n - 1$$

$$65\ 536 = 4^n$$

$$\log 65\ 536 = \log 4^n$$

$$= n \log 4$$

$$\frac{\log 65\ 536}{\log 4} = n$$

$$8 = n$$

So 8 terms give a sum of 21 845

Exercise 1.05 Geometric series

1 Find the sum of 10 terms of each geometric series

a $6 + 24 + 96 +$

b $3 + 15 + 75 +$

2 Find the sum of 8 terms of each series

a $-1 + 7 - 49 +$

b $8 + 24 + 72 +$

3 Find the sum of 15 terms of each series

a $4 + 8 + 16 +$

b $\frac{3}{4} - \frac{3}{8} + \frac{3}{16} -$ (to 1 decimal place)

4 Evaluate

a $2 + 10 + 50 + \quad + 6250$

b $18 + 9 + 4\frac{1}{2} + \quad + \frac{9}{64}$

c $3 + 21 + 147 + \quad + 7203$

d $\frac{3}{4} + 2\frac{1}{4} + 6\frac{3}{4} + \ldots + 182\frac{1}{4}$

e $-3 + 6 - 12 + \quad + 384$

ISBN 97807043220

5 For the series $7 + 14 + 28 + \quad$ find

a the 9th term

b the sum of the first 9 terms

6 Find the sum of 30 terms of the series $109 \; + 109^{2} + 109^{3} + \quad$ correct to 2 decimal places

7 Find the sum of 25 terms of the series $1 + 1.12 + 1.12^{2} + \quad$ correct to 2 decimal places

8 Find the value of n if the sum of n terms of the series $11 + 33 + 99 + \quad$ is equal to 108 251

9 How many terms of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ \quad$ give a sum of $\frac{1023}{1024}$?

10 The common ratio of a geometric series is 4 and the sum of the first 5 terms is 3069 Find the first term

11 Find the number of terms needed to be added for the sum to exceed 1 000 000 in the series $4 + 16 + 64 +$

12 **a** Find the sum of 10 terms of the series $2 + 4 + 8 +$

b Find the sum of 10 terms of the series $1 + 3 + 5 +$

c Find the sum of the first 10 terms of the series $3 + 7 + 13 +$

PUZZLES

1 A poor girl saved a rich king from drowning one day. The king offered the girl a reward of a sum of money in 30 daily payments He gave her a choice of payment:

Choice 1 \$1 the first day, \$2 the second dy, \$3 the third day and s on.

Choice 2 1 cent the first day, 2 cents the second dy, 4 cents the third day and s on, the payment doubling each day.

How much money would the girl receive for each choice? Which plan would give the girl more money?

2 Can you solve Fibonaccis problem ?

A man entered an orchard through 7 guarded gates and gathered a certain number of apples As he left the orchard he gave the guard at the first gate half the apples he had and 1 apple more He repeated this process for each of the remaining 6 guards and eventually left the orchard with 1 apple How many apples did he gather ? (He did not give away any half-apples)

ISBN 97807043220

1.06 Limiting sum of an infinite geometric series

In some geometric sequences the sum becomes very large as n increases for exampl, the series $2 + 4 + 8 + 16 + 32 +$ We say these series **diverge** (their sum is infinite)

In other geometric sequences howeve, such as $8 + 4 + 2 + 1 +$ the sum does not increase greatly after a few terms but approaches some constant valu. We say these series **converge** (they have a **limiting sum** that is a specific value sometimes called the **sum to infinity**)

WS

Limiing sum o an ininie geomeic eie

Geomeic eie owod

EXAMPLE 13

a Find the sum of 15 terms of $2 + 6 + 18 +$

b By evaluating the sum of 10 terms and 20 terms correct to 4 decimal places for the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ estimate its limiting su.

Solution

a $a = 2, r = 3, n = 15$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{15} = \frac{2(3^{15} - 1)}{3 - 1}$$

$$= \frac{2(3^{15} - 1)}{2}$$

$$= 3^{15} - 1$$

$$= 14\,348\,906$$

b $a = 2, r = \frac{1}{2}$: $S_n = \frac{a(1 - r^n)}{1 - r}$

Sum to 10 terms

$$S_{10} = \frac{2\left[1 - \left(\frac{1}{2}\right)^{10}\right]}{1 - \frac{1}{2}}$$

$$= \frac{2\left[1 - \frac{1}{2^{10}}\right]}{\frac{1}{2}}$$

$$= 39961$$

Sum to 20 terms

$$S_{20} = \frac{2\left[1 - \left(\frac{1}{2}\right)^{20}\right]}{1 - \frac{1}{2}}$$

$$= \frac{2\left[1 - \frac{1}{2^{20}}\right]}{\frac{1}{2}}$$

$$= 40000$$

The limiting sum is 4

ISBN 97807043220

Can you see why the series 2 + 6 + 18 + ... does not have a limiting sum and the series $2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ has a limiting sum?

Its because of the common rati. Only geometric series with common ratios that are fractions $|r| < 1$ will have a limiting su.

For $S_n=\frac{a(1-r^n)}{1-r}$

As $n \to \infty$ $r^n \to 0$ when $-1 < r < 1$

We write $\lim_{n\to\infty} r^n = 0$

Limiting sum of a geometric series

$$S=\frac{a}{1-r} \text{ when } |r|<1.$$

Proof

$$S_n=\frac{a(1-r^n)}{1-r}$$

For $|r|<1$, $\lim_{n\to\infty} r^n = 0$

$$S_\infty=\frac{a(1-0)}{1-r}$$
$$=\frac{a}{1-r}$$

EXAMPLE 14

a Find the limiting sum of the series $2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

b Find the sum to infinity of the series $6+2+\frac{2}{3}+\ldots$

c Does the series $\frac{3}{4}+\frac{15}{16}+1\frac{11}{64}+\ldots$ have a limiting sum?

Solution

a $a=2, r=\frac{1}{2}$

Since $|r|<1$ the series has a limiting su.

$$S=\frac{a}{1-r}$$
$$=\frac{2}{1-\frac{1}{2}}$$
$$=\frac{2}{\frac{1}{2}}$$
$$=2\times\frac{2}{1}$$
$$=4$$

So the limiting sum is 4

ISBN 97807043220

b $a = 6$

$2 \div 6 = \frac{1}{3}$ and $\frac{2}{3} \div 2 = \frac{1}{3}$ so $r = \frac{1}{3}$

Since $|r| < 1$ the series has a limiting sum

$$S = \frac{a}{1-r}$$
$$= \frac{6}{1-\frac{1}{3}}$$
$$= \frac{6}{\frac{2}{3}}$$
$$= 6 \times \frac{3}{2}$$
$$= 9$$

So the limiting sum is 9

c For $\frac{3}{4} + \frac{15}{16} + 1\frac{11}{64} +$

$$r = \frac{15}{16} \div \frac{3}{4} \text{ or } 1\frac{11}{64} \div \frac{15}{16}$$
$$= 1\frac{1}{4}$$

Since $|r| > 1$ this series does not have a limiting sum

DID YOU KNOW?

A series involving π and e

Here is an interesting series involving π

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

Gottfried Wilhelm Leibniz (1646–1716) discovered this result It is interesting that while π is an irrational number, it can be written as the sum of rational numbes.

Here is another interesting series involving e

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

Remember 2 $= 2 \times 1$, $3! = 3 \times 2 \times 1$, $4! = 4 \times 3 \times 2 \times 1$ and so on

Research these and other serie.

ISBN 97807043220

Exercise 1.06 Limiting sum of an infinite geometric series

1 Which series has a limiting sum? Find the limiting sum where it exists

a $9 + 3 + 1 +$ **b** $\frac{1}{4} + \frac{1}{2} + 1 +$ **c** $16 - 4 + 1 -$

d $\frac{2}{3} + \frac{7}{9} + \frac{49}{54} +$ **e** $1 + \frac{2}{3} + \frac{4}{9} +$ **f** $\frac{5}{8} + \frac{1}{8} + \frac{1}{40} +$

g $-6 + 36 - 216 +$ **h** $-2\frac{1}{4} + 1\frac{7}{8} - 1\frac{27}{48} +$ **i** $\frac{1}{9} + \frac{1}{6} + \frac{1}{4} +$

j $2 - \frac{4}{5} + \frac{8}{25} -$

2 Find the limiting sum of each series

a $40 + 20 + 10 +$ **b** $320 + 80 + 20 +$ **c** $100 - 50 + 25 -$

d $6 + 3 + 1\frac{1}{2} +$ **e** $\frac{2}{5} + \frac{6}{35} + \frac{18}{245} +$ **f** $72 - 24 + 8 -$

g $-12 + 2 - \frac{1}{3} +$ **h** $\frac{3}{4} - \frac{1}{2} + \frac{1}{3} -$ **i** $12 + 9 + 6\frac{3}{4} +$

j $-\frac{2}{3} + \frac{5}{12} - \frac{25}{96} +$

3 Find the difference between the limiting sum and the sum of 6 terms of each series correct to 2 significant figures

a $56 - 28 + 14 -$ **b** $72 + 24 + 8 +$ **c** $1 + \frac{1}{5} + \frac{1}{25} +$

d $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ **e** $1\frac{1}{4} + \frac{15}{16} + \frac{45}{64} +$

4 A geometric series has limiting sum 6 and common ratio $\frac{1}{3}$ Evaluate the first term of the series

5 A geometric series has a limiting sum of 5 and first term 3 Find the common rati.

6 The limiting sum of a geometric series is $9\frac{1}{3}$ and the common ratio is $\frac{2}{5}$ Find the first term of the series

7 A geometric series has limiting sum 40 and its first term is 5 Find the common ratio of the series

8 A geometric series has limiting sum $-6\frac{2}{5}$ and first term -8 Find its common rati.

9 The limiting sum of a geometric series is $-\frac{3}{10}$ and its first term is $-\frac{1}{2}$ Find the common ratio of the series

10 The second term of a geometric series is 2 and its limiting sum is 9 Find the values of first term a and common ratio r

11 A geometric series has 3rd term 12 and 4th term -3 Find a r and the limiting sum

ISBN 97807043220

12 A geometric series has 2nd term $\frac{2}{3}$ and 4th term $\frac{8}{27}$ Find a r and its limiting sum

13 The 3rd term of a geometric series is 54 and the 6th term is $11\frac{83}{125}$
Evaluate a r and the limiting sum

14 The 2nd term of a geometric series is $\frac{4}{15}$ and the 5th term is $\frac{32}{405}$
Find the values of a and r and its limiting sum

15 The limiting sum of a geometric series is 5 and the 2nd term is $1\frac{1}{5}$
Find the first term and the common ratio

16 The series $x + \frac{x}{4} + \frac{x}{16} +$ has a limiting sum of $\frac{7}{8}$ Evaluate x

17 **a** For what values of k does the limiting sum exist for the series $k + k^2 + k^3 +$?

b Find the limiting sum of the series when $k = -\frac{2}{3}$

c Evaluate k if the limiting sum of the series is 3

18 Show that in any geometric series the difference between the limiting sum and the sum of n terms is $\frac{ar^n}{1-r}$

ISBN 97807043220

1. TEST YOURSELF

WS

Fomula shet: Meauemen Sequence and eie

Qz

Pacice quiz

For Questions **1** to **3** select the correct answer **A B C** or **D**

1 The sum of n terms of a geometric sequence is

A $S_n = \frac{n}{2}[2a + (n-1)d]$ **B** $S_n = \frac{a}{1-r}$

C $S_n = \frac{a(1-r^n)}{1-r}$ **D** $S_n = \frac{n}{2}(a+l)$

2 The limiting sum of an infinite geometric series exists when

A $r > 1$ **B** $|r| > 1$ **C** $|r| < 1$ **D** $r < 1$

3 The nth term of the statement that can be proved 12 ,6, is

A $9 + 3n$ **B** $15 - 3n$ **C** $9 - 3n$ **D** $12n - 15$

4 Find a formula for the nth term of each sequence

a 9, 13, 17, **b** 7, 0, –7, **c** 2, 6, 18,

d 200, 50, $12\frac{1}{2}$ **e** –2, 4, –8,

5 For the series 156 + 145 + 134 +

a Find the 15th term

b Find the sum of 15 terms

c Find the sum of 14 terms

d Write a relationship between T_{15} S_{15} and S_{14}

e Find the value of n for the first negative term of the series

6 Find whether each sequence is

i arithmetic **ii** geometric **iii** neither.

a 97, 93, 89, **b** $\frac{2}{3}$ $\frac{1}{2}$ $\frac{3}{8}$

c $\sqrt{5}$ $\sqrt{20}, \sqrt{45}$ **d** –16, –04 .,

e 34, .5, 1.6, **f** 48, 24, 12,

g $-\frac{1}{5}, 1, -5,$ **h** 105, 100, 95,

i $1\frac{1}{2}, 1\frac{1}{4}, 1,$ **j** $\log x$ $\log x^2$ $\log x^3$

ISBN 97807043220

7 The nth term of the sequence 8 1, 8, is 543 Evaluate n

8 The 11th term of an arithmetic sequence is 97 and the 6th term is 32
Find the first term and common difference

9 A sequence has nth term given by $T_n = n^3 - 5$ Fin:

a the 4th term **b** the sum of 4 terms **c** which term is 5827

10 A sequence has terms 5 x 4 , Evaluate x if the sequence is

a arithmetic **b** geometric

11 If x, $2x + 3$ and $5x$ are the first 3 terms of an arithmetic series calculate the value of x

12 Find the 20th term of

a 3, 10, 17, **b** 101, 98, 95, **c** 03, .6,0.9,

13 Find the limiting sum of the series 81 + 27 + 9 +

14 For each series find the formula for the sum of n terms

a 5 + 9 + 13 + **b** $1 + 107 + 107^2 +$

15 a For what values of x does the geometric series $1 + x + x^2 +$ have a limiting sum?

b Find the limiting sum when $x = \frac{3}{5}$

c Evaluate x when the limiting sum is $1\frac{1}{2}$

16 The first term of an arithmetic series is 4 and the sum of 10 terms is 265
Find the common difference

17 If $x + 2$, $7x - 2$ and $15x + 6$ are consecutive terms in a geometric sequence evaluate x

18 Evaluate 8 + 14 + 20 + + 122.

19 a Calculate the sum of all the multiples of 7 from 1 to 100

b Calculate the sum of all numbers from 1 to 100 that are not multiples of 7

20 The sum of n terms of the series 214 + 206 + 198 + is 2760 Evaluate n

21 Evaluate n if the nth term of the sequence 4 1, 6, is 236 196

ISBN 97807043220

1. CHALLENGE EXERCISE

1 The nth term of a sequence is given by $T_n = \dfrac{n^2}{n+1}$

a What is the 9th term of the sequence?

b Which term is equal to $18\frac{1}{20}$?

2 For the series $\dfrac{3\pi}{4}$ π $\dfrac{5\pi}{4}$ find the exact value of

a the common difference

b the 7th term

c the sum of 6 terms

3 Evaluate the sum of the first 20 terms of the series

a $3 + 5 + 9 + 17 + 33 + 65 +$

b $5 - 2 + 10 - 8 + 15 - 32 +$

4 Which term of the sequence $\dfrac{7}{9}$ $\dfrac{14}{45}$ $\dfrac{28}{225}$ is equal to $\dfrac{224}{28\,125}$?

5 Find the sum of all integers between 1 and 200 that are not multiples of 9

6 Find the values of n for which $S_n > 2499$ for the series $20 + 4 + \dfrac{4}{5} +$

7 The sum of the first 5 terms of a geometric series is 77 and the sum of the next 5 terms is -2464

a Find the first term and common ratio of the series

b Find the 4th term of the series

8 **a** Find the limiting sum of the series $1 + \cos^2 x + \cos^4 x +$ where $\cos^2 x \neq 0, 1$.

b Why does this series have a limiting sum?

ISBN 97807043220

FUNCTIONS

2.

TRANSFORMATIONS OF FUNCTIONS

In ths chapter you wll explore transformaions on the graph of the funcion $y = f x$ that move or stretch the functon We have already met some transformtions of funtion in Year 11.

For example we learned that the graph of $y = -f x$ s a reflecton of the graph of $y = f x$ n the x-axs $y = k$ sn x s the graph of $y =$ sn x but stretched verically to ive an ampitude of k, and $y = \cos\ x + b$ s the graph of $y = \cos x$ shfted b unts to the rght

You ill also look at both grapical and algebric soluions of equaions uing the transformtions of functons

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- understand and apply translatons and dlatons of functons
- apply combnatons of transformaions to funcions
- use transformaions to sketch the graphs of ifferent types of funtions
- solve equatons and nequaltes graphcally and algebracally

TERMINOLOGY

dilation The process of stretching or compressing the graph of a function horizontally or vertically.

paramete: a constant in the equation of a function that determines the properties of that function and its graph for example the parameters for $y = mx + c$ are m (gradient) and c (y-intercept)

scale facto: The value of k by which the graph of a function is dilated

transformatio: A general name for the process of changing the graph of a function by moving reflecting or stretching it

translatio: The process of shifting the graph of a function horizontally and/or vertically without changing its size or shape

2.01 Vertical translations of functions

INVESTIGATION

VERTICAL TRANSLATIONS

Some graphics calculators or graphing software use a dynamic feature to show how a constant c (a **parameter**) changes the graph of a function

Use dynamic geometry software to explore the effect of c on each graph below. If you dont have dynamic softwar, substitute different values for c into the equation Use positive and negative values integers and fraction.

1 $f(x) = x + c$

2 $f(x) = x^2 + c$

3 $f(x) = x^3 + c$

4 $f(x) = x^4 + c$

5 $f(x) = e^x + c$

6 $f(x) = \ln x + c$

7 $f(x) = \frac{1}{x} + c$

8 $f(x) = |x| + c$

How does the value of c transform the graph? What is the difference between positive and negative values of c?

Notice that c shifts the graph up and down without changing its size or shape We call this a **vertical translation** (a shift along the y-axis)

ISBN 97807043220

Vertical translation

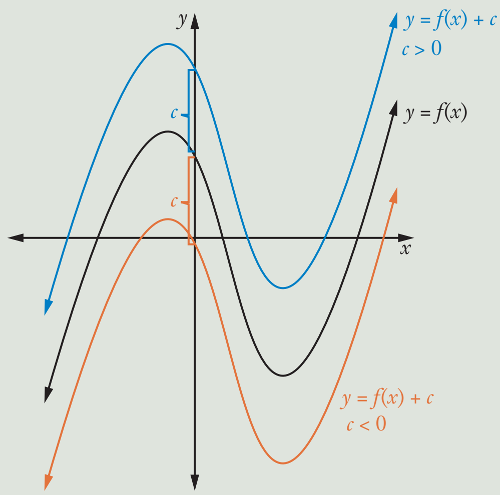
For the function $y = f(x)$

$y = f(x) + c$ translates the graph vertically (along the y-axis)

If $c > 0$ the graph is translated upwards by c units

If $c < 0$ the graph is translated downward.

A vertical translation changes the y values of the function



n Year 11, we learned that y sn $x + c$ s the graph of y sn x shfted up c unts

EXAMPLE 1

a Explain how the graph of $y = x^2 + 2$ is related to the graph of $y = x^2$

b If the graph of the function $y = x^2 + 7x + 1$ is translated 4 units down find the equation of the transformed function

c The point $P(3\ \ –2)$ lies on the function $y = f(x)$ Find the transformed point (the image of P) if the function is translated

i 6 units down **ii** 8 units up

Solution

a The graph of $y = x^2 + 2$ is a vertical translation 2 units up from the original (parent) function $y = x^2$

b For a vertical translation 4 units down

$y = f(x) + c$ where $c = –4$

$y = x^2 + 7x + 1 - 4$

$= x^2 + 7x - 3$

The equation of the transformed function is $y = x^2 + 7x - 3$

c **i** $P(3\ \ –2)$ is translated 6 units down so subtract 6 from the y value

The transformed point is $(3\ \ -2 - 6) \equiv (3, -8)$

ii $P(3\ \ –2)$ is translated 8 units up so add 8 to the y value

The transformed point is $(3\ \ -2 + 8) \equiv (3, 6.$

For ponts we use ≡' dentcal to rather than

ISBN 97807043220

EXAMPLE 2

a Sketch the graph of $y = x^3 - 3$.

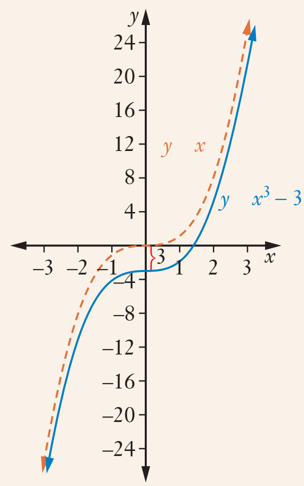
b **i** State the relationship of $y = \frac{1}{x} - 2$ to $y = \frac{1}{x}$

ii State the domain and range of $y = \frac{1}{x} - 2$

iii Sketch the graph of $y = \frac{1}{x} - 2$.

Solution

a A vertical translation of -3 units shifts the function $y = x^3$ down to the graph of $y = x^3 - 3$.
If you need to find some points on the graph of $y = x^3 - 3$ you could subtract 3 from y values of $y = x^3$



b **i** $y = \frac{1}{x} - 2$ is a vertical translation 2 units down of $y = \frac{1}{x}$

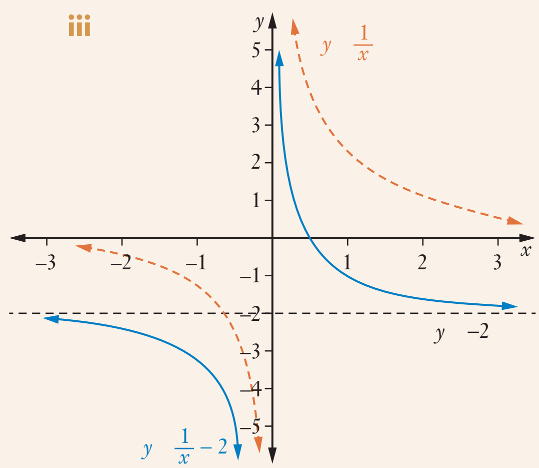
ii Since $x \neq 0$ domain is $(-\infty, 0) \cup (0, \infty)$

Since $\frac{1}{x} \neq 0$, $\frac{1}{x} - 2 \neq -2$

So range is $(-\infty, -2) \cup (-2, \infty)$

Since the horizontal asymptote is at $y = -2$ we sketch it as a dotted line

iii



ISBN 97807043220

Exercise 2.01 Vertical translations of functions

1 Describe how each constant affects the graph of $y = x^2$

a $y = x^2 + 3$ **b** $y = x^2 - 7$

c $y = x^2 - 1$ **d** $y = x^2 + 5$

2 Describe how each constant affects the graph of $y = x^3$

a $y = x^3 + 1$ **b** $y = x^3 - 4$ **c** $y = x^3 + 8$

3 Describe how the graph of $y = \frac{1}{x}$ transforms to the graph of $y = \frac{1}{x} + 9$

4 Find the equation of each translated function

a $y = x^2$ is translated 3 units downwards

b $f(x) = 2^x$ is translated 8 units upwards

c $y = |x|$ is translated 1 unit upwards

d $y = x^3$ is translated 4 units downwards

e $f(x) = \log x$ is translated 3 units upwards

f $y = \frac{2}{x}$ is translated 7 units downwards

5 Describe the relationship between the graph of $f(x) = x^4$ and

a $f(x) = x^4 - 1$ **b** $f(x) = x^4 + 6$

6 Find the equation of the transformed function if

a $y = 2x^3 + 3$ is translated

i 5 units down **ii** 3 units up

b $y = |x| - 4$ is translated

i 1 unit up **ii** 2 units down

c $y = e^x + 2$ is translated

i 1 unit down **ii** 3 units up

d $y = \log_e x - 1$ is translated

i 11 units up **ii** 7 units down

7 If $P = (1, -3)$ lies on the function $y = f(x)$ find the transformed (image) point of P if the function is translated

a 2 units up **b** 6 units down **c** m units up

8 Find the original point P on the function $y = f(x)$ if the coordinates of its transformed image are $(-1\ 2)$ when the function is translate:

a 1 unit up **b** 3 units down

9 Sketch each set of functions on the same number plane

a $y = x^2$ $y = x^2 + 2$ and $y = x^2 - 3$

b $y = 3^x$ and $y = 3^x - 4$

c $y = |x|$ and $y = |x| - 3$

ISBN 97807043220

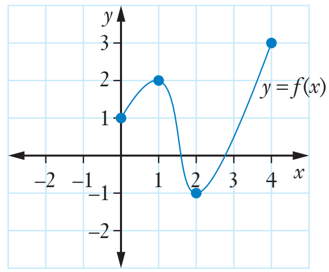
10 a Describe the **transformation** of $y = \dfrac{1}{x}$ into $y = \dfrac{1}{x} + 1$.

b Sketch the graph of $y = \dfrac{1}{x} + 1$.

11 The graph shows $y = f(x)$ Sketch the graph o:

a $y = f(x) - 1$

b $y = f(x) + 2$



12 a Show that $\dfrac{3x+1}{x} = \dfrac{1}{x} + 3$

b Hence or otherwise sketch the graph of $y = \dfrac{3x+1}{x}$

2.02 Horizontal translations of functions

WS
Transltions f uncion

Trasltios o uncion

WS
Gaphing anslaions o uncion

INVESTIGATION

HORIZONTAL TRANSLATIONS

Use a graphics calculator or graphing software to explore the affect of parameter b on each graph below. If you dn't have dynamic softwre, substitute different values for b into the equation Use positive and negative value, integers and fractions for b

1 $f(x) = (x + b)^2$

2 $f(x) = (x + b)^3$

3 $f(x) = (x + b)^4$

4 $f(x) = e^{x+b}$

5 $f(x) = \ln(x + b)$

6 $f(x) = \dfrac{1}{x+b}$

7 $f(x) = |x + b|$

How does the graph change as the value of b changes?

What is the difference between positive and negative values of b?

Notice that the parameter shifts the graph to the left or right without changing its size or shape We call this a **horizontal translation** (it shifts the function along the x-axis)

For a horizontal translation the shift is in the opposite direction from the sign of b

To understand why this happen, we change the subject of the equation to x since the translation is a shift along the x-axis For exampl:

$$y = (x + 5)^3$$
$$\sqrt[3]{y} = x + 5$$
$$\sqrt[3]{y} - 5 = x$$

This is a shift of 5 units to the left

ISBN 97807043220

Horizontal translations

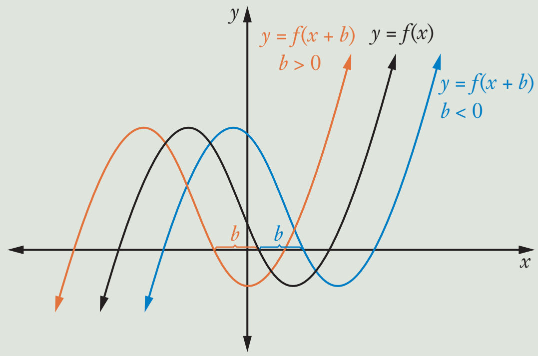
For the function $y = f(x)$

$y = f(x + b)$ translates the graph horizontally (along the x-axis)

If $b > 0$ the graph is translated to the left by b units

If $b < 0$ the graph is translated to the right

A horizontal translation changes the x values of the function



n Year 1, we learned that *y* tan ($x + b$) s the graph of *y* tan *x* shfted left *b* unts

EXAMPLE 3

a What is the relationship of $f(x) = \log_2 (x + 3)$ to $f(x) = \log_2 x$?

b If the graph $y = (x - 4)^3$ is translated 7 units to the right find the equation of the transformed function

c The point P(2 5) lies on the function $y = f(x)$ Find the corresponding (image) point of P given a horizontal translation with $b = 1$.

d The point Q (3, –4) on the graph of $y = f(x - 2)$ is the image of point P(x y) on $y = f(x)$ Find the coordinates of P

Solution

a $f(x) = \log_2 (x + 3)$ is a horizontal translation 3 units to the left from the parent function $f(x) = \log_2 x$

b If $y = (x - 4)^3$ is translated 7 units to the right

$y = f(x + b)$ where $b = -7$

$y = (x - 4 - 7)^3 = (x - 11)^3$

So the equation of the transformed function is $y = (x - 11)^3$

c $y = f(x + b)$ describes a horizontal translation (along the x-axis)

When $b = 1$, x values shift 1 unit to the left

Image of $P \equiv (2 - 1, 5) \equiv (1, 5)$

ISBN 97807043220

d $y = f(x - 2)$ is a horizontal translation 2 units to the right of $y = f(x)$

So $(x\ \ y)$ becomes $(x + 2, y)$

But $Q(3\ \ -4)$ is the image of $P(x\ \ y)$

So $(x + 2, y) \equiv (3, -4)$

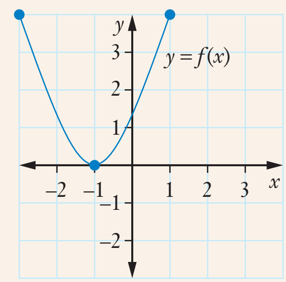
So $x + 2 = 3, y = -4$

$x = 1, y = -4$

So $P \equiv (1, -4)$

EXAMPLE 4

a The graph of $y = f(x)$ shown is transformed into $y = f(x + b)$ Sketch the transformed graph if $b = -3$



b Sketch the graph of

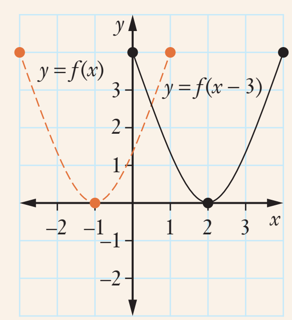
i $y = |x + 3|$

ii $y = \dfrac{1}{x - 2}$

Solution

a The graph $y = f(x + b)$ where $b = -3$ describes a horizontal translation of 3 units to the right

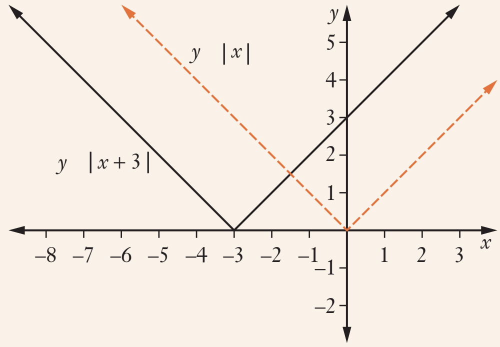
The transformed graph is 3 units to the right of the original function



ISBN 97807043220

b **i** The function $y = |x + 3|$ is in the form $y = f(x + b)$ where $b = 3$.

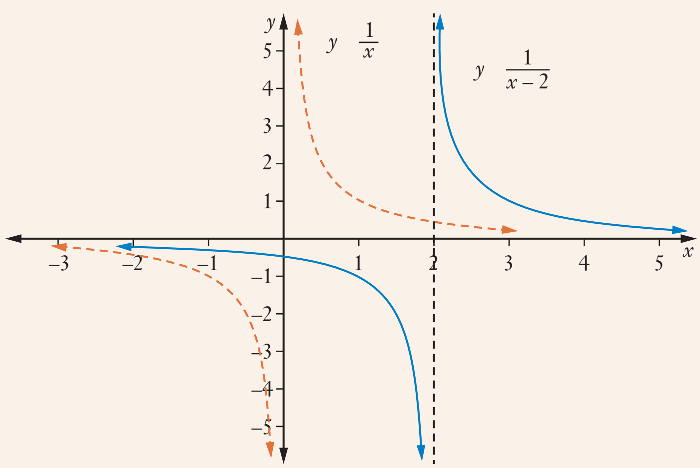
Since $b > 0$, $y = |x|$ is shifted 3 units to the left



If you need to find some points on the graph of $y = |x + 3|$ you could subtract 3 from x values of $y = |x|$

ii $y = \dfrac{1}{x-2}$ is in the form $y = f(x + b)$ where $b = -2$

Since $b < 0$, $y = \dfrac{1}{x}$ is shifted 2 units to the right



Exercise 2.02 Horizontal translations of functions

1 Describe how each constant affects the graph of $y = x^2$

a $y = (x - 4)^2$ **b** $y = (x + 2)^2$

2 Describe how each constant affects the graph of $y = x^3$

a $y = (x - 5)^3$ **b** $y = (x + 3)^3$

3 Find the equation of each translated graph

a $y = x^2$ translated 3 units to the left **b** $f(x) = 2^x$ translated 8 units to the right

c $y = |x|$ translated 1 unit to the left **d** $y = x^3$ translated 4 units to the right

e $f(x) = \log x$ translated 3 units left

ISBN 97807043220

4 Describe how $y = \dfrac{1}{x}$ transforms to $y = \dfrac{1}{x-3}$

5 Describe the relationship between $f(x) = x^4$ and

a $f(x) = (x + 2)^4$ **b** $f(x) = (x - 5)^4$

6 Find the equation if

a $y = -x^2$ is translated

i 4 units to the left **ii** 8 units to the right

b $y = |x|$ is translated

i 3 units to the right **ii** 4 units to the left

c $y = e^{x+2}$ is translated

i 4 units to the left **ii** 7 units to the right

d $y = \log_2 (x - 3)$ is translated

i 2 units to the right **ii** 3 units to the left

7 If $P = (1, -3)$ lies on the function $y = f(x)$ find the image point of P if the function is transformed to $y = f(x + b)$ where

a $b = -4$ **b** $b = 9$ **c** $b = t$

8 Find the original point on the function $y = f(x)$ if the coordinates of its image are $(-1, 2)$ when the function is translated

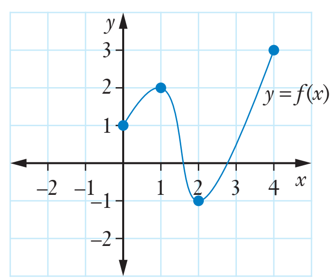
a 4 units to the left **b** 8 units to the right

9 Sketch on the same number plane

a $y = x^3$ and $y = (x + 1)^3$ **b** $f(x) = \ln x$ and $f(x) = \ln (x + 2)$

10 The graph shown is $y = f(x)$ Sketch the graph o:

a $y = f(x - 1)$ **b** $y = f(x + 3)$



11 Find the equation of the transformed function if $f(x) = x^5$ is translated

a 5 units down **b** 3 units to the right

c 2 units up **d** 7 units to the left

12 The point $P(3\ \ -2)$ is the image of a point on $y = f(x)$ after it has been translated 4 units to the left Find the original poin.

ISBN 97807043220

2.03 Vertical dilations of functions

A **dilation** stretches or compresses a function changing its size and shap.

INVESTIGATION

VERTICAL DILATION

Explore the effect of parameter k on each graph below. If you dn't have dynamic software substitute different values for k into the equation Use positive and negative values integers and fractions for k

1 $f(x) = kx$ **2** $f(x) = kx^2$ **3** $f(x) = kx^3$

4 $f(x) = kx^4$ **5** $f(x) = ke^x$ **6** $f(x) = k \ln x$

7 $f(x) = k\left(\frac{1}{x}\right)$ **8** $f(x) = k\,|\,x\,|$

How does the graph change as the value of k changes?

What is the difference between positive and negative values of k?

Notice that k stretches the graph up and down along the y-axis and changes its shape We call this **vertical dilation** The value of the parameter k controls the amount of stretching (expanding) or shrinking (compressing)

We call k the **scale factor**

Vertical dilations

For the curve $y = f(x)$

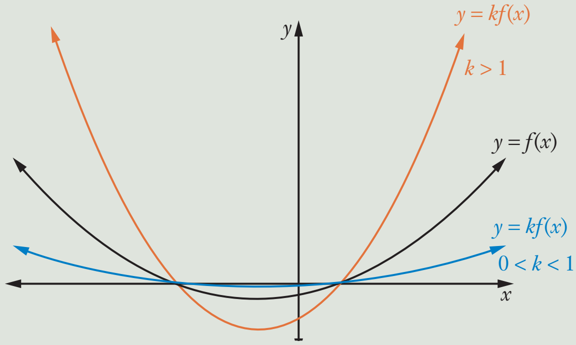
$y = kf(x)$ dilates the curve vertically (along the y-axis) by a scale factor of k

If $k > 1$ the graph is stretche, or expandd.

If $0 < k < 1$ the graph is shrun, or compressd.

A vertical dilation changes the y values of the function

n Year 11, we learned that y $k \cos x$ s the graph of y $\cos x$ stretched vericaly to gve an ampltude of k.



ISBN 97807043220

EXAMPLE 5

a The function $y = x^7$ is dilated vertically by a factor of 3 Find the equation of the transformed function

b Describe how the function $f(x) = \frac{\log_2 x}{2}$ is related to the function $f(x) = \log_2 x$

c Find the scale factor of each dilation of a function and state whether the dilation stretches or compresses the graph

i $y = 7x^2$ **ii** $y = \frac{e^x}{5}$

Solution

a If a function $y = f(x)$ has a vertical dilation with factor k the equation of its transformed function is $y = kf(x)$

So if the function $y = x^7$ has a vertical dilation with factor 3 the equation of the transformed function is $y = 3x^7$

Since $k > 1$ the function is stretched verticall.

b $f(x) = \frac{\log_2 x}{2}$

$= \frac{1}{2}\log_2 x$

So the function is in the form $y = kf(x)$ where $k = \frac{1}{2}$

Since $0 < k < 1$ the function is compressed verticall.

So $f(x) = \frac{\log_2 x}{2}$ is the result of $f(x) = \log_2 x$ being dilated (compressed) vertically by a scale factor of $\frac{1}{2}$

c The function $y = kf(x)$ has scale factor k

i $y = 7x^2$ has scale factor 7 (stretched)

ii $y = \frac{e^x}{5}$

$= \frac{1}{5}e^x$

Scale factor is $\frac{1}{5}$ (compressed)

ISBN 97807043220

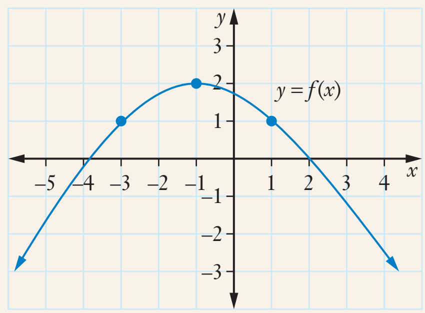
EXAMPLE 6

a The point $N = (-1\ 8)$ lies on the function $y = f(x)$ Find the image of N on the function $y = kf(x)$ when

i $k = 5$ **ii** $k = \frac{1}{2}$

b A function $y = f(x)$ is transformed to $y = kf(x)$ If the image of point A on the transformed function is $(-6\ 12$, find the coordinates of A when $k = 3$.

c The graph shown is $y = f(x)$.
Sketch the graph of $y = 2f(x)$



d Sketch the graphs of $y = x^2$ and $y = \frac{x^2}{2}$ on the same set of axes

Solution

a $y = kf(x)$ describes a vertical dilation (along the y-axis)

So the y values of the parent function will change

i When $k = 5$: y values are multiplied by a factor of 5

Image of $N \equiv (-1, 8 \times 5) \equiv (-1\ 40)$

ii When $k = \frac{1}{2}$ y values will be multiplied by a factor of $\frac{1}{2}$ (or divided by 2)

Image of $N \equiv \left(-1\ \ 8 \times \frac{1}{2}\right) \equiv (-1\ 4)$

b When $k = 3$, $(x\ \ y)$ becomes $(x, 3y)$

$(x, 3y) \equiv (-6, 12)$

$x = -6$

$3y = 12$

$y = 4$

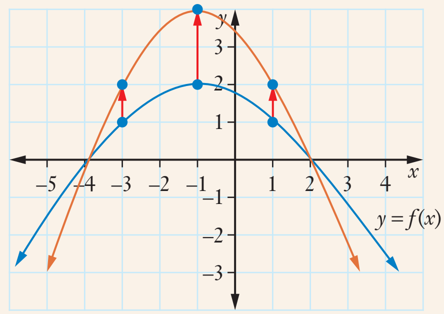
So $A \equiv (-6\ 4)$

c The graph of $y = 2f(x)$ is a vertical dilation of $y = f(x)$ with factor 2

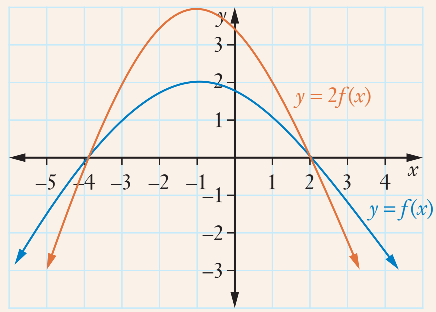
So each y value is doubled and the graph is twice as high as the original graph
For example

$y = 1$ becomes $y = 2$

$y = 2$ becomes $y = 4$

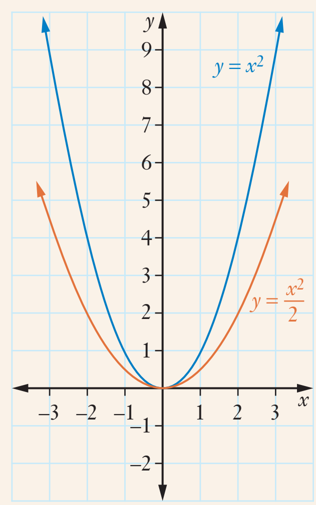


The transformed graph is still a parabola
However it is higher (stretched) and narrower than the original graph



d $y = \dfrac{x^2}{2}$ is a vertical dilation of $y = x^2$ with scale factor $\dfrac{1}{2}$
This halves the y values

$(-3, 9)$	becomes	$\left(-3\ \ 4\frac{1}{2}\right)$
$(-2\ 4)$	becomes	$(-2, 2)$
$(-1, 1)$	becomes	$\left(-1\ \ \frac{1}{2}\right)$
$(0, 0)$	becomes	$(0, 0)$
$(1, 1)$	becomes	$\left(-1\ \ \frac{1}{2}\right)$
$(2, 4)$	becomes	$(2, 2)$
$(3, 9)$	becomes	$\left(3\ \ 4\frac{1}{2}\right)$



ISBN 97807043220

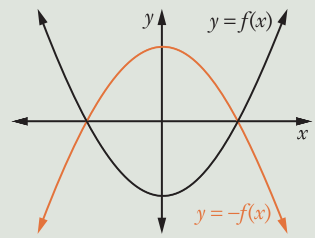
Reflections in the *x*-axis

You studied reflections in Year 11 in Chapte 5, *Further functions*

Reflections in the *x*-axis

$y = -f(x)$ is a reflection of the curve $y = f(x)$ in the x-axis

This is also a vertical dilation with scale factor $k = -1$



EXAMPLE 7

a Point $P(2\ 4)$ is on the function $y = f(x)$ Find the image of P on the function $y = -f(x)$

b Sketch the vertical dilation of $f(x) = \frac{1}{x}$ with scale factor -1

Solution

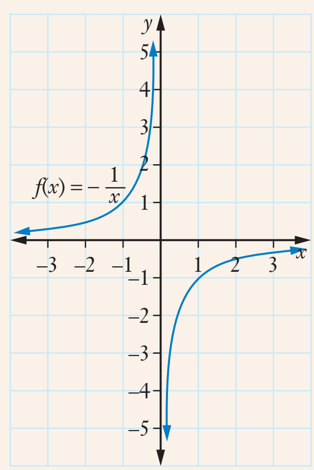
a The function $y = -f(x)$ is a reflection in the x-axis

The y values are multiplied by -1

Image of $P \equiv (2, 4 \times [-1]) \equiv (2, -4)$

b A vertical stretch with scale factor -1

is a reflection of $f(x) = \frac{1}{x}$ in the x-axis



Exercise 2.03 Vertical dilations of functions

1 Describe how the constant affects each transformed graph given the parent functio, and state the scale factor.

a $y = x$

i $y = 6x$ **ii** $y = \frac{x}{2}$ **iii** $y = -x$

b $y = x^2$

i $y = 2x^2$ **ii** $y = \frac{x^2}{6}$ **iii** $y = -x^2$

c $y = x^3$

i $y = 4x^3$ **ii** $y = \frac{x^3}{7}$ **iii** $y = \frac{4x^3}{3}$

d $y = x^4$

i $y = 9x^4$ **ii** $y = \frac{x^4}{3}$ **iii** $y = \frac{3x^4}{8}$

e $y = |x|$

i $y = 5|x|$ **ii** $y = \frac{|x|}{8}$ **iii** $y = -|x|$

f $f(x) = \log x$

i $f(x) = 9 \log x$ **ii** $f(x) = -\log x$ **iii** $f(x) = \frac{2\log x}{5}$

2 Find the equation of each transformed graph and state its domain and range

a $y = x^2$ dilated vertically with a scale factor of 6

b $y = \ln x$ dilated vertically with a scale factor of $\frac{1}{4}$

c $f(x) = |x|$ reflected in the x-axis

d $f(x) = e^x$ dilated vertically with a scale factor of 4

e $y = \frac{1}{x}$ dilated vertically with a scale factor of 7

3 Find the equation of each transformed function after the vertical dilation given

a $y = 3^x$ with scale factor 5 **b** $f(x) = x^2$ with scale factor $\frac{1}{3}$

c $y = x^3$ with scale factor -1 **d** $y = \frac{1}{x}$ with scale factor $\frac{1}{2}$

e $y = |x|$ with scale factor $\frac{2}{3}$

4 Point $M = (3\ 6)$ lies on the graph of $y = f(x)$ Find the coordinates of the image of M when $f(x)$ is

a dilated vertically with a factor of 4 **b** reflected in the x-axis

c dilated vertically with a factor of 12 **d** dilated vertically with a factor of $\frac{5}{6}$

5 The coordinates of the image of $X(x\ y)$ are $(4\ 12)$ when $y = f(x)$ is vertically dilated Find the coordinates of X if the scale factor is

a 3 **b** 2 **c** $\frac{1}{3}$ **d** $\frac{3}{4}$ **e** -1

ISBN 97807043220

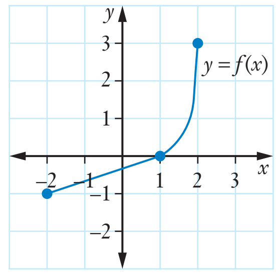
6 Sketch each pair of functions on the same set of axes

a $f(x) = \log_2 x$ and $f(x) = 2\ \log_2 x$ **b** $y = 3^x$ and $y = 2\ 3^x$

c $y = \frac{1}{x}$ and $y = \frac{3}{x}$ **d** $y = |x|$ and $y = 2|x|$

e $y = x^3$ and $y = -x^3$

7 Points on a function $y = f(x)$ are shown on the graph Sketch the graph of the transformed function showing the image points given a vertical stretch with facto:



a 3 **b** $\frac{1}{2}$ **c** -1

8 Sketch the graph of $y = 2\sqrt{1 - x^2}$

2.04 Horizontal dilations of functions

WS
Dilaions o uncion

WS
Advanced graph

INVESTIGATION

HORIZONTAL DILATIONS

Use dynamic geometry software to explore the affect of parameter a on each graph below. If you dn't have dynamic softwre, substitute different values for a into the equation Use positive and negative value, integers and fractions for a

1 $f(x) = ax$ **2** $f(x) = (ax)^2$ **3** $f(x) = (ax)^3$

4 $f(x) = (ax)^4$ **5** $f(x) = e^{ax}$ **6** $f(x) = \ln ax$

7 $f(x) = \frac{1}{ax}$ **8** $f(x) = |ax|$

How does a transform the graph as the value of a changes?

What is the difference between positive and negative values of a?

Notice that with **horizontal dilations** the higher the value of a the more the graph is compressed along the x-axis from left and right This is inverse variation and the scale factor for horizontal dilations is $\frac{1}{a}$

This is because horizontal dilation affects the x values of the function To see tis, we change the subject of the function to x For exampl:

$$y = (3x)^3$$
$$\sqrt[3]{y} = 3x$$
$$\frac{\sqrt[3]{y}}{3} = x$$
$$\text{or } x = \frac{1}{3}\sqrt[3]{y}$$

n Year 1, we learned that y sn ax is the graph of y sn x compressed horzontally to gve a perod of $\frac{2\pi}{a}$.

This shows a scale factor of $\frac{1}{3}$

Like horizontal translations a horizontal stretch works the opposite way to what you would expect because the equation is in the form $y = f(x)$ rather than $x = f(y)$

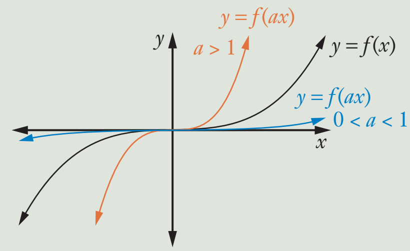
Horizontal dilations

For the curve $y = f(x)$

$y = f(ax)$ stretches the curve horizontally (along the x-axis) by a scale factor of $\frac{1}{a}$

If $a > 1$ the graph is compresse.

If $0 < a < 1$ the graph is stretche.



EXAMPLE 8

a Describe how the function $f(x) = x^3$ is related to the function $f(x) = (4x)^3$

b The function $y = \ln x$ is dilated horizontally by a scale factor of 2 Find the equation of the transformed function

c Find the scale factor of each function and state whether it stretches or compresses the graph

i $y = e^{3x}$ **ii** $f(x) = \left|\frac{x}{4}\right|$

Solution

a The function $y = f(ax)$ is a horizontal dilation of $y = f(x)$ with scale factor $\frac{1}{a}$

So the function $f(x) = (4x)^3$ is a horizontal dilation of $f(x) = x^3$ with scale factor $\frac{1}{4}$

b If $y = \ln x$ is dilated horizontally by a scale factor of 2

$$\frac{1}{a} = 2$$

$$a = \frac{1}{2}$$

So $y = \ln\left(\frac{1}{2}x\right)$ or $y = \ln\frac{x}{2}$

ISBN 97807043220

c **i** $y = e^{3x}$ is in the form $y = f(ax)$ where $f(x) = e^x$

This is a horizontal dilation with $a = 3$.

Scale factor $= \frac{1}{a} = \frac{1}{3}$ (stretched)

ii $f(x) = \left|\frac{x}{4}\right|$ can be written as $f(x) = \left|\frac{1}{4}x\right|$

The function is in the form $y = f(ax)$ where $f(x) = |x|$

This is a horizontal dilation with $a = \frac{1}{4}$

Scale factor $= \frac{1}{a}$

$= \frac{1}{\frac{1}{4}}$

$= 4$ (compressed)

EXAMPLE 9

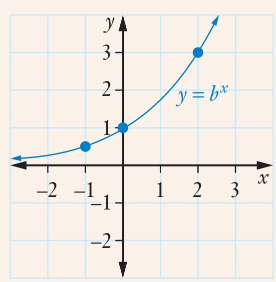
a The points $P(-3\ 4)$ and $Q(9\ 0)$ lie on the function $y = f(x)$ Find the coordinates of the images of P and Q for the function $y = f(ax)$ when

i $a = 3$ **ii** $a = \frac{1}{5}$

b When the function $y = f(x)$ is transformed to $y = f(ax)$ the coordinates of the image of $N(x\ y)$ are $(16, -5)$ Find the coordinates of N when

i $a = 4$ **ii** $a = \frac{1}{2}$

c The graph of $y = b^x$ shown is transformed to $y = b^{2x}$ Sketch the graph of the transformed function



d State the scale factor if the graph $y = |x|$ is transformed to $y = \left|\frac{x}{2}\right|$ and sketch both graphs on the same set of axes

ISBN 97807043220

Solution

a The function $y = f(ax)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{a}$

i When $a = 3$ scale factor is $\frac{1}{3}$

All x values are multiplied by $\frac{1}{3}$ (divided by 3)

Image of $P \equiv \left(-3 \times \frac{1}{3},\ 4\right) \equiv (-1\ \ 4)$

Image of $Q \equiv \left(9 \times \frac{1}{3},\ 0\right) \equiv (3, 0)$

ii When $a = \frac{1}{5}$ scale factor is $\frac{1}{\frac{1}{5}}$ or 5

All x values are multiplied by 5

Image of $P \equiv (-3 \times 5\ \ 4) \equiv (-15\ \ 4)$

Image of $Q \equiv (9 \times 5, 0) \equiv (45, 0)$

b We multiply all x values by scale factor $\frac{1}{a}$

i When $a = 4$ scale factor is $\frac{1}{4}$

So $(x\ \ y)$ becomes $\left(x \times \frac{1}{4}\ \ y\right) = \left(\frac{x}{4}\ \ y\right)$

$\left(\frac{x}{4}\ \ y\right) \equiv (16, -5)$

$\frac{x}{4} = 16$

$x = 64$

$y = -5$

So $N \equiv (64\ \ -5)$

ii When $a = \frac{1}{2}$ scale factor is $\frac{1}{\frac{1}{2}}$ or 2

So $(x\ \ y)$ becomes $(x \times 2\ \ y) \equiv (2x\ \ y)$

$(2x\ \ y) \equiv (16, -5)$

$2x = 16$

$x = 8$

$y = -5$

So $N \equiv (8, -5)$

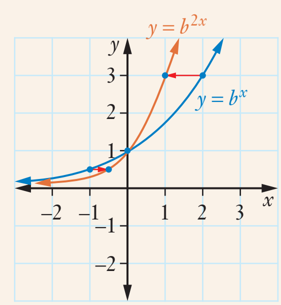
c The graph of $y = b^{2x}$ describes a horizontal dilation of $y = b^x$ with scale factor $\frac{1}{2}$

So we halve the x values

$x = -1$ becomes $x = -\frac{1}{2}$

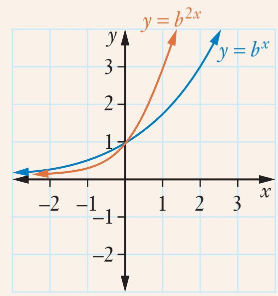
$x = 0$ becomes $x = 0$

$x = 2$ becomes $x = 1$



ISBN 97807043220

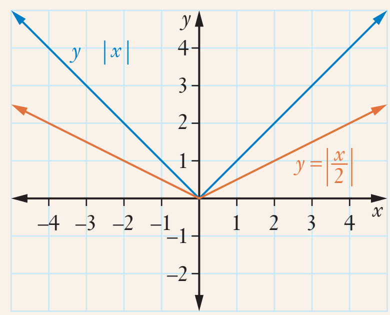
The transformed function is still in the shape of an exponential function but it has changed shape and siz.



d The graph $y = \left|\frac{x}{2}\right|$ is a horizontal dilation of $y = |x|$ with a scale factor $\frac{1}{2}$ or 2

We double the x values

(−3, 3)	becomes	(−6, 3)
(−2, 2)	becomes	(−4 2)
(−1, 1)	becomes	(−2, 1)
(0, 0)	becomes	(0, 0)
(1, 1)	becomes	(2, 1)
(2, 2)	becomes	(4, 2)
(3, 3)	becomes	(6, 3)



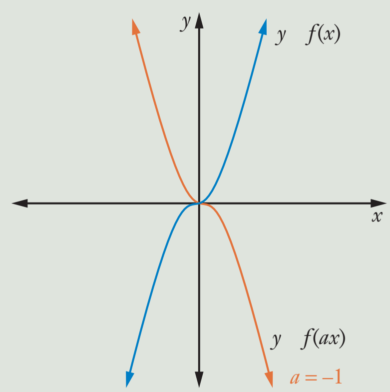
Reflections in the *y*-axis

You studied reflections in the y-axis in Year 11 in Chapter5, *Further functions*

Reflections in the *y*-axis

$y = f(-x)$ is a reflection of the curve $y = f(x)$ in the y-axis

This is a horizontal stretch with scale factor $a = \frac{1}{-1} = -1$



Notice that for even functions $y = f(x) = f(-x)$

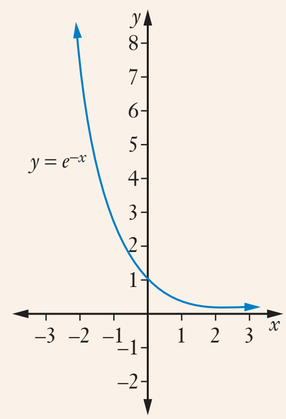
Even functions are already symmetrical about the y-axis so the reflected graph is the same as the original graph

EXAMPLE 10

Sketch the graph of the horizontal dilation of $y = e^x$ with scale factor -1

Solution

The horizontal dilation with scale factor -1 is a reflection of $y = e^x$ in the y-axis



Exercise 2.04 Horizontal dilations of functions

1 Describe the transformation that the constant makes on $f(x) = x^4$ and state the scale factor.

a $f(x) = (8x)^4$ **b** $f(x) = \left(\frac{x}{5}\right)^4$

c $f(x) = \left(\frac{3x}{7}\right)^4$ **d** $f(x) = (-x)^4$

2 Describe whether the constant describes a horizontal or vertical dilation and state the scale factor.

a $y = x^2$

i $y = (2x)^2$ **ii** $y = (5x)^2$ **iii** $y = \left(\frac{x}{3}\right)^2$

b $y = x^3$

i $y = 4x^3$ **ii** $y = \left(\frac{x}{2}\right)^3$ **iii** $y = (-x)^3$

c $y = x^4$

i $y = (7x)^4$ **ii** $y = \frac{x^4}{8}$ **iii** $y = \left(\frac{3x}{4}\right)^4$

d $y = |x|$

i $y = |5x|$ **ii** $y = \left|\frac{x}{2}\right|$ **iii** $y = \left|\frac{3x}{5}\right|$

e $y = 5^x$

i $y = 5^{3x}$ **ii** $y = -5^x$ **iii** $y = 5^{\frac{x}{2}}$

f $f(x) = \log x$

i $f(x) = 8 \log x$ **ii** $f(x) = \log(-x)$ **iii** $f(x) = \log \frac{x}{7}$

ISBN 97807043220

3 Find the equation of each transformed graph and state its domain and range

a $f(x) = |x|$ is dilated horizontally with a scale factor of $\frac{1}{5}$

b $y = x^2$ is dilated horizontally with a scale factor of 3

c $y = x^3$ is reflected in the y-axis

d $y = e^x$ is dilated vertically with a scale factor $\frac{1}{9}$

e $y = \log_4 x$ is reflected in the x-axis

4 Point X (–2 7) lies on $y = f(x)$ Find the coordinates of the image of X on $y = f(ax)$ given

a $a = 2$ **b** $a = -1$ **c** $a = \frac{1}{3}$

5 The function $y = f(x)$ is transformed into the function $y = f(ax)$ The coordinates of the image point of $(x\ \ y)$ on the original function are (–24 1) on the transformed functio. Find the values of $(x\ \ y)$ if

a $a = 3$ **b** $a = 2$ **c** $a = \frac{1}{4}$

6 Sketch each pair of functions on the same set of axes

a $f(x) = \ln x$ and $f(x) = \ln (2x)$

b $y = 2^x$ and $y = 2^{\frac{x}{3}}$

c $y = \frac{1}{x}$ and $y = \frac{1}{3x}$

d $y = |x|$ and $y = |2x|$

e $f(x) = x^2$ and $f(x) = (3x)^2$

f $y = \ln x$ and $y = \ln (-x)$

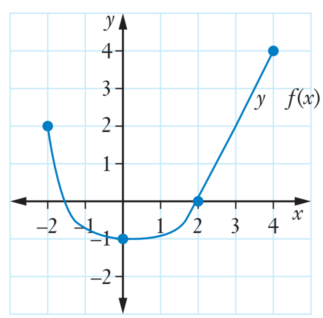
7 Sketch the graphs of $y = e^x$ $y = e^{2x}$ and $y = 2e^x$ on the same set of axes

8 Explain why a reflection in the y-axis does not change the graph of

a $y = x^2$ **b** $f(x) = |x|$

9 Sketch the graph of $y = f(ax)$ given the graph of $y = f(x)$ shown whe:

a $a = \frac{1}{2}$ **b** $a = 2$



ISBN 97807043220

Combinaion
o
anomtios

2.05 Combinations of transformations

A function can have any combination of the different types of transformations acting on it

Transformations

For the curve $y = f(x)$

$y = f(x) + c$ translates the function verticaly:

- up if $c > 0$
- down if $c < 0$

$y = f(x + b)$ translates the function horizontally:

- to the left if $b > 0$
- to the right if $b < 0$

$y = kf(x)$ dilates the function vertically with scale factor k

- stretches if $k > 1$
- compresses if $0 < k < 1$
- reflects the function in the x-axis if $k = -1$

$y = f(ax)$ dilates the function horizontally with scale factor $\frac{1}{a}$

- compresses if $a > 1$
- stretches if $0 < a < 1$
- reflects the function in the y-axis if $a = -1$

EXAMPLE 11

a Find the equation of the transformed function if $y = x^4$ is shifted 2 units down and 5 units to the left

b Find the equation of the transformed function if $y = e^x$ is dilated vertically by a scale factor 3 and translated horizontally 2 units to the right

Solution

a Starting with $y = x^4$

A vertical translation 2 units down gives $y = x^4 - 2$.

A horizontal translation 5 units to the left gives $b = 5$.

So the equation becomes $y = (x + 5)^4 - 2$.

ISBN 97807043220

Notice that we could do this the other way around

A horizontal translation 5 units to the left gives $y = (x + 5)^4$

A vertical translation 2 units down gives $y = (x + 5)^4 - 2$.

b Starting with $y = e^x$

A vertical dilation of scale factor 3 gives $y = 3e^x$

A horizontal translation 2 units to the right gives $b = -2$

So the equation becomes $y = 3e^{x-2}$

Notice that we could do this the other way around

A horizontal translation 2 units to the right gives $y = e^{x-2}$

A vertical dilation of scale factor 3 gives $y = 3e^{x-2}$

When the transformations are *both* vertical or *both* horizontal then the order is importan.

EXAMPLE 12

a When the function $y = x^2$ is translated 3 units up (vertically) and vertically dilated by scale factor 4 the equation of the transformed function is $y = 4x^2 + 3$ Find the order in which the transformations were done

b The equation of the transformed function is $y = (2x + 5)^3$ when the function $y = x^3$ is horizontally dilated by scale factor $\frac{1}{2}$ and translated 5 units (horizontally) to the left In which order were the transformations done?

c The equation of the transformed function is $y = \ln [3(x - 2)]$ when the function $y = \ln x$ is horizontally dilated by scale factor $\frac{1}{3}$ and translated 2 units (horizontally) to the right In which order were the transformations done ?

Solution

a Starting with $y = x^2$

A vertical translation 3 units up gives $y = x^2 + 3$.

A vertical dilation by scale factor 4 gives $y = 4(x^2 + 3)$.

This is not the equation of the transformed function

ISBN 97807043220

Try the other way aroun:

A vertical dilation by scale factor 4 gives $y = 4x^2$

A vertical translation 3 units up gives $y = 4x^2 + 3$.

So the correct order is the vertical dilation then the vertical translatio.

b Starting with $y = x^3$

A horizontal dilation of scale factor $\frac{1}{2}$ gives $a = 2$ so the equation is $y = (2x)^3$

A horizontal translation 5 units to the left gives $b = 5$ so $y = [2(x + 5)]^3$

This is not the equation of the transformed function

Try the other way aroun:

A horizontal translation 5 units to the left gives $y = (x + 5)^3$

A horizontal dilation of scale factor $\frac{1}{2}$ gives $a = 2$ so the equation is $y = (2x + 5)^3$

So the correct order is the horizontal translation then the horizontal dilatio.

c Starting with $y = \ln x$

A horizontal dilation of scale factor $\frac{1}{3}$ gives $y = \ln (3x)$

A horizontal translation 2 units to the right gives $y = \ln [3(x - 2)]$

So the correct order is the horizontal dilation then the horizontal translatio.

Doing the horizontal dilation first gives $y = f(a(x + b))$ while doing the horizontal translation first gives $y = f(ax + b)$

We can state the order we want to perform the transformation.

EXAMPLE 13

Find the equation of the function if $y = x^2$ is first horizontally dilated with scale factor $\frac{1}{2}$ then translated 3 units to the right

Solution

A horizontal dilation with scale factor $\frac{1}{2}$ gives $a = 2$.

So $y = x^2$ becomes $y = (2x)^2$

A horizontal translation 3 units to the right gives $b = -3$

So $y = (2x)^2$ transforms to $y = [2(x - 3)]^2$

Remember to put brackets around $x - 3$

We can combine all the transformations into a single expressio:

ISBN 97807043220

Equation of a transformed function

$y = kf(a(x + b)) + c$ where a b c and k are constants is a transformation of $y = f(x)$

- a horizontal dilation of scale factor $\frac{1}{a}$
- a horizontal translation of b
- a vertical dilation of k
- a vertical translation of c

Order of transformations

For $y = kf(a(x + b)) + c$

1 do horizontal dilation (a) then horizontal translation (b)

2 do vertical dilation (k) then vertical translation (c)

It doesnt matter whether you do horizontal or vertical transformations firs.

Notice that the horizontal dilation and translation parameters a and b are inside the brackets (they change x values) and the vertical dilation and translation parameters k and c are outside the brackets (they change the y values)

EXAMPLE 14

a Describe the transformations of $y = e^x$ in the correct order to produce the transformed function $y = \frac{1}{2}e^{x+} \;\; - 3$.

b Describe the transformations of $y = x^2$ in order that give the transformed function $y = 3(2x - 6)^2 + 1$.

c Find the equation of the transformed function if $y = f(x)$ undergoes a vertical dilation with factor 5 a horizontal dilation with factor -1 a translation 4 units to the right and 9 units down

Solution

a For $y = \frac{1}{2}e^{x+} \;\; - 3$:

Horizontal transformations (a and b) No dilation $b = 1$ gives a translation 1 unit left

Vertical transformations (k and c) dilation of scale factor $\frac{1}{2}$ and translation 3 units down
Correct order is

1 Horizontal translation 1 unit left

2 Vertical dilation of scale factor $\frac{1}{2}$

3 Vertical translation 3 units down

Because verical transformtions can be done frst the order 2–3–1 s also possble

b For $y = 3(2x - 6)^2 + 1$:

First put the equation in the form $y = kf(a(x + b)) + c$

$y = 3(2x - 6)^2 + 1$

$= 3[2(x - 3)]^2 + 1$

Horizontal transformations dilation $a = 2$ and translation $b = -3$

Vertical transformation: dilation $k = 3$ and translation $c = 1$

1 Horizontal dilation of scale factor $\frac{1}{2}$

2 Horizontal translation 3 units right

3 Vertical dilation of scale factor 3

4 Vertical translation 1 unit up

The order 3–4–1–2 s also possble

Alternative method There is another possible orde, if you notice that $y = 3(2x - 6)^2 + 1$ is of the form $y = kf(ax + b) + c$ where the $(ax + b)$ is not factorised so we can do the horizontal translation first then horizontal dilatio.

The horizontal translation is 6 units right ($b = -6$) followed by a horizontal dilation of scale factor $\frac{1}{2}$ then 3 and 4 as abov.

c We require $y = kf(a(x + b)) + c$

Horizontal transformations dilation $a = -1$ and translation $b = -4$

Vertical transformation: dilation $k = 5$ and translation $c = -9$

Horizontal transformations $y = kf(-1(x - 4)) + c$

Add vertical transformations $y = 5f(-(x - 4)) - 9$

This answer can also be written as $y = 5f(-x + 4) - 9$ or $y = 5f(4 - x) - 9$

Domain and range

We can find the domain and range of functions without drawing their graph.

Effect of transformations on domain and range

Horizontal transformations change x values so affect the domain

Vertical transformations change y values so affect the range

ISBN 97807043220

EXAMPLE 15

Find the domain and range of

a $f(x) = -3(x-2)^2 + 5$ **b** $y = 5\sqrt{2x+1}$

Solution

a $y = x^2$ has domain $(-\infty\ \ \infty)$ and range $[0\ \ \infty)$

Horizontal transformations affect the domain

No horizontal dilation

Horizontal translation 2 units right domain of $x-2$ is $(-\infty\ \ \infty)$ so domain of $f(x)$ is unchanged

Vertical transformations affect the rang:

Vertical dilatio, scale factor -3 Range of y is $[0, \infty)$ so range of $3y$ is 3 times as much so no change for $[,\ \ \infty)$

But the – sign in -3 means the y is reflected in the x-axis so range of $-3y$ is $(-\infty, 0$.

Vertical translation 5 units u: Range of $-3y$ is $(-\infty\ \ 0]$ so range of $-3y + 5$ is $(-\infty, 5]$.

So $y = -3(x-2)^2 + 5$ has domain $(-\infty\ \ \infty)$ and range $(-\infty, 5]$.

b $y = \sqrt{x}$ has domain $[0\ \ \infty)$ and range $[0\ \ \infty)$

Horizontal transformations affect the domain

Domain of $2x + 1$ is $[0, \infty)$ so $2x + 1 \geq 0$

$2x \geq -1$

$x \geq -\frac{1}{2}$

Vertical transformations affect the rang:

Vertical dilatio, scale factor5: Range of $\sqrt{2x+1}$ is $[0, \infty)$ so range of $5\sqrt{2x+1}$ is 5 times as much so unchange.

No vertical translation

So $y = 5\sqrt{2x+1}$ has domain $\left[-\frac{1}{2}\ \ \infty\right)$ and range $[0\ \ \infty)$

Exercise 2.05 Combinations of transformations

1 The point (2 –6) lies on the function $y = f(x)$ Find the coordinates of its image if the function is

a horizontally translated 3 units to the right and vertically translated 5 units down

b translated 4 units up and 3 units to the left

c translated 7 units to the right and 9 units up

d translated 11 units down and 4 units to the left

2 Find the equation of the transformed function where $f(x) = x^5$ is reflected

a in the x-axis and vertically dilated with scale factor 4

b in the y-axis and horizontally dilated with scale factor 3

3 Find the equation of each transformed function

a $y = x^3$ is translated 3 units down and 4 units to the left

b $f(x) = |x|$ is translated 9 units up and 1 unit to the right

c $f(x) = x$ is dilated vertically with a scale factor of 3 and translated down 6 units

d $y = e^x$ is reflected in the x-axis and translated up 2 units

e $y = x^3$ is horizontally dilated by a scale factor of $\frac{1}{2}$ and translated down 5 units

f $f(x) = \frac{1}{x}$ is vertically dilated by a factor of 2 and horizontally dilated by a factor of 3

g $f(x) = \sqrt{x}$ is reflected in the y-axis vertically dilated by a scale factor of 3 and horizontally dilated by a scale factor of $\frac{1}{2}$

h $y = \ln x$ is horizontally dilated by a scale factor of 3 and translated upwards by 2 units

i $f(x) = \log_2 x$ is horizontally dilated by a scale factor of $\frac{1}{4}$ and vertically dilated by a scale factor of 3

j $y = x^2$ is horizontally dilated by a scale factor of 2 and translated down 3 units

4 Describe the transformations to $y = x^3$ in the correct order if the transformed function has equation

a $y = (x - 1)^3 + 7$ **b** $y = 4x^3 - 1$ **c** $y = -5x^3 - 3$

d $y = 2(x + 7)^3$ **e** $y = 6(2x - 4)^3 + 5$ **f** $y = 2(3x + 9)^3 - 10$

5 Describe the transformations in their correct order for each of the functions from

a $y = \log x$ to $y = 2\log(x + 3) - 1$ **b** $f(x) = x^2$ to $f(x) = -(3x)^2 + 9$

c $y = e^x$ to $y = 2e^{5x} - 3$ **d** $f(x) = \sqrt{x}$ to $f(x) = 4\sqrt{x-7} + 1$

e $y = |x|$ to $y = |-2(x + 1)| - 1$ **f** $y = \frac{1}{x}$ to $y = -\frac{1}{2x} + 8$

ISBN 97807043220

6 The point (8 −12) lies on the function $y = f(x)$ Find the coordinates of the image point when the function is transformed into

a $y = 3f(x - 1) + 5$ **b** $y = -f(2x) - 7$ **c** $y = 2f(x + 3) - 1$

d $y = 6f(-x) + 5$ **e** $y = -2f(2x - 4) - 3$

7 Given the function $y = f(x)$ find the coordinates of the image of $(x\ \ y)$ if the function is

a translated 6 units down and 3 units to the right

b reflected in the y-axis and translated 6 units up

c vertically dilated with scale factor 2 and translated 5 units to the left

d horizontally dilated with scale factor 3 and translated 5 units up

e reflected in the x-axis vertically dilated with scale factor , translated 6 units to the left horizontally dilated with scale factor 5 and translated 1 unit down

8 Find the equation of the transformed function if $y = f(x)$ is

a translated 2 units down and 1 unit to the left

b translated 5 units to the right and 3 units up

c reflected in the x-axis and translated 4 units to the right

d reflected in the y-axis and translated up 2 units

e reflected in the x-axis and horizontally dilated with a factor of 4

f vertically dilated by a scale factor of 2 and translated 2 units down

9 Find the equation of the transformed function using the correct order of transformations for $y = kf(a(x + b)) + c$

a $f(x) = \frac{1}{x}$ is reflected in the y-axis translated up 3 units and dilated vertically by a scale factor of 9

b $y = x^2$ is translated down by 6 units and by 2 units to the left and is horizontally dilated with scale factor $\frac{1}{5}$

c $f(x) = \ln x$ has a vertical dilation with factor 8 a vertical translation of 3 dow, a horizontal dilation with factor 2 and a horizontal translation of 5 to the right

d $y = \sqrt{x}$ has a vertical translation of 4 up a horizontal translation of 4 to the lef, a reflection in the y-axis and a vertical dilation with factor 9

e $f(x) = |x|$ is translated up by 7 units dilated horizontally by a factor of $\frac{1}{6}$ and reflected in the x-axis

f $y = x^3$ is translated 4 units to the left then dilated horizontally with scale factor $\frac{1}{4}$

g $y = 2^x$ is translated up by 5 units translated 2 units to the righ, then is vertically dilated with scale factor 6

10 Find the domain and range of each function

a $f(x) = (x + 3)^2 + 5$ **b** $y = 5|-2x| - 2$ **c** $f(x) = \dfrac{1}{2x - 4} + 1$

d $y = 4^{3x} + 2$ **e** $f(x) = 3 \log (3x - 6) - 5$

11 **a** By completing the square write the equation for the parabola $y = x^2 + 2x - 7$ in the form $y = (x + a)^2 + b$

b Describe the transformations on $y = x^2$ that result in the function $y = x^2 + 2x - 7$.

12 Describe the transformations that change $y = x^2$ into the function $y = x^2 - 10x - 3$.

13 The function $y = f(x)$ is transformed to the function $y = kf(a(x + b)) + c$

Find the coordinates of the image point of $(x\ \ y)$ when

a $c = 5$, $b = -3$, $k = 2$ and $a = \dfrac{1}{2}$

b $c = -2$, $b = 6$, $k = -1$ and $a = 3$

14 **a** Find the equation of the transformed graph if $x^2 + y^2 = 9$ is translated 3 units to the right and 4 units up

b The circle $x^2 + y^2 = 1$ is transformed into the circle $x^2 - 4x + y^2 + 6y + 12 = 0$. Describe how the circle is transformed

WS

Gaphing anomed uncion

2.06 Graphs of functions with combined transformations

We can find points and sketch the graphs of functions that are changed by a combination of transformations Translations are the easiest transformations to use since they shift the graph while keeping it the same size and shape

EXAMPLE 16

Sketch the graph of $y = (x - 2)^2 - 5$.

Solution

$y = (x - 2)^2 - 5$ is transformed from $y = x^2$ by a horizontal translation of 2 units to the right and a vertical translation of 5 units down

The vertex (turning point) of parabola $y = x^2$ is (0, 0.

So the vertex of $y = (x - 2)^2 - 5$ is $(0 + 2, 0 - 5) \equiv (2, -5)$

Sketching the graph we keep the shape of $y = x^2$ and shift it to the new vertex

We can find the intercepts for a more accurate grap.

ISBN 97807043220

For x-intercepts $y = 0$

$$0 = (x - 2)^2 - 5$$

$$5 = (x - 2)^2$$

$$\pm\sqrt{5} = x - 2$$

$$2 \pm \sqrt{5} = x$$

For y-intercepts $x = 0$

$$y = (0 - 2)^2 - 5$$

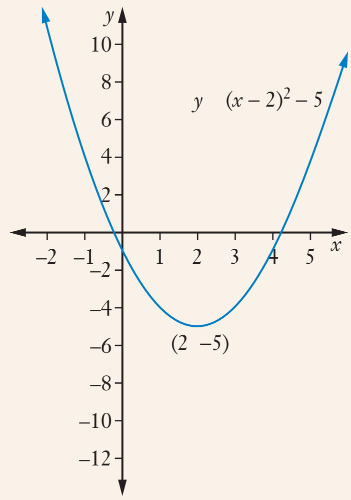
$$= 4 - 5$$

$$= -1$$

So the x-intercepts are approximately 42   −02

To find other points on the grap, you can transform points on $y = x^2$ the same way as for the vertex

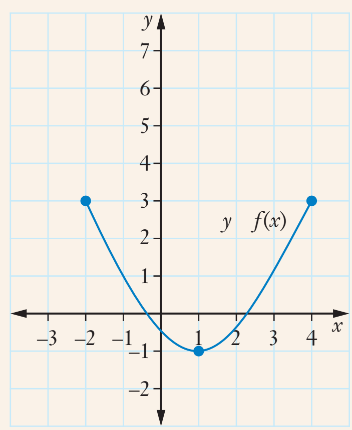
Sketch the graph using a scale on each axis that will show the information For exampl, the vertex is at (2 −5) so the y values must go down as far as $y = -5$



EXAMPLE 17

The graph $y = f(x)$ shown is reflected in the y-axis dilated vertically with a scale factor of 2 and translated 1 unit up

Sketch the graph of the transformed function



ISBN 97807043220

Solution

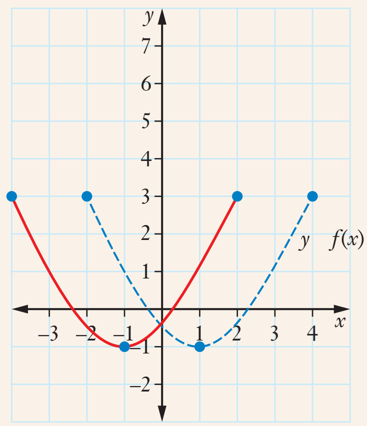
A reflection in the y-axis is a horizontal dilation with scale factor -1

Multiply each x value by -1

$x = 1$ becomes $x = -1$

$x = 4$ becomes $x = -4$

$x = -2$ becomes $x = 2$

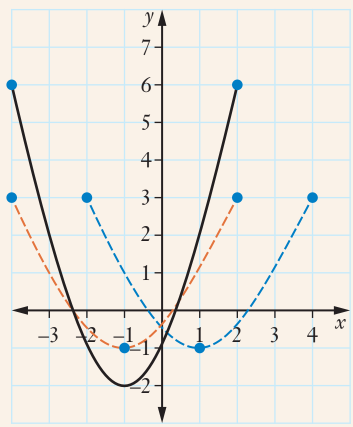


For a vertical dilation with scale factor 2

Multiply each y value by 2

$y = -1$ becomes $y = -2$

$y = 3$ becomes $y = 6$

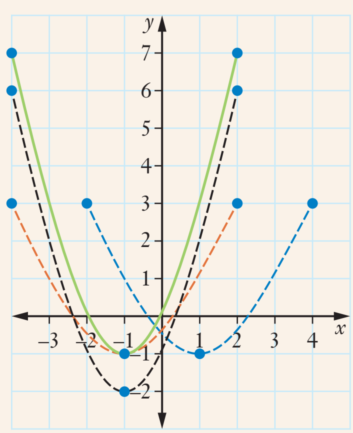


For a vertical translation 1 unit up

Add 1 to y values

$y = 6$ becomes $y = 7$

$y = -2$ becomes $y = -1$

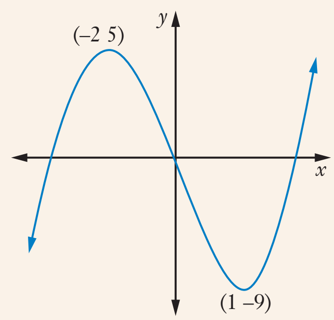


ISBN 97807043220

In the previous example we took one transformation at a tim. In the next exampe, we take transformations together (in the correct order) and plot images of key points on the original (parent) curve

EXAMPLE 18

a The function $y = f(x)$ is sketched below with stationary (turning) points as shown



i Describe the transformations if $y = f(x)$ is transformed to $y = 3f(x + 1) - 2$ and how they change the coordinates $(x\ \ y)$ of the parent function

ii Find the coordinates of the image of each stationary point when the function is transformed

iii Sketch the graph of $y = 3f(x + 1) - 2$.

b **i** Describe the transformations if $y = |x|$ is transformed to $y = -\left|\frac{x}{2}\right| + 3$ and the image of point $(x\ \ y)$ on the parent function

ii Sketch the transformed function

Solution

a **i** Transformations (in order) ar:

A horizontal translation 1 unit to the left

So $(x\ \ y)$ becomes $(x - 1, y)$

A vertical dilation scale factor :

So $(x - 1, y)$ becomes $(x - 1, 3y)$

A vertical translation 2 units down

So $(x - 1, 3y)$ becomes $(x - 1, 3y - 2)$

ii For (–2, 5:

Image becomes $(-2 - 1, 3 \times 5 - 2) \equiv (-3, 13.$

For $(1\ \ -9)$

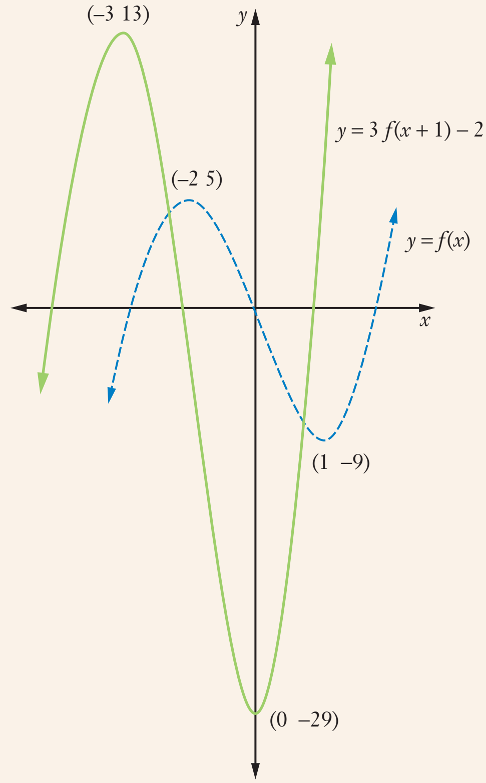
Image becomes $(1 - 1, 3 \times [-9] - 2) \equiv (0, -29)$

ISBN 97807043220

iii $y = f(x)$ passes through (0 0.

Image becomes $(0 - 1, 3 \times 0 - 2)$
$\equiv (-1, -2)$

Sketch the graph showing this information using a suitable scale on each axis For exampl, the y values must go up to 13 and down to −29



b **i** Transformations (in order) ar:

A horizontal dilation scale factor :

So $(x\ \ y)$ becomes $(2x\ \ y)$

A vertical dilation scale factor −1 (reflection in the x-axis)

So $(2x\ \ y)$ becomes $(2x\ \ -y)$

A vertical translation 3 units up

So $(2x\ \ -y)$ becomes $(2x\ \ -y + 3)$.

ii The intercepts of $y = |x|$ are at (0, 0.

Image of (0 0) is $(2\ \times 0, -0 + 3) \equiv (0, 3.$

We can find the intercepts on $y = -\left|\frac{x}{2}\right| + 3$

ISBN 97807043220

For x-intercept $y = 0$

$$0 = -\left|\frac{x}{2}\right| + 3$$

$$-3 = -\left|\frac{x}{2}\right|$$

$$3 = \left|\frac{x}{2}\right|$$

$$\pm 3 = \frac{x}{2}$$

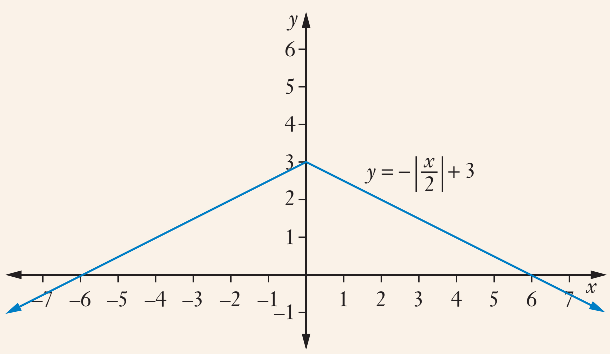
$$\pm 6 = x$$

For y-intercept $x = 0$:

$$y = -\left|\frac{0}{2}\right| + 3$$

$$= 3$$

Sketching this information using an appropriate scale gives the graph



Exercise 2.06 Graphs of functions with combined transformations

1 Given $f(x) = x^2$ sketch the graph o:

a $f(x) = x^2 + c$ when

i $c > 0$ **ii** $c < 0$

b $f(x) = (x + b)^2$ when

i $b > 0$ **ii** $b < 0$

c $f(x) = kx^2$ when

i $k > 1$ **ii** $0 < k < 1$ **iii** $k = -1$

d $f(x) = (ax)^2$ when

i $a > 1$ **ii** $0 < a < 1$ **iii** $a = -1$

2 Sketch the graph of the transformed function if the parabola $y = x^2$ is transformed into

a $y = (x + 2)^2 + 4$ **b** $y = (x - 3)^2 - 1$ **c** $y = (x - 1)^2 + 3$

d $y = -(x + 1)^2 - 2$ **e** $y = 2(x - 1)^2 - 4$

ISBN 97807043220

3 Sketch the graph of the transformed function if the cubic function $y = x^3$ is transformed into

a $y = (x - 1)^3 + 2$ **b** $y = (x - 2)^3 - 3$ **c** $y = -(x + 1)^3 + 4$

d $y = 2(x + 3)^3 - 5$ **e** $y = 3(x - 1)^3 - 2$

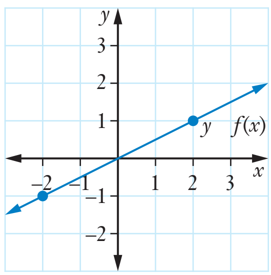
4 A cubic function has stationary points at (6 1) and $(-3, -2)$

a Find the images of these points if the function is transformed to $y = -2f(3x) + 1$.

b Sketch the graph of the transformed function

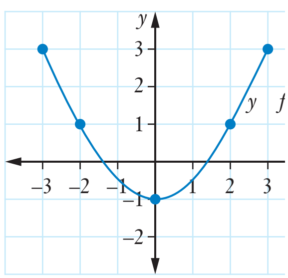
5 Given each function $y = f(x)$ sketch the graph of the transformed functio.

a



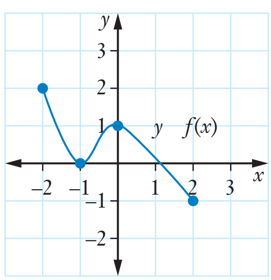
i $y = 3f(x - 1)$ **ii** $y = -f(2x) + 3$

b



i $y = 3f(x + 3) - 2$ **ii** $y = -2f\left(\frac{x}{4}\right) + 3$

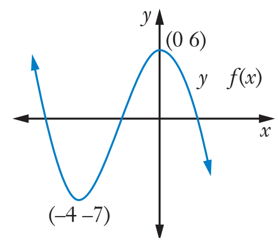
c



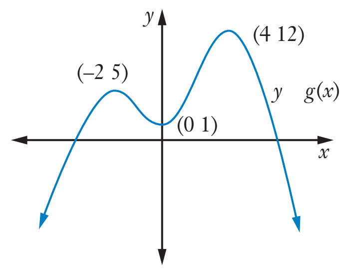
i $y = 2f(-x) - 1$ **ii** $y = -3f(2x + 4) + 2$

ISBN 97807043220

6 For the function $y = f(x)$ with turning points as shown sketch the transformed function if it is vertically dilated with scale factor 3 translated 4 units dow, and horizontally translated 2 units to the left



7 For the function $y = g(x)$ with turning points as shown sketch the graph of the transformed function $y = -g[2(x-1)] - 5$.



8 Sketch the graph of

a $y = -3(x-2)^3 + 1$

b $y = 2e^{x+1} - 4$

c $f(x) = 3\sqrt{x-2} - 1$

d $y = 2|3x| + 4$

e $y = -(3x)^2 + 1$

9 Sketch the graph of

a $y = 3 - 2\ln x$

b $f(x) = -2e^x + 1$

c $y = 1 - (x+1)^3$

d $y = \dfrac{2}{x-1} + 3$

e $y = -2(x-3) + 1$

10 a The coordinates of the image of $(x\ y)$ when $y = f(x)$ is transformed to $y = 3f(x-2) + 1$ are (–3 2. Find the original point $(\ x\ y)$

b Sketch the graph of the original function $y = f(x)$ if $y = 3f(x-2) + 1$ is a cubic function with turning points (–3, 2) and (2, –4)

11 The coordinates of the image point of the vertex $(x\ y)$ of a parabola are (–24 18) when $y = f(x)$ is transformed as shown below. Find the coordinates of the original point $(x\ y)$ and sketch the graph of the original quadratic function

a $y = 3f(x-2) - 5$

b $y = -5f[3(x+1)]$

c $y = 2f(2x-6) - 3$

WS

Solving inequaliies gaphically

2.07 Equations and inequalities

We can use the graphs of transformed functions to solve equation.

EXAMPLE 19

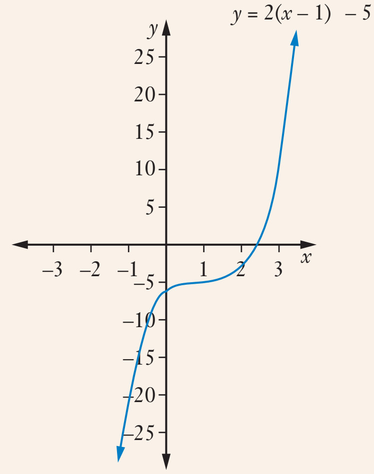
The graph of the cubic function $y = 2(x - 1)^3 - 5$ is shown

a Solve graphically

i $2(x - 1)^3 - 5 = 0$

ii $2(x - 1)^3 - 5 = 10$

b Solve each of the equations in part **a** algebraically.



Solution

a **i** The solution of $2(x - 1)^3 - 5 = 0$ is where $y = 0$ (x-intercepts)

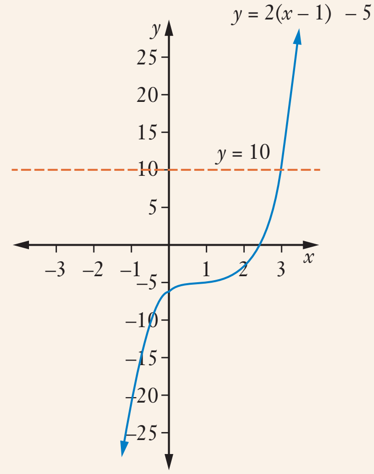
From the graph the x-intercept is 24

The solution is $x = 24$

ii Draw the line $y = 10$ on the graph

The solution of $2(x - 1)^3 - 5 = 10$ is where the line intersects the graph

The solution is $x = 29$



b **i**
$$
\begin{aligned}
2(x - 1)^3 - 5 &= 0 \\
2(x - 1)^3 &= 5 \\
(x - 1)^3 &= 25 \\
x - 1 &= \sqrt[3]{2\ 5} \\
x &= \sqrt[3]{2\ 5} + 1 \\
&= 236
\end{aligned}
$$

ii
$$
\begin{aligned}
2(x - 1)^3 - 5 &= 10 \\
2(x - 1)^3 &= 15 \\
(x - 1)^3 &= 75 \\
x - 1 &= \sqrt[3]{7\ 5} \\
x &= \sqrt[3]{7\ 5} + 1 \\
&= 296
\end{aligned}
$$

ISBN 97807043220

We can use transformed functions to find solutions to practical question.

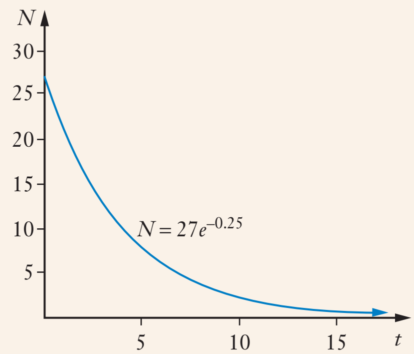
EXAMPLE 20

The graph of $N = 27e^{-025\,t}$ shows the number N of cases of measles over t weeks in a country region

a Use the graph to find the solution to $27e^{-025\,t} = 10$.

b State the meaning of this solution

c Solve the equation algebraically.

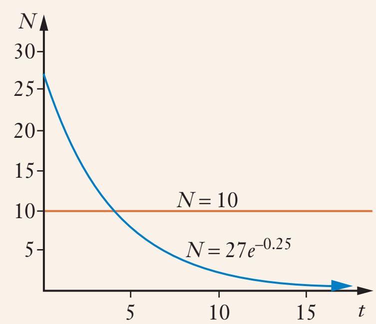


Solution

a Draw the line $N = 10$ on the graph

The solution will be where the line intersects the graph

The solution is $t = 4$



b This solution means that after 4 weeks there will be 10 cases of measles

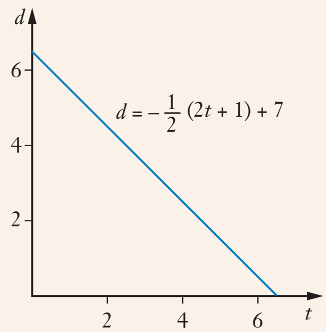
c

$$
\begin{aligned}
27e^{-025\,t} &= 10 \\
e^{-025\,t} &= \frac{10}{27} \\
\ln e^{-025\,t} &= \ln \frac{10}{27} \\
-025\,t &= -099325 \\
t &= \frac{-099325}{-025} \\
&= 397300 \\
&\approx 397
\end{aligned}
$$

We can solve inequalities graphicall.

EXAMPLE 21

a The graph is of the function $d = -\frac{1}{2}(2t + 1) + 7$ where d is the distance (in cm) of a marble at t seconds as it rolls towards a barrier.



Solve graphically and explain the solutions

i $-\frac{1}{2}(2t + 1) + 7 = 4$

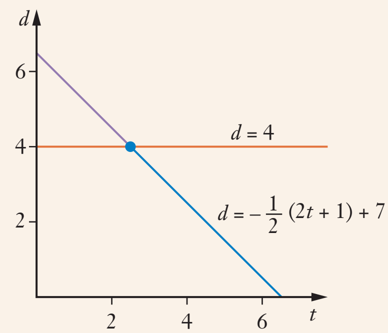
ii $-\frac{1}{2}(2t + 1) + 7 \geq 4$

b Sketch the graph of $y = 2(x + 3)^2 - 5$ and solve graphically

i $2(x + 3)^2 - 5 = 3$ **ii** $2(x + 3)^2 - 5 < 3$

Solution

a Draw the line $d = 4$ across the graph



i From the graph the solution of $-\frac{1}{2}(2t + 1) + 7 = 4$ is $x = 2.5$.

This means that at 25 seconds the marble is 4 cm from the barrier.

ii The solution of $-\frac{1}{2}(2t + 1) + 7 \geq 4$ is all the t values on and above the line $d = 4$ shown in purpl.

For this part of the graph $t \leq 2.5$.

Because $t \geq 0$ (time is never negative) $0 \leq t \leq 25$ is the solution

This means that for the first 25 seconds the marble is 4 cm or more from the barrier.

b The function $y = 2(x + 3)^2 - 5$ is a transformation of $y = x^2$

The vertex of $y = x^2$ is (0, 0.

The image of (0 0) is $(0 - 3, 0 \times 2 - 5) \equiv (-3, -5)$

ISBN 97807043220

For x-intercept $y = 0$

$$0 = 2(x+3)^2 - 5$$

$$5 = 2(x+3)^2$$

$$25 = (x+3)^2$$

$$\pm\sqrt{2\ 5} = x + 3$$

$$\pm\sqrt{2\ 5} - 3 = x$$

$$-14 \quad -46 = x$$

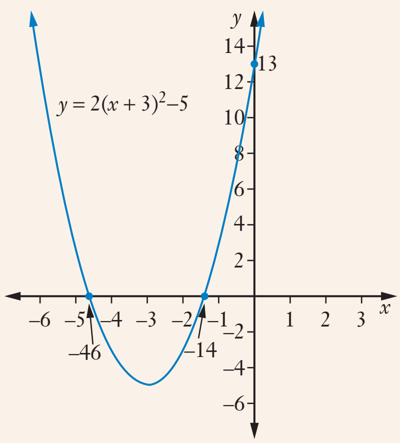
For y-intercept $x = 0$

$$y = 2(0+3)^2 - 5$$

$$= 2(9) - 5$$

$$= 13$$

Sketch the graph using a suitable scale on the axes

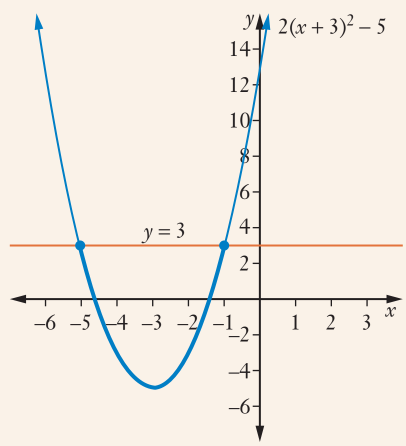


i Draw the line $y = 3$.

From the graph the solution of $2(x+3)^2 - 5 = 3$ is $x = -5, -1$

ii The solution of $2(x+3)^2 - 5 < 3$ is all x values below the line $y = 3$.

From the graph the solution of $2(x+3)^2 - 5 < 3$ is $-5 < x < -1$

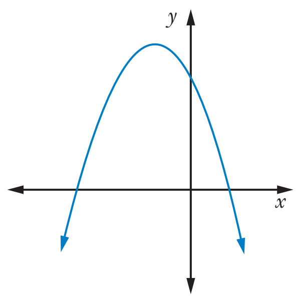


ISBN 97807043220

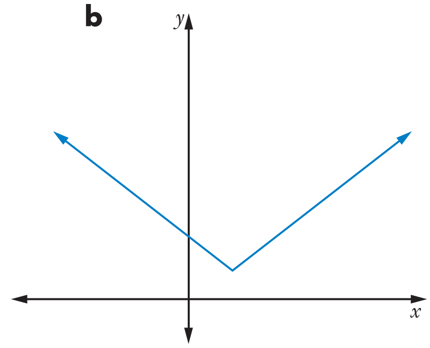
Exercise 2.07 Equations and inequalities

1 For each function $y = f(x)$ state how many solutions there are for the equation $f(x) = 0$.

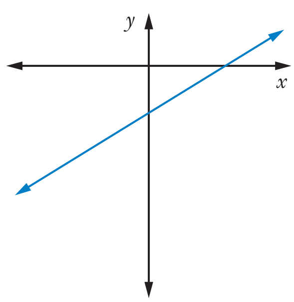
a



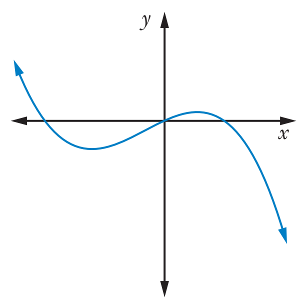
b



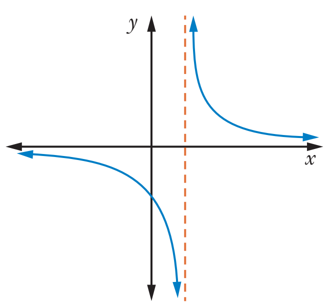
c



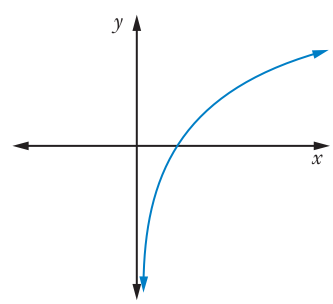
d



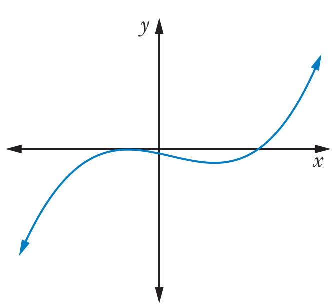
e



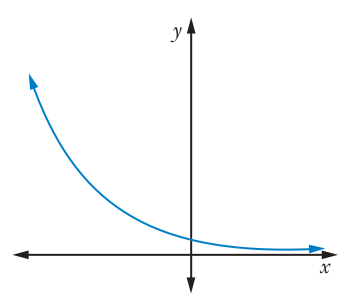
f



g

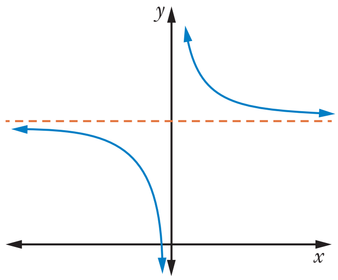


h

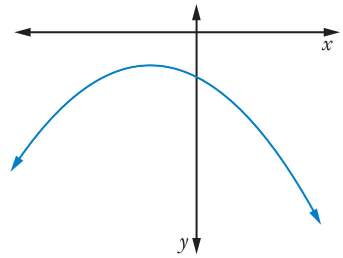


ISBN 97807043220

i



j



2 The graph of the quadratic function $f(x) = -2(x + 1)^2 + 3$ is shown

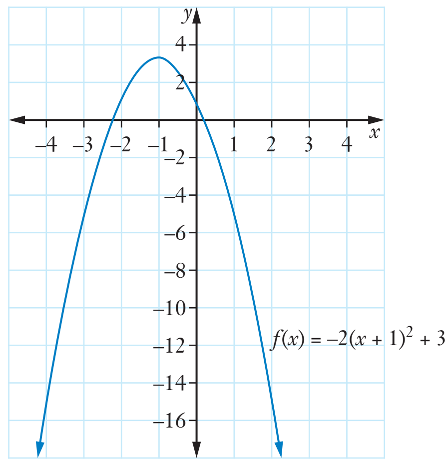
a Solve graphically

i $-2(x + 1)^2 + 3 = 1$

ii $-2(x + 1)^2 + 3 = -2$

iii $-2(x + 1)^2 + 3 = 0$

b Solve $-2(x + 1)^2 + 3 = 0$ algebraically.



3 The graph of the linear function $f(x) = 3(4x - 5) - 2$ is shown
Use the graph to solve

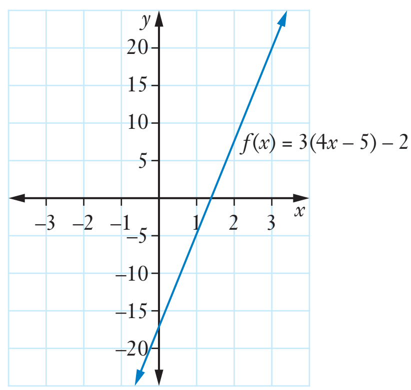
a $3(4x - 5) - 2 = 0$

b $3(4x - 5) - 2 = 5$

c $3(4x - 5) - 2 = -15$

d $3(4x - 5) - 2 > 10$

e $3(4x - 5) - 2 \le 20$



ISBN 97807043220

4 **a** Sketch the graph of the cubic function $y = -(x + 3)^3 + 1$.

b Solve graphically

i $-(x + 3)^3 + 1 = 0$

ii $-(x + 3)^3 + 1 = -10$

iii $-(x + 3)^3 + 1 = -20$

c Solve $-(x + 3)^3 + 1 = 0$ algebraically.

5 **a** Sketch the graph of $y = 3|x - 2| + 4$

b How many solutions does the equation $3|x - 2| + 4 = 1$ have?

c Solve $3|x - 2| + 4 = 10$ graphically and check your solutions algebraically.

6 **a** Sketch the graph of the function $f(x) = \dfrac{2}{x-3} - 4$ showing asymptote.

b Solve the equation $\dfrac{2}{x-3} - 4 = -5$

c Solve $\dfrac{2}{x-3} - 4 = -2$

7 The formula for the area of a garden with side x metres is given by $A = -3(x - 2)^2 + 18$.

a Draw the graph of the area of the garden

b From the graph solve the equation $-3(x - 2)^2 + 18 = 10$.

8 A factory has costs according to the formula $C = 2(x + 1)^2 + 3$ where C stands for costs in \$1000s and x is the number of products made

a Draw the graph of the costs

b Find the factory overhead (cost when no products are made)

c Solve $2(x + 1)^2 + 3 = 20$ from the graph and explain your answer.

9 Loudness in decibels (dB) is given by $\text{dB} = 10\log\left(\dfrac{x}{I}\right)$ where I is a constant

a Sketch the graph of the function given $I = 2$.

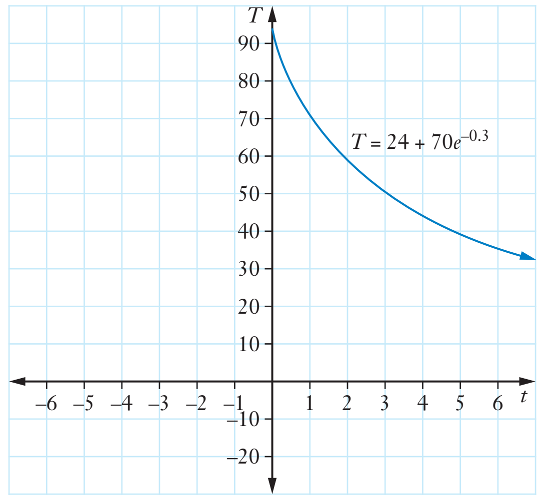
b From the graph solve the equation

i $10\log\left(\dfrac{x}{I}\right) = 5$

ii $10\log\left(\dfrac{x}{I}\right) = 2$

ISBN 97807043220

10 According to Newtons law of coolin, the temperature T of an object as it cools over time t minutes is given by the formula $T = A + Be^{-kt}$ The graph shown is for the formula $T = 24 + 70e^{-0.3t}$ for a metal ball that has been heated and is now cooling down



a From the graph solve these equations and explain what the solutions mea.

i $24 + 70e^{-0.3t} = 50$ **ii** $24 + 70e^{-0.3t} = 30$

b Solve these equations algebraically

i $24 + 70e^{-0.3t} = 80$ **ii** $24 + 70e^{-0.3t} = 26$

c What temperature will the object approach as t becomes large?

Can you give a reason for this?

11 a Sketch the graph of $y = (x - 1)^2 - 2$.

b From the graph solv:

i $(x - 1)^2 - 2 = 2$ **ii** $(x - 1)^2 - 2 \geq 2$ **iii** $(x - 1)^2 - 2 < 2$

12 a Sketch the graph of $f(x) = -(2x + 4)^2 + 1$.

b From the graph solv:

i $-(2x + 4)^2 + 1 = -3$ **ii** $-(2x + 4)^2 + 1 > -3$ **iii** $-(2x + 4)^2 + 1 \leq -3$

2. TEST YOURSELF

Qz Practice quiz

For Questions **1** to **3** choose the correct answer **A** **B** **C** or **D**

1 The function $y = f(x)$ transformed to $y = f(x - 8)$ is

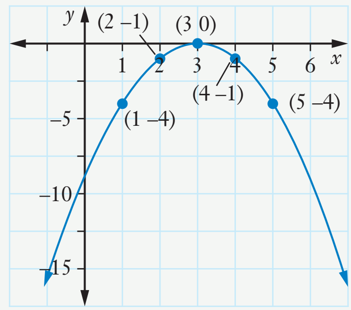
A a vertical translation 8 units up

B a horizontal translation 8 units to the right

C a vertical translation 8 units down

D a horizontal translation 8 units to the left

2 The graph below is a transformation of $y = x^2$
Find its equation



A $y = (-x + 3)^2$ **B** $y = (-x - 3)^2$

C $y = -(x + 3)^2$ **D** $y = -(x - 3)^2$

3 Find the coordinates of the image of $(x\ \ y)$ when the function $y = f(x)$ is transformed to $y = -2f(x + 1) + 4$

A $(x + 1, -2y - 4)$ **B** $(x + 1, -2y + 4)$

C $(x - 1, -2y + 4)$ **D** $(-x + 1, 2y + 4)$

4 **a** Draw the graph of $y = e^{x-1} - 2$.

b Use the graph to solve $e^{x-1} - 2 = 8$.

c Solve $e^{x-1} - 2 = 20$ algebraically.

5 The point (24 36) lies on the graph of $y = f(x)$ Find the coordinates of its image point if the function is transformed to

a $y = 3f(4x) - 1$ **b** $y = f[3(x + 2)] + 4$ **c** $y = 5f(-x) - 3$

d $y = -2f(x + 7) - 3$ **e** $y = -f(2x - 8) + 5$

ISBN 97807043220

6 Find the equation of each transformed function

a $y = x^3$ is translated

i 3 units up **ii** 7 units to the left

b $y = |x|$ is dilated

i vertically with scale factor 3

ii horizontally with scale factor 2

c $f(x) = \ln x$ is dilated vertically with factor 5 and reflected in the y-axis

d $f(x) = \frac{1}{x}$ is reflected in the x-axis and translated 4 units to the right

e $f(x) = 3^x$ is dilated vertically with scale factor 9 dilated horizontally with scale factor $\frac{1}{3}$ and translated 6 units down and 2 units to the right

7 a State the meaning of the constants a b c and k in the function $y = kf(a(x + b)) + c$ and the effect they have on the graph of the function $y = f(x)$

b Describe the effect on the graph of the function if

i $k = -1$ **ii** $a = -1$

8 Show that if $y = x^2$ is dilated vertically with scale factor 3 reflected in the x-axis and translated 1 unit up the transformed function is eve.

9 a Draw the graph of $y = 2(x - 3) + 5$.

b From the graph solv:

i $2(x - 3) + 5 \le 7$ **ii** $2(x - 3) + 5 > 9$

10 The population of a city over time t years is given by $P = 2e^{0.4(t + 1)}$ where P is population in 10 000s

a Sketch the graph of the population

b Use the graph to solve $2e^{0.4(t + 1)} = 5$ and explain the meaning of the solutio.

11 Find the equation of the transformed function if $f(x) = x^4$ is horizontally translated 4 units to the left

12 If (8 2) lies on the graph of $y = f(x)$ find the coordinates of the image of this point when the function is transformed to $y = -4f[2(x + 1)] - 3$.

13 Solve graphically (and also algebraically for part **a**)

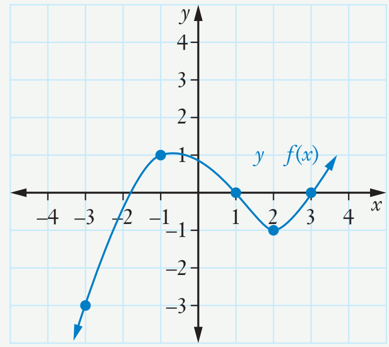
a $2(3x - 6)^2 - 5 = 9$ **b** $2(3x - 6)^2 - 5 > 9$ **c** $2(3x - 6)^2 - 5 \le 9$

14 The function $y = f(x)$ is transformed to $y = -7f(x - 3) - 4$

a Find the coordinates of the image of $(x\ \ y)$

b If the image point is (–3 3, find the value of x and y

15 From the graph of $y = f(x)$ shown draw the graph o:



a $y = 2f(x - 1)$

b $y = -f(x) - 2$

16 By drawing the graph of $y = 2(x + 1)^2 - 8$ solv:

a $2(x + 1)^2 - 8 \le 0$

b $2(x + 1)^2 - 8 > 0$

17 Sketch on the same set of axes

a $y = x^2$ and $y = -4x^2 + 3$

b $y = |x|$ and $y = -|x - 1| + 2$

c $f(x) = e^x$ and $f(x) = \dfrac{e^{x+2}}{2} - 1$

d $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x+2} + 1$

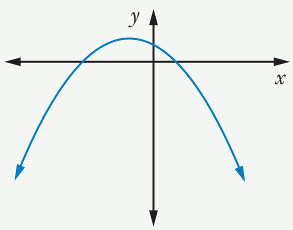
e $y = x^3$ and $y = 2(x - 3)^3 + 1$

f $f(x) = \ln x$ and $f(x) = \ln(-x) + 5$

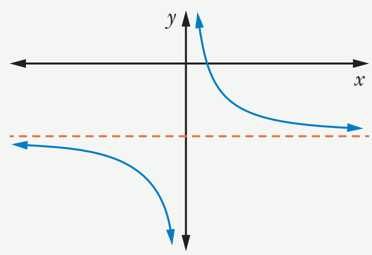
g $y = \sqrt{x}$ and $y = 2\sqrt{x + 4} - 1$

18 Find the number of solutions of $f(x) = 0$ given the graph of each function $y = f(x)$

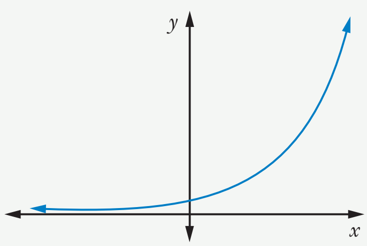
a



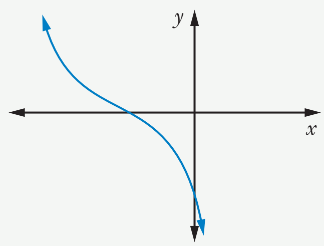
b



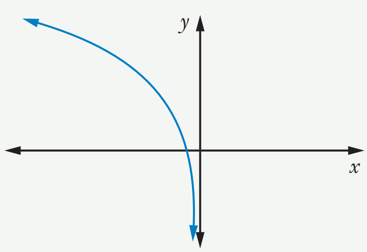
c



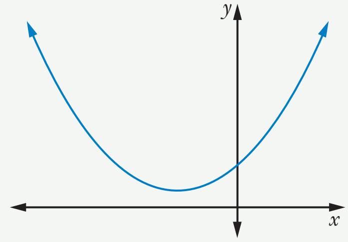
d



e

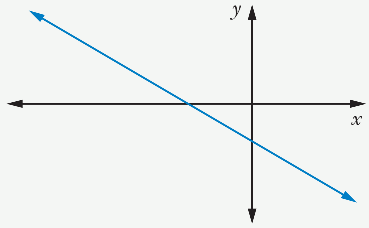


f

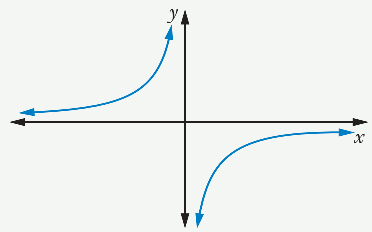


ISBN 97807043220

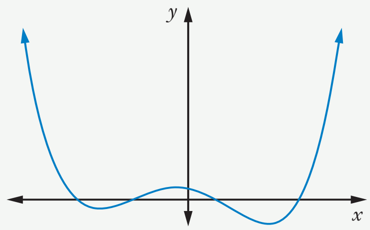
g



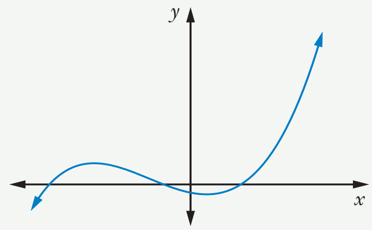
h



i



j



19 **a** Show that $x^2 + y^2 = r^2$ is not a function and describe its graph

b Find 2 functions that together form $x^2 + y^2 = r^2$

c By applying a vertical dilation with scale factor a to both these functions what shape does the combination of these stretched functions make?

20 The point $(x\ \ y)$ lies on the function $y = f(x)$ The image of $(\ x\ \ y)$ is the point $(12\ \ 6)$ when the function is transformed to $y = -6f(2x + 8)$ Find the coordinates of $(\ x\ \ y)$

21 **a** Draw the graph of $y = (x - 2)^2 + 1$.

b From the graph solv:

i $(x - 2)^2 + 1 = 10$ **ii** $(x - 2)^2 + 1 > 10$ **iii** $(x - 2)^2 + 1 \le 10$

22 Point $(x\ \ y)$ lies on $y = f(x)$ Find the image of $(\ x\ \ y)$ if the function is transformed to

a $y = 3f(x + 1) - 5$ **b** $y = -2f[2(x - 6)] + 4$

c $y = 5f(-x) - 3$ **d** $y = -3f(-3x + 9) - 1$

23 State whether the function $y = f(x)$ is stretched or compressed if it is dilated

a vertically with scale factor 7 **b** horizontally with scale factor $\frac{1}{6}$

c horizontally with scale factor 3 **d** vertically with scale factor $\frac{1}{4}$

e horizontally with scale factor $\frac{7}{6}$

24 Find the domain and range of

a $y = 3(x - 7)^2 - 10$ **b** $y = -|x + 1| + 2$ **c** $y = -\frac{2}{x - 3} - 5$

ISBN 97807043220

2. CHALLENGE EXERCISE

1 A ball is thrown into the air from a height of 1 m reaches its maximum height of 3 m after 1 second and after 2 seconds it is 1 m high

a The path of the ball follows the shape of a parabola Find the equation of the height h of the ball over time t seconds

b After how long does the ball fall to the ground?

c Put the function in the form $h = kf[a(t + b)] + c$ and describe the transformations to change $h = t^2$ into this equation

2 a If $(4\ \ -3)$ lies on the function $y = f(x)$ find the coordinates of its image poin.

i P on $y = 3f(x + 3) + 1$

ii Q on $y = -f(2x) - 3$

iii R on $y = f(2x - 2) + 1$

b Find the equation of the linear function passing through P that is perpendicular to QR

c If $y = x$ is transformed into this linear function describe the transformation.

3 a Show that $\dfrac{2x-7}{x-3} = -\dfrac{1}{x-3} + 2$

b Sketch the graph of $y = \dfrac{2x-7}{x-3}$ and state its domain and range

c Solve

i $\dfrac{2x-7}{x-3} \geq 0$ **ii** $\dfrac{2x-7}{x-3} < 2$

4 a If $y = \dfrac{1}{x}$ is dilated horizontally with scale factor 2 explain why the equation of the transformed function is the same as if it was dilated vertically with scale factor 2

b Is this the same result for the function $y = \dfrac{1}{x^2}$? Why?

5 a What is the equation of the axis of symmetry of the quadratic function $f(x) = ax^2 + bx + c$?

b What types of transformations on this function will change the axis of symmetry?

c Find the equation of the axis of symmetry of the quadratic function

i $f(x) = 2(x + 1)^2 - 2$

ii $y = -(x - 3)^2 + 7$

iii $y = k(x + b)^2 + c$

v $y = k(ax + b)^2 + c$

ISBN 97807043220

6 The function $y = \sin x$ in the domain $[0\ 2\pi]$ is transformed by a reflection in the x-axis a vertical dilation scale factor , a horizontal dilation scale factor 2 and a vertical translation 1 unit down

- **a** Find the equation of the transformed function
- **b** State the amplitude period and centre of the transformed functio.

7 The circle $x^2 + 4x + y^2 - 6y + 12 = 0$ is transformed by a vertical translation 3 units down and a horizontal translation 5 units right Find the equation of the transformed circl.

8 The function $y = 2^x$ is transformed to $y = 3(2^{-3x-6}) - 5$ Describe the transformations applied to the function

9 The polynomial $P(x) = x^3 - 3x - 2$ is translated up 2 units and then reflected in the y-axis Find the equation of the transposed polynomia.

3.

TRIGONOMETRIC FUNCTIONS

In ths chapter you wll study the effect of transformtions on rigonomeric funtions and solve trgonometrc equatons graphcally and algebracally.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- apply and understand the effect of ifferent transforations ontrigonomtric fuctions
- solve trgonometrc equatons graphcally and algebracally

TERMINOLOGY

amplitude The height from the centre of a sine or cosine function to the maximum or minimum values (peaks and troughs of its graph respectively) For $y = k \sin ax$ and $y = k \cos ax$ the amplitude is k

centre The mean value of a sine or cosine function that is equidistant from the maximum and minimum values For $y = k \sin ax + c$ and $y = k \cos ax + c$ the centre is c

period The length of one cycle of a periodic function on the x-axis before the function repeats itsel.
For $y = k \sin ax$ and $y = k \cos ax$ the period is $\frac{2\pi}{a}$

phase A horizontal shift (translation.
For $y = k \sin [a(x + b)]$ and $y = k \cos [a(x + b)]$ the phase is b that is the graphs of $y = k \sin ax$ and $y = k \cos ax$ respectively are shifted b units to the left

WS
Trafrming igonomeic uncion

WS
Skeching peiodic uncions ampliude and peiod

WS
Skeching peiodic uncions phae and vetical sht

3.01 Transformations of trigonometric functions

The transformations you studied in Chapter 2 *Transformations of functions* can be applied to the trigonometric functions

Vertical dilations

A vertical dilation of $y = f(x)$ is $y = kf(x)$ with scale factor k

EXAMPLE 1

a Describe the transformation if $y = \cos x$ is transformed to $y = 3 \cos x$

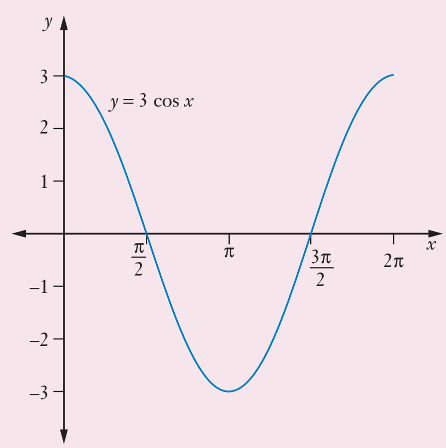
b Sketch the graph of $y = 3 \cos x$ in the domain $[0\ 2\ \pi]$ and state its range

Solution

a $y = 3 \cos x$ is a vertical dilation of $y = \cos x$ with scale factor 3

b A vertical dilation multiplies the y values by 3

$(0, 1)$	becomes	$(0\ 3)$
$\left(\frac{\pi}{2}\ 0\right)$	becomes	$\left(\frac{\pi}{2}\ 0\right)$
$(\pi\ -1)$	becomes	$(\pi\ -3)$
$\left(\frac{3\pi}{2}, 0\right)$	becomes	$\left(\frac{3\pi}{2}, 0\right)$
$(2\pi, 1)$	becomes	$(2\pi, 3)$



The range is $[-3, 3]$.

Notice that $y = 3 \cos x$ has amplitude 3

ISBN 97807043220

Amplitude as a vertical dilation

$y = k \sin x$ or $y = k \cos x$ has **amplitude** k (a vertical dilation with scale factor k)

- If $k > 1$ the function is stretche.
- If $0 < k < 1$ the function is compresse.
- If $k = -1$ the function is reflected in the x-axis

Note y k tan x does not have an ampltude

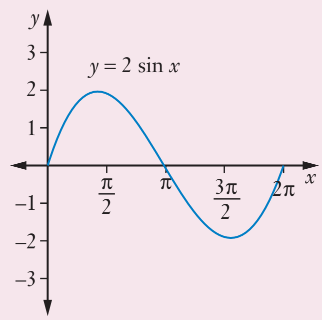
EXAMPLE 2

a Sketch the graph of $f(x) = 2 \sin x$ in the domain [0 2 π]

b Find an equation for a cosine function reflected in the x-axis with amplitude 7

Solution

a This is a vertical dilation of $f(x) = \sin x$ with scale factor 2 (it has amplitude 2)



b $y = \cos x$ is reflected in the x-axis with scale factor $k = -1$

$y = -\cos x$

Amplitude $k = 7$

So $y = -7 \cos x$

Horizontal dilations

A horizontal dilation of $y = f(x)$ is $y = f(ax)$ with scale factor $\dfrac{1}{a}$

EXAMPLE 3

a Describe the transformation if $f(x) = \sin x$ is transformed to $y = \sin 2x$

b Draw the graph of $f(x) = \sin 2x$ in the domain [0 2 π]

ISBN 97807043220

Solution

a $f(x) = \sin 2x$ is a horizontal dilation of $f(x) = \sin x$ with scale factor $\frac{1}{2}$

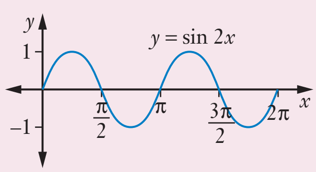
b A horizontal dilation multiplies the x values by $\frac{1}{2}$ (or divides them by 2)

(0 0)	becomes	(0 0)
$\left(\frac{\pi}{2}, 1\right)$	becomes	$\left(\frac{\pi}{4}, 1\right)$
$(\pi, 0)$	becomes	$\left(\frac{\pi}{2}, 0\right)$
$\left(\frac{3\pi}{2}, -1\right)$	becomes	$\left(\frac{3\pi}{4}, -1\right)$
$(2\pi, 0)$	becomes	$(\pi, 0)$

Notice that these image points lie in the domain [0 π] and not [0 2 π]

The **period** of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$

To sketch the function in the domain [, 2 π] we repeat the sine curve from $x = \pi$ to 2π



Notice that a horizontal dilation compresses the graph of $y = \sin x$ which changes its period The function $y = \sin 2x$ has 2 complete sine function cycles in the domain [0 2 π]

Period as a horizontal dilation

$y = \sin ax$ has **period** $\frac{2\pi}{a}$

$y = \cos ax$ has **period** $\frac{2\pi}{a}$

$y = \tan ax$ has **period** $\frac{\pi}{a}$

- If $a > 1$ the function is compressed horizontall.
- If $0 < a < 1$ the function is stretched horizontall.
- If $a = -1$ the function is reflected in the y-axis

ISBN 97807043220

EXAMPLE 4

a Find the period of each function

i $y = \cos x$ **ii** $f(x) = \sin 5x$ **iii** $y = \tan 2x$

b Sketch each graph in the domain [0 2 π]

i $y = \tan \frac{x}{2}$ **ii** $y = \sin(-x)$

Solution

a **i** $y = \cos x$ has period 2π

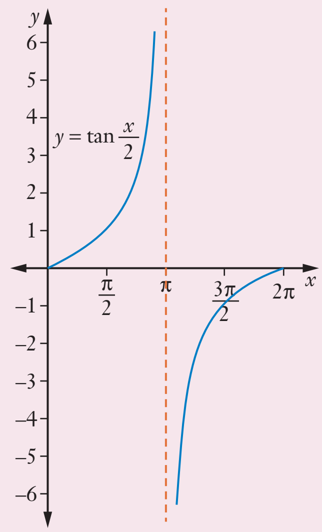
ii $f(x) = \sin 5x$ has period $\frac{2\pi}{5}$

iii $y = \tan 2x$ has period $\frac{\pi}{2}$

b **i** $y = \tan \frac{x}{2}$ is a horizontal dilation of $y = \tan x$

It has period $\frac{\pi}{\frac{1}{2}}$ or 2π

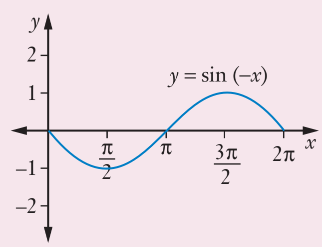
So there will be one cycle of the tan function in the domain [0 2 π]



ii $y = \sin(-x)$ is a reflection of $y = \sin x$ in the y-axis so $a = -1$

This will change the x value. Transforming points in the domain [−2π 0] will give image points in the domain [0 2 π]

(0 0)	becomes	$(0 \times -1, 0) \equiv (0, 0)$
$\left(-\frac{\pi}{2}, -1\right)$	becomes	$\left(\frac{\pi}{2}, -1\right)$
$(-\pi, 0)$	becomes	$(\pi, 0)$
$\left(-\frac{3\pi}{2}, 1\right)$	becomes	$\left(\frac{3\pi}{2}, 1\right)$
$(-2\pi, 0)$	becomes	$(2\pi, 0)$



ISBN 97807043220

Vertical translations

A vertical translation of $y = f(x)$ is $y = f(x) + c$

EXAMPLE 5

Sketch the graph of $y = \cos x + 2$ in the domain $[-\pi \ \ \pi]$

Solution

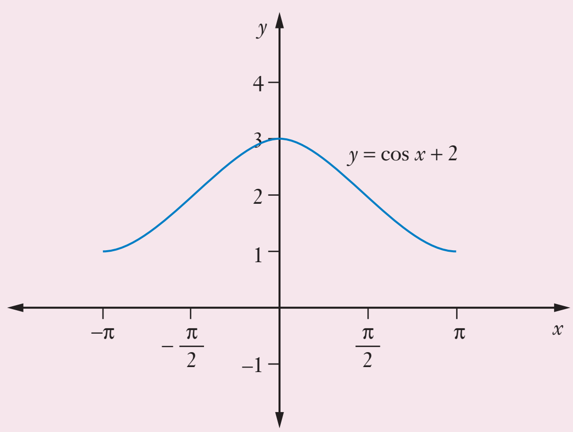
$y = f(x) + c$ is a vertical translation of $y = f(x)$

So $y = \cos x + 2$ is a vertical translation of $y = \cos x$ up 2 units

This changes the y values by adding 2 to each

The domain is $[-\pi \ \ \pi]$

$(-\pi \ \ -1)$	becomes	$(-\pi \ \ -1 + 2) \equiv (-\pi, 1)$
$\left(-\dfrac{\pi}{2}, 0\right)$	becomes	$\left(-\dfrac{\pi}{2}, 2\right)$
$(0, 1)$	becomes	$(0 \ \ 3)$
$\left(\dfrac{\pi}{2}, 0\right)$	becomes	$\left(\dfrac{\pi}{2}, 2\right)$
$(\pi \ \ -1)$	becomes	$(\pi, 1)$



Notice that the centre of the function is 2

Centre as a vertical translation

The **centre** of $y = \sin x + c$ and $y = \cos x + c$ is c

- If $c > 0$ the centre is translated upward.
- If $c < 0$ the centre is translated downward.

ISBN 97807043220

EXAMPLE 6

a Find the centre of the function

i $f(x) = \sin x - 7$ **ii** $y = \cos x + 4$

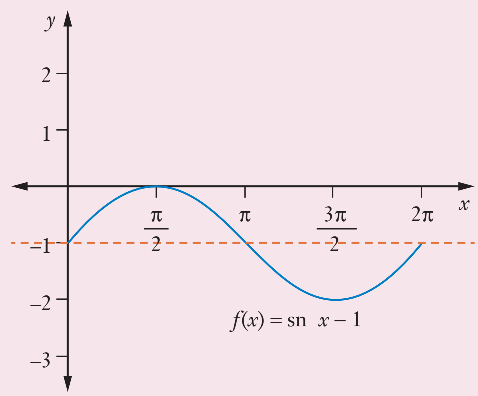
b Sketch the graph of $f(x) = \sin x - 1$.

Solution

a **i** The centre is -7

ii The centre is 4

b This is a vertical translation 1 unit down of $f(x) = \sin x$ The centre is -1



Horizontal translations

A horizontal translation of $y = f(x)$ is given by $y = f(x + b)$

EXAMPLE 7

Sketch the graph of $y = \sin\left(x - \frac{\pi}{2}\right)$ in the domain $[0\ 2\pi]$

Solution

$y = \sin\left(x - \frac{\pi}{2}\right)$ is a horizontal translation of $y = \sin x$ by $\frac{\pi}{2}$ units to the right

We change the x values by adding $\frac{\pi}{2}$

But since we need the *transformed* values of x to be in the domain $[0\ 2\pi]$ our *original* values need to be in the domain $\left[0 - \frac{\pi}{2}\ 2\pi - \frac{\pi}{2}\right] \equiv \left[-\frac{\pi}{2}\ \frac{3\pi}{2}\right]$

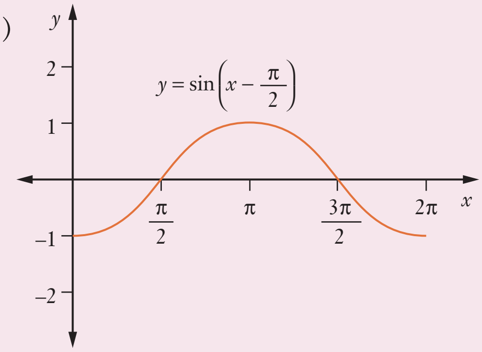
$\left(-\frac{\pi}{2}, -1\right)$ becomes $\left(-\frac{\pi}{2}+\frac{\pi}{2}, -1\right) \equiv (0, -1)$

$(0\ 0)$ becomes $\left(0+\frac{\pi}{2}, 0\right) \equiv \left(\frac{\pi}{2}, 0\right)$

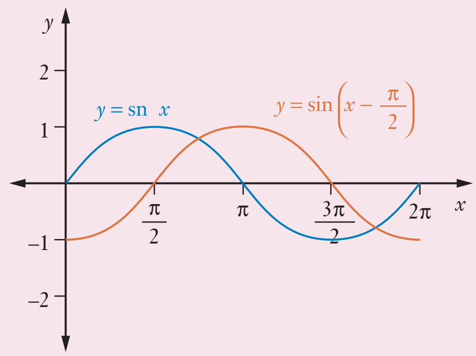
$\left(\frac{\pi}{2}, 1\right)$ becomes $\left(\frac{\pi}{2}+\frac{\pi}{2}, 1\right) \equiv (\pi, 1)$

$(\pi, 0)$ becomes $\left(\pi+\frac{\pi}{2}, 0\right) \equiv \left(\frac{3\pi}{2}, 0\right)$

$\left(\frac{3\pi}{2}, -1\right)$ becomes $\left(\frac{3\pi}{2}+\frac{\pi}{2}, -1\right) \equiv (2\pi\ -1)$



Instead of finding points you could sketch $y = \sin x$ and then shift it $\frac{\pi}{2}$ units to the right



Phase as a horizontal translation

The **phase** of $y = \sin(x + b)$, $y = \cos(x + b)$ and $y = \tan(x + b)$ is b

- If $b > 0$ the phase shift is to the lef.
- If $b < 0$ the phase shift is to the righ.

Phase shi o igonomeic uncion

EXAMPLE 8

a Explain the meaning of $\frac{\pi}{4}$ in the equation $y = \tan\left(x + \frac{\pi}{4}\right)$

b Sketch the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$ in the domain $[0\ 2\ \pi]$

Solution

a The function $y = \tan\left(x + \frac{\pi}{4}\right)$ has a phase of $\frac{\pi}{4}$ (to the left)

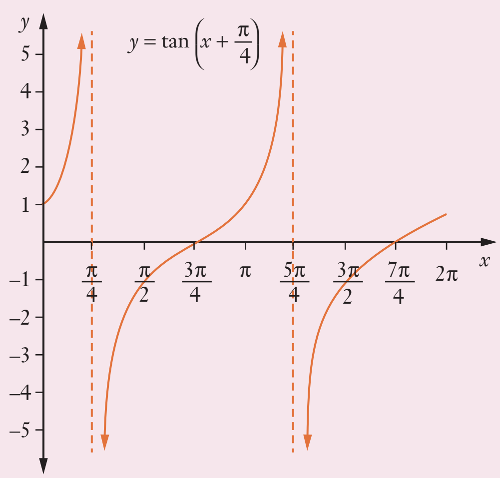
ISBN 97807043220

b To find points on the transformed grap, subtract $\frac{\pi}{4}$ from x values

But since we need the *transformed* values of x to be in the domain $[0\ 2\ \pi]$,

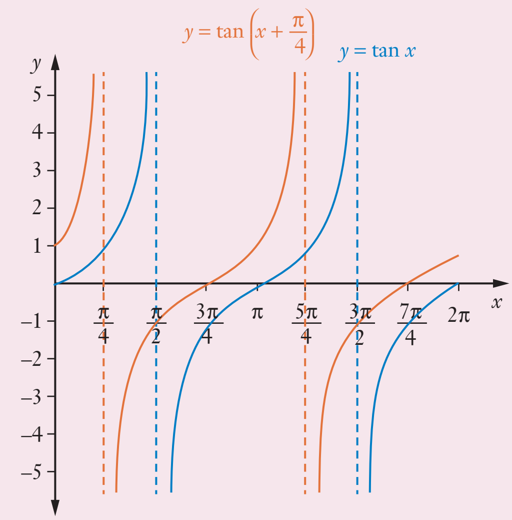
our *original* values need to be in the domain $\left[0+\frac{\pi}{4}\ 2\ \pi+\frac{\pi}{4}\right] \equiv \left[\frac{\pi}{4}\ \frac{9\pi}{4}\right]$

$\left(\frac{\pi}{4}, 1\right)$	becomes	$(0, 1)$
Undefined at $x = \frac{\pi}{2}$	becomes	Undefined at $x = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$
$\left(\frac{3\pi}{4}, -1\right)$	becomes	$\left(\frac{\pi}{2}, -1\right)$
$(\pi, 0)$	becomes	$\left(\frac{3\pi}{4}, 0\right)$
$\left(\frac{5\pi}{4}, 1\right)$	becomes	$(\pi, 1)$
Undefined at $x = \frac{3\pi}{2}$	becomes	Undefined at $x = \frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$
$\left(\frac{7\pi}{4}, -1\right)$	becomes	$\left(\frac{3\pi}{2}, -1\right)$
$(2\pi, 0)$	becomes	$\left(\frac{7\pi}{4}, 0\right)$
$\left(\frac{9\pi}{4}, -1\right)$	becomes	$(2\pi, 1)$



ISBN 97807043220

Instead of finding points you could sketch $y = \tan x$ and then shift it $\frac{\pi}{4}$ units to the left



Exercise 3.01 Transformations of trigonometric functions

1 Describe whether each transformation of a trigonometric function changes its amplitude perio, centre or phae.

a horizontal translation **b** vertical dilation

c horizontal dilation **d** vertical translation

2 Sketch the graph of each function in the domain [0 2 π]

a $y = 5 \sin x$ **b** $f(x) = 2 \tan x$ **c** $y = -\cos x$

d $f(x) = -2 \sin x$ **e** $y = -\tan x$

3 Sketch the graph of each function in the domain [0 2 π]

a $y = \sin x + 1$ **b** $y = \tan x - 2$ **c** $f(x) = \cos x - 3$

4 Sketch the graph of each function in the domain [0 2 π]

a $y = \cos 4x$ **b** $y = \sin \frac{x}{2}$

c $f(x) = \tan 2x$ **d** $y = \tan \frac{x}{4}$

5 Sketch the graph of each function in the domain [0 2 π]

a $y = \cos(x + \pi)$ **b** $y = \tan\left(x - \frac{\pi}{2}\right)$ **c** $y = \sin\left(x - \frac{\pi}{4}\right)$

 ISBN 97807043220

6 Find the equation of the transformation of $y = \sin x$ if the transformed function has

a amplitude 9

b a reflection in the x-axis

c centre –4

d period π

e a phase shift of π units to the right

7 Find the equation of the transformation of $y = \cos x$ if the transformed function has

a amplitude 4

b a phase of $\frac{\pi}{3}$ units

c centre 8

d period $\frac{\pi}{2}$

e a vertical dilation with scale factor 7

8 Find the equation of the transformation of $y = \tan x$ if the transformed function has

a period 2π

b a shift of $\frac{\pi}{6}$ units to the right

c a reflection in the y-axis

9 Sketch each graph in the domain $[-\pi \ \ \pi]$

a $y = 3 \sin x$

b $y = \tan(-x)$

c $f(x) = \cos 2x$

d $y = \sin(x - \pi)$

e $f(x) = -\cos x$

Gaphing igonomeic uncion

Skeching peiodic uncion

3.02 Combined transformations of trigonometric functions

We can put all the information about trigonometric functions togethe.

General equation of trigonometric functions

Function	Amplitude	Period	Phase	Centre
$y = k\sin[a(x+b)] + c$	k	$\frac{2\pi}{a}$	b Shift left if $b > 0$ Shift right if $b < 0$	c Shift up if $c > 0$ Shift down if $c < 0$
$y = k\cos[a(x+b)] + c$	k	$\frac{2\pi}{a}$		
$y = k\tan[a(x+b)] + c$	No amplitude	$\frac{\pi}{a}$		

EXAMPLE 9

a Sketch each function in the domain [0 2 π]

i $y = 4\sin\frac{x}{2} + 1$ **ii** $y = 3\cos\left(x - \frac{\pi}{4}\right)$

b Find the equation of a cosine function that has amplitude 5 period 4 π centre −2 and a phase of 2 units to the left

Solution

a i $y = 4\sin\frac{x}{2} + 1$ has amplitude 4

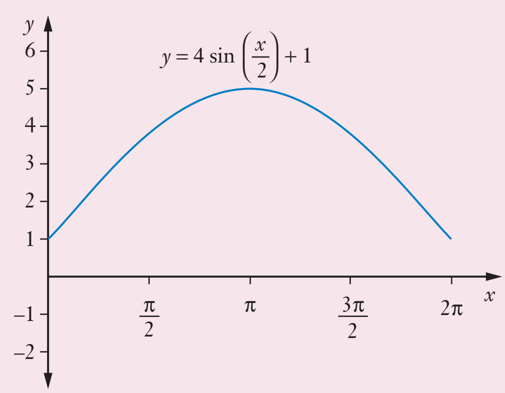
period $\frac{2\pi}{\frac{1}{2}} = 4\pi$ and centre 1

Period 4π means only half the sine function curve will be in the domain [0 2 π]

Centre 1 and amplitude 4 means

Minimum $1 - 4 = -3$

Maximum $1 + 4 = 5$



ISBN 97807043220

ii $y = 3\cos\left(x - \frac{\pi}{4}\right)$ has amplitude 3 period 2π phase $\frac{\pi}{4}$ to the right and centre 0

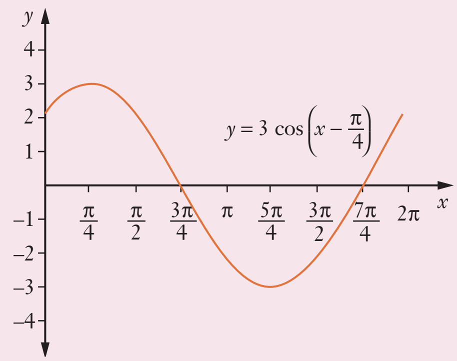
A phase is a horizontal translation so it changes the x values and the domain

Add $\frac{\pi}{4}$ to each x value

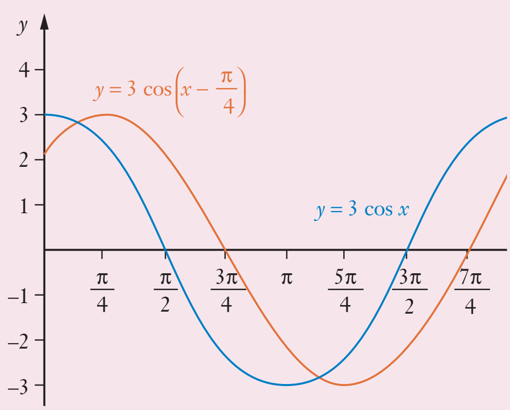
But since we need the *transformed* values of x to be in the domain $[0\ 2\pi]$,

our *original* values need to be in the domain $\left[0 - \frac{\pi}{4}\ 2\pi - \frac{\pi}{4}\right] \equiv \left[-\frac{\pi}{4}\ \frac{7\pi}{4}\right]$

$\left(-\frac{\pi}{4}, 2\ 1\right)$	becomes	$(0, .1)$
$(0\ 3)$	becomes	$\left(\frac{\pi}{4}, 3\right)$
$\left(\frac{\pi}{4}, 2\ 1\right)$	becomes	$\left(\frac{\pi}{2}, 2\ 1\right)$
$\left(\frac{\pi}{2}, 0\right)$	becomes	$\left(\frac{3\pi}{4}, 0\right)$
$\left(\frac{3\pi}{4}, -2\ 1\right)$	becomes	$(\pi\ -21)$
$(\pi\ -3)$	becomes	$\left(\frac{5\pi}{4}, -3\right)$
$\left(\frac{5\pi}{4}, -2\ 1\right)$	becomes	$\left(\frac{3\pi}{2}, -2\ 1\right)$
$\left(\frac{3\pi}{2}, 0\right)$	becomes	$\left(\frac{7\pi}{4}, 0\right)$
$\left(\frac{7\pi}{4}, 2\ 1\right)$	becomes	$(2\pi, .1)$



Alternatively, sketch $y = 3\cos x$ and then shift it $\frac{\pi}{4}$ units to the right



ISBN 97807043220

b $y = k \cos [a(x + b)] + c$

Amplitude $k = 5$

Period $\frac{2\pi}{a} = 4\pi$

$2\pi = 4\pi a$

$\frac{1}{2} = a$

Phase $b = 2$

Centre $c = -2$

The equation is $y = 5 \cos \left[\frac{1}{2}(x + 2)\right] - 2$

Exercise 3.02 Combined transformations of trigonometric functions

1 Sketch the graph of each function in the domain [0 2 π]

a $y = 2 \sin x - 3$

b $y = -\tan 2x$

c $f(x) = \cos\left(x + \frac{\pi}{2}\right) + 1$

d $y = \sin\left(-\frac{x}{2}\right) + 2$

e $f(x) = 3 \cos 2x - 2$

2 **a** Find the equation of the transformed function if $y = \sin x$ is vertically dilated with scale factor 5 horizontally dilated with scale factor $\frac{1}{3}$ vertically translated 6 units down and horizontally translated 5 units to the left

b Describe each transformation as a change in period amplitud, centre or phase of the function

3 Find the equation of the transformed function of $y = \cos x$ if it is

a vertically dilated with scale factor 4 horizontally dilated with scale factor $\frac{1}{6}$ vertically translated 2 units up and horizontally translated $\frac{\pi}{3}$ units to the right

b reflected in the x-axis reflected in the y-axis translated 5 units down and π units to the left

4 Sketch each graph in the domain [$-\pi$ π]

a $y = 3 \sin 2x$

b $y = 2 \tan \frac{x}{2} + 1$

c $f(x) = -2 \cos 3x$

d $y = 5 \sin x - 3$

e $y = \cos(-2x) + 1$

ISBN 97807043220

5 Describe the features of each function in terms of amplitude perio, cente, and phse.

a $y = 3 \tan 4x - 5$

b $y = 8 \cos (x + \pi) - 3$

c $y = 5 \sin [2(x - 3)] + 1$

6 Find the equation of each function

a a sine function with amplitude 7 period π phase of 1 unit to the right and centre -3

b a cosine function with amplitude 1 a reflection in the x-axis period $\frac{2\pi}{5}$ and centre 2

c a tangent function with period 2π a reflection in the x-axis and a phase of 2 units to the left

d $y = \sin x$ with a vertical dilation scale factor 4 a reflection in the y-axis a horizontal dilation scale factor 3 a vertical translation 2 units up and a horizontal translation 5 units to the left

7 Describe the features of $y = k \operatorname{cosec} [a(x + b)] + c$

8 Find the equation of the transformed function if $y = \tan x$ is translated 3 units to the right and then dilated horizontally with scale factor $\frac{1}{4}$

9 The water depth at a harbour entrance is 5 m at low tide and 25 m at high tide The time between each low tide is around 12 hours

a Find the centre of the tidal motion

b What is the amplitude and period?

c Write an equation for the water depth D metres in terms of time t hours as a cosine function

10 Find an equation for blood pressure B as a sine function of tim, t minutes if the maximum blood pressure is 120 and the minimum is 80 with a heart rate of 60 beats per minute

ISBN 97807043220

WS
Solving igonomeic equaion gaphically

WS
Trigonomric equaion

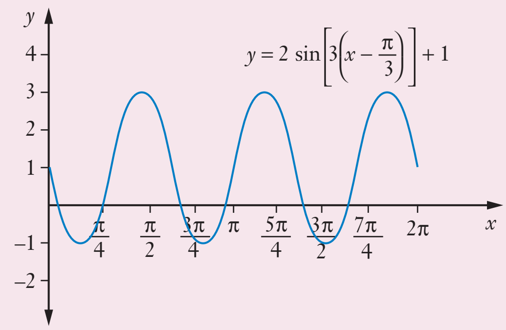
3.03 Trigonometric equations

Graphical solutions

We can use the work on transformations to help solve trigonometric equations graphicall.

EXAMPLE 10

a The graph of the trigonometric function $y = 2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1$ is shown for [0 2 π]



Find the number of solutions to the trigonometric equation $2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1 = 0$ for $[0, 2\pi]$

b **i** Sketch the graphs of $y = \frac{x}{4} - 1$ and $y = 3\cos x - 2$ for $[0, 2\pi]$

ii Find the number of solutions to the equation $3\cos x - 2 = \frac{x}{4} - 1$ for $[0, 2\pi]$

iii Solve the equation graphically.

Solution

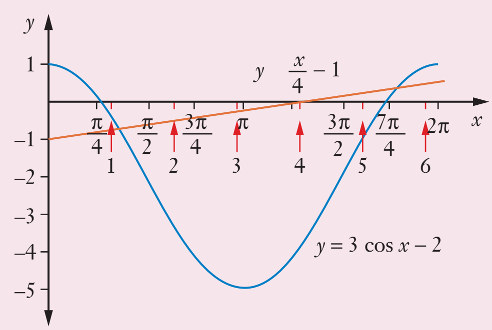
a To solve $2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1 = 0$ graphically, we find the x-intercepts

The function has 6 x-intercepts in the domain [0 2 π]

So the equation has 6 solutions

b **i** $y = 3\cos x - 2$ has amplitude 3 and centre –2

$y = \frac{x}{4} - 1$ is a linear function with x-intercept 4 and y-intercept –1



ISBN 97807043220

ii The solutions to $3 \cos x - 2 = \frac{x}{4} - 1$ are shown by where the graphs $y = 3 \cos x - 2$ and $y = \frac{x}{4} - 1$ intersect

The graphs intersect in 2 places in [0 2 π] so the equation has 2 solutions

iii The graphs intersect just after $x = 1$ and $x = \frac{7\pi}{4} \approx 5.5$.

A precise graph drawn on graph paper or using technology would show that the solutions are $x \approx 11$ and 56

Algebraic solutions

EXAMPLE 11

Solve for [0°, 360°]

a $\sin 3x = \frac{1}{2}$ **b** $\cos (2x - 60°) = \frac{1}{\sqrt{2}}$

Solution

a For the domain [0°, 360°]

$0° \leq x \leq 360°$

$0° \leq 3x \leq 1080°$

So the new domain is [0°, 1080°] (3 revolutions of the circle)

$\sin 3x = \frac{1}{2}$ sn $x > 0$ n 1st and 2nd quadrants

$3x = 30°, 180° - 30°\ \ 360° + 30°\ \ 360° + (180° - 30°)\ 720° + 30°\ \ 720° + (180° - 30°)$

$= 30°, 150°\ \ 390°, 510°, 750°\ \ 870°$

$x = 10°, 50°, 130°, 170°, 250°\ \ 290°$

b $0° \leq x \leq 360°$

$0° \leq 2x \leq 720°$

$-60° \leq 2x - 60° \leq 660°$

So the new domain is [–60° 660°] (2 revolutions of the circle starting at –60°)

$\cos (2x - 60°) = \frac{1}{\sqrt{2}}$ cos $x > 0$ n 1st and 4th quadrants

$2x - 60° = -45°\ \ 45°, 360° - 45°, 360° + 45°$

(720° – 45° is outside the domain)

$= -45°\ \ 45°, 315°\ \ 405°$

$2x = 15°, 105°, 375°\ \ 465°$

$x = 75\ °, 5.5\ \ °, 18.5\ \ °, 23.5\ \ °$

ISBN 97807043220

EXAMPLE 12

Solve each equation for [0 2 π]

a $6 \cos 2x - 3 = 0$

b $\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}$

Solution

a For the domain [0 2 π]

$0 \le \quad x \le 2\pi$

$0 \le 2x \le 4\pi$ so when solving for $2x$ we need to go around the circle twice

$$6 \cos 2x - 3 = 0$$

$$6 \cos 2x = 3$$

$$\cos 2x = \frac{1}{2}$$

cos $x > 0$ n 1st and 4th quadrants

$$2x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 2\pi + 2\pi - \frac{\pi}{3} \text{ as } 0 \le 2x \le 4\pi$$

$$= \frac{\pi}{3} \quad \frac{5\pi}{3} \quad \frac{7\pi}{3} \quad \frac{11\pi}{3}$$

$$x = \frac{\pi}{6} \quad \frac{5\pi}{6} \quad \frac{7\pi}{6} \quad \frac{11\pi}{6}$$

b $0 \le \quad x \quad \le 2\pi$

$$-\frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{7\pi}{4}$$

So the new domain is $\left[-\frac{\pi}{4} \quad \frac{7\pi}{4}\right]$ $\left(1 \text{ revolution of the circle starting at } -\frac{\pi}{4}\right)$

$$\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}$$

tan $x > 0$ n 1st and 3rd quadrants

$$x - \frac{\pi}{4} = \frac{\pi}{3} \quad \pi + \frac{\pi}{3}$$

$$= \frac{\pi}{3} \quad \frac{4\pi}{3}$$

$$x = \frac{\pi}{3} + \frac{\pi}{4} \quad \frac{4\pi}{3} + \frac{\pi}{4}$$

$$= \frac{7\pi}{12} \quad \frac{19\pi}{12}$$

ISBN 97807043220

Exercise 3.03 Trigonometric equations

1 By drawing the graph of $y = 2 \sin 3x$ in the domain $[0\ 2\ \pi]$

a find the number of solutions of $2 \sin 3x = 1$

b solve $2 \sin 3x = 1$ graphically

2 **a** Sketch the graphs of $y = -\cos x + 3$ and $y = x - 1$ for $[0, 2\pi]$

b Solve

i $-\cos x + 3 = x - 1$ **ii** $-\cos x + 3 = 2$

3 Solve for $[0°, 360°]$

a $2 \sin 2x = 1$ **b** $\tan 3x = -1$ **c** $\cos (x + 90°) = \frac{\sqrt{3}}{2}$

d $\tan (x - 45°) = \sqrt{3}$ **e** $\sin (x + 60°) = 0$

4 Solve for $[0\ 2\ \pi]$

a $\tan 2x = \sqrt{3}$ **b** $2 \cos 3x + 1 = 0$ **c** $4 \sin^2\left(x - \frac{\pi}{3}\right) = 3$

d $2 \cos^2 2x - 1 = 0$ **e** $\cos (x + \pi) = 1$

5 Solve for $[-\pi\ \ \pi]$

a $\tan 3x = 1$ **b** $\cos (x + \frac{\pi}{4}) = \frac{1}{\sqrt{2}}$ **c** $\sin 2x = -1$

d $\cos (x - \frac{\pi}{2}) = 0$ **e** $\tan^2 4x = 0$

6 Solve for $[0\ 2\ \pi]$

a $\cos 2\left(x - \frac{\pi}{2}\right) = \frac{1}{2}$ **b** $2 \sin\left(3x + \frac{3\pi}{2}\right) = 1$

7 The function $T = 15 \cos \frac{\pi t}{6} + 20$ models the average monthly temperatures in Nelson Springs starting in Januar.

a Find the amplitude period and centre of the functio.

b Solve $15 \cos \frac{\pi t}{6} + 20 = 35$ and explain the meaning of the solutions

8 A set of tidal waves has a maximum height of 20 m and a minimum height of 6 m The waves break every 10 seconds

a Find the equation of a sine function that describes the motion of the waves

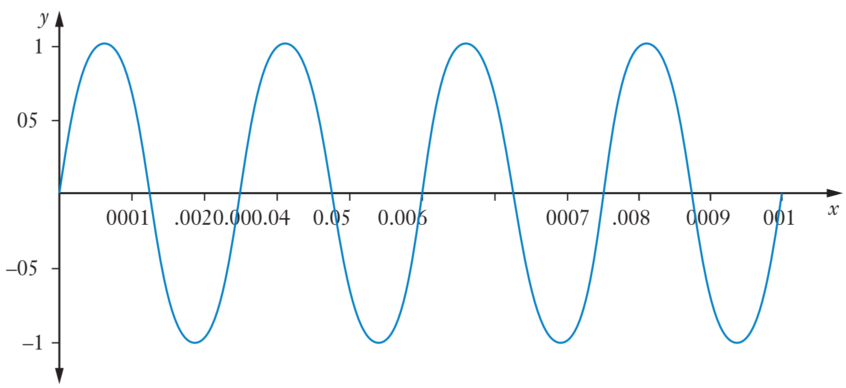
b Find the first 4 times that the waves reach their maximum height

c Find the first time that the waves reach their minimum height

d When will the height of the waves be in the centre?

ISBN 97807043220

9 Sound waves have the shape of sine functions The graph below shows the sound wave that occurs when playing the note A above middle C on a pian. Its equation is $y = \sin(880\pi x)$ where x is time in seconds



a Find the amplitude and period of this sound wave

b Use the graph to solve for [0 .01:

i $\sin(880\pi x) = 05$ **ii** $\sin(880\pi x) = 0$

c Solve algebraically for [0 .01] (to 2 significant figures:

i $\sin(880\pi x) = 05$ **ii** $\sin(880\pi x) = 0$

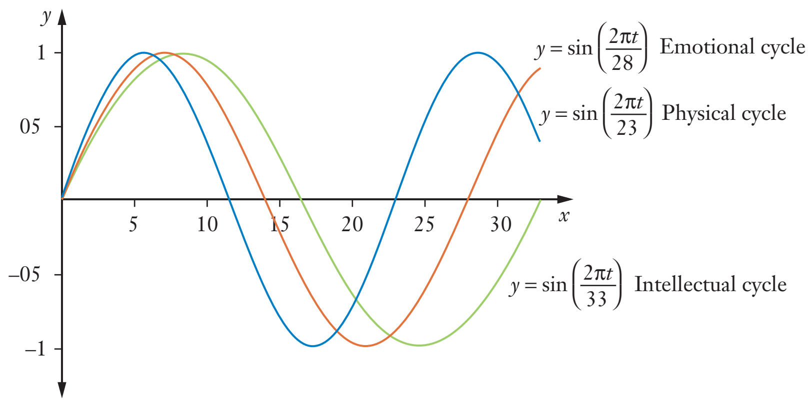
d The higher the amplitude the more volume the sound has (it is louder. (The word amplifier comes from this property.) Find the equation of the note A that is 3 times as loud as the one drawn

e A note that is higher in pitch has a higher frequency (more cycles) than a lower note Draw a rough sketch of a middle C note with the same volume as the A note above C

f The unit of measurement for frequency is hertz (Hz) the number of wave cycles of a sound in 1 second What is the frequency in hertz of note A ?

ISBN 97807043220

10 Biorhythms is a theory that emotional physical and mental activity in humans can be modelled by 3 sine functions physical $y = \sin\dfrac{2\pi t}{23}$ emotional $y = \sin\dfrac{2\pi t}{28}$ and intellectual $y = \sin\dfrac{2\pi t}{33}$ where t is time in days starting from your date of birt. It was first developed by German doctor Wilhelm Fliess in 1878 but became popular in the 1970s when computers were able to chart the 3 biorhythms Their graphs are sketched belw.



a What is the period of each function? What does this mean?

b When do the physical and intellectual graphs intersect?

c When do the emotional and physical graphs intersect?

d Biorhythms are supposed to be at optimal levels when $y = 1$ (maximum points) Estimate the range of times when all 3 biorhythms are near optimal levels together.

11 A vertical spring is pulled down and then let go It bounces back up and down again according to the equation $h = 12 \cos t + 15$ where h is the height of the spring in cm and t is time in seconds

a Describe the significance of the 15 in the equation

b What are the maximum and minimum heights of the spring?

c What is the height of the spring after π seconds?

d At what times will the spring be at its minimum height?

WS

Fomula shet: Funcion Trigonomry and Vts

Qz

Pacice quiz

For Questions **1** to **3** choose the correct answer **A B C** or **D**

1 The function $y = 2 \cos 3x - 7$ has

A amplitude 2 period 3 and centre -7

B amplitude 7 period $\frac{1}{3}$ and centre 2

C amplitude 2 period $\frac{2\pi}{3}$ and centre 7

D amplitude 2 period $\frac{2\pi}{3}$ and centre -7

2 The equation of a function with phase π units to the left is

A $y = \tan(x + \pi)$

B $y = \tan(\pi x)$

C $y = \tan x + \pi$

D $y = \tan(x - \pi)$

3 The solution of $\cos 2x = 1$ in the domain $[0\ 2\ \pi]$ is

A $x = 0, 2\pi$

B $x = 0, \pi, 2\pi$

C $x = \frac{\pi}{2}\ \frac{3\pi}{2}$

D $x = \pi$

4 Sketch the graph of each function in the domain $[0\ 2\ \pi]$

a $y = 3 \cos x$

b $y = \tan \frac{x}{2}$

c $y = \sin x - 2$

d $y = -\sin x$

e $y = 3 \cos 2x - 1$

ISBN 97807043220

5 The function $h = 3\cos\left(\frac{2\pi t}{3}\right) + 10$ shows the water level in a lock in metres over time t hours

a Find the maximum and minimum levels of water and when they occur.

b Solve $3\cos\left(\frac{2\pi t}{3}\right) + 10 = 11$ and explain what the solution means

6 Simplify

a $2\cot^2 x + 2$

b $\tan A \operatorname{cosec} A$

c $(\sec A + \tan A)(\sec A - \tan A)$

d $\sin(180° - x)$

7 Solve for [0 2 π]

a $4\cos^2 x = 3$

b $2\sin 2x = 1$

c $\cos\left(x - \frac{\pi}{2}\right) = -1$

d $\tan^2\left(x + \frac{\pi}{6}\right) = 3$

8 A persons blood pressure has a maximum pressure of 135 and a minimum pressure of 85 and the heartbeat is 70 beats per minute

a Write an equation showing this blood pressure y as a sine function of time t minutes

b Draw a graph showing this function

ISBN 97807043220

9 Solve for [$-\pi$ π]

a $2\cos 2x = 1$ **b** $\tan\left(x - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{3}}$ **c** $\sin\left(x + \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}$

10 The high tide mark on a cliff above a river is 80 m and the low tide mark is 50 m The average time between high tides in the river is approximately 13 hours

a Write the equation for the height of the tides as a sine functio.

b Find the times when the river is halfway between high and low tides

11 Find the equation of each transformed function of $y = \cos x$ that has

a period π **b** amplitude 5 **c** a reflection in the x-axis

d a phase of $\frac{\pi}{6}$ units to the right **e** centre 4

12 a Sketch the graph of $y = 2\sin\frac{x}{2} - 1$ for $[0, 2\pi]$

b From the graph solve $2\sin\frac{x}{2} - 1 = 0$

c Solve $2\sin\frac{x}{2} - 1 = 0$ algebraically.

13 State the amplitude perio, centre and phase of each functin.

a $y = 2\sin 3x - 1$ **b** $y = \cos\left(\frac{x}{2} + \pi\right)$ **c** $y = -3\tan\left(5x - \frac{\pi}{4}\right)$

14 Solve for [0°, 360°]

a $\tan(x + 45°) = 1$ **b** $\sqrt{2}\cos(x - 20°) + 1 = 0$ **c** $3\sin[2(x + 10°)] - 2 = 0$

15 Solve for [–180°, 180°]

a $5\tan 2x = -5$ **b** $\cos[3(x - 30°)] + 1 = 0$

ISBN 97807043220

3. CHALLENGE EXERCISE

1 a Find the amplitude period and phase of the function $y = 2\cos\left(2x - \frac{\pi}{2}\right)$

b Solve $2\cos\left(2x - \frac{\pi}{2}\right) = \sqrt{3}$ for $[0, 2\pi]$

2 Find the equation of

a a cosine function with amplitude 8 period 2π and centre 4

b a sine function with amplitude 2 period $\frac{\pi}{4}$ phase $\frac{\pi}{3}$ units to the right and centre 3

c a tangent function with period 2π and phase $\frac{\pi}{2}$ units to the left

3 Sketch the graph of $y = 3\sec 2x$ for $[0, 2\pi]$

4 Solve $\operatorname{cosec} 2x = \sqrt{2}$ for $[0, 2\pi]$

5 a Find the equation of the transformation of $y = \cos x$ if it has a vertical translation 5 units down a horizontal translation $\frac{\pi}{6}$ units to the right then a vertical stretc, scale factor 4 and a horizontal stretch scale factor .

b Find the amplitude perio, centre and phase of this transformed functin.

ISBN 97807043220

Practice set 1

In Questions **1** to **7** select the correct answer **A B C** or **D**

1 For what values of r does the limiting sum of a geometric series exist?

A $|r| > 1$ **B** $|r| < 1$ **C** $|r| \geq 1$ **D** $|r| \leq 1$

2 The transformation of $y = f(x)$ to $y = 3f(2x)$ is

A Vertical dilation scale factor , horizontal dilation scale factor 2

B Horizontal dilation scale factor 3 vertical dilation scale factor 2

C Vertical dilation scale factor , horizontal dilation scale factor $\frac{1}{2}$

D Horizontal dilation scale factor 3 vertical dilation scale factor $\frac{1}{2}$

3 Simplify $\frac{\sin\theta}{\cos^2\theta\sec\theta}$

A $\cot\theta$ **B** $\tan^2\theta$ **C** $\tan\theta$ **D** $\cot^2\theta$

4 The nth term of the sequence 7 4, 33, … is

A $7n$ **B** 7^{n-1} **C** $7n - 1$ **D** 7^n

5 $y = \cos(x + \pi) + 3$ has a phase shift of

A π units to the right **B** 3 units up

C 3 units down **D** π units to the left

6 Find the limiting sum of $\frac{3}{5} + \frac{2}{5} + \frac{4}{15} +$

A $\frac{1}{5}$ **B** $\frac{9}{10}$ **C** $1\frac{4}{5}$ **D** $\frac{6}{15}$

7 The formula for the sum $1 + 1.03 + 103^2 +\quad + 103^{n-1}$ is

A $S = \frac{103(103^{n-} - 1)}{103 - 1}$ **B** $S = \frac{103(103^{n} - 1)}{103 - 1}$

C $S = \frac{103^{n-} - 1}{103 - 1}$ **D** $S = \frac{103^{n} - 1}{103 - 1}$

8 **a** Sketch the graphs of $y = x^2$ and $y = -(x + 2)^2$ on the same set of axes

b Describe the transformations that changed $y = x^2$ into the transformed function

9 Solve each equation for [0° 360°.

a $\tan 2x + 1 = 0$ **b** $2\cos 3x = 1$ **c** $2\sin(x - 90°) = \sqrt{3}$

d $\tan(x - 180°) = \sqrt{3}$ **e** $2\cos^2(x + 45°) = 1$

ISBN 97807043220

10 Describe the amplitude perio, centre and phase shift of each functin.

a $y = 4\cos 5x$ **b** $y = -2\sin\left(x - \dfrac{\pi}{6}\right) + 1$ **c** $y = \tan\left(\dfrac{x}{4} + 2\right)$

11 Find in index form the 10th term of $\dfrac{3}{4}\ \dfrac{3}{16}\ \dfrac{3}{64}\ \ldots$

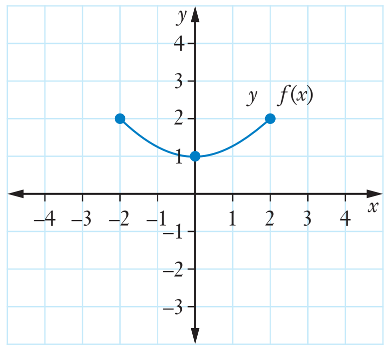
12 Copy the graph $y = f(x)$ and sketch

a $y = 2f(x)$

b $y = f(x) + 1$

c $y = f(x - 2)$

d $y = f(2x)$



13 The 2nd term of a geometric sequence is 52 and the 4th term is 13
Find 2 sequences that satisfy these requirements

14 Find which term -370 is in the series $17 + 8 - 1 -$

15 Describe the transformations on $y = x^3$ if the equation of the transformed function is $y = 4(x - 1)^3 - 3$ and state whether any dilations stretch or compress the graph of the function

16 Solve each equation for [0° 360°.

a $6\sin^2 x - 7\sin x + 2 = 0$ **b** $2\sin[2(x - 30)] = 1$

17 A moving sculpture has a ball on the end of a wire that oscillates backwards and forwards between 2 points The equation of the distance d cm of the ball from the centre of the sculpture at time t seconds is given by $d = 6\cos(2\pi t) + 10$.

a Find the centre of motion and the maximum distance of the ball from this centre in both directions

b How long does it take the ball to complete one complete cycle between the 2 points?

18 Find the equation of the function if $y = \sqrt{x}$ is dilated vertically with scale factor 4 dilated horizontally with scale factor $\dfrac{1}{3}$ translated vertically 1 unit down and translated horizontally 7 units to the left

ISBN 97807043220

19 **a** Find the 50th term of 3 , 1, …

b Calculate the sum of 50 terms

20 The nth term of a series is given by $7n - 3$.

a Find the first 3 terms and the 12th term

b Evaluate the sum of the first 20 terms

c Which term is equal to 200?

21 **a** Sketch the graph of $f(x) = 5\left|x - 2\right| - 3$.

b From the graph solve each equatio.

i $5\left|x - 2\right| - 3 = 2$ **ii** $5\left|x - 2\right| - 3 = 7$ **iii** $5\left|x - 2\right| - 3 = -3$

c State the domain and range of $f(x)$

22 Find the exact value of

a cos 120° **b** sin 300° **c** tan 225°

d cos (–135°) **e** tan 690°

23 The 4th term of an arithmetic sequence is 18 and the 8th term is 62
Find the formula for the general term of the sequence

24 Evaluate x if $\sec x = \operatorname{cosec}(2x - 30°)$

25 Sketch the graph of each function in the domain $[0\ 2\ \pi]$

a $y = -7\cos x$ **b** $y = 2\sin x$ **c** $y = \cos x + 1$

d $y = \tan\left(x + \frac{\pi}{2}\right)$ **e** $y = 3\cos 2x$ **f** $y = -4\sin\frac{x}{2} + 3$

26 Prove each identity.

a $\cot x \sec x = \operatorname{cosec} x$ **b** $\sin^2 x \operatorname{cosec}^2 x - \sin^2 x = \cos^2 x$

27 Find the equation of the transformed function of $y = \sin x$ if the function has

a amplitude 2

b period 4π

c centre –3

d a reflection in the x-axis and amplitude 5

e a phase shift $\frac{\pi}{2}$ units to the left

f amplitude 5 period $6\,\pi$ centre 1 and a phase shift of π units to the right

28 The geometric series $x + x^2 + x^3 + \ldots$ has a sum to infinity of 5 Find the value of x

ISBN 97807043220

29 Solve each equation for $[0\ 2\ \pi]$

a $\cos x = 062$ **b** $\tan^2 x = 1$ **c** $2 \sin x - 1 = 0$

d $\cos x = 0$ **e** $4 \sin^2 x = 3$ **f** $2 \cos 2x + 1 = 0$

30 Find the first value of n for which the sum of the sequence 20 ,08, … is greater than 2485

31 Evaluate $\frac{2}{25} + \frac{2}{125} + \frac{2}{625} +$

32 The average temperature T over t months is given by $T = 20 \cos \frac{\pi t}{6} + 18$ where January is $t = 0$

a Find the amplitude period and centre of the functio. What do these features mean in terms of maximum and minimum temperatures and cycles?

b Find the month with an average temperature of −2°C

c What is the month with the highest average temperature?

d What is the average temperature in September?

e When is the average temperature 18°C?

33 Solve graphically $(x - 1)^2 - 4 \leq 0$

34 Show that $f(x) = 3x^2 - 2$ is an even function

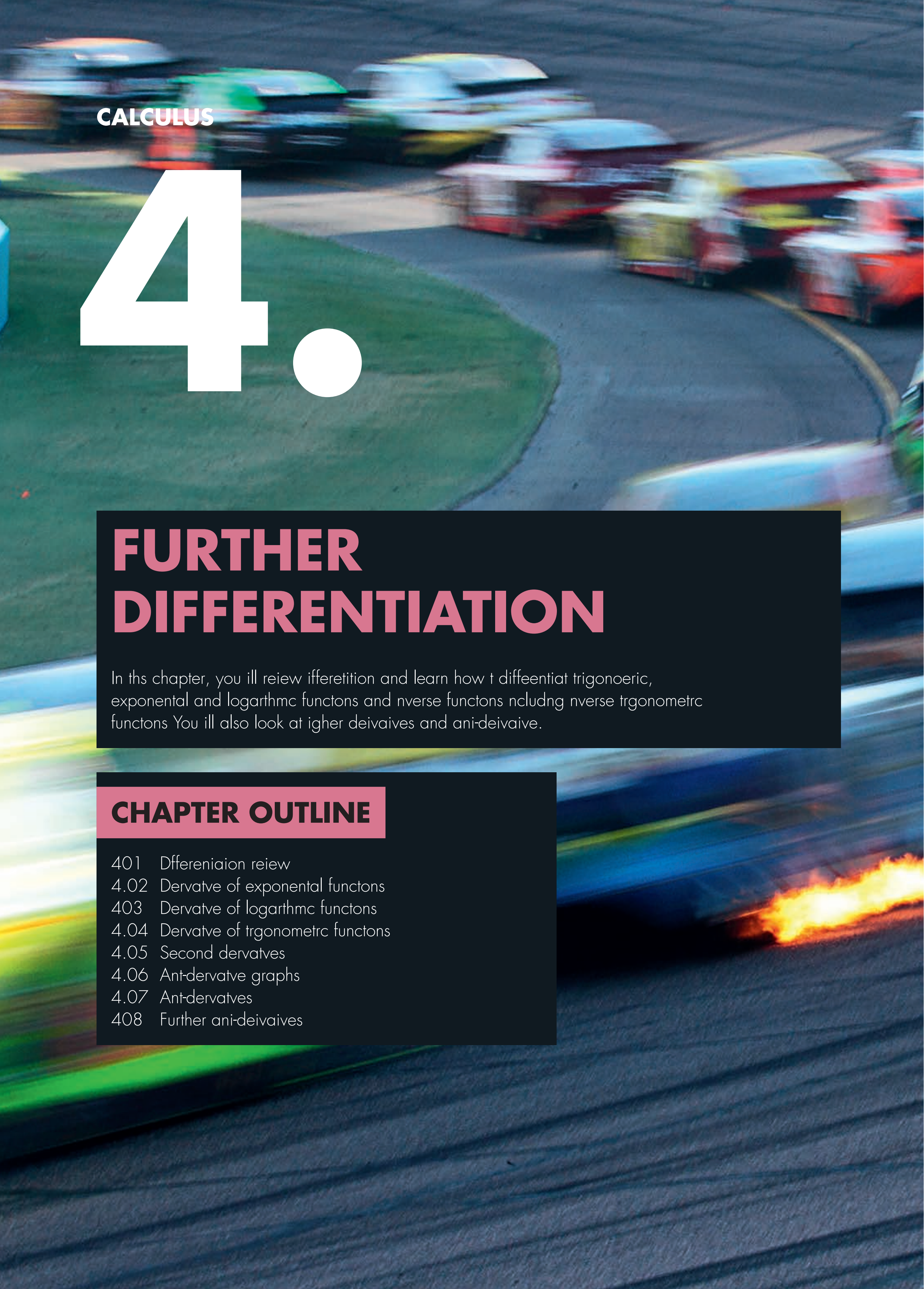
35 **a** Show that log 3 log , log 27 … are terms of an arithmetic sequene.

b Find the exact sum of 20 terms of the sequence

36 **a** Sketch the graph of $y = 3(x - 2)^2 - 4$

b From the graph solve each inequalit.

i $3(x - 2)^2 - 4 \geq 8$ **ii** $3(x - 2)^2 - 4 < 8$



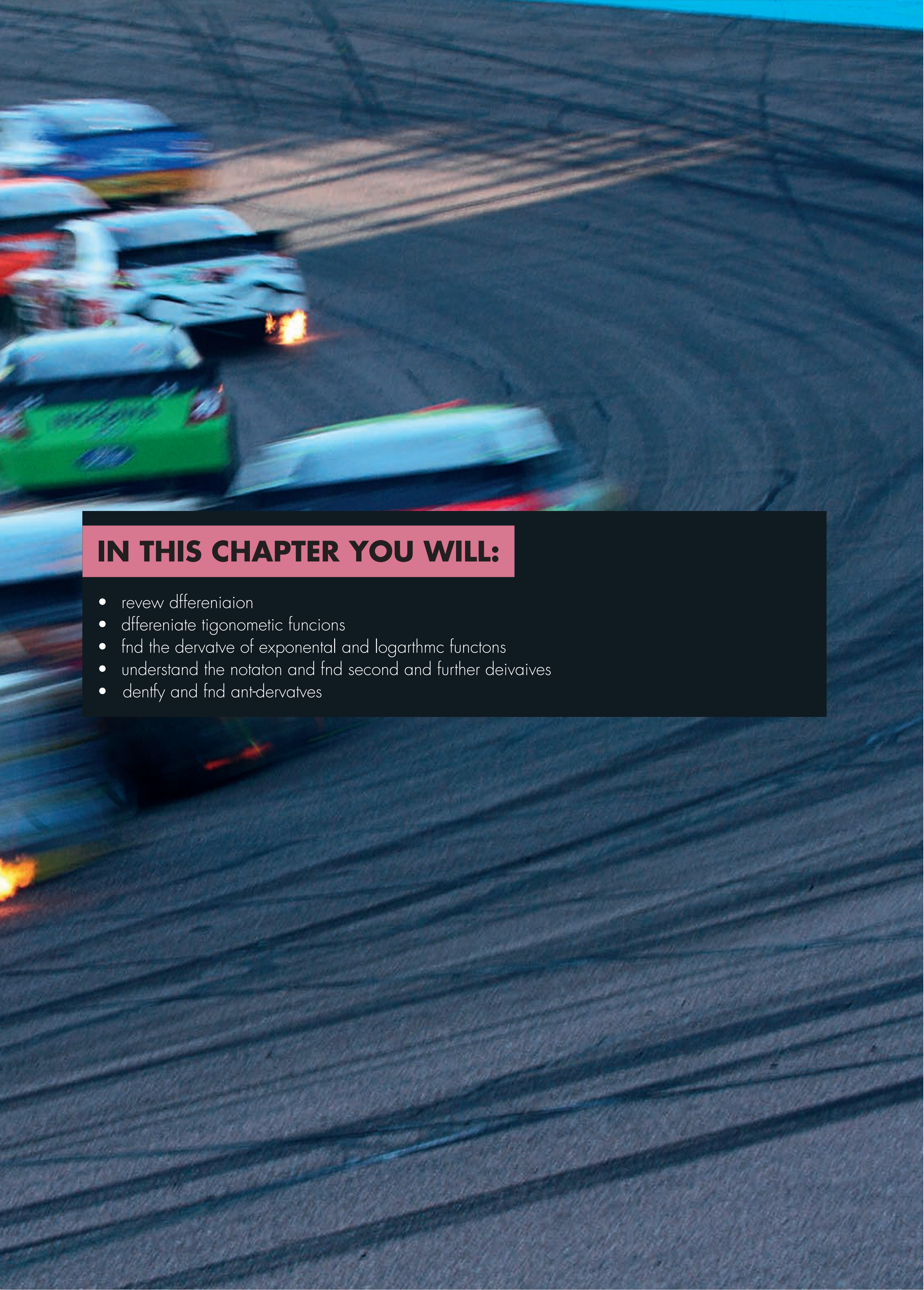
CALCULUS

4.

FURTHER DIFFERENTIATION

In ths chapter, you ill reiew ifferetition and learn how t diffeentiat trigonoeric, exponental and logarthmc functons and nverse functons ncludng nverse trgonometrc functons You ill also look at igher deivaives and ani-deivaive.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- revew dffereniaion
- dffereniate tigonometic funcions
- fnd the dervatve of exponental and logarthmc functons
- understand the notaton and fnd second and further deivaives
- dentfy and fnd ant-dervatves

TERMINOLOGY

anti-derivative: A function $F(x)$ whose derivative is $f(x)$ that i, $F'(x) = f(x)$ Also called the **primitive** or **integral** function

anti-differentiatio: The process of finding the original function given its derivative

second derivative: The derivative $f''(x)$ or $\frac{d^2y}{dx^2}$ the derivative of the derivative $f\ (x)$ or $\frac{dy}{dx}$

4.01 Differentiation review

Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-}$$

Product rule

If $y = uv$ then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ or $y' = u'v + v'u$

Quotient rule

If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ or $y' = \frac{u'v - v'u}{v^2}$

Rates of change

The **average rate of change** between 2 points $(x\ \ y\ )$ and $(x_2\ \ y_2)$ is the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The **instantaneous rate of change** at point $(x\ \ y)$ is the derivative $f'(x)$ or $\frac{dy}{dx}$

EXAMPLE 1

Water is pumped into a dam according to the formula $Q = 3t^3 + 2t^2 + 270$ where Q is the amount of water in kL and t is time in hours Fin:

a the amount of water in the dam after 6 hours

b the average rate at which the water is pumped into the dam between 3 and 6 hours

c the rate of change after 6 hours

ISBN 97807043220

Solution

a $Q = 3t^3 + 2t^2 + 270$

When $t = 6$

$Q = 3(6)^3 + 2(6)^2 + 270$

$= 990$

So there is 990 kL of water in the dam after 6 hours

b When $t = 3$

$Q = 3(3)^3 + 2(3)^2 + 270$

$= 369$

$$\text{Average rate of change} = \frac{Q_2 - Q_1}{t_2 - t_1} = \frac{990 - 369}{6 - 3} = \frac{621}{3} = 207$$

So average rate of change is 207 kL h^{-}

c $\frac{dQ}{dt} = 9t^2 + 4t$

When $t = 6$

$\frac{dQ}{dt} = 9(6)^2 + 4(6)$

$= 348$

So the rate of increase after 6 hours is 348 kL h^{-}

Exercise 4.01 Differentiation review

1 Differentiate each function

a $3x^4 - 2x^3 + 7x - 4$ **b** $2x + 5$ **c** $6x^2 - 3x - 2$

2 Find the derivative $f'(x)$ given $f(x) = 4x^5 + 9x^2$

3 Find $\frac{dx}{dt}$ if $x = 2\pi t^3 - 3t^2 + 1$.

4 Find $f'(-2)$ when $f(x) = 8x^3 + 5x - 2$.

5 Differentiate

a x^{-5} **b** $x^{\frac{2}{3}}$ **c** $\frac{1}{x^2}$

d $\sqrt[4]{x}$ **e** $-\frac{5}{x^4}$

6 Find the derivative of $y = \sqrt[3]{x}$ at the point where $x = 8$.

7 Differentiate

a $(3x-1)^7$ **b** $(x^2-x+2)^3$ **c** $\sqrt{7x-2}$

d $\dfrac{1}{3x-2}$ **e** $\sqrt[3]{x^2-3}$

8 Find the derivative of

a $x^2(x+4)$ **b** $(2x-1)(6x+5)$ **c** $4x(x^2+1)$

d $(4x+3)(x^2-1)^2$ **e** $2x^3\sqrt{x+1}$

9 Differentiate

a $\dfrac{2x+3}{x-5}$ **b** $\dfrac{x^3}{4x-7}$ **c** $\dfrac{x^2+3}{2x-3}$

d $\dfrac{3x+1}{(2x+9)^2}$ **e** $\dfrac{3x+4}{\sqrt{2x-1}}$

10 Find the gradient of the tangent to the curve

a $y=x^2-2x+5$ at the point where $x=-2$

b $f(x)=x^3-3$ at the point $(-1,-4)$

11 Find the gradient of the normal to the curve

a $f(x)=3x^4+x^2-2$ at the point where $x=-1$

b $y=x^2+x-3$ at the point $(-3, 3)$

12 Find the equation of the tangent to the curve

a $y=2x^2-5x-6$ at the point $(3\ \ -3)$

b $y=5x^3-2x^2-x$ at the point where $x=2$

13 Find the equation of the normal to the curve

a $f(x)=x^3+2x^2-3x-5$ at the point $(-1,-1)$

b $y=x^2-3x+1$ at the point where $x=3$

14 For the curve $y=x^2-8x+15$ find any values of x for which $\dfrac{dy}{dx}=0$

15 Find the coordinates of the points at which the curve $y=x^3-2$ has a tangent with gradient 12

16 Function $f(x)=x^2+x-4$ has a tangent parallel to the line $3x+y-4=0$ at point P Find the equation of the tangent at P

17 Find the coordinates of P if the gradient of the tangent to $y=\sqrt{x}$ is $\dfrac{1}{4}$ at point P

ISBN 97807043220

18 For the curve $y = \dfrac{5x-3}{4x+1}$ at the point where $x = 0$ find the equation o:

a the tangent **b** the normal

19 Find a formula for the rate of change $\dfrac{dQ}{dt}$ given

a $Q = 3t^2 + 8$ **b** $Q = \dfrac{2}{t-3}$ **c** $Q = \sqrt[3]{2x+3}$

20 The mass M in kg of a snowball as it rolls down a hill over time t seconds is given by $M = t^2 + 3t + 4$

a Find the average rate at which the mass changes between

i 2 and 5 seconds **ii** 6 and 8 seconds

b Find the rate at which the mass is changing after

i 5 seconds **ii** a minute

21 According to Boyles La, the pressure of a gas in pascals (Pa) is given by the formula $P = \dfrac{k}{V}$ where k is a constant and V is the volume of the gas in m^3 If $k = 250$ for a certain gas find the rate of change in the pressure when $V = 107$

22 The height of a ball in metres is given by $h = 4t - 2t^2$ where t is time in seconds

a Find the height after

i 1 s **ii** 15 s

b How long does it take for the ball to reach the ground?

c Find the velocity of the ball after

i 05 s **ii** 1 s **iii** 2 s

4.02 Derivative of exponential functions

WS
Deivaive o exponenial uncion

You learned how to differentiate $y = e^x$ in Year 1, in Chapte 8, *Exponential and logarithmic functions*

Differentiation rules for e^x

$$\frac{d}{dx} e^x = e^x$$

$$\text{If } y = e^{f(x)} \text{ then } \frac{dy}{dx} = f'(x)\, e^{f(x)}$$

EXAMPLE 2

a If $f(x) = 3e^x$ find the equation of the tangent to the curve at $(, 3 \quad e^2)$

b Differentiate

i x^2e^x **ii** e^{8x} **iii** e^{5x-2}

Solution

a

$$f(x) = 3e^x$$
$$f'(x) = 3e^x$$

At $(2\ 3\ e^2)$

$$f'(2) = 3e^2$$

So $m = 3e^2$

Equation

$$\begin{aligned} y - y \ &= m(x - x\ ) \\ y - 3e^2 &= 3e^2(x-2) \\ &= 3e^2x - 6e^2 \\ y &= 3e^2x - 3e^2 \end{aligned}$$

(or $3e^2x - y - 3e^2 = 0$)

b **i** $y' = u'v + v'u$

where $u = x^2$ and $v = e^x$

$u' = 2x \qquad v' = e^x$

$$\begin{aligned} y' &= 2xe^x + e^xx^2 \\ &= xe^x(2+x) \end{aligned}$$

ii

$$\begin{aligned} \frac{dy}{dx} &= ae^{ax} \\ &= 8e^{8x} \end{aligned}$$

iii

$$\begin{aligned} \frac{dy}{dx} &= f'(x)e^{f\ x} \\ &= 5e^{5x-2} \end{aligned}$$

We can differentiate other exponential function.

EXAMPLE 3

Differentiate 2^x

Solution

$$\begin{aligned} 2 &= e^{\ln 2} \\ 2^x &= (e^{\ln 2})^x \\ &= e^{x \ln 2} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \ln 2\ e^{x \ln 2} \\ &= \ln 2 \times 2^x \\ &= 2^x \ln 2 \end{aligned}$$

ln 2 s a constant

ISBN 97807043220

Derivative of a^x

$$\text{If } y = a^x \text{ then } \frac{dy}{dx} = a^x \ln a$$

The proof of this has the same steps as in the previous example

Exercise 4.02 Derivative of exponential functions

1 Differentiate

a e^{7x}	**b** e^{-x}	**c** e^{6x-2}	**d** $e^{x\ +}$
e $e^{x\ +5x+7}$	**f** e^{5x}	**g** e^{-2x}	**h** e^{10x}
i $e^{2x} + x$	**j** $x^2 + 2x + e^{1-x}$	**k** $(x + e^{4x})^5$	xe^{2x}
m $\dfrac{e^{3x}}{x^2}$	**n** $x^3 e^{5x}$	**o** $\dfrac{e^{2x+1}}{2x+5}$	

2 If $f(x) = e^{3x-2}$ find the exact value of $f'(1)$

3 Find the derivative of

a 3^x **b** 10^x **c** 2^{3x-4}

4 Find the gradient of the tangent to the curve $y = e^{5x}$ at the point where $x = 0$

5 Find the equation of the tangent to the curve $y = e^{2x} - 3x$ at the point (0 1.

6 For the curve $y = e^{3x}$ at the point where $x = 1$ find the exact gradient o:

a the tangent **b** the normal

7 For the curve $y = e^x$ at the point (1 e) find the equation o:

a the tangent **b** the normal

8 Find the equation of the tangent to the curve $y = 4^{x+1}$ at the point (0 4.

9 The population of a city is given by $P = 24\,500e^{0038\,t}$ where t is time in years

a Find the population after

i 5 years **ii** 10 years

b Find the average rate of change in population between

i the 1st and 5th years **ii** the 5th and 10th years

c Find the rate of change in population after

i 5 years **ii** 10 years

10 The displacement of a particle is given by $s = 10e^{2t} - 5t$ cm after t minutes

a Find the average rate of change in displacement between 1 and 5 minutes

b Find the rate of change in displacement after

i 1 minute **ii** 2 minutes **iii** 8 minutes

11 A radioactive substance has a mass of $M = 20e^{-0021\,t}$ in grams over time t years

a Find the initial mass

b Find the mass after 50 years

c Find the average rate of change in mass between 50 and 100 years

d Find the rate of change in mass after

i 50 years **ii** 100 years **iii** 200 years

12 An object moves according to the formula $x = 3e^{2t}$ where x is displacement in cm and t is time in s

a Find the displacement at 5 s

b Find the velocity at 5 s

INVESTIGATION

DERIVATIVE OF A LOGARITHMIC FUNCTION

Draw the derivative (gradient) function of a logarithm function

What is the shape of the derivative function?

WS

Deivaive o logaihmic uncion

WS

Exponenial and logaihmic uncion

4.03 Derivative of logarithmic functions

Logarithm rules

$$\text{If } y = a^x \text{ then } \log_a y = x$$

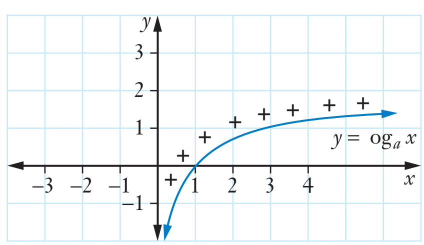
$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

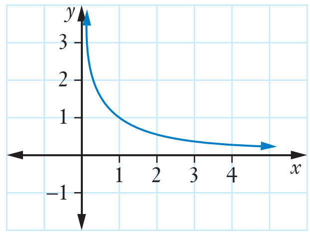
$$\log_a x = \frac{\log_b x}{\log_b a}$$

To find the derivative of a logarithmic functio, notice that the gradient of the function is always positive but is decreasing



ISBN 97807043220

The derivative function of a logarithmic function is a hyperbola



There is a special rule for $y = \ln x$

Derivative of $y = \ln x$

$$\text{If } y = \ln x \text{ then } \frac{dy}{dx} = \frac{1}{x} \text{ where } x > 0.$$

Proof

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Given $y = \ln x = \log_e x$

Then $x = e^y$

$$\frac{dx}{dy} = e^y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{e^y}$$

$$= \frac{1}{x}$$

EXAMPLE 4

a Differentiate $(\ln x + 1)^3$

b Find the equation of the tangent to the curve $y = \ln x$ at the point (3, ln 3.

Solution

a $(\ln x + 1)^3$ is a composite function in the form $y = [f(x)]^n$

$$\frac{dy}{dx} = f'(x)\, nf(x)^{n-1}$$
$$= \frac{1}{x} \times 3(\ln x + 1)^2$$
$$= \frac{3(\ln x + 1)^2}{x}$$

b $\dfrac{dy}{dx} = \dfrac{1}{x}$

At (3 ln 3)

$$\frac{dy}{dx} = \frac{1}{3}$$

So $m = \dfrac{1}{3}$

Equation

$$y - y\ = m(x - x\ )$$
$$y - \ln 3 = \frac{1}{3}(x - 3)$$
$$3y - 3\ln 3 = x - 3$$
$$0 = x - 3y - 3 + 3\ln 3$$

Chain rule

If $y = \ln f(x)$ then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$ where $f(x) > 0$

Proof

$y = \ln f(x)$ is a composite function

Let $y = \ln u$ and $u = f(x)$

$$\frac{dy}{du} = \frac{1}{u} \quad \text{and} \quad \frac{du}{dx} = f'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times f'(x)$$
$$= \frac{1}{f(x)} \times f'(x)$$
$$= \frac{f'(x)}{f(x)}$$

ISBN 97807043220

EXAMPLE 5

a Differentiate

i $\ln(x^2 - 3x + 1)$ **ii** $\ln\left(\dfrac{x+1}{3x-4}\right)$

b Find the gradient of the normal to the curve $y = \ln(x^3 - 5)$ at the point where $x = 2$.

Solution

a **i** $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

$$= \frac{2x-3}{x^2-3x+1}$$

ii It is easier to simplify first using log laws

$$y = \ln\left(\frac{x+1}{3x-4}\right)$$

$$= \ln(x+1) - \ln(3x-4)$$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$= \frac{1}{x+1} - \frac{3}{3x-4}$$

$$= \frac{1(3x-4)}{(x+1)(3x-4)} - \frac{3(x+1)}{(3x-4)(x+1)}$$

$$= \frac{3x-4-3(x+1)}{(x+1)(3x-4)}$$

$$= \frac{3x-4-3x-3}{(x+1)(3x-4)}$$

$$= \frac{-7}{(x+1)(3x-4)}$$

b $\dfrac{dy}{dx} = \dfrac{3x^2}{x^3-5}$

When $x = 2$

$$\frac{dy}{dx} = \frac{3(2)^2}{2^3-5}$$

$$= 4$$

$$m = 4$$

The normal is perpendicular to the tangent

$$m\,m_2 = -1$$

$$4m_2 = -1$$

$$m_2 = -\frac{1}{4}$$

ISBN 97807043220

We can differentiate logarithmic functions with a different bas, a

EXAMPLE 6

Differentiate $y = \log_2 x$

Solution

$$y = \log_2 x$$
$$= \frac{\ln x}{\ln 2} \text{ using the change of base law}$$
$$= \frac{1}{\ln 2} \ln x$$

$$\frac{dy}{dx} = \frac{1}{\ln 2} \times \frac{1}{x}$$
$$= \frac{1}{x \ln 2}$$

ln 2 s a constant

Derivative of $\log_a x$

$$\text{If } y = \log_a x \text{ then } \frac{dy}{dx} = \frac{1}{x \ln a}$$

The proof of this has the same steps as in the above example

Exercise 4.03 Derivative of logarithmic functions

1 Differentiate

a	$x + \ln x$	**b**	$1 - \ln 3x$	**c**	$\ln (3x + 1)$
d	$\ln (x^2 - 4)$	**e**	$\ln (5x^3 + 3x - 9)$	**f**	$\ln (5x + 1) + x^2$
g	$3x^2 + 5x - 5 + \ln 4x$	**h**	$\ln (8x - 9) + 2$	**i**	$\ln (2x + 4)(3x - 1)$
j	$\ln\left(\frac{4x+1}{2x-7}\right)$	**k**	$(1 + \ln x)^5$		$(\ln x - x)^9$
m	$(\ln x)^4$	**n**	$(x^2 + \ln x)^6$	**o**	$x \ln x$
p	$\frac{\ln x}{x}$	**q**	$(2x + 1)\ln x$	**r**	$x^3 \ln (x + 1)$
s	$\ln (\ln x)$	**t**	$\frac{\ln x}{x - 2}$	**u**	$\frac{e^{2x}}{\ln x}$
v	$e^x \ln x$	**w**	$5(\ln x)^2$		

2 Find $f'(1)$ if $f(x) = \ln \sqrt{2 - x}$

3 Find the derivative of $\log_{10} x$

4 Find the equation of the tangent to the curve $y = \ln x$ at the point (2 ln 2.

ISBN 97807043220

5 Find the equation of the tangent to the curve $y = \ln (x - 1)$ at the point where $x = 2$.

6 Find the gradient of the normal to the curve $y = \ln (x^4 + x)$ at the point (1 ln 2.

7 Find the exact equation of the normal to the curve $y = \ln x$ at the point where $x = 5$.

8 Find the equation of the tangent to the curve $y = \ln (5x + 4)$ at the point where $x = 3$.

9 Find the derivative of $\log_3 (2x + 5)$

10 Find the equation of the normal to the curve $y = \log_2 x$ at the point where $x = 2$.

11 The formula for the time t in years for kangaroo population growth on Kangaroo Island is given by $t = \dfrac{\ln\left(\dfrac{P}{20\,000}\right)}{0021}$

- **a** What is the initial population?
- **b** Find correct to one decimal place the time it takes for the population to grow to
  - **i** 25 000
  - **ii** 50 000
- **c** Change the subject of the equation to P
- **d** Find correct to the nearest whole number the average rate of change in population between 2 and 5 years
- **e** Find correct to the nearest whole number the rate at which the population is growing after
  - **i** 3 years
  - **ii** 5 years
  - **iii** 10 years

CLASS INVESTIGATION

DERIVATIVE OF TRIGONOMETRIC FUNCTIONS

1 Draw the derivative (gradient) function of sine cosine and tangent function.

What is the shape of the derivative function of each graph?

2 By substituting values of x in radians close to 0 find approximations to $\lim_{x \to 0} \dfrac{\sin x}{x}$ $\lim_{x \to 0} \dfrac{\tan x}{x}$ and $\lim_{x \to 0} \dfrac{\cos x}{x}$

3 Differentiate by first principles to find the derivative of each trigonometric function using the above limits The sine function will use the EXT trigonometric identity $\sin (A + B) = \sin A \cos B + \cos A \sin B$

ISBN 97807043220

Deivaive o igonomeic uncion

Trigonomric uncions and gadien

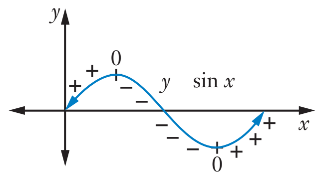
Futhr igonomeic equaion

Difretiting igonomeic uncion

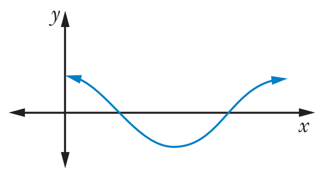
4.04 Derivative of trigonometric functions

Derivative of sin *x*

We can sketch the derivative (gradient) function of $y = \sin x$



The sketch of the gradient function is $y = \cos x$



Derivative of sin *x*

$$\text{If } y = \sin x \text{ then } \frac{dy}{dx} = \cos x$$

Proof

This proof uses trigonometric results from the investigation on the previous page

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} \\
&= \sin x \lim_{h\to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \lim_{h\to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \times 0 + \cos x \times 1 \\
&= \cos x
\end{aligned}$$

ISBN 97807043220

EXAMPLE 7

a Differentiate $y = x \sin x$

b Find the equation of the tangent to the curve $y = \sin x$ at the point (π, 0.

Solution

a $y = x \sin x$ is in the form $y = uv$

where $u = x$ and $v = \sin x$

$u' = 1$ and $v' = \cos x$

$y' = u'v + v'u$

$= 1 \times \sin x + \cos x \times x$

$= \sin x + x \cos x$

b $\dfrac{dy}{dx} = \cos x$

At (π, 0)

$\dfrac{dy}{dx} = \cos \pi$

$= -1$

So $m = -1$

Equation

$y - y \ = m(x - x\ )$

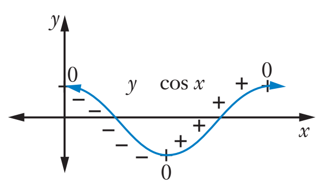
$y - 0 = -1(x - \pi)$

$y = -x + \pi$

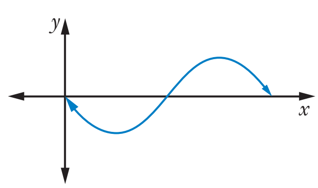
or $x + y - \pi = 0$

Derivative of cos *x*

We can sketch the derivative (gradient) function of $y = \cos x$



The sketch of the gradient function below is $y = -\sin x$



Derivative of cos *x*

If $y = \cos x$ then $\dfrac{dy}{dx} = -\sin x$

ISBN 97807043220

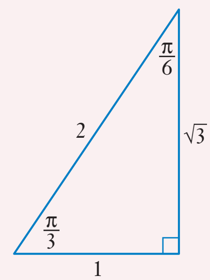
You can prove this in a similar way to the derivative of $y = \sin x$ A simpler proof involves changing $\cos x$ into $\sin\left(\frac{\pi}{2} - x\right)$ and using the derivative of $y = \sin x$

EXAMPLE 8

a Find the derivative of $y = \cos x$ at the point where $x = \frac{\pi}{3}$

b Find the equation of the tangent to $y = \cos x$ at this point

Solution



a $\frac{dy}{dx} = -\sin x$

When $x = \frac{\pi}{3}$

$$\frac{dy}{dx} = -\sin\frac{\pi}{3}$$

$$= -\frac{\sqrt{3}}{2}$$

b When $x = \frac{\pi}{3}$

$$y = \cos\frac{\pi}{3}$$

$$= \frac{1}{2}$$

Equation

$$y - y \ = m(x - x \ )$$

$$y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$

$$2y - 1 = -\sqrt{3}\left(x - \frac{\pi}{3}\right) \qquad \text{(multiplying both sides by 2)}$$

$$= -\sqrt{3}x + \frac{\pi\sqrt{3}}{3}$$

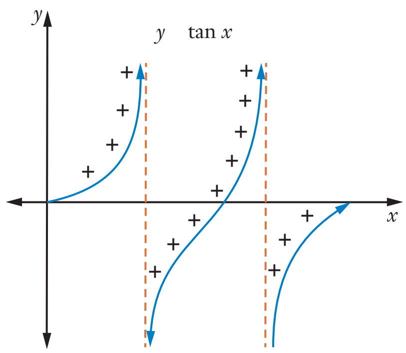
$$6y - 3 = -3\sqrt{3}x + \pi\sqrt{3} \qquad \text{(multiplying both sides by 3)}$$

$$3\sqrt{3}x + 6y - 3 - \pi\sqrt{3} = 0$$

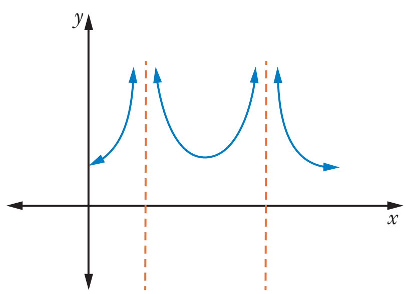
ISBN 97807043220

Derivative of tan x

We can sketch the derivative (gradient) function of $y = \tan x$ Notice that the gradient function will have asymptotes in the same place as the original graph because this is where the tangent is vertical and the gradient is undefined



The gradient function is $y = \sec^2 x$, where $\sec x = \dfrac{1}{\cos x}$



Derivative of tan x

$$\text{If } y = \tan x \text{ then } \frac{dy}{dx} = \sec^2 x$$

You can prove this in a similar way to the derivative of $y = \sin x$ A simpler proof involves changing $\tan x$ into $\dfrac{\sin x}{\cos x}$ and using the quotient rule

ISBN 97807043220

EXAMPLE 9

a Differentiate $y = \dfrac{\tan x}{3x^2}$

b Find the gradient of the tangent to the curve $f(x) = \tan x$ at the point where $x = \dfrac{\pi}{4}$

Solution

a $y = \dfrac{\tan x}{3x^2}$ is in the form $y = \dfrac{u}{v}$

$u = \tan x$ and $v = 3x^2$

$u' = \sec^2 x$ $\quad v' = 6x$

$$y' = \frac{u'v - v'u}{u^2}$$

$$= \frac{\sec^2 x \times 3x^2 - 6x \times \tan x}{(3x^2)^2}$$

$$= \frac{3x(x\sec^2 x - 2\tan x)}{9x^4}$$

$$= \frac{x\sec^2 x - 2\tan x}{3x^3}$$

b $\dfrac{dy}{dx} = \sec^2 x$

At $x = \dfrac{\pi}{4}$

$$\frac{dy}{dx} = \sec^2 \frac{\pi}{4}$$

$$= (\sqrt{2})^2$$

$$= 2$$

Chain rule

If $y = \sin f(x)$ then $\dfrac{dy}{dx} = f'(x) \cos f(x)$

If $y = \cos f(x)$ then $\dfrac{dy}{dx} = -f'(x) \sin f(x)$

If $y = \tan f(x)$ then $\dfrac{dy}{dx} = f'(x) \sec^2 f(x)$

ISBN 97807043220

Here is the proof for $y = \sin f(x)$ The others are similr.

Proof

$y = \sin f(x)$ is a composite function

where $y = \sin u$ and $u = f(x)$

$$\frac{dy}{du} = \cos u \text{ and } \frac{du}{dx} = f'(x)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \cos u \times f'(x) \\ &= f'(x) \cos u \\ &= f'(x) \cos f(x)\end{aligned}$$

EXAMPLE 10

a Differentiate each function

i $y = \sin 7x$ **ii** $y = \cos\left(4x^3 + \frac{\pi}{3}\right)$ **iii** $y = \tan(5x - \pi)$

b Find the gradient of the normal to the curve $f(x) = \cos \frac{x}{2}$ at the point where $x = \pi$

Solution

a **i** $\frac{dy}{dx} = f'(x) \cos f(x)$

$= 7 \cos 7x$

ii $\frac{dy}{dx} = -f'(x) \sin f(x)$

$= -12x^2 \sin\left(4x^3 + \frac{\pi}{3}\right)$

iii $\frac{dy}{dx} = f'(x) \sec^2 f(x)$

$= 5 \sec^2(5x - \pi)$

b $\frac{dy}{dx} = -f'(x) \sin f(x)$

$= -\frac{1}{2} \sin \frac{x}{2}$

At $x = \pi$

$\frac{dy}{dx} = -\frac{1}{2} \sin \frac{\pi}{2}$

$= -\frac{1}{2} \times 1$

$= -\frac{1}{2}$

So $m = -\frac{1}{2}$

Normal is perpendicular to the tangent

$m\, m_2 = -1$

$-\frac{1}{2} m_2 = -1$

$m_2 = 2$

While trigonometric functions are usually expressed in radians we can differentiate angles *in degrees* by using the conversion $\pi = 180°$

EXAMPLE 11

Differentiate $y = \sin x°$

Solution

$180° = \pi$ radians

$$1° = \frac{\pi}{180}$$

$$x° = \frac{\pi x}{180}$$

So $y = \sin x°$ becomes

$$y = \sin \frac{\pi x}{180}$$

$$\frac{dy}{dx} = f'(x) \cos f(x)$$

$$= \frac{\pi}{180} \cos \frac{\pi x}{180}$$

$$= \frac{\pi}{180} \cos x°$$

We can also differentiate composite functions involving trigonometric function.

EXAMPLE 12

Differentiate

a $\tan (e^x)$

b $\ln (\cos x)$

Solution

a $\frac{dy}{dx} = f'(x) \sec^2 f(x)$

$= e^x \sec^2 (e^x)$

b $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

$= \frac{-\sin x}{\cos x}$

$= -\tan x$

Exercise 4.04 Derivative of trigonometric functions

1 Differentiate

a $\sin 4x$
b $\cos 3x$
c $\tan 5x$
d $\tan (3x + 1)$
e $\cos (-x)$
f $3 \sin x$
g $4 \cos (5x - 3)$
h $2 \cos (x^3)$
i $7 \tan (x^2 + 5)$
j $\sin 3x + \cos 8x$
k $\tan (\pi + x) + x^2$
$x \tan x$
m $\sin 2x \tan 3x$
n $\frac{\sin x}{2x}$
o $\frac{3x+4}{\sin 5x}$

ISBN 97807043220

p $(2x + \tan 7x)^9$ **q** $\sin^2 x$ **r** $3 \cos^3 5x$

s $e^x - \cos 2x$ **t** $\sin(1 - \ln x)$ **u** $\sin(e^x + x)$

v $\ln(\sin x)$ **w** $e^{3x} \cos 2x$ **x** $\dfrac{e^{2x}}{\tan 7x}$

2 Find the gradient of the tangent to the curve $y = \tan 3x$ at the point where $x = \dfrac{\pi}{9}$

3 Find the equation of the tangent to the curve $y = \sin(\pi - x)$ at the point $\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$ in exact form

4 Differentiate $\ln(\cos x)$

5 Find the exact gradient of the normal to $y = \sin 3x$ at the point where $x = \dfrac{\pi}{18}$

6 Differentiate $e^{\tan x}$

7 Find the equation of the normal to the curve $y = 3 \sin 2x$ at the point where $x = \dfrac{\pi}{8}$ in exact form

8 Show that $\dfrac{d}{dx}[\ln(\tan x)] = \tan x + \cot x$

9 Differentiate each function

a $y = \tan x°$ **b** $y = 3 \cos x°$ **c** $y = \dfrac{\sin x°}{5}$

10 Find the derivative of $\cos x \sin^4 x$

11 The population of salmon in a salmon farm grows and reduces as fish are born and sold

The population is given by $P = 225 \cos \dfrac{2\pi t}{9} + 750$ where t is time in days

a What is the centre of the population?

b What is the minimum number of salmon in the farm at any one time?

c What is the maximum population?

d At what times is the population 700?

e At what rate is the population changing after

i 3 days? **ii** a week? **iii** 10 days? **v** 18 days?

f At what times is the population growing at the rate of 25 fish per day?

12 The tide was measured over time at a beach at Merimbula and given the formula $D = 8 \sin \dfrac{\pi t}{6} + 9$ where D is depth of water in metres and t is time in hours

a How deep was the water

i initially? **ii** after 5 hours?

b When was the water 10 m deep?

c At what rate was the depth changing after

i 3 hours? **ii** 11 hours? **iii** 12 hours?

d At what times was the depth of water decreasing by 3 m h^{-} ?

4.05 Second derivatives

WS The econd deivaive

WS Fis and econd deivaive

Second derivative

Differentiating $f(x)$ gives $f'(x)$ the first derivativ.

Differentiating $f'(x)$ gives $f''(x)$ the **second derivative**

It is also possible to differentiate further.

Using function notation differentiating several times gives $f'(x), f''(x), f'''(x)$ and so on

Using $\frac{dy}{dx}$ notation differentiating several times gives $\frac{d^2y}{dx^2}$ $\frac{d^3y}{dx^3}$ and so on

The notation $\frac{d^2y}{dx^2}$ comes from $\frac{d^2}{dx^2}(y)$

EXAMPLE 13

a Find the first 4 derivatives of $f(x) = x^3 - 4x^2 + 3x - 2$.

b Find the second derivative of $y = (2x + 5)^7$

c If $f(x) = 4\cos 3x$ show that $f''(x) = -9f(x)$

Solution

a
$$f'(x) = 3x^2 - 8x + 3$$
$$f''(x) = 6x - 8$$
$$f'''(x) = 6$$
$$f''''(x) = 0$$

b
$$\frac{dy}{dx} = f'(x) \times nf(x)^{n-1}$$
$$= 2 \times 7(2x+5)^6$$
$$= 14(2x+5)^6$$
$$\frac{d^2y}{dx^2} = f'(x) \times nf(x)^{n-1}$$
$$= 2 \times 6 \times 14(2x+5)^5$$
$$= 168(2x+5)^5$$

c
$$f'(x) = -f'(x) \times \sin f(x)$$
$$= -3 \times 4\sin 3x$$
$$= -12\sin 3x$$
$$f''(x) = f'(x) \times \cos f(x)$$
$$= 3 \times (-12\cos 3x)$$
$$= -36\cos 3x$$
$$= -9(4\cos 3x)$$
$$= -9f(x) \qquad \text{since } f(x) = 4\cos 3x$$

ISBN 97807043220

Exercise 4.05 Second derivatives

1 Find the first 4 derivatives of $x^7 - 2x^5 + x^4 - x - 3$.

2 If $f(x) = x^9 - 5$ find $f''(x)$

3 Find $f'(x)$ and $f''(x)$ if $f(x) = 2x^5 - x^3 + 1$.

4 Find $f'(1)$ and $f''(-2)$ given $f(t) = 3t^4 - 2t^3 + 5t - 4$

5 Find the first 3 derivatives of $x^7 - 2x^6 + 4x^4 - 7$.

6 Find the first and second derivatives of $y = 2x^2 - 3x + 3$.

7 If $f(x) = x^4 - x^3 + 2x^2 - 5x - 1$, find $f'(-1)$ and $f''(2)$

8 Find the first and second derivatives of x^{-4}

9 If $g(x) = \sqrt{x}$ find $g''(4)$

10 Given $h = 5t^3 - 2t^2 + t + 5$ find $\frac{d^2h}{dt^2}$ when $t = 1$.

11 Find any values of x for which $\frac{d^2y}{dx^2} = 3$ given $y = 3x^3 - 2x^2 + 5x$

12 Find all values of x for which $f''(x) > 0$ given that $f(x) = x^3 - x^2 + x + 9$

13 Find the first and second derivatives of $(4x - 3)^5$

14 Find $f'(x)$ and $f''(x)$ if $f(x) = \sqrt{2 - x}$

15 Find the first and second derivatives of $f(x) = \frac{x + 5}{3x - 1}$

16 Find $\frac{d^2v}{dt^2}$ if $v = (t + 3)(2t - 1)^2$

17 Find the value of b in $y = bx^3 - 2x^2 + 5x + 4$ if $\frac{d^2y}{dx^2} = -2$ when $x = \frac{1}{2}$

18 Find $f''(1)$ if $f(t) = t(2t - 1)^7$

19 Find the value of b if $f(x) = 5bx^2 - 4x^3$ and $f''(-1) = -3$

20 If $y = e^{4x} + e^{-4x}$ show that $\frac{d^2y}{dx^2} = 16y$

21 Prove that $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ given $y = 3e^{2x}$

ISBN 97807043220

22 Show that $\frac{d^2y}{dx^2} = b^2y$ for $y = ae^{bx}$

23 Find the value of n if $y = e^{3x}$ satisfies the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + ny = 0$

24 Show that $\frac{d^2y}{dx^2} = -25y$ if $y = 2 \cos 5x$

25 Given $f(x) = -2 \sin x$ show that $f''(x) = -f(x)$

26 If $y = 2 \sin 3x - 5 \cos 3x$ show that $\frac{d^2y}{dx^2} = -9y$

27 Find values of a and b if $\frac{d^2y}{dx^2} = ae^{3x} \cos 4x + be^{3x} \sin 4x$ given $y = e^{3x} \cos 4x$

28 Find the exact value of $f''(2)$ if $f(x) = x\sqrt{3x-4}$

29 The displacement of a particle moving in a straight line is given by $x = 2t^3 - 5t^2 + 7t + 8$, where x is in metres and t is in seconds

a Find the initial displacement

b Find the displacement after 3 seconds

c Find the velocity after 3 seconds

d Find the acceleration after 3 seconds

30 The height in cm of a pendulum as it swings is given by $h = 8 \cos \pi t + 12$ where t is time in seconds

a What is the height of the pendulum after 3 s?

b What is the maximum and minimum height of the pendulum?

c What is the velocity of the pendulum after

i 1 s? **ii** 15 s ?

d What is the acceleration of the pendulum

i initially? **ii** after 1 s? **iii** after 15 s ?

ISBN 97807043220

4.06 Anti-derivative graphs

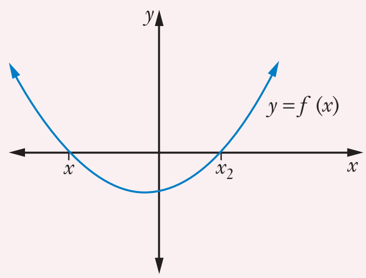
The process of finding the original function $y = f(x)$ given the derivative $y = f'(x)$ is called **anti-differentiation** and the original function is called the **anti-derivative** function also called the **primitive** or **integral function**

WS
Shape o cuvs

Anideivaive gaph

EXAMPLE 14

Sketch the graph of the anti-derivative (primitive function) given the graph of the derivative function below and an initial condition or starting poin, of 0,2).



Solution

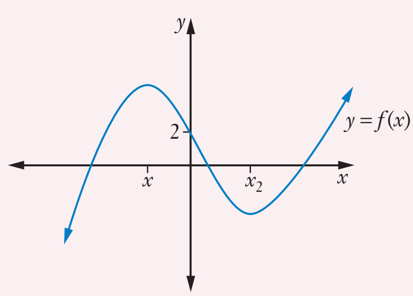
Remember that when you sketch a derivative function the x-intercepts are where the original function has zero gradient or stationary (turning) point.

On this graph the stationary points are at $x = x_1$ and $x = x_2$

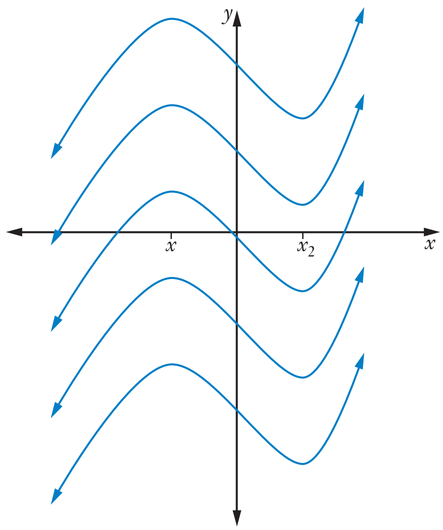
Above the x-axis shows where the original function has a positive gradient (it is increasing) On this grap, this is where $x < x$ and $x > x_2$

Below the x-axis shows where the original function has a negative gradient (it is decreasing) On this grap, this is where $x < x < x_2$

We can sketch this information together with the point (,):



We are not given enough information to sketch a unique grap. There is no way of knowing what the y values of the stationary points are or the stretch or compression of the graph Alo, if we are not given a fixed point on the function we could sketch many graphs that satisfy the information from the derivative function

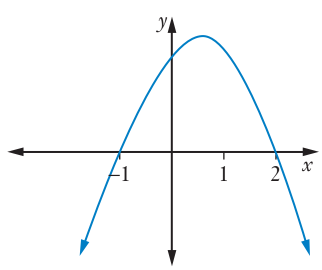


The anti-derivative gives a **family** of curves

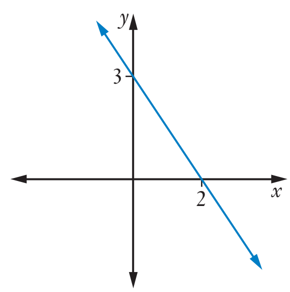
Exercise 4.06 Anti-derivative graphs

1 For each function graphed sketch the graph of the anti-derivative function given it passes through

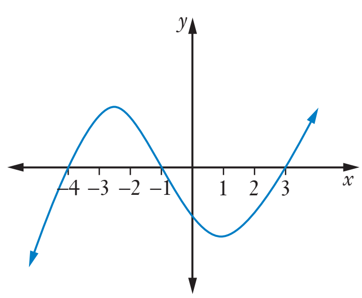
a (0 –1)



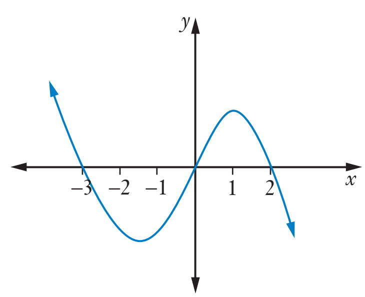
b (1, 2)



c (0 3)

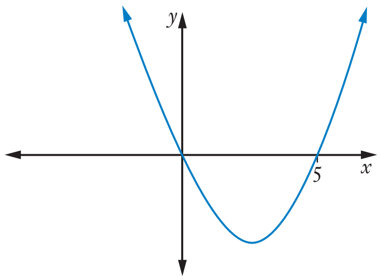


d (–1, –1)



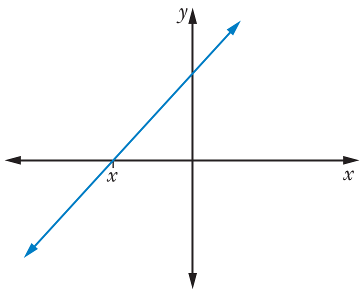
ISBN 97807043220

e (0, 1)

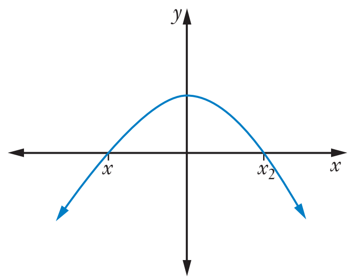


2 Sketch a family of graphs that could represent the anti-derivative function of each graph

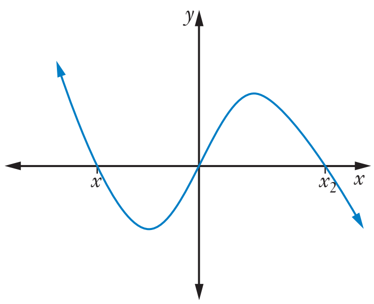
a



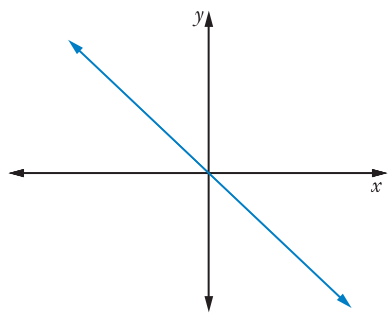
b



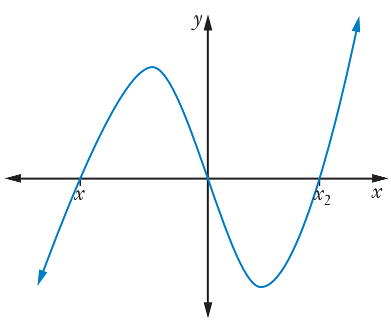
c



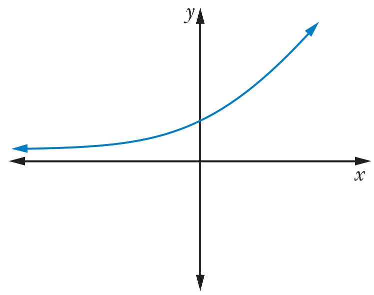
d



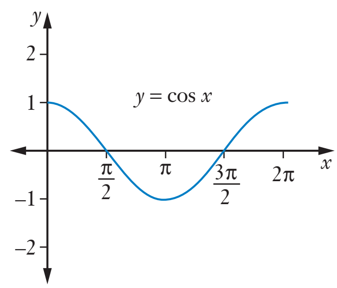
e



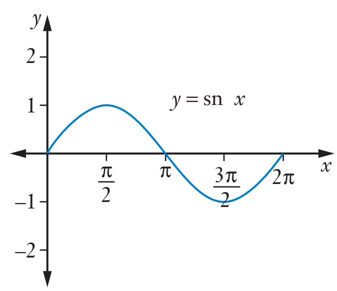
3 The anti-derivative function of the graph below passes through (0 −1) Sketch its grap.



4 Sketch the graph of the anti-derivative function of $y = \cos x$ given that it passes through (0 0.



5 Sketch a family of anti-derivative functions for the graph below.



INVESTIGATION

ANTI-DERIVATIVE OF $y = x^n$

1 Differentiate

a x^2 **b** $x^2 + 5$ **c** $x^2 - 7$ **d** $x^2 + 4$ **e** $x^2 - 75$

What would be the anti-derivative of $2x$?

2 Differentiate

a x^3 **b** $x^3 + 1$ **c** $x^3 + 6$ **d** $x^3 - 2$ **e** $x^3 - 14$

What would be the anti-derivative of $3x^2$?

ISBN 97807043220

3 Differentiate

a x^4 **b** $x^4 - 3$ **c** $x^4 + 2$ **d** $x^4 + 10$ **e** $x^4 - 1$

What would be the anti-derivative of $4x^3$?

4 Differentiate

a x^n **b** $x^n + 7$ **c** $x^n + 9$ **d** $x^n - 5$ **e** $x^n - 2$

What would be the anti-derivative of nx^{n-} ?

Can you find a general rule for anti-derivatives that would work for these examples?

4.07 Anti-derivatives

WS Anideivaive

WS Anideivaive

WS Anidifretition

Since anti-differentiation is the reverse of differentiation we can find the equation of an anti-derivative function

Anti-derivative of x^n

$$\text{If } \frac{dy}{dx} = x^n \text{ then } y = \frac{1}{n+1}x^{n+1} + C \text{ where } C \text{ is a constant}$$

Proof

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+} \ + C\right) = \frac{(n+1)x^n}{n+1}$$

$$= x^n$$

We can apply the same rules to anti-derivatives as we use for derivative. Here are some of the main ones we use

Anti-derivative rules

If $\dfrac{dy}{dx} = k$ then $y = kx$

If $\dfrac{dy}{dx} = kx^n$ then $y = \dfrac{1}{n+1}kx^{n+1} + C$

If $\dfrac{dy}{dx} = f(x) + g(x)$ then $y = F(x) + G(x) + C$ where $F(x)$ and $G(x)$ are the anti-derivatives of $f(x)$ and $g(x)$ respectively.

EXAMPLE 15

Find the anti-derivative of $x^4 - 4x^3 + 9x^2 - 6x + 5$.

Solution

If $f(x) = x^4 - 4x^3 + 9x^2 - 6x + 5$

$$F(x) = \frac{1}{5}x^5 - 4 \times \frac{1}{4}x^4 + 9 \times \frac{1}{3}x^3 - 6 \times \frac{1}{2}x^2 + 5x + C$$

Ant-dervatve of 5 s 5 x.

$$= \frac{x^5}{5} - x^4 + 3x^3 - 3x^2 + 5x + C$$

If we have some information about the anti-derivative function we can use this to evaluate the constant C

The anideivaive uncion

EXAMPLE 16

a The gradient of a curve is given by $\frac{dy}{dx} = 6x^2 + 8x$ If the curve passes through the point (1 −3) find its equatio.

b If $f''(x) = 6x + 2$ and $f'(1) = f(-2) = 0$ find $f(3)$

Solution

a $\frac{dy}{dx} = 6x^2 + 8x$

So $y = 6 \times \frac{1}{3}x^3 + 8 \times \frac{1}{2}x^2 + C$

$= 2x^3 + 4x^2 + C$

Substitute (1 −3)

$-3 = 2(1)^3 + 4(1)^2 + C$

$= 6 + C$

$-9 = C$

Equation is $y = 2x^3 + 4x^2 - 9$

ISBN 97807043220

b $f''(x) = 6x + 2$

$$f'(x) = 6 \times \frac{1}{2}x^2 + 2 \times \frac{1}{1}x + C$$

$$= 3x^2 + 2x + C$$

Since $f'(1) = 0$

$$0 = 3(1)^2 + 2(1) + C$$

$$= 5 + C$$

$$-5 = C$$

So $f'(x) = 3x^2 + 2x - 5$

$$f(x) = 3 \times \frac{1}{3}x^3 + 2 \times \frac{1}{2}x^2 - 5 \times \frac{1}{1}x + D$$

$$= x^3 + x^2 - 5x + D$$

Since $f(-2) = 0$

$$0 = (-2)^3 + (-2)^2 - 5(-2) + D$$

$$= -8 + 4 + 10 + D$$

$$= 6 + D$$

$$-6 = D$$

Equation is $f(x) = x^3 + x^2 - 5x - 6$

$$f(3) = 3^3 + 3^2 - 5(3) - 6$$

$$= 27 + 9 - 15 - 6$$

$$= 15$$

Chain rule

If $\frac{dy}{dx} = (ax + b)^n$ then $y = \frac{1}{a(n+1)}(ax + b)^{n+1} + C$ where C is a constant $a \neq 0$ and $n \neq -1$

Proof

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+} + C\right) = \frac{a(n+1)(ax+b)^n}{a(n+1)}$$

$$= (ax + b)^n$$

EXAMPLE 17

a Find the anti-derivative of $(3x + 7)^8$

b The gradient of a curve is given by $\frac{dy}{dx} = (2x - 3)^4$ If the curve passes through the point (2 −7) find its equatio.

Solution

a $\frac{dy}{dx} = (3x + 7)^8$

$$y = \frac{1}{a(n+1)}(ax + b)^{n+1} + C$$

$$= \frac{1}{3(8+1)}(3x + 7)^{8+1} + C$$

$$= \frac{1}{27}(3x + 7)^9 + C$$

$$= \frac{(3x+7)^9}{27} + C$$

b $\frac{dy}{dx} = (2x - 3)^4$

$$y = \frac{1}{a(n+1)}(ax + b)^{n+1} + C$$

$$= \frac{1}{2(4+1)}(2x - 3)^{4+1} + C$$

$$= \frac{1}{10}(2x - 3)^5 + C$$

Substitute (2 −7)

$$-7 = \frac{1}{10}(2 \times 2 - 3)^5 + C$$

$$= \frac{1}{10}(1)^5 + C$$

$$= \frac{1}{10} + C$$

$$-7\frac{1}{10} = C$$

So the equation is $y = \frac{1}{10}(2x - 3)^5 - 7\frac{1}{10}$

$$= \frac{(2x-3)^5 - 71}{10}$$

ISBN 97807043220

General chain rule

If $\dfrac{dy}{dx} = f'(x)[f(x)]^n$ then $y = \dfrac{1}{n+1}[f(x)]^{n+1} + C$ where C is a constant and $n \neq -1$

Proof

$$\frac{d}{dx}\left(\frac{1}{n+1}[f(x)]^{n+} + C\right) = \frac{1}{n+1}f'(x)(n+1)[f(x)]^{n+1-1}$$
$$= f'(x)[f(x)]^n$$

EXAMPLE 18

Find the anti-derivative of

a $8x^3(2x^4 - 1)^5$

b $x^2(x^3 + 2)^7$

Solution

a Given $f(x) = 2x^4 - 1$

$$f'(x) = 8x^3$$
$$\frac{dy}{dx} = 8x^3(2x^4 - 1)^5$$
$$= f'(x)[f(x)]^n$$
$$y = \frac{1}{n+1}f(x)^{n+1} + C$$
$$= \frac{1}{5+1}(2x^4 - 1)^{5+1} + C$$
$$= \frac{1}{6}(2x^4 - 1)^6 + C$$
$$= \frac{(2x^4 - 1)^6}{6} + C$$

b Given $f(x) = x^3 + 2$

$$f'(x) = 3x^2$$
$$\frac{dy}{dx} = x^2(x^3 + 2)^7$$
$$= \frac{1}{3} \times 3x^2(x^3 + 2)^7$$
$$= \frac{1}{3}f'(x)[f(x)]^n$$
$$y = \frac{1}{3} \times \frac{1}{n+1}f(x)^{n+1} + C$$
$$= \frac{1}{3} \times \frac{1}{7+1}(x^3 + 2)^{7+1} + C$$
$$= \frac{1}{24}(x^3 + 2)^8 + C$$
$$= \frac{(x^3 + 2)^8}{24} + C$$

ISBN 97807043220

Exercise 4.07 Anti-derivatives

1 Find the anti-derivative of

a $2x - 3$ **b** $x^2 + 8x + 1$ **c** $x^5 - 4x^3$

d $(x - 1)^2$ **e** 6 **f** $(3x + 2)^5$

g $8(2x - 7)^4$

2 Find $f(x)$ if

a $f'(x) = 6x^2 - x$ **b** $f'(x) = x^4 - 3x^2 + 7$ **c** $f'(x) = x - 2$

d $f'(x) = (x + 1)(x - 3)$ **e** $f'(x) = x^{\frac{\ }{2}}$

3 Express y in terms of x if

a $\frac{dy}{dx} = 5x^4 - 9$ **b** $\frac{dy}{dx} = x^{-4} - 2x^{-2}$ **c** $\frac{dy}{dx} = \frac{x^3}{5} - x^2$

d $\frac{dy}{dx} = \frac{2}{x^2}$ **e** $\frac{dy}{dx} = x^3 - \frac{2x}{3} + 1$

4 Find the anti-derivative of

a $\sqrt{x}$ **b** x^{-3} **c** $\frac{1}{x^8}$

d $x^{-\frac{\ }{2}} + 2x^{-\frac{2}{3}}$ **e** $x^{-7} - 2x^{-2}$

5 Find the anti-derivative of

a $2x(x^2 + 5)^4$ **b** $3x^2(x^3 - 1)^9$ **c** $8x(2x^2 + 3)^3$

d $15x^4(x^5 + 1)^6$ **e** $x(x^2 - 4)^7$ **f** $x^5(2x^6 - 7)^8$

g $(2x - 1)(x^2 - x + 3)^4$ **h** $(3x^2 + 4x - 7)(x^3 + 2x^2 - 7x)^{10}$

i $(x - 3)(x^2 - 6x - 1)^5$

6 If $\frac{dy}{dx} = x^3 - 3x^2 + 5$ and $y = 4$ when $x = 1$ find an equation for y in terms of x

7 If $f'(x) = 4x - 7$ and $f(2) = 5$ find an equation for $y = f(x)$

8 Given $f'(x) = 3x^2 + 4x - 2$ and $f(-3) = 4$ find the value of $f(1)$

9 Given that the gradient of the tangent to a curve is given by $\frac{dy}{dx} = 2 - 6x$ and the curve passes through (–2 3, find the equation of the cure.

10 If $\frac{dx}{dt} = (t - 3)^2$ and $x = 7$ when $t = 0$ find x when $t = 4$

11 Given $\frac{d^2y}{dx^2} = 8$ and $\frac{dy}{dx} = 0$ and $y = 3$ when $x = 1$ find the equation of y in terms of x

12 If $\frac{d^2y}{dx^2} = 12x + 6$ and $\frac{dy}{dx} = 1$ at the point $(-1, -2)$ find the equation of the curv.

ISBN 97807043220

13 If $f''(x) = 6x - 2$ and $f'(2) = f(2) = 7$ find the equation of the function $y = f(x)$

14 Given $f''(x) = 5x^4$ $f'(0) = 3$ and $f(-1) = 1$, find $f(2)$

15 A curve has $\frac{d^2y}{dx^2} = 8x$ and the tangent at $(-2\ \ 5)$ has an angle of inclination of 45° with the x-axis Find the equation of the curv.

16 The tangent to a curve with $\frac{d^2y}{dx^2} = 2x - 4$ makes an angle of inclination of 135° with the x-axis at the point $(2\ \ -4)$ Find its equatio.

17 A function has a tangent parallel to the line $4x - y - 2 = 0$ at the point $(0\ \ -2)$ and $f''(x) = 12x^2 - 6x + 4$ Find the equation of the functio.

18 A curve has $\frac{d^2y}{dx^2} = 6$ and the tangent at $(-1\ \ 3)$ is perpendicular to the line $2x + 4y - 3 = 0$ Find the equation of the curv.

19 A function has $f'(1) = 3$ and $f(1) = 5$ Evaluate $f(-2)$ given $f''(x) = 6x + 18$.

20 The velocity of an object is given by $\frac{dx}{dt} = 6t - 5$ If the object has initial displacement of -2 find the equation for the displacemen.

21 The acceleration of a particle is given by $\frac{d^2x}{dt^2} = 24t^2 - 12t + 6$ m s^{-2} Its velocity $\frac{dx}{dt} = 0$ when $t = 1$ and its displacement $x = -3$ when $t = 0$ Find the equation for its displacemen.

4.08 Further anti-derivatives

Anti-derivative of exponential functions

$$\text{If } \frac{dy}{dx} = e^x \text{ then } y = e^x + C$$

Chain rule

$$\text{If } \frac{dy}{dx} = e^{ax+b} \text{ then } y = \frac{1}{a}e^{ax+b} + C$$

$$\text{If } \frac{dy}{dx} = f'(x)e^{f\ x} \text{ then } y = e^{f\ x} + C$$

Proof (by differentiation)

$$\frac{d}{dx}\left(\frac{1}{a}e^{ax+b} + C\right) = \frac{1}{a} \times ae^{ax+b}$$
$$= e^{ax+b}$$

$$\frac{d}{dx}[e^{f\ x} + C] = f'(x)e^{f\ x}$$

EXAMPLE 19

a Find the anti-derivative of $e^{4x} + 1$.

b Find the equation of the function $y = f(x)$ given $f'(x) = 6e^{3x}$ and $f(2) = 2e^6$

Solution

a $\frac{1}{a}e^{ax+b} + C = \frac{1}{4}e^{4x} + C$

b
$$\begin{aligned} f'(x) &= 6e^{3x} \\ f(x) &= 6 \times \frac{1}{3}e^{3x} + C \\ &= 2e^{3x} + C \end{aligned}$$

If $f(2) = 2e^6$
$$\begin{aligned} 2e^6 &= 2e^{3\times 2} + C \\ &= 2e^6 + C \\ 0 &= C \end{aligned}$$
So $f(x) = 2e^{3x}$

Anti-derivative of $\frac{1}{x}$

$$\text{If } \frac{dy}{dx} = \frac{1}{x} \text{ then } y = \ln|x| + C$$

Chain rule

$$\text{If } \frac{dy}{dx} = \frac{f'(x)}{f(x)} \text{ then } y = \ln|f(x)| + C$$

Proof

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ for $x > 0$ because $\ln x$ is defined only for $x > 0$

So the anti-derivative of $\frac{1}{x}$ when $x > 0$ is $\ln x$

Suppose $x < 0$

Then $\ln(-x)$ is defined because $-x$ is positive

ISBN 97807043220

$$\frac{d}{dx}[\ln(-x)] = \frac{f'(x)}{f(x)}$$

$$= \frac{-1}{-x}$$

$$= \frac{1}{x} \qquad x < 0$$

So if $\frac{dy}{dx} = \frac{1}{x}$ then $y = \begin{cases} \ln x + C & \text{if } x > 0 \\ \ln(-x) + C & \text{if } x < 0 \end{cases}$

or more simply, $\quad y = \ln|x| + C$

EXAMPLE 20

a Find the anti-derivative of $\frac{3}{x}$

b Find the equation of the function that has $\frac{dy}{dx} = \frac{6x}{x^2 - 5}$ and passes through (3 3 ln 4.

Solution

a $\frac{dy}{dx} = \frac{3}{x}$

$$= 3 \times \frac{1}{x}$$

$$y = 3 \ln |x|$$

b $\frac{dy}{dx} = \frac{6x}{x^2 - 5}$

$$= 3 \times \frac{2x}{x^2 - 5}$$

$$= 3 \times \frac{f'(x)}{f(x)} \text{ where } f(x) = x^2 - 5$$

$$y = 3 \ln f|x| + C$$

$$= 3 \ln |x^2 - 5| + C$$

Substitute (3 3 ln 4:

$$3 \ln 4 = 3 \ln |3^2 - 5| + C$$

$$= 3 \ln 4 + C$$

$$0 = C$$

So $y = 3 \ln |x^2 - 5|$

Anti-derivatives of trigonometric functions

If $\dfrac{dy}{dx} = \cos x$ then $y = \sin x + C$ since $\dfrac{d}{dx}(\sin x) = \cos x$

If $\dfrac{dy}{dx} = \sin x$ then $y = -\cos x + C$ since $\dfrac{d}{dx}(\cos x) = -\sin x$ so $\dfrac{d}{dx}(-\cos x) = \sin x$

If $\dfrac{dy}{dx} = \sec^2 x$ then $y = \tan x + C$ since $\dfrac{d}{dx}(\tan x) = \sec^2 x$

Chain rule

If $\dfrac{dy}{dx} = \cos(ax + b)$ then $y = \dfrac{1}{a}\sin(ax + b) + C$

If $\dfrac{dy}{dx} = \sin(ax + b)$ then $y = -\dfrac{1}{a}\cos(ax + b) + C$

If $\dfrac{dy}{dx} = \sec^2(ax + b)$ then $y = \dfrac{1}{a}\tan(ax + b) + C$

If $\dfrac{dy}{dx} = f'(x)\cos f(x)$ then $y = \sin f(x) + C$

If $\dfrac{dy}{dx} = f'(x)\sin f(x)$ then $y = -\cos f(x) + C$

If $\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$ then $y = \tan f(x) + C$

Proof

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{a}\sin(ax+b)+C\right] &= \frac{1}{a}\times a\cos(ax+b)\\ &= \cos(ax+b)\end{aligned}$$

The other results can be proved similarly.

EXAMPLE 21

a Find the anti-derivative of $\cos 3x$

b Find the equation of the curve that passes through $\left(\dfrac{\pi}{4}, 3\right)$ and has $\dfrac{dy}{dx} = \sec^2 x$

Solution

a
$$\begin{aligned}y &= \frac{1}{a}\sin(ax+b) + C\\ &= \frac{1}{3}\sin 3x + C\end{aligned}$$

ISBN 97807043220

b $y = \tan x + C$

Substitute $\left(\frac{\pi}{4}, 3\right)$

$3 = \tan \frac{\pi}{4} + C$

$= 1 + C$

$2 = C$

So $y = \tan x + 2$

Exercise 4.08 Further anti-derivatives

1 Find the anti-derivative of

a $\sin x$ **b** $\sec^2 x$ **c** $\cos x$

d $\sec^2 7x$ **e** $\sin (2x - \pi)$

2 Anti-differentiate

a e^x **b** e^{6x} **c** $\frac{1}{x}$

d $\frac{3}{3x-1}$ **e** $\frac{x}{x^2+5}$

3 Find the anti-derivative of

a $e^x + 5$ **b** $\cos x + 4x$ **c** $x + \frac{1}{x}$

d $8x^3 - 3x^2 + 6x - 3 + x^{-}$ **e** $\sin 5x - \sec^2 9x$

4 Find the equation of a function with $\frac{dy}{dx} = \cos x$ and passing through $\left(\frac{\pi}{2}, -4\right)$

5 Find the equation of the function that has $f'(x) = \frac{5}{x}$ and $f(1) = 3$.

6 A function has $\frac{dy}{dx} = 4 \cos 2x$ and passes through the point $\left(\frac{\pi}{6}, 2\sqrt{3}\right)$

Find the exact equation of the function

7 A curve has $f''(x) = 27e^{3x}$ and has $f(2) = f'(2) = e^6$ Find the equation of the curv.

8 The rate of change of a population over time t years is given by $\frac{dP}{dt} = 1350e^{0.054t}$
If the initial population is 35 000 fin:

a the equation for population

b the population after 10 years

ISBN 97807043220

9 The velocity of a particle is given by $\frac{dx}{dt} = 3e^{3t}$ and the particle has an initial displacement of 5 metres Find the equation for displacement of the particl.

10 A pendulum has acceleration given by $\frac{d^2x}{dt^2} = -9 \sin 3t$ initial displacement 0 cm and initial velocity 3 cm s^{-}

a Find the equation for its velocity.

b Find the displacement after 2 seconds

c Find the times when the pendulum has displacement 0 cm

Summary of differentiation rules

Rule	Chain rule
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-}$
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}[e^{f(x)}] = f'(x)e^{f(x)}$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}[\sin f(x)] = f'(x)\cos f(x)$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}[\cos f(x)] = -f'(x)\sin f(x)$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}[\tan f(x)] = f'(x)\sec^2 f(x)$

Product rule $\frac{d}{dx}(uv) = u'v + v'u$

Quotient rule $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$

ISBN 97807043220

4. TEST YOURSELF

For Questions **1** to **4** choose the correct answer **A B C** or **D**

Qz
Pacice quiz

1 The anti-derivative of sin $6x$ is

A $\frac{1}{6}\cos 6x$ **B** $6\cos 6x$ **C** $-6\cos 6x$ **D** $-\frac{1}{6}\cos 6x$

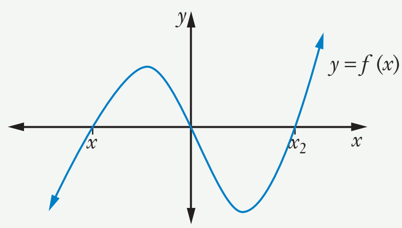
2 The gradient of the tangent to the curve $y = e^{2x} + x$ at $(0, 1)$ i:

A 2 **B** 3 **C** e **D** 1

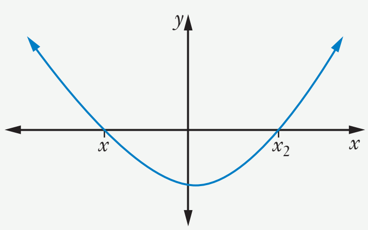
3 If $y = \cos 2x$ then $\frac{d^2y}{dx^2}$ is :

A $-4y$ **B** $-2y$ **C** $4y$ **D** y

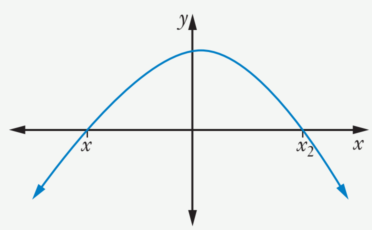
4 The graph of the derivative $y = f'(x)$ is shown Which of the following graphs could be the graph of $y = f(x)$?



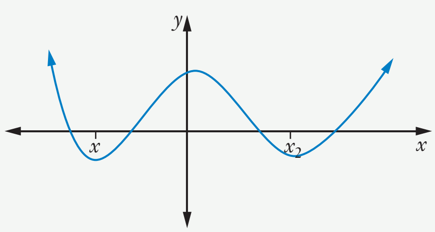
A



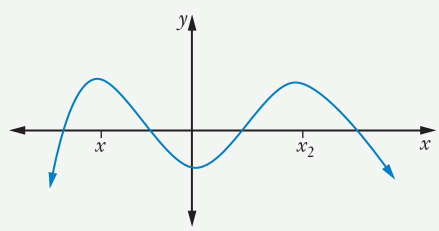
B



C



D



5 Differentiate

a e^{5x} **b** $2e^{1-x}$ **c** $\ln 4x$ **d** $\ln(4x + 5)$

e xe^{x} **f** $\frac{\ln x}{x}$ **g** $(e^{x} + 1)^{10}$

6 Differentiate

a $\cos x$ **b** $2\sin x$ **c** $\tan x + 1$ **d** $x\sin x$

e $\frac{\tan x}{x}$ **f** $\cos 3x$ **g** $\tan 5x$

ISBN 97807043220

7 Find the equation of the tangent to the curve $y = 2 + e^{3x}$ at the point where $x = 0$

8 Find the equation of the tangent to the curve $y = \sin 3x$ at the point $\left(\frac{\pi}{4} \; \frac{1}{\sqrt{2}}\right)$

9 If $x = \cos 2t$ show that $\frac{d^2x}{dt^2} = -4x$

10 Find the exact gradient of the normal to the curve $y = x - e^{-x}$ at the point where $x = 2.$

11 Find the anti-derivative of

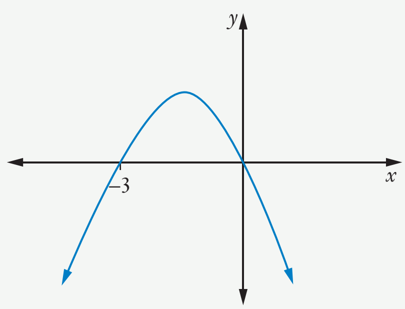
a $10x^4 - 4x^3 + 6x - 3$ **b** e^{5x} **c** $\sec^2 9x$

d $\frac{1}{x+5}$ **e** $\cos 2x$ **f** $\sin\left(\frac{x}{4}\right)$

12 Find the gradient of the tangent to the curve $y = 3\cos 2x$ at the point where $x = \frac{\pi}{6}$

13 A curve has $\frac{dy}{dx} = 6x^2 + 12x - 5$ If the curve passes through the point (, -3) find the equation of the curve

14 Sketch the graph of the anti-derivative of the following function given that the anti-derivative passes through (0 4.



15 Find the equation of the normal to the curve $y = \ln x$ at the point (2 ln 2.

16 Find the equation of the normal to the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$

17 Differentiate

a $(5x^2 + 7)^4$ **b** $4x(2x - 3)^7$ **c** $\frac{5x-1}{3x+4}$

d $2x^3e^x$ **e** $\frac{\tan 3x}{x+1}$

18 If $f''(x) = 15x + 12$ and $f(2) = f'(2) = 5$ find the equation of $y = f(x)$

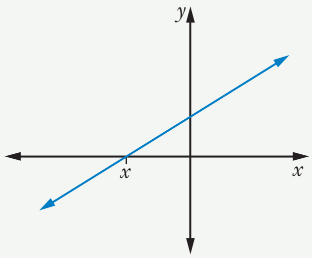
19 If $f(x) = 3x^5 - 2x^4 + x^3 - 2$ fin:

a $f(-1)$ **b** $f'(-1)$ **c** $f''(-1)$

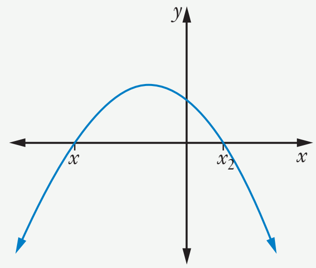
ISBN 97807043220

20 Sketch an example of the graph of an anti-derivative function for each graph

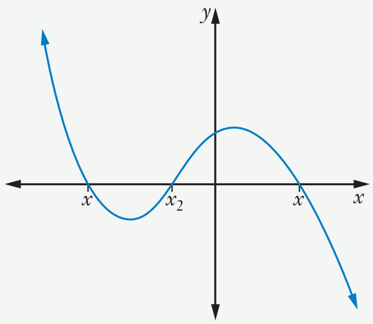
a



b



c



21 A function has $f'(3) = 5$ and $f(3) = 2$ If $f''(x) = 12x - 6$ find the equation of the functio.

22 Find the anti-derivative of

a $x^3(3x^4 - 5)^6$ **b** $3x(x^2 + 1)^9$

ISBN 97807043220

4. CHALLENGE EXERCISE

1 Find the exact gradient of the tangent to the curve $y = e^{x + \ln x}$ at the point where $x = 1$.

2 Find the first and second derivatives of $\dfrac{5-x}{(4x^2+1)^3}$

3 Find the anti-derivative of

a $2xe^x$ **b** $x^2 \sin(x^3)$

4 Differentiate $e^{x \sin 2x}$

5 A curve passes through the point (0 −1) and the gradient at any point is given by $(x + 3)(x - 5)$ Find the equation of the curv.

6 The rate of change of V with respect to t is given by $\dfrac{dV}{dt} = (2t - 1)^2$

If $V = 5$ when $t = \dfrac{1}{2}$ find V when $t = 3$.

7 Find the derivative of $y = \dfrac{x \log_e x}{e^x}$

8 **a** Differentiate $\ln(\tan x)$

b Find the anti-derivative of $\tan x$

9 Find the anti-derivative of

a $x^2 \sin(x^3 - \pi)$ **b** xe^x

ISBN 97807043220

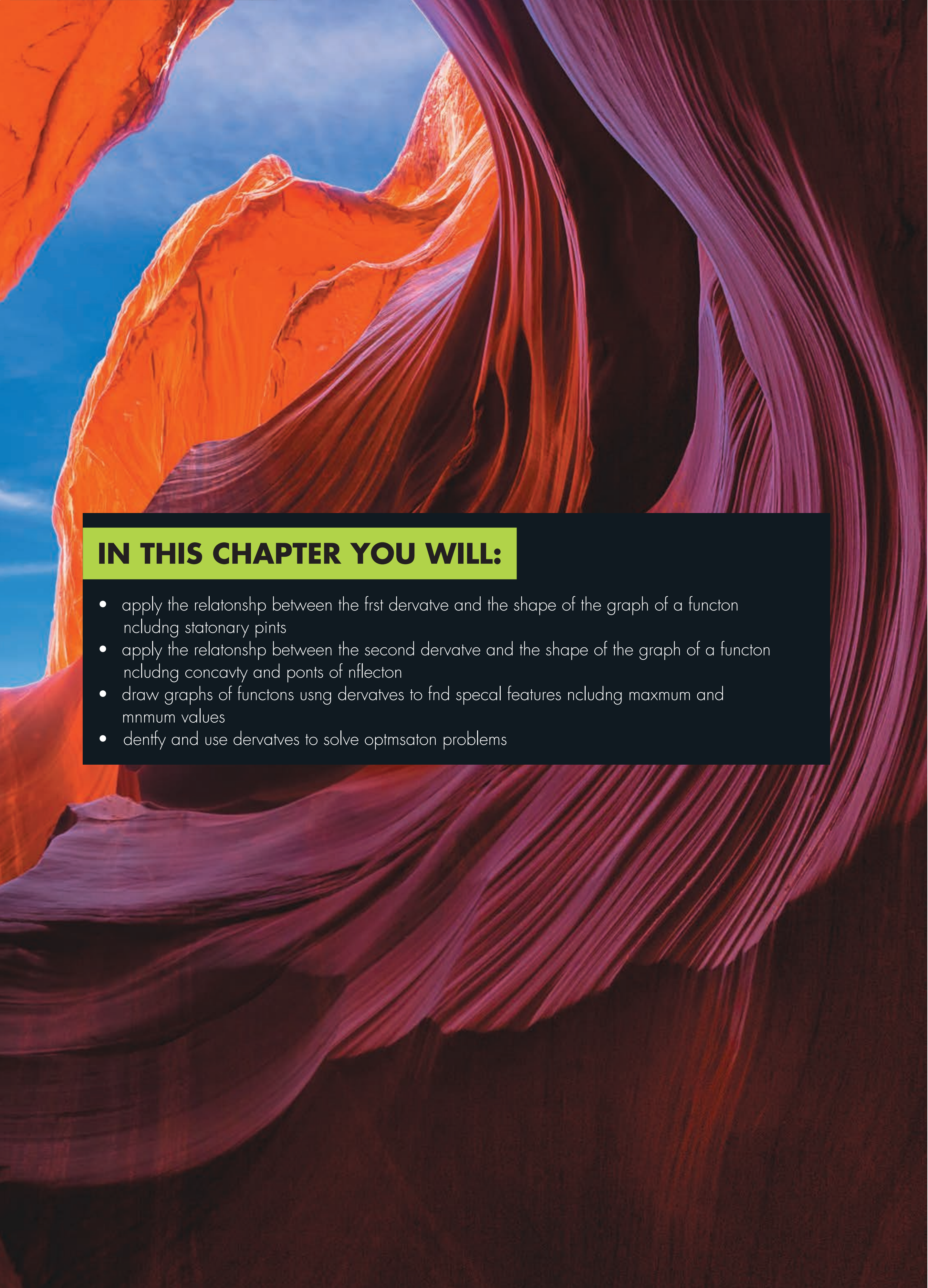
CALCULUS

5.

GEOMETRICAL APPLICATIONS OF DIFFERENTIATION

We can use irst and second deivaives to ind the shape of funcion,incluing speial features such as statonary pint, and draw thir graph. Wewill also usedifferntation to solve prctical optmsaton problems

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- apply the relatonshp between the frst dervatve and the shape of the graph of a functon ncludng statonary pints
- apply the relatonshp between the second dervatve and the shape of the graph of a functon ncludng concavty and ponts of nflecton
- draw graphs of functons usng dervatves to fnd specal features ncludng maxmum and mnmum values
- dentfy and use dervatves to solve optmsaton problems

TERMINOLOGY

concavit: The shape of a curve as it bend; it can be concave up or concave down

global maximum or minimum The absolute highest or lowest value of a function over a given domain

horizontal point of inflection A stationary point where the concavity of the curve changes

local maximum or minimum a relatively high or low value of a function shown graphically as a turning point

maximum point A stationary point where the curve reaches a peak

minimum point A stationary point where the curve reaches a trough

monotonic increasing or **decreasing** A function that is always increasing or decreasing

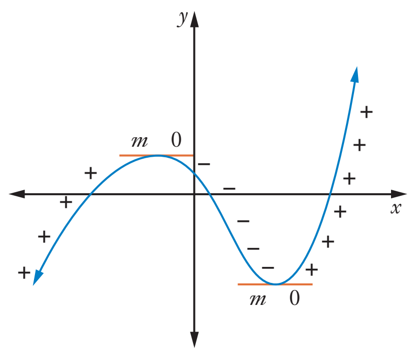
point of inflection A point at which the curve is neither concave upwards nor downwards but where the concavity changes

stationary point A point on the graph of $y = f(x)$ where the tangent is horizontal and its gradient $f\ (x) = 0$ It could be a maximum point minimum point or a horizontal point of inflection

5.01 Increasing and decreasing curves

WS
The sign o he deivaive

You have already seen how the derivative describes the shape of a curv.



Sign of the first derivative

If $f'(x) > 0$ the graph of $y = f(x)$ is increasing

If $f'(x) < 0$ the graph of $y = f(x)$ is decreasing

If $f'(x) = 0$ the graph of $y = f(x)$ has a **stationary point**

Sometimes a curve is **monotonic increasing** or **decreasing** (*always* increasing or decreasing)

Monotonic increasing or decreasing functions

A curve is monotonic increasing if $f'(x) > 0$ for all x

A curve is monotonic decreasing if $f'(x) < 0$ for all x

ISBN 97807043220

EXAMPLE 1

a Find all x values for which the curve $f(x) = x^2 - 4x + 1$ is increasing

b Find any stationary points on the curve $y = x^3 - 48x - 7$.

Solution

a $f'(x) = 2x - 4$

For increasing curve

$f'(x) > 0$

$2x - 4 > 0$

$2x > 4$

$x > 2$

So the curve is increasing for $x > 2$.

b $y' = 3x^2 - 48$

For stationary points

$y' = 0$

$3x^2 - 48 = 0$

$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

When $x = 4$

$y = 4^3 - 48(4) - 7$

$= -135$

When $x = -4$

$y = (-4)^3 - 48(-4) - 7$

$= 121$

So the stationary points are (4 –135) and (–4, 121.

Exercise 5.01 Increasing and decreasing curves

1 For what x values is the function $f(x) = -2x^2 + 8x - 1$ increasing?

2 Find all values of x for which the curve $y = 2x^2 - x$ is decreasing

3 Find the domain over which the function $f(x) = 4 - x^2$ is increasing

4 Find values of x for which the curve $y = x^2 - 3x - 4$ is

a decreasing **b** increasing **c** stationary

5 Show that the function $f(x) = -2x - 7$ is always (monotonic) decreasing

6 Prove that $y = x^3$ is monotonic increasing for all $x \neq 0$

7 Find the stationary point on the curve $f(x) = x^3$

8 Find all x values for which the curve $y = 2x^3 + 3x^2 - 36x + 9$ is stationary.

9 Find all stationary points on the curve

a $y = x^2 - 2x - 3$ **b** $f(x) = 9 - x^2$

c $y = 2x^3 - 9x^2 + 12x - 4$ **d** $y = x^4 - 2x^2 + 1$

10 Find any stationary points on the curve $y = (x - 2)^4$

11 Find any values of x for which the curve $y = 2x^3 - 21x^2 + 60x - 3$ is stationary

12 The function $f(x) = 2x^2 + px + 7$ has a stationary point at $x = 3$ Evaluate p

13 Evaluate a and b if $y = x^3 - ax^2 + bx - 3$ has stationary points at $x = -1$ and $x = 2$.

14 **a** Find the derivative of $y = x^3 - 3x^2 + 27x - 3$.

b Show that the curve is monotonic increasing for all values of x

15 Sketch a function with $f'(x) > 0$ for $x < 2$, $f'(2) = 0$ and $f'(x) < 0$ when $x > 2$.

16 Sketch a curve with $\frac{dy}{dx} < 0$ for $x < 4$, $\frac{dy}{dx} = 0$ when $x = 4$ and $\frac{dy}{dx} > 0$ for $x > 4$

17 Sketch a curve with $\frac{dy}{dx} > 0$ for all $x \neq 1$ and $\frac{dy}{dx} = 0$ when $x = 1$.

18 Sketch a function that has $f'(x) > 0$ in the domain $(-\infty\ \ -2) \cup (5, \infty)$, $f'(x) = 0$ for $x = -2$ and $x = 5$, and $f'(x) < 0$ in the domain $(-2, 5$.

19 A function has $f(3) = 2$ and $f'(3) < 0$ Show this information on a sketc.

20 The derivative of a function is positive at the point $(-2, -1)$ Show this information on a graph

21 Find the stationary points on the curve $y = (3x - 1)(x - 2)^4$

22 Differentiate $y = x\sqrt{x+1}$ Hence find the stationary point on the curv, giving the exact coordinates

23 The curve $f(x) = ax^4 - 2x^3 + 7x^2 - x + 5$ has a stationary point at $x = 1$.
Find the value of a

24 Show that $f(x) = \sqrt{x}$ has no stationary points

25 Show that $f(x) = \frac{1}{x^3}$ has no stationary points

ISBN 97807043220

5.02 Stationary points

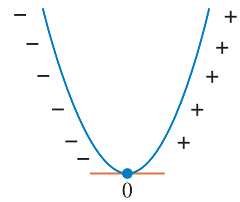
In Year 1, Chapte 6, *Introduction to calculus* you learned about 3 types of stationary point: minimum point maximum point and horizontal point of inflectio.

WS
Saionay poin

Minimum and maximum turning points

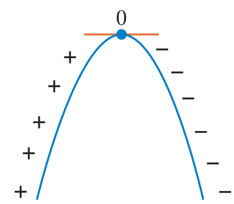
At a local **minimum point** the curve is decreasing on the LHS and increasing on the RHS

x	LHS	Minimum	RHS
$f\,'(x)$	< 0	0	> 0

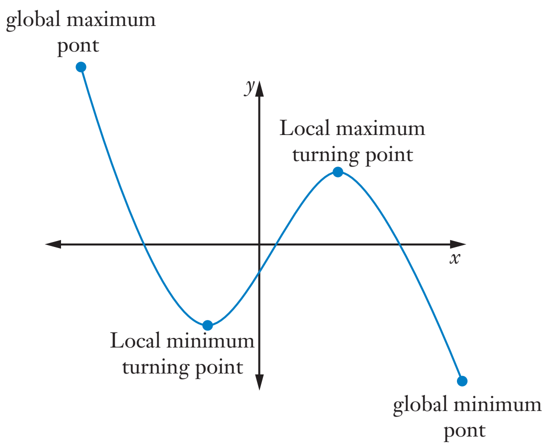


At a local **maximum point** the curve is increasing on the LHS and decreasing on the RHS

x	LHS	Minimum	RHS
$f\,'(x)$	> 0	0	< 0



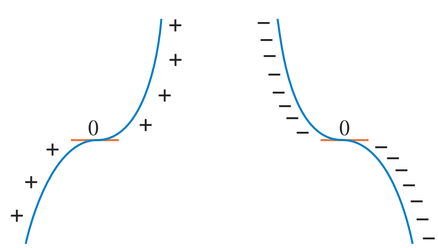
These stationary points are called **local maximum or minimum** points because they are not necessarily the **global maximum or minimum** points on the curve



Horizontal point of inflection

These curves are increasing or decreasing on **both** sides of the horizontal **point of inflection**
It is not a turning point since the curve does not turn around at this point

We will learn more about points of inflection in the next section



ISBN 97807043220

EXAMPLE 2

Find any stationary points on the curve $f(x) = 2x^3 - 15x^2 + 24x - 7$ and determine their nature

> deterine thir natur' means ind what type of statonary pint they are

Solution

$f'(x) = 6x^2 - 30x + 24$

For stationary points

$$
\begin{aligned}
f'(x) &= 0 \\
6x^2 - 30x + 24 &= 0 \\
6(x^2 - 5x + 4) &= 0 \\
x^2 - 5x + 4 &= 0 \\
(x-1)(x-4) &= 0 \\
x = 1, \qquad x &= 4
\end{aligned}
$$

So there are 2 stationary points where $x = 1$ and $x = 4$

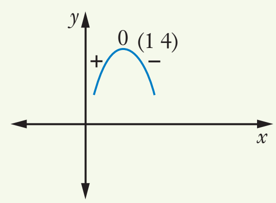
$f(1) = 2(1)^3 - 15(1)^2 + 24(1) - 7$

$= 4$

So (1 4) is a stationary poin.

To determine its natur, choose a point close to 1, 4) on the LHS and HS, for exaple, $x = 0$ and $x = 2$ and test the sign of $f'(x)$

x	0	1	2
f (x)	24	0	−12
	+		−



Positive to negative so (, 4) is a maximum poit.

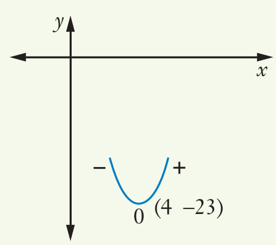
$f(4) = 2(4)^3 - 15(4)^2 + 24(4) - 7$

$= -23$

So (4 −23) is a stationary point

To determine its natur, chooe, for examle, $x = 2$ and $x = 5$.

x	2	4	5
f (x)	−12	0	24
	−		+



Negative to positive so (, −23) is a minimum point

 ISBN 97807043220

Exercise 5.02 Stationary points

1 Find the stationary point on the curve $y = x^2 - 1$ and show that it is a minimum point

2 Find the stationary point on the curve $y = x^4$ and determine its type

3 The function $f(x) = 7 - 4x - x^2$ has one stationary point Find its coordinates and show that it is a maximum turning point

4 Find the turning point on the curve $y = 3x^2 + 6x + 1$ and determine its nature

5 For the curve $y = (4 - x)^2$ find the turning point and determine its natur.

6 The curve $y = x^3 - 6x^2 + 5$ has 2 turning points Find them and use the derivative to determine their nature

7 Find the turning points on the curve $y = x^3 - 3x^2 + 5$ and determine their nature

8 Find any stationary points on the curve $f(x) = x^4 - 2x^2 - 3$ What type of stationary points are they?

9 The curve $y = x^3 - 3x + 2$ has 2 stationary points Find their coordinates and determine their typ.

10 The curve $y = x^5 + mx^3 - 2x^2 + 5$ has a stationary point at $x = -1$ Find the value of m

11 For a certain function $f'(x) = 3 + x$ For what value of x does the function have a stationary point? What type of stationary point is it?

12 A curve has $f'(x) = x(x + 1)$ For what x values does the curve have stationary points? What type are they?

13 **a** Differentiate $P = 2x + \dfrac{50}{x}$ with respect to x

b Find any stationary points on the curve and determine their nature

14 For the function $A = \dfrac{h^2 - 2h + 5}{8}$ find any stationary points and determine their natur.

15 Find any stationary points on the function $V = 40r - \pi r^3$ correct to 2 decimal places and determine their nature

16 Find any stationary points on the curve $S = 2\pi r + \dfrac{120}{r}$ correct to 2 decimal places and determine their nature

17 **a** Differentiate $A = x\sqrt{3600 - x^2}$

b Find any stationary points on $A = x\sqrt{3600 - x^2}$ (to 1 decimal place) and determine their nature

Concaviy

Shape o cuvs

5.03 Concavity and points of inflection

The first derivative $f'(x)$ is the rate of change of the function $y = f(x)$

Similarly, the second derivative $f''(x)$ is the rate of change of the first derivative $f'(x)$

This means the relationship between $f''(x)$ and $f'(x)$ is the same as the relationship between $f'(x)$ and $f(x)$

Relationship between 1st and 2nd derivatives

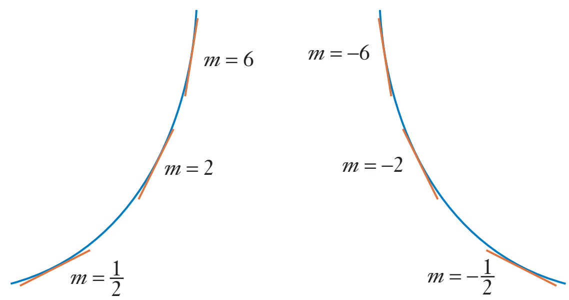
If $f''(x) > 0$ then $f'(x)$ is increasing

If $f''(x) < 0$ then $f'(x)$ is decreasing

If $f''(x) = 0$ then $f'(x)$ is stationary.

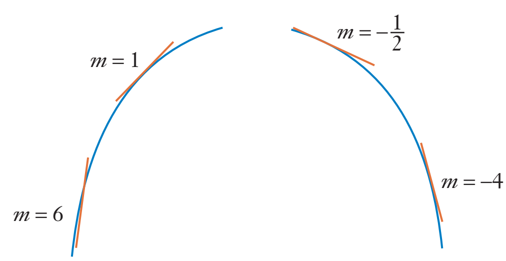
The sign of the second derivative shows the shape of the graph

If $f''(x) > 0$ then $f'(x)$ is increasing
This means that the gradient of the tangent is increasing



Notice the upward shape of these curves The curve lies above the tangens. We say that the curve is **concave upwards**

If $f''(x) < 0$ then $f'(x)$ is decreasing
This means that the gradient of the tangent is decreasing



Notice the downward shape of these curves The curve lies below the tangens. We say that the curve is **concave downwards**

Sign of 2nd derivative

If $f''(x) > 0$ the curve is concave upward.

If $f''(x) < 0$ the curve is concave downward.

ISBN 97807043220

EXAMPLE 3

Find the domain over which the curve $f(x) = 2x^3 - 7x^2 - 5x + 4$ is concave downwards

Solution

$f'(x) = 6x^2 - 14x - 5$

$f''(x) = 12x - 14$

For concave downwards

$f''(x) < 0$

$12x - 14 < 0$

$12x < 14$

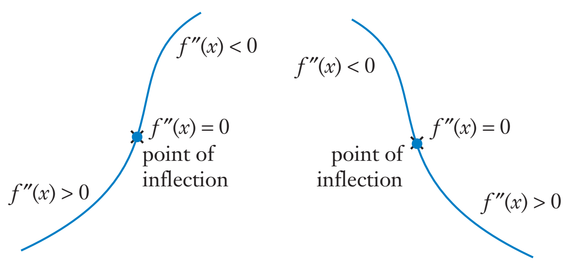
$x < \frac{14}{12}$

$x < 1\frac{1}{6}$

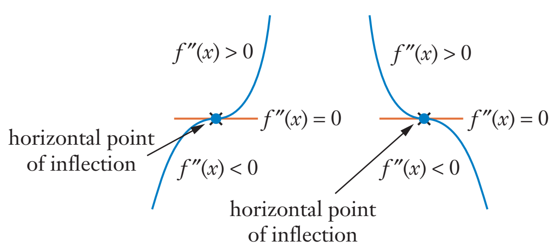
So the domain over which the curve is concave downwards is $(-\infty, 1\frac{1}{6})$

Points of inflection

At the point where $f''(x) = 0$, $f'(x)$ is constant This means that the gradient of the tangent is neither increasing nor decreasing This happens when the curve goes from being concave upwards to concave downwards or concave downwards to concave upwards We say that the curve is changing **concavity** at a **point of inflection** The diagrams above show a point of inflection and the change in concavity as the curve changes shape



The diagrams on the right show a **horizontal point of inflection** that occurs at a stationary point Notice that the tangent is horizontal at the point of inflection



Points of inflection

If $f''(x) = 0$ and concavity change, it is a **point of inflection**

If $f'(x) = 0$ also, it is a **horizontal point of inflection**

ISBN 97807043220

EXAMPLE 4

a Find the point of inflection on the curve $y = x^3 - 6x^2 + 5x + 9$

b Does the function $y = x^4$ have a point of inflection?

Solution

a $y' = 3x^2 - 12x + 5$

$y'' = 6x - 12$

For point of inflection $y'' = 0$ and concavity changes

$6x - 12 = 0$

$x = 2$

Check that concavity changes by choosing values on the LHS and RHS for example $x = 1$ and $x = 3$ and testing the sign of the second derivative y''

x	1	2	3
y''	−6	0	6
	−		+

Since concavity changes (negative to positive) there is a point of inflection at $x = 2$.

When $x = 2$:

$y = 2^3 - 6(2)^2 + 5(2) + 9$

$= 3$

So (2 3) is a point of inflectio.

b $\frac{dy}{dx} = 4x^3$

$\frac{d^2y}{dx^2} = 12x^2$

For point of inflection $\frac{d^2y}{dx^2} = 0$ and concavity changes

$12x^2 = 0$

$x^2 = 0$

$x = 0$

ISBN 97807043220

Check that concavity changes by choosing values on the LHS and RHS for exampl,

$x = \pm 1$ and test the sign of $\frac{d^2y}{dx^2}$

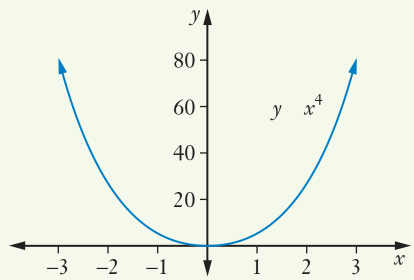
x	-1	0	1
$\frac{d^2y}{dx^2}$	12	0	12
	+		+

Since concavity doesnt change (both sides are positive) (, 0) is not a point of inflectin.

So $y = x^4$ does not have a point of inflection

We can see this by drawing the graph of $y = x^4$

This graph has a turning point at (0 0.



Exercise 5.03 Concavity and points of inflection

1 For what values of x is the curve $y = x^3 + x^2 - 2x - 1$ concave upwards?

2 Find all values of x for which the function $f(x) = (x - 3)^3$ is concave downwards

3 Prove that the curve $y = 8 - 6x - 4x^2$ is always concave downwards

4 Show that the curve $y = x^2$ is always concave upwards

5 Find the domain over which the curve $f(x) = x^3 - 7x^2 + 1$ is concave downwards

6 Find any points of inflection on the curve $g(x) = x^3 - 3x^2 + 2x + 9$

7 Find the points of inflection on the curve $y = x^4 - 6x^2 + 12x - 24$

8 Find the stationary point on the curve $y = x^3 - 2$ and show that it is a point of inflection

9 Determine whether there are any points of inflection on the curve

a $y = x^6$ **b** $y = x^7$ **c** $y = x^5$

d $y = x^9$ **e** $y = x^{12}$

10 Sketch a curve that is always concave up

11 Sketch a curve where $f''(x) < 0$ for $x > 1$ and $f''(x) > 0$ for $x < 1$.

12 Find any points of inflection on the curve $y = x^4 - 8x^3 + 24x^2 - 4x - 9$

13 Show that $f(x) = \frac{2}{x^2}$ is concave upwards for all $x \neq 0$

14 For the function $f(x) = 3x^5 - 10x^3 + 7$

a Find any points of inflection

b Find which of these points are horizontal points of inflection (stationary points)

15 **a** Show that the curve $y = x^4 + 12x^2 - 20x + 3$ has no points of inflection

b Describe the concavity of the curve

16 If $y = ax^3 - 12x^2 + 3x - 5$ has a point of inflection at $x = 2$ evaluate a

17 Evaluate p if $f(x) = x^4 - 6px^2 - 20x + 11$ has a point of inflection at $x = -2$

18 The curve $y = 2ax^4 + 4bx^3 - 72x^2 + 4x - 3$ has points of inflection at $x = 2$ and $x = -1$. Find the values of a and b

19 The curve $y = x^6 - 3x^5 + 21x - 8$ has 2 points of inflection

a Find these points of inflection

b Show that they are not stationary points

WS

The shape o a cuve

5.04 Interpreting rates of change graphically

We can find out more about the shape of a graph if we combine the results from the first and second derivatives

EXAMPLE 5

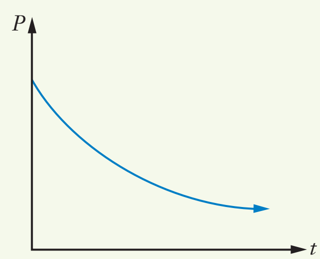
a For a particular curve $f(2) = -1, f'(2) > 0$ and $f''(2) < 0$ Draw the shape of the curve at this point

b The curve below shows the population (P) of unemployed people over time t months

i Describe the sign of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$

ii How is the population of unemployed people changing over time?

iii Is the rate of change of unemployment increasing or decreasing?



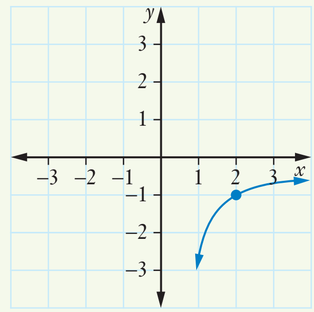
ISBN 97807043220

Solution

a $f(2) = -1$ means that the point (2 −1) lies on the curve

If $f'(2) > 0$ the curve is increasing at this poin.

If $f''(2) < 0$ the curve is concave downwards at this poin.



b **i** The curve is decreasing so $\frac{dP}{dt} < 0$

The curve is concave upwards so $\frac{d^2P}{dt^2} > 0$

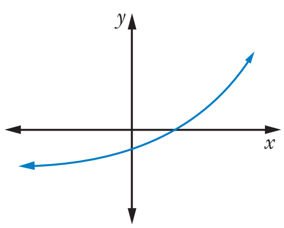
ii Since the curve is decreasing the number of unemployed people is decreasin.

iii Since the curve is concave upwards the (negative) gradient is increasin.
This means that the rate of change of unemployment is increasing

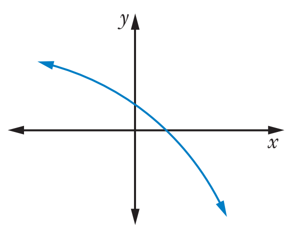
Exercise 5.04 Interpreting rates of change graphically

1 For each curve describe the sign of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

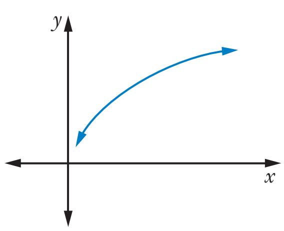
a



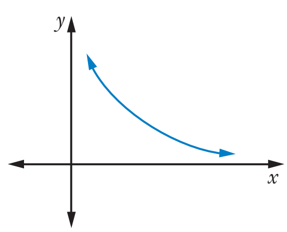
b



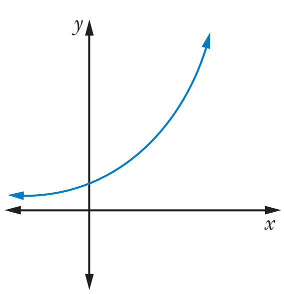
c



d

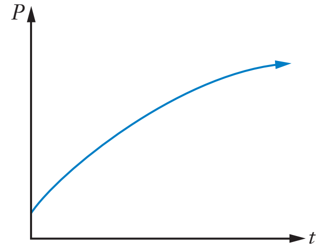


e



ISBN 97807043220

2 The curve below shows the population of a colony of sea lions



a Describe the sign of the first and second derivatives

b Is the rate of change of the sea lion population increasing?

3 Inflation is increasing but the rate of increase is slowin. Draw a graph to show this trend

4 Draw a sketch to show the shape of each curve

a $f'(x) < 0$ and $f''(x) < 0$

b $f'(x) > 0$ and $f''(x) < 0$

c $f'(x) < 0$ and $f''(x) > 0$

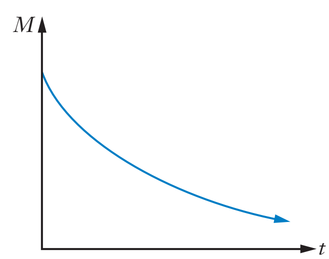
d $f'(x) > 0$ and $f''(x) > 0$

5 The size of classes at a local TAFE college is decreasing and the rate at which this is happening is decreasing Draw a graph to show thi.

6 As an iceblock melts the rate at which it melts increase. Draw a graph to show this information

7 The graph shows the decay of a radioactive substance

Describe the sign of $\frac{dM}{dt}$ and $\frac{d^2M}{dt^2}$



ISBN 97807043220

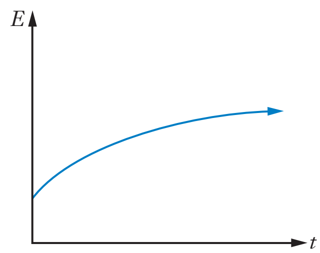
8 The population P of fish in a certain lake was studied over time At the start of the study the number of fish was 2500

a During the study, $\frac{dP}{dt} < 0$ What does this say about the number of fish during the study?

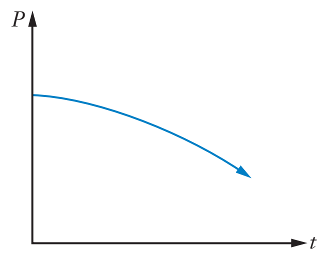
b If at the same time $\frac{d^2P}{dt^2} > 0$ what can you say about the population rate of change ?

c Sketch the graph of the population P against t

9 The graph shows the level of education of youths in a certain rural area over the past 100 years Describe how the level of education has changed over this period of time Include mention of the rate of chang.



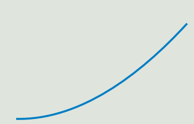
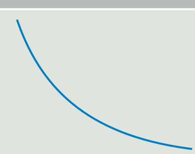
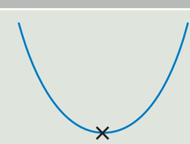
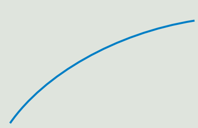
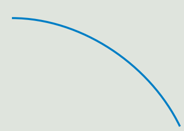
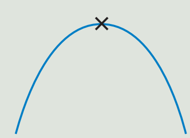
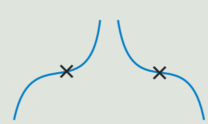
10 The graph shows the number of students in a high school over several years Describe how the school population is changing over time including the rate of chang.



5.05 Stationary points and the second derivative

Putting the first and second derivatives together gives this summary of the shape of a curve

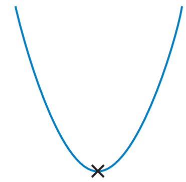
Shape of a curve and the derivatives

	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$
$\frac{d^2y}{dx^2} > 0$			
$\frac{d^2y}{dx^2} < 0$			
$\frac{d^2y}{dx^2} = 0$			

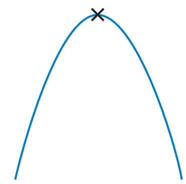
ISBN 97807043220

We can use the table to find the requirements for stationary point.

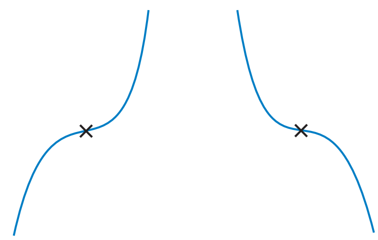
If $f'(x) = 0$ and $f''(x) > 0$ there is a minimum turning point (concave upwards)



If $f'(x) = 0$ and $f''(x) < 0$ there is a maximum turning point (concave downwards)



If $f'(x) = 0$ and $f''(x) = 0$ and concavity changes then there is a horizontal point of inflection



Stationary points and the derivatives

Minimum turning point $f'(x) = 0$ and $f''(x) > 0$

Maximum turning point $f'(x) = 0$ and $f''(x) < 0$

Horizontal point of inflection $f'(x) = 0, f''(x) = 0$ and concavity changes

Now we can use the second derivative to determine the nature of stationary points

EXAMPLE 6

a Find the stationary points on the curve $f(x) = 2x^3 - 3x^2 - 12x + 7$ and distinguish between them

b Find the stationary point on the curve $y = 2x^5 - 3$ and determine its nature

Solution

a $f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

ISBN 97807043220

For stationary points

$f'(x) = 0$

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$x = -1, x = 2$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 7$

$= 14$

$f''(-1) = 12(-1) - 6$

$= -18$

< 0 (concave downwards)

So (−1 14) is a maximum turning poin.

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 7$

$= -13$

$f''(2) = 12(2) - 6$

$= 18$

> 0 (concave upwards)

So (2 −13) is a minimum turning point

b $y' = 10x^4$

$y'' = 40x^3$

For stationary points

$y' = 0$

$10x^4 = 0$

$x^4 = 0$

$x = 0$

When $x = 0$

$y = 2(0)^5 - 3$

$= -3$

$y'' = 40(0)^3$

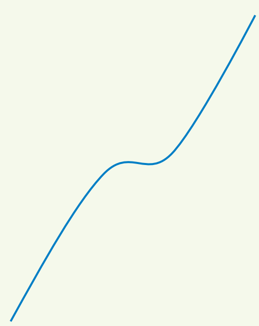
$= 0$

Check that concavity changes by choosing values on the LHS and RHS for exampl, $x = \pm 1$

x	−1	0	1
y''	−40	0	40
	−		+

Since concavity changes (, −3) is a horizontal point of inflection

The table also tells us that the curve changes from concave downwards to concave upwards



Exercise 5.05 Stationary points and the second derivative

1 Find the stationary point on the curve $y = x^2 - 2x + 1$ and determine its nature

2 Find the stationary point on the curve $f(x) = 3x^4 + 1$ and determine what type of point it is

3 Find the stationary point on the curve $y = 3x^2 - 12x + 7$ and show that it is a minimum turning point

4 Determine the stationary point on $y = x - x^2$ and show that it is a maximum point

5 Show that $f(x) = 2x^3 - 5$ has a horizontal point of inflection and find its coordinates

6 Does the function $f(x) = 2x^5 + 3$ have a stationary point? If it does determine its natur.

7 Find any stationary points on $f(x) = 2x^3 + 15x^2 + 36x - 50$ and determine their nature

8 Find the stationary points on the curve $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ and determine whether they are maximum or minimum points

9 Find any stationary points on the curve $y = (4x^2 - 1)^4$ and determine their nature

10 **a** Find any stationary points on the curve $y = 2x^3 - 27x^2 + 120x$ and distinguish between them

b Find any points of inflection on the curve

11 Find any stationary points on the curve $y = (x - 3)\sqrt{4 - x}$ and determine their nature

12 Find any stationary points on the curve $f(x) = x^4 + 8x^3 + 16x^2 - 1$ and determine their nature

13 The curve $y = ax^2 - 4x + 1$ has a stationary point where $x = \frac{1}{2}$

a Find the value of a

b Hence or otherwis, find the stationary point and determine its natue.

14 The curve $y = x^3 - mx^2 + 5x - 7$ has a stationary point where $x = -1$ Find the value of m

15 The curve $y = ax^3 + bx^2 - x + 5$ has a point of inflection at $(1\ \ -2)$
Find the values of a and b

ISBN 97807043220

5.06 Curve sketching

We can sketch the graph of a function by using special features such as intercept, stationary points and points of inflection Here is a summary of strategies for sketching a curv.

Cuve keching

Futhr crve keching

Cuve keching

Cuve skeching wih deivaive

Futhr gaph

Cuve keching

Cuve keching

Sketching curves

- Find stationary points $\left(\dfrac{dy}{dx}=0\right)$ and determine their natur.
- Find points of inflection $\left(\dfrac{d^2y}{dx^2}=0\right)$ and check that concavity change.
- Find any x-intercepts ($y=0$) and y-intercepts ($x=0$)
- Find domain and range
- Find any asymptotes or other discontinuities
- Find limiting behaviour of the function
- Use the symmetry of the function where possible
  - check if the function is even $f(-x)=f(x)$
  - check if the function is odd $f(-x)=-f(x)$

EXAMPLE 7

a Find any stationary points and points of inflection on the curve $f(x)=x^3-3x^2-9x+1$ and hence sketch the curve

b Sketch the curve of the composite function $y=f(g(x))$ where $f(x)=2x+1$ and $g(x)=x^3$ showing any important feature.

Solution

a $f'(x)=3x^2-6x-9$

$f''(x)=6x-6$

For stationary points

$f'(x)=0$

$3x^2-6x-9=0$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3,\qquad x=-1$

$f(3)=3^3-3(3)^2-9(3)+1$

$=-26$

$f''(3)=6(3)-6$

$=12$

>0 (concave upwards)

So (3 –26) is a minimum turning point

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 1$

$\quad = 6$

$f''(-1) = 6(-1) - 6$

$\quad = -12$

$\quad < 0$ (concave downwards)

So (−1 6) is a maximum turning poin.

For points of inflection

$f''(x) = 0$

$6x - 6 = 0$

$6x = 6$

$x = 1$

Check concavity changes by choosing values on LHS and RHS eg $x = 0$ and $x = 2$.

x	0	1	2
$f''(x)$	−6	0	6

Since concavity changes $x = 1$ is at a point of inflection

$f(1) = 1^3 - 3(1)^2 - 9(1) + 1$

$\quad = -10$

So (1, −10) is a point of inflection

For x-intercept $y = 0$

$0 = x^3 - 3x^2 - 9x + 1$

This has no factors so we cant find the x-intercepts

For y-intercept $x = 0$

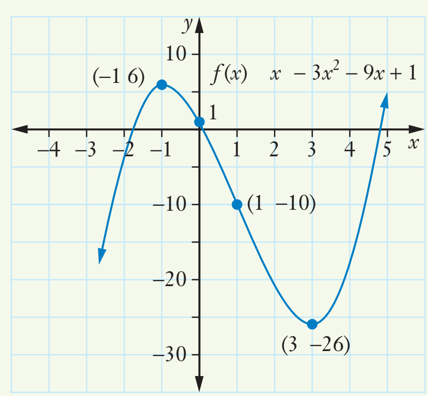
$f(0) = 0^3 - 3(0)^2 - 9(0) + 1$

$\quad = 1$

$f(x) = x^3 - 3x^2 - 9x + 1$ is a cubic function with no symmetry or discontinuities

It is not an even or odd function

Notice that the point of inflection at (1, −10) is not a stationary point It is the point where the graph naturally changes concavity.



ISBN 97807043220

b $y = f(g(x))$

$= 2x^3 + 1$

$\frac{dy}{dx} = 6x^2$

$\frac{d^2y}{dx^2} = 12x$

For stationary points

$\frac{dy}{dx} = 0$

$6x^2 = 0$

$x^2 = 0$

$x = 0$

When $x = 0$

$y = 2(0)^3 + 1$

$= 1$

$\frac{d^2y}{dx^2} = 12(0)$

$= 0$

Check concavity either side

x	−1	0	1
$\frac{d^2y}{dx^2}$	−12	0	12

Since concavity changes (, 1) is a horizontal point of inflection

For x-intercepts $y = 0$

$0 = 2x^3 + 1$

$-1 = 2x^3$

$-05 = x^3$

$\sqrt[3]{-05} = x$

$-08 \approx x$

For y-intercept $x = 0$

$y = 2(0)^3 + 1$

$= 1$

This is (0 1, the point of inflectin.

This is a cubic function We can make the graph more accurate by finding some extra points

When $x = -1$

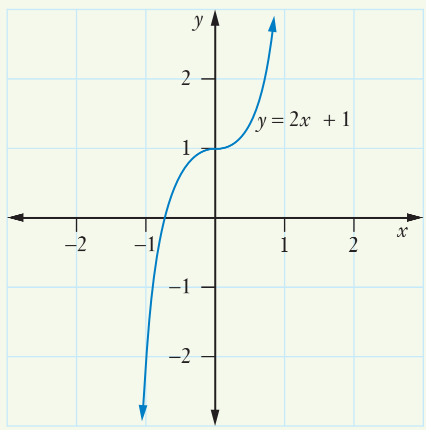
$y = 2(-1)^3 + 1$

$= -1$

When $x = 1$:

$y = 2(1)^3 + 1$

$= 3$



You can use derivatives to help sketch other function, for example trigonometrc, exponential and logarithmic graphs

ISBN 97807043220

EXAMPLE 8

Sketch the curve $y = xe^x$ showing any important feature.

Solution

$y = xe^x$

$y' = u'v + v'u$ where $u = x$ and $v = e^x$

$u' = 1 \qquad v' = e^x$

$y' = 1 \times e^x + e^x \times x$

$\quad = e^x(1 + x)$

$y'' = u'v + v'u$ where $u = e^x$ and $v = 1 + x$

$\qquad u' = e^x \qquad v' = 1$

$y'' = e^x \times (1 + x) + 1 \times e^x$

$\quad = e^x(2 + x)$

For stationary points

$$\begin{aligned} y' &= 0 \\ e^x(1 + x) &= 0 \\ 1 + x &= 0 \qquad (e^x \neq 0) \\ x &= -1 \end{aligned}$$

When $x = -1$

$$\begin{aligned} y &= -1e^{-} \\ &= -\frac{1}{e} \\ y'' &= e^{-}\ (2 + -1) \\ &= \frac{1}{e} \\ &> 0 \qquad \text{(concave upwards)} \end{aligned}$$

So $\left(-1\ -\frac{1}{e}\right)$ is a minimum turning point

For x-intercepts $y = 0$

$0 = xe^x$

$x = 0 \qquad (e^x \neq 0)$

For y-intercepts $x = 0$

$y = 0e^0$

$\quad = 0$

The general exponential function $y = a^x$ has an asymptote at the x-axis

Limiting behaviour as $x \to \pm\infty$

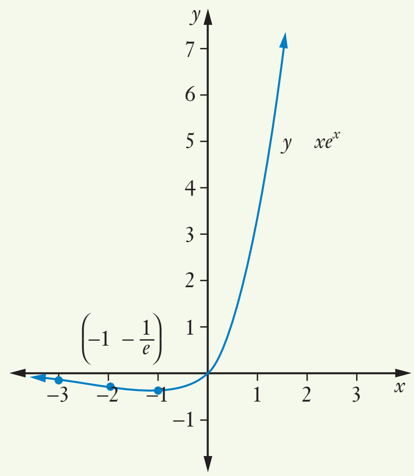
As $x \to \infty$ $xe^x \to \infty$ since x and e^x are both becoming large as x becomes large

As $x \to -\infty$ $x \to -\infty$ but $e^x \to 0$ when x is negative $\left(\text{since } e^{-x} = \frac{1}{e^x}\right)$

So $xe^x \to 0^-$ (it approaches zero from the negative side)

ISBN 97807043220

We can sketch this information on a grap.



When the second derivative is hard to find we can use the first derivative to check the type of stationary points

EXAMPLE 9

Find any stationary points and sketch the function $y = x\sqrt{16 - x^2}$

Solution

$$y = x\sqrt{16 - x^2}$$

$$y' = u'v + v'u \qquad u = x \qquad v = \sqrt{16 - x^2} = (16 - x^2)^{\frac{1}{2}}$$

$$u' = 1 \qquad v' = -2x \times \frac{1}{2}(16 - x^2)^{-\frac{1}{2}}$$

$$= -\frac{x}{\sqrt{16 - x^2}}$$

$$y' = 1 \times \sqrt{16 - x^2} + \left(-\frac{x}{\sqrt{16 - x^2}}\right) \times x$$

$$= \sqrt{16 - x^2} - \frac{x^2}{\sqrt{16 - x^2}}$$

ISBN 97807043220

For stationary points

$$y' = 0$$

$$\sqrt{16-x^2} - \frac{x^2}{\sqrt{16-x^2}} = 0$$

$$16 - x^2 - x^2 = 0 \qquad \text{(multiplying both sides by } \sqrt{16-x^2}\text{)}$$

$$16 - 2x^2 = 0$$

$$16 = 2x^2$$

$$8 = x^2$$

$$\pm\sqrt{8} = x$$

When $x = \sqrt{8}$

$y = \sqrt{8} \times \sqrt{16-(\sqrt{8})^2}$

$= \sqrt{8} \times \sqrt{8}$

$= 8$

So $(\sqrt{8}\ \ 8)$ is a stationary poin.

When $x = -\sqrt{8}$

$y = -\sqrt{8} \times \sqrt{16-(-\sqrt{8})^2}$

$= -\sqrt{8} \times \sqrt{8}$

$= -8$

So $(-\sqrt{8}\ \ -8)$ is a stationary point

Since the second derivative is hard to find we can check the first derivative on LHS and RHS of $\pm\sqrt{8} \approx 28$ to see where the curve is increasing and decreasin.

x	2	2.8	3
y'	+23	0	–08

Positive to negative so $(\sqrt{8}\ \ 8)$ is a maximum turning poin.

x	–3	–28	–2
y'	–08	0	+23

Negative to positive so $(-\sqrt{8}\ \ -8)$ is a minimum turning point

For x-intercepts $y = 0$

$0 = x\sqrt{16-x^2}$

$$x = 0, \sqrt{16-x^2} = 0$$

$$16 - x^2 = 0$$

$$16 = x^2$$

$$\pm 4 = x$$

ISBN 97807043220

For y-intercept $x = 0$

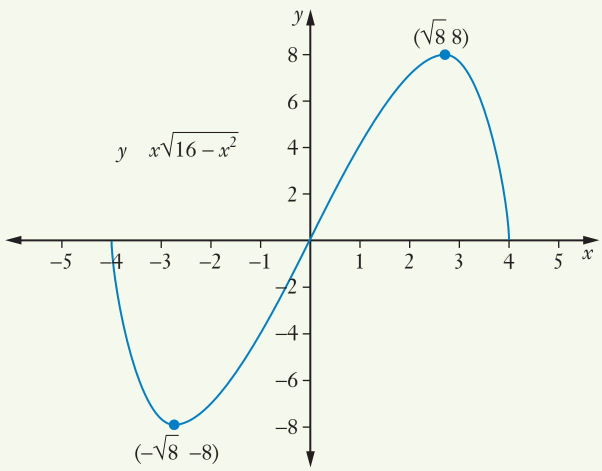
$y = 0\sqrt{16 - 0^2}$

$= 0$

Domain $\sqrt{16 - x^2} \geq 0$

This simplifies to $-4 \leq x \leq 4$ or $[-4\ 4]$ by solving the inequality or by noticing that the graph of $y = \sqrt{16 - x^2}$ is a semicircle with radius 4

We can sketch this information on a grap.



Exercise 5.06 Curve sketching

1 Find the stationary point on the curve $f(x) = x^2 - 3x - 4$ and determine its type
Find the x- and y-intercepts of the graph of $f(x)$ and sketch the curve

2 Sketch the graph of $y = 6 - 2x - x^2$ showing the stationary poin.

3 Find the stationary point on the curve of the composite function $y = f(g(x))$ where $f(x) = x^3$ and $g(x) = x - 1$ and determine its nature Hence sketch the curv.

4 Sketch the graph of $y = x^4 + 3$ showing any stationary point.

5 Find the stationary point on the curve $y = x^5$ and show that it is a point of inflection
Hence sketch the curve

ISBN 97807043220

6 Sketch the graph of $f(x) = x^7$

7 Find any stationary points on the curve $y = 2x^3 - 9x^2 - 24x + 30$ and sketch its graph

8 **a** Determine any stationary points on the curve $y = x^3 + 6x^2 - 7$.

b Find any points of inflection on the curve

c Sketch the curve

9 Find any stationary points and points of inflection on the curve $y = f(x) + g(x)$ where $f(x) = x^3 - 7x^2 - 1$ and $g(x) = x^2 + 4$ and hence sketch the curve

10 Find any stationary points and points of inflection on the curve $y = 2 + 9x - 3x^2 - x^3$ Hence sketch the curve

11 Sketch the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$ showing all stationary point.

12 Find all stationary points and points of inflection on the curve $y = (2x + 1)(x - 2)^4$ Sketch the curve

13 Show that the curve $y = \dfrac{2}{1+x}$ has no stationary points By considering the domain and range of the function sketch the curv.

14 Sketch in the domain $[0\ 2\ \pi]$ showing all stationary point:

a $y = \cos 2x$ **b** $y = 5 \sin 4x$

15 Draw the graph of each function showing stationary point, points of inflection and other features

a $y = x^2 \ln x$ **b** $y = \dfrac{x}{e^x}$ **c** $y = \dfrac{1}{x^2 - 1}$

ISBN 97807043220

5.07 Global maxima and minima

WS

Geae and lea value

A curve may have local maximum and minimum turning points but the absolute highest and lowest values of a function over a given domain are called the **global maximum or minimum values** of the function

EXAMPLE 10

Find the global maximum and minimum values of y for the function $f(x) = x^4 - 2x^2 + 1$ in the domain [–2, 3.

Solution

$f'(x) = 4x^3 - 4x$

$f''(x) = 12x^2 - 4$

For stationary points

$$\begin{aligned} f'(x) &= 0 \\ 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ 4x(x + 1)(x - 1) &= 0 \end{aligned}$$

$x = 0, \quad x = -1, \quad x = 1$

$$\begin{aligned} f(0) &= 0^4 - 2(0)^2 + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} f''(0) &= 12(0)^2 - 4 \\ &= -4 \\ &< 0 \quad \text{(concave downward)} \end{aligned}$$

So (0 1) is a maximum turning poin.

$$\begin{aligned} f(-1) &= (-1)^4 - 2(-1)^2 + 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f''(-1) &= 12(-1)^2 - 4 \\ &= 8 \\ &> 0 \quad \text{(concave upward)} \end{aligned}$$

So (–1 0) is a minimum turning poin.

$$\begin{aligned} f(1) &= 1^4 - 2(1)^2 + 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f''(1) &= 12(1)^2 - 4 \\ &= 8 \\ &> 0 \quad \text{(concave upward)} \end{aligned}$$

So (1 0) is a minimum turning poin.

ISBN 97807043220

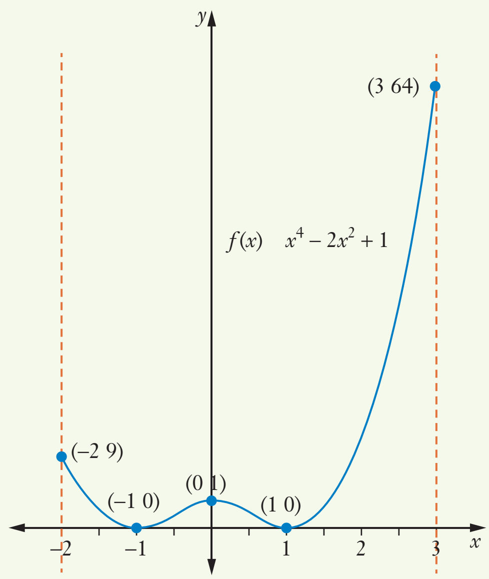
At the endpoints of the domain

$$f(-2) = (-2)^4 - 2(-2)^2 + 1$$
$$= 9$$
$$f(3) = 3^4 - 2(3)^2 + 1$$
$$= 64$$

Checking we also notice that $f(x) = x^4 - 2x^2 + 1$ is an even function

$$f(-x) = (-x)^4 - 2(-x)^2 + 1$$
$$= x^4 - 2x^2 + 1$$
$$= f(x)$$

Drawing this information



In the domain [–2 3, the global maximum value is 64 and the global minimum value is0.

ISBN 97807043220

Exercise 5.07 Global maxima and minima

1 Sketch the graph of $y = x^2 + x - 2$ in the domain [–2 2] and find the maximum value of y in this domain

2 Sketch the graph of $f(x) = 9 - x^2$ over the domain [–4 2. Hence find the maximum and minimum values of the function over this domain

3 Find the maximum value of $y = x^2 - 4x + 4$ in the domain [–3, 3].

4 Sketch the graph of $f(x) = 2x^3 + 3x^2 - 36x + 5$ for $-3 \le x \le 3$ showing any stationary points Find the global maximum and minimum values of the functio.

5 Find the global maximum for $y = x^5 - 3$ in the domain [–2, 1].

6 Sketch the curve $f(x) = 3x^2 - 16x + 5$ for $0 \le x \le 4$ and find its global maximum and minimum

7 Find the local and global maximum and minimum of $f(x) = 3x^4 + 4x^3 - 12x^2 - 3$ in the domain [–2, 2.

8 Sketch $y = x^3 + 2$ over the domain [–3 3] and find its global minimum and maximu.

9 Sketch $y = \sqrt{x+5}$ for $-4 \le x \le 4$ and find its maximum and minimum values

10 Show that $y = \dfrac{1}{x-2}$ has no stationary points Find its maximum and minimum values in the domain [–3, 3].

INVESTIGATION

THE LARGEST DISC

One disc 20 cm in diameter and one 10 cm in diameter are cut from a disc of cardboard 30 cm in diameter. Can you find the largest disc that can be cut from the remainder of the cardboard?

5.08 Finding formulas for optimisation problems

Optimisation problems involve finding maximum or minimum values For exampl, a salesperson wants to maximise profit a warehouse manager wants to maximise storage a driver wants to minimise petrol consumption a farmer wants to maximise paddock size

To solve an optimisation proble, we must first find a formula for the quantity that we are trying to maximise or minimise

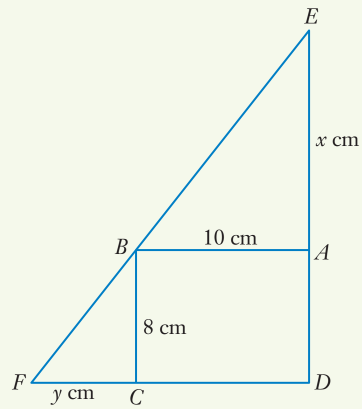
EXAMPLE 11

a A rectangular prism has a base with length twice its width Its volume is 300 cm^3 Show that the surface area is given by $S = 4x^2 + \frac{900}{x}$

b $ABCD$ is a rectangle with $AB = 10$ cm and $BC = 8$ cm Length $AE = x$ cm and $CF = y$ cm

i Show that $xy = 80$

ii Show that triangle EDF has area given by $A = 80 + 5x + \frac{320}{x}$

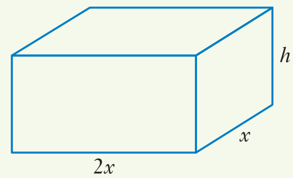


 ISBN 97807043220

Solution

a Volum:

$$V = lwh$$
$$= 2x \times x \times h$$
$$= 2x^2h$$
$$V = 300$$
$$300 = 2x^2h$$
$$\frac{300}{2x^2} = h \qquad [1]$$



Surface area

$$S = 2(lw + wh + lh)$$
$$= 2(2x^2 + xh + 2xh)$$
$$= 2(2x^2 + 3xh)$$
$$= 4x^2 + 6xh$$

Substitute [1]

$$S = 4x^2 + 6x \times \frac{300}{2x^2}$$
$$= 4x^2 + \frac{900}{x}$$

b **i** Triangles AEB and CBF are similar.

So $\frac{10}{y} = \frac{x}{8}$

$$xy = 80 \qquad [1]$$

ii Side $FD = y + 10$ and side $ED = x + 8$

Since $xy = 80$

$$y = \frac{80}{x} \qquad [2]$$

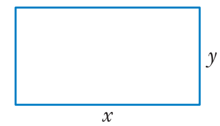
Area

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(y + 10)(x + 8)$$
$$= \frac{1}{2}(xy + 8y + 10x + 80)$$
$$= \frac{1}{2}\left(80 + 8 \times \frac{80}{x} + 10x + 80\right) \qquad \text{substituting [1] and [2]}$$
$$= \frac{1}{2}\left(160 + \frac{640}{x} + 10x\right)$$
$$= 80 + \frac{320}{x} + 5x$$

ISBN 97807043220

Exercise 5.08 Finding formulas for optimisation problems

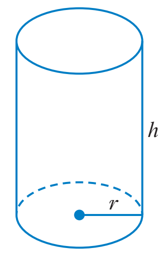
1 The area of a rectangle is to be 50 m^2 Show that its perimeter is given by the equation $P = 2x + \frac{100}{x}$



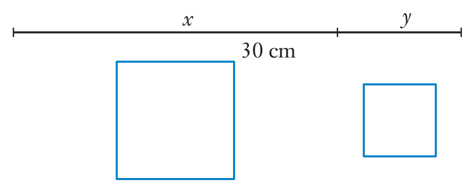
2 A rectangular paddock on a farm is to have a fence with a 120 m perimeter. Show that the area of the paddock is given by $A = 60x - x^2$

3 The product of 2 numbers is 20 Show that the sum of the numbers is $S = x + \frac{20}{x}$

4 A closed cylinder is to have a volume of 400 cm^3 Show that its surface area is $S = 2\pi r^2 + \frac{800}{r}$



5 A 30 cm length of wire is cut into 2 pieces and each piece bent to form a square as shown



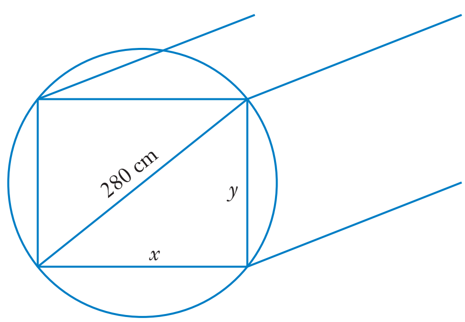
a Show that $y = 30 - x$

b Show that the total area of the 2 squares is given by $A = \frac{x^2 - 30x + 450}{8}$

6 A timber post with a rectangular cross-sectional area is to be cut out of a log with a diameter of 280 mm as shown

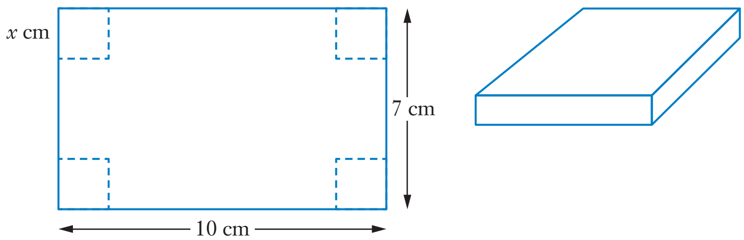
a Show that $y = \sqrt{78\,400 - x^2}$

b Show that the cross-sectional area is given by $A = x\sqrt{78\,400 - x^2}$



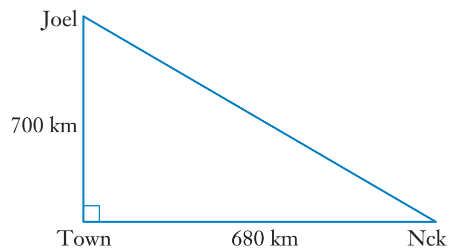
ISBN 97807043220

7 A 10 cm by 7 cm rectangular piece of cardboard has equal square corners with side x cm cut out The sides are folded up to make an open box as shon. Show that the volume of the box is $V = 70x - 34x^2 + 4x^3$

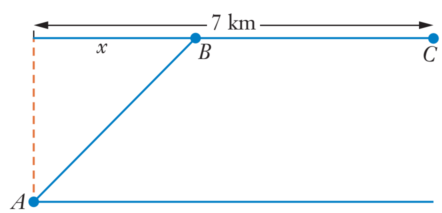


8 A travel agency calculates the expense E per person of organising a holiday in a group of x people as $E = 200 + 400x$ The cost C for each person taking a holiday is $C = 900 - 100x$ Show that the profit to the travel agency on a holiday with a group of x people is given by $P = 700x - 500x^2$

9 Joel is 700 km north of a town travelling towards it at an average speed of 75 km h^{-} Nick is 680 km east of the town travelling towards it at 80 km h^{-}
Show that after t hours the distance between Joel and Nick is given by
$d = \sqrt{952400 - 213800t + 12025t^2}$



10 Taylor swims from point A to point B across a 500 m wide river, then walks along the river bank to point C The distance along the river bank is 7 km If she swims at 5 km h^{-} and walks at 4 km h^{-} show that the time taken to reach point C is given by $t = \dfrac{\sqrt{x^2 + 025}}{5} + \dfrac{7 - x}{4}$



ISBN 97807043220

Opimiaion poblem

Futhr opimiaion poblem

Sating maxima and minima poblem

Applicaion o opimiaion

Maximum volume

Applicaion o deivaive poblem

Applicaion o deivaive aignmen

Second deivaive aignmen

Second deivaive poblem

5.09 Optimisation problems

You can use derivatives to find the maximum or minimum value of a formul.

Always check that an answer gives a maximum or minimum value

EXAMPLE 12

The equation for the expense per year, E (in units of \$10 000) of running a certain business is given by $E = x^2 - 6x + 12$ where x is the number (in 100s) of items manufactured

a Find the expense of running the business if no items are manufactured

b Find the number of items needed to minimise the expense of the business

c Find the minimum expense of the business

Solution

a When $x = 0$

$E = 0^2 - 6(0) + 12$

$= 12$ (expense is in units of \$10 000)

So the expense of running the business when no items are manufactured is $12 \times \$10\,000 = \$120\,000$ per year.

b For stationary points

$$\frac{dE}{dx} = 0$$

$$\frac{dE}{dx} = 2x - 6 = 0$$

$$2x = 6$$

$$x = 3 \qquad (x \text{ is in 100s of items})$$

$$\frac{d^2E}{dx^2} = 2$$

$$> 0 \qquad \text{(concave upwards)}$$

So $x = 3$ gives a minimum value

$3 \times 100 = 300$

So 300 items manufactured each year will give the minimum expense

c When $x = 3$:

$E = 3^2 - 6(3) + 12$

$= 3$

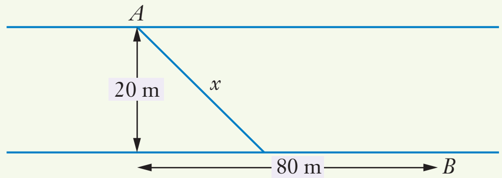
So the minimum expense per year is $3 \times \$10\,000 = \$30\,000$

ISBN 97807043220

EXAMPLE 13

a The council wants to make a rectangular swimming area at the beach using the seashore on one side and a length of 300 m of shark-proof netting for the other 3 sides What are the dimensions of the rectangle that encloses the greatest area ?

b Kristyn is at point A on one side of a 20 m wide river and needs to get to point B on the other side 80 m along the bank as shown Kristyn swims to any point on the other bank and then runs along the side of the river to point B

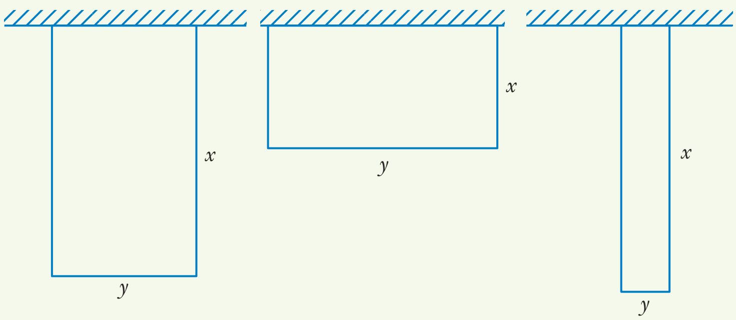


If she can swim at 7 km h^{-} and run at 11 km h^{-} find x the distance she swims to the nearest metre to minimise her total travel tim.

Opimiaion poblem

Solution

a Many different rectangles could have a perimeter of 300 m Let the length of the rectangle be y and the width be x



Perimeter $2\ x + y = 300$ m

$$y = 300 - 2x \qquad [1]$$

Area

$$A = xy$$

$$= x(300 - 2x) \qquad \text{substituting [1]}$$

$$= 300\ x - 2x^2$$

$$\frac{dA}{dx} = 300 - 4x$$

For stationary points

$$\frac{dA}{dx} = 0$$

$$300 - 4x = 0$$

$$300 = 4x$$

$$75 = x$$

ISBN 97807043220

$$\frac{d^2A}{dx^2} = -4$$

< 0 Concave downwards

So $x = 75$ gives maximum area

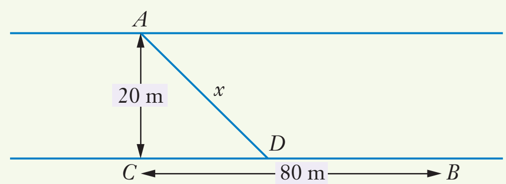
When $x = 75$ Substituting into [1]

$y = 300 - 2(75)$

$= 150$

So the dimensions that give the maximum area are 150 m × 75 m

b First we need to find a formula for the time Kristyn takes to run the distance $AD + DB$



$AD = x$ so find DB

$DB = 80 - CD$

By Pythagoras theorem $x^2 = 20^2 + CD^2$

$x^2 - 20^2 = CD^2$

$CD = \sqrt{x^2 - 400}$

$DB = 80 - \sqrt{x^2 - 400}$

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

$$s = \frac{d}{t}$$

$$st = d$$

$$t = \frac{d}{s}$$

Time taken to swim AD

$$t = \frac{x}{7}$$

Time taken to run DB

$$t_2 = \frac{80 - \sqrt{x^2 - 400}}{11}$$

Total tim:

$$t = t + t_2$$

$$= \frac{x}{7} + \frac{80 - \sqrt{x^2 - 400}}{11}$$

$$= \frac{11x}{77} + \frac{7(80 - \sqrt{x^2 - 400})}{77}$$

$$= \frac{11x + 560 - 7(x^2 - 400)^{\frac{}{2}}}{77}$$

ISBN 97807043220

$$\frac{dt}{dx} = \frac{11 - 7 \times 2x \times \frac{1}{2}(x^2 - 400)^{-\frac{1}{2}}}{77}$$

$$= \frac{11 - 7x(x^2 - 400)^{-\frac{1}{2}}}{77}$$

For minimum time $\frac{dt}{dx} = 0$

$$\frac{11 - 7x(x^2 - 400)^{-\frac{1}{2}}}{77} = 0$$

$$11 - 7x(x^2 - 400)^{-\frac{1}{2}} = 0$$

$$11 = 7x(x^2 - 400)^{-\frac{1}{2}}$$

$$11 = \frac{7x}{\sqrt{x^2 - 400}}$$

$$11\sqrt{x^2 - 400} = 7x$$

$$121(x^2 - 400) = 49x^2 \qquad \text{squaring both sides}$$

$$121x^2 - 48\,400 = 49x^2$$

$$72x^2 = 48\,400$$

$$x^2 = 672222\ldots$$

$$x = \sqrt{672222}$$

$$\approx 259$$

To check that t is a minimum

x	25	25.9	26
$\frac{dt}{dx}$	–0009	0	0.0006

Since the function is decreasing on LHS and increasing on RHS t is a minimum at $x = 259$

So Kristyn should swim a distance of 259 m to minimise her total travel time

Exercise 5.09 Optimisation problems

1 The height in metre, of a ball is given by the equation $h = 16t - 4t^2$ where t is time in seconds Find when the ball will reach its maximum heigh, and what the maximum height will be

2 The cost per hour of a bike ride is given by the formula $C = x^2 - 15x + 70$ where x is the distance travelled in km Find the distance that gives the minimum cos.

3 The perimeter of a rectangle is 60 m and its length is x m

a Show that the area of the rectangle is given by the equation $A = 30x - x^2$

b Hence find the maximum area of the rectangle

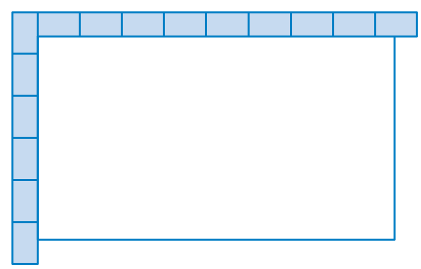
4 A farmer wants to make a rectangular paddock with an area of 4000 m^2 To minimise fencing costs she wants the paddock to have a minimum perimeter.

a Show that the perimeter is given by the equation $P = 2x + \dfrac{8000}{x}$

b Find the dimensions of the rectangle that will give the minimum perimeter, correct to 1 decimal place

c Calculate the cost of fencing the paddock at $4.75 per metr.

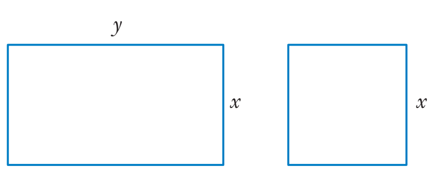
5 Bill wants to put a small rectangular vegetable garden in his backyard using 2 existing walls as part of its border. He has 8 m of garden edging for the border on the other 2 sides Find the dimensions of the garden bed that will give the greatest area



6 Find 2 numbers whose sum is 28 and whose product is a maximum

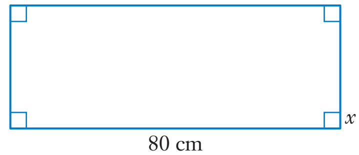
7 The difference of 2 numbers is 5 Find these numbers if their product is to be minimu.

8 A piece of wire 10 m long is broken into 2 parts which are bent into the shape of a rectangle and a square as shown Find the dimensions x and y that make the total area a maximum



ISBN 97807043220

9 A box is made from an 80 cm by 30 cm rectangle of cardboard by cutting out 4 equal squares of side x cm from each corner. The edges are turned up to make an open box



a Show that the volume of the box is given by the equation $V = 4x^3 - 220x^2 + 2400x$

b Find the value of x that gives the box its greatest volume

c Find the maximum volume of the box

10 The formula for the surface area of a cylinder is given by $S = 2\pi r(r + h)$ where r is the radius of its base and h is its height

a Show that if the cylinder holds a volume of 54π m^3 the surface area is given by the equation $S = 2\pi r^2 + \dfrac{108\pi}{r}$

b Hence find the radius that gives the minimum surface area

11 A silo in the shape of a cylinder is required to hold 8600 m^3 of wheat

a Find an equation for the surface area of the silo in terms of the base radius

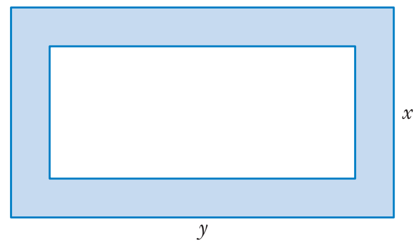
b Find the minimum surface area required to hold this amount of wheat to the nearest square metre

12 A rectangle is cut from a circular disc of radius 6 cm

a Show that the formula for the area of the rectangle is $A = x\sqrt{144 - x^2}$

b Find the area of the largest rectangle that can be produced

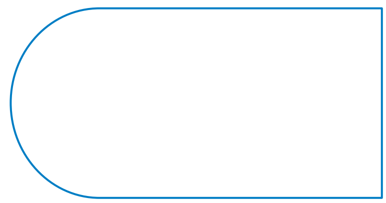
13 A poster consists of a photograph bordered by a 5 cm margin The area of the poster is to be 400 cm^2



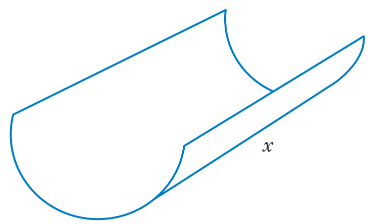
a Show that the area of the photograph is given by the equation $A = 500 - 10x - \dfrac{4000}{x}$

b Find the maximum area possible for the photograph

14 A surfboard is in the shape of a rectangle and semicircle as show. The perimeter is to be m. Find the maximum area of the surfboard correct to 2 decimal places



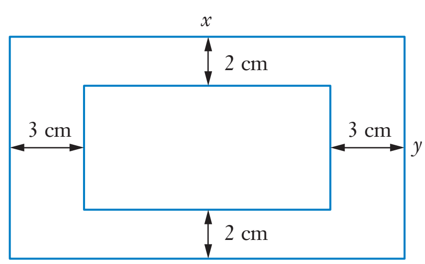
15 A half-pipe is to be made from a rectangular piece of metal of length x m The perimeter of the rectangle is 30 m



a Find the dimensions of the rectangle that will give the maximum surface area

b Find the height from the ground up to the top of the half-pipe with this maximum area correct to 1 decimal plac.

16 The picture frame shown has a border of 2 cm at the top and bottom and 3 cm at the sides If the total area of the border is to be 100 cm^2 find the maximum area of the fram.



17 A 3 m piece of wire is cut into 2 pieces and bent around to form a square and a circle Find the size of the 2 lengths correct to 2 decimal place, that will make the total area of the square and circle a minimum

ISBN 97807043220

18 Two cars are travelling along roads that intersect at right angles to one anothe. One starts 200 km away and travels towards the intersection at 80 km h^{-} while the other starts at 120 km away and travels towards the intersection at 60 km h^{-}

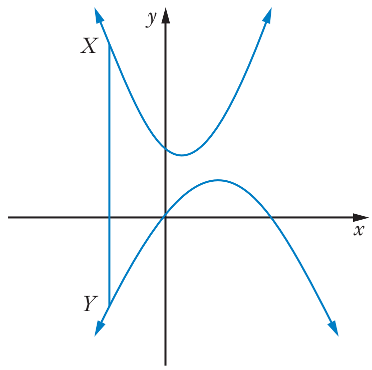
a Show that their distance apart after t hours is given by $d^2 = 10\,000t^2 - 46\,400t + 54\,400$

b Hence find their minimum distance apart

19 X is a point on the curve $y = x^2 - 2x + 5$ Point Y lies directly below X and is on the curve $y = 4x - x^2$

a Show that the distance d between X and Y is $d = 2x^2 - 6x + 5$.

b Find the minimum distance between X and Y



20 A truck travels 1500 km at an hourly cost given by $s^2 + 9000$ cents where s is the average speed of the truck

a Show that the cost for the trip is given by $C = 1500\left(s + \dfrac{9000}{s}\right)$

b Find to the nearest km h^{-} the speed that minimises the cost of the tri.

c Find the cost of the trip to the nearest dollar.

ISBN 97807043220

CLASS CHALLENGE

HERON'S PROBLEM

One boundary of a farm is a straight river bank and on the farm stands a hous. Some distance away there is a shed Each is sited away from the river ban. Each morning the farmer takes a bucket from his house to the river, fills it with watr, and carries the water to the shed

Find the position on the river bank that will allow him to walk the shortest distance from house to river to shed Furthe, describe how the farmer could solve the problem on the ground with the aid of a few stakes for sighting

LEWIS CARROLL'S PROBLEM

After a battle at least 95% of the combatants had lost a tooth at least 90% had lost an eye at least 80% had lost an ar, and at least 75% had lost a lg. At least how many had lost all four?

ISBN 97807043220

5. TEST YOURSELF

For Questions **1–4** choose the correct answer **A B C** or **D**

Pacice quiz

Deivaive indawod

Second deivaive indawod

1 A maximum turning point has

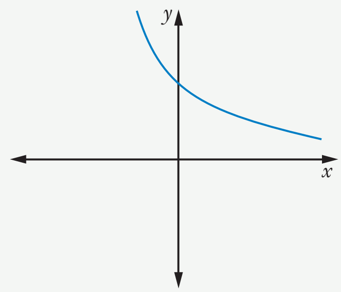
A $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ **B** $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$

C $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$ **D** $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

2 For the graph shown

A $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$ **B** $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$

C $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$ **D** $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$



3 For a horizontal point of inflection

A $f''(x) = 0$ **B** $f'(x) = 0$ and $f''(x) = 0$

C $f''(x) = 0$ and concavity changes **D** $f'(x) = 0, f''(x) = 0$ and concavity changes

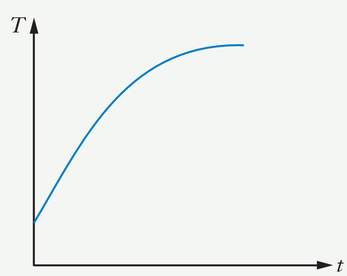
4 The graph below shows temperature T at time t Which statement describes the shape of the graph?

A The temperature is increasing and the rate of change in temperature is increasing

B The temperature is decreasing and the rate of change in temperature is increasing

C The temperature is increasing and the rate of change in temperature is decreasing

D The temperature is decreasing and the rate of change in temperature is decreasing



5 Find the stationary points on the curve $y = x^3 + 6x^2 + 9x - 11$ and determine their nature

6 Find all x values for which the curve $y = 2x^3 - 7x^2 - 3x + 1$ is concave upwards

7 The height in metres of an object thrown up into the air is given by $h = 20t - 2t^2$ where t is time in seconds Find the maximum height that the object reache.

ISBN 97807043220

8 Find the domain over which the curve $y = 5 - 6x - 3x^2$ is decreasing

9 Find the point of inflection on the curve $y = 2x^3 - 3x^2 + 3x - 2$.

10 A soft drink manufacturer wants to minimise the amount of aluminium in its cans while still holding 375 mL of soft drink Given that 375 mL has a volume of 375 cm ^{3}

a show that the surface area of a can is given by $S = 2\pi r^2 + \dfrac{750}{r}$

b find the radius of the can that gives the minimum surface area

11 For the function $y = 3x^4 + 8x^3 + 6x^2$

a find any stationary points

b determine their nature

c sketch the curve for the domain $[-3, 3]$

d find the maximum and minimum values of the function in this domain

12 A rectangular prism with a square base is to have a surface area of 250 cm^2

a Show that the volume is given by $V = \dfrac{125x - x^3}{2}$

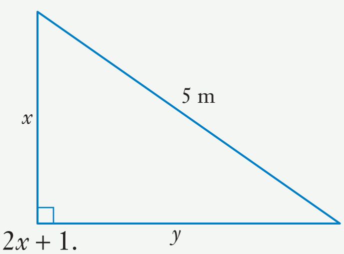
b Find the dimensions that will give the maximum volume

13 The cost to a business of manufacturing x products a week is given by $C = x^2 - 300x + 9000$ Find the number of products that will give the minimum cost each week

14 A 5 m length of timber is used to border a triangular garden bed with the other sides of the garden against the house walls

a Show that the area of the garden is $A = \dfrac{1}{2}x\sqrt{25 - x^2}$

b Find the greatest possible area of the garden bed



15 Find any points of inflection on the curve $f(x) = x^4 - 6x^3 + 2x + 1$.

16 Find the maximum value of the curve $y = x^3 + 3x^2 - 24x - 1$ in the domain [–5, 6.

17 A function has $f'(2) < 0$ and $f''(2) < 0$ Sketch the shape of the function near $x = 2$.

18 Sketch the graph of the function $f(x) = xe^{2x}$ showing all features

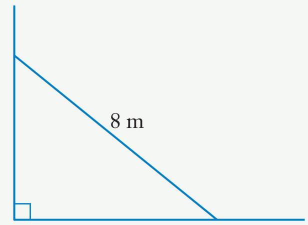
19 Sketch the graph of the function $y = 2 \cos 4x$ in the domain [0 π.

ISBN 97807043220

5. CHALLENGE EXERCISE

1 Sketch the curve $y = x(x - 2)^3$ showing any stationary points and points of inflection

2 Find the maximum possible area if an 8 m length of fencing is placed across a corner to enclose a triangular space



3 Find the greatest and least values of $f(x) = 4x^3 - 3x^2 - 18x$ in the domain [–2, 3.

4 Show that the function $f(x) = 2(5x - 3)^3$ has a horizontal point of inflection at (06 0.

5 Two circles have radii r and s such that $r + s = 25$.
Show that the sum of areas of the circles is least when $r = s$

6 Find the equation of a curve that is always concave upwards with a stationary point at (–1 2) and y-intercept 3

7 **a** Show that $y = x^n$ has a stationary point at (0 0) where n is a positive integer.

b If n is even show that (, 0) is a minimum turning poit.

c If n is odd show that (, 0) is a point of inflectin.

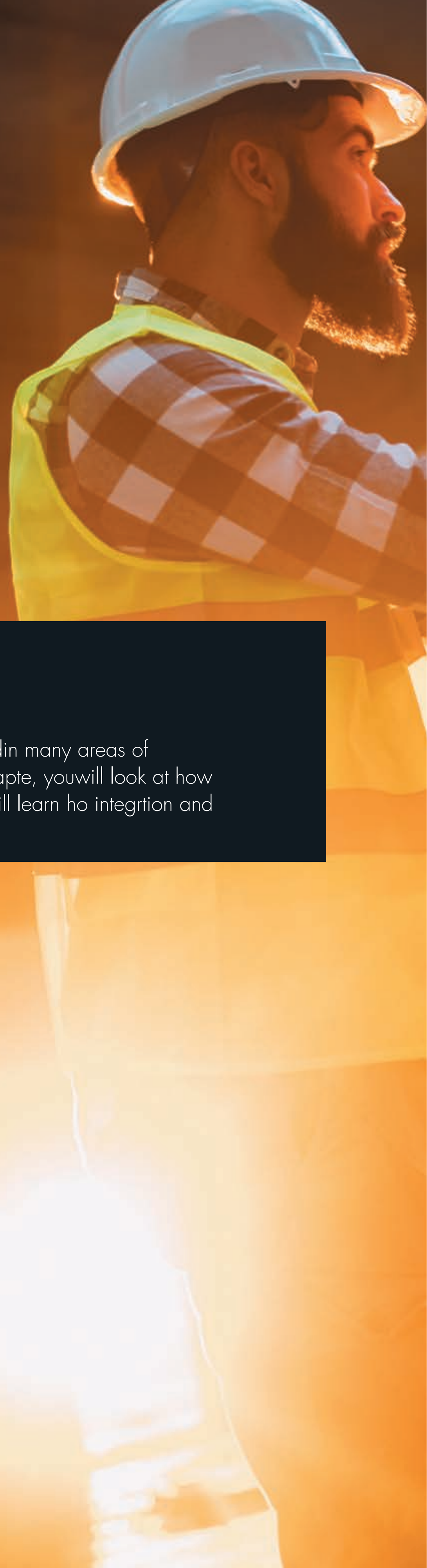
8 Find the minimum and maximum values of $y = \dfrac{x+3}{x^2 - 9}$ in the domain [–2, 2.

9 The cost of running a car at an average speed of V km h^{-} is given by $c = 100 + \dfrac{V^2}{75}$ cents per hour. Find the average speed (to the nearest km h^{-}) at which the cost of a 1000 km trip is a minimum

ISBN 97807043220

CALCULUS

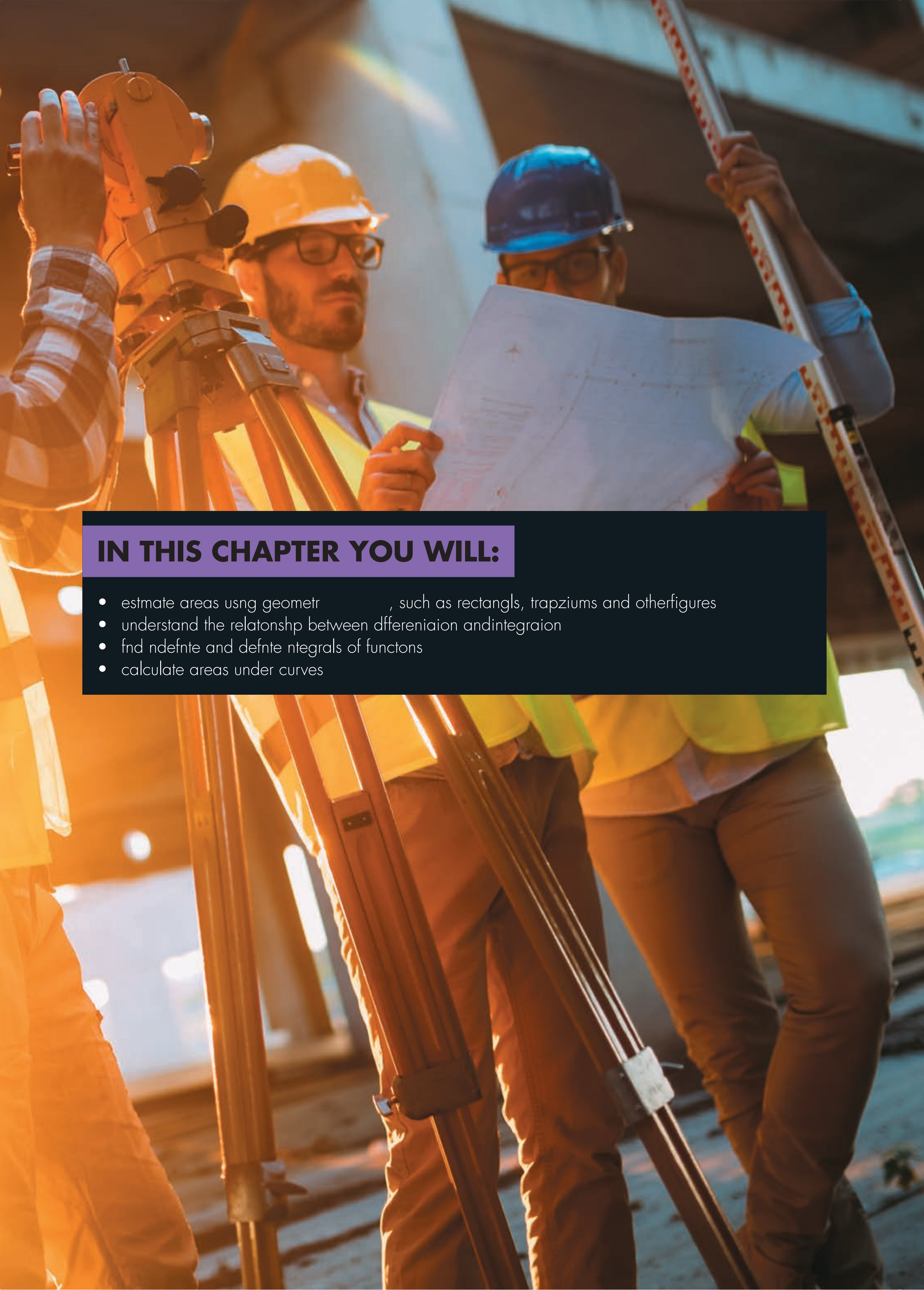
6.



INTEGRATION

Integraton s the process of fndng an area under a curv. Tisis usedin many areas of knowledge such as survein, phyics and the soial sience. In tis chapte, youwill look at how to fnd both approxmate and exact areas under a curve and you ill learn ho integrtion and dffereniaion are relate.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- estmate areas usng geometr , such as rectangls, trapziums and otherfigures
- understand the relatonshp between dffereniaion andintegraion
- fnd ndefnte and defnte ntegrals of functons
- calculate areas under curves

TERMINOLOGY

definite integra: The integral or anti-derivative $y = F(x)$ used to find the area between the curve $y = f(x)$ the x-axis and boundaries $x = a$ and $x = b$ given by $\int_a^b f(x)\,dx = F(b) - F(a)$

indefinite integra: A general anti-derivative $\int f(x)\,dx$

integra: An anti-derivativ.

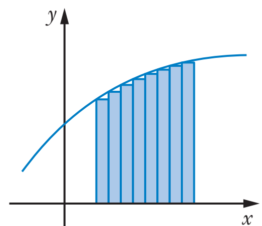
integratio: The process of finding an anti-derivative

trapezoidal rul: A formula for approximating area under a curve by using a trapezium

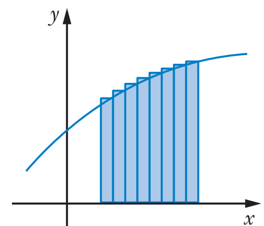
WS
Aeas using ecangle

6.01 Approximating areas under a curve

Mathematicians since the time of Archimedes have used rectangles to approximate irregular areas In more recent times we use the number plane to find areas enclosed between a curve and the x-axis We call this the **area under the curve**



inner rectangles



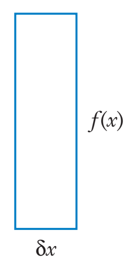
outer rectangles

The first diagram has inner or left rectangles that are below the curve because the top left corners of the rectangles touch the curv.

The second diagram has outer or right rectangles that are above the curve because the top right corners of the rectangles touch the curve

The more rectangles we have the more accurately they approximate the area under the curv.

Integral notation



The diagram at right shows one of the rectangles The height of each rectangle is $f(x)$ and its width is δx so its area is $f(x)\ \delta x$ So the sum of all the rectangles is $\Sigma f(x)\ \delta x$ for the different values of x

We can approximate the area under the curve using a large number of rectangles by making the width of each rectangle very small

Taking an infinite number of rectangle, $\delta x \to 0$

$$\text{Area} = \lim_{\delta x \to 0}\left(\sum f(x)\delta x\right)$$
$$= \int f(x)\,dx$$

δx s 'delta x and means a small change n x.
δ s the Greek letter for d for dfference

We use the **integral** symbol $\int$ to stand for the sum of rectangles (the symbol is an S for sum)

We call $\int f(x)\,dx$ an **indefinite integral**

If we are finding the area under the curve $y = f(x)$ between $x = a$ and $x = b$ we can write $\int_a^b f(x)\,dx$

We call $\int_a^b f(x)\,dx$ a **definite integral**

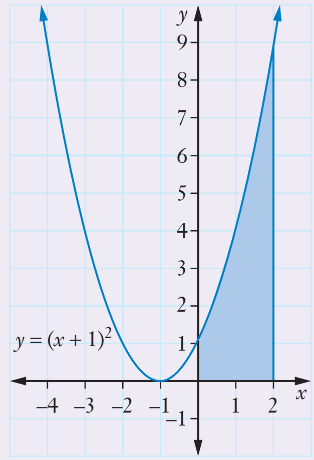
ISBN 97807043220

EXAMPLE 1

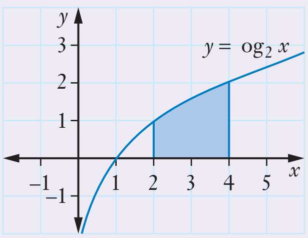
a Find an approximation to the shaded area by using

i 4 inner rectangles

ii 4 outer rectangles

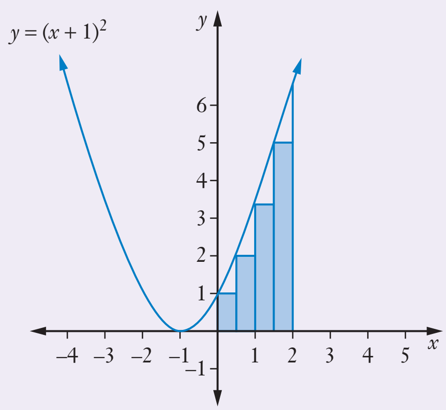


b Find the shaded area below by using a trapezium



Solution

a **i** Using inner rectangles the top left corners touch the curve and they lie below the curve



Each rectangle has height $f(x)$ and width 05 units

Height of 1st rectangle

$f(0) = (0 + 1)^2 = 1$

Area $= 1 \times 05 \ = 05$

Height of 2nd rectangle

$f(05) \ = (05 \ + 1)^2 = 225$

Area $= 225 \ \times 05 \ = 1.125$

Height of 3rd rectangle

$f(1) = (1 + 1)^2 = 4$

Area $= 4 \times 05 \ = 2$

Height of 4th rectangle

$f(15) \ = (1.5 + 1)^2 = 625$

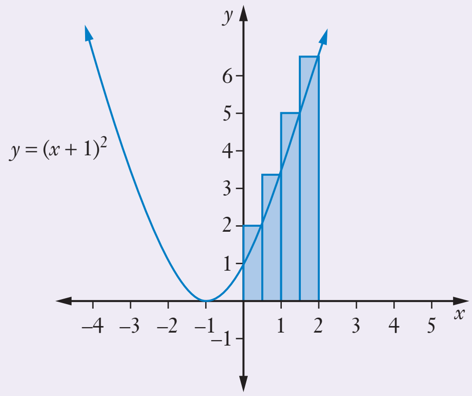
Area $= 625 \ \times 05 \ = 3.125$

Total area $= 05 \ + 1.125 + 2 + 3.125 = 675$

So area is 675 units ^{2}

ISBN 97807043220

ii Using outer rectangles the top right corners touch the curve and they go above the curve



Height of 1st rectangle

$f(05) = (05 + 1)^2 = 225$

Area = $225 \times 05 = 1.125$

Height of 2nd rectangle

$f(1) = (1 + 1)^2 = 4$

Area = $4 \times 05 = 2$

Height of 3rd rectangle

$f(15) = (1.5 + 1)^2 = 625$

Area = $625 \times 05 = 3.125$

Height of 4th rectangle

$f(2) = (2 + 1)^2 = 9$

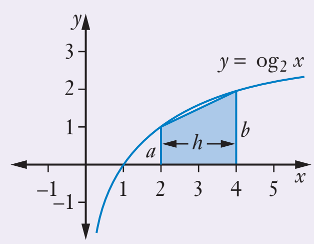
Area = $9 \times 05 = 45$

Total area = $1.125 + 2 + 3.125 + 45 = 1075$

So area is 1075 units ^{2}

b Area of a trapezium

$A = \frac{1}{2}h(a + b)$ where h = perpendicular height and a and b are parallel sides



Here the trapezium has parallel sides at $x = 2$ and $x = 4$

$a = f(2)$

$= \log_2 2$

$= 1$

$b = f(4)$

$= \log_2 4$

$= 2$

$h = 4 - 2$

$= 2$

$A = \frac{1}{2}h(a + b)$

$= \frac{1}{2}(2)(1 + 2)$

$= 3$

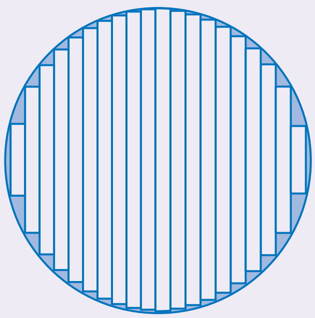
So area is 3 units^2

ISBN 97807043220

DID YOU KNOW?

Archimedes

Integration has been of interest to mathematicians since very early times **Archimedes** (287–212 BCE) found the area of enclosed curves by cutting them into very thin layers and finding their sum He found the formula for the volume of a sphere this way. He also found an estimation of π correct to 2 decimal place.



Area within curve

Archimedes

TECHNOLOGY

Areas under a curve

We can use a spreadsheet to find approximate areas under a curve using rectangle. Using technology allows us to find sums of large numbers of rectangles without needing to do many calculations This gives a more accurate approximation to the area under a cure.

For example we can use a spreadsheet to find the approximate area under the curve $y = (x + 1)^2$ between $x = 0$ and $x = 2$ from Example 1 **a i**

We find the y values using the formula **=(A2+1)^2** (copy the formula down the column)

The width is **=A3–A2** (copy this value down the column)

The area is **=B2*C2** (copy the formula down the column)

	A	B	C	D
1	x	y	Width	Area
2	0	1	0.5	0.5
3	0.5	2.25	0.5	1.125
4	1	4	0.5	2
5	1.5	6.25	0.5	3.125
6				
7			Total area	6.75

We can use the spreadsheet to find the area using a much larger number of rectangle, for example 20 or 4.

	A	B	C	D	E	F	G	H	I
1	x	y	Width	Area		x	y	Width	Area
2	0	1	0.1	0.1		0	1	0.05	0.05
3	0.1	1.21	0.1	0.121		0.05	1.1025	0.05	0.055125
4	0.2	1.44	0.1	0.144		0.1	1.21	0.05	0.0605
5	0.3	1.69	0.1	0.169		0.15	1.3225	0.05	0.066125
6	0.4	1.96	0.1	0.196		0.2	1.44	0.05	0.072
7	0.5	2.25	0.1	0.225		0.25	1.5625	0.05	0.078125
8	0.6	2.56	0.1	0.256		0.3	1.69	0.05	0.0845
9	0.7	2.89	0.1	0.289		0.35	1.8225	0.05	0.091125
10	0.8	3.24	0.1	0.324		0.4	1.96	0.05	0.098
11	0.9	3.61	0.1	0.361		0.45	2.1025	0.05	0.105125
12	1	4	0.1	0.4		0.5	2.25	0.05	0.1125
13	1.1	4.41	0.1	0.441		0.55	2.4025	0.05	0.120125
14	1.2	4.84	0.1	0.484		0.6	2.56	0.05	0.128
15	1.3	5.29	0.1	0.529		0.65	2.7225	0.05	0.136125
16	1.4	5.76	0.1	0.576		0.7	2.89	0.05	0.1445
17	1.5	6.25	0.1	0.625		0.75	3.0625	0.05	0.153125
18	1.6	6.76	0.1	0.676		0.8	3.24	0.05	0.162
19	1.7	7.29	0.1	0.729		0.85	3.4225	0.05	0.171125
20	1.8	7.84	0.1	0.784		0.9	3.61	0.05	0.1805
21	1.9	8.41	0.1	0.841		0.95	3.8025	0.05	0.190125
22	2	9	0.1	0.9		1	4	0.05	0.2
23						1.05	4.2025	0.05	0.210125
24						1.1	4.41	0.05	0.2205
25			Total area	9.17		1.15	4.6225	0.05	0.231125
26						1.2	4.84	0.05	0.242
27						1.25	5.0625	0.05	0.253125
28						1.3	5.29	0.05	0.2645
29						1.35	5.5225	0.05	0.276125
30						1.4	5.76	0.05	0.288
31						1.45	6.0025	0.05	0.300125
32						1.5	6.25	0.05	0.3125
33						1.55	6.5025	0.05	0.325125
34						1.6	6.76	0.05	0.338
35						1.65	7.0225	0.05	0.351125
36						1.7	7.29	0.05	0.3645
37						1.75	7.5625	0.05	0.378125
38						1.8	7.84	0.05	0.392
39						1.85	8.1225	0.05	0.406125
40						1.9	8.41	0.05	0.4205
41						1.95	8.7025	0.05	0.435125
42						2	9	0.05	0.45
43									
44								Total area	8.9175
45									

Use technology with a larger number of rectangles for a more accurate area to this question

ISBN 97807043220

We can find the area under the same curve by using different method.

We can use other shapes to find areas under a curv.

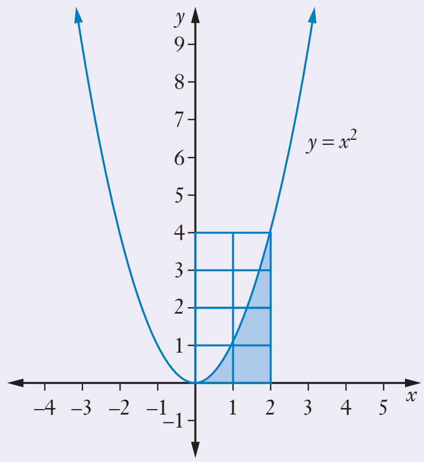
EXAMPLE 2

Find an approximation to the area under the curve $y = x^2$ between $x = 0$ and $x = 2$ by using

a squares

b a triangle

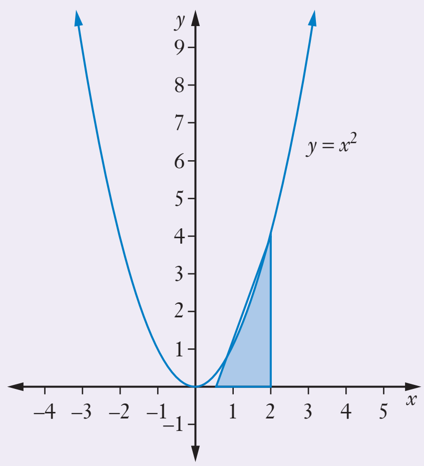
Solution



a On the grid each square is 1 square uni. By counting and approximating squares

$A \approx 3$

So area is 3 units^2



b Using a triangle

$b = 2 - 05 \ = 1.5$

$h = f(2)$

$= 2^2$

$= 4$

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 1\ 5 \times 4$

$= 3$

So area is 3 units^2

ISBN 97807043220

Exercise 6.01 Approximating areas under a curve

1 Find an approximation to the area under the curve $y = x^2 + 2x$ between $x = 1$ and $x = 2$ by using

a 2 inner rectangles **b** 2 outer rectangles

2 Find an approximation (to 2 decimal places) to the area under the curve $y = \frac{2}{x+1}$ from $x = 1$ to $x = 3$ using

a 2 inner rectangles **b** 2 outer rectangles

c 4 inner rectangles **d** 4 outer rectangles

3 Use a trapezium to find an approximate area under the curve

a $f(x) = x^2$ between $x = 2$ and $x = 3$

b $y = \ln x$ between $x = 4$ and $x = 7$

c $f(x) = x^3 + 1$ between $x = 0$ and $x = 4$

d $f(x) = \sin x$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$ (give answer in exact form)

e $y = 9 - x^2$ between $x = 1$ and $x = 2$

4 Find the approximate area under the curve $f(x) = x^3 + 3$ between $x = 0$ and $x = 4$ by using

a 2 inner rectangles **b** 2 outer rectangles **c** a trapezium

5 Use a trapezium to find each area under the curve

a $y = \frac{1}{x}$ between $x = 1$ and $x = 7$

b $y = x^2 + 5$ between $x = 0$ and $x = 1$

c $f(x) = \cos x$ between $x = 0$ and $x = \frac{\pi}{3}$ (in exact form)

d $y = e^x$ between $x = 1$ and $x = 4$ (in exact form)

e $f(x) = x(x - 4)(x - 9)$ between $x = 2$ and $x = 3$

6 **a** Sketch the graph of $y = 1 - x^2$ and shade the area under the curve (enclosed between the curve and the x-axis)

b Find this approximate area by using a triangle

7 Find the approximate area under the curve $y = \sqrt{x-1}$ between $x = 2$ and $x = 5$ by using

a 6 inner rectangles **b** 6 outer rectangles

c a trapezium **d** squares

8 Find the exact area under the curve $y = \sqrt{25 - x^2}$

ISBN 97807043220

9 **a** Find the exact area under the curve $y = \sqrt{9 - x^2}$

b Find the approximate area under the curve $y = \sqrt{9 - x^2}$

i between $x = 1$ and $x = 2$ using a trapezium

ii between $x = 0$ and 3 using 3 outer rectangles

10 Use a triangle to find the approximate area under the curve

a $y = x^2$ between $x = 0$ and $x = 4$

b $y = \sqrt{x}$ between $x = 0$ and $x = 3$

c $y = \cos x$ between $x = 0$ and $x = \dfrac{\pi}{2}$

11 Find the approximate area under each curve by using

i 4 inner rectangles **ii** 4 outer rectangles

a $y = -x^2 + 4x$

b $y = \sin x$ in the domain $[0 \quad \pi]$ (in exact form)

12 Find the approximate area under the curve $y = x^2 + 5$ between $x = 0$ and $x = 5$ (using technology where available) using

a 10 inner rectangles **b** 10 outer rectangles

6.02 Trapezoidal rule

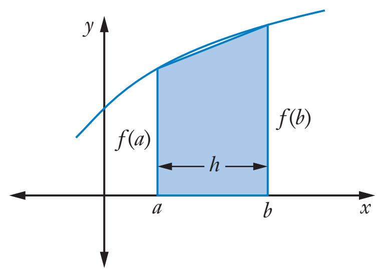
WS Trapezoidal ule

A trapezium usually gives a much closer approximation to the area under a curve than a rectangle does

The **trapezoidal rule** is a formula that uses a trapezium to find the area under a curve

$$A = \frac{1}{2}h[f(a) + f(b)] \text{ where } h = b - a$$

$$= \frac{1}{2}(b - a)[f(a) + f(b)]$$



Trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b - a)[f(a) + f(b)]$$

ISBN 97807043220

EXAMPLE 3

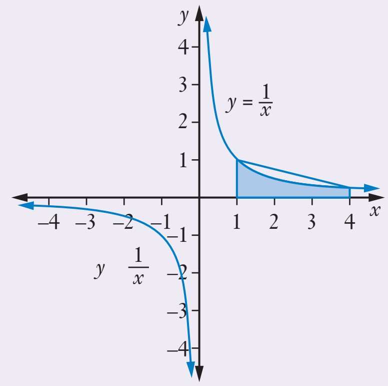
Use the trapezoidal rule to find an approximation for

a $\int^{4} \frac{1}{x}\,dx$

b $\int_{0} x^{3}\,dx$ using 2 subintervals

Solution

a $\int^{4} \frac{1}{x}\,dx$ is the area under the curve as shaded in the diagram



$f(x) = \frac{1}{x}$ $a = 1$ and $b = 4$

$$\int_{a}^{b} f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)]$$

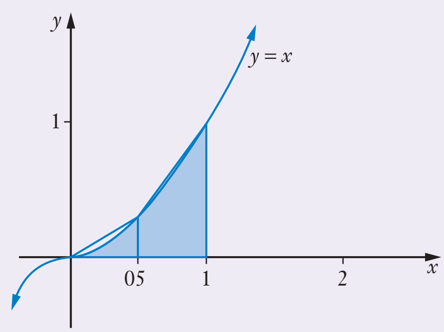
$$\int^{4} \frac{1}{x}\,dx \approx \frac{1}{2}(4-1)[f(1)+f(4)]$$

$$= \frac{1}{2}(3)\left[\frac{1}{1}+\frac{1}{4}\right]$$

$$= \frac{15}{8}$$

$$= 1\frac{7}{8}$$

b 2 subintervals means 2 trapezia
We use the trapezoidal formula twic.



$f(x) = x^{3}$ and $h = 05$

$$\int_{a}^{b} f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)]$$

$$\int_{0} x^{3}\,dx = \int_{0}^{0.5} x^{3}\,dx + \int_{0.5} x^{3}\,dx$$

$$\approx \frac{1}{2}(05\ -0)[f(0)+f(05)]\ +\frac{1}{2}(1-05)[\ f(05)\ +f(1)]$$

$$= \frac{1}{2}(05)[0^{\ 3} + 05^{\ 3}] + \frac{1}{2}(05)[05^{\ \ 3} + 1^{3}]$$

$$= 03125$$

ISBN 97807043220

There is a more general trapezoidal rule when using several subintervals or trapezia

Trapezoidal rule for *n* subintervals

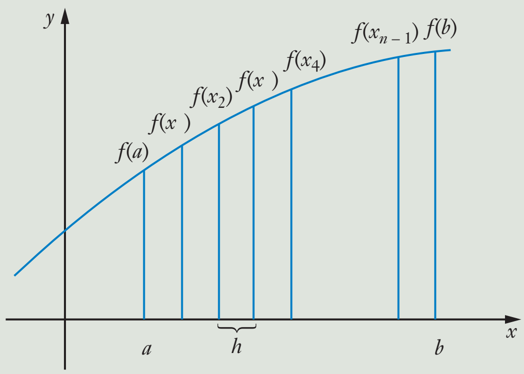
Given n subintervals (trapezia)

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\Big[f(a)+f(b)+2\big\{f(x_1)+ \quad + f(x_{n-1})\big\}\Big]$$

where $a = x_0$ and $b = x_n$ and the values of x_0 x x_2 x_n are found by dividing the interval $a \le x \le b$ into n equal subintervals of width $h = \frac{b-a}{n}$

Since $h = \frac{b-a}{n}$ the formula can also be written as

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\Big[f(a)+f(b)+2\big\{f(x_1)+ \quad + f(x_{n-1})\big\}\Big]$$



Proof

Interval $b - a$ is divided into n trapezia

So the width of each trapezium is $h = \frac{b-a}{n}$

$$\int_a^b f(x)\,dx \approx \frac{1}{2}h[f(a)+f(x\ )] + \frac{1}{2}h[f(x\ )+f(x_2)] + \frac{1}{2}h[f(x_2)+f(x_3)] + \quad + \frac{1}{2}h[f(x_{n-1})+f(b)]$$

$$= \frac{h}{2}[f(a)+f(x\ )+f(x\ )+f(x_2)+f(x_2)+f(x_3)+ \quad + f(x_{n-1})+f(b)]$$

$$= \frac{h}{2}(f(a)+2f(x\ )+2f(x_2)+2f(x_3)+ \quad + 2f(x_{n-1})+f(b))$$

$$= \frac{h}{2}\Big[f(a)+f(b)+2\big\{f(x_1)+ \quad + f(x_{n-1})\big\}\Big]$$

ISBN 97807043220

EXAMPLE 4

Trapezoidal ule

a Use the trapezoidal rule with 4 subintervals to find an approximation for $\int_2^3 \frac{2}{x-1}\,dx$ correct to 3 decimal places

b Use the trapezoidal rule with 7 subintervals to find an approximation for $\int_0^{14} (t^2+3)\,dt$

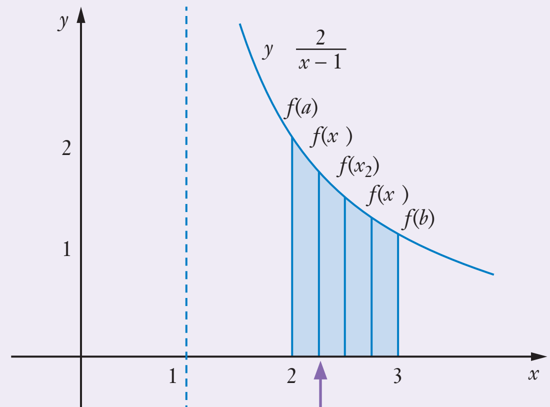
Solution

a From the question and diagram

$a = 2, b = 3, n = 4$

The width of each trapezium is

$$h = \frac{b-a}{n}$$
$$= \frac{3-2}{4}$$
$$= \frac{1}{4}$$
$$= 025$$



Note that these x values ncrease by h 025 the wdth of each trapezum

Substituting into the general trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+f(b)+2\{f(x_1)+ \quad + f(x_{n-1})\}\right]$$

$$\int_2^3 \frac{2}{x-1}\,dx \approx \frac{025}{2}\left[f(2)+f(3)+2\{f(225\ )+f(2\ 5)+f(275)\}\right]$$

$$=0125\left[\frac{2}{2-1}+\frac{2}{3-1}+2\left(\frac{2}{225\ -1}+\frac{2}{2\ 5-1}+\frac{2}{275\ -1}\right)\right]$$

$$= 0125[2\ \ +1+2(16\ \ +1.3333\quad +11428\quad)]$$

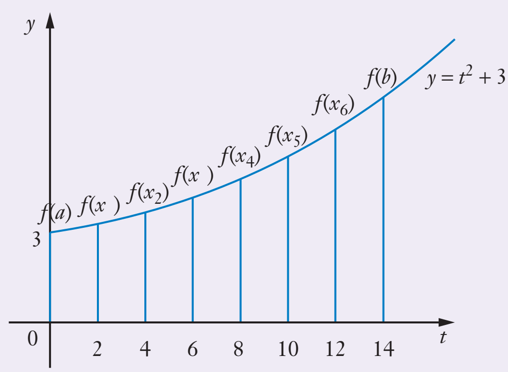
$$= 139404$$

$$\approx 1394$$

b $a = 0, b = 14, n = 7.$

The width of each trapezium is

$$h = \frac{14-0}{7}$$
$$= \frac{14}{7}$$
$$= 2$$



 ISBN 97807043220

Substituting into the general trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+f(b)+2\left\{f(x_1)+\ \ +\ \ f(x_{n-1})\right\}\right]$$

$$\int_0^{14}(t^2+3)\,dt \approx \frac{2}{2}\left[f(0)+f(14)+2\left\{f(2)+f(4)+f(6)+f(8)+f(10)+f(12)\right\}\right]$$

$$= [(0^2+3)+(14^2+3)+2\{(2^2+3)+(4^2+3)+(6^2+3)+(8^2+3)$$
$$+(10^2+3)+(12^2+3)\}]$$

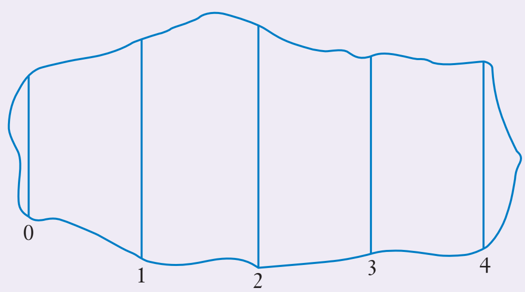
$$= 966$$

We can use the trapezoidal rule to find irregular area.

EXAMPLE 5

A surveyor needs to find the area of the irregular piece of land shown

Use the trapezoidal rule to find its approximate area



Solution

We can use the values in the diagram or put them in the table belo.

x	0	1	2	3	4
$f(x)$	3.7	5.9	6.4	5.1	4.9

$a = 0$, $b = 4$, $n = 4$

From the diagram or table the height of each trapezium is .

$$\text{Area} \approx \frac{h}{2}\left[f(a)+f(b)+2\left\{f(x_1)+\ \ +f(x_{n-1})\right\}\right]$$

$$= \frac{1}{2}\left[f(0)+f(4)+2\left\{f(1)+f(2)+f(3)\right\}\right]$$

$$= \frac{1}{2}\left[3\ 7+4\,9+2\left\{5\,9+6\,4+5\,1\right\}\right]$$

$$= 21.7$$

So the area of the land is approximately 217 m ^{2}

Exercise 6.02 Trapezoidal rule

1 Use the trapezoidal rule to find an approximation for each integral

a $\int^{2} x^2\,dx$ **b** $\int_0^{2} (x^3+1)\,dx$ **c** $\int^{5} \frac{dx}{x}$ **d** $\int^{2} \frac{dx}{x+3}$

2 Find an approximation to $\int^{3} x^3\,dx$ using the trapezoidal rule with

a 1 subinterval **b** 2 subintervals

3 Use the trapezoidal rule with 2 trapezia to find an approximation to

a $\int_2^{3} \log x\,dx$ **b** $\int_0^{2} \frac{dx}{x+4}$

4 Find an approximation to

a $\int^{4} \log x\,dx$ using 3 trapezia **b** $\int_0^{2} (x^2-x)\,dx$ using 4 trapezia

c $\int_0 \sqrt{x}\,dx$ using 5 subintervals **d** $\int^{5} \frac{dx}{x^2}$ using 4 subintervals

e $\int_3^{6} \frac{dx}{x-1}$ using 6 trapezia

5 Given the table of values find the approximate value of each definite integra.

a $\int^{9} f(x)\,dx$

x	1	3	5	7	9
$f(x)$	3.2	5.9	8.4	11.6	20.1

b $\int^{4} f(t)\,dt$

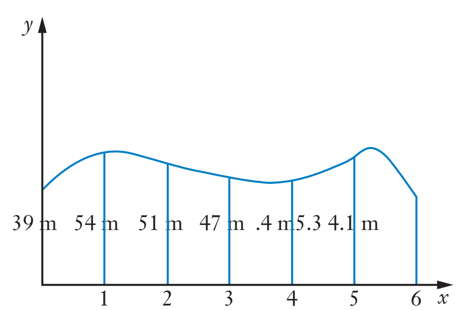
t	1	2	3	4
$f(t)$	8.9	6.5	4.1	2.9

c $\int_2^{14} f(x)\,dx$

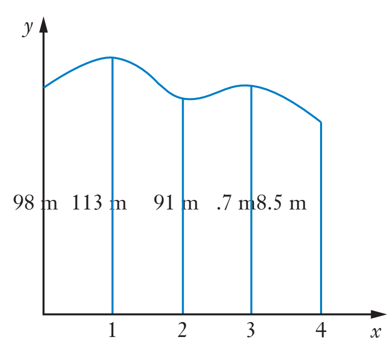
x	2	4	6	8	10	12	14
$f(x)$	25.1	37.8	52.3	89.3	67.8	45.4	39.9

6 Use the trapezoidal rule to find the approximate area of each irregular figure below.

a

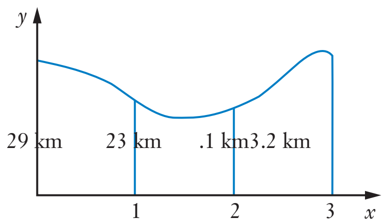


b

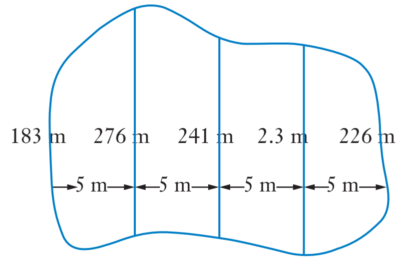


ISBN 97807043220

c



d



6.03 Definite integrals

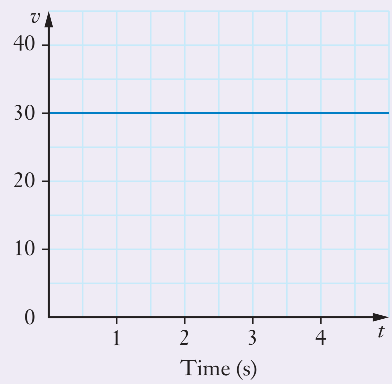
We can link the area under a graph to calculu.

WS Deinie inegal

WS Finding deinie inegal

EXAMPLE 6

This graph shows the velocity of an object over time as it travels at a constant 30 m s⁻



a Find the distance it travels in

i 1 s **ii** 2 s **iii** 3 s

b Find the area under the line between

i $t = 0$ and $t = 1$ **ii** $t = 0$ and $t = 2$ **iii** $t = 0$ and $t = 3$

Solution

a **i** $s = \frac{d}{t}$, so $d = st$

In 1 s the object travels $30 \times 1 = 30$ m

ii In 2 s the object travels $30 \times 2 = 60$ m

iii In 3 s the object travels $30 \times 3 = 90$ m

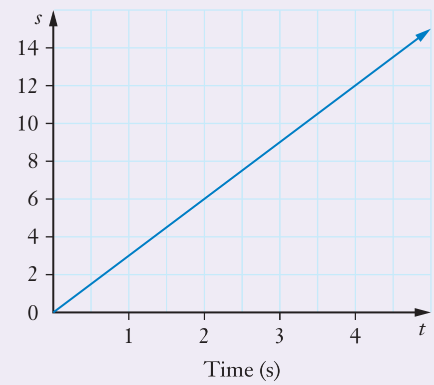
b **i** The area is $30 \times 1 = 30$ units^2

ii The area is $30 \times 2 = 60$ units^2

iii The area is $30 \times 3 = 90$ units^2

EXAMPLE 7

This graph shows the speed of an object increasing at a steady rate



a Find the distance travelled in

i 1 s

ii 4 s

b Find the area under the graph between

i $t = 0$ and $t = 1$

ii $t = 0$ and $t = 4$

Solution

a **i** At $t = 0, s = 0$, and at $t = 1, s = 3$.

$$\text{Average speed} = \frac{0+3}{2}$$
$$= 1.5 \text{ m s}^{-}$$

In 1 s the object travels
$15 \ \times 1 = 1.5$ m.

ii At $t = 0, s = 0$, and at $t = 4, s = 12$.

$$\text{Average speed} = \frac{0+12}{2}$$
$$= 6 \text{ m s}^{-}$$

In 4 s the object travels
$6 \times 4 = 24$ m

b **i** $A = \frac{1}{2}bh$

$$= \frac{1}{2} \times 1 \times 3$$
$$= 15 \text{ units}^2$$

ii $A = \frac{1}{2}bh$

$$= \frac{1}{2} \times 4 \times 12$$
$$= 24 \text{ units}^2$$

The graphs in the last 2 examples show velocity (rate of change of displacement) and speed (rate of change of distance) against time The area under each curve gave the information about the original variable This is the anti-derivatie.

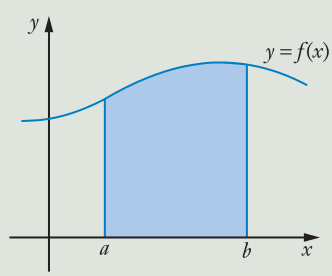
In the same way, the area under any rate of change graph will give the original variabe, or the anti-derivative

Fundamental theorem of calculus

The area enclosed by the curve $y = f(x)$ the x-axis and the lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is the anti-derivative of function $f(x)$



ISBN 97807043220

Proof

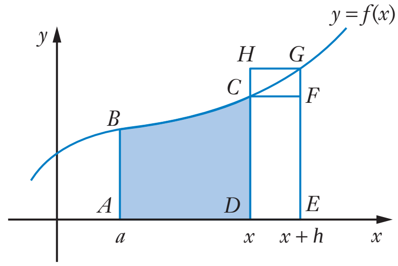
Consider a continuous curve $y = f(x)$ for all values of $x > a$

Let area $ABCD$ be $A(x)$

Let area $ABGE$ be $A(x + h)$

Then area $DCGE$ is $A(x + h) - A(x)$

Area $DCFE <$ area $DCGE <$ area $DHGE$



$$f(x) \times h < A(x+h) - A(x) \qquad < f(x+h) \times h$$

$$f(x) < \frac{A(x+h) - A(x)}{h} \qquad < f(x+h)$$

$$\lim_{h \to 0} f(x) < \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} < \lim_{h \to 0} f(x+h)$$

$$f(x) < A'(x) < f(x)$$

Ths s the formula for the deivaive of A x) from frst prncples, from Year 11, Chapter 6 *ntroducton to cacuus* .

So $A'(x) = f(x)$

$A(x)$ is *an* anti-derivative of $f(x)$

Let $F(x)$ be the anti-derivative of $f(x)$ with a constant term of 0

Then $A(x) = F(x) + C$ [1]

Now $A(x)$ is the area under $y = f(x)$ between a and x

$A(a) = 0$

Substitute in [1]

$A(a) = F(a) + C$

$0 = F(a) + C$

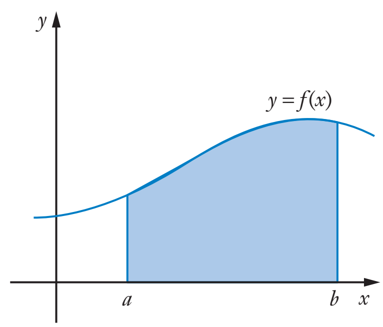
$-F(a) = C$

$A(x) = F(x) - F(a)$

If $x = b$ where $b > a$

$A(b) = F(b) - F(a)$

So $\int_b^a f(x)\,dx = F(b) - F(a)$



EXAMPLE 8

Evaluate

a $\int_3^4 (2x+1)\,dx$ **b** $\int_0^5 3x^2\,dx$ **c** $\int_0^2 (-3x^2)\,dx$ **d** $\int_{-} x^3\,dx$

Solution

You learned in Chapter , *Further differentiation* that the anti-derivative of x^n is $\frac{1}{n+1}x^{n+} + C$ Let $F(x)$ be the anti-derivative where $C = 0$

ISBN 97807043220

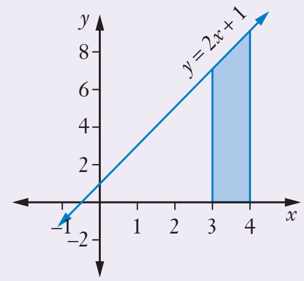
a $\int_3^4 (2x+1)\,dx = \left[x^2 + x\right]_3^4$ The anti-derivative

$= (4^2 + 4) - (3^2 + 3)$ $F(b) - F(a)$

$= 20 - 12$

$= 8$

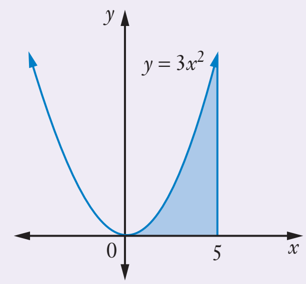
The graph shows the area that ths ntegral calculates
Can you fnd ts area an easer way ?



b $\int_0^5 3x^2\,dx = \left[x^3\right]_0^5$

$= 5^3 - 0^3$

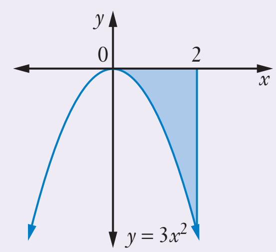
$= 125$



c $\int_0^2 (-3x^2)\,dx = \left[-x^3\right]_0^2$

$= -2^3 - (-0^3)$

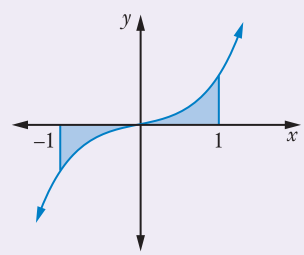
$= -8$



d $\int_{-} x^3\,dx = \left[\frac{x^4}{4}\right]_{-}$

$= \frac{1^4}{4} - \frac{(-1)^4}{4}$

$= 0$



We can also find the definite integral of x^n when n is a fraction or negative

EXAMPLE 9

Evaluate

a $\int^2 \frac{2x-3}{x^3}\,dx$ **b** $\int^8 \sqrt[3]{x}\,dx$

ISBN 97807043220

Solution

a $\displaystyle\int^{2}\frac{2x-3}{x^3}\,dx=\int^{2}\frac{2x}{x^3}-\frac{3}{x^3}\,dx$

$$=\int^{2}\frac{2}{x^2}-\frac{3}{x^3}\,dx$$

$$=\int^{2}(2x^{-2}-3x^{-3})\,dx$$

$$=\left[\frac{2x^{-1}}{-1}-\frac{3x^{-2}}{-2}\right]^{2}$$

$$=\left[-\frac{2}{x}+\frac{3}{2x^2}\right]^{2}$$

$$=\left[-\frac{2}{2}+\frac{3}{2(2)^2}\right]-\left[-\frac{2}{1}+\frac{3}{2(1)^2}\right]$$

$$=-1+\frac{3}{8}+2-\frac{3}{2}$$

$$=-\frac{1}{8}$$

b $\displaystyle\int^{8}\sqrt[3]{x}\,dx=\int^{8}x^{\frac{}{3}}\,dx$

$$=\left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]^{8}$$

$$=\left[\frac{3x^{\frac{4}{3}}}{4}\right]^{8}$$

$$=\left[\frac{3\sqrt[3]{x^4}}{4}\right]^{8}$$

$$=\frac{3\sqrt[3]{8^4}}{4}-\frac{3\sqrt[3]{1^4}}{4}$$

$$=\frac{3\times16}{4}-\frac{3\times1}{4}$$

$$=11\frac{1}{4}$$

We can use the definite integral to find original information given a rate of chang.

EXAMPLE 10

The velocity of a particle is given by $v = 8t^3 - 3t^2 + 6t + 1$ cm s^{-}
Find the change in displacement in the first 3 seconds

Solution

$$v=\frac{dx}{dt}=8t^3-3t^2+6t+1$$

$$x=\int_0^3(8t^3-3t^2+6t+1)\,dt$$

$$=\left[8\frac{t^4}{4}-3\frac{t^3}{3}+6\frac{t^2}{2}+t\right]_0^3$$

$$=\left[2t^4-t^3+3t^2+t\right]_0^3$$

$$=[2(3)^4-3^3+3(3)^2+3]-[2(0)^4-0^3+3(0)^2+0]$$

$$=165$$

So the change in displacement in the first 3 seconds is 165 cm

DID YOU KNOW?

Differentiation vs integration

Many mathematicians in the 17th century were interested in the problem of finding areas under a curve The Englishman **Isaac Barrow** (1630–77) is said to be the first to discover that differentiation and integration are inverse operations This discovery is called the **fundamental theorem of calculus**

Barrow was an outstanding Greek scholar as well as making contributions in the areas of mathematics theolog, astronomy and physcs. Howver, when he was a scholboy, he was so often in trouble that his father was overheard saying to God in his prayers that if he decided to take one of his children he could best spare Isaa.

Another English mathematician named Isaac Sir Isaac Newton (1643 –1727) was also a scientist and astronomer, and helped to discover calculs. He was not interested in his school work but spent most of his time inventing thing, such as a water clock and sundil.

Newton left school at 14 to manage the family estate after his sepfatherdied. However, he spent so much time reading that he was sent back to school He went on to university and developed the theories in mathematics and science that have made him famous today.

Exercise 6.03 Definite integrals

1 Evaluate each definite integral

a $\int_0^2 4x\,dx$ **b** $\int^3 (2x+1)\,dx$ **c** $\int_{-}^6 3x^2\,dx$

d $\int^2 (4t-7)\,dt$ **e** $\int_{-} (6y+5)\,dy$ **f** $\int_0^3 6x^2\,dx$

g $\int^2 (x^2+1)\,dx$ **h** $\int_0^2 4x^3\,dx$ **i** $\int_{-}^4 (3x^2-2x)\,dx$

2 Evaluate

a $\int_{-} x^2\,dx$ **b** $\int_{-2}^3 (x^3+1)\,dx$ **c** $\int_{-2}^2 x^5\,dx$

d $\int^4 \sqrt{x}\,dx$ **e** $\int_0 (x^3-3x^2+4x)\,dx$ **f** $\int^2 (2x-1)^2\,dx$

g $\int_{-} (y^3+y)\,dy$ **h** $\int_3^4 (2-x)^2\,dx$ **i** $\int_{-2}^2 4t^3\,dt$

j $\int_2^4 \frac{x^2}{3}\,dx$ **k** $\int^3 \frac{5x^4}{x}\,dx$ $\int_2^4 \frac{x^4-3x}{x}\,dx$

m $\int^2 \frac{4x^3+x^2+5x}{x}\,dx$ **n** $\int_3^5 \frac{x^3-2x^2+3x}{x}\,dx$ **o** $\int_3^4 \frac{x^2+x+3}{3x^5}\,dx$

3 For each velocity function find the change in displacement between 2 and 4 second.

a $v = 3t^2 + 7$ m s^{-} **b** $v = 8t - 5$ km h^{-} **c** $v = 4t^3 + 2t + 3$ cm s^{-}

d $v = (t+3)^2$ m s^{-} **e** $v = 5 - 6t + 9t^2$ cm s^{-}

ISBN 97807043220

4 A high-power hose fills an empty swimming pool at the rate of $r = 25 + 4t^3$ L min^{-}
Find the volume to the nearest litre after

a 5 minutes **b** 15 minutes **c** half an hour

INVESTIGATION

AREAS

Look at the results of definite integrals in the examples and exercises Sketch the graphs where possible and shade in the areas found

Can you see why the definite integral sometimes gives a negative answer?

Can you see why it will sometimes be zero?

6.04 Indefinite integrals

WS Indeinie inegal

To find the indefinite integral $\int f(x)\,dx$ we find the anti-derivative of the functio.

WS Inegaing uncion

Integral of x^n

$$\int x^n\,dx = \frac{1}{n+1}x^{n+} + C \text{ where } n \neq -1$$

EXAMPLE 11

Find each indefinite integral

a $\int 5x^9\,dx$

b $\int\left(\frac{1}{x^3}+\sqrt{x}\right)dx$

Solution

a $\int 5x^9\,dx = 5\int x^9\,dx$

$$= 5\times\frac{x^{10}}{10}+C$$

$$= \frac{x^{10}}{2}+C$$

b $\int\left(\frac{1}{x^3}+\sqrt{x}\right)dx = \int\left(x^{-3}+x^{\frac{}{2}}\right)dx$

$$= \frac{x^{-2}}{-2}+\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+C$$

$$= -\frac{1}{2x^2}+\frac{2\sqrt{x^3}}{3}+C$$

DID YOU KNOW?

John Wallis

English clergyman and mathematician **John Wallis** (1616–1703) found that the **area under the curve** $y = 1 + x + x^2 + x^3 +$ is given by

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} +$$

He found this result independently of the fundamental theorem of calculus

We can use the indefinite integral to find original information given a rate of chang.

EXAMPLE 12

The rate of air flow into a container is given by $R = 4 + 3t^2$ mm^3 s$^-$
If there is initially no air in the container, find the volume of air in it after 12 secons.

Solution

$R = \frac{dV}{dt} = 4 + 3t^2$

$V = \int (4 + 3t^2)\, dt$

$= 4t + t^3 + C$

When $t = 0$, $V = 0$

$0 = 4(0) + 0^3 + C$

$= C$

So $V = 4t + t^3$

When $t = 12$:

$V = 4(12) + 12^3$

$= 1776$

So the container will hold 1776 mm^3 of air after 12 seconds

Exercise 6.04 Indefinite integrals

1 Find each indefinite integral

a $\int x^2\, dx$

b $\int 3x^5\, dx$

c $\int 2x^4\, dx$

d $\int (m + 1)\, dm$

e $\int (t^2 - 7)\, dt$

f $\int (h^7 + 5) dh$

g $\int (y - 3)\, dy$

h $\int (2x + 4)\, dx$

i $\int (b^2 + b)\, db$

ISBN 97807043220

2 Find

a $\int (x^2 + 2x + 5)\,dx$

b $\int (4x^3 - 3x^2 + 8x - 1)\,dx$

c $\int (6x^5 + x^4 + 2x^3)\,dx$

d $\int (x^7 - 3x^6 - 9)\,dx$

e $\int (2x^3 + x^2 - x - 2)\,dx$

f $\int (x^5 + x^3 + 4)\,dx$

g $\int (4x^2 - 5x - 8)\,dx$

h $\int (3x^4 - 2x^3 + x)\,dx$

i $\int (6x^3 + 5x^2 - 4)\,dx$

j $\int (3x^{-4} + x^{-3} + 2x^{-2})\,dx$

3 Find each indefinite integral

a $\int \frac{dx}{x^8}$

b $\int x^{\overline{3}}\,dx$

c $\int \frac{x^6 - 3x^5 + 2x^4}{x^3}\,dx$

d $\int (1 - 2x)^2\,dx$

e $\int (x - 2)(x + 5)\,dx$

f $\int \frac{3}{x^2}\,dx$

g $\int \frac{dx}{x^3}$

h $\int \frac{4x^3 - x^5 - 3x^2 + 7}{x^5}\,dx$

i $\int (y^2 - y^{-7} + 5)\,dy$

j $\int (t^2 - 4)(t - 1)\,dt$

k $\int \sqrt{x}\,dx$

$\int \frac{2}{t^5}\,dt$

m $\int \sqrt[3]{x}\,dx$

n $\int x\sqrt{x}\,dx$

o $\int \sqrt{x}\left(1 + \frac{1}{\sqrt{x}}\right)dx$

4 The rate of change of the angle sum S of a polygon with n sides is a constant 180°
If $S = 360°$ when $n = 4$ find S when $n = 7$.

5 For a certain graph the rate of change of y values with respect to its x values is given by $R = 3x^2 - 2x + 1$ If the graph passes through the point (-1 3, find its equatin.

6 The rate of change in velocity over time is given by $\frac{dx}{dt} = 4t + t^2 - t^3$
If the initial velocity is 2 cm s^{-} find the displacement after 15 .

7 The rate of flow of water into a dam is given by $R = 500 + 20t$ L h^{-} If there is 15 000 L of water initially in the dam how much water will there be in the dam after 10 hours?

6.05 Chain rule

You found the anti-derivative of $y = (ax + b)^n$ in Chapter 4 *Further differentiation*
We can write this as an integra.

Chain rule for $(ax + b)^n$

$$\int (ax + b)^n\,dx = \frac{(ax + b)^{n+}}{a(n + 1)} + C \quad \text{where } n \neq -1$$

EXAMPLE 13

Find

a $\int (5x-9)^3\,dx$ **b** $\int_2^3 (3-x)^8\,dx$ **c** $\int^3 \sqrt{4x-3}\,dx$

Solution

a
$$\int (5x-9)^3\,dx = \frac{(5x-9)^4}{5\times 4} + C$$
$$= \frac{(5x-9)^4}{20} + C$$

b
$$\int_2^3 (3-x)^8\,dx = \left[\frac{(3-x)^9}{-1\times 9}\right]_2^3$$
$$= \left[-\frac{(3-x)^9}{9}\right]_2^3$$
$$= -\frac{(3-3)^9}{9} - \left(-\frac{(3-2)^9}{9}\right)$$
$$= 0 + \frac{1}{9}$$
$$= \frac{1}{9}$$

c
$$\int^3 \sqrt{4x-3}\,dx = \int^3 (4x-3)^{\frac{1}{2}}\,dx$$
$$= \left[\frac{(4x-3)^{\frac{3}{2}}}{4\times\frac{3}{2}}\right]^3$$
$$= \left[\frac{\sqrt{(4x-3)^3}}{6}\right]^3$$
$$= \frac{\sqrt{(4\times 3-3)^3}}{6} - \frac{\sqrt{(4\times 1-3)^3}}{6}$$
$$= \frac{\sqrt{9^3}}{6} - \frac{\sqrt{1^3}}{6}$$
$$= \frac{27}{6} - \frac{1}{6}$$
$$= 4\frac{1}{3}$$

In Chapter 4 *Further differentiation* you also learned about the general chain rule for $f'(x)[f(x)]^n$

Chain rule for $f'(x)[f(x)]^n$

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+} + C \quad \text{where } n \neq -1$$

ISBN 97807043220

EXAMPLE 14

a Find $\int x(x^2+1)^3\,dx$

b Find the exact value of $\int^2 x^2\sqrt{x^3-1}\,dx$

Solution

a

$f(x) = x^2 + 1$

$f'(x) = 2x$

$$\int x(x^2+1)^3\,dx$$

$$= \int \frac{1}{2} \times 2x(x^2+1)^3\,dx$$

$$= \frac{1}{2}\int 2x(x^2+1)^3\,dx$$

$$= \frac{1}{2} \times \frac{1}{4}(x^2+1)^4 + C$$

$$= \frac{1}{8}(x^2+1)^4 + C$$

applyng the chan rule formula

b

$f(x) = x^3 - 1$

$f'(x) = 3x^2$

$$\int^2 x^2\sqrt{x^3-1}\,dx$$

$$= \int^2 \frac{1}{3} \times 3x^2(x^3-1)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\int^2 3x^2(x^3-1)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\left[\frac{1}{\frac{3}{2}}(x^3-1)^{\frac{3}{2}}\right]^2$$

$$= \frac{1}{3}\left[\frac{2}{3}\sqrt{(x^3-1)^3}\right]^2$$

$$= \frac{2}{9}\left[\sqrt{(x^3-1)^3}\right]^2$$

$$= \frac{2}{9}\left(\sqrt{(2^3-1)^3} - \sqrt{(1^3-1)^3}\right)$$

$$= \frac{2}{9}\left(\sqrt{7^3} - \sqrt{0^3}\right)$$

$$= \frac{2}{9}(7\sqrt{7})$$

$$= \frac{14}{9}\sqrt{7}$$

ISBN 97807043220

Exercise 6.05 Chain rule

1 Find each indefinite integral

a $\int (3x-4)^2\,dx$ **b** $\int (x+1)^4\,dx$ **c** $\int (5x-1)^9\,dx$

d $\int (3y-2)^7\,dy$ **e** $\int (4+3x)^4\,dx$ **f** $\int (7x+8)^{12}\,dx$

g $\int (1-x)^6\,dx$ **h** $\int \sqrt{2x-5}\,dx$ **i** $2\int (3x+1)^{-4}\,dx$

j $\int 3(x+7)^{-2}\,dx$ **k** $\int \frac{1}{2(4x-5)^3}\,dx$ $\int \sqrt[3]{4x+3}\,dx$

m $\int (2-x)^{\overline{-\;-}\,2}\,dx$ **n** $\int \sqrt{(t+3)^3}\,dt$ **o** $\int \sqrt{(5x+2)^5}\,dx$

2 Evaluate

a $\int^2 (2x+1)^4\,dx$ **b** $\int_0 (3y-2)^3\,dy$ **c** $\int^2 (1-x)^7\,dx$

d $\int_0^2 (3-2x)^5\,dx$ **e** $\int_0 \frac{(3x-1)^2}{6}\,dx$ **f** $\int_4^5 (5-x)^6\,dx$

g $\int_3^6 \sqrt{x-2}\,dx$ **h** $\int_0^2 \frac{5}{(2n+1)^3}\,dn$ **i** $\int^4 \frac{2}{\sqrt{(5x-4)^3}}\,dx$

3 Find each indefinite integral

a $\int 4x^3(x^4+5)^2\,dx$ **b** $\int 2x(x^2-3)^5\,dx$

c $\int 3x^2(x^3+1)^3\,dx$ **d** $\int (2x+3)(x^2+3x-2)^4\,dx$

e $\int x(3x^2-7)^6\,dx$ **f** $\int x^2(4-5x^3)^2\,dx$

g $\int 4x^5(2x^6-3)^4\,dx$ **h** $\int 3x(5x^2+3)^7\,dx$

i $\int (x+2)(x^2+4x)^5\,dx$ **j** $\int (3x^2-2)(3x^3-6x-2)^3\,dy$

4 Evaluate

a $\int_0^2 x(2x^2+3)^2\,dx$ **b** $\int_0 x^2(x^3-1)^5\,dx$

c $\int^2 x^4(x^5+2)^3\,dx$ **d** $\int_0 x^3(5-x^4)^7\,dx$

e $\int_2^4 3x(x^2+2)^4\,dx$ **f** $\int_{-} 5x^2(2x^3-7)^3\,dx$

g $\int_{-}^0 (x-1)(x^2-2x+3)^6\,dx$ **h** $4\int_0 (x^2+2)(x^3+6x-1)^2\,dx$

i $5\int_{-2}^2 x^2(x^3-1)(x^6-2x^3-1)^4\,dx$

5 A function has $\frac{dy}{dx} = x^2(x^3-2)^4$ and passes through the point (1 4. Find its equatin.

6 The velocity of an object is given by $v = x(x^2-3)^4$ m s$^-$ If the displacement is 0 after 2 s fin:

a the equation for the displacement

b the displacement after 3 s

ISBN 97807043220

6.06 Integration involving exponential functions

We can write the anti-derivative of e^x as an integral

Integral of e^x

$$\int e^x\, dx = e^x + C$$

EXAMPLE 15

Evaluate $\int_0^2 4e^x\, dx$

Solution

$$\begin{aligned}\int_0^2 4e^x\, dx &= 4\left[e^x\right]_0^2\\ &= 4(e^2 - e^0)\\ &= 4(e^2 - 1)\end{aligned}$$

Integral of a^x

$$\int a^x\, dx = \frac{1}{\ln a}a^x + C$$

EXAMPLE 16

Find $\int 2^x\, dx$

Solution

$$\int a^x\, dx = \frac{1}{\ln a}a^x + C$$

$$\int 2^x\, dx = \frac{1}{\ln 2}2^x + C$$

Chain rule for e^{ax+b}

$$\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + C$$

ISBN 97807043220

EXAMPLE 17

a Find $\int (e^{2x} - e^{-x})\,dx$

b Evaluate $\int^{2} 5e^{3x}\,dx$

Solution

a
$$\int (e^{2x} - e^{-x})\,dx = \frac{1}{2}e^{2x} - \frac{1}{-1}e^{-} + C$$
$$= \frac{1}{2}e^{2x} + e^{-x} + C$$

b
$$\int^{2} 5e^{3x}\,dx = \left[5 \times \frac{1}{3}e^{3x}\right]^{2}$$
$$= \left[\frac{5e^{3x}}{3}\right]^{2}$$
$$= \frac{5e^{3\times 2}}{3} - \frac{5e^{3\times 1}}{3}$$
$$= \frac{5e^{6}}{3} - \frac{5e^{3}}{3}$$
$$= \frac{5e^{3}}{3}(e^{3} - 1)$$

Exercise 6.06 Integration involving exponential functions

1 Find each indefinite integral

a $\int e^{4x}\,dx$ **b** $\int e^{-x}\,dx$ **c** $\int e^{5x}\,dx$

d $\int e^{-2x}\,dx$ **e** $\int e^{4x+1}\,dx$ **f** $\int -3e^{5x}\,dx$

g $\int e^{2t}\,dt$ **h** $\int (e^{7x} - 2)\,dx$ $\int (e^{x-3} + x)\,dx$

2 Evaluate in exact form

a $\int_{0} e^{5x}\,dx$ **b** $\int_{0}^{2} -e^{-x}\,dx$ **c** $\int^{4} 2e^{3x+4}\,dx$

d $\int_{2}^{3} (3x^{2} - e^{2x})\,dx$ **e** $\int_{0}^{2} (e^{2x} + 1)\,dx$ **f** $\int^{2} (e^{x} - x)\,dx$

g $\int_{0}^{3} (e^{2x} - e^{-x})\,dx$

3 Evaluate correct to 2 decimal places

a $\int^{3} e^{-x}\,dx$ **b** $\int_{0}^{2} 2e^{3y}\,dy$ **c** $\int_{5}^{6} (e^{x+5} + 2x - 3)\,dx$

d $\int_{0} (e^{3t+4} - t)\,dt$ **e** $\int^{2} (e^{4x} + e^{2x})\,dx$

4 Find the indefinite integral of

a 5^{x} **b** 7^{3x} **c** 3^{2x-1}

ISBN 97807043220

5 **a** Differentiate x^2e^x

b Hence find $\int x(2+x)e^x\,dx$

6 A function has $f'(x) = x^2e^{2x}$ and passes through the point (0 0. Find the equation of the function

7 A particle moves so that its velocity over time t is given by $v = 2e^t - 1$ m s$^-$
If displacement $x = 10$ when $y = 0$ find x when $t = 3$.

6.07 Integration involving logarithmic functions

We can write the anti-derivatives that involve the logarithmic function as an integra.

WS Inegal involving exponenial and logaihmic uncion

WS Inegaion o $\frac{}{x}$

WS Inegaion o ecipocal

WS Difretition and inegaion involving exponenial and logaihmic uncion

Integral of $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln|x| + C \quad \text{where } x \neq 0$$

EXAMPLE 18

Evaluate $\int^5 \frac{3}{x}\,dx$

Solution

$$\begin{aligned}\int^5 \frac{3}{x}\,dx &= \left[3\ln|x|\right]^5 \\ &= 3\ln 5 - 3\ln 1 \\ &= 3\ln 5 - 3\times 0 \\ &= 3\ln 5\end{aligned}$$

Integral of $\frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \quad \text{where } f(x) \neq 0$$

EXAMPLE 19

a Find $\int \frac{x^2}{x^3+7}\,dx$

b Find the exact value of $\int_0^3 \frac{x+1}{x^2+2x+4}\,dx$

ISBN 97807043220

Solution

a $f(x) = x^3 + 7,\ f'(x) = 3x^2$

$$\begin{aligned}\int \frac{x^2}{x^3+7}\,dx &= \int \frac{1}{3} \times \frac{3x^2}{x^3+7}\,dx\\ &= \frac{1}{3}\int \frac{3x^2}{x^3+7}\,dx\\ &= \frac{1}{3}\ln|x^3+7| + C\end{aligned}$$

applyng the formula

b $f(x) = x^2 + 2x + 4,\ f'(x) = 2x + 2$

$$\begin{aligned}\int_0^3 \frac{x+1}{x^2+2x+4}\,dx &= \frac{1}{2}\int_0^3 \frac{2x+2}{x^2+2x+4}\,dx \qquad \text{as } x+1 = \frac{1}{2}(2x+2)\\ &= \frac{1}{2}\left[\ln|x^2+2x+4|\right]_0^3\\ &= \frac{1}{2}[\ln|3^2+2(3)+4| - \ln|0^2+2(0)+4|]\\ &= \frac{1}{2}[\ln 19 - \ln 4]\\ &= \frac{1}{2}\ln\left(\frac{19}{4}\right)\end{aligned}$$

Exercise 6.07 Integration involving logarithmic functions

1 Find the integral of each function

a $\dfrac{2}{2x+5}$ **b** $\dfrac{4x}{2x^2+1}$ **c** $\dfrac{5x^4}{x^5-2}$ **d** $\dfrac{1}{2x}$ **e** $\dfrac{2}{x}$

f $\dfrac{5}{3x}$ **g** $\dfrac{2x-3}{x^2-3x}$ **h** $\dfrac{x}{x^2+2}$ **i** $\dfrac{3x}{x^2+7}$ **j** $\dfrac{x+1}{x^2+2x-5}$

2 Find

a $\displaystyle\int \frac{4}{4x-1}\,dx$ **b** $\displaystyle\int \frac{dx}{x+3}$ **c** $\displaystyle\int \frac{x^2}{2x^3-7}\,dx$

d $\displaystyle\int \frac{x^5}{2x^6+5}\,dx$ **e** $\displaystyle\int \frac{x+3}{x^2+6x+2}\,dx$

3 Evaluate correct to one decimal place

a $\displaystyle\int^3 \frac{2}{2x+5}\,dx$ **b** $\displaystyle\int_2^5 \frac{dx}{x+1}$ **c** $\displaystyle\int^7 \frac{x^2}{x^3+2}\,dx$

d $\displaystyle\int_0^3 \frac{4x+1}{2x^2+x+1}\,dx$ **e** $\displaystyle\int_3^4 \frac{x-1}{x^2-2x}\,dx$

ISBN 97807043220

4 **a** Show that $\dfrac{3x+3}{x^2-9} = \dfrac{1}{x+3} + \dfrac{2}{x-3}$ **b** Hence find $\displaystyle\int \frac{3x+3}{x^2-9}\,dx$

5 **a** Show that $\dfrac{x-6}{x-1} = 1 - \dfrac{5}{x-1}$ **b** Hence find $\displaystyle\int \frac{x-6}{x-1}\,dx$

6 A function has $f'(x) = \dfrac{x^2}{3x^3-1}$ and passes through the point (1 0. Find the equation of the function

7 A particle has velocity $v = \dfrac{5t}{t^2+4}$ m s^{-} Find its displacement after 5 s if its initial displacement is 4 m

8 The number of people with measles is increasing at the rate given by $R = \dfrac{x^2}{3x^3+1}$ people/week If 3 people had measles initiall, find the number with measles after 8 weeks

6.08 Integration involving trigonometric functions

WS Inegals o igonomeic uncion

WS Mixed inegaion queion

Integrals of trigonometric functions

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

EXAMPLE 20

WS Finding indeinie inegals

WS Finding indeinie inegals 2

WS Finding deinie inegal

Find the exact value of $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$

Solution

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sec^2 x\,dx &= [\tan x]_0^{\frac{\pi}{3}} \\ &= \tan\frac{\pi}{3} - \tan 0 \\ &= \sqrt{3}\end{aligned}$$

ISBN 97807043220

Chain rule for trigonometric functions

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

EXAMPLE 21

a Find $\int \sin 3x\,dx$

b Find $\int \cos x°\,dx$

c Find the exact value of $\int_0^{\frac{\pi}{8}} \sin 2x\,dx$

Solution

a $\int \sin 3x\,dx = -\frac{1}{3}\cos 3x + C$

c
$$\begin{aligned}\int_0^{\frac{\pi}{8}} \sin 2x\,dx &= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{8}}\\
&= -\frac{1}{2}\cos\left(2\times\frac{\pi}{8}\right) - \left[-\frac{1}{2}\cos(2\times 0)\right]\\
&= -\frac{1}{2}\cos\frac{\pi}{4} + \frac{1}{2}\cos 0\\
&= -\frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{1}{2}\\
&= -\frac{1}{2\sqrt{2}} + \frac{1}{2}\\
&= \frac{2-\sqrt{2}}{4}\end{aligned}$$

b
$$\begin{aligned}\int \cos x°\,dx &= \int \cos\left(\frac{\pi x}{180}\right)dx\\
&= \frac{1}{\frac{\pi}{180}}\sin\left(\frac{\pi x}{180}\right) + C\\
&= \frac{180}{\pi}\sin x° + C\end{aligned}$$

Exercise 6.08 Integration involving trigonometric functions

1 Find the integral of each function

a $\cos x$	**b** $\sin x$	**c** $\sec^2 x$
d $\frac{\sin x°}{4}$	**e** $\sin 3x$	**f** $-\sin 7x$
g $\sec^2 5x$	**h** $\cos(x+1)$	**i** $\sin(2x-3)$
j $\cos(2x-1)$	**k** $\sin(\pi - x)$	$\cos(x+\pi)$
m $2\sec^2 7x$	**n** $4\sin\left(\frac{x}{2}\right)$	**o** $3\sec^2\left(\frac{x}{3}\right)$

ISBN 97807043220

2 Evaluate each definite integral giving exact answers where appropriat.

a $\int_0^{\frac{\pi}{2}} \cos x \, dx$ **b** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx$ **c** $\int_{\frac{\pi}{2}}^{\pi} \sin \frac{x}{2} \, dx$

d $\int_0^{\frac{\pi}{2}} \cos 3x \, dx$ **e** $\int_0^{\frac{}{2}} \sin(\pi x) \, dx$ **f** $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$

g $\int_0^{\frac{\pi}{12}} 3 \cos 2x \, dx$ **h** $\int_0^{\frac{\pi}{10}} -\sin(5x) \, dx$

3 Find

a $\int (\cos x \cos \frac{\pi}{3} - \sin x \sin \frac{\pi}{3}) \, dx$

b $\int (\sin \pi \cos x - \cos \pi \sin x) \, dx$

4 A curve has $\frac{dy}{dx} = \cos 4x$ and passes through the point $(\pi \ \frac{\pi}{4})$.

Find the equation of the curve

5 A pendulum swings at the rate given by $\frac{dx}{dt} = 12\pi \cos \frac{2\pi t}{3}$ cm s^{-}

It starts 2 cm to the right of the origin

a Find the equation of the displacement of the pendulum

b Find the exact displacement after

i 1 s **ii** 5 s

6 The rate at which the depth of water changes in a bay is given by $R = 4\pi \sin \frac{\pi t}{6}$ m h^{-}

a Find the equation of the depth of water d over time t hours if the depth is 2 m initially.

b Find the depth after 2 hours

c Find the highest lowest and centre of depth of wate.

d What is the period of the depth of water?

ISBN 97807043220

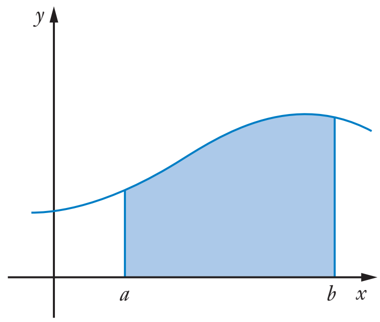
Sating aea

Aea unde a cuve

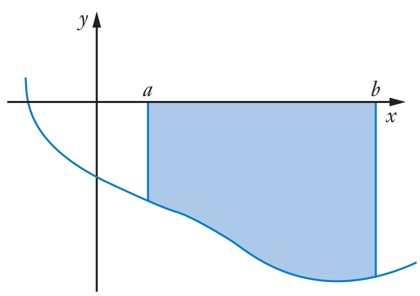
6.09 Areas enclosed by the x-axis

The definite integral gives the **signed** area under a curve

Areas above the x-axis give a positive definite integral



Areas below the x-axis give a negative definite integral



So to find areas below the x-axis we take the absolute value of the definite integra.

Area under a curve

$$\text{Area} = \left|\int_a^b f(x)\,dx\right|$$

It is important to sketch the graph to see where the area is in relation to the x-axis

ISBN 97807043220

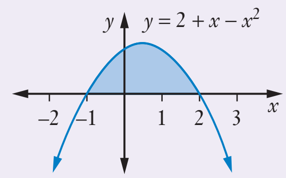
EXAMPLE 22

a Find the area enclosed by the curve $y = 2 + x - x^2$ and the x-axis

b Find the area bounded by the curve $y = x^2 - 4$ and the x-axis

Solution

a Sketch the graph of $y = 2 + x - x^2$ and shade the area enclosed between the curve and the x-axis



The area is above the x-axis so the definite integral will be positive

$$\text{Area} = \int_{-}^{2} 2 + x - x^2 \, dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-}^{2}$$

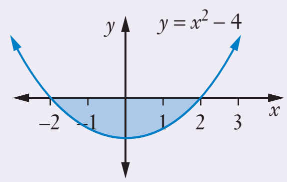
$$= \left(2(2) + \frac{2^2}{2} - \frac{2^3}{3} \right) - \left(2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3} \right)$$

$$= \left(4 + 2 - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} + \frac{1}{3} \right)$$

$$= 4\frac{1}{2}$$

So the area is $4\frac{1}{2}$ units^2

b Sketch the graph of $y = x^2 - 4$



The definite integral will be negative because the area is below the x-axis

$$\int_{-2}^{2} (x^2 - 4)\, dx = \left[\frac{x^3}{3} - 4x \right]_{-2}^{2}$$

$$= \left(\frac{2^3}{3} - 4(2) \right) - \left(\frac{(-2)^3}{3} - 4(-2) \right)$$

$$= \left(\frac{8}{3} - 8 \right) - \left(-\frac{8}{3} + 8 \right)$$

$$= -10\frac{2}{3}$$

$$\text{Area} = \left| -10\frac{2}{3} \right|$$

$$= 10\frac{2}{3} \text{ units}^2$$

ISBN 97807043220

Aea unde a cuve

EXAMPLE 23

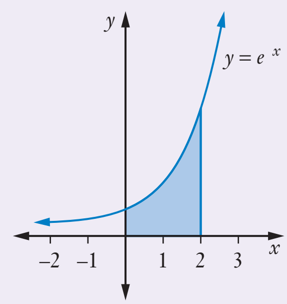
a Find the exact area enclosed between the curve $y = e^{3x}$ the x-axis and the lines $x = 0$ and $x = 2$.

b Find the exact area enclosed between the hyperbola $y = \frac{1}{x}$ the x-axis and the lines $x = 1$ and $x = 2$.

c Find the area enclosed between the curve $y = \cos x$ the x-axis and the lines $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

Solution

a Sketch the graph of $y = e^{3x}$

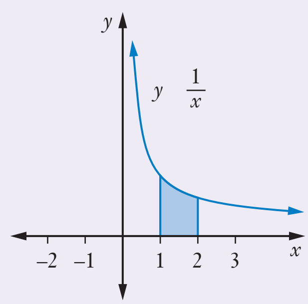


The definite integral will be positive because the area is above the x-axis

$$\begin{aligned}\text{Area} &= \int_0^2 e^{3x}\,dx \\ &= \left[\frac{1}{3}e^{3x}\right]_0^2 \\ &= \frac{1}{3}e^{3\times 2} - \frac{1}{3}e^{3\times 0} \\ &= \frac{1}{3}e^{6} - \frac{1}{3}e^{0} \\ &= \frac{1}{3}(e^{6} - 1)\end{aligned}$$

So the area is $\frac{1}{3}(e^6 - 1)$ units^2

b Sketch the graph of $y = \frac{1}{x}$ For $x = 1$ and $x = 2$ we only need to sketch the graph in the 1st quadrant



The definite integral will be positive because the area is above the x-axis

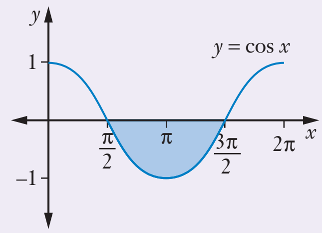
$$\begin{aligned}\text{Area} &= \int^2 \frac{dx}{x} \\ &= \big[\ln|x|\big]^2 \\ &= \ln 2 - \ln 1 \\ &= \ln 2 - 0 \\ &= \ln 2\end{aligned}$$

Absolute value not requred as 2 and 1 are postve

So the area is $\ln 2$ units^2

ISBN 97807043220

c Sketch the graph of $y = \cos x$



The definite integral will be negative because the area is below the x-axis

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, dx = [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)$$

$$= -1 - 1$$

$$= -2$$

$$\text{Area} = |-2|$$

$$= 2 \text{ units}^2$$

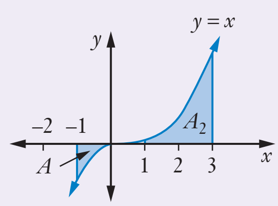
If the area has some parts above the x-axis and some below the x-axis we need to find these separately.

EXAMPLE 24

Find the area enclosed between the curve $y = x^3$ the x-axis and the lines $x = -1$ and $x = 3$.

Solution

Sketch the graph of $y = x^3$



There are 2 areas marked A and A_2 on the diagram

A is below the x-axis so the integral will be negative

A_2 is above the x-axis so the integral will be positive

A

$$\int_{-}^{0} x^3 \, dx = \left[\frac{x^4}{4}\right]_{-}^{0}$$

$$= \frac{0^4}{4} - \frac{(-1)^4}{4}$$

$$= -\frac{1}{4}$$

$$\text{So } A = \left|-\frac{1}{4}\right|$$

$$= \frac{1}{4} \text{ units}^2$$

$$A_2 = \int_0^3 x^3 \, dx$$

$$= \left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{3^4}{4} - \frac{0^4}{4}$$

$$= \frac{81}{4} \text{ units}^2$$

$$\text{Total area} = A + A_2$$

$$= \frac{1}{4} + \frac{81}{4}$$

$$= 20\frac{1}{2} \text{ units}^2$$

ISBN 97807043220

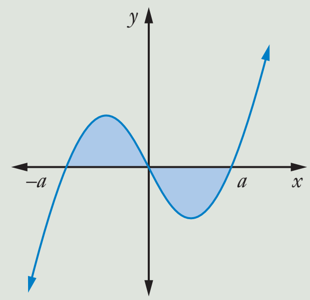
Odd and even functions

Some functions have special properties that we can use to find their areas

Odd functions

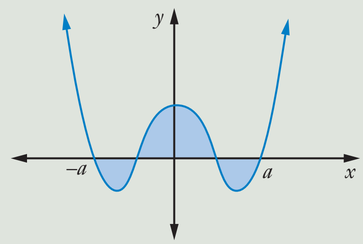
If $f(-x) = -f(x)$ then $\int_{-a}^{a} f(x)\,dx = 0$

The postve and negatve areas cancel each other out



Even functions

If $f(-x) = f(x)$ then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



EXAMPLE 25

Find the area between the curve

a $y = x^3$ the x-axis and the lines $x = -2$ and $x = 2$

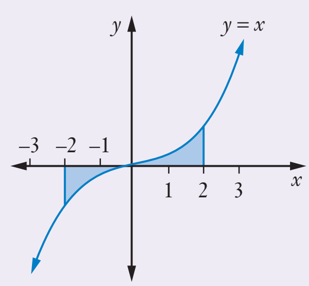
b $y = x^2$ the x-axis and the lines $x = -4$ and $x = 4$

Solution

a Sketch the graph of $y = x^3$ and shade the area bounded by the curve the x-axis and the boundaries $x = \pm 2$

$y = x^3$ is an odd function since $f(-x) = -f(x)$

This means that the shaded areas are symmetrical We can find the area between $x = 0$ and $x = 2$.



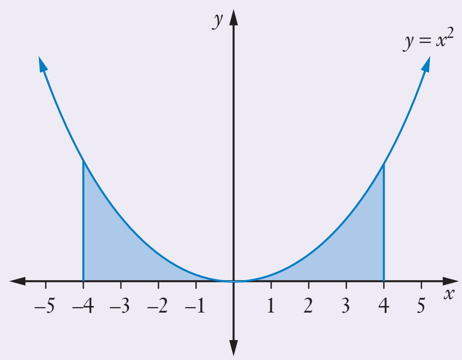
The total area will be twice this area

ISBN 97807043220

$$\text{Area} = 2\int_0^2 x^3\,dx$$
$$= 2\left[\frac{x^4}{4}\right]_0^2$$
$$= 2\left(\frac{2^4}{4} - \frac{0^4}{4}\right)$$
$$= 8$$

So area is 8 units^2

b Sketch the graph of $y = x^2$ and shade the area enclosed between the curve the x-axis and the lines $x = \pm 4$



$y = x^2$ is an even function since $f(-x) = f(x)$

$$\text{Area} = \int_{-4}^{4} x^2\,dx$$
$$= 2\int_0^4 x^2\,dx$$
$$= 2\left[\frac{x^3}{3}\right]_0^4$$
$$= 2\left(\frac{4^3}{3} - \frac{0^3}{3}\right)$$
$$= 2 \times \frac{64}{3}$$
$$= 42\frac{2}{3}$$

So area is $42\frac{2}{3}$ units^2

Exercise 6.09 Areas enclosed by the x-axis

1. Find the area enclosed between the curve $y = 1 - x^2$ and the x-axis
2. Find the area bounded by the curve $y = x^2 - 9$ and the x-axis
3. Find the area enclosed between the curve $y = x^2 + 5x + 4$ and the x-axis
4. Find the area enclosed between the curve $y = x^2 - 2x - 3$ and the x-axis
5. Find the area bounded by the curve $y = -x^2 + 9x - 20$ and the x-axis
6. Find the area enclosed between the curve $y = -2x^2 - 5x + 3$ and the x-axis
7. Find the area enclosed between the curve $y = x^3$ the x-axis and the lines $x = 0$ and $x = 2$.

ISBN 97807043220

8 Find the area enclosed between the curve $y = x^4$ the x-axis and the lines $x = -1$ and $x = 1$.

9 Find the area enclosed between the curve $y = x^3$ the x-axis and the lines $x = -2$ and $x = 2$.

10 Find the area enclosed between the curve $y = x^3$ the x-axis and the lines $x = -3$ and $x = 2$.

11 Find the exact area enclosed by the curve $y = 2e^{2x}$ the x-axis and the lines $x = 1$ and $x = 2$.

12 Find the exact area bounded by the curve $y = e^{4x-3}$ the x-axis and the lines $x = 0$ and $x = 1$.

13 Find the area enclosed by the curve $y = x + e^{-x}$ the x-axis and the lines $x = 0$ and $x = 2$, correct to 2 decimal places

14 Find the area bounded by the curve $y = e^{5x}$ the x-axis and the lines $x = 0$ and $x = 1$, correct to 3 significant figures

15 Find the area enclosed between the curve $y = \sin x$ and the x-axis in the domain $[0\ 2\ \pi]$

16 Find the exact area bounded by the curve $y = \cos 3x$ the x-axis and the lines $x = 0$ and $x = \frac{\pi}{12}$

17 Find the area enclosed between the curve $y = \sec^2 \frac{x}{4}$ the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$ correct to 2 decimal place.

18 Find the area bounded by the curve $y = 3x^2$ the x-axis and the lines $x = -1$ and $x = 1$.

19 Find the area enclosed between the curve $y = x^2 + 1$, the x-axis and the lines $x = -2$ and $x = 2$.

20 Find the area enclosed between the curve $y = x^2$ the x-axis and the lines $x = -3$ and $x = 2$.

21 Find the area enclosed between the curve $y = x^2 + x$ and the x-axis

22 Find the area enclosed between the curve $y = \frac{1}{x^2}$ the x-axis and the lines $x = 1$ and $x = 3$.

23 Find the area enclosed between the curve $y = \frac{2}{(x-3)^2}$ the x-axis and the lines $x = 0$ and $x = 1$.

24 Find the exact area between the curve $y = \frac{1}{x}$ the x-axis and the lines $x = 2$ and $x = 3$.

25 Find the exact area bounded by the curve $y = \frac{1}{x-1}$ the x-axis and the lines $x = 4$ and $x = 7$.

26 Find the area bounded by the curve $y = \frac{x}{x^2+1}$ the x-axis and the lines $x = 2$ and $x = 4$, correct to 2 decimal places

27 Find the area bounded by the curve $y = \sqrt{x}$ the x-axis and the line $x = 4$

28 Find the area bounded by the curve $y = \sqrt{x+2}$ the x-axis and the line $x = 7$.

29 Use the trapezoidal rule with 4 subintervals to find the area bounded by the curve $y = \ln x$ the x-axis and the line $x = 5$ correct to 2 decimal place.

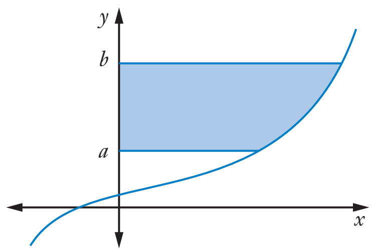
30 Find the area bounded by the x-axis the curve $y = x^3$ and the lines $x = -a$ and $x = a$

ISBN 97807043220

6.10 Areas enclosed by the *y*-axis

WS
Aea o inegaion

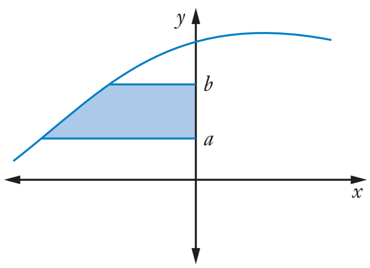
We can find an area bounded by a graph and the y-axis by writing the equation in the form $x = f(y)$



The definite integral gives the **signed** area

Areas to the right of the y-axis give a positive definite integral

Areas to the left of the y-axis give a negative definite integral



Area bounded by a curve and the *y*-axis

For the curve $x = f(y)$

$$\text{Area} = \left| \int_a^b f(y)\,dy \right|$$

EXAMPLE 26

a Find the area enclosed by the curve $x = y^2$ the y-axis and the lines $y = 1$ and $y = 3$.

b Find the area enclosed by the curve $y = x^2$ the y-axis and the lines $y = 0$ and $y = 4$ in the first quadrant

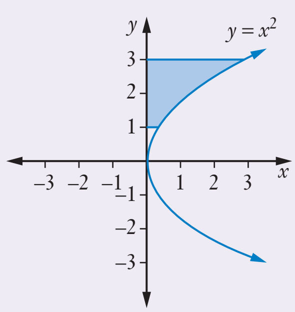
Solution

a Sketch the graph of $x = y^2$ and shade the area bounded by the curve the y-axis and the lines $y = 1$ and $y = 3$.

This is the same shape as the parabola $y = x^2$ with the x and y values swapped

For example when $x = 1, y = \pm 1$ when $x = 4, y = \pm 2$

The area is to the right of the y-axis so the integral will be positive

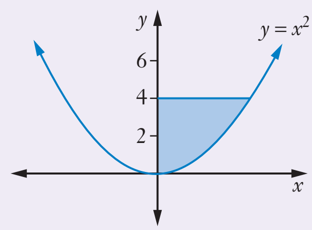


ISBN 97807043220

$$\text{Area} = \int_a^b f(y)\,dy$$
$$= \int^3 y^2\,dy$$
$$= \left[\frac{y^3}{3}\right]^3$$
$$= \frac{3^3}{3} - \frac{1^3}{3}$$
$$= 8\frac{2}{3}$$

So the area is $8\frac{2}{3}$ units^2

b Sketch the graph of $y = x^2$ and shade the area enclosed between the curve the y-axis and the lines $y = 0$ and $y = 4$



The area is to the right of the y-axis so the integral will be positive

Change the subject of the equation to x

$$y = x^2$$
$$\pm\sqrt{y} = x$$

In the first quadrant

$$x = \sqrt{y}$$
$$= y^{\frac{1}{2}}$$

$$\text{Area} = \int_a^b f(y)\,dy$$
$$= \int_0^4 y^{\frac{1}{2}}\,dy$$
$$= \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$
$$= \left[\frac{2\sqrt{y^3}}{3}\right]_0^4$$
$$= \frac{2\sqrt{4^3}}{3} - \frac{2\sqrt{0^3}}{3}$$
$$= 5\frac{1}{3}$$

So the area is $5\frac{1}{3}$ units^2

ISBN 97807043220

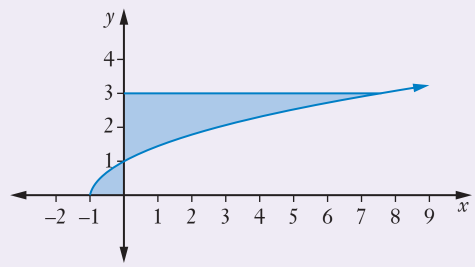
EXAMPLE 27

Find the area enclosed between the curve $y = \sqrt{x+1}$ the y-axis and the lines $y = 0$ and $y = 3$.

Solution

The area between $y = 0$ and $y = 1$ is to the left of the y-axis so the integral will be negative

The area between $y = 1$ and $y = 3$ is to the right of the y-axis so the integral will be positive



Changing the subject

$$y = \sqrt{x+1}$$
$$y^2 = x + 1$$
$$y^2 - 1 = x$$
$$\int_0 (y^2 - 1)\,dy = \left[\frac{y^3}{3} - y\right]_0$$
$$= \left(\frac{1^3}{3} - 1\right) - \left(\frac{0^3}{3} - 0\right)$$
$$= -\frac{2}{3}$$

$$A = \left|-\frac{2}{3}\right|$$
$$= \frac{2}{3}\text{ units}^2$$

$$A_2 = \int^3 (y^2 - 1)\,dy$$
$$= \left[\frac{y^3}{3} - y\right]^3$$
$$= \left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right)$$
$$= 6\frac{2}{3}\text{ units}^2$$

$$\text{Total area} = A + A_2$$
$$= \frac{2}{3} + 6\frac{2}{3}$$
$$= 7\frac{1}{3}\text{ units}^2$$

Exercise 6.10 Areas enclosed by the y-axis

1 Find the area bounded by the y-axis the curve $x = y^2$ and the lines $y = 0$ and $y = 4$

2 Find the area enclosed between the curve $x = y^3$ the y-axis and the lines $y = 1$ and $y = 3$.

3 Find the area in the first quadrant enclosed between the curve $y = x^2$ the y-axis and the lines $y = 1$ and $y = 4$

4 Find the area between the lines $y = x - 1$, $y = 0$, $y = 1$ and the y-axis

5 Find the area bounded by the line $y = 2x + 1$, the y-axis and the lines $y = 3$ and $y = 4$

6 Find the area bounded by the curve $y = \sqrt{x}$ the y-axis and the lines $y = 1$ and $y = 2$.

7 Find the area bounded by the curve $x = y^2 - 2y - 3$ and the y-axis

8 Find the area bounded by the curve $x = -y^2 - 5y - 6$ and the y-axis

9 Find the area enclosed by the curve $y = \sqrt{3x - 5}$ the y-axis and the lines $y = 2$ and $y = 3$.

10 Find the area in the first quadrant enclosed between the curve $y = \dfrac{1}{x^2}$ the y-axis and the lines $y = 1$ and $y = 4$

11 Find the area enclosed between the curve $y = x^3$ the y-axis and the lines $y = 1$ and $y = 8$.

12 Find the area enclosed between the curve $y = x^3 - 2$ and the y-axis between $y = -1$ and $y = 25$

13 Find the area in the second quadrant enclosed between the lines $y = 4$ and $y = 1 - x$

14 Find the area enclosed between the y-axis and the curve $x = y(y - 2)$

15 Find the area in the first quadrant bounded by the curve $y = x^4 + 1$, the y-axis and the lines $y = 1$ and $y = 3$ correct to 2 significant figure.

16 Find the area between the curve $y = \ln x$ the y-axis and the lines $y = 2$ and $y = 4$ correct to 3 significant figures

WS Sums and difrencs f aea

WS Calculaing phyical aea

WS Calculaing aea beween cuvs

WS Aea beween cuvs

WS Aea beween cuvs 2

6.11 Sums and differences of areas

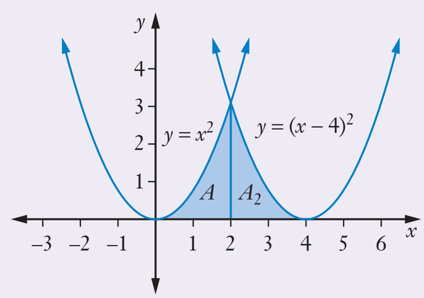
EXAMPLE 28

a Find the area enclosed between the curves $y = x^2$ $y = (x - 4)^2$ and the x-axis

b Find the area enclosed between the curve $y = x^2$ and the line $y = x + 2$.

Solution

a Sketch the graphs of $y = x^2$ and $y = (x - 4)^2$ (translation of $y = x^2$ by 4 units to the right) and shade the area enclosed between the curves and the x-axis



Find the x values of their intersection ($x = 2$ from the graph or by solving simultaneous equations)

Shaded area $= A \;+ A_2$

$$= \int_0^2 x^2\,dx + \int_2^4 (x - 4)^2\,dx$$

ISBN 97807043220

Area A

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{2^3}{3} - \frac{0^3}{3}$$

$$= 2\frac{2}{3}$$

Area A_2

$$\int_2^4 (x-4)^2\,dx = \left[\frac{(x-4)^3}{3}\right]_2^4$$

$$= \frac{(4-4)^3}{3} - \frac{(2-4)^3}{3}$$

$$= 0 + \frac{8}{3}$$

$$= 2\frac{2}{3}$$

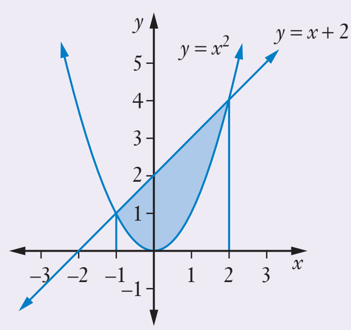
Total area $= A \;\; + A_2$

$$= 2\frac{2}{3} + 2\frac{2}{3}$$

$$= 5\frac{1}{3}$$

So area is $5\frac{1}{3}$ units^2

b Sketch the graphs of $y = x^2$ and $x + 2$ and shade the area enclosed between them



We can find the x values of the points of intersection of the functions from the graph or by solving simultaneous equations

$y = x^2$ [1]

$y = x + 2$ [2]

Substituting [1] into [2]

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2, \qquad x = -1$$

Notice that between $x = -1$ and $x = 2$ the graph of $y = x + 2$ is *above* the graph of $y = x^2$

So we can find the area by integrating $(x + 2)$ and x^2 between $x = -1$ and $x = 2$ and then finding their difference

$$A = \int_{-}^{2} (x+2)\, dx - \int_{-}^{2} x^2\, dx = \int_{-}^{2} (x + 2 - x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-}^{2}$$

$$= \left[\frac{2^2}{2} + 2(2) - \frac{2^3}{3} \right] - \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1^3)}{3} \right]$$

$$= \frac{10}{3} - \left(\frac{-7}{6} \right)$$

$$= \frac{9}{2}$$

$$= 4\frac{1}{2}$$

So area is $4\frac{1}{2}$ units^2

Exercise 6.11 Sums and differences of areas

1 Find the area bounded by the line $y = 1$ and the curve $y = x^2$

2 Find the area enclosed between the line $y = 2$ and the curve $y = x^2 + 1$.

3 Find the area enclosed by the curve $y = x^2$ and the line $y = x$

4 Find the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$.

5 Find the area enclosed between the curve $y = x^2$ and the line $y = x + 6$

6 Find the area bounded by the curve $y = x^3$ and the line $y = 4x$

7 Find the area enclosed between the curves $y = (x - 1)^2$ and $y = (x + 1)^2$ and the x-axis

8 Find the area enclosed between the curve $y = x^2$ and the line $y = -6x + 16$.

9 Find the area enclosed between the curve $y = x^3$ the x-axis and the line $y = -3x + 4$

10 Find the area enclosed by the curves $y = (x - 2)^2$ and $y = (x - 4)^2$

ISBN 97807043220

11 Find the area enclosed between the curves $y = x^2$ and $y = x^3$

12 Find the area enclosed by the curves $y = x^2$ and $x = y^2$

13 Find the area bounded by the curve $y = x^2 + 2x - 8$ and the line $y = 2x + 1$.

14 Find the area bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$.

15 Find the exact area enclosed between the curve $y = \sqrt{4 - x^2}$ and the line $x - y + 2 = 0$

16 Find the exact area in the first quadrant between the curve $y = \dfrac{1}{x}$ the x-axis and the lines $y = x$ and $x = 2$.

17 Find the exact area bounded by the curves $y = \sin x$ and $y = \cos x$ in the domain $[0\ 2\ \pi]$

18 Find the exact area enclosed between the curve $y = e^{2x}$ and the lines $y = 1$ and $x = 2$.

19 Find the exact area enclosed by the curve $y = \sin x$ and the line $y = \dfrac{1}{2}$ for $[0, 2\pi]$

Summary of integration rules

Rule	Chain rule
$\int x^n\,dx = \dfrac{1}{n+1}x^{n+} \ +C$	$\int f'(x)[f(x)]^n\,dx = \dfrac{1}{n+1}\left[f(x)^{n+}\ \right]+C$
$\int e^x\,dx = e^x + C$	$\int e^{ax+b}\,dx = \dfrac{1}{a}e^{ax+b} + C$
$\int a^x\,dx = \dfrac{1}{\ln a}\,a^x + C$	
$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int \dfrac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + C$
$\int \cos x\,dx = \sin x + C$	$\int \cos(ax+b)\,dx = \dfrac{1}{a}\sin(ax+b) + C$
$\int \sin x\,dx = -\cos x + C$	$\int \sin(ax+b)\,dx = -\dfrac{1}{a}\cos(ax+b) + C$
$\int \sec^2 x\,dx = \tan x + C$	$\int \sec^2(ax+b)\,dx = \dfrac{1}{a}\tan(ax+b) + C$

ISBN 97807043220

[redacted]

WS
Deivaive and Inegals

Qz
Pacice quiz

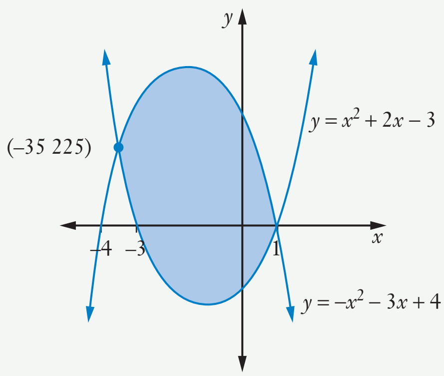
For Questions **1** to **4** choose the correct answer **A** **B** **C** or **D**

1 Find $\int \sin(6x)\, dx$

A $\frac{1}{6}\cos(6x) + C$ **B** $6\cos(6x) + C$

C $-6\cos(6x) + C$ **D** $-\frac{1}{6}\cos(6x) + C$

2 Find the shaded area below.



A $\int_{-4}(-x^2 - 3x + 4)\, dx - \int_{-4}(x^2 + 2x - 3)\, dx$

B $\int_{-3\ 5}(-x^2 - 3x + 4)\, dx - \int_{-3\ 5}(x^2 + 2x - 3)\, dx$

C $\int_{-3}(-x^2 - 3x + 4)\, dx + \int_{-3}(x^2 + 2x - 3)\, dx$

D $\int_{-3\ 5}(-x^2 - 3x + 4)\, dx + \int_{-3\ 5}(x^2 + 2x - 3)\, dx$

3 Find $\int 4e^{3x}\, dx$

A $\frac{4}{3}e^{3x} + C$ **B** $\frac{3}{4}e^{3x} + C$ **C** $12e^{3x} + C$ **D** $\frac{1}{12}e^{3x} + C$

4 Find $\int \frac{x}{x^2 + 3}\, dx$

A $\frac{2}{(x^2 + 3)^2} + C$ **B** $2\ln|x^2 + 3| + C$

C $\frac{1}{2(x^2 + 3)^2}$ **D** $\frac{1}{2}\ln|x^2 + 3| + C$

5 **a** Use the trapezoidal rule with 2 subintervals to find an approximation to $\int^{2} \frac{dx}{x^2}$

b Use integration to find the exact value of $\int^{2} \frac{dx}{x^2}$

6 Find the integral of

a $3x + 1$ **b** $\frac{5x^2 - x}{x}$ **c** $\sqrt{x}$

d $(2x + 5)^7$ **e** $x^3(3x^4 - 2)^4$

ISBN 97807043220

7 Find $\int 3^x \, dx$

8 Find the approximate area under the curve $f(x) = x^3$ between $x = 1$ and $x = 3$ by using

a 4 inner rectangles **b** 4 outer rectangles **c** a trapezium

9 Evaluate

a $\int_0^2 (x^3 - 1)\, dx$ **b** $\int_{-} x^5 \, dx$ **c** $\int_0 (3x - 1)^4 \, dx$

d $\int_0 x^2(x^3 - 5)^2 \, dx$ **e** $\int_{-}^{2} 3x(x^2 + 1)^3 \, dx$

10 Find the area enclosed between the curve $y = \ln x$ the y-axis and the lines $y = 1$ and $y = 3$.

11 Find the area bounded by the curve $y = x^2$ the x-axis and the lines $x = -1$ and $x = 2$.

12 Find $\int \sin x° \, dx$

13 Find the area enclosed between the curves $y = x^2$ and $y = 2 - x^2$

14 Find the indefinite integral of

a e^{4x} **b** $\dfrac{x}{x^2 - 9}$ **c** e^{-x}

d $\dfrac{1}{x + 4}$ **e** $(x - 3)(x^2 - 6x + 1)^8$

15 Evaluate $\int^2 \dfrac{3x^4 - 2x^3 + x^2 - 1}{x^2} dx$

16 Find the exact area in the first quadrant bounded by $x^2 + y^2 = 9$, the y-axis and the lines $y = 0$ and $y = 3$.

17 Find the area bounded by the curve $y = x^3$ the y-axis and the lines $y = 0$ and $y = 1$.

18 Find the integral of $(7x + 3)^{11}$

19 Find the area bounded by the curve $y = x^2 - x - 2$, the x-axis and the lines $x = 1$ and $x = 3$.

20 Find the exact area bounded by the curve $y = e^{2x}$ the x-axis and the lines $x = 2$ and $x = 5$.

21 Use the trapezoidal rule with 4 strips to find the area bounded by the curve $y = \ln(x^2 - 1)$, the x-axis and the lines $x = 3$ and $x = 5$.

22 Evaluate $\int_0^4 (3t^2 - 6t + 5)\, dt$

23 Find the indefinite integral of

a $\sin 2x$ **b** $3 \cos x$ **c** $\sec^2 5x$ **d** $1 + \sin x$

24 Find the area bounded by the curve $y = x^2 + 2x - 15$ and the x-axis

25 The rate at which a metal cools is given by $R = -16e^{-0.4t}$ degrees min^{-}
If the temperature is initially 215°C fin:

a the equation for the temperature T of the metal

b the temperature to the nearest degre, of the metal aftr:
i 5 minutes **ii** half an hour

26 Evaluate

a $\int_0^{\frac{\pi}{4}} \cos x\, dx$ **b** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x\, dx$

27 Find

a $\int 5(2x-1)^4\, dx$ **b** $\int \frac{3x^5}{4}\, dx$

28 Find the exact area bounded by the curve $y = \sin x$ the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$

29 Find

a $\int x^2(x^3-2)^5\, dx$ **b** $\int x(5x^2+2)^4\, dx$

c $\int 5x^3(2x^4-1)^2\, dx$ **d** $\int (x+2)(x^2+4x-3)^3\, dx$

30 Find the area bounded by the curve $y = \cos 2x$ the x-axis and the lines $x = 0$ and $x = \pi$

31 Find (in exact form) the approximate area bounded by the curve $y = \sqrt{x-2}$, the x-axis and the line $x = 4$ usin:

a a triangle **b** 2 inner rectangles **c** 2 outer rectangles

32 Find $f(x)$ given $f'(x)$ and a point on the graph of $f(x)$

a $f'(x) = 3x(2x^2-1)^4$ and passing through (1 3)

b $f'(x) = \sec^2 2x$ and passing through $\left(\frac{\pi}{6}\ \frac{\sqrt{3}}{2}\right)$

c $f'(x) = e^{5x}$ and passing through $(0\ \frac{1}{5})$

d $f'(x) = x^3(x^4-15)^3$ and passing through (2 0)

e $f'(x) = \frac{3x^3}{x^4+1}$ and passing through (0 2)

33 The velocity of a particle is given by $v = \frac{t^2}{\sqrt{t^3+9}}$ m s^{-}

If the initial displacement is –2 m fin:

a the equation for displacement

b the displacement after 5 s

c when the displacement is 10 m

ISBN 97807043220

6. CHALLENGE EXERCISE

1 **a** Show that $f(x) = x^3 + x$ is an odd function

b Hence find the value of $\int_{-2}^{2} f(x)\,dx$

c Find the total area between $y = f(x)$ the x-axis and the lines $x = -2$ and $x = 2$.

2 **a** Show that $\sec x \operatorname{cosec} x = \dfrac{\sec^2 x}{\tan x}$

b Hence or otherwis, find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec} x \sec x\,dx$

3 Find the area enclosed between the curves $y = (x - 1)^2$ and $y = 5 - x^2$

4 Find the exact value of $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2 2x\,dx$

5 Evaluate $\int_0 \dfrac{x}{(3x^2 - 4)^2}\,dx$

6 Use the trapezoidal rule with 4 subintervals to find the area enclosed between the curve $y = \dfrac{3}{x - 2}$ the y-axis and the lines $y = 1$ and $y = 3$.

7 **a** Sketch the curve $y = x(x - 1)(x + 2)$

b Find the total area enclosed between the curve and the x-axis

8 Find the area bounded by the parabola $y = x^2$ and the line $y = 4 - x$ correct to 2 decimal places

9 **a** Find the derivative of $x\sqrt{x+3}$

b Hence find $\int \dfrac{x+2}{\sqrt{x+3}}\,dx$

10 **a** Find $\dfrac{d}{dx}(x^2 \ln x)$

b Hence find the exact value of $\int^{3} 2x(1 + 2\ln x)\,dx$

11 Find the area enclosed between the curves $y = \sqrt{x}$ and $y = x^3$

12 **a** Find the sum of 50 terms of the sequence $2^0, 2^{02}, 2^{0.4}, 2^{0.6}$

b Hence use 50 inner rectangles to find the approximate area under the curve $y = 2^x$ between $x = 0$ and $x = 10$.

c Find this approximate area by using 100 outer rectangles

ISBN 97807043220

Practice set 2

In Questions **1** to **6** select the correct answer **A B C** or **D**

1 The area of a rectangle with sides x and y is 45 Its perimeter P is given by

A $P = x + 45x^2$

B $P = x + \dfrac{45}{x}$

C $P = 2x + \dfrac{90}{x}$

D $P = 2x + \dfrac{45}{x}$

2 The area enclosed between the curve $y = x^3 - 1$, the y-axis and the lines $y = 1$ and $y = 2$ is given by

A $\int^2 (x^3 - 1)\,dy$

B $\int^2 (y + 1)\,dy$

C $\int^2 (\sqrt[3]{y} + 1)\,dy$

D $\int^2 (\sqrt[3]{y + 1})\,dy$

3 Find $\int 4x^2(5x^3 + 4)^7\,dx$

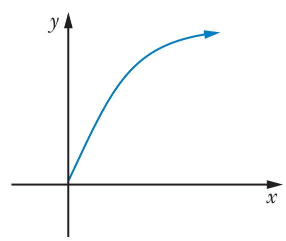
A $\dfrac{4(5x^3 + 4)^8}{15} + C$

B $\dfrac{(5x^3 + 4)^8}{30} + C$

C $\dfrac{(5x^3 + 4)^8}{2} + C$

D $\dfrac{(5x^3 + 4)^8}{120} + C$

4 For the curve shown which inequalities are correct ?



A $\dfrac{dy}{dx} > 0, \dfrac{d^2y}{dx^2} > 0$

B $\dfrac{dy}{dx} > 0, \dfrac{d^2y}{dx^2} < 0$

C $\dfrac{dy}{dx} < 0, \dfrac{d^2y}{dx^2} > 0$

D $\dfrac{dy}{dx} < 0, \dfrac{d^2y}{dx^2} < 0$

5 A cone with base radius r and height h has a volume of 300 cm^3
Its slant height l is given b:

A $l = \sqrt{\dfrac{\pi h^3 + 900}{\pi h}}$

B $l = \sqrt{\dfrac{h^2 + 900}{\pi h}}$

C $l = \sqrt{\dfrac{h^2 + 810\,000}{\pi h}}$

D $l = \sqrt{\dfrac{h^3 + 900}{\pi h}}$

ISBN 97807043220

6 The rate at which a waterfall is flowing over a cliff is given by $R = 4t + 3t^2$ m^3 s$^-$ Find the amount of water flowing after a minute if the amount of water is 10 970 m^3 after 20 seconds

A 223 220 m^3 **B** 8800 m^3

C 225 370 m^3 **D** 226 250 m^3

7 Find all values of x for which the curve $y = (2x - 1)^2$ is decreasing

8 Find $\int (3x^2 - 2x + 1)\, dx$

9 Find the maximum value of the curve $y = x^2 + 3x - 4$ in the domain [–1, 4.

10 For the graph of $y = 8 \sin 3x + 5$ fin:

a the amplitude **b** the period **c** the centre

11 The area of a rectangle is 4 m^2 Find its minimum perimete.

12 If $y = \sin 7x$ show that $\frac{d^2y}{dx^2} = -49y$

13 Find the anti-derivative of $3x^8 + 4x$

14 Sketch the curve $y = x^3 - 3x^2 - 9x + 2$ showing all stationary points and points of inflection

15 Find the area enclosed between the curve $y = x^2 - 1$ and the x-axis

16 Find $\int \frac{3x}{2x^2 - 5}\, dx$

17 If $f(x) = x^3 - 2x^2 + 5x - 9$ find $f'(3)$ and $f''(-2)$

18 Evaluate $\int^3 (6x^2 + 4x)\, dx$

19 Find the domain over which the curve $y = 3x^3 + 7x^2 - 3x - 1$ is concave upwards

20 Evaluate $\int^2 x\sqrt{3x^2 - 3}\, dx$

21 Find $\int \sec^2 x(\tan x + 1)^3\, dx$

22 **a** If $f(x) = 2x^4 - x^3 - 7x + 9$ find $f(1)$ $f'(1)$ and $f''(1)$

b What is the geometrical significance of these results? Illustrate by a sketch of $y = f(x)$ at $x = 1$.

23 A piece of wire of length 4 m is cut into 2 parts One part is bent to form a rectangle with sides x and $3x$ and the other part is bent to form a square with sides y

- **a** Prove that the total area of the rectangle and square is given by $A = 7x^2 - 4x + 1$.
- **b** Find the dimensions of the rectangle and square when the area has the least value

24 Given the function $f(x) = x^2$ find the equation of the transformed function if $y = f(x)$ is translated 5 units up 4 units to the lef, stretched horizontally by a factor of 2 and stretched vertically by a factor of 3

25 The gradient function of a curve is given by $f'(x) = 4x - 3$. If $f(2) = -3$ find $f(-1)$

26 Evaluate $\int_0^3 (2x+1)\,dx$

27 The following table gives values for $f(x) = \frac{1}{x^2}$

x	1	2	3	4	5
$f(x)$	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$

Use the table together with the trapezoidal rule to evaluate $\int^5 \frac{dx}{x^2}$ correct to 3 decimal places

28
- **a** Find the stationary point on the curve $y = (x - 2)^3$ and determine its nature
- **b** Hence sketch the curve

29 Two families travelling on holidays drive along roads that intersect at right angle. One family is initially 230 km from the intersection and drives towards the intersection at an average of 65 km h⁻ The other family is initially 125 km from the intersection and travels towards it at an average of 80 km h⁻

- **a** Show that their distance apart after t hours is given by $d^2 = 10\,625t^2 - 49\,900t + 68\,525$
- **b** Hence find how long it will take them to reach their minimum distance apart
- **c** Find their minimum distance apart

30 For the sequence 100 −50 2, … fid:

- **a** the 10th term
- **b** the sum of the first 10 terms
- **c** the limiting sum

ISBN 97807043220

31 A rectangle is cut from a circular disc of radius 15 cm Find the area of the largest rectangle that can be produced

32 Find $\int (3x+5)^7 \, dx$

33 Evaluate $\int^3 \frac{dx}{x}$ correct to 3 decimal places

34 Find the stationary points on the curve $f(x) = x^4 - 2x^2 + 3$ and distinguish between them

35 Evaluate $\int^2 \sqrt{5x-1} \, dx$ as a fraction

36 Find the area enclosed between the curve $y = (x - 1)^2$ and the line $y = 4$

37 If a function has a stationary point at $(-1, 2)$ and $f''(x) = 2x - 4$ find $f(2)$

38 Find the area enclosed between the curves $y = x^2$ and $y = -x^2 + 2x + 4$

39 Differentiate $x^3 + e^{2x}$

40 Water is flowing out of a pool at the rate given by $R = -20$ litres per minute If the volume of water in the pool is initially 8000 L fin:

- **a** the volume after 5 minutes
- **b** how long it will take to empty the pool

41 Find the exact value of $\int_0^3 3xe^{x\ +}\ dx$

42 The velocity of a particle is given by $v = 12t^2 + 4t + 80$ m s^{-} If the particle is initially 3 m to the right of the origin find its displacement after 5 .

43 The graph of $y = f(x)$ has a stationary point at (3 2. If $f''(x) = 6x - 8$ find the equation of $f(x)$

44 Find the derivative of $\ln (4x + 3)^3$

45 Find $\int \frac{2x+1}{3x^2+3x-2} dx$

46 Differentiate

a $\frac{x}{e^{2x}}$ **b** $\log_3 x$

47 Find the equation of the tangent to $y = e^{x+1}$ at the point where $x = -1$

48 Find the stationary point on the curve $y = xe^{2x}$ and determine its nature

ISBN 97807043220

49 Find the equation of $y = f(x)$ passing through $(\pi\ 1)$ and with $f'(x) = -6 \sin 3x$

50 Find $\int_0^{\frac{\pi}{2}} \sin 2x\, dx$

51 Differentiate

a $\ln (\sin x)$ **b** $\tan (e^{5x} + 1)$

52 Find an approximation to $\int_0^{\frac{\pi}{4}} \tan x\, dx$ correct to 3 decimal places by using a triangle

53 Find the area under the curve $y = 4 - x^2$ by using

a 4 inner rectangles **b** 4 outer rectangles

54 Differentiate each function

a $e^x \sin x$ **b** $\tan^3 x$ **c** $2 \cos\left(3x - \frac{\pi}{2}\right)$

55 Find the equation of the tangent to the curve $y = \tan 3x$ at the point where $x = \frac{\pi}{4}$

56 Differentiate

a $\sin^3 (e^x)$ **b** $\tan (\ln x + 1)$

57 Find the exact area bounded by the curve $y = \ln (x + 4)$ the y-axis and the lines $y = 0$ and $y = 1$.

58 Find the anti-derivative of each function

a e^{3x} **b** $\sec^2 \pi x$ **c** $\frac{1}{2x}$

d $\cos\left(\frac{x}{5}\right)$ **e** $\sin 8x$

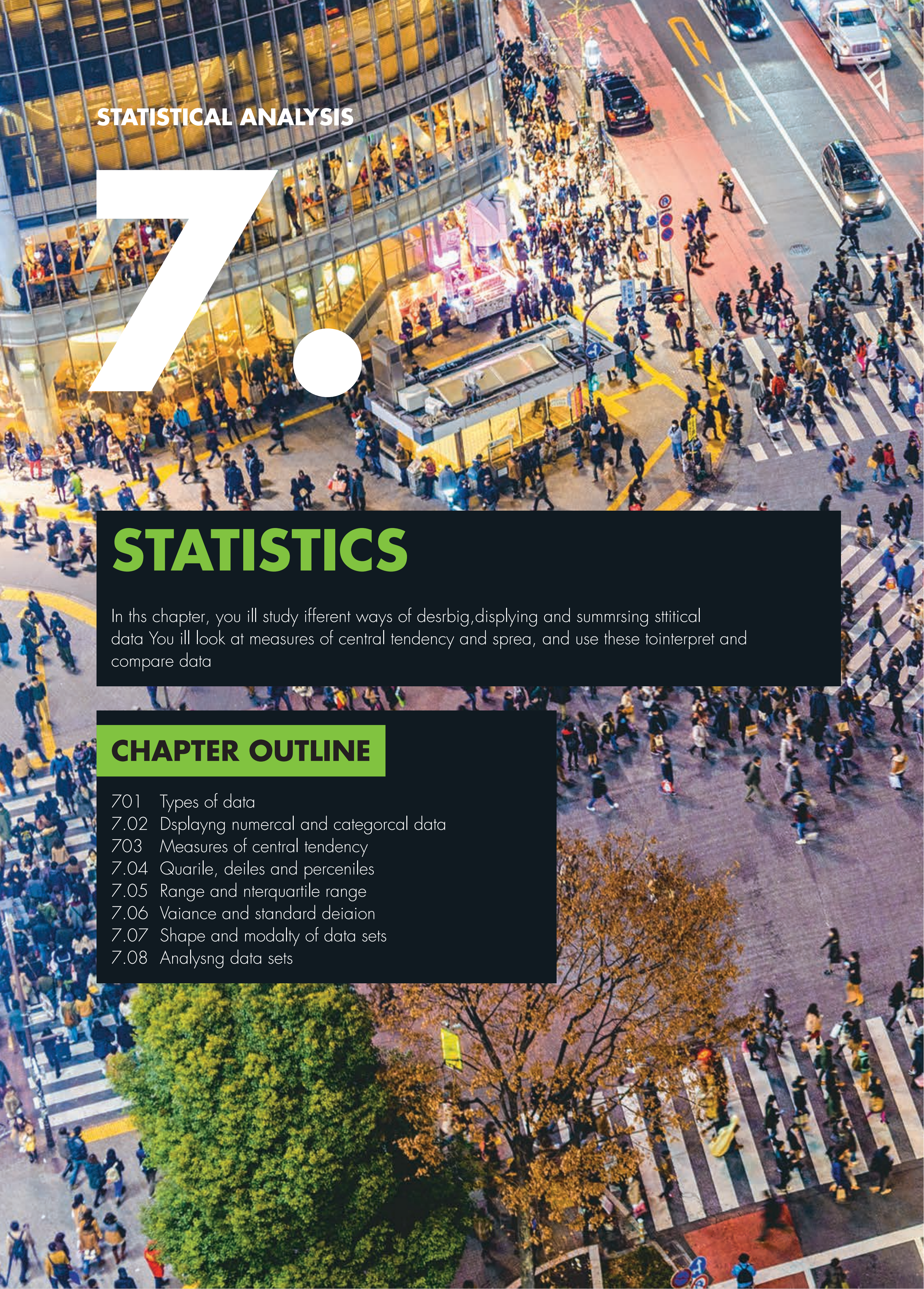
59 Find $\int \frac{3x^2 - 2x + 5}{x^2} dx$

60 Find $\int (e^{5x} - \sin \pi x)\, dx$

61 Find the exact area enclosed between the curve $y = e^x$ the x-axis the y-axis and the line $x = 2$.

62 Evaluate $\int_{\frac{\pi}{3}}^{\pi} \cos\left(\frac{x}{2} + \pi\right) dx$

ISBN 97807043220



STATISTICAL ANALYSIS

7.

STATISTICS

In ths chapter, you ill study ifferent ways of desrbig,displying and summrsing sttitical data You ill look at measures of central tendency and sprea, and use these tointerpret and compare data

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- dentfy dfferent types of data
- dsplay data n tables and graphs
- calculate measures of central tendency the mean medan and mode
- calculate measures of spread the range quantles nterquartile rane, vriance and standard devaton
- dentfy outlers
- recognse dfferent modaiies and shapes of data sets
- dentfy bas n data
- compare 2 sets of data

TERMINOLOGY

bar chart Graph with vertical or horizontal columns also called a **column graph**
bimodal A graph with 2 peak.
box plot Graphical display of five-number summary, also called a **box-and-whisker plot**
categorical data Data that are named by categories
continuous data Numerical data that can take any value that lies within an interval
decile One of the values that divide a data set into 10 equal parts
discrete data Numerical data that can only take specific distinct values
dot plot A column graph of dot.
five-number summary The lowest and highest values media, and lower and upper quartiles of a data set
frequency polygon Frequency line graph
histogram Bar chart of frequencies with no gap between columns
interquartile range Measure of spread the difference between the upper and lower quartiles
mean Average scoe, calculated by dividing the sum of scores by the total number of scores
median The middle score when all scores are placed in order.
modality The number of peaks in a set of dat.
mode The score with the highest frequenc.
multimodal Having many peaks in a set of data
nominal data Categorical data that is listed by name with no order.
numerical data Data whose values are numbers
ogive Cumulative frequency polygon
ordinal data Categorical data that can be ordered
outlier A score that is clearly apart from other scores – it may be much higher or lower than the other scores
Pareto chart A chart containing both a bar chart and a line graph where individual values are represented in descending order by the bars and the cumulative total is represented by the line graph
percentile One of the values that divide a data set into 100 equal parts
pie chart Circular graph showing categories as sectors
quantile One of the values that divide a data set into equal parts
quartile One of the values that divide a data set into 4 equal parts
range Difference between the highest and lowest scores
skewness The shape or asymmetry of a graph to one side
standard deviation Measure of the spread of data values from the mean The square root of variance
stem-and-leaf plot Graphical display of tens (stem) and units (leaves)
symmetrical distribution A distribution where the left and right sides are mirror images of each other.
two-way table A table that combines the effects of 2 separate variables (usually categorical)
variance Measure of spread the square of standard deviation

7.01 Types of data

WS Saiical daa machup

There are many different types of data for exampl, the type of public transport people use to go to work the heights of basketball players or the marks students gain in an exa.

There are 2 main types of data

- **Categorical data** uses categories described by words or symbols
- **Numerical data** uses numbers or quantities

ISBN 97807043220

These types can be divided further

Categorical and numerical data

Categorical data

- **Nominal data** which cannot be put in order
- **Ordinal data** which can be ordered

Numerical data

- **Discrete data** which can be counted as separate values
- **Continuous data** which is measured along a smooth scale

For example

- Public transport – bus trai, trm, ferry – is **categorical nominal** data since it cannot be put into an order.
- Ratings – strongly disagree disagre, agre, strongly agree – are **categorical ordinal** since they can be put in order.
- Shoe sizes – 6 $6\frac{1}{2}, 7, 7\frac{1}{2}, \ldots$ – are **numerical discrete** since they can be counted
- Heights of basketball players – 181 cm 17.64 c, 12.1 cm … – are **numerical continuous** since they are along a smooth scale

EXAMPLE 1

Describe each type of data

a The breeds of dogs

b Exam marks

c The volume of water in a dam

d Audience size for TV programs

e Makes of cars

f Months of the year

Solution

a Categorical nominal

b Numerical discrete

c Numerical continuous

d Numerical discrete

e Categorical nominal

f Categorical ordinal

Exercise 7.01 Types of data

1 State whether each type of data is categorical (C) or numerical (N)

a Length of a fence
b Number of koalas in captivity
c Shoe size
d Colour
e Area of land
f Scores on a test
g Number of lollies in a packet
h Gender
i Speed
j Type of swimming strokes
k Attendance at a football match
Meals on a menu
m Width of a building
n Age
o Weight
p Ranking of quality of a movie
q Surface area of a balloon as it is blown up
r Shirt sizes
s Type of sports offered at a school
t Length of a swimming race

2 State whether each type of data is categorical nominal (N) categorical ordinal (O, numerical discrete (D) numerical continuous (C.

a Survey of radio stations Excellent very goo, god, por, very poor
b Weight of truck loads
c Make and model of motorbikes
d Eye colour
e Volume of water in rivers
f Scores on a maths exam
g Number of jellybeans in a packet
h Nationality
i Acceleration
j Olympic sports
k Concert attendance
Choice of desserts on a menu
m Types of trees in a park

3 Give 3 examples of

a categorical data
b numerical data
c numerical discrete data
d categorical ordinal data
e numerical continuous data
f categorical nominal data

INVESTIGATION

DATA COLLECTION

Certain organisations are specially set up to collect and analyse data The Australian Bureau of Statistics (ABS) collects all sorts of data including the organisation of a regular censu.

The census attempts to collect details of every person living in Australia on a particular day. Questions asked include where a person livs, occupaton, saary, number of chldren, religion and marital status Governments and other organisations use this data to plan future policies in areas such as education transpor, housig. For examle, if the number of children in a certain region is increasing then extra schools could be planned in that are.

ISBN 97807043220

There is evidence that a census was done back in ancient Roman times Investigate the methods that the Romans or some other ancient civilisation used for collecting data and writing reports

What information do you think a census should collect? Is there information that you think that is not useful or invades privacy and therefore should not be collected?

Go to the ABS website and find out more about what this organisation doe. Other worldwide organisations such as the World Health Organization (WHO) and the United Nations also collect data Research these and other organisations that collect dat, such as universities and the CSIR

7.02 Displaying numerical and categorical data

Data can be displayed in many different ways using tables and graphs

WS Gaphing daa

WS Tw-way ables and Paeo chas

WS Evey pitre ells a soy

WS Cumulaive equency gaph

Numerical data

Frequency tables **histograms** **frequency polygons** and other graphs can be used to display numerical data

EXAMPLE 2

a For the following Year 12 English essay marks (out of 1):

8, 4, 5, 4, 8, 6, 7, 8, 9, 5, 6, 7, 7, 5, 4, 6, 7, 9, 3, 5, 5

i draw a frequency distribution table

ii draw a histogram for this data

iii draw a frequency polygon on the same set of axes as the histogram

v how many scores are less than 5?

v what percentage of scores are over 6?

b The assessment scores for a Year 12 mathematics class are belw.

75, 53, 58, 71, 68, 51, 60, 87, 62, 62, 89, 65, 69, 47, 70, 72, 75, 68, 76, 83, 62, 88, 94, 53, 85

i Draw a frequency table that shows the results of the class test using groups of 40–49 50–59 and so o.

ii Add a column for class centre and cumulative frequency.

iii Draw a cumulative frequency histogram and a cumulative frequency polygon (ogive)

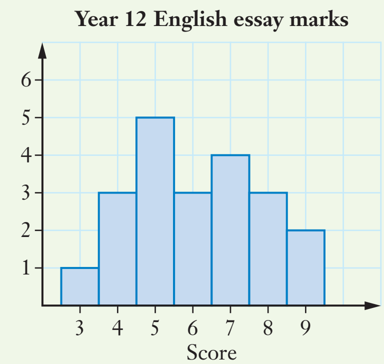
v What percentage of students scored less than 60?

Solution

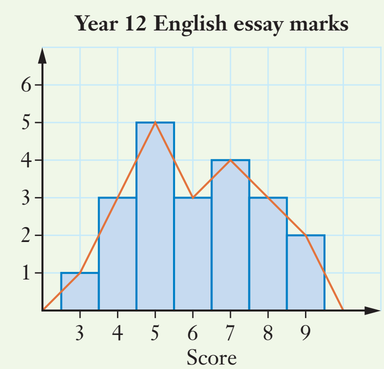
a **i** The scores range from 3 to 9 We arrange them in a table as shwn.

Score	Tally	Frequency
3	|	1
4	|||	3
5	~~||||~~	5
6	|||	3
7	||||	4
8	|||	3
9	||	2

ii The histogram is a bar chart or column graph where the centre of the column is lined up with the score and the columns join together.



iii The frequency polygon is a line graph as shown It starts and ends on the horizontal axis



v Reading from either the table or the graph scores of 3 and 4 are less than . There is one score of 3 and 3 scores of 4

So there are 1 + 3 = 4 scores less than 5

v There are 4 + 3 + 2 = 9 scores over 6 out of a total of 21 scores

$$\frac{9}{21} \times 100\% \approx 429\%$$

ISBN 97807043220

b **i**

Scores	Tally	Frequency
40–49	|	1
50–59	||||	4
60–69	~~||||~~ |||	8
70–79	~~||||~~ |	6
80–89	~~||||~~	5
90–99	|	1

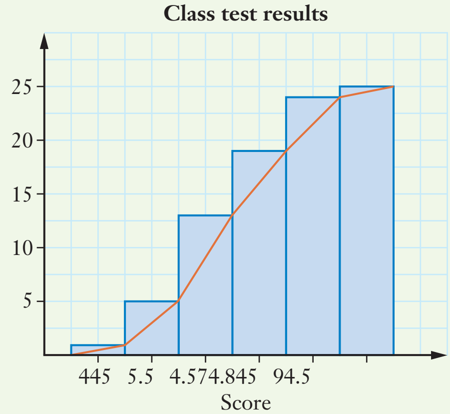
ii The class centre is the average of the highest and lowest possible score in each group For exampl, $\dfrac{40+49}{2}=44\ 5$

Add each score to the previous total for cumulative frequencies

Scores	Class centre	Frequency	Cumulative frequency
40–49	44.5	1	1
50–59	54.5	4	5
60–69	64.5	8	13
70–79	74.5	6	19
80–89	84.5	5	24
90–99	94.5	1	25

iii Use the class centres for the scores on the graph The cumulative frequency polygon or **ogive** starts at the bottom left of the first column and ends at the top right corner of the last column

Class test results



v 5 students scored less than 60

$\dfrac{5}{25}\times 100\% = 20\%$

You can also draw **stem-and-leaf plots** to show discrete data

EXAMPLE 3

The heartbeat rates in beats per minute of a sample of hospital patients were taken

75, 53, 58, 71, 68, 51, 60, 87, 62, 62, 89, 65, 69, 47, 70, 72, 75, 68, 76, 83, 62, 88, 94, 53, 85

Draw a stem-and-leaf plot to display these scores

Solution

On the left of a vertical line put in the 10s for the scores (the stem. On the rigt, place the unit for each score in order (the leaf)

For example for a score of 68 show 6 | 8

Stem	Leaf
4	7
5	1 3 3 8
6	0 2 2 2 5 8 8 9
7	0 1 2 5 5 6
8	3 5 7 8 9
9	4

Note The stem-and-leaf plot keeps the actual scores whereas grouping them into a frequency distribution table loses this individual information

Categorical data

We can display categorical data in different tables and graph, including **two-way tables** **bar charts** **pie charts** and **Pareto charts**

EXAMPLE 4

In a survey of Year 12 studens, it was found that 47 students had a dog but not a at, 19 had both a dog and a cat 32 had a cat but not a do, and 54 had neither a dog nor a ct.

a Draw a two-way table showing this data

b Find the percentage of students who have

- **i** both a dog and cat
- **ii** a cat but not a dog
- **iii** neither a cat nor a dog

ISBN 97807043220

Solution

a A two-way table separates out the students with dogs from those with cats

	Has a dog	Does not have a dog
Has a cat	19	32
Does not have a cat	47	54

Note that this table could be the other way around with the cats at the top and the dogs down the side

b There are $19 + 32 + 47 + 54 = 152$ students altogether.

i 19 students out of 152 have both a dog and a cat $\frac{19}{152} \times 100\% \approx 125\%$

ii 32 students out of 152 have a cat but not a dog $\frac{32}{152} \times 100\% \approx 211\%$

iii 54 students out of 152 have neither a dog nor a cat $\frac{54}{152} \times 100\% \approx 355\%$

EXAMPLE 5

The table shows the eye colour of students

Colour	Frequency
Blue	7
Brown	19
Green	4
Grey	5

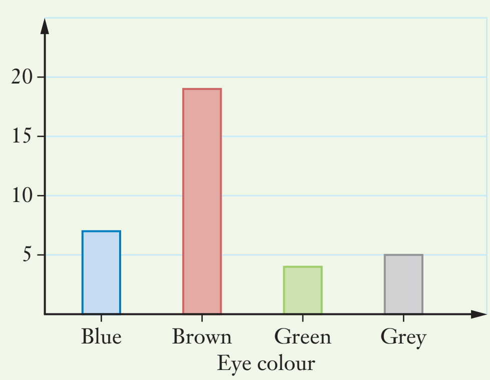
Represent this data in

a a bar chart

b a pie chart

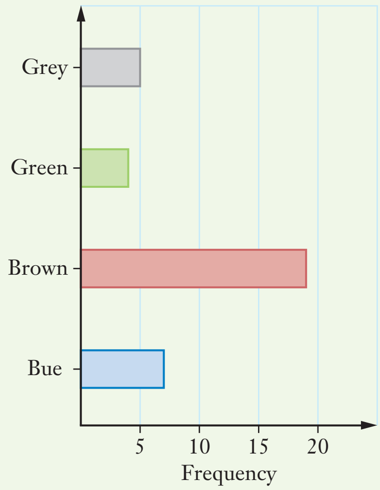
Solution

a Unlike a histogram in a bar chart the columns do not need to join u.



ISBN 97807043220

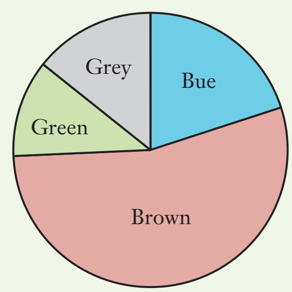
You can also draw the data as a horizontal bar chart like thi.



b A pie chart is a circle divided into portions (sectors) Since the angle inside a circle is 360° each frequency is a proportion of 360.

There were 35 students surveyed

Colour	Frequency	Angle
Blue	7	$\frac{7}{35} \times 360° \approx 72°$
Brown	19	$\frac{19}{35} \times 360° \approx 195°$
Green	4	$\frac{4}{35} \times 360° \approx 41°$
Grey	5	$\frac{5}{35} \times 360° \approx 51°$



Note The number of degrees calculated adds to only 359° because the answers are not exact This will not greatly affect the pie chat.

ISBN 97807043220

A **Pareto chart** is useful for displaying categorical data from the most to the least important

WS

Paeo chat emplae

EXAMPLE 6

The table shows a survey groups preferences for types of TV shos.

a Arrange the table in descending order of frequency and add a percentage frequency column and a cumulative percentage frequency column

b Draw a Pareto chart to show this data

Type	Frequency
News	68
Drama	78
Comedy	73
Reality	107
Sport	174
	500

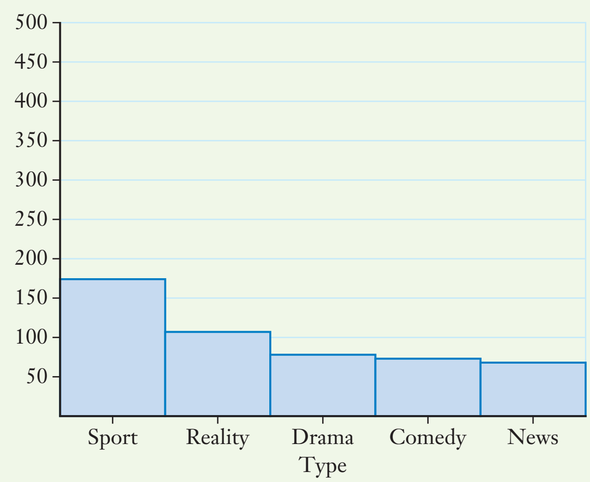
Solution

a

Type	Frequency	Percentage frequency	Cumulative percentage frequency
Sport	174	348%	348%
Reality	107	214%	562%
Drama	78	156%	718%
Comedy	73	146%	864%
News	68	136%	100%
Total	500		

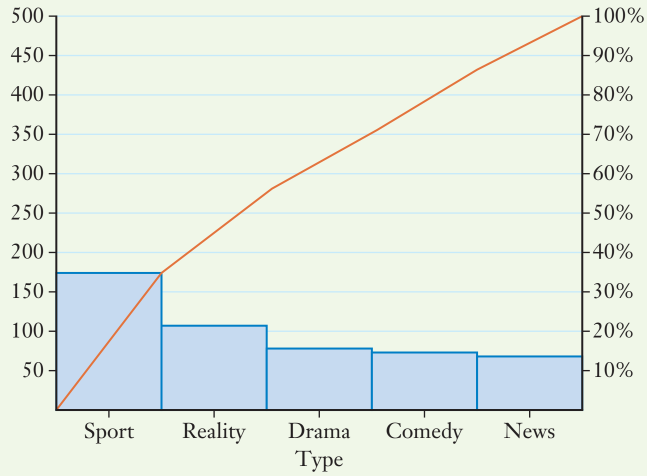
For example percentage frequency for Sport $= \frac{174}{500} \times 100\% = 348\%$

b Step 1 Draw a bar chart of the frequencies using the left axis



ISBN 97807043220

Step 2 Draw a line graph of cumulative percentages using the right axis



Saiical gaph

INVESTIGATION

GRAPHS AND SPREADSHEETS

You can draw different types of graph, including Pareto chars, using a spreadshet.
Enter the data into the spreadsheet highlight the table and select the chart you want to us.
If you are not sure of how to do this search for online tutorial.

DID YOU KNOW?

Vilfredo Pareto

The Pareto chart is named after **Vilfredo Pareto** (1848–1923) an economis, sociologit, engineer and philosopher. The chart can be used as a tool for quality contol.

Research the Pareto chrt, the Pareto principle and the 80/20rule. Find examples of its uses

Exercise 7.02 Displaying numerical and categorical data

1 For each set of scores on the next page

- **i** draw a frequency distribution table
- **ii** draw a histogram and frequency polygon
- **iii** find the highest and lowest scores (groups for parts **d** and **e**)
- **v** find the most frequent score (group for parts **d** and **e**)

ISBN 97807043220

a Results of a class quiz

8, 6, 5, 7, 6, 8, 3, 2, 6, 5, 8, 4, 7, 3, 8, 7, 5, 6, 5, 8, 6, 4, 9, 6, 5

b The number of people ordering pizzas each night

15, 12, 17, 18, 18 ,15, 16, 13, 15, 17, 18, 12, 17, 14, 16, 15, 17, 18, 19, 15, 15, 12

c The number of people attending a gym

110, 112, 114, 109, 112, 113, 108, 110, 113, 112, 113, 110, 109, 110, 110, 112, 114, 114, 112, 114, 113

d The results of an assessment task

45, 79, 65, 48, 69, 50, 62, 74, 38, 69, 88, 96, 90, 58, 52, 68, 63, 61, 79, 74, 50, 65, 77, 91, 56, 77, 63, 81, 90, 59, 67, 50, 61

(Use groups of 30–39 40–49 and so o.)

e The heights of students (in cm) in a Year 12 clas:

159, 173, 182, 166, 172, 179, 181, 163, 178, 169, 183, 158, 162, 167, 174, 175, 180, 174, 176, 159, 161, 171, 174, 179, 180, 159, 157

(Use groups of 155–159 160–16, 165–169 and so n.)

2 For each data set

i add a cumulative frequency column and class centre where necessary

ii sketch a cumulative frequency histogram and ogive (cumulative frequency polygon)

a Number of cars in a school car park

Number of cars	Frequency
10	4
11	8
12	11
13	9
14	5

b Results of a science experiment

Score	Frequency
1	7
2	1
3	3
4	0
5	2
6	5

c Number of sales made in a shoe shop

Sales	Class centre	Frequency
0–4		6
5–9		2
10–14		3
15–19		5
20–24		8
25–29		9
30–34		5

d Results of an assessment task

Scores	Class centre	Frequency
0–19		3
20–39		2
40–59		7
60–79		6
80–99		1

ISBN 97807043220

3 The table shows the number of daily rescues at a beach over a period of time

a Draw a histogram showing this data

b How many times were more than 6 rescues made?

c What was the most common number of daily rescues during the survey?

Rescues	Frequency
4	1
5	3
6	6
7	4
8	5
9	0
10	2

4 The volume of traffic on a stretch of highway was measured and the results are shown in the table

a Draw a histogram to show this data

b What was the percentage volume of traffic between 21 and 40 minutes?

Volume/min	Frequency
1–10	7
11–20	10
21–30	8
31–40	5
41–50	4

5 **a** Draw a two-way table for the following data

- 27 people play soccer but not tennis
- 35 people play tennis but not soccer
- 28 play neither sport
- 12 play both sports

b What percentage of people play

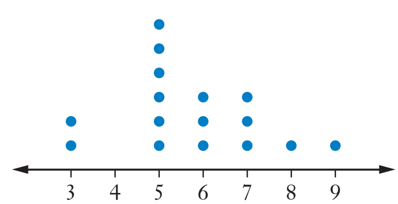
i both sports? **ii** neither sport?

c What percentage of people who play at least one of these sports play only soccer but not tennis?

d What fraction of people who play soccer play tennis as well?

e What percentage of tennis players do not also play soccer?

6 Lauren surveyed her friends and had them rank a film from 1 to 10 The **dot plot** shows the results of her survey.



a How many friends did Lauren survey?

b What was the most common ranking?

c Draw the results in a histogram

d What percentage of rankings were above 4?

e What fraction of Laurens friends ranked the film below 4 ?

ISBN 97807043220

7 The stem-and-leaf plot shows the weights (in kg) of a group of people surveyed at a local gym

Stem	Leaf
5	4 6 8 9
6	1 3 7 8
7	0 3 4 5 5 9
8	1 2 2 4 7 8 9
9	3 5
10	2 6 7

a Arrange these weights in a frequency distribution table using groups of 50–59 60–69 and so on and include class centre and cumulative frequency columns

b Draw an ogive of this data

c How many people weighed

i 80 kg or more? **ii** less than 60 kg?

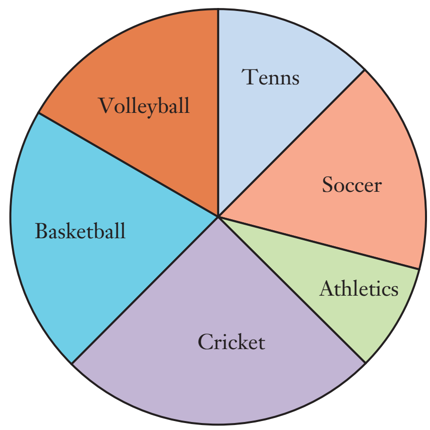
d What percentage of people surveyed weighed from 70 kg to 89 kg?

e What fraction of people weighed between 50 kg and 80 kg?

8 The pie chart shows the number of students taking different school sports

a What percentage of students play cricket?

b There are 720 students at the school who play these sports By measuring the angles in the pie chart complete a table showing the frequencies for the different sports



9 The table shows the results of a survey of university students asking what degree they were doing

a What percentage of students were studying law?

b What percentage of students were studying medicine or music?

c Draw a pie chart showing this information

Degree	Frequency
Medicine	104
Arts	87
Music	58
Science	93
Economics	79
Law	101

10 The two-way table shows the results of a survey into the protective effect of vaccination on a new virus

	Vaccinated	Not vaccinated
Infected	11	76
Not infected	159	58

a How many people in the survey were infected with the virus?

b What percentage of people surveyed were vaccinated?

c What percentage of vaccinated people had the virus?

d How many people with the virus were not vaccinated?

ISBN 97807043220

11 The two-way table shows the number of people taking part in a trial of a new medication to prevent asthma

	Taking medication	Control group
Asthmatic	104	105
Not asthmatic	112	109

a How many people took part in the trial?

b What percentage of people were asthmatic?

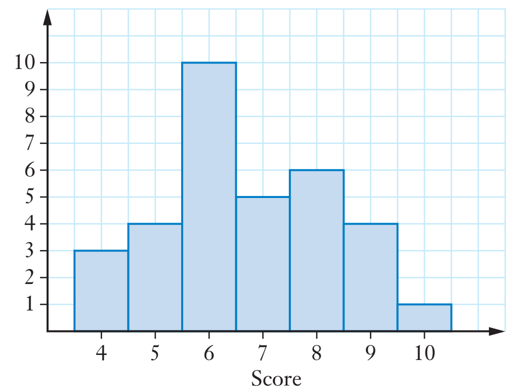
c What percentage of asthmatic people were in the control group?

d How many people who were not asthmatic took part in the trial?

e What fraction of the non-asthmatic people took medication?

12 The frequency histogram shows the scores on a maths quiz

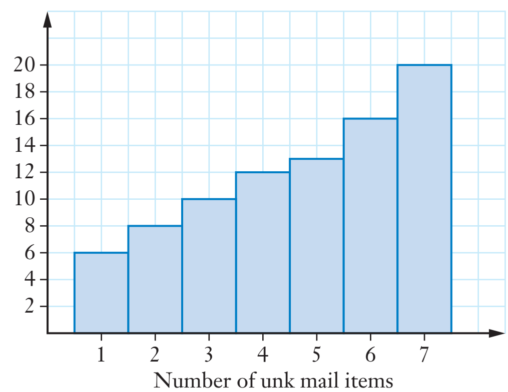
Draw a cumulative frequency histogram and polygon



13 The cumulative frequency histogram shows data collected in a survey on the number of junk mail items that people received daily in their inbox

a Draw a frequency distribution table to show the number of junk mail items people receive daily.

b Construct a frequency histogram to show this data



14 Explain how a stem-and-leaf plot and grouped frequency distribution table can be used for the same data What are the advantages and disadvantages of each ?

ISBN 97807043220

15 Draw a Pareto chart for each set of data

a A survey into why people like a movie

Reason	Votes
Acting	33
Storyline	29
Music	12
Characters	26

b Customer complaints about an Internet provider

Complaint	Frequency
Internet speed	34
Cost	61
Data allowance	59
Technical difficulties	46

c Votes for best café in a subur:

Café	Votes
Coffee Haus	32
Coffee Bean	48
Café Focus	21
Jumping Bean	63
Caffeine Café	36

7.03 Measures of central tendency

When we analyse data we try to find a'typic' 'nomal'or 'avrage' core. For example we might want to know the average crowd size at football matches through the season You would usually expect to find this score somewhere in the centre of the dat. There are 3 **measures of central tendency** the mean the mode and the media.

WS Meaue o cenal endency

Saiical meaue

Saiic om a equency able

WS Mean median and mode

The mean

$$\text{Mean} = \frac{\text{Sum of scores}}{\text{Total number of scores}}$$

$$\bar{x} = \frac{\Sigma x}{n}$$

The **mean** has symbol $\bar{x}$ n is the number of scores and Σx is the sum of scores Not: Σ is the Greek letter 'sigm' and is used in mathematics to stand for a su.

The symbol $\bar{x}$ usually represents the mean of a **sample** For the mean of a **population** the correct symbol is μ the Greek letter m.

ISBN 97807043220

EXAMPLE 7

There are 5 children in a family, aged 3,19, 11, 17 ad 10. Find the mean of thei ages.

Solution

$$\bar{x} = \frac{\Sigma x}{n}$$

$$= \frac{13+19+11+17+10}{5}$$

There are 5 chldren so n 5

$$= \frac{70}{5}$$

$$= 14$$

So the mean age of the children is 14

The mean can also be calculated using a calculators statistics mod.

	Casio scientific	Sharp scientific
Place your calculator in statistical mode	MODE STAT 1 VAR	MODE STAT =
Clear the statistical memory.	SHIFT 1 EDIT DEL-A	2ndF DEL
Enter data	SHIFT 1 Data to get table 13 = 19 = etc to enter in column AC to leave table	13 M+ 19 M+ etc
Calculate mean	SHIFT 1 VAR $\bar{x}$ =	RCL $\bar{x}$
Check the number of scores	SHIFT 1 VAR n =	RCL n
Change back to normal mode	MODE COMP	MODE 0

For the mean of larger data sets it is easier to sort the data into a frequency distribution table to add up the scores

The mean of data in a frequency table

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$

where Σfx is the sum of each score × its frequency and Σf is the sum of frequencies

ISBN 97807043220

EXAMPLE 8

Find the mean number of hours that members of a class practise piano each week
The number of hours for each student is

1, 4, 6, 1, 3, 6, 2, 1, 1, 3, 2, 5, 6, 6, 1, 2, 6, 2, 5, 6, 6, 2, 3, 6, 2

Solution

First draw up a frequency distribution table for the hours of practice Include an *fx* column for multiplying each score by its frequency.

The table gives us a quick way of finding the sum of scores For exampl, we know from the table that there are 6 lots of 2 so we can use $2 \times 6 = 12$ The sum of the *fx* column gives us the sum of all scores

Hours (x)	Frequency (f)	Score $\times$ frequency (fx)
1	5	5
2	6	12
3	3	9
4	1	4
5	2	10
6	8	48
	$\Sigma f = 25$	$\Sigma fx = 88$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$
$$= \frac{88}{25}$$
$$= 352$$

So the mean time students practise each week is 352 hours

On a calculator

Operation	Casio scientific	Sharp scientific
Clear the statistical memory first (see previous example previous page.		
Enter data	SHIFT MODE scroll down to STAT, Frequency? ON SHIFT 1 Data to get table 1 = 2 = etc to enter in x column 5 = 6 = etc to enter in FREQ column AC to leave table	1 2ndF STO 5 M+ 2 2ndF STO 6 M+ etc
Calculate mean	SHIFT 1 VAR $\bar{x}$ =	RCL $\bar{x}$

ISBN 97807043220

Although grouped data is not completely accurate because we dont know exactly what scores are in each group we can still calculate an estimate of the mea.

EXAMPLE 9

From the table find the mean commuting time that a sample of people take to travel to work

Minutes	Frequency
0–8	3
9–17	5
18–26	7
27–35	8
36–44	2

Solution

Add **class centre** and fx columns to the table

Use the class centres as the scores when calculating fx

Minutes	Class centre (x)	Frequency (f)	Score × frequency (fx)
0–8	4	3	12
9–17	13	5	65
18–26	22	7	154
27–35	31	8	248
36–44	40	2	80
		$\Sigma f = 25$	$\Sigma fx = 559$

$$\begin{aligned}\bar{x} &= \frac{\Sigma fx}{\Sigma f}\\ &= \frac{559}{25}\\ &\approx 2236\end{aligned}$$

The mean time taken to travel to work is 2236 minutes

The mode

The **mode** is the most frequent score

ISBN 97807043220

There is no mode if all the scores are different or there could be several scores with the same frequency.

EXAMPLE 10

a Find the mode of these scores

7, 4, 3, 5, 7, 1, 2

b Find the mode for these shoes sold at a shoe store

Shoe size	5	$5\frac{}{2}$	6	$6\frac{}{2}$	7	$7\frac{}{2}$	8	$8\frac{}{2}$	9	$9\frac{}{2}$
Frequency	8	9	15	28	53	61	58	29	12	10

Solution

a There are two 7s and only one of the other scores so the mode is .

b The shoe size with the highest frequency is $7\frac{1}{2}$ (there were 61 of them)

So the mode is $7\frac{1}{2}$

With grouped dat, instead of finding the mode we find the **modal class**

EXAMPLE 11

Find the modal class in this data set showing the ages of people at a caravan park

Scores	Frequency
10–19	2
20–29	0
30–39	1
40–49	5
50–59	7
60–69	3

Solution

The group or class with the highest frequency is 50–59

While we do not know the individual score with the highest frequency, we say the modal class is 50–9.

The mode is useful when looking at trends such as the most popular types of clothing It can also be used for categorical data

The median

The **median** is the middle score when all scores are in order.

If there are 2 middle scores the median is the average of those score.

ISBN 97807043220

EXAMPLE 12

Find the median age of a group of people in a band

18, 15, 20, 18, 17, 16, 11, 13

Solution

Put the ages in order.

11, 13, 15, 16, 17, 18, 18, 20

There are 2 middle ages 16 and 1, so we find their averae.

$$\text{Median} = \frac{16+17}{2} = 165$$

So the median age of the band members is 165

The median can also be calculated using a calculators statistics mod.

Operation	Casio scientific
Clear the statistical memory.	
Enter data	SHIFT 1 Data to get table 18 = 15 = etc to enter in colum. AC to leave table
Calculate the median	SHIFT 1 MinMax med =
Change back to normal mode	MODE COMP

You can find the median of data in a frequency tabl. If there is a large number of scors, you can find the position of the middle score using a cumulative frequency column

EXAMPLE 13

Find the median of this data set

Score	Frequency
5	3
6	2
7	4
8	7
9	6
10	3

ISBN 97807043220

Solution

Add a column for cumulative frequencies

Score	Frequency	Cumulative frequency
5	3	3
6	2	5
7	4	9
8	7	16
9	6	22
10	3	25

There are 25 scores so the position of the middle score is $\frac{25+1}{2}$ = 13th

The 10th to 16th scores are 8 so the 13th score is .

You can check tis by wrtng the scores n order.

The median is 8

The position of the median

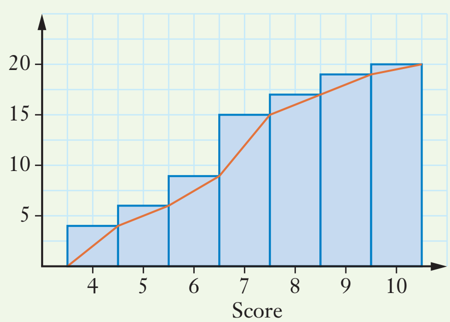
The median of n scores is the $\frac{n+1}{2}$ th score

If n is even then the median is the average of the 2 middle scores on both sides of the $\frac{n+1}{2}$ th position

Another way to find the median is from an ogive (cumulative frequency polygon) We simply use the halfway point on the cumulative frequency axis of the graph

EXAMPLE 14

Find the median from the cumulative frequency polygon below.

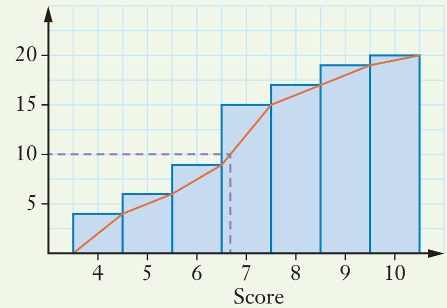


ISBN 97807043220

Solution

There are 20 scores in the data set
so the halfway point is at the 10th score
as shown on the cumulative frequency axis
The dotted line meets the ogive inside
the 7 column

The median is 7



Outliers

Sometimes a set of data contains a score that is unusual compared with the other score. This unusual or extreme value is called an **outlier**

EXAMPLE 15

The prices of houses sold in the town of Greenfield in a particular week are

\$355 000, \$420 000, \$320 000, \$285 000, \$390 000, \$1 200 000, \$415 000, \$320 000, \$435 000 \$380 000

a Is there an outlier? Why do you think an outlier may be in this data?

b Find the mean house price with and without the outlier.

c Find the median with and without the outlier.

d Find the mode with and without the outlier.

Solution

a The outlier is \$1 200 000 as this is much higher than the other prices It may be that there is one special house in the area that is much larger than the others or a certain street with huge houses in it that is unusual for the area

b Using a calculator

With the outlie, the mean house price is \$452 00.

Notce the bg dfference

Without the outlie, the mean house price is \$368 8889.

c With the outlie, the median house price is \$385 00.

Notce the small dfference

Without the outlie, the median house price is \$380 00.

d The mode is \$320 000 in both cases since this is the most frequent price with or without the outlier.

ISBN 97807043220

Note When real estate agents talk about house price, they usually use the median price since this is not as affected by outliers as the mean

INVESTIGATION

OUTLIERS

Which measures of central tendency do outliers tend to affect most?

Find other examples of data that contain outliers and find the mean mode and media. How do they change if the outlier is removed? Should outliers be looked at closely and discarded or is there a place for them?

Exercise 7.03 Measures of central tendency

1 For each data set find

i the mean **ii** the mode **iii** the median

a Number of people auditioning for parts in a play 5 ,7, 6 , 6, 1

b Number of minutes for an ambulance to respond to a call 1 ,6, 8 ,4, 6, 4, 5

c Ages of students on a basketball team 15, 18, 14, 19, 18, 17, 11

d Scores on a class quiz 4, 6, 5, 4, 7, 8

e Prices of petrol (in dollars) 143, .66,1.55, 1.49 1.2, 1.1, 149, 1.38

2 Find the mode of each data set

a

Hair colour	Frequency
Brown	28
Blond	21
Red	8
Black	12
Grey	17

b

Type of cat	Frequency
Siamese	18
Burmese	12
Russian blue	9
Tabby	32
Ginger	26
Persian	19

ISBN 97807043220

3 For each data set find

i the mean **ii** the median **iii** the mode

a Judges scores on a dance contest

Score	Frequency
3	3
4	4
5	2
6	7
7	6
8	2
9	3

b Number of matches in each match box surveyed

Score	Frequency
50	1
51	6
52	5
53	3
54	4
55	2

c Results in a History assignment

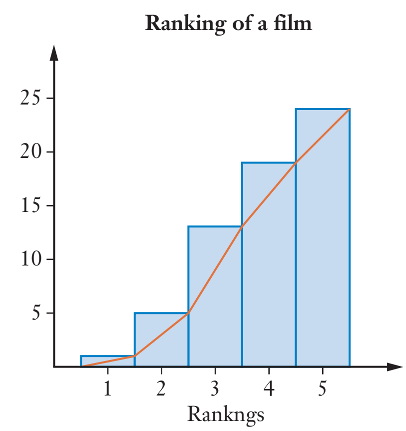
Score	Frequency
14	4
15	2
16	1
17	4
18	3
19	5
20	6

d Attendances at hockey matches

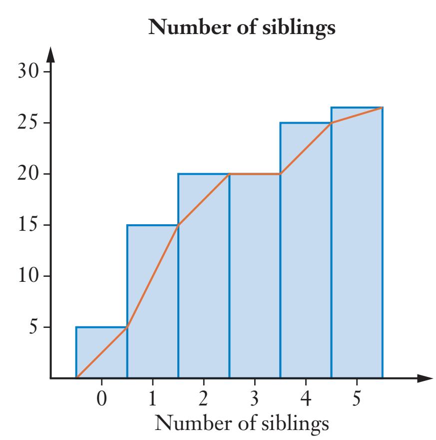
Attendance	Frequency
100	3
101	0
102	2
103	1
104	6
105	5

4 Find the median from each ogive

a

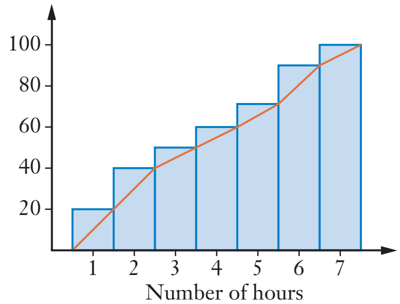


b

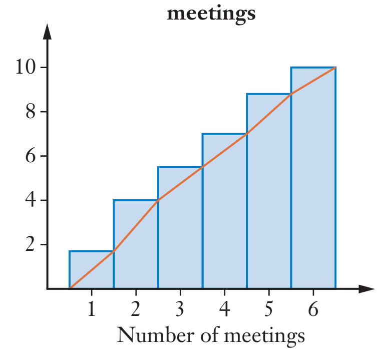


ISBN 97807043220

c



d



5 For each data set find

i the mean **ii** the modal class

a Games of chess played each week by members of a chess club

Score	Frequency
2–4	5
5–7	4
8–10	7
11–13	4
14–16	3
17–19	2

b Hours per week that gymnasts train

Score	Frequency
0–4	3
5–9	2
10–14	6
15–19	8
20–24	9
25–29	5

c Results in a Legal Studies exam

Score	Frequency
10–24	4
25–39	0
40–54	1
55–69	5
70–84	9
85–99	8

d Time it takes for computers to boot u:

Time (s)	Frequency
20–24	12
25–29	8
30–34	9
35–39	7
40–44	8
45–49	11
50–54	12
55–59	6

ISBN 97807043220

6 For each data set

- **i** add a cumulative frequency column
- **ii** draw a cumulative frequency polygon
- **iii** find the median from the graph (estimate for parts **c** and **d**)

a Number of athletes representing their school over a 30-year period

Athletes	Frequency
1	5
2	6
3	4
4	8
5	5
6	2

b Number of lollies in a bag

Number	Frequency
45	3
46	5
47	1
48	7
49	3
50	1

c Hours a week worked by employees in a cafe

Hours	Frequency
1–5	7
6–10	5
11–15	3
16–20	6
21–25	7
26–30	2

d Time to complete a rac:

Time (min)	Frequency
2.5–2.8	3
2.9–3.2	2
3.3–3.6	0
3.7–4.0	6
4.1–4.4	1
4.5–4.8	4
4.9–5.2	4

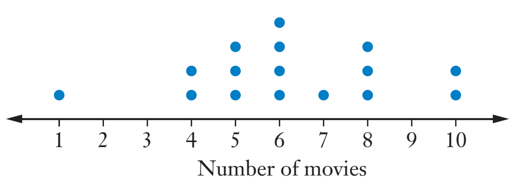
7 For each data set

- **i** draw a frequency distribution table including cumulative frequency
- **ii** find the mean
- **iii** find the mode or modal class
- **v** draw an ogive
- **v** find the median from the ogive

a Home runs scored over a baseball season

4, 6, 5, 8, 8, 6, 5, 3, 4, 9, 6, 3, 5, 6, 5, 4, 7, 5, 8, 5, 6, 2, 3

b Number of movies seen in a year



ISBN 97807043220

c Ages of people living in a block of units (use classes of 20–29 30–39 and so on:

Stem	Leaf
2	1 3 5 5 8 9
3	0 2 4 5 6 6 7 8 8 9
4	2 3 3 6 8
5	0 1 1 1 4 5 5 7 8 8 9
6	3 3 4 5 6 7 7
7	1 5 6
8	1 2 2 4 5 6

8 For each set of data fin:

i the outlier

ii the mean mode and median

iii the mean mode and median without the outlier

a Weights (in kg) of people in a lif:

69, 75, 58, 77, 32, 68, 60, 64, 59

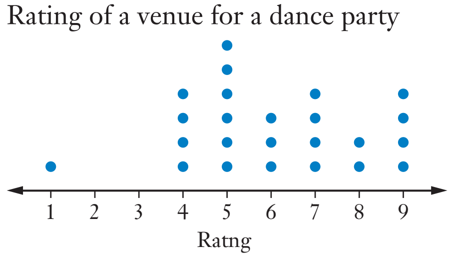
b Number of questions attempted in an exam

Questions	Frequency
1	1
2	0
3	0
4	1
5	3
6	6
7	5
8	3
9	4

c Ages of people at a family party

Stem	Leaf
2	3 5 6
3	2 7
4	1 3
5	0 0 3 4
6	3 5
7	
8	
9	7

d Rating of a venue for a dance party



9 For the set of times (in minutes) students are recorded as late for school shown below, find the outlier. Which measures of central tendency (man, meian, mode) does it change ?

5, 3, 6, 4, 7, 1, 6, 8, 7, 9, 6, 5, 8, 6, 7, 4, 5, 7, 4

ISBN 97807043220

10 The stem-and-leaf plot below shows the results of a class test

Stem	Leaf
1	7
2	
3	8 9
4	4 5 6
5	1 3 4 6 7 8
6	4 5 5 7 7 9 9 9 9
7	0 2 3 3 4 5 8
8	3 4 5 7
9	0 1 1 3

a State which score is an outlier and what this means

b Draw a frequency table including a cumulative frequency column using groups 10–19 20–2, 30–39 and so n.

c Use the table to estimate the mean and find the modal class

i with the outlier included **ii** without the outlier included

d Draw an ogive excluding the outlier and find the median

INVESTIGATION

LIMITATIONS OF CENTRAL TENDENCY

Find the mean mode and median of each set of dat. What do you notice ?

Set 1 5, 6, 7, 7, 8, 9

Set 2 1, 2, 7, 7, 12, 13

How do the 2 sets of data differ? Can we find out by using the measures of central tendency? How else could we describe how they are different from each other?

WS

Quatil, deciles and pecenile

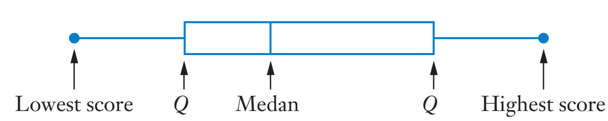
Boxand whiske plos

7.04 Quartiles, deciles and percentiles

The measures of central tendency give us good information about data sets but they don't describe the spread of data As we have sen, the median divides data sets so that half the values lie below the median and half lie above it A measure that divides a data set into parts of equal size is called a **quantile** The median gives only a very rough description of the data set but with more divisions we can describe the dat's spread in more detal.

Quartiles

A **quartile** divides a data set into quarters



 ISBN 97807043220

The 1st quartile (Q) is called the **lower quartile** and the 3rd quartile (Q_3) is called the **upper quartile** The 2nd quartile (Q_2) is the median

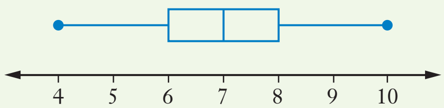
A **box plot** (also called a **box-and-whisker plot**) gives a way of showing a five-number summary the quartiles and highest and lowest scores

EXAMPLE 16

a Find Q Q_2 and Q_3 for the number of runs in a softball game

4, 3, 7, 8, 7, 9, 5, 6, 8, 3, 9

b Find Q Q_3 the media, the highest and lowest score for this daa.



c Find the quartiles of the data in this frequency table

Score	Frequency
1	1
5	3
6	5
7	8
8	11
9	13
10	9

Solution

a Put the 11 scores in order.

3, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9

Find Q_2 (the median) in the usual way the 6th score is 7

3, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9

Q is the middle of the scores below the median the 3rd score .

If a quarile falls between 2 scores we take ther average

3, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9

Q_3 is the middle of the scores above the median the 9th score .

So Q $= 4$, $Q_2 = 7$, $Q_3 = 8$

ISBN 97807043220

On a calculator

Operation	Casio Scientific
Clear the statistical memory.	
Enter data	SHIFT 1 Data to get table 4 = 3 = etc to enter in colum. AC to leave table
Calculate Q	SHIFT 1 MinMax Q =
Calculate Q_2 (median)	SHIFT 1 MinMax med =
Calculate Q_3	SHIFT 1 MinMax Q_3 =

b From the box plot

Lowest score = 4, Q = 6 Median = 7, Q_3 = 8 Highest score = 10.

c Add a cumulative frequency column to the table

Score	Frequency	Cumulative frequency
1	1	1
5	3	4
6	5	9
7	8	17
8	11	28
9	13	41
10	9	50

There are 50 scores so the median is the $\frac{50+1}{2}$ or 255th score (average of 25th and 26th scores)

So $Q_2 = 8$ reading from the cumulative frequency colum.

Q is the $\frac{25+1}{2}$ or 13th score (middle of the 1st 25 scores)

So $Q = 7$ reading from the cumulative frequency colum.

Q_3 is the $\frac{50+25+1}{2}$ or 38th score (middle of the last 25 scores halfway between the 26th to 50th scores)

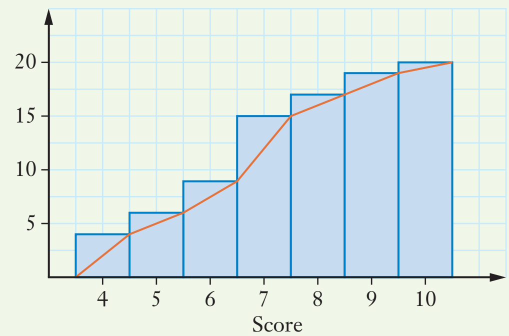
So $Q_3 = 9$ reading from the cumulative frequency colum.

So $Q = 7$, $Q_2 = 8$, $Q_3 = 9$

ISBN 97807043220

EXAMPLE 17

Find the median and upper and lower quartiles from the ogive



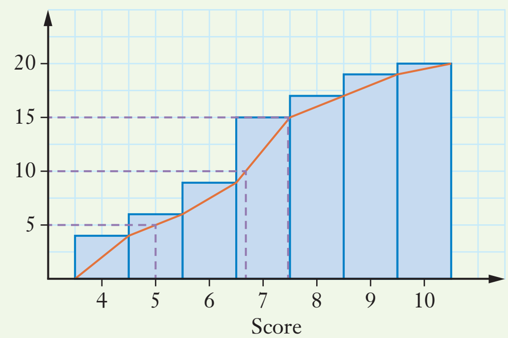
Solution

There are 20 scores

The median is halfway

$$\frac{1}{2} \times 20 = 10$$

So the median is 7 reading across from 10 on the cumulative frequency ais.



To find the 1st (lower) quartil:

$$\frac{1}{4} \times 20 = 5$$

So reading across from 5 Q = 5

To find the 3rd (upper) quartil:

$$\frac{3}{4} \times 20 = 15$$

So reading across from 15 $Q_3 = 7.5$.

Deciles and percentiles

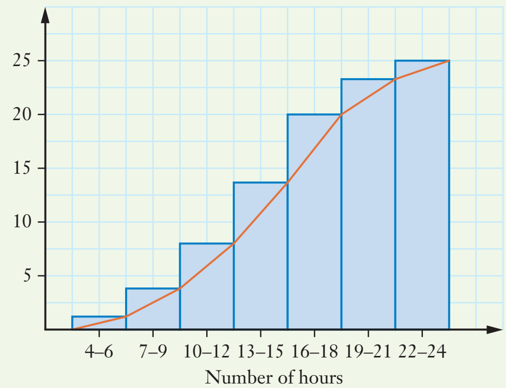
For a more detailed description of the spread we can divide the data set into smaller parts

Deciles divide the data set into 10 parts and **percentiles** divide data sets into 100 parts

Deciles and pecenile

EXAMPLE 18

The ogive shows the number of hours that a rock group rehearses each week over 25 weeks The scores have been sorted into groups



Use the ogive to estimate

a the 35th percentile

b the 60th percentile

c the 7th decile

We can only **estmate** because the scores have been grouped nto classes

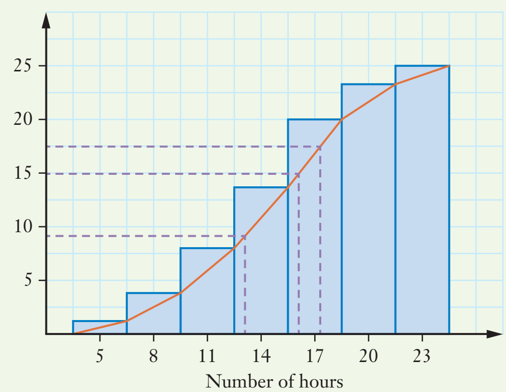
Solution

Redraw the ogive using **class centres** for number of hours and use the cumulative frequency axis to find answers

a The 35th percentile is the score that separates the lower 35% of the scores

35% of 25 scores = 875th score

The 35th percentile ≈ 13.



b The 60th percentile separates the lower 60% of the scores

60% of 25 scores = 15th score

The 60th percentile ≈ 16.

c The 7th decile separates the lower $\frac{7}{10}$ of the scores

$\frac{7}{10} \times 25$ scores = 175th score

The 7th decile ≈ 17.

You can use a graphics calculator or software to draw graphs and find quartile, deciles and percentiles more accurately.

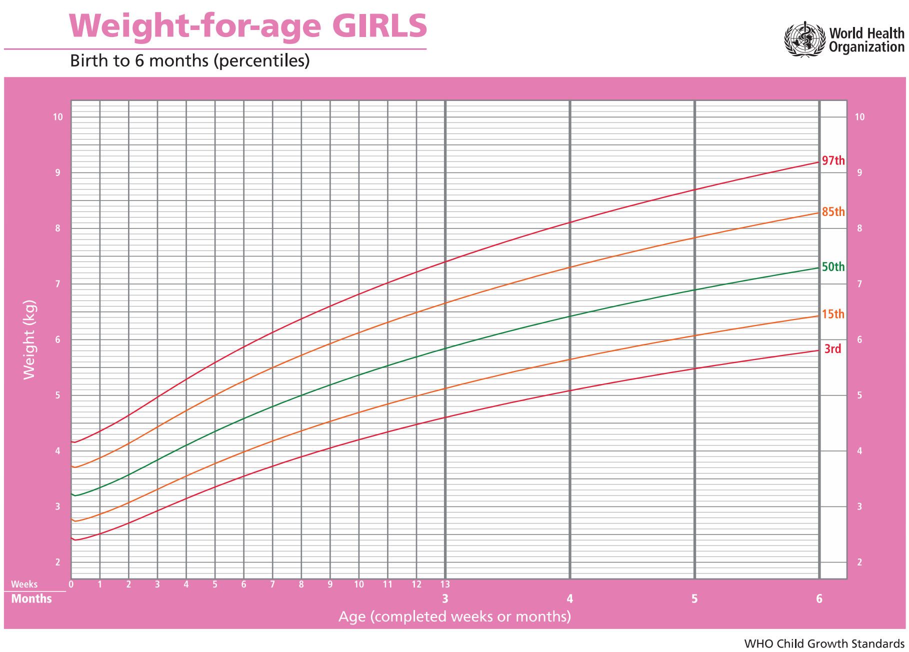
ISBN 97807043220

INVESTIGATION

RESEARCHING QUANTILES

Research the words *quartile* *decile* and *percentile* When were they first used? Where are they used now? There are other measures such as tercile and quintil. What are they ?

Percentiles are used in many applications including graphs of infants and childrens growth rates



hp://www.who.it/childroth/tandrds/ct_fa_grls_p_0_6.pf ?ua

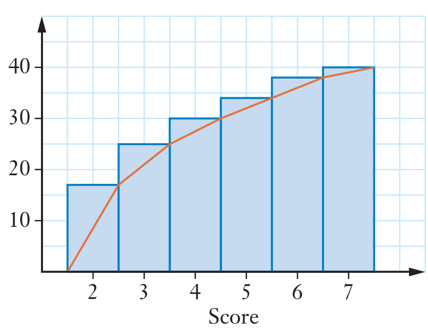
Research and compare different types of growth charts used in Australia and those of the World Health Organization (WHO.

Exercise 7.04 Quartiles, deciles and percentiles

1 For each set of data fin:

i the 1st quartile **ii** the 2nd quartile **iii** the 3rd quartile

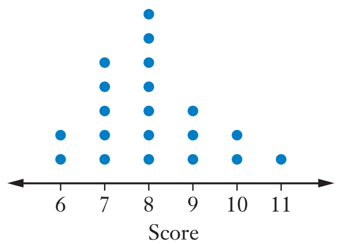
a



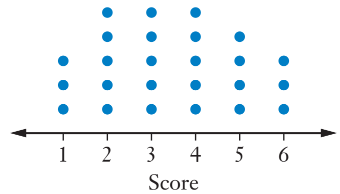
b

Score	Frequency
10	3
11	5
12	4
13	6
14	0
15	2

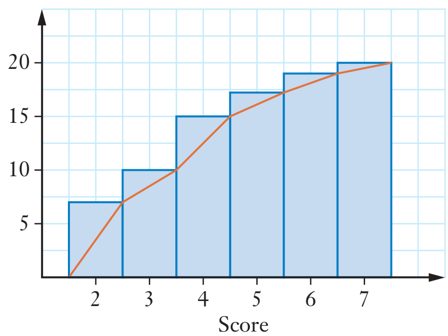
c



d



2 Find the 1st and 3rd quartiles for the following data



3 For the following data find

a the 23rd 55th and 91st percentiles

b the 2nd and 8th decile

Score	Frequency
23	13
24	19
25	23
26	21
27	9
28	15

4 For the dot plot

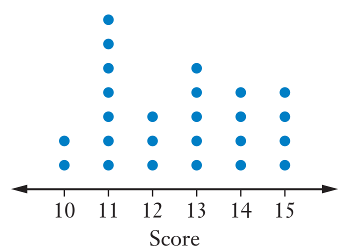
a draw a cumulative frequency polygon

b find

i the 1st quartile **ii** the 3rd quartile

iii the 35th percentile **v** the 7th decile

v the 1st decile



5 John measured the weights of children in a particular year at school and organised his findings in a table Estimate the weight that i:

a the median

b the 1st quartile

c the 3rd quartile

d the 60th percentile

Weight (kg)	Frequency
30–34	1
35–39	9
40–44	8
45–49	5
50–54	2

ISBN 97807043220

6 The number of different dress sizes in Huangs Sportwear shop was counted in a stocktake and the results set out in a tabl.

Size	Frequency
8	12
10	23
12	20
14	21
16	13
18	11

a How many dresses were counted?

b What percentage of dresses in the store were size 12?

c Find the median dress size

d Find the 3rd quartile

e What percentile is size 14?

7 Antonietta surveyed a number of people to find out how many pets they have Her results are show.

Number of pets	Frequency
0	7
1	11
2	3
3	2
4	1

a What percentage of the people surveyed had 2 pets?

b Draw a cumulative frequency polygon for this data

c From the graph fin:

i the median

ii the 1st quartile

iii the 3rd quartile

8 Abdul measured the reaction times of a group of drivers and placed his results in a table

Time (s)	Frequency
0.65–0.69	2
0.70–0.74	14
0.75–0.79	19
0.80–0.84	8
0.85–0.89	7

a What was the mean reaction time?

b What percentage of people surveyed reacted within 075 and 079 seconds ?

c Draw an ogive to show this data

d Use the graph to estimate

i the 30th percentile

ii the median reaction time

iii reaction times between the 1st and 3rd quartiles

9 For each data set fin:

i the median **ii** the lower quartile **iii** the upper quartile

a The number of dogs at a pound over several days

36, 79, 38, 29, 45, 83, 85, 47, 51, 72, 64

b The number of flying hours that Alexis had during a helicopter flying cours:

3, 4, 9, 8, 14, 17, 15, 11, 12

c The distance (in km) travelled by a taxi during several shifts

1283, 14.2, 13.7,99.5,137.5, 203.4 154., 1153, 19.3, 125.4

d The number of people attending a choir rehearsal over several weeks

15, 14, 12, 16, 15, 19, 17, 18

ISBN 97807043220

Range and inequtile ange

Inequtile ange

Inequtile ange

Saiical machup

Cumulaive equency gaph

7.05 Range and interquartile range

The **range** and **interquartile range** measure the spread of data

Range and interquartile range

$$\text{Range} = \text{highest score} - \text{lowest score}$$

$$\text{Interquartile range} = Q_3 - Q$$

EXAMPLE 19

Find the range and interquartile range of these scores 8 1,5,15, 20 2, 7, 16, 9

Solution

$$\begin{aligned}\text{Range} &= \text{highest score} - \text{lowest score}\\ &= 21 - 5\\ &= 16\end{aligned}$$

Put the 9 scores in order to find the quartiles

5, 8, 9, 13, 15, 16, 17, 20, 21

$Q_2 = 15$

5, 8, | 9, 13, 15, 16, 17, | 20, 21

$Q = \dfrac{8+9}{2} = .5 \qquad Q_3 = \dfrac{17+20}{2} = 185$

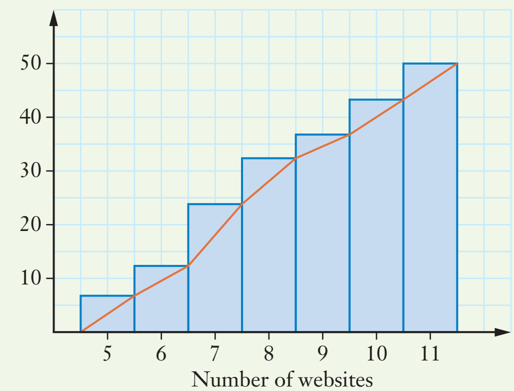
$$\begin{aligned}\text{Interquartile range} &= Q_3 - Q\\ &= 18.5 - 8.5\\ &= 10\end{aligned}$$

ISBN 97807043220

EXAMPLE 20

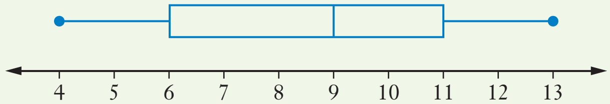
a This set of data shows the results of a survey into the number of travel websites people visit regularly. Fid:

i the range **ii** the interquartile range



b From this box plot fin:

i the median **ii** the range **iii** the interquartile range



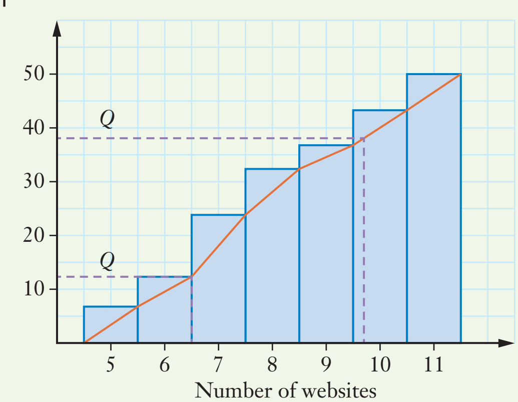
Solution

a **i** Range = 11 – 5 = 6 highest – lowest

ii There are 50 scores

$\frac{1}{4} \times 50 = 12.5$ so Q will be the 125th score

$\frac{3}{4} \times 50 = 375$ so Q_3 will be the 375th score



Q = 65 and Q_3 = 10

Interquartile range = $Q_3 - Q$

= 10 – 65

= 3.5

b **i** Median = 9 **ii** Range = 13 – 4 = 9 **iii** Interquartile range = 11 – 6 = 5

ISBN 97807043220

Oulie

Outliers

Outliers are extreme scores They affect the range because an outlier will be the highest or lowest score Howeve, outliers do not affect the interquartile range because the interquartile range does not depend on the highest or lowest scores

Some outliers are more obvious than others There is a formal definition of outlier that allows us to test if its an outlier rather than just deciding by inspectio.

Outlier

A score is an outlier if it is more than 15 times the interquartile range (IQR) below Q or above Q_3

An outlier is below $Q_1 - 15 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$

EXAMPLE 21

a For the scores 5, 2, 9, 10, 6, 7, 6, 5, 10, 9, 5, 7, 8, 7, 6, determine if 2 is an outlier.

b For this table of data find a score that looks like an outlier and use the definition to determine if it is an outlier.

Score	Frequency
1	1
2	0
3	0
4	0
5	3
6	5
7	8
8	11
9	13
10	9

Solution

a $Q = 5$ and $Q_3 = 9$

$$\begin{aligned}\text{IQR} &= Q_3 - Q \\ &= 9 - 5 \\ &= 4\end{aligned}$$

$$\begin{aligned}15 \times \text{IQR} &= 1.5 \times 4 \\ &= 6\end{aligned}$$

$$\begin{aligned}Q - 1.5 \times \text{IQR} &= 5 - 6 \\ &= -1\end{aligned}$$

$$\begin{aligned}Q + 15 \times \text{IQR} &= 9 + 6 \\ &= 15\end{aligned}$$

Any outlier would have to be less than -1 or greater than 15 So 2 is not an outlie.

ISBN 97807043220

b A score of 1 looks like an outlier.

From Example 16c

$Q\ \ = 7$ and $Q_3 = 9$

IQR $= Q_3 - Q$

$= 9 - 7$

$= 2$

$15\ \times$ IQR $= 1.5 \times 2$

$= 3$

$Q\ \ - 1.5 \times \text{IQR} = 7 - 3$

$= 4$

$Q_3 + 1.5 \times \text{IQR} = 9 + 3$

$= 12$

Any outlier would have to be less than 4 or greater than 12 So 1 is an outlie.

Exercise 7.05 Range and interquartile range

1 Find the range of each data set

a 7, 4, 9, 8, 11, 4, 3, 19, 7, 16

b 56, 89, 43, 99, 45, 28, 37, 78

c 103, 108, 99, 112, 126, 87, 101, 123

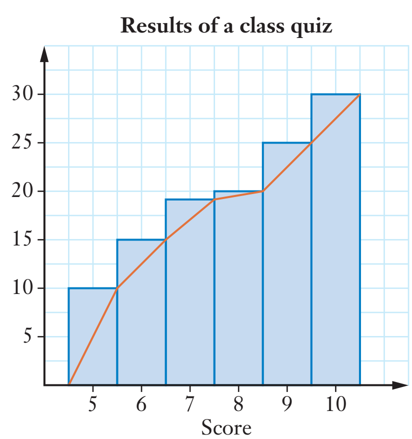
d

Score	Frequency
8	5
9	3
10	7
11	0
12	8
13	7

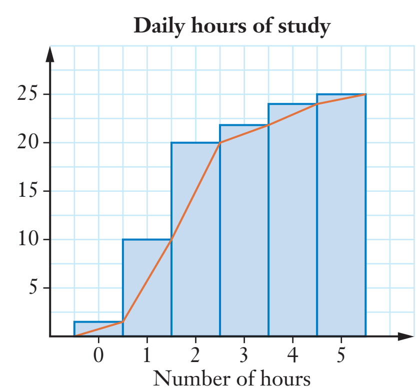
2 For each set of data fin:

i the median **ii** the range **iii** the interquartile range

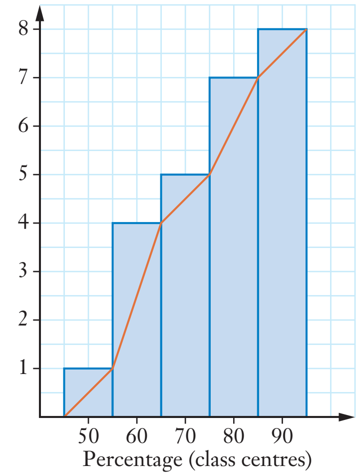
a



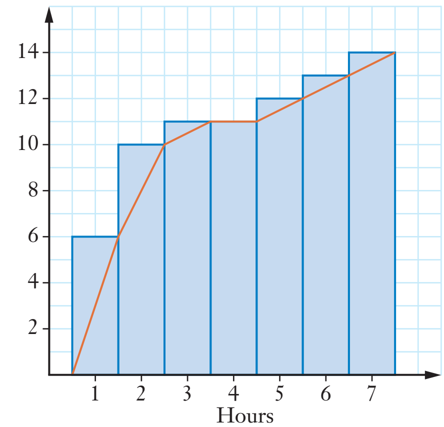
b



c Percentage participation in a basketball game



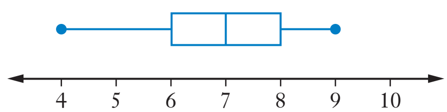
d Hours of study the day before an exam



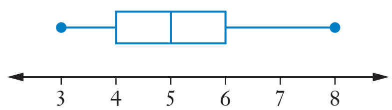
3 For each set of data fin:

i the median **ii** the range **iii** the interquartile range

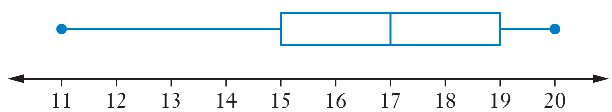
a



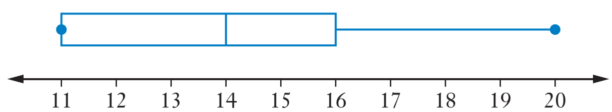
b



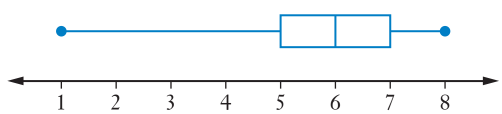
c



d



e



4 For this set of data fin:

a the mean
b the mode
c the median
d the range
e the interquartile range

Rain (mm)	Frequency
5	4
6	7
7	8
8	3
9	3

ISBN 97807043220

5 Find a potential outlier in each set of data and use the definition to see if it really is an outlier.

a 100, 7. 93, 54, 32, 66, 53, 97, 51, 80

b 11, 5, 7, 19, 5, 3, 7, 5, 6, 10, 11. 2, 5, 7, 4, 6, 1

c

Score	Frequency
1	1
2	0
3	0
4	0
5	5
6	4

7.06 Variance and standard deviation

Vriance and andad deviaion

Saiical calculaions

Saiical meaue puzzle

Variance is another measure of spread It measures how far the scores in a data set are from the mean o the data. You studied variance and standard deviation when studying discrete probability distributions in Year 1, Chapter 10 *Discrete probability distributions*

The formula for variance σ^2 is

$$\sigma^2 = \frac{\Sigma(x-\overline{x})^2}{n}$$

However, you do not have to use it as the calculatr's statistical mode can calculate it more easily. The following example will show you what the above formula mens, but youdon't have to learn it

EXAMPLE 22

The data below shows the times (in minutes) taken for a fire engine to reach the site of a fire Find the variance for this dat.

4, 7, 3, 9, 4, 5, 1, 3, 5, 9

Solution

First we need to find the mean

$$\overline{x} = \frac{\Sigma x}{n} = \frac{50}{10} = 5$$

Now we find the difference between each score and the mean Then we square each difference because we only want positive values This is shown in the table next pae.

ISBN 97807043220

x	$x-\bar{x}$	$(x-\bar{x})^2$
1	$1-5=-4$	16
3	$3-5=-2$	4
3	$3-5=-2$	4
4	$4-5=-1$	1
4	$4-5=-1$	1
5	$5-5=0$	0
5	$5-5=0$	0
7	$7-5=2$	4
9	$9-5=4$	16
9	$9-5=4$	16
		$\Sigma(x-\bar{x})^2=62$

Variance is the mean of these squared difference.

$$\sigma^2 = \frac{\Sigma(x-\bar{x})^2}{n}$$
$$= \frac{62}{10}$$
$$= 6\ 2$$

Standard deviation is another measure of spread and it is simply the square root of varianc.

For the data in the previous example the standard deviation is $\sqrt{6\ 2} \approx 2\ 49$

We use s for standard deviation of a sample and σ (the lowercase Greek sigma) for the standard deviation of a **population** In this course we will use s most of the time

The formula for standard deviation σ, i:

$$\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$$

This example shows how to calculate standard deviation and variance using the calculators statistical mode

EXAMPLE 23

The table shows the number of hours of karate practice that students at a karate club do each week

Find correct to one decimal place

a the standard deviation

b the variance

Practice times (h)	Frequency
2	2
3	4
4	3
5	4
6	1
7	2
8	4

ISBN 97807043220

Solution

a

Operation	Casio scientific	Sharp scientific
Clear the statistical memory.	SHIFT 1 Edit Del-A	2ndF DEL
Enter data	SHIFT 1 Data to get table 2 = 3 = etc to enter in x column 2 = 4 = etc to enter in FREQ column AC to leave table	2 2ndF STO 2 M+ 3 2ndF STO 4 M+ etc
Calculate standard deviation	SHIFT 1 VAR sx =	RCL sx

$s = 20774 \quad \approx 2.1$

b Variance $= s^2$

$= 20774 \quad ^2$

$= 4.3157 \ldots$

≈ 43

Exercise 7.06 Variance and standard deviation

1 For each set of data find

i the mean **ii** the standard deviation

a Number of minutes kept on hold on the telephone 7 ,9, 8 ,6, 2, 4, 5

b Travel time (in minutes) to get into the cit: 2, 5,67, 54 6, 8,59, 70, 59, 41

c Height of children (in cm) 101, 112, 131, 122, 130, 143, 152, 107, 112

d Number of repetitions on gym equipment 8 ,9, 5 ,7, 6 ,8, 9, 6, 3, 6

e Age of performers in a play 18, 19, 17, 16, 20, 18, 15, 19, 14, 20

2 For each data set fin:

i the standard deviation **ii** the variance

a Weights (in kg: 51, 67, 64, 53, 60, 48, 58, 49, 61, 71, 67, 58

b Class quiz results 4, 6, 5, 3, 7, 9, 8, 10, 4, 6, 7, 6, 5, 8, 6, 7, 9, 10, 5, 4, 8

c Time spent waiting in a queue (in mins: 11, 14, 15, 25, 31, 54, 36, 39, 31, 41, 44, 50

d Weight of crates (in kg: 8, 8,56, 91 6, 73, 55

e Response time (in mins) for helicopter rescue 1 ,7, 3 ,6, 5 ,4, 8, 9, 3

ISBN 97807043220

3 For each data set find

i the mean **ii** the standard deviation **iii** the variance

a Number of books rad:

Books	Frequency
5	3
6	5
7	6
8	2
9	1
10	3
11	5
12	4

b Piano practice time per week

Practice (h)	Frequency
1	3
2	0
3	2
4	5
5	7
6	3
7	2

c Weight of luggag:

Weight (kg)	Frequency
31	3
32	0
33	2
34	5
35	7
36	3

d Results of a half-yearly exam

Score	Frequency
10–19	1
20–29	4
30–39	8
40–49	12
50–59	15
60–69	11
70–79	7

4 In a taste test the people surveyed had to rank a new biscuit on a scale from 1 to 5 The table shows the results of the survy.

a What is the range?

b Find the interquartile range

c Find the standard deviation

Rank	Frequency
1	5
2	11
3	18
4	21
5	9

5 A Year 12 Art class received these results for their major wrk.

a Calculate

i the mean **ii** the standard deviation

b Remove the outlier and calculate

i the mean **ii** the standard deviation

Class	Frequency
20–29	1
30–39	0
40–49	2
50–59	4
60–69	7
70–79	9
80–89	8
90–99	7

6 The data shows the ages of students in an MBA course

20, 25, 31, 34, 17, 27, 29, 53, 20, 31, 19, 23, 30, 29, 18, 25

a Show that there is an outlier.

b Calculate the standard deviation

i with the outlier **ii** without the outlier

ISBN 97807043220

7 Jane recorded the number of crocodile sightings each week in a region of the Northern Territory over several week.

5, 8, 9, 4, 11, 7, 9, 15, 17, 10, 8, 5, 9, 12

a What was the mean number of crocodiles sighted per week?

b Find the standard deviation

c Calculate the variance

7.07 Shape and modality of data sets

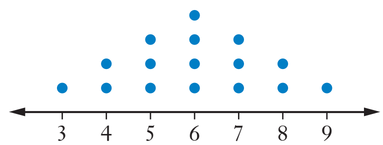
Shape o a diibuion

The shape o a equency diibuion

Shape o diibuion

The measures of central tendency and spread are called **summary statistics** and they help us make decisions about data Other features of data can help with these decision, and one of these features is the **shape** of the data The shape often gives us an idea of the centre and spread even before we measure them while the **modality** describes the number of peaks in the distribution of data

A data set where the mean mode and median are equal has a **symmetrical distribution**

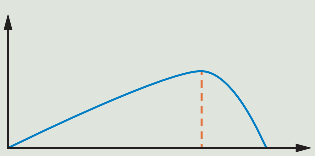


Notice how symmetrical this dot plot is

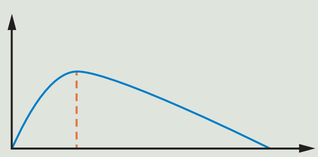
Other graphs that are not so symmetrical can be described by their **skewness**

The shape of a statistical distribution

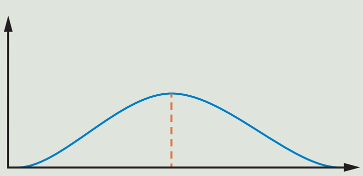
This distribution is **negatively skewed** as most of the area is to the left (or negative direction) of the centre We can say that the 'tai' points to the low scores in the negative direction



This distribution is **positively skewed** as most of the area is to the right (or positive direction) of the centre We can say that the 'tai' points to the high scores in the positive direction



This distribution is **symmetrical**

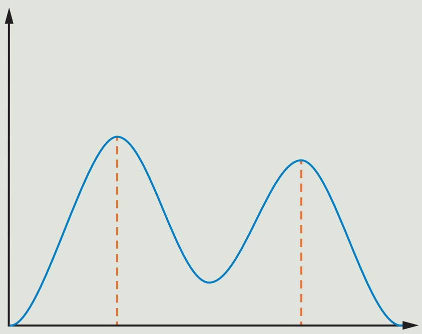


All these graphs are also called **unimodal** since they only have one peak.

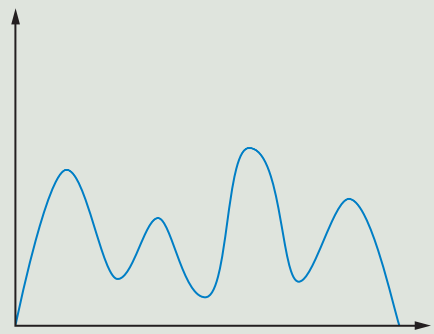
ISBN 97807043220

The modality of a statistical distribution

This graph is described as **bimodal** because it has 2 peaks



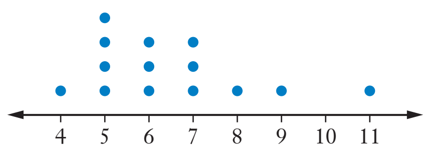
This graph is described as **multimodal** because it has many peaks



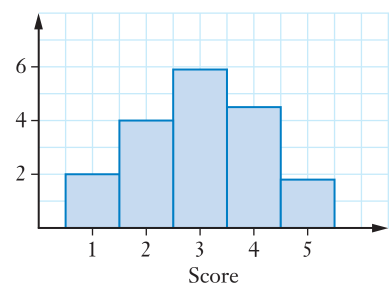
Exercise 7.07 Shape and modality of data sets

1 Describe the shape and modality of each graph

a



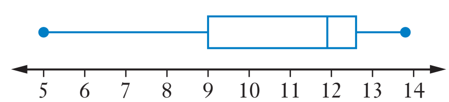
b



c

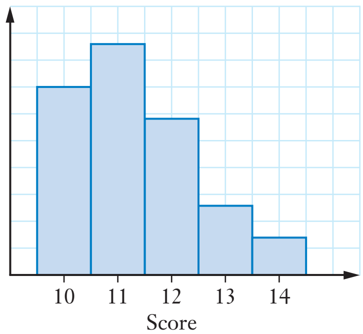
Stem	Leaf
1	3 4 4 5 5 7 9
2	0 1 1 7 8 8 9 9
3	2 2 5 6 6 8 9 9
4	1 1 3 5
5	0 0 3
6	3 4 5 8
7	8 9

d

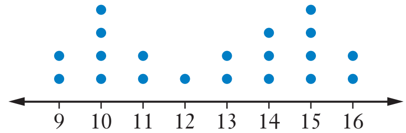


ISBN 97807043220

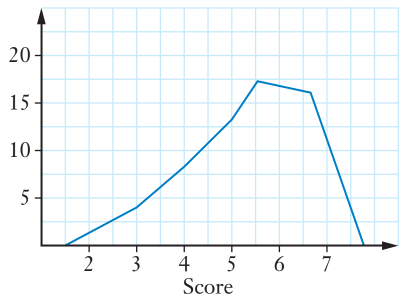
e



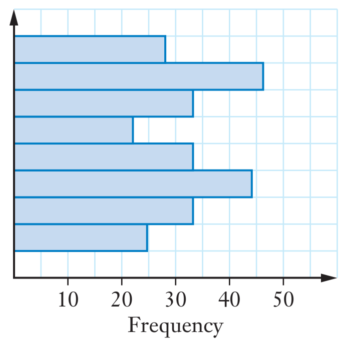
f



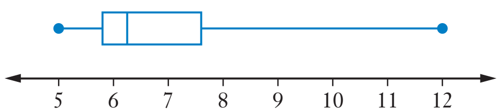
g



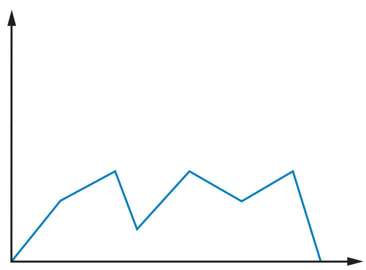
h



i



j



2 a Draw a dot plot for this data

5, 9, 4, 8, 9, 10, 7, 5, 3, 9, 7, 8, 6, 12, 8, 9

b Describe the shape of the dot plot

3 Describe the shape and modality of each data set

a

Score	Frequency
1	7
2	9
3	5
4	3
5	1
6	1

b

Score	Frequency
12	3
13	7
14	11
15	14
16	9
17	2

c

Score	Frequency
10–14	3
15–19	6
20–24	7
25–29	4
30–34	7
35–39	5

ISBN 97807043220

d

Score	Frequency
6	1
7	2
8	5
9	8
10	7
11	4

e

Score	Frequency
50– 59	2
60– 69	4
70– 79	6
80– 89	3
90– 99	1

4 Draw graphs with the following shapes

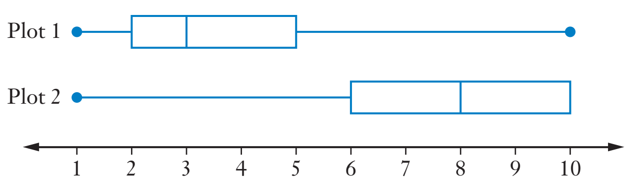
a bimodal **b** skewed negatively

c symmetrical **d** skewed positively

5 Describe the shape and modality of each set of data in the back-to-back stem-and-leaf plot

Set 1		Set 2
4	1	3 3 4
7 7 6 3	2	1 2 5 6 8 9 9
8 7 5 5 2 0	3	0 2 2 5 6
8 6 6	4	3 6 7
9 8 8 7 1 1	5	0 1
5 4 0	6	3
1 1	7	

6 **a** Describe the shape of the distributions summarised by the parallel box plots



b What is the difference between their medians?

c Find the difference in their interquartile ranges

7 The heights of a number of students were measured and the results are below.

159, 175, 181, 153, 177, 168, 175, 163, 155, 184, 167, 179, 157, 149, 160, 171, 180, 160, 162, 169, 163, 179, 145, 187, 161, 148, 182, 151, 150, 178

a Draw a frequency distribution table for the heights using groups of 145–14, 150–154 155–159 and so o.

b Describe the type of distribution for this data

8 Draw a box plot that describes a symmetrical distribution

9 This table shows the results of an assessment task

a Find

i the mean

ii the median

iii the mode

b Describe the shape of the distribution

Score	Frequency
4	3
5	5
6	6
7	9
8	6
9	5
10	3

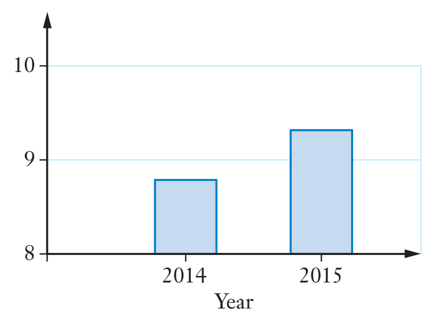
10 Choose a random sample of about 50 people and collect data on the number of siblings (brothers and sisters) each one has Graph the data and describe the shape and modality of the graph

INVESTIGATION

MISLEADING GRAPHS

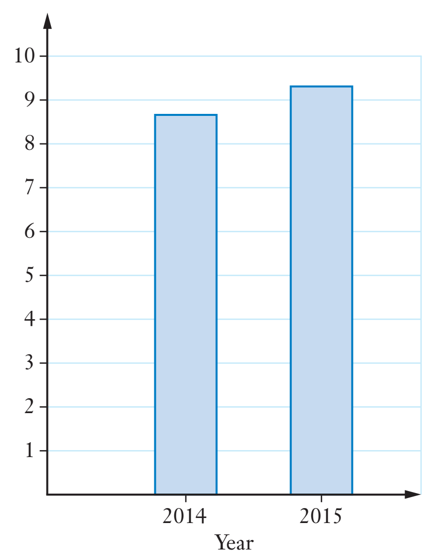
Sydneys median house price increased from $886 408 in 2014 to $929 842 in 201.

The column graph shows this information Looking at the graph you would think that this was a huge price rise because the second column is almost twice as tall as the first column



Now look at this graph What is the difference between the 2 graphs? Which one do you think shows the information better? Is one of the graphs misleading? Why?

Search online for other misleading graphs Collect them into a portfolio and share with the class Write an account of why each one is misleading and how you could change it to give a better reading of the information



Statistics can be misleading in different ways In the investigation this was caused by the scale on the graph Sometimes the measures of central tendency or spread can be misleading as well

Double box plo

WS Compaing ciy empeaue

WS Compaing wod lengh

WS Compaing posscrs

7.08 Analysing data sets

EXAMPLE 24

The table shows the heights of students in a Year 12 clas.

a Find the mean and standard deviation of the heights

b Are the mean and standard deviation misleading for this data? Why?

Height (cm)	Class centre	Frequency
150–154	152	3
155–159	157	18
160–164	162	27
165–169	167	31
170–174	172	12
175–179	177	15
180–184	182	25
185–189	187	11
190–194	192	3

Solution

a Using a calculator

$\overline{x} = 1706$

$s = 102$

b Looking at the table the data looks to be bimoda. This might be because the survey is for both males and females

If this is the case the mean of 17.6 is misleading because it does’t tell us about differences in male and female heights The spread may be less than 0.2 if we split the data into male and female data

ISBN 97807043220

Comparing two or more sets of data

Sometimes we need to compare different data sets to see how similar or different they are

EXAMPLE 25

a Two surveys were made into the number of people attending an outdoor cinem: one in 2019 and one in 2020

For the 2019 survey, the mean was 112 and the standard deviation was67.

For the 2020 survey, the mean was 95 and the standard deviation was 19.

Describe how these results differ.

b This back-to-back stem-and-leaf plot shows the results of tests of the life of 2 brands of batteries batteries (measured in hours)

Buzz		Eternity
9 9 1	4	5
5 2 1 1 1 0	5	1 3 4 9
7 7 4 0	6	1 2 3
4 2 0	7	0 1 2 8
4 0	8	1 2 5 5 7
	9	0

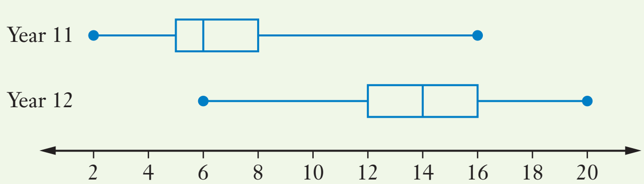
i Describe the shape of the distribution for each brand

ii Find the mean result for each brand

iii Find the standard deviation for each brand

v Compare the results for the 2 brands of batteries

c The parallel box plots below show the results of 2 surveys into the number of hours 2 groups of students study each week



i What is the median number of hours studied for each Year group?

ii Calculate the range for each group

iii What is the interquartile range for each group?

v What is the highest number of hours studied in each group surveyed?

v What is the main difference between the 2 groups?

ISBN 97807043220

Solution

a The mean was lower in the second survey, so it looks as f, on averge, fewer people were going to the movies in 2020 than in 2019

The standard deviation was higher in the second survey, so there was a greater variation in the number of people going to the movies

b **i** Buzz is slightly positively skewed and Eternity is approximately bimodal

ii Using a calculator

The mean for Buzz is 604

The mean for Eternity is 694

iii The standard deviation for Buzz is 123 and the standard deviation for Eternity is 140

v The mean was higher on Eternity so these batteries had longer lives overall

The standard deviation was slightly higher on Eternity, showing slightly more variability in the life of these batteries That s, the battery lives were more spread out than for Buzz but there was't a big difference between the 2 brans.

c **i** Year 1: median is .

Year 1: median is 1.

ii Year 1: range $= 16 - 2 = 14$

Year 1: range $= 20 - 6 = 14$

iii Year 1: interquartile range $= 8 - 5 = 3$

Year 1: interquartile range $= 16 - 12 = 4$

v Year 1: highest hours $= 16$

Year 1: highest hours $= 20$

v Year 12 students generally study for more hour.

CLASS DISCUSSION

ANALYSING DATA SETS

Why do you think the surveys in the example give different results? Are they taken from the same population? How could you tell? What other information could help you decide?

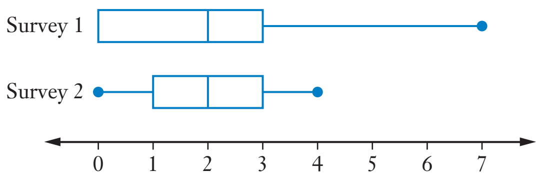
Find other examples online in newspapers or in magazines that compare 2 or more sets of data Is the information taken from the same or different populations ? Can you tell?

Put these examples in a portfolio and present a report to the class

ISBN 97807043220

Exercise 7.08 Analysing data sets

1 The parallel box plots show the results of 2 surveys into the number of children in families



a What was the largest number of children in a family in

i survey 1? **ii** survey 2?

b What is the median number of children in the families in

i survey 1? **ii** survey 2?

c Find the interquartile range for

i survey 1 **ii** survey 2

d Do you think the surveys were taken from the same population or from different populations?

2 The heights (in centimetres) of people were taken from 2 samples

Sample 1 158, 169, 177, 147, 160, 153, 167, 164, 171, 159, 162, 166, 172, 148, 157, 176, 155, 170, 149, 150

Sample 2 181, 166, 179, 181, 164, 173, 168, 165, 176, 181, 157, 162, 178, 182, 157, 172, 180, 173, 159, 151

a Plot this information in a back-to-back stem-and-leaf plot

b Find the median of each sample

c Find the range of each sample

d Do you think the samples were taken from the same population?

3 Surveys were taken at 3 different banks to measure the length of time customers waited before being served The results are shown in the tabe.

Time (min)	Bank 1	Bank 2	Bank 3
0–2	29	59	2
3–5	38	26	8
6–8	15	12	11
9–11	9	3	21
12–14	5	0	28
15–17	3	0	20
18–20	1	0	10

a Find **i** the mean and **ii** the median waiting times for customers at each bank

b Find the standard deviation for each bank

c Do you think there is a significant difference in waiting times at the banks?

ISBN 97807043220

4 Mrs Spells piano students earned the following marks in their piano examination:

Class 1: 91, 86, 74, 92, 85, 89, 63, 71, 80, 91, 85, 72, 54, 78

Class 2: 97, 87, 69, 91, 88, 89, 93, 94, 71, 79, 84, 85, 88

a Sketch parallel box plots showing this information

b For each class fin:

i the median | **ii** the interquartile range

iii the mean | **v** the range

5 Two speed cameras at different locations recorded speeds (in km h^{-1}) of vehicles travelling over the speed limit

Camera : 85, 66, 75, 69, 72, 83, 80, 69, 74, 77, 73, 74, 90, 84, 65, 73, 69, 89, 76, 103

Camera : 122, 142, 120, 118, 116, 135, 140, 123, 135, 124, 120, 119, 138, 131, 122, 119, 125, 130, 130, 113

a Draw a back-to-back stem-and-leaf plot to show this data

b Find the mean speeds recorded by each camera

c What do you think was the speed limit at the site where each camera was placed?

6 Jon sat for the HSC in one year and scored 56 4, 1,53, 41 and 35 for his maths assessments He resat his HSC the next year and his maths assessment scores were 7, 58, 67, 74, 59 and68.

a By how much did his mean scores increase the second year?

b What was the difference in the median scores?

c Calculate the difference in the range of scores for each year.

d By how much does the standard deviation differ over the 2 years?

7 These 2 sets of scores have the same median

Test : 1, 2, 4, 6, 7, 9, 10 **Test :** 4, 5, 5, 6, 6, 7, 8

a What is the mean of

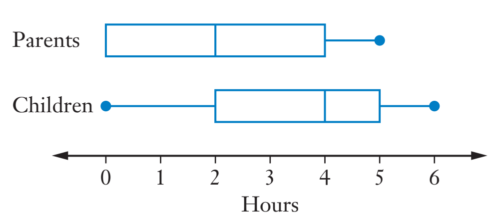
i Test 1? | **ii** Test 2?

b Calculate the standard deviation of

i Test 1 | **ii** Test 2

c Describe how the 2 sets of scores differ.

8 The parallel box plots show the results of 2 surveys into the number of hours people spend watching TV each da.



a What is the median for each group?

b For each group fin:

i the interquartile range | **ii** the range

c What is the highest number of hours of TV watched?

ISBN 97807043220

9 For the back-to-back stem-and-leaf plot fin:

a the median of each test

b the mean of each test

c the standard deviation of each test

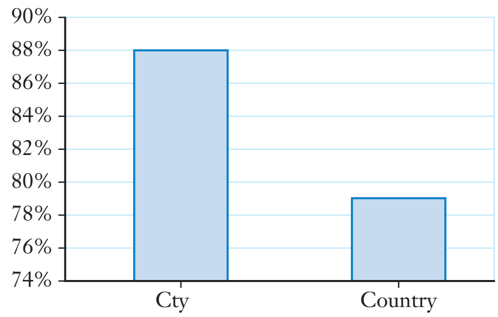
d the difference in range between the 2 tests

Science		English
	4	3
2 1 1 0	5	8 9
8 6 5 3 3 1	6	0 1 2 2 5
9 9 6 3 2	7	4 6
1 0 0	8	1 2 3
4 2	9	7

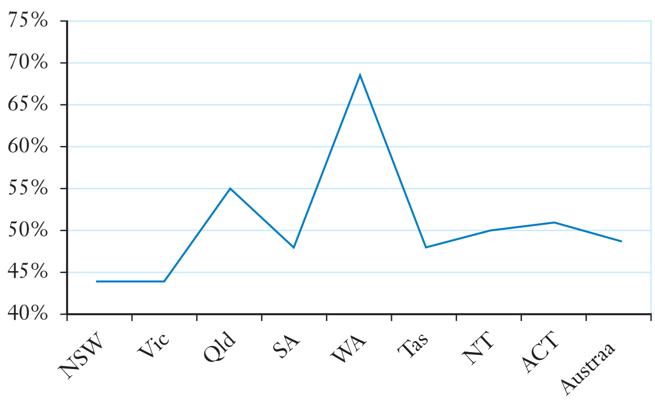
10 The graph compares access to the Internet for city and country households in 2015

a Describe how this graph is misleading

b Redraw the graph so it is not misleading



11 The graph shows the average annual growth in incomes from 2006 to 2011



Redraw the graph so it is not misleading

INVESTIGATION

COMPARING SURVEY RESULTS

Conduct a survey among your friends Make up your own topi, such as what sports they play, subjects they study or their heighs. Alternatiely, you could carry out an experiment such as counting numbers of people travelling in cars or measuring the time taken for the same journey to school on different days

To check your result, take another sample and do the same survey or experimet. Are the new results the same? Can you explain why?

ISBN 97807043220

7. TEST YOURSELF

Pacice quiz

Saiic eview

Saiic owod

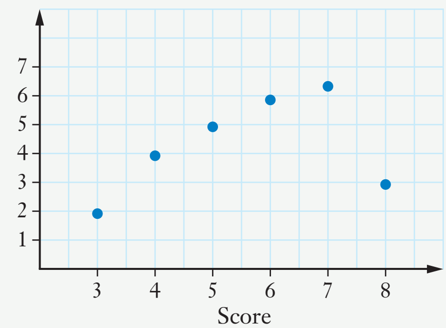
For Questions **1** to **6** select the correct answer **A B C** or **D**

1 What type of data is a persons body temperature ?

A categorical ordinal **B** numerical continuous
C numerical discrete **D** categorical nominal

2 This frequency distribution is

A symmetrical
B positively skewed
C negatively skewed
D multimodal



3 What type of data is a flowers colour ?

A categorical ordinal **B** numerical continuous
C numerical discrete **D** categorical nominal

4 The median of scores 7, 3, 2, 6, 4, 3, 8 9 is:

A 6 **B** 4 **C** 3 **D** 5

5 Which measure is most affected by an outlier?

A median **B** interquartile range
C mode and interquartile range **D** mean and range

6 Test 1 has a mean of 4 and standard deviation of .. Test 2 has a mean of 4 and a standard deviation of 25 Which statement below is true when comparing the centres and spreads for both tests?

A The centres are the same and test 2 has a larger spread than test 1
B The spreads are the same and test 2 has a higher centre than test 1
C The spreads are the same and test 2 has a lower centre than test 1
D The centres are the same and test 2 has a smaller spread than test 1

7 Find the mode median and range of this data se:

8, 6, 8, 4, 5, 6, 8, 5, 7, 4, 7, 8, 6, 8, 9

ISBN 97807043220

8 The table shows the results of a survey to find the distance people must travel to work

Distance (km)	Frequency
0–4	6
5–9	8
10–14	4
15–19	7
20–24	3

a Find the modal class

b Find the mean

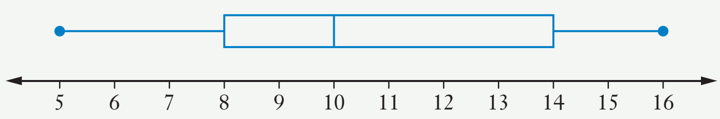
c What is the median?

d Find the standard deviation and variance

9 Find the mean and standard deviation of this data set

16, 34, 29, 80, 65, 77, 91, 58, 67, 40

10 From this box plot fin:



a the median

b the range

c the 3rd quartile

d the interquartile range

11 The table shows the results of a survey into the number of years that people keep a car before selling it

Years	Frequency
0–2	15
3–5	37
6–8	13
9–11	18
12–14	17

a Draw a cumulative frequency polygon to show this data

b Estimate the median age of the cars

c Estimate

i the 20th percentile **ii** the 3rd quartile

iii the 91st percentile **v** the 3rd decile **v** the 9th decile

12 The table records the number of times people visited a doctor in the past 6 months

Visits	Frequency
0	9
1	8
2	5
3	1
4	3
5	2

a Find the mean number of visits

b Find the median number of visits

c What is the range?

d Find the interquartile range

13 Most families have 2 children' Is this statement about a mea, median or mode ?

14 Nikola conducted a survey to find the number of people in each car that travelled across the Anzac Bridge one mornin. These were her resuts:

Occupants	Frequency
1	43
2	32
3	12
4	8
5	5

a Draw a cumulative frequency polygon to illustrate the results

b From the polygon fin:

i the median number of people in a car

ii the interquartile range

ISBN 97807043220

15 The table shows the results of Mr Cheungs history clas.

Score	Frequency
6	7
7	6
8	3
9	6
10	2

a Find the mean score

b Find the standard deviation

c What is the mode?

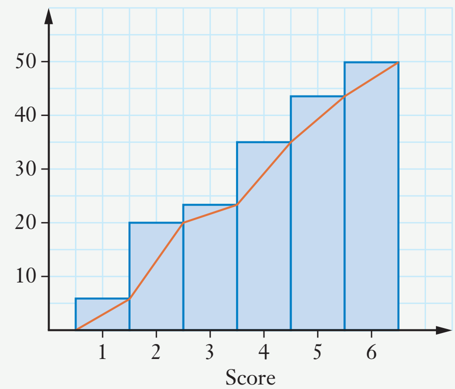
16 Find the mean mode and median of this data se:

45, 49, 49, 48, 43, 45, 41, 40, 49, 48, 44, 40, 42

17 From the graph fin:

a the interquartile range

b the median



18 A class test gave the following scores

15, 19, 12, 2, 19, 16, 13, 18, 11, 15, 17, 11, 18, 14, 14, 16, 18, 14, 12

a Find

i the range **ii** the mean

iii the mode **v** the median

b Show that one score is an outlier. Which is it?

c Find without this outlie:

i the range **ii** the mean

iii the mode **v** the median

d Does the outlier have much effect on all these measures?

19 The two-way table shows the results of a survey into the number of pets microchipped at a veterinary surgery.

	Male	Female	Total
Cats	369	473	842
Dogs	578	664	1242
Total	947	1137	2084

a What percentage of microchipped animals were

i female? **ii** female cats? **iii** male dogs? **v** dogs?

b Draw this information in a Pareto chart

ISBN 97807043220

20 The table shows the reasons employees gave for leaving their jobs in a large organisation Draw a Pareto chart of this dat.

Work too difficult	185
Boring work	139
Not paid enough	104
Not getting on with co-workers	56
Unsuitable hours	116

21 A back-to-back stem-and-leaf plot shows the number of mushrooms found in 2 regions of a forest in New Zealand

North		West
	2	2 5 6
5 4 2	3	4 8 9
8 7 6 3 3	4	0 0 1 3 7
6 5 4 4 2 2	5	1 1 2 3 4 4 7 8

a Calculate the median number of mushrooms recorded in each region

b Find the mean and standard deviation of North region

c Find the mean and standard deviation of West regin.

d Compare and contrast the 2 regions

22 Below are the results of 2 English assessments

Term : 8, 7, 9, 8, 6, 5, 8, 7, 7, 5, 9, 9

Term : 5, 7, 8, 4, 6, 6, 5, 5, 5, 4, 7

a Draw a box plot for each set of data

b What is the median of each assessment?

c Find the interquartile range of each set

d Find the mean and standard deviation for each assessment

e Compare and contrast the 2 assessments

ISBN 97807043220

7. CHALLENGE EXERCISE

1 The table shows the heights of students in Year 2.

Height (cm)	Frequency
150–154	7
155–159	5
160–164	15
165–169	9
170–174	8
175–179	15
180–184	6
185–189	2

a Describe the modality of the distribution
Can you explain this?

b Find the mean height and the variance

2 The mean of a data set is 45 The table containing the data has a frequency missing What is it ?

Score	Frequency
2	2
3	5
4	
5	3
6	2
7	4

3 **a** Sketch a box plot for the following scores
7, 9, 5, 6, 8, 11, 25, 14, 16

b What does this plot show? What is its shape?
Can you make a more symmetrical box plot by taking out one of the scores?

4 For the scores 7, 11, 15, 19, 23

a Find

i the mean **ii** the standard deviation

b Copy and complete this table

Score x	$x - \bar{x}$	$(x - \bar{x})^2$
7		
11		
15		
19		
23		
	$\Sigma(x - \bar{x}) =$	$\Sigma(x - \bar{x})^2 =$

c Find the standard deviation by using the formula $\sigma = \sqrt{\dfrac{\Sigma(x - \bar{x})^2}{n}}$

ISBN 97807043220

5 The mean of 7 scores is 25 If an extra score of 28 is also include, what will be the new mean?

6 Compare and contrast each pair of data sets

a Set A has a mean of 54 and a standard deviation of .6

Set B has a mean of 76 and a standard deviation of 21

b Set A has a mean of 1.6 and a standard deviation of .7

Set B has a mean of 213 and a standard deviation of 92

7 A Year 12 class received the following scores out of 10 on their maths quiz

a Find the mean and standard deviation of the scores

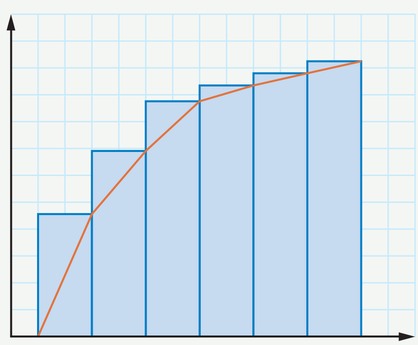
b Draw a box plot for this set of scores

c Describe the shape of the distribution

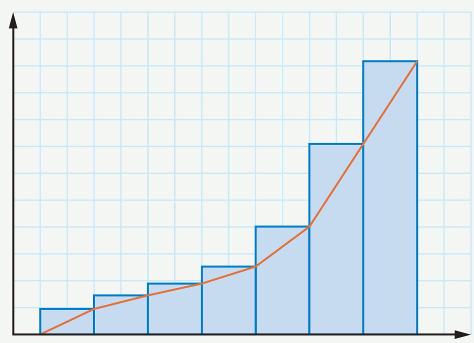
Score	Frequency
4	7
5	11
6	15
7	9
8	4
9	2
10	1

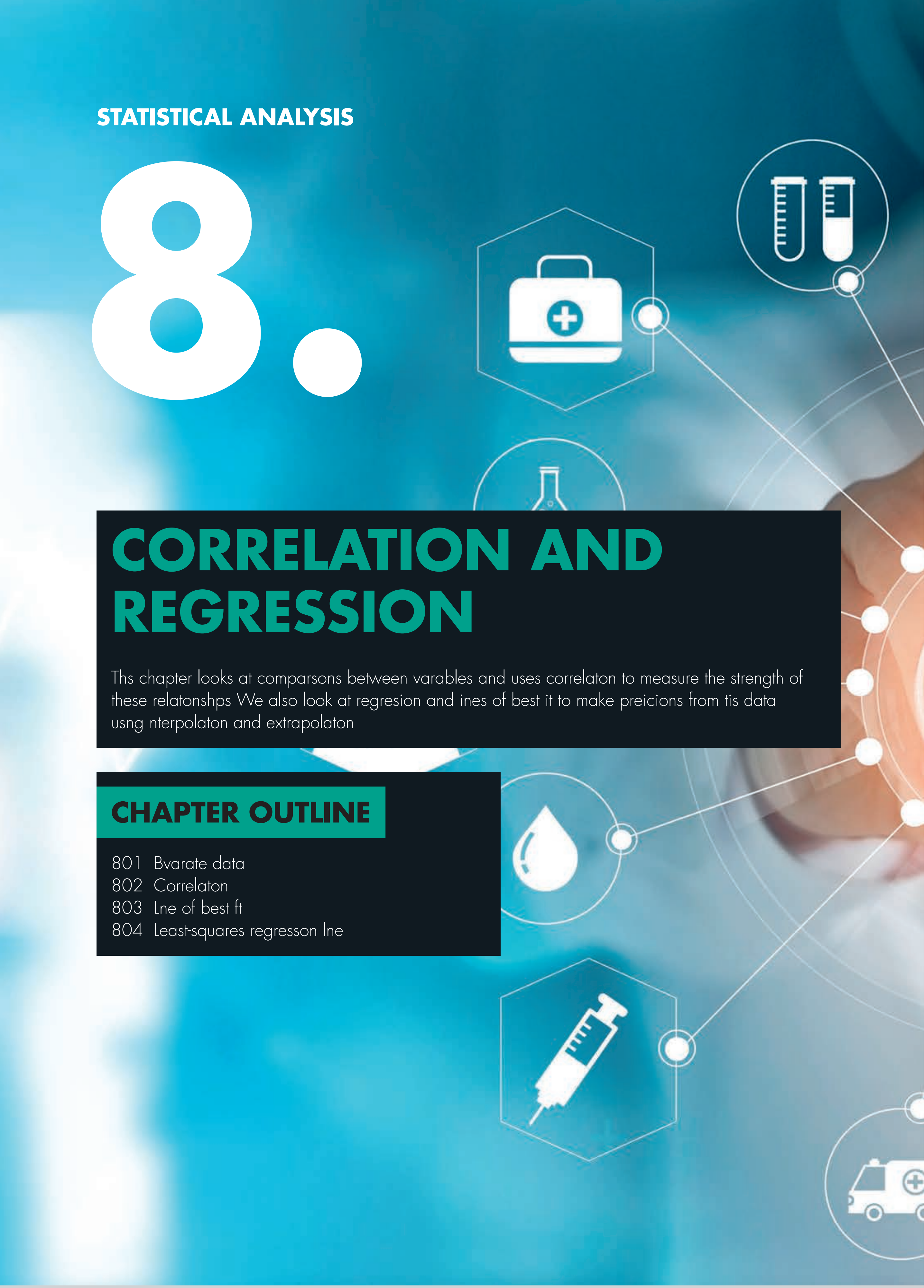
8 Describe the shape of each distribution given the ogiv.

a



b





STATISTICAL ANALYSIS

8.

CORRELATION AND REGRESSION

Ths chapter looks at comparsons between varables and uses correlaton to measure the strength of these relatonshps We also look at regresion and ines of best it to make preicions from tis data usng nterpolaton and extrapolaton

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- nterpret scatterplots of bvarate data
- look for correlaton n bvarate data and calculate Pearsons correlaion coeffcient
- apply lnes of best ft ncludng the least-squares regresson lne
- nterpolate and extrapolate from data

TERMINOLOGY

bivariate daa: Data relating to 2 variables that have been measured from the same data set

extrapolatin: Making predictions from a model using values outside the range of the original data set

interpolation Making predictions from a model using values lying within the range of the original data set

least-squares regression line A line of best fit where the squares of the distances from each point in the scatterplot to the line are minimised

line of best fit A line drawn through a scatterplot that best models the relationship between 2 variables in bivariate data

Pearsn's correlation coefficent: A calculated value r that measures how closely 2 variables are related in a linear relationship Its value is always between −1 and 1

scatterplot A graph showing the value of 2 variables in a bivariate data set

8.01 Bivariate data

WS Scaeplo

WS A page o caeplo

WS Body meauemen

In Chapter 7 *Statistics* we looked at data with one variabl.

Bivariate data measures 2 variables on the same data set to see if they correlate with (are related to) each other. For exampe, we might want to see if a peron's level of education correlates with the amount of money the person earns

We draw **scatterplots** to graph bivariate data

EXAMPLE 1

This table shows the bivariate data for the number of days each year the temperature in a city was over 30°C and the number of people admitted to the local hospital for heat stroke

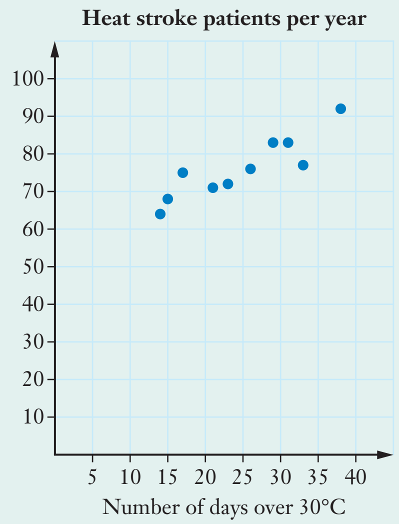
Year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Number of days over 30°C	14	17	15	23	21	26	31	33	29	38
Number of patients	64	75	68	72	71	76	83	77	83	92

Draw a scatterplot for this data

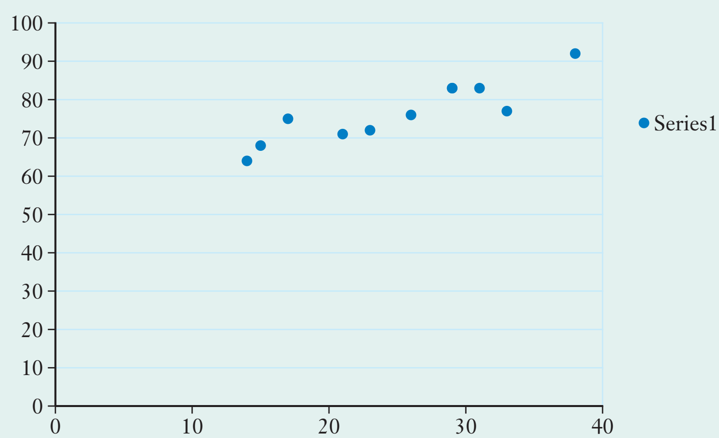
ISBN 97807043220

Solution

We draw the graph with number of days over 30°C on the horizontal axis and number of patients on the vertical axis



You could also put the data from the table into a spreadsheet and choose the scatterplot chart

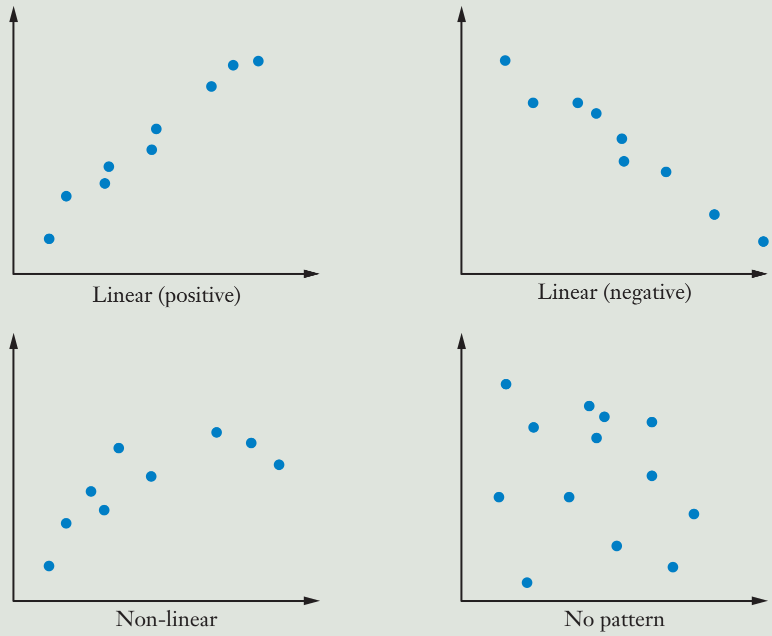


We look for patterns in the scatterplot to see if the 2 variables are relate. For instane, in the example above there seems to be a linear pattern (it is roughly a straight-line graph)

Not all relationships are linear. We can have scatterplots that have curves or non-linear shapes or no shape at al.

ISBN 97807043220

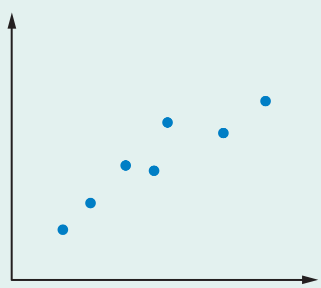
Shape of scatterplots



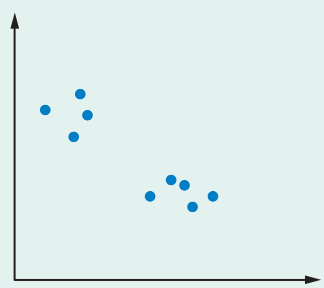
EXAMPLE 2

Describe the shape of each scatterplot

a



b



Solution

a This data is linear (positive)

b This data is linear (negative)

ISBN 97807043220

Exercise 8.01 Bivariate data

1 **Biometric data** is bivariate data drawn from body measurements such as arm length or height This table shows the heights and weights of 10 people surveyd.

Height (m)	161.8	175.3	159.5	182.3	166.4	167.9	186.4	164.7	154.8	171.2
Weight (kg)	59.4	73.7	55.3	74.8	63.5	68.2	73.5	62.1	49.9	83.6

a Draw a scatterplot to show this data
b Describe the shape of the scatterplot
c Describe the relationship between height and weight (if possible)

2 A group of 8 people were surveyed at random about how many people in their family lived at home They were also asked about the number of bedrooms in their hoe.

Number of people in family	4	3	6	7	2	1	5	8
Number of bedrooms	3	2	5	4	2	3	4	6

a Draw a scatterplot showing the data
b Describe the shape of the scatterplot
c Describe the relationship (if any) between the number of people in the family and the number of bedrooms in the home

3 A survey of 10 people measured their IQ with the amount they earned each week

IQ	114	127	95	130	123	141	136	83	109	96
Amount earned per week ($)	689	945	510	874	751	769	553	350	1250	884

a Draw a scatterplot for this data
b Describe the shape of the scatterplot
c Describe any relationship between IQ and amount earned

4 The table shows the amount of sleep students have and their exam results

Hours of sleep	6.5	9	7.5	6	7	10	11.5	4	8	8.5
Exam results (%)	87	76	43	87	69	55	60	78	94	72

a Draw a scatterplot of this data
b Describe the shape of the scatterplot
c What is the relationship between the amount of sleep and exam results?

ISBN 97807043220

5 The table shows the amount of study time and exam results for some students

Hours of study/week	15	26	12	17	5	10	2	8	20	3
Exam results (%)	78	86	70	80	63	67	43	77	92	58

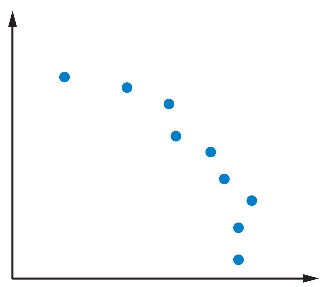
a Draw a scatterplot of the data

b Describe the form of the scatterplot

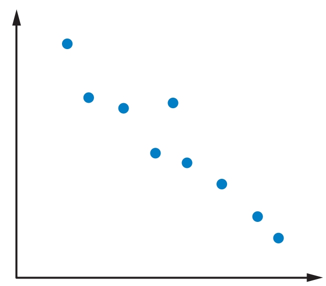
c What is the relationship between the amount of study and the exam results?

6 Describe the pattern of each set of bivariate data

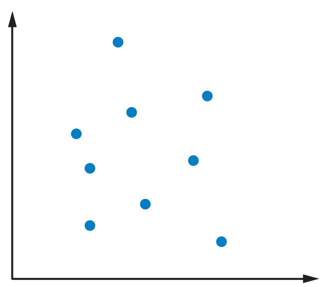
a



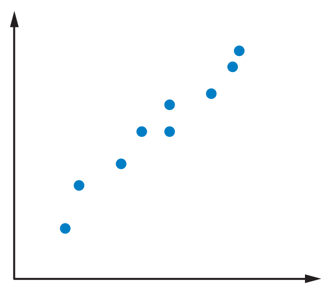
b



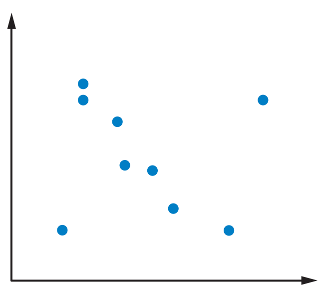
c



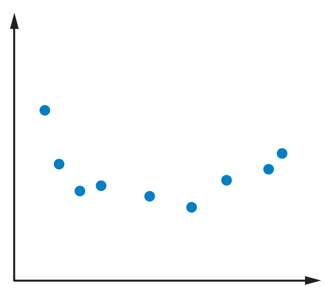
d



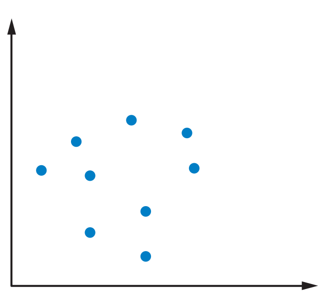
e



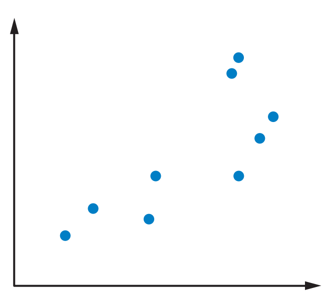
f



g

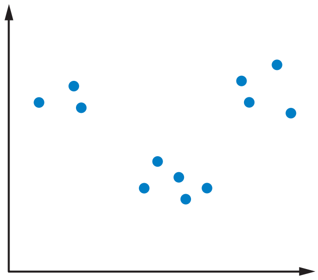


h

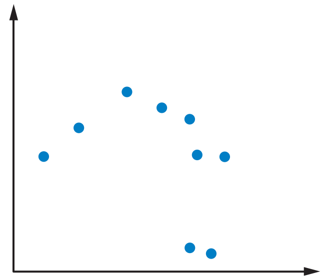


ISBN 97807043220

i



j



7 **a** Draw a scatterplot for this set of bivariate data

x	8	3	7	10	2	9	4	5	9	1
y	2	7	1	2	8	4	5	7	3	7

b Describe the shape of the data

8 State whether each scatterplot has

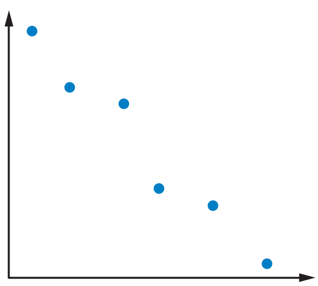
A a positive linear relationship

B a negative linear relationship

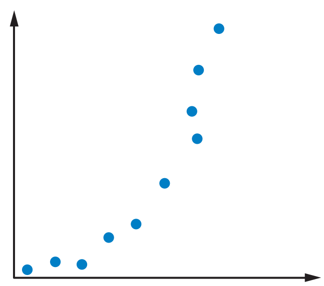
C a non-linear relationship

D little or no relationship

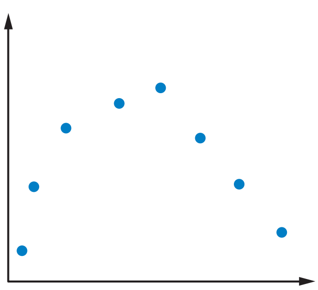
a



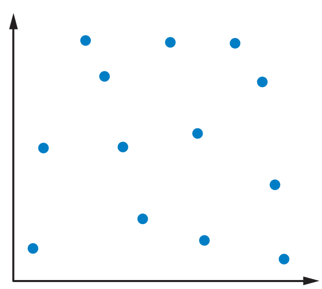
b



c

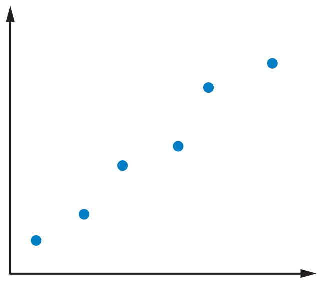


d

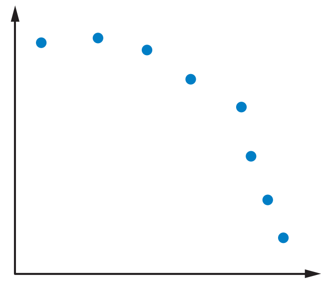


ISBN 97807043220

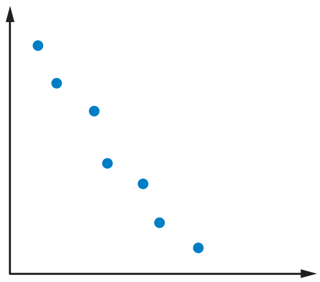
e



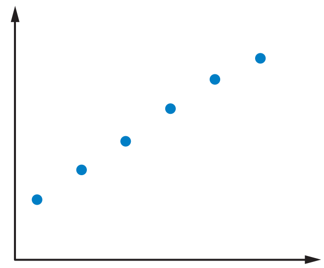
f



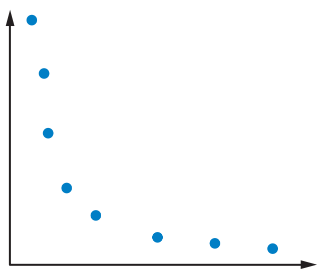
g



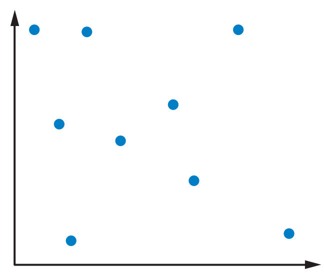
h



i



j



Coelaion

Coelaion maching game

Coelaion

8.02 Correlation

Correlation measures how well 2 variables are related if there seems to be a linear relationship between them The relationship could be strog, moderate or wak.

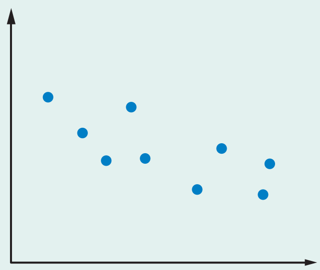
Linear scatterplots

ISBN 97807043220

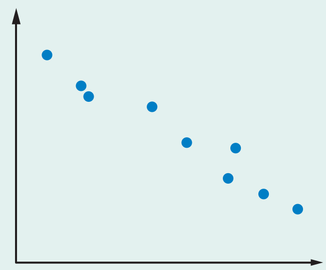
EXAMPLE 3

Describe the strength of the linear pattern in each scatterplot as strong moderate or wea.

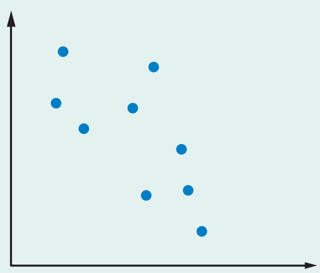
a



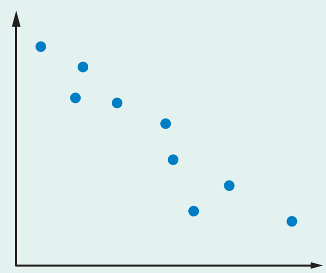
b



c



d



Solution

a The linear pattern is moderate

b The linear pattern is strong

c The linear pattern is weak

d The linear pattern is strong

Pearson's correlation coefficient

The terms strong moderate or weak are very general and not very accurat. We use a measurement (r) called the **Pearsn's correlation coefficient** to determine how closely related variables are in a linear relationship

The formula for r is complex but you can use a calculator or spreadsheet to calculate i.

The correlation coefficient always lies between −1 and 1

Correlation coefficient

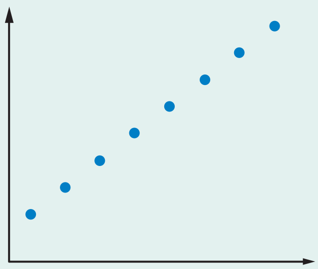
- $-1 \leq r \leq 1$ for all correlation coefficients
- $0 < r \leq 1$ for a scatterplot with positive direction where 1 is perfect positive correlation
- $-1 \leq r < 0$ for a scatterplot with negative direction where −1 is perfect negative correlation
- $r = 0$ means no correlation

ISBN 97807043220

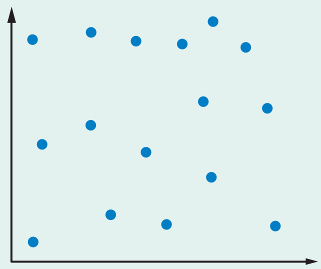
EXAMPLE 4

Match each scatterplot with its correct correlation coefficient

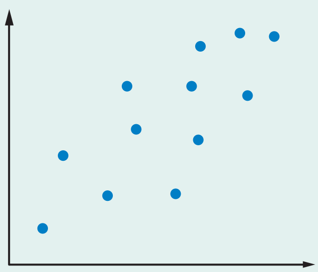
a



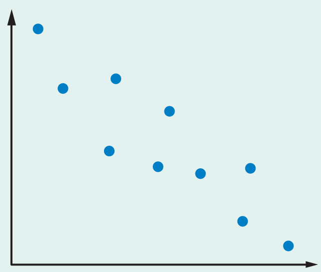
b



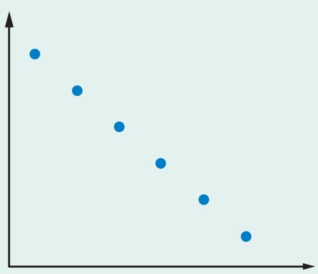
c



d



e



A 05 **B** 1 **C** –1 **D** –04 **E** 0

Solution

a B **b** E **c** A **d** D **e** C

DID YOU KNOW?

Karl Pearson

Karl Pearson (1857–1936) an Englishma, developed the formula for the correlation coefficient r The coefficiet's full name is **Pearsn's product moment correlation coefficient** Karl Pearson was a mathematician and statisticia, but he also studied history, law and German literatue.

Research Karl Pearson to find out more about his life and studes.

ISBN 97807043220

EXAMPLE 5

A group of students was surveyed for the number of hours that they studied and their result in a maths exam Find the correlation coefficient for this bivariate dat.

Number of hours studied	6	5	12	8	15	9	14
Exam results (%)	71	46	74	67	76	77	83

Solution

Operation	Casio Scientific	Sharp Scientific
Place your calculator in statistical mode	MODE 2 STAT2: A +BX	MODE 1 STAT 1 LINE
Clear the statistical memory.	SHIFT 1 3 Edit 2 Del-A	2ndF DEL
Enter data	SHIFT 1 2 Data 6 = 5 = etc for 1st column 71 = 46 = etc for 2nd column AC	6 2ndF STO 71 M+ 5 2ndF STO 46 M+ etc
Calculate r	SHIFT 1 5 : Reg 3 : r =	ALPHA r =
Change back to normal mode	MODE 1 COMP	MODE 0

$r = 0738$ (correct to 3 decimal places)

ISBN 97807043220

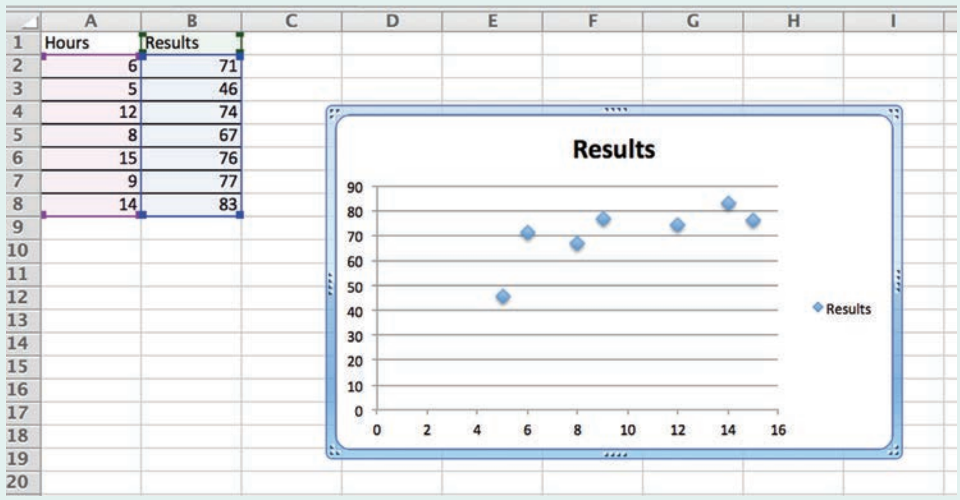
EXAMPLE 6

a Enter the data from Example 5 above into a spreadsheet and draw a scatterplot

b Use the spreadsheet to find the Pearsons correlation coefficient for the dat.

Solution

a Put the values from the table into a spreadsheet and select scatterplot from the charts



b In an empty cell type =**PEARSON(A2A8B2B8**)

$r = 0738$ (to 3 decimal places)

Causality

Two variables can have a high correlation without one *causing* the other. For exampe, does a persons height cause them to weigh more? Possibly, since most tall people would have heavir, longer bones Howeve, there are other causes for higher weight that aen't related to heght.

EXAMPLE 7

For each set of bivariate data find which ones have a causal relationshi.

a Number of people in a family and the number of TVs

b Speed of a boat and time taken to travel across a lake

Solution

a Not causal the number of people in a family doesnt determine how many TVs there ae.

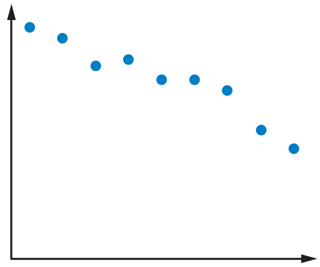
b Causal The speed of a boat will determine the time taken to travel across a lake (higher speed means less time)

 ISBN 97807043220

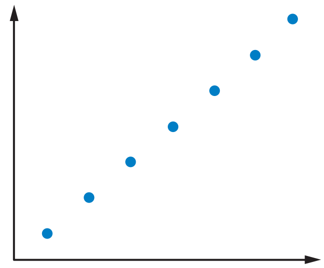
Exercise 8.02 Correlation

1 Match each graph with the correct correlation coefficient

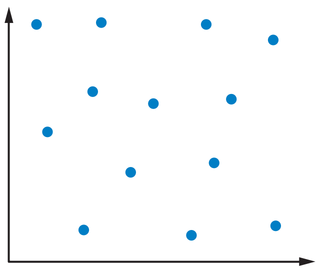
a



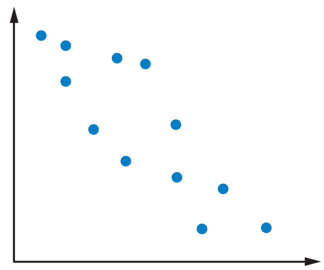
b



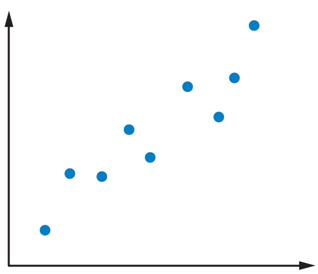
c



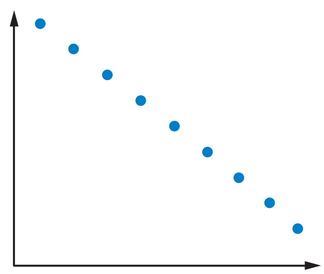
d



e



f



A –05 **B** 1 **C** 08 **D** –1

E –09 **F** 0 **G** –07

2 For each table of values

i draw a scatterplot

ii find the correlation coefficient

a

Height (m)	1.72	1.85	1.61	1.74	1.59	1.79
Weight (kg)	91.3	85.2	58.3	61.9	74.5	102.6

b

Height (cm)	167	180	174	171	154	190
Speed (m s^{-1})	3.7	2.1	3.4	2.2	2.8	4.1

c

Study time (h)	13	21	8	11	18	17
Results (%)	45	89	81	67	74	53

ISBN 97807043220

d

Temperature (°C)	15	18	21	24	26	30	35
Attendance at beach	28	19	54	88	190	245	108

e

Forest cleared (ha)	41	58	87	99	132	168
Number of birds	1200	854	530	201	157	92

f

Number of cars in car park	1100	1450	1809	2004	2234	2569	2871	2906
Pollution (ppm)	1.21	1.54	1.78	2.34	2.99	3.35	4.76	5.97

g

Age	15	19	27	34	49	57
Annual income ($)	2 851	12 600	27 890	38 740	41 834	29 450

h

Exercise (h/week)	14	8	2	10	6	4	32
Weight (kg)	51.8	87.2	74.8	68.4	62.1	63.9	58.9

i

Height (m)	1.59	1.77	1.64	1.78	1.89	1.42
Shoe size	5	7	6	10	9.5	4

j

Exam results (%)	68	92	38	51	77	84
Hours of sleep	7	6	8	6.5	9	7.5

3 For the bivariate data in question 2 which do you think have causality ?

4 Find the correlation coefficient of each set of data correct to 2 decimal place.

a

x	y
3	7
5	9
4	3
11	7
15	12
8	4
9	1

b

x	y
5	67
6	49
3	81
9	23
11	55
8	91
4	61

ISBN 97807043220

c

x	y
8	11
4	8
7	11
2	4
9	12
14	16
23	23

d

x	y
5	21
3	28
6	19
5	17
9	21
4	26
11	15
15	18
9	12

5 Determine whether each pair of variables are likely to have a causal relationship

a Population of a city and pollution

b Head circumference and weight

c Hours training and fitness

d Weight and health

e Size of house and number of pets

f Size of house and selling price

INVESTIGATION

CAUSALITY

Discuss whether each pair of variables have a high correlation and if they d, whether one variable causes the other.

1 A persons height and shoe size

2 A persons smoking and lung cancer

3 Amount of study and success in an exam

4 Mathematical and musical ability

5 The number of people at a party and the amount of food and drink consumed

6 The amount of time sunbaking and the incidence of skin cancer

7 The amount of time practising basketball and the number of baskets scored in a game

8 Results in English and Maths exams

9 The length of a persons leg and their walking speed

10 The temperature and the number of people swimming at the beach

Discuss other relationships between variables Can you find other examples of highly correlated variables where one causes the other? Can you find examples of highly correlated variables where there is no causality?

Discuss causality in each situation described below.

1 The time a sales representative has been with a company and the number of sales gives a correlation coefficient of –06

2 Height of basketball players and number of baskets scored have a correlation coefficient of 087

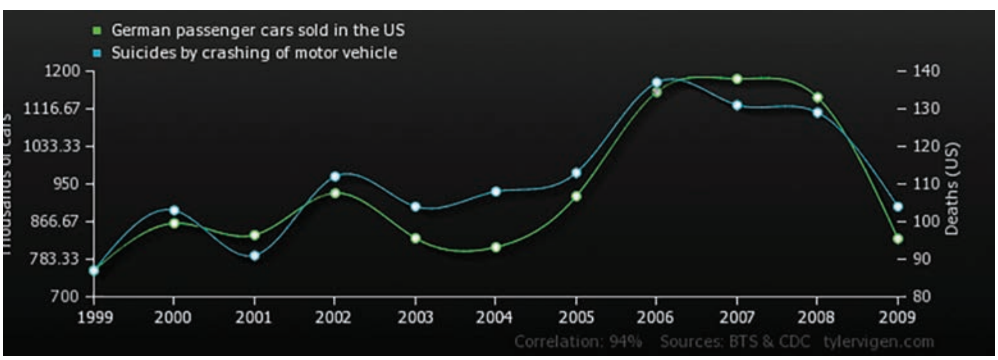
3 Height and self-esteem have a correlation coefficient of 032

4 Temperature and growth of grass have a correlation coefficient of –075

5 Number of hours study and results in the HSC have a correlation coefficient of 085

Collect data from the Internet newspapers or magazine, or do your own experiments to compare two sets of data Draw a scatterplot and find the correlation coefficien, thn, if there is a high correlation investigate causality. Is the correlation positive or negative? Is it linear?

Here is an example of a high correlation of totally unrelated variables German cars sold vs suicides by car crashes in the US each year.



http://tylervigencom/page?page 2

There are many other examples of correlated data that have no causality.

ISBN 97807043220

8.03 Line of best fit

Statistical data is rarely perfect but we can often see trends in a scatterplo. If there seems to be a linear correlation we can draw a **regression line** and find its equation We can then use this line to make predictions

The easiest regression line to find is the **line of best fit**

Using a ruler, we draw the line that represents as many points as possibe. We try to draw a line where about half the points are above the line and half are below it so that the distance between the line and the points is kept to a minimum

WS Lines o bes i

WS Lines o i

WS Heigh v hoe ize

EXAMPLE 8

A ball is rolled down a ramp and its velocity is measured over time

t (s)	1	2	3	4	5	6
v (m s^{-1})	0.7	1.5	3.6	6.1	7.8	9.9

a Draw a scatterplot of the data and draw a line of best fit

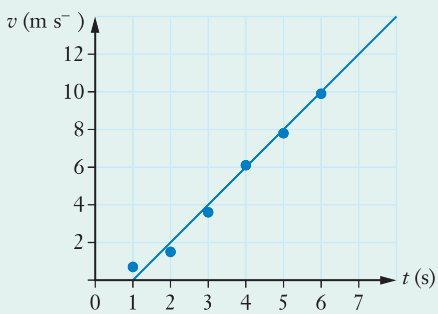
b Use the line of best fit to find the velocity after 35 seconds

c Find the equation of this line

d Use the equation to find the velocity after 10 seconds

Solution

a



b After 35 seconds the velocity is 5 m s $^{-}$

c Choose 2 points on the line sa,(2, 2) an (4 6).

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{4-2} = \frac{4}{2} = 2$$

Equation $y - y \ = m(x - x\ )$

$v - 2 = 2(t - 2)$ using $m = 2$ and (2 2)

$= 2t - 4$

$v = 2t - 2$

d When $t = 10$

$v = 2(10) - 2$

$= 18$

So after 10 seconds the velocity is 18 m s⁻

Interpolation and extrapolation

Interpolation is using a model to make predictions about values lying **within** the range of the original data set

Extrapolation is using a model to make predictions about values **outside** the range of the original data set

In the above example finding v when $t = 35$ is interpolation while finding v when $t = 10$ is extrapolation

DID YOU KNOW?

Regression

The word **regression** comes from the Latin *regressio* meaning'a retu'.

Sir Francis Galton (1822–1911) was the first to use this name He created the concepts of correlation and regression and used statistics to develop questionnaires and surveys on human differences

Research correlatio, regression and Sir Francis Gaton.

CLASS DISCUSSION

EXTRAPOLATION

Extrapolation is not always accurate For exampl, a formula may work well at firt, then the conditions may change in a way that means the formula is no longer a good model

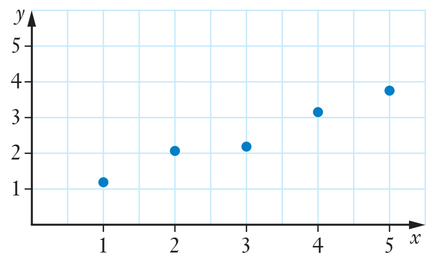
Think of examples of bivariate data from the previous examples Can you always extrapolate answers from a line of best fit? Why? What are the risks?

ISBN 97807043220

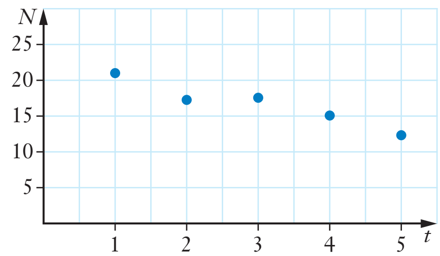
Exercise 8.03 Line of best fit

1 Copy each scatterplot draw a line of best fit and find its equatio.

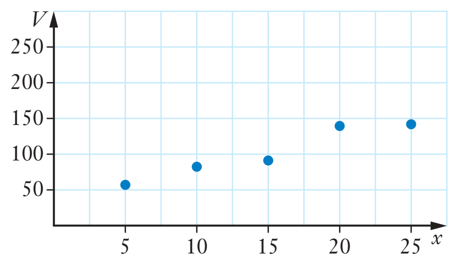
a



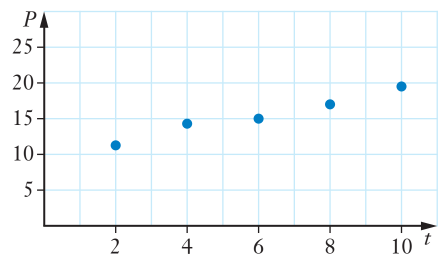
b



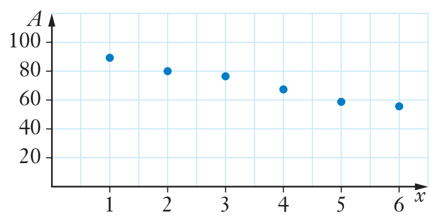
c



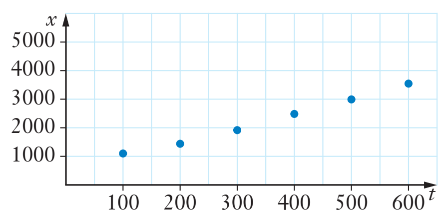
d



e



f



2 The following table shows the results of an experiment testing the temperature of a liquid as it cools down

t (min)	5	10	15	20	25	30
T (°C)	87	78	69	56	53	41

a Plot this data on a number plane and draw a line of best fit

b Find the equation of the line

c Use the equation to find the temperature after

i 17 minutes **ii** 35 minutes

d If room temperature is 23°C is the line of best fit a good model for the cooling of the liquid? Why?

ISBN 97807043220

3 The population of birds in a particular area was sampled over several years

***t* (years)**	1	2	3	4	5	6
P	1030	983	968	954	915	899

a Plot this data on a number plane and draw a line of best fit

b Find the equation of the line

c Use the equation to find the population of birds after 7 years

d At this rate of decline after how many years would you expect there to be no more birds in this area?

4 The effect of a dose of medicine on a childs temperature over time was measured from a sample of children with the following result.

***t* (min)**	2	4	6	8	10
***T* (°C)**	395	391	389	382	377

a Plot this data on a number plane and draw a line of best fit

b Find the equation of the line

c Use the equation to find the childs temperature after 15 minute.

d Is this equation reliable as a measure of temperature after a longer time sa, 1 hour ?

5 This table shows the results of a survey into the number of people who attend a new restaurant over a number of weeks

***t* (weeks)**	1	2	3	4	5	6
No of people	76	114	163	187	228	274

a Draw a scatterplot and sketch a line of best fit

b Find the equation of this line

c Use the equation to find the number of people you would expect to attend the restaurant after 10 weeks

d Is this equation a good model for the number of people attending the restaurant?

ISBN 97807043220

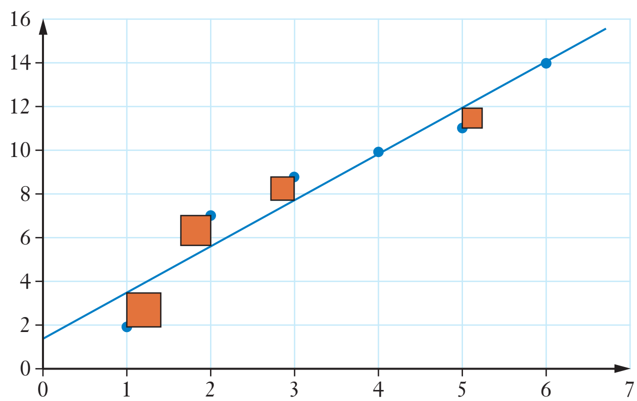
8.04 Least-squares regression line

eaquae egeion line

eaquae egeion line

The line of best fit relies on our eyes and ruler for its accuracy.

There are several different models of regression lines that try to give a more accurate result The most popular model is called the **least-squares regression line** It uses a line of best fit for which the squares of the distances from each point in the scatterplot to the line are minimised (see the diagram below) Squaring the distances takes away any negative values (a similar technique to finding standard deviation)



The equation of the least-squares regression line is given by $y = mx + c$ where $m = r\dfrac{s_y}{s_x}$ and $c = \bar{y} - m\bar{x}$ where r = correlation coefficient $\bar{x}$ and $\bar{y}$ are the sample means and s_x and s_y are the sample standard deviations

However, you dn't need to use these formulas because the regression line can be found using a scientific calculator, graphics calculatr, online calculator or spreadseet.

EXAMPLE 9

The table shows the results of a survey into the number of cigarettes people smoke during the year and the number of days they are absent from work

Cigarettes/year	2000	3000	4000	5000	6000	7000	8000	9000	10 000
Absences	23	27	54	49	63	81	107	128	147

Find the least-squares regression line by using

a a calculator

b a spreadsheet

ISBN 97807043220

Solution

a

Operation	Casio scientific	Sharp scientific
Enter data	SHIFT 1 2 Data 2000 = 3000 = etc for 1st column 23 = 27 = etc for 2nd column AC	2000 2ndF STO 23 M+ 3000 2ndF STO 27 M+ et.
Calculate a	SHIFT 1 5 Reg 1 A =	ALPHA a =
Calculate b	SHIFT 1 5 Reg 2 B =	ALPHA b =

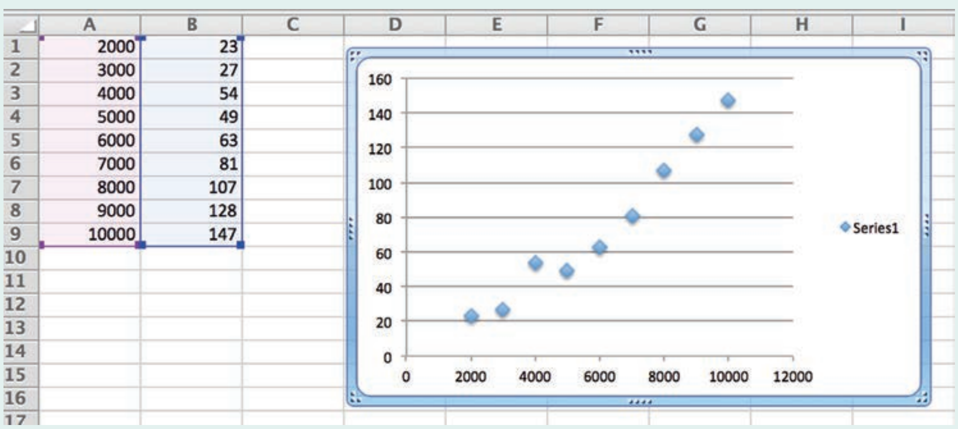
$a = -1826$

$b = 00156$

On a scientific calculator, the equation is in the form $y = a + bx$ or $y = bx + a$

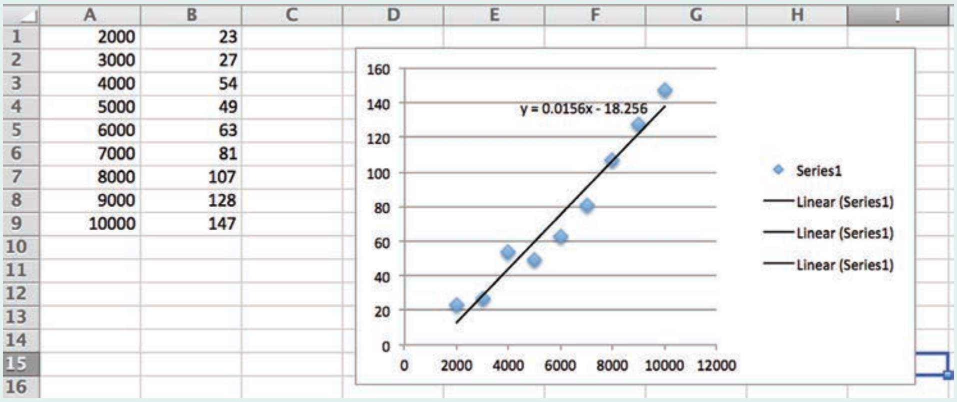
So $y = 00156\,x - 1826$

b Enter the 2 columns in a spreadsheet and draw a scatterplot



In Chart Layout select Trendlie, then Linear trendlne. You can display the equation of the line by going to Trendline agan, selecting Trendline options and selecting Display equation on chart

ISBN 97807043220



The equation of the least-squares regression line is $y = 00156\,x - 18256$

Exercise 8.04 Least-squares regression line

1 For each data set fin:

- **i** the correlation coefficient
- **ii** the gradient of the least-squares regression line

a

x	1	2	3	4	5	6	7
y	3	4	7	10	11	15	17

b

x	2	4	6	8	10	12	14	16
y	8	11	19	29	34	41	45	67

2 For each data set

- **i** find the equation of the least-squares regression line
- **ii** sketch the scatterplot and regression line on the same axes

a Age of machine and breakdown rates

Age (years)	1	2	3	4	5
Breakdowns	0	2	5	9	15

b Length of time in office and popularity of a political party

Time (years)	1	2	3	4	5	6
Popularity (%)	52.3	43.8	43.7	42.1	37.9	37.6

c Length of drought and yield of crops

Time (years)	1	2	3	4	5	6
Yield (t)	107.3	101.8	100.2	87.6	63.5	47.1

d Engine size of cars and number of accidents

Size (L)	1.3	1.8	2.0	2.1	2.4	3.8
Accidents	459	447	513	519	506	625

e Age and number wearing glasses

Age	10	20	30	40	50	60	70	80
Glasses	34	28	41	56	87	105	156	209

3 A block of ice was taken out of a freezer and left to thaw. The results are in the table below.

Time t (min)	5	10	15	20	25	30	35	40
Mass m (kg)	23.7	18.8	11.3	8.7	6.2	5.5	2.3	1.5

a Find the correlation coefficient Is there a high correlation ? Is it positive or negative? What does this mean?

b Find the equation of the least-squares regression line

c Use the equation to estimate the mass of the ice after

i 18 minutes **ii** an hour

d Discuss why extrapolation may not be useful in this situation

4 This table shows the weight of gemstones sold at an auction and their selling price

Weight (carat)	0.05	0.8	1.5	1.7	2.5	2.8	3.1	4.0
Price ($)	144	672	1245	1478	2100	2500	2881	3215

a Draw a scatterplot for this data

b Find the correlation coefficient Is there a high correlation between the weight of a gemstone and its cost? Is it positive or negative?

c Find the equation of the least-squares regression line

d How much would you expect to pay for a 2 carat gemstone?

e How much would you expect a gemstone to weigh if it cost $10 000?

5 The table shows the results of a survey into ages and earnings of a group of people

Age	15	32	19	28	43	67
Earnings/week ($)	689	1205	840	1154	1587	986

a Draw a scatterplot for this data

b Find the correlation coefficient

c Find the equation of the least-squares regression line

d From this equation find the earnings of a 50-year-ol.

e Is this equation a good model to extrapolate?

ISBN 97807043220

8. TEST YOURSELF

For Questions **1** to **5** select the correct answer **A B C** or **D**

Pacice quiz

Fiing he daa indawod

1 Which variables are not correlated?

A Hand size and height **B** Hours of study and exam results

C Height and hours of employment **D** Speed of car and fuel economy

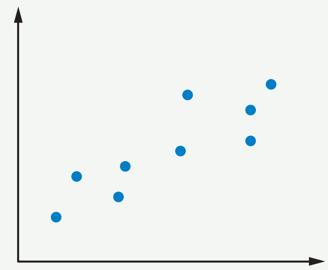
2 Describe the correlation in the scatterplot shown

A Weak negative correlation

B Moderate positive correlation

C Moderate negative correlation

D Strong positive correlation



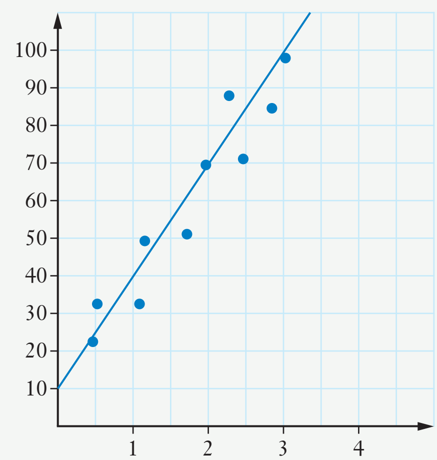
3 Find the equation of the line of best fit

A $y = 60x + 10$

B $y = 10x + 30$

C $y = 30x - 10$

D $y = 30x + 10$



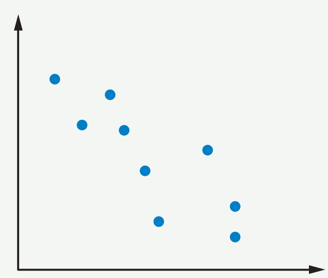
4 Estimate the correlation coefficient of this bivariate data

A –05

B 05

C –1

D 1



5 Using the equation of a line of best fit to predict the value of a variable within the domain of the data set is called

A extrapolation **B** causality **C** interpolation **D** correlation

ISBN 97807043220

6 Make a scatterplot of this table of bivariate data the:

x	1	2	3	4	5	6	7	8	9
y	17	21	24	29	36	43	44	52	58

a draw a line of best fit and find its equation

b draw a least-squares regression line and find its equation

7 For each scatterplot state whether it ha:

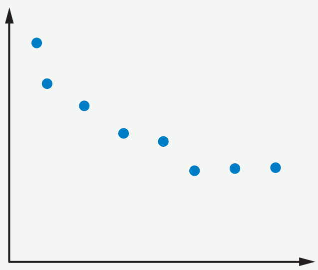
A a positive linear correlation

B a negative linear correlation

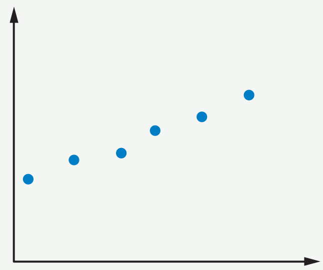
C a non-linear correlation

D little or no correlation

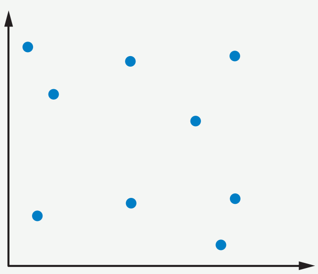
a



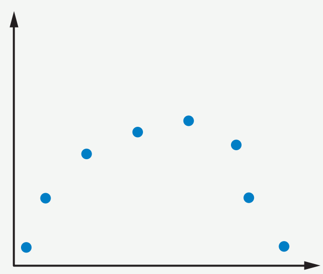
b



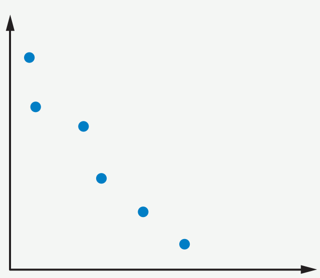
c



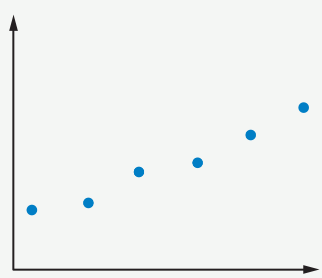
d



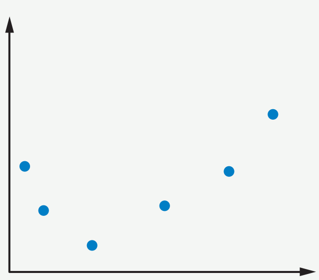
e



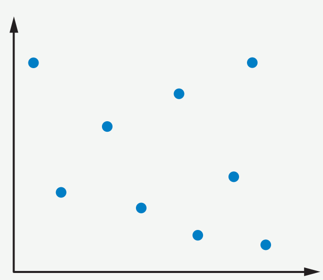
f



g



h



ISBN 97807043220

8 A group of students was surveyed to find whether there was a correlation between a students music and maths assessment mark. The results are in the table beow.

Music	79	58	91	93	65	43	39	64	82	51
Maths	62	63	82	79	73	57	29	52	76	40

a Find the correlation coefficient

b Find the equation of the least-squares regression line

c Using the equation find the maths assessment mark for a student who scores 60 in music

d Find the music assessment mark for a student who scores 70 in maths

9 Determine whether each pair of variables are likely to have a causal relationship

a Height and amount of food eaten

b Number of years playing sport and weight

c Height and arm length

d Number of chickens and number of eggs

e Size of bookshelves and number of books

ISBN 97807043220

8. CHALLENGE EXERCISE

1 The equation of the least-squares regression line is given by $y = mx + c$ where $m = r\dfrac{s_y}{s_x}$ and $c = \bar{y} - m\bar{x}$ where r = correlation coefficient $\bar{x}$ and $\bar{y}$ are the sample means and s_x and s_y are the sample standard deviations Evaluate r and $\bar{y}$ given the equation of the least-squares regression line is $y = 2x + 4$, $\bar{x} = 1.2$, $s_x = 18$ and $s_y = 45$

2 Sketch a scatterplot that shows a linear correlation of

a –1 **b** approximately 08 **c** 0

d approximately –02 **e** 1

3 The table below shows the results of an experiment into the volume of water evaporating from a body of water at different temperatures

T **(°C)**	10	15	20	25	30	35	40	45
V **(L)**	0.5	1.3	2.9	5.8	10.3	15.7	39.8	76.1

a Draw a scatterplot to show this data

b Why would a least-squares regression line not give a good approximation for this data?

c Use technology or otherwise to find an equation that might approximately model this data

4 The table shows heights and shoe sizes of several males

Shoe size	4	5	6	7	8	9	10	11
Height (m)	1.54	1.65	1.68	1.73	1.59	1.82	1.89	1.95

a Draw a scatterplot to show this data

b Find the correlation coefficient

c Find the equation of the least-squares regression line

d The male in the sample with shoe size 8 is found to be an outlier for this data For the sample without this outlier, fid:

i the correlation coefficient

ii the equation of the least-squares regression line

e Is this equation a good model for shoe sizes and height?

ISBN 97807043220

Practice set 3

In Questions **1** to **5** select the correct answer **A B C** or **D**

1 Find the median of 3, 10, 1, 4, , 6.

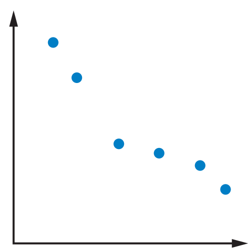
A 5 **B** 1 **C** 4 **D** 6

2 Which pair of variables are not correlated?

A Size of house and size of family
B Height and foot size
C Number of babies born and number of nappies used
D Distance travelled and average speed over 2 hours

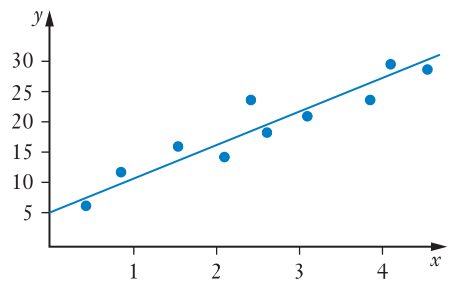
3 Describe the correlation in the scatterplot

A Strong positive linear correlation
B Moderate positive linear correlation
C Moderate negative linear correlation
D Strong negative linear correlation



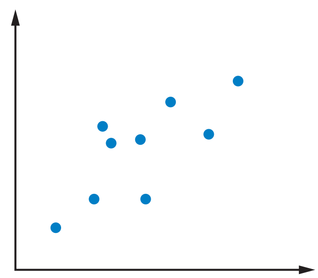
4 The equation of this line of best fit is closest to

A $y = 10x + 5$
B $y = 5x - 5$
C $y = 10x - 5$
D $y = 5x + 5$



5 The correlation coefficient of the bivariate data shown on this scatterplot is closest to

A –05
B 05
C –1
D 1



ISBN 97807043220

6 These scores are the results of a maths quiz

7, 9, 5, 8, 9, 5, 6, 8, 7, 9, 5, 5, 7, 6, 5

a Complete a frequency distribution table for these scores

b Draw a frequency polygon and histogram for this data

c Draw a cumulative frequency histogram and polygon

d Find the median

e Find the interquartile range

f Draw a box plot to show the five-number summary for this data

7 For this table of bivariate data

x	1	2	3	4	5	6	7	8
y	18	23	29	38	41	46	52	60

a draw a line of best fit and find its equation

b draw a least-squares regression line and find its equation

8 Shoppers were asked what they most liked about the shopping centre
Draw a Pareto chart for the survey results

Variety of shops	34
Amenities	11
Child-friendly	28
Parking	27

9 Find the mean standard deviation and variance of the scores ,5, 9 ,6, 5 8, 7.

10 The table shows students scores on a maths test

a Find the mean and standard deviation

b Show that the score of 3 is an outlier.

c Find the mean and standard deviation excluding the outlier.

Score	Frequency
3	1
4	0
5	0
6	2
7	4
8	6
9	8
10	3

11 Find the mean mod, median and range of the scores8, 9 ,7, 6 5, 6.

ISBN 97807043220

12 The table below shows the results of a survey into the ratings of a TV sho.

Rating	Frequency
1	1
2	7
3	15
4	19
5	10

a How many people were surveyed?

b Find correct to 2 decimal places the mean and standard deviation

c Find the median

d Draw a box plot for these results

13 Find the correlation coefficient for this set of data correct to 2 decimal place.

x	3	7	4	8	12	2
y	15	11	9	8	7	18

14 Draw an example of statistical data that is

a positively skewed

b negatively skewed

c symmetrical

d bimodal

e multimodal

15 Describe each data set as categorical nominal categorical ordina, quantitative discrete or quantitative continuous

a Types of trees planned for a park

b Length of road between towns

c Survey ratings of Poor, God, Verygood, Excellent

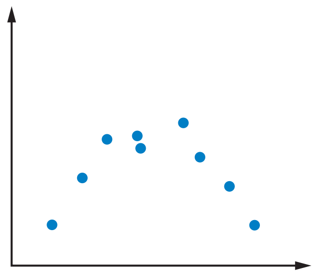
d Dress sizes

e Test scores

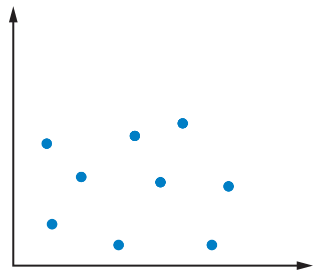
ISBN 97807043220

16 Match each scatterplot to its description

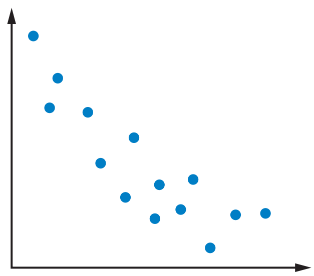
a



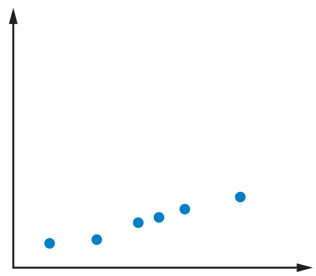
b



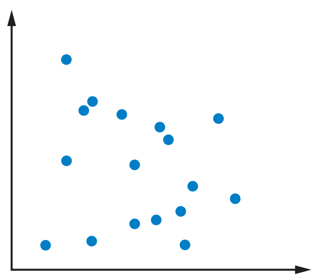
c



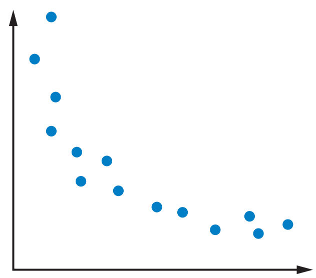
d



e



f



A positive linear correlation

B negative linear correlation

C non-linear correlation

D no correlation

ISBN 97807043220

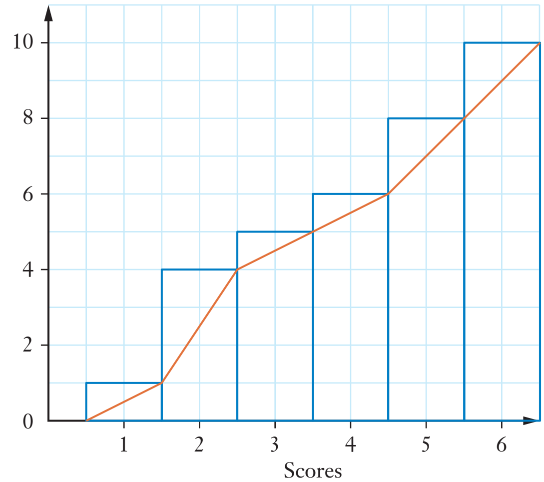
17 For this cumulative frequency polygon fin:

a the median

b the first quartile

c the 60th percentile

d the 4th decile



18 Find all stationary points and points of inflection on the graph of the function $f(x) = 2x^3 - 6x^2 - 48x + 17$.

19 Solve each equation for [0 2 π]

a $2 \sin x = 1$

b $\tan^2 x = 1$

c $2 \cos 2x + 1 = 0$

20 For the sequence 3 , 1, … fnd:

a the 100th term

b the sum of the first 100 terms

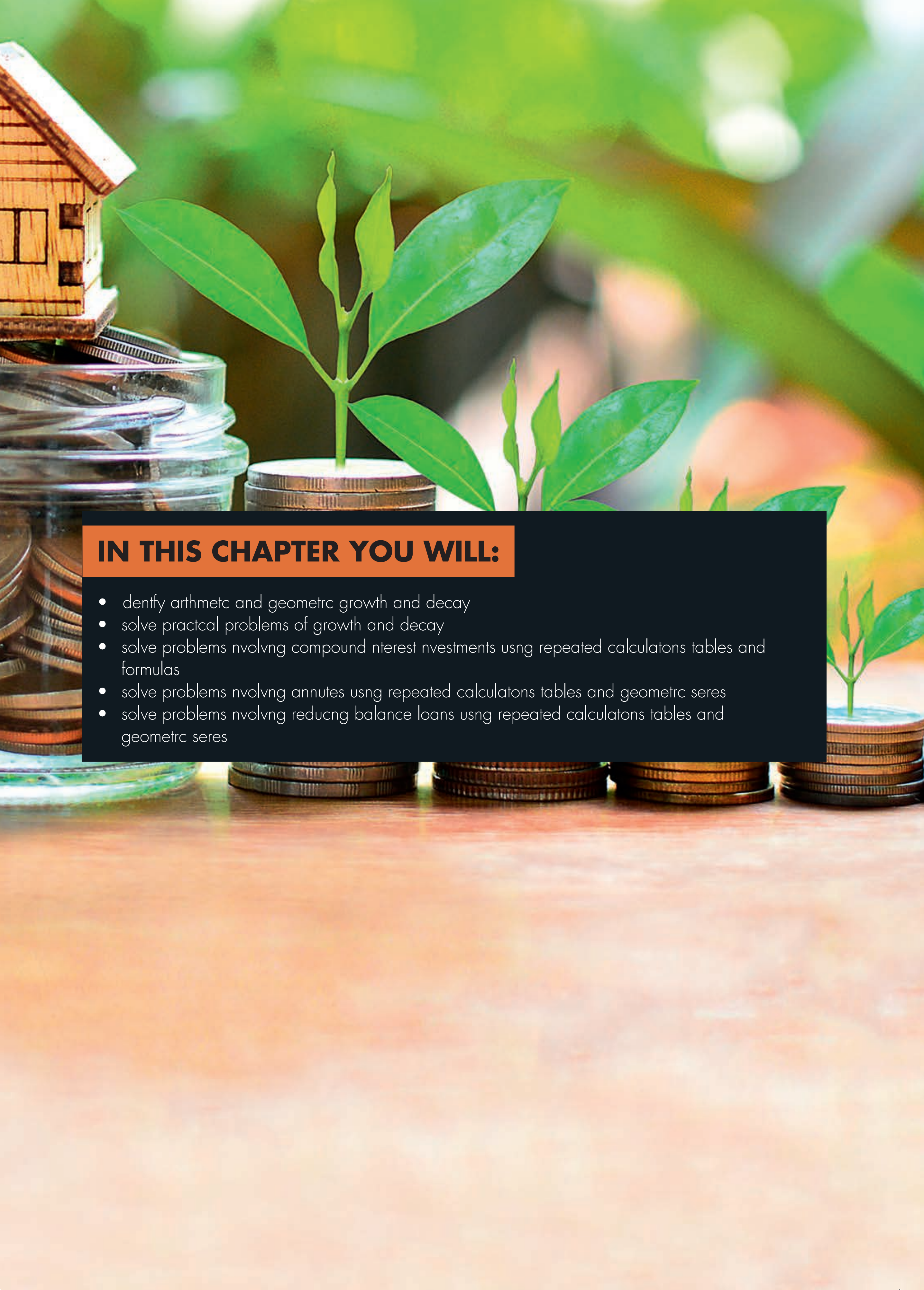
ISBN 97807043220

9.

INVESTMENTS, ANNUITIES AND LOANS

Seres and sequences have many applcatons Fnancal mathematcs s an mportant part of everyday iing as we put moneyin the ban, pay off crdit cars, take out superannutin, buy houses and cars and many other thngs In ths chapter you wll study the fnances of nvestments annutes and loans and see how they relate to seres

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- dentfy arthmetc and geometrc growth and decay
- solve practcal problems of growth and decay
- solve problems nvolvng compound nterest nvestments usng repeated calculatons tables and formulas
- solve problems nvolvng annutes usng repeated calculatons tables and geometrc seres
- solve problems nvolvng reducng balance loans usng repeated calculatons tables and geometrc seres

TERMINOLOGY

annuit: An investment for a fixed period of time where payments are made or received regularly.

compound interest The interest earned on both the principal and previous interest payments of an investment

future valu: The total value at the close of an investment including all payments and interest earned

present valu: A single payment (called the principal) that will produce a future value over a given time

reducing balance loan A loan that is repaid by making regular payments with interest calculated on the amount still owing (the reducing balance of the loan) after each payment

superannuatin: A fixed portion of income that is invested regularly to provide a lump sum or pension when a person retires from the paid workforce an example of an annuity.

9.01 Arithmetic growth and decay

WS

Aihmeic gowh and deay

We can use arithmetic sequences and series to describe **growth** (increase) and **decay** (decrease) in practical problems This is sometimes called **discrete linear growth and decay**

EXAMPLE 1

A stack of cans on a display at a supermarket has 5 cans on the top row. The next row down has 2 more cans and the next one has 2 more cans and so on

a Calculate the number of cans in the 11th row down

b If there are 320 cans in the display altogether, how many rows are there?

Solution

a The first row has 5 cans the 2nd row has 7 can, the 3rd row 9 cans and so n. This forms an arithmetic sequence with $a = 5$ and $d = 2$.

For the 11th row, we want $n = 11$:

$$T_n = a + (n - 1)d$$

$$T_{11} = 5 + (11 - 1) \times 2$$

$$= 5 + 10 \times 2$$

$$= 25$$

So there are 25 cans in the 11th row.

ISBN 97807043220

b If there are 320 cans altogether, this is the sum of cans in all rows

$$S_n = 320$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$320 = \frac{n}{2}[2 \times 5 + (n-1) \times 2]$$

$$= \frac{n}{2}(10 + 2n - 2)$$

$$= \frac{n}{2}(2n + 8)$$

$$= n^2 + 4n$$

$$0 = n^2 + 4n - 320$$

$$= (n-16)(n+20)$$

$$n - 16 = 0, \quad n + 20 = 0$$

$$n = 16, \qquad n = -20$$

Since n must be a positive integer, then $n = 16$.

There are 16 rows of cans

We could subsitute n 16 nto S_n to check ths

Exercise 9.01 Arithmetic growth and decay

1 A market gardener plants daffodil bulbs in rows starting with a row of 45 bulbs
Each successive row has 5 more bulbs than the row before

- **a** Calculate the number of bulbs in the 34th row.
- **b** Which row would be the first to have more than 100 bulbs in it?
- **c** The market gardener plants 10 545 bulbs altogether. How many rows are there?

2 A stack of logs has 1 on the top then 3 on the next row down and each successive row has 2 more logs than the one on top of i.

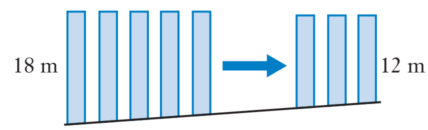
- **a** How many logs are in the 20th row?
- **b** Which row has 57 logs?
- **c** If there are 1024 logs altogether, how many rows are in the stack?

3 A set of books is stacked in layers where each layer contains 3 books fewer than the layer below. There are 6 books in the top laer, 9 in the next ayer, 12 in the next and so on There are n layers altogether.

a Write down the number of books in the bottom laye.

b Show that there are $\frac{3}{2}n(n + 3)$ books in the stack altogether.

4 A timber fence is to be built on sloping land with the shortest piece of timber 12 m and the longest 18 m There are 61 pieces of timber in the fene.



a What is the difference in height between each piece of timber?

b Assuming no wastage what length of timber is needed for the fence altogether ?

5 A sculpture consists of a set of poles set in a row, with the tallest pole2.4 m hih, the next pole 21 the next one .8 and so o, down to the last pole which is0.6 m hih.

a How many poles are in the sculpture?

b The poles are made of timber. What length of timber is there altogether in the poles?

6 Johanna has \$2000 in a term deposit that earns simple interest of 25% pa How much money does she have including interes, aftr:

a 1 year? **b** 2 years? **c** 3 years?

d 10 years? **e** 30 years?

7 Each house in a row of terraced houses is to have a new fence The houses are on a hill so the first fence will be 1 m high the second will be .05 m hig, the third1.1 m high and so on

a How high will the fence need to be for the 6th house?

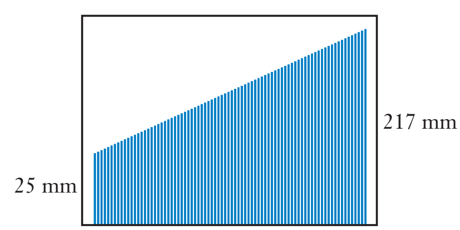
b If the height of the last fence is 135 m how many houses are there ?

8 At a courier company, there are different price categories for different weights of parces. The 1st category is parcels in the range 0–05 kg then .5–1 k, then 11.5 kg and so n.

a What is the 10th weight category?

b Which category is 85–9 kg ?

9 A logo is made with vertical lines equally spaced as shown The shortest line is 25 m, the longest is 217 mm and the sum of the lengths of all the lines is 5929 mm

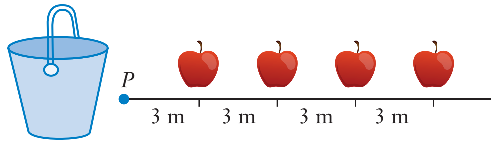


a How many lines are in the logo?

b Find the difference in length between adjacent lines

ISBN 97807043220

10 In a game a child starts at point P and runs and picks up an apple 3 m away. She then runs back to P and puts the apple in a bucket The child then runs to get the next apple 6 m away, and runs back to P to place it in the bucket This continues until she has all the apples in the bucket



a How far does the child run from P to pick up the kth apple?

b How far does the child run to fetch all k apples including return trips to P?

c The child runs 270 m to fetch all the apples and return them to the bucket How many apples are there?

9.02 Geometric growth and decay

We can use geometric sequences and series to describe growth and decay in practical problems. This is called **geometric growth and decay** It is also called exponential growth and decay because $T_n = ar^{n-1}$ is an exponential function You studied exponential growth and decay involving e^x in the Year 11 coure.

WS Geomeic gowh and deay

WS Pacical applicaion o seies

EXAMPLE 2

A layer of tinting for a car window lets in 95% of light

a What percentage of light is let in by

i 2 layers of tinting? **ii** 3 layers of tinting? **iii** 10 layers of tinting?

b How many layers will let in 40% of light?

Solution

a **i** 1 layer lets in 95% of light

So 2 layers lets in 95% × 95% of light

$$95\% \times 95\% = 095 \times 095$$
$$= 09025$$
$$= 9025\%$$

So 2 layers lets in 9025% of light

ii 1 layer lets in 95% or 095 of light

2 layers lets in $095 \times 095 = 095^2$ of light

3 layers lets in $095^2 \times 095 = 095^3$ of light

$$095^3 \approx 0857$$
$$= 857\%$$

So 3 layers lets in 857% of light

ISBN 97807043220

iii The number of layers forms the geometric sequence $095\ .95^{2}\ .95^{3} \ldots$ with $a = 095,\ r = 095$

For 10 layers $n = 10$.

$$\begin{aligned}
T_n &= ar^{n-1} \\
T_{10} &= 095(095)^{10-1} \\
&= 095(095)^{9} \\
&= 095^{10} \\
&\approx 05987 \\
&= 5987\%
\end{aligned}$$

So 10 layers lets in 5987% of light

b We want to find n when the nth term is 40% or 04

$$\begin{aligned}
T_n &= ar^{n-1} \\
04 &= 095(095)^{n-1} \\
&= 095^{n} \\
\log 04 &= \log 095^{n} \\
&= n \log 095 \\
\frac{\log 04}{\log 095} &= n \\
179 &\approx n
\end{aligned}$$

So around 18 layers of tinting will let in 40% of light

EXAMPLE 3

A car bought for \$35 000 depreciates (loses value) by 12% pa

a Find its value after

i 1 year **ii** 2 years **iii** 3 years

b Write the value of the car as a sequenc.

c Find what the car is worth after 10 years

d When will the value of the car drop below \$15 000? Answer to the nearest yea.

ISBN 97807043220

Solution

a **i** After 1 year the car is worth \$35 000 – 12% of \$35 000

$\$35\,000 - 12\%$ of $\$35\,000 = \$35\,000(1 - 12\%)$

$= \$35\,000(1 - 012)$

$= \$35\,000(088)$

$= \$30\,800$

So the car is worth \$30 800 after 1 year.

ii After 2 years the car is worth \$30 800 – 12% of \$30 800

$\$30\,800 - 12\%$ of $\$30\,800 = \$30\,800(1 - 12\%)$

$= \$30\,800(088)$

$= \$27\,104$

So the car is worth \$27 104 after 2 years

iii After 3 years the car is worth \$27 104 – 12% of \$27 104

$\$27\,104 - 12\%$ of $\$27\,104 = \$27\,104(088)$

$= \$23\,85152$

So the car is worth \$23 85152 after 3 years

b 1st year = \$30 800 2nd year = \$27 104 3rd year = \$23 85152, …

So 30 800 27 10, 23 81.2, … is a geometric sequence with $a = 30\,800$ and $r = 088$

c When $n = 10$

$T_n = ar^{n-1}$

$T_{10} = 30\,800\,(088)^{10-1}$

$= 30\,800\,(088)^{9}$

$= 974753$

So the car is worth \$974753 after 10 years

d We want $T_n < 15\ 000$

$$30\ 800(088)^{n-1} < 15\ 000$$
$$088^{n-1} < 0487$$
$$\log 088^{n-1} < \log 0487$$
$$(n-1)\log 088 < \log 0487$$
$$(n-1) > \frac{\log 0\ 487}{\log 0\ 8\ 8} \quad \text{(The inequality reverses because } \log x < 0 \text{ for } 0 < x < 1\text{)}$$
$$n > \frac{\log 0\ 487}{\log 0\ 8\ 8} + 1$$
$$> 663$$

We can subsitute n 7 nto T_n to check ths

So it will take approximately 7 years for the car to be worth less than \$15 000

We can also use the limiting sum to model some types of problem.

EXAMPLE 4

a Write $.\dot{5}$ as a fraction

b A ball is dropped from a height of 1 metre and bounces up to $\frac{1}{3}$ of its height

It continues bouncing rising $\frac{1}{3}$ of its height on each bounce

i Draw a diagram showing the motion

ii What is the total distance through which the ball travels?

Solution

a $0\dot{5} = 05555555\ldots$

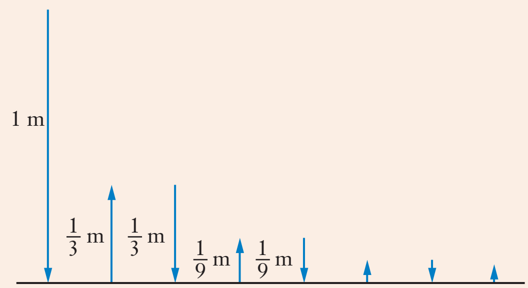
$$= \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} +$$

This is a geometric series with $a = \frac{5}{10} = \frac{1}{2}$ and $r = \frac{1}{10}$

$$S = \frac{a}{1-r}$$
$$= \frac{\frac{1}{2}}{1-\frac{1}{10}}$$
$$= \frac{\frac{1}{2}}{\frac{9}{10}}$$
$$= \frac{5}{9}$$

ISBN 97807043220

b **i**



ii Notice that there is a series for the ball coming downwards and another series upwards There is more than one way of calculating the total distane. Here is one way of solving it

$$\text{Total distance} = 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{9} + \frac{1}{9} + \frac{1}{27} + \frac{1}{27} +$$

$$= 1 + 2\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \quad\right)$$

$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{3}$

$$S = \frac{a}{1-r}$$

$$= \frac{\frac{1}{3}}{1-\frac{1}{3}}$$

$$= \frac{1}{2}$$

$$\text{Total distance} = 1 + 2\left(\frac{1}{2}\right)$$

$$= 2$$

So the ball travels 2 metres altogether.

INVESTIGATION

LIMITING SUM APPLICATIONS

1 In the above example in theory will the ball ever stop ?

2 Kim owes \$1000 on her credit card If she pays back 10% of the amount owing each month she will never finish paying it of. Is this true or false ?

Exercise 9.02 Geometric growth and decay

1 Water evaporates from a pond at an average rate of 7% each wee.

a What percentage of water is left in the pond after

i 1 week? **ii** 2 weeks? **iii** 3 weeks?

b What percentage is left after 15 weeks?

c If there was no rain approximately how long (to the nearest week) would it take for the pond to only have 25% of its water left?

2 The price of shares in a particular company is falling by an average of 2% each day.

a What percentage of their initial value do they have after 2 days?

b Approximately how many days will it take for the shares to halve in value?

c After how many days will the shares be worth 10% of their initial value?

3 A painting appreciates (increases its value) by 16% pa It is currently worth $20 00.

a How much will it be worth in

i 1 year? **ii** 2 years? **iii** 3 years?

b How much will it be worth in 11 years?

c How long will it take for it to be worth $50 000?

4 A southern brown bandicoot population in Western Australia is decreasing by 5% each yar.

a What percentage of the population is left after 5 years?

b After how many years will the population be only 50% of its current level?

c How many years will it take for the population to decrease by 80%?

5 Write each recurring decimal as a fractio.

a $0\dot{4}$ **b** $0\dot{7}$ **c** $1\dot{2}$ **d** $0\dot{2}\dot{5}$ **e** $2\dot{8}\dot{1}$

f $02\dot{3}$ **g** $14\dot{7}$ **h** $101\dot{5}$ **i** $01\dot{3}\dot{2}$ **j** $2\dot{3}6\dot{1}$

6 A frog jumps 05 metres It then jumps .1 m and on each subsequent jump it travels .2 of the previous distance Find the total distance through which the frog jump.

7 A tree grows by $\frac{4}{5}$ of each previous years growt.

If it was initially 3 m high find the ultimate height of the tree

ISBN 97807043220

8 An 8 cm seedling grows by 48 cm in the first week and then keeps growing by .6 of its previous weeks growt. How tall will it grow ?

9 An object rolls 05 m in the first second Then each second aftr, it rolls by $\frac{5}{6}$ of its previous roll Find how far it will roll altogethe.

10 A 100 m cliff erodes by $\frac{2}{7}$ of its height each year.

a What will the height of the cliff be after 10 years?

b After how many years will the cliff be less than 50 m high?

11 A lamb grows by $\frac{2}{5}$ of its previous growth each month If a lamb is 45 cm tal:

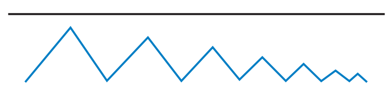
a how tall will it be after 6 months?

b what will its final height be?

12 A weight on an elastic string drops down 60 cm and then bounces back to $\frac{2}{3}$ of its initial height It keeps bouncin, each time rising back to $\frac{2}{3}$ of its previous height What is the total distance through which the weight travels?

13 Mary bounces a ball dropping it from .5 m on its first bounc. It then rises up to $\frac{2}{5}$ of its height on each bounce Find the distance through which the ball travel.

14 A roadside wall has a zigzag pattern on it as shown The two longest lines are each 2 m long then the next two lines are $1\frac{3}{4}$ m long and the lines in each subsequent pair are $\frac{7}{8}$ of the length of the previous pair. Find the total length of the lins.



15 Frankie receives a text message that she is asked to send to 8 friends Frankie forwards this text on to 8 friends and each of them sends it on to 8 friends and so o.

a Describe the number of people receiving the text as a sequence (including Frankies text.

b How many people would receive the message in the 9th round of texts?

c How many people would have received the text altogether if it is sent 9 times?

9.03 Compound interest

An **investment** is money that is put into the bank or used to pay for something that will increase in value or **appreciate** in the futur. An investment can include real estte,art, jewellery or antiques

The amount invested is called the **present value** or **principal** The amount the investment is worth after a period of time is called the **future value**

Future value of investments (FV)

Money in the bank earns interest Some investments earn simple interes, but most earn **compound interest** For exampl, a term deposit or investment account gives the option of adding the interest back into the account (compound interest) or taking the interest as cash or another investment (simple interest)

EXAMPLE 5

Mahmoud invests $7000 into a term deposit account for 3 years where it earns 5% ..

a How much interest does he earn over the 3 years if the interest is paid into the term deposit account at the end of each year?

b What is the future value of the investment?

Solution

a Interest is $5\% = 005$

The interest is added to the principal each time it is paid

$$\begin{aligned}\text{Amount after 1 year} &= \$7000 + 005 \times \$7000\\ &= \$7000(1 + 005)\\ &= \$7000(105)\\ &= \$7350\end{aligned}$$

ISBN 97807043220

$$\begin{aligned}\text{Amount after 2 years} &= \$7350 + 005 \quad \times \$7350\\ &= \$7350(105)\\ &= \$771750\end{aligned}$$

$$\begin{aligned}\text{Amount after 3 years} &= \$771750 \ + 005 \ \times \$771750\\ &= \$771750(105)\\ &\approx \$810338\end{aligned}$$

$$\begin{aligned}\text{Amount of interest} &= \$810338 \ - \$7000\\ &= \$110338\end{aligned}$$

b The future value is \$810338

We also use compound interest to calculate the future value of other investment.

EXAMPLE 6

Rachel and Wade buy a house in Sydney for \$1 250 00. House prices in that area go up by an average of 115% pa What is their house worth after 2 years ?

Solution

Interest is 115% = 0.115

$$\begin{aligned}\text{Amount after 1 year} &= \$1\ 250\ 000 + 0.115 \times \$1\ 250\ 000\\ &= \$1\ 250\ 000(1 + 0115)\\ &= \$1\ 250\ 000(1115)\\ &= \$1\ 393\ 750\end{aligned}$$

$$\begin{aligned}\text{Amount after 2 years} &= \$1\ 393\ 750(1115)\\ &= \$1\ 554\ 03125\end{aligned}$$

So value after 2 years is \$1 554 03125

Compound interest tables simplify calculations The values in the table are called **future value interest factors** as they give the future values of an investment of \$1 at a certain interest rate and time

Future value interest factors on $1							
Periods	**1%**	**2%**	**5%**	**8%**	**10%**	**15%**	**20%**
1	1.0100	1.0200	1.0500	1.0800	1.1000	1.1500	12000
2	1.0201	1.0404	1.1025	1.1664	1.2100	1.3225	14400
3	1.0303	1.0612	1.1576	1.2597	1.3310	1.5209	17280
4	1.0406	1.0824	1.2155	1.3605	1.4641	1.7490	20736
5	1.0510	1.1041	1.2763	1.4693	1.6105	2.0114	24883
6	1.0615	1.1262	1.3401	1.5869	1.7716	2.3131	29860
7	1.0721	1.1487	1.4071	1.7138	1.9487	2.6600	35832
8	1.0829	1.1717	1.4775	1.8509	2.1436	3.0590	42998
9	1.0937	1.1951	1.5513	1.9990	2.3579	3.5179	51598
10	1.1046	1.2190	1.6289	2.1589	2.5937	4.0456	61917
11	1.1157	1.2434	1.7103	2.3316	2.8531	4.6524	74301
12	1.1268	1.2682	1.7959	2.5182	3.1384	5.3503	89161
13	1.1381	1.2936	1.8856	2.7196	3.4523	6.1528	106993
14	1.1495	1.3195	1.9799	2.9372	3.7975	7.0757	128392
15	1.1610	1.3459	2.0789	3.1722	4.1772	8.1371	154070
16	1.1726	1.3728	2.1829	3.4259	4.5950	9.3576	184884
17	1.1843	1.4002	2.2920	3.7000	5.0545	10.7613	221861
18	1.1961	1.4282	2.4066	3.9960	5.5599	12.3755	266233

To see how the table work, we can use the example of Mahmod's term depoit.

EXAMPLE 7

Mahmoud invests $7000 into a term deposit account for 3 years where it earns 5% pa
Use the table to find the future value of the investment

Solution

From the table

For 3 years $n = 3$ and interest is 5%

Finding the column for 3 years at 5% gives 11576

11576 is the future value on $1

So future value on $7000 = $7000 × 11576

= $810320

ISBN 97807043220

Notice that the table gives a slightly different answer from Example 6 This is because the future value interest factors are rounded to 4 decimal places So using a table is quicker but not as accurate

We can use the table for investments of less than a yea. The value of n stands for time periods not year.

EXAMPLE 8

Stephanie invests \$2000 into a term deposit account for 5 months where it earns 12% pa paid monthly. Use the table to find the future value of the investmet.

Solution

Interest is 12% pa

pa or per annum means each year

So interest per month = 12% ÷ 12 = 1%

The value across from $n = 5$ months in the 1% column is 10510

Future value on \$2000 = \$2000 × 10510

= \$2102

Present value of investments (PV)

Sometimes you want to know how much you would need to invest now to end up with a certain amount in the future For exampl, you may be saving up for a holiday or a deposit for a houe. This value you need to invest now to achieve a future value is called the **present value**

EXAMPLE 9

Geordie wants to invest enough money now so that he will have \$5000 in 4 years time to buy a car. Use the table of future value interest factors to calculate how much present value he would need to invest if the interest rate is 5% pa

Solution

a We use $n = 4$ and 5% We know FV = 5000 and we want to find the present value

Let PV = x

From the table the value across from $n = 4$ in the 5% column is 12155

This is the future value on \$1

So $FV = x \times 1.2155$ or $1.2155x$

But $FV = 5000$

So $12155\ x = 5000$

$$x = \frac{5000}{12155}$$
$$= 411353$$

So Geordie needs to invest a present value of $411353 to have $5000 in 4 years time

Exercise 9.03 Compound interest

1 Calculate the future value if $6500 is invested for

- **a** 2 years at 3% pa
- **b** 3 years at 25% pa
- **c** 4 years at 41% pa
- **d** 3 years at 18% pa
- **e** 2 years at 53% pa

2 Calculate the future value of each investment

- **a** $2500 for 3 years at 45% pa
- **b** $10 000 for 4 years at 62% pa
- **c** $3400 for 5 years at 35% pa
- **d** $5000 for 3 years at 6% pa
- **e** $80 000 for 2 years at 45% pa

3 Christian and Kate buy a house for $750 000 What is the house worth after 3 years if its value increases by 6% pa ?

4 Aparna bought a diamond ring for $3000 How much was it worth 3 years later if it appreciated by 58% pa ?

5 A painting bought for $15 000 appreciates by 9% pa What is its future value after 4 years?

6 The present value of a necklace is $950 What is its future value after 4 years if it appreciates at 3% pa ?

7 Hien deposits $4500 into a tour fund where it earns interest of 29% pa What will be the future value of the tour fund after 3 years?

8 Use the table of future value interest factors on page 536 to calculate the future value of each investment

- **a** $800 for 7 years at 5% pa
- **b** $2000 for 10 years at 1% pa
- **c** $5000 for 6 years at 20% pa
- **d** $60 000 for 5 years at 10% pa
- **e** $100 000 for 8 years at 15% pa
- **f** $67325 for 6 years at 5% pa
- **g** $124953 for 4 years at 1% pa

ISBN 97807043220

h $3000 for 3 months at 12% pa paid monthly

i $1000 for 6 months at 12% pa paid monthly

j $3500 for 10 months at 12% pa paid monthly

9 Use the table of future value interest factors on page 536 to calculate the present value if the future value is $10 000 after

a 7 years at 2% pa

b 5 years at 15% pa

c 10 years at 8% pa

d 3 years at 1% pa

e 4 years at 5% pa

9.04 Compound interest formula

WS Compound inee

The calculations on compound interest follow a pattern called a **recurrence relation**

EXAMPLE 10

Patrick invests $2000 at the beginning of the year at 6% pa Find a formula for the amount in the bank at the end of *n* years

Solution

6% = 006

Amount after 1 year

A = \$2000 + 006 of \$2000

= \$2000(1 + 006)

= \$2000(106)

Amount after 2 years

$A_2 = A + 006\,A$

= [\$2000(106)] + 006 × [\$2000(106)]

= [\$2000(106)](1 + 006)

= [\$2000(106)](106)

= \$2000(106)^2

Amount after 3 years

$A_3 = A_2 + 006\,A_2$

$= [\$2000(106)^{2}] + 006 \times [\$2000(106)^{2}]$

$= [\$2000(106)^{2}](1 + 006)$

$= [\$2000(106)^{2}](106)$

$= \$2000(106)^{3}$

The recurrence relation is $A_{n+1} = A_n + 006\,A_n$

$A \quad A_2 \; A_3$. is a geometric sequence with $a = 2000(106)$ and $r = 106$

$T_n = ar^{n-1}$

$= 2000(106)(106)^{n-1}$

$= 2000(106)^{n}$

So the amount after n years is $\$2000(106)^{n}$

Compound interest

$A = P(1 + r)^n$

where P = principal (present value)

r = interest rate per period as a decimal

n = number of periods

A = future value

EXAMPLE 11

Find the amount that will be in the bank after 6 years if \$2000 is invested at 12% pa with interest paid

a yearly **b** quarterly **c** monthly

Solution

$P = 2000$

a $r = 12\% = 012, \; n = 6$

$A = P(1 + r)^n$

$= 2000(1 + 012)^{6}$

$= 2000(112)^{6}$

$= 394765$

So the amount is \$394765

ISBN 97807043220

b For quarterly interest the annual interest rate is divided by .

$r = 012 \div 4 = 003$

Interest is paid 4 times a year.

$n = 6 \times 4 = 24$

$A = P(1 + r)^n$

$= 2000(1 + 003)^{24}$

$= 2000(103)^{24}$

$= 406559$

So the amount is \$406559

c For monthly interest the annual interest rate is divided by 1.

$r = 012 \div 12 = 001$

Interest is paid 12 times a year.

$n = 6 \times 12 = 72$

$A = P(1 + r)^n$

$= 2000(1 + 001)^{72}$

$= 2000(101)^{72}$

$= 409420$

So the amount is \$409420

We can find the present value using the compound interest formul.

EXAMPLE 12

Geoff wants to invest enough money to pay for a \$10 000 holiday in 7 years time
If interest is 25% pa what present value does Geoff need to invest now ?

Solution

$A = 10\,000$, $r = 25\%$ or 0025, $n = 7$

We want to find the present value P

$A = P(1 + r)^n$

$10\,000 = P(1 + 0025)^7$

$= P(1025)^7$

$\frac{10000}{1025^7} = P$

$P \approx 841265$

The present value to invest is \$841265

ISBN 97807043220

We can use the compound interest formula to find the interest rate or time period by rearranging the formula

EXAMPLE 13

a Silvana invested \$1800 at 6% pa interest and it grew to \$272.6.
For how many years was the money invested if interest was paid twice a year?

b Find the interest rate if a \$1500 investment is worth \$173891 after 5 years

Solution

a $P = 1800$ and $A = 272266$

Interest is paid twice a year

$r = 006 \div 2 = 003$

$$\begin{aligned} A &= P(1+r)^n \\ 272266 &= 1800(1+003)^n \\ &= 1800(103)^n \\ \frac{272266}{1800} &= 103^n \\ 151259 &= 103^n \\ \log(.51259) &= \log(103)^n \\ &= n\log(103) \\ \frac{\log(151259)}{\log(103)} &= n \\ 14 &\approx n \end{aligned}$$

Since interest is paid in twice a year, the number of years will be $14 \div 2 = 7$.

So the money was invested for 7 years

b $P = 1500, A = 173891, \ n = 5$

$$\begin{aligned} A &= P(1+r)^n \\ 173891 &= 1500(1+r)^5 \\ \frac{173891}{1500} &= (1+r)^5 \\ 115927 &= (1+r)^5 \end{aligned}$$

$$\begin{aligned} \sqrt[5]{115927} &= 1+r \\ 103 &= 1+r \\ 003 &= r \\ r &= 3\% \end{aligned}$$

So the interest rate is 3% pa

ISBN 97807043220

Exercise 9.04 Compound interest formula

1 Find the amount of money in the bank after 10 years if

a \$500 is invested at 4% pa **b** \$7500 is invested at 7% pa

c \$8000 is invested at 8% pa **d** \$5000 is invested at 65% pa

e \$2500 is invested at 78% pa

2 Sam banks \$1500 where it earns interest at the rate of 6% pa Find the amount after 5 years if interest is paid

a annually **b** twice a year **c** quarterly

3 Chantelle banks \$3000 in an account that earns 5% pa Find the amount in the bank after 10 years if interest is paid

a quarterly **b** monthly

4 Reza put \$350 in the bank where it earns interest of 8% pa Find the amount there will be in the account after 2 years if interest is paid

a annually **b** monthly

5 How much money will there be in an investment account after 3 years if the present value is \$850 and interest of 45% pa is pai:

a twice a year? **b** quarterly?

6 Find the amount of money there will be in a bank after 8 years if \$1000 earns interest of 7% pa with interest pai:

a twice a year **b** quarterly **c** monthly

7 Tanya left \$2500 in a credit union account for 4 year, with interest of5.5%pa. paid yearly.

a How much money did she have in the account at the end of that time?

b What would be the difference in the future value if interest was paid quarterly?

8 **a** Find the amount of money there will be after 15 years if Hannah banks \$6000 and it earns 9% pa interes, paid quartery.

b How much more money will Hannah have than if interest was paid annually?

9 How much money will be in a bank account after 5 years if \$500 earns 65% pa with interest paid monthly?

10 Find the amount of interest earned over 4 years if \$1400 earns 6% pa paid quarterl.

11 How much money will be in a credit union account after 8 years if \$8000 earns 75% pa interest paid monthly ?

12 Elva wins a lottery and invests \$500 000 in an account that earns 8% pa with interest paid monthly. How much will be in the account after 12 years?

ISBN 97807043220

13 Calculate the principal invested for 4 years at 5% pa to achieve a future value o:

a \$5000 **b** \$675 **c** \$12 000 **d** \$28950 **e** \$12 800

14 What present value is required to accumulate to \$5400 in 3 years with interest of 58% pa paid quarterly ?

15 How many years ago was an investment made if \$5000 was invested at 6% pa paid monthly and it is now worth \$635245 ?

16 Find the number of years that \$10 000 was invested at 8% pa with interest paid twice a year if there is now \$18 72981 in the bank

17 Jude invested \$4500 five years ago at x% pa Evaluate x if the amount in his bank account is now

a \$631148 **b** \$574327 **c** \$661198 **d** \$616539 **e** \$676646

18 Hamish is given the choice of a bank account in which interest is paid annually or quarterly. If he deposits \$120, find the difference in the amount of interest paid over 3 years if interest is 7% pa

19 Kate has \$4000 in a bank account that pays 5% pa with interest paid annuall, and Rachel has \$4000 in a different account paying 4% quarterly. Which person will receive more interest over 5 years and by how much ?

20 A bank offers investment account A at 8% .. with interest paid twice a year and account B with interest paid at 6% pa at monthly interval. If Georgia invests \$5000 over 6 years which account pays more interest ? How much more does it pay?

21 A hairdresser earns \$36 400 for the first year of work His salary increases each year by 2.

a What is his salary in his
  i 5th year of work? **ii** 8th year of work?

b When will his salary reach \$60 000?

22 Yuron earns \$120 000 in his 1st yea, then his salary goes up b 3.5% each year after tat.

a How much does Yuron earn in hs:
  i 3rd year? **ii** 12th year? **iii** 20th year?

b What is the first year in which Yuron earns over \$300 000?

23 Masae invests \$5000 in a bank account

a How many years at 2% pa interest will it take for her investment to grow t:
  i \$5410? **ii** \$7000?

b At what interest rate would the investment grow to \$6000 after
  i 6 years? **ii** 10 years?

ISBN 97807043220

24 **a** Use the table of future value interest factors on page 536 to find the interest factor for an investment on $1 over 9 years at 8% pa

b Prove this interest factor is correct by using the compound interest formula

25 Show that the future value interest factor of 15209 is true for an investment over 3 years at 15% pa

9.05 Annuities

A better way to build up money faster is to make regular contributions to an investment This is called an **annuity** a name that comes from the same Latin word'anns' as annul, meaning yealy. However contributions to an annuity could be made more frequently than this For exampl, payments into **superannuation** can be made every week or fortnight when an employee is paid

WS Invemen and annuiies

WS Annuiy able poblem

WS Annuiy: peadhee

Future value of an annuity

EXAMPLE 14

Stevies grandparents put $100 into a bank account for her on her first birthda. They deposit $100 into the account on each birthday until Stevie is 18 and give her the total amount for her 18th birthday.

How much is in the account at the end of 3 years if interest is 2% pa ?

Solution

Amount at the end of the 1st year

Since $100 is deposited at the end of the 1st year on Stevies 1st birthda, it earns no interest in that year.

A = $100

Amount at the end of the 2nd year

A_2 = $100 + 002 × $100

= $100(1 + 002)

= $100(102)

= $102

But *another* $100 is deposited at the end of the 2nd year on Stevies 2nd birthda.

So amount = $102 + $100

= $202

Amount at the end of the 3rd year

A_3 = $202(102)

= $20604

But *another* $100 is deposited at the end of the 3rd year on Stevies 3rd birthda.

So amount = $20604 + $100

= $30604

So after 3 years the annuity is worth $30604

This is the $300 put in by Stevies grandparents plus interest of $604

WS

Fuue value
able o
annuiie

ISBN 97807043220

It is assumed that annuity payments are made at the *end* of each period unless stated otherwis. If they are made at the *beginning* of each period then the calculations would be differen.

You can use the table on the previous page to calculate annuitie. You can also download a copy from NelsonNet

EXAMPLE 15

Stevies grandparents put \$100 into a bank account for her on each birthda, with the final deposit on her 18th birthday. They give Stevie the total amount of the money for her 18th birthay.

a Use the table of future value of annuities factors to calculate how much is in the account after 3 years if interest is 2% pa

b How much will Stevie receive on her 18th birthday?

Solution

a From the table

The value across from $n = 3$ years in the 2% column is 30604

This is the future value on \$1

Future value on \$100 = \$100 × 30604

= \$30604

b The value across from $n = 18$ years in the 2% column is 214123

Future value on \$100 = \$100 × 214123

= \$214123

So Stevie will receive \$214123 on her 18th birthday.

We can use the table of future values of an annuity to calculate how much to contribute regularly to achieve a particular future value

EXAMPLE 16

a Christopher wants to save a certain amount at the end of each year for 5 years until he has \$20 000 to buy a car. If the interest rate is 3%p., find the amount of each annual contribution Christopher needs to make

b Alexis wants to save up a \$50 000 deposit for a home over 7 years She wants to make contributions at the end of each quarter. Interest is 8%p., paid quartely. What size contribution would she make?

Solution

a From the future value for annuities table on the previous page the value across from $n = 5$ in the 3% column is 53091

If we call the contribution x

Future value = $x \times 53091$

So $53091\,x = 20\ 000$

$$x = \frac{20\,000}{53091}$$

$$= 376712$$

So each contribution is \$376712

b The contribution is quarterly, or 4 times a yer.

Interest rate = 8% ÷ 4 = 2%

Alexis makes 4 contributions each year for 7 years

Number of periods = $7 \times 4 = 28$

From the table the value across from $n = 28$ in the 2% column is 370512

If we call the contribution x

Future value = $x \times 370512$

So $370512\,x = 50\ 000$

$$x = \frac{50\,000}{370512}$$

$$= 134948$$

So each contribution is \$134948

Annuities with regular withdrawals

Another type of annuity is a sum of money earning compound interest that has regular withdrawals or payouts coming out of it

EXAMPLE 17

Yasmin retires with a lump sum superannuation payment of \$145 00. She puts the money into a financial management company that guarantees 12% pa on her annuit, with interest paid monthly. Yasmin withdraws \$1800 at the end of each month as a penion.

Find what Yasmn's annuity is worth after 3 monhs.

Solution

Monthly interest = 12% ÷ 12 = 1%

Amount at the end of 1st month = \$145 000(1 + 001)

= \$145 000(101)

= \$146 450

ISBN 97807043220

But Yasmin withdraws $1800

So amount = $146 450 – $1800

= $144 650

Amount at the end of 2nd month = $144 650(101)

= $146 09650

But Yasmin withdraws $1800

So amount = $146 09650 – $1800

= $144 29650

Amount at the end of 3rd month = $144 29650(101)

= $145 73947

But Yasmin withdraws $1800

So amount = $144 73947 – $1800

= $143 93947

So after 3 months Yasmn's annuity is worth $143 3947.

Notice that Yasmn's annuity is gradually decreasng. If she took a little less money out each month she could keep the value of her annuity at around $145 000 or increase its value a little Try doing the above example with different values to see if Yasmin could draw a pension while keeping her lump sum the same

TECHNOLOGY

Annuities and spreadsheets

1 The formula for the future value of an annuity is $FV = a\left[\frac{(1+r)^n - 1}{r}\right]$ where

a = regular contribution r = interest rate and n = number of periods

You can use this formula to draw up a spreadsheet for future values of an annuit.

Does this formula look familiar? It comes from the sum of a geometric series

We can use this formula to write a table of future values in a spreadshee.

Using rows 1 and 2 for headings we can put 1 in A3 and the formula =**1+A3** in A. Drag this formula down the column for the periods 1 ,3, …

We will use $a = 1$ and $r = 005$ (5% interest)

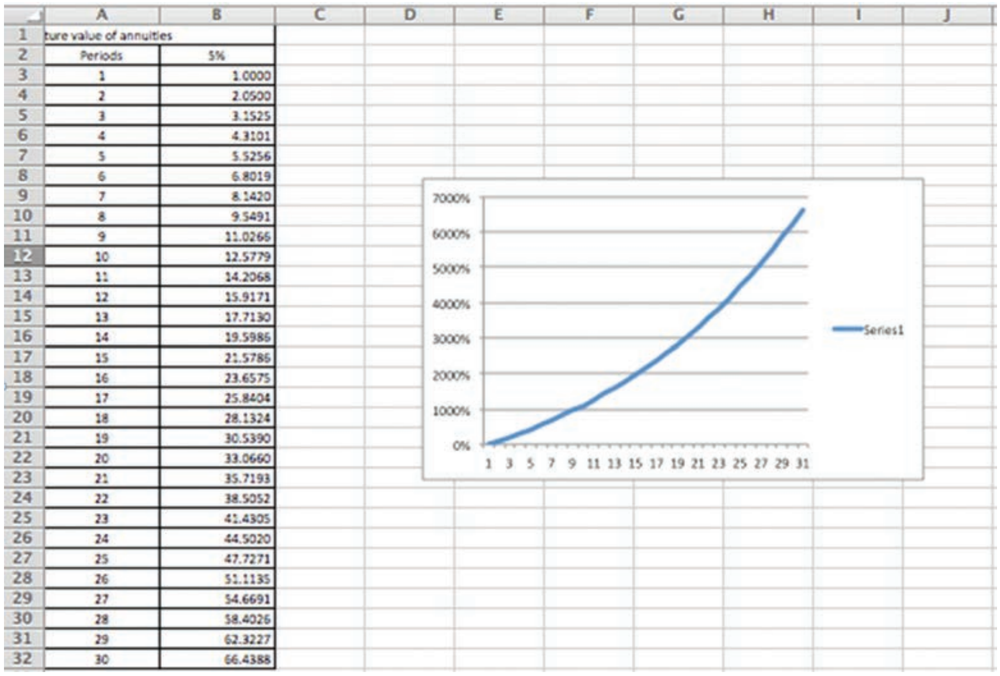
In B3 put the formula =**((1+005)^A3–1)/005** and drag it down the column

This gives a set of values for the future value of an annuity at 5% pa

Now highlight the column of future values and select the line graph from Charts

Can you find future values from the graph?

Change the formula to a different interest rate For exampl, use0.08 instead of0.05 in the formula and drag it down the column How does this change the graph ? Try other interest rate changes and look at how the graph changes



	A	B
1	ture value of annuities	
2	Periods	5%
3	1	1.0000
4	2	2.0500
5	3	3.1525
6	4	4.3101
7	5	5.5256
8	6	6.8019
9	7	8.1420
10	8	9.5491
11	9	11.0266
12	10	12.5779
13	11	14.2068
14	12	15.9171
15	13	17.7130
16	14	19.5986
17	15	21.5786
18	16	23.6575
19	17	25.8404
20	18	28.1324
21	19	30.5390
22	20	33.0660
23	21	35.7193
24	22	38.5052
25	23	41.4305
26	24	44.5020
27	25	47.7271
28	26	51.1135
29	27	54.6691
30	28	58.4026
31	29	62.3227
32	30	66.4388

2 Use the formula $PV = a\left[\dfrac{(1+r)^n - 1}{r(1+r)^n}\right]$ for the present value of an annuity to draw up a spreadsheet and graph for present value interest factors using similar steps

For example for 5% interes, in B3 use the formula **=((105)^A3–1)/(005*(105)^A3)** for the value of 1 in A, then drag the formula down the column

How does the graph change if you change the interest rate?

Exercise 9.05 Annuities

1 Calculate the future value of an annuity with a yearly contribution (at the end of each year) of

a \$5000 for 2 years at 25% pa

b \$1200 for 3 years at 4% pa

c \$875 for 3 years at 36% pa

d \$10 000 for 2 years at 41% pa

e \$2000 for 3 years at 32% pa

ISBN 97807043220

Use the future value table for annuities on page 546 to answer Questions 2 to 7

2 Find the future value of an annuity with annual contributions of

a	\$6300 for 7 years at 4% pa	**b**	\$980 for 5 years at 6% pa
c	\$7500 for 10 years at 5% pa	**d**	\$49575 for 4 years at 3% pa
e	\$20 500 for 12 years at 2% pa	**f**	\$64712 for 6 years at 1% pa
g	\$800 for 15 years at 7% pa	**h**	\$598 for 14 years at 9% pa
i	\$15 000 for 11 years at 13% pa	**j**	\$160 000 for 8 years at 4% pa

3 Find the future value of an annuity with contributions of

a \$400 a month for 2 years at 12% pa paid monthly

b \$940 a quarter for 5 years at 8% pa paid quarterly

c \$2500 twice a year for 8 years at 14% pa paid every 6 months

d \$550 three times a year for 5 years at 6% pa paid every 4 months

e \$587 a month for 18 months at 12% pa paid monthly

4 At the end of each year, Alicia puts \$3500 into a superannuation fund where it earns 9% pa How much will she have in superannuation after 30 years ?

5 The future value of an annuity is \$35 000 after 12 years If interest is 6% ., find the amount of each yearly contribution

6 Find the amount of each annual contribution needed to give a future value of

a \$8450 after 5 years at 7% pa

b \$25 000 after 8 years at 3% pa

c \$10 000 after 7 years at 4% pa

d \$3200 after 5 years at 2% pa

e \$1 000 000 after 20 years at 5% pa

7 Emlynn wants to put aside a regular amount of money each month for 2 years at 12% pa paid monthl, so she will have \$8000 to pay for a film-making couse. How much will she need to contribute?

8 Ilona wins \$50 000 in a lottery and invests it in a holiday fund annuity where she withdraws \$5000 at the end of each year. The annuity pays interest of 4 .a.

a What is the value of her annuity after

i 1 year? **ii** 2 years? **iii** 3 years?

b What will the annuity be worth after 3 years if Ilona decides to withdraw \$4000 each year instead?

c What will the annuity be worth after 3 years if the interest is 27% and Ilona withdraws \$4000 each year?

ISBN 97807043220

9 Dave puts his $125 000 superannuation payout into an annuity and takes out a pension of $500 a month The annuity pays interest of 12%p., paid montly.

What is the value of the annuity after

a 1 month? **b** 2 months? **c** 3 months?

e 3 months if interest is 6% pa ?

d 3 months if Dave decides to take a pension of $1000 each month?

10 Graph A shows an annuity of $15 000 earning 1% interest per mont, with a regular withdrawal of $500 per month Graph B shows the same $15 000 annuity paying interest of 2% per month with a regular withdrawal of $500 each month

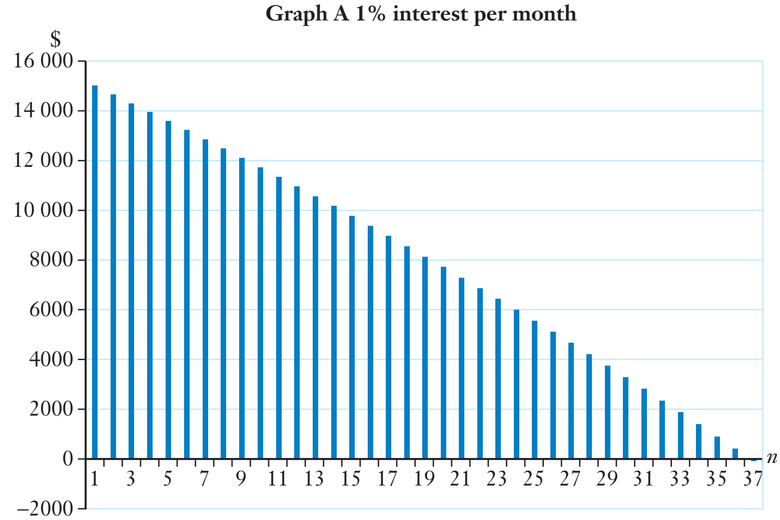
For each graph determin:

a after how many months the annuity will be worth $8000

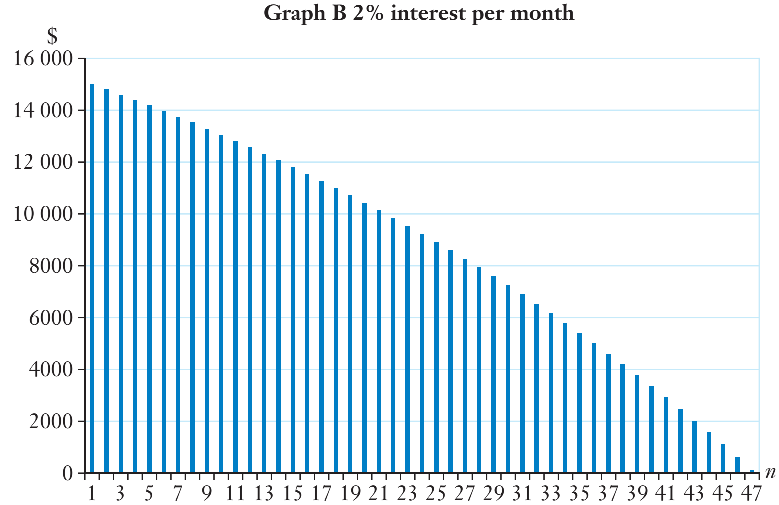
b what the annuity will be worth after a year

c how long it will take for the annuity to run out

Graph A 1% interest per month



Graph B 2% interest per month



ISBN 97807043220

Use the future value table on p 546 to answer Questions 11 and 1.

11 Ryuji pays \$2000 into a superannuation fund at the end of each year at 6% .. interet. How many payments would Ryuji make for his superannuation to be greater tha:

a \$50 000? **b** \$80 000?

12 Find the interest rate if \$1500 is invested at the end of each year grows to

a \$11 28495 after 6 years **b** \$17 19585 after 10 years

9.06 Annuities and geometric series

WS

Annuiies and geomeic eie

Annuiie

Annuiie

EXAMPLE 18

A sum of \$1500 is invested at the end of each year in a superannuation fund If interest is aid at 6% p.a., how much money will be available at the end of 25 years ?

Solution

It is easier to keep track of each annual contribution separately.

Use $A = P(1 + r)^n$ with $P = 1500$ and $r = 006$

The 1st contribution goes in at the end of the 1st year, so it only earns interest for 24 yeas.

$A \ = 1500(1 + 006)^{24}$

$\quad = 1500(106)^{24}$

The 2nd contribution goes in at the end of the 2nd year, so it earns interest for 23 yeas.

$A_2 = 1500(106)^{23}$

Similarly, the 3rd contribution earns interest for 22 yeas.

$A_3 = 1500(106)^{22}$

This pattern continues until the final contribution

The 25th contribution goes in at the end of the 25th year, so it earns interest for 0 yeas.

$A_{25} = 1500(106)^{0}$

The future value is the total of all these contributions together with their interest

$\text{FV} = A \ + A_2 + A_3 + \quad + A_{25}$

$\quad = 1500(106)^{24} + 1500(106)^{23} + 1500(106)^{22} + \quad + 1500(106)^{0}$

$\quad = 1500(106)^{0} + 1500(106) \quad + 1500(106)^{2} + \quad + 1500(106)^{24}$

$\quad = 1500(106^{0} + 106 \quad + 106^{2} + \quad + 106^{24})$ (factorising)

$106^{0} + 106 \quad + 106^{2} + \quad + 106^{24}$ is a geometric series with $a = 106^{0} = 1$, $r = 106$ and $n = 25$

ISBN 97807043220

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$

$$S_{25} = \frac{1(106^{25} - 1)}{(106 - 1)}$$

$$\approx 5486$$

So FV $\approx 1500(5486)$

$= 82\ 29677$

So the total amount of superannuation after 25 years is \$82 29677

In the previous example the contributions were made at the *end* of each year. If they were made at the *beginning* of each year, they would all earn an extra year of interet. The 1st contribution would be invested for 25 years the 2nd for 24 year, and so on until the last contribution for 1 yer.

EXAMPLE 19

An amount of \$50 is put into an investment account at the end of each month If interest is paid at 12% pa paid monthl, how much is in the account at the end of 10 years ?

Solution

We use the compound interest formula where $P = 50$

$r = 012 \div 12 = 001, \; n = 10 \times 12 = 120$

The 1st contribution goes in at the end of the 1st month so it only earns interest for 119 month.

$A \;= 50(1 + 001)^{119}$

$\;= 50(101)^{119}$

The 2nd contribution goes in at the end of the 2nd month so it earns interest for 118 month.

$A_2 = 50(101)^{118}$

The 3rd contribution earns interest for 117 months

$A_3 = 50(101)^{117}$

This pattern continues until the final contribution

The 120th contribution earns interest for 0 months

$A_{120} = 50(101)^{0}$

$\text{FV} = A \; + A_2 + A_3 + \quad + A_{120}$

$= 50(101)^{119} + 50(101)^{118} + 50(101)^{117} + \quad + 50(101)^{0}$

$= 50(101)^{0} + 50(101) \; + 50(101)^{2} + \quad + 50(101)^{119}$

$= 50\left(101^{0} + 1.01 \; + 1.01^2 + \quad + 1.01^{119}\right)$ (factorising)

ISBN 97807043220

$101^{0} + 1.01 + 1.01^{2} + \quad + 1.01^{119}$ is a geometric series with $a = 1.01^{0}$ or 1, $r = 101$ and $n = 120$

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$

$$S_{120} = \frac{1(101^{120} - 1)}{(101 - 1)}$$

$$\approx 23004$$

So $\text{FV} \approx 50(23004)$

$= 11\ 501.93$

So the total amount after 10 years is \$11 50193

Exercise 9.06 Annuities and geometric series

1 A sum of \$1500 is invested at the end of each year for 15 years at 8% pa
Find the amount of superannuation available at the end of the 15 years

2 Liam wants to save up \$15 000 for a car in 5 years time He invests \$2000 at the end of each year in an account that pays 75% pa interes. How much more will Liam have to pay at the end of 5 years to make up the \$15 000?

3 A school invests \$5000 at the end of each year at 6% pa to go towards a new librar.
How much will the school have after 10 years?

4 Jacqueline puts aside \$500 at the end of each year for 5 years If the money is invested at 65% pa how much will she have at the end of the 5 years ?

5 Miguels mother invests \$200 for him each birthday up to and including his 18th birthday. The money earns 6 .a. How much money will Miguel have on his 18th birthday?

6 Xuan is saving up for a holiday. She invests \$800 at the end of each year at7.5%pa.
How much will she have for her holiday after 5 years time ?

7 A couple saves \$3000 at the *beginning* of each year towards a deposit on a house
If the interest rate is 5% pa how much will the couple have saved after 6 years ?

8 Lucia saves up \$2000 each year and at the end of the year she invests it at 6% pa

a She does this for 10 years What is her investment worth ?

b Lucia continues investing \$2000 a year for 5 more years What is the future value of her investment?

9 Jodie starts work in 2019 and puts $1000 in a superannuation fund at the end of the year. She keeps putting in this same amount at the end of every year until she retires at the end of 2036 If interest is paid at 10% ., calculate how much Jodie will have when she retires

10 Bol invests $1000 at the *beginning* of each year. The interest rate is 8 .a.

a How much will her investment be worth after 6 years?

b How much more would Bols investment be worth after 6 years if she had invested $1200 each year?

11 Asam cannot decide whether to invest $1000 at the end of each year for 15 years or $500 for 30 years in a superannuation fund If the interest rate is 5% ., which would be the better investment for Asam?

12 Pooja is saving up to go overseas in 8 years time She invests $1000 at the end of each year at 7% pa and estimates that the trip will cost her around $10 00. Will she have enough? If so how much over will it be ? If she doesnt have enoug, how much will she need to add to this money to make it up to the $10 000?

13 Mila puts aside $20 at the *beginning* of each month for 3 years How much will she have then if the investment earns 82% pa paid monthly ?

14 **a** Find the future value on an investment of $1 at the end of each year for 19 years at 7% pa using the table of future values of an annuity on page 54.

b Prove that this table value is correct

9.07 Reducing balance loans

Reducing balance loas

Reducing balance loan: peadhee

Reducible loan

oan epaymen able

People take out loans for many reasons – to buy items such as a car, boat or furnitue, to consolidate debts to buy a hom, and for home renovatios. A home loan is called a **mortgage**

An investment or annuity increases in value over time while a loan decreases as the loan is paid off This is called a **reducing balance loan** The amount of time taken to pay off the loan is called the term of the loan A reducing balance loan is similar to an annuity with regular withdrawas.

EXAMPLE 20

Trang borrows $8000 over 3 years to buy furnitur. Interest on the loan is1.25% per month and monthly repayments are $27732 Find the amount owing afte:

a 1 month **b** 2 months **c** 3 months

Solution

Use $A = P(1 + r)^n$ with $P = 8000$ and $r = 125\% \ = 00125$

Let A_n be the amount owing after n months

ISBN 97807043220

a 1st month Amount owing is \$8000 plus interest less the repaymen.

$A \;= 8000(1 + 00125) \quad - 27732$

$\quad = 8000(10125) \; - 27732$

$\quad = 782268$

Amount owing = \$782268

b 2nd month Amount owing is \$782.68 plus interest less the repaymen.

$A_2 = 782268(10125) \quad - 27732$

$\quad = 764314$

Amount owing = \$764314

c 3rd month Amount owing is \$764.14 plus interest less the repaymen.

$A_3 = 764314(10125) \quad - 27732$

$\quad = 746136$

Amount owing = \$746136

If you know the term of the loan and the amount of the regular contributions you can calculate the amount of interest owing

You can use a loan repayments table to calculate the amount you need to contribute to pay off a loan Here is a table that gives the monthly loan repayments on a \$1000 loa.

	Term (years)					
Interest rate (% ..)	**5**	**10**	**15**	**20**	**25**	**30**
2	\$17.53	\$9.20	\$6.44	\$5.06	\$4.24	\$3.70
25	\$17.75	\$9.43	\$6.67	\$5.30	\$4.49	\$3.95
3	\$17.97	\$9.66	\$6.91	\$5.55	\$4.74	\$4.22
35	\$18.19	\$9.89	\$7.15	\$5.80	\$5.01	\$4.49
4	\$18.42	\$10.12	\$7.40	\$6.06	\$5.28	\$4.77
45	\$18.64	\$10.36	\$7.65	\$6.33	\$5.56	\$5.07
5	\$18.87	\$10.61	\$7.91	\$6.60	\$5.85	\$5.37
55	\$19.10	\$10.85	\$8.17	\$6.88	\$6.14	\$5.68
6	\$19.33	\$11.10	\$8.44	\$7.16	\$6.44	\$6.00
65	\$19.57	\$11.35	\$8.71	\$7.46	\$6.75	\$6.32
7	\$19.80	\$11.61	\$8.99	\$7.75	\$7.07	\$6.65
75	\$20.04	\$11.87	\$9.27	\$8.06	\$7.39	\$6.99
8	\$20.28	\$12.13	\$9.56	\$8.36	\$7.72	\$7.34

ISBN 97807043220

EXAMPLE 21

a Piri wants to borrow \$350 000 over 30 years to buy a unit but she is not sure she can afford to pay the monthly repayments If interest is .5% per mont, calculae:

i the amount of each monthly repayment

ii the total amount Piri would pay

b Hamish borrows \$25 000 over 5 years to buy a car. Interest is 2% per monh. Fnd:

i the amount of each monthly repayment

ii the total amount Hamish pays

iii the flat rate of interest on the loan

Solution

a **i** From the table 30 years at .5% .. gives 5.7.

This is on a loan of \$1000 so for \$350 000 we multiply the value by 35.

$\$507 \times 350 = \$177450$

So Piri would pay \$177450 each month

ii 30 years = 30 × 12 = 360 months

$$\text{Total amount repaid} = \$177450 \times 360$$
$$= \$638\ 820$$

b **i** From the table 5 years at 2% .. gives \$7.3.

This is on a loan of \$1000 so for \$25 000 we multiply the value by 2.

$\$1753 \times 25 = \$43825$

So Hamish pays \$43825 each month

ii 5 years = 5 × 12 = 60 months

$$\text{Total amount repaid} = \$43825 \times 60$$
$$= \$26\ 295$$

iii $\text{Interest} = \$26\ 295 - \$25\ 000$

$= \$1295$

$$\frac{1295}{25000} \times 100\% = 518\%$$

So the flat rate of interest is 518%

ISBN 97807043220

You can use the table to do other calculation.

EXAMPLE 22

a The monthly repayments on a loan of \$70 000 at 5% pa are \$55.7. Find the term of the loan

b A \$150 000 loan with a term of 20 years has monthly instalments of \$1119 Find the interest rate

Solution

a Let the value in the table be x

The table is for loans of \$1000

$\$70\,000 \div \$1000 = 70$

$70 \times x = \$55370$

$$x = \frac{\$55370}{70}$$

$$\approx \$791$$

Looking at the table in the 5% row, \$791 is in the 15 year column

So the term of the loan is 15 years

b Let the value in the table be x

$\$150\,000 \div \$1000 = 150$

$150 \times x = \$1119$

$$x = \frac{\$1119}{150}$$

$$\approx \$746$$

Looking at the table in the 20 year column \$.46 is in the .5% ro.

So the interest rate is 65% pa

INVESTIGATION

FINANCIAL CALCULATORS

Most bank and other financial websites have calculators rather than tables for loan repayments values of investments and annuitie. Search the websites of banks or general websites that have these and try using these calculators

Exercise 9.07 Reducing balance loans

1 Calculate the amount owing after 3 months on a loan of

a \$20 000 at 09% per month with repayments of \$43287 per month

b \$3500 at 13% per month with repayments of \$15157 per month

c \$100 000 at 22% per month with repayments of \$220322 per month

d \$2000 at 2% per month with repayments of \$10574 per month

e \$45 800 at 12% pa with repayments of \$50.30 per month

2 For each loan below, fid:

  i the total amount repaid
  ii total amount of interest paid
  iii the flat interest rate of interest pa

a $5000 over 3 years with a monthly payment of $16607
b $15 900 over 5 years with a monthly payment of $40376
c $80 000 over 12 years with a monthly payment of $110962
d $235 000 over 25 years with a monthly payment of $90709
e $1348 over 2 years with a monthly payment of $7127

Use the table of loan repayments on page 557 to answer the rest of the questions

3 Find the amount of the monthly repayment on a loan of

a $8 000 over 5 years at 6% pa
b $15 000 over 5 years at 8% pa
c $72 000 over 10 years at 75% pa
d $430 000 over 20 years at 4% pa
e $312 000 over 15 years at 55% pa
f $137 000 over 25 years at 35% pa
g $49 000 over 10 years at 7% pa
h $765 000 over 30 years at 25% pa
i $925 000 over 25 years at 2% pa
j $1 000 000 over 30 years at 55% pa

4 Markus takes out a mortgage of $680 500 over 20 years at 35% interest

a Find his monthly repayment
b Find the total amount he will pay.
c How much interest does he pay?
d Calculate the flat rate of interest over the whole loan

5 Find the term of each loan given the monthly payments of

a $8112 for a $4 000 loan at 8% pa
b $777 for a $75 000 loan at 45% pa
c $93795 for a $169 000 loan at 3% pa
d $156025 for a $395 000 loan at 25% pa
e $23220 for a $20 000 loan at 7% pa
f $313125 for a $625 000 loan at 35% pa
g $1302 for a $120 000 loan at 55% pa
h $180964 for a $281 000 loan at 2% pa
i $7215 for a $6 500 loan at 6% pa
j $47415 for a $81 750 loan at 35% pa

ISBN 97807043220

6 Find the interest rate of each loan if the monthly instalment is

- **a** \$5799 for a \$3000 loan for 5 years
- **b** \$61965 for an \$81 000 loan for 15 years
- **c** \$230736 for a \$456 000 loan for 20 years
- **d** \$257156 for a \$212 000 loan for 10 years
- **e** \$651536 for a \$947 000 loan for 20 years
- **f** \$17820 for a \$9000 loan for 5 years
- **g** \$270970 for a \$686 000 loan for 30 years
- **h** \$1422 for a \$300 000 loan for 25 years
- **i** \$681473 for an \$845 500 loan for 20 years
- **j** \$312724 for a \$422 600 loan for 15 years

9.08 Loans and geometric series

We can apply the formulas for compound interest and geometric series to work out the amount of the regular repayments of a reducing balance loan

WS

Loans and geomeic eie

WS

Applicaion o seies

EXAMPLE 23

Find the amount of each monthly repayment on a loan of \$20 000 at 12% pa over 4 year.

Solution

Let M stand for the monthly repayment

Number of payments is $4 \times 12 = 48$

Monthly interest is $12\% \div 12 = 1\% = 001$

Each month we add interest and subtract the repaymen.

Amount owing after 1 month

$A \;= 20\,000(1 + 001) \;\; - M$

$\;\;\; = 20\,000(101) \;\; - M$

Amount owing after 2 months

$A_2 = A\,(101) \;\; - M$

$\;\;\; = [20\,000(101) \;\; - M](101) \;\; - M$

$\;\;\; = 20\,000(101)^2 - M(101) \;\; - M$ (expanding brackets)

$\;\;\; = 20\,000(101)^2 - M(101 \;\; + 1)$ (factorising)

Amount owing after 3 months

$A_3 = A_2(101) \quad - M$

$\quad = [20\,000(101)^2 - M(101 \quad + 1)](101) \quad - M$

$\quad = 20\,000(101)^3 - M(101 \quad + 1)(101) \quad - M$ (expanding brackets)

$\quad = 20\,000(101)^3 - M(101^2 + 1.01\ ) - M$

$\quad = 20\,000(101)^3 - M(101^2 + 1.01 \ + 1)$ (factorising)

Continuing this pattern after 48 months the amount owing i:

$A_{48} = 20\,000(101)^{48} - M(101^{47} + 1.01^{46} + 1.01^{45} + \quad + 1.01^2 + 1.01 \ + 1)$

But the loan is paid out after 48 months

So $A_{48} = 0$

$0 = 20\,000(101)^{48} - M(101^{47} + 1.01^{46} + 1.01^{45} + \quad + 1.01^2 + 1.01 \ + 1)$

$M(101^{47} + 1.01^{46} + 1.01^{45} + \quad + 1.01^2 + 1.01 \ + 1) = 20\,000(101)^{48}$

$$M = \frac{20000(1\,0\,1)^{48}}{1.01^{47} + 1.01^{46} + 1.01^{45} + \quad + 1.01^2 + 1.01^1 + 1}$$

$$= \frac{20000(1\,0\,1)^{48}}{1 + 1\,0\,1^1 + 1\,0\,1^2 + 1.01^3 + \ldots + 1.01^{46} + 1.01^{47}}$$

The denominator is a geometric series with $a = 1$, $r = 101$ and $n = 48$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{48} = \frac{1(101^{48} - 1)}{101 - 1}$$

$$= \frac{101^{48} - 1}{001}$$

≈ 61223

$$M \approx \frac{20000(1\,0\,1)^{48}}{61223}$$

$= 52668$

So the monthly repayment is \$52668

We can use this method to find loan repayments for more complex question.

ISBN 97807043220

EXAMPLE 24

A store charges 9% pa for loan, and repayments do not have to be made until the 4th month Ivan buys \$8000 worth of furniture and pays it off over 3 year.

a How much does Ivan owe after 3 months?

b What are his monthly repayments?

c How much does Ivan pay altogether?

Solution

a Number of payments $= 3 \times 12 - 3 = 33$ (3 months of no repayments)

Monthly interest rate $= 009 \div 12 = 00075$

Let M stand for the monthly repayment

The first repayment is made in the 4th month

After 3 months the amount owing is

$$A = P(1 + r)^n$$

$$A_3 = 8000(1 + 00075)^3$$

$$= 8000(10075)^3$$

$$= 818135$$

So the amount owing after 3 months is \$818135

b Amount owing after 4 months

$$A_4 = A_3(10075) - M$$

$$= [8000(10075)^3](10075) - M$$

$$= 8000(10075)^4 - M$$

Amount owing after 5 months

$$A_5 = A_4(10075) - M$$

$$= [8000(10075)^4 - M](10075) - M$$

$$= 8000(10075)^5 - M(10075) - M$$

$$= 8000(10075)^5 - M(10075 + 1) \quad \text{(factorising)}$$

Continuing this pattern after 36 months the amount owing will b:

$$A_{36} = 8000(10075)^{36} - M(10075^{32} + 10075^{31} + 10075^{30} + \quad + 10075 + 1)$$

But the loan is paid out after 36 months

So $A_{36} = 0$

$$0 = 8000(100075)^{36} - M(10075^{32} + 10075^{31} + \quad + 10075 + 1)$$

$$M(10075^{32} + 10075^{31} + \quad + 10075 \quad + 1) = 8000(10075)^{36}$$

$$M = \frac{8000(1\ 0075)^{36}}{1.0075^{32} + 1.0075^{31} + \quad + 10075 \quad + 1}$$

$$= \frac{8000(1\ 0075)^{36}}{1 + 1\ 0075^{1} + .0075^{\ 2} + \quad + 10075^{\ 32}}$$

The denominator is a geometric series with $a = 1$, $r = 10075$ and $n = 33$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{33} = \frac{1(10075^{\ 33} - 1)}{10075 \ - 1}$$

$$= \frac{10075^{\ 33} - 1}{00075}$$

$$\approx 372849$$

$$M = \frac{8000(1\ 0075)^{36}}{372849}$$

$$\approx 28079$$

So the monthly repayment is $28079

c $28079 $\times$ 33 = $926607

So Ivan pays $926607 altogether.

Exercise 9.08 Loans and geometric series

1 An amount of $3000 is borrowed at 22% pa and paid off over 5 years with yearly repayments How much is each repayment ?

2 The sum of $20 000 is borrowed at 18% pa interest calculated monthly over 8 year. How much are the monthly repayments?

3 David borrows $5000 from the bank and pays back the loan in monthly instalments over 4 years If the loan incurs interest of 15% .. calculated monthy, find the amount of each instalment

4 Tri and Mai mortgage their house for $150 00.

- **a** Find the amount of the monthly repayments they will have to make if the mortgage is over 25 years with interest at 6% pa compounded monthl.
- **b** If they want to pay their mortgage out after 15 years what monthly repayments would they need to make?

5 A loan of $6000 is paid back in equal annual instalments over 3 years If the interest is 125% pa find the amount of each annual instalmen.

6 Santi buys a car for $38 000 paying a 10% deposit and taking out a loan for the balanc. If the loan is over 5 years with interest of 15% monthly, fid:

- **a** the amount of each monthly loan repayment
- **b** the total amount that Santi paid for the car.

ISBN 97807043220

7 A \$2000 loan is offered at 18% pa with interest charged monthl, over 3 yers.

a If no repayment need be paid for the first 2 months find the amount of each repayment

b How much will be paid back altogether?

8 Breanna thinks she can afford a mortgage payment of \$800 each month
How much can she borrow, to the nearest \$10, over 25 years at11.5 .a. ?

9 Get Rich Bank offers a mortgage at $7\frac{1}{2}$% pa over 10 years and Capital Bank offers a mortgage at $5\frac{1}{2}$% pa over 25 year, both with interest calculated monthy.

a Find the amount of the monthly repayments for each bank on a loan of \$80 000

b Find the difference in the total amount paid on each mortgage

10 Majed buys a \$35 000 car. He puts down a 5% deposit and pays the balance back in monthly instalments over 4 years at 12% pa

a Find the amount of the monthly payments

b Find the total amount that Majed pays for the car.

11 Amy borrowed money over 7 years at 155% pa and she pays \$1200 a mont.
How much did she borrow?

12 NSW Bank offers loans at 9% pa with no repayments for the first 3 month, while Sydney Bank offers loans at 7% pa Compare these loans on an amount of \$5000 over 3 years and state which bank offers the better loan and why.

13 Danny buys a home cinema system for \$10 000 He pays a \$1500 deposit and borrows the balance at 18% pa over 4 year.

a Find the amount of each monthly repayment

b How much did Danny pay altogether?

14 A store offers furniture on hire purchase at 20% pa over 5 year, with no repayments for 6 months Ali buys furniture worth \$12 00.

a How much does Ali owe after 6 months?

b What are the monthly repayments?

c How much does Ali pay for the furniture altogether?

15 A loan of \$6000 over 5 years at 15% pa interes, charged monthy, is paid back in 5 annual instalments

a What is the amount of each instalment?

b How much is paid back altogether?

16 **a** Using the table of loan repayments on page 557 find the amount of the monthly payments on a \$1000 loan over 10 years at 45% pa

b Show that this table value is correct

[redacted]

In Questions **1** to **3** select the correct answer **A B C** or **D**

WS

Fomula shet: Meauemen Sequence and eie

Qz

Pacice quiz

1 An amount of \$2500 invested at 6% pa for 5 years with interest paid twice a year has a future value of

A $\$2500(106)^{5}$ **B** $\$2500(103)^{5}$

C $\$2500(103)^{10}$ **D** $\$2500(106)^{10}$

2 An investment has a present value of \$68 000 and a future value of \$79 500 Find the flat interest rate on the investment

A 115% **B** 169%

C 855% **D** 145%

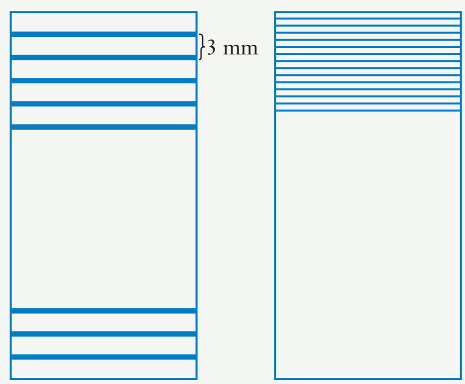
3 A tree is planted when it is 12 m tall Every year its growth is $\frac{3}{8}$ of its previous years height Find how tall the tree will gro.

A 192 m **B** 2 m **C** 25 m **D** 32 m

4 A loan of \$22 000 at 12% p . is paid off in monthly payments of \$26.9. Find the amount owing after 3 months

5 Zac puts \$1500 into a savings account that earns 37% pa How much will Zac have in the account after 3 years?

6 A bamboo blind has 30 slats It is attached to the window at the top and when the blind is down the gap between each slat and the nex, and between the top slat and the top of the window, is 3 m. When the blind i up, the slats have no gaps between them



a Show that when the blind is up the bottom slat rises 90 mm

b How far does the next slat rise?

c Explain briefly why the distances the slats rise when the blind is up form an arithmetic sequence

d Find the distance the 17th slat from the bottom rises

e What is the sum of the distances that all slats rise?

7 Cristina borrows \$62 500 over 5 years with monthly repayments of \$2022 Fin:

a the total amount Cristina pays

b the amount of interest she pays

c the flat interest rate on the total loan

ISBN 97807043220

8 Use the table of future values of an investment on page 536 to find the future value of

- **a** \$595 over 4 years at 5% pa
- **b** \$5 000 over 9 years at 8% pa
- **c** \$165120 over 6 years at 10% pa
- **d** \$13 500 over 11 years at 2% pa
- **e** \$9 485 over 18 years at 1% pa

9 Murat earned \$20 000 in one year. At the beginning of the 2nd year he received a salary increase of \$450 He now receives the same increase each yea.

- **a** What will his salary be after 10 years?
- **b** How much will Murat earn altogether over the 10 years?

10 Ana puts her superannuation payout of \$186 900 into an account that earns 3% pa paid monthly. She withdraws \$2500 at the end of each month as a pensin.
Find the amount in the account after

- **a** 1 month
- **b** 2 months
- **c** 3 months

11 Gerri wants to contribute a certain amount of money at the end of each year into a superannuation fund so that she will have \$200 000 at the end of 25 years If the fund averages 13% pa find the amount of the money Gerri would contribute each yea.

12 A supermarket stacks boxes with 20 boxes in the bottom stack 18 boxes in the next stac, 16 in the next and so on

- **a** How many stacks are there?
- **b** How many boxes are there?

13 Convert each recurring decimal to a fraction

- **a** $0\dot{4}$
- **b** $07\dot{2}$
- **c** $1\dot{5}\dot{7}$

14 Find the amount invested in a bank account at 95% pa if the balance in the account is \$586091 after 6 years

15 Haylee borrows \$50 000 for farm machinery at 18% pa over 5 years and makes equal yearly repayments on the loan at the end of each year.

- **a** How much does she owe at the end of the first year, just before she makes the first repayment?
- **b** How much is each yearly repayment?

16
- **a** If \$2000 is invested at 45% pa how much will it be worth after 4 years ?
- **b** If interest is paid quarterly, how much would the investment be worth after 4 years?

17 Pedro borrows \$200 000 to buy a house If the interest is 6% .. compounded monthly and the loan is over 20 years

- **a** how much is each monthly repayment?
- **b** how much does Pedro pay altogether?

ISBN 97807043220

18 **a** Find the annual contribution needed for an annuity to have a future value of $12 000 after 4 years at 5% pa if the contribution is made at the *beginning* of the year.

b Find the single investment that would need to be invested at the same interest rate now to have this future value

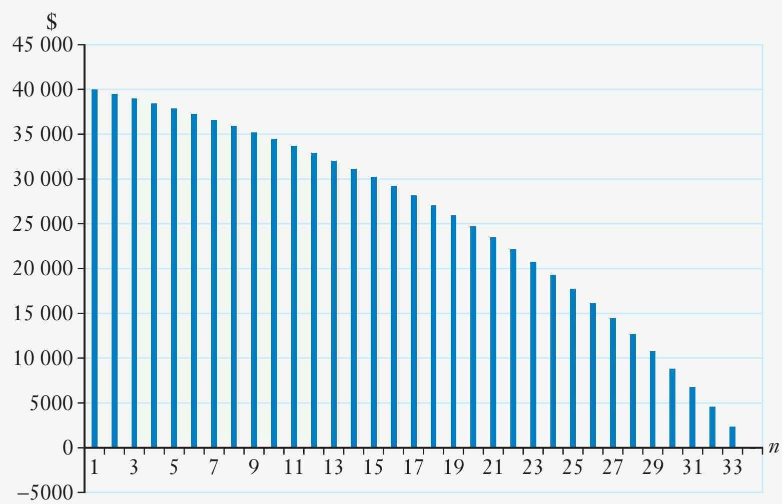
19 Every week during a typing course Jamal improves his typing speed by 3 words per minute until he reaches 60 words per minute by the end of the course

a If he can type 18 words per minute in the first week of the course how many words per minute can he type by week 8?

b How many weeks does the course run for?

20 A ball drops from a height of 12 metres then bounces back to $\frac{3}{5}$ of this height On the next bounce it bounces up to $\frac{3}{5}$ of this height and so on Through what distance will the ball travel ?

21 The table shows an annuity of $40 000 earning 5% pa with withdrawals of $2500 at the end of each year.



a What is the annuity worth after

i 10 years? **ii** 20 years?

b When is the annuity worth

i $30 000? **ii** $10 000?

c How long will it take for the annuity to run out?

ISBN 97807043220

9. CHALLENGE EXERCISE

1 Jane puts $300 into an account at the beginning of each year to pay for her daughters education in 5 years time If 6% .. interest is paid quartery, how much money will Jane have at the end of the 5 years?

2 A factory sells shoes at $60 a pair. For 10 pairs of shoes there is a discout, whereby each pair costs $58 For 20 pair, the cost is $56 a pair and so n. Fnd:

a the price of each pair of shoes on an order of 100 pairs of shoes

b the total price on an order of 60 pairs of shoes

3 Find the amount of money in a bank account if $5000 earns 85% pa for 4 year, then 65% pa for 3 year, with interest paid monthly for all 7 yeas.

4 A metal is heated to 500°C A minute later it cools to 425 °C then a minute later it cools down to 36125 °C If the metal continues to cool in the same wa, fnd:

a its temperature after

i 10 minutes **ii** 15 minutes

b how long it will take to cool down to

i 200° **ii** 100°

5 Lukas puts $1000 into a superannuation account at the beginning of each year where it earns 6% pa He retires and collects the superannuation at the end of 25 year.

a How much will the first $1000 be worth at the end of 25 years in index form ?

b When Lukas deposits the second $1000 at the end of the 2nd year, how much will it be worth after 25 years?

c How much will the third $1000 be worth after 25 years?

d How much will the final $1000 be worth that Lukas deposits at the beginning of the 25th year?

e Show that the total amount in the account after 25 years is $1000 \times \dfrac{106\,(106^{25} - 1)}{006}$

f Find the amount that Lukas will have at the end of the 25 years

6 Kim borrows $10 000 over 3 years at a rate of 1% interest compounded each month If she pays off the loan in three equal annual instalments fin:

a the amount Kim owes after one month

b the amount she owes after the first year, just before she pays the first instalment

c the amount of each instalment

d the total amount of interest Kim pays

7 A superannuation fund paid 6% pa for the first 10 years and then 10% .. after tht. If Thanh put $5000 into this fund at the beginning of each yea, how much would she have at the end of 25 years?

STATISTICAL ANALYSIS

10. CONTINUOUS PROBABILITY DISTRIBUTIONS

In ths chapter you wll expand the work you have done on dscrete probablty dstrbutons n Year 1. Youwill study cotinuous probblitydisribtios inclding the norma ditriuton.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- recognse contnuous random varables
- understand the properies of a probaiity denity funcion(PD)
- fnd cumulatve dstrbuton functons CDF
- fnd probabltes of contnuous data
- calculate measures of central tendency and spread for contnuous probablty dstrbutons
- recognse the normal istibuion andidenifyits properties
- calculate probabltes and quantles for normal istibuions
- understand the standard normal istibuion and *z*-scores
- apply the normal istibuion to soling pracical problems

TERMINOLOGY

continuous random variale: A random variable that can have any value along a continuum for example the height of a basketball playe.

cumulative distribution functin: A function $F(x)$ for the probability $P(X \le x)$

empirical rule The percentage probabilities (68% 95% 9.7%) that normally-distributed scores will lie withi 1, 2, and 3 stanard deviations, respectively, from the men.

normal distribution A continuous probability distribution in which the mean mode and median are at the centre of a symmetrical bell-shaped graph

probability density function A function of a continuous random variable whose integral gives the probability $P(X \le x)$

random variabe: A variable whose values are based on a chance experiment for example the number of road accidents in an hour

uniform probability distribution A probability distribution in which every outcome has the same probability

***z*-score** Measures how many standard deviations above or below the mean a score is

Pobabiliy deniy uncion

Pobabiliy deniy uncion

10.01 Probability density functions

EXAMPLE 1

This table gives the results of a survey of different times that runners take to complete a race

Time (min)	Frequency
0–<4	6
4–<8	8
8–<12	11
12–<16	4
16–<20	2
20–<24	1

a Add a column of relative frequencies

b Sketch a frequency histogram for the relative frequencies

c Estimate each probability

i $P(X < 12)$ **ii** $P(X \ge 16)$ **iii** $P(4 \le X < 8)$

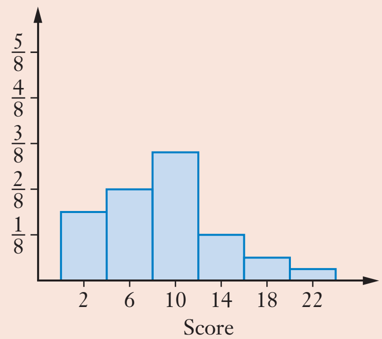
Solution

a

Time (min)	Frequency	Relative frequency
0–<4	6	$\frac{6}{32} = \frac{3}{16}$
4–<8	8	$\frac{8}{32} = \frac{1}{4}$
8–<12	11	$\frac{11}{32}$
12–<16	4	$\frac{4}{32} = \frac{1}{8}$
16–<20	2	$\frac{2}{32} = \frac{1}{16}$
20–<24	1	$\frac{1}{32}$

ISBN 97807043220

b



c **i** $P(X < 12) = \frac{6}{32} + \frac{8}{32} + \frac{11}{32}$

$= \frac{25}{32}$

ii $P(X \geq 16) = \frac{2}{32} + \frac{1}{32}$

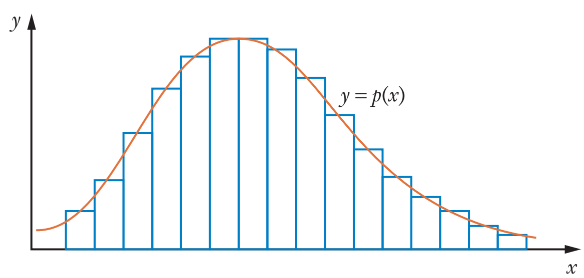
$= \frac{3}{32}$

iii $P(4 \leq X < 8) = \frac{1}{4}$

The times in the above example are values of a **continuous random variable** but sorted into groups While we can estimate probabilities using relative frequeny, we use other methods when dealing with continuous data

Continuous probability distributions

With continuous data we ca't really draw a histogram as we did in Example 1 or there would be an 'infinit' number of columns with'zeo' widts. Instad, the probability distribution is a continuous curve



A **continuous probability distribution** is represented by a function $P(X = x)$ or $p(x)$ called a **probability density function** (PDF) where X is the random variable As with discrete probability distributions the sum of all probabilities must be .

With a continuous probablity distribution, we cannot calculate the probability for a single outcome so $P(X = x) = 0$ Instea, we can only calculate the probability for a range of values such as $P(4 \leq X < 8)$

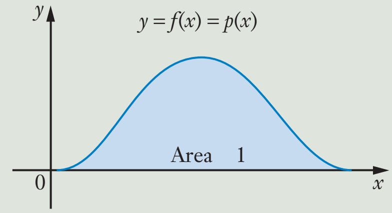
ISBN 97807043220

Area under a probability density function

The area under a probability density function is 1

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

where $f(x) \geq 0$ (since $0 \leq p(x) \leq 1$)



EXAMPLE 2

a A function is given by $f(x) = \begin{cases} \dfrac{3x^2}{26} & \text{for } 1 \leq x \leq 3 \\ 0 & \text{for all other } x \end{cases}$

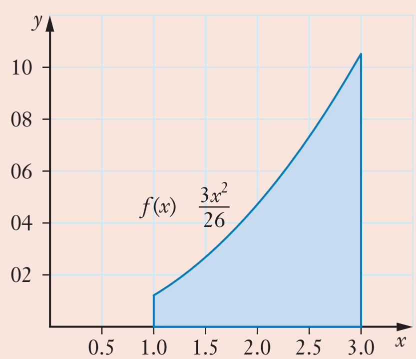
Show that it is a continuous probability distribution

b A function is given by $f(x) = ax^2$ defined for the domain [0 5.
Find the value of a for which this is a probability density function

Solution

a For a continuous probability distribution the area under the curve must be .

Drawing the graph notice that the area will be 0 for all x values outside $1 \leq x \leq 3$.

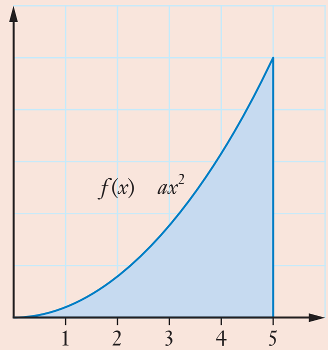


$$\begin{aligned}
\text{Area} &= \int^{3} \frac{3x^2}{26}dx \\
&= \frac{1}{26}\int^{3} 3x^2 dx \\
&= \frac{1}{26}\left[x^3\right]^3 \\
&= \frac{1}{26}[3^3 - 1^3] \\
&= \frac{1}{26}[26] \\
&= 1
\end{aligned}$$

So $f(x)$ is a continuous probability distribution

ISBN 97807043220

b Drawing the graph gives a parabola in the domain [0 5.



A PDF has area 1

So $\int_0^5 ax^2\,dx = 1$

$$\int_0^5 ax^2\,dx = a\int_0^5 x^2\,dx$$

$$= a\left[\frac{x^3}{3}\right]_0^5$$

$$= a\left[\frac{5^3}{3} - \frac{0^3}{3}\right]$$

$$= a\left[\frac{125}{3}\right]$$

$$= \frac{125a}{3}$$

So $\frac{125a}{3} = 1$

$$a = \frac{3}{125}$$

Exercise 10.01 Probability density functions

1 State whether each random variable is discrete or continuous

a The size of T-shirts worn by people

b The speed of cars as they pass a certain point

c The volume of water in a dam

d The number of seats on an aeroplane

e The weight of babies born in August

2 Which of the following functions describe continuous probability distributions?

a $f(x) = 02$ in the domain [1, 6]

b $f(x) = \frac{x}{12}$ in the domain [0 6]

c $f(x) = \begin{cases} \frac{x^3}{324} & \text{for } 0 \le x \le 3 \\ 0 & \text{for all other } x \end{cases}$

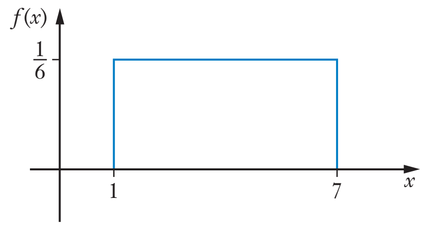
d $f(x) = \frac{x^2}{21}$ in the interval $1 \le x \le 4$

e $f(x) = \begin{cases} \frac{x}{8} & \text{for } 1 \le x \le 8 \\ 0 & \text{for all other } x \end{cases}$

ISBN 97807043220

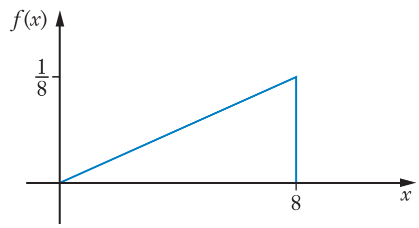
3 Which of the following graphs are of probability density functions?

a

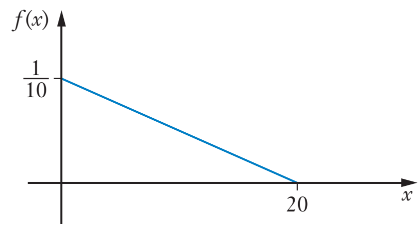


b

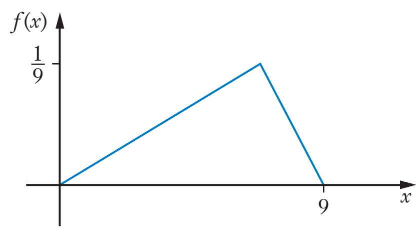
c



d



e



4 A probability density function has the equation $f(x) = \dfrac{x^4}{3355}$ over the domain $[2 \;\; b]$. Evaluate b

5 Given the continuous probability distribution $f(x) = \begin{cases} kx^3 & \text{for } 0 \leq x \leq 5 \\ 0 & \text{for all other } x \end{cases}$

find the value of k

6 A probability density function is given by $f(x) = ae^x$ over a certain domain Find the exact value of a if the domain is

a [1, 3] **b** [1, 7] **c** [0 4]

7 A function is given by $f(x) = \dfrac{x^2}{72}$

Over what domain starting at $x = 0$ is this a probability density function?

8 A PDF is given by $f(x) = \dfrac{2x^5}{87\,381}$ over the interval $1 \leq x \leq b$ Find the value of b

 ISBN 97807043220

10.02 Calculating probabilities

WS Pobabiliy deniy uncion

WS Uniom and iangula pobabiliy deniy uncion

Since $P(X = x) = 0$ for continuous probability distributions we can only find the probability of a **range** of values $P(a \le X \le b)$

Also since $P(X = a) = 0$ and $P(X = b) = 0$ it makes no difference whether we use $\le$ or $<$ $\ge$ or $>$

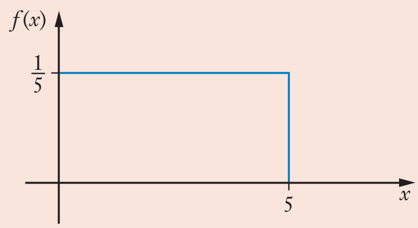
$P(a < X < b) = P(a \le X \le b)$

EXAMPLE 3

For the probability density function fin:

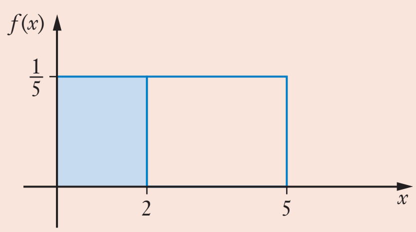
a $P(X \le 2)$

b $P(1 < X < 4)$



Solution

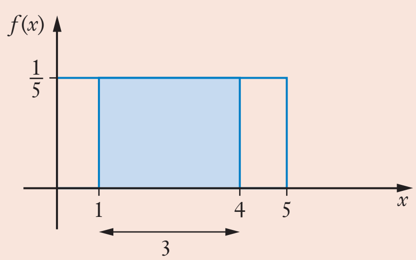
a $P(X \le 2)$ is the shaded area between $x = 0$ and $x = 2$.



$$P(X \le 2) = 2 \times \frac{1}{5}$$
$$= \frac{2}{5}$$

b $P(1 < X < 4)$ is the shaded area between $x = 1$ and $x = 4$

Notice that $P(1 < X < 4) = P(1 \le X \le 4)$ since $P(X = 1) = P(X = 4) = 0$

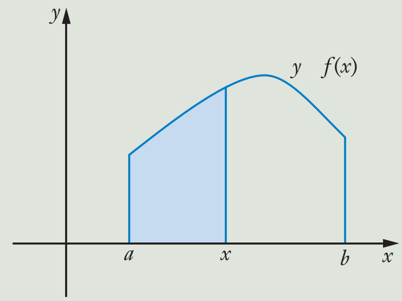


$$P(1 < X < 4) = 3 \times \frac{1}{5}$$
$$= \frac{3}{5}$$

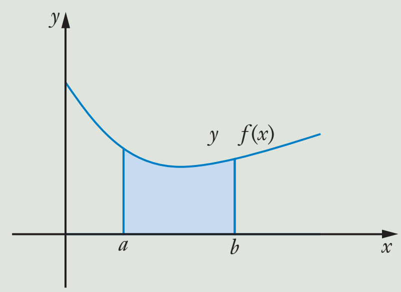
ISBN 97807043220

Probabilities in probability density functions

$P(X \le x) = \int_a^x f(x)\,dx$ where $y = f(x)$ is a PDF defined in the domain $[a \;\; b]$



$P(a \le X \le b) = \int_a^b f(x)\,dx$ where $y = f(x)$ is a PDF and a and b are in the defined domain



EXAMPLE 4

A function is given by $f(x) = \dfrac{3x^2}{117}$ defined in the domain [2 5. Fid:

a $P(X \le 4)$

b $P(3 \le X \le 4)$

Solution

a $P(X \le 4) = \int_2^4 \frac{3x^2}{117}\,dx$ (since domain is [2, 5])

$$= \frac{1}{117}\int_2^4 3x^2\,dx$$

$$= \frac{1}{117}\left[x^3\right]_2^4$$

$$= \frac{1}{117}(4^3 - 2^3)$$

$$= \frac{56}{117}$$

ISBN 97807043220

b
$$P(3 \le X \le 4) = \int_3^4 \frac{3x^2}{117}\,dx$$
$$= \frac{1}{117}\int_3^4 3x^2\,dx$$
$$= \frac{1}{117}\left[x^3\right]_3^4$$
$$= \frac{1}{117}(4^3 - 3^3)$$
$$= \frac{37}{117}$$

Uniform distributions

In Year 1, Chapter10, *Discrete probability distributions* you learned that with a **uniform probability distribution** every outcome has the same probabilit.

EXAMPLE 5

A continuous probability function $y = f(x)$ is uniform in the domain [5 15.

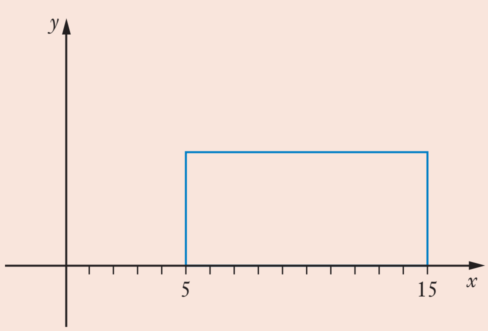
a Sketch the probability density function

b Find

i $P(X \ge 8)$ **ii** $P(7 \le X \le 10)$ **iii** $P(8 < X < 11)$

Solution

a A uniform distribution has all equal probabilities so will have the same height This gives a rectangle



PDF has area 1

$A = bh$

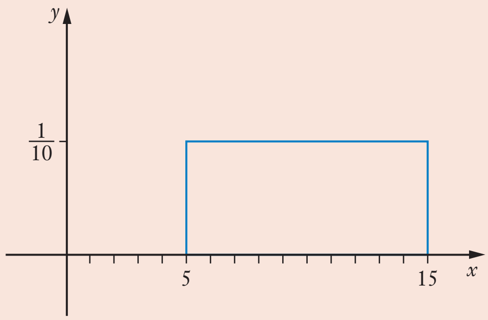
$= (15 - 5) \times h$

$= 10h$

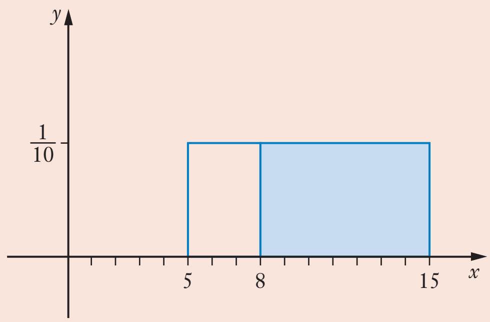
So $10h = 1$

$h = \frac{1}{10}$

ISBN 97807043220



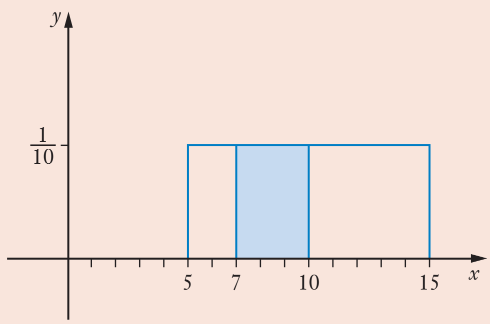
b **i**



$$P(X \geq 8) = (15 - 8) \times \frac{1}{10}$$

$$= \frac{7}{10}$$

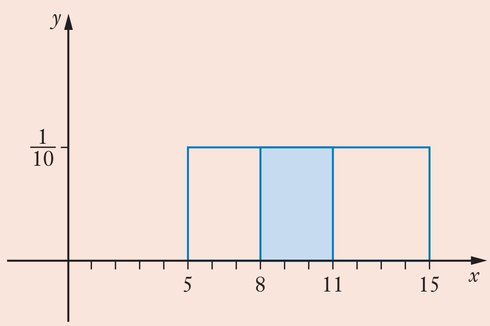
ii



$$P(7 \leq X \leq 10) = (10 - 7) \times \frac{1}{10}$$

$$= \frac{3}{10}$$

iii



$$P(8 < X < 11) = (11 - 8) \times \frac{1}{10}$$

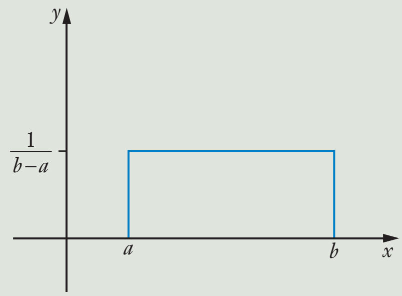
$$= \frac{3}{10}$$

Notice in the example that intervals with the same width have the same probability.

ISBN 97807043220

Uniform continuous probability distributions

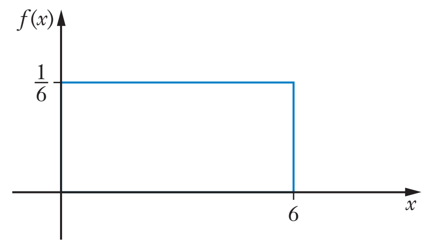
$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for all other } x \text{ values} \end{cases}$$



Equal intervals along the x-axis will have the same probability.

Exercise 10.02 Calculating probabilities

1 For the continuous probability distribution graphed fin:



a $P(X \le 3)$

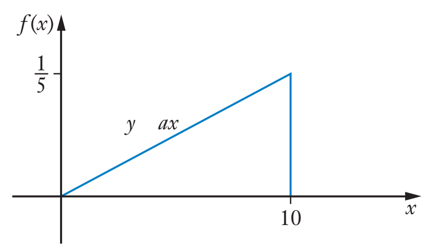
b $P(1 \le X \le 2)$

c $P(1 \le X \le 4)$

d $P(X < 4)$

e $P(X \ge 4)$

2 A probability density function is shown



a Find the equation of the linear function $y = ax$

b Find

i $P(X < 9)$

ii $P(X \le 3)$

iii $P(4 \le X \le 7)$

v $P(2 < X < 6)$

v $P(X > 5)$

3 The continuous probability distribution is defined by $f(x) = ax^2$ in the domain [0 5.

a Evaluate a

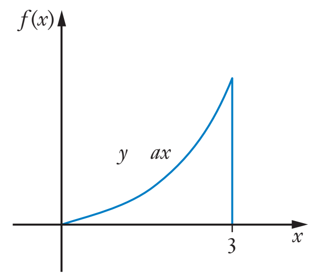
b Find

i $P(X \le 3)$ **ii** $P(1 < X < 4)$ **iii** $P(X > 2)$

v $P(X < 1)$ **v** $P(3 \le X < 4)$

ISBN 97807043220

4 The continuous random variable X has the PDF shown



a Evaluate a

b Find

i $P(1 \le X \le 3)$ **ii** $P(X < 2)$

iii $P(1 \le X \le 2)$ **v** $P(X \le 1)$

5 A continuous probability function is given by $f(x) = ke^x$ defined on the domain [,].

a Find the exact value of k

b Find each exact probability

i $P(2 \le X \le 5)$ **ii** $P(X < 4)$ **iii** $P(X \ge 3)$

6 **a** Show that $y = \sin x$ is a probability density function in the domain $\left[0 \quad \frac{\pi}{2}\right]$

b Find each exact probability

i $P\left(X \le \frac{\pi}{3}\right)$ **ii** $P\left(0 < X < \frac{\pi}{4}\right)$ **iii** $P\left(X > \frac{\pi}{6}\right)$

7 **a** Show that the uniform distribution $f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for all other } x \text{ values} \end{cases}$

is a probability density function

b If $a = 3$ and $b = 7$ fin:

i $P(X \le 6)$ **ii** $P(X \ge 5)$ **iii** $P(5 \le X \le 6)$

WS Cumulaive disibuion uncion

10.03 Cumulative distribution function

In the previous section you integrated the PDF each time to find $P(X \le x)$ a cumulative probability. The **cumulative distribution function** is a general formula for finding $P(X \le x)$ directly.

Cumulative distribution function (CDF)

The cumulative distribution function is given by $F(x) = \int_a^x f(x)\,dx$ where $y = f(x)$ is a PDF defined in the domain $[a \quad b]$

ISBN 97807043220

EXAMPLE 6

A continuous probability function is given by $f(x) = \dfrac{4x^3}{255}$ defined in the domain [1 4.

a Find the cumulative distribution function

b Use the CDF to find

i $P(X \leq 3)$

ii $P(X < 16)$

Solution

a $F(x) = \int_a^x f(x)\,dx$ where $f(x) = \dfrac{4x^3}{255}$ is a PDF defined in the domain [1 4.

$$= \int^x \frac{4x^3}{255}\,dx$$

$$= \frac{1}{255}\int^x 4x^3\,dx$$

$$= \frac{1}{255}\left[x^4\right]^x$$

$$= \frac{1}{255}(x^4 - 1^4)$$

$$= \frac{1}{255}(x^4 - 1)$$

$$= \frac{x^4 - 1}{255}$$

b Using $F(x) = \dfrac{x^4 - 1}{255}$ to find $P(X \leq x)$

i For $P(X \leq 3)$

$$F(3) = \frac{3^4 - 1}{255}$$

$$= \frac{81 - 1}{255}$$

$$= \frac{80}{255}$$

$$= \frac{16}{51}$$

So $P(X \leq 3) = \dfrac{16}{51}$

ii For $P(X < 1.6)$

$$F(16) = \frac{1\,6^4 - 1}{255}$$

$$= 00218$$

So $P(X < 16) = 00218$

We can use the cumulative distribution function to find probabilities such as $P(X \geq a)$ or $P(a \leq X \leq b)$

EXAMPLE 7

A continuous probability function is given by $f(x) = \frac{3x^2}{335}$ defined in the domain [2 7.

a Find the cumulative distribution function

b Use the CDF to find

i $P(X \geq 4)$ **ii** $P(35 \leq X \leq 62)$

Solution

a $F(x) = \int_a^x f(x)\,dx$ where $f(x) = \frac{3x^2}{335}$ is a PDF defined in the domain [2 7.

$$= \int_2^x \frac{3x^2}{335}\,dx$$

$$= \frac{1}{335}\int_2^x 3x^2\,dx$$

$$= \frac{1}{335}\left[x^3\right]_2^x$$

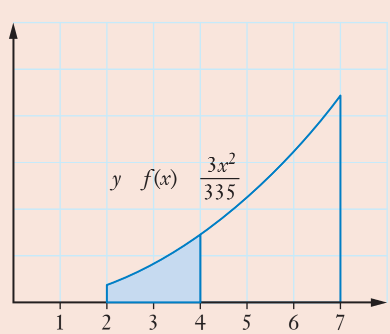
$$= \frac{1}{335}(x^3 - 2^3)$$

$$= \frac{1}{335}(x^3 - 8)$$

$$= \frac{x^3 - 8}{335}$$

b We use $F(x) = \frac{x^3 - 8}{335}$ to find $P(X \leq x)$

i



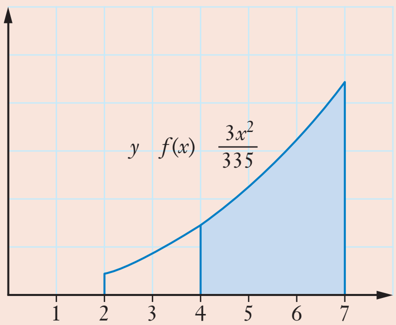
To find $P(X \geq 4)$ first find $P(X \leq 4)$

The shaded part of the PDF is $P(X \leq 4)$

$$P(X \leq 4) = \frac{4^3 - 8}{335}$$

$$= \frac{56}{335}$$

ISBN 97807043220



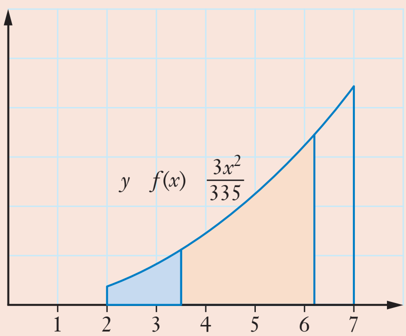
Area under the curve is 1

$P(X \geq 4) = 1 - P(X \leq 4)$

$= 1 - \frac{56}{335}$

$= \frac{279}{335}$

ii



$P(X \leq 35)$ is the area to the left of $x = 3.5$

$P(X \leq 62)$ is the area to the left of $x = 62$

So $P(35 \leq X \leq 62) = P(X \leq 62) - P(X \leq 3.5)$

$= \frac{6\ 2^3 - 8}{335} - \frac{3\ 5^3 - 8}{335}$

$= 0688 - 0104$

$= 0583$

Mode of a continuous probability distribution

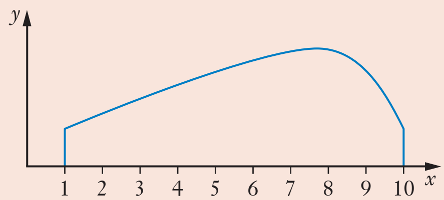
We sometimes want to know what the highest probability i. This is the **mode**

Mode

The mode is the maximum point of the probability density function

EXAMPLE 8

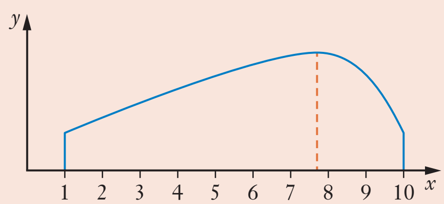
a Find the mode of the continuous probability distribution shown below.



b A continuous probability distribution is defined on the interval $1 \le x \le 5$ and has equation $f(x) = \frac{3x(6-x)}{92}$ Find the mod.

Solution

a



The highest point of the PDF is at $x = 77$

So the mode is 77

b $f(x) = \frac{3x(6-x)}{92}$

$= \frac{3}{92}(6x - x^2)$

The function is a parabola with $a < 0$ so will have a maximum turning point
We use calculus to see if this point lies within the defined domain [,].
(We could also use $x = -\frac{b}{2a}$ for the axis of symmetry of a parabola)

$f'(x) = \frac{3}{92}(6 - 2x)$

For stationary points

$f'(x) = 0$

$\frac{3}{92}(6 - 2x) = 0$

$6 - 2x = 0$

$6 = 2x$

$3 = x$

$x = 3$ lies in the domain [1, 5.

$f''(x) = \frac{3}{92}(-2)$

$= -\frac{3}{46}$

< 0

Concave down so a maximum turning point

So the mode is 3

ISBN 97807043220

Exercise 10.03 Cumulative distribution function

1 Find the cumulative distribution function for each continuous probability distribution

a $f(x)=\dfrac{x^2}{9}$ defined in the domain [0 3]

b $f(x)=\dfrac{4x^3}{1296}$ defined in the domain [0 6]

c $f(x)=\dfrac{e^x}{e^4-1}$ in the interval $0 \le x \le 4$

d $f(x)=\dfrac{4(x-2)^3}{625}$ in the domain [2 7]

e $f(x)=\dfrac{3x(8-x)}{135}$ in the domain [2 5]

2 a Find the cumulative distribution function for $f(x)=\begin{cases} \dfrac{5x^4}{7776} & \text{for } 1 \le x \le 6 \\ 0 & \text{for all other values} \end{cases}$

b Find

i $P(X \le 3)$ **ii** $P(X \le 2)$ **iii** $P(X < 5)$

v $P(X > 4)$ **v** $P(2 \le X \le 4)$

3 A continuous probability distribution is given by $f(x)=\dfrac{4x^3}{2320}$ in the domain [3 7.

a Find the cumulative distribution function

b Find

i $P(X \le 4)$ **ii** $P(X \le 6)$ **iii** $P(X \ge 5)$

v $P(X > 4)$ **v** $P(4 \le X < 6)$

4 A continuous probability distribution is defined by $f(x)=\dfrac{2e^{2x}}{e^{10}-1}$ in the domain [0 5.

a Find the cumulative distribution function

b Calculate each probability correct to 2 significant figures

i $P(X \le 2)$ **ii** $P(X \le 4)$ **iii** $P(X > 3)$

v $P(X \ge 28)$ **v** $P(2 \le X \le 4)$

5 a Evaluate a if $f(x)=ax^3$ is a continuous probability distribution defined in the domain [0 9.

b Find the cumulative distribution function

c Find

i $P(X \le 5)$ **ii** $P(X \le 4)$ **iii** $P(X > 8)$

v $P(X \ge 3)$ **v** $P(2 \le X \le 6)$

6 **a** Find the exact value of a if $f(x) = \dfrac{a}{x}$ is a continuous probability distribution defined in the domain [1 6.

b Find the cumulative distribution function

c Find to 2 decimal places

i $P(X \leq 3)$ **ii** $P(X \leq 2)$ **iii** $P(X > 5)$

v $P(X \geq 4)$ **v** $P(2 \leq X \leq 5)$

7 **a** Show that $y = \cos x$ is a probability density function in the domain $\left[\dfrac{3\pi}{2}, 2\pi\right]$

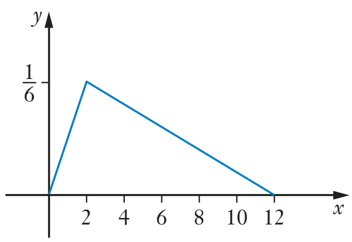
b Find the cumulative distribution function

c Find each probability in exact form

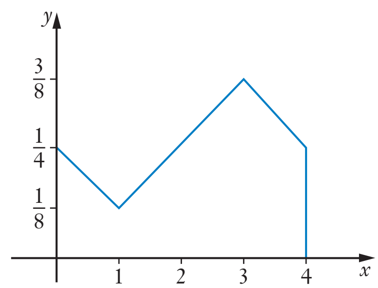
i $P\left(X \leq \dfrac{5\pi}{3}\right)$ **ii** $P\left(X \geq \dfrac{7\pi}{4}\right)$ **iii** $P\left(\dfrac{5\pi}{3} \leq X \leq \dfrac{11\pi}{6}\right)$

8 Find the mode of each continuous probability distribution

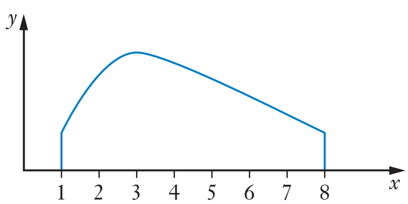
a



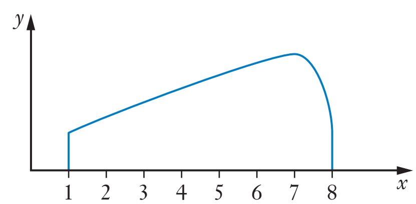
b



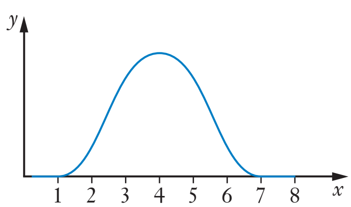
c



d



e



f $f(x) = -\dfrac{3(x^2 - 8x - 9)}{434}$ defined in the domain [0 7]

g $f(x) = \dfrac{4e^{4x}}{e^8(e^{16} - 1)}$ defined in the interval $2 \leq x \leq 6$

 ISBN 97807043220

h $f(x) = -\dfrac{3(x^2 - 16x + 15)}{1100}$ defined in the domain [1 11]

i $f(x) = \dfrac{2(2x^3 - 33x^2 + 168x + 3)}{2105}$ defined in the interval $0 \le x \le 5$

j $f(x) = \dfrac{3x^2}{342}$ defined in the interval $1 \le x \le 7$

9 **a** Find the mode of the function $f(x) = -\dfrac{3}{22}(x^2 - 6x + 5)$ defined on the domain [2 4.

b Find the cumulative distribution function

c Find $P(X \le a)$ where a is the mode

10 The times that athletes took to finish a race varied between 3 and 7 minutes and are represented by the continuous probability function $f(x) = \dfrac{1}{116}(x^3 - 9x^2 + 24x + 1)$ defined in the domain [3 7.

a Find the cumulative distribution function

b Find the probability that an athlete will finish this race

i in less than 5 minutes

ii in 4 minutes or more

iii in between 4 and 5 minutes

c What is the most likely time in which an athlete would finish the race?

10.04 Quantiles

WS

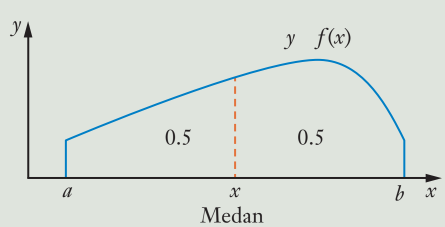
Quanile

Median

For a continuous probability distribution the **median** is the value of x that splits the distribution into halves Because the PDF has an area of , the area on each side of the median is $\frac{1}{2}$

Median

The median lies at the point x where $\int_a^x f(x)\,dx = 05$ given $y = f(x)$ is a PDF defined in the domain $[a\ \ b]$



The median o a coninuou pobabiliy diibuion

EXAMPLE 9

Find the median of the continuous probability distribution defined as $f(x) = \frac{x^2}{21}$ in the domain [1 4.

Solution

For $f(x) = \frac{x^2}{21}$ defined in the domain [1 4, first find the cumulative distribution function (CDF)

$$\int^x \frac{x^2}{21}\,dx = \frac{1}{21}\int^x x^2 dx$$

$$= \frac{1}{21}\left[\frac{x^3}{3}\right]^x$$

$$= \frac{1}{21}\left(\frac{x^3}{3} - \frac{1^3}{3}\right)$$

$$= \frac{1}{21}\left(\frac{x^3-1}{3}\right)$$

$$= \frac{x^3-1}{63}$$

For the median

$$\int_a^x f(x)\,dx = 0\ 5$$

$$\frac{x^3-1}{63} = 05$$

$$x^3 - 1 = 31.5$$

$$x^3 = 325$$

$$x = \sqrt[3]{32\ 5}$$

$$\approx 3.2$$

So the median is 32

Ths means $P\,X < 32)\quad 05$

Quartiles, deciles and percentiles

You learned about quartile, deciles and percentiles in Chapter7, *Statistics* They are values that separate a proportion of a set of data For exampl, Q > bottom 25% of scores Q_3 > bottom 75% of scores 2nd decile > bottom 20% of scores and 67th percentile > bottom 67% of scores

EXAMPLE 10

A continuous probability distribution is defined as $f(x) = \frac{x^4}{11\,605}$ in the domain [4 9.

Find correct to 2 decimal place:

a the 1st quartile

b the 38th percentile

c the 7th decile

ISBN 97807043220

Solution

First find the CD.

$$\int_4^x \frac{x^4}{11\,605}\,dx = \frac{1}{11605}\int_4^x x^4\,dx$$

$$= \frac{1}{11\,605}\left[\frac{x^5}{5}\right]_4^x$$

$$= \frac{1}{11\,605}\left(\frac{x^5}{5} - \frac{4^5}{5}\right)$$

$$= \frac{1}{11\,605}\left(\frac{x^5 - 1024}{5}\right)$$

$$= \frac{x^5 - 1024}{58\,025}$$

a 1st quartile 25%

$$\int_a^x f(x)\,dx = 025$$

$$\frac{a^5 - 1024}{58\,025} = 025$$

$$a^5 - 1024 = 14\ 50625$$

$$a^5 = 15\ 53025$$

$$a \approx 689$$

So the 1st quartile is 689

Ths means $P X < 689)$ 025

b 38th percentile 38%

$$\int_a^x f(x)\,dx = 038$$

$$\frac{a^5 - 1024}{58\,025} = 038$$

$$a^5 - 1024 = 22\ 0495$$

$$a^5 = 23\ 0735$$

$$a \approx 746$$

So the 38th percentile is 746

Ths means $P X < 746$ 038

c 7th decile 70%

$$\int_a^x f(x)\,dx = 07$$

$$\frac{a^5 - 1024}{58\,025} = 07$$

$$a^5 - 1024 = 40\ 6175$$

$$a^5 = 41\ 6415$$

$$a \approx 839$$

So the 7th decile is 839

Ths means $P X < 839)$ 07

ISBN 97807043220

Exercise 10.04 Quantiles

1 Find the median of each continuous random variable correct to 2 decimal places

a $f(x) = \dfrac{3x^2}{511}$ defined on the interval $1 \le x \le 8$

b $f(x) = \dfrac{4x^3}{2401}$ defined in the domain [0 7]

c $f(x) = \dfrac{5x^4}{16\,807}$ in the interval $0 \le x \le 7$

d $f(x) = \dfrac{3(x-3)^2}{16}$ in the domain [1 5]

e $f(x) = \dfrac{(3x+1)^2}{244}$ in the interval $0 \le x \le 4$

f $f(x) = \dfrac{4x^3}{6560}$ defined in the domain [1 9]

g $f(x) = \dfrac{3x^2}{1034}$ defined in the domain [3 11]

h $f(x) = \dfrac{6x^5}{15\,625}$ in the interval $0 \le x \le 5$

i $f(x) = \dfrac{(2x-1)^4}{16\,105}$ in the domain [1 6]

j $f(x) = \dfrac{x(x^2-3)^3}{3570}$ defined in the domain [2 4]

2 For each continuous probability distribution fin:

i the 1st quartile **ii** the 2nd decile **iii** the 77th percentile

a $f(x) = \dfrac{3x^2}{973}$ defined in the domain [3 10]

b $f(x) = \dfrac{x^3}{324}$ defined in the interval $0 \le x \le 6$

c $f(x) = \dfrac{5x^4}{3124}$ defined in the interval $1 \le x \le 5$

3 For the continuous probability distribution $f(x) = \dfrac{3x^2}{512}$ defined in the domain [0 8, fid:

a the median

b the 35th percentile

4 For the continuous probability distribution $f(x) = \dfrac{x^2}{168}$ defined on the interval $2 \le x \le 8$, find

a the median **b** the 1st quartile **c** the 3rd quartile

d the 67th percentile **e** the 14th percentile **f** the 8th decile

ISBN 97807043220

5 For the continuous probability distribution defined as $f(x) = \frac{x^2}{576}$ on the interval $0 \le x \le 12$, fin:

a the 20th percentile

b the median

c the 3rd quartile

6 For the continuous probability distribution $f(x) = \frac{x^3}{1020}$ defined in the interval $2 \le x \le 8$, find

a the cumulative probability function

b $P(X \le 5)$

c $P(X > 4)$

d $P(3 \le X \le 7)$

e the median

f the 3rd quartile

g the 9th decile

h the 23rd percentile

10.05 Normal distribution

WS

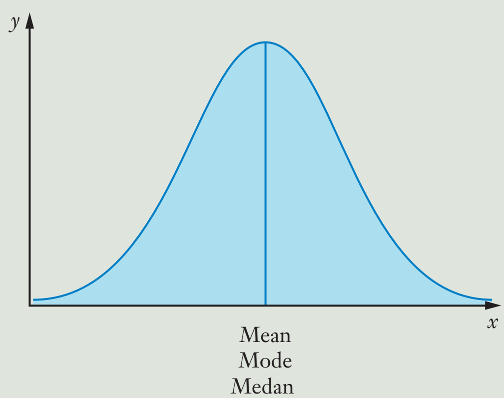
The nomal diibuion

The **normal distribution** is a special continuous probability distribution Its probability density function is often called a **bell curve** because of its shape

Normal distribution

The normal distribution is a symmetrical bell-shaped function

The mean mode and median are equa, at the centre of the probability density functin.



There are many examples of data that are normally distributed such as I, birth weigts, ages reaction times and exam results in a schoo.

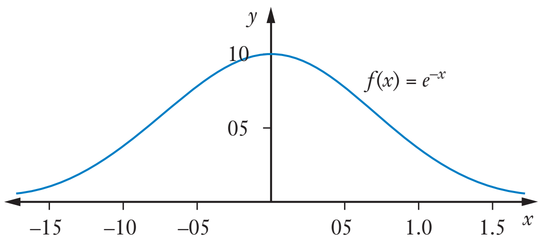
We use the population mean μ and standard deviation σ for the normal distribution

ISBN 97807043220

TECHNOLOGY

The normal distribution

A good estimate for the **shape** of the normal distribution is $f(x) = e^{-x}$



Use your graphing techniques and technology to sketch this function

Given a population that is normally distributed with mean $\mu = 10$ and standard deviation $\sigma = 2$ use technology to sketch the graph of

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)}{2\sigma}}$$

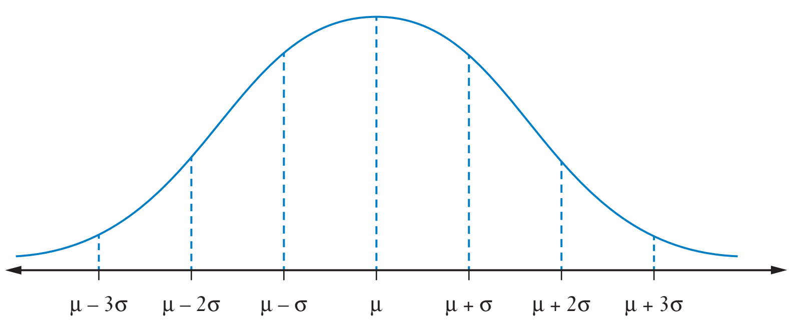
This is the actual equation of the normal distribution

What do you notice about this graph? Use technology and the trapezoidal rule with many subintervals to approximate the area under the graph What do you find ?

Research the normal distribution and its features

Graphing a normal distribution

In a normal distribution most of the data lies within 3 standard deviations of the mea, with the mean in the centre

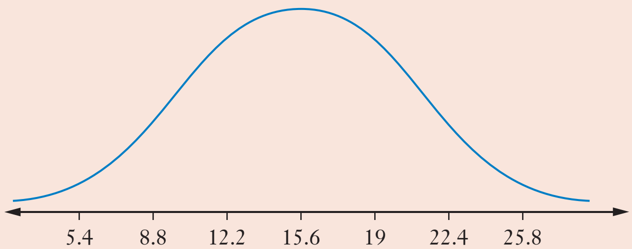


ISBN 97807043220

EXAMPLE 11

a A set of data is normally distributed with mean 83 and standard deviation 12
Sketch the probability distribution function

b A normal distribution has the probability density function below. Find its mean and standard deviation



Solution

a

$\mu + \sigma = 8.3 + 1.2$

$= 95$

$\mu + 2\sigma = 8.3 + 2(12)$

$= 107$

$\mu + 3\sigma = 8.3 + 3(12)$

$= 11.9$

$\mu - \sigma = 8.3 - 1.2$

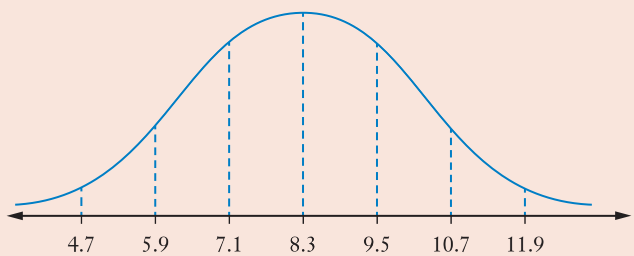
$= 7.1$

$\mu - 2\sigma = 8.3 - 2(12)$

$= 59$

$\mu - 3\sigma = 8.3 - 3(12)$

$= 47$



b

$\mu = 15.6$ (centre of the distribution)

$\mu + \sigma = 19$

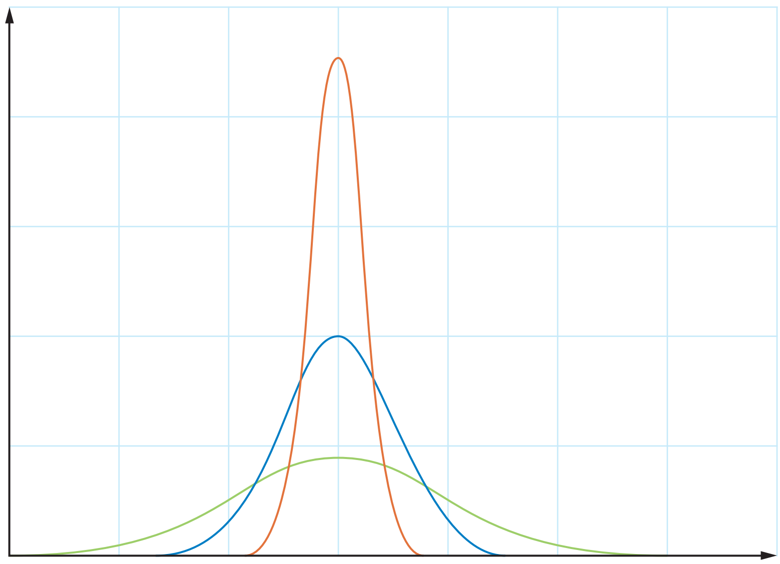
$156 + \sigma = 19$

$\sigma = 19 - 15.6$

$= 34$ (check that all the differences between all values on the graph are 34)

ISBN 97807043220

The normal distribution can have different shapes depending on the size of the standard deviation In the diagra, the green curve shows the normal distribution with the highest standard deviation



Standard normal distribution

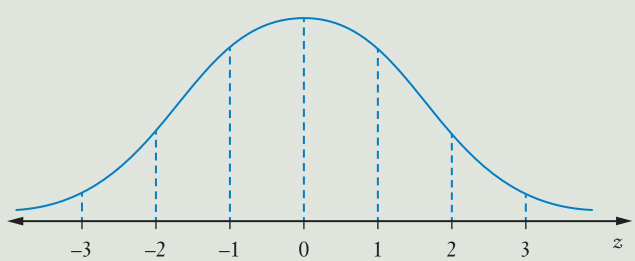
It is difficult to find probabilities in the normal distribution by integration because the equation of the cumulative distribution function is complicated To get around tis, tables of probabilities have been developed from the **standard normal distribution** This is a normal curve that has been transformed so that the mean is 0 and the standard deviation is 1 The values of a standard normal distribution are called z rather than x also known as ***z*-scores** or **standardised scores**

Standard normal distribution

$\mu = 0$

$\sigma = 1$

Area under the curve is 1



Probability tables for the standard normal distribution

WS
Nomal disibuion able

Probability tables for the standard normal distribution begin next page A copy can also be downloaded from *NelsonNet* Values represent the area to the left of (or less than) the z-score Row labels show the z-score to one decimal place Column labels show the second decimal plac.

ISBN 97807043220

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
–34	00003	.00030.0000.003		0.003	0.003	00003	.00030.0003	0.0002		
–33	00005	.00050.0000.004		0.004	0.004	00004	.00040.0004	0.0003		
–32	00007	.00070.0000.006		0.006	0.006	00006	.00050.0005	0.0005		
–31	00010	.00090.0000.009		0.008	0.008	00008	.00080.0007	0.0007		
–30	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
–29	00019	.00180.0010.007		0.016	0.016	00015	.00150.0014	0.0014		
–28	00026	.00250.0020.003		0.023	0.022	00021	.00210.0020	0.0019		
–27	00035	.00340.0030.002		0.031	0.030	00029	.00280.0027	0.0026		
–26	00047	.00450.0040.003		0.041	0.040	00039	.00380.0037	0.0036		
–25	00062	.00600.0050.007		0.055	0.054	00052	.00510.0049	0.0048		
–24	00082	.00800.0070.005		0.073	0.071	00069	.00680.0066	0.0064		
–23	00107	.01040.0100.009		0.096	0.094	00091	.00890.0087	0.0084		
–22	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
–21	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
–20	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
–19	00287	.02810.0270.028		0.062	0.256	00250	.02440.0239	0.0233		
–18	00359	.03510.0340.036		0.029	0.322	00314	.03070.0301	0.0294		
–17	00446	.04360.0420.048		0.009	0.401	00392	.03840.0375	0.0367		
–16	00548	.05370.0520.056		0.005	0.495	00485	.04750.0465	0.0455		
–15	00668	.06550.0640.060		0.018	0.606	00594	.05820.0571	0.0559		
–14	00808	.07930.0770.074		0.049	0.735	00721	.07080.0694	0.0681		
–13	00968	.09510.0930.098		0.001	0.885	00869	.08530.0838	0.0823		
–12	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
–11	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
–10	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
–09	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
–08	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–07	02420	.23890.2350.237		0.296	0.266	02236	.22060.2177	0.2148		
–06	02743	.27090.2670.263		0.211	0.578	02546	.25140.2483	0.2451		
–05	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–04	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–03	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–02	04207	.41680.4120.400		0.452	0.013	03974	.39360.3897	0.3859		
–01	04602	.45620.4520.443		0.443	0.404	04364	.43250.4286	0.4247		
–00	05000	.49600.4920.480		0.440	0.801	04761	.47210.4681	0.4641		

ISBN 97807043220

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
01	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
02	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
03	06179	.62170.6250.623		0.631	0.368	06406	.64430.6480	0.6517		
04	06554	.65910.6620.664		0.600	0.736	06772	.68080.6844	0.6879		
05	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
06	07257	.72910.7320.737		0.789	0.422	07454	.74860.7517	0.7549		
07	07580	.76110.7640.763		0.704	0.734	07764	.77940.7823	0.7852		
08	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
09	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
10	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
11	08643	.86650.8680.878		0.829	0.749	08770	.87900.8810	0.8830		
12	08849	.88690.8880.897		0.825	0.944	08962	.89800.8997	0.9015		
13	09032	.90490.9060.902		0.999	0.115	09131	.91470.9162	0.9177		
14	09192	.92070.9220.926		0.951	0.265	09279	.92920.9306	0.9319		
15	09332	.93450.9350.930		0.982	0.394	09406	.94180.9429	0.9441		
16	09452	.94630.9470.944		0.995	0.505	09515	.95250.9535	0.9545		
17	09554	.95640.9570.952		0.991	0.599	09608	.96160.9625	0.9633		
18	09641	.96490.9650.964		0.971	0.678	09686	.96930.9699	0.9706		
19	09713	.97190.9720.972		0.938	0.744	09750	.97560.9761	0.9767		
20	09772	.97780.9780.978		0.993	0.798	09803	.98080.9812	0.9817		
21	09821	.98260.9830.984		0.938	0.842	09846	.98500.9854	0.9857		
22	09861	.98640.9860.981		0.975	0.878	09881	.98840.9887	0.9890		
23	09893	.98960.9890.991		0.904	0.906	09909	.99110.9913	0.9916		
24	09918	.99200.9920.995		0.927	0.929	09931	.99320.9934	0.9936		
25	09938	.99400.9940.993		0.945	0.946	09948	.99490.9951	0.9952		
26	09953	.99550.9950.997		0.959	0.960	09961	.99620.9963	0.9964		
27	09965	.99660.9960.998		0.969	0.970	09971	.99720.9973	0.9974		
28	09974	.99750.9970.997		0.977	0.978	09979	.99790.9980	0.9981		
29	09981	.99820.9980.993		0.984	0.984	09985	.99850.9986	0.9986		
30	09987	.99870.9980.998		0.988	0.989	09989	.99890.9990	0.9990		
31	09990	.99910.9990.991		0.992	0.992	09992	.99920.9993	0.9993		
32	09993	.99930.9990.994		0.994	0.994	09994	.99950.9995	0.9995		
33	09995	.99950.9990.996		0.996	0.996	09996	.99960.9996	0.9997		
34	09997	.99970.9990.997		0.997	0.997	09997	.99970.9997	0.9998		

ISBN 97807043220

EXAMPLE 12

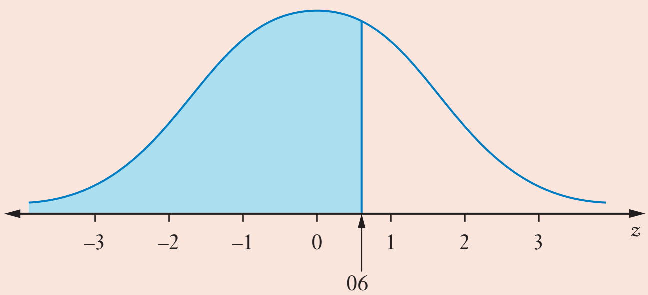
For a standard normal distribution use the table to fin:

a $P(z \le 06)$ **b** $P(z \le -183)$ **c** $P(z < 234)$

d $P(z \ge -27)$ **e** $P(-03 \le z \le 14)$

Solution

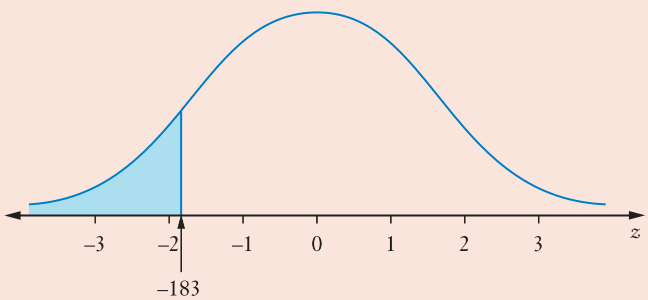
a The table gives the area under the PDF for the standard normal distribution



Find 06 in the left column of the table

$P(z \le 06) = 07257$

b

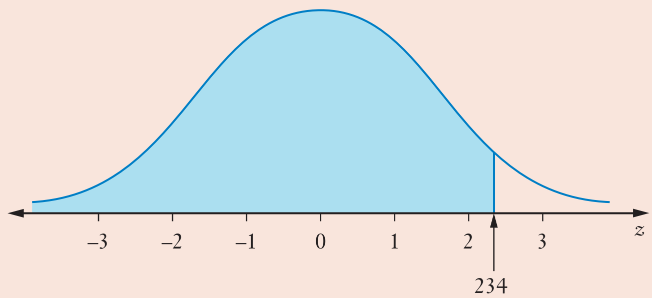


For –183 find –18 in the left column of the table and the entry under 003 in this row.

$P(z \le -183) = 00336$

ISBN 97807043220

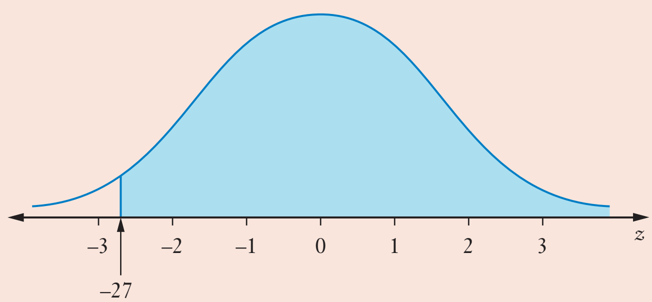
c



For 234 find .3 in the left column of the table and the entry under .04 in this ro.

$P(z < 234) = 09904$

d



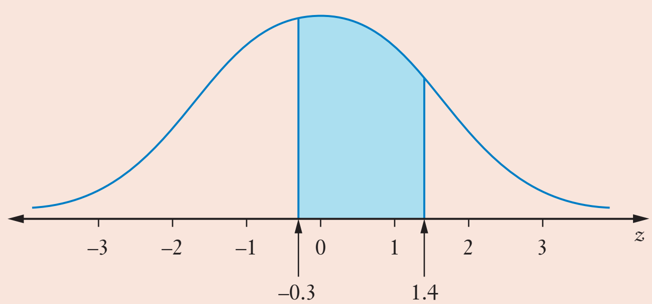
Find –27 in the left column of the table

$P(z \leq -27) = 00035$

$P(z \geq -27) = 1 - 00035$

$= 09965$

e



$P(-03 \leq z \leq 14) = P(z \leq 14) - P(z \leq -03)$

$= 09192 - 03821$

$= 05371$

ISBN 97807043220

Quartiles, deciles and percentiles

EXAMPLE 13

For a standard normal distribution use the table on .633 to fin:

a the median **b** the lower quartile **c** the upper quartile

d the 84th percentile **e** the 6th decile

Solution

a The median separates the bottom half of the scores

We want to find the score with the cumulative probability of .5 in the tabl.

Median = 0 (this is the same value as the mean)

b The lower quartile separates the bottom 025 of the data

In the table there are 2 values close to 025

$P(z \leq -067) = 02514$ and $P(z \leq -068) = 02483$

02514 is closer to 025 so the lower quartile is approximately -067

(Note $02514 - 025 = 00014$, and $.25 - 02483 = 00017$)

c The upper quartile separates the bottom 075 of the data

In the table there are 2 values close to 075

$P(z \leq 067) = 07486$ and $P(z \leq 068) = 07517$

07486 is closer to 075 so the upper quartile is approximately 067

Notice that the values for the lower and upper quartiles are ±067 because the normal distribution is symmetrical

d The 84th percentile separates the bottom 084 of the data

In the table there are 2 values close to 084

$P(z \leq 099) = 08389$ and $P(z \leq 1) = 08413$

08389 is closer to 084 so the 84th percentile is approximately 099

e The 6th decile separates the bottom 06 of the data

In the table there are 2 values close to 06

$P(z \leq 025) = 05987$ and $P(z \leq 026) = 06026$

05897 is closer to 06 so the 6th decile is approximately 025

ISBN 97807043220

EXAMPLE 14

Shade the area of the normal distribution where the values are

a above the top 20% of data **b** above the top 10% of data

Solution

a $P(z > a) = 20\%$, so $P(z \leq a) = 80\%$ (08 or the 8th decile)

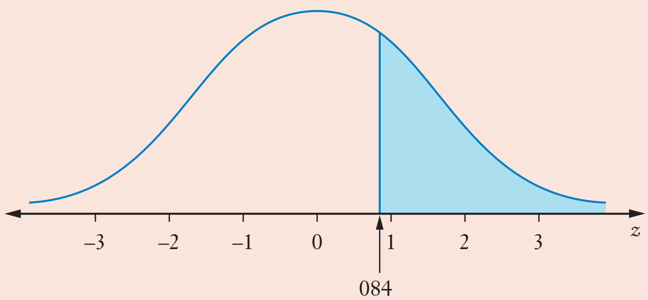
From the table

$P(z \leq 084) = 07995$ and $P(z \leq 085) = 08023$

07995 is closer to 08 so the 8th decile is approximately 084

This means all values to the left of 084 lie below 80%

So the top 20% lies to the right of 084



b $P(z > a) = 10\%$, so $P(z \leq a) = 90\%$ (09 or the 9th decile)

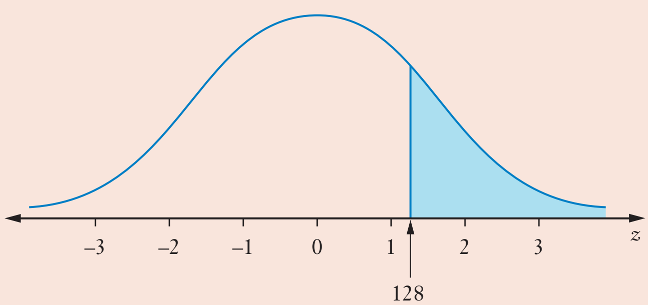
From the table

$P(z \leq 128) = 08997$ and $P(z \leq 129) = 09015$

08997 is closer to 09 so the 9th decile is approximately 128

This means all values to the left of 128 lie below 90%

So the top 10% lies to the right of 128



ISBN 97807043220

Exercise 10.05 Normal distribution

1 Draw a probability density function for the normal distribution with

a mean 9 and standard deviation 2

b mean 86 and standard deviation 03

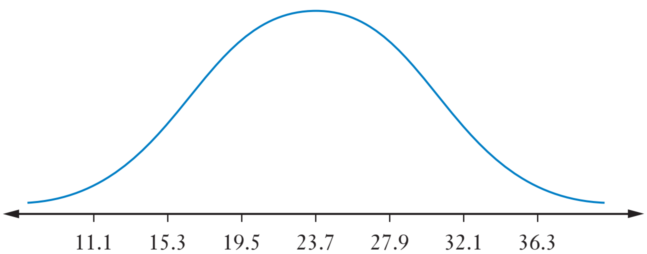
c mean 115 and standard deviation 14

d mean 27 and standard deviation 25

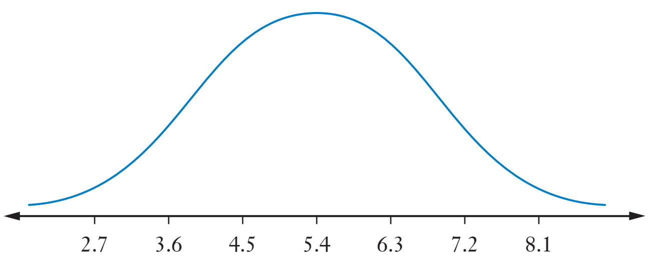
e mean 1152 and standard deviation 32

2 What is the mean and standard deviation of each normal distribution?

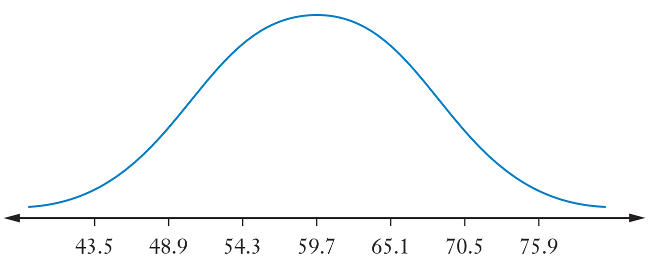
a



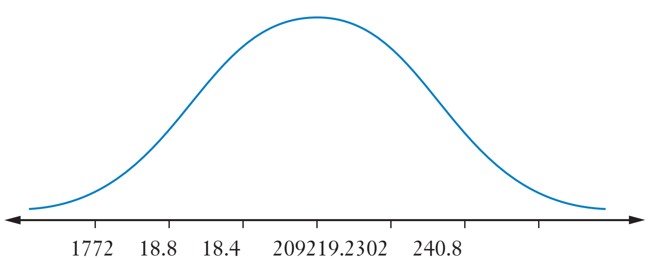
b



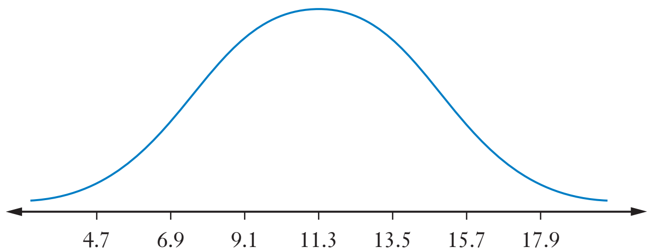
c



d



e



3 Draw the probability density function for a standard normal distribution

4 Use the probability table for a standard normal distribution on pages 633–4 to find

a $P(z \le 0)$ **b** $P(z \le 1)$

c $P(z \le 2)$ **d** $P(z \le 3)$

e $P(z \le -1)$ **f** $P(z \le -2)$

g $P(z \le -3)$ **h** $P(z \le 1.5)$

i $P(z < -267)$ **j** $P(z \le 309)$

5 Use the probability table to find

a $P(z \ge -046)$ **b** $P(z > 2.11)$

c $P(z \ge -201)$ **d** $P(-24 \le z \le -176)$

e $P(-22 \le z \le 22)$ **f** $P(121 < z < 189)$

g $P(-145 \le z \le 3.1)$ **h** $P(-1 \le z \le 1)$

i $P(-2 \le z \le 2)$ **j** $P(-3 \le z \le 3)$

6 Use the probability table to find

a the 8th decile **b** the 3rd quartile

c the 29th percentile **d** the 2nd decile

e the 89th percentile **f** the 12th percentile

g the 3rd decile **h** the 1st quartile

i the 63rd percentile

7 Sketch a normal curve and draw on it the area where values lie in the

a bottom 30% **b** bottom 15%

c top 67% **d** top 24%

e top 12%

ISBN 97807043220

10.06 Empirical rule

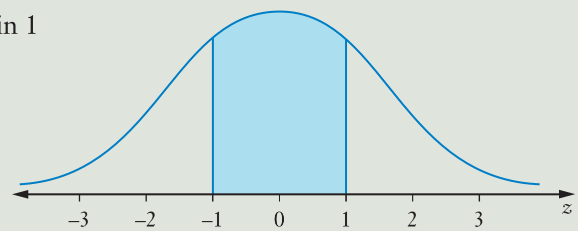
In Question 5 of the previous exercise you calculated $P(-1 \leq z \leq 1)$, $P(-2 \leq z \leq 2)$ and $P(-3 \leq z \leq 3)$ the probability that values in a normal distribution will fall within , 2 or 3 standard deviations of the mean respectively. These probabilities are part of the **empirical rule**

WS The empiical ule

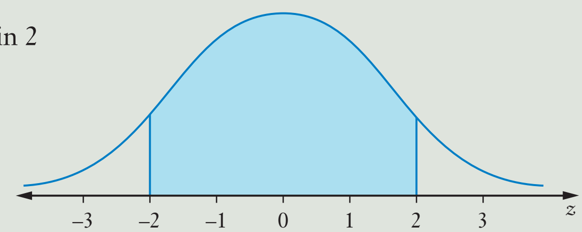
WS The nomal cuve

Empirical rule

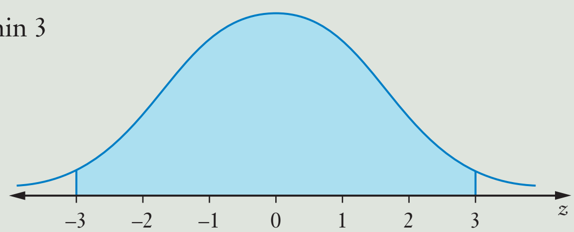
Approximately 68% of scores lie within 1 standard deviation of the mean with z-scores between –1 and 1



Approximately 95% of scores lie within 2 standard deviations of the mean with z-scores between –2 and 2



Approximately 997% of scores lie within 3 standard deviations of the mean with z-scores between –3 and 3



EXAMPLE 15

a A data set is normally distributed with mean 20 and standard deviation 3

What percentage of scores lie between

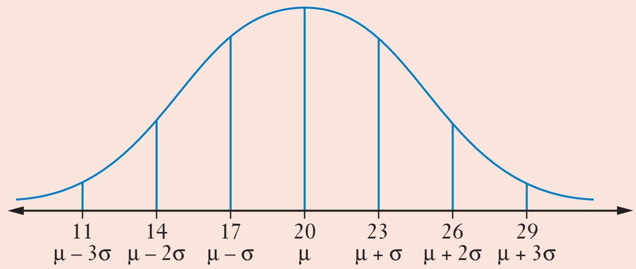
i 17 and 23? **ii** 14 and 26? **iii** 11 and 29?

b A normal distribution has $\mu = 652$ and $\sigma = 13$ What percentage of scores lie between

i 613 and 652 ? **ii** 652 and 665 ? **iii** 626 and 691 ?

Solution

a We can draw the PDF for the normal curv.



i Scores between 17 and 23 are within 1 standard deviation of the mean

So about 68% of scores lie between 17 and 23

ii Scores between 14 and 26 are within 2 standard deviations of the mean

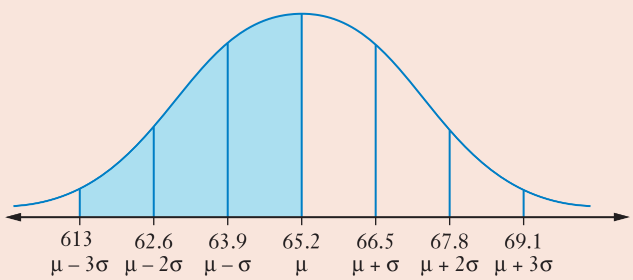
So about 95% of scores lie between 14 and 26

iii Scores between 11 and 29 are within 3 standard deviations of the mean

So about 997% of scores lie between 11 and 29

b We can draw the PDF for the normal curv.

i



Scores between 613 and 691 are within 3 standard deviations of the mean (997% of data)

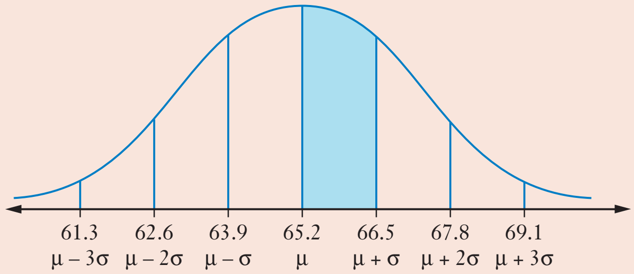
Scores between 613 and 652 are half this area

997% ÷ 2 = 4985%

So 4985% of scores lie between 613 and 652

ISBN 97807043220

ii



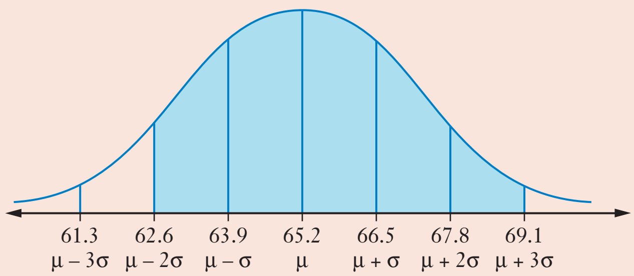
Scores between 639 and 665 are within 1 standard deviation of the mean (68% of data)

Scores between 652 and 665 are half this area

$68\% \div 2 = 34\%$

So 34% of scores lie between 652 and 665

iii



Scores between 626 and 652 are half the area within 2 standard deviations of the mean $\left(\frac{1}{2} \times 95\%\right)$

Scores between 652 and 691 are half the area within 3 standard deviations of the mean $\left(\frac{1}{2} \times 997\%\right)$

$$\begin{aligned}\text{Total area} &= \frac{1}{2} \times 95\% + \frac{1}{2} \times 997\% \\ &= 475\% + 4985\% \\ &= 9735\%\end{aligned}$$

So 9735% of scores lie between 626 and 691

ISBN 97807043220

Exercise 10.06 Empirical rule

1 What is the approximate percentage of data in a normal distribution that lies within

a 1 standard deviation of the mean?

b 2 standard deviations of the mean?

c 3 standard deviations of the mean?

2 A set of data has a mean of 15 and a standard deviation of 15 If the data set is normally distributed find the percentage of data that lies betwee:

a 135 and 165 **b** 12 and 18 **c** 105 and 195

3 A set of data is normally distributed with a mean of 84 and a standard deviation of 09 Find the percentage of data that lies between

a 75 and 93 **b** 66 and 102 **c** 57 and 111

4 A set of data is normally distributed with mean 18 and standard deviation 2 Find the percentage of data that lies between

a 16 and 20 **b** 14 and 22 **c** 12 and 24

d 16 and 18 **e** 18 and 24 **f** 12 and 22

5 A normal distribution has a mean of 65 and standard deviation 4

a Sketch the graph of its PDF.

b Find the percentage of data that lies between

i 57 and 73 **ii** 61 and 65 **iii** 65 and 77

v 57 and 69 **v** 61 and 73

6 A normal distribution has a mean of 97 and standard deviation 21 Find the percentage of data that lies between

a 76 and 118 **b** 97 and 118 **c** 97 and 139

d 55 and 97 **e** 34 and 118

WS The andad nomal crve

WS z coe

WS Aea unde he nomal cuve

z coe

10.07 z-scores

We can transform any normal distribution into a standard normal distribution by using z-scores

z-scores

To convert a raw scor, x into a z-score use the formul:

$$z = \frac{x - \mu}{\sigma}$$

where μ is the mean and σ is the standard deviation of the distribution

ISBN 97807043220

EXAMPLE 16

A data set is normally distributed with mean 15 and standard deviation 2

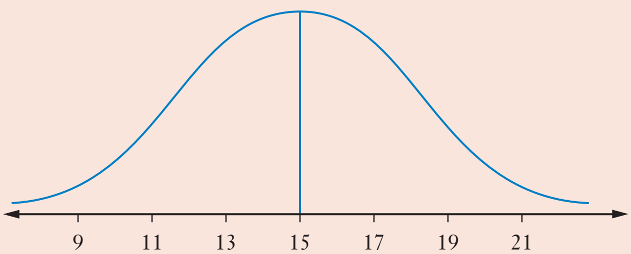
a Draw the probability density function for this distribution

b Use z-scores to convert each x value to standardised scores

i $x = 19$ **ii** $x = 15$ **iii** $x = 9$

Solution

a Drawing the PDF gives the curve below.



b $\mu = 15$ and $\sigma = 2$

i For $x = 19$

$$z = \frac{x - \mu}{\sigma} = \frac{19 - 15}{2} = 2$$

Ths s because 19 s 2 standard devatons above the mean

ii For $x = 15$:

$$z = \frac{x - \mu}{\sigma} = \frac{15 - 15}{2} = 0$$

Ths s because 15 s the mean

iii For $x = 9$

$$z = \frac{x - \mu}{\sigma} = \frac{9 - 15}{2} = -3$$

ISBN 97807043220

EXAMPLE 17

A data set is normally distributed with mean 238 and standard deviation 12

a Find the z-score for each raw score and describe where it is on the standard normal distribution

i 257 **ii** 206 **iii** 283

b Which of the scores from part **a** are very unlikely?

c Find the value of a raw score whose z-score is −182 correct to 2 decimal place.

Solution

a i For $x = 257$

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{25.7-23.8}{1\ 2}$$
$$\approx 1.58$$

A score of 257 lies 158 standard deviations above (to the right of) the mean

ii For $x = 206$

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{20.6-23.8}{1\ 2}$$
$$\approx -267$$

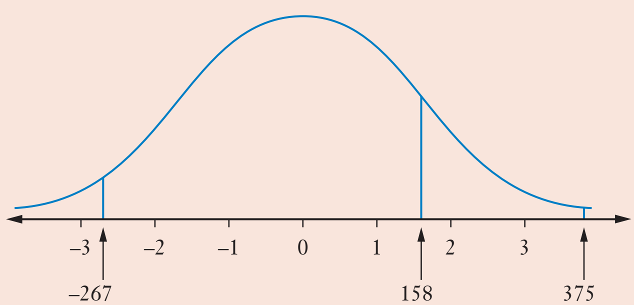
A score of 206 lies 267 standard deviations below (to the left of) the mean

iii For $x = 283$

$$z = \frac{28.3-23.8}{1\ 2}$$
$$\approx 375$$

A score of 283 lies 375 standard deviations above (to the right of) the mean

b We can draw each score on the standard normal distributio.



997% of scores lie within 3 standard deviations of the mean
So a z-score of 375 is very unlikely.

This means that a score of 283 is very unlikely.

ISBN 97807043220

c

$$z = \frac{x - \mu}{\sigma}$$

$$-182 = \frac{x - 23\ 8}{1\ 2}$$

$$-2184 = x - 238$$

$$21616 = x$$

So the raw score is 2162

Exercise 10.07 *z*-scores

1 A data set is normally distributed with mean 18 and standard deviation 13

a Find the z-score for each raw score

i 18 **ii** 193 **iii** 206 **v** 219

v 167 **v** 154 **vii** 141

b Which raw score has a z-score of

i 15 ? **ii** –21 ?

2 The length of fish caught in a fishing competition had a mean of 531 cm and standard deviation 87

a If the lengths of fish almost certainly lie within 3 standard deviations of the mean between which lengths would the fish almost certainly lie?

b Find the z-score for each length

i 531 cm **ii** 618 cm **iii** 444 cm **v** 705 cm

v 357 cm **v** 792 cm **vii** 27 cm **viii** 65 cm

3 A sample of overnight temperatures at Thredbo in June showed a mean temperature of 68°C and a standard deviation of 11

a Almost all temperatures lie within 3 standard deviations of the mean Within what range do almost all temperatures lie?

b Find the z-score for each temperature

i 68°C **ii** 79°C **iii** 9°C **v** 101°C

v 57°C **v** 46°C **vii** 35°C **viii** 6°C

4 A survey showed that the mean volume of juice in an orange is 664 mL with a standard deviation of 58

a The volume of juice very probably lies within 2 standard deviations of the mean Between which 2 volumes do they lie?

b Find the z-scores for each volume

i 664 mL	**ii** 722 mL	**iii** 78 mL	**v** 838 mL
v 606 mL	**v** 548 mL	**vii** 49 mL	**viii** 90 mL

c Find the volume that has a z-score of

i 12	**ii** 29	**iii** –06	**v** –23

5 a Find the z-score for each raw score below if the mean is 68 and the standard deviation is 45

i 80	**ii** 532	**iii** 786	**v** 621
v 90	**v** 597	**vii** 827	**viii** 564

b From your answers to part **a** which scores are most unlikely?

c Which scores in part **a** lie within 2 standard deviations of the mean?

d Which scores in part **a** lie within 3 standard deviations of the mean?

6 The mean diameter of a batch of circular discs is 142 mm with standard deviation 14

a What is the z-score for a disc with a diameter of

i 16 mm?	**ii** 12 mm?

b Find the diameter of a disc with z-score

i –21	**ii** 13	**iii** 32
v –076	**v** 195	

7 A set of data that is normally distributed has a mean of 23 and standard deviation 2 Which raw score has a z-score of 25 ?

8 A data set that is normally distributed has a standard deviation of 45 A score of 39 has a z-score of 27 What is the mean ?

9 Find the standard deviation of a normally distributed data set if the mean is 89 and a raw score of 59 has z-score of –06

10 A set of data that is normally distributed has a mean of 534 and standard deviation of 56 Find the raw score that has a z-score of

a 0	**b** –2	**c** 1
d 28	**e** –17	

ISBN 97807043220

11 The standard deviation of a normal distribution is 33 and the z-score of 45 is −1. Calculate the mean

12 The mean of a normally distributed data set is 16 and standard deviation is 19

a Find the scores between which

i 95% of data lies **ii** 68% of data lies **iii** 997% of data lies

b Calculate the z-score of

i 20 **ii** 135

c Find the raw score that has a z-score of

i −3 **ii** 11

13 A normal distribution has a mean of 1047 and standard deviation 51

a Find the scores that lie within 1 standard deviation of the mean

b Calculate the z-score of

i 80 **ii** 103

c Which score has a z-score of

i 2? **ii** −13 ?

10.08 Applications of the normal distribution

The normal distribution is often used in quality control and predicting outcomes

WS Applying he nomal diibuion

WS Nomal disibuion woded poblems

WS Nomal disibuion woded poblem 2

WS Coninuou andom vaiable aignmen

WS Coninuou andom vaiable poblem

EXAMPLE 18

a A company produces 1 kg packets of sugar. A quality control check found that the weight of the packets was normally distributed with a mean weight of 0995 kg and standard deviation 003 kg The company policy is to reject any packet with a weight outside 2 standard deviations from the mean

i What is the smallest weight allowed by the company?

ii What is the largest weight allowed?

iii What percentage of packets will be rejected?

v What percentage of large packets will be rejected?

b The mean shelf life of a spice is 134 weeks and the standard deviation is 18 weeks

i Would a shelf life of 20 weeks be unusual? Why?

ii Find the z-score for a shelf life of 155 weeks

iii What percentage of shelf lives would be expected to be between 134 and 155 weeks?

ISBN 97807043220

Solution

a $\mu = 0995$

i Smallest weight allowed

$\mu - 2\sigma = 0995 - 2 \times 003$

$= 0935$

ii Largest weight allowed

$\mu + 2\sigma = 0995 + 2 \times 003$

$= 1055$

iii About 95% of weights lie within 2 standard deviations of the mean so 5% will lie outside this area

The company rejects 5% of the packets of sugar.

v Since the normal distribution is symmetrical the 5% is made up of .5% of larger and 25% of smaller packets

So the company rejects 25% of larger packets

b **i** $\mu = 134$

$\mu - 3\sigma = 134 - 3 \times 1.8$

$= 8$

$\mu + 3\sigma = 134 + 3 \times 1.8$

$= 18.8$

So the shelf life of spices almost certainly lies between 8 and 188 weeks

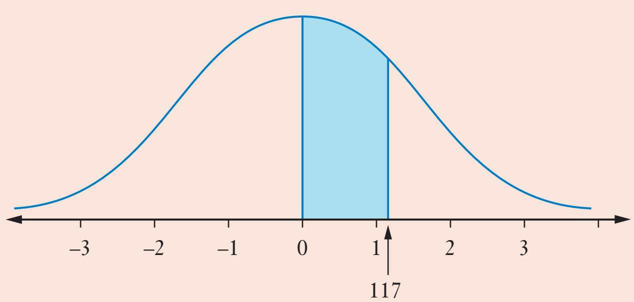
A shelf life of 20 weeks is outside this range so it would be unusua.

ii $z = \dfrac{x - \mu}{\sigma}$

$= \dfrac{15.5 - 13.4}{1\ 8}$

$= 1.17$

iii



$\mu = 134$ so its z-score $= 0$

$P(134 \le X \le 155) = P(0 \le z \le 1.17)$

$= P(X \le 1.17) - P(X \le 0)$

$= 08790 - 05000$ (using the table on pages 633–634)

$= 0379$

So 379% of shelf lives would be expected to be between 134 and 155 weeks

ISBN 97807043220

Using z-scores also allows us to compare 2 data sets

EXAMPLE 19

In Year 7 at a school the mean weight of students was 9.4 kg and the standard deviation was 38 kg In Yea 8, the mean wa 63.5 kg and the standard deviation ws 1. kg.

John in Year7, and Deng in Yar 8, both weighed68 kg. Which student was heavier in relation to his Year?

Solution

For John

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{68 - 59.4}{3\ 8}$$

$$\approx 2263$$

For Deng

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{68 - 63.5}{1\ 7}$$

$$\approx 2647$$

A z-score of 2647 is higher than a z-score of 2263

So Deng weighed more in relation to his Year group than Jon.

Exercise 10.08 Applications of the normal distribution

1 A machine at the mint produces coins with a mean diameter of 24 mm and a standard deviation of 02 mm

a What percentage of coins will have a diameter between

- **i** 234 mm and 246 mm ?
- **ii** 24 mm and 244 mm ?
- **iii** 238 mm and 242 mm ?
- **v** 234 mm and 24 mm ?
- **v** 234 mm and 242 mm ?

b A coin selected at random has a diameter of 248 mm

- **i** What is its z-score?
- **ii** Is this diameter unusual? What could you say about this?

c

- **i** Convert a diameter of 237 mm to a z-score
- **ii** Find the probability of producing a coin with a diameter of less than 237 mm
- **iii** Find the probability of producing a coin with a diameter between 237 mm and 239 mm

2 The maximum temperature in April is normally distributed with a mean of 2.3°C and a standard deviation of 34°C

a What percentage of the time in April would you expect the temperature to be between

i 219° and 287° ? **ii** 185° and 321° ? **iii** 151° and 355° ?

v 253° and 287° ? **v** 219° and 321° ?

b The temperature drops to 14°

i What is its z-score? **ii** Is this temperature unusual? Why?

c The temperature rises to 38°

i What is its z-score? **ii** Is this temperature unusual? Why?

3 A company surveyed the amount of time it took to assemble its product The times were normally distributed with mean 8 minutes and standard deviation 17 minutes

a What is the z-score for a time of 15 minutes? Would this be an acceptable amount of time to assemble the product? Why?

b Between what times would 95% of times lie?

c What percentage of times would lie between 46 and 131 minutes ?

d **i** Find the z-score for a time of 7 minutes 30 seconds

ii What percentage of times would lie between 7 minutes 30 seconds and 8 minutes?

e Find the probability of times lying between

i 6 and 8 minutes **ii** 6 and 97 minutes **iii** 5 and 10 minutes

f What percentage of times would lie between

i 3 and 12 minutes? **ii** 7 and 11 minutes? **iii** 35 and 119 minutes ?

4 A certain brand of perfume is sold in 20 mL bottles The volume of perfume in the bottles was tested in a quality control check The mean was found to be 9.9 mL with a standard deviation of 04 mL

a What percentage of bottles have a volume between

i 191 mL and 207 mL ?

ii 199 mL and 211 mL ?

iii 20 mL and 21 mL?

b Between which two volumes do 997% of bottles lie ?

c Comment on a bottle that has a volume of 23 mL

d Find the probability that a bottle of perfume will have a volume between 189 and 193 mL

ISBN 97807043220

5 A farmer is only allowed to deliver standard size apples to the markets The mean diameter must be 75 cm with a standard deviation of 03 cm Only apples within 2 standard deviations of the mean are allowed

a What percentage of apples are allowed?

b What is the largest diameter allowed?

c What is the smallest diameter allowed?

6 A certain brand of car battery has a mean life of 31 years with a standard deviation of 03 years

a What is the minimum life you could reasonably expect from a battery of this type?

b What percentage of batteries would have a life between

i 28 and 34 years ? **ii** 25 and 35 years ?

c Would a life of over 4 years be unusual? Why?

7 A Jack Russell terrier has a mean height of 28 cm and a standard deviation of 0833 cm

a What range of heights (to 1 decimal place) would you expect for this breed of dog?

b What is the z-score of a height of 272 cm ?

c What percentage of dogs would be between 272 cm and 30 cm tall ?

d Would a dog 24 cm tall be typical of this breed? Why?

8 A factory produces 5 kg bags of bread mix In a quality chec, the mean weight of a bag of bread mix was found to be 495 kg with a standard deviation of .15 k. The factory rejects bags that weigh less than 465 kg or more than 525 kg

a What percentage of bags does the factory reject?

b What percentage of bags does the factory reject because their weight is too small?

c The manager of the factory decided that too many bags are being rejected and that in future only those outside the normal range of 3 standard deviations would be rejected

i What percentage will be rejected?

ii What weights will be rejected?

9 A company manufactures steel rods with a mean diameter of 106 cm and a standard deviation of 05 cm The manufacturer rejects rods that are outside acceptable limis.

a If the company rejects 03% of rods fin:

i the percentage of rods it accepts

ii the largest diameter it will accept

iii the smallest diameter it will accept

b In one batch of rods the diameters are 1.9 c,9.6 m, 8.3cm, 11.4 cm and 126 cm Which ones will be rejected ?

ISBN 97807043220

10 A manufacturer of canned fruit guarantees that the minimum weight in each can is 250 g A random check showed that the mean weight was 22.5 g with a standard deviation of 04 g Comment on this guarante. Is it realistic ? Why?

11 A butcher shop advertises that it will give a free leg of lamb to any customer who can prove that a packet of mince with a mean weight of 1 kg weighs less than 980 g If the mean weight is 1 kg with a standard deviation of 10 g what percentage of customers should expect to receive a free leg of lamb?

12 The width of a type of door almost certainly lies within the range 814 mm to 826 mm (3 standard deviations of the mean)

a What is the standard deviation?

b What is the mean width of the door?

c Within what widths do 95% of these doors lie?

13 The mean time for a ferry to travel from one port to another is 31 hours About 475% of the time the ferry takes between 31 hours and 35 hours

a What is the standard deviation?

b What is the minimum time you would expect the ferry to take?

14 Xavier takes 78 minutes to drive to Epping and 65 minutes to drive to the city. The mean time to Epping is 753 minutes with a standard deviation of 26 while the mean time to the city is 627 minutes with a standard deviation of 17

a Calculate the z-scores for Xaviers trips to Epping and the cit.

b Which was the longer trip in comparison with the mean?

15 Kieran scored 78 in an exam where the mean was 695 and the standard deviation was 85 Cameron scored 71 in an exam where the mean was 612 and the standard deviation was 48

a Find Kierans z-score

b Calculate Camerons z-score

c Which student scored higher in comparison with the other students in each exam?

ISBN 97807043220

10. TEST YOURSELF

WS
Fomula hee: Saisics and pobabiliy

Qz
Pacice quiz

For Questions **1** to **3** choose the correct answer **A B C** or **D**

1 Which function does not describe a continuous probability distribution?

A $f(x) = \frac{x^2}{21}$ for the interval $1 \le x \le 4$ **B** $f(x) = \frac{e^x}{e^3 - 1}$ in the domain [0 3]

C $f(x) = \frac{x^4}{625}$ in the domain [1 5] **D** $f(x) = \frac{4x^3}{625}$ in the interval $0 \le x \le 5$

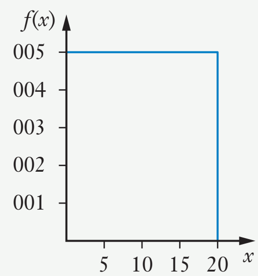
2 The percentage of z-scores between -2 and 2 is

A 95% **B** 68% **C** 475% **D** 4985%

3 Which random variable is not continuous?

A The temperature of different freezers
B The mass of rocks found at the site of a volcano
C The length of the arms of people
D Shoe sizes of people

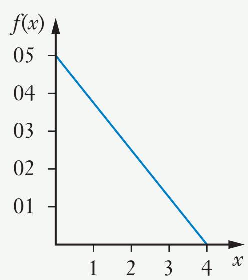
4 For the uniform continuous probability distribution shown fin:



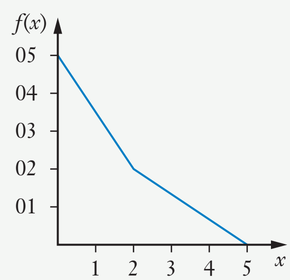
a $P(X \le 15)$ **b** $P(X \le 8)$
c $P(7 \le X \le 18)$ **d** $P(4 < X < 13)$
e $P(X \ge 6)$ **f** the median
g the 18th percentile **h** the 89th percentile
i the 6th decile **j** the 3rd quartile

5 State whether each graph represents a probability density function

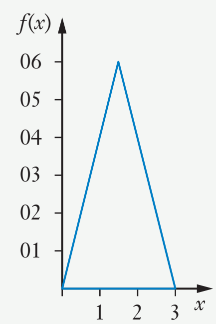
a



b



c



6 Find the median of the continuous random variable $f(x) = \frac{3x^2}{124}$ defined on the interval $1 \le x \le 5$ Answer correct to 2 decimal placs.

7 **a** Find the cumulative distribution function for $f(x) = \begin{cases} \frac{3x^2}{511} & \text{for } 1 \le x \le 8 \\ 0 & \text{for all other } x \end{cases}$

b Find

i $P(X \le 3)$ **ii** $P(X \le 5)$ **iii** $P(X > 6)$

v $P(X \ge 4)$ **v** $P(2 \le X \le 7)$

8 For each continuous probability distribution fin:

i the mode **ii** the median

a $f(x) = \frac{2x}{15}$ defined in the domain [1 4]

b $f(x) = \frac{x^2}{243}$ defined in the interval $0 \le x \le 9$

9 The birth weights of babies born at St Johns Hospital were measured and found to be normally distributed with mean 32 kg and standard deviation 031

a Find the range of weights in which 997% of the weights of these babies would lie

b Find the z-score (to 2 decimal places) for a weight of

i 39 kg **ii** 35 kg

c Use the standard normal probability table on pages 633–634 to find the probability that a baby born at the hospital would have a birth weight between 35 kg and 39 kg

10 A function is given by $f(x) = \frac{3x^5}{2048}$ Over what domain starting at $x = 0$ is this a probability density function?

11 Draw a probability density function for the normal distribution with

a mean 15 and standard deviation 05 **b** mean 34 and standard deviation 02

12 A factory produces 2 kg bags of nails In a quality chec, the mean weight of a bag of nails was found to be 195 kg with a standard deviation of .08 k. The factory rejects bags that weigh less than 171 kg or more than 219 kg

a What percentage of bags does the factory reject?

b After a complaint the manager decided to reject bags that weigh less than .87 kg or more than 211 kg What percentage will be rejected ?

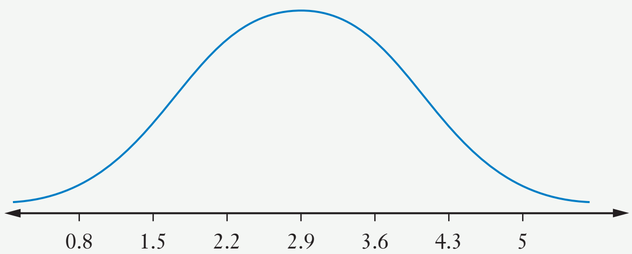
13 For the PDF $f(x) = \frac{3x^2}{316}$ defined on the interval $3 \le x \le 7$ fin:

a the median **b** the 3rd quartile **c** the 4th decile

d the 63rd percentile **e** the 28th percentile

ISBN 97807043220

14 Find the mean and standard deviation of this normal distribution



15 Find the cumulative distribution function for each continuous probability distribution

a $f(x) = \dfrac{x^4}{625}$ defined on the interval $0 \le x \le 5$

b $f(x) = \dfrac{x^6}{117\,649}$ defined on $[0\ 7]$

c $f(x) = \dfrac{e^x}{e^6 - 1}$ in the interval $0 \le x \le 6$

d $f(x) = \dfrac{x}{40}$ for $[1, 9]$

16 Use the probability table for a standard normal distribution on pages 633– 634 to find

a	$P(z \le 054)$	**b**	$P(z \le 1.32)$	**c**	$P(z \le -3)$
d	$P(z \le -071)$	**e**	$P(z \ge -1)$	**f**	$P(z \ge 25)$
g	$P(z \ge -108)$	**h**	$P(-23\ \ \le z \le -109)$	**i**	$P(11\ \ \le z \le 3.11)$

17 A set of data is normally distributed with mean 125 and standard deviation 15
Find the percentage of data that lies between

a	95 and 155	**b**	125 and 14	**c**	11 and 17
d	10 and 15	**e**	12 and 13		

18 A probability density function is given by $f(x) = ae^x$ over a certain domain
Find the exact value of a if the domain is

a $[0\ 5]$ **b** $[1, 4]$

19 Shade on the standard normal distribution the area where values lie

a in the bottom 20%

b in the bottom 32%

c in the top 15%

d in the top 30%

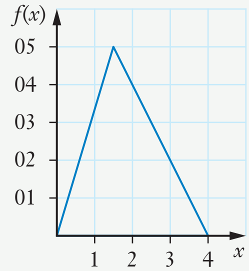
e between the bottom 10% and the top 40%

ISBN 97807043220

20 Find the mode of each continuous probability distribution

a $f(x) = \dfrac{x}{30}$ defined in the domain [2 8]

b



c $f(x) = \dfrac{4}{189}(x^3 - 9x^2 + 24x)$ defined in the domain [0 3]

21 Klare took 93 s to finish a race where the mean was 83 s and the standard deviation was 12 Simon took .9 s to run in the next heat where the mean was .1 s and the standard deviation was 08

a Find Klares z-score

b Calculate Simons z-score

c Which person had the better time in comparison with the other runners in their heat?

22 Show that $f(x) = \dfrac{x^3}{600}$ defined on the interval $1 \le x \le 7$ is a probability density function

23 Circular tables are made with a mean radius of 11 m with standard deviation 002 m

a Find the z-score for a table with a radius of

i 115 m **ii** 107 m

b Find the radius of a table with z-score

i 1 **ii** −2 **iii** 31 **v** −063 **v** 127

24 A normal distribution has a standard deviation of 16 A score of 7.9 has a z-score of −13 What is the mean ?

25 The standard deviation of a normal distribution is 19 and the z-score of 524 is 17 Calculate the mean

ISBN 97807043220

10. CHALLENGE EXERCISE

1 A probability density function is given by $f(x) = \dfrac{5x^4}{3124}$ over the interval $a \le x \le b$

Find the value of a and b if the median of the PDF is 4353 031

2 A normal distribution has mean μ and standard deviation σ
Evaluate μ and σ given that 95% of scores lie between 124 and 14

3 Find the mode of the continuous probability distribution $f(x) = \dfrac{4}{249}(x+2)(x-4)^2$ defined in the domain [0 3.

4 Find the cumulative distribution function for the continuous probability distribution
$f(x) = \dfrac{3x(x^2+1)^2}{62\,000}$ defined on the domain [3 7.

5 For a normal distribution $P(X \le 238) = 9192\%$ and $P(X \le 17.15) = 3085\%$
Find the mean and standard deviation

6 A good model for a normal distribution is the function $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)}{2\sigma}}$

Use technology to show that this is a probability density function given $\mu = 6$ and $\sigma = 1$.

ISBN 97807043220

Practice set 4

In Questions **1** to **5** select the correct answer **A B C** or **D**

1 An amount of \$6000 is invested at 35% pa with interest paid quarterl. Find the balance after 10 years

A \$850145 **B** \$846359 **C** \$654616 **D** \$689996

2 Which function does **not** describe a continuous probability distribution?

A $f(x) = \dfrac{3x^2}{124}$ for the interval $1 \le x \le 5$

B $f(x) = \dfrac{8x(x^2+1)^3}{1000}$ in the domain [0 3]

C $f(x) = 4x^3$ in the domain [0 1]

D $f(x) = 2 \cos 2x$ in the interval $0 \le x \le \dfrac{\pi}{4}$

3 The percentage of scores between z-scores of -3 and 3 is

A 95% **B** 68% **C** 997% **D** 4985%

4 The percentage of data in a normal distribution that lies within 2 standard deviations of the mean is

A 68% **B** 95% **C** 997% **D** 34%

5 The median of a continuous probability distribution $f(x)$ defined in the domain $[a \ b]$ is

A x where $\int_a^b f(x)\,dx = x$

B x at the maximum value of $f(x)$

C x where $\int_a^x f(x)\,dx = 05$

D x where $\int_a^b 0\ 5\,dx = F(x)$

6 Find each integral

a $\int e^{3x}\,dx$ **b** $\int (4x-3)\,dx$ **c** $\int \sec^2 4x\,dx$ **d** $\int \dfrac{dx}{x-3}$

7 Use the table of future values of an annuity on page 546 to answer each question

a Mahmoud wants to deposit \$1000 at the end of each year for 20 years so he has a nest egg when he retires from work If interest is 7% ., find how much Mahmoud will have

b Georgina wants to save a certain amount at the end of each quarter for 3 years so that she will have \$10 000 for an overseas trip If interest is 8% .. paid quarterly, how much will Georgina need to save each quarter?

ISBN 97807043220

8 Alice promises her son a sum of money on his 18th birthday, made up of \$10 for his 1st year of life \$15 for his 2nd yea, \$20 for his 3rd year and so on up to 18 year. How much will her son receive?

9 A plant grows so that it increases its height each month by 02 of its previous months heigh. If it grows to 3m, find its height in the first moth.

10 **a** Find the cumulative distribution function for $f(x) = \frac{x^2}{114}$ defined for [1 7.

b Find as a fraction

i $P(X \leq 5)$ **ii** $P(X \leq 2)$ **iii** $P(X > 3)$

v $P(X \geq 4)$ **v** $P(3 \leq X \leq 6)$

c Find correct to 2 decimal places

i the median **ii** the 93rd percentile

11 A farmer places 20 bales of hay in a row in the shed He then stacks 17 on top of these then 14 in the next row up and so o, continuing with this patten.

a How many bales of hay are in the top row?

b How many rows are there?

c How many bales of hay are stacked in the shed?

12 Ryanna has \$120 000 in a superannuation trust fun. She withdraws \$1600 each month as a pension

a If the trust fund earns 6% pa paid monthl, find the amount left in the fund afer:

i 1 month **ii** 2 months **iii** 3 months

b **i** Show that the amount left after n months is given by

$120\,000(1005)^n - 1600(1 + 1005 + 1005^2 + \quad + 1005^{n-1})$

ii How much will be in the trust fund after 20 months?

13 I put \$2000 in the bank where it earns interest at the rate of 12% ., paid quarterly. How much will there be in my account after 3 years?

14 Find the median of the continuous random variable $f(x) = \frac{4x^3}{609}$ defined on the interval $2 \leq x \leq 5$.

15 Express 0 1 7 as a fraction

16 Find any stationary points and points of inflection on the graph of the function $y = x^3 - 6x^2 - 15x + 1$.

17 I borrow \$5000 at 18% interest pa compounded monthly and make equal monthly payments over 3 years at the end of which the loan is fully paid ou.
Find the amount of each monthly payment

ISBN 97807043220

18 The volume of blood in adult humans is normally distributed with mean 47 L and standard deviation 04

a What would be the range of blood volumes for 95% of adults?

b What percentage of blood volumes would lie between 43 L and 51 L ?

c Use the table of standard normal probabilities on pages 633–634 to find the z-score for a volume of

i 6 L **ii** 52 L **iii** 39 L **v** 41 L

d Find the probability for the blood volume X

i $P(X \le 6)$ **ii** $P(X \le 39)$ **iii** $P(X \ge 41)$

v $P(X \ge 52)$ **v** $P(39 \le X \le 52)$

e Find the percentage of blood volumes that lie between

i 4 L and 5 L **ii** 42 L and 52 L **iii** 36 L and 58 L

19 Use the repayments table for reducing balance loans on $1000 on page 557 to find the monthly repayments for a loan of

a $25 000 at 4% pa over 5 years **b** $100 000 at 25% pa over 25 years

c $128 500 at 6% pa over 15 years **d** $2400 at 75% pa over 10 years

20 Find the cumulative distribution function for each continuous probability distribution

a $f(x) = \frac{3(x+2)^2}{335}$ defined in the domain [0 5]

b $f(x) = \frac{x^3}{156}$ defined in the domain [1 5]

c $f(x) = 2 \cos x$ in the interval $0 \le x \le \frac{\pi}{6}$

21 Find the mode of the continuous probability distribution $f(x) = \frac{4x - x^2}{9}$ defined in the domain [1 4.

22 Use the probability tables for a standard normal distribution on pages 633–4 to find

a $P(z \le 1.35)$ **b** $P(z \ge -088)$ **c** $P(z \le -1)$

d $P(z \ge 204)$ **e** $P(-312 \le z \le 281)$

23 Li scored 72% in her first maths exam in which the class mean was 69% with standard deviation 08 She scored 65% in her second exam with class mean 55% and standard deviation 12 In which exam did Li do better in relation to her class ?

24 I borrow $10 000 over 5 years at 185% monthly interest How much do I need to pay each month?

ISBN 97807043220

25 A function is given by $f(x) = \dfrac{x^4}{1555}$ Over what domain starting at $x = 1$ is this a probability density function?

26 A data set is normally distributed with mean 125 and standard deviation 13
Find the raw score for each z-score

a 04 **b** –15 **c** 296 **d** –3

27 Find the mode of the continuous probability distribution $f(x) = \dfrac{3x^2}{512}$ defined in the domain [0 8.

28 A normal distribution has a standard deviation of 25 A score of 19 has a z-score of 02 What is the mean?

29 A factory produces 350 mL cans of soft drink In a quality chec, the mean volume of soft drink in cans was found to be 3498 mL with a standard deviation of .2 m. The factory rejects cans that are outside 2 standard deviations of the mean

a Find the percentage and volumes of cans that the factory rejects

b The factory manager decides they are rejecting too many cans and that only cans whose volume is outside 3 standard deviations of the mean will be rejected What percentage and volumes of cans will the factory reject?

30 A sum of $2500 is put into a bank account where it earns 24% pa
Find the amount in the bank after 4 years if interest is paid

a annually **b** quarterly **c** monthly

ANSWERS

Answers are based on full calculator values and only rounded at the en, even when different parts of a question require rounding This gives more accurate answes. Answers based on reading graphs may not be accuate.

Chapter 1

Exercse 101

1 **a** 14 1, 20 **b** 23, 28, 33
c 44 5, 66 **d** 85 8, 75
e 1, −1, −3 **f** −15 −24 −33
g $2, 2\frac{}{2}, 3$ **h** 31, .7,4.3
32 −64 128 **j** $\frac{27}{320}$ $\frac{81}{1280}$ $\frac{243}{5120}$

2 **a** 1456 **b** 63 **c** 78
d 126 **e** 91 **f** 441

3 $\frac{1}{16}$ $\frac{1}{32}$ $\frac{1}{64}$ **4** 38 5, 6, 83

5 21 3, 5,89, 144

6

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Exercse 102

1 **a** $y = 13$ **b** $x = -4$ **c** $x = 72$
d $b = 11$ **e** $x = 7$ **f** $x = 42$
g $k = 2$ **h** $x = 1$ $t = -2$
j $t = 3$

2 **a** 46 **b** 78 **c** 94
d −6 **e** 67

3 **a** 590 **b** −850 **c** 414
d 1610 **e** −397

4 **a** −110 **b** 124 **c** −83
d 37 **e** $15\frac{4}{5}$

5 $T_n = 2n + 1$

6 **a** $T_n = 8n + 1$ **b** $T_n = 2n + 98$
c $T_n = 3n + 3$ **d** $T_n = 6n + 74$
e $T_n = 4n - 25$ **f** $T_n = 20 - 5n$
g $T_n = \frac{n+6}{8}$ **h** $T_n = -2n - 28$
$T_n = 1.2n + 2$ **j** $T_n = \frac{3n-1}{4}$

7 28th term **8** 54th term
9 30th term **10** 15th term
11 Yes **12** No
13 Yes **14** $n = 13$
15 $n = 30, 31, 32, \ldots$ **16** −2
17 103 **18** 785
19 **a** $d = 8$ **b** 87
20 $d = 9$ **21** $a = 12, d = 7$
22 173 **23** $a = 5$
24 280 **25** 1133

26 **a** $T_2 - T\ = T_3 - T_2 = d = \log_5 x$
b $80 \log_5 x$ or $\log_5 x^{80}$ **c** 86

27 **a** $T_2 - T\ = T_3 - T_2 = d = \sqrt{3}$
b $50\sqrt{3}$

28 26 **29** $122b$ **30** 38th term

Exercse 103

1 **a** 375 **b** 555 **c** 480
2 **a** 2640 **b** 4365 **c** 240
3 **a** 2050 **b** −2575
4 **a** −4850 **b** 4225
5 **a** 28 875 **b** 3276 **c** −1419
d 6426 **e** 6604 **f** 598
g −2700 **h** 11 704 −290
j 1284
6 21 **7** 8 **8** 11

9 8 and 13 terms
10 $a = 14, d = 4$ **11** $a = -3, d = 5$
12 2025 **13** 3420 **14** 1010
15 **a** $(2x + 4) - (x + 1) = (3x + 7) - (2x + 4) = x + 3$
b $25(51x + 149)$
16 1290 **17** 16
18 $S_n = S_{n-1} + T_n$
So $S_n - S_{n-1} = T_n$
19 **a** 816 **b** 4234

Exercse 104

1 **a** No **b** Ye, $r = -\frac{3}{4}$ **c** Ye, $r = \frac{2}{7}$
d No **e** No **f** No
g Ye, $r = 03$ **h** Ye, $r = -\frac{3}{5}$ No
j Ye, $r = -8$
2 **a** $x = 196$ **b** $y = -48$
c $a = \pm 12$ **d** $y = \frac{2}{3}$
e $x = 2$ **f** $p = \pm 10$
g $y = \pm 21$ **h** $m = \pm 6$
$x = 4 \pm 3\sqrt{5}$ **j** $k = 1 \pm 3\sqrt{7}$
k $t = \pm\frac{1}{6}$ $t = \pm\frac{2}{3}$
3 **a** $T_n = 5^{n-1}$ **b** $T_n = 102^{\,n-1}$
c $T_n = 9^{n-1}$ **d** $T_n = 2 \times 5^{n-1}$
e $T_n = 6 \times 3^{n-1}$ **f** $T_n = 8 \times 2^{n-1} = 2^{n+2}$
g $T_n = \frac{1}{4} \times 4^{n-1} = 4^{n-2}$ **h** $T_n = 1000(-01)^{\,n-1}$
$T_n = -3(-3)^{n-1} = (-3)^n$ **j** $T_n = \frac{1}{3}\left(\frac{2}{5}\right)^{n-}$
4 **a** 1944 **b** 9216 **c** −8192
d 3125 **e** $\frac{64}{729}$
5 **a** 256 **b** 26 244 **c** 1369
d −768 **e** $\frac{3}{1024}$
6 **a** 234 375 **b** 2688 **c** −81 920
d $\frac{2187}{156\,250}$ **e** 27
7 **a** 3×2^{9} **b** 7^{9} **c** 104^{20}
d $\frac{1}{4}\left(\frac{1}{2}\right)^{19} = \frac{1}{2^{21}}$ **e** $\left(\frac{3}{4}\right)^{20}$

8 11^{49} **9** 6th **10** 5th
11 No **12** 7th **13** 11th
14 9th **15** $n = 5$ **16** $r = 3$
17 **a** $r = -6$ **b** −18
18 $a = \frac{1}{10}$ $r = \pm 2$
19 $n = 7$ **20** $208\frac{2}{7}$

Exercse 105

1 **a** 2 097 150 **b** 7 324 218
2 **a** 720 600 **b** 26 240
3 **a** 131 068 **b** 05
4 **a** 7812 **b** $35\frac{55}{64}$ **c** 8403
d 273 **e** 255
5 **a** 1792 **b** 3577
6 14858 **7** 13333 **8** $n = 9$
9 10 terms **10** $a = 9$ **11** 10 terms
12 **a** 2046 **b** 100 **c** 2146

Puzzes

1 Choice 1 gives \$46500 Choice 2 gives \$10 737 41823 so choice 2 is better.
2 382 apples

Exercse 106

1 **a** Ye, 13 $\frac{1}{2}$ **b** No
c Ye, 12 $\frac{4}{5}$ **d** No
e Ye, 3 **f** Ye, $\frac{25}{32}$
g No **h** Ye, $-1\frac{5}{22}$
No **j** Ye, 1 $\frac{3}{7}$
2 **a** 80 **b** $426\frac{2}{3}$ **c** $66\frac{1}{3}$ **d** 12
e $\frac{7}{10}$ **f** 54 **g** $-10\frac{2}{7}$ **h** $\frac{9}{20}$
48 **j** $-\frac{16}{39}$
3 **a** 058 **b** 015 **c** 0000080
d 0016 **e** 089

4 $a = 4$ **5** $r = \frac{2}{5}$ **6** $a = 5\frac{3}{5}$

7 $r = \frac{7}{8}$ **8** $r = -\frac{1}{4}$ **9** $r = -\frac{2}{3}$

10 $a = 3, r = \frac{2}{3}$ or $a = 6, r = \frac{1}{3}$

11 $a = 192, r = -\frac{1}{4}$ $S = 153\frac{3}{5}$

12 $a = 1, r = \frac{2}{3}$ $S = 3$ or $a = -1, r = -\frac{2}{3}$ $S = -\frac{3}{5}$

13 $a = 150, r = \frac{3}{5}$ $S = 375$

14 $a = \frac{2}{5}$ $r = \frac{2}{3}$ $S = 1\frac{1}{5}$

15 $a = 3, r = \frac{2}{5}$ or $a = 2, r = \frac{3}{5}$

16 $x = \frac{21}{32}$

17 **a** $k| < 1$ **b** $-\frac{2}{5}$ **c** $k = \frac{3}{4}$

18 See worked solutions

Test yourslf 1

1 C **2** C **3** B

4 **a** $T_n = 4n + 5$ **b** $T_n = 14 - 7n$

c $T_n = 2 \times 3^{n-1}$ **d** $T_n = 200\left(\frac{1}{4}\right)^{n-}$

e $T_n = (-2)^n$

5 **a** 2 **b** 1185 **c** 1183

d $T_{15} = S_{15} - S_4$ $S_{15} = S_4 + T_{15}$

e $n = 16$

6 **a** i **b** ii **c** i **d** iii

e i **f** ii **g** ii **h** i

i **j** i

7 $n = 108$

8 $a = -33$ $d = 13$

9 **a** 59 **b** 80 **c** 18th

10 **a** $x = 25$ **b** $x = \pm 15$

11 $x = 3$

12 **a** 136 **b** 44 **c** 6

13 $121\frac{}{2}$

14 **a** $S_n = n(2n + 3)$ **b** $S_n = \frac{107^n - 1}{007}$

15 **a** $x| < 1$ **b** $2\frac{}{2}$ **c** $x = \frac{1}{3}$

16 $d = 5$

17 $x = -\frac{2}{17}$ 2

18 1300

29 **a** 735 **b** 4315

20 $n = 20$

21 $n = 11$

Chaenge exercse 1

1 **a** 81 **b** 19th

2 **a** $\frac{\pi}{4}$ **b** $\frac{9\pi}{4}$ **c** $\frac{33\pi}{4}$

3 **a** 2 097 170 **b** –698 775

4 6th **5** 17 823

6 $n \geq 5$

7 **a** $a = 7, r = -2$ **b** –56

8 **a** $\operatorname{cosec}^2 x$

b $r = \cos^2 x$

$-1 < \cos x < 1$ where $\cos^2 x \neq 0, 1$

So $0 < \cos^2 x < 1$

Since $|r| < 1$ the series has a limiting su.

Chapter 2

Exercse 201

1 **a** Vertical translation 3 units up

b Vertical translation 7 units down

c Vertical translation 1 unit down

d Vertical translation 5 units up

2 **a** Vertical translation 1 unit up

b Vertical translation 4 units down

c Vertical translation 8 units up

3 Vertical translation 9 units up

4 **a** $y = x^2 - 3$ **b** $f(x) = 2^x + 8$

c $y = x| + 1$ **d** $y = x^3 - 4$

e $f(x) = \log x + 3$ **f** $y = \frac{2}{x} - 7$

5 **a** Vertical translation 1 unit down

b Vertical translation 6 units up

6 **a** $y = 2x^3 - 2$ **ii** $y = 2x^3 + 6$

b $y = x| - 3$ **ii** $y = x| - 6$

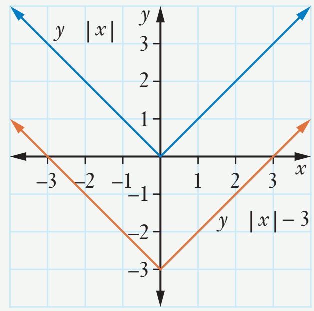
c $y = e^x + 1$ **ii** $y = e^x + 5$

d $f(x) = \log x + 10$ **ii** $f(x) = \log x - 8$

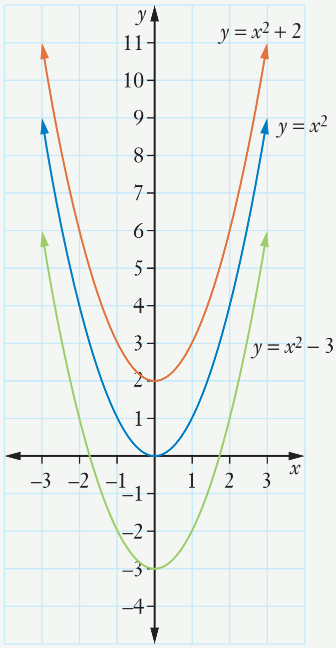
7 **a** $(1 \ -1)$ **b** $(1 \ -9)$ **c** $(1 \ -3 + m)$

8 **a** (−1, 1) **b** (−1, 5)

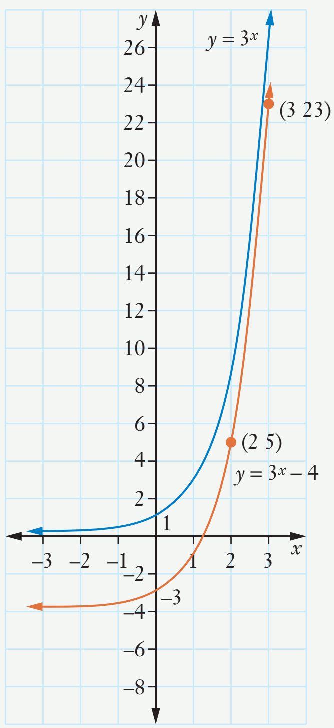
c



9 **a**

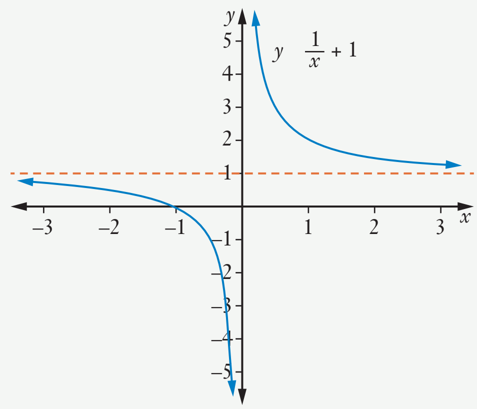


b

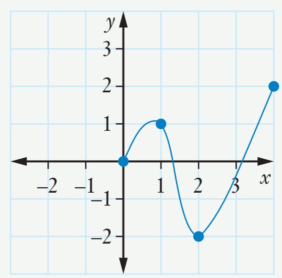


10 **a** Vertical translation 1 unit up

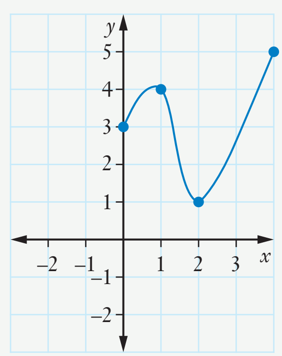
b



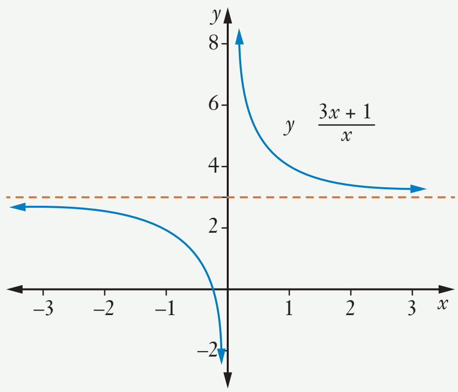
11 **a**



b



12 b



Exercse 202

1 a Horizontal translation 4 units to the right

b Horizontal translation 2 units to the left

2 a Horizontal translation 5 units to the right

b Horizontal translation 3 units to the left

3 a $y = (x + 3)^2$ **b** $f(x) = 2^{x-8}$

c $y = |x + 1|$ **d** $y = (x - 4)^3$

e $f(x) = \log(x + 3)$

4 Horizontal translation 3 units to the right

5 a Horizontal translation 2 units to the left

b Horizontal translation 5 units to the right

6 a $y = -(x + 4)^2$ **ii** $y = -(x - 8)^2$

b $y = |x - 3|$ **ii** $y = |x + 4|$

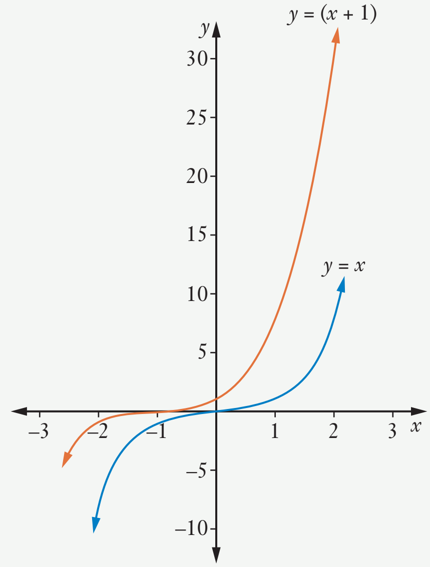
c $y = e^{x+6}$ **ii** $y = e^{x-5}$

d $f(x) = \log_2(x - 5)$ **ii** $f(x) = \log_2 x$

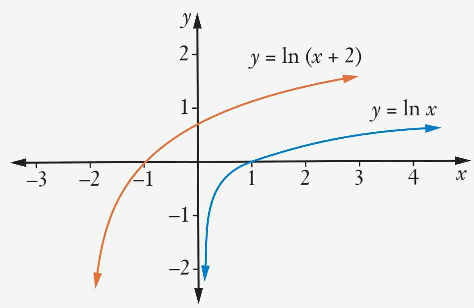
7 a $(5\ \ -3)$ **b** $(-8, -3)$ **c** $(1 + t\ \ -3)$

8 a $(3\ \ 2)$ **b** $(-9\ \ 2)$

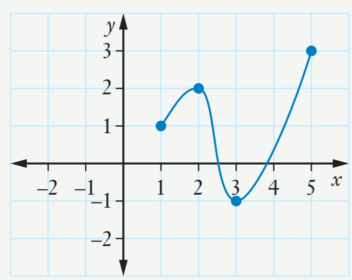
9 a



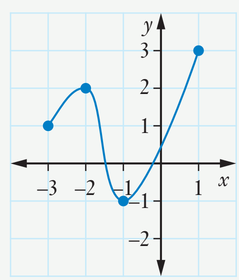
b



10 a



b



11 a $f(x) = x^5 - 5$ **b** $f(x) = (x - 3)^5$

c $f(x) = x^5 + 2$ **d** $f(x) = (x + 7)^5$

12 $(7\ \ -2)$

Exercse 203

1 **a** Vertical dilation scale factor 6 (stretched)

ii Vertical dilation scale factor $\frac{1}{2}$ (compressed)

iii Vertical dilation scale factor −1 (reflection in x-axis)

b Vertical dilation scale factor 2 (stretched)

ii Vertical dilation scale factor $\frac{1}{6}$ (compressed)

iii Vertical dilation scale factor −1 (reflection in x-axis)

c Vertical dilation scale factor 4 (stretched)

ii Vertical dilation scale factor $\frac{1}{7}$ (compressed)

iii Vertical dilation scale factor $\frac{4}{3}$ (stretched)

d Vertical dilation scale factor 9 (stretched)

ii Vertical dilation scale factor $\frac{1}{3}$ (compressed)

iii Vertical dilation scale factor $\frac{3}{8}$ (compressed)

e Vertical dilation scale factor 5 (stretched)

ii Vertical dilation scale factor $\frac{1}{8}$ (compressed)

iii Vertical dilation scale factor −1 (reflection in x-axis)

f Vertical dilation scale factor 9 (stretched)

ii Vertical dilation scale factor −1 (reflection in x-axis)

iii Vertical dilation scale factor $\frac{2}{5}$ (compressed)

2 **a** $y = 6x^2$ domain $(-\infty \ \ \infty)$ range $[0 \ \ \infty)$

b $y = \frac{\ln x}{4}$ domain $(0 \ \ \infty)$ range $(-\infty \ \ \infty)$

c $f(x) = -|x|$ domain $(-\infty \ \ \infty)$ range $(-\infty \ \ 0]$

d $f(x) = 4e^x$ domain $(-\infty \ \ \infty)$ range $(0 \ \ \infty)$

e $y = \frac{7}{x}$ domain $(-\infty \ \ 0) \cup (0 \ \ \infty)$ range $(-\infty \ \ 0) \cup (0 \ \ \infty)$

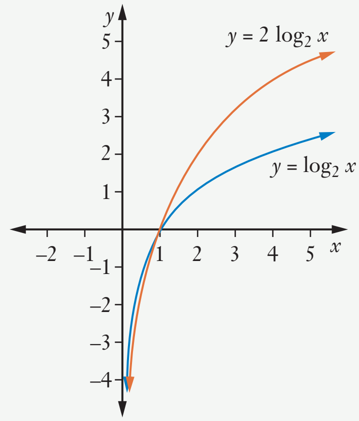
3 **a** $y = 5 \cdot 3^x$ **b** $f(x) = \frac{x^2}{3}$ **c** $y = -x^3$

d $y = \frac{1}{2x}$ **e** $y = \frac{2|x|}{3}$

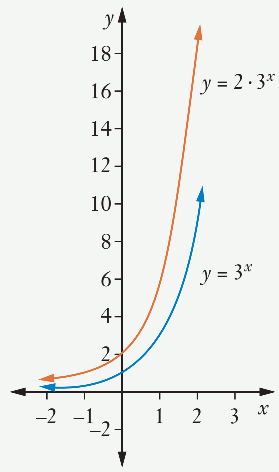
4 **a** (3 24) **b** (3 −6) **c** (3 72) **d** (3 5)

5 **a** (4 4) **b** (4 6) **c** (4 36) **d** (4 16)
e (4 −12)

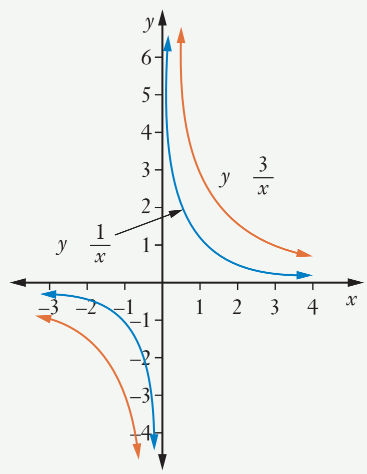
6 **a**



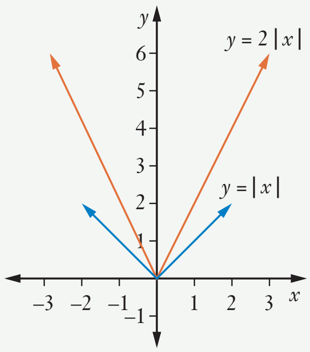
b



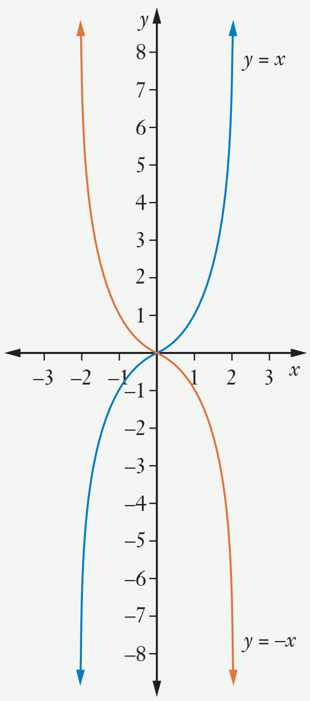
c



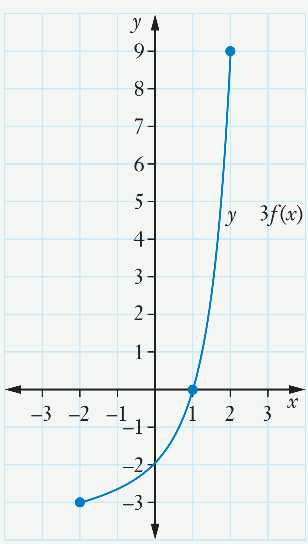
d



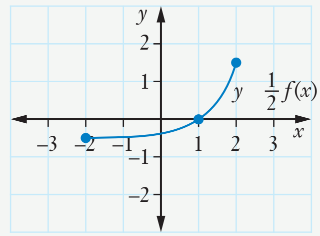
e



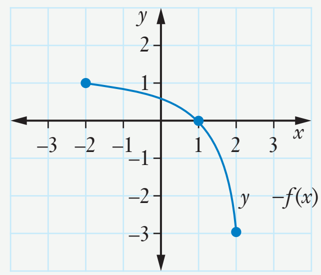
7 a



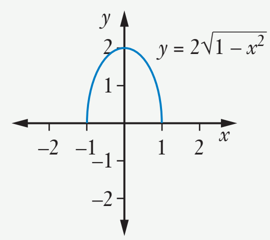
b



c



8



Exercse 204

1 a Horizontal dilation scale factor $\frac{1}{8}$ (compressed)

b Horizontal dilation scale factor 5 (stretched)

c Horizontal dilation scale factor $\frac{7}{3}$ (stretched)

d Horizontal dilation scale factor -1 (reflection in y-axis) Note that this is the same graph as $f(x) = x^4$

2 a Horizontal dilation scale factor $\frac{1}{2}$ (compressed)

ii Horizontal dilation scale factor $\frac{1}{5}$ (compressed)

iii Horizontal dilation scale factor 3 (stretched)

b Vertical dilation scale factor 4 (stretched)

ii Horizontal dilation scale factor 2 (stretched)

iii Horizontal dilation scale factor -1 (reflection in y-axis)

c Horizontal dilation scale factor $\frac{1}{7}$ (compressed)

ii Vertical dilation scale factor $\frac{1}{8}$ (compressed)

iii Horizontal dilation scale factor $\frac{4}{3}$ (stretched)

d Horizontal dilation scale factor $\frac{1}{5}$ (compressed) (or vertical dilation scale factor 5 stretched)

ii Horizontal dilation scale factor 2 (stretched) (or vertical dilation scale factor $\frac{1}{2}$ compressed)

iii Horizontal dilation scale factor $\frac{5}{3}$ (stretched) (or vertical dilation scale factor $\frac{3}{5}$ compressed)

e Horizontal dilation scale factor $\frac{1}{3}$ (compressed)

ii Vertical dilation scale factor –1 (reflection in x-axis)

iii Horizontal dilation scale factor 2 (stretched)

f Vertical dilation scale factor 8 (stretched)

ii Horizontal dilation scale factor –1 (reflection in y-axis)

iii Horizontal dilation scale factor 7 (stretched)

3 **a** $f(x) = \ 5x|$ domain $(-\infty \ \ \infty)$ range $[0 \ \ \infty)$

b $y = \left(\frac{x}{3}\right)^2$ domain $(-\infty \ \ \infty)$ range $[0 \ \ \infty)$

c $y = (-x)^3$ domain $(-\infty \ \ \infty)$ range $(-\infty \ \ \infty)$

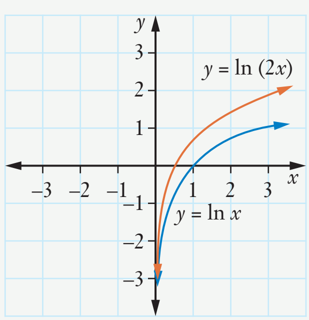
d $y = \frac{e^x}{9}$ domain $(-\infty \ \ \infty)$ range $(0 \ \ \infty)$

e $y = -\log_4 x$ domain $(0 \ \ \infty)$ range $(-\infty \ \ \infty)$

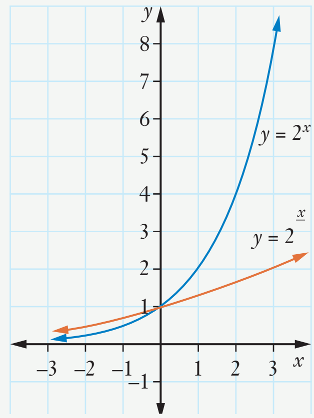
4 **a** (–1 7) **b** (2 7) **c** (–6 7)

5 **a** (–72 1) **b** (–48 1) **c** (–6 1)

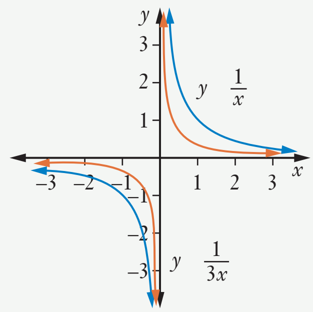
6 **a**



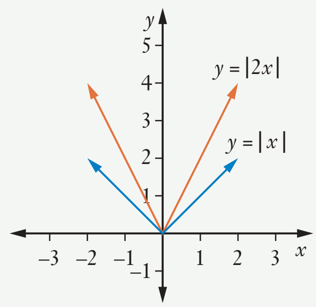
b



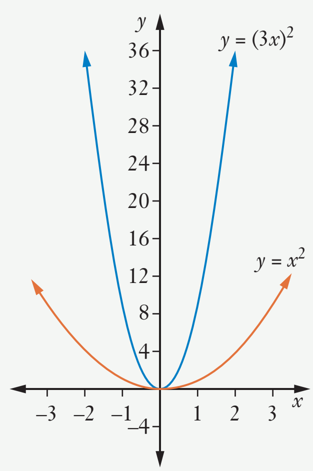
c



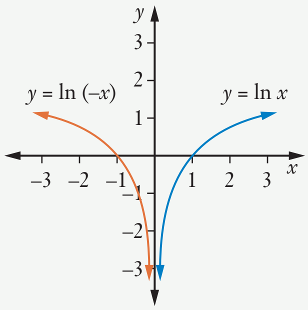
d



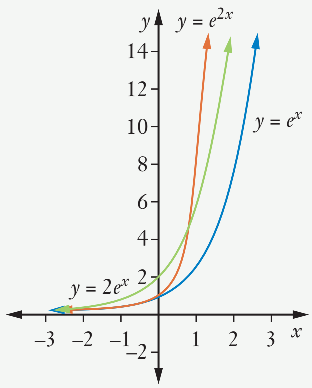
e



f



7

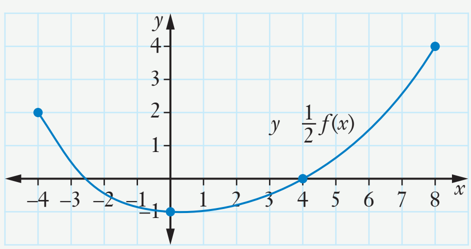


8 A reflection in y-axis transforms $y = f(x)$ into $y = f(-x)$

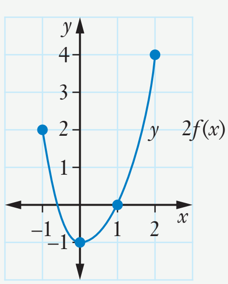
a Since $y = x^2$ is an even function $f(x) = f(-x)$ so a reflection in the y-axis doesnt change the function

b Since $y = \;x|$ is an even function $f(x) = f(-x)$ so a reflection in the y-axis doesnt change the function

9 a



b



Exercse 205

1 a (5 −11) **b** (−1, −2)

c (9 3) **d** (−2, −17)

2 a $f(x) = -\,4x^5$ **b** $f(x) = -\frac{1}{243}x^5$

3 a $y = (x+4)^3 - 3$ **b** $f(x) = \;x - 1| + 9$

c $f(x) = 3x - 6$ **d** $y = -e^x + 2$

e $y = (2x)^3 - 5$ **f** $f(x) = \frac{6}{x}$

g $f(x) = 3\sqrt{-2x}$ **h** $y = \ln\frac{x}{3} + 2$

$f(x) = 3\log_2 4x$ **j** $y = \left(\frac{x}{2}\right)^2 - 3$

4 a Horizontal translation 1 unit to the right vertical translation 7 units up

b Vertical dilation scale factor , vertical translation 1 unit down

c Vertical dilation scale factor , reflection in x-axis vertical translation 3 units down

d Horizontal translation 7 units to the left vertical dilation scale factor 2

e Rewrite as $y = 6[2(x-2)]^3 + 5$ Horizontal dilation scale factor $\frac{1}{2}$ horizontal translation 2 units to the right vertical dilation scale factor 6 vertical translation 5 units up

f Rewrite as $y = 2[3(x+3)]^3 - 10$ Horizontal dilation scale factor $\frac{1}{3}$ horizontal translation 3 units to the left vertical dilation scale factor 2 vertical translation 10 units down

5 a Horizontal translation 3 units to the left vertical dilation scale factor 2 vertical translation 1 unit down

b Horizontal dilation scale factor $\frac{1}{3}$ reflection in x-axis vertical translation 9 units up

c Horizontal dilation scale factor $\frac{1}{5}$ vertical dilation scale factor 2 vertical translation 3 units down

d Horizontal translation 7 units to the right vertical dilation scale factor 4 vertical translation 1 unit up

e Reflection in y-axis horizontal dilation scale factor $\frac{1}{2}$ horizontal translation 1 unit to the left vertical translation 1 unit down

f Horizontal dilation scale factor $\frac{1}{2}$ reflection in x-axis vertical translation 8 units up

6 **a** $(9\ \ -31)$ **b** $(4\ 5)$
c $(5\ \ -25)$ **d** $(-8, -67)$
e Change to $y = -2f[2(x-2)] - 3$ $(6, 21)$

7 **a** $(x+3, y-6)$ **b** $(-x\ \ y+6)$
c $(x-5, 2y)$ **d** $(3x\ \ y+5)$
e $(5x-30, -8y-1)$

8 **a** $y = f(x+1) - 2$ **b** $y = f(x-5) + 3$
c $y = -f(x-4)$ **d** $y = f(-x) + 2$
e $y = -f\left(\frac{x}{4}\right)$ **f** $y = 2f(x) - 2$

9 **a** $f(x) = -\frac{9}{x} + 3$ **b** $y = 5(x+2)^2 - 6$
c $f(x) = 8\ln\left[\frac{1}{2}(x-5)\right] - 3$
d $y = 9\sqrt{-(x+4)} + 4$ **e** $f(x) = -|6x| + 7$
f $y = [4(x+4)]^3 = 64(x+4)^3$
g $y = 6(2^{x-2} + 5)$

10 **a** Domain $(-\infty\ \ \infty)$ range $[,\ \ \infty)$
b Domain $(-\infty\ \ \infty)$ range $[-2, \infty)$
c Domain $(-\infty\ \ 2) \cup (2\ \ \infty)$, range $(-\infty, 1) \cup (1, \infty)$
d Domain $(-\infty\ \ \infty)$ range $(,\ \ \infty)$
e Domain $(2\ \ \infty)$ range $(-\infty\ \ \infty)$

11 **a** $y = (x+1)^2 - 8$
b Horizontal translation 1 unit to the left vertical translation 8 units down

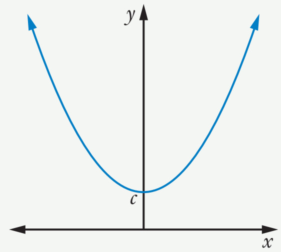
12 Horizontal translation 5 units to the right vertical translation 28 units down

13 **a** $(2x+3, 2y+5)$ **b** $\left(\frac{x}{3} - 6\ \ -y-2\right)$

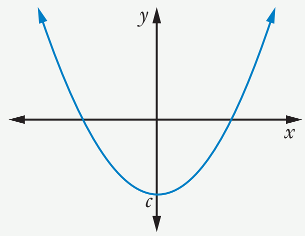
14 **a** Circle $(x-3)^2 + (y-4)^2 = 9$ or $x^2 - 6x + y^2 - 8x + 16 = 0$
b Translated 2 units to the righ, 3 units down

Exercse 206

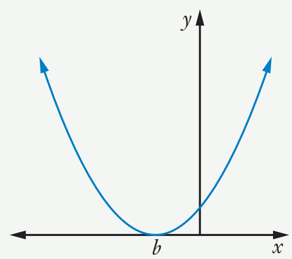
1 **a**



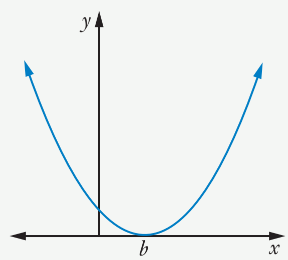
ii



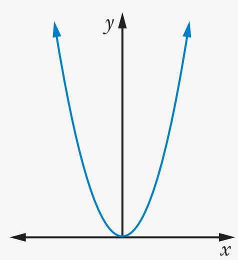
b



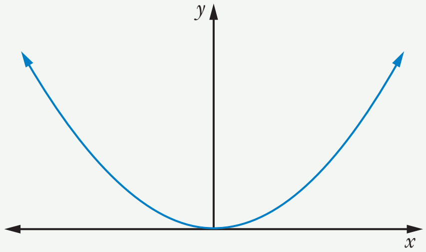
ii



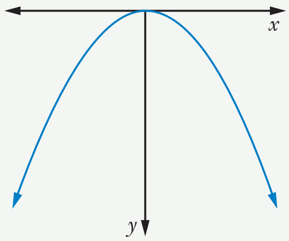
c



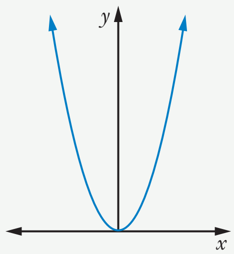
ii



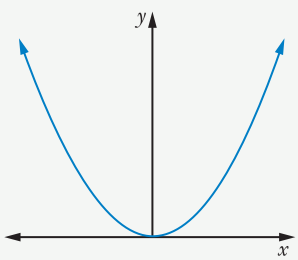
iii



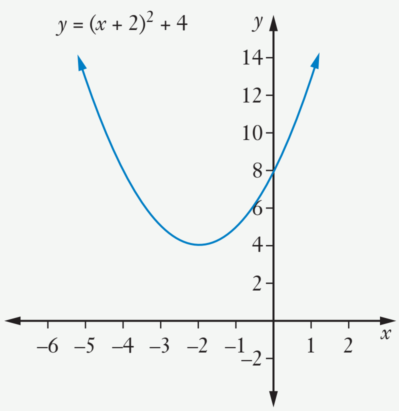
d i, iii



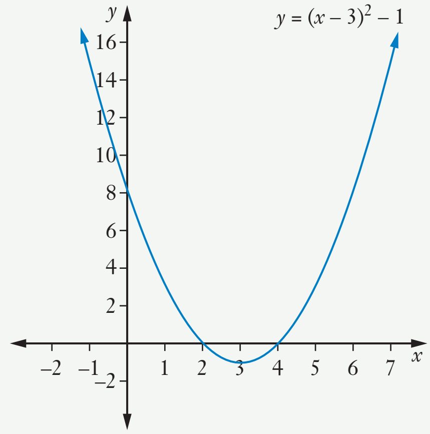
ii



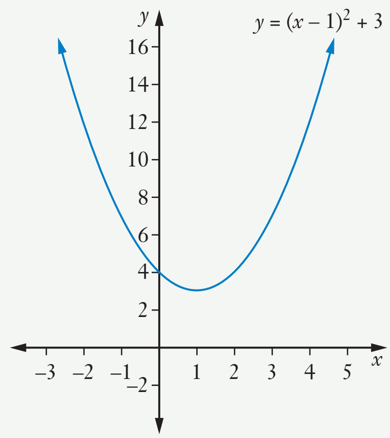
2 a



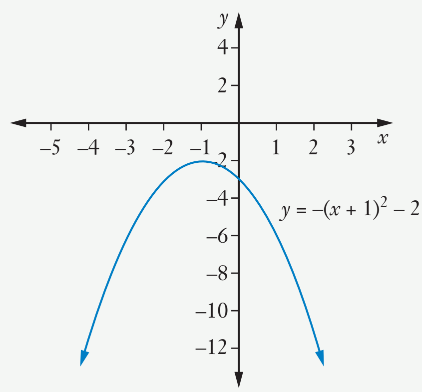
b



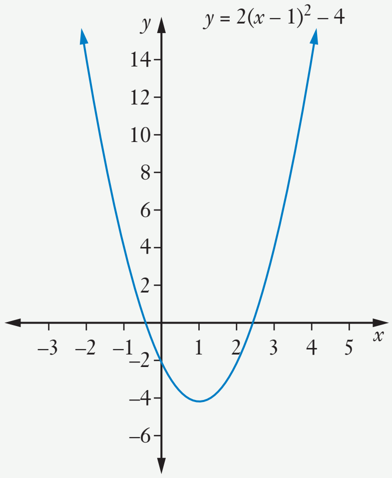
c



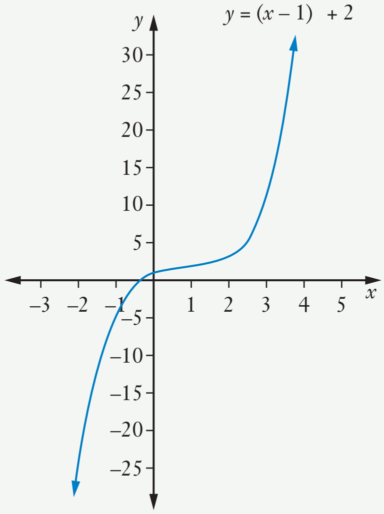
d



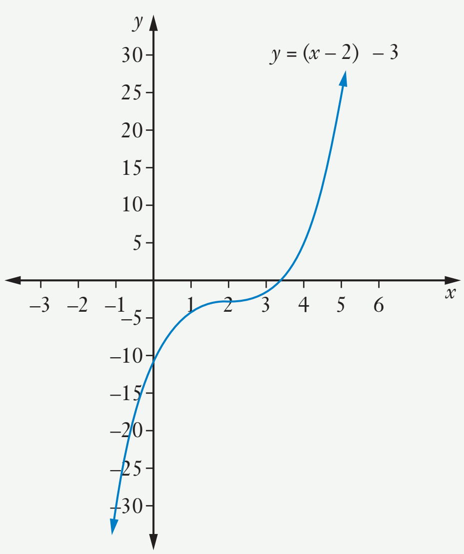
e



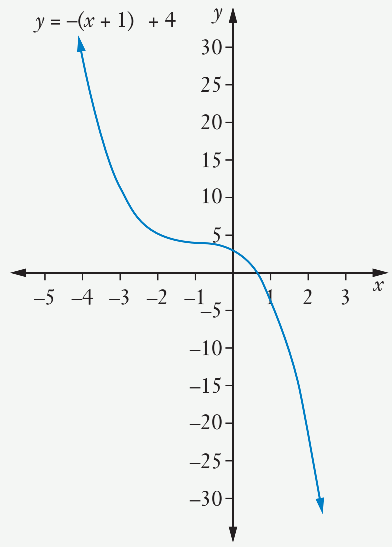
3 a



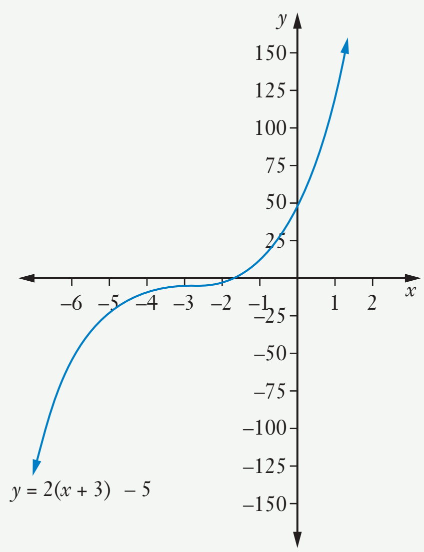
b



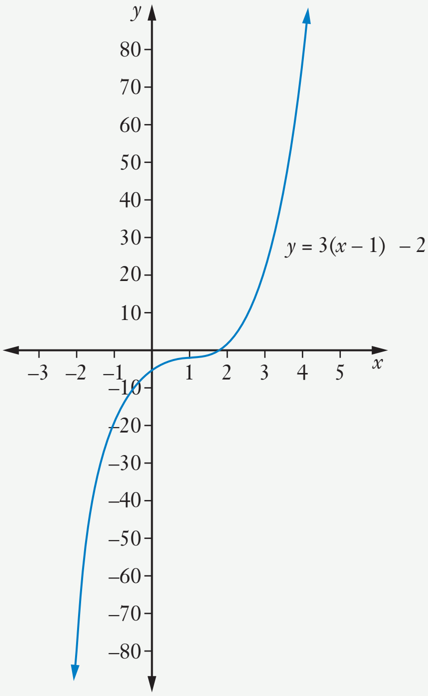
c



d

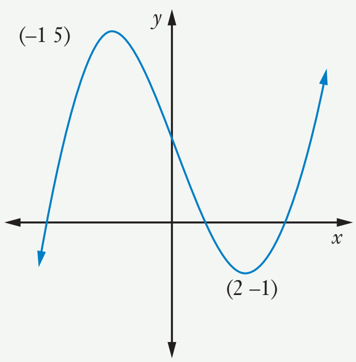


e

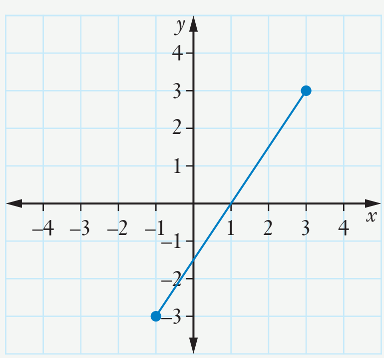


4 a (2 −1) and (−1, 5)

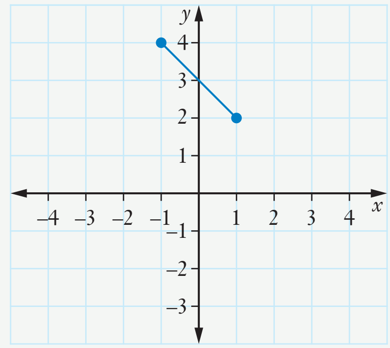
b



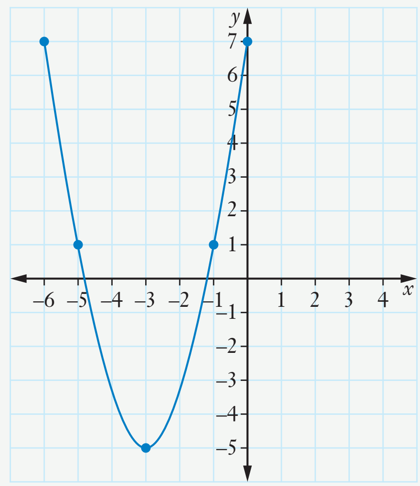
5 a

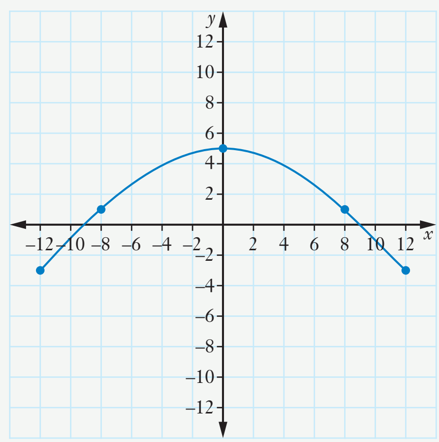


ii

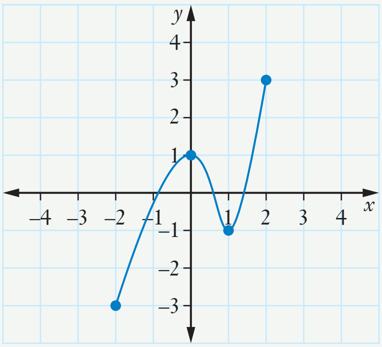


b

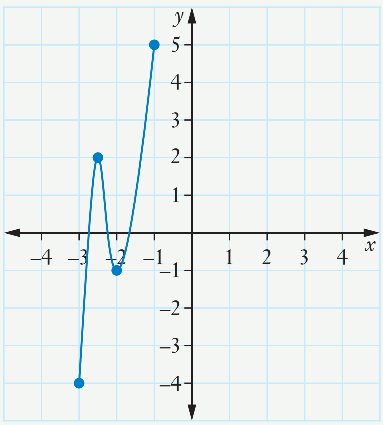




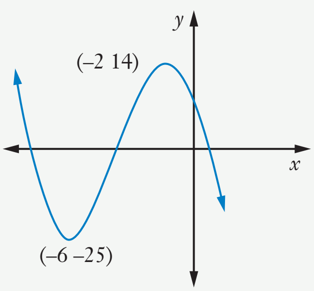
c



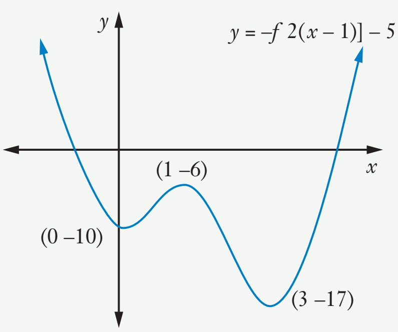
ii



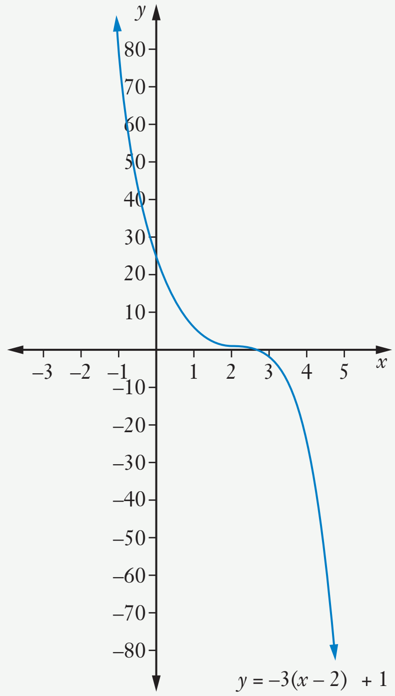
6



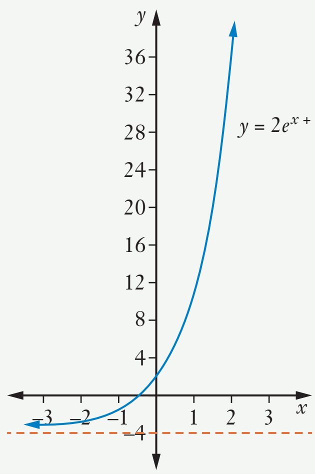
7



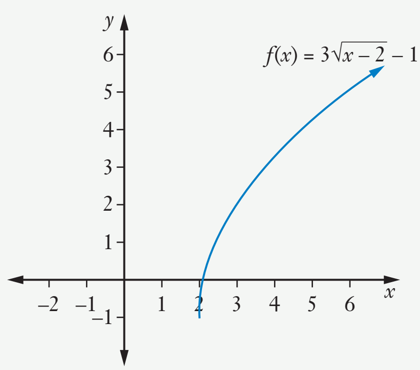
8 **a**



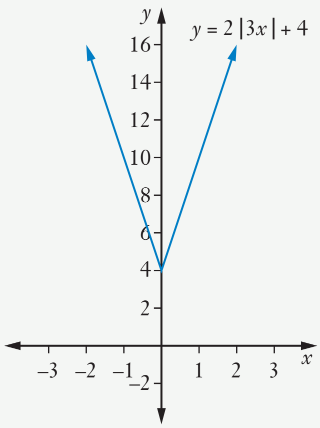
b



c

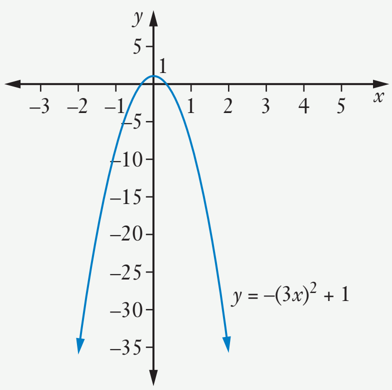


d



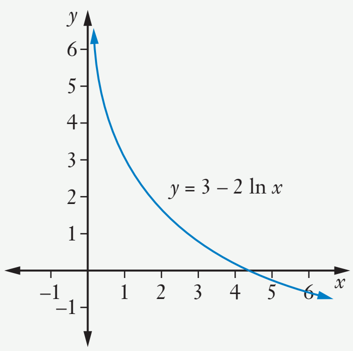
y
x
$y = 2\,|3x| + 4$

e



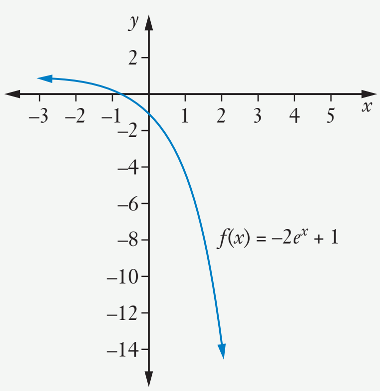
y
x
$y = -(3x)^2 + 1$

9 a



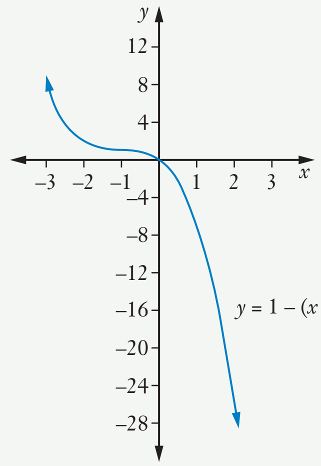
y
x
$y = 3 - 2\ln x$

b



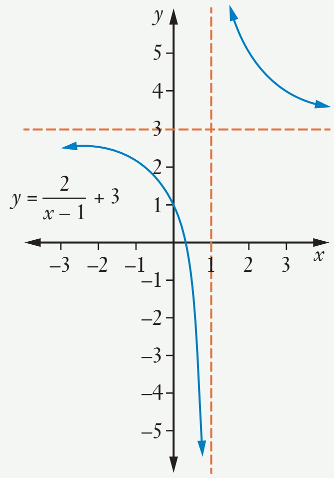
y
x
$f(x) = -2e^x + 1$

c



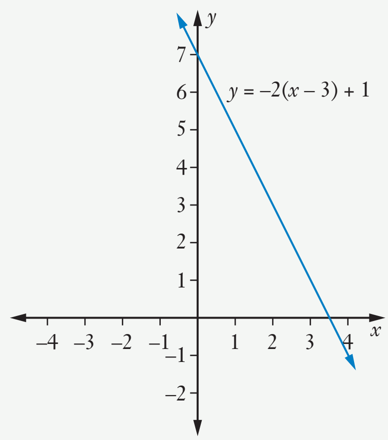
y
x
$y = 1 - (x + 1)$

d



y
x
$y = \dfrac{2}{x - 1} + 3$

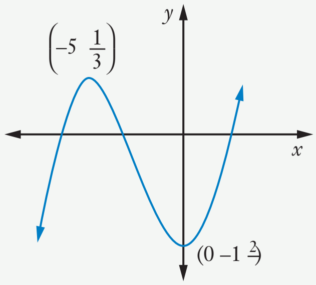
e



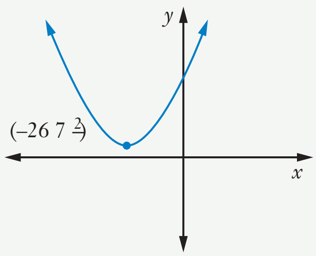
y
x
$y = -2(x - 3) + 1$

10 **a** $(-5, \frac{1}{3})$

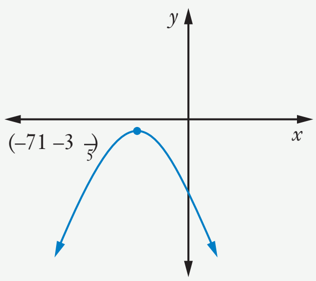
b



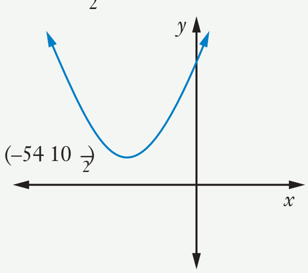
11 **a** $(-26\ 7\frac{2}{3})$



b (−71 −36)



c $(-54\ 10\frac{}{2})$



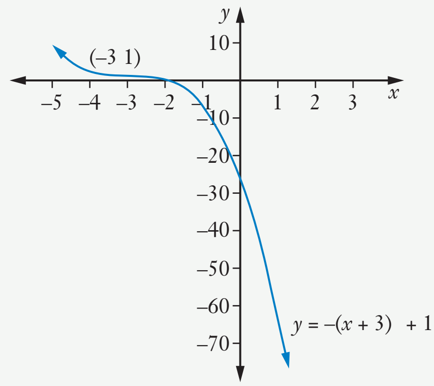
Exercse 207

1 **a** 2 **b** 0 **c** 1 **d** 3
e 0 **f** 1 **g** 2 **h** 0
1 **j** 0

2 **a** $x = -2\ 0$ **ii** $x = 06\ \ -26$
iii $x = -22\ 0.2$ **b** $x = 02,\ -22$

3 **a** $x = 14$ **b** $x = 19$ **c** $x = 02$
d $x > 22$ **e** $x \le 31$

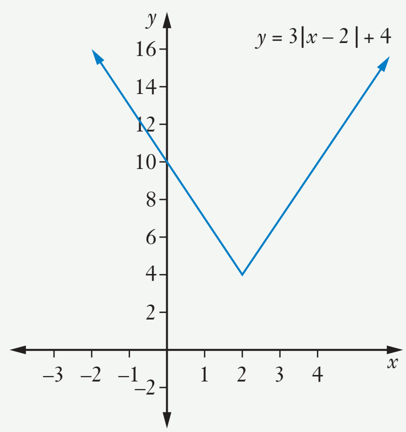
4 **a**



b $x = -2$ **ii** $x = -08$ **iii** $x = -03$

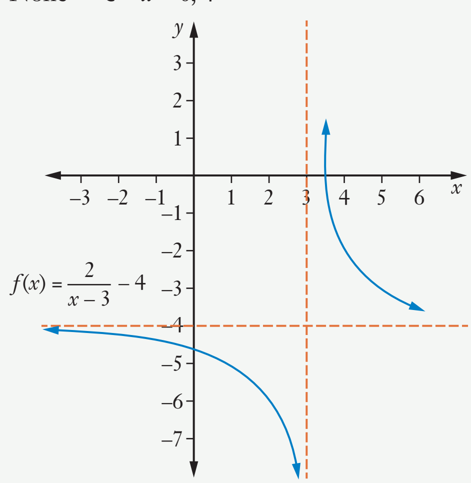
c $x = -2$

5 **a**



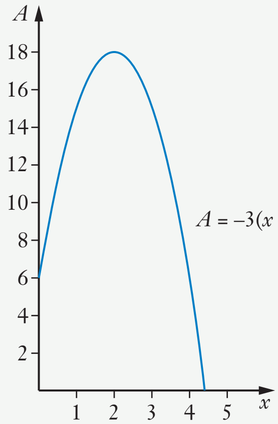
b None **c** $x = 0, 4$

6 **a**



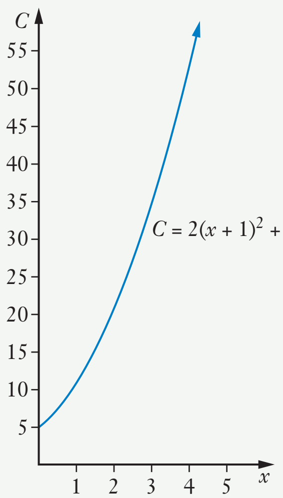
b $x = 1$ **c** $x = 4$

7 a



b $x = 04 .6$

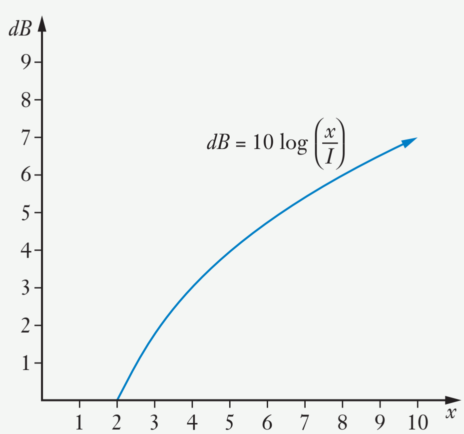
8 a



b \$5000

c The costs are \$20 000 when 2 products are made

9 a



b $x = 6$ **ii** $x = 3$

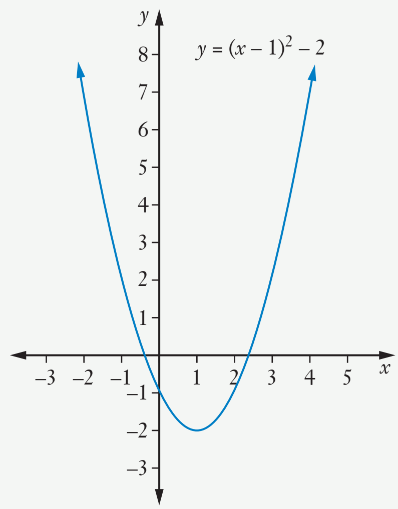
10 a $t = 35$ After3.5 minutes the temperature is 50°C

ii $t = 8$ After 8 minutes the temperature is 30°C

b $t = .74$ **ii** $t = 1185$

c 24°C (room temperature)

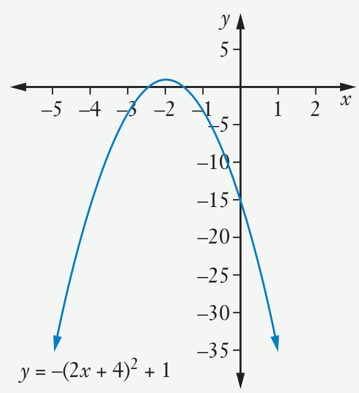
11 a



b $x = 3, -1$ **ii** $x \leq -1, x \geq 3$

iii $-1 < x < 3$

12 a



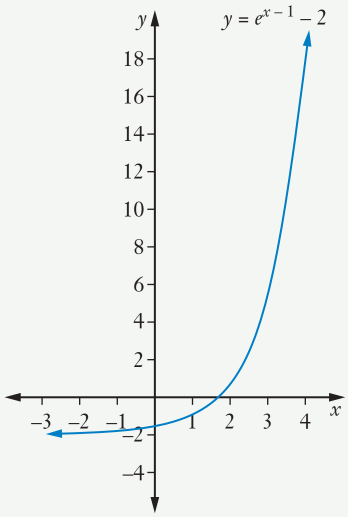
b $x = -3, -1$ **ii** $-3 < x < -1$

iii $x \leq -3, x \geq -1$

Test yourslf 2

1 B **2** D **3** C

4 a



b $x = 3.1$ **c** $x = 409$

5 a (6 107) **b** (6 40)

c (–24 177) **d** (17 –75)

e (16 –31)

6 a $y = x^3 + 3$ **ii** $y = (x + 7)^3$

b $y = 3\ x|$ **ii** $y = \left|\frac{x}{2}\right|$

c $f(x) = 5 \ln(-x)$ **d** $f(x) = -\frac{1}{x-4}$

e $f(x) = 9{\cdot}3^{3x-2} - 6$

7 a c is a vertical translation c units up if $c > 0$ or down if $c < 0$

b is a horizontal translation b units to the right if $b < 0$ and to the left if $b > 0$

k is a vertical dilation with scale factor k stretched if $k > 1$ and compressed if $0 < k < 1$.

a is a horizontal dilation with scale factor $\frac{1}{a}$ compressed if $a > 1$ and stretched if $0 < a < 1$.

b Reflection in the x-axis

ii Reflection in the y-axis

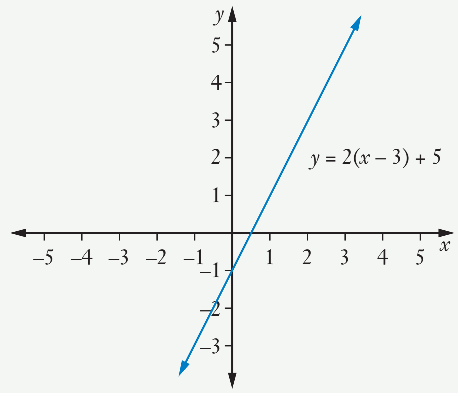
8 $f(x) = -3x^2 + 1$

$f(-x) = -3(-x)^2 + 1$

$= -3x^2 + 1$

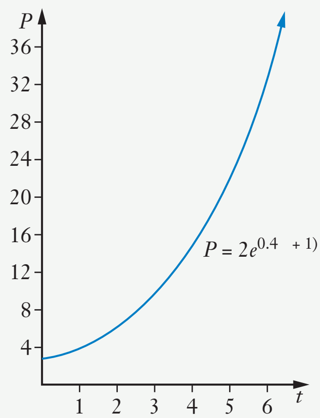
$= f(x)$ so even

9 a



b $x \le 4$ **ii** $x > 5$

10 a



b $t = 13$ It takes .3 years for the population to reach 50 000

11 $f(x) = (x + 4)^4$

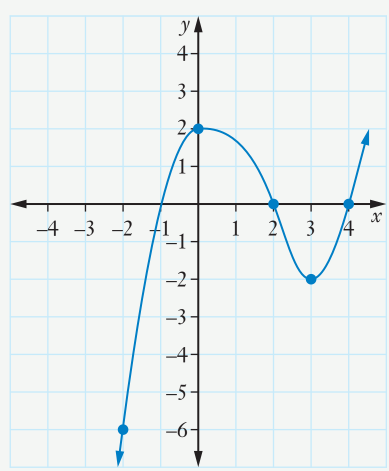
12 (3 –11)

13 a $x = 288, .12$ **b** $x < 1.12, x > 288$

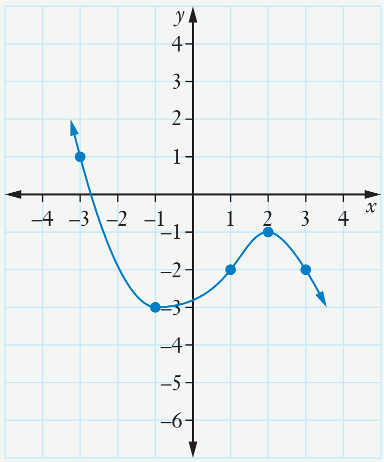
c $112\ \le x \le 288$

14 a $(x + 3, -7y - 4)$ **b** $x = -6, y = -1$

15 a

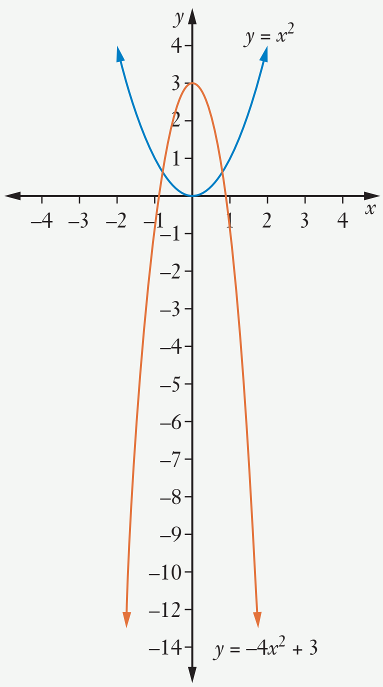


b

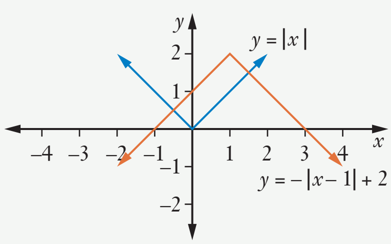


16 **a** $-3 \le x \le 1$ **b** $x < -3, x > 1$

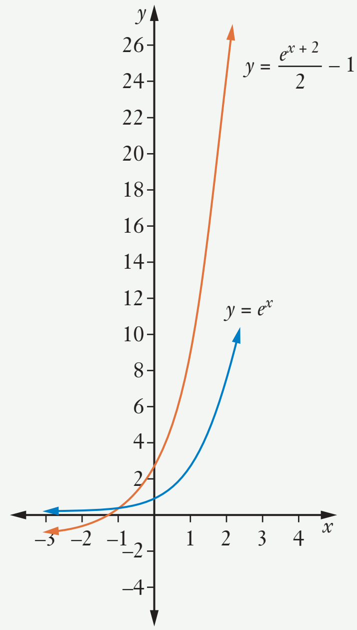
17 **a**



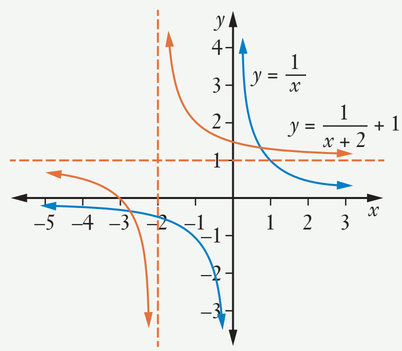
b



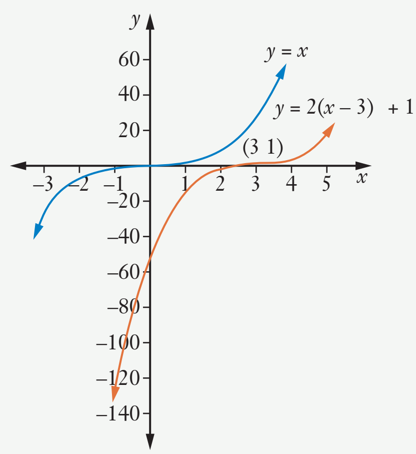
c



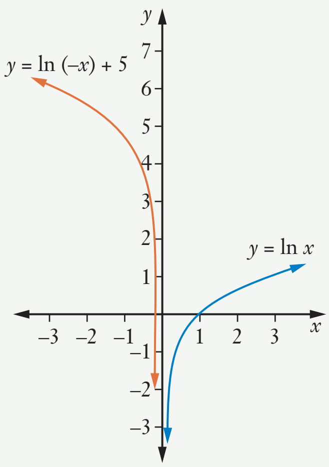
d



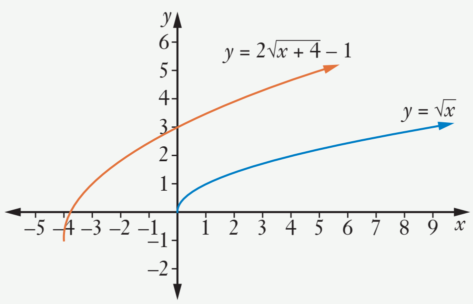
e



f

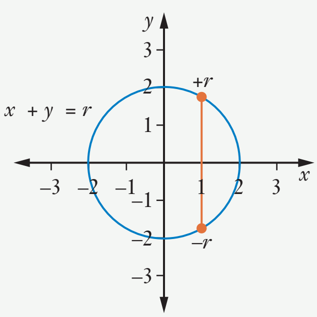


g



18 **a** 2 **b** 1 **c** 0 **d** 1
e 1 **f** 0 **g** 1 **h** 0
4 **j** 3

19 **a** The circle is not a function since a vertical line cuts the graph in more than one point

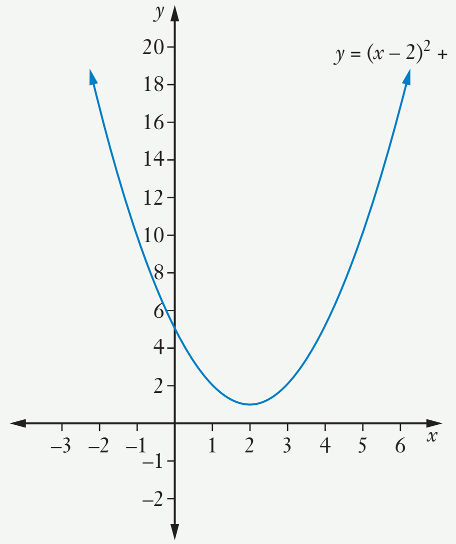


b $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$

c An ellipse (oval)

20 (32 −1)

21 **a**



b $x = -1\ 5$ **ii** $x < -1, x > 5$
iii $-1 \le x \le 5$

22 **a** $(x-1, 3y-5)$ **b** $\left(\frac{1}{2}[x+6], -2y+4\right)$

c $(-x\ 5y-3)$ **d** $\left(-\frac{1}{3}[x+3], -3y-1\right)$

23 **a** Stretched **b** Compressed
c Stretched **d** Compressed
e Stretched

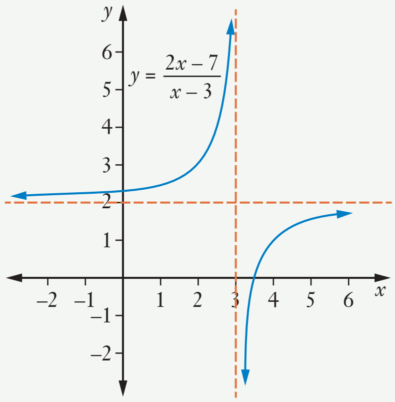
24 **a** Domain $(-\infty\ \infty)$ range $[-10\ \infty)$
b Domain $(-\infty\ \infty)$ range $(-\infty\ 2]$
c Domain $(-\infty, 3) \cup (3, \infty)$; range $(-\infty\ -5) \cup (-5, \infty)$

Chaenge exercse 2

1 **a** $h = -2t^2 + 4t + 1$ **b** 22 seconds
c $h = -2(t-1)^2 + 3$ Horizontal translation 1 unit to the right reflection in the x-axis vertical dilation scale factor 2 vertical translation 3 units up

2 **a** (1 −8) **ii** (2 0) **iii** (3 −2)
b $x - 2y - 17 = 0$
c Horizontal dilation with scale factor 2 and horizontal translation 17 units to the right OR vertical dilation with scale factor $\frac{1}{2}$ and vertical translation $\frac{17}{2}$ units down

3 **b**



c $x < 3, x \geq 35$ **ii** $x > 3$

4 **a** A horizontal dilation with scale factor $\frac{1}{a} = 2$

$a = \frac{1}{2}$

$y = \frac{1}{(ax)} = \frac{1}{\frac{1}{2}x} = \frac{2}{x}$

A vertical dilation with scale factor $k = 2$

$y = \left(\frac{1}{x}\right) = 2\left(\frac{1}{x}\right) = \frac{2}{x}$

So these transformations have the same effect on $y = \frac{1}{x}$

b No Horizontal dilation gives $y = \frac{4}{x}$ vertical dilation gives $y = \frac{2}{x}$

5 **a** $x = -\frac{b}{2a}$ **b** Horizontal translation

c $x = -1$ **ii** $x = 3$

iii $x = -b$ **v** $x = -\frac{b}{a}$

6 **a** $y = -3 \sin \frac{x}{2} - 1$

b Amplitude 3 period 4π centre -1

7 $x^2 - 6x + y^2 + 8 = 0$

8 Reflection in y-axis horizontal dilation scale factor $\frac{1}{3}$ horizontal translation 2 units to the lef, vertical dilation scale factor 3 vertical translation 5 units down

9 $y = -x^3 + 3x$

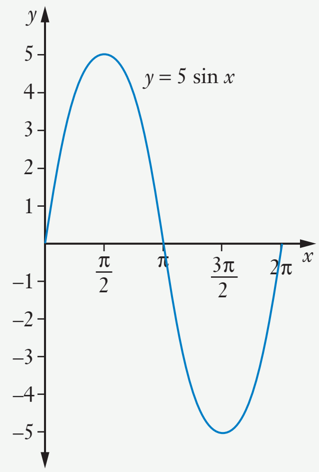
Chapter 3

Exercse 301

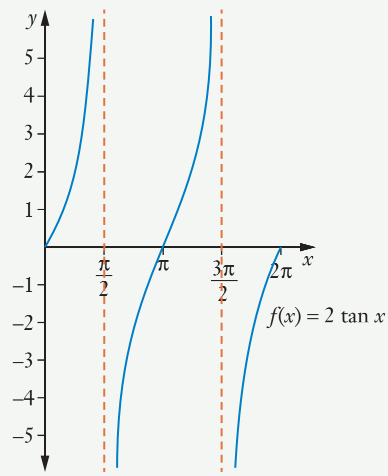
1 **a** Phase shift **b** Amplitude

c Period **d** Centre

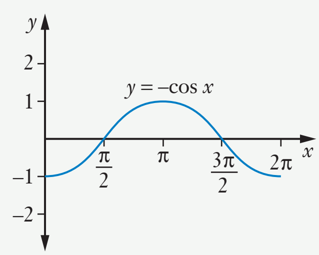
2 **a**



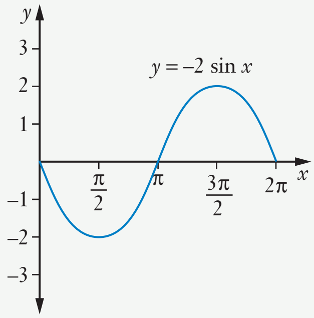
b



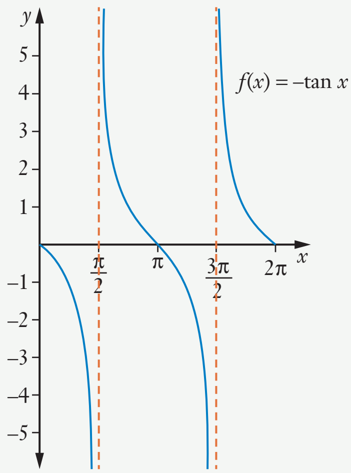
c



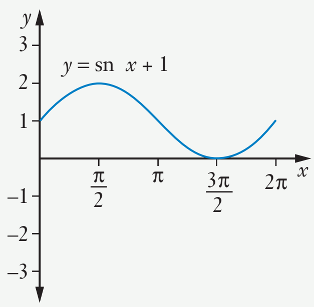
d


$y = -2 \sin x$

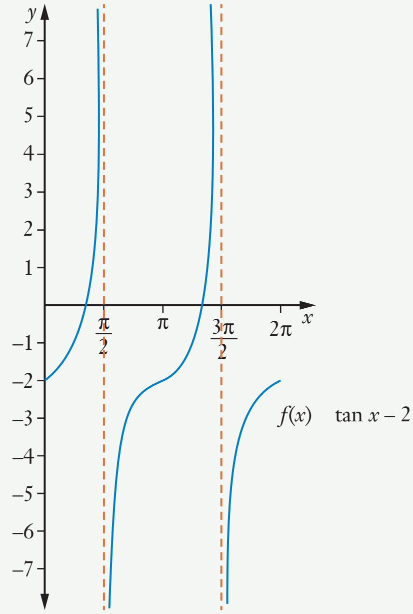
e


$f(x) = -\tan x$

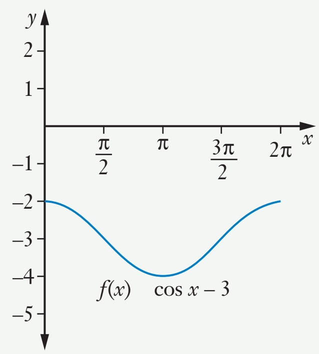
3 a


$y = \text{sn } x + 1$

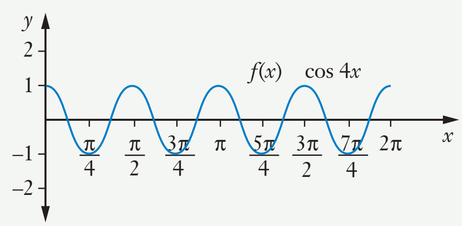
b


$f(x) \quad \tan x - 2$

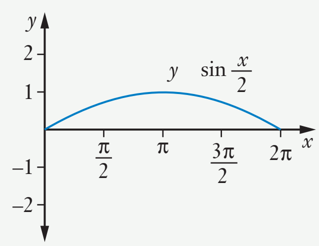
c


$f(x) \quad \cos x - 3$

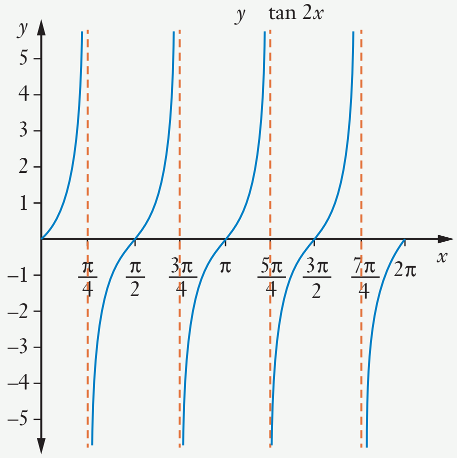
4 a


$f(x) \quad \cos 4x$

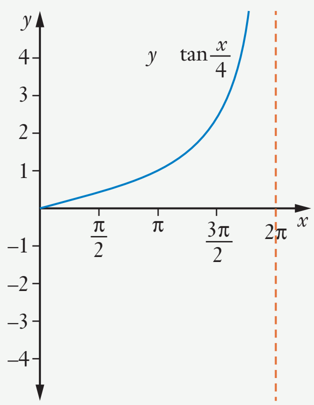
b


$y \quad \sin \frac{x}{2}$

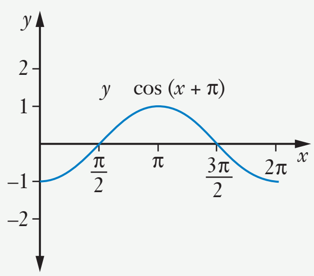
c



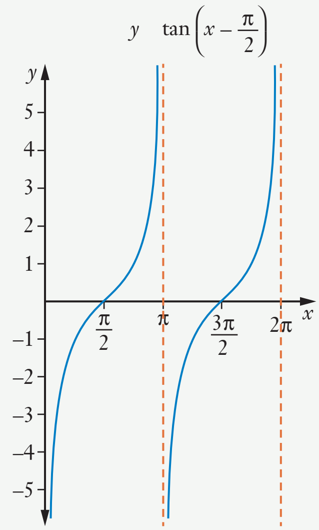
d



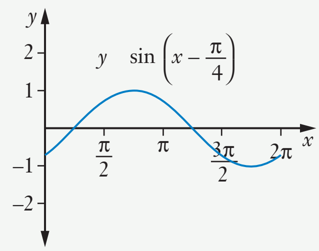
5 a



b



c



6 a $y = 9 \sin x$ **b** $y = -\sin x$

c $y = \sin x - 4$ **d** $y = \sin 2x$

e $y = \sin (x - \pi)$

7 a $y = 4 \cos x$ **b** $y = \cos \left(x + \frac{\pi}{3}\right)$

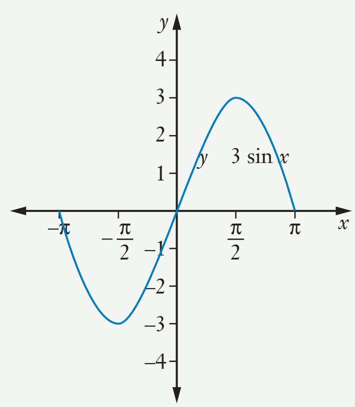
c $y = \cos x + 8$ **d** $y = \cos 4x$

e $y = 7 \cos x$

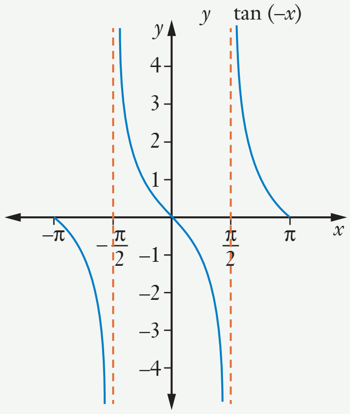
8 a $y = \tan \frac{x}{2}$ **b** $y = \tan \left(x - \frac{\pi}{6}\right)$

c $y = \tan (-x)$

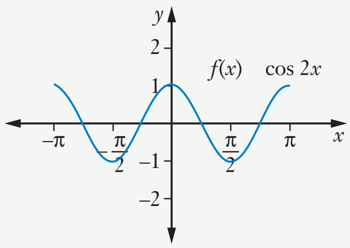
9 a



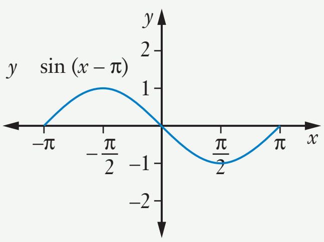
b



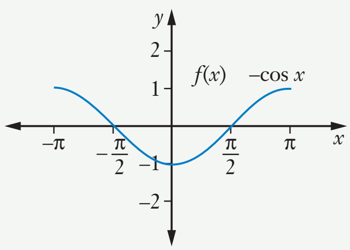
c



d

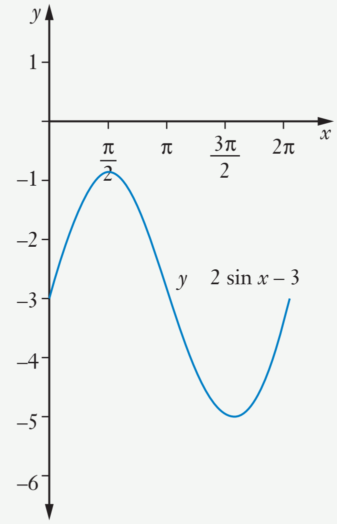


e

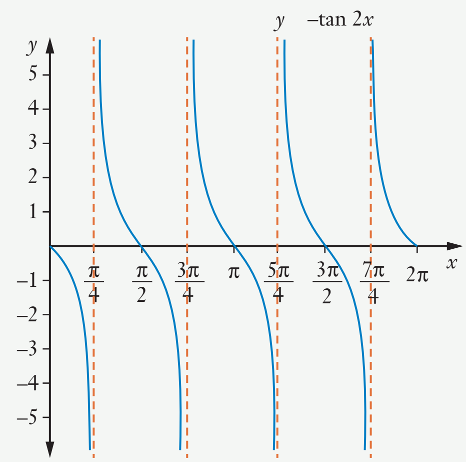


Exercse 302

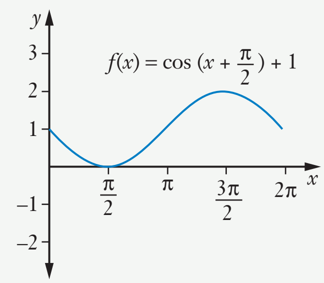
1 a



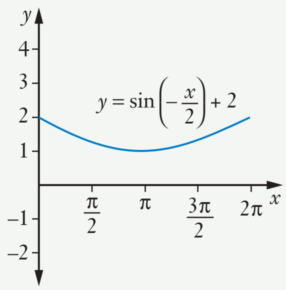
b



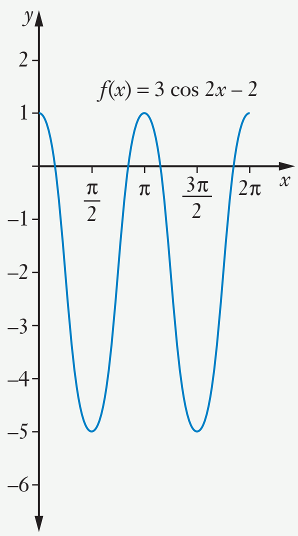
c



d



e



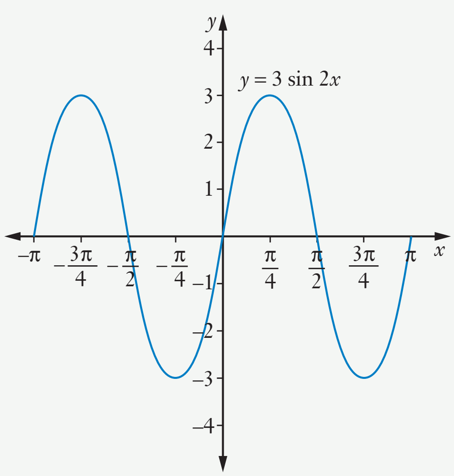
2 a $y = 5\sin[3(x+5)] - 6$

b Amplitude 5 period $\frac{2\pi}{3}$ centre −6 phase shift 5 units to the left

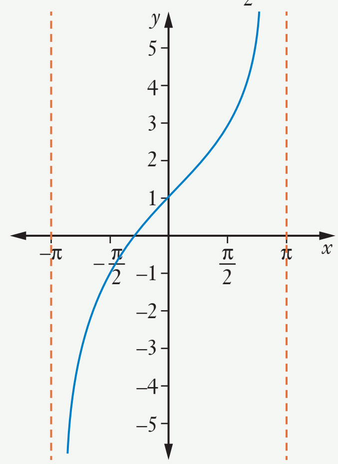
3 a $y = 4\cos\left[6(x - \frac{\pi}{3})\right] + 2$

b $y = -\cos(x + \pi) - 5$

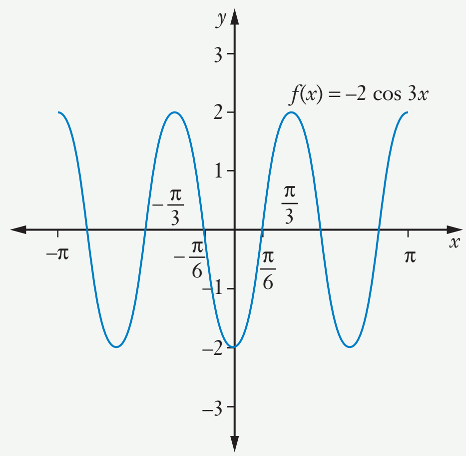
4 a



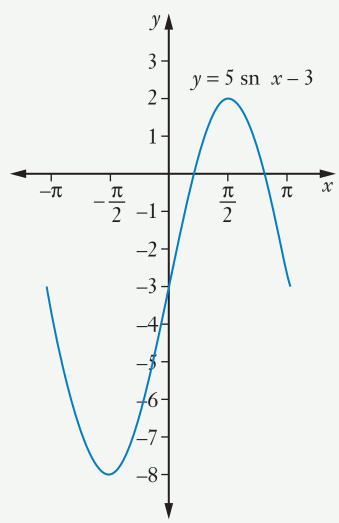
b



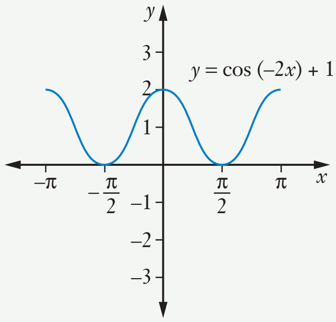
c



d



e



5 **a** No amplitude period $\frac{\pi}{4}$ centre -5

b Amplitude 8 period 2π phase shift π units to the left centre -3

c Amplitude 5 period π phase shift 3 units to the right centre 1

6 **a** $y = 7\sin[2(x-1)] - 3$

b $y = -\cos 5x + 2$

c $y = -\tan\left[\frac{1}{2}(x+2)\right]$

d $y = 4\sin\left[-\frac{1}{3}(x+5)\right] + 2$

7 No amplitude period $\frac{2\pi}{a}$ phase shift b units to the right when $b < 0$ to the left when $b > 0$, centre c

8 $y = \tan(4x - 3)$

9 **a** 15 m

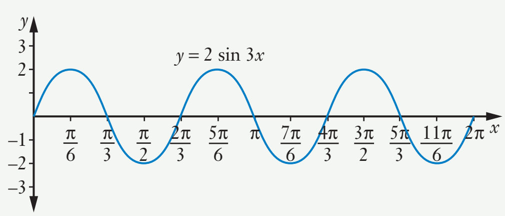
b Amplitude 10 period 12

c $D = 10\cos\frac{\pi t}{6} + 15$

10 $B = 20\sin\frac{\pi t}{30} + 100$

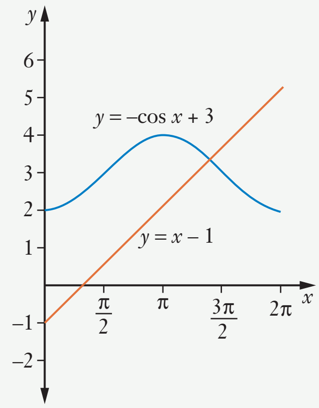
Exercse 303

1



a 6 **b** $x = 02, .9,2.3, 3\ 4., 5.1$

2 **a**



b $x = .4$ **ii** $x = 0, 2\pi$

3 **a** $x = 15°\ 75°\ 195°\ 255°$

b $x = 45°\ 105°\ 165°\ 225°\ 285°\ 345°$

c $x = 240°\ 300°$ **d** $x = 105°\ 285°$

e $x = 120°\ 300°$

4 **a** $x = \frac{\pi}{6}\ \frac{2\pi}{3}\ \frac{7\pi}{6}\ \frac{5\pi}{3}$

b $x = \frac{2\pi}{9}\ \frac{4\pi}{9}\ \frac{8\pi}{9}\ \frac{10\pi}{9}\ \frac{14\pi}{9}\ \frac{16\pi}{9}$

c $x = 0, \frac{2\pi}{3}\ \pi\ \frac{5\pi}{3}\ 2\pi$

d $x = \frac{\pi}{8}\ \frac{3\pi}{8}\ \frac{5\pi}{8}\ \frac{7\pi}{8}\ \frac{9\pi}{8}\ \frac{11\pi}{8}\ \frac{13\pi}{8}\ \frac{15\pi}{8}$

e $x = \pi$

5 **a** $x = -\frac{11\pi}{12}\ -\frac{7\pi}{12}\ -\frac{\pi}{4}\ \frac{\pi}{12}\ \frac{5\pi}{12}\ \frac{3\pi}{4}$

b $x = -\frac{\pi}{2}\ 0$ **c** $x = -\frac{\pi}{4}\ \frac{3\pi}{4}$

d $x = -\pi\ 0, \pi$

e $x = -\pi\ -\frac{3\pi}{4}\ -\frac{\pi}{2}\ -\frac{\pi}{4}\ 0, \frac{\pi}{4}\ \frac{\pi}{2}\ \frac{3\pi}{4}\ \pi$

6 **a** $x = \frac{\pi}{3}\ \frac{2\pi}{3}\ \frac{4\pi}{3}\ \frac{5\pi}{3}$

b $x = \frac{2\pi}{9}\ \frac{4\pi}{9}\ \frac{8\pi}{9}\ \frac{10\pi}{9}\ \frac{14\pi}{9}\ \frac{16\pi}{9}\ \frac{19\pi}{9}$

7 **a** Amplitude 15 period 1, centre 20

b $t = 0\ 1, 4, \ldots$ monhs. Average temperature of 35° occurs in January of each year.

8 **a** $y = 7\sin\frac{\pi t}{5} + 13$

b $t = 25\ 1., 25,32.5$ seconds

c 75 seconds

d 0 , 0,15, 20 25, … seconds

9 a Amplitude 1 period $\frac{1}{440}$

b x = 00002 .000,0.005, 0.032, 00045 .005,0.07, 0.075 0.0093

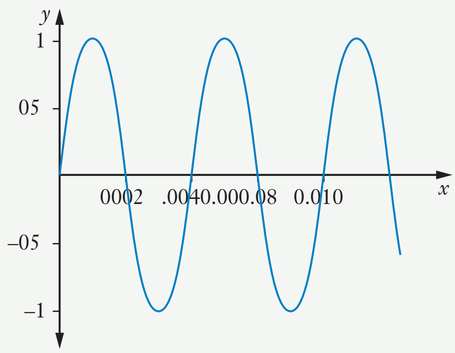
ii x = 0 .00,0.001, 0.0035 0.045, 00056 .006,0.08, 0.009

c x = 000019 .0009,0.005, 0.032, 00047 .005,0.000, 0.078 0.0093

ii x = 0 .001,0.003, 0.034 0.045, 00057 .006,0.000, 0.0091

d $y = 3 \sin (880\pi x)$

e



f 440 Hz

10 a Physical 23 days emotional 28 day, intellectual 33 days

b 7 days 21 days **c** 6 days 19 day, 32 days

d around 7 days

11 a 15 is the centre of motion (equilibrium of spring)

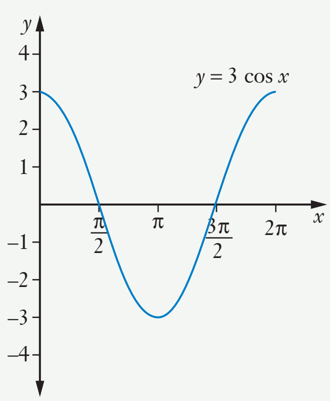
b 27 cm maximum 3 cm minimum

c 3 cm **d** π, 3π 5π … seconds

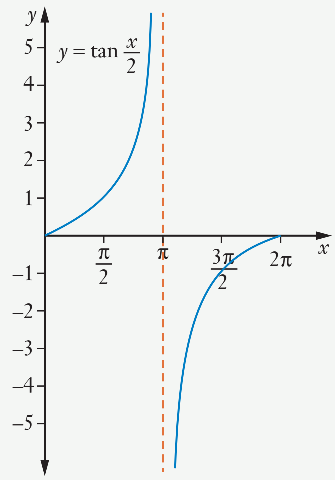
Test yourslf 3

1 D **2** A **3** B

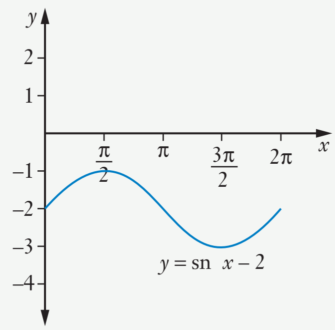
4 a



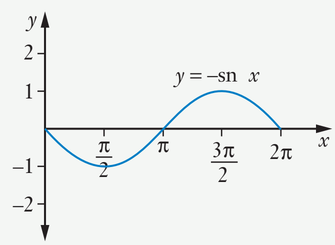
b



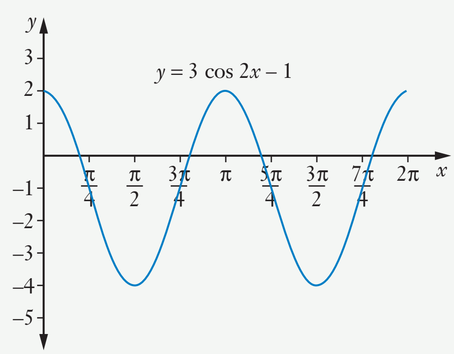
c



d



e



5 **a** Maximum 13 m $t = 0, 3, 6, 9, \ldots$ h; minimum 7 m $t = 15, .5,7.5 \ldots$h

b $t = 06$.,36, .46., 8., …h; times when water level in the lock is 11 m

6 **a** $2\ \text{cosec}^2\, x$ **b** $\sec A$

c 1 **d** $\sin x$

7 **a** $x = \frac{\pi}{6}\ \frac{5\pi}{6}\ \frac{7\pi}{6}\ \frac{11\pi}{6}$

b $x = \frac{\pi}{12}\ \frac{5\pi}{12}\ \frac{13\pi}{12}\ \frac{17\pi}{12}$

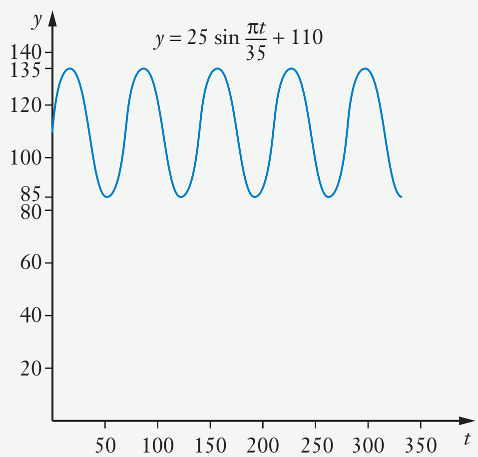
c $x = \frac{3\pi}{2}$

d $x = \frac{\pi}{6}\ \frac{\pi}{2}\ \frac{7\pi}{6}\ \frac{3\pi}{2}$

e $x = 115, .99,4.29, 5.13$

8 **a** $y = 25 \sin \frac{\pi t}{35} + 110$

b



9 **a** $x = \pm\frac{\pi}{6}\ \pm\frac{5\pi}{6}$ **b** $x = \frac{\pi}{12}\ -\frac{11\pi}{12}$

c $x = \pm\frac{\pi}{4}$

10 **a** $h = 15 \sin \frac{2\pi t}{13} + 65$

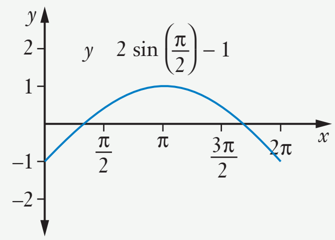
b 0 ., 3,1.5, … hours

11 **a** $y = \cos 2x$ **b** $y = 5 \cos x$

c $y = -\cos x$ **d** $y = \cos\left(x - \frac{\pi}{6}\right)$

e $y = \cos x + 4$

12 **a**



b $x = 1, 5.2$ **c** $x = \frac{\pi}{3}\ \frac{5\pi}{3}$

13 **a** Amplitude 2 period $\frac{2\pi}{3}$ centre −1

b Amplitude 1 period 4 π centre , phase shift 2π units to the left

c No amplitude reflection in x-axis period $\frac{\pi}{5}$ centre 0 phase shift $\frac{\pi}{20}$ to the right

14 **a** $x = 0°\ 180°\ 360°$ **b** $x = 155°\ 245°$

c $x = 10°\ 54\ \ 59°\ 6\ \ 190°\ 54\ \ 239°\ 6$

15 **a** $x = -1125\ °\ -225\ °\ 6.5\ \ °, 15.5\ \ °$

b $x = 90°\ -30°\ -150°$

Chaenge exercise 3

1 **a** Amplitude 2 period π phase shift $\frac{\pi}{4}$ units to the right

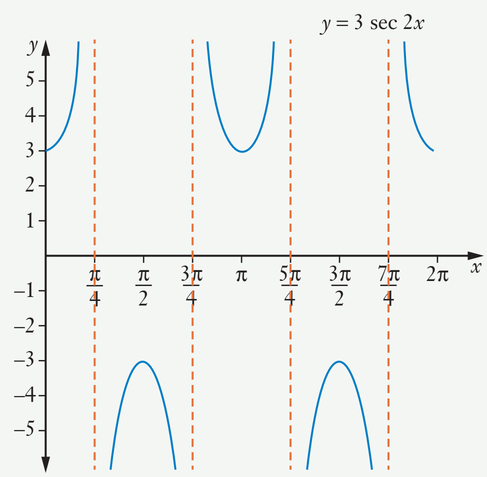
b $x = \frac{\pi}{6}\ \frac{\pi}{3}\ \frac{7\pi}{6}\ \frac{4\pi}{3}$

2 **a** $y = 8 \cos x + 4$

b $y = 2 \sin\left[8\left(x - \frac{\pi}{3}\right)\right] + 3$

c $y = \tan\left[\frac{1}{2}\left(x + \frac{\pi}{2}\right)\right]$

3



4 $x = \frac{\pi}{8} \quad \frac{3\pi}{8} \quad \frac{9\pi}{8} \quad \frac{11\pi}{8}$

5 **a** $y = 4\cos\left(\frac{1}{3}\left[x - \frac{\pi}{6}\right]\right) - 20$

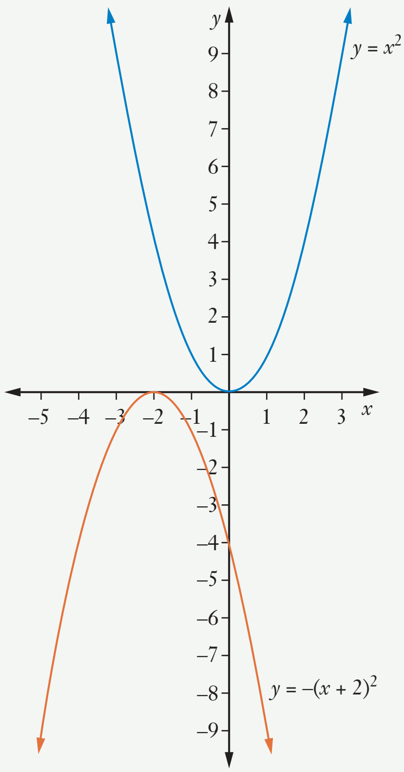
b Amplitude 4 period 6π centre -20 phase shift $\frac{\pi}{6}$ to the right

Practce set 1

1 B **2** C **3** C

4 D **5** D **6** C **7** D

8 **a**



b Reflection in x-axis horizontal translation 2 units to the left

9 **a** $x = 67°\ 30\ ,\ 157°\ 30\quad 247°\ 30\ ,\ 337°\ 30$

b $x = 20°\ \ 100°\ \ 140°\ 220\ °\ \ 260°\ \ 340°$

c $x = 150°\ \ 210°$

d $x = 60°\ \ 240°$

e $x = 0°\ \ 90°\ \ 180°\ \ 270°\ \ 360°$

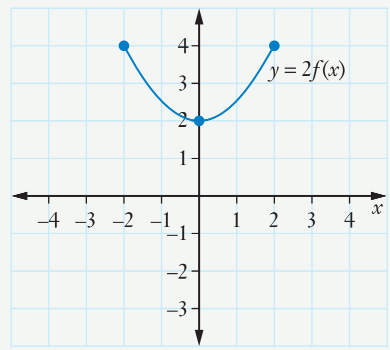
10 **a** Amplitude 4 period $\frac{2\pi}{5}$ centre 0

b Amplitude 2 reflection in x-axis centre , phase shift $\frac{\pi}{6}$ units to the right

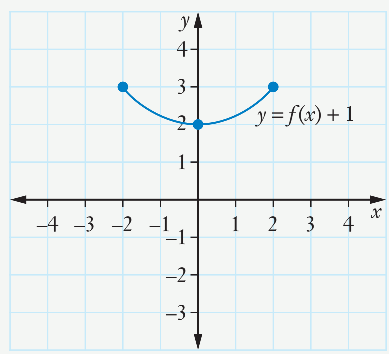
c Period 4π phase shift 8 units to the left

11 $\frac{3}{4^{10}}$

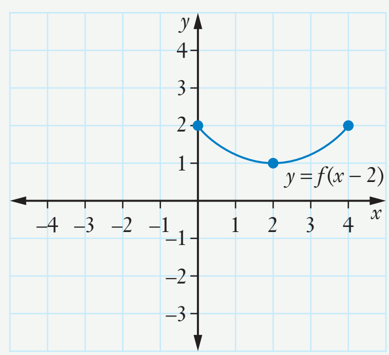
12 **a**



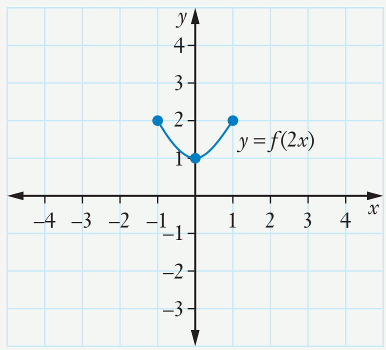
b



c



d



13 ±104 5, ±26 1, …

14 44th term

15 Vertical dilatio, scale factor 4 (stretc), horizontal translation 1 unit to the right vertical translation 3 units down

16 **a** $x = 30°\ 41°\ 49\ ,\ 138°\ 11\ \ 150°$

b $x = 45°\ 105°\ 225°\ 285°$

17 **a** Centre 10 max distance 16 cm 4 cm

b 1 second

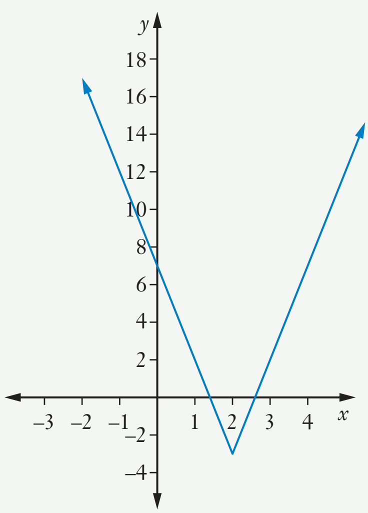
18 $y = 4\sqrt{3(x+7)} - 1$

19 **a** 199 **b** 5050

20 **a** $T\ = 4$, $T_2 = 11$, $T_3 = 18$, $T_{12} = 81$

b 1410 **c** 29th term

21 **a**



b $x = 1, 3$ **ii** $x = 0\ 4$ **iii** $x = 2$

c Domain $(-\infty\ \ \infty)$ range $[-3, \infty)$

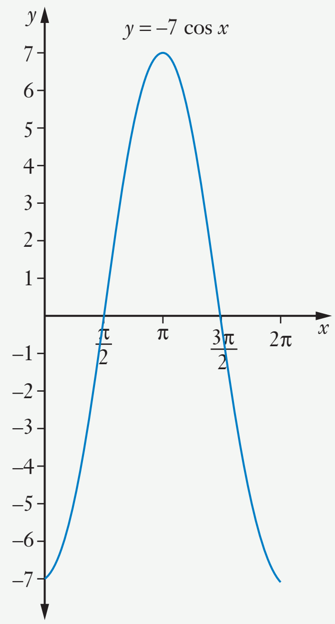
22 **a** $-\frac{1}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** 1

d $-\frac{1}{\sqrt{2}}$ **e** $-\frac{1}{\sqrt{3}}$

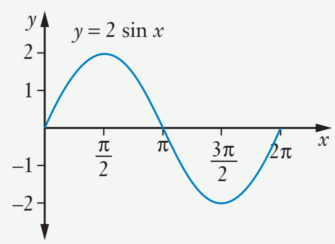
23 $T_n = 11n - 26$

24 $x = 40°$

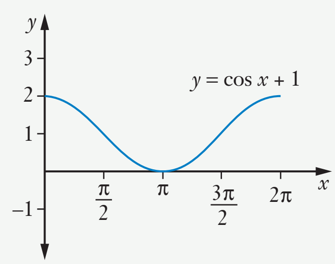
25 **a**



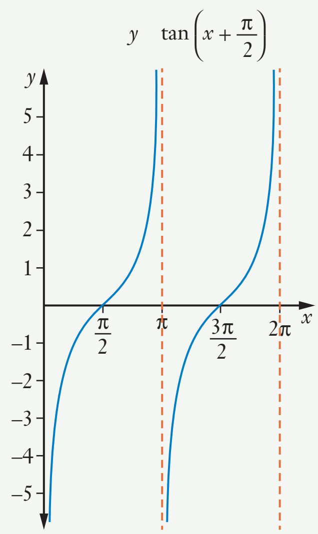
b



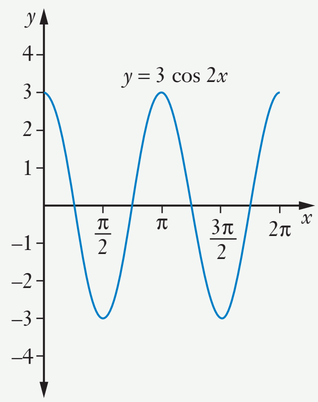
c



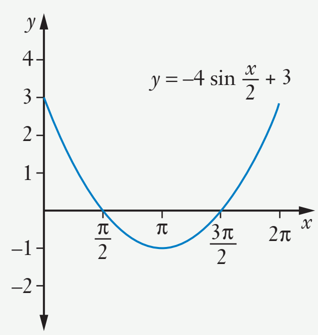
d



e



f



26 Proofs (see worked solutions)

27 **a** $y = 2\sin x$ **b** $y = \sin\frac{x}{2}$

c $y = \sin x - 3$ **d** $y = -5\sin x$

e $y = \sin\left(x + \frac{\pi}{2}\right)$

f $y = 5\sin\left[\frac{1}{3}(x - \pi)\right] + 1$

28 $x = \frac{5}{6}$

29 **a** $x = 09.38$ **b** $x = \frac{\pi}{4} \quad \frac{3\pi}{4} \quad \frac{5\pi}{4} \quad \frac{7\pi}{4}$

c $x = \frac{\pi}{6} \quad \frac{5\pi}{6}$ **d** $x = \frac{\pi}{2} \quad \frac{3\pi}{2}$

e $x = \frac{\pi}{3} \quad \frac{2\pi}{3} \quad \frac{4\pi}{3} \quad \frac{5\pi}{3}$

f $x = \frac{\pi}{3} \quad \frac{2\pi}{3} \quad \frac{4\pi}{3} \quad \frac{5\pi}{3}$

30 $n = 4$ **31** $\frac{1}{10}$

32 **a** Centre 18 amplitude 2, so maximum temperature is 38°C minimum −2°C Period is 12 so it is an annual cycl.

b July **c** January

d 8°C **e** April and October

33 $-1 \le x \le 3$

34 $f(-x) = 3(-x)^2 - 2$

$= 3x^2 - 2$

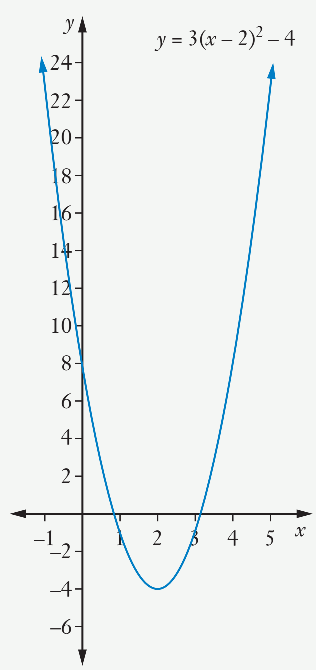
$= f(x)$ so even

35 **a** $\log 3 \log 3^2, \log 3^3 \ldots = \log 3, 2\log 3, 3\log 3 \ldots$

Arithmetic series since $2\log 3 - \log 3 = 3\log 3 - 2\log 3 = \log 3$

b $210 \log 3$

36 **a**



b $x \le 0, x \ge 4$ **ii** $0 < x < 4$

Chapter 4

Exercse 401

1 **a** $12x^3 - 6x^2 + 7$ **b** 2
c $12x - 3$

2 $20x^4 + 18x$ **3** $6\pi t^2 - 6t$ **4** $f\ (-2) = 101$

5 **a** $-5x^{-6}$ **b** $\frac{2}{3}x^{-\,-\,3}$ **c** $-\frac{2}{x^3}$
d $\frac{1}{4\sqrt[4]{x^3}}$ **e** $\frac{20}{x^5}$

6 $\frac{1}{12}$

7 **a** $21(3x-1)^6$ **b** $3(2x-1)(x^2-x+2)^2$
c $\frac{7}{2\sqrt{7x-2}}$ **d** $-\frac{3}{(3x-2)^2}$
e $\frac{2x}{3\sqrt[3]{(x^2-3)^2}}$

8 **a** $3x^2 + 8x$ **b** $24x + 4$ **c** $12x^2 + 4$
d $4(x^2 - 1)(5x^2 + 3x - 1)$ **e** $\frac{x^2(7x+6)}{\sqrt{x+1}}$

9 **a** $-\frac{13}{(x-5)^2}$ **b** $\frac{x^2(8x-21)}{(4x-7)^2}$
c $\frac{2(x^2-3x-3)}{(2x-3)^2}$ **d** $\frac{-6x+23}{(2x+9)^3}$
e $\frac{3x-7}{\sqrt{(2x-1)^3}}$

10 **a** -6 **b** 3

11 **a** $\frac{1}{14}$ **b** $\frac{1}{5}$

12 **a** $7x - y - 24 = 0$ **b** $51x - y - 72 = 0$

13 **a** $x - 4y - 3 = 0$ **b** $x + 3y - 6 = 0$

14 $x = 4$ **15** $(2\ 6, (2, -10)$

16 $3x + y + 8 = 0$ **17** $(4\ 2)$

18 **a** $17x - y - 3 = 0$ **b** $x + 17y + 51 = 0$

19 **a** $6t$ **b** $-\frac{2}{(t-3)^2}$ **c** $\frac{2}{3\sqrt[3]{(2x+3)^2}}$

20 **a** 10 kg s⁻ **ii** 17 kg s⁻
b 13 kg s⁻ **ii** 123 kg s⁻

21 –218 Pa/m ^{3}

22 **a** 2 m **ii** 15 m **b** 2 s
c 2 m s⁻ **ii** 0 m s⁻ **iii** –4 m s⁻

Exercse 402

1 **a** $7e^{7x}$ **b** $-e^{-x}$ **c** $6e^{6x-2}$
d $2xe^{x\ \ +}$ **e** $(3x^2 + 5)e^{x\ \ +5x+7}$
f $5e^{5x}$ **g** $-2e^{-2x}$ **h** $10e^{0\,x}$
$2e^{2x} + 1$ **j** $2x + 2 - e^{1-x}$
k $5(1 + 4e^{4x})(x + e^{4x})^4$ $e^{2x}(2x+1)$
m $\frac{e^{3x}(3x-2)}{x^3}$ **n** $x^2e^{5x}(5x+3)$
o $\frac{4e^{2x+1}(x+2)}{(2x+5)^2}$

2 $3e$

3 **a** $3^x \ln 3$ **b** $10^x \ln 10$ **c** $3(2^{3x-4}) \ln 2$

4 5 **5** $x + y - 1 = 0$

6 **a** $3e^3$ **b** $-\frac{1}{3e^3}$

7 **a** $y = 2ex - e$ **b** $x + 2ey - 2e^2 - 1 = 0$

8 $x \ln 4 - y + 4 = 0$

9 **a** 29 627 **ii** 35 826
b 1044 people/year **ii** 1240 people/year
c 1126 people/year **ii** 1361 people/year

10 **a** 55 042 cm min⁻
b 1428 cm min⁻ **ii** 1087 cm min⁻
iii 177 722 2054 cm min⁻

11 **a** 20 g **b** 7 g
c –0091 g/year
d –0147 g/year **ii** –0051 g/year
iii –00063 g/year

12 **a** 66 0794 cm **b** 132 1588 cm s⁻

Exercse 403

1 **a** $1 + \frac{1}{x}$ **b** $-\frac{1}{x}$ **c** $\frac{3}{3x+1}$
d $\frac{2x}{x^2-4}$ **e** $\frac{15x^2+3}{5x^3+3x-9}$
f $\frac{10x^2+2x+5}{5x+1}$ **g** $6x + 5 + \frac{1}{x}$
h $\frac{8}{8x-9}$ $\frac{6x+5}{(x+2)(3x-1)}$
j $\frac{-30}{(4x+1)(2x-7)}$ **k** $\frac{5}{x}(1 + \ln x)^4$
$9\left(\frac{1}{x} - 1\right)(\ln x - x)^8$ **m** $\frac{4}{x}(\ln x)^3$

n $6\left(2x+\frac{1}{x}\right)(x^2+\ln x)^5$

o $1+\ln x$ **p** $\frac{1-\ln x}{x^2}$

q $\frac{2x\ln x+2x+1}{x}$

r $3x^2\ln(x+1)+\frac{x^3}{x+1}$ **s** $\frac{1}{x\ln x}$

t $\frac{x-2-x\ln x}{x(x-2)^2}$ **u** $\frac{e^{2x}(2x\ln x-1)}{x(\ n\ x)^2}$

v $e^x\left(\frac{1}{x}+\ln x\right)$ **w** $\frac{10\ln x}{x}$

2 $f\ (1)=-\frac{1}{2}$ **3** $\frac{1}{x\ln 10}$

4 $x-2y-2+2\ln 2=0$ **5** $x-y-2=0$

6 $-\frac{2}{5}$ **7** $5x+y-\ln 5-25=0$

8 $5x-19y+19\ln 19-15=0$

9 $\frac{2}{(2x+5)\ln 3}$

10 $(2\ln 2)\,x+y-1-4\ln 2=0$

11 **a** 20 000

b 106 years **ii** 436 years

c $P=20\,000\,e^{0.021}$ **d** 452 kangaroos/year

e 447 kangaroos/year

ii 466 kangaroos/year

iii 518 kangaroos/year

Exercse 404

1 **a** $4\cos 4x$ **b** $-3\sin 3x$

c $5\sec^2 5x$ **d** $3\sec^2(3x+1)$

e $\sin(-x)$ **f** $3\cos x$

g $-20\sin(5x-3)$ **h** $-6x^2\sin(x^3)$

$14x\sec^2(x^2+5)$ **j** $3\cos 3x-8\sin 8x$

k $\sec^2(\pi+x)+2x$ $x\sec^2 x+\tan x$

m $3\sin 2x\sec^2 3x+2\tan 3x\cos 2x$

n $\frac{x\cos x-\sin x}{2x^2}$

o $\frac{3\sin 5x-5(3x+4)\cos 5x}{\sin^2 5x}$

p $9(2+7\sec^2 7x)(2x+\tan 7x)^8$

q $2\sin x\cos x=\sin 2x$ **r** $-45\sin 5x\cos^2 5x$

s $e^x+2\sin 2x$ **t** $-\frac{1}{x}\cos(1-\ln x)$

u $(e^x+1)\cos(e^x+x)$ **v** $\frac{\cos x}{\sin x}=\cot x$

w $e^{3x}(3\cos 2x-2\sin 2x)$

x $\frac{e^{2x}(2\tan 7x-7\sec^2 7x)}{\tan^2 7x}$

2 12 **3** $6\sqrt{3}x-12y+6-\pi\sqrt{3}=0$

4 $-\frac{\sin x}{\cos x}=-\tan x$ **5** $-\frac{2}{3\sqrt{3}}=-\frac{2\sqrt{3}}{9}$

6 $\sec^2 x\,e^{\tan x}$

7 $8x+24\sqrt{2}y-72-\pi=0$

8 Proof (see worked solutions)

9 **a** $\frac{\pi}{180}\sec^2 x°$ **b** $-\frac{\pi}{60}\sin x°$ **c** $\frac{\pi}{900}\cos x°$

10 $\sin^3 x\,(4\cos^2 x-\sin^2 x)$

11 **a** 750 **b** 525 **c** 975

d 26 ., 16,15.4 … days

e –136 fish/day **ii** 155 fish/day

iii –101 fish/day **v** 0 fish/day

f 023 .2,9.3,1327, … days

12 **a** 9 m **ii** 13 m

b 02, .8, 2.2,17.8, … h

c 0 m h⁻ **ii** 36 m h⁻ **iii** 42 m h⁻

d 15, 1.5, 3.5,22.5, … h

Exercse 405

1 $7x^6-10x^4+4x^3-1;\ 42x^5-40x^3+12x^2;$
$210x^4-120x^2+24x\ \ 840x^3-240x+24$

2 $72x^7$

3 $f\ (x)=10x^4-3x^2\ \ f''(x)=40x^3-6x$

4 $f\ \ (1)=11, f''(-2)=168$

5 $7x^6-12x^5+16x^3\ \ 42x^5-60x^4+48x^2;$
$210x^4-240x^3+96x$

6 $\frac{dy}{dx}=4x-3,\ \frac{d^2y}{dx^2}=4$

7 $f\ (-1)=-16, f''(2)=40$ **8** $-4x^{-5}\ \ 20x^{-6}$

9 $-\frac{1}{32}$ **10** 26

11 $x=\frac{7}{18}$ **12** $x>\frac{1}{3}$

13 $20(4x-3)^4\ \ 320(4x-3)^3$

14 $f\ (x) = -\dfrac{1}{2\sqrt{2-x}}$ $f''(x) = -\dfrac{1}{4\sqrt{(2-x)^3}}$

15 $f\ (x) = -\dfrac{16}{(3x-1)^2}$ $f''(x) = \dfrac{96}{(3x-1)^3}$

16 $\dfrac{d^2v}{dt^2} = 24t + 16$ **17** $b = \dfrac{2}{3}$

18 196 **19** $b = -27$

20 $\dfrac{dy}{dx} = 4e^{4x} - 4e^{-4x}$

$\dfrac{d^2y}{dx^2} = 16e^{4x} + 16e^{-4x} = 16y$

21, 22 Proofs (see worked solutions)

23 $n = -15$

24 $y = 2\cos 5x$ $\dfrac{dy}{dx} = -10\sin 5x$

$\dfrac{d^2y}{dx^2} = -50\cos 5x = -25y$

25 $f(x) = -2\sin x$ $f\ (x) = -2\cos x$

$f''(x) = 2\sin x = -f(x)$

26 $y = 2\sin 3x - 5\cos 3x$ $\dfrac{dy}{dx} = 6\cos 3x + 15\sin 3x$

$\dfrac{d^2y}{dx^2} = -18\sin 3x + 45\cos 3x = -9y$

27 $a = -7$ $b = -24$ **28** $f''(2) = \dfrac{3}{4\sqrt{2}} = \dfrac{3\sqrt{2}}{8}$

29 **a** 8 m **b** 38 m **c** 31 m s^{-}

d 26 m s^{-2}

30 **a** 4 cm

b Maximum 20 cm minimum 4 cm

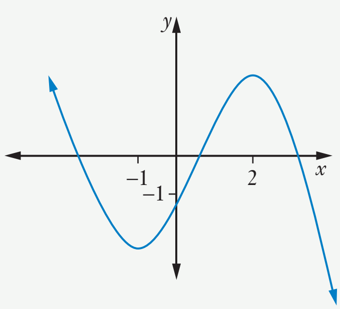
c 0 cm s^{-} **ii** 251 cm s^{-}

d $-8\pi^2$ or -79 cm s^{-2} **ii** $8\pi^2$ or 79 cm s^{-2}

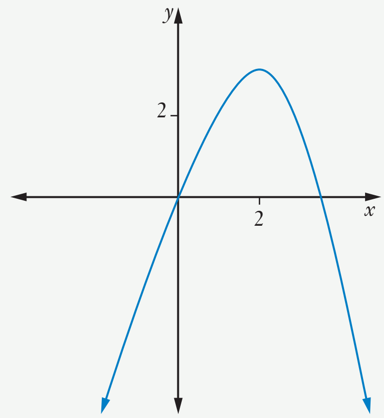
iii 0 cm s^{-2} **e** $\dfrac{d^2h}{dt^2} = -\pi^2(h-12)$

Exercse 406

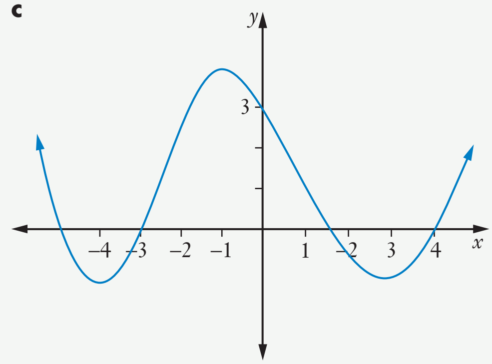
1 **a**



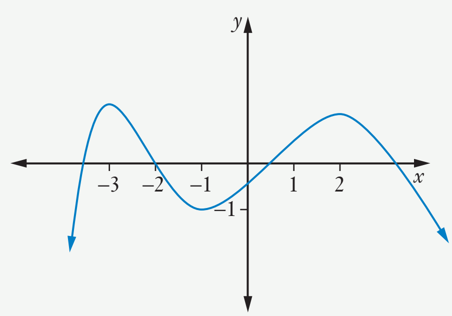
b



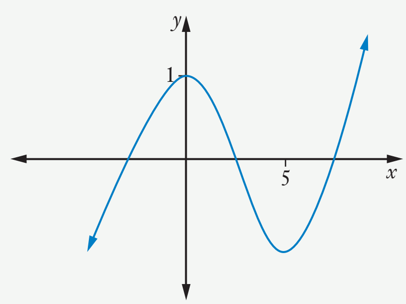
c



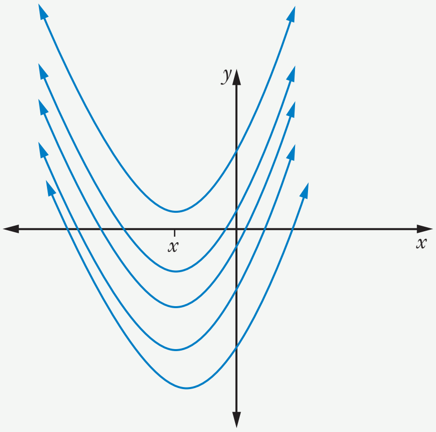
d



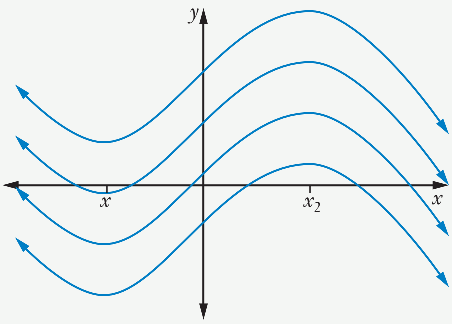
e



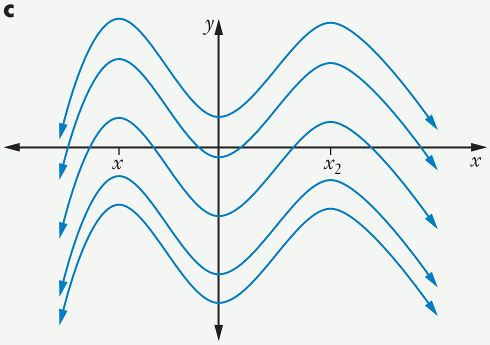
2 a



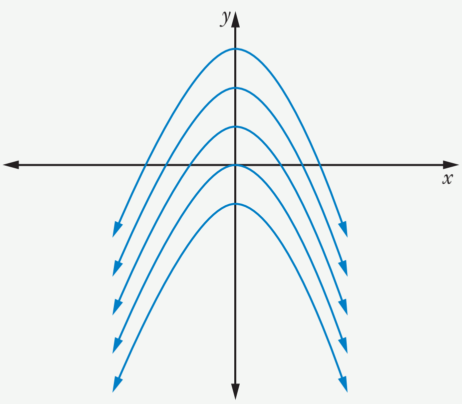
b



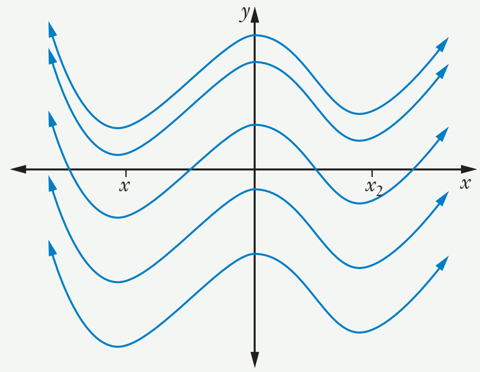
c



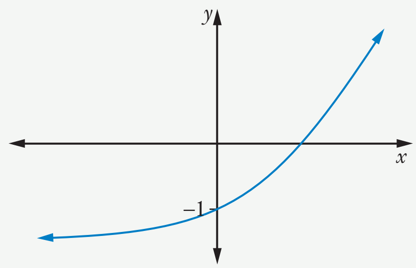
d



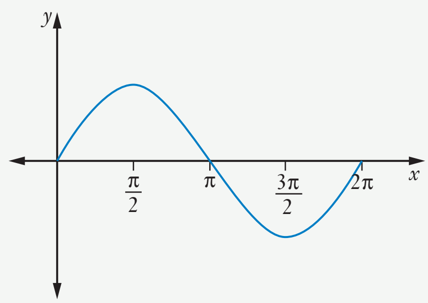
e



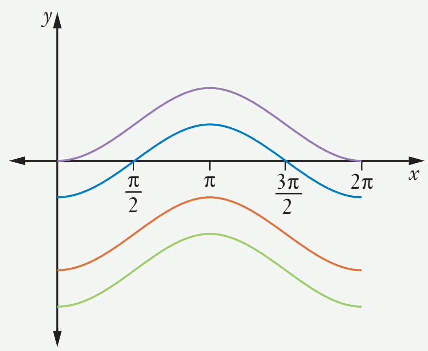
3



4



5



Exercse 407

1 a $x^2 - 3x + C$ **b** $\frac{x^3}{3} + 4x^2 + x + C$

c $\frac{x^6}{6} - x^4 + C$ **d** $\frac{(x-1)^3}{3} + C$

e $6x + C$ **f** $\frac{(3x+2)^6}{18} + C$

g $\frac{4(2x-7)^5}{5} + C$

2 a $f(x) = 2x^3 - \frac{x^2}{2} + C$

b $f(x) = \frac{x^5}{5} - x^3 + 7x + C$

c $f(x) = \frac{x^2}{2} - 2x + C$

d $f(x) = \frac{x^3}{3} - x^2 - 3x + C$ **e** $f(x) = \frac{2x^{\frac{3}{2}}}{3} + C$

3 a $y = x^5 - 9x + C$ **b** $y = -\frac{x^{-3}}{3} + 2x^{-} + C$

c $y = \frac{x^4}{20} - \frac{x^3}{3} + C$ **d** $y = -\frac{2}{x} + C$

e $y = \frac{x^4}{4} - \frac{x^2}{3} + x + C$

4 a $\frac{2\sqrt{x^3}}{3} + C$ **b** $-\frac{x^{-2}}{2} + C$

c $-\frac{1}{7x^7} + C$ **d** $2x^{\overline{2}} + 6x^{\overline{3}} + C$

e $-\frac{x^{-6}}{6} + 2x^{-} + C$

5 a $\frac{(x^2+5)^5}{5} + C$ **b** $\frac{(x^3-1)^{10}}{10} + C$

c $\frac{(2x^2+3)^4}{2} + C$ **d** $\frac{3(x^5+1)^7}{7} + C$

e $\frac{(x^2-4)^8}{16} + C$ **f** $\frac{(2x^6-7)^9}{108} + C$

g $\frac{(x^2-x+3)^5}{5} + C$ **h** $\frac{(x^3+2x^2-7x)^{11}}{11} + C$

$\frac{(x^2-6x-1)^6}{12} + C$

6 $y = \frac{x^4}{4} - x^3 + 5x - \frac{1}{4}$

7 $f(x) = 2x^2 - 7x + 11$ **8** $f(1) = 8$

9 $y = 2x - 3x^2 + 19$ **10** $x = 16\frac{}{3}$

11 $y = 4x^2 - 8x + 7$ **12** $y = 2x^3 + 3x^2 + x - 2$

13 $f(x) = x^3 - x^2 - x + 5$ **14** $f(2) = 205$

15 $y = \frac{4x^3}{3} - 15x - 14\frac{}{3}$

16 $y = \frac{x^3}{3} - 2x^2 + 3x - 4\frac{2}{3}$

17 $f(x) = x^4 - x^3 + 2x^2 + 4x - 2$

18 $y = 3x^2 + 8x + 8$

19 $f(-2) = 77$ **20** $x = 3t^2 - 5t - 2$

21 $x = 2t^4 - 2t^3 + 3t^2 - 8t - 3$

Exercse 408

1 a $-\cos x + C$ **b** $\tan x + C$ **c** $\sin x + C$

d $\frac{1}{7}\tan 7x + C$ **e** $-\frac{1}{2}\cos(2x - \pi) + C$

2 a $e^x + C$ **b** $\frac{1}{6}e^{6x} + C$ **c** $\ln|x| + C$

d $\text{n}\,|3x-1| + C$ **e** $\frac{1}{2}\ln|x^2+5| + C$

3 a $e^x + 5x + C$ **b** $\sin x + 2x^2 + C$

c $\frac{x^2}{2} + \ln|x| + C$

d $2x^4 - x^3 + 3x^2 - 3x + \ln|x| + C$

e $-\frac{1}{5}\cos 5x - \frac{1}{9}\tan 9x + C$

4 $y = \sin x - 5$ **5** $f(x) = 5\ln|x| + 3$

6 $y = 2\sin 2x + \sqrt{3}$

7 $f(x) = 3e^{3x} - 8e^6x + 14e^6$

8 a $P = 25\,000e^{0.054} + 10\,000$ **b** 52 900

9 $x = e^3 + 4$

10 a $\frac{dx}{dt} = 3\cos 3t$

b –03 cm **c** $0, \frac{\pi}{3}\ \frac{2\pi}{3}\ \pi\ \frac{4\pi}{3}, \ldots$ s

Test yourslf 4

1 D **2** B **3** A **4** C

5 a $5e^{5x}$ **b** $-2e^{1-x}$ **c** $\frac{1}{x}$

d $\frac{4}{4x+5}$ **e** $e^x(x+1)$ **f** $\frac{1-\ln x}{x^2}$

g $10e^x(e^x+1)^9$

6 a $-\sin x$ **b** $2\cos x$ **c** $\sec^2 x$

d $x\cos x + \sin x$ **e** $\frac{x\sec^2 x - \tan x}{x^2}$

f $-3\sin 3x$ **g** $5\sec^2 5x$

7 $3x - y + 3 = 0$

8 $12x + 4\sqrt{2}y - 4 - 3\pi = 0$

9 $\dfrac{d^2x}{dt^2} = -4\cos 2t$

10 **a** $\dfrac{1}{\sqrt{1-x^2}}$ **b** $\dfrac{3}{1+9x^2}$

c $-\dfrac{10}{\sqrt{1-25x^2}}$

11 $-\dfrac{e^2}{e^2+1}$

12 **a** $2x^5 - x^4 + 3x^2 - 3x + C$

b $\dfrac{1}{5}e^{5x} + C$

c $\dfrac{1}{9}\tan 9x + C$

d $\text{n}|x+5| + C$

e $\dfrac{1}{2}\sin 2x + C$

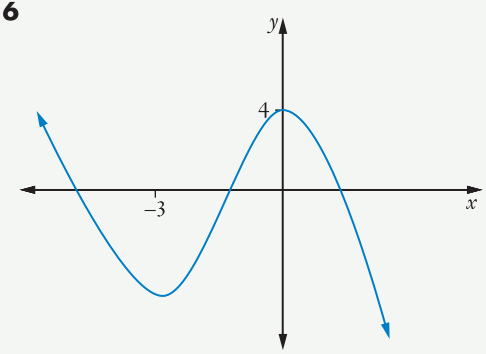
f $-4\cos\dfrac{x}{4} + C$

13 $-3\sqrt{3}$

14 $y = 2x^3 + 6x^2 - 5x - 33$

15 $\dfrac{1}{5\sqrt[5]{(x-3)^4}}$

16



17 $2x + y - \ln 2 - 4 = 0$

18 $4x + 8y - 8 - \pi = 0$

19 **a** $\dfrac{1}{\sqrt{1-x^2}}$ **b** $-\dfrac{1}{\sqrt{25-x^2}}$ **c** $\dfrac{1}{1+x^2}$

d $\dfrac{4}{\sqrt{1-16x^2}}$ **e** $\dfrac{2}{4+x^2}$

20 **a** $40x(5x^2+7)^3$

b $4(16x-3)(2x-3)^6$

c $\dfrac{23}{(3x+4)^2}$ **d** $2x^2e^x(x+3)$

e $\dfrac{3(x+1)\sec^2 3x - \tan 3x}{(x+1)^2}$

21 **a** 0

b $\sin^{-} x + \cos^{-} x = \dfrac{\pi}{2}$ $\dfrac{d}{dx}\left(\dfrac{\pi}{2}\right) = 0$

22 $f(x) = \dfrac{5x^3}{2} + 6x^2 - 49x + 59$

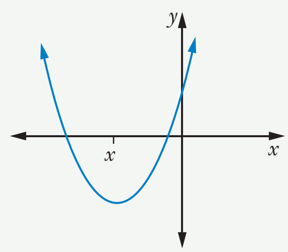
23 **a** -8 **b** 26 **c** -90

24 **a** $f^{-}(x) = x^2 - 1$ **b** $P = (3, 8)$

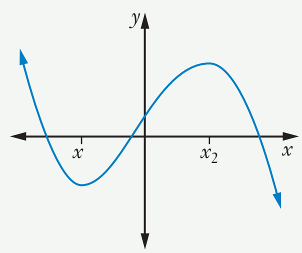
c $6x - y - 10 = 0$

25 $\dfrac{x}{1+x^2} + \tan^{-} x$

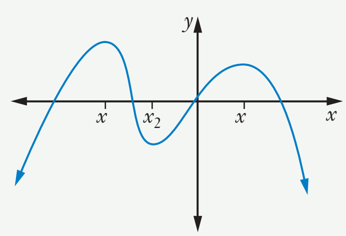
26 **a**



b



c



27 $f(x) = 2x^3 - 3x^2 - 31x + 68$

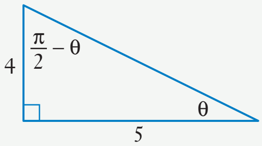
28 $4x - 6\sqrt{3}y + \sqrt{3}\pi - 6 = 0$

29 **a** $\dfrac{(3x^4-5)^7}{84} + C$ **b** $\dfrac{3(x^2+1)^{10}}{20} + C$

Chaenge exercse 4

1 $2e$

2 **a**



b 0

c $\theta + \frac{\pi}{2} - \theta$

3 $\frac{20x^2 - 120x - 1}{(4x^2+1)^4}$ $\frac{-24(20x^3 - 140x^2 - 3x + 5)}{(4x^2+1)^5}$

4 **a** $e^{x}\ \ + C$ **b** $-\frac{1}{3}\cos(x^3) + C$

5 $(2x\cos 2x + \sin 2x)e^{x\sin 2x}$

6 $y = \frac{x^3}{3} - x^2 - 15x - 1$

7 **a** $\frac{2x}{\sqrt{1-x^4}}$ **b** $\frac{e^x}{1+e^{2x}}$

c $\frac{\cos x - \sin x}{\sin x + \cos x}$

8 $25\frac{5}{6}$

9 $\frac{1 + \ln x - x\ln x}{e^x}$

10 **a** $\theta = \tan^{-}\ (4t)$ **b** 3° 32

11 **a** $x = y + e^y$ **b** $P = (1 + e, 1)$

c $x - (1 + e)y = 0$

12 **a** $\frac{\sec^2 x}{\tan x}$ or $\frac{1}{\sin x \cos x}$ **b** $-\ln|\cos x| + C$

13 **a** $-\frac{\cos\left(x^3 - \pi\right)}{3} + C$ **b** $\frac{e^x}{2} + C$

Chapter 5

Exercise 501

1 $x < 2$ **2** $x < \frac{1}{4}$

3 $(-\infty\ 0)$

4 **a** $x < 1.5$ **b** $x > 1.5$ **c** $x = 1.5$

5 $f\ (x) = -2 < 0$ for all x

6 $y\ = 3x^2 > 0$ for all $x \neq 0$

7 (0 0)

8 $x = -3\ 2$

9 **a** (1 –4) **b** (0 9)

c (1 1) and (, 0) **d** (0 1, 1, 0) and –1, 0)

10 (2 0)

11 $x = 2, 5$

12 $p = -12$

13 $a = 1\frac{}{2}$ $b = -6$

14 **a** $\frac{dy}{dx} = 3x^2 - 6x + 27$

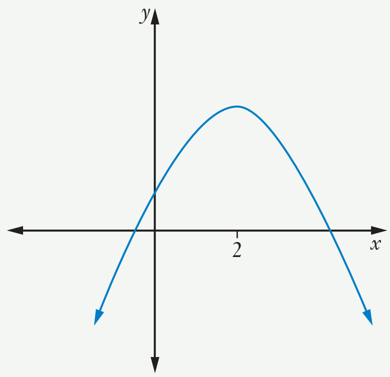
b The quadratic function has $a > 0$

$b^2 - 4ac = -288 < 0$

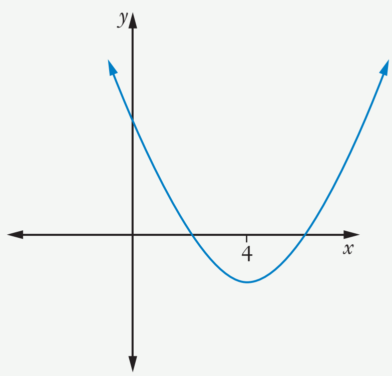
So $3x^2 - 6x + 27 > 0$ for all x

The function is monotonic increasing for all x

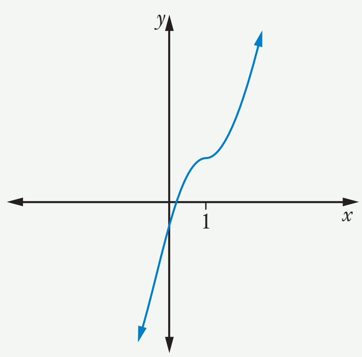
15



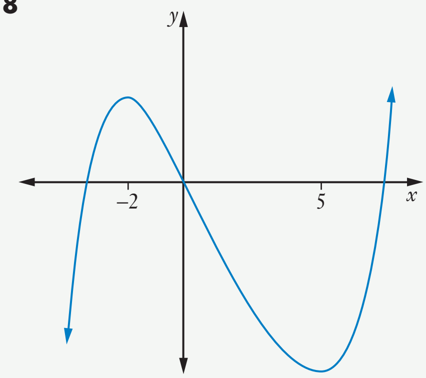
16



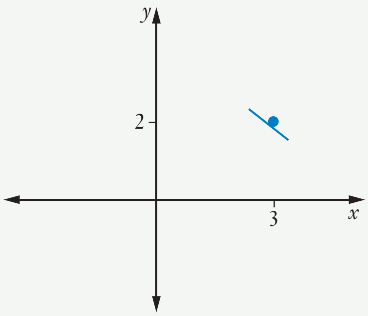
17



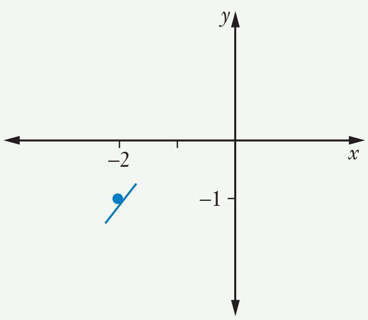
18



19



20



21 (2 0) and $\left(\frac{2}{3}\ 3\ \frac{13}{81}\right)$

22 $\frac{3x+2}{2\sqrt{x+1}}$ $\left(-\frac{2}{3}\ -\frac{2\sqrt{3}}{9}\right)$ **23** $a = -175$

24 $y\ = \frac{1}{2\sqrt{x}} \neq 0$ **25** $y\ = -\frac{3}{x^4} \neq 0$

Exercse 502

1 (0 –1; show $y\ < 0$ on LHS $y\ > 0$ on RHS

2 (0 0, minimum

3 (–2 11; show $f\ (x) > 0$ on LHS and $f\ (x) < 0$ on RHS

4 (–1 –2, minimum **5** (4 0) minimum

6 (0 5) maximu, 4, –27) minimum

7 (0 5) maximu, 2, 1) minimum

8 (0 –3) maximu, 1, –4) minium, (–1 –4) minimum

9 (1 0) minimu, (1, 4) maximum

10 $m = -3$ **11** $x = -3$ minimum

12 $x = 0$ minimum $x = -1$ maximum

13 **a** $\frac{dP}{dx} = 2 - \frac{50}{x^2}$

b (–5 –20) maximu, 5, 20) minimum

14 $\left(1\ \frac{1}{2}\right)$ minimum

15 (206 5.94) maximu, (2.6, 54.94) minimum

16 (437 5.92) minimu, (4.7, 54.92) maximum

17 **a** $\frac{3600-2x^2}{\sqrt{3600-x^2}}$

b (424 1800) maximu, (– 424 –1800) minimum

Exercse 503

1 $x > -\frac{1}{3}$ **2** $x < 3$

3 $y'' = -8 < 0$ **4** $y'' = 2 > 0$

5 $\left(-\infty\ 2\ \frac{}{3}\right)$ **6** (1 9)

7 (1 –17) and (–, –41)

8 (0 –2: $y'' < 0$ on LHS $y'' > 0$ on RHS

9 **a** No

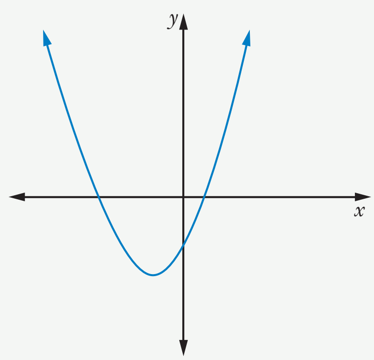
b Yes – point of inflection at (, 0)

c Yes –point of inflection at (, 0)

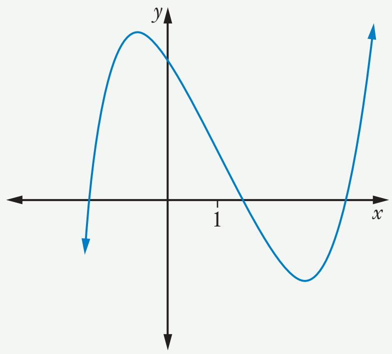
d Yes – point of inflection at (, 0)

e No

10



11



12 None (2 31) is not a point of inflection since concavity does not change

13 Show that $\frac{12}{x^4} > 0$ for all $x \neq 0$

14 **a** (0 7, 1, 0) and –1, 14) **b** (0 7)

15 **a** $12x^2 + 24 \neq 0$ and there are no points of inflection

b The curve is always concave upwards

16 $a = 2$ **17** $p = 4$ **18** $a = 3, b = -3$

19 **a** (0 –8, 2, 2)

b $\frac{dy}{dx} = 6x^5 - 15x^4 + 21$

At (0 –8:

$\frac{dy}{dx} \neq 0$

At (2 2:

$\frac{dy}{dx} \neq 0$

Exercse 504

1 **a** $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} > 0$ **b** $\frac{dy}{dx} < 0, \frac{d^2y}{dx^2} < 0$

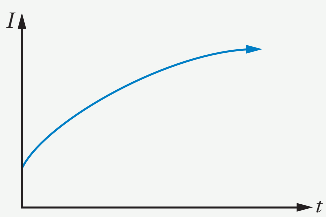
c $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} < 0$ **d** $\frac{dy}{dx} < 0, \frac{d^2y}{dx^2} > 0$

e $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} > 0$

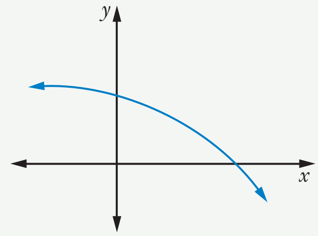
2 **a** $\frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$

b No the rate is decreasin.

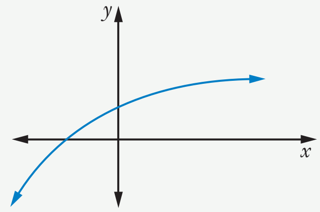
3



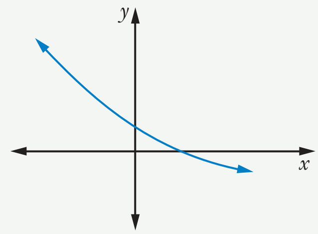
4 **a**



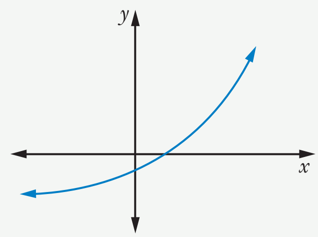
b



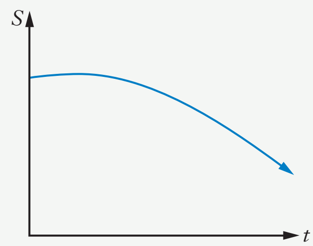
c



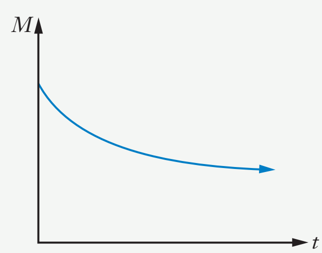
d



5



6

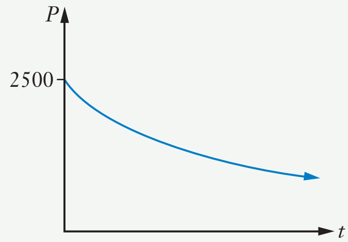


7 $\frac{dM}{dt} < 0, \frac{d^2M}{dt^2} > 0$

8 **a** The number of fish is decreasing

b The population rate is increasing

c



9 The level of education is increasing but the rate of increase is slowing down

10 The population is decreasing and the rate of change in population is decreasing

Exercse 505

1 (1 0, minimum

2 (0 1, minimum (flat)

3 (2 –5; $y'' = 6 > 0$ so minimum

4 (05 .25; $y'' = -2 < 0$ so maximum

5 (0 –5; $f''(x) < 0$ on LHS $f''(x) > 0$ on RHS

6 Yes – point of inflection at (, 3)

7 (–2 –78) minimu, (3, –77) maximum

8 (0 1) maximu, (1, –4) minium, (2 –31) minimum

9 (0 1) maximu, 05, 0) minium, (–05 0) minimum

10 **a** (4 176) maximu, 5, 175) minimum

b (45 17.5)

11 (367 .38, maximum

12 (0 –1) minimu, (2, 15) maxium, (–4 –1) minimum

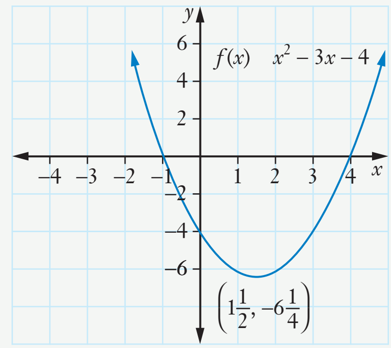
13 **a** $a = 4$ **b** $\left(\frac{1}{2}\ 0\right)$ minimum

14 $m = -4$

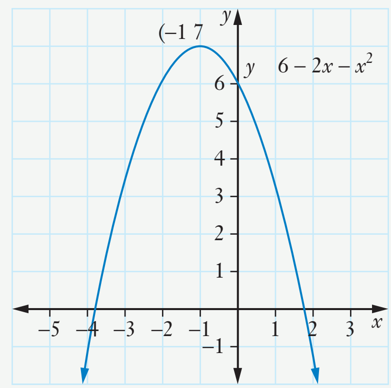
15 $a = 3, b = -9$

Exercse 506

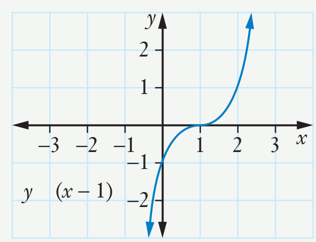
1



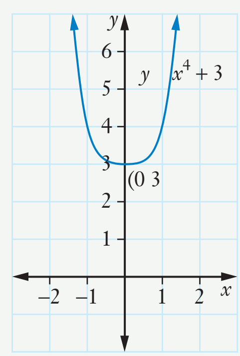
2



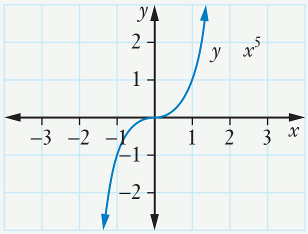
3 (1 0) point of inflection



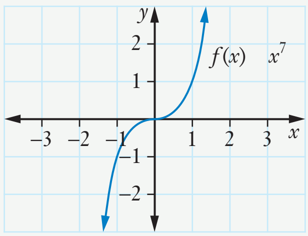
4



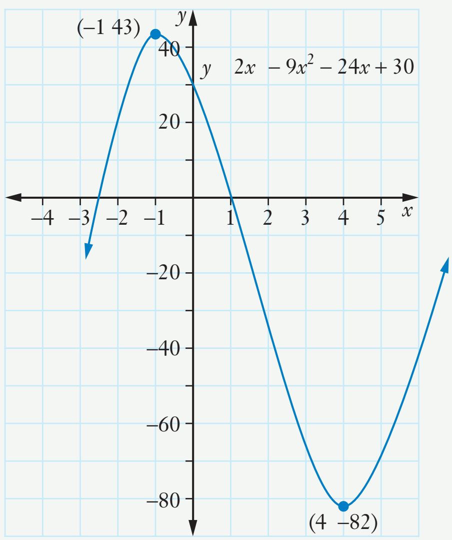
5 (0 0)



6



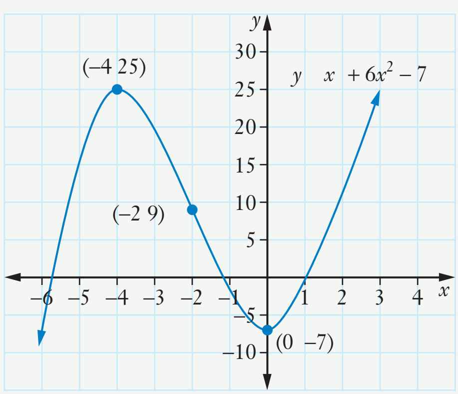
7 (–1 43) maximu, 4, –82) minimum



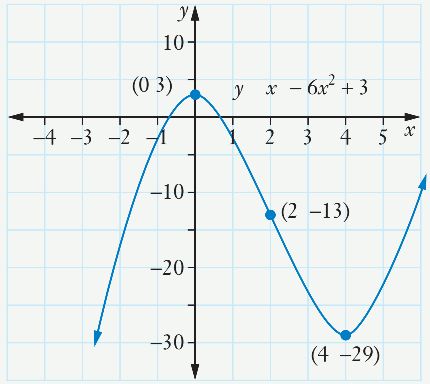
8 **a** (0 –7) minimu, (4, 25) maximum

b (–2 9)

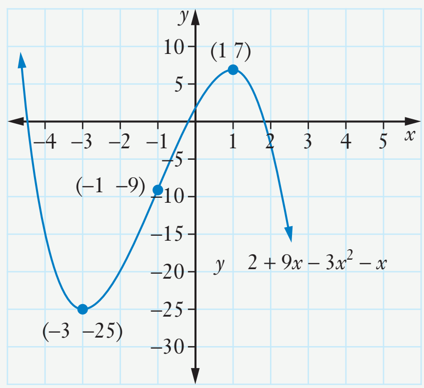
c



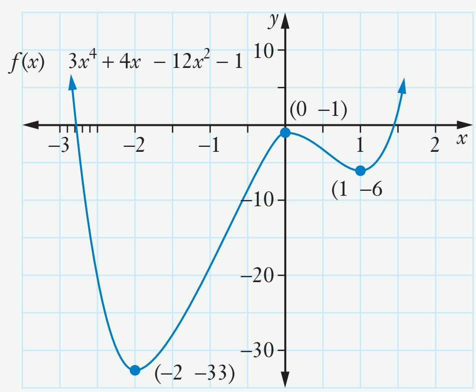
9 (0 3) maximu, 2, –13) point of inflecton,
(4 –29) minimum



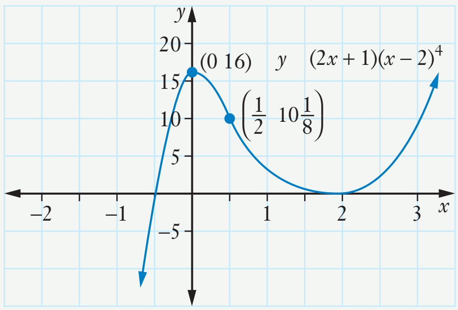
10 (–3 –25) minimu, (1, –9) point of inflecton,
(1 7) maximum



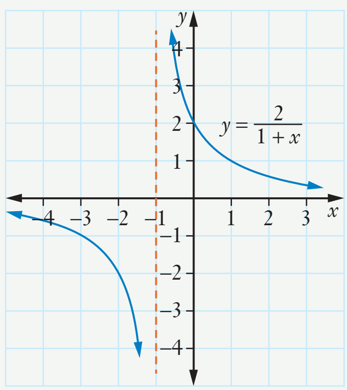
11 (–2 –33) minimu, 0, –1) maxium,
(1 –6) minimum



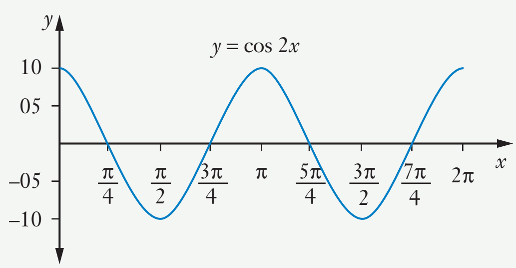
12 (0 16) maximu, $\left(\frac{1}{2}, 10\frac{1}{8}\right)$ point of inflection
(2 0) minimum



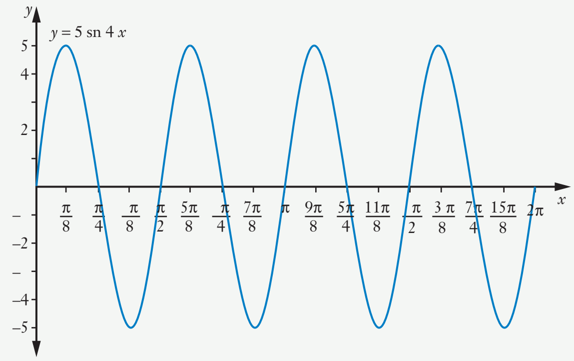
13 $\frac{dy}{dx} = -\frac{2}{(1+x)^2} \neq 0$ for any x



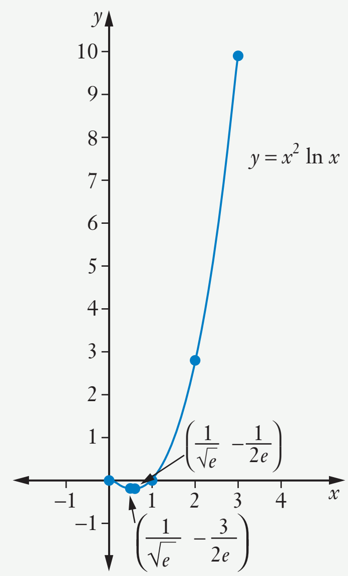
14 a



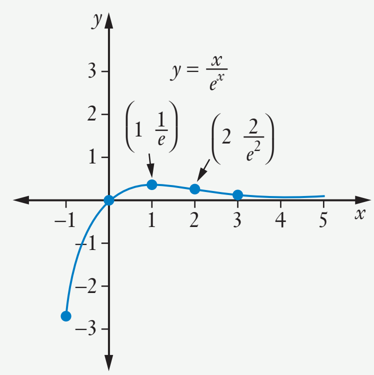
b



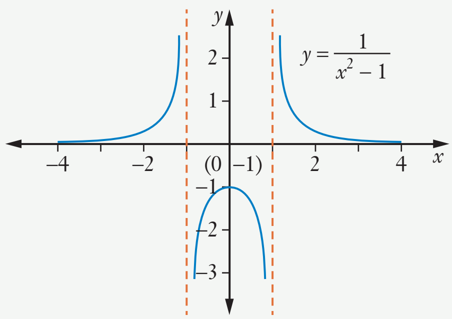
15 a



b

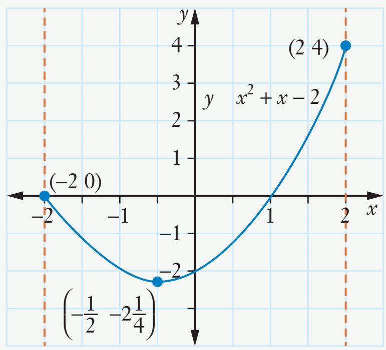


c

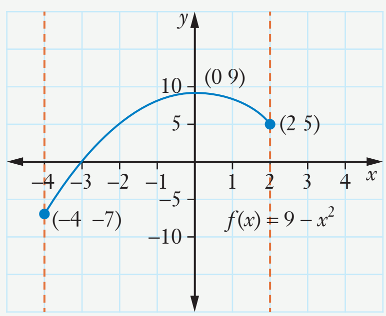


Exercse 507

1 Maximum value is 4

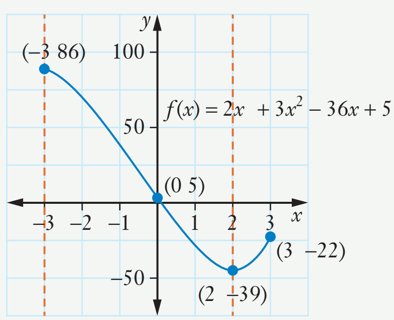


2 Maximum value is 9 minimum value is –.



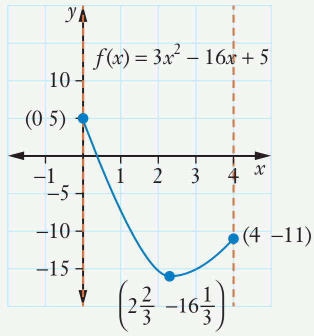
3 Maximum value is 25

4 Maximum value is 86 minimum value is –3.



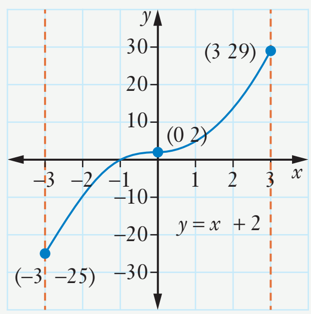
5 Maximum value is –2

6 Maximum value is 5 minimum value is $-16\frac{1}{3}$

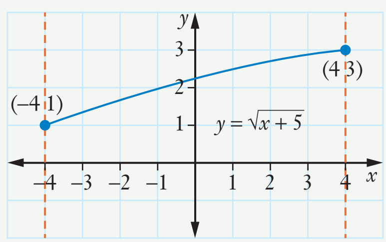


7 Global maximum 29 local maximum –, global minimum –35 local minimum –3, –8

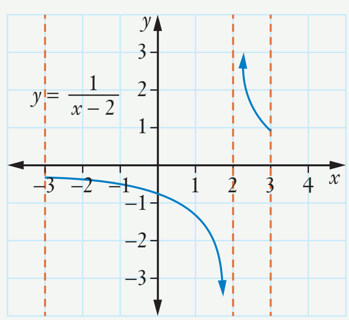
8 Minimum –25 maximum 29



9 Maximum 3 minimum 1



10 Maximum ∞ minimum –∞



nvestgaton

The disc has radius $\frac{30}{7}$ cm (This result uses Stewarts theorem – research thi.)

Exercse 508

See worked solutions for full proofs

1 $\frac{50}{x} = y$

$P = 2x + 2y$

2 $y = 60 - x$

$A = xy$

3 $\frac{20}{x} = y$

$S = x + y$

4 $\frac{400}{\pi r^2} = h$

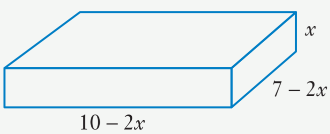
$S = 2\pi r^2 + 2\pi rh$

5 **a** $x + y = 30$

b $A = \left(\frac{1}{4}x\right)^2 + \left(\frac{1}{4}y\right)^2$

6 **a** $x^2 + y^2 = 280^2 = 78\ 400$ **b** $A = xy$

7

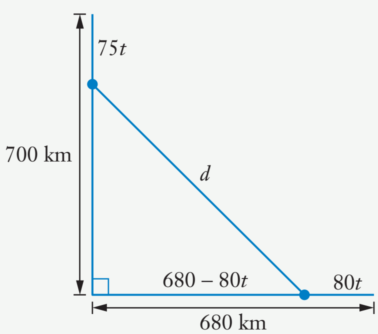


$V = x(10 - 2x)(7 - 2x)$

8 Profit per person = Cost – Expenses

$= (900 - 100x) - (200 + 400x)$

9



$d^2 = (700 - 75t)^2 + (680 - 80t)^2$

10 Distance AB $d = \sqrt{x^2 + 0\ 5^2}$

$t = \frac{\sqrt{x^2 + 025}}{5}$

Distance BC $d = 7 - x$

$t = \frac{7 - x}{4}$

Exercse 509

See worked solutions for full proofs

1 2 s 16 m

2 75 km

3 **a** $y = 30 - x$

b Maximum area is 225 m^2

4 **a** $\frac{4000}{x} = y$

$P = 2x + 2y$

b 632 m by 632 m **c** \$12 32288

5 4 m by 4 m

6 14 and 14

7 –25 and 25

8 $x = 125$ m, $y = 125$ m

9 **a** $V = x(30 - 2x)(80 - 2x)$

b $x = 6\frac{2}{3}$ cm **c** 74074 cm^3

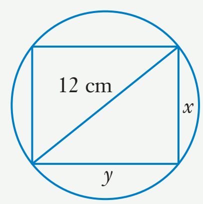
10 **a** $\frac{54}{r^2} = h$

$S = 2\pi r(r + h)$

b Radius is 3 m

11 **a** $S = 2\pi r^2 + \frac{17\,200}{r}$ **b** 2324 m^2

12 **a**



$x^2 + y^2 = 12^2$

$A = xy = x\sqrt{144 - x^2}$

b 72 cm^2

13 **a** $\frac{400}{x} = y$

$A = (x - 10)(y - 10)$

b 100 cm^2

14 112 m^2

15 **a** 75 m by 75 m **b** 24 m

16 16017 cm^2

17 168 m .32 m

18 **a** $d^2 = (200 - 80t)^2 + (120 - 60t)^2$

b Minimum distance 24 km

19 **a** $d = (x^2 - 2x + 5) - (4x - x^2)$

$= 2x^2 - 6x + 5$

b 05

20 a $s = \dfrac{d}{t}$

So $t = \dfrac{d}{s}$

$= \dfrac{1500}{s}$

Cost of trip taking t hours

$C = (s^2 + 9000)t$

$= (s^2 + 9000)\dfrac{1500}{s}$

$= 1500\left(s + \dfrac{9000}{s}\right)$

b 95 km h^{-} **c** \$2846

Test yourslf 5

1 A **2** C **3** D **4** C

5 (–3 –11) maximu, (1, –15) minimum

6 $x > 1\frac{}{6}$

7 50 m

8 $x > -1$

9 $(\frac{1}{2}\ -1)$

10 a $\dfrac{375}{\pi r^2} = h$

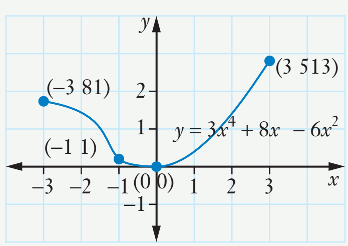
$S = 2\pi r^2 + 2\pi rh$

b 39 cm

11 a (0 0) and (–, 1)

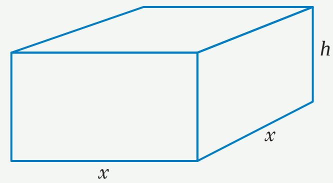
b (0 0) minimu, (1, 1) horizontal point of inflection

c



d Maximum value 513 minimum value 0

12 a



$\dfrac{125 - x^2}{2x} = h$

$V = x^2 h$

b 645 cm by 645 cm by 645 cm

13 150 products

14 a $y = \sqrt{25 - x^2}$

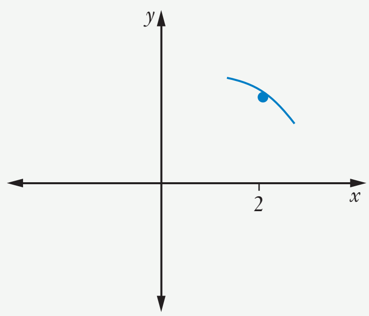
$A = \dfrac{1}{2}xy$

b 625 m ^{2}

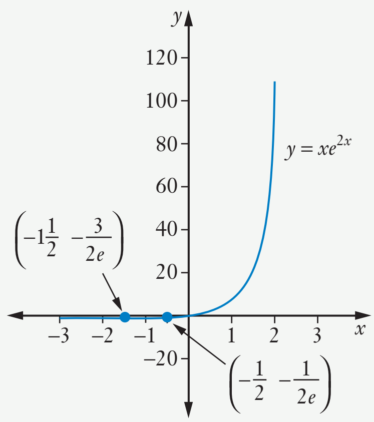
15 (0 1) and (, –74)

16 179

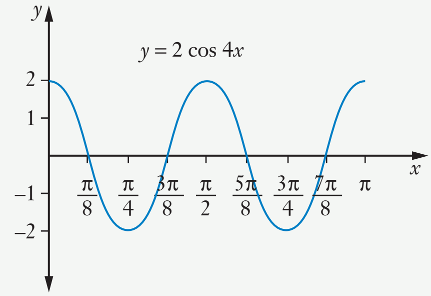
17



18

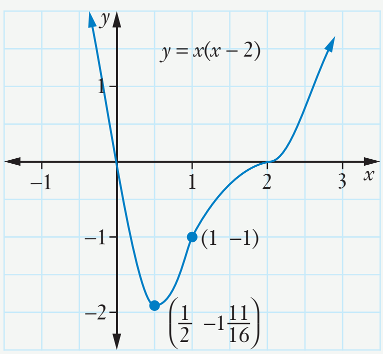


19



Chaenge exercse 5

1



2 16 m^2

3 27 −2025

4 $f\ (06)\ = f\ (06)\ = 0$ and concavity changes

5 Proof See Worked Solutins. $r = s = 125$

6 $y = x^2 + 2x + 3$

7 **a** $y\ = 0$ at (0 0)

b $y'' > 0$ on LHS and RHS

c $y'' < 0$ on LHS $y'' > 0$ on RHS

8 Minimum −1 maximum $-\frac{1}{5}$

9 87 km h^{-}

Chapter 6

Exercse 601

1 **a** 4125 units ^{2} **b** 6625 units ^{2}

2 **a** 117 units ^{2} **b** 167 units ^{2} **c** 127 units ^{2}

d 152 units ^{2}

3 **a** 65 units ^{2} **b** 5 units^2 **c** 132 units^2

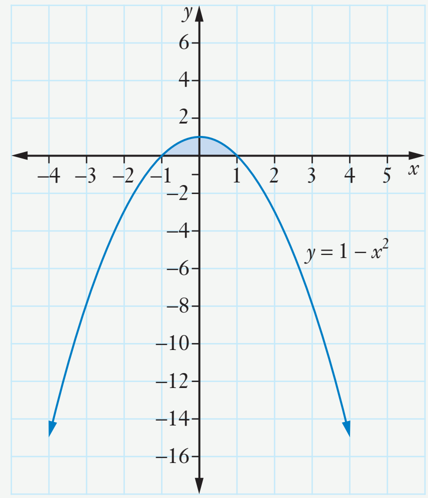
d $\frac{\pi\left(1+\sqrt{2}\right)}{8\sqrt{2}} = \frac{\pi}{16}\left(2+\sqrt{2}\right)$ units^2 **e** 65 units ^{2}

4 **a** 28 units^2 **b** 156 units^2 **c** 140 units^2

5 **a** $3\frac{3}{7}$ units^2 **b** 55 units ^{2} **c** $\frac{\pi}{4}$ units^2

d $\frac{3}{2}(e + e^4)$ **e** 23 units^2

6 **a**



b 1 unit^2

7 **a** 441 units ^{2} **b** 491 units ^{2}

c 45 units ^{2} **d** 45 units ^{2}

8 $\frac{25\pi}{2}$ units^2

9 **a** $\frac{9\pi}{2}$ units^2

b 25 units ^{2} **ii** 81 units ^{2}

10 **a** 224 units ^{2} **b** 33 units ^{2} **c** 1 unit^2

11 **a** 6 units^2 **ii** 14 units^2

b $\frac{\pi\sqrt{2}}{4}$ units^2 **ii** $\frac{\pi\left(\sqrt{2}+2\right)}{4}$ units^2

12 **a** 60625 units ^{2} **b** 73125 units ^{2}

Exercse 602

1 **a** 25 **b** 10

c 24 **d** 0225

2 **a** 28 **b** 22

3 **a** 039 **b** 041

4 **a** 108 **b** 075 **c** 065

d 094 **e** 092

5 **a** 751 **b** 165 **c** 6502

6 **a** 289 m ^{2} **b** 3925 m ^{2} **c** 745 km ^{2}

d 49225 m ^{2}

Exercse 603

1 **a** 8 **b** 10 **c** 217
d –1 **e** 10 **f** 54
g $3\frac{1}{3}$ **h** 16 50

2 **a** $\frac{2}{3}$ **b** $21\frac{1}{4}$ **c** 0
d $4\frac{2}{3}$ **e** $1\frac{1}{4}$ **f** $4\frac{1}{3}$
g 0 **h** $2\frac{1}{3}$ 0
j $6\frac{2}{9}$ **k** 100 54
m $15\frac{5}{6}$ **n** $22\frac{2}{3}$ **o** 00126

3 **a** 70 m **b** 38 km **c** 258 cm
d $72\frac{2}{3}$ m **e** 142 cm

4 **a** 750 L **b** 51 000 L **c** 810 750

Exercse 604

1 **a** $\frac{x^3}{3}+C$ **b** $\frac{x^6}{2}+C$ **c** $\frac{2x^5}{5}+C$
d $\frac{m^2}{2}+m+C$ **e** $\frac{t^3}{3}-7t+C$ **f** $\frac{h^8}{8}+5h+C$
g $\frac{y^2}{2}-3y+C$ **h** x^2+4x+C $\frac{b^3}{3}+\frac{b^2}{2}+C$

2 **a** $\frac{x^3}{3}+x^2+5x+C$
b $x^4-x^3+4x^2-x+C$
c $x^6+\frac{x^5}{5}+\frac{x^4}{2}+C$
d $\frac{x^8}{8}-\frac{3x^7}{7}-9x+C$
e $\frac{x^4}{2}+\frac{x^3}{3}-\frac{x^2}{2}-2x+C$
f $\frac{x^6}{6}+\frac{x^4}{4}+4x+C$
g $\frac{4x^3}{3}-\frac{5x^2}{2}-8x+C$
h $\frac{3x^5}{5}-\frac{x^4}{2}+\frac{x^2}{2}+C$
$\frac{3x^4}{2}+\frac{5x^3}{3}-4x+C$
j $-x^{-} -\frac{x^{-2}}{2}-2x^{-} +C$

3 **a** $-\frac{1}{7x^7}+C$ **b** $\frac{3x^{\frac{4}{3}}}{4}+C$
c $\frac{x^4}{4}-x^3+x^2+C$ **d** $x-2x^2+\frac{4x^3}{3}+C$
e $\frac{x^3}{3}+\frac{3x^2}{2}-10x+C$ **f** $-\frac{3}{x}+C$
g $-\frac{1}{2x^2}+C$
h $-\frac{4}{x}-x+\frac{3}{2x^2}-\frac{7}{4x^4}+C$
$\frac{y^3}{3}+\frac{y^{-6}}{6}+5y+C$
j $\frac{t^4}{4}-\frac{t^3}{3}-2t^2+4t+C$
k $\frac{2\sqrt{x^3}}{3}+C$ $-\frac{1}{2t^4}+C$ **m** $\frac{3\sqrt[3]{x^4}}{4}+C$
n $\frac{2\sqrt{x^5}}{5}+C$ **o** $\frac{2\sqrt{x^3}}{3}+x+C$

4 900° **5** $y=x^3-x^2+x+6$
6 –11 07925 cm s⁻ **7** 21 000 L

Exercse 605

1 **a** $\frac{(3x-4)^3}{9}+C$ **b** $\frac{(x+1)^5}{5}+C$
c $\frac{(5x-1)^{10}}{50}+C$ **d** $\frac{(3y-2)^8}{24}+C$
e $\frac{(4+3x)^5}{15}+C$ **f** $\frac{(7x+8)^{13}}{91}+C$
g $-\frac{(1-x)^7}{7}+C$ **h** $\frac{\sqrt{(2x-5)^3}}{3}+C$
$-\frac{2(3x+1)^{-3}}{9}+C$ **j** $-3(x+7)^{-}+C$
k $-\frac{1}{16(4x-5)^2}+C$ $\frac{3\sqrt[3]{(4x+3)^4}}{16}+C$

m $-2(2-x)^{\overline{2}} + C$ **n** $\dfrac{2\sqrt{(t+3)^5}}{5} + C$

o $\dfrac{2\sqrt{(5x+2)^7}}{35} + C$

2 **a** 2882 **b** $-1\frac{1}{4}$ **c** $-\frac{1}{8}$

d $60\frac{2}{3}$ **e** $\frac{1}{6}$ **f** $\frac{1}{7}$

g $4\frac{2}{3}$ **h** $1\frac{1}{5}$ $\frac{3}{5}$

3 **a** $\frac{1}{3}(x^4+5)^3 + C$ **b** $\frac{1}{6}(x^2-3)^6 + C$

c $\frac{1}{4}(x^3+1)^4 + C$ **d** $\frac{1}{5}(x^2+3x-2)^5 + C$

e $\frac{1}{42}(3x^2-7)^7 + C$ **f** $-\frac{1}{45}(4-5x^3)^3 + C$

g $\frac{1}{15}(2x^6-3)^5 + C$ **h** $\frac{3}{80}(5x^2+3)^8 + C$

$\frac{1}{12}(x^2+4x)^6 + C$ **j** $\frac{1}{12}(3x^3-6x-2)^4 + C$

4 **a** $108\frac{2}{3}$ **b** $-\frac{1}{18}$

c 66 81275 **d** $10\ 159\frac{1}{32}$

e 564 5376 **f** $-1236\frac{2}{3}$

g $-19\ 839\frac{3}{14}$ **h** $96\frac{4}{9}$

–474 618 5653

5 $y = \frac{1}{15}((x^3-2)^5 + 61)$

6 **a** $x = \frac{1}{10}(x^2-3)^5 - 1]$ **b** 7775 m

Exercse 606

1 **a** $\frac{1}{4}e^{4x} + C$ **b** $-e^{-x} + C$ **c** $\frac{1}{5}e^{5x} + \mathrm{C}$

d $-\frac{1}{2}e^{-2x} + C$ **e** $\frac{1}{4}e^{4x+1} + C$ **f** $-\frac{3}{5}e^{5x} + C$

g $\frac{1}{2}e^{2} + C$ **h** $\frac{1}{7}e^{7x} - 2x + C$

$e^{x-3} + \frac{x^2}{2} + C$

2 **a** $\frac{1}{5}(e^5 - 1)$ **b** $e^{-2} - 1 = \frac{1}{e^2} - 1$

c $\frac{2e^7}{3}(e^9 - 1)$ **d** $19 - \frac{1}{2}e^4(e^2 - 1)$

e $\frac{1}{2}e^4 + 1\frac{1}{2}$ **f** $e^2 - e - 1\frac{1}{2}$

g $\frac{1}{2}e^6 + e^{-} - 1\frac{1}{2}$

3 **a** 032 **b** 26829

c 37 85568 **d** 34685

e 75519

4 **a** $\frac{1}{\ln 5}5^x + C$ **b** $\frac{1}{3\ln 7}7^{3x} + C$

c $\frac{1}{2\ln 3}3^{2x-1} + C$

5 **a** $x(2+x)e^x$ **b** $x^2e^x + C$

6 $f(x) = \frac{1}{6}(e^{2x} - 1)$ **7** $2e^3 + 5$ m

Exercse 607

1 **a** $\ln|2x+5| + C$ **b** $\ln|2x^2+1| + C$

c $\ln|x^5-2| + C$

d $\frac{1}{2}\ln|x| + C$ or $\frac{1}{2}\ln|2x| + C$

e $2\ln|x| + C$ **f** $\frac{5}{3}\ln|x| + C$

g $\ln|x^2-3x| + C$ **h** $\frac{1}{2}\ln|x^2+2| + C$

$\frac{3}{2}\ln|x^2+7| + C$ **j** $\frac{1}{2}\ln|x^2+2x-5| + C$

2 **a** $\ln|4x-1| + C$ **b** $\ln|x+3| + C$

c $\frac{1}{6}\ln|2x^3-7| + C$ **d** $\frac{1}{12}\ln|2x^6+5| + C$

e $\frac{1}{2}\ln|x^2+6x+2| + C$

3 **a** 05 **b** 07 **c** 16

d 31 **e** 05

4 **a** RHS = LHS

b $\ln|x+3| + 2\ln|x-3| + C$

5 **a** RHS = LHS

b $x - 5\ln|x-1| + C$

6 $f(x) = \frac{1}{9}\ln\left|\frac{3x^3-1}{2}\right|$ **7** 895 m

8 4

Exercse 608

1 **a** $\sin x + C$ **b** $-\cos x + C$

c $\tan x + C$ **d** $-\frac{45}{\pi}\cos x° + C$

e $-\frac{1}{3}\cos 3x + C$ **f** $\frac{1}{7}\cos 7x + C$

g $\frac{1}{5}\tan 5x + C$ **h** $\sin (x + 1) + C$

$-\frac{1}{2}\cos (2x - 3) + C$ **j** $\frac{1}{2}\sin (2x - 1) + C$

k $\cos (\pi - x) + C = -\cos x + C$

$\sin (x + \pi) + C = -\sin x + C$

m $\frac{2}{7}\tan 7x + C$ **n** $-8\cos \frac{x}{2} + C$

o $9\tan \frac{x}{3} + C$

2 **a** 1 **b** $\sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

c $\frac{2}{\sqrt{2}} = \sqrt{2}$ **d** $-\frac{1}{3}$

e $\frac{1}{\pi}$ **f** $\frac{1}{2}$

g $\frac{3}{4}$ **h** $-\frac{1}{5}$

3 **a** $\sin\left(x + \frac{\pi}{3}\right) + C$ or $\frac{\sin x + \sqrt{3}\cos x}{2} + C$

b $\cos (\pi - x) + C = -\cos x + C$

4 $y = \frac{1}{4}\sin 4x + \frac{\pi}{4}$

5 **a** $x = 18\sin\frac{2\pi t}{3} + 2$ cm

b $9\sqrt{3} + 2$ cm **ii** $-9\sqrt{3} + 2$ cm

6 **a** $d = -24\cos\frac{\pi t}{6} + 26$ **b** 14 m

c 50 m 2 m 26 m **d** 12 h

Exercse 609

1 $1\frac{1}{3}$ units^2 **2** 36 units^2 **3** 45 units ^{2}

4 $10\frac{2}{3}$ units^2 **5** $\frac{1}{6}$ units^2 **6** 143 units ^{2}

7 4 units^2 **8** 04 units ^{2} **9** 8 units^2

10 2425 units ^{2} **11** $e^2(e^2 - 1)$ units^2

12 $\frac{1}{4}(e - e^{-})$ units^2 **13** 286 units ^{2}

14 295 units ^{2} **15** 4 units^2

16 $\frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$ units^2 **17** 086 units ^{2}

18 2 units^2 **19** $9\frac{1}{3}$ units^2

20 $11\frac{2}{3}$ units^2 **21** $\frac{1}{6}$ units^2

22 $\frac{2}{3}$ units^2 **23** $\frac{1}{3}$ units^2

24 $\ln 3 - \ln 2 = \ln 15$ units ^{2}

25 $\ln 2$ units^2

26 061 units ^{2} **27** $5\frac{1}{3}$ units^2

28 18 units^2 **29** 398 units ^{2}

30 $\frac{a^4}{2}$ units^2

Exercse 610

1 $21\frac{1}{3}$ units^2 **2** 20 units^2

3 $4\frac{2}{3}$ units^2 **4** 15 units ^{2}

5 $1\frac{1}{4}$ units^2 **6** $2\frac{1}{3}$ units^2

7 $10\frac{2}{3}$ units^2 **8** $\frac{1}{6}$ units^2

9 $3\frac{7}{9}$ units^2 **10** 2 units^2

11 $11\frac{1}{4}$ units^2 **12** 60 units^2

13 45 units ^{2} **14** $1\frac{1}{3}$ units^2

15 19 units ^{2} **16** 472 units ^{2}

Exercise 611

1 $1\frac{1}{3}$ units^2 **2** $1\frac{1}{3}$ units^2

3 $\frac{1}{6}$ units^2 **4** $10\frac{2}{3}$ units^2

5 $20\frac{5}{6}$ units^2 **6** 8 units^2

7 $\frac{2}{3}$ units^2 **8** $166\frac{2}{3}$ units^2

9 $\frac{5}{12}$ units^2 **10** $\frac{2}{3}$ units^2

11 $\frac{1}{12}$ units^2 **12** $\frac{1}{3}$ units^2

13 36 units^2 **14** $2\frac{2}{3}$ units^2

15 $(\pi - 2)$ units^2

16 $\frac{1}{2} + \ln 2$ units^2

17 $2\sqrt{2}$ units^2

18 $\frac{1}{2}(e^4 - 5)$ units^2

19 $\sqrt{3} - \frac{\pi}{3} = \frac{3\sqrt{3} - \pi}{3}$ units^2

Test yourslf 6

1 D **2** B **3** A **4** D

5 **a** 0535 **b** 05

6 **a** $\frac{3x^2}{2} + x + C$ **b** $\frac{5x^2}{2} - x + C$

c $\frac{2\sqrt{x^3}}{3} + C$ **d** $\frac{(2x+5)^8}{16} + C$

e $\frac{(3x^4 - 2)^5}{60} + C$

7 $\frac{3^x}{\ln 3} + C$

8 **a** 14 units^2 **b** 27 units^2 **c** 28 units^2

9 **a** 2 **b** 0 **c** $2\frac{1}{5}$

d $6\frac{7}{9}$ **e** $228\frac{3}{8}$

10 $e(e^2 - 1)$ units^2

11 3 units^2

12 $-\frac{180}{\pi}\cos x° + C$

13 $2\frac{2}{3}$ units^2

14 **a** $\frac{1}{4}e^{4x} + C$ **b** $\frac{1}{2}\ln|x^2 - 9| + C$

c $-e^{-x} + C$ **d** $\ln|x + 4| + C$

e $\frac{(x^2 - 6x + 1)^9}{18} + C$

15 $4\frac{1}{2}$

16 $\frac{9\pi}{4}$ units^2

17 $\frac{3}{4}$ units^2

18 $\frac{(7x+3)^{12}}{84} + C$

19 3 units^2

20 $\frac{1}{2}e^4(e^6 - 1)$ units^2

21 536 units ^{2}

22 36

23 **a** $-\frac{1}{2}\cos 2x + C$ **b** $3\sin x + C$

c $\frac{1}{5}\tan 5x + C$ **d** $x - \cos x + C$

24 $85\frac{1}{3}$ units^2

25 **a** $T = 40e^{-0.4} + 175$

b 180° **ii** 175°

26 **a** $\frac{1}{\sqrt{2}}$ **b** $\sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

27 **a** $\frac{(2x-1)^5}{2} + C$ **b** $\frac{x^6}{8} + C$

28 $\frac{1}{\sqrt{2}}$ units^2

29 **a** $\frac{1}{18}(x^3 - 2)^6 + C$ **b** $\frac{1}{50}(5x^2 + 2)^5 + C$

c $\frac{5}{24}(2x^4 - 1)^3 + C$ **d** $\frac{1}{8}(x^2 + 4x - 3)^4 + C$

30 2 units^2

31 **a** $\sqrt{2}$ units^2 **b** 1 unit^2

c $1 + \sqrt{2}$ units^2

32 **a** $f(x) = \frac{1}{20}[3(2x^2 - 1)^5 + 57]$

b $f(x) = \frac{1}{2}\tan 2x$

c $f(x) = \frac{1}{5}e^{5x}$

d $f(x) = \frac{1}{16}(x^4 - 15)^4 - 1]$

e $f(x) = \frac{3}{4}\ln(x^4 + 1) + 2$

33 **a** $x = \frac{2\sqrt{t^3 + 9}}{3} - 4$ **b** 37 m

c 76 s

Chaenge exercse 6

1 **a** Show $f(-x) = -f(x)$

b 0 **c** 12 units^2

2 **a** RHS = LHS **b** $\ln\sqrt{3} = \frac{\ln 3}{2}$

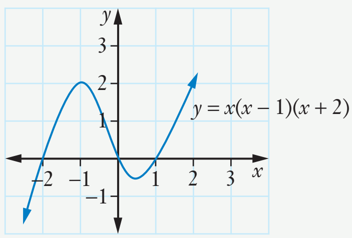
3 9 units^2

4 $\frac{3 - \sqrt{3}}{6}$

5 $\frac{1}{8}$

6 73 units ^{2}

7 **a**



b $3\frac{}{12}$ units^2

8 1168 units ^{2}

9 **a** $\dfrac{3(x+2)}{2\sqrt{x+3}}$ **b** $\dfrac{2x\sqrt{x+3}}{3} + C$

10 **a** $x(1 + 2 \ln x)$ **b** $18 \ln 3$

11 $\dfrac{5}{12}$ units^2

12 **a** 68797 **b** 137593 **c** 152762

13 **a** 68797 **b** 13759 units ^{2}

c 15276 units ^{2}

Practice set 2

1 C **2** D **3** B

4 B **5** A **6** C

7 $x < \dfrac{1}{2}$

8 $x^3 - x^2 + x + C$

9 24

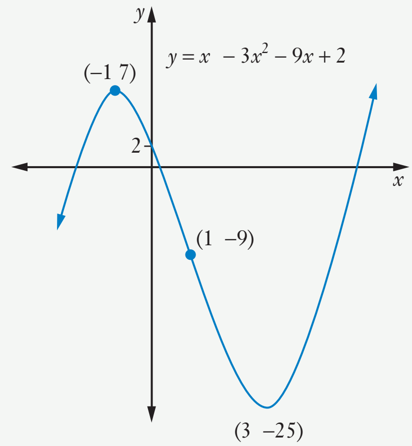
10 **a** 8 **b** $\dfrac{2\pi}{3}$ **c** 5

11 8 m

12 $\dfrac{d^2y}{dx^2} = 7(-7 \sin 7x)$

13 $\dfrac{x^9}{3} + 2x^2 + C$

14



15 $1\frac{1}{3}$ units^2

16 $\dfrac{3}{4}$ ln $2x^2 - 5| + C$

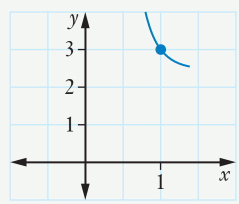
17 $f\ (3) = 20; f''(-2) = -16$

18 68 **19** $-\dfrac{7}{9}\ \infty$ **20** 3

21 $\dfrac{1}{4}(\tan x + 1)^4 + C$

22 **a** $f(1) = 3, f\ (1) = -2\ f''(1) = 18$

b Curve is decreasing and concave upwards at (1, 3.



23 **a** $P = 8x + 4y = 4$

$4y = 4 - 8x$

$y = 1 - 2x$

$A = 3x^2 + y^2$

b Rectangle $\frac{2}{7}$ m $\times \frac{6}{7}$ m square with sides $\frac{3}{7}$ m

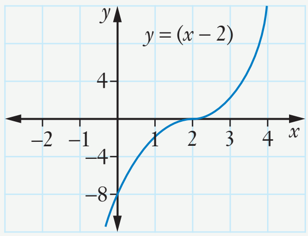
24 $y = 3\left[\frac{1}{2}(x+4)\right]^2 + 5$

25 $f(-1) = 0$

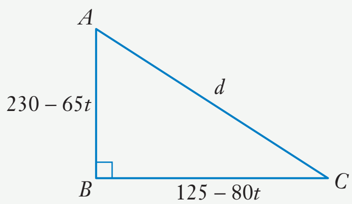
26 12 **27** 0944

28 **a** (2 0; point of inflection

b



29 **a**



$d^2 = (230 - 65t)^2 + (125 - 80t)^2$

b 23 h **c** 998 km

30 **a** $-\frac{25}{128}$ **b** $66\frac{77}{128}$ **c** $66\frac{2}{3}$

31 450 cm^2 **32** $\frac{(3x+5)^8}{24} + C$

33 1099

34 (0 3) maximu, 1, 2) minium, (–1 2) minimum

35 $2\frac{8}{15}$ **36** $10\frac{2}{3}$ units^2

37 $f(2) = -16$ **38** 9 units^2 **39** $3x^2 + 2e^{2x}$

40 **a** 7900 L **b** 400 min (6 h 40 min)

41 $\frac{3e}{2}(e^9 - 1)$ **42** 953 m

43 $f(x) = x^3 - 4x^2 - 3x + 20$

44 $\frac{12}{4x+3}$

45 $\frac{1}{3}\ln(3x^2 + 3x - 2) + C$

46 **a** $\frac{1-2x}{e^{2x}}$ **b** $\frac{1}{x\ln 3}$

47 $x - y + 2 = 0$

48 $\left(-\frac{1}{2}\ -\frac{1}{2e}\right)$ minimum

49 $y = 2\cos 3x + 3$

50 1

51 **a** $\cot x$ **b** $5e^{5x}\sec^2(e^5 + 1)$

52 0393

53 **a** 6 units^2 **b** 14 units^2

54 **a** $e^x(\sin x + \cos x)$ **b** $3\tan^2 x \sec^2 x$

c $-6\sin(3x - \frac{\pi}{2})$

55 $12x - 2y - 2 - 3\pi = 0$

56 **a** $3e^x \sin^2(e^x)\cos(e^x)$ **b** $\frac{\sec^2(\ln x + 1)}{x}$

57 $(5 - e)$ units^2

58 **a** $\frac{1}{3}e^{3x} + C$ **b** $\frac{1}{\pi}\tan \pi x + C$

c $\frac{1}{2}\ln x + C$ **d** $5\sin\left(\frac{x}{5}\right) + C$

e $-\frac{1}{8}\cos 8x + C$

59 $3x - 2\ln x - \frac{5}{x} + C$

60 $\frac{1}{5}e^{5x} + \frac{1}{\pi}\cos \pi x + C$ **61** $(e^2 - 1)$ units^2

62 –1

Chapter 7

Note Answers obtained from reading graphs are approximate

Exercse 701

1 **a** N **b** N **c** N
d C **e** N **f** N
g N **h** C N
j C **k** N C
m N **n** N **o** N
p N **q** N **r** N
s C **t** N

2 **a** O **b** C **c** N
d N **e** C **f** D
g D **h** N C
j N **k** D N
m N

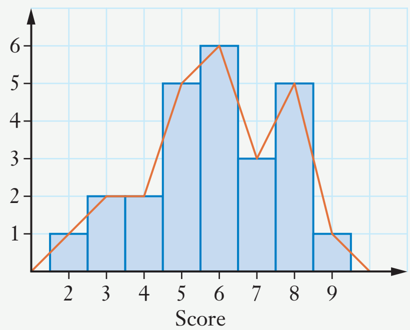
3 **a** eg types of spor, hair colour
b eg height, test scors, prices
c eg ranking, test scors, clothing sizes
d eg karate belt colou, size of take-away coffee cups Olympic medals
e eg race time, lengh, rainfall
f eg types of tree, housig, cars

Exercse 702

1 a

Score	Frequency
2	1
3	2
4	2
5	5
6	6
7	3
8	5
9	1

ii



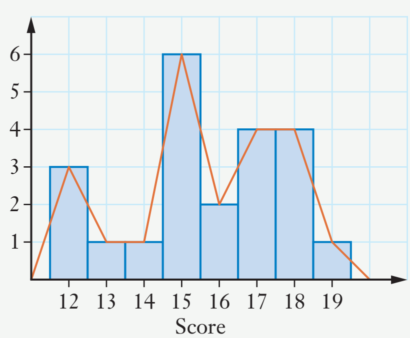
iii Highest 9 lowest 2

v 6

b

Pizzas	Frequency
12	3
13	1
14	1
15	6
16	2
17	4
18	4
19	1

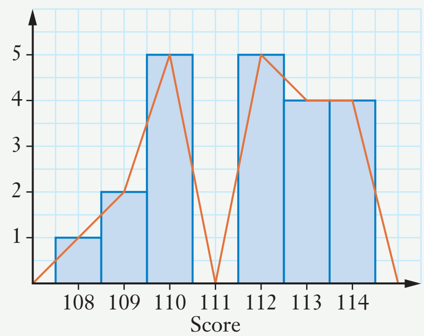
ii



iii Highest 19 lowest 12 **v** 15

c

Gym attendance	Frequency
108	1
109	2
110	5
111	0
112	5
113	4
114	4



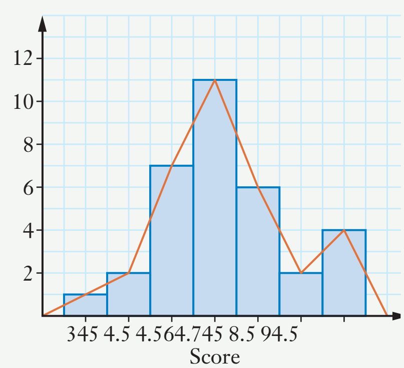
iii Highest 114 lowest 108

v 110 and 112

d

Results	Class centre	Frequency
30–39	345	1
40–49	445	2
50–59	545	7
60–69	64.5	11
70–79	745	6
80–89	845	2
90–99	945	4

ii

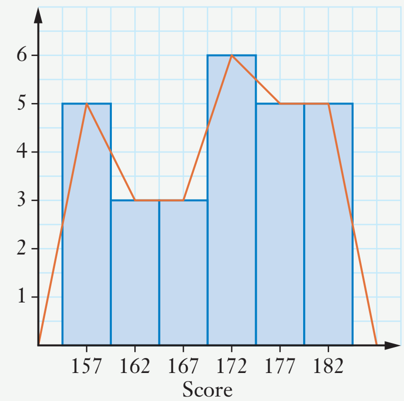


iii Highest 945 lowest 3.5 **v** 645

e

Height (cm)	Class centre	Frequency
155–159	157	5
160–164	162	3
165–169	167	3
170–174	172	6
175–179	177	5
180–184	182	5

ii

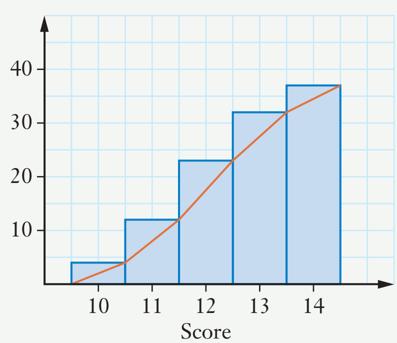


iii Highest 182 lowest 157 **v** 172

2 a

Number of cars	Frequency	Cumulative frequency
10	4	4
11	8	12
12	11	23
13	9	32
14	5	37

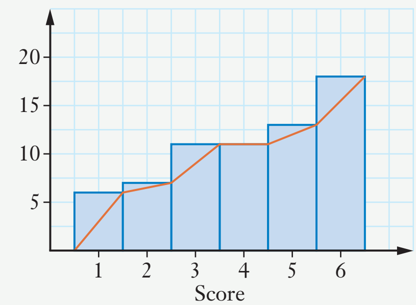
ii



b

Score	Frequency	Cumulative frequency
1	7	7
2	1	8
3	3	11
4	0	11
5	2	13
6	5	18

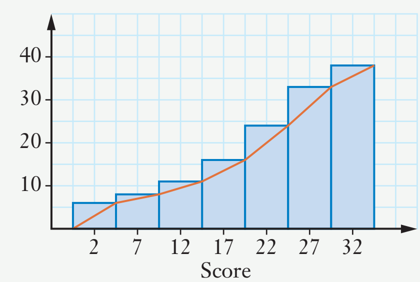
ii



c

Sales	Class centre	Frequency	Cumulative frequency
0–4	2	6	6
5–9	7	2	8
10–14	12	3	11
15–19	17	5	16
20–24	22	8	24
25–29	27	9	33
30–34	32	5	38

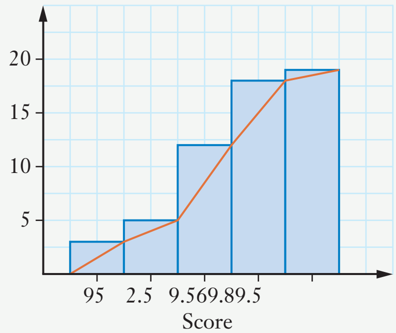
ii



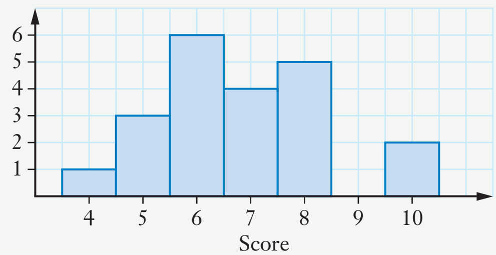
d

Scores	Class centre	Frequency	Cumulative frequency
0–19	9.5	3	3
20–39	29.5	2	5
40–59	49.5	7	12
60–79	69.5	6	18
80–99	89.5	1	19

ii

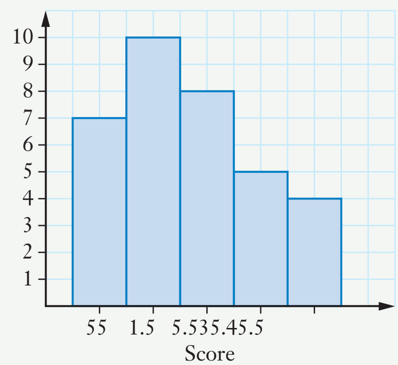


3 a



b 11 times **c** 6 rescues

4 a



b 382%

5 a

	Play soccer	Do not play soccer
Play tennis	12	35
Do not play tennis	27	28

b 118% **ii** 275%

c 365% **d** $\frac{4}{13}$ **e** 745%

6 a 16 **b** 5

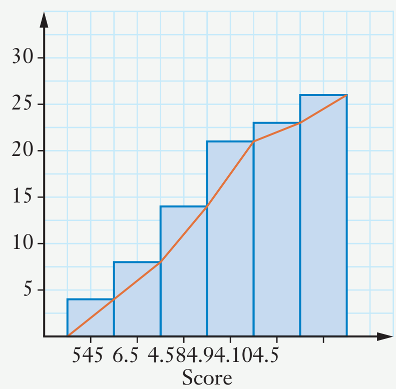
c

d 875% **e** $\frac{1}{8}$

7 a

Weight (kg)	Class centre	Frequency	Cumulative frequency
50–59	54.5	4	4
60–69	64.5	4	8
70–79	74.5	6	14
80–89	84.5	7	21
90–99	94.5	2	23
100–109	104.5	3	26

b



c 12 **ii** 4

d 50% **e** $\frac{7}{13}$

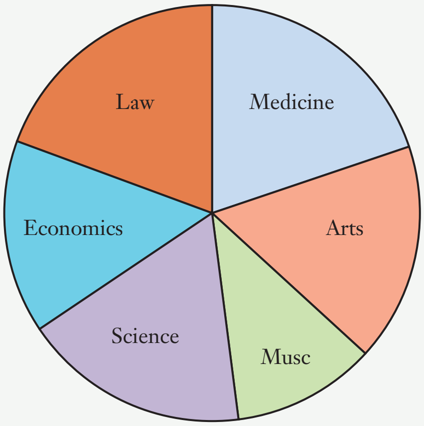
8 a 25%

b

Sport	Frequency
Tennis	90
Soccer	120
Athletics	60
Cricket	180
Basketball	150
Volleyball	120

9 **a** 193% **b** 31%

c



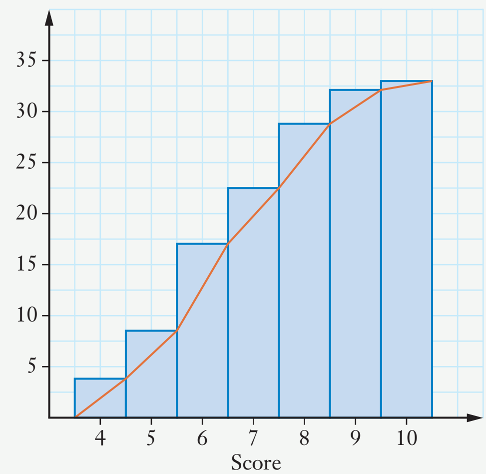
10 **a** 87 **b** 559%

c 65% **d** 76

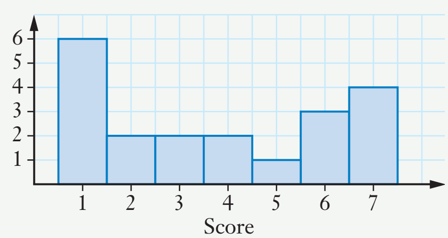
11 **a** 430 **b** 486% **c** 502%

d 221 **e** $\frac{112}{221}$

12



13 **a**

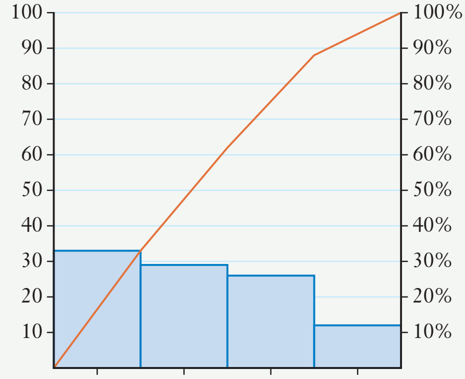


b

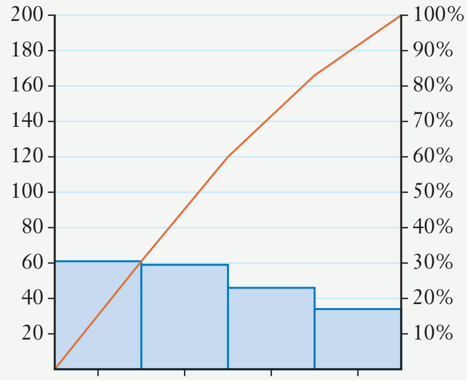
Junk mail items	Frequency
1	6
2	2
3	2
4	2
5	1
6	3
7	4

14 Stem-and-leaf plots list individual scores so retain all details Not easy to dra, and can be long A grouped frequency distribution table groups scores so individual data is lost Easy to draw and compact

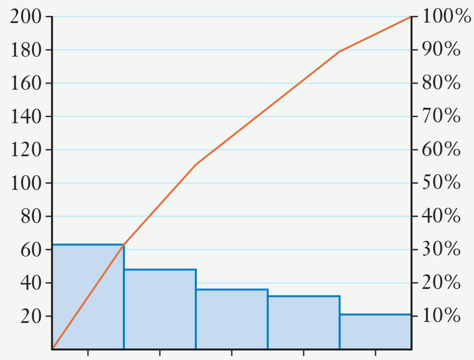
15 **a**



b



c



Exercse 703

1 a 5 **ii** 5 **iii** 5

b 5 **ii** 4 **iii** 5

c 16 **ii** 18 **iii** 17

d 57 **ii** 4 **iii** 55

e 151 **ii** 149 **iii** 149

2 a Brown **b** Tabby

3 a 6 **ii** 6 **iii** 6

b 524 **ii** 52 **iii** 51

c 1756 **ii** 18 **iii** 20

d 1033 **ii** 104 **iii** 104

4 a 3 **b** 15

c 35 **d** 3

5 a 924 **ii** 8–10

b 17 **ii** 20–24

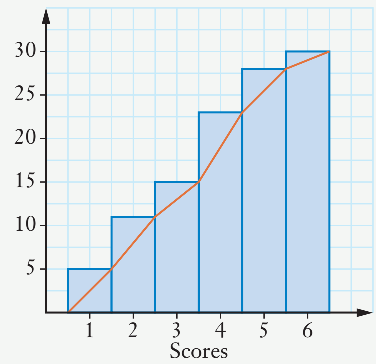
c 687 **ii** 70–84

d 39 **ii** 20–24 50 –54 (bimodal)

6 a

Athletes	Frequency	Cumulative frequency
1	5	5
2	6	11
3	4	15
4	8	23
5	5	28
6	2	30

ii

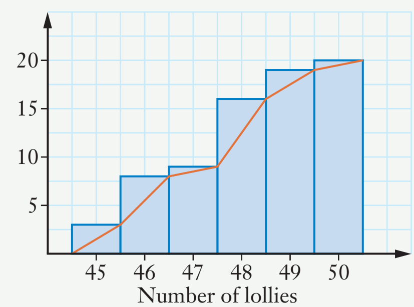


iii 35

b

Lollies	Frequency	Cumulative frequency
45	3	3
46	5	8
47	1	9
48	7	16
49	3	19
50	1	20

ii

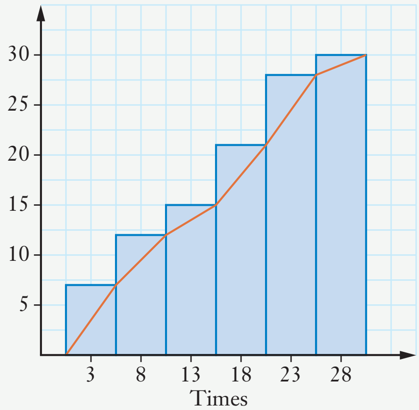


iii 48

c

Time (h)	Class centre	Frequency	Cumulative frequency
1–5	3	7	7
6–10	8	5	12
11–15	13	3	15
16–20	18	6	21
21–25	23	7	28
26–30	28	2	30

ii

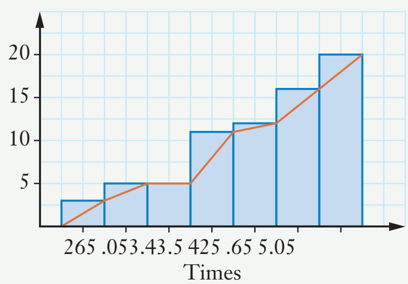


iii 155

d

Time (min)	Class centre	Frequency	Cumulative frequency
25 –28	2.65	3	3
29 –32	3.05	2	5
33 –36	3.45	0	5
37 –40	3.85	6	11
41 –44	4.25	1	12
45 –48	4.65	4	16
49 –52	5.05	4	20

ii



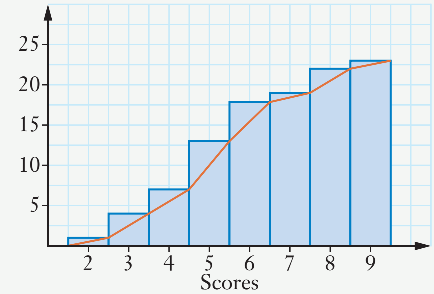
iii 40

7 a

Score	Frequency	Cumulative frequency
2	1	1
3	3	4
4	3	7
5	6	13
6	5	18
7	1	19
8	3	22
9	1	23

ii 535 **iii** 5

v



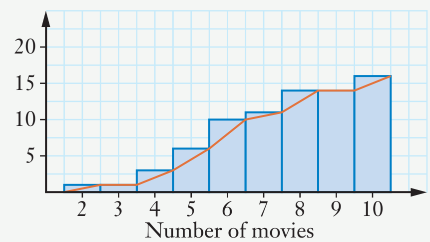
v 55

b

Number of movies	Frequency	Cumulative frequency
2	1	1
3	0	1
4	2	3
5	3	6
6	4	10
7	1	11
8	3	14
9	0	14
10	2	16

ii 625 **iii** 6

v



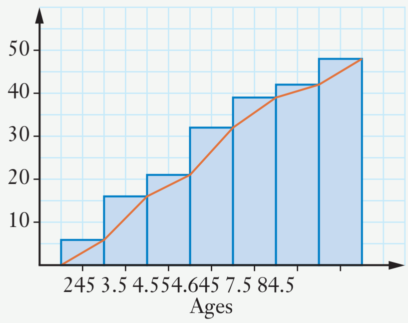
v 65

c

Ages	Class interval	Frequency	Cumulative frequency
20–29	24.5	6	6
30–39	34.5	10	16
40–49	44.5	5	21
50–59	54.5	11	32
60–69	64.5	7	39
70–79	74.5	3	42
80–89	84.5	6	48

ii 545 **iii** 50–59

v



v 545

8 **a** 32 **ii** Mean = 624 no mod, median = 64

iii Mean = 6625 no mod, median = 66

b 1 **ii** Mean = 66 mode = 6 median = 7

iii Mean = 68 mode = 6 median = 7

c 97 **ii** Mean = 471 mode = 50, median = 465

iii Mean = 432 mode = 50 median = 43

d 3 **ii** Mean = 61 mode = 5 median = 6

iii Mean = 63 mode = 5 median = 6

9 Outlier 1 It changes the mea.

10 **a** 17 One student had a very low score on the test

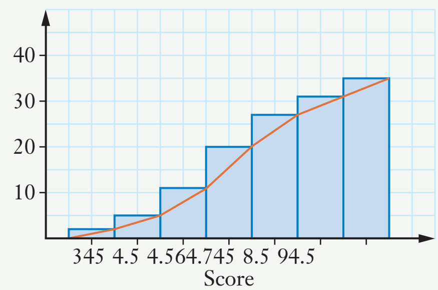
b

Class	Class centre	Frequency	Cumulative frequency
10–19	14.5	1	1
20–29	24.5	0	1
30–39	34.5	2	3
40–49	44.5	3	6
50–59	54.5	6	12
60–69	64.5	9	21
70–79	74.5	7	28
80–89	84.5	4	32
90–99	94.5	4	36

c Mean = 656 modal class = 60–69

ii Mean = 671 modal class = 60–69

d



Median = 66

Exercse 704

1 **a** 2 **ii** 3 **iii** 45

b 11 **ii** 12 **iii** 13

c 7 **ii** 8 **iii** 9

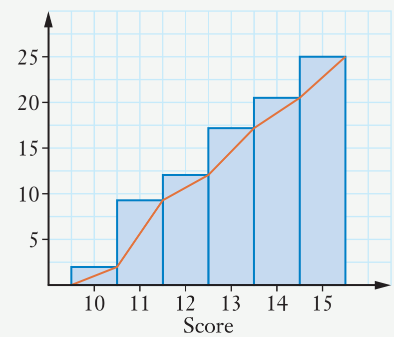
d 2 **ii** 3 **iii** 5

2 Q = 2, Q_3 = 45

3 **a** 23rd percentile = 24 55th percentile = 255, 91st percentile = 28

b 2nd decile = 24 8th decile = 27

4 **a**



b 11 **ii** 14 **iii** 12

v 135 **v** 105

5 **a** 42 **b** 37 **c** 38

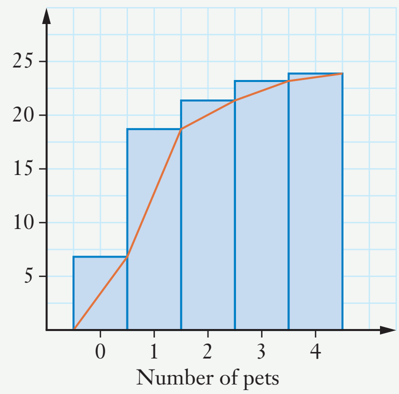
d 43 **e** 37–47

6 **a** 100 **b** 20% **c** 12

d 14 **e** 69th

7 **a** 125%

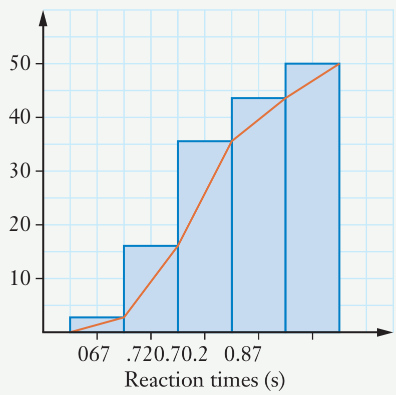
b



c 1 **ii** 0 **iii** 15

8 **a** 0774 **b** 38%

c



d 074 **ii** 077 **iii** 072 –082

9 **a** 51 **ii** 38 **iii** 79

b 11 **ii** 6 **iii** 145

c 1329 **ii** 1153 **iii** 1546

d 155 **ii** 145 **iii** 175

Exercse 705

1 **a** 16 **b** 71

c 39 **d** 5

2 **a** 65 **ii** 5 **iii** 4

b 2 **ii** 5 **iii** 1

c 40 **ii** 100 **iii** 55

d 2 **ii** 6 **iii** 1

3 **a** 7 **ii** 5 **iii** 2

b 5 **ii** 5 **iii** 2

c 17 **ii** 9 **iii** 4

d 14 **ii** 19 **iii** 5

e 6 **ii** 7 **iii** 2

4 **a** 676 **b** 7 **c** 7

d 4 **e** 15

5 **a** No outlier **b** 19 **c** 1

Exercse 706

1 **a** 54 **ii** 21

b 525 **ii** 146

c 1233 **ii** 161

d 63 **ii** 18

e 176 **ii** 196

2 **a** 72 **ii** 521

b 196 **ii** 38

c 135 **ii** 1832

d 14 **ii** 1966

e 23 **ii** 52

3 **a** 84 **ii** 24 **iii** 58

b 44 **ii** 17 **iii** 29

c 341 **ii** 15 **iii** 239

d 512 **ii** 149 **iii** 222

4 **a** 4 **b** 15 **c** 114

5 **a** 734 **ii** 163

b Ye, 20 –29 748 **ii** 144

6 **a** Ye, 3.

$Q_3 + 1.5 \times \text{IQR} = 305 \ + 1575 \ = 4625$

$Q \ - 15\ \text{IQR} \ = 20 - 1575 \ = 425$

So 53 is outside 425 –4625

b 84 **ii** 53

7 **a** 92 **b** 35 **c** 126

Exercse 707

1 **a** Positively skewed unimodal

b Symmetrical unimodal

c Positively skewed multimodal

d Negatively skewed unimodal

e Positively skewed unimodal

f Bimodal

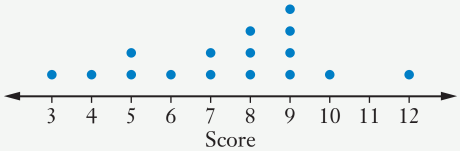
g Negatively skewed unimodal

h Bimodal

Positively skewed unimodal

j Multimodal

2 a



b Positively skewed

3 a Positively skewed unimodal

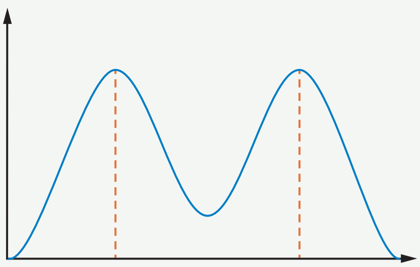
b Symmetrical unimodal

c Bimodal

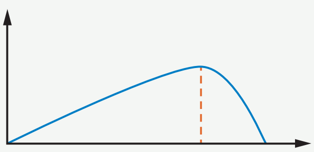
d Negatively skewed unimodal

e Symmetrical unimodal

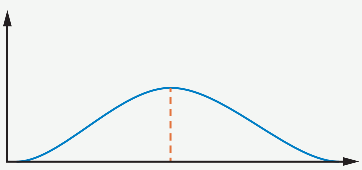
4 a



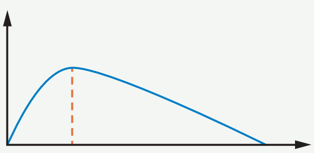
b



c



d



5 Set 1 bimodal Set : positively skewe, unimodal

6 a Plot 1 positively skewed
Plot 2 negatively skewed

b 5 **c** 1

7 a

Score	Class centre	Frequency
145–149	147	3
150–154	152	3
155–159	157	3
160–164	162	6
165–169	167	3
170–174	172	1
175–179	177	6
180–184	182	4
185–189	187	1

b Bimodal

8

9 a 7 **ii** 7 **iii** 7

b Symmetrical

10 Class discussion

Exercse 708

1 a 7 **ii** 4

b 2 **ii** 2

c 3 **ii** 2

d Class discussion

2 a

Sample 1		Sample 2
9 8 7	14	
9 8 7 5 3 0	15	1 7 7 9
9 7 6 4 2 0	16	2 4 5 6 8
7 6 2 1 0	17	2 3 3 6 8 9
	18	0 1 1 1 2

b Sample 1 161 Sample : 17.5

c Sample 1 30 Sample : 31

d Class discussion

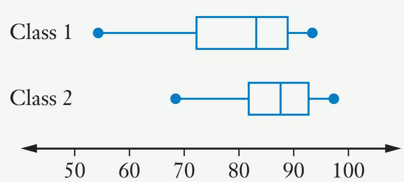
3 a Bank 1 508 Bank : .7, Bank3: 1.95

ii Bank 1 4 Bank : , Bank3: 13

b Bank 1 41 Bank : ., Bank3:4.4

c Class discussion

4 a



b Class 1 825 Class : 88

ii Class 1 17 Class : 1.5

iii Class 1 794 Class : 8.8

v Class 1 38 Class : 28

5 a

Camera 1		Camera 2
9 9 9 6 5	6	
7 6 5 4 4 3 3 2	7	
9 5 4 3 0	8	
0	9	
3	10	
	11	3 6 8 9 9
	12	0 0 2 2 3 4 5
	13	0 0 1 5 5 8
	14	0 2

b Camera 1 773 kmh ⁻ Camera 2 1261 kmh ⁻

c Camera 1 60 kmh ⁻ Camera 2 110 kmh ⁻

6 a 175 **b** 17 **c** 10

d 267

7 a 557 **ii** 586

b 316 **ii** 125

c The average test score is similar in both tests but test 1 has a wider spread of scores

8 a Parents 2 Childre: 4

b Parents 4 Childre: 3

ii Parents 5 Childre: 6 **c** 6

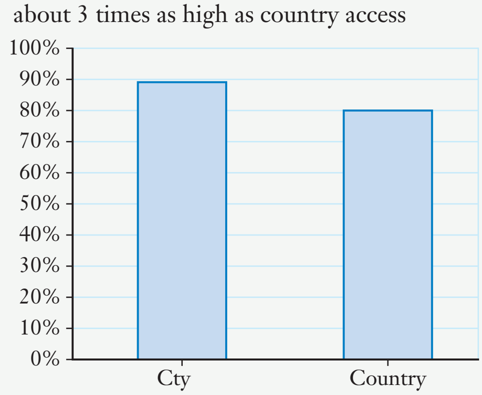
9 a Science 70 Englis: 6.5

b Science 698 Englis: 6.8

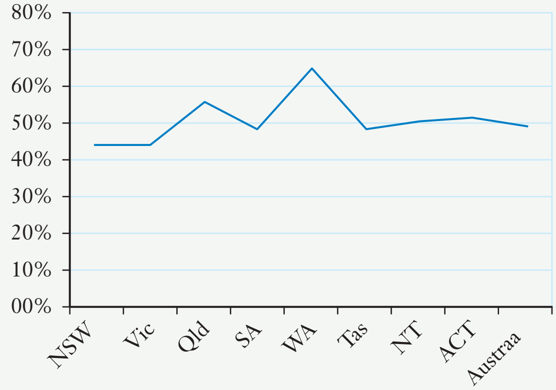
c Science 128 Englis: 1.4 **d** 10

10 a The graph makes it look as if city access is about 3 times as high as country access

b



11



Test yourslf 7

1 B **2** C **3** D

4 D **5** D **6** A

7 Mode 8 median , range 5

8 a 5–9 **b** 1075 **c** 95

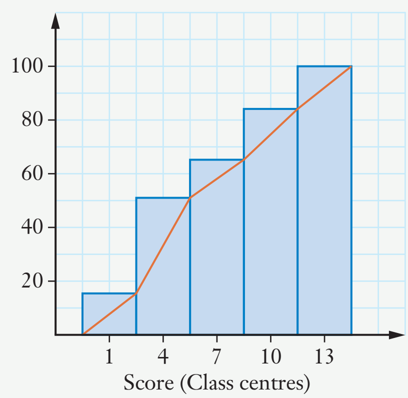
d 66 **ii** 4397

9 Mean 557 standard deviation 2.5

10 a 10 **b** 11

c 14 **d** 6

11 a



b 55

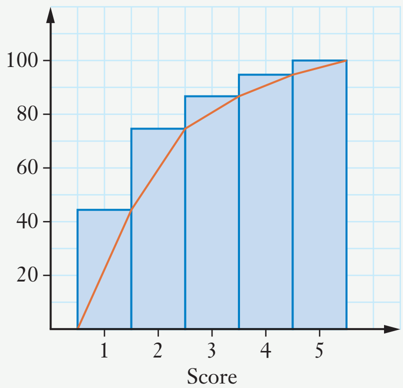
c 3 **ii** 10 **iii** 13

v 4 **v** 12

12 a 15 **b** 1 **c** 5 **d** 2

13 Mode

14 **a**



b 2 **ii** 15

15 **a** 76 **b** 135 **c** 6

16 Mean 448 mode 4, median 45

17 **a** 3 **b** 35

18 **a** 17 **ii** 1442 **iii** 14 18

v 15

b 2

c 8 **ii** 151 **iii** 14 18

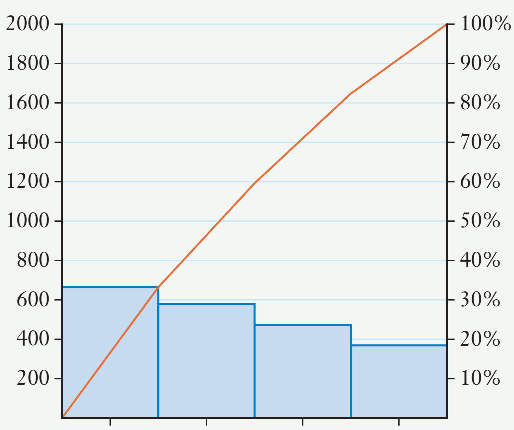
v 15

d Affects range and mean

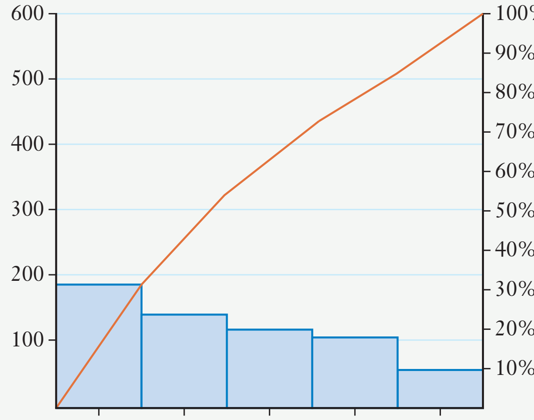
19 **a** 546% **ii** 227% **iii** 277%

v 596%

b



20



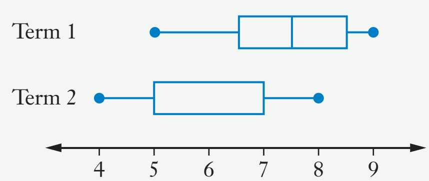
21 **a** North = 4., West = 43

b Mean 465 standard deviation .8

c Mean 434 standard deviation 1.7

d Slightly more mushrooms found in the North region on average spread higher in West region so results more variable

22 **a**



b Term : .; Ter 2: 5

c Term : ; Ter 2: 2

d Term : mean ., standard deviation14;
Term : mean ., standard deviation1.2

e Students did better on average in Term1. The 2 assessments had a similar spread

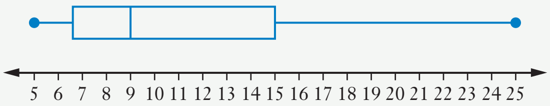
Chaenge exercse 7

1 **a** Bimodal Female heights may have their own mode and male heights have their own (higher) mode

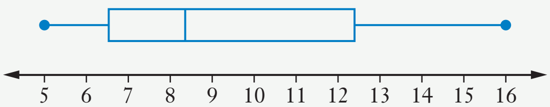
b Mean 1683 variance 9.2

2 4

3 a



b Positively skewed There is an outlier at 5.



Without the outlier it is still slightly positively skewed but it is more symmetrica.

4 a 15 **ii** 566

b

Score x	$x - \overline{x}$	$(x - \overline{x})^2$
7	−8	64
11	−4	16
15	0	0
19	4	16
23	8	64
	$\Sigma(x - \overline{x})$ 0	$\Sigma(x - \overline{x})^2$ 160

c 566

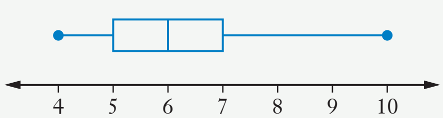
5 25375

6 a Set B has a higher average or centre and more consistent results (less spread out than set A.

b Set B has a higher average and is less consistent (more spread out)

7 a Mean 604 standard deviation .41

b



c Positively skewed

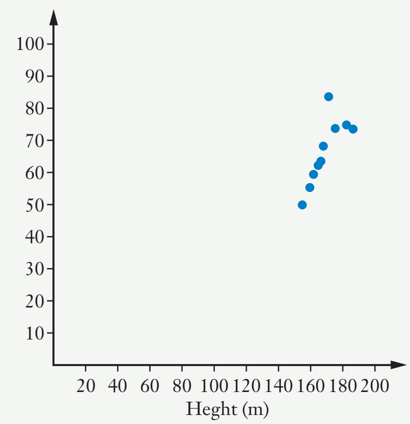
8 a Positively skewed

b Negatively skewed

Chapter 8

Exercse 801

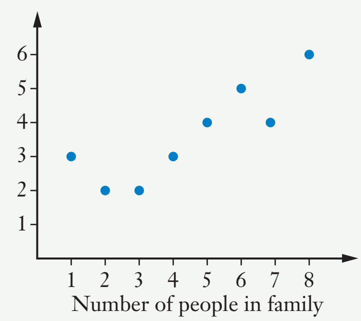
1 a



b The shape seems to be linear in a positive direction

c As height increases weight increase.

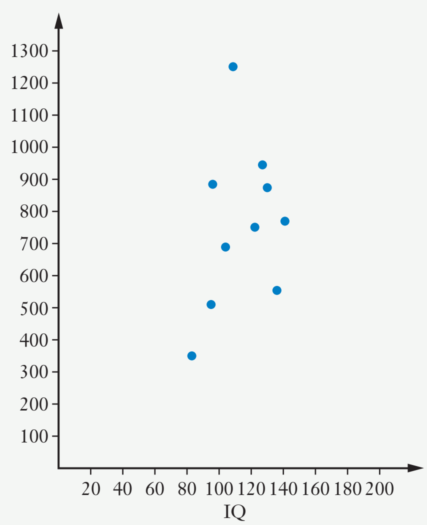
2 a



b The scatterplot is non-linear, or linear (positive) with outliers

c With some exception, the more family members the more bedrooms in the hom.

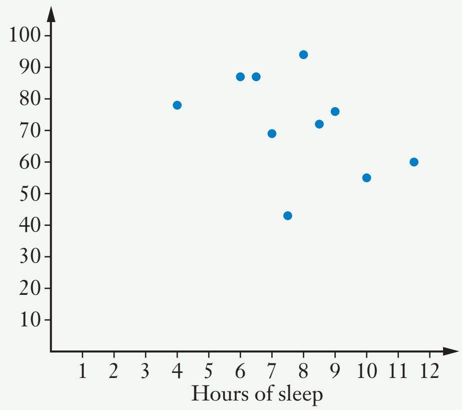
3 a



b The scatterplot has no pattern or is linear (positive) with outliers

c With some exception, the higher the Q, the more the person earns

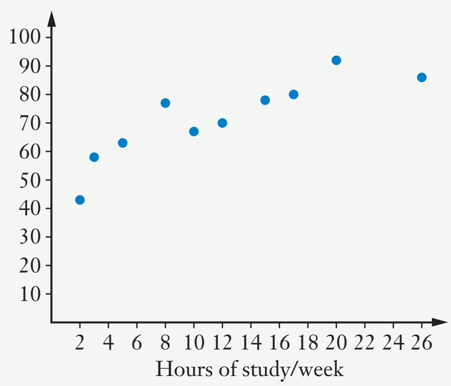
4 a



b The scatterplot has no pattern

c There is little or no correlation found between amount of sleep and exam results

5 a



b The shape is linear in a positive direction

c As the number of hours of study increases exam results increase

6 a Non-linear

b Linear in negative direction

c No pattern

d Linear in positive direction

e Linear in negative direction with outliers

f Non-linear

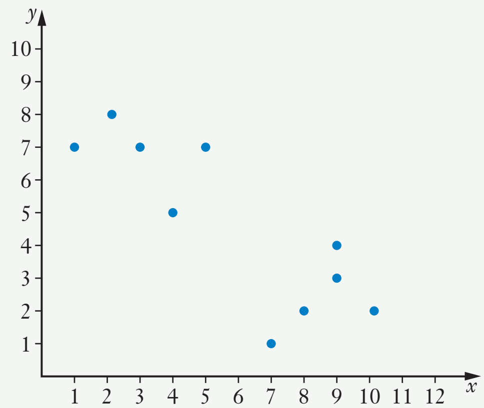
g No pattern

h Linear in positive direction

Non-linear

j Non-linear

7 a



b Linear in a negative directon

8 a B **b** C **c** C **d** D

e A **f** C **g** B **h** A

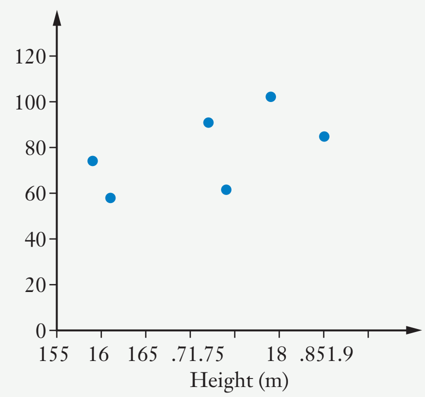
C **j** D

Exercse 802

1 a G **b** B **c** F **d** A

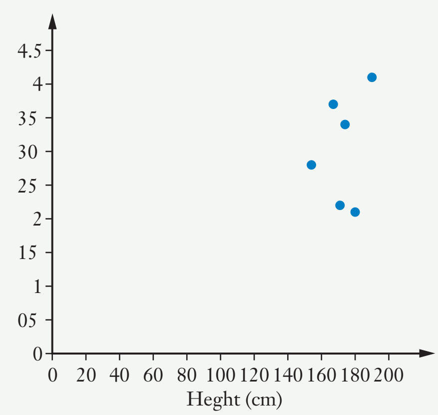
e C **f** D

2 a



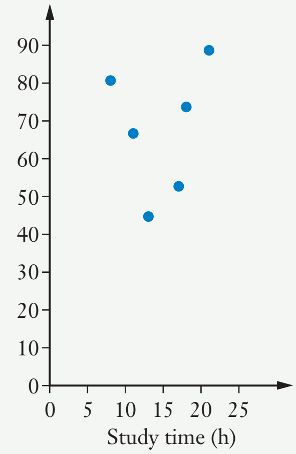
ii 0571

b



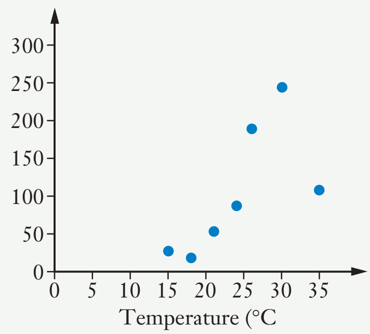
ii 0284

c



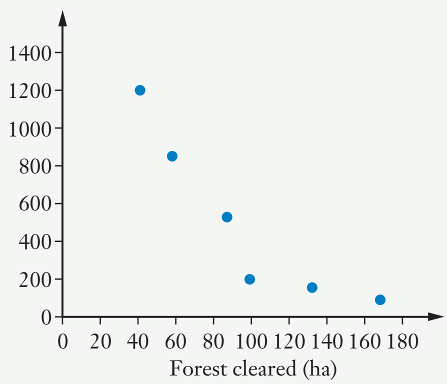
ii 0181

d



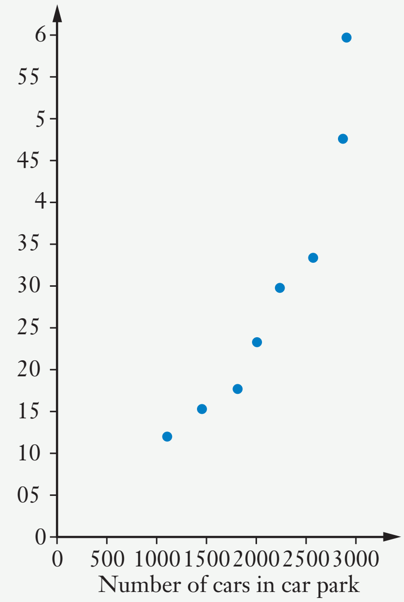
ii 0685

e



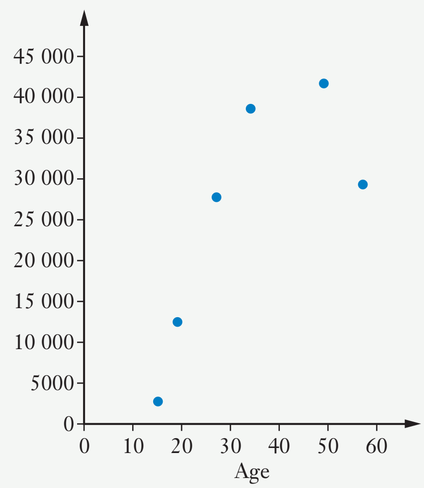
ii –0908

f



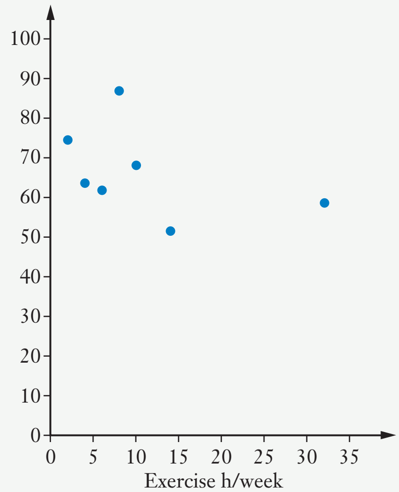
ii 0930

g

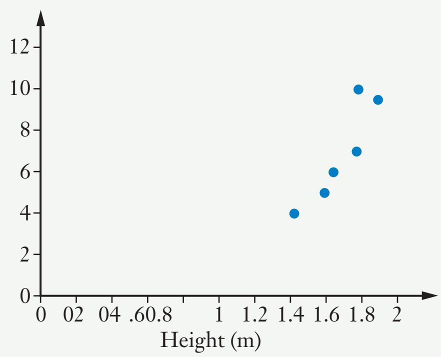


ii 0749

h

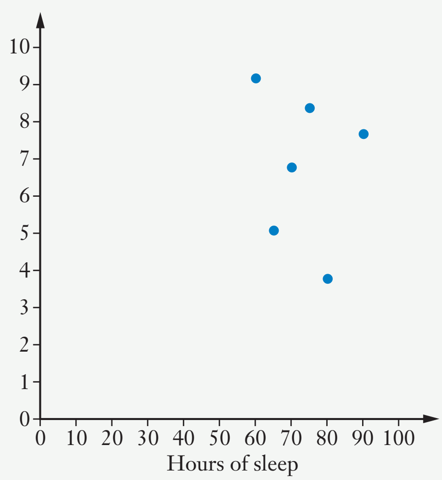


ii –0430



ii 0905

j



ii –0182

3 d ,f, i (Class discussion)

4 **a** 040 **b** –039 **c** 099 **d** –060

5 a c and f (Class discussion)

Exercse 803

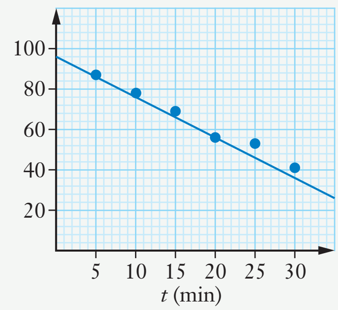
All answers are approximate

1 **a** $y = 06\,x + 07$ **b** $N = -2t + 23$

c $V = 56\,x + 32$ **d** $P = 14\,t + 63$

e $A = -8x + 98$ **f** $x = 52\,t + 480$

2 **a**

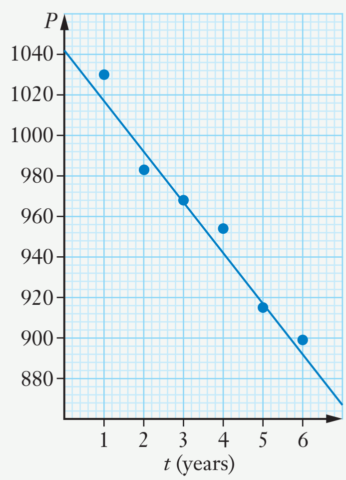


b $T = -2t + 95$

c 61°C **ii** 25°C

d Good model when interpolating but not for extrapolating because it can only cool to room temperature (23°C) but not below this

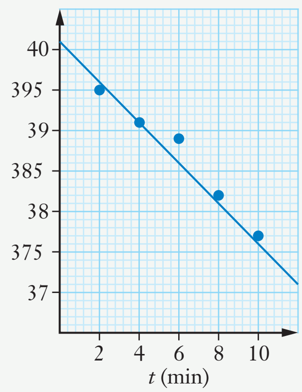
3 **a**



b $P = -25t + 1045$ **c** 870

d 42 years

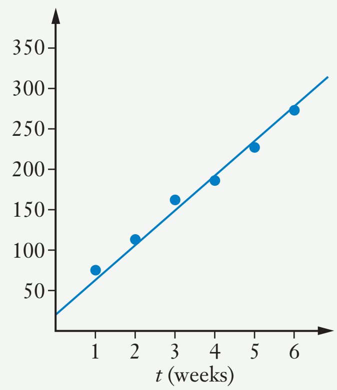
4 a



b $T = -024\ t + 401$ **c** 365 °

d No (class discussion)

5 a



b $N = 375\ t + 425$ **c** 418

d No because the number of people cannot keep increasing forever.

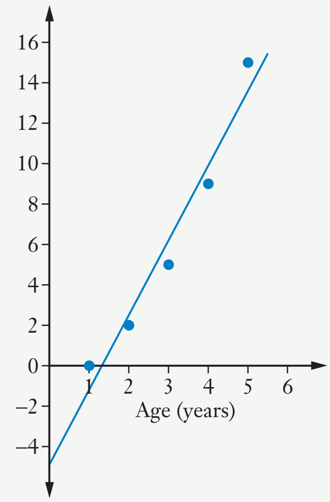
Exercse 804

1 a $r = 09923$ **ii** $m = 243$

b $r = 0976$ **ii** $m = 389$

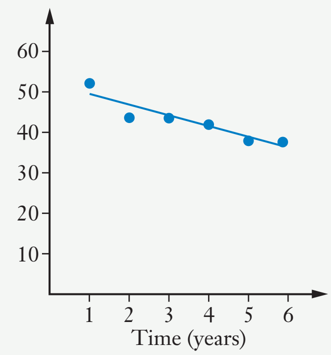
2 a $y = 37\ x - 49$

ii



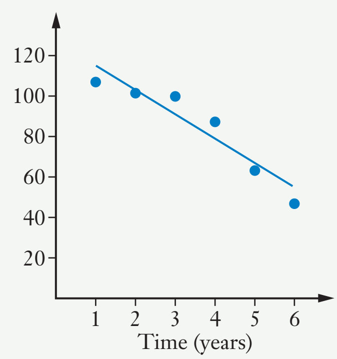
b $y = -265\ x + 5218$

ii



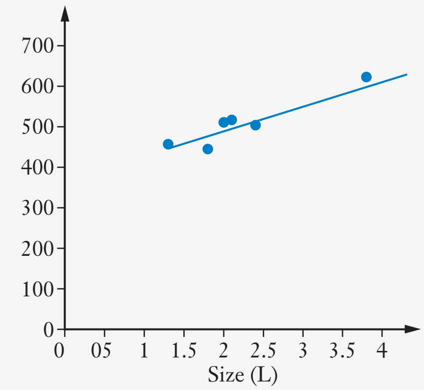
c $y = -1224\ x + 12743$

ii



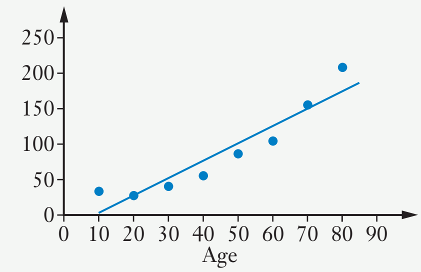
d $y = 6988\ x + 35543$

ii



e $y = 249\,x - 2236$

ii



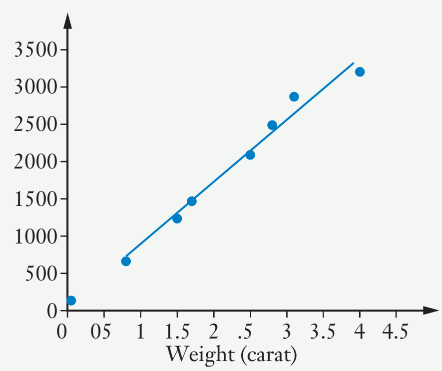
3 **a** $r = -0954$ which shows a high negative correlation This means that the regression line has a negative gradient and that the mass decreases as time increases

b $m = -0614\,t + 2356$

c 125 kg **ii** −133 kg

d The answer to **c ii** shows that extrapolation is not useful in this situation The ice has melted to 0 volume before an hour has passed and the equation is no longer relevant

4 **a**



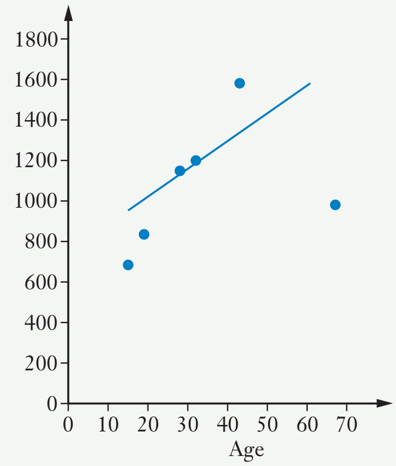
b 0993 This is a high positive correlatin.

c $y = 8317\,x + 692$

d $173259

e 1194 carats

5 **a**



b 0395

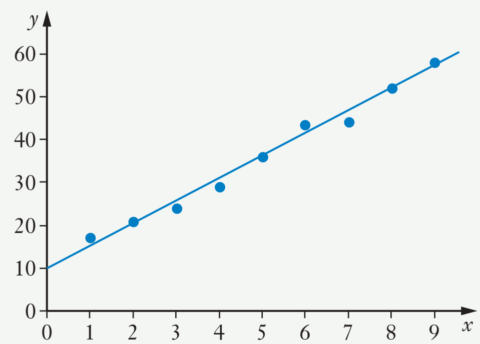
c $y = 66\,x + 8535$ **d** $118192

e No For exampl. using the equation to find the age of a person earning $1800 would give a 145-year-ol!

Test yourslf 8

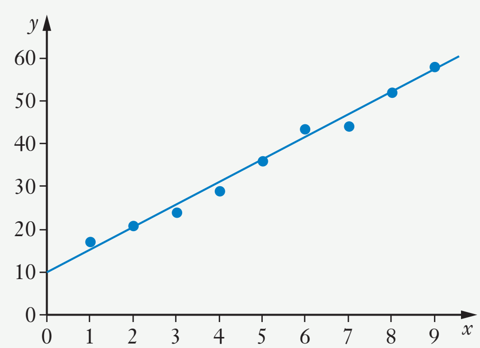
1 C **2** B **3** D **4** A **5** C

6 **a**



$y = 5x + 10$

b



$y = 51833\,x + 10083$

7 **a** C **b** A **c** D **d** C
e B **f** A **g** C **h** D

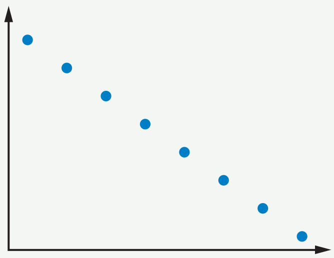
8 **a** 084 **b** $y = 0758\,x + 1088$
c 56 **d** 78

9 **c, d** are causal (Class discussion)

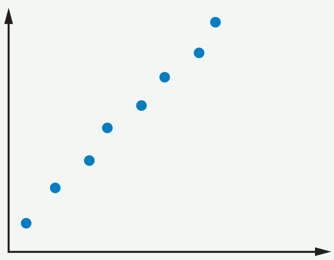
Chaenge exercse 8

1 $r = 08$, $\bar{y} = 64$

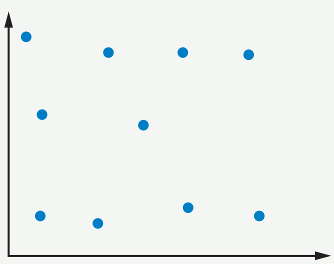
2 **a**



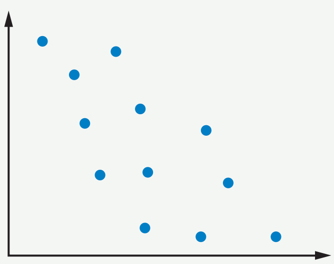
b



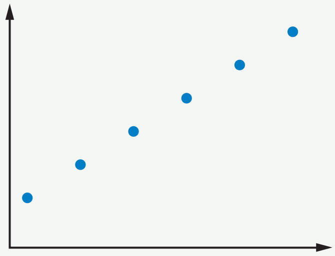
c



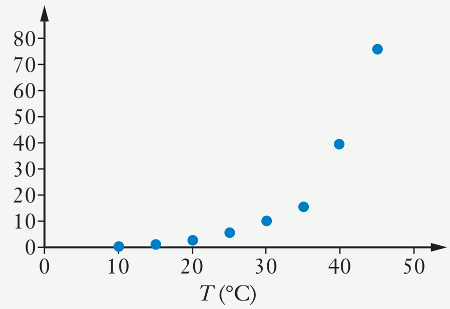
d



e

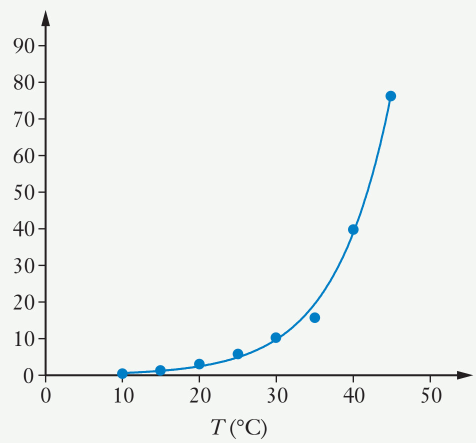


3 **a**



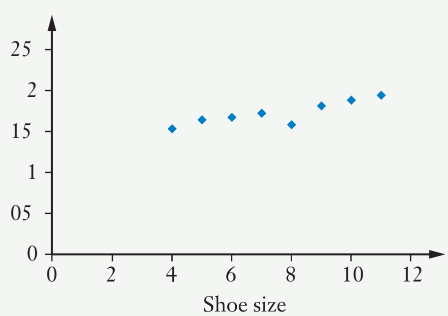
b The correlation is non-linear, so a least squares regression line would not be a good approximation

c



$V = 01563\ e^{01379\ T}$

4 **a**



b 0876

c $y = 0052\,x + 134$

d 0992

ii $y = 0054\,x + 135$

e No Shoe sizes cannot keep increasing

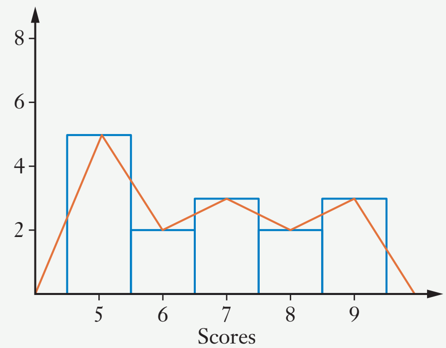
Practice set 3

1 A **2** A **3** D **4** D **5** B

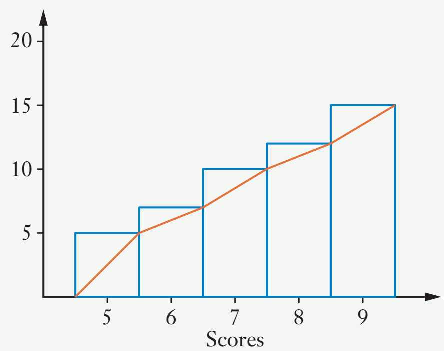
6 a

Score	Frequency
5	5
6	2
7	3
8	2
9	3

b



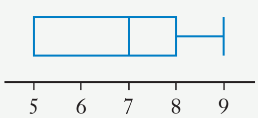
c



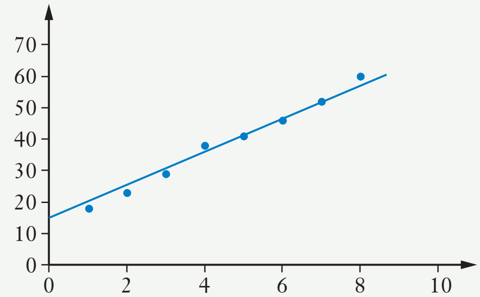
d 7

e 3

f

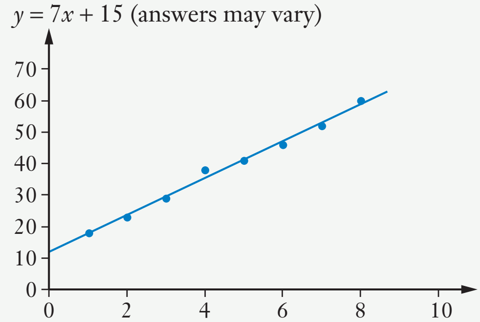


7 a



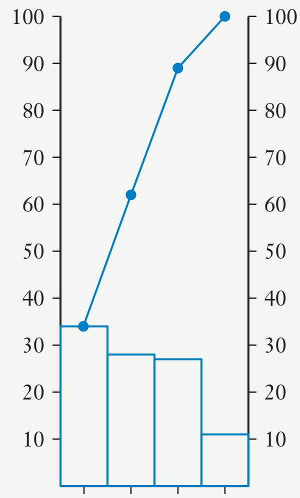
$y = 7x + 15$ (answers may vary)

b



$y = 5869\,x + 11964$

8



9 Mean 6375 standard deviation .8, variance 348

10 a Mean 804 standard deviation .54

b 3 is not between 4 and 12 so 3 is an outlie.

c Mean 826 standard deviation .15

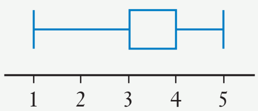
11 Mean 643 mode , median6, range 5

12 **a** 52

b Mean 358 standard deviation .01

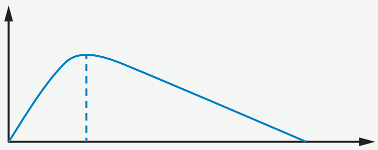
c 4

d

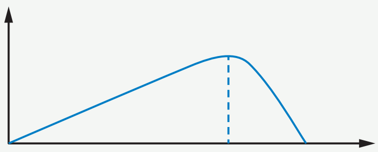


13 –082

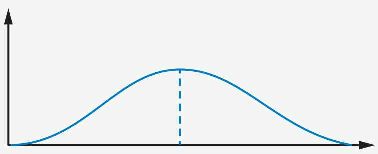
14 **a**



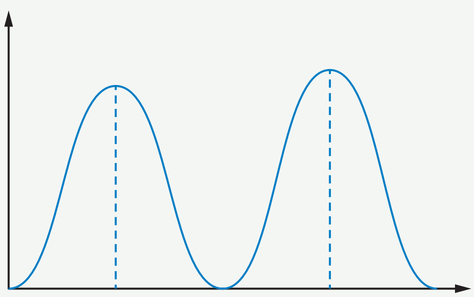
b



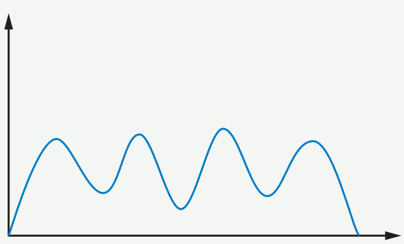
c



d



e



15 **a** categorical nominal

b quantitative continuous

c categorical ordinal

d quantitative discrete

e quantitative discrete

16 **a** C **b** D **c** B **d** A

e D **f** C

17 **a** 35 **b** 21 **c** 45 **d** 25

18 (4 –143) min (–2 73) ma, 1, –35) point of inflection

19 **a** $x = \frac{\pi}{6}\ \frac{5\pi}{6}$

b $x = \frac{\pi}{4}\ \frac{3\pi}{4}\ \frac{5\pi}{4}\ \frac{7\pi}{4}$

c $x = \frac{\pi}{3}\ \frac{2\pi}{3}\ \frac{4\pi}{3}\ \frac{5\pi}{3}$

20 **a** 399 **b** 20 100

Chapter 9

Exercse 901

1 **a** 210 **b** 13th **c** 57

2 **a** 39 **b** 29th **c** 32

3 **a** $3n + 3$

b $S_n = \frac{1}{2}n[2 \times 6 + (n - 1) \times 3]$

4 **a** 001 m **b** 915 m

5 **a** 7 **b** 105 m

6 **a** \$2050 **b** \$2100 **c** \$2150

d \$2500 **e** \$3500

7 **a** 125 m **b** 8

8 **a** 45 –5 kg **b** 18th

9 **a** 49 **b** 4 mm

10 **a** $3k$ m **b** $3k(k + 1)$ m **c** 9

Exercise 902

1 **a** 93% **ii** 8649% **iii** 8044%

b 3367% **c** 19 weeks

2 **a** 9604% **b** 34 days **c** 114 days

3 **a** \$23 200 **ii** \$26 912

iii \$31 21792

b \$102 34529 **c** 62 years

4 **a** 774% **b** 135 years **c** 314 years

5 **a** $\frac{4}{9}$ **b** $\frac{7}{9}$ **c** $1\frac{2}{9}$ **d** $\frac{25}{99}$

e $2\frac{9}{11}$ **f** $\frac{7}{30}$ **g** $1\frac{43}{90}$ **h** $1\frac{7}{450}$

$\frac{131}{990}$ **j** $2\frac{361}{999}$

6 0625 m **7** 15 m

8 20 cm **9** 3 m
10 **a** 484 m **b** After 3 years
11 **a** 747 cm **b** 75 cm
12 300 cm **13** 35 m
14 32 m
15 **a** 1 , 4, 12, …
b 16 777 216 people
c 19 173 961 people

Exercse 903

1 **a** $689585 **b** $699979 **c** $763337
d $685736 **e** $720726
2 **a** $285292 **b** $12 72032 **c** $403813
d $595508 **e** $87 362
3 $893 262 **4** $355286
5 $21 17372 **6** $106923
7 $490296
8 **a** $112568 **b** $220920 **c** $14 930
d $96 630 **e** $305 900 **f** $90222
g $130026 **h** $309090 $106150
j $386610
9 **a** $870549 **b** $497166 **c** $463199
d $970591 **e** $822707

Exercse 904

1 **a** $74012 **b** $14 75364
c $17 27140 **d** $938569
e $529819
2 **a** $200734 **b** $201587 **c** $202028
3 **a** $493086 **b** $494103
4 **a** $40824 **b** $41051
5 **a** $97140 **b** $97212
6 **a** $173399 **b** $174221 **c** $174783
7 **a** $309706 **b** $1347
8 **a** $22 80081 **b** $94592
9 $69141 **10** $177658
11 $14 54976 **12** $1 301 69462
13 **a** $411351 **b** $55532 **c** $987243
d $23817 **e** $10 53059
14 $454328 **15** 4 years
16 8 years
17 **a** $x = 7$ **b** $x = 5$ **c** $x = 8$
d $x = 65$ **e** $x = 85$

18 $768 **19** Kate $22437
20 Account A $84.94
21 **a** $39 40053 **ii** $41 812 16
b 27th year
22 **a** $128 547 **ii** $175 19637
iii $230 70016
b 28th year
23 **a** 4 years **ii** 17 years
b 31% **ii** 18%
24 **a** 19990
b $A = 108^{9}$
$= 19990$ to 4 decimal places
25 FV interest factor is 15209
$A = 115^{3}$
$= 15209$ to 4 decimal places

Exercse 905

1 **a** $10 125 **b** $374592 **c** $272063
d $20 410 **e** $619405
2 **a** $49 75929 **b** $552436
c $94 33425 **d** $207402
e $274 94805 **f** $398108
g $20 10320 **h** $15 55948
$327 21450 **j** $1 474 272
3 **a** $10 78940 **b** $22 83956
c $69 72025 **d** $951137
e $11 51383
4 $477 07625
5 $207470
6 **a** $146939 **b** $281142 **c** $126610
d $61491 **e** $30 24255
7 $29659
8 **a** $47 000 **ii** $43 880 **iii** $40 63520
b $43 75680 **c** $41 83342
9 **a** $125 750 **b** $126 50750
c $127 27258 **e** $125 37688
d $125 75753
10 **Graph A: a** 18 months **b** $10 400
c 36 months (3 years)
Graph B: a 27 months (2 years 3 months)
b $12 200
c 46 months (3 years 10 months)

11 **a** 18 **b** 21
12 **a** 9% **b** 3%

Exercse 906

1 $40 72817 **2** $338322
3 $65 90397 **4** $284682
5 $618113 **6** $464671
7 $21 42603
8 **a** $26 36159 **b** $46 55194
9 $45 59917
10 **a** $792280 **b** $158456
11 $500 for 30 years ($11 64086 better off)
12 Ye, $29.80 over **13** $81872
14 **a** $3738
b Proof (see worked solutions)

Exercse 907

1 **a** $19 23454 **b** $317764
c $99 99013 **d** $179881
e $45 65971
2 **a** $597852 **ii** $97852
iii 65%
b $24 22560 **ii** $832560
iii 105%
c $159 78528 **ii** $79 78528
iii 83%
d $272 127 **ii** $37 127
iii 063%
e $171048 **ii** $36248
iii 134%
3 **a** $15464 **b** $30420 **c** $85464
d $260580 **e** $254904 **f** $68637
g $56889 **h** $302175 $3922
j $5680
4 **a** $394690 **b** $947 256
c $266 756 **d** 196%
5 **a** 5 years **b** 10 years **c** 20 years
d 30 years **e** 10 years **f** 25 years
g 10 years **h** 15 years 10 years
j 20 years
6 **a** 6% **b** 45% **c** 2%
d 8% **e** 55% **f** 7%
g 25% **h** 3% 75%
j 4%

Exercse 908

1 $104762 **2** $39446 **3** $13915
4 **a** $96645 **b** $126579
5 $251959
6 **a** $86846 **b** $55 90760
7 **a** $7781 **b** $264542
8 $78 700
9 **a** Get Rich $94961 Capital Bank $49.27
b $33 42780 more through Capital Bank
10 **a** $87560 **b** $43 77880
11 $61 29220
12 NSW Bank totals $579125 Sydney Bank totals $555788 so Sydney Bank is better.
13 **a** $24969 **b** $13 48512
14 **a** $13 25113 **b** $37407
c $20 19978
15 **a** $183568 **b** $917841
16 **a** $1036
b Proof (see worked solutions)

Test yourslf 9

1 C **2** B **3** A
4 $21 98094 **5** $167274
6 **a** Each slat rises 3 mm so the bottom one rises up 30×3 mm or 90 mm
b 87 mm
c 90 8, 4, … is an arithmetic sequence with $a = 90$ $d = -3$
d 42 mm **e** 1395 mm
7 **a** $121 320 **b** $58 820 **c** 188%
8 **a** $72322 **b** $9995 **c** $292527
d $16 78590 **e** $11 34501
9 **a** $24 050 **b** $220 250
10 **a** $184 86725 **b** $182 82942
c $180 78649
11 $128519
12 **a** 10 stacks **b** 110 boxes
13 **a** $\frac{4}{9}$ **b** $\frac{13}{18}$ **c** $1\frac{19}{33}$
14 $340001

15 **a** \$59 000 **b** \$15 98889

16 **a** \$238504 **b** \$239203

17 **a** \$143286 **b** \$343 88691

18 **a** \$265156 **b** \$987243

19 **a** 39 words/min **b** 15 weeks

20 48 m

21 **a** \$33 000 **ii** \$23 000

b 14 years **ii** 28 years

c 32 years

Chaenge exercse 9

1 \$179979

2 **a** \$40 **b** \$2880

3 \$852253

4 **a** 1158 ° **ii** 514 °

b 66 min **ii** 109 min

5 **a** $\$1000(106)^{25}$ **b** $\$1000(106)^{24}$

c $\$1000(106)^{23}$ **d** \$1000(106)

e $S_{25} = \dfrac{1000 \times 106 \times (106^{25} - 1)}{106 - 1}$

f \$58 15638

6 **a** \$10 100 **b** \$11 26825

c \$421241 **d** \$263722

7 \$466 56374

Chapter 10

Exercse 1001

1 **a** Discrete **b** Continuous

c Continuous **d** Discrete

e Continuous

2 a, d **3** a, d

4 $b = 7$ **5** $k = \dfrac{4}{625}$

6 **a** $a = \dfrac{1}{e(e^2 - 1)}$

b $a = \dfrac{1}{e(e^6 - 1)}$ **c** $a = \dfrac{1}{e^4 - 1}$

7 [0 6] **8** $b = 8$

Exercse 1002

1 **a** $\dfrac{1}{2}$ **b** $\dfrac{1}{6}$ **c** $\dfrac{1}{2}$

d $\dfrac{2}{3}$ **e** $\dfrac{1}{3}$

2 **a** $y = \dfrac{x}{50}$

b $\dfrac{81}{100}$ **ii** $\dfrac{9}{100}$ **iii** $\dfrac{33}{100}$

v $\dfrac{8}{25}$ **v** $\dfrac{3}{4}$

3 **a** $a = \dfrac{3}{125}$

b $\dfrac{27}{125}$ **ii** $\dfrac{63}{125}$ **iii** $\dfrac{117}{125}$

v $\dfrac{1}{125}$ **v** $\dfrac{37}{125}$

4 **a** $a = \dfrac{4}{81}$

b $\dfrac{80}{81}$ **ii** $\dfrac{16}{81}$ **iii** $\dfrac{5}{27}$

v $\dfrac{1}{81}$

5 **a** $k = \dfrac{1}{e(e^5 - 1)}$

b $\dfrac{e(e^3 - 1)}{e^5 - 1}$ **ii** $\dfrac{e^3 - 1}{e^5 - 1}$ **iii** $\dfrac{e^2(e^3 - 1)}{e^5 - 1}$

6 **a** $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_0^{\frac{\pi}{2}}$

$= -\cos\dfrac{\pi}{2} - (-\cos 0)$

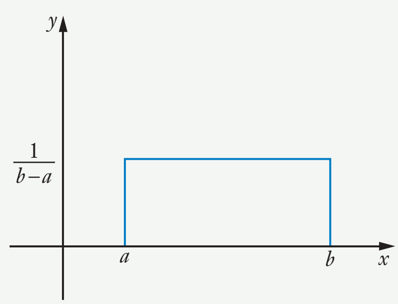
$= -0 + 1$

$= 1$

So $y = \sin x$ is a PDF in the domain $\left[0 \ \dfrac{\pi}{2}\right]$

b $\dfrac{1}{2}$ **ii** $1 - \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2} - 1}{\sqrt{2}}$ **iii** $\dfrac{\sqrt{3}}{2}$

7 **a**



$A = bh$

$= (b - a) \times \dfrac{1}{b - a}$

$= 1$ so PDF

b $\frac{3}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{4}$

Exercse 1003

1 **a** $F(x)=\frac{x^3}{27}$ **b** $F(x)=\frac{x^4}{1296}$

c $F(x)=\frac{e^x-1}{e^4-1}$ **d** $F(x)=\frac{(x-2)^4}{625}$

e $F(x)=\frac{12x^2-x^3-40}{135}$

2 **a** $F(x)=\frac{x^5-1}{7776}$

b $\frac{121}{3338}$ **ii** $\frac{31}{7776}$ **iii** $\frac{781}{1944}$

v $\frac{2251}{2592}$ **v** $\frac{31}{243}$

3 **a** $F(x)=\frac{x^4-81}{2320}$

b $\frac{35}{464}$ **ii** $\frac{243}{464}$ **iii** $\frac{111}{145}$

v $\frac{429}{464}$ **v** $\frac{13}{29}$

4 **a** $F(x)=\frac{e^{2x}-1}{e^{10}-1}$

b 00024 **ii** 014 **iii** 098

v 099 **v** 013

5 **a** $a=\frac{4}{6561}$ **b** $F(x)=\frac{x^4}{6561}$

c $\frac{625}{6561}$ **ii** $\frac{256}{6561}$ **iii** $\frac{2465}{6561}$

v $\frac{80}{81}$ **v** $\frac{1280}{6561}$

6 **a** $a=\frac{1}{\ln 6}$ **b** $F(x)=\frac{\ln x}{\ln 6}$

c 061 **ii** 039 **iii** 010

v 023 **v** 051

7 **a** Show that $\int_{\frac{3\pi}{2}}^{2\pi}\cos x\,dx=1$

b $F(x)=\sin x+1$

c $\frac{2-\sqrt{3}}{2}$ **ii** $\frac{1}{\sqrt{2}}$

iii $\frac{\sqrt{3}-1}{2}$

8 **a** 2 **b** 3 **c** 3 **d** 7

e 4 **f** 4 **g** 6 **h** 8

4 **j** 7

9 **a** 3

b $F(x)=-\frac{1}{22}(x^3-9x^2+15x-2)$

c $\frac{1}{2}$

10 **a** $F(x)=\frac{1}{464}(x^4-12x^3+48x^2+4x-201)$

b $\frac{9}{29}$ **ii** $\frac{393}{464}$ **iii** $\frac{73}{464}$

c 7 minutes

Exercse 1004

1 **a** 635 **b** 589 **c** 609 **d** 3

e 311 **f** 757 **g** 879 **h** 445

529 **j** 373

2 **a** 647 **ii** 605 **iii** 919

b 424 **ii** 401 **iii** 562

c 379 **ii** 362 **iii** 475

3 **a** 635 **b** 564

4 **a** 638 **b** 512 **c** 728 **d** 702

e 428 **f** 744

5 **a** 702 **b** 952 **c** 1090

6 **a** $F(x)=\frac{x^4-16}{4080}$ **b** $\frac{203}{1360}$

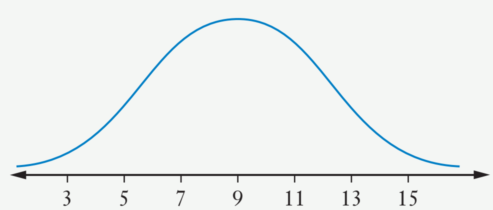
c $\frac{16}{17}$ **d** $\frac{29}{51}$

e 673 **f** 745

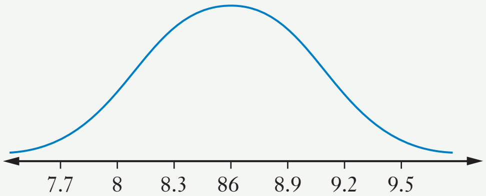
g 779 **h** 556

Exercse 1005

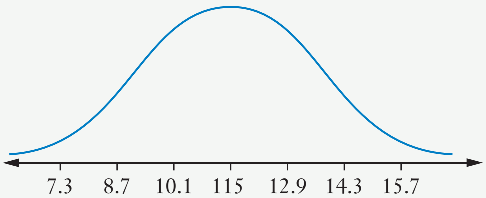
1 **a**



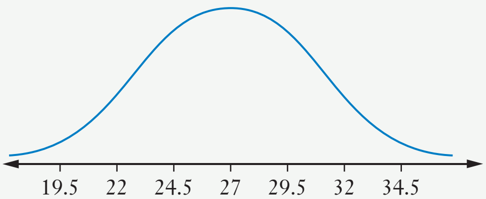
b



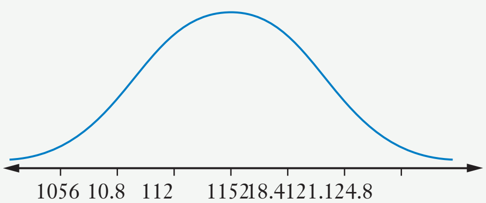
c



d

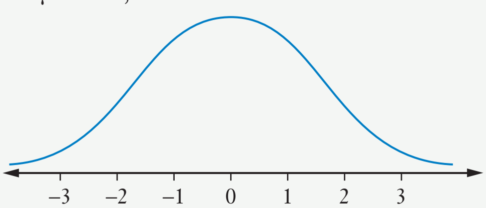


e



2 **a** $\mu = 237$, $\sigma = 42$ **b** $\mu = 54$, $\sigma = 09$
c $\mu = 597$ $\sigma = 54$ **d** $\mu = 209$ $\sigma = 106$
e $\mu = 11.3$, $\sigma = 22$

3

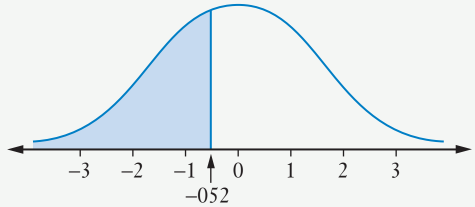


4 **a** 05 **b** 08413 **c** 09772
d 09987 **e** 01587 **f** 00228
g 00013 **h** 09332 00038
j 0999

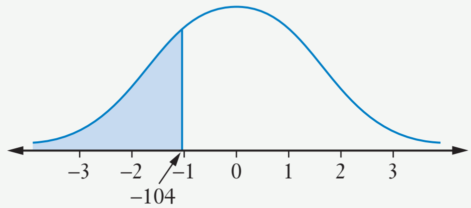
5 **a** 06772 **b** 00174 **c** 09778
d 0031 **e** 09722 **f** 00837
g 09255 **h** 06826 09544
j 09974

6 **a** 084 **b** 067 **c** –053
d –084 **e** 123
f –117 or –118
g –052 **h** –067 033

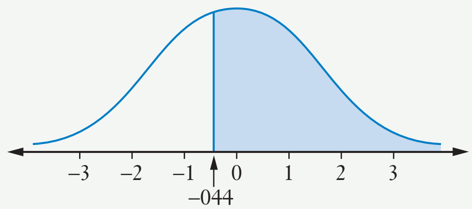
7 **a**



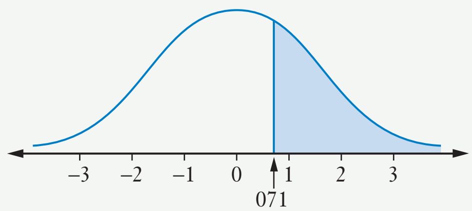
b



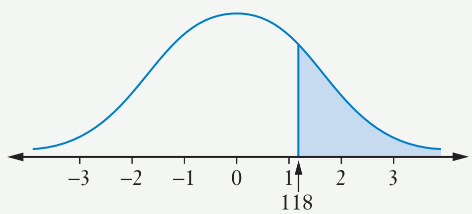
c



d



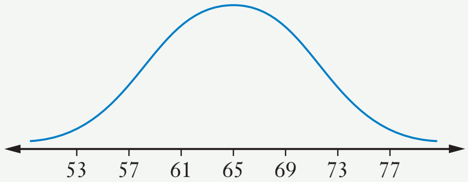
e



Exercse 1006

1 **a** 68% **b** 95% **c** 997%
2 **a** 68% **b** 95% **c** 997%
3 **a** 68% **b** 95% **c** 997%
4 **a** 68% **b** 95% **c** 997%
d 34% **e** 4985% **f** 9735%

5 **a**



b 95% **ii** 34% **iii** 4985%
v 815% **v** 815%

6 **a** 68% **b** 34% **c** 475%
d 475% **e** 8385%

Exercse 1007

1 **a** 0 **ii** 1 **iii** 2
v 3 **v** –1 **v** –2
v –3
b 1995 **ii** 1527

2 **a** 27 cm to 792 cm
b 0 **ii** 1 **iii** –1
v 2 **v** –2 **v** 3
v –3 **viii** 14

3 **a** 35 ° to 101 °
b 0 **ii** 1 **iii** 2
v 3 **v** –1 **v** –2
v –3 **viii** –07

4 **a** 548 mL to 78 mL
b 0 **ii** 1 **iii** 2
v 3 **v** –1 **v** –2
v –3 **viii** 41
c 7336 mL **ii** 8322 mL
iii 6292 mL **v** 5306 mL

5 **a** 27 **ii** –33 **iii** 24
v –13 **v** 49 **v** –18
v 33 **viii** –26
b 532 90 and 8.7 **c** 621 and 597
d 80 7., 21,59.7 and56.4

6 **a** 129 **ii** –157
b 1126 mm **ii** 1602 mm
iii 1868 mm **v** 13136 mm
v 1693 mm

7 28 **8** 2685 **9** 50

10 **a** 534 **b** 422 **c** 59
d 6908 **e** 4388

11 483

12 **a** 122 to 198 **ii** 141 to 179
iii 103 to 217
b 21 **ii** –13
c 103 **ii** 1809

13 **a** 996 to 1098
b –48 **ii** –03
c 1149 **ii** 9807

Exercse 1008

1 **a** 997% **ii** 475% **iii** 68%
v 4985% **v** 8385%
b 4 **ii** Ye: outside normal range
c –15 **ii** 00668 **iii** 02417

2 **a** 68% **ii** 95% **iii** 997%
v 34% **v** 815%
b –33 **ii** Ye: outside normal range
c 37 **ii** Ye: outside normal range

3 **a** 41 – no outside normal range
b 46 min to 114 min
c 9735%
d –029 **ii** 1141%
e 0381 **ii** 0721 **iii** 08418
f 989% **ii** 6832% **iii** 985%

4 **a** 95% **ii** 4985% **iii** 3983%
b 187 mL to 211 mL
c Unusual – outside range
d 00606

5 **a** 95% **b** 81 cm **c** 69 cm

6 **a** 22 years
b 68% **ii** 8854%
c Yes – outside normal range

7 **a** 255 cm to 305 cm
b –096 **c** 8233%
d No – outside normal range

8 **a** 5% **b** 25% **c** 03%
ii greater than 54 kg less than .5 kg

9 **a** 997% **ii** 121 cm **iii** 91 cm
b 83 cm 1.6 cm

10 Ye: Minimum weight within normal limits is 25.3 .

11 25%

12 **a** 2 mm **b** 820 mm
c 816 mm to 824 mm

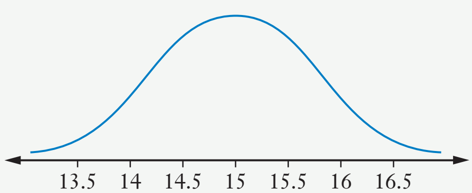
13 **a** 02 h **b** 25 h
14 **a** Epping 104 city .35 **b** City
15 **a** 1 **b** 204 **c** Cameron

Test yourslf 10

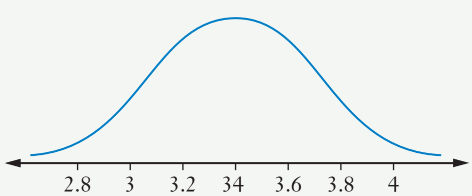
1 C **2** A **3** D
4 **a** 075 **b** 04 **c** 055
d 045 **e** 07 **f** 10
g 36 **h** 178 12
j 15
5 **a** Yes **b** Yes **c** No
6 398
7 **a** $F(x) = \dfrac{x^3 - 1}{511}$

b $\dfrac{26}{511}$ **ii** $\dfrac{124}{511}$ **iii** $\dfrac{296}{511}$
v $\dfrac{64}{73}$ **v** $\dfrac{335}{511}$

8 **a** 4 **ii** 29
b 9 **ii** 71
9 **a** 227 kg to 413 kg
b 226 **ii** 097
c 01541
10 [0 4]
11 **a**

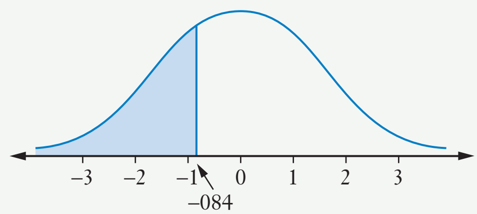


b

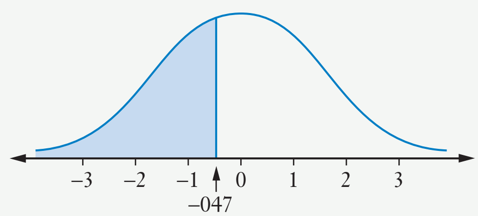


12 **a** 03% **b** 185%
13 **a** 57 **b** 64 **c** 535
d 609 **e** 487
14 Mean 29 standard deviation .7
15 **a** $F(x) = \dfrac{x^5}{3125}$ **b** $F(x) = \dfrac{x^7}{823543}$

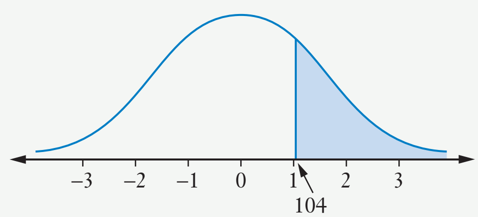
c $F(x) = \dfrac{e^x - 1}{e^6 - 1}$ **d** $F(x) = \dfrac{x^2 - 1}{80}$
16 **a** 07054 **b** 09066 **c** 00013
d 02389 **e** 08413 **f** 00062
g 08599 **h** 01272 01348
17 **a** 95% **b** 34% **c** 8385%
d 905% **e** 2586%
18 **a** $a = \dfrac{1}{e^5 - 1}$ **b** $a = \dfrac{1}{e(e^3 - 1)}$
19 **a**



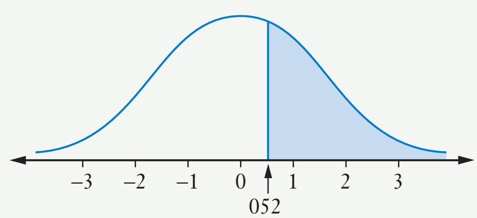
b



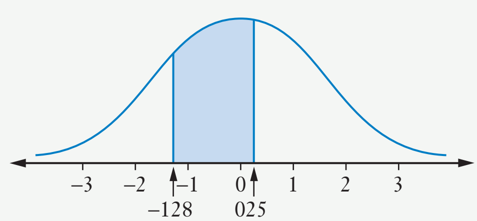
c



d



e



20 **a** 8 **b** 15 **c** 2
21 **a** 083 **b** 1
c Klare

22 $F(x) = \frac{1}{600}\left[\frac{x^4}{4}\right]^7 = 1$ so PDF

23 **a** 25 **ii** –15
b 112 m **ii** 106 m **iii** 116 m
v 109 m **v** 113 m

24 8998

25 4917

Chaenge exercse 10

1 $a = 1, b = 5$

2 $\mu = 132,\ \sigma = 04$

3 0

4 $F(x) = \frac{(x^2+1)^3 - 1000}{124000}$

5 Mean 189 standard deviation .5

6 Show area is 1

Practice set 4

1 A **2** B **3** C **4** B **5** C

6 **a** $\frac{1}{3}e^{3x} + C$ **b** $2x^2 - 3x + C$
c $\frac{1}{4}\tan 4x + C$ **d** $\ln|x-3| + C$

7 **a** $40 99550 **b** $74560

8 $945

9 24 m

10 **a** $F(x) = \frac{x^3 - 1}{342}$
b $\frac{62}{171}$ **ii** $\frac{7}{342}$ **iii** $\frac{158}{171}$
v $\frac{31}{38}$ **v** $\frac{21}{38}$
c 556 **ii** 683

11 **a** 2 **b** 7 **c** 77

12 **a** $119 000 **ii** $117 995
iii $116 98498
b See worked solutions
ii $99 02088

13 $285152

14 423

15 $\frac{8}{45}$

16 (–1 9) maximu, 5, –99) minimum (, –45) point of inflection

17 $18076

18 **a** 39 L to 55 L **b** 68%
c 325 **ii** 125 **iii** –2 **v** –15
d 09994 **ii** 00228 **iii** 09332
v 01056 **v** 08716
e 7333% **ii** 7888% **iii** 994%

19 **a** $46050 **b** $449
c $108454 **d** $2849

20 **a** $F(x) = \frac{(x+2)^3 - 8}{335}$ **b** $F(x) = \frac{x^4 - 1}{624}$
c $F(x) = 2 \sin x$

21 2

22 **a** 09115 **b** 08106
c 01587 **d** 00207
e 09966

23 2nd exam

24 $27733

25 [1 6]

26 **a** 1302 **b** 1055 **c** 16348
d 86

27 8

28 185

29 **a** 5% reject cans below 34.4 and above 35.2 mL
b 03% reject cans below 34.2 and above 3504 mL

30 **a** $274878 **b** $275111 **c** $275163