MATHS IN FOCUS 11 MATHEMATICS EXTENSION 1

FULLY WORKED SOLUTIONS

Chapter 1: Algebraic techniques

Exercise 1.01 Index laws

Question 1

a $5^3 \times 2^2 = 5 \times 5 \times 5 \times 2 \times 2 = 500$

b $3^4 + 8^2 = 3 \times 3 \times 3 \times 3 + 8 \times 8 = 145$

c $\left(\frac{1}{4}\right)^3 = \frac{1^3}{4^3} = \frac{1}{64}$

d $\sqrt[3]{27} = 3$

e $\sqrt[4]{16} = 2$

Question 2

a $3.7^2 = 13.69... \approx 13.7$

b $1.06^{1.5} = 1.091... \approx 1.1$

c $2.3^{-0.2} = 0.8465... \approx 0.8$

d $\sqrt[3]{19} = 2.668... \approx 2.7$

e $\sqrt[3]{34.8 - 1.2 \times 43.1} = -2.567... \approx -2.6$

f $\frac{1}{\sqrt[3]{0.99 + 5.61}} = 0.533... \approx 0.5$

Question 3

a $a^6 \times a^9 \times a^2 = a^{6+9+2} = a^{17}$

b $y^3 \times y^{-8} \times y^5 = y^{3-8+5} = y^0 = 1$

c $a^{-1} \times a^{-3} = a^{-1-3} = a^{-4} = \dfrac{1}{a^4}$

d $w^{\frac{1}{2}} \times w^{\frac{1}{2}} = w^{\frac{1}{2}+\frac{1}{2}} = w \ = w$

e $x^6 \div x = x^{6-1} = x^5$

f $p^3 \div p^{-7} = p^{3-(-7)} = p^{3+7} = p^{10}$

g $\dfrac{y^{11}}{y^5} = y^{11-5} = y^6$

h $\left(x^7\right)^3 = x^{7\times 3} = x^{21}$

i $\left(2x^5\right)^2 = 2^2 x^{5\times 2} = 4x^{10}$

j $\left(3y^{-2}\right)^4 = 3^4 y^{-2\times 4} = 81y^{-8} = \dfrac{81}{y^8}$

k $a^3 \times a^5 \div a^7 = a^{3+5-7} = a^1 = a$

l $\left(\dfrac{x^2}{y^9}\right)^5 = \dfrac{x^{2\times 5}}{y^{9\times 5}} = \dfrac{x^{10}}{y^{45}}$

m $\dfrac{w^6 \times w^7}{w^3} = \dfrac{w^{6+7}}{w^3} = \dfrac{w^{13}}{w^3} = w^{13-3} = w^{10}$

n $\dfrac{p^2 \times \left(p^3\right)^4}{p^9} = \dfrac{p^2 \times p^{3\times 4}}{p^9} = \dfrac{p^2 \times p^{12}}{p^9} = \dfrac{p^{2+12}}{p^9} = \dfrac{p^{14}}{p^9} = p^{14-9} = p^5$

o $\dfrac{x^6 \div x^7}{x^2} = \dfrac{x^{6-7}}{x^2} = \dfrac{x^{-1}}{x^2} = x^{-1-2} = x^{-3} = \dfrac{1}{x^3}$

p $\dfrac{a^2 \times \left(b^2\right)^6}{a^4 \times b^9} = \dfrac{a^2 \times b^{2\times 6}}{a^4 b^9} = \dfrac{a^2 \times b^{12}}{a^4 b^9} = a^{2-4} \times b^{12-9} = a^{-2} \times b^3 = \dfrac{b^3}{a^2}$

q $\dfrac{\left(x^2\right)^{-3}\times\left(y^3\right)^2}{x^{-1}\times y^4}=\dfrac{x^{2\times-3}\times y^{3\times 2}}{x^{-1}y^4}=\dfrac{x^{-6}\times y^6}{x^{-1}y^4}=x^{-6-(-1)}\times y^{6-4}=x^{-5}\times y^2=\dfrac{y^2}{x^5}$

Question 4

a $x^5\times x^9=x^{5+9}=x^{14}$

b $a^{-1}\times a^{-6}=a^{-1-6}=a^{-7}=\dfrac{1}{a^7}$

c $\dfrac{m^7}{m^3}=m^{7-3}=m^4$

d $k^{13}\times k^6\div k^9=k^{13+6-9}=k^{10}$

e $a^{-5}\times a^4\times a^{-7}=a^{-5+4-7}=a^{-8}=\dfrac{1}{a^8}$

f $x^{\frac{2}{5}}\times x^{\frac{3}{5}}=x^{\frac{2}{5}+\frac{3}{5}}=x\;=x$

g $\dfrac{m^5\times n^4}{m^4\times n^2}=m^{5-4}\times n^{4-2}=m^1\times n^2=mn^2$

h $\dfrac{p^{\frac{1}{2}}\times p^{\frac{1}{2}}}{p^2}=\dfrac{p^{\frac{1}{2}+\frac{1}{2}}}{p^2}=\dfrac{p^1}{p^2}=p^{1-2}=p^{-1}=\dfrac{1}{p}$

i $\left(3x^{11}\right)^2=3^2\times x^{11\times 2}=9x^{22}$

j $\dfrac{\left(x^4\right)^6}{x^3}=\dfrac{x^{4\times 6}}{x^3}=\dfrac{x^{24}}{x^3}=x^{24-3}=x^{21}$

Question 5

a $\left(pq^3\right)^5 = p^5 \times q^{3\times 5} = p^5q^{15}$

b $\left(\dfrac{a}{b}\right)^8 = \dfrac{a^8}{b^8}$

c $\left(\dfrac{4a}{b^4}\right)^3 = \dfrac{4^3 \times a^3}{b^{4\times 3}} = \dfrac{64a^3}{b^{12}}$

d $\left(7a^5b\right)^2 = 7^2 \times a^{5\times 2} \times b^2 = 49a^{10}b^2$

e $\dfrac{\left(2m^7\right)^3}{m^4} = \dfrac{2^3 \times m^{7\times 3}}{m^4} = \dfrac{8m^{21}}{m^4} = 8m^{21-4} = 8m^{17}$

f $\dfrac{xy^3 \times \left(xy^2\right)^4}{xy} = \dfrac{xy^3 \times x^4y^{2\times 4}}{xy} = \dfrac{x^{1+4} \times y^{3+8}}{xy} = \dfrac{x^5y^{11}}{xy} = x^{5-1}y^{11-1} = x^4y^{10}$

g $\dfrac{\left(2k^8\right)^4}{\left(6k^3\right)^3} = \dfrac{2^4k^{8\times 4}}{6^3k^{3\times 3}} = \dfrac{16k^{32}}{216k^9} = \dfrac{2}{27}k^{32-9} = \dfrac{2k^{23}}{27}$

h $\left(2y^5\right)^7 \times \dfrac{y^{12}}{8} = \dfrac{2^7y^{5\times 7} \times y^{12}}{8} = \dfrac{128y^{35} \times y^{12}}{8} = 16y^{35+12} = 16y^{47}$

i $\left(\dfrac{a^6 \times a^4}{a^{11}}\right)^{-3} = \left(\dfrac{a^{11}}{a^{6+4}}\right)^3 = \left(\dfrac{a^{11}}{a^0}\right)^3 = \left(a^{11-10}\right)^3 = \left(a\right)^3 = a^3$

j $\left(\dfrac{5xy^9}{x^8 \times y^3}\right)^3 = \dfrac{5^3x^3y^{9\times 3}}{x^{8\times 3}y^{3\times 3}} = \dfrac{125x^3y^{27}}{x^{24}y^9} = 125x^{3-24}y^{27-9} = 125x^{-21}y^{18} = \dfrac{125y^{18}}{x^{21}}$

Question 6

$(2)^3\left(\dfrac{3}{4}\right)^2 = 2^3 \times \left(\dfrac{3}{2^2}\right)^2 = 2^3 \times \dfrac{3^2}{2^{2\times 2}} = 9 \times 2^{-1} = \dfrac{9}{2}$ or 4.5

Question 7

$$\frac{\left(\frac{2}{3}\right)^3\left(\frac{1}{9}\right)^2}{\left(\frac{2}{3}\right)\left(\frac{1}{9}\right)^5}=\frac{\left(\frac{2^3}{3^3}\right)\left(\frac{1}{3^2}\right)^2}{\left(\frac{2}{3}\right)\left(\frac{1}{3^2}\right)^5}=\frac{\left(\frac{2^3}{3^3}\right)\frac{1}{3^{2\times 2}}}{\left(\frac{2}{3}\right)\frac{1}{3^{2\times 5}}}=\frac{\frac{2^3}{3^{4+3}}}{\frac{2}{3^{10+1}}}=\frac{2^3}{3^7}\times\frac{3^{11}}{2}=2^2\times 3^4=324$$

Question 8

$$\frac{\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^3}{\left(\frac{1}{4}\right)^4}=\frac{\frac{1}{2^2}\times\frac{1}{3^3}}{\left(\frac{1}{2^2}\right)^4}=\frac{1}{2^2}\times\frac{1}{3^3}\times 2^{2\times 4}=\frac{64}{27}=2\frac{10}{27}$$

Question 9

a $\dfrac{a^{11}b^8}{a^8b^7}=a^{11-8}b^{8-7}=a^3b$

b $\left(\dfrac{2}{5}\right)^3\left(\dfrac{5}{8}\right)=\dfrac{2^3}{5^3}\times\dfrac{5}{2^3}=\dfrac{1}{5^2}=\dfrac{1}{25}$

Question 10

a $\dfrac{p^5q^8r^4}{p^4q^6r^2}=p^{5-4}q^{8-6}r^{4-2}=pq^2r^2$

b $\left(\dfrac{7}{8}\right)\left(\dfrac{2}{3}\right)^2\left(\dfrac{3}{4}\right)^2=\dfrac{7}{8}\times\dfrac{2^2}{3^2}\times\left(\dfrac{3}{2^2}\right)^2=\dfrac{7}{8}\times\dfrac{2^2}{3^2}\times\dfrac{3^2}{2^{2\times 2}}=\dfrac{7}{8}\times\dfrac{1}{2^2}=\dfrac{7}{32}$

Question 11

$$\left(\left(\left(\frac{2}{3}\right)^{\frac{1}{6}}\right)^4\right)^3=\left(\frac{2}{3}\right)^{\frac{1}{6}\times 4\times 3}=\left(\frac{2}{3}\right)^2=\frac{4}{9}$$

Question 12

$$\frac{\left(\frac{1}{2}\right)^3\left(\frac{2}{3}\right)^6}{\left(\frac{2}{3}\right)^4}=\left(\frac{1}{2}\right)^3\left(\frac{2}{3}\right)^{6-4}=\frac{1}{2^8}\times\left(\frac{2}{3}\right)^2=\frac{1}{2^3}\times\frac{2^2}{3^2}=\frac{1}{18}$$

Question 13

$$\frac{\left(\frac{1}{3}\right)^4\left(\frac{2}{9}\right)^7}{\left(\frac{1}{3}\right)^5\left(\frac{2}{9}\right)^5}=\left(\frac{1}{3}\right)^{4-5}\left(\frac{2}{9}\right)^{7-5}=\left(\frac{1}{3}\right)^{-1}\left(\frac{2}{9}\right)^2=3\times\frac{4}{81}=\frac{4}{27}$$

Question 14

$$\frac{\left(\frac{1}{3}\right)^{-5}}{\left(\frac{1}{3}\right)^{-9}}=\left(\frac{1}{3}\right)^{-5+9}=\left(\frac{1}{3}\right)^4=\frac{1}{81}$$

Question 15

$$\frac{\left(\frac{3}{4}\right)^4\left(\frac{1}{9}\right)^6}{\left(\frac{3}{4}\right)^3\left(\left(\frac{1}{9}\right)^2\right)^2}=\frac{\left(\frac{3}{4}\right)^4\left(\frac{1}{9}\right)^6}{\left(\frac{3}{4}\right)^3\left(\frac{1}{9}\right)^{2\times 2}}=\frac{\left(\frac{3}{4}\right)^4\left(\frac{1}{9}\right)^6}{\left(\frac{3}{4}\right)^3\left(\frac{1}{9}\right)^4}=\left(\frac{3}{4}\right)^{4-3}\left(\frac{1}{9}\right)^{6-4}=\left(\frac{3}{4}\right)\left(\frac{1}{9}\right)^2=\frac{1}{108}$$

Question 16

$$\frac{\left(\frac{1}{9}\right)^6\times\left(\frac{3}{4}\right)^3}{\left(\frac{1}{9}\right)^5\times\left(\frac{3}{4}\right)^2}=\left(\frac{1}{9}\right)^{6-5}\times\left(\frac{3}{4}\right)^{3-2}=\frac{1}{9}\times\frac{3}{4}=\frac{1}{12}$$

Exercise 1.02 Zero and negative indices

Question 1

a $3^{-3}=\frac{1}{3^3}=\frac{1}{27}$

b $4^{-1}=\frac{1}{4}=\frac{1}{4}$

c $7^{-3}=\frac{1}{7^3}=\frac{1}{343}$

d $10^{-4}=\frac{1}{10^4}=\frac{1}{10\ 000}$

e $2^{-8}=\frac{1}{2^8}=\frac{1}{256}$

f $6^0=1$

g $2^{-5}=\frac{1}{2^5}=\frac{1}{32}$

h $3^{-4}=\frac{1}{3^4}=\frac{1}{81}$

i $7^{-1}=\frac{1}{7}=\frac{1}{7}$

j $9^{-2}=\frac{1}{9^2}=\frac{1}{81}$

k $2^{-6}=\frac{1}{2^6}=\frac{1}{64}$

l $3^{-2}=\frac{1}{3^2}=\frac{1}{9}$

m $4^0=1$

n $6^{-2}=\frac{1}{6^2}=\frac{1}{36}$

o $5^{-3}=\frac{1}{5^3}=\frac{1}{125}$

p $10^{-5}=\frac{1}{10^5}=\frac{1}{100\ 000}$

q $2^{-7}=\frac{1}{2^7}=\frac{1}{128}$

r $2^0=1$

s $8^{-2}=\frac{1}{8^2}=\frac{1}{64}$

t $4^{-3}=\frac{1}{4^3}=\frac{1}{64}$

Question 2

a $2^0 = 1$

b $\left(\frac{1}{2}\right)^{-4} = \left(\frac{2}{1}\right)^4 = 2^4 = 16$

c $\left(\frac{2}{3}\right)^{-1} = \left(\frac{3}{2}\right)^1 = \frac{3}{2}$

d $\left(\frac{5}{6}\right)^{-2} = \left(\frac{6}{5}\right)^2 = \frac{6^2}{5^2} = \frac{36}{25}$

e $\left(\frac{x+2y}{3x-y}\right)^0 = 1$

f $\left(\frac{1}{5}\right)^{-3} = \left(\frac{5}{1}\right)^3 = 5^3 = 125$

g $\left(\frac{3}{4}\right)^{-1} = \left(\frac{4}{3}\right)^1 = \frac{4}{3}$

h $\left(\frac{1}{7}\right)^{-2} = \left(\frac{7}{1}\right)^2 = 7^2 = 49$

i $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{3^3}{2^3} = \frac{27}{8}$

j $\left(\frac{1}{2}\right)^{-5} = \left(\frac{2}{1}\right)^5 = 2^5 = 32$

k $\left(\frac{3}{7}\right)^{-1} = \left(\frac{7}{3}\right)^1 = \frac{7}{3}$

l $\left(\frac{8}{9}\right)^0 = 1$

m $\left(\frac{6}{7}\right)^{-2} = \left(\frac{7}{6}\right)^2 = \frac{7^2}{6^2} = \frac{49}{36}$

n $\left(\frac{9}{10}\right)^{-2} = \left(\frac{10}{9}\right)^2 = \frac{10^2}{9^2} = \frac{100}{81}$

o $\left(\frac{6}{11}\right)^0 = 1$

p $\left(-\frac{1}{4}\right)^{-2} = \left(-\frac{4}{1}\right)^2 = (-4)^2 = 16$

q $\left(-\frac{2}{5}\right)^{-3} = \left(-\frac{5}{2}\right)^3 = -\frac{5^3}{2^3} = -\frac{125}{8}$

r $\left(-3\frac{2}{7}\right)^{-1} = \left(-\frac{23}{7}\right)^{-1} = \left(-\frac{7}{23}\right)^1 = -\frac{7}{23}$

s $\left(-\frac{3}{8}\right)^0 = 1$

t $\left(-1\frac{1}{4}\right)^{-2} = \left(-\frac{5}{4}\right)^{-2} = \left(-\frac{4}{5}\right)^2 = \frac{16}{25}$

Question 3

a $\frac{1}{m^3}=m^{-3}$

b $\frac{1}{x}=x^{-1}$

c $\frac{1}{p^7}=p^{-7}$

d $\frac{1}{d^9}=d^{-9}$

e $\frac{1}{k^5}=k^{-5}$

f $\frac{1}{x^2}=x^{-2}$

g $\frac{2}{x^4}=2x^{-4}$

h $\frac{3}{y^2}=3y^{-2}$

i $\frac{1}{2z^6}=\frac{1}{2}\times\frac{1}{z^6}=\frac{1}{2}z^{-6}$

j $\frac{3}{5t^8}=\frac{3}{5}\times\frac{1}{t^8}=\frac{3}{5}t^{-8}$

k $\frac{2}{7x}=\frac{2}{7}\times\frac{1}{x}=\frac{2}{7}x^{-1}$

l $\frac{5}{2m^6}=\frac{5}{2}\times\frac{1}{m^6}=\frac{5}{2}m^{-6}$

m $\frac{2}{3y^7}=\frac{2}{3}\times\frac{1}{y^7}=\frac{2}{3}y^{-7}$

n $\frac{1}{(3x+4)^2}=(3x+4)^{-2}$

o $\frac{1}{(a+b)^8}=(a+b)^{-8}$

p $\frac{1}{x-2}=(x-2)^{-1}$

q $\frac{1}{(5p+1)^3}=(5p+1)^{-3}$

r $\frac{2}{(4t-9)^5}=2(4t-9)^{-5}$

s
$$\begin{aligned}\frac{1}{4(x+1)^{11}}&=\frac{1}{4}\times\frac{1}{(x+1)^{11}}\\&=\frac{1}{4}\times(x+1)^{-11}\\&=\frac{(x+1)^{-11}}{4}\end{aligned}$$

t
$$\begin{aligned}\frac{5}{9(a+3b)^7}&=\frac{5}{9}\times\frac{1}{(a+3b)^7}\\&=\frac{5}{9}\times(a+3b)^{-7}\\&=\frac{5(a+3b)^{-7}}{9}\end{aligned}$$

Question 4

a $t^{-5} = \frac{1}{t^5}$

b $x^{-6} = \frac{1}{x^6}$

c $y^{-3} = \frac{1}{y^3}$

d $n^{-8} = \frac{1}{n^8}$

e $w^{-10} = \frac{1}{w^{10}}$

f $2x^{-} = 2\times\frac{1}{x} = \frac{2}{x}$

g $3m^{-4} = 3\times\frac{1}{m^4} = \frac{3}{m^4}$

h $5x^{-7} = 5\times\frac{1}{x^7} = \frac{5}{x^7}$

i $(2x)^{-3} = \frac{1}{(2x)^3} = \frac{1}{8x^3}$

j $(4n)^{-1} = \frac{1}{(4n)^1} = \frac{1}{4n}$

k $(x+1)^{-6} = \frac{1}{(x+1)^6}$

l $(8y+z)^{-1} = \frac{1}{(8y+z)} = \frac{1}{8y+z}$

m $(k-3)^{-2} = \frac{1}{(k-3)^2}$

n $(3x+2y)^{-9} = \frac{1}{(3x+2y)^9}$

o $\left(\frac{1}{x}\right)^{-5} = \left(\frac{x}{1}\right)^5 = (x)^5 = x^5$

p $\left(\frac{1}{y}\right)^{-10} = \left(\frac{y}{1}\right)^{10} = (y)^{10} = y^{10}$

q $\left(\frac{2}{p}\right)^{-1} = \left(\frac{p}{2}\right)^1 = \frac{p}{2}$

r $\left(\frac{1}{a+b}\right)^{-2} = \left(\frac{a+b}{1}\right)^2 = \frac{(a+b)^2}{1^2} = (a+b)^2$

s $\left(\frac{x+y}{x-y}\right)^{-1} = \left(\frac{x-y}{x+y}\right)^1 = \frac{x-y}{x+y}$

t $\left(\frac{2w-z}{3x+y}\right)^{-7} = \left(\frac{3x+y}{2w-z}\right)^7$

Exercise 1.03 Fractional indices

Question 1

a $81^{\frac{1}{2}} = \sqrt{81} = 9$

b $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$

c $16^{\frac{1}{2}} = \sqrt{16} = 4$

d $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

e $49^{\frac{1}{2}} = \sqrt{49} = 7$

f $1000^{\frac{1}{3}} = \sqrt[3]{1000} = 10$

g $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$

h $64^{\frac{1}{2}} = \sqrt{64} = 8$

i $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

j $1^{\frac{1}{7}} = \sqrt[7]{1} = 1$

k $81^{\frac{1}{4}} = \sqrt[4]{81} = 3$

l $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$

m $0^{\frac{1}{8}} = \sqrt[8]{0} = 0$

n $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

o $343^{\frac{1}{3}} = \sqrt[3]{343} = 7$

p $128^{\frac{1}{7}} = \sqrt[7]{128} = 2$

q $256^{\frac{1}{4}} = \sqrt[4]{256} = 4$

r $125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = (5)^2 = 25$

s $4^{\frac{5}{2}} = \left(\sqrt{4}\right)^5 = 2^5 = 32$

t $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

u $9^{\frac{3}{2}} = \left(\sqrt{9}\right)^3 = 3^3 = 27$

v $8^{-\frac{1}{3}} = \dfrac{1}{8^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{8}} = \dfrac{1}{2}$

w $9^{-\frac{1}{2}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

x $16^{-\frac{1}{4}} = \dfrac{1}{\sqrt[4]{16}} = \dfrac{1}{2}$

y $64^{-\frac{2}{3}} = \dfrac{1}{64^{\frac{2}{3}}} = \dfrac{1}{\left(2^6\right)^{\frac{2}{3}}} = \dfrac{1}{2^4} = \dfrac{1}{16}$

Question 2

a $23^{\frac{1}{4}} \approx 2.19$

b $\sqrt[4]{45.8} \approx 2.60$

c $\sqrt[7]{1.24+4.3^2} \approx 1.53$

d $\dfrac{1}{\sqrt[5]{12.9}} \approx 0.60$

e $\sqrt[8]{\dfrac{3.6-1.4}{1.5+3.7}} \approx 0.90$

f $\dfrac{\sqrt[4]{5.9\times 3.7}}{8.79-1.4} \approx 0.29$

Question 3

a $y^{\frac{1}{3}} = \sqrt[3]{y}$

b $x^{\frac{1}{6}} = \sqrt[6]{x}$

c $a^{\frac{1}{2}} = \sqrt{a}$

d $t^{\frac{1}{9}} = \sqrt[9]{t}$

e $y^{\frac{2}{3}} = \sqrt[3]{y^2}$

f $x^{\frac{3}{4}} = \sqrt[4]{x^3}$

g $b^{\frac{2}{5}} = \sqrt[5]{b^2}$

h $a^{\frac{4}{7}} = \sqrt[7]{a^4}$

i $x^{-\frac{1}{2}} = \dfrac{1}{x^{\frac{1}{2}}} = \dfrac{1}{\sqrt{x}}$

j $d^{-\frac{1}{3}} = \dfrac{1}{\sqrt[3]{d}}$

k $x^{-\frac{1}{8}} = \dfrac{1}{\sqrt[8]{x}}$

l $y^{-\frac{1}{3}} = \dfrac{1}{\sqrt[3]{y}}$

m $a^{-\frac{1}{4}} = \dfrac{1}{\sqrt[4]{a}}$

n $z^{-\frac{3}{4}} = \dfrac{1}{\left(\sqrt[4]{z}\right)^3} = \dfrac{1}{\sqrt[4]{z^3}}$

o $y^{-\frac{3}{5}} = \dfrac{1}{\left(\sqrt[5]{y}\right)^3} = \dfrac{1}{\sqrt[5]{y^3}}$

p $(2x+5)^{\frac{1}{2}} = \sqrt{2x+5}$

q $(6q+r)^{\frac{1}{3}} = \sqrt[3]{6q+r}$

r $(a+b)^{\frac{1}{9}} = \sqrt[9]{a+b}$

s $(3x-1)^{-\frac{1}{2}} = \dfrac{1}{(3x-1)^{\frac{1}{2}}} = \dfrac{1}{\sqrt{3x-1}}$

t $(x+7)^{-\frac{2}{5}} = \dfrac{1}{\sqrt[5]{(x+7)^2}} = \dfrac{1}{\left(\sqrt[5]{x+7}\right)^2}$

Question 4

a $\sqrt{t} = t^{\frac{1}{2}}$

b $\sqrt[5]{y} = y^{\frac{1}{5}}$

c $\sqrt{x^3} = x^{\frac{3}{2}}$

d $\sqrt[3]{9-x} = (9-x)^{\frac{1}{3}}$

e $\sqrt{4s+1} = (4s+1)^{\frac{1}{2}}$

f $\sqrt{(3x+1)^5} = (3x+1)^{\frac{1}{2}\times 5} = (3x+1)^{\frac{5}{2}}$

g $\dfrac{1}{\sqrt{2t+3}} = \dfrac{1}{(2t+3)^{\frac{}{2}}} = (2t+3)^{-\frac{1}{2}}$

h $\dfrac{1}{\sqrt{(5x-y)^3}} = \dfrac{1}{(5x-y)^{\frac{1}{2}\times 3}} = \dfrac{1}{(5x-y)^{\frac{3}{2}}} = (5x-y)^{-\frac{3}{2}}$

i $\dfrac{1}{\sqrt[3]{(x-2)^2}} = \dfrac{1}{(x-2)^{\frac{1}{3}\times 2}} = \dfrac{1}{(x-2)^{\frac{2}{3}}} = (x-2)^{-\frac{2}{3}}$

j $\dfrac{1}{2\sqrt{y+7}} = \dfrac{1}{2(y+7)^{\frac{1}{2}}} = \dfrac{1}{2} \times \dfrac{1}{(y+7)^{\frac{1}{2}}} = \dfrac{1}{2}(y+7)^{-\frac{1}{2}}$

k $\dfrac{5}{\sqrt[3]{x+4}} = \dfrac{5}{(x+4)^{\frac{1}{3}}} = 5 \times \dfrac{1}{(x+4)^{\frac{1}{3}}} = 5(x+4)^{-\frac{1}{3}}$

l $\dfrac{2}{3\sqrt{y^2-1}} = \dfrac{2}{3(y^2-1)^{\frac{1}{2}}} = \dfrac{2}{3} \times \dfrac{1}{(y^2-1)^{\frac{1}{2}}} = \dfrac{2}{3}(y^2-1)^{-\frac{1}{2}}$

m $\dfrac{3}{5\sqrt[4]{(x^2+2)^3}} = \dfrac{3}{5(x^2+2)^{3\times\frac{1}{4}}} = \dfrac{3}{5} \times \dfrac{1}{(x^2+2)^{\frac{3}{4}}} = \dfrac{3}{5}(x^2+2)^{-\frac{3}{4}}$

Question 5

a $x\sqrt{x} = x \times x^{\frac{1}{2}} = x^{1+\frac{1}{2}} = x^{\frac{3}{2}}$

b $\dfrac{\sqrt{x}}{x} = \dfrac{x^{\frac{1}{2}}}{x^{1}} = x^{\frac{1}{2}-} = x^{-\frac{1}{2}} = \dfrac{1}{x^{\frac{1}{2}}}$

c $\dfrac{x}{\sqrt[3]{x}} = \dfrac{x^{1}}{x^{\frac{1}{3}}} = x^{1-\frac{1}{3}} = x^{\frac{2}{3}}$

d $\dfrac{x^{2}}{\sqrt[3]{x}} = \dfrac{x^{2}}{x^{\frac{1}{3}}} = x^{2-\frac{1}{3}} = x^{\frac{5}{3}}$

e $x\sqrt[4]{x} = x \times x^{\frac{1}{4}} = x^{1+\frac{1}{4}} = x^{\frac{5}{4}}$

Question 6

a $(a-2b)^{-\frac{1}{3}} = \dfrac{1}{(a-2b)^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{a-2b}}$

b $(y-3)^{-\frac{2}{3}} = \dfrac{1}{(y-3)^{\frac{2}{3}}} = \dfrac{1}{\sqrt[3]{(y-3)^{2}}}$

c $4(6a+1)^{-\frac{4}{7}} = 4 \times \dfrac{1}{(6a+1)^{\frac{4}{7}}} = \dfrac{4}{\sqrt[7]{(6a+1)^{4}}}$

d $\dfrac{(x+y)^{-\frac{5}{4}}}{3} = \dfrac{1}{3} \times \dfrac{1}{(x+y)^{\frac{5}{4}}} = \dfrac{1}{3\sqrt[4]{(x+y)^{5}}}$

e $\dfrac{6(3x+8)^{-\frac{2}{9}}}{7} = \dfrac{6}{7} \times \dfrac{1}{(3x+8)^{\frac{2}{9}}} = \dfrac{6}{7\sqrt[9]{(3x+8)^{2}}}$

Exercise 1.04 Simplifying algebraic expressions

Question 1

a $9a - 6a = 3a$

b $5z - 4z = z$

c $4b - b = 3b$

d $2r - 5r = -3r$

e $-4y + 3y = -y$

f $-2x - 3x = -5x$

g $2a - 2a = 0$

h $-4k + 7k = 3k$

i $3t + 4t + 2t = 9t$

j $8w - w + 3w = 10w$

k $4m - 3m - 2m = -m$

l $x + 3x - 5x = -x$

m $8h - h - 7h = 0$

n $3b - 5b + 4b + 9b = 11b$

o $-5x + 3x - x - 7x = -10x$

p $6x - 5y - y = 6x - 6y$

q $8a + b - 4b - 7a = a - 3b$

r $xy + 2y + 3xy = 4xy + 2y$

s $2ab^2 - 5ab^2 - 3ab^2 = -6ab^2$

t $m^2 - 5m - m + 12 = m^2 - 6m + 12$

u $p^2 - 7p + 5p - 6 = p^2 - 2p - 6$

v $ab + 2b - 3ab + 8b = -2ab + 10b$

w $ab + bc - ab - ac + bc = 2bc - ac$

x $a^5 - 7x^3 + a^5 - 2x^3 + 1 = 2a^5 - 9x^3 + 1$

y $x^3 - 3xy^2 + 4x^2y - x^2y + xy^2 + 2y^3 = x^3 - 2xy^2 + 3x^2y + 2y^3$

Question 2

a $5 \times 2b = 10b$

b $2x \times 4y = 8xy$

c $5p \times 2p = 10p^2$

d $-3z \times 2w = -6wz$

e $-5a \times (-3b) = 15ab$

f $x \times 2y \times 7z = 14xyz$

g $8ab \times 6c = 48abc$

h $4d \times 3d = 12d.^2$

i $3a \times 4a \times a = 12a^3$

j $\left(-3y\right)^3 = \left(-3\right)^3 y^3 = -27y^3$

k $\left(2x^2\right)^5 = 2^5 x^{2 \times 5} = 32x^{10}$

l $2ab^3 \times 3a = 6a^2b^3$

m $5a^2b \times (-2ab) = -10a^3b^2$

n $7pq^2 \times 3p^2q^2 = 21p^3q^4$

o $5ab \times a^2b^2 = 5a^3b^3$

p $4h^3 \times (-2h^7) = -8h^{10}$

q $k^3p \times p^2 = k^3p^3$

r $\left(-3t^3\right)^4 = \left(-3\right)^4 t^{3 \times 4} = 81t^{12}$

s $7m^6 \times (-2m^5) = -14m^{11}$

t $-2x^2 \times 3x^3y \times (-4xy^2) = 24x^6y^3$

Question 3

a $30x \div 5 = 6x$

b $2y \div y = 2$

c $\dfrac{8a^2}{2} = 4a^2$

d $\dfrac{8a^2}{a} = 8a^{2-1} = 8a$

e $\dfrac{8a^2}{2a} = 4a^{2-1} = 4a$

f $\dfrac{xy}{2x} = \dfrac{y}{2}$

g $12p^3 \div 4p^2 = 3p^{3-2} = 3p$

h $\dfrac{3a^2b^2}{6ab} = \dfrac{1}{2}a^{2-1}b^{2-1} = \dfrac{1}{2}ab = \dfrac{ab}{2}$

i $\dfrac{20x}{15xy} = \dfrac{4}{3}x^{1-1}y^{-1} = \dfrac{4}{3y}$

j $\dfrac{-9x^7}{3x^4} = -\dfrac{3}{1}x^{7-4} = -3x^3$

k $-15ab \div (-5b) = 3a$

l $\dfrac{2ab}{6a^2b^3} = \dfrac{1}{3}a^{1-2}b^{1-3} = \dfrac{1}{3}a^{-1}b^{-2} = \dfrac{1}{3ab^2}$

m $\dfrac{-8p}{4pqs} = -\dfrac{2}{1}p^{1-1}q^{-1}s^{-1} = -2q^{-1}s^{-1} = -\dfrac{2}{qs}$

n $14cd^2 \div 21c^3d^3 = \dfrac{14cd^2}{21c^3d^3} = \dfrac{2}{3}c^{1-3}d^{2-3} = \dfrac{2}{3}c^{-2}d^{-1} = \dfrac{2}{3c^2d}$

o $\dfrac{2xy^2z^3}{4x^3y^2z} = \dfrac{1}{2}x^{1-3}y^{2-2}z^{3-1} = \dfrac{1}{2}x^{-2}z^2 = \dfrac{z^2}{2x^2}$

p $\dfrac{42p^5q^4}{7pq^3} = \dfrac{6}{1}p^{5-1}q^{4-3} = 6p^4q$

q $5a^9b^4c^{-2} \div 20a^5b^{-3}c^{-1} = \dfrac{5a^9b^4c^{-2}}{20a^5b^{-3}c^{-1}} = \dfrac{1}{4}a^{9-5}b^{4--3}c^{-2--1} = \dfrac{1}{4}a^4b^7c^{-1} = \dfrac{a^4b^7}{4c}$

r $\dfrac{2\left(a^{-5}\right)^2b^4}{4a^{-9}\left(b^2\right)^{-1}} = \dfrac{2a^{-5\times 2}b^4}{4a^{-9}b^{2\times -1}} = \dfrac{1}{2}a^{-10--9}b^{4--2} = \dfrac{1}{2}a^{-1}b^6 = \dfrac{b^6}{2a}$

s $-5x^4y^7z \div 15xy^8z^{-2} = \dfrac{-5x^4y^7z}{15xy^8z^{-2}} = -\dfrac{1}{3}x^{4-1}y^{7-8}z^{1--2} = -\dfrac{1}{3}x^3y^{-1}z^3 = -\dfrac{x^3z^3}{3y}$

t $-9\left(a^4b^{-1}\right)^3 \div -18a^{-1}b^3 = \dfrac{-9\left(a^4b^{-1}\right)^3}{-18a^{-1}b^3} = \dfrac{-9a^{4\times 3}b^{-1\times 3}}{-18a^{-1}b^3} = \dfrac{1}{2}a^{12--1}b^{-3-3} = \dfrac{1}{2}a^{13}b^{-6} = \dfrac{a^{13}}{2b^6}$

Exercise 1.05 Expansion

Question 1

$2(x-4)=2x-8$

Question 2

$3(2h+3)=6h+9$

Question 3

$-5(a-2)=-5a+10$

Question 4

$x(2y+3)=2xy+3x$

Question 5

$x(x-2)=x^2-2x$

Question 6

$2a(3a-8b)=6a^2-16ab$

Question 7

$ab(2a+b)=2a^2b+ab^2$

Question 8

$5n\left(n-4\right)=5n^2-20n$

Question 9

$3x^2y\left(xy+2y^2\right)=3x^3y^2+6x^2y^3$

Question 10

$3+4\left(k+1\right)=3+4k+4=4k+7$

Question 11

$2\left(t-7\right)-3=2t-14-3=2t-17$

Question 12

$y\left(4y+3\right)+8y=4y^2+3y+8y=4y^2+11y$

Question 13

$9-5\left(b+3\right)=9-5b-15=-5b-6$

Question 14

$3-\left(2x-5\right)=3-2x+5=8-2x$

Question 15

$5\left(3-2m\right)+7\left(m-2\right)=15-10m+7m-14=1-3m$

Question 16

$2(h+4)+3(2h-9)=2h+8+6h-27=8h-19$

Question 17

$3(2d-3)-(5d-3)=6d-9-5d+3=d-6$

Question 18

$a(2a+1)-(a^2+3a-4)=2a^2+a-a^2-3a+4=a^2-2a+4$

Question 19

$x(3x-4)-5(x+1)=3x^2-4x-5x-5=3x^2-9x-5$

Question 20

$2ab(3-a)-b(4a-1)=6ab-2a^2b-4ab+b=2ab-2a^2b+b$

Question 21

$5x-(x-2)-3=5x-x+2-3=4x-1$

Question 22

$8-4(2y+1)+y=8-8y-4+y=4-7y$

Question 23

$(a+b)-(a-b)=a+b-a+b=2b$

Question 24

$2(3t-4)-(t+1)+3=6t-8-t-1+3=5t-6$

Exercise 1.06 Binomial Products

Question 1

$(a+5)(a+2)=a^2+5a+2a+10=a^2+7a+10$

Question 2

$(x+3)(x-1)=x^2+3x-x-3=x^2+2x-3$

Question 3

$(2y-3)(y+5)=2y^2+10y-3y-15=2y^2+7y-15$

Question 4

$(m-4)(m-2)=m^2-4m-2m+8=m^2-6m+8$

Question 5

$(x+4)(x+3)=x^2+4x+3x+12=x^2+7x+12$

Question 6

$(y+2)(y-5)=y^2+2y-5y-10=y^2-3y-10$

Question 7

$(2x-3)(x+2)=2x^2+4x-3x-6=2x^2+x-6$

Question 8

$(h-7)(h-3)=h^2-3h-7h+21=h^2-10h+21$

Question 9

$(x+5)(x-5)=x^2+5x-5x-25=x^2-25$

Question 10

$(5a-4)(3a-1)=15a^2-5a-12a+4=15a^2-17a+4$

Question 11

$(2y+3)(4y-3)=8y^2-6y+12y-9=8y^2+6y-9$

Question 12

$(x-4)(y+7)=xy-4y+7x-28$

Question 13

$(x^2+3)(x-2)=x^3-2x^2+3x-6$

Question 14

$(n+2)(n-2)=n^2-2n+2n-4=n^2-4$

Question 15

$(2x+3)(2x-3)=4x^2-6x+6x-9=4x^2-9$

Question 16

$(4+7y)(4-7y)=16-28y+28y-49y^2=16-49y^2$

Question 17

$(a+2b)(a-2b)=a^2-2ab+2ab-4b^2=a^2-4b^2$

Question 18

$(3x-4y)(3x+4y)=9x^2-12xy+12xy-16y^2=9x^2-16y^2$

Question 19

$(x+3)(x-3)=x^2-3x+3x-9=x^2-9$

Question 20

$(y-6)(y+6)=y^2-6x+6x-36=y^2-36$

Question 21

$(3a+1)(3a-1)=9a^2-3a+3a-1=9a^2-1$

Question 22

$(2z-7)(2z+7)=4z^2-14z+14z-49=4z^2-49$

Question 23

$(x+9)(x-2y+2)=x^2-2xy+2x+9x-18y+18=x^2-2xy+11x-18y+18$

Question 24

$(b-3)(2a+2b-1)=2ab+2b^2-b-6a-6b+3=2b^2+2ab-7b-6a+3$

Question 25

$(x+2)(x^2-2x+4)=x^3-2x^2+4x+2x^2-4x+8=x^3+8$

Question 26

$(a-3)(a^2+3a+9)=a^3+3a^2+9a-3a^2-9a-27=a^3-27$

Question 27

$(a+9)^2=(a+9)(a+9)=a^2+9a+9a+81=a^2+18a+81$

Question 28

$(k-4)^2=(k-4)(k-4)=k^2-4k-4k+16=k^2-8k+16$

Question 29

$(x+2)^2=(x+2)(x+2)=x^2+2x+2x+4=x^2+4x+4$

Question 30

$(y-7)^2=(y-7)(y-7)=y^2-7y-7y+49=y^2-14y+49$

Question 31

$(2x+3)^2=(2x+3)(2x+3)=4x^2+6x+6x+9=4x^2+12x+9$

Question 32

$$\left(2t-1\right)^2=\left(2t-1\right)\left(2t-1\right)=4t^2-2t-2t+1=4t^2-4t+1$$

Question 33

$$\left(3a+4b\right)^2=\left(3a+4b\right)\left(3a+4b\right)=9a^2+12ab+12ab+16b^2=9a^2+24ab+16b^2$$

Question 34

$$\left(x-5y\right)^2=\left(x-5y\right)\left(x-5y\right)=x^2-5xy-5xy+25y^2=x^2-10xy+25y^2$$

Question 35

$$\left(2a+b\right)^2=\left(2a+b\right)\left(2a+b\right)=4a^2+2ab+2ab+b^2=4a^2+4ab+b^2$$

Question 36

$$\left(a-b\right)\left(a+b\right)=a^2-ba+ba-b^2=a^2-b^2$$

Question 37

$$\left(a+b\right)^2=\left(a+b\right)\left(a+b\right)=a^2+ab+ab+b^2=a^2+2ab+b^2$$

Question 38

$$\left(a-b\right)^2=\left(a-b\right)\left(a-b\right)=a^2-ab-ab+b^2=a^2-2ab+b^2$$

Question 39

$$\left(a+b\right)\left(a^2-ab+b^2\right)=a^3-a^2b+ab^2+a^2b-ab^2+b^3=a^3+b^3$$

Question 40

$$\left(a-b\right)\left(a^2+ab+b^2\right)=a^3+a^2b+ab^2-a^2b-ab^2-b^3=a^3-b^3$$

Exercise 1.07 Special products

Question 1

$$(t+4)^2 = (t+4)(t+4) = t^2 + 4t + 4t + 16 = t^2 + 8t + 16$$

Question 2

$$(z-6)^2 = (z-6)(z-6) = z^2 - 6z - 6z + 36 = z^2 - 12z + 36$$

Question 3

$$(x-1)^2 = (x-1)(x-1) = x^2 - x - x + 1 = x^2 - 2x + 1$$

Question 4

$$(y+8)^2 = (y+8)(y+8) = y^2 + 8y + 8y + 64 = y^2 + 16y + 64$$

Question 5

$$(q+3)^2 = (q+3)(q+3) = q^2 + 3q + 3q + 9 = q^2 + 6q + 9$$

Question 6

$$(k-7)^2 = (k-7)(k-7) = k^2 - 7k - 7k + 49 = k^2 - 14k + 49$$

Question 7

$$(n+1)^2 = (n+1)(n+1) = n^2 + n + n + 1 = n^2 + 2n + 1$$

Question 8

$$\begin{aligned}(2b+5)^2 &= (2b+5)(2b+5)\\ &= 4b^2+10b+10b+25\\ &= 4b^2+20b+25\end{aligned}$$

Question 9

$$(3-x)^2 = (3-x)(3-x) = 9-3x-3x+x^2 = x^2-6x+9$$

Question 10

$$(3y-1)^2 = (3y-1)(3y-1) = 9y^2-3y-3y+1 = 9y^2-6y+1$$

Question 11

$$(x+y)^2 = (x+y)(x+y) = x^2+xy+xy+y^2 = x^2+2xy+y^2$$

Question 12

$$\begin{aligned}(3a-b)^2 &= (3a-b)(3a-b)\\ &= 9a^2-3ab-3ab+b^2\\ &= 9a^2-6ab+b^2\end{aligned}$$

Question 13

$$\begin{aligned}(4d+5e)^2 &= (4d+5e)(4d+5e)\\ &= 16d^2+20de+20de+25e^2\\ &= 16d^2+40de+25e^2\end{aligned}$$

Question 14

$(t+4)(t-4)=t^2-4t+4t-16=t^2-16$

Question 15

$(x+3)(x-3)=x^2-3x+3x-9=x^2-9$

Question 16

$(p+1)(p-1)=p^2+p-p-1=p^2-1$

Question 17

$(r+6)(r-6)=r^2+6r-6r-36=r^2-36$

Question 18

$(x-10)(x+10)=x^2+10x-10x-100=x^2-100$

Question 19

$(2a+3)(2a-3)=4a^2+6a-6a-9=4a^2-9$

Question 20

$(x-5y)(x+5y)=x^2+5xy-5xy-25y^2=x^2-25y^2$

Question 21

$(4a+1)(4a-1)=16a^2+4a-4a-1=16a^2-1$

Question 22

$$\left(7-3x\right)\left(7+3x\right)=49+21x-21x-9x^2=49-9x^2$$

Question 23

$$\left(x^2+2\right)\left(x^2-2\right)=x^4-2x^2+2x^2-4=x^4-4$$

Question 24

$$\begin{aligned}\left(x^2+5\right)^2&=\left(x^2+5\right)\left(x^2+5\right)\\&=x^4+5x^2+5x^2+25\\&=x^4+10x^2+25\end{aligned}$$

Question 25

$$\begin{aligned}\left(3ab-4c\right)\left(3ab+4c\right)&=9a^2b^2-12abc+12abc-16c^2\\&=9a^2b^2-16c^2\end{aligned}$$

Question 26

$$\begin{aligned}\left(x+\frac{2}{x}\right)^2&=\left(x+\frac{2}{x}\right)\left(x+\frac{2}{x}\right)\\&=x^2+\frac{2x}{x}+\frac{2x}{x}+\frac{4}{x^2}\\&=x^2+2+2+\frac{4}{x^2}\\&=x^2+4+\frac{4}{x^2}\end{aligned}$$

Question 27

$$\left(a-\frac{1}{a}\right)\left(a+\frac{1}{a}\right)=a^2-\frac{a}{a}+\frac{a}{a}-\frac{1}{a^2}$$
$$=a^2-1+1-\frac{1}{a^2}$$
$$=a^2-\frac{1}{a^2}$$

Question 28

$$\left(x+[y-2]\right)\left(x-[y-2]\right)=x^2-x(y-2)+x(y-2)-(y-2)^2$$
$$=x^2-(y-2)^2$$
$$=x^2-(y-2)(y-2)$$
$$=x^2-\left(y^2-2y-2y+4\right)$$
$$=x^2-\left(y^2-4y+4\right)$$
$$=x^2-y^2+4y-4$$

Question 29

$$\left([a+b]+c\right)^2=\left([a+b]+c\right)\left([a+b]+c\right)$$
$$=(a+b)^2+c(a+b)+c(a+b)+c^2$$
$$=(a+b)(a+b)+ca+cb+ca+cb+c^2$$
$$=a^2+ab+ab+b^2+2ac+2cb+c^2$$
$$=a^2+2ab+b^2+2ac+2cb+c^2$$

Question 30

$$\left([x+1]-y\right)^2=\left([x+1]-y\right)\left([x+1]-y\right)$$
$$=(x+1)^2-y(x+1)-y(x+1)+y^2$$
$$=(x+1)(x+1)-yx-y-yx-y+y^2$$
$$=x^2+x+x+1-2yx-2y+y^2$$
$$=x^2+2x+1-2yx-2y+y^2$$

Question 31

$$\begin{aligned}(a+3)^2-(a-3)^2 &= (a+3)(a+3)-(a-3)(a-3)\\ &= a^2+3a+3a+9-\left(a^2-3a-3a+9\right)\\ &= a^2+3a+3a+9-a^2+3a+3a-9\\ &= 12a\end{aligned}$$

Question 32

$$\begin{aligned}16-(z-4)(z+4) &= 16-\left(z^2-4z+4z-16\right)\\ &= 16-z^2+4z-4z+16\\ &= 32-z^2\end{aligned}$$

Question 33

$$\begin{aligned}2x+(3x+1)^2-4 &= 2x+(3x+1)(3x+1)-4\\ &= 2x+9x^2+3x+3x+1-4\\ &= 9x^2+8x-3\end{aligned}$$

Question 34

$$\begin{aligned}(x+y)^2-x(2-y) &= (x+y)(x+y)-2x+xy\\ &= x^2+xy+xy+y^2-2x+xy\\ &= x^2+3xy-2x+y^2\end{aligned}$$

Question 35

$$\begin{aligned}(4n-3)(4n+3)-2n^2+5 &= 16n^2-12n+12n-9-2n^2+5\\ &= 14n^2-4\end{aligned}$$

Question 36

$$
\begin{aligned}
(x-4)^3 &= (x-4)(x-4)(x-4) \\
&= \left(x^2-4x-4x+16\right)(x-4) \\
&= \left(x^2-8x+16\right)(x-4) \\
&= x^3-4x^2-8x^2+32x+16x-64 \\
&= x^3-12x^2+48x-64
\end{aligned}
$$

Question 37

$$
\begin{aligned}
\left(x-\frac{1}{x}\right)^2-\left(\frac{1}{x}\right)^2+2 &= \left(x-\frac{1}{x}\right)\left(x-\frac{1}{x}\right)-\frac{1}{x^2}+2 \\
&= x^2-\frac{x}{x}-\frac{x}{x}+\frac{1}{x^2}-\frac{1}{x^2}+2 \\
&= x^2-1-1+\frac{1}{x^2}-\frac{1}{x^2}+2 \\
&= x^2
\end{aligned}
$$

Question 38

$$
\begin{aligned}
\left(x^2+y^2\right)^2-4x^2y^2 &= \left(x^2+y^2\right)\left(x^2+y^2\right)-4x^2y^2 \\
&= x^4+x^2y^2+x^2y^2+y^4-4x^2y^2 \\
&= x^4+y^4-2x^2y^2
\end{aligned}
$$

Question 39

$$
\begin{aligned}
(2a+5)^3 &= (2a+5)(2a+5)(2a+5) \\
&= \left(4a^2+10a+10a+25\right)(2a+5) \\
&= \left(4a^2+20a+25\right)(2a+5) \\
&= 8a^3+20a^2+40a^2+100a+50a+125 \\
&= 8a^3+60a^2+150a+125
\end{aligned}
$$

Exercise 1.08 Factorisation

Question 1

$2y+6=2(y+3)$

Question 2

$5x-10=5(x-2)$

Question 3

$3m-9=3(m-3)$

Question 4

$8x+2=2(4x+1)$

Question 5

$24-18y=6(4-3y)$

Question 6

$x^2+2x=x(x+2)$

Question 7

$m^2-3m=m(m-3)$

Question 8

$2y^2 + 4y = 2y(y+2)$

Question 9

$15a - 3a^2 = 3a(5-a)$

Question 10

$ab^2 + ab = ab(b+1)$

Question 11

$4x^2y - 2xy = 2xy(2x-1)$

Question 12

$3mn^3 + 9mn = 3mn(n^2+3)$

Question 13

$8x^2z - 2xz^2 = 2xz(4x-z)$

Question 14

$6ab + 3a - 2a^2 = a(6b+3-2a)$

Question 15

$5x^2 - 2x + xy = x(5x-2+y)$

Question 16

$3q^5 - 2q^2 = q^2\left(3q^3 - 2\right)$

Question 17

$5b^3 + 15b^2 = 5b^2\left(b + 3\right)$

Question 18

$6a^2b^3 - 3a^3b^2 = 3a^2b^2\left(2b - a\right)$

Question 19

$x\left(m+5\right)+7\left(m+5\right)=\left(m+5\right)\left(x+7\right)$

Question 20

$2\left(y-1\right)-y\left(y-1\right)=\left(y-1\right)\left(2-y\right)$

Question 21

$4\left(7+y\right)-3x\left(7+y\right)=\left(7+y\right)\left(4-3x\right)$

Question 22

$6x\left(a-2\right)+5\left(a-2\right)=\left(a-2\right)\left(6x+5\right)$

Question 23

$x\left(2t+1\right)-y\left(2t+1\right)=\left(2t+1\right)\left(x-y\right)$

Question 24

$$a(3x-2)+2b(3x-2)-3c(3x-2)=(3x-2)(a+2b-3c)$$

Question 25

$$6x^3+9x^2=3x^2(2x+3)$$

Question 26

$$3pq^5-6q^3=3q^3(pq^2-2)$$

Question 27

$$15a^4b^3+3ab=3ab(5a^3b^2+1)$$

Question 28

$$4x^3-24x^2=4x^2(x-6)$$

Question 29

$$35m^3n^4-25m^2n=5m^2n(7mn^3-5)$$

Question 30

$$24a^2b^5+16ab^2=8ab^2(3ab^3+2)$$

Question 31

$$2\pi r^2+2\pi rh=2\pi r(r+h)$$

Question 32

$$(x-3)^2+5(x-3)=(x-3)((x-3)+5)=(x-3)(x-3+5)=(x-3)(x+2)$$

Question 33

$$y^2(x+4)+2(x+4)=(x+4)(y^2+2)$$

Question 34

$$a(a+1)-(a+1)^2=(a+1)(a-(a+1))=(a+1)(a-a-1)=(a+1)(-1)=-(a+1)$$

Exercise 1.09 Factorisation by grouping in pairs

Question 1

$$2x+8+bx+4b=2(x+4)+b(x+4)=(x+4)(2+b)$$

Question 2

$$ay-3a+by-3b=a(y-3)+b(y-3)=(y-3)(a+b)$$

Question 3

$$x^2+5x+2x+10=x(x+5)+2(x+5)=(x+2)(x+5)$$

Question 4

$$m^2-2m+3m-6=m(m-2)+3(m-2)=(m-2)(m+3)$$

Question 5

$$ad-ac+bd-bc=a(d-c)+b(d-c)=(a+b)(d-c)$$

Question 6

$$x^3+x^2+3x+3=x^2(x+1)+3(x+1)=(x+1)(x^2+3)$$

Question 7

$$5ab-3b+10a-6=b(5a-3)+2(5a-3)=(5a-3)(b+2)$$

Question 8

$$2xy - x^2 + 2y^2 - xy = x(2y - x) + y(2y - x) = (2y - x)(x + y)$$

Question 9

$$ay + a + y + 1 = a(y + 1) + 1(y + 1) = (a + 1)(y + 1)$$

Question 10

$$x^2 + 5x - x - 5 = x(x + 5) - 1(x + 5) = (x + 5)(x - 1)$$

Question 11

$$y + 3 + ay + 3a = 1(y + 3) + a(y + 3) = (y + 3)(1 + a)$$

Question 12

$$m - 2 + 4y - 2my = m - 2my - 2 + 4y = m(1 - 2y) - 2(1 - 2y) = (1 - 2y)(m - 2)$$

Question 13

$$2x^2 + 10xy - 3xy - 15y^2 = 2x(x + 5y) - 3y(x + 5y) = (x + 5y)(2x - 3y)$$

Question 14

$$a^2b + ab^3 - 4a - 4b^2 = ab(a + b^2) - 4(a + b^2) = (a + b^2)(ab - 4)$$

Question 15

$$5x - x^2 - 3x + 15 = x(5 - x) + 3(-x + 5) = (5 - x)(x + 3)$$

Question 16

$x^4+7x^3-4x-28=x^3(x+7)-4(x+7)=(x+7)(x^3-4)$

Question 17

$7x-21-xy+3y=7(x-3)-y(x-3)=(x-3)(7-y)$

Question 18

$4d+12-de-3e=4(d+3)-e(d+3)=(d+3)(4-e)$

Question 19

$3x-12+xy-4y=3(x-4)+y(x-4)=(x-4)(3+y)$

Question 20

$2a+6-ab-3b=2(a+3)-b(a+3)=(a+3)(2-b)$

Question 21

$x^3-3x^2+6x-18=x^2(x-3)+6(x-3)=(x-3)(x^2+6)$

Question 22

$pq-3p+q^2-3q=p(q-3)+q(q-3)=(q-3)(p+q)$

Question 23

$3x^3-6x^2-5x+10=3x^2(x-2)-5(x-2)=(x-2)(3x^2-5)$

Question 24

$$4a-12b+ac-3bc=4(a-3b)+c(a-3b)=(a-3b)(4+c)$$

Question 25

$$xy+7x-4y-28=x(y+7)-4(y+7)=(y+7)(x-4)$$

Question 26

$$x^4-4x^3-5x+20=x^3(x-4)-5(x-4)=(x-4)(x^3-5)$$

Question 27

$$4x^3-6x^2+8x-12=2x^2(2x-3)+4(2x-3)=(2x-3)(2x^2+4)=2(2x-3)(x^2+2)$$

Question 28

$$\begin{aligned}3a^2+9a+6ab+18b&=3(a^2+3a+2ab+6b)\\&=3(a(a+3)+2b(a+3))\\&=3(a+3)(a+2b)\end{aligned}$$

Question 29

$$5y-15+10xy-30x=5(y-3+2xy-6x)=5(1(y-3)+2x(y-3))=5(y-3)(1+2x)$$

Question 30

$$\pi r^2+2\pi r-3r-6=\pi r(r+2)-3(r+2)=(r+2)(\pi r-3)$$

Exercise 1.10 Factorising trinomials

Question 1

$x^2+4x+3=(x+3)(x+1)$

Question 2

$y^2+7y+12=(y+3)(y+4)$

Question 3

$m^2+2m+1=(m+1)(m+1)=(m+1)^2$

Question 4

$t^2+8t+16=(t+4)(t+4)=(t+4)^2$

Question 5

$z^2+z-6=(z+3)(z-2)$

Question 6

$x^2-5x-6=(x-6)(x+1)$

Question 7

$v^2-8v+15=(v-5)(v-3)$

Question 8

$t^2 - 6t + 9 = (t-3)(t-3) = (t-3)^2$

Question 9

$x^2 + 9x - 10 = (x+10)(x-1)$

Question 10

$y^2 - 10y + 21 = (y-7)(y-3)$

Question 11

$m^2 - 9m + 18 = (m-3)(m-6)$

Question 12

$y^2 + 9y - 36 = (y+12)(y-3)$

Question 13

$x^2 - 5x - 24 = (x-8)(x+3)$

Question 14

$a^2 - 4a + 4 = (a-2)(a-2) = (a-2)^2$

Question 15

$x^2 + 14x - 32 = (x+16)(x-2)$

Question 16

$y^2 - 5y - 36 = (y-9)(y+4)$

Question 17

$n^2 - 10n + 24 = (n-6)(n-4)$

Question 18

$x^2 - 10x + 25 = (x-5)(x-5) = (x-5)^2$

Question 19

$p^2 + 8p - 9 = (p+9)(p-1)$

Question 20

$k^2 - 7k + 10 = (k-5)(k-2)$

Question 21

$x^2 + x - 12 = (x+4)(x-3)$

Question 22

$m^2 - 6m - 7 = (m-7)(m+1)$

Question 23

$q^2 + 12q + 20 = (q+10)(q+2)$

Question 24

$d^2 - 4d - 5 = (d-5)(d+1)$

Exercise 1.11 Further trinomials

Question 1

$2a^2 + 11a + 5 = (2a + 1)(a + 5)$

Question 2

$5y^2 + 7y + 2 = (5y + 2)(y + 1)$

Question 3

$3x^2 + 10x + 7 = (3x + 7)(x + 1)$

Question 4

$3x^2 + 8x + 4 = (3x + 2)(x + 2)$

Question 5

$2b^2 - 5b + 3 = (2b - 3)(b - 1)$

Question 6

$7x^2 - 9x + 2 = (7x - 2)(x - 1)$

Question 7

$3y^2 + 5y - 2 = (3y - 1)(y + 2)$

Question 8

$2x^2 + 11x + 12 = (2x + 3)(x = 4)$

Question 9

$5p^2 + 13p - 6 = (5p - 2)(p + 3)$

Question 10

$6x^2 + 13x + 5 = (2x + 1)(3x + 5)$

Question 11

$2y^2 - 11y - 6 = (2y + 1)(y - 6)$

Question 12

$10x^2 + 3x - 1 = (2x + 1)(5x - 1)$

Question 13

$8t^2 - 14t + 3 = (2t - 3)(4t - 1)$

Question 14

$6x^2 - x - 12 = (2x - 3)(3x + 4)$

Question 15

$6y^2 + 47y - 8 = (y + 8)(6y - 1)$

Question 16

$4n^2 - 11n + 6 = (4n-3)(n-2)$

Question 17

$8t^2 + 18t - 5 = (4t-1)(2t+5)$

Question 18

$12q^2 + 23q + 10 = (3q+2)(4q+5)$

Question 19

$4r^2 + 11r - 3 = (4r-1)(r+3)$

Question 20

$4x^2 - 4x - 15 = (2x+3)(2x-5)$

Question 21

$6y^2 - 13y + 2 = (y-2)(6y-1)$

Question 22

$6p^2 - 5p - 6 = (3p+2)(2p-3)$

Question 23

$8x^2 + 31x + 21 = (8x+7)(x+3)$

Question 24

$12b^2-43b+36=(3b-4)(4b-9)$

Question 25

$6x^2-53x-9=(6x+1)(x-9)$

Question 26

$9x^2+30x+25=(3x+5)(3x+5)=(3x+5)^2$

Question 27

$16y^2+24y+9=(4y+3)(4y+3)=(4y+3)^2$

Question 28

$25k^2-20k+4=(5k-2)(5k-2)=(5k-2)^2$

Question 29

$36a^2-12a+1=(6a-1)(6a-1)=(6a-1)^2$

Question 30

$49m^2+84m+36=(7m+6)(7m+6)=(7m+6)^2$

Exercise 1.12 Perfect squares

Question 1

$y^2 - 2y + 1 = (y-1)(y-1) = (y-1)^2$

Question 2

$x^2 + 6x + 9 = (x+3)(x+3) = (x+3)^2$

Question 3

$m^2 + 10m + 25 = (m+5)(m+5) = (m+5)^2$

Question 4

$t^2 - 4t + 4 = (t-2)(t-2) = (t-2)^2$

Question 5

$x^2 - 12x + 36 = (x-6)(x-6) = (x-6)^2$

Question 6

$4x^2 + 12x + 9 = (2x+3)(2x+3) = (2x+3)^2$

Question 7

$16b^2 - 8b + 1 = (4b-1)(4b-1) = (4b-1)^2$

Question 8

$9a^2 + 12a + 4 = (3a+2)(3a+2) = (3a+2)^2$

Question 9

$25x^2 - 40x + 16 = (5x-4)(5x-4) = (5x-4)^2$

Question 10

$49y^2 + 14y + 1 = (7y+1)(7y+1) = (7y+1)^2$

Question 11

$9y^2 - 30y + 25 = (3y-5)(3y-5) = (3y-5)^2$

Question 12

$16k^2 - 24k + 9 = (4k-3)(4k-3) = (4k-3)^2$

Question 13

$25x^2 + 10x + 1 = (5x+1)(5x+1) = (5x+1)^2$

Question 14

$81a^2 - 36a + 4 = (9a-2)(9a-2) = (9a-2)^2$

Question 15

$49m^2 + 84m + 36 = (7m+6)(7m+6) = (7m+6)^2$

Question 16

$$t^2+t+\frac{1}{4}=\left(t+\frac{1}{2}\right)\left(t+\frac{1}{2}\right)=\left(t+\frac{1}{2}\right)^2$$

Question 17

$$x^2-\frac{4x}{3}+\frac{4}{9}=\left(x-\frac{2}{3}\right)\left(x-\frac{2}{3}\right)=\left(x-\frac{2}{3}\right)^2$$

Question 18

$$9y^2+\frac{6y}{5}+\frac{1}{25}=\left(3y+\frac{1}{5}\right)\left(3y+\frac{1}{5}\right)=\left(3y+\frac{1}{5}\right)^2$$

Question 19

$$x^2+2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)=\left(x+\frac{1}{x}\right)^2$$

Question 20

$$25k^2-20+\frac{4}{k^2}=\left(5k-\frac{2}{k}\right)\left(5k-\frac{2}{k}\right)=\left(5k-\frac{2}{k}\right)^2$$

Exercise 1.13 Difference of two squares

Question 1

$a^2 - 4 = (a+2)(a-2)$

Question 2

$x^2 - 9 = (x+3)(x-3)$

Question 3

$y^2 - 1 = (y+1)(y-1)$

Question 4

$x^2 - 25 = (x-5)(x+5)$

Question 5

$4x^2 - 49 = (2x-7)(2x+7)$

Question 6

$16y^2 - 9 = (4y-3)(4y+3)$

Question 7

$1 - 4z^2 = (1-2z)(1+2z)$

Question 8

$25t^2-1=(5t-1)(5t+1)$

Question 9

$9t^2-4=(3t-2)(3t+2)$

Question 10

$9-16x^2=(3-4x)(3+4x)$

Question 11

$x^2-4y^2=(x-2y)(x+2y)$

Question 12

$36x^2-y^2=(6x-y)(6x+y)$

Question 13

$4a^2-9b^2=(2a-3b)(2a+3b)$

Question 14

$x^2-100y^2=(x-10y)(x+10y)$

Question 15

$4a^2-81b^2=(2a-9b)(2a+9b)$

Question 16

$$(x+2)^2-y^2=((x+2)-y)((x+2)+y)=(x+2-y)(x+2+y)$$

Question 17

$$\begin{aligned}(a-1)^2-(b-2)^2&=((a-1)-(b-2))((a-1)+(b-2))\\&=(a-1-b+2)(a-1+b-2)\\&=(a+1-b)(a-3+b)\end{aligned}$$

Question 18

$$z^2-(1+w)^2=(z-(1+w))(z+(1+w))=(z-1-w)(z+1+w)$$

Question 19

$$x^2-\frac{1}{4}=\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)$$

Question 20

$$\frac{y^2}{9}-1=\left(\frac{y}{3}+1\right)\left(\frac{y}{3}-1\right)$$

Question 21

$$\begin{aligned}(x+2)^2-(2y+1)^2&=((x+2)-(2y+1))((x+2)+(2y+1))\\&=(x+2-2y-1)(x+2+2y+1)\\&=(x+1-2y)(x+3+2y)\end{aligned}$$

Question 22

$x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x - 1)(x + 1)$

Question 23

$9x^6 - 4y^2 = \left(3x^3 - 2y\right)\left(3x^3 + 2y\right)$

Question 24

$x^4 - 16y^4 = \left(x^2 - 4y^2\right)\left(x^2 + 4y^2\right) = \left(x - 2y\right)\left(x + 2y\right)\left(x^2 + 4y^2\right)$

Exercise 1.14 Mixed factorisation

Question 1

$$4a^3 - 36a = 4a\left(a^2 - 9\right) = 4a(a+3)(a-3)$$

Question 2

$$2x^2 - 18 = 2\left(x^2 - 9\right) = 2(x+3)(x-3)$$

Question 3

$$3p^2 - 3p - 36 = 3\left(p^2 - p - 12\right) = 3(p-4)(p+3)$$

Question 4

$$5y^2 - 5 = 5\left(y^2 - 1\right) = 5(y-1)(y+1)$$

Question 5

$$5a^2 - 10a + 5 = 5\left(a^2 - 2a + 1\right) = 5(a-1)(a-1) = 5(a-1)^2$$

Question 6

$$3z^3 + 27z^2 + 60z = 3z\left(z^2 + 9z + 20\right) = 3z(z+5)(z+4)$$

Question 7

$$9ab - 4a^3b^3 = ab\left(9 - 4a^2b^2\right) = ab(3-2ab)(3+2ab)$$

Question 8

$x^3 - x = x(x^2 - 1) = x(x-1)(x+1)$

Question 9

$6x^2 + 8x - 8 = 2(3x^2 + 4x - 4) = 2(3x-2)(x+2)$

Question 10

$y^2(y+5) - 16(y+5) = (y+5)(y^2 - 16) = (y+5)(y-4)(y+4)$

Question 11

$$\begin{aligned} x^4 + 8x^3 - x^2 - 8x &= x^3(x+8) - x(x+8) \\ &= (x^3 - x)(x+8) \\ &= x(x^2-1)(x+8) \\ &= x(x-1)(x+1)(x+8) \end{aligned}$$

Question 12

$y^6 - 4 = (y^3 - 2)(y^3 + 2)$

Question 13

$x^3 - 3x^2 - 10x = x(x^2 - 3x - 10) = x(x-5)(x+2)$

Question 14

$$\begin{aligned} x^3 - 3x^2 - 9x + 27 &= x^2(x-3) - 9(x-3) \\ &= (x-3)(x^2-9) \\ &= (x-3)(x-3)(x+3) \\ &= (x-3)^2(x+3) \end{aligned}$$

Question 15

$$4x^2y^3 - y = y(4x^2y^2 - 1) = y(2xy-1)(2xy+1)$$

Question 16

$$24 - 6b^2 = 6(4-b^2) = 6(2-b)(2+b)$$

Question 17

$$18x^2 + 33x - 30 = 3(6x^2 + 11x - 10) = 3(3x-2)(2x+5)$$

Question 18

$$3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)(x-1) = 3(x-1)^2$$

Question 19

$$\begin{aligned} x^3 + 2x^2 - 25x - 50 &= x^2(x+2) - 25(x+2) \\ &= (x+2)(x^2-25) \\ &= (x+2)(x-5)(x+5) \end{aligned}$$

Question 20

$$z^3 + 6z^2 + 9z = z\left(z^2 + 6z + 9\right) = z\left(z+3\right)\left(z+3\right) = z\left(z+3\right)^2$$

Question 21

$$3y^2 + 30y + 75 = 3\left(y^2 + 10y + 25\right) = 3\left(y+5\right)^2$$

Question 22

$$ab^2 - 9a = a\left(b^2 - 9\right) = a\left(b+3\right)\left(b-3\right)$$

Question 23

$$4k^3 + 40k^2 + 100k = 4k\left(k^2 + 10k + 25\right) = 4k\left(k+5\right)\left(k+5\right) = 4k\left(k+5\right)^2$$

Question 24

$$\begin{aligned} 3x^3 + 9x^2 - 3x - 9 &= 3\left(x^3 + 3x^2 - x - 3\right) \\ &= 3\left(x^2\left(x+3\right) - 1\left(x+3\right)\right) \\ &= 3\left(x+3\right)\left(x^2 - 1\right) \\ &= 3\left(x+3\right)\left(x+1\right)\left(x-1\right) \end{aligned}$$

Question 25

$$\begin{aligned} 4a^3b + 8a^2b^2 - 4ab^2 - 2a^2b &= 2ab\left(2a^2 + 4ab - 2b - a\right) \\ &= 2ab\left[2a\left(a+2b\right) - 1\left(a+2b\right)\right] \\ &= 2ab\left(2a-1\right)\left(a+2b\right) \end{aligned}$$

Exercise 1.15 Simplifying algebraic fractions

Question 1

$$\frac{5a+10}{5}=\frac{5(a+2)}{5}=a+2$$

Question 2

$$\frac{6t-3}{3}=\frac{3(2t-1)}{3}=2t-1$$

Question 3

$$\frac{8y+2}{6}=\frac{2(4y+1)}{6}=\frac{4y+1}{3}$$

Question 4

$$\frac{8}{4d-2}=\frac{8}{2(2d-1)}=\frac{4}{2d-1}$$

Question 5

$$\frac{x^2}{5x^2-2x}=\frac{x^2}{x(5x-2)}=\frac{x}{5x-2}$$

Question 6

$$\frac{y-4}{y^2-8y+16}=\frac{y-4}{(y-4)(y-4)}=\frac{1}{y-4}$$

Question 7

$$\frac{2ab-4a^2}{a^2-3a}=\frac{2a(b-2a)}{a(a-3)}=\frac{2(b-2a)}{a-3}$$

Question 8

$$\frac{s^2+s-2}{s^2+5s+6}=\frac{(s+2)(s-1)}{(s+2)(s+3)}=\frac{s-1}{s+3}$$

Question 9

$$\frac{b^4-1}{b^2-1}=\frac{(b^2-1)(b^2+1)}{b^2-1}=b^2+1$$

Question 10

$$\frac{2p^2+7p-15}{6p-9}=\frac{(2p-3)(p+5)}{3(2p-3)}=\frac{p+5}{3}$$

Question 11

$$\frac{a^2-1}{a^2+2a-3}=\frac{(a-1)(a+1)}{(a-1)(a+3)}=\frac{a+1}{a+3}$$

Question 12

$$\frac{3(x-2)+y(x-2)}{x^2-4}=\frac{(x-2)(3+y)}{(x-2)(x+2)}=\frac{3+y}{x+2}$$

Question 13

$$\frac{x^3+3x^2-9x-27}{x^2+6x+9}=\frac{x^2(x+3)-9(x+3)}{(x+3)(x+3)}=\frac{(x+3)(x^2-9)}{(x+3)(x+3)}=\frac{(x+3)(x-3)}{(x+3)}=x-3$$

Question 14

$$\frac{2p^2-3p-2}{2p^2+p}=\frac{(2p+1)(p-2)}{p(2p+1)}=\frac{p-2}{p}$$

Question 15

$$\frac{ay-ax+by-bx}{2ay-by-2ax+bx}=\frac{a(y-x)+b(y-x)}{y(2a-b)-x(2a-b)}=\frac{(y-x)(a+b)}{(2a-b)(y-x)}=\frac{a+b}{2a-b}$$

Exercise 1.16 Operations with algebraic fractions

Question 1

a $\frac{x}{2}+\frac{3x}{4}=\frac{2x}{4}+\frac{3x}{4}=\frac{5x}{4}$

b $\frac{y+1}{5}+\frac{2y}{3}=\frac{3(y+1)}{15}+\frac{10y}{15}=\frac{3y+3+10y}{15}=\frac{13y+3}{15}$

c $\frac{a+2}{3}-\frac{a}{4}=\frac{4(a+2)}{12}-\frac{3a}{12}=\frac{4a+8-3a}{12}=\frac{a+8}{12}$

d $\frac{p-3}{6}+\frac{p+2}{2}=\frac{p-3}{6}+\frac{3(p+2)}{6}=\frac{p-3+3p+6}{6}=\frac{4p+3}{6}$

e $\frac{x-5}{2}-\frac{x-1}{3}=\frac{3(x-5)}{6}+\frac{-2(x-1)}{6}=\frac{3x-15-2x+2}{6}=\frac{x-13}{6}$

Question 2

a

$$\frac{3x-6}{5}\times\frac{x-1}{3}=\frac{3(x-2)}{5}\times\frac{10}{x-2}=\frac{30(x-2)}{5(x-2)}=\frac{30}{5}=6$$

b

$$\frac{a^2-4}{3}\times\frac{5b}{a+2}=\frac{(a-2)(a+2)}{3}\times\frac{5b}{a+2}=\frac{5b(a-2)}{3}$$

c

$$\begin{aligned}\frac{t^2+3t-10}{xy^2}\div\frac{5t-10}{2xy}&=\frac{t^2+3t-10}{xy^2}\times\frac{2xy}{5t-10}\\&=\frac{(t+5)(t-2)}{xy^2}\times\frac{2xy}{5(t-2)}\\&=\frac{2(t+5)}{5y}\end{aligned}$$

d

$$\begin{aligned}\frac{2a-6}{2x+4}\times\frac{5x+10}{4}&=\frac{2(a-3)}{2(x+2)}\times\frac{5(x+2)}{4}\\&=\frac{5(a-3)}{4}\end{aligned}$$

e

$$\begin{aligned}\frac{5x+10-xy-2y}{15}\div\frac{7x+14}{3}&=\frac{5x+10-xy-2y}{15}\times\frac{3}{7x+14}\\&=\frac{5(x+2)-y(x+2)}{15}\times\frac{3}{7(x+2)}\\&=\frac{(x+2)(5-y)}{15}\times\frac{3}{7(x+2)}\\&=\frac{3(x+2)(5-y)}{15\times7(x+2)}\\&=\frac{5-y}{35}\end{aligned}$$

f

$$\frac{3}{b+2}\times\frac{b^2+2b}{6a-3}=\frac{3}{b+2}\times\frac{b(b+2)}{3(2a-1)}=\frac{b}{2a-1}$$

g

$$\begin{aligned}\frac{3ab^2}{5xy}\div\frac{12ab-6a}{x^2y+2xy^2}&=\frac{3ab^2}{5xy}\times\frac{x^2y+2xy^2}{12ab-6a}\\&=\frac{3ab^2}{5xy}\times\frac{xy(x+2y)}{6a(2b-1)}\\&=\frac{b^2(x+2y)}{10(2b-1)}\end{aligned}$$

h

$$\frac{ax-ay+bx-by}{x^2-y^2}\times\frac{x^2y+xy^2}{ab^2+a^2b}=\frac{a(x-y)+b(x-y)}{(x+y)(x-y)}\times\frac{xy(x+y)}{ab(b+a)}$$
$$=\frac{(a+b)(x-y)}{(x+y)(x-y)}\times\frac{xy(x+y)}{ab(b+a)}$$
$$=\frac{xy}{ab}$$

i

$$\frac{x^2-6x+9}{x^2-25}\div\frac{x^2-5x+6}{x^2+4x-5}=\frac{x^2-6x+9}{x^2-25}\times\frac{x^2+4x-5}{x^2-5x+6}$$
$$=\frac{(x-3)(x-3)}{(x+5)(x-5)}\times\frac{(x+5)(x-1)}{(x-3)(x-2)}$$
$$=\frac{(x-3)(x-1)}{(x-5)(x-2)}$$

j

$$\frac{p^2-4}{q^2+2q+1}\times\frac{5q+5}{3p+6}=\frac{(p+2)(p-2)}{(q+1)(q+1)}\times\frac{5(q+1)}{3(p+2)}$$
$$=\frac{5(p-2)}{3(q+1)}$$

Question 3

a

$$\frac{2}{x}+\frac{3}{x}=\frac{5}{x}$$

b

$$\begin{aligned}\frac{1}{x-1}-\frac{2}{x}&=\frac{x}{x(x-1)}-\frac{2(x-1)}{x(x-1)}\\&=\frac{x-2(x-1)}{x(x-1)}\\&=\frac{x-2x+2}{x(x-1)}\\&=\frac{-x+2}{x(x-1)}\end{aligned}$$

c

$$1+\frac{3}{a+b}=\frac{a+b}{a+b}+\frac{3}{a+b}=\frac{a+b+3}{a+b}$$

d

$$\begin{aligned}x-\frac{x^2}{x+2}&=\frac{x(x+2)}{x+2}-\frac{x^2}{x+2}\\&=\frac{x(x+2)-x^2}{x+2}\\&=\frac{x^2+2x-x^2}{x+2}\\&=\frac{2x}{x+2}\end{aligned}$$

e

$$p-q+\frac{1}{p+q}=\frac{(p-q)(p+q)}{p+q}+\frac{1}{p+q}=\frac{p^2-q^2+1}{p+q}$$

f

$$\begin{aligned}\frac{1}{x+1}+\frac{1}{x-3}&=\frac{x-3}{(x+1)(x-3)}+\frac{x+1}{(x-3)(x+1)}\\&=\frac{x-3+x+1}{(x-3)(x+1)}\\&=\frac{2x-2}{(x-3)(x+1)}\\&=\frac{2(x-1)}{(x-3)(x+1)}\end{aligned}$$

g

$$\begin{aligned}\frac{2}{x^2-4}-\frac{3}{x+2}&=\frac{2}{(x+2)(x-2)}-\frac{3(x-2)}{(x+2)(x-2)}\\&=\frac{2-3x+6}{(x-2)(x+2)}\\&=\frac{-3x+8}{(x-2)(x+2)}\end{aligned}$$

h

$$\begin{aligned}\frac{1}{a^2+2a+1}+\frac{1}{a+1}&=\frac{1}{(a+1)(a+1)}+\frac{a+1}{(a+1)(a+1)}\\&=\frac{1+a+1}{(a+1)(a+1)}\\&=\frac{a+2}{(a+1)^2}\end{aligned}$$

Question 4

a

$$\begin{aligned}\frac{a^2-5a}{y^2-4y+4}\div\frac{3a-15}{y^2-4}\times\frac{y^2-y-2}{5ay}&=\frac{a(a-5)}{(y-2)(y-2)}\times\frac{(y+2)(y-2)}{3(a-5)}\times\frac{(y+1)(y-2)}{5ay}\\&=\frac{(y+2)(y+1)}{15y}\end{aligned}$$

b

$$\begin{aligned}\frac{3}{x-3}+\frac{2x+8}{x^2-9}\times\frac{x^2+3x}{4x-16}&=\frac{3}{x-3}+\frac{2(x+4)}{(x-3)(x+3)}\times\frac{x(x+3)}{4(x-4)}\\&=\frac{3}{x-3}+\frac{x(x+4)}{2(x-3)(x-4)}\\&=\frac{6(x-4)}{2(x-3)(x-4)}+\frac{x(x+4)}{2(x-3)(x-4)}\\&=\frac{6(x-4)+x(x+4)}{2(x-3)(x-4)}\\&=\frac{6x-24+x^2+4x}{2(x-3)(x-4)}\\&=\frac{x^2+10x-24}{2(x-3)(x-4)}\end{aligned}$$

c

$$\begin{aligned}\frac{5b}{2b+6}\div\frac{b^2}{b^2+b-6}-\frac{b}{b+1}&=\frac{5b}{2(b+3)}\times\frac{(b+3)(b-2)}{b^2}-\frac{b}{b+1}\\&=\frac{5(b-2)}{2b}-\frac{b}{b+1}\\&=\frac{(5b-10)(b+1)}{2b(b+1)}-\frac{2b^2}{2b(b+1)}\\&=\frac{(5b^2-5b-10)-2b^2}{2b(b+1)}\\&=\frac{3b^2-5b-10}{2b(b+1)}\end{aligned}$$

d

$$\begin{aligned}\frac{x^2-8x+15}{5x^2+10x}\div\frac{x^2-9}{10x^2}\times\frac{x^2+5x+6}{2x-10}&=\frac{(x-3)(x-5)}{5x(x+2)}\times\frac{10x^2}{(x-3)(x+3)}\times\frac{(x+3)(x+2)}{2(x-5)}\\&=x\end{aligned}$$

Question 5

a

$$\frac{5}{x^2-4}-\frac{3}{x-2}-\frac{2}{x+2}=\frac{5}{(x-2)(x+2)}-\frac{3(x+2)}{(x-2)(x+2)}-\frac{2(x-2)}{(x-2)(x+2)}$$
$$=\frac{5-3(x+2)-2(x-2)}{(x-2)(x+2)}$$
$$=\frac{5-3x-6-2x+4}{(x-2)(x+2)}$$
$$=\frac{3-5x}{(x-2)(x+2)}$$

b

$$\frac{2}{p^2+pq}+\frac{3}{pq-q^2}=\frac{2}{p(p+q)}+\frac{3}{q(p-q)}$$
$$=\frac{2q(p-q)}{pq(p+q)(p-q)}+\frac{3p(p+q)}{pq(p-q)(p+q)}$$
$$=\frac{2q(p-q)+3p(p+q)}{pq(p+q)(p-q)}$$
$$=\frac{2pq-2q^2+3p^2+3pq}{pq(p+q)(p-q)}$$
$$=\frac{3p^2+5pq-2q^2}{pq(p+q)(p-q)}$$

c

$$\frac{a}{a+b}-\frac{b}{a-b}+\frac{1}{a^2-b^2}=\frac{a(a-b)}{(a+b)(a-b)}-\frac{b(a+b)}{(a-b)(a+b)}+\frac{1}{(a+b)(a-b)}$$
$$=\frac{a(a-b)-b(a+b)+1}{(a+b)(a-b)}$$
$$=\frac{a^2-ab-ab-b^2+1}{(a+b)(a-b)}$$
$$=\frac{a^2-2ab-b^2+1}{(a+b)(a-b)}$$

Exercise 1.17 Substitution

Question 1

a $ab = 3.1 \times (-2.3) = -7.1$

b $3b = 3 \times (-2.3) = -6.9$

c $5a^2 = 5 \times 3.1^2 = 48.1$

d $ab^3 = 3.1 \times (-2.3)^3 = -37.7$

e $(a+b)^2 = (3.1-2.3)^2 = 0.6$

f $\sqrt{a-b} = \sqrt{3.1+2.3} = 2.3$

g $-b^2 = -(-2.3)^2 = -5.3$

Question 2

$T = a+(n-1)d = -4+(18-1)\times 3 = -4+17\times 3 = 47$

Question 3

$y = mx + c = 3 \times (-2) + (-1) = -7$

Question 4

$h = 100t - 5t^2 = 100 \times 5 - 5 \times 5^2 = 375$

Question 5

$v = -gt = -9.8 \times 20 = -196$

Question 6

$y = 2^x + 3 = 2^{1.3} + 3 = 5.5$

Question 7

$$S = 2\pi r\left(r + h\right) = 2\pi \times 5\left(5 + 7\right) = 377$$

Question 8

$A = \pi r^2 = \pi \times 9.5^2 = A = 284$

Question 9

$u_n = ar^{n-1} = u_4 = 5 \times (-2^{4-1}) = -40$

Question 10

$$V = \frac{1}{3}lbh = \frac{1}{3} \times 4.7 \times 5.1 \times 6.5 = 51.935$$

Question 11

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - \left(-2\right)}{-1 - 3} = \frac{7}{-4} = -1\frac{3}{4}$$

Question 12

$$A = \frac{1}{2}h\left(a + b\right) = \frac{1}{2} \times 7 \times \left(2.5 + 3.9\right) = 22.4$$

Question 13

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3} \times \pi \times 7.6^3 = 1838.8$$

Question 14

$$v = u + at = \frac{1}{4} + \frac{3}{5} \times \frac{5}{6} = \frac{3}{4}$$

Question 15

$$S = \frac{a}{1-r} = \frac{5}{1-\frac{2}{3}} = 15$$

Question 16

$$c = \sqrt{a^2 + b^2} = \sqrt{6^2 + 8^2} = 10$$

Question 17

$$y = \sqrt{16 - x^2} = \sqrt{16 - 2^2} = \sqrt{12} = 2\sqrt{3}$$

Question 18

$$E = mc^2 = 8.3 \times 1.7^2 = 23.987$$

Question 19

$$A = P\left(1 + \frac{r}{100}\right)^n = 200\left(1 + \frac{12}{100}\right)^5 = 352.47$$

Question 20

$$S = \frac{a\left(r^n - 1\right)}{r - 1} = \frac{3\left(2^5 - 1\right)}{2 - 1} = 93$$

Exercise 1.18 Simplifying surds

Question 1

a $\sqrt{12}=\sqrt{4\times3}=\sqrt{4}\sqrt{3}=2\sqrt{3}$

b $\sqrt{63}=\sqrt{9\times7}=\sqrt{9}\sqrt{7}=3\sqrt{7}$

c $\sqrt{24}=\sqrt{4\times6}=\sqrt{4}\sqrt{6}=2\sqrt{6}$

d $\sqrt{50}=\sqrt{25\times2}=\sqrt{25}\sqrt{2}=5\sqrt{2}$

e $\sqrt{72}=\sqrt{36\times2}=\sqrt{36}\sqrt{2}=6\sqrt{2}$

f $\sqrt{200}=\sqrt{100\times2}=\sqrt{100}\sqrt{2}=10\sqrt{2}$

g $\sqrt{48}=\sqrt{16\times3}=\sqrt{16}\sqrt{3}=4\sqrt{3}$

h $\sqrt{75}=\sqrt{25\times3}=\sqrt{25}\sqrt{3}=5\sqrt{3}$

i $\sqrt{32}=\sqrt{16\times2}=\sqrt{16}\sqrt{2}=4\sqrt{2}$

j $\sqrt{54}=\sqrt{9\times6}=\sqrt{9}\sqrt{6}=3\sqrt{6}$

k $\sqrt{112}=\sqrt{16\times7}=\sqrt{16}\sqrt{7}=4\sqrt{7}$

l $\sqrt{300}=\sqrt{100\times3}=\sqrt{100}\sqrt{3}=10\sqrt{3}$

m $\sqrt{128}=\sqrt{64\times2}=\sqrt{64}\sqrt{2}=8\sqrt{2}$

n $\sqrt{243}=\sqrt{81\times3}=\sqrt{81}\sqrt{3}=9\sqrt{3}$

o $\sqrt{245}=\sqrt{49\times5}=\sqrt{49}\sqrt{5}=7\sqrt{5}$

p $\sqrt{108}=\sqrt{36\times3}=\sqrt{36}\sqrt{3}=6\sqrt{3}$

q $\sqrt{99}=\sqrt{9\times11}=\sqrt{9}\sqrt{11}=3\sqrt{11}$

r $\sqrt{125}=\sqrt{25\times5}=\sqrt{25}\sqrt{5}=5\sqrt{5}$

Question 2

a $2\sqrt{27} = 2\sqrt{9\times3} = 2\sqrt{9}\sqrt{3} = 2\times3\sqrt{3} = 6\sqrt{3}$

b $5\sqrt{80} = 5\sqrt{16\times5} = 5\sqrt{16}\sqrt{5} = 5\times4\sqrt{5} = 20\sqrt{5}$

c $4\sqrt{98} = 4\sqrt{49\times2} = 4\sqrt{49}\sqrt{2} = 4\times7\sqrt{2} = 28\sqrt{2}$

d $2\sqrt{28} = 2\sqrt{4\times7} = 2\sqrt{4}\sqrt{7} = 2\times2\sqrt{7} = 4\sqrt{7}$

e $8\sqrt{20} = 8\sqrt{4\times5} = 8\sqrt{4}\sqrt{5} = 8\times2\sqrt{5} = 16\sqrt{5}$

f $4\sqrt{56} = 4\sqrt{4\times14} = 4\sqrt{4}\sqrt{14} = 4\times2\sqrt{14} = 8\sqrt{14}$

g $8\sqrt{405} = 8\sqrt{81\times5} = 8\sqrt{81}\sqrt{5} = 8\times9\sqrt{5} = 72\sqrt{5}$

h $15\sqrt{8} = 15\sqrt{4\times2} = 15\sqrt{4}\sqrt{2} = 15\times2\sqrt{2} = 30\sqrt{2}$

i $7\sqrt{40} = 7\sqrt{4\times10} = 7\sqrt{4}\sqrt{10} = 7\times2\sqrt{10} = 14\sqrt{10}$

j $8\sqrt{45} = 8\sqrt{9\times5} = 8\sqrt{9}\sqrt{5} = 8\times3\sqrt{5} = 24\sqrt{5}$

Question 3

a $3\sqrt{2} = \sqrt{9}\sqrt{2} = \sqrt{9\times2} = \sqrt{18}$

b $2\sqrt{5} = \sqrt{4}\sqrt{5} = \sqrt{4\times5} = \sqrt{20}$

c $4\sqrt{11} = \sqrt{16}\sqrt{11} = \sqrt{16\times11} = \sqrt{176}$

d $8\sqrt{2} = \sqrt{64}\sqrt{2} = \sqrt{64\times2} = \sqrt{128}$

e $5\sqrt{3} = \sqrt{25}\sqrt{3} = \sqrt{25\times3} = \sqrt{75}$

f $4\sqrt{10} = \sqrt{16}\sqrt{10} = \sqrt{16\times10} = \sqrt{160}$

g $3\sqrt{13} = \sqrt{9}\sqrt{13} = \sqrt{9\times13} = \sqrt{117}$

h $7\sqrt{2} = \sqrt{49}\sqrt{2} = \sqrt{49\times2} = \sqrt{98}$

i $11\sqrt{3} = \sqrt{121}\sqrt{3} = \sqrt{121\times3} = \sqrt{363}$

j $12\sqrt{7} = \sqrt{144}\sqrt{7} = \sqrt{144\times7} = \sqrt{1008}$

Question 4

a $\sqrt{x} = 3\sqrt{5}$

RHS: $\sqrt{9}\sqrt{5} = \sqrt{9\times5} = \sqrt{45}$

$x = 45$

b $2\sqrt{3} = \sqrt{x}$

LHS: $\sqrt{4}\sqrt{3} = \sqrt{4\times3} = \sqrt{12}$

$x = 12$

c $3\sqrt{7} = \sqrt{x}$

LHS: $\sqrt{9}\sqrt{7} = \sqrt{9\times7} = \sqrt{63}$

$x = 63$

d $5\sqrt{2} = \sqrt{x}$

LHS: $\sqrt{25}\sqrt{2} = \sqrt{25\times2} = \sqrt{50}$

$x = 50$

e $2\sqrt{11} = \sqrt{x}$

LHS: $\sqrt{4}\sqrt{11} = \sqrt{4\times11} = \sqrt{44}$

$x = 44$

f $\sqrt{x} = 7\sqrt{3}$

RHS: $\sqrt{49}\sqrt{3} = \sqrt{49\times3} = \sqrt{147}$

$x = 147$

g $4\sqrt{19} = \sqrt{x}$

LHS: $\sqrt{16}\sqrt{19} = \sqrt{16\times19} = \sqrt{304}$

$x = 304$

h $\sqrt{x} = 6\sqrt{23}$

RHS: $\sqrt{36}\sqrt{23} = \sqrt{36\times23} = \sqrt{828}$

$x = 828$

i $5\sqrt{31} = \sqrt{x}$

LHS: $\sqrt{25}\sqrt{31} = \sqrt{25\times31} = \sqrt{775}$

$x = 775$

j $\sqrt{x} = 8\sqrt{15}$

RHS: $\sqrt{64}\sqrt{15} = \sqrt{64 \times 15} = \sqrt{960}$

$x = 960$

Exercise 1.19 Operations with surds

Question 1

a $\sqrt{5}+2\sqrt{5}=3\sqrt{5}$

b $3\sqrt{2}-2\sqrt{2}=\sqrt{2}$

c $\sqrt{3}+5\sqrt{3}=6\sqrt{3}$

d $7\sqrt{3}-4\sqrt{3}=3\sqrt{3}$

e $\sqrt{5}-4\sqrt{5}=-3\sqrt{5}$

f $4\sqrt{6}-\sqrt{6}=3\sqrt{6}$

g $\sqrt{2}-8\sqrt{2}=-7\sqrt{2}$

h $\sqrt{5}+4\sqrt{5}+3\sqrt{5}=8\sqrt{5}$

i $\sqrt{2}-2\sqrt{2}-3\sqrt{2}=-4\sqrt{2}$

j $\sqrt{5}+\sqrt{45}=\sqrt{5}+3\sqrt{5}=4\sqrt{5}$

k $\sqrt{8}-\sqrt{2}=2\sqrt{2}-\sqrt{2}=\sqrt{2}$

l $\sqrt{3}+\sqrt{48}=\sqrt{3}+4\sqrt{3}=5\sqrt{3}$

m $\sqrt{12}-\sqrt{27}=2\sqrt{3}-3\sqrt{3}=-\sqrt{3}$

n $\sqrt{50}-\sqrt{32}=5\sqrt{2}-4\sqrt{2}=\sqrt{2}$

o $\sqrt{28}+\sqrt{63}=2\sqrt{7}+3\sqrt{7}=5\sqrt{7}$

p $2\sqrt{8}-\sqrt{18}=4\sqrt{2}-3\sqrt{2}=\sqrt{2}$

q $3\sqrt{54}+2\sqrt{24}=9\sqrt{6}+4\sqrt{6}=13\sqrt{6}$

r $\sqrt{90}-5\sqrt{40}-2\sqrt{10}=3\sqrt{10}-10\sqrt{10}-2\sqrt{10}=-9\sqrt{10}$

s $4\sqrt{48}+3\sqrt{147}+5\sqrt{12}=16\sqrt{3}+21\sqrt{3}+10\sqrt{3}=47\sqrt{3}$

t $3\sqrt{2}+\sqrt{8}-\sqrt{12}=3\sqrt{2}+2\sqrt{2}-2\sqrt{3}=5\sqrt{2}-2\sqrt{3}$

u $\sqrt{63}-\sqrt{28}-\sqrt{50}=3\sqrt{7}-2\sqrt{7}-5\sqrt{2}=\sqrt{7}-5\sqrt{2}$

v $\sqrt{12}-\sqrt{45}-\sqrt{48}-\sqrt{5}=2\sqrt{3}-3\sqrt{5}-4\sqrt{3}-\sqrt{5}=-2\sqrt{3}-4\sqrt{5}$

Question 2

a $\sqrt{7}\times\sqrt{3}=\sqrt{21}$

b $\sqrt{3}\times\sqrt{5}=\sqrt{15}$

c $\sqrt{2}\times3\sqrt{3}=3\sqrt{6}$

d $5\sqrt{7}\times2\sqrt{2}=10\sqrt{14}$

e $-3\sqrt{3}\times2\sqrt{2}=-6\sqrt{6}$

f $5\sqrt{3}\times2\sqrt{3}=10\sqrt{9}=30$

g $-4\sqrt{5}\times3\sqrt{11}=-12\sqrt{55}$

h $2\sqrt{7}\times\sqrt{7}=2\sqrt{49}=14$

i $2\sqrt{3}\times5\sqrt{12}=10\sqrt{36}=60$

j $\sqrt{6}\times\sqrt{2}=\sqrt{12}=2\sqrt{3}$

k $\left(\sqrt{2}\right)^2=2$

l $\left(2\sqrt{7}\right)^2=4\times7=28$

m $\sqrt{3}\times\sqrt{5}\times\sqrt{2}=\sqrt{30}$

n $2\sqrt{3}\times\sqrt{7}\times\left(-\sqrt{5}\right)=-2\sqrt{105}$

o $\sqrt{2}\times\sqrt{6}\times3\sqrt{3}=3\sqrt{36}=18$

Question 3

a $\dfrac{4\sqrt{12}}{2\sqrt{2}}=2\sqrt{6}$

b $\dfrac{12\sqrt{18}}{3\sqrt{6}} = 4\sqrt{3}$

c $\dfrac{5\sqrt{8}}{10\sqrt{2}} = \dfrac{\sqrt{4}}{2} = \dfrac{2}{2} = 1$

d $\dfrac{16\sqrt{2}}{2\sqrt{12}} = \dfrac{8}{\sqrt{6}}$

e $\dfrac{10\sqrt{30}}{5\sqrt{10}} = 2\sqrt{3}$

f $\dfrac{2\sqrt{2}}{6\sqrt{20}} = \dfrac{1}{3\sqrt{10}}$

g $\dfrac{4\sqrt{2}}{8\sqrt{10}} = \dfrac{1}{2\sqrt{5}}$

h $\dfrac{\sqrt{3}}{3\sqrt{15}} = \dfrac{1}{3\sqrt{5}}$

i $\dfrac{\sqrt{2}}{\sqrt{8}} = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$

j $\dfrac{3\sqrt{15}}{6\sqrt{10}} = \dfrac{\sqrt{3}}{2\sqrt{2}}$

k $\dfrac{5\sqrt{12}}{5\sqrt{8}} = \dfrac{\sqrt{3}}{\sqrt{2}}$

l $\dfrac{15\sqrt{18}}{10\sqrt{10}} = \dfrac{3\sqrt{9}}{2\sqrt{5}} = \dfrac{9}{2\sqrt{5}}$

m $\dfrac{\sqrt{15}}{2\sqrt{6}} = \dfrac{\sqrt{5}}{2\sqrt{2}}$

n $\left(\sqrt{\dfrac{2}{3}}\right)^2 = \dfrac{2}{3}$

o $\left(\sqrt{\dfrac{5}{7}}\right)^2 = \dfrac{5}{7}$

Question 4

a $\sqrt{2}\left(\sqrt{5}+\sqrt{3}\right)=\sqrt{10}+\sqrt{6}$

b $\sqrt{3}\left(2\sqrt{2}-\sqrt{5}\right)=2\sqrt{6}-\sqrt{15}$

c $4\sqrt{3}\left(\sqrt{3}+2\sqrt{5}\right)=4\sqrt{9}+8\sqrt{15}=12+8\sqrt{15}$

d $\sqrt{7}\left(5\sqrt{2}-2\sqrt{3}\right)=5\sqrt{14}-2\sqrt{21}$

e $-\sqrt{3}\left(\sqrt{2}-4\sqrt{6}\right)=-\sqrt{6}+4\sqrt{18}=-\sqrt{6}+12\sqrt{2}$

f $\sqrt{3}\left(5\sqrt{11}+3\sqrt{7}\right)=5\sqrt{33}+3\sqrt{21}$

g $-3\sqrt{2}\left(\sqrt{2}+4\sqrt{3}\right)=-3\sqrt{4}-12\sqrt{6}=-6-12\sqrt{6}$

h $\sqrt{5}\left(\sqrt{5}-5\sqrt{3}\right)=\sqrt{25}-5\sqrt{15}=5-5\sqrt{15}$

i $\sqrt{3}\left(\sqrt{12}+\sqrt{10}\right)=\sqrt{36}+\sqrt{30}=6+\sqrt{30}$

j $2\sqrt{3}\left(\sqrt{18}+\sqrt{3}\right)=2\sqrt{54}+2\sqrt{9}=6\sqrt{6}+6$

k $-4\sqrt{2}\left(\sqrt{2}-3\sqrt{6}\right)=-4\sqrt{4}+12\sqrt{12}=-8+24\sqrt{3}$

l $-7\sqrt{5}\left(-3\sqrt{20}+2\sqrt{3}\right)=21\sqrt{100}-14\sqrt{15}=210-14\sqrt{15}$

m $10\sqrt{3}\left(\sqrt{2}-2\sqrt{12}\right)=10\sqrt{6}-20\sqrt{36}=10\sqrt{6}-120$

n $-\sqrt{2}\left(\sqrt{5}+2\right)=-\sqrt{10}-2\sqrt{2}$

o $2\sqrt{3}\left(2-\sqrt{12}\right)=4\sqrt{3}-2\sqrt{36}=4\sqrt{3}-12$

Question 5

a $\left(\sqrt{2}+3\right)\left(\sqrt{5}+3\sqrt{3}\right)=\sqrt{10}+3\sqrt{6}+3\sqrt{5}+9\sqrt{3}$

b $$\begin{aligned}\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{2}-\sqrt{7}\right)&=\sqrt{10}-\sqrt{35}-\sqrt{4}+\sqrt{14}\\&=\sqrt{10}-\sqrt{35}-2+\sqrt{14}\end{aligned}$$

c $$\begin{aligned}\left(\sqrt{2}+5\sqrt{3}\right)\left(2\sqrt{5}-3\sqrt{2}\right)&=2\sqrt{10}-3\sqrt{4}+10\sqrt{15}-15\sqrt{6}\\&=2\sqrt{10}-6+10\sqrt{15}-15\sqrt{6}\end{aligned}$$

d $$\begin{aligned}\left(3\sqrt{10}-2\sqrt{5}\right)\left(4\sqrt{2}+6\sqrt{6}\right)&=12\sqrt{20}+18\sqrt{60}-8\sqrt{10}-12\sqrt{30}\\&=24\sqrt{5}+36\sqrt{15}-8\sqrt{10}-12\sqrt{30}\end{aligned}$$

e $$\begin{aligned}\left(2\sqrt{5}-7\sqrt{2}\right)\left(\sqrt{5}-3\sqrt{2}\right)&=2\sqrt{25}-6\sqrt{10}-7\sqrt{10}+21\sqrt{4}\\&=10-13\sqrt{10}+42\\&=52-13\sqrt{10}\end{aligned}$$

f $$\begin{aligned}\left(\sqrt{5}+6\sqrt{2}\right)\left(3\sqrt{5}-\sqrt{3}\right)&=3\sqrt{25}-\sqrt{15}+18\sqrt{10}-6\sqrt{6}\\&=15-\sqrt{15}+18\sqrt{10}-6\sqrt{6}\end{aligned}$$

g $$\begin{aligned}\left(\sqrt{7}+\sqrt{3}\right)\left(\sqrt{7}-\sqrt{3}\right)&=\sqrt{49}-\sqrt{21}+\sqrt{21}-\sqrt{9}\\&=7-3\\&=4\end{aligned}$$

h $$\begin{aligned}\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)&=\sqrt{4}+\sqrt{6}-\sqrt{6}-\sqrt{9}\\&=2-3\\&=-1\end{aligned}$$

i $$\begin{aligned}\left(\sqrt{6}+3\sqrt{2}\right)\left(\sqrt{6}-3\sqrt{2}\right)&=\sqrt{36}-3\sqrt{12}+3\sqrt{12}-9\sqrt{4}\\&=6-18\\&=-12\end{aligned}$$

j $$\begin{aligned}\left(3\sqrt{5}+\sqrt{2}\right)\left(3\sqrt{5}-\sqrt{2}\right)&=9\sqrt{25}-3\sqrt{10}+3\sqrt{10}-\sqrt{4}\\&=45-2\\&=43\end{aligned}$$

k $\left(\sqrt{8}-\sqrt{5}\right)\left(\sqrt{8}+\sqrt{5}\right)=\sqrt{64}+\sqrt{40}-\sqrt{40}-\sqrt{25}$

$$=8-5$$
$$=3$$

l $\left(\sqrt{2}+9\sqrt{3}\right)\left(\sqrt{2}-9\sqrt{3}\right)=\sqrt{4}-9\sqrt{6}+9\sqrt{6}-81\sqrt{9}$

$$=2-243$$
$$=-241$$

m $\left(2\sqrt{11}+5\sqrt{2}\right)\left(2\sqrt{11}-5\sqrt{2}\right)=4\sqrt{121}-10\sqrt{22}+10\sqrt{22}-25\sqrt{4}$

$$=44-50$$
$$=-6$$

n $\left(\sqrt{5}+\sqrt{2}\right)^2=\left(\sqrt{5}+\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)$

$$=\sqrt{25}+\sqrt{10}+\sqrt{10}+\sqrt{4}$$
$$=5+2\sqrt{10}+2$$
$$=7+2\sqrt{10}$$

o $\left(2\sqrt{2}-\sqrt{3}\right)^2=\left(2\sqrt{2}-\sqrt{3}\right)\left(2\sqrt{2}-\sqrt{3}\right)$

$$=4\sqrt{4}-2\sqrt{6}-2\sqrt{6}+\sqrt{9}$$
$$=8-4\sqrt{6}+3$$
$$=11-4\sqrt{6}$$

p $\left(3\sqrt{2}+\sqrt{7}\right)^2=\left(3\sqrt{2}+\sqrt{7}\right)\left(3\sqrt{2}+\sqrt{7}\right)$

$$=9\sqrt{4}+3\sqrt{14}+3\sqrt{14}+\sqrt{49}$$
$$=18+6\sqrt{14}+7$$
$$=25+6\sqrt{14}$$

q $\left(2\sqrt{3}+3\sqrt{5}\right)^2=\left(2\sqrt{3}+3\sqrt{5}\right)\left(2\sqrt{3}+3\sqrt{5}\right)$

$$=4\sqrt{9}+6\sqrt{15}+6\sqrt{15}+9\sqrt{25}$$
$$=12+12\sqrt{15}+45$$
$$=57+12\sqrt{15}$$

r $\left(\sqrt{7}-2\sqrt{5}\right)^2=\left(\sqrt{7}-2\sqrt{5}\right)\left(\sqrt{7}-2\sqrt{5}\right)$

$$\begin{aligned}&=\sqrt{49}-2\sqrt{35}-2\sqrt{35}+4\sqrt{25}\\&=7-4\sqrt{35}+20\\&=27-4\sqrt{35}\end{aligned}$$

s $\left(2\sqrt{8}-3\sqrt{5}\right)^2=\left(2\sqrt{8}-3\sqrt{5}\right)\left(2\sqrt{8}-3\sqrt{5}\right)$

$$\begin{aligned}&=4\sqrt{64}-6\sqrt{40}-6\sqrt{40}+9\sqrt{25}\\&=32-12\sqrt{40}+45\\&=77-24\sqrt{10}\end{aligned}$$

t $\left(3\sqrt{5}+2\sqrt{2}\right)^2=\left(3\sqrt{5}+2\sqrt{2}\right)\left(3\sqrt{5}+2\sqrt{2}\right)$

$$\begin{aligned}&=9\sqrt{25}+6\sqrt{10}+6\sqrt{10}+4\sqrt{4}\\&=45+12\sqrt{10}+8\\&=53+12\sqrt{10}\end{aligned}$$

Question 6

a $\left(3\sqrt{2}\right)^2=9\times 2=18$

b $2\left(3\sqrt{2}\right)^3=2\times 27\times\sqrt{8}=54\times 2\sqrt{2}=108\sqrt{2}$

c $\left(2\times 3\sqrt{2}\right)^3=\left(6\sqrt{2}\right)^3=216\times\sqrt{8}=216\times 2\sqrt{2}=432\sqrt{2}$

d $\left(3\sqrt{2}+1\right)^2=\left(3\sqrt{2}+1\right)\left(3\sqrt{2}+1\right)$

$$\begin{aligned}&=9\sqrt{4}+3\sqrt{2}+3\sqrt{2}+1\\&=18+6\sqrt{2}+1\\&=19+6\sqrt{2}\end{aligned}$$

e $\left(3\sqrt{2}+3\right)\left(3\sqrt{2}-3\right)=9\sqrt{4}-9\sqrt{2}+9\sqrt{2}-9$

$$=18-9=9$$

Question 7

a $\left(2\sqrt{5}+1\right)^2 = a+\sqrt{b}$

$\text{LHS:}\left(2\sqrt{5}+1\right)\left(2\sqrt{5}+1\right)=4\sqrt{25}+2\sqrt{5}+2\sqrt{5}+1=20+4\sqrt{5}+1$

$=21+4\sqrt{5}=21+\sqrt{16}\sqrt{5}=21+\sqrt{80}$

$a=21, b=80$

b $\left(2\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}-3\sqrt{5}\right)=a+b\sqrt{10}$

$\text{LHS: }\left(2\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}-3\sqrt{5}\right)=2\sqrt{4}-6\sqrt{10}-\sqrt{10}+3\sqrt{25}$

$=4-7\sqrt{10}+15$

$=19-7\sqrt{10}$

$a=19, b=-7$

Question 8

a $\left(\sqrt{a+3}-2\right)\left(\sqrt{a+3}+2\right)=\sqrt{(a+3)^2}+2\sqrt{a+3}-2\sqrt{a+3}-4$

$=a+3-4$

$=a-1$

b $\left(\sqrt{p-1}-\sqrt{p}\right)^2=\left(\sqrt{p-1}-\sqrt{p}\right)\left(\sqrt{p-1}-\sqrt{p}\right)$

$=\sqrt{(p-1)^2}-\sqrt{p(p-1)}-\sqrt{p(p-1)}+\sqrt{p^2}$

$=p-1-2\sqrt{p(p-1)}+p$

$=2p-1-2\sqrt{p(p-1)}$

Question 9

$\left(2\sqrt{7}-\sqrt{3}\right)\left(2\sqrt{7}+\sqrt{3}\right)=k$

$\text{LHS: }\left(2\sqrt{7}-\sqrt{3}\right)\left(2\sqrt{7}+\sqrt{3}\right)=4\sqrt{49}+2\sqrt{21}-2\sqrt{21}-\sqrt{9}$

$=28-3$

$=25$

$k=25$

Question 10

$$\left(2\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-3\sqrt{y}\right)=2\sqrt{x^2}-6\sqrt{xy}+\sqrt{xy}-3\sqrt{y^2}$$
$$=2x-5\sqrt{xy}-3y$$

Question 11

$$\left(2\sqrt{3}-\sqrt{5}\right)^2=a-\sqrt{b}$$
$$\text{LHS: }\left(2\sqrt{3}-\sqrt{5}\right)\left(2\sqrt{3}-\sqrt{5}\right)=4\sqrt{9}-2\sqrt{15}-2\sqrt{15}+\sqrt{25}$$
$$=12-4\sqrt{15}+5$$
$$=17-\sqrt{16}\sqrt{15}$$
$$=17-\sqrt{240}$$
$$a=17, b=240$$

Question 12

$$\left(7\sqrt{2}-3\right)^2=a+b\sqrt{2}$$
$$\text{LHS: }\left(7\sqrt{2}-3\right)\left(7\sqrt{2}-3\right)=49\sqrt{4}-21\sqrt{2}-21\sqrt{2}+9$$
$$=98-42\sqrt{2}+9$$
$$=107-42\sqrt{2}$$
$$a=107, b=-42$$

Exercise 1.20 Rationalising the denominator

Question 1

a $\dfrac{1}{\sqrt{7}}\times\dfrac{\sqrt{7}}{\sqrt{7}}=\dfrac{\sqrt{7}}{\sqrt{49}}=\dfrac{\sqrt{7}}{7}$

b $\dfrac{\sqrt{3}}{2\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{\sqrt{6}}{2\sqrt{4}}=\dfrac{\sqrt{6}}{4}$

c $\dfrac{2\sqrt{3}}{\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}=\dfrac{2\sqrt{15}}{\sqrt{25}}=\dfrac{2\sqrt{15}}{5}$

d $\dfrac{6\sqrt{7}}{5\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{6\sqrt{14}}{5\sqrt{4}}=\dfrac{6\sqrt{14}}{10}=\dfrac{3\sqrt{14}}{5}$

e $\dfrac{1+\sqrt{2}}{\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}=\dfrac{\sqrt{3}+\sqrt{6}}{\sqrt{9}}=\dfrac{\sqrt{3}+\sqrt{6}}{3}$

f $\dfrac{\sqrt{6}-5}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{\sqrt{12}-5\sqrt{2}}{\sqrt{4}}=\dfrac{2\sqrt{3}-5\sqrt{2}}{2}$

g $\dfrac{\sqrt{5}+2\sqrt{2}}{\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}=\dfrac{\sqrt{25}+2\sqrt{10}}{\sqrt{25}}=\dfrac{5+2\sqrt{10}}{5}$

h $\dfrac{3\sqrt{2}-4}{2\sqrt{7}}\times\dfrac{\sqrt{7}}{\sqrt{7}}=\dfrac{3\sqrt{14}-4\sqrt{7}}{2\sqrt{49}}=\dfrac{3\sqrt{14}-4\sqrt{7}}{14}$

i $\dfrac{8+3\sqrt{2}}{4\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}=\dfrac{8\sqrt{5}+3\sqrt{10}}{4\sqrt{25}}=\dfrac{8\sqrt{5}+3\sqrt{10}}{20}$

j $\dfrac{4\sqrt{3}-2\sqrt{2}}{7\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}=\dfrac{4\sqrt{15}-2\sqrt{10}}{7\sqrt{25}}=\dfrac{4\sqrt{15}-2\sqrt{10}}{35}$

Question 2

a

$$\frac{4}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{4\sqrt{3}-4\sqrt{2}}{\sqrt{9}-\sqrt{4}}$$
$$=\frac{4\sqrt{3}-4\sqrt{2}}{3-2}$$
$$=4\sqrt{3}-4\sqrt{2}$$
$$=4\left(\sqrt{3}-\sqrt{2}\right)$$

b

$$\frac{\sqrt{3}}{\sqrt{2}-7}\times\frac{\sqrt{2}+7}{\sqrt{2}+7}=\frac{\sqrt{6}+7\sqrt{3}}{\sqrt{4}-49}$$
$$=\frac{\sqrt{6}+7\sqrt{3}}{2-49}$$
$$=\frac{-\left(\sqrt{6}+7\sqrt{3}\right)}{47}$$

c

$$\frac{2\sqrt{3}}{\sqrt{5}+2\sqrt{6}}\times\frac{\sqrt{5}-2\sqrt{6}}{\sqrt{5}-2\sqrt{6}}=\frac{2\sqrt{15}-4\sqrt{18}}{\sqrt{25}-4\sqrt{36}}$$
$$=\frac{2\sqrt{15}-12\sqrt{2}}{5-24}$$
$$=\frac{-\left(2\sqrt{15}-12\sqrt{2}\right)}{19}$$
$$=\frac{-2\left(\sqrt{15}-6\sqrt{2}\right)}{19}$$

d

$$\frac{\sqrt{3}-4}{\sqrt{3}+4}\times\frac{\sqrt{3}-4}{\sqrt{3}-4}=\frac{\sqrt{9}-4\sqrt{3}-4\sqrt{3}+16}{\sqrt{9}-16}$$
$$=\frac{3-8\sqrt{3}+16}{3-16}$$
$$=\frac{19-8\sqrt{3}}{-13}$$
$$=\frac{8\sqrt{3}-19}{13}$$

e

$$\frac{\sqrt{2}+5}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\frac{\sqrt{6}+\sqrt{4}+5\sqrt{3}+5\sqrt{2}}{\sqrt{9}-\sqrt{4}}$$
$$=\frac{\sqrt{6}+2+5\sqrt{3}+5\sqrt{2}}{3-2}$$
$$=\sqrt{6}+2+5\sqrt{3}+5\sqrt{2}$$

f $$\frac{3\sqrt{3}+\sqrt{2}}{2\sqrt{5}+3\sqrt{2}}\times\frac{2\sqrt{5}-3\sqrt{2}}{2\sqrt{5}-3\sqrt{2}}=\frac{6\sqrt{15}-9\sqrt{6}+2\sqrt{10}-3\sqrt{4}}{4\sqrt{25-9\sqrt{4}}}$$
$$=\frac{6\sqrt{15}-9\sqrt{6}+2\sqrt{10}-6}{20-18}$$
$$=\frac{6\sqrt{15}-9\sqrt{6}+2\sqrt{10}-6}{2}$$

Question 3

a $$\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{2}-1}=\frac{\sqrt{2}-1}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}+\frac{\sqrt{2}+1}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}$$
$$=\frac{\sqrt{2}-1+\sqrt{2}+1}{\sqrt{4}-\sqrt{2}+\sqrt{2}-1}$$
$$=\frac{2\sqrt{2}}{2-1}=2\sqrt{2}$$

b $$\frac{\sqrt{2}}{\sqrt{2}-\sqrt{3}}-\frac{3}{\sqrt{2}+\sqrt{3}}=\frac{\sqrt{2}\left(\sqrt{2}+\sqrt{3}\right)}{\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)}-\frac{3\left(\sqrt{2}-\sqrt{3}\right)}{\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{2}+\sqrt{3}\right)}$$
$$=\frac{\left(\sqrt{4}+\sqrt{6}\right)-\left(3\sqrt{2}-3\sqrt{3}\right)}{\sqrt{4}-\sqrt{6}+\sqrt{6}-\sqrt{9}}$$
$$=\frac{2+\sqrt{6}-3\sqrt{2}+3\sqrt{3}}{2-3}$$
$$=-\left(2+\sqrt{6}-3\sqrt{2}+3\sqrt{3}\right)$$
$$=-2-\sqrt{6}+3\sqrt{2}-3\sqrt{3}$$

c $$t+\frac{1}{t},t=\sqrt{3}-2$$

$$\sqrt{3}-2+\frac{1}{\sqrt{3}-2}\times\frac{\sqrt{3}+2}{\sqrt{3}+2}=\sqrt{3}-2+\frac{\sqrt{3}+2}{\sqrt{9}-4}$$
$$=\sqrt{3}-2+\frac{\sqrt{3}+2}{3-4}$$
$$=\sqrt{3}-2+\frac{\sqrt{3}+2}{-1}$$
$$=\sqrt{3}-2-\sqrt{3}-2$$
$$=-4$$

d $z^2 + \frac{1}{z^2},\ z = 1 + \sqrt{2}$

$$\begin{aligned}\left(1+\sqrt{2}\right)^2 - \frac{1}{\left(1+\sqrt{2}\right)^2} &= \left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right) - \frac{1}{\left(1+\sqrt{2}\right)\left(1+\sqrt{2}\right)}\\
&= 1+\sqrt{2}+\sqrt{2}+\sqrt{4} - \frac{1}{1+\sqrt{2}+\sqrt{2}+\sqrt{4}}\\
&= 1+2\sqrt{2}+2 - \frac{1}{1+2\sqrt{2}+2}\\
&= 3+2\sqrt{2} - \frac{1}{3+2\sqrt{2}} \times \frac{3-2\sqrt{2}}{3-2\sqrt{2}}\\
&= 3+2\sqrt{2} - \frac{3-2\sqrt{2}}{9-4\sqrt{4}}\\
&= 3+2\sqrt{2} - \frac{3-2\sqrt{2}}{9-8}\\
&= 3+2\sqrt{2} - \frac{3-2\sqrt{2}}{1}\\
&= 3+2\sqrt{2} - 3+2\sqrt{2}\\
&= 4\sqrt{2}\end{aligned}$$

e

$$\begin{aligned}\frac{\sqrt{2}+3}{\sqrt{2}} + \frac{1}{\sqrt{3}} &= \frac{\sqrt{2}+3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} + \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\\
&= \frac{\sqrt{4}+3\sqrt{2}}{\sqrt{4}} + \frac{\sqrt{3}}{\sqrt{9}}\\
&= \frac{2+3\sqrt{2}}{2} + \frac{\sqrt{3}}{3}\\
&= \frac{3\left(2+3\sqrt{2}\right)}{6} + \frac{2\sqrt{3}}{6}\\
&= \frac{6+9\sqrt{2}+2\sqrt{3}}{6}\end{aligned}$$

f

$$\begin{aligned}\frac{\sqrt{3}}{\sqrt{2}+3}+\frac{\sqrt{2}}{\sqrt{3}} &= \frac{\sqrt{3}}{\sqrt{2}+3}\times\frac{\sqrt{2}-3}{\sqrt{2}-3}+\frac{\sqrt{2}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\
&= \frac{\sqrt{6}-3\sqrt{3}}{\sqrt{4}-9}+\frac{\sqrt{6}}{\sqrt{9}}\\
&= \frac{\sqrt{6}-3\sqrt{3}}{2-9}+\frac{\sqrt{6}}{3}\\
&= \frac{3\left(\sqrt{6}-3\sqrt{3}\right)}{3\times(-7)}+\frac{7\sqrt{6}}{7\times 3}\\
&= \frac{-3\sqrt{6}+9\sqrt{3}+7\sqrt{6}}{21}\\
&= \frac{4\sqrt{6}+9\sqrt{3}}{21}\end{aligned}$$

g

$$\begin{aligned}\frac{\sqrt{5}}{\sqrt{6}+2}-\frac{2}{5\sqrt{3}} &= \frac{\sqrt{5}}{\sqrt{6}+2}\times\frac{\sqrt{6}-2}{\sqrt{6}-2}-\frac{2}{5\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\
&= \frac{\sqrt{30}-2\sqrt{5}}{\sqrt{36}-4}-\frac{2\sqrt{3}}{5\sqrt{9}}\\
&= \frac{\sqrt{30}-2\sqrt{5}}{6-4}-\frac{2\sqrt{3}}{5\times 3}\\
&= \frac{15\left(\sqrt{30}-2\sqrt{5}\right)}{15\times 2}-\frac{2\times 2\sqrt{3}}{2\times 15}\\
&= \frac{15\sqrt{30}-30\sqrt{5}-4\sqrt{3}}{30}\end{aligned}$$

h

$$\begin{aligned}\frac{\sqrt{2}+7}{4+\sqrt{3}}-\frac{\sqrt{2}}{4-\sqrt{3}} &= \frac{\left(\sqrt{2}+7\right)\left(4-\sqrt{3}\right)}{\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)}-\frac{\sqrt{2}\left(4+\sqrt{3}\right)}{\left(4-\sqrt{3}\right)\left(4+\sqrt{3}\right)}\\
&= \frac{\left(4\sqrt{2}-\sqrt{6}+28-7\sqrt{3}\right)-\left(4\sqrt{2}+\sqrt{6}\right)}{\left(4-\sqrt{3}\right)\left(4+\sqrt{3}\right)}\\
&= \frac{4\sqrt{2}-\sqrt{6}+28-7\sqrt{3}-4\sqrt{2}-\sqrt{6}}{16-\sqrt{9}}\\
&= \frac{28-7\sqrt{3}-2\sqrt{6}}{16-3}\\
&= \frac{28-7\sqrt{3}-2\sqrt{6}}{13}\end{aligned}$$

i

$$\frac{\sqrt{5}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\frac{2+\sqrt{3}}{\sqrt{3}+1}=\frac{\sqrt{5}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}-\frac{2+\sqrt{3}}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$=\frac{\sqrt{15}+\sqrt{10}-\sqrt{6}-\sqrt{4}}{\sqrt{9}-\sqrt{4}}-\frac{2\sqrt{3}-2+\sqrt{9}-\sqrt{3}}{\sqrt{9}-1}$$

$$=\frac{\sqrt{15}+\sqrt{10}-\sqrt{6}-\sqrt{4}}{3-2}-\frac{\sqrt{3}-2+3}{3-1}$$

$$=\frac{2\left(\sqrt{15}+\sqrt{10}-\sqrt{6}-2\right)}{2\times1}-\frac{\sqrt{3}+1}{2}$$

$$=\frac{2\sqrt{15}+2\sqrt{10}-2\sqrt{6}-4-\sqrt{3}-1}{2}$$

$$=\frac{2\sqrt{15}+2\sqrt{10}-2\sqrt{6}-5-\sqrt{3}}{2}$$

Question 4

a

$$\frac{3}{2\sqrt{5}}=\frac{\sqrt{a}}{b}$$

LHS: $\frac{3}{2\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{3\sqrt{5}}{2\sqrt{25}}=\frac{\sqrt{9}\sqrt{5}}{2\times5}=\frac{\sqrt{45}}{10}$

$a=45,\ b=10$

b

$$\frac{\sqrt{3}}{4\sqrt{2}}=\frac{a\sqrt{6}}{b}$$

LHS: $\frac{\sqrt{3}}{4\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{6}}{4\sqrt{4}}=\frac{\sqrt{6}}{4\times2}=\frac{\sqrt{6}}{8}$

$a=1,\ b=8$

c

$$\frac{2}{\sqrt{5}+1}=a+b\sqrt{5}$$

LHS: $\frac{2}{\sqrt{5}+1}\times\frac{\sqrt{5}-1}{\sqrt{5}-1}=\frac{2\left(\sqrt{5}-1\right)}{\sqrt{25}-1}=\frac{2\left(\sqrt{5}-1\right)}{5-1}=\frac{2\left(\sqrt{5}-1\right)}{4}=\frac{\sqrt{5}}{2}-\frac{1}{2}$

$a=-\frac{1}{2}\quad b=\frac{1}{2}$

d $\dfrac{2\sqrt{7}}{\sqrt{7}-4}=a+b\sqrt{7}$

LHS: $\dfrac{2\sqrt{7}}{\sqrt{7}-4}\times\dfrac{\sqrt{7}+4}{\sqrt{7}+4}=\dfrac{2\sqrt{49}+8\sqrt{7}}{\sqrt{49}-16}=\dfrac{14+8\sqrt{7}}{7-16}=\dfrac{14+8\sqrt{7}}{-9}=-\dfrac{14}{9}-\dfrac{8\sqrt{7}}{9}$

$a=-1\dfrac{5}{9},\ b=-\dfrac{8}{9}$

e $\dfrac{\sqrt{2}+3}{\sqrt{2}-1}=a+\sqrt{b}$

LHS:
$$\begin{aligned}\frac{\sqrt{2}+3}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}&=\frac{\sqrt{4}+\sqrt{2}+3\sqrt{2}+3}{\sqrt{4}-1}\\&=\frac{2+4\sqrt{2}+3}{2-1}\\&=\frac{5+\sqrt{16}\sqrt{2}}{1}\\&=5+\sqrt{32}\end{aligned}$$

$a=5,\ \ b=32$

Question 5

$$\begin{aligned}\frac{\sqrt{2}-1}{\sqrt{2}+1}+\frac{4}{\sqrt{2}}&=\frac{\sqrt{2}-1}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}+\frac{4}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\\&=\frac{\left(\sqrt{2}-1\right)\left(\sqrt{2}-1\right)}{\sqrt{4}-1}+\frac{4\sqrt{2}}{\sqrt{4}}\\&=\frac{2-\sqrt{2}-\sqrt{2}+1}{2-1}+\frac{4\sqrt{2}}{2}\\&=\frac{3-2\sqrt{2}}{1}+2\sqrt{2}\\&=3-2\sqrt{2}+2\sqrt{2}\\&=3\end{aligned}$$

So rational

Question 6

a $x+\dfrac{1}{x},\ x=\sqrt{3}+2$

$$\begin{aligned}
\sqrt{3}+2+\frac{1}{\sqrt{3}+2} &= \sqrt{3}+2+\frac{1}{\sqrt{3}+2}\times\frac{\sqrt{3}-2}{\sqrt{3}-2} \\
&= \sqrt{3}+2+\frac{\sqrt{3}-2}{\sqrt{9}-4} \\
&= \sqrt{3}+2+\frac{\sqrt{3}-2}{3-4} \\
&= \sqrt{3}+2+\frac{\sqrt{3}-2}{-1} \\
&= \sqrt{3}+2-\sqrt{3}+2 \\
&= 4
\end{aligned}$$

b $x^2+\dfrac{1}{x^2},\ x=\sqrt{3}+2$

$$\begin{aligned}
\left(\sqrt{3}+2\right)^2+\frac{1}{\left(\sqrt{3}+2\right)^2} &= \left(\sqrt{3}+2\right)\left(\sqrt{3}+2\right)+\frac{1}{\left(\sqrt{3}+2\right)\left(\sqrt{3}+2\right)} \\
&= \sqrt{9}+2\sqrt{3}+2\sqrt{3}+4+\frac{1}{\sqrt{9}+2\sqrt{3}+2\sqrt{3}+4} \\
&= 3+4\sqrt{3}+4+\frac{1}{3+4\sqrt{3}+4} \\
&= 7+4\sqrt{3}+\frac{1}{7+4\sqrt{3}}\times\frac{7-4\sqrt{3}}{7-4\sqrt{3}} \\
&= 7+4\sqrt{3}+\frac{7-4\sqrt{3}}{49-16\sqrt{9}} \\
&= 7+4\sqrt{3}+\frac{7-4\sqrt{3}}{49-48} \\
&= 7+4\sqrt{3}+\frac{7-4\sqrt{3}}{1} \\
&= 7+4\sqrt{3}+7-4\sqrt{3} \\
&= 14
\end{aligned}$$

c $\left(x+\frac{1}{x}\right)^2, \ x=\sqrt{3}+2$

$$\begin{aligned}\left(\sqrt{3}+2+\frac{1}{\sqrt{3}+2}\right)^2 &= \left(\sqrt{3}+2+\frac{1}{\sqrt{3}+2}\times\frac{\sqrt{3}-2}{\sqrt{3}-2}\right)^2 \\ &= \left(\sqrt{3}+2+\frac{\sqrt{3}-2}{\sqrt{9}-4}\right)^2 \\ &= \left(\sqrt{3}+2+\frac{\sqrt{3}-2}{3-4}\right)^2 \\ &= \left(\sqrt{3}+2+\frac{\sqrt{3}-2}{-1}\right)^2 \\ &= \left(\sqrt{3}+2-\sqrt{3}+2\right)^2 \\ &= (4)^2 \\ &= 16\end{aligned}$$

Test yourself 1

Question 1

$$\frac{\sqrt{3}}{2\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}}=\frac{\sqrt{21}}{2\times 7}$$
$$=\frac{\sqrt{21}}{14}$$
$$=\frac{2\sqrt{21}}{28}$$

$\therefore$ B, C

Question 2

$$\frac{x-3}{5}-\frac{x+1}{4}=\frac{4(x-3)}{20}-\frac{5(x+1)}{20}$$
$$=\frac{4x-12-5x-5}{20}$$
$$=\frac{-x-17}{20}$$
$$=\frac{-(x+17)}{20}$$

$\therefore$ D

Question 3

$$x^3-4x^2-x+4=x^2(x-4)-(x-4)$$
$$=(x^2-1)(x-4)$$
$$=(x+1)(x-1)(x-4)$$

$\therefore$ A, D

Question 4

$$\begin{aligned}3\sqrt{2}+2\sqrt{98} &= 3\sqrt{2}+2\sqrt{49\times 2}\\ &= 3\sqrt{2}+2\sqrt{49}\sqrt{2}\\ &= 3\sqrt{2}+14\sqrt{2}\\ &= 17\sqrt{2}\end{aligned}$$

$\therefore$ C

Question 5

$$\begin{aligned}\frac{3}{x^2-4}+\frac{2}{x-2}-\frac{1}{x+2} &= \frac{3}{(x+2)(x-2)}+\frac{2(x+2)}{(x-2)(x+2)}-\frac{1(x-2)}{(x+2)(x-2)}\\ &= \frac{3+2x+4-x+2}{(x+2)(x-2)}\\ &= \frac{x+9}{(x+2)(x-2)}\end{aligned}$$

$\therefore$ C

Question 6

$5ab-2a^2-7ab-3a^2=-5a^2-2ab$

$\therefore$ B

Question 7

$$\frac{\sqrt{80}}{\sqrt{27}}=\frac{\sqrt{16\times 5}}{\sqrt{9\times 3}}=\frac{\sqrt{16}\sqrt{5}}{\sqrt{9}\sqrt{3}}=\frac{4\sqrt{5}}{3\sqrt{3}}$$

$\therefore$ A

Question 8

$$\begin{aligned}(3x-2y)^2 &= (3x-2y)(3x-2y)\\ &= 9x^2-6xy-6xy+4y^2\\ &= 9x^2-12xy+4y^2\end{aligned}$$

$\therefore$ D

Question 9

a $7^{-2}=\frac{1}{7^2}=\frac{1}{49}$

b $5^{-1}=\frac{1}{5}=\frac{1}{5}$

c $9^{-\frac{1}{2}}=\frac{1}{9^{\frac{1}{2}}}=\frac{1}{\sqrt{9}}=\frac{1}{3}$

Question 10

a $x^5\times x^7\div x^3=x^{5+7-3}=x^9$

b $\left(5y^3\right)^2=5^2y^{3\times 2}=25y^6$

c $\frac{\left(a^5\right)^4b^7}{a^9b}=\frac{a^{20}b^7}{a^9b}=a^{11}b^6$

d $\left(\frac{2x^6}{3}\right)^3=\frac{2^3x^{6\times 3}}{3^3}=\frac{8x^{18}}{27}$

e $\left(\frac{ab^4}{a^5b^6}\right)^0=1$

Question 11

a $36^{\frac{1}{2}} = \sqrt{36} = 6$

b $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

c $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

d $49^{-\frac{1}{2}} = \frac{1}{49^{\frac{1}{2}}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

e $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$

f $(-3)^0 = 1$

Question 12

a $a^{14} \div a^9 = a^{14-9} = a^5$

b $\left(x^5 y^3\right)^6 = x^{5\times 6} y^{3\times 6} = x^{30} y^{18}$

c $p^6 \times p^5 \div p^2 = p^{6+5-2} = p^9$

d $\left(2b^9\right)^4 = 2^4 b^{9\times 4} = 16b^{36}$

e $\frac{\left(2x^7\right)^3 y^2}{x^{10} y} = \frac{2^3 x^{21} y^2}{x^{10} y} = 8x^{11} y$

Question 13

a $\sqrt{n} = n^{\frac{1}{2}}$

b $\dfrac{1}{x^5} = x^{-5}$

c $\dfrac{1}{x+y} = (x+y)^{-1}$

d $\sqrt[4]{x+1} = (x+1)^{\frac{1}{4}}$

e $\sqrt[7]{a+b} = (a+b)^{\frac{1}{7}}$

f $\dfrac{2}{x} = 2x^{-1}$

g $\dfrac{1}{2x^3} = \dfrac{x^{-3}}{2} = \dfrac{1}{2}x^{-3}$

h $\sqrt[3]{x^4} = x^{\frac{4}{3}}$

i $\sqrt[7]{(5x+3)^9} = (5x+3)^{\frac{9}{7}}$

j $\dfrac{1}{\sqrt[4]{m^3}} = \dfrac{1}{m^{\frac{3}{4}}} = m^{-\frac{3}{4}}$

Question 14

a $a^{-5} = \frac{1}{a^5}$

b $n^{\frac{1}{4}} = \sqrt[4]{n}$

c $(x+1)^{\frac{1}{2}} = \sqrt{x+1}$

d $(x-y)^{-1} = \frac{1}{x-y}$

e $(4t-7)^{-4} = \frac{1}{(4t-7)^4}$

f $(a+b)^{\frac{1}{5}} = \sqrt[5]{a+b}$

g $x^{-\frac{1}{3}} = \frac{1}{x^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{x}}$

h $b^{\frac{3}{4}} = \left(\sqrt[4]{b}\right)^3 = \sqrt[4]{b^3}$

i $(2x+3)^{\frac{4}{3}} = \left(\sqrt[3]{2x+3}\right)^4 = \sqrt[3]{(2x+3)^4}$

j $x^{-\frac{3}{2}} = \frac{1}{x^{\frac{3}{2}}} = \frac{1}{\left(\sqrt{x}\right)^3} = \frac{1}{\sqrt{x^3}}$

Question 15

$$\left(\frac{9}{25}\right)^2\left(1\frac{2}{3}\right)^4 = \frac{9^2}{25^2}\times\left(\frac{5}{3}\right)^4 = \frac{9^2}{25^2}\times\frac{5^4}{3^4} = 1$$

Question 16

$$\left(\frac{1}{3}\right)^4\left(\frac{3}{4}\right)^3 = \frac{1}{3^4}\times\frac{3^3}{4^3} = \frac{1}{3\times 4^3} = \frac{1}{192}$$

Question 17

a $\sqrt{x} = x^{\frac{1}{2}}$

b $\frac{1}{y} = y^{-1}$

c $\sqrt[6]{x+3} = (x+3)^{\frac{1}{6}}$

d $\frac{1}{(2x-3)^{11}} = (2x-3)^{-11}$

e $\sqrt[3]{y^7} = y^{\frac{7}{3}}$

Question 18

a $x^{-3} = \frac{1}{x^3}$

b $(2a+5)^{-1} = \frac{1}{2a+5}$

c $\left(\frac{a}{b}\right)^{-5} = \left(\frac{b}{a}\right)^{5}$

Question 19

a $5y - 7y = -2y$

b $\dfrac{3a+12}{3} = \dfrac{3(a+4)}{3} = a+4$

c $-2k^3 \times 3k^2 = -6k^5$

d $\dfrac{x}{3} + \dfrac{y}{5} = \dfrac{5x}{15} + \dfrac{3y}{15} = \dfrac{5x+3y}{15}$

e $4a - 3b - a - 5b = 3a - 8b$

f $\sqrt{8} + \sqrt{32} = 2\sqrt{2} + 4\sqrt{2} = 6\sqrt{2}$

g $3\sqrt{5} - \sqrt{20} + \sqrt{45} = 3\sqrt{5} - 2\sqrt{5} + 3\sqrt{5} = 4\sqrt{5}$

Question 20

a $x^2 - 36 = (x+6)(x-6)$

b $a^2 + 2a - 3 = (a+3)(a-1)$

c $4ab^2 - 8ab = 4ab(b-2)$

d $5y - 15 + xy - 3x = 5(y-3) + x(y-3) = (y-3)(5+x)$

e $4n - 2p + 6 = 2(2n - p + 3)$

Question 21

a $b+3(b-2)=b+3b-6=4b-6$

b $(2x-1)(x+3)=2x^2+6x-x-3=2x^2+5x-3$

c $5(m+3)-(m-2)=5m+15-m+2=4m+17$

d $(4x-3)^2=(4x-3)(4x-3)=16x^2-12x-12x+9=16x^2-24x+9$

e $(p-5)(p+5)=p^2-5p+5p-25=p^2-25$

f $7-2(a+4)-5a=7-2a-8-5a=-7a-1$

g $\sqrt{3}\left(2\sqrt{2}-5\right)=2\sqrt{6}-5\sqrt{3}$

h $\left(3+\sqrt{7}\right)\left(\sqrt{3}-2\right)=3\sqrt{3}-6+\sqrt{21}-2\sqrt{7}$

Question 22

a $\dfrac{4a-12}{5b^3}\times\dfrac{10b}{a^2-9}=\dfrac{4(a-3)}{5b^3}\times\dfrac{10b}{(a-3)(a+3)}=\dfrac{8}{b^2(a+3)}$

b $\dfrac{5m+10}{m^2-m-2}\div\dfrac{m^2-4}{3m+3}=\dfrac{5(m+2)}{(m-2)(m+1)}\times\dfrac{3(m+1)}{(m+2)(m-2)}=\dfrac{15}{(m-2)^2}$

Question 23

$V=s^3=54^{3}=15.464$ units^{3}

Question 24

a $\left(2\sqrt{5}+\sqrt{3}\right)\left(2\sqrt{5}-\sqrt{3}\right)=4\sqrt{25}-2\sqrt{15}+2\sqrt{15}-\sqrt{9}$

$$=20-3$$

$$=17$$

b

$$\frac{3\sqrt{3}}{2\sqrt{5}+\sqrt{3}}\times\frac{2\sqrt{5}-\sqrt{3}}{2\sqrt{5}-\sqrt{3}}=\frac{6\sqrt{15}-3\sqrt{9}}{4\sqrt{25}-2\sqrt{15}+2\sqrt{15}-\sqrt{9}}$$

$$=\frac{6\sqrt{15}-9}{20-3}$$

$$=\frac{6\sqrt{15}-9}{17}$$

Question 25

$$\frac{3}{x-2}+\frac{1}{x+3}-\frac{2}{x^2+x-6}=\frac{3\left(x+3\right)}{\left(x-2\right)\left(x+3\right)}+\frac{1\left(x-2\right)}{\left(x+3\right)\left(x-2\right)}-\frac{2}{\left(x+3\right)\left(x-2\right)}$$

$$=\frac{3x+9+x-2-2}{\left(x-2\right)\left(x+3\right)}$$

$$=\frac{4x+5}{\left(x-2\right)\left(x+3\right)}$$

Question 26

a $ab^2=4\times\left(-3\right)^2=36$

b $a-bc=4-\left(-3\right)\left(-2\right)=-2$

c $\sqrt{a}=\sqrt{4}=2$

d $\left(bc\right)^3=\left(-3\times-2\right)^3=\left(6\right)^3=216$

e $c\left(2a+3b\right)=-2\left(2\times4+3\times-3\right)=-2\left(8-9\right)=-2\left(-1\right)=2$

Question 27

a $\dfrac{3\sqrt{12}}{6\sqrt{15}}=\dfrac{\cancel{3\times 2\sqrt{3}}}{\cancel{6\sqrt{3}}\sqrt{5}}=\dfrac{1}{\sqrt{5}}$

b $\dfrac{4\sqrt{32}}{2\sqrt{2}}=\dfrac{4\times 4\sqrt{2}}{2\sqrt{2}}=4\times 2=8$

Question 28

$d=5t^2=5\times 1.5^2=11.25$

Question 29

a $\dfrac{2}{5\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}=\dfrac{2\sqrt{3}}{5\sqrt{9}}=\dfrac{2\sqrt{3}}{5\times 3}=\dfrac{2\sqrt{3}}{15}$

b $\dfrac{1+\sqrt{3}}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{\sqrt{2}+\sqrt{6}}{\sqrt{4}}=\dfrac{\sqrt{2}+\sqrt{6}}{2}$

Question 30

a
$$\begin{aligned}\left(3\sqrt{2}-4\right)\left(\sqrt{3}-\sqrt{2}\right)&=3\sqrt{6}-3\sqrt{4}-4\sqrt{3}-4\sqrt{2}\\&=3\sqrt{6}-6-4\sqrt{3}-4\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}\left(\sqrt{7}+2\right)^2&=\left(\sqrt{7}+2\right)\left(\sqrt{7}+2\right)\\&=\sqrt{49}+2\sqrt{7}+2\sqrt{7}+4\\&=7+4\sqrt{7}+4\\&=11+4\sqrt{7}\end{aligned}$$

Question 31

a $3x^2 - 27 = 3\left(x^2 - 9\right) = 3(x-3)(x+3)$

b $6x^2 - 12x - 18 = 6\left(x^2 - 2x - 3\right) = 6(x-3)(x+1)$

c $5y^2 - 30y + 45 = 5\left(y^2 - 6y + 9\right) = 5(y-3)^2$

Question 32

a $\dfrac{3x^4 y}{9xy^5} = \dfrac{x^3}{3y^4}$

b $\dfrac{5}{15x-5} = \dfrac{5}{5(3x-1)} = \dfrac{1}{3x-1}$

Question 33

a $\left(3\sqrt{11}\right)^2 = 3^2 \times \left(\sqrt{11}\right)^2 = 9 \times 11 = 99$

b $\left(2\sqrt{3}\right)^3 = 2^3 \times \left(\sqrt{3}\right)^3 = 8 \times 3\sqrt{3} = 24\sqrt{3}$

Question 34

a $(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$

b $(a+b)^2 = (a+b)(a+b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$

Question 35

a $a^2 - 2ab + b^2 = (a-b)(a-b) = (a-b)^2$

b $a^2 - b^2 = (a-b)(a+b)$

Question 36

$$
\begin{aligned}
x+\frac{1}{x} &= \sqrt{3}+1+\frac{1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1} \\
&= \sqrt{3}+1+\frac{\sqrt{3}-1}{\sqrt{9}-1} \\
&= \sqrt{3}+1+\frac{\sqrt{3}-1}{2} \\
&= \frac{2\left(\sqrt{3}+1\right)}{2}+\frac{\sqrt{3}-1}{2} \\
&= \frac{2\sqrt{3}+2+\sqrt{3}-1}{2} \\
&= \frac{3\sqrt{3}+1}{2}
\end{aligned}
$$

Question 37

a $\frac{4}{a}+\frac{3}{b}=\frac{4b}{ab}+\frac{3a}{ab}=\frac{4b+3a}{ab}$

b
$$
\begin{aligned}
\frac{x-3}{2}-\frac{x-2}{5} &= \frac{5(x-3)}{10}-\frac{2(x-2)}{10} \\
&= \frac{5x-15-2x+4}{10} \\
&= \frac{3x-11}{10}
\end{aligned}
$$

Question 38

$$\begin{aligned}\frac{3}{\sqrt{5}+2}-\frac{\sqrt{2}}{2\sqrt{2}-1}&=\frac{3}{\sqrt{5}+2}\times\frac{\sqrt{5}-2}{\sqrt{5}-2}-\frac{\sqrt{2}}{2\sqrt{2}-1}\times\frac{2\sqrt{2}+1}{2\sqrt{2}+1}\\&=\frac{3\left(\sqrt{5}-2\right)}{\sqrt{25}-4}-\frac{\sqrt{2}\left(2\sqrt{2}+1\right)}{4\sqrt{4}-1}\\&=\frac{3\sqrt{5}-6}{5-4}-\frac{2\sqrt{4}+\sqrt{2}}{4\times2-1}\\&=\frac{7\left(3\sqrt{5}-6\right)}{7\times1}-\frac{2\times2+\sqrt{2}}{7}\\&=\frac{21\sqrt{5}-42-4-\sqrt{2}}{7}\\&=\frac{21\sqrt{5}-46-\sqrt{2}}{7}\end{aligned}$$

Question 39

a $3\sqrt{8}=3\times2\sqrt{2}=6\sqrt{2}$

b $-2\sqrt{2}\times4\sqrt{3}=-8\sqrt{6}$

c $\sqrt{108}-\sqrt{48}=\sqrt{36\times3}-\sqrt{16\times3}=6\sqrt{3}-4\sqrt{3}=2\sqrt{3}$

d $\dfrac{8\sqrt{6}}{2\sqrt{18}}=\dfrac{4}{\sqrt{3}}$

e $5a\times(-3b)\times(-2a)=30a^2b$

f $\dfrac{2m^3n}{6m^2n^5}=\dfrac{m}{3n^4}$

g $3x-2y-x-y=2x-3y$

Question 40

a $2\sqrt{2}\left(\sqrt{3}+\sqrt{2}\right)=2\sqrt{6}+2\sqrt{4}=2\sqrt{6}+4$

b $\left(5\sqrt{7}-3\sqrt{5}\right)\left(2\sqrt{2}-\sqrt{3}\right)=10\sqrt{14}-5\sqrt{21}-6\sqrt{10}+3\sqrt{15}$

c $\left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right)=9-3\sqrt{2}+3\sqrt{2}-\sqrt{4}=9-2=7$

d
$$\begin{aligned}\left(4\sqrt{3}-\sqrt{5}\right)\left(4\sqrt{3}+\sqrt{5}\right)&=16\sqrt{9}+4\sqrt{15}-4\sqrt{15}-\sqrt{25}\\&=16\times 3-5\\&=43\end{aligned}$$

e
$$\begin{aligned}\left(3\sqrt{7}-\sqrt{2}\right)^2&=\left(3\sqrt{7}-\sqrt{2}\right)\left(3\sqrt{7}-\sqrt{2}\right)\\&=9\sqrt{49}-3\sqrt{14}-3\sqrt{14}+\sqrt{4}\\&=9\times 7-6\sqrt{14}+2\\&=63-6\sqrt{14}+2\\&=65-6\sqrt{14}\end{aligned}$$

Question 41

a $\dfrac{3}{\sqrt{7}}\times\dfrac{\sqrt{7}}{\sqrt{7}}=\dfrac{3\sqrt{7}}{\sqrt{49}}=\dfrac{3\sqrt{7}}{7}$

b $\dfrac{\sqrt{2}}{5\sqrt{3}}\times\dfrac{\sqrt{3}}{\sqrt{3}}=\dfrac{\sqrt{6}}{5\sqrt{9}}=\dfrac{\sqrt{6}}{15}$

c $\dfrac{2}{\sqrt{5}-1}\times\dfrac{\sqrt{5}+1}{\sqrt{5}+1}=\dfrac{2\left(\sqrt{5}+1\right)}{\sqrt{25}-1}=\dfrac{2\left(\sqrt{5}+1\right)}{5-1}=\dfrac{2\left(\sqrt{5}+1\right)}{4}=\dfrac{\sqrt{5}+1}{2}$

d $\dfrac{2\sqrt{2}}{3\sqrt{2}+\sqrt{3}}\times\dfrac{3\sqrt{2}-\sqrt{3}}{3\sqrt{2}-\sqrt{3}}=\dfrac{2\sqrt{2}\left(3\sqrt{2}-\sqrt{3}\right)}{9\sqrt{4}-\sqrt{9}}=\dfrac{6\sqrt{4}-2\sqrt{6}}{18-3}=\dfrac{12-2\sqrt{6}}{15}$

e $$\frac{\sqrt{5}+\sqrt{2}}{4\sqrt{5}-3\sqrt{3}}\times\frac{4\sqrt{5}+3\sqrt{3}}{4\sqrt{5}+3\sqrt{3}}=\frac{\left(\sqrt{5}+\sqrt{2}\right)\left(4\sqrt{5}+3\sqrt{3}\right)}{16\sqrt{25}-9\sqrt{9}}$$
$$=\frac{4\sqrt{25}+3\sqrt{15}+4\sqrt{10}+3\sqrt{6}}{80-27}$$
$$=\frac{20+3\sqrt{15}+4\sqrt{10}+3\sqrt{6}}{53}$$

Question 42

a $$\frac{3x}{5}-\frac{x-2}{2}=\frac{6x}{10}-\frac{5(x-2)}{10}=\frac{6x-5x+10}{10}=\frac{x+10}{10}$$

b $$\frac{a+2}{7}+\frac{2a-3}{3}=\frac{3(a+2)}{21}+\frac{7(2a-3)}{21}$$
$$=\frac{3a+6+14a-21}{21}$$
$$=\frac{17a-15}{21}$$

c $$\frac{1}{x^2-1}-\frac{2}{x+1}=\frac{1}{(x-1)(x+1)}-\frac{2(x-1)}{(x+1)(x-1)}$$
$$=\frac{1-2x+2}{(x+1)(x-1)}$$
$$=\frac{3-2x}{(x+1)(x-1)}$$

d $$\frac{4}{k^2+2k-3}+\frac{1}{k+3}=\frac{4}{(k+3)(k-1)}+\frac{1(k-1)}{(k+3)(k-1)}$$
$$=\frac{4+k-1}{(k+3)(k-1)}$$
$$=\frac{3+k}{(k+3)(k-1)}$$
$$=\frac{1}{k-1}$$

e

$$\frac{\sqrt{3}}{\sqrt{2}+\sqrt{5}}-\frac{5}{\sqrt{3}-\sqrt{2}}=\frac{\sqrt{3}}{\sqrt{2}+\sqrt{5}}\times\frac{\sqrt{2}-\sqrt{5}}{\sqrt{2}-\sqrt{5}}-\frac{5}{\sqrt{3}-\sqrt{2}}\times\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

$$=\frac{\sqrt{3}\left(\sqrt{2}-\sqrt{5}\right)}{\sqrt{4}-\sqrt{25}}-\frac{5\left(\sqrt{3}+\sqrt{2}\right)}{\sqrt{9}-\sqrt{4}}$$

$$=\frac{\sqrt{6}-\sqrt{15}}{2-5}-\frac{5\sqrt{3}+5\sqrt{2}}{3-2}$$

$$=\frac{\sqrt{6}-\sqrt{15}}{-3}-\frac{5\sqrt{3}+5\sqrt{2}}{1}$$

$$=\frac{-\left(\sqrt{6}-\sqrt{15}\right)}{3}-\frac{3\left(5\sqrt{3}+5\sqrt{2}\right)}{3}$$

$$=\frac{-\sqrt{6}+\sqrt{15}-15\sqrt{3}-15\sqrt{2}}{3}$$

Question 43

a $\sqrt{108}-\sqrt{12}=\sqrt{n}$

LHS: $\sqrt{36\times3}-\sqrt{4\times3}=6\sqrt{3}-2\sqrt{3}=4\sqrt{3}=\sqrt{16}\sqrt{3}=\sqrt{48}$

$n=48$

b $\sqrt{112}+\sqrt{7}=\sqrt{n}$

LHS: $\sqrt{16\times7}+\sqrt{7}=4\sqrt{7}+\sqrt{7}=5\sqrt{7}=\sqrt{25}\sqrt{7}=\sqrt{175}$

$n=175$

c $2\sqrt{8}+\sqrt{200}=\sqrt{n}$

LHS: $2\sqrt{4\times2}+\sqrt{100\times2}=4\sqrt{2}+10\sqrt{2}=14\sqrt{2}=\sqrt{196}\sqrt{2}=\sqrt{392}$

$n=392$

d $4\sqrt{147}+3\sqrt{75}=\sqrt{n}$

LHS: $4\sqrt{49\times3}+3\sqrt{25\times3}=28\sqrt{3}+15\sqrt{3}=43\sqrt{3}=\sqrt{1849}\sqrt{3}=\sqrt{5547}$

$n=5547$

e $2\sqrt{245}+\dfrac{\sqrt{180}}{2}=\sqrt{n}$

LHS: $2\sqrt{49\times5}+\dfrac{\sqrt{36\times5}}{2}=14\sqrt{5}+\dfrac{6\sqrt{5}}{2}=14\sqrt{5}+3\sqrt{5}=17\sqrt{5}=\sqrt{289}\sqrt{5}=\sqrt{1445}$

$n=1445$

Challenge exercise 1

Question 1

$$64^{-\frac{2}{3}} = \frac{1}{64^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{64^2}} = \frac{1}{16}$$

Question 2

$$2\left(2^k - 1\right) + 2^{k+1} = 2\left(2^{k+1} - 1\right)$$

$$\text{LHS: } 2\left(2^k - 1\right) + 2^{k+1} = 2^{k+1} - 2 + 2^{k+1} = 2 \times 2^{k+1} - 2 = 2\left(2^{k+1} - 1\right) = \text{RHS}$$

$$\therefore 2\left(2^k - 1\right) + 2^{k+1} = 2\left(2^{k+1} - 1\right)$$

Question 3

$$\frac{a}{b^3c^2} = \frac{\left(\frac{2}{5}\right)^4}{\left(\left(-\frac{1}{3}\right)^3\right)^3\left(\left(\frac{3}{5}\right)^2\right)^2} = \left(\frac{2}{5}\right)^4 \div \left(-\frac{1}{3}\right)^9 \div \left(\frac{3}{5}\right)^4 = \frac{2^4}{5^4} \times \left(-\frac{3^9}{1^9}\right) \times \frac{5^4}{3^4} = -2^4\,3^5$$

Question 4

a $4ab\left(a - 2b\right) - 2a^2\left(b - 3a\right) = 4a^2b - 8ab^2 - 2a^2b + 6a^3 = 2a^2b - 8ab^2 + 6a^3$

b $\left(y^2 - 2\right)\left(y^2 + 2\right) = y^4 - 2y^2 + 2y^2 - 4 = y^4 - 4$

c
$$\begin{aligned}
\left(2x - 5\right)^3 &= \left(2x - 5\right)\left(2x - 5\right)\left(2x - 5\right) \\
&= \left(2x - 5\right)\left(4x^2 - 10x - 10x + 25\right) \\
&= \left(2x - 5\right)\left(4x^2 - 20x + 25\right) \\
&= 8x^3 - 40x^2 + 50x - 20x^2 + 100x - 125 \\
&= 8x^3 - 60x^2 + 150x - 125
\end{aligned}$$

Question 5

$$\sqrt{3}+1+\frac{1}{2\sqrt{5}-3}\times\frac{2\sqrt{5}+3}{2\sqrt{5}+3}=\sqrt{3}+1+\frac{2\sqrt{5}+3}{4\sqrt{25}-9}$$
$$=\sqrt{3}+1+\frac{2\sqrt{5}+3}{20-9}$$
$$=\sqrt{3}+1+\frac{2\sqrt{5}+3}{11}$$
$$=\frac{11\left(\sqrt{3}+1\right)}{11}+\frac{2\sqrt{5}+3}{11}$$
$$=\frac{11\sqrt{3}+11+2\sqrt{5}+3}{11}$$
$$=\frac{11\sqrt{3}+14+2\sqrt{5}}{11}$$

Question 6

$$\frac{2\sqrt{3}}{7\sqrt{6}-\sqrt{54}}=\frac{2\sqrt{3}}{7\sqrt{6}-\sqrt{9\times6}}=\frac{2\sqrt{3}}{7\sqrt{6}-3\sqrt{6}}=\frac{2\sqrt{3}}{4\sqrt{6}}=\frac{1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2\sqrt{4}}=\frac{\sqrt{2}}{4}$$

Question 7

a $\quad (x+4)^2+5(x+4)=(x+4)\left((x+4)+5\right)=(x+4)(x+9)$

b $\quad x^4-x^2y-6y^2=\left(x^2+2y\right)\left(x^2-3y\right)$

c $\quad a^2b-2a^2-4b+8=a^2(b-2)-4(b-2)$
$$=(b-2)\left(a^2-4\right)$$
$$=(b-2)(a-2)(a+2)$$

Question 8

$$\frac{2xy+2x-6-6y}{4x^2-16x+12}=\frac{2\left(xy+x-3-3y\right)}{4\left(x^2-4x+3\right)}$$
$$=\frac{x\left(y+1\right)-3\left(y+1\right)}{2\left(x-3\right)\left(x-1\right)}$$
$$=\frac{\left(y+1\right)\left(x-3\right)}{2\left(x-1\right)\left(x-3\right)}=\frac{\left(y+1\right)}{2\left(x-1\right)}$$

Question 9

$$\frac{\left(a+1\right)^3}{a^2-1}=\frac{\left(a+1\right)^3}{\left(a+1\right)\left(a-1\right)}=\frac{\left(a+1\right)^2}{a-1}$$

Question 10

$$\frac{4}{x^2}-\frac{a^2}{b^2}=\left(\frac{2}{x}\right)^2-\left(\frac{a}{b}\right)^2=\left(\frac{2}{x}-\frac{a}{b}\right)\left(\frac{2}{x}+\frac{a}{b}\right)$$

Question 11

a

$$\begin{aligned}\left(2x-1\right)^3&=\left(2x-1\right)\left(2x-1\right)\left(2x-1\right)\\&=\left(2x-1\right)\left(4x^2-2x-2x+1\right)\\&=\left(2x-1\right)\left(4x^2-4x+1\right)\\&=8x^3-8x^2+2x-4x^2+4x-1\\&=8x^3-12x^2+6x-1\end{aligned}$$

b

$$\frac{6x^2+5x-4}{8x^3-12x^2+6x-1}=\frac{\left(2x-1\right)\left(3x+4\right)}{\left(2x-1\right)^3}=\frac{3x+4}{\left(2x-1\right)^2}$$

Question 12

$$V = \pi r^2 h$$
$$9 = \pi r^2 \times 16$$
$$\frac{9}{16\pi} = r^2$$
$$\sqrt{\frac{9}{16\pi}} = r$$
$$r = \frac{3}{4\sqrt{\pi}}$$

Question 13

$$\begin{aligned} s &= u + \frac{1}{2}at^2 \\ &= 2 + \frac{1}{2} \times \sqrt{3} \times \left(2\sqrt{3}\right)^2 \\ &= 2 + \frac{1}{2} \times \sqrt{3} \times 4 \times 3 \\ &= 2 + 6\sqrt{3} \end{aligned}$$

Question 14

a

$$\begin{aligned} \left(\sqrt{x} + x\right)^2 &= \left(\sqrt{x} + x\right)\left(\sqrt{x} + x\right) \\ &= x + x\sqrt{x} + x\sqrt{x} + x^2 \\ &= x + 2x\sqrt{x} + x^2 \\ &= x + 2x^{\frac{3}{2}} + x^2 \end{aligned}$$

b

$$\begin{aligned} \left(\sqrt[3]{a} + \sqrt[3]{b}\right)\left(\sqrt[3]{a} - \sqrt[3]{b}\right) &= \left(a^{\frac{1}{3}} + b^{\frac{1}{3}}\right)\left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right) \\ &= a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} - b^{\frac{2}{3}} \\ &= a^{\frac{2}{3}} - b^{\frac{2}{3}} \end{aligned}$$

c $\left(p+\frac{1}{\sqrt{p}}\right)^2=\left(p+p^{-\frac{1}{2}}\right)\left(p+p^{-\frac{1}{2}}\right)$

$$= p^2 + p^{\frac{1}{2}} + p^{\frac{1}{2}} + p^{-1}$$

$$= p^2 + 2p^{\frac{1}{2}} + p^{-1}$$

d $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)$

$$= x + x^0 + x^0 + x^{-1}$$

$$= x + 1 + 1 + x^{-1}$$

$$= x + x^{-} + 2$$

Question 15

$$\frac{a^3b^2}{c^2}=\frac{\left(\left(\frac{3}{4}\right)^2\right)^3\left(\left(\frac{2}{3}\right)^3\right)^2}{\left(\left(\frac{1}{2}\right)^4\right)^2}$$

$$=\left(\frac{3}{4}\right)^6\times\left(\frac{2}{3}\right)^6\div\left(\frac{1}{2}\right)^8$$

$$=\frac{3^6}{4^6}\times\frac{2^6}{3^6}\times\frac{2^8}{1^8}$$

$$=\frac{2^{14}}{4^6}=\frac{2^{14}}{2^{12}}=2^2=4$$

MATHS IN FOCUS 11
MATHEMATICS EXTENSION 1

FULLY WORKED SOLUTIONS

Chapter 2: Equations and inequalities

Exercise 2.01 Equations

Question 1

$t+4=-1$

$t+4-4=-1-4$

$t=-5$

Question 2

$$\begin{aligned} z+1.7&=-3.9 \\ z+1.7-17&=-3.9-1.7 \\ z&=-5.6 \end{aligned}$$

Question 3

$y-3=-2$

$y-3+3=-2+3$

$y=1$

Question 4

$w-2.6=4.1$

$w-2.6+2\ 6=4.1+2\ 6$

$w=6.7$

Question 5

$$5 = x - 7$$
$$-x + 5 - 5 = x - x - 7 - 5$$
$$-x = -12$$
$$x = 12$$

Question 6

$$1.5x = 6$$
$$\frac{1.5x}{1.5} = \frac{6}{1.5}$$
$$x = 4$$

Question 7

$$5y = \frac{1}{3}$$
$$\frac{5y}{5} = \frac{1}{3 \times 5}$$
$$y = \frac{1}{15}$$

Question 8

$$\frac{b}{7} = 5$$
$$7 \times \frac{b}{7} = 5 \times 7$$
$$b = 35$$

Question 9

$$-2 = \frac{n}{8}$$

$$-2 \times 8 = \frac{n}{8} \times 8$$

$$-16 = n$$

$$n = -16$$

Question 10

$$\frac{r}{6} = \frac{2}{3}$$

$$\frac{r}{6} \times 6 = \frac{2}{3} \times 6$$

$$r = 4$$

Question 11

$$2y + 1 = 19$$

$$2y + 1 - 1 = 19 - 1$$

$$2y = 18$$

$$\frac{2y}{2} = \frac{18}{2}$$

$$y = 9$$

Question 12

$$33 = 4k + 9$$

$$33 - 9 = 4k + 9 - 9$$

$$24 = 4k$$

$$\frac{24}{4} = \frac{4k}{4}$$

$$k = 6$$

Question 13

$$7d - 2 = 12$$
$$7d - 2 + 2 = 12 + 2$$
$$7d = 14$$
$$\frac{7d}{7} = \frac{14}{7}$$
$$d = 2$$

Question 14

$$-2 = 5x - 27$$
$$-2 + 27 = 5x - 27 + 27$$
$$25 = 5x$$
$$\frac{25}{5} = \frac{5x}{5}$$
$$x = 5$$

Question 15

$$\frac{y}{3} + 4 = 9$$
$$\frac{y}{3} + 4 - 4 = 9 - 4$$
$$\frac{y}{3} \times 3 = 5 \times 3$$
$$y = 15$$

Question 16

$$\frac{x}{2}-3=7$$
$$\frac{x}{2}-3+3=7+3$$
$$\frac{x}{2}\times 2=10\times 2$$
$$x=20$$

Question 17

$$\frac{m}{5}+7=11$$
$$\frac{m}{5}=11-7$$
$$\frac{m}{5}\times 5=4\times 5$$
$$m=20$$

Question 18

$$3x+5=17$$
$$3x=17-5$$
$$\frac{3x}{3}=\frac{12}{3}$$
$$x=4$$

Question 19

$$4a+7=-21$$
$$4a=-21-7$$
$$\frac{4a}{4}=\frac{-28}{4}$$
$$a=-7$$

Question 20

$$7y - 1 = 20$$
$$7y = 20 + 1$$
$$\frac{7y}{7} = \frac{21}{7}$$
$$y = 3$$

Question 21

$$3(x + 2) = 15$$
$$3x + 6 = 15$$
$$\frac{3x}{3} = \frac{9}{3}$$
$$x = 3$$

Question 22

$$-2(3a + 1) = 8$$
$$-6a - 2 = 8$$
$$\frac{-6a}{-6} = \frac{10}{-6}$$
$$a = -1\frac{2}{3}$$

Question 23

$$7t + 4 = 3t - 12$$
$$7t - 3t = -12 - 4$$
$$\frac{4t}{4} = \frac{-16}{4}$$
$$t = -4$$

Question 24

$$x-3=6x-9$$
$$x-6x=-9+3$$
$$\frac{-5x}{-5}=\frac{-6}{-5}$$
$$x=1\frac{1}{5}$$

Question 25

$$2(a-2)=4-3a$$
$$2a-4=4-3a$$
$$\frac{5a}{5}=\frac{8}{5}$$
$$a=1\frac{3}{5}$$

Question 26

$$5b+2=-3(b-1)$$
$$5b+2=-3b+3$$
$$\frac{8b}{8}=\frac{1}{8}$$
$$b=\frac{1}{8}$$

Question 27

$$3(t+7)=2(2t-9)$$
$$3t+21=4t-18$$
$$\frac{-t}{-1}=\frac{-39}{-1}$$
$$t=39$$

Question 28

$$2+5(p-1)=5p-(p-2)$$
$$2+5p-5=5p-p+2$$
$$5p-3=4p+2$$
$$p=5$$

Question 29

$$3.7x+1.2=5.4x-6.3$$
$$3.7x-5.4x=-6.3-1.2$$
$$\frac{-1.7x}{-1.7}=\frac{-7.5}{-1.7}$$
$$x=4\frac{7}{17}$$

Question 30

$$\frac{b}{5}=\frac{2}{3}$$
$$3b=10$$
$$b=\frac{10}{3}=3\frac{1}{3}$$

Question 31

$$\frac{5x}{4}=\frac{11}{7}$$
$$35x=44$$
$$x=\frac{44}{35}=1\frac{9}{35}$$

Question 32

$$\frac{x}{3}-4=8$$
$$\frac{x}{3}=8+4$$
$$\frac{x}{3}\times 3=12\times 3$$
$$x=36$$

Question 33

$$\frac{x+5}{7}=\frac{2}{7}$$
$$\frac{x+5}{7}\times 7=\frac{2}{7}\times 7$$
$$x+5=2$$
$$x=-3$$

Question 34

$$\frac{y}{2}=-\frac{3}{5}$$
$$5y=-6$$
$$y=-\frac{6}{5}=-1\frac{1}{5}$$

Question 35

$$\frac{x}{9}-\frac{2}{3}=7$$
$$\frac{x}{9}=7+\frac{2}{3}$$
$$\frac{x}{9}=\frac{23}{3}$$
$$3x=207$$
$$x=69$$

Question 36

$$\frac{w-3}{2}=5$$
$$w-3=10$$
$$w=13$$

Question 37

$$\frac{2t}{5}-\frac{t}{3}=2$$
$$\frac{6t-5t}{15}=2$$
$$\frac{t}{15}=2$$
$$t=30$$

Question 38

$$\frac{x}{4}+\frac{1}{2}=4$$
$$\frac{x+2}{4}=4$$
$$x+2=16$$
$$x=14$$

Question 39

$$\frac{x}{5}-\frac{x}{2}=\frac{3}{10}$$
$$\frac{2x-5x}{10}=\frac{3}{10}$$
$$-3x=3$$
$$x=-1$$

Question 40

$$\frac{x+4}{3}+\frac{x}{2}=1$$
$$\frac{2x+8+3x}{6}=1$$
$$5x+8=6$$
$$5x=-2$$
$$x=-\frac{2}{5}$$

Question 41

$$\frac{p-3}{2}+\frac{2p}{3}=2$$
$$\frac{3p-9+4p}{6}=2$$
$$7p-9=12$$
$$7p=21$$
$$p=3$$

Question 42

$$\frac{t+3}{7}+\frac{t-1}{3}=4$$
$$\frac{3t+9+7t-7}{21}=4$$
$$10t+2=84$$
$$10t=82$$
$$t=8\frac{1}{5}$$

Question 43

$$\frac{x+5}{9}-\frac{x+2}{5}=1$$
$$\frac{5x+25-9x-18}{45}=1$$
$$-4x+7=45$$
$$-4x=38$$
$$x=-\frac{38}{4}=-9\frac{1}{2}$$

Question 44

$$\frac{q-1}{3}-\frac{q-2}{4}=2$$
$$\frac{4q-4-3q+6}{12}=2$$
$$q+2=24$$
$$q=22$$

Question 45

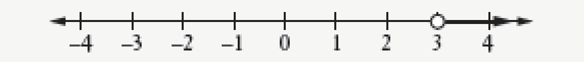
$$\frac{x+3}{5}+2=\frac{x+7}{2}$$
$$\frac{x+3+10}{5}=\frac{x+7}{2}$$
$$2x+26=5x+35$$
$$-3x=9$$
$$x=-3$$

Exercise 2.02 Inequalities

Question 1

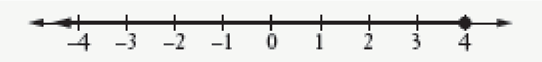
a

$$x+4>7$$
$$x+4-4>7-4$$
$$x>3$$



b

$$y-3\le 1$$
$$y-3+3\le 1+3$$
$$y\le 4$$



Question 2

a

$$5t>35$$
$$\frac{5t}{5}>\frac{35}{5}$$
$$t>7$$

b

$$3x-7\ge 2$$
$$3x\ge 9$$
$$x\ge 3$$

c

$$2(p+5)>8$$
$$2p+10>8$$
$$2p>-2$$
$$p>-1$$

d

$$4-(x-1)\le 7$$
$$4-x+1\le 7$$
$$-x\le 2$$
$$-x\times(-1)\le 2\times(-1)$$
$$x\ge -2$$

e

$$3y+5>2y-4$$
$$y>-9$$

f

$$2a-6\le 5a-3$$
$$-3a\le 3$$
$$\frac{-3a}{-3}\ge\frac{3}{-3}$$
$$a\ge -1$$

g

$$3+4y\ge -2(1-y)$$
$$3+4y\ge -2+2y$$
$$2y\ge -5$$
$$y\ge -2\frac{1}{2}$$

h

$$2x+9<1-4(x+1)$$
$$2x+9<1-4x-4$$
$$6x<-12$$
$$x<-2$$

i

$$\frac{a}{2} \le -3$$

$$\frac{a}{2} \times 2 \le -3 \times 2$$

$$a \le -6$$

j

$$8 > \frac{2y}{3}$$

$$8 \times 3 > \frac{2y}{3} \times 3$$

$$24 > 2y$$

$$12 > y$$

$$y < 12$$

k

$$\frac{b}{2} + 5 < -4$$

$$\frac{b}{2} < -9$$

$$\frac{b}{2} \times 2 < -9 \times 2$$

$$b < -18$$

l

$$\frac{x}{3} - 4 > 6$$

$$\frac{x}{3} > 10$$

$$x > 30$$

m

$$\frac{1}{4}+\frac{x}{5}\le 1$$
$$\frac{5+4x}{20}\le 1$$
$$5+4x\le 20$$
$$4x\le 15$$
$$x\le \frac{15}{4}$$
$$x\le 3\frac{3}{4}$$

n

$$\frac{m}{4}-3>\frac{2}{3}$$
$$\frac{m-12}{4}\times 12>\frac{2}{3}\times 12$$
$$3m-36>8$$
$$3m>44$$
$$m>\frac{44}{3}$$
$$m>14\frac{2}{3}$$

o

$$\frac{2b}{5}-\frac{1}{2}\ge 6$$
$$\frac{4b-5}{10}\ge 6$$
$$4b-5\ge 60$$
$$4b\ge 65$$
$$b\ge \frac{65}{4}$$
$$b\ge 16\frac{1}{4}$$

p

$$\frac{r-3}{2}\le -6$$
$$r-3\le -12$$
$$r\le -9$$

q

$$\frac{z+1}{9}+2>3$$
$$\frac{z+1}{9}>1$$
$$z+1>9$$
$$z>8$$

r

$$\frac{w}{6}+\frac{2w+5}{3}<4$$
$$\frac{w+4w+10}{6}<4$$
$$5w+10<24$$
$$5w<14$$
$$w<\frac{14}{5}$$
$$w<2\frac{4}{5}$$

s

$$\frac{x+1}{2}-\frac{x-2}{3}\geq 7$$
$$\frac{3x+3-2x+4}{6}\geq 7$$
$$x+7\geq 42$$
$$x\geq 35$$

t

$$\frac{t+2}{7}-\frac{t+3}{2}\leq 2$$
$$\frac{2t+4-7t-21}{14}\leq 2$$
$$\frac{-5t-17}{14}\leq 2$$
$$-5t-17\leq 28$$
$$-5t\leq 45$$
$$\frac{-5t}{-5}\geq\frac{45}{-5}$$
$$t\geq -9$$

u

$$\frac{q-2}{3} < 2 + \frac{3q}{4}$$
$$\frac{q-2}{3} \times 12 < \frac{8+3q}{4} \times 12$$
$$4q - 8 < 24 + 9q$$
$$-5q < 32$$
$$\frac{-5q}{-5} > \frac{32}{-5}$$
$$q > -6\frac{2}{5}$$

v

$$\frac{2x}{3} - \frac{x-1}{2} > \frac{2}{9}$$
$$\frac{12x - 9x + 9}{18} > \frac{4}{18}$$
$$3x > -5$$
$$x > \frac{-5}{3}$$
$$x > -1\frac{2}{3}$$

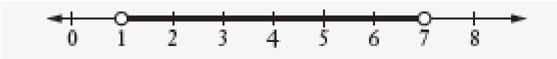
w

$$\frac{2b-5}{8} + 3 \le \frac{b+6}{12}$$
$$\frac{6b - 15 + 72}{24} \le \frac{2b+12}{24}$$
$$6b + 57 \le 2b + 12$$
$$4b \le -45$$
$$b \le -11\frac{1}{4}$$

Question 3

a

$$3 < x + 2 < 9$$
$$3 - 2 < x + 2 - 2 < 9 - 2$$
$$1 < x < 7$$

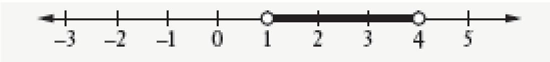


b

$$-4 \le 2p < 10$$
$$\frac{-4}{2} \le \frac{2p}{2} < \frac{10}{2}$$
$$-2 \le p < 5$$

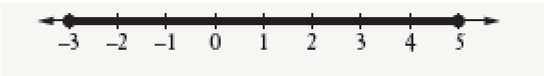
c

$$2 < 3x - 1 < 11$$
$$2 + 1 < 3x - 1 + 1 < 11 + 1$$
$$\frac{3}{3} < \frac{3x}{3} < \frac{12}{3}$$
$$1 < x < 4$$



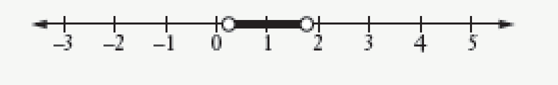
d

$$-6 \le 5y + 9 \le 34$$
$$-6 - 9 \le 5y + 9 - 9 \le 34 - 9$$
$$\frac{-15}{5} \le \frac{5y}{5} \le \frac{25}{5}$$
$$-3 \le y \le 5$$



e

$$-2 < 3(2y-1) < 7$$
$$-2 < 6y-3 < 7$$
$$-2+3 < 6y-3+3 < 7+3$$
$$\frac{1}{6} < \frac{6y}{6} < \frac{10}{6}$$
$$\frac{1}{6} < y < 1\frac{2}{3}$$



Exercise 2.03 Absolute value

Question 1

a $|7| = 7$, since $7 \geq 0$

b $|-5| = -(-5) = 5$

c $|-6| = -(-6) = 6$

d $0 \geq 0$, so 0

e $2 \geq 0$, so 2

f $|-11| = -(-11) = 11$

g $|-2| = 2, |3| = 3$

$2 \times 3 = 6$

h $|-8| = 8$

$3 \times 8 = 24$

i $|-5| \times |-5| = 5 \times 5 = 25$

j $|-5| \times |-5| \times |-5| = 5 \times 5 \times 5 = 125$

Question 2

a $3 + 2 = 5$

b $3 - 4 = -1$

c $|-2| = 2$

d $|-14| = 14$

e $3 + 1 = 4$

f $5 - 2 \times 36 = -67$

g $|-7| = 7$

h $3 \times 4 = 12$

i $2 \times 3 - 3 \times 4 = -6$

j $2 + 8 = 10$

Question 3

a $|5 - 2| = |3| = 3$

b $|-1 - 2| = |-3| = 3$

c $|-2 - (-3)| = |1| = 1$

d $|4 - 7| = |-3| = 3$

e $|-1 - (-2)| = |1| = 1$

Question 4

a a

b $-a$, $-a$ has the same effect as $|a|$

c 0, $|0| = 0$

d $3a$, $|3a| = 3|a| = 3a$

e $-3a$, $|3a| = 3|a| = 3(-a) = -3a$

f 0, $|3 \times 0| = |0| = 0$

g $-a + 1$, When $a > -1, a + 1 > 0$, so $|a + 1| = a + 1$

h $-a - 1$, When $a < -1, a + 1 < 0$, so $|a + 1| = -(a + 1) = -a - 1$

i $x - 2$, When $x > 2, x - 2 > 0$, so $|x - 2| = x - 2$

Question 5

a $|a + b| = |2 + 4| = |6| = 6$

$|a| + |b| = |2| + |4| = 2 + 4 = 6$

$|a + b| = |a| + |b|$

$\therefore |a+b| \le |a| + |b|$

b $|a + b| = |-1 + (–2)| = |–3| = 3$

$|a| + |b| = |–1| + |–2| = 1 + 2 = 3$

$|a + b| = |a| + |b|$

$\therefore |a+b| \le |a| + |b|$

c $|a + b| = |–2 + 3| = |1| = 1$

$|a| + |b| = |–2| + |3| = 2 + 3 = 5$

$|a + b| < |a| + |b|$

$\therefore |a+b| \le |a| + |b|$

d $|a + b| = |–4 + 5| = |1| = 1$

$|a| + |b| = |–4| + |5| = 4 + 5 = 9$

$|a + b| < |a| + |b|$

$\therefore |a+b| \le |a| + |b|$

e $|a + b| = |–7 + –3| = |–10| = 10$

$|a| + |b| = |–7| + |–3| = 7 + 3 = 10$

$|a + b| = |a| + |b|$

$\therefore |a+b| \le |a| + |b|$

Question 6

a $\sqrt{5^2} = \sqrt{25} = 5 = |5|$

b $\sqrt{(-2)^2} = \sqrt{4} = 2 = |-2|$

c $\sqrt{(-3)^2} = \sqrt{9} = 3 = |-3|$

d $\sqrt{4^2} = \sqrt{16} = 4 = |4|$

e $\sqrt{(-9)^2} = \sqrt{81} = 9 = |-9|$

Question 7

a $|x+5| = x+5$ for $x+5 \geq 0 \Rightarrow x \geq -5$

$|x+5| = -(x+5) = -x-5$ for $x+5 < 0 \Rightarrow x < -5$

b $|b-3| = b-3$ for $b-3 \geq 0 \Rightarrow b \geq 3$

$|b-3| = -(b-3) = -b+3$ for $b-3 < 0 \Rightarrow b < 3$

c $|a+4| = a+4$ for $a+4 \geq 0 \Rightarrow a \geq -4$

$|a+4| = -(a+4) = -a-4$ for $a+4 < 0 \Rightarrow a < -4$

d $|2y-6| = 2y-6$ for $2y-6 \geq 0 \Rightarrow y \geq 3$

$|2y-6| = -(2y-6) = -2y+6$ for $2y-6 < 0 \Rightarrow y < 3$

e $|3x+9| = 3x+9$ for $3x+9 \geq 0 \Rightarrow x \geq -3$

$|3x+9| = -(3x+9) = -3x-9$ for $3x+9 < 0 \Rightarrow x < -3$

f $|4-x| = 4-x$ for $4-x \geq 0 \Rightarrow x \leq 4$

$|4-x| = -(4-x) = x-4$ for $4-x < 0 \Rightarrow x > 4$

g $|2k+1| = 2k+1$ for $2k+1 \geq 0 \Rightarrow k \geq -\frac{1}{2}$

$|2k+1| = -(2k+1) = -2k-1$ for $2k+1 < 0 \Rightarrow k < -\frac{1}{2}$

h $|5x-2| = 5x-2$ for $5x-2 \geq 0 \Rightarrow x \geq \frac{2}{5}$

$|5x-2| = -(5x-2) = -5x+2$ for $5x-2 < 0 \Rightarrow x < \frac{2}{5}$

i $|a+b| = a+b$ for $a+b \geq 0 \Rightarrow a \geq -b$

$|a+b| = -(a+b) = -a-b$ for $a+b < 0 \Rightarrow a < -b$

Question 8

$x = \pm 3$

$|3| = 3$, $|-3| = 3$

Question 9

$\frac{|n|}{n} = \frac{n}{n} = 1$ for $|n| > 0 \Rightarrow n > 0$

$\frac{|n|}{n} = \frac{-n}{n} = -1$ for $n < 0$

Question 10

$$\frac{x-2}{|x-2|}$$

$$\frac{x-2}{x-2} = 1, x > 2$$

$$\frac{x-2}{-x+2} = -1, x < 2$$

Exercise 2.04 Equations involving absolute values

Question 1

a $|5| = 5$, $|-5| = 5$

b $|8| = 8$, $|-8| = 8$

c $|0| = 0$

Question 2

a

$$|x+2| = 7$$
$$x + 2 = \pm 7$$
$$x = 5, -9$$

b

$$|n-1| = 3$$
$$n - 1 = \pm 3$$
$$n = 4, -2$$

c

$$|2x+3| = 9$$
$$2x + 3 = \pm 9$$
$$2x = \pm 9 - 3$$
$$x = \frac{\pm 9 - 3}{2}$$
$$x = -6, 3$$

d

$$71x|-34=$$
$$7x-1=\pm 34$$
$$7x=\pm 34+1$$
$$x=\frac{\pm 34+1}{7}$$
$$x=-4\frac{5}{7},5$$

e

$$\frac{x}{3}=\pm 4\Rightarrow x=\pm 12$$

Question 3

a

$$|8x-5|=11$$
$$8x-5=\pm 11$$
$$8x=5\pm 11$$
$$x=\frac{5\pm 11}{8}$$
$$x=-\frac{3}{4},2$$

b

$$|5-3n|=1$$
$$5-3n=\pm 1$$
$$-3n=-5\pm 1$$
$$n=\frac{-5\pm 1}{-3}$$
$$x=1\frac{1}{3},2$$

c

$$16 = |5t+4|$$
$$5t+4 = \pm 16$$
$$5t = -4 \pm 16$$
$$t = \frac{-4 \pm 16}{5}$$
$$t = -4, 2\frac{2}{5}$$

d

$$21 = |9-2y|$$
$$9-2y = \pm 21$$
$$-2y = -9 \pm 21$$
$$y = \frac{-9 \pm 21}{-2}$$
$$y = -6, 15$$

e

$$|3x+2| - 7 = 0$$
$$3x+2 = \pm 7$$
$$3x = -2 \pm 7$$
$$x = \frac{-2 \pm 7}{3}$$
$$x = -3, 1\frac{2}{3}$$

Exercise 2.05 Exponential equations

Question 1

a

$$2^n = 16$$
$$2^n = 2^4$$
$$n = 4$$

b

$$3^y = 243$$
$$3^y = 3^5$$
$$y = 5$$

c

$$2^m = 512$$
$$2^m = 2^9$$
$$m = 9$$

d

$$10^x = 100\ 000$$
$$10^x = 10^5$$
$$x = 5$$

e

$$6^m = 1$$
$$6^m = 6^0$$
$$m = 0$$

f

$$4^x = 64$$
$$4^x = 4^3$$
$$x = 3$$

g

$$\begin{aligned} 4^x + 3 &= 19 \\ 4^x &= 16 \\ 4^x &= 4^2 \\ x &= 2 \end{aligned}$$

h

$$\begin{aligned} 5(3^x) &= 45 \\ 3^x &= 9 \\ 3^x &= 3^2 \\ x &= 2 \end{aligned}$$

i

$$\begin{aligned} 4^x &= 4 \\ 4^x &= 4^1 \\ x &= 1 \end{aligned}$$

j

$$\begin{aligned} \frac{6^k}{2} &= 18 \\ 6^k &= 36 \\ 6^k &= 6^2 \\ k &= 2 \end{aligned}$$

Question 2

a

$$3^{2x} = 81$$
$$9^x = 9^2$$
$$x = 2$$

b

$$2^{5x-1} = 16$$
$$\frac{(2^5)^x}{2} = 16$$
$$2^{5x} = 32$$
$$2^{5x} = 2^5$$
$$5x = 5$$
$$x = 1$$

c

$$4^{x+3} = 4$$
$$x + 3 = 1$$
$$x = -2$$

d

$$3^{n-2} = 1$$
$$3^{n-2} = 3^0$$
$$n - 2 = 0$$
$$n = 2$$

e

$$7^{2x+1} = 7$$
$$2x + 1 = 1$$
$$x = 0$$

f

$$3^{x-3} = 27$$
$$3^{x-3} = 3^3$$
$$x - 3 = 3$$
$$x = 6$$

g

$$5^{3y+2} = 125$$
$$5^{3y+2} = 5^3$$
$$3y + 2 = 3$$
$$y = \frac{1}{3}$$

h

$$7^{3x-4} = 49$$
$$7^{3x-4} = 7^2$$
$$3x - 4 = 2$$
$$x = 2$$

i

$$2^{4x} = 256$$
$$2^{4x} = 2^8$$
$$4x = 8$$
$$x = 2$$

j

$a = 3$

$$9^{3a+1} = 9$$
$$3a + 1 = 1$$
$$3a = 0$$
$$a = 0$$

Question 3

a

$$\begin{aligned} 4^m &= 2 \\ 2^{2m} &= 2 \\ 2m &= 1 \\ m &= \frac{1}{2} \end{aligned}$$

b

$$\begin{aligned} 27^x &= 3 \\ 3^{3x} &= 3 \\ 3x &= 1 \\ x &= \frac{1}{3} \end{aligned}$$

c

$$\begin{aligned} 125^x &= 5 \\ 5^{3x} &= 5 \\ 3x &= 1 \\ x &= \frac{1}{3} \end{aligned}$$

d

$$\begin{aligned} \left(\frac{1}{49}\right)^k &= 7 \\ 7^{-2k} &= 7 \\ -2k &= 1 \\ k &= -\frac{1}{2} \end{aligned}$$

e

$$\begin{aligned} \left(\frac{1}{1000}\right)^k &= 100 \\ 10^{-3k} &= 10^2 \\ -3k &= 2 \\ k &= -\frac{2}{3} \end{aligned}$$

f

$$\begin{aligned} 16^n &= 8 \\ 2^{4n} &= 2^3 \\ 4n &= 3 \\ n &= \frac{3}{4} \end{aligned}$$

g

$$\begin{aligned} 25^x &= 125 \\ 5^{2x} &= 5^3 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

h

$$\begin{aligned} 64^n &= 16 \\ 4^{3n} &= 4^2 \\ 3n &= 2 \\ n &= \frac{2}{3} \end{aligned}$$

i

$$\begin{aligned} \left(\frac{1}{4}\right)^{3k} &= 2 \\ 4^{-3k} &= 2 \\ 2^{-6k} &= 2^1 \\ -6k &= 1 \\ k &= -\frac{1}{6} \end{aligned}$$

j

$$\begin{aligned} 8^{x-} &= 4 \\ 2^{3(x-1)} &= 2^2 \\ 3x-3 &= 2 \\ x &= \frac{5}{3} \end{aligned}$$

Question 4

a

$$\begin{aligned}2^{4x+1} &= 8^x\\ 2^{4x+1} &= 2^{3x}\\ 4x+1 &= 3x\\ x &= -1\end{aligned}$$

b

$$\begin{aligned}3^{5x} &= 9^{x-2}\\ 3^{5x} &= 3^{2x-4}\\ 5x &= 2x-4\\ 3x &= -4\\ x &= \frac{-4}{3} = -1\frac{1}{3}\end{aligned}$$

c

$$\begin{aligned}7^{2k+3} &= 7^{k-1}\\ 2k+3 &= k-1\\ k &= -4\end{aligned}$$

d

$$\begin{aligned}4^{3n} &= 8^{n+3}\\ 2^{6n} &= 2^{3n+9}\\ 6n &= 3n+9\\ 3n &= 9\\ n &= 3\end{aligned}$$

e

$$\begin{aligned}6^{x-5} &= 216^x\\ 6^{x-5} &= 6^{3x}\\ x-5 &= 3x\\ -2x &= 5\\ x &= -\frac{5}{2} = -2\frac{1}{2}\end{aligned}$$

f

$$16^{2x-1} = 4^{x-4}$$
$$4^{4x-2} = 4^{x-4}$$
$$4x - 2 = x - 4$$
$$3x = -2$$
$$x = -\frac{2}{3}$$

g

$$27^{x+3} = 3^{x}$$
$$3^{3x+9} = 3^{x}$$
$$3x + 9 = x$$
$$2x = -9$$
$$x = -\frac{9}{2} = -4\frac{1}{2}$$

h

$$\left(\frac{1}{2}\right)^{x} = \left(\frac{1}{64}\right)^{2x+3}$$
$$2^{-x} = 2^{-6(2x+3)}$$
$$-x = -12x - 18$$
$$11x = -18$$
$$x = -\frac{18}{11} = -1\frac{7}{11}$$

i

$$\left(\frac{3}{4}\right)^{x} = \left(\frac{27}{64}\right)^{2x-3}$$
$$\left(\frac{3}{4}\right)^{x} = \left(\frac{3}{4}\right)^{3(2x-3)}$$
$$x = 6x - 9$$
$$-5x = -9$$
$$x = \frac{9}{5} = 1\frac{4}{5}$$

Question 5

a

$$4^m = \sqrt{2}$$
$$2^{2m} = 2^{\frac{1}{2}}$$
$$2m = \frac{1}{2}$$
$$m = \frac{1}{4}$$

b

$$\left(\frac{9}{25}\right)^{k+3} = \sqrt{\frac{3}{5}}$$
$$\left(\frac{3}{5}\right)^{2(k+3)} = \left(\frac{3}{5}\right)^{\frac{1}{2}}$$
$$2k + 6 = \frac{1}{2}$$
$$2k = -5\frac{1}{2}$$
$$k = -2\frac{3}{4}$$

c

$$\frac{1}{\sqrt{2}} = 4^{2x-5}$$
$$2^{-\frac{1}{2}} = 2^{4x-10}$$
$$-\frac{1}{2} = 4x - 10$$
$$8x = 19$$
$$x = \frac{19}{8} = 2\frac{3}{8}$$

d

$$3^k = 3\sqrt{3}$$
$$3^k = 3^{1+\frac{1}{2}}$$
$$k = 1 + \frac{1}{2} = 1\frac{1}{2}$$

e

$$\left(\frac{1}{27}\right)^{3n+1} = \frac{\sqrt{3}}{81}$$

$$3^{-3\ 3n+1)} = \frac{3^{\frac{1}{2}}}{3^4}$$

$$3^{-3(3n+1)} = 3^{\frac{1}{2}-4}$$

$$-9n - 3 = \frac{-7}{2}$$

$$-9n = -\frac{1}{2}$$

$$n = \frac{1}{18}$$

f

$$\left(\frac{2}{5}\right)^{3n+1} = \left(\frac{5}{2}\right)^{-n}$$

$$\left(\frac{2}{5}\right)^{3n+1} = \left(\frac{2}{5}\right)^{n}$$

$$3n + 1 = n$$

$$2n = -1$$

$$n = -\frac{1}{2}$$

g

$$32^{-x} = \left(\frac{1}{16}\right)$$

$$\left(\frac{1}{2}\right)^{5x} = \left(\frac{1}{2}\right)^{4}$$

$$5x = 4$$

$$x = \frac{4}{5}$$

h

$$9^{2b+5} = 3^b\sqrt{3}$$
$$3^{2(2b+5)} = 3^{b+\frac{1}{2}}$$
$$4b + 10 = b + \frac{1}{2}$$
$$3b = -9\frac{1}{2}$$
$$b = -\frac{19}{6} = -3\frac{1}{6}$$

i

$$81^{x+1} = \sqrt{3^x}$$
$$3^{4(x+1)} = 3^{\frac{x}{2}}$$
$$4x + 4 = \frac{x}{2}$$
$$8x = x - 8$$
$$7x = -8$$
$$x = -\frac{8}{7} = -1\frac{1}{7}$$

Question 6

a

$$x^3 = 27$$
$$x^3 = 3^3$$
$$x = 3$$

b

$$y^2 = 64$$
$$y^2 = (\pm 8)^2$$
$$y = \pm 8$$

c

$$n^4 = 16$$
$$n^4 = (\pm 2)^4$$
$$n = \pm 2$$

d

$$x^2 = 20$$
$$x^2 = \left(\pm\sqrt{20}\right)^2$$
$$x^2 = \left(\pm 2\sqrt{5}\right)^2$$
$$x = \pm 2\sqrt{5}$$

e

$$p^3 = 1000$$
$$p^3 = (10)^3$$
$$p = 10$$

f

$$2x^2 = 50$$
$$x^2 = 25$$
$$x^2 = (\pm 5)^2$$
$$x = \pm 5$$

g

$$6y^4 = 486$$
$$y^4 = 81$$
$$y^4 = (\pm 3)^4$$
$$y = \pm 3$$

h

$$w^3 + 7 = 15$$
$$w^3 = 8$$
$$w^3 = 2^3$$
$$w = 2$$

i

$$6n^2 - 4 = 92$$
$$6n^2 = 96$$
$$n^2 = 16$$
$$n^2 = (\pm 4)^2$$
$$n = \pm 4$$

Question 7

a

$$p^2 = 45$$
$$p = \pm\sqrt{45}$$
$$p = \pm 6.71$$

b

$$x^3 = 100$$
$$x = \sqrt[3]{100}$$
$$x = 4.64$$

c

$$n^5 = 240$$
$$n = \sqrt[5]{240}$$
$$n = 2.99$$

d

$$2x^2 = 70$$
$$x^2 = 35$$
$$x = \pm\sqrt{35}$$
$$x = \pm 5.92$$

e

$$4y^3 + 7 = 34$$
$$4y^3 = 27$$
$$y = \sqrt[3]{\frac{27}{4}}$$
$$y = 1.89$$

f

$$\frac{d^4}{3} = 14$$
$$d^4 = 42$$
$$d = \pm\sqrt[4]{42}$$
$$d = \pm 2.55$$

g

$$\frac{k^2}{2} - 3 = 7$$
$$\frac{k^2}{2} = 10$$
$$k^2 = 20$$
$$k = \pm\sqrt{20}$$
$$k = \pm 4.47$$

h

$$\frac{x^3 - 1}{5} = 2$$
$$x^3 - 1 = 10$$
$$x^3 = 11$$
$$x = \sqrt[3]{11}$$
$$x = 2.22$$

i

$$2y^2 - 9 = 20$$
$$2y^2 = 29$$
$$y^2 = \frac{29}{2}$$
$$y = \pm\sqrt{\frac{29}{2}}$$
$$y = \pm 3.81$$

Question 8

a

$$x^{-1} = 5$$
$$\frac{1}{x} = 5$$
$$1 = 5x$$
$$x = \frac{1}{5}$$

b

$$a^{-3} = 8$$
$$\frac{1}{a^3} = 8$$
$$1 = 8a^3$$
$$a = \sqrt[3]{\frac{1}{8}}$$
$$a = \frac{1}{2}$$

c

$$y^{-5} = 32$$
$$\left(\frac{1}{y}\right)^5 = 2^5$$
$$\frac{1}{y} = 2$$
$$y = \frac{1}{2}$$

d

$$x^{-2}+1=50$$
$$\left(\frac{1}{x}\right)^2=49$$
$$\left(\frac{1}{x}\right)^2=\left(\pm7\right)^2$$
$$\frac{1}{x}=\pm7$$
$$x=\pm\frac{1}{7}$$

e

$$2n^{-}=3$$
$$\left(\frac{1}{n}\right)^1=\left(\frac{3}{2}\right)^1$$
$$n=\frac{2}{3}$$

f

$$a^{-3}=\frac{1}{8}$$
$$\left(\frac{1}{a}\right)^3=\left(\frac{1}{2}\right)^3$$
$$a=2$$

g

$$x^{-2}=\frac{1}{4}$$
$$\left(\frac{1}{x}\right)^2=\left(\pm\frac{1}{2}\right)^2$$
$$x=\pm2$$

h

$$b^{-1}=\frac{1}{9}$$
$$\left(\frac{1}{b}\right)^1=\left(\frac{1}{9}\right)^1$$
$$b=9$$

i

$$x^{-2} = 2\frac{1}{4}$$

$$\left(\frac{1}{x}\right)^2 = \left(\frac{9}{4}\right)^1$$

$$\left(\frac{1}{x}\right)^2 = \left(\pm\frac{3}{2}\right)^2$$

$$x = \pm\frac{2}{3}$$

j

$$b^{-4} = \frac{16}{81}$$

$$\left(\frac{1}{b}\right)^4 = \left(\pm\frac{2}{3}\right)^4$$

$$b = \pm\frac{3}{2} = \pm1\frac{1}{2}$$

Puzzle

Question 1

All months have 28 days. Some months have more days as well.

Question 2

10

Question 3

Bottle $1.05; cork 5 cents

Question 4

16 each time

Question 5

Friday

Exercise 2.06 Solving quadratic equations by factorisation

Question 1

$y^2 + y = 0$
$y(y+1) = 0$
$\therefore y = 0, -1$

Question 2

$b^2 - b - 2 = 0$
$(b-2)(b+1) = 0$
$\therefore b = 2, -1$

Question 3

$p^2 + 2p - 15 = 0$
$(p-3)(p+5) = 0$
$\therefore p = 3, -5$

Question 4

$t^2 - 5t = 0$
$t(t-5) = 0$
$\therefore t = 0, 5$

Question 5

$x^2 + 9x + 14 = 0$

$(x+2)(x+7) = 0$

$\therefore x = -2, -7$

Question 6

$q^2 - 9 = 0$

$(q+3)(q-3) = 0$

$\therefore q = \pm 3$

Question 7

$x^2 - 1 = 0$

$(x+1)(x-1) = 0$

$\therefore x = \pm 1$

Question 8

$a^2 + 3a = 0$

$a(a+3) = 0$

$\therefore a = 0, -3$

Question 9

$2x^2 + 8x = 0$

$2x(x+4) = 0$

$\therefore x = 0, -4$

Question 10

$$4x^2 - 1 = 0$$
$$(2x - 1)(2x + 1) = 0$$
$$\therefore x = \pm\frac{1}{2}$$

Question 11

$$3x^2 + 7x + 4 = 0$$
$$3x^2 + 3x + 4x + 4 = 0$$
$$3x(x + 1) + 4(x + 1) = 0$$
$$(x + 1)(3x + 4) = 0$$
$$\therefore x = -1 \quad -\frac{4}{3}$$

Question 12

$$2y^2 + y - 3 = 0$$
$$2y^2 - 2y + 3y - 3 = 0$$
$$2y(y - 1) + 3(y - 1) = 0$$
$$(y - 1)(2y + 3) = 0$$
$$\therefore y = 1 \quad -\frac{3}{2}$$

Question 13

$$8b^2 - 10b + 3 = 0$$
$$8b^2 - 4b - 6b + 3 = 0$$
$$4b(2b - 1) - 3(2b - 1) = 0$$
$$(2b - 1)(4b - 3) = 0$$
$$\therefore b = \frac{1}{2} \quad \frac{3}{4}$$

Question 14

$$x^2 - 3x = 10$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$
$$\therefore x = 5, -2$$

Question 15

$$3x^2 = 2x$$
$$3x^2 - 2x = 0$$
$$x(3x-2) = 0$$
$$\therefore x = 0, \frac{2}{3}$$

Question 16

$$2x^2 = 7x - 5$$
$$2x^2 - 7x + 5 = 0$$
$$2x^2 - 2x - 5x + 5 = 0$$
$$2x(x-1) - 5(x-1) = 0$$
$$(x-1)(2x-5) = 0$$
$$\therefore x = 1, \frac{5}{2}$$

Question 17

$$x(5-x) = 0$$
$$x = 0, x = 5$$

Question 18

$$y^2 = y + 2$$
$$y^2 - y - 2 = 0$$
$$(y+1)(y-2) = 0$$
$$\therefore\ y = -1, 2$$

Question 19

$$8n = n^2 + 15$$
$$n^2 - 8n + 15 = 0$$
$$(n-3)(n-5) = 0$$
$$\therefore\ n = 3, 5$$

Question 20

$$12 = 7x - x^2$$
$$x^2 - 7x + 12 = 0$$
$$(x-3)(n-4) = 0$$
$$\therefore\ x = 3, 4$$

Question 21

$$m^2 = 6 - 5m$$
$$m^2 + 5m - 6 = 0$$
$$(m-1)(m+6) = 0$$
$$\therefore\ m = 1, -6$$

Question 22

$$x(x+1)(x+2)=0$$
$$\therefore x=0,-1,-2$$

Question 23

$$(y-1)(y+5)(y+2)=0$$
$$\therefore y=1,-5,-2$$

Question 24

$$(x+3)(x-1)=32$$
$$x^2+2x-3-32=0$$
$$x^2+2x-35=0$$
$$(x-5)(x+7)=0$$
$$\therefore x=5,-7$$

Question 25

$$(m-3)(m-4)=20$$
$$m^2-7m+12-20=0$$
$$m^2-7m-8=0$$
$$(m-8)(m+1)=0$$
$$\therefore m=8,-1$$

Exercise 2.07 Solving quadratic equations by completing the square

Question 1

a

$$x+1=\pm\sqrt{7}$$
$$x=\pm\sqrt{7}-1$$

b

$$y+5=\pm\sqrt{5}$$
$$y=\pm\sqrt{5}-5$$

c

$$a-3=\pm\sqrt{6}$$
$$a=3\pm\sqrt{6}$$

d

$$x-2=\pm\sqrt{13}$$
$$x=2\pm\sqrt{13}$$

e

$$2y+3=\pm\sqrt{2}$$
$$y=\frac{\pm\sqrt{2}-3}{2}$$

Question 2

a

$$h + 2 = \pm\sqrt{15}$$
$$h = \pm\sqrt{15} - 2$$
$$h = -5.9,\ 1.9$$

b

$$a - 1 = \pm\sqrt{8}$$
$$a = 1 \pm \sqrt{8}$$
$$a = -1.8,\ 3.8$$

c

$$x - 4 = \pm\sqrt{17}$$
$$x = 4 \pm \sqrt{17}$$
$$x = -0.1,\ 8.1$$

d

$$y + 7 = \pm\sqrt{21}$$
$$h = -7 \pm \sqrt{21}$$
$$h = -11.6,\ -2.4$$

e

$$3x - 1 = \pm\sqrt{12}$$
$$3x = 1 \pm \sqrt{12}$$
$$x = \frac{1 \pm \sqrt{12}}{3}$$
$$x = -0.8,\ 1.5$$

Question 3

a

$$x^2+4x-1=0$$
$$x^2+4x+4-4-1=0$$
$$(x+2)^2=5$$
$$x+2=\pm\sqrt{5}$$
$$x=\pm\sqrt{5}-2$$

b

$$a^2-6a+2=0$$
$$a^2-6a+9-9+2=0$$
$$(a-3)^2=7$$
$$a-3=\pm\sqrt{7}$$
$$a=\pm\sqrt{7}+3$$

c

$$y^2-8y-7=0$$
$$y^2-8y+16-16-7=0$$
$$(y-4)^2=23$$
$$y-4=\pm\sqrt{23}$$
$$y=\pm\sqrt{23}+4$$

d

$$x^2+2x-12=0$$
$$x^2+2x+1-1-12=0$$
$$(x+1)^2=13$$
$$x+1=\pm\sqrt{13}$$
$$x=\pm\sqrt{13}-1$$

e

$$p^2+14p+5=0$$
$$p^2+14p+49-49+5=0$$
$$(p+7)^2=44$$
$$p+7=\pm\sqrt{44}$$
$$p=\pm2\sqrt{11}-7$$

f

$$x^2-10x-3=0$$
$$x^2-10x+25-25-3=0$$
$$(x-5)^2=28$$
$$x-5=\pm\sqrt{28}$$
$$x=\pm2\sqrt{7}+5$$

g

$$y^2+20y+12=0$$
$$y^2+20y+100-100+12=0$$
$$(y+10)^2=88$$
$$y+10=\pm\sqrt{88}$$
$$x=\pm2\sqrt{22}-10$$
$$x=2(\pm\sqrt{22}-5)$$

h

$$x^2-2x-1=0$$
$$x^2-2x+1-1-1=0$$
$$(x-1)^2=2$$
$$x-1=\pm\sqrt{2}$$
$$x=\pm\sqrt{2}+1$$

i

$$n^2+24n+7=0$$
$$n^2+24n+144-144+7=0$$
$$(n+12)^2=137$$
$$n+12=\pm\sqrt{137}$$
$$n=\pm\sqrt{137}-12$$

Question 4

a

$$x^2 - 2x - 5 = 0$$
$$x^2 - 2x + 1 - 1 - 5 = 0$$
$$(x - 1)^2 = 6$$
$$x - 1 = \pm\sqrt{6}$$
$$x = \pm\sqrt{6} + 1$$
$$x = 3.45,\ -1.45$$

b

$$x^2 + 12x + 34 = 0$$
$$x^2 + 12x + 36 - 36 + 34 = 0$$
$$(x + 6)^2 = 2$$
$$x + 6 = \pm\sqrt{2}$$
$$x = \pm\sqrt{2} - 6$$
$$x = -4.59,\ -7.41$$

c

$$q^2 + 18q - 1 = 0$$
$$q^2 + 18q + 81 - 81 - 1 = 0$$
$$(q + 9)^2 = 82$$
$$q + 9 = \pm\sqrt{82}$$
$$q = \pm\sqrt{82} - 9$$
$$q = 0.0554,\ -18.1$$

d

$$x^2 - 4x - 2 = 0$$
$$x^2 - 4x + 4 - 4 - 2 = 0$$
$$(x - 2)^2 = 6$$
$$x - 2 = \pm\sqrt{6}$$
$$x = \pm\sqrt{6} + 2$$
$$x = 445,0.-449$$

e

$$b^2+16b+50=0$$
$$b^2+16b+64-64+50=0$$
$$(b+8)^2=14$$
$$b+8=\pm\sqrt{14}$$
$$b=\pm\sqrt{14}-8$$
$$b=-4.26,\ -11.7$$

f

$$x^2-24x+112=0$$
$$x^2-24x+144-144+112=0$$
$$(x-12)^2=32$$
$$x-12=\pm\sqrt{32}$$
$$x=\pm\sqrt{32}+12$$
$$x=17.7,\ 6.34$$

g

$$r^2-22r-7=0$$
$$r^2-22r+121-121-7=0$$
$$(r-11)^2=128$$
$$r-11=\pm\sqrt{128}$$
$$r=\pm\sqrt{128}+11$$
$$r=22.3,\ -0.314$$

h

$$x^2+8x+5=0$$
$$x^2+8x+16-16+5=0$$
$$(x+4)^2=11$$
$$x+4=\pm\sqrt{11}$$
$$x=\pm\sqrt{11}-4$$
$$x=-0.683,\ -7{,}32$$

i

$$a^2+6a-1=0$$
$$a^2+6a+9-9-1=0$$
$$(a+3)^2=10$$
$$a+3=\pm\sqrt{10}$$
$$a=\pm\sqrt{10}-3$$
$$a=0.162,\ -6,16$$

Exercise 2.08 Solving quadratic equations by quadratic formula

Question 1

a

$$a = 1, b = 6, c = 2$$
$$y = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 2}}{2 \times 1}$$
$$= \frac{-6 \pm \sqrt{28}}{2}$$
$$y = \frac{-6 + \sqrt{28}}{2} = -0.354$$
$$y = \frac{-6 \pm \sqrt{28}}{2} = -5.65$$

b

$$a = 2, b = -5, c = 3$$
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 2 \times 3}}{2 \times 2}$$
$$= \frac{5 \pm 1}{4}$$
$$x = \frac{5 + 1}{4} = 1.5$$
$$x = \frac{5 - 1}{4} = 1$$

c

$$a = 1, b = -1, c = -9$$
$$b = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times -9}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{37}}{2}$$
$$b = \frac{1 - \sqrt{37}}{2} = -2.54$$
$$b = \frac{1 + \sqrt{37}}{2} = 3.54$$

d

$$a = 2, b = -1, c = -1$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 2 \times -1}}{2 \times 2}$$

$$= \frac{1 \pm 3}{4}$$

$$x = \frac{1-3}{4} = -0.5$$

$$x = \frac{1+3}{4} = 1$$

e

$$a = -8, b = 1, c = 3$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times -8 \times 3}}{2 \times -8}$$

$$= \frac{-1 \pm \sqrt{97}}{-16}$$

$$x = \frac{-1 + \sqrt{97}}{-16} = -0.553$$

$$x = \frac{-1 - \sqrt{97}}{-16} = 0.678$$

f

$$a = 1, b = 8, c = -2$$

$$n = \frac{-8 \pm \sqrt{8^2 - 4 \times 1 \times -2}}{2 \times 1}$$

$$= \frac{-8 \pm \sqrt{72}}{2}$$

$$n = \frac{-8 + \sqrt{72}}{2} = 0.243$$

$$n = \frac{-8 - \sqrt{72}}{2} = -8.24$$

g

$$a=1, b=7, c=10$$

$$m=\frac{-7\pm\sqrt{7^2-4\times1\times10}}{2\times1}$$

$$=\frac{-7\pm3}{2}$$

$$m=\frac{-7+3}{2}=-2$$

$$m=\frac{-7-3}{2}=-5$$

h

$$a=1, b=-7, c=0$$

$$x=\frac{-(-7)\pm\sqrt{(-7)^2-4\times1\times0}}{2\times1}$$

$$=\frac{7\pm7}{2}$$

$$x=\frac{7+7}{2}=7$$

$$x=\frac{7-7}{2}=0$$

i

$$x^2+5x-6=0$$

$$a=1, b=5, c=-6$$

$$x=\frac{-5\pm\sqrt{5^2-4\times1\times-6}}{2\times1}$$

$$=\frac{-5\pm7}{2}$$

$$x=\frac{-5+7}{2}=1$$

$$x=\frac{-5-7}{2}=-6$$

Question 2

a

$$a=1, b=1, c=-4$$
$$x=\frac{-1\pm\sqrt{1^2-4\times1\times-4}}{2\times1}$$
$$=\frac{-1\pm\sqrt{17}}{2}$$

b

$$a=3, b=-5, c=1$$
$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4\times3\times1}}{2\times3}$$
$$=\frac{5\pm\sqrt{13}}{6}$$

c

$$a=1, b=-4, c=-3$$
$$q=\frac{-(-4)\pm\sqrt{(-4)^2-4\times1\times-3}}{2\times1}$$
$$=\frac{4\pm\sqrt{28}}{2}$$
$$=\frac{4\pm2\sqrt{7}}{2}$$
$$=2\pm\sqrt{7}$$

d

$$a=4, b=12, c=1$$
$$h=\frac{-12\pm\sqrt{12^2-4\times4\times1}}{2\times4}$$
$$=\frac{-12\pm\sqrt{128}}{8}$$
$$=\frac{-12\pm8\sqrt{2}}{8}$$
$$=\frac{-3\pm2\sqrt{2}}{2}$$

e

$$a = 3, b = -8, c = 2$$

$$s = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= \frac{8 \pm \sqrt{40}}{6}$$

$$= \frac{8 \pm 2\sqrt{10}}{6}$$

$$= \frac{4 \pm \sqrt{10}}{3}$$

f

$$a = 1, b = 11, c = -3$$

$$x = \frac{-11 \pm \sqrt{11^2 - 4 \times 1 \times -3}}{2 \times 1}$$

$$= \frac{-11 \pm \sqrt{133}}{2}$$

g

$$a = 6, b = 5, c = -2$$

$$d = \frac{-5 \pm \sqrt{5^2 - 4 \times 6 \times -2}}{2 \times 6}$$

$$= \frac{-5 \pm \sqrt{73}}{12}$$

h

$$x^2 - 2x - 7 = 0$$

$$a = 1, b = -2, c = -7$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times -7}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{32}}{2}$$

$$= \frac{2 \pm 4\sqrt{2}}{2}$$

$$= 1 \pm 2\sqrt{2}$$

i

$$t^2 - t - 1 = 0$$
$$a = 1, b = -1, c = -1$$
$$t = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times -1}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

Exercise 2.09 Formulas and equations

Question 1

$100.6 = 17.3 + 9.8t$

$9.8t = 82.7$

$t = \frac{83.3}{9.8}$

$t = 8.5$

Question 2

$1625 = \frac{26}{2}(3 + l)$

$3250 = 78 + 26l$

$26l = 3172$

$l = \frac{3172}{26} = 122$

Question 3

$36 = \frac{1}{2}b \times 9$

$72 = 9b$

$b = 8$

Question 4

$120 = \frac{1}{2} \times 5 \times (a + 7)$

$240 = 5a + 35$

$5a = 205$

$a = 41$

Question 5

$$5 \times 3 - 2y - 7 = 0$$
$$8 - 2y = 0$$
$$2y = 8$$
$$y = 4$$

Question 6

$$140 = \pi r^2$$
$$r^2 = \frac{140}{\pi}$$
$$r = \sqrt{\frac{140}{\pi}} = 6.68$$

Question 7

$$25.1 = \frac{1}{2} \times x \times 7.8$$
$$50.2 = 7.8x$$
$$x = \frac{50.2}{7.8} = 6.44$$

Question 8

$$326.25 = \frac{150 \times 14.5 \times n}{100}$$
$$32625 = 2175n$$
$$n = \frac{32625}{2175} = 15$$

Question 9

$$-\frac{5}{6} = \frac{7 - y_1}{-3 - 1}$$

$$-\frac{5}{6} = \frac{7 - y_1}{-4}$$

$$20 = 6(7 - y\ )$$

$$20 = 42 - 6y_1$$

$$6y\ = 22$$

$$y_1 = 3\frac{2}{3}$$

Question 10

$$232 = 2\pi \times 4.5 \times (4.5 + h)$$

$$232 = 9\pi(4.5 + h)$$

$$4.5 + h = \frac{232}{9\pi}$$

$$h = \frac{232}{9\pi} - 4.5$$

$$h = 3.7$$

Question 11

a

$$\frac{65}{1.6^2} = 25.39$$

b

$$21.5 = \frac{w}{1.8^2}$$
$$w = 21.5 \times 1.8^2$$
$$w = 69.66$$

c

$$19.7 = \frac{73.8}{h^2}$$
$$h^2 = \frac{73.8}{19.7}$$
$$h = \sqrt{\frac{73.8}{19.7}}$$
$$= 1.94$$

Question 12

$$12\ 000 = 15\ 000(1 - r)^3$$
$$(1 - r)^3 = \frac{12\ 000}{15\ 000} = 0.8$$
$$1 - r = \sqrt[3]{0.8}$$
$$r = 1 - \sqrt[3]{0.8}$$
$$= 0.072$$

Question 13

$$-2 = \frac{x_1 + 5}{2}$$
$$-4 = x\ \ + 5$$
$$x_1 = -9$$

Question 14

$$23 = 5t^2$$
$$t^2 = \frac{23}{5} = 4.6$$
$$t = \sqrt{46}\ = .14$$ (time is positive)

Question 15

$$5 = x^2 + 1$$
$$x^2 = 4$$
$$x = \pm 2$$

Question 16

$$56.3 = 4\pi r^2$$
$$r^2 = \frac{56.3}{4\pi}$$
$$r = \sqrt{\frac{56.3}{4\pi}}$$
$$= 2.12$$ (radius must be positive)

Question 17

$$24.6 = \frac{1}{2} \times r^2 \times 0.45$$
$$r^2 = \frac{2 \times 24.6}{0.45}$$
$$r = \sqrt{\frac{2 \times 24.6}{0.45}}$$
$$= 10.46 \quad \text{(radius must be positive)}$$

Question 18

$$3 = \frac{2}{x^3 - 1}$$
$$x^3 - 1 = \frac{2}{3}$$
$$x^3 = 1\frac{2}{3}$$
$$x = \sqrt[3]{1\frac{2}{3}}$$
$$= 1.19$$

Question 19

$$4 = \sqrt{2x + 5}$$
$$16 = 2x + 5$$
$$2x = 11$$
$$x = 5.5$$

Question 20

$$150 = \frac{4}{3}\pi r^3$$

$$r^3 = \frac{3\times150}{4\pi}$$

$$r = \sqrt[3]{\frac{3\times150}{4\pi}}$$

$$= 3.30$$

Exercise 2.10 Linear simultaneous equations

Question 1

$a-b=-2$ [1]

$a+b=4$ [2]

[1] + [2]

$2a=2$
$a=1$

Substitute $a=1$ in [2].

$1+b=4$
$b=3$

Question 2

$5x+2y=12$ [1]

$3x-2y=4$ [2]

[1] + [2]

$8x=16$
$x=2$

Substitute $x=2$ in [1].

$10+2y=12$
$2y=2$
$y=1$

Question 3

$4p-3q=11$ [1]

$5p+3q=7$ [2]

[1] + [2]

$9p=18$
$p=2$

Substitute $p=2$ in [2].

$10+3q=7$
$3q=-3$
$q=-1$

Question 4

$y=3x-1$ [1]

$y=2x+5$ [2]

Substitute [1] in [2].

$3x-1=2x+5$
$x=1+5$
$x=6$

Substitute $x=6$ in [1].

$y=3\times6-1$
$y=17$

Question 5

$2x + 3y = -14$ [1]

$x + 3y = -4$ [2]

2 × [2]

$2x + 6y = -8$ [3]

[3] – [1]

$3y = 6$
$y = 2$

Substitute $y = 2$ in [2].

$x + 3 \times 2 = -4$
$x + 6 = -4$
$x = -10$

Question 6

$7t + v = 22$ [1]

$4t + v = 13$ [2]

[1] – [2]

$3t = 9$
$t = 3$

Substitute $t = 3$ in [2].

$4 \times 3 + v = 13$
$12 + v = 13$
$v = 1$

Question 7

$4x+5y+2=0$ [1]

$4x+y+10=0$ [2]

[1] – [2]

$$4y-8=0$$
$$y=2$$

Substitute $y=2$ in [2].

$$4x+2+10=0$$
$$4x=-12$$
$$x=-3$$

Question 8

$2x-4y=28$ [1]

$2x-3y=-11$ [2]

[2] – [1]

$y=-39$ [3]

Substitute $y=-39$ in [2].

$$2x-3\times-39=-11$$
$$2x=-128$$
$$x=-64$$

Question 9

$5x - y = 19$ [1]

$2x + 5y = -14$ [2]

$2 \times$ [1]

$10x - 2y = 38$ [3]

$5 \times$ [2]

$10x + 25y = -70$ [4]

[4] – [3]

$27y = -108$
$y = -4$

Substitute $y = -4$ in [1].

$5x + 4 = 19$
$5x = 15$
$x = 3$

Question 10

$5m + 4n = 22$ [1]

$m - 5n = -13$ [2]

5 × [2]

$5m - 25n = -65$ [3]

[1] – [3

$29n = 87$
$n = 3$

Substitute $n = 3$ in [1].

$5m + 4 \times 3 = 22$
$5m = 10$
$m = 2$

Question 11

$4w_1 + 3w_2 = 11$ [1]

$3w_1 + w_2 = 2$ [2]

3 × [2]

$9w_1 + 3w_2 = 6$ [3]

[3] – [1]

$5w = -5$
$w = -1$

Substitute $w = -1$ in [2]

$-3 + w_2 = 2$
$w_2 = 5$

Question 12

$3a - 4b = -16$ [1]

$2a + 3b = 12$ [2]

2 × [1]

$6a - 8b = -32$ [3]

3 × [2]

$6a + 9b = 36$ (4)

[4] – [3]

$17b = 68$
$b = 4$

Substitute $b = 4$ in [4].

$6a + 9 \times 4 = 36$
$6a = 0$
$a = 0$

Question 13

$5p+2q+18=0$ [1]

$2p-3q+11=0$ [2]

$2 \times [1]$

$10p+4q+36=0$ [3]

$5 \times [2]$

$10p-15q+55=0$ [4]

$[3]-[4]$

$19q-19=0$
$q=1$

Substitute $q=1$ in (1)

$5p+2\times 1+18=0$
$5p+20=0$
$p=-4$

Question 14

$7x_1 + 3x_2 = 4$ [1]

$3x_1 + 5x_2 = -2$ [2]

3 × [1]

$21x_1 + 9x_2 = 12$ [3]

7 × [2]

$21x_1 + 35x_2 = -14$ [4]

[4] – [3]

$$26x_2 = -26$$
$$x_2 = -1$$

Substitute $x_2 = -1$ in (1).

$$7x\ + 3 \times -1 = 4$$
$$7x\ = 7$$
$$x_1 = 1$$

Question 15

$9x-2y=-1$ [1]

$7x-4y=9$ [2]

$2 \times$ [1]

$18x-4y=-2$ [3]

[3] – [2]

$11x=-11$
$x=-1$

Substitute $x=-1$ in [1].

$9\times -1-2y=-1$
$-2y=8$
$y=-4$

Question 16

$5s - 3t - 13 = 0$ [1]

$3s - 7t - 13 = 0$ [2]

3 × [1]

$15s - 9t - 39 = 0$ [3]

5 × [2]

$15s - 35t - 65 = 0$ [4]

[3] – [4]

$26t + 26 = 0$
$t = -1$

Substitute $t = -1$ in [1].

$5s - 3 \times -1 - 13 = 0$
$5s - 10 = 0$
$s = 2$

Question 17

$3a-2b=-6$ [1]

$a-3b=-2$ [2]

3 × [2]

$3a-9b=-6$ [3]

[1] – [3]

$7b=0$
$b=0$

Substitute $b=0$ in [1].

$3a-2\times 0=-6$
$3a=-6$
$a=-2$

Question 18

$3k-2h=-14$ [1]

$2k-5h=-13$ [2]

2 × [1]

$6k-4h=-28$ [3]

3 × [2]

$6k-15h=-39$ [4]

[3] – [4]

$11h=11$
$h=1$

Substitute $h=1$ in [1].

$3k-2\times 1=-14$
$3k=-12$
$k=-4$

Problem

23 adults and 16 children.

$x + y = 39$ [1]

$29x + 17y = 939$ [2]

29 × [1]

$29x + 29y = 1131$ [3]

[3] – [2]

$12y = 192$
$y = 16$

Substitute $y = 16$ in [1].

$x + 16 = 39$
$x = 23$

Exercise 2.11 Non-linear simultaneous equations

Question 1

$y = x^2$ [1]

$y = x$ [2]

Substitute [1] in [2].

$$\begin{aligned} x^2 &= x \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ x &= 0, x = 1 \end{aligned}$$

From [2], $x = 0, y = 0$ and $x = 1, y = 1$.

Question 2

$y = x^2$ [1]

$2x + y = 0$ [2]

Substitute [1] in [2].

$$\begin{aligned} 2x + x^2 &= 0 \\ x(x+2) &= 0 \\ x &= 0, x = -2 \end{aligned}$$

From [1], $x = 0, y = 0$ and $x = -2, y = (-2)^2 = 4$.

Question 3

$x^2 + y^2 = 9$ [1]

$x + y = 3 \Rightarrow y = 3 - x$ [2]

Substitute [2] in [1].

$$x^2 + (3 - x)^2 = 9$$
$$x^2 + 9 - 6x + x^2 = 9$$
$$2x^2 - 6x = 0$$
$$2x(x - 3) = 0$$
$$x = 0, x = 3$$

From [2]

$x = 0, y = 3 - 0 = 3$

$x = 3, y = 3 - 3 = 0$

Question 4

$x - y = 7 \Rightarrow y = x - 7$ [1]

$xy = -12$ [2]

Substitute [1] in [2].

$$x(x - 7) = -12$$
$$x^2 - 7x + 12 = 0$$
$$(x - 3)(x - 4) = 0$$
$$x = 3, x = 4$$

From [1]

$x = 3, y = 3 - 7 = -4$

$x = 4, y = 4 - 7 = -3$

Question 5

$y = x^2 + 4x$ [1]

$2x - y - 1 = 0 \Rightarrow y = 2x - 1$ [2]

Substitute [1] in [2].

$$x^2 + 4x = 2x - 1$$
$$x^2 + 2x + 1 = 0$$
$$(x+1)^2 = 0$$
$$x = -1$$

From [2]

$x = -1, y = 2 \times -1 - 1 = -3$

Question 6

$y = x^2$ [1]

$6x - y - 9 = 0 \Rightarrow y = 6x - 9$ [2]

Substitute [1] in [2].

$$x^2 = 6x - 9$$
$$x^2 - 6x + 9 = 0$$
$$(x-3)^2 = 0$$
$$x = 3$$

From [2]

$x = 3, y = 6 \times 3 - 9 = 9$

Question 7

$x = t^2$ [1]

$x + t - 2 = 0 \Rightarrow t = -x + 2$ [2]

Substitute [2] in [1].

$$x = (-x + 2)^2$$
$$x = x^2 - 4x + 4$$
$$x^2 - 5x + 4 = 0$$
$$(x - 1)(x - 4) = 0$$
$$x = 1, x = 4$$

From [2]

$$x = 1, t = -1 + 2 = 1$$
$$x = 4, t = -4 + 2 = -2$$

Question 8

$m^2 + n^2 = 16$ [1]

$m + n + 4 = 0 \Rightarrow n = -m - 4$ [2]

Substitute [2] in [1].

$$m^2 + (-m - 4)^2 = 16$$
$$2m^2 + 4m + 16 = 16$$
$$2m(m + 4) = 0$$
$$m = 0, m = -4$$

From [2]

$$m = 0, n = 0 - 4 = -4$$
$$m = -4, n = 4 - 4 = 0$$

Question 9

$xy = 2$ [1]

$y = 2x$ [2]

Substitute [2] in [1].

$$x(2x) = 2$$
$$2x^2 = 2$$
$$x = \pm 1$$

From [2]

$$x = -1, y = 2 \times -1 = -2$$
$$x = 1, y = 2 \times 1 = 2$$

Question 10

$y = x^3$ [1]

$y = x^2$ [2]

Substitute [1] in [2].

$$x^3 = x^2$$
$$x^3 - x^2 = 0$$
$$x^2(x-1) = 0$$
$$x = 0, x = 1$$

From [2]

$$x = 0, y = 0$$
$$x = 1, y = 1$$

Question 11

$y = x - 1$ [1]

$y = x^2 - 3$ [2]

Substitute [2] in [1].

$$
\begin{aligned}
&x^2 - 3 = x - 1 \\
&x^2 - x - 2 = 0 \\
&(x-2)(x+1) = 0 \\
&x = -1, x = 2
\end{aligned}
$$

From [1]

$$
\begin{aligned}
&x = -1, y = -1 - 1 = -2 \\
&x = 2, y = 2 - 1 = 1
\end{aligned}
$$

Question 12

$y = x^2 + 1$ [1]

$y = 1 - x^2$ [2]

Substitute [1] in [2].

$$
\begin{aligned}
&x^2 + 1 = 1 - x^2 \\
&2x^2 = 0 \\
&x = 0
\end{aligned}
$$

From [1]

$x = 0, y = 0 + 1 = 1$

Question 13

$y = x^2 - 3x + 7$ [1]

$y = 2x + 3$ [2]

Substitute [1] in [2].

$$x^2 - 3x + 7 = 2x + 3$$
$$x^2 - 5x + 4 = 0$$
$$(x-1)(x-4) = 0$$
$$x = 1, x = 4$$

From [2]

$$x = 1, y = 2\times 1 + 3 = 5$$
$$x = 4, y = 2\times 4 + 3 = 11$$

Question 14

$xy = 1$ [1]

$4x - y + 3 = 0 \Rightarrow y = 4x + 3$ [2]

Substitute [2] in [1].

$$x\times(4x+3) = 1$$
$$4x^2 + 3x = 1$$
$$4x^2 + 3x - 1 = 0$$
$$(4x-1)(x+1) = 0$$
$$x = \frac{1}{4}, x = -1$$

From [2]

$$x = \frac{1}{4}, y = 4\times\frac{1}{4} + 3 = 4$$
$$x = -1, y = 4\times -1 + 3 = -1$$

Question 15

$h = t^2$ [1]

$h = (t+1)^2$ [2]

Substitute [2] in [1].

$$(t+1)^2 = t^2$$
$$t^2 + 2t + 1 = t^2$$
$$2t + 1 = 0$$
$$t = -\frac{1}{2}$$

From [1]

$$t = -\frac{1}{2} \;\; h = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$$

Question 16

$x + y = 2 \Rightarrow y = 2 - x$ [1]

$2x^2 + xy - y^2 = 8$ [2]

Substitute [1] in [2].

$$2x^2 + x(2-x) - (2-x)^2 = 8$$
$$2x^2 + 2x - x^2 - 4 + 4x - x^2 = 8$$
$$6x - 4 = 8$$
$$6x = 12$$
$$x = 2$$

From [1]

$$x = 2, y = 2 - 2 = 0$$

Question 17

$y = x^3$ [1]

$y = x^2 + 6x$ [2]

Substitute [1] in [2].

$$x^3 = x^2 + 6x$$
$$x^3 - x^2 - 6x = 0$$
$$x(x^2 - x - 6) = 0$$
$$x(x+2)(x-3) = 0$$
$$x = 0, x = -2, x = 3$$

From [1]

$$x = 0, y = 0$$
$$x = -2, y = (-2)^3 = -8$$
$$x = 3, y = 3^3 = 27$$

Question 18

Case 1

$y = |x| \Rightarrow y = x, x \geq 0$ [1]

$y = x^2$ [2]

Substitute [2] in [1].

$$\begin{aligned}
&x^2 = x \\
&x^2 - x = 0 \\
&x(x-1) = 0 \\
&x = 0, x = 1
\end{aligned}$$

From [1]

$$\begin{aligned}
&x = 0, y = 0 \\
&x = 1, y = 1
\end{aligned}$$

Case 2

$y = |x| \Rightarrow y = -x, x < 0$ [1]

$y = x^2$ [2]

Substitute [2] in [1].

$$\begin{aligned}
&x^2 = -x \\
&x^2 + x = 0 \\
&x(x+1) = 0 \\
&x = 0, x = -1
\end{aligned}$$

From [1]

$x = -1, y = 1$

Question 19

$y = x^2 - 7x + 6$ [1]

$24x + 4y - 23 = 0$ [2]

Substitute [1] in [2].

$$24x + 4(x^2 - 7x + 6) - 23 = 0$$
$$4x^2 - 4x + 1 = 0$$
$$(2x - 1)^2 = 0$$
$$x = \frac{1}{2}$$

From [1]

$$x = \frac{1}{2}, y = \left(\frac{1}{2}\right)^2 - 7 \times \frac{1}{2} + 6 = 2\frac{3}{4}$$

Question 20

$x^2 + y^2 = 1$ [1]

$5x + 12y + 13 = 0$ [2]

From [2]: $y = -\frac{5}{12}x - \frac{13}{12}$ [3]

Substitute [3] into [1].

$$x^2 + \left(-\frac{5}{12}x - \frac{13}{12}\right)^2 = 1$$

$$x^2 + \frac{1}{144}(5x+13)^2 = 1$$

$144x^2 + 25x^2 + 130x + 169 = 144$

$169x^2 + 130x + 25 = 0$

$$x = \frac{-130 \pm \sqrt{(130)^2 - 4(169)(25)}}{2(169)}$$

$$= \frac{-130 \pm \sqrt{0}}{338} = \frac{-65}{169} = -\frac{5}{13}$$

From [3]

$$y = -\frac{5}{12}\left(-\frac{5}{13}\right) - \frac{13}{12} = \frac{25}{156} - \frac{13}{12} = \frac{25-169}{156} = -\frac{144}{156} = -\frac{12}{13}$$

Exercise 2.12 Simultaneous equations with three unknown variables

Question 1

$x = -2$ [1]

$2x - y = 4$ [2]

$x - y + 6z = 0$ [3]

Substitute $x = -2$ in [2].

$-4 - y = 4 \Rightarrow y = -8$

Substitute $x = -2, y = -8$, in [3].

$-2 - (-8) + 6z = 0 \Rightarrow z = -1$

Question 2

$a = -2$ [1]

$2a - 3b = -1$ [2]

$a - b + 5c = 9$ [3]

Substitute $a = -2$ in [2].

$-4 - 3b = -1 \Rightarrow b = -1$

Substitute $a = -2, b = -1$, in [3].

$-2 - (-1) + 5c = 9 \Rightarrow c = 2$

Question 3

$2a+b+c=1$ [1]

$a+b=-2$ [2]

$c=7$ [3]

Substitute $c=7$ in [1].

$2a+b+7=1 \Rightarrow 2a+b=-6$ [4]

[4] – [2]

$a=-6-(-2)=-4$

Substitute $a=-4$ in [2].

$-4+b=-2 \Rightarrow b=2$

Question 4

$a+b+c=0$ [1]

$a-b+c=-4$ [2]

$2a-3b-c=-1$ [3]

[1] – [2]

$2b=4 \Rightarrow b=2$

Substitute $b=2$ in [1] and [3].

$a+c=-2$ [4]

$2a-c=5$ [5]

[4] + [5]

$3a=3 \Rightarrow a=1$

From [1]

$1+c=-2 \Rightarrow c=-3$

Question 5

$x+y-z=7$ [1]

$x+y+2z=1$ [2]

$3x+y-2z=19$ [3]

[2] – [1]

$3z=-6 \Rightarrow z=-2$

Substitute $z=-2$ in [1] and [3].

$x+y=5$ [4]

$3x+y=15$ [5]

[5] – [4]

$2x=10 \Rightarrow x=5$

From [4]

$5+y=5 \Rightarrow y=0$

Question 6

$2p+5q-r=25$ [1]

$2p-2q-r=-24$ [2]

$3p-q+5r=4$ [3]

[1] – [2]

$7q=49 \Rightarrow q=7$

Substitute $q=7$ in [1] and [3].

$2p-r=-10$ [4]

$3p+5r=11$ [5]

[5] + 5 × [4]

$3p+10p=11-50$
$p=-3$

From [4]

$2\times -3-r=-10 \Rightarrow r=4$

Question 7

$2x - y + 3z = 9$ [1]

$3x + y - 2z = -2$ [2]

$3x - y + 5z = 14$ [3]

[1] + [2]

$5x + z = 7$ [4]

[2] + [3]

$6x + 3z = 12$ [5]

3 × [4] – [5]

$9x = 9 \Rightarrow x = 1$

Substitute $x = 1$ in [4].

$5 \times 1 + z = 7 \Rightarrow z = 2$

From [1]

$$2 \times 1 - y + 3 \times 2 = 9$$
$$8 - y = 9$$
$$y = -1$$

Question 8

$x - y - z = 1$ [1]

$2x + y - z = -9$ [2]

$2x - 3y - 2z = 7$ [3]

[2] – [3]

$4y + z = -16$ [4]

[2] – 2 × [1]

$3y + z = -11$ [5]

[4] – [5]

$y = -5$

Substitute $y = -5$ in [5].

$3 \times -5 + z = -11 \Rightarrow z = 4$

From [1]

$$x - (-5) - 4 = 1$$
$$x = 0$$

Question 9

$3h + j - k = -3$ [1]

$h + 2j + k = -3$ [2]

$5h - 3j - 2k = -13$ [3]

[1] + [2]

$4h + 3j = -6$ [4]

2 × [1] – [3]

$h + 5j = 7$ [5]

4 × [5] – [4]

$17j = 34 \Rightarrow j = 2$

Substitute $j = 2$ in [5].

$h + 5 \times 2 = 7 \Rightarrow h = -3$

From [1]

$$3 \times -3 + 2 - k = -3$$
$$-7 - k = -3 \Rightarrow k = -4$$

Question 10

$2a-7b+3c=7$ [1]

$a+3b+2c=-4$ [2)]

$4a+5b-c=9$ [3]

2 × [2] – [1]

$13b+c=-15$ [4]

4 × [2] – [3]

$7b+9c=-25$ [5]

9 × [4] – [5]

$110b=-110 \Rightarrow b=-1$

Substitute $b=-1$ in [4].

$13\times-1+c=-15 \Rightarrow c=-2$

From [2]

$$a+3\times-1+2\times-2=-4$$
$$a-7=-4 \Rightarrow a=3$$

Exercise 2.13 Quadratic inequalities

Question 1

$x^2 + 3x = x(x+3)$
$x(x+3) = 0 \Rightarrow x = 0, x = -3$

The *x*-intercepts are $x = 0, x = -3$.

Test $x = -1$, which is a point between the *x*-intercepts.

$x^2 + 3x = (-1)^2 + 3 \times -1 = -2 < 0$, as required.

So the solution is in the interval $-3 < x < 0$.

Question 2

$y^2 - 4y = y(y-4)$
$y(y-4) = 0 \Rightarrow y = 0, y = 4$

The *y*-intercepts are $y = 0, y = 4$.

Test $y = 1$, which is a point between the *y*-intercepts.

$y^2 - 4y = 1^2 - 4 \times 1 = -3 < 0$, as required.

So the solution is the interval $0 < y < 4$.

Question 3

$n^2 - n = n(n-1)$
$n(n-1) = 0 \Rightarrow n = 0, n = 1$

The n-intercepts are $n = 0, n = 1$.

Test $n = \frac{1}{2}$, which is a point between the n-intercepts.

$$n^2 - n = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4} < 0$$

We require $n^2 - n \geq 0$, so the solution is outside the interval $0 < n < 1$.

Test a point to the left of the 1st intercept.

$n = -1$, $n^2 - n = 1 + 1 = 2 > 0$, as required.

Test a point to the right of the 2nd intercept.

$n = 2$, $n^2 - n = 4 - 2 = 2 > 0$, as required.

So the solution is $n \leq 0$, $n \geq 1$

Question 4

$x^2 - 4 = (x-2)(x+2)$
$(x-2)(x+2) = 0 \Rightarrow x = -2, x = 2$

The *x*-intercepts are $x = -2, x = 2$.

Test $x = 0$, which is a point between the *x*-intercepts.

$x^2 - 4 = 0^2 - 4 = -4 < 0$

We require $x^2 - 4 \geq 0$, so the solution is outside the interval $-2 \leq x \leq 2$.

Test a point to the left of the 1st intercept.

$x = -3$, $x^2 - 4 = 9 - 4 = 5 \geq 0$, as required.

Test a point to the right of the 2nd intercept.

$x = 3$, $x^2 - 4 = 9 - 4 = 5 \geq 0$, as required.

So the solution is $x \leq -2$, $x \geq 2$.

Question 5

$1-n^2=(1-n)(1+n)$
$(1-n)(1+n)=0 \Rightarrow n=-1, n=1$

The n-intercepts are $n=-1, n=1$.

Test $n=0$, which is a point between the n-intercepts.

$1-n^2=1-0^2=1>0$

We require $1-n^2<0$, so the solution is outside the interval $-1 \le n \le 1$.

Test a point to the left of the 1st intercept.

$n=-2, 1-n^2=1-(-2)^2=-3<0$, as required.

Test a point to the right of the 2nd intercept.

$n=2, 1-n^2=1-2^2=-3<0$, as required.

So the solution is $n<-1, n>1$.

Question 6

$n^2+2n-15=(n-3)(n+5)$
$(n-3)(n+5)=0 \Rightarrow n=-5, n=3$

The n-intercepts are $n=-5, n=3$.

Test $n=0$, which is a point between the n-intercepts.

$n^2+2n-15=0+0-15=-15<0$, as required.

Test a point to the left of the 1st intercept.

$n=-10, n^2+2n-15=100-20-15=65>0$, which is not required.

Test a point to the right of the 2nd intercept.

$n=10, n^2+2n-15=100+20-15=105>0$, which is not required.

So the solution is the interval $-5 \le n \le 3$.

Question 7

$$c^2 - c - 2 = (c-2)(c+1)$$
$$(c-2)(c+1) = 0 \Rightarrow c = -1, c = 2$$

The *n*-intercepts are $c = -1, c = 2$.

Test $c = 0$, which is a point between the *c*-intercepts.

$c^2 - c - 2 = 0 - 0 - 2 = -2 < 0$

We require $c^2 - c - 2 > 0$, so the solution is outside the interval $-1 < c < 2$.

Test a point to the left of the 1st intercept.

$c = -2$, $c^2 - c - 2 = 4 + 2 - 2 = 4 > 0$, as required.

Test a point to the right of the 2nd intercept.

$c = 3$, $c^2 - c - 2 = 9 - 3 - 2 = 4 > 0$, as required.

So the solution is $c < -1$, $c > 2$.

Question 8

$$x^2 + 6x + 8 = (x+2)(x+4)$$
$$(x+2)(x+4) = 0 \Rightarrow x = -4, x = -2$$

The *x*-intercepts are $x = -4, x = -2$.

Test $x = -1$, which is a point between the *x*-intercepts.

$x^2 + 6x + 8 = 1 - 6 + 2 = -3 < 0$, as required.

Test a point to the left of the 1st intercept.

$x = -5$, $x^2 + 6x + 8 = 25 - 30 + 8 = 3 > 0$, which is not required.

Test a point to the right of the 2nd intercept.

$x = 0$, $x^2 + 6x + 8 = 0 + 0 + 8 = 8 > 0$, which is not required.

So the solution is the interval $-4 \le x \le -2$.

Question 9

$x^2 - 9x + 20 = (x-4)(x-5)$
$(x-4)(x-5) = 0 \Rightarrow x = 4, x = 5$

The x-intercepts are $x = 4, x = 5$.

Test $x = 4.5$, which is a point between the x-intercepts.

$x^2 - 9x + 20 = 4.5^2 - 9 \times 4.5 + 20 = -0.25 < 0$, as required.

Test a point to the left of the 1st intercept.

$x = 0$, $x^2 - 9x + 20 = 0 - 0 + 20 = 20 > 0$, which is not required.

Test a point to the right of the 2nd intercept.

$x = 10$, $x^2 - 9x + 20 = 100 - 90 + 20 = 30 > 0$, which is not required.

So the solution is the interval $4 < x < 5$.

Question 10

$2b^2+5b+2=(2b+1)(b+2)$

$(2b+1)(b+2)=0 \Rightarrow b=-2, b=-\frac{1}{2}$

The b-intercepts are $b=-2$ $b=-\frac{1}{2}$.

Test $b=-1$, which is a point between the b-intercepts.

$2b^2+5b+2=2-5+2=-1<0$

We require $2b^2+5b+2\geq 0$, so the solution is outside the interval $-2\leq b\leq -\frac{1}{2}$.

Test a point to the left of the 1st intercept.

$b=-3$, $2b^2+5b+2=2\times 9-15+2=5>0$, as required.

Test a point to the right of the 2nd intercept.

$b=0$, $2b^2+5b+2=2\times 0+0+0\geq 0$, as required.

So the solution is $b\leq -2$, $b\geq -\frac{1}{2}$.

Question 11

$1-2a-3a^2=-(3a-1)(a+1)$

$-(3a-1)(a+1)=0 \Rightarrow a=-1, a=\dfrac{1}{3}$

The *a*-intercepts are $a=-1$ $a=\dfrac{1}{3}$.

Test $a=0$, which is a point between the *a*-intercepts.

$1-2a-3a^2=1-0-0=1>0$, which is not required.

We require $1-2a-3a^2<0$, so the solution is outside the interval $-1<a<\dfrac{1}{3}$.

Test a point to the left of the 1st intercept.

$a=-2$, $1-2a-3a^2=1+4-12=-7<0$, as required.

Test a point to the right of the 2nd intercept.

$a=1$, $1-2a-3a^2=1-2-3=-4<0$, as required.

So the solution is $a<-1$, $a>\dfrac{1}{3}$.

Question 12

$2y^2 - y - 6 = (2y+3)(y-2)$

$(2y+3)(y-2) = 0 \Rightarrow y = -1\frac{1}{2}, y = 2$

The *y*-intercepts are $y = -1\frac{1}{2}, y = 2$.

Test $y = 0$, which is a point between the *y*-intercepts.

$2y^2 - y - 6 = 2 - 0 - 6 = -4 < 0$

We require $2y^2 - y - 6 > 0$, so the solution is outside the interval $-1\frac{1}{2} \le y \le 2$.

Test a point to the left of the 1st intercept.

$y = -2$, $2y^2 - y - 6 = 8 + 2 - 6 = 4 > 0$, as required.

Test a point to the right of the 2nd intercept.

$y = 3$, $2y^2 - y - 6 = 18 - 3 - 6 = 9 > 0$, as required.

So the solution is $y < -1\frac{1}{2}, y > 2$.

Question 13

$3x^2 - 5x + 2 = (3x - 2)(x - 1)$

$(3x - 2)(x - 1) = 0 \Rightarrow x = \frac{2}{3}, x = 1$

The x-intercepts are $x = \frac{2}{3}, x = 1$.

Test $x = \frac{3}{4}$, which is a point between the x-intercepts.

$3x^2 - 5x + 2 = \frac{27}{16} - \frac{15}{4} + 2 = -\frac{1}{16} < 0$

We require $3x^2 - 5x + 2 \geq 0$, so the solution is outside the interval $\frac{2}{3} \leq x \leq 1$.

Test a point to the left of the 1st intercept.

$x = 0$, $3x^2 - 5x + 2 = 0 - 0 + 2 = 2 \geq 0$, as required.

Test a point to the right of the 2nd intercept.

$x = 2$, $3x^2 - 5x + 2 = 12 - 10 + 2 = 4 \geq 0$, as required.

So the solution is $x \leq \frac{2}{3}, x \geq 1$.

Question 14

$6-13b-5b^2=-(5b-2)(b+3)$

$-(5b-2)(b+3)=0 \Rightarrow b=-3, b=\dfrac{2}{5}$

The *a*-intercepts are $b=-3$ $b=\dfrac{2}{5}$.

Test $b=0$, which is a point between the *b*-intercepts.

$6-13b-5b^2=6-0-0=6>0$, which is not required.

We require $6-13b-5b^2<0$, so the solution is outside the interval $-3<b<\dfrac{2}{5}$.

Test a point to the left of the 1st intercept.

$b=-4$, $6+52-80=-22<0$, as required.

Test a point to the right of the 2nd intercept.

$b=1$, $6-13-5=-12<0$, as required.

So the solution is $b<-3$, $b>\dfrac{2}{5}$.

Question 15

$6x^2+11x+3=(3x+1)(2x+3)$

$(3x+1)(2x+3)=0 \Rightarrow x=-\frac{1}{3}, x=-\frac{3}{2}=-1\frac{1}{2}$

The x-intercepts are $x=-1\frac{1}{2}, x=-\frac{1}{3}$.

Test $x=-1$, which is a point between the x-intercepts.

$6x^2+11x+3=6-11+3=-2<0$, as required.

Test a point to the left of the 1st intercept.

$x=-2, 6x^2+11x+3=24-22+3=5>0$, as required.

Test a point to the right of the 2nd intercept.

$x=0, 6x^2+11x+3=0+0+3=3>0$, as required.

So the solution is the interval $-1\frac{1}{2} \le x \le -\frac{1}{3}$.

Question 16

$y^2 + y \le 12 \Rightarrow y^2 + y - 12 \le 0$
$y^2 + y - 12 = (y-3)(y+4)$
$(y-3)(y+4) = 0 \Rightarrow y = -4, y = 3$

The y-intercepts are $y = -4, y = 3$.

Test $y = 0$, which is a point between the y-intercepts.

$y^2 + y - 12 = 0 + 0 - 12 = -12 < 0$, as required.

Test a point to the left of the 1st intercept.

$y = -5$, $y^2 + y - 12 = 25 - 5 - 12 = 8 > 0$, which is not required.

Test a point to the right of the 2nd intercept.

$y = 4$, $y^2 + y - 12 = 16 + 4 - 12 = 8 > 0$, which is not required.

So the solution is the interval $-4 \le y \le 3$.

Question 17

$x^2 > 16 \Rightarrow x^2 - 16 > 0$
$x^2 - 16 = (x-4)(x+4)$
$(x-4)(x+4) = 0 \Rightarrow x = -4, x = 4$

The *x*-intercepts are $x = -4, x = 4$.

Test $x = 0$, which is a point between the *x*-intercepts.

$x^2 - 16 = 0 - 16 = -16 < 0$, which is not required.

Test a point to the left of the 1st intercept.

$x = -5$, $x^2 - 16 = 25 - 16 = 9 > 0$, as required.

Test a point to the right of the 2nd intercept.

$x = 5$, $x^2 - 16 = 25 - 16 = 9 > 0$, as required.

So the solution is $x < -4$, $x > 4$.

Question 18

$a^2 \leq 1 \Rightarrow a^2 - 1 \leq 0$
$a^2 - 1 = (a-1)(a+4)$
$(a-1)(a+1) = 0 \Rightarrow a = -1, a = 1$

The *x*-intercepts are $a = -1, a = 1$.

Test $a = 0$, which is a point between the *a*-intercepts.

$a^2 - 1 = 0 - 1 = -1 < 0$, as required

Test a point to the left of the 1st intercept.

$a = -2$, $a^2 - 1 = 4 - 1 = 3 > 0$, which is not required.

Test a point to the right of the 2nd intercept.

$a = 2$, $a^2 - 1 = 4 - 1 = 3 > 0$, which is not required.

So the solution is the interval $-1 \leq a \leq 1$.

Question 19

$x^2 < x + 6 \Rightarrow x^2 - x - 6 < 0$
$x^2 - x - 6 = (x+2)(x-3)$
$(x+2)(x-3) = 0 \Rightarrow x = -2, x = 3$

The *x*-intercepts are $x = -2, x = 3$.

Test $x = 0$, which is a point between the *x*-intercepts.

$x^2 - x - 6 = 0 - 0 - 6 = -6 < 0$, as required

Test a point to the left of the 1st intercept.

$x = -3$, $x^2 - x - 6 = 9 + 3 - 6 = 6 > 0$, which is not required.

Test a point to the right of the 2nd intercept.

$x = 4$, $x^2 - x - 6 = 16 - 4 - 6 = 6 > 0$, which is not required.

So the solution is $-2 < x < 3$.

Question 20

$x^2 \geq 2x + 3 \Rightarrow x^2 - 2x - 3 \geq 0$

$x^2 - 2x - 3 = (x - 3)(x + 1)$

$(x - 3)(x + 1) = 0 \Rightarrow x = 3, x = -1$

The x-intercepts are –1, 3.

Test $x = 0$, a point between the intercepts.

$0^2 \geq 2(0) + 3$, $0 \geq 3$, incorrect, so not required.

Therefore $x \leq -1$ and $x \geq 3$.

Question 21

$x^2 < 2x \Rightarrow x^2 - 2x < 0$
$x^2 - 2x = x(x-2)$
$x(x-2) = 0 \Rightarrow x = 0, x = 2$

The x-intercepts are $x = 0, x = 2$.

Test $x = 1$, which is a point between the x-intercepts.

$x^2 - 2x = 1 - 2 = -1 < 0$, as required.

Test a point to the left of the 1st intercept.

$x = -1$, $x^2 - 2x = 1 + 2 = 3 > 0$, which is not required.

Test a point to the right of the 2nd intercept.

$x = 3$, $x^2 - 2x = 9 - 6 = 3 > 0$, which is not required.

So the solution is the interval $0 < x < 2$.

Question 22

$2a^2 \leq 5a - 3 \Rightarrow 2a^2 - 5a + 3 \leq 0$

$2a^2 - 5a + 3 = (2a - 3)(a - 1)$

$(2a - 3)(a - 1) = 0 \Rightarrow a = 1, a = \dfrac{3}{2}$

The a-intercepts are 1, $\dfrac{3}{2}$.

Test $\dfrac{5}{4}$, which is a point between the intercepts.

$2\left(\dfrac{5}{4}\right)^2 \leq 5\left(\dfrac{5}{4}\right) + 3,\ \dfrac{25}{8} \leq \dfrac{25}{4} + 3,\ \dfrac{25}{8} \leq \dfrac{37}{4} = \dfrac{74}{8}$, true.

Therefore $1 \leq a \leq \dfrac{3}{2}$.

Question 23

$5y^2 + 6y \geq 8 \Rightarrow 5y^2 + 6y - 8 \geq 0$

$5y^2 + 6y - 8 = (5y - 4)(y + 2)$

$(5y - 4)(y + 2) = 0 \Rightarrow y = -2, y = \dfrac{4}{5}$

The *y*-intercepts are $y = -2$ $y = \dfrac{4}{5}$.

Test $y = 0$, which is a point between the *y*-intercepts.

$5y^2 + 6y - 8 = 0 + 0 - 8 = -8 < 0$, which is not required.

Test a point to the left of the 1st intercept.

$y = -3$, $5y^2 + 6y - 8 = 45 - 18 - 8 = 35 > 0$, as required.

Test a point to the right of the 2nd intercept.

$y = 1, 5y^2 + 6y - 8 = 5 + 6 - 8 = 3 > 0$, as required.

So the solution is $y \leq -2$, $y \geq \dfrac{4}{5}$.

Question 24

$6m^2 > 15 - m \Rightarrow 6m^2 + m - 15 \geq 0$
$6m^2 + m - 15 = (3m+5)(2m-3)$
$(3m+5)(2m-3) = 0 \Rightarrow m = -\dfrac{5}{3}, m = \dfrac{3}{2}$

The *m*-intercepts are $m = -\dfrac{5}{3}\ m = \dfrac{3}{2}$.

Test $m = 0$, which is a point between the *m*-intercepts.

$6m^2 + m - 15 = 0 + 0 - 15 = -15 < 0$, which is not required.

Test a point to the left of the 1st intercept.

$m = -2$, $6m^2 + m - 15 = 24 - 2 - 15 = 7 > 0$, as required.

Test a point to the right of the 2nd intercept.

$m = 2$, $6m^2 + m - 15 = 24 + 2 - 15 = 11 > 0$, as required.

So the solution is $m < -\dfrac{5}{3}, m > \dfrac{3}{2}$.

Exercise 2.14 Inequalities involving the unknown in the denominator

Question 1

$$\frac{1}{y} \times y^2 < y^2$$
$$y < y^2$$
$$y^2 - y > 0$$

$$y^2 - y = y(y-1)$$
$$y(y-1) = 0 \Rightarrow y = 0, y = 1$$

Test $y = \frac{1}{2}$, a point between the *y*-intercepts.

$y^2 - y = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} < 0$, which is not required.

Test a point to the left of the 1st intercept.

$y = -1, y^2 - y = 1 + 1 = 2 > 0$, as required.

Test a point to the right of the 2nd intercept.

$y = 2$, $y^2 - y = 4 + 1 = 5 > 0$, as required.

The solution is $y < 0$, $y > 1$.

Question 2

$$\frac{1}{x}\times x^2 > 2x^2$$
$$x > 2x^2$$
$$2x^2 - x < 0$$

$$2x^2 - x = x(2x-1)$$
$$x(2x-1) = 0 \Rightarrow x = 0, x = \frac{1}{2}$$

Test $x = \frac{1}{4}$, a point between the x-intercepts.

$2x^2 - x = 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} = -\frac{1}{8} < 0$, as required.

Test a point to the left of the 1st intercept.

$x = -1$, $2x^2 - x = 2(-1)^2 - \frac{1}{4} = 1\frac{3}{4} > 0$, not required.

Test a point to the right of the 2nd intercept.

$x = 1$, $2x^2 - x = 2(1)^2 - \frac{1}{4} = 1\frac{3}{4} > 0$, not required.

The solution is $0 < x < \frac{1}{2}$.

Question 3

$$\frac{3}{x}\times x^2 < 2\times x^2$$

$$3x < 2x^2$$

$$2x^2 - 3x > 0$$

$$2x^2 - 3x = x(2x-3)$$

$$x(2x-3) = 0 \Rightarrow x = 0, x = \frac{3}{2}$$

Test $x = \frac{1}{2}$, a point between the *x*-intercepts.

$2x^2 - 3x = 2\left(\frac{1}{2}\right)^2 - 3\times\frac{1}{2} = -1 < 0$, not required.

Test a point to the left of the 1st intercept.

$x = -1$, $2x^2 - 3x = 2(-1)^2 - 3\times -1 = 5 > 0$, required.

Test a point to the right of the 2nd intercept.

$x = 2$, $2x^2 - x = 2(2)^2 - 3\times 2 = 2 > 0$, required.

The solution is $x < 0$, $x > \frac{3}{2}$.

Question 4

$m \neq 0$

$\frac{2}{m} \times m^2 \geq 7 \times m^2$

$2m \geq 7m^2$

$7m^2 - 2m \leq 0$

$7m^2 - 2m = m(7m - 2)$

$m(7m-2) = 0 \Rightarrow m = 0, m = \frac{2}{7}$

Test $m = \frac{1}{7}$, a point between the m-intercepts.

$7m^2 - 2m = 7\left(\frac{1}{7}\right)^2 - 2 \times \frac{1}{7} = -\frac{1}{7} < 0$, as required.

Test a point to the left of the 1st intercept.

$m = -1, 7m^2 - 2m = 7(-1)^2 - 2 \times -1 = 9 > 0$, required.

Test a point to the right of the 2nd intercept.

$m = 1, 7m^2 - 2m = 7(1)^2 - 2 \times 1 = 5 > 0$, required.

The solution is $0 < m \leq \frac{2}{7}$.

Question 5

$\frac{3}{x} \times x^2 > -5 \times x^2$

$3x > -5x^2$

$5x^2 + 3x > 0$

$5x^2 + 3x = x(5x+3)$

$x(5x+3) = 0 \Rightarrow x = -\frac{3}{5}, x = 0$

Test $x = -\frac{1}{5}$, a point between the *x*-intercepts.

$5x^2 + 3x = 5\left(-\frac{1}{5}\right)^2 + 3 \times -\frac{1}{5} = -\frac{2}{5} < 0$, not required.

Test a point to the left of the 1st intercept.

$x = -1$, $5x^2 + 3x = 5(-1)^2 + 3 \times -1 = 2 > 0$,as required.

Test a point to the right of the 2nd intercept.

$x = 1$, $5x^2 + 3x = 5(1)^2 + 3 \times 1 = 8 > 0$,as required.

The solution is $x < -\frac{3}{5}$, $x > 0$.

Question 6

$b \neq 0$

$\frac{2}{b} \times b^2 \leq -1 \times b^2$

$2b \leq -b^2$

$b^2 + 2b \leq 0$

$b^2 + 2b = b(b+2)$

$b(b+2) = 0 \Rightarrow b = -2, b = 0$

Test $b = -1$, a point between the *b*-intercepts.

$b^2 + 2b = (-1)^2 + 2 \times -1 = -1 < 0$, as required.

Test a point to the left of the 1st intercept.

$b = -3$, $b^2 + 2b = (-3)^2 + 2 \times -3 = 3 > 0$, not required.

Test a point to the right of the 2nd intercept.

$b = 1$, $b^2 + 2b = (1)^2 + 2 \times 1 = 3 > 0$, not required.

The solution is $-2 \leq b < 0$.

Question 7

$$\frac{1}{x-1}\times(x-1)^2 > 4\times(x-1)^2$$
$$x-1>4x^2-8x+4$$
$$4x^2-9x+5<0$$

$$4x^2-9x+5=(4x-5)(x-1)$$
$$(4x-5)(x-1)=0 \Rightarrow x=1, x=\frac{5}{4}$$

Test $x=\frac{9}{8}$, a point between the x-intercepts.

$4x^2-9x+5=4\left(\frac{9}{8}\right)^2-9\times\frac{9}{8}+5=-\frac{1}{16}<0$, as required.

Test a point to the left of the 1st intercept.

$x=0$, $4x^2-9x+5=0-0+5=5>0$, not required.

Test a point to the right of the 2nd intercept.

$x=2$, $4x^2-9x+5=16-18+5=3>0$, not required.

The solution is $1<x<1\frac{1}{4}$.

Question 8

$$\frac{1}{z+3}\times(z+3)^2<-5\times(z+3)^2$$

$$z+3<-5z^2-30z-45$$

$$5z^2+31z+48>0$$

$$5z^2+31z+48=(5z+16)(z+3)$$

$$(5z+16)(z+3)=0\Rightarrow z=-\frac{16}{5}=-3\frac{1}{5},z=-3$$

Test $x=-3.1$, a point between the *z*-intercepts.

$5z^2+31z+48=5(-3.1)^2+31(-3.1)+48=-0.05<0$, as required.

Test a point to the left of the 1st intercept.

$z=-4, 5z^2+31z+48=80-124+48=4>0$, not required.

Test a point to the right of the 2nd intercept.

$z=0, 5z^2+31z+48=0+0+48=48>0$, not required.

The solution is $-3\frac{1}{5}<z<-3$.

Question 9

$x \neq 2$

$$\frac{3}{x-2} \times (x-2)^2 \geq 4 \times (x-2)^2$$
$$3x - 6 \geq 4x^2 - 16x + 16$$
$$4x^2 - 19x + 22 \leq 0$$

$$4x^2 - 19x + 22 = (4x - 11)(x - 2)$$
$$(4x - 11)(x - 2) = 0 \Rightarrow x = 2, x = 2\frac{3}{4}$$

Test $x = 2\frac{1}{2}$, a point between the *x*-intercepts.

$4x^2 - 19x + 22 = 4\left(2\frac{1}{2}\right)^2 - 19 \times 2\frac{1}{2} + 22 = -\frac{1}{2} < 0$, as required.

Test a point to the left of the 1st intercept.

$x = 0$, $4x^2 - 19x + 22 = 0 - 0 + 22 = 22 > 0$, not required.

Test a point to the right of the 2nd intercept.

$x = 3$, $4x^2 - 19x + 22 = 36 - 57 + 22 = 1 > 0$, not required.

The solution is $2 < x \leq 2\frac{3}{4}$.

Question 10

$$\frac{-1}{2-x}\times(2-x)^2 < 6\times(2-x)^2$$
$$-2+x < 6x^2-24x+24$$
$$6x^2-25x+26>0$$

$$6x^2-25x+26=(6x-13)(x-2)$$
$$(6x-13)(x-2)=0 \Rightarrow x=2, x=2\frac{1}{6}$$

Test $x=2.1$, a point between the x-intercepts.

$6x^2-25x+26=4\left(2.1\right)^2-19\times 2.1+22=-0.26<0$, not required.

Test a point to the left of the 1st intercept.

$x=0$, $6x^2-25x+26=0-0+22=22>0$, as required.

Test a point to the right of the 2nd intercept.

$x=3$, $6x^2-25x+26=54-75+26=5>0$, as required.

The solution is $x<2$, $x>2\frac{1}{6}$.

Question 11

$x \neq -4$

$\dfrac{5}{x+4} \times (x+4)^2 \leq -9 \times (x+4)^2$

$5x + 20 \leq -9x^2 - 72x - 144$

$9x^2 + 77x + 164 \leq 0$

$9x^2 + 77x + 164 \leq 0 = (9x+41)(x+4)$

$(9x+41)(x+4) = 0 \Rightarrow x = -4\dfrac{5}{9}, x = -4$

Test $x = -4.1$, a point between the *x*-intercepts.

$9x^2 + 77x + 164 = 4\left(-4.1\right)^2 + 77 \times -4.1 + 164 = -84.46 < 0$, as required.

Test a point to the left of the 1st intercept.

$x = -5$, $9x^2 + 77x + 164 = 225 - 385 + 164 = 4 > 0$, as required.

Test a point to the right of the 2nd intercept.

$x = 0$, $9x^2 + 77x + 164 = 0 + 0 + 164 = 164 > 0$, as required.

The solution is $-4\dfrac{5}{9} \leq x < -4$.

Question 12

$3x-4\neq 0 \Rightarrow x\neq \frac{4}{3}$

$\frac{2}{3x-4}\times(3x-4)^2>5\times(3x-4)^2$
$6x-8>45x^2-120x+80$
$45x^2-126x+88<0$

Use quadratic formula to solve $45x^2-126x+88=0$

$x=\frac{-(-126)\pm\sqrt{(-126)^2-4\times 45\times 88}}{2\times 45}$
$x=\frac{126\pm 6}{90}$
$x=1\frac{1}{3}, x=1\frac{7}{15}$

Test $x=1.4$, a point between the *x*-intercepts.

$45x^2-126x+88=45(1.4)^2-126(1.4)+88=-0.2<0$, as required.

Test a point to the left of the 1st intercept.

$x=0$, $45x^2-126x+88=0-0+88=88>0$, not required.

Test a point to the right of the 2nd intercept.

$x=2$, $45x^2-126x+88=180-252+88=16>0$, not required.

The solution is $1\frac{1}{3}<x\leq 1\frac{7}{15}$.

Question 13

$2a+5 \neq 0 \Rightarrow a \neq -2\frac{1}{2}$

$\frac{-3}{2a+5} \times (2a+5)^2 < 2 \times (2a+5)^2$
$-6a - 15 < 8a^2 + 40a + 50$
$8a^2 + 46a + 65 > 0$

Use quadratic formula to solve $8a^2 + 46a + 65 = 0$

$a = \frac{-46 \pm \sqrt{46^2 - 4 \times 8 \times 65}}{2 \times 8}$
$= \frac{-46 \pm 6}{16}$
$a = -3\frac{1}{4}, a = -2\frac{1}{2}$

Test $a = -3$, a point between the *x*-intercepts.

$8a^2 + 46a + 65 = 8(-3)^2 + 46(-3) + 65 = -1 < 0$, not required.

Test a point to the left of the 1st intercept.

$a = -4$, $8a^2 + 46a + 65 = 45(-4)^2 + 46(-4) + 65 = 601 > 0$, as required.

Test a point to the right of the 2nd intercept.

$a = 0$, $8a^2 + 46a + 65 = 0 + 0 + 65 = 65 > 0$, as required.

The solution is $a < -3\frac{1}{4}, a > -2\frac{1}{2}$.

Question 14

$2x-1\neq 0 \Rightarrow x\neq \dfrac{1}{2}$

$$\frac{x}{2x-1}\times(2x-1)^2>5\times(2x-1)^2$$
$$2x^2-x>20x^2-20x+5$$
$$18x^2-19x+5<0$$

$$18x^2-19x+5=(9x-5)(2x-1)$$
$$(9x-5)(2x-1)=0$$
$$x=\frac{1}{2}=0.5, x=\frac{5}{9}=0.56$$

Test $x=0.55$, a point between the x-intercepts.

$18x^2-19x+5=18(0.55)^2-19(0.55)+5=-0.005<0$, as required.

Test a point to the left of the 1st intercept.

$x=0$, $18x^2-19x+5=0-0+5=5>0$, not required.

Test a point to the right of the 2nd intercept.

$x=1$, $18x^2-19x+5=18-19+5=4>0$, not required.

The solution is $\dfrac{1}{2}<x<\dfrac{5}{9}$.

Question 15

$y+1 \neq 0 \Rightarrow y \neq -1$

$$\frac{y}{y+1} \times (y+1)^2 < 2 \times (y+1)^2$$
$$y^2 + y < 2y^2 + 4y + 2$$
$$y^2 + 3y + 2 > 0$$

$$y^2 + 3y + 2 = (y+2)(y+1)$$
$$(y+2)(y-1) = 0$$
$$y = -2, y = -1$$

Test $y = -1.5$, a point between the *y*-intercepts.

$y^2 + 3y + 2 = (-1.5)^2 + 3(-1.5) + 2 - 0.25 < 0$, not required.

Test a point to the left of the 1st intercept.

$y = -3, y^2 + 3y + 2 = (-3)^2 + 3(-3) + 2 = 2 > 0$, as required.

Test a point to the right of the 2nd intercept.

$y = 0, y^2 + 3y + 2 = 0 + 0 + 2 = 2 > 0$, as required.

The solution is $y < -2, y > -1$.

Question 16

$x-4\neq 0 \Rightarrow x\neq 4$

$$\frac{3x+1}{x-4}\times(x-4)^2 \geq \frac{1}{3}\times(x-4)^2$$
$$(3x+1)(x-4)\geq \frac{1}{3}(x^2-8x+16)$$
$$9x^2-33x-12\geq x^2-8x+16$$
$$8x^2-25x-28\geq 0$$

Use quadratic formula to solve $8x^2-25x-28=0$

$$x=\frac{-(-25)\pm\sqrt{(-25)^2-4\times 8\times -28}}{2\times 8}$$
$$=\frac{25\pm 39}{16}$$
$$x=-\frac{7}{8}, x=4$$

Test $x=0$, a point between the *x*-intercepts.

$8x^2-25x-28=0-0-28=-28<0$, not required.

Test a point to the left of the 1st intercept.

$x=-1, 8x^2-25x-28=8+25-28=5>0$, as required.

Test a point to the right of the 2nd intercept.

$x=5, 8x^2-25x-28=200-125-28=47>0$, as required.

The solution is $x\leq -\frac{7}{8}, x>4$.

Question 17

$2p-9\neq 0 \Rightarrow p\neq 4\frac{1}{2}$

$\frac{8p+7}{2p-9}\times(2p-9)^2>5\times(2p-9)^2$

$(8p+7)(2p-9)>5(2p-9)^2$

$16p^2-58p-63\geq 20p^2-180p+405$

$2p^2-61p+234<0$

Use quadratic formula to solve $2p^2-61p+234=0$

$p=\frac{-(-61)\pm\sqrt{(-61)^2-4\times2\times234}}{2\times2}$

$=\frac{61\pm43}{4}$

$p=4\frac{1}{2}, p=26$

Test $p=10$, a point between the p-intercepts.

$2p^2-61p+234=200-610+234=-176<0$, as required.

Test a point to the left of the 1st intercept.

$p=0, 2p^2-61p+234=0-0+234=234>0$, not required.

Test a point to the right of the 2nd intercept.

$p=30, 2p^2-61p+234=1800-1830+234=204>0$, not required.

The solution is $4\frac{1}{2}<p<26$.

Question 18

$5x+1 \neq 0 \Rightarrow x \neq -\dfrac{1}{5}$

$$\frac{x-2}{5x+1} \times (5x+1)^2 \leq \frac{3}{4} \times (5x+1)^2$$
$$4(x-2)(5x+1) \leq 3(5x+1)^2$$
$$20x^2 - 36x - 8 \leq 75x^2 + 30x + 3$$
$$5x^2 + 6x + 1 \geq 0$$

$$5x^2 + 6x + 1 = (5x+1)(x+1)$$
$$(5x+1)(x+1) = 0$$
$$x = -1 \ \ x = -\frac{1}{5}$$

Test $x = -0.5$, a point between the *x*-intercepts.

$5x^2 + 6x + 1 = 1.25 - 3 + 1 = -0.75 < 0$, not required.

Test a point to the left of the 1st intercept.

$x = -2, \quad 5x^2 + 6x + 1 = 20 - 12 + 1 = 9 > 0$, as required.

Test a point to the right of the 2nd intercept.

$x = 5$, $8x^2 - 25x - 28 = 200 - 125 - 28 = 47 > 0$, as required.

The solution is $x \leq -1 \ \ x > -\dfrac{1}{5}$.

Question 19

$x \neq 0$

$$\frac{x^2-5}{x} \times x^2 < -4 \times x^2$$
$$x(x^2-5) < -4x^2$$
$$x^3+4x^2-5x<0$$
$$x(x^2+4x-5)<0$$
$$x(x-1)(x+5)<0$$
$$x(x-1)(x+5)=0$$
$$x=-5, x=0, x=1$$

Test $x = 0.5$, a point between the *x*-intercepts 0 and 1.

$x^3+4x^2-5x=-1.375<0$, as required.

Test $x=-1$, a point between the *x*-intercepts 0 and –5.

$x^3+4x^2-5x=8>0$, not required.

Test $x=-10$, a point to the left of –5.

$x^3+4x^2-5x=-550<0$, as required.

Test $x=2$, a point to the right of 1.

$x^3+4x^2-5x=14>0$, not required.

The solution is $x<-5, 0<x<1$.

Question 20

$3x-2\neq 0$

$x\neq \frac{2}{3}$

$\frac{2x^2}{3x-2}\leq -1$

$2x^2\leq -3x+2$

$2x^2+3x-2\leq 0$

$(x+2)(2x-1)\leq 0$

$(x+2)(2x-1)=0\Rightarrow x=-2, x=\frac{1}{2}$

Test $x=0$, a point between –2 and $\frac{1}{2}$.

$2x^2+3x-2=-2<0$, as required.

Test $x=-10$, a point to the left of –2.

$2x^2+3x-2=-102<0$, as required.

Try $x=1$, a point to the right of $\frac{2}{3}$, to see if $\frac{2x^2}{3x-2}\leq -1$.

$x=1, \frac{2x^2}{3x-2}=1$, not less than –1.

Try $x=\frac{1}{3}$, a point to the left of $\frac{2}{3}$, to see if $\frac{2x^2}{3x-2}\leq -1$.

$x=\frac{1}{3}\ \frac{2x^2}{3x-2}=-\frac{2}{9}$, not less than –1.

Try $x=0.6$, a point between $\frac{1}{2}$ and $\frac{2}{3}$, to see if $\frac{2x^2}{3x-2}\leq -1$.

$x=0.6, \frac{2x^2}{3x-2}=-3.6<-1$, not less than –1.

The solution is $x\leq -2 \quad \frac{1}{2}\leq x<\frac{2}{3}$.

Question 21

$7x+4 \neq 0 \Rightarrow x \neq -\dfrac{4}{7}$

$\dfrac{3x^2}{7x+4} < -2$

$3x^2 < -14x - 8$

$3x^2 + 14x + 8 < 0$

$(3x+2)(x+4) < 0$

$(3x+2)(x+4) = 0 \Rightarrow x = -4, x = -\dfrac{2}{3}$

Try $x = -\dfrac{13}{21}$, a point between $-\dfrac{2}{3}$ and $-\dfrac{4}{7}$., to see if $\dfrac{3x^2}{7x+4} \leq -2$.

$x = -\dfrac{13}{21}, \dfrac{3x^2}{7x+4} = -3.4$, which is less than –2.

Try $x = -1$, a point between -4 and $-\dfrac{2}{3}$, to see if $\dfrac{3x^2}{7x+4} \leq -2$.

$x = -1, \dfrac{3x^2}{7x+4} = -1$, which is greater than –2.

Try $x = -10$, a point less than -4, to see if $\dfrac{3x^2}{7x+4} \leq -2$.

$x = -10, \dfrac{3x^2}{7x+4} = -4.5$, which is less than –2.

The solution is $x < -4 \quad -\dfrac{2}{3} < x < -\dfrac{4}{7}$.

Question 22

$x-1 \neq 0 \Rightarrow x \neq 1$

$\frac{2x(x-4)}{x-1} \leq 7$

$2x^2 - 8x \leq 7x - 7$

$2x^2 - 15x + 7 \leq 0$

$(2x-1)(x-7) \leq 0$

$(2x-1)(x-7) = 0 \Rightarrow x = \frac{1}{2}, x = 7$

Try $x = 0$, a point less than $\frac{1}{2}$, to see if $\frac{2x(x-4)}{x-1} \leq 7$.

$x = 0, \frac{2x(x-4)}{x-1} = 0$, which is less than 7.

Try $x = \frac{3}{4}$, a point between $\frac{1}{2}$ and 1, to see if $\frac{2x(x-4)}{x-1} \leq 7$.

$x = \frac{3}{4}$ $\frac{2x(x-4)}{x-1} = -0.56$, which is less than 7.

Try $x = 2$, a point between 1and 7, to see if $\frac{2x(x-4)}{x-1} \leq 7$.

$x = 2, \frac{2x(x-4)}{x-1} = -8$, which is less than 7.

Try $x = 10$, a point greater than 7, to see if $\frac{2x(x-4)}{x-1} \leq 7$.

$x = 10, \frac{2x(x-4)}{x-1} = 13.3$, which is greater than 7.

The solution is $x \leq \frac{1}{2}, \quad 1 < x < 7$.

Exercise 2.15 Inequalities involving absolute values

Question 1

a $-4 < a < 4$

The distance from the origin to point a is less than 4.

b $k \geq 1, k \leq -1$

The distance from the origin to point k is greater than or equal to 1.

c $x > 6, x < -6$

The distance from the origin to point x is greater than 6.

d $-10 \leq p \leq 10$

The distance from the origin to point p is less than or equal to 10.

e $a > 14, a < -14$

The distance from the origin to point a is greater than to 14.

f $-12 < y < 12$

The distance from the origin to point y is less than 1.

g $b \geq 20, b \leq -20$

The distance from the origin to point b is greater than or equal to 20.

Question 2

a

$$2a < -4 \Rightarrow a < -2$$

$$2a > 4 \Rightarrow a > 2$$

b

$$-1 \le x - 5 \le 1$$
$$-1 + 5 \le x \le 1 + 5$$
$$4 \le x \le 6$$

c

$$-11 < 4y + 3 < 11$$
$$-11 - 3 < 4y + 3 - 3 < 11 - 3$$
$$-14 < 4y < 8$$
$$-\frac{14}{4} < y < \frac{8}{4}$$
$$-3\frac{1}{2} < y < 2$$

d

$$2x - 3 \le -15$$
$$2x \le -12$$
$$x \le -6$$

$$2x - 3 \ge 15$$
$$2x \ge 18$$
$$x \ge 9$$

e

$$-2 \le \frac{a}{2} - 3 \le 2$$
$$-2 + 3 \le \frac{a}{2} - 3 + 3 \le 2 + 3$$
$$1 \le \frac{a}{2} \le 5$$
$$2 \le a \le 10$$

Question 3

a $y \le -\frac{4}{5}, y \ge 2$

$$5y - 3 \le -7$$
$$5y \le -4$$
$$y \le -\frac{4}{5}$$

$$5y - 3 \ge 7$$
$$5y \ge 10$$
$$y \ge 2$$

b $-2 < a < -\frac{1}{3}$

$$-5 < 7 + 6a < 5$$
$$-5 - 7 < 6a < 5 - 7$$
$$-12 < 6a < -2$$
$$-2 < a < -\frac{1}{3}$$

c $-1.4 \le t \le 2$

$$-17 \le 10t - 3 \le 17$$
$$-17 + 3 \le 10t \le 17 + 3$$
$$-14 \le 10t \le 20$$
$$-1.4 \le t \le 2$$

d $-3 < x < 11$

$$14 > |2x - 8| \Rightarrow |2x - 8| < 14$$
$$-14 < 2x - 8 < 14$$
$$-14 + 8 < 2x < 14 + 8$$
$$-6 < 2x < 22$$
$$-3 < x < 11$$

e $n \le -1, n \ge 3\frac{2}{5}$

$6 - 5n \le -11$
$-5n \le -17$
$n \ge 3\frac{2}{5}$

$6 - 5n \ge 11$
$-5n \ge 5$
$n \le -1$

Test Yourself 2

Question 1

C

$$a=1, b=-5, c=-1$$
$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4\times1\times-1}}{2\times1}$$
$$=\frac{5\pm\sqrt{29}}{2}$$

Question 2

A, D

$$200=4\pi r^2$$
$$r=\sqrt{\frac{200}{4\pi}}=\sqrt{\frac{50}{\pi}}=5\sqrt{\frac{2}{\pi}}$$

Question 3

B

$x-y=7$ [1]

$x+2y=1$ [2]

[2] – [1]

$$3y=-6$$
$$y=-2$$

Substitute $y=-2$ in [1]

$$x-(-2)=7$$
$$x=5$$

Question 4

C

$x \neq 2$

$$\frac{2x}{x-2} \times (x-2)^2 \geq 1(x-2)^2$$
$$2x(x-2) \geq x^2 - 2x + 4$$
$$x^2 - x - 2 \geq 0$$
$$(x+1)(x-2) \geq 0$$
$$(x+1)(x-2) = 0 \Rightarrow x = -1, x = 2$$

Test $x = 0$, a point between the *x*-intercepts.

$x = 0, \quad x^2 - x - 2 = -2 < 0$, not required.

Test $x = -2$, a point to the left of –1.

$x = -2, \quad x^2 - x - 2 = 4 + 2 - 2 = 4 > 0$, as required.

Test $x = 3$, a point to the right of 2.

$x = 3, \quad x^2 - x - 2 = 9 - 3 - 2 = 4 > 0$, as required.

The solution is $x \leq -2, x > 2$.

Question 5

a

$$8 = 3b - 22$$
$$8 + 22 = 3b$$
$$30 = 3b$$
$$b = 10$$

b

$$\frac{a}{4} - \frac{a+2}{3} = 9$$
$$12 \times \frac{a}{4} - 12 \times \frac{a+2}{3} = 12 \times 9$$
$$3a - 4(a+2) = 108$$
$$-a - 8 = 108$$
$$-a = 116$$
$$a = -116$$

c

$$4(3x+1) = 11x - 3$$
$$12x + 4 = 11x - 3$$
$$12x - 11x = -3 - 4$$
$$x = -7$$

d

$x+3 \neq 0 \Rightarrow x \neq -3$

$$\frac{-4}{x+3} \times (x+3)^2 \leq 3(x+3)^2$$
$$-4(x+3) \leq 3x^2 + 18x + 27$$
$$3x^2 + 22x + 39 \geq 0$$
$$(3x+13)(x+3) \geq 0$$
$$(3x+13)(x+3) = 0 \Rightarrow x = -4\frac{1}{3}, x = -3$$

Test $x = -5$, a point less than $x = -4\frac{1}{3}$.

$x = -5, \quad 3x^2 + 22x + 39 = 4 > 0$, as required.

Test $x = 4$, a point to the right of 3.

$x = 3, \quad 3x^2 + 22x + 39 > 0$, as required.

Test $x = -4$, a point between $-4\frac{1}{3}$ and -3.

$x = -4, \quad 3x^2 + 22x + 39 = -1 < 0$, is not required.

The solution is $x \leq -4\frac{1}{3}, x > -3$.

e

$$3p + 1 \leq p + 9$$
$$3p - p \leq 9 - 1$$
$$2p \leq 8$$
$$p \leq 4$$

Question 6

a

$$A = 1000 \times \left(1 + \frac{6}{100}\right)^4 = 1262.48$$

b

$$12450 = P \times \left(1 + \frac{55}{100}\right)^7$$

$$12450 = .45468 \quad P$$

$$P = \frac{12450}{1.45468} = 8558.59$$

Question 7

a

$x - y + 7 = 0 \qquad [1]$

$3x - 4y + 26 = 0 \qquad [2]$

3 × [1] – [2]

$$\begin{aligned} y - 5 &= 0 \\ y &= 5 \end{aligned}$$

Substitute $y = 5$ in [1]

$$\begin{aligned} x - 5 + 7 &= 0 \\ x &= -2 \end{aligned}$$

b

$xy = 4 \qquad [1]$

$2x - y - 7 = 0 \Rightarrow y = 2x - 7 \qquad [2]$

Substitute [2] in [1].

$$\begin{aligned} x(2x - 7) &= 4 \\ 2x^2 - 7x - 4 &= 0 \\ (2x + 1)(x - 4) &= 0 \\ x = -\frac{1}{2}, x &= 4 \end{aligned}$$

From [2]

$x = -\frac{1}{2}, \quad y = -1 - 7 = -8$

$x = 4, \quad y = 8 - 7 = 1$

Question 8

a

$$3^{x+2} = 3^4$$
$$x + 2 = 4$$
$$x = 2$$

b

$$(2^4)^y = 2^1$$
$$2^{4y} = 2^1$$
$$4y = 1$$
$$y = \frac{1}{4}$$

Question 9

a

$$(3b - 1)^2 = 25$$
$$9b^2 - 6b + 1 = 25$$
$$3b^2 - 2b - 8 = 0$$
$$(3b + 4)(b - 2) = 0$$
$$b = 2$$
$$3b + 4 = 0 \Rightarrow b = -1\frac{1}{3}$$

b

$$2x - 7 \le -1$$
$$2x \le 6$$
$$x \le 3$$

$$2x - 7 \ge 1$$
$$2x \ge 8$$
$$x \ge 4$$

Question 10

a

$$A = \frac{1}{2} \times 6 \times (5+7) = 36$$

b

$$40 = \frac{1}{2} \times 5 \times (4+b)$$
$$16 = 4 + b$$
$$b = 12$$

Question 11

$$(2x-1)(x-1) = 0$$
$$x = 1$$
$$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$

Question 12

$$-2 < 3y + 1 \leq 10$$
$$-2 - 1 < 3y \leq 10 - 1$$
$$-3 < 3y \leq 9$$
$$-1 < y \leq 3$$

Question 13

a

$$a = 1, b = 7, c = 2$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 1 \times 2}}{2 \times 1}$$

$$= \frac{-7 \pm \sqrt{41}}{2}$$

$$x = \frac{-7 - \sqrt{41}}{2} = -6.70$$

$$x = \frac{-7 + \sqrt{41}}{2} = -0.298$$

b

$$a = 1, b = -2, c = -9$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times -9}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{40}}{2}$$

$$x = \frac{2 - \sqrt{40}}{2} = -2.16$$

$$x = \frac{2 + \sqrt{40}}{2} = 4.16$$

c

$$a = 3, b = 2, c = -4$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 3 \times -4}}{2 \times 3}$$

$$= \frac{-2 \pm \sqrt{52}}{6}$$

$$x = \frac{-2 + \sqrt{52}}{6} = 0.869$$

$$x = \frac{-2 - \sqrt{52}}{6} = -1.54$$

Question 14

a

$$A = 4\pi \times 7.8^2 = 764.5$$

b

$$102.9 = 4\pi r^2$$
$$r = \sqrt{\frac{102.9}{4\pi}}$$
$$r = 2.9$$

Question 15

$$\frac{x-3}{7} - \frac{3}{4} > 9$$
$$\frac{x-3}{7} > 9\frac{3}{4}$$
$$x-3 > 7 \times \frac{39}{4}$$
$$x > 3 + \frac{273}{4}$$
$$x > 3 + 68\frac{1}{4}$$
$$x > 71\frac{1}{4}$$

Question 16

$(x-9)(x-2)>0$
$(x-9)(x-2)=0 \Rightarrow x=2, x=9$

Test $x=0$, which is to the left of $x=2$.

$x=0, \quad x^2-11x+18=18>0$, as required.

Test $x=10$, which is to the right of $x=9$.

$x=10, \quad x^2-11x+18=8>0$, as required.

Hence, $x<2, \quad x>9$.

Question 17

$x^2+y^2=16$ [1]

$3x+4y-20=0 \Rightarrow y=\frac{1}{4}(20-3x)$ [2]

Substitute [2] in [1].

$$x^2+\left(\frac{1}{4}(20-3x)\right)^2=16$$
$$16x^2+(20-3x)^2=256$$
$$16x^2+400-120x+9x^2=256$$
$$25x^2-120x+144=0$$
$$(5x-12)^2=0$$
$$5x=12$$
$$x=2.4$$

Substitute $x=2.4$ in [2].

$$y=\frac{1}{4}(20-3\times 2.4)$$
$$y=3.2$$

Question 18

a

$$V = \frac{4}{3}\pi \times 8^3 = 2144.66 \approx 2100$$

b

$$250 = \frac{4}{3}\pi \times r^3$$
$$r = \sqrt[3]{\frac{3\times 250}{4\pi}} = 3.90$$

Question 19

a B

$$a = 1, b = -6, c = 9$$
$$\Delta = (-6)^2 - 4\times 1\times 9$$
$$= 0$$

$\Delta = 0$, so there is one solution.

b A

$$2x - 3 = \pm 7$$
$$2x = \pm 7 + 3$$
$$2x = 10, -4$$
$$x = 5, -2$$

So there are two solutions.

c A

$$a = 1, b = -1, c = -5$$
$$\Delta = (-1)^2 - 4\times 1\times -5$$
$$= 21 > 0$$

$\Delta > 0$, so there are two solutions.

d C

$$a = 2, b = -1, c = 4$$
$$\Delta = (-1)^2 - 4 \times 2 \times 4$$
$$= -31 < 0$$

$\Delta < 0$, so there are no solutions.

e B

$$3x + 2 = 7$$
$$3x = 5$$
$$x = \frac{5}{3}$$

So one solution.

Question 20

$a + b = 5$ [1]

$2a + b + c = 4$ [2]

$a - b - c = 5$ [3]

[2] + [3]

$3a = 9 \Rightarrow a = 3$

Substitute $a = 3$ in [1].

$3 + b = 5 \Rightarrow b = 2$

From [3]

$3 - 2 - c = 5 \Rightarrow c = -4$

Question 21

$3n + 5 < -5$

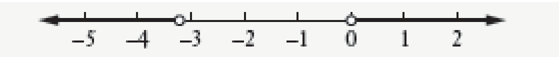
$3n < -10$

$n < -3\frac{1}{3}$

$3n + 5 > 5$

$3n > 0$

$n > 0$



Question 22

$t \le \frac{2}{5}, \quad t \ge 2\frac{2}{3}$

$3t - 8 \ne 0 \Rightarrow t \ne \frac{8}{3}$

$\frac{7t+4}{3t-8} \ge -1$

$7t + 4 \ge -3t + 8$

$10t \ge 4$

$t \ge \frac{2}{5}$

Try a value less than $\frac{2}{5}$, $t = 0$, $\frac{7t+4}{3t-8} = -\frac{1}{2} > -1$

Try a value greater than $\frac{8}{3}$, $t = 3$, $\frac{7t+4}{3t-8} = 25 > -1$

Try a value between $\frac{2}{5}$ and $\frac{8}{3}$, $t = 2$, $\frac{7t+4}{3t-8} = -9 < -1$

The solution is $t \le \frac{2}{5}, \quad t > 2\frac{2}{3}$.

Question 23

$\left(3^2\right)^{2x+1} = \left(3^3\right)^x$
$3^{4x+2} = 3^{3x}$
$4x + 2 = 3x$
$x = -2$

Question 24

a

$$2(3y - 5) > y + 5$$
$$6y - 10 > y + 5$$
$$5y > 15$$
$$y > 3$$

b

$$n^2 + 3n \le 0$$
$$n(n+3) \le 0$$
$$n(n+3) = 0 \Rightarrow n = -3, \quad n = 0$$

Try $n = -4$, which is less than –3.

$n = -4, \quad n^2 + 3n = 16 - 12 = 4 > 0$, not required.

Try $n = 1$, which is greater than 0.

$n = 1, \quad n^2 + 3n = 1 + 3 = 4 > 0$, not required.

Try $n = -1$, which is between 0 and –3.

$n = -1, \quad n^2 + 3n = 1 - 3 = -2 < 0$, as required.

The solution is $-3 \le n \le 0$.

c

$$3^{2x+1}=3^3$$
$$2x+1=3$$
$$x=2$$

d

$$5x^3-1=39$$
$$5x^3=40$$
$$x^3=8$$
$$x=2$$

e

$$(5x-4)^2=121$$
$$25x^2-40x-105=0$$
$$5x^2-8x+21=0$$
$$(5x+7)(x-3)=0$$
$$5x+7=0 \Rightarrow x=-1\frac{2}{5}$$
$$x-3=0 \Rightarrow x=3$$

f

$$2t+1\le -3$$
$$2t\le -4$$
$$t\le -2$$

$$2t+1\ge 3$$
$$2t\ge 2$$
$$t\ge 1$$

g

$$x^2 + 2x - 8 \le 0$$
$$(x-2)(x+4) \le 0$$
$$(x-2)(x+4) = 0 \Rightarrow x = -4, \quad x = 2$$

Try $x = -5$, which is less than –4.

$x = -5, \quad x^2 + 2x - 8 = 25 - 10 - 8 = 7 > 0$, not required.

Try $x = 3$, which is greater than 2.

$x = 3, \quad x^2 + 2x - 8 = 9 + 6 - 8 = 7 > 0$, not required.

Try $x = 0$, which is between –4 and 2.

$x = 0, \quad x^2 + 2x - 8 = 0 + 0 - 8 = -8 < 0$, as required.

The solution is $-4 \le x \le 2$.

h

$$\left(2^3\right)^{x+} = \left(2^2\right)^x$$
$$2^{3x+3} = 2^{2x}$$
$$3x + 3 = 2x$$
$$x = -3$$

i

$$(y-2)(y+2) > 0$$
$$(y-2)(y+2) = 0 \Rightarrow y = -2, y = 2$$

Try $y = -3$, which is less than –2.

$y = -3, \quad y^2 - 4 = 9 - 4 = 5 > 0$, as required.

Try $y = 3$, which is greater than 2.

$y = 3, \quad y^2 - 4 = 9 - 4 = 5 > 0$, as required.

Try $y = 0$, which is between –2 and 2.

$y = 0, \quad y^2 - 4 = 0 - 4 = -4 < 0$, not required.

The solution is $y < -2, \quad y > 2$.

j

$$1-x^2 \le 0$$
$$(1-x)(1+x) \le 0$$
$$(1-x)(1+x) = 0 \Rightarrow x = -1, x = 1$$

Try $x = -2$, which is less than –1.

$x = -2, \quad 1-x^2 = 1-4 = -3 < 0$, as required.

Try $x = 2$, which is greater than 1.

$x = 2, \quad 1-x^2 = 1-4 = -3 < 0$, as required.

Try $x = 0$, which is between –1 and 1.

$x = 0, \quad 1-x^2 = 1-0 = 1 > 0$, not required.

The solution is $x \le -1, \quad x \ge 1$.

k

$$\left(3^3\right)^{2x-1} = 3^2$$
$$3(2x-1) = 2$$
$$6x-3 = 2$$
$$6x = 5$$
$$x = \frac{5}{6}$$

l

$$-5 \le 4b-3 \le 5$$
$$-5+3 \le 4b \le 5+3$$
$$-2 \le 4b \le 8$$
$$-\frac{1}{2} \le b \le 2$$

m

$$x^2 - 2x - 3 < 0$$
$$(x+1)(x-3) \le 0$$
$$(x+1)(x-3) = 0 \Rightarrow x = -1, x = 3$$

Try $x = -2$, which is less than –1.

$x = -2, \quad x^2 - 2x - 3 = 4 + 4 - 3 = 5 > 0$, not required.

Try $x = 4$, which is greater than 3.

$x = 4, \quad x^2 - 2x - 3 = 16 - 8 - 3 = 5 > 0$, not required.

Try $x = 0$, which is between –1 and 3.

$x = 0, \quad x^2 - 2x - 3 = 0 + 0 - 3 = -3 < 0$, as required.

The solution is $-1 < x < 3$.

n

$$m^2 + m - 6 \ge 0$$
$$(m+3)(m-2) \ge 0$$
$$(m+3)(m-2) = 0 \Rightarrow m = -3, m = 2$$

Try $m = -4$, which is less than –3.

$m = -4, \quad m^2 + m - 6 = 16 - 4 - 6 = 6 > 0$, as required.

Try $m = 3$, which is greater than 2.

$m = 3, \quad m^2 + m - 6 = 9 + 3 - 6 = 6 > 0$, as required.

Try $m = 0$, which is between –3 and 2.

$m = 0, \quad m^2 + m - 6 = 0 + 0 - 6 = -6 < 0$, not required.

The solution is $m \le -3, \quad m \ge 2$.

o

$t \neq 0$

$$\frac{2t-3}{t} < 5$$
$$2t-3 < 5t$$
$$-3t < 3$$
$$t > -1$$

Try a value less than -1 , $t = -2$, $\frac{2t-3}{t} = 3.5$, which is less than 5.

Try a value between -1 and 0, $t = -\frac{1}{2}$, $\frac{2t-3}{t} = 8$, which is greater than 5.

Try a value greater than 0, $t = 1$, $\frac{2t-3}{t} = -1$, which is less than 5.

The solution is $t < -1, \quad t > 0$.

p

$y - 1 \neq 0 \Rightarrow y \neq 1$

$$\frac{y+1}{y-1} > 2$$
$$y+1 > 2y-2$$
$$3 > y$$
$$y < 3$$

Try a value less than 1, $y = 0$, $\frac{y+1}{y-1} = -1$, which is not greater than 2.

Try a value between 1 and 3, $y = 2$, $\frac{y+1}{y-1} = 3$, which is greater than 2, as required.

Try a value greater than 3, $y = 4$, $\frac{y+1}{y-1} = 1\frac{2}{3}$, which is greater than 2.

The solution is $1 < y < 3$.

q

$2n-4 \neq 0 \Rightarrow n \neq 2$

$$\frac{n}{2n-4} \geq 3$$
$$n \geq 6n-12$$
$$-5n \geq -12$$
$$n \leq 2\frac{2}{5}$$

Try a value less than 2, $n=0$, $\frac{n}{2n-4}=0$, which is not greater than 3.

Try a value between 2 and $2\frac{2}{5}$, $n=2\frac{1}{5}$, $\frac{n}{2n-4}=5\frac{1}{2}$, which is greater than 3, as required.

Try a value greater than $2\frac{2}{5}$, $n=3$, $\frac{n}{2n-4}=1\frac{1}{2}$, which is not greater than 3.

The solution is $2<n\leq 2\frac{2}{5}$.

r $2x+1 \neq 0 \Rightarrow x \neq -\frac{1}{2}$

$$\frac{3x-2}{2x+1} \leq -1$$
$$3x-2 \leq -2x-1$$
$$5x \leq 1$$
$$x \leq \frac{1}{5}$$

Try a value less than $-\frac{1}{2}$, $x=-1$, $\frac{3x-2}{2x+1}=5$, which is not less than –1.

Try a value between $-\frac{1}{2}$ and $\frac{1}{5}$, $x=0$, $\frac{3x-2}{2x+1}=-2$, which is less than –1, as required.

Try a value greater than $\frac{1}{5}$, $x=1$, $\frac{3x-2}{2x+1}=\frac{1}{3}$, which is less than –1.

The solution is $-\frac{1}{2}<x\leq\frac{1}{5}$.

Challenge exercise 2

Question 1

$a^{3y-5} = a^{-2}$
$3y - 5 = -2$
$3y = 3$
$y = 1$

Question 2

$x^2 - a^2 > 0$
$(x-a)(x+a) > 0$
$(x-a)(x+a) = 0 \Rightarrow x = -a, x = a$

Try $x = -2a$, which is less than $-a$.

$x = -2a, \quad 4a^2 - a^2 = 3a^2 > 0$, as required.

Try $x = 2a$, which is greater than a.

$x = 2a, \quad 4a^2 - a^2 = 3a^2 > 0$, as required.

Try $x = 0$, which is between $-a$ and a.

$x = 0, \quad 0 - a^2 = -a^2 < 0$, not required.

The solution is $x < -a \quad x > a$.

Question 3

$a=1, b=-6, c=-3$

$x=\dfrac{-(-6)\pm\sqrt{(-6)^2-4\times1\times-3}}{2\times1}$

$=\dfrac{6\pm\sqrt{48}}{2}$

$=\dfrac{6\pm4\sqrt{3}}{2}$

$=3\pm2\sqrt{3}$

$3\pm2\sqrt{3}=a+b\sqrt{3}$

$a=3,\quad b=\pm2$

Question 4

$\dfrac{2}{x-1}-\dfrac{1}{x+1}=1$

$\dfrac{2(x+1)-1(x-1)}{(x-1)(x+1)}=1$

$x+3=(x-1)(x+1)$

$x+3=x^2-1$

$x^2-x-4=0$

$a=1, b=-1, c=-4$

$x=\dfrac{1\pm\sqrt{1+16}}{2}=\dfrac{1\pm\sqrt{17}}{2}$

$x=\dfrac{1-\sqrt{17}}{2}\approx-156,\ x=\dfrac{1+\sqrt{17}}{2}\approx2.56$

Question 5

$y \neq 0$

$$\frac{6-2y}{y} \geq y-3$$
$$6-2y \geq y^2-3y$$
$$0 \geq y^2-y-6$$
$$y^2-y-6 \leq 0$$
$$(y-3)(y+2) \leq 0$$
$$(y-3)(y+2)=0 \Rightarrow y=-2, y=3$$

Try a value less than -2, $y=-3$, $\frac{6-2y}{y}=-4$,and $y-3=-6$.

So, $\frac{6-2y}{y} \geq y-3$, is true for $y \leq -2$.

Try a value between –2 and 0, $y=-1$, $\frac{6-2y}{y}=-8$,and $y-3=-4$.

So, $\frac{6-2y}{y} \geq y-3$, is not true for $y=-3$.

Try a value between 0 and 3, $y=1$, $\frac{6-2y}{y}=4$, and $y-3=-2$.

So, $\frac{6-2y}{y} \geq y-3$, is true for $0 < y \leq 3$.

Try a value greater than 3, $y=4$, $\frac{6-2y}{y}=-0.5$, and $y-3=1$.

So, $\frac{6-2y}{y} \geq y-3$, is not true for $y > 3$.

The solution is $y \leq -2, \quad 0 < y \leq 3$.

Question 6

a

$$\begin{aligned}&x^5-9x^3-8x^2+72\\&=x^3(x^2-9)-8(x^2-9)\\&=(x^2-9)(x^3-8)\\&=(x-3)(x+3)(x^3-8)\end{aligned}$$

b $x = \pm 3, 2$

$$\begin{aligned}&(x-3)(x+3)(x-2)(x^2+2x+4)=0\\&x-3=0 \Rightarrow x=3\\&x+3=0 \Rightarrow x=-3\\&x-2=0 \Rightarrow x=2\end{aligned}$$

x^2+2x+4 has no factors, since its discriminant is negative $(4-16=-12)$

Question 7

$y=x^3+x^2$ [1]

$y=x+1$ [2]

Substitute [1] in [2].

$$\begin{aligned}&x^3+x^2=x+1\\&x^3+x^2-x-1=0\\&x^2(x+1)-(x+1)=0\\&(x+1)(x^2-1)=0\\&(x+1)(x+1)(x-1)=0\\&(x+1)^2(x-1)=0\\&x=-1, x=1\end{aligned}$$

Substitute $x=-1$ in [2].

$y=-1+1=0$

Substitute $x=1$ in [2].

$y=1+1=2$

Question 8

$$x^2 - 8x + b = (x-4)^2 = x^2 - 8x + 16$$

So b = 16.

$$x^2 - 8x - 1 = 0$$
$$x^2 - 8x + 16 - 1 = 16$$
$$(x-4)^2 = 17$$
$$x - 4 = \pm\sqrt{17}$$
$$x = 4 \pm \sqrt{17}$$

Question 9

For $x > 3$

$$\frac{x-3}{3-x} = x$$
$$\frac{x-3}{-(x-3)} = x$$
$$x = -1$$

Reject this answer, since $x > 3$

For $x < 3$

$$\frac{-(x-3)}{3-x} = x$$
$$\frac{3-x}{3-x} = x$$
$$x = 1$$

So $x = 1$

Question 10

$$(x-4)(x-1) \le 28$$
$$x^2 - 5x + 4 \le 28$$
$$x^2 - 5x - 24 \le 0$$
$$(x-8)(x+3) \le 0$$
$$(x-8)(x+3) = 0 \Rightarrow x = -3, x = 8$$

Test $x = -4$, which is less than –3.

$x^2 - 5x - 24 = 16 + 20 - 24 = 12 > 0$, not required.

Test $x = 10$, which is greater than 8.

$x^2 - 5x - 24 = 100 - 50 - 24 = 26 > 0$, not required.

Test $x = 0$, which is between –3 and 8.

$x^2 - 5x - 24 = 0 - 0 - 24 = -24 < 0$, as required.

The solution is $-3 \le x \le 8$.

Question 11

$$x^{\frac{3}{2}} = 8^{-1} = 2^{-3}$$
$$\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = \left(2^{-3}\right)^{\frac{2}{3}}$$
$$x^1 = 2^{-2}$$
$$x = \frac{1}{2^2} = \frac{1}{4}$$

Question 12

$$x^2 - 2ax - b = (x-a)^2 - a^2 - b$$

$$(x-a)^2 - a^2 - b = 0$$

$$(x-a)^2 = a^2 + b$$

$$x - a = \pm\sqrt{a^2 + b}$$

$$x = a \pm \sqrt{a^2 + b}$$

Question 13

$$\frac{y^2-5y+2}{3y-2}\ge y$$

$$3y-2\ne 0 \Rightarrow y\ne \frac{2}{3}$$

$$y^2-5y+2\ge y(3y-2)$$

$$y^2-5y+2\ge 3y^2-2y$$

$$2y^2+3y-2\le 0$$

$$(2y-1)(y+2)\le 0$$

$$(2y-1)(y+2)=0 \Rightarrow y=-2, y=\frac{1}{2}$$

Test $y=-3<-2$ to check $\frac{y^2-5y+2}{3y-2}\ge y$.

$\frac{9+15+2}{-9-7}\ge -3, \quad -1\frac{5}{8}\ge -3$, is true.

Test $y=0$, where $-2<y<\frac{1}{2}$.

$\frac{0-0+2}{0-2}\ge 0, \quad -1\ge 0$, is not true.

Test $y=0.6$, where $\frac{1}{2}<y<\frac{2}{3}$.

$\frac{0.36-3+2}{1.8-2}\ge 0.6, \quad 3.2\ge 0.6$, is true.

Test $y=1$, where $y>\frac{2}{3}$.

$\frac{1-5+2}{3-2}\ge 1, \quad -2\ge 1$, is not true.

The solution is $y\le -2 \quad \frac{1}{2}\le y<\frac{2}{3}$.

Question 14

$3x^2 = 8(2x-1)$

$3x^2 - 16x + 8 = 0$

$a = 3, b = -16, c = 8$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4\times 3\times 8}}{2\times 3}$$

$$= \frac{16 \pm \sqrt{160}}{6}$$

$$= \frac{16 \pm 4\sqrt{10}}{6}$$

$$= \frac{8 \pm 2\sqrt{10}}{3}$$

$$= \frac{2(4 \pm \sqrt{10})}{3}$$

Question 15

$(2x-1)^2 = (5-x)^2$

$4x^2 - 4x + 1 = 25 - 10x + x^2$

$3x^2 + 6x - 24 = 0$

$x^2 + 2x - 8 = 0$

$(x-2)(x+4) = 0$

$x = 2, x = -4$

MATHS IN FOCUS 11
MATHEMATICS EXTENSION 1

FULLY WORKED SOLUTIONS

Chapter 3 Permutations and combinations

Exercise 3.01 Counting techniques

Question 1

$26 \times 26 \times 26 \times 26$

456 976

Question 2

$26 \times 26 \times 10 \times 10$

67 600

Question 3

$26^5 \times 10^4$

118 813 760 000

Question 4

26×10

260

Question 5

$26^{10} \times 10^{15}$

Question 6

10^3

1000

Question 7

10^6

1 000 000

Question 8

$12 \times 5 \times 5$

300

Question 9

$8 \times 2 \times 4$

64

Question 10

Total number of PIN 10^4

$P(\text{correct guess}) = \frac{3}{10000}$

Question 11

a $7 \times 4 \times 3$

84

b $P(\text{most favourite}) = \frac{1}{84}$

Question 12

a 10^8

100 000 000

b 10^3

1 000

Question 13

Serial numbers = 26^{10} = 14 116 709 563 376 > 20 000 000, so there will be enough serial numbers.

Question 14

Number of passwords = $26^2 \times 10^5$

$P(\text{correct guess}) = \frac{1}{26^2 \times 10^5} = \frac{1}{67\ 600\ 000}$

Question 15

$10^x > 3.5 \times 10^6$

$x = 7$

Question 16

Number of serial numbers

$26^3 \times 10^4 \times 26^4$

8×10^{13}

Yes there are enough combinations

Question 17

Bridal combinations

$12 \times 18 \times 24$

5184

$P(\text{same}) = \dfrac{1}{5184}$

Question 18

$3 \times 2 \times 1 = 6$

Question 19

$20 \times 19 \times 18 = 6840$

Question 20

$6 \times 5 \times 4 \times 3 = 360$

Question 21

$200 \times 199 \times 198 = 7\,880\,400$

Question 22

$7 \times 6 \times 5 = 210$

Question 23

$3^5 = 243$

Question 24

a $100 \times 99 = 9900$

b $P(\text{set ordered pair}) = \dfrac{1}{9900}$

Question 25

Number of triples

$10 \times 9 \times 8 = 720$

$P(\text{CAB}) = \dfrac{1}{720}$

Exercise 3.02 The pigeonhole principle

Question 1

There are 4 different block colours, so at most, 4 different colours can be chosen before a colour is repeated.

Hence, 4 + 1 = 5 blocks must be chosen to ensure there will be two blocks with the same colour, since the fifth block must be one of the colours already drawn.

Red	Blue	Green	Yellow	
1	1	1	1	First 4 blocks
1				Fifth block

Or

Red	Blue	Green	Yellow	
1	1	1	1	First 4 blocks
	1			Fifth block

Or

Red	Blue	Green	Yellow	
1	1	1	1	First 4 blocks
		1		Fifth block

Or

Red	Blue	Green	Yellow	
1	1	1	1	First 4 blocks
			1	Fifth block

Question 2

5 different states are to be represented. At most, 5 committee members can be chosen where each person comes from a different state. This means the committee must comprise 5 + 1 = 6 members to ensure that there will be at least two members from the same state. The sixth member comes from a state that another committee member also is from.

Question 3

There are 9 different sports offered by the school, so at most, 9 students can be surveyed where each takes a different sport. This means 9 + 1 = 10 students need to be surveyed to ensure that at least two students are from the same sporting group.

Question 4

There are 3 types of animals, so if there are 3 + 1 = 4 animals, at least 2 must be the same type.

Question 5

There are 2 types of socks – black and white – so only 2 + 1 = 3 need to be chosen to get a pair of the same colour.

Question 6

There are 4 suits, so only 4 + 1 = 5 need to be chosen for at least 2 cards to be the same suit.

Question 7

There are 4 eye colours, so if 4 + 1 = 5 people are chosen, at least 2 must have the same eye colour.

Question 8

$n = 83, k = 4, \frac{n}{k} = \frac{83}{4} = 20\frac{3}{4}$. There is at least one barrel with at least $20\frac{3}{4}$ oranges.

Want $\frac{n}{k} > 20\frac{3}{4}$, so take x = 21 oranges.

Small	Medium	Large	Extra large	
1	1	1	1	First set of 4 oranges
1	1	1	1	Second set of 4 oranges
1	1	1		21st set of 3 oranges Any 3 of 4 possibilities

20 sets

Question 9

There are 1024 cars, so $n = 1024$

There are 13 sections (letters A to M) for car parking, so k =13.

$$\frac{n}{k} = \frac{1024}{13} = 78.7692\ldots.$$

There is at least one section with at least 78.7692… cars.

Round this up to 79 cars.

Question 10

There are 129 dairy cows, so let $n = 129$.

There are 3 pens, so let $k = 3$.

$$\frac{n}{k} = \frac{129}{3} = 43.$$

There is at least one pen with at least 43 cows.

Question 11

There were 2495 people, so $n = 2495$.

There are 9 parks, so $k = 9$.

$$\frac{n}{k} = \frac{2495}{9} = 277.2.$$

There was at least one park with at least 277.2 people.

Round this up to 278 people.

Question 12

Remainders can be 0, 1, 2, 3, 4, 5, 6, so take $k = 7$.

There are 30 numbers to be divided for remainders, so $n = 30$.

$$\frac{n}{k} = \frac{30}{7} = 4\frac{2}{7}$$

This means that there are at least $4\frac{2}{7}$ numbers that have the same remainder.

Round the value up to 5.

There are at least 5 numbers from 1 to 30 that have at least the same remainder.

Question 13

$$\frac{n}{8} > 28$$
$$n > 224$$

$n = 225$

Question 14

Let n be the number of people.

$$\frac{n}{35} = \frac{450}{35} = 12.86$$

This means there are at least 12.86 different foods that are preferred by at least 35 people.

Round 12.86 up to 13, so there are 13 different food types.

Question 15

There are 3 different groups: cinema, concert and dinner.

Let n be the number of friends and let $k = 3$ be the number of groups.

For 'at least 3', we require that $\frac{n}{k} > 2$.

$\frac{n}{3} > 2$

$n > 6$

$n = 7$

7 friends are required.

Exercise 3.03 Factorial notation

Question 1

a $6! = 720$

b $10! = 3\ 628\ 800$

c $0! = 1$

d $8! - 7! = 35\ 280$

e $5 \times 4! = 120$

f $\frac{7!}{4!} = 210$

g $\frac{12!}{5!} = 3\ 991\ 680$

h $\frac{13!}{4!9!} = 715$

i $\frac{8!}{3!5!} = 56$

j $\frac{11!}{4!7!} = 330$

Question 2

$9! = 362\ 880$

Question 3

$6! = 720$

Question 4

$12! = 479\ 001\ 600$

Question 5

$5! = 120$

Question 6

a $11! = 39\ 916\ 800$

b $12! = 479\ 001\ 600$

Question 7

$8! = 40\ 320$

Question 8

$7! =5\ 040$

Question 9

$3! = 6$

Question 10

$6! = 720$

Question 11

$7! = 5\ 040$

Question 12

$15! = 1.3 \times 10^{12}$

Question 13

a $11! = 39\ 916\ 800$

b $10! = 3\ 628\ 800$

Question 14

a $6! = 720$

b $5! = 120$

Question 15

$7! = 5\ 040$

Question 16

a $P(\text{Heart}) = \frac{1}{4}$

b $P(\text{Order}) = \frac{1}{4!} = \frac{1}{24}$

Question 17

a $12! = 479\ 001\ 600$

b $P(\text{pink}) = \frac{1}{12}$

Question 18

Number of arrangements

$5! = 120$

$P(\text{correct}) = \frac{1}{120}$

Question 19

13! = 6 227 020 800

Question 20

a

$$\frac{8!}{4!}$$
$$=\frac{8\times7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1}$$
$$=8\times7\times6\times5$$

b

$$\frac{11!}{6!}$$
$$=\frac{11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{6\times5\times4\times3\times2\times1}$$
$$=11\times10\times9\times8\times7$$

c

$$\frac{n!}{r!}$$
$$=\frac{n\times(n-1)\times(n-2)\times...\times r!}{r!}$$
$$=n\times(n-1)\times(n-2)\times...\times(r+1)$$

d

$$\frac{n!}{(n-r)!}$$
$$=\frac{n\times(n-1)\times(n-2)\times...\times(n-r+1)\times(n-r)!}{(n-r)!}$$
$$=n\times(n-1)\times(n-2)\times...\times(n-r+1)$$

Exercise 3.04 Permutations

Question 1

a

$$^{6}P_{3} = \frac{6!}{(6-3)!}$$
$$= 120$$

b

$$^{5}P_{2} = \frac{5!}{(5-2)!}$$
$$= 20$$

c

$$^{8}P_{3} = \frac{8!}{(8-3)!}$$
$$= 336$$

d

$$^{10}P_{7} = \frac{10!}{(10-7)!}$$
$$= 604\ 800$$

e

$$^{9}P_{6} = \frac{9!}{(9-6)!}$$
$$= 60\ 480$$

f

$$^{7}P_{5} = \frac{7!}{(7-5)!}$$
$$= 2520$$

g

$$^{8}P_{6} = \frac{8!}{(8-6)!}$$
$$= 20\ 160$$

h

$$
\begin{aligned}
{}^{11}P_8 &= \frac{11!}{(11-8)!} \\
&= 6\ 652\ 800
\end{aligned}
$$

i

$$
\begin{aligned}
{}^{9}P &= \frac{9!}{(9-1)!} \\
&= 9
\end{aligned}
$$

j

$$
\begin{aligned}
{}^{6}P_6 &= \frac{6!}{(6-6)!} \\
&= 720
\end{aligned}
$$

Question 2

a ${}^{26}P_2 = 650$

b ${}^{26}P_3 = 15\ 600$

c ${}^{26}P_4 = 358\ 800$

d ${}^{26}P_5 = 7\ 893\ 600$

Question 3

a $9 \times {}^{9}P_2 = 648$

b $6 \times {}^{9}P_2 = 432$

c $2 \times {}^{9}P_2 = 144$

Question 4

a ${}^5P_2 = 20$

b $1\times {}^4P = 4$

c $3\times {}^4P = 12$

d Even = 20 – 12 = 8

Question 5

a 4! = 24

b ${}^4P_3 = 24$

Question 6

a $9\times {}^9P_3 = 4536$

b $4\times {}^9P_3 = 2016$

c $7\times {}^9P_3 = 3528$

Question 7

a 5! = 120

b 2 × 4! = 48

c 4 × 4! = 96

d 3 × 4! = 72

e ${}^5P_3 = 60$

Question 8

a 12! = 479 001 600

b $^{12}P_3 = 1320$

Question 9

a $^{8}P_2 = 56$

b $^{8}P_3 = 336$

c $^{8}P_4 = 1680$

Question 10

a CENTIPEDE, EEE

$$\frac{9!}{3!}$$
$$= 60480$$

b ALGEBRA, AA

$$\frac{7!}{2!}$$
$$= 2520$$

c TELEVISION, EE II

$$\frac{10!}{2!2!}$$
$$= 907200$$

d ANTARCTICA, AAA TT CC

$$\frac{10!}{3!2!2!}$$
$$= 151200$$

e DONOR, OO

$$\frac{5!}{2!}$$
$$= 60$$

f BASKETBALL, BB AA LL

$$\frac{10!}{2!2!2!}$$
$$= 453600$$

g GREEDY, EE

$$\frac{6!}{2!}$$
$$= 360$$

h DUTIFUL, UU

$$\frac{7!}{2!}$$
$$= 2520$$

i MANUFACTURER, AA UU RR

$$\frac{12!}{2!2!2!}$$
$$= 59875200$$

j AEROPLANE, AA EE

$$\frac{9!}{2!2!}$$
$$= 90720$$

Question 11

a 4! = 24

b 7! = 5040

c 8! = 40 320

d 10! = 3 628 800

e 11! = 39 916 800

Question 12

a 4 friends = 3!

= 6

b 7 friends = 6!

= 720

c 8 friends = 7!

= 5040

d 10 friends = 9!

= 362 880

e 11 friends = 10!

= 3 628 800

Question 13

As for in a circle (n – 1)!, but need to divide by 2 for symmetry (clockwise, anticlockwise).

a (10 – 1)! = 9! = 362 880, then ÷ 2, = 181 440

b (12 – 1)! = 11! = 39 916 800, then ÷ 2, = 19 958 400

c (9 – 1)! = 8! = 40 320, then ÷ 2, = 20 160

d (11 – 1)! = 10! = 3 628 800, then ÷ 2, = 1 814 400

e (13 – 1)! = 12! = 479 001 600, then ÷ 2, = 239 500 800

Question 14

a 6! = 720

b 5! = 120

Question 15

a 9! = 362 880

b 8! = 40 320

Question 16

a 10! = 3 628 800

b 9! = 362 880

c $\frac{9!}{2}$ allowing for symmetry (clockwise, anticlockwise)

= 181 440

Question 17

a 4! = 24

b ${}^4P_2 = 12$

c ${}^4P_3 = 24$

Question 18

a (7 – 1)! = 6! = 720

b (6 – 1)! × 2! = 5! × 2! = 120 × 2 = 240

c answer **a** – answer **b** = 720 – 240 = 480

d (5 – 1)! × 3! = 4! × 3! = 24 × 6 = 144

Question 19

a $10! = 3\ 628\ 800$

b $1 \times 9! = 362\ 880$

c $2 \times 5! \times 5! = 28\ 800$

Question 20

Number of arrangements = 9!

= 362880

Arrangements with two people sitting together = 2 × 8!

= 80648

$$P(\text{two people together}) = \frac{80\ 640}{362\ 880} = \frac{2}{9}$$

Question 21

a 20!

b $5! \times 8! \times 7! \times 3!$

c $P(\text{cookbook}) = \frac{7}{20}$

Question 22

a $\dfrac{5!}{2!}$

$= 60$

b $\dfrac{4 \times 4!}{2!} = \dfrac{4 \times 24}{2} = 48$

c $\dfrac{3 \times 4!}{2!} = \dfrac{3 \times 24}{2} = 36$

d Number less than 40 000

$1 \times 4! = 24 \dfrac{1 \times 4!}{2!} = \dfrac{1 \times 24}{2} = 12$

P(less than 40 000) = $\dfrac{12}{60} = \dfrac{1}{5}$

Question 23

Number of arrangements of the letters of LAPTOP

$\dfrac{6!}{2!} = 360$

Number of arrangements with T as the starting letter

$\dfrac{5!}{2!} = 60$

P(start with T) = $\dfrac{60}{360} = \dfrac{1}{6}$

Question 24

PHYSICAL

${}^{8}P_{3} = 336$

$P(\text{CAL}) = \dfrac{1}{336}$

Question 25

a 8! = 40320

b seat the 1 person first.

$6 \times 7! = 6 \times 5040 = 30\ 240$

c seat the 2 people first

${}^{6}P_{2} \times 6! = 30 \times 720 = 21\ 600$

Question 26

VALUED

3 letter arrangements

a ${}^{5}P_{2} = 20$

b $3 \times {}^{5}P_{2} = 60$

Question 27

THEORY

a 6! = 720

b $1 \times 5! = 120$

c $4 \times 4! \times 2 = 4 \times 24 \times 2 = 192$

Question 28

a $x!$

b $(x - 1)!$

c $2!(x - 2)!$

d $3!(x - 3)!$

e answer to part **b** – answer to part **c**

$(x - 1)! - 2!(x - 2)!$

$= (x - 1)(x - 2)! - 2!(x - 2)!$

$= (x - 1 - 2)(x - 2)!$

$= (x - 3)(x - 2)!$

Question 29

a

$$\frac{{}^8P_3}{3!}$$

$$=\frac{\frac{8!}{(8-3)!}}{3!}$$

$$=\frac{8!}{5!3!}$$

$$=\frac{8!}{5!(8-5)!}$$

$$=\frac{{}^8P_5}{5!}$$

b

$$\frac{{}^nP_r}{r!}$$

$$=\frac{\frac{n!}{(n-r)!}}{r!}$$

$$=\frac{n!}{(n-r)!r!}$$

$$=\frac{n!}{(n-r)!(n-(n-r))!}$$

$$=\frac{\frac{n!}{(n-(n-r))!}}{(n-r)!}$$

$$=\frac{{}^nP_{n-r}}{(n-r)!}$$

Question 30

$$^{n+1}P_r = \frac{(n+1)!}{(n+1-r)!}$$

$$\begin{aligned}
{}^nP_r + r\,{}^nP_{r-1} &= \frac{n!}{(n-r)!} + r\frac{n!}{(n-[r-1])} \\
&= \frac{n!}{(n-r)!} + \frac{rn!}{(n-r+1)!} \\
&= \frac{(n+1-r)n!}{(n+1-r)(n-r)!} + \frac{rn!}{(n-r+1)!} \\
&= \frac{(n+1-r)n!}{(n+1-r)!} + \frac{rn!}{(n-r+1)!} \\
&= \frac{n.n! + n! - r.n! + r.n!}{(n-r+1)!} \\
&= \frac{n.n! + n!}{(n-r+1)!} \\
&= \frac{(n+1)n!}{(n-r+1)!} \\
&= \frac{(n+1)!}{(n-r+1)!}
\end{aligned}$$

$$\therefore {}^{n+}P_r = {}^nP_r + r\,{}^nP_{r-1}$$

Exercise 3.05 Combinations

Question 1

a $\binom{9}{5} = \frac{9!}{(9-5)!5!} = 126$

b $\binom{12}{7} = \frac{12!}{(12-7)!7!} = 792$

c $\binom{8}{3} = \frac{8!}{(8-3)!3!} = 56$

d $\binom{10}{4} = \frac{10!}{(10-4)!4!} = 210$

e $\binom{11}{5} = \frac{11!}{(11-5)!5!} = 462$

Question 2

a **i** ${}^{10}C_0 = 1$

ii ${}^{7}C_0 = 1$

iii $\binom{14}{0} = 1$

iv ${}^{9}C_9 = 1$

v $\binom{11}{11} = 1$

b **i** ${}^{n}C_0 = 1$

ii ${}^{n}C_n = 1$

Question 3

a $^{8}C_{6} = 28$

b $^{9}C_{6} = 84$

c $^{11}C_{6} = 462$

d $^{15}C_{6} = 5005$

e $^{20}C_{6} = 38760$

Question 4

a

R_1R_2	R_2R_3	R_3B_1	B_1B_2	B_2B_3
R_1R_3	R_2B_1	R_3B_2	B_1B_3	
R_1B_1	R_2B_2	R_3B_3		
R_1B_2	R_2B_3			
R_1B_3				

Total number of arrangements = 15

b $^{20}C_{7} = 77\ 520$

Question 5

$^{20}C_{5} = 15\ 504$

Question 6

$^{10}C_{6} = 210$

Question 7

$^{52}C_5 = 2\ 598\ 960$

Question 8

a $^{10}P_3 = 720$

b $^{10}C_3 = 120$

Question 9

a $^{14}P_3 = 2184$

b $^{14}C_3 = 364$

Question 10

$^{23}C_6 = 100\ 947$

Question 11

$^{20}C_4 = 4845$

Question 12

$^{27}C_3 = 2925$

Question 13

$^{35}C_8 = 23\ 535\ 820$

Question 14

a $^{12}C_5 = 792$

b Teams with Erik

$^{11}C_4 = 330$

$P(\text{Erik}) = \dfrac{330}{792} = \dfrac{5}{12}$

C Teams with Erik and Jens

$^{10}C_3 = 120$

$P(\text{Erik and Jens}) = \dfrac{120}{792} = \dfrac{5}{33}$

Question 15

a $^{23}C_6 = 100\ 947$

b $^{11}C_6 = 462$

c $^{12}C_6 = 924$

d $^{11}C_3 \times {}^{12}C_3 = 36\ 300$

e $^{22}C_5 = 26\ 334$

f $^{23}C_6 - {}^{22}C_5 = 74\ 613$

g $^{11}C_2 \times {}^{12}C_4 = 27\ 225$

Question 16

$^{15}C_2 = 105$

$105 \times 1 = \$105$

Question 17

a $^{25}C_9 = 2\ 042\ 975$

b $^{11}C_9 = 55$

c $^{11}C_5 \times {}^{14}C_4 = 462\ 462$

d $^{11}C_7 \times {}^{14}C_2 = 30\ 030$

Question 18

a $^{15}C_0 = 3003$

b **i** $^{8}C_4 \times {}^{7}C_4 = 2450$

ii $^{8}C_6 \times {}^{7}C_2 = 588$

iii $^{8}C_7 \times {}^{7}C_1 = 56$

iv $^{8}C_3 \times {}^{7}C_5 = 1176$

Question 19

a $^{52}C_0 = 1.58 \times 10^{10}$

b $^{13}C_0 = 286$

c $^{13}C_7 \times {}^{39}C_3 = 15\ 682\ 524$

d $^{26}C_0 = 5\ 311\ 735$

e $^{4}C_4 \times {}^{48}C_6 = 12\ 271\ 512$

Question 20

a $^{17}C_7 \times {}^{21}C_5 = 395\ 747\ 352$

b $^{17}C_9 \times {}^{21}C_3 = 32\ 332\ 300$

c $^{17}C_{10} \times {}^{21}C_2 = 4\ 084\ 080$

d $^{21}C_4 \times {}^{17}C_8 = 145\ 495\ 350$

e $^{21}C_6 \times {}^{17}C_6 = 671\ 571\ 264$

Question 21

a $^{22}C_7 = 170\ 544$

b $^{9}C_7 = 36$

c $^{8}C_3 \times {}^{9}C_2 \times {}^{5}C_2 = 20\ 160$

d $^{9}C_4 \times {}^{5}C_1 \times {}^{8}C_2 = 17\ 640$

e $^{8}C_4 \times {}^{5}C_2 \times {}^{9}C_1 = 6300$

Question 22

a $^{7}C_6 = 7$

b $^{19}C_6 = 27\ 132$

c $^{7}C_3 \times {}^{6}C_1 \times {}^{12}C_2 = 13\ 860$

d $^{7}C_1 \times {}^{12}C_4 \times {}^{6}C_1 = 20\ 790$

e $^{12}C_3 \times {}^{6}C_1 \times {}^{7}C_2 = 27\ 720$

Question 23

a $^5C_4 = 5$

b $^5C_2 \times {}^9C_2 = 360$

c $^9C_4 = 126$

Question 24

a $^{12}C_5 = 792$

b $^{12}C_7 = 792$

c

$$\begin{aligned} ^{12}C_5 &= \frac{12!}{(12-5)!5!} \\ &= \frac{12!}{7!5!} \\ &= \frac{12!}{(12-7)!7!} \\ &= {}^{12}C_7 \end{aligned}$$

Question 25

$^9C_6 = 84$

$^8C_6 + {}^8C_5 = 28 + 56 = 84$

$\therefore {}^9C_6 = {}^8C_6 + {}^8C_5$

Question 26

$^{13}C_7 = 1716$

$^{13}C_6 = 1716$

$\therefore {}^{13}C_7 = {}^{13}C_6$

Question 27

$$\binom{10}{4} = 210$$

$$\binom{9}{4} + \binom{9}{3} = 126 + 84 = 210$$

$$\therefore \binom{10}{4} = \binom{9}{4} + \binom{9}{3}$$

Question 28

$$\begin{aligned}\binom{n}{r} &= \frac{n!}{(n-r)!r!} \\ &= \frac{n!}{(n-r)!(n-(n-r))!} \\ &= \binom{n}{n-r}\end{aligned}$$

Question 29

$$\begin{aligned}{}^nP_r &= \frac{n!}{(n-r)!} \\ &= r!\frac{n!}{(n-r)!r!} \\ &= r\ \ {}^nC_r\end{aligned}$$

Question 30

$$\binom{n}{k}=\frac{n!}{(n-k)!k!}$$

$$\begin{aligned}\binom{n-1}{k-1}+\binom{n-1}{k} &= \frac{(n-1)!}{(n-1-[k-1])!(k-1)!}+\frac{(n-1)!}{(n-1-k)!k!}\\ &= \frac{(n-1)!}{(n-k)!(k-1)!}+\frac{(n-1)!}{(n-1-k)!k!}\\ &= \frac{k(n-1)!}{k(n-k)!(k-1)!}+\frac{(n-k)(n-1)!}{(n-k)(n-1-k)!k!}\\ &= \frac{k(n-1)!}{(n-k)!\,k!}+\frac{(n-k)(n-1)!}{(n-k)!k!}\\ &= \frac{(k+n-k)(n-1)!}{(n-k)!\,k!}\\ &= \frac{n(n-1)!}{(n-k)!\,k!}\\ &= \frac{n!}{(n-k)!\,k!}\\ &= \binom{n}{k}\end{aligned}$$

Exercise 3.06 Pascal's triangle and binomial coefficients

Question 1

a

$$\begin{aligned}\binom{9}{5} &= \frac{9!}{(9-5)!5!}\\ &= \frac{9!}{4!5!}\\ &= \frac{12!}{(9-4)!4!}\\ &= \binom{9}{4}\end{aligned}$$

b

$$\begin{aligned}{}^7C_2 &= \frac{7!}{(7-2)!2!}\\ &= \frac{7!}{2!5!}\\ &= \frac{12!}{(7-5)!5!}\\ &= {}^7C_5\end{aligned}$$

c

$$\begin{aligned}\binom{12}{5} &= \frac{12!}{(12-5)!5!}\\ &= \frac{12!}{7!5!}\\ &= \frac{12!}{(12-7)!7!}\\ &= \binom{12}{7}\end{aligned}$$

Question 2

a

$$
\begin{aligned}
{}^{7}C_{5} &= \frac{7!}{(7-5)!5!} \\
&= \frac{7\times 6!}{2!5!} \\
&= \frac{2\times 6!}{2!5!} + \frac{5\times 6!}{2!5!} \\
&= \frac{6!}{1!5!} + \frac{6!}{2!4!} \\
&= {}^{6}C_{5} + {}^{6}C_{4}
\end{aligned}
$$

b

$$
\begin{aligned}
\binom{10}{6} &= \frac{10!}{(10-6)!6!} \\
&= \frac{10\times 9!}{4!6!} \\
&= \frac{6\times 9!}{4!6!} + \frac{4\times 9!}{4!6!} \\
&= \frac{9!}{4!5!} + \frac{9!}{3!6!} \\
&= \binom{9}{5} + \binom{9}{6}
\end{aligned}
$$

c

$$
\begin{aligned}
\binom{7}{3} &= \frac{7!}{(7-3)!3!} \\
&= \frac{7\times 6!}{4!3!} \\
&= \frac{4\times 6!}{4!3!} + \frac{3\times 6!}{4!3!} \\
&= \frac{6!}{3!3!} + \frac{6!}{4!2!} \\
&= \binom{6}{3} + \binom{6}{2}
\end{aligned}
$$

Question 3

$$\binom{n}{1} = \frac{n!}{(n-1)!1!}$$
$$= \frac{n!}{(n-1)!(n-(n-1))!}$$
$$= \binom{n}{n-1}$$

Question 4

${}^{7}C_{x} = {}^{7}C_{2}$

${}^{n}C_{k} = {}^{n}C_{n-k}$

so $x = 7 - 2 = 5$ and ${}^{7}C_{5} = {}^{7}C_{2}$

So $x = 5$

Question 5

${}^{12}C_{3} = {}^{12}C_{y}$

${}^{n}C_{k} = {}^{n}C_{n-k}$

so $y = 12 - 3 = 9$ and ${}^{12}C_{3} = {}^{12}C_{9}$

So $y = 9$

Question 6

${}^{11}C_{a} = {}^{11}C_{8}$

${}^{n}C_{k} = {}^{n}C_{n-k}$

so $a = 11 - 8 = 3$ and ${}^{11}C_{3} = {}^{11}C_{8}$

So $a = 3$

Question 7

$$\binom{n}{6} = \binom{10}{5} + \binom{10}{6}$$

$$^{n}C_{k} = {}^{n-1}C_{k-} + {}^{n-1}C_{k}$$

so $\binom{11}{6} = \binom{10}{5} + \binom{10}{6}$

So $n = 11$

Question 8

$$^{20}C_{7} = {}^{19}C_{k} + {}^{19}C_{7}$$

$$^{n}C_{k} = {}^{n-1}C_{k-} + {}^{n-1}C_{k}$$

so $^{20}C_{7} = {}^{19}C_{6} + {}^{19}C_{7}$

So $k = 6$

Question 9

a $(a + x)^4$

$= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$

b $(a + x)^6$

$= a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$

c $(a + x)^5$

$= a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5$

d $(2a + 1)^3$

$= 8a^3 + 12a^2 + 6a + 1$

e $(x - 2)^7$

$= x^7 - 14x^6 + 84x^5 - 280x^4 + 560x^3 - 672x^2 + 448x - 128$

f $(4x^2 + 3)^4$

$$= \binom{4}{0}(4x^2)^4 + \binom{4}{1}(4x^2)^3 \times 3 + \binom{4}{2}(4x^2)^2 \times 3^2 + \binom{4}{3}(4x^2) \times 3^3 + \binom{4}{4}3^4$$

$= 256x^8 + 768x^6 + 864x^4 + 432x^2 + 81$

g $(3 - 2x)^6$

$$= \binom{6}{0}3^6 - \binom{6}{1}3^5 \times 2x + \binom{6}{2}3^4 \times (2x)^2 - \binom{6}{3}3^3 \times (2x)^3 + \binom{6}{4}3^2 \times (2x)^4 - \binom{6}{5}3 \times (2x)^5 + \binom{6}{6}(2x)^6$$

$= 729 - 2916x + 4860x^2 - 4320x^3 + 2160x^4 - 576x^5 + 64x^6$

h $(4a - 5b)^3$

$$= \binom{3}{0}(4a)^3 - \binom{3}{1}(4a)^2(5b) + \binom{3}{2}(4a)(5b)^2 - \binom{3}{3}(5b)^3$$

$= 64a^3 - 240a^2b + 300ab^2 - 125b^3$

i $(2 + 3m)^5$

$$= \binom{5}{0}2^5 + \binom{5}{1}2^4(3m) + \binom{5}{2}2^3(3m)^2 + \binom{5}{3}2^2(3m)^3 + \binom{5}{4}2(3m)^4 + \binom{5}{5}(3m)^5$$

$= 32 + 240m + 720m^2 + 1080m^3 + 810m^4 + 243m^5$

j $(1 - 2x)^8$

$$= \binom{8}{0}1^8 - \binom{8}{1}1^7(2x) + \binom{8}{2}1^6(2x)^2 - \binom{8}{3}1^5(2x)^3 + \binom{8}{4}1^4(2x)^4 - \binom{8}{5}1^3(2x)^5 + \binom{8}{6}1^2(2x)^6 - \binom{8}{7}1(2x)^7 + \binom{8}{8}(2x)^8$$

$= 1 - 16x + 112x^2 - 448x^3 + 1120x^4 - 1792x^5 + 1792x^6 - 1024x^7 + 256x^8$

Question 10

a $\left(\sqrt{2}+1\right)^5$

$$=\binom{5}{0}\left(\sqrt{2}\right)^5+\binom{5}{1}\left(\sqrt{2}\right)^4(1)+\binom{5}{2}\left(\sqrt{2}\right)^3(1)^2+\binom{5}{3}\left(\sqrt{2}\right)^2(1)^3+\binom{5}{4}\left(\sqrt{2}\right)(1)^4+\binom{5}{5}(1)^5$$
$$=4\sqrt{2}+20+20\sqrt{2}+20+5\sqrt{2}+1$$
$$=29\sqrt{2}+41$$

b $\left(\sqrt{3}-1\right)^6$

$$=\binom{6}{0}\left(\sqrt{3}\right)^6-\binom{6}{1}\left(\sqrt{3}\right)^5(1)+\binom{6}{2}\left(\sqrt{3}\right)^4(1)^2-\binom{6}{3}\left(\sqrt{3}\right)^3(1)^3$$
$$+\binom{6}{4}\left(\sqrt{3}\right)^2(1)^4-\binom{6}{5}\left(\sqrt{3}\right)(1)^5+\binom{6}{6}(1)^6$$
$$=27-54\sqrt{3}+135-60\sqrt{3}+45-6\sqrt{3}+1$$
$$=208-120\sqrt{3}$$

c $\left(\sqrt{3}+\sqrt{5}\right)^4$

$$=\binom{4}{0}\left(\sqrt{3}\right)^4+\binom{4}{1}\left(\sqrt{3}\right)^3\left(\sqrt{5}\right)+\binom{4}{2}\left(\sqrt{3}\right)^2\left(\sqrt{5}\right)^2+\binom{4}{3}\left(\sqrt{3}\right)\left(\sqrt{5}\right)^3+\binom{4}{4}\left(\sqrt{5}\right)^4$$
$$=9+12\sqrt{15}+90+20\sqrt{15}+25$$
$$=124+32\sqrt{15}$$

d $\left(3+\frac{x}{2}\right)^4$

$$=\binom{4}{0}(3)^4+\binom{4}{1}(3)^3\left(\frac{x}{2}\right)+\binom{4}{2}(3)^2\left(\frac{x}{2}\right)^2+\binom{4}{3}(3)\left(\frac{x}{2}\right)^3+\binom{4}{4}\left(\frac{x}{2}\right)^4$$
$$=81+54x+\frac{27}{2}x^2+\frac{3}{2}x^3+\frac{1}{16}x^4$$

e $\left(x+\frac{1}{x}\right)^5$

$$=\binom{5}{0}x^5+\binom{5}{1}x^4\left(\frac{1}{x}\right)+\binom{5}{2}x^3\left(\frac{1}{x}\right)^2+\binom{5}{3}x^2\left(\frac{1}{x}\right)^3+\binom{5}{4}x\left(\frac{1}{x}\right)^4+\binom{5}{5}\left(\frac{1}{x}\right)^5$$
$$=x^5+5x^3+10x+10x^{-1}+5x^{-3}+x^{-5}$$

f $\left(1-\frac{x}{2}\right)^3$

$$=\binom{3}{0}1^3-\binom{3}{1}1^2\left(\frac{x}{2}\right)+\binom{3}{2}1\left(\frac{x}{2}\right)^2-\binom{3}{3}\left(\frac{x}{2}\right)^3$$
$$=1-\frac{3x}{2}+\frac{3x^2}{4}-\frac{x^3}{8}$$

g $\left(\frac{a}{3}-\frac{b}{2}\right)^3$

$$=\binom{3}{0}\left(\frac{a}{3}\right)^3-\binom{3}{1}\left(\frac{a}{3}\right)^2\left(\frac{b}{2}\right)+\binom{3}{2}\left(\frac{a}{3}\right)\left(\frac{b}{2}\right)^2-\binom{3}{3}\left(\frac{b}{2}\right)^3$$
$$=\frac{a^3}{27}-\frac{a^2b}{6}+\frac{ab^2}{4}-\frac{b^3}{8}$$

Question 11

$$\left(\sqrt{2}+3\right)^3$$
$$=\binom{3}{0}\left(\sqrt{2}\right)^3+\binom{3}{1}\left(\sqrt{2}\right)^2 3+\binom{3}{2}\left(\sqrt{2}\right)3^2+\binom{3}{3}3^3$$
$$=2\sqrt{2}+18+27\sqrt{2}+27$$
$$=45+29\sqrt{2}$$
$$a=45, b=29$$

Question 12

$$\left(2-\sqrt{5}\right)^4$$
$$=\binom{4}{0}2^4-\binom{4}{1}2^3\left(\sqrt{5}\right)+\binom{4}{2}2^2\left(\sqrt{5}\right)^2-\binom{4}{3}2\left(\sqrt{5}\right)^3+\binom{4}{4}\left(\sqrt{5}\right)^4$$
$$=16-32\sqrt{5}+120-40\sqrt{5}+25$$
$$=161-72\sqrt{5}$$
$$a=161,\ \ b=-72$$

Question 13

$$\left(\sqrt{3}-1\right)^5$$
$$=\binom{5}{0}\left(\sqrt{3}\right)^5-\binom{5}{1}\left(\sqrt{3}\right)^4(1)+\binom{5}{2}\left(\sqrt{3}\right)^3(1)^2-\binom{5}{3}\left(\sqrt{3}\right)^2(1)^3+\binom{5}{4}\left(\sqrt{3}\right)(1)^4-\binom{5}{5}(1)^5$$
$$=4\sqrt{3}-20+20\sqrt{3}-20+5\sqrt{3}-1$$
$$=44\sqrt{3}-76$$
$$=-76+\sqrt{5808}$$
$$a=-76,\ b=5808$$

Question 14

$$\left(\sqrt{2}+\sqrt{3}\right)^3$$
$$=\binom{3}{0}\left(\sqrt{2}\right)^3+\binom{3}{1}\left(\sqrt{2}\right)^2\left(\sqrt{3}\right)+\binom{3}{2}\left(\sqrt{2}\right)\left(\sqrt{3}\right)^2+\binom{3}{3}\left(\sqrt{3}\right)^3$$
$$=2\sqrt{2}+6\sqrt{3}+9\sqrt{2}+3\sqrt{3}$$
$$=11\sqrt{2}+9\sqrt{3}$$
$$a=11, b=9$$

Test Yourself 3

Question 1

$2 \times 3! = 12$

C

Question 2

D

Question 3

There are 5 different colours.

So by the pigeonhole principle 6 must be chosen to ensure 2 are the same colour.

$\therefore$ B

Question 4

$$\begin{aligned}&(x-3)^5\\&=\binom{5}{0}(x)^5-\binom{5}{1}(\sqrt{3})^4(3)+\binom{5}{2}(x)^3(3)^2-\binom{5}{3}(x)^2(3)^3+\binom{5}{4}(x)(3)^4-\binom{5}{5}(3)^5\\&=x^5-15x^4+90x^3-270x^2+405x-243\end{aligned}$$

Question 5

a $11! = 39\,916\,800$

b 2 people not together = $11! - 2!10! = 32\,659\,200$

c P(2 people sit together)

$$\begin{aligned}&=\frac{2!10!}{11!}\\&=\frac{2}{11}\end{aligned}$$

Question 6

a

$$
\begin{aligned}
{}^{11}C_3 &= \frac{11!}{(11-3)!3!} \\
&= \frac{11!}{8!(11-3)!} \\
&= {}^{11}C_8
\end{aligned}
$$

b

$$
\begin{aligned}
{}^{10}C \ &= \frac{10!}{(10-1)!1!} \\
&= \frac{10!}{9!1!} \\
&= \frac{10!}{(10-1)!9!} \\
&= {}^{10}C_9
\end{aligned}
$$

c

$$
\begin{aligned}
{}^{9}C_7 &= \frac{9!}{(9-7)!7!} \\
&= \frac{9\times 8!}{2!7!} \\
&= \frac{7\times 8!}{2!7!} + \frac{2\times 8!}{2!7!} \\
&= \frac{8!}{2!6!} + \frac{8!}{1!7!} \\
&= \frac{8!}{(8-6)!6!} + \frac{8!}{(8-7)!7!} \\
&= {}^{8}C_6 + {}^{8}C_7
\end{aligned}
$$

d

$$\begin{aligned}\binom{11}{6} &= \frac{11!}{(11-6)!6!} \\ &= \frac{11\times 10!}{5!6!} \\ &= \frac{5\times 10!}{5!6!} + \frac{6\times 10!}{5!6!} \\ &= \frac{10!}{4!6!} + \frac{10!}{5!5!} \\ &= \frac{10!}{(10-6)!6!} + \frac{10!}{(10-5)!5!} \\ &= \binom{10}{5} + \binom{10}{6}\end{aligned}$$

Question 7

a $^{22}C_5 = 26\ 334$

b $^{12}C_2\,^{10}C_3 = 7920$

Question 8

$$\begin{aligned}&(2x+3y)^4 \\ &= \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(3y) + \binom{4}{2}(2x)^2(3y)^2 + \binom{4}{3}(2x)(3y)^3 + \binom{4}{4}(3y)^4 \\ &= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4\end{aligned}$$

Question 9

The number of hairs on head is up to 150 000.

So up to 150 000 people could each have a different number of hairs on their head.

Population is 256 840 which is greater than 150 000, so at least 2 must have the same number of hairs on their head.

Question 10

$$\frac{9!}{3!}=60\ 480$$

Question 11

a 7! = 5040

b 6! = 720

Question 12

a $\left(\sqrt{5}-2\right)^6$

$$=\binom{6}{0}\left(\sqrt{5}\right)^6-\binom{6}{1}\left(\sqrt{5}\right)^5(2)+\binom{6}{2}\left(\sqrt{5}\right)^4(2)^2-\binom{6}{3}\left(\sqrt{5}\right)^3(2)^3$$
$$+\binom{6}{4}\left(\sqrt{5}\right)^2(2)^4-\binom{6}{5}\left(\sqrt{5}\right)(2)^5+\binom{6}{6}(2)^6$$
$$=125-300\sqrt{5}+1500-800\sqrt{5}+1200-192\sqrt{5}+64$$
$$=2889-1292\sqrt{5}$$

b $\left(x-\frac{3}{x^2}\right)^4$

$$=\binom{4}{0}(x)^4-\binom{4}{1}(x)^3\left(\frac{3}{x^2}\right)+\binom{4}{2}(x)^2\left(\frac{3}{x^2}\right)^2-\binom{4}{3}(x)\left(\frac{3}{x^2}\right)^3+\binom{4}{4}\left(\frac{3}{x^2}\right)^4$$
$$=x^4-12x+\frac{54}{x^2}-\frac{108}{x^5}+\frac{81}{x^8}$$

c $\left(2-3\sqrt{2}\right)^7$

$$=\binom{7}{0}2^7-\binom{7}{1}2^6\left(3\sqrt{2}\right)+\binom{7}{2}2^5\left(3\sqrt{2}\right)^2-\binom{7}{3}2^4\left(3\sqrt{2}\right)^3$$
$$+\binom{7}{4}2^3\left(3\sqrt{2}\right)^4-\binom{7}{5}2^2\left(3\sqrt{2}\right)^5+\binom{7}{6}2\left(3\sqrt{2}\right)^6-\binom{7}{7}\left(3\sqrt{2}\right)^7$$
$$=128-1344\sqrt{2}+12\ 096-30\ 240\sqrt{2}+90\ 720-81\ 648\sqrt{2}+81\ 648-17\ 496\sqrt{2}$$
$$=184\ 592-130\ 728\sqrt{2}$$

Question 13

a $\quad {}^{10}P_3 = 720$

b $\quad {}^{10}C_3 = 120$

Question 14

a $\quad {}^{26}C_5 = 65\ 780$

b $\quad {}^{16}C_3\,{}^{10}C_2 = 25\ 200$

c $\quad {}^{10}C_5 = 252$

Question 15

$$\begin{aligned}
&\left(\sqrt{5}-\sqrt{3}\right)^5\\
&=\binom{5}{0}\left(\sqrt{5}\right)^5-\binom{5}{1}\left(\sqrt{5}\right)^4\left(\sqrt{3}\right)+\binom{5}{2}\left(\sqrt{5}\right)^3\left(\sqrt{3}\right)^2-\binom{5}{3}\left(\sqrt{5}\right)^2\left(\sqrt{3}\right)^3+\binom{5}{4}\left(\sqrt{5}\right)\left(\sqrt{3}\right)^4-\binom{5}{5}\left(\sqrt{3}\right)^5\\
&=25\sqrt{5}-125\sqrt{3}+150\sqrt{5}-150\sqrt{3}+45\sqrt{5}-9\sqrt{3}\\
&=220\sqrt{5}-284\sqrt{3}\\
&a=220,\ b=-284
\end{aligned}$$

Question 16

${}^{6}P_3 = 120$

Question 17

a PERMUTATION, TT

$$\frac{11!}{2!} = 19\ 958\ 400$$

b COMBINATION, OO II NN

$$\frac{11!}{2!2!2!} = 4\ 989\ 600$$

c FACTORIAL, AA

$$\frac{9!}{2!} = 181\ 440$$

d PROBABILITY, BB II

$$\frac{11!}{2!2!} = 9\ 979\ 200$$

e SELECTION, EE

$$\frac{9!}{2!} = 181\ 440$$

Question 18

There are 750 bricks, so n = 750.

They are sorted into 7 lots, so k = 7

At least one lot will contain at least $\frac{n}{k} = \frac{750}{7} = 107\frac{1}{7}$

Round up to 108.

At least one lot will contain at least 108 bricks.

Question 19

a $\binom{6}{4} = 15$

b ${}^{9}P_{7} = 181\ 440$

Question 20

$26^4 \times 9^2 = 37\ 015\ 056$

Question 21

a $0! = 1$

b

$$\begin{aligned}\binom{n}{0} &= \frac{n!}{(n-0)!0!} \\ &= \frac{n!}{n!(n-n)!} \\ &= \binom{n}{n}\end{aligned}$$

Question 22

Divide the square into 16 smaller squares each with area of 1 cm^2.

For 3 points to form a triangle with area less than 1 cm^2, they must lie within the same smaller square (since the triangle inside will have a smaller area than the smaller square's area).

When placing 16 points inside the square, it is possible that each could lie in a different smaller square. Placing the next 16 points could also result in each being in a different smaller square. This means that now the smaller squares must have at least 2 points inside.

The next (33rd) point must go into one of the 16 smaller squares, so even if there were only 2 points in each smaller square previously, now there must be 3 points in at least one of the smaller squares.

So it is possible to form a triangle from these 3 points (out of the 33 points) with an area of less than 1 cm^2.

Challenge exercise 3

Question 1

Pairings to make 9 are (1, 8), (2, 7), (3, 6), (4, 5) by the pigeonhole principle if 5 cards are chosen there must be at least one of the above combinations must be selected.

Question 2

MATHEMATICS

Number of possible randomly-generated 4-letter 'words' $= {}^{11}P_4 = 7920$ (including repeated words due to M, A, T appearing more than once in the word MATHEMATICS)

Number of ways to make the word 'CAME' = 1 × 2 × 2 × 1 = 4

P('CAME') = $\dfrac{4}{7920} = \dfrac{1}{1980}$

Question 3

$$\frac{{}^nC_k}{{}^nC_{k-1}}$$

$$=\frac{\dfrac{n!}{(n-k)!k!}}{\dfrac{n!}{(n-(k-1))!(k-1)!}}$$

$$=\frac{\dfrac{1}{(n-k)!k!}}{\dfrac{1}{(n-(k-1))!(k-1)!}}$$

$$=\frac{(n-(k-1))!(k-1)!}{(n-k)!k!}$$

$$=\frac{(n-(k-1))(n-k)!(k-1)!}{(n-k)!k(k-1)!}$$

$$=\frac{(n-(k-1))}{k}$$

$$=\frac{n-k+1}{k}$$

Question 4

1, 2, 3, 3, 7

a $\frac{5!}{2} = 60$

b 1st digit must be 3, 3 or 7

$3 \times 4 \times 3 \times 2 \times 1 = 72$

Question 5

a Arranging 6 different charms plus a clasp = 7 things.

In a circle = (7 – 1)!, but needs to be divided by 2 to allow for symmetry (clockwise, anticlockwise)

$$\frac{(7-1)!}{2} = \frac{6!}{2} = \frac{720}{2} = 360$$

b In a circle = (6 – 1)!, but needs to be divided by 2 to allow for symmetry (clockwise, anticlockwise)

$$\frac{(6-1)!}{2} = \frac{5!}{2} = \frac{120}{2} = 60$$

Question 6

a ${}^{20}C_5 \, {}^{10}C_3 = 1\,860\,480$

b $P(\text{Patrick}) = \frac{{}^{19}C_4 \, {}^{10}C_3}{{}^{20}C_5 \, {}^{10}C_3} = \frac{1}{4}$

c $P(\text{Patrick and Alexis}) = \frac{{}^{19}C_4 \, {}^{9}C_2}{{}^{20}C_5 \, {}^{10}C_3} = \frac{3}{40}$

d $P(\text{not Patrick and not Alexis}) = \frac{{}^{19}C_5 \, {}^{9}C_3}{{}^{20}C_5 \, {}^{10}C_3} = \frac{21}{40}$

Question 7

a $(n-1)!$

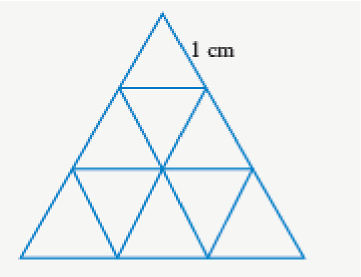
b $(n-1-k+1)!\ k! = (n-k)!\ k!$

Question 8

$$(1-0.01)^3$$
$$=\binom{3}{0}1^3-\binom{3}{1}1^2(0.01)+\binom{3}{2}1(0.01)^2-\binom{3}{3}(0.01)^3$$
$$=1-0.03+0.0003-0.000001$$
$$=0.9703$$

Question 9

Divide the equilateral triangle into 9 smaller equilateral triangles each with sides of 1 cm.



For 2 points to have a distance apart of less than 1 cm, they must lie within the same smaller triangle (since the distance between 2 points inside the triangle must be less than the length of the side of the triangle).

When placing 9 points inside the original triangle, it is possible that each could lie in a different smaller triangle.

The next (10th) point must go into one of the 9 smaller triangles, so now there must be 2 points in at least one of the smaller triangles.

So it is possible to have those 2 points less than 1 cm apart.

If the 2 points happen to lie on the outside lines of the same smaller triangle, they could be

1 cm apart.

MATHS IN FOCUS 11
MATHEMATICS EXTENSION 1

FULLY WORKED SOLUTIONS

Chapter 4: Functions

Exercise 4.01 Functions

Question 1

a The set of ordered pairs is (Wade, black), (Scott, blond), (Geoff, grey), (Deng, black), (Mila, brown), (Stevie, blond)

It is a many-to-one relation because two different elements of the domain correspond with one element of the range. (Scott, blond) and (Stevie, blond); (Wade, black) and (Deng, black)

b The set of ordered pairs is (1, 1), (1, 4), (2, 3), (3, 1), (4, 4)

It is a many-to-many relation because more than one element of the domain corresponds with more than one element of the range.

c The set of ordered pairs is (1, A), (2, D), (3, A), (4, B), (5, C).

It is a many-to-one because more than one element of the domain corresponds with the same element in the range. (1, A), (3, A)

d The set of ordered pairs is (3, 5), (5, –2), (5, 2), (8, –7), (9, 3), (5, 6), (8, 0)

It is a one-to-many because there is an element in the domain that corresponds with more than one element of the range. (5, –2), (5, 2), (5, 6); (8, –7), (8, 0)

e The set of ordered pairs is (1, 9), (2, 15), (3, 27), (4, 33), (5, 45)

It is one-to-one because each element in the domain corresponds with a unique element in the range.

Question 2

a A vertical line intersects the graph at only one point, so it is a function.

A horizontal line crosses the graph at only one point, so the function is one-to-one.

b A vertical line intersects the graph at more than one point, so it is not a function.

c A vertical line intersects the graph at more than one point, so it is not a function.

d A vertical line intersects the graph at only one point, so it is a function.

A horizontal line crosses the graph at more than one point, so the function is not one-to-one.

e A vertical line intersects the graph at only one point, so it is a function.

A horizontal line crosses the graph at more than one point, so the function is not one-to-one.

f A vertical line intersects the graph at only one point, so it is a function.

A horizontal line crosses the graph at more than one point, so the function is not one-to-one.

g A vertical line intersects the graph at more than one point, so it is not a function.

h A vertical line intersects the graph at only one point, so it is a function.

A horizontal line crosses the graph at more than one point, so the function is not one-to-one.

i Each element in the domain corresponds with a unique element in the range, so it is a function.

Two elements in the domain correspond with the same element in the range. (1, 3), (3, 3). So it is not one-to-one.

j One element in the domain corresponds with two elements in the range. (2, –1), (2, 7). So, not a function.

k Function. Not one-to-one.

Two elements in the domain correspond with the same element in the range. (1, 4), (5, 4). So it is not one-to-one.

l Not a function. One element in the domain corresponds with two elements in the range. (2, 1), (2, 4)

m Each element in the domain corresponds with a unique element in the range, so it is a function.

Each element in the range corresponds with a unique element in the domain, so it is one-to-one.

n Not a function.

Four elements in the domain correspond with one element in the range.
(Paul, football), (Hamish, football), (Jacob, football), (Pierre, football)

Two elements in the domain correspond with one element in the range. (Ben, tennis), (Pierre, tennis)

Two elements in the domain correspond with one element in the range.
(Lee, badminton), (Lien, badminton)

o Each element in the domain corresponds with a unique element in the range, so it is a function.

Two elements in the domain correspond with the same element in the range. (A, 3), (D, 3); (C, 7), (F, 7); (B, 4), (G, 4). So it is not one-to-one.

Question 3

a {–3, –1, 0, 1, 6}

The independent variable is the first element of each ordered pair.

b {–2, 4, 5, 8}

The dependent variable is the second element of each ordered pair.

c Many-to-one

Two elements in the domain correspond with the same element in the range. (–3, 4), (1, 4)

d Yes. A many-to-one relation is always a function.

Exercise 4.02 Function notation

Question 1

$f(1) = 1 + 3 = 4$

$f(-3) = -3 + 3 = 0$

Question 2

$h(0) = 0^2 - 2 = -2$

$h(2) = 2^2 - 2 = 2$

$h(-4) = (-4)^2 - 2 = 14$

Question 3

$f(5) = -5^2 = -25$

$f(-1) = -(-1)^2 = -1$

$f(3) = -3^2 = -9$

$f(-2) = -(-2)^2 = -4$

Question 4

$f(0) = 0^4 - 0^2 + 1 = 0$

$f(-2) = (-2)^4 - (-2)^2 + 1 = 16 - 4 + 1 = 13$

$f(0) + f(-2) = 1 + 13 = 14$

Question 5

$f(-3) = 2(-3)^3 - 5(-3) + 4 = -54 + 15 + 4 = -35$

Question 6

$f(x) = 13$ means $2x - 5 = 13$

$\Rightarrow 2x = 18$

$\Rightarrow x = 9$

Question 7

$f(x) = 28$ means $x^2 + 3 = 28$

$\Rightarrow x^2 = 25$

$\Rightarrow x = \pm 5$

Question 8

Require the value(s) of x so that $3^x = \frac{1}{27}$

$3^x = \frac{1}{3^3} = 3^{-3}$, so $x = -3$

Question 9

Require the value(s) of z so that $|2z + 3| = 5$.

Square both sides to make both sides positive.

$(2z + 3)^2 = 25$

$4z^2 + 12z + 9 = 25$

$4z^2 + 12z - 16 = 0$

$z^2 + 3z - 4 = 0$

$(z - 1)(z + 4) = 0$

$z = 1, -4$

Question 10

$f(p) = 2(p) - 9 = 2p - 9$

$f(x + h) = 2(x + h) - 9 = 2x + 2h - 9$

Question 11

$g(x - 1) = (x - 1)^2 + 2(x - 1) + 3$

$= (x^2 - 2x + 1) + (2x - 2) + 3$

$= x^2 + 2$

Question 12

$f(k) = k^2 - 1 = (k - 1)(k + 1)$

Question 13

a $f(t) = 0 \Rightarrow t^2 - 2t + 1 = 0$

$(t - 1)^2 = 0$

$t = 1$ second

b $f(t) = 9 \Rightarrow t^2 - 2t + 1 = 9$

$t^2 - 2t - 8 = 0$

$(t - 4)(t + 2) = 0$

$t = 4, -2$

Question 14

$f(b) = b^4 + b^2 - 5$

$f(-b) = (-b)^4 + (-b)^2 - 5 = b^4 + b^2 - 5$

$f(b) - f(-b) = (b^4 + b^2 - 5) - (b^4 + b^2 - 5) = 0$

Question 15

Since $x = 5$ is greater than 1, use $f(x) = x^3$, so $f(5) = 5^3 = 125$

Since $x = 1$ is less than or equal to 1, use $f(x) = x$, so $f(1) = 1$

Since $x = -1$ is less than or equal to 1, use $f(x) = x$, so $f(-1) = -1$

Question 16

Since $x = 2$ is greater than 1, use $f(x) = 2x - 4$, so $f(2) = 2(2) - 4 = 0$

$x = -2$ is less than -1, so use $f(x) = x^2$, to get $f(-2) = (-2)^2 = 4$

$x = -1$ is in $-1 \leq x \leq 1$, so use $f(x) = x + 3$, to get $f(-1) = -1 + 3 = 2$

Hence, $f(2) - f(-2) + f(-1) = 0 - 4 + 2 = -2$

Question 17

$x = 3$ is greater than 0, so use $g(x) = x + 1$, to get $g(3) = 3 + 1 = 4$

$x = 0$ satisfies $x \geq 0$, so use $g(x) = x + 1$, to get $g(0) = 0 + 1 = 1$

$x = -2$ is less than 0, so use $g(x) = -2x + 1$, to get $g(-2) = -2(-2) + 1 = 5$

Hence, $g(3) + g(0) + g(-2) = 4 + 1 + 5 = 10$

Question 18

$x = 3$ is greater than 2, so use $f(x) = x$ to get $f(3) = 3$

$x = 2$ satisfies $-2 \le x \le 2$, so use $f(x) = x^2$, to get $f(2) = 2^2 = 4$

$x = -3$ is less than -2, so use $f(x) = 4$, to get $f(-3) = 4$

Hence, $f(3) - f(2) + 2f(-3) = 3 - 4 + 2(4) = 7$

Question 19

$x = -1$ is less than 2, so use $f(x) = 2x^2 + 3x - 1$, to get $f(-1) = 2(-1)^2 + 3(-1) - 1 = -2$

$x = 3$ is greater than 2, so use $f(x) = x^3 - 1$, to get $f(3) = 3^3 - 1 = 26$

Hence, $f(-1) - f(3) = -2 - 26 = -28$

Question 20

$$\begin{aligned} f(x+h) - f(x) &= [(x+h)^2 - 5(x+h) + 4] - [x^2 - 5x + 4] \\ &= x^2 + 2xh + h^2 - 5x - 5h + 4 - x^2 + 5x - 4 \\ &= 2xh + h^2 - 5h \end{aligned}$$

Question 21

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{2(x+h)^2 + (x+h) - (2x^2 + x)}{h} \\ &= \frac{2x^2 + 4xh + 2h^2 + x + h - 2x^2 - x}{h} \\ &= \frac{4xh + 2h^2 + h}{h} \\ &= 4x + 2h + 1 \end{aligned}$$

Question 22

$f(x) - f(c) = (5x - 4) - (5c - 4)$

$= 5x - 5c$

$= 5(x - c)$

Question 23

For all values of k, $k^2 \geq 0$.

Hence, $f(k^2) = 3k^2 + 5$

Question 24

a $x = 0$ satisfies $x \leq 0$, so use $f(x) = x^2 - x + 2$, to get $f(0) = 0^2 - 0 + 2 = 2$

b Both $x = 1$ and $x = 2$ are in the interval $0 < x < 3$, so $f(2) - f(1) = 5 - (5) = 0$

c $-n^2 \leq 0$ for all values of n.

Hence,

$f(-n^2) = (-n^2)^2 - (-n^2) + 2$

$= n^4 + n^2 + 2$

Question 25

a $f(2) = \dfrac{2^2 - 2(2) - 3}{2 - 3} = \dfrac{-3}{-1} = 3$

b Division cannot be performed if the value of the denominator is zero.
Hence, we require that $x - 3 \neq 0$.

So we require $x \neq 3$.

c Approach $x = 3$ using values greater than $x = 3$

$x = 3.5$, $y = f(3.5) = 4.5$

$x = 3.2$, $y = f(3.2) = 4.2$

$x = 3.1$, $y = f(3.1) = 4.1$

Approach $x = 3$ using values smaller than $x = 3$

$x = 2.5$, $y = f(2.5) = 3.5$

$x = 2.8$, $y = f(2.8) = 3.8$

$x = 2.9$, $y = f(2.9) = 3.9$

As we approach $x = 3$, the value of y approaches 4.

Exercise 4.03 Properties of functions

Question 1

a x-intercept, let $y = 0$, $0 = 3x - 2 \Rightarrow x = \frac{2}{3}$

y-intercept, let $x = 0$, $y = 3 \times 0 - 2 = -2$

b x-intercept, let $y = 0$, $2x - 5 \times 0 + 20 = 0 \Rightarrow x = -10$

y-intercept, let $x = 0$, $2 \times 0 - 5y + 20 = 0 \Rightarrow y = 4$

c x-intercept, let $f(x) = 0$, $x + 3 \times 0 - 12 = 0 \Rightarrow x = 12$

y-intercept, let $x = 0$, $0 + 3 \times y - 12 = 0 \Rightarrow y = 4$

d x-intercept, let $f(x) = 0$

$0 = x^2 + 3x$

$x(x + 3) = 0$

$x = 0$, $x = -3$

y-intercept, let $x = 0$, $f(0) = 0^2 + 3 \times 0 = 0$

e x-intercept, let $f(x) = 0$

$0 = x^2 - 4$

$x^2 = 4$

$x = -2$, $x = 2$

y-intercept, let $x = 0$, $f(0) = 0^2 - 4 = -4$

f x-intercept, let $p(x) = 0$

$0 = x^2 + 5x + 6$

$(x + 2)(x + 3) = 0$

$x = -2$, $x = -3$

y-intercept, let $x = 0$, $p(0) = 0^2 + 5 \times 0 + 6 = 6$

g x-intercept, let $y = 0$

$0 = x^2 - 8x + 15$

$(x - 3)(x - 5) = 0$

$x = 3, x = 5$

y-intercept, let $x = 0$, $y = 0^2 - 8 \times 0 + 15 = 15$

h x-intercept, let $p(x) = 0$

$0 = x^3 + 5$

$\mathrm{x} = -5^{\frac{1}{3}} = -\sqrt[3]{5}$

y-intercept, let $x = 0$, $p(0) = 0^3 + 5 = 5$

i x-intercept, let $y = 0$

$0 = \dfrac{x+3}{x}$

$0 = x + 3$

$x = -3$

$0 = x + 3$

$x = -3$

y-intercept, let $x = 0$, $y = \dfrac{0+3}{0} = \dfrac{3}{0}$. This is undefined.

j x-intercept, let $g(x) = 0$

$0 = 9 - x^2$

$0 = (x - 3)(x + 3)$

$x = -3, x = 3$

y-intercept let $x = 0$, $g(0) = 9 - 0^2 = 9$

Question 2

a $f(x) = 0 \Rightarrow 3x - 6 = 0, x = 2$

b x-intercept, let $f(x) = 0 \Rightarrow x = 2$

y-intercept let $x = 0$, $f(0) = 3 \times 0 - 6 = -6$

Question 3

$f(-x) = (-x)^2 - 2$

$= x^2 - 2$

$= f(x)$

$f(x)$ is an even function because it satisfies the condition $f(-x) = f(x)$.

Question 4

a $f(x) = (x^2)^3 + 1 = x^6 + 1$

b $[f(x)]^2 = (x^3 + 1)^2 = x^6 + 2x^3 + 1$

c $f(-x) = (-x)^3 + 1 = -x^3 + 1$

d $f(-x) \neq f(x)$, so $f(x)$ is not even.
$f(-x) \neq -f(x)$, so $f(x)$ is not odd.

e $f(x) = 0 \Rightarrow x^3 + 1 = 0$
$x^3 = -1 \Rightarrow x = -1$

f x-intercept, let $f(x) = 0 \Rightarrow x = -1$, using the answer from **e**.
y-intercept let $x = 0$, $f(0) = 0^3 + 1 = 1$

Question 5

$g(-x) = (-x)^8 + 3(-x)^4 - 2(-x)^2 = x^8 + 3x^4 - 2x^2 = g(x)$
So even

Question 6

$f(-x) = -x = -1 \times x = -1 \times f(x) = -f(x)$
So odd

Question 7

$f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$

So even

Question 8

$f(-x) = 4(-x) - (-x)^3 = -4x + x^3 = -(4x - x^3) = -f(x)$

So odd

Question 9

a $f(-x) = (-x)^4 + (-x)^2 = x^4 + x^2 = f(x)$

So even

b $f(x)$ is even, so $f(-x) = f(x)$

Hence, $f(x) - f(-x) = 0$

Question 10

a Odd

$$f(-x)=\frac{(-x)^3}{(-x)^4-(-x)^2}=\frac{-x^3}{x^4-x^2}=-\frac{x^3}{x^4-x^2}=-f(x)$$

b Neither

$$f(-x)=\frac{1}{(-x)^3-1}=\frac{1}{-x^3-1}=-\frac{1}{x^3+1}$$

c Even

$$f(-x)=\frac{3}{(-x)^2-4}=\frac{3}{x^2-4}=f(x)$$

d Neither

$$f(-x)=\frac{-x-3}{-x+3}=\frac{x+3}{x-3}$$

e Neither

$$f(-x)=\frac{(-x)^3}{(-x)^5-(-x)^2}=\frac{-x^3}{-x^5-x^2}=\frac{x^3}{x^5+x^2}$$

Question 11

a Require $f(-x) = k(-x)^n = f(x)$

$(-x)^n = x^2$ for even powers of x. That is, for $n = 2, 4, 6, 8, \ldots$

b Require $f(-x) = k(-x)^n = -f(x)$

$(-x)^n = -x^n$ for odd powers of x. That is, for $n = 1, 3, 5, 7, \ldots$

Question 12

a No

For $f(x)$ to be even, all powers of x must be even.

The power of x^n is even for $n = 2, 4, 6, 8, \ldots$, but x is always odd powered.

b Yes, when n is odd $(1, 3, 5, \ldots)$

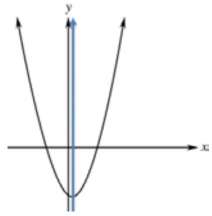
For $f(x)$ to be odd, all powers of x must be odd.1

The power of x^n is odd for $n = 1, 3, 5, \ldots$, and x is always odd powered.

Question 13

a **i** $(0, \infty)$

For all x values to the right of the origin, y increases as x increases.



ii $(-\infty, 0)$

For all x values to the left of the origin, y decreases as x increases.

iii Even

The function is symmetrical about the y-axis.

b The turning point is at $x = 2$.

i $(-\infty, 2)$

For all x values less than $x = 2$, y increases as x increases.

ii $(2, \infty)$

For all x values greater than $x = 2$, y decreases as x increases.

iii Neither

The function is not symmetrical about the y-axis, so it is not even.

Rotating the graph 180° does not produce the original graph, so it is not odd.

c **i** $(-2, 2)$

For all x values from –2 to 2, y increases as x increases.

ii $(-\infty, -2) \cup (2, \infty)$

For all x values less than –2 and for all x values greater than 2, y decreases as x increases.

iii Neither

The function is not symmetrical about the y-axis, so it is not even.

Rotating the graph 180° does not produce the original graph, so it is not odd.

d **i** $(-\infty, 0) \cup (0, \infty)$

For all x values less than 0 and for all x values greater than 0, y increases as x increases.

ii None

There is no interval where y decreases as x increases.

iii Odd

Rotating the graph 180° produces the original graph.

e **i** None

There is no interval where y increases as x increases.

ii $(-\infty, \infty)$

For all x values, y decreases as x increases.

iii Neither

The function is not symmetrical about the y-axis, so it is not even.

Rotating the graph 180° does not produce the original graph, so it is not odd.

Question 14

a The graph of the function is a parabola and is defined for all values of x, so the domain is $(-\infty, \infty)$.

The minimum value of the function is $y = 1$ when $x = 0$. For all other values of x we have $y > 1$. So the range is $[1, \infty)$.

b The graph of the function is a cubic and is defined for all values of x, so the domain is $(-\infty, \infty)$.
The function is one-to-one and all values of x can be used, so the range is $(-\infty, \infty)$.

c In the real number system, $\sqrt{x}$ requires that $x \geq 0$, so the domain is $[0, \infty)$.

The minimum value of the function is $y = 0$ when $x = 0$.
For all other values $x > 0$, $y > 0$. So the range is $[0, \infty)$

d The function is defined when $x + 5 \geq 0$. That is, when $x \geq -5$, so the domain is $[-5, \infty)$.

The minimum value of the function is $y = 0$ when $x = 0$. For all other values of $x > -5$, $y > 0$. So the range is $[0, \infty)$.

e The function is defined when $2x - 6 \geq 0$. That is, when $x \geq 3$, so the domain is $[3, \infty)$.

The minimum value of the function is $y = 0$ when $x = 3$ For all other values of $x > 3$, $y < 0$. So the range is $(-\infty, 0]$.

Question 15

a $f(3) = (3 - 2)^2 = 1$

b $f(-5) = (-5 - 2)^2 = 49$

c $(x - 2)^2 = 0 \Rightarrow x = 2$

d At the x-intercept, $f(x) = 0$. This occurs at $x = 2$

At the y-intercept, $x = 0$. $f(0) = (0 - 2)^2 = 4$

e The graph of $f(x)$ is a parabola and is defined for all values of x.
So the domain is $(-\infty, \infty)$.

The minimum value of $f(x)$ is 0 when $x = 2$. For all other values of x, $f(x) \geq 0$.
So the range is $[0, \infty)$.

f $f(-x) = (-x - 2)^2$ or $f(-x) = (-x - 2)^2 = [-(x + 2)]^2 = (x + 2)^2$

$f(-x) = (-x - 2)^2$

g The graph of the function is a parabola with axis of symmetry about the line $x = 2$.

It not symmetrical about the y-axis, so it is not even.

Rotating the graph 180° does not produce the original graph, so it is not odd.

Exercise 4.04 Linear functions

Question 1

a There are 12 months in one year, so in x years, the total number of months, N, is $12 \times x = 12x$ months.

b Each lot consists of 2 litres, so for n lots, the total amount, A, of juice is $2 \times n = 2n$

c Each litre of petrol costs $1.50, so for x litres, the total cost. c, is $1.50 \times x = 1.50x$

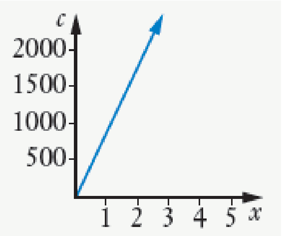
d Each team consists of 4 people. For x teams, the total number of people, y, is $4 \times x = 4x$

e Each can of peachesholds 400 g. For x cans, the total weight, w, is $400 \times x = 400x$

Question 2

Each refrigerator costs $850. For x refrigerators, the total cost c is $c = 850x$

Refrigerators ***x***	0	1	2	3	4	5
Cost $c	0	850	1700	2550	3400	4250



Question 3

a Let $y = 0$: $0 = x - 2 \Rightarrow x = 2$. The x-intercept is 2.

Let $x = 0$: $y = 0 - 2 \Rightarrow y = -2$. The y-intercept is -2.

b Let $y = 0$: $0 = 3x + 9 \Rightarrow x = -3$. The x-intercept is -3.

Let $x = 0$: $y = 3 \times 0 + 9 \Rightarrow y = 9$. The y-intercept is 9.

c Let $y = 0$: $0 = 4 - 2x \Rightarrow x = 2$. The x-intercept is 2.

Let $x = 0$: $y = 4 - 2 \times 0 = 4$. The y-intercept is 4.

d Let $y = 0$: $0 = 2x + 3 \Rightarrow x = -\frac{3}{2}$. The x-intercept is $-\frac{3}{2}$.

Let $x = 0$: $y = 2 \times 0 + 3 \Rightarrow y = 3$. The y-intercept is 3.

e Let $y = 0$: $0 = 5x - 4 \Rightarrow x = \frac{4}{5}$. The x-intercept is $\frac{4}{5}$.

Let $x = 0$: $y = 5 \times 0 - 4 \Rightarrow y = -4$. The y-intercept is -4.

f Let $y = 0$: $0 = 10x + 5 \Rightarrow x = -\frac{1}{2}$. The x-intercept is $-\frac{1}{2}$.

Let $x = 0$: $y = 10 \times 0 + 5 \Rightarrow y = 5$. The y-intercept is 5.

g Let $y = 0$: $x + 0 - 2 = 0 \Rightarrow x = 2$. The x-intercept is 2.

Let $x = 0$: $0 + y - 2 = 0 \Rightarrow y = 2$. The y-intercept is 2.

h Let $y = 0$: $2x - 0 + 4 = 0 \Rightarrow x = -2$. The x-intercept is -2.

Let $x = 0$: $2 \times 0 - y + 4 = 0 \Rightarrow y = 4$. The y-intercept is 4.

i Let $y = 0$: $x - 0 + 3 = 0 \Rightarrow x = -3$. The x-intercept is -3.

Let $x = 0$: $0 - y + 3 = 0 \Rightarrow y = 3$. The y-intercept is 3.

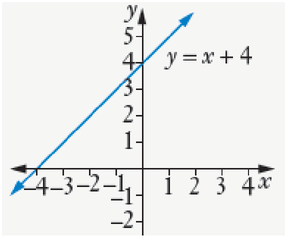
j Let $y = 0$: $3x - 6 \times 0 - 2 = 0 \Rightarrow x = \frac{2}{3}$. The x-intercept is $\frac{2}{3}$.

Let $x = 0$: $3 \times 0 - 6y - 2 = 0 \Rightarrow y = -\frac{1}{3}$. The y-intercept is $-\frac{1}{3}$.

Question 4

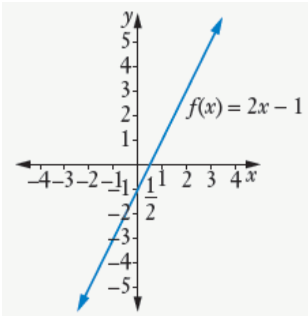
a Let $y = 0$: $0 = x + 4 \Rightarrow x = -4$. The x-intercept is -4

Let $x = 0$: $y = 0 + 4 \Rightarrow y = 4$. The y-intercept is 4



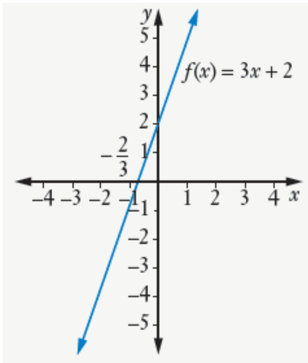
b Let $f(x) = 0$: $0 = 2x - 1 \Rightarrow x = \frac{1}{2}$. The x-intercept is $\frac{1}{2}$.

Let $x = 0$: $f(x) = 2 \times 0 - 1 \Rightarrow f(x) = -1$. The y-intercept is -1



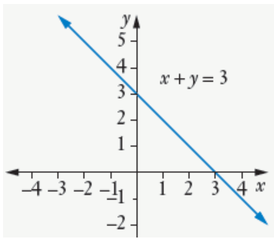
c Let $f(x) = 0$: $0 = 3x + 2 \Rightarrow x = -\frac{2}{3}$. The x-intercept is $-\frac{2}{3}$.

Let $x = 0$: $f(x) = 3 \times 0 + 2 \Rightarrow f(x) = 2$. The y-intercept is 2.



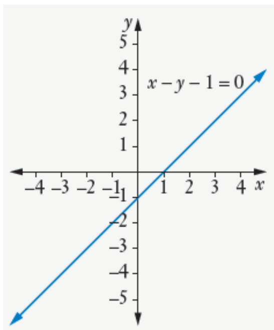
d Let $y = 0$: $x + 0 = 3 \Rightarrow x = 3$. The x-intercept is 3.

Let $x = 0$: $0 + y = 3 \Rightarrow y = 3$. The y-intercept is 3.



e Let $y = 0$: $x - 0 - 1 = 0 \Rightarrow x = 1$. The x-intercept is 1.

Let $x = 0$: $0 - y - 1 = 0 \Rightarrow y = -1$. The y-intercept is -1.



Question 5

a The function is neither vertical nor horizontal, so its domain and its range is $(-\infty, \infty)$.

b There is no restriction on the value of x, so the domain is $(-\infty, \infty)$.

For all x values, the y-value is 2. So the range is range [2]

c x is restricted to be -4, so the domain is $[-4]$.

There is no restriction on the value of y, so the range is $(-\infty, \infty)$.

d $x - 2 = 0 \Rightarrow x = 2$

x is restricted to be 2, so the domain is [2].

There is no restriction on the value of y, so the range is $(-\infty, \infty)$.

e $3 - y = 0 \Rightarrow y = 3$

There is no restriction on the value of x, so the domain is $(-\infty, \infty)$.

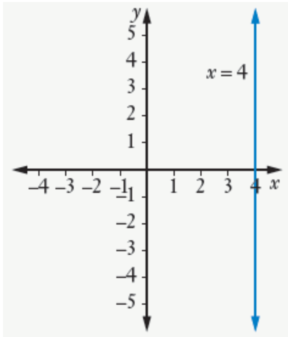
y is restricted to be 3, so the range is [3].

Question 6

a $x = 4$ represents a vertical line passing through the point 4 on the x axis.

x is restricted to be 4, so the domain is [4].

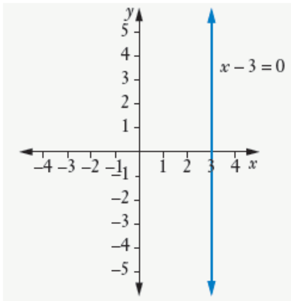
There is no restriction on the value of y, so the range is $(-\infty, \infty)$.



b $x - 3 = 0 \Rightarrow x = 3$. This represents a vertical line passing through the point 3 on the x axis.

x is restricted to be 3, so the domain is [3].

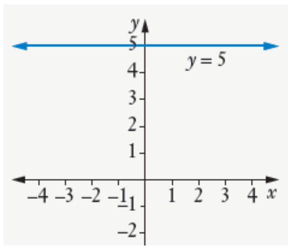
There is no restriction on the value of y, so the range is $(-\infty, \infty)$.



c $y = 5$ represents a horizontal line passing through the point 5 on the y axis.

There is no restriction on the value of x, so the domain is $(-\infty, \infty)$.

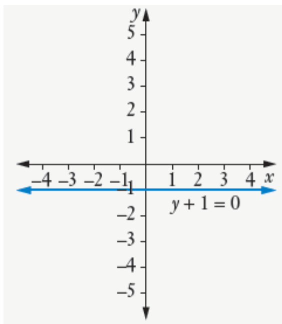
y is restricted to be 5, so the range is [5].



d $y + 1 = 0 \Rightarrow y = -1$ represents a horizontal line passing through the point –1 on the y axis.

There is no restriction on the value of x, so the domain is $(-\infty, \infty)$.

y is restricted to be –1, so the range is [–1].



Question 7

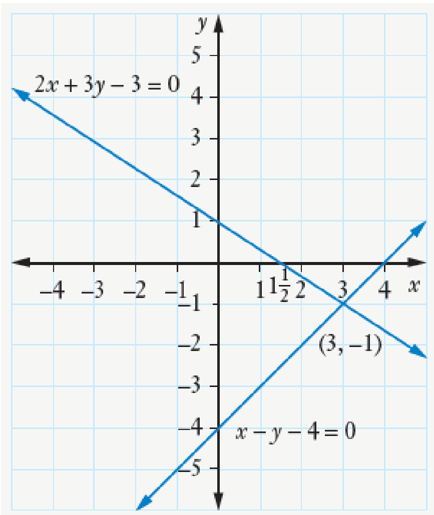
a Each box contains $\frac{144}{4} = 36$ cans. For x boxes, the number of cans, N, is

$N = 36 \times x = 36x$

b For 28 boxes, $x = 28$. So $N = 36x \times 28 = 1008$ cans

c For 612 cans, $N = 612$. So $N = 36x \Rightarrow 612 = 36x$. Hence, $x = \frac{612}{36} = 17$ boxes.

Question 8



Find the x-intercept and the y-intercept of $x - y - 4 = 0$

Let $y = 0$:

$x - 0 - 4 = 0 \Rightarrow x = 4$. The x-intercept is 4.

Let $x = 0$:

$0 - y - 4 = 0 \Rightarrow y = -4$. The y-intercept is –4.

Find the x-intercept and the y-intercept of $2x + 3y - 3 = 0$.

Let $y = 0$:

$2x + 3 \times 0 - 3 \Rightarrow x = \frac{3}{2}$. The x-intercept is $\frac{3}{2}$.

Let $x = 0$:

$2 \times 0 + 3y - 3 = 0 \Rightarrow y = 1$. The y-intercept is 1

The point of intersection is (3, –1).

Exercise 4.05 The gradient of a straight line

Question 1

Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

a $(x_1, y_1) = (3, 2)$, $(x_2, y_2) = (1, -2)$

$m = \dfrac{-2-2}{1-3} = 2$

b $(x_1, y_1) = (0, 2)$, $(x_2, y_2) = (3, 6)$

$m = \dfrac{6-2}{3-0} = \dfrac{4}{3} = 1\dfrac{1}{3}$

c $(x_1, y_1) = (-2, 3)$, $(x_2, y_2) = (4, -5)$

$m = \dfrac{-5-3}{4-(-2)} = -\dfrac{8}{6} = -\dfrac{4}{3} = -1\dfrac{1}{3}$

d $(x_1, y_1) = (2, -5)$, $(x_2, y_2) = (-3, 7)$

$m = \dfrac{7-(-5)}{-3-2} = -\dfrac{12}{5} = -2\dfrac{2}{5}$

e $(x_1, y_1) = (2, 3)$, $(x_2, y_2) = (-1, 1)$

$m = \dfrac{1-3}{-1-2} = \dfrac{2}{3}$

f $(x_1, y_1) = (-5, 1)$, $(x_2, y_2) = (3, 0)$

$m = \dfrac{0-1}{3-(-5)} = -\dfrac{1}{8}$

g $(x_1, y_1) = (-2, -3)$, $(x_2, y_2) = (-4, 6)$

$m = \dfrac{6-(-3)}{-4-(-2)} = -\dfrac{9}{2} = -4\dfrac{1}{2}$

h $(x_1, y_1) = (-1, 3)$, $(x_2, y_2) = (-7, 7)$

$m = \dfrac{7-3}{-7-(-1)} = -\dfrac{4}{6} = -\dfrac{2}{3}$

i $(x_1, y_1) = (1, -4)$, $(x_2, y_2) = (5, 5)$

$m = \dfrac{5-(-4)}{5-1} = \dfrac{9}{4} = 2\dfrac{1}{4}$

Question 2

Use $m = \tan \theta$ correct to 1 decimal place.

a $m = \tan 25° = 0.5$

b $m = \tan 82° = 7.1$

c $m = \tan 68° = 2.5$

d $m = \tan 100° = -5.7$

e $m = \tan 130° = -1.2$

f $m = \tan 164° = -0.3$

Question 3

In $y = mx + c$, m is the gradient and c is the y-intercept.

a **i** 3 **ii** 5

b **i** 2 **ii** 1

c **i** 6 **ii** –7

d Write $y = -x$ as $y = -1x + 0$

i –1 **ii** 0

e **i** –4 **ii** 3

f **i** 1 **ii** –2

g $f(x) = -2x + 6$

i –2 **ii** 6

h $y = -1x + 1$

i –1 **ii** 1

i $y = 9x + 0$

i 9 **ii** 0

Question 4

Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

a x-intercept $(x_1, y_1) = (3, 0)$, y-intercept $(x_2, y_2) = (0, -1)$

$$m = \frac{-1-0}{0-3} = \frac{1}{3}$$

b x-intercept $(x_1, y_1) = (5, 0)$, passing through $(x_2, y_2) = (2, 4)$

$$m = \frac{4-0}{2-5} = -\frac{4}{3} = -1\frac{1}{3}$$

c $(x_1, y_1) = (1, 1)$, y-intercept $(x_2, y_2) = (-2, 7)$

$$m = \frac{7-1}{-2-1} = -\frac{6}{3} = -2$$

d x-intercept $(x_1, y_1) = (-3, 0)$, passing through $(x_2, y_2) = (2, 3)$

$$m = \frac{3-0}{2-(-3)} = \frac{3}{5}$$

e Passing through the origin $(x_1, y_1) = (0, 0)$, passing through $(x_2, y_2) = (-3, -1)$

$$m = \frac{-1-0}{-3-0} = \frac{1}{3}$$

Question 5

a $\tan\theta = 2$, $\theta = \tan^{-1} 2 = 63.44° = 63°26'$

b $\tan\theta = 1.7$, $\theta = \tan^{-1} 1.7 = 59.53° = 59°32'$

c $\tan\theta = 6$, $\theta = \tan^{-1} 6 = 80.54° = 80°32'$

d $\tan\theta = -5$, $\theta = \tan^{-1}(-5) = -78.69°$

Angle with positive x-axis is $(180 - 78.69)° = 101.31° = 101°.19'$

e $\tan\theta = -0.85$, $\theta = \tan^{-1}(-0.85) = -40.36°$

Angle with positive x-axis is $(180 - 40.36)° = 139.64° = 139°.38'$

f $\tan\theta = -1.2$, $\theta = \tan^{-1}(-1.2) = -50.19°$

Angle with positive x-axis is $(180 - 50.19)° = 129.81° = 129°.48'$

Question 6

First, write each equation in the form $y = mx + c$; m is the gradient and c is the y-intercept.

a	$y = -2x + 3$	**i**	$m = -2$	**ii**	$c = 3$
b	$y = -5x - 6$	**i**	$m = -5$	**ii**	$c = -6$
c	$y = 6x - 1$	**i**	$m = 6$	**ii**	$c = -1$
d	$y = x + 4$	**i**	$m = 1$	**ii**	$c = 4$
e	$y = -2x + \frac{1}{2}$	**i**	$m = -2$	**ii**	$c = \frac{1}{2}$
f	$y = 3x + \frac{3}{2}$	**i**	$m = 3$	**ii**	$c = 1\frac{1}{2}$
g	$y = -\frac{1}{3}x - 2$	**i**	$m = -\frac{1}{3}$	**ii**	$c = -2$
h	$y = -\frac{4}{5}x + 2$	**i**	$m = -\frac{4}{5}$	**ii**	$c = 2$
i	$y = \frac{7}{2}x - \frac{1}{2}$	**i**	$m = 3\frac{1}{2}$	**ii**	$c = -\frac{1}{2}$

Question 7

Where possible, express each equation in the form $y = mx + c$; m is the gradient.

a $y = -2x - 1$, $m = -2$

b $y = 2$ is a horizontal line passing through 2 on the y-axis. Hence, its gradient is 0.

c $y = -x - 1$, $m = -1$

d $y = -3x + 8$, $m = -3$

e $y = 2x + 5$, $m = 2$

f $y = -\frac{1}{4}x + 3$, $m = -\frac{1}{4}$

g $y = \frac{3}{2}x + 2$, $m = \frac{3}{2} = 1\frac{1}{2}$

h $y = \frac{5}{4}x - \frac{15}{4}$, $m = \frac{5}{4} = 1\frac{1}{4}$

i $y = \frac{2}{3}x + 3$, $m = \frac{2}{3}$

j $y = \frac{1}{5}x - 1$, $m = \frac{1}{5}$

k $y = \frac{2}{7}x + 5$, $m = \frac{2}{7}$

l $y = -\frac{3}{5}x - 2$, $m = -\frac{3}{5}$

m $y = -\frac{1}{14}x + \frac{1}{6}$, $m = -\frac{1}{14}$

n $y = 15x - 40$, $m = 15$

o $y = -\frac{3}{2}x - 3$, $m = -\frac{3}{2} = -1\frac{1}{2}$

Question 8

Use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Given $(x_1, y_1) = (8, y_1)$, $(x_2, y_2) = (-1, 3)$, $m = 2$

$$2 = \frac{3 - y_1}{-1 - 8}$$

$-18 = 3 - y_1$

$y_1 = 21$

Question 9

Given $(x_1, y_1) = (2, -1)$, $(x_2, y_2) = (x, 0)$, $m = -5$

$$-5 = \frac{0 - (-1)}{x - 2}$$

$-5(x - 2) = 1$

$x - 2 = -\dfrac{1}{5}$

$x = 1\dfrac{4}{5}$ or 1.8

Question 10

Given $(x_1, y_1) = (4, 2)$, $(x_2, y_2) = (x, -3)$, $m = -1$

$$-1 = \frac{3 - 2}{x - 4}$$

$-1(x - 4) = -5$

$x - 4 = 5$

$x = 9$

Question 11

a $P = kd$, where k is the constant of proportionality.

Given $d = 450$, $P = 150$

$150 = 450k \Rightarrow k = \frac{150}{450} = \frac{1}{3}$

$P = \frac{1}{3}d$

b When $d = 840$, $P = \frac{1}{3} \times 840 = 280$. Mario earns 280 points.

c When $P = 57$, $57 = \frac{1}{3} \times d \Rightarrow d = 3 \times 57 = 171$. Mario spent \$171

Question 12

Gradient m_{AB} of side AB

$m_{AB} = \frac{5-2}{1-(-1)} = \frac{3}{2} = 1\frac{1}{2}$

Gradient m_{BC} of side BC

$m_{BC} = \frac{5-5}{6-1} = 0$

Gradient m_{CD} of side CD

$m_{CD} = \frac{5-2}{6-4} = \frac{3}{2} = 1\frac{1}{2}$

Gradient of side DA

$m_{DA} = \frac{2-2}{4-(-1)} = 0$

$m_{AB} = m_{CD}$, $m_{BC} = m_{DA}$

This means opposite sides are parallel, so $ABCD$ is a parallelogram.

Exercise 4.06 Finding a linear equation

Question 1

a Use $y = mx + c$, where m is the gradient, c is the y-intercept.

$y = 4x - 1$

b Use $y - y_1 = m(x - x_1)$, with $m = -3$ and $(x_1, y_1) = (0, 4)$

$y - 4 = -3(x - 0)$

$y - 4 = -3x$

$y = -3x + 4$

c $y = mx + c$, with $m = 5$, and the y-intercept, c, is 0.

$y = 5x$

d $y - y_1 = m(x - x_1)$, with $m = 4$ and $(x_1, y_1) = (-5, 0)$

$y - 0 = 4[x - (-5)]$

$y = 4x + 20$ or $4x - y + 20 = 0$

e $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (0, 3)$

$$m = \frac{3-0}{0-1} = -3$$

$y - y_1 = m(x - x_1)$, with $m = -3$ and $(x_1, y_1) = (1, 0)$

$y - 0 = -3(x - 1)$

$y = -3x + 3$ or $3x + y - 3 = 0$

f $(x_1, y_1) = (3, 0)$, $(x_2, y_2) = (0, -4)$

$$m = \frac{-4-0}{0-3} = \frac{4}{3} = 1\frac{1}{3}$$

$y - y_1 = m(x - x_1)$, with $m = \frac{4}{3}$ and $(x_1, y_1) = (3, 0)$

$$y - 0 = \frac{4}{3}(x - 3)$$

$$y = \frac{4}{3}x - 4 \text{ or } 4x - 3y - 12 = 0$$

Question 2

a $(x_1, y_1) = (2, 5)$, $(x_2, y_2) = (-1, 1)$

$$m = \frac{1-5}{-1-2} = \frac{4}{3} = 1\frac{1}{3}$$

$y - y_1 = m(x - x_1)$, with $m = \frac{4}{3}$ and $(x_1, y_1) = (2, 5)$

$y - 5 = \frac{4}{3}(x - 2)$

$3y - 15 = 4x - 8$

$4x - 3y + 7 = 0$ or $y = \frac{4}{3}x + \frac{7}{3}$

b $(x_1, y_1) = (0, 1)$, $(x_2, y_2) = (-4, -2)$

$$m = \frac{-2-1}{-4-0} = \frac{3}{4}$$

$y - y_1 = m(x - x_1)$, with $m = \frac{3}{4}$ and $(x_1, y_1) = (0, 1)$

$y - 1 = \frac{3}{4}(x - 0)$

$y = \frac{3}{4}x + 1$ or $3x - 4y + 4 = 0$

c $(x_1, y_1) = (-2, 1)$, $(x_2, y_2) = (3, 5)$

$$m = \frac{5-1}{3-(-2)} = \frac{4}{5}$$

$y - y_1 = m(x - x_1)$, with $m = \frac{4}{5}$ and $(x_1, y_1) = (-2, 1)$

$y - 1 = \frac{4}{5}(x + 2)$

$y = \frac{4}{5}x + \frac{13}{5}$ or $4x - 5y + 13 = 0$

d $(x_1, y_1) = (3, 4)$, $(x_2, y_2) = (-1, 7)$

$$m = \frac{7-4}{-1-3} = -\frac{3}{4}$$

$y - y_1 = m(x - x_1)$, with $m = -\frac{3}{4}$ and $(x_1, y_1) = (3, 4)$.

$y - 4 = -\frac{3}{4}(x - 3)$

$y = -\frac{3}{4}x + \frac{25}{4}$ or $3x + 4y - 25 = 0$

e $(x_1, y_1) = (-4, -1)$, $(x_2, y_2) = (-2, 0)$

$$m = \frac{0-(-1)}{-2-(-4)} = \frac{1}{2}$$

$y - y_1 = m(x - x_1)$, with $m = \frac{1}{2}$ and $(x_1, y_1) = (-4, -1)$

$y + 1 = \frac{1}{2}(x + 4)$

$y = \frac{1}{2}x + 1$ or $x - 2y + 2 = 0$

Question 3

a $(x_1, y_1) = (2, 0)$, $(x_2, y_2) = (3, -4)$

$$m = \frac{-4-0}{3-2} = -4$$

b $y - y_1 = m(x - x_1)$, with $m = -4$ and $(x_1, y_1) = (2, 0)$

$y - 0 = -4(x - 2)$

$y = -4x + 8$ or $4x + y - 8 = 0$

Question 4

a The equation of a horizontal line that passes through a point with y-coordinate k is $y = k$.

For the point (2, 3), $k = 3$, so the equation is $y = 3$.

b The equation of a vertical line that passes through a point with x-coordinate k is $x = k$.

For the point (–1, 2), $k = -1$, so the equation is $x = -1$

Question 5

a The line passes through (0, 0), so its y-intercept is 0.

b The gradient is –2, so $m = -2$. Use $y = mx + c$, with $m = -2$ and $c = 0$

$y = -2x$

Question 6

a Each level adds 15 points to the score, so for x levels, $15x$ points are added.

The player starts with 20 points, so the total number of points, P, for x levels is $15x + 20$.

$P = 15x + 20$

b **i** $x = 24$, $P = 15 \times 24 + 20 = 380$ points

ii $x = 55$, $P = 15 \times 55 + 20 = 845$ points

iii 3725

c **i** $P = 2195$, $2195 = 15 \times x + 20$

$$x = \frac{2195 - 20}{15} = 145 \text{ levels}$$

ii $P = 7700$, $7700 = 15 \times x + 20$

$$x = \frac{7700 - 20}{15} = 512 \text{ levels}$$

iii $P = 12\,665$, $12\,665 = 15 \times x + 20$

$$x = \frac{12\,665 - 20}{15} = 843 \text{ levels}$$

Question 7

a Total weekly fixed costs \$1500 + \$3000 + \$2500 = \$7000

Without fixed costs, the cost to produce n TVs is $250n$

The total cost, c, to produce n TVs is $c = 250n + 7000$

b **i** $n = 100$, $c = 250 \times 100 + 7000 = \$32\,000$

ii $n = 270$, $c = 250 \times 270 + 7000 = \$74\,500$

iii $n = 1200$, $c = 250 \times 1200 + 7000 = \$307\,000$

c **i** $c = 52\ 000$

$52\ 000 = 250 \times n + 700$

$n = \dfrac{52\ 000 - 7000}{250} = 180$

ii $c = 78\ 250$

$78\ 250 = 250 \times n + 7000$

$n = \dfrac{78\ 250 - 7000}{250} = 285$

iii $c = 367\ 000$

$367\ 000 = 250 \times n + 7000$

$n = \dfrac{367\ 000 - 7000}{250} = 1440$

d The cost, c, for producing n TVs is $c = 250n + 7000$

The revenue from selling each TV at $950 is $950n$

Break even occurs when revenue is the same as the costs, c.

$950n = 250n + 7000$

$700n = 7000$

$n = 10$ TVs

Question 8

a Each hour, 8 litres of water evaporates. After h hours, $8h$ litres have evaporated.

There is 450 litres of water initially, so the amount, A litres, of water lost due to evaporation after h hours is $A = 450 – 8h$

b **i** $h = 3$: $A = 450 - 8 \times 3 = 426$ L

ii $h = 24$: $A = 450 - 8 \times 24 = 258$ L

c 56.25 h or 2 days 8 hours 15 minutes

We require h when $A = 0$

$0 = 450 – h$

$h = \dfrac{450}{8} = 56.25$

56.25 hours is 2 days 8.25 hours or 2 days 8 hours 15 minutes

Question 9

a $C = 20 - 1.69x$

Each single purchased is \$1.69, so the cost for x singles is $1.69x$

Deduct this from the initial \$20 credit, so the amount of credit, C, remaining is $C = 20 - 1.69x$

b Require x when $C = 0$

$0 = 20 - 1.69x$

$x = \frac{20}{1.69} = 11.83$

There is enough credit to purchase 11 songs, but not enough for 12 songs.

Question 10

a $A = 20\,000 - 320x$

The monthly repayment is \$320. After x months, the total payments amount to $320x$.

The amount, A, remaining from the money owed after x months is $A = 20000 - 320x$

b **i** $x = 5$, $A = 20000 - 320 \times 5 = \$18\,400$

ii 1 year = 12 months

$x = 12$, $A = 20\,000 - 320 \times 12 = \$16\,160$

iii 5 years is $5 \times 12 = 60$ months

$x = 60$, $A = 20\,000 - 320 \times 60 = \$800$

c Require x when $A = 0$

$0 = 20\,000 - 320x$

$x = \frac{20\,000}{320} = 62.5$

Repayments are paid monthly, so 63 months is required. This is 5 years and 3 months.

Question 11

a For x balloons sold, the revenue is $5x$.

Profit, P, is revenue less overhead costs.

$P = 5x - 100$

b $x = 300$, $P = 5 \times 300 - 100 = \$1400$

c $P = 1055$,

$1055 = 5x - 100$

$$x = \frac{1055+100}{5} = \frac{1155}{5} = 231$$

d Require profit to be zero.

$0 = 5x - 100 \Rightarrow x = 20$

Exercise 4.07 Parallel and perpendicular lines

Question 1

a Write the equation as $y = -3x + 4$

The gradient is $m = -3$

The required line has the same gradient.

b Write the equation as $y = -3x + 4$

The gradient is $m_1 = -3$.

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$-3 \times m_2 = -1 \Rightarrow m_2 = \frac{1}{3}$

c $(x_1, y_1) = (3, 5)$, $(x_2, y_2) = (-1, 2)$

$m = \frac{2-5}{1-3} = \frac{3}{4}$

The gradient is $m = \frac{3}{4}$.

The required line has the same gradient, $m = \frac{3}{4}$.

d $(x_1, y_1) = (3, 0)$, $(x_2, y_2) = (0, 2)$

$m = \frac{2-0}{0-3} = -\frac{2}{3}$

The gradient is $m_1 = -\frac{2}{3}$.

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$-\frac{2}{3} \times m_2 = -1 \Rightarrow m_2 = \frac{3}{2} = 1\frac{1}{2}$

e The angle made with the positive x-axis is 135°.

The gradient is $m_1 = \tan 135° = -1$

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$-1 \times m_2 = -1 \Rightarrow m_2 = 1$

f Write the equation as $y = \frac{6}{5}x + \frac{4}{5}$.

The gradient is $m_1 = \frac{6}{5}$.

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$\frac{6}{5} \times m_2 = -1 \Rightarrow m_2 = -\frac{5}{6}$

g Write the equation as $y=\frac{1}{3}x+\frac{7}{3}$.

The gradient is $m=\frac{1}{3}$

The gradient of the parallel line has the same gradient, $m=\frac{1}{3}$.

h $(x_1, y_1) = (4, -2)$, $(x_2, y_2) = (3, 3)$

$m=\frac{3-(-2)}{3-4}=-5$

The gradient is $m_1 = -5$.

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$-5\times m_2=-1 \Rightarrow m_2=\frac{1}{5}$

Question 2

a Use $y - y_1 = m(x - x_1)$, with $m = 1$ and $(x_1, y_1) = (2, 3)$

$y - 3 = 1(x - 2)$

$y = x + 1$ or $x - y + 1 = 0$

b Write the equation as $y=\frac{1}{3}x-\frac{7}{3}$

Use $y - y_1 = m(x - x_1)$, with $m=\frac{1}{3}$ and $(x_1, y_1) = (-1, 5)$

$y - 5 = \frac{1}{3}(x + 1)$

$y=\frac{1}{3}x+\frac{16}{3}$ or $x - 3y + 16 = 0$

c Use $y - y_1 = m(x - x_1)$, with $m = -1$ and $(x_1, y_1) = (5, 0)$

$y - 0 = -1(x - 5)$

$y = -x + 5$ or $x + y - 5 = 0$

d The gradient of $y = 2x$ is $m_1 = 2$.

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$2 \times m_2 = -1 \Rightarrow m_2 = -\frac{1}{2}$

Use $y - y_1 = m(x - x_1)$, with $m = -\frac{1}{2}$ and $(x_1, y_1) = (3, -4)$

$y + 4 = -\frac{1}{2}(x - 3)$

$y = -\frac{1}{2}x - \frac{5}{2}$ or $x + 2y + 5 = 0$

e The gradient of $2x + y = 3 = 0$ is $m_1 = -2$

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$-2 \times m_2 = -1 \Rightarrow m_2 = \frac{1}{2}$

Use $y - y_1 = m(x - x_1)$, with $m = \frac{1}{2}$ and $(x_1, y_1) = (-2, 1)$.

$y - 1 = \frac{1}{2}(x + 2)$

$y = \frac{1}{2}x + 2$ or $x - 2y + 4 = 0$

f The gradient of $3x - y - 5 = 0$ is $m_1 = 3$.

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$3 \times m_2 = -1 \Rightarrow m_2 = -\frac{1}{3}$

Use $y - y_1 = m(x - x_1)$, with $m = -\frac{1}{3}$ and $(x_1, y_1) = (7, -2)$.

$y + 2 = -\frac{1}{3}(x - 7)$

$y = -\frac{1}{3}x + \frac{1}{3}$ or $x + 3y - 1 = 0$

g The gradient of $4x - 3y + 2 = 0$ is $m_1 = \frac{4}{3}$.

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$\frac{4}{3} \times m_2 = -1 \Rightarrow m_2 = -\frac{3}{4}$

Use $y - y_1 = m(x - x_1)$, with $m = -\frac{3}{4}$ and $(x_1, y_1) = (-3, -1)$.

$y + 1 = -\frac{3}{4}(x + 3)$

$y = -\frac{3}{4}x - \frac{13}{4}$ or $3x + 4y + 13 = 0$

h The gradient of $x + y + 3 = 0$ is $m = -1$.

The gradient of the line parallel to this line is $m = -1$.

Use $y - y_1 = m(x - x_1)$, with $m = -1$ and $(x_1, y_1) = (0, 0)$.

$y - 0 = -1(x - 0)$

$y = -x$

i The gradient of $5x - y - 2 = 0$ is $m = 5$.

The gradient of the line parallel to this line is $m = 5$.

Use $y - y_1 = m(x - x_1)$, with $m = 5$ and $(x_1, y_1) = (3, 7)$.

$y - 7 = 5\,(x - 3)$

$y = 5x - 8$ or $5x - y - 8 = 0$

j The gradient of $x - 2y = 9$ is $m_1 = \frac{1}{2}$.

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$\frac{1}{2} \times m_2 = -1 \Rightarrow m_2 = -2$

Use $y - y_1 = m(x - x_1)$, with $m = -2$ and $(x_1, y_1) = (0, -2)$.

$y + 2 = -2(x - 0)$

$y = -2x - 2$ or $2x + y + 2 = 0$

k The gradient of $3x + 2y - 1 = 0$ is $m_1 = -\frac{3}{2}$.

The gradient, m_2, of the perpendicular line satisfies $m_1 \times m_2 = -1$.

$-\frac{3}{2} \times m_2 = -1 \Rightarrow m_2 = \frac{2}{3}$

Use $y - y_1 = m(x - x_1)$, with $m = \frac{2}{3}$ and $(x_1, y_1) = (-2, 4)$.

$y - 4 = \frac{2}{3}(x + 2)$

$y = \frac{2}{3}x + \frac{16}{3}$ or $2x - 3y + 16 = 0$.

Question 3

Two lines are parallel if they have the same gradient. That is, $m_1 = m_2$.

The gradient of $y = 3x - 2$ is $m_1 = 3$.

$6x - 2y - 9 = 0$ can be transposed to $y = 3x - 4\frac{1}{2}$, with gradient $m_2 = 3$.

Hence, $m_1 = m_2$.

Question 4

Two lines are perpendicular if the product of the two gradients is –1. That is, $m_1 \times m_2 = -1$.

$x + 5y = 0$ can be transposed to $y = -\frac{1}{5}x$, with gradient $m_1 = -\frac{1}{5}$.

$y = 5x + 3$ has gradient $m_2 = 5$.

$m_1 \times m_2 = -\frac{1}{5} \times 5 = -1$

Question 5

Two lines are parallel if they have the same gradient. That is, $m_1 = m_2$.

$6x - 5y + 1 = 0$ can be transposed to $y = \frac{6}{5}x + \frac{1}{5}$, with gradient $m_1 = \frac{6}{5}$.

$6x - 5y - 4 = 0$ can be transposed to $y = \frac{6}{5}x + \frac{3}{5}$, with gradient $m_2 = \frac{6}{5}$.

Hence, $m_1 = m_2$.

Question 6

Two lines are perpendicular if the product of the two gradients is –1. That is, $m_1 \times m_2 = -1$.

$7x + 3y + 2 = 0$ can be transposed to $y = -\frac{7}{3}x - \frac{2}{3}$, with gradient $m_1 = -\frac{7}{3}$.

$3x - 7y = 0$ can be transposed to $y = \frac{3}{7}x$, with gradient $m_2 = \frac{3}{7}$.

$m_1 \times m_2 = -\frac{7}{3} \times \frac{3}{7} = -1$

Question 7

For the two lines to be perpendicular, the product of the gradient of the line $y = kx - 1$ with the gradient of the line $3x - 2y + 5 = 0$ must be -1.

Gradient of $y = kx - 1$ is $m_1 = k$.

$3x - 2y + 5 = 0$ can be transposed to $y = \frac{3}{2}x + \frac{5}{2}$, which has gradient $m_2 = \frac{3}{2}$.

Require $m_1 \times m_2 = -1$.

$$k \times \frac{3}{2} = -1 \Rightarrow k = -\frac{2}{3}$$

Question 8

Find the gradient of the line joining the two given points.

$(x_1, y_1) = (3, -1)$, $(x_2, y_2) = (2, -5)$

Gradient is $m_1 = \frac{-5-(-1)}{2-3} = 4$.

$8x - 2y - 3 = 0$ can be transposed to $y = 4x - \frac{3}{2}$, which has gradient $m_2 = 4$.

$m_1 = m_2$, so the two lines are parallel.

Question 9

Gradient of line connecting A to B.

$(x_1, y_1) = (-3, -2)$, $(x_2, y_2) = (-1, 4)$

Gradient $m_{AB} = \dfrac{4-(-2)}{-1-(-3)} = 3$.

Gradient of line connecting B to C.

$(x_1, y_1) = (-1, 4)$, $(x_2, y_2) = (7, -1)$

Gradient $m_{BC} = \dfrac{-1-4}{7-(-1)} = -\dfrac{5}{8}$

Gradient of line connecting C to D.

$(x_1, y_1) = (7, -1)$, $(x_2, y_2) = (5, -7)$

Gradient $m_{CD} = \dfrac{-7-(-1)}{5-7} = 3$

Gradient of line connecting A to D.

$(x_1, y_1) = (-3, -2)$, $(x_2, y_2) = (5, -7)$

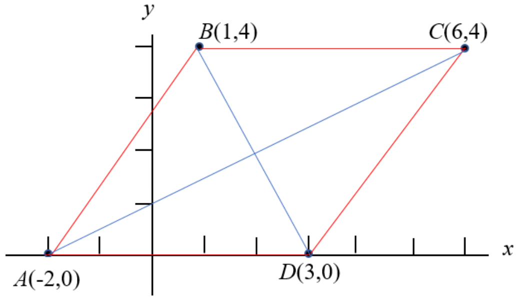
Gradient $m_{AD} = \dfrac{-7-(-2)}{5-(-3)} = -\dfrac{5}{8}$.

$m_{AB} = m_{CD}$ and $m_{BC} = m_{AD}$

This shows that *ABCD* is a parallelogram, with *AB* and *CD* parallel sides, and *BC* and *AD* parallel sides.

Question 10

We will show that BD is perpendicular to AC by showing that the product of the gradients is –1.



Gradient of line connecting B to D.

$(x_1, y_1) = (1, 4)$, $(x_2, y_2) = (3, 0)$

Gradient $m_{BD} = \dfrac{0-4}{3-1} = -2$

Gradient of line connecting A to C

$(x_1, y_1) = (-2, 0)$, $(x_2, y_2) = (6, 4)$

Gradient $m_{AC} = \dfrac{4-0}{6-(-2)} = \dfrac{1}{2}$

$m_{BD} \times m_{AC} = -2 \times \dfrac{1}{2} = -1$

Hence, the diagonals are perpendicular to each other.

Question 11

Let m_1 be the gradient of the line joining $(2, -1)$ to $(-5, -7)$.

Let m_2 be the gradient of the perpendicular to this line.

$(x_1, y_1) = (2, -1)$, $(x_2, y_2) = (-5, -7)$

Gradient $m_1 = \dfrac{-7-(-1)}{-5-2} = \dfrac{6}{7}$

$m_1 \times m_2 = -1$ so $\dfrac{6}{7} \times m_2 = -1 \Rightarrow m_2 = -\dfrac{7}{6}$

Use $y - y_1 = m(x - x_1)$, with $m = -\dfrac{7}{6}$ and $(x_1, y_1) = (6, -3)$.

$y + 3 = -\dfrac{7}{6}(x - 6)$

$y = -\dfrac{7}{6}x + 4$ or $7x + 6y - 24 = 0$

Exercise 4.08 Quadratic functions

Question 1

a Let $y = 0$

$x^2 + 2x = 0$

$x(x + 2) = 0$

$x = 0$

$x + 2 = 0 \Rightarrow x = -2$

x-intercepts are 0 and –2.

Let $x = 0$

$y = 0^2 + 2 \times 0 = 0$

The y-intercept is 0.

b Let $y = 0$

$-x^2 + 3x = 0$

$-x(x - 3) = 0$

$x = 0$

$x - 3 = 0 \Rightarrow x = 3$

x-intercepts are 0 and 3.

Let $x = 0$

$y = -0^2 + 3 \times 0 = 0$

The y-intercept is 0.

c Let $f(x) = 0$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

x-intercepts are 1 and –1.

Let $x = 0$

$f(0) = 0^2 - 1 = -1$

The y-intercept is –1.

d Let $y = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x - 2 = 0 \Rightarrow x = 2$

$x + 1 = 0 \Rightarrow x = -1$

x-intercepts are 2 and –1

Let $x = 0$

$y = 0^2 - 0 - 2 = -2$

The y-intercept is –2.

e Let $y = 0$

$x^2 - 9x + 8 = 0$

$(x - 8)(x - 1) = 0$

$x - 8 = 0 \Rightarrow x = 8$

$x - 1 = 0 \Rightarrow x = 1$

x-intercepts are 1 and 8.

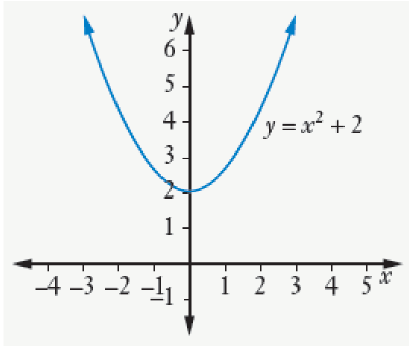
Let $x = 0$

$y = 0^2 - 9 \times 0 + 8 = 8$

The y-intercept is 8.

Question 2

a $a > 0$, so the graph is concave upward.



Let $y = 0$

$x^2 + 2 = 0$ has no solution, so there are no x-intercepts.

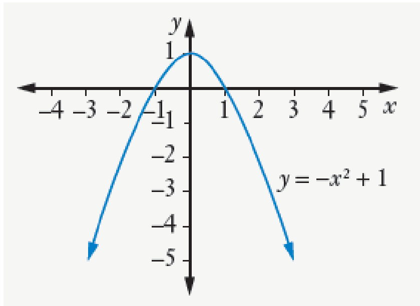
Let $x = 0$.

$y = 0^2 + 2 = 2$, so the y-intercept is 2.

The function is even, so the graph is symmetrical about the y-axis.

This means the y-intercept is also the minimum turning point.

b $a < 0$, so the graph is concave downward.



Let $y = 0$

$-x^2 + 1 = 0 \Rightarrow x = \pm 1$

The x-intercepts are $x = \pm 1$.

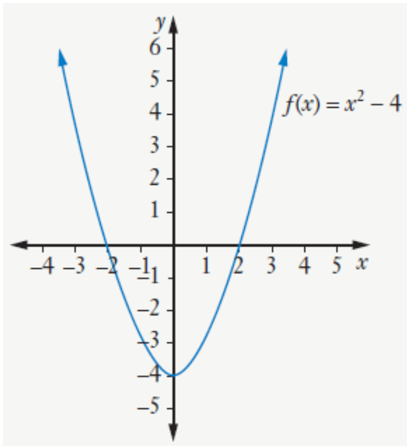
Let $x = 0$

$y = -0^2 + 1 = 1$, so the y-intercept is 1.

The function is even, so the graph is symmetrical about the y-axis.

This means the y-intercept is also the maximum turning point.

c $a > 0$, so the graph is concave upward.



Let $y = 0$

$x^2 - 4 = 0 > x = \pm 2$

The x-intercepts are $x = \pm 2$.

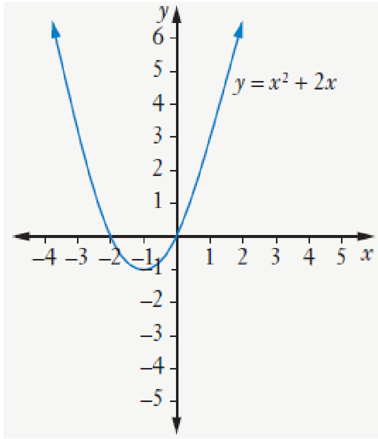
Let $x = 0$

$y = -0^2 - 4 = -4$, so the y-intercept is -4.

The function is even, so the graph is symmetrical about the y-axis.

This means the y-intercept is also the minimum turning point.

d $a > 0$, so the graph is concave upward.



Let $y = 0$

$x^2 + 2x = 0$

$x(x + 2) = 0 \Rightarrow x = 0, x = -2$

The x-intercepts are 0 and –2.

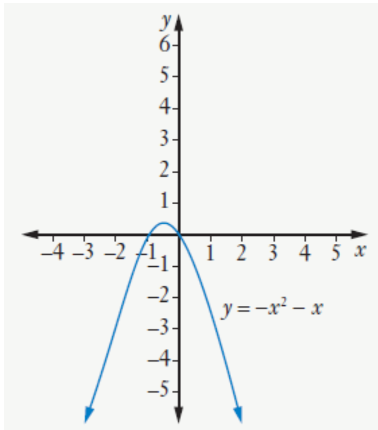
Let $x = 0$

$y = 0^2 + 2 \times 0 = 0$, so the y-intercept is 0.

The x-coordinate of the minimum turning point is at $x = -1$, midway between the x-intercepts.

When $x = -1$, $y = (-1)^2 + 2 \times (-1) = -1$. The minimum turning point is at (–1, –1).

e $a < 0$, so the graph is concave downward.



Let $y = 0$

$-x^2 - x = 0$

$x(x + 1) = 0 \Rightarrow x = 0, x = -1$

The x-intercepts are 0 and –1.

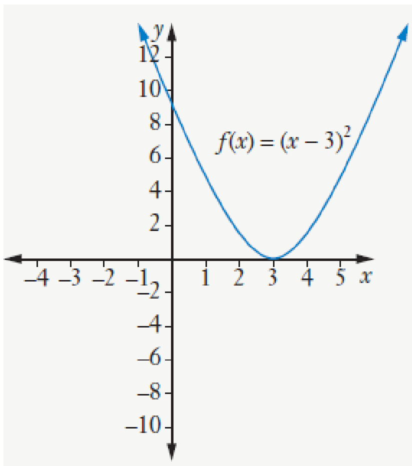
Let $x = 0$

$y = -0^2 - 0 = 0$, so the y-intercept is 0.

The x-coordinate of the minimum turning point is at $x = -\frac{1}{2}$, midway between the x-intercepts.

When $x = -\frac{1}{2}$, $y = -\left(-\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{1}{4}$. The maximum turning point is at $\left(-1\ \frac{1}{4}\right)$.

f $a > 0$, so the graph is concave upward.



Let $f(x) = 0$

$(x - 3)^2 = 0 \Rightarrow x = 3$

Since there is only one x-intercept, the graph does not intersect the x-axis.

$x = 3$ is the minimum turning point.

When $x = 3$, $f(x) = (3 - 3)^2 = 0$.
The turning point is at (3, 0).

Let $x = 0$

$f(0) = (0 - 3)^2 = 9$, so the y-intercept is 9.

g $a > 0$, so the graph is concave upward.

Let $f(x) = 0$

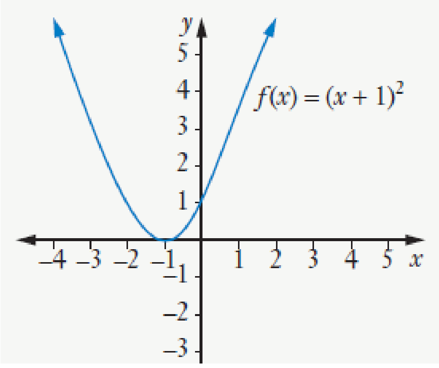
$(x + 1)^2 = 0 \Rightarrow x = -1$

Since there is only one x-intercept, the graph does not intersect the x-axis.

$x = -1$ is the minimum turning point.

Let $x = 0$

$f(0) = (0 + 1)^2 = 1$, so the y-intercept is 1.



When $x = -1$, $y = (-1 + 1)^2 = 0^2 = 0$. The minimum turning point is at $(-1, 0)$.

h $a > 0$, so the graph is concave upward.

Let $y = 0$

$x^2 + 3x - 4 = 0$

$(x + 4)(x - 1) = 0 \Rightarrow x = -4, x = 1$

The x-intercepts are -4 and 1.

Let $x = 0$

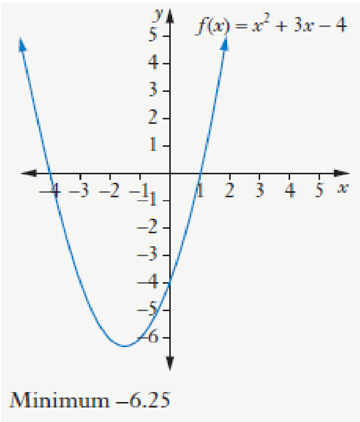
$y = 0^2 + 3 \times 0 - 4 = -4$, so the y-intercept is -4.

The x-coordinate of the minimum turning point is at

$x = -1\frac{1}{2}$, midway between the x-intercepts.

When $x = -1\frac{1}{2}$, $y = \left(-\frac{3}{2}\right)^2 + 3\times\left(-\frac{3}{2}\right) - 4 = -6\frac{1}{4}$.

The minimum turning point is at $\left(-1\frac{1}{2}, -6\frac{1}{4}\right)$.



i $a > 0$, so the graph is concave upward.

Let $y = 0$

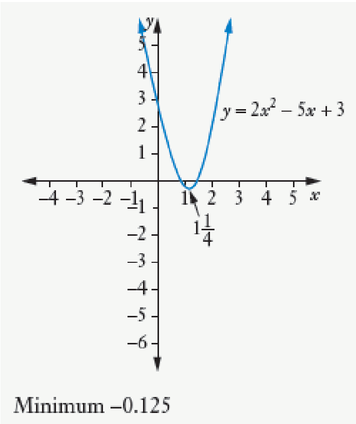
$2x^2 - 5x + 3 = 0$

$(2x + 1)(x - 3) = 0 \Rightarrow x = -\frac{1}{2}, x = 3$

The x-intercepts are $-\frac{1}{2}$ and 3.

Let $x = 0$

$y = 2 \times 0^2 - 5 \times 0 + 3 = 3$, so the y-intercept is 3.



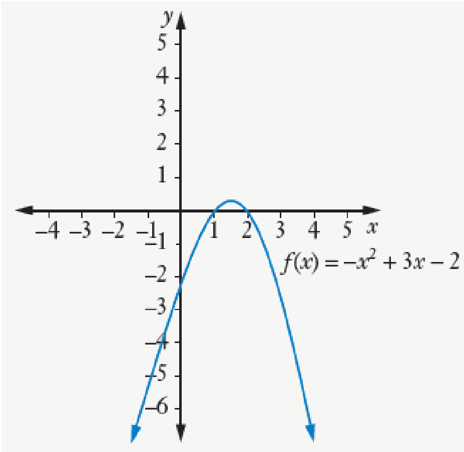
The x-coordinate of the minimum turning point is at

$x = \frac{1}{2}\left(-\frac{1}{2} + 3\right) = 1\frac{1}{4}$, midway between the x-intercepts.

When $x = 1\frac{1}{4}$, $y = 2\left(\frac{5}{4}\right)^2 - 5\times\frac{5}{4} + 3 = -\frac{1}{8}$. The minimum turning point is at

$\left(1\frac{1}{4}, -\frac{1}{8}\right)$.

j $a < 0$, so the graph is concave downward.



Let $f(x) = 0$

$-x^2 + 3x - 2 = 0$

$x^2 - 3x + 2 = 0$

$(x - 1)(x - 2) = 0 \Rightarrow x = 1, x = 2$

The x-intercepts are 1 and 2.

Let $x = 0$

$f(0) = -0^2 + 3 \times 0 - 2 = -2$, so the y-intercept is -2.

The x-coordinate of the minimum turning point is at $x=\frac{1}{2}(1+2)=1\frac{1}{2}$, midway between the x-intercepts.

When $x=1\frac{1}{2}$, $y=-\left(\frac{3}{2}\right)^2+3\times\frac{3}{2}-2=\frac{1}{4}$. The minimum turning point is at $\left(1\frac{1}{2},\frac{1}{4}\right)$.

Question 3

a **i** Let $y = 0$

$x^2 - 7x + 12 = 0$

$(x - 4)(x - 3) = 0 \Rightarrow x = 3, x = 4$

The x-intercepts are 3 and 4.

Let $x = 0$

$y = 0^2 - 7 \times 0 + 12 = 12$, so the y-intercept is 12.

ii All values of x can be used, so the domain is $(-\infty, \infty)$.

The x-coordinate of the turning point is at $x=\frac{1}{2}(3+4)=3\frac{1}{2}$, midway between the x-intercepts.

When $x=3\frac{1}{2}$, $y=\left(\frac{7}{2}\right)^2-y\times\frac{7}{2}+12=-\frac{1}{4}$.

$a > 0$, so the graph is concave upward.

The minimum turning point is the minimum y-value for the range.

The range is $\left[-\frac{1}{4}\ \infty\right)$.

b **i** Let $f(x) = 0$

$x^2 + 4x = 0$

$x(x + 4) = 0 \Rightarrow x = 0, x = -4$

The x-intercepts are 0 and –4.

Let $x = 0$

$f(0) = 0^2 + 4 \times 0 = 0$, so the y-intercept is 0.

ii All values of x can be used, so the domain is $(-\infty, \infty)$.

The x-coordinate of the turning point is at $x = \frac{1}{2}\left(0 + (-4)\right) = -2$, midway between the x-intercepts.

When $x = -2$, $y = (-2)^2 + 4 \times (-2) = -4$

$a > 0$, so the graph is concave upward.

The y-value for the minimum turning point is also the minimum y-value for the range.

The range is $[-4, \infty)$.

c **i** Let $y = 0$

$x^2 - 2x - 8 = 0$

$(x - 4)(x + 2) = 0 \Rightarrow x = -2, x = 4$

The x-intercepts are –2 and 4.

Let $x = 0$

$y = 0^2 - 2 \times 0 - 8 = -8$, so the y-intercept is –8.

ii All values of x can be used, so the domain is $(-\infty, \infty)$.

The x-coordinate of the turning point is at $x = \frac{1}{2}(-2 + 4) = 1$, midway between the x-intercepts.

When $x = 1$, $y = 1^2 - 2 \times 1 - 8 = -9$

$a > 0$, so the graph is concave upward.

The minimum turning point is the minimum y-value for the range.
The range is $[-9, \infty)$.

d **i** Let $y = 0$

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0 \Rightarrow x = 3$

The x-intercept is 3.

Let $x = 0$

$y = (0-3)^2 = 9$, so the y-intercept is 9.

ii All values for x can be used, so the domain is $(-\infty, \infty)$.

Since there is only one x-intercept, it represents the x-coordinate of the turning point.

At $x = 3$, $y = (0-3)^2 = 0$.

$a > 0$, so it is a minimum turning point, which is also the minimum y-value for the range.

So the range is $[0, \infty)$.

e **i** Let $f(x) = 0$

$4 - x^2 = 0$

$(2-x)(2+x) = 0 \Rightarrow x = 2, x = -2$

The x-intercepts are -2 and 2.

Let $x = 0$

$f(0) = 4 - 0^2 = 4$, so the y-intercept is 4.

ii All values of x can be used, so the domain is $(-\infty, \infty)$.

The x-coordinate of the turning point is at $x = \frac{1}{2}(-2+2) = 0$, midway between the x-intercepts.

When $x = 0$, $y = 4$

$a < 0$, so the graph is concave downward. This means $(0, 4)$ is a maximum turning point.

Hence the range is $(-\infty, 4]$.

Question 4

a $x^2 - 5$ is valid for all x values, so the domain is $(-\infty, \infty)$.

$a > 0$, so the turning point is a minimum.

The minimum y value in $y = x^2 - 5$ is -5 when $x = 0$, so the range is $[-5, \infty)$.

b All values of x can be used, so the domain is $(-\infty, \infty)$.

Let $f(x) = 0$

$x^2 - 6x = 0$

$x(x - 6) = 0 \Rightarrow x = 0, x = 6$

$a > 0$, so the turning point is a minimum.

This occurs at $x = \frac{1}{2}(0 + 6) = 3$

The corresponding y-value is $y = 3^2 - 6 \times 3 = -9$

So the range is $[-9, \infty)$.

c All values of x can be used, so the domain is $(-\infty, \infty)$.

Let $f(x) = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0 \Rightarrow x = 2, x = -1$

$a > 0$, so the turning point is a minimum.

This occurs at $x = \frac{1}{2}(2 + (-1)) = \frac{1}{2}$

The corresponding y-value is $y = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2 = -2\frac{1}{4}$

So the range is $\left[-2\frac{1}{4}, \infty\right)$.

d All values of x can be used, so the domain is $(-\infty, \infty)$.

$a < 0$, so the turning point is a maximum.

This occurs at $x^2 = 0 \Rightarrow x = 0, y = 0$

So the range is $(-\infty, 0]$.

e All values of x can be used, so the domain is $(-\infty, \infty)$.

Let $f(x) = 0$

$(x - 7)^2 = 0 \Rightarrow x = 7$

$a > 0$ and there is only one x-intercept, so the turning point is a minimum and is on the x-axis.

When $x = 7$, $y = 0$, so the range is $[0, \infty)$.

Question 5

a The depth corresponds to the y-value of the turning point.

The graph is symmetrical about the y-axis, so the maximum occurs at $x = 0$.

When $x = 0$, $y = 6$, so the depth is 6 m.

b The width is the distance between the two x-intercepts.

Let $y = 0$

$-3x^2 + 6 = 0$

$x = \pm\sqrt{2}$

The distance required is $2\sqrt{2} \approx 2.8$ m

Exercise 4.09 Axis of symmetry

Question 1

Use a = 1, b = 2

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{2}{2\times 1}$ = –1

When $x = -1$, $y = (-1)^2 + 2 \times (-1) = -1$

Minimum value is –1.

Question 2

Use a = 1, b = 0

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{0}{2\times 1}$ = 0

When $x = 0$, $y = (0)^2 - 4 = -4$

Minimum value is –4.

Question 3

Use a = 4, b = −3

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{-3}{2\times 4} = \frac{3}{8}$

When $x = \frac{3}{8}, y = 4\left(\frac{3}{8}\right)^2 - 3\times\frac{3}{8} + 1 = \frac{7}{16}$

Minimum point is $\left(\frac{3}{8}\ \frac{7}{16}\right)$.

Question 4

Use a = −1, b = 2

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{2}{2\times -1}$ = 1

When $x = 1$, $y = -(1)^2 + 2 \times 1 - 7 = -6$

Maximum value is –6.

Question 5

Use $a = -2$, $b = -4$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{-4}{2\times -2} = -1$

When $x = -1$, $y = -2(-1)^2 - 4 \times -1 + 5 = 7$

Vertex is at $(-1, 7)$.

Question 6

Use $a = 1$, $b = 3$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{-3}{2\times 1} = -\frac{3}{2} = -1.5$

When $x = -1.5$, $y = (-1.5)^2 + 3 \times (-1.5) + 2 = -0.25$

Minimum value is -0.25.

Question 7

a Use $a = 1$, $b = 6$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{6}{2\times1} = -3$

When $x = -3$, $y = (-3)^2 + 6 \times (-3) - 3 = -12$

Vertex at $(-3, -12)$.

b Use $a = -1$, $b = -8$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{-8}{2\times-1} = -4$

When $x = -4$, $y = -(-4)^2 - 8 \times (-4) + 1 = 17$

Vertex at $(-4, 17)$.

c Use $a = 3$, $b = 18$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{18}{2\times3} = -3$

When $x = -3$, $y = 3(-3)^2 + 18 \times (-3) + 4 = -23$

Vertex at $(-3, -23)$.

d Use $a = -2$, $b = 5$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{5}{2\times-2} = \frac{5}{4} = 1.25$

When $x = 1.25$, $y = -2(1.25)^2 + 5 \times 1.25 = 3.125$

Vertex at $(1.25, 3.125)$.

e Use $a = 4$, $b = 10$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{10}{2\times4} = -\frac{10}{8} = -\frac{5}{4} = -1.25$

When $x = -1.25$, $y = 4(-1.25)^2 + 10 \times (-1.25) - 7 = -13.25$

Vertex at $(-1.25, -13.25)$.

Question 8

a **i** Use $a = 1, b = 2$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{2}{2 \times 2} = -1$

ii When $x = -1, y = (-1)^2 + 2 \times (-1) - 2 = -3$

$a > 0$, so the minimum value is –3.

iii Vertex is at $(-1, -3)$.

b **i** Use $a = -2, b = 4$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{4}{2 \times -2} = 1$

ii When $x = 1, y = -2(1)^2 + 4 \times 1 - 1 = 1$

$a < 0$, so the maximum value is 1.

iii Vertex is at $(1, 1)$.

Question 9

a Use $a = 1, b = 2$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{2}{2 \times 1} = -1$

When $x = -1, y = (-1)^2 + 2 \times -1 + 1 = 0$

Turning point at $(-1, 0)$

$a > 0$, so the turning point is a minimum.

b Use $a = 1, b = -8$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{-8}{2 \times 1} = 4$

When $x = 4, y = (4)^2 - 8 \times 4 - 7 = -23$

Turning point at $(4, -23)$

$a > 0$, so the turning point is a minimum.

c Use $a = 1$, $b = 4$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{4}{2\times 1} = -2$

When $x = -2$, $y = (-2)^2 + 4 \times (-2) - 3 = -7$

Turning point at $(-2, -7)$

$a > 0$, so the turning point is a minimum.

d Use $a = 1$, $b = -2$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{-2}{2\times 1} = 1$

When $x = 1$, $y = (1)^2 - 2 \times 1 = -1$

Turning point at $(1, -1)$.

$a > 0$, so the turning point is a minimum.

e Use $a = 1$, $b = -4$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{-4}{2\times 1} = 2$

When $x = 2$, $y = (2)^2 - 4 \times 2 - 7 = -11$

Turning point at $(2, -11)$.

$a > 0$, so the turning point is a minimum.

f Use $a = 2$, $b = 1$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{1}{2\times 2} = -\frac{1}{4} = -0.25$

When $x = -0.25$, $y = 2(-0.25)^2 - 0.25 - 3 = -3.125$

Turning point at $(-0.25, -3.125)$.

$a > 0$, so the turning point is a minimum.

g Use $a = -1$, $b = -2$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{-2}{2\times -1} = -1$

When $x = -1$, $y = -1^2 - 2 \times (-1) + 5 = 6$

Turning point at $(-1, 6)$.

$a < 0$, so the turning point is a maximum.

h Use $a = -2$, $b = 8$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{8}{2\times -2} = 2$

When $x = 2$, $y = -2 \times 2^2 + 8 \times 2 + 3 = 11$

Turning point at (2, 11).

$a < 0$, so the turning point is a maximum.

i Use $a = -3$, $b = 3$

Axis of symmetry at $x = -\frac{b}{2a} = -\frac{3}{2\times -3} = \frac{3}{6} = \frac{1}{2} = 0.5$

When $x = 0.5$, $y = -3 \times 0.5^2 + 3 \times 0.5 + 7 = 7.75$

Turning point at (0.5, 7.75).

$a < 0$, so the turning point is a maximum.

Question 10

a **i** $a = 1$, $b = 4$, $c = 4$

$$x = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 4}}{2 \times 1} = -2$$

The x-intercept is –2.

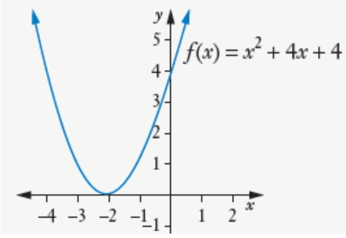
ii $a > 0$, so the function has a minimum value.

Minimum at $x = -\frac{b}{2a} = -\frac{4}{2 \times 1} = -2$

When $x = -2$, $y = (-2)^2 + 4 \times (-2) + 4 = 0$

Minimum value is 0.

iii



iv $y = f(x)$ is 0 at $x = -2$

b **i** $a = 1$, $b = -2$, $c = -3$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times (-3)}}{2 \times 1} = \frac{2 \pm 4}{2}$$

$$x = \frac{2+4}{2} = 3 \text{ and } x = \frac{2-4}{2} = -1$$

The x-intercepts are –1 and 3.

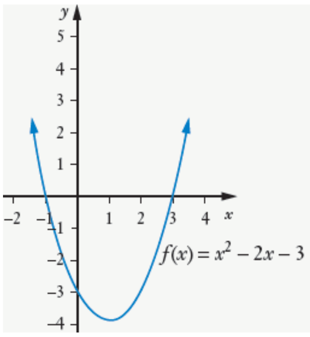
ii $a > 0$, so the function has a minimum value.

Minimum at $x = -\frac{b}{2a} = -\frac{-2}{2 \times 1} = 1$

When $x = 1$, $y = 12 - 2 \times 1 - 3 = -4$

Minimum value is –4

iii



iv $y = f(x)$ is 0 at $x = -1$ and at $x = 3$

c **i** $a = 1$, $b = -6$, $c = 1$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 1 \times 1}}{2 \times 1} = \frac{6 \pm 4\sqrt{2}}{2} \approx 3 \pm 2.83$$

$x = 3 + 2.83 = 5.83$ and $x = 3 - 2.83 = 0.17$

The x-intercepts are 5.83 and 0.17.

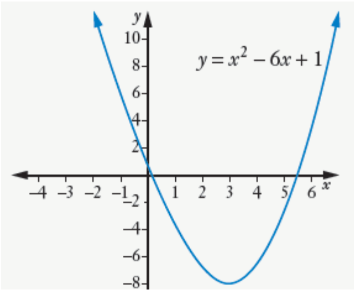
ii $a > 0$, so the function has a minimum value.

Minimum at $x = -\frac{b}{2a} = -\frac{-6}{2 \times 1} = 3$

When $x = 3$, $y = 3^2 - 6 \times 3 + 1 = -8$

Minimum value is –8

iii



iv $y = f(x)$ is 0 at $x \approx 0.2$ and at $x \approx 5.8$

d **i** $a = -1$, $b = -2$, $c = 6$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times (-1) \times 6}}{2 \times (-1)} = \frac{2 \pm 2\sqrt{7}}{-2} = -1 \mp \sqrt{7} \approx -1 \mp 2.65$$

$x = -1 - 2.65 = -3.65$ and $x = -1 + 2.65 = 1.65$

The x-intercepts are –3.65 and 1.65.

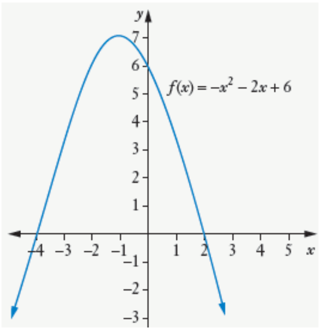
ii $a < 0$, so the function has a maximum value.

Maximum at $x = -\dfrac{b}{2a} = -\dfrac{-2}{2 \times -1} = -1$

When $x = -1$, $y = -1(-1)^2 - 2 \times (-1) + 6 = 7$

Maximum value is 7.

iii



iv $y = f(x)$ is 0 at $x \approx -3.7$ and at $x \approx 1.7$.

e **i** $a = -1, b = -1, c = 3$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times (-1) \times 3}}{2 \times (-1)} = \frac{1 \pm \sqrt{13}}{-2}$$

$x \approx -2.30$ and $x \approx 1.31$

The x-intercepts are –2.30 and 1.31.

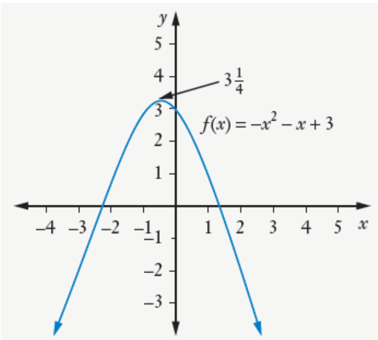
ii $a < 0$, so the function has a maximum value.

Maximum at $x = -\frac{b}{2a} = -\frac{-1}{2 \times -1} = -\frac{1}{2} = -0.5$

When $x = 0.5$, $y = -(-0.5)^2 - (-0.5) + 3 = 3.25$

Maximum value is 3.25.

iii



iv $y = f(x)$ is 0 at $x \approx -2.3$ and at $x \approx 1.3$.

Question 11

a $a = 1$, $b = -2$, $c = 5$

Minimum at $x = -\dfrac{b}{2a} = -\dfrac{-2}{2\times 1} = 1$

When $x = 1$, $y = 1^2 - 2 \times 1 + 5 = 4$

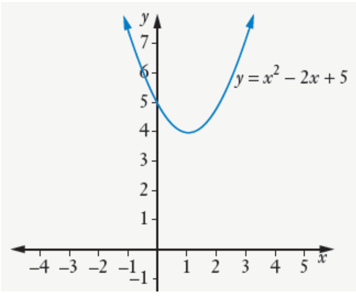
Minimum value is 4.

b $x = \dfrac{-(-2)\pm\sqrt{(-2)^2 - 4\times 1\times 5}}{2\times 1} = \dfrac{1\pm\sqrt{-16}}{2}$

There are no x-intercepts because $\sqrt{-16}$ cannot be evaluated.

Hence, there are no solutions to the quadratic equation.

c



Question 12

a $a = -2$, $b = 1$, $c = -4$

Maximum at $x = -\frac{b}{2a} = -\frac{1}{2\times -2} = \frac{1}{4} = 0.25$

When $x = 0.25$, $f(x) = -2(0.25)^2 + 0.25 - 4 = -3.875$

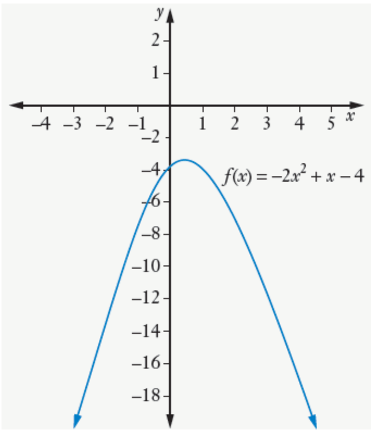
Maximum value is -3.875.

b $x = \frac{-1 \pm \sqrt{1^2 - 4\times(-2)\times(-4)}}{2\times(-2)} = \frac{-1\pm\sqrt{-31}}{-4}$

There are no x-intercepts because $\sqrt{-31}$ cannot be evaluated.

Hence, there are no solutions to the quadratic equation.

c



Question 13

For a function $f(x)$ to be even, $f(-x) = f(x)$

For $f(x) = -x^2$,

$f(-x) = -(-x)^2 = -x^2 = f(x)$

Hence, $f(x)$ is even.

Question 14

a Let $y = f(x)$

$f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$

The function is even.

b $f(-x) = (-x)^2 - 3 = x^2 - 3 = f(x)$

The function is even.

c Let $y = f(x)$

$f(-x) = -2(-x)^2 = -2x^2 = f(x)$

The function is even.

d $f(-x) = (-x)^2 - 3(-x) = x^2 + 3x \neq f(x)$

The function is not even.

e Let $y = f(x)$

$f(-x) = (-x)^2 + (-x) = x^2 - x \neq f(x)$

The function is not even.

f Let $y = f(x)$

$f(-x) = (-x)^2 - 4 = x^2 - 4 = f(x)$

The function is even.

g Let $y = f(x)$

$f(-x) = (-x)^2 - 2(-x) - 3 = x^2 + 2x - 3 \neq f(x)$

The function is not even.

h Let $y = f(x)$

$$\begin{aligned} f(-x) &= (-x)^2 - 5(-x) + 4 \\ &= x^2 + 5x + 4 \\ &\neq f(x) \end{aligned}$$

The function is not even.

i $p(-x) = (-x + 1)^2 \neq p(x)$

The function is not even.

Question 15

a Let $d = f(w)$

$$f(-w) = -\frac{(-w)^2}{800} + 200 = -\frac{w^2}{800} + 200 = f(w)$$

The function is even.

b The depth is the y-coordinate of the turning point.

The function $d = f(w)$ is symmetrical about the d-axis, so the maximum occurs at $w = 0$.

When $w = 0$, $d = -\frac{0^2}{800} + 200 = 200$.

The depth is 200 m.

c The width of the span is the length between the two w-intercepts.

Let $d = 0$ so that $-\frac{w^2}{800} + 200 = 0$

$w^2 = 160\ 000 \Rightarrow w = \pm 400$

The two w-intercepts are –400 and 400.

The distance between them is 400 – (–400) = 800 metres.

d Use $w = 400 - 10 = 390$. Then the depth is

$$-\frac{390^2}{800} + 200 = .875 \quad \text{metres.}$$

e Solve for w using $d = 100$

$$-\frac{w^2}{800} + 200 = 100$$

$$-\frac{w^2}{800} = -100$$

$w^2 = 80\ 000$

$w = 282.84$

Span is $2 \times 282.84 = 565.69$ metres

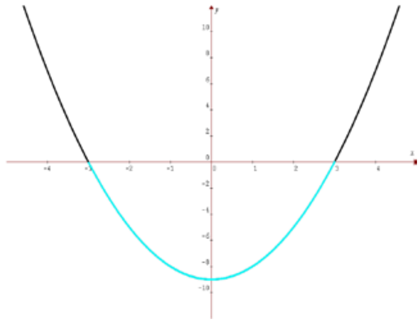
Question 1

Draw $y = x^2 - 9$

Require values of x that produce y values above the x-axis.

The x-intercepts are –3, 3.

$x < -3$ and $x > 3$



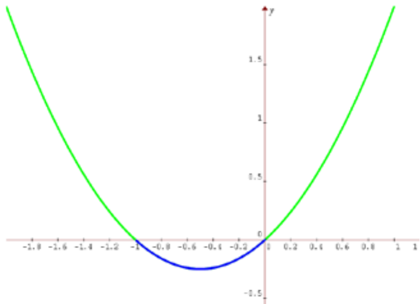
Question 2

Draw $y = n^2 + n$

Require values of n that produce y values on or below the n-axis.

The n-intercepts are 0, –1.

$-1 \leq n \leq 0$



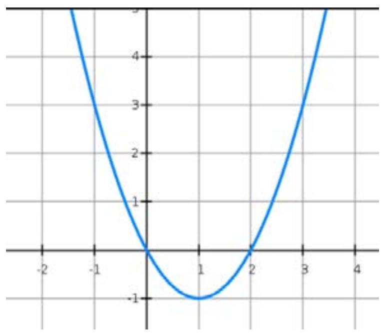
Question 3

Draw $y = a^2 - 2a$

Require values of a that produce y values on or above the a-axis.

The a-intercepts are 0, 2.

$a \leq 0, a \geq 2$



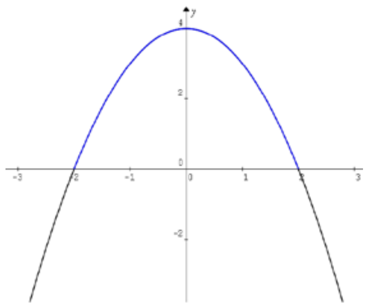
Question 4

Draw $y = 4 - x^2$

Require values of x that produce y values below the x-axis.

The x-intercepts are –2, 2.

$x < -2, x > 2$



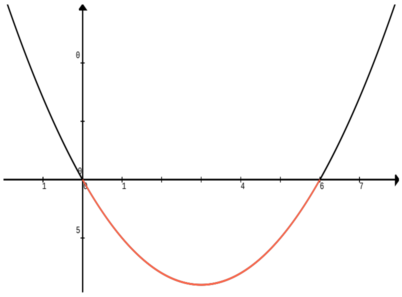
Question 5

Draw $t = y^2 - 6y$

Require values of y that produce t values on or below the y-axis.

The y-intercepts are 0, 6.

$0 \le y \le 6$



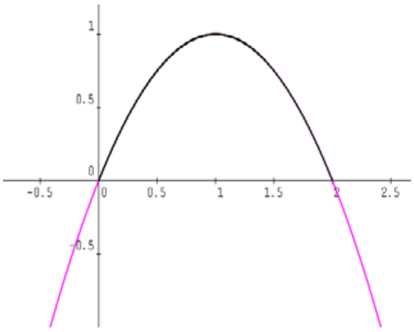
Question 6

Draw $y = 2t - t^2$

Require values of t that produce y values above the t-axis.

The t-intercepts are 0, 2.

$0 < t < 2$



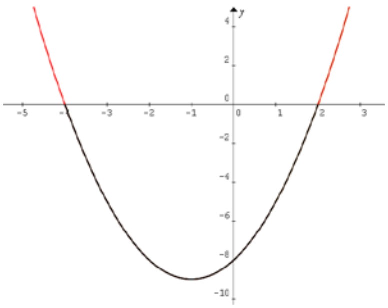
Question 7

Draw $y = x^2 + 2x - 8 = (x + 4)(x - 2)$

Require values of x that produce y values above the x-axis.

The x-intercepts are 2, –4.

$x < -4,\ x > 2$



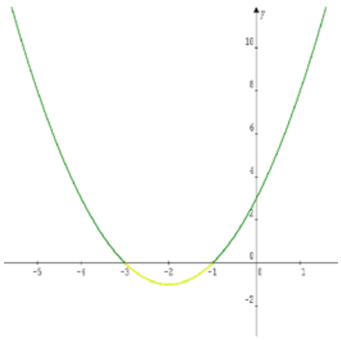
Question 8

Draw $y = p^2 + 4p + 3 = (p + 3)(p + 1)$

Require values of p that produce y values on or above the p-axis.

The p-intercepts are –3, –1.

$p \le -3,\ p \ge -1$



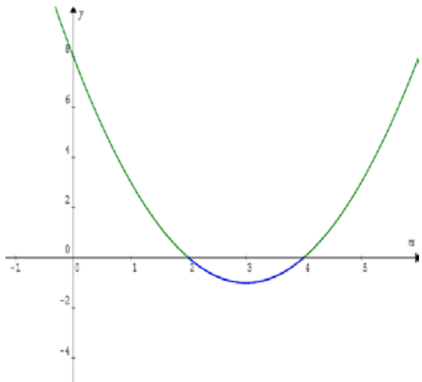
Question 9

Draw $y = m^2 - 6m + 8 = (m-2)(m-4)$

Require values of m that produce y values above the m-axis.

The m-intercepts are 2, 4.

$m < 2, m > 4$



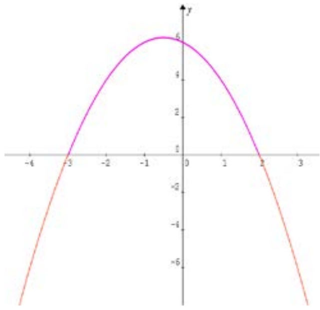
Question 10

Draw $y = 6 - x - x^2 = -(x+3)(x-2)$

Require values of x that produce y values on or below the x-axis.

The x-intercepts are –3, 2.

$x \leq -3, x \geq 2$



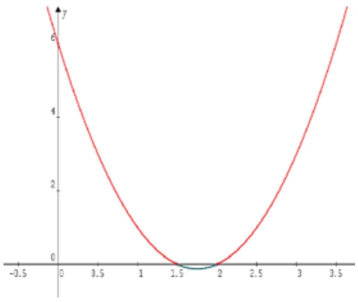
Question 11

Draw $y = 2h^2 - 7h + 6 = (2h-3)(h-2)$

Require values of h that produce y values below the h-axis.

The h-intercepts are 1.5, 2.

$1.5 < h < 2$



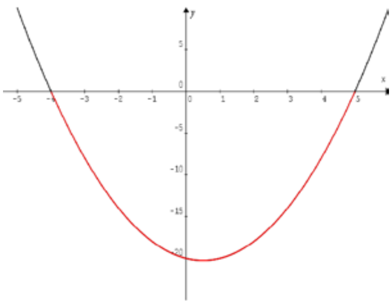
Question 12

Draw $y = x^2 - x - 20 = (x+4)(x-5)$

Require values of x that produce y values on or below the x-axis.

The x-intercepts are –4, 5.

$-4 \leq x \leq 5$



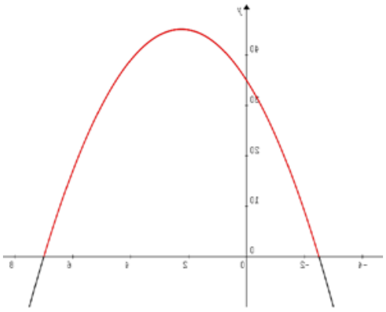
Question 13

Draw $y = 35 + 9k - 2k^2 = -(2k + 5)(k - 7)$

Require values of k that produce y values on or above the k-axis.

The k-intercepts are –2.5, 7.

$-2.5 \leq k \leq 7$



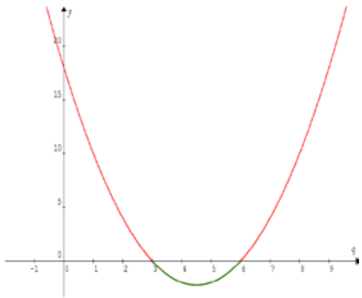
Question 14

Draw $y = q^2 - 9q + 18 = (q - 6)(q - 3)$

Require values of q that produce y values above the q-axis.

The q-intercepts are 3, 6.

$q < 3, q > 6$



Question 15

Let $y = (x + 2)^2$

For all values of x, $y \geq 0$

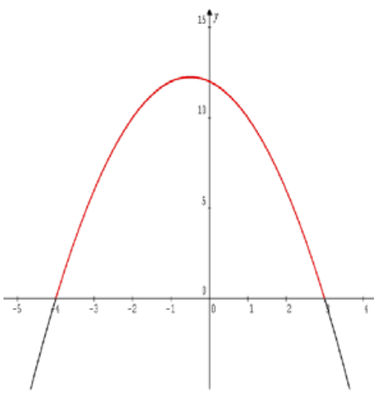
Question 16

Draw $y = 12 - n - n^2 = -(n + 4)(n - 3)$

Require values of n that produce y values on or below the n-axis.

The n-intercepts are –4, 3.

$n \leq -4, \geq 3$



Question 17

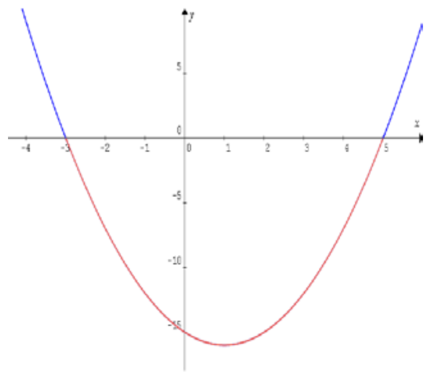
$x^2 - 2x < 15 \Rightarrow x^2 - 2x - 15 < 0$

Draw $y = x^2 - 2x - 15 = (x - 5)(x + 3)$

Require values of x that produce y values below the x-axis.

The n-intercepts are –3, 5.

$-3 < x < 5$



Question 18

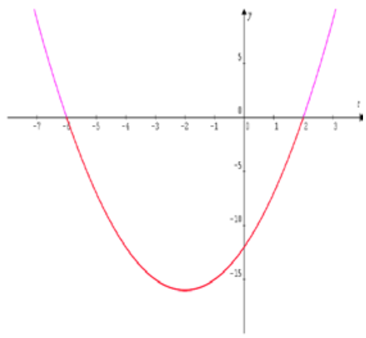
$-t^2 \geq 4t - 12 \Rightarrow t^2 + 4t - 12 \leq 0$

Draw $y = t^2 + 4t - 12 = (t - 2)(t + 6)$

Require values of t that produce y values on or below the t-axis.

The t-intercepts are –6, 2.

$-6 \leq t \leq 2$



Question 19

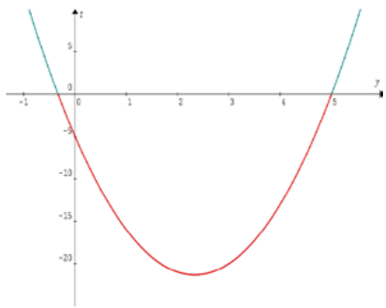
$3y^2 > 14y + 5 \Rightarrow 3y^2 - 14y - 5 > 0$

Draw $t = 3y^2 - 14y - 5 = (3y + 1)(y - 5)$

Require values of y that produce t values on or below the y-axis.

The y-intercepts are $-\frac{1}{3}$, 5.

$y < -\frac{1}{3},\ y > 5$



Question 20

$(x - 3)(x + 1) = x^2 - 2x - 3$

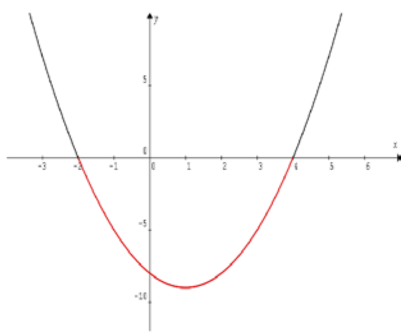
Require $x^2 - 2x - 3 \geq 5$ or $x^2 - 2x - 8 \geq 0$

Draw $y = x^2 - 2x - 8 = (x - 4)(x + 2)$

Require values of x that produce y values on or above the x-axis.

The x-intercepts are –2, 4.

$x \leq -2,\ x \geq 4$



Exercise 4.11 The discriminant

Question 1

a $a = 1, b = -4, c = -1$

$\Delta = b^2 - 4ac$

$= (-4)^2 - 4 \times 1 \times (-1)$

$= 20$

b $a = 2, b = 3, c = 7$

$\Delta = b^2 - 4ac$

$= 3^2 - 4 \times 2 \times 7$

$= -47$

c $a = -4, b = 2, c = -1$

$\Delta = b^2 - 4ac$

$= (2)^2 - 4 \times (-4) \times (-1)$

$= -12$

d $a = 6, b = -1, c = -2$

$\Delta = b^2 - 4ac$

$= (-1)^2 - 4 \times 6 \times (-2)$

$= 49$

e $a = -1, b = -3, c = 0$

$\Delta = b^2 - 4ac$

$= (-3)^2 - 4 \times (-1) \times 0$

$= 9$

f $a = 1, b = 0, c = 4$

$\Delta = b^2 - 4ac$

$= 0^2 - 4 \times 1 \times 4$

$= -16$

g $a = 1, b = -2, c = 1$

$\Delta = b^2 - 4ac$

$= (-2)^2 - 4 \times 1 \times 1$

$= 0$

h $a = -3, b = -2, c = 5$

$\Delta = b^2 - 4ac$

$= (-2)^2 - 4 \times (-3) \times 5$

$= 64$

i $a = -2, b = 1, c = 2$

$\Delta = b^2 - 4ac$

$= 1^2 - 4 \times (-2) \times 2$

$= 17$

Question 2

a $a = 1, b = -1, c = 4$

$\Delta = b^2 - 4ac$

$= (-1)^2 - 4 \times 1 \times -4$

$= 17$

$\Delta > 0$ and it is not a perfect square, so there are two unequal real irrational roots.

b $a = 2, b = 3, c = 6$

$\Delta = b^2 - 4ac$

$= 3^2 - 4 \times 2 \times 6$

$= -39$

$\Delta < 0$, so there are no real roots.

c $a = 1, b = -9, c = 20$

$\Delta = b^2 - 4ac$

$= (-9)^2 - 4 \times 1 \times 20$

$= 1$

$\Delta > 0$ and it is a perfect square, so there are two unequal real rational roots.

d $a = 1, b = 6, c = 9$

$\Delta = b^2 - 4ac$

$= 6^2 - 4 \times 1 \times 9$

$= 0$

$\Delta = 0$, so there are two equal real rational roots.

e $a = 2, b = -5, c = -1$

$\Delta = b^2 - 4ac$

$= (-5)^2 - 4 \times 2 \times -1$

$= 33$

$\Delta > 0$ and it is not a perfect square, so there are two unequal real irrational roots.

f $a = -1, b = 2, c = -5$

$\Delta = b^2 - 4ac$

$= 2^2 - 4 \times -1 \times -5$

$= -16$

$\Delta < 0$, so there are no real roots.

g $a = -2, b = -5, c = 3$

$\Delta = b^2 - 4ac$

$= (-5)^2 - 4 \times -2 \times 3$

$= 49$

$\Delta > 0$ and it is a perfect square, so there are two unequal real rational roots.

h $a = -5, b = 2, c = -6$

$\Delta = b^2 - 4ac$

$= 2^2 - 4 \times -5 \times -6$

$= -116$

$\Delta < 0$, so there are no real roots.

i $a = -1, b = 1, c = 0$

$\Delta = b^2 - 4ac$

$= 1^2 - 4 \times -1 \times 0$

$= 1$

$\Delta = 0$ and it is a perfect square, so there are two unequal real rational roots.

Question 3

$a = 1, b = 2, c = p$

$\Delta = b^2 - 4ac$

$\quad = 2^2 - 4 \times 1 \times p$

$\quad = 4 - 4p$

For equal roots, $\Delta = 0$

$4 - 4p = 0 \Rightarrow p = 1$

Question 4

$a = 1, b = k, c = 1$

$\Delta = b^2 - 4ac$

$\quad = k^2 - 4 \times 1 \times 1$

$\quad = k^2 - 4$

For equal roots, $\Delta = 0$

$k^2 - 4 = 0 \Rightarrow k = \pm 2$

Question 5

$a = 2, b = 1, c = b + 1$

$\Delta = b^2 - 4ac$

$\quad = 1^2 - 4 \times 2 \times (b + 1)$

$\quad = -(8b + 7)$

For real roots, $\Delta \geq 0$

$-(8b + 7) \geq 0 \Rightarrow b \leq -\frac{7}{8}$

Question 6

$a = p, b = 4, c = 2$

$\Delta = b^2 - 4ac$

$\quad = 4^2 - 4 \times p \times 2$

$\quad = 16 - 8p$

For no real roots, $\Delta < 0$

$16 - 8p < 0 \Rightarrow p > 2$

Question 7

$a = k + 2, b = 1, c = -3$

$\Delta = b^2 - 4ac$

$\quad = 1^2 - 4 \times (k + 2) \times -3$

$\quad = 12k + 25$

For two real unequal roots, $\Delta > 0$

$12k + 25 > 0 \Rightarrow k > -2\frac{1}{12}$

Question 8

$a = 3, b = -1, c = 7$

$\Delta = b^2 - 4ac$

$\quad = (-1)^2 - 4 \times 3 \times 7$

$\quad = -83$

$\Delta < 0$, so there are no x-intercepts, so the entire graph lies above the y-axis.

That is, $3x^2 - x + 7 > 0$ for all values of x.

Question 9

Find any points of intersection by solving the two equations simultaneously.

$y = 2x + 6 \qquad [1]$

$y = x^2 + 3 \qquad [2]$

Substitute [2] in [1]:

$x^2 + 3 = 2x + 6$

$x^2 - 2x - 3 = 0$

$b^2 - 4ac = (-2)^2 - 4(1)(-3)$

$\quad = 16 > 0$

So there are 2 points of intersection.

Question 10

$3x + y - 4 = 0$ [1]

$y = x^2 + 5x + 3$ [2]

From [1]:

$y = -3x + 4$ [3]

Substitute [2] in [3]:

$x^2 + 5x + 3 = -3x + 4$

$x^2 + 8x - 1 = 0$

$b^2 - 4ac = 8^2 - 4(1)(-1) = 68 > 0$

So there are 2 points of intersection.

Question 11

$y = - x - 4$ [1]

$y = x^2$ [2]

Substitute [2] in [1]:

$x^2 = - x - 4$

$x^2 + x + 4 = 0$

$b^2 - 4ac = 12 - 4(1)(4) = -15 < 0$

So there are no points of intersection.

Question 12

$y = 5x - 2$ [1]

$y = x^2 + 3x - 1$ [2]

Substitute [2] in [1]:

$x^2 + 3x - 1 = 5x - 2$

$x^2 - 2x + 1 = 0$

$b^3 - 4ac = (-2)^2 - 4(1)(1) = 0$

So there is 1 point of intersection.

$\therefore$ the line is a tangent to the parabola.

Question 13

$a = 1$, $b = k + 1$, $c = 4$

$\Delta = b^2 - 4ac$

$\quad = (k + 1)^2 - 4 \times 1 \times 4$

$\quad = (k + 1)^2 - 16$

For real roots, $\Delta \geq 0$

$(k + 1)^2 - 16 \geq 0$

$(k + 1)^2 \geq 16$

$k + 1 \geq 4 \Rightarrow k \geq 3$

$k + 1 \leq -4 \Rightarrow k \leq -5$

Question 14

$a = k$, $b = 3k$, $c = 9$

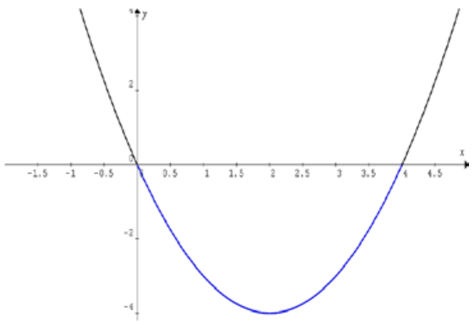
$\Delta = b^2 - 4ac$

$\quad = (3k)^2 - 4 \times k \times 9$

$\quad = 9k^2 - 36k$

The function is always positive, which means there are no x-intercepts.

Hence we require $\Delta < 0$.



$9k^2 - 36k < 0$

$k^2 - 4k < 0$

$k(k - 4) < 0$

$0 < k < 4$

Question 15

$m < -3$, $m > 3$

$a = 1$, $b = -2m$, $c = 9$

$\Delta = b^2 - 4a$

$\quad = (-2m)^2 - 4 \times 1 \times 9$

$\quad = 4m^2 - 36$

We require $\Delta > 0$

$4m^2 - 36 > 0$

$4m^2 > 9$

$m < -3$, $m > 3$

Exercise 4.12 Finding a quadratic equation

Question 1

a Substitute for x and d to find k.

$40 = k \times 100^2$

$k = \dfrac{40}{100^2} = 0.004$

b $x = 80$

$d = 0.004x^2 = 0.004 \times 80^2 = 25.6$

The braking distance is 25.6 m.

c $x = 50$

$d = 0.004x^2 = 0.004 \times 50^2 = 10$

The car will require 10 m to stop, which is less than the distance to the dog.

Hence, the car will not hit the dog.

d $x = 110$

$d = 0.004x^2 = 0.004 \times 110^2 = 48.4$

The car will require 48.4 m to stop, which is more than the distance to the dog.

Hence, the car will hit the dog.

Question 2

a $A = kx^2$, where k is the proportionality constant.

$x = 5$, $A = 125$

$125 = k \times 5^2 \Rightarrow k = 5$

Hence, $A = 5x^2$

b When $x = 4.2$,

$A = 5 \times 4.2^2 = 88.2$

The area is 88.2 cm^2.

c When $A = 250$,

$250 = 5 \times x^2$

$x^2 = 50$

$x = 7.1$

The length is 7.1 cm.

Question 3

a Substitute for h in $V = \pi r^2 h$

$V = 8\pi r^2$

b When $r = 5$,

$V = 8\pi \times 5^2 = 200\pi \approx 628.3$

The volume is 628.3 cm^3

c When $V = 100$,

$100 = 8\pi \times r^2$

$r = \sqrt{\dfrac{100}{8\pi}} \approx 2$

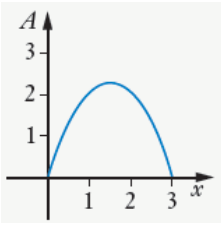
The radius is approximately 2 cm.

Question 4

a Area = length × width

$A = x(3 - x) = 3x - x^2$

b



c $A = x(3 - x)$

x-intercepts at 0, 3.
Maximum turning point at

$x = \frac{1}{2}(0+3) = 1.5$

d When $x = 1.5$, the maximum value of A is $3x - x^2 = 3 \times 1.5 - 1.5^2$
$= 2.25$ units

Question 5

a $y = ax^2 + bx + c$

Substitute (0, –5)	$c = -5$	[1]
Substitute (2, –3)	$4a + 2b + c = -3$	
	$4a + 2b - 5 = -3 \Rightarrow 4a + 2b = 2$	[2]
Substitute (–3, 7)	$9a - 3b + c = 7$	
	$9a - 3b - 5 = 7 \Rightarrow 9a - 3b = 12$	[3]
3 × [2]	$12a + 6b = 6$	[4]
2 × [3]	$18a - 6b = 24$	[5]
[4] + [5]	$30a = 30$	
	$a = 1$	

Substitute $a = 1$ in [2].

$4 \times 1 + 2b = 2$

$b = -1$

The equation of the parabola is $y = x^2 - x - 5$.

b $y = ax^2 + bx + c$

Substitute (1, –2)	$a + b + c = -2$	[1]
Substitute (3, 0)	$9a + 3b + c = 0$	[2]
Substitute (–2, 10)	$4a - 2b + c = 10$	[3]
[2] – [1]	$8a + 2b = 2 \Rightarrow 4a + b = 1$	[4]
[3] – [1]	$3a - 3b = 12 \Rightarrow a - b = 4$	[5]
[4] + [5]	$5a = 5$ $a = 1$	

Substitute $a = 1$ in [4].

$4 \times 1 + b = 1$

$b = -3$

The equation of the parabola is $y = x^2 - 3x$.

c $y = ax^2 + bx + c$

Substitute (–2, 21)	$4a - 2b + c = 21$	[1]
Substitute (1, 6)	$a + b + c = 6$	[2]
Substitute (–1, 12)	$a - b + c = 12$	[3]
[1] – [2]	$3a - 3b = 15 \Rightarrow a - b = 5$	[4]
[1] – [3]	$3a - b = 9$	[5]
[5] – [4]	$2a = 4$	
	$a = 2$	

Substitute $a = 2$ in [4].

$2 - b = 5$

$b = -3$

Substitute $a = 2$, $b = -3$ in [2].

$2 - 3 + c = 6$

$c = 7$

The equation of the parabola is $y = 2x^2 - 3x + 7$.

d $y = ax^2 + bx + c$

Substitute (2, 3)	$4a + 2b + c = 3$	[1]
Substitute (1, –4)	$a + b + c = -4$	[2]
Substitute (–1, –12)	$a - b + c = -12$	[3]
[1] – [2]	$3a + b = 7$	[4]
[1] – [3]	$3a + 3b = 15 \Rightarrow a + b = 5$	[5]
[4] – [5]	$2a = 2$	
	$a = 1$	

Substitute $a = 1$ in [4].

$3 + b = 7$

$b = 4$

Substitute $a = 1$, $b = 4$ in [2].

$1 + 4 + c = -4$

$c = -9$

The equation of the parabola is $y = x^2 + 4x - 9$.

e $y = ax^2 + bx + c$

Substitute (0, 1)	$c = 1$	[1]
Substitute (–2, 1)	$4a - 2b + c = 1$	
Using [1]:	$c = 1 \Rightarrow 2a - b = 0$	[2]
Substitute (2, –7)	$4a + 2b + c = -7$	
Using [1]:	$c = 1 \Rightarrow 2a + b = -4$	[3]
[2] + [3]	$4a = -4$	
	$a = -1$	

Substitute $a = -1$ in [2].

$-2 - b = 0$

$b = -2$

The equation of the parabola is $y = -x^2 - 2x + 1$.

Question 6

a Let $h = at^2 + bt = c$

When $t = 0$, $h = 10$, so $c = 10$.

Hence $h = at^2 + bt + 10$

When $t = 1$, $h = 22.5$

$a + b + 10 = 22.5 \Rightarrow a + b = 12.5$ [1]

When $t = 4$, $h = 0$

$16a + 4b + 10 = 0 \Rightarrow 16a + 4b = -10$ [2]

[2] – 4 × [1] $\Rightarrow 12a = -60$

$\Rightarrow a = -5$

Substitute $\Rightarrow a = -5$ in [1].

$\Rightarrow -5 + b = 12.5$

$b = 17.5$

The equation for the height of the ball is $h = -5t + 17.5t + 10$.

b When $t = 2$, $h = -5 \times 2^2 + 17.5 \times 2 + 10 = 25$

The height of the ball after 2 seconds is 25 m.

c The ball is in line with the cliff when $h = 10$.

$-5t^2 + 17.5t + 10 = 10$

$-5t^2 + 17.5t = 0$

$t(t - 3.5) = 0$

Solving for t gives

$t = 0$ and $t = 3.5$

$t = 0$ is the initial position of the ball.

So the required time is 3.5 seconds.

Question 7

a Let $y = ax^2 + bx + c$

When $x = 0$, $y = 8$, so $c = 8$.

Hence $y = ax^2 + bx$.

When $x = 7.5$, $y = 0$

$7.5^2a + 7.5b + 8 = 0 \Rightarrow 56.25a + 7.5b = -8$ [1]

When $x = -7.5$, $y = 0$

$7.5^2a - 7.5b + 8 = 0 \Rightarrow 56.25a - 7.5b = -8$ [2]

$[1] + [2] \Rightarrow 112.5a = -16$

$$a = -\frac{16}{112.5} = -\frac{32}{225}$$

Substitute $a = -\frac{32}{225}$ in [1] to get $b = 0$.

The equation of the parabola is $y = -\frac{32}{225}x^2 + 8$.

b $x = 3$, $y = -\frac{32}{225} \times 3^2 + 8 = 6.72$

The depth is 6.72 cm.

c $y = 5$, $-\frac{32}{225}x^2 + 8 = 5$

$x^2 = 21.09375 \Rightarrow x \approx 4.59$

The depth is approximately 4.59 cm

Question 8

a $y = ax^2 + bx + c$

Substitute (0, 0) $c = 0 \Rightarrow y = ax^2 + bx$

Substitute (3, –3) $9a + 3b = -3$ [1]

Substitute (–1, 5) $a - b = 5$ [2]

[1] + 3 × [2] $12a = 12$

$a = 1$

Substitute $a = 1$ in [2].

$1 - b = 5$

$b = -4$

The equation of the parabola is $y = x^2 - 4x$.

b **i** $x = 5, y = 5^2 - 4 \times 5 = 5$

ii $x = -4, y = (-4)^2 - 4 \times (-4) = 32$

c $y = -4, x^2 - 4x = -4$

$x^2 - 4x + 4 = 0$

$(x - 2)^2 = 0 \Rightarrow x = 2$

d $y = -4, x^2 - 4x = 2$

$x^2 - 4x - 2 = 0$

Solve using quadratic formula.

$a = 1, b = -4, c = -2$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times (-2)}}{2 \times 1}$$
$$= \frac{4 \pm \sqrt{24}}{2}$$
$$= \frac{4 \pm 2\sqrt{6}}{2}$$
$$= 2 \pm \sqrt{6}$$

Question 9

a $f(x) = ax^2 + bx + c$

Substitute (0, 7)	$c = 7 \Rightarrow f(x) = ax^2 + bx + 7$	
Substitute (1, 10)	$a + b + 7 = 10 \Rightarrow a + b = 3$	[1]
Substitute (–1, 6)	$a - b + 7 = 6 \Rightarrow a - b = -1$	[2]

[1] + [2] $\quad 2a = 4$

$a = 2$

Substitute $a = 1$ in [1].

$1 + b = 3$

$b = 2$

The equation of the quadratic function is $f(x) = x^2 + 2x + 7$.

b $f(-5) = 5^2 + 2 \times (-5) + 7 = 22$

c $a = 1 > 0$

$\Delta = b^2 - 4ac$

$a = 1, b = 2, c = 7$

$\Delta = 2^2 - 4 \times 1 \times 7 = -24$

$\Delta < 0$, which means there are no x-intercepts. Hence $f(x) > 0$ for all values of x.

Question 10

$y = ax^2 + bx + c$

(0, 0) means $c = 0$, hence $y = ax^2 + bx$.

Axis of symmetry at $x = 1$ means x-intercepts are 0 and 2.

At $x = 1, y = -2$, so $a + b = -2$ [1]

At $x = 2, y = 0$, so $4a + 2b = 0 \Rightarrow 2a + b = 0$ [2]

[2] – [1] $\Rightarrow a = 2$

Substitute $a = 2$ in [1].

$2 + b = -2 \Rightarrow b = -4$

The equation of the parabola is $y = 2x^2 - 4x$.

Question 11

$y = ax^2 + bx + c$

(0, 4) means $c = 4$, hence $y = ax^2 + bx + 4$

Axis of symmetry at $x = 3$ means x-intercepts are 0 and 6.

At $x = 3$, $y = 13$, so $9a + 3b + 4 = 13 \Rightarrow 3a + b = 3$ [1]

At $x = 6$, $y = 0$, so $36a + 6b = 0 \Rightarrow 6a + b = 0$ [2]

[2] – [1] $\Rightarrow$ $3a = -3$

$a = -1$

Substitute $a = -1$ in [1].

$-3 + b = 3$

$b = 6$

The equation of the parabola is $y = -x^2 + 6x + 4$.

Exercise 4.13 Cubic functions

Question 1

a $y = 0, x^3 - 1 = 0 \Rightarrow x = 1$

$x = 0, y = 0^3 - 1 = -1$

x-intercept is 1, y-intercept is –1

b $y = 0, -x^3 + 8 = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$

$x = 0, y = -0^3 + 8 = 8$

x-intercept is 2, y-intercept is 8

c $y = 0, (x + 5)^3 = 0 \Rightarrow x = -5$

$x = 0, y = (0 + 5)^3 = 125$

x-intercept is –5, y-intercept is 125

d $f(x) = 0, -(x - 4)^3 = 0 \Rightarrow x = 4$

$x = 0, f(0) = -(0 - 4)^3 = 64$

x-intercept is –4, y-intercept is 64

e $f(x) = 0, 3(x + 7)^3 - 3 = 0 \Rightarrow (x + 7)^3 = 1 \Rightarrow x + 7 = 1 \Rightarrow x = -6$

$x = 0, f(0) = 3(0 + 7)^3 - 3 = 1026$

x-intercept is –6, y-intercept is 1026

f $y = 0, (x - 2)(x - 1)(x + 5) = 0 \Rightarrow x = 2, x = 1, x = -5$

$x = 0, y = (0 - 2)(0 - 1)(0 + 5) = 10$

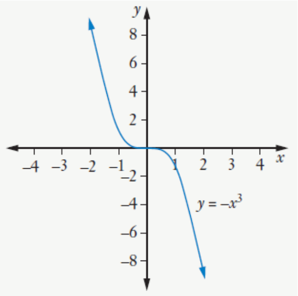
x-intercept is 2, 1, –5, y-intercept is 10

Question 2

a x-intercept is 0, y-intercept is 0

The graph is a reflection of $y = x^3$ about the x-axis.

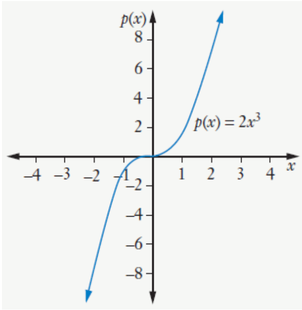
There is a point of inflection at (0, 0).



b x-intercept is 0, y-intercept is 0

The graph is a dilation from the x-axis of the graph of $y = x^3$ by a factor of 2.

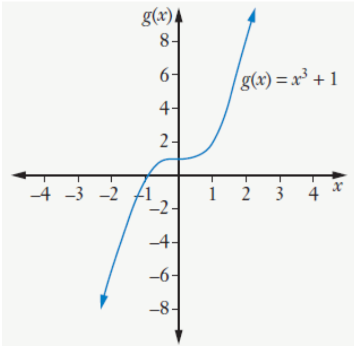
There is a point of inflection at (0, 0).



c x-intercept is –1, y-intercept is 1

The graph is a vertically upward translation by 1 unit of the graph of $y = x^3$.

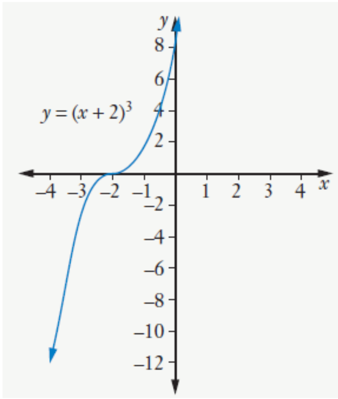
There is a point of inflection at (0, 1).



d x-intercept is –2, y-intercept is 8

The graph is a horizontal translation of the graph of $y = x^3$ by 2 units left.

There is a point of inflection at (–2, 0).



e

$$-(x-3)^3 + 1 = 0$$

$$(x-3)^3 = 1$$

$$x - 3 = 1$$

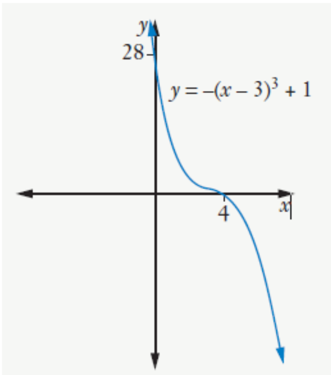
$$x = 4$$

x-intercept is 4.

The y-intercept is $y = -(0 - 3)^3 + 1 = 28$

The graph is a horizontal translation of the graph of $y = x^3$ by 3 units right, followed by a reflection about the x-axis and then a vertical translation up of 1 unit.

There is a point of inflection at (4, 0).

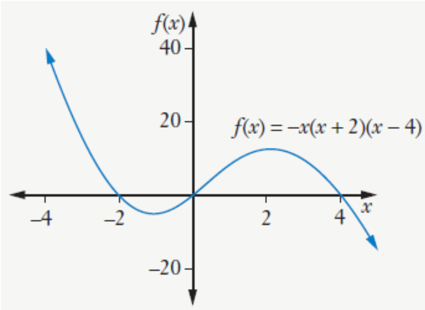


f x-intercepts at –2, 0, 4

The y-intercept is 0.

For $x < -2$ and for $0 < x < 4$, $f(x) > 0$

For $-2 < x < 0$ and for $x > 4$, $f(x) < 0$

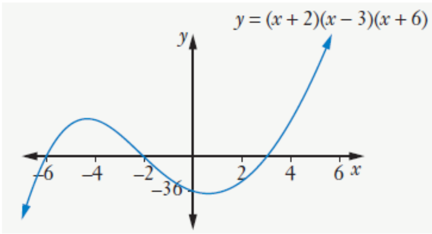


g x-intercepts at –2, 3, –6

The y-intercept is $2 \times (-3) \times 6 = -36$

For $x < -6$ and for $-2 < x < 3$, $f(x) < 0$

For $-6 < x < -2$ and for $x > 3$, $f(x) > 0$



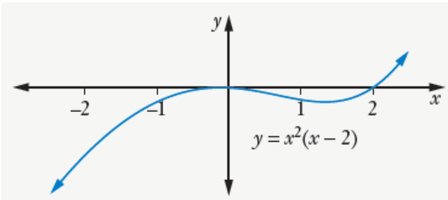
h x-intercepts at 0, 2

0 is a repeated intercept, hence the curve touches the x-axis at that point.

The y-intercept is 0.

For $x > 6$, $f(x) > 0$

For $x \leq 6$, $f(x) \leq 0$



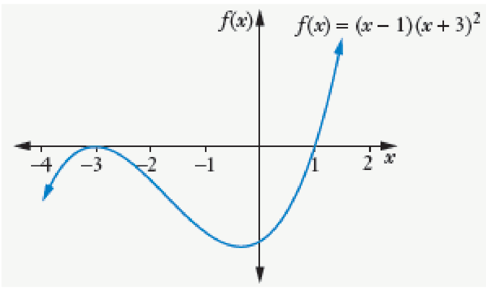
i x-intercept at 1, –3

–3 is a repeated intercept, hence the curve touches the x-axis at that point.

The y-intercept is $-1 \times 3^2 = -9$.

For $x > 1$, $f(x) > 0$

For $x \leq 1$, $f(x) \leq 0$



Question 3

a Comparing $y = 8x^3 + 1$ with $y = k(x - b)^3 + c$, the point of inflection is at (b, c), which is $(0, 1)$.

b Comparing $y = -x^3 + 27$ with $y = k(x - b)^3 + c$, the point of inflection is at (b, c), which is $(0, 27)$.

c Comparing $f(x) = (x + 2)^3$ with $y = k(x - b)^3 + c$, the point of inflection is at (b, c), which is $(-2, 0)$.

d Comparing $y = 2(x - 1)^3 - 16$ with $y = k(x - b)^3 + c$, the point of inflection is at (b, c), which is $(1, -16)$.

e Comparing $f(x) = -(x + 1)^3 + 1$ with $y = k(x - b) + c$, the point of inflection is at (b, c), which is $(-1, 1)$.

Question 4

a $2x^3 - 5 = 0$

$2x^3 = 5$

$x^3 = 2.5$

$x = \sqrt[3]{2.5} = 1.3572 \approx 1.4$

b $(x - 1)^3 + 2 = 0$

$(x - 1)^3 = -2$

$x - 1 = \sqrt[3]{-2}$

$x = \sqrt[3]{-2} + 1 = -0.2599 \approx -0.3$

c $-3x^3 + 1 = 0$

$-3x^3 = -1$

$x^3 = \frac{1}{3}$

$x = \sqrt[3]{\frac{1}{3}} = 0.6933 \approx 0.7$

d $2(x + 3)^3 - 3 = 0$

$2(x + 3)^3 = 3$

$(x + 3)^3 = 1.5$

$x + 3 = \sqrt[3]{1.5}$

$x = \sqrt[3]{1.5} - 3 = -1.8552 \approx -1.9$

e $-3(2x - 1)^3 + 2 = 0$

$-3(2x - 1)^3 = -2$

$(2x - 1)^3 = \frac{2}{3}$

$2x - 1 = \sqrt[3]{\frac{2}{3}}$

$2x = \sqrt[3]{\frac{2}{3}} + 1$

$x = \frac{\sqrt[3]{\frac{2}{3}} + 1}{2} = 0.9367 \approx 0.9$

Question 5

a The graph is a vertical translation down of $y = x^3$ by 64 units.

It is an increasing curve, with x-intercept at 4, and y-intercept at –64.

There is a point of inflection at (0, –64).

b The graph is a horizontal translation to the right of $y = x^3$ by 3 units.

This is followed by a reflection about the x-axis.

It is a decreasing curve, with x-intercept 3 and y-intercept at 27.

There is a point of inflection at (3, 0).

c The x-intercepts are 0, –2, –4. The y-intercept is 0.

From the negative side of the x-axis, the graph is increasing to a maximum turning point between $x = –4$ and –2.

It then decreases to a minimum turning point between $x = –2$ and 0, then increases on the positive side of the x-axis.

d The x-intercepts are –3, –1, 4. The y-intercept is 24.

From the negative side of the x-axis, the graph is decreasing to a minimum turning point between $x = –3$ and –1, then increasing to a maximum turning point between $x = –1$ and 4. It then decreases on the positive side of the x-axis.

e The x-intercepts are –5, 0. The y-intercept is 0.

From the negative side of the x-axis, the graph is increasing to a maximum turning point at $x = –5$.

It then decreases to a minimum turning point between $x = –5$ and 0, then increases on the positive side of the x-axis.

Question 6

a $x = 1.7$

b $x = –1.3$

c $x = 1.7$

d $x = –1$.

e $x = –0.8$

f $x = –2, 0, 1$

Question 7

a $120 = k \times 3.5^3 \Rightarrow k \approx 0.8$

The equation is $V = 2.8x^3$.

b $V = 2.8 \times 6^3 = 604.5$

The volume is 604.5 cm^3.

c 250 = 2.8x3

$$x = \left(\frac{250}{28}\right)^{\frac{1}{3}} \approx 4.5$$

The length is 4.5 cm.

Question 8

a $v = 4.2r^3$

$v = kr^3$

$7238 = k \times 12^3$

$$k = \frac{7238}{12^3} \approx 4.2$$

The equation is $v = 4.2r^3$.

b $v = 4.2 \times 2.5^3 \simeq 65.4$

The volume is 65.4 mm^3.

c $7000 = 4.2r^3$

$$r = \left(\frac{7000}{42}\right)^{\frac{1}{3}} \approx 11.9$$

The radius is 11.9 mm.

Question 9

$f(-x) = -(-x)^3 = -(-x^3) = -f(x)$

So $f(x)$ is an odd function.

Question 10

a $y = f(-x) = 3(-x)^3 = -3x^3 = -f(x)$

y is an odd function.

b $y = f(-x) = (-x + 1)^3 \neq -f(x)$

y is not an odd function.

c $f(-x) = -2(-x)^3 - 1 = 2x^3 - 1 \neq -f(x)$

$f(x)$ is not an odd function.

d $y = f(-x) = -5(-x)^3 = 5x^3 = -f(x)$

y is an odd function.

e $y = f(-x) = (-x - 2)^3 + 3 \neq -f(x)$

y is not an odd function.

Therefore, **a** and **d** are odd functions.

Question 11

a $y = kx^3 + c$

(1, 2) means $k = 2$ because it is the dilation factor in the y-direction.

Substituting (1, 2) in $y = kx^3 + c$ gives $2 = k + c \Rightarrow c = 0$.

The equation is $y = 2x^3$.

b Substituting (0, 5) gives $c = 5$.

Substituting (2, –3) gives

$8k + c = -3$

$8k + 5 = -3$

$k = -1$

The equation is $y = -x^3 + 5$.

c

Substituting (1, 4) gives	$k + c = -4$	[1]
Substituting (–2, 23) gives	$-8k + c = 23$	[2]
[1] – [2] $\Rightarrow$	$9k = -27 \Rightarrow k = -3$	
Substitute $k = -3$ in [1].	$-3 + c = -4 \Rightarrow c = -1$	

The equation is $y = -3x^3 - 1$.

d Substituting (1, –2) gives $k + c = -2$ [1]

Substituting (2, 33) gives $8k + c = 33$ [2]

[2] – [1] $\Rightarrow$ $7k = 35 \Rightarrow k = 5$

Substitute $k = 5$ in [1]. $5 + c = -2 \Rightarrow c = -7$

The equation is $y = 5x^3 - 7$.

e Substituting (2, –29) gives $8k + c = -29$ [1]

Substituting (–3, 111) gives $-27k + c = 111$ [2]

[1] – [2] $\Rightarrow$ $35k = -140 \Rightarrow k = -4$

Substitute $k = -4$ in [1]. $-32 + c = -29 \Rightarrow c = 3$

The equation is $y = -4x^3 + 3$.

Question 12

a x-intercepts are 2, 3, –5

$y = k\,(x - 2)(x - 3)(x + 5)$

Substitute (–2, –120)

$k(-2 - 2)(-2 - 3)(-2 + 5) = -120$

$60k = -120$

$k = -2$

The equation is $y = -2(x - 2)(x - 3)(x + 5)$.

b x-intercepts are –1, 4, 6

$y = k\,(x + 1)(x - 4)(x - 6)$

Substitute (3, 96)

$k(3 + 1)(3 - 4)(3 - 6) = 96$

$12k = 96$

$k = 8$

The equation is $y = 8\,(x + 1)(x - 4)(x - 6)$.

c x-intercepts are 1, 3 and $k = -3$

$y = -3(x - a)(x - 1)(x - 3)$

Substitute (0, –27)

$-3(-a)(-1)(-3) = -27$

$9a = -27$

$a = -3$

The equation is $y = -3(x + 3)(x - 1)(x - 3)$.

Exercise 4.14 Polynomial functions

Question 1

The degree is the highest power of x.

a 7

b 4

c 1

d 11

e 3

f 0 ($3 = 3x^0$)

Question 2

a $P(2) = 2^3 - 7 \times 2^2 + 2 - 1 = -19$

b $P(-1) = (-1)^3 - 7 \times (-1)^2 + (-1) - 1 = -10$

c $P(0) = 0^3 - 7 \times 0^2 + 0 - 1 = -1$

Question 3

a $P(-11) = -11 + 5 = -6$

b $Q(3) = 2 \times 3 - 1 = 5$

c $P(2) + Q(-2) = (2 + 5) + [2 \times (-2) - 1] = 2$

d $P(x) + Q(x) = 3x + 4$

Power of x is 1.

e $P(x) \times Q(x) = 2x^2 + 9x - 5$

Highest power of x is 2.

Question 4

a The highest power of x is 5, so the degree is 5.

b The constant term does not contain a power of x. The constant term is 4.

c The coefficient of a power includes the sign of the coefficient.

Hence, the coefficient is –3.

d 0, since we can include the term $0x^2$.

Question 5

a $P(x) = (x - 3)(x + 3)$

$x = \pm 3$

b $x + 5 = 0 \Rightarrow x = -5$

c $P(x) = (x + 2)(x - 1)$

$x = -2, 1$

d $P(x) = (x - 4)^2$

$x = 4$

e $P(x) = x^3 - 2x^2 + 5x = x\,(x^2 - 2x + 5)$

The discriminant of $x^2 - 2x + 5$ is -16, so $x^2 - 2x + 5$ has no solutions.

Hence the zeros of $P(x)$ are 0.

Question 6

The following are not polynomials.

a $\frac{1}{x} = x^{-1}$ has a negative power.

b 3^x is not allowed. The power must be 0 or a positive whole value.

f x^{-2} has a negative power.

Question 7

a $a + 1 = 1 \Rightarrow a = 0$

b $b - 7 = 3 \Rightarrow b = 10$

c $c + 5 = -1 \Rightarrow c = -6$

d $a + 1 = 0 \Rightarrow a = -1$

e $a + 1 = 5 \Rightarrow a = 4$

Question 8

a $2x + 5 = 0 \Rightarrow x = -2\frac{1}{2}$

b $x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0 \Rightarrow x = -1, 2$

c $P(x)$ has degree 1 and $R(x)$ has degree 3.

Hence the degree of $P(x) + R(x)$ is 3, being the higher of these two degrees.

d $P(x)$ has degree 1 and $Q(x)$ has degree 2.

The leading term of $P(x) \cdot Q(x)$ will be of degree $1 + 2 = 3$.

e The leading term of $Q(x)$ is x^2 and the leading term of $R(x)$ is x^3.

Hence, the leading term of $Q(x) \cdot R(x)$ will be $x^2 \times x^3 = x^5$.

Question 9

a $\Delta = b^2 - 4ac = -8$
$-8 < 0, \therefore f(x)$ has no zeros.

b $3x^2 \times 3x = 9x^3$

c $1 + (-3) = -2$

d The terms of $f(x) \cdot g(x)$ that contain x^1 are $-2x \times (-3) = 6x$ and $1 \times 3x = 3x$.
The sum is $6x + 3x = 9x$, so the coefficient is 9.

e $f(x) + g(x) = 3x^2 + x - 2 = (3x - 2)(x + 1)$
$3x - 2 = 0 \Rightarrow x = \frac{2}{3}$
$x + 1 = 0 \Rightarrow x = -1$

Question 10

a $P(x) = (x - 3)(x + 3)$; 2 roots

b The discriminant is less than 0; 0 roots.

c $\Delta = (-3)^2 - 4 \times 1 \times (-7) = 37$
The discriminant is positive, so 2 roots.

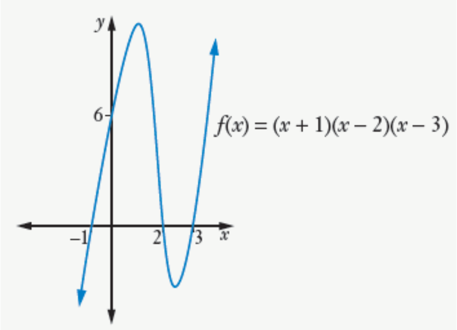
d $\Delta = 1^2 - 4 \times 2 \times 3 = -23$
The discriminant is negative; 0 roots.

e $\Delta = (-5)^2 - 4 \times 3 \times (-2) = 49$
The discriminant is positive, so 2 roots.

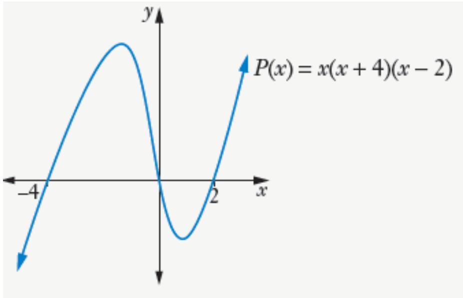
f $x = 0$; $\quad x - 1 = 0 \Rightarrow x = 1$
$x + 4 = 0 \Rightarrow x = -4 \quad x + 6 = 0 \Rightarrow x = -6$
4 roots

Question 11

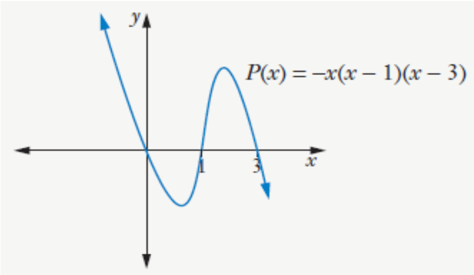
a x-intercepts –1, 2, 3; y-intercept 6



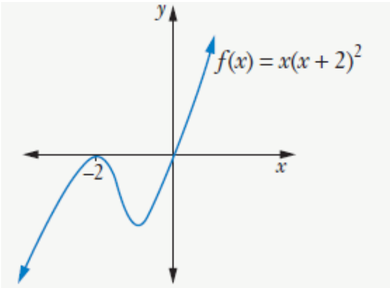
b x-intercepts –4, 0, 2; y-intercept 0



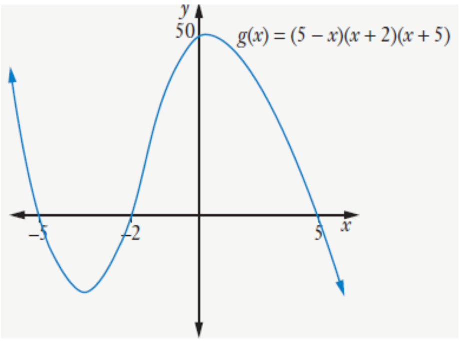
c x-intercepts 0, 1, 3; y-intercept 0



d x-intercepts –2, 0; y-intercept 0



e x-intercepts –5, –2, 5; y-intercept 50

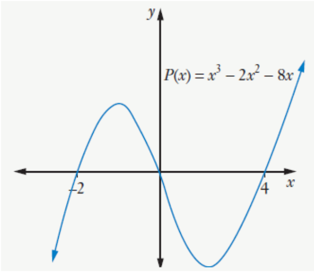


Question 12

a **i** $P(x) = x(x^2 - 2x = 8)$

$= x(x - 4)(x + 2)$

ii

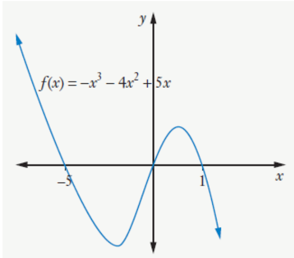


Increases to maximum turning point, then decreases to minimum turning point, then increases

b **i** $f(x) = -x(x^2 + 4x - 5)$

$= -x(x + 5)(x - 1)$

ii

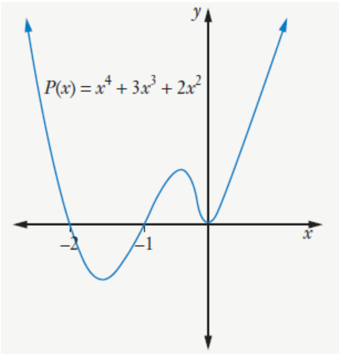


Decreases to minimum turning point, then increases to maximum turning point, then decreases.

c **i** $P(x) = x^2 (x^2 + 3x + 2)$

$= x^2 (x + 1)(x + 2)$

ii



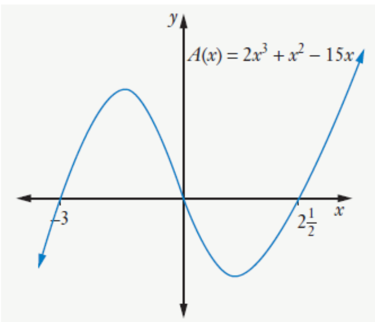
Decreases to minimum turning point, then increases to maximum turning point, then decreases to minimum turning point, then increases.

Increases to maximum turning point, then decreases to minimum turning point, then increases.

d **i** $A(x) = x(2x^2 + x - 15)$

$= x(2x - 5)(x + 3)$

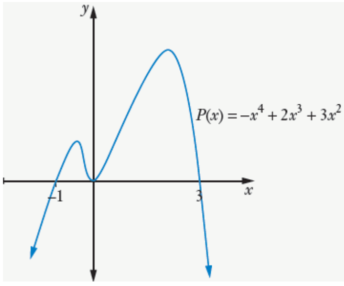
ii



e **i** $P(x) = -x^2\,(x^2 - 2x - 3)$

$= -x^2\,(x - 3)(x + 1)$

ii



Increases to maximum turning point, then decreases to minimum turning point, then increases to maximum point, then decrease

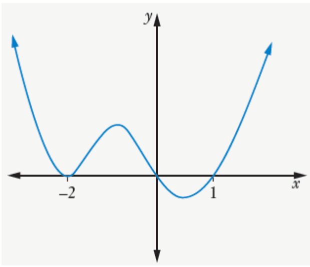
Question 13

a $x = 0$

$x - 1 = 0 \Rightarrow x = 1$

$x + 2 = 0 \Rightarrow x = -2$

b y-intercept is 0

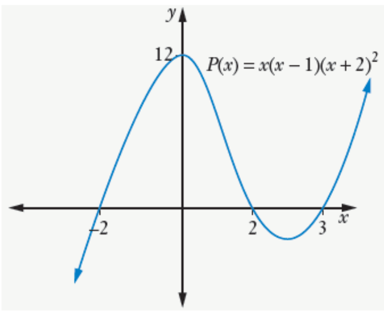


Question 14

a $(x - 3)(x - 2)(x + 2) = (x - 3)(x^2 - 4)$

$= x^3 - 4x - 3x^2 + 12$

b x-intercepts –2, 2, 3; y-intercept 12

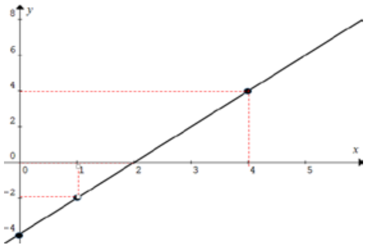


Question 1

a **i** $x = 2$

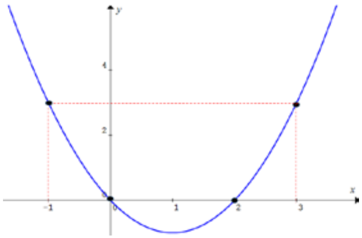
ii $x = 1$

iii $x = 4$



b **i** $x = 0, 2$

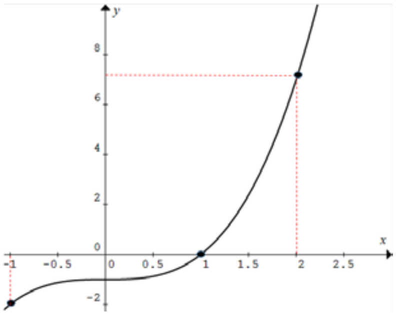
ii $x = 3, -1$



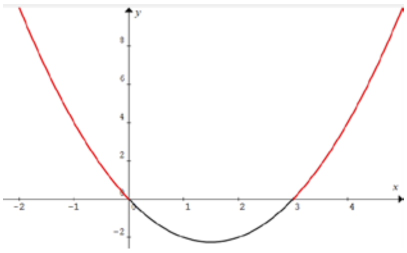
c **i** $x = 1$

ii $x = 2$

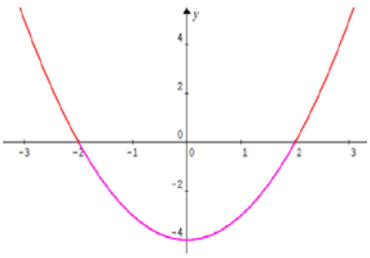
iii $x = -1$



d **i** $x < 0, x > 3$

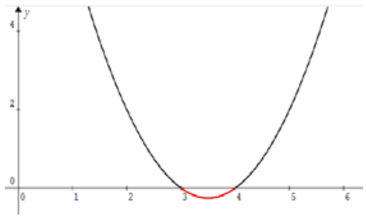


ii $-2 \le x \le 2$



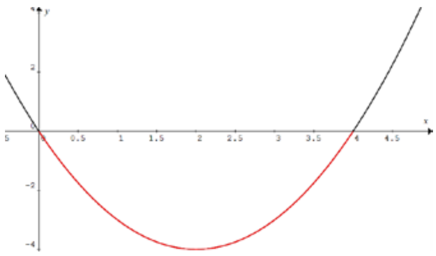
iii $x < 3, x > 4$

Write the inequality as
$x^2 - 7x + 12 > 0$



iv $0 < m < 4$

Write the inequality as
$m^2 - 4m < 0$



Question 2

a $y = x + 3$ [1]

$y = 2x + 2$ [2]

Substitute [2] in [1].

$2x + 2 = x + 3$

$x = 1$

Substitute $x = 1$ in [1].

$y = 1 + 3$

$y = 4$

The point of intersection is (1, 4).

b $y = 3x - 1$ [1]

$y = 5x + 1$ [2]

Substitute [2] in [1].

$5x + 1 = 3x - 1$

$2x = -2$

$x = -1$

Substitute $x = -1$ in [1].

$y = 3 \times (-1) - 1$

$y = -4$

The point of intersection is (–1, –4).

c $x + 2y - 4 = 0$ [1]

$2x - y = 0$ [2]

2 × [1] − [2].

$5y - 10 = 0 \Rightarrow y = 2$

Substitute $y = 2$ in [1].

$x + 2 \times 2 - 4 = 0 \Rightarrow x = 0$

The point of intersection is (0, 2).

d $3x + y - 2 = 0$ [1]

$2x - 3y - 5 = 0$ [2]

3 × [1] + [2].

$11x - 11 = 0 \Rightarrow x = 1$

Substitute $x = 1$ in [1].

$3 \times 1 + y - 2 = 0 \Rightarrow y = -1$

The point of intersection is (1, –1).

e $4x - 3y - 5 = 0$ [1]

$7x - 2y - 12 = 0$ [2]

3 × [2] − 2 × [1].

$13x - 26 = 0 \Rightarrow x = 2$

Substitute $x = 2$ in [1].

$4 \times 2 - 3y - 5 = 0 \Rightarrow y = 1$

The point of intersection is (2, 1).

Question 3

a $y = x^2$ [1]

$y = x$ [2]

Substitute [1] in [2].

$x^2 = x$

$x(x - 1) = 0$

$x = 0, x = 1$

Substitute $x = 0$ in [1] or [2] to get $y = 0$.

Substitute $x = 1$ in [1] or [2] to get $y = 1$.

The points of intersection are (0, 0), (1, 1).

b $y = x^2$ [1]

$y = 4$ [2]

Substitute [1] in [2].

$x^2 = 4 \Rightarrow x = \pm 2$

The points of intersection are (–2, 4), (2, 4).

c $y = x^2$ [1]

$y = x + 2$ [2]

Substitute [1] in [2].

$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$x = -1, x = 2$

Substitute $x = -1$ in [1] or [2] to get $y = 1$.

Substitute $x = 2$ in [1] or [2] to get $y = 4$.

The points of intersection are (–1, 1), (2, 4).

d $y = x^2$ [1]

$y = -2x + 3$ [2]

Substitute [1] in [2].

$x^2 = -2x + 3$

$x^2 + 2x - 3 = 0$

$(x + 3)(x - 1) = 0$

$x = -3, x = 1$

Substitute $x = 1$ in [1] or [2] to get $y = 1$.

Substitute $x = -3$ in [1] or [2] to get $y = 9$.

The points of intersection are (–3, 9), (1, 1).

e $y = x^2 - 5$ [1]

$y = 4x$ [2]

Substitute [1] in [2].

$x^2 - 5 = 4x$

$x^2 - 4x - 5 = 0$

$(x - 5)(x + 1) = 0$

$x = -1, x = 5$

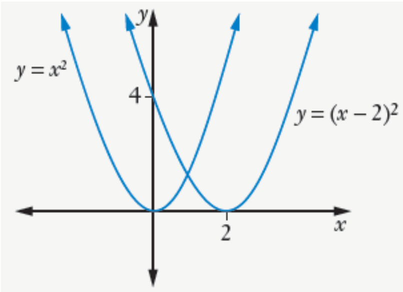
Substitute $x = -1$ in [1] or [2] to get $y = -4$.

Substitute $x = 5$ in [1] or [2] to get $y = 20$.

The points of intersection are (–1, –4), (5, 20).

Question 4

a



b 1

c

$f(x) = x^2$ [1]

$f(x) = (x-2)^2$ [2]

Substitute [2] in [1].

$(x-2)^2 = x^2$

$x - 2 = \pm x$

$x - 2 = x$, not possible

$x - 2 = -x \Rightarrow x = 1$

Substitute $x = 1$ in [1] or [2] to get $f(1) = 1$.

The point of intersection is (1, 1).

Question 5

$f(x) = x^2$ [1]

$f(x) = (x+2)^2$ [2]

Substitute [2] in [1].

$(x+2)^2 = x^2$

$x + 2 = \pm x$

$x + 2 = x$, not possible

$x + 2 = -x \Rightarrow x = -1$

Substitute $x = -1$ in [1] or [2] to get $f(1) = 1$.

The point of intersection is (–1, 1).

Question 6

$y = x^2 - 5$ [1]

$y = 2x^2 + 5x + 1$ [2]

Substitute [2] in [1].

$2x^2 + 5x + 1 = x^2 - 5$

$x^2 + 5x + 6 = 0$

$(x+3)(x+2) = 0$

$x = -3,\ x = -2$

Substitute $x = -3$ in [1] to get $y = (-3)^2 - 5 = 4$

Substitute $x = -2$ in [1] to get $y = (-2)^2 - 5 = -1$

The points of intersection are (–3, 4), (–2, –1).

Question 7

$y = 3x^2 - 4x - 4$ [1]

$y = 5x^2 - 2$ [2]

Substitute [2] in [1].

$5x^2 - 2 = 3x^2 - 4x - 4$

$2x^2 + 4x + 2 = 0$

$x^2 + 2x + 1 = 0$

$(x + 1)(x + 1) = 0$

$x = -1$

Substitute $x = -1$ in [2] to get $y = 5 \times (-1)^2 - 2 = 3$.

The points of intersection are (–1, 3).

Question 8

a $y = 10x$ [1]

$y = 3x + 980$ [2]

Substitute [1] in [2].

$10x = 3x + 980$

$7x = 980$

$x = 140$

Substitute $x = 140$ in [1] to get $y = 10 \times 140 = 1400$.

The breakeven point is to sell 140 roses at a cost of $1400.

b $x = 189$

Income: $x = 10 \times 189 = 1890$

Costs: $x = 3 \times 189 + 980 = 1547$

Profit is income less costs = 1890 – 1547 = $343

c $x = 45$

Income: $x = 10 \times 45 = 450$

Costs: $x = 3 \times 45 + 980 = 1115$

Income less costs = 450 – 1115 = –665

The loss is $665.

Question 9

Let x be the number of calculators.

The cost function is $y = 3x + 852$.

The income is $y = 15x$.

To break even,

$15x = 3x + 852$

$12x = 852$

$x = 71$

71 calculators must be sold to break even.

Question 10

a Let x be the number of cupcakes.

The cost function is $y = 1x + 264$.

The income is $y = 5x$.

b To break even,

$5x = 1x + 264$

$4x = 264$

$x = 66$

66 cupcakes must be sold daily to break even.

c $x = 250$

Income: $x = 5 \times 250 = 1250$

Costs: $x = 250 + 264 = 514$

Profit is income less costs = 1250 – 514 = $736

d $x = 50$

Income $x = 5 \times 50 = 250$

Costs $x = 50 + 264 = 314$

Income less costs = 250 – 314 = –64

The loss is $64.

Question 11

a Let x, y be the length and perimeter respectively.

Then $y = kx$, where k is the constant of proportionality.

Substitute $x = 5$, $y = 90$.

$90 = k \times 5 \Rightarrow k = 18$

The equation is $y = 18x$.

b $108 = k \times 3^2 \Rightarrow k = 12$

$x = 3$, $y = 108$

The equation is $y = 12x^2$.

c $12x^2 = 18x$

$2x^2 - 3x = 0$

$x(2x - 3) = 0$

$x = 0$, $x = 1\frac{1}{2}$

A side can't have zero length, so the required length is $1\frac{1}{2}$.

Question 1

Monic means the coefficient of the leading term is 1.

The number with no x variable assigned to it is 5.

C

Question 2

$$-\frac{b}{2a}=-\frac{2}{2\times -1}=1$$

The equation is $x = 1$

$x = 1$, $f(1) = 1 + 2 \times 1 - 1^2 = 2$

The turning point is at (1, 2).

A

Question 3

x-intercept $y = 0$, $2x - 6 = 0 \Rightarrow x = 3$

y-intercept $x = 0$, $-3y - 6 = 0 \Rightarrow y = -2$

B

Question 4

The equation is a vertical line passing through $x = -2$.

B

Question 5

The x-intercepts are –1, 0, 2. The y-intercept is 0.

The graph is positive in $0 < x < 2$ [Try $x = 1$ in each option to check].

The graph is negative in $-1 < x < 0$ [Try $x = -\frac{1}{2}$ in each option to check].

D

Question 6

a $f(-2) = (-2)^2 - 3(-2) - 4 = 6$

b $f(a) = a^2 - 3a - 4$

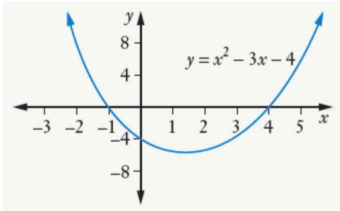
c $x^2 - 3x - 4 = 0$

$(x + 1)(x - 4) = 0$

$x = -1, 4$

Question 7

a Equation of axis of symmetry



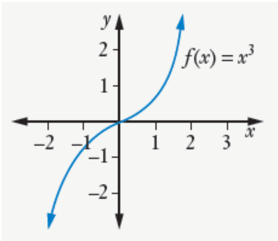
$$-\frac{b}{2a} = -\frac{-3}{2\times1} = \frac{3}{2} = 1.5$$

$x = 1.5$

y-coordinate of turning point at $x = 1.5$ is $(1.5)^2 - 3 \times 1.5 - 4 = -6.25$

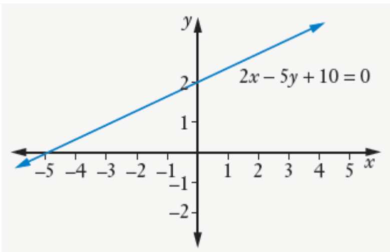
Domain: $(-\infty, \infty)$; Range: $\left[-6\frac{1}{4}, \infty\right)$

b



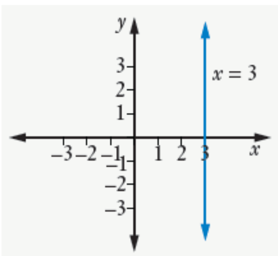
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

c



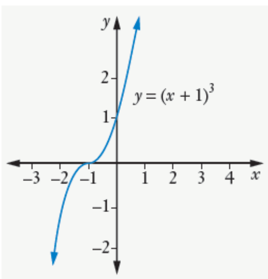
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

d



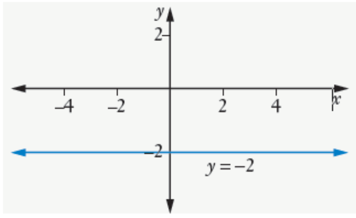
Domain: [3]; Range: $(-\infty, \infty)$

e



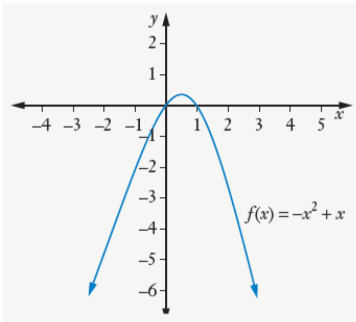
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

f



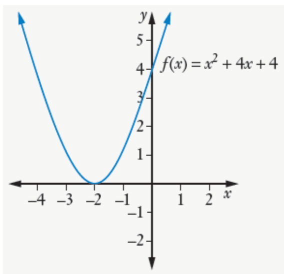
Domain: $(-\infty, \infty)$; Range [–2]

g



Domain: $(-\infty, \infty)$; Range: $(-\infty, 0.25]$

h



Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

Question 8

a $f(2) = 3 \times 2 - 4 = 2$

b $3x - 4 = 7 \Rightarrow x = 3\frac{2}{3}$

c $3x - 4 = 0 \Rightarrow x = 1\frac{1}{3}$

Question 9

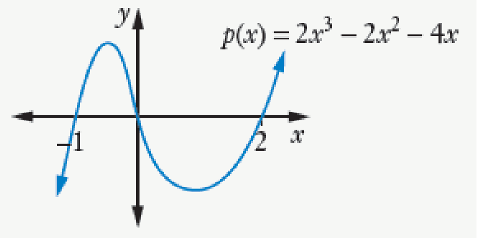
$P(x) = 2x(x^2 - x - 2)$

$\quad = 2x(x - 2)(x + 1)$

x-intercepts at $-1, 0, 2$

$x = 0, y = 0$

y-intercept at 0



Question 10

a $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-1)}{-2-3} = -\frac{6}{5} = -1\frac{1}{5}$

b Write the equation in the form $y = mx + c$

$y = 2x + 1$

The gradient is 2.

c Write the equation in the form $y = m_1x + c$.

$y = -\frac{5}{3}x + \frac{8}{3}$

$m_1 = -\frac{5}{3}$

$m_1 \times m_2 = -1$

$-\frac{5}{3} \times m_2 = -1 \Rightarrow m_2 = \frac{3}{5}$

d $m = \tan 45° = 1$

Question 11

a $-\frac{b}{2a} = -\frac{-4}{2\times1} = 2$

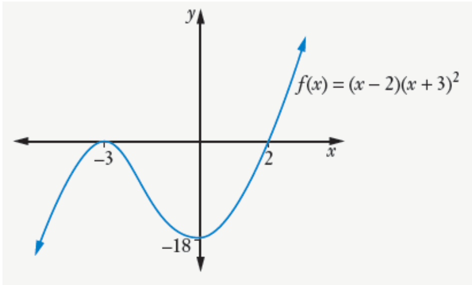
b When $x = 2$, $y = 2^2 - 4 \times 2 + 1 = -3$

The minimum value is –3.

Question 12

x-intercepts: 2, –3 (repeated root, graph touches the *x*-axis at this point)

y-intercept: $x = -2 \times 3^2 = -18$



Question 13

a The degree is the highest power of *x*, namely 3.

b –3

c $P(x) = x(x^2 + 2x - 3)$

$= x(x + 3)(x - 1)$

$x = 0, –3, 1$

d The leading term is the term with the highest power of *x*, namely x^3.

Question 14

a x-intercept: $y = 0, 2x + 20 = 0 \Rightarrow x = -10$

y-intercept: $x = 0, -5y + 20 = 0 \Rightarrow y = 4$

b x-intercept: $y = 0$

$x^2 - 5x - 14 = 0$

$(x - 7)(x + 2) = 0$

$x = -2, x = 7$

y-intercept: $x = 0, y = -14$

c x-intercept: $y = 0$

$(x + 2)^3 = 0 \Rightarrow x = -2$

y-intercept: $x = 0, y = 2^3 = 8$

d x-intercept: $y = 0, 2x - 10 = 0 \Rightarrow x = 5$

y-intercept: $x = 0, -5y - 10 = 0 \Rightarrow y = -2$

Question 15

$y = 2x + 3$ [1]

$x - 5y + 6 = 0$ [2]

Substitute [1] in [2].

$x - 5(2x + 3) + 6 = 0$

$x - 10x - 15 + 6 = 0$

$-9x - 9 = 0$

$x = -1$

Substitute $x = -1$ in [1].

$y = 2 \times 1 + 3 = 5$

The point of intersection is (–1, 1).

Question 16

a $-\frac{b}{2a} = -\frac{-1}{2\times -2} = -\frac{1}{4}$

The equation of the axis of symmetry is $x = -\frac{1}{4}$.

b $x = -\frac{1}{4}$

$$
\begin{aligned}
y &= -2\left(-\frac{1}{4}\right)^2 - \left(-\frac{1}{4}\right) + 6 \\
&= -\frac{1}{8} + \frac{1}{6} + 6 \\
&= 6\frac{1}{8}
\end{aligned}
$$

Question 17

From Question 16, y has a maximum value of $6\frac{1}{8}$.

Domain: $(-\infty\ -\infty)$, Range: $\left(-\infty, 6\frac{1}{8}\right]$

Question 18

a D

$\Delta = (-1)^2 - 4 \times 2 \times 3 = -23$

$\Delta < 0$

b B

$\Delta = (-10)^2 - 4 \times 1 \times (-25) = 200$

$\Delta > 0$

c C

$\Delta = (-10)^2 - 4 \times 1 \times 25 = 0$

$\Delta = 0$

d B

$\Delta = 7^2 - 4 \times 3 \times (-2) = 73$

$\Delta > 0.$

e A

$\Delta = (-1)^2 - 4 \times 6 \times (-2) = 49$

$\Delta > 0$ and is a square number.

Question 19

a $m = 7$

$y - 3 = 7(x - 2)$

$y - 3 = 7x - 14$

$7x - y - 11 = 0$

b $5x + y - 3 = 0 \Rightarrow y = -5x + 3$

$m = -5$

$y - 1 = -5(x - 1)$

$y - 1 = -5x + 5$

$5x + y - 6 = 0$

c $2x - 3y + 6 = 0 \Rightarrow y = \frac{2}{3}x + 2$

$m_1 = \frac{2}{3} \quad m_2 = -\frac{3}{2}$

$y - 0 = -\frac{3}{2}(x - 0)$

$y = -\frac{3}{2}x$

$3x + 2y = 0$

d $m = \frac{4-1}{-2-3} = -\frac{3}{5}$

Use (3, 1)

$y - 1 = -\frac{3}{5}(x - 3)$

$5y - 5 = -3x + 9$

$3x + 5y - 14 = 0$

e $m = \frac{1}{3}$

Use (3, 0)

$y - 0 = \frac{1}{3}(x - 3)$

$3y = x - 3$

$x - 3y - 3 = 0$

Question 20

$a = 2$, $b = –18$, $c = 40$

$f(4) = 0 \Rightarrow$	$16a + 4b + c = 0$	[1]
$f(5) = 0 \Rightarrow$	$25a + 5b + c = 0$	[2]
$f(-1) = 60 \Rightarrow$	$a - b + c = 60$	[3]
[2] – [1]	$9a + b = 0$	[4]
[1] – [3]	$15a + 5b = -60 \Rightarrow 3a + b = -12$	[5]
[4] – [5]	$6a = 12 \Rightarrow a = 12$	
From [4],	$9 \times 2 + b = 0 \Rightarrow b = -18$	
From [3],	$2 - (-18) + c = 60 \Rightarrow c = 40$	

Question 21

a $y = f(x)$

$f(-x) = (-x)^2 - 1 = x^2 - 1 = f(x)$

Even

b $y = f(x)$

$f(-x) = -x + 1$

Neither even nor odd

c $y = f(x)$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

Odd

d $y = f(x)$

$f(-x) = (-x + 1)^2$

Neither even nor odd

e $y = f(x)$

$f(-x) = -5(-x)^3 = -5 \times (-x^3) = -(-5x^3) = -f(x)$

Odd

Question 22

$f(x) = x^3 - x$

$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x)$

Odd

Question 23

Gradient connecting points

$$m_1 = \frac{3-4}{3-(-1)} = -\frac{1}{4}$$

$4x - y - 6 = 0 \Rightarrow y = 4x - 6$

Gradient is $m_2 = 4$.

$m_1 \times m_2 = -\frac{1}{4} \times 4 = -1$

$\therefore$ lines are perpendicular.

Question 24

$a = -1 < 0$

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 3^2 - 4 \times (-1) \times (-4) \\ &= -7 \\ &< 0\end{aligned}$$

$\therefore -4 + 3x - x^2 < 0$ for all x.

Question 25

a $xy = 7 \Rightarrow y = \dfrac{7}{x}$ [1]

$3x - 5y - 2 = 0$ [2]

Substitute [1] into [2].

$3x - 5\left(\dfrac{7}{x}\right) - 1 = 0$

$3x^2 - x - 35 = 0$

$\Delta = (-1)^2 - 4 \times 3 \times -35 = 4321 > 0$

$\Delta > 0$, hence there are 2 points of intersection.

b $y = 3x - 3$ [1]

$x^2 + y^2 = 9$ [2]

Substitute [1] into [2].

$x^2 + (3x - 3)^2 = 9$

$10x^2 - 18x = 0$

$2x(5x - 9) = 0$

$x = 0$, $x = 2\dfrac{4}{5}$

There are 2 points of intersection.

c $x - 2y - 3 = 0 \Rightarrow y = \dfrac{1}{2}x - \dfrac{3}{2}$ [1]

$x^2 + y^2 = 1$ [2]

Substitute [1] into [2].

$x^2 + \left(\dfrac{1}{2}x - \dfrac{3}{2}\right)^2 = 1$

$x^2 + \dfrac{1}{4}x^2 - \dfrac{3}{2}x + \dfrac{9}{4} = 1$

$5x^2 - 6x + 5 = 0$

$\Delta = (-6)^2 - 4 \times 5 \times -5 = -64 < 0$

$\Delta < 0$, hence there are no points of intersection.

d $y = x^2$ [1]

$y = 4x - 4$ [2]

Substitute [1] into [2].

$x^2 = 4x - 4$

$x^2 - 4x + 4 = 0$

$(x - 2)^2 = 0$

$x = 2$

There is one point of intersection.

e $y = \frac{2}{x}$ [1]

$y = 3x + 1$ [2]

Substitute [1] into [2].

$\frac{2}{x} = 3x + 1$

$3x^2 + x - 2 = 0$

$(3x - 2)(x + 1) = 0$

$x = -1, x = \frac{2}{3}$

There are two points of intersection.

Question 26

The gradient of $y = 5x - 7$ is $m_1 = 5$.

$10x - 2y = 1 \Rightarrow y = 5x - \frac{1}{2}$

The gradient is $m_2 = 5$.

$m_1 = m_2$. The 2 lines have the same gradient, so they are parallel.

Question 27

$g(x) = -x^2 + 9x - 20 = 0$

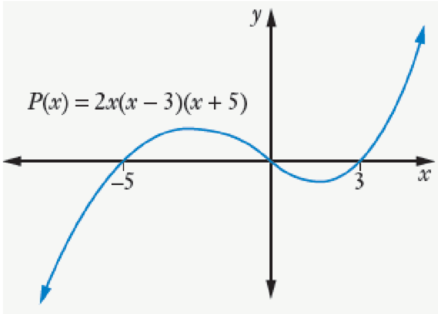
$x^2 - 9x + 20 = 0$

$(x - 4)(x - 5) = 0$

$x = 4, x = 5$

Question 28

x-intercepts at 0, 3, –5; y-intercept at 0



Question 29

$P(x) = x(x^2 - 4x + 4) = x(x - 2)^2$

$x = 0, x = 2$

Question 30

$$\frac{2-(-4)}{x-3} = -5$$

$-5(x - 3) = 6$

$$x - 3 = -\frac{6}{5} = -1\frac{1}{5}$$

$$x = 1\frac{4}{5}$$

Question 31

$f(5) = 2 \times 5 = 10$

$f(5) = 0^2 - 3 = -3$

$f(1) = 2 \times 1 = 2$

$f(5) - f(0) + f(1) = 10 - (-3) + 2 = 15$

Question 32

a $2^2 = 4$

b $2 - (-3) = 5$

c $3^2 = 9$

d 3

e $2 - 0 = 2$

Question 33

a $y = ax^2 + bx + c$

$(-2, 18)$,	$4a - 2b + c = 18$	[1]
$(3, -2)$,	$9a + 3b + c = -2$	[2]
$(1, 0)$	$a + b + c = 0$	[3]
[2] – [1]	$5a + 5b = -20 \Rightarrow a + b = -4$	[4]
[1] – [3]	$3a - 3b = 18 \Rightarrow a + b = 6$	[5]
[4] + [5]	$2a = 2 \Rightarrow a = 1$	
From [5],	$1 - b = 6 \Rightarrow b = -5$	
From [3],	$1 - 5 + c = 0 \Rightarrow c = 4$	

The equation is $y = x^2 - 5x + 4$.

b $y = A(x - 3)(x + 2)$

$(0, 12)$, $12 = A(0 - 3)(0 + 2)$

$-6A = 12$

$A = -2$

The equation is

$$y = -2(x - 3)(x + 2)$$
$$= -2(x^2 - x - 6)$$
$$= -2x^2 + 2x + 12$$

Question 34

a $A = kx^2$

$448 = k \times 8^2$

$k = \frac{448}{64} = 7$

$A = 7x^2$

b $A = 7 \times 10^2 = 700$ cm^2

c $1093.75 = 7x^2$

$x^2 = 156.25$

$x = 12.5$ cm

Question 35

a function; many-to-one

b not a function; many-to-many

c not a function; one-to-many

d function; one-to-one

e not a function; one-to-many

Question 36

$f(x) = kx^3 + 5$

(1, 2) $2 = k \times 1^3 + 5 \Rightarrow k = -3$

$f(x) = -3x^3 + 5$

Question 37

$12x = 7x + 15$

$5x = 15$

$x = 3$

$y = 12 \times 3 = 36$

Break-even point is (3, 36).

Question 38

a $m_1 = \frac{1}{2}$, $m_1 \times m_2 = -1 \Rightarrow m_2 = -2$

(1, –1) $y - (-1) = -2(x - 1)$

$y + 1 = -2x + 2$

$2x + y - 1 = 0$

b $y = 0,\ 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

Question 39

Want the discriminant to be negative.

$\Delta = 3^3 - 4 \times m \times (-4) = 9 + 16m$

$\Delta < 0 \Rightarrow 9 + 16m < 0$

$m < -\frac{9}{16}$

Question 40

a $y = 1 - 2x$ [1]

$y = 3x - 4$ [2]

Substitute [2] in [1].

$3x - 4 = 1 - 2x$

$5x = 5$

$x = 1$

From [1], $y = 1 - 2 \times 1 = -1$.

The point of intersection is (1, –1).

b $y = x^2 - x$ [1]

$y = 2x - 2$ [2]

Substitute [1] in [2].

$x^2 - x = 2x - 2$

$x^2 - 3x + 2 = 0$

$(x - 2)(x - 1) = 0$

$x = 1, x = 2$

From [1], when $x = 1$, $y = 1^2 - 1 = 0$

From [1], when $x = 2$, $y = 2^2 - 2 = 2$

The points of intersection are (1, 0), (2, 2).

c $y = x^2$ [1]

$y = 2x^2 - 9$ [2]

Substitute [2] in [1].

$2x^2 - 9 = x^2$

$x^2 = 9$

$x = \pm 3$

From [1], when $x = 3$, $y = 3^2 = 9$

From [1], when $x = -3$, $y = (-3)^2 = 9$

The points of intersection are (–3, 9), (3, 9).

Question 41

$3x - 4y + 5 = 0 \Rightarrow y = \frac{3}{4}x + \frac{5}{4}$

$(0, 0),\ m = \frac{3}{4}$

$y - 0 = \frac{3}{4}(x - 0)$

$y = \frac{3}{4}x \Rightarrow 3x - 4y = 0$

Question 42

Gradient of line connecting points is

$m = \frac{5-(-2)}{0-3} = -\frac{7}{3}$

$m_1 \times m_2 = -1 \Rightarrow m_2 = \frac{3}{7}$

$(0, -2), y - (-2) = \frac{3}{7}(x - 0)$

$y + 2 = \frac{3}{7}x \Rightarrow 3x - 7y - 14 = 0$

Question 43

a $A = kd$

$10.8 = k \times 87 \Rightarrow k \approx 0.12$

$A = 0.12d$

b $A = 0.12 \times 250 = 31$ litres

c $35.5 = 0.12d$

$d = \frac{35.5}{0.12} \approx 296$ km

Question 44

a $y = x^2 - 3x$

$x(x - 3) \le 0$

x-intercepts at 0, 3 and the graph is concave upwards.

$y \le 0$ for $0 \le x \le 3$

b $y = n^2 - 9$

$n^2 - 9 > 0$

$(n - 3)(n + 3) > 0$

n-intercepts at –3, 3 and the graph is concave upwards.

$y > 0$, for $n < -3$, $n > 3$

c $t = 4 - y^2$

$4 - y^2 \ge 0$

$(2 - y)(2 + y) \ge 0$

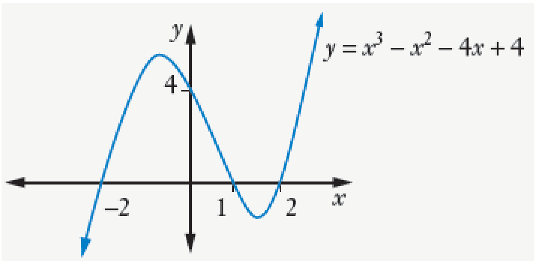
y-intercepts at –2, 2 and the graph is concave downwards.

$t > 0$ for $-2 \le y \le 2$

Question 45

a
$$\begin{aligned} x^3 - x^2 - 4x + 4 &= x^2(x - 1) - 4(x - 1) \\ &= (x - 1)(x^2 - 4) \\ &= (x - 1)(x - 2)(x + 2) \end{aligned}$$

$x = 1$, $x = -2$, $x = 2$



b x-intercepts are the values of x for which $f(x) = 0$.

$x = 1$, $x = -2$, $x = 2$

c $f(0) = 4$, so the y-intercept is 4.

d **i** A horizontal line passing through $f(x) = 1$ intersects the graph in three places, so there are three solutions.

ii A horizontal line passing through $f(x) = -2$ intersects the graph in one place, so there is one solution.

Challenge exercise 4

Question 1

$3b^2 - 7b + 1 = 7$

$3b^2 - 7b - 6 = 0$

$(3b + 2)(b - 3) = 0$

$b = -\frac{2}{3}, b = 3$

Question 2

$(x + 2)^2 - 1 = 0$

$(x + 2)^2 = 1$

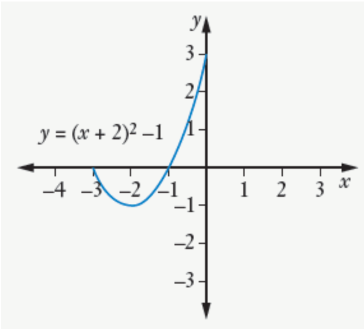
$x + 2 = \pm 1$

$x = -2 \pm 1$

The x-intercepts are –1, –3.

The y-intercept is $(0 + 2)^2 - 1 = 3$.

The graph is a horizontal translation 2 units left of the graph of $y = x^2$, followed by a vertical translation of 1 unit down.



Question 3

The gradients of any pair of points is the same.

Gradient of line joining $(-3k, 1)$ to $(k - 1, k - 3)$ is $m_1 = \frac{k-3-1}{k-1-(-3k)} = \frac{k-4}{4k-1}$

Gradient of line joining $(-3k, 1)$ to $(k - 4, k - 5)$ is $m_2 = \frac{k-5-1}{k-4-(-3k)} = \frac{k-6}{4k-4}$

$m_1 = m_2$, hence $\frac{k-4}{4k-1} = \frac{k-6}{4k-4}$

$(k - 4)(4k - 4) = (k - 6)(4k - 1)$

$4k^2 - 20k + 16 = 4k^2 - 25k + 6$

$5k = -10$

$k = -2$

Question 4

$2x + 5y + 19 = 0$ [1]

$4x - 3y - 1 = 0$ [2]

$2 \times [1] - [2]$ $13y + 39 = 0 \Rightarrow y = -3$

From [2], $4x - 3 \times (-3) - 1 = 0 \Rightarrow x = -2$

The point of intersection is (–2, –3).

Write $3x - 2y + 1 = 0$ as $y = \frac{3}{2}x + \frac{1}{2}$, so $m_1 = \frac{3}{2}$

$m_1 \times m_2 = -1 \Rightarrow m_2 = -\frac{2}{3}$

Using (–2, –3),

$y - (-3) = -\frac{2}{3}(x - (-2))$

$y + 3 = -\frac{2}{3}(x + 2)$

$3y + 9 = -2x - 4$

$2x + 3y + 13 = 0$

Question 5

Substitute (3, 4) into each equation.

$3a - 4 - 2 = 0$ $\Rightarrow a = 2$

$3b - 20 + 11 = 0$ $\Rightarrow b = 3$

Question 6

a $k^2 - 2k - 3 = 0$, with $k = 3x - 2$

$(k-3)(k+1) = 0$

$k = 3 \Rightarrow 3x - 2 = 3, x = 1\frac{2}{3}$

$k = -1 \Rightarrow 3x - 2 = -1, x = \frac{1}{3}$

b $y^2 - 26y + 25 = 0$, with $y = 2^x$

$(y - 25)(y - 1) = 0$

$y = 25 \Rightarrow 5^x = 5^2, x = 2$

$y = 1 \Rightarrow 5^x = 5^0, x = 0$

c $y^2 - 10y + 16 = 0$, with $y = 2^x$

$(y - 8)(y - 2) = 0$

$y = 2 \Rightarrow 2^x = 2^1, x = 1$

$y = 8 \Rightarrow 2^x = 2^3, x = 3$

d $2y^2 - 5y + 2 = 0$, with $y = 2^x$

$(2y-1)(y-2) = 0$

$y = \frac{1}{2} \Rightarrow 2^x = 2^{-}\ , x = -1$

$y = 2 \Rightarrow 2^x = 2^1$

e $y^2 - 5y + 6 = 0$, with $y = \left(x + \frac{1}{x}\right)$

$(y - 2)(y - 3) = 0$

$y = 2 \Rightarrow x + \frac{1}{x} = 2$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1$

$y = 3 \Rightarrow x + \frac{1}{x} = 3$

$x^2 - 3x + 1 = 0$

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4\times1\times1}}{2\times1} = 1.5 \pm 1.118$

$x = 2.6, x = 0.38$

Question 7

$2x - y + 5 = 0$ [1]

$x + 2y - 5 = 0$ [2]

$2 \times [1] + [2]$ $5x + 5 = 0 \Rightarrow x = -1$

From [2], $-1 + 2y - 5 = 0 \Rightarrow y = -3$

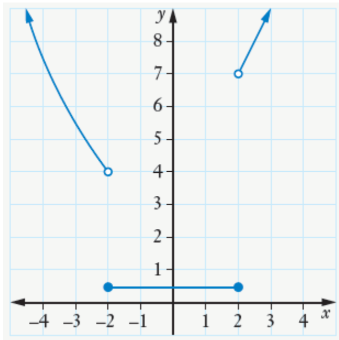
The point of intersection is (–1, –3).

The gradient joining (1, 3) to (–1, –3) is

$$\frac{-3-(-3)}{-1-1} = 0$$

Since the gradient is zero, the required equation is a horizontal line passing through $y = 3$.

Question 8

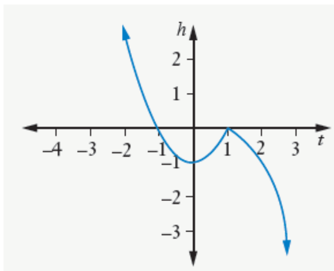


$f(3) = 2 \times 3 + 3 = 9$

$f(-4) = (-4)^2 = 16$

$f(0) = 1$

Question 9



$h(2) = 1 - 2^2 = -3$

$h(-1) = (-1)^2 - 1 = 0$

$h(0) = 0^2 - 1 = -1$

$h(2) + h(-1) - h(0) = -3 + 0 - (-1) = -2$

Question 10

$2x^3 - 2x^2 - 12x = 0$

$2x(x^2 - x - 6) = 0$

$x = 0$ or $x^2 - x - 6 = 0$

$x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0$

$x = -2, x = 3$

Question 11

$\Delta = (-k)^2 - 4 \times 2 \times (k - 2)$

$= k^2 - 8k + 16$

$= (k - 4)^2$

The discriminant is a perfect square and is always positive, so the solutions to the quadratic are real rational.

Question 12

The discriminant must be negative.

$\Delta = (-1)^2 - 4 \times 1 \times (3p - 2) = 9 - 12p$

Want $9 - 12p < 0 \Rightarrow p > \frac{3}{4}$.

Question 13

$f((-a)^2) = 2(-a^2) - 1 = 2a^2 - 1 = f(a^2)$

Question 14

(–2, 0), (0, 5) Gradient of line joining the intercepts is $\frac{5}{2}$.

The gradient of the perpendicular line is $-\frac{2}{5}$.

Use (3, –4) with $-\frac{2}{5}$.

$y-(-4)=-\frac{2}{5}(x-3)$

$2x+5y+14=0$

Question 15

$x^2 = x^3$

$x^3 - x^2 = 0$

$x^2(x - 1) = 0$

$x = 0, x = 1$

$x = 0, y = 0^2 = 0$

$x = 1, y = 1^2 = 1$

The points of intersection are (0, 0), (1, 1).

Question 16

Substitute (0, 1) to get	$d = 1$	
Substitute (1, 3) to get	$a + b + c + 1 = 3 \Rightarrow a + b + c = 2$	[1]
Substitute (–1, 3) to get	$-a + b - c + 1 = 3 \Rightarrow a + b - c = 2$	[2]
Substitute (2, 15) to get	$8a + 4b + 4c + 1 = 15 \Rightarrow 8a + 4b + 2c = 14$	[3]
[1] + [2]	$2b = 4 \Rightarrow b = 2$	
[3] + 2 × [2]	$6a + 6b = 18$	
	$6a + 12 = 18$	
	$a = 1$	
From [1],	$1 + 2 + c = 2 \Rightarrow c = -1$	

The equation is $y = x^3 = 2x^2 - x + 1$.

Question 17

The discriminant is $\Delta = (-2p)^2 - 4 \times 1 \times p^2 = 0$

$\Delta = 0$, hence there are equal roots.

Question 18

$y + \frac{25}{7} = 10$, where $y = x^2 + 1$

$y^2 + 25 = 10y$

$y^2 - 10y + 25 = 0$

$(y - 5)^2 = 0$

$y = 5$

$x^2 + 1 = 5$

$x^2 = 4$

$x = \pm 2$

Question 19

The discriminant is

$\Delta = (2k)^2 - 4 \times 1 \times (k + 5)$

$\quad = 4k^2 - 4k - 20$

$\quad = 4(k^2 - k - 5)$

Solve $k^2 - k - 5 = 0$ using quadratic formula to get $k = \frac{1 \pm \sqrt{21}}{2}$.

These represent the intercepts of the graph of the function $f(k) = k^2 - k - 5$.

We require $\Delta \geq 0$, that is, $f(k) \geq 0$

This occurs when $k \leq \frac{1-\sqrt{21}}{2}$ and $k \geq \frac{1+\sqrt{21}}{2}$.

Question 20

The x-intercepts are –2, 1, 6, hence we have

$P(x) = (x + 2)(x - 1)(x - 6)$

MATHS IN FOCUS 11 MATHEMATICS EXTENSION 1

WORKED SOLUTIONS

Chapter 5: Trigonometry

Exercise 5.01 Trigonometric ratios

Question 1

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{5}{13}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{12}{13}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{12}{5}$$

Question 2

$$\cos\beta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5}$$

$$\sin\beta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5}$$

$$\tan\beta = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$$

Question 3

$$a^2 + b^2 = c^2$$

$$5^2 + 7^2 = c^2$$

$$\sqrt{74} = c = \text{hypotenuse}$$

$$\cos\beta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{5}{\sqrt{74}}$$

$$\sin\beta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{7}{\sqrt{74}}$$

$$\tan\beta = \frac{\text{opposite}}{\text{adjacent}} = \frac{7}{5}$$

Question 4

$a^2 + b^2 = c^2$

$5^2 + b^2 = 9^2$

$b^2 = 9^2 - 5^2$

$b = \sqrt{56} = \text{opposite}$

$\cos x = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{5}{9}$

$\sin x = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{56}}{9}$

$\tan x = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{56}}{5}$

Question 5

$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$

$a^2 + b^2 = c^2$

$4^2 + 3^2 = c^2$

$c = \sqrt{25}$

$c = 5 = \text{hypotenuse}$

$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5}$

$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5}$

Question 6

$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{2}{3}$

$a^2 + b^2 = c^2$

$2^2 + b^2 = 3^2$

$b^2 = 3^2 - 2^2$

$b = \sqrt{5} = \text{opposite}$

$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{5}}{2}$

$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{5}}{3}$

Question 7

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{6}$$

$$a^2 + b^2 = c^2$$

$$1^2 + b^2 = 6^2$$

$$b^2 = 6^2 - 1^2$$

$$b = \sqrt{35} = \text{adjacent}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{35}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{35}}{6}$$

Question 8

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = 07 \ = \frac{0.7}{1} = \frac{7}{10}$$

$$a^2 + b^2 = c^2$$

$$7^2 + b^2 = 10^2$$

$$b^2 = 10^2 - 7^2$$

$$b = \sqrt{51} = \text{opposite}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{51}}{7}$$

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{51}}{10}$$

Question 9

a $AB^2 + CB^2 = AC^2$

$1^2 + 1^2 = AC^2$

$AC = \sqrt{2} = \text{hypotenuse}$

b $\angle BCA = \angle BAC$ (opposite angles in isosceles triangle)

$\angle BCA + \angle BAC + \angle ABC = 180°$

$\angle BAC + \angle BAC + \angle ABC = 180°$

$2\angle BAC + 90° = 180°$

$2\angle BAC = 90°$

$\angle BAC = 45°$

c $\cos 45° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$

$\tan 45° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1$

$\sin 45° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$

Question 10

a $BC^2 + AC^2 = AB^2$

$1^2 + AC^2 = 2^2$

$AC^2 = 2^2 - 1^2$

$AC = \sqrt{3}$

b $\cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}}$

$\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}$

c $\cos 60° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2}$

$\tan 60° = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\sin 60° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$

Question 11

a 47°

b 82°

c 19°

d 77°

e 52°

Question 12

a 47° 13′

b 81° 46′

c 19° 26′

d 76° 37′

e 52° 30′

Question 13

a $77°45 = 77\frac{45}{60} = 77.75°$

b $65°30 = 65\frac{30}{60} = 65.5°$

c $24°51 = 24\frac{51}{60} = 24.85°$

d $68°21 = 68\frac{21}{60} = 68.35°$

e $82°31 = 82\frac{31}{60} = 82.517°$

Question 14

a $59.53° = 59\frac{32}{60} = 59°32'$

b $72.231° = 72\frac{14}{60} = 72°14'$

c $85.887° = 85\frac{53}{60} = 85°53'$

d $46.9° = 46\frac{54}{60} = 46°54'$

e $73.213° = 73\frac{13}{60} = 73°13'$

Question 15

a $\sin 39°25 = 0.635$

b $\cos 45°51 = 0.697$

c $\tan 18°43 = 0.339$

d $\sin 68°06 = 0.928$

e $\tan 54°20 = 1.393$

Question 16

a $\sin\theta = 0.298$

$\theta = \sin^{-}\ 0.298$

$\theta = 17°20'$

b $\tan\theta = 0.683$

$\theta = \tan^{-}\ 0.683$

$\theta = 34°20'$

c $\cos\theta = 0.827$

$\theta = \cos^{-}\ 0.827$

$\theta = 34°12'$

d $\tan\theta = 1.056$

$\theta = \tan^{-}\ 1.056$

$\theta = 46°34'$

e $\cos\theta = 0.188$

$\theta = \cos^{-}\ 0.188$

$\theta = 79°10'$

Exercise 5.02 Finding a side of a right-angled triangle

Question 1

a

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 31°43 = \frac{x}{12}$$

$$12 \times \sin 31°43 = x$$

$$x = 6.30862... \approx 6.3$$

b

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 51°14 = \frac{y}{8.9}$$

$$8.9 \times \cos 51°14 = x$$

$$y = 5.5727... \approx 5.6$$

c

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 45°39 = \frac{b}{3.8}$$

$$3.8 \times \tan 45°39 = b$$

$$b = 3.8872... \approx 3.9$$

d

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 29°51 = \frac{x}{11.2}$$

$$11.2 \times \sin 29°51 = x$$

$$x = 5.5745... \approx 5.6\,\text{m}$$

e

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 67°22 = \frac{m}{7.6}$$

$$76 \times \cos 67°22 = m$$

$$m = 2.9247... \approx 2.9$$

f

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 21°45 = \frac{5}{x}$$

$$x \times \sin 21°45 = 5$$

$$x = \frac{5}{\sin 21°45'}$$

$$x = 13.4931... \approx 13.5$$

g

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 57°27 = \frac{5.4}{y}$$

$$y \times \cos 57°27 = 5.4$$

$$y = \frac{5.4}{\cos 57°27'}$$

$$y = 10.0365... \approx 10.0$$

h

$$\tan 71°12 = \frac{9.6}{p}$$

$$p \times \tan 71°12 = 9.6$$

$$p = \frac{9.6}{\tan 71°12'}$$

$$p = 3.2681... \approx 3.3$$

i

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 61°50 = \frac{x}{10.7}$$

$$10.7 \times \cos 61°50 = x$$

$$x = 5.0508... \approx 5.1\text{ cm}$$

j

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 33°42 = \frac{18.9}{t}$$

$$t \times \tan 33°42 = 18.9$$

$$t = \frac{18.9}{\tan 33°42'}$$

$$t = 28.3393... \approx 28.3$$

k

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 31°12 = \frac{x}{5.4}$$

$$5.4 \times \tan 31°12 = x$$

$$x = 3.2703... \approx 3.3$$

l

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 37°22 = \frac{x}{4.7}$$

$$47 \times \sin 37°22 = x$$

$$x = 28524.9 \approx \text{cm}$$

m

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 72°18 = \frac{6.3}{x}$$

$$x \times \cos 72°18 = 6.3$$

$$x = \frac{6.3}{\cos 72°18'}$$

$$x = 2072142.7 \approx \text{cm}$$

n

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 63°14 = \frac{x}{23}$$

$$23 \times \sin 63°14 = x$$

$$x = 2053552.5 \approx \text{mm}$$

o

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 39°47 = \frac{3.7}{y}$$

$$y \times \tan 39°47 = 3.7$$

$$y = \frac{3.7}{\tan 39°47'}$$

$$y = 4.4435... \approx 4.4 \text{ m}$$

p

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 46°5 = \frac{14.3}{k}$$

$$k \times \cos 46°5 = 14.3$$

$$k = \frac{14.3}{\cos 46°5'}$$

$$k = 2061672.6 \approx \text{cm}$$

q

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 74°29 = \frac{h}{4.8}$$

$$48 \times \tan 74°29 = h$$

$$h = 17.2887... \approx 17.3 \text{ m}$$

r

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 68°41 = \frac{0.45}{d}$$

$$d \times \cos 68°41 = 0.45$$

$$d = \frac{0.45}{\cos 68°41'}$$

$$d = 1.2378... \approx 1.2 \text{ m}$$

s

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 19°17 \ = \frac{5.75}{x}$$

$$x \times \sin 19°17 \ = 5.75$$

$$x = \frac{5.75}{\sin 19°17'}$$

$$x = 17.4115... \approx 17.4 \text{ cm}$$

t

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 6°3 \ = \frac{17.3}{b}$$

$$b \times \tan 6°3 \ = 17.3$$

$$b = \frac{17.3}{\tan 6°3'}$$

$$b = 163.2281... \approx 163.2 \text{ m}$$

Question 2

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 60° = \frac{2.7}{x}$$

$$x \times \tan 60° = 2.7$$

$$x = \frac{2.7}{\tan\ 60°} = 1.5588... \approx 1.6 \text{m}$$

Question 3

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 73° = \frac{l}{6.2}$$

$$62 \ \times \tan 73 \ ° = l$$

$$l = 2027922.3\approx \text{cm}$$

Question 4

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 67° = \frac{12.8}{x}$$

$$x \times \sin 67° = 12.8$$

$$x = \frac{12.8}{\sin 67°} = 13.9054... \approx 13.9 \text{m}$$

Question 5

a

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 64°12 = \frac{DE}{6}$$

$$6 \times \tan 64°12 = DE$$

$$DE = 12.4115...$$

$$DE = 12.4\,\text{cm}$$

$$CE = DE + CD$$

$$CE = 12.4 + 6$$

$$CE = 18.4\,\text{cm}$$

b

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 64°12 = \frac{6}{AE}$$

$$AE \times \cos 64°12 = 6$$

$$AE = \frac{6}{\cos 64°12'}$$

$$AE = 13.7857...$$

$$AE = 13.8\,\text{cm}$$

Question 6

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 43°36 = \frac{x}{14.5}$$

$$14.5 \times \cos 43°36 = x$$

$$x = 10.5004...$$

$$x = 10.5\text{ cm}$$

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 43°36 = \frac{y}{14.5}$$

$$14.5 \times \sin 43°36 = y$$

$$y = 9.9994...$$

$$y = 10.0\text{ cm}$$

Question 7

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 56°44 = \frac{26}{h}$$

$$h \times \cos 56°44 = 26$$

$$h = \frac{26}{\cos 56°44'}$$

$$h = 47.3988...$$

$$\text{hypotenuse} = 47.4\text{ mm}$$

Question 8

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 59°54 \ = \frac{10.2}{h}$$

$$h \times \cos 59°54 \ = 10.2$$

$$h = \frac{10.2}{\cos 59°54'}$$

$$h = 20.3385...$$

$$AC = 20.3 \text{ m}$$

Question 9

a

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 46° = \frac{5.3}{AC}$$

$$AC \times \sin 46° = 5.3$$

$$AC = \frac{5.3}{\sin 46°}$$

$$AC = 7.3678...$$

$$AC = 7.4 \text{ cm}$$

b

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 54° = \frac{5.3}{CB}$$

$$CB \times \sin 54° = 5.3$$

$$CB = \frac{5.3}{\sin 54°}$$

$$CB = 6.5511...$$

$$CB = 6.6 \text{ cm}$$

c

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 46° = \frac{5.3}{AD}$$

$$AD \times \tan 46° = 5.3$$

$$AD = \frac{5.3}{\tan 46°}$$

$$AD = 5.1181...$$

$$AD = 5.1 \text{ cm}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 54° = \frac{5.3}{BD}$$

$$BD \times \tan 54° = 5.3$$

$$BD = \frac{5.3}{\tan 54°}$$

$$BD = 3.8506...$$

$$BD = 3.9 \text{ cm}$$

$$AD + BD = AB$$

$$5.1 \text{ cm} + 3.9 \text{ cm} = AB$$

$$AB = 9.0 \text{ cm}$$

Question 10

a

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 28°23 = \frac{6}{s}$$

$$s \times \sin 28°23 = 6$$

$$s = \frac{6}{\sin 28°23'}$$

$$s = 12.6217...$$

$$\text{side} = 12.6\,\text{cm}$$

b

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 28°23 = \frac{6}{o}$$

$$o \times \tan 28°23 = 6$$

$$o = \frac{6}{\tan 28°23'} = 11.1044... = 11.1\text{ cm}$$

$$\text{diagonal} = 2 \times o = 2 \times 11.1\text{ cm} = 22.2\text{ cm}$$

Question 11

Let intersection where diagonals cross = E

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 57°29 = \frac{AE}{7.9}$$

$$79 \times \tan 57°29 = AE$$

$$\begin{aligned} AE &= 12.3925... \\ &= 12.4\text{ cm} \end{aligned}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 72°51 = \frac{EC}{7.9}$$

$$7.9 \times \tan 72°51 = EC$$

$$EC = 25.5998...$$

$$EC = 25.6\text{ cm}$$

$$\begin{aligned} AC &= AE + EC \\ &= 124\text{ cm} + 25.6\text{ cm} \\ &= 38.0\text{ cm} \end{aligned}$$

Exercise 5.03 Finding an angle in a right-angled triangle

Question 1

a

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan x = \frac{5}{6}$$
$$x = \tan^{-1}\frac{5}{6}$$
$$x = 39°48'20''$$
$$x = 39°48'$$

b

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos\alpha = \frac{9}{11}$$
$$\alpha = \cos^{-1}\frac{9}{11}$$
$$\alpha = 35°05'48''$$
$$\alpha = 35°06'$$

c

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin\theta = \frac{8}{13}$$
$$\theta = \sin^{-1}\frac{8}{13}$$
$$\theta = 37°58'47''$$
$$\theta = 37°59'$$

d

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos\alpha = \frac{5.9}{9.3}$$
$$\alpha = \cos^{-1}\frac{5.9}{9.3}$$
$$\alpha = 50°37'25''$$
$$\alpha = 50°37'$$

e

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan\alpha = \frac{4.6}{5.7}$$
$$\alpha = \tan^{-1}\frac{4.6}{5.7}$$
$$\alpha = 38°54'14''$$
$$\alpha = 38°54'$$

f

$$\sin\beta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin\beta = \frac{6.5}{8.4}$$
$$\beta = \sin^{-1}\frac{6.5}{8.4}$$
$$\beta = 50°41'50''$$
$$\beta = 50°42'$$

g

$$\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos x = \frac{3.9}{5.5}$$
$$x = \cos^{-1}\frac{3.9}{5.5}$$
$$x = 44°50'20''$$
$$x = 44°50'$$

h

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan\theta = \frac{4.6}{7.7}$$
$$\theta = \tan^{-1}\frac{4.6}{7.7}$$
$$\theta = 30°51'15''$$
$$\theta = 30°51'$$

i

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin\alpha = \frac{5.8}{11.7}$$
$$\alpha = \sin^{-1}\frac{5.8}{11.7}$$
$$\alpha = 29°43'3''$$
$$\alpha = 29°43'$$

j

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{14.9}{21.3}$$
$$\theta = \cos^{-1}\frac{14.9}{21.3}$$
$$\theta = 45°36'38''$$
$$\theta = 45°37'$$

k

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan\alpha = \frac{3.8}{2.4}$$
$$\alpha = \tan^{-1}\frac{3.8}{2.4}$$
$$\alpha = 57°43'27''$$
$$\alpha = 57°43'$$

l

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin\theta = \frac{5.7}{8.3}$$
$$\theta = \sin^{-1}\frac{5.7}{8.3}$$
$$\theta = 43°22'23''$$
$$\theta = 43°22'$$

m

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin\theta = \frac{6.9}{11.3}$$
$$\theta = \sin^{-1}\frac{6.9}{11.3}$$
$$\theta = 37°38'03''$$
$$\theta = 37°38'$$

n

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{3}{7}$$
$$\theta = \cos^{-1}\frac{3}{7}$$
$$\theta = 64°37'23''$$
$$\theta = 64°37'$$

o

$$\tan\beta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan\beta = \frac{11.6}{5.1}$$
$$\beta = \tan^{-1}\frac{11.6}{5.1}$$
$$\beta = 66°16'01''$$
$$\beta = 66°16'$$

p

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos\alpha = \frac{13}{15}$$
$$\alpha = \cos^{-1}\frac{13}{15}$$
$$\alpha = 29°55'35''$$
$$\alpha = 29°56'$$

q

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{4.4}{7.6}$$
$$\theta = \cos^{-1}\frac{4.4}{7.6}$$
$$\theta = 54°37'24''$$
$$\theta = 54°37'$$

r

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin\alpha = \frac{8.4}{14.3}$$
$$\alpha = \sin^{-1}\frac{8.4}{14.3}$$
$$\alpha = 35°58'25''$$
$$\alpha = 35°58'$$

s

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan\theta = \frac{5}{3}$$
$$\theta = \tan^{-1}\frac{5}{3}$$
$$\theta = 59°02'10''$$
$$\theta = 59°02'$$

t

$$\cos\gamma = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos\gamma = \frac{10.3}{18.9}$$
$$\gamma = \cos^{-1}\frac{10.3}{18.9}$$
$$\gamma = 56°58'37''$$
$$\gamma = 56°59'$$

Question 2

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin\theta = \frac{12.3}{20}$$
$$\theta = \sin^{-1}\frac{12.3}{20}$$
$$\theta = 37°57'06''$$
$$\theta = 37°57'$$

Question 3

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan\theta = \frac{5.6}{13.7}$$
$$\theta = \tan^{-1}\frac{5.6}{13.7}$$
$$\theta = 22°13'57'$$
$$\theta = 22°14'$$

Question 4

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin\theta = \frac{18}{30}$$
$$\theta = \sin^{-1}\frac{18}{30}$$
$$\theta = 36°52'11''$$
$$\theta = 36°52'$$

Question 5

$$\cos\angle DAB = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos\angle DAB = \frac{4.5}{7}$$
$$\angle DAB = \cos^{-1}\frac{4.5}{7}$$
$$\angle DAB = 49°59'41''$$
$$\angle DAB = 50°$$

Question 6

a

$$AC^2 + DE^2 = AE^2$$
$$7^2 + 9^2 = AE^2$$
$$AE = \sqrt{130}$$
$$AE = 11.4 \text{ cm}$$

b

$$\tan \angle DEA = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan \angle DEA = \frac{7}{9}$$
$$\angle DEA = \tan^{-1} \frac{7}{9}$$
$$\angle DEA = 37°52'29''$$
$$\angle DEA = 37°52'$$

Question 7

$$\cos \beta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos \beta = \frac{52}{74.5}$$
$$\beta = \cos^{-1} \frac{52}{74.5}$$
$$\beta = 45°44'08''$$
$$\beta = 45°44'$$

$$\cos \alpha = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos \alpha = \frac{52}{61.3}$$
$$\alpha = \cos^{-1} \frac{52}{61.3}$$
$$\alpha = 31°58'27''$$
$$\alpha = 31°58'$$

Question 8

a

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan \theta = \frac{h}{15}$$
$$15 \times \tan 41 = h$$
$$h = 13.0393...$$
$$h = 13 \text{ m}$$

b

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan \theta = \frac{13}{6}$$
$$\theta = \tan^{-1} \frac{13}{6}$$
$$\theta = 65°13'29''$$
$$\theta = 65°13'$$

Question 9

a

$$\tan \angle AED = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan \angle AED = \frac{5}{1}$$

$$\angle AED = \tan^{-} 5$$

$$\angle AED = 78°41'24''$$

$$\angle AED = 78°41'$$

$$\angle BEC + \angle AED + \angle AED = 180°$$

$$180° - \angle AED - \angle AED = \angle BEC$$

$$\angle BEC = 180° - 78°41 - 90°$$

$$\angle BEC = 11°19'$$

b

$$\tan \angle BEC = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 11°19 = \frac{5}{EC}$$

$$EC \times \tan 11°19 = 5$$

$$EC = \frac{5}{\tan 11°19'}$$

$$EC = 24.98$$

$$EC = 25 \text{ cm}$$

$$DC = EC + DE$$

$$DC = 25 \text{ cm} + 1 \text{ cm}$$

$$DC = 26 \text{ cm}$$

Question 10

a

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 39° = \frac{w}{15.9}$$

$$159 \times \tan 39° = w$$

$$w = 12.8755...$$

$$w = 12.9 \text{ m}$$

b

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{12.9}{15.9 - 7.4}$$

$$\tan\theta = \frac{12.9}{8.5}$$

$$\theta = \tan^{-1}\frac{12.9}{8.5}$$

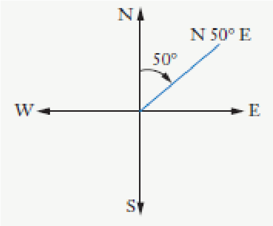
$$\theta = 56°37'6''$$

$$\theta = 56°37'$$

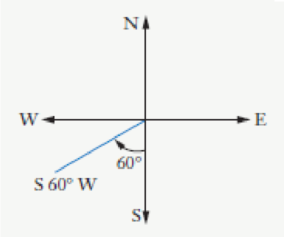
Exercise 5.04 Applications of trigonometry

Question 1

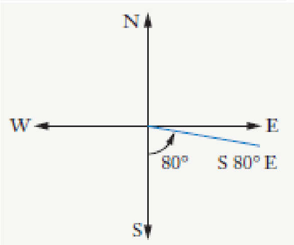
a



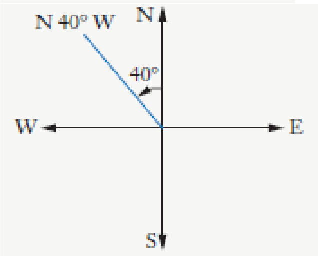
b



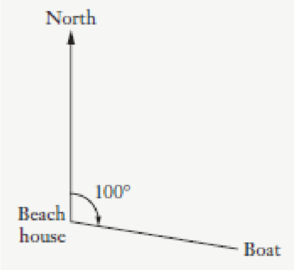
c



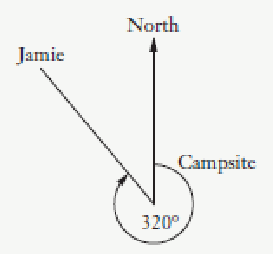
d



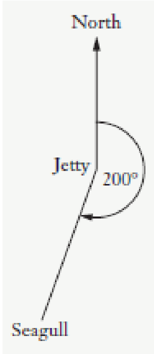
e



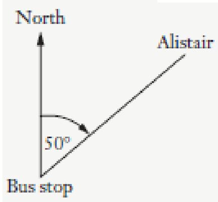
f



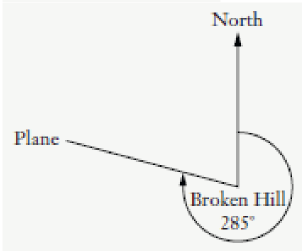
g



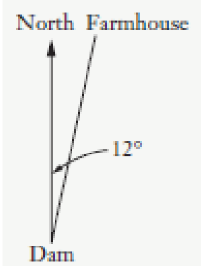
h



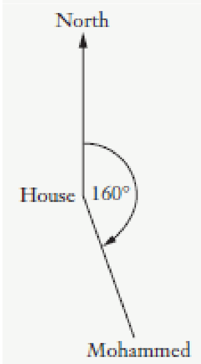
i



j



k



Question 2

a **i** S 35° E

ii 180° – 35° = 145°

b **i** 90° – 10° = 80°

N 80° E

ii 080°

c **i** N 23° W

ii 360° – 23° = 337°

d **i** S

ii 180°

Question 3

260° – 180° = 080° (Alternate angles)

Question 4

180° + 30° = 210°

Question 5

360° – 340° = 20° (360°)

180° – 20° = 160°

Question 6

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 42°12 = \frac{h}{11.5}$$

$$11.5 \times \tan 42°12 = h$$

$$h = 10.4275...$$

$$h = 10.4 \text{ m}$$

Question 7

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 39°20 = \frac{h}{25.8}$$

$$258 \times \tan 39°20 = h$$

$$h = 21.1421...$$

$$h = 21 \text{ m}$$

Question 8

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin 52° = \frac{100}{w}$$
$$w = \frac{100}{\sin 52°}$$
$$w = 126.902$$
$$w = 126.9\,\text{m}$$

Question 9

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan\theta = \frac{4.2}{1.3}$$
$$\theta = \tan^{-1}\frac{4.2}{1.3}$$
$$\theta = 72°48'05''$$
$$\theta = 72°48'$$

Question 10

a

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos 65° = \frac{A}{2500}$$
$$2500\times\cos 65° = A$$
$$A = 1056.5456...$$
$$A = 1056.5\text{ km}$$

b

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos 25° = \frac{A}{2500}$$
$$2500\times\cos 25° = A$$
$$A = 2265.7694...$$
$$A = 2265.8\ \text{ km}$$

c 180°+ 65° = 245°

Question 11

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 39°44 \ = \frac{h}{100}$$

$$100 \times \tan 39°44 \ = h$$

$$h = 83.1199...$$

$$h = 83.1 \text{ m}$$

Question 12

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 65° = \frac{d}{2}$$

$$2 \times \sin 65° = d$$

$$d = 1.8126...$$

$$d = 1.8 \text{ km}$$

Question 13

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 46°08 \ = \frac{8}{A}$$

$$A \times \cos 46°08 \ = 8$$

$$A = \frac{8}{\cos 46°08'}$$

$$A = 11.5443...$$

$$A = 12 \text{ m}$$

Question 14

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{2.4}{1.3}$$

$$\theta = \tan^{-1}\frac{2.4}{1.3}$$

$$\theta = 61°33'25''$$

$$\theta = 62°$$

$$\begin{aligned}\text{Bearing} &= 62° + 180° \\ &= 242°\end{aligned}$$

Question 15

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{390}{560}$$

$$\theta = \tan^{-1}\frac{390}{560}$$

$$\theta = 34°51'16''$$

$$\theta = 35°$$

$$\text{Bearing} = 035°$$

Question 16

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 67°13\ = \frac{h}{10}$$

$$10 \times \sin 67°13 = h$$

$$h = 9.2197...$$

$$h = 9.2\,\text{m}$$

Question 17

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 59°42\ = \frac{h}{100}$$

$$100 \times \tan 59°42\ = h$$

$$h = 171.1294...$$

$$h = 171\text{m}$$

Question 18

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 40° = \frac{7.5}{d}$$

$$d \times \cos 40° = 7.5$$

$$d = \frac{7.5}{\cos 40°}$$

$$d = 9.7905...$$

$$d = 9.8\text{ km}$$

Question 19

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{20}{15.8}$$

$$\theta = \tan^{-1}\frac{20}{15.8}$$

$$\theta = 51°41'29''$$

$$\theta = 51°41'$$

Question 20

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 72°25' = \frac{h}{1.8}$$

$$1.8 \times \tan 72°25' = h$$

$$h = 5.6800...$$

$$h = 5.7\ \text{m}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 72°25' = \frac{2.6 + 5.7}{x}$$

$$x \times \tan 72°25' = 8.3$$

$$x = \frac{8.3}{\tan 72°25'}$$

$$x = 2.6302...$$

$$x = 2.6\,\text{m}$$

Question 21

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{15.9}{100}$$

$$\theta = \tan^{-1}\frac{15.9}{100}$$

$$\theta = 9°02'04''$$

$$\theta = 9°02'$$

Question 22

195° – 180° = 15°

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 15° = \frac{d}{2000}$$

$$2000 \times \cos 15° = d$$

$$d = 1931.8516...$$

$$d = 1931.9 \text{ km}$$

Question 23

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 25°41 = \frac{15}{d}$$

$$d \times \sin 25°41 = 15$$

$$d = \frac{15}{\sin 25°41'}$$

$$d = 34.6103...$$

$$d = 35 \text{m}$$

Question 24

$$\tan\theta = \frac{2.7}{1.6}$$

$$\theta = \tan^{-1}\frac{2.7}{1.6}$$

$$\theta = 59°20'57''$$

$$\theta = 59°$$

$$\text{Bearing} = \theta + 90°$$

$$\text{Bearing} = 59° + 90°$$

$$\text{Bearing} = 149°$$

Question 25

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 38°19\ = \frac{h}{250}$$

$$1250 \times \tan 38°19\ = h$$

$$h = 197.5562...$$

$$h = 198\,\text{m}$$

Question 26

$$\theta = \text{Bearing} - 90°$$

$$\theta = 127° - 90°$$

$$\theta = 37°$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 37° = \frac{3.6}{d}$$

$$d \times \tan 37° = 3.6$$

$$d = \frac{3.6}{\tan 37°}$$

$$d = 4.7773...$$

$$d = 4.8\text{ km}$$

Question 27

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin 22°32\ = \frac{h}{24}$$

$$24 \times \sin 22°32\ = h$$

$$h = 9.1973...$$

$$h = 9.2\,\text{m}$$

Question 28

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan\theta = \frac{5.8}{7.8}$$
$$\theta = \tan^{-1}\frac{5.8}{7.8}$$
$$\theta = 36°38'03''$$
$$\theta = 37°$$
$$\text{Bearing} = \theta + 180°$$
$$\text{Bearing} = 37° + 180°$$
$$\text{Bearing} = 217°$$

Question 29

a

$$\theta = 90° - \text{Bearing}$$
$$\theta = 90° - 75°$$
$$\theta = 15°$$
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin 15° = \frac{n}{4.7}$$
$$4.7 \times \sin 15° = n$$
$$n = 1.2164...$$
$$n = 1.2\,\text{km}$$

b

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos 25° = \frac{a}{2.9}$$
$$29 \times \cos 25° = a$$
$$a = 2.6282...$$
$$a = 2.63\ \text{km}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\cos 15° = \frac{b}{4.7}$$
$$4.7 \times \cos 15° = b$$
$$b = 4.5498...$$
$$b = 4.55\ \text{km}$$
$$\text{Home} = a + b$$
$$\text{Home} = 2.63 + 4.55$$
$$\text{Home} = 7.18\,\text{km}$$
$$= 7.2\,\text{km}$$

Question 30

a

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 71° = \frac{h}{4.5}$$

$$45 \times \tan 71° = h$$

$$h = 13.0689...$$

$$h = 13.1\text{m}$$

b

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{13.1}{10.8}$$

$$\theta = \tan^{-1}\frac{13.1}{10.8}$$

$$\theta = 50°25'48''$$

$$\theta = 50°26'$$

Exercise 5.05 The sine rule

Question 1

a

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\frac{x}{\sin 57°14'}=\frac{8}{\sin 49°15'}$$

$$x=\frac{8\times\sin 57°14'}{\sin 49°15'}$$

$$x=8.8798...$$

$$x=8.9$$

b

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\frac{y}{\sin 38°56'}=\frac{14.7}{\sin 78°42'}$$

$$y=\frac{14.7\times\sin 38°56'}{\sin 78°42'}$$

$$y=9.4203...$$

$$y=9.4$$

c

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\frac{a}{\sin 105°31'}=\frac{6.1}{\sin 35°49'}$$

$$a=\frac{6.1\times\sin 105°31'}{\sin 35°49'}$$

$$a=10.0439...$$

$$a=10.0$$

d

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\frac{b}{\sin 41°22'}=\frac{8.5}{\sin 31°40'}$$

$$b=\frac{8.5\times\sin 41°22'}{\sin 31°40'}$$

$$b=10.7003...$$

$$b=10.7$$

e

$$180°-104°-23°=53°$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\frac{d}{\sin 53°}=\frac{9.7}{\sin 104°}$$

$$d=\frac{97\ \times\sin 53\ °}{\sin 104°}$$

$$d=7.9839...$$

$$d=8.0$$

Question 2

a

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$
$$\frac{\sin\theta}{7}=\frac{\sin 28°9'}{4.2}$$
$$\sin\theta=\frac{7\times\sin 28°9'}{4.2}$$
$$\theta=\sin^{-1}\left(\frac{7\times\sin 28°9}{42}\right)$$
$$\theta=51°50'28''$$
$$\theta=51°50',128°10'$$

b

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$
$$\frac{\sin\alpha}{5.7}=\frac{\sin 43°52'}{4.5}$$
$$\sin\alpha=\frac{5.7\times\sin 43°52'}{4.5}$$
$$\alpha=\sin^{-1}\left(\frac{5.7\times\sin 43°52}{45}\right)$$
$$\alpha=61°22'31''$$
$$\alpha=61°23',118°37'$$

c

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$
$$\frac{\sin x}{9}=\frac{\sin 32°4'}{7}$$
$$\sin x=\frac{9\times\sin 32°4'}{7}$$
$$x=\sin^{-1}\left(\frac{9\times\sin 32°4}{7}\right)$$
$$x=43°02'47'',180°-43°02'47''$$
$$x=43°03',136°57'$$

d

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$
$$\frac{\sin\theta}{4.9}=\frac{\sin 21°31'}{3.7}$$
$$\sin\theta=\frac{4.9\times\sin 21°31'}{3.7}$$
$$\theta=\sin^{-1}\left(\frac{4.9\times\sin 21°31}{37}\right)$$
$$\theta=29°03'36''$$
$$\theta=29°04'$$
$$180°-29°04\ =\theta$$
$$\theta=150°56'$$

e

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$
$$\frac{\sin\beta}{8.7}=\frac{\sin 53°12'}{10.9}$$
$$\sin\beta=\frac{8.7\times\sin 53°12'}{10.9}$$
$$\beta=\sin^{-1}\left(\frac{8.7\times\sin 53°12}{109}\right)$$
$$\beta=39°43'33''$$
$$\beta=39°44'$$
$$180°-53°12'-39°44'=\alpha$$
$$\alpha=87°04'$$

Question 3

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \theta}{4.6} = \frac{\sin 33°47'}{3.2}$$

$$\sin \theta = \frac{4.6 \times \sin 33°47'}{3.2}$$

$$\theta = \sin^{-1}\left(\frac{4.6 \times \sin 33°47}{32}\right)$$

$$\theta = 53°03'77''$$

$$\theta = 53°04'$$

$$180° - 53°04 = \angle A$$

$$\angle A = 126°56'$$

Question 4

a

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{s}{\sin 32°} = \frac{18.9}{\sin 48°}$$

$$s = \frac{189 \times \sin 32°}{\sin 48°}$$

$$s = 13.4771...$$

$$s = 13.5 \text{ mm}$$

b

$$180° - 48° - 32° = \theta°$$

$$\theta° = 100°$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{l}{\sin 100°} = \frac{18.9}{\sin 48°}$$

$$l = \frac{189 \times \sin 100°}{\sin 48°}$$

$$l = 25.0460...$$

$$l = 25.0 \text{ mm}$$

Question 5

a

$$180° - 86° - 51° = \theta°$$

$$\theta° = 43°$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{s}{\sin 43°} = \frac{2.1}{\sin 51°}$$

$$s = \frac{21 \times \sin 43°}{\sin 51°}$$

$$s = 1.8428...$$

$$s = 1.8 \text{m}$$

b

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{l}{\sin 86°} = \frac{2.1}{\sin 51°}$$

$$l = \frac{2.1 \times \sin 86°}{\sin 51°}$$

$$l = 2.6956...$$

$$l = 2.7 \text{ m}$$

Question 6

$180° - 63° - 48° = \theta°$

$\theta° = 69°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{XZ}{\sin 69°} = \frac{5.4}{\sin 63°}$$

$$XZ = \frac{54 \times \sin 69°}{\sin 63°}$$

$XZ = 5.6580...$

$XZ = 5.7\,\text{cm}$

Question 7

a

$180° - 47° - 53° = A°$

$A° = 80°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{AB}{\sin 53°} = \frac{12.7}{\sin 80°}$$

$$AB = \frac{127 \times \sin 53°}{\sin 80°}$$

$AB = 10.2991...$

$AB = 10.3\text{ m}$

b

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{AC}{\sin 47°} = \frac{12.7}{\sin 80°}$$

$$AC = \frac{127 \times \sin 47°}{\sin 80°}$$

$AC = 9.4314...$

$AC = 9.4\text{ m}$

Question 8

a

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin \angle P}{14.7} = \frac{\sin 62°29'}{15}$$

$$\sin \angle P = \frac{147 \times \sin 62°29'}{15}$$

$$\angle P = \sin^{-1}\left(\frac{147 \times \sin 62°29}{15}\right)$$

$\angle P = 60°21'31''$

$\angle P = 60°22'$

b

$180° - \angle P - \angle R = \angle Q$

$180° - 60°22' - 62°29' = \angle Q$

$\angle Q = 57°09'$

Question 9

$\angle ABC = \angle BCA \quad (\text{base angles of isosceles triangle})$

$180° = \angle ABC + \angle BCA + \angle BAC$

$180° - \angle BAC = \angle ABC + \angle ABC$

$180° - 52° = 2\angle ABC$

$128° = 2\angle ABC$

$64° = \angle ABC = \angle BCA$

$\angle BCA + \angle ACD = 180° \quad (\text{supplementary angles} = 180°)$

$64° + \angle ACD = 180°$

$\angle ACD = 116°$

$\angle CAD = 180° - \angle ACD - \angle CDA \quad (\text{angle sum of a triangle})$

$\angle CAD = 180° - 116° - 32° = 32°$

$\angle BAD = \angle BAC + \angle CAD$

$\angle BAD = 52° + 32° = 84°$

a

$AB = AC = 8.3 \text{ cm}$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{AD}{\sin 116°} = \frac{8.3}{\sin 32°}$$

$$AD = \frac{8.3 \times \sin 116°}{\sin 32°}$$

$AD = 1407751.1 \approx \text{cm}$

b

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{BD}{\sin 84°} = \frac{8.3}{\sin 32°}$$

$$BD = \frac{83 \times \sin 84°}{\sin 32°}$$

$BD = 15.5769\ldots \approx 15.6 \text{ cm}$

Question 10

$\angle ABC = \angle BCA = \angle CAB = 60°$ (property of equilateral triangles)

$\angle CAB = \angle DAB + \angle DAC$

$60° = 26°15 + \angle DAC$

$\angle DAC = 33°45'$

$180° - \angle BCA - \angle DAC = \angle ADC$ (angle sum of a triangle)

$180° - 60° - 33°45 = \angle ADC$

$\angle ADC = 86°15'$

a

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{AD}{\sin 60°} = \frac{63}{\sin 86°15'}$$

$$AD = \frac{63 \times \sin 60°}{\sin 86°15'}$$

$$AD = 5467665.7 \text{ mm}$$

b

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{DC}{\sin 33°45'} = \frac{63}{\sin 86°15'}$$

$$DC = \frac{63 \times \sin 33°45'}{\sin 86°15'}$$

$$DC = 3507603.1 \text{ mm}$$

Question 11

a

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$

$$\frac{\sin\angle B}{7.5}=\frac{\sin 67^\circ}{7.2}$$

$$\sin\angle B=\frac{75\times\sin 67^\circ}{7.2}$$

$$\angle B=\sin^{-1}\left(\frac{75\times\sin 67^\circ}{72}\right)$$

$$\angle B=73^\circ 30'28'',180^\circ-73^\circ 30'28''$$

$$\angle B=74^\circ,106^\circ$$

b

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$

$$\frac{\sin\angle B}{8.4}=\frac{\sin 92^\circ}{10.7}$$

$$\sin\angle B=\frac{84\times\sin 92^\circ}{10.7}$$

$$\angle B=\sin^{-1}\left(\frac{84\times\sin 92^\circ}{107}\right)$$

$$\angle B=51^\circ 40'50''$$

$$\angle B=52^\circ$$

c

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$

$$\frac{\sin\angle B}{8.3}=\frac{\sin 29^\circ}{4.9}$$

$$\sin\angle B=\frac{83\times\sin 29^\circ}{4.9}$$

$$\angle B=\sin^{-1}\left(\frac{83\times\sin 29^\circ}{49}\right)$$

$$\angle B=55^\circ 12'21''$$

$$\angle B=55^\circ \text{ or } 125^\circ$$

Exercise 5.06 The cosine rule

Question 1

a $c^2 = a^2 + b^2 - 2ab\cos C$

$m^2 = 5^2 + 8^2 - 2\times 5\times 8\cos 46°39'$

$m = \sqrt{25 + 64 - 80\cos 46°39'}$

$m = 5.8381...$

$m = 5.8$

b $c^2 = a^2 + b^2 - 2ab\cos C$

$b^2 = 10^2 + 8^2 - 2\times 10\times 8\cos 69°14'$

$b = \sqrt{100 + 64 - 160\cos 69°14'}$

$b = 10.3571...$

$b = 10.4\text{ m}$

c $c^2 = a^2 + b^2 - 2ab\cos C$

$h^2 = 5.4^2 + 5.7^2 - 2\times 5.4\times 5.7\cos 83°19'$

$h = \sqrt{29.16 + 32.49 - 61.56\cos 83°19'}$

$h = 7.3814...$

$h = 7.4\,\text{cm}$

d $c^2 = a^2 + b^2 - 2ab\cos C$

$n^2 = 9.3^2 + 11.8^2 - 2\times 9.3\times 11.8\cos 101°45'$

$n = \sqrt{86.49 + 139.24 - 219.48\cos 101°45'}$

$n = 16.4446...$

$n = 16.4$

e $c^2 = a^2 + b^2 - 2ab\cos C$

$y^2 = 8.7^2 + 15.9^2 - 2\times 8.7\times 15.9\cos 28°52'$

$y = \sqrt{7569 + 25.81 \quad 26.66\cos 28 \;\, 52° \quad '}$

$y = 9.2852...$

$y = 9.3$

Question 2

a

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$
$$\cos\theta=\frac{7^2+6^2-6^2}{2\times7\times6}$$
$$\theta=\cos^{-1}\left(\frac{49}{84}\right)$$
$$\theta=54°18'52''$$
$$\theta=54°19'$$

b

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$
$$\cos\theta=\frac{5.9^2+8.1^2-7.3^2}{2\times5.9\times8.1}$$
$$\theta=\cos^{-1}\left(\frac{4713}{9558}\right)$$
$$\theta=60°27'20''$$
$$\theta=60°27'$$

c

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$
$$\cos x=\frac{4.2^2+3.6^2-3.8^2}{2\times4.2\times3.6}$$
$$x=\cos^{-1}\left(\frac{1616}{3024}\right)$$
$$x=57°41'50''$$
$$x=57°42'$$

d

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$
$$\cos\beta=\frac{5.3^2+6.1^2-10.4^2}{2\times5.3\times6.1}$$
$$\beta=\cos^{-1}\left(\frac{-4286}{6466}\right)$$
$$\beta=131°31'03''$$
$$\beta=131°31'$$

e

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$
$$\cos\theta=\frac{7.6^2+8.2^2-9.5^2}{2\times7.6\times8.2}$$
$$\theta=\cos^{-1}\left(\frac{3475}{12464}\right)$$
$$\theta=73°48'40''$$
$$\theta=73°49'$$

Question 3

$$c^2=a^2+b^2-2ab\cos C$$
$$AC^2=12.9^2+23.8^2-2\times12.9\times23.8\cos125°$$
$$AC=\sqrt{166.41+566.44-614.04\cos125°}$$
$$AC=32.9400...$$
$$AC=32.94\,\text{mm}$$

Question 4

Other interior angle = 180° – 79°25′ = 100°35′

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$x^2 = 11^2 + 5^2 - 2\times 11\times 5\cos 79°25'$$
$$x = \sqrt{121 + 25 - 110\cos 79°25'}$$
$$x = 11.2159...$$
$$x = 11.2\,\text{mm}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$y^2 = 11^2 + 5^2 - 2\times 11\times 5\cos 100°35'$$
$$y = \sqrt{121 + 25 - 110\cos 100°35'}$$
$$y = 12.9...$$
$$y = 12.9\text{ mm}$$

Question 5

a

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$AC^2 = 12^2 + 10.4^2 - 2\times 12\times 10.4\cos 63°57'$$
$$AC = \sqrt{144 + 108.16 - 249.6\cos 63°57'}$$
$$AC = 11.9392...$$
$$AC = 11.9\text{ cm}$$

b

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos\angle DAC = \frac{9.7^2 + 11.9^2 - 8.4^2}{2\times 9.7\times 11.9}$$
$$\angle DAC = \cos^{-1}\left(\frac{16514}{23086}\right)$$
$$\angle DAC = 44°11'29''$$
$$\angle DAC = 44°11'$$

c

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos\angle ADC = \frac{9.7^2 + 8.4^2 - 11.9^2}{2\times 9.7\times 8.4}$$
$$\angle ADC = \cos^{-1}\left(\frac{2304}{16296}\right)$$
$$\angle ADC = 82°12'13''$$
$$\angle ADC = 82°12'$$

Question 6

$\angle XYZ = \angle YZX$

$$\cos\angle XYZ = \frac{7.3^2 + 5.9^2 - 7.3^2}{2 \times 7.3 \times 5.9}$$

$$\angle XYZ = \cos^{-1}\left(\frac{3481}{8614}\right)$$

$\angle XYZ = 66°09'52''$

$\angle XYZ = \angle YZX = 66°10'$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\angle YXZ = \frac{7.3^2 + 7.3^2 - 5.9^2}{2 \times 7.3 \times 7.3}$$

$$\angle YXZ = \cos^{-1}\left(\frac{7177}{10658}\right)$$

$\angle YXZ = 47°40'14''$

$\angle YXZ = 47°40'$

Question 7

a $c^2 = a^2 + b^2 - 2ab\cos C$

$NP^2 = 8.9^2 + 12^2 - 2 \times 12 \times 8.9 \cos 119°15'$

$NP = \sqrt{79.21 + 144 - 213.6\cos 119°15'}$

$NP = 18.0991...$

$NP = 18.1\text{mm}$

b

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\angle NOP = \frac{12.7^2 + 15.6^2 - 18.1^2}{2 \times 12.7 \times 15.6}$$

$$\angle NOP = \cos^{-1}\left(\frac{7704}{39624}\right)$$

$\angle NOP = 78°47'02''$

$\angle NOP = 78°47'$

Question 8

a $c^2 = a^2 + b^2 - 2ab\cos C$

$AC^2 = 8.4^2 + 3.7^2 - 2\times 3.7\times 8.4\cos 42°08'$

$AC = \sqrt{70.56 + 13.69 - 62.16\cos 42°08'}$

$AC = 6.1768...$

$AC = 6.2\,\text{cm}$

b $c^2 = a^2 + b^2 - 2ab\cos C$

$AD^2 = 9.9^2 + 6.2^2 - 2\times 9.9\times 6.2\cos 101°38'$

$AD = \sqrt{98.01 + 38.44 - 122.76\cos 101°38'}$

$AD = 12.6816...$

$AD = 12.7\,\text{cm}$

Question 9

Interior angle of a regular pentagon = 108°

$c^2 = a^2 + b^2 - 2ab\cos C$

$AD^2 = 8^2 + 8^2 - 2\times 8\times 8\cos 108°$

$AD = \sqrt{64 + 64 - 128\cos 108°}$

$AD = 12.9442...$

$AD = 12.9\text{ cm}$

Question 10

a Interior angle of a regular pentagon = 120°

$\angle ABC = 120°$

$\angle BAC = \angle BCA = 30° \ \left(\text{base angles of isosceles triangle}\right)$

$\angle ACD = 90°$

$c^2 = a^2 + b^2 - 2ab\cos C$

$AC^2 = 5.5^2 + 5.5^2 - 2\times 5.5\times 5.5\cos 90°$

$AC = \sqrt{3025\ +3.25\quad 0.5\cos 120\qquad ^\circ}$

$AC = 9.5262...$

$AC = 9.5 \text{ cm}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$AC^2 = 9.5^2 + 5.5^2 - 2\times 9.5\times 5.5\cos 90°$

$AC = \sqrt{90.25 + 30.25 - 104.5\cos 90°}$

$AC = 10.9999...$

$AC = 11 \text{ cm}$

b

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\angle ADF = \frac{9.5^2 + 11^2 - 5.5^2}{2\times 9.5\times 11}$$

$$\angle ADF = \cos^{-1}\left(\frac{181}{209}\right)$$

$$\angle ADF = 29°59'58''$$

$$\angle ADF = 30°$$

Exercise 5.07 Area of a triangle

Question 1

a

$$A=\frac{1}{2}ab\sin C$$
$$A=\frac{1}{2}\times 5\times 4\times \sin 48°27'$$
$$A=7.4837...$$
$$A=7.5\text{ cm}^2$$

b

$$A=\frac{1}{2}ab\sin C$$
$$A=\frac{1}{2}\times 8\times 9\times \sin 63°49'$$
$$A=32.3059...$$
$$A=32.3\text{ units}^2$$

c

$$A=\frac{1}{2}ab\sin C$$
$$A=\frac{1}{2}\times 3\times 7\times \sin 109°$$
$$A=9.9279...$$
$$A=9.9\,\text{mm}^2$$

d

$$A=\frac{1}{2}ab\sin C$$
$$A=\frac{1}{2}\times 8.3\times 7.6\times \sin 73°19'$$
$$A=30.2123...$$
$$A=30.2\text{ units}^2$$

e

$$A=\frac{1}{2}ab\sin C$$
$$A=\frac{1}{2}\times 5.8\times 3.4\times \sin 139°57'$$
$$A=6.3444...$$
$$A=6.3\,\text{cm}^2$$

Question 2

OA = 4 cm (radii of a circle are equal)

$$A=\frac{1}{2}ab\sin C$$
$$A=\frac{1}{2}\times 4\times 4\times \sin 110°$$
$$A=7.5175...$$
$$A=7.5\text{ cm}^2$$

Question 3

$$A=\frac{1}{2}ab\sin C$$
$$A=\left(\frac{1}{2}\times3.5\times4.8\times\sin 67°13\right)\times 2$$
$$A=15.4891...$$
$$A=15.5\text{ cm}^2$$

Question 4

$$A=\frac{1}{2}ab\sin C$$
$$A=\left(\frac{1}{2}\times5\times8\times\sin 119°30\right)\times 2$$
$$A=34.8142...$$
$$A=34.8\text{cm}^2$$

Question 5

$$A=\frac{1}{2}ab\sin C$$
$$A=\frac{1}{2}\times1.7\times1.5\times\sin 65°$$
$$A=1.1555...$$
$$A=1.2\text{m}^2$$

Question 6

a $AE = AB$, $\therefore$ isosceles triangle

$EB = 10.5$

$$\angle AEB=\angle ABE=(180°-84°)\div 2$$
$$\angle AEB=\angle ABE=48°$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$
$$\frac{AE}{\sin 48°}=\frac{10.5}{\sin 85°}$$
$$AE=\frac{105\ \times\sin 48\ °}{\sin 85°}$$
$$AE=7.8328...$$
$$AE=7.8\text{m}$$

b $$A=\frac{1}{2}ab\sin C+lb$$

$$A=\frac{1}{2}\times7.8\times7.8\times\sin 84°$$
$$+10.5\times14.3$$
$$=180.6583...$$
$$=180.7\text{m}^2$$

Question 7

a

$\angle BAC + \angle ABC + \angle ACB = 180°$
$\angle ACB + 44° + 58° = 180°$

$\angle ACB = 78°$

$\angle ACB + \angle ACD = 180°$

$\angle ACD + 78° = 180°$

$\angle ACD = 102°$

$\frac{a}{\sin A} = \frac{b}{\sin B}$

$\frac{AC}{\sin 36°} = \frac{9.4}{\sin 102°}$

$AC = \frac{94 \times \sin 36°}{\sin 102°}$

$AC = 5.6486...$

$AC = 5.6 \text{ cm}$

b

$A = \frac{1}{2} ab \sin C$

$A = \frac{1}{2} \times 9.4 \times 6.7 \times \sin 36°$

$A = 18.5093...$

$A = 18.5 \text{ cm}^2$

c

$\frac{a}{\sin A} = \frac{b}{\sin B}$

$\frac{BC}{\sin 58°} = \frac{5.6}{\sin 44°}$

$BC = \frac{5.6 \times \sin 58°}{\sin 44°}$

$BC = 6.8365...$

$BC = 6.8 \text{ cm}$

$A = \frac{1}{2} ab \sin C$

$A = \frac{1}{2} \times 6.8 \times 5.6 \times \sin 78°$

$A = 18.8863....$

$A = 18.9 \text{ cm}^2$

Question 8

$A = \frac{1}{2}\ ab \sin C$

$= \frac{1}{2} \times 5 \times 5 \times \sin (60°)$

$= \frac{25}{2} \times \frac{\sqrt{3}}{2}$

$= \frac{25\sqrt{3}}{4} \text{ cm}^2$

Exercise 5.08 Mixed problems

Question 1

a

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$
$$\cos\theta = \frac{39^2+68^2-52^2}{2\times 39\times 68}$$
$$\theta = \cos^{-1}\left(\frac{39^2+68^2-52^2}{2\times 39\times 68}\right)$$
$$\theta = 50°$$
$$90° - \theta = \text{bearing}$$
$$90° - 50° = \text{bearing}$$
$$\text{bearing} = 40°$$

b

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$
$$\cos\theta = \frac{52^2+68^2-39^2}{2\times 52\times 68}$$
$$\theta = \cos^{-1}\left(\frac{52^2+68^2-39^2}{2\times 52\times 68}\right)$$
$$\theta = 35°$$
$$270° + \theta = \text{bearing}$$
$$270° + 35° = \text{bearing}$$
$$\text{bearing} = 305°$$

Question 2

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{h}{\sin 54°37'} = \frac{12.8}{\sin(180° - 54°37' - 85°58')}$$
$$h = \frac{12.8\times\sin 54°37'}{\sin 39°25'}$$
$$h = 16.4\,\text{m}$$

Question 3

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$
$$\cos\theta = \frac{8^2+11^2-5.5^2}{2\times 8\times 11}$$
$$\theta = \cos^{-1}\left(\frac{8^2+11^2-5.5^2}{2\times 8\times 11}\right)$$
$$\theta = 284466\quad °$$
$$\approx 28°$$

Question 4

a 324° – 270° = 54°

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{d}{\sin 49°} = \frac{1.3}{\sin 54°}$$
$$d = \frac{1.3 \times \sin 49°}{\sin 54°}$$
$$d = 1.21 \text{km}$$

b $\frac{1.21}{80} \times 60 = 1 \text{minute}$

Question 5

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{d}{\sin 20°} = \frac{80}{\sin 55°}$$
$$d = \frac{80 \times \sin 20°}{\sin 55°}$$
$$d = 33.4 \text{m}$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{d}{\sin 75°} = \frac{33.4}{\sin 90°}$$
$$d = \frac{334 \times \sin 75°}{\sin 90°}$$
$$d = 32 \text{m}$$

Question 6

166° – 65° = 101°

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$x^2 = 50^2 + 85^2 - 2 \times 50 \times 85 \cos 101°$$
$$x = \sqrt{2500 + 7225 - 8500 \cos 101°}$$
$$x = 107 \text{m}$$

Question 7

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{h - 4.1}{\sin 65°} = \frac{3.9}{\sin 54°}$$

$$h - 41 = \frac{39 \times \sin 65°}{\sin 54°}$$

$$h = \frac{39 \times \sin 65°}{\sin 54°} + 4.1$$

$$h = 8.5$$

Question 8

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{d}{\sin 22°} = \frac{15.7}{\sin 130°}$$

$$h = \frac{157 \times \sin 22°}{\sin 130°}$$

$$h = 7.7\,\text{km}$$

Question 9

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 65°} = \frac{3.8}{\sin 40°}$$

$$a = \frac{3.8 \times \sin 65°}{\sin 40°}$$

$$a = 5.4\,\text{km}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{b}{\sin 75°} = \frac{3.8}{\sin 40°}$$

$$b = \frac{3.8 \times \sin 75°}{\sin 40°}$$

$$b = 5.7\,\text{km}$$

Question 10

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{d}{\sin 34°} = \frac{852}{\sin 15°}$$
$$b = \frac{852 \times \sin 34°}{\sin 15°}$$
$$b = 1841\,\text{km}$$

Question 11

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos \angle DAB = \frac{8^2 + 8^2 - 11.3^2}{2 \times 8 \times 8}$$
$$\angle DAB = \cos^{-1}\left(\frac{8^2 + 8^2 - 11.3^2}{2 \times 8 \times 8}\right)$$
$$\angle DAB = 89°52'$$

Question 12

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{d}{\sin 97°} = \frac{8.7}{\sin 61°}$$
$$b = \frac{87 \times \sin 97°}{\sin 61°}$$
$$b = 9.9\,\text{km}$$

Question 13

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$x^2 = 83.7^2 + 105.6^2 - 2 \times 83.7 \times 105.6 \cos 119°$$
$$x = \sqrt{83.7^2 + 105.6^2 - 2 \times 83.7 \times 105.6 \cos 119°}$$
$$x = 163.5\,\text{km}$$

Question 14

360° – 130° – 149° = 81°

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$x^2 = 1280^2 + 3215^2 - 2\times1280\times3215\cos 81°$$
$$x = \sqrt{1280^2 + 3215^2 - 2\times1280\times3215\cos 81°}$$
$$x = 3269\,\text{km}$$

Question 15

a

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$AC^2 = 11.3^2 + 4.6^2 - 2\times4.6\times11.3\cos 78°$$
$$AC = \sqrt{11.3^2 + 4.6^2 - 2\times4.6\times11.3\cos 78°}$$
$$AC = 11.3\,\text{cm}$$

b

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin\angle ADC}{11.3} = \frac{\sin 23°30'}{6.4}$$
$$\sin\angle ADC = \frac{11.3\times\sin 23°30'}{6.4}$$
$$\angle ADC = \sin^{-1}\left(\frac{11.3\times\sin 23°30}{64}\right)$$
$$\angle ADC = 44°45', 180 - 44°45'$$
$$= 44°45', 135°15'$$

Question 16

$$\frac{\sin A}{a}=\frac{\sin B}{b}$$

$$\frac{\sin\alpha}{875}=\frac{\sin 34°}{630}$$

$$\sin\alpha=\frac{875\times\sin 34°}{630}$$

$$\alpha=\sin^{-1}\left(\frac{875\times\sin 34°}{630}\right)$$

$$\alpha=51°$$

180° – 51° – 34° = x

x = 95°

360 – 95° – 124° = θ

θ = 141°

Question 17

a

$$c^2=a^2+b^2-2ab\cos C$$

$$BD^2=7.2^2+7.2^2-2\times 7.2\times 7.2\cos 107°$$

$$BD=\sqrt{7.2^2+7.2^2-2\times 7.2\times 7.2\cos 107°}$$

$$BD=11.6\,\text{cm}$$

b

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

$$\cos\angle DCB=\frac{8.9^2+10.4^2-11.6^2}{2\times 8.9\times 10.4}$$

$$\angle DCB=\cos^{-1}\left(\frac{8.9^2+10.4^2-11.6^2}{2\times 8.9\times 10.4}\right)$$

$$\angle DCB=73°14'$$

Question 18

a

$$\frac{\sin\theta}{2.3} = \frac{\sin 92^\circ}{4}$$

$$\sin\theta = \frac{23 \times \sin 92^\circ}{4}$$

$$\theta = \sin^{-1}\left(\frac{23 \times \sin 92^\circ}{4}\right)$$

$$\theta = 35^\circ 05'$$

b **i**

$$\frac{l}{\sin 92^\circ} = \frac{2.3}{\sin 31^\circ}$$

$$l = \frac{23 \times \sin 92^\circ}{\sin 31^\circ}$$

$$l = 4.5\,\text{m}$$

ii

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 52^\circ 55} = \frac{4}{\sin 92^\circ}$$

$$a = \frac{4 \times \sin 52^\circ 55'}{\sin 92^\circ}$$

$$a = 3.2\,\text{m}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{b}{\sin 57^\circ} = \frac{2.3}{\sin 31^\circ}$$

$$b = \frac{23 \times \sin 57^\circ}{\sin 31^\circ}$$

$$b = 3.75\,\text{m}$$

$b - a = d$

Question 19

a The insert is a right-angled triangle of height 8 cm.

Let the base of the triangle have length *c*, which is the diagonal of the bottom face of the box.

$$c^2 = 25^2 + 11^2 = 746$$
$$c = \sqrt{746}$$

$$\text{Area} = \frac{1}{2} \times \sqrt{746} \times 8$$

$$= 109.2520 \ldots$$

$$\approx 109 \text{ cm}^2$$

b

$$\tan\theta = \frac{8}{\sqrt{746}}$$
$$\theta = 16.325...$$
$$= 16°19'31.26''$$
$$\approx 16°20'$$

Question 20

$$\sin\frac{\theta}{2} = \frac{7}{13}$$
$$\frac{\theta}{2} = 32.5789...$$
$$\theta \approx 65°0'$$

Question 21

a

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{h}{\sin 32°} = \frac{15}{\sin 58°}$$
$$h = \frac{15 \times \sin 32°}{\sin 58°}$$
$$h \approx 9.37 \text{ m}$$

b Let angle of elevation be θ.

$$\tan\theta = \frac{9.37}{20}$$
$$\theta = 251030 \quad °$$
$$\theta \approx 25°$$

Question 22

a

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$
$$\frac{d}{\sin 34°}=\frac{100}{\sin 94°}$$
$$d=\frac{100\times\sin 34°}{\sin 94°}$$
$$d=56\,\text{m}$$

b

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$
$$\frac{h}{\sin 58°}=\frac{56}{\sin 32°}$$
$$h=\frac{56\times\sin 58°}{\sin 32°}$$
$$h=89.7\,\text{m}$$

Question 23

a

$$\sin 25°=\frac{h}{114}$$
$$h=114\times\sin 25°$$
$$h=48\,\text{m}$$

b

$$\sin 22°=\frac{48}{h}$$
$$h=\frac{48}{\sin 22°}$$
$$h\approx 128.6\,\text{m}$$

c

$$c^2=a^2+b^2-2ab\cos C$$
$$x^2=128.6^2+114^2-2\times 128.6\times 114\cos 47°$$
$$x=\sqrt{128.6^2+114^2-2\times 128.6\times 114\cos 47°}$$
$$x=97.7\ \text{m}$$

Question 24

$$950\ \text{m/h}\div 3=\frac{950}{3}$$
$$\frac{a}{\sin A}=\frac{b}{\sin B}$$
$$\frac{x}{\sin 17°}=\frac{3000}{\sin 73°}$$
$$x=\frac{3000\times\sin 17°}{\sin 73°}=917.192\ 044\ 4\,\text{m}$$
$$\frac{y}{\sin 12°}=\frac{3000}{\sin 78°}$$
$$y=\frac{3000\times\sin 12°}{\sin 78°}=637.669\ 685\ \text{m}$$
$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$
$$\cos\theta=\frac{638^2+917^2-(950\div 3)^2}{2\times 638\times 917}$$
$$\theta=\cos^{-1}\left(\frac{638^2+917^2-(950\div 3)^2}{2\times 638\times 917}\right)=11°10'$$

Exercise 5.09 Radians

Question 1

a $\frac{\pi}{5}\times\frac{180}{\pi}=36°$

b $\frac{2\pi}{3}\times\frac{180}{\pi}=120°$

c $\frac{5\pi}{4}\times\frac{180}{\pi}=225°$

d $\frac{7\pi}{6}\times\frac{180}{\pi}=210°$

e $3\pi\times\frac{180}{\pi}=540°$

f $\frac{7\pi}{9}\times\frac{180}{\pi}=140°$

g $\frac{4\pi}{3}\times\frac{180}{\pi}=240°$

h $\frac{7\pi}{3}\times\frac{180}{\pi}=420°$

i $\frac{\pi}{9}\times\frac{180}{\pi}=20°$

j $\frac{5\pi}{18}\times\frac{180}{\pi}=50°$

Question 2

a $135\times\frac{\pi}{180}=\frac{3\pi}{4}$

b $30\times\frac{\pi}{180}=\frac{\pi}{6}$

c $150\times\frac{\pi}{180}=\frac{5\pi}{6}$

d $240\times\frac{\pi}{180}=\frac{4\pi}{3}$

e $300\times\frac{\pi}{180}=\frac{5\pi}{3}$

f $63\times\frac{\pi}{180}=\frac{7\pi}{20}$

g $15\times\frac{\pi}{180}=\frac{\pi}{12}$

h $450\times\frac{\pi}{180}=\frac{5\pi}{2}$

i $225\times\frac{\pi}{180}=\frac{5\pi}{4}$

j $120\times\frac{\pi}{180}=\frac{2\pi}{3}$

Question 3

a $56\times\frac{\pi}{180}=0.98$

b $68\times\frac{\pi}{180}=1.19$

c $127\times\frac{\pi}{180}=2.22$

d $289\times\frac{\pi}{180}=5.04$

e $312\times\frac{\pi}{180}=5.45$

Question 4

a $18°34 \times \dfrac{\pi}{180} = 0.32$

b $35°12 \times \dfrac{\pi}{180} = 0.61$

c $101°56 \times \dfrac{\pi}{180} = 1.78$

d $88°29 \times \dfrac{\pi}{180} = 1.54$

e $50°39 \times \dfrac{\pi}{180} = 0.88$

Question 5

a $1.09 \times \dfrac{180}{\pi} = 62°27'$

b $0.768 \times \dfrac{180}{\pi} = 44°00'$

c $1.16 \times \dfrac{180}{\pi} = 66°28'$

d $0.99 \times \dfrac{180}{\pi} = 56°43'$

e $0.32 \times \dfrac{180}{\pi} = 18°20'$

f $3.2 \times \dfrac{180}{\pi} = 183°21'$

g $2.7 \times \dfrac{180}{\pi} = 154°42'$

h $4.31 \times \dfrac{180}{\pi} = 246°57'$

i $5.6 \times \dfrac{180}{\pi} = 320°51'$

j $0.11 \times \dfrac{180}{\pi} = 6°18'$

Question 6

a $\text{si}0.32 \quad 0{=}.34$

b $\cos 1.5 = 0.07$

c $\text{ta}0.06 \quad 0{=}.06$

d $\cos 0.580.8{=}3$

e $\tan 2.29 = -1.14$

f $\sin 2.8 = 0.33$

g $\tan 5.3 = -1.50$

h $\cos 4.70.0{=}6$

i $\cos 3.9 = -0.73$

j $\sin 2.98 = 0.16$

Question 7

a $\sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

b $\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

c $\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$

d $\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$

e $\tan\left(\frac{\pi}{4}\right)=\frac{1}{1}=1$

f $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$

g $\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

h $\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

i $\tan\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{1}=\sqrt{3}$

Exercise 5.10 Length of an arc

Question 1

a
$l = r\theta$
$l = 4 \times \pi$
$l = 4\pi\,\text{cm}$

b
$l = r\theta$
$l = 3 \times \frac{\pi}{3}$
$l = \pi\ \text{m}$

c
$l = r\theta$
$l = 10 \times \frac{5\pi}{6} = \frac{50\pi}{6}$
$l = \frac{25\pi}{3}\ \text{cm}$

d
$\theta = 30° \times \frac{\pi}{180} = \frac{\pi}{6}$
$l = r\theta$
$l = 3 \times \frac{\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2}\text{cm}$

e
$\theta = 45° \times \frac{\pi}{180} = \frac{\pi}{4}$
$l = r\theta$
$l = 7 \times \frac{\pi}{4} = \frac{7\pi}{4}\ \text{mm}$

Question 2

a
$l = r\theta$
$l = 1.5 \times 0.43$
$l = 0.65\,\text{m}$

b
$l = r\theta$
$l = 3.21 \times 1.22$
$l = 3.92\,\text{cm}$

c
$\theta = 55° \times \frac{\pi}{180} = \frac{11\pi}{36}$
$l = r\theta$
$l = 72\ \times \frac{11\pi}{36}$
$l = 6.91\text{mm}$

d
$\theta = 23°12\ \times \frac{\pi}{180} = 0.404...$
$l = r\theta$
$l = 5.9 \times 0.404...$
$l = 2.39\,\text{cm}$

e
$\theta = 82°35\ \times \frac{\pi}{180} = 1.441...$
$l = r\theta$
$l = 2.1 \times 1.441...$
$l = 3.03\,\text{m}$

Question 3

$\theta = 29°51\ \times \frac{\pi}{180} = 0.520...$
$l = r\theta$
$l = 3.4 \times 0.520...$
$l = 1.8\,\text{m}$

Question 4

$l = r\theta$

$\frac{3\pi}{2} = \frac{\pi}{5} \times \theta$

$\frac{3\pi}{2} \div \frac{\pi}{5} = \theta$

$\theta = 7.5\,\text{m}$

Question 5

$l = r\theta$

$\frac{2\pi}{7} = 3\theta$

$\theta = \frac{2\pi}{21}$

Question 6

$C = 2\pi r$

$300 = 2\pi r$

$\frac{300}{2\pi} = r$

$r = \frac{150}{\pi}$

$l = r\theta$

$l = \frac{150}{\pi} \times \frac{\pi}{6} = 25\,\text{mm}$

Question 7

$A = \pi r^2$

$60 = \pi r^2$

$\frac{60}{\pi} = r^2$

$r = \sqrt{\frac{60}{\pi}}$

$l = r\theta$

$8 = \sqrt{\frac{60}{\pi}} \times \theta$

$8 \div \sqrt{\frac{60}{\pi}} = \theta$

$\theta = 1.83$

Question 8

$\theta = 40° \times \frac{\pi}{180} = \frac{2\pi}{9}$

$C = 2\pi r$

$124 = 2\pi r$

$r = \frac{124}{2\pi} = \frac{62}{\pi}$

$l = r\theta$

$l = \frac{62}{\pi} \times \frac{2\pi}{9} = 13\frac{7}{9}$ mm

Question 9

a

$c = 2r \times \sin\frac{\theta}{2}$

$25 = 2r \times \sin\frac{\pi}{12}$

$r = \dfrac{25}{2\sin\frac{\pi}{12}}$

$r = 48.3\,\text{mm}$

b

$l = r\theta$

$l = 483 \times \frac{\pi}{6}$

$l = 25.3\,\text{mm}$

Question 10

$SA = A = \frac{\theta}{2\pi}\pi r^2 = \frac{\pi}{6\pi} \times \pi \times 5^2 = \frac{25\pi}{6}\text{cm}$

$C_{\text{cone}} = \text{Arc length of sector}$

$\quad = 2\pi \times 5 \times \frac{\frac{\pi}{3}}{2\pi} = \frac{5\pi}{3}$

$r_{\text{cone}} = \frac{5\pi}{3} \div 2\pi = \frac{5}{6}$

$a^2 + b^2 = c^2$

$\left(\frac{5}{6}\right)^2 + h^2 = 5^2$

$h^2 = 5^2 - \left(\frac{5}{6}\right)^2$

$h = \sqrt{25 - \frac{25}{36}} = \sqrt{\frac{875}{36}} = \frac{5\sqrt{35}}{6}$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3} \times \pi \times \left(\frac{5}{6}\right)^2 \times \frac{5\sqrt{35}}{6} = \frac{\pi \times 25 \times 5\sqrt{35}}{3 \times 36 \times 6} = \frac{125\pi\sqrt{35}}{648}\ \text{cm}^3$

Exercise 5.11 Area of a sector

Question 1

a

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 4^2 \times \pi$$
$$A = 8\pi\,\text{cm}^2$$

b

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 3^2 \times \frac{\pi}{3}$$
$$A = \frac{3\pi}{2}\ \text{m}^2$$

c

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 10^2 \times \frac{5\pi}{6}$$
$$A = \frac{500\pi}{12} = \frac{125\pi}{3}\ \text{cm}^2$$

d

$$30°\times\frac{\pi}{180} = \frac{\pi}{6}$$
$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 3^2 \times \frac{\pi}{6}$$
$$A = \frac{9\pi}{12} = \frac{3\pi}{4}\text{cm}^2$$

e

$$45°\times\frac{\pi}{180} = \frac{\pi}{4}$$
$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 7^2 \times \frac{\pi}{4}$$
$$A = \frac{49\pi}{8}\text{mm}^2$$

Question 2

a

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 1.5^2 \times 0.43 = 0.48\,\text{m}^2$$

b

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 3.21^2 \times 1.22 = 6.29\,\text{cm}^2$$

c

$$55°\times\frac{\pi}{180} = \frac{11\pi}{36}$$
$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 7.2^2 \times \frac{11\pi}{36} = 24.88\,\text{mm}^2$$

d

$$23°12\ \times\frac{\pi}{180} = 0.404...$$
$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 5.9^2 \times 0.404... = 7.05\,\text{cm}^2$$

e

$$82°35\ \times\frac{\pi}{180} = 1.441...$$
$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 2.1^2 \times 1.441... = 3.18\,\text{m}^2$$

Question 3

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2} \times 4.3^2 \times 1.8$$
$$A = 16.6\,\text{m}^2$$

Question 4

$$A = \frac{1}{2}r^2\theta$$
$$20 = \frac{1}{2} \times 3^2 \times \theta$$
$$20 = 4.5 \times \theta$$
$$\theta = 4.4$$

Question 5

$$A = \frac{1}{2}r^2\theta$$
$$6\pi = \frac{1}{2} \times r^2 \times \frac{\pi}{3}$$
$$6\pi = r^2 \times \frac{\pi}{6}$$
$$r^2 = 6\pi \div \frac{\pi}{6}$$
$$r = \sqrt{36}$$
$$r = 6$$

Question 6

a

$$30 \times \frac{\pi}{180} = \frac{\pi}{6}$$
$$l = r\theta$$
$$l = 7 \times \frac{\pi}{6}$$
$$l = \frac{7\pi}{6}$$

b

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2} \times 7^2 \times \frac{\pi}{6}$$
$$A = \frac{49\pi}{12}$$

Question 7

$C = 2\pi r$

$185 = 2\pi r$

$r = \dfrac{185}{2\pi}$

$A = \dfrac{1}{2}r^2\theta$

$A = \dfrac{1}{2} \times \left(\dfrac{185}{2\pi}\right)^2 \times \dfrac{\pi}{5} = \dfrac{34\,225}{40\pi} = \dfrac{6845}{8\pi}$

Question 8

$A = \pi r^2$

$200 = \pi r^2$

$r^2 = \dfrac{200}{\pi}$

$r = \sqrt{\dfrac{200}{\pi}}$

$A = \dfrac{1}{2}r^2\theta$

$A = \dfrac{1}{2} \times \left(\sqrt{\dfrac{200}{\pi}}\right)^2 \times \dfrac{3\pi}{4} = \dfrac{600}{8} = 75\,\text{cm}^2$

Question 9

$l = r\theta$

$4.2 = 5.7 \times \theta$

$\theta = \dfrac{14}{19}$

$A = \dfrac{1}{2}r^2\theta$

$A = \dfrac{1}{2} \times (5.7)^2 \times \dfrac{14}{19} = 11.97\,\text{cm}^2$

Question 10

$l = r\theta$

$\frac{\pi}{5} = r\theta$

$r = \frac{\pi}{5\theta}$

$A = \frac{1}{2}r^2\theta$

$\frac{3\pi}{10} = \frac{1}{2} \times \left(\frac{\pi}{5\theta}\right)^2 \times \theta$

$\frac{6\pi}{10} = \frac{\pi^2}{25\theta^2} \times \theta$

$15\pi = \frac{\pi^2}{\theta}$

$\frac{15}{\pi} = \frac{1}{\theta}$

$\theta = \frac{\pi}{15}$

$r = \frac{\pi}{5} \div \theta$

$r = \frac{\pi}{5} \div \frac{\pi}{15}$

$r = 3\text{cm}$

Question 11

a $l = r\theta$

$l = 3 \times \frac{\pi}{7} = \frac{3\pi}{7}$ cm

b $A = \frac{1}{2}r^2\theta$

$A = \frac{1}{2} \times (3)^2 \times \frac{\pi}{7} = \frac{9\pi}{14}$ cm^2

Question 12

a $l = r\theta$

$l = 5\times\frac{\pi}{6} = \frac{5\pi}{6}$ cm

$c = 2r\times\sin\left(\frac{\theta}{2}\right) = 2\times5\times\sin\left(\frac{\pi}{12}\right) = 2.6$ cm

b $A = \frac{1}{2}r^2\theta$

$A = \frac{1}{2}\times(5)^2\times\frac{\pi}{6} = \frac{25\pi}{12}$ cm^2

c $\sin\left(\frac{\pi}{12}\right) = \frac{\text{opp}}{5}$

$5\sin\left(\frac{\pi}{12}\right) = \text{opp}$

Length of chord = $2\times5\sin\left(\frac{\pi}{12}\right)\approx 2.6\,\text{cm}$

Question 13

a $45\times\frac{\pi}{180} = \frac{\pi}{4}$

$c = 2r\times\sin\left(\frac{\theta}{2}\right)$

$8 = 2\times r\times\sin\left(\frac{\pi}{8}\right)$

$r = \frac{8}{2\sin\left(\frac{\pi}{8}\right)} = 10.5\,\text{mm}$

b $A = \frac{1}{2}r^2\theta$

$A = \frac{1}{2}\times(105\ )^2\times\frac{\pi}{4}$

$= 42.9$ mm^2

Question 14

a

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times(4)^2\times\frac{\pi}{4}$$
$$A = \frac{16\pi}{8}\text{cm}^2$$
$$A = 2\pi\,\text{cm}^2$$

b

$$c = 2r\times\sin\left(\frac{\theta}{2}\right)$$
$$c = 2\times4\times\sin\left(\frac{\pi}{8}\right)$$
$$r = 8\sin\left(\frac{\pi}{8}\right) = 3.1\text{cm}$$

c

$$A = \frac{1}{2}ab\sin C$$
$$A = \frac{1}{2}\times4\times4\times\sin\frac{\pi}{4}$$
$$= \frac{1}{2}\times4\times4\times\frac{1}{\sqrt{2}}$$
$$= 4\sqrt{2}\ \text{cm}^2$$

d Area of shaded minor segment

$$= 2\pi - 4\sqrt{2}$$

Question 15

a

$$c = 2r\times\sin\left(\frac{\theta}{2}\right)$$
$$15 = 2\times12\times\sin\left(\frac{\theta}{2}\right)$$
$$15 = 24\sin\left(\frac{\theta}{2}\right)$$
$$\frac{\theta}{2} = \sin^{-}\left(\frac{15}{24}\right)$$
$$\theta = 2\sin^{-1}\left(\frac{15}{24}\right)$$
$$\theta = 1.35$$
$$\theta = 1.35\times\frac{180}{\pi} = 77^\circ\,22'$$

b

$$A = \frac{1}{2}ab\sin C$$
$$A = \frac{1}{2}\times12\times12\times\sin 77^\circ 22'$$
$$A = 70.3\ \text{cm}^2$$

c

$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times(12)^2\times1.35$$
$$A = 9722\ \ -7.3$$
$$A = 26.96\,\text{cm}^2$$

d

$$A_{\text{crcle}} = \pi r^2 = \pi\times12^2$$
$$A_{\text{maj seg cut off}} = 144\pi - 26.96$$
$$= 425.43\,\text{cm}^2$$

Question 16

$$\theta = 100 \times \frac{\pi}{180} = \frac{5\pi}{9}$$

$$l = r\theta$$

$$l = 4 \times \frac{5\pi}{9} = \frac{20\pi}{9} \text{ cm}$$

$$P = 4 + 4 + \frac{20\pi}{9} = \left(8 + \frac{20\pi}{9}\right) \text{ cm}$$

$$\approx 14.98 \text{ cm}$$

Question 17

$$V = A \times w$$

$$A = \frac{1}{2} r^2 \theta$$

$$V = \frac{1}{2} r^2 \theta \times w$$

$$V = \frac{1}{2} \times (15)^2 \times \frac{\pi}{6} \times 6$$

$$A = \frac{225\pi}{2} \text{ cm}^3$$

Test Yourself 5

Question 1

$$l = r\theta$$

$$\frac{25\pi}{8} = r \times \frac{5\pi}{4}$$

$$r = \frac{25\pi}{8} \div \frac{5\pi}{4} = \frac{5}{2} = 2.5 \text{ cm}$$

C

Question 2

A, C

Question 3

180° + 35° = 225° D

S 35° W C

Question 4

$$a^2 + b^2 = c^2$$

$$3^2 + 5^2 = c^2$$

$$c^2 = 34$$

$$c = \sqrt{34}$$

$$\cos\theta = \frac{5}{\sqrt{34}}$$

$$\sin\theta = \frac{3}{\sqrt{34}}$$

Question 5

a $\sin 39°54\ = 0.64$

b $\tan 61°30\ = 1.84$

c $\cos 19°2\ = 0.95$

d $\sin 0.14 = 0.14$

e $\tan 3.5 = 0.37$

Question 6

a $\sin\theta = 0.72$

$\theta = \mathrm{si}0.72$

$\theta = 46°03'$

b $\cos\theta = 0.286$

$\theta = \cos 0.286$

$\theta = 73°23'$

c $\tan\theta = \dfrac{5}{7}$

$\theta = \tan^{-1}\dfrac{5}{7}$

$\theta = 35°32'$

Question 7

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 35 = \frac{100}{h}$$

$$h = \frac{100}{\cos 35}$$

$$h = 122\ \text{km}$$

Question 8

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan 30 = \frac{5}{AB}$$

$$AB = \frac{5}{\tan 30} = 5 \div \frac{1}{\sqrt{3}} = 5\sqrt{3}$$

Question 9

a

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{x}{\sin 32°14'} = \frac{11.8}{\sin 91°53'}$$

$$x = \frac{11.8 \times \sin 32°14'}{\sin 91°53'}$$

$$x = 6.3\,\text{cm}$$

b

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$x^2 = 3.4^2 + 7.1^2 - 2 \times 3.4 \times 7.1 \times \cos 106°25'$$

$$x = \sqrt{3.4^2 + 7.1^2 - 2 \times 3.4 \times 7.1 \times \cos 106°25'}$$

$$x = 8.7\,\text{m}$$

Question 10

a $0.75 \times \frac{180}{\pi} = 42°58'$

b $1.3 \times \frac{180}{\pi} = 74°29'$

c $3.95 \times \frac{180}{\pi} = 226°19'$

d $4.2 \times \frac{180}{\pi} = 240°39'$

e $5.66 \times \frac{180}{\pi} = 324°18'$

Question 11

a

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{8.3}{19.7}$$

$$\theta = \cos^{-}\left(\frac{83}{197}\right) = 65°05'$$

b

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\theta = \frac{5^2 + 8^2 - 9^2}{2 \times 5 \times 8}$$

$$\theta = \cos^{-1}\left(\frac{5^2 + 8^2 - 9^2}{2 \times 5 \times 8}\right) = 84°1'$$

c

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin\theta}{17.3} = \frac{\sin 31°41'}{14.2}$$

$$\sin\theta = \frac{17.3 \times \sin 31°41'}{14.2}$$

$$\theta = \sin^{-1}\left(\frac{17.3 \times \sin 31°41}{142}\right)$$

$$\theta = 39°47', 180° - 39°47'$$

$$= 39°47', 140°13'$$

Question 12

$$A = \frac{1}{2}ab\sin C$$

$$A = \frac{1}{2} \times 8.7 \times 15.4 \times \sin 103° = 65.3\,\text{cm}^2$$

Question 13

a

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{1.8}{3.2}$$

$$\theta = \tan^{-1}\left(\frac{18}{32}\right)$$

$$\theta = 29° + 180°$$

$$\text{Bearing} = 209°$$

b

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan\theta = \frac{3.2}{1.8}$$

$$\theta = \tan^{-1}\left(\frac{32}{18}\right)$$

$$\theta = 61°$$

$$\text{Bearing} = 90° - 61° = 029°$$

Question 14

a

$$\angle ABC + \angle ADC + \angle BAD = 180°$$
$$39° + 42° + \angle BAD = 180°$$

$$\angle BAD = 99°$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{AD}{\sin 39°} = \frac{20}{\sin 99°}$$

$$AD = \frac{20\sin 39°}{\sin 99°}$$

b Let the height of the pole be *h*.

$$\sin 42° = \frac{h}{AD}$$

$$h = AD\sin 42°$$

$$= \frac{20\sin 39° \sin 42°}{\sin 99°}$$

$$\approx 8.5 \text{ m}$$

Question 15

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$x^2 = 2500^2 + 1800^2 - 2\times 2500\times 1800\times \cos 85°$$
$$x = \sqrt{2500^2 + 1800^2 - 2\times 2500\times 1800\times \cos 85°}$$
$$x \approx 2951\text{km}$$

Question 16

a $60°\times\dfrac{\pi}{180} = \dfrac{\pi}{3}$

b $45°\times\dfrac{\pi}{180} = \dfrac{\pi}{4}$

c $150°\times\dfrac{\pi}{180} = \dfrac{5\pi}{6}$

d $180°\times\dfrac{\pi}{180} = \pi$

e $20°\times\dfrac{\pi}{180} = \dfrac{\pi}{9}$

Question 17

a
$$l = r\theta$$
$$l = 5\times\frac{\pi}{6} = \frac{5\pi}{6}\text{ cm}$$

b
$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 5^2\times\frac{\pi}{6} = \frac{25\pi}{12}\text{ cm}^2$$

Question 18

a $\tan\left(\dfrac{\pi}{3}\right) = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

b $\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{\sqrt{3}}{2}$

c $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{1}{\sqrt{2}}$

d $\tan\left(\dfrac{\pi}{6}\right) = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{1}{\sqrt{3}}$

e $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{1}{\sqrt{2}}$

f $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{1}{2}$

g $\tan\left(\dfrac{\pi}{4}\right) = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{1}{1} = 1$

h $\cos\left(\dfrac{\pi}{3}\right) = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{1}{2}$

i $\sin\left(\dfrac{\pi}{3}\right) = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{\sqrt{3}}{2}$

Question 19

a

$$C = 2\pi r$$
$$8\pi = 2\pi r$$
$$r = \frac{8\pi}{2\pi} = 4$$
$$A = \frac{1}{2}r^2\theta$$
$$A = \frac{1}{2}\times 4^2 \times \frac{\pi}{7} = \frac{8\pi}{7}\text{ cm}^2$$

b

$$A = \frac{1}{2}r^2\left(\theta - \sin\theta\right)$$
$$A = \frac{1}{2}\times 4^2 \times \left(\frac{\pi}{7} - \sin\frac{\pi}{7}\right)$$
$$A = 0.12\text{ cm}^2$$

Question 20

The shaded figure is a right-angled triangle.

Let its base length be *c*.

$$c^2 = 12^2 + 4^2 = 160$$
$$c = \sqrt{160}$$
$$\tan\alpha = \frac{\sqrt{160}}{10}$$
$$\alpha = 51.6711...$$
$$\alpha \approx 51°40'$$

Question 21

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin\angle M}{14.9} = \frac{\sin 43°49'}{12.7}$$
$$\sin\angle M = \frac{149\times\sin 43°49'}{12.7}$$
$$\angle M = \sin^{-1}\left(\frac{149\times\sin 43°49}{127}\right)$$
$$\angle M = 54°19' \text{ or } 125°41'$$

Challenge exercise 5

Question 1

70 × 2 = 140 km

80 × 2 = 160 km

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos\theta = \frac{140^2 + 160^2 - 218^2}{2\times140\times160}$$
$$\theta = \cos^{-1}\left(\frac{140^2 + 160^2 - 218^2}{2\times140\times160}\right)$$
$$\theta = 92°58'$$

Question 2

Let y be the longest side in the diagram.

The third angle in the obtuse-angled triangle is 180° – 125° – 28° = 27°.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{y}{\sin 125°} = \frac{15}{\sin 27°}$$
$$y = \frac{15\sin 125°}{\sin 27°} = 27.065...$$

In the bigger right-angled triangle,

$$\sin 28° = \frac{x}{27.065...}$$
$$x = 27065\sin 8 \quad 1^{\circ}2\approx.7\ \text{cm}$$

Question 3

a

$$\angle ACB + \angle ACD = 180°$$
$$\angle ACB + 81°14 \ = 180°$$
$$\angle ACB = 98°46'$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{AC}{\sin 39°53'} = \frac{25.3}{\sin 41°21'}$$
$$AC = \frac{25.3 \times \sin 39°53'}{\sin 41°21'} \text{ cm}$$

b

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{h}{\sin 81°14'} = \frac{AC}{\sin 74°12'}$$
$$h = \frac{AC \times \sin 81°14'}{\sin 74°12'} = 25.2 \text{ cm}$$

Question 4

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{y}{\sin 52°} = \frac{58}{\sin 85°}$$
$$y = \frac{58 \times \sin 52°}{\sin 85°}$$
$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin 43° = \frac{h}{\frac{58 \times \sin 52°}{\sin 85°}}$$
$$h = \frac{58 \times \sin 52°}{\sin 85°} \times \sin 43° = 31 \text{m}$$

Question 5

a

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan 35° = \frac{h}{93}$$
$$h = 93 \times \tan 35° = 65 \text{m}$$

b

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan \theta = \frac{65}{124}$$
$$\theta = \tan^{-} \frac{65}{124} = 27°42'$$

Question 6

a $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

$\tan 65° = \dfrac{100}{x}$

$x = \dfrac{100}{\tan 65°}$

$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

$\tan 69° = \dfrac{100}{y}$

$y = \dfrac{100}{\tan 69°}$

$c^2 = a^2 + b^2 - 2ab\cos C$

$d^2 = \left(\dfrac{100}{\tan 65°}\right)^2 + \left(\dfrac{100}{\tan 69°}\right)^2 - 2\left(\dfrac{100}{\tan 65°}\right)\left(\dfrac{100}{\tan 69°}\right)\cos 40°$

$d = \sqrt{\left(\dfrac{100}{\tan 65°}\right)^2 + \left(\dfrac{100}{\tan 69°}\right)^2 - 2\left(\dfrac{100}{\tan 65°}\right)\left(\dfrac{100}{\tan 69°}\right)\cos 40°}$

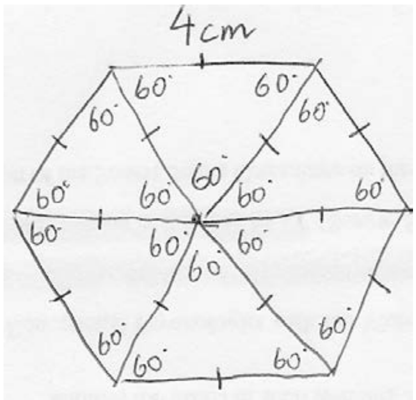
$d \approx 30.1\text{m}$

b speed = distance ÷ time

speed= 30 m ÷ 60 seconds

speed = 0.5 m s^{-1}

Question 7



Divide hexagon into 6 congruent isosceles triangles by drawing an interval from the centre of the hexagon to every vertex.

Each angle at the centre of the pentagon = $\dfrac{360°}{6}$ = 60°.

Each equal angle in each isosceles triangle

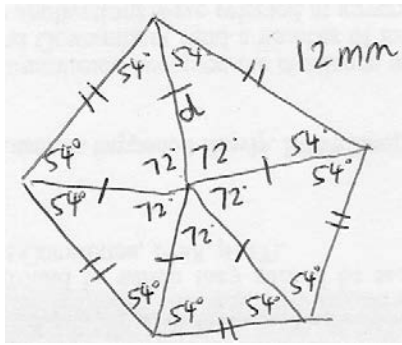
$= \dfrac{180° - 60°}{2} = 60°$

$\therefore$ Each triangle is equilateral (all angles 60°) with sides of length 4 cm.

Area of 1 triangle = $\dfrac{1}{2} \times 4 \times 4 \times \sin 60° = 8 \times \dfrac{\sqrt{3}}{2} = 4\sqrt{3}$

Area of hexagon = $6 \times 4\sqrt{3} = 24\sqrt{3} = 41.5692 \ldots \approx 42\text{ cm}^2$

Question 8



Divide pentagon into 5 congruent isosceles triangles by drawing an interval from the centre of the pentagon to every vertex.

Each angle at the centre of the pentagon = $\dfrac{360°}{5}$ = 72°.

Each equal angle in each isosceles triangle

$= \dfrac{180° - 72°}{2} = 54°$

Let d be the length of one of the equal sides of a triangle.

$$\frac{d}{\sin 54°} = \frac{12}{\sin 72°}$$

$$d = \frac{12 \sin 54°}{\sin 72°}$$

Area of 1 triangle = $\dfrac{1}{2} \times d \times d \times \sin 72° = \dfrac{1}{2} d^2 \sin 72°$

Area of pentagon = $5 \times \dfrac{1}{2} d^2 \sin 72° = \dfrac{5}{2}\left(\dfrac{12 \sin 54°}{\sin 72°}\right)^2 \sin 72° = 247.7487 \ldots \approx 247.7\text{ mm}^2$

Question 9

$$A_{\text{sector}} = \frac{r^2\theta}{2}$$
$$24.3 = r^2\,\frac{\theta}{2}$$
$$\theta r^2 = 48.6$$
$$\theta = \frac{48.6}{r^2}$$
$$l = r\theta$$
$$8.9 = r\theta$$
$$\theta = \frac{8.9}{r}$$
$$\frac{48.6}{r^2} = \frac{8.9}{r}$$
$$48.6 = 8.9r$$
$$r = \frac{48.6}{8.9}$$
$$\theta = 89\ \div \frac{48.6}{8.9} = \frac{79.21}{48.6}$$
$$A = \frac{r^2}{2}\left(\theta - \sin\theta\right)$$
$$A = \frac{1}{2}\times\left(\frac{48.6}{8.9}\right)^2\times\left(\frac{79.21}{48.6} - \sin\frac{79.21}{48.6}\right) = 9.4\,\text{cm}^2$$

Question 10

a

$$l = r\theta$$
$$l = 11\times\frac{\pi}{9}$$
$$l = \frac{11\pi}{9}$$
$$l = 3.84\,\text{cm}$$

b

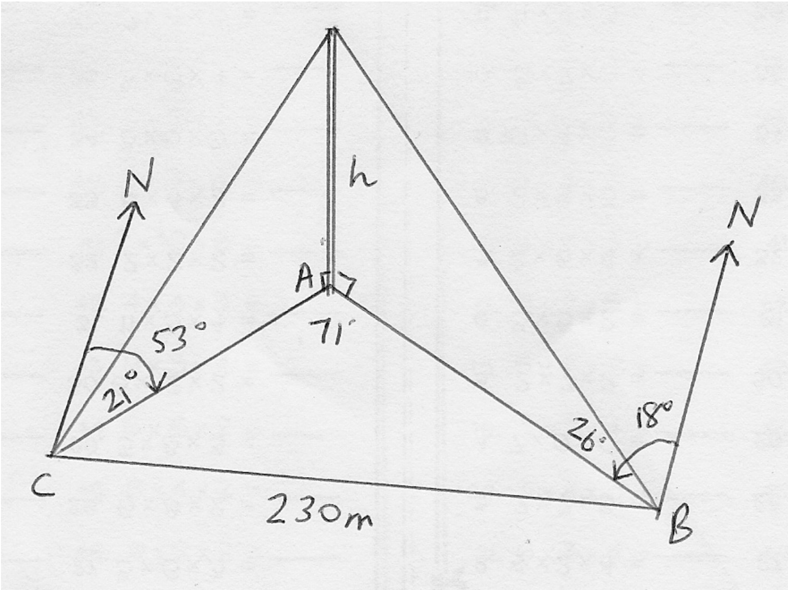
$$A = \frac{1}{2}ab\sin C - r^2\,\frac{\theta}{2}$$
$$A = \frac{1}{2}\times 11\times 4\times\sin\frac{\pi}{2} - 11^2\times\frac{\pi}{18}$$
$$A = 0.88\,\text{cm}^2$$

c

$$P = BC + CD + BD$$
$$P = 11 + 11 + 3.84$$
$$P = 25.84\ \text{cm}$$

Question 11

Diagram shows information, where h is the height of the tower. At point B, tower is on a bearing of 342°, or 360° – 342° = 18° west from north.



For the left triangle:

$$\tan 21° = \frac{h}{AC}$$

$$AC = \frac{h}{\tan 21°}$$

For the right triangle:

$$\tan 26° = \frac{h}{AB}$$

$$AB = \frac{h}{\tan 26°}$$

$BC^2 = AB^2 + AC^2 - 2\,AB \times AC\cos 71°$

$$230^2 = \left(\frac{h}{\tan 26°}\right)^2 + \left(\frac{h}{\tan 21°}\right)^2 - 2\left(\frac{h}{\tan 26°}\right)\left(\frac{h}{\tan 21°}\right)\cos 71°$$

$$52900 = \left(\frac{h^2}{\tan^2 26°}\right) + \left(\frac{h^2}{\tan^2 21°}\right) - 2h^2\left(\frac{\cos 71°}{\tan 26° \tan 21°}\right)$$

$$52900 = h^2\left[\frac{1}{\tan^2 26°} + \frac{1}{\tan^2 21°} - \frac{2\cos 71°}{\tan 26° \tan 21°}\right]$$

$$52900 = h^2(7.5123...)$$

$$h^2 = \frac{52900}{7.5123...} = 7041.719...$$

$h = \sqrt{7041.719...}$ = 83.9149 … ≈ 84 m

Question 12

a

$$\text{Arc of one hour} = \frac{2\pi}{12} = \frac{\pi}{6}$$

$$\theta = \frac{\pi}{6} \times 5 = \frac{5\pi}{6}$$

$$l = r\theta$$

$$l = 12 \times \frac{5\pi}{6} = \frac{60\pi}{6} = 10\pi \approx 31.4\,\text{cm}$$

b

$$\theta = \frac{\pi}{6} \times 2 = \frac{\pi}{3}$$

$$A = \frac{\theta}{2} r^2$$

$$A = \frac{1}{2} \times \frac{\pi}{3} \times 12^2 = 75.4\ \text{cm}^2$$

MATHS IN FOCUS 11
MATHEMATICS EXTENSION 1

WORKED SOLUTIONS

Chapter 6: Polynomials and inverse functions

Exercise 6.01 Division of polynomials

Question 1

$(3x^2+2x+5)\div(x+4)$

$$
\begin{array}{r}
3x-10 \\
x+4\overline{\smash{)}3x^2+2x+5} \\
\underline{3x^2+12x\phantom{{}+5}} \\
-10x+5 \\
\underline{-10x-40} \\
45
\end{array}
$$

$3x^2+2x+5=(x+4)(3x-10)+45$

Question 2

$(x^2+5x-2)\div(x+1)$

$$
\begin{array}{r}
x+4 \\
x+1\overline{\smash{)}x^2+5x-2} \\
\underline{x^2+x\phantom{{}-2}} \\
4x-2 \\
\underline{4x+4} \\
-6
\end{array}
$$

$x^2+5x-2=(x+1)(x+4)-6$

Question 3

$(x^2 - 7x + 4) \div (x - 1)$

$$
\begin{array}{r|l}
 & x - 6 \\
\hline
x - 1 & x^2 - 7x + 4 \\
 & \underline{x^2 - x} \\
 & \quad -6x + 4 \\
 & \quad \underline{-6x + 6} \\
 & \qquad -2
\end{array}
$$

$x^2 - 7x + 4 = (x - 1)(x - 6) - 2$

Question 4

$(x^3 + x^2 + 2x - 1) \div (x - 3)$

$$
\begin{array}{r|l}
 & x^2 + 4x + 14 \\
\hline
x - 3 & x^3 + x^2 + 2x - 1 \\
 & \underline{x^3 - 3x^2} \\
 & \quad 4x^2 + 2x \\
 & \quad \underline{4x^2 - 12x} \\
 & \qquad 14x - 1 \\
 & \qquad \underline{14x - 42} \\
 & \qquad\quad 41
\end{array}
$$

$x^3 + x^2 + 2x - 1 = (x - 3)(x^2 + 4x + 14) + 41$

Question 5

$(4x^2+2x-3)\div(2x+3)$

$$
\begin{array}{r|l}
 & \quad\quad\; 2x-2 \\
\cline{2-2}
2x+3 & 4x^2+2x-3 \\
 & \underline{4x^2+6x} \\
 & \quad\; -4x-3 \\
 & \quad\; \underline{-4x-6} \\
 & \quad\quad\quad\;\; 3
\end{array}
$$

$4x^2+2x-3=(2x-2)(2x+3)+3$

Question 6

$(x^3+x^2-x-3)\div(x-2)$

$$
\begin{array}{r|l}
 & \quad\quad\; x^2+3x+5 \\
\cline{2-2}
x-2 & x^3+\;x^2-x-3 \\
 & \underline{x^3-2x^2} \\
 & \quad 3x^2-x \\
 & \quad \underline{3x^2-6x} \\
 & \quad\quad\quad 5x-3 \\
 & \quad\quad\quad \underline{5x-10} \\
 & \quad\quad\quad\quad\;\; 7
\end{array}
$$

$x^3+x^2-x-3=(x-2)(x^2+3x+5)+7$

Question 7

$(x^4 - x^3 - 2x^2 + x - 3) \div (x + 4)$

$$
\begin{array}{r|l}
 & x^3 - 5x^2 + 18x - 71 \\
\hline
x + 4 & x^4 - x^3 - 2x^2 + x - 3 \\
 & \underline{x^4 + 4x^3} \\
 & -5x^3 - 2x^2 \\
 & \underline{-5x^3 - 20x^2} \\
 & 18x^2 + x \\
 & \underline{18x^2 + 72x} \\
 & -71x - 3 \\
 & \underline{-71x - 284} \\
 & 281
\end{array}
$$

$x^4 - x^3 - 2x^2 + x - 3 = (x + 4)(x^3 - 5x^2 + 18x - 71) + 281$

Question 8

$(4x^3 - 2x^2 + 6x - 1) \div (2x + 1)$

$$
\begin{array}{r|l}
 & 2x^2 - 2x + 4 \\
\hline
2x + 1 & 4x^3 - 2x^2 + 6x - 1 \\
 & \underline{4x^3 + 2x^2} \\
 & -4x^2 + 6x \\
 & \underline{-4x^2 - 2x} \\
 & 8x - 1 \\
 & \underline{8x + 4} \\
 & -5
\end{array}
$$

$4x^3 - 2x^2 + 6x - 1 = (2x + 1)(2x^2 - 2x + 4) - 5$

Question 9

$(3x^5 - 2x^4 - 3x^3 + x^2 - x - 1) \div (x + 2)$

$$
\begin{array}{r}
3x^4 - 8x^3 + 13x^2 - 25x + 49 \\
x + 2 \overline{\smash{\big)}\, 3x^5 - 2x^4 - 3x^3 + x^2 - x - 1} \\
\underline{3x^5 + 6x^4} \\
-8x^4 - 3x^3 \\
\underline{-8x^4 - 16x^3} \\
13x^3 + x^2 \\
\underline{13x^2 + 26x^2} \\
-25x^2 - x \\
\underline{-25x^2 - 50x} \\
49x - 1 \\
\underline{49x + 98} \\
-99
\end{array}
$$

$$3x^5 - 2x^4 - 3x^3 + x^2 - x - 1 = (x + 2)(3x^4 - 8x^3 + 13x^2 - 25x + 49) - 99$$

Question 10

$(x^4 - 2x^2 + 5x + 4) \div (x - 3)$

$$
\begin{array}{r}
x^3 + 3x^2 + 7x + 26 \\
x - 3 \overline{\smash{\big)}\, x^4 + 0x^3 - 2x^2 + 5x + 4} \\
\underline{x^4 - 3x^3} \\
3x^3 - 2x^2 \\
\underline{3x^3 - 9x^2} \\
7x^2 + 5x \\
\underline{7x^2 - 21x} \\
26x + 4 \\
\underline{26x - 78} \\
82
\end{array}
$$

$$x^4 - 2x^2 + 5x + 4 = (x - 3)(x^3 + 3x^2 + 7x + 26) + 82$$

Question 11

$(2x^3+4x^2-x+8)\div(x^2+3x+2)$

$$
\begin{array}{r}
x-5 \\
x^2+3x+2\,\overline{\big)\,2x^3+4x^2-x+8} \\
\underline{2x^3+6x^2+4x} \\
-2x^2-5x+8 \\
\underline{-2x^2-6x-4} \\
x+12
\end{array}
$$

$$2x^3+4x^2-x+8=(2x-2)(x^2+3x+2)+x+12$$

Question 12

$(x^4-2x^3+4x^2+2x+5)\div(x^2+2x-1)$

$$
\begin{array}{r}
x^2-4x+13 \\
x^2+2x-1\,\overline{\big)\,x^4-2x^3+4x^2+2x+5} \\
\underline{x^4+2x^3-x^2} \\
-4x^3+5x^2+2x \\
\underline{-4x^3-8x^2+4x} \\
13x^2-2x+5 \\
\underline{13x^2+26x-13} \\
-28x+18
\end{array}
$$

$$x^4-2x^3+4x^2+2x+5=(x^2+2x-1)(x^2-4x+13)-28x+18$$

Question 13

$$(3x^5 - 2x^3 + x - 1) \div (x + 1)$$

$$\begin{array}{r}
3x^4 - 3x^3 + x^2 - x + 2 \\
x+1 \overline{\smash{\big)}\, 3x^5 + 0x^4 - 2x^3 + 0x^2 + x - 1} \\
\underline{3x^5 + 3x^4} \qquad\qquad\qquad\qquad\quad \\
-3x^4 - 2x^3 \qquad\qquad\qquad \\
\underline{-3x^4 - 3x^3} \qquad\qquad\qquad \\
x^3 + 0x^2 \qquad\qquad \\
\underline{x^3 + x^2} \qquad\qquad \\
-x^2 + x \qquad \\
\underline{-x^2 - x} \qquad \\
2x - 1 \\
\underline{2x + 2} \\
-3
\end{array}$$

$$3x^5 - 2x^3 + x - 1 = (x + 1)(3x^4 - 3x^3 + x^2 - x + 2) - 3$$

Question 14

$$(x^3 - 3x^2 + 3x - 1) \div (x^2 + 5)$$

$$\begin{array}{r}
x - 5 \\
x^2+5 \overline{\smash{\big)}\, x^3 - 3x^2 + 3x - 1} \\
\underline{x^3 + 0x^2 + 5x} \quad\;\; \\
-3x^2 - 2x - 1 \\
\underline{-3x^2 + 0x - 15} \\
-2x + 14
\end{array}$$

$$x^3 - 3x^2 + 3x - 1 = (x^2 + 5)(x - 3) - 2x + 14$$

Question 15

$(2x^4 - 5x^3 + 2x^2 + 2x - 5) \div (x^2 - 2x)$

$$
\begin{array}{r@{}l}
 & \phantom{2x^4 - 5x^3 + {}}2x^2 - x \\
x^2 - 2x \big) & \overline{2x^4 - 5x^3 + 2x^2 + 2x - 5} \\
 & \underline{2x^4 - 4x^3} \\
 & -x^3 + 2x^2 \\
 & -\underline{x^3 + 2x^2} \\
 & \phantom{2x^4 - 5x^3 + {}} 0 + 2x - 5
\end{array}
$$

$2x^4 - 5x^3 + 2x^2 + 2x - 5 = (x^2 - 2x)(2x^2 - x) + 2x - 5$

Exercise 6.02 Remainder and factor theorems

Question 1

a $(x^3 - 2x^2 + x + 5) \div (x - 4)$

$P(4) = 4^3 - 2 \times 4^2 + 4 + 5$
$= 41$

b $(x^2 + 5x + 3) \div (x + 2)$

$P(-2) = (-2)^2 + 5 \times (-2) + 3$
$= -3$

c $(2x^3 - 4x - 1) \div (x + 3)$

$P(-3) = 2 \times (-3)^3 - 4 \times (-3) - 1$
$= -43$

d $(3x^5 + 2x^2 - x + 4) \div (x - 5)$

$P(5) = 3 \times 5^5 + 2 \times 5^2 - 5 + 4$
$= 9424$

e $(5x^3 + 2x^2 + 2x - 9) \div (x - 1)$

$P(1) = 5 \times 1^3 + 2 \times 1^2 + 2 \times 1 - 9$
$= 0$

f $(x^4 - x^3 + 3x^2 - x - 1) \div (x + 2)$

$P(-2) = (-2)^4 - (-2)^3 + 3 \times (-2)^2 - (-2) - 1$
$= 37$

g $(2x^2 + 7x - 2) \div (x + 7)$

$P(-7) = 2 \times (-7)^2 + 7 \times (-7) - 2$
$= 47$

h $(x^7 + 5x^3 - 1) \div (x - 3)$

$P(3) = 3^7 + 5 \times 3^3 - 1$
$= 2321$

i $(2x^6 - 3x^2 + x + 4) \div (x + 5)$

$P(-5) = 2 \times (-5)^6 - 3 \times (-5)^2 + (-5) + 4$
$= 31\ 174$

j $(3x^4 - x^3 - x^2 - x - 7) \div (x + 1)$

$P(-1) = 3 \times (-1)^4 - (-1)^3 - (-1)^2 - (-1) - 7$
$= -3$

Question 2

a $P(x) = 4x^2 - 10x + k$

$P(1) = 3$

$5 \times 1^2 - 10 \times (-1) + k = 3$

$-5 + k = 3$

$k = 8$

b $P(x) = x^3 - (k-1)x^2 + kx + 4$

$P(-2) = -14$

$-14 = (-2)^3 - (k-1)(-2)^2 + 5 \times k \times (-2) + 4$

$-14 = -8 - 4(k-1) - 10k + 4$

$-14 = -14k$

$k = 1$

c $P(x) = 2x^5 + 7x^2 + 1 + k$

$P(-6) = 0$

$0 = 2 \times (-6)^5 + 7 \times (-6)^2 + 1 + k$

$0 = -15552 + 252 + 1 + k$

$k = 15299$

d $2x^4 - kx^3 + 3x^2 + x - 3$

Is divisible by $x - 3$ means $P(3) = 0$

$0 = 2 \times 3^4 - k \times 3^3 + 3 \times 3^2 + 3 - 3$

$0 = 162 - 27k + 27$

$27k = 189$

$k = 7$

e $P(x) = 2x^4 - 3x^2 + 5$

$P(k) = 25$

$25 = 2k^4 - 3k^2 + 5$

$2k^4 - 3k^2 - 20 = 0$

$(2k^2 + 5)(k^2 - 4) = 0$

$2k^2 + 5 = 0$ No real solutions

$k^2 - 4 = 0$

$k = \pm 2$

Question 3

a $f(x) = x^3 - 4x^2 + x + 6$

$f(2) = 2^3 - 4 \times 2^2 + 2 + 6 = 0$

b Yes it is as $f(2) = 0$

c

$$
\begin{array}{r}
x^2 - 2x - 3 \\
x - 2 \overline{\smash{\big)} x^3 - 4x^2 + x + 6} \\
\underline{x^3 - 2x^2} \\
-2x^2 + x \\
\underline{-2x^2 + 4x} \\
-3x + 6 \\
\underline{-3x + 6} \\
0
\end{array}
$$

$x^3 - 4x^2 + x + 6 = (x - 2)(x^2 - 2x - 3)$

d $f(x) = x^3 - 4x^2 + x + 6$

$f(x) = (x - 3)(x^2 - 2x - 3)$

$f(x) = (x - 3)(x - 2)(x + 1)$

Question 4

a $P(x) = x^4 + 3x^3 - 9x^2 - 27x$

$P(-3) = (-3)^4 + 3(-3)^3 - 9(-3)^2 - 27(-3)$

$= 81 - 81 - 81 + 81$

$= 0$

$x + 3$ is a factor of $P(x)$.

b

$$
\begin{array}{r}
x^3 - 9x \\
x + 3 \overline{\smash{\big)} x^4 + 3x^3 - 9x^2 - 27x} \\
\underline{x^4 + 3x^3} \\
0 - 9x^2 - 27x \\
\underline{-9x^2 - 27x} \\
0
\end{array}
$$

$P(x) = x^4 + 3x^3 - 9x^2 - 27x$

$P(x) = x(x + 3)(x^2 - 9)$

$P(x) = x(x - 3)(x + 3)^2$

Question 5

$P(x) = ax^3 - 4bx^2 + x - 4$
$P(3) = 89,\ P(-1) = -3$
Using $P(3) = 89$
$89 = 27a - 36b + 3 - 4$
$90 = 27a - 36b$
$3a - 4b = 10$
Using $P(-1) = -3$
$-3 = -a - 4b - 5$
$2 = -a - 4b$
$a = -2 - 4b$
sub into $3a - 4b = 10$
$3(-2 - 4b) - 4b = 10$
$-6 - 12b - 4b = 10$
$-16b = 16$
$b = -1$
$a = -2 - 4b$
$a = -2 + 4$
$a = 2$
$a = 2, b = -1$

Question 6

$f(x) = ax^2 - 3x + 1$
$g(x) = x^3 - 3x^2 + 2$
$f(-1) = g(-1)$
$a(-1)^2 - 3(-1) + 1 = (-1)^3 - 3(-1)^2 + 2$ $P(x) = x^5 - 2x^4 + 7x^2 - 3x + 5$
$a + 3 + 1 = -1 - 3 + 2$
$a + 4 = -2$
$a = -6$

Question 7

a

$$\begin{aligned} P(3) &= 3^5 - 2\times 3^4 + 7\times 3^2 - 3\times 3 + 5 \\ &= 243 - 162 + 63 - 9 + 5 \\ &= 140 \end{aligned}$$

As $P(3) \neq 0$, x cannot be a factor of $P(x)$

b

$$Q(x) = 2x^3 - 5x + k$$
$$Q(3) = 2\times 3^3 - 5\times 3 + k$$

For $x - 3$ to be a factor $Q(3) = 0$

$Q(3) = 0$

$0 = 54 - 15 + k$

$k = -39$

Question 8

a

$$P(x)=x^3+ax^2+bx+2$$
$$P(-1)=P(2)=0$$
$$P(-1)=(-1)^3+a(-1)^2+b(-1)+2=0$$
$$-1+a-b+2=0$$
$$a-b=-1$$
$$P(2)=2^3+a\times 2^2+b\times 2+2=0$$
$$8+4a+2b+2=0$$
$$4a+2b=-10$$
$$2a+b=-5$$

Solving simultaneously

$$a-b=-1$$
$$2a+b=-5$$
$$3a=-6$$
$$a=-2$$
$$-2-b=-1$$
$$b=-1$$
$$a=-2, b=-1$$

b

$$P(x)=x^3-2x^2-x+2$$
$$P(1)=0$$

$$\begin{array}{r|l} & x^2-x-2 \\ \hline x-1 & x^3-2x^2-x+2 \\ & \underline{x^3-x^2} \\ & \quad -x^2-x \\ & \quad \underline{-x^2+x} \\ & \qquad -2x-2 \\ & \qquad \underline{-2x-2} \\ & \qquad\qquad 0 \end{array}$$

$$P(x)=\left(x-1\right)\left(x^2-x-2\right)$$
$$P(x)=\left(x-1\right)(x+1)(x-2)$$

Question 9

a

$$f(x) = ax^4 + bx^3 + 15x^2 + 9x + 2$$
$$f(2) = 216,\ f(-1) = 0$$
$$f(2) = a \times 2^4 + b \times 2^3 + 15 \times 2^2 + 9 \times 2 + 2 = 216$$
$$16a + 8b + 60 + 18 + 2 = 216$$
$$16a + 8b + 80 = 216$$
$$16a + 8b = 136$$
$$2a + b = 17$$
$$f(-1) = a \times (-1)^4 + b \times (-1)^3 + 15 \times (-1)^2 + 9 \times (-1) + 2 = 0$$
$$a - b + 15 - 9 + 2 = 0$$
$$a - b + 8 = 0$$
$$a - b = -8$$

Solve simultaneously

$$2a + b = 17$$
$$a - b = -8$$
$$3a = 9$$
$$a = 3$$
$$6 + b = 17$$
$$b = 11$$
$$a = 3, b = 11$$

b

$$f(x) = 3x^4 + 11x^3 + 15x^2 + 9x + 2$$

$$\begin{array}{r} 3x^3 + 8x^2 + 7x + 2 \\ x+1 \overline{)\, 3x^4 + 11x^3 + 15x^2 + 9x + 2} \\ \underline{3x^4 + 3x^3} \qquad\qquad\qquad\quad \\ 8x^3 + 15x^2 \qquad\qquad \\ \underline{8x^3 + 8x^2} \qquad\qquad \\ 7x^2 + 9x \qquad \\ \underline{7x^2 + 7x} \qquad \\ 2x + 2 \\ \underline{2x + 2} \\ 0 \end{array}$$

$$f(x) = (x+1)(3x^3 + 8x^2 + 7x + 2)$$

c

$$g(x) = 3x^3 + 8x^2 + 7x + 2$$
$$g(-1) = 3\times(-1)^3 + 8\times(-1)^2 + 7\times(-1) + 2$$
$$= -3 + 8 - 7 + 2$$
$$= 0$$

$x + 1$ is a factor of $g(x)$

d

$$g(x) = 3x^3 + 8x^2 + 7x + 2$$

$$\begin{array}{r@{}l}
 & \underline{\;3x^2 + 5x + 2} \\
x+1 \big) & 3x^3 + 8x^2 + 7x + 2 \\
 & \underline{3x^3 + 3x^2} \\
 & \qquad 5x^2 + 7x \\
 & \qquad \underline{5x^2 + 5x} \\
 & \qquad\qquad 2x + 2 \\
 & \qquad\qquad \underline{2x + 2} \\
 & \qquad\qquad\quad 0
\end{array}$$

$$3x^2 + 5x + 2 = (3x + 2)(x + 1)$$
$$f(x) = (3x + 2)(x + 1)^3$$

Question 10

a

$$P(x) = x^2 - 2x - 8$$
$$P(x) = (x - 4)(x + 2)$$

b

$$P(x) = x^3 + x^2 - 2x$$
$$P(x) = x(x^2 + x - 2)$$
$$P(x) = x(x + 2)(x - 1)$$

c

$$f(x) = x^3 + x^2 - 10x + 8$$
$$f(1) = 1 + 1 - 10 + 8$$
$$= 0$$

$x - 1$ is a factor of $f(x)$

$$\begin{array}{r|l} & x^2 + 2x - 8 \\ \hline x - 1 & x^3 + x^2 - 10x + 8 \\ & \underline{x^3 - x^2} \\ & \quad 2x^2 - 10x \\ & \quad \underline{2x^2 - 2x} \\ & \qquad -8x + 8 \\ & \qquad \underline{-8x + 8} \\ & \qquad\qquad 0 \end{array}$$

$$f(x) = (x - 1)(x^2 + 2x - 8)$$
$$f(x) = (x - 1)(x + 4)(x - 2)$$

d

$$g(x) = x^3 + 4x^2 - 11x - 30$$
$$g(3) = 27 + 36 - 33 - 30$$
$$= 0$$

$x - 3$ is a factor of $g(x)$

$$\begin{array}{r|l} & x^2 + 7x + 10 \\ \hline x - 3 & x^3 + 4x^2 - 11x - 30 \\ & \underline{x^3 - 3x^2} \\ & \quad 7x^2 - 11x \\ & \quad \underline{7x^2 - 21x} \\ & \qquad 10x - 30 \\ & \qquad \underline{10x - 30} \\ & \qquad\qquad 0 \end{array}$$

$$g(x) = (x - 3)(x^2 + 7x + 10)$$
$$g(x) = (x - 3)(x + 2)(x + 5)$$

e

$$G(x)=x^3-11x^2+31x-21$$
$$G(1)=1-11+31-21$$
$$=0$$

$x-1$ is a factor of $G(x)$

$$\begin{array}{r}
x^2-10x+21 \\
x-1\overline{\big)x^3-11x^2+31x-21} \\
\underline{x^3-x^2} \\
-10x^2+31x \\
\underline{-10x^2+10x} \\
21x-21 \\
\underline{21x-21} \\
0
\end{array}$$

$$G(x)=(x-1)(x^2-10x+21)$$
$$G(x)=(x-1)(x-3)(x-7)$$

f

$$P(x)=x^3-12x^2+17x+90$$
$$P(-2)=-8-48-34+90$$
$$=0$$

$x+2$ is a factor of $P(x)$

$$\begin{array}{r}
x^2-14x+45 \\
x+2\overline{\big)x^3-12x^2+17x+90} \\
\underline{x^3+2x^2} \\
-14x^2+17x \\
\underline{-14x^2-28x} \\
45x+90 \\
\underline{45x+90} \\
0
\end{array}$$

$$P(x)=(x+2)(x^2-14x+48)$$
$$P(x)=(x+2)(x-5)(x-9)$$

g

$$Q(x) = x^3 - 7x^2 + 16x - 12$$
$$Q(2) = 8 - 28 + 32 - 12$$
$$= 0$$

$x - 2$ is a factor of $Q(x)$

$$\begin{array}{r}
x^2 - 5x + 6 \\
x - 2 \overline{\smash{\big)}\, x^3 - 7x^2 + 16x - 12} \\
\underline{x^3 - 2x^2} \qquad\qquad\quad \\
-5x^2 + 16x \qquad \\
\underline{-5x^2 + 10x} \qquad \\
6x - 12 \\
\underline{6x - 12} \\
0
\end{array}$$

$$Q(x) = (x - 2)(x^2 - 5x + 6)$$
$$Q(x) = (x - 2)^2(x - 3)$$

h

$$R(x) = x^4 + 6x^3 + 9x^2 + 4x$$
$$R(-1) = 1 - 6 + 9 - 4$$
$$= 0$$

$x + 1$ is a factor of $R(x)$

$$\begin{array}{r}
x^3 + 5x^2 + 4x \\
x + 1 \overline{\smash{\big)}\, x^4 + 6x^3 + 9x^2 + 4x} \\
\underline{x^4 + x^3} \qquad\qquad\quad \\
5x^3 + 9x^2 \qquad \\
\underline{5x^3 + 5x^2} \qquad \\
4x^2 + 4x \\
\underline{4x^2 + 4x} \\
0
\end{array}$$

$$R(x) = x(x + 1)(x^2 + 5x + 4)$$
$$R(x) = x(x + 1)^2(x + 4)$$

Question 11

a

$$P(x) = x^3 - 7x + 6$$
$$P(1) = 1 - 7 + 6$$
$$= 0$$

$x - 1$ is a factor of $P(x)$

$$\begin{array}{r|l} & x^2 + x - 6 \\ \hline x - 1 & x^3 + 0x^2 - 7x + 6 \\ & \underline{x^3 - x^2} \\ & \quad x^2 - 7x \\ & \quad \underline{x^2 - x} \\ & \qquad -6x + 6 \\ & \qquad \underline{-6x - 6} \\ & \qquad\quad 0 \end{array}$$

$$P(x) = (x-1)(x^2 + x - 6)$$
$$P(x) = (x-1)(x+3)(x-2)$$

b

Zeros of $P(x)$ are -3, 1, 2

c

Yes $(x - 2)(x + 3)$ is a factor of $P(x)$ as both $(x - 2)$ and $(x + 3)$ are factors of $P(x)$.

Question 12

$f(x) = x^4 + 10x^3 + 23x^2 - 34x - 120$

a

$(x+5)(x-2) = x^2 + 3x - 10$

$$
\begin{array}{r l}
 & \phantom{x^4 + 10x^3 +{}} x^2 + 7x + 12 \\
x^2 + 3x - 10 \big) & x^4 + 10x^3 + 23x^2 - 34x - 120 \\
 & \underline{x^4 + 3x^3 - 10x^2} \\
 & \phantom{x^4 +{}} 7x^3 + 33x^2 - 34x \\
 & \phantom{x^4 +{}} \underline{7x^3 + 21x^2 - 70x} \\
 & \phantom{x^4 + 10x^3 +{}} 12x^2 + 36x - 120 \\
 & \phantom{x^4 + 10x^3 +{}} \underline{12x^2 + 36x - 120} \\
 & \phantom{x^4 + 10x^3 + 23x^2 +{}} 0
\end{array}
$$

Therefore $x^2 + 3x - 10$ is a factor of $f(x)$

Therefore $(x+5)(x-2)$ is a factor of $f(x)$

b

$f(x) = (x+5)(x-2)(x^2 + 7x + 12)$

$f(x) = (x+5)(x-2)(x+3)(x+4)$

Question 13

$$P(x) = x^4 + 3x^3 - 13x^2 - 51x - 36$$

As $P(x)$ has zeros –3, 4

$(x + 3)$ and $(x - 4)$ are factors of $P(x)$

$$(x+3)(x-4) = x^2 - x - 12$$

$$\begin{array}{r}
x^2 + 4x + 3 \\
x^2 - x - 12 \overline{\smash{\big)}\, x^4 + 3x^3 - 13x^2 - 51x - 36} \\
\underline{x^4 - x^3 - 12x^2} \qquad\qquad \\
4x^3 - x^2 - x \qquad\quad \\
\underline{4x^3 - 4x^2 - 48x} \qquad\quad \\
3x^2 - 3x - 36 \\
\underline{3x^2 - 3x - 36} \\
0 \quad
\end{array}$$

$$P(x) = (x+3)(x-4)(x^2 + 4x + 3)$$
$$P(x) = (x+3)^2(x-4)(x+1)$$

Question 14

$$P(x) = x^3 - 3x^2 - 34x + 120$$

a

$$P(-6) = -216 - 108 + 204 + 120$$
$$= 0$$

–6 is a zero of $P(x)$

$$P(5) = 125 - 75 - 170 + 120$$
$$= 0$$

5 is a zero of $P(x)$

b As $P(x)$ is monic the third linear term is also monic.

From the other 2 factors

$-6 \times 5 \times y = 120$

$y = -4$

the third factor is $(x - 4)$

$P(x) = (x+6)(x-5)(x-4)$

Question 15

a

$$x^2 + 4x - 3 \equiv a(x+1)^2 + b(x+1) + c$$
$$x^2 + 4x - 3 \equiv ax^2 + 2ax + a + bx + b + c$$

Equating coefficients

$a = 1$

$2a + b = 4$

$a + b + c = -3$

$2 + b = 4$

$b = 2$

$1 + 2 + c = -3$

$c = -6$

$a = 1, b = 2, c = -6$

b

$$2x^2 - 3x + 1 \equiv a(x+2)^2 + b(x+2) + c$$
$$2x^2 - 3x + 1 \equiv ax^2 + 4ax + 4a + bx + 2b + c$$

Equating coefficients

$a = 2$

$4a + b = -3$

$4a + 2b + c = 1$

$8 + b = -3$

$b = -11$

$8 - 22 + c = 1$

$c = 15$

$a = 2, b = -11, c = 15$

c

$x^2 - x - 2 \equiv a(x-1)^2 + b(x-1) + c$

$x^2 - x - 2 \equiv ax^2 - 2ax + a + bx - b + c$

Equating coefficients

$a = 1$

$-2a + b = -1$

$a - b + c = -2$

$-2 + b = -1$

$b = 1$

$1 - 1 + c = -2$

$c = -2$

$a = 1, b = 1, c = -2$

d

$x^2 + x + 6 \equiv a(x-3)^2 + b(x-3) + c$

$x^2 + x + 6 \equiv ax^2 - 6ax + 9a + bx - 3b + c$

Equating coefficients

$a = 1$

$-6a + b = 1$

$9a - 3b + c = 6$

$-6 + b = 1$

$b = 7$

$9 - 21 + c = 6$

$c = 18$

$a = 1, b = 7, c = 18$

e

$$3x^2 - 5x - 2 \equiv a(x+1)^2 + b(x-1) + c$$
$$3x^2 - 5x - 2 \equiv ax^2 + 2ax + a + bx - b + c$$
Equating coefficients
$$a = 3$$
$$2a + b = -5$$
$$a - b + c = -2$$
$$6 + b = -5$$
$$b = -11$$
$$3 + 11 + c = -2$$
$$c = -16$$
$$a = 3, b = -11, c = -16$$

f

$$x^3 + 3x^2 - 2x + 1 \equiv ax^3 + b(x-1)^2 + cx + d$$
$$x^3 + 3x^2 - 2x + 1 \equiv ax^3 + bx^2 - 2bx + b + cx + d$$
Equating coefficients
$$a = 1$$
$$b = 3$$
$$-2b + c = -2$$
$$b + d = 1$$
$$-6 + c = -2$$
$$c = 4$$
$$3 + d = 1$$
$$d = -2$$
$$a = 1, b = 3, c = 4, d = -2$$

Question 16

$P(x)$ has zeros – 3, 0 and 4 and is monic

$$P(x) = x(x+3)(x-4)$$
$$P(x) = x(x^2 - x - 12)$$
$$P(x) = x^3 - x^2 - 12x$$

Question 17

$P(x) = ax^3 - bx^2 + cx - 8$

$P(2) = P(-1) = 0, P(3) = 28$

Using $P(2) = 0$

$8a - 4b + 2c - 8 = 0$

Using $P(-1) = 0$

$-a - b - c - 8 = 0$

Using $P(3) = 28$

$27a - 9b + 3c - 8 = 28$

This gives us the 3 equations

$8a - 4b + 2c = 8$

$a + b + c = -8$

$9a - 3b + c = 12$

$6a - 6b = 24$

$8a - 4b = 20$

$12a - 12b = 48$

$24a - 12b = 60$

$12a = 12$

$a = 1$

$12 - 12b = 48$

$-12b = 36$

$b = -3$

$8 + 12 + 2c = 8$

$2c = -12$

$c = -6$

$a = 1, b = -3, c = -6$

Question 18

$P(x)$ has leading term $2x^4$ and zeros – 2, 0, 1 and 3

$P(x) = 2x(x+2)(x-1)(x-3)$

$P(x) = x(x+2)(x^2 - 4x3)$

$P(x) = 2x(x^3 - 4x^2 + 3x + 2x^2 - 8x + 6)$

$P(x) = 2x(x^3 - 2x^2 - 5x + 6)$

$P(x) = 2x^4 - 4x^3 - 10x^2 + 12x$

Question 19

Let $P(x)$ be a polynomial of degree 2 with zeros a_1 and a_2

Then

$$P(x) = k(x - a_1)(x - a_2)$$

Suppose $P(x)$ has another distinct zero a_3

Then

$$P(a_3) = 0$$

These 3 roots give us the polynomial

$$P(x) = k(x - a_1)(x - a_2)(x - a_3)$$

Which gives us a polynomial of degree 3.

Question 20

Let $P(x)$ be a polynomial of degree 3 with zeros a_1 a_2 and a_3

Then

$$P(x) = k(x - a_1)(x - a_2)(x - a_3)$$

Suppose $P(x)$ has another distinct zero a_4

Then

$$P(a_4) = 0$$

These 4 roots give us the polynomial

$$P(x) = k(x - a_1)(x - a_2)(x - a_3)(x - a_4)$$

Which gives us a polynomial of degree 4.

Question 1

a

$P(x)=x^3-4x^2+x+6$

$P(-1)=0$

$\therefore\ x+1$ is a factor of $P(x)$

$$
\begin{array}{r@{}l}
 & x^2-5x+6 \\
x+1 & \overline{)x^3-4x^2+x+6} \\
 & \underline{x^3+x^2} \\
 & -5x^2+x \\
 & \underline{-5x^2-5x} \\
 & 6x+6 \\
 & \underline{6x+6} \\
 & 0
\end{array}
$$

$P(x)=(x+1)(x^2-5x+6)$

$P(x)=(x+1)(x-3)(x-2)$

Zeros of $P(x)$ are -1, 2, 3.

b

$R(x)=x^3-3x^2-x+3$

$R(1)=0$

$\therefore\ x-1$ is a factor of $R(x)$

$$
\begin{array}{r@{}l}
 & x^2-2x-3 \\
x-1 & \overline{)x^3-3x^2-x+3} \\
 & \underline{x^3-x^2} \\
 & -2x^2-x \\
 & \underline{-2x^2+2x} \\
 & -3x+3 \\
 & \underline{-3x+3} \\
 & 0
\end{array}
$$

$P(x)=(x-1)(x^2-2x-3)$

$P(x)=(x-1)(x-3)(x+1)$

Zeros of $P(x)$ are -1, 1, 3.

c

$$P(x) = x^3 - 3x^2 + 6x + 8$$
$$P(1) = 0$$
$\therefore\ x - 1$ is a factor of $P(x)$

$$\begin{array}{rl}
 & x^2 - 2x - 8 \\
x-1 \!\!\!& \overline{)\,x^3 - 3x^2 - 6x + 8} \\
 & \underline{x^3 - x^2} \\
 & \quad -2x^2 - 6x \\
 & \quad \underline{-2x^2 + 2x} \\
 & \qquad\quad -8x + 8 \\
 & \qquad\quad \underline{-8x + 8} \\
 & \qquad\qquad 0
\end{array}$$

$$P(x) = (x-1)(x^2 - 2x - 8)$$
$$P(x) = (x-1)(x-4)(x+2)$$
Zeros of $P(x)$ are -2, 1, 4.

d

$$f(x) = x^3 + x^2 - 16x + 20$$
$$f(2) = 0$$
$\therefore\ x - 2$ is a factor of $f(x)$

$$\begin{array}{rl}
 & x^2 + 3x - 10 \\
x-2 \!\!\!& \overline{)\,x^3 + x^2 - 16x + 20} \\
 & \underline{x^3 - 2x^2} \\
 & \quad 3x^2 - 16x \\
 & \quad \underline{3x^2 - 6x} \\
 & \qquad -10x + 20 \\
 & \qquad \underline{-10x + 20} \\
 & \qquad\qquad 0
\end{array}$$

$$f(x) = (x-2)(x^2 + 3x - 10)$$
$$f(x) = (x+1)(x+5)(x-2)$$
Zeros of $f(x)$ are -5, 2.

e

$$P(x) = x^3 - 11x^2 + 23x + 35$$
$$P(-1) = 0$$
$\therefore$ $x+1$ is a factor of $P(x)$

$$\begin{array}{r|l} & x^2 - 12x + 35 \\ \hline x+1 & x^3 - 11x^2 + 23x + 35 \\ & \underline{x^3 + x^2} \\ & \quad -12x^2 + 23x \\ & \quad \underline{-12x^2 - 12x} \\ & \qquad\quad 35x + 35 \\ & \qquad\quad \underline{35x + 35} \\ & \qquad\qquad\quad 0 \end{array}$$

$$P(x) = (x+1)(x^2 - 12x + 35)$$
$$P(x) = (x+1)(x-5)(x-7)$$
Zeros of $P(x)$ are -1, 5, 7.

f

$$P(x) = x^3 + 7x^2 - 17x + 9$$
$$P(1) = 0$$
$\therefore$ $x-1$ is a factor of $P(x)$

$$\begin{array}{r|l} & x^2 + 8x - 9 \\ \hline x-1 & x^3 + 7x^2 - 17x + 9 \\ & \underline{x^3 - x^2} \\ & \quad 8x^2 - 17x \\ & \quad \underline{8x^2 - 8x} \\ & \qquad\quad -9x + 9 \\ & \qquad\quad \underline{-9x + 9} \\ & \qquad\qquad\quad 0 \end{array}$$

$$P(x) = (x-1)(x^2 + 8x - 9)$$
$$P(x) = (x-1)(x+9)(x-1)$$
Zeros of $P(x)$ are -9, 1.

g

$f(x) = x^4 - 7x^2 + 6x$

$P(-1) = 0$

$\therefore x+1$ is a factor of $P(x)$

$$
\begin{array}{r|l}
 & x^3 - x^2 - 6x \\
\hline
x+1 & x^4 + 0x^3 - 7x^2 - 6x \\
 & \underline{x^4 - x^3} \\
 & \quad -x^3 - 7x^2 \\
 & \quad \underline{-x^3 - x^2} \\
 & \qquad -6x^2 - 6x \\
 & \qquad \underline{-6x^2 - 6x} \\
 & \qquad\qquad 0
\end{array}
$$

$f(x) = (x+1)(x^3 - x^2 - 6x)$

$f(x) = x(x+1)(x-3)(x+2)$

Zeros of $f(x)$ are $-2,\ -1,\ 0,\ 3.$

h

$Q(x)=x^4-x^3-7x^2+x+6$

$Q(1)=0$

$\therefore$ $x-1$ is a factor of $Q(x)$

$$\begin{array}{r}
x^3-7x-6 \\
x-1\overline{)x^4-x^3-7x^2+x+6} \\
\underline{x^4-x^3} \qquad\qquad\qquad \\
0-7x^2+x \quad \\
\underline{-7x^2+7x} \quad \\
-6x+6 \\
\underline{-6x+6} \\
0 \quad
\end{array}$$

$Q(x)=(x-1)(x^3-7x-6)$

Let $f(x)=x^3-7x-6$

$f(-1)=0$

$\therefore$ $x+1$ is a factor of $f(x)$

$$\begin{array}{r}
x^2-x-6 \\
x+1\overline{)x^3+0x^2-7x-6} \\
\underline{x^3+x^2} \qquad\qquad \\
-x^2-7x \quad \\
\underline{-x^2-x} \quad \\
-6x-6 \\
-\underline{6x-6} \\
0 \quad
\end{array}$$

$f(x)=(x+1)(x^2-x-6)$

$f(x)=(x+1)(x-3)(x+2)$

$\therefore$

$Q(x)=(x-1)(x+1)(x-3)(x+2)$

Zeros of $Q(x)$ are -2, -1, 1, 3.

i

$$f(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$$
$$f(1) = 0$$
$\therefore$ $x-1$ is a factor of $f(x)$

$$\begin{array}{r}
x^3 - x^2 - 4x + 4 \\
x-1\overline{)\,x^4 - 2x^3 - 3x^2 + 8x - 4} \\
\underline{x^4 - x^3} \qquad\qquad\qquad\quad \\
-x^3 - 3x^2 \qquad\qquad\quad \\
\underline{-x^2 + x^2} \qquad\qquad\quad \\
-4x^2 + 8x \qquad\quad \\
\underline{-4x^2 + 4x} \qquad\quad \\
4x - 4 \\
\underline{4x - 4} \\
0 \quad
\end{array}$$

$$f(x) = (x-1)(x^3 - x^2 - 4x + 4)$$
Let $P(x) = x^3 - x^2 - 4x + 4$
$$P(1) = 0$$
$\therefore$ $x-1$ is a factor of $P(x)$

$$\begin{array}{r}
x^2 - 4 \\
x+1\overline{)\,x^3 - x^2 - 4x + 4} \\
\underline{x^3 - x^2} \qquad\qquad\quad \\
0 - 4x + 4 \\
\underline{-4x + 4} \\
0 \quad
\end{array}$$

$$P(x) = (x-1)(x^2 - 4)$$
$$P(x) = (x+1)(x-2)(x+2)$$
$\therefore$
$$f(x) = (x-1)^2(x+2)(x-2)$$
Zeros of $f(x)$ are -2, 1, 2.

j

$P(x) = x^4 + 3x^3 - 15x^2 - 19x + 30$

$P(1) = 0$

$\therefore\ x - 1$ is a factor of $P(x)$

$$
\begin{array}{r}
x^3 + 4x^2 - 11x - 30 \\
x - 1 \overline{\smash{\big)}\, x^4 + 3x^3 - 15x^2 - 19x + 30} \\
\underline{x^4 - x^3} \qquad\qquad\qquad\qquad\quad \\
4x^3 - 15x^2 \qquad\qquad\quad \\
\underline{4x^3 - 4x^2} \qquad\qquad\quad \\
-11x^2 - 19x \qquad\quad \\
\underline{-11x^2 + 11x} \qquad\quad \\
-30x + 30 \\
\underline{-30x + 30} \\
0
\end{array}
$$

$P(x) = (x - 1)(x^3 + 4x^2 - 11x - 30)$

Let $f(x) = x^3 + 4x^2 - 11x - 30$

$f(-2) = 0$

$\therefore\ x + 2$ is a factor of $f(x)$

$$
\begin{array}{r}
x^2 + 2x - 15 \\
x + 2 \overline{\smash{\big)}\, x^3 + 4x^2 - 11x - 30} \\
\underline{x^3 + 2x^2} \qquad\qquad\quad \\
2x^2 - 11x \qquad\quad \\
\underline{2x^2 + 4x} \qquad\quad \\
-15x - 30 \\
\underline{-15x - 30} \\
0
\end{array}
$$

$f(x) = (x + 2)(x^2 + 2x - 15)$

$f(x) = (x + 1)(x + 5)(x - 3)$

$\therefore$

$P(x) = (x - 1)(x + 2)(x + 5)(x - 3)$

Zeros of $P(x)$ are $-5, -2, 1, 3.$

Question 2

a

$$x^3+x^2-5x+3=0$$

Let $f(x)=x^3+x^2-5x+3$

$f(1)=0$

$\therefore x-1$ is a factor of $f(x)$

$$\begin{array}{r}
x^2+2x-3 \\
x-1\overline{\big)x^3+x^2-5x+3} \\
\underline{x^3-x^2} \\
2x^2-5x \\
\underline{2x^2-2x} \\
-3x+3 \\
\underline{-3x+3} \\
0
\end{array}$$

$f(x)=(x-1)(x^2+2x-3)$

$f(x)=(x-1)(x+3)(x-1)$

$(x-1)(x+3)(x-1)=0$

$x=\ -3,\ 1$

b

$$x^3-3x^2-x+3=0$$

Let $f(x)=x^3-3x^2-x+3$

$f(1)=0$

$\therefore x-1$ is a factor of $f(x)$

$$\begin{array}{r}
x^2-2x-3 \\
x-1\overline{\big)x^3-3x^2-x+3} \\
\underline{x^3-x^2} \\
-2x^2-x \\
\underline{-2x^2+2x} \\
-3x+3 \\
\underline{-3x+3} \\
0
\end{array}$$

$f(x)=(x-1)(x^2-2x-3)$

$f(x)=(x-1)(x-3)(x+1)$

$(x-1)(x-3)(x+1)=0$

$x=-1, 1, 3$

c

$$x^3 - 9x^2 + 26x - 24 = 0$$
Let $f(x) = x^3 - 9x^2 + 26x - 24$
$f(2) = 0$
$\therefore x - 2$ is a factor of $f(x)$

$$\begin{array}{r l}
 & x^2 - 7x + 12 \\
x - 2 \,\big) & \overline{x^3 - 9x^2 + 26x - 24} \\
 & \underline{x^3 - 2x^2} \\
 & \quad -7x^2 + 26x \\
 & \quad \underline{-7x^2 + 14x} \\
 & \qquad\qquad 12x - 24 \\
 & \qquad\qquad \underline{12x - 24} \\
 & \qquad\qquad\qquad 0
\end{array}$$

$f(x) = (x - 2)(x^2 - 7x + 12)$
$f(x) = (x - 2)(x - 3)(x - 4)$
$(x - 2)(x - 3)(x - 4) = 0$
$x = 2, 3, 4$

d

$$x^3 - 2x^2 - 13x - 10 = 0$$
Let $f(x) = x^3 - 2x^2 - 13x - 10$
$f(-1) = 0$
$\therefore x + 1$ is a factor of $f(x)$

$$\begin{array}{r l}
 & x^2 - 3x - 10 \\
x + 1 \,\big) & \overline{x^3 - 2x^2 - 13x - 10} \\
 & \underline{x^3 + x^2} \\
 & \quad -3x^2 - 13x \\
 & \quad \underline{-3x^2 - 3x} \\
 & \qquad\quad -10x - 10 \\
 & \qquad\quad \underline{-10x - 10} \\
 & \qquad\qquad\quad 0
\end{array}$$

$f(x) = (x + 1)(x^2 - 3x - 10)$
$f(x) = (x + 1)(x - 5)(x + 2)$
$(x + 1)(x - 5)(x + 2) = 0$
$x = -2, -1, 5$

e

$$x^3-10x^2+23x-14=0$$
Let $f(x)=x^3-10x^2+23x-14$
$f(1)=0$
$\therefore x-1$ is a factor of $f(x)$

$$\begin{array}{r l}
 & x^2-9x+14 \\
x-1 \big) & \overline{x^3-10x^2+23x-14} \\
 & \underline{x^3-x^2} \\
 & \quad -9x^2+23x \\
 & \quad \underline{-9x^2+9x} \\
 & \qquad\qquad 14x-14 \\
 & \qquad\qquad \underline{14x-14} \\
 & \qquad\qquad\qquad 0
\end{array}$$

$f(x)=(x-1)(x^2-9x+14)$
$f(x)=(x-1)(x-2)(x-7)$
$(x-1)(x-2)(x-7)=0$
$x=1,\ 2,\ 7$

f

$$x^3-13x-12=0$$
Let $f(x)=x^3-13x-12$
$f(-1)=0$
$\therefore x+1$ is a factor of $f(x)$

$$\begin{array}{r l}
 & x^2-x-12 \\
x+1 \big) & \overline{x^3+0x^2-13x-12} \\
 & \underline{x^3+x^2} \\
 & \quad -x^2-13x \\
 & \quad \underline{-x^2-x} \\
 & \qquad -12x-12 \\
 & \qquad \underline{-12x-12} \\
 & \qquad\qquad 0
\end{array}$$

$f(x)=(x+1)(x^2-x-12)$
$f(x)=(x+1)(x-4)(x+3)$
$(x+1)(x-4)(x+3)=0$
$x=-3,-1,4$

g

$$x^4 - 9x^3 + 11x^2 + 21x = 0$$

Let $f(x) = x^4 - 9x^3 + 11x^2 + 21x$

$f(-1) = 0$

$\therefore\ x + 1$ is a factor of $f(x)$

$$
\begin{array}{r l}
 & \quad\quad\quad x^3 - 10x^2 + 21x \\
x+1 & \overline{\big)\, x^4 - 9x^3 + 11x^2 + 21x} \\
 & \underline{x^4 + x^3} \\
 & \quad -10x^3 + 11x^2 \\
 & \quad \underline{-10x^2 - 10x^2} \\
 & \quad\quad\quad 21x^2 + 21x \\
 & \quad\quad\quad \underline{21x^2 + 21x} \\
 & \quad\quad\quad\quad\quad 0
\end{array}
$$

$f(x) = (x + 1)(x^3 - 10x^2 + 21x)$

$f(x) = x(x + 1)(x - 3)(x - 7)$

$x(x + 1)(x - 3)(x - 7) = 0$

$x = -1,\ 0,\ 3,\ 7$

h

$$x^4 + x^3 - 16x^2 - 4x + 48 = 0$$

Let $f(x) = x^4 + x^3 - 16x^2 - 4x + 48$

$f(-2) = 0$

$\therefore x + 2$ is a factor of $f(x)$

$$
\begin{array}{r}
x^3 - x^2 - 14x + 24 \\
x + 2\,\big)\overline{x^4 + x^3 - 16x^2 - 4x + 48} \\
\underline{x^4 + 2x^3} \\
-x^3 - 16x^2 \\
\underline{-x^3 - 2x^2} \\
-14x^2 - 4x \\
\underline{-14x - 28x} \\
24x + 48 \\
\underline{24x + 48} \\
0
\end{array}
$$

$f(x) = (x-1)(x^3 + 4x^2 - 11x - 30)$

Let $P(x) = x^3 - x^2 - 14x + 24$

$P(2) = 0$

$\therefore x - 2$ is a factor of $f(x)$

$$
\begin{array}{r}
x^2 + x - 12 \\
x - 2\,\big)\overline{x^3 - x^2 - 14x + 24} \\
\underline{x^3 - 2x^2} \\
x^2 - 14x \\
\underline{x^2 - 2x} \\
-12x + 24 \\
-\underline{12x + 24} \\
0
\end{array}
$$

$P(x) = (x-2)(x^2 + x - 12)$

$P(x) = (x-2)(x+4)(x-3)$

$\therefore$

$f(x) = (x+2)(x-2)(x+4)(x-3)$

$(x+2)(x-2)(x+4)(x-3) = 0$

$x = -4, -2, 2, 3$

i $x^4 - 5x^2 + 4 = 0$

is a quadratic equation with respect to x^2

$(x^2 - 1)(x^2 - 4) = 0$

$(x + 1)(x - 1)(x + 2)(x - 2) = 0$

$x = -2, -1, 1, 2$

j

$x^4 - x^3 - 13x^2 + x + 12 = 0$

Let $f(x) = x^4 - x^3 - 13x^2 + x + 12$

$f(1) = 0$

$\therefore x - 1$ is a factor of $f(x)$

$$
\begin{array}{r}
x^3 - 13x - 12 \\
x - 1 \overline{) x^4 - x^3 - 13x^2 + x + 12} \\
\underline{x^4 - x^3 \quad\quad\quad\quad\quad\quad} \\
0 - 13x^2 + x \quad\quad \\
\underline{-13x^2 + 13x} \quad\quad \\
-12x + 12 \\
\underline{-12x + 12} \\
0
\end{array}
$$

$f(x) = (x - 1)(x^3 - 13x - 12)$

From **2f** we know

$(x^3 - 13x - 12) = (x + 1)(x - 4)(x + 3)$

$f(x) = (x - 1)(x + 1)(x - 4)(x + 3)$

$(x - 1)(x + 1)(x - 4)(x + 3) = 0$

$x = -3, -1, 1, 4$

Question 3

a

$$2x^3 - 3x^2 - 3x + 2 = 0$$

Let $f(x) = 2x^3 - 3x^2 - 3x + 2$

$f(-1) = 0$

$\therefore x + 1$ is a factor of $f(x)$

$$\begin{array}{r} 2x^2 - 5x + 2 \\ x+1 \overline{\smash{\big)}\, 2x^3 - 3x^2 - 3x + 2} \\ \underline{2x^3 + 2x^2} \\ -5x^2 - 3x \\ \underline{-5x^2 - 5x} \\ 2x + 2 \\ \underline{2x + 2} \\ 0 \end{array}$$

$f(x) = (x+1)(2x^2 - 5x + 2)$

$f(x) = (x+1)(2x-1)(x-2)$

$(x+1)(2x-1)(x-2) = 0$

$x = -1, \frac{1}{2}, 2$

b

$2x^3 - 3x^2 - 2x + 3 = 0$

Let $f(x) = 2x^3 - 3x^2 - 2x + 3$

$f(1) = 0$

$\therefore x - 1$ is a factor of $f(x)$

$$\begin{array}{r}
2x^2 - x - 3 \\
x-1 \overline{\smash{\big)} 2x^3 - 3x^2 - 2x + 3} \\
\underline{2x^3 - 2x^2} \\
-x^2 - 2x \\
\underline{-x^2 + x} \\
-3x + 3 \\
\underline{-3x + 3} \\
0
\end{array}$$

$f(x) = (x-1)(2x^2 - x - 3)$

$f(x) = (x-1)(2x-3)(x+1)$

$(x-1)(2x-3)(x+1) = 0$

$x = -1, \frac{3}{2}, 1$

c

$5x^3 - 4x^2 - 11x - 2 = 0$

Let $f(x) = 5x^3 - 4x^2 - 11x - 2$

$f(-1) = 0$

$\therefore x + 1$ is a factor of $f(x)$

$$\begin{array}{r}
5x^2 - 9x - 2 \\
x+1 \overline{\smash{\big)} 5x^3 - 4x^2 - 11x - 2} \\
\underline{5x^3 + 5x^2} \\
-9x^2 - 11x \\
\underline{-9x^2 - 9x} \\
-2x - 2 \\
\underline{-2x - 2} \\
0
\end{array}$$

$f(x) = (x+1)(5x^2 - 9x - 2)$

$f(x) = (x+1)(5x+1)(x-2)$

$(x+1)(5x+1)(x-2) = 0$

$x = -1, -\frac{1}{5}, 2$

d

$$4x^3 - 25x^2 + 49x - 30 = 0$$

Let $f(x) = 4x^3 - 25x^2 + 49x - 30$

$f(2) = 0$

$\therefore\ x - 2$ is a factor of $f(x)$

$$
\begin{array}{r}
4x^2 - 17x + 15 \\
x - 2\,\overline{\big)\,4x^3 - 25x^2 + 49x - 30} \\
\underline{4x^3 - 8x^2} \qquad\qquad\quad \\
-17x^2 + 49x \qquad\quad \\
\underline{-17x^2 + 34x} \qquad\quad \\
15x - 30 \\
\underline{15x - 30} \\
0
\end{array}
$$

$f(x) = (x - 2)(4x^2 - 17x + 15)$

$f(x) = (x - 2)(4x - 5)(x - 3)$

$(x - 2)(4x - 5)(x - 3) = 0$

$x = \dfrac{5}{4}, 2, 3$

e

$$6x^3 - 13x^2 + 9x - 2 = 0$$

Let $f(x) = 6x^3 - 13x^2 + 9x - 2$

$f(1) = 0$

$\therefore\ x - 1$ is a factor of $f(x)$

$$
\begin{array}{r}
6x^2 - 7x + 2 \\
x - 2\,\overline{\big)\,6x^3 - 13x^2 + 9x - 2} \\
\underline{6x^3 - 6x^2} \qquad\qquad\quad \\
-7x^2 + 9x \qquad\quad \\
\underline{-7x^2 + 7x} \qquad\quad \\
2x - 2 \\
\underline{2x - 2} \\
0
\end{array}
$$

$f(x) = (x - 1)(6x^2 - 7x + 2)$

$f(x) = (x - 1)(3x - 2)(2x - 1)$

$(x - 1)(3x - 2)(2x - 1) = 0$

$x = \dfrac{1}{2}, \dfrac{2}{3}, 1$

Question 4

$P(x) = x^4 - 6x^3 - 19x^2 + 84x + 180$

$P(-2) = 0$

$\therefore\ x+2$ is a factor of $P(x)$

$$
\begin{array}{r}
x^3 - 8x^2 - 3x + 90 \\
x+2\overline{)x^4 - 6x^3 - 19x^2 + 84x + 180} \\
\underline{x^4 + 2x^3} \qquad\qquad\qquad\qquad \\
-8x^3 - 19x^2 \qquad\qquad\quad \\
\underline{-8x^3 - 16x^2} \qquad\qquad\quad \\
-3x^2 + 84x \qquad\quad \\
\underline{-3x - 6x} \qquad\quad \\
90x + 180 \\
\underline{90x + 180} \\
0
\end{array}
$$

$P(x) = (x+2)(x^3 - 8x^2 - 3x + 90)$

Let $f(x) = x^3 - 8x^2 - 3x + 90$

$f(-3) = 0$

$\therefore\ x+3$ is a factor of $f(x)$

$$
\begin{array}{r}
x^2 - 11x + 30 \\
x+3\overline{)x^3 - 8x^2 - 3x + 90} \\
\underline{x^3 + 3x^2} \qquad\qquad\qquad \\
-11x^2 - 3x \qquad\quad \\
\underline{-11x^2 - 33x} \qquad\quad \\
30x - 30 \\
\underline{30x - 30} \\
0
\end{array}
$$

$f(x) = (x+3)(x^2 - 11x + 30)$

$f(x) = (x+3)(x-5)(x-6)$

$\therefore$

$P(x) = (x+2)(x+3)(x-5)(x-6)$

Zeros of $P(x)$ are

$-3, -2, 5, 6$

Question 5

$2x^4 - 5x^3 + 54x - 2 = 0$

Let $f(x) = 2x^4 - 5x^3 + 54x - 2$

$f(1) = 0$

$\therefore\ x - 1$ is a factor of $f(x)$

$$
\begin{array}{r l}
 & 2x^3 - 3x^2 - 3x + 2 \\
x - 1 \big) & 2x^4 - 5x^3 + 0x^2 + 54x - 2 \\
 & \underline{2x^4 - 2x^3} \\
 & \quad -3x^3 + 0x^2 \\
 & \quad \underline{-3x^3 + 3x^2} \\
 & \qquad -3x^2 + 5x \\
 & \qquad \underline{-3x + 3x} \\
 & \qquad\quad 2x - 2 \\
 & \qquad\quad \underline{2x - 2} \\
 & \qquad\qquad 0
\end{array}
$$

$f(x) = (x - 1)(2x^3 - 3x^2 - 3x + 2)$

Let $P(x) = 2x^3 - 3x^2 - 3x + 2$

$P(-1) = 0$

$\therefore\ x + 1$ is a factor of $f(x)$

$$
\begin{array}{r l}
 & 2x^2 - 5x + 2 \\
x + 1 \big) & 2x^3 - 3x^2 - 3x + 2 \\
 & \underline{2x^3 + 2x^2} \\
 & \quad -5x^2 - 3x \\
 & \quad \underline{-5x^2 - 5x} \\
 & \qquad 2x + 2 \\
 & \qquad \underline{2x + 2} \\
 & \qquad\quad 0
\end{array}
$$

$P(x) = (x + 1)(2x^2 - 5x + 2)$

$P(x) = (x + 1)(2x - 1)(x - 2)$

$\therefore\ f(x) = (x - 1)(x + 1)(2x - 1)(x - 2)$

$(x - 1)(x + 1)(2x - 1)(x - 2) = 0$

$x = -1, \dfrac{1}{2}, 1, 2$

Exercise 6.04 Roots and coefficients of polynomial equations

Question 1

a $x^2 - 2x + 8 = 0$

i $\alpha + \beta = 2$

ii $\alpha\beta = 8$

b $3x^2 + 6x - 2 = 0$

i $\alpha + \beta = -\frac{6}{3} = -2$

ii $\alpha\beta = -\frac{2}{3}$

c $x^2 + 7x + 1 = 0$

i $\alpha + \beta = -7$

ii $\alpha\beta = 1$

d $4x^2 - 9x - 12 = 0$

i $\alpha + \beta = \frac{9}{4}$

ii $\alpha\beta = -\frac{12}{4} = -3$

e $5x^2 + 15x = 0$

i $\alpha + \beta = -\frac{15}{5} = -3$

ii $\alpha\beta = 0$

Question 2

a $x^3 + x^2 - 2x + 8 = 0$

i $\alpha + \beta + \gamma = -1$

ii $\alpha\beta + \alpha\gamma + \beta\gamma = -2$

iii $\alpha\beta\gamma = -8$

b $x^3 - 3x^2 + 5x - 2 = 0$

i $\alpha + \beta + \gamma = 3$

ii $\alpha\beta + \alpha\gamma + \beta\gamma = 5$

iii $\alpha\beta\gamma = 2$

c $2x^3 - x^2 + 6x + 2 = 0$

i $\alpha + \beta + \gamma = \frac{1}{2}$

ii $\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{6}{2} = -3$

iii $\alpha\beta\gamma = -\frac{2}{2} = -1$

d $-x^3 - 3x^2 - 11 = 0$

i $\alpha + \beta + \gamma = -3$

ii $\alpha\beta + \alpha\gamma + \beta\gamma = 0$

iii $\alpha\beta\gamma = -11$

e $x^3 + 7x - 3 = 0$

i $\alpha + \beta + \gamma = 0$

ii $\alpha\beta + \alpha\gamma + \beta\gamma = 7$

iii $\alpha\beta\gamma = 3$

Question 3

a $x^4 + 2x^3 - x^2 - x + 5 = 0$

i $\alpha + \beta + \gamma + \delta = -2$

ii $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -1$

iii $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 1$

iv $\alpha\beta\gamma\delta = 5$

b $x^4 - x^3 - 3x^2 + 2x - 7 = 0$

i $\alpha + \beta + \gamma + \delta = 1$

ii $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -3$

iii $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -2$

iv $\alpha\beta\gamma\delta = -7$

c $-x^4 + x^3 + 3x^2 - 2x + 4 = 0$

i $\alpha + \beta + \gamma + \delta = 1$

ii $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -3$

iii $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -2$

iv $\alpha\beta\gamma\delta = -4$

d $2x^4 - 2x^3 - 4x^2 + 3x - 2 = 0$

i $\alpha + \beta + \gamma + \delta = \frac{2}{2} = 1$

ii $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{-4}{2} = -2$

iii $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{3}{2}$

iv $\alpha\beta\gamma\delta = \frac{-2}{2}\ -1$

e $2x^4 - 22x^3 + 7 = 0$

i $\alpha + \beta + \gamma + \delta = \frac{12}{2} = 6$

ii $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 0$

iii $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 0$

iv $\alpha\beta\gamma\delta = \frac{7}{2}$

Question 4

α and β are the roots of $x^2 - 5x - 5 = 0$

a $\alpha + \beta = 5$

b $\alpha\beta = -5$

c
$$\frac{1}{\alpha} + \frac{1}{\beta}$$
$$= \frac{\alpha + \beta}{\alpha\beta}$$
$$= \frac{5}{-5}$$
$$= -1$$

d $\alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= 5^2 - 2 \times (-5)$

$= 35$

Question 5

α, β and γ are roots of $2x^3 + 5x^2 - x - 3 = 0$

a $\alpha\beta\gamma = \frac{3}{2}$

b $\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{1}{2}$

c $\alpha + \beta + \gamma = -\frac{5}{2}$

d

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$$
$$=\frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$$
$$=\frac{-\frac{1}{2}}{\frac{3}{2}}=-\frac{1}{3}$$

e $(\alpha + 1)(\beta + 1)(\gamma + 1)$

$= (\alpha\beta + \alpha + \beta + 1)(\gamma + 1)$

$= \alpha\beta\gamma + \alpha\gamma + \beta\gamma + \gamma + \alpha\beta + \alpha + \beta + 1$

$= \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \beta\gamma + \alpha + \beta + \gamma + 1$

$$=\frac{3}{2}-\frac{1}{2}-\frac{5}{2}+1$$

$$=-\frac{1}{2}$$

Question 6

α, β, γ and δ are roots of $x^4 - 2x^3 + 5x - 3 = 0$

a $\alpha\beta\gamma\delta = -3$

b $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -5$

c

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha\beta\gamma\delta}$$
$$=\frac{5}{3}$$

Question 7

$x^2 - 3x + k - 2 = 0$

Let $\alpha = -4$

$\alpha + \beta = 3$

$-4 + \beta = 3$

$\beta = 7$

$\alpha\beta = k - 2$

$-4 \times 7 = k - 2$

$k = -28 + 2$

$= -26$

OR substitute $x = -4$ into equation:

$(-4)^2 - 3(-4) + k - 2 = 0$

$16 + 12 + k - 2 = 0$

$26 + k = 0$

$k = -26.$

Question 8

$x^3 - 5x^2 - x + 21 = 0$

Let $\gamma = 3$

$\alpha + \beta + \gamma = 5$

$\alpha + \beta + 3 = 5$

$\alpha + \beta = 2$

$\alpha\beta\gamma = -21$

$3\alpha\beta = -21$

$\alpha\beta = -7$

$\alpha + \beta = 2$, $\alpha\beta = -7$

Question 9

$P(x) = 2x^3 - 7x^2 + 4x + 1$

Let $\alpha = 1$

$\alpha + \beta + \gamma = \frac{7}{2}$

$1 + \beta + \gamma = \frac{7}{2}$

$\beta + \gamma = \frac{5}{2}$

Sum of other roots is $\frac{5}{2}$.

$\alpha\beta\gamma = -\frac{1}{2}$

$\beta\gamma = -\frac{1}{2}$

Product of other roots is $-\frac{1}{2}$.

Question 10

$x^2 - (k + 2)x + k + 1 = 0$

$\alpha + \beta = k + 2$

$\alpha\beta = k + 1$

a Let 2 roots be α

$2\alpha = k + 2$

$\alpha^2 = k + 1$

$k = 2\alpha - 2$

$\alpha^2 = 2\alpha - 2 + 1$

$\alpha^2 - 2\alpha + 1 = 0$

$(\alpha - 1)^2 = 0$

$\alpha = 1$

$k = 2 - 2 = 0$

b Let $\beta = 5$

$\alpha + 5 = k + 2$

$k = \alpha + 3$

$5\alpha = k + 1$

$5\alpha = \alpha + 3 + 1$

$4\alpha = 4$

$\alpha = 1$

$k = 4$

c Let $\beta = \alpha + 1$

$\alpha + \alpha + 1 = k + 2$

$2\alpha = k + 1$

$k = 2\alpha - 1$

$\alpha(\alpha + 1) = k + 1$

$\alpha^2 + \alpha = 2\alpha$

$\alpha^2 - \alpha = 0$

$\alpha(\alpha - 1) = 0$

$\alpha = 0, 1$

$\alpha = 0, k = -1$

$\alpha = 1, k = 1$

d Let $\beta = 2\alpha$

$\alpha + 2\alpha = k + 2$

$3\alpha = k + 2$

$k = 3\alpha - 2$

$\alpha \times 2\alpha = k + 1$

$2\alpha^2 = 3\alpha - 2 + 1$

$2\alpha^2 - 3\alpha + 1 = 0$

$(2\alpha - 1)(\alpha - 1) = 0$

$\alpha = \frac{1}{2}, 1$

$\alpha = \frac{1}{2}, k = -\frac{1}{2}$

$\alpha = 1, k = 1$

$k = -\frac{1}{2}, 1$

e Let $\beta = \frac{1}{\alpha}$

$\alpha + \frac{1}{\alpha} = k + 2$

$\alpha \times \frac{1}{\alpha} = k + 1$

$1 = k + 1$

$k = 0$

Question 11

$x^3 + ax^2 + bx + 24 = 0$

Let $f(x) = x^3 + ax^2 + bx + 24$

$f(4) = 0$, $f(-2) = 0$

$f(4) = 64 + 16a + 4b + 24 = 0$

$16 + 4a + b + 6 = 0$

$4a + b = -22$

$f(-2) = -8 + 4a - 2b + 24 = 0$

$-4 + 2a - b + 12 = 0$

$2a - b = -8$

$6a = -30$

$a = -5$

$-10 - b = -8$

$b = -2$

$a = -5$, $b = -2$

Question 12

$P(x) = x^4 - 2x^3 + 7x - 6$

a $P(1) = 1^4 - 2 \times 1^3 + 7 \times 1 - 6$

$= 1 - 2 + 7 - 6$

$= 0$

$\therefore$ 1 is a zero of $P(x)$

b $\alpha + \beta + \gamma + 1 = 2$

$\alpha + \beta + \gamma = 1$

$\alpha\beta\gamma \times 1 = -6$

$\alpha\beta\gamma = -6$

Question 13

$ax^4 - 2x^3 - 8x + 16 = 0$

Let $P(x) = ax^4 - 2x^3 - 8x + 16$

$P(2) = 0$

$P(2) = 16a - 16 - 16 + 16 = 0$

$16a = 16$

$a = 1$

Let the 4 roots be

$2, 2, \alpha, \beta$

$2 + 2 + \alpha + \beta = 2$

$\alpha + \beta = -2$

Question 14

$x^3 - px^2 - qx + 30 = 0$

a three roots are 3, 5 and α

$3 \times 5 \times \alpha = -30$

$15\alpha = -30$

$\alpha = -2$

b $\alpha + \beta + \gamma = -(-p)$

$3 + 5 - 2 = p$

$p = 6$

$\alpha\beta + \alpha\gamma + \beta\gamma = -q$

$15 - 6 - 10 = -q$

$-1 = -q$

$q = 1$

$p = 6, q = 1$

Question 15

$x^4 + 2x^3 - 18x - 5 = 0$

roots $\alpha, \beta, \gamma, \delta$

Let $\alpha\beta = -5$

$\alpha\beta\gamma\delta = -5$

$\therefore \gamma\delta = 1$

Question 16

$x^4 + x^3 + 7x^2 + 14x - 1 = 0$

roots α, β, γ and δ

Let $\alpha + \beta = 4$

$\alpha + \beta + \gamma + \delta = -1$

$4 + \gamma + \delta = -1$

$\gamma + \delta = -5$

Question 17

$x^3 - 3x^2 + 4$; roots α, α and β

$\alpha + \alpha + \beta = 3$

$\beta = 3 - 2\alpha$ [1]

$\alpha\alpha + \alpha\beta + \alpha\beta = 0$

$\alpha^2 + 2\alpha\beta = 0$

Sub in for β from [1]

$\alpha^2 + 2\alpha(3 - 2\alpha) = 0$

$\alpha^2 + 6\alpha - 4\alpha^2 = 0$

$-3\alpha^2 + 6\alpha = 0$

$3\alpha^2 - 6\alpha = 0$

$3\alpha(\alpha - 2) = 0$

$\alpha = 0, 2$

$\alpha = 0$ cannot be a root

$\alpha = 2$

$\beta = 3 - 2(2) = -1$

So the roots are 2, 2 and –1.

Question 18

$12x^3 - 4x^2 - 3x + 1 = 0$

Sum of two roots = 0 $\Rightarrow \alpha = -\beta$

$\alpha - \alpha + \gamma = \frac{4}{12}$

$\gamma = \frac{1}{3}$

$\alpha(-\alpha)\gamma = -\frac{1}{12}$

$-\alpha^2 \times \frac{1}{3} = -\frac{1}{12}$

$\alpha^2 = \frac{1}{4}$

$\alpha = \pm\frac{1}{2}$

Solutions $\pm\frac{1}{2}, \frac{1}{3}$

Question 19

$6x^4 + 5x^3 - 24x^2 - 15x + 18 = 0$

Roots α, β, γ and δ

Sum of two roots = 0 $\Rightarrow \beta = -\alpha$

$\gamma + \delta = -\frac{5}{6}$

$\alpha(-\alpha) + \alpha\gamma + \alpha\delta - \alpha\gamma - \alpha\delta + \gamma\delta = -\frac{24}{6} \Rightarrow -\alpha^2 + \gamma\delta = -4$

$\alpha(-\alpha)\gamma + \alpha(-\alpha)\delta + \alpha\gamma\delta - \alpha\gamma\delta = \frac{15}{6} \Rightarrow -\alpha^2(\gamma + \delta) = \frac{15}{6}$

$-\alpha^2\left(-\frac{5}{6}\right) = \frac{15}{6}$

$\alpha^2 = 3$

$\alpha = \pm\sqrt{3}$

Using $-\alpha^2 + \gamma\delta = -4$, $-3 + \gamma\delta = -4 \Rightarrow \gamma\delta = -1$

Using $\gamma + \delta = -\frac{5}{6}$, $\gamma = -\frac{5}{6} - \delta$

Sub $\gamma = -\frac{5}{6} - \delta$ into $\gamma\delta = -1$

$-\delta(-\frac{5}{6} + \delta) = -1$

$\delta^2 + \frac{5}{6}\delta = 1$

$6\delta^2 + 5\delta - 6 = 0$

$(3\delta - 2)(2\delta + 3) = 0$

$\delta = -\frac{3}{2}\ \ \frac{2}{3}$

Solutions $\pm\sqrt{3}$, $-\frac{3}{2}\ \ \frac{2}{3}$

Question 20

$x^3 + mx^2 - 3x - 18 = 0$

Two roots are equal: α, α and β

$\alpha^2 + 2\alpha\beta = -3$

$\alpha^2\beta = 18$

$\beta = \frac{18}{\alpha^2}$

$\alpha^2 + 2\alpha\frac{18}{\alpha^2} + 3 = 0$

$\alpha^3 + 3\alpha + 36 = 0$

Let $f(\alpha) = \alpha^3 + 3\alpha + 36$

$f(-3) = 0$

$\therefore \alpha + 3$ is a factor of $f(\alpha)$

$$
\begin{array}{r}
\alpha^2 - 3\alpha + 12 \\
\alpha + 3 \overline{\big)\ \alpha^3 + 0\alpha^2 + 3\alpha + 36} \\
\underline{\alpha^3 + 3\alpha^2} \qquad\qquad \\
-3\alpha^2 + 3\alpha \qquad \\
\underline{-3\alpha^2 - 9\alpha} \qquad \\
12\alpha + 36 \\
\underline{12\alpha + 36} \\
0
\end{array}
$$

$f(\alpha) = (\alpha + 3)(\alpha^2 - 3\alpha + 12)$

However $\alpha^2 - 3\alpha + 12$ has no real solutions, $\therefore$ $\alpha = -3$

$\beta = \frac{18}{(-3)^2} = \frac{18}{9} = 2$

Sum of the roots one at a time

$\alpha + \alpha + \beta = -m$

$-3 - 3 + 2 = -m$

$m = 4$

Question 1

$P(x) = x^3 - 3x^2 - 4x + 12$

a

$$\begin{array}{r|l} & x^2 - x - 6 \\ x-2 & x^3 - 3x^2 - 4x + 12 \\ & \underline{x^3 - 2x^2} \\ & \quad -x^2 - 4x \\ & \quad \underline{-x^2 + 2x} \\ & \qquad -6x + 12 \\ & \qquad \underline{-6x + 12} \\ & \qquad\qquad 0 \end{array}$$

Remainder is 0 therefore $x - 2$ is a factor of $P(x)$.

b $P(x) = (x - 2)(x^2 - x - 6) = (x - 2)(x - 3)(x + 2)$

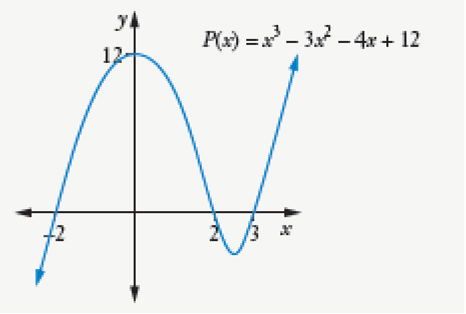
c y-intercept occurs when $x = 0$; y-intercept = 12

x-intercept occurs when $y = 0$.

$0 = (x - 2)(x - 3)(x + 2)$

$x = \pm 2, 3$

$P(x)$ has a leading positive coefficient.



Question 2

a $P(x) = x^3 + 3x^2 - 10x - 24$

y-intercept occurs when $x = 0$; y-intercept = -24

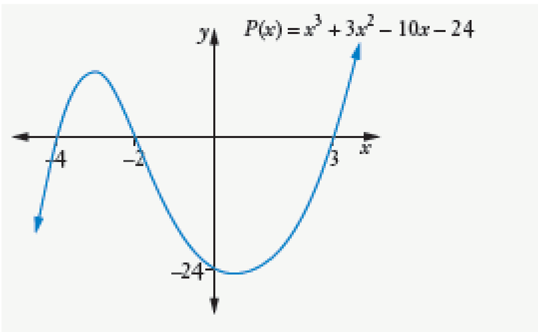
x-intercept occurs when $y = 0$.

$P(3) = 0$; $x - 3$ is a factor of $P(x)$

$$
\begin{array}{r}
x^2 + 6x + 8 \\
x - 3 \overline{\big) x^3 + 3x^2 - 10x + 24} \\
\underline{x^3 - 3x^2} \qquad\qquad \\
6x^2 - 10x \qquad \\
\underline{6x^2 - 18x} \qquad \\
8x - 24 \\
\underline{8x - 24} \\
0
\end{array}
$$

$$P(x) = (x - 3)(x^2 + 6x + 8)$$
$$P(x) = (x - 3)(x + 2)(x + 4)$$
$$x = -4, -2, \ 3$$

$P(x)$ has a leading positive coefficient.



b $P(x) = x^3 + x^2 - 9x - 9$

y-intercept occurs when $x = 0$; y-intercept = –9

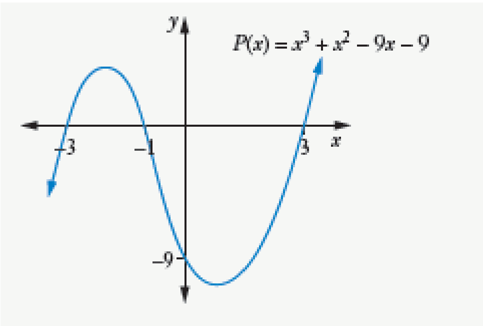
x-intercept occurs when $y = 0$.

$P(-1) = 0$; $x + 1$ is a factor of $P(x)$

$$\begin{array}{r l}
 & x^2-9 \\
x+1 \,\big) & x^3+3x^2-10x+24 \\
 & \underline{x^3+x^2} \\
 & 0-9x-9 \\
 & \underline{-9x-9} \\
 & 0
\end{array}$$

$$P(x) = (x+1)(x^2-9)$$
$$P(x) = (x+1)(x-3)(x+3)$$
$$x = -3,\ -1,\ 3$$

$P(x)$ has a leading positive coefficient.



c $P(x) = 12 - 19x + 8x^2 - x^3$

y-intercept occurs when $x = 0$; y-intercept = 12

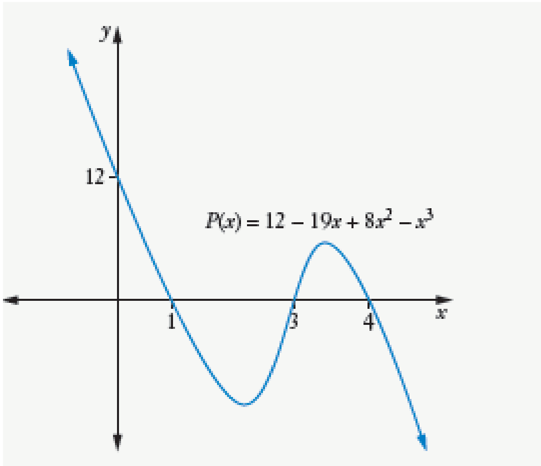
x-intercept occurs when $y = 0$

$P(1) = 0$; $x - 1$ is a factor of $P(x)$

$$\begin{array}{r l}
 & -x^2 + 7x - 12 \\
x-1 \big) & -x^3 + 8x^2 - 19x + 12 \\
 & \underline{-x^3 + x^2} \\
 & \quad 7x^2 - 19x \\
 & \quad \underline{7x^2 - 7x} \\
 & \qquad -12x + 12 \\
 & \qquad \underline{-12x + 12} \\
 & \qquad\qquad 0
\end{array}$$

$$P(x) = (x-1)(-x^2 + 7x - 12)$$
$$P(x) = (x-1)(-x+3)(x-4)$$
$$x = 1,\ 3,\ 4$$

$P(x)$ has a leading negative coefficient.



d $P(x) = x^3 - 13x + 12$

y-intercept occurs when $x = 0$; y-intercept = 12

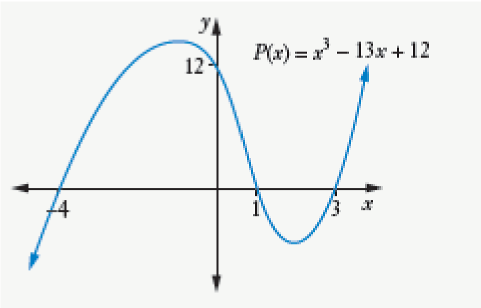
x-intercept occurs when $y = 0$

$P(1) = 0$; $x - 1$ is a factor of $P(x)$

$$\begin{array}{r l}
 & x^2 + x - 12 \\
x - 1 \,\big) & \overline{x^3 + 0x^2 - 13x + 12} \\
 & \underline{x^3 - x^2} \\
 & \quad x^2 - 13x \\
 & \quad \underline{x^2 - x} \\
 & \qquad -12x + 12 \\
 & \qquad \underline{-12x + 12} \\
 & \qquad\qquad 0
\end{array}$$

$P(x) = (x-1)(x^2 + x - 12)$
$P(x) = (x+1)(x+4)(x-3)$
$x = -4, 1, 3$

$P(x)$ has a leading positive coefficient.



e $P(x) = -x^3 + 2x^2 + 9x - 18$

y-intercept occurs when $x = 0$; y-intercept = -18

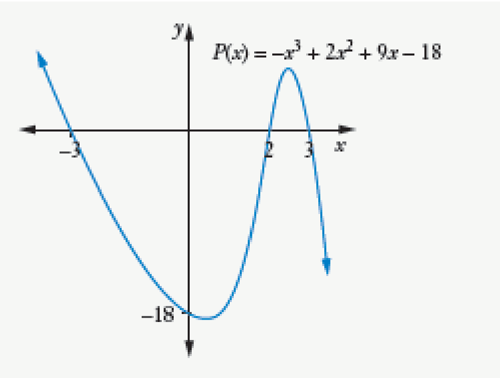
x-intercept occurs when $y = 0$

$P(2) = 0$; $x - 2$ is a factor of $P(x)$

$$\begin{array}{r}
-x^2+9 \\
x-2\,\big)\overline{-x^3+2x^2+9x-18} \\
\underline{-x^3+2x^2} \\
0+9x-18 \\
\underline{+9x-18} \\
0
\end{array}$$

$$P(x) = (x-2)(-x^2+9)$$
$$P(x) = (x+1)(3-x)(x+3)$$
$$x = -3,\ 2,\ 3$$

$P(x)$ has a leading negative coefficient.



f $P(x) = x^3 + 2x^2 - 4x - 8$

y-intercept occurs when $x = 0$; y-intercept $= -8$

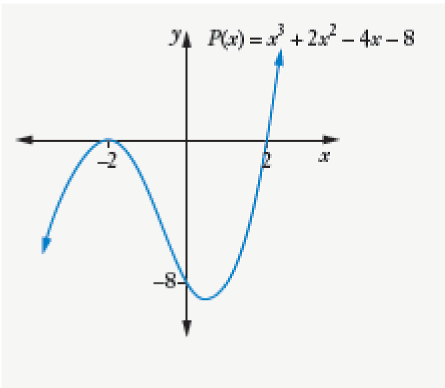
x-intercept occurs when $y = 0$

$P(2) = 0$; $x - 2$ is a factor of $P(x)$

$$
\begin{array}{r}
x^2 + 4x + 4 \\
x - 2 \enclose{longdiv}{x^3 + 2x^2 - 4x - 8} \\
\underline{x^3 - 2x^2} \qquad\qquad \\
4x^2 - 4x \qquad \\
\underline{6x^2 - 8x} \qquad \\
4x - 8 \\
\underline{4x - 8} \\
0
\end{array}
$$

$$P(x) = (x - 2)(x^2 + 4x + 4)$$
$$P(x) = (x - 3)(x + 2)^2$$
$$x = \pm 2$$

$P(x)$ has a leading positive coefficient.



g $P(x) = x^3 - 5x^2 + 8x - 4$

y-intercept occurs when $x = 0$; y-intercept $= -4$

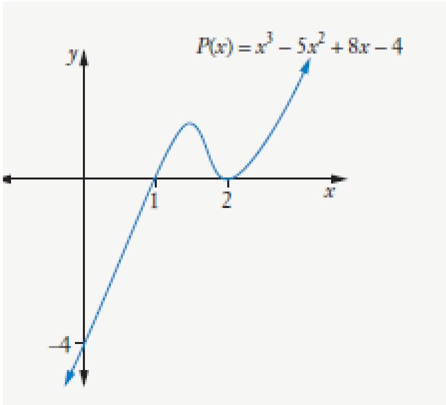
x-intercept occurs when $y = 0$

$P(1) = 0$; $x - 1$ is a factor of $P(x)$

$$\begin{array}{r@{}l}
 & \quad\; x^2 - 4x + 4 \\
x-1 \big) & \overline{x^3 - 5x^2 + 8x - 4} \\
 & \underline{x^3 - x^2} \\
 & \quad -4x^2 + 8x \\
 & \quad \underline{-4x^2 + 4x} \\
 & \qquad\quad 4x - 4 \\
 & \qquad\quad \underline{4x - 4} \\
 & \qquad\qquad 0
\end{array}$$

$$P(x) = (x-1)(x^2 - 4x + 4)$$
$$P(x) = (x-3)(x-2)^2$$
$$x = 1, 2$$

$P(x)$ has a leading positive coefficient.



h $P(x) = x^3 + x^2 - 5x + 3$

y-intercept occurs when $x = 0$; y-intercept = 3

x-intercept occurs when $y = 0$

$P(1) = 0$; $x - 1$ is a factor of $P(x)$

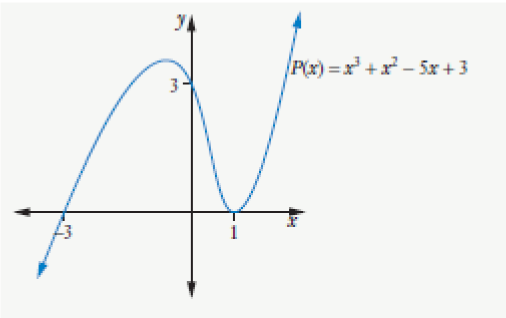
$$
\begin{array}{r}
x^2 + 2x - 3 \\
x-1 \overline{\big) x^3 + x^2 - 5x + 3} \\
\underline{x^3 - x^2} \qquad\qquad \\
2x^2 - 5x \qquad \\
\underline{2x^2 + 2x} \qquad \\
-3x + 3 \\
\underline{-3x + 3} \\
0
\end{array}
$$

$P(x) = (x-1)(x^2 + 2x - 3)$

$P(x) = (x+3)(x-1)^2$

$x = -3,\ 1$

$P(x)$ has a leading positive coefficient.



i $f(x) = 16x + 12x^2 - x^4$

y-intercept occurs when $x = 0$; y-intercept = 0

x-intercept occurs when $f(x) = 0$

$f(0) = 0$, $f(4) = 0$

Let $P(x) = 16 + 12x - x^3$

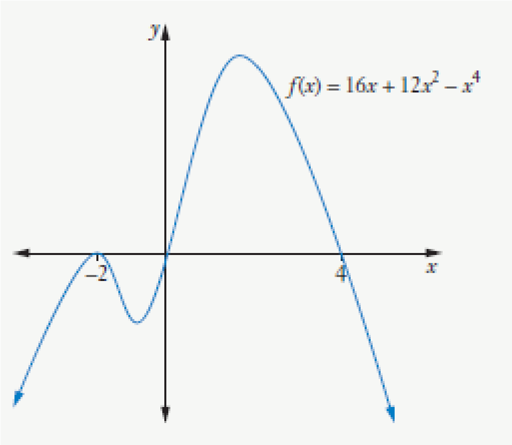
$f(x) = xP(x)$

$P(4) = 0$; $x - 4$ is a factor of $P(x)$

$$\begin{array}{r}
-x^2-4x-4 \\
x-4\overline{)-x^3+0x^2+12x+16} \\
\underline{-x^3+4x^2} \qquad\qquad \\
-4x^2+12x \quad\;\; \\
\underline{-4x^2+16x} \quad\;\; \\
-4x+16 \\
\underline{-4x+16} \\
0
\end{array}$$

$$P(x) = -(x-4)(x^2+4x+4)$$
$$P(x) = (x-4)(x+2)^2$$
$$f(x) = -x(x-4)(x+2)^2$$
$$x = -2, 0, 4$$

$f(x)$ has a leading negative coefficient.



j $P(x) = x^4 - 2x^2 + 1$

y-intercept occurs when $x = 0$; y-intercept = 1

x-intercept occurs when $P(x) = 0$

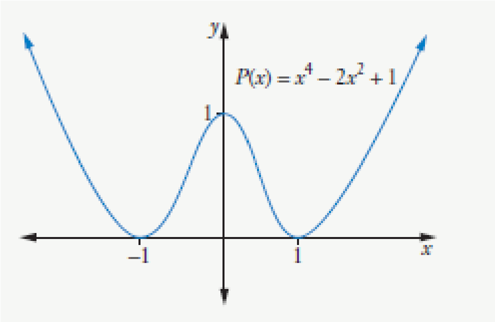
This is a quadratic in terms of x^2.

$P(x) = (x^2 - 1)(x^2 - 1) = (x - 1)(x + 1)(x - 1)(x + 1)$

$= (x - 1)^2(x + 1)^2$

$x = \pm 1$

$P(x)$ has a leading positive coefficient.

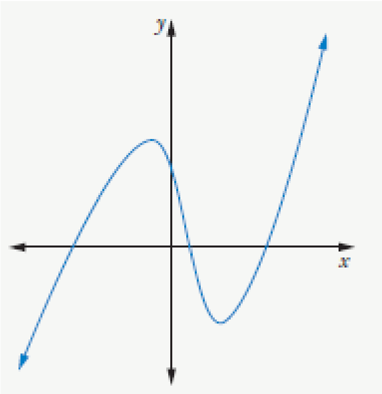


Question 3

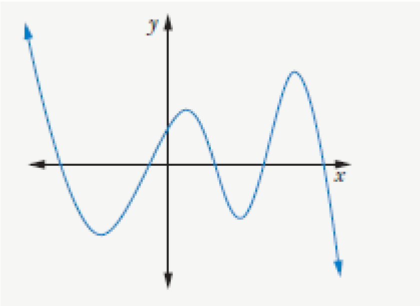
a	**i**	Positive	**ii**	Even
b	**i**	Negative	**ii**	Odd
c	**i**	Negative	**ii**	Even
d	**i**	Negative	**ii**	Odd
e	**i**	Positive	**ii**	Odd
f	**i**	Positive	**ii**	Even
g	**i**	Positive	**ii**	Odd
h	**i**	Negative	**ii**	Even
i	**i**	Positive	**ii**	Odd
j	**i**	Positive	**ii**	Even

Question 4

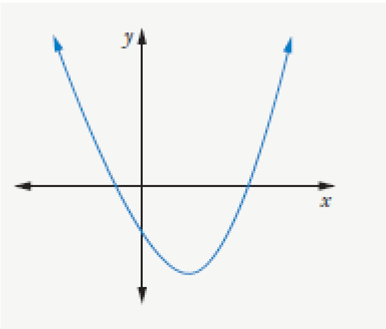
a



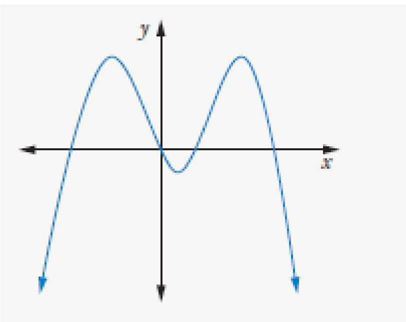
b



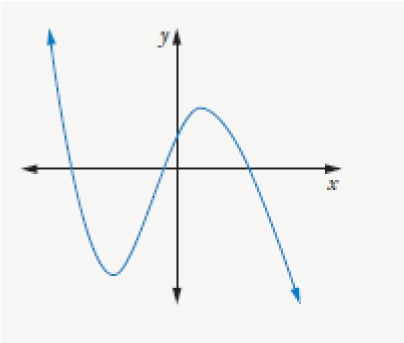
c



d



e



Exercise 6.06 Multiple roots

Question 1

a $P(x) = x^2 - 6x + 9 = (x - 3)^2$

$x = 3$, a double root.

b $P(x) = x^3 - 9x^2 + 14x = x(x^2 - 9x + 14) = x(x - 2)(x - 7)$

$x = 0, 2, 7$, single roots.

c $P(x) = x^3 - 3x^2 = x^2(x - 3)$

$x = 0$, a double root, and $x = 3$, a single root.

d

$$f(x) = x^3 - 2x^2 - 4x + 8$$
$$f(2) = 0$$
$$\therefore x - 2 \text{ is a factor of } f(x)$$

$$\begin{array}{r|l} & x^2 - 4 \\ \hline x-2 & x^3 - 2x^2 - 4x + 8 \\ & \underline{x^3 - 2x^2} \\ & 0 - 4x + 8 \\ & \underline{-4x + 8} \\ & 0 \end{array}$$

$$f(x) = (x-2)(x^2-4)$$
$$f(x) = (x-2)(x-2)(x+2)$$

$x = -2$, a single root, and $x = 2$, a double root.

e $P(x) = x^3 - 6x^2 + 12x - 8$

$P(2) = 0$

$\therefore x - 2$ is a factor of $P(x)$

$$
\begin{array}{r|l}
 & x^2 - 4x + 4 \\
\hline
x-2 & x^3 - 6x^2 + 12x - 8 \\
 & \underline{x^3 - 2x^2} \\
 & \quad -4x^2 + 12x \\
 & \quad \underline{-4x^2 + 8x} \\
 & \qquad\quad 4x - 8 \\
 & \qquad\quad \underline{4x - 8} \\
 & \qquad\qquad 0
\end{array}
$$

$P(x) = (x-2)(x^2 - 4x + 4)$

$P(x) = (x-2)(x-2)(x-2)$

$x = 2$, a triple root.

f

$A(x) = x^4 - 4x^3 + 5x^2 - 2x$

$A(1) = 0$

$x - 1$ is a factor of $A(x)$

$$
\begin{array}{r|l}
 & x^3 - 3x^2 + 2x \\
\hline
x-1 & x^4 - 4x^3 + 5x^2 - 2x \\
 & \underline{x^4 - x^3} \\
 & \quad -3x^3 + 5x^2 \\
 & \quad \underline{-3x^3 + 3x^2} \\
 & \qquad 2x^2 + 4 - 2x \\
 & \qquad \underline{2x^2 - 2x} \\
 & \qquad\quad 0
\end{array}
$$

$A(x) = x(x-1)(x^2 - 3x + 2)$

$A(x) = x(x-1)^2(x-2)$

$x = 0$, 2, single roots, and $x = 1$, a double root.

g

$P(x) = x^4 - 4x^3 - 2x^2 + 12x + 9$

$P(-1) = 0$

$\therefore\ x+1$ is a factor of $P(x)$

$$
\begin{array}{r}
x^3 - 5x^2 + 3x + 9 \\
x+1 \overline{\big)\, x^4 - 4x^3 - 2x^2 + 12x + 9} \\
\underline{x^4 + x^3} \qquad\qquad\qquad\qquad \\
-5x^3 - 2x^2 \qquad\qquad\quad \\
\underline{-5x^3 - 5x^2} \qquad\qquad\quad \\
3x^2 + 12x \qquad\quad \\
\underline{3x^2 + 3x} \qquad\quad \\
9x + 9 \\
\underline{9x + 9} \\
0
\end{array}
$$

$P(x) = (x-1)(x^3 - 5x^2 + 3x + 9)$

Let $f(x) = x^3 - 5x^2 + 3x + 9$

$f(-1) = 0$

$\therefore\ x+1$ is a factor of $f(x)$

$$
\begin{array}{r}
x^2 - 6x + 9 \\
x+1 \overline{\big)\, x^3 - 5x^2 + 3x + 9} \\
\underline{x^3 + x^2} \qquad\qquad\qquad \\
-6x^2 + 3x \qquad\quad \\
\underline{-6x^2 - 6x} \qquad\quad \\
9x - 6 \\
\underline{9x - 6} \\
0
\end{array}
$$

$f(x) = (x+1)(x^2 - 6x + 9)$

$f(x) = (x+1)(x-3)(x-3)$

$\therefore$

$P(x) = (x+1)^2(x-3)^2$

$x = -1, 3$, double roots.

h $Q(x) = x^5 - 8x^4 + 16x^3 = x^3(x^2 - 8x + 16)$

$= x^3(x - 4)^2$

$x = 0$, a triple root, and $x = 4$, a double root.

i

$$P(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$$
$$P(1) = 0$$
$\therefore x - 1$ is a factor of $P(x)$

$$\begin{array}{r} x^3 + 3x^2 - 9x + 5 \\ x-1 \overline{\big) x^4 + 2x^3 - 12x^2 + 14x - 5} \\ \underline{x^4 - x^3} \qquad\qquad\qquad\quad \\ 3x^3 - 12x^2 \qquad\qquad \\ \underline{3x^3 - 3x^2} \qquad\qquad \\ -9x^2 + 14x \qquad \\ \underline{-9x^2 + 9x} \qquad \\ 5x - 5 \\ \underline{5x - 5} \\ 0 \end{array}$$

$$P(x) = (x-1)(x^3 + 3x^2 - 9x + 5)$$
Let $f(x) = x^3 + 3x^2 - 9x + 5$
$$f(1) = 0$$
$\therefore x - 1$ is a factor of $f(x)$

$$\begin{array}{r} x^2 + 4x - 5 \\ x-1 \overline{\big) x^3 + 3x^2 - 9x + 5} \\ \underline{x^3 - x^2} \qquad\qquad\quad \\ 4x^2 - 9x \qquad \\ \underline{4x^2 - 4x} \qquad \\ -5x + 5 \\ \underline{-5x + 5} \\ 0 \end{array}$$

$$f(x) = (x-1)(x^2 + 4x - 5)$$
$$f(x) = (x-1)(x-1)(x+5)$$
$\therefore$
$$P(x) = (x-1)^3(x+5)$$

$x = -5$, a single root, and $x = 1$, a triple root.

j $f(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$

$$f(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$$
$$f(1) = 0$$
$\therefore x - 1$ is a factor of $f(x)$

$$\begin{array}{r}
x^3 + 6x^2 + 12x + 8 \\
x - 1 \overline{\smash{\big)} x^4 + 5x^3 + 6x^2 - 4x - 8} \\
\underline{x^4 - x^3 } \\
6x^3 + 6x^2 \\
\underline{6x^3 - 6x^2} \\
12x^2 - 4x \\
\underline{12x^2 - 12x} \\
8x - 8 \\
\underline{8x - 8} \\
0
\end{array}$$

$$f(x) = (x - 1)(x^3 + 6x^2 + 12x + 8)$$
Let $g(x) = x^3 + 6x^2 + 12x + 8$
$$g(-2) = 0$$
$\therefore x + 2$ is a factor of $g(x)$

$$\begin{array}{r}
x^2 + 4x + 4 \\
x + 2 \overline{\smash{\big)} x^3 + 6x^2 + 12x + 8} \\
\underline{x^3 + 2x^2} \\
4x^2 + 12x \\
\underline{4x^2 + 8x} \\
4x + 8 \\
\underline{4x + 8} \\
0
\end{array}$$

$$g(x) = (x + 2)(x^2 + 4x + 4)$$
$$= (x + 2)(x + 2)^2$$
$$= (x + 2)^3$$
$$\therefore f(x) = (x - 1)(x + 2)^3$$

$x = -2$, a triple root, and $x = 1$, a single root.

Question 2

$P(x) = (x + 4)^2$

Yes, it is unique as $P(x)$ must be monic.

Question 3

a Polynomial of degree 3 with a triple root at $x = 1$.

$P(x) = a(x - 1)^3$, where $a \in \mathbf{R}$

It is not unique as $a \in \mathbf{R}$.

b $P(2) = 5$

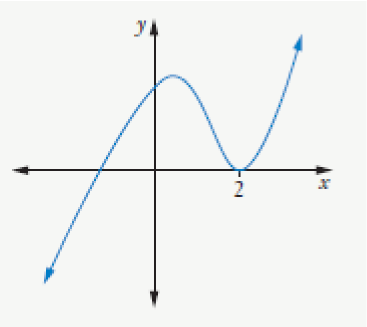
$5 = a(2 - 1)^3$

$a = 5$

$P(x) = 5(x - 1)^3$

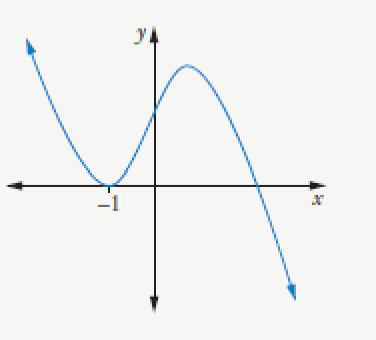
Question 4

Double root at $x = 2$, and leading term $2x^3$ (positive).



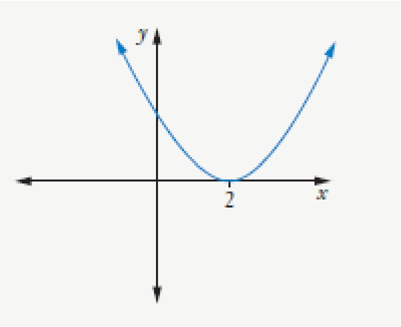
Question 5

Double root at $x = -1$, and leading term $-x^3$ (negative).



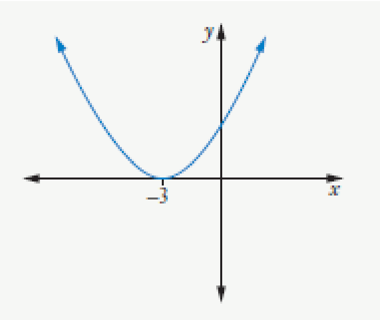
Question 6

Double root at $x = 2$, and leading term x^4 (positive).



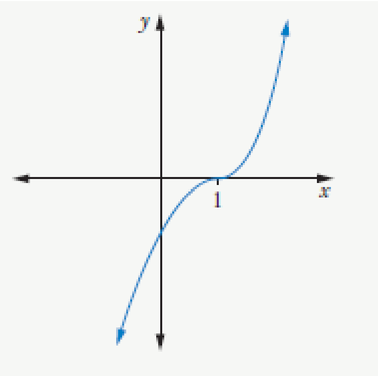
Question 7

Double root at $x = -3$, and leading term x^6 (positive).



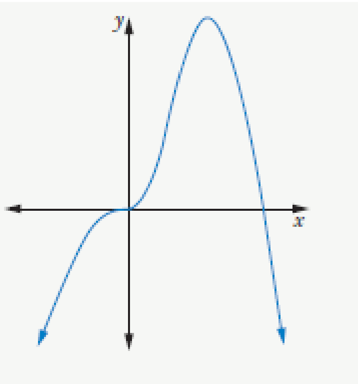
Question 8

Triple root at $x = 1$, and leading term x^3 (positive).



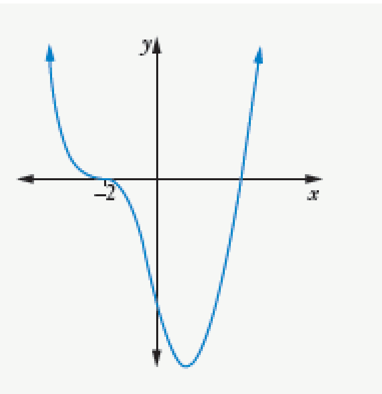
Question 9

Triple root at $x = 0$, and leading term $-x^4$ (negative).



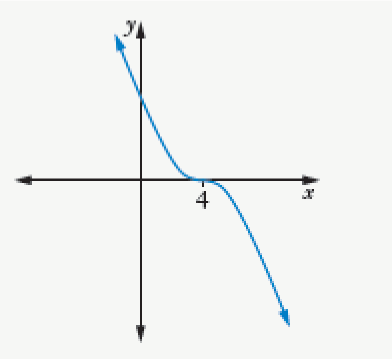
Question 10

Triple root at $x = -2$, and leading term x^8 (positive).



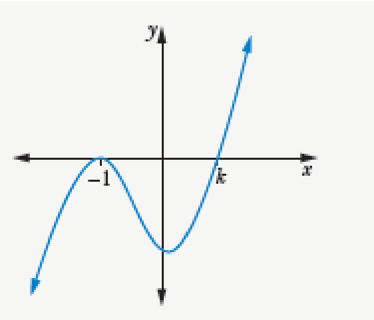
Question 11

Triple root at $x = 4$, and leading term $-4x^3$ (negative).



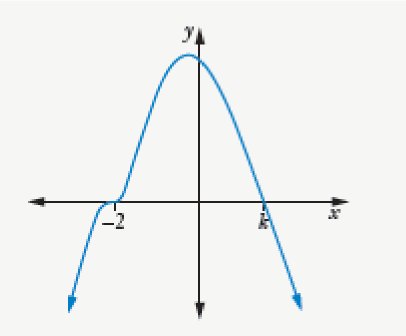
Question 12

A cubic function with positive leading coefficient starts negative and touches the x-axis at the double root. It then becomes positive as x becomes very large so it must cross the x-axis again. So there is another root at k as shown on the graph. Alternatively, k could be on the left side of –1, so that the curve touches the x-axis as a minimum point at $x = -1$.



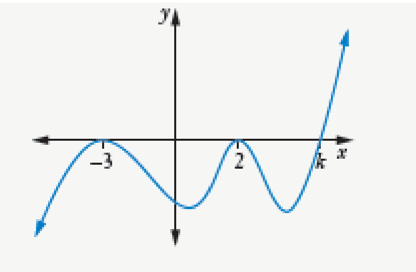
Question 13

An even polynomial with negative leading coefficient is negative at both ends. The triple root has a point of inflection so the curve must cross the *x*-axis to turn negative again. So there is another root at *k* as shown on the graph. Alternatively, *k* could be on the left side of –2 so that the curve decreases to a point of inflection at $x = -2$.



Question 14

An odd polynomial with positive leading coefficient starts negative and touches the *x*-axis at both the double roots. It then becomes positive as *x* becomes very large so it must cross the *x*-axis again. So there is another root at *k* as shown on the graph. Alternatively, *k* could be on the left side of 2 or –3, so that the curve touches the *x*-axis as a minimum point at $x = 2$ or -3.



Exercise 6.07 The inverse of a function

Question 1

a $y = 3x$

$x = 3y \Rightarrow y^{-} = \frac{x}{3}$

b $y = -x$

$x = -y \Rightarrow y^{-} = -x$

c $f(x) = \frac{x}{5}$

$x = \frac{y}{5} \Rightarrow y^{-} = 5x$

$f^{-1}(x) = 5x$

d $y = \sqrt[3]{x}$

$x = \sqrt[3]{y} \Rightarrow x^3 = y$

$y^{-1} = x^3$

e $y = 7x$

$x = 7y \Rightarrow y^{-} = \frac{x}{7}$

f $f(x) = x + 1$

$x = y + 1 \Rightarrow y = x - 1$

$f^{-1}(x) = x - 1$

g $y = x - 5$

$x = y - 5 \Rightarrow y^{-1} = x + 5$

h $f(x) = x + 3$

$x = y + 3 \Rightarrow y = x - 3$

$f^{-1}(x) = x - 3$

i $y = x^3$

$x = y^3 \Rightarrow \sqrt[3]{x} = y$

$y^{-1} = \sqrt[3]{x}$

j $y = x^5$

$x = y^5 \Rightarrow \sqrt[5]{x} = y$

$y^{-1} = \sqrt[5]{x}$

k $f(x) = x - 9$

$x = y - 9 \Rightarrow y = x + 9$

$f^{-1}(x) = x + 9$

l $f(x) = 5 - x$

$x = 5 - y \Rightarrow y = 5 - x$

$f^{-1}(x) = 5 - x$

m $y = -3x$

$x = -3y \Rightarrow y^{-1} = -\dfrac{x}{3}$

n $y = x^2$

$x = y^2 \Rightarrow y^{-1} = \pm\sqrt{x}$

o $y = \sqrt[7]{x}$

$x = \sqrt[7]{x} \Rightarrow x^7 = y$

$y^{-1} = x^7$

p $y = \dfrac{x}{9}$

$x = \dfrac{y}{9} \Rightarrow y^{-1} = 9x$

q $y = x^8$

$x = y^8 \Rightarrow \sqrt[8]{x} = y\,;\ y^{-1} = \pm\sqrt[8]{x}$

Question 2

a $y = x^3 + 5$

$x = y^3 + 5$

$y^3 = x - 5$

$y^{-1} = \sqrt[3]{x-5}$

b $y = x^7 - 1$

$x = y^7 - 1$

$y^7 = x + 1$

$y^{-1} = \sqrt[7]{x+1}$

c $y = \sqrt[3]{x-2}$

$x = \sqrt[3]{y-2}$

$x^3 = y - 2$

$y^{-1} = x^3 + 2$

d

$$y = \frac{2}{x}$$

$$x = \frac{2}{y}$$

$$y^{-1} = \frac{2}{x}$$

e

$$y = \frac{3}{x+5}$$

$$x = \frac{3}{y+5}$$

$$y + 5 = \frac{3}{x}$$

$$y^{-1} = \frac{3}{x} - 5 = \frac{3-5x}{x}$$

f

$$y = \frac{x+1}{2}$$
$$x = \frac{y+1}{2}$$

$2x = y + 1$

$y^{-1} = 2x - 1$

g

$$f(x) = \sqrt{x+2}$$
$$x = \sqrt{y+2}$$

$x^2 = y + 2$

$y^{-1} = x^2 - 2$

h $y = \sqrt[3]{x-7}$

$x = \sqrt[3]{y-7}$

$x^3 = y - 7$

$y^{-1} = x^3 + 7$

i

$$y = \frac{3}{\sqrt{x}}$$
$$x = \frac{3}{\sqrt{y}}$$
$$\sqrt{y} = \frac{3}{x}$$
$$y^{-1} = \frac{9}{x^2}$$

j $y = 3x^5 - 2$

$x = 3y^5 - 2$

$3y^5 = x + 2$

$$y^5 = \frac{x+2}{3}$$

$$y^{-1} = \sqrt[5]{\frac{x+2}{3}}$$

k

$$f(x) = 2\sqrt{x} + 5$$

$$x = 2\sqrt{y} + 5$$

$$2\sqrt{y} = x - 5$$

$$\sqrt{y} = \frac{x-5}{2}$$

$$y^{-1} = \left(\frac{x-5}{2}\right)^2$$

l

$$y = 3\sqrt[3]{2x+1}$$

$$x = 3\sqrt[3]{2y+1}$$

$$\sqrt[3]{2y+1} = \frac{x}{3}$$

$$2y+1 = \frac{x^3}{27}$$

$$2y = \frac{x^3 - 27}{27}$$

$$y^{-1} = \frac{x^3 - 27}{54}$$

m

$$y = 2x^4$$

$$x = 2y^4$$

$$y^4 = \frac{x}{2}$$

$$y^{-1} = \pm\sqrt[4]{\frac{x}{2}}$$

n

$$y = x^2 + 5$$
$$x = y^2 + 5$$
$$y^2 = x - 5$$
$$y^{-1} = \pm\sqrt{x-5}$$

o

$$y = x^6 - 3$$
$$x = y^6 - 3$$
$$y^6 = x + 3$$
$$y^{-1} = \pm\sqrt[6]{x+3}$$

p

$$y = x^2 + 8x$$
$$x = y^2 + 8y$$
$$y^2 + 8y + 16 = x + 16$$
$$(y+4)^2 = x + 16$$
$$y + 4 = \pm\sqrt{x+16}$$
$$y^{-1} = -4 \pm \sqrt{x+16}$$

q

$$y = 4x - x^2$$
$$x = 4y - y^2$$
$$y^2 - 4y = -x$$
$$y^2 - 4y + 4 = 4 - x$$
$$(y-2)^2 = 4 - x$$
$$y - 2 = \pm\sqrt{4-x}$$
$$y^{-1} = 2 \pm \sqrt{4-x}$$

r

$$y = x^2 - 2x + 3$$
$$x = y^2 - 2y + 3$$
$$y^2 - 2y + 1 + 2 = x$$
$$(y-1)^2 + 2 = x$$
$$(y-1)^2 = x - 2$$
$$y - 1 = \pm\sqrt{x-2}$$
$$y^{-1} = 1 \pm \sqrt{x-2}$$

s

$$y = x^2 + 10x - 1$$
$$x = y^2 + 10y - 1$$
$$y^2 + 10y + 25 - 26 = x$$
$$(y+5)^2 = x + 26$$
$$y + 5 = \pm\sqrt{x+26}$$
$$y^{-1} = -5 \pm \sqrt{x+26}$$

t

$$y = x^2 - 6x - 3$$
$$x = y^2 - 6y - 3$$
$$y^2 - 6y + 9 - 12 = x$$
$$(y-3)^2 = x + 12$$
$$y - 3 = \pm\sqrt{x+12}$$
$$y^{-1} = 3 \pm \sqrt{x+12}$$

u

$$y = x^2 + 12x - 11$$
$$x = y^2 + 12y - 11$$
$$y^2 + 12y + 36 = x + 47$$
$$(y+6)^2 = x + 47$$
$$y + 6 = \pm\sqrt{x+47}$$
$$y^{-1} = -6 \pm \sqrt{x+47}$$

Exercise 6.08 Graphing the inverse of a function

Question 1

a $f(x) = 2x + 1$

A line with gradient 2 and y-intercept 1, (0, 1).

x-intercept (y = 0): $0 = 2x + 1$

$-1 = 2x \Rightarrow x = -\frac{1}{2}$, $(-\frac{1}{2}, 0)$

Inverse of $y = 2x + 1$

$x = 2y + 1$

$x - 1 = 2y$

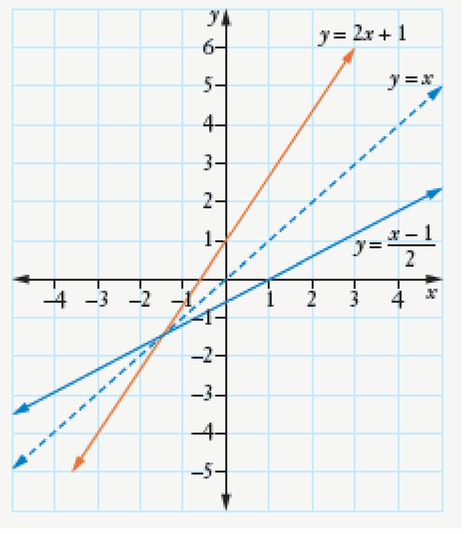
$y = \frac{x-1}{2} = \frac{1}{2}x - \frac{1}{2}$

A line with gradient $\frac{1}{2}$ and y-intercept $-\frac{1}{2}$, $(0, -\frac{1}{2})$.

x-intercept (y = 0): $0 = \frac{x-1}{2}$

$0 = x - 1$

$x = 1$ (1, 0)



b $y = x^3 - 1$

A cubic curve with point of inflection and y-intercept at -1, $(0, -1)$.

x-intercept ($y = 0$): $0 = x^3 - 1$

$1 = x^3$

$x = 1$, $(1, 0)$

Inverse of $y = x^3 - 1$

$x = y^3 - 1$

$x + 1 = y^3$

$y = \sqrt[3]{x+1}$

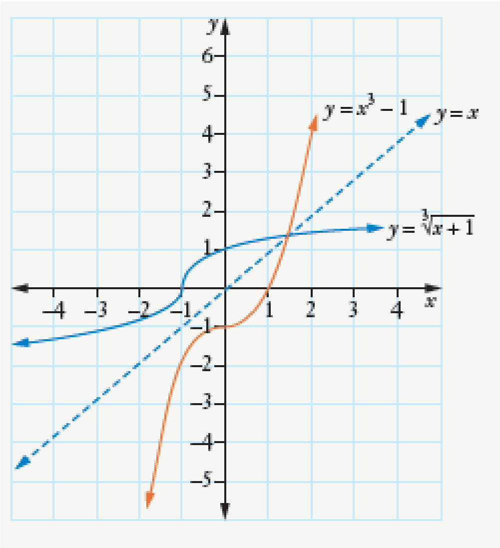
y-intercept ($x = 0$): $y = \sqrt[3]{1}$

$y = 1$ $(0, 1)$

x-intercept ($y = 0$): $0 = \sqrt[3]{x+1}$

$0 = x + 1$

$x = -1$, $(-1, 0)$



c $f(x) = \frac{x}{4}$

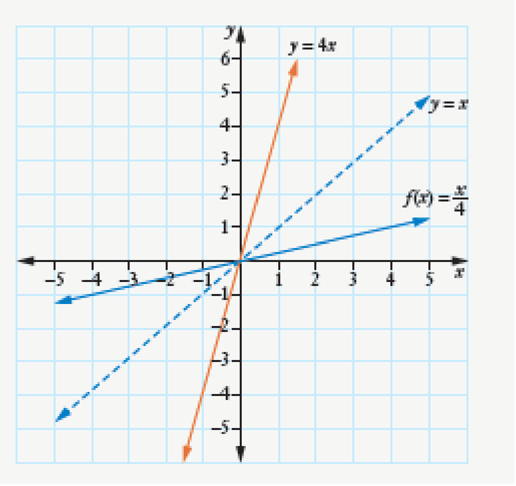
A line with gradient $\frac{1}{4}$ and y-intercept 0, (0, 0).

Inverse of $y = \frac{x}{4}$

$x = \frac{y}{4}$

$y = 4x$

A line with gradient 4 and y-intercept 0, (0, 0).



d $y = \sqrt{x+1}$

Domain: $x + 1 \geq 0 \Rightarrow x \geq -1$, $[-1, \infty)$

Range: $y \geq 0$, $[0, \infty)$

y-intercept ($x = 0$): $y = \sqrt{1} = 1$, $(0, 1)$

x-intercept ($y = 0$): $0 = \sqrt{x+1}$

$0 = x + 1$

$x = -1$, $(-1, 0)$

Inverse of $y = \sqrt{x+1}$

$x = \sqrt{y+1}$

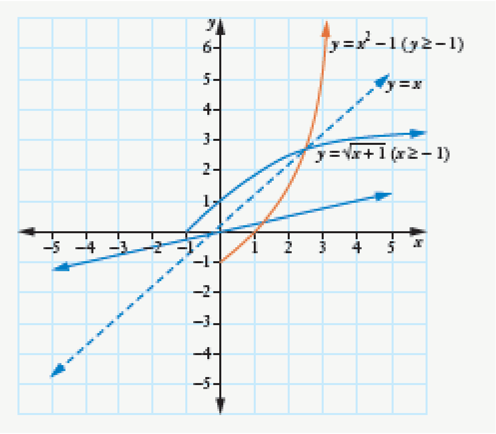
$x^2 = y + 1$

$y = x^2 - 1$

A parabola with vertex and y-intercept at $(0, -1)$.

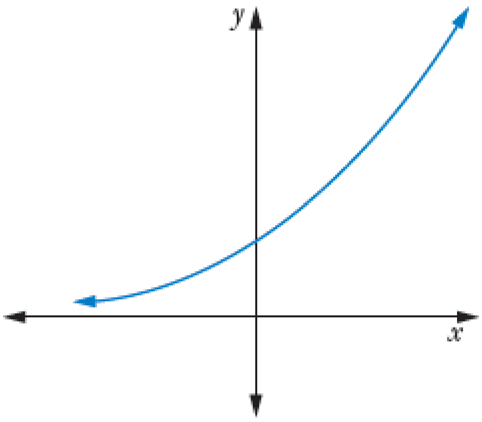
Domain and range are the range and domain of $y = \sqrt{x+1}$.

Domain: $[0, \infty)$, range: $[-1, \infty)$



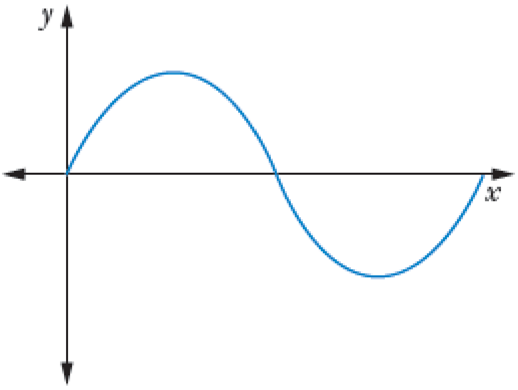
Question 2

a



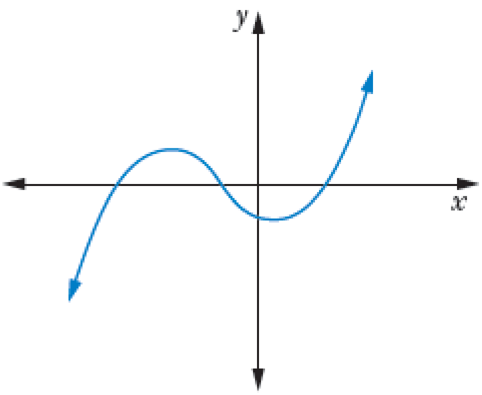
Yes, it passes the horizontal line test.

b



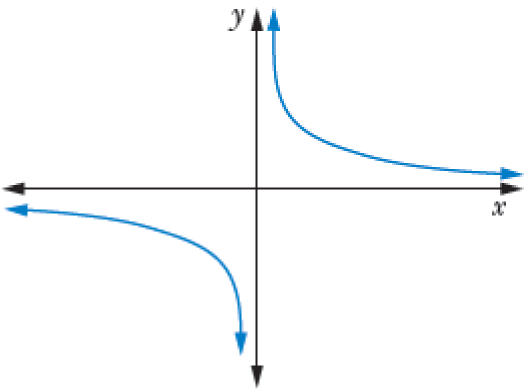
No it fails the horizontal line test.

c



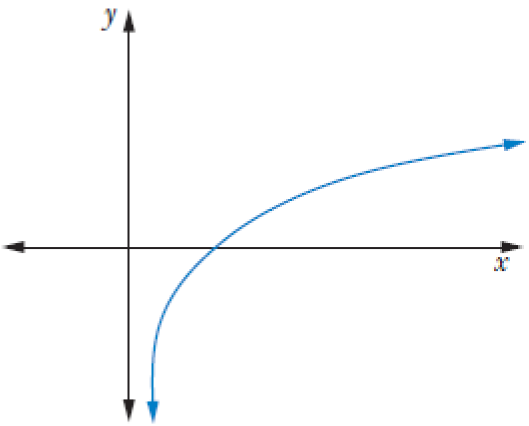
No, it fails the horizontal line test.

d



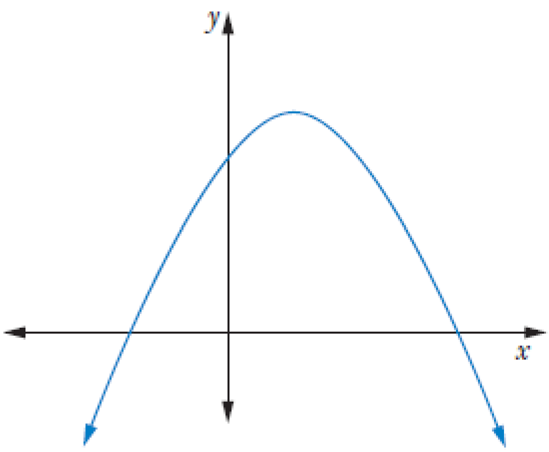
Yes, it passes the horizontal line test.

e



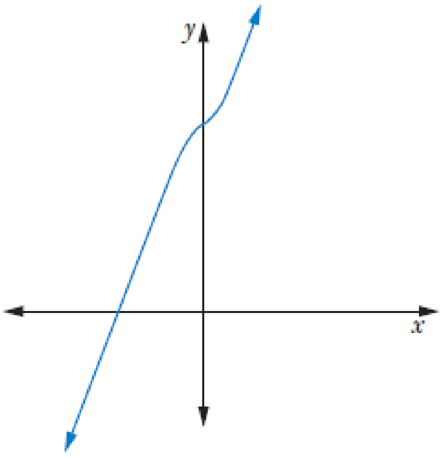
Yes, it passes the horizontal line test.

f



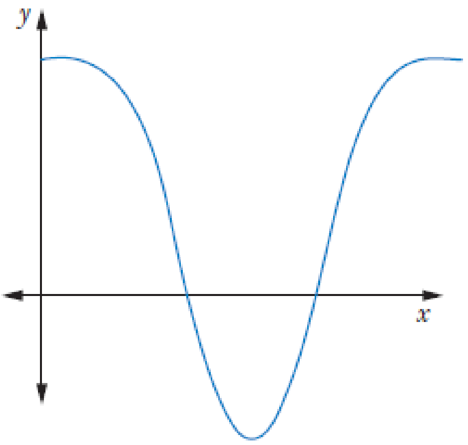
No, it fails the horizontal line test.

g



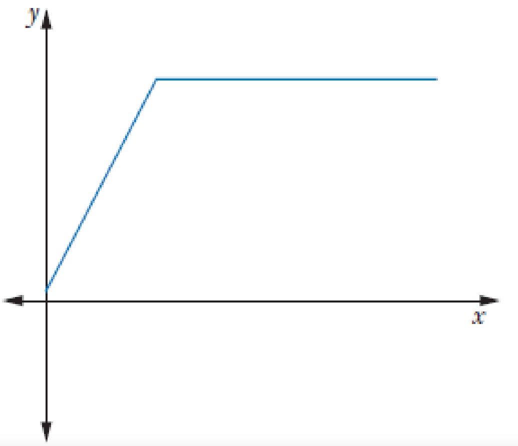
Yes, it passes the horizontal line test.

h



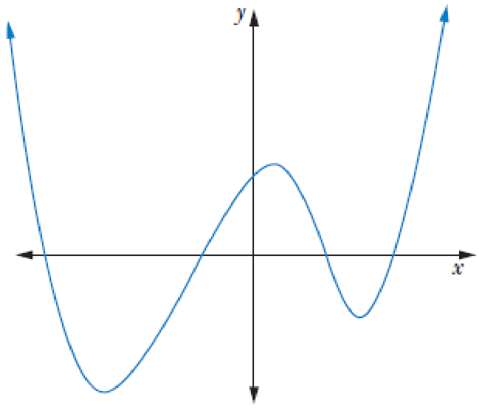
No, it fails the horizontal line test.

i



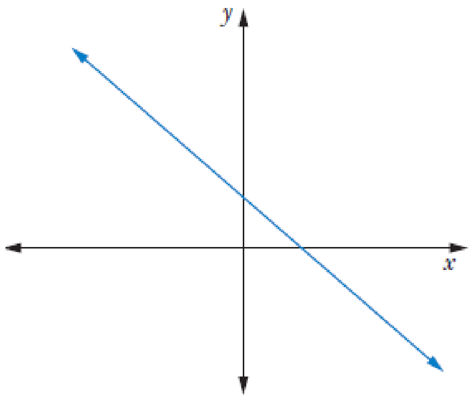
No, it fails the horizontal line test.

j



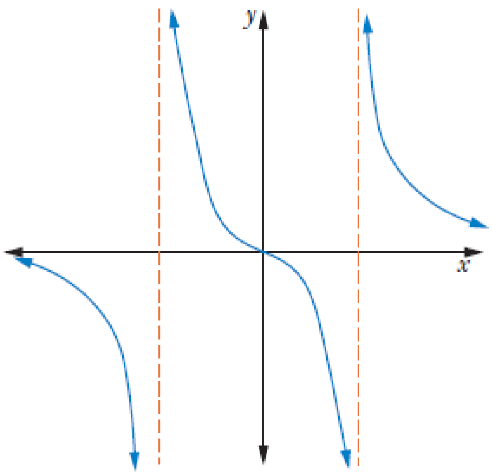
No, it fails the horizontal line test.

k



Yes, it passes the horizontal line test.

l



No, it fails the horizontal line test.

Exercise 6.09 Inverse functions

Question 1

A $f(x) = 5x - 7$

Does have an inverse function.

B $y = \dfrac{4}{x}$

Does have an inverse function.

C $y = x^2 + 1$

Does not have an inverse function.

D $y = x^3$

Does have an inverse function.

Question 2

a $y = x^3$

$x = y^3 \Rightarrow y^{-1} = \sqrt[3]{x}$

Domain = $(-\infty, \infty)$, range = $(-\infty, \infty)$

b $y = 3x - 2$

$x = 3y - 2 \Rightarrow 3y = x + 2$

$$y^{-1} = \frac{x+2}{3}$$

Domain = $(-\infty, \infty)$, range = $(-\infty, \infty)$

c $f(x) = \dfrac{2}{x}$

$f^{-1}(x) = \dfrac{2}{x}$

Domain = $(-\infty, 0) \cup (0, \infty)$, range = $(-\infty, 0) \cup (0, \infty)$

d

$$y = \frac{1}{x+1}$$
$$x = \frac{1}{y+1}$$

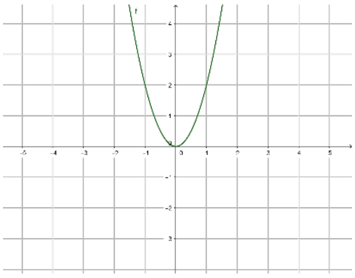
$$y + 1 = \frac{1}{x}$$

$$y^{-1} = \frac{1}{x} - 1$$

Domain = $(-\infty, 0) \cup (0, \infty)$, range = $(-\infty, -1) \cup (-1, \infty)$

Question 3

a $y = 2x^2$

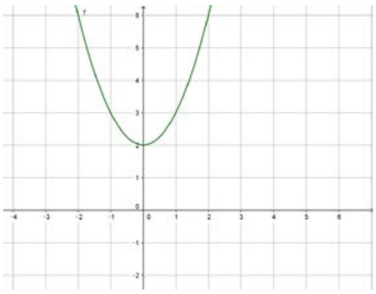


$\frac{x}{2} = y^2$

$y^{-1} = \sqrt{\frac{x}{2}}$

Domain = [0, ∞), range = [0, ∞)

b $y = x^2 + 2$



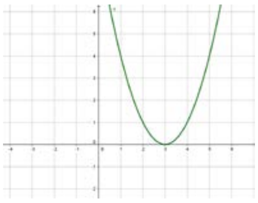
$x = y^2 + 2$

$y^2 = x - 2$

$y^{-1} = \sqrt{x-2}$

Domain = [2, ∞), range = [0, ∞)

c $y = (x - 3)^2$

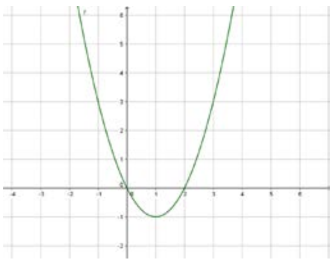


$x = (y - 3)^2$

$$y - 3 = \sqrt{x}$$
$$y^{-1} = 3 + \sqrt{x}$$

Domain = $[0, \infty)$, range = $[3, \infty)$

d $y = x^2 - 2x$



$x = y^2 - 2y$

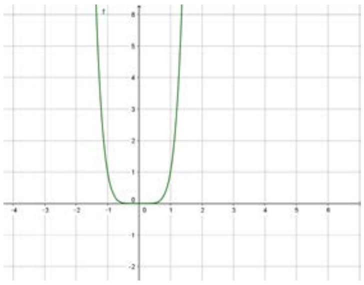
$y^2 - 2y + 1 = x + 1$

$(y - 1)^2 = x + 1$

$$y - 1 = \sqrt{x + 1}$$
$$y^{-1} = 1 + \sqrt{x + 1}$$

Domain = $[-1, \infty)$, range = $[1, \infty)$

e $y = x^6$

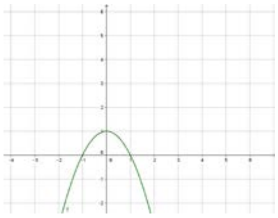


$x = y^6$

$y^{-1} = \sqrt[6]{x}$

Domain = [0, ∞), range = [0, ∞)

f $y = 1 - x^2$



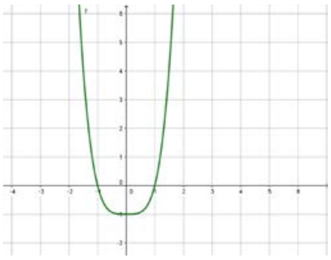
$x = 1 - y^2$

$y^2 = 1 - x$

$y^{-1} = -\sqrt{1-x}$

Domain = (–∞, 1], range = (–∞, 0]

g $y = x^4 - 1$



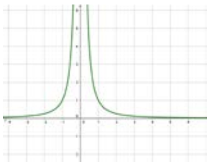
$x = y^4 - 1$

$y^4 = x + 1$

$y^{-1} = \sqrt[4]{x+1}$

Domain = [–1, ∞), range = [0, ∞)

h $y = \dfrac{1}{x^2}$



$x = \dfrac{1}{y^2}$

$y^2 = \dfrac{1}{x}$

$y^{-1} = -\dfrac{1}{\sqrt{x}}$

Domain = (0, ∞), range = (–∞, 0)

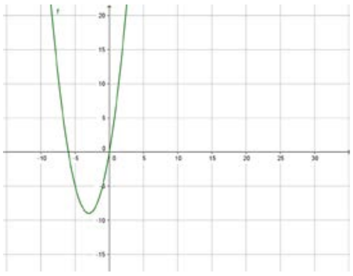
Question 4

a $y = x^2 + 6x$

$y = x(x + 6)$

x-intercepts 0, –6.

By symmetry turning point occurs when $x = -3$, $(-3, -9)$.



y is monotonic increasing for $x \geq -3$, that is, $[-3, \infty)$.

b $x = y^2 + 6y$

$y^2 + 6y + 9 = x + 9$

$(y + 3)^2 = x + 9$

$$y + 3 = \sqrt{x+9}$$
$$y^{-1} = -3 + \sqrt{x+9}$$

Domain = $[-9, \infty)$, range = $[-3, \infty)$

c $y = x^2 + 6x$ is monotonic decreasing for $x \leq -3$, that is $(-\infty, -3]$.

d $x = y^2 + 6y$

$y^2 + 6y + 9 = x + 9$

$(y + 3)^2 = x + 9$

$$y + 3 = -\sqrt{x+9}$$
$$y^{-1} = -3 - \sqrt{x+9}$$

Domain = $[-9, \infty)$, range = $(-\infty, -3]$

Question 5

a $y = x^2$

Monotonic decreasing $x < 0$

$x = y^2$

$$y^{-1} = -\sqrt{x}$$

b $y = 3x^2 - 1$

Monotonic decreasing $x < 0$

$x = 3y^2 - 1$

$3y^2 = x + 1$

$$y^2 = \frac{x+1}{3}$$
$$y^{-1} = -\sqrt{\frac{x+1}{3}}$$

c $f(x) = (x - 2)^4$

Monotonic decreasing $x < 2$

$x = (y - 2)^4$

$$y - 2 = -\sqrt[4]{x}$$
$$y^{-1} = 2 - \sqrt[4]{x}$$

d $y = \dfrac{3}{x^2}$

Monotonic decreasing $x > 0$

$$x = \frac{3}{y^2}$$
$$y^2 = \frac{3}{x}$$

$$y^{-1} = \sqrt{\frac{3}{x}}$$

e $y = \dfrac{2}{x^4}$

Monotonic decreasing $x > 0$

$$x = \frac{2}{y^4}$$

$$y^4 = \frac{2}{x}$$

$$y^{-1} = \sqrt[4]{\frac{2}{x}}$$

Question 6

a $f(x) = x + 7$

$x = y + 7 \Rightarrow y = x - 7$

$f^{-1}(x) = x - 7$

$f^{-1}(f(x)) = (x + 7) - 7 = x$

$f(f^{-1}(x)) = (x - 7) + 7 = x$

$\therefore f^{-1}(f(x)) = f(f^{-1}(x)) = x$ as required.

b $y = 3x$

$x = 3y \Rightarrow y = \dfrac{x}{3}$

$f^{-1}(x) = \dfrac{x}{3}$

$f^{-1}(f(x)) = \dfrac{3x}{3} = x$

$f(f^{-1}(x)) = 3 \times \dfrac{x}{3} = x$

$\therefore f^{-1}(f(x)) = f(f^{-1}(x)) = x$ as required.

c $y = \sqrt{x}$

$x = \sqrt{y} \Rightarrow y = x^2$

$f^{-1}(x) = x^2$

$f^{-1}(f(x)) = \left(\sqrt{x}\right)^2 = x$

$f(f^{-1}(x)) = \sqrt{x^2} = x$

$\therefore f^{-1}(f(x)) = f(f^{-1}(x)) = x$ as required.

d $y = 3x + 1$

$x = 3y + 1 \Rightarrow 3y = x - 1$

$f^{-}(x) = \dfrac{x-1}{3}$

$f^{-1}(f(x)) = \dfrac{(3x+1)-1}{3} = \dfrac{3x}{3} = x$

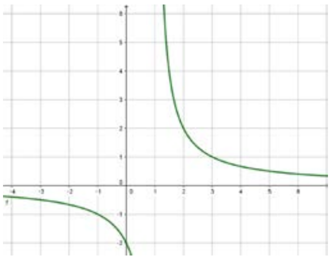
$f(f^{-1}(x)) = 3\left(\dfrac{x-1}{3}\right)+1 = x - 1 + 1 = \mathrm{x}$

$\therefore f^{-1}(f(x)) = f(f^{-1}(x)) = x$ as required.

Question 7

$y = \frac{2}{x-1}$

a



Domain $(-\infty,1)\cup(1,\infty)$, range $(-\infty,0)\cup(0,\infty)$

b

$$y = \frac{2}{x-1}$$
$$x = \frac{2}{y-1}$$

$$y-1 = \frac{2}{x}$$

$$y^{-1} = \frac{2}{x}+1$$

c Domain $(-\infty,0)\cup(0,\infty)$, range $(-\infty,1)\cup(1,\infty)$

Test Yourself 6

Question 1

Graph shows zeros at –1 and 2.

x = –1 is a double root (turning point)

x = 2 is a triple root (horizontal point of inflection)

Function is decreasing as $x \to \infty$. so leading coefficient must be negative.

So the equation must be: $P(x) = -(x + 1)^2(x + 2)^3$

C

Question 2

$y = \dfrac{1}{x-3}$

$x = \dfrac{1}{y-3} \Rightarrow y - 3 = \dfrac{1}{x}$

$y = \dfrac{1}{x} + 3$

B

Question 3

Consecutive roots are α and α + 1.

$\alpha + \beta = -\dfrac{b}{a}$

$\alpha + \alpha + 1 = -\dfrac{3}{1} \Rightarrow 2\alpha + 1 = -3$

$2\alpha = -4 \Rightarrow \alpha = -2$

$\alpha\beta = \dfrac{c}{a}$

$\alpha(\alpha + 1) = \dfrac{k-1}{1} = k - 1$

Substitute $\alpha = -2$:

$-2(-2 + 1) = k - 1$

$-2(-1) = k - 1$

$2 = k - 1$

$3 = k$

D

Question 4

$P(x) = x^4 + 4x^3 - 14x^2 - 36x + 45$

$P(1) = 0$

$\therefore\ x - 1$ is a factor of $P(x)$

$$
\begin{array}{r l}
 & \quad x^3 + 5x^2 - 9x + 45 \\
x - 1 \big) & \overline{x^4 + 4x^3 - 14x^2 - 36x + 45} \\
 & \underline{x^4 - x^3} \\
 & \quad 5x^3 - 14x^2 \\
 & \quad \underline{5x^3 - 5x^2} \\
 & \qquad -9x^2 - 36x \\
 & \qquad \underline{-9x^2 + 9x} \\
 & \qquad\quad -45x + 45 \\
 & \qquad\quad \underline{-45x + 45} \\
 & \qquad\qquad 0
\end{array}
$$

$P(x) = (x - 1)(x^3 + 5x^2 - 9x + 45)$

Let $f(x) = x^3 + 5x^2 - 9x + 45$

$f(-5) = 0$

$\therefore\ x + 5$ is a factor of $f(x)$

$$
\begin{array}{r l}
 & \qquad\quad x^2 - 9 \\
x + 5 \big) & \overline{x^3 + 5x^2 - 9x + 45} \\
 & \underline{x^3 + 5x^2} \\
 & \quad 0 - 9x + 45 \\
 & \quad \underline{-x + 45} \\
 & \qquad\quad 0
\end{array}
$$

$f(x) = (x + 5)(x^2 - 9)$

$f(x) = (x + 5)(x - 3)(x + 3)$

$\therefore$

$P(x) = (x - 1)(x + 5)(x - 3)(x + 3)$

Question 5

α, β and γ are the roots of $x^3 - 3x^2 + x - 9 = 0$

a $\quad \alpha + \beta + \gamma = -\frac{b}{a} = \frac{-(-3)}{1} = 3$

b $\quad \alpha\beta\gamma = -\frac{d}{a} = \frac{-(-9)}{1} = 9$

c $\quad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{1}{1} = 1$

d $\quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} = \frac{1}{9}$

Question 6

$P(x)$ is a monic with roots -2, 1 and 6.

$P(x) = (x + 2)(x - 1)(x - 6)$

$= (x^2 + x - 2)(x - 6)$

$= x^3 - 5x^2 - 8x + 12$

Question 7

a

$(x^4 + x^3 - 19x^2 - 49x - 30) \div (x^2 - 2x - 15)$

$$
\begin{array}{r}
x^2 + 3x + 2 \\
x^2 - 2x - 15 \overline{\big) x^4 + x^3 - 19x^2 - 49x - 30} \\
\underline{x^4 - 2x^3 - 15x^2} \qquad\qquad \\
3x^3 - 4x^2 - 49x \qquad \\
\underline{3x^3 - 6x^2 - 45x} \qquad \\
2x^2 - 4x - 30 \\
\underline{2x^2 - 4x - 30} \\
0
\end{array}
$$

$P(x) = (x^2 - 2x - 15)(x^2 + 3x + 2)$

b $\quad P(x) = (x - 5)(x + 3)(x + 2)(x + 1)$

Question 8

$f(x) = 3 - 2x$

$x = 3 - 2y \Rightarrow 2y = 3 - x$

$$f^{-}(x) = \frac{3 - x}{2}$$

Question 9

$P(x) = x^3 + 2x^2 - 3x$

a Degree $P(x) = 3$ (from x^3)

b coefficient of $x = -3$ (from $-3x$)

c $P(x) = x(x^2 + 2x - 3) = x(x + 3)(x - 1)$

Zeros are -3, 0 and 1.

d Leading term x^3.

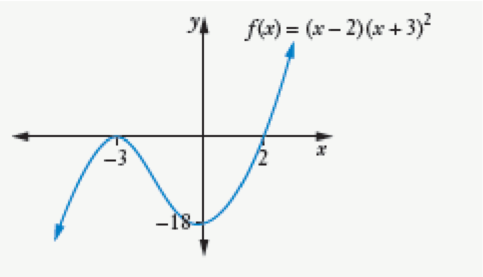
Question 10

$f(x) = (x - 2)(x + 3)^2$

y-intercept occurs when $x = 0$, $(0, -18)$

x-intercepts occur when $y = 0$; 2 and -3.

$f(x)$ has a leading positive coefficient.



Question 11

$ax^4 + 3x^3 - 48x^2 + 60x = 0$

a Let $P(x) = ax^4 + 3x^3 - 48x^2 + 60x$

$P'(\mathrm{x}) = 4ax^3 + 9x^2 - 96x + 60$

$P'(2) = 0$

$32a + 36 - 192 + 60 = 0$

$32a = 96$

$a = 3$

b $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = \frac{-3}{3} = -1$

$4 + \alpha + \delta = -1$

$\alpha + \delta = -5$

Question 12

a $y = \sqrt{x-1}$

Domain $[1, \infty)$, range $[0, \infty)$

b

$$x = \sqrt{y-1}$$
$$y - 1 = x^2$$
$$y^{-1} = x^2 + 1$$

Domain $[0, \infty)$, range $[1, \infty)$

Question 13

$y=\dfrac{1}{x+2}$

a Domain $(-\infty,-2)\cup(-2,\infty)$, range $(-\infty,0)\cup(0,\infty)$

b

$$x=\frac{1}{y+2}$$
$$y+2=\frac{1}{x}$$
$$y^{-1}=\frac{1}{x}-2$$

c Domain $(-\infty,0)\cup(0,\infty)$, range $(-\infty,-2)\cup(-2,\infty)$

Question 14

Let $P(x) = x^3 - 7x^2 + 5x - 4$

$P(7) = -343 - 343 - 35 - 4 \neq 0$

$\therefore$ $x + 7$ is not a factor of $P(x)$.

Question 15

$x^4 + 2x^3 - 8x^2 - 18x - 9 = 0$

Roots α, $-\alpha$, β, and γ.

$\alpha - \alpha + \beta + \gamma = -\frac{b}{a} = -\frac{2}{1} = -2 \Rightarrow \beta + \gamma = -2$ [1]

$\alpha(-\alpha) + \alpha\beta + \alpha\gamma - \alpha\beta - \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{-8}{1} = -8 \Rightarrow -\alpha^2 + \beta\gamma = -8$ [2]

$\alpha(-\alpha)\beta + \alpha(-\alpha)\gamma + \alpha\beta\gamma - \alpha\beta\gamma = -\frac{d}{a} = \frac{-(-18)}{1} = 18 \Rightarrow -\alpha^2(\gamma + \beta) = 18$

$-\alpha^2(-2) = 18$ (substituting in from [1])

$\alpha^2 = 9$

$\alpha = \pm 3$

Using [2]: $-9 + \beta\gamma = -8 \Rightarrow \beta\gamma = 1$

Using $\gamma + \beta = -2 \Rightarrow \gamma = -2 - \beta$

Sub $\gamma = -2 - \beta$ into $\gamma\beta = -1$

$-\beta(-2 - \beta) = -1$

$\beta^2 + 2\beta = -1$

$\beta^2 + 2\beta + 1 = 0$

$(\beta + 1)^2 = 0$

$\beta = -1$

$\beta\gamma = 1$

$-\gamma = 1$

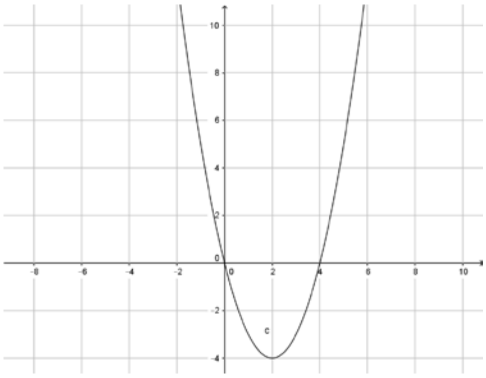
$\gamma = -1$

Solutions

$\pm 3, -1$

Question 16

a $y = x^2 - 4x$



a The function is monotonic increasing for $x \geq 2$; or $[2, \infty)$

b

$$x = y^2 - 4y$$
$$y^2 - 4y + 4 = x + 4$$
$$(y-2)^2 = x + 4$$
$$y - 2 = \sqrt{x+4}$$
$$y = 2 + \sqrt{x+4}$$

Question 17

$p(x) = x^3 - 1$, $q(x) = 2x + 5$

a $p^{-1}(x)$

$x = y^3 - 1$

$y^3 = x + 1$

$$p^{-1}(x) = \sqrt[3]{x+1}$$
$$p^{-1}(7) = \sqrt[3]{7+1}$$
$$p^{-1}(7) = 2$$

b $q^{-1}(x)$

$x = 2y + 5$

$2y = \mathrm{x} - 5$

$$q^{-1}(x) = \frac{x-5}{2}$$

$p(3) = 3^3 - 1 = 26$

$q^{-1}(p(3)) = q^{-1}(26) = \dfrac{26-5}{2} = \dfrac{21}{5} = 10\dfrac{1}{2}$

Question 18

$f(x) = ax^2 + bx + c$

$f(4) = f(5) = 0$, $f(-1) = 60$

$f(x) = d(x - 4)(x - 5)$

Using $f(-1) = 60$: $60 = d(-5)(-6)$

$60 = 30d$

$d = 2$

$f(x) = 2(x - 4)(x - 5)$

$= 2(x^2 - 9x + 20)$

$= 2x^2 - 18x + 40$

$a = 2$, $b = -18$, $c = 40$

Question 19

$y = x^3 - 3x^2 - 10x + 24$

y-intercept occurs when $x = 0$; y-intercept = 24, (0, 24)

x-intercept occurs when $y = 0$

$y(2) = 0$

$$
\begin{array}{r|l}
 & x^2 - x - 12 \\
\hline
x-2 & x^3 - 3x^2 - 10x + 24 \\
 & \underline{x^3 - 2x^2} \\
 & -x^2 - 10x \\
 & \underline{-x^2 + 2x} \\
 & -12x + 24 \\
 & \underline{-12x + 24} \\
 & 0
\end{array}
$$

$y = (x-2)(x^2 - x - 12)$

$y = (x-2)(x+3)(x-4)$

Zeros of y are -3, 2, 4.

x-intercepts ate (–3, 0), (2, 0) and (4, 0).

Question 20

$$
\begin{array}{r}
3x^4+6x^3+5x^2+18x+36 \\
x-2\overline{\big)\,3x^5+0x^4-7x^3+8x^2+0x-5} \\
\underline{3x^5-6x^4} \qquad\qquad\qquad\qquad\quad \\
6x^4-7x^3 \qquad\qquad\qquad\quad \\
\underline{6x^4-12x^3} \qquad\qquad\qquad \\
5x^3+8x^2 \qquad\qquad\quad \\
\underline{5x^2-10x^2} \qquad\qquad \\
18x^2+0x \qquad\quad \\
\underline{18x^2-36x} \qquad \\
36x-5 \quad \\
\underline{36x-72} \\
67
\end{array}
$$

$$p(x)=(x-2)(3x^4+6x^3+5x^2+18x+36)+67$$

Question 21

$P(x) = 8x^3 - 5kx + 9$

$P(2) = 3$

$3 = 64 - 10k + 9$

$10k = 70$

$k = 7$

Question 22

$P(x)=x^5+2x^4+x^3-x^2-2x-1$

$P(1)=0$

$\therefore x-1$ is a factor of $P(x)$

$$
\begin{array}{r|l}
 & x^4+3x^3+4x^2+3x+1 \\
\hline
x-1 & x^5+2x^4+x^3-x^2-2x-1 \\
 & \underline{x^5-x^4} \\
 & \quad 3x^4+x^3 \\
 & \quad \underline{3x^4-3x^3} \\
 & \qquad 4x^3-x^2 \\
 & \qquad \underline{4x^3-4x^2} \\
 & \qquad\quad 3x^2-2x \\
 & \qquad\quad \underline{3x^2-3x} \\
 & \qquad\qquad x-1 \\
 & \qquad\qquad \underline{x-1} \\
 & \qquad\qquad\quad 0
\end{array}
$$

$P(x)=(x-1)(x^4+3x^3+4x^2+3x+1)$

Let $Q(x)=x^4+3x^3+4x^2+3x+1$

$Q(-1)=0$

$$
\begin{array}{r|l}
 & x^3+2x^2+2x+1 \\
\hline
x+1 & x^4+3x^3+4x^2+3x+1 \\
 & \underline{x^4+x^3} \\
 & \quad 2x^3+4x^2 \\
 & \quad \underline{2x^3+2x^2} \\
 & \qquad 2x^2+3x \\
 & \qquad \underline{2x^2+2x} \\
 & \qquad\quad x+1 \\
 & \qquad\quad \underline{x+1} \\
 & \qquad\qquad 0
\end{array}
$$

$P(x)=(x-1)(x+1)(x^3+2x^2+2x+1)$

Let $f(x) = x^3 + 2x^2 + 2x + 1$

$f(-1) = 0$

$\therefore x + 1$ is a factor of $f(x)$

$$\begin{array}{r l}
 & x^2 + x + 1 \\
x+1 \big) & \overline{x^3 + 2x^2 + 2x + 1} \\
 & \underline{x^3 + x^2} \\
 & \quad x^2 + 2x \\
 & \quad \underline{x^2 + x} \\
 & \qquad x + 1 \\
 & \qquad \underline{x + 1} \\
 & \qquad\quad 0
\end{array}$$

$f(x) = (x + 1)(x^2 + x + 1)$

$\therefore P(x) = (x - 1)(x + 1)^2(x^2 + x + 1)$

Question 23

$f(x) = x^2 - 4$

Monotonic decreasing $x < 0$

$x = y^2 - 4 \Rightarrow y^2 = x + 4$

$y^{-1} = -\sqrt{x + 4}$

Question 24

$g(x) = -x^2 + 9x - 20$

$g(x) = -(x^2 - 9x + 20)$

$g(x) = -(x - 5)(x - 4)$

zeros of $g(x)$ are $x = 4, 5$

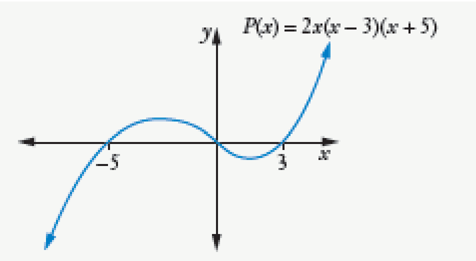
Question 25

$P(x) = 2x(x - 3)(x + 5)$

y-intercept occurs when $x = 0$; y-intercept = 0, (0, 0)

x-intercepts occur when $y = 0$; x-intercepts = 0, 3, –5.

$P(x)$ has a leading positive coefficient.



Question 26

$P(x) = x^3 + 2x^2 - 3x + k$

$P(2) = -4$

$-4 = 8 + 8 - 6 + k$

$10 + k = -4$

$k = -14$

Question 27

$x^4 - 7x^3 + 5x^2 - x + 3 = 0$

Roots α, β, γ and δ.

Let $\alpha + \beta = 3$

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} = \frac{-(-7)}{1} = 7$$

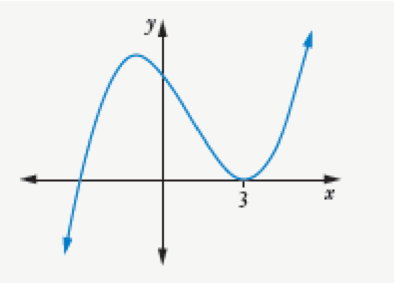
$3 + \gamma + \delta = 7$

$\gamma + \delta = 4$

The sum of the other 2 roots is 4.

Question 28

Leading term $3x^3$ (positive) and double root at $x = 3$.

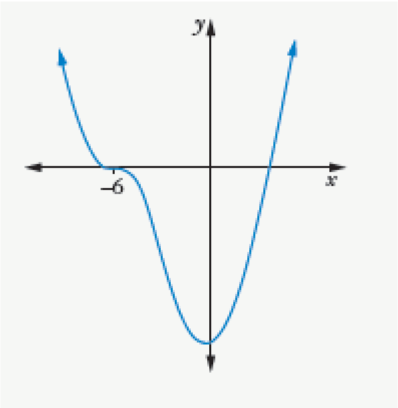


Question 29

$P(x)$ has a triple root at $x = –6$.

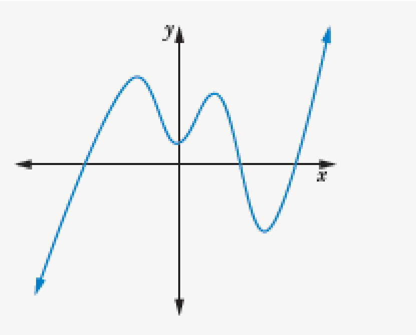
a $P(x) = (x + 6)^3Q(x)$

b Leading co-efficient 3 (positive) and degree 4.



Question 30

Leading term $3x^5$ (positive).



Question 31

$f(x) = x^3$

$x = y^3 \Rightarrow y = \sqrt[3]{x}$

$f^{-}(x) = \sqrt[3]{x}$

$f(f^{-}(x)) = \left(\sqrt[3]{x}\right)^3 = x$

$f^{-}(f(x)) = \sqrt[3]{x^3} = x$

$\therefore f(f^{-1}(x)) = f^{-1}(f(x)) = x$ as required.

Challenge exercise 6

Question 1

a

$$P(u)=u^3-4u^2+5u-2$$
$$P(1)=0$$
$$\therefore u-1 \text{ is a factor of } P(u)$$

$$\begin{array}{r}
u^2-3u+2 \\
u-1\overline{)u^3-4u^2+5u-2} \\
\underline{u^3-u^2} \\
-3u^2+5u \\
\underline{-3u^2+3u} \\
2u-2 \\
\underline{2u-2} \\
0
\end{array}$$

$$P(u)=(u-1)(u^2-3u+2)$$
$$P(u)=(u-1)(u-1)(u-2)$$
$$=(u-1)^2(u-2)$$

b $(x-1)^3-4(x-1)^2+5(x-1)-2=0$

Let $u = x - 1$

$u^3 - 4x^2 + 5x - 2 = 0$

From part **a**, $u = 1, 2$.

When $u = 1$, $1 = x - 1 \Rightarrow x = 2$

When $u = 2$, $2 = x - 1 \Rightarrow x = 3$

Solution $x = 2, 3$.

Question 2

a

$$f(u) = u^3 - 13u^2 + 39u - 27$$
$$f(1) = 0$$
$$\therefore u - 1 \text{ is a factor of } f(u)$$

$$
\begin{array}{r}
u^2 - 12u + 27 \\
u - 1\overline{)u^3 - 13u^2 + 39u - 27} \\
\underline{u^3 - u^2} \\
-12u^2 + 39u \\
\underline{-12u^2 + 12u} \\
27u - 27 \\
\underline{27u - 27} \\
0
\end{array}
$$

$$f(u) = (u-1)(u^2 - 12u + 27)$$
$$f(x) = (u-1)(u-3)(u-9)$$

b $3^{3x} - 13 \times 3^{2x} + 39 \times 3^x - 27 = 0$

Let $u = 3^x$

This becomes $u^3 - 13u^2 + 39u - 27 = 0$

$(u-1)(u-3)(u-9) = 0$

$u = 1, 3, 9$

$3^x = 1, 3^x = 3, 3^x = 9$

$x = 0, 1, 2$

Question 3

$P(x) = x^3 + 5x^2 + 4x - 1$

$3x + y + 4 = 0 \Rightarrow y = -3x - 4$

Intercept

$-3x - 4 = x^3 + 5x^2 + 4x - 1$

$x^3 + 5x^2 + 7x + 3 = 0$

Let $f(x) = x^3 + 5x^2 + 7x + 3$

$f(-1) = 0$

$$\begin{array}{r|l} & x^2+4x+3 \\ \hline x+1 & x^3+5x^2+7x+3 \\ & \underline{x^3+x^2} \\ & \quad 4x^2+7x \\ & \quad \underline{4x^2+4x} \\ & \qquad 3x+3 \\ & \qquad \underline{3x+3} \\ & \qquad\quad 0 \end{array}$$

$$f(x) = (x+1)(x^2+4x+3)$$
$$P(x) = (x+1)(x+1)(x+3)$$

Intercepts $x = -1, -3$

When $x = -1$ $y = -3(-1) - 4 = -1$.

When $x = -3$, $y = -3(-3) - 4 = 5$.

The 2 intercepts are $(-1, -1)$, $(-3, 5)$

Question 4

$(6x^2 - 3x + 1) \div (3x - 2)$

$$\begin{array}{r} 2x+\frac{1}{3} \\ 3x-2\,\overline{)\,6x^2-3x+1} \\ \underline{6x^2-4x} \\ x+1 \\ \underline{x-\frac{2}{3}} \\ \frac{5}{3} \end{array}$$

$$6x^2 - 3x + 1 = (3x-2)\left(2x+\frac{1}{3}\right)+\frac{5}{3}$$

Question 5

$P(x) = ax^3 + bx^2 + cx + d$

$P(1) = 8$, $P(2) = 17$, $P(-1) = -4$, $P(0) = 5$

$P(0) = 5 \Rightarrow d = 5$

$P(1) = a + b + c + 5 = 8$	[1]
$P(-1) = -a + b - c + 5 = -4$	[2]
$2b + 10 = 4$	[1] + [2]
$2b = -6 \Rightarrow b = -3$	
$P(2) = 8a - 12 + 2c + 5 = 17$	
$8a + 2c = 24$	[3]
$a + c + 2 = 8$	from [2]
$a + c = 6 \Rightarrow 2a + 2c = 12$	[4]
$6a = 12$	[3] – [4]
$a = 2$	
$c = 4$	from [4]

$a = 2$, $b = -3$, $c = 4$, $d = 5$

Question 6

α, β and γ are roots of $2x^3 + 8x^2 - x + 6 = 0$

a $\alpha\beta\gamma = -\frac{d}{a} = \frac{-6}{2} = -3$

b $\alpha^2 + \beta^2 + \gamma^2$

$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$\alpha + \beta + \gamma = -\frac{b}{a} = \frac{-8}{2} = -4$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{-1}{2} = -\frac{1}{2}$

$\alpha^2 + \beta^2 + \gamma^2 = (-4)^2 - 2\left(-\frac{1}{2}\right)$

$= 16 + 1$

$= 17$

Question 7

$(x + 1)(x - 2) = x^2 - x - 2$

$$
\begin{array}{r|l}
 & 2x + 1 \\
x^2 - x - 2 & 2x^3 - x^2 + ax - 2 \\
 & \underline{2x^3 - 2x^2 - 4x} \\
 & x^2 + (a+4)x - 2 \\
 & \underline{x^2 - x - 2} \\
 & 0
\end{array}
$$

To be a factor

$a + 4 - (-1) = 0$

$a - 5 = 0$

$a = -5$

Question 8

Let $x - k$ be a factor of $P(x)$.

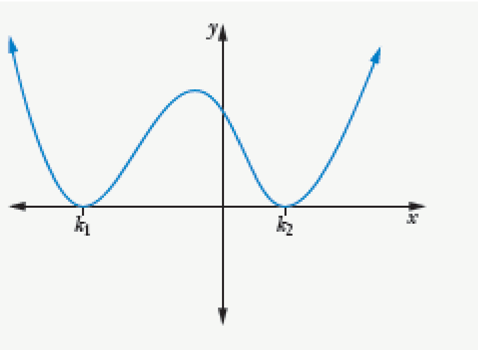
$\therefore\ P(x) = (x - k)Q(x)$

When $x = k$

$P(k) = (k - k)Q(k) = 0$

$\therefore\ P(k) = 0$

Question 9



MATHS IN FOCUS 11
MATHEMATICS EXTENSION 1

WORKED SOLUTIONS

Chapter 7: Further functions

Exercise 7.01 The hyperbola

Question 1

a $D = \dfrac{k}{x}$.

When $x = 2$, $D = 80$.

$80 = \dfrac{k}{2}$

$k = 160$

$\therefore D = \dfrac{160}{x}$

b When $x = 0.8$,

$D = \dfrac{160}{0.8} = 200$ mm

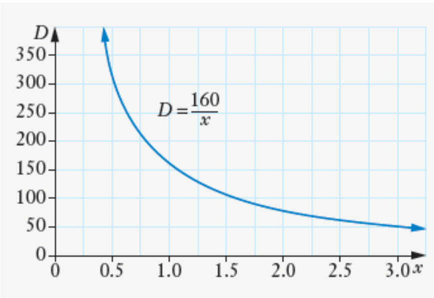
c When $D = 115.3$,

$115.3 = \dfrac{160}{x}$

$x = \dfrac{160}{115.3} = 1.3876\ldots \approx 1.4$ mm

d

x	0.5	1	1.5	2	2.5	3
D	320	160	106.7	80	64	53.3



Question 2

a $C = \dfrac{k}{n}$.

When $x = 128$, $C = 2$.

$2 = \dfrac{k}{128}$

$k = 256$

$\therefore C = \dfrac{256}{n}$

b When $n = 100$,

$C = \dfrac{256}{100} = \$2.56$

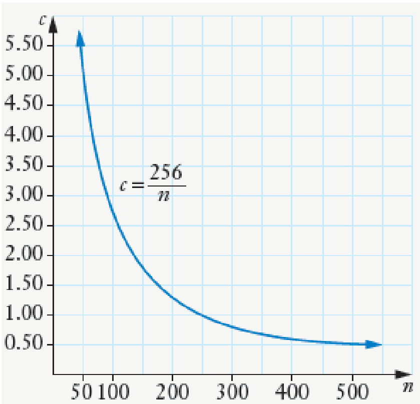
c When $C = \$0.5$,

$0.5 = \dfrac{256}{n}$

$n = \dfrac{256}{0.5} = 512$ boxes

d

n	100	200	300	400	512
C **($)**	2.56	1.28	0.85	0.64	0.5



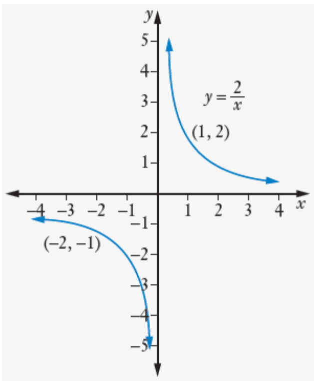
Question 3

a **i** Domain $(-\infty, 0) \cup (0, \infty)$

Range $(-\infty, 0) \cup (0, \infty)$

ii No y-intercept

iii

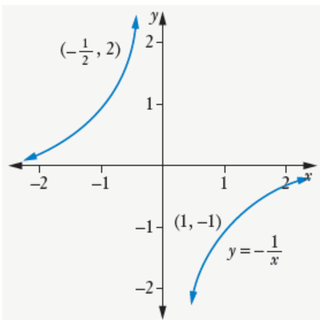


b **i** Domain $(-\infty, 0) \cup (0, \infty)$

Range $(-\infty, 0) \cup (0, \infty)$

ii No y-intercept

iii

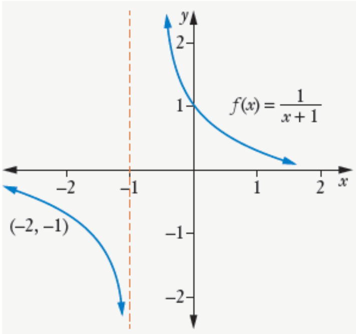


c **i** Domain $(-\infty,-1)\cup(-1,\infty)$

Range $(-\infty,0)\cup(0,\infty)$

ii $f(0) = \dfrac{1}{0+1} = 1$, y-intercept 1.

iii Vertical asymptote at $x = -1$.

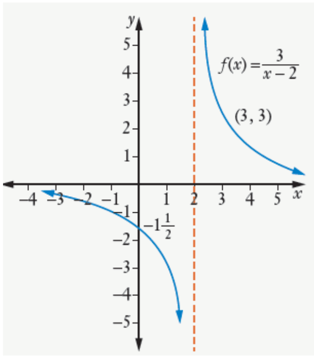


d **i** Domain $(-\infty,2)\cup(2,\infty)$

Range $(-\infty,0)\cup(0,\infty)$

ii $f(0) = \dfrac{3}{-2} = -1\dfrac{1}{2}$, y-intercept $-1\dfrac{1}{2}$

iii Vertical asymptote at $x = 2$.



e **i** $3x + 6 \neq 0$

$3x \neq -6$

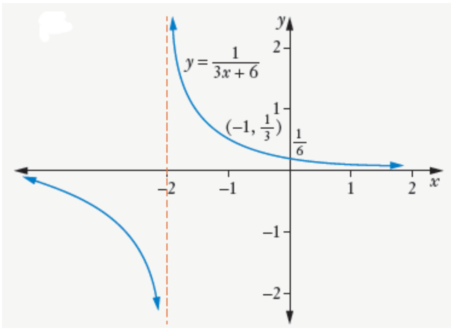
$x \neq -2$

Domain $(-\infty,-2)\cup(-2,\infty)$

Range $(-\infty,0)\cup(0,\infty)$

ii $f(0) = \dfrac{1}{0+6} = \dfrac{1}{6}$, y-intercept $\dfrac{1}{6}$.

iii Vertical asymptote at $x = -2$.

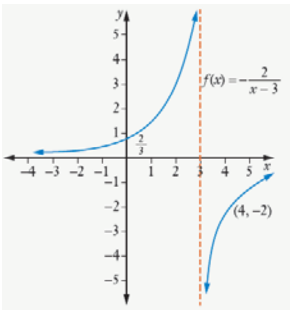


f **i** Domain $(-\infty,3)\cup(3,\infty)$

Range $(-\infty,0)\cup(0,\infty)$

ii $f(0) = -\dfrac{2}{-3} = \dfrac{2}{3}$, y-intercept $\dfrac{2}{3}$.

iii Vertical asymptote at $x = 3$.

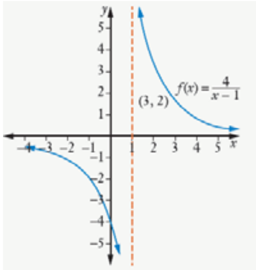


g **i** Domain $(-\infty,1)\cup(1,\infty)$

Range $(-\infty,0)\cup(0,\infty)$

ii $f(0) = \dfrac{4}{-1} = -4$, y-intercept -4.

iii Vertical asymptote at $x = 1$.

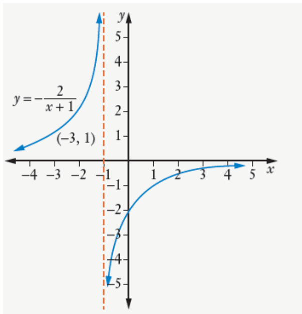


h **i** Domain $(-\infty,-1)\cup(-1,\infty)$

Range $(-\infty,0)\cup(0,\infty)$

ii $f(0) = -\dfrac{2}{1} = -2$, y-intercept -2.

iii Vertical asymptote at $x = -1$.



i **i** $6x - 3 \neq 0$

$6x \neq 3$

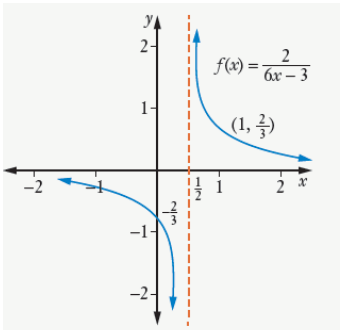
$x \neq \dfrac{1}{2}$

Domain $\left(-\infty,\dfrac{1}{2}\right)\cup\left(\dfrac{1}{2},\infty\right)$

Range $(-\infty,0)\cup(0,\infty)$

ii $f(0) = \dfrac{2}{0-3} = -\dfrac{2}{3}$, y-intercept $-\dfrac{2}{3}$.

iii Vertical asymptote at $x = \dfrac{1}{2}$.



Question 4

$f(x) = \dfrac{2}{x}$

$f(-x) = \dfrac{2}{-x} = -\dfrac{2}{x} = -f(x)$

$\therefore$ $f(x)$ is odd.

Question 5

a **i** Yes

ii Neither

$$f(x) = -\frac{2}{x+1} = -\frac{2}{(-x)+1} = \frac{2}{x-1} \neq f(x) \text{ or } -f(x)$$

iii Not continuous at $x = -1$.

b Vertical asymptote $x = -1$, horizontal asymptote $y = 0$.

c Domain $(-\infty, -1) \cup (-1, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

Exercise 7.02 Absolute value functions

Question 1

a x-intercept

$0 = |x| + 7$

$|x| = -7$

No solution, no x-intercept.

y-intercept

$y = |0| + 7 = 7$

b x-intercept

$0 = |x| - 2$

$|x| = 2$

$x = \pm 2$

y-intercept

$y = |0| - 2 = -2$

c x-intercept

$0 = 5|x|$

$|x| = 0$

$x = 0$

y-intercept

$y = 5|0| = 0$

d x-intercept

$0 = -|x| + 3$

$|x| = 3$

$x = \pm 3$

y-intercept

$y = -|0| + 3 = 3$

e x-intercept

$0 = -|x + 6|$

$x + 6 = 0$

$x = -6$

y-intercept

$y = |0 + 6| = 6$

f x-intercept

$0 = |3x - 2|$

$3x - 2 = 0$

$3x = 2$

$x = \frac{2}{3}$

y-intercept

$y = |0 - 2| = 2$

g x-intercept

$0 = |5x + 4|$

$5x + 4 = 0$

$5x = -4$

$x = -\frac{4}{5}$

y-intercept

$y = |0 + 4| = 4$

h x-intercept

$0 = |7x - 1|$

$7x - 1 = 0$

$7x = 1$

$x = \frac{1}{7}$

y-intercept

$y = |0 - 1| = 1$

i x-intercept

$0 = |2x| + 9$

$|2x| = -9$

No solution.

y-intercept

$y = |0| + 9 = 9$

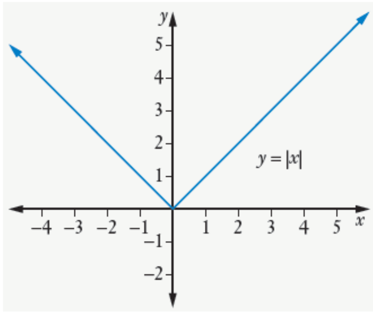
Question 2

a x-intercept

0

y-intercept

0



b x-intercept

$0 = |x| + 1$

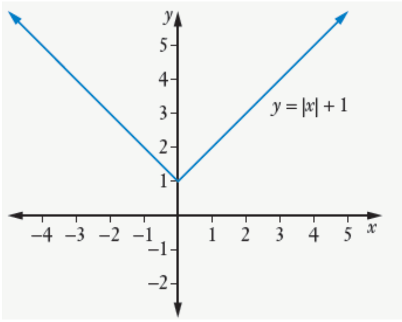
$|x| = -1$

No solution.

y-intercept

$y = |0| + 1 = 1$

$$f(x) = \begin{cases} x+1 & \text{for } x \geq 0 \\ -x+1 & \text{for } x < 0 \end{cases}$$



c x-intercept

$0 = |x| - 3$

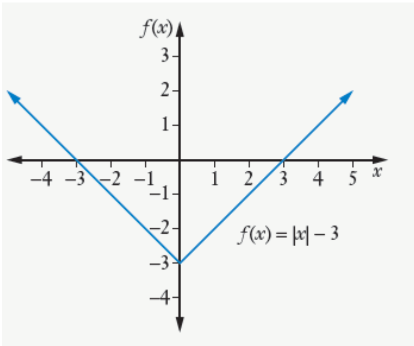
$|x| = 3$

$x = \pm 3$

y-intercept

$y = |0| - 3 = -3$

$$f(x) = \begin{cases} x-3 & \text{for } x \geq 0 \\ -x-3 & \text{for } x < 0 \end{cases}$$



d x-intercept

$0 = 2|x|$

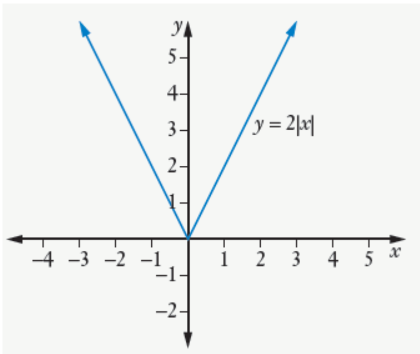
$|x| = 0$

$x = 0$

y-intercept

$y = -2|0| = 0$

$$f(x) = \begin{cases} 2x & \text{for } x \geq 0 \\ -2x & \text{for } x < 0 \end{cases}$$



e x-intercept

$0 = -|x|$

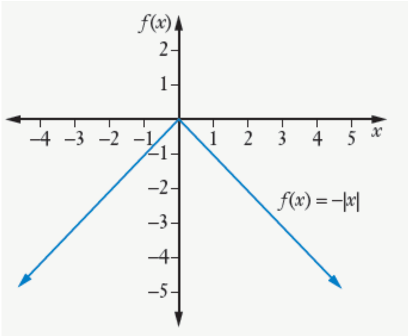
$|x| = 0$

$x = 0$

y-intercept

$y = -|0| = 0$

$$f(x) = \begin{cases} -x & \text{for } x \geq 0 \\ x & \text{for } x < 0 \end{cases}$$



f x-intercept

$0 = |x + 1|$

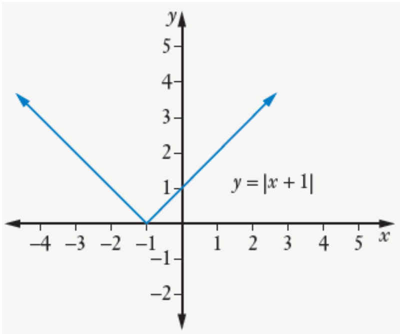
$x + 1 = 0$

$x = -1$

y-intercept

$y = |0 + 1| = 1$

$$f(x) = \begin{cases} x+1 & \text{for } x \geq -1 \\ -x-1 & \text{for } x < -1 \end{cases}$$



g x-intercept

$0 = -|x - 1|$

$|x - 1| = 0$

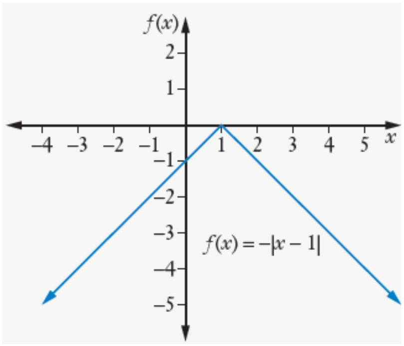
$x - 1 = 0$

$x = 1$

y-intercept

$y = -|0 - 1| = -1$

$$f(x) = \begin{cases} -x+1 & \text{for } x \geq 1 \\ x-1 & \text{for } x < 1 \end{cases}$$



h x-intercept

$0 = |2x - 3|$

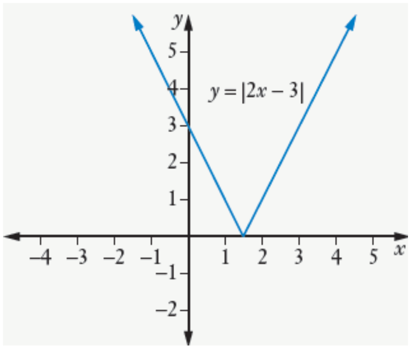
$2x - 3 = 0$

$x = \frac{3}{2} = 1\frac{1}{2}$

y-intercept

$y = |0 - 3| = 3$

$$f(x) = \begin{cases} 2x-3 & \text{for } x \geq 1\frac{1}{2} \\ -2x+3 & \text{for } x < 1\frac{1}{2} \end{cases}$$



i x-intercept

$0 = |3x| + 1$

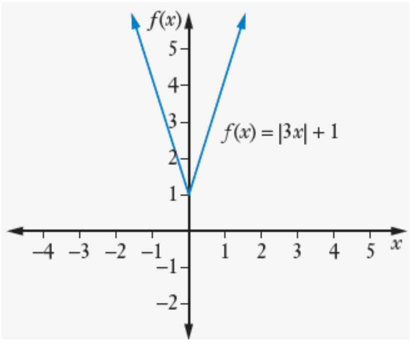
$|3x| = -1$

No solution

y-intercept

$y = |0| + 1 = 1$

$$f(x) = \begin{cases} 3x+1 & \text{for } x \geq 0 \\ -3x-1 & \text{for } x < 0 \end{cases}$$



Question 3

a Domain $(-\infty, \infty)$, range $[0, \infty)$

b Domain $(-\infty, \infty)$.
$|x| \geq 0$ so $|x| - 8 \geq -8$, so range $[-8, \infty)$.

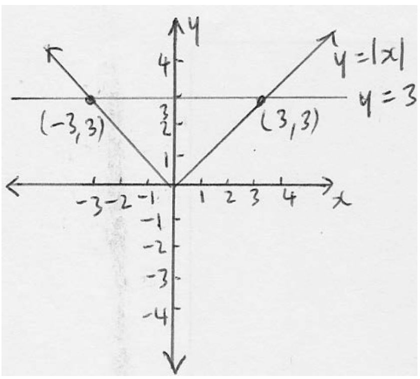
c Domain $(-\infty, \infty)$, range $[0, \infty)$

d Domain $(-\infty, \infty)$.
$|x| \geq 0$ so $2|x| - 3 \geq -3$, so range $[-3, \infty)$.

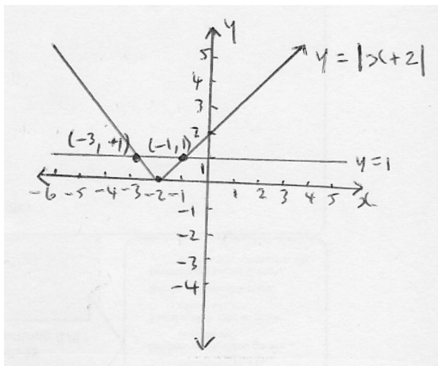
e Domain $(-\infty, \infty)$.
$|x - 3| \geq 0$ so $-|x - 3| \leq 0$, so range $(-\infty, 0]$

Question 4

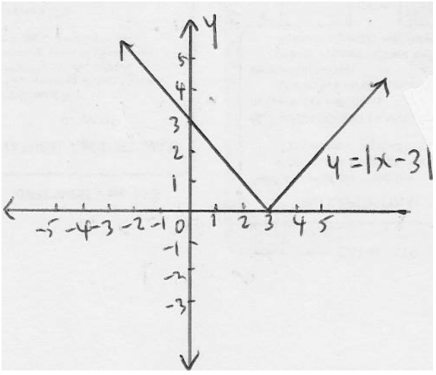
a $x = \pm 3$



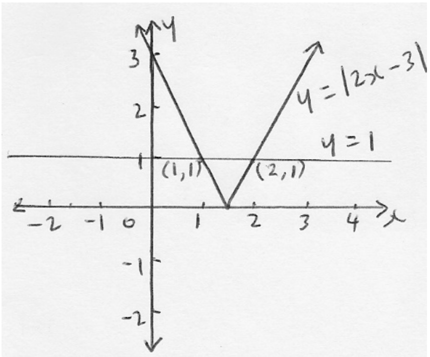
b $x = -3, -1$



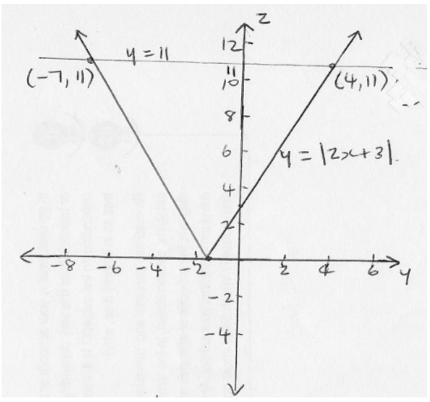
c $x = 3$



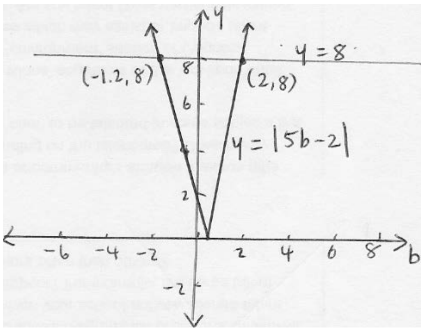
d $x = 1, 2$



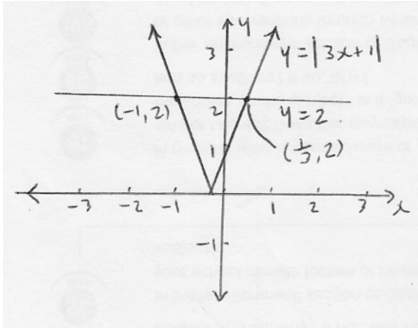
e $x = -7, 4$



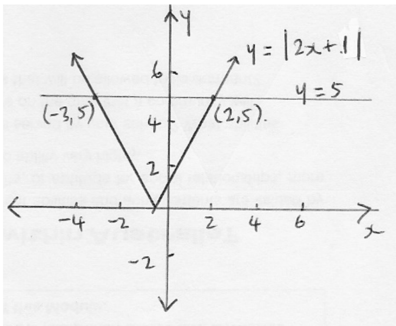
f $x = 2, -1.2$



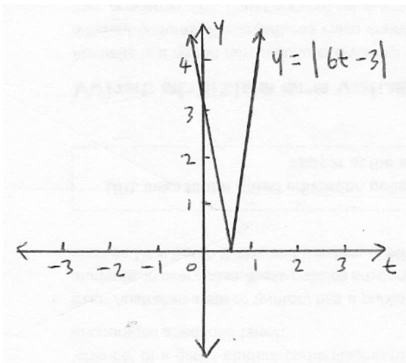
g $x = \dfrac{1}{3}, -1$



h $x = 2, -3$

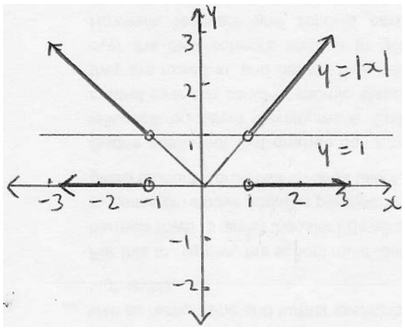


i $t = \dfrac{1}{2}$

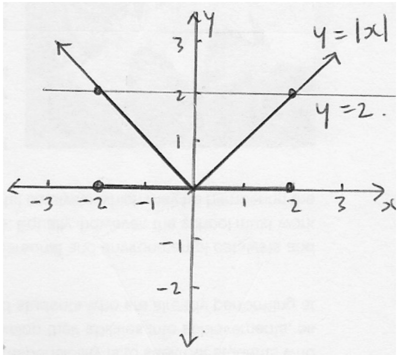


Question 5

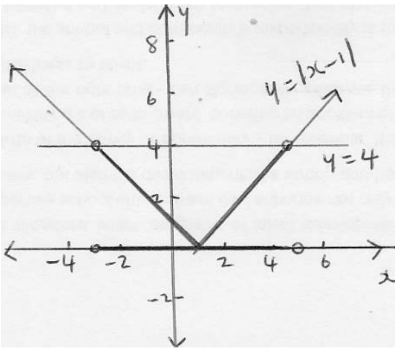
a $x < -1, x > 1$



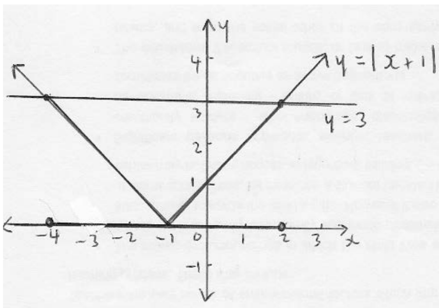
b $-2 \le x \le 2$



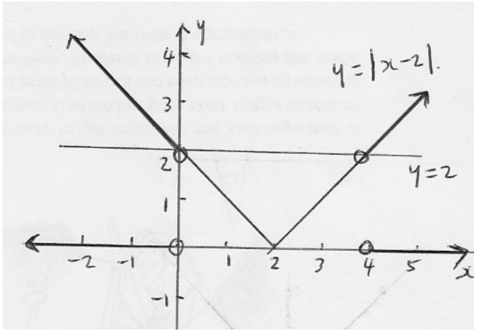
c $-3 < x < 5$



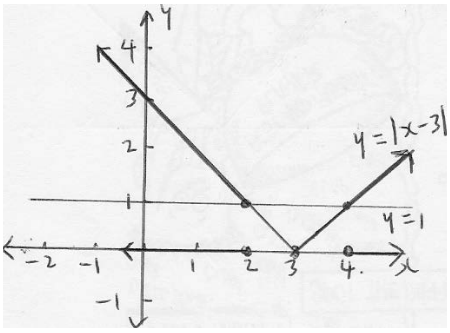
d $-4 \le x \le 2$



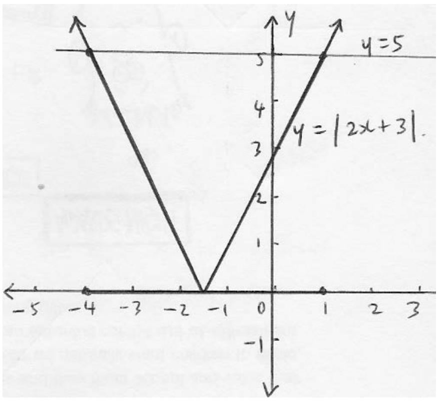
e $x < 0, x > 4$



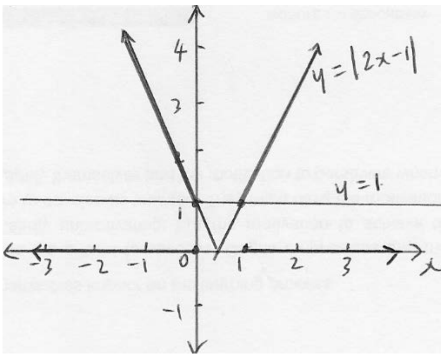
f $x \le 2, x \ge 4$



g $-4 \le x \le 1$



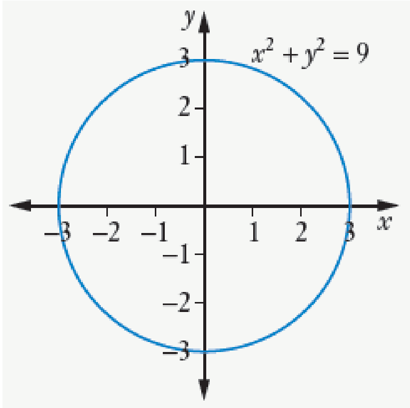
h $x \le 0, x \ge 1$



Question 1

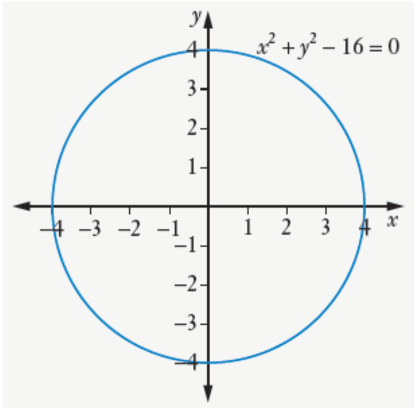
a **i** Circle, centre (0, 0), radius $\sqrt{9} = 3$

ii Domain [–3, 3], range [–3, 3]



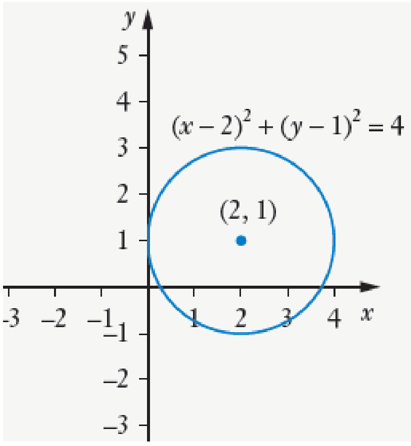
b **i** Circle, centre (0, 0), radius $\sqrt{16} = 4$

ii Domain [–4, 4], range [–4, 4]



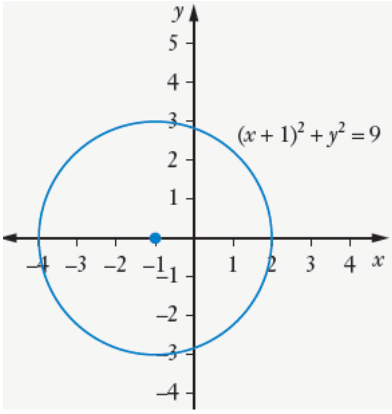
c **i** Circle, centre (2, 1), radius $\sqrt{4} = 2$

ii Domain [0, 4], range [–1, 3]



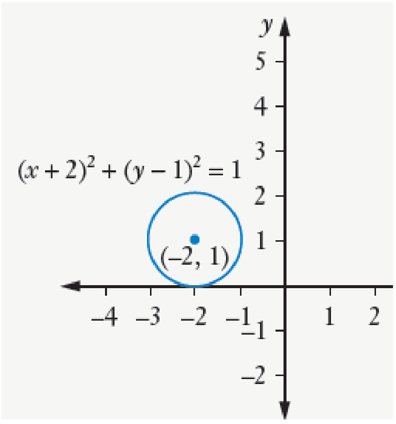
d **i** Circle, centre (–1, 0), radius $\sqrt{9} = 3$

ii Domain [–4, 2], range [–3, 3]



e **i** Circle, centre (–2, 1), radius $\sqrt{1} = 1$

ii Domain [–3, –1], range [0, 2]

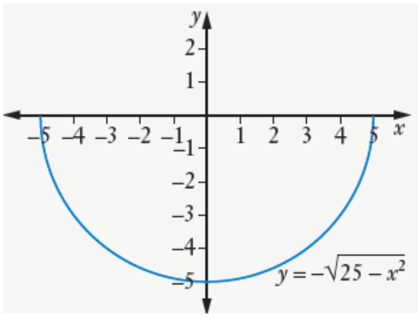


Question 2

a **i** $-\sqrt{}$ so below x-axis.

ii Radius $\sqrt{25} = 5$

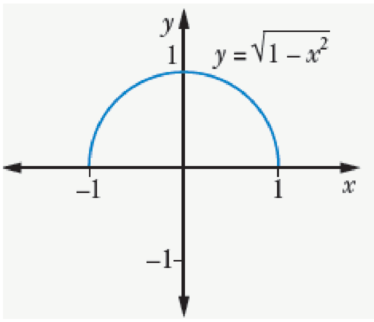
iii Domain [–5, 5], range [–5, 0]



b **i** $+\sqrt{}$ so above x-axis.

ii Radius $\sqrt{1} = 1$

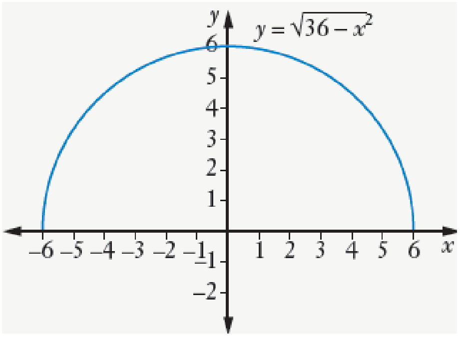
iii Domain [–1, 1], range [0, 1]



c **i** $+\sqrt{\ }$ so above x-axis.

ii Radius $\sqrt{36} = 6$

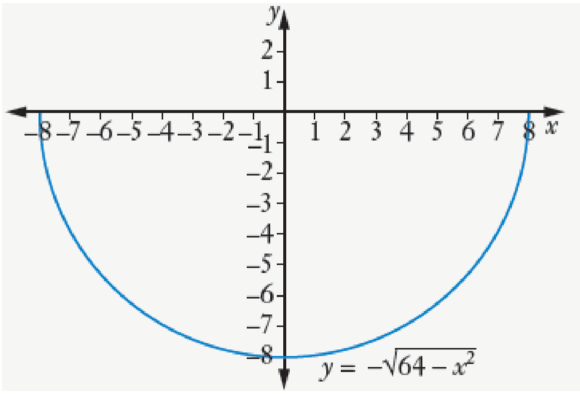
iii Domain [–6, 6], range [0, 6]



d **i** $-\sqrt{\ }$ so below x-axis.

ii Radius $\sqrt{64} = 8$

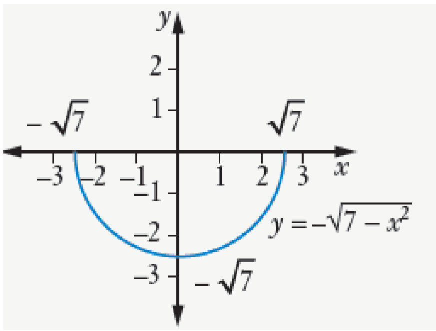
iii Domain [–8, 8], range [–8, 0]



e **i** $-\sqrt{\ }$ so below x-axis.

ii Radius $\sqrt{7}$

iii Domain $[-\sqrt{7}, \sqrt{7}]$, range $[-\sqrt{7}, 0]$



Question 3

a $x^2 + y^2 = 100$

Radius $= \sqrt{100} = 10$

Centre (0, 0)

b $x^2 + y^2 = 5$

Radius $= \sqrt{5}$

Centre (0, 0)

c Radius $= \sqrt{16} = 4$

Centre (4, 5)

d Radius $= \sqrt{49} = 7$

Centre (5, –6)

e Radius $= \sqrt{81} = 9$

Centre (0, 3)

Question 4

a $x^2 + y^2 = 4^2$

$x^2 + y^2 = 16$

b $(x - 3)^2 + (y - 2)^2 = 5^2$

$x^2 - 6x + 9 + y^2 - 4y + 4 = 25$

$x^2 - 6x + y^2 - 4y - 12 = 0$

c $(x + 1)^2 + (y - 5)^2 = 3^2$

$x^2 + 2x + 1 + y^2 - 10y + 25 = 9$

$x^2 + 2x + y^2 - 10y + 17 = 0$

d $(x - 2)^2 + (y - 3)^2 = 6^2$

$x^2 - 4x + 4 + y^2 - 6y + 9 = 36$

$x^2 - 4x + y^2 - 6y - 23 = 0$

e $(x + 4)^2 + (y - 2)^2 = 5^2$

$x^2 + 8x + 16 + y^2 - 4y + 4 = 25$

$x^2 + 8x + y^2 - 4y - 5 = 0$

f $x^2 + (y + 2)^2 = 1^2$

$x^2 + y^2 + 4y + 4 = 1$

$x^2 + y^2 + 4y + 3 = 0$

g $(x - 4)^2 + (y - 2)^2 = 7^2$

$x^2 - 8x + 16 + y^2 - 4y + 4 = 49$

$x^2 - 8x + y^2 - 4y - 29 = 0$

h $(x + 3)^2 + (y + 4)^2 = 9^2$

$x^2 + 6x + 9 + y^2 + 8y + 16 = 81$

$x^2 + 6x + y^2 + 8y - 56 = 0$

i $(x + 2)^2 + y^2 = \left(\sqrt{5}\right)^2$

$x^2 + 4x + 4 + y^2 = 5$

$x^2 + 4x + y^2 - 1 = 0$

j $(x + 4)^2 + (y + 7)^2 = \left(\sqrt{3}\right)^2$

$x^2 + 8x + 16 + y^2 + 14y + 49 = 3$

$x^2 + 8x + y^2 + 14y + 62 = 0$

Question 5

a $x^2 - 4x + 4 + y^2 - 2y + 1 - 4 - 4 - 1 = 0$

$(x - 2)^2 + (y - 1)^2 = 9$

Radius $\sqrt{9} = 3$, centre (2, 1)

b $x^2 + 8x + 16 + y^2 - 4y + 4 - 5 = 16 + 4$

$(x + 4)^2 + (y - 2)^2 = 25$

Radius $\sqrt{25} = 5$, centre (–4, 2)

c $x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y - 1)^2 = 1$

Radius $\sqrt{1} = 1$, centre (0, 1)

d $x^2 - 10x + 25 + y^2 + 6y + 9 - 2 = 25 + 9$

$(x - 5)^2 + (y + 3)^2 = 36$

Radius $\sqrt{36} = 6$, centre $(5, -3)$

e $x^2 + 2x + 1 + y^2 - 2y + 1 = 1$

$(x + 1)^2 + (y - 1)^2 = 1$

Radius $\sqrt{1} = 1$, centre $(-1, 1)$

f $x^2 - 12x + 36 + y^2 = 36$

$(x - 6)^2 + y^2 = 36$

Radius $\sqrt{36} = 6$, centre $(6, 0)$

g $x^2 + 6x + 9 + y^2 - 8y + 16 = 9 + 16$

$(x + 3)^2 + (y - 4)^2 = 25$

Radius $\sqrt{25} = 5$, centre $(-3, 4)$

h $x^2 + 20x + 100 + y^2 - 4y + 4 + 40 = 100 + 4$

$(x + 10)^2 + (y - 2)^2 = 64$

Radius $\sqrt{64} = 8$, centre $(-10, 2)$

i $x^2 - 14x + 49 + y^2 + 2y + 1 + 25 = 49 + 1$

$(x - 7)^2 + (y + 1)^2 = 25$

Radius $\sqrt{25} = 5$, centre $(7, -1)$

j $x^2 + 2x + 1 + y^2 + 4y + 4 - 5 = 1 + 4$

$(x + 1)^2 + (y + 2)^2 = 10$

Radius $\sqrt{10}$, centre $(-1, -2)$

Question 6

a $x^2 - 6x + 9 + y^2 + 2y + 1 - 6 = 9 + 1$

$(x - 3)^2 + (y + 1)^2 = 16$

Centre (3, –1), radius $\sqrt{16} = 4$

b $x^2 - 4x + 4 + y^2 - 10y + 25 + 4 = 4 + 25$

$(x - 2)^2 + (y - 5)^2 = 25$

Centre (2, 5), radius $\sqrt{25} = 5$

c $x^2 + 2x + 1 + y^2 + 12y + 36 - 12 = 1 + 36$

$(x + 1)^2 + (y + 6)^2 = 49$

Centre (–1, –6), radius $\sqrt{49} = 7$

d $x^2 - 8x + 16 + y^2 - 14y + 49 + 1 = 16 + 49$

$(x - 4)^2 + (y - 7)^2 = 64$

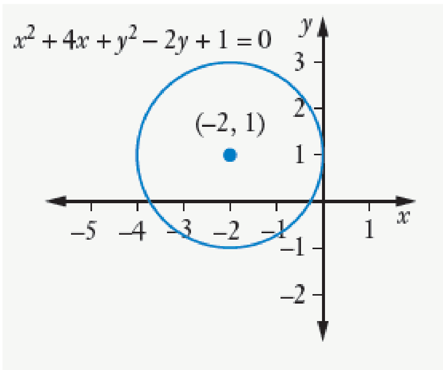
Centre (4, 7), radius $\sqrt{64} = 8$

Question 7

$x^2 + 4x + 4 + y^2 - 2y + 1 = 4$

$(x + 2)^2 + (y - 1)^2 = 4$

Circle, centre (–2, 1), radius $\sqrt{4} = 2$



Exercise 7.04 Reflections of functions

Question 1

a **i** $y=-f(x)=-(x^2-2)$
$=-x^2+2$

ii $y=f(-x)=(-x)^2-2$
$=x^2-2$

iii $y=-f(-x)=-(x^2-2)$
$=-x^2+2$

b **i** $y=-f(x)=-(x+1)^3$

ii $y=f(-x)=(-x+1)^3$

iii $y=-f(-x)=-\left[(-x+1)^3\right]=-(-x+1)^3$

c **i** $y=-f(x)=-(5x-3)=-5x+3$

ii $y=f(-x)=5(-x)-3=-5x-3$

iii $y=-f(-x)=-(-5x-3)=5x+3$

d **i** $y=-f(x)=-|2x+5|$

ii $y=f(-x)=|2(-x)+5|=|-2x+5|$

iii $y=-f(-x)=-|-2x+5|$

e **i** $y=-f(x)=-\dfrac{1}{x-1}$

ii $y=f(-x)=\dfrac{1}{-x-1}=-\dfrac{1}{x+1}$

iii $y=-f(-x)=-\left(-\dfrac{1}{x+1}\right)=\dfrac{1}{x+1}$

Question 2

a $-f(x)$ takes the negative of each y-value in $f(x)$, so it is a reflection in the x-axis.

b $f(-x)$ takes the negative of each x-value in $f(x)$, so it is a reflection in the y-axis.

c $f(-x)$ reflects $f(x)$ in the y-axis and $-f(-x)$ reflects $f(-x)$ in the x-axis.

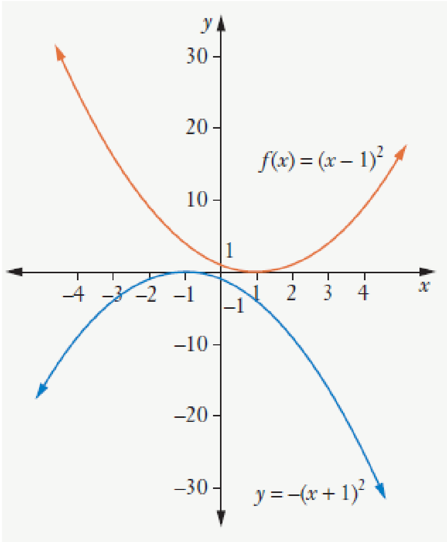
Question 3

$f(-x)=(-x-1)^2=\left[-(x+1)\right]^2=(x+1)^2$ and
$-f(-x)=-(x+1)^2$.

The graph of $y=-f(-x)$ is the graph of $f(x)=\left(x-1\right)^2$ reflected in the y-axis and x-axis.

$-f(-x)$ has x-intercept at $-(x+1)^2=0 \Rightarrow x=-1$.

The y-intercept is $-(0+1)^2=-1$



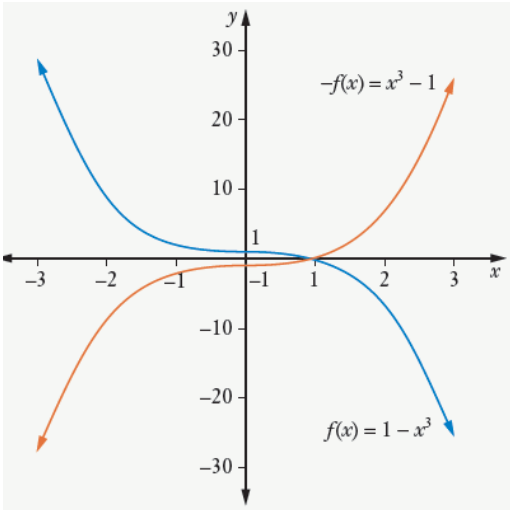
Question 4

$f(x)=1-x^3$ has an x-intercept at $1-x^3=0 \Rightarrow x=1$
.

The y-intercept is $1-0^3=1$

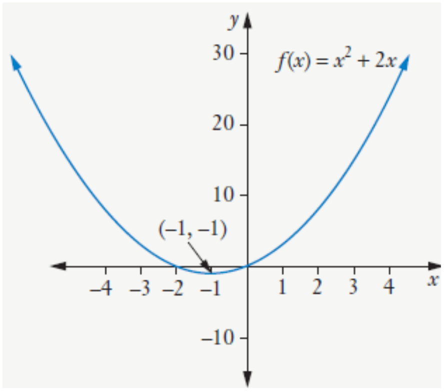
$$\begin{aligned}-f(x)&=-(1-x^3)\\&=-1+x^3\\&=x^3-1\end{aligned}$$

This is a reflection of $f(x)$ in the x-axis.



Question 5

a $f(x) = x^2 + 2x = x(x+2)$



x-intercepts at $x(x+2) = 0 \Rightarrow x = 0, x = -2$.

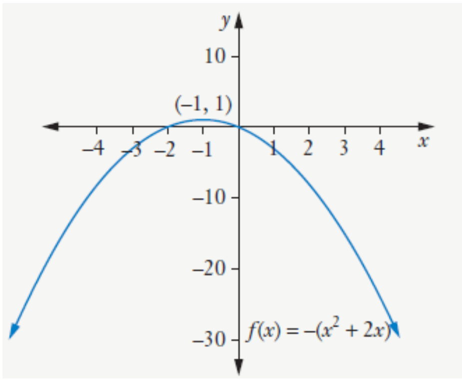
The y-intercept is 0.

The axis of symmetry is midway between the x-intercepts, that is, at $x = -1$.

$f(-1) = (-1)^2 + 2 \times (-1) = -1$

The parabola is concave upward, and the turning point is at (–1, –1)

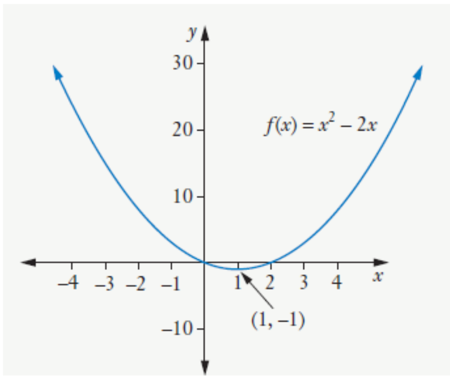
b $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis. The axis of symmetry and the x-intercepts remain the same.
The turning point (–1, –1) becomes (–1, 1), and the shape is now concave downward.



c $y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis. The new x-intercepts are the negative of the old, so (–2, 0) and (0, 0) become (2, 0) and (0, 0).

The axis of symmetry $x = -1$ becomes $x = 1$ and the turning point (–1, –1) becomes (1, –1).

The shape remains concave upward.

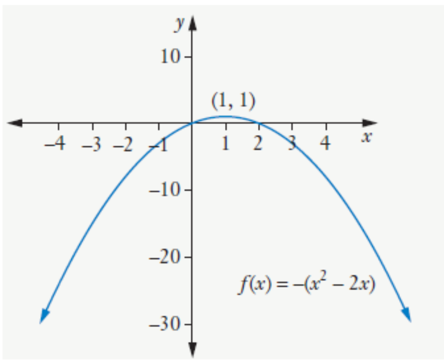


d $y = -f(-x)$ reflects $y = f(-x)$ in the x-axis.

The axis of symmetry, $x = 1$, and the x-intercepts (2, 0) and (0, 0), are the same as those of $y = f(-x)$.

The turning point (1, –1) becomes (1, 1).

The shape is now concave downward.



Question 6

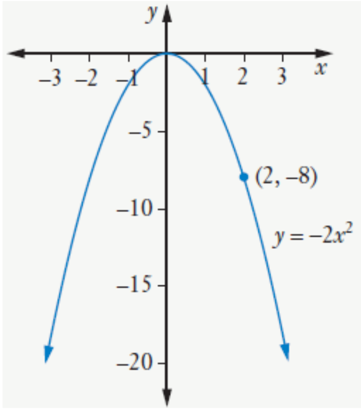
a $f(-x) = 2(-x)^2 = 2x^2 = f(x)$, so $f(x)$ is an even function.

b **i** $f(-x) = 2(-x)^2 = 2x^2$

The equation is $y = 2x^2$

ii $-f(x) = -(2x^2) = -2x^2$

The equation is $y = -2x^2$



c $-f(-x) = -(2x^2) = -2x^2$

The x-intercept and y-intercept are both at 0, so the turning point is at (0, 0).

The graph is concave downward, and the maximum turning point is at (0, 0).

Question 7

a $f(-x) = -(-x)^3 = x^3 = -f(x)$, so $f(x)$ is an odd function.

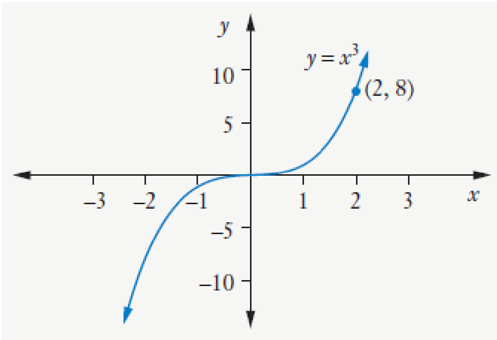
b **i** $-f(x) = -(-x^3) = x^3$

The equation is $y = x^3$

ii $-f(-x) = -(x^3) = -x^3$

The equation is $y = -x^3$

c $y = f(-x) = x^3$



Question 8

a $f(x) = x^3 - 7x^2 + 12x = x(x^2 - 7x + 12) = x(x-3)(x-4)$

x-intercepts

$f(x) = 0$

$x = 0, x = 3, x = 4$

y-intercept

$x = 0$

$f(x) = 0^3 - 7 \times 0^2 + 12 \times 0 = 0$

The y-intercept is 0.

$f(x) = x(x-3)(x-4)$

For $x > 4$, $f(x) > 0$

For $3 < x < 4$, $f(x) < 0$

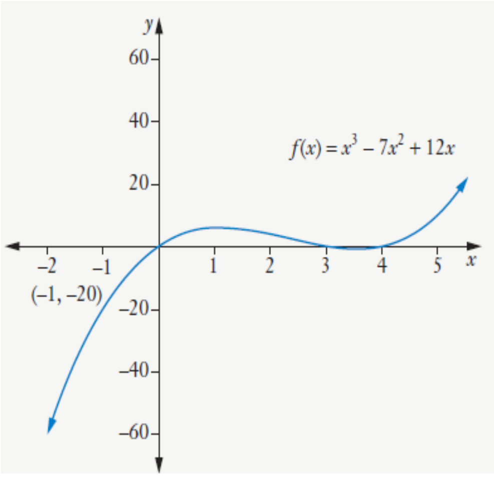
For $0 < x < 3$, $f(x) > 0$

For $x < 0$, $f(x) < 0$

Approximate turning points by evaluating $f(x)$ for values of x midway between x-intercepts.

Try $x = 1, f(1) = 1(1-3)(1-4) = 8$

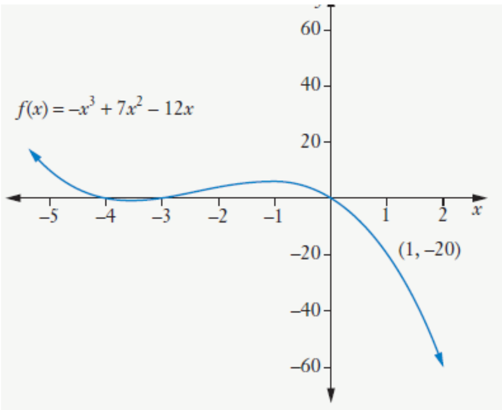
Try $x = 3.5, f(3.5) = 3.5(3.5-3)(3.5-4) = -0.875$



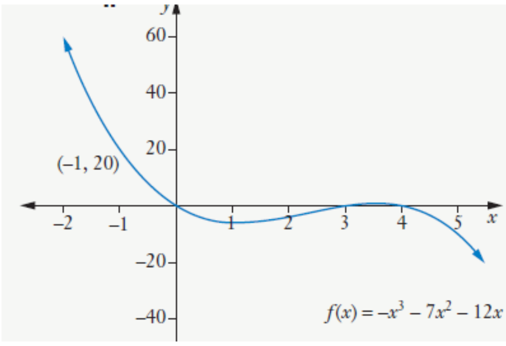
b **i** $f(-x)$ reflects the graph of $f(x)$ in the y-axis. Thus, the x-intercepts are the negative of the x-intercepts of $f(x)$.

Thus, $x = 0, x = 3, x = 4$ become $x = 0, x = -3, x = -4$.

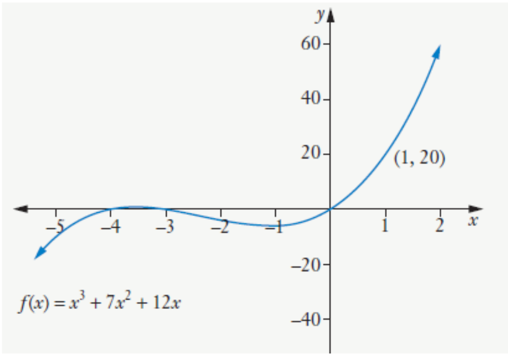
The approximate y values of the turning points remain the same.



ii $-f(x)$ reflects the graph of $f(x)$ in the x-axis. The x-intercepts remain the same, but the y values of the turning points become the negative of the y values of $f(x)$.



iii $-f(-x)$ reflects the graph of $f(-x)$ in the x-axis. The x-intercepts remain the same, but the y values of the turning points become the negative of the y values of $f(-x)$.



Question 9

a $P(x) = (x^3 - 2x^2) - (x^2 - 4)$

$= x^2(x-2) - (x-2)(x+2)$

$= (x-2)(x^2 - x + 2)$

$= (x-2)(x-2)(x+1)$

$= (x+1)(x-2)^2$

b x-intercepts

$P(x) = 0$

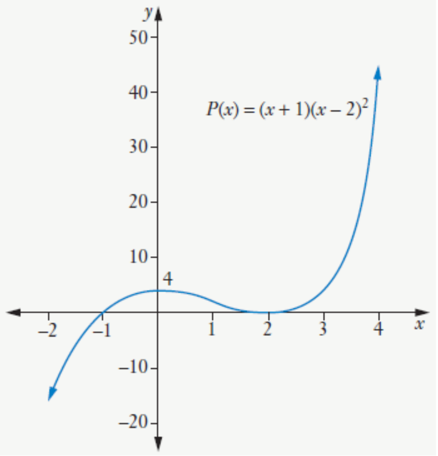
$(x+1)(x-2)^2 = 0$

$x = -1, x = 2$

y-intercept

$x = 0$

$y = (0+1)(0-2)^2 = 4$



x-intercepts

$x = 2$ is a repeated root, which means the graph of $P(x)$ touches the x-axis at $x = 2$.

$P(x) > 0$ for $x > 2$ and for $-1 < x < 2$.

Approximate the turning point by evaluating $P(x)$ for values of x midway between x-intercepts.

Try $x = 0.5$, $P(0.5) = (0.5+1)(0.5-2)^2 = 2.25$

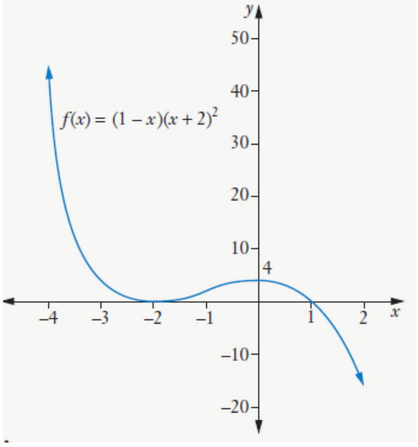
c **i** The graph of $y = P(-x)$ reflects the graph of $y = P(x)$ in the y-axis.

The x-intercepts are the negative of the x-intercepts of $P(x)$.

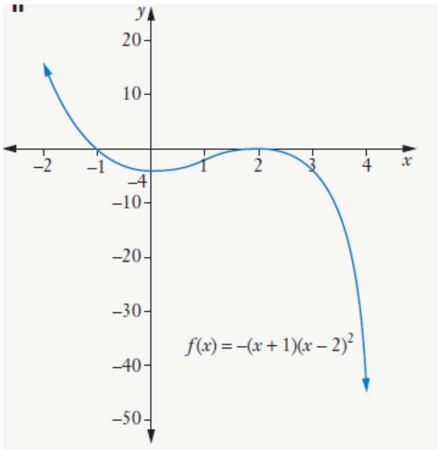
Thus, $x = -1, x = 2$ become $x = 1, x = -2$.

The y-intercept remains unchanged.

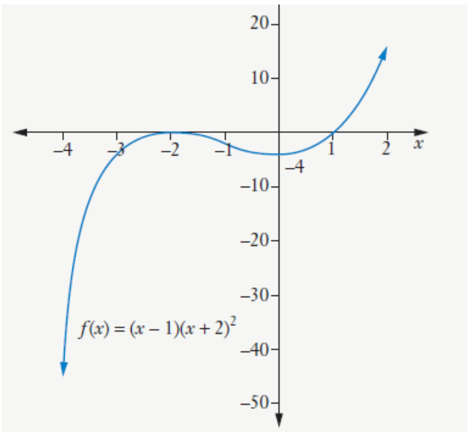
The y value of the turning points remain the same.



ii $y=-P(x)$ reflects the graph of $y=P(x)$ in the x-axis. The x-intercepts remain the same, but the y value of the turning point becomes the negative of the y value of $y=P(x)$.



iii $y=-P(-x)$ reflects the graph of $y=P(-x)$ in the x-axis.
The x-intercepts remain the same, but the y value of the turning point becomes the negative of the y value of $y=P(-x)$.



Exercise 7.05 Combined and composite functions

Question 1

a **i** $y = 4x + 1 + 2x^2 + x$
$= 2x^2 + 5x + 1$

ii $y = 4x + 1 - (2x^2 + x)$
$= 4x + 1 - 2x^2 - x$
$= -2x^2 + 3x + 1$

iii $y = (4x + 1)(2x^2 + x)$
$= 8x^3 + 4x^2 + 2x^2 + x$
$= 8x^3 + 6x^2 + x$

iv $y = \dfrac{4x+1}{2x^2+x}$

b **i** $y = x^4 + 5x - 4 + x^3 + 5$
$= x^4 + x^3 + 5x + 1$

ii $y = x^4 + 5x - 4 - x^3 - 5$
$= x^4 - x^3 + 5x - 9$

iii $y = (x^4 + 5x - 4)(x^3 + 5)$
$= x^7 + 5x^4 + 5x^4 + 25x$
$- 4x^3 - 20$
$= x^7 + 10x^4 - 4x^3 + 25x - 20$

iv $y = \dfrac{x^4+5x-4}{x^3+5}$

c **i** $y = x^2 + 3 + 5x^2 - 7x - 2$
$= 6x^2 - 7x + 1$

ii $y = x^2 + 3 - (5x^2 - 7x - 2)$
$= x^2 + 3 - 5x^2 + 7x + 2$
$= -4x^2 + 7x + 5$

iii $y = (x^2 + 3)(5x^2 - 7x - 2)$
$= 5x^4 - 7x^3 - 2x^2 + 15x^2$
$- 21x - 6$
$= 5x^4 - 7x^3 + 13x^2 - 21x - 6$

iv $y = \dfrac{x^2+3}{5x^2-7x-2}$

d **i** $y = 3x^2 + 2x - 1 + x^2 - x + 5$
$= 4x^2 + x + 4$

ii $y = 3x^2 + 2x - 1 - (x^2 - x + 5)$
$= 3x^2 + 2x - 1 - x^2 + x - 5$
$= 2x^2 + 3x - 6$

iii $y = (3x^2 + 2x - 1)(x^2 - x + 5)$
$= 3x^4 - 3x^3 + 15x^2 + 2x^3$
$- 2x^2 + 10x - x^2 + x - 5$
$= 3x^4 - x^3 + 12x^2 + 11x - 5$

iv $y = \dfrac{3x^2+2x-1}{x^2-x+5}$

e **i** $y = 4x^5 + 7 + 3x - 4$
$= 4x^5 + 3x + 3$

ii $y = 4x^5 + 7 - 3x + 4$
$= 4x^5 - 3x + 11$

iii $y = (4x^5 + 7)(3x - 4)$
$= 12x^6 - 16x^5 + 21x - 28$

iv $y = \dfrac{4x^5+7}{3x-4}$

Question 2

a	**i**	1	**ii**	1	**iii**	1 + 1 = 2
b	**i**	2	**ii**	2	**iii**	2 + 1 = 3
c	**i**	2	**ii**	1	**iii**	2 + 2 = 4
d	**i**	3	**ii**	3	**iii**	3 + 1 = 4

Question 3

a	**i**	4 – 7 = –3	**ii**	4 – (–7) = 11	**iii**	4 × (–7) = –28
b	**i**	1 + (–5) = –4	**ii**	1 – (–5) = 6	**iii**	1 × (–5) = –5
c	**i**	25 – 3 = 22	**ii**	25 – (–3) = 28	**iii**	25 × (–3) = –75
d	**i**	7	**ii**	7	**iii**	No constant term

Question 4

a $f(x) + g(x) = x + 2 + x - 4$

$= 2x - 2$

Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

b $f(x) + g(x) = 2x^2 + x - 1 - x - 1$

$= 2x^2 - 2$

Domain $(-\infty, \infty)$, range $[-2, \infty)$

c $f(x) + g(x) = x^3 + x + 2$

Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

d $f(x) + g(x) = x^2 - 1 + x - 1$

$= x^2 + x - 2$

Vertex of parabola:

$$x = \frac{-1}{2(1)} = -\frac{1}{2}$$

$$y = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) - 2$$

$$= -2\frac{1}{4}$$

Domain $(-\infty, \infty)$, range $\left[-2\frac{1}{4}, \infty\right)$

Question 5

a $f(x) - g(x) = 3x + 2 - x + 1$

$= 2x + 3$

Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

b $f(x) + g(x) = x^2 - 1 - x + 1$

$= x^2 - x$

Vertex of parabola:

$$x = \frac{-(-1)}{2(1)} = \frac{1}{2}$$

$$y = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$$

Domain $(-\infty, \infty)$, range $\left[-\frac{1}{4}\ \infty\right)$

c $f(x) - g(x) = x^3 + x - x - 2$

$= x^3 - 2$

Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

d $f(x) - g(x) = 3x^2 - x - 1 - x^2 - x - 3$

$= 2x^2 - 2x - 4$

Vertex of parabola:

$$x = \frac{-(-2)}{2(2)} = \frac{1}{2}$$

$$y = 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) - 4$$
$$= -4\frac{1}{2}$$

Domain $(-\infty, \infty)$, range $\left[-4\frac{1}{2}, \infty\right)$

Question 6

a $f(x)g(x) = (x + 2)(x - 4)$

$= x^2 - 2x - 8$

Vertex of parabola:

$$x = \frac{-(-2)}{2(1)} = 1$$

$$y = 1^2 - 2(1) - 8$$
$$= -9$$

Domain $(-\infty, \infty)$, range $[-9\ \infty)$

b $f(x)g(x) = (x - 5)(x + 5)$

$= x^2 - 25$

Domain $(-\infty, \infty)$, range $[-2,\ \infty)$

c $f(x)g(x) = x^2 \times x$

$= x^3$

Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

Question 7

a $\dfrac{f(x)}{g(x)} = \dfrac{5}{x-4}$

Domain $(-\infty, 4) \cup (4, \infty)$

b $\dfrac{f(x)}{g(x)} = \dfrac{x-1}{x+1}$

Domain $(-\infty, -1) \cup (-1, \infty)$

c $\dfrac{f(x)}{g(x)} = \dfrac{2x}{x-3}$

Domain $(-\infty, 3) \cup (3, \infty)$

d $\dfrac{f(x)}{g(x)} = \dfrac{x+3}{x^3}$

Domain $(-\infty, 0) \cup (0, \infty)$

Question 8

a $f(g(x)) = (x^2 + 1)^2 = x^4 + 2x^2 + 1$

b $f(g(x)) = (5x - 3)^3$

$= (5x - 3)^3(5x - 3)^2$

$= (5x - 3)(25x^2 - 30x + 9)$

$= 125x^3 - 150x^2 + 45x - 75x^2 + 90x - 27$

$= 125x^3 - 225x^2 + 135x - 27$

OR $f(g(x)) = (5x - 3)^3$

$= {}^3C_0\,(5x)^3 + {}^3C_1\,(5x)^2(-3) + {}^3C_2\,(5x)(-3)^2 + {}^3C_3\,(-3)^3$

$= 1(125x^3) + 3(25x^2)(-3) + 3(5x)(9) + 1(-27)$

$= 125x^3 - 225x^2 + 135x - 27$

c $f(g(x)) = (x^2 - 3x + 2)^7$

d $f(g(x)) = \sqrt{2x-1}$

e $f(g(x)) = \sqrt[3]{x^4 + 7x^2 - 4}$

f $f(g(x)) = 3(2x + 1) = 6x + 3$

g $f(g(x)) = 2x^3 - 7$

h $f(g(x)) = 6x^2 - 5$

i $f(g(x)) = 2(3x)^2 = 2(9x^2) = 18x^2$

j $f(g(x)) = 4(x^2 + 3)^2 + 1$

$= 4(x^4 + 6x^2 + 9) + 1$

$= 4x^4 + 24x^2 + 36 + 1$

$= 4x^4 + 24x^2 + 37$

Question 9

a $f(g(x)) = (x-1)^2 = x^2 - 2x + 1$

Vertex of parabola:

$$x = \frac{-(-2)}{2(1)} = 1$$

$$y = (1-1)^2 = 0$$

Domain $(-\infty, \infty)$, range $[0, \infty)$

b $f(g(x)) = (x+5)^3$

Domain $(-\infty, \infty)$, range $(-\infty, \infty)$

c $f(g(x)) = \sqrt{x-2}$

$x - 2 \geq 0$

$x \geq 2$

Domain $[2, \infty)$, range $[0, \infty)$

d $f(g(x)) = -\sqrt{3x+9}$

$3x + 9 \geq 0$

$3x \geq -9$

$x \geq -3$

Domain $[-3, \infty)$, range $(-\infty, 0]$

e $f(g(x)) = \sqrt{4-x^2}$

Equation of a semicircle, above x-axis. centre (0, 0), radius $\sqrt{4} = 2$

Domain $[-2, 2]$, range $[0, 2]$

f $f(g(x)) = -\sqrt{1-x^2}$

Equation of a semicircle, below x-axis. centre (0, 0), radius $\sqrt{1} = 1$

Domain $[-1, 1]$, range $[-1, 0]$

Question 10

a $f(g(x)) = \sqrt{x^3}$

b $g(f(x)) = \left(\sqrt{x}\right)^3$

Question 11

a $f(x)g(x) = \dfrac{x^2+3}{x}$

b $f(g(x)) = \dfrac{1}{x^2+3}$

c $\dfrac{f(x)}{g(x)} = \dfrac{1}{x\left(x^2+3\right)} = \dfrac{1}{x^3+3x}$

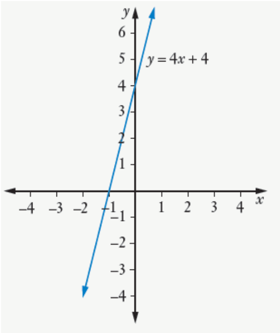
d $\dfrac{g(x)}{f(x)} = \dfrac{x^2+3}{\frac{1}{x}} = \left(x^2+3\right)x = x^3+3x$

Question 1

a $f(x) = 3x + 5$

$g(x) = x - 1$

$f(x) + g(x) = 4x + 4$



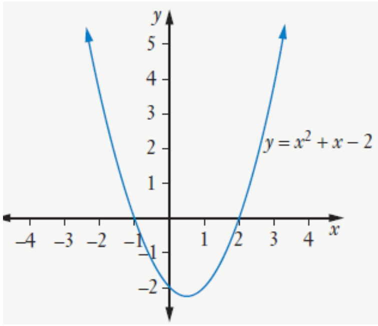
b $f(x) = x - 2$

$g(x) = x^2$

$f(x) + g(x) = x^2 + x - 2 = (x + 2)(x - 1)$

y-intercept = –2

x-intercepts = –2, 1



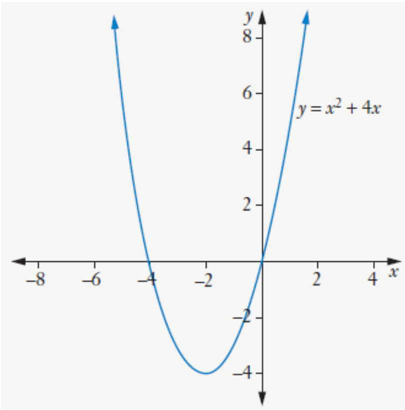
c $f(x) = 3x$

$g(x) = x^2 + x$

$f(x) + g(x) = x^2 + 4x = x(x + 4)$

y-intercept = 0

x-intercepts = 0, –4



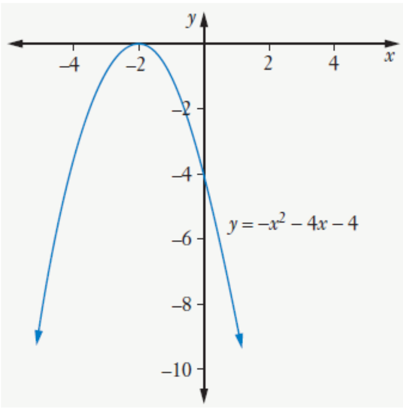
d $f(x) = -x - 3$

$g(x) = -x^2 - 3x - 1$

$f(x) + g(x) = -x^2 - 4x - 4 = -(x + 2)^2$

y-intercept = –4

x-intercepts = –2



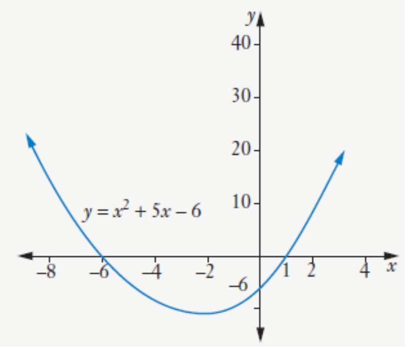
e $f(x) = 3x - 4$

$g(x) = x^2 + 2x - 2$

$f(x) + g(x) = x^2 + 5x - 6 = (x + 6)(x - 1)$

y-intercept $= -6$

x-intercepts $= -6, 1$



f $f(x) = -x^2 - 5$

$g(x) = 2x + 1$

$f(x) + g(x) = -x^2 + 2x - 4$

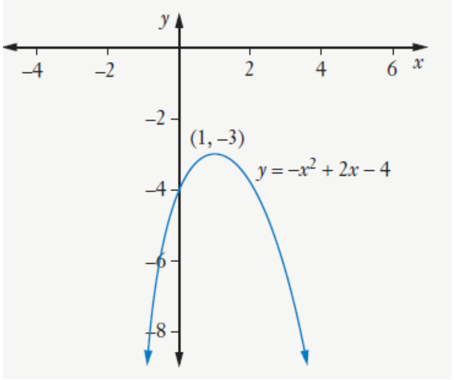
$\Delta < 0$ $\therefore$ no real solutions

Vertex

$$x = -\frac{b}{2a} = -\frac{2}{2(-1)} = 1$$

$y = -1^2 + 2(1) - 4 = -3$

Turning point $(1, -3)$

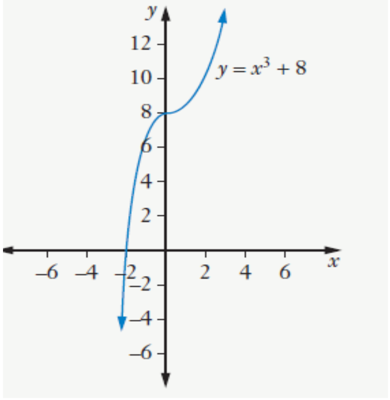


g $f(x) = -x + 1$

$g(x) = x^3 + x + 7$

$f(x) + g(x) = x^3 + 8$

This is the $f(x) = x^3$ shifted up 8 units.



h $f(x) = x^3 - 4x - 1$

$g(x) = x^2 - 3$

$f(x) + g(x) = x^3 + x^2 - 4x - 4$

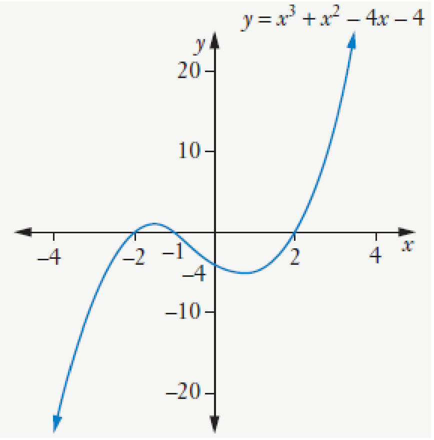
$= x^2(x + 1) - 4(x + 1)$

$= (x + 1)(x^2 - 4)$

$= (x + 1)(x + 2)(x - 2)$

y-intercept $= -4$

x-intercepts $= -1, -2, 2$

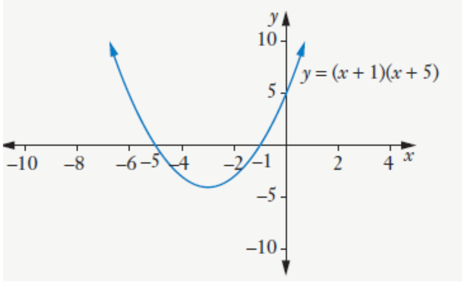


Question 2

a $f(x) = x + 1$

$g(x) = x + 5$

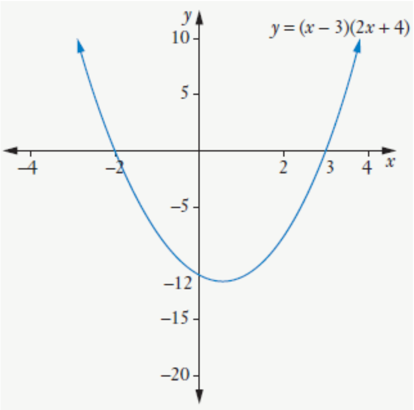
$f(x) \cdot g(x) = (x + 1)(x + 5)$



b $f(x) = x - 3$

$g(x) = 2x + 4$

$f(x) \cdot g(x) = (x - 3)(2x + 4)$



c $f(x) = x^2 - 1$

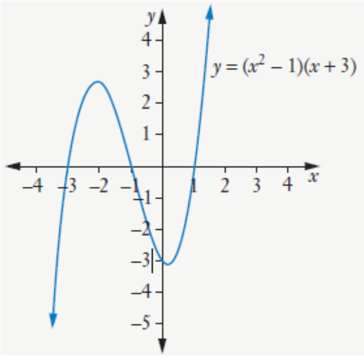
$g(x) = x + 3$

$f(x) \cdot g(x) = (x^2 - 1)(x + 3)$

$= (x + 1)(x - 1)(x + 3)$

y-intercept = –3

x-intercept = –1, 1, –3



d $f(x) = -x - 2$

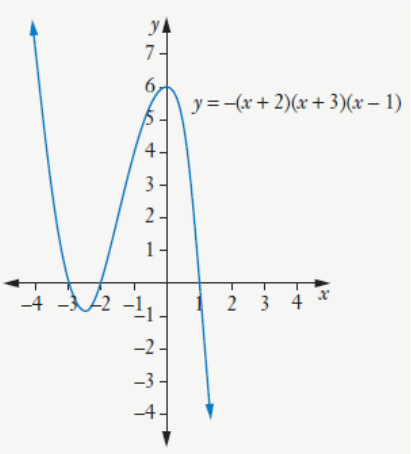
$g(x) = x^2 + 2x - 3$

$f(x) \cdot g(x) = (-x - 2)(x^2 + 2x - 3)$

$= -(x + 2)(x + 3)(x - 1)$

y-intercept = 6

x-intercept = –3, –2, 1



e $f(x) = x^2$

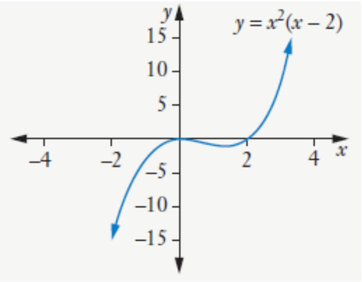
$g(x) = x - 2$

$f(x) \cdot g(x) = x^2(x - 2)$

y-intercept = 0

x intercept = 0, 2

Double root at $x = 0$



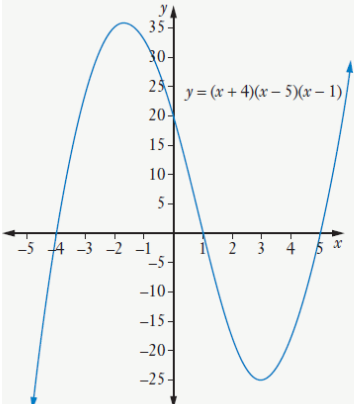
f $f(x) = x + 4$

$g(x) = x^2 - 6x + 5$

$f(x) \cdot g(x) = (x + 4)(x - 5)(x - 1)$

y-intercept = 20

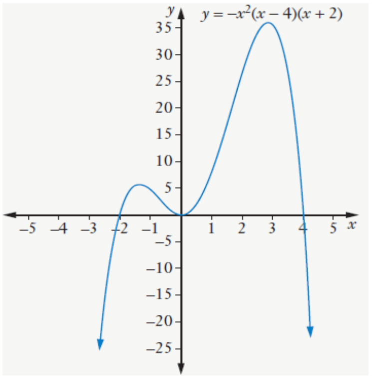
x-intercept = −4, 1, 5



g $f(x) = -x^2$

$g(x) = x^2 - 2x - 8$

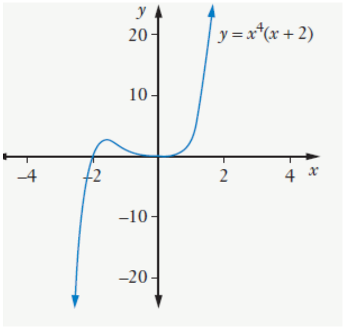
$f(x) \cdot g(x) = -x^2(x - 4)(x + 2)$



h $f(x) = x^3$

$g(x) = x^2 + 2x$

$f(x) \cdot g(x) = x^4(x + 2)$

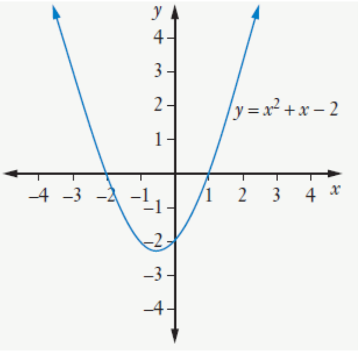


Question 3

$f(x) = x + 1,\ g(x) = x^2 - 3$

a $f(x) + g(x) = x^2 + x - 2$

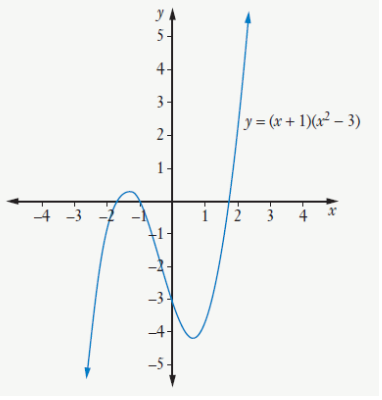
$= (x + 2)(x - 1)$



b $f(x) \cdot g(x) = (x + 1)(x^2 - 3)$

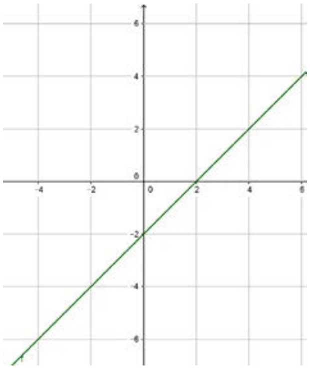
y-intercept = −3

x-intercept = $-1,\ \pm\sqrt{3}$

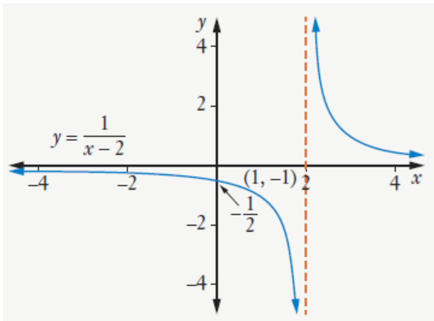


Question 1

a $f(x) = x - 2$

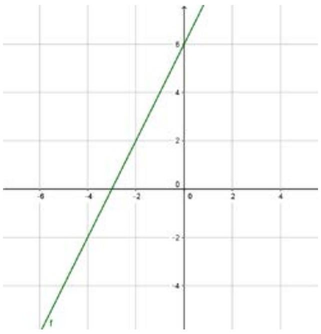


$y = \dfrac{1}{f(x)}$

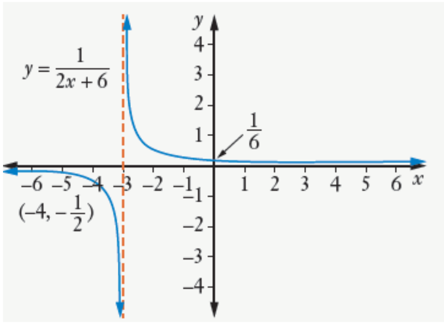


Domain $(-\infty, 2) \cup (2, \infty)$; Range $(-\infty, 0) \cup (0, \infty)$

b $f(x) = 2x + 6$



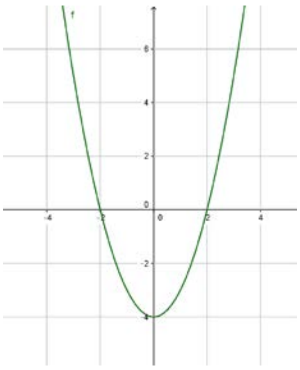
$y = \dfrac{1}{f(x)}$



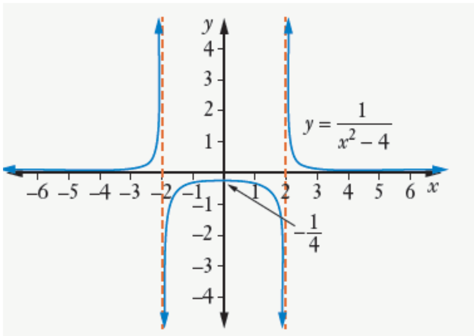
Domain $(-\infty, -3) \cup (-3, \infty)$

Range $(-\infty, 0) \cup (0, \infty)$

c $f(x) = x^2 - 4$



$y = \dfrac{1}{f(x)}$

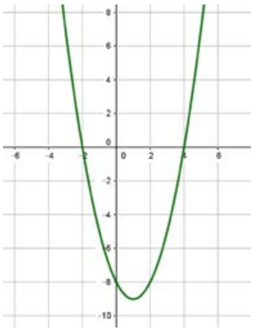


Domain $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

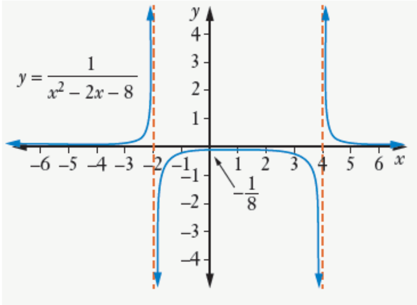
Range $\left(-\infty, -\dfrac{1}{4}\right) \cup (0, \infty)$

d $f(x) = x^2 - 2x - 8$

$= (x - 4)(x + 2)$



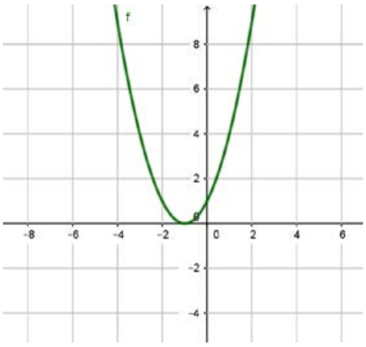
$y = \dfrac{1}{f(x)}$



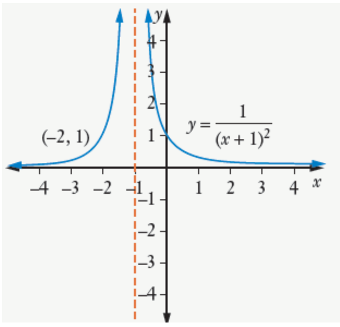
Domain $(-\infty,-2)\cup(-2,4)\cup(4,\infty)$

Range $(-\infty,-\frac{1}{9})\cup(0,\infty)$

e $f(x) = (x + 1)^2$

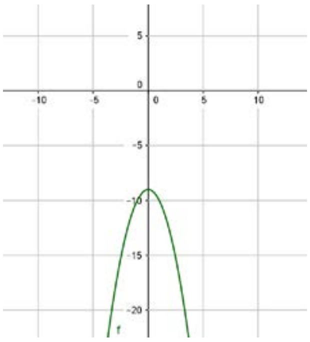


$y = \dfrac{1}{f(x)}$

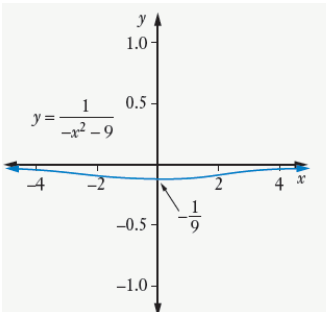


Domain $(-\infty,-1)\cup(-1,\infty)$; Range $(0,\infty)$

f $f(x) = -x^2 - 9$

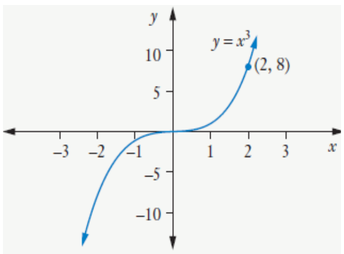


$y = \dfrac{1}{f(x)}$

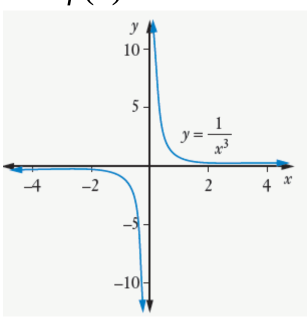


Domain $(-\infty\ \infty)$; Range $[-\frac{1}{9},0)$

g $f(x) = x^3$

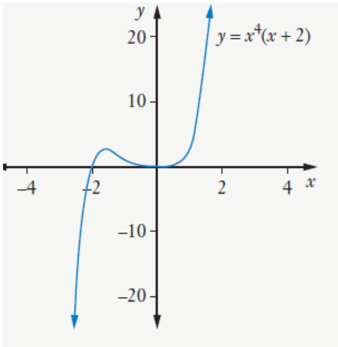


$y = \dfrac{1}{f(x)}$

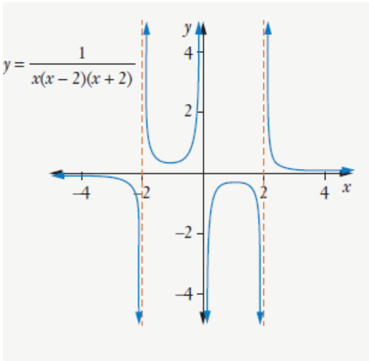


Domain $(-\infty,0)\cup(0,\infty)$; Range $(-\infty,0)\cup(0,\infty)$

h $f(x) = x(x - 2)(x + 2)$



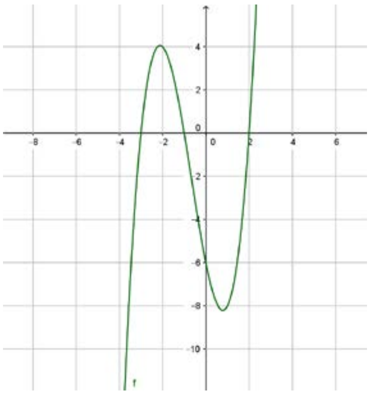
$y = \dfrac{1}{f(x)}$



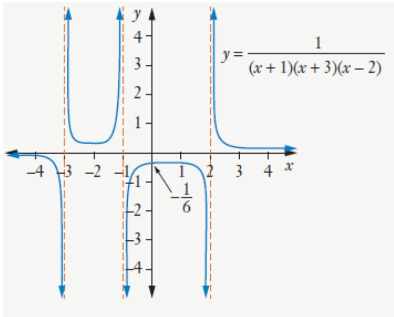
Domain $(-\infty,-2)\cup(-2,0)\cup(0,2)\cup(2,\infty)$

Range $(-\infty,0)\cup(0,\infty)$

i $f(x) = (x + 1)(x + 3)(x - 2)$



$y = \dfrac{1}{f(x)}$

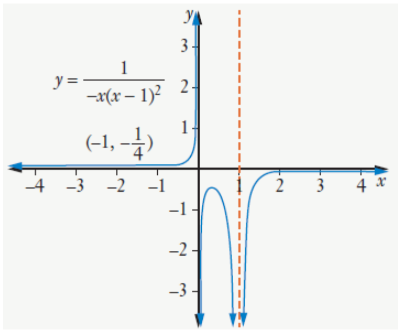


Domain $(-\infty,-3)\cup(-3,-1)\cup(-1,2)\cup(2,\infty)$

Range $(-\infty,0)\cup(0,\infty)$

j $f(x) = -x(x - 1)^2$

$y = \dfrac{1}{f(x)}$

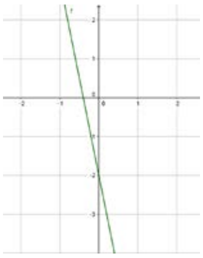


Domain $(-\infty,0)\cup(0,1)\cup(1,\infty)$

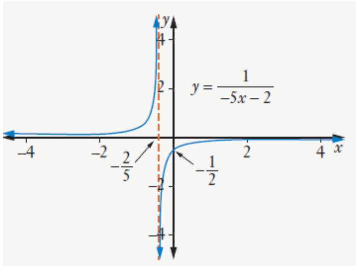
Range $(-\infty,0)\cup(0,\infty)$

Question 2

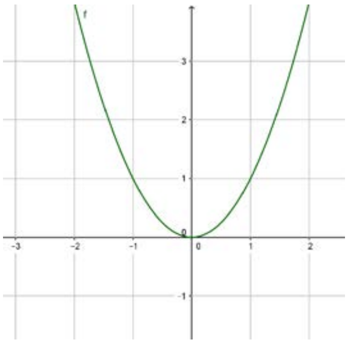
a $f(x) = -5x - 2$



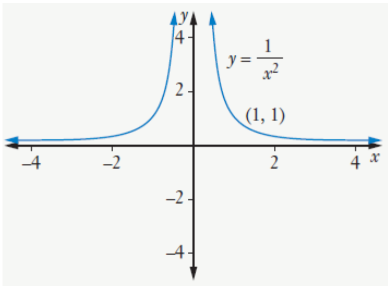
$y = \dfrac{1}{f(x)}$



b $f(x) = x^2$

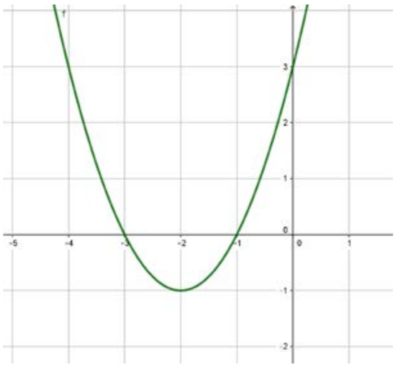


$y = \dfrac{1}{f(x)}$

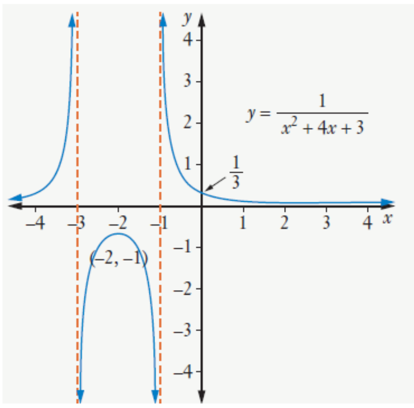


c $f(x) = x^2 + 4x + 3$

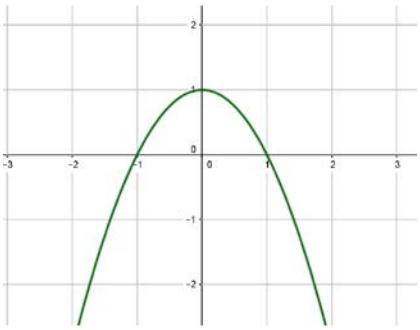
$= (x + 3)(x + 1)$



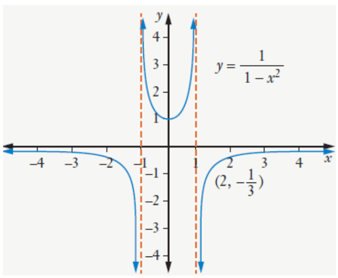
$y = \dfrac{1}{f(x)}$



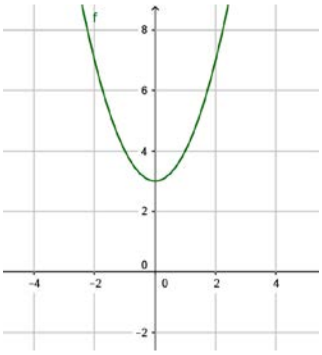
d $f(x) = 1 - x^2$



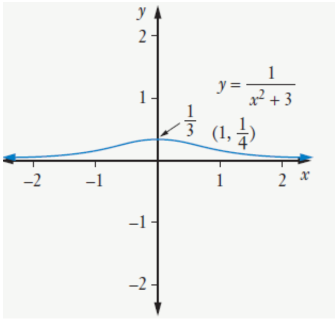
$y = \dfrac{1}{f(x)}$



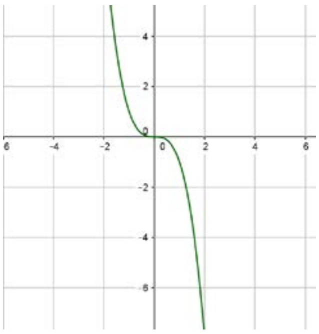
e $f(x) = x^2 + 3$



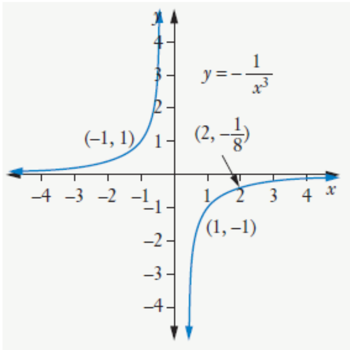
$$y = \frac{1}{f(x)}$$



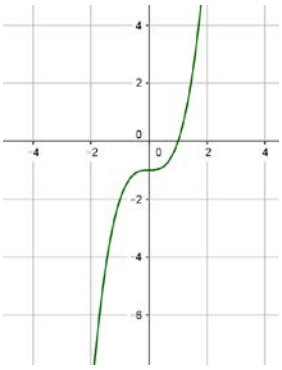
f $f(x) = -x^3$



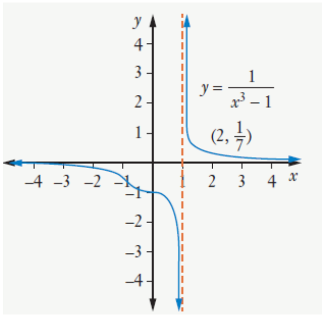
$$y = \frac{1}{f(x)}$$



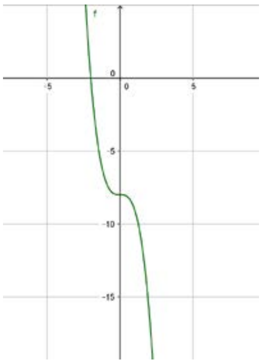
g $f(x) = x^3 - 1$



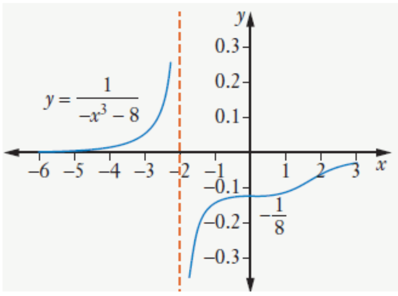
$$y = \frac{1}{f(x)}$$



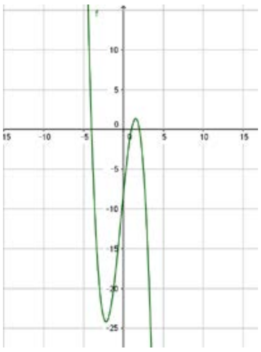
h $f(x) = -x^3 - 8$



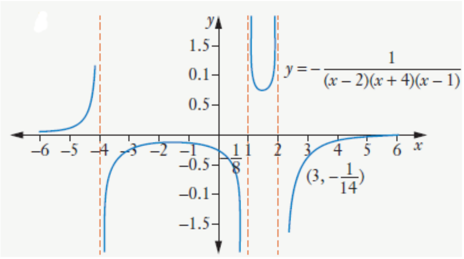
$$y = \frac{1}{f(x)}$$



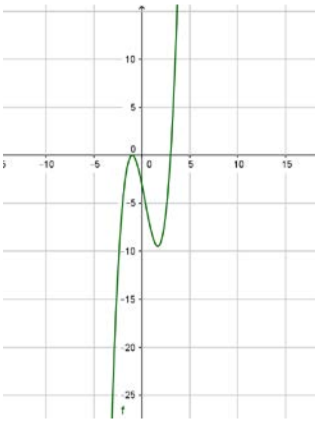
i $f(x) = -(x-2)(x+4)(x-1)$



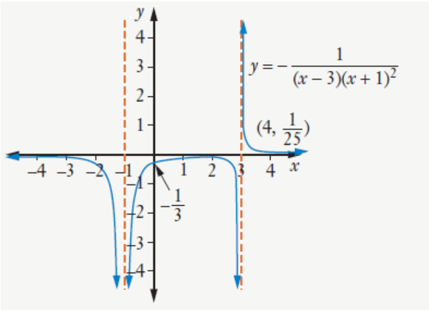
$$y = \frac{1}{f(x)}$$



j $f(x) = (x-3)(x+1)^2$

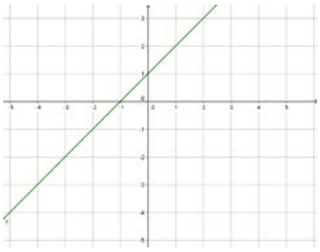


$$y = \frac{1}{f(x)}$$

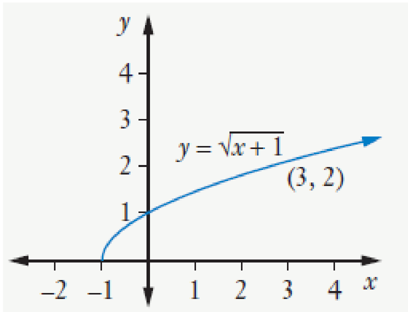


Question 1

a $f(x) = x + 1$

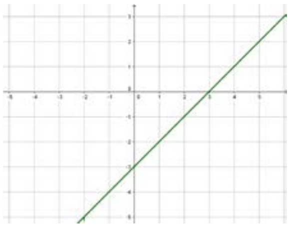


$y = \sqrt{f(x)}$

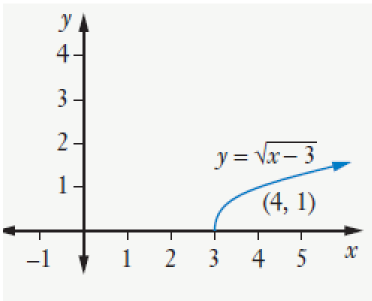


Domain [–1, ∞); Range [0, ∞)

b $f(x) = x - 3$

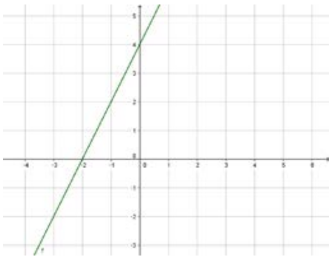


$y = \sqrt{f(x)}$

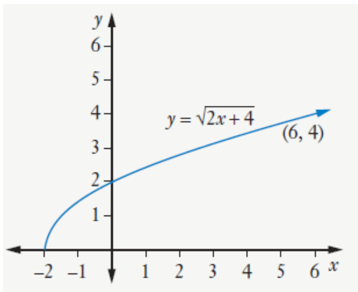


Domain [3, ∞); Range [0, ∞)

c $f(x) = 2x + 4$

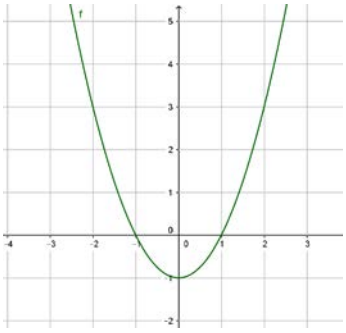


$y = \sqrt{f(x)}$

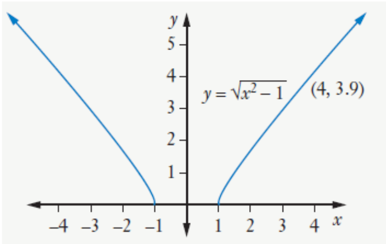


Domain [–2, ∞); Range [0, ∞)

d $f(x) = x^2 - 1$



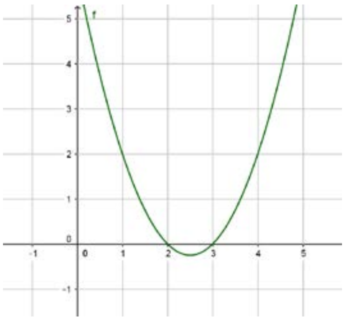
$y = \sqrt{f(x)}$



Domain $(-\infty, -1] \cup [1, \infty)$; Range $[0, \infty)$

e $f(x) = x^2 - 5x + 6$

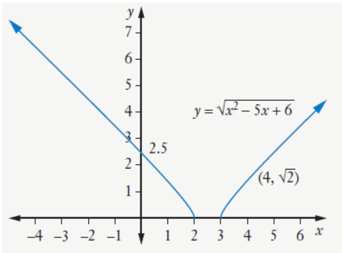
$= (x - 3)(x - 2)$



$y = \sqrt{f(x)}$

y-intercept

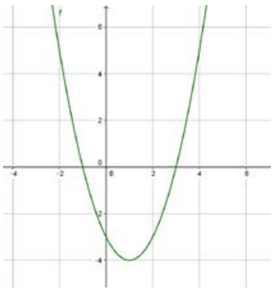
$y = \sqrt{6} \approx 2\ 45$



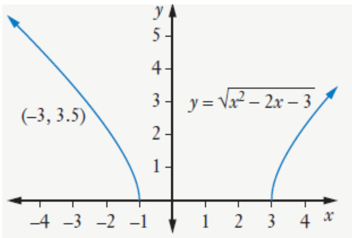
Domain $(-\infty, 2] \cup [3, \infty)$; Range $[0, \infty)$

f $f(x) = x^2 - 2x - 3$

$= (x - 3)(x + 1)$

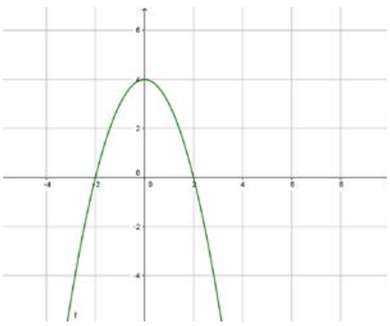


$y = \sqrt{f(x)}$

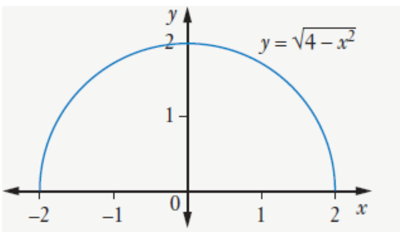


Domain $(-\infty, -1] \cup [3, \infty)$; Range $[0, \infty)$

g $f(x) = 4 - x^2$

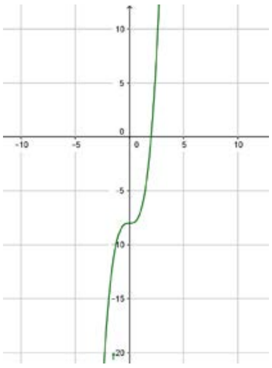


$y = \sqrt{f(x)}$

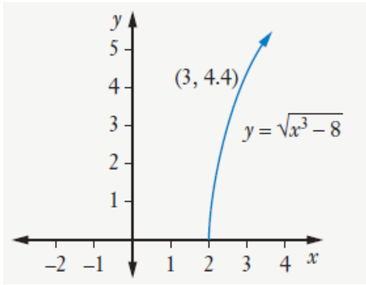


Domain $[-2, 2]$; Range $[0, 2]$

h $f(x) = x^3 - 8$

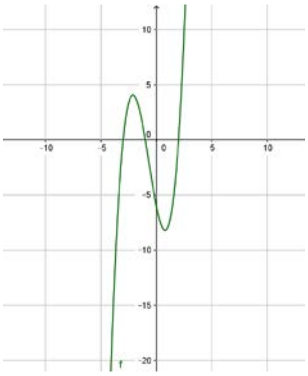


$y = \sqrt{f(x)}$

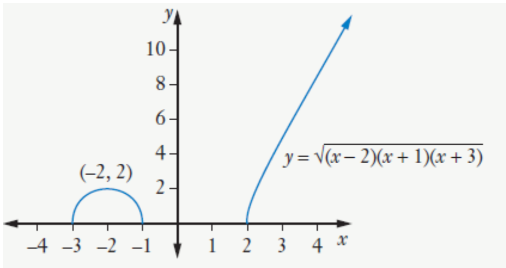


Domain $[2, \infty)$; Range $[0, \infty)$

i $f(x) = (x - 2)(x + 1)(x + 3)$

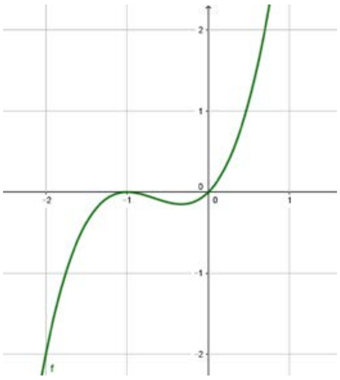


$y = \sqrt{f(x)}$

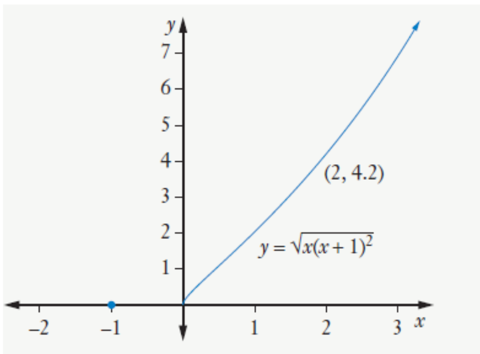


Domain $[-3, -1] \cup [2, \infty)$; Range $[0, \infty)$

j $f(x) = x(x + 1)^2$



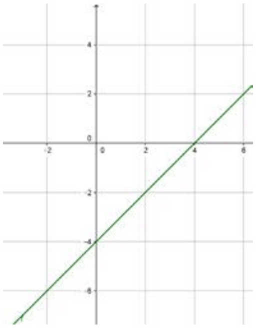
$y = \sqrt{f(x)}$

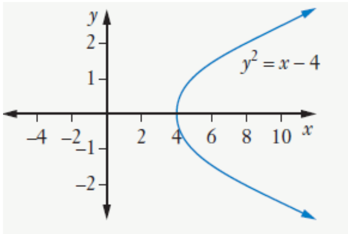


Domain $[-1] \cup [0, \infty)$; Range $[0, \infty)$

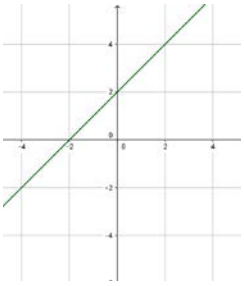
Question 2

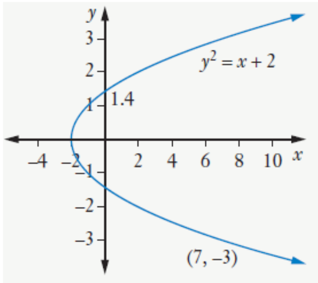
a $f(x) = x - 4$ $\qquad$ $y^2 = f(x)$



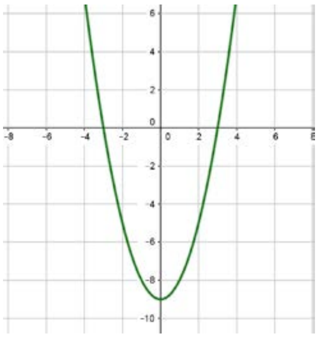


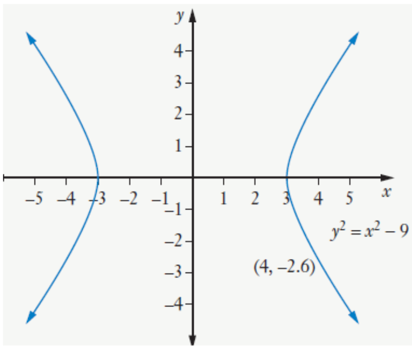
b $f(x) = x + 2$ $\qquad$ $y^2 = f(x)$



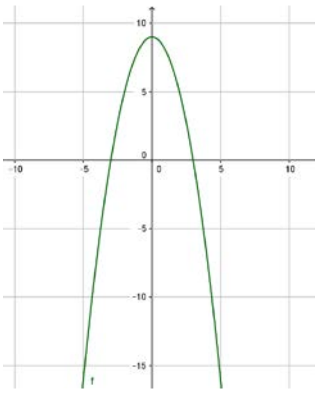


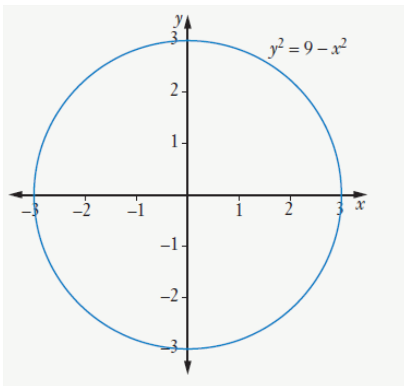
c $f(x) = x^2 - 9$ $\qquad$ $y^2 = f(x)$





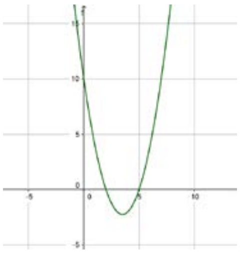
d $f(x) = 9 - x^2$ $\qquad$ $y^2 = f(x)$



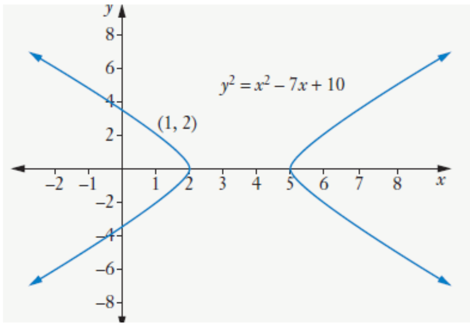


e $f(x) = x^2 - 7x + 10$

$\quad = (x - 5)(x - 2)$

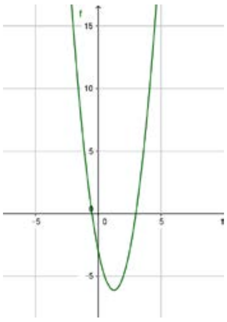


$y^2 = f(x)$

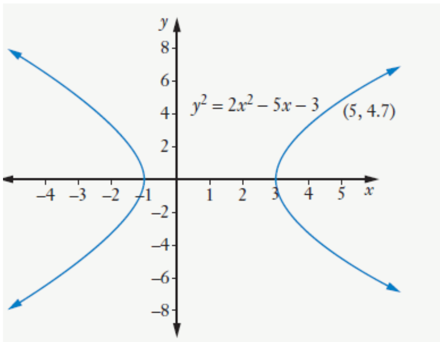


f $f(x) = 2x^2 - 5x - 3$

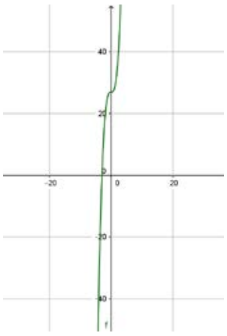
$\quad = (2x + 1)(x - 3)$



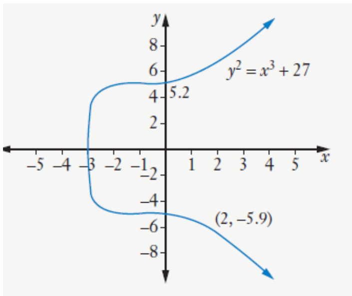
$y^2 = f(x)$



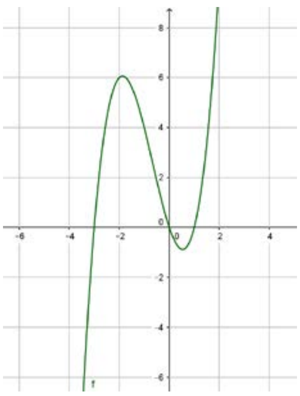
g $f(x) = x^3 + 27$



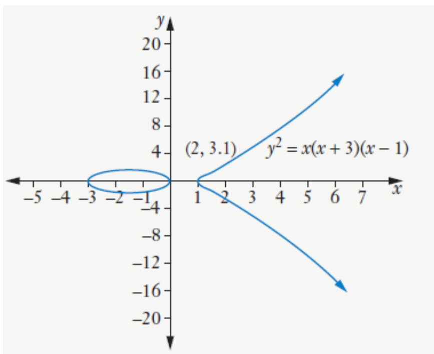
$y^2 = f(x)$



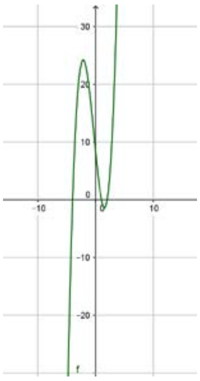
h $f(x) = x(x + 3)(x - 1)$

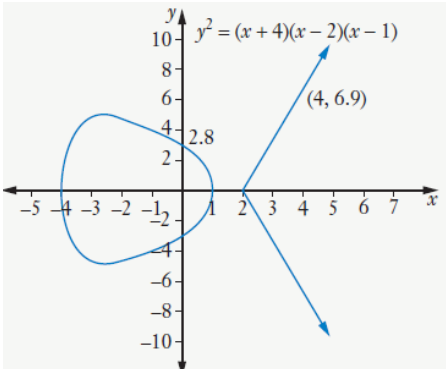


$y^2 = f(x)$

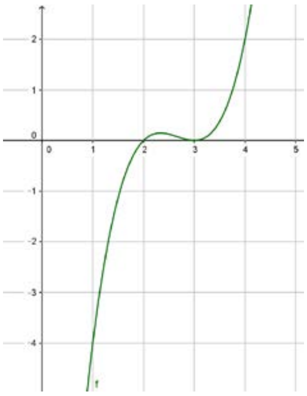


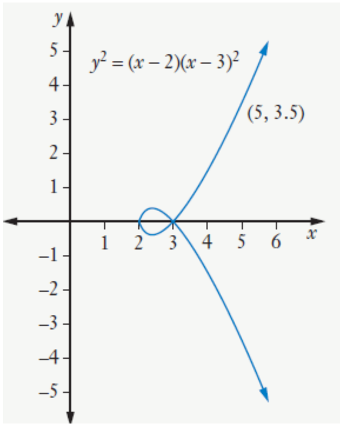
i $f(x) = (x + 4)(x - 2)(x - 1)$ $y^2 = f(x)$





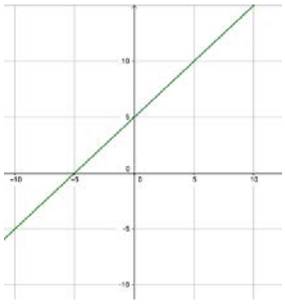
j $f(x) = (x - 2)(x - 3)^2$ $y^2 = f(x)$



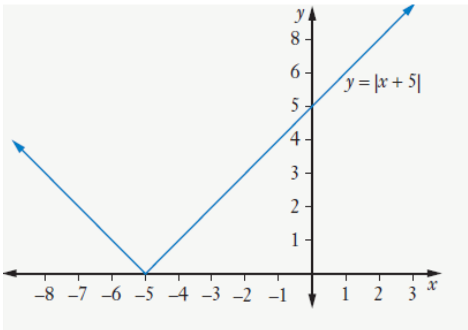


Question 1

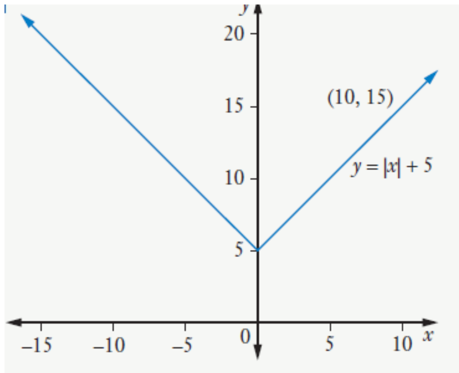
a $f(x) = x + 5$



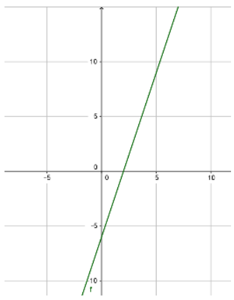
i $y = |f(x)|$



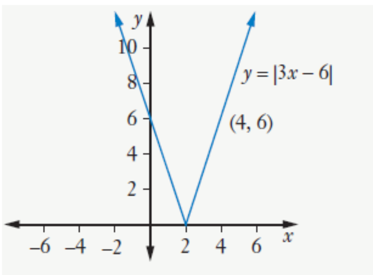
ii $y = f(|x|)$



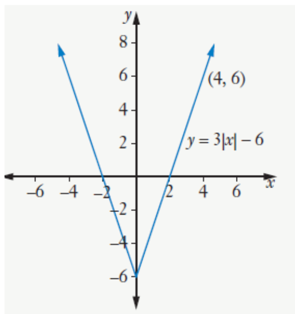
b $f(x) = 3x - 6$



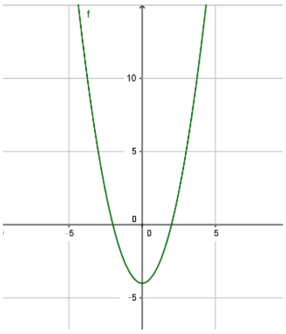
i $y = |f(x)|$



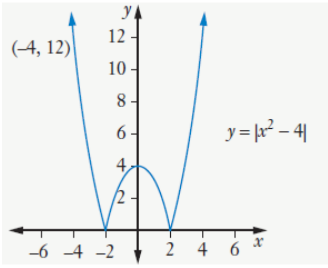
ii $y = f(|x|)$



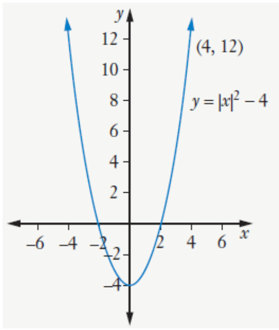
c $f(x) = x^2 - 4$



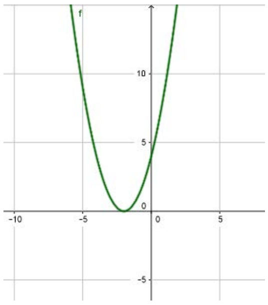
i $y = |f(x)|$



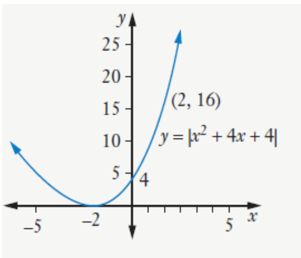
ii $y = f(|x|)$



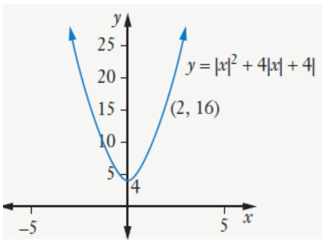
d $f(x) = x^2 + 4x + 4$



i $y = |f(x)|$

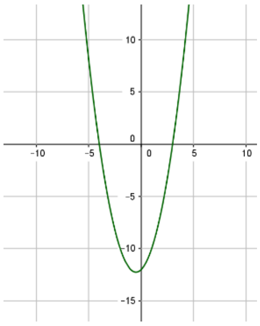


ii $y = f(|x|)$

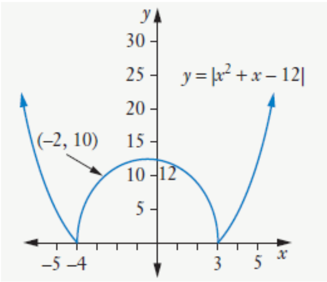


e $f(x) = x^2 + x - 12$

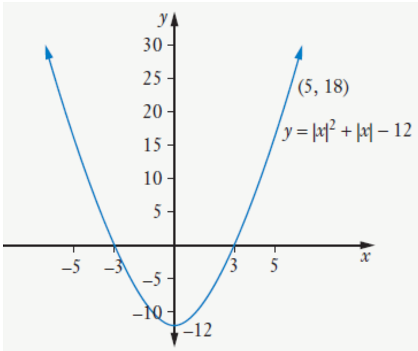
$\quad\quad = (x + 4)(x - 3)$



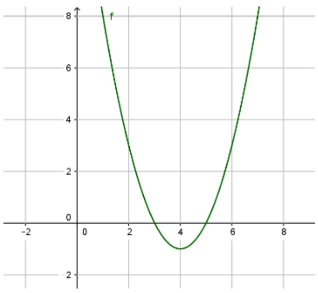
i $y = |f(x)|$



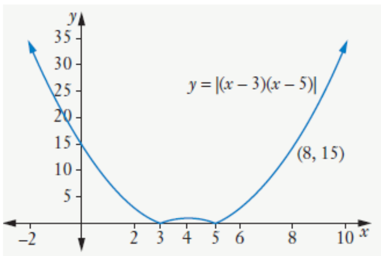
ii $y = f(|x|)$



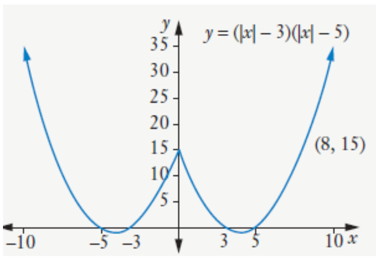
f $f(x) = (x - 3)(x - 5)$



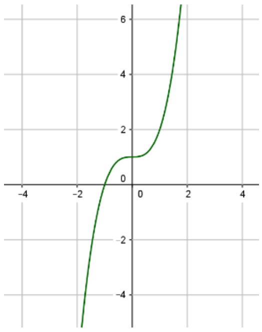
i $y = |f(x)|$



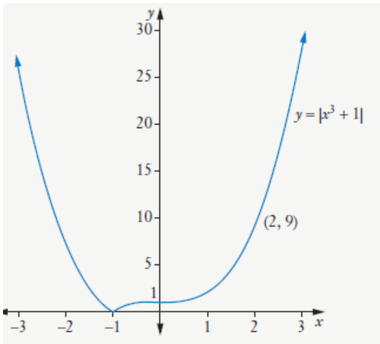
ii $y = f(|x|)$



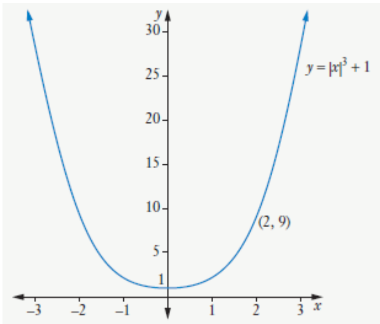
g $f(x) = x^3 + 1$



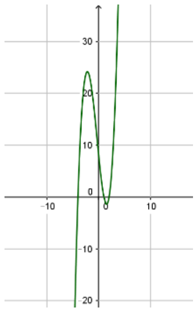
i $y = |f(x)|$



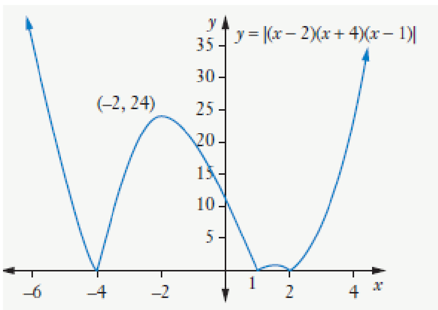
ii $y = f(|x|)$



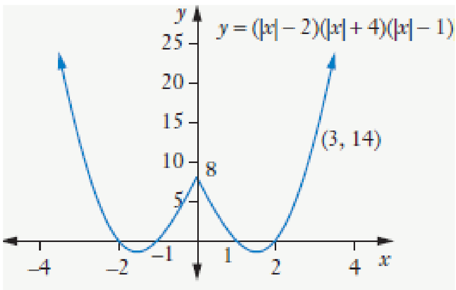
h $f(x) = (x - 2)(x + 4)(x - 1)$



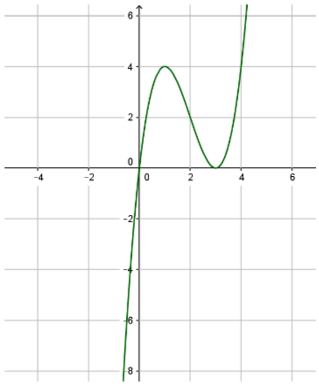
i $y = |f(x)|$



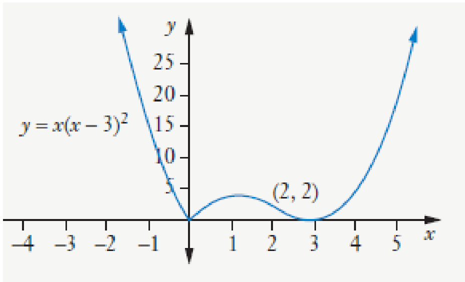
ii $y = f(|x|)$



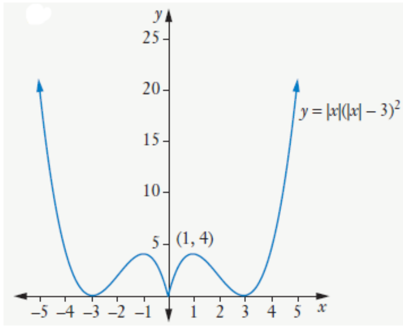
i $f(x) = x(x-3)^2$



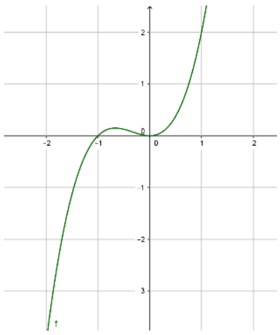
i $y = |f(x)|$



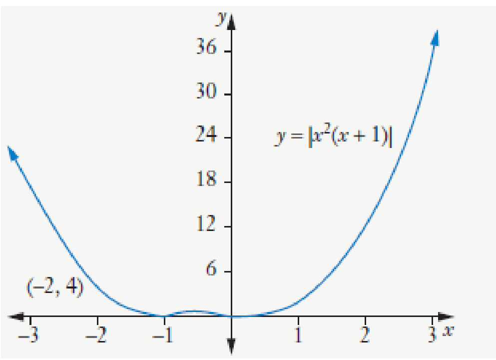
ii $y = f(|x|)$



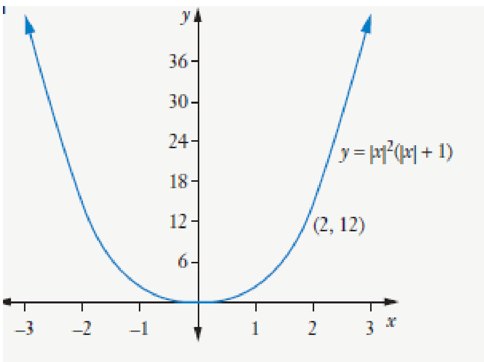
j $f(x) = x^2(x+1)$



i $y = |f(x)|$



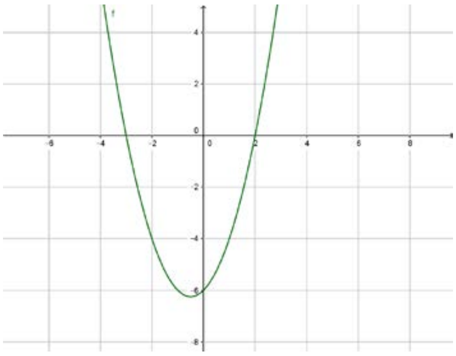
ii $y = f(|x|)$



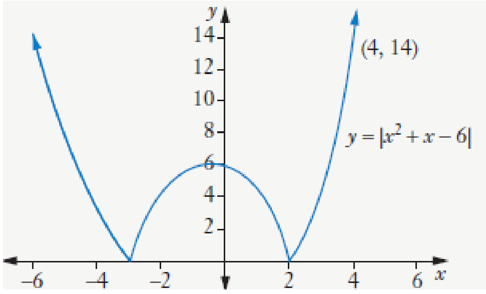
Question 2

$f(x) = x^2 + x - 6$

$\quad = (x + 3)(x - 2)$



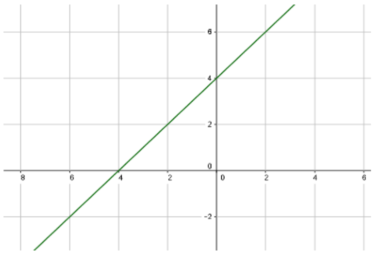
$y = |f(x)|$



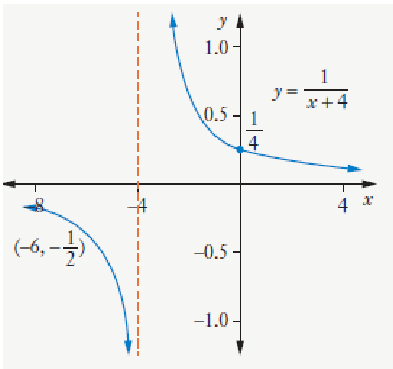
Domain $(-\infty, \infty)$; Range $[0, \infty)$

Question 3

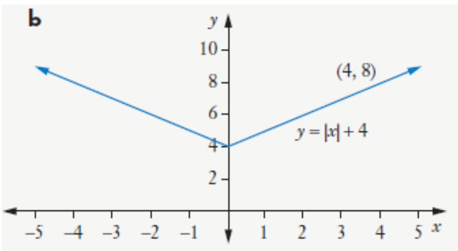
$f(x) = x + 4$



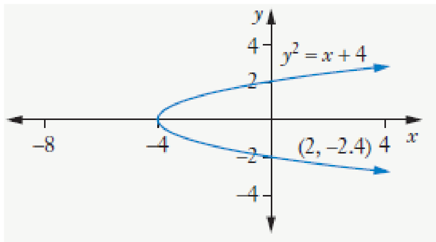
a $\quad y = \dfrac{1}{f(x)}$



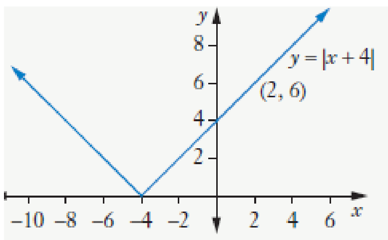
b $\quad y = f(|x|)$



c $\quad y^2 = f(x)$

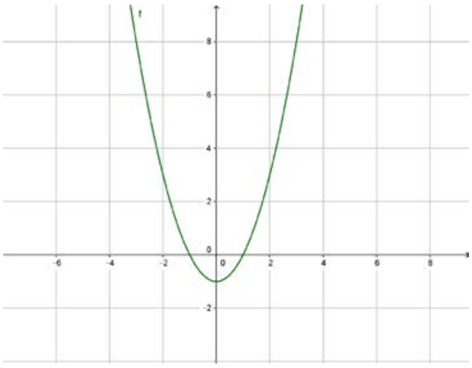


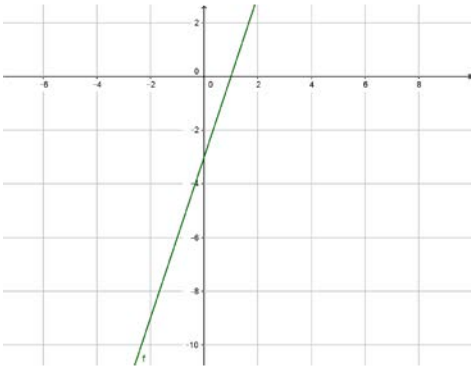
d $\quad y = |f(x)|$



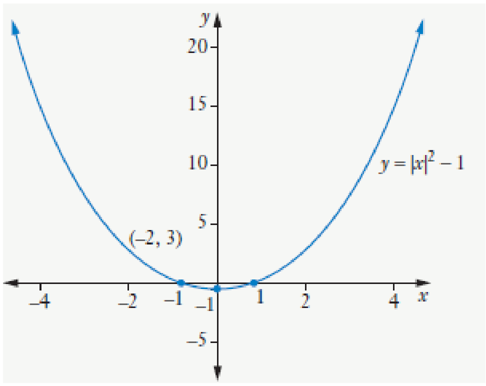
Question 4

$f(x) = x^2 - 1$ $\qquad$ $g(x) = 3x - 3$

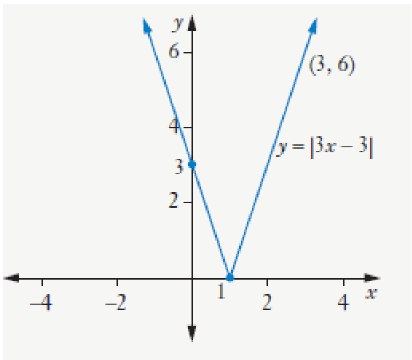




a $\quad y = f(|x|)$

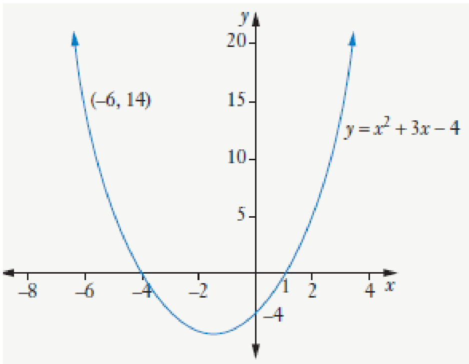


b $\quad y = |g(x)|$

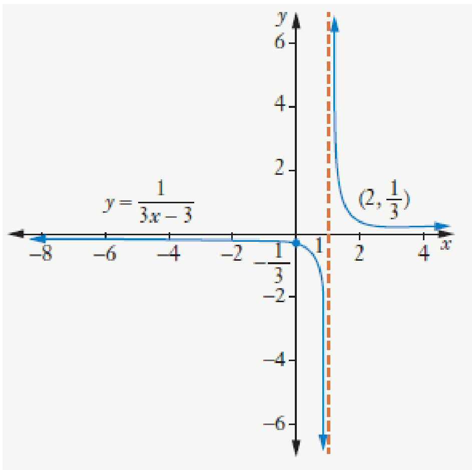


c $\quad f(x) + g(x) = x^2 + 3x - 4$

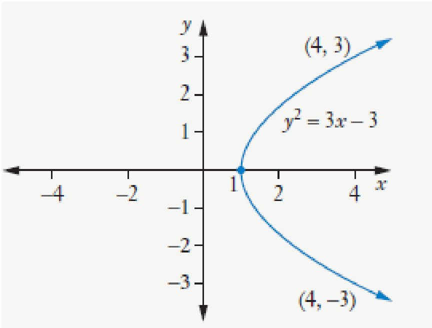
$\qquad\qquad = (x + 4)(x - 1)$



d $\quad y = \dfrac{1}{g(x)}$

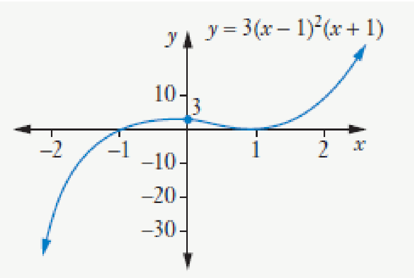


e $y^2 = g(x)$

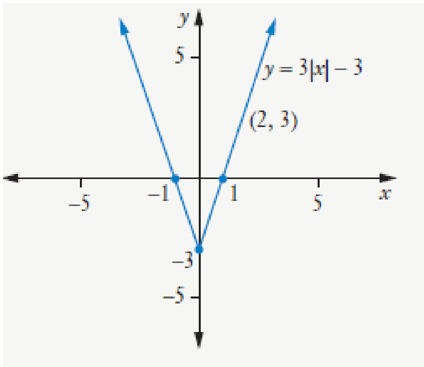


f $y = g(x)\,f(x)$

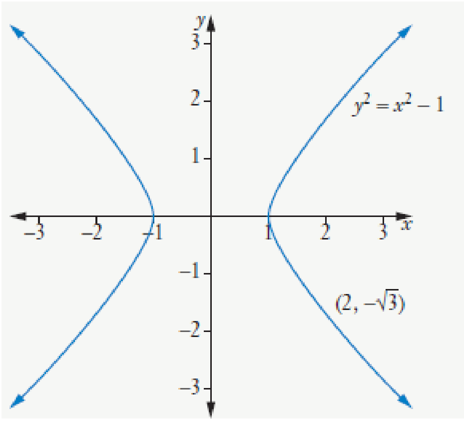
$= 3(x-1)^2(x+1)$



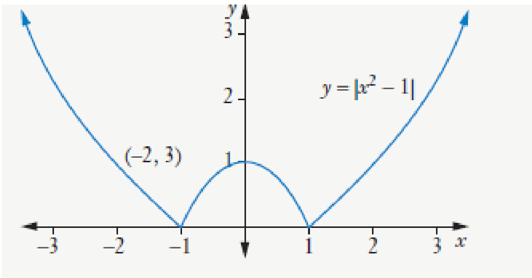
g $y = g(|x|)$



h $y^2 = f(x)$



i $y = |f(x)|$



Exercise 7.10 Parametric equations of a function

Question 1

a $x = 2t \Rightarrow t = \frac{1}{2}x$ [1]

$y = 3t - 4$ [2]

Substitute for *t*, [1] in [2].

$$y = 3\left(\frac{1}{2}x\right) - 4$$

$$y = \frac{3x}{2} - 4$$

$$2y = 3x - 8$$

$$3x - 2y - 8 = 0$$

b $x = 5q + 1$ [1]

$y = 2q \Rightarrow q = \frac{1}{2}y$ [2]

Substitute for *q*, [2] in [1].

$$x = 5\left(\frac{1}{2}y\right) + 1$$

$$x = \frac{5y}{2} + 1$$

$$2x = 5y + 2$$

$$2x - 5y - 2 = 0$$

c $x = 3n - 2 \Rightarrow n = \frac{1}{3}(x + 2)$ [1]

$y = 2n - 5$ [2]

Substitute for *n*, [1] in [2].

$$y = 2\left(\frac{1}{3}(x + 2)\right) - 5$$

$$(y + 5) = \frac{2}{3}(x + 2)$$

$$3(y + 5) = 2(x + 2)$$

$$3y + 15 = 2x + 4$$

$$2x - 3y - 11 = 0$$

d $x = 7p + 3$ [1]

$y = 2p + 1 \Rightarrow p = \frac{1}{2}(y - 1)$ [2]

Substitute for *p*, [2] in [1].

$$x = 7\left(\frac{1}{2}(y-1)\right) + 3$$
$$(x-3) = \frac{7}{2}(y-1)$$
$$2(x-3) = 7(y-1)$$
$$2x - 6 = 7y - 7$$
$$2x - 7y + 1 = 0$$

e $x = -6t \Rightarrow t = -\frac{1}{6}x$ [1]

$y = 3t + 2$ [2]

Substitute for *t*, [1] in [2].

$$y = 3\left(-\frac{1}{6}x\right) + 2$$
$$(y-2) = -\frac{1}{2}x$$
$$2(y-2) = -x$$
$$2y - 4 = -x$$
$$x + 2y - 4 = 0$$

Question 2

a $x = t$ [1]

$y = t^2 - 3t + 6$ [2]

Substitute for *t*, [1] in [2].

$y = x^2 - 3x + 6$

b $x = r - 3 \Rightarrow r = x + 3$ [1]

$y = r^2 + 1$ [2]

Substitute for *r*, [1] in [2].

$$\begin{aligned} y &= (x+3)^2 + 1 \\ &= x^2 + 6x + 9 + 1 \\ &= x^2 + 6x + 10 \end{aligned}$$

c $x = 2p + 3 \Rightarrow p = \frac{1}{2}(x-3)$ [1]

$y = p^2$ [2]

Substitute for *p*, [1] in [2].

$$\begin{aligned} y &= \left(\frac{1}{2}(x-3)\right)^2 \\ &= \frac{1}{4}(x-3)^2 \end{aligned}$$

$4y = x^2 - 6x + 9$

$x^2 - 6x - 4y + 9 = 0$

d $x = 3s + 1 \Rightarrow s = \frac{1}{3}(x-1)$ [1]

$y = s^2 + 2s$ [2]

Substitute for *s*, [1] in [2].

$$\begin{aligned} y &= \left(\frac{1}{3}(x-1)\right)^2 + \frac{2}{3}(x-1) \\ &= \frac{1}{9}(x-1)^2 + \frac{2}{3}(x-1) \end{aligned}$$

$9y = (x-1)^2 + 6(x-1)$

$9y = x^2 - 2x + 1 + 6x - 6$

$x^2 + 4x - 9y - 5 = 0$

e $x = 4k - 7 \Rightarrow k = \frac{1}{4}(x+7)$ [1]

$y = 2k^2 + k$ [2]

Substitute for *k*, [1] in [2].

$y = 2\left(\frac{1}{4}(x+7)\right)^2 + \frac{1}{4}(x+7)$

$y = \frac{1}{8}\left(x^2 + 14x + 49\right) + \frac{1}{4}x + \frac{7}{4}$

$8y = x^2 + 14x + 49 + 2x + 14$

$x^2 + 16x - 8y + 63 = 0$

Question 3

a $x = 2t \Rightarrow t = \frac{1}{2}x$ [1]

$y = 3t - 1$ [2]

Substitute for t, [1] in [2].

$$y = 3\left(\frac{1}{2}x\right) - 1$$

$$y = \frac{3}{2}x - 1$$

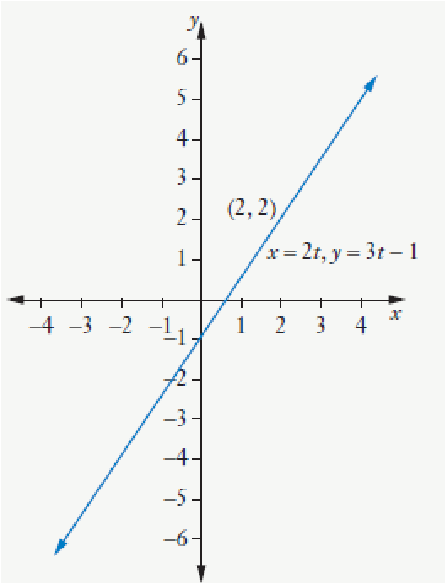
$$3x - 2y - 2 = 0$$

x-intercept

$y = 0,\quad 3x - 2 = 0 \Rightarrow x = \frac{2}{3}$

y-intercept

$x = 0,\quad -2y - 2 = 0 \Rightarrow y = -1$



b $x = q + 4 \Rightarrow q = x - 4$ [1]

$y = 3q + 6$ [2]

Substitute for q, [1] in [2].

$$y = 3(x - 4) + 6$$

$$y = 3x - 6$$

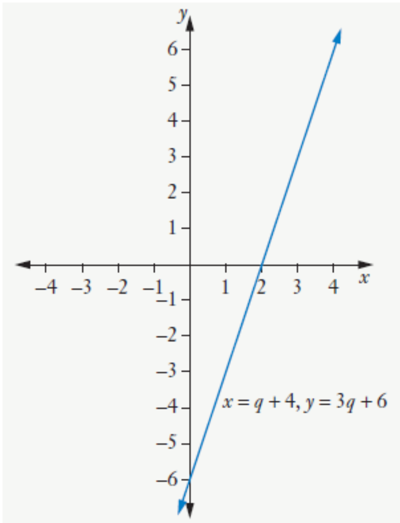
$$3x - y - 6 = 0$$

x-intercept

$y = 0,\quad 3x - 6 = 0 \Rightarrow x = 2$

y-intercept

$x = 0,\quad -y - 6 = 0 \Rightarrow y = -6$



c $x = p + 1 \Rightarrow p = x - 1$ [1]

$y = p^2 - 3p$ [2]

Substitute for *p*, [1] in [2].

$y = (x-1)^2 - 3(x-1)$

$y = x^2 - 2x + 1 - 3x + 3$

$y = x^2 - 5x + 4$

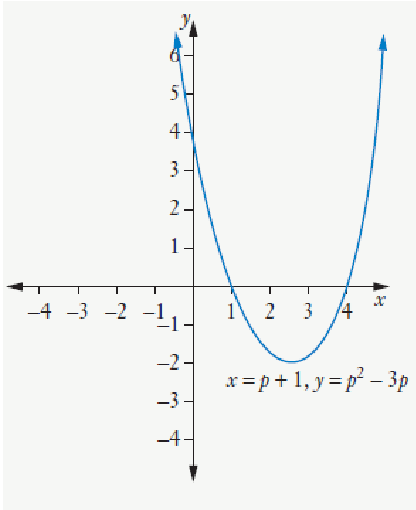
$y = (x-1)(x-4)$

x-intercept

$y = 0, \quad (x-1)(x-4) = 0 \Rightarrow x = 1, x = 4$

y-intercept

$x = 0, \quad y = 4$



d $x = p - 2 \Rightarrow p = x + 2$ [1]

$y = p^2 - 1$ [2]

Substitute for *p*, [1] in [2].

$y = (x+2)^2 - 1$

$y = x^2 + 4x + 4 - 1$

$y = x^2 + 4x + 3$

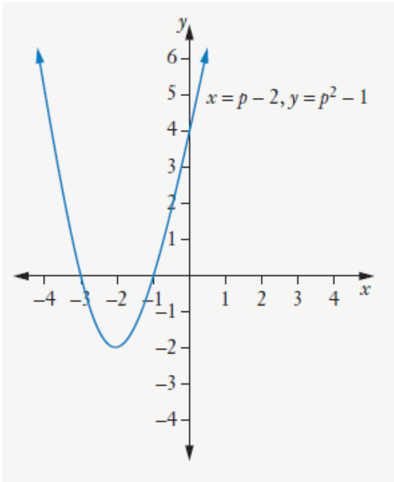
$y = (x+1)(x+3)$

x-intercept

$y = 0, \quad (x+1)(x+3) = 0 \Rightarrow x = -1, x = -3$

y-intercept

$x = 0, \quad y = 3$



e $x = p + 4 \Rightarrow p = x - 4$ [1]

$y = p^2 - 1$ [2]

Substitute for *p*, [1] in [2].

$y = (x-4)^2 - 1$

$y = x^2 - 8x + 16 - 1$

$y = x^2 - 8x + 15$

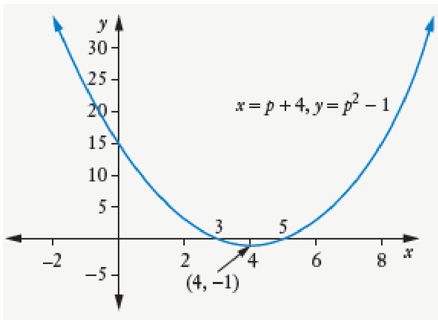
$y = (x-3)(x-5)$

x-intercept

$y = 0, \quad (x-3)(x-5) = 0 \Rightarrow x = 3, x = 5$

y-intercept

$x = 0, \quad y = 15$



Question 4

a $x = 6\cos\alpha \Rightarrow x^2 = 36\cos^2\alpha$ [1]

$y = 6\sin\alpha \Rightarrow y^2 = 36\sin^2\alpha$ [2]

[1] + [2]

$x^2 + y^2 = 36\cos^2\alpha + 36\sin^2\alpha = 36(\cos^2\alpha + \sin^2\alpha) = 36$

b $x = -2\cos\theta \Rightarrow x^2 = 4\cos^2\theta$ [1]

$y = -2\sin\theta \Rightarrow y^2 = 4\sin^2\theta$ [2]

[1] + [2]

$x^2 + y^2 = 4\cos^2\theta + 4\sin^2\theta = 4(\cos^2\theta + \sin^2\theta) = 4$

c $x = \sin\beta \Rightarrow x^2 = \sin^2\beta$ [1]

$y = \cos\beta + 1 \Rightarrow (y-1)^2 = \cos^2\beta$ [2]

[1] + [2]

$x^2 + (y-1)^2 = \sin^2\beta + \cos^2\beta = 1$

$x^2 + y^2 - 2y + 1 = 1$

$x^2 + y^2 - 2y = 0$

d $x = 5 + 4\cos\theta \Rightarrow (x-5)^2 = 16\cos^2\theta$ [1]

$y = 2 + 4\sin\theta \Rightarrow (y-2)^2 = 16\sin^2\theta$ [2]

[1] + [2]

$(x-5)^2 + (y-2)^2 = 16\cos^2\theta + 16\sin^2\theta = 16(\cos^2\theta + \sin^2\theta) = 16$

$x^2 - 10x + 25 + y^2 - 4y + 4 = 16$

$x^2 - 10x + y^2 - 4y + 13 = 0$

e $x = 3 + \cos\theta \Rightarrow (x-3)^2 = \cos^2\theta$ [1]

$y = \sin\theta - 2 \Rightarrow (y+2)^2 = \sin^2\theta$ [2]

[1] + [2]

$(x-3)^2 + (y+2)^2 = \cos^2\theta + \sin^2\theta = 1$

$x^2 - 6x + 9 + y^2 + 4y + 4 = 1$

$x^2 - 6x + y^2 + 4y + 12 = 0$

Question 5

$x = p$ [1]

$x - y + 1 = 0 \Rightarrow y = x + 1$ [2]

Substitute for *x*, [1] in [2].

$y = p + 1$

Question 6

a $3x - 4y + 24 = 0$ [1]

$x = 4t$ [2]

Substitute for *x*, [2] in [1].

$3 \times 4t - 4y + 24 = 0$

$4y = 12t + 24$

$y = 3t + 6$

b $x + 7y - 21 = 0$ [1]

$x = t + 5$ [2]

Substitute for *x*, [2] in [1].

$t + 5 + 7y - 21 = 0$

$7y = -t + 16$

$y = \frac{1}{7}(-t + 16)$

c $y = x^2$ [1]

$x = 2t - 1$ [2]

Substitute for *x*, [2] in [1].

$y = (2t - 1)^2$

$y = 4t^2 - 4t + 1$

d $y = 2x^2 - 3x + 4$ [1]

$x = 1 - t$ [2]

Substitute for *x*, [2] in [1].

$$\begin{aligned} y &= 2(1-t)^2 - 3(1-t) + 4 \\ &= 2(1 - 2t + t^2) - 3 + 3t + 4 \\ &= 2 - 4t + 2t^2 - 3 + 3t + 4 \end{aligned}$$

$y = 2t^2 - t + 3$

e $x^2 + y^2 = 9$ [1]

$x = 3\cos t$ [2]

Substitute for *x*, [2] in [1].

$(3\cos t)^2 + y^2 = 9$

$9\cos^2 t + y^2 = 9$

$$\begin{aligned} y^2 &= 9 - 9\cos^2 t \\ &= 9(1 - \cos^2 t) \\ &= 9\sin^2 t \end{aligned}$$

$y = \sqrt{9\sin^2 t} = 3\sin t$

Question 7

$(x-h)^2+(y-k)^2=r^2$ describes a circle of radius r, centre at (h, k).

a $x=1+\cos\alpha \Rightarrow (x-1)^2=\cos^2\alpha$ [1]

$y=\sin\alpha-2 \Rightarrow (y+2)^2=\sin^2\alpha$ [2]

[1] + [2]

$(x-1)^2+(y+2)^2=\cos^2\alpha+\sin^2\alpha=1=1^2$

The circle has radius 1, centre at (1, –2).

b $x=-3+5\cos\beta \Rightarrow (x+3)^2=25\cos^2\beta$ [1]

$y=2+5\sin\beta \Rightarrow (y-2)^2=25\sin^2\beta$ [2]

[1] + [2]

$(x+3)^2+(y-2)^2=25\cos^2\beta+25\sin^2\beta=25(\cos^2\beta+\sin^2\beta)=5^2$

The circle has radius 5, centre at (–3, 2).

c $x=4-2\cos\theta \Rightarrow (x-4)^2=4\cos^2\theta$ [1]

$y=-2+2\sin\theta \Rightarrow (y+2)^2=4\sin^2\theta$ [2]

[1] + [2]

$(x-4)^2+(y+2)^2=4\cos^2\theta+4\sin^2\theta=4(\cos^2\theta+\sin^2\theta)=2^2$

The circle has radius 2, centre at (4, –2).

d $x=6+7\cos\theta \Rightarrow (x-6)^2=49\cos^2\theta$ [1]

$y=-5+7\sin\theta \Rightarrow (y+5)^2=49\sin^2\theta$ [2]

[1] + [2]

$(x-6)^2+(y+5)^2=49\cos^2\theta+49\sin^2\theta=49(\cos^2\theta+\sin^2\theta)=7^2$

The circle has radius 7, centre at (6, –5).

e $x=-2\cos\theta-8 \Rightarrow (x+8)^2=4\cos^2\theta$ [1]

$y=2\sin\theta+9 \Rightarrow (y-9)^2=4\sin^2\theta$ [2]

[1] + [2]

$(x+8)^2+(y-9)^2=4\cos^2\theta+4\sin^2\theta=4(\cos^2\theta+\sin^2\theta)=2^2$

The circle has radius 2, centre at (–8, 9).

Question 8

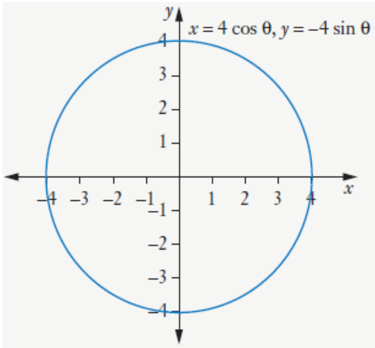
a $x = 4\cos\theta \Rightarrow x^2 = 16\cos^2\theta$ [1]

$y = -4\sin\theta \Rightarrow y^2 = 16\sin^2\theta$ [2]

[1] + [2]

$x^2 + y^2 = 16\cos^2\theta + 16\sin^2\theta = 16(\cos^2\theta + \sin^2\theta) = 4^2$

The graph is a circle, radius 4, centre at (0, 0).



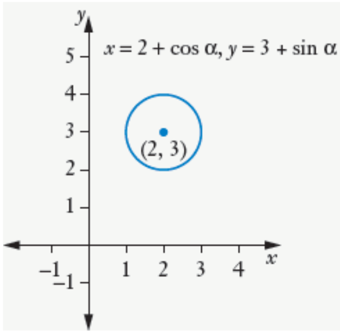
b $x = 2 + \cos\alpha \Rightarrow (x-2)^2 = \cos^2\alpha$ [1]

$y = 3 + \sin\alpha \Rightarrow (y-3)^2 = \sin^2\alpha$ [2]

[1] + [2]

$(x-2)^2 + (y-3)^2 = \cos^2\alpha + \sin^2\alpha = 1^2$

The graph is a circle, radius 1, centre at (2, 3).



Question 9

a $x^2 + y^2 = 4$ is the equation of a circle of radius 2, centre (0, 0).

The parametric form of $x^2 + y^2 = r^2$ is

$x = r\cos\theta$

$y = r\sin\theta$

So $x^2 + y^2 = 2^2$

$x = 2\cos\theta, \quad y = 2\sin\theta$

b $x^2 + 4x + y^2 - 5 = 0$

$x^2 + 4x + 4 + y^2 - 5 - 4 = 0$

$(x+2)^2 + y^2 - 9 = 0$

$(x+2)^2 + y^2 = 9$

Let $x + 2 = 3\cos\theta \Rightarrow x = 3\cos\theta - 2$.

Let $y = 3\sin\theta$.

c $x^2 - 6x + y^2 + 2y - 26 = 0$

$(x-3)^2 - 9 + (y+1)^2 - 1 - 26 = 0$

$(x-3)^2 + (y+1)^2 = 6^2$

Let $x - 3 = 6\cos\theta \Rightarrow x = 3 + 6\cos\theta$

Let $y + 1 = 6\sin\theta \Rightarrow y = -1 + 6\sin\theta$

d $x^2 - 8x + y^2 - 10y - 8 = 0$

$(x-4)^2 - 16 + (y-5)^2 - 25 - 8 = 0$

$(x-4)^2 + (y-5)^2 = 7^2$

Let $x - 4 = 7\cos\theta \Rightarrow x = 4 + 7\cos\theta$

Let $y - 5 = 7\sin\theta \Rightarrow y = 5 + 7\sin\theta$

e $x^2 + 2x + y^2 - 4y + 1 = 0$

$(x+1)^2 - 1 + (y-2)^2 - 4 + 1 = 0$

$(x+1)^2 + (y-2)^2 = 2^2$

Let
$x + 1 = 2\cos\theta \Rightarrow x = -1 + 2\cos\theta$

Let $y - 2 = 2\sin\theta \Rightarrow y = 2 + 2\sin\theta$

Question 10

a $x = 4p \Rightarrow p = \frac{1}{4}x$ [1]

$y = 2p^2 - 3$ [2]

Substitute for *p*, [1] in [2].

$$y = 2\left(\frac{1}{4}x\right)^2 - 3 = 2 \times \frac{x^2}{16} - 3 = \frac{x^2}{8} - 3$$

$8y = x^2 - 24$

$x^2 - 8y - 24 = 0$

b *x*-intercept

$y = 0, \quad x^2 - 8 \times 0 - 24 = 0 \Rightarrow x = \pm\sqrt{24} = \pm 2\sqrt{6}$ or ±4.9

y-intercept

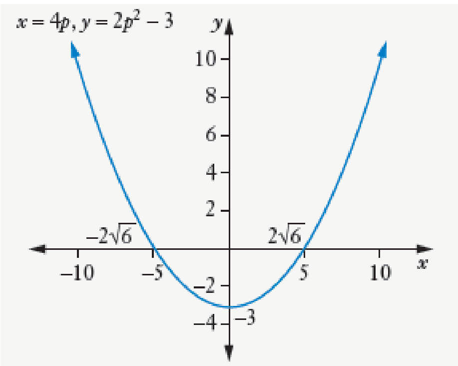
$x = 0, \quad 0^2 - 8y - 24 = 0 \Rightarrow y = -3$

c Domain $(-\infty, \infty)$; range $[-3, \infty)$

All real values of *x* can be used, so the domain is $(-\infty, \infty)$.

$y = \frac{x^2}{8} - 3$ is concave upward and has a turning point at (0, –3), so the minimum value of *y* is –3. Hence, the range is $[-3, \infty)$.

d



Test Yourself 7

Question 1

$x \neq 4$

Domain $(-\infty, 4) \cup (4, \infty)$

B

Question 2

$(x - 1)^2 + (y + 2)^2 = 3^2 = 9$

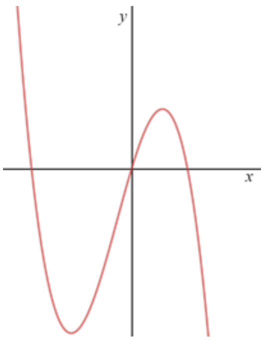
A

Question 3

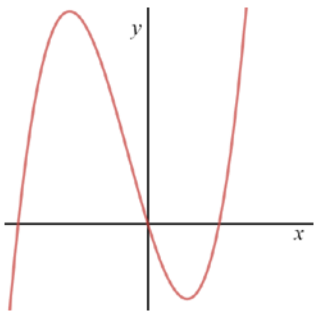
A

$f(-x)$ reflects the graph in the y-axis and $-f(-x)$ reflects the graph of $f(-x)$ in the x-axis.

$y = f(-x)$



$y = -f(-x)$



Question 4

a $A = \frac{k}{n}$

When $n = 5$, $A = 30$.

$30 = \frac{k}{5}$

$k = 150$

$A = \frac{150}{n}$

b **i** When $n = 10$:

$A = \frac{150}{10}$

$= 15 \text{ cm}^2$

ii When $n = 8$:

$A = \frac{150}{8}$

$= 18.75 \text{ cm}^2$

c **i** When $A = 16.67$:

$16.67 = \frac{150}{n}$

$n = \frac{150}{16.67}$

$= 8.9982 \ldots$

≈ 9 people

ii When $A = 25$:

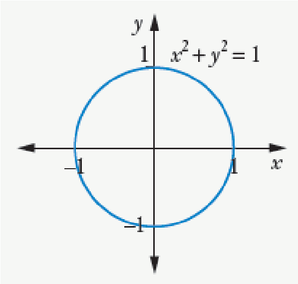
$25 = \frac{150}{n}$

$n = \frac{150}{25}$

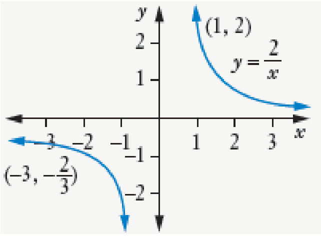
$= 6$ people

Question 5

a Circle, centre (0, 0), radius $\sqrt{1} = 1$



b Hyperbola



c $y = |x + 2|$

x-intercept ($y = 0$)

$0 = |x + 2|$

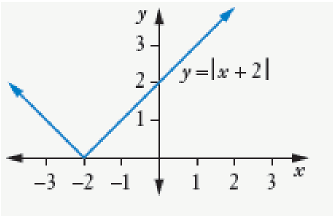
$x + 2 = 0$

$x = –2$

y-intercept ($x = 0$)

$y = |0 + 2|$

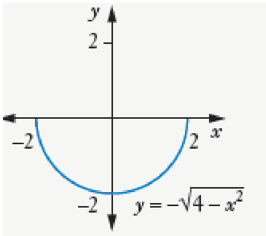
$= 2$



$$y = \begin{cases} x+2 & \text{for } x \geq 2 \\ -x-2 & \text{for } x < 2 \end{cases}$$

d $y = -\sqrt{4 - x^2}$

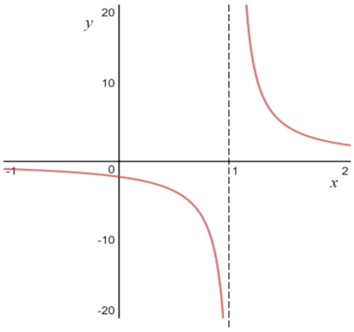
Semicircle, below x-axis, centre (0, 0), radius $\sqrt{4} = 2$



e $f(x)=\dfrac{2}{x-1}$ has a vertical asymptote at $x=1$ and a horizontal asymptote $y=0$.

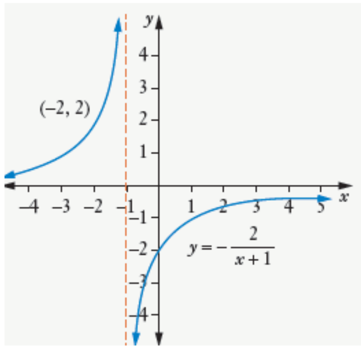
There is no x-intercept or y-intercept because the axes are asymptotes.

As $x\to+\infty,\ f(x)\to 0$ and as $x\to-\infty,\ f(x)\to 0$.



$y=f(-x)$ reflects the graph of $f(x)$ in the y-axis.

$$y=\frac{2}{-x-1}=-\frac{2}{x+1}$$



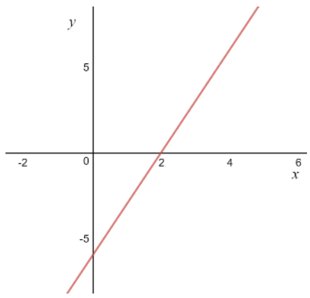
f $y=f(x)=3x-6$ is a straight line.

x-intercept

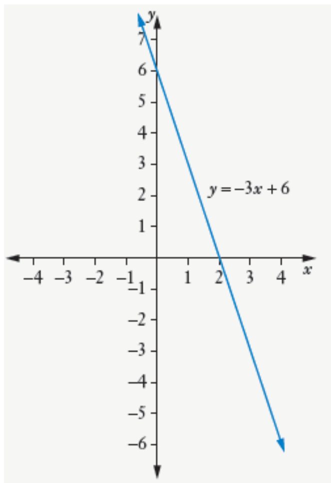
$y=0,\ 0=3x-6\Rightarrow x=2$

y-intercept

$x=0,\ y=3\times 0-6=-6$



$y=-f(x)=-(3x-6)=-3x+6$ is a reflection of the graph $y=f(x)$ in the x-axis.



g $y = f(x) = x^2 + x = x(x+1)$ is a parabola, concave upward.

x-intercepts

$x(x+1) = 0 \Rightarrow x = 0, x = -1$

y-intercept

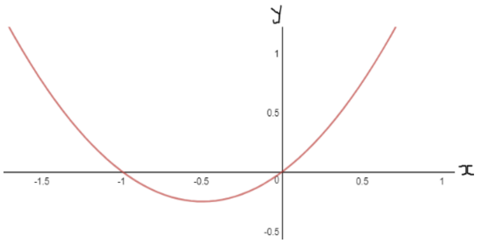
$x = 0, y = 0$

Axis of symmetry is at $x = -0.5$, half way between the x-intercepts, 0 and –1.

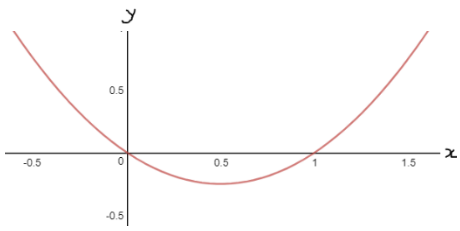
$f(-0.5) = 0.5^2 - 0.5 = -0.25$

The turning point is at (–0.5, –0.25)

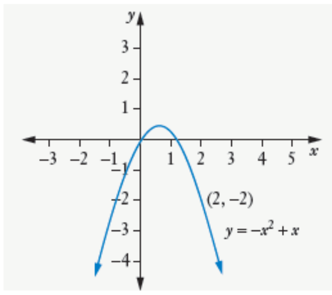
$y = f(x)$



$y = f(-x)$ is a reflection of the graph $y = f(x)$ in the y-axis.



$y = -f(-x)$ is a reflection of the graph $y = -f(x)$ in the x-axis.



Question 6

$x^2 - 6x + 9 + y^2 - 2y + 1 - 6 = 9 + 1$

$(x - 3)^2 + (y - 1)^2 = 16$

Radius $\sqrt{16} = 4$, centre (3, 1).

Question 7

a $f(x) + g(x) = x^3 + 3x - 1$

b $f(x)\,g(x) = x^3(3x - 1)$

$= 3x^4 - x^3$

c $f(g(x)) = (3x - 1)^3$

$= (3x - 1)(3x - 1)^2$

$= (3x - 1)(9x^2 - 6x + 1)$

$= 27x^3 - 18x^2 + 3x - 9x^2 + 6x - 1$

$= 27x^3 - 27x^2 + 9x - 1$

d $g(f(x)) = 3x^3 - 1$

Question 8

a $x-3y+6=0$ [1]

$x=3p+5$ [2]

Substitute for *x*, [2] in [1].

$$3p+5-3y+6=0$$
$$3y=3p+11$$
$$y=\frac{3p+11}{3} \text{ or } y=p+3\frac{2}{3}$$

b $y=x^2-2x+5$ [1]

$x=-2p$ [2]

Substitute for *x*, [2] in [1].

$$y=(-2p)^2-2\times(-2p)+5$$
$$=4p^2+4p+5$$

c $x^2+y^2=81$ [1]

$x=9\cos p$ [2]

Substitute for *x*, [2] in [1].

$$(9\cos p)^2+y^2=81$$
$$81\cos^2 p+y^2=81$$
$$y^2=81-81\cos^2 p$$
$$=81(1-\cos^2 p)$$
$$y^2=81\sin^2 p, \quad \text{using } 1-\cos^2 p=\sin^2 p$$
$$y=9\sin p$$

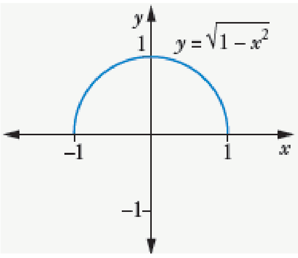
Question 9

a No, it fails the vertical line test.

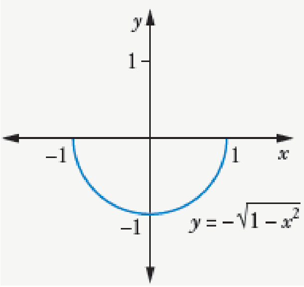
b $y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$

c $y = \sqrt{1-x^2}$ is a semicircle, above the x-axis, with radius $\sqrt{1} = 1$.



$y = -\sqrt{1-x^2}$ is a semicircle, below the x-axis, with radius $\sqrt{1} = 1$.



Question 10

a Circle, centre (0, 0), radius $\sqrt{16} = 4$.

Domain [–4, 4], range [–4, 4]

b Hyperbola, $x \neq 2$

Domain $(-\infty, 2) \cup (2, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

c $|x| \geq 0$, so $|x + 3| \geq 3$

Domain $(-\infty, \infty)$, range $[3, \infty)$

d $y = \sqrt{9-x^2}$

Semicircle, above the x-axis, centre (0, 0), with radius $\sqrt{9} = 3$.

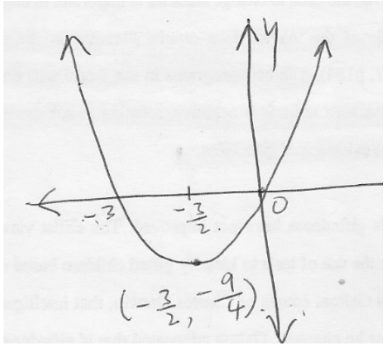
Domain [–3, 3], range [–3, 3]

Question 11

a $y = \dfrac{1}{x^2+3x} = \dfrac{1}{x(x+3)}$

Vertical asymptotes at $x = 0$ and $x = -3$.

Graph $f(x) = x^2 + 3x$ first.

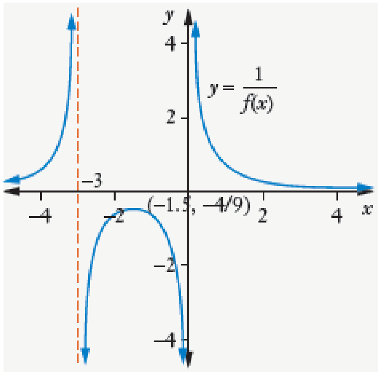


Vertex:

$x = \dfrac{-3}{2(1)} = -\dfrac{3}{2}$

$y = \left(-\dfrac{3}{2}\right)^2 + 3\left(-\dfrac{3}{2}\right) = \dfrac{9}{4} - \dfrac{9}{2} = -\dfrac{9}{4}$

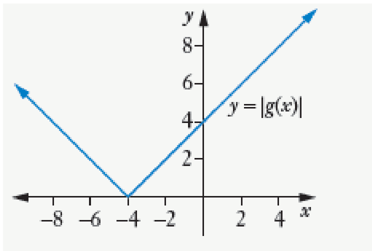
Now graph $y = \dfrac{1}{x^2+3x}$.



Vertical asymptotes at $x = -3$, $x = 0$.

$\therefore$ Vertex on $y = \dfrac{1}{f(x)}$ is $\left(-\dfrac{3}{2}, -\dfrac{4}{9}\right)$.

b $y = |x + 4|$



x-intercept ($y = 0$)

$0 = |x + 4|$

$x + 4 = 0$

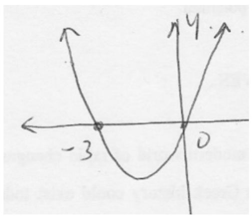
y-intercept ($x = 0$)

$y = |0 + 4|$

$= 4$

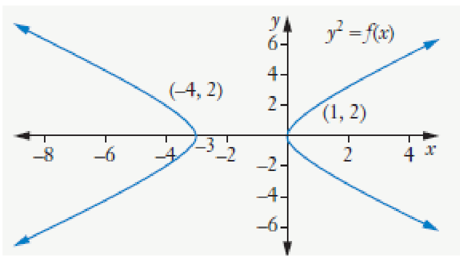
$$y = \begin{cases} x+4 & \text{for } x \geq -4 \\ -x-4 & \text{for } x < -4 \end{cases}$$

c $y^2 = x^2 + 3x$



$y = \pm\sqrt{x^2+3x} = \pm\sqrt{x(x+3)}$

First graph $f(x) = x^2 + 3x$.

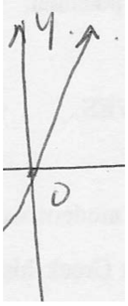


$y = \pm\sqrt{f(x)}$ is defined for $x \leq -3$ and $x \geq 0$ only.

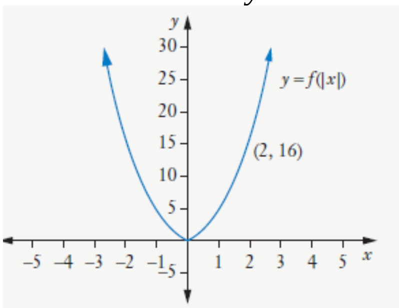
Domain $(-\infty, -3] \cup [0, \infty)$.

d $y = \left|x^2\right| + 3|x|$.

First graph $f(x) = x^2 + 3x$ for $x \geq 0$.



Then reflect in the y-axis.



e $y^2 = x + 4$

$y = \pm\sqrt{x+4}$

$x + 4 \geq 0$

$x \geq -4$

Domain $[-4, \infty)$.

x-intercept ($y = 0$)

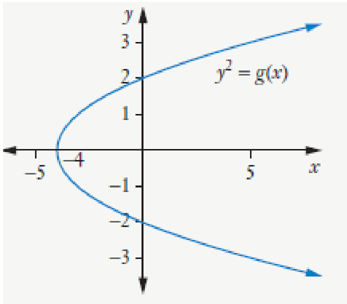
$0 = x + 4$

$x = -4$

y-intercept ($x = 0$)

$y^2 = 4$

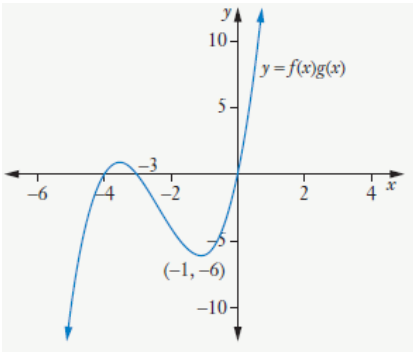
$y = \pm 2$



f $y = (x^2 + 3x)(x + 4)$

$\quad = x(x + 3)(x + 4)$

Cubic curve, x-intercepts 0, –3, –4



Question 12

$f(x) + g(x) = x^2 - 4x + 2x - 3$

$\qquad\qquad\quad = x^2 - 2x - 3$

Parabola with vertex

$x = \dfrac{-(-2)}{2(1)} = 1$

$y = 1^2 - 2(1) - 3 = -4$

Domain $(-\infty, \infty)$, range $[-4, \infty)$.

Question 13

a $x = 3t$ [1]

$y = t - 2 \Rightarrow t = y + 2$ [2]

Substitute for *t*, [2] in [1].

$x = 3(y+2)$

$x = 3y + 6$

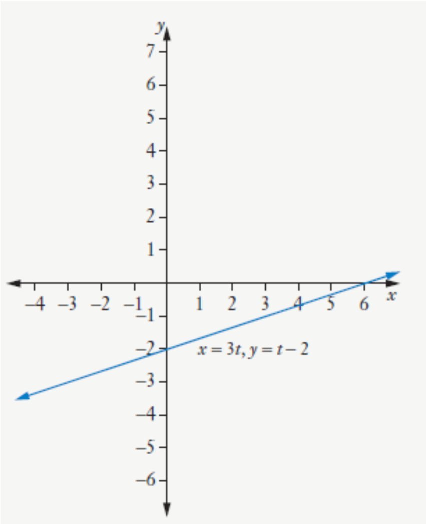
$x - 3y - 6 = 0$

This is the equation of a straight line.

x-intercept

$y = 0, \quad x - 6 = 0 \Rightarrow x = 6$

y-intercept $x = 0, \quad -3y - 6 = 0 \Rightarrow y = -2$



b $x = 3\cos\theta \Rightarrow x^2 = 9\cos^2\theta$ [1]

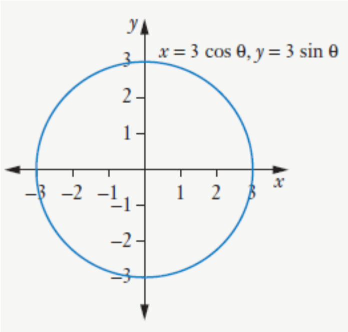
$y = 3\sin\theta \Rightarrow y^2 = 9\sin^2\theta$ [2]

[1] + [2]

$x^2 + y^2 = 9\cos^2\theta + 9\sin^2\theta = 9(\cos^2\theta + \sin^2\theta) = 9$

$x^2 + y^2 = 3^2$

This is the equation of a circle, radius 3, centre at (0, 0).



c $x = 2p \Rightarrow p = \frac{1}{2}x$ [1]

$y = 4p^2 + 8p$ [2]

Substitute for *p*, [1] in [2].

$y = 4\left(\frac{1}{2}x\right)^2 + 8 \times \frac{1}{2}x = x^2 + 4x = x(x+4)$

This is the equation of a parabola.

x-intercepts

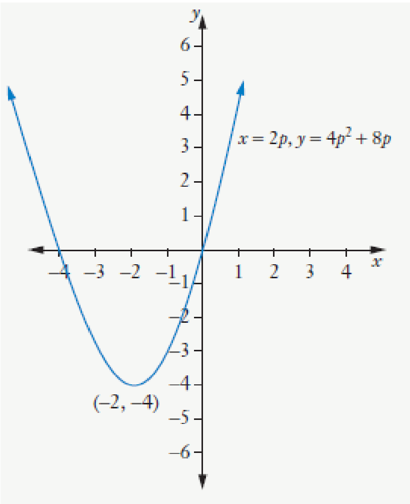
$y = 0, \quad x(x+4) = 0 \Rightarrow x = 0, x = -4$

y-intercept: $x = 0, y = -2$

Axis of symmetry is at $x = -2$, half way between the *x*-intercepts, 0 and –4.

$f(-2) = (-2)^2 + 4(-2) = -4$

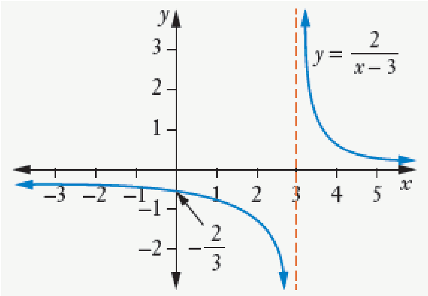
The turning point is at (–2, –4).



Question 14

a Domain $(-\infty, 3) \cup (3, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

b Vertical asymptote at $x = 3$.



y-intercept ($x = 0$)

$$y = \frac{2}{0-3} = -\frac{2}{3}$$

Question 15

a $x^2 - 8x + y^2 + 6y = 0$

$$x = 4 + 5\cos\theta \Rightarrow (x-4)^2 = 25\cos^2\theta \qquad [1]$$

$$y = -3 + 5\sin\theta \Rightarrow (y+3)^2 = 25\sin^2\theta \qquad [2]$$

[1] + [2]

$$(x-4)^2 + (y+3)^2 = 25\cos^2\theta + 25\sin^2\theta$$
$$(x-4)^2 + (y+3)^2 = 25(\cos^2\theta + \sin^2\theta)$$
$$(x-4)^2 + (y+3)^2 = 5^2$$
$$x^2 - 8x + 16 + y^2 + 6y + 9 = 25$$
$$x^2 - 8x + y^2 + 6y = 0$$

b $(x-4)^2 + (y+3)^2 = 5^2$

This is the equation of a circle, radius 5, centre at $(4, -3)$.

Question 16

a x-intercept ($y = 0$)

$0 = |x + 1|$

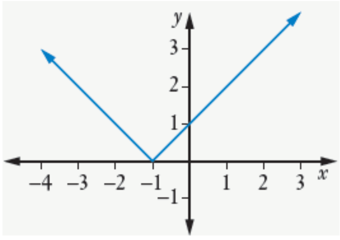
$x + 1 = 0$

$x = -1$

y-intercept ($x = 0$)

$y = |0 + 1|$

$= 1$



$$y = \begin{cases} x+1 & \text{for } x \geq -1 \\ -x-1 & \text{for } x < -1 \end{cases}$$

b **i** $|x + 1| = 3$

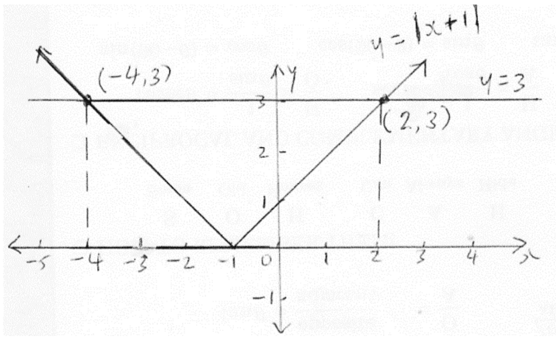
$x = -4, 2$

ii $|x + 1| < 3$

$-4 < x < 2$

iii $|x + 1| > 3$

$x < -4, x > 2$



Question 17

$y = |x - 3|$

x-intercept ($y = 0$)

$0 = |x - 3|$

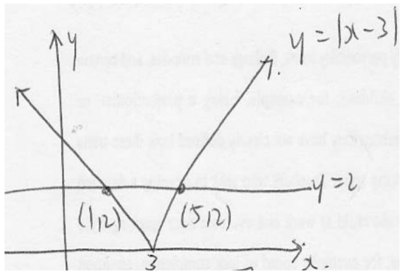
$x - 3 = 0$

$x = 3$

y-intercept ($x = 0$)

$y = |0 - 3|$

$= 3$



$$y = \begin{cases} x-3 & \text{for } x \geq 3 \\ -x+3 & \text{for } x < 3 \end{cases}$$

$x = 1, 5$

Question 18

a Centre (0, 0), radius $\sqrt{100} = 10$

b Centre (3, 2), radius $\sqrt{121} = 11$

c $x^2 + 6x + 9 + y^2 + 2y + 1 = 9$

$(x + 3)^2 + (y + 1)^2 = 9$

Centre (–3, –1), radius $\sqrt{9} = 3$

Question 19

a x-intercepts ($y = 0$)

$0 = x^3 - 4x$

$= x(x^2 - 4)$

$= x(x + 2)(x - 2)$

$x = 0, -2, 2$

y-intercept ($x = 0$)

$y = 0^3 - 4(0)$

$= 0$

b x-intercepts ($y = 0$)

$0 = -\frac{2}{x+1}$

No x-intercept

y-intercept ($x = 0$)

$y = -\frac{2}{0+1}$

$= -2$

c x-intercepts ($y = 0$)

$x^2 = 9$

$x = \pm 3$

y-intercept ($x = 0$)

$y^2 = 9$

$y = \pm 3$

d x-intercepts ($y = 0$)

$0 = \sqrt{25 - x^2}$

$0 = 25 - x^2$

$x^2 = 25$

$x = \pm 5$

y-intercept ($x = 0$)

$y = \sqrt{25 - 0}$

$= 5$

e x-intercepts ($y = 0$)

$0 = |x - 2| + 3$

$-3 = |x - 2|$

No x-intercepts

y-intercept ($x = 0$)

$y = |0 - 2| + 3$

$= 5$

Question 20

a $f(x) + g(x) = 2x^2 + x - 6 + 5x^3 + 1 = 5x^3 + 2x^2 + x - 5$

Degree 3

b $f(x)\,g(x) = (2x^2 + x - 6)(5x^3 + 1)$

Leading term $= 2x^2 \times 5x^3 = 10x^5$

c $f(x) - g(x) = 2x^2 + x - 6 - 5x^3 - 1$

$= -5x^3 + 2x^2 + x - 7$

Constant term -7

d **i** $[g(x)]^2 = (5x^3 + 1)^2 = 25x^6 + 10x^3 + 1$

ii $g(x^2) = 5(x^2)^3 + 1 = 5x^6 + 1$

iii $\frac{1}{f(x)} = \frac{1}{2x^2 + x - 6}$

Question 1

a $y^2 = x(x-2)(x-4)$

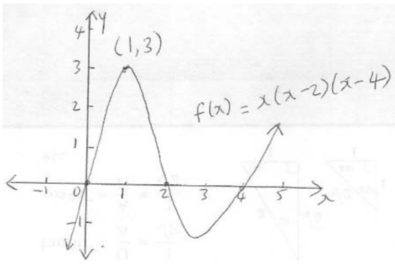
$y = \pm\sqrt{x(x-2)(x-4)}$

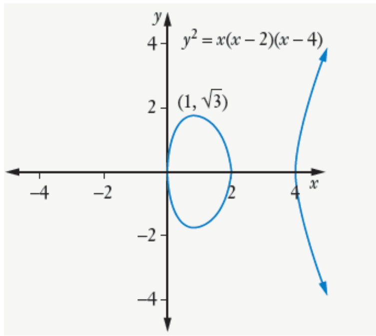
First graph $y = x(x-2)(x-4)$

x-intercepts ($y = 0$)	y-intercept ($x = 0$)
$0 = x(x-2)(x-4)$	$y = 0(0-2)(0-4)$
$x = 0, 2, 4$	$= 0$

$y = \pm\sqrt{x(x-2)(x-4)}$ exists only for $x(x-2)(x-4) \geq 0$

Domain $[0, 2] \cup [4, \infty)$, range $(-\infty, \infty)$





b $y^2 = (x-1)^3$

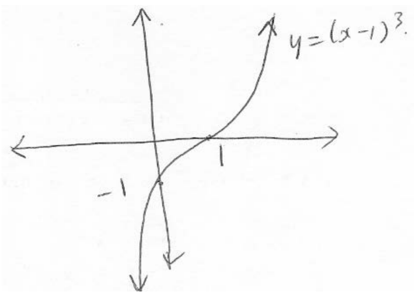
$y = \pm\sqrt{(x-1)^3}$

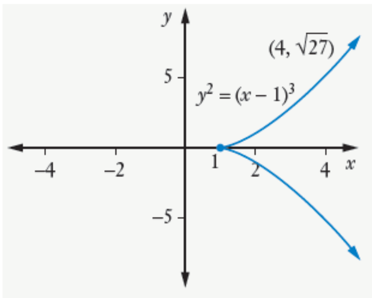
First graph $y = (x-1)^3$

x-intercepts ($y = 0$)	y-intercept ($x = 0$)
$0 = (x-1)^3$	$y = (0-1)^3$
$x - 1 = 0$	$= -1$
$x = 1$	

$y = \pm\sqrt{(x-1)^3}$ exists only for $(x-1)^3 \geq 0$.

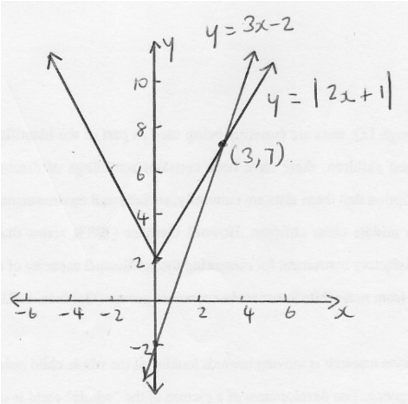
Domain $[1, \infty)$, range $(-\infty, \infty)$





Question 2

$x = 3$.



Question 3

a $f(x) = |x| + 3x - 4$

$$f(x) = \begin{cases} x + 3x - 4 = 4x - 4 & x \geq 0 \\ -x + 3x - 4 = 2x - 4 & x < 0 \end{cases}$$

$$y = f(x) = 4x - 4, \quad x \geq 0$$

y-intercept ($x = 0$)

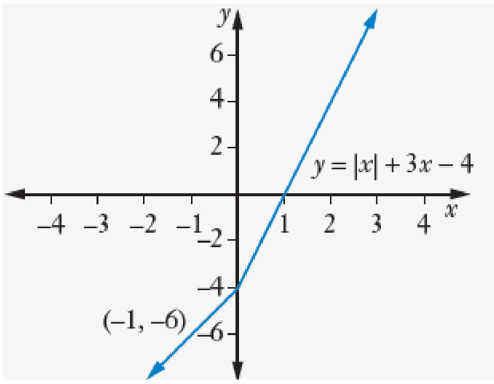
$y = 0 – 4 = 4$

x-intercept ($x = 0$)

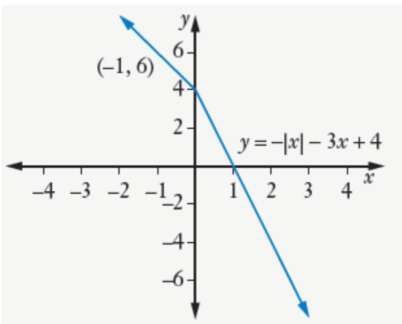
$0 = 4x – 4$

$4x = 4$

$x = 1$



b $y = –f(x)$ is the graph in **a** reflected in the x-axis.



c $y = \dfrac{1}{f(x)}$ has a vertical asymptote at $x = 1$ (where $f(x) = 0$).

As $x \to \infty, \dfrac{1}{f(x)} \to 0$.

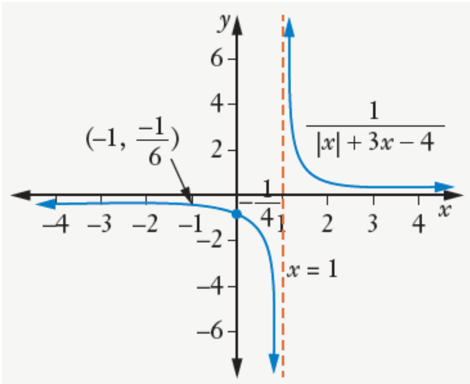
As $x \to -\infty, \dfrac{1}{f(x)} \to 0$.

As $x \to 1^+, \dfrac{1}{f(x)} \to \infty$

As $x \to 1^-, \dfrac{1}{f(x)} \to -\infty$, and

y-intercept ($x = 0$)

$$y = \frac{1}{|0| + 3(0) - 4} = -\frac{1}{4}$$



Question 4

a $a = \dfrac{k}{b^2}$

When $b = 3$, $a = 2$.

$2 = \dfrac{k}{3^2} = \dfrac{k}{9}$

$k = 18$

$a = \dfrac{18}{b^2}$

b When $b = 2$:

$a = \dfrac{18}{2^2} = 4.5$

c When $a = 10$:

$10 = \dfrac{18}{b^2}$

$b^2 = \dfrac{18}{10} = 1.8$

$b = \sqrt{1.8}$

$= 1.3416\ldots$

≈ 1.34

Question 5

$x^2 - 1 \neq 0$

$x^2 \neq 1$

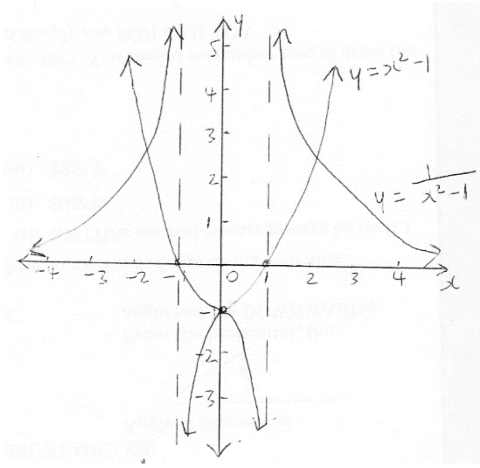
$x = \pm 1$

Domain $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Range $(-\infty, -1) \cup (0, \infty)$

Question 6

Graph $f(x) = x^2 - 1$ first, then $y = \dfrac{1}{f(x)} = \dfrac{1}{x^2 - 1}$.



Parabola, concave up, vertex (0, –1).

x-intercepts (y = 0)

$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

y-intercept (x = 0)

$y = 0^2 - 1 = -1$

Now graph $y = \dfrac{1}{f(x)} = \dfrac{1}{x^2 - 1}$.

Vertical asymptotes at $x = \pm 1$.

x-intercepts (y = 0)

$$0 = \frac{1}{x^2 - 1}$$

No solution.

y-intercept (x = 0)

$$y = \frac{1}{0^2 - 1} = -1$$

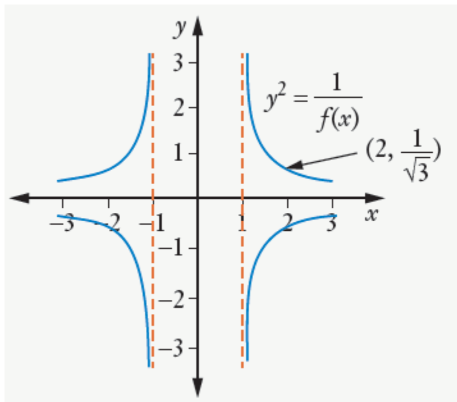
As $x \to -1^-, y \to \infty$, and as $x \to -1^+, y \to -\infty$

As $x \to 1^-, y \to -\infty$, and as $x \to 1^+, y \to \infty$

As $x \to \pm\infty, y \to 0$.

Now graph $y^2 = \dfrac{1}{f(x)} = \dfrac{1}{x^2 - 1}$

$$y = \pm \frac{1}{\sqrt{x^2 - 1}}$$



This function exists for only those values of x for which the graph of $y = \dfrac{1}{x^2 - 1}$ is on or above the x-axis, that is, for $x < -1$ and $x > 1$.

The graph of $y = -\dfrac{1}{\sqrt{x^2 - 1}}$ is a reflection in the x-axis of the graph of $y = \dfrac{1}{\sqrt{x^2 - 1}}$.

Question 7

$x^2 + 3x + 2.25 + y^2 - 2y + 1 - 3 = 2.25 + 1$

$(x + 1.5)^2 + (y - 1)^2 = 6.25$

Centre $(-1.5, 1)$, radius $\sqrt{6.25} = 2.5$

Question 8

$x^2 + 4x + 4 + y^2 - 8y + 16 - 5 = 4 + 16$

$(x + 2)^2 + (y - 4)^2 = 25$

Centre $(-2, 4)$

$x^2 - 2x + 1 + y^2 + 10y + 25 + 10 = 1 + 25$

$(x - 1)^2 + (y + 5)^2 = 16$

Centre $(1, -5)$

Line passing through $(-2, 4)$ and $(1, -5)$.

Gradient $= \dfrac{-5-4}{1-(-2)} = \dfrac{-9}{3} = -3$

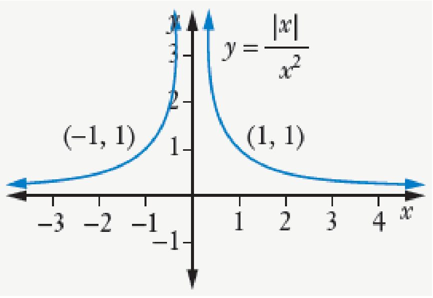
Equation: $y - 4 = -3(x - [-2]) = -3x - 6$

$3x + y + 2 = 0$

Question 9

$y = \dfrac{x}{x^2} = \dfrac{1}{x}$ for $x \geq 0$

$y = \dfrac{-x}{x^2} = -\dfrac{1}{x}$ for $x < 0$



Question 10

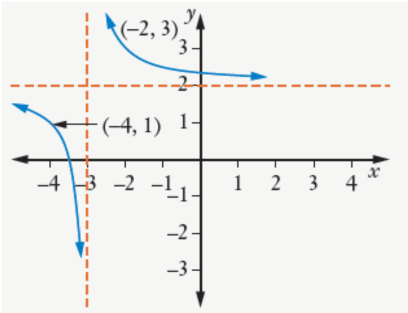
a RHS = $2 + \dfrac{1}{x+3} = \dfrac{2(x+3)+1}{x+3} = \dfrac{2x+6+1}{x+3} = \dfrac{2x+7}{x+3}$

= LHS

b Domain $(-\infty, -3) \cup (-3, \infty)$

Range $(-\infty, 2) \cup (2, \infty)$

c Vertical asymptote at $x = -3$.



Question 11

$x^2 - 2x + 1 + y^2 + 4y + 4 + 1 = 1 + 4$

$(x - 1)^2 + (y + 2)^2 = 4$

Centre (1, –2)

$x^2 - 2x + 1 + y^2 + 4y + 4 - 4 = 1 + 4$

$(x - 1)^2 + (y + 2)^2 = 9$

Centre (1, –2)

Both circles have centre (1, –2), so they are concentric.

Question 12

Domain $(-\infty, 0) \cup (0, \infty)$

$x^2 \geq 0$ so $\dfrac{1}{x^2} > 0$

So $1 - \dfrac{1}{x^2} < 1$

Range $(-\infty, 1)$

Horizontal asymptote $y = 1$

Vertical asymptote $x = 0$

Even function.

x-intercepts ($y = 0$)

$0 = 1 - \dfrac{1}{x^2}$

$\dfrac{1}{x^2} = 1$

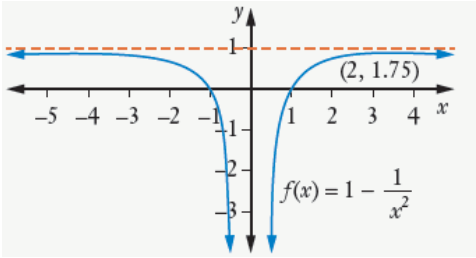
$x^2 = 1$

$x = \pm 1$

No y-intercept as $x = 0$ is a vertical asymptote.

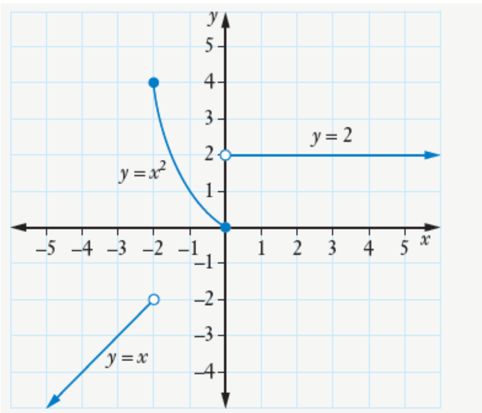
As $x \to \infty, y \to 1$, and as $x \to -\infty, y \to 1$

As $x \to 0$ $y \to -\infty$.



Question 13

a

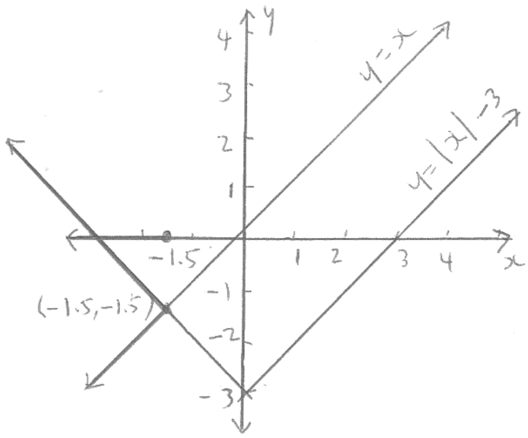


b Discontinuous at $x = -2$ and $x = 0$.

c Domain $(-\infty, \infty)$; Range $(-\infty, 2) \cup [0, 4]$

Question 14

$x \le -1.5$



Question 15

a $x = 2\cos\theta \Rightarrow \left(\dfrac{x}{2}\right)^2 = \cos^2\theta$ [1]

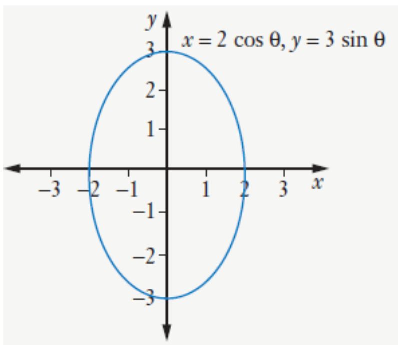
$y = 3\sin\theta \Rightarrow \left(\dfrac{y}{3}\right)^2 = \sin^2\theta$ [2]

[1] + [2]

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2\theta + \sin^2\theta$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

b This equation describes an ellipse.



When $y = 0$, $x^2 = 4 \Rightarrow x = \pm 2$

When $x = 0$, $y^2 = 9 \Rightarrow y = \pm 3$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

$$y = \pm 3\sqrt{1 - \frac{x^2}{4}}$$

The domain is $-2 \le x \le 2$ and the range is $-3 \le y \le 3$.

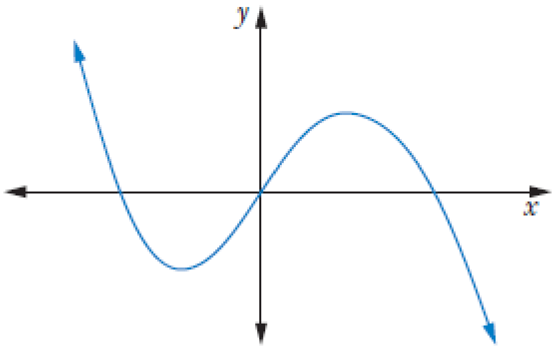
MATHS IN FOCUS 11 MATHEMATICS EXTENSION 1

WORKED SOLUTIONS

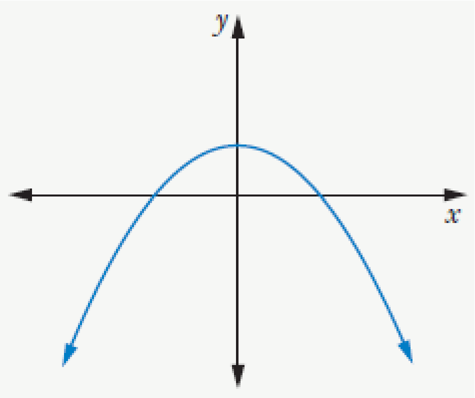
Chapter 8: Introduction to calculus

Exercise 8.01 Gradient of a curve

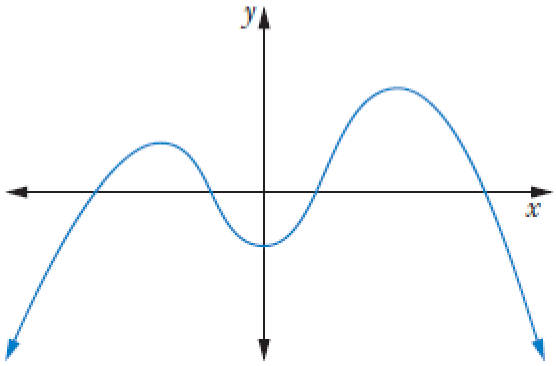
Question 1



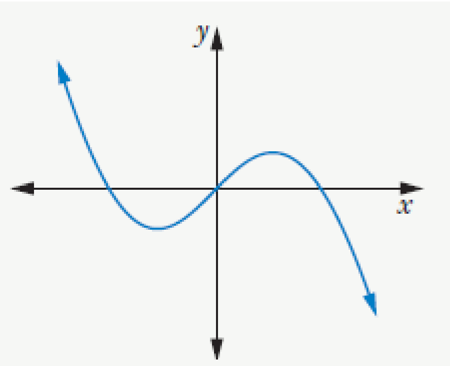
Decreases rapidly, then slows down and turns around at minimum turning point. It then increases, becoming rapid and then slows down to maximum turning point. It then decreases, becoming more rapid.



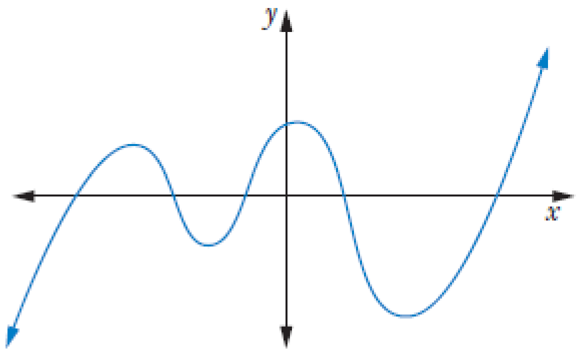
Question 2



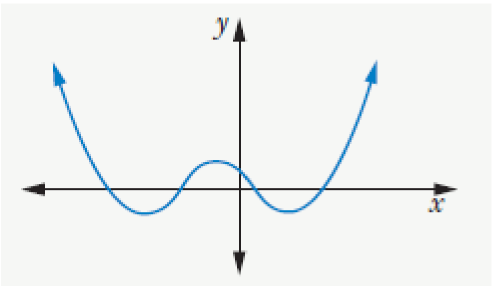
Increases rapidly then slows down and turns around at maximum point, then decreases, becoming more rapid, then slows down and turns around at minimum turning point. It then increases, becoming rapid and then slowing down to maximum turning point. It then decreases, becoming more rapid.



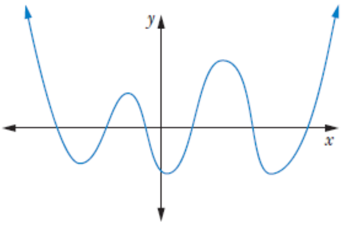
Question 3



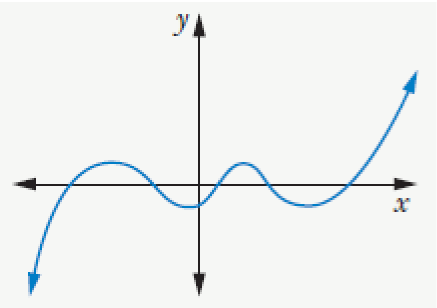
Increases rapidly then slows down and turns around at maximum point, then decreases, becoming rapid, then slows down and turns around at minimum turning point. It then increases, becoming rapid and then slowing down to maximum turning point. It then decreases, becoming more rapid then slows down to minimum turning point, then increases and becomes more rapid.



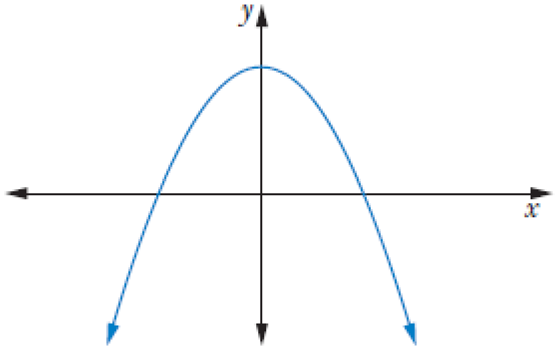
Question 4



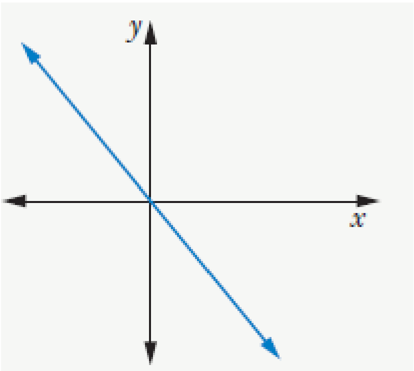
Decreases rapidly then slows down and turns around at minimum point, then increases, becoming more rapid, then slows down and turns around at maximum turning point. It then decreases to minimum turning point, increases to Maximum turning point, decreases to minimum turning point, then increases, becoming more rapid.



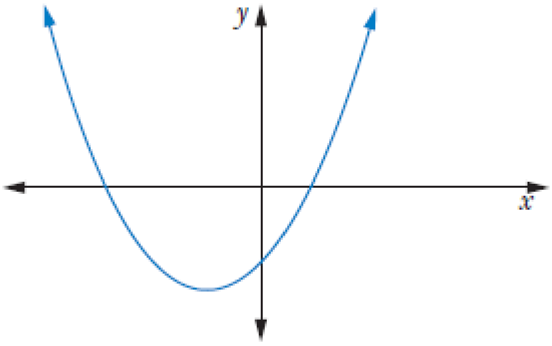
Question 5



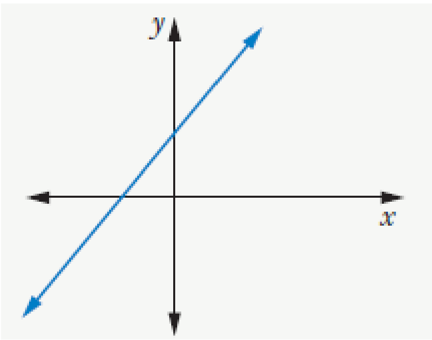
Increases rapidly then slows down and turns around at maximum point, then decreases slowly at first and becomes more rapid.



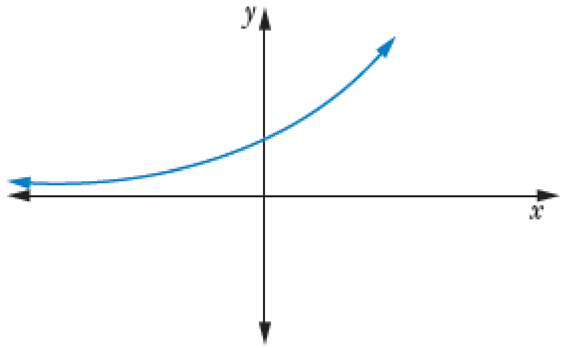
Question 6



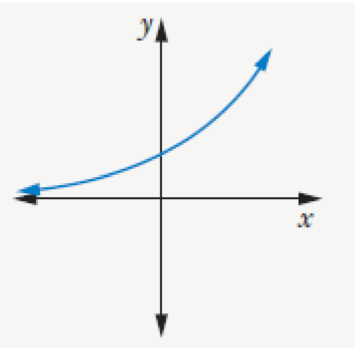
Decreases rapidly then slows down and turns around at minimum point, then increases slowly at first and becomes more rapid.



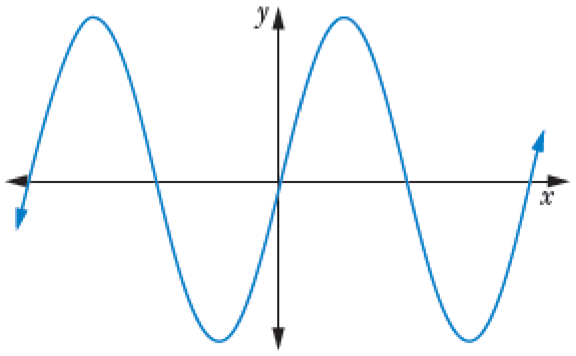
Question 7



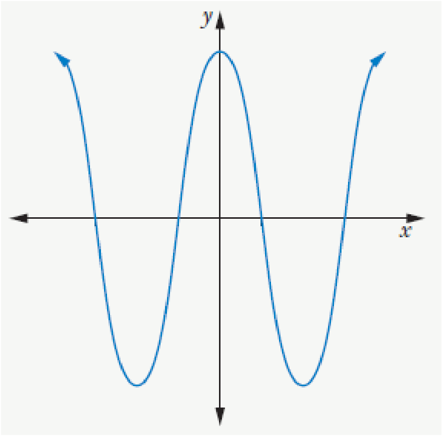
The curve is always increasing. However, it starts increasing slowly and the increase becomes more rapid.



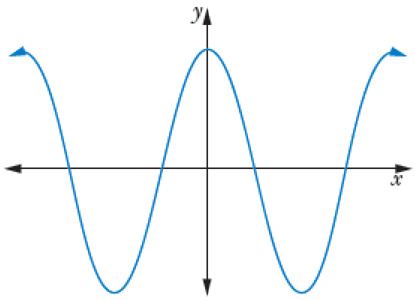
Question 8



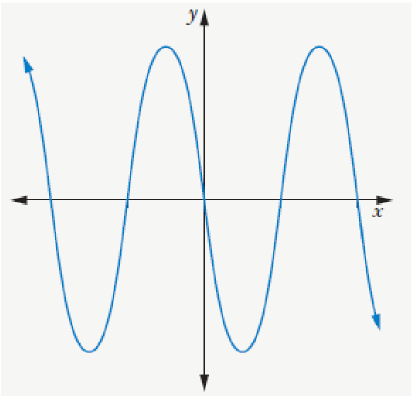
Increases rapidly then slows down and turns around at maximum point, then decreases, becoming more rapid, then slows down and turns around at minimum turning point. It then increases, speeding up, then the whole shape repeats after $x = 0$.



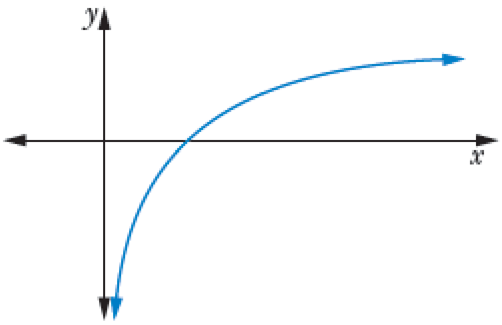
Question 9



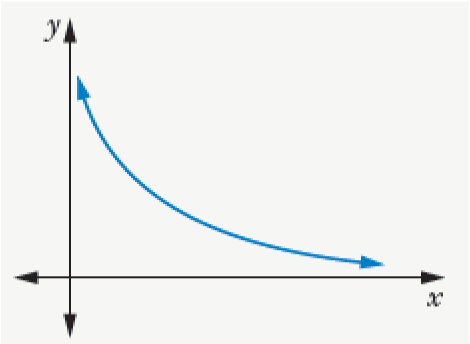
Increases rapidly then slows down and turns around at maximum point, then decreases, becoming more rapid, then slows down and turns around at minimum turning point. It then increases, speeding up, then slows down and turns at a maximum turning point. It then decreases, speeding up. The shape starts to repeat after the 3rd *x*-intercept.



Question 10



The curve is always increasing, but it starts steeply then slows down.



Exercise 8.02 Differentiability

Question 1

Not differentiable at $x = 0$.

Question 2

Not differentiable at $x = x_1$.

Question 3

No points.

Question 4

Not differentiable at $x = 0$.

Question 5

Not differentiable at $x = x_1, x_2$.

Question 6

Not differentiable at $x = 0$.

Question 7

Not differentiable at $x = –3$.

Question 8

Not differentiable at $x = 2$.

Question 9

Not differentiable at $x = –2, 3$.

Question 10

Not differentiable [–1, 0].

Question 11

Not differentiable at $x = 0$.

Exercise 8.03 Differentiation from first principles

Question 1

a

$y = x^4 + 1$

$y(1.01) = 1.01^4 + 1$

$= 1.0406 + 1$

$= 2.0406$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{204060401 - 2}{1.01 - 1} \\ &\approx 4.06 \end{aligned}$$

b

$y(0.999) = 1.996$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1.996 - 2}{0.999 - 1} \\ &= 3.994 \end{aligned}$$

c The gradient is 4

Question 2

$f(x) = x^3 + x$

a $x = 2.1$

$f(2.1) = 2.1^3 + 2.1 = 11.361$

$$\frac{f(x+h)-f(x)}{h}$$
$$=\frac{11.361-10}{0.1}$$
$$=13.61$$

b $f(2.01) = 2.01^3 + 2.01 = 10.130601$

$$\frac{f(x+h)-f(x)}{h}$$
$$=\frac{10.130601-10}{0.01}$$
$$=13.0601$$

c $f(1.99) = 9.87$

$$\frac{f(x+h)-f(x)}{h}$$
$$=\frac{9.87-10}{-0.01}$$
$$=12.9401$$

d Tangent at point (2,10) = 13

Question 3

$f(x) = x^2 - 4$

$f(3) = 5$

Point (3, 5)

$f(3.01) = 5.0601$

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{5.0601-5}{0.01}$$
$$= 6.01$$

$f(2.99) = 4.9401$

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{4.9401-5}{-0.01}$$
$$= 5.99$$

Gradient = 6

Question 4

$f(x) = x^2 + x + 5$

a $f(2) = 2^2 + 2 + 5 = 11$

b $f(2 + h) = (2 + h)^2 + (2 + h) + 5$

$= 4 + 4h + h^2 + 2 + h + 5$

$= h^2 + 5h + 11$

c $f(2 + h) - f(2)$

$= 11 + 5h + h^2 - 11$

$= h^2 + 5h$

d

$$\frac{f(2+h)-f(2)}{h}$$
$$=\frac{5h+h^2}{h}$$
$$=5+h$$

e $f'(2) = \lim_{h\to 0}(5+h)$

$= 5$

Question 5

$f(x) = 4x^2 - 3$

a $f(-1) = 4(-1)^2 - 3 = 1$

b $f(-1 + h) - f(-1) = 4(-1 + h)^2 - 3 - 1$

$= 4 - 8h + 4h^2 - 4$

$= -8h + 4h^2$

c

$$\lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{-8h + 4h^2}{h}$$

$$= \lim_{h \to 0} -8 + 4h$$

$= -8$

Question 6

$y = x^2 - 1$

a $f(3) = 3^2 - 1 = 8$

b $f(3 + h) - f(3)$

$= (3 + h)^2 - 1 - 8$

$= 9 + 6h + h^2 - 9$

$= 6h + h^2$

c

$$f'(3) = \lim_{h \to 0} \frac{6h + h^2}{h}$$

$$= \lim_{h \to 0} 6 + h$$

$$= 6$$

Question 7

$f(x) = 4 - 3x - 5x^2$

a $f(1) = -4$

$$
\begin{aligned}
f'(1) &= \lim_{h\to 0}\frac{f(1+h) - f(1)}{h}\\
&= \lim_{h\to 0}\frac{4-3(1+h)-5(1+h)^2-(-4)}{h}\\
&= \lim_{h\to 0}\frac{4-3-3h-5(1+2h+h^2)+4}{h}\\
&= \lim_{h\to 0}\frac{4-3-3h-5-10h-5h^2+4}{h}\\
&= \lim_{h\to 0}\frac{-13h-5h^2}{h}\\
&= \lim_{h\to 0}(-13-5h)\\
&= -13
\end{aligned}
$$

b $f(-2) = -10$

$$
\begin{aligned}
f'(-2) &= \lim_{h\to 0}\frac{f(-2+h) - f(-2)}{h}\\
&= \lim_{h\to 0}\frac{4-3(-2+h)-5(-2+h)^2-(-10)}{h}\\
&= \lim_{h\to 0}\frac{4+6-3h-5(4-4h+h^2)+10}{h}\\
&= \lim_{h\to 0}\frac{4+6-3h-20+20h-5h^2+10}{h}\\
&= \lim_{h\to 0}\frac{17h-5h^2}{h}\\
&= \lim_{h\to 0}(17-5h)\\
&= 17
\end{aligned}
$$

Question 8

$f(x) = x^2$

a $f(x + h) = x^2 + 2xh + h^2$

b $f(x + h) - f(x) = x^2 + 2xh + h^2 - x^2$

$= 2xh + h^2$

c

$$\frac{f(x+h)-f(x)}{h} = \frac{2xh + h^2}{h}$$
$$= 2x + h$$

d

$$f'(x) = \lim_{h\to 0}(2x+h)$$
$$= 2x$$

Question 9

$f(x) = 2x^2 - 7x + 3$

a $f(x + h) = 2(x + h)^2 - 7(x + h) + 3$

$= 2(x^2 + 2xh + h^2) - 7x - 7h + 3$

$2x^2 + 4xh + 2h^2 - 7x - 7h + 3$

b $f(x + h) - f(x) = 2x^2 + 4xh + 2h^2 - 7x - 7h + 3 - (2x^2 - 7x + 3)$

$= 4xh + 2h^2 - 7h$

c

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{4xh + 2h^2 - 7h}{h}$$
$$= 4x - 2h - 7$$

d

$$f'(x) = \lim_{h\to 0}(4x-2h-7)$$
$$= 4x - 7$$

Question 10

a $f(x) = x^2$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} \\
&= \lim_{h\to 0}\frac{2xh+h^2}{h} \\
&= \lim_{h\to 0}(2x+h) \\
&= 2x
\end{aligned}$$

When $x = 1$, $f'(x) = 2$

b $f(x) = x^2 + x$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)^2+(x+h)-(x^2+x)}{h} \\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2+x+h-x^2-x}{h} \\
&= \lim_{h\to 0}\frac{2xh+h^2+h}{h} \\
&= \lim_{h\to 0}(2x+h+1) \\
&= 2x+1
\end{aligned}$$

When x = 2, $f'(x) = 5$

c $f(x) = 2x^2 - 5$

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to0}\frac{2(x+h)^2-5-(x^2-5)}{h}$$
$$= \lim_{h\to0}\frac{2x^2+4xh+2h^2-5-x^2+5}{h}$$
$$= \lim_{h\to0}\frac{4xh+2h^2}{h}$$
$$= \lim_{h\to0}(4x+2h)$$
$$= 4x$$

When x = –3, $f'(x)$ = -12

d $f(x) = 3x^2 + 3x + 1$

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to0}\frac{3(x+h)^2+3(x+h)+1-(3x^2+3x+1)}{h}$$
$$= \lim_{h\to0}\frac{3x^2+6xh+3h^2+3x+3h+1-3x^2-3x-1}{h}$$
$$= \lim_{h\to0}\frac{6xh+3h^2+3h}{h}$$
$$= \lim_{h\to0}(6x+3h+3)$$
$$= 6x+3$$

When $x = 2$, $f'(x) = 15$

e $f(x) = x^2 - 7x - 4$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^2 - 7(x+h) - 4 - (x^2 - 7x - 4)}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 7x - 7h - 4 - x^2 + 7x + 4}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2 - 7h}{h}$$
$$= \lim_{h \to 0} (2x + h - 7)$$
$$= 2x - 7$$

When $x = -1$, $f'(x) = -9$

Question 11

a $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$
$$= \lim_{h \to 0} (2x + h)$$
$$= 2x$$

b $f(x) = x^2 + 5x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^2 + 5(x+h) - (x^2 + 5x)}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 + 5x + 5h - x^2 - 5x}{h}$$

$$= \lim_{h\to 0}\frac{2xh + h^2 + 5h}{h}$$

$$= \lim_{h\to 0}(2x + h + 5)$$

$$= 2x + 5$$

c $f(x) = 4x^2 - 4x - 3$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{4(x+h)^2 - 4(x+h) - 3 - (4x^2 - 4x - 3)}{h}$$

$$= \lim_{h\to 0}\frac{4x^2 + 8xh + 4h^2 - 4x - 4h - 3 - 4x^2 + 4x + 3}{h}$$

$$= \lim_{h\to 0}\frac{8xh + 4h^2 - 4h}{h}$$

$$= \lim_{h\to 0}(8x + 4h - 4)$$

$$= 8x - 4$$

d $f(x) = 5x^2 - x - 1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{5(x+h)^2 - (x+h) - 1 - (5x^2 - x - 1)}{h}$$

$$= \lim_{h\to 0}\frac{5x^2 + 10xh + 5h^2 - x - h - 1 - 5x^2 + x + 1}{h}$$

$$= \lim_{h\to 0}\frac{10xh + 5h^2 - h}{h}$$

$$= \lim_{h\to 0}(10x + 5h - 1)$$

$$= 10x - 1$$

Exercise 8.04 Short methods of differentiation

Question 1

a $\frac{d}{dx}(x+4)=1$

b $\frac{d}{dx}(5x-9)=5$

c $\frac{d}{dx}(x^2+3x+4)=2x+3$

d $\frac{d}{dx}(5x^2-x-8)=10x-1$

e $\frac{d}{dx}(x^3+2x^2-7x-3)=3x^2+4x-7$

f $\frac{d}{dx}(2x^3-7x^2+7x-1)=6x^2-14x+7$

g $\frac{d}{dx}(3x^4-2x^2+5x)=12x^3-4x+5$

h $\frac{d}{dx}(x^6-5x^5-2x^4)=6x^5-25x^4-8x^3$

i $\frac{d}{dx}(2x^5-4x^3+x^2-2x+4)=10x^4-12x^2+2x-2$

j $\frac{d}{dx}(4x^{10}-7x^9)=40x^9-63x^8$

Question 2

a $x(2x + 1) = 2x^2 + x$

$$\frac{d}{dx}\left(2x^2 + x\right) = 4x + 1$$

b $(2x - 3)^2 = 4x^2 - 12x + 9$

$$\frac{d}{dx}\left(4x^2 - 12x + 9\right) = 8x - 12$$

c $(x + 4)(x - 4) = x^2 - 16$

$$\frac{d}{dx}\left(x^2 - 16\right) = 2x$$

d $(2x^2 - 3)^2 = 4x^4 - 12x^2 + 9$

$$\frac{d}{dx}\left(4x^4 - 12x^2 + 9\right) = 16x^3 - 24x$$

e $(2x + 5)(x^2 - x + 1) = 2x^3 - 2x^2 + 2x + 5x^2 - 5x + 5$

$= 2x^3 + 3x^2 - 3x + 5$

$$\frac{d}{dx}\left(2x^3 + 3x^2 - 3x + 5\right) = 6x^2 + 6x - 3$$

Question 3

a $\frac{d}{dx}\left(\frac{x^2}{6}-x\right)=\frac{x}{3}-1$

b $\frac{d}{dx}\left(\frac{x^4}{2}-\frac{x^3}{3}+4\right)=2x^3-x^2$

c

$$\frac{x^6}{3}\left(x^2-3\right)=\frac{x^8}{3}-x^6$$
$$\frac{d}{dx}\left(\frac{x^8}{3}-x^6\right)=\frac{8x^7}{3}-6x^5$$

d

$$\frac{2x^3+5x}{x}=2x^2+5$$
$$\frac{d}{dx}\left(2x^2+5\right)=4x$$

e

$$\frac{x^2+2x}{4x}=\frac{x}{4}+\frac{1}{2}$$
$$\frac{d}{dx}\left(\frac{x}{4}+\frac{1}{2}\right)=\frac{1}{4}$$

f

$$\frac{2x^5-3x^4+6x^3-2x^2}{3x^2}=\frac{2}{3}x^3-x^2+2x-\frac{2}{3}$$
$$\frac{d}{dx}\left(\frac{2}{3}x^3-x^2+2x-\frac{2}{3}\right)=2x^2-2x+2$$

Question 4

$f(x) = 8x^2 - 7x + 4$

$f'(x) = 16x - 7$

Question 5

$y = x^4 - 2x^3 + 5$

$$\frac{dy}{dx} = 4x^3 - 6x^2$$

$$\begin{aligned}\frac{dy}{dx}(-2) &= 4(-2)^3 - 6(-2)^2 \\ &= -56\end{aligned}$$

Question 6

$y = 6x^{10} - 5x^8 + 7x^5 - 3x + 8$

$$\frac{dy}{dx} = 60x^9 - 40x^7 + 35x^4 - 3$$

Question 7

$S = 5t^2 - 20\text{t}$

$$\frac{ds}{dt} = 10t - 20$$

Question 8

$g(x) = 5x^4$

$g'(x) = 20x^3$

Question 9

$V = 15t^2 - 9$

$$\frac{dv}{dt} = 30t$$

Question 10

$h = 40t - 2t^2$

$$\frac{dh}{dt} = 40 - 4t$$

Question 11

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

Question 12

$f(x) = 2x^3 - 3x + 4$

$f'(x) = 6x^2 - 3$

$f'(1) = 6 - 3 = 3$

Question 13

$f(x) = x^2 - x + 5$

$f'(x) = 2x - 1$

a $f'(3) = 2 \times 3 - 1 = 5$

b $f'(-2) = 2 \times (-2) - 1 = -5$

c $f'(x) = 7$

$\Rightarrow 7 = 2x - 1$

$8 = 2x$

$x = 4$

Question 14

$y = x^3 - 7$

$$\frac{dy}{dx} = 3x^2$$

a $\quad \frac{dy}{dx}(2) = 3 \times 2^2 = 12$

b

$$\frac{dy}{dx} = 12$$
$$3x^2 = 12$$
$$x^2 = 4$$
$$x = \pm 2$$

Question 15

$g(t) = 3t^3 - 4t^2 - 2t + 1$

$g'(t) = 9t^2 - 8t - 2$

$g'(2) = 9(2)^2 - 8(2) - 2 = 36 - 16 - 2 = 18$

Exercise 8.05 Derivatives and indices

Question 1

a $\frac{d}{dx}\left(x^{-3}\right)=-3x^{-3-1}=-3x^{-4}$

b $\frac{d}{dx}\left(x^{1.4}\right)=1.4x^{1.4-1}=1.4x^{0.4}$

c $\frac{d}{dx}\left(6x^{0.2}\right)=6\times 0.2x^{0.2-1}=1.2x^{-0.8}$

d $\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{\frac{1}{2}-}=\frac{1}{2}x^{-\frac{1}{2}}$

e $\frac{d}{dx}\left(2x^{\frac{1}{2}}-3x^{-}\right)=2\times\frac{1}{2}x^{\frac{1}{2}-}-3\times(-1)x^{-1-1}=x^{-\frac{1}{2}}+3x^{-2}$

f $\frac{d}{dx}\left(3x^{\frac{1}{3}}\right)=3\times\frac{1}{3}x^{\frac{1}{3}-}=x^{-\frac{2}{3}}$

g $\frac{d}{dx}\left(8x^{\frac{3}{4}}\right)=8\times\frac{3}{4}x^{\frac{3}{4}-}=6x^{-\frac{1}{4}}$

h $\frac{d}{dx}\left(-2x^{-\frac{}{2}}\right)=-2\times\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-}=x^{-\frac{3}{2}}$

Question 2

a $\frac{1}{x} = x^{-1}$

$$\frac{d}{dx}\left(x^{-1}\right) = -x^{-2} = -\frac{1}{x^2}$$

b $5\sqrt{x} = 5x^{\frac{1}{2}}$

$$\frac{d}{dx}\left(5x^{\frac{1}{2}}\right) = \frac{5}{2}x^{-\frac{1}{2}} = \frac{5}{2\sqrt{x}}$$

c $\sqrt[6]{x} = x^{\frac{1}{6}}$

$$\frac{d}{dx}\left(x^{\frac{1}{6}}\right) = \frac{1}{6}x^{-\frac{5}{6}} = \frac{1}{6\sqrt[6]{x^5}}$$

d $\frac{2}{x^5} = 2x^{-5}$

$$\frac{d}{dx}\left(2x^{-5}\right) = -10x^{-6} = -\frac{10}{x^6}$$

e $-\frac{5}{x^3} = -5x^{-3}$

$$\frac{d}{dx}\left(-5x^{-3}\right) = 15x^{-4} = \frac{15}{x^4}$$

F $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$$\frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2\sqrt{x^3}}$$

g $\frac{1}{2x^6} = \frac{1}{2}x^{-6}$

$$\frac{d}{dx}\left(\frac{1}{2}x^{-6}\right) = -3x^{-7} = -\frac{3}{x^7}$$

h $x\sqrt{x} = x^{\frac{3}{2}}$

$$\frac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$$

i $\frac{2}{3x} = \frac{2}{3}x^{-1}$

$$\frac{d}{dx}\left(\frac{2}{3}x^{-1}\right) = -\frac{2}{3}x^{-2} = -\frac{2}{3x^2}$$

j $\frac{1}{4x^2} + \frac{3}{x^4} = \frac{1}{4}x^{-2} + 3x^{-4}$

$$\frac{d}{dx}\left(\frac{1}{4}x^{-2} + 3x^{-4}\right) = -\frac{1}{2}x^{-3} - 12x^{-5}$$
$$= -\frac{1}{2x^3} - \frac{12}{x^5}$$
$$= -\frac{x^2}{2x^5} - \frac{24}{2x^5}$$
$$= -\frac{x^2 + 24}{2x^5}$$

Question 3

$$y = \sqrt[3]{x} = x^{\frac{1}{3}}$$
$$y' = \frac{1}{3}x^{-\frac{2}{3}}$$
$$y'(27) = \frac{1}{3}(27)^{-\frac{2}{3}}$$
$$= \frac{1}{3} \times \frac{1}{9}$$
$$= \frac{1}{27}$$

Question 4

$x = \frac{12}{t} = 12t^{-1}$

$\frac{dx}{dt} = -12t^{-2}$

$\frac{dx}{dt}(2) = -12(2)^{-2}$

$= -3$

Question 5

$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$

$f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$

$f'(16) = \frac{1}{4}(16)^{-\frac{2}{3}}$

$= \frac{1}{4} \times \frac{1}{8}$

$= \frac{1}{32}$

Question 6

$y = \frac{3}{2x^2} = \frac{3}{2}x^{-2}$

$\frac{dy}{dx} = -3x^{-2}$

$\frac{dy}{dx}(1) = -3(1)^{-2}$

$= -3$

Question 7

$y=\left(x+\sqrt{x}\right)^2=x^2+2x\sqrt{x}+x$

$=x^2+2x^{\frac{3}{2}}+x$

$\frac{dy}{dx}=2x+3x^{\frac{1}{2}}+1$

$=2x+3\sqrt{x}+1$

Question 8

$f(x)=\frac{\sqrt{x}}{2}=\frac{1}{2}x^{\frac{1}{2}}$

$f'(x)=\frac{1}{4}x^{-\frac{1}{2}}$

$f'(4)=\frac{1}{4}(4)^{-\frac{1}{2}}$

$=\frac{1}{8}$

Question 9

a $\frac{\sqrt{x}}{x}=x^{-\frac{1}{2}}$

$\frac{d}{dx}\left(x^{-\frac{1}{2}}\right)=-\frac{1}{2}x^{-\frac{3}{2}}=-\frac{1}{2\sqrt{x^3}}$

b

$y'=-\frac{1}{2\sqrt{x^3}}$

$y'(4)=-\frac{1}{2\sqrt{(4)^3}}$

$=-\frac{1}{16}$

Question 10

$f(x) = 3\sqrt{x} = 3x^{\frac{1}{2}}$

$f'(x) = \frac{3}{2}x^{-\frac{1}{2}} = -\frac{3}{2\sqrt{x}}$

Let $f'(a) = \frac{3}{4}$

$\frac{3}{4} = -\frac{3}{2\sqrt{a}}$

$2 = \sqrt{a}$

$a = 4$

Question 11

$y = \frac{2}{x} = 2x^{-1}$

$y' = -2x^{-2}$

Let $y' = -\frac{2}{25}$

$-\frac{2}{25} = -2x^{-2}$

$\frac{1}{25} = \frac{1}{x^2}$

$x = \pm 5$

$x = 5 \quad y = \frac{2}{5}$

$x = -5 \quad y = -\frac{2}{5}$

$\therefore$ The points of contact are $\left(5, \frac{2}{5}\right), \left(-5, -\frac{2}{5}\right)$.

Exercise 8.06 Tangents and normals

Question 1

a $y = x^3 - 3x$

$y' = 3x^2 - 3$

$y'(5) = 3 \times 5^2 - 3 = 72$

b $f(x) = x^2 + x - 4$

$f'(x) = 2x + 1$

$f'(-7) = 2 \times (-7) + 1 = -13$

c $f(x) = 5x^3 - 4x - 1$

$f'(x) = 15x^2 - 4$

$f'(-1) = 15 \times (-1)^2 - 4 = 11$

d $y = 5x^2 + 2x + 3$

$y' = 10x + 2$

$y'(-2) = 10 \times (-2) + 2 = -18$

e $y = 2x^9$

$y' = 18x^8$

$y'(1) = 18$

f $f(x) = x^3 - 7$

$f'(x) = 3x^2$

$f'(3) = 27$

g $v = 2t^2 + 3t - 5$

$$\frac{dv}{dt} = 4t + 3$$

$$\frac{dv}{dt}(2) = 4(2) + 3 = 11$$

h $Q = 3r^3 - 2r^2 + 8r - 4$

$$\frac{dQ}{dr} = 9r^2 - 4r + 8$$
$$\frac{dQ}{dr}(4) = 9(4)^2 - 4(4) + 8 = 136$$

i $h = t^4 - 4t$

$$\frac{dh}{dt} = 4t^3 - 4$$
$$\frac{dh}{dt}(0) = 4(0)^3 - 4 = -4$$

j $f(t) = 3t^5 - 8t^3 + 5t$

$f'(t) = 15t^4 - 24t^2 + 5$

$f'(2) = 15(2)^4 - 24(2)^2 + 5 = 149$

Question 2

a $f(x) = 2x^3 + 2x - 1$

$$f'(x) = 6x^2 + 2$$
$$f'(-2) = 6(-2)^2 + 2 = 26$$
$$m_n = -\frac{1}{26}$$

b $y = 3x^2 + 5x - 2$

$$y' = 6x + 5$$
$$y'(-5) = 6(-5) + 5 = -25$$
$$m_n = \frac{1}{25}$$

c $f(\mathrm{x}) = x^2 - 2x - 7$

$$f'(x) = 2x - 2$$
$$f'(-9) = 2(-2) - 2 = -20$$
$$m_n = \frac{1}{20}$$

d $y = x^3 + x^2 + 3x - 2$

$$y' = 3x^2 + 2x + 3$$
$$y'(-4) = 3(-4)^2 + 2(-4) + 3 = 43$$
$$m_n = -\frac{1}{43}$$

e $f(x) = x^{10}$

$$f'(x) = 10x^9$$
$$f'(-1) = 10(-1)^9 = -10$$
$$m_n = \frac{1}{10}$$

f $y = x^2 + 7x - 5$

$$y' = 2x + 7$$
$$y'(-7) = 2(-7) + 7 = -7$$
$$m_n = \frac{1}{7}$$

g $A = 2x^3 + 3x^2 - x + 1$

$$\frac{dA}{dx} = 6x^2 + 6x - 1$$
$$\frac{dA}{dx}(3) = 6(3)^2 + 6(3) - 1 = 71$$
$$m_n = -\frac{1}{71}$$

h $f(a) = 3a^2 - 2a - 6$

$$f'(a) = 6a - 2$$
$$f'(-3) = 6(-3) - 2 = -20$$
$$m_n = \frac{1}{20}$$

i $V = h^3 - 4h + 9$

$$\frac{dv}{dh} = 3h^2 - 4$$
$$\frac{dv}{dh}(2) = 3(2)^2 - 4 = 8$$
$$m_n = -\frac{1}{8}$$

j $g(x) = x^4 - 2x^2 + 5x - 3$

$g'(x) = 4x^3 - 4x + 5$

$g'(-1) = 4(-1)^3 - 4(-1) + 5 = 5$

$m_n = -\frac{1}{5}$

Question 3

a $y = x^2 + 1$

$y' = 2x$

i $y'(3) = 6$

$m_t = 6$

ii $m_n = -\frac{1}{6}$

b $f(x) = 5 - x^2$

$f'(x) = -2x$

i $f'(-4) = 8$

$m_t = 8$

ii $m_n = -\frac{1}{8}$

c $y = 2x^5 - 7x^2 + 4$

$y' = 10x^4 - 14^x$

i $y'(-1) = 24$

$m_t = 24$

ii $m_n = -\frac{1}{24}$

d $p(x) = x^6 - 3x^4 - 2x + 8$

$p'(x) = 6x^5 - 12x^3 - 2$

i $p'(-1) = -8$

$m_t = -8$

ii $m_n = \frac{1}{8}$

e $f(x) = 4 - x - x^2$

$f'(x) = -1 - 2x$

i $f'(-6) = 11$

$m_t = 11$

ii $m_n = -\frac{1}{11}$

Question 4

a $y = x^4 - 5x + 1$

$y' = 4x^3 - 5$

$y'(2) = 4(2)^3 - 5 = 27$

$y - y_1 = m\left(x - x_1\right)$

$y - 7 = 27(x - 2)$

$y - 7 = 27x - 54$

$27x - y - 47 = 0$

b $f(x) = 5x^3 - 3x^2 - 2x + 6$

$f'(x) = 15x^2 - 6x - 2$

$f'(1) = 15(1)^2 - 6(1) - 2 = 7$

$y - y_1 = m(x - x_1)$

$y - 6 = 7(x - 1)$

$y - 6 = 7x - 7$

$7x - y - 1 = 0$

c $y = x^2 + 2x - 8$

$y' = 2x + 2$

$y'(-3) = 2(-3) + 2 = -4$

$y - y_1 = m(x - x_1)$

$y + 5 = -4(x + 3)$

$y + 5 = -4x - 12$

$4x + y + 17 = 0$

d $y = 3x^3 + 1$

$y(2) = 3(2)^3 + 1 = 25$

$y' = 9x^2$

$y'(2) = 9(2)^2 = 36$

$y - y_1 = m(x - x_1)$

$y - 25 = 36(x - 2)$

$y - 25 = 36x - 72$

$36x - y - 47 = 0$

e $v = 4t^4 - 7t^3 - 2$

$v(2) = 4(2)^4 - 7(2)^3 - 2 = 6$

$v' = 16t^3 - 21t^2$

$v'(2) = 16(2)^3 - 21(2)^2 = 44$

$y - y_1 = m(x - x_1)$

$v - 6 = 44(t - 2)$

$v - 6 = 44t - 88$

$44t - v - 82 = 0$

Question 5

a $f(x) = x^3 - 3x + 5$

$f'(x) = 3x^2 - 3$

$f'(3) = 3(3)2– 3 = 24$

$m_t = 2, \quad m_n = -\frac{1}{24}$

$y - y_1 = m(x - x_1)$

$y - 23 = -\frac{1}{24}(x - 3)$

$24y - 552 = -x + 3$

$x + 24y - 555 = 0$

b $y = x^2 - 4x - 5$

$y' = 2x - 4$

$y'(-2) = 2(-2) - 4 = -8$

$m_t = -8 \quad m_n = \frac{1}{8}$

$y - y_1 = m(x - x_1)$

$$y - 7 = \frac{1}{8}(x + 2)$$
$$8y - 56 = x + 2$$
$$x - 8y + 58 = 0$$

c $f(x) = 7x - 2x^2$

$f(6) = 7(6) - 2(6)^2 = -30$

$f'(x) = 7 - 4x$

$f'(6) = 7 - 4(6) = -17$

$m_t = -17, \ m_n = \frac{1}{17}$

$y - y_1 = m(x - x_1)$

$$y + 30 = \frac{1}{17}(x - 6)$$
$$17y + 510 = x - 6$$
$$x - 17y - 516 = 0$$

d $y = 7x^2 - 3x - 3$

$y' = 14x - 3$

$y'(-3) = 14(-3) - 3 = -45$

$$m_t = -4, \quad m_n = \frac{1}{45}$$

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 69 = \frac{1}{45}\left(x + 3\right)$$
$$45y - 3105 = x + 3$$
$$x - 45y + 3108 = 0$$

e $y = x^4 - 2x^3 + 4x + 1$

$y(1) = 1 - 2 + 4 + 1 = 4$

$y' = 4x^3 - 6x^2 + 4$

$y'(1) = 4(1)^3 - 6(1)^2 + 4 = 2$

$$m_t = 2 \quad m_n = -\frac{1}{2}$$

$$y - y_1 = m\left(x - x_1\right)$$

$$y - 4 = -\frac{1}{2}\left(x - 1\right)$$
$$2y - 8 = -x + 1$$
$$x + 2y - 9 = 0$$

Question 6

a $f(x) = 4x^2 - x + 8$

$f'(x) = 8x - 1$

i $f'(1) = 8(1) - 1 = 7$

$m_t = 7$

$y - y_1 = m\left(x - x_1\right)$

$y - 11 = 7(x - 1)$

$y - 11 = 7x - 7$

$7x - y + 4 = 0$

ii $m_n = -\frac{1}{7}$

$y - y_1 = m_n\left(x - x_1\right)$

$y - 11 = -\frac{1}{7}\left(x - 1\right)$

$7y - 77 = -x + 1$

$x + 7y - 78 = 0$

b $y = x^3 - 2x^2 - 5x$

$y' = 3x^2 - 4x - 5$

i $y'(-3) = 3(-3)^2 - 4(-3) - 5 = 34$

$m_t = 34$

$y - y_1 = m\left(x - x_1\right)$

$y + 30 = 34(x + 3)$

$y + 30 = 34x + 102$

$34x - y + 72 = 0$

ii $m_n = -\dfrac{1}{34}$

$y - y_1 = m_n\left(x - x_1\right)$

$$y + 30 = -\frac{1}{34}\left(x + 3\right)$$
$$34y + 1020 = -x - 3$$
$$x + 34y + 1023 = 0$$

c $F(x) = x^5 - 5x^3$

$F(1) = 1^5 - 5(1)^3 = 1 - 5 = -4$

$F'(x) = 5x^4 - 15x^2$

i $F'(1) = 5(1)^4 - 15(1)^2 = -10$

$m_t = -10$

$y - y_1 = m\left(x - x_1\right)$

$y + 4 = -10(x - 1)$

$y + 4 = -10x + 10$

$10x + y - 6 = 0$

ii $m_n = \frac{1}{10}$

$y - y_1 = m_n\left(x - x_1\right)$

$y + 4 = -\frac{1}{10}\left(x - 1\right)$

$10y + 40 = x - 1$

$x - 10y - 41 = 0$

d $y = x^2 - 8x + 7$

$y' = 2x - 8$

i $y'(3) = 2(3) - 8 = -2$

$m_t = -2$

$y - y_1 = m\left(x - x_1\right)$

$y + 8 = -2(x - 3)$

$y + 8 = -2x + 6$

$2x + y + 2 = 0$

ii $m_n = \frac{1}{2}$

$y - y_1 = m_n\left(x - x_1\right)$

$y + 8 = \frac{1}{2}\left(x - 3\right)$

$2y + 16 = x - 3$

$x - 2y - 19 = 0$

Question 7

$y = x^3 - 27x - 5$

$\frac{dy}{dx} = 3x^2 - 27$

$0 = 3x^2 - 27$

$3x^2 = 27$

$x^2 = 9$

$x = \pm 3$

Question 8

$y = x^3 + 1$

$y' = 3x^2$

Let $y' = 3$

$3 = 3x^2$

$x^2 = 1$

$x = \pm 1$

$y(1) = 1 + 1 = 2$

$y(-1) = -1 + 1 = 0$

Coordinates are $(-1, 0)$, $(1, 2)$

Question 9

$f(x) = x^2 + 4x - 12$

$f'(x) = 2x + 4$

Let $f'(x) = -6$

$-6 = 2x + 4$

$2x = -10$

$x = -5$

$f(-5) = (-5)^2 + 4(-5) - 12 = 25 - 20 - 12 = -7$

$P = (-5, -7)$

Question 10

$y = 4x^2 + 1$

$y' = 8x$

Tangent to x-axis, $y' = 0$

$0 = 8x$

$x = 0$

$y(0) = 1$

$P = (0, 1)$

Question 11

$y = 5x^2 - 3x$

$y' = 10x - 3$

Line $7x - y + 3 = 0$

$y = 7x + 3$

$m = 7$

$y' = 7$

$7 = 10x - 3$

$10x = 10$

$x = 1$

$y(1) = 5 - 3 = 2$

$Q = (1, 2)$

Question 12

$y = x^2 + 4x - 1$

$y' = 2x + 4$

Line $4x + 2y + 7 = 0$

$2y = -4x - 7$

$y = \frac{-4x-7}{2}$

$m = -2$

$m_n = \frac{1}{2}$

$y' = \frac{1}{2}$

$\frac{1}{2} = 2x + 4$

$2x = -\frac{7}{2}$

$x = -\frac{7}{4}$

$y(-\frac{7}{4}) = (-\frac{7}{4})^2 + 4(-\frac{7}{4}) - 1 = -\frac{79}{16}$

$S = (-\frac{7}{4}, -\frac{79}{16})$

Question 13

a $y = 3x^2 - 4$

$y' = 6x$

Let $y' = 6$

$6 = 6x$

$x = 1$

$y(1) = 3 - 4 = -1$

$A = (1, -1)$

b $y - y_1 = m\left(x - x_1\right)$

$y + 1 = 6(x - 1)$

$y + 1 = 6x - 6$

$6x - y - 7 = 0$

Question 14

$h = 3t^2 - 2t + 5$

$h' = 6t - 2$

$h'(2) = 6(2) - 2 = 10$

$h - h_1 = m\left(t - t_1\right)$

$h(2) = 3(2)^2 - 2(2) + 5 = 12 - 4 + 5 = 13$

$h - 13 = 10(t - 2)$

$h - 13 = 10t - 20$

$10t - h - 7 = 0$

Question 15

$f(x) = 2x^2 - 8x + 3$

$f'(x) = 4x - 8$

Line $4x - 2y + 1 = 0$

$2y = 4x + 1$

$$y = \frac{4x+1}{2}$$

$m = 2$

$f'(x) = 2$

$2 = 4x - 8$

$4x = 10$

$$x = \frac{5}{2}$$

$$f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 8\left(\frac{5}{2}\right) + 3 = -\frac{9}{2}$$

$$y - y_1 = m\left(x - x_1\right)$$

$$y + \frac{9}{2} = 2\left(x - \frac{5}{2}\right)$$

$$y + \frac{9}{2} = 2x - 5$$

$$2x - y - \frac{9}{2} - 5 = 0$$

$$2x - y - \frac{19}{2} = 0$$

$$4x - 2y - 19 = 0$$

Question 16

$$y=\frac{1}{x^3}$$
$$y=x^{-3}$$
$$y'=-3x^{-4}$$
$$y'(2)=-3(2)^{-4}=-\frac{3}{16}$$
$$y-y_1=m\left(x-x_1\right)$$
$$y-\frac{1}{8}=-\frac{3}{16}\left(x-2\right)$$
$$16y-2=-3x+6$$
$$3x+16y-8=0$$

Question 17

$$f(x)=6\sqrt{x}$$
$$f(x)=6x^{\frac{1}{2}}$$
$$f'(x)=3x^{-\frac{1}{2}}$$
$$f(9)=6(9)^{\frac{1}{2}}=18$$
$$f'(9)=3(9)^{-\frac{1}{2}}=1$$
$$y-y_1=m\left(x-x_1\right)$$
$$y-18=1\left(x-9\right)$$
$$y-18=x-9$$
$$x-y+9=0$$

Question 18

$$y = \frac{4}{x}$$

$$y = 4x^{-1}$$

$$y' = -4x^{-2}$$

$$y'(8) = -\frac{4}{64} = -\frac{1}{16}$$

$$y - y_1 = m\left(x - x_1\right)$$

$$y - \frac{1}{2} = -\frac{1}{16}\left(x - 8\right)$$

$$16y - 8 = -x + 8$$

$$x + 16y - 16 = 0$$

Question 19

$$y = \sqrt{x}$$

$$y = x^{\frac{1}{2}}$$

$$y' = \frac{1}{2}x^{-\frac{1}{2}}$$

$$y' = \frac{1}{6}$$

$$\frac{1}{6} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{1}{3} = \frac{1}{\sqrt{x}}$$

$$\sqrt{x} = 3$$

$$x = 9$$

$$y(9) = (9)^{\frac{1}{2}} = 3$$

$A = (9, 3)$

Exercise 8.07 Chain rule

Question 1

a

$$y=(x+3)^4$$

$$u=x+3 \qquad y=u^4$$

$$\frac{du}{dx}=1 \qquad \frac{dy}{du}=4u^3$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=1\times4(x+3)^3$$

$$=4(x+3)^3$$

b

$$y=(2x-1)^3$$

$$u=2x-1 \qquad y=u^3$$

$$\frac{du}{dx}=2 \qquad \frac{dy}{du}=3u^2$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=2\times3(2x-1)^2$$

$$=6(2x-1)^2$$

c

$$y=(5x^2-4)^7$$

$$u=5x^2-4 \qquad y=u^7$$

$$\frac{du}{dx}=10x \qquad \frac{dy}{du}=7u^6$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=10x\times7(5x^2-4)^6$$

$$=70x(5x^2-4)^6$$

d

$$y=(8x+3)^6$$

$u=8x+3 \qquad y=u^6$

$\dfrac{du}{dx}=8 \qquad \dfrac{dy}{du}=6u^5$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$
$$=8\times6(8x+3)^5$$
$$=48(8x+3)^5$$

e

$$y=(1-x)^5$$

$u=1-x \qquad y=u^5$

$\dfrac{du}{dx}=-1 \qquad \dfrac{dy}{du}=5u^4$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$
$$=-1\times5(1-x)^4$$
$$=-5(1-x)^4$$

f

$$y=3(5x+9)^9$$

$u=5x+9 \qquad y=3u^9$

$\dfrac{du}{dx}=5 \qquad \dfrac{dy}{du}=27u^8$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$
$$=5\times27(5x+9)^8$$
$$=135(5x+9)^8$$

g

$$y = 2(x-4)^2$$

$u = x - 4 \qquad y = 2u^2$

$\dfrac{du}{dx} = 1 \qquad \dfrac{dy}{du} = 4u$

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx} \times \frac{dy}{du} \\ &= 1 \times 4(x-4) \\ &= 4(x-4)\end{aligned}$$

h

$$y = \left(2x^3 + 3x\right)^4$$

$u = 2x^3 + 3x \qquad y = u^4$

$\dfrac{du}{dx} = 6x^2 + 3 \qquad \dfrac{dy}{du} = 4u^3$

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx} \times \frac{dy}{du} \\ &= \left(6x^2 + 3\right) \times 4\left(2x^3 + 3x\right)^3 \\ &= 4\left(6x^2 + 3\right)\left(2x^3 + 3x\right)^3\end{aligned}$$

i

$$y = \left(x^2 + 5x - 1\right)^8$$

$u = x^2 + 5x - 1 \qquad y = u^8$

$\dfrac{du}{dx} = 2x + 5 \qquad \dfrac{dy}{du} = 8u^7$

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx} \times \frac{dy}{du} \\ &= (2x+5) \times 8\left(x^2 + 5x - 1\right)^7 \\ &= 8(2x+5)\left(x^2 + 5x - 1\right)^7\end{aligned}$$

j

$$y=\left(x^6-2x^2+3\right)^8$$

$u=x^6-2x^2+3$ $\quad\quad$ $y=u^6$

$\dfrac{du}{dx}=6x^5-4x$ $\quad\quad$ $\dfrac{dy}{du}=6u^5$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=\left(6x^5-4x\right)\times 6\left(x^6-2x^2+3\right)^5$$

$$=6\left(6x^5-4x\right)\left(x^6-2x^2+3\right)^5$$

k

$$y=\left(3x-1\right)^{\frac{1}{2}}$$

$u=3x-1$ $\quad\quad$ $y=u^{\frac{1}{2}}$

$\dfrac{du}{dx}=3$ $\quad\quad$ $\dfrac{dy}{du}=\dfrac{1}{2}u^{-\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=3\times\frac{1}{2}\left(3x-1\right)^{-\frac{1}{2}}$$

$$=\frac{3}{2\sqrt{3x-1}}$$

l

$$y=\left(4-x\right)^{-2}$$

$u=4-x$ $\quad\quad$ $y=u^{-2}$

$\dfrac{du}{dx}=-1$ $\quad\quad$ $\dfrac{dy}{du}=-2u^{-3}$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=-1\times\left(-2\right)\left(4-x\right)^{-3}$$

$$=2\left(4-x\right)^{-3}$$

m

$$y=\left(x^2-9\right)^{-3}$$

$u=x^2-9 \qquad y=u^{-3}$

$\dfrac{du}{dx}=2x \qquad \dfrac{dy}{du}=-3u^{-4}$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=2x\times(-3)\left(x^2-9\right)^{-4}$$

$$=-6x\left(x^2-9\right)^{-4}$$

n

$$y=(5x+4)^{\frac{1}{3}}$$

$u=5x+4 \qquad y=u^{\frac{1}{3}}$

$\dfrac{du}{dx}=5 \qquad \dfrac{dy}{du}=\dfrac{1}{3}u^{-\frac{2}{3}}$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=5\times\frac{1}{3}(5x+4)^{-\frac{2}{3}}$$

$$=\frac{5}{3}(5x+4)^{-\frac{2}{3}}$$

o

$$y=\left(x^3-7x^2+x\right)^{\frac{3}{4}}$$

$u=x^3-7x^2+x \qquad y=u^{\frac{3}{4}}$

$\dfrac{du}{dx}=3x^2-14x+1 \qquad \dfrac{dy}{du}=\dfrac{3}{4}u^{-\frac{1}{4}}$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=\left(3x^2-14x+1\right)\times\frac{3}{4}\left(x^3-7x^2+x\right)^{-\frac{1}{4}}$$

$$=\frac{3}{4}\left(3x^2-14x+1\right)\left(x^3-7x^2+x\right)^{-\frac{1}{4}}$$

p

$$y = \sqrt{3x+4}$$
$$y = (3x+4)^{\frac{1}{2}}$$
$$u = 3x+4 \qquad\qquad y = u^{\frac{1}{2}}$$
$$\frac{du}{dx} = 3 \qquad\qquad \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$
$$= 3 \times \frac{1}{2}(3x+4)^{-\frac{1}{2}}$$
$$= \frac{3}{2\sqrt{3x+4}}$$

q

$$y = \frac{1}{5x-2}$$
$$y = (5x-2)^{-1}$$
$$u = 5x-2 \qquad\qquad y = u^{-1}$$
$$\frac{du}{dx} = 5 \qquad\qquad \frac{dy}{du} = -u^{-2}$$
$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$
$$= 5 \times (-1)(5x-2)^{-2}$$
$$= -5(5x-2)^{-2}$$
$$= \frac{-5}{(5x-2)^2}$$

r

$$y = \frac{1}{\left(x^2+1\right)^4}$$

$$y = \left(x^2+1\right)^{-4}$$

$u = x^2 + 1 \qquad y = u^{-4}$

$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = -4u^{-5}$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$= 2x \times (-4)\left(x^2+1\right)^{-5}$$

$$= -8x\left(x^2+1\right)^{-5}$$

$$= \frac{-8x}{\left(x^2+1\right)^5}$$

s

$$y = \sqrt[3]{(7-3x)^2}$$

$$y = (7-3x)^{\frac{2}{3}}$$

$u = 7 - 3x \qquad y = u^{\frac{2}{3}}$

$\frac{du}{dx} = -3 \qquad \frac{dy}{du} = \frac{2}{3}u^{-\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$= -3 \times \frac{2}{3}(7-3x)^{-\frac{1}{3}}$$

$$= -2(7-3x)^{-\frac{1}{3}}$$

$$= \frac{-2}{\sqrt[3]{7-3x}}$$

t

$$y=\frac{5}{\sqrt{4+x}}$$

$$y=\left(4+x\right)^{-\frac{1}{2}}$$

$$u=4+x \qquad y=5u^{-\frac{1}{2}}$$

$$\frac{du}{dx}=1 \qquad \frac{dy}{du}=-\frac{5}{2}u^{-\frac{3}{2}}$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=1\times\left(-\frac{5}{2}\right)\left(4+x\right)^{-\frac{3}{2}}$$

$$=\frac{-5}{2\sqrt{\left(4+x\right)^{3}}}$$

u

$$y=\frac{1}{2\sqrt{3x-1}}$$

$$y=\frac{1}{2}\left(3x-1\right)^{-\frac{1}{2}}$$

$$u=3x-1 \qquad y=\frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{du}{dx}=3 \qquad \frac{dy}{du}=-\frac{1}{4}u^{-\frac{3}{2}}$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=3\times\left(-\frac{1}{4}\right)\left(3x-1\right)^{-\frac{3}{2}}$$

$$=\frac{-3}{4\sqrt{\left(3x-1\right)^{3}}}$$

v

$$y=\frac{3}{4(2x+7)^9}$$

$$y=(2x+7)^{-9}$$

$$u=2x+7 \qquad y=\frac{3}{4}u^{-9}$$

$$\frac{du}{dx}=2 \qquad \frac{dy}{du}=-\frac{27}{4}u^{-10}$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=2\times\left(-\frac{27}{4}\right)(2x+7)^{-10}$$

$$=-\frac{27}{2}(2x+7)^{-10}$$

$$=\frac{-27}{2(2x+7)^{10}}$$

w

$$y=\frac{1}{x^4-3x^3+3x}$$

$$y=\left(x^4-3x^3+3x\right)^{-1}$$

$$u=x^4-3x^3+3x \qquad y=u^{-1}$$

$$\frac{du}{dx}=4x^3-9x^2+3 \qquad \frac{dy}{du}=-u^{-2}$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=\left(4x^3-9x^2+3\right)\times(-1)\left(x^4-3x^3+3x\right)^{-2}$$

$$=-\left(4x^3-9x^2+3\right)\left(x^4-3x^3+3x\right)^{-2}$$

$$=-\frac{4x^3-9x^2+3}{\left(x^4-3x^3+3x\right)^2}$$

x

$$y=\sqrt[3]{(4x+1)^4}$$

$$y=(4x+1)^{\frac{4}{3}}$$

$$u=4x+1 \qquad y=u^{\frac{4}{3}}$$

$$\frac{du}{dx}=4 \qquad \frac{dy}{du}=\frac{4}{3}u^{\frac{1}{3}}$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=4\times\frac{4}{3}(4x+1)^{\frac{1}{3}}$$

$$=\frac{16}{3}(4x+1)^{\frac{1}{3}}$$

$$=\frac{16}{3}\sqrt[3]{4x+1}$$

y

$$y=\frac{1}{\sqrt[4]{(7-x)^5}}$$

$$y=(7-x)^{-\frac{5}{4}}$$

$$u=7-x \qquad y=u^{-\frac{5}{4}}$$

$$\frac{du}{dx}=-1 \qquad \frac{dy}{du}=-\frac{5}{4}u^{-\frac{9}{4}}$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=-1\times\left(-\frac{5}{4}\right)(7-x)^{-\frac{9}{4}}$$

$$=\frac{5}{4\sqrt[4]{(7-x)^9}}$$

Question 2

$y=(3x-2)^3$

$u=3x-2$ $\qquad$ $y=u^3$

$\dfrac{du}{dx}=3$ $\qquad$ $\dfrac{dy}{du}=3u^2$

$\dfrac{dy}{dx}=\dfrac{du}{dx}\times\dfrac{dy}{du}$

$=3\times3(3x-2)^2$

$=9(3x-2)^2$

$\dfrac{dy}{dx}(1)=9(3(1)-2)^2=9$

$m=9$

Question 3

$f(x)=2\left(x^2-3\right)^5$

$y=2\left(x^2-3\right)^5$

$u=x^2-3$ $\qquad$ $y=2u^5$

$\dfrac{du}{dx}=2x$ $\qquad$ $\dfrac{dy}{du}=10u^4$

$\dfrac{dy}{dx}=\dfrac{du}{dx}\times\dfrac{dy}{du}$

$=2x\times10\left(x^2-3\right)^4$

$=20x\left(x^2-3\right)^4$

$f'(x)=20x\left(x^2-3\right)^4$

$f'(2)=20(2)\left((2)^2-3\right)^4=40$

Question 4

$y=\sqrt{x-3}$

$y=(x-3)^{\frac{1}{2}}$

$u=x-3 \qquad y=u^{\frac{1}{2}}$

$\frac{du}{dx}=1 \qquad \frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$

$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$

$=1\times\frac{1}{2}(x-3)^{-\frac{1}{2}}$

$=\frac{1}{2\sqrt{x-3}}$

$\frac{dy}{dx}=\frac{1}{2}$

$\frac{1}{2}=\frac{1}{2\sqrt{x-3}}$

$2\sqrt{x-3}=2$

$\sqrt{x-3}=1$

$x-3=1$

$x=4$

$y(4)=\sqrt{4-3}=1$

$N=(4,1)$

Question 5

$$f(x)=\frac{1}{4x-1}$$
$$y=\frac{1}{4x-1}$$
$$y=\left(4x-1\right)^{-1}$$
$$u=4x-1 \qquad y=u^{-1}$$
$$\frac{du}{dx}=4 \qquad \frac{dy}{du}=-u^{-2}$$
$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$
$$=4\times\left(-1\right)\left(4x-1\right)^{-2}$$
$$=-4\left(4x-1\right)^{-2}$$
$$=\frac{-4}{\left(4x-1\right)^{2}}$$
$$f'(x)=\frac{-4}{\left(4x-1\right)^{2}}$$
$$f'(x)=\frac{-4}{49}$$
$$\frac{-4}{49}=\frac{-4}{\left(4x-1\right)^{2}}$$
$$\left(4x-1\right)^{2}=49$$
$$4x-1=\pm 7$$
$$4x-1=7$$
$$4x=8$$
$$x=2$$
$$4x-1=-7$$
$$4x=-6$$
$$x=-\frac{3}{2}$$
$$x=-\frac{3}{2},\ 2$$

Question 6

$y=\left(2x+1\right)^4$

$u=2x+1$ $\qquad$ $y=u^4$

$\frac{du}{dx}=2$ $\qquad$ $\frac{dy}{du}=4u^3$

$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$

$=2\times4\left(2x+1\right)^3$

$=8\left(2x+1\right)^3$

$\frac{dy}{dx}(-1)=8\left(2(-1)+1\right)^3=-8$

$m=-8$

$y(-1)=\left(2(-1)+1\right)^4=1$

$y-y_1=m\left(x-x_1\right)$

$y-1=-8(x+1)$

$y-1=-8x-8$

$8x+y+7=0$

Question 7

$y=(2x-1)^8$

$u=2x-1$ $\qquad$ $y=u^8$

$\dfrac{du}{dx}=2$ $\qquad$ $\dfrac{dy}{du}=8u^7$

$\dfrac{dy}{dx}=\dfrac{du}{dx}\times\dfrac{dy}{du}$

$=2\times8(2x-1)^7$

$=16(2x-1)^7$

$\dfrac{dy}{dx}(1)=16(2(1)-1)^7=16$

$m=16$

$y-y_1=m(x-x_1)$

$y-1=16(x-1)$

$y-1=16x-16$

$16x-y-15=0$

Question 8

$y=(3x-4)^3$

$u=3x-4 \qquad y=u^3$

$\frac{du}{dx}=3 \qquad \frac{dy}{du}=3u^2$

$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$

$=3\times3(3x-4)^2$

$=9(3x-4)^2$

$\frac{dy}{dx}(1)=9(3(1)-4)^2=9$

$m_t=9$

$m_n=-\frac{1}{9}$

$y-y_1=m(x-x_1)$

$y+1=-\frac{1}{9}(x-1)$

$9y+9=-x+1$

$9y+8+x=0$

Question 9

$y=\left(x^2+1\right)^4$

$u=x^2+1$ $\qquad\qquad$ $y=u^4$

$\dfrac{du}{dx}=2x$ $\qquad\qquad$ $\dfrac{dy}{du}=4u^3$

$\dfrac{dy}{dx}=\dfrac{du}{dx}\times\dfrac{dy}{du}$

$=2x\times4\left(x^2+1\right)^3$

$=8x\left(x^2+1\right)^3$

$\dfrac{dy}{dx}(1)=8(1)\left((1)^2+1\right)^3=64$

$m_t=64$

$m_n=-\dfrac{1}{64}$

$y-y_1=m\left(x-x_1\right)$

$y-16=-\dfrac{1}{64}\left(x-1\right)$

$64y-1024=-x+1$

$x+64y-1025=0$

Question 10

a $f(x) = \dfrac{1}{2x+3} = (2x+3)^{-1}$

$f'(x) = -1(2x + 3)^{-2} \times 2$

$= \dfrac{-2}{(2x+3)^2}$

$f(-1) = \dfrac{1}{2(-1)+3} = \dfrac{1}{1} = 1$ (–1, 1)

$f'(-1) = \dfrac{-2}{\left[2(-1)+3\right]^2} = \dfrac{-2}{1^2} = -2$

Equation of tangent

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= -2(x - -1) \\ &= -2(x+1) \\ &= -2x - 2 \\ 2x + y + 1 &= 0 \end{aligned}$$

b Gradient of normal = $\dfrac{-1}{-2}$ = 2

Equation of normal

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{2}(x - -1) \\ 2(y-1) &= x + 1 \\ 2y - 2 &= x + 1 \\ 0 &= x - 2y + 3 \end{aligned}$$

Exercise 8.08 Product rule

Question 1

a

$$y = x^3(2x+3)$$

$u = x^3$ $\qquad$ $v = 2x+3$

$u' = 3x^2$ $\qquad$ $v' = 2$

$$\begin{aligned} y' &= u'v + v'u \\ &= 3x^2 \times (2x+3) + 2 \times x^3 \\ &= 2x^3 + 3x^2(2x+3) \\ &= x^2(2x + 3[2x+3]) \\ &= x^2(2x+6x+9) \\ &= x^2(8x+9) \\ &= 8x^3 + 9x^2 \end{aligned}$$

b

$$y = (3x-2)(2x+1)$$

$u = 3x-2$ $\qquad$ $v = 2x+1$

$u' = 3$ $\qquad$ $v' = 2$

$y' = u'v + v'u$

$= 3 \times (2x+1) + 2 \times (3x-2)$

$= 6x + 3 + 6x - 4$

$= 12x - 1$

c

$$y = 3x(5x+7)$$

$u = 3x$ $\qquad$ $v = 5x+7$

$u' = 3$ $\qquad$ $v' = 5$

$y' = u'v + v'u$

$= 3 \times (5x+7) + 5 \times 3x$

$= 15x + 21 + 15x$

$= 30x + 21$

d

$$y = 4x^4\left(3x^2 - 1\right)$$

$u = 4x^4 \qquad v = 3x^2 - 1$

$u' = 16x^3 \qquad v' = 6x$

$$\begin{aligned}
y' &= u'v + v'u \\
&= 16x^3 \times \left(3x^2 - 1\right) + 6x \times 4x^4 \\
&= 48x^5 - 16x^3 + 24x^5 \\
&= 72x^5 - 16x^3
\end{aligned}$$

e

$$y = 2x\left(3x^4 - 1\right)$$

$u = 2x \qquad v = 3x^4 - x$

$u' = 2 \qquad v' = 12x^3 - 1$

$$\begin{aligned}
y' &= u'v + v'u \\
&= 2 \times \left(3x^4 - x\right) + \left(12x^3 - 1\right) \times 2x \\
&= 6x^4 - 2x + 24x^4 - 2x \\
&= 30x^4 - 4x
\end{aligned}$$

f

$$y = x^2\left(x+1\right)^3$$

$u = x^2 \qquad v = \left(x+1\right)^3$

$u' = 2x \qquad v' = 3\left(x+1\right)^2$

$$\begin{aligned}
y' &= u'v + v'u \\
&= 2x \times \left(x+1\right)^3 + 3\left(x+1\right)^2 \times x^2 \\
&= \left(x+1\right)^2\left(2x\left(x+1\right) + 3x^2\right) \\
&= \left(x+1\right)^2\left(2x^2 + 2x + 3x^2\right) \\
&= \left(x+1\right)^2\left(5x^2 + 2x\right) \\
&= x\left(x+1\right)^2\left(5x+2\right)
\end{aligned}$$

g

$y = 4x(3x-2)^5$

$u = 4x \qquad\qquad = (3x-2)^5$

$u' = 4 \qquad\qquad v' = 15(3x-2)^4$

$$
\begin{aligned}
y' &= u'v + v'u \\
&= 4 \times (3x-2)^5 + 15(3x-2)^4 \times 4x \\
&= 4(3x-2)^5 + 60x(3x-2)^4 \\
&= (3x-2)^4\left(4(3x-2)+60x\right) \\
&= (3x-2)^4(12x-8+60x) \\
&= (72x-8)(3x-2)^4 \\
&= 8(9x-1)(3x-2)^4
\end{aligned}
$$

h

$y = 3x^4(4-x)^3$

$u = 3x^4 \qquad\qquad v = (4-x)^3$

$u' = 12x^3 \qquad\qquad v' = -3(4-x)^2$

$$
\begin{aligned}
y' &= u'v + v'u \\
&= 12x^3 \times (4-x)^3 + -3(4-x)^2 \times 3x^4 \\
&= 12x^3(4-x)^3 - 9x^4(4-x)^2 \\
&= 3x^3(4-x)^2\left(4(4-x)-3x\right) \\
&= 3x^3(4-x)^2(16-4x-3x) \\
&= 3x^3(16-7x)(4-x)^2
\end{aligned}
$$

i

$$y=(x+1)(2x+5)^4$$

$u=x+1 \qquad v=(2x+5)^4$

$u'=1 \qquad v'=8(2x+5)^3$

$$\begin{aligned}
y' &= u'v+v'u \\
&= 1\times(2x+5)^4+8(2x+5)^3\times(x+1) \\
&= (2x+5)^4+8(x+1)(2x+5)^3 \\
&= (2x+5)^3([2x+5]+8[x+1]) \\
&= (2x+5)^3(2x+5+8x+8) \\
&= (10x+13)(2x+5)^3
\end{aligned}$$

Question 2

$$y=2x(3x-2)^4$$

$u=2x \qquad v=(3x-2)^4$

$u'=2 \qquad v'=12(3x-2)^3$

$$\begin{aligned}
y' &= u'v+v'u \\
&= 2\times(3x-2)^4+12(3x-2)^3\times 2x \\
&= 2(3x-2)^4+24x(3x-2)^3 \\
&= (3x-2)^3(2(3x-2)+24x) \\
&= (3x-2)^4(6x-4+24x) \\
&= (30x-4)(3x-2)^4
\end{aligned}$$

$$y'(1)=(30(1)-4)(3(1)-2)^4=26$$

Question 3

$f(x)=(2x+3)(3x-1)^5$

$y=(2x+3)(3x-1)^4$

$u=2x+3 \qquad v=(3x-1)^5$

$u'=2 \qquad v'=15(3x-1)^4$

$y'=u'v+v'u$

$=2\times(3x-1)^5+15(3x-1)^4\times(2x+3)$

$=(3x-1)^4\left(2\,[3x-1]+15[2x+3]\right)$

$=(3x-1)^4(6x-2+30x+45)$

$=(3x-1)^4(36x+43)$

$f'(x)=(3x-1)^4(36x+43)$

$f'(1)=(3(1)-1)^4(36(1)+43)=1264$

Question 4

$y=x\sqrt{2x+5}$

$y=x(2x+5)^{\frac{1}{2}}$

$u=x \qquad v=(2x+5)^{\frac{1}{2}}$

$u'=1 \qquad v'=\dfrac{1}{2}(2x+5)^{-\frac{1}{2}}\times 2=(2x+5)^{-\frac{1}{2}}$

$y'=u'v+v'u$

$=1\times(2x+5)^{\frac{1}{2}}+(2x+5)^{-\frac{1}{2}}\times x$

$=\sqrt{2x+5}+\dfrac{x}{\sqrt{2x+5}}$

$y'(1)=\dfrac{1}{\sqrt{2(1)+5}}+\sqrt{2(1)+5}$

$=\dfrac{1}{\sqrt{7}}+\sqrt{7}$

$=\dfrac{\sqrt{7}}{7}+\dfrac{7\sqrt{7}}{7}$

$=\dfrac{8\sqrt{7}}{7}$

Question 5

$x=(2t-5)(t+1)^3$

$u=2t-5 \qquad v=(t+1)^3$

$u'=2 \qquad v'=3(t+1)^2$

$x'=u'v+v'u$

$=2\times(t+1)^3+3(t+1)^2\times(2t-5)$

$=(t+1)^2(2[t+1]+3[2t-5])$

$=(t+1)^2(2t+2+6t-15)$

$=(t+1)^2(8t-13)$

$x'(3)=((3)+1)^2(8(3)-13)=176$

Question 6

$y=x^2(2x-1)^4$

$u=x^2 \qquad v=(2x-1)^4$

$u'=2x \qquad v'=8(2x-1)^3$

$y'=u'v+v'u$

$=2x\times(2x-1)^4+8(2x-1)^3\times x^2$

$=(2x-1)^3(2x(2x-1)+8x^2)$

$=(2x-1)^3(4x^2-2x+8x^2)$

$=(2x-1)^3(12x^2-2x)$

$=2x(2x-1)^3(6x-1)$

$y'(1)=2(1)(2(1)-1)^3(6(1)-1)=10$

$m=10$

$y-y_1=m(x-x_1)$

$y-1=10(x-1)$

$y-1=10x-10$

$10x-y-9=0$

Question 7

$h = (t+1)^2 (t-1)^7$

$u = (t+1)^2 \qquad\qquad v = (t-1)^7$

$u' = 2(t+1) \qquad\qquad v' = 7(t-1)^6$

$h' = u'v + v'u$

$= 2(t+1)\times(t-1)^7 + 7(t-1)^6 \times (t+1)^2$

$= (t+1)(t-1)^6 \left(2[t-1] + 7[t+1]\right)$

$= (t-1)^6 (t+1)(2t-2+7t+7)$

$= (t-1)^6 (t+1)(9t+5)$

$h'(2) = ((2)-1)^6 ((2)+1)(9(2)+5) = 69$

$m = 69$

$h - h_1 = m(t - t_1)$

$h - 9 = 69(t - 2)$

$h - 9 = 69t - 138$

$69t - h - 129 = 0$

Question 8

$y = 2x(x+3)^2$

$u = 2x \qquad v = (x+3)^2$

$u' = 2 \qquad v' = 2(x+3)$

$y' = u'v + v'u$

$= 2 \times (x+3)^2 + 2(x+3) \times 2x$

$= 2(x+3)^2 + 4x(x+3)$

$= 2x^2 + 12x + 18 + 4x^2 + 12x$

$= 6x^2 + 24x + 18$

$y'(1) = 14$

$14 = 6x^2 + 24x + 18$

$6x^2 + 24x + 4 = 0$

$3x^2 + 12x + 2 = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-12 \pm \sqrt{12^2 - 4 \times 3 \times 2}}{2 \times 3}$

$= \dfrac{-12 \pm \sqrt{120}}{6}$

$= \dfrac{-12 \pm 2\sqrt{30}}{6}$

$= \dfrac{-6 \pm \sqrt{30}}{3}$

Question 9

$f(x) = (4x-1)(3x+2)^2$

$y = (4x-1)(3x+2)^2$

$u = 4x-1$ $\quad\quad$ $v = (3x+2)^2$

$u' = 4$ $\quad\quad$ $v' = 6(3x+2)$

$y' = u'v + v'u$

$= 4\times(3x+2)^2 + 6(3x+2)\times(4x-1)$

$= 4(3x+2)^2 + 6(4x-1)(3x+2)$

$= 2(3x+2)\left[2(3x+2)+3(4x-1)\right]$

$= 2(3x+2)(6x+4+12x-3)$

$= 2(3x+2)(18x+1)$

$f(-1) = (4(-1)-1)(3(-1)+2)^2 = -5$

$f'(-1) = 2(3(-1)+2)(18(-1)+1) = 34$

$m = 34$

$y - y_1 = m(x - x_1)$

$y + 5 = 34(x + 1)$

$y + 5 = 34x + 34$

$34x - y + 29 = 0$

Exercise 8.09 Quotient rule

Question 1

a

$$y = \frac{1}{2x-1}$$

$u = 1 \qquad v = 2x-1$

$u' = 0 \qquad v' = 2$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{0\times(2x-1) - 2\times 1}{(2x-1)^2}$$

$$= \frac{-2}{(2x-1)^2}$$

b

$$y = \frac{3x}{x+5}$$

$u = 3x \qquad v = x+5$

$u' = 3 \qquad v' = 1$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{3\times(x+5) - 1\times 3x}{(x+5)^2}$$

$$= \frac{3x + 15 - 3x}{(x+5)^2}$$

$$= \frac{15}{(x+5)^2}$$

c

$$y=\frac{x^3}{x^2-4}$$

$u=x^3 \qquad v=x^2-4$

$u'=3x^2 \qquad v'=2x$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{3x^2\times\left(x^2-4\right)-2x\times x^3}{\left(x^2-4\right)^2}$$

$$=\frac{3x^4-12x^2-2x^4}{\left(x^2-4\right)^2}$$

$$=\frac{x^4-12x^2}{\left(x^2-4\right)^2}$$

$$=\frac{x^2\left(x^2-12\right)}{\left(x^2-4\right)^2}$$

d

$$y=\frac{x-3}{5x+1}$$

$u=x-3 \qquad v=5x+1$

$u'=1 \qquad v'=5$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{1\times\left(5x+1\right)-5\times\left(x-3\right)}{\left(5x+1\right)^2}$$

$$=\frac{5x+1-5x+15}{\left(5x+1\right)^2}$$

$$=\frac{16}{\left(5x+1\right)^2}$$

e

$$y=\frac{x-7}{x^2}$$

$u=x-7 \qquad v=x^2$

$u'=1 \qquad v'=2x$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{1\times x^2-2x\times(x-7)}{\left(x^2\right)^2}$$

$$=\frac{x^2-2x^2+14x}{x^4}$$

$$=\frac{14x-x^2}{x^4}$$

$$=\frac{14-x}{x^3}$$

f

$$y=\frac{5x+4}{x+3}$$

$u=5x+4 \qquad v=x+3$

$u'=5 \qquad v'=1$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{5\times(x+3)-1\times(5x+4)}{(x+3)^2}$$

$$=\frac{5x+15-5x-4}{(x+3)^2}$$

$$=\frac{11}{(x+3)^2}$$

g

$$y = \frac{x}{2x^2 - 1}$$

$u = x \qquad v = 2x^2 - 1$

$u' = 1 \qquad v' = 4x$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{1 \times \left(2x^2 - 1\right) - 4x \times x}{\left(2x^2 - 1\right)^2}$$

$$= \frac{2x^2 - 1 - 4x^2}{\left(2x^2 - 1\right)^2}$$

$$= \frac{-1 - 2x^2}{\left(2x^2 - 1\right)^2}$$

h

$$y = \frac{x + 4}{x - 2}$$

$u = x + 4 \qquad v = x - 2$

$u' = 1 \qquad v' = 1$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{1 \times (x - 2) - 1 \times (x + 4)}{(x - 2)^2}$$

$$= \frac{x - 2 - x - 4}{(x - 2)^2}$$

$$= \frac{-6}{(x - 2)^2}$$

i

$$y=\frac{2x+7}{4x-3}$$

$u=2x+7$ $\quad$ $v=4x-3$

$u'=2$ $\quad$ $v'=4$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{2\times(4x-3)-4\times(2x+7)}{(4x-3)^2}$$

$$=\frac{8x-6-8x-28}{(4x-3)^2}$$

$$=\frac{-34}{(4x-3)^2}$$

j

$$y=\frac{x+5}{3x+1}$$

$u=x+5$ $\quad$ $v=3x+1$

$u'=1$ $\quad$ $v'=3$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{1\times(3x+1)-3\times(x+5)}{(3x+1)^2}$$

$$=\frac{3x+1-3x-15}{(3x+1)^2}$$

$$=\frac{-14}{(3x+1)^2}$$

k

$$y=\frac{x+1}{3x^2-7}$$

$u=x+1$ $\qquad$ $v=3x^2-7$

$u'=1$ $\qquad$ $v'=6x$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{1\times\left(3x^2-7\right)-6x\times\left(x+1\right)}{\left(3x^2-7\right)^2}$$

$$=\frac{3x^2-7-6x^2-6x}{\left(3x^2-7\right)^2}$$

$$=\frac{-3x^2-6x-7}{\left(3x^2-7\right)^2}$$

l

$$y=\frac{2x^2}{2x-3}$$

$u=2x^2$ $\qquad$ $v=2x-3$

$u'=4x$ $\qquad$ $v'=2$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{4x\times\left(2x-3\right)-2\times 2x^2}{\left(2x-3\right)^2}$$

$$=\frac{8x^2-12x-4x^2}{\left(2x-3\right)^2}$$

$$=\frac{4x^2-12x}{\left(2x-3\right)^2}$$

$$=\frac{4x\left(x-3\right)}{\left(2x-3\right)^2}$$

m

$$y = \frac{x^2+4}{x^2-5}$$

$u = x^2 + 4 \qquad\qquad v = x^2 - 5$

$u' = 2x \qquad\qquad v' = 2x$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{2x \times \left(x^2-5\right) - 2x \times \left(x^2+4\right)}{\left(x^2-5\right)^2}$$

$$= \frac{2x\left(x^2-5-x^2-4\right)}{\left(x^2-5\right)^2}$$

$$= \frac{-18x}{\left(x^2-5\right)^2}$$

n

$$y = \frac{x^3}{x+4}$$

$u = x^3 \qquad\qquad v = x + 4$

$u' = 3x^2 \qquad\qquad v' = 1$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{3x^2 \times \left(x+4\right) - 1 \times x^3}{\left(x+4\right)^2}$$

$$= \frac{3x^3 + 12x^2 - x^3}{\left(x+4\right)^2}$$

$$= \frac{2x^3 + 12x^2}{\left(x+4\right)^2}$$

$$= \frac{2x^2\left(x+6\right)}{\left(x+4\right)^2}$$

o

$$y=\frac{x^3+2x-1}{x+3}$$

$u=x^3+2x-1 \qquad v=x+3$

$u'=3x^2+2 \qquad v'=1$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{\left(3x^2+2\right)\times\left(x+3\right)-1\times\left(x^3+2x-1\right)}{\left(x+3\right)^2}$$

$$=\frac{3x^3+9x^2+2x+6-x^3-2x+1}{\left(x+3\right)^2}$$

$$=\frac{2x^3+9x^2+7}{\left(x+3\right)^2}$$

p

$$y=\frac{x^2-2x-1}{3x+4}$$

$u=x^2-2x-1 \qquad v=3x+4$

$u'=2x-2 \qquad v'=3$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{\left(2x-2\right)\times\left(3x+4\right)-3\times\left(x^2-2x-1\right)}{\left(3x+4\right)^2}$$

$$=\frac{6x^2+2x-8-3x^2+6x+3}{\left(3x+4\right)^2}$$

$$=\frac{3x^2+8x-5}{\left(3x+4\right)^2}$$

q

$$y=\frac{2x}{(x+5)^{\frac{1}{2}}}$$

$u=2x$ $\qquad$ $v=(x+5)^{\frac{1}{2}}$

$u'=2$ $\qquad$ $v'=\dfrac{1}{2\sqrt{x+5}}$

Determining v': let $y=(x+5)^{\frac{1}{2}}$

$u=x+5$ $\qquad$ $y=u^{\frac{1}{2}}$

$\dfrac{du}{dx}=1$ $\qquad$ $\dfrac{dy}{du}=\dfrac{1}{2}u^{-\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=1\times\frac{1}{2}(x+5)^{-\frac{1}{2}}$$

$$=\frac{1}{2\sqrt{x+5}}$$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{2\times(x+5)^{\frac{1}{2}}-\frac{1}{2\sqrt{x+5}}\times 2x}{(x+5)}$$

$$=\frac{2(x+5)^{\frac{1}{2}}-x(x+5)^{-\frac{1}{2}}}{x+5}$$

Can further be simplified by

$$=\frac{\frac{2\times(x+5)}{(x+5)^{\frac{1}{2}}}-\frac{x}{(x+5)^{\frac{1}{2}}}}{(x+5)}$$

$$=\frac{\frac{2x+10-x}{(x+5)^{\frac{1}{2}}}}{(x+5)}$$

$$=\frac{x+10}{(x+5)^{\frac{3}{2}}}$$

r

$$y=\frac{x-1}{(7x+2)^4}$$

$u=x-1$ $\quad$ $v=(7x+2)^4$

$u'=1$ $\quad$ $v'=28(7x+2)^3$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{1\times(7x+2)^4-28(7x+2)^3\times(x-1)}{\left((7x+2)^4\right)^2}$$

$$=\frac{(7x+2-28x+28)(7x+2)^3}{(7x+2)^8}$$

$$=\frac{30-21x}{(7x+2)^5}$$

$$=\frac{3(10-7x)}{(7x+2)^5}$$

s

$$y=\frac{3x+1}{\sqrt{x+1}}$$

$$y=\frac{3x+1}{(x+1)^{\frac{1}{2}}}$$

$$u=3x+1 \qquad v=(x+1)^{\frac{1}{2}}$$

$$u'=3 \qquad v'=\frac{1}{2\sqrt{x+1}}$$

Determining v': let $y=(x+1)^{\frac{1}{2}}$

$$u=x+1 \qquad y=u^{\frac{1}{2}}$$

$$\frac{du}{dx}=1 \qquad \frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=1\times\frac{1}{2}(x+1)^{-\frac{1}{2}}$$

$$=\frac{1}{2\sqrt{x+1}}$$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{3\times(x+1)^{\frac{1}{2}}-\frac{1}{2\sqrt{x+1}}\times(3x+1)}{(x+1)}$$

$$=\frac{\frac{6\times(x+1)}{2(x+1)^{\frac{1}{2}}}-\frac{3x+1}{2(x+1)^{\frac{1}{2}}}}{(x+1)}$$

$$=\frac{\frac{6x+6-3x-1}{2(x+1)^{\frac{1}{2}}}}{(x+1)}$$

$$=\frac{3x+5}{2(x+1)^{\frac{3}{2}}}$$

t

$$y=\frac{\sqrt{x-1}}{2x-3}$$

$$y=\frac{(x-1)^{\frac{1}{2}}}{2x-3}$$

$$u=(x-1)^{\frac{1}{2}} \qquad v=2x-3$$

$$u'=\frac{1}{2(x-1)^{\frac{1}{2}}} \qquad v'=2$$

Determining u' : let $y=(x-1)^{\frac{1}{2}}$

$$u=x-1 \qquad y=u^{\frac{1}{2}}$$

$$\frac{du}{dx}=1 \qquad \frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=1\times\frac{1}{2}(x-1)^{-\frac{1}{2}}$$

$$=\frac{1}{2\sqrt{x-1}}$$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{\frac{1}{2(x-1)^{\frac{1}{2}}}\times(2x-3)-2\times(x-1)^{\frac{1}{2}}}{(2x-3)^2}$$

$$=\frac{\frac{2x-3}{2(x-1)^{\frac{1}{2}}}-\frac{4x-4}{2(x-1)^{\frac{1}{2}}}}{(2x-3)^2}$$

$$=\frac{1-2x}{2(2x-3)^2(x-1)^{\frac{1}{2}}}$$

$$=\frac{1-2x}{2(2x-3)^2\sqrt{x-1}}$$

Question 2

$$y = \frac{2x}{3x+1}$$

$u = 2x \qquad v = 3x+1$

$u' = 2 \qquad v' = 3$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{2\times(3x+1) - 3\times 2x}{(3x+1)^2}$$

$$= \frac{6x+2-6x}{(3x+1)^2}$$

$$= \frac{2}{(3x+1)^2}$$

$$y'(1) = \frac{2}{(3(1)+1)^2} = \frac{2}{16} = \frac{1}{8}$$

Question 3

$$f(x) = \frac{4x+5}{2x-1}$$

$$y = \frac{4x+5}{2x-1}$$

$u = 4x+5 \qquad v = 2x-1$

$u' = 4 \qquad v' = 2$

$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{4\times(2x-1) - 2\times(4x+5)}{(2x-1)^2}$$

$$= \frac{8x-4-8x-10}{(2x-1)^2}$$

$$= \frac{-14}{(2x-1)^2}$$

$$f'(x) = \frac{-14}{(2x-1)^2}$$

$$f'(2) = \frac{-14}{(2(2)-1)^2} = -\frac{14}{9}$$

Question 4

$$y=\frac{4x-1}{2x-1}$$

$u=4x-1$ $\quad\quad$ $v=2x-1$

$u'=4$ $\quad\quad$ $v'=2$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{4\times(2x-1)-2\times(4x-1)}{(2x-1)^2}$$

$$=\frac{8x-4-8x+2}{(2x-1)^2}$$

$$=\frac{-2}{(2x-1)^2}$$

$$y'(x)=-2$$

$$-2=\frac{-2}{(2x-1)^2}$$

$$1=(2x-1)^2$$

$$2x-1=\pm1$$

$$2x=1\pm1$$

$$x=0,\ 1$$

Question 5

$$f(x)=\frac{2x}{x+3}$$

$$y=\frac{2x}{x+3}$$

$u=2x \qquad\qquad v=x+3$

$u'=2 \qquad\qquad v'=1$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{2\times(x+3)-1\times 2x}{(x+3)^2}$$

$$=\frac{2x+6-2x}{(x+3)^2}$$

$$=\frac{6}{(x+3)^2}$$

$$f'(x)=\frac{6}{(x+3)^2}$$

$$f'(x)=\frac{1}{6}$$

$$\frac{1}{6}=\frac{6}{(x+3)^2}$$

$$(x+3)^2=36$$

$$x+3=\pm 6$$

$$x=-3\pm 6$$

$$x=-9,\ 3$$

Question 6

$$y=\frac{x}{x+2}$$

$u=x \qquad\qquad v=x+2$

$u'=1 \qquad\qquad v'=1$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{1\times(x+2)-1\times(x)}{(x+2)^2}$$

$$=\frac{2}{(x-2)^2}$$

$$y'(4)=\frac{2}{(4-2)^2}=\frac{1}{18}$$

$$m=\frac{1}{18}$$

$$y-y_1=m(x-x_1)$$

$$y-\frac{2}{3}=\frac{1}{18}(x-4)$$

$$18y-12=x-4$$

$$x-18y+8=0$$

Question 7

$$y=\frac{x^2-1}{x+3}$$

$u=x^2-1 \qquad v=x+3$

$u'=2x \qquad v'=1$

$$y'=\frac{u'v-v'u}{v^2}$$

$$=\frac{2x\times(x+3)-1\times(x^2-1)}{(x+3)^2}$$

$$=\frac{2x^2+6x-x^2+1}{(x+3)^2}$$

$$=\frac{x^2+6x+1}{(x+3)^2}$$

$$y'(2)=\frac{(2)^2+6(2)+1}{((2)+3)^2}=\frac{17}{25}$$

$$m=\frac{17}{25}$$

$$y(2)=\frac{(2)^2-1}{(2)+3}=\frac{3}{5}$$

$$y-y_1=m(x-x_1)$$

$$y-\frac{3}{5}=\frac{17}{25}(x-2)$$

$$25y-15=17x-34$$

$$17x-25y-19=0$$

Exercise 8.10 Rates of change

Question 1

a

$$h = 20t - 4t^2$$
$$\frac{dh}{dt} = 20 - 8t$$

b

$$D = 5t^3 + 2t^2 + 1$$
$$\frac{dD}{dt} = 15t^2 + 4t$$

c

$$A = 16x - 2x^2$$
$$\frac{dA}{dx} = 16 - 4x$$

d

$$x = 3t^5 - t^4 + 2t - 3$$
$$\frac{dx}{dt} = 15t^4 - 4t^3 + 2$$

e

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dr} = 4\pi r^2$$

f

$$S = 2\pi r + \frac{50}{r^2}$$
$$S = 2\pi r + 50r^{-2}$$
$$\frac{dS}{dr} = 2\pi + 50 \times (-2) r^{-2-1}$$
$$\frac{dS}{dr} = 2\pi - 100r^{-3}$$
$$\frac{dS}{dr} = 2\pi - \frac{100}{r^3}$$

g

$$D=\sqrt{x^2-4}$$

$$u=x^2-4 \qquad D=u^{\frac{1}{2}}$$

$$\frac{du}{dx}=2x \qquad \frac{dD}{du}=\frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dD}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=2x\times\frac{1}{2}\left(x^2-4\right)^{-\frac{1}{2}}$$

$$=\frac{x}{\sqrt{x^2-4}}$$

h

$$S=800r+\frac{400}{r}$$

$$S=800r+400r^{-1}$$

$$\frac{dS}{dr}=800+400\times(-1)r^{-1-1}$$

$$\frac{dS}{dr}=800-400r^{-2}$$

$$\frac{dS}{dr}=800-\frac{400}{r^2}$$

Question 2

a $h(3) = 3^3 - 7(3) + 5 = 27 - 21 + 5 = 11$

$h(4) = 4^3 - 7(4) + 5 = 64 - 28 + 5 = 41$

Average rate of change $=\frac{41-11}{4-3}=\frac{30}{1}=30$

b

$$h=t^3-7t+5$$

$$\frac{dh}{dt}=3t^2-7$$

$$\frac{dh}{dt}(3)=3(3)^2-7=20$$

Question 3

$$V = t^2 + 3t$$
$$\frac{dV}{dt} = 2t + 3$$
$$\frac{dV}{dt}(5) = 2(5) + 3 = 13 \text{ L/s}$$

Question 4

a **i** $M(1) = 1 – 2(1)^2 + 100 = 99$

$M(3) = 3 – 2(3)^2 + 100 = 85$

Average rate of change $= \frac{85-99}{3-1} = \frac{-14}{2} = –7$ g/min

ii $M(2) = 2 – 2(2)^2 + 100 = 94$

$M(5) = 5 – 2(5)^2 + 100 = 55$

Average rate of change $= \frac{55-94}{5-2} = \frac{-39}{3} = –13$ g/min

b

$$M = t - 2t^2 + 100$$
$$\frac{dM}{dt} = 1 - 4t$$
$$\frac{dM}{dt}(5) = 1 - 4(5) = -19$$

The ice block will be melting at 19 g/min.

Question 5

$$S = t^3 - t^2 + 5t + 1$$
$$\frac{dS}{dt} = 3t^2 - 2t + 5$$
$$\frac{dS}{dt}(8) = 3(8)^2 - 2(8) + 5 = 181 \text{ cm}^2\text{s}^{-1}$$

Question 6

$A = 4t^2 + t$

$\frac{dA}{dt} = 8t + 1$

$\frac{dA}{dt}(5) = 8(5) + 1 = 41$

Question 7

$d = 10t^2 + 5t + 11$

a **i** $d_0 = 11$ km

ii $d(3) = 10(3)^2 + 5(3) + 11 = 116$ km

iii $d(5) = 10(5)^2 + 5(5) + 11 = 286$ km

b $v = \frac{dd}{dt} = 20t + 5$

i $v(3) = 20(3) + 5 = 65$ km/h

ii $v(5) = 20(5) + 5 = 105$ km/h

Question 8

$P = 100V^{-1}$

$\frac{dP}{dV} = -100V^{-2}$

$\frac{dP}{dV}(20) = -100(20)^{-2}$

$= \frac{-100}{400}$

$= -\frac{1}{4}$

Question 9

a $v = \dfrac{dx}{dt} = 3t^2 - 9$

$v(3) = 3(3^2) - 9 = 18$ cm s^{-1}

b $a = \dfrac{dv}{dt} = 6t$

$a(2) = 6(2) = 12$ cm s^{-2}

c $x(0) = 0^3 - 9(0) = 0$

So when $t = 0$ (initially), $x = 0$ (at the origin).

Other times at the origin:

$0 = t^3 - 9t$

$0 = t(t^2 - 9)$

$0 = t(t + 3)(t - 3)$

$t = 0, -3, 3$

$t = 3$ s $(t > 0)$

d $6t = 30$

$t = 5$ s

Question 10

a $v = \dfrac{ds}{dt} = 4t - 8$

$v(0) = 4(0) - 8 = -8$ m s^{-1}

b $a = \dfrac{dv}{dt} = 4$, a constant

Acceleration is 4 m s^{-2}.

c $s(5) = 2(5)^2 - 8(5) + 3 = 13$ m

d $v = 0$

$4t - 8 = 0$

$4t = 8$

$t = 2$ s

e $s(2) = 2(2)^2 - 8(2) + 3$

$= -5$ m

Exercise 8.11 Related rates of change

Question 1

a

$$y = x^4 \qquad \frac{dx}{dt} = 2$$

$$\frac{dy}{dx} = 4x^3$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= 4x^3 \times 2 \ = 8x^3$$

b

$$y = 3x^3 + 7 \qquad \frac{dx}{dt} = 6$$

$$\frac{dy}{dx} = 9x^2$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= 9x^2 \times 6 = 54x^2$$

c

$$y = x^2 - x - 2 \qquad \frac{dx}{dt} = -3$$

$$\frac{dy}{dx} = 2x - 1$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= (2x - 1) \times -3$$

$$= -6x + 3$$

Question 2

a

$$y = 2x^3 + 3x - 7 \qquad \frac{dx}{dt} = 3$$

$$\frac{dy}{dx} = 6x^2 + 3$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= \left(6x^2 + 3\right) \times 3$$

$$= 18x^2 + 9$$

$$\frac{dy}{dt}(4) = 18(4)^2 + 9 = 297$$

b

$$y = (3x+1)^3 \qquad \frac{dx}{dt} = -4$$

$$\frac{dy}{dx} = 9(3x+1)^2$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= 9(3x+1)^2 \times -4$$

$$= -36(3x+1)^2$$

$$\frac{dy}{dt}(4) = -36(3(4)+1)^2 = -6084$$

c

$$y = (5-x)^5 \qquad \frac{dx}{dt} = 4$$

$$\frac{dy}{dx} = -5(5-x)^4$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$= -5(5-x)^4 \times 4$$

$$= -20(5-x)^4$$

$$\frac{dy}{dt}(4) = -20(5-4)^4 = -20$$

Question 3

$y = x^3 + 5x - 4 \qquad \dfrac{dy}{dt} = 6$

$\dfrac{dy}{dx} = 3x^2 + 5$

$\dfrac{dx}{dy} = \dfrac{1}{3x^2 + 5}$

$\dfrac{dx}{dt} = \dfrac{dx}{dy} \times \dfrac{dy}{dt}$

$= \left(\dfrac{1}{3x^2 + 5}\right) \times 6$

$= \dfrac{6}{3x^2 + 5}$

$\dfrac{dx}{dt}(2) = \dfrac{6}{3(2)^2 + 5} = \dfrac{6}{17}$

Question 4

$y = x^2 + x \qquad \dfrac{dy}{dt} = -5$

$\dfrac{dy}{dx} = 2x + 1$

$\dfrac{dx}{dy} = \dfrac{1}{2x + 1}$

$\dfrac{dx}{dt} = \dfrac{dx}{dy} \times \dfrac{dy}{dt}$

$= \left(\dfrac{1}{2x + 1}\right) \times -5$

$= \dfrac{-5}{2x + 1}$

$\dfrac{dx}{dt}(4) = \dfrac{-5}{2(4) + 1} = -\dfrac{5}{9}$

Question 5

$x=3t^2-t \qquad \frac{dy}{du}=6$

$\frac{dx}{dt}=6t-1$

$\frac{dx}{du}=\frac{dt}{du}\times\frac{dx}{dt}$

$=6\times(6t-1)$

$=36t-6$

$\frac{dx}{dt}(12)=36(12)-6=426$

Question 6

$V=5\pi r^2 \qquad \frac{dr}{dt}=2$

$\frac{dV}{dr}=10\pi r$

$\frac{dV}{dt}=\frac{dr}{dt}\times\frac{dV}{dr}$

$=2\times(10\pi r)$

$=20\pi r$

$\frac{dV}{dt}(4.6)=20\pi(4.6)=289$

Question 7

$A=16x-2x^2 \qquad \frac{dx}{dt}=11$

$\frac{dA}{dx}=16-4x$

$\frac{dA}{dt}=\frac{dx}{dt}\times\frac{dA}{dx}$

$=11\times(16-4x)$

$=176-44x$

$\frac{dA}{dt}(3)=176-44(3)=44$

Question 8

$V = x^3 + 4x^2 - 3x + 4 \qquad \frac{dV}{dt} = 10$

$\frac{dV}{dx} = 3x^2 + 8x - 3$

$\frac{dx}{dV} = \frac{1}{3x^2 + 8x - 3}$

$\frac{dx}{dt} = \frac{dV}{dt} \times \frac{dx}{dV}$

$= 10\left(\frac{1}{3x^2 + 8x - 3}\right)$

$= \frac{10}{3x^2 + 8x - 3}$

$\frac{dx}{dt}(1) = \frac{10}{3(1)^2 + 8(1) - 3} = \frac{10}{8} = \frac{5}{4}$

Question 9

$V = s^3 \qquad \frac{ds}{dt} = 0.12$

$\frac{dV}{ds} = 3s^2$

$\frac{dV}{dt} = \frac{ds}{dt} \times \frac{dV}{ds}$

$= 0.12 \times \left(3s^2\right)$

$= 0.36s^2$

$\frac{dV}{dt}(150) = 0.36(150)^2 = 8100 \text{ mm}^3/\text{s}$

Question 10

$V = \pi r^2 h \qquad \frac{dr}{dt} = 1.2 \times 10^{-3}$

$V = 2000\pi r^2$

$\frac{dV}{dr} = 4000\pi r$

$\frac{dV}{dt} = \frac{dr}{dt} \times \frac{dV}{dr}$

$= 1.2 \times 10^{-3} \times (4000\pi r)$

$= 4.8\pi r$

$\frac{dV}{dt}(19) = 4.8\pi(19) = 287 \text{ mm}^3/\text{s}$

Question 11

$S = 4\pi r^2 \qquad \frac{dr}{dt} = 1.3$

$\frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = \frac{dr}{dt} \times \frac{dS}{dr}$

$= 1.3 \times (8\pi r)$

$= 10.4\pi r$

$\frac{dS}{dt}(6.3) = 10.4\pi(6.3) = 205.84 \text{ cm}^2/\text{s}$

Question 12

$V = \frac{6\pi h^2}{7} \qquad \frac{dh}{dt} = 2.3$

$\frac{dV}{dh} = \frac{12\pi h}{7}$

$\frac{dV}{dt} = \frac{dh}{dt} \times \frac{dV}{dh}$

$= 23 \times \left(\frac{12\pi h}{7}\right)$

$\frac{dV}{dt}(12.9) = 2.3 \times \left(\frac{12\pi(1.9\ \ )}{7}\right) = 159.79 \text{ cm}^3/\text{s}$

Question 13

$y = 2x^2 - 7x + 9 \qquad \frac{dx}{dt} = 8$

$\frac{dy}{dx} = 4x - 7$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= (4x - 7) \times 8$

$= 36x - 56$

$\frac{dy}{dt}(3) = 36(3) - 56 = 40$ units per second

Question 14

$V = x^3 \qquad \frac{dx}{dt} = -0.8$

$\frac{dV}{dx} = 3x^2$

$\frac{dV}{dt} = \frac{dx}{dt} \times \frac{dV}{dx}$

$= -0.8 \times (3x^2)$

$= -2.4x^2$

$\frac{dV}{dt}(120) = -2.4(120)^2 = -34560 \text{ mm}^3\text{/s}$

Rate of decrease is 34 560 mm^3/s.

Question 15

$N = x^2 + 7x \qquad \frac{dx}{dt} = -2$

$\frac{dN}{dx} = 2x + 7$

$\frac{dN}{dt} = \frac{dN}{dx} \times \frac{dx}{dt}$

$= (2x + 7) \times -2$

$= -4x - 14$

$\frac{dN}{dt}(150) = -4(150) - 14 = -614$

Decreasing by 614 radios per week.

Question 16

$V = 8x^3 - 5x^2 - 3x - 1 \qquad \frac{dx}{dt} = 4.2$

$\frac{dV}{dx} = 24x^2 - 10x - 3$

$\frac{dV}{dt} = \frac{dx}{dt} \times \frac{dV}{dx}$

$= 4.2\left(24x^2 - 10x - 3\right)$

$\frac{dV}{dt}(-4.7) = 4.2\left(24(-4.7)^2 - 10(-4.7) - 3\right)$

$= 2411.47 \text{ cm}^3/\text{s}$

$\approx 2411.5 \text{ cm}^3/\text{s}$

Question 17

$V = 053\ \pi d^2 \qquad \frac{dd}{dt} = -0.02$

$\frac{dV}{dd} = 1.06\pi d$

$\frac{dV}{dt} = \frac{dd}{dt} \times \frac{dV}{dd}$

$= -0.02 \times (1.06\pi d)$

$= -0.0212\pi d$

$\frac{dV}{dt}(167) = -0.0212\pi(167) = -11.12\,\text{mm}^3/\text{s}$

The volume of the tyre will be decreasing at 11.12 mm3/s.

Question 18

$N = 5x^2 + 3x \qquad \frac{dx}{dt} = -5$

$\frac{dN}{dx} = 10x + 3$

$\frac{dN}{dt} = \frac{dN}{dx} \times \frac{dx}{dt}$

$= (10x + 3) \times -5$

$= -50x - 15$

$\frac{dN}{dt}(55) = -50(55) - 15 = -2765$

Decreasing by 2765 rabbits per day.

Question 19

$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = 115$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dV} = \frac{1}{4\pi r^2}$

$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

$= 115\left(\frac{1}{4\pi r^2}\right)$

$= \frac{115}{4\pi r^2}$

$\frac{dr}{dt}(3) = \frac{115}{4\pi(3)^2} = 1.02$ cm/s

Question 20

$P = x^2 - 3000x + 100 \qquad \frac{dP}{dt} = 15000$

$\frac{dP}{dx} = 2x - 3000$

$\frac{dx}{dP} = \frac{1}{2x - 3000}$

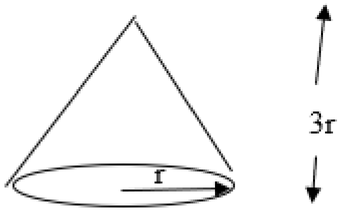
$\frac{dx}{dt} = \frac{dP}{dt} \times \frac{dx}{dP}$

$= 15000\left(\frac{1}{2x - 3000}\right)$

$= \frac{15000}{2x - 3000}$

$\frac{dx}{dt}(5000) = \frac{15000}{2(5000) - 3000} = 2.14$ houses per year

Question 21



$V = \frac{1}{3}\pi r^2 h$, but $h = 3r$

so $V = \frac{1}{3}\pi r^2 (3r) = \pi r^3$

$\frac{dV}{dr} = 3\pi r^2$

$\frac{dr}{dV} = \frac{1}{3\pi r^2}$

Given $\frac{dV}{dt} = -1.4$

$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

$= -14 \times \frac{1}{3\pi r^2}$

$= \frac{-1.4}{3\pi r^2}$

$\frac{dr}{dt}(3.7) = \frac{-1.4}{3\pi(3.7)^2} \approx -0.01$ cm/s

The radius is decreasing by 0.01 cm/s.

Question 22

$$V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$$

$$\frac{dV}{dr} = 4\pi r^2 \qquad \frac{dS}{dr} = 8\pi r$$

$$\frac{dr}{dt} = 0.3$$

a

$$\frac{dS}{dt} = \frac{dr}{dt} \times \frac{dS}{dr}$$

$$= 0.3 \times 8\pi r$$

$$\frac{dS}{dt}(88) = 0.3 \times 8\pi(88) = 663.5 \text{ mm}^2/\text{s}$$

b

$$\frac{dV}{dt} = \frac{dr}{dt} \times \frac{dV}{dr}$$

$$= 0.3 \times 4\pi r^2$$

$$\frac{dV}{dt} = 1.2\pi r^2$$

$$\frac{dV}{dt}(88) = 1.2\pi(88)^2 = 29194.2 \text{ mm}^3/\text{s}$$

Question 23

$$V = s^3 \qquad S = 6s^2$$

$$\frac{dV}{ds} = 3s^2 \qquad \frac{dS}{ds} = 12s$$

$$\frac{ds}{dV} = \frac{1}{3s^2} \qquad \frac{dV}{dt} = 23$$

$$\frac{dS}{dt} = \frac{dV}{dt} \times \frac{ds}{dV} \times \frac{dS}{ds}$$

$$= 23 \times \frac{1}{3s^2} \times 12s$$

$$= \frac{92}{s}$$

$$\frac{dS}{dt}(140) = \frac{92}{140} = 0.66 \text{ mm}^2/\text{s}$$

Question 24

$V=\frac{4}{3}\pi r^3 \qquad S=4\pi r^2$

$\frac{dV}{dr}=4\pi r^2 \qquad \frac{dS}{dr}=8\pi r$

$\frac{dr}{dS}=\frac{1}{8\pi r} \qquad \frac{dS}{dt}=1.9$

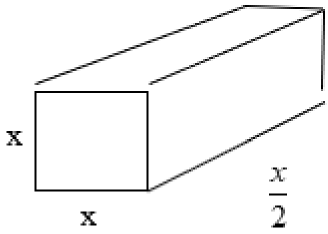
$\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dS}\times\frac{dS}{dt}$

$=4\pi r^2\times\frac{1}{8\pi r}\times 1.9$

$=\frac{1.9r}{2}$

$\frac{dV}{dt}(0.6)=\frac{1.9(0.6)}{2}=0.57\ \text{mm}^3/\text{s}$

Question 25



$$S = 2(x \times x) + 4\left(x \times \frac{x}{2}\right) = 2x^2 + 2x^2 = 4x^2$$

$$\frac{dS}{dx} = 8x$$

$$V = x \times x \times \frac{x}{2} = \frac{x^3}{2}$$

$$\frac{dV}{dx} = \frac{3x^2}{2}$$

$$\frac{dx}{dV} = \frac{2}{3x^2}$$

$$\frac{dV}{dt} = -12$$

$$\frac{dS}{dt} = \frac{dV}{dt} \times \frac{dS}{dx} \times \frac{dx}{dV}$$

$$= -12 \times 8x \times \frac{2}{3x^2}$$

$$= -\frac{64}{x}$$

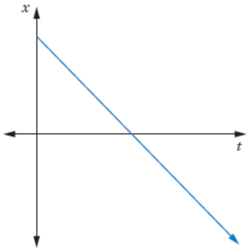
$$\frac{dS}{dt}(2.1) = -\frac{64}{2.1} = -30.48 \text{ cm}^2/\text{s}$$

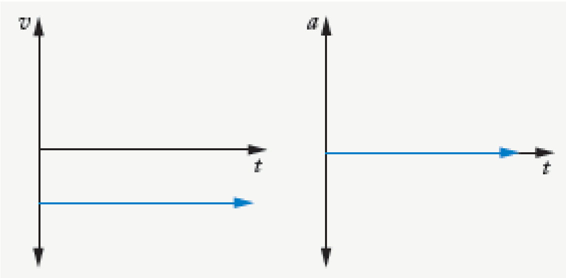
The surface area will be decreasing at 30.48 cm^2/s.

Exercise 8.12 Motion in a straight line

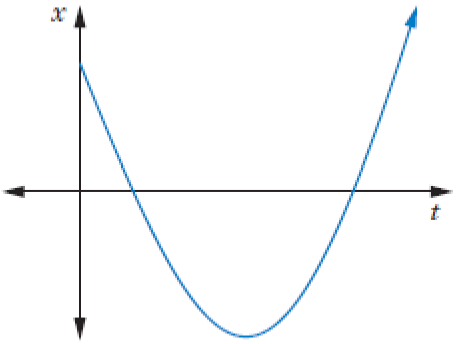
Question 1

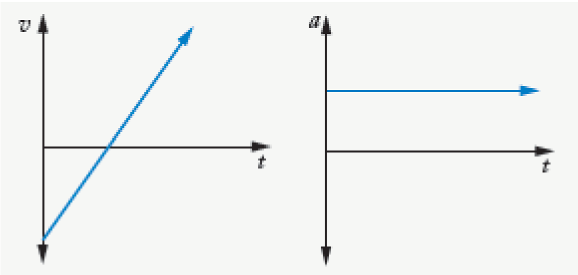
a



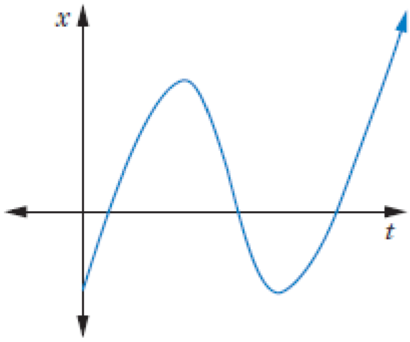


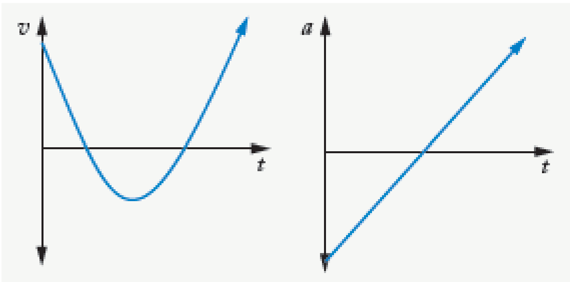
b



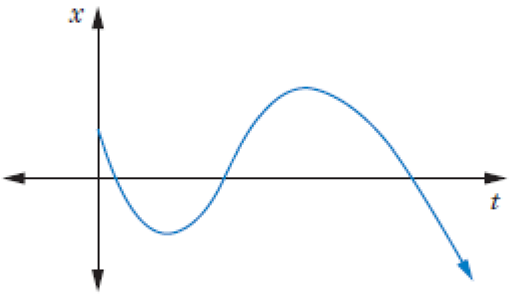


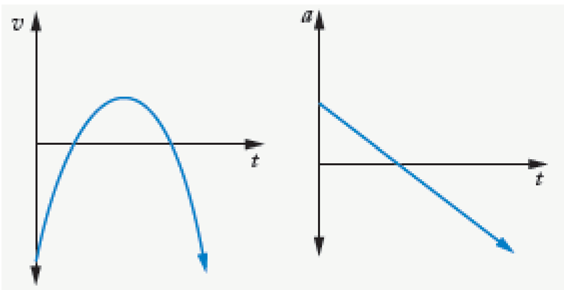
c



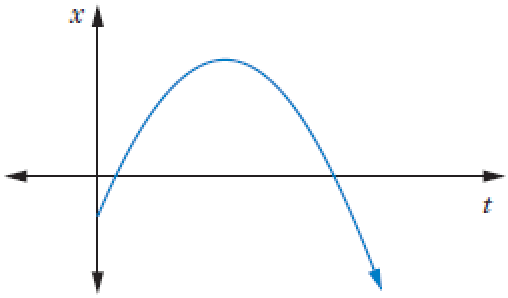


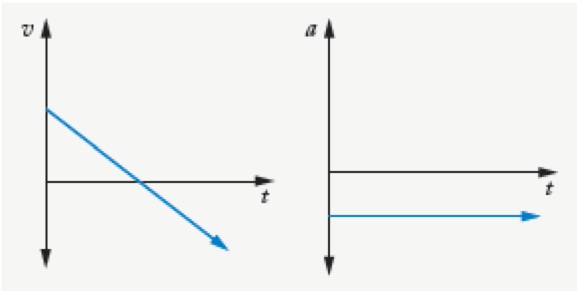
d





e





Question 2

a The particle is at the origin t_2, t_4, t_6.

b The particle is at rest from 0 to t_1, and at the turning points t_3, t_5.

c The particle furthest from the origin t_5.

Question 3

a Particle is at rest at 0, t_2, t_4, t_6.

b Acceleration = 0 at t_1, t_3, t_5.

c Speed is greatest at t_5.

d **i** At rest, accelerating to the left.

ii Moving to the left with zero acceleration.

Question 4

a Pendulum is at rest at $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

b At the origin at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

Question 5

a At the origin, with positive velocity (moving to the right).

b To the right of the origin, at rest.

c To the left of the origin, with negative velocity (moving to the left).

d To the right of the origin, with negative velocity (moving to the left).

e To the left of the origin, at rest.

Question 6

$h = 40t - 5t^2 + 4$

a $h(0) = 4$ m

b

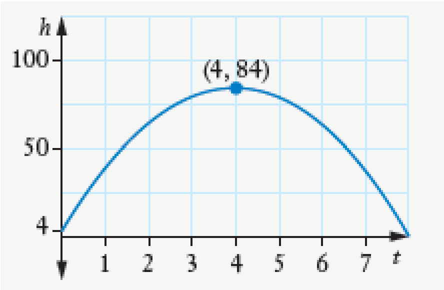
$$\dot{h} = 40 - 10t$$
$$\dot{h}(0) = 40\ m/s$$

c $h(1) = 40(1) - 5(1)^2 + 4 = 39$ m

d

$$\dot{h} = 0$$
$$0 = 40 - 10t$$
$$10t = 40$$
$$t = 4$$
$$h(4) = 40(4) - 5(4)^2 + 4 = 84 \text{ m}$$

e



Question 7

$x = 2 - t - t^2$

a $x(0) = 2$

b $x = 0$

$0 = 2 - t - t^2$

$0 = (2 + t)(t - 1)$

$t = -2, 1$

At $t = 1$ second.

c $x(2) = 2 - (2) - (2)^2 = -4$ cm

d $2 - (-4) = 6$ cm

e

$$\dot{x} = -1 - 2t$$
$$\dot{x}(3) = -1 - 2(3) = -7 \text{ cm/s}$$

Question 8

$x = t^3 + 6t^2 - 2t + 1$

a

$$v = 3t^2 + 12t - 2$$
$$a = 6t + 12$$

b $x(5) = (5)^3 + 6(5)^2 - 2(5) + 1 = 266$ m

c $\dot{x}(5) = 3(5)^2 + 12(5) - 2 = 133$ m/s

d $\ddot{x}(5) = 6(5) + 12 = 42$ m/s^2

Question 9

$x=(4t-3)^5$

a

$$\dot{x}=20(4t-3)^4$$
$$\ddot{x}=320(4t-3)^3$$

b

$$x=(4t-3)^5$$
$$x(1)=(4(1)-3)^5=1\text{ cm}$$
$$\dot{x}=20(4t-3)^4$$
$$\dot{x}(1)=20((1)t-3)^4=20\text{ cm/s}$$
$$\ddot{x}=320(4t-3)^3$$
$$\ddot{x}(1)=320(4(1)-3)^3=320\text{ cm/s}^2$$

c It is accelerating away from the origin to the right.

Question 10

a

$$s=ut+\frac{1}{2}gt^2$$
$$s=5t-5t^2$$
$$v=5-10t$$

b

$$v=5-10t$$
$$v(10)=5-10(10)=-95\text{ m/s}$$

c

$$v=5-10t$$
$$a=-10=g$$

Question 11

$$s=\frac{2t-5}{3t+1}$$

$u=2t-5 \qquad v=3t+1$

$u'=2 \qquad v'=3$

$$\dot{s}=\frac{u'v-v'u}{v^2}$$

$$=\frac{2\times(3t+1)-3\times(2t-5)}{(3t+1)^2}$$

$$=\frac{6t+2-6t+15}{(3t+1)^2}$$

$$=\frac{17}{(3t+1)^2}$$

$$=17(3t+1)^{-2}$$

$$\ddot{s}=-102(3t+1)^{-3}$$

Question 12

$x=t^3-4t^2+3t$

a

$\dot{x}=3t^2-8t+3$

$\dot{x}(0)=3$ m/s

b

$0=t^3-4t^2+3t$

$0=t(t-1)(t-3)$

$t=0,1,3$

At the origin when t = 0, 1, 3.

c

$\ddot{x}=6t-8$

$\ddot{x}(3)=6(3)-8=10$ m/s^2

Question 13

$h = 7 + 6t - t^2$

a $h(0) = 7$ m

b

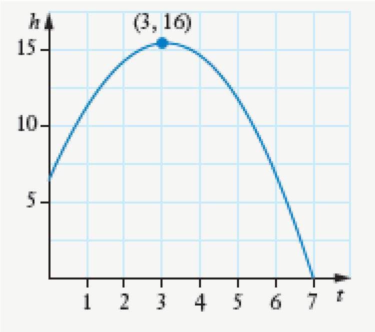
$$\dot{h} = 6 - 2t$$
$$0 = 6 - 2t$$
$$2t = 6$$
$$t = 3$$
$$h(3) = 7 + 6(3) - (3)^2 = 16 \text{ m}$$

c

$$0 = 7 + 6t - t^2$$
$$0 = (t - 7)(-1 - t)$$
$$t = -1, 7$$

Reaches the ground in 7 seconds.

d



e

$$h(4) = 7 + 6(4) - (4)^2 = 15 \text{ m}$$

Travel = (16 – 7) + (16 – 15) = 10 m

Question 14

$x = 15t - 3t^2$

a

$$\dot{x} = 15 - 6t$$
$$0 = 15 - 6t$$
$$6t = 15$$
$$t = \frac{5}{2}$$
$$x\left(\frac{5}{2}\right) = 15\left(\frac{5}{2}\right) - 3\left(\frac{5}{2}\right)^2 = 18\frac{3}{4} \text{ m}$$

b

$x = 3t(5 - t)$

at origin when $t = 0, 5$

$\dot{x}(5) = 15 - 6(5) = -15$ m/s

c 5 s

Question 15

a

$x = 2t^3 - 3t^2 + 42t$

$0 = 2t^3 - 3t^2 + 42t$

$0 = t\left(2t^2 - 3t + 42\right)$

$t = 0$ is a solution

Consider $2t^2 - 3t + 42$

$\Delta = b^2 - 4ac$

$= 9 - 4 \times 2 \times 42$

< 0

$\therefore$ No solution

$\therefore$ Particle will never return to the origin.

b

Particle is at rest when $\dot{x} = 0$.

$\dot{x} = 6t^2 - 6t + 42$

$0 = 6t^2 - 6t + 42$

Consider $6t^2 - 6t + 42$

$\Delta = b^2 - 4ac$

$= 36 - 4 \times 6 \times 42$

< 0

$\therefore$ No solution

$\therefore$ Particle will never be at rest.

Question 16

$x = t\sqrt{49 - t^2}$

a Particle travels for $49 - t^2 \geq 0$

$49 \geq t^2$

$-7 \geq t \geq 7$

But $t \geq 0$

It travels for 7 seconds.

b

$$x = t\sqrt{49 - t^2}$$

$$u = t \qquad v = \sqrt{49 - t^2}$$

$$u' = 1 \qquad v' = \frac{-t}{\sqrt{49 - t^2}}$$

$$\dot{x} = u'v + v'u$$

$$= 1 \times \sqrt{49 - t^2} + \left(\frac{-t}{\sqrt{49 - t^2}}\right) \times t$$

$$= \frac{49 - t^2 - t^2}{\sqrt{49 - t^2}}$$

$$= \frac{49 - 2t^2}{\sqrt{49 - t^2}}$$

$$\dot{x} = 0$$

$$0 = \frac{49 - 2t^2}{\sqrt{49 - t^2}}$$

$$0 = 49 - 2t^2$$

$$2t^2 = 49$$

$$t^2 = \frac{49}{2}$$

$$t = \frac{7}{\sqrt{2}}$$

c

$$x\left(\frac{7}{\sqrt{2}}\right)=\left(\frac{7}{\sqrt{2}}\right)\sqrt{49-\left(\frac{7}{\sqrt{2}}\right)^2}$$

$$=\frac{7}{\sqrt{2}}\sqrt{\frac{49}{2}}$$

$$=\frac{49}{2}$$

As it returns to origin total distance is twice this amount

d = 49 cm

Exercise 8.13 Multiple roots of polynomial equations

Question 1

a

$$(x-4)^2 = x^2 - 8x + 16$$

$$P(x) = x^3 - 7x^2 + 8x + 16$$

$$\begin{array}{r} x+1 \\ x^2-8x+16\overline{\smash{\big)}\,x^3-7x^2+8x+16} \\ \underline{x^3-8x^2+16x} \\ x^2-8x+16 \\ \underline{x^2-8x+16} \\ 0 \end{array}$$

$\therefore$ $(x-4)^2$ is a factor of $P(x)$

b $P(x) = (x + 1)(x - 4)^2$

c

$$P(x) = (x+1)(x-4)^2$$

$$P(4) = ((4)+1)((4)-4)^2 = 0$$

$$P'(x) = 3x^2 - 14x + 8$$

$$P'(4) = 3(4)^2 - 14(4) + 8 = 0$$

$\therefore$ $P(4) = P'(4) = 0$

Question 2

a

$$(x+3)^3 = x^3+9x^2+27x+27$$
$$f(x) = x^4+7x^3+9x^2-27x-54$$

$$\begin{array}{r l}
 & x-2 \\ \cline{2-2}
x^3+9x^2+27x+27 \big) & x^4+7x^3+9x^2-27x-54 \\
 & \underline{x^4+9x^3+27x^2+27x} \\
 & \quad -2x^3-18x^2-54x-54 \\
 & \quad \underline{-2x^3-18x^2-54x-54} \\
 & \qquad 0
\end{array}$$

$\therefore$ $(x + 3)^3$ is a factor of $f(x)$

b $f(x) = (x - 2)(x + 3)^3$

c

$$f(x) = (x-2)(x+3)^3$$
$$f(-3) = ((-3)-2)((-3)+3)^3 = 0$$
$$f'(x) = 4x^3+21x^2+18x-27$$
$$f'(-3) = 4(-3)^3+21(-3)^2+18(-3)-27 = 0$$

$\therefore$ $f(-3) = f'(-3) = 0$

Question 3

a $P(x)=(x-k)^3 Q(x)$

b

$$P(x)=(x-k)^3 Q(x)$$
$$P(k)=(k-k)^3 Q(k)=0$$
$$P'(x)=3(x-k)^2 Q(x)+(x-k)^3 Q'(x)$$
$$=(x-k)^2\left[3Q(x)+(x-k)Q\ (x)\right]$$
$$P'(k)=(k-k)^2\left[3Q(x)+(k-k)Q'(x)\right]=0$$

Question 4

a

$$P(x) = x^3 + x^2 - 8x - 12$$

$$\begin{array}{r|l} & x^2 + 4x + 4 \\ \hline x - 3 & x^3 + x^2 - 8x - 12 \\ & \underline{x^3 - 3x^2} \\ & \quad 4x^2 - 8x \\ & \quad \underline{4x^2 - 12x} \\ & \qquad 4x - 12 \\ & \qquad \underline{4x - 12} \\ & \qquad\quad 0 \end{array}$$

$$P(x) = (x-3)(x^2+4x+4)$$

$$P(x) = (x-3)(x+2)^2$$

b A root of multiplicity 1 at $x = 3$.

A root of multiplicity 2 at $x = -2$.

c

$$P(x) = x^3 + x^2 - 8x - 12$$

$$P(-2) = (-2)^3 + (-2)^2 - 8(-2) - 12 = 0$$

$$P'(x) = 3x^2 + 2x - 8$$

$$P'(-2) = 3(-2)^2 + 2(-2) - 8 = 0$$

$$\therefore\ P(-2) = P'(-2) = 0$$

Question 5

a

$$P(x) = x^5 - 2x^4 + x^3$$
$$P(x) = x^3\left(x^2 - 2x + 1\right)$$
$$P(x) = x^3\left(x - 1\right)^2$$

b $x = 0$ is a root of multiplicity 3.

$x = 1$ is a root of multiplicity 2.

c

$$P(x) = x^5 - 2x^4 + x^3$$
$$P'(x) = 5x^4 - 8x^3 + 3x^2$$
$$P(0) = 0$$
$$P'(0) = 0$$
$$\therefore\ P(0) = \ P\ (x) = 0$$
$$P(1) = (1)^5 - 2(1)^4 + (1)^3 \ = 0$$
$$P'(1) = 5(1)^4 - 8(1)^3 + 3(1)^2 \ = 0$$
$$\therefore\ P(1) = \ P\ (1) = 0$$

Question 6

$$P(x) = \left(x - 5\right)^3 Q(x)$$
$$P'(x) = 3\left(x - 5\right)^2 Q(x) + \left(x - 5\right)^3 Q'(x)$$
$$= \left(x - 5\right)^2 \left[3Q(x) + \left(x - 5\right)Q\ (x)\right]$$

$\therefore$ $P\ '\ (x)$ has a double root at $x = 5$.

Question 7

$$P(x)=\left(x+3\right)^{6}Q(x)$$
$$P'(x)=6\left(x+3\right)^{5}Q(x)+\left(x+3\right)^{6}Q'(x)$$
$$=\left(x+3\right)^{5}\left[6Q(x)+\left(x+3\right)Q\ (x)\right]$$

$\therefore$ $P'(x)$ has a root of multiplicity 5 at $x = –3$.

Question 8

$$P(x)=\left(x-p\right)^{n}Q(x)$$
$$P'(x)=n\left(x-p\right)^{n-}\ Q(x)+\left(x-p\right)^{n}Q'(x)$$
$$=\left(x-p\right)^{n-}\ \left[nQ(x)+\left(x-p\right)Q\ (x)\right]$$

$\therefore$ P’(x) has a root of multiplicity n – 1 at x = p

Question 9

a

$$P(x)=x^{3}+5x^{2}+3x-9$$

$$\begin{array}{r|l} & x-1 \\ \hline x^{2}+6x+9 & x^{3}+5x^{2}+3x-9 \\ & \underline{x^{3}+6x^{2}+9x} \\ & \quad -x^{2}-6x-9 \\ & \quad \underline{-x^{2}-6x-9} \\ & \qquad 0 \end{array}$$

$$P(x)=\left(x-1\right)\left(x^{2}+6x+9\right)=\left(x-1\right)\left(x+3\right)^{2}$$

b $x = –3$ is a root of multiplicity 2.

c Multiplicity 1

Test Yourself 8

Question 1

$$\frac{d}{dx}\left(\frac{2}{3}x^{-4}\right)=\frac{2}{3}\times(-4)x^{-4-1}=-\frac{8}{3x^5}$$

C

Question 2

$y=3x\left(x^3-5\right)$

$u=3x \qquad v=x^3-5$

$u'=3 \qquad v'=3x^2$

$y'=u'v+v'u$

$=3\times\left(x^3-5\right)+3x^2\times3x$

$=3x^3-15+9x^3$

$=12x^3-15$

B

Question 3

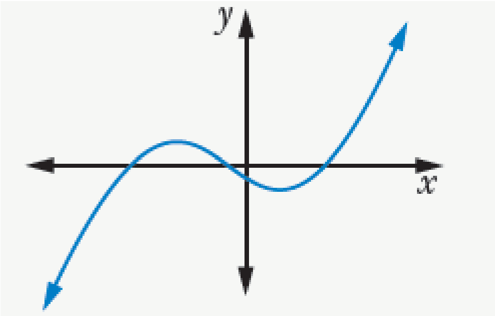
D

Question 4

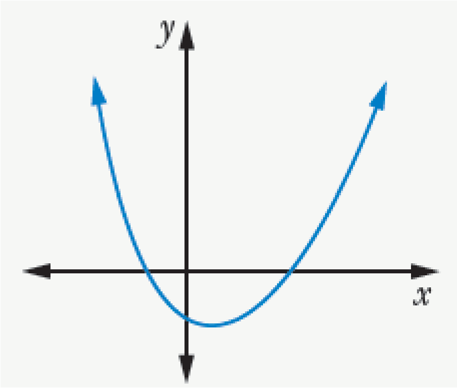
B, C

Question 5

a



b



Question 6

$$y = 5x^2 - 3x + 2$$

$$y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{5(x+h)^2 - 3(x+h) + 2 - (5x^2 - 3x + 2)}{h}$$

$$= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 3x - 3h + 2 - 5x^2 + 3x - 2}{h}$$

$$= \lim_{h \to 0} \frac{10xh + 5h^2 - 3h}{h}$$

$$= \lim_{h \to 0} (10x + 5h - 3)$$

$$= 10x - 3$$

Question 7

a

$$\frac{d}{dx}\left(7x^6-3x^3+x^2-8x-4\right)$$
$$=7\times6x^{6-1}-3\times3x^{3-1}+1\times2x^{2-1}-8x^{1-1}$$
$$=42x^5-9x^2+2x-8$$

b $\frac{d}{dx}\left(3x^{-4}\right)=3\times(-4)x^{-4-1}=-12x^{-5}$

c

$$y=\frac{2}{(x+1)^4}$$
$$y=2(x+1)^{-4}$$
$$y'=2\times(-4)(x+1)^{-4-1}\times1$$
$$=-8(x+1)^{-5}$$
$$=-\frac{8}{(x+1)^5}$$

d

$$y=x^2\sqrt{x}$$
$$=x^{\frac{5}{2}}$$
$$y'=\frac{5}{2}x^{\frac{3}{2}}$$
$$=\frac{5x\sqrt{x}}{2}$$

e

$$y=\left(x^2+4x-2\right)^9$$

$u=x^2+4x-2 \qquad y=u^9$

$\dfrac{du}{dx}=2x+4 \qquad \dfrac{dy}{du}=9u^8$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$
$$=(2x+4)\times 9\left(x^2+4x-2\right)^8$$
$$=9(2x+4)\left(x^2+4x-2\right)^8$$
$$=18(x+2)\left(x^2+4x-2\right)^8$$

f

$$y=\frac{3x-2}{2x+1}$$

$u=3x-2 \qquad v=2x+1$

$u'=3 \qquad v'=2$

$$y'=\frac{u'v-v'u}{v^2}$$
$$=\frac{3\times(2x+1)-2\times(3x-2)}{(2x+1)^2}$$
$$=\frac{6x+3-6x+4}{(2x+1)^2}$$
$$=\frac{7}{(2x+1)^2}$$

g

$$y=x^3(3x+1)^6$$

$u=x^3 \qquad v=(3x+1)^6$

$u'=3x^2 \qquad v'=18(3x+1)^5$

$$y'=u'v+v'u$$
$$=3x^2\times(3x+1)^6+18(3x+1)^5\times x^3$$
$$=3x^2(3x+1)^6+18x^3(3x+1)^5$$
$$=3x^2(3x+1)^5\left[(3x+1)+6x\right]$$
$$=3x^2(3x+1)^5(9x+1)$$

Question 8

$v = 2t^2 - 3t - 4$

$\frac{dv}{dt} = 4t - 3$

Question 9

$y = x^3 + 3x^2 + x - 5$

$y' = 3x^2 + 6x + 1$

$y'(1) = 3(1)^2 + 6(1) + 1 = 10$

Question 10

$h = 60t - 3t^2$

$\frac{dh}{dt} = 60 - 6t$

$\frac{dh}{dt}(3) = 60 - 18 = 42$

Question 11

a Not differentiable at $x = -2$.

b Not differentiable at $x = 1$.

c Not differentiable at $x = 2$.

Question 12

a

$$y = \frac{4}{x} = 4x^{-1}$$

$$y' = -4x^{-2} = -\frac{4}{x^2}$$

b

$$f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$$

$$f'(x) = \frac{1}{5}x^{-\frac{4}{5}} = \frac{1}{5\sqrt[5]{x^4}}$$

c

$$f(x) = 2(4x+9)^4$$

$$y = 2(4x+9)^4$$

$u = 4x+9$ $\qquad$ $y = 2u^4$

$\frac{du}{dx} = 4$ $\qquad$ $\frac{dy}{du} = 8u^3$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$= 4 \times 8(4x+9)^3$$

$$= 32(4x+9)^3$$

d

$$y = (3x+2)(x-1)^3$$

$u = 3x+2$ $\qquad$ $v = (x-1)^3$

$u' = 3$ $\qquad$ $v' = 3(x-1)^2$

$$y' = u'v + v'u$$

$$= 3 \times (x-1)^3 + 3(x-1)^2 \times (3x+2)$$

$$= 3(x-1)^2(x-1+3x+2)$$

$$= 3(x-1)^2(4x+1)$$

e

$$y = \frac{x^3 - 3}{2x + 5}$$

$u = x^3 - 3 \qquad v = 2x + 5$

$u' = 3x^2 \qquad v' = 2$

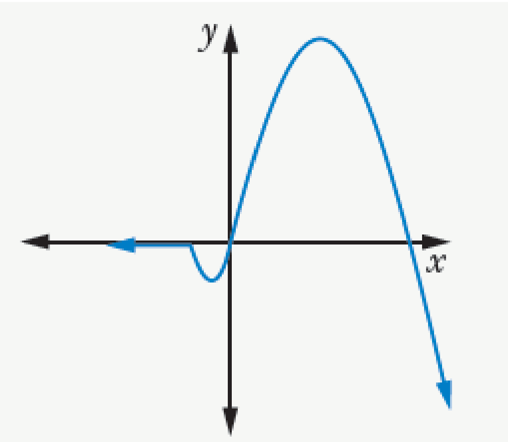
$$y' = \frac{u'v - v'u}{v^2}$$

$$= \frac{3x^2 \times (2x + 5) - 2 \times (x^3 - 3)}{(2x + 5)^2}$$

$$= \frac{6x^3 + 15x^2 - 2x^3 + 6}{(2x + 5)^2}$$

$$= \frac{4x^3 + 15x^2 + 6}{(2x + 5)^2}$$

Question 13



Question 14

$y = x^2 + 5x - 3$

$y' = 2x + 5$

$y'(2) = 4 + 5 = 9$

$y - y_1 = m(x - x_1)$

$y - 11 = 9(x - 2)$

$y - 11 = 9x - 18$

$9x - y - 7 = 0$

Question 15

$y = x^2 - x + 1$

$y' = 2x - 1$

$y' = 3$

$3 = 2x - 1$

$2x = 4$

$x = 2$

$y(2) = 4 - 2 + 1 = 3$

(2, 3)

Question 16

$S = 4\pi r^2$

$\frac{dS}{dr} = 8\pi r$

Question 17

$f(x) = x^2 - 3x + 1$

$x = 1$

$f(1) = 1^2 - 3(1) + 1 = 1 - 3 + 1 = -1$

$x = 1.1$

$f(1.1) = (1.1)^2 - 3(1.1) + 1 = 1.21 - 3.3 + 1 = -1.09$

$$m = \frac{f(1) - f(1.1)}{1 - 1.1}$$
$$= \frac{-1 + 1.09}{1 - 1.1}$$
$$= -0.9$$

Question 18

$y = 2x^3 - 9x^2 - 60x + 3$

$$y' = 6x^2 - 18x - 60$$
$$= 6(x-5)(x+2)$$
$$y' = 0$$
$$0 = 6(x-5)(x+2)$$
$$x = -2,\ 5$$
$$y(-2) = 2(-2)^3 - 9(-2)^2 - 60(-2) + 3 = 71$$
$$(-2, 71)$$
$$y(5) = 2(5)^3 - 9(5)^2 - 60(5) + 3 = -272$$
$$(5, -272)$$

Question 19

$y = x^2 + 2x - 5$

$y' = 2x + 2$

$y' = 4$

$4 = 2x + 2$

$2x = 2$

$x = 1$

$y(1) = (1)^2 + 2(1) - 5 = 1 + 2 - 5 = -2$

$y - y_1 = m(x - x_1)$

$y + 2 = 4(x - 1)$

$y + 2 = 4x - 4$

$4x - y - 6 = 0$

Question 20

a

$$s = ut + \frac{1}{2}at^2$$
$$\frac{ds}{dt} = u + at$$

b

$$5 = 7 - 10t$$
$$10t = 2$$
$$t = \frac{1}{5}$$

Question 21

$$y = \frac{1}{3x} = \frac{1}{3}x^{-1}$$

$$y' = -\frac{1}{3}x^{-2} = -\frac{1}{3x^2}$$

$$y\left(\frac{1}{6}\right) = \frac{1}{3\left(\frac{1}{6}\right)} = 2$$

$$y'\left(\frac{1}{6}\right) = -\frac{1}{3}\left(\frac{1}{6}\right)^{-2} = -12$$

$$y - y_1 = m\ (x - x_1)$$

$$y - 2 = -12(x - \frac{1}{6})$$

$$y - 2 = -12x + 2$$

$$12x + y - 4 = 0$$

Question 22

$h = 4t - t^2$

a **i** $h(0) = 0$ m

ii $h(2) = 4(2) - (2)^2 = 8 - 4 = 4$ m

iii $h(3) = 4(3) - (3)^2 = 12 - 9 = 3$ m

iv $h(3.5) = 4(3.5) - (3.5)^2 = 14 - 12.25 = 1.75$ m

b **i** $h(1) = 4(1) - 1^2 = 4 - 1 = 3$

$h(2) = 4(2) - 2^2 = 8 - 4 = 4$

Average rate of change = $\frac{4-3}{2-1} = \frac{1}{1} = 1$ m s^{-1}

ii $h(2) = 4$

$h(3) = 4(3) - 3^2 = 12 - 9 = 3$

Average rate of change = $\frac{3-4}{3-2} = \frac{-1}{1} = -1$ m s^{-1}

c $h' = 4 - 2t$

i $h'(0) = 4$ m/s

ii $h'(2) = 4 - 2(2) = 0$ m/s

iii $h'(3) = 4 - 2(3) = -2$ m/s

Question 23

$f(x) = x^2 - 3x + 5$

a $f(x + h) = (x + h)^2 - 3(x + h) + 5$

$= x^2 + 2xh + h^2 - 3x - 3h + 5$

b $f(x + h) - f(x)$

$= x^2 + 2xh + h^2 - 3x - 3h + 5 - (x^2 - 3x + 5)$

$= 2xh + h^2 - 3h$

c

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{2xh+h^2-3h}{h} \\
&= \lim_{h\to 0}(2x+h-3) \\
&= 2x-3
\end{aligned}
$$

Question 24

a $f(x) = (4x - 3)^5$

$f(1) = (4(1) - 3)^5 = 1$

b $f'(x) = 20(4x - 3)^4$

$f'(1) = 20(4(1) - 3)^4 = 20$

Question 25

$f(x) = (x-3)^9$

$y = (x-3)^9$

$u = x-3 \qquad y = u^9$

$\frac{du}{dx} = 1 \qquad \frac{dy}{du} = 9u^8$

$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$

$= 9(x-3)^8$

$f'(x) = 9(x-3)^8$

$f'(4) = 9((4)-3)^8 = 9$

Question 26

a

$$y = 3\left(x^2 - 6x + 1\right)^4$$

$u = x^2 - 6x + 1 \qquad y = 3u^4$

$\dfrac{du}{dx} = 2x - 6 \qquad \dfrac{dy}{du} = 12u^3$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$= \left(2x - 6\right) \times 12\left(x^2 - 6x + 1\right)^3$$

$$= 12\left(2x - 6\right)\left(x^2 - 6x + 1\right)^3$$

$$= 24\left(x - 3\right)\left(x^2 - 6x + 1\right)^3$$

b

$$y = \frac{2}{\sqrt{3x - 1}}$$

$$y = \left(3x - 1\right)^{-\frac{1}{2}}$$

$u = 3x - 1 \qquad y = 2u^{-\frac{1}{2}}$

$\dfrac{du}{dx} = 3 \qquad \dfrac{dy}{du} = -u^{-\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$= 3 \times -\left(3x - 1\right)^{-\frac{3}{2}}$$

$$= \frac{-3}{\sqrt{\left(3x - 1\right)^3}}$$

Question 27

$V=\frac{4}{3}\pi r^3 \qquad \frac{dV}{dt}=35$

$\frac{dV}{dr}=4\pi r^2$

$\frac{dr}{dV}=\frac{1}{4\pi r^2}$

$\frac{dr}{dt}=\frac{dV}{dt}\times\frac{dr}{dV}$

$=35\left(\frac{1}{4\pi r^2}\right)$

$\frac{dr}{dt}(12)=\frac{35}{4\pi(12)^2}$

$=0.0193\text{ mm/s}$

Question 28

$x=5+6t-3t^2$

$\dot{x}=6-6t$

a **i** $x(0) = 5$ cm

ii $\dot{x}(0)=6\text{ cm/s}$

b $\dot{x}(1)=6-6(1)=0$

c From the answer to part **b** it is at rest at $t = 1$.

d Maximum displacement occurs when $\dot{x}$ = 0.

$t = 1$

As it is a negative coefficient parabola it is a maximum.

$x(1)=5+6(1)-3(1)^2=8\text{ cm}$

e $\ddot{x}=-6\text{ cm/s}^2$

Therefore it is moving with constant acceleration

Question 29

$x = 4t^2 - 5t^3$

a $x(0) = 0$

$$\dot{x} = 8t - 15t^2$$
$$\dot{x}(0) = 0$$
$$\ddot{x} = 8 - 30t$$
$$\ddot{x}(0) = 8$$

Initial displacement = 0

Initial velocity = 0

Initial acceleration = 8 m/s^2

b Particle at the origin when $x = 0$.

$$0 = 4t^2 - 5t^3$$
$$0 = t^2(4 - 5t)$$
$$4 - 5t = 0$$
$$5t = 4$$
$$t = \frac{4}{5}$$

$t = 0, \frac{4}{5}$

c $\dot{x}(2) = 8(2) - 15(4) = 16 - 60 = -44$ m/s

$\ddot{x}(2) = 8 - 30(2) = -52$ m/s^2

Question 30

$x = t^3 - 12t^2 + 36t - 9$

a Particle is at rest when $\dot{x} = 0$.

$$\begin{aligned}\dot{x} &= 3t^2 - 24t + 36 \\ &= 3\left(t^2 - 8t + 12\right) \\ &= 3(t-6)(t-2)\end{aligned}$$

$\dot{x} = 0$ when t = 2, 6.

b **i** $x(1) = (1)^3 - 12(1)^2 + 36(1) - 9 = 1 - 12 + 36 - 9 = 16$ cm

ii

$$\dot{x} = 3t^2 - 24t + 36$$
$$\dot{x}(1) = 3(1)^2 - 24(1) + 36 = 15 \text{ cm/s}$$

iii

$$\ddot{x} = 6t - 24$$
$$\ddot{x}(1) = 6(1) - 24 = -18 \text{ cm/s}^2$$

c After 1 second the particle is moving away from the origin to the right but slowing down.

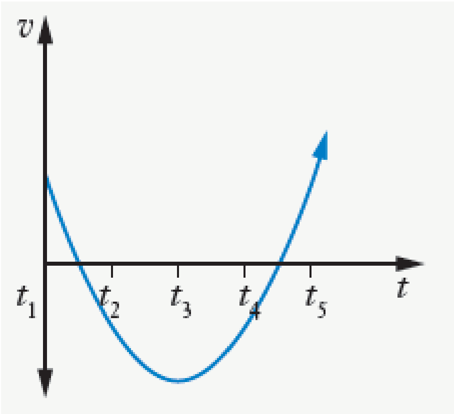
Question 31

a t_1, t_3, t_5, are t-intercepts of graph

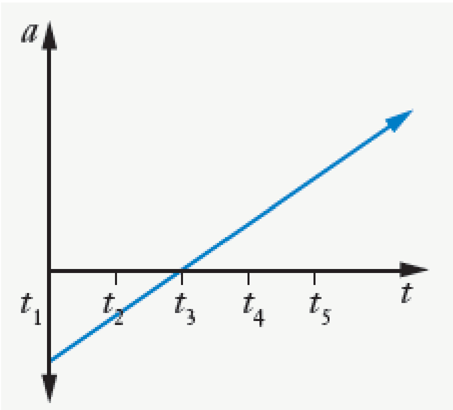
b t_2, t_4, at turning points where gradient is 0

c t_5, where graph is steepest

d **i**



ii



Question 32

$h = 20t - 5t^2$

a $h(1) = 20(1) - 5(1)^2 = 15$ m

b Maximum height occurs when $\dot{h} = 0$.

$$\dot{h} = 20 - 10t$$
$$0 = 20 - 10t$$
$$10t = 20$$
$$t = 2$$

As h is a parabola with a negative leading coefficient this is a maximum.

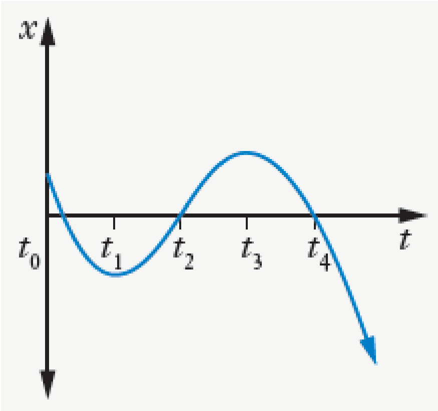
$h(2) = 20(2) - 5(2)^2 = 20$ m

c Time of flight equals 2 times the time to reach maximum.

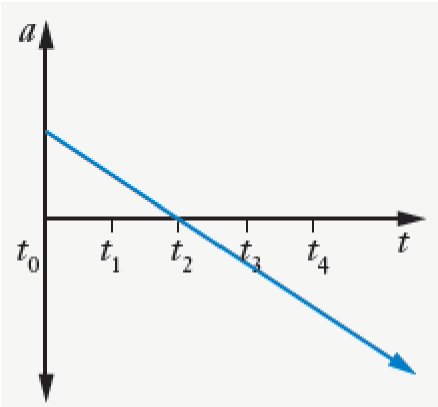
4 s

Question 33

a **i**



ii



b Particle at rest at t_1, t_3.

Question 34

$P(x) = (x - b)^7$

a $P(b) = (b - b)^7 = 0$

$P'(x) = 7(x - b)^6$

$P'(b) = 7(b - b)^6 = 0$

$P(b) = P'(b) = 0$

b $P(x) = x^7 + 3x^6 + ax^5 + x^4 + 3x^3 + bx^2 - x + 1$

$P(1) = (1)^7 + 3(1)^6 + a(1)^5 + (1)^4 + 3(1)^3 + b(1)^2 - (1) + 1 = 0$

$1 + 3 + a + 1 + 3 + b - 1 + 1 = 0$

$8 + a + b = 0$

$a + b = -8$ [1]

$P'(x) = 7x^6 + 18x^5 + 5ax^4 + 4x^3 + 9x^2 + 2bx - 1 = 0$

$P'(1) = 7(1)^6 + 18(1)^5 + 5a(1)^4 + 4(1)^3 + 9(1)^2 + 2b(1) - 1 = 0$

$7 + 18 + 5a + 4 + 9 + 2b - 1 = 0$

$37 + 5a + 2b = 0$

$5a + 2b = -37$ [2]

$\underline{2a + 2b = -16}$ [3] = [1] × 2

$3a \quad = -21$ [2] – [3]

$a = -7$

Sub into [1]:

$-7 + b \quad -8$

$b = -1$

So $a = -7$, $b = -1$.

Question 35

a $f(5) = 5^3 - 7(5)^2 - 5(5) + 75 = 125 - 175 - 25 + 75 = 0$

b From part **a** $f(5) = 0$

$f'(x) = 3x^2 - 14x - 5 = (3x + 1)(x - 5)$

$f'(5) = (3(5) + 1)((5) - 5) = 0$

$f(5) = f'(5) = 0$

c $x = 5$ is at least a double root

d

$$P(x) = x^3 - 7x^2 - 5x + 75$$

$$\begin{array}{r l}
 & x^2 - 2x - 15 \\
\cline{2-2}
x - 5 \big) & x^3 - 7x^2 - 5x + 75 \\
 & \underline{x^3 - 5x^2} \\
 & -2x^2 - 5x \\
 & \underline{-2x^2 + 10x} \\
 & -15x + 75 \\
 & \underline{-15x + 75} \\
 & 0
\end{array}$$

$$P(x) = (x - 5)(x^2 - 2x - 15)$$

$$P(x) = (x + 3)(x - 5)^2$$

$$P(x) = (x + 3)(x - 5)^2$$

Challenge exercise 8

Question 1

$y = x(x - 1)(x + 2)$

Cuts the x-axis when $y = 0$.

$x = -2, 0, 1$

$y = x^3 + x^2 - 2x$

$y' = 3x^2 + 2x - 2$

$y'(-2) = 3(-2)^2 + 2(-2) - 2 = 12 - 4 - 2 = 6$

$m = 6$

$y - y_1 = m(x - x_1)$

$y - 0 = 6(x + 2)$

$y = 6x + 12$

$6x - y + 12 = 0$

$y'(0) = 3(0)^2 + 2(0) - 2 = 0 + 0 - 2 = -2$

$m = -2$

$y - y_1 = m(x - x_1)$

$y - 0 = -2(x - 0)$

$y = -2x$

$2x + y = 0$

$y'(1) = 3(1)^2 + 2(1) - 2 = 3 + 2 - 2 = 3$

$m = 3$

$y - y_1 = m(x - x_1)$

$y - 0 = 3(x - 1)$

$y = 3x - 3$

$3x - y - 3 = 0$

Question 2

a $y = x^3 - 6$

$y' = 3x^2$

$y' = 12$

$12 = 3x^2$

$x^2 = 4$

$x = \pm 2$

$y(2) = (2)^3 - 6 = 8 - 6 = 2$

$y(-2) = (-2)^3 - 6 = -8 - 6 = = -14$

$(-2, -14), (2, 2)$

b

$$m_t = 12$$
$$m_n = -\frac{1}{12}$$

$$y - y_1 = m_n(x - x_1)$$

$$y - 2 = -\frac{1}{12}(x - 2)$$
$$-12y + 24 = x - 2$$
$$x + 12y - 26 = 0$$

$$m_t = 12$$
$$m_n = -\frac{1}{12}$$

$$y - y_1 = m_n(x - x_1)$$

$$y + 14 = -\frac{1}{12}(x + 2)$$
$$-12y - 168 = x + 2$$
$$x + 12y + 170 = 0$$

Question 3

$y = x^2 + 1$

$y(2) = (2)^2 + 1 = 4 + 1 = 5$

$y' = 2x$

$y'(2) = 2(2) = 4$

$m_t = 4$

$m_n = -\frac{1}{4}$

$y - y_1 = m_n\left(x - x_1\right)$

$y - 5 = -\frac{1}{4}(x-2)$

$-4y + 20 = x - 2$

$x + 4y - 22 = 0$

Solve $x + 4y - 22 = 0$ and $y = x^2 + 1$ simultaneously.

$x + 4(x^2 + 1) - 22 = 0$

$4x^2 + x - 18 = 0$

$(4x + 9)(x - 2) = 0$

$x = 2,\ -\frac{9}{4}$

We already have $x = 2$ as a solution so use $x = -\frac{9}{4}$

$y(-\frac{9}{4}) = (-\frac{9}{4})^2 + 1 = 6\frac{1}{16}$

$P = \left(-2\frac{1}{4}, 6\frac{1}{16}\right)$

Question 4

$y = x^4 - nx^2 + 3x - 2$

$y' = 4x^3 - 2nx + 3$

when $x = -2$, $y' = 3$

$3 = 4(-2)^3 - 2n(-2) + 3$

$3 = -32 + 4n + 3$

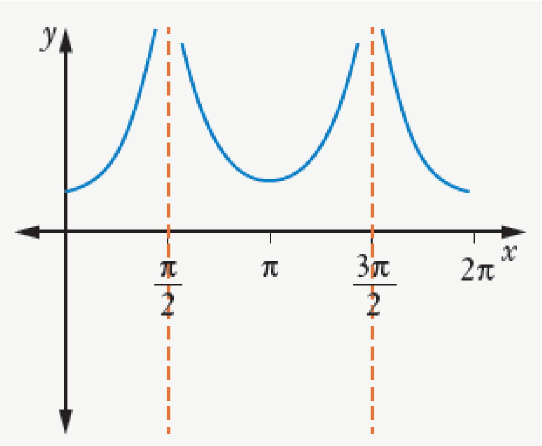
$4n = 32$

$n = 8$

Question 5

a Not differentiable at $x = \frac{\pi}{2} \ \frac{3\pi}{2}$.

b



Question 6

a

$$S=\pi\left(\frac{d}{2}\right)^2 \qquad \frac{dd}{dt}=0.1$$

$$\frac{dS}{dd}=\frac{\pi d}{2}$$

$$\frac{dS}{dt}=\frac{dd}{dt}\times\frac{dS}{dd}$$

$$=0.1\times\left(\frac{\pi d}{2}\right)=0.05\pi d$$

$$\frac{dS}{dt}(50)=0.05\pi(50)=7.85\ \text{mm}^2/\text{h}$$

b Maximum occurs when d = 200 (as it is always increasing until this point).

$$S=\pi\left(\frac{200}{2}\right)^2=31\,416\ \text{mm}^2$$

Question 7

$$y=\sqrt{x^2-3}$$

$$y=\left(x^2-3\right)^{\frac{1}{2}}$$

$$u=x^2-3 \qquad y=u^{\frac{1}{2}}$$

$$\frac{du}{dx}=2x \qquad \frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{du}{dx}\times\frac{dy}{du}$$

$$=2x\times\frac{1}{2}\left(x^2-3\right)^{-\frac{1}{2}}$$

$$=\frac{x}{\sqrt{x^2-3}}$$

$$\frac{dy}{dx}(5)=\frac{(5)}{\sqrt{(5)^2-3}}=\frac{5}{\sqrt{22}}=\frac{5\sqrt{22}}{22}$$

Question 8

$y = 3\sqrt{x-1}$

$y = 3(x+1)^{\frac{1}{2}}$

$y(8) = 3((8)+1)^{\frac{1}{2}} = 9$

$u = x+1 \qquad y = 3u^{\frac{1}{2}}$

$\frac{du}{dx} = 1 \qquad \frac{dy}{du} = \frac{3}{2}u^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$

$= 1 \times \frac{3}{2}(x+1)^{-\frac{1}{2}}$

$= \frac{3}{2\sqrt{x+1}}$

$\frac{dy}{dx}(8) = \frac{3}{2\sqrt{(8)+1}} = \frac{1}{2}$

$m_t = \frac{1}{2}$

$m_n = -2$

$y - y_1 = m(x - x_1)$

$y - 9 = -2(x-8)$

$y - 9 = -2x + 16$

$2x + y - 25 = 0$

Question 9

a Solve $6x - 8y + 1 = 0$ and $y = 2x^2$ simultaneously.

$6x - 8(2x^2) + 1 = 0$

$6x - 16x^2 + 1 = 0$

$16x^2 - 6x - 1 = 0$

$(8x + 1)(2x - 1) = 0$

$$8x + 1 = 0$$
$$8x = -1$$
$$x = -\frac{1}{8}$$
$$2x - 1 = 0$$
$$2x = 1$$
$$x = \frac{1}{2}$$
$$8y = 6x + 1$$
$$y = \frac{6x + 1}{8}$$

$$y\left(-\frac{1}{8}\right) = \frac{6\left(-\frac{1}{8}\right) + 1}{8} = \frac{1}{32}$$

Using $y = 2x^2$

$$y' = 4x$$
$$y'\left(-\frac{1}{8}\right) = 4\left(-\frac{1}{8}\right) = -\frac{1}{2}$$
$$m_1 = -\frac{1}{2}$$

$$y - y_1 = m_1(x - x_1)$$
$$y - \frac{1}{32} = -\frac{1}{2}\left(x + \frac{1}{8}\right)$$
$$2y - \frac{1}{16} = -x - \frac{1}{8}$$
$$16x + 32y + 1 = 0$$

$$y\left(\frac{1}{2}\right)=\frac{6\left(\frac{1}{2}\right)+1}{8}=\frac{1}{2}$$

Using $y = 2x^2$

$$y' = 4x$$

$$y'\left(\frac{1}{2}\right)=4\left(\frac{1}{2}\right)=2$$

$$m_2 = 2$$

$$y-y_1=m_2\left(x-x_1\right)$$

$$y-\frac{1}{2}=2\left(x-\frac{1}{2}\right)$$

$$y-\frac{1}{2}=2x-1$$

$$2x-y-\frac{1}{2}=0$$

$$4x-2y-1=0$$

b $m_1 \times m_2 = 2\times\left(-\frac{1}{2}\right)=-1$

Therefore the 2 lines are perpendicular.

Question 10

$$f(x)=\frac{2}{x^3-8x^2+12x}$$

$$f(x)=\frac{2}{x\left(x-6\right)\left(x-2\right)}$$

$f(x)$ is not differentiable at x = 0, 2, 6.

Question 11

$S = 4\pi r^2 \qquad \frac{dS}{dt} = 0.3$

$\frac{dS}{dr} = 8\pi r$

a

$$\frac{dr}{dS} = \frac{1}{8\pi r}$$

$$\frac{dr}{dt} = \frac{dS}{dt} \times \frac{dr}{dS}$$

$$= 03 \times \frac{1}{8\pi r} = \frac{0.3}{8\pi r}$$

$$\frac{dr}{dt}(5) = \frac{03}{8\pi(5)} = 0.0024 \text{ cm/s}$$

b

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$= 4\pi r^2 \times \frac{0.3}{8\pi r} = \frac{0.3r}{2}$$

$$\frac{dV}{dt}(5) = \frac{0.3(5)}{2} = 0.75 \text{ cm}^3\text{/s}$$

Question 12

$y = x^2 - x - 4$

$y' = 2x - 1$

$y'(2) = 2(2) - 1 = 4 - 1 = 3$

$y(2) = (2)^2 - (2) - 4 = 4 - 2 - 4 = -2$

$y' = -1$

$-1 = 2x - 1$

$x = 0$

$y(0) = (0)^2 - (0) - 4 = 0 - 0 - 4 = -4$

$$m = \frac{rise}{run} = \frac{-2-(-4)}{2-0} = 1$$

$$y - y_1 = m(x - x_1)$$

$y + 4 = 1(x - 0)$

$y + 4 = x$

$x - y - 4 = 0$

Question 13

$f(x) = ax^2 + bx + c$

$f(2) = 4$	[1]
$f'(1) = 0$	[2]
$f'(-3) = 8$	[3]
$f'(x) = 2ax + b$	[4]
$0 = 2a + b$	[5] From [2] and [4]
$8 = -6a + b$	[6] From [3] and [4]
$-8 = 8a$	[5] – [6]
$a = -1$	
$b = 2$	Substituting into [5]
$4 = -(2)^2 + 2(2) + c$	From [1]
$c = 4$	

$a = -1$, $b = 2$, $c = 4$

Question 14

a $f(x + h) = (x + h)^3$

$= (x + h)(x + h)^2$

$= (x + h)(x^2 + 2xh + h^2)$

$= x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3$

$= x^3 + 3x^2h + 3xh^2 + h^3$

OR

$f(x + h) = (x + h)^3$

$= {}^3C_0\, x^3 + {}^3C_1\, x^2h + {}^3C_2\, xh^2 + {}^3C_3\, h^3$

$= 1x^3 + 3x^2h + 3xh^2 + 1h^3$

$= x^3 + 3x^2h + 3xh^2 + h^3$

b

$$f(x) = x^3$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0}\left(3x^2 + 3xh + h^2\right)$$

$$= 3x^2$$

Question 15

$$f(x)=\frac{1}{x}$$

a **i**

$$f(x)\left(1.1\right)=\frac{1}{\left(1.1\right)}=0.9091$$

$$\frac{y_2-y_1}{x_2-x_1}$$
$$=\frac{0.9091-1}{1.1-1}$$
$$=\frac{-0.0909}{0.1}$$
$$=-0.909$$

ii

$$f(x)\left(1.01\right)=\frac{1}{\left(1.01\right)}=0.990\ 099$$

$$\frac{y_2-y_1}{x_2-x_1}$$
$$=\frac{0990\ 099\ -1}{1.01-1}$$
$$=\frac{-0.009\ 900\ 99}{0.01}$$
$$=-0.99$$

iii

$$f(x)\left(0.99\right)=\frac{1}{\left(0.99\right)}=1.0101$$

$$\frac{y_2-y_1}{x_2-x_1}$$
$$=\frac{1.0101-1}{0.99-1}$$
$$=\frac{0.0101}{-0.01}$$
$$=-1.01$$

b gradient = –1

c

$$\frac{1}{x+h}-\frac{1}{x}$$
$$=\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}$$
$$=\frac{x-x-h}{x(x+h)}$$
$$=\frac{-h}{x(x+h)}$$

d

$$f(x)=\frac{1}{x}$$
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{-h}{x(x+h)}}{h}$$
$$=\lim_{h\to 0}\frac{-1}{x(x+h)}$$
$$=-\frac{1}{x^2}$$

Question 16

$$V=21x+x^2 \qquad \frac{dV}{dt}=15$$
$$\frac{dV}{dx}=21+2x$$
$$\frac{dx}{dV}=\frac{1}{21+2x}$$
$$\frac{dx}{dt}=\frac{dV}{dt}\times\frac{dx}{dV}$$
$$=\frac{15}{21+2x}$$
$$\frac{dx}{dt}(20)=\frac{15}{21+2(20)}=0.25\,\text{cm/s}$$

Question 17

$x = (t^3 + 1)^6$

a $x(0) = [0^3 + 1]^6 = (1)^6 = 1$ m

$$\dot{x} = 18t^2\left(t^3+1\right)^5$$

$$\dot{x}(0) = 18(0)^2\left(0^3+1\right)^5 = 0$$

b

$$\dot{x} = 18t^2\left(t^3+1\right)^5$$

$$u = 18t^2 \qquad v = \left(t^3+1\right)^5$$

$$u' = 36t \qquad v' = 15t^2\left(t^3+1\right)^4$$

$$\ddot{x} = u'v + v'u$$

$$= 36t \times \left(t^3+1\right)^5 + 15t^2\left(t^3+1\right)^4 \times 18t^2$$

$$= 36t\left(t^3+1\right)^5 + 270t^4\left(t^3+1\right)^4$$

$$\ddot{x}(2) = 270(2)^4\left((2)^3+1\right)^4 + 36(2)\left((2)^3+1\right)^5 = 3.26\times10^7 \text{ cm/s}$$

c $x = (t^3 + 1)^6$

As $t \geq 0$

$t^3 \geq 0$

$t^3 + 1 > 0$

Therefore the particle can never be at the origin.

Question 18

a $y = 3x + 1$

i To find y^{-1}

$x = 3y + 1$

ii $3y = x - 1$

$$y = \frac{x-1}{3}$$

iii $y = \frac{x}{3} - \frac{1}{3}$

$$\frac{dy}{dx} = \frac{1}{3}$$

iv $x = 3y + 1$

$$\frac{dx}{dy} = 3$$

v $\frac{dy}{dx} \times \frac{dx}{dy} = \frac{1}{3} \times 3 = 1$

b $f(x) = 3x^5$

i $f^{-1}(x) \Rightarrow x = 3y^5$

ii

$$x = 3y^5$$

$$y^5 = \frac{x}{3}$$

$$y = \sqrt[5]{\frac{x}{3}}$$

iii $y = \text{y} \left(\frac{x}{3}\right)^{\frac{1}{5}}$

$$\frac{dy}{dx} = \frac{1}{15}\left(\frac{x}{3}\right)^{-\frac{4}{5}}$$

iv $x = 3y^5$

$$\frac{dx}{dy} = 15y^4$$

But y = $\left(\frac{x}{3}\right)^{\frac{1}{5}}$

So $\frac{dx}{dy} = 15\left[\left(\frac{x}{3}\right)^{\frac{1}{5}}\right]^4 = 15\left(\frac{x}{3}\right)^{\frac{4}{5}}$

v

$$\frac{dy}{dx} \times \frac{dx}{dy} = \frac{1}{15}\left(\frac{x}{3}\right)^{-\frac{4}{5}} \times 15\left(\frac{x}{3}\right)^{\frac{4}{5}}$$

$$= \frac{1}{15} \times 15 \times \left(\frac{x}{3}\right)^{-\frac{4}{5}+\frac{4}{5}}$$

$$= 1$$

MATHS IN FOCUS 11
MATHEMATICS EXTENSION 1

WORKED SOLUTIONS

Chapter 9: Probability

Exercise 9.01 Set notation and Venn diagrams

Question 1

a {H, T}

b {1, 2, 3, 4, 5}

c {1, 2, 3, 4, 5, 6}

d {red, green, yellow, blue}

e {1, 2, 3, 4, 5, 6, 7, 8}

Question 2

a **i** {2, 4}

ii {1, 2, 3, 4, 5, 6}

b **i** {red, white}

ii {red, yellow, white}

c **i** { }

ii {4, 5, 6, 7, 8, 9, 10, 11, 12, 15}

d **i** {brown, blue}

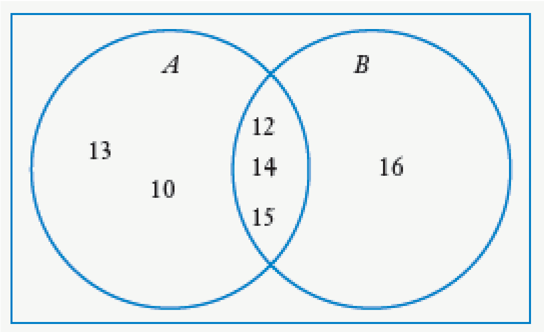
ii {Blue, green, brown, hazel, grey}

e **i** { }

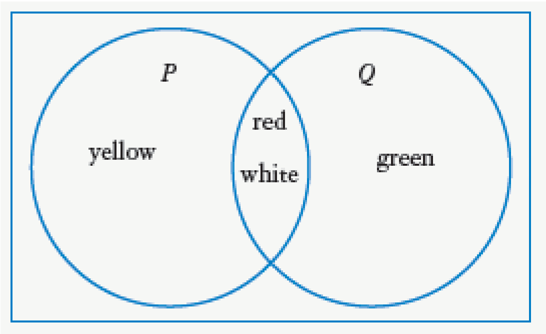
ii {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Question 3

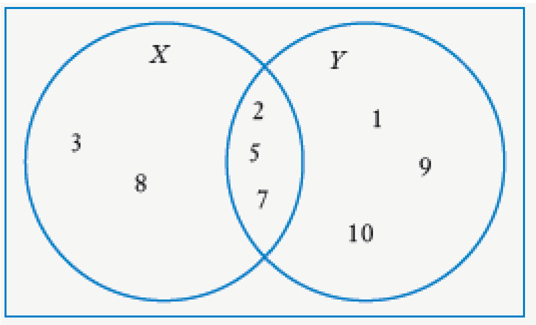
a



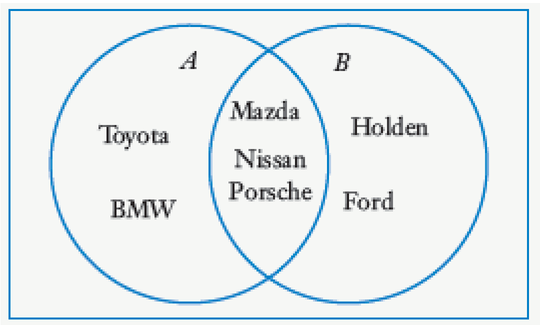
b



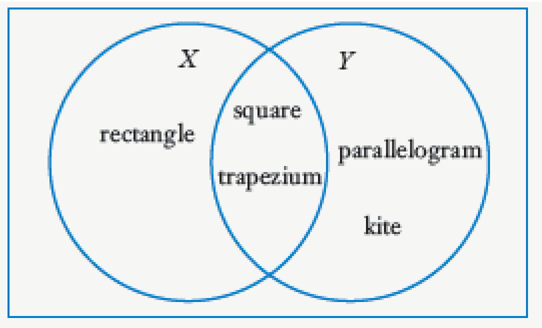
c



d



e



Question 4

a False. Not equally likely.

b False. Not equally likely.

c False. Independent events.

d False. Independent events.

e False. Not equally likely.

Question 5

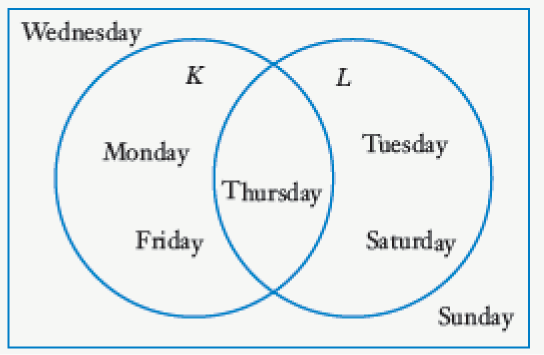
a {1, 2, 3, 4, 5, 6}

b {2, 4, 6}

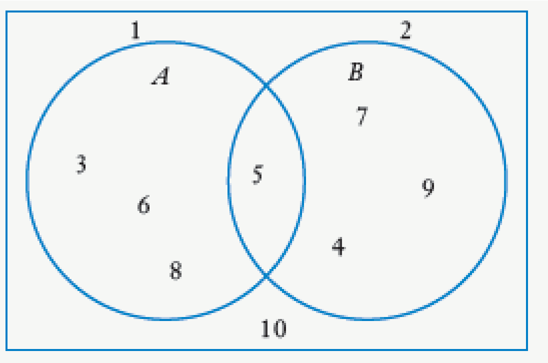
c $\frac{3}{6}=\frac{1}{2}=0.5$

Question 6

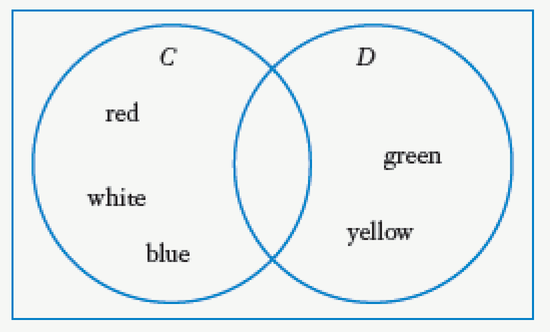
a



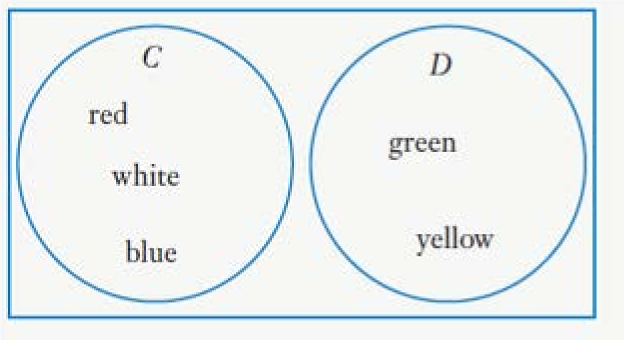
b



c



or



Exercise 9.02 Relative frequency

Question 1

a

Score	Frequency	Relative frequency
4	6	$\frac{6}{23}$
5	4	$\frac{4}{23}$
6	1	$\frac{1}{23}$
7	7	$\frac{7}{23}$
8	2	$\frac{2}{23}$
9	3	$\frac{3}{23}$

b **i** $\frac{2}{23}$

ii $\frac{6}{23}+\frac{4}{23}+\frac{1}{23}=\frac{11}{23}$

iii $\frac{4}{23}+\frac{1}{23}+\frac{7}{23}+\frac{2}{23}+\frac{3}{23}=\frac{17}{23}$

c **i** 7 days

ii 6 days

Question 2

a

Number of days	Frequency	Relative frequency
1	3	$\frac{3}{21}\times 100 = 15\%$
2	6	$\frac{7}{21}\times 100 = 30\%$
3	1	$\frac{1}{21}\times 100 = 5\%$
4	7	$\frac{7}{21}\times 100 = 35\%$
5	2	$\frac{2}{21}\times 100 = 10\%$
6	1	$\frac{1}{21}\times 100 = 5\%$
Total	**21**	

b 4 days

c **i** 15%

ii 10%

iii 35% + 5% = 40%

iv 35% + 10% + 5% = 50%

v 15% + 30% = 45%

Question 3

a

Class	Frequency	Relative frequency
0 – 19	9	$\frac{9}{50} = 0.18$
20 – 39	12	$\frac{12}{50} = 0.24$
40 – 59	18	$\frac{18}{50} = 0.36$
60 – 79	7	$\frac{7}{50} = 0.14$
80 – 99	4	$\frac{4}{50} = 0.08$
Total	**50**	

b **i** 0.24

ii 0.14 + 0.08 = 0.22

iii 0.18 + 0.24 = 0.42

Question 4

a

Velocity (m/s)	**Frequency**	**Relative frequency**
2 – 4	2	$\frac{2}{20}=\frac{1}{10}$
5 – 7	7	$\frac{7}{20}$
8 – 10	4	$\frac{4}{20}=\frac{1}{5}$
11 – 13	1	$\frac{1}{20}$
14 – 16	6	$\frac{6}{20}=\frac{3}{10}$
Total	**20**	

b **i** $\frac{7}{20}$

ii $\frac{1}{20}$

iii $\frac{1}{5}$

iv $\frac{1}{20}+\frac{6}{20}=\frac{7}{20}$

v $\frac{2}{20}+\frac{7}{20}+\frac{4}{20}=\frac{13}{20}$

c **i** $\frac{2}{20}+\frac{7}{20}=\frac{9}{20}$

ii $\frac{7}{20}+\frac{4}{20}+\frac{1}{20}+\frac{6}{20}=\frac{18}{20}=\frac{9}{10}$

iii $\frac{4}{20}+\frac{1}{20}+\frac{6}{20}=\frac{11}{20}$

Question 5

a $\frac{3}{30} \times 100 = 10\%$

b

Sales/min	**Frequency**	**Relative frequency**
0	4	$\frac{4}{30} \times 100 = 13.3\%$
1	12	$\frac{12}{30} \times 100 = 40\%$
2	6	$\frac{6}{30} \times 100 = 20\%$
3	3	$\frac{3}{30} \times 100 = 10\%$
4	0	$\frac{0}{30} \times 100 = 0\%$
5	5	$\frac{5}{30} \times 100 = 1.7\%$
Total	**30**	

c 1

d **i** 20%

ii 16.7%

iii 10% + 0% + 16.7% = 26.7%

Question 6

a

Score	**Frequency**	**Relative frequency**
3	1	$\frac{1}{20}$
4	4	$\frac{4}{20}=\frac{1}{5}$
5	3	$\frac{3}{20}$
6	3	$\frac{3}{20}$
7	3	$\frac{3}{20}$
8	2	$\frac{2}{20}=\frac{1}{10}$
9	4	$\frac{4}{20}=\frac{1}{5}$
Total	**20**	

b **i** $\frac{3}{20}$

ii $\frac{2}{20}+\frac{4}{20}=\frac{6}{20}=\frac{3}{10}$

iii $\frac{1}{20}+\frac{4}{20}=\frac{5}{20}=\frac{1}{4}$

iv $\frac{1}{20}+\frac{4}{20}+\frac{3}{20}+\frac{3}{20}+\frac{3}{20}=\frac{14}{20}=\frac{7}{10}$

Question 7

a

Score	Frequency	Relative frequency
4	1	$\frac{1}{20} = 0.05$
5	5	$\frac{5}{20} = 0.25$
6	3	$\frac{3}{20} = 0.15$
7	6	$\frac{6}{20} = 0.30$
8	2	$\frac{2}{20} = 0.1$
9	3	$\frac{3}{20} = 0.15$
Total	**20**	

b **i** 0.1

ii 0.15 + 0.30 + 0.1 + 0.15 = 0.7

iii 0.05 + 0.25 + 0.15 = 0.45

iv 0.25 + 0.15 + 0.30 + 0.1 + 0.15 = 0.95

v 1 – 0.15 = 0.85

Question 8

a

Ages	**Frequency**	**Relative frequency**
10 – 19	3	$\frac{3}{25}\times 100 = 12\%$
20 – 29	4	$\frac{4}{25}\times 100 = 16\%$
30 – 39	8	$\frac{8}{25}\times 100 = 32\%$
40 – 49	5	$\frac{5}{25}\times 100 = 20\%$
50 – 59	5	$\frac{5}{25}\times 100 = 20\%$
Total	**25**	

b **i** 32%

ii 12%

iii 20% + 20% = 40%

c **i** 12% + 16% + 32% = 60%

ii 20%

iii 16% + 32% + 20% = 68%

iv 32% + 20% + 20% = 72%

v 12% + 16% + 32% + 20% = 80%

Question 9

a Total = 28, February

b $\frac{4}{28}=\frac{1}{7}$

c $\frac{8}{28}+\frac{4}{28}+\frac{2}{28}=\frac{14}{28}\times 100=50\%$

d

Food (kg)	**Frequency**	**Relative frequency**
0 – 15	3	$\frac{3}{28}$
15 – 29	11	$\frac{11}{28}$
30 – 44	8	$\frac{8}{28}=\frac{2}{7}$
45 – 59	4	$\frac{4}{28}=\frac{1}{7}$
60 – 74	2	$\frac{2}{28}=\frac{1}{14}$
Total	**28**	

e **i** $\frac{8}{28}=\frac{2}{7}$

ii $\frac{4}{28}+\frac{2}{28}=\frac{6}{28}=\frac{3}{14}$

iii $\frac{3}{28}+\frac{11}{28}=\frac{14}{28}=\frac{1}{2}$

iv $\frac{11}{28}+\frac{8}{28}+\frac{4}{28}=\frac{23}{28}$

v $\frac{8}{28}+\frac{4}{28}+\frac{2}{28}=\frac{14}{28}=\frac{1}{2}$

Exercise 9.03 Theoretical probability

Question 1

a $P(\text{Peter}) = \frac{1}{30}$

b $P(\text{not Peter}) = 1 - P(\text{Peter}) = 1 - \frac{1}{30} = \frac{29}{30}$

Question 2

a $P(\text{ace of diamonds}) = \frac{1}{52}$

b $P(\text{not ace of diamonds}) = 1 - P(\text{ace of diamonds}) = 1 - \frac{1}{52} = \frac{51}{52}$

Question 3

$P(\text{correct newspaper}) = \frac{1}{6}$

Question 4

a $P(\text{winning}) = \frac{5}{200} = \frac{1}{40}$

b $P(\text{not winning}) = 1 - P(\text{winning}) = 1 - \frac{1}{40} = \frac{39}{40}$

Question 5

$P(\text{win first prize}) = \frac{10}{200\,000} = \frac{1}{20\,000}$

Question 6

a $P(\text{white}) = \dfrac{8}{6+8} = \dfrac{8}{14} = \dfrac{4}{7}$

b $P(\text{red}) = \dfrac{6}{6+8} = \dfrac{6}{14} = \dfrac{3}{7}$

Question 7

$P(\text{brown}) = \dfrac{3}{20+14+3} = \dfrac{3}{37}$

Question 8

a $P(\text{not on time}) = 1 - P(\text{on time}) = 1 - \dfrac{18}{33} = \dfrac{15}{33} = \dfrac{5}{11}$

b $P(\text{on time}) = \dfrac{18}{33} \times 352 = 192$

Question 9

a $P(\text{green}) = \dfrac{11}{5+4+11} = \dfrac{11}{20}$

b $P(\text{yellow or green}) = \dfrac{11+4}{5+4+11} = \dfrac{15}{20} = \dfrac{3}{4}$

Question 10

a $P(\text{not pink}) = 1 - P(\text{pink}) = 1 - \dfrac{7}{9} = \dfrac{2}{9}$

b $P(\text{pink}) = \dfrac{7}{9} \times 189 = 147$

Question 11

$P(\text{no disability}) = 100\% - P(\text{disability}) = 100\% - 0.2\% = 99.8\%$

Question 12

a $P(6) = \frac{1}{6}$

b $P(\text{even}) = \frac{3}{6} = \frac{1}{2}$

c $P(\text{number less than 3}) = \frac{2}{6} = \frac{1}{3}$

Question 13

a $P(80 \text{ or } 90) = \frac{2}{124} = \frac{1}{62}$

b $P(\text{multiple of 10}) = \frac{12}{124} = \frac{3}{31}$

c $P(\text{odd number}) = \frac{62}{124} = \frac{1}{2}$

d $P(\text{less than 100}) = \frac{99}{124}$

Question 14

a $P(\text{not breaking down}) = 100\% - P(\text{breakng down}) = 100\% - 1.5\% = 98.5$

b **i** $P(\text{break down}) = 1.5\% \times 2600 = 39$

ii $P(\text{no break dwn}) = 98.5\% \times 2600 = 2561$

Question 15

$P(\text{at least one head}) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$

Question 16

a $P(\text{odd number}) = \frac{8}{15}$

b $P(\text{number less than 8}) = \frac{7}{15}$

c $P(8) = \frac{1}{15}$

Question 17

a $P(\text{heads}) = \frac{1}{2}$

b $P(\text{heads}) = 1$

Question 18

a $P(\text{soccer}) = \frac{7}{12+7+3+5+4} = \frac{7}{31}$

b $P(\text{squash or swimming}) = \frac{3+4}{12+7+3+5+4} = \frac{7}{31}$

c $P(\text{tennis}) = \frac{12}{12+7+3+5+4} = \frac{12}{31}$

Question 19

$$P(\text{red or yellow}) = \frac{29+21}{29+17+21+19} = \frac{50}{86} = \frac{25}{43}$$

Question 20

$$P(\text{white}) = \frac{2}{9} \times 153 = 34$$

Question 21

$$P(\text{tails}) = \frac{1}{3}$$

Question 22

a $P(2) = \frac{1}{6}$

b $P(4) = \frac{2}{6} = \frac{1}{3}$

c $P(\text{less than } 5) = \frac{5}{6}$

Question 23

$$P(\text{at least 1 germinating}) = \frac{4}{49} + \frac{8}{49} + \frac{16}{49} + \frac{18}{49} = \frac{46}{49}$$

Question 24

a $P(\text{no friends}) = 1 - P(\text{friends}) = 1 - \frac{1}{15} - \frac{4}{15} - \frac{6}{15} - \frac{2}{15} = \frac{2}{15}$

b $P(\text{at least 1 friend}) = \frac{1}{15} + \frac{4}{15} + \frac{6}{15} + \frac{2}{15} = \frac{13}{15}$

Question 25

$A \cap B = 0$

Exercise 9.04 Addition rule of probability

Question 1

a $P(\text{divisible by } 3) = \frac{6}{20} = \frac{3}{10}$

b $P(\text{Less than 10 or divisible by } 3) = \frac{9+3}{20} = \frac{12}{20} = \frac{3}{5}$

c $P(\text{composite number}) = \frac{11}{20}$

d $P(\text{composite number or greater than } 12) = \frac{14}{20} = \frac{7}{10}$

Question 2

a $P(\text{multiple of } 5) = \frac{10}{50} = \frac{1}{5}$

b $P(\text{odd number}) = \frac{25}{50} = \frac{1}{2}$

c $P(\text{odd number or multiple of } 5) = \frac{30}{50} = \frac{3}{5}$

d $P(\text{even number or greater than } 40) = \frac{30}{50} = \frac{3}{5}$

e $P(\text{less than } 20) = \frac{19}{50}$

Question 3

a $P(\text{vowel}) = \frac{5}{26}$

b $P(\text{vowel or letter from random}) = \frac{9}{26}$

c $P(\text{consonant or letter from movies}) = \frac{24}{26} = \frac{12}{13}$

Question 4

a $P(\text{less than } 30) = \frac{29}{100}$

b $P(\text{odd number or greater than } 70) = \frac{65}{100} = \frac{13}{20}$

c $P(\text{divisible by } 5 \text{ or less than } 20) = \frac{36}{100} = \frac{9}{25}$

Question 5

a $P(\text{even number or less than } 10) = \frac{27}{45} = \frac{3}{5}$

b $P(\text{between } 1 \text{ and } 15 \text{ or divisble by } 6) = \frac{20}{45} = \frac{4}{9}$

c $P(\text{greater than } 30 \text{ or odd number}) = \frac{30}{45} = \frac{2}{3}$

Question 6

a $P(\text{sing and dance}) = \frac{19+16-28}{28} = \frac{6}{28} = \frac{3}{14}$

b $P(\text{only sing }) = \frac{19-6}{28} = \frac{13}{28}$

c $P(\text{only dance }) = \frac{15-6}{28} = \frac{9}{28}$

Question 7

$63+59-80 = 42$ both brown eyes and dark hair

a $P(\text{dark hair not brown eyes}) = \frac{63-42}{80} = \frac{21}{80}$

b $P(\text{brown eyes not dark hair}) = \frac{59-42}{80} = \frac{17}{80}$

c $P(\text{brown eyes and dark hair}) = \frac{42}{80} = \frac{21}{40}$

Question 8

$13+9-20 = 2$ both camera and computer

a $P(\text{camera and computer}) = \frac{2}{20} = \frac{1}{10}$

b $P(\text{computer only}) = \frac{13-2}{20} = \frac{11}{20}$

c $P(\text{camera only}) = \frac{9-2}{20} = \frac{7}{20}$

Question 9

$54+31-75=10$ both Geography and History

a $P(\text{only Geography})=\frac{31-10}{75}=\frac{21}{75}=\frac{7}{25}$

b $P(\text{both Geography and History})=\frac{10}{75}=\frac{2}{15}$

c $P(\text{History not Geography})=\frac{54-10}{75}=\frac{44}{75}$

Question 10

$14+12-20=6$ dogs do both

a $P(\text{both walk to heel and stay})=\frac{6}{20}=\frac{3}{10}$

b $P(\text{walk to heel but not stay})=\frac{14-6}{20}=\frac{8}{20}=\frac{2}{5}$

c $P(\text{stay but not walk to heel})=\frac{12-6}{20}=\frac{6}{20}=\frac{3}{10}$

Question 1

$$P(\text{one head}) = \frac{1}{2}$$
$$P(\text{two heads}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Question 2

$$P(\text{one tail}) = \frac{1}{2}$$
$$P(\text{three tails}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

Question 3

$$P(\text{one picture}) = \frac{1}{2}$$
$$P(\text{two picture}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Question 4

$$P(\text{red}) = \frac{5}{2+5+4} = \frac{5}{11}$$
$$P(\text{two red}) = \frac{5}{11} \times \frac{5}{11} = \frac{25}{121}$$

Question 5

a

$$P(\text{breakdown}) = 0.21$$
$$P(\text{both breakdown}) = 0.21 \times 0.21 = 0.0441$$

b

$$P(\text{no breakdown}) = 1 - P(\text{breakdown}) = 1 - 0.21 = 0.79$$
$$P(\text{neither breakdown}) = 0.79 \times 0.79 = 0.6241$$

Question 6

$$P(\text{flowering}) = 93\%$$
$$P(\text{three flowering}) = 93\% \times 93\% \times 93\% = 80.4\%$$

Question 7

$$P(\text{target}) = 69\%$$
$$P(\text{three targets}) = 69\% \times 69\% \times 69\% = 32.9\%$$

Question 8

a

$$P(\text{not albino}) = 1 - P(\text{albino}) = 1 - \frac{2}{33} = \frac{31}{33}$$
$$P(\text{no albinos}) = \frac{31}{33} \times \frac{31}{33} \times \frac{31}{33} = \frac{29791}{35937}$$

b

$$P(\text{albino}) = \frac{2}{33}$$
$$P(\text{all albinos}) = \frac{2}{33} \times \frac{2}{33} \times \frac{2}{33} = \frac{8}{35937}$$

c $P(\text{at least 1 albino}) = 1 - P(\text{no albino}) = 1 - \frac{29\,781}{35\,937} = \frac{6146}{35\,937}$

Question 9

a $P(\text{jam}) = \frac{1}{2400}$

b $P(2 \text{ jams}) = \frac{1}{2400} \times \frac{1}{2400} = \frac{1}{5\,760\,000}$

c $P(\text{both not jam}) = \frac{2399}{2400} \times \frac{2399}{2400} = \frac{5\,755\,201}{5\,760\,000}$

Question 10

a

$$P(6) = \frac{1}{6}$$

$$P(5\,\text{sixes}) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{7776}$$

b

$$P(\text{no } 6) = 1 - P(6) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(\text{no sixes}) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{3125}{7776}$$

c $P(\text{at least one } 6) = 1 - P(\text{no sixes}) = 1 - \frac{3125}{7776} = \frac{4651}{7776}$

Question 11

a $P(\text{both faulty}) = \frac{3}{5000} \times \frac{3}{5000} = \frac{9}{25\ 000\ 000}$

b

$$P(\text{not faulty}) = 1 - P(\text{faulty}) = 1 - \frac{3}{5000} = \frac{4997}{5000}$$

$$P(\text{neither faulty}) = \frac{4997}{5000} \times \frac{4997}{5000} = \frac{24\ 970\ 009}{25\ 000\ 000}$$

c $P(\text{at least one faulty}) = 1 - P(\text{none faulty}) = 1 - \frac{24\ 970\ 009}{25\ 000\ 000} = \frac{29\ 991}{25\ 000\ 000}$

Question 12

a

$$P(\text{odd}) = \frac{5}{10} = \frac{1}{2}$$

$$P(2\text{ odd}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

b

$$P(\text{divisible by } 3) = \frac{3}{10}$$

$$P(2\text{ divisible by } 3) = \frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$$

c

$$P(\text{less than } 4) = \frac{3}{10}$$

$$P(2\text{ less than } 4) = \frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$$

Question 13

a

$$P(\text{target}) = 85\%$$
$$P(3\text{ targets}) = 85\% \times 85\% \times 85\% = 61.4\%$$

b

$$P(\text{not target}) = 1 - P(\text{ target}) = 100\% - 85\% = 15\%$$
$$P(3\text{ not targets}) = 15\% \times 15\% \times 15\% = 0.34\%$$

c $P(\text{at least one target}) = 1 - P(\text{ no targets}) = 100\% - 0.34\% = 99.66\%$

Question 14

a

$$P(\text{no tail}) = P(\text{head}) = \frac{1}{2}$$
$$P(\text{no tails}) = \left(\frac{1}{2}\right)^n = \frac{1^n}{2^n} = \frac{1}{2^n}$$

b $P(\text{at least 1 tail}) = 1 - P(\text{no tails}) = 1 - \frac{1}{2^n} = \frac{2^n}{2^n} - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

Question 15

a

$$P(\text{green}) = \frac{6}{6+8} = \frac{6}{14} = \frac{3}{7}$$

$$P(\text{both green}) = \frac{3}{7} \times \frac{3}{7} = \frac{9}{49}$$

b

$$P(\text{green}) = \frac{6}{6+8} = \frac{6}{14} = \frac{3}{7}$$

$$P(\text{green after one selected}) = \frac{5}{5+8} = \frac{5}{13}$$

$$P(\text{green without replacement}) = \frac{3}{7} \times \frac{5}{13} = \frac{15}{91}$$

Question 16

$$P(\text{first win}) = \frac{10}{250} = \frac{1}{25}$$

$$P(\text{second win given first win}) = \frac{9}{249} = \frac{3}{83}$$

$$P(\text{win first and second}) = \frac{1}{25} \times \frac{3}{83} = \frac{3}{2075}$$

Question 17

$$P(\text{first red}) = \frac{20}{20+25} = \frac{20}{45} = \frac{4}{9}$$

$$P(\text{second red given first red}) = \frac{19}{44}$$

$$P(\text{red first and second}) = \frac{4}{9} \times \frac{19}{44} = \frac{76}{396} = \frac{19}{99}$$

Question 18

a

$$P(\text{less than } 10) = \frac{9}{100}$$

$$P(\text{second less than } 10) = \frac{8}{99}$$

$$P(2 \text{ less than } 10) = \frac{9}{100} \times \frac{8}{99} = \frac{72}{9900} = \frac{2}{275}$$

b

$$P(\text{even}) = \frac{50}{100} = \frac{1}{2}$$

$$P(\text{second even}) = \frac{49}{99}$$

$$P(2 \text{ even}) = \frac{1}{2} \times \frac{49}{99} = \frac{49}{198}$$

c

$$P(\text{neither multiple of } 5) = 1 - P(\text{multiple of } 5) = 1 - \frac{1}{5} = \frac{4}{5}$$

$$P(\text{second neither multiple of } 5) = \frac{80-1}{100-1} = \frac{79}{99}$$

$$P(2 \text{ neither multiple of } 5\text{s}) = \frac{4}{5} \times \frac{79}{99} = \frac{316}{495}$$

Question 19

a

$$P(\text{green}) = \frac{23}{23+19} = \frac{23}{42}$$
$$P(\text{second green}) = \frac{23-1}{42-1} = \frac{22}{41}$$
$$P(\text{both green}) = \frac{23}{42} \times \frac{22}{41} = \frac{506}{1722} = \frac{253}{861}$$

b

$$P(\text{red}) = \frac{19}{23+19} = \frac{19}{42}$$
$$P(\text{second red}) = \frac{19-1}{42-1} = \frac{18}{41}$$
$$P(\text{both red}) = \frac{19}{42} \times \frac{18}{41} = \frac{342}{1722} = \frac{57}{287}$$

Question 20

$$P(\text{apple}) = \frac{8}{8+9+3} = \frac{8}{20} = \frac{2}{5}$$
$$P(\text{second apple}) = \frac{8-1}{20-1} = \frac{7}{19}$$
$$P(\text{both apple}) = \frac{2}{5} \times \frac{7}{19} = \frac{14}{95}$$

Question 1

a $P(\mathrm{T})P(\mathrm{T})P(\mathrm{T})=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

b

$$P(\mathrm{H})P(\mathrm{T})P(\mathrm{T})=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$
$$P(\mathrm{T})P(\mathrm{H})P(\mathrm{T})=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$
$$P(\mathrm{T})P(\mathrm{T})P(\mathrm{H})=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$
$$=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

c $P(\text{at least 1 head})=1-P(\mathrm{TTT})=1-\frac{1}{8}=\frac{7}{8}$

Question 2

a $P(8, 8)=\frac{1}{30}\times\frac{1}{30}=\frac{1}{900}$

b $P(3, 18)=\frac{1}{30}\times\frac{1}{30}=\frac{1}{900}$

c

$$P(3, 18)=\frac{1}{30}\times\frac{1}{30}=\frac{1}{900}$$
$$P(18, 3)=\frac{1}{30}\times\frac{1}{30}=\frac{1}{900}$$
$$=\frac{1}{900}+\frac{1}{900}=\frac{2}{900}=\frac{1}{450}$$

Question 3

a $P(\text{RR}) = \frac{5}{13} \times \frac{5}{13} = \frac{25}{169}$

b

$$P(\text{RB}) = \frac{5}{13} \times \frac{8}{13} = \frac{40}{169}$$
$$P(\text{BR}) = \frac{8}{13} \times \frac{5}{13} = \frac{40}{169}$$
$$= \frac{40}{169} + \frac{40}{169} = \frac{80}{169}$$

Question 4

a

$$\text{Not white} = \text{O}$$
$$P(\text{OOO}) = 65\% \times 65\% \times 65\% = 27.5\%$$

b

$$P(\text{WWO}) = 35\% \times 35\% \times 65\% = 7.96\%$$
$$P(\text{OWW}) = 65\% \times 35\% \times 35\% = 7.96\%$$
$$P(\text{WOW}) = 35\% \times 65\% \times 35\% = 7.96\%$$
$$= 7.96\% + 7.96\% + 7.96\% = 23.9\%$$

c $P(\text{at least 1 white}) = 1 - P(\text{OOO}) = 1 - 27.5\% = 72.5\%$

Question 5

a

$$P(\text{ROR})=\frac{3}{10}\times\frac{7}{10}\times\frac{3}{10}=\frac{63}{1000}$$
$$P(\text{RRO})=\frac{3}{10}\times\frac{3}{10}\times\frac{7}{10}=\frac{63}{1000}$$
$$P(\text{ORR})=\frac{7}{10}\times\frac{3}{10}\times\frac{3}{10}=\frac{63}{1000}$$
$$\frac{63}{1000}+\frac{63}{1000}+\frac{63}{1000}=\frac{189}{1000}$$

b

$$P(\text{OOR})=\frac{7}{10}\times\frac{7}{10}\times\frac{3}{10}=\frac{147}{1000}$$
$$P(\text{ROO})=\frac{3}{10}\times\frac{7}{10}\times\frac{7}{10}=\frac{147}{1000}$$
$$P(\text{ORO})=\frac{7}{10}\times\frac{3}{10}\times\frac{7}{10}=\frac{147}{1000}$$
$$\frac{147}{1000}+\frac{147}{1000}+\frac{147}{1000}=\frac{441}{1000}$$

c

No rain = O

$$P(\text{at least one rain})=1-P(\text{OOO})=1-\frac{7}{10}\times\frac{7}{10}\times\frac{7}{10}=1-\frac{343}{1000}=\frac{657}{1000}$$

Question 6

a

No variegated leaf = O

$$P(\text{VVO})=0.85\times0.85\times0.15=0.108$$
$$P(\text{VOV})=0.85\times0.15\times0.85=0.108$$
$$P(\text{OVV})=0.15\times0.85\times0.85=0.108$$
$$=0.108+0.108+0.108=0.325$$

b $P(\text{OOO})=0.15\times0.15\times0.15=0.0034$

c $P(\text{at least one plant})=1-P(\text{OOO})=1-0.15\times0.15\times0.15=1-0.0034=0.997$

Question 7

a

$$P(\text{YB}) = \frac{3}{9} \times \frac{2}{9} = \frac{6}{81} = \frac{2}{27}$$
$$P(\text{BY}) = \frac{2}{9} \times \frac{3}{9} = \frac{6}{81} = \frac{2}{27}$$
$$= \frac{2}{27} + \frac{2}{27} = \frac{4}{27}$$

b

$$P(\text{YB}) = \frac{3}{9} \times \frac{2}{8} = \frac{6}{72} = \frac{1}{12}$$
$$P(\text{BY}) = \frac{2}{9} \times \frac{3}{8} = \frac{6}{72} = \frac{1}{12}$$
$$= \frac{1}{12} + \frac{1}{12} = \frac{2}{12} = \frac{1}{6}$$

Question 8

a $P(\text{W}) = \frac{4}{100} = \frac{1}{25}$

b $P(\text{WW}) = \frac{1}{25} \times \frac{3}{99} = \frac{3}{2475} = \frac{1}{825}$

c

$$P(\text{WL}) = \frac{1}{25} \times \frac{96}{99} = \frac{96}{2475} = \frac{32}{825}$$
$$P(\text{LW}) = \frac{96}{100} \times \frac{4}{99} = \frac{384}{9900} = \frac{32}{825}$$
$$= \frac{32}{825} + \frac{32}{825} = \frac{64}{825}$$

d $P(\text{LL}) = \frac{96}{100} \times \frac{95}{99} = \frac{9120}{9900} = \frac{152}{165}$

e $P(\text{at least one prize}) = 1 - P(\text{LL}) = 1 - \frac{96}{100} \times \frac{95}{99} = 1 - \frac{152}{165} = \frac{13}{165}$

Question 9

a

$$\text{B and }1: P(\text{F}) = \frac{8}{8+7} = \frac{8}{15}$$

$$\text{B and }2: P(\text{F}) = \frac{6}{6+9} = \frac{6}{15} = \frac{2}{5}$$

$$P(\text{FF}) = \frac{8}{15} \times \frac{2}{5} = \frac{16}{75}$$

b

$$\text{B and }1: P(\text{F}) = \frac{8}{8+7} = \frac{8}{15}$$

$$\text{B and }1: P(\text{M}) = \frac{7}{8+7} = \frac{7}{15}$$

$$\text{B and }2: P(\text{F}) = \frac{6}{6+9} = \frac{6}{15} = \frac{2}{5}$$

$$\text{B and }2: P(\text{M}) = \frac{9}{6+9} = \frac{9}{15} = \frac{3}{5}$$

$$P(\text{MF}) = \frac{7}{15} \times \frac{2}{5} = \frac{14}{75}$$

$$P(\text{FM}) = \frac{8}{15} \times \frac{3}{5} = \frac{24}{75}$$

$$= \frac{14}{75} + \frac{24}{75} = \frac{38}{75}$$

Question 10

a

$$P(\text{LW}) = \frac{3}{5} \times \frac{3}{4} = \frac{9}{20}$$

$$P(\text{WL}) = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$$

$$= \frac{9}{20} + \frac{1}{10} = \frac{11}{20}$$

b $P(\text{LL}) = \frac{3}{5} \times \frac{1}{4} = \frac{3}{20}$

Question 11

a

Not faulty $= \mathrm{O}$

$$P(\mathrm{FOO}) = \frac{3}{100} \times \frac{97}{100} \times \frac{97}{100} = \frac{28\,227}{1\,000\,000}$$

$$P(\mathrm{OFO}) = \frac{97}{100} \times \frac{3}{100} \times \frac{97}{100} = \frac{28\,227}{1\,000\,000}$$

$$P(\mathrm{OOF}) = \frac{97}{100} \times \frac{97}{100} \times \frac{3}{100} = \frac{28\,227}{1\,000\,000}$$

$$= \frac{28\,227}{1\,000\,000} + \frac{28\,227}{1\,000\,000} + \frac{28\,227}{1\,000\,000} = \frac{84\,681}{1\,000\,000}$$

b

Not faulty $= \mathrm{O}$

$$P(\mathrm{OOO}) = \frac{97}{100} \times \frac{97}{100} \times \frac{97}{100} = \frac{912\,673}{1\,000\,000}$$

c $$P(\mathrm{FFF}) = \frac{3}{100} \times \frac{3}{100} \times \frac{3}{100} = \frac{27}{1\,000\,000}$$

Question 12

a $P(\mathrm{OO}) = 42\% \times 42\% = 17.6\%$

b

$P(\mathrm{CN}) = 46\% \times 12\% = 5.5\%$

$P(\mathrm{NC}) = 12\% \times 46\% = 5.5\%$

$= 5.5\% + 5.5\% = 11\%$

c $P(\mathrm{CC}) = 46\% \times 46\% = 21.2\%$

Question 13

a

$$P(\text{XN}) = \frac{31}{55} \times \frac{19}{55} = \frac{589}{3025}$$

$$P(\text{XA}) = \frac{31}{55} \times \frac{5}{55} = \frac{155}{3025}$$

$$P(\text{AX}) = \frac{5}{55} \times \frac{31}{55} = \frac{155}{3025}$$

$$P(\text{NX}) = \frac{19}{55} \times \frac{31}{55} = \frac{589}{3025}$$

$$= \frac{589}{3025} + \frac{155}{3025} + \frac{589}{3025} + \frac{155}{3025} = \frac{1488}{3025}$$

b $$P(\text{NN}) = \frac{5}{55} \times \frac{5}{55} = \frac{25}{3025} = \frac{1}{121}$$

Question 14

a

Asia $=$ O

$$P(\text{OO}) = \frac{12}{51} \times \frac{11}{50} = \frac{121}{5100} = \frac{22}{425}$$

b

$$\begin{aligned} P(\text{at least one Australian}) &= 1 - P(\text{no Australian}) \\ &= 1 - \frac{19}{51} \times \frac{18}{50} \\ &= \frac{368}{425} \end{aligned}$$

c $$P(\text{EE}) = \frac{7}{51} \times \frac{6}{50} = \frac{42}{2550} = \frac{21}{1275} = \frac{7}{425}$$

Question 15

a $P(\text{MM}) = \frac{34}{66} \times \frac{33}{65} = \frac{1122}{4290} = \frac{17}{65}$

b

$$P(\text{MW, MM}) = \frac{19}{66} \times \frac{21}{65} = \frac{399}{4290} = \frac{133}{1430}$$

$$P(\text{MM, MW}) = \frac{21}{66} \times \frac{19}{65} = \frac{399}{4290} = \frac{133}{1430}$$

$$= \frac{133}{1430} + \frac{133}{1430} = \frac{266}{1430} = \frac{133}{715}$$

c $P(\text{Married, Married}) = \frac{32}{66} \times \frac{31}{65} = \frac{992}{4290} = \frac{496}{2145}$

Question 16

a $P(6,6,6) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

b

$$\text{Not } 6 = \text{O}$$

$$P(6,6,\text{O}) = \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{216}$$

$$P(6,\text{O},6) = \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{5}{216}$$

$$P(\text{O},6,6) = \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{5}{216}$$

$$= \frac{5}{216} + \frac{5}{216} + \frac{5}{216} = \frac{15}{216} = \frac{5}{72}$$

c

$$P(\text{at least one } 6) = 1 - P(\text{no 6s}) = 1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$$

$$= 1 - \frac{125}{216} = \frac{91}{216}$$

Question 17

a

$$P(\text{D, E}) = \frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$$
$$P(\text{E, D}) = \frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$$
$$= \frac{1}{20} + \frac{1}{20} = \frac{2}{20} = \frac{1}{10}$$

b

Not D or E = O

$$P(\text{OO}) = \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$$

c $P(\text{at least one D}) = 1 - P(\text{no D}) = 1 - \frac{4}{5} \times \frac{3}{4} = 1 - \frac{3}{5} = \frac{2}{5}$

Question 18

a $P(\text{BB}) = \frac{5}{9} \times \frac{5}{9} = \frac{25}{81}$

b

$$P(\text{BG}) = \frac{5}{9} \times \frac{4}{9} = \frac{20}{81}$$
$$P(\text{GB}) = \frac{4}{9} \times \frac{5}{9} = \frac{20}{81}$$
$$= \frac{20}{81} + \frac{20}{81} = \frac{40}{81}$$

c $P(\text{at least one girl}) = 1 - P(\text{no girl}) = 1 - \frac{5}{9} \times \frac{5}{9} = \frac{56}{81}$

Exercise 9.07 Conditional probability

Question 1

a $P(B|W)=\frac{9}{9+7}=\frac{9}{16}$

b $P(W|W)=\frac{7}{9+7}=\frac{7}{16}$

Question 2

$P(B|G)=\frac{13}{13+15-1}=\frac{13}{27}$

Question 3

a $P(6|6)=\frac{1}{6}$

b $P(\text{sum}\geq 8|3)=\frac{2}{6}=\frac{1}{3}$

Question 4

$P(W2|W1)=\frac{P(W2\cap W1)}{P(W1)}=\frac{39\%}{52\%}=75\%$

Question 5

$P(T2|T1)=\frac{P(T2\cap T1)}{P(T1)}=\frac{0.65}{0.75}=0867 =8.7\%$

Question 6

$$P(E2|\ E1)=\frac{P(E2\cap E1)}{P(E1)}=\frac{45\%}{80\%}=56.25\%$$

Question 7

a $P(H\mid S)=\dfrac{P(S\cap H)}{P(S)}=\dfrac{3}{10}\div\dfrac{8}{10}=\dfrac{3}{8}$

b $P(\overline{S}\mid H)=\dfrac{P(\overline{S}\cap H)}{P(H)}=\dfrac{2}{10}\div\dfrac{5}{10}=\dfrac{2}{5}$

Question 8

a $P(O\,|\,7)=\dfrac{10-1}{20-1}=\dfrac{9}{19}$

b $P(<5\,|\,12)=\dfrac{4}{20-1}=\dfrac{4}{19}$

c $P(\text{divisable by }3\,|\,6)=\dfrac{6-1}{20-1}=\dfrac{5}{19}$

Question 9

a

$9+7-12=4$ had chips and pie

$P(C\mid P)=\dfrac{P(C\cap P)}{P(P)}=\dfrac{4}{12}\div\dfrac{9}{12}=\dfrac{4}{9}$

b $P(\overline{P}\mid C)=\dfrac{P(C\cap\overline{P})}{P(C)}=\dfrac{3}{12}\div\dfrac{7}{12}=\dfrac{3}{7}$

Question 10.

a $P(S \mid F)=\dfrac{P(S\cap F)}{P(F)}=\dfrac{5}{25}\div\dfrac{9+5}{25}=\dfrac{5}{25}\div\dfrac{14}{25}=\dfrac{5}{14}$

b $P(\overline{F} \mid S)=\dfrac{P(S\cap \overline{F})}{P(S)}=\dfrac{25-3-9-5}{25}\div\dfrac{25-3-9}{25}=\dfrac{8}{25}\div\dfrac{13}{25}=\dfrac{8}{13}$

Question 11

a **i** $P(S \mid T)=\dfrac{P(S\cap T)}{P(T)}=\dfrac{23}{99}\div\dfrac{31}{99}=\dfrac{23}{31}$

ii $P(S \mid \overline{T})=\dfrac{P(S\cap \overline{T})}{P(\overline{T})}=\dfrac{65}{99}\div\dfrac{68}{99}=\dfrac{65}{68}$

b **i** $P(T \mid S)=\dfrac{P(S\cap T)}{P(S)}=\dfrac{23}{99}\div\dfrac{88}{99}=\dfrac{23}{88}$

ii $P(T \mid \overline{S})=\dfrac{P(\overline{S}\cap T)}{P(\overline{S})}=\dfrac{8}{99}\div\dfrac{11}{99}=\dfrac{8}{11}$

Question 12

	Women	Men	Total
Permanent	23	38	61
Casual	79	64	143
Total	102	102	204

a $P(P\,|\,W)=\dfrac{P(P\cap W)}{P(W)}=\dfrac{23}{204}\div\dfrac{102}{204}=\dfrac{23}{102}$

b $P(C\,|\,M)=\dfrac{P(C\cap M)}{P(M)}=\dfrac{64}{204}\div\dfrac{102}{204}=\dfrac{64}{102}=\dfrac{32}{51}$

c $P(M\,|\,C)=\dfrac{P(C\cap M)}{P(C)}=\dfrac{64}{204}\div\dfrac{143}{204}=\dfrac{64}{143}$

d $P(M\,|\,P)=\dfrac{P(M\cap P)}{P(P)}=\dfrac{38}{204}\div\dfrac{61}{204}=\dfrac{38}{61}$

Question 13

a $P(M\,|\,S)=\dfrac{P(M\cap S)}{P(S)}=\dfrac{14}{35}\div\dfrac{8+14}{35}=\dfrac{14}{35}\div\dfrac{22}{35}=\dfrac{14}{22}=\dfrac{7}{11}$

b $P(\overline{S}\,|\,M)=\dfrac{P(M\cap\overline{S})}{P(M)}=\dfrac{35-8-14}{35}\div\dfrac{35-8}{35}=\dfrac{13}{35}\div\dfrac{27}{35}=\dfrac{13}{27}$

Question 14

a $P(T\,|>50)=\dfrac{P(T\cap>50)}{P(>50)}=\dfrac{34}{433}\div\dfrac{122}{433}=\dfrac{34}{122}=\dfrac{17}{61}$

b $P(25-50\,|\,\overline{T})=\dfrac{P(\overline{T}\cap 25-50)}{P(\overline{T})}=\dfrac{105}{433}\div\dfrac{278}{433}=\dfrac{105}{278}$

c $P\left(\overline{T} \mid < 25\right) = \dfrac{P\left(\overline{T} \cap < 25\right)}{P\left(< 25\right)} = \dfrac{85}{433} \div \dfrac{138}{433} = \dfrac{85}{138}$

d $P\left(> 50 \mid T\right) = \dfrac{P\left(T \cap > 50\right)}{P\left(T\right)} = \dfrac{34}{433} \div \dfrac{155}{433} = \dfrac{34}{155}$

e $P\left(T \mid > 25\right) = \dfrac{P\left(T \cap > 25\right)}{P\left(> 25\right)} = \dfrac{68+34}{433} \div \dfrac{173+122}{433} = \dfrac{102}{433} \div \dfrac{295}{433} = \dfrac{102}{295}$

Question 15

a $P\left(W\right) = 45\%$

b

$$
\begin{aligned}
P\left(H \mid W\right) &= \frac{P(H \cap W)}{P(W)} \\
&= \frac{076 \times .42}{0.76 \times 0.42 + 0.45 \times 0.58} \\
&= \frac{0.3192}{03192 + .261} \\
&= \frac{0.3192}{0.5802} \\
&\approx 55\%
\end{aligned}
$$

c

$$
\begin{aligned}
P\left(A \mid L\right) &= \frac{P(A \cap L)}{P(L)} \\
&= \frac{0.58 \times 0.55}{0.58 \times 0.55 + 0.42 \times 0.24} \\
&= \frac{0.319}{0.319 + 0.1008} \\
&= \frac{0.319}{0.4198} \\
&\approx 76\%
\end{aligned}
$$

Question 16

$$P(M \mid D) = \frac{P(M \cap D)}{P(D)}$$
$$= \frac{39\% \times 2\%}{61\% \times 3\% + 39\% \times 2\%}$$
$$= \frac{0.0078}{0.0261}$$
$$\approx 29.9\%$$

Question 17

a $P(\overline{B} \mid R) = 100\% - 21\% = 79\%$

b $P(R \mid B) = \dfrac{P(B \cap R)}{P(B)} = \dfrac{21\% \times 46\%}{21\% \times 46\% + 85\% \times 54\%} = \dfrac{0.0966}{0.5556} = 17.4\%$

c $P(\overline{R} \mid B) = \dfrac{P(\overline{R} \cap B)}{P(B)} = \dfrac{85\% \times 54\%}{21\% \times 46\% + 85\% \times 54\%} = 82.6\%$

d $P(R \mid \overline{B}) = \dfrac{P(R \cap \overline{B})}{P(\overline{B})} = \dfrac{79\% \times 46\%}{1 - P(B)} = \dfrac{0.3634}{1 - 0.5556} = 81.8\%$

Question 18

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$
$$0.67 = \frac{P(A \cap B)}{0.31}$$
$$P(A \cap B) = 0.67 \times 0.31$$
$$P(A \cap B) = 0.2077$$

Question 19

$P(L)\ P(M) = 0.17 \times 0.12$

$= 0.0204$

$= P(L \cap M)$

So L and M are independent.

Question 20

$P(X \cup Y) = P(X) + P(Y) – P(X \cap Y)$

$0.594 = 0.3 + 0.42 – P(X \cap Y)$

$= 0.72 – P(X \cap Y)$

$–0.126 = –P(X \cap Y)$

$0.126 = P(X \cap Y)$

$P(X)\ P(Y) = 0.3 \times 0.42$

$= 0.126$

$= P(X \cap Y)$

So X and Y are independent.

Question 1

$$\begin{aligned} P(\text{at least one } 1) &= 1 - P(\text{no } 1\text{s}) \\ &= 1 - \frac{5}{6} \times \frac{5}{6} \\ &= 1 - \frac{25}{36} \\ &= \frac{11}{36} \end{aligned}$$

C

Question 2

$$P(\text{BW}) = \frac{5}{12} \times \frac{7}{11} = \frac{35}{132}$$
$$P(\text{WB}) = \frac{7}{12} \times \frac{5}{11} = \frac{35}{132}$$
$$= \frac{35}{132} + \frac{35}{132} = \frac{70}{132} = \frac{35}{66}$$

D

Question 3

$A \cap B$

B

Question 4

Score	Frequency
8	5
9	2
10	9
11	8
12	1
Total	**25**

$\frac{8}{25} = 0.32 = 32\%$

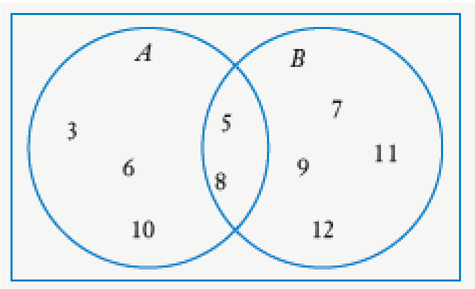
A, D

Question 5

a **i** $A \cup B = \{3,5,6,7,8,9,10,11,12\}$

ii $A \cap B = \{5,8\}$

b



Question 6

a {HH, HT, TH, TT}

b {red, white, blue}

Question 7

a

Face	**Frequency**	**Relative frequency**
1	17	$\frac{17}{100}$
2	21	$\frac{21}{100}$
3	14	$\frac{14}{100}=\frac{7}{50}$
4	20	$\frac{20}{100}=\frac{1}{5}$
5	18	$\frac{18}{100}=\frac{9}{50}$
6	10	$\frac{10}{100}=\frac{1}{10}$
Total	**100**	

b **i** $\frac{7}{50}$

ii $\frac{18+10}{100}=\frac{28}{100}=\frac{7}{25}$

iii $\frac{1}{10}$

iv $\frac{21}{100}+\frac{17}{100}=\frac{38}{100}=\frac{19}{50}$

v $\frac{21}{100}+\frac{17}{100}+\frac{7}{50}=\frac{52}{100}=\frac{13}{25}$

Question 8

a $P(\text{GGG}) = 93\% \times 93\% \times 93\% = 80.4\%$

b

$$P(\text{G}\overline{\text{G}}\overline{\text{G}}) = 93\% \times 7\% \times 7\% = 0.45\%$$
$$P(\overline{\text{G}}\text{G}\overline{\text{G}}) = 7\% \times 93\% \times 7\% = 0.45\%$$
$$P(\overline{\text{G}}\overline{\text{G}}\text{G}) = 7\% \times 7\% \times 93\% = 0.45\%$$
$$= 3 \times 0.45\% = 1.4\%$$

c $P(\text{at least one}) = 100\% - P(\text{none}) = 100\% - 7\% \times 7\% \times 7\% = 99.97\%$

Question 9

a

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

b **i** $\frac{6}{36} = \frac{1}{6}$

ii $\frac{6}{36} = \frac{1}{6}$

iii $\frac{18}{36} = \frac{1}{2}$

Question 10

a **i** $P(W) = \frac{5}{200} = \frac{1}{40}$

ii $P(\overline{W}) = \frac{195}{200} = \frac{39}{40}$

b

$$P(W\,2\,1\overline{W}\;) = \frac{5}{199}$$

$$P(W\overline{W}) = \frac{5}{200} \times \frac{195}{199} = \frac{975}{39800}$$
$$P(\overline{W}W) = \frac{195}{200} \times \frac{5}{199} = \frac{975}{39800}$$
$$= \frac{975}{39800} + \frac{975}{39800} = \frac{1950}{39800} = \frac{39}{796}$$

Question 11

a $P(G \cap \overline{H}) = \frac{11-3}{30} = \frac{8}{30} = \frac{4}{15}$

b $P(G \cap H) = \frac{3}{30} = \frac{1}{10}$

c $P(G \mid H) = \frac{P(G \cap H)}{P(H)} = \frac{1}{10} \div \frac{17}{30} = \frac{3}{17}$

d $P(H \mid G) = \frac{P(G \cap H)}{P(G)} = \frac{1}{10} \div \frac{11}{30} = \frac{3}{11}$

Question 12

False; the events are independent.

Question 13

a $\dfrac{50}{100}=\dfrac{1}{2}$

b $\dfrac{29}{100}$

c $\dfrac{20}{100}=\dfrac{1}{5}$

d $\dfrac{29+20-5}{100}=\dfrac{44}{100}=\dfrac{11}{25}$

e $\dfrac{50+29-15}{100}=\dfrac{64}{100}=\dfrac{16}{25}$

Question 14

a $P\left(W1,W2\right)=\dfrac{3}{5}\times\dfrac{2}{3}=\dfrac{6}{15}=\dfrac{2}{5}$

b

$$P\left(W1,L2\right)=\frac{3}{5}\times\frac{1}{3}=\frac{3}{15}=\frac{1}{5}$$

$$P\left(L1,W2\right)=\frac{2}{5}\times\frac{2}{3}=\frac{4}{15}$$

$$=\frac{4}{15}+\frac{1}{5}=\frac{7}{15}$$

c $P\left(L1,W2\right)=\dfrac{2}{5}\times\dfrac{1}{3}=\dfrac{2}{15}$

Question 15

$$P\left(\text{BW}\right)=\frac{5}{12}\times\frac{7}{11}=\frac{35}{132}$$

$$P\left(\text{WB}\right)=\frac{7}{12}\times\frac{5}{11}=\frac{35}{132}$$

$$=\frac{35}{132}+\frac{35}{132}=\frac{70}{132}=\frac{35}{66}$$

Question 16

$$\frac{1}{7}\times\frac{1}{8}=\frac{1}{56}$$

Question 17

a $P(\text{BBB})=4.5\%\times 4.5\%\times 4.5\%=0.009\%$

b $P(\text{at least one})=100\%-P(\text{none})=100\%-95.5\%\times 95.5\%\times 95.5\%=12.9\%$

Question 18

a **i** $P(\text{YY})=\frac{4}{13}\times\frac{3}{12}=\frac{12}{156}=\frac{1}{13}$

ii

$$P(\text{RB})=\frac{3}{13}\times\frac{6}{12}=\frac{18}{156}=\frac{3}{26}$$
$$P(\text{BR})=\frac{6}{13}\times\frac{3}{12}=\frac{18}{156}=\frac{3}{26}$$
$$=\frac{3}{26}+\frac{3}{26}=\frac{6}{26}=\frac{3}{13}$$

iii $P(\text{BB})=\frac{6}{13}\times\frac{5}{12}=\frac{30}{156}=\frac{5}{26}$

b **i** $P(Y\mid B)=\frac{4}{12}=\frac{1}{3}$

ii $P(R\mid Y)=\frac{3}{12}=\frac{1}{4}$

Question 19

a

$$8+9-12=5\text{ both}$$

$$P(\text{Both})=\frac{5}{12}$$

b $P(\text{MI9})=\frac{9-5}{12}=\frac{4}{12}=\frac{1}{3}$

Question 20

a $P(\text{L})=1-P(\text{W})-P(\text{D})=1-\frac{2}{5}-\frac{3}{8}=\frac{9}{40}$

b **i**

$$P(\text{WD})=\frac{2}{5}\times\frac{3}{8}=\frac{6}{40}=\frac{3}{20}$$

$$P(\text{DW})=\frac{3}{8}\times\frac{2}{5}=\frac{6}{40}=\frac{3}{20}$$

$$=\frac{3}{20}+\frac{3}{20}=\frac{6}{20}=\frac{3}{10}$$

ii

$$P(\text{LD})=\frac{9}{40}\times\frac{3}{8}=\frac{27}{320}$$

$$P(\text{DL})=\frac{3}{8}\times\frac{9}{40}=\frac{27}{320}$$

$$=\frac{27}{320}+\frac{27}{320}=\frac{54}{320}=\frac{27}{160}$$

iii $P(\text{WW})=\frac{2}{5}\times\frac{2}{5}=\frac{4}{25}$

Question 21

a $P(911) = \frac{1}{10} \times \frac{1}{20} = \frac{1}{200}$

b $P(<100) = \frac{9}{10} \times \frac{9}{20} = \frac{81}{200}$

c $P(300-500) = \frac{2}{10} \times \frac{11}{20} = \frac{22}{200} = \frac{11}{100}$

Question 22

a

$$1 - \frac{2}{3} = \frac{1}{3} \div 5 = \frac{1}{15}$$
$$P(2) = \frac{1}{15}$$

b $P(\text{even}) = P(2) + P(4) + P(6) = \frac{1}{15} + \frac{1}{15} + \frac{2}{3} = \frac{4}{5}$

Question 23

a $P(W) = \frac{3}{150} = \frac{1}{50}$

b $P(\bar{W}W) = \frac{147}{150} \times \frac{3}{149} = \frac{441}{22\,350} = \frac{147}{7450}$

c $P(WW) = \frac{3}{150} \times \frac{2}{149} = \frac{6}{22\,350} = \frac{1}{3725}$

d $P(\bar{W}\bar{W}) = \frac{147}{150} \times \frac{146}{149} = \frac{21\,462}{22\,350} = \frac{3577}{3725}$

Question 24

a

$$P(\text{RB}) = \frac{8}{19} \times \frac{5}{19} = \frac{40}{361}$$
$$P(\text{BR}) = \frac{5}{19} \times \frac{8}{19} = \frac{40}{361}$$
$$= \frac{40}{361} + \frac{40}{361} = \frac{80}{361}$$

b

$$P(\text{RB}) = \frac{8}{19} \times \frac{5}{18} = \frac{40}{342} = \frac{20}{171}$$
$$P(\text{BR}) = \frac{5}{19} \times \frac{8}{18} = \frac{40}{342} = \frac{20}{171}$$
$$= \frac{20}{171} + \frac{20}{171} = \frac{40}{171}$$

Question 25

a

$$7 + 5 - 9 = 3 \text{ both}$$
$$P(P \cap \overline{I}) = \frac{5-3}{9} = \frac{2}{9}$$

b $P(P \cap I) = \frac{3}{9} = \frac{1}{3}$

c $P(P \mid I) = \frac{3}{7}$

Question 26

a $P(\text{AB}) = \frac{3}{5} \times \frac{7}{10} = \frac{21}{50}$

b $P(\bar{\text{A}}\bar{\text{B}}) = \frac{2}{5} \times \frac{3}{10} = \frac{6}{50} = \frac{3}{25}$

c

$$P(\bar{\text{A}}\text{B}) = \frac{2}{5} \times \frac{7}{10} = \frac{14}{50} = \frac{7}{25}$$
$$P(\text{A}\bar{\text{B}}) = \frac{3}{5} \times \frac{3}{10} = \frac{9}{50}$$
$$= \frac{9}{50} + \frac{7}{25} = \frac{23}{50}$$

Question 1

a

$$25+15-35=5\text{ both}$$
$$P(\text{M})=\frac{20}{35}\times\frac{19}{34}=\frac{380}{1190}=\frac{38}{119}$$

b

$$P(\text{F})\times P(\text{both})=\frac{10}{35}\times\frac{5}{34}=\frac{50}{1190}=\frac{5}{119}$$
$$P(\text{both})\times P(\text{F})=\frac{5}{35}\times\frac{10}{34}=\frac{50}{1190}=\frac{5}{119}$$

P(one likes both and one only goes to league game) = $\frac{5}{119}+\frac{5}{119}=\frac{10}{119}$

Question 2

$$P(\text{L})=1-P(\text{W})-P(\text{D})=1-0.5-0.2=0.3$$

a $P(\text{DD})=0.2\times0.2=0.04$

b

$$\text{O}=\text{lose or draw}$$
$$P(\text{WO})=0.5\times0.5=0.25$$
$$P(\text{OW})=0.5\times0.5=0.25$$
$$P(\text{WW})=0.5\times0.5=0.25$$
$$=0.25\times3=0.75$$

c $P(\text{OO})=0.5\times0.5=0.25$

Question 3

$$P(\text{A, A, A, A, O}) = \frac{4}{52}\times\frac{3}{51}\times\frac{2}{50}\times\frac{1}{49} = \frac{24}{6\,497\,400} = \frac{1}{270\,725}$$

$$= 5\times\frac{1}{270\,725} = \frac{1}{54\,145}$$

Question 4

a $P(\text{ace or heart}) = \frac{4+13-1}{52} = \frac{16}{52} = \frac{4}{13}$

b $P(\text{diamond or odd}) = \frac{13+16-4}{52} = \frac{25}{52}$

c $P(\text{jack or spade}) = \frac{4+13-1}{52} = \frac{16}{52} = \frac{4}{13}$

Question 5

No, all numbers are equally likely to win.

Question 6

a

$30-19-2=9$ just sport

$P(\text{just sport}) = \frac{9}{30} = \frac{3}{10}$

b $P(\text{just sport}) = \frac{9}{30}\times\frac{8}{29} = \frac{72}{870} = \frac{12}{145}$

Question 7

a $P\left(2\text{ heads and double }6\right)=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{6}\times\frac{1}{6}=\frac{1}{144}$

b $P\left(2\text{ heads and double}\right)=\frac{1}{2}\times\frac{1}{2}\times\frac{5}{6}\times\frac{1}{6}=\frac{5}{144}$

c One way to score 13 is combinations of 5, 5, 3:

$$P(5, 5, 3)=\frac{1}{144}\times\frac{1}{144}\times\frac{5}{144}=\frac{5}{2\,985\,984}$$
$$P(3, 5, 5)=\frac{5}{144}\times\frac{1}{144}\times\frac{1}{144}=\frac{5}{2\,985\,984}$$
$$P(5, 3, 5)=\frac{1}{144}\times\frac{5}{144}\times\frac{1}{144}=\frac{5}{2\,985\,984}$$

Other way to score 13 is combinations of 4, 4, 5.

$$P\left(2\text{ tails and double }6\right)=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{6}\times\frac{1}{6}=\frac{1}{144}$$

$$P(4, 4, 5) = P(4, 5, 4) = P(5, 4, 4) =\frac{1}{144}\times\frac{1}{144}\times\frac{1}{144}=\frac{1}{2\,985\,984}$$

$$P(\text{score of }13) = \frac{15}{2\,985\,984}+\frac{3}{2\,985\,984}=\frac{18}{2\,985\,984}=\frac{1}{165\,888}$$

Question 8

$P(B) = 0.52$ (52%) where B = boy

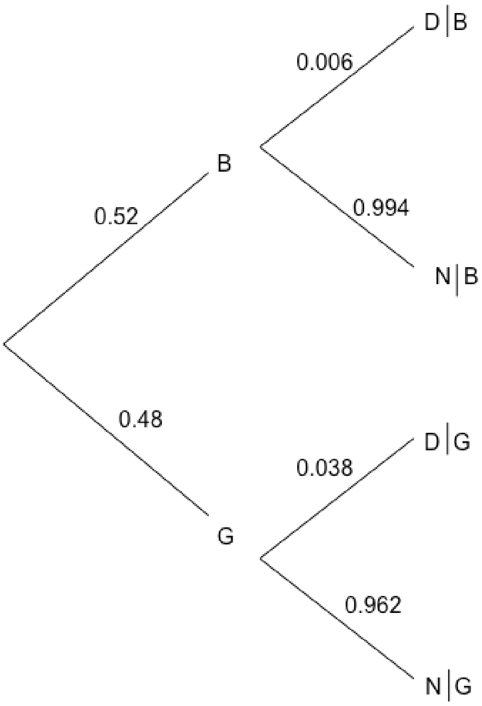
$P(G) = 1 - 0.52 = 0.48$ where G = girl

$P(D|G) = 0.038$ (3.8%) where D = defective

$P(N|G) = 1 - 0.038 = 0.962$ where N = not defective

$P(D|B) = 0.006$ (0.6%) where D = defective

$P(N|B) = 1 - 0.006 = 0.994$ where N = not defective



$$P(B|D) = \frac{P(B \cap D)}{P(D)}$$

$$= \frac{052 \times .006}{0.52 \times 0.006 + 0.48 \times 0.038}$$

$= 0.146$

$= 14.6\%$

MATHS IN FOCUS 11 MATHEMATICS EXTENSION 1

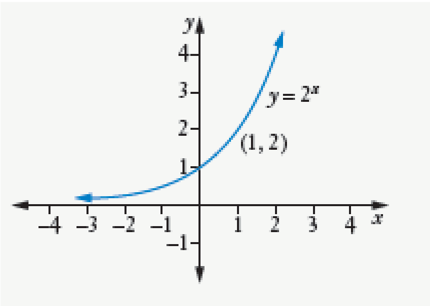
WORKED SOLUTIONS

Chapter 10: Exponential and logarithmic functions

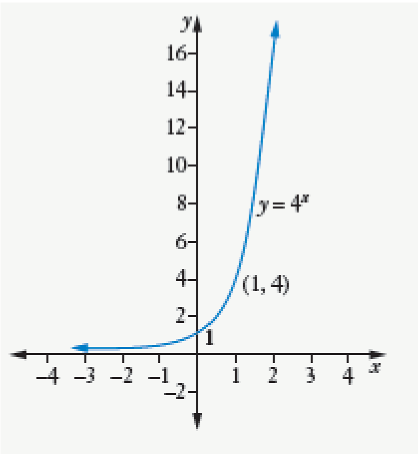
Exercise 10.01 Exponential functions

Question 1

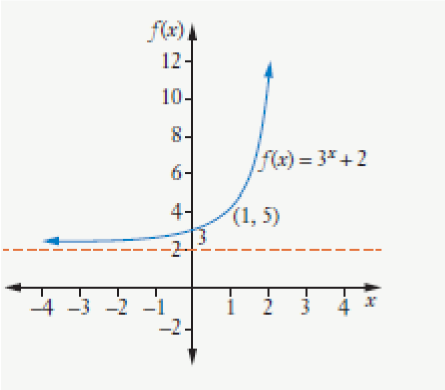
a (0, 1), (1, 2)



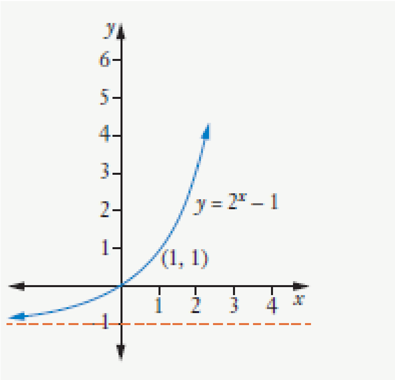
b (0, 1), (1, 4)



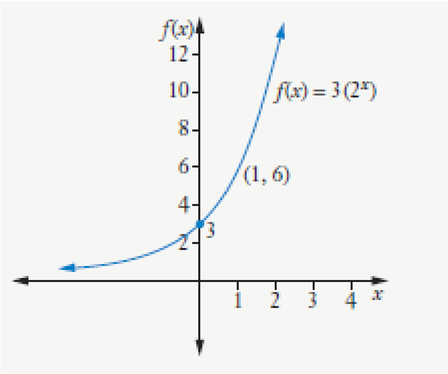
c (0, 3), (1, 5)



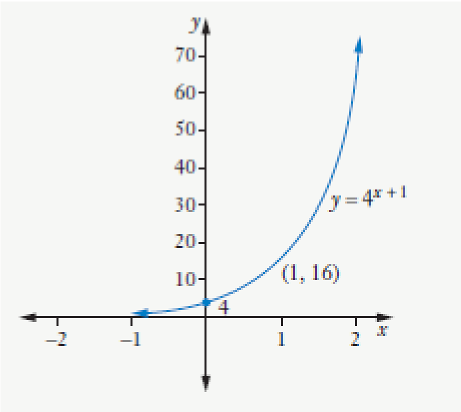
d (0, 0), (1, 1)



e (0, 3), (1, 6)



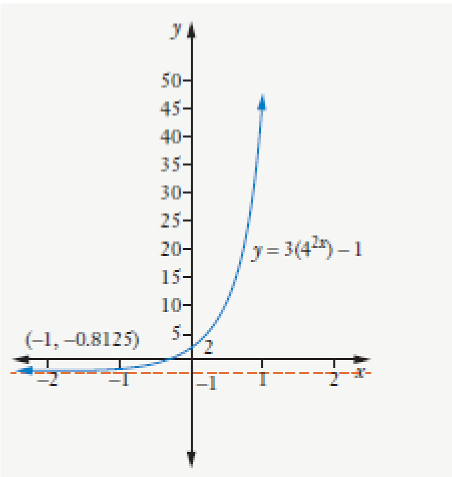
f (–1, 1) (0, 4)



g $y = 3(4^{2x}) - 1$

The ‘3’ gives a steeper graph and the ‘– 1’ shifts the graph down 1 unit from the origin, so the horizontal asymptote is $x = -1$. y-intercept 2.

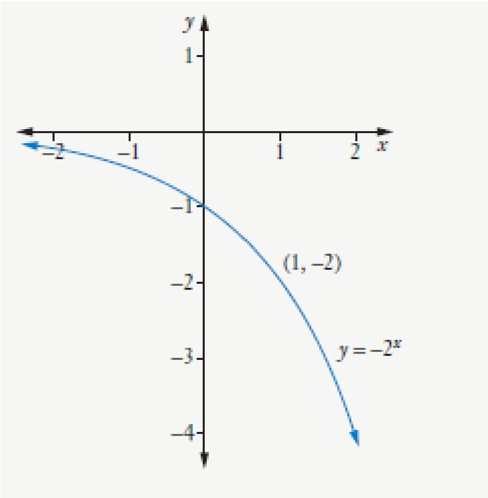
(–1, –0.8125)



h $f(x) = -2^x$

This is a reflection of the graph of $y = 2^x$ in the x-axis. y-intercept -1.

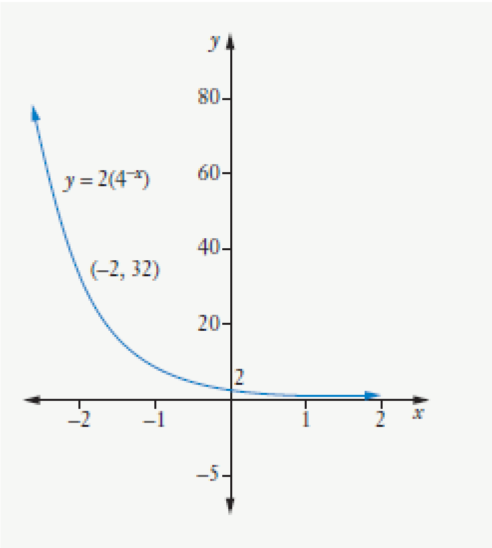
$(1, -2)$



i $y = 2(4^{-x})$

The '2' gives a steeper graph and the '$-x$' reflects the graph across the y-axis. y-intercept $2 \times 1 = 2$.

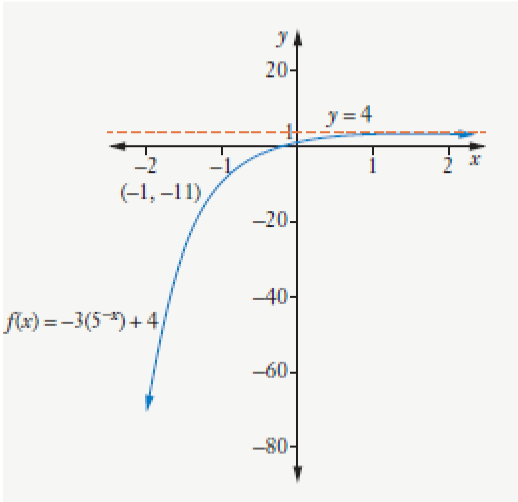
$(-2, 32)$



j $y = -3(5^{-x}) + 4$

The '–3' gives a steeper graph and reflects it across the *x*-axis. The '–*x*' reflects the graph across the *y*-axis and the '+ 4' shifts the graph up 4 units from the origin. Horizontal asymptote $y = 4$. *y*-intercept $-3 \times 1 + 4 = 1$.

(–1, –11)



Question 2

a Domain: $(-\infty, \infty)$, range: $(0, \infty)$

b Domain: $(-\infty, \infty)$, range: $(5, \infty)$

c Domain: $(-\infty, \infty)$, range: $(0, \infty)$

d Domain: $(-\infty, \infty)$, range: $(-\infty, 1)$

Question 3

a $f(g(x)) = y = 2^{3x-4}$

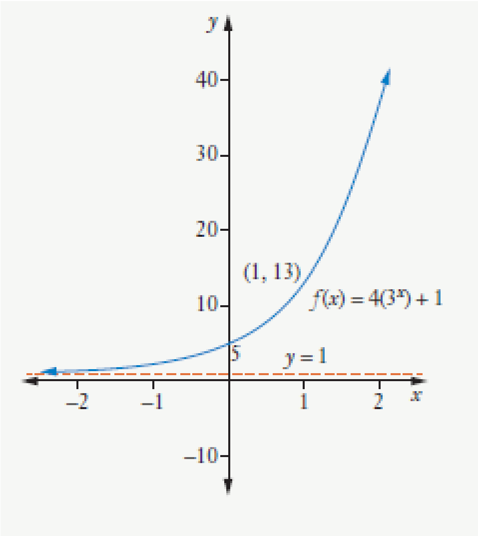
b $g(f(x)) = y = 3(2^x) - 4$

Question 4

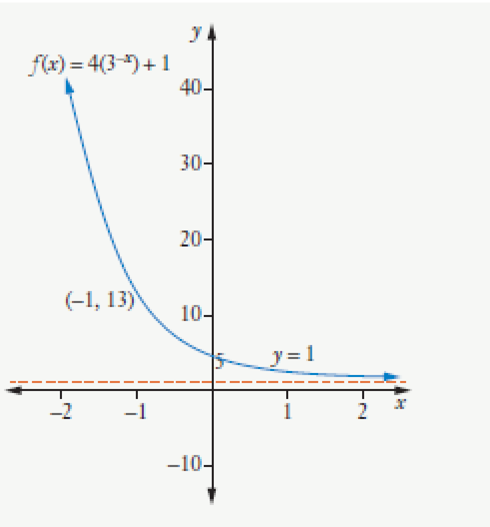
a $y = 4(3^x) + 1$

The '4' gives a steeper graph and the '+ 1' shifts the graph up 1 unit from the origin. Horizontal asymptote $y = 1$. y-intercept $4 \times 1 + 1 = 5$.

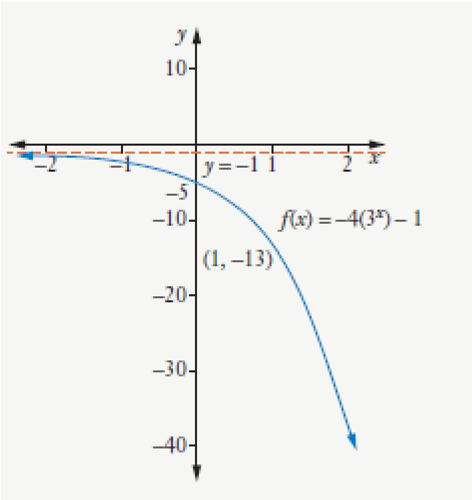
(1, 13)



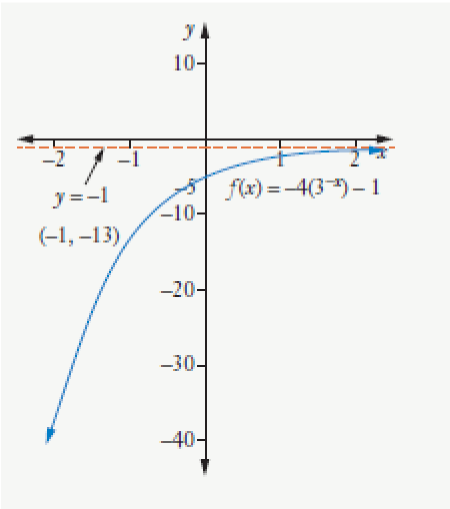
b **i** This is the graph in **a** reflected across the y-axis.



ii This is the graph in **a** reflected across the *x*-axis.

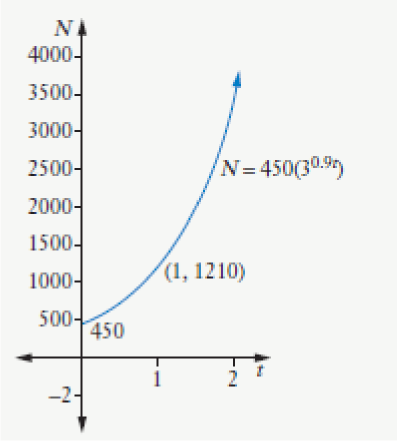


iii This is the graph in **a** reflected across the *x*- and *y*-axes.



Question 5

a (0, 450), (1, 1210), $t \geq 0$ as t represents time.



b When $t = 0$, $N = 450(3^0) = 450$. Initial sales = 450.

c **i** When $t = 3$, $N = 450(3^{0.9 \times 3}) = 8738.56\ldots$

≈ 8739.

ii When $t = 5$, $N = 450(3^{0.9 \times 5}) = 63\ 133.25\ldots$

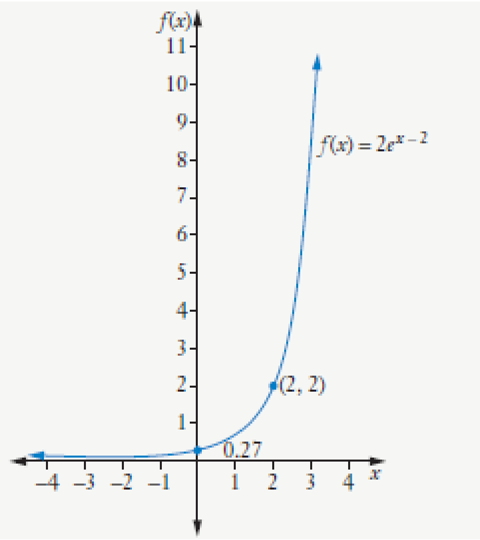
$\approx 63\ 133$.

iii When $t = 10$, $N = 450(3^{0.9 \times 10}) = 8\ 857\ 350$.

Question 1

$x = 2, y = 2$

$x = 3, y = 2e$



Question 2

a $e^{1.5} = 4.48$

b $e^{-2} = 0.14$

c $2e^{0.3} = 2.70$

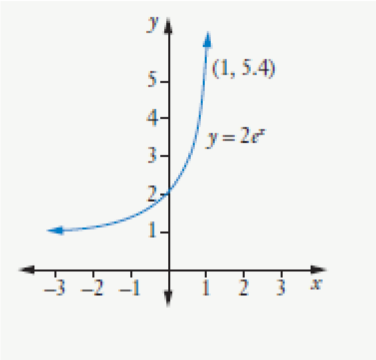
d $\frac{1}{e^3} = 0.05$

e $-3e^{-3.1} = -0.14$

Question 3

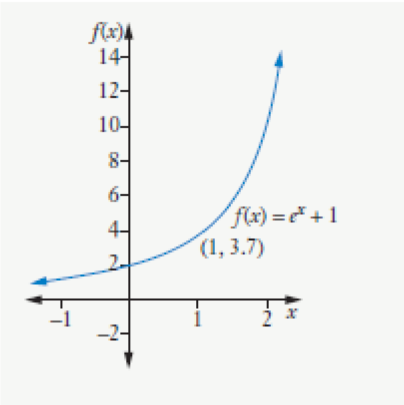
a $x = 0, y = 2$

$x = 1, y = 5.4$

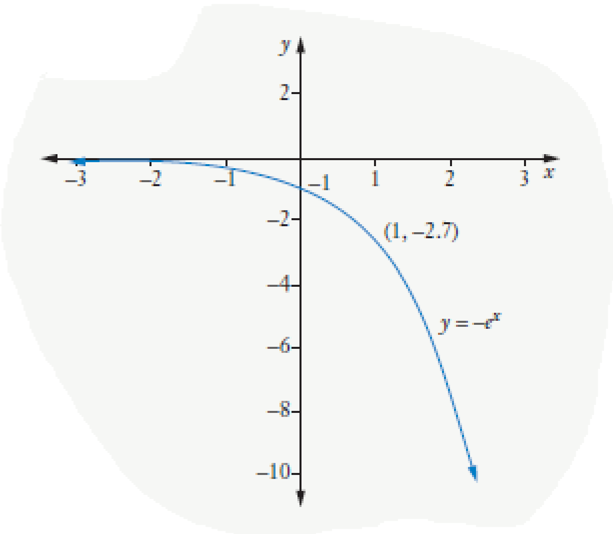


b $x = 0, y = 2$

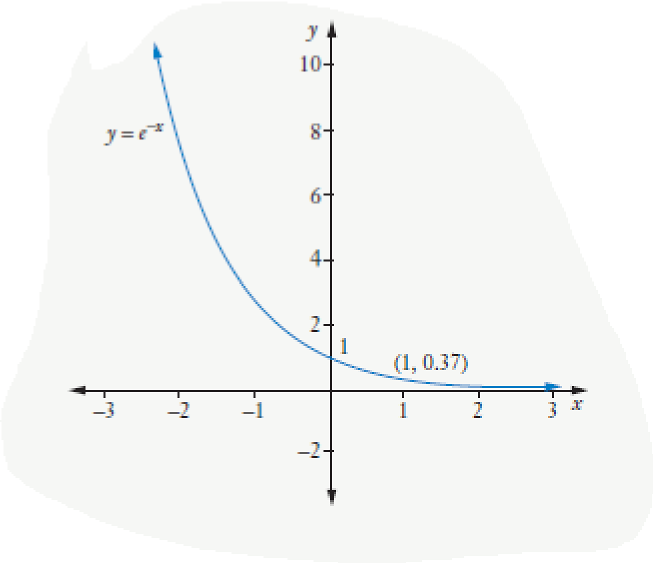
$x = 1, y = 3.7$



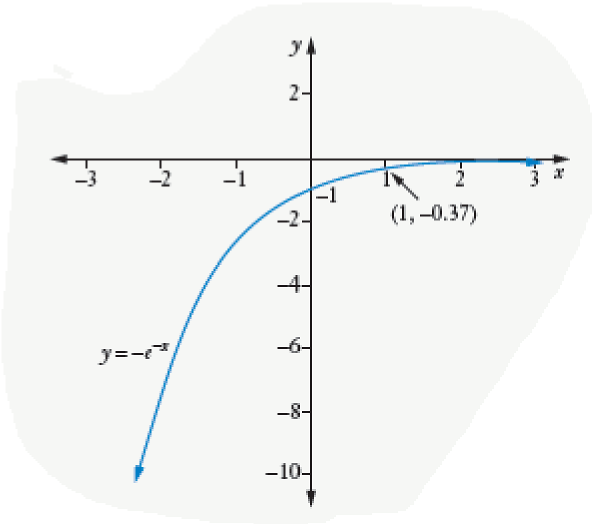
c $y = -e^x$ is a reflection of $y = e^x$ in the x-axis.



d $y = e^{-x}$ is a reflection of $y = e^x$ in the y-axis.



e $y = -e^{-x}$ is a reflection of $y = e^x$ in the x- and y-axes.



Question 4

Domain $(-\infty, \infty)$

Range $(-2, \infty)$

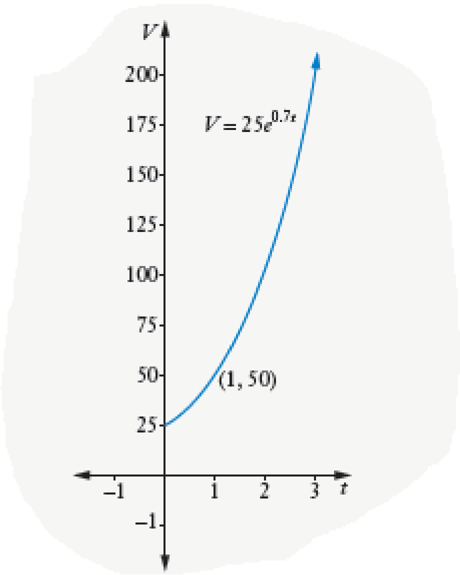
Question 5

a $f(g(x)) = e^{x^3+3}$

b $g(f(x)) = \left(e^x\right)^3 + 3 = e^{3x} + 3$

Question 6

a (0, 25), (1, 68), $t \geq 0$ as t represents time.



b **i** When $t = 3$, $V = 25(e^{0.7 \times 3}) = 204.1542\ldots$

≈ 204.2.

ii When $t = 8$, $V = 25(e^{0.7 \times 8}) = 6760.6601\ldots$

≈ 6760.7.

c No, because it predicts that the volume never stops increasing.

Question 7

a When $t = 10$, $M = 150(e–^{0.014 \times 10}) = 130.4037\ldots$

≈ 130.4 g

b When $t = 50$, $M = 150(e^{-0.014 \times 50}) = 74.4877\ldots$

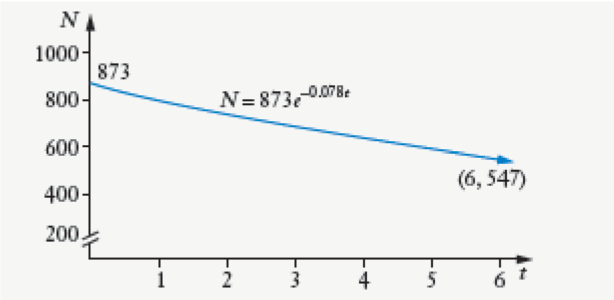
≈ 74.5 g

c When $t = 250$, $M = 150(e^{-0.014 \times 250}) = 4.5296\ldots$

≈ 4.5 g

Question 8

a (0, 873), (6, 547), $t \geq 0$ as t represents time.



b **i** When $t = 0$, $N = 873(e^{-0.078 \times 0}) = 873$

ii When $t = 5$, $N = 873(e^{-0.078 \times 5}) = 591.0706\ldots$

≈ 591

iii When $t = 10$, $N = 873(e^{-0.078 \times 10}) = 400.1884\ldots$

≈ 400

Question 9

a When $t = 0$, $T = 23 + 125(e^{-0.06 \times 0}) = 23 + 125 = 148°\text{C}$

b **i** When $t = 2$, $T = 23 + 125(e^{-0.06 \times 2}) = 133.8650 \ldots$

$\approx 133.9°\text{C}$

ii When $t = 5$, $T = 23 + 125(e^{-0.06 \times 5}) = 115.6022 \ldots$

$\approx 115.6°\text{C}$

iii When $t = 10$, $T = 23 + 125(e^{-0.06 \times 10}) = 91.6014 \ldots$

$\approx 91.6°\text{C}$

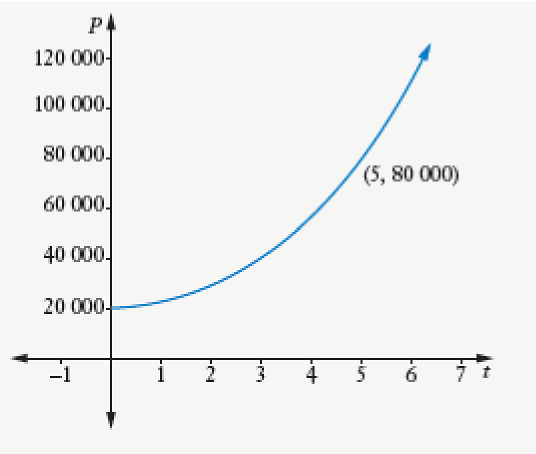
iv When $t = 120$, $T = 23 + 125(e^{-0.06 \times 120}) = 23.0933 \ldots$

$\approx 23.1°\text{C}$

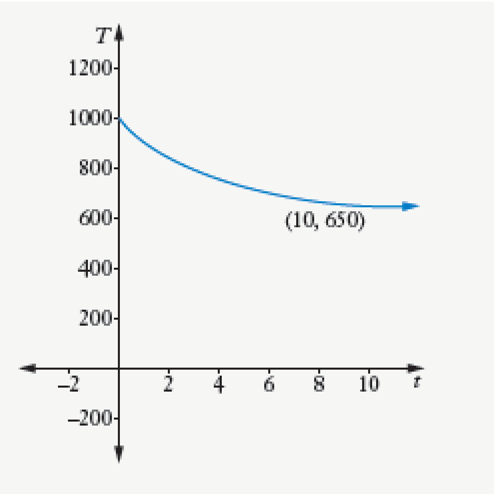
c As $t \to \infty$, $125(e^{-0.06t}) \to 0$, so $T \to 23$.

The limiting temperature is 23°C, which is room temperature.

Question 10



Question 11



Exercise 10.03 Differentiation of exponential functions

Question 1

a $\frac{d}{dx}\left(9e^x\right)=9e^x$

b $\frac{d}{dx}\left(-e^x\right)=-e^x$

c $\frac{d}{dx}\left(e^x+x^2\right)=e^x+2\times x^{2-1}=e^x+2x$

d

$$\frac{d}{dx}\left(2x^3-3x^2+5x-e^x\right)=3\times 2x^{3-1}-2\times 3x^{2-1}+5x^{1-1}-e^x$$
$$=6x^2-6x+5-e^x$$

e $\frac{d}{dx}\left(\left(e^x+1\right)^3\right)=3\times e^x\times\left(e^x+1\right)^{3-1}=3e^x\left(e^x+1\right)^2$

f $\frac{d}{dx}\left(\left(e^x+5\right)^7\right)=7\times e^x\times\left(e^x+5\right)^{7-1}=7e^x\left(e^x+5\right)^6$

g $\frac{d}{dx}\left(\left(2e^x-3\right)^2\right)=2\times 2e^x\times\left(2e^x-3\right)^{2-1}=4e^x\left(2e^x-3\right)$

h

$$\frac{d}{dx}\left(xe^x\right)=x\times e^x+1\times e^x$$
$$=xe^x+e^x$$
$$=e^x\left(x+1\right)$$

i

$$\frac{d}{dx}\left(\frac{e^x}{x}\right)=\frac{e^x\times x-e^x\times 1}{\left(x\right)^2}$$
$$=\frac{xe^x-e^x}{x^2}$$
$$=\frac{e^x\left(x-1\right)}{x^2}$$

j

$$\frac{d}{dx}\left(x^2e^x\right)=2x\times e^x+x^2e^x$$
$$=2xe^x+x^2e^x$$
$$=xe^x\left(2+x\right)$$

k

$$\frac{d}{dx}\left(e^x\left(2x+1\right)\right)=2\left(2x+1\right)^0\times e^x+\left(2x+1\right)e^x$$
$$=2e^x+\left(2x+1\right)e^x$$
$$=e^x\left(2+2x+1\right)$$
$$=e^x\left(2x+3\right)$$

l

$$\frac{d}{dx}\left(\frac{e^x}{7x-3}\right)=\frac{e^x\times\left(7x-3\right)-7\times e^x}{\left(7x-3\right)^2}$$
$$=\frac{e^x\left(7x-3\right)-7e^x}{\left(7x-3\right)^2}$$
$$=\frac{e^x\left(7x-3-7\right)}{\left(7x-3\right)^2}$$
$$=\frac{e^x\left(7x-10\right)}{\left(7x-3\right)^2}$$

m

$$\frac{d}{dx}\left(\frac{5x}{e^x}\right)=\frac{5\times e^x-5x\times e^x}{\left(e^x\right)^2}$$
$$=\frac{5e^x-5xe^x}{e^{2x}}$$
$$=\frac{5e^x\left(1-x\right)}{e^{2x}}$$
$$=\frac{5\left(1-x\right)}{e^x}$$

Question 2

a $\frac{dy}{dx} = 2e^{2x}$

b $\frac{dy}{dx} = -e^{-x}$

c $\frac{dy}{dx} = 2 \times 3e^{3x} = 6e^{3x}$

d $\frac{dy}{dx} = -7e^{7x}$

e $\frac{dy}{dx} = -3 \times 2e^{2x} + 2x \ = -6e^{2x} + 2x$

f $\frac{dy}{dx} = 2e^{2x} - -2e^{-2x}$

$= 2e^{2x} + 2e^{-2x}$

$= 2\left(e^{2x} + e^{-2x}\right)$

g $\frac{dy}{dx} = -5e^{-x} - 3$

h $u = x,\ v = e^{4x}$

$\frac{du}{dx} = 1,\ \frac{dv}{dx} = 4e^{4x}$

$\frac{dy}{dx} = 1 \times \left(e^{4x}\right) + \left(4e^{4x}\right) \times x$

$= e^{4x}\left(1 + 4x\right)$

i $u = 2e^{3x} - 3$, $v = x + 1$

$$\frac{du}{dx} = 6e^{3x}, \ \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{6e^{3x}(x+1) - 1 \times (2e^{3x} - 3)}{(x+1)^2}$$
$$= \frac{6xe^{3x} + 6e^{3x} - 2e^{3x} + 3}{(x+1)^2}$$
$$= \frac{6xe^{3x} + 4e^{3x} + 3}{(x+1)^2}$$

j $u = 9e^{3x} + 2$, $\frac{du}{dx} = 27e^{3x}$

$y = u^5$

$$\frac{dy}{du} = 5u^4$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 5u^4 \times 27e^{3x}$$
$$= 5(9e^{3x} + 2)^4 \times 27e^{3x}$$
$$= 135e^{3x}(9e^{3x} + 2)^4$$

Question 3

$$f(x) = x^3 + 3x - e^x$$
$$f'(x) = 3x^2 + 3 - e^x$$
$$f'(1) = 3(1)^2 + 3 - e^1$$
$$= 3 + 3 - e^1$$
$$= 6 - e$$

Question 4

$y = e^x$
$y' = e^x$
$m = e^1$
$m = e$

Question 5

$y = e^{2x}$
$y' = 2e^{2x}$
When $x = 5$, $y' = 2e^{2\times 5} = 2e^{10}$
Gradient of normal $= -\dfrac{1}{2e^{10}}$

Question 6

$y = 4e^x$
$y' = 4e^x$
$m = 4e^{1.6}$
$m = 19.81$

Question 7

$y = -e^x$
$x = 1 \ \ y = -e$
$y' == -e^x$
$m = -e^1$
$m = -e$
$y - y_1 = m(x - x_1)$
$y - -e = -e(x - 1)$
$y + e = -ex + e$
$y = -ex$

Question 8

$y = e^x$

$x = 3 \;\; y = e^3$

$y' = e^x$

$m = e^3$

$\text{Gradient of normal} = -\dfrac{1}{e^3}$

$y - y_1 = m\left(x - x_1\right)$

$y - e^3 = -\dfrac{1}{e^3}\left(x - 3\right)$

$e^3\left(y - e^3\right) = -\left(x - 3\right)$

$e^3 y - e^6 = -x + 3$

$x + e^3 y - 3 - e^6 = 0$

Question 9

a

$P = 3e^{1.4} + 12\ 569$

$P\left(0\right) = 3e^0 + 12\ 569$

$P\left(0\right) = 3 + 12\ 569$

$P\left(0\right) = 12\ 572$

b **i**

$P = 3e^{1.4} + 12\ 569$

$P' = 3 \times 1.4e^{1.4t} = 4.2e^{1.4t}$

$P'\left(3\right) = 4.2e^{1.4 \times 3}$

$P'\left(3\right) = 280 \;\; \text{insects/week}$

ii

$P = 3e^{1.4} + 12\ 569$

$P' = 42\ e^{1.4t}$

$P'\left(7\right) = 4.2e^{1.4 \times 7}$

$P'\left(7\right) = 75\ \ 742 \;\; \text{insects/week}$

Question 10

a

$$x = 2e^{4t}$$
$$x(0) = 2e^{0}$$
$$x(0) = 2 \text{ m}$$

b

$$x = 2e^{4t}$$
$$v = x' = 8e^{4t}$$
$$v(10) = 8e^{40} \text{ ms}^{-1}$$

c

$$v = x' = 8e^{4t}$$
$$a = v' = 32e^{4t}$$
$$a(2) = 32e^{8}$$
$$a(2) \approx 95\ 390.7 \text{ ms}^{-2}$$

Question 11

a

$$x = 6e^{-0.34t} - 5$$
$$x(0) = 6e^{0} - 5$$
$$x(0) = 6 - 5$$
$$x(0) = 1 \text{ cm}$$

b

$$x = 6e^{-0.34t} - 5$$
$$v = x \ = 6 \times (-0.34)e^{-0.34t}$$
$$= -2.04e^{-0.34t}$$
$$v(0) = -2.04e^{0}$$
$$v(0) = -2.04 \text{ cms}^{-1}$$

c

$$x = 6e^{-0.34t} - 5$$
$$x(4) = 6e^{-034\ 4} \ -5$$
$$= 6e^{-136} \ -5$$
$$x(4) \approx -3.46 \text{ cm}$$

d

$$v = x' = -204\ e^{-0.34t}$$
$$v(9) = -2.04e^{-0.34\ 9)}$$
$$= -2.04e^{-3.06}$$
$$\approx -0.096 \text{ cm s}^{-1}$$

e

$$v = x' = -204\ e^{-0.34t}$$
$$a = v \ = -2.04 \times -0.34e^{-0.34t} = 0.6936e^{-0.34t}$$
$$a(2) = 0.6936e^{-0.34\ 2)}$$
$$\approx 0.35 \text{ cm s}^{-2}$$

Question 12

a **i** When $t = 3$, $V = 3e^{0.8 \times 3} = 33.0695 \ldots$

$\approx 33.1 \text{ mm}^3$

ii When $t = 5$, $V = 3e^{0.8 \times 5} = 163.79445 \ldots$

$\approx 163.8 \text{ mm}^3$

b $\dfrac{dV}{dt} = 3(0.8 \times e^{0.8t}) = 2.4e^{0.8t}$

i When $t = 3$, $\dfrac{dV}{dt} = 2.4e^{0.8 \times 3} = 26.4556 \ldots$

$\approx 26.5 \text{ mm}^3 \text{ s}^{-1}$

ii When $t = 5$, $\dfrac{dV}{dt} = 2.4e^{0.8 \times 5} = 131.03556 \ldots$

$\approx 131.0 \text{ mm}^3 \text{ s}^{-1}$

Question 13

a **i** When $t = 5$, $P = 34\,500e^{0.025 \times 5} = 39\,093.62163\ldots$

$\approx 39\,094$

ii When $t = 10$, $P = 34\,500e^{0.025 \times 10} = 44\,298.87688\ldots$

$\approx 44\,299$

b $\dfrac{dP}{dt} = 34\,500(0.025 \times e^{0.025t}) = 862.5e^{0.025t}$

i When $t = 5$, $\dfrac{dP}{dt} = 862.5e^{0.025 \times 5} = 977.34\ldots$

≈ 977 people/year

ii When $t = 10$, $\dfrac{dP}{dt} = 862.5e^{0.025 \times 10} = 1107.471922\ldots$

≈ 1107 people/year

Question 14

a **i** When $t = 1$, $D = 3e^{-0.017 \times 1} = 2.9494 \ldots$

≈ 2.95 m

ii When $t = 2$, $D = 3e^{-0.017 \times 2} = 2.8997 \ldots$

≈ 2.90 m

iii When $t = 3$, $D = 3e^{-0.017 \times 3} = 2.8508 \ldots$

≈ 2.85 m

b $\frac{dD}{dt} = 3(-0.017 \times e^{-0.017t}) = -0.051e^{-0.017t}$

i When $t = 1$, $\frac{dD}{dt} = -0.051e^{-0.017 \times 1} = -0.05014 \ldots$

≈ -0.050 m/month

ii When $t = 2$, $\frac{dD}{dt} = -0.051e^{-0.017 \times 2} = -0.04929 \ldots$

≈ -0049 m/month

iii When $t = 3$, $\frac{dD}{dt} = -0.051e^{-0.017 \times 3} = -0.04846 \ldots$

≈ -0048 m/month

Exercise 10.04 Logarithms

Question 1

a

$$\log_2 16 = x$$
$$16 = 2^x$$
$$2^4 = 2^x$$
$$x = 4$$

b

$$\log_4 16 = x$$
$$16 = 4^x$$
$$4^2 = 4^x$$
$$x = 2$$

c

$$\log_5 125 = x$$
$$125 = 5^x$$
$$5^3 = 5^x$$
$$x = 3$$

d

$$\log_3 3 = x$$
$$3 = 3^x$$
$$3 = 3^x$$
$$x = 1$$

e

$$\log_7 49 = x$$
$$49 = 7^x$$
$$7^2 = 7^x$$
$$x = 2$$

f

$$\log_7 7 = x$$
$$49 = 7^x$$
$$7 \ \ = 7^x$$
$$x = 1$$

g

$$\log_5 1 = x$$
$$1 = 5^x$$
$$5^0 = 5^x$$
$$x = 0$$

h

$$\log_2 128 = x$$
$$128 = 2^x$$
$$2^7 = 2^x$$
$$x = 7$$

i

$$\log_8 8 = x$$
$$8 = 8^x$$
$$8 \ \ = 8^x$$
$$x = 1$$

Question 2

a

$$2^{\log_2 3} = x$$
$$\log_2 x = \log_2 3$$
$$x = 3$$

b

$$7^{\log_7 4} = x$$
$$\log_7 x = \log_7 4$$
$$x = 4$$

c

$$3^{\log\ 29} = x$$
$$\log_3 x = \log_3 29$$
$$x = 29$$

Question 3

a

$$\log_2 8 = x$$
$$8 = 2^x$$
$$2^3 = 2^x$$
$$x = 3$$
$$3 \times x$$
$$3 \times 3 = 9$$

b

$$\log_5 25 = x$$
$$25 = 5^x$$
$$5^2 = 5^x$$
$$x = 2$$
$$x + 1$$
$$2 + 1 = 3$$

c

$$\log_3 81 = x$$
$$81 = 3^x$$
$$3^4 = 3^x$$
$$x = 4$$
$$3 - x$$
$$3 - 4 = -1$$

d

$$\log_3 27 = x$$
$$27 = 3^x$$
$$3^3 = 3^x$$
$$x = 3$$
$$4 \times x$$
$$4 \times 3 = 12$$

e

$$\log_0 10000 = x$$
$$10000 = 10^x$$
$$10^4 = 10^x$$
$$x = 4$$
$$2 \times x$$
$$2 \times 4 = 8$$

f

$$\log_4 64 = x$$
$$64 = 4^x$$
$$4^3 = 4^x$$
$$x = 3$$
$$1 + x$$
$$1 + 3 = 4$$

g

$$\log_4 64 = x$$
$$64 = 4^x$$
$$4^3 = 4^x$$
$$x = 3$$
$$3 \times x + 5$$
$$3 \times 3 + 5 = 14$$

h

$$\log_3 9 = x$$
$$9 = 3^x$$
$$3^2 = 3^x$$
$$x = 2$$
$$\frac{x}{2}$$
$$\frac{2}{2} = 1$$

i

$$\log_8 64 = x$$
$$64 = 8^x$$
$$8^2 = 8^x$$
$$x = 2$$
$$\log_2 8 = y$$
$$8 = 2^y$$
$$2^3 = 2^y$$
$$y = 3$$
$$\frac{x+4}{y}$$
$$\frac{2+4}{3} = 2$$

Question 4

a

$$\log_2 \frac{1}{2} = x$$

$$\frac{1}{2} = 2^x$$

$$2^{-} = 2^x$$

$$x = -1$$

b

$$\log_3 \sqrt{3} = x$$

$$\sqrt{3} = 3^x$$

$$3^{\frac{1}{2}} = 3^x$$

$$x = \frac{1}{2}$$

c

$$\log_4 2 = x$$

$$2 = 4^x$$

$$2^1 = 2^{2x}$$

$$1 = 2x$$

$$x = \frac{1}{2}$$

d

$$\log_5 \frac{1}{25} = x$$

$$\frac{1}{25} = 5^x$$

$$5^{-2} = 5^x$$

$$x = -2$$

e

$$\log_7 \sqrt[4]{7} = x$$
$$\sqrt[4]{7} = 7^x$$
$$7^{\frac{1}{4}} = 7^x$$
$$x = \frac{1}{4}$$

f

$$\log_3 \frac{1}{\sqrt[3]{3}} = x$$
$$\frac{1}{\sqrt[3]{3}} = 3^x$$
$$3^{-\frac{1}{3}} = 3^x$$
$$x = -\frac{1}{3}$$

g

$$\log_4 \frac{1}{2} = x$$
$$\frac{1}{2} = 4^x$$
$$2^{-1} = 2^{2x}$$
$$-1 = 2x$$
$$x = -\frac{1}{2}$$

h

$$\log_8 2 = x$$
$$2 = 8^x$$
$$2^1 = 2^{3x}$$
$$1 = 3x$$
$$x = \frac{1}{3}$$

i

$$\log_6 6\sqrt{6} = x$$
$$6\sqrt{6} = 6^x$$
$$6 \times 6^{\frac{1}{2}} = 6^x$$
$$6^{1\frac{1}{2}} = 6^x$$
$$x = 1\frac{1}{2}$$

j

$$\log_2 \frac{\sqrt{2}}{4} = x$$
$$\frac{\sqrt{2}}{4} = 2^x$$
$$2^{\frac{1}{2}} \times 2^{-2} = 2^x$$
$$2^{-1\frac{1}{2}} = 2^x$$
$$x = -1\frac{1}{2}$$

Question 5

a $\log_{0} 1200 = 3.08$

b $\log_{0} 875 = 2.94$

c $\log_{e} 25 = 3.22$

d $\ln 144.\cancel{9}4$

e $5 \ln 8 = 10.40$

f $\log_{0} 350 + 4.5 = 7.04$

g $\dfrac{\log_{0} 15}{2} = 0.59$

h $\ln 9.8 + \log_{0} 17 = 3.51$

i $\dfrac{\log_{0} 30}{\log_{e} 30} = 0.43$

Question 6

a $3^x = y,\ \log_3 y = x$

b $5^x = z, \log_5 z = x$

c $x^2 = y, \log_x y = 2$

d $2^b = a, \log_2 a = b$

e $b^3 = d,\ \log_b d = 3$

f $y = 8^x, \log_8 y = x$

g $y = 6^x, \log_6 y = x$

h $y = e^x,\ \log_e y = x$

i $y = a^x, \log_a y = x$

j $Q = e^x,\ \log_e Q = x$

Question 7

a $\log_3 5 = x,\ 3^x = 5$

b $\log_a 7 = x,\ a^x = 7$

c $\log_3 a = b,\ 3^b = a$

d $\log_x y = 9,\ x^9 = y$

e $\log_a b = y,\ a^y = b$

f $y = \log_2 6,\ 2^y = 6$

g $y = \log_3 x,\ 3^y = x$

h $y = \log_{0} 9,\ 10^y = 9$

i $y = \ln 4,\ y = \log_e 4,\ e^y = 4$

Question 8

a

$$\log_{10} x = 6$$
$$10^6 = x$$
$$x = 1000000$$

b

$$\log_3 x = 5$$
$$3^5 = x$$
$$x = 243$$

c

$$\log_x 343 = 3$$
$$x^3 = 343$$
$$x = \sqrt[3]{343}$$
$$x = 7$$

d

$$\log_x 64 = 6$$
$$x^6 = 64$$
$$x = \sqrt[6]{64}$$
$$x = 2$$

e

$$\log_5 \frac{1}{5} = x$$
$$5^x = \frac{1}{5}$$
$$5^x = 5^{-1}$$
$$x = -1$$

f

$$\log_x \sqrt{3} = \frac{1}{2}$$
$$x^{\frac{1}{2}} = \sqrt{3}$$
$$x^{\frac{1}{2}} = 3^{\frac{1}{2}}$$
$$x = 3$$

g

$$\ln x = 3.8$$
$$\log_e x = 3.8$$
$$e^{3.8} = x$$
$$x = 44.7$$

h

$$3\log_{0} x - 2 = 10$$
$$3\log_{0} x = 12$$
$$\log_{0} x = 4$$
$$10^4 = x$$
$$x = 10000$$

i

$$\log_4 x = \frac{3}{2}$$
$$4^{\frac{3}{2}} = x$$
$$x = 8$$

Question 9

$$\log_y 125 = 3$$
$$y^3 = 125$$
$$y^3 = 5^3$$
$$y = 5$$

Question 10

$$\log_{0} x = 1.65$$
$$10^{1.65} = x$$
$$x = 44.7$$

Question 11

$\log_e b = 0.894$
$e^{0.894} = b$
$b = 2.44$

Question 12

$\log_2 1 = x$
$2^x = 1$
$2^x = 2^0$
$x = 0$

$\log_a 1 = x$
$a^x = 1$
$a^x = a^0$
$x = 0$

Question 13

$\log_5 5 = x$
$5^x = 5$
$5^x = 5^1$
$x = 1$

$\log_a a = x$
$a^x = a$
$a^x = a^1$
$x = 1$

Question 14

a

$$\ln e = x$$
$$\log_e e = x$$
$$e^x = e^1$$
$$x = 1$$

b **i** $\log_e e^3 = 3$

ii $\log_e e^2 = 2$

iii $\ln_e e^5 = 5$

iv $\log_e \sqrt{e} = \frac{1}{2}$

v $\log_e \frac{1}{e} = -1$

vi $e^{\ln 2} = 2$

vii $e^{\ln 3} = 3$

viii $e^{\ln 5} = 5$

ix $e^{\ln 7} = 7$

x $e^{\ln 1} = 1$

xi $e^{\ln e} = e$

Question 15

a

$$A = 85 - 55\log_{0}(t+2)$$
$$A(0) = 85 - 55\log_{0}(0+2)$$
$$A(0) = 68.4$$

b **i**

$$A = 85 - 55\log_{10}(t+2)$$
$$A(1) = 85 - 55\log_{10}(1+2)$$
$$A(1) = 58.8$$

ii

$$A = 85 - 55\log_{0}(t+2)$$
$$A(3) = 85 - 55\log_{0}(3+2)$$
$$A(3) = 46.6$$

c

$$A = 85 - 55\log_{0}(t+2)$$
$$30 = 85 - 55\log_{0}(t+2)$$
$$-55 = -55\log_{0}(t+2)$$
$$1 = \log_{0}(t+2)$$
$$(t+2) = 10^{1}$$
$$t+2 = 10$$
$$t = 8 \text{ weeks}$$

Question 16

a

$$\text{pH} = -\log\left[\text{H}^+\right]$$
$$\text{pH} = -\log[0.0035]$$
$$\text{pH} = .5\ \text{acidic}$$

b

$$\text{pH} = -\log\left[\text{H}^+\right]$$
$$\text{pH} = -\log\left[10^{-7}\right]$$
$$\text{pH} = 7\ \text{neutral}$$

c

$$\text{pH} = -\log\left[\text{H}^+\right]$$
$$\text{pH} = -\log\left[10^{\ 9}\right]$$
$$\text{pH} = 9\ \text{alkaline}$$

d

$$\text{pH} = -\log\left[\text{H}^+\right]$$
$$\text{pH} = -\log[0.01]$$
$$\text{pH} = 2\ \text{acidic}$$

e

$$\text{pH} = -\log\left[\text{H}^+\right]$$
$$\text{pH} = -\log\left[1.2\times10^{-12}\right]$$
$$\text{pH} = 1.9\ \text{alkaline}$$

f

$$\text{pH} = -\log\left[\text{H}^+\right]$$
$$\text{pH} = -\log[0.00001]$$
$$\text{pH} = 5\ \text{acidic}$$

Question 17

a $f\left(g\left(x\right)\right)=\log\left(2x-7\right)$

b $g\left(f\left(x\right)\right)=2\left(\log x\right)-7=2\log x-7$

Exercise 10.05 Logarithm laws

Question 1

a $\log_a 4 + \log_a y = \log_a 4y$

b $\log_a 4 + \log_a 5 = \log_a 4 \times 5 = \log_a 20$

c $\log_a 12 - \log_a 3 = \log_a \frac{12}{3} = \log_a 4$

d $\log_a b - \log_a 5 = \log_a \frac{b}{5}$

e

$$3\log_x y + \log_x z = \log_x y^3 + \log_x z$$
$$= \log_x y^3 \times z = \log_x y^3 z$$

f

$$2\log_k 3 + 3\log_k y = \log_k 3^2 + \log_k y^3$$
$$= \log_k 3^2 \times y^3$$
$$= \log_k 9y^3$$

g

$$5\log_a x - 2\log_a y = \log_a x^5 - \log_a y^2$$
$$= \log_a x^5 \div y^2$$
$$= \log_a \frac{x^5}{y^2}$$

h

$$\log_a x + \log_a y - \log_a z = \log_a x \times y \div z$$
$$= \log_a \frac{xy}{z}$$

i

$$\log_0 a + 4\log_0 b + 3\log_0 c = \log_{10} a + \log_{10} b^4 + \log_0 c^3$$
$$= \log_0 a \times b^4 \times c^3$$
$$= \log_{10} ab^4c^3$$

j

$$3\log_3 p + \log_3 q - 2\log_3 r = \log_3 p^3 + \log_3 q - \log_3 r^2$$
$$= \log_3 p^3 \times q \div r^2$$
$$= \log_3 \frac{p^3 q}{r^2}$$

k $\log_4 \frac{1}{n} = \log_4 n^{-} \ = -\log_4 n$

l $\log_x \frac{1}{6} = \log_x 6^{-} \ = -\log_x 6$

Question 2

a

$$\log_5 5^2$$
$$= 2\log_5 5$$
$$= 2 \times 1$$
$$= 2$$

b

$$\log_7 7^6$$
$$= 6\log_7 7$$
$$= 6 \times 1$$
$$= 6$$

Question 3

a

$$\log_7 10$$
$$= \log_7 2 \times 5$$
$$= \log_7 2 + \log_7 5$$
$$= 0.36 + 0.83$$
$$= 1.19$$

b

$$\log_7 0.4$$
$$= \log_7 2 \div 5$$
$$= \log_7 2 - \log_7 5$$
$$= 0.36 - 0.83$$
$$= -0.47$$

c

$$\log_7 20$$
$$= \log_7 2 \times 2 \times 5$$
$$= \log_7 2 + \log_7 2 + \log_7 5$$
$$= 0.36 + 0.36 + 0.83$$
$$= 1.55$$

d

$$\log_7 25$$
$$= \log_7 5 \times 5$$
$$= \log_7 5 + \log_7 5$$
$$= 0.83 + 0.83$$
$$= 1.66$$

e

$$\log_7 8$$
$$= \log_7 2 \times 2 \times 2$$
$$= \log_7 2 + \log_7 2 + \log_7 2$$
$$= 0.36 + 0.36 + 0.36$$
$$= 1.08$$

f

$$\begin{aligned}&\log_7 14\\&= \log_7 2\times 7\\&= \log_7 2+\log_7 7\\&= 0.36+1\\&= 1.36\end{aligned}$$

g

$$\begin{aligned}&\log_7 50\\&= \log_7 2\times 5\times 5\\&= \log_7 2+\log_7 5+\log_7 5\\&= 0.36+0.83+0.83\\&= 2.02\end{aligned}$$

h

$$\begin{aligned}&\log_7 35\\&= \log_7 5\times 7\\&= \log_7 5+\log_7 7\\&= 0.83+1\\&= 1.83\end{aligned}$$

i

$$\begin{aligned}&\log_7 98\\&= \log_7 7\times 7\times 2\\&= \log_7 7+\log_7 7+\log_7 2\\&= 1+1+0.36\\&= 2.36\end{aligned}$$

Question 4

a

$$\begin{aligned}&\log_5 50-\log_5 2\\&= \log_5 \frac{50}{2}\\&= \log_5 25\\&= \log_5 5^2\\&= 2\end{aligned}$$

b

$$\begin{aligned}&\log_2 16+\log_2 4\\&=\log_2 16\times 4\\&=\log_2 64\\&=\log_2 2^6\\&=6\end{aligned}$$

c

$$\begin{aligned}&\log_4 2+\log_4 8\\&=\log_4 2\times 8\\&=\log_4 16\\&=\log_4 4^2\\&=2\end{aligned}$$

d

$$\begin{aligned}&\log_5 500-\log_5 4\\&=\log_5 \frac{500}{4}\\&=\log_5 125\\&=\log_5 5^3\\&=3\end{aligned}$$

e

$$\begin{aligned}&\log_9 117-\log_9 13\\&=\log_9 \frac{117}{13}\\&=\log_9 9\\&=1\end{aligned}$$

f

$$\begin{aligned}&\log_8 32+\log_8 16\\&=\log_8 32\times 16\\&=\log_8 512\\&=\log_8 8^3\\&=3\end{aligned}$$

g

$$\begin{aligned}&3\log_2 2+2\log_2 4\\&=\log_2 2^3+2\log_2 4^2\\&=\log_2 8\times 16\\&=\log_2 128\\&=\log_2 2^7\\&=7\end{aligned}$$

h

$$\begin{aligned}&2\log_4 6-\left(2\log_4 3+\log_4 2\right)\\&=\log_4 6^2-\left(\log_4 3^2+\log_4 2\right)\\&=\log_4 36-\left(\log_4 3^2\times 2\right)\\&=\log_4 26-\log_4 18\\&=\log_4 \frac{36}{18}\\&=\log_4 2\\&=\log_4 4^{\frac{1}{2}}\\&=\frac{1}{2}\end{aligned}$$

i

$$\begin{aligned}&\log_6 4-2\log_6 12\\&=\log_6 4-\log_6 12^2\\&=\log_6 \frac{4}{144}\\&=\log_6 \frac{1}{36}\\&=\log_6 6^{-2}\\&=-2\end{aligned}$$

j

$$\begin{aligned}&2\log_3 6+\log_3 18-3\log_3 2\\&=\log_3 6^2+\log_3 18-\log_3 2^3\\&=\log_3 36\times 18\div 8\\&=\log_3 81\\&=\log_3 3^4\\&=4\end{aligned}$$

Question 5

a

$$\begin{aligned}&\log_a 15\\&=\log_a 5\times 3\\&=\log_a 5+\log_a 3\\&=x+y\end{aligned}$$

b

$$\begin{aligned}&\log_a 0.6\\&=\log_a 3\div 5\\&=\log_a 3-\log_a 5\\&=x-y\end{aligned}$$

c

$$\begin{aligned}&\log_a 27\\&=\log_a 3\times 3\times 3\\&=\log_a 3+\log_a 3+\log_a 3\\&=x+x+x\\&=3x\end{aligned}$$

d

$$\begin{aligned}&\log_a 25\\&=\log_a 5\times 5\\&=\log_a 5+\log_a 5\\&=y+y\\&=2y\end{aligned}$$

e

$$\begin{aligned}&\log_a 9\\&=\log_a 3\times 3\\&=\log_a 3+\log_a 3\\&=x+x\\&=2x\end{aligned}$$

f

$$\begin{aligned}&\log_a 75\\&=\log_a 3\times5\times5\\&=\log_a 3+\log_a 5+\log_a 5\\&=x+y+y\\&=x+2y\end{aligned}$$

g

$$\begin{aligned}&\log_a 3a\\&=\log_a 3\times a\\&=\log_a 3+\log_a a\\&=x+1\end{aligned}$$

h

$$\begin{aligned}&\log_a \frac{a}{5}\\&=\log_a a\div5\\&=\log_a a-\log_a 5\\&=1-y\end{aligned}$$

i

$$\begin{aligned}&\log_a 9a\\&=\log_a 3\times3\times a\\&=\log_a 3+\log_a 3+\log_a a\\&=x+x+1\\&=2x+1\end{aligned}$$

Question 6

a

$$\begin{aligned}&\log_a xy\\&=\log_a x\times y\\&=\log_a x+\log_a y\\&=p+q\end{aligned}$$

b

$$\begin{aligned}&\log_a y^3\\&= 3\log_a y\\&= 3\times q\\&= 3q\end{aligned}$$

c

$$\begin{aligned}&\log_a \frac{y}{x}\\&= \log_a y \div x\\&= \log_a y - \log_a x\\&= q - p\end{aligned}$$

d

$$\begin{aligned}&\log_a x^2\\&= 2\log_a x\\&= 2\times p\\&= 2p\end{aligned}$$

e

$$\begin{aligned}&\log_a xy^5\\&= \log_a x\times y^5\\&= \log_a x + \log_a y^5\\&= \log_a x + 5\log_a y\\&= p + 5\times q\\&= p + 5q\end{aligned}$$

f

$$\begin{aligned}&\log_a \frac{x^2}{y}\\&= \log_a x^2 \div y\\&= \log_a x^2 - \log_a y\\&= 2\log_a x - \log_a y\\&= 2\times p - q\\&= 2p - q\end{aligned}$$

g

$$\begin{aligned}&\log_a ax\\&=\log_a a\times x\\&=\log_a a+\log_a x\\&=1+p\end{aligned}$$

h

$$\begin{aligned}&\log_a \frac{a}{y^2}\\&=\log_a a\div y^2\\&=\log_a a-\log_a y^2\\&=\log_a a-2\log_a y\\&=1-2\times q\\&=1-2q\end{aligned}$$

i

$$\begin{aligned}&\log_a a^3y\\&=\log_a a^3\times y\\&=\log_a a^3+\log_a y\\&=3\log_a a+\log_a y\\&=3\times1+q\\&=3+q\end{aligned}$$

j

$$\begin{aligned}&\log_a \frac{x}{ay}\\&=\log_a x\div a\div y\\&=\log_a x-\log_a a-\log_a y\\&=p-1-q\end{aligned}$$

Question 7

a

$$\log_a \frac{c}{b}$$
$$= \log_a c \div b$$
$$= \log_a c - \log_a b$$
$$= 4.7 - 3.4$$
$$= 1.3$$

b

$$\log_a bc^2$$
$$= \log_a b \times c^2$$
$$= \log_a b + \log_a c^2$$
$$= \log_a b + 2\log_a c$$
$$= 3.4 + 2 \times 4.7$$
$$= 12.8$$

c

$$\log_a (bc)^2$$
$$= \log_a b^2 \times c^2$$
$$= \log_a b^2 + \log_a c^2$$
$$= 2\log_a b + 2\log_a c$$
$$= 2 \times 3.4 + 2 \times 4.7$$
$$= 16.2$$

d

$$\log_a abc$$
$$= \log_a a \times b \times c$$
$$= \log_a a + \log_a b + \log_a c$$
$$= 1 + 3.4 + 4.7$$
$$= 9.1$$

e

$$\begin{aligned}&\log_a a^2c\\&=\log_a a^2\times c\\&=\log_a a^2+\log_a c\\&=2\log_a a+\log_a c\\&=2\times1+4.7\\&=6.7\end{aligned}$$

f

$$\begin{aligned}&\log_a b^7\\&=\log_a b^7\\&=7\log_a b\\&=7\times3.4\\&=23.8\end{aligned}$$

g

$$\begin{aligned}&\log_a \frac{a}{c}\\&=\log_a a\div c\\&=\log_a a-\log_a c\\&=1-4.7\\&=-3.7\end{aligned}$$

h

$$\begin{aligned}&\log_a a^3\\&=3\log_a a\\&=3\times1\\&=3\end{aligned}$$

i

$$\begin{aligned}&\log_a bc^4\\&=\log_a b+\log_a c^4\\&=\log_a b+4\log_a c\\&=3.4+4\times4.7\\&=22.2\end{aligned}$$

Question 8

a

$$\log_4 12 = \log_4 x + \log_4 3$$
$$\log_4 12 = \log_4 x \times 3$$
$$\log_4 12 = \log_4 3x$$
$$12 = 3x$$
$$x = 4$$

b

$$\log_3 4 = \log_3 y - \log_3 7$$
$$\log_3 4 = \log_3 y \div 7$$
$$\log_3 4 = \log_3 \frac{y}{7}$$
$$4 = \frac{y}{7}$$
$$y = 28$$

c

$$\log_a 6 = \log_a x - 3\log_a 2$$
$$\log_a 6 = \log_a x - \log_a 2^3$$
$$\log_a 6 = \log_a x \div 8$$
$$\log_a 6 = \log_a \frac{x}{8}$$
$$6 = \frac{x}{8}$$
$$x = 48$$

d

$$\log_2 81 = 4\log_2 x$$
$$\log_2 81 = \log_2 x^4$$
$$81 = x^4$$
$$3^4 = x^4$$
$$x = 3$$

e

$$\log_x 54 = \log_x k + 2\log_x 3$$
$$\log_x 54 = \log_x k + \log_x 3^2$$
$$\log_x 54 = \log_x k \times 9$$
$$\log_x 54 = \log_x 9k$$
$$54 = 9k$$
$$k = 6$$

Question 9

a

$$L = 10\log\left(\frac{I}{I_0}\right)$$
$$L = \log_{10}\left(\frac{I}{I_0}\right)^{10}$$
$$\left(\frac{I}{I_0}\right)^{10} = 10^L$$
$$\frac{I^{10}}{I_0^{\ 10}} = 10^L$$
$$I^{\,0} = 10^L I_0^{\ 10}$$
$$I = \sqrt[0]{10^L I_0^{\ 10}}$$
$$I = 10^{\frac{L}{0}} I_0$$

b

$$I = 10^{\frac{db}{0}} \times I_0$$
$$I = 10^{\frac{45}{0}} \times I_0$$
$$I = 31\,62.8 \quad I_0$$

Question 10

a **i**

$$A = 100 - 50\log(t+1)$$
$$A - 100 = -50\log(t+1)$$
$$\frac{A-100}{-50} = \log(t+1)$$
$$\log(t+1) = \frac{100-A}{50}$$

ii

$$\log(t+1) = \frac{100-A}{50}$$
$$\log_{10}(t+1) = \frac{100-A}{50}$$
$$t+1 = 10^{\frac{100-A}{50}}$$
$$t = 10^{\frac{100-A}{50}} - 1$$

b **i**

$$A = 100 - 50\log(t+1)$$
$$A = 100 - 50\log(3+1)$$
$$A = 100 - 50\log(4)$$
$$A = 69.9$$

ii

$$t = 10^{\frac{100-A}{50}} - 1$$
$$t = 10^{\frac{100-75}{50}} - 1$$
$$t = 10^{\frac{1}{2}} - 1$$
$$t = 2.2$$

Question 11

a $\log_4 9 = \frac{\log 9}{\log 4} = 1.58496\ldots$

≈ 1.58

b $\log_6 25 = \frac{\log 25}{\log 6} = 1.7964\ldots$

≈ 1.80

c $\log_9 200 = \frac{\log 200}{\log 9} = 2.4113\ldots$

≈ 2.41

d $\log_2 12 = \frac{\log 12}{\log 2} = 3.5849\ldots$

≈ 3.58

e $\log_3 23 = \frac{\log 23}{\log 3} = 2.8540\ldots$

≈ 2.85

f $\log_8 250 = \frac{\log 250}{\log 8} = 2.65526\ldots$

≈ 2.66

g $\log_5 9.5 = \frac{\log 9.5}{\log 5} = 1.3988\ldots$

≈ 1.40

h $2\log_4 23.4 = 2 \times \frac{\log 23.4}{\log 4} = 4.5484\ldots$

≈ 4.55

i $7 - \log_7 108 = 7 - \frac{\log 108}{\log 7} = 4.5938\ldots \approx 4.59$

j $3\log_{11} 340 = 3\times\frac{\log 340}{\log 11}$ = 7.2925 …

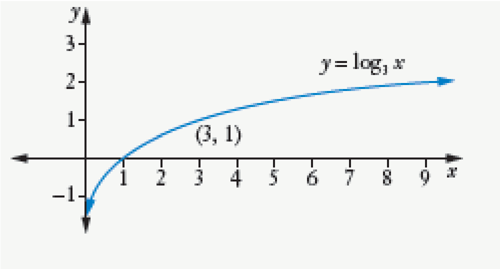
≈ 7.29

Exercise 10.06 Logarithmic functions

Question 1

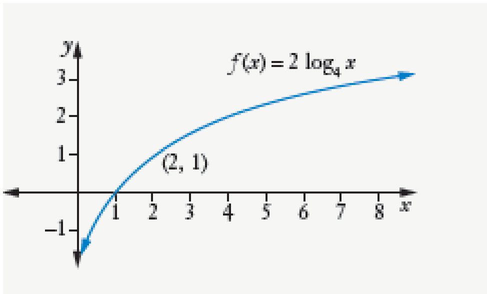
a

$$y = \log_3 x$$
$$x = 3^y$$
$$\text{when } y = 0; x = 3^0 \Rightarrow x = 1 \ (1,0)$$
$$\text{when } y = 1; \ x = 3 \ \Rightarrow \ x = 3 \ (3, 1)$$



b

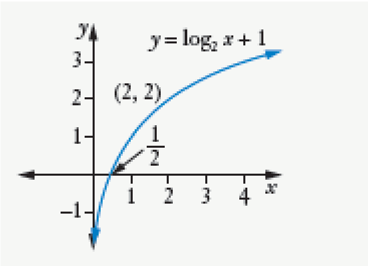
$$f\left(x\right) = 2\log_4 x$$
$$y = \log_4 x^2$$
$$x^2 = 4^y$$
$$x = 4^{\frac{y}{2}}$$
$$\text{when } y = 0; \ x = 4^{\frac{0}{2}} \Rightarrow x = 1 \ (1, 0)$$
$$\text{when } y = 1; \ x = 4^{\frac{1}{2}} \Rightarrow x = 2 \ (2, 1)$$



c

$$y = \log_2 x + 1$$
$$y - 1 = \log_2 x$$
$$x = 2^{y-1}$$
when $y = 1$; $x = 2^{1-1} \Rightarrow x = 1 \quad (1, 1)$

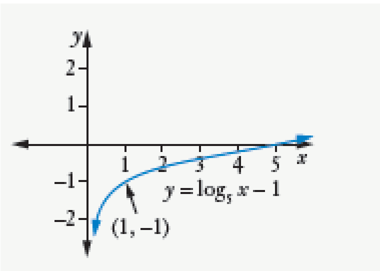
when $y = 2$; $x = 2^{2-1} \Rightarrow x = 2 \quad (2, 2)$



d

$$y = \log_5 x - 1$$
$$y + 1 = \log_5 x$$
$$x = 5^{y+1}$$
when $y = -1$; $x = 5^{-1+1} \Rightarrow x = 1 \quad (1, -1)$

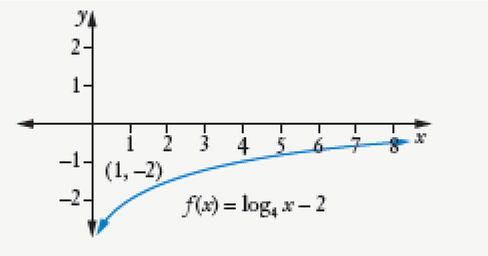
when $y = 0$; $x = 5^{0+1} \Rightarrow x = 5 \quad (5, 0)$



e

$$y = \log_4 x - 2$$
$$y + 2 = \log_4 x$$
$$x = 4^{y+2}$$
when $y = -2;\ x = 4^{-2+2} \Rightarrow x = 1\ \ (1, -2)$

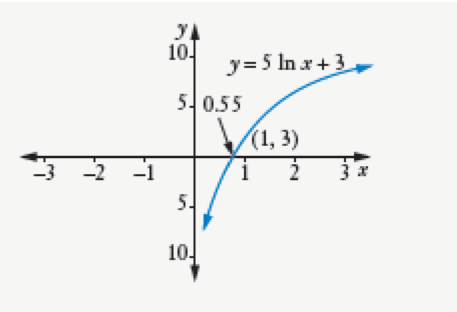
when $y = -1;\ x = 4^{-1+2} \Rightarrow x = 4\quad (4, -1)$



f

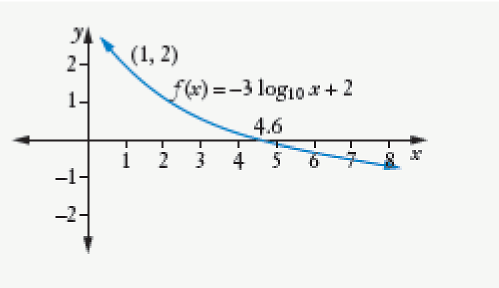
$$y = 5\ln x + 3$$
$$y - 3 = \log_e x^5$$
$$y - 3 = \log_e x^5$$
$$x^5 = e^{y-3}$$
$$x = \sqrt[5]{e^{y-3}}$$
$$x = e^{\frac{y-3}{5}}$$
when $y = 3;\ x = e^{\frac{3-3}{5}} \Rightarrow x = 1\ \ (1, 3)$

when $y = 1;\ x = e^{\frac{1-3}{5}} \Rightarrow x = e^{\frac{-2}{5}} \Rightarrow x = 0.67\quad (0.67, 1)$



g

$$f\left(x\right)=-3\log_{0}\ x+2$$
$$y-2=\log_{0}\ x^{-3}$$
$$y-2=\log_{10}\frac{1}{x^{3}}$$
$$\frac{1}{x^{3}}=10^{y-2}$$
$$x^{3}=\frac{1}{10^{y-2}}$$
$$x=\sqrt[3]{\frac{1}{10^{y-2}}}$$
$$x=10^{\frac{2-y}{3}}$$
$$\text{when } y=2;\ x=10^{\frac{2-2}{3}}\Rightarrow x=1\ \ (1,\ 2)$$
$$\text{when } y=-1;\ x=10^{\frac{2+1}{3}}=10^{\frac{3}{3}}\Rightarrow x=10\quad(10,-1)$$



Question 2

$y = x$

$y = 1, x = 1 \quad (1, 1)$

$y = 0, x = 0 \quad (0, 0)$

$y = 10^x$

$x = 0, y = 1 \quad (0, 1)$

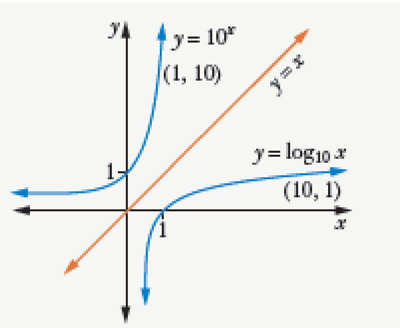
$x = 1, y = 10 \quad (1, 10)$

$y = \log_0 x$

$x = 10^y$

$y = 0, x = 1 \quad (1, 0)$

$y = 1, x = 10 \quad (10, 1)$

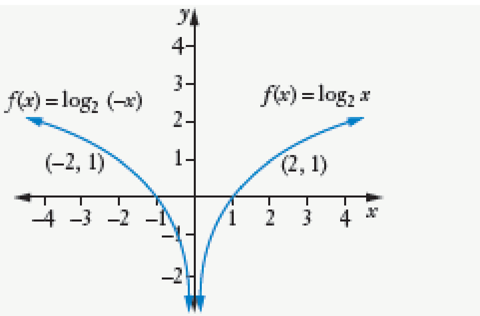


Curves are reflections of each other about the line $y = x$.

Question 3

$y = \log_2 x$ goes through points (1, 0) and (2, 1).

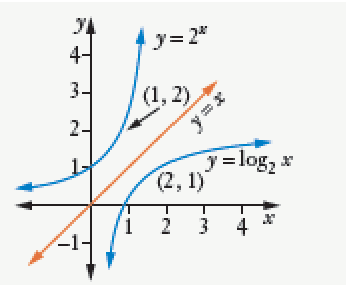
$y = \log_2 (-x)$ is a reflection of $y = \log_2 x$ in the y-axis and goes through points (–1, 0) and (–2, 1).



Question 4

a

$$y = \log_2 x$$
$$x = 2^y$$
$$y = 0, x = 1 \quad (1, 0)$$
$$y = 1, x = 2 \quad (2, 1)$$
$$y = 2^x$$
$$x = 0, y = 1 \quad (0, 1)$$
$$x = 1, y = 2 \quad (1, 2)$$



b The inverse function is $y = 2^x$.

Question 5

a

$y = \log_7 x$

$x = 7^y$

inverse $y = 7^x$

b

$y = \log_9 x$

$x = 9^y$

inverse $y = 9^x$

c

$y = \log_e x$

$x = e^y$

inverse $y = e^x$

d

$y = 2^x$

$\log_2 y = x$

inverse $\log_2 x = y$

e

$y = 6^x$

$\log_6 y = x$

inverse $\log_6 x = y$

f

$y = e^x$

$\log_e y = x$

inverse $\log_e x = y$

or $\ln x = y$

Question 6

a $5.6 – 4.8 = 0.8$

$10^{0.8} = 6.3095 \ldots \approx 6$

b $6.3 – 4.3 = 2$

$10^{2} = 100$

c $5.6 – 4.3 = 1.3$

$10^{1.3} = 19.9526 \ldots \approx 20$

d $5.5 – 5.2 = 0.3$

$10^{0.3} = 1.9952 \ldots \approx 2$

e $6.3 – 4.2 = 2.1$

$10^{2.1} = 125.8925 \ldots \approx 126$

Question 7

a $23 – 20 = 3$

$10^{3} = 1000$

b $41 – 40 = 1$

$10^{1} = 10$

c $66.5 – 65.2 = 1.3$

$10^{1.3} = 19.9526 \ldots \approx 20$

d $88.9 – 85.4 = 3.5$

$10^{3.5} = 3162.28 \ldots \approx 3162$

e $58.\ 6 – 52.3 = 6.3$

$10^{6.3} = 1\ 995\ 262.32 \ldots \approx 1\ 9995\ 262$

Exercise 10.07 Exponential equations

Question 1

a

$$4^x = 9$$
$$x = \log_4 9$$
$$x = \frac{\log 9}{\log 4}$$
$$x = 1.6$$

b

$$3^x = 5$$
$$x = \log_3 5$$
$$x = \frac{\log 5}{\log 3}$$
$$x = 1.5$$

c

$$7^x = 14$$
$$x = \log_7 14$$
$$x = \frac{\log 14}{\log 7}$$
$$x = 1.4$$

d

$$2^x = 15$$
$$x = \log_2 15$$
$$x = \frac{\log 15}{\log 2}$$
$$x = 3.9$$

e

$$5^x = 34$$
$$x = \log_5 34$$
$$x = \frac{\log 34}{\log 5}$$
$$x = 2.2$$

f

$$6^x = 60$$
$$x = \log_6 60$$
$$x = \frac{\log 60}{\log 6}$$
$$x = 2.3$$

g

$$2^x = 76$$
$$x = \log_2 76$$
$$x = \frac{\log 76}{\log 2}$$
$$x = 6.2$$

h

$$4^x = 50$$
$$x = \log_4 50$$
$$x = \frac{\log 50}{\log 4}$$
$$x = 2.8$$

i

$$3^x = 23$$
$$x = \log_3 23$$
$$x = \frac{\log 23}{\log 3}$$
$$x = 2.9$$

j

$$
\begin{aligned}
9^x &= 210 \\
x &= \log_9 210 \\
x &= \frac{\log 210}{\log 9} \\
x &= 2.4
\end{aligned}
$$

Question 2

a

$$
\begin{aligned}
2^x &= 6 \\
x &= \log_2 6 \\
x &= \frac{\log 6}{\log 2} \\
x &= 2.58
\end{aligned}
$$

b

$$
\begin{aligned}
5^y &= 15 \\
y &= \log_5 15 \\
y &= \frac{\log 15}{\log 5} \\
y &= 1.68
\end{aligned}
$$

c

$$
\begin{aligned}
3^x &= 20 \\
x &= \log_3 20 \\
x &= \frac{\log 20}{\log 3} \\
x &= 2.73
\end{aligned}
$$

d

$$
\begin{aligned}
7^m &= 32 \\
m &= \log_7 32 \\
m &= \frac{\log 32}{\log 7} \\
m &= 1.78
\end{aligned}
$$

e

$$4^k = 50$$
$$k = \log_4 50$$
$$k = \frac{\log 50}{\log 4}$$
$$k = 2.82$$

f

$$3^t = 4$$
$$t = \log_3 4$$
$$t = \frac{\log 4}{\log 3}$$
$$t = 1.26$$

g

$$8^x = 11$$
$$x = \log_8 11$$
$$x = \frac{\log 11}{\log 8}$$
$$x = 1.15$$

h

$$2^p = 57$$
$$p = \log_2 57$$
$$p = \frac{\log 57}{\log 2}$$
$$p = 5.83$$

i

$$4^x = 81.3$$
$$x = \log_4 81.3$$
$$x = \frac{\log 81.3}{\log 4}$$
$$x = 3.17$$

j

$$6^{n} = 102.6$$
$$n = \log_6 102.6$$
$$n = \frac{\log 102.6}{\log 6}$$
$$n = 2.58$$

Question 3

a

$$3^{x+} = 8$$
$$x + 1 = \log_3 8$$
$$x = \frac{\log 8}{\log 3} - 1$$
$$x = 0.9$$

b

$$5^{3n} = 71$$
$$3n = \log_5 71$$
$$n = \frac{\log 71}{\log 5} \div 3$$
$$n = 0.9$$

c

$$2^{x-3} = 12$$
$$x - 3 = \log_2 12$$
$$x = \frac{\log 12}{\log 2} + 3$$
$$x = 6.6$$

d

$$4^{2n-1}=7$$
$$2n-1=\log_4 7$$
$$2n=\frac{\log 7}{\log 4}+1$$
$$n=\left(\frac{\log 7}{\log 4}+1\right)\div 2$$
$$n=1.2$$

e

$$7^{5x+2}=11$$
$$5x+2=\log_7 11$$
$$5x=\frac{\log 11}{\log 7}-2$$
$$x=\left(\frac{\log 11}{\log 7}-2\right)\div 5$$
$$x=-0.2$$

f

$$8^{3-n}=5.7$$
$$3-n=\log_8 5.7$$
$$-n=\frac{\log 5.7}{\log 8}-3$$
$$n=\left(\frac{\log 5.7}{\log 8}-3\right)\div -1$$
$$n=2.2$$

g

$$2^{x+2}=18.3$$
$$x+2=\log_2 18.3$$
$$x=\frac{\log 18.3}{\log 2}-2$$
$$x=2.2$$

h

$$3^{7k-3} = 32.9$$
$$7k - 3 = \log_3 32.9$$
$$7k = \frac{\log 2.9}{\log 3} + 3$$
$$k = \left(\frac{\log 2.9}{\log 3} + 3\right) \div 7$$
$$k = 0.9$$

i

$$9^{\frac{x}{2}} = 50$$
$$\frac{x}{2} = \log_9 50$$
$$x = \frac{\log 50}{\log 9} \times 2$$
$$x = 3.6$$

Question 4

a

$$e^x = 200$$
$$x = \ln 200$$
$$x = 5.30$$

b

$$e^{3t} = 5$$
$$3t = \ln 5$$
$$t = \ln 5 \div 3$$
$$t = 0.536$$

c

$$2e^t = 75$$
$$e^t = 37.5$$
$$t = \ln 37.5$$
$$t = 3.62$$

d

$$45 = e^x$$
$$\ln 45 = x$$
$$x = 3.81$$

e

$$3000 = 100e^n$$
$$30 = e^n$$
$$\ln 30 = n$$
$$n = 3.40$$

f

$$100 = 20e^{3t}$$
$$5 = e^{3t}$$
$$\ln 5 = 3t$$
$$\ln 5 \div 3 = t$$
$$t = 0.536$$

g

$$2000 = 50e^{0.15t}$$
$$40 = e^{0.15t}$$
$$\ln 40 = 0.15t$$
$$\ln 40 \div 0.15 = t$$
$$t = 24.6$$

h

$$15000 = 2000e^{0.03k}$$
$$7.5 = e^{0.03k}$$
$$\ln 7.5 = 0.03k$$
$$\ln 7.5 \div 0.03 = k$$
$$k = 67.2$$

i

$$3Q = Qe^{0.02t}$$
$$3 = e^{0.02t}$$
$$\ln \quad = 0.02\ t$$
$$\ln \quad \div 0.02\ = t$$
$$t = 54.9$$

Question 5

a **i**

$$A = 850(1.025)^n$$
$$A(0) = 850(1.025)^0$$
$$A(0) = \$850$$

ii

$$A = 850(1.025)^n$$
$$A(7) = 850(1.025)^7$$
$$A(7) = \$1010.38$$

b

$$A = 850(1.025)^n$$
$$1000 = 850(1.025)^n$$
$$\frac{1000}{850} = (1.025)^n$$
$$\log_{1.025}\frac{1000}{850} = n$$
$$n = \frac{\log\frac{1000}{850}}{\log 1.025}$$
$$n = 6.6 \text{ years}$$

Question 6

a **i**

$$P = 35\,000e^{0.024t}$$
$$P(0) = 35\,000e^{0024\ \times 0}$$
$$P(0) = 35\,000$$

ii

$$P = 35\ 000e^{0.024t}$$
$$P(10) = 35\ 000e^{0024\ \times 10}$$
$$P(10) = 44\ \ 494$$

iii

$$P = 35\,000e^{0.024t}$$
$$P(50) = 35\,000e^{0024\ \times 50}$$
$$P(50) = 116\ 204$$

b **i**

$$P = 35\ 000e^{0.024t}$$
$$80\ 000 = 35\ 000e^{0.024t}$$
$$\frac{80\ 000}{35\ 000} = e^{0.024t}$$
$$\ln\frac{80\ 000}{35\ 000} = 0.024t$$
$$t = \ln\frac{80\ 000}{35\ 000} \div 0.024$$
$$t = 34.4 \text{ years}$$

ii

$$P = 35\ 000e^{0.024t}$$
$$200\ 000 = 35\ 000e^{0.024t}$$
$$\frac{200\ 000}{35\ 000} = e^{0.024t}$$
$$\ln\frac{200\ 000}{35\ 000} = 0.024t$$
$$t = \ln\frac{200\ 000}{35\ 000} \div 0.024$$
$$t = 72.6 \text{ years}$$

Question 7

a **i**

$$N = 8900e^{-0.048t}$$
$$N(0) = 8900e^{-0048 \times 0}$$
$$N(0) = 8900$$

ii

$$N = 8900e^{-0.048t}$$
$$N(5) = 8900e^{-0048 \times 5}$$
$$N(5) = 7001$$

iii

$$N = 8900e^{-0.048t}$$
$$N(70) = 8900e^{-0048 \times 70}$$
$$N(70) = 309$$

b **i**

$$N = 8900e^{-0.048t}$$
$$5000 = 8900e^{-0.048t}$$
$$\frac{5000}{8900} = e^{-0.048t}$$
$$\ln\frac{5000}{8900} = -0.048t$$
$$t = \ln\frac{5000}{8900} \div -0.048$$
$$t = 12 \text{ years}$$

ii

$$N = 8900e^{-0.048t}$$
$$200 = 8900e^{-0.048t}$$
$$\frac{200}{8900} = e^{-0.048t}$$
$$\ln\frac{200}{8900} = -0.048t$$
$$t = \ln\frac{200}{8900} \div -0.048$$
$$t = 79 \text{ years}$$

Question 8

a

$$M = 100e^{-0.00003t}$$
$$M(0) = 100e^{-000003 \times 0}$$
$$M(0) = 100 \text{ g}$$

b

$$M = 100e^{-0.00003t}$$
$$M(50) = 100e^{-000003 \times 50}$$
$$M(50) = 99.85 \text{ g}$$

c

$$M = 100e^{-0.00003t}$$
$$M(500) = 100e^{-000003 \times 500}$$
$$M(500) = 98.5 \text{ g}$$

d

$$M = 100e^{-0.00003t}$$
$$50 = 100e^{-0.00003t}$$
$$\frac{50}{100} = e^{-0.00003t}$$
$$\ln\left(\frac{1}{2}\right) = -0.00003t$$
$$t = \ln\left(\frac{1}{2}\right) \div (-0.00003)$$
$$t = 23\,105 \text{ years}$$

Question 9

a

$$T = 18 + 12e^{0.002t}$$
$$T(5) = 18 + 12e^{0002 \times 5}$$
$$T(5) = 30.1°\text{C}$$

b

$$T = 18 + 12e^{0.002t}$$
$$50 = 18 + 12e^{0.002t}$$
$$32 = 12e^{0.002t}$$
$$\frac{32}{12} = e^{0.002t} \qquad \backslash$$
$$\ln\frac{32}{12} = 0.002t$$
$$t = \ln\frac{32}{12} \div 0.002$$
$$t = 490.4 \text{ hours}$$

Question 10

a **i**

$$x = 5e\ \ + 23$$
$$x(0) = 5e^{0} + 23$$
$$x(0) = 28 \text{ cm}$$

ii

$$x = 5e\ \ + 23$$
$$x' = v = 5e^{t}$$
$$v(20) = 5e^{20} \text{ cm s}^{-1}$$

iii

$$x = 5e\ \ + 23$$
$$x(6) = 5e^{6} + 23$$
$$x(6) = 2040.1 \text{ cm}$$

iv

$$x = 5e\ \ + 23$$
$$85 = 5e\ \ + 23$$
$$62 = 5e^t$$
$$\frac{62}{5} = e^t$$
$$\ln\frac{62}{5} = t$$
$$t = 2.52\text{ s}$$

v

$$x = 5e\ \ + 23$$
$$x' = v = 5e^t$$
$$1000 = 5e^t$$
$$200 = e^t$$
$$\ln 200 = t$$
$$t = 5.30\text{ s}$$

b

$$x = 5e\ \ + 23$$
$$x' = v = 5e^t$$
$$v' = a = 5e^t$$
$$a = 5e^t$$
$$= 5e\ \ + 23 - 23$$
$$= x - 23$$

c

$$a = x - 23$$
$$a = 85 - 23$$
$$a = 62\text{ cm s}^{-2}$$

Question 11

a

$$y = e^{2x}$$
$$\ln y = 2x$$
$$x = \frac{\ln y}{2}$$
$$\text{inverse: } y = \frac{\ln x}{2}$$

b

$$y = \ln(x+1)$$
$$e^y = x+1$$
$$x = e^y - 1$$
$$\text{inverse: } y = e^x - 1$$

c

$$f(x) = e^{3x} + 1$$
$$y = e^{3x} + 1$$
$$y - 1 = e^{3x}$$
$$\ln(y-1) = 3x$$
$$x = \frac{\ln(y-1)}{3}$$
$$\text{inverse: } y = \frac{\ln(x-1)}{3}$$

Question 12

a domain: $(-\infty, \infty)$, range: $(0, \infty)$

b

$$f(x) = 3^x$$
$$y = 3^x$$
$$\log_3 y = x$$
$$\text{inverse: } y = \log_3 x$$

c domain: $(0, \infty)$ range: $(-\infty, \infty)$

Question 1

a

$$N = 80e^{0.002t}$$
$$N(0) = 80e^{0002 \times 0}$$
$$N(0) = 80 \text{ birds}$$

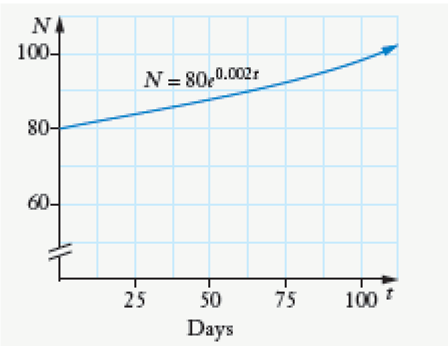
b

$$N = 80e^{0.002t}$$
$$N(30) = 80e^{0002 \times 30}$$
$$N(30) = 85 \text{ birds}$$

c

$$N = 80e^{0.002t}$$
$$500 = 80e^{0.002t}$$
$$\frac{500}{80} = e^{0.002t}$$
$$\ln\frac{500}{80} = 0.002t$$
$$t = \ln\frac{500}{80} \div 0.002$$
$$t = 916.29$$
916 days

d



Question 2

a

$$N = N_0 e^{0.32t}$$
$$N = 20\ 000 e^{0.32t}$$
$$N(5) = 20\ 000 e^{032\ \times 5}$$
$$N(5) \approx 99\ 061$$

b

$$N = N_0 e^{0.32t}$$
$$200\ 000 = 20\ 000 e^{0.32t}$$
$$10 = e^{0.32t}$$
$$\ln 10 = 0.32t$$
$$t = \ln 10.32$$
$$t = 7 \text{ hours}$$

Question 3

a

$$\text{when } t = 0, M = 100$$
$$M = 100e^{-kt}$$
$$\text{when } t = 5, M = 95$$
$$95 = 100e^{-5k}$$
$$0.95 = e^{-5k}$$
$$\ln 0.95 = -5k$$
$$k = 0.01$$
$$\therefore M = 100e^{-0.01t}$$

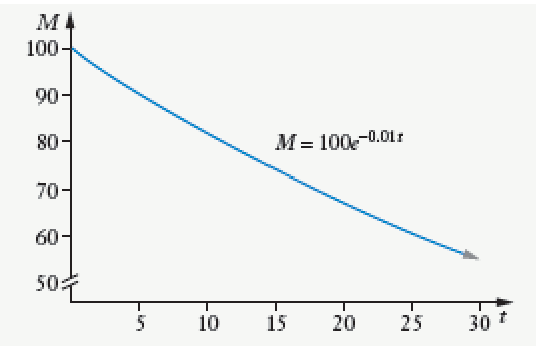
b

$$M = 100e^{-0.01t}$$
$$M(10) = 100e^{-001 \times 10}$$
$$M(10) = 90.25 \text{ kg}$$

c

$$M = 100e^{-0.01t}$$
$$50 = 100e^{-0.01t}$$
$$\frac{1}{2} = e^{-0.01t}$$
$$\ln\frac{1}{2} = -0.01t$$
$$t = 67.6 \text{ years}$$

d



Question 4

a

$$A = A_0 e^{-kt}$$
when $A = 100, t = 0$
$$100 = A_0 e^{-k \times 0}$$
$$100 = A_0$$
when $A = 65, t = 5$
$$65 = 100e^{-k \times 5}$$
$$0.65 = e^{-5k}$$
$$\ln 0.65 = -5k$$
$$k = \frac{\ln 0.65}{-5}$$
$$A = 100e^{\frac{\ln 0.65}{5}t}$$
$$A(12) = 100e^{\frac{\ln 0.65}{5} \times 12}$$
$$A(12) = 35.6 \text{ L}$$

b

$$A = 100e^{\frac{\ln 0.65}{5}t}$$
$$10 = 100e^{\frac{\ln 0.65}{5}t}$$
$$0.1 = e^{\frac{\ln 0.65}{5}t}$$
$$\ln 0.1 = \frac{\ln 0.65}{5} \times t$$
$$t = \ln 0.1 \div \frac{\ln 0.65}{5}$$
$$t = 26.7 \text{ min}$$

Question 5

a

$$P = P_0 e^{kt}$$
$$\text{when } t = 0, P = 5000$$
$$5000 = P_0 e^{k \times 0}$$
$$5000 = P_0$$

b

$$P = 5000e^{kt}$$
$$\text{when } t = 3, P = 8000$$
$$8000 = 5000e^{k \times 3}$$
$$1.6 = e^{3k}$$
$$\ln 1.6 = 3k$$
$$k = \frac{\ln 1.6}{3}$$
$$k = 0.157$$

c

$$P = 5000e^{\frac{\ln 1.6}{3}t}$$
$$\text{when } t = 6$$
$$P = 5000e^{\frac{\ln 1.6}{3} \times 6}$$
$$P = 12\ 800 \text{ units}$$

d

$$P = 5000e^{\frac{\ln 1.6}{3}t}$$
$$\text{when } P = 20\ 000$$
$$20\ 000 = 5000e^{\frac{\ln 1.6}{3} \times t}$$
$$4 = e^{\frac{\ln 1.6}{3} \times t}$$
$$\ln 4 = \frac{\ln 1.6}{3} \times t$$
$$t = \ln 4 \div \frac{\ln 1.6}{3}$$
$$t \approx 8.8 \text{ years}$$

Question 6

$D = D_0 e^{-kt}$

when $D = 3, t = 0$

$3 = D_0 e^{-k \times 0}$

$D_0 = 3$

$D = 3e^{-kt}$

when $D = 2.7, t = 20$

$2.7 = 3e^{-k \times 20}$

$0.9 = e^{-k \times 20}$

$\ln 0.9 = -20k$

$k = \dfrac{\ln 0.9}{-20}$

$D = 3e^{\frac{\ln 0.9}{20}t}$

when $t = 50$

$D = 3e^{-\frac{\ln 0.9}{20} \times 50}$

$D = 2.3 \text{ million } \text{m}^2$

Question 7

a

$$\frac{dP}{dt} = 0.069P$$

$$P = P_0 e^{kt}$$

$$P = 50\ 000e^{0.069t}$$

b

$$P = 50\ 000e^{0.069t}$$

$$P(5) = 50000e^{0069\ \times 5}$$

$$P(5) = 70\ 599$$

c

$$P = 50000e^{0.069t}$$

$$P' = 3450e^{0.069t}$$

$$P'(5) = 3450e^{0069\ \times 5}$$

$$P'(5) = 4871 \text{ people/year}$$

d

$$P = 50\ 000e^{0.069t}$$

$$300\ 000 = 50\ 000e^{0.069t}$$

$$6 = e^{0.069t}$$

$$\ln 6 = 0.069t$$

$$t = \ln 6 \div 0069$$

$$t = 26 + 2015$$

$$t = 2041$$

Question 8

a

$$T = T_0 e^{-kt}$$
$$T_0 = 90, T = 81, t = 10$$
$$81 = 90e^{-k \times 10}$$
$$0.9 = e^{-10k}$$
$$\ln 0.9 = -10k$$
$$k = \frac{\ln 0.9}{-10}$$
$$T = 90e^{-\frac{\ln 0.9}{-10}t}$$
$$T = 90e^{-\frac{\ln 0.9}{10} \times 30}$$
$$T = 6561 \text{ °C}$$

b

$$T = 90e^{-\frac{\ln 0.9}{-10}t}$$
$$30 = 90e^{-\frac{\ln 0.9}{10} \times t}$$
$$\frac{1}{3} = e^{-\frac{\ln 0.9}{10} \times t}$$
$$\ln \frac{1}{3} = -\frac{\ln 0.9}{10} \times t$$
$$t = \ln \frac{1}{3} \div -\frac{\ln 0.9}{10}$$
t = 1 hour 44 min

Question 9

a $S = Ae^{kt}$

$A = 150, S = 125, t = 3$

$125 = 150e^{k\times 3}$

$\frac{5}{6} = e^{3k}$

$\ln\frac{5}{6} = 3k$

$k = \frac{\ln\frac{5}{6}}{3}$

$S = 150e^{\frac{\ln\frac{5}{6}}{3}t}$

$S = 150e^{\frac{\ln\frac{5}{6}}{3}\times 8} = 92 \text{ kg}$

b

$S = 150e^{\frac{\ln\frac{5}{6}}{3}t}$

$S' = -912\ e^{\frac{\ln\frac{5}{6}}{3}t}$

$S'(8) = -9.12e^{\frac{\ln\frac{5}{6}}{3}\times 8} = -5.6 \text{ kg/h}$

Reducing at a rate of 5.6 kg/h.

c

$S = 150e^{\frac{\ln\frac{5}{6}}{3}t}$

$50 = 150e^{\frac{\ln\frac{5}{6}}{3}t}$

$\frac{1}{3} = e^{\frac{\ln\frac{5}{6}}{3}t}$

$\ln\frac{1}{3} = \frac{\ln\frac{5}{6}}{3}\times t$

$t = \ln\frac{1}{3} \div \frac{\ln\frac{5}{6}}{3} = 18 \text{ hours}$

Question 10

a

$$M = M_0 e^{-kt}$$
$$M_0 = 200$$
$$M = 195, t = 10$$
$$195 = 200e^{-k \times 10}$$
$$0.975 = e^{-k \times 10}$$
$$\ln 0.975 = -10k$$
$$k = \frac{\ln 0.975}{-10}$$
$$k = 0.00253$$

b

$$M = 200e^{\frac{\ln 0.975}{-10}t}$$
$$M(15) = 200e^{\frac{\ln 0.975}{0} \times 15} = 192.5 \text{ g}$$

c

$$M = 200e^{\frac{\ln 0.975}{-10}t}$$
$$M' = 200e^{\frac{\ln 0.975}{10} \times t}$$
$$M' = -0506\, e^{\frac{\ln 0.975}{10} \times t}$$
$$M'(15) = -0.506e^{\frac{\ln 0.975}{0} \times 15} = 0.49 \quad \text{g/year}$$

d

$$M = 200e^{\frac{\ln 0.975}{10}t}$$
$$100 = 200e^{\frac{\ln 0.975}{10}t}$$
$$0.5 = e^{\frac{\ln 0.975}{10}t}$$
$$\ln 0.5 = \frac{\ln 0.975}{10} \times t$$
$$t = \ln 0.5 \div \frac{\ln 0.975}{10}$$
$$t = 273.8 \text{ years}$$

Question 11

a

$$B = B_0 e^{kt}$$

$$B_0 = 15\ 000,\ B = 25\ 000, t = 7$$

$$25\ 000 = 15\ 000 e^{k \times 7}$$

$$\frac{5}{3} = e^{k \times 7}$$

$$\ln\frac{5}{3} = 7k$$

$$k = \frac{\ln\frac{5}{3}}{7}$$

$$B = 15\ 000 e^{\frac{\ln\frac{5}{3}}{7}t} \approx 15\ 000 e^{0.0730t}$$

b

$$B = 15\ 000 e^{0.0730t}$$

$$B(12) = 15\ 000 e^{00730\ \times 12} \approx 36\ 008$$

c

$$B = 15\ 000 e^{0.0730t}$$

$$5\ 000\ 000 = 15\ 000 e^{0.0730t}$$

$$\frac{1000}{3} = e^{0.0730t}$$

$$\ln\frac{1000}{3} = 0.0730t$$

$$t = \ln\frac{1000}{3} \div 0.0730$$

$$t \approx 79.6 \text{ hours}$$

Question 12

$P = P_0 e^{kt}$

$t = 3$

$1.2P_0 = P_0 e^{3k}$

$1.2 = e^{3k}$

$\ln 1.2 = 3k$

$k = \dfrac{\ln 1.2}{3}$

$2 = e^{\frac{\ln 1.2}{3} \times t}$

$\ln 2 = \dfrac{\ln 1.2}{3} \times t$

$t = \ln 2 \div \dfrac{\ln 1.2}{3}$

≈ 11.4 years

Question 13

a

$T = T_0 e^{kt}$

$0.5 = e^{k \times 1600}$

$\ln 0.5 = 1600k$

$k = \ln 0.5 \div 1600$

$T = T_0 e^{-0.0004332t}$

$T \div T_0 = e^{-00004332 \times 500}$

$= 80.5\%$

$100\% - 80.5\% = 19.5\%$

b

$T = T_0 e^{-0.0004332t}$

$0.25 = e^{-0.0004332t}$

$\ln 0.20.\cancel{00}04332 \qquad t$

$t = \ln 0.25 \div \left(-0.0004332\right)$

$t \approx 3200$ years

Question 14

a $P = P_0e^{-kt}$

$$\frac{dP}{dt} = -kP_0e^{-k} = -kP$$

b When $t = 0$, $P = P_0e^{-k(0)} = P_0e^0 = P_0$

When $t = 4$, $P = P_0 - 10\%P_0 = 0.9P_0$

Substituting: $0.9P_0 = P_0e^{-k(4)}$

$0.9 = e^{-4k}$

$\ln 0.9 = -4k$

$\frac{\ln 0.9}{-4} = k$

$0.02634\ldots = k$

So $P = P_0e^{-0.02634\ldots t}$

When $t = 10$, $P = P_0e^{-0.02634\ldots\times 10}$

$= P_0(0.7684\ldots)$

Decrease from $P_0 = 1 - 0.7684\ldots = 0.2315\ldots \approx 23\%$

So the decline in population is 23%

c $\frac{dP}{dt} = -0.02634\ldots P_0e^{-0.02634\ldots t}$

When $t = 10$, $\frac{dP}{dt} = -0.02634\ldots P_0e{-}0.02634\ldots(10) = -0.0208\ldots P_0$

$\approx -2\%P_0$

So the rate of decline in population is 2% per year.

d A fall of 20% = $0.8P_0$

Substituting: $0.8P_0 = P_0e^{-0.02634\ldots t}$

$0.8 = e^{-0.02634\ldots t}$

$\ln 0.8 = -0.02634\ldots\, t$

$$\frac{\ln 0.8}{-0.02634...} = t$$

$t = 8.4716 \ldots \approx 8.5$

So it takes 8.5 years to fall by 20%.

Question 15

$W = W_0e^{-kt}$
$0.4 = e^{-k \times 5}$
$\ln 0.4 = -5k$
$k = \frac{\ln 0.4}{-5} \approx -0.1833$
$W = W_0e^{-0.1833t}$
$0.1 = e^{-0.1833t}$
$\ln 0.1 = -0.1833t$
$t = \ln 0.1 \div (-0.1833)$
$t \approx 12.6$ min

Question 16

$S = S_0e^{-kt}$
$0.85 = e^{-k \times 3}$
$\ln 0.85 = -3k$
$k = \frac{\ln 0.85}{-3} \approx 0.0542$
$S = S_0e^{0.0542t}$
$0.5 = e^{0.0542t}$
ln0.0.0542 t
$t =$ ln0.0.0542
$\approx$ 12.8 years

Question 17

a

$$A = A_0 e^{-kt}$$
$$t = 1,\ A = 120,\ A_0 = 150$$
$$120 = 150e^{-k \times 1}$$
$$\ln 0.8 = -k$$
$$k = -\ln 0.8$$
$$A = 150e^{\ln 0.8 \times t}$$
when $t = 3$
$$A = 150e^{\ln .8 \times 3}$$
$$A = 76.8 \text{ mg/dL}$$

b

$$A = 150e^{\ln 0.8 \times t}$$
when $A = 20$
$$20 = 150e^{\ln 0.8 \times t}$$
$$\frac{2}{15} = e^{\ln 0.8 \times t}$$
$$\ln \frac{2}{15} = \ln 0.8 \times t$$
$$t = \ln \frac{2}{15} \div \ln 0.8$$
$$t = 9 \text{ hours}$$

Question 18

$C = C_0 e^{-kt}$

$\dfrac{dC}{dt} = -kC$

when $t = 5$, $C \div C_0 = 0.6$

$0.6 = e^{-k \times 5}$

$\ln 0.6 = -5k$

$k = \dfrac{\ln 6}{-5}$

$C = C_0 e^{\frac{\ln 6}{5} t}$

when $C \div C_0 = 0.2$

$0.2 = e^{\frac{\ln 6}{5} t}$

$\ln 0.2 = \dfrac{\ln 6}{5} \times t$

$t = \ln 0.2 \div \dfrac{\ln 6}{5}$

$t = 15.8$ s

Question 19

$P = P_0 e^{kt}$

when $t = 2$, $P \div P_0 = 1.1$

$1.1 = e^{k \times 2}$

$\ln 1.1 = 2k$

$k = \dfrac{\ln 1.1}{2}$

$P = P_0 e^{\frac{\ln 1.1}{2} t}$

when $P \div P_0 = 1.5$

$1.5 = e^{\frac{\ln 1.1}{2} t}$

$\ln 1.5 = \dfrac{\ln 1.1}{2} \times t$

$t = \ln 1.5 \div \dfrac{\ln 1.1}{2}$

$t \approx 8.5$ years

Question 20

$Q = Ae^{kt}$

$Q' = kAe^{k} \;\; = kQ$

Exercise 10.09 Further exponential growth and decay

Question 1

a

$$\frac{dx}{dt} = 2Ae^{2t}$$
$$= 2\left(100 + Ae^{2} \; -100\right)$$
$$= 2\left(x - 100\right)$$

b

$$x = 100 + Ae^{2t}$$
$$180 = 100 + Ae^{2\times 3}$$
$$80 = Ae^{6}$$
$$A = \frac{80}{e^{6}} = 0.198$$

c

$$x = 100 + Ae^{2t}$$
$$150 = 100 + 0.198e^{2t}$$
$$50 = 0.198e^{2t}$$
$$e^{2t} = \frac{50}{0.198}$$
$$2t = \ln\frac{50}{0.198}$$
$$t = \ln\frac{50}{0.198} \div 2 = 2.76$$

Question 2

a

$$\frac{dN}{dt}=0.14Ae^{0.14t}$$
$$=0.14\left(45+Ae^{0.14}\ -45\right)$$
$$=0.14\left(N-45\right)$$

b

$$N=45+Ae^{0.14t}$$
$$82=45+Ae^{014\ \times2}$$
$$37=Ae^{0.28}$$
$$A=\frac{37}{e^{0.28}}$$
$$=27.96$$

c

N = 45 + 27.96$e^{0.14t}$

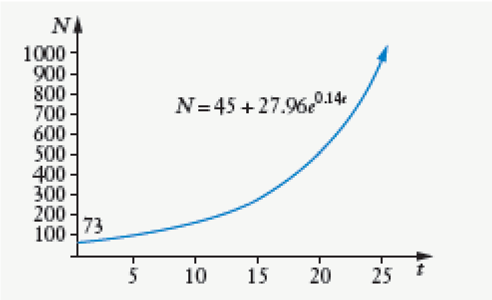
= 45 + 27.96$e^{0.14 \times 5}$

= 101.31

d

$$N=45+2.96\ \ e^{0.14t}$$
$$120=45+27.96e^{0.14t}$$
$$75=2.96\ \ e^{0.14t}$$
$$\frac{75}{27.96}=e^{0.14t}$$
$$\ln\frac{75}{27.96}=0.14t$$
$$t=\ln\frac{75}{27.96}\div0.14=7.05$$

e



N
1000
900
800
700
600
500
400
300
200
100
73
$N = 45 + 27.96e^{0.14t}$
5
10
15
20
25
t

Question 3

a $\frac{dv}{dt} = kAe^{kt}$

$= k(5000 + Ae^{kt} – 5000)$

$= k(V – 5000)$

b $V = 5000 + Ae^{kt}$

When $V = 87\ 000$, $t = 0$.

$87\ 000 = 5000 + Ae^{k \times 0}$

$82\ 000 = Ae^0$

$A = 82\ 000$

$V = 5000 + 82\ 000e^{kt}$

When $V = 129\ 00$, $t = 10$.

$129\ 000 = 5000 + 82\ 000e^{k \times 10}$

$124\ 000 = 82\ 000e^{10k}$

$\frac{124\ 000}{82\,000} = \frac{62}{41} = e^{10k}$

$\ln \frac{62}{41} = 10k$

$k = \ln \frac{62}{41} \div 10 = 0.0414$

c $V = 5000 + 82\ 000e^{0.0414t}$

When $t = 3 \times 24 = 72$ hours,

$V = 5000 + 82\ 000e^{0.0414 \times 72} = 1\ 615\ 609.47$ kL

d $V = 5000 + 82\,000e^{0.0414t}$

When $V = 4\,200\,000$ kL,

$4\,200\,000 = 5000 + 82\,000e^{0.0414t}$

$4\,195\,000 = 82\,000e^{0.0414t}$

$$\frac{4\,195\,000}{82\,000} = \frac{4195}{82} = e^{0.0414t}$$

$$\ln\frac{4195}{82} = 0.0414t$$

$$t = \ln\frac{4195}{82} \div 0.0414 = 95.047 \text{ hours}$$

$\approx$ 3 days 23 hours

Question 4

a $\frac{dN}{dt}\left(kAe^{kt}\right)$

$= k(P + Ae^{kt} – P)$

$= k(N – P)$

b $N = P + Ae^{kt}$

Given $\frac{dN}{dt} = k(N – 1000)$, so $P = 1000$.

When $t = 0$, $N = 1500$; $1500 = 1000 + Ae^{k \times 0}$

$1500 = 1000 + A$

$A = 500$

When $t = 5$, $N = 2200$; $2200 = 1000 + 500e^{k \times 5}$

$1200 = 500e^{5k}$

$\frac{1200}{500} = 2.4 = e^{5k}$

$\ln 2.4 = 5k$

$k = \frac{\ln 2.4}{5}$

$N = 1000 + 500e^{\frac{\ln 2.4}{5}t}$

When $N = 2500$; $2500 = 1000 + 500e^{\frac{\ln 2.4}{5}t}$

$1500 = 500e^{\frac{\ln 2.4}{5}t}$

$3 = e^{\frac{\ln 2.4}{5}t}$

$\ln 3 = \frac{\ln 2.4}{5}t$

$t = \frac{5 \times \ln 3}{\ln 2.4} = 6.27442$

≈ 6.27

Question 5

a $T = 18 + Ae^{-kt}$

When $t = 0$, $T = 80$.

$80 = 18 + Ae^{k \times 0}$

$80 = 18 + A$

$A = 62$

$T = 18 + 62e^{-kt}$

When $t = 15$, $T = 68$.

$68 = 18 + 62e^{-k \times 15}$

$50 = 62e^{-15k}$

$\frac{50}{62} = e^{-15k}$

$\ln\frac{50}{62} = -15k$

$k = \ln\frac{50}{62} \div -15 = 0.0143$

$T = 18 + 62e^{-0.0143t}$

b $T = 18 + 62e^{-0.0143t}$

When $T = 30$

$30 = 18 + 62e^{-0.0143t}$

$12 = 62e^{-0.0143t}$

$\frac{12}{62} = \frac{6}{31} = e^{-0.0143t}$

$\ln\frac{6}{31} = -0.0143t$

$t = \ln\frac{6}{31} \div -0.0143 = 114.5$ minutes

c As $t \to \infty$, $e^{-0.0143t} \to 0$, $\therefore\ T \to 18$ (room temperature)

Question 6

a $P = 950 + Ae^{kt}$

When $t = 0$, $P = 14\ 000$; $14\ 000 = 950 + Ae^{k \times 0}$

$13\ 050 = Ae^{0}$

$A = 13\ 050$

When $t = 6$, $P = 20\ 000$; $20\ 000 = 950 + 13\ 050e^{k \times 6}$

$19\ 050 = 13\ 050e^{6k}$

$$\frac{19\ 050}{13\ 050} = \frac{127}{87} = e^{6k}$$

$$\ln\frac{127}{87} = 6k$$

$$k = \ln\frac{127}{87} \div 6 = 0.063$$

$\therefore\ P = 950 + 13\ 050e^{0.063t}$

When $t = 10$; $P = 950 + 13\ 050e^{0.063 \times 10}$

$= 25\ 464$

b $P = 950 + 13\ 050e^{0.063t}$

When $P = 1\ 000\ 000$;

$1\ 000\ 000 = 950 + 13\ 050e^{0.063t}$

$999\ 050 = 13\ 050e^{0.063t}$

$$\frac{999\ 050}{13\ 050} = \frac{689}{9} = e^{0.063t}$$

$$\ln\frac{689}{9} = 0.063t$$

$$t = \ln\frac{689}{9} \div 0.063$$

$= 68.8$ weeks

Question 7

a $T = –10 + Ae^{kt}$

When $t = 0$, $T = 14$.

$14 = –10 + Ae^{k \times 0}$

$24 = Ae^0$

$A = 24$

When $t = 25$, $T = 11$.

$11 = –10 + 24e^{k \times 25}$

$21 = 24e^{25k}$

$0.875 = e^{25k}$

$\ln 0.875 = 25k$

$k = \dfrac{\ln 0.875}{25}$

$\therefore T = -10 + 24e^{\frac{\ln 0.875}{25}t}$

When $t = 5 \times 60 = 300$

$T = -10 + 24e^{\frac{\ln 0.875}{25} \times 300}$

$= –5.2°$

b $T = -10 + 24e^{\frac{\ln 0.875}{25}t}$

When $T = –8$; $-8 = -10 + 24e^{\frac{\ln 0.875}{25} \times t}$

$2 = 24e^{\frac{\ln 0.875}{25} \times t}$

$\dfrac{1}{12} = e^{\frac{\ln 0.875}{25} \times t}$

$\ln \dfrac{1}{12} = \dfrac{\ln 0.875}{25} \times t$

$t = \ln \dfrac{1}{12} \div \dfrac{\ln 0.875}{25}$

$t = 465$ s

$t = 8$ min

Question 8

a

$$\frac{dv}{dt} = -kAe^{-kt}$$
$$= -k\left(P + Ae^{-k} \ -P\right)$$
$$= -k\left(v - P\right)$$

b

$v = P + Ae^{-kt}$

When $v = 0, P = 500, t = 0.$

$0 = 500 + Ae^{-k \times 0}$

$-500 = Ae^{0}$

$\therefore A = -500$

When $t = 5, v = 21.$

$21 = 500 - 500e^{-k \times 5}$

$-479 = -500e^{-k \times 5}$

$0.958 = e^{-k \times 5}$

$\ln 0.958 = -5k$

$k = \dfrac{\ln 0.958}{-5}$

$\therefore k = 0.00858$

c

$v = 500 - 500e^{\frac{\ln 0.958}{5}t}$

When $t = 20,$

$v = 500 - 500e^{\frac{\ln 0.958}{5} \times 20}$

v = 78.85 ms^{-1}

d

As $t \to \infty$

$e^{\frac{\ln 0.958}{5}t} \to 0$

$\therefore v \to 500$ ms^{-1}

Question 9

a

$$T = C + Ae^{-kt}$$

When $T = -14, C = 25, t = 0$.

$$-14 = 25 + Ae^{-k\times 0}$$

$$-39 = Ae^{0}$$

$$A = -39$$

When $t = 40, T = -5$.

$$-5 = 25 - 39e^{-k \times 40}$$

$$-30 = -39e^{-k \times 40}$$

$$\frac{10}{13} = e^{-40k}$$

$$\ln\frac{10}{13} = -40k$$

$$k = \ln\frac{10}{13} \div -40$$

$$k = 0.00656$$

$$\therefore T = 25 - 39e^{-0.00656t}$$

When $t = 2 \times 60 = 120$

$$T = 25 - 39e^{-000656 \times 120}$$

$$T = 7.2°\text{C}$$

b

$$T = 25 - 39e^{-0.00656t}$$

When $T = 0$,

$$0 = 25 - 39e^{-0.00656t}$$

$$-25 = -39e^{-0.00656t}$$

$$\frac{25}{39} = e^{-0.00656t}$$

$$\ln\frac{25}{39} = -0.0065t$$

$$t = \ln\frac{25}{39} \div -0.0065$$

$$t = 68\text{ s}$$

Question 10

a

$N = P + Ae^{kt}$

When $N = 3000, P = 1800, t = 0.$

$3000 = 1800 + Ae^{k \times 0}$

$1200 = Ae^{0}$

$A = 1200$

When $t = 3, N = 3400.$

$3400 = 1800 + 1200e^{k \times 3}$

$1600 = 1200e^{k \times 3}$

$\frac{4}{3} = e^{3k}$

$\ln\frac{4}{3} = 3k$

$k = \ln\frac{4}{3} \div 3$

$k = 0.0959$

$\therefore N = 1800 + 1200e^{0.0959t}$

When $t = 5,$

$N = 1800 + 1200e^{00959 \times 5}$

$N = 3738$

b

$N = 1800 + 1200e^{0.0959t}$

When $N = 8000,$

$8000 = 1800 + 1200e^{0.0959 \times t}$

$6200 = 1200e^{0.0959 \times t}$

$\frac{31}{6} = e^{0.0959t}$

$\ln\frac{31}{6} = 0.0959t$

$t = \ln\frac{31}{6} \div 0.0959$

$t = 17.1$ years

Question 11

$X = C + Ae^{-kt}$

When $X = 0, C = 7.9, t = 0.$

$0 = 7.9 + Ae^{-k \times 0}$

$-79 = Ae^{0}$

$A = -7.9$

When $t = 2, X = 2.7.$

$2.7 = 7.9 - 7.9e^{-k \times 2}$

$-5.2 = -7.9e^{-k \times 2}$

$\frac{52}{79} = e^{-2k}$

$\ln \frac{52}{79} = -2k$

$k = \ln \frac{52}{79} \div -2$

$k = 0.2091$

$\therefore X = 7.9 - 7.9e^{-0.2091t}$

When $t = 5,$

$X = 7.9 - 7.9e^{-02091 \times 5}$

$X = 5.1$

Question 12

a $P = T + Ae^{kt}$

When $P = 18\ 000, T = 10\ 000, t = 0$.

$18\ 000 = 10\ 000 + Ae^{k\times 0}$

$8000 = Ae^{0}$

$A = 8000$

When $t = 4, P = 25\ 000$.

$25\ 000 = 10\ 000 + 8000e^{k\times 4}$

$15\ 000 = 8000e^{4k}$

$1.875 = e^{4k}$

$\ln 1.875 = 4k$

$k = \dfrac{\ln 1.875}{4}$

$\therefore P = 10\ 000 + 8000e^{\frac{\ln 1.875}{4}t}$

When $P = 40\ 000$

$40\ 000 = 10\ 000 + 8000e^{\frac{\ln 1.875}{4}t}$

$30\ 000 = 8000e^{\frac{\ln 1.875}{4}t}$

$3.75 = e^{\frac{\ln 1.875}{4}t}$

$\ln 3.75 = \dfrac{\ln 1.875}{4} \times t$

$t = \ln 3.75 \div \dfrac{\ln 1.875}{4}$

$t = 8.4$ years

b $P = 10\ 000 + 8000e^{\frac{\ln 1.875}{4}t}$

When $P = 18\ 000 \times 2 = 36\ 000$

$36\ 000 = 10\ 000 + 8000e^{\frac{\ln 1.875}{4}t}$

$26\ 000 = 8000e^{\frac{\ln 1.875}{4}t}$

$3.25 = e^{\frac{\ln 1.875}{4}t}$

$\ln 3.25 = \dfrac{\ln 1.875}{4} \times t$

$t = \ln 3.25 \div \dfrac{\ln 1.875}{4}$

$t = 7.5$ years

Question 13

$P = 200 + 1600e^{-kt}$

When $t = 0$,

$P = 200 + 1600e^{-k \times 0}$

$P = 200 + 1600$

$P = 1800$

When $t = 56, P = 1800 \div 2 = 900.$

$900 = 200 + 1600e^{-k \times 56}$

$700 = 1600e^{-56k}$

$0.4375 = e^{-56k}$

$\ln 0.4375 = -56k$

$k = \frac{\ln 0.4375}{-56}$

$\therefore P = 200 + 1600e^{\frac{\ln 0.4375}{56}t}$

When $P = 1800 \div 4 = 450,$

$450 = 200 + 1600e^{\frac{\ln 0.4375}{56}t}$

$250 = 1600e^{\frac{\ln 0.4375}{56}t}$

$0.15625 = e^{\frac{\ln 0.4375}{56}t}$

$\ln 0.15625 = \frac{\ln 0.4375}{56} \times t$

$t = \ln 0.15625 \div \frac{\ln 0.4375}{56}$

$t = 125.7$ years

Question 14

$T = P + Ae^{-kt}$

When $t = 0, T = 95, P = 25.$

$95 = 25 + Ae^{-k \times 0}$

$70 = Ae^{0}$

$A = 70$

When $t = 2.5, T = 95 \times .75 = 71.25$

$71.25 = 25 + 70e^{-k \times 2.5}$

$46.25 = 70e^{-2.5k}$

$\frac{37}{56} = e^{-2.5k}$

$\ln\frac{37}{56} = -2.5k$

$k = \ln\frac{37}{56} \div -2.5$

$k = 0.1658$

$\therefore T = 25 + 70e^{0.1658t}$

When $T = 30,$

$30 = 25 + 70e^{0.1658t}$

$5 = 70e^{0.1658t}$

$\frac{1}{14} = e^{0.1658t}$

$\ln\frac{1}{14} = 0.1658t$

$t = \ln\frac{1}{14} \div 0.1658$

t = 16 min

Question 15

$v = 100 + 280e^{-kt}$

When $t = 0$,

$v = 100 + 280e^{-k \times 0}$

$v = 380$

When $t = 50, v = 380 \times 80\% = 304.$

$304 = 100 + 280e^{-k \times 50}$

$204 = 280e^{-50k}$

$\frac{51}{70} = e^{-50k}$

$\ln\frac{51}{70} = -50k$

$k = \ln\frac{51}{70} \div -50$

$k = 0.0063$

$\therefore v = 100 + 280e^{-0.0063t}$

When $t = 3 \times 60 = 180$,

$v = 100 + 280e^{-00063 \times 180}$

$v = 189.5$

% decrease = $\frac{380 - 189.5}{380} \times 100 \approx 50.1\%$

Question 16

a $P = 800 + 2000e^{kt}$

When $t = 0$,

$P = 800 + 2000e^{k \times 0}$

$P = 2800$

b When $t = 5$, $P = 2800 \times 120\% = 3360$.

$3360 = 800 + 2000e^{-k \times 5}$

$2560 = 2000e^{-5k}$

$1.28 = e^{-5k}$

$\ln 1.28 = -5k$

$k = \dfrac{\ln 1.28}{-5}$

$\therefore P = 800 + 2000e^{\frac{\ln 1.28}{5}t}$

When $P = 2800 \times 2 = 5600$,

$5600 = 800 + 2000e^{\frac{\ln 1.28}{5}t}$

$4800 = 2000e^{\frac{\ln 1.28}{5}t}$

$2.4 = e^{\frac{\ln 1.28}{5}t}$

$\ln 2.4 = \dfrac{\ln 1.28}{5} \times t$

$t = \ln 2.4 \div \dfrac{\ln 1.28}{5}$

$t = 17.7$ years

c $P = 800 + 2000e^{\frac{\ln 1.28}{5}t}$

When $P = 7500$,

$7500 = 800 + 2000e^{\frac{\ln 1.28}{5}t}$

$6700 = 2000e^{\frac{\ln 1.28}{5}t}$

$3.35 = e^{\frac{\ln 1.28}{5}t}$

$\ln 3.35 = \dfrac{\ln 1.28}{5} \times t$

$t = \ln 3.35 \div \dfrac{\ln 1.28}{5}$

$t \approx 24.5$ years

Question 17

a $Q = 50 + 80e^{-kt}$

When $t = 0$,

$Q = 50 + 80e^{-k \times 0}$

$Q = 130$

When $t = 10, T = 130 \times 96\% = 124.8$

$124.8 = 50 + 80e^{-k \times 10}$

$74.8 = 80e^{-10k}$

$0.935 = e^{-10k}$

$\ln 0.935 = -10k$

$k = \dfrac{\ln 0.935}{-10}$

$\therefore Q = 50 + 80e^{\frac{\ln 0.935}{10}t}$

When $t = 50$,

$Q = 50 + 80e^{\frac{\ln 0.935}{10} \times 50}$

$Q = 107.2$

% decrease = $\dfrac{130 - 107.2}{130} \times 100 \approx 17.6\%$

b $Q = 50 + 80e^{\frac{\ln 0.935}{10}t}$

When $Q = 130 \times 60\% = 78$,

$78 = 50 + 80e^{\frac{\ln 0.935}{10} \times t}$

$28 = 80e^{\frac{\ln 0.935}{10} \times t}$

$0.35 = e^{\frac{\ln 0.935}{10} \times t}$

$\ln 0.35 = \dfrac{\ln 0.935}{10} \times t$

$t = \ln 0.35 \div \dfrac{\ln 0.935}{10}$

$t \approx 156$ years

Question 18

a $P = 500 + 6500e^{-kt}$

When $t = 0$,

$P = 500 + 6500e^{-k \times 0}$

$P = 7000$

When $t = 5, P = 7000 \times 82\% = 5740.$

$5740 = 500 + 6500e^{-k \times 5}$

$5240 = 6500e^{-5k}$

$\frac{262}{325} = e^{-5k}$

$\ln\frac{262}{325} = -5k$

$k = \ln\frac{262}{325} \div -5$

$k = 0.0431$

$\therefore P = 500 + 6500e^{-0.0431t}$

When $t = 10$,

$P = 500 + 6500e^{-00431 \times 10}$

$P = 4724$

$\% \text{ decrease} = \frac{7000 - 4724}{7000} \times 100 = 3.5\%$

b $P = 500 + 6500e^{-0.0431t}$

When $P = 7000 \times 10\% = 700,$

$700 = 500 + 6500e^{-0.0431t}$

$200 = 6500e^{-0.0431t}$

$\frac{2}{65} = e^{-0.0431t}$

$\ln\frac{2}{65} = -0.0431t$

$t = \ln\frac{2}{65} \div -0.0431$

$t = 80.8$ years

Test Yourself 10

Question 1

$\log_a 15 - \log_a 3$

$= \log_a \frac{15}{3}$

$= \log_a 5$

D

Question 2

$a^x = y$

$\log_a y = x$

B

Question 3

$5^x = 4$

$\log_5 4 = x$

$x = \frac{\log 4}{\log 5} \text{or} \frac{\ln 4}{\ln 5}$

A, C

Question 4

a $\log_2 8 = \log_2 2^3 = 3$

b $\log_7 7 = 1$

c $\log_0 1000 = \log_0 10^3 = 3$

d $\log_9 81 = \log_9 9^2 = 2$

e $\log_e e = 1$

f $\log_4 64 = \log_4 4^3 = 3$

g $\log_9 3 = \log_9 9^{\frac{1}{2}} = \frac{1}{2}$

h $\log_2 \frac{1}{2} = \log_2 2^{-} = -1$

i $\log_5 \frac{1}{25} = \log_5 5^{-2} = -2$

j $\ln e^3 = 3$

Question 5

a $e^2 - 1 = 6.39$

b $\log_0 95 = \frac{\log 95}{\log 10} = 1.98$

c $\log_e 26 = \ln 26 = 3.26$

d $\log_4 7 = \frac{\log 7}{\log 4} = 1.40$

e $\log_4 3 = \frac{\log 3}{\log 4} = 0.792$

f $\ln 50 = 3.91$

g $e + 3 = 5.72$

h $\frac{5e^3}{\ln 4} = 72.4$

Question 6

a $e^{\ln 6} = 6$

b $e^{\ln 2} = 2$

Question 7

a $\log_3 a = x;\ 3^x = a$

b $\ln b = y;\ \log_e b = y;\ e^y = b$

c $\log c = z; \log_0 c = z;\ 10^z = c$

Question 8

a

$$\begin{aligned}&\log_7 6\\&= \log_7(2\times 3)\\&= \log_7 2 + \log_7 3\\&= 0.36 + 0.56\\&= 0.92\end{aligned}$$

b

$$\begin{aligned}&\log_7 8\\&= \log_7(2^3)\\&= 3\log_7 2\\&= 3\times 0.36\\&= 1.08\end{aligned}$$

c

$$\begin{aligned}&\log_7 1.5\\&= \log_7\left(\frac{3}{2}\right)\\&= \log_7 3 - \log_7 2\\&= 0.56 - 0.36\\&= 0.2\end{aligned}$$

d

$$\begin{aligned}&\log_7 14\\&= \log_7(2\times 7)\\&= \log_7 2 + \log_7 7\\&= 0.36 + 1\\&= 1.36\end{aligned}$$

e

$$\begin{aligned}&\log_7 3.5\\&= \log_7\left(\frac{7}{2}\right)\\&= \log_7 7 - \log_7 2\\&= 1 - 0.36\\&= 0.64\end{aligned}$$

Question 9

a

$$3^x = 8$$
$$\log_3 8 = x$$
$$x = \frac{\log 8}{\log 3}$$
$$x = 1.9$$

b

$$2^{3x-4} = 3$$
$$\log_2 3 = 3x - 4$$
$$3x = \frac{\log 3}{\log 2} + 4$$
$$x = \left(\frac{\log 3}{\log 2} + 4\right) \div 3$$
$$x = 1.9$$

c

$$\log_x 81 = 4$$
$$81 = x^4$$
$$x = \sqrt[4]{81}$$
$$x = 3$$

d

$$\log_6 x = 2$$
$$x = 6^2$$
$$x = 36$$

Question 10

$$12 = 10e^{0.01t}$$
$$1.2 = e^{0.01t}$$
$$\ln 1.2 = 0.01t$$
$$t = \ln 1.2 \div 0.01$$
$$t = 18.2$$

Question 11

$\log_9 8$

$= \dfrac{\log 8}{\log 9}$

$= 0.9$

Question 12

a $D = e^{-kt}$

When $t = 80$, $D = 0.9$,

$0.9 = e^{-k \times 80}$

$\ln 0.9 = -80k$

$k = \dfrac{\ln 0.9}{-80}$

$\therefore D = e^{\frac{\ln 0.9}{80} t}$

When $t = 500$,

$D = e^{\frac{\ln 0.9}{80} \times 500}$

$D = 0.518$

% decay $= 100 - 51.8 = 48.2\%$

b $D = e^{\frac{\ln 0.9}{80} t}$

When $D = 0.25$,

$0.25 = e^{\frac{\ln 0.9}{80} \times t}$

$\ln 0.25 = \dfrac{\ln 0.9}{80} \times t$

$t = \ln 0.25 \div \dfrac{\ln 0.9}{80}$

$t = 1052.6$ years

Question 13

a

$$5\log_a x + 3\log_a y$$
$$= \log_a x^5 + \log_a y^3$$
$$= \log_a x^5 y^3$$

b

$$2\log_x k - \log_x 3 + \log_x p$$
$$= \log_x k^2 - \log_x 3 + \log_x p$$
$$= \log_x k^2 \div 3 \times p$$
$$= \log_x \frac{k^2 p}{3}$$

Question 14

a $\log_{0} 4.5 \approx 0.65$

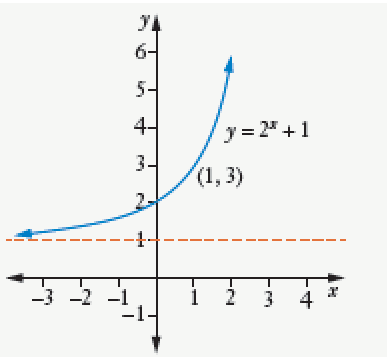
b $\ln 3.7 \approx 1.3$

Question 15

$y = 2^x + 1$

When $x = 0, y = 2$ (0, 2);

when $x = 1, y = 3$ (1, 3)



Domain: $(-\infty, \infty)$, range: $(1, \infty)$; horizontal asymptote $y = 1$.

Question 16

a $2^x = 9$

$$\log_2 9 = x$$
$$x = \frac{\log 9}{\log 2}$$
$$x = 3.17$$

b $3^x = 7$

$$\log_3 7 = x$$
$$x = \frac{\log 7}{\log 3}$$
$$x = 1.77$$

c $5^{x+1} = 6$

$$\log_5 6 = x + 1$$
$$x = \frac{\log 6}{\log 5} - 1$$
$$x = 0.11$$

d $4^{2y} = 11$

$$\log_4 11 = 2y$$
$$y = \frac{\log 11}{\log 4} \div 2$$
$$y = 0.86$$

e $8^{3n-2} = 5$

$$3n - 2 = \log_8 5$$
$$= \frac{\ln 5}{\ln 8}$$
$$3n = \frac{\ln 5}{\ln 8} + 2$$
$$n = \frac{\frac{\ln 5}{\ln 8} + 2}{3}$$
$$= 0.9246...$$
$$\approx 0.92$$

f $\log_x 16 = 4$

$$4^x = 16$$
$$x = 2$$

g $\log_3 y = 3$

$$3^3 = y$$
$$y = 27$$

h $\log_7 n = 2$

$$7^2 = n$$
$$n = 49$$

i

$$\log_x 64 = \frac{1}{2}$$
$$x^{\frac{1}{2}} = 64$$
$$x = 64^2$$
$$x = 4096$$

j

$$\log_8 m = \frac{1}{3}$$
$$m = 8^{\frac{1}{3}} = 2$$

Question 17

a $2^x = y;\ \log_2 y = x$

b $5^a = b;\ \log_5 b = a$

c $10^x = y;\ \log_0\ y = x;\ \log y = x$

d $e^x = z;\ \log_e z = x;\ \ln z = x$

e $3^{x+}\ = y;\ \log_3 y = x + 1$

Question 18

a $P = Ae^{kt}$

When $t = 0, P = 8500.$

$8500 = Ae^{k \times 0}$

$A = 8500$

When $t = 5, P = 12\ 000.$

$12\ 000 = 8500e^{k \times 5}$

$\frac{24}{17} = e^{5k}$

$\ln\frac{24}{17} = 5k$

$k = \ln\frac{24}{17} \div 5$

$k = 0.06897$

$\therefore P = 8500e^{0.06897t}$

When $t = 10,$

$P = 8500e^{006897 \times 10}$

$P = 16\ 941$

b $P = 8500e^{0.06897t}$

$P' = 5862\ e^{0.06897t}$

When $t = 10,$

$P' = 5862\ e^{006897 \times 10}$

P'=1168 birds/year

c $P = 8500e^{0.06897t}$

When $P = 30\ 000,$

$30\ 000 = 8500e^{0.06897t}$

$\frac{60}{17} = e^{0.06897t}$

$\ln\frac{60}{17} = 0.06897t$

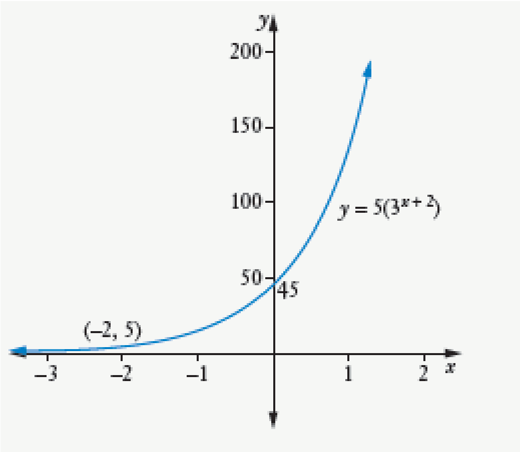
$t = \ln\frac{60}{17} \div 0.06897$

t = 18.3 years

Question 19

a $y = 5(3^{x+2})$

(0, 45), (–1, 15), (–2, 5)



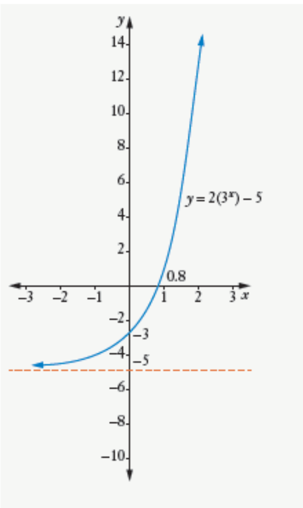
b $y = 2(3^x) - 5$

Horizontal asymptote at $y = -5$.

(0, –3), $\left(-1, -4\frac{1}{3}\right)$,

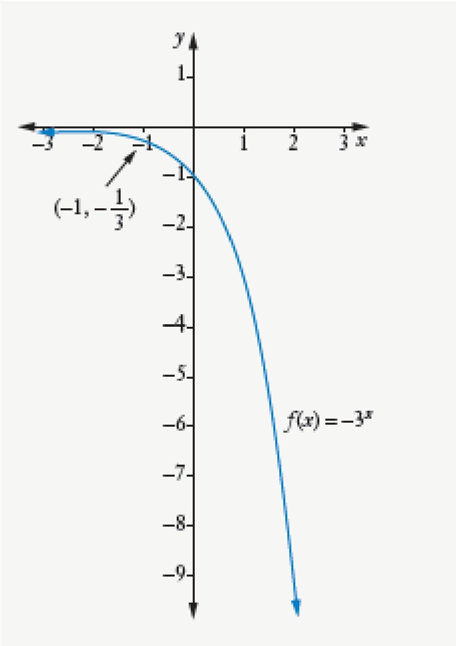
x-intercept ($y = 0$): $0 = 2(3^x) - 5 \Rightarrow 2(3^x) = 5$

$$3^x = 2.5$$
$$\ln 3^x = \ln 2.5$$
$$x \ln 3 = \ln 2.5$$
$$x = \frac{\ln 2.5}{\ln 3} \approx 0.8$$



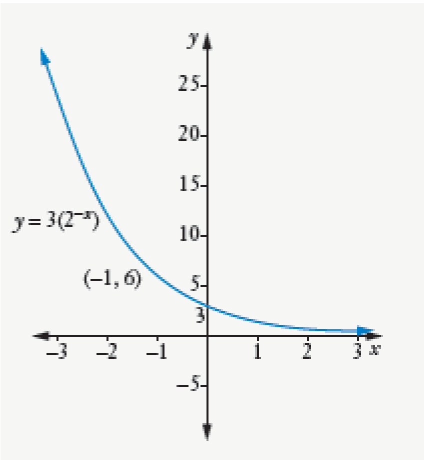
c $f(x) = -3^x$ is the graph of $f(x) = 3^x$ reflected in the x-axis.

$(0, -1), \left(-1\ -\frac{1}{3}\right)$



d $y = 3(2^{-x})$ is the graph of $y = 3(2^x)$ reflected in the x-axis.

$(0, 3), \left(-1, 6\right)$



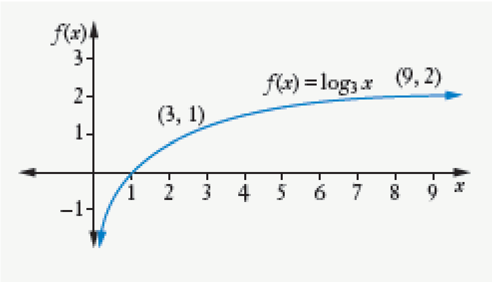
Question 20

a

$$f(x) = \log_3 x$$
$$y = \log_3 x$$
$$x = 3^y$$

When $y = 0, x = 1$ (1, 0);

when $y = 1, x = 3$ (3, 1).

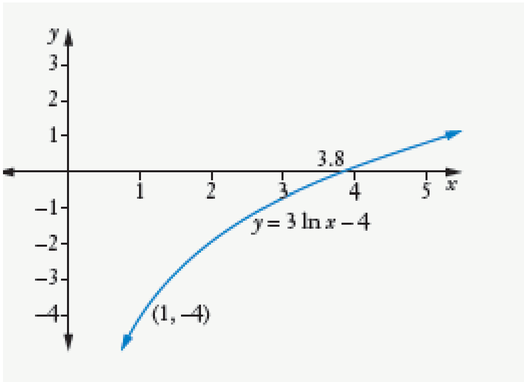


b

$$f(x) = \log_3 x$$
$$y = 3\ln x - 4$$
$$y + 4 = \log_e x^3$$
$$x^3 = e^{y+4}$$
$$x = \sqrt[3]{e^{y+4}}$$
$$x = e^{\frac{y+4}{3}}$$

When $y = -4, x = 1$ $(1, -4)$;

when $y = 0, x = 3.8$ (3.8, 0).



Question 21

a

$$\begin{aligned}&\log_x 6\\&=\log_x(2\times 3)\\&=\log_x 2+\log_x 3\\&=a+b\end{aligned}$$

b

$$\begin{aligned}&\log_x 1.5\\&=\log_x(3\div 2)\\&=\log_x 3-\log_x 2\\&=b-a\end{aligned}$$

c

$$\begin{aligned}&\log_x 8\\&=\log_x(2\times 2\times 2)\\&=\log_x 2+\log_x 2+\log_x 2\\&=a+a+a\\&=3a\end{aligned}$$

d

$$\begin{aligned}&\log_x 18\\&=\log_x(3\times 3\times 2)\\&=\log_x 3+\log_x 3+\log_x 2\\&=b+b+a\\&=a+2b\end{aligned}$$

e

$$\begin{aligned}&\log_x 27\\&=\log_x(3\times 3\times 3)\\&=\log_x 3+\log_x 3+\log_x 3\\&=b+b+b\\&=3b\end{aligned}$$

Question 22

a

$$L = 10\log\left(\frac{I}{I_0}\right)$$

When $I = 5500I_0$,

$$L = 10\log\left(\frac{5500I_0}{I_0}\right)$$
$$= 10\log 5500$$
$$\approx 37.4 \text{ dB}$$

b

$$L = 10\log\left(\frac{I}{I_0}\right)$$

When $L = 32$,

$$32 = 10\log\left(\frac{I}{I_0}\right)$$
$$3.2 = \log\left(\frac{I}{I_0}\right)$$
$$10^{3.2} = \frac{I}{I_0}$$
$$I \approx 1585I_0$$

Question 23

a $\log_a \frac{1}{x} = \log_a x^{-} \quad = -\log_a x$

b $\log_e \frac{1}{y} = \ln y^{-} \quad = -\ln y$

Question 24

a

$$\log_6 12 + \log_6 3$$
$$= \log_6 (12 \times 3)$$
$$= \log_6 36$$
$$= \log_6 6^2$$
$$= 2$$

b

$$\log 25 + \log 4$$
$$= \log_0 (25 \times 4)$$
$$= \log_0 100$$
$$= \log_0 10^2$$
$$= 2$$

c

$$2 \log_4 8$$
$$= \log_4 8^2$$
$$= \log_4 64$$
$$= \log_4 4^3$$
$$= 3$$

d

$$\log_8 72 - \log_8 9$$
$$= \log_8 (72 \div 9)$$
$$= \log_8 8$$
$$= 1$$

e

$$\log 53000 - \log 53$$
$$= \log_0 (53000 \div 53)$$
$$= \log_0 1000$$
$$= \log_0 10^3$$
$$= 3$$

Question 25

$P = P_0 e^{kt}$

When $t = 25, P \div P_0 = 2.$

$2 = e^{k \times 25}$

$\ln 2 = 25k$

$k = \dfrac{\ln 2}{25}$

$\therefore P = P_0 e^{\frac{\ln 2}{25} t}$

When $P \div P_0 = 3,$

$3 = e^{\frac{\ln 2}{25} t}$

$\ln 3 = \dfrac{\ln 2}{25} \times t$

$t = \ln 3 \div \dfrac{\ln 2}{25}$

$t \approx 39.6$ years

Question 26

a

$e^x = 15$

$x = \ln 15$

$x = 2.7$

b

$2.7^x = 21.8$

$\log_{2.7} 21.8 = x$

$x = \dfrac{\log 21.8}{\log 2.7}$

$x = 3.1$

c

$10^x = 128.7$

$\log_0 128.7 = x$

$x = \dfrac{\log 128.7}{\log 10}$

$x = 2.1$

Question 27

a **i** $A = 5280(1.019)^n$

When $t = 0,\ A = 5280\left(1.019\right)^0$

$A = 5280$

ii $A = 5280(1.019)^n$

When $t = 3,\ A = 5280\left(1.019\right)^3$

$A = 5586.71$

iii $A = 5280(1.019)^n$

When $t = 4,\ A = 5280\left(1.019\right)^4$

$A = 5692.86$

b **i** $A = 5280(1.019)^n$

When $A = 6000$,

$6000 = 5280\left(1.019\right)^n$

$\frac{25}{22} = \left(1.019\right)^n$

$\log_{1.019}\frac{25}{22} = n$

$n = \frac{\log\frac{25}{22}}{\log 1.019}$

$n = 6.8$ years

ii $A = 5280(1.019)^n$

When $A = 10\ 000$,

$10\ 000 = 5280\left(1.019\right)^n$

$\frac{125}{66} = \left(1.019\right)^n$

$\log_{1.019}\frac{125}{66} = n$

$n = \frac{\log\frac{125}{66}}{\log 1.019}$

$n = 33.9$ years

Question 28

a $\frac{dy}{dx} = 3e^{3x}$

b $\frac{dy}{dx} = -2e^{-2x}$

c $\frac{dy}{dx} = 4 \times 5e^{4x} = 20e^{4x}$

d $\frac{dy}{dx} = 8 \times -2e^{8x} + 15x^2 = -16e^{8x} + 15x^2$

e $u = x^2$, $v = e^{2x}$

$\frac{du}{dx} = 2x$, $\frac{dv}{dx} = 2e^{2x}$

$$\frac{dy}{dx} = e^{2x} \times 2x + x^2 \times 2e^{2x}$$
$$\frac{dy}{dx} = 2xe^{2x}(1+x)$$

f $u = 4e^{3x} - 1$; $\frac{du}{dx} = 12e^{3x}$

$y = u^9$, $\frac{dy}{du} = 9u^8$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$\frac{dy}{dx} = 9u^8 \times 12e^{3x}$$
$$= 9\left(4e^{3x} - 1\right)^8 \times 12e^{3x}$$
$$= 108e^{3x}\left(4e^{3x} - 1\right)^8$$

g $u = x$, $v = e^{2x}$

$$\frac{du}{dx} = 1, \ \frac{dv}{dx} = 2e^{2x}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{e^{2x}(1) - x\left(2e^{2x}\right)}{\left(e^{2x}\right)^2} \\ &= \frac{e^{2x}(1-2x)}{\left(e^{2x}\right)^2} \\ &= \frac{1-2x}{e^{2x}}\end{aligned}$$

Question 29

a $N = 1118 - 37e^{0.032t}$

When $t = 0$,
$N = 1118 - 37e^{0032 \times 0}$
$N = 1118 - 37$
$N = 1081$

b $N = 1118 - 37e^{0.032t}$

When $t = 5$,
$N = 1118 - 37e^{0032 \times 5}$
$N = 1075$

c **i** $N = 1118 - 37e^{0.032t}$

When $N = 500$,
$500 = 1118 - 37e^{0.032 \times t}$
$-618 = -37e^{0.032t}$
$\frac{618}{37} = e^{0.032t}$
$\ln\frac{618}{37} = 0.032t$
$t = \ln\frac{618}{37} \div 0.032$
$t \approx$ 88 years

ii $N = 1118 - 37e^{0.032t}$

When $N = 100$,
$100 = 1118 - 37e^{0.032 \times t}$
$-1018 = -37e^{0.032t}$
$\frac{1018}{37} = e^{0.032t}$
$\ln\frac{1018}{37} = 0.032t$
$t = \ln\frac{1018}{37} \div 0.032$
$t \approx$ 103.6 years

Question 30

a $y = e^x + x$

$$\frac{dy}{dx} = e^x + 1$$

b $y = -4e^x$

$$\frac{dy}{dx} = -4e^x$$

c $y = 3e^{-x}$

$$\frac{dy}{dx} = 3\times\left(-e^{-x}\right) = -3e^{-x}$$

d $y = (3 + e^x)^9$

$$\begin{aligned}\frac{dy}{dx} &= 9\times e^x\times\left(3+e^x\right)^{9-1}\\ &= 9e^x\left(3+e^x\right)^8\end{aligned}$$

e $y = 3x^5e^x$

$$\begin{aligned}\frac{dy}{dx} &= 3x^5e^x + 15x^4e^x\\ &= 3x^4e^x\left(x+5\right)\end{aligned}$$

f

$$y = \frac{e^x}{7x-2}$$

$$\frac{dy}{dx} = \frac{e^x\times\left(7x-2\right)-e^x\times\left(7\right)}{\left(7x-2\right)^2}$$

$$\frac{dy}{dx} = \frac{7xe^x - 2e^x - 7e^x}{\left(7x-2\right)^2}$$

$$\frac{dy}{dx} = \frac{7xe^x - 9e^x}{\left(7x-2\right)^2}$$

$$\frac{dy}{dx} = \frac{e^x\left(7x-9\right)}{\left(7x-2\right)^2}$$

Question 31

6.7 – 4.7 = 2

$10^2 = 100$ times

Question 32

8.5 – 7.2 = 1.3

$10^{1.3} = 19.9526\ \ldots$

≈ 20 times

Question 33

a $f\left(h(x)\right) = \log_e\left(6x^2 - 1\right)$

b $g\left(h(x)\right) = e^{6x^2 - 1}$

c $h\left(g(x)\right) = 6\left(e^x\right)^2 - 1 = 6e^{2x} - 1$

d $f\left(g(x)\right) = \log_e\left(e^x\right) = x$

e $g\left(f(x)\right) = e^{\mathrm{og}_e x} = x$

f

$$f(x) = \log_e x$$
$$e^y = x$$
$$f^{-}(x) = y = e^x$$

Question 34

a $T = 25 + Ae^{-kt}$

$$\frac{dT}{dt} = -kAe^{-kt}$$
$$= -k\left(25 + Ae^{-k} \ -25\right)$$
$$= -k\left(T - 25\right)$$

b $T = 25 + Ae^{-kt}$

$$When\ T = 320, t = 0.\ 320 = 25 + Ae^{-k \times 0}$$
$$295 = Ae^{0}$$
$$A = 295$$
$$T = 25 + 295e^{-kt}$$
$$\text{When } T = 285, t = 3.\ 285 = 25 + 295e^{-k \times 3}$$
$$260 = 295e^{-3k}$$
$$\frac{52}{59} = e^{-3k}$$
$$\ln\frac{52}{59} = -3k$$
$$k = \ln\frac{52}{59} \div -3$$
$$k = 0.042$$
$$\therefore T = 25 + 295e^{-0.042t}$$

c $T = 25 + 295e^{-0.042t}$

$$\text{When } t = 30,\ T = 25 + 295e^{-0042\ \times 30}$$
$$T = 1084\ ℃$$

d $T = 25 + 295e^{-0.042t}$

$$\text{When } T = 30,\ 30 = 25 + 295e^{-0.042 \times t}$$
$$5 = 295e^{-0.042t}$$
$$\frac{1}{59} = e^{-0.042t}$$
$$\ln\frac{1}{59} = -0.042t$$
$$t = \ln\frac{1}{59} \div \left(-0.042\right)$$
$$t = 97 \text{ min}$$

Challenge exercise 10

Question 1

a

$$\begin{aligned}&\log_b 6b\\&=\log_b 6+\log_b b\\&=\log_b(2\times 3)+\log_b b\\&=\log_b 2+\log_b 3+\log_b b\\&=0.6+1.1+1\\&=2.7\end{aligned}$$

b

$$\begin{aligned}&\log_b 8b\\&=\log_b 8+\log_b b\\&=\log_b\left(2^3\right)+\log_b b\\&=3\log_b 2+1\\&=3\times 0.6+1\\&=2.8\end{aligned}$$

c

$$\begin{aligned}&\log_b 1.5\ b^2\\&=\log_b 1.5+\log_b b^2\\&=\log_b\left(\frac{3}{2}\right)+2\\&=\log_b 3-\log_b 2+2\\&=1.1-0.6+2\\&=2.5\end{aligned}$$

Question 2

$y = \log_e x$

$x = e^y$

when $y = 0, x = 1$

$y = \log_0 x$

$x = 10^y$

when $y = 0, x = 1$

intersection $= (1, 0)$

Question 3

a

$P = P_0 e^{0.0151t}$

when $P \div P_0 = 1.35$

$1.35 = e^{0.0151t}$

$\ln 1.35 = 0.0151t$

$t = \ln 1.35 \div 0.0151$

$t \approx 19.9$ years

b

$P = P_0 e^{0.0151t}$

when $t = 10$

$P = P_0 e^{00151 \times 10}$

$P = 116\ P_0$

Increase $= 1.16 - 1 = 0.16 = 16\%$

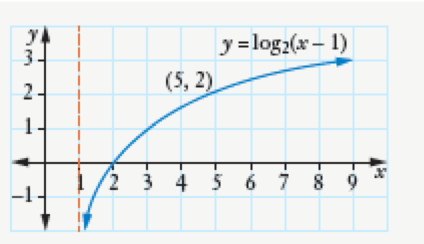
Question 4

$2^y = x - 1$

$x = 2^y + 1$

When $y = 0, x = 2$ (2, 0);

when $y = 1, x = 3$ (3, 1).



Domain : $(1, \infty)$, range : $(-\infty, \infty)$.

Question 5

$u = 3^x$

$3^{2x} - 3^x - 2 = 0$

$\left(3^x\right)^2 - 3^x - 2 = 0$

$u^2 - u - 2 = 0$

$(u-2)(u+1) = 0$

1

$u - 2 = 0$

$3^x = 2$

$x = \log_3 2$

$x = \dfrac{\log 2}{\log 3}$

$x = 0.63$

2

$u + 1 = 0$

$3^x = -1$

$x = \log_3 -1$

no solution

So $x = 0.63$

Question 6

a

$$\begin{aligned} \mathrm{pH} &= -\log\left[\mathrm{H}^+\right] \\ &= \log 1 - \log\left[\mathrm{H}^+\right] \\ &= \log \frac{1}{\mathrm{H}^+} \end{aligned}$$

b

$$\begin{aligned} \mathrm{pH} &= -\log\left[\mathrm{H}^+\right] \\ -\mathrm{pH} &= \log\left[\mathrm{H}^+\right] \\ 10^{-\mathrm{pH}} &= \mathrm{H}^+ \\ \frac{1}{10^{\mathrm{pH}}} &= \mathrm{H}^+ \end{aligned}$$

c **i**

$$\begin{aligned} \mathrm{H}^+ &= \frac{1}{10^{\mathrm{pH}}} \\ \mathrm{H}^+ &= \frac{1}{10^{6.3}} \\ \mathrm{H}^+ &= 0.000\,000\,5 \end{aligned}$$

ii

$$\begin{aligned} \mathrm{H}^+ &= \frac{1}{10^{\mathrm{pH}}} \\ \mathrm{H}^+ &= \frac{1}{10^{7.7}} \\ \mathrm{H}^+ &= 0.000\,000\,02 \end{aligned}$$

Question 7

a $y = 8 + \log_2 (x + 2)$

$y - 8 = \log_2 (x + 2)$

$2^{y-8} = x + 2$

$x = 2^{y-8} - 2$

$x = 2(2^{y-9} - 1)$

b **i** $y = 8 + \log_2 (5 + 2)$

$= 8 + \log_2 (7)$

$= 8 + \dfrac{\ln 7}{\ln 2}$

$= 10.8073 \ldots \approx 10.81$

ii $x = 2^{1-8} - 2$

$= 2^{-7} - 2$

$= -1.9921875$

≈ -1.99

Question 8

a

$$y = 3e^x - 5$$
$$y' = 3e^x$$
$$y'(2) = 3e^2$$
$$m = 3e^2$$
$$y - y_1 = m(x - x_1)$$
$$y - (3e^2 - 5) = 3e^2(x - 2)$$
$$y - 3e^2 + 5 = 3e^2 x - 6e^2$$
$$y - 3e^2 + 5 - 3e^2 x + 6e^2 = 0$$
$$3e^2 x - y - 3e^2 - 5 = 0$$

b

$$y = 3e^x - 5$$
$$y' = 3e^x$$
$$y'(2) = 3e^2$$
$$m = 3e^2$$
$$m \times n = -1$$
$$3e^2 \times n = -1$$
$$n = \frac{-1}{3e^2}$$
$$y - y_1 = m(x - x_1)$$
$$y - (3e^2 - 5) = \frac{-1}{3e^2}(x - 2)$$
$$y - 3e^2 + 5 = \frac{-x}{3e^2} + \frac{2}{3e^2}$$
$$y - 3e^2 + 5 + \frac{x}{3e^2} - \frac{2}{3e^2} = 0$$
$$\frac{x}{3e^2} + y - \frac{2}{3e^2} + 5 - 3e^2 = 0$$
$$x + 3e^2 y - 2 + 15e^2 - 9e^4 = 0$$

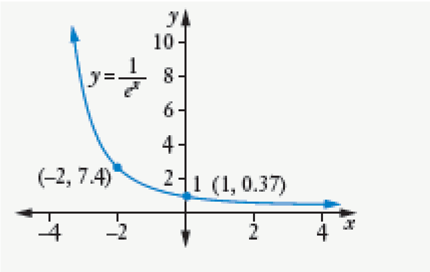
Question 9

a

$$y = \frac{1}{e^x}$$

When $x = 0, y = 1 \quad (0, 1)$;

when $x = 1, y = 0.37 \quad (1, 0.37)$.



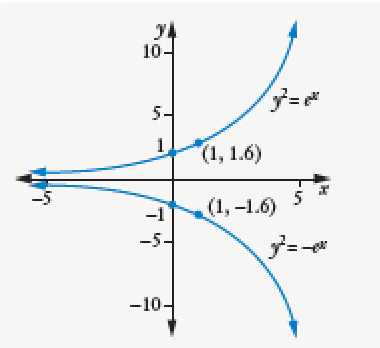
b

$$y^2 = e^x$$

$$y = \sqrt{e^x}$$

When $x = 0, y = \pm 1 \quad (0,1), (0, -1)$;

when $x = 1, y = \pm 1.65 \quad (1, 1.65), (1, -1.65)$.



Question 10

$$N = \frac{kN_0}{bN_0 + \left(k - bN_0\right)e^{-kt}}$$

$$\frac{dN}{dt} = \frac{-kN_0\left[-k\left(k - bN_0\right)e^{-k}\ \right]}{\left(bN_0 + \left(k - bN_0\right)e^{-kt}\right)^2}$$

$$= \frac{k^2 N_0\left(k - bN_0\right)e^{-kt}}{\left(bN_0 + \left(k - bN_0\right)e^{-kt}\right)^2}$$

$$= \frac{k^2 N_0\left[bN_0 + \left(k - bN_0\right)e^{-k}\ - bN_0\right]}{\left(bN_0 + \left(k - bN_0\right)e^{-kt}\right)^2}$$

$$= \frac{k^2 N_0\left[bN_0 + \left(k - bN_0\right)e^{-k}\ \right] - bk^2 N_0^2}{\left(bN_0 + \left(k - bN_0\right)e^{-kt}\right)^2}$$

$$= \frac{k^2 N_0}{\left(bN_0 + \left(k - bN_0\right)e^{-kt}\right)} - \frac{bk^2 N_0^2}{\left(bN_0 + \left(k - bN_0\right)e^{-kt}\right)^2}$$

$$= k\left[\frac{kN_0}{\left(bN_0 + \left(k - bN_0\right)e^{-k}\ \right)}\right] - b\left[\frac{kN_0}{\left(bN_0 + \left(k - bN_0\right)e^{-k}\ \right)}\right]^2$$

$$= kN - bN^2$$

MATHS IN FOCUS 11
MATHEMATICS EXTENSION 1

WORKED SOLUTIONS

Chapter 11: Trigonometric functions

Exercise 11.01 Angles of any magnitude

Question 1

a $\cos \theta > 0$

1st and 4th quadrants

b $\tan \theta > 0$

1st and 3rd quadrants

c $\sin \theta > 0$

1st and 2nd quadrants

d $\tan \theta < 0$

2nd and 4th quadrants

e $\sin \theta < 0$

3rd and 4th quadrants

f $\cos \theta < 0$

2nd and 3rd quadrants

g $\sin \theta < 0$ and $\tan \theta > 0$

3rd quadrant

h $\cos \theta < 0$ and $\tan \theta < 0$

2nd quadrant

i $\cos \theta > 0$ and $\tan \theta < 0$

4th quadrant

j sin θ < 0 and tan θ < 0

4th quadrant

Question 2

a 240° is in the 3rd quadrant.

b $\cos 240° = -\cos 60°$

$$= -\frac{1}{2}$$

Question 3

a 315° is in the 4th quadrant.

b $\sin 315° = -\sin 45°$

$$= -\frac{1}{\sqrt{2}}$$

Question 4

a 120° is in the 2nd quadrant.

b $\tan 120° = -\tan 60°$

$$= -\sqrt{3}$$

Question 5

a –225° is in the 2nd quadrant

b $\sin(-225°) = \sin 45°$

$$= \frac{1}{\sqrt{2}}$$

Question 6

a $-330°$ is in the 1st quadrant.

b $\cos(-330°) = \cos 30°$

$$= \frac{\sqrt{3}}{2}$$

Question 7

a $\tan 225° = \tan 45°$

$$= 1$$

b $\cos 315° = \cos 45°$

$$= \frac{1}{\sqrt{2}}$$

c $\tan 300° = -\tan 60$

$$= -\sqrt{3}$$

d $\sin 150° = \sin 30°$

$$= \frac{1}{2}$$

e $\cos 120° = -\cos 60°$

$$= -\frac{1}{2}$$

f $\sin 210° = -\sin 30°$

$$= -\frac{1}{2}$$

g $\cos 330° = \cos 30°$

$$= \frac{\sqrt{3}}{2}$$

h tan 150° = –tan 30°

$$= -\frac{1}{\sqrt{3}}$$

i sin 300° = –sin 60°

$$= -\frac{\sqrt{3}}{2}$$

j cos 135° = –cos 45°

$$= -\frac{1}{\sqrt{2}}$$

Question 8

a cos (–225°) = cos (135°)

= –cos 30°

$$= -\frac{1}{\sqrt{2}}$$

b cos (–210°) = cos (150°)

= –cos (30°)

$$= -\frac{\sqrt{3}}{2}$$

c tan (–300°) = tan (60°)

$$= \sqrt{3}$$

d cos (–150°) = cos (210°)

= –cos (30)

$$= -\frac{\sqrt{3}}{2}$$

e sin (–60°) = –sin (60°)

$$= -\frac{\sqrt{3}}{2}$$

f tan (–240°) = tan (120°)

$= -\tan(60°)$

$= -\sqrt{3}$

g cos (–300°) = cos (60)

$= \frac{1}{2}$

h tan (– 30°) = – tan (30°)

$= -\frac{1}{\sqrt{3}}$

i cos (–45°) = cos (45°)

$= \frac{1}{\sqrt{2}}$

j sin (–135°) = – sin (135°)

$= -\sin(45°)$

$= -\frac{1}{\sqrt{2}}$

Question 9

a cos 570° = cos 210°

$= -\cos 30°$

$= -\frac{\sqrt{3}}{2}$

b tan 420° = tan 60°

$= \sqrt{3}$

c sin 480° = sin 120°

$= \sin 60°$

$= \frac{\sqrt{3}}{2}$

d $\cos 660° = \cos 300°$

$= \cos 60°$

$= \frac{1}{2}$

e $\sin 690° = \sin 330°$

$= -\sin 30°$

$= -\frac{1}{2}$

f $\tan 600° = \tan 240°$

$= \tan 60°$

$= \sqrt{3}$

g $\sin 495° = \sin 135°$

$= \sin 45°$

$= \frac{1}{\sqrt{2}}$

h $\cos 405° = \cos 45°$

$= \frac{1}{\sqrt{2}}$

i $\tan 675° = \tan 315°$

$= -\tan 45°$

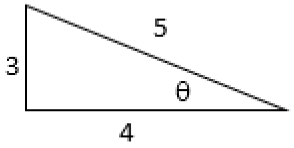
$= -1$

j $\sin 390° = \sin 30$

$= \frac{1}{2}$

Question 10

tan θ = $\frac{3}{4}$



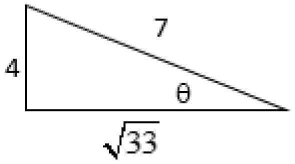
cos θ < 0 ⇒ θ is in the 3rd quadrant

sin θ = $-\frac{3}{5}$

cos θ = $-\frac{4}{5}$

Question 11

sin θ = $\frac{4}{7}$



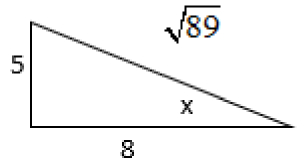
tan θ < 0 ⇒ θ is in the 2nd quadrant

cos θ = $-\frac{\sqrt{33}}{7}$

tan θ = $-\frac{4}{\sqrt{33}}$

Question 12

$\tan x = -\dfrac{5}{8}$

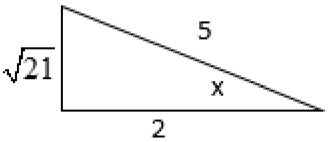


$\sin x < 0$ and $\tan x < 0 \Rightarrow x$ is in the 4th quadrant

$\cos x = \dfrac{8}{\sqrt{89}}$

Question 13

$\cos x = \dfrac{2}{5}$



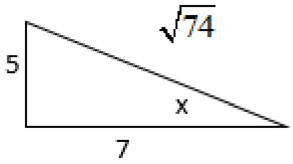
$\tan x < 0 \Rightarrow x$ is in the 4th quadrant

$\sin x = -\dfrac{\sqrt{21}}{5}$

$\tan x = -\dfrac{\sqrt{21}}{2}$

Question 14

$\tan x = \frac{5}{7}$



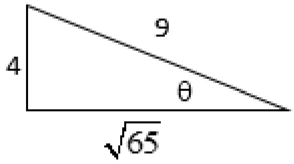
$\sin x > 0$ and $\cos x < 0 \Rightarrow x$ is in the 2nd quadrant

$\cos x = -\frac{7}{\sqrt{74}}$

$\sin x = \frac{5}{\sqrt{74}}$

Question 15

$\sin \theta = -\frac{4}{9}$



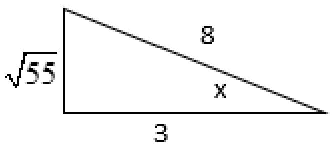
$270° < \theta < 360° \Rightarrow \theta$ is in the 4th quadrant

$\cos \theta = \frac{\sqrt{65}}{9}$

$\tan \theta = -\frac{4}{\sqrt{65}}$

Question 16

$\cos x = -\frac{3}{8}$



$180° < x < 270° \Rightarrow x$ is in the 3rd quadrant

$\sin x = -\frac{\sqrt{55}}{8}$

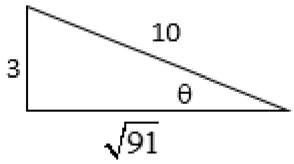
$\tan x = \frac{\sqrt{55}}{3}$

Question 17

$\sin x = 0.3$ and $\tan x < 0 \Rightarrow x$ is in the 2nd quadrant.

a $\sin x = \frac{3}{10}$

b



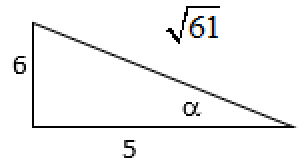
$\cos \theta = -\frac{\sqrt{91}}{10}$

$\tan \theta = -\frac{3}{\sqrt{91}}$

Question 18

tan α = –1.2

270° < α < 360° ⇒ α is in the 4th quadrant



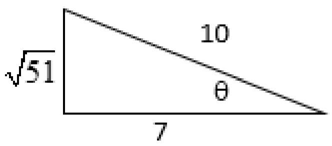
$\cos \alpha = \dfrac{5}{\sqrt{61}}$

$\sin \alpha = -\dfrac{6}{\sqrt{61}}$

Question 19

cos θ = –0.7

90° < θ < 180° ⇒ θ is in the 2nd quadrant



$\sin \theta = \dfrac{\sqrt{51}}{10}$

$\tan \theta = -\dfrac{\sqrt{51}}{7}$

Question 20

a $\sin(180° - \theta) = \sin\theta$

b $\cos(360° - x) = \cos x$

c $\tan(180° + \beta) = \tan\beta$

d $\sin(180° + \alpha) = -\sin\alpha$

e $\tan(360° - \theta) = -\tan\theta$

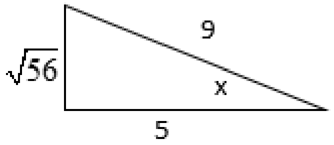
f $\sin(-\theta) = -\sin\theta$

g $\cos(-\alpha) = \cos\alpha$

h $\tan(-x) = -\tan x$

Exercise 11.02 Trigonometric identities

Question 1



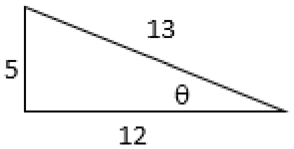
sec $x = \dfrac{9}{5}$

cot $x = \dfrac{5}{\sqrt{56}}$

cosec $x = \dfrac{9}{\sqrt{56}}$

Question 2

$\sin\theta = \dfrac{5}{13}$



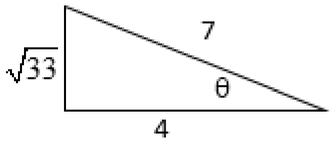
cosec $\theta = \dfrac{13}{5}$

sec $\theta = \dfrac{13}{12}$

cot $\theta = \dfrac{12}{5}$

Question 3

$\cos\theta = \frac{4}{7}$



$\operatorname{cosec}\theta = \frac{7}{\sqrt{33}}$

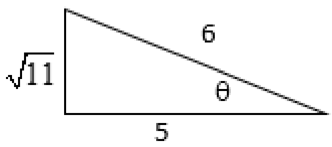
$\sec\theta = \frac{7}{4}$

$\cot\theta = \frac{4}{\sqrt{33}}$

Question 4

$\sec\theta = -\frac{6}{5}$

$\sin\theta > 0$ and $\sec\theta < 0 \Rightarrow \theta$ is in the 2nd quadrant



$\tan\theta = -\frac{\sqrt{11}}{5}$

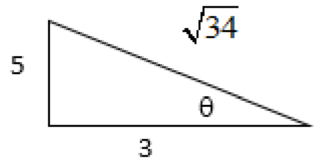
$\operatorname{cosec}\theta = \frac{6}{\sqrt{11}}$

$\cot\theta = -\frac{5}{\sqrt{11}}$

Question 5

cot θ = 0.6

cot θ > 0 and cosec θ < 0 ⇒ θ is in the 3rd quadrant



cosec θ = $-\dfrac{\sqrt{34}}{5}$

sin θ = $-\dfrac{5}{\sqrt{34}}$

sec θ = $-\dfrac{\sqrt{34}}{3}$

tan θ = $\dfrac{5}{3}$

Question 6

sin 67° = cos (90° – 67°) = cos 23°

Question 7

sec 82° = cosec (90° – 82°) = cosec 8°

Question 8

tan 48° = cot (90° – 48°) = cot 42°

Question 9

a cos 61° + sin 29°

= cos 61° + cos (90° – 24°)

= cos 61° + cos 61°

= 2 cos 61°

b sec θ – cosec (90° – θ)

= sec θ – sec θ

= 0

c tan 70° + cot 20° – 2 tan 70°

= tan 70° + tan (90° – 70°) – 2 tan 70°

= tan 70° + tan 70° – 2 tan 70°

= 0

d

$$\begin{aligned}&\frac{\sin 55°}{\cos 35°}\\&=\frac{\cos(90°-55°)}{\cos 35°}\\&=\frac{\cos 35°}{\cos 35°}\\&=1\end{aligned}$$

e

$$\begin{aligned}&\frac{\cot 25°+\tan 65°}{\cot 25°}\\&=\frac{\cot 25°+\cot(90°-65°)}{\cot 25°}\\&=\frac{\cot 25°+\cot 25°}{\cot 25°}\\&=\frac{2\cot 25°}{\cot 25°}\\&=2\end{aligned}$$

Question 10

$\sin 80 = \cos (90 - x)$

$\cos (90 - 80) = \cos (90 - x)$

$\cos(10) = \cos(90 - x)$

$10 = 90 - x$

$x = 80$

Question 11

$\tan 22 = \cot (90 - y)$

$\cot (90 - 22) = \cot (90 - y)$

$\cot 68 = \cot (90 - y)$

$68 = 90 - y$

$y = 22$

Question 12

$\cos 49 = \sin (p + 10)$

$\sin (90 - 49) = \sin (p + 10)$

$\sin (41) = \sin (p + 10)$

$41 = p + 10$

$p = 31$

Question 13

sin 35 = cos (b + 30)

cos (90 – 35) = cos (b + 30)

cos 55 = cos (b + 30)

55 = b + 30

b = 25

Question 14

cot (2t + 5) = tan (3t – 15)

cot (2t + 5) = cot (90 – [3t– 15])

2t + 5 = 90 – 3t + 15

5t + 5 = 105

5t = 100

t = 20

Question 15

tan (15 – k) = cot (2k + 60)

tan (15 – k) = tan (90 – [2k + 60])

15 – k = 90 – 2k – 60

15 – k = 30 – 2k

k + 15 = 30

k = 15

Question 16

a

$$\begin{aligned}&\tan\theta\cos\theta\\&=\frac{\sin\theta}{\cos\theta}\cos\theta\\&=\sin\theta\end{aligned}$$

b

$$\begin{aligned}&\tan\theta\,\mathrm{cosec}\,\theta\\&=\frac{\sin\theta}{\cos\theta}\frac{1}{\sin\theta}\\&=\frac{1}{\cos\theta}\\&=\sec\theta\end{aligned}$$

c

$$\begin{aligned}&\sec x\,\mathrm{cosec}\;x\\&=\frac{1}{\cos x}\frac{\cos x}{\sin x}\\&=\frac{1}{\sin x}\\&=\mathrm{cosec}\,x\end{aligned}$$

d $1 - \sin^2 x = \cos^2 x$

e

$$\begin{aligned}&\sqrt{1-\cos^2\alpha}\\&=\sqrt{\sin^2\alpha}\\&=\sin\alpha\end{aligned}$$

f $\cot^2 x + 1 = \mathrm{cosec}^2\, x$

g $1 + \tan^2 x = \sec^2 x$

h $\sec^2\theta - 1 = \tan^2\theta$

i $5\cot^2\theta + 5 = 5\,(\cot^2\theta + 1) = 5\,\mathrm{cosec}^2\,\theta$

j

$$\frac{1}{\mathrm{cosec}^2 x}=\sin^2 x$$

k $\sin^2 \alpha \,.\, \text{cosec}^2 \alpha$

$$= \sin^2 \alpha . \frac{1}{\sin^2 \alpha}$$
$$= 1$$

l

$$\cot\theta - \cot\theta\cos^2\theta$$
$$= \cot\theta\left(1 - \cos^2\theta\right)$$
$$= \cot\theta\left(\sin^2\theta\right)$$
$$= \frac{\cos\theta}{\sin\theta}\sin^2\theta$$
$$= \cos\theta\sin\theta$$

Question 17

a To prove $\cos^2 x - 1 = -\sin^2 x$

LHS = $\cos^2 x - 1$

= $-(1 - \cos^2 x)$

= $-\sin^2 x$

= RHS

b To prove $\sec\theta + \tan\theta = \dfrac{1 + \sin\theta}{\cos\theta}$

$$\text{LHS} = \sec\theta + \tan\theta$$
$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$
$$= \frac{1 + \sin\theta}{\cos\theta}$$
$$= \text{RHS}$$

c To prove $3+3\tan^2\alpha=\dfrac{3}{1-\sin^2\alpha}$

$$\begin{aligned}\text{LHS}&=3+3\tan^2\alpha\\&=3\,(1+\tan^2\alpha)\\&=3\sec^2\alpha\\&=\frac{3}{\cos^2\alpha}\\&=\frac{3}{1-\sin^2\alpha}\\&=\text{RHS}\end{aligned}$$

d To prove $\sec^2 x-\tan^2 x=\operatorname{cosec}^2 x-\cot^2 x$

LHS $=\sec^2 x-\tan^2 x$

$=1+\tan^2 x-\tan^2 x$

$=1$

RHS $=\operatorname{cosec}^2 x-\cot^2 x$

$=1+\cot^2 x-\cot^2 x$

$=1$

= LHS

e To prove $(\sin x-\cos x)^3=\sin x-\cos x-2\sin^2 x\cos x+2\sin x\cos^2 x$

LHS $=(\sin x-\cos x)^3$

$=\sin^3 x-3\sin^2 x\cos x+3\sin x\cos^2 x-\cos^3 x$

$=\sin x(\sin^2 x+\cos^2 x)+2\sin x\cos^2 x-\cos x(\sin^2 x+\cos^2 x)-2\sin^2 x\cos x$

$=\sin x-\cos x-2\sin^2 x\cos x+2\sin x\cos^2 x$

= RHS

f To prove $\cot\theta + 2\sec\theta = \dfrac{1-\sin^2\theta + 2\sin\theta}{\sin\theta\cos\theta}$

$$\begin{aligned}\text{LHS} &= \cot\theta + 2\sec\theta \\ &= \frac{\cos\theta}{\sin\theta} + \frac{2}{\cos\theta} \\ &= \frac{\cos^2\theta + 2\sin\theta}{\sin\theta\cos\theta} \\ &= \frac{1-\sin^2\theta + 2\sin\theta}{\sin\theta\cos\theta} \\ &= \text{RHS}\end{aligned}$$

g To prove $\cos^2(90 - \theta)\cot\theta = \sin\theta\cos\theta$

$$\begin{aligned}\text{LHS} &= \cos^2(90-\theta)\ \cot\ \theta \\ &= \sin^2\theta\ \cot\theta \\ &= \sin^2\theta\ \frac{\cos\theta}{\sin\theta} \\ &= \sin\theta\cos\theta \\ &= \text{RHS}\end{aligned}$$

h To prove $(\text{cosec}\ x + \cot x)(\text{cosec}\ x - \cot x) = 1$

LHS = $(\text{cosec}\ x + \cot x)(\text{cosec}\ x - \cot x)$

$= \text{cosec}^2\ x - \cot^2 x$

$= 1 + \cot^2 x - \cot^2 x$

$= 1$

= RHS

i To prove $\dfrac{1-\sin^2\theta\cos^2\theta}{\cos^2\theta} = \tan^2\theta + \cos^2\theta$

$$\begin{aligned}\text{LHS} &= \frac{1-\sin^2\theta\cos^2\theta}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta\cos^2\theta}{\cos^2\theta} \\ &= \sec^2\theta - \sin^2\theta \\ &= \tan^2\theta + 1 - \sin^2\theta \\ &= \tan^2\theta + \cos^2\theta \\ &= \text{RHS}\end{aligned}$$

Exercise 11.03 Further trigonometric identities

Question 1

a $\sin(a - b) = \sin a \cos b - \cos a \sin b$

b $\cos(p + q) = \cos p \cos q - \sin p \sin q$

c $\tan(\alpha + \beta)$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

d $\sin(x + 20°) = \sin x \cos 20° + \cos x \sin 20°$

e $\tan(48° + x)$

$$= \frac{\tan 48° + \tan x}{1 - \tan 48° \tan x}$$

f $\cos(2\theta - \alpha) = \cos 2\theta \cos\alpha + \sin 2\theta \sin\alpha$

g $\cos(x + 75°) = \cos x \cos 75° - \sin x \sin 75°$

h $\tan(5x - 7y)$

$$= \frac{\tan 5x - \tan 7y}{1 + \tan 5x \tan 7y}$$

i $\sin(4\alpha - \beta) = \sin 4\alpha \cos\beta - \cos 4\alpha \sin\beta$

Question 2

a sin a cos b + cos a sin b = sin (a + b)

b $\dfrac{\tan 36° + \tan 29°}{1 - \tan 36° \tan 29°}$ = tan (36° + 29°) = tan 65°

c cos 28° cos 27° – sin 28° sin 27° = cos (28° + 27°) = cos 55°

d sin 2x cos 3y + cos 2x sin 3y = sin (2x + 3y)

e $\dfrac{\tan 3\theta - \tan\theta}{1 + \tan 3\theta \tan\theta}$ = tan (3θ – θ) = tan (2θ)

f sin 74 cos 42 – cos 74. sin 42 = sin (74 – 42) = sin 32°

g sin (45° + 30°) + sin (45° – 30°)

= sin 45° cos 30° + cos 45° sin 30° + sin 45° cos 30° – cos 45° sin 30°

= 2 sin 45° cos 30°

$= 2\left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\sqrt{3}}{\sqrt{2}}$

$= \dfrac{\sqrt{3}}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{6}}{2}$

h sin (x + y) – sin (x – y)

= sin x cos y + cos x sin y – (sin x cos y – cos x sin y)

= 2 cos x sin y

i cos (x – y) – cos (x + y)

= cos x cos y + sin x sin y – (cos x cos y – sin x sin y)

= 2 sin x sin y

j cos (m + n) + cos (m – n)

= cos m cos n – sin m sin n + cos m cos n + sin m sin n

= 2 cos m cos n

Question 3

a $\dfrac{2t}{1-t^2} = \tan A$

b $\dfrac{1-t^2}{1+t^2} = \cos A$

c $\tan(2A) = \dfrac{2\tan A}{1-\tan^2 A}, \quad A = 10°$

$$\frac{2\tan 10°}{1-\tan^2 10°} = \tan 20°$$

d

$$\frac{1-\tan^2\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)} = \cos(A), \quad \text{using } \cos(A) = \frac{1-t^2}{1+t^2}, \quad t = \tan\left(\frac{A}{2}\right)$$

$A = 50°$

$$\frac{1-\tan^2 25°}{1+\tan^2 25°} = \cos 50°$$

e

$$\frac{2\tan(A)}{1+\tan^2(A)} = \sin(2A), \text{ using } \sin(A) = \frac{2t}{1+t^2}, \quad t = \tan\left(\frac{A}{2}\right)$$

f

$$\frac{1-\tan^2\left(\frac{A}{2}\right)}{1+\tan^2\left(\frac{A}{2}\right)} = \cos(A), \text{ using } \cos(A) = \frac{1-t^2}{1+t^2}, \quad t = \tan\left(\frac{A}{2}\right)$$

Question 4

a sin 75°

= sin (45° + 30°)

= sin 45° cos 30° + cos 45° sin 30°

$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

b cos 15°

= cos (45° − 30°)

= cos 45° cos 30° + sin 45° sin 30°

$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

c tan 75°

= tan (45° + 30°)

$$=\frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$$

$$=\frac{1+\frac{1}{\sqrt{3}}}{1-1\times\frac{1}{\sqrt{3}}}$$

$$=\frac{\frac{3}{3}+\frac{\sqrt{3}}{3}}{\frac{3}{3}-\frac{\sqrt{3}}{3}}$$

$$=\frac{3+\sqrt{3}}{3-\sqrt{3}}$$

$$=\frac{3+\sqrt{3}}{3-\sqrt{3}}\times\frac{3+\sqrt{3}}{3+\sqrt{3}}$$

$$=\frac{9+3\sqrt{3}+3\sqrt{3}+3}{9-3}$$

$$=\frac{12+6\sqrt{3}}{6}$$

$$=2+\sqrt{3}$$

d tan 105°

= tan (60° + 45°)

$$=\frac{\tan 60° + \tan 45°}{1 - \tan 60° \tan 45°}$$

$$=\frac{\sqrt{3}+1}{1-\sqrt{3}}$$

$$=\frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$=\frac{1+\sqrt{3}}{1-\sqrt{3}}\times\frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$=\frac{1+\sqrt{3}+\sqrt{3}+3}{1-3}$$

$$=\frac{4+2\sqrt{3}}{-2}$$

$$=-2-\sqrt{3}$$

e cos 105°

= cos (60° + 45°)

= cos 60° cos 45° – sin 60° sin 45°

$$= \frac{\sqrt{2}}{2} \times \frac{1}{2} - \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

f sin 15°

= sin (45° – 30°)

= sin 45° cos 30° – cos 45° sin 30°

$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

g sin 105° = sin (180° – 75°) = sin 75°

$= \frac{\sqrt{6} + \sqrt{2}}{4}$ (from **a**)

h tan 285° = tan (285° – 180°) = tan 105°

$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}$ (from **d**)

i sin (x + 30°) + cos (x + 30°)

= sin x cos 30° + cos x sin 30° + cos x cos 30° – sin x sin 30°

$$= \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x$$
$$= \left(\frac{\sqrt{3} - 1}{2} \right) \sin x + \left(\frac{\sqrt{3} + 1}{2} \right) \cos x$$

j cos (45° – y) + cos (45° + y)

= 2 cos 45° cos y (from 2**j**)

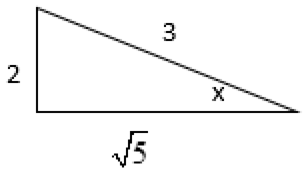
$$= 2\frac{\sqrt{2}}{2}\cos y$$
$$= \sqrt{2}\cos y$$

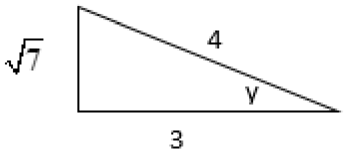
Question 5

$\dfrac{\tan(x+y)+\tan(x-y)}{1-\tan(x+y)\tan(x-y)}$ = tan [(x + y) + (x – y)] = tan 2x

Question 6

sin x = $\dfrac{2}{3}$, cos y = $\dfrac{3}{4}$





cos x =.$\dfrac{\sqrt{5}}{3}$, tan x = $\dfrac{2}{\sqrt{5}}$, sin y = $\dfrac{\sqrt{7}}{4}$, tan y = $\dfrac{\sqrt{7}}{3}$

a sin (x + y)

= sin x cos y + cos x sin y

$$= \frac{2}{3}\times\frac{3}{4}+\frac{\sqrt{5}}{3}\times\frac{\sqrt{7}}{4}$$
$$= \frac{6}{12}+\frac{\sqrt{35}}{12}$$
$$= \frac{6+\sqrt{35}}{12}$$

b cos ($x - y$)

$= \cos x \cos y + \sin x \sin y$

$$= \frac{\sqrt{5}}{3} \times \frac{3}{4} + \frac{2}{3} \times \frac{\sqrt{7}}{4}$$

$$= \frac{3\sqrt{5}}{12} + \frac{2\sqrt{7}}{12}$$

$$= \frac{3\sqrt{5} + 2\sqrt{7}}{12}$$

c tan ($x + y$)

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \frac{\frac{2\sqrt{5}}{5} + \frac{\sqrt{7}}{3}}{1 - \frac{2\sqrt{5}}{5} \times \frac{\sqrt{7}}{3}}$$

$$= \frac{\frac{6\sqrt{5}}{15} + \frac{5\sqrt{7}}{15}}{\frac{15 - 2\sqrt{35}}{15}}$$

$$= \frac{6\sqrt{5} + 5\sqrt{7}}{15 - 2\sqrt{35}}$$

$$= \frac{6\sqrt{5} + 5\sqrt{7}}{15 - 2\sqrt{35}} \times \frac{15 + 2\sqrt{35}}{15 + 2\sqrt{35}}$$

$$= \frac{90\sqrt{5} + 12\sqrt{175} + 75\sqrt{7} + 10\sqrt{245}}{225 - 140}$$

$$= \frac{90\sqrt{5} + 60\sqrt{7} + 75\sqrt{7} + 70\sqrt{5}}{85}$$

$$= \frac{160\sqrt{5} + 135\sqrt{7}}{85}$$

$$= \frac{32\sqrt{5} + 27\sqrt{7}}{17}$$

Question 7

a $\sin 2\theta = \sin(\theta + \theta)$

$= \sin\theta\cos\theta + \cos\theta\sin\theta$

$= 2\sin\theta\cos\theta$

b $\cos 2\theta = \cos(\theta + \theta)$

$= \cos\theta\cos\theta - \sin\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta$

c $\tan 2\theta = \tan(\theta + \theta)$

$$= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$= \frac{2\tan\theta}{1 - \tan^2\theta}$$

Question 8

a $\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta\cos\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$

$= 3\sin\theta\cos^2\theta - \sin^3\theta$

b $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta$

$= \cos^3\theta - 3\sin^2\theta\cos\theta$

c $\tan 3\theta = \tan(2\theta + \theta)$

$$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta}$$

$$= \frac{\dfrac{2\tan\theta}{1-\tan^2\theta} + \dfrac{\tan\theta\left(1-\tan^2\theta\right)}{1-\tan^2\theta}}{\dfrac{1-\tan^2\theta}{1-\tan^2\theta} - \dfrac{2\tan\theta}{1-\tan^2\theta}\times\tan\theta}$$

$$= \frac{2\tan\theta + \tan\theta\left(1-\tan^2\theta\right)}{1-\tan^2\theta - 2\tan^2\theta}$$

$$= \frac{\tan\theta\left(3-\tan^2\theta\right)}{1-3\tan^2\theta}$$

Question 9

a

$$\frac{\tan 7\theta - \tan 3\theta}{1+\tan 7\theta\ \tan 3\theta} = \tan(7\theta - 3\theta) = \tan 4\theta$$

b $\sin 4\theta = \sin(7\theta - 3\theta)$

$= \sin 7\theta \cos 3\theta - \cos 7\theta \sin 3\theta$

Question 10

$\cos 9x = \cos(7x + 2x)$

$= \cos 7x \cos 2x - \sin 7x \sin 2x$

Question 11

a $$\frac{2\tan 30°}{1+\tan^2 30°} = \sin\left(2\times 30°\right) = \sin 60° = \frac{\sqrt{3}}{2}$$

b $$\frac{1-\tan^2\ 22.5°}{1+\tan^2\ 22.5°} = \cos\left(2\times 22.5°\right) = \cos 45° = \frac{1}{\sqrt{2}}$$

c $$\frac{1-\tan^2\ 30°}{1+\tan^2\ 30°} = \cos\left(2\times 30°\right) = \cos 60° = \frac{1}{2}$$

d $$\frac{2\tan 60°}{1-\tan^2 60°} = \tan\left(2\times 60°\right) = \tan\left(120°\right) = -\sqrt{3}$$

Question 12

a $\sin 3a \sin 2b = \frac{1}{2}\left[\cos(3a-2b)-\cos(3a+2b)\right]$

b $\cos 5y \sin 3z = \frac{1}{2}\left[\sin(5y+3z)-\sin(5y-3z)\right]$

c $\cos 2p \cos 3q = \frac{1}{2}\left[\cos(2p+3q)+\cos(2p-3q)\right]$

d $\sin 4x \cos 9y = \frac{1}{2}\left[\sin(4x+9y)+\sin(4x-9y)\right]$

e $\cos 7x \cos 2x$

$$= \frac{1}{2}\left[\cos(7x+2x)+\cos(7x-2x)\right]$$
$$= \frac{1}{2}(\cos 9x+\cos 5x)$$

f $\sin 4y \sin y$

$$= \frac{1}{2}\left[\cos(4y-y)-\cos(4y+y)\right]$$
$$= \frac{1}{2}(\cos 3y-\cos 5x)$$

g $\sin 6a \cos 5a$

$$= \frac{1}{2}\left[\sin(6a+5a)+\sin(6a-5a)\right]$$
$$= \frac{1}{2}(\sin 11a+\sin a)$$

h $\cos 2x \sin 5x$

$$= \frac{1}{2}\left[\sin(2x+5x)-\sin(2x-5x)\right]$$
$$= \frac{1}{2}(\sin 7x-\sin(-3x))$$
$$= \frac{1}{2}(\sin 7x+\sin 3x)$$

Question 13

a cos 23° cos 22° – sin 23° sin 22° = cos (23° + 22°) = cos 45°

$$= \frac{\sqrt{2}}{2}$$

b

$$\frac{\tan 85° - \tan 25°}{1 + \tan 85° \tan 25°}$$
$$= \tan(85° - 25°)$$
$$= \tan 60°$$
$$= \sqrt{3}$$

c sin 180° cos 60° + cos 180° sin 60°

$$= 0 \times \frac{1}{2} + (-1) \times \frac{\sqrt{3}}{2}$$
$$= -\frac{\sqrt{3}}{2}$$

Alternative solution

sin 180° cos 60° + cos 180° sin 60° = sin (180° + 60°) = –sin 60°

$$= -\frac{\sqrt{3}}{2}$$

d cos 290° cos 80° + sin 290° sin 80° = cos (290° – 80°) = cos 210° = – cos30°

$$= -\frac{\sqrt{3}}{2}$$

e

$$\frac{\tan 11° + \tan 19°}{1 - \tan 11° \tan 19°}$$
$$= \tan(11° + 19°)$$
$$= \tan 30°$$
$$= \frac{1}{\sqrt{3}}$$

f cos 165° cos 15°

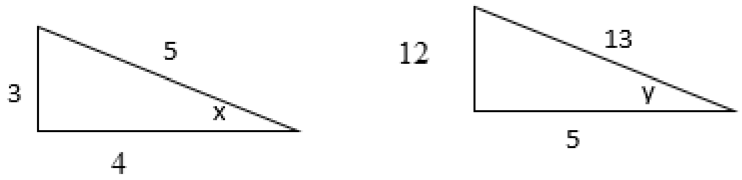
$$= \frac{1}{2}\left[\cos(165° + 15°) + \cos(165° - 15°)\right]$$
$$= \frac{1}{2}(\cos 180° + \cos 150°)$$
$$= \frac{1}{2}\left(-1 - \frac{\sqrt{3}}{2}\right)$$
$$= \frac{-2 - \sqrt{3}}{4}$$

g sin 105° cos 75°

$$= \frac{1}{2}\left[\sin(105° + 75°) + \sin(105° - 75°)\right]$$
$$= \frac{1}{2}(\sin 180° + \sin 30°)$$
$$= \frac{1}{2}\left(0 + \frac{1}{2}\right)$$
$$= \frac{1}{4}$$

Question 14

$\sin x = \frac{3}{5}$, $\cos y = \frac{5}{13}$



a $\cos x = \frac{4}{5}$

b $\sin y = \frac{12}{13}$

c $\sin(x - y) = \sin x \cos y - \cos x \sin y$

$= \frac{3}{5} \times \frac{5}{13} - \frac{4}{5} \times \frac{12}{13} = \frac{15}{65} - \frac{48}{65} = \frac{-33}{65}$

d $\tan y = \frac{12}{5}$

e $\tan x = \frac{3}{4}$

$\tan(x + y)$

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4} \times \frac{12}{5}}$$

$$= \frac{\frac{15}{20} + \frac{48}{20}}{\frac{20}{20} - \frac{36}{20}}$$

$$= \frac{\frac{63}{20}}{\frac{-16}{20}}$$

$$= \frac{-63}{16} = -3\frac{15}{16}$$

Question 15

a cos (x + y) + cos (x – y) = 2 cos x cos y (from 2**j**)

b cos 50° cos 65°

$$=\frac{1}{2}\left[\cos\left(50°+65°\right)+\cos\left(50°-65°\right)\right]$$
$$=\frac{1}{2}\left(\cos 115°+\cos\left(-15°\right)\right)$$
$$=\frac{1}{2}\left(\cos 15°-\cos 65°\right)$$

Question 16

a cosec A

$$=\frac{1}{\sin A}=\frac{1}{\frac{2t}{1+t^2}}=\frac{1+t^2}{2t}$$

b sec A

$$=\frac{1}{\cos A}=\frac{1}{\frac{1-t^2}{1+t^2}}=\frac{1+t^2}{1-t^2}$$

c cot A

$$\frac{1}{\tan A}=\frac{1}{\frac{2t}{1-t^2}}=\frac{1-t^2}{2t}$$

d sin A + cos A

$$=\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}=\frac{1+2t-t^2}{1+t^2}$$

e 1 + tan A

$$1+\frac{2t}{1-t^2}=\frac{1+2t-t^2}{1-t^2}$$

f $1 + \tan A \tan \dfrac{A}{2}$

$$=1+\left(\frac{2t}{1-t^2}\right)\times t=1+\frac{2t^2}{1-t^2}=\frac{1-t^2+2t^2}{1-t^2}=\frac{1+t^2}{1-t^2}$$

g $3 \cos A + 4 \sin A$

$$=3\times\frac{1-t^2}{1+t^2}+4\times\frac{2t}{1+t^2}=\frac{3(1-t^2)}{1+t^2}+\frac{8t}{1+t^2}=\frac{3+8t-3t^2}{1+t^2}$$

h

$$\frac{1+\sin A+\cos A}{1+\sin A-\cos A}=\frac{1+\dfrac{2t}{1+t^2}+\dfrac{1-t^2}{1+t^2}}{1+\dfrac{2t}{1+t^2}-\dfrac{1-t^2}{1+t^2}}$$

$$=\frac{1+t^2+2t+1-t^2}{1+t^2+2t-1+t^2}$$

$$=\frac{2+2t}{2t+2t^2}$$

$$=\frac{1+t}{t(1+t)}$$

$$=\frac{1}{t}$$

i $\tan A + \sec A$

$$\tan A+\frac{1}{\cos A}=\frac{2t}{1-t^2}+\frac{1}{\dfrac{1-t^2}{1+t^2}}$$

$$=\frac{2t}{1-t^2}+\frac{1+t^2}{1-t^2}$$

$$=\frac{1+2t+t^2}{1-t^2}$$

$$=\frac{(1+t)^2}{(1-t)(1+t)}$$

$$=\frac{1+t}{1-t}$$

j sin 2A = 2 sin A cos A

$$= 2 \times \frac{2t}{1+t^2} \times \frac{1-t^2}{1+t^2}$$
$$= \frac{4t(1-t^2)}{(1+t^2)^2}$$

Question 17

a sin (x + y) + sin (x – y)

= sin x cos y + cos x sin y + sin x cos y – cos x sin y

= 2 sin x cos y

b cos (x + y) – cos (x – y)

= –[cos (x – y) – cos (x + y)]

= –cos x cos y – sin x sin y + (cos x cos y – sin x sin y)

= –2 sin x cos y

c sin (x – y) – sin (x + y)

= sin x cos y – cos x sin y – (sin x cos y + cos x sin y)

= –2 cos x sin y

d tan (x + y) + tan (x – y)

$$= \frac{\tan x + \tan y}{1 - \tan x \tan y} + \frac{\tan x - \tan y}{1 + \tan x \tan y}$$
$$= \frac{\left(\tan x + \tan y\right)\left(1 + \tan x \tan y\right) + \left(1 - \tan x \tan y\right)\left(\tan x - \tan y\right)}{\left(1 - \tan x \tan y\right)\left(1 + \tan x \tan y\right)}$$
$$= \frac{\tan x + \tan y + \tan^2 x \tan y + \tan x \tan^2 y + \tan x - \tan y - \tan^2 x \tan y + \tan x \tan^2 y}{1 - \tan^2 x \tan^2 y}$$
$$= \frac{2\tan x + 2\tan x \tan^2 y}{1 - \tan^2 x \tan^2 y}$$
$$= \frac{2\tan x\left(1 + \tan^2 y\right)}{1 - \tan^2 x \tan^2 y}$$

Question 18

a $2\cos 3x \sin 3x = \sin(3x + 3x) - \sin(3x - 3x)$

$= \sin 6x$

b $\cos^2 7y - \sin^2 7y = \cos(2 \times 7y)$

$= \cos 14y$

c $\frac{2\tan 5\theta}{1-\tan^2\theta} = \tan 10\theta$

d $1 - 2\sin^2 y = \cos 2y$

e $\sin 6\theta \cos 6\theta$

$$= \frac{1}{2}\left(\sin[6\theta + 6\theta] + \sin[6\theta - 6\theta]\right)$$

$$= \frac{1}{2}\sin 12\theta$$

f $(\sin x + \cos x)^2 = \sin^2 x + 2\sin x \cos x + \cos^2 x$

$= 1 + 2\sin x \cos x$

$= 1 + \sin 2x$

g $2\cos^2 3\alpha - 1 = \cos 6\alpha$

h $1 - 2\sin^2 40° = \cos 80°$

i $\frac{2\tan\beta}{1-\tan^2\beta} = \tan 2\beta$

j $(\sin 3x - \cos 3x)^2 = \sin^2 3x - 2\sin 3x \cos 3x + \cos^2 3x$

$= 1 - 2\sin 3x \cos 3x$

$=1 - \sin 6x$

Question 19

a

$$\frac{\sin 2x}{1+\cos 2x}$$
$$=\frac{2\sin x\cos x}{1+2\cos^2 x-1}$$
$$=\frac{2\sin x\cos x}{2\cos^2 x}$$
$$=\frac{\sin x}{\cos x}$$
$$=\tan x$$

b tan 15°

$$=\frac{\sin 30°}{1+\cos 30°}$$
$$=\frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}}$$
$$=\frac{1}{2+\sqrt{3}}$$
$$=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}$$
$$=\frac{2-\sqrt{3}}{4-3}$$
$$=2-\sqrt{3}$$

Question 20

tan $2x$

$$= \frac{2\tan x}{1-\tan^2 x}$$

$$\tan 45° = \frac{2\tan 22.5°}{1-\tan^2 22.5°}$$

$$1-\tan^2 22.5° = 2\tan 22.5°$$

$$\tan^2 22.5° + 2\tan 22.5° - 1 = 0$$

Is a quadratic in terms of tan 22.5°

$$\tan 22.5° = \frac{-2 \pm \sqrt{2^2 - 4\times 1\times(-1)}}{2}$$

$$= \frac{-2 \pm \sqrt{8}}{2}$$

$$= -1 \pm \sqrt{2}$$

As x is in the first quadrant we only take the positive solution

$$\tan 22.5° = -1 + \sqrt{2}$$

Question 21

a

$$\text{RHS} = \frac{1}{2}\sin 2\theta \tan\theta$$

$$= \frac{1}{2} 2\sin\theta\cos\theta\frac{\sin\theta}{\cos\theta}$$

$$= \sin^2\theta = \text{LHS}$$

b

$$\text{RHS} = \frac{1-\cos\theta}{\sin\theta}$$

$$= \frac{1-\left(2\cos^2\frac{\theta}{2}-1\right)}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{-2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= -\cot\frac{\theta}{2}$$

$$= \tan\frac{\theta}{2} = \text{LHS}$$

c

$$\frac{1+\sin A-\cos A}{1+\sin A+\cos A} = \frac{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}, \quad t = \tan\left(\frac{A}{2}\right)$$

$$= \frac{1+t^2+2t-1+t^2}{1+t^2+2t+1-t^2}$$

$$= \frac{2t+2t^2}{2t}$$

$$= \frac{t(1+t)}{2(1+t)}$$

$$= t$$

$$= \tan\left(\frac{A}{2}\right)$$

Question 22

$$
\begin{aligned}
\text{RHS} &= \sin 11\theta \sin 3\theta \\
&= \sin(7\theta+4\theta)\sin(7\theta-4\theta) \\
&= (\sin 7\theta \cos 4\theta + \cos 7\theta \sin 4\theta)(\sin 7\theta \cos 4\theta - \cos 7\theta \sin 4\theta) \\
&= \sin^2 7\theta \cos^2 4\theta - \cos^2 7\theta \sin^2 4\theta \\
&= \sin^2 7\theta(1-\sin^2 4\theta) - (1-\sin^2 7\theta)\sin^2 4\theta \\
&= \sin^2 7\theta - \sin^2 7\theta \sin^2 4\theta - \sin^2 4\theta + \sin^2 7\theta \sin^2 4\theta \\
&= \sin^2 7\theta - \sin^2 4\theta \\
&= \text{LHS}
\end{aligned}
$$

Question 23

$$
\begin{aligned}
\text{LHS} &= \cos 3\theta \\
&= \cos(2\theta+\theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta \\
&= \cos^3\theta - \sin^2\theta\cos\theta - 2\sin^2\theta\cos\theta \\
&= \cos^3\theta - 3\sin^2\theta\cos\theta \\
&= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta \\
&= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\
&= 4\cos^3\theta - 3\cos\theta \\
&= \text{RHS}
\end{aligned}
$$

Question 24

$\sin 2A - \cos 2A = 2\sin A\cos A - (\cos^2 A - \sin^2 A)$

$= 2\sin A\cos A - \cos^2 A + \sin^2 A$

$$=2\times\frac{2t}{1+t^2}\times\frac{1-t^2}{1+t^2}-\left(\frac{1-t^2}{1+t^2}\right)^2+\left(\frac{2t}{1+t^2}\right)^2$$

$$=\frac{4t\left(1-t^2\right)}{\left(1+t^2\right)^2}-\frac{\left(1-t^2\right)^2}{\left(1+t^2\right)^2}+\frac{4t^2}{\left(1+t^2\right)^2}$$

$$=\frac{4t-4t^3-1+2t^2-t^4+4t^2}{\left(1+t^2\right)^2}$$

$$=\frac{4t-4t^3-1+6t^2-t^4}{\left(1+t^2\right)^2}$$

Exercise 11.04 Radians

Question 1

a

$$\operatorname{cosec}\left(\frac{\pi}{4}\right)=\frac{1}{\sin\left(\frac{\pi}{4}\right)}$$

$$=\frac{1}{\frac{1}{\sqrt{2}}}=\sqrt{2}$$

b

$$\sec\left(\frac{\pi}{6}\right)=\frac{1}{\cos\left(\frac{\pi}{6}\right)}$$

$$=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}$$

c

$$\cot\left(\frac{\pi}{3}\right)=\frac{1}{\tan\left(\frac{\pi}{3}\right)}$$

$$=\frac{1}{\sqrt{3}}$$

d

$$\frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}=\tan\frac{\pi}{3}$$

$$=\frac{\sqrt{3}}{1}=\sqrt{3}$$

e

$$1-\cos^2\frac{\pi}{4}=\sin^2\frac{\pi}{4}$$

$$=\left(\frac{1}{\sqrt{2}}\right)^2=\frac{1}{2}$$

f

$$\tan\frac{\pi}{3}\cos\frac{\pi}{3}$$
$$=\sqrt{3}\times\frac{1}{2}=\frac{\sqrt{3}}{2}$$

g

$$\sqrt{1+\tan^2\frac{\pi}{4}}=\sqrt{\sec^2\frac{\pi}{4}}=\sec\frac{\pi}{4}=\frac{1}{\cos\frac{\pi}{4}}$$
$$=\frac{1}{\frac{1}{\sqrt{2}}}=\sqrt{2}$$

h

$$\operatorname{cosec}^2\frac{\pi}{6}-1=\frac{1}{\sin^2\frac{\pi}{6}}-1$$
$$=\frac{1}{\left(\frac{1}{2}\right)^2}-1$$
$$=4-1=3$$

i

$$\frac{\cot\frac{\pi}{5}+\tan\frac{3\pi}{10}}{\cot\frac{\pi}{5}}=\frac{\tan\left(\frac{\pi}{2}-\frac{\pi}{5}\right)+\tan\frac{3\pi}{10}}{\tan\left(\frac{\pi}{2}-\frac{\pi}{5}\right)}$$
$$=\frac{\tan\frac{3\pi}{10}+\tan\frac{3\pi}{10}}{\tan\frac{3\pi}{10}}=\frac{2\tan\frac{3\pi}{10}}{\tan\frac{3\pi}{10}}$$
$$=2$$

Question 2

a

$$\frac{3\pi}{4}=\frac{4\pi}{4}-\frac{\pi}{4}$$
$$=\pi-\frac{\pi}{4}$$

b $\frac{3\pi}{4}$ is in the second quadrant.

c

$$\cos\frac{3\pi}{4}=-\cos\frac{\pi}{4}$$
$$=-\frac{1}{\sqrt{2}}$$

Question 3

a

$$\frac{5\pi}{6}=\frac{6\pi}{6}-\frac{\pi}{6}$$
$$=\pi-\frac{\pi}{6}$$

b $\frac{5\pi}{6}$ is in the second quadrant.

c

$$\sin\frac{5\pi}{6}=\sin\frac{\pi}{6}$$
$$=\frac{1}{2}$$

Question 4

a

$$\frac{7\pi}{4} = \frac{8\pi}{4} - \frac{\pi}{4}$$
$$= 2\pi - \frac{\pi}{4}$$

b $\frac{7\pi}{4}$ is in the fourth quadrant.

c

$$\tan\frac{7\pi}{4} = -\tan\frac{\pi}{4}$$
$$= -1$$

Question 5

a

$$\frac{4\pi}{3} = \frac{3\pi}{3} + \frac{\pi}{3}$$
$$= \pi + \frac{\pi}{3}$$

b $\frac{4\pi}{3}$ is in the third quadrant.

c

$$\cos\frac{4\pi}{3} = -\cos\frac{\pi}{3}$$
$$= -\frac{1}{2}$$

Question 6

a

$$\frac{5\pi}{3} = \frac{6\pi}{3} - \frac{\pi}{3}$$
$$= 2\pi - \frac{\pi}{3}$$

b $\frac{5\pi}{3}$ is in the fourth quadrant.

c

$$\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3}$$
$$= -\frac{\sqrt{3}}{2}$$

Question 7

a **i**

$$\frac{13\pi}{6} = \frac{12\pi}{6} + \frac{\pi}{6}$$
$$= 2\pi + \frac{\pi}{6}$$

ii $\frac{13\pi}{6}$ is in the first quadrant.

iii

$$\cos\frac{13\pi}{6} = \cos\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{2}$$

b **i**

$$\sin\frac{9\pi}{4}=\sin\frac{\pi}{4}$$
$$=\frac{\sqrt{2}}{2}$$

ii

$$\tan\frac{7\pi}{3}=\tan\frac{\pi}{3}$$
$$=\sqrt{3}$$

iii

$$\cos\frac{11\pi}{4}=\cos\left(\pi-\frac{\pi}{4}\right)=-\cos\frac{\pi}{4}$$
$$=-\frac{\sqrt{2}}{2}$$

iv

$$\tan\frac{19\pi}{6}=\tan\frac{\pi}{6}$$
$$=\frac{1}{\sqrt{3}}$$

v

$$\sin\frac{10\pi}{3}=\sin\frac{4\pi}{3}=-\sin\frac{\pi}{3}$$
$$=-\frac{\sqrt{3}}{2}$$

Question 8

a

	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{3}$	$\frac{8\pi}{3}$	$\frac{10\pi}{3}$	$\frac{11\pi}{3}$
sin	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
cos	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
tan	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$

b

	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	$\frac{9\pi}{4}$	$\frac{11\pi}{4}$	$\frac{13\pi}{4}$	$\frac{15\pi}{4}$
sin	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
cos	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
tan	1	–1	1	–1	1	–1	1	–1

c

	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$	$\frac{13\pi}{6}$	$\frac{17\pi}{6}$	$\frac{19\pi}{6}$	$\frac{23\pi}{6}$
sin	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
cos	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
tan	$\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{3}$

Question 9

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
sin	0	1	0	–1	0	1	0	–1	0
cos	1	0	–1	0	1	0	–1	0	1
tan	0	Not defined	0	Not defined	0	Not defined	0	Not defined	0

Question 10

a

$$\sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{3}+\cos\frac{\pi}{4}\sin\frac{\pi}{3}$$

$$=\frac{1}{\sqrt{2}}\times\frac{1}{2}+\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}$$

$$=\frac{1}{2\sqrt{2}}+\frac{\sqrt{3}}{2\sqrt{2}}$$

$$=\frac{1+\sqrt{3}}{2\sqrt{2}}$$

b

$$\cos\left(\frac{\pi}{6}-\frac{\pi}{3}\right)=\cos\frac{\pi}{6}\cos\frac{\pi}{3}+\sin\frac{\pi}{6}\sin\frac{\pi}{3}$$

$$=\frac{\sqrt{3}}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{\sqrt{3}}{2}$$

$$=\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}$$

$$=\frac{\sqrt{3}}{2}$$

c

$$\sin\frac{2\pi}{3}\cos\frac{\pi}{4}-\cos\frac{2\pi}{3}\sin\frac{\pi}{4}$$

$$=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\left(-\frac{1}{2}\right)\times\frac{1}{\sqrt{2}}$$

$$=\frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}$$

$$=\frac{1+\sqrt{3}}{2\sqrt{2}}$$

d $2\sin\frac{\pi}{8}\cos\frac{\pi}{8}=\sin\left(2\times\frac{\pi}{8}\right)$, using $2\sin A\cos A=\sin 2A$

$=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

e

$\tan A = \frac{2t}{1-t^2}$

$\frac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}}=\tan\left(2\times\frac{\pi}{6}\right)$, using $\tan A=\frac{2t}{1-t^2}$, $\quad t=\frac{1}{2}A$

$=\tan\frac{\pi}{3}$

$=\sqrt{3}$

Question 11

a

$\cos\frac{\pi}{9}\cos\frac{\pi}{5}-\sin\frac{\pi}{9}\sin\frac{\pi}{5}$

$=\cos\left(\frac{\pi}{9}+\frac{\pi}{5}\right)$

$=\cos\frac{14\pi}{45}$

b

$\sin\frac{5\pi}{7}\cos\frac{\pi}{8}-\cos\frac{5\pi}{7}\sin\frac{\pi}{8}$

$=\sin\left(\frac{5\pi}{7}-\frac{\pi}{8}\right)$

$=\sin\frac{33\pi}{56}$

c

$$\frac{\tan\pi - \tan\frac{\pi}{5}}{1+\tan\pi\tan\frac{\pi}{5}} = \tan\left(\pi - \frac{\pi}{5}\right)$$
$$= \tan\frac{4\pi}{5}$$
$$= -\tan\frac{\pi}{5}$$

d

$$\sin\frac{\pi}{11}\cos\frac{\pi}{9} = \frac{1}{2}\left[\sin\left(\frac{\pi}{11}+\frac{\pi}{9}\right)+\sin\left(\frac{\pi}{11}-\frac{\pi}{9}\right)\right]$$
$$= \frac{1}{2}\left[\sin\left(\frac{20\pi}{99}\right)+\sin\left(-\frac{2\pi}{99}\right)\right]$$
$$= \frac{1}{2}\left[\sin\frac{20\pi}{99}-\sin\frac{2\pi}{99}\right]$$

e

$$\frac{2\tan\frac{\pi}{7}}{1+\tan^2\frac{\pi}{7}} = \sin\left(2\times\frac{\pi}{7}\right), \text{ using } \sin A = \frac{2t}{1+t^2}, \quad t = \tan\left(\frac{A}{2}\right)$$

$$= \sin\frac{2\pi}{7}$$

Question 12

a **i** $\frac{\pi}{6}+\frac{\pi}{4}=\frac{2\pi}{12}+\frac{3\pi}{12}=\frac{5\pi}{12}$

ii

$$\tan\frac{5\pi}{12}=\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$$

$$=\frac{\tan\frac{\pi}{6}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{6}\tan\frac{\pi}{4}}$$

$$=\frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}\times 1}$$

$$=\frac{1+\sqrt{3}}{\sqrt{3}-1}$$

b **i** $\frac{2\pi}{3}+\frac{\pi}{4}=\frac{8\pi}{12}+\frac{3\pi}{12}=\frac{11\pi}{12}$

ii

$$\cos\frac{11\pi}{12}=\cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)$$

$$=\cos\frac{2\pi}{3}\cos\frac{\pi}{4}-\sin\frac{2\pi}{3}\sin\frac{\pi}{4}$$

$$=-\frac{1}{2}\times\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$$

$$=-\frac{1+\sqrt{3}}{2\sqrt{2}}$$

c **i** $\frac{9\pi}{4}-\frac{5\pi}{3}=\frac{27\pi}{12}-\frac{20\pi}{12}=\frac{7\pi}{12}$

ii

$$\sin\frac{7\pi}{12}=\sin\left(\frac{9\pi}{4}-\frac{5\pi}{3}\right)$$
$$=\sin\frac{9\pi}{4}\cos\frac{5\pi}{3}-\cos\frac{9\pi}{4}\sin\frac{5\pi}{3}$$
$$=\frac{1}{\sqrt{2}}\times\frac{1}{2}-\frac{1}{\sqrt{2}}\times-\frac{\sqrt{3}}{2}$$
$$=\frac{1+\sqrt{3}}{2\sqrt{2}}$$

d **i**

$$\sin\frac{\pi}{12}=\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$
$$=\sin\frac{\pi}{3}\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}$$
$$=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

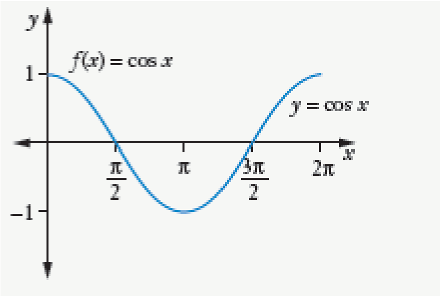
ii

$$\cos\frac{13\pi}{12}=\cos\left(\frac{4\pi}{3}-\frac{\pi}{4}\right)$$
$$=\cos\frac{4\pi}{3}\cos\frac{\pi}{4}+\sin\frac{4\pi}{3}\sin\frac{\pi}{4}$$
$$=-\frac{1}{2}\times\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$$
$$=-\frac{1+\sqrt{3}}{2\sqrt{2}}$$

Exercise 11.05 Trigonometric functions

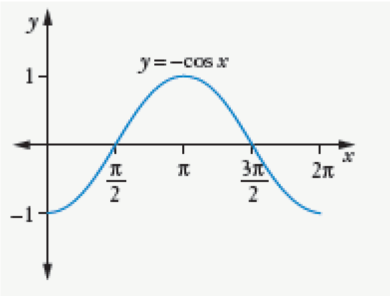
Question 1

a $y = \cos x$



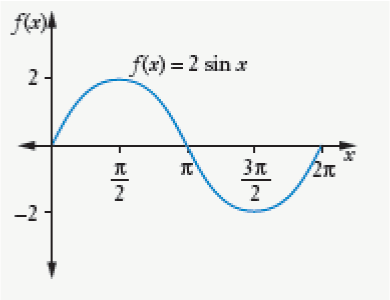
b $y = –f(x)$ is the reflection of $y = f(x)$ in the x-axis.

$y = –\cos x$

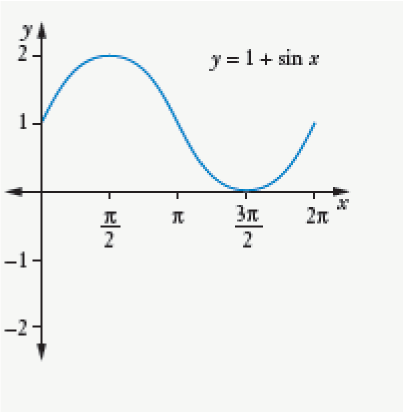


Question 2

a $f(x) = 2 \sin x$



b $y = 1 + \sin x$



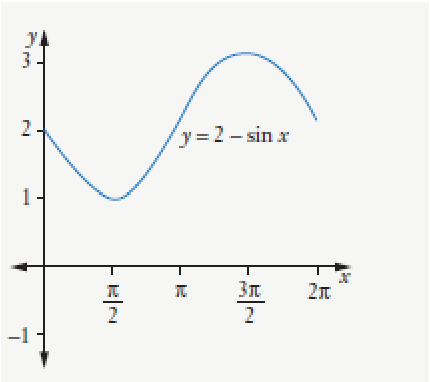
c $y = -\sin x$ is a reflection of $y = \sin x$ in the x-axis.

$y = 2 - \sin x$ is a translation of $y = -\sin x$ shifted up 2 units.

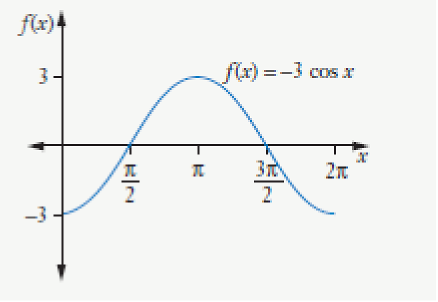
It has centre 2 and amplitude 1.

Minimum value = 2 – 1 = 1, Maximum value = 2 + 1 = 3

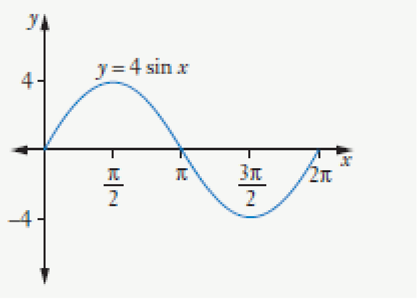
The graph is between $y = 1$ and 3.



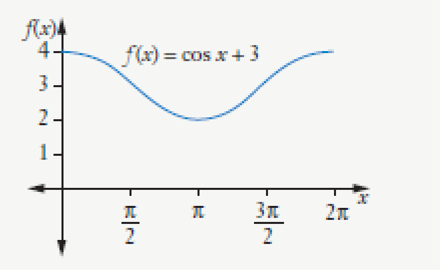
d $f(x) = -3 \cos x$



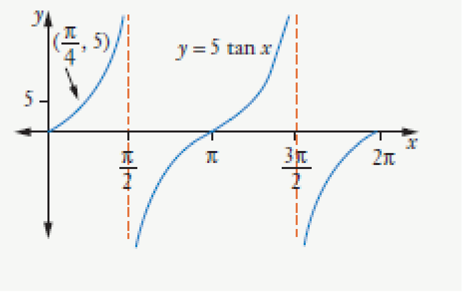
e $y = 4 \sin x$



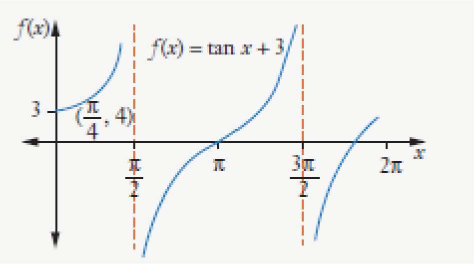
f $f(x) = \cos x + 3$



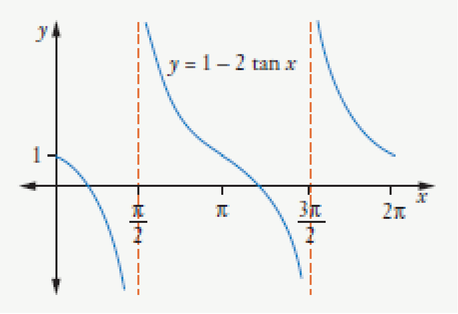
g $y = 5 \tan x$



h $f(x) = \tan x + 3$

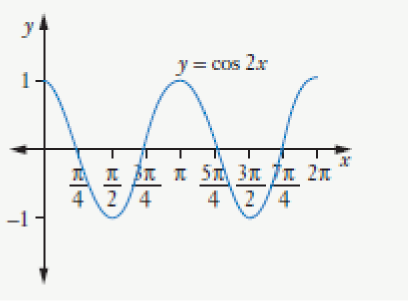


i $y = 1 - 2\tan x$

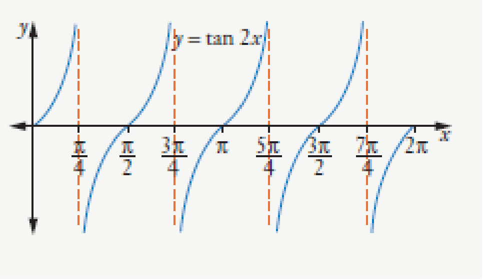


Question 3

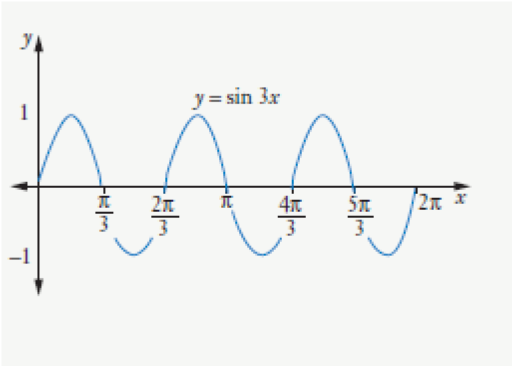
a $y = \cos 2x$



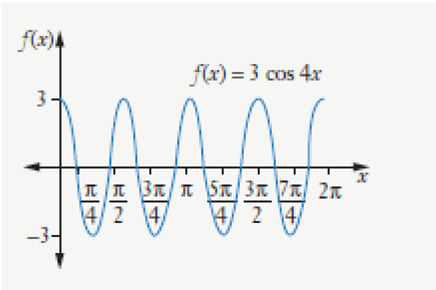
b $y = \tan 2x$



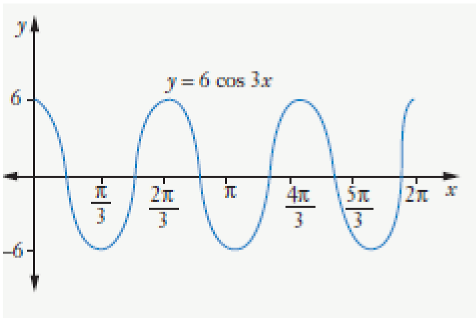
c $y = \sin 3x$



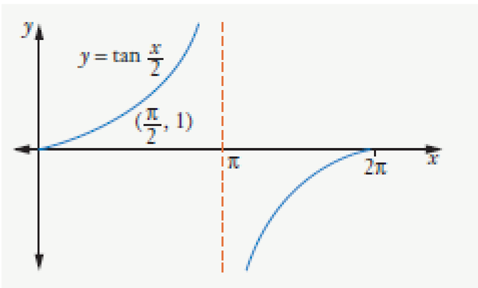
d $f(x) = 3 \cos 4x$



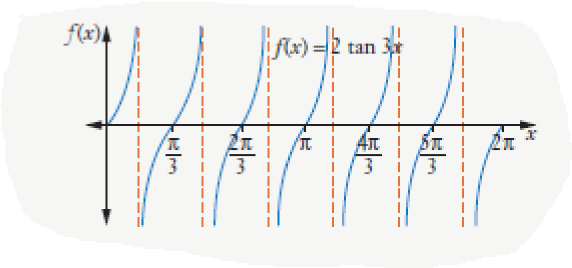
e $y = 6 \cos 3x$



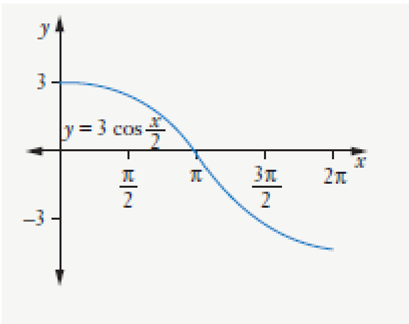
f $y = \tan \dfrac{x}{2}$



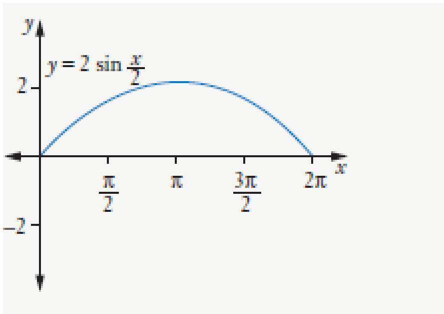
g $f(x) = 2\tan 3x$



h $y = 3\cos\dfrac{x}{2}$



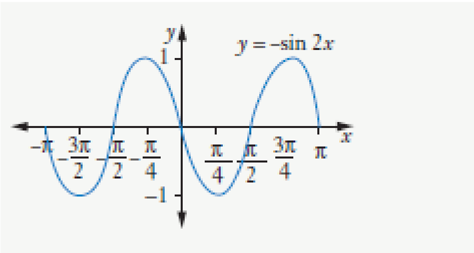
i $y = 2\sin\dfrac{x}{2}$



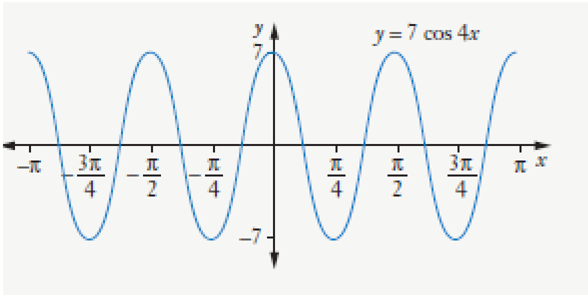
Question 4

a $y = –\sin 2x$ is the reflection of $y = \sin x$ in the x-axis.

Amplitude = 1, period = $\dfrac{2\pi}{2}$ or π.

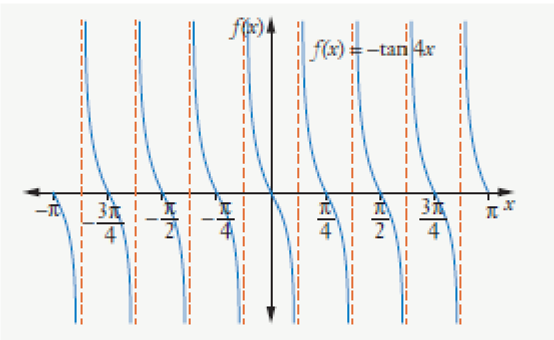


b $y = 7 \cos 4x$

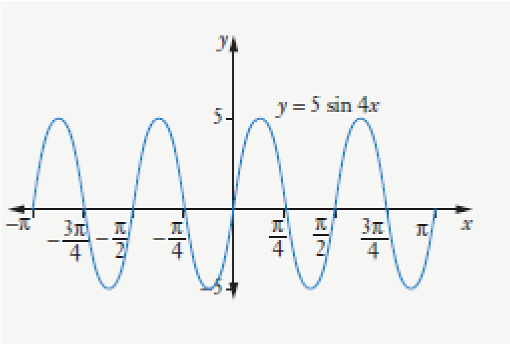


c $y = –\tan 4x$ is the reflection of $y = \tan 4x$ in the x-axis.

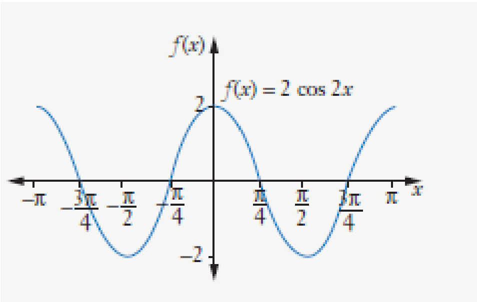
Period = $\dfrac{\pi}{4}$.



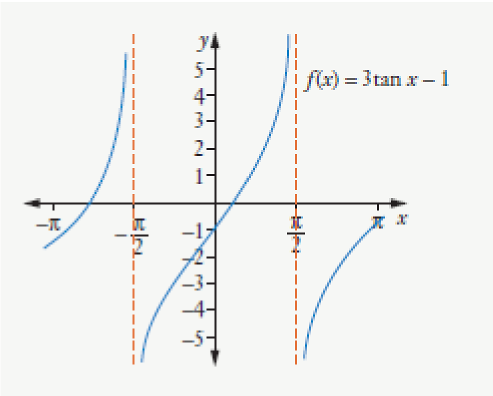
d $y = 5 \sin 4x$



e $f(x) = 2 \cos 2x$

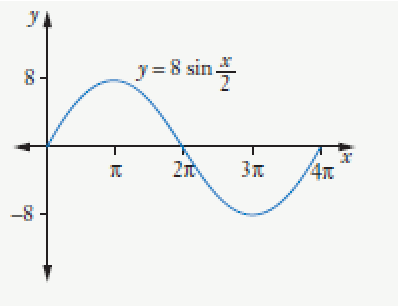


f $y = 3 \tan x - 1$ has no amplitude, period π and centre -1.



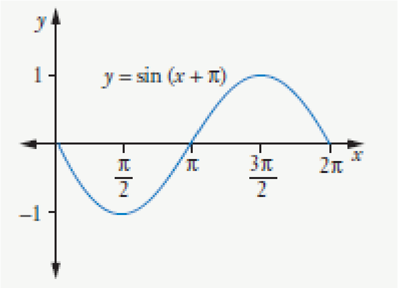
Question 5

$$y = 8\sin\frac{x}{2}$$

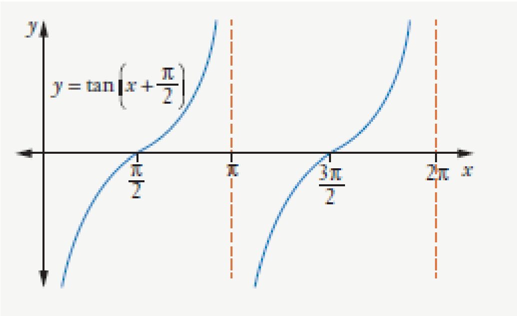


Question 6

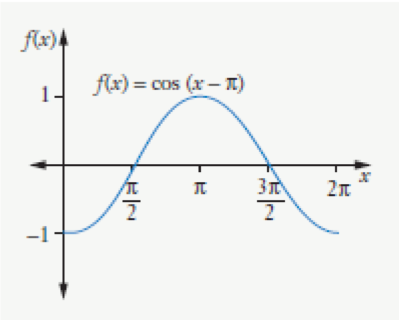
a $y = \sin(x + \pi)$



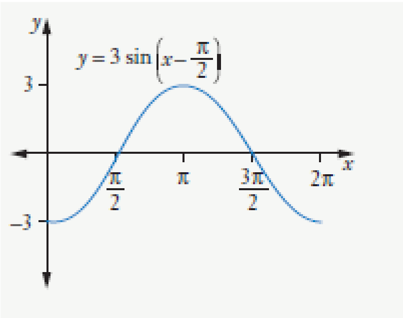
b $y = \tan\left(x + \frac{\pi}{2}\right)$



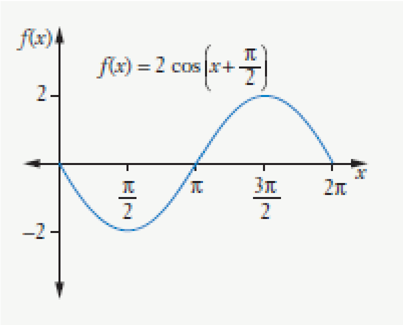
c $y = \cos(x - \pi)$



d $y = 3\sin\left(x - \dfrac{\pi}{2}\right)$

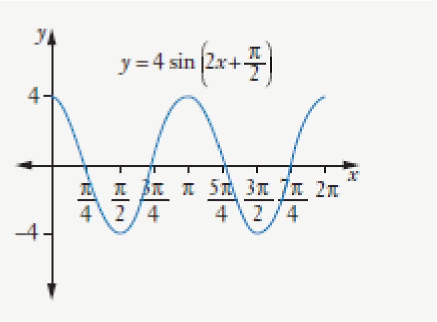


e $f(x) = 2\cos\left(x + \dfrac{\pi}{2}\right)$

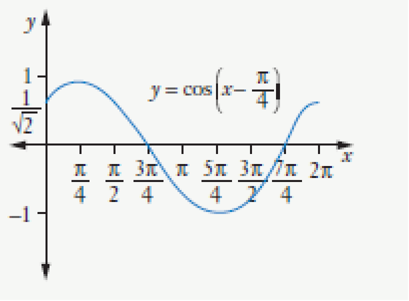


f $y = 4\sin\left(2x + \frac{\pi}{2}\right)$

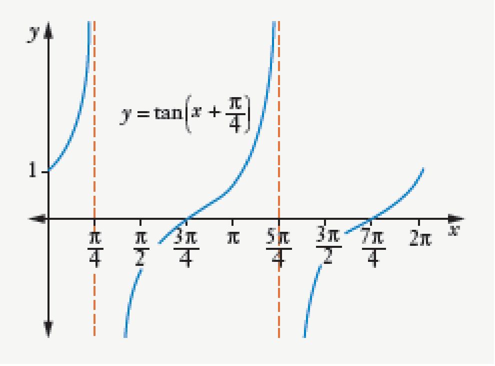
$\Rightarrow y = 4\sin\left(2\left(x + \frac{\pi}{4}\right)\right)$



g $y = \cos\left(x - \frac{\pi}{4}\right)$

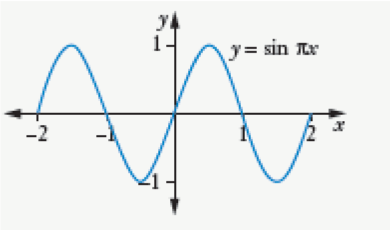


h $y = \tan\left(x + \frac{\pi}{4}\right)$

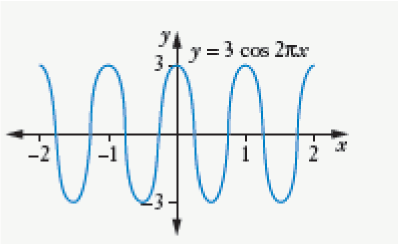


Question 7

a $y = \sin \pi x$ has amplitude 1 and period $\dfrac{2\pi}{\pi}$ or 2.



b $y = 3 \cos 2\pi x$



Question 8

a $y = k \sin ax$ has amplitude k, period $\frac{2\pi}{a}$, phase 0 and centre 0.

So $y = 5 \sin 2x$ has:

i amplitude 5

ii period $\frac{2\pi}{2} = \pi$

iii centre 0

iv phase 0

b $y = k \cos (x + b)$ has amplitude k, period $\frac{2\pi}{a}$, phase b and centre 0.

So $f(x) = -\cos (x - \pi)$ has:

i amplitude 1

ii period $\frac{2\pi}{1} = 2\pi$

iii centre 0

iv phase $-\pi$

c $y = k \tan ax + c$ has no amplitude, period $\frac{\pi}{a}$, phase 0 and centre c.

So $y = 2 \tan (4x) - 2$ has:

i no amplitude

ii period $\frac{\pi}{4}$

iii centre -2

iv phase 0

d $y = k \sin(ax + b) + c$ has amplitude k, period $\frac{2\pi}{a}$, phase b and centre c.

So $y = 3 \sin\left(x + \frac{\pi}{4}\right) + 1$ has:

i amplitude 3

ii period $\frac{2\pi}{1} = 2\pi$

iii centre 1

iv phase $\frac{\pi}{4}$

e $y = k \cos[a(x + b)] + c$ has amplitude k, period $\frac{2\pi}{a}$, phase b and centre c.

So $y = 8 \cos(\pi x - 2) - 3 = 8 \cos[\pi\left(x - \frac{2}{\pi}\right)] - 3$ has:

i amplitude 8

ii Period $\frac{2\pi}{\pi} = 2$

iii Centre –3

iv Phase $-\frac{2}{\pi}$

f $y = k \tan[a(x + b)] + c$ has no amplitude, period $\frac{\pi}{a}$, phase b and centre c.

So $f(x) = 3 \tan(5x + \frac{\pi}{2}) + 2 = 3 \tan[5\left(x + \frac{\pi}{10}\right)] + 2$ has:

i no amplitude

ii Period $\frac{\pi}{5}$

iii Centre 2

iv Phase $\frac{\pi}{10}$

Question 9

a $y = 4 \sin x - 1$ has amplitude 4 and centre –1.

Domain = $(-\infty, \infty)$ since x can be an angle of any magnitude.

Range: Minimum value = –1 – 4 = – 5

Maximum value = –1 + 4 = 3

So range = [–5, 3]

b $f(x) = -3 \cos 5x + 7$ has amplitude 3 and centre 7.

Domain = $(-\infty, \infty)$ since x can be an angle of any magnitude.

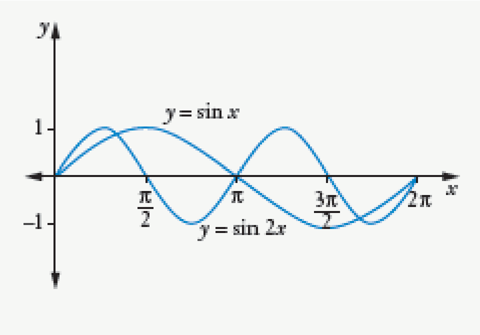
Range: Minimum value = 7 – 3 = 4

Maximum value = 7 + 3 = 10

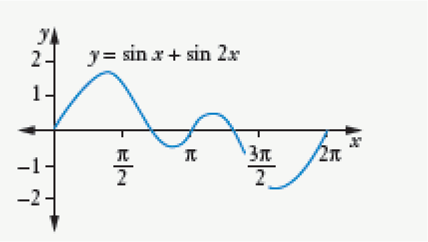
So range = [4, 10]

Question 10

a $y = \sin x$, $y = \sin 2x$

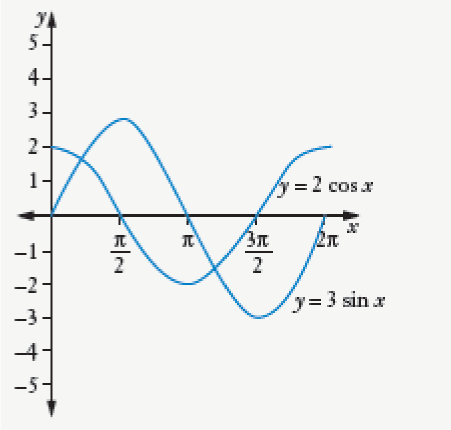


b $y = \sin x + \sin 2x$

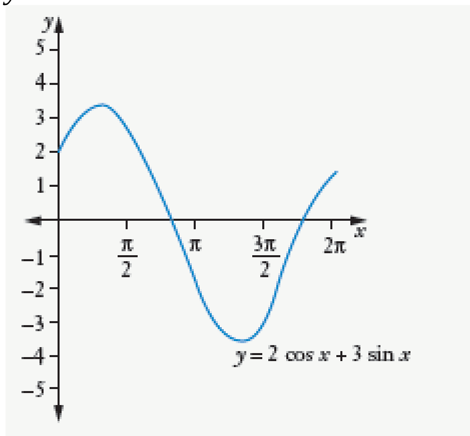


Question 11

a $y = 2 \cos x, y = 3 \sin x$

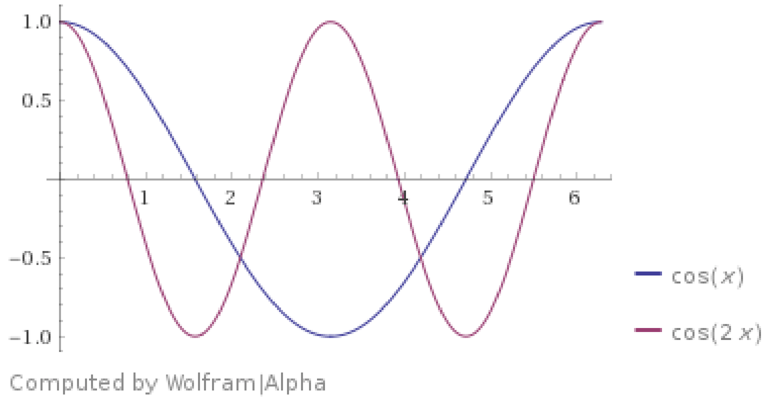


b $y = 2 \cos x + 3 \sin x$



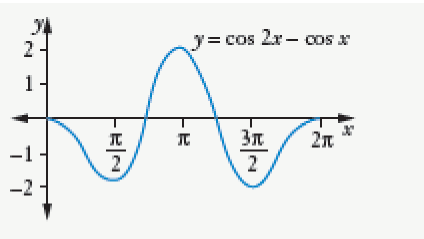
Question 12

Sketch $y = \cos x$ and $y = \cos 2x$ (period $\frac{2\pi}{2} = \pi$) on the same number plane.

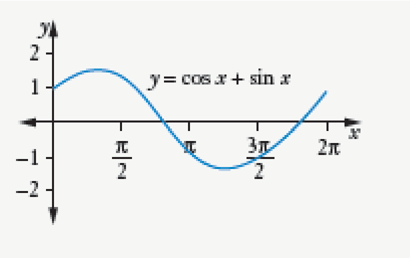


To find $y = \cos 2x - \cos x$ we can use a table of values or subtract values on the curves at different points.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
cos 2*x*	1	0	–1	0	1	0	–1	0	1
cos *x*	1	$\frac{1}{\sqrt{2}} =$ 0.71	0	$-\frac{1}{\sqrt{2}} =$ –0.71	–1	$-\frac{1}{\sqrt{2}} =$ –0.71	0	$\frac{1}{\sqrt{2}} =$ 0.71	1
cos 2*x* – cos *x*	0	–0.71	–1	0.71	2	0.71	–1	–0.71	0



Question 13



Exercise 11.06 Trigonometric equations

Question 1

a $\sin\theta = 0.35$

$\theta = \sin^{-1}(0.35)$

$\theta = 20°29', 159°31'$

b $\cos\theta = -\frac{1}{2}$

$\theta = 120°, 240°$

c $\tan\theta = -1$

$\theta = 135°, 315°$

d $\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = 60°, 120°$

e $\tan\theta = -\frac{1}{\sqrt{3}}$

$\theta = 150°, 330°$

f

$2\cos\theta = \sqrt{3}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\theta = 30°, 330°$

g $\tan 2\theta = \sqrt{3}$

$2\theta = 60°, 240°, 420°, 600°$

$\theta = 30°, 120°, 210°, 300°$

h $2\cos 2\theta - 1 = 0$

$2\cos 2\theta = 1$

$\cos 2\theta = \frac{1}{2}$

$2\theta = 60°, 300°, 420°, 660°$

$\theta = 30°, 150°, 210°, 330°$

i $2\sin 3\theta = -1$

$\sin 3\theta = -\frac{1}{2}$

$3\theta = 210°, 330°, 570°, 690°, 930°, 1050°$

$\theta = 70°, 110°, 190°, 230°, 310°, 350°$

j $\tan^2 3\theta = 1$

$\tan 3\theta = \pm 1$

$3\theta = 45°, 135°, 225°, 315°, 405°, 495°, 585°, 675°, 765°, 855°, 945°, 1035°$

$\theta = 15°, 45°, 75°, 105°, 135°, 165°, 195°, 225°, 255°, 285°, 315°, 345°$

k $\sin^2 x = 1$

$\sin x = \pm 1$

$x = 90°, 270°$

l $2\cos^2 x - \cos x = 0$

$\cos x\,(2\cos x - 1) = 0$

$\cos x = 0$ $\qquad$ $2\cos x = 1$

$x = 90°, 270°$ $\qquad$ $\cos x = \frac{1}{2}$

$x = 60°, 300°$

$x = 60°, 90°, 270°, 300°$

Question 2

a $\cos x = 1$

$x = 0°, 360°$

b $\sin x + 1 = 0$

$\sin x = -1$

$x = 270°$

c $\cos^2 x = 1$

$\cos x = \pm 1$

$x = 0°, 180°, 360°$

d $\sin x = 1$

$x = 90°$

e $\tan x = 0$

$x = 0°, 180°, 360°$

f $\sin^2 x + \sin x = 0$

$\sin x(\sin x + 1) = 0$

$\sin x = 0$	$\sin x + 1 = 0$
$x = 0°, 180°, 360°$	$\sin x = -1$
	$x = 270°$

$\therefore\ x = 0°, 180°, 270°, 360°$

g $\cos^2 x - \cos x = 0$

$\cos x(\cos x - 1) = 0$

$\cos x = 0$	$\cos x - 1 = 0$
$x = 90°, 270°$	$\cos x = 1$
	$x = 0°, 360°$

$\therefore\ x = 0°, 90°, 270°, 360°$

h $\tan^2 x = \tan x$

$\tan^2 x - \tan x = 0$

$\tan x(\tan x - 1) = 0$

$\tan x = 0$ $\quad\quad$ $\tan x - 1 = 0$

$x = 0°, 180°$ $\quad\quad$ $\tan x = 1$

$x = 45°, 225°$

$\therefore\ x = 0°, 45°, 180°, 225°$

h $\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

$\therefore\ x = 60°, 120°, 240°, 300°$

Question 3

a $\sin x = 0$

$x = 0, \pi, 2\pi$

b $\tan 2x = 0$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

c $\sin x = -1$

$x = \frac{3\pi}{2}$

d $\cos x - 1 = 0$

$\cos x = 1$

$x = 0, 2\pi$

e $\cos x = -1$

$x = \pi$

Question 4

a $\cos\theta = 0.187$

$\theta \approx \pm 79°13'$

b

$$\sin\theta = \frac{1}{2}$$
$$\theta = 30°,\ 180° - 30°$$
$$\theta = 30°, 150°$$

c

$$\tan\theta = 1$$
$$\theta = 45°,\ -180° + 45°$$
$$\theta = 45°,\ -135°$$

d

$$\sin\theta = -\frac{\sqrt{3}}{2}$$
$$\theta = -60°,\ -180° + 60°$$
$$\theta = -60°,\ -120°$$

e

$$\tan\theta = -\frac{1}{\sqrt{3}}$$
$$\theta = -30°,\ 180° - 30°$$
$$\theta = -30°,\ 150°$$

f

$$3\tan^2\theta = 1$$
$$\tan^2\theta = \frac{1}{3}$$
$$\tan\theta = \pm\frac{1}{\sqrt{3}}$$
$$\theta = \pm 30°,\ \pm(180° - 30°)$$
$$\theta = \pm 30°,\ \pm 150°$$

g

$$\tan\theta + 1 = 0$$
$$\tan\theta = -1$$
$$\theta = -45°,\ \ 180° - 45°$$
$$\theta = -45°,\ \ 135°$$

h

$$\tan 2\theta = 1$$
$$2\theta = 45°,\ \ 180° + 45°,\ \ -180° + 45°,\ \ -360° + 45°$$
$$2\theta = 45°,\ 225°,\ -135°,\ -315°$$
$$\theta = 22.5°, 112.5°, -67.5°, -157.5°$$

Question 5

a

$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}\ \ \frac{5\pi}{3}$$

b

$$\sin x = -\frac{1}{\sqrt{2}}$$
$$x = \frac{5\pi}{4}\ \ \frac{7\pi}{4}$$

c

$$\tan x = 1$$
$$x = \frac{\pi}{4}\ \ \frac{5\pi}{4}$$

d

$$\tan x = \sqrt{3}$$
$$x = \frac{\pi}{3}\ \ \frac{4\pi}{3}$$

e

$$\cos x = -\frac{\sqrt{3}}{2}$$
$$x = \frac{5\pi}{6} \quad \frac{7\pi}{6}$$

Question 6

a

$$2\sin x = \sqrt{3}$$
$$\sin x = \frac{\sqrt{3}}{2}$$
$$x = \frac{\pi}{3} \quad \frac{2\pi}{3}$$

b

$$2\cos x = 0$$
$$\cos x = 0$$
$$x = \pm\frac{\pi}{2}$$

c

$$3\tan^2 x = 1$$
$$\tan^2 x = \frac{1}{3}$$
$$\tan x = \pm\frac{1}{\sqrt{3}}$$
$$x = \pm\frac{\pi}{6} \quad \pm\frac{5\pi}{6}$$

Question 7

$$2\cos x = -1$$
$$\cos x = -\frac{1}{2}$$
$$x = \pm\frac{2\pi}{3} \quad \pm\frac{4\pi}{3}$$

Question 8

a

$$\tan^2 x + \tan x = 0$$
$$\tan x(\tan x + 1) = 0$$
$$\tan x = 0, -1$$
$$x = 0, \pi, 2\pi \text{ and } \frac{3\pi}{4}, \frac{7\pi}{4}$$
$$x = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi$$

b

$$\sin^2 x - \sin x = 0$$
$$\sin x(\sin x - 1) = 0$$
$$\sin x = 0, 1$$
$$x = 0, \pi, 2\pi \text{ and } \frac{\pi}{2}$$
$$x = 0, \frac{\pi}{2}, \pi, 2\pi$$

c

$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$
$$2\cos x + 1 = 0, \cos x - 1 = 0$$
$$\cos x = -\frac{1}{2}, \cos x = 1$$
$$x = \frac{2\pi}{3}, \frac{4\pi}{3} \text{ and } 0, 2\pi$$
$$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$$

d $4\sin^2 x = 1$

$\sin^2 x = \frac{1}{4}$

$\sin x = \pm\sqrt{\frac{1}{4}}$

$\sin x = \pm\frac{1}{2}$

$x = \frac{\pi}{6}\ \pi-\frac{\pi}{6},\pi+\frac{\pi}{6},2\pi-\frac{\pi}{6}$

$x = \frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

e

$$\begin{aligned}
&\tan x\cos x+\tan x=0\\
&\tan x(\cos x+1)=0\\
&\tan x=0, \qquad \cos x=-1\\
&x=0\ \pi,2\pi \qquad x=\pi\\
&x=0,\pi,2\pi
\end{aligned}$$

f $\sin^2 x + 2\cos x - 2 = 0$

$1 - \cos^2 x + 2\cos x - 2 = 0$

$-\cos^2 x + 2\cos x - 1 = 0$

$0 = \cos^2 x - 2\cos x + 1$

$0 = (\cos x - 1)^2$

$\cos x - 1 = 0$

$\cos x = 1$

$x = 0, 2\pi$

Question 1

a Amplitude = $\frac{8.5-5.5}{2}=\frac{3}{2}$

Period = 1 year

b 5:30 p.m.

Question 2

a 1300 crimes

b **i** highest number 1600

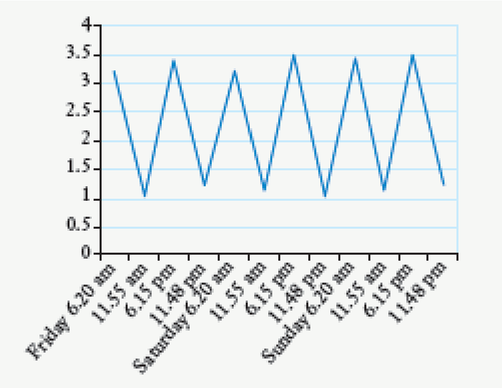
ii lowest number 1100

c Amplitude = $\frac{1600-1100}{2}=250$,

Period 9 – 1 = 8 years or 18 – 9 = 9 years

Question 3

a



b Period 12 hours, amplitude 1.25

c Height 2.5 m

Question 1

a $y = \arcsin x$

Domain $[-1, 1]$

Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

b $y = \tan^{-1} x$

Domain $(-\infty, \infty)$

Range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

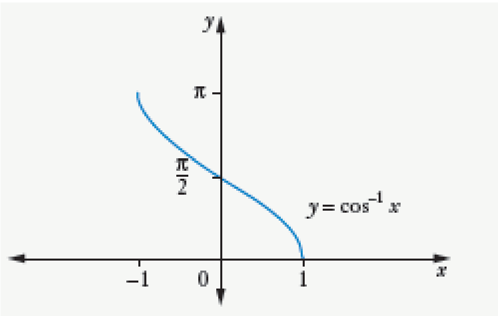
c $f(x) = \cos^{-1} x$

Domain $[-1, 1]$

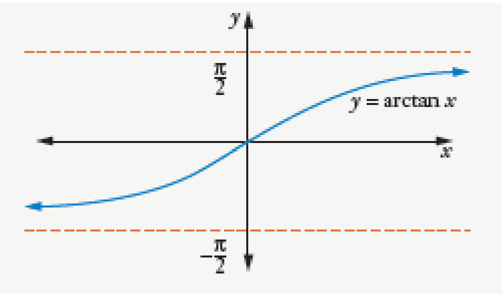
Range $[0, \pi]$

Question 2

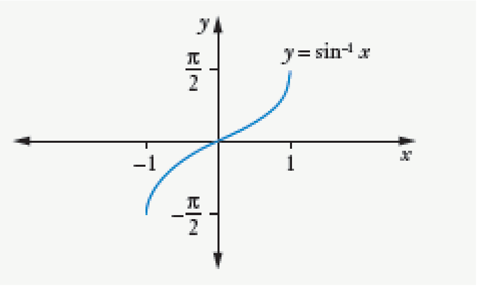
a $y = \cos^{-1} x$



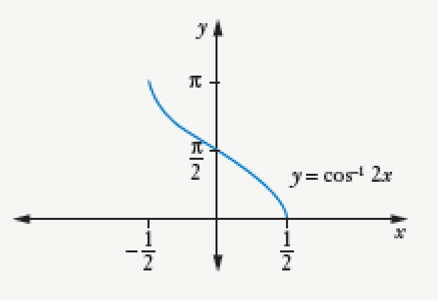
b $y = \arctan x$



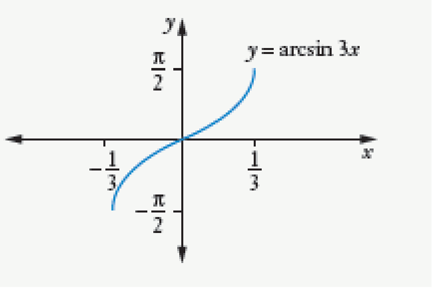
c $y = \sin^{-1} x$



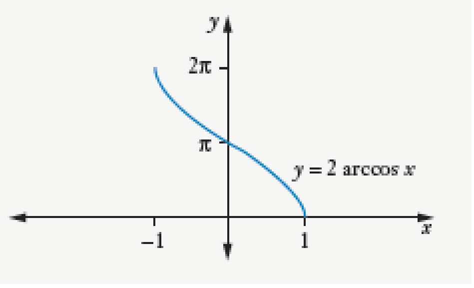
d $y = \cos^{-1} 2x$



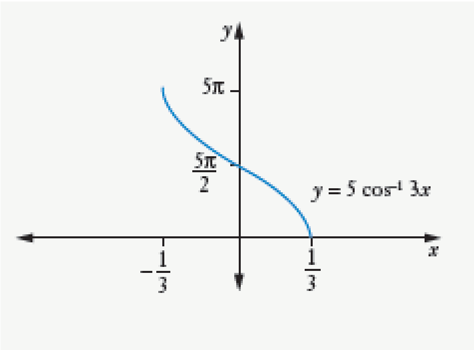
e $y = \arcsin 3x$



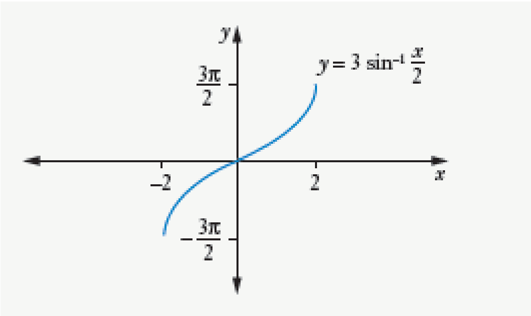
f $y = 2 \arccos x$



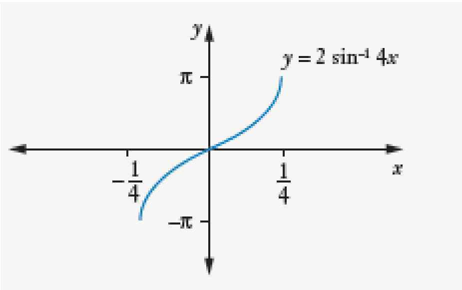
g $y = 5\cos^{-1} 3x$



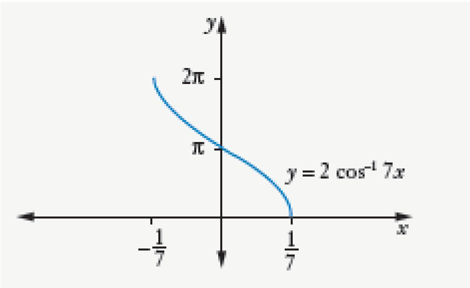
h $y = 3\sin^{-1}\left(\dfrac{x}{2}\right)$



i $2\sin^{-1} 4x$



j $y = 2\cos^{-1} 7x$



Question 3

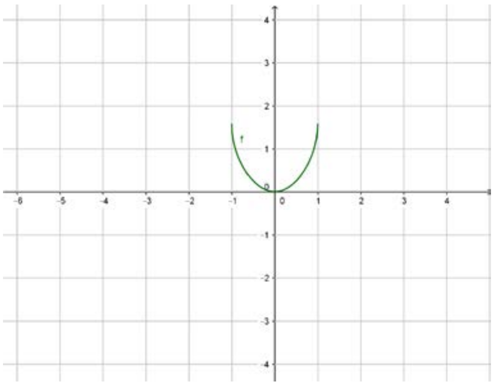
a $y = \arcsin (x^2)$

The domain of arcsin is [–1, 1] so x^2 must be in [–1, 1].

But $x^2 \geq 0$ so x must be in [–1, 1] for x^2 to be in [–1, 1].

Domain [–1, 1]

Since $x^2 \geq 0$, x^2 must be in [0, 1].



So $\arcsin (x^2)$ must be in $\left[0 \;\; \frac{\pi}{2}\right]$.

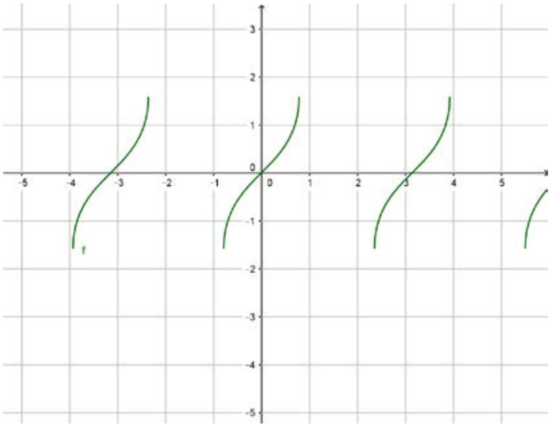
Range $\left[0 \;\; \frac{\pi}{2}\right]$

b $y = \sin^{-1}(\tan x)$

The domain of $\sin^{-1}$ is $[-1, 1]$ so $\tan x$ must be in $[-1, 1]$.

So x must be in $\left[-\frac{\pi}{4}\ \ \frac{\pi}{4}\right]$ for $\tan x$ to be in $[-1, 1]$.

Domain $\left[-\frac{\pi}{4}\ \ \frac{\pi}{4}\right]$



So $\sin^{-1}(\tan x)$ must be in $\left[-\frac{\pi}{2}\ \ \frac{\pi}{2}\right]$.

Range $\left[-\frac{\pi}{2}\ \ \frac{\pi}{2}\right]$

Exercise 11.09 Properties of inverse trigonometric functions

Question 1

a $\sin^{-1} 1 = \frac{\pi}{2}$

b $\tan^{-1} 0 = 0$

c $\cos^{-1} 1 = 0$

d

$$\sin^{-1}\left(\frac{1}{2}\right)$$
$$y = \sin^{-1} x$$
$$x = \sin y$$
$$\frac{1}{2} = \sin y$$
$$y = \frac{\pi}{6}$$

e

$$\tan^{-}(-1)$$
$$y = \tan^{-1} x$$
$$x = \tan y$$
$$-1 = \tan y$$
$$y = -\frac{\pi}{4}$$

f

$$\sin^{-}(-1)$$
$$y = \sin^{-1} x$$
$$x = \sin y$$
$$-1 = \sin y$$
$$y = -\frac{\pi}{2}$$

g

$\cos^{-} 0$

$y = \cos^{-1} x$

$x = \cos y$

$0 = \cos y$

$y = \frac{\pi}{2}$

h

$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$y = \cos^{-1} x$

$x = \cos y$

$\frac{1}{\sqrt{2}} = \cos y$

$y = \frac{\pi}{4}$

i

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$y = \tan^{-1} x$

$x = \tan y$

$\frac{1}{\sqrt{3}} = \tan y$

$y = \frac{\pi}{6}$

j

$\tan^{-}\left(-\sqrt{3}\right)$

$y = \tan^{-1} x$

$x = \tan y$

$-\sqrt{3} = \tan y$

$y = -\frac{\pi}{3}$

k

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$y = \sin^{-1} x$$

$$x = \sin y$$

$$\frac{1}{\sqrt{2}} = \sin y$$

$$y = \frac{\pi}{4}$$

l

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$y = \cos^{-1} x$$

$$x = \cos y$$

$$-\frac{\sqrt{3}}{2} = \cos y$$

$$y = \pi - \frac{\pi}{6}$$

$$y = \frac{5\pi}{6}$$

Question 2

a $\tan(\cos^{-1} 1) = \tan 0 = 0$

b $\cos[\cos^{-1}(-1)] = \cos \pi = -1$

c $\cos^{-1}(\sin \pi) = = \cos^{-1} 0 = \frac{\pi}{2}$

d $\cos\left[\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] = \frac{1}{\sqrt{2}}$

e $\sin(\tan^{-1} 1) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

f $\tan\left[\cos^{-1}\left(\frac{1}{2}\right)\right] = \tan \frac{\pi}{3} = \sqrt{3}$

g $\sin^{-1}(\tan 0) = \sin^{-1} 0 = 0$

h $\tan\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = \tan -\frac{\pi}{3} = -\sqrt{3}$

i $\cos\left[\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \cos -\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

j $\tan[\tan^{-1}(-1)] = -1$

Question 3

a $\sin^{-1} 0.4 = 0.41$

b $\tan^{-1} 1.72 = 1.04$

c $\cos^{-1} 0.569 = 0.97$

d $\sin^{-1}(-0.6) = -0.64$

e $\tan^{-1}(-3.7) = -1.31$

Question 4

a $\sin^{-1}(\sin 0.67) = 0.67$

b $\tan^{-1}[\tan(-0.14)] = -0.14$

c $\cos^{-1}(\cos 1.64) = 1.64$

d $\sin(\cos^{-1} 0.26) = 0.97$

e $\tan[\sin^{-1}(-0.67)] = -0.90$

Question 5

a $\cos^{-1}(-1) + \cos^{-1} 1 = \pi + 0 = \pi$

b $\sin^{-1} 1 + \sin^{-1}(-1) = \frac{\pi}{4} + \frac{-\pi}{4} = 0$

c $\tan^{-1} 1 + \tan^{-1}(-1) = \frac{\pi}{4} + \frac{-\pi}{4} = 0$

d $\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

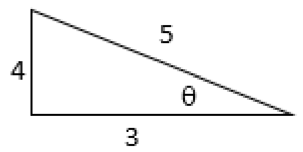
e $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

f $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

Question 6

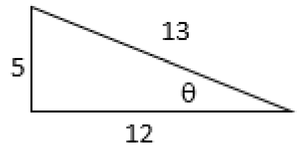
a $\sin\left[\sin^{-1}\left(\frac{4}{5}\right)\right] = \frac{4}{5}$

b $\cos\left[\sin^{-1}\left(\frac{4}{5}\right)\right]$



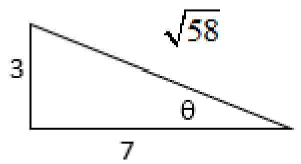
From the diagram $\cos\theta = \frac{3}{5}$

c $\tan\left[\cos^{-1}\left(\frac{12}{13}\right)\right]$



From the diagram $\tan\theta = \frac{5}{12}$

d $\sin\left[\tan^{-1}\left(\frac{3}{7}\right)\right]$



From the diagram $\sin\theta = \frac{3}{\sqrt{58}}$

e $\cos^{-1}\left[\sin\left(\frac{\pi}{4}\right)\right] = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

f $\tan^{-1}(\cos\pi) = \tan^{-1} 1 = -\frac{\pi}{4}$

Question 7

a $\tan^{-1}(-1) = -\tan^{-1} 1$

$$\text{LHS} = \tan^{-}(-1) = -\frac{\pi}{4}$$

$$\text{RHS} = -\tan^{-1}(1) = -\frac{\pi}{4} = \text{LHS}$$

b $\sin^{-1}(-1) = -\sin^{-1} 1$

$$\text{LHS} = \sin^{-}(-1) = -\frac{\pi}{2}$$

$$\text{RHS} = -\sin^{-1}(1) = -\frac{\pi}{2} = \text{LHS}$$

c $\tan^{-1}(-3\ ) = -\tan^{-1} 3$

$$\text{LHS} = \tan^{-}(-3) = -1\ 249$$

$$\text{RHS} = -\tan^{-1} 3 = -1.249 = \text{LHS}$$

d $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right)$

$$\text{LHS} = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$\text{RHS} = \pi - \cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3} = \text{LHS}$$

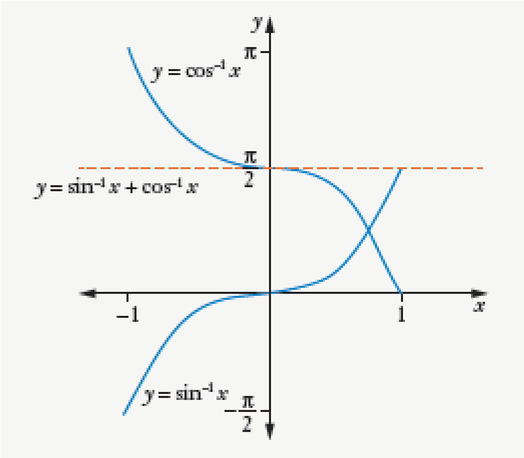
e $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$$\text{LHS} = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$$

$$\text{RHS} = -\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4} = \text{LHS}$$

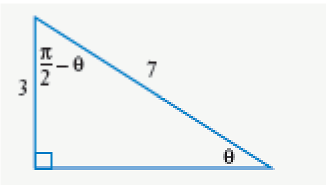
Question 8

a, **b**



Question 9

a Let $\sin^{-1}\left(\dfrac{3}{7}\right) = \theta$, $\cos^{-1}\left(\dfrac{3}{7}\right) = \dfrac{\pi}{2} - \theta$



Then $\sin\theta = \dfrac{3}{7}$ and $\cos\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{3}{7}$.

From the diagram $\theta + \dfrac{\pi}{2} - \theta = \dfrac{\pi}{2}$

$\therefore \sin^{-1}\left(\dfrac{3}{7}\right) + \cos^{-1}\left(\dfrac{3}{7}\right) = \dfrac{\pi}{2}$

b

$$\text{Let } \alpha = \sin^{-1}\left(-\frac{5}{9}\right)$$
$$\Rightarrow \sin\alpha = -\frac{5}{9}$$
$$\Rightarrow -\sin\alpha = \frac{5}{9}$$
$$\Rightarrow -\sin^{-}\left(\frac{5}{9}\right) = \alpha$$
$$\therefore \sin^{-1}\left(-\frac{5}{9}\right) = -\sin^{-1}\left(\frac{5}{9}\right)$$

c

$$\text{Let } \alpha = \cos^{-1}\left(-\frac{2}{5}\right)$$
$$\Rightarrow \cos\alpha = -\frac{2}{5}$$
$$\text{but } \alpha \in [0, \pi]$$
$$\Rightarrow \alpha = \pi - \cos^{-}(\alpha)$$
$$\Rightarrow \alpha = \pi - \cos^{-1}\left(\frac{2}{5}\right)$$
$$\therefore \cos^{-1}\left(-\frac{2}{5}\right) = \pi - \cos^{-1}\left(\frac{2}{5}\right)$$

d

$$\text{Let } \alpha = \tan^{-1}\left(-\frac{7}{10}\right)$$
$$\Rightarrow \tan\alpha = -\frac{7}{10}$$
$$\Rightarrow -\tan\alpha = \frac{7}{10}$$
$$\Rightarrow -\tan^{-}\left(\frac{7}{10}\right) = \alpha$$
$$\therefore \tan^{-1}\left(-\frac{7}{10}\right) = -\tan^{-1}\left(\frac{7}{10}\right)$$

Question 10

a $f(x) = \sin^{-1} x$

Let $\alpha = \sin^{-1} x$

$\Rightarrow x = \sin \alpha$

$x = -\sin(-\alpha)$

$\alpha = -\sin^{-1}(-x)$

$\therefore f(-x) = -f(x)$

$\therefore f(x) = \sin^{-1} x$ is an odd function

$f(x) = \tan^{-1} x$

Let $\alpha = \tan^{-1} x$

$\Rightarrow x = \tan \alpha$

$x = -\tan(-\alpha)$

$\alpha = -\tan^{-1}(-x)$

$\therefore f(-x) = -f(x)$

$\therefore f(x) = \tan^{-1} x$ is an odd function

b $f(x) = \cos^{-1} x$

Let $\alpha = \cos^{-1} x$

$\Rightarrow x = \cos \alpha$

$-x = -\cos(\alpha)$

Due to the restriction of the range

$-\alpha = \pi - \cos(x)$

$f(-x) = \pi - \cos(x)$

As $f(-x) \neq -f(x)$ or $f(x)$

$f(x) = \cos^{-1} x$ is neither odd or even.

Question 11

a $\cos^{-1}\left[\cos\left(\frac{\pi}{2}\right)\right]=\cos^{-1}(0)=\frac{\pi}{2}$

True

b $\sin\left[\sin^{-1}(-1)\right]=\sin\left(-\frac{\pi}{2}\right)=-1$

True

c $\tan^{-1}(\tan\pi)=\tan^{-1}0=0\neq\pi$

False

d $\cos\left[\arccos\left(\frac{1}{2}\right)\right]=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

True

e $\tan(\tan^{-1}12)=\tan(1.4877\ldots)=12$

True

f $\arctan\left[\tan\left(\frac{2\pi}{3}\right)\right]=\arctan\left[\tan\left(\pi-\frac{\pi}{3}\right)\right]$

$=\arctan\left[-\tan\left(\frac{\pi}{3}\right)\right]$

$=\arctan\left(-\sqrt{3}\right)$

$=-\frac{\pi}{3}\neq\frac{2\pi}{3}$

False

Question 12

a $y = -2\tan^{-1} 7x$

$f(-x) = -2\tan^{-1} [7(-x)]$

$= -2\tan^{-1} (-7x)$

$= -2[-\tan^{-1} 7x]$

$= -[-2\tan^{-1} 7x]$

$= -f(x)$

So it is an odd function

b $y = \tan^{-1} x$ has domain $(-\infty, \infty)$

So $y = \tan^{-1} 7x$ has domain $(-\infty, \infty)$

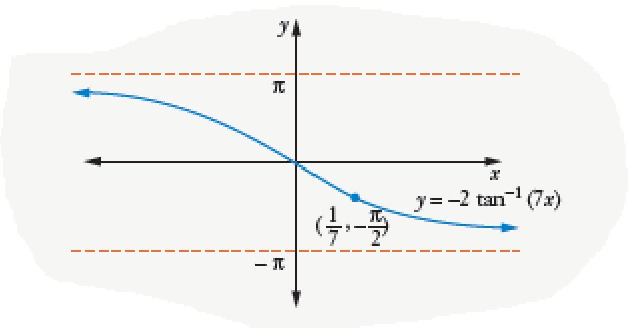
$y = \tan^{-1} 7x$ has range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

So $-\frac{\pi}{2} < \tan^{-1} 7x < \frac{\pi}{2}$

$-\pi < 2\tan^{-1} 7x < \pi$

So $y = 2\tan^{-1} 7x$ has range $[-\pi, \pi]$

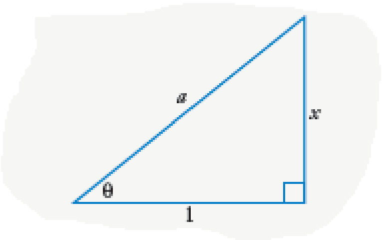
$y = -2\tan^{-1} 7x$ is a reflection of $y = 2\tan^{-1} 7x$ in the x-axis.



Question 13

Let $\tan^{-1} x = \theta$.

Then $\tan\theta = x = \dfrac{x}{1}$



By Pythagoras' theorem:

$c^2 = a^2 + b^2 = x^2 + 1$

$c = \sqrt{x^2+1}$

$\cos(\tan^{-1} x) = \cos\theta = \dfrac{1}{\sqrt{x^2+1}}$

Question 14

a $y = \sin^{-1} x$ has domain [–1, 1]

So $\sin^{-1}(\sin a)$ has domain [–1, 1]

This means $-1 \le \sin a \le 1$.

But this is true for all real a.

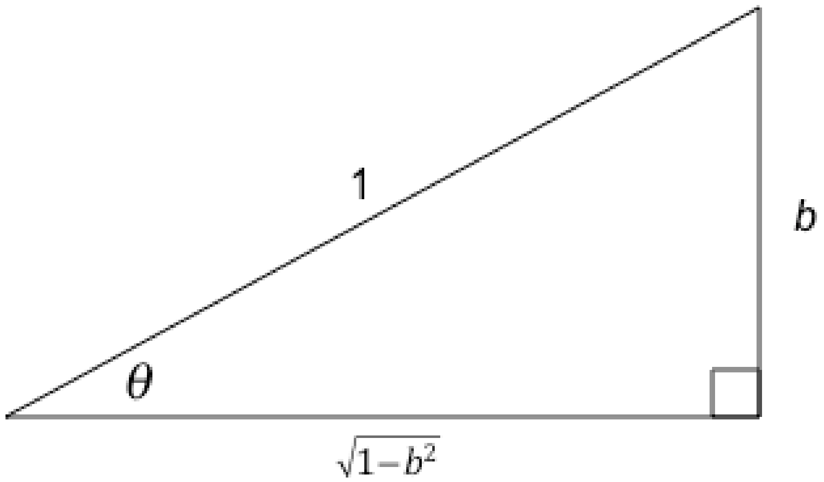
So $\sin^{-1}(\sin a)$ is defined for all real a.

b $y = \sin^{-1} x$ has domain [–1, 1]

So $\sin^{-1} b$ is defined for $-1 \le b \le 1$.

Let $\sin^{-1} b = \theta$.

Then $\sin\theta = b = \dfrac{b}{1}$



By Pythagoras' theorem:

$c^2 = a^2 + b^2$

$1 = a^2 + b^2$

$1 - b^2 = a^2$

$\sqrt{1-b^2} = a$

$\sin(\sin^{-1} b) = \sin\theta = \dfrac{b}{1} = b$

So $\sin(\sin^{-1} b) = b$ for $-1 \le b \le 1$.

Question 1

Amplitude = $\frac{3.5-2.5}{2}$ = 0.5

Period = 30 – 15 = 15 minutes

$\therefore$ D

Question 2

$\cos\frac{2\pi}{3}=-\frac{1}{2}$

D

Question 3

Amplitude = 3, period = $\frac{\pi}{2}$

y = 3 sin 4x

B

Question 4

$-1 \le 5x \le 1$, so $-\frac{1}{5} \le x$ and $x \le \frac{1}{5}$

Domain $[-\frac{1}{5}\ \ \frac{1}{5}]$

Range [0, 2π]

B

Question 5

$\text{cosec}^2 x = 1 + \cot^2 x$

C

Question 6

a $\cos 315° = \frac{1}{\sqrt{2}}$

b $\sin(-60°) = -\sin(60°) = -\frac{\sqrt{3}}{2}$

c $\tan 120° = -\tan 60° = -\sqrt{3}$

Question 7

a $\sin x = \frac{\sqrt{3}}{2}$, $x = 60°, 120°$

b $\tan x = 1$, $x = 45°, 225°$

c $2\cos x + 1 = 0$

$2\cos x = -1$

$\cos x = -\frac{1}{2}$, $x = 120°, 240°$

d $\sin^2 x = \frac{3}{4}$

$\sin x = \pm\frac{\sqrt{3}}{2}$, $x = 60°, 120°, 240°, 300°$

e $\tan 2x = \frac{1}{\sqrt{3}}$

$2x = 30°, 210°, 390°, 570°$

$x = 15°, 105°, 195°, 285°$

Question 8

a tan $x = -1$

$$x = \frac{3\pi}{4} \quad \frac{7\pi}{4}$$

b $2 \sin x = 1$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} \quad \frac{5\pi}{6}$$

c $\tan^2 x = 3$

$$\tan x = \pm\sqrt{3}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

d $\cos x = 1$

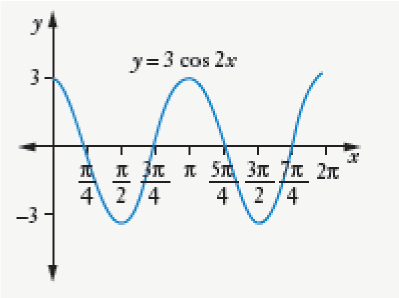
$x = 0, 2\pi$

e $\sin x = -1$

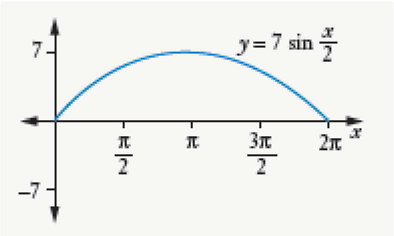
$$x = \frac{3\pi}{2}$$

Question 9

a $y = 3\cos 2x$

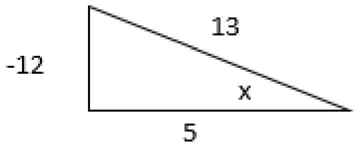


b $y = 7\sin\left(\dfrac{x}{2}\right)$



Question 10

$\sin x = -\dfrac{12}{13}$, $\cos x > 0$



As $\sin x < 0$ and $\cos x > 0$, x is in the fourth quadrant.

$\cos x = \dfrac{5}{13}$, $\tan x = -\dfrac{12}{5}$

Question 11

a $\cos(180 + \theta) = -\cos\theta$

b $\tan(-\theta) = -\tan\theta$

c $\sin(\pi - \theta) = \sin\theta$

d $\tan x \cos x = \dfrac{\sin x}{\cos x}\cos x = \sin x$

e

$$\begin{aligned}&\sqrt{4-4\sin^2 A}\\&=\sqrt{4\cos^2 A}\\&=2\cos A\end{aligned}$$

f $\cos(90 - x) = \sin x$

g $\cot\beta.\tan\beta = 1$

h $\sin a\cos b - \cos a\sin b = \sin(a - b)$

i $2\sin x\cos x = \sin 2x$

j $\frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]=\sin x\cos y$

k $1 - 2\sin^2 10x = \cos 20x$

Question 12

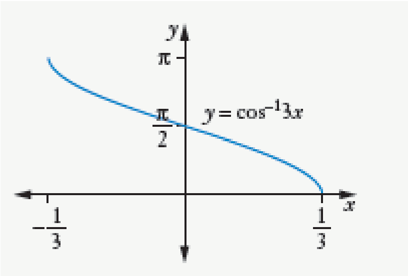
a $\sin\frac{5\pi}{4}=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$

b $\cos\frac{5\pi}{6}=-\cos\frac{\pi}{6}=-\frac{\sqrt{3}}{2}$

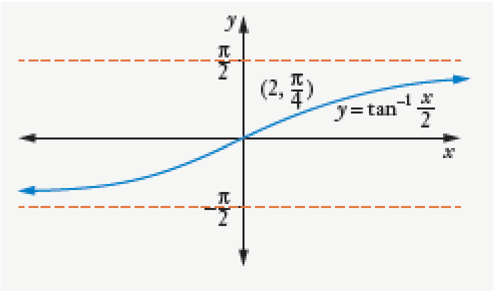
c $\tan\frac{4\pi}{3}=\tan\frac{\pi}{3}=\sqrt{3}$

Question 13

a $y = \cos^{-1} 3x$



b $y = \tan^{-1} \dfrac{x}{2}$



Question 14

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\pi - \cos^{-}\left(\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\therefore \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Question 15

a $\tan^{-1} 1 = \dfrac{\pi}{4}$

b $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$

c $\cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right) = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

d $\cos(\sin^{-1} 1) = \cos\dfrac{\pi}{2} = 0$

e $\tan^{-1}\left[\tan\left(\dfrac{5\pi}{6}\right)\right] = \tan^{-1}\left[\tan\left(\pi - \dfrac{\pi}{6}\right)\right]$

$= \tan^{-1}\left[-\tan\left(\dfrac{\pi}{6}\right)\right]$

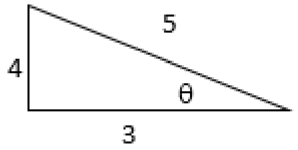
$= \tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

$= -\dfrac{\pi}{6}$

Question 16

a $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3} + \dfrac{\pi}{6} = \dfrac{\pi}{2}$

b $\tan\left(\cos^{-1}\dfrac{3}{5}\right)$

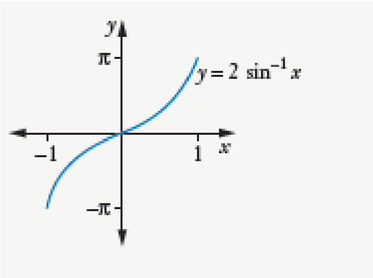


$\tan\left(\cos^{-1}\dfrac{3}{5}\right) = \dfrac{4}{3}$

Question 17

a Domain [–1, 1], range [–π, π]

b



Question 18

$$\text{LHS} = \frac{2\cos^2\theta}{1-\sin\theta}$$

$$= \frac{2(1-\sin^2\theta)}{1-\sin\theta}$$

$$= \frac{2(1-\sin\theta)(1+\sin\theta)}{1-\sin\theta}$$

$$= 2(1+\sin\theta)$$

$$= 2+2\sin\theta = \text{RHS}$$

Question 19

sin b = cos $(2b - 30)°$

cos $(90 - b)°$ = cos $(2b - 30)°$

$90 - b = 2b - 30$

$3b = 120$

$b = 40°$

Question 20

a $\dfrac{2\tan 22.5°}{1-\tan^2 22.5°}$ = tan (2 × 22.5°) = tan 45° = 1

b cos 20° cos 10° – sin 20° sin 10° = cos (20° + 10°)

= cos 30° = $\dfrac{\sqrt{3}}{2}$

c sin 45° cos 45° = $\dfrac{1}{\sqrt{2}} \times \dfrac{1}{\sqrt{2}} = \dfrac{1}{2}$

Question 21

a $\dfrac{1-\tan^2 105°}{1+\tan^2 105°}$ = cos (2× 105°) = cos 210°

= –cos 300° (180° + 30° = 210°, 3rd quadrant)

$= -\dfrac{\sqrt{3}}{2}$

b $\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ = tan (2 ×$\dfrac{\pi}{8}$) = tan $\dfrac{\pi}{4}$ = 1

Question 22

a tan A = $\dfrac{2t}{1-t^2}$

So cot A = $\dfrac{1-t^2}{2t}$

b $\sin A - \cos A = \dfrac{2t}{1+t^2} - \dfrac{1-t^2}{1+t^2}$

$= \dfrac{2t-(1-t^2)}{1+t^2}$

$= \dfrac{2t-1+t^2}{1+t^2}$

c $4 - 3\sec A = 4 - 3 \times \dfrac{1+t^2}{1-t^2}$ since $\cos A = \dfrac{1-t^2}{1+t^2}$

$= 4 - \dfrac{3+3t^2}{1-t^2}$

$= \dfrac{4\left(1-t^2\right)-\left(3+3t^2\right)}{1-t^2}$

$= \dfrac{4-4t^2-3-3t^2}{1-t^2}$

$= \dfrac{1-7t^2}{1-t^2}$

d $\dfrac{1+\tan A}{\sec A} = \dfrac{1+\dfrac{2t}{1-t^2}}{\dfrac{1+t^2}{1-t^2}}$

$= \dfrac{\dfrac{1-t^2+2t}{1-t^2}}{\dfrac{1+t^2}{1-t^2}}$

$= \dfrac{1-t^2+2t}{1-t^2} \div \dfrac{1+t^2}{1-t^2}$

$= \dfrac{1-t^2+2t}{1-t^2} \times \dfrac{1-t^2}{1+t^2}$

$= \dfrac{1-t^2+2t}{1+t^2}$

Question 23

$y = -2\cos\left(3x + \frac{\pi}{12}\right) + 5 = -2\cos\left[3\left(x + \frac{\pi}{36}\right)\right] + 5$

Period $= \frac{2\pi}{3}$, amplitude 2, centre 5, phase $\frac{\pi}{36}$

Question 24

a $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

So $\sec\frac{\pi}{4} = \frac{\sqrt{2}}{1} = \sqrt{2}$

b $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

So $\cot\frac{\pi}{6} = \frac{\sqrt{3}}{1} = \sqrt{3}$

c $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

So $\operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}$

d $\dfrac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2} \div \frac{1}{2} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$

e $\cos^{-1}\left(\sin\frac{3\pi}{4}\right) = \cos^{-1}\frac{1}{\sqrt{2}}$ since $\sin\frac{3\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ (2nd quadrant)

$= \frac{\pi}{4}$

f $\tan^{-1}\left(\tan\frac{11\pi}{6}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ since $\tan\frac{11\pi}{6} = -\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ (4th quadrant)

$= -\frac{\pi}{6}$

Question 25

a $y = –6 \sin (2x) + 5$ has centre 5 and amplitude 6.

Minimum value = 5 – 6 = –1

Maximum value = 5 + 6 = 11

Domain = $(-\infty, \infty)$, range = [–1, 11]

b $f(x) = 4 \cos x – 3$ has centre –3 and amplitude 4.

Minimum value = –3 – 4 = – 7

Maximum value = – 3 + 4 = 1

Domain = $(-\infty, \infty)$, range = [–7, 1]

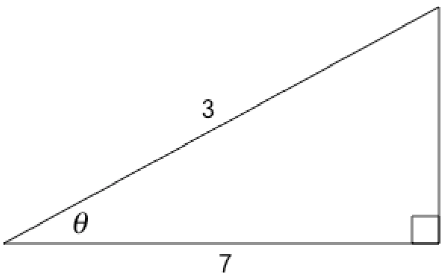
Challenge exercise 11

Question 1

a $\sin 600° = \sin 240° = -\sin 60° = -\dfrac{\sqrt{3}}{2}$

b $\tan(-405°) = \tan(-45°) = -\tan 45° = -1$

c Let $\cos^{-1}\left(-\dfrac{3}{7}\right) = \theta$, then $\cos\theta = -\dfrac{3}{7}$



By Pythagoras' theorem

$c^2 = a^2 + b^2$

$7^2 = a^2 + 3^2$

$49 = a^2 + 9$

$40 = a^2$

$\sqrt{40} = a$

$\sqrt{4} \times \sqrt{10} = a$

$2\sqrt{10} = a$

$\cos^{-1}\left(-\dfrac{3}{7}\right) = \pi - \theta$ (2nd quadrant since range of $\cos^{-1} x$ is $[0, \pi]$)

So $\tan\left[\cos^{-1}\left(-\dfrac{3}{7}\right)\right] = \tan(\pi - \theta) = -\tan\theta = -\dfrac{2\sqrt{10}}{3}$

Question 2

$2 \cos (\theta + 10°) = -1$

$\cos (\theta + 10°) = -\frac{1}{2}$

$\theta + 10° = 120°, 240°$

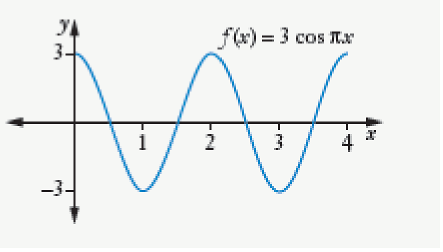
$\theta = 110°, 230°$

Question 3

$f(x) = 3 \cos \pi x$

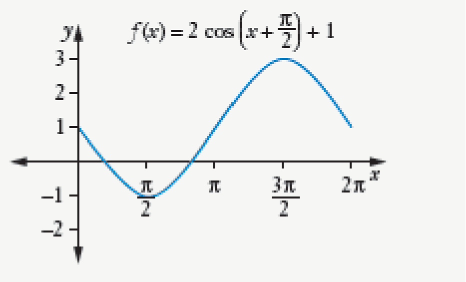
a amplitude = 3, period = $\frac{2\pi}{\pi} = 2$

b

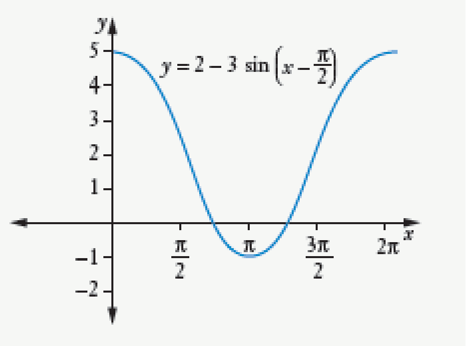


Question 4

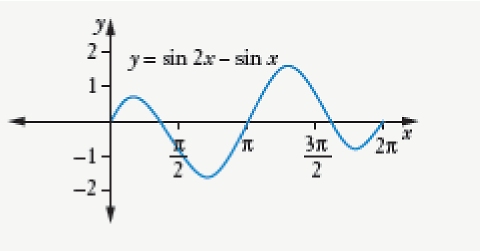
a $f(x) = 2\cos\left(x + \frac{\pi}{2}\right) + 1$



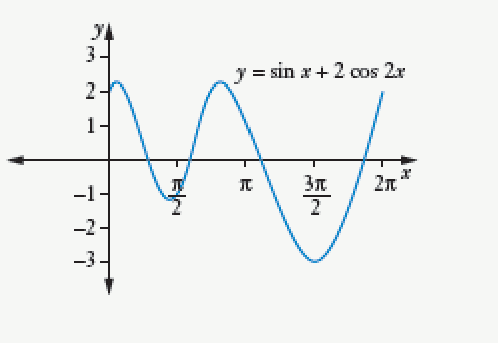
b $y = 2 - 3\sin\left(x - \frac{\pi}{2}\right)$



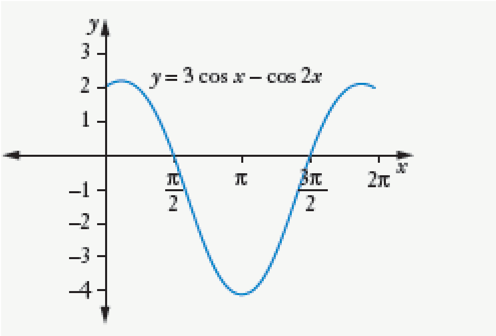
c $y = \sin 2x - \sin x$



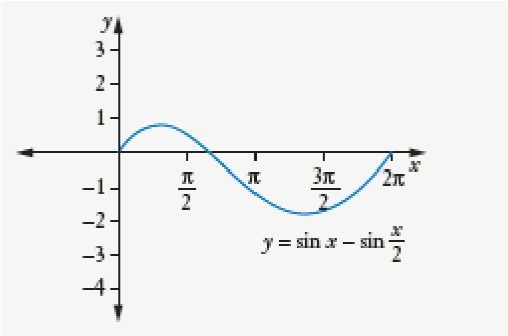
d $y = \sin x + 2\cos 2x$



e $y = 3\cos x - \cos 2x$



f $y = \sin x - \sin \dfrac{x}{2}$



Question 5

$\cos^2 x - \cos x = 0$

$\cos x (\cos x - 1)$

$\cos x = 0 \qquad\qquad \cos x = 1$

$x = \frac{\pi}{2}\ \frac{3\pi}{2} \qquad\qquad x = 0,\ 2\pi$

$\therefore x = 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$

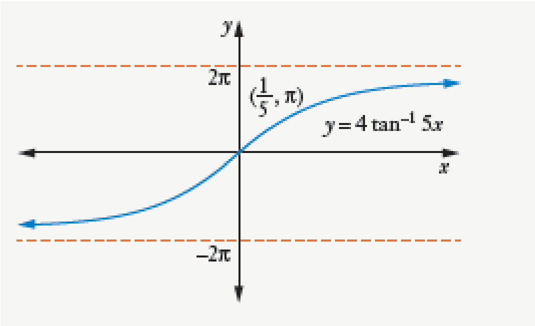
Question 6

$\sin 120° + \cos 135°$

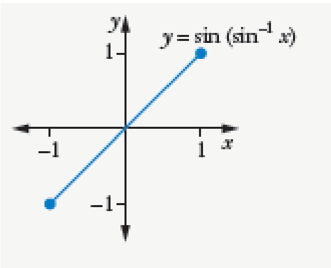
$$= \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{3} - \sqrt{2}}{2}$$

Question 7

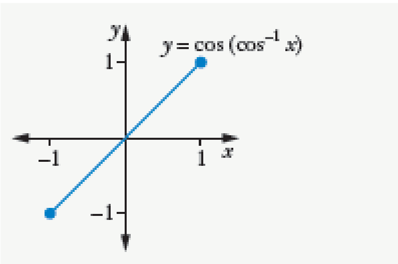
a $y = 4\tan^{-1} 5x$



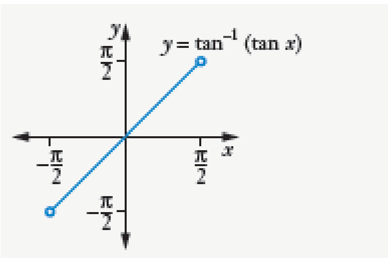
b $y = \sin(\sin^{-1} x)$



c $y = \cos(\cos^{-1} x)$

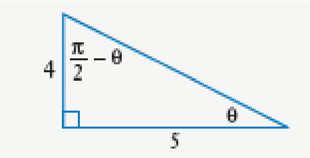


d $y = \tan(\tan^{-1} x)$



Question 8

Let $\tan^{-1}\left(\frac{4}{5}\right)=\theta$, $\tan^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}-\theta$



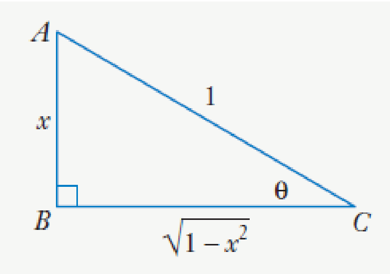
Then $\tan\theta = \frac{4}{5}$ and $\tan\left(\frac{\pi}{2}-\theta\right) = \frac{5}{4}$

$\therefore \tan^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{5}{4}\right) = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2}$

Question 9

Let $\sin^{-1} x = \theta$

Then $\sin^{-1}\frac{x}{1} = \theta$



By Pythagoras' theorem,

$BC=\sqrt{1-x^2}$

$\cos\theta=\frac{\sqrt{1-x^2}}{1}$

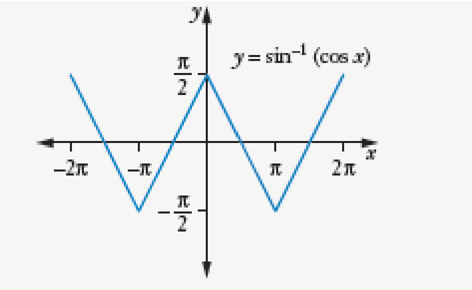
$=\sqrt{1-x^2}$

$\therefore\ \theta=\cos^{-1}\sqrt{1-x^2}$

$\therefore\ \sin^{-1}x=\cos^{-1}\sqrt{1-x^2}$

Question 10

$y = \sin^{-1} (\cos x)$



Question 11

$y = x \cos^{-1} x$

Domain $[-1, 1]$

Range $\left[-\pi \quad \frac{\pi}{2}\right]$

Question 12

$$\begin{aligned}
\text{LHS} &= \frac{\cos\theta(\sin\theta+\cos\theta)}{(1+\sin\theta)(1-\sin\theta)} \\
&= \frac{\cos\theta(\sin\theta+\cos\theta)}{1-\sin^2\theta} \\
&= \frac{\cos\theta(\sin\theta+\cos\theta)}{\cos^2\theta} \\
&= \frac{\sin\theta+\cos\theta}{\cos\theta} \\
&= \frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\cos\theta} \\
&= 1+\tan\theta = \text{RHS}
\end{aligned}$$

Question 13

LHS = $\cos 6x \cos 4x - \sin 6x \sin 4x$

$= \cos (6x + 4x)$

$= \cos 10x$

$= 2 \cos^2 5x - 1$ = RHS

Question 14

Period $\pi = \dfrac{2\pi}{a}$, $\pi a = 2\pi$, $a = 2$

So $y = \sin 2x$

Centre –2 gives $y = \sin 2x - 2$

Range [–5, 1]:

Given amplitude k:

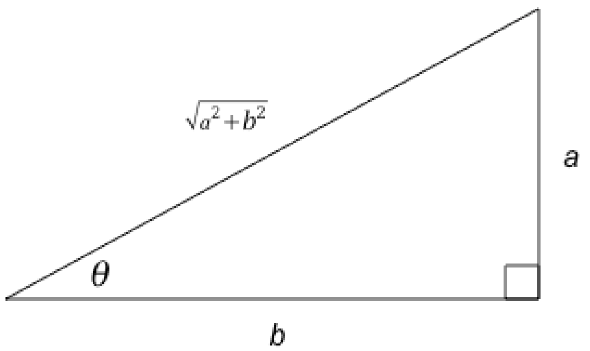
Minimum value $-2 - k = -5$, $-2 + 5 = k$, $3 = k$

Maximum value $-2 + k = 1$

So the equation is $y = 3 \sin 2x - 2$

Question 15

Let $\tan^{-1}\left(\frac{a}{b}\right) = \theta$, then $\tan\theta = \frac{a}{b}$



By Pythagoras' theorem:

$c^2 = a^2 + b^2$

$c = \sqrt{a^2+b^2}$

$\cos\left[\tan^{-1}\left(\frac{a}{b}\right)\right] = \cos\theta = \frac{b}{\sqrt{a^2+b^2}}$

Question 16

$\frac{11\pi}{12} = \frac{12\pi}{12} - \frac{\pi}{12} = \pi - \frac{\pi}{12}$, (2nd quadrant)

So $\sin \frac{11\pi}{12} = \sin\left(\pi - \frac{\pi}{12}\right) = \sin \frac{\pi}{12}$

$= \sin\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$

$= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4}$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

MATHS IN FOCUS 11 MATHEMATICS EXTENSION 1

WORKED SOLUTIONS

Chapter 12: Discrete probability distributions

Exercise 12.01 Random variables

Question 1

a Rating is a number of stars, so discrete.

b Speed is a continuous variable

c The 36 possible outcomes in rolling two dice can be listed, so discrete.

d The number on the raffle ticket is unique and can be listed, so discrete.

e Weight is a continuous variable.

f The size of jeans is specific and can be listed, so discrete.

g Temperature is a continuous variable.

h The amount of water used, whilst specific, need not be a value that can be measured exactly, so continuous.

i The number of cars can be counted and listed, so discrete.

j The number of cities can be counted and listed, so discrete.

k The number of heads occurring can be counted and listed, so discrete.

l The number of right and wrong responses in the test can be counted and listed, so discrete.

Question 2

a $X = \{0, 1\}$

b $X = \{0, 1, 2, \ldots ,10\}$

c $X = \{1, 2, 3, \ldots, 20\}$

d $X = \{0, 1, 2, \ldots 31\}$

e $\mathrm{X} = \{2, 3, 4, \ldots ,12\}$

Exercise 12.02 Discrete probability distributions

Question 1

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

There are 36 possible outcomes.

The number of outcomes for each sum can be determined diagonally.

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

Question 2

a $\left(0, \frac{1}{2}\right)\left(1, \frac{1}{2}\right)$

There are two possible outcomes, no heads (X = 0), heads (X =1), with the probability of each being $\frac{1}{2}$.

b $\left(0,\frac{1}{4}\right),\left(1,\frac{1}{2}\right),\left(2,\frac{1}{4}\right)$

There are 4 possible outcomes.

(tail, tail) (X = 0) $P(X = 0) = \frac{1}{4}$

(tail, head), (head, tail) (X = 1) $P(X = 1) = \frac{2}{4} = \frac{1}{2}$

(head, head) (X = 2) $P(X = 2) = \frac{1}{4}$

c $\left(0,\frac{1}{8}\right),\left(1,\frac{3}{8}\right),\left(2,\frac{3}{8}\right),\left(3,\frac{1}{8}\right)$

There are 8 possible outcomes.

(tail, tail, tail) (X = 0) $P(X = 0) = \frac{1}{8}$

(tail, tail, head), (tail, head, tail), (head, tail, tail), (X = 1) $P(X = 1) = \frac{3}{8}$

(head, head, tail), (head, tail, head), (tail, head, head), (X = 2) $P(X = 2) = \frac{3}{8}$

(head, head, head), (X =3) $P(X = 3) = \frac{1}{8}$

Question 3

a Total number of bags sampled = 8 + 9 + 21 + 9 + 6 = 53

X	48	49	50	51	52
$P(X = x)$	0.15	0.11	0.4	0.17	0.11

$P(X = 48) = \frac{8}{53} = 0.15$, $P(X = 49) = \frac{9}{53} = 0.11$, $P(X = 50) = \frac{21}{53} = 0.4$

$P(X = 51) = \frac{9}{53} = 0.17$, $P(X = 52) = \frac{6}{53} = 0.11$

b **i** At least 50 jelly beans means 50, 51, or 52 jelly beans.

This is the sum of the individual probabilities.

0.4 + 0.17 + 0.11 = 0.68

ii Fewer than 51 jelly beans means 48, 49, or 50 jelly beans.

This is the sum of the individual probabilities.

Or the probability of 51 and 52 beans subtracted from 1.

1 – 0.17 – 0.11 = 0.72

Question 4

a The sum of the probabilities should be 1.

$$p(3)=\frac{3-2}{6}=\frac{1}{6},\ p(4)=\frac{4-2}{6}=\frac{2}{6},\ p(5)=\frac{5-2}{6}=\frac{3}{6}$$

$$p(3)+p(4)+p(5)=\frac{1}{6}+\frac{2}{6}+\frac{3}{6}=1$$

b

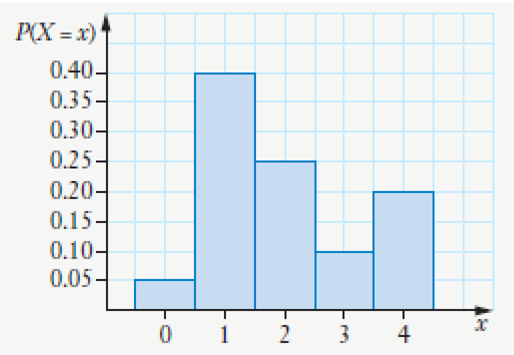
x	3	4	5
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

c **i** $P(X>3)=p(4)+p(5)=\frac{5}{6}$

ii $P(X=\text{odd})=p(3)+p(5)=\frac{1}{6}+\frac{3}{6}=\frac{2}{3}$

iii $P(3\le X<5)=p(3)+p(4)=\frac{1}{6}+\frac{2}{6}=\frac{1}{2}$

Question 5



Question 6

a

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b This is a uniform probability distribution because each probability is the same and the sum of the probabilities is 1.

c **i** $\frac{1}{2}$

$$P(X=4)+P(X=5)+P(X=6)=\frac{3}{6}=\frac{1}{2}$$

ii $\frac{1}{3}$

$$P(X=1)+P(X=2)=\frac{2}{6}=\frac{1}{3}$$

iii $\frac{1}{2}$

$$P(X=2)+P(X=3)+P(X=4)=\frac{3}{6}=\frac{1}{2}$$

Question 7

a No

The sum of the probabilities is not 1. It is $\frac{6}{5}$.

b Yes

The sum of the probabilities is 1.

c No

$$P(X=0)+P(X=1)+P(X=2)=\frac{1}{2}+1+\frac{3}{4}=2\frac{1}{4}$$

The sum of the probabilities is not 1.

Question 8

a The sum of the probabilities is 1.

$$p(1) = 2k,\ \ p(2) = 3k,\ \ p(3) = 4k,\ \ p(4) = 5k$$
$$2k + 3k + 4k + 5k = 1$$
$$14k = 1$$
$$k = \frac{1}{14}$$

b The sum of the probabilities is 1.

$$0.2 + k + 0.15 + 0.34 + 0.12 = 1$$
$$0.81 + k = 1$$
$$k = 0.19$$

c The sum of the probabilities is 1.

$$k + \frac{1}{10} + 0 + \frac{1}{5} + \frac{3}{10} + \frac{2}{5} = 1$$
$$k + 1 = 1$$
$$k = 0$$

Question 9

a

X	1	2	3	4
$P(X = x)$	$\frac{k}{6}$	$\frac{4k}{7}$	$\frac{9k}{8}$	$\frac{16k}{9}$

$$P(X = 1) = p(1) = \frac{k}{6}$$

$$P(X = 2) = p(2) = \frac{4k}{7}$$

$$P(X = 3) = p(3) = \frac{9k}{8}$$

$$P(X = 4) = p(4) = \frac{16k}{9}$$

b The sum of the probabilities is 1.

$$\frac{k}{6} + \frac{4k}{7} + \frac{9k}{8} + \frac{16k}{9} = 1$$

$$\frac{504k + 1728k + 3402k + 5376k}{3024} = 1$$

$$\frac{11010k}{3024} = 1$$

$$k = \frac{3024}{11010} = \frac{504}{1835}$$

Question 10

a

Y	0	1	2	3
$P(Y = y)$	$\frac{5}{9}$	$\frac{5}{18}$	$\frac{5}{36}$	$\frac{1}{36}$

(1,1) (2,1) (3,1) (4,1) (5,1) (6,1)

(1,2) (2,2) (3,2) (4,2) (5,2) (6,2)

(1,3) (2,3) (3,3) (4,3) (5,3) (6,3)

(1,4) (2,4) (3,4) (4,4) (5,4) (6,4)

(1,5) (2,5) (3,5) (4,5) (5,5) (6,5)

(1,6) (2,6) (3,6) (4,6) (5,6) (6,6)

Number uppermost is a 6, $P(X = 1) = \frac{10}{36} = \frac{5}{18}$

Numbers uppermost are doubles except for double 6, $P(X = 2) = \frac{5}{36}$

Numbers uppermost are double 6, $P(X = 3) = \frac{1}{36}$

Number combination is none of the above, $P(X = 0) = 1 - \frac{5}{18} - \frac{5}{36} - \frac{1}{36} = \frac{5}{9}$

b **i** $P(X = 3) = \frac{1}{36}$

ii At least \$2 means winning \$2 or \$3.

$$P(X = 2) + P(X = 3) = \frac{5}{36} + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$$

iii Less than \$3 is no win or 1 win or 2 wins.

$$P(X = 0) + P(X = 1) + P(X = 2) = \frac{5}{9} + \frac{5}{18} + \frac{5}{36} = \frac{35}{36}$$

Question 11

The sum of the probabilities is 1.

$$2p + 3p + 5p + p = 1$$
$$11p = 1$$
$$p = \frac{1}{11}$$

Question 12

a

X	0	1	2	3	5
$P(X = x)$	$\frac{13}{25}$	$\frac{19}{100}$	$\frac{1}{10}$	$\frac{9}{100}$	$\frac{1}{10}$

Number on card is less than 20, $P(X = 1) = \frac{19}{100}$

Number on card is less than 90, $P(X = 2) = \frac{10}{100} = \frac{1}{10}$

Number on card is between 61 and 69, $P(X = 3) = \frac{9}{100}$

Number on card ends in 0, $P(X = 5) = \frac{10}{100} = \frac{1}{10}$

Number on card is none of the above, $P(X = 0) = 1 - \frac{19}{100} - \frac{1}{10} - \frac{9}{100} - \frac{1}{10} = \frac{13}{25}$

b $P(X = 3) + P(X = 5) = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$

c $1 - P(X = 5) = 1 - \frac{1}{10} = \frac{9}{10}$

Question 13

a $P(X = 6) = \dfrac{2}{7}$

b

$$\mathrm{P}(X = 6) + P(X = 8) + P(X = 10)$$
$$= \frac{2}{7} + \frac{1}{14} + \frac{1}{7} = \frac{1}{2}$$

c

$$P(X = 9) + P(X = 10)$$
$$= \frac{3}{14} + \frac{1}{7} = \frac{5}{14}$$

d

$$P(X = 7) + P(X = 5) + P(X = 6)$$
$$= \frac{1}{14} + \frac{3}{14} + \frac{2}{7} = \frac{4}{7}$$

e

$$P(X = 7) + P(X = 8)$$
$$= \frac{1}{14} + \frac{1}{14} = \frac{1}{7}$$

f

$$P(X = 7) + P(X = 8) + P(X = 9)$$
$$= \frac{1}{14} + \frac{1}{14} + \frac{3}{14} = \frac{5}{14}$$

g

$$P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9)$$
$$= \frac{2}{7} + \frac{1}{14} + \frac{1}{14} + \frac{3}{14} = \frac{9}{14}$$

Question 14

a $P(Q=8)=\frac{3}{16}$

b

$$P(Q=4)+P(Q=6)+P(Q=8)+P(Q=10)$$
$$=\frac{1}{4}+\frac{1}{8}+\frac{3}{16}+\frac{3}{16}=\frac{3}{4}$$

c

$$P(Q=4)+P(Q=6)$$
$$=\frac{1}{4}+\frac{1}{8}=\frac{3}{8}$$

d

$$P(Q=4)+P(Q=6)+P(Q=8)+P(Q=10)$$
$$=\frac{1}{4}+\frac{1}{8}+\frac{3}{16}+\frac{3}{16}=\frac{3}{4}$$

e

$$P(Q=0)+P(Q=2)$$
$$=\frac{1}{16}+\frac{3}{16}=\frac{1}{4}$$

f

$$P(Q=2)+P(Q=4)+P(Q=6)+P(Q=8)$$
$$=\frac{3}{16}+\frac{1}{4}+\frac{1}{8}+\frac{3}{16}=\frac{3}{4}$$

Question 15

a

x	0	1	2
$P(X = x)$	$\frac{994\,009}{1\,000\,000}$	$\frac{5982}{1\,000\,000}$	$\frac{9}{1\,000\,000}$

$$P(\text{faulty machine}) = \frac{3}{1000},\ P(\text{machine not faulty}) = \frac{997}{1000}$$

$$P(X = 0) = P(\text{no faulty machine}) = \frac{997}{1000} \times \frac{997}{1000} = \frac{994\,009}{1\,000\,000}$$

$$P(X = 1) = P(\text{one faulty machine}) = 2 \times \frac{3}{1000} \times \frac{997}{1000} = \frac{5982}{1\,000\,000}$$

$$P(X = 2) = P(\text{two faulty machine}) = \frac{3}{1000} \times \frac{3}{1000} = \frac{9}{1\,000\,000}$$

b **i** $\frac{5982}{1\,000\,000}$

$$P(X = 1) = \frac{5982}{1\,000\,000}$$

ii $\frac{5991}{1\,000\,000}$

$$P(X = 1) + P(X = 2) = \frac{5982}{1\,000\,000} + \frac{9}{1\,000\,000} = \frac{5991}{1\,000\,000}$$

Question 16

a

x	0	1	2
$P(X=x)$	$\frac{25}{49}$	$\frac{20}{49}$	$\frac{4}{49}$

$$P(\text{white ball}) = \frac{6}{21} = \frac{2}{7},\ P(\text{not a white ball}) = \frac{5}{7}$$

$$P(X = 0) = P(\text{no white balls}) = \frac{5}{7} \times \frac{5}{7} = \frac{25}{49}$$

$$P(X = 1) = P(\text{one white ball}) = 2 \times \frac{5}{7} \times \frac{2}{7} = \frac{20}{49}$$

$$P(X = 2) = P(\text{two white balls}) = \frac{2}{7} \times \frac{2}{7} = \frac{4}{49}$$

b

x	0	1	2
$P(X=x)$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{1}{14}$

$$P(X = 0) = P(\text{no white balls}) = \frac{15}{21} \times \frac{14}{20} = \frac{1}{2}$$

$$P(X = 1) = P(\text{one white ball}) = 2 \times \frac{6}{21} \times \frac{15}{20} = \frac{3}{7}$$

$$P(X = 2) = P(\text{two white balls}) = \frac{6}{21} \times \frac{5}{20} = \frac{1}{14}$$

Question 17

a The circle has 8 sectors of equal size, with $P(\text{one sector}) = \frac{1}{8}$.

b The sectors representing probability are not the same size.

c

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

$P(1) = \frac{1}{2}$, $P(2) = P(3) = P(4) = P(5) = \frac{1}{8}$

Question 18

x	0	1	2	3
$P(X=x)$	68.1%	27.9%	3.8%	0.2%

$P(\text{green}) = 0.12$ $\quad$ $P(\text{not green}) = 0.88$

no green lights $P(0) = 0.88 \times 0.88 \times 0.88 \approx 0.681 = 68.1\%$

1 green light $P(1) = 3 \times 0.12 \times 0.88 \times 0.88 \approx 0.279 = 27.9\%$

2 green lights $P(2) = 3 \times 0.12 \times 0.12 \times 0.88 \approx 0.038 = 3.8\%$

3 green lights $P(3) = 0.12 \times 0.12 \times 0.12 \approx 0.0017 = 0.2\%$

Question 19

x	0	1	2	3	4
$P(X=x)$	5.76%	24%	37.47%	26%	6.77%

$P(\text{boy}) = 0.51 \qquad P(\text{girl}) = 0.49$

No boys $P(X = 0) = 0.49^4 \approx 0.0576 = 5.76\%$

One boy $P(X = 1) = 4 \times 0.51 \times 0.49^3 \approx 0.24 = 24\%$

Two boys $P(X = 2) = 6 \times 0.51^2 \times 0.49^2 \approx 0.3747 = 37.47\%$

Three boys $P(X = 3) = 4 \times 0.51^3 \times 0.49 \approx 0.26 = 26\%$

Four boys $P(X = 4) = 0.51^4 \approx 0.0677 = 6.77\%$

Question 20

a

x	0	1	2
$P(X=x)$	$\frac{893}{990}$	$\frac{19}{198}$	$\frac{1}{495}$

$P(\text{win prize}) = \dfrac{5}{100}$, $P(\text{no prize won}) = \dfrac{95}{100}$

No prize $P(X = 0) = \dfrac{95}{100} \times \dfrac{94}{99} = \dfrac{893}{990}$

One prize $P(X = 1) = 2 \times \dfrac{5}{100} \times \dfrac{95}{99} = \dfrac{19}{198}$

Two prizes $P(X = 2) = \dfrac{5}{100} \times \dfrac{4}{99} = \dfrac{1}{495}$

b $P(X = 1) + P(X = 2) = \dfrac{19}{198} + \dfrac{1}{495} = \dfrac{97}{990}$

Exercise 12.03 Mean or expected value

Question 1

a $0\times\frac{1}{4}+1\times\frac{1}{2}+2\times\frac{1}{4}=1$

b $1\times0.31+2\times0.16+3\times0.15+4\times0.2+5\times0.18=2.78$

c $1\times\frac{1}{8}+2\times\frac{1}{16}+3\times\frac{1}{4}+4\times\frac{3}{8}+5\times\frac{3}{16}=3\frac{7}{16}$

d $p(0)=\frac{1}{8},\ p(1)=\frac{1}{4},\ p(4)=\frac{5}{8}$

Expected value: $0\times\frac{1}{8}+1\times\frac{1}{4}+4\times\frac{5}{8}=2\frac{3}{4}$

e $1\times\frac{1}{2}+2\times\frac{2}{8}+4\times\frac{1}{4}=2$

Question 2

a **i** Sum of probabilities is 1.

$$k+\frac{1}{5}+\frac{3}{10}+\frac{2}{5}=1$$

$$k=\frac{1}{10}$$

ii $1\times\frac{1}{10}+2\times\frac{1}{5}+3\times\frac{3}{10}+4\times\frac{2}{5}=3$

b **i** $p(0)=3k$, $p(1)=4k$, $p(2)=5k$

$$p(0)+p(1)+p(2)=3k+4k+5k=1$$
$$12k=1$$
$$k=\frac{1}{12}$$

ii

$$0\times3k+1\times4k+2\times5k$$
$$=14k$$
$$=14\times\frac{1}{12}=1\frac{1}{6}$$

c **i**

$$0.1+0.02+0.17+0.24+k+0.32=1$$
$$k+0.85=1$$
$$k=0.15$$

ii

$$1\times0.1+2\times0.02+3\times0.17+4\times0.24+5\times k+6\times0.32$$
$$=3.53+5k$$
$$=4.28$$

Question 3

a

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$0\times\frac{1}{4}+1\times\frac{1}{2}+2\times\frac{1}{4}=1$$

b

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

$$2\times\frac{1}{36}+3\times\frac{2}{36}+4\times\frac{3}{36}+5\times\frac{4}{36}+6\times\frac{5}{36}+7\times\frac{6}{36}+8\times\frac{5}{36}+9\times\frac{4}{36}+10\times\frac{3}{36}+11\times\frac{2}{36}+12\times\frac{1}{36}=7$$

c

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8}=1\frac{1}{2}$$

d

x	0	1	2	3
$P(X=x)$	0.999^3	$3\times 0.001\times 0.999^2$	$3\times 0.001^2\times 0.999$	0.001^3

Expected value

$$0\times 0.999^3+1\times 3\times 0.001\times 0.999^2+2\times 3\times 0.001^2\times 0.999+3\times 0.001^3$$

$$=0003\ =\frac{3}{1000}$$

e **i**

x	0	1	2
$P(X=x)$	$\frac{144}{361}$	$\frac{168}{361}$	$\frac{49}{361}$

$$P(\text{red})=\frac{7}{19}$$

$$P(X\ =0)=\frac{12}{19}\times\frac{12}{19}=\frac{144}{361}$$

$$P(X\ =1)=2\times\frac{7}{19}\times\frac{12}{19}=\frac{168}{361}$$

$$P(X\ =2)=\frac{7}{19}\times\frac{7}{19}=\frac{49}{361}$$

Expected value

$$0\times\frac{144}{361}+1\times\frac{168}{361}+2\times\frac{49}{361}=\frac{266}{361}$$

$$=\frac{14}{19}$$

ii $P(X=0)=\frac{12}{19}\times\frac{11}{18}=\frac{132}{342}$

$P(X=1)=2\times\frac{7}{19}\times\frac{12}{18}=\frac{168}{342}$

$P(X=2)=\frac{7}{19}\times\frac{6}{18}=\frac{42}{342}$

Expected value

$$0\times\frac{132}{342}+1\times\frac{168}{342}+2\times\frac{42}{342}=\frac{252}{342}$$
$$=\frac{14}{19}$$

Question 4

p = 0.3, q = 0.1

Sum of probabilities is 1.

$p+0.25+0.35+q=1 \Rightarrow p+q=0.4$ [1]

Expected value is 6.35.

$3\times p+7\times 0.25+8\times 0.35+9\times q=6.35$

$3p+9q=1.8 \Rightarrow p+3q=0.6$ [2]

[2] – [1]: $2q=0.2 \Rightarrow q=0.1$

From [1]: $p+0.1=0.4 \Rightarrow p=0.3$

Question 5

Sum of probabilities is 1.

$$a+\frac{1}{8}+b+\frac{1}{4}+\frac{3}{8}=1 \Rightarrow 4a+4b=1 \qquad [1]$$

Expected value is $3\frac{1}{4}$.

$$1\times a+2\times\frac{1}{8}+3\times b+4\times\frac{1}{4}+5\times\frac{3}{8}=3\frac{3}{4} \Rightarrow 8a+24b=5 \qquad [2]$$

[2] – 2 × [1]: $16b=3 \Rightarrow b=\frac{3}{16}$

From [1]: $4a+4\times\frac{3}{16}=1 \Rightarrow a=\frac{1}{16}$

Question 6

a

x	1	2	3	4
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$P(X=x)=\frac{1}{4}$ for $x=1,2,3,4$

b $E(X)=1\times\frac{1}{4}+2\times\frac{1}{4}+3\times\frac{1}{4}+4\times\frac{1}{4}=2\frac{1}{2}$

Question 7

X	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E(X) = 0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8}=1\frac{1}{2}$$

Question 8

a

X	0	1	2	3
$P(X = x)$	$\frac{27}{1331}$	$\frac{216}{1331}$	$\frac{576}{1331}$	$\frac{512}{1331}$

$$E(X) = 0\times\frac{27}{1331}+1\times\frac{216}{1331}+2\times\frac{576}{1331}+3\times\frac{512}{1331}=\frac{2904}{1331}=2\frac{2}{11}$$

b

X	0	1	2	3
$P(X = x)$	$\frac{6}{990}=\frac{1}{165}$	$\frac{144}{990}=\frac{8}{55}$	$\frac{504}{990}=\frac{28}{55}$	$\frac{336}{990}=\frac{56}{165}$

$$E(X) = 0\times\frac{1}{165}+1\times\frac{8}{55}+2\times\frac{28}{55}+3\times\frac{56}{165}=\frac{360}{165}=2\frac{2}{11}$$

Question 9

a

x	0	1	2	3
$P(X=x)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

b 56 cents

$$E(X)=0\times\frac{2}{3}+1\times\frac{1}{6}+2\times\frac{1}{9}+3\times\frac{1}{18}=\frac{5}{9}$$

This is $\frac{500}{9}=56$ cents

c Lose 44 cents

$\$1-\$0.56=\$0.44$

Question 10

a

x	0	1	2	3	4
$P(X=x)$	$\frac{81}{625}$	$\frac{216}{625}$	$\frac{216}{625}$	$\frac{96}{625}$	$\frac{16}{625}$

$$P(X=0)=\left(\frac{3}{5}\right)^4=\frac{81}{625}$$

$$P(X=1)=4\times\frac{2}{5}\times\left(\frac{3}{5}\right)^3=\frac{216}{625}$$

$$P(X=2)=6\times\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^2=\frac{216}{625}$$

$$P(X=3)=4\times\left(\frac{2}{5}\right)^3\times\frac{3}{5}=\frac{96}{625}$$

$$P(X=4)=\left(\frac{2}{5}\right)^4=\frac{16}{625}$$

b $E(X)=0\times\frac{81}{625}+1\times\frac{216}{625}+2\times\frac{216}{625}+3\times\frac{96}{625}+3\times\frac{16}{625}=1\frac{359}{625}$

c She will keep her job

The expected value is greater than 1, which is more than 1 call per hour.

Question 11

Win 42 cents

X	-1	2	3
$P(X=x)$	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{1}{6}$

$E(X)=-1\times\frac{7}{12}+2\times\frac{1}{4}+3\times\frac{1}{6}=\frac{5}{12}$

This is $\frac{500}{12}=42$ cents

Exercise 12.04 Variance and standard deviation

Question 1

a **i** $E(X)=1\times 0.17+2\times 0.24+3\times 0.12+4\times 0.13+5\times 0.23+6\times 0.11=3.34$

$$\begin{aligned}\text{Var}(X)&=\sum\left(x^2p(x)\right)-\mu^2\\&=1^2\times 0.17+2^2\times 0.24+3^2\times 0.12+4^2\times 0.13\\&\quad+5^2\times 0.23+6^2\times 0.11-3.34^2\\&=14.007-11.1556\\&=2.8514\end{aligned}$$

Standard deviation $\sqrt{2.8514}=1.69$

ii Variance is 2.8514 (from **i**)

b **i** $E(X)=0\times\frac{1}{7}+1\times\frac{3}{7}+2\times\frac{1}{14}+3\times\frac{5}{14}=1\frac{9}{14}$

$$\begin{aligned}\text{Var}(X)&=\sum\left(x^2p(x)\right)-\mu^2\\&=1^2\times\frac{3}{8}+2^2\times\frac{1}{4}+3^2\times\frac{1}{8}+4^2\times\frac{1}{16}+5^2\times\frac{3}{16}-\left(\frac{241}{196}\right)^2\\&=\frac{55}{14}-\left(\frac{23}{14}\right)^2\\&=\frac{241}{196}=1.23\end{aligned}$$

Standard deviation $\sqrt{1.23}=1.11$

ii Variance is 1.23 (from **i**)

c **i** $E(X)=1\times\frac{3}{8}+2\times\frac{1}{4}+3\times\frac{1}{8}+4\times\frac{1}{16}+5\times\frac{3}{16}=2.4375$

$$\begin{aligned}\text{Var}(X)&=\sum\left(x^2p(x)\right)-\mu^2\\&=1^2\times\frac{3}{8}+2^2\times\frac{1}{4}+3^2\times\frac{1}{8}+4^2\times\frac{1}{16}+5^2\times\frac{3}{16}-2.4375^2\\&=88175\ -.9414\\&=2.25\end{aligned}$$

Standard deviation $\sqrt{2.2461}=1.50$

ii Variance is 2.25 (from **i**)

Question 2

a $E(X) = 1\times 0.09 + 4\times 0.18 + 7\times 0.26 + 9\times 0.32 + 10\times 0.15 = 7.01$

$$\begin{aligned}\text{Var}(X) &= \sum\left(x^2 p(x)\right) - \mu^2 \\ &= 1^2\times 0.09 + 4^2\times 0.18 + 7^2\times 0.26 + 9^2\times 0.32 + 10^2\times 0.15 - 7.01^2 \\ &= 7.49\end{aligned}$$

Standard deviation $\sqrt{7.49} = 2.74$

b $E(X) = 0\times\frac{1}{9} + 2\times\frac{1}{3} + 4\times\frac{5}{9} = 2.89$

$\text{Var}(X) = 0^2\times\frac{1}{9} + 2^2\times\frac{1}{3} + 4^2\times\frac{5}{9} - 2.89^2$

$= 1.87$

Standard deviation $\sqrt{1.87} = 1.37$

Question 3

$\frac{2}{5} + \frac{1}{10} + \frac{3}{20} + \frac{1}{20} + n = 1$

$n + \frac{14}{20} = 1$

$n = \frac{3}{10}$

$E(X) = 1\times\frac{2}{5} + 2\times\frac{1}{10} + 3\times\frac{3}{20} + 4\times\frac{1}{20} + 5\times\frac{3}{10} = 2.75$

$\text{Var}(X) = 1^2\times\frac{2}{5} + 2^2\times\frac{1}{10} + 3^2\times\frac{3}{20} + 4^2\times\frac{1}{20} + 5^2\times\frac{3}{10} - 2.75^2 = 2.89$

Question 4.

a Sum of probabilities is 1.

$a + b + 0.17 + 0.2 + 0.3 = 1 \Rightarrow a + b = 0.33$ [1]

$E(X) = 3.32$

$1 \times a + 2 \times b + 3 \times 0.17 + 4 \times 0.2 + 5 \times 0.3 = 3.32 \Rightarrow a + 2b = 0.51$ [2]

[2] – [1]: $b = 0.51 - 0.33 = 0.18$

From [1]: $a + 0.18 = 0.33 \Rightarrow a = 0.15$

b $\text{Var}(X) = 1^2 \times 0.15 + 2^2 \times 0.18 + 3^2 \times 0.17 + 4^2 \times 0.2 + 5^2 \times 0.3 - 3.32^2 = 2.08$

c Standard deviation $\sqrt{2.08} = 1.44$

Question 5

a **i**

X	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E(X) = 0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8}=1.5$$

ii $$\text{Var}(X) = 0^2\times\frac{1}{8}+1^2\times\frac{3}{8}+2^2\times\frac{3}{8}+3^2\times\frac{1}{8}-1.5^2=0.75$$

Standard deviation $\sqrt{0.75}=0.87$

iii From **ii**, variance = 0.75

b **i**

X	0	1	2
$P(X = x)$	$\frac{66}{231}$	$\frac{120}{231}$	$\frac{45}{231}$

$$E(X) = 0\times\frac{66}{231}+1\times\frac{120}{231}+2\times\frac{45}{231}=0.91$$

ii $$\text{Var}(X) = 0^2\times\frac{66}{231}+1^2\times\frac{120}{231}+2^2\times\frac{45}{231}-0.91^2=0.47$$

Standard deviation $\sqrt{0.47}=0.69$

iii From **ii**, variance = 0.47

Question 6

a $P(X = x) = \frac{1}{5} = 0.2$

$E(X) = 1\times0.2 + 2\times0.2 + 3\times0.2 + 4\times0.2 + 5\times0.2 = 3$

b $\text{Var}(X) = 1^2\times0.2 + 2^2\times0.2 + 3^2\times0.2 + 4^2\times0.2 + 5^2\times0.2 - 3^2 = 2$

Standard deviation $\sqrt{2} = 1.41$

c Variance = 2 (From **b**.)

Question 7

a

x	0	1	2
$P(X=x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

(1,1) (2,1) (3,1) (4,1) (5,1) (6,1)

(1,2) (2,2) (3,2) (4,2) (5,2) (6,2)

(1,3) (2,3) (3,3) (4,3) (5,3) (6,3)

(1,4) (2,4) (3,4) (4,4) (5,4) (6,4)

(1,5) (2,5) (3,5) (4,5) (5,5) (6,5)

(1,6) (2,6) (3,6) (4,6) (5,6) (6,6)

$$P(\text{no sixes}) = \frac{25}{36}$$

$$P(\text{one six}) = \frac{10}{36} = \frac{5}{18}$$

$$P(\text{two sixes}) = \frac{1}{36}$$

b $$E(X) = 0 \times \frac{25}{36} + 1 \times \frac{5}{18} + 2 \times \frac{1}{36} = 0.33$$

$$\text{Var}(X) = 0^2 \times \frac{25}{36} + 1^2 \times \frac{5}{18} + 2^2 \times \frac{1}{36} - \left[E(X)\right]^2 = 0.28$$

Standard deviation $\sqrt{0.28} = 0.53$

Question 8

a $P(\text{black jellybean}) = 0.04$, $P(\text{not a lack jellybean}) = 0.96$

X	0	1	2
***P*(*X* = *x*)**	0.92	0.074	0.0016

$E(X) = 0 \times 0.92 + 1 \times 0.074 + 2 \times 0.0016 = 0.08$

b $\text{Var}(X) = 0^2 \times 0.92 + 1^2 \times 0.074 + 2^2 \times 0.0016 - 0.08^2 = 0.08$

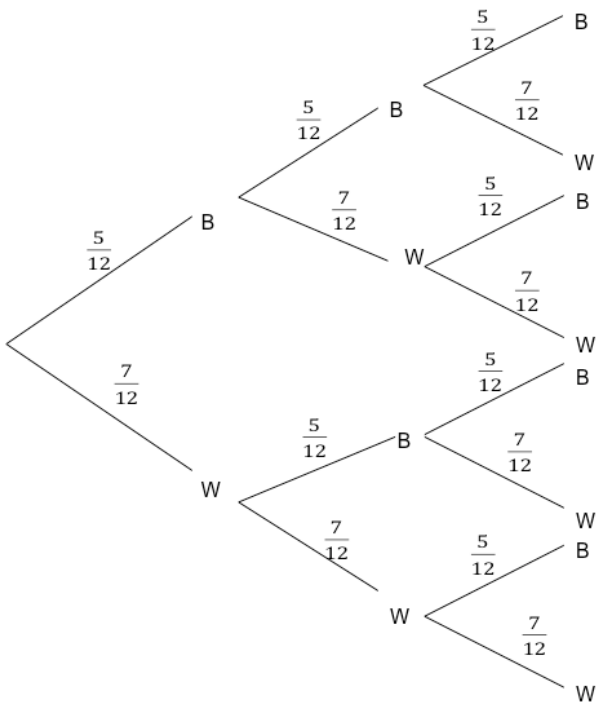
Standard deviation $\sqrt{0.08} = 0.28$

c Variance = 0.08 (From **b**)

Question 9

a Random variable for number of blue balls: $X = \{0, 1, 2, 3\}$

$P(B) = \frac{5}{12}$, $P(W) = \frac{7}{12}$



$P(X = 0) = P(WWW)$

$= \frac{7}{12}\times\frac{7}{12}\times\frac{7}{12} = \frac{343}{1728}$

$P(X = 1) = P(BWW) + P(WBW) + P(WWB)$

$= \frac{5}{12}\times\frac{7}{12}\times\frac{7}{12}+\frac{7}{12}\times\frac{5}{12}\times\frac{7}{12}+\frac{7}{12}\times\frac{7}{12}\times\frac{5}{12} = \frac{735}{1728}$

$P(X = 2) = P(BBW) + P(BWB) + P(WBB)$

$= \frac{5}{12}\times\frac{5}{12}\times\frac{7}{12}+\frac{5}{12}\times\frac{7}{12}\times\frac{5}{12}+\frac{7}{12}\times\frac{5}{12}\times\frac{5}{12} = \frac{525}{1728}$

$P(X = 3) = P(BBB)$

$= \frac{5}{12}\times\frac{5}{12}\times\frac{5}{12} = \frac{125}{1728}$

Probability distribution:

x	0	1	2	3
$P(X = x)$	$\frac{343}{1728}$	$\frac{735}{1728}$	$\frac{525}{1728}$	$\frac{125}{1728}$

$$\mu = \sum xp(x)$$

$$= 0\times\frac{343}{1728}+1\times\frac{735}{1728}+2\times\frac{525}{1728}+3\times\frac{125}{1728} = 1.25$$

$$\sigma^2 = \sum(x^2 p(x)) - \mu^2$$

$$= 0^2\times\frac{343}{1728}+1^2\times\frac{735}{1728}+2^2\times\frac{525}{1728}+3^2\times\frac{125}{1728} - 1.25^2 = 0.73$$

b Without replacement:

$P(B) = \frac{5}{12}$, $P(W) = \frac{7}{12}$

2nd ball after B: $P(B) = \frac{4}{11}$, $P(W) = \frac{7}{11}$

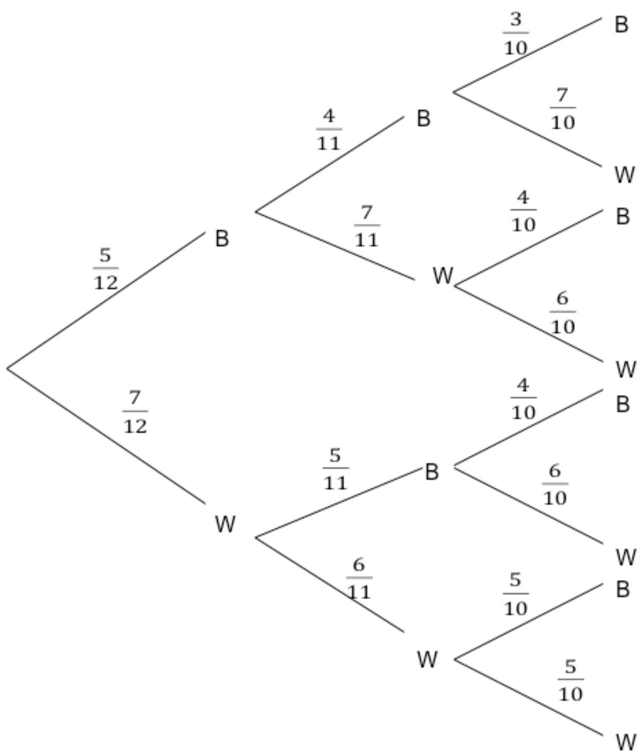
2nd ball after W: $P(B) = \frac{5}{11}$, $P(W) = \frac{6}{11}$

3rd ball after BB: $P(B) = \frac{3}{10}$, $P(W) = \frac{7}{10}$

3rd ball after BW: $P(B) = \frac{4}{10}$, $P(W) = \frac{6}{10}$

3rd ball after WB: $P(B) = \frac{4}{10}$, $P(W) = \frac{6}{10}$

3rd ball after WW: $P(B) = \frac{5}{10}$, $P(W) = \frac{5}{10}$



$P(X = 0) = P(WWW)$

$= \frac{7}{12} \times \frac{6}{11} \times \frac{5}{10} = \frac{7}{44}$

$P(X = 1) = P(BWW) + P(WBW) + P(WWB)$

$= \frac{5}{12} \times \frac{7}{11} \times \frac{6}{10} + \frac{7}{12} \times \frac{5}{11} \times \frac{6}{10} + \frac{7}{12} \times \frac{6}{11} \times \frac{5}{10} = \frac{21}{44}$

$P(X = 2) = P(BBW) + P(BWB) + P(WBB)$

$= \frac{5}{12} \times \frac{4}{11} \times \frac{7}{10} + \frac{5}{12} \times \frac{7}{11} \times \frac{4}{10} + \frac{7}{12} \times \frac{5}{11} \times \frac{4}{10} = \frac{7}{22}$

$P(X = 3) = P(BBB)$

$= \frac{5}{12} \times \frac{4}{11} \times \frac{3}{10} = \frac{1}{22}$

Probability distribution:

x	0	1	2	3
$\boldsymbol{P(X = x)}$	$\frac{7}{44}$	$\frac{21}{44}$	$\frac{7}{22}$	$\frac{1}{22}$

$\mu = \sum xp(x)$

$= 0\times\frac{7}{44}+1\times\frac{21}{44}+2\times\frac{7}{22}+3\times\frac{1}{22} = 1.25$

$\sigma^2 = \sum(x^2p(x))-\mu^2$

$= 0^2\times\frac{7}{44}+1^2\times\frac{21}{44}+2^2\times\frac{7}{22}+3^2\times\frac{1}{22} - 1.25^2 = 0.6$

Question 10

a **i**

x	0	1	2
$P(X=x)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

$P(0\text{ aces}) = \frac{48}{52}\times\frac{47}{51} = \frac{188}{221}$

$P(1\text{ ace}) = \frac{4}{52}\times\frac{48}{51}+\frac{48}{52}\times\frac{4}{51} = \frac{32}{221}$

$P(2\text{ aces}) = \frac{4}{52}\times\frac{3}{51} = \frac{1}{221}$

ii

$E(X) = 0\times\frac{188}{221}+1\times\frac{32}{221}+2\times\frac{1}{221} = 0.15$

$\text{Var}(X) = 0^2\times\frac{188}{221}+1^2\times\frac{32}{221}+2^2\times\frac{1}{221}-0.15^2 = 0.14$

Standard deviation $\sqrt{0.14} = 0.37$

b **i**

y	0	5	10
$P(Y=y)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

ii

$$E(X) = 0\times\frac{188}{221} + 5\times\frac{32}{221} + 10\times\frac{1}{221} = 0.77$$

$$\text{Var}(X) = 0^2\times\frac{188}{221} + 5^2\times\frac{32}{221} + 10^2\times\frac{1}{221} - 0.77^2 = 3.48$$

Standard deviation $\sqrt{3.48} = 1.87$

Test Yourself 12

Question 1

$0.16+0.08+0.14+0.21+0.17=0.76$

C

Question 2

$E(X)=\sum(X=x)\times p(x)$

D

Question 3

The sum of the probabilities is 1.

$0.28+0.16+0.04+0.1+t+0.25=1$
$t+0.83=1$
$t=0.17$

B

Question 4

The sum of the probabilities must be 1.

Options A and C have the sum of the probabilities greater than 1.

Option B has the sum of the probabilities less than 1.

Option D has the sum of the probabilities equal to 1.

D

Question 5

a $X = \{0, 1, 2, 3, 4, 5\}$

In 5 rolls of a die, a 6 can occur 0, 1, 2, 3, 4 or 5 times.

b $X = \{0, 1, 2, 3, \ldots, 10\}$

In 10 tosses of a coin, heads can occur 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 or 10 times.

c $X = \{0, 1, 2, 3, \ldots, 30\}$

There are 30 days in November.

The temperature can rise above 300 on the 1st, 2nd, …., or 30th day.

d $X = \{0, 1, 2\}$

On 2 rolls of the dice, a double can occur 0, 1 or 2 times.

e $X = \{0, 1, 2, \ldots 9\}$

From 9 cards drawn, a red card can appear 0, 1, 2, …, or 9 times.

Question 6

The sum of the probabilities must be 1.

$$5k + 3k + 4k - 1 + 2k - 3 + 6k = 1$$
$$20k = 5$$
$$k = \frac{1}{4}$$

Question 7

$$p(1) = \frac{1}{15},\ p(2) = \frac{2}{15},\ p(3) = \frac{1}{5},\ p(4) = \frac{4}{15},\ p(5) = \frac{1}{3}$$

Mean: $1 \times \frac{1}{15} + 2 \times \frac{2}{15} + 3 \times \frac{1}{5} + 4 \times \frac{4}{15} + 5 \times \frac{1}{3} = 3.67$

Variance: $1^2 \times \frac{1}{15} + 2^2 \times \frac{2}{15} + 3^2 \times \frac{1}{5} + 4^2 \times \frac{4}{15} + 5^2 \times \frac{1}{3} - 3.67^2 = 1.56$

Standard deviation: $\sqrt{1.56} = 1.25$

Question 8

a $\frac{1}{12}$

b $P(X=4)+P(X=7)=\frac{1}{6}+\frac{5}{12}=\frac{7}{12}$

c $P(X=7)+P(X=8)+P(X=9)=\frac{5}{12}+\frac{1}{3}+\frac{1}{12}=\frac{5}{6}$

d $P(X=4)+P(X=7)+P(X=8)=\frac{1}{6}+\frac{5}{12}+\frac{1}{3}=\frac{11}{12}$

e $P(X=8)+P(X=9)=\frac{1}{3}+\frac{1}{12}=\frac{5}{12}$

Question 9

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$P(0\text{ tails})=P(X=0)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

$P(1\text{ tail})=P(X=1)=\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{1}{2}$

$P(2\text{ tails})=P(X=2)=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

Question 10

a and **c**

For **a**, all outcomes have the same probability. $P(\text{tails}) = P(\text{heads})$.
So this probability distribution is uniform.

For **b**, getting heads can occur with three combinations, head/head, head/tail, tail/head.

Not all combinations obtained with tossing two coins have the same probability.
So this probability distribution is not uniform.

For **c**, $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$.
So this probability distribution is uniform.

For **d**, $P(6) = \frac{1}{6}$ and $P(\text{not } 6) = \frac{5}{6}$.
So this probability distribution is not uniform.

Question 11

a Discrete

The number of heads can be counted and listed.

b Continuous

The distance can take on a range of values that may not be whole numbers or unique.

c Discrete

The number of correct and incorrect responses can be listed.

d Continuous

The masses of pebbles may take on a range of non-integer values.

Question 12

$E(X) = 0 \times 0.31 + 1 \times 0.22 + 2 \times 0.18 + 3 \times 0.24 + 4 \times 0.05 = 1.5$

$\text{Var}(X) = 0^2 \times 0.31 + 1^2 \times 0.22 + 2^2 \times 0.18 + 3^2 \times 0.24 + 4^2 \times 0.05 - 1.5^2 = 1.65$

Standard deviation $\sqrt{1.65} = 1.28$

Question 13

a

x	1	2	3	4	5	6	7
$P(X=x)$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$

b Yes. Each outcome has the same probability, so it is a uniform distribution.

c **i** $P(X=6) + P(X=7) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}$

ii $P(X=1) + P(X=2) + P(X=3) = \frac{3}{7}$

iii $P(X=4) + P(X=5) + P(X=6) + P(X=7) = \frac{4}{7}$

d $E(X) = 1 \times \frac{1}{7} + 2 \times \frac{1}{7} + 3 \times \frac{1}{7} + 4 \times \frac{1}{7} + 5 \times \frac{1}{7} + 6 \times \frac{1}{7} + 7 \times \frac{1}{7} = 4$

Question 14

a **i** $f(3)=\frac{3-1}{10}=\frac{1}{5}$

ii $f(5)=\frac{5-4}{5}=\frac{1}{5}$

iii $f(9)=\frac{9}{15}=\frac{3}{5}$

b $f(3)+f(5)+f(9)=\frac{1}{5}+\frac{1}{5}+\frac{3}{5}=1$

So it is a probability distribution.

Question 15

a

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$P(0\text{ tails})=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$, $P(1\text{ tail})=\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{1}{2}$, $P(2\text{ tails})=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

b No. This is not a uniform distribution because each outcome is not equally likely.

c **i** $P(1\text{tail})=\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}=\frac{1}{2}$

ii $P(1\text{ tail})+P(2\text{ tails})=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$

Question 16

a Yes

$$0.2+0.07+0.15+0.2+0.3+0.08=1$$

The sum of the probabilities is 1.

b No

$$f(0)+f(2)+f(3)=\frac{1}{6}+\frac{3}{6}+\frac{4}{6}=\frac{8}{6}>1$$

The sum of the probabilities is not 1.

c No

$$\frac{1}{8}+\frac{1}{4}+\frac{1}{2}+\frac{1}{16}+\frac{3}{16}=\frac{18}{16}>1$$

The sum of the probabilities is not 1.

Question 17

a

X	0	1.5	2
***P*(*X* = *x*)**	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

$$E(X)=0\times\frac{3}{4}+1.5\times\frac{1}{8}+2\times\frac{1}{8}=0.44$$

b Jonas loses 56 cents

$$\$1-\$0.44=\$0.56$$

In the long term, the player will lose 56 cents.

Question 18

a 21% + 14% + 47% + 18% = 100% = 1

So discrete probability function.

b $E(X) = 3\times0.21+5\times0.14+6\times0.47+9\times0.18 = 5.77$

$\text{Var}(X) = 3^2\times0.21+5^2\times0.14+6^2\times0.47+9^2\times0.18-5.77^2 = 3.5971$

Question 19

a Sum of probabilities must be 1.

$$\frac{1}{8}+n+\frac{1}{16}+\frac{3}{8}+\frac{5}{16}+\frac{1}{16}=1$$
$$n+\frac{15}{16}=1$$
$$n=\frac{1}{16}$$

b Sum of probabilities must be 1.

$$0.27+0.51+0.14+n=1$$
$$n+0.92=1$$
$$n=0.08$$

c Sum of probabilities must be 1.

$$P(1)+P(2)+P(3)=1$$
$$n+3n+5n=1$$
$$9n=1$$
$$n=\frac{1}{9}$$

Question 20

Sum of probabilities is 1.

$0.2+0.2+0.3+a+b=1$

$a+b=0.3$ [1]

$E(X)=3.8$.

$2\times0.2+3\times0.2+4\times0.3+5a+6b=3.8$

$5a+6b=1.6$ [2]

[2] – 5 × [1]: $b=0.1$

From [1]: $a+0.1=0.3\Rightarrow a=0.2$

Question 21

a

y	\$2	–\$1	\$2
$P(Y=y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b $E(Y)=2\times\frac{1}{4}-1\times\frac{1}{2}+2\times\frac{1}{4}=0.5$

In the long term, the player will lose \$1 – \$0.50 = \$0.50.

Challenge exercise 12

Question 1

Sum of probabilities is 1.

$a+b+c+0.16+0.24=1$

$a+b+c=0.6$ [1]

$E(X)=2.94$

$a+2b+3c+4\times0.16+5\times0.24=2.94$

$a+2b+3c=1.1$ [2]

$\mathrm{Var}(X)=2.2564$

$a+4b+9c+16\times0.16+25\times0.24-2.94^2=2.2564$

$a+4b+9c=2.34$ [3]

[2] – [1]: $b+2c=0.5$ [4]

[3] – [1]: $3b+8c=1.74$ [5]

4 × [4] – [5]: $b=0.26$

From [4]: $0.26+2c=0.5\Rightarrow c=0.12$

From [1]: $a+0.26+0.12=0.6\Rightarrow a=0.22$

Question 2

Sum of probabilities is 1.

$a+0.3+0.4+0.1=1\Rightarrow a=1-0.8=0.2$

$E(X)=2.9$

$a+0.6+0.4k+0.5=2.9$

$\mathrm{Var}(X)=4.3044$

$a+1.2+0.4k^2+2.5-2.9^2=1.89$

$a+0.4k^2=6.6\Rightarrow k=4$

Question 3

Sum of probabilities is 1.

$l + 0.82 = 1 \Rightarrow l = 0.18$

$E(X) = 3.34$

$0.4 + 0.36 + 0.3k + 6 \times 0.18 = 3.34$

$0.3k = 1.5$

$k = 5$

Question 4.

a $0.2 + 0.2 + 0.2 + 0.2 + 0.2$

$= 0.2 \times 5$

$= 1$ so discrete probability distribution

b Yes

Each outcome has the same probability.

c **i** $P(X = 1) + P(X = 2) + P(X = 3) = 3 \times 0.2 = 0.6$

ii $P(X = 3) + P(X = 4) + P(X = 5) = 3 \times 0.2 = 0.6$

iii $P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 4 \times 0.2 = 0.8$

d $E(X) = 1 \times 0.2 + 2 \times 0.2 + 3 \times 0.2 + 4 \times 0.2 + 5 \times 0.2 = 3$

$\text{Var}(X) = 1 \times 0.2 + 4 \times 0.2 + 9 \times 0.2 + 16 \times 0.2 + 25 \times 0.2 - 3^2 = 2$

e $P(X = 1) = 0.35$, $P(X = 3) = P(X = 4) = P(X = 5) = 0.2$

So $0.35 + P(X = 2) + 0.2 + 0.2 + 0.2 = 1$

$P(X = 2) + 0.95 = 1$

$P(X = 2) = 0.05$

Question 5

a $4+15+23+59+19=120$

b

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{30}$	$\frac{1}{8}$	$\frac{23}{120}$	$\frac{59}{120}$	$\frac{19}{120}$

Each probability is the ratio of the frequency and the total number of people.

$P(X=1)=\frac{4}{120}=\frac{1}{30}$, $P(X=2)=\frac{15}{120}=\frac{1}{8}$, and so on.

c No: mean 3.62

$$E(X)=1\times\frac{1}{30}+2\times\frac{1}{8}+3\times\frac{23}{120}+4\times\frac{59}{120}+5\times\frac{19}{120}=3.62$$

The sample mean is 3.62 and the population mean is 2.5.

This is an error of $P(X=2)=\frac{3.62-2.5}{2.5}\times 100\%=45\%$, which is significant.

d Yes, standard deviation 1

$$\text{Var}(X)=1\times\frac{1}{30}+4\times\frac{1}{8}+9\times\frac{23}{120}+16\times\frac{59}{120}+25\times\frac{19}{120}-3.62^2=0.9789$$

Standard deviation is $\sqrt{09789}$ =.99

This is a good estimate of the population estimate of 1.

e Factors such as who participated in the survey and their location can influence the mean.

Since the standard deviation measures the spread from the mean, it can minimise the effect of bias.