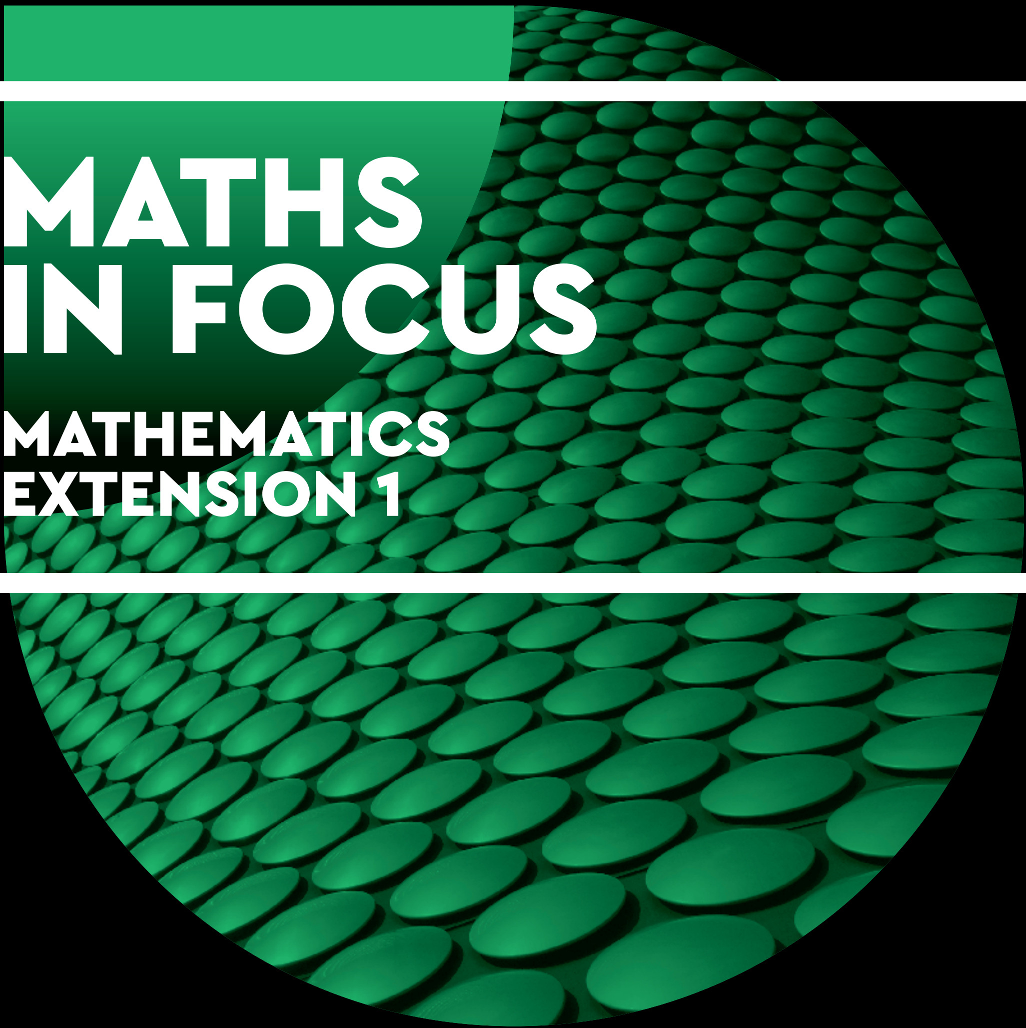
MATHS IN FOCUS

MATHEMATICS EXTENSION 1



YEAR 12

Margaret Grove

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PREFACE

Maths in Focus 12 Mathematics Extension 1 has been rewritten for the new Mathematics Extension 1 syllabus (2017). In this 3rd edition of the book, teachers will find those familiar features that have made Maths in Focus a leading senior mathematics series, such as clear and abundant worked examples in plain English, comprehensive sets of graded exercises, chapter *Test Yourself* and *Challenge* exercises, Investigations, and practice sets of mixed revision and exam-style questions.

The Mathematics Extension 1 course is designed for students who intend to study mathematics at university, possibly majoring in the subject.

This book covers the Year 12 content of the course, which includes the Year 12 Mathematics Advanced course. The specific Mathematics Extension 1 content is labelled EXT1. The theory follows a logical order, although some topics may be learned in any order. We have endeavoured to produce a practical text that captures the spirit of the course, providing relevant and meaningful applications of mathematics.

The *NelsonNet* student and teacher websites contain additional resources such as worksheets, video tutorials and topic tests. We wish all teachers and students using this book every success in embracing the new senior mathematics course.

ABOUT THE AUTHOR

Margaret Grove has spent over 30 years teaching HSC Mathematics, most recently at Bankstown TAFE College. She has written numerous senior mathematics texts and study guides over the past 25 years, including the bestselling *Maths in Focus* series for Mathematics and Mathematics Extension 1.

Margaret thanks her family, especially her husband Geoff, for their support in writing this book.

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Gaspare Carrozza and **Haroon Ha** from Homebush Boys High School wrote many of the *NelsonNet* worksheets.

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Roger Walter wrote the *ExamView* questions.

Shane Scott, **Brandon Pettis** and **George Dimitriadis** wrote the worked solutions to all exercise sets.

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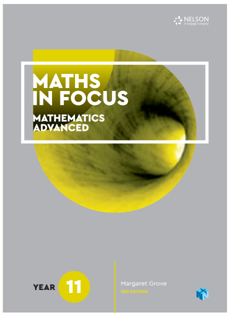
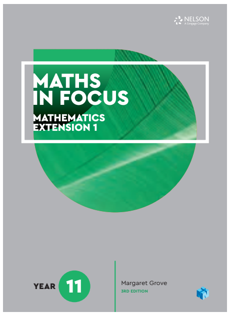
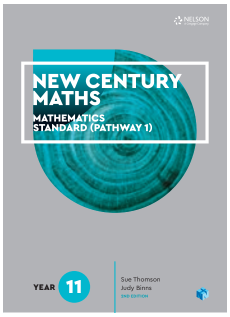
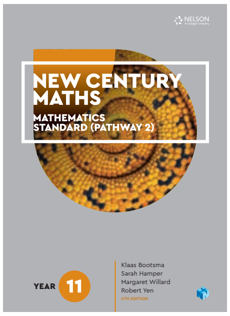
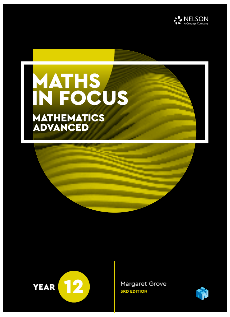
MATHS IN FOCUS AND NEW CENTURY MATHS 11–12

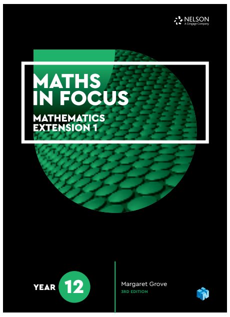


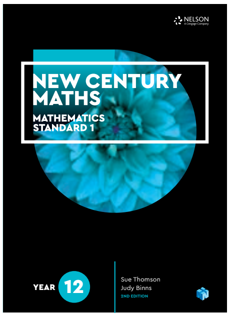


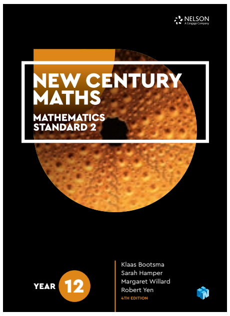


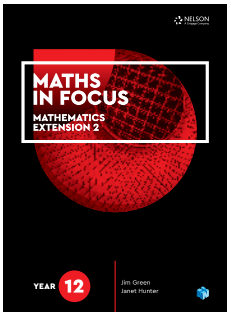










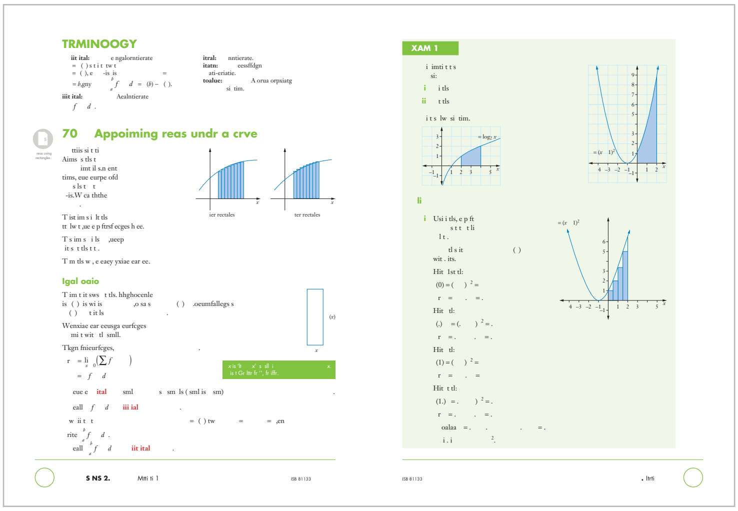


ABOUT THIS BOOK

AT THE BEGINNING OF EACH CHAPTER

- Each chapter begins on a double-page spread showing the **Chapter contents** and a list of chapter outcomes

- **Terminology** is a chapter glossary that previews the key words and phrases from within the chapter



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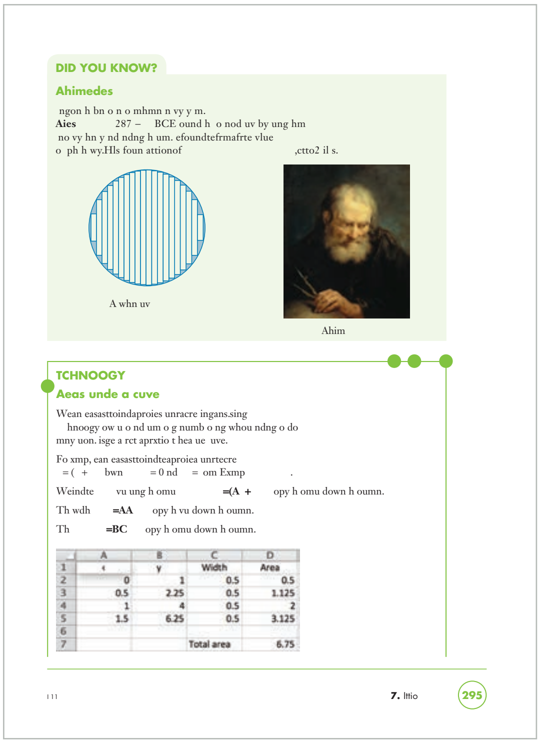
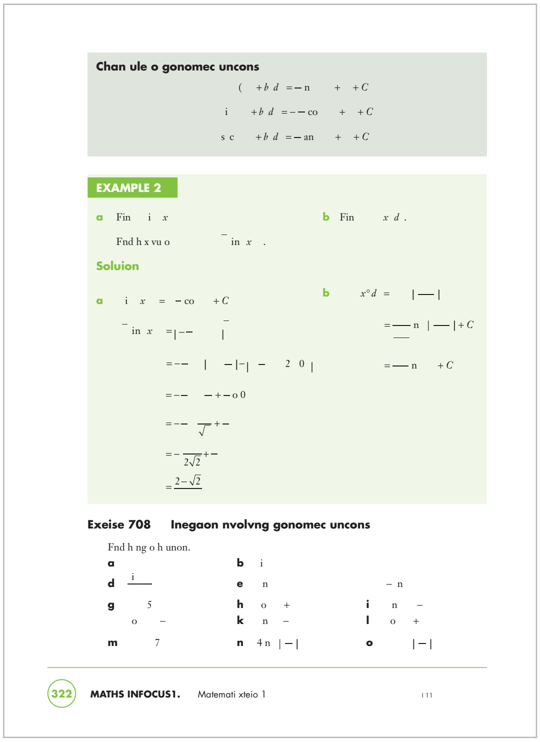
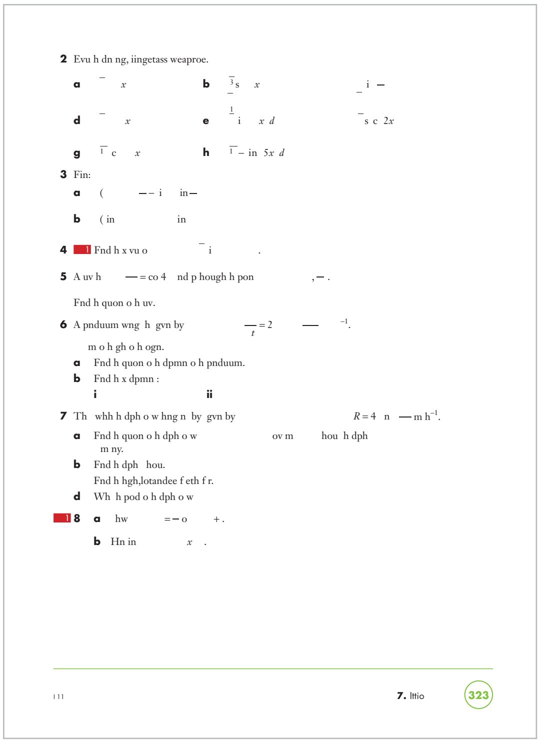
IN EACH CHAPTER

- Important facts and formulas are highlighted in a shaded box.
- Important words and phrases are printed in red and listed in the Terminology chapter glossary.
- The specific Mathematics Extension 1 content is labelled EXT1.
- Graded exercises include exam-style problems and realistic applications.
- Worked solutions to all exercise questions are provided on the *NelsonNet* teacher website.
- **Investigations** explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.
- **Did you know?** contains interesting facts and applications of the mathematics learned in the chapter.







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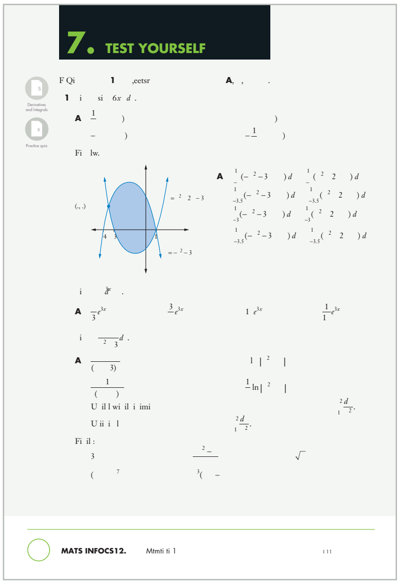
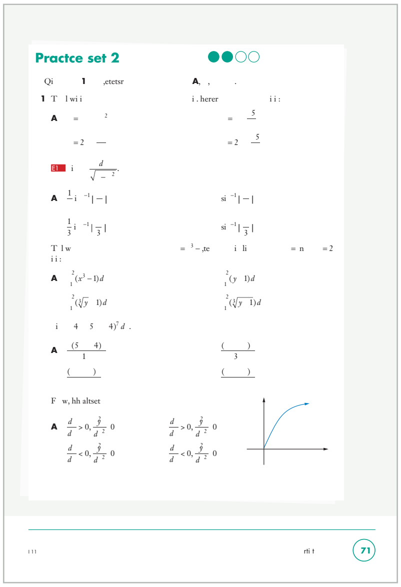
AT THE END OF EACH CHAPTER

- **Test Yourself** contains chapter revision exercises.
- If you have trouble completing the *Test Yourself* exercises, you need to go back and revise the chapter before trying the exercises again.
- **Challenge Exercise** contains chapter extension questions. Attempt these only after you are confident with the *Test Yourself* exercises, because these are more difficult and are designed for students who understand the topic really well.
- **Practice sets** (after several chapters) provide a comprehensive variety of mixed exam-style questions from various chapters, including short-answer, free-response and multiple-choice questions.



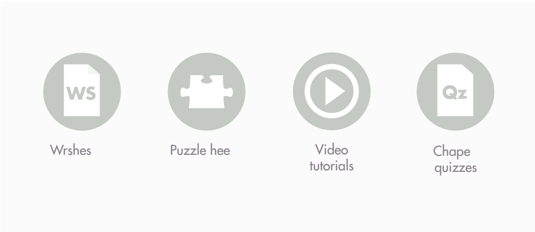
AT THE END OF THE BOOK

- **Answers** and **Index** (worked solutions on the teacher website).



NELSONNET STUDENT WEBSITE

Margin icons link to print (PDF) and multimedia resources found on the *NelsonNet* student website, **www.nelsonnet.com.au**. These include:



- **Worksheets** and **puzzle sheets** that are write-in enabled PDFs
- **Video tutorials:** worked examples explained by ‘flipped classroom’ teachers
- ***ExamView* quizzes:** interactive and self-marking

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NELSONNET TEACHER WEBSITE

The *NelsonNet* teacher website, also at **www.nelsonnet.com.au**, contains:

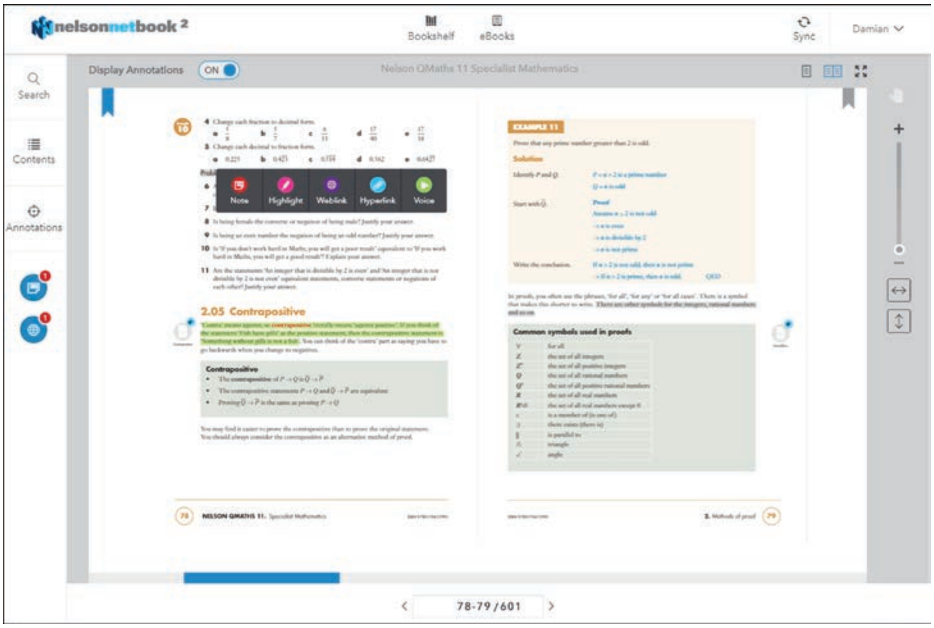
- A **teaching program**, in Microsoft Word and PDF formats
- **Topic tests**, in Microsoft Word and PDF formats
- **Worked solutions** to each exercise set
- **Chapter PDFs** of the textbook
- ***ExamView*** exam-writing software and questionbanks
- **Resource finder:** search engine for *NelsonNet* resources

Note: Complimentary access to these resources is only available to teachers who use this book as a core educational resource in their classroom. Contact your Cengage Education Consultant for information about access codes and conditions.

NELSONNETBOOK

NelsonNetBook is the web-based interactive version of this book found on *NelsonNet*.

- To each page of NelsonNetBook you can add notes, voice and sound bites, highlighting, weblinks and bookmarks
- **Zoom** and **Search** functions
- Chapters can be customised for different groups of students.



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STUDY SKILLS

The Year 11 course introduces the basics of topics such as calculus that are then applied in the Year 12 course. You will struggle in the HSC if you don't set yourself up to revise the Year 11 topics as you learn new Year 12 topics. Your teachers will be able to help you build up and manage good study habits. Here are a few hints to get you started. There is no right or wrong way to learn. Different styles of learning suit different people. There is also no magical number of hours a week that you should study, as this will be different for every student. But just listening in class and taking notes is not enough, especially when learning material that is totally new.

If a skill is not practised within the first 24 hours, up to 50% can be forgotten. If it is not practised within 72 hours, up to 85–90% can be forgotten! So it is really important that, whatever your study timetable, new work must be looked at soon after it is presented to you.

With a continual succession of new work to learn and retain, this is a challenge. But the good news is that you don't have to study for hours on end!

IN THE CLASSROOM

In order to remember, first you need to focus on what is being said and done.

According to an ancient proverb:

I hear and I forget
I see and I remember
I do and I understand.

If you chat to friends and just take notes without really paying attention, you aren't giving yourself a chance to remember anything and will have to study harder at home.

If you are unsure of something that the teacher has said, the chances are that others are also not sure. Asking questions and clarifying things will ultimately help you gain better results, especially in a subject like mathematics where much of the knowledge and skills depends on being able to understand the basics.

Learning is all about knowing what you know and what you don't know. Many students feel like they don't know anything, but it's surprising just how much they know already. Picking up the main concepts in class and not worrying too much about other less important parts can really help. The teacher can guide you on this.

Here are some pointers to get the best out of classroom learning:

- Take control and be responsible for your own learning
- Clear your head of other issues in the classroom
- Active, not passive, learning is more memorable
- Ask questions if you don't understand something

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- Listen for cues from the teacher
- Look out for what are the main concepts.

Note-taking varies from class to class, but here are some general guidelines:

- Write legibly
- Use different colours to highlight important points or formulas
- Make notes in textbooks (using pencil if you don't own the textbook)
- Use highlighter pens to point out important points
- Summarise the main points
- If notes are scribbled, rewrite them at home.

AT HOME

You are responsible for your own learning and nobody else can tell you how best to study. Some people need more revision time than others, some study better in the mornings while others do better at night, and some can work at home while others prefer a library.

- Revise both new and older topics regularly
- Have a realistic timetable and be flexible
- Summarise the main points
- Revise when you are fresh and energetic
- Divide study time into smaller rather than longer chunks
- Study in a quiet environment
- Have a balanced life and don't forget to have fun!

If you are given exercises out of a textbook to do for homework, consider asking the teacher if you can leave some of them till later and use these for revision. It is not necessary to do every exercise at one sitting, and you learn better if you can spread these over time.

People use different learning styles to help them study. The more variety the better, and you will find some that help you more than others. Some people (around 35%) learn best visually, some (25%) learn best by hearing and others (40%) learn by doing.

- Summarise on cue cards or in a small notebook
- Use colourful posters
- Use mind maps and diagrams
- Discuss work with a group of friends
- Read notes out aloud
- Make up songs and rhymes
- Do exercises regularly
- Role-play teaching someone else

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ASSESSMENT TASKS AND EXAMS

You will cope better in exams if you have practised doing sample exams under exam conditions. Regular revision will give you confidence, and if you feel well prepared this will help get rid of nerves in the exam. You will also cope better if you have had a reasonable night's sleep before the exam.

One of the biggest problems students have with exams is in timing. Make sure you don't spend too much time on questions you're unsure about, but work through and find questions you can do first.

Divide the time up into smaller chunks for each question and allow some extra time to go back to questions you couldn't do or finish. For example, in a 3-hour exam with 50 questions, allow around 3 minutes for each question. This will give an extra half hour at the end to tidy up and finish off questions. Alternatively, in a 3-hour exam with questions worth a total of 100 marks, allow around 1.5 minutes per mark.

- Read through and ensure you know how many questions there are
- Divide your time between questions with extra time at the end
- Don't spend too much time on one question
- Read each question carefully, underlining key words
- Show all working out, including diagrams and formulas
- Cross out mistakes with a single line so it can still be read
- Write legibly

AND FINALLY…

Study involves knowing what you don't know, and putting in a lot of time into concentrating on these areas. This is a positive way to learn. Rather than just saying 'I can't do this', say instead 'I can't do this *yet*', and use your teachers, friends, textbooks and other ways of finding out.

With the parts of the course that you do know, make sure you can remember these easily under exam pressure by putting in lots of practice.

Remember to look at new work

today, tomorrow, in a week, in a month.

Some people hardly ever find time to study while others give up their outside lives to devote their time to study. The ideal situation is to balance study with other aspects of your life, including going out with friends, working and keeping up with sport and other activities that you enjoy.

Good luck with your studies!

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MATHEMATICAL VERBS

A glossary of 'doing words' commonly found in mathematics problems

analyse: study in detail the parts of a situation

apply: use knowledge or a procedure in a given situation

classify, identify: state the type, name or feature of an item or situation

comment: express an observation or opinion about a result

compare: show how two or more things are similar or different

construct: draw an accurate diagram

describe: state the features of a situation

estimate: make an educated guess for a number, measurement or solution, to find roughly or approximately

evaluate, calculate: find the value of a numerical expression, for example, 3×8^2 or $4x + 1$ when $x = 5$

expand: remove brackets in an algebraic expression for exampl, expanding $3(2 \quad y + 1)$ gives $6y + 3$

explain: describe why or how

factorise: opposite to **expand**, to insert brackets by taking out a common factor, for example, factorising $6y + 3$ gives $3(2y + 1)$

give reasons: show the rules or thinking used when solving a problem. See also **justify**

increase: make larger

interpret: find meaning in a mathematical result

justify: give reasons or evidence to support your argument or conclusion. See also **give reasons**

rationalise: make rational, remove surds

show that, prove: (in questions where the answer is given) use calculation, procedure or reasoning to prove that an answer or result is true

simplify: give a result in its most basic, shortest, neatest form, for example, simplifying a ratio or algebraic expression

sketch: draw a rough diagram that shows the general shape or ideas, less accurate than **construct**

solve: find the value(s) of an unknown pronumeral in an equation or inequality

substitute: replace a variable by a number and evaluate

verify: check that a solution or result is correct, usually by substituting back into the equation or referring back to the problem

write, state: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

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FINANCIAL MATHEMATICS

1.

SEQUENCES AND SERIES

A sequence is a set of numbers that form a pattern. Many sequences occur in real life – the growth of plants, savings in the bank, populations, clearing of forests and so on. In the Year 11 course you looked at how things can grow or decay (decrease) exponentially. In this chapter you will look at two other types of patterns that apply to real-life applications.

You will also learn about proving by mathematical induction in this chapter.

CHAPTER OUTLINE

1.01 General sequences and series
1.02 Arithmetic sequences
1.03 Arithmetic series
1.04 Geometric sequences
1.05 Geometric series
1.06 Limiting sum of an infinite geometric series
1.07 EXT1 Proof by mathematical induction

IN THIS CHAPTER YOU WILL:

- identify the difference between a sequence and a series
- identify the difference between arithmetic and geometric sequences and series
- find the nth term of arithmetic and geometric sequences
- find the sum to n terms of arithmetic and geometric series
- understand and apply the limiting sum formula for infinite geometric series
- EXT1 understand and apply the proof by mathematical induction

TERMINOLOGY

arithmetic sequence: A list of numbers where the difference between successive terms is a constant (called the common difference)
arithmetic series: A sum of the terms forming an arithmetic sequence
common difference: The constant difference between successive terms of an arithmetic sequence
common ratio: The constant multiplier of successive terms in a geometric sequence
geometric sequence: A list of numbers where the ratio of successive terms is a constant (called the common ratio)
geometric series: A sum of the terms forming a geometric sequence
limiting sum: The limit, where it exists, of a geometric series as $n \to \infty$
EXT1 **mathematical induction:** A method of proving a statement true for positive integers n. Also called inductive proof, this proof relies on proving a statement true for a specific value of n then generalising to all cases
recurrence relation: An equation that defines a term of a sequence or series by referring to its previous term(s)
sequence: A list of numbers where each term of the sequence is related to the previous term by a particular pattern
series: The sum of terms of a sequence of numbers
term: A value of a sequence

1.01 General sequences and series

WS Sequences and series

WS Classifying sequences

Sequences

A **sequence** is an ordered list of numbers, called **terms** of the sequence, which follow a pattern. Some patterns are easy to see and some are more difficult to find.

EXAMPLE 1

Find the next 3 terms in the sequence:

a 14, 17, 20, … **b** 5, 10, 20, 40, … **c** 5, 1, –3, …

Solution

a For the sequence 14, 17, 20, … we add 3 to each term for the next term.

$14 + 3 = 17$ and $17 + 3 = 20$.

Following this pattern, the next 3 terms are 23, 26 and 29.

b For 5, 10, 20, 40, … we multiply each term by 2 for the next term.

$5 \times 2 = 10$, $10 \times 2 = 20$, $20 \times 2 = 40$.

So the next 3 terms are 80, 160 and 320.

c For the sequence 5, 1, –3, … we subtract 4 (or add –4) to each term for the next term.

$5 - 4 = 1$ and $1 - 4 = -3$.

So the next 3 terms are –7, –11 and –15.

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Series

A **series** is a sum of terms that form a sequence.

EXAMPLE 2

Find the sum of the series with 5 terms:

a $8 + 15 + 22 + \ldots$ **b** $4 + 8 + 16 + \ldots$

Solution

a We add 7 to each term in the series $8 + 15 + 22 + \ldots$ to find the next term.

So the series with 5 terms is $8 + 15 + 22 + 29 + 36$.

$$\begin{aligned}\text{Sum} &= 8 + 15 + 22 + 29 + 36 \\ &= 110\end{aligned}$$

b We multiply each term in the series $4 + 8 + 16 + \ldots$ by 2 to find the next term.

So the series with 5 terms is $4 + 8 + 16 + 32 + 64$.

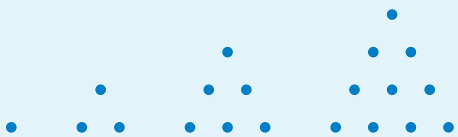
$$\begin{aligned}\text{Sum} &= 4 + 8 + 16 + 32 + 64 \\ &= 124\end{aligned}$$

DID YOU KNOW?

Polygonal numbers

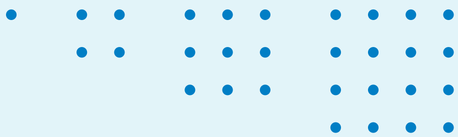
Around 500 BC the Pythagoreans explored different polygonal numbers:

Triangular numbers: $1 + 2 + 3 + 4 + \ldots$



1    $1 + 2$ $1 + 2 + 3$ $1 + 2 + 3 + 4$

Square numbers: $1 + 3 + 5 + 7 + \ldots$



1    $1 + 3$ $1 + 3 + 5$ $1 + 3 + 5 + 7$

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Exercise 1.01 General sequences and series

1 Find the next 3 terms in each sequence.

a 5, 8, 11, ...

b 8, 13, 18, ...

c 11, 22, 33, …

d 100, 95, 90, ...

e 7, 5, 3, …

f 12, 3, –6, ...

g $\frac{1}{2}, 1, 1\frac{1}{2}, \ldots$

h 1.3, 1.9, 2.5, …

i 2, – 4, 8, –16, ...

j $\frac{1}{5}, \frac{3}{20}, \frac{9}{80}, \ldots$

2 Find the sum of each series if it has 6 terms.

a 4 + 12 + 36 + …

b 1 + 2 + 4 + …

c 3 + 7 + 11 + …

d –6 + 12 – 24 + …

e 1 + 4 + 9 + 16 + …

f 1 + 8 + 27 + 64 + ...

3 Find the next 3 terms of the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

4 Find the next 4 terms in the series 3 + 6 + 11 + 18 + 27 + …

5 What are the next 5 terms in the sequence 1, 1, 2, 3, 5, 8, 13, ...?

6 Complete the next 3 rows in Pascal's triangle:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

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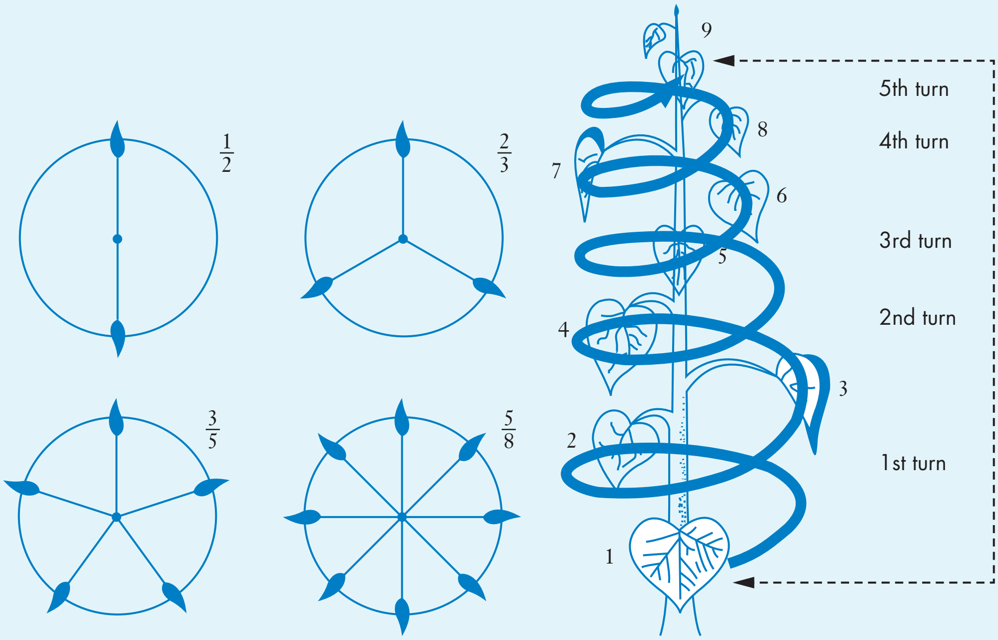
DID YOU KNOW?

Fibonacci numbers

The numbers 1, 1, 2, 3, 5, 8, … are called **Fibonacci** numbers after **Leonardo Fibonacci** (1170–1250). These numbers occur in many natural situations.

For example, when new leaves grow on a plant's stem, they spiral around the stem. The ratio of the number of turns to the number of spaces between successive leaves gives the sequence of fractions $\frac{1}{1}\ \frac{1}{2}\ \frac{2}{3}\ \frac{3}{5}\ \frac{5}{8}\ \frac{8}{13}\ \frac{13}{21}\ \frac{21}{34}\ \frac{34}{55}$



The Fibonacci ratio is the number of turns divided by the number of spaces.

Research Fibonacci numbers and find out where else they appear in nature.

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Arithmetic sequences

Arithmetic progressions

1.02 Arithmetic sequences

In an **arithmetic sequence**, each term is a constant amount more than the previous term. The constant is called the **common difference**, d.

EXAMPLE 3

Find the common difference of the arithmetic sequence:

a 5, 9, 13, 17, … **b** 85, 80, 75, …

Solution

a For this sequence, $9 - 5 = 4$, $13 - 9 = 4$ and $17 - 13 = 4$.

So common difference $d = 4$.

b For this sequence, $80 - 85 = -5$ and $75 - 80 = -5$.

So common difference $d = -5$.

A **recurrence relation** is an equation that defines a term of a sequence by referring to its previous term. In any arithmetic sequence, a term is d more than the previous term. We can write this as a recurrence relation:

$$T_n = T_{n-1} + d \text{ where } T_n \text{ is the } n\text{th term of the sequence}$$

or $T_n - T_{n-1} = d$

EXAMPLE 4

a If 5, x, 31, … is an arithmetic sequence, find x.

b **i** Evaluate k if $k + 2$, $3k + 2$, $6k - 1$, … is an arithmetic sequence.

ii Write down the first 3 terms of the sequence.

iii Find the common difference d.

Solution

a For an arithmetic sequence,

$T_2 - T_1 = d$ and $T_3 - T_2 = d$

So $T_2 - T_1 = T_3 - T_2$

$$\begin{aligned} x - 5 &= 31 - x \\ 2x - 5 &= 31 \\ 2x &= 36 \\ x &= 18 \end{aligned}$$

Note: x is called the **arithmetic mean** because $x = \dfrac{5+31}{2}$.

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b **i** For an arithmetic sequence,

$$T_2 - T_1 = T_3 - T_2$$
$$(3k + 2) - (k + 2) = (6k - 1) - (3k + 2)$$
$$3k + 2 - k - 2 = 6k - 1 - 3k - 2$$
$$2k = 3k - 3$$
$$2k + 3 = 3k$$
$$3 = k$$

ii Substituting $k = 3$ into the terms of the sequence:

$$T_1 = k + 2 = 3 + 2 = 5 \qquad T_2 = 3k + 2 = 3(3) + 2 = 11 \qquad T_3 = 6k - 1 = 6(3) - 1 = 17$$

iii The sequence is 5, 11, 17, …

$$d = 11 - 5 \text{ or } 17 - 11 = 6$$

So $d = 6$.

The general term of an arithmetic sequence

Given an arithmetic sequence with 1st term $T_1 = a$ and common difference d:

$$T_1 = a \qquad T_2 = T_1 + d = a + d \qquad T_3 = T_2 + d = (a + d) + d = a + 2d \qquad T_4 = T_3 + d = (a + 2d) + d = a + 3d$$

Notice that the multiple of d is one less than the number of the term. So the multiple of d for the nth term T_n is $n - 1$.

*n*th term of an arithmetic sequence

$$T_n = a + (n - 1)d$$

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EXAMPLE 5

a Find the 20th term of the sequence 3, 10, 17, …

b Find a formula for the nth term of the sequence 2, 8, 14, …

c Find the first positive term of the sequence –50, –47, –44, …

Solution

a $a = 3, d = 7, n = 20$

$$T_n = a + (n - 1)d$$
$$T_{20} = 3 + (20 - 1) \times 7$$
$$= 3 + 19 \times 7$$
$$= 136$$

b $a = 2, d = 6$

$$T_n = a + (n - 1)d$$
$$= 2 + (n - 1) \times 6$$
$$= 2 + 6n - 6$$
$$= 6n - 4$$

c $a = -50, d = 3$

For the first positive term,

$$T_n > 0$$
$$a + (n - 1)d > 0$$
$$-50 + (n - 1) \times 3 > 0$$
$$-50 + 3n - 3 > 0$$
$$3n - 53 > 0$$
$$3n > 53$$
$$n > 17.66...$$

So $n = 18$ gives the first positive term.

$$T_{18} = -50 + (18 - 1) \times 3$$
$$= 1$$

So the first positive term is 1.

EXAMPLE 6

The 5th term of an arithmetic sequence is 37 and the 8th term is 55. Find the common difference and the first term of the sequence.

Solution

$T_n = a + (n - 1)d$

Given $T_5 = 37$:

$a + (5 - 1)d = 37$

$a + 4d = 37$ [1]

Given $T_8 = 55$:

$a + (8 - 1)d = 55$

$a + 7d = 55$ [2]

Solve [1] and [2] simultaneously:

$3d = 18$ [2] – [1]

$d = 6$

Substitute $d = 6$ into [1]:

$a + 4(6) = 37$

$a + 24 = 37$

$a = 13$

So the common difference is 6 and the first term is 13.

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Exercise 1.02 Arithmetic sequences

1 Find the value of the pronumeral in each arithmetic sequence.

a $5, 9, y, \ldots$ **b** $8, 2, x \ldots$ **c** $45, x, 99, \ldots$

d $16, b, 6, \ldots$ **e** $x, 14, 21, \ldots$ **f** $32, x - 1, 50, \ldots$

g $3, 5k + 2, 21, \ldots$ **h** $x, x + 3, 2x + 5, \ldots$ **i** $t - 5, 3t, 3t + 1, \ldots$

j $2t - 3, 3t + 1, 5t + 2, \ldots$

2 Find the 15th term of each sequence.

a $4, 7, 10, \ldots$ **b** $8, 13, 18, \ldots$ **c** $10, 16, 22, \ldots$

d $120, 111, 102, \ldots$ **e** $-3, 2, 7, \ldots$

3 Find the 100th term of each sequence.

a $-4, 2, 8, \ldots$ **b** $41, 32, 23, \ldots$ **c** $18, 22, 26, \ldots$

d $125, 140, 155, \ldots$ **e** $-1, -5, -9, \ldots$

4 What is the 25th term of each sequence?

a $-14, -18, -22, \ldots$ **b** $0.4, 0.9, 1.4, \ldots$ **c** $1.3, 0.9, 0.5, \ldots$

d $1, 2\frac{1}{2}, 4, \ldots$ **e** $1\frac{2}{5}, 2, 2\frac{3}{5}, \ldots$

5 Find the formula for the nth term of the sequence 3, 5, 7, …

6 Find the formula for the nth term of each sequence.

a $9, 17, 25, \ldots$ **b** $100, 102, 104, \ldots$ **c** $6, 9, 12, \ldots$

d $80, 86, 92, \ldots$ **e** $-21, -17, -13, \ldots$ **f** $15, 10, 5, \ldots$

g $\frac{7}{8}, 1, 1\frac{1}{8}, \ldots$ **h** $-30, -32, -34, \ldots$ **i** $3.2, 4.4, 5.6, \ldots$

j $\frac{1}{2}, 1\frac{1}{4}, 2, \ldots$

7 Find which term of 3, 7, 11, … is equal to 111.

8 Which term of the sequence 1, 5, 9, … is 213?

9 Which term of the sequence 15, 24, 33, … is 276?

10 Which term of the sequence 25, 18, 11, … is equal to −73?

11 Is 0 a term of the sequence 48, 45, 42, …?

12 Is 270 a term of the sequence 3, 11, 19, …?

13 Is 405 a term of the sequence 0, 3, 6, …?

14 Find the first value of n for which the terms of the sequence 100, 93, 86, … is less than 20.

15 Find the values of n for which the terms of the sequence −86, −83, −80, … are positive.

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16 Find the first negative term of the sequence 54, 50, 46, …

17 Find the first term that is greater than 100 in the sequence 3, 7, 11, …

18 The first term of an arithmetic sequence is −7 and the common difference is 8. Find the 100th term.

19 The first term of an arithmetic sequence is 15 and the 3rd term is 31.

a Find the common difference.

b Find the 10th term of the sequence.

20 The first term of an arithmetic sequence is 3 and the 5th term is 39. Find the common difference.

21 The 2nd term of an arithmetic sequence is 19 and the 7th term is 54. Find the first term and common difference.

22 Find the 20th term in an arithmetic sequence with 4th term 29 and 10th term 83.

23 The common difference of an arithmetic sequence is 6 and the 5th term is 29. Find the first term of the sequence.

24 If the 3rd term of an arithmetic sequence is 45 and the 9th term is 75, find the 50th term of the sequence.

25 The 7th term of an arithmetic sequence is 17 and the 10th term is 53. Find the 100th term of the sequence.

26 **a** Show that $\log_5 x, \log_5 x^2, \log_5 x^3, \ldots$ is an arithmetic sequence.

b Find the 80th term.

c If $x = 4$, evaluate the 10th term correct to 1 decimal place.

27 **a** Show that $\sqrt{3}, \sqrt{12}, \sqrt{27}, \ldots$ is an arithmetic sequence.

b Find the 50th term in simplest form.

28 Find the 25th term of $\log_2 4, \log_2 8, \log_2 16, \ldots$

29 Find the 40th term of $5b, 8b, 11b, \ldots$

30 Which term is $213y$ of the sequence $28y, 33y, 38y, \ldots$?

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1.03 Arithmetic series

The sum of an **arithmetic series** with n terms is given by the formula:

WS Arithmetic series

WS Arithmetic series

Aihmeic series

WS Arithmetic series assignment

WS Arithmetic series extension

Sum of an arithmetic series with *n* terms

$S_n = \frac{n}{2}(a + l)$ where a = 1st term and l = last (nth) term

Proof

Let the last or nth term be l.

$$S_n = a + (a + d) + (a + 2d) + \ldots + l \qquad [1]$$

Writing this around the other way:

$$S_n = l + (l - d) + (l - 2d) + \ldots. + a \qquad [2]$$

[1] + [2]

$$2S_n = (a + l) + (a + l) + (a + l) + \ldots. + (a + l) \ n \text{ times}$$

$$= n(a + l)$$

$$S_n = \frac{n}{2}(a + l)$$

We can find a more general formula if we substitute $T_n = a + (n - 1)d$ for l:

Sum of an arithmetic series with *n* terms

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Proof

$$S_n = \frac{n}{2}(a + l)$$

$$= \frac{n}{2}[a + a + (n - 1)d]$$

$$= \frac{n}{2}[2a + (n - 1)d]$$

We can also use these formulas to find the sum of the first n terms of an arithmetic sequence (also called the ***n*****th partial sum**).

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EXAMPLE 7

a Evaluate 9 + 14 + 19 + … + 224.

b For what value of n is the sum of the series 2 + 11 + 20 + … equal to 618?

c The 6th term of an arithmetic sequence is 23 and the sum of the first 10 terms is 210. Find the sum of the first 20 terms of the sequence.

Solution

a $a = 9, d = 5, T_n = 224$

$$T_n = a + (n-1)d$$
$$224 = 9 + (n-1) \times 5$$
$$= 9 + 5n - 5$$
$$= 5n + 4$$
$$220 = 5n$$
$$44 = n$$

$$S_n = \frac{n}{2}(a + l)$$
$$= \frac{44}{2}(9 + 224)$$
$$= 5126$$

b $a = 2, d = 9, S_n = 618$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$618 = \frac{n}{2}[2 \times 2 + (n-1) \times 9]$$
$$1236 = n(4 + 9n - 9)$$
$$= n(9n - 5)$$
$$= 9n^2 - 5n$$

$$0 = 9n^2 - 5n - 1236$$
$$= (n - 12)(9n + 103)$$

(or use the quadratic formula)

$$n - 12 = 0, 9n + 103 = 0$$
$$n = 12$$

($9n + 103 = 0$ gives a negative value of n)

c

$$T_n = a + (n-1)d$$
$$T_6 = a + (6-1)d = 23$$
$$a + 5d = 23 \quad [1]$$
$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$S_{10} = \frac{10}{2}[2a + (10-1)d] = 210$$
$$5(2a + 9d) = 210$$
$$2a + 9d = 42 \quad [2]$$

[1] × 2: $2a + 10d = 46$ [3]

[3] − [2]: $d = 4$

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Substitute $d = 4$ in [1]

$$a + 5(4) = 23$$

$$a + 20 = 23$$

$$a = 3$$

Substitute $a = 3$, $d = 2$, $n = 20$ into the formula for S_n.

$$S_{20} = \frac{20}{2}[2(3) + (20 - 1)4]$$

$$= 10(6 + 19 \times 4)$$

$$= 820$$

Exercise 1.03 Arithmetic series

1 Find the sum of 15 terms of each series.

a $4 + 7 + 10 + \ldots$ **b** $2 + 7 + 12 + \ldots$ **c** $60 + 56 + 52 + \ldots$

2 Find the sum of 30 terms of each series.

a $1 + 7 + 13 + \ldots$ **b** $15 + 24 + 33 + \ldots$ **c** $95 + 89 + 83 + \ldots$

3 Find the sum of 25 terms of each series.

a $-2 + 5 + 12 + \ldots$ **b** $5 - 4 - 13 - \ldots$

4 Find the sum of 50 terms of each series.

a $50 + 44 + 38 + \ldots$ **b** $11 + 14 + 17 + \ldots$

5 Evaluate each arithmetic series.

a $15 + 20 + 25 + \ldots + 535$ **b** $9 + 17 + 25 + \ldots + 225$

c $5 + 2 - 1 - \ldots - 91$ **d** $81 + 92 + 103 + \ldots + 378$

e $229 + 225 + 221 + \ldots + 25$ **f** $-2 + 6 + 14 + \ldots + 94$

g $0 - 9 - 18 - \ldots - 216$ **h** $79 + 81 + 83 + \ldots + 229$

i $14 + 11 + 8 + \ldots - 43$ **j** $1\frac{1}{2} + 1\frac{3}{4} + 2 + \ldots + 25\frac{1}{4}$

6 How many terms of the series $45 + 47 + 49 + \ldots$ are needed to give a sum of 1365?

7 For what value of n is the sum of the arithmetic series $5 + 9 + 13 + \ldots$ equal to 152?

8 How many terms of the series $80 + 73 + 66 + \ldots$ are needed to give a sum of 495?

9 How many terms of the series $20 + 18 + 16 + \ldots$ are needed to give a sum of 104?

10 The sum of the first 5 terms of an arithmetic sequence is 110 and the sum of the first 10 terms is 320. Find the first term and the common difference.

11 The sum of the first 5 terms of an arithmetic sequence is 35 and the sum of the next 5 terms is 160. Find the first term and the common difference.

12 Find S_{25}, given an arithmetic series with 8th term 16 and 13th term 81.

13 The sum of 12 terms of an arithmetic series is 186 and the 20th term is 83. Find the sum of 40 terms of the series.

14 The sum of the first 4 terms of an arithmetic series is 42 and the sum of the 3rd and 7th term is 46. Find the sum of the first 20 terms.

15 **a** Show that $x + 1, 2x + 4, 3x + 7, \ldots$ are the first 3 terms in an arithmetic sequence.

b Find the sum of the first 50 terms of the sequence.

16 The 20th term of an arithmetic series is 131 and the sum of the 6th to 10th terms inclusive is 235. Find the sum of the first 20 terms.

17 The sum of 50 terms of an arithmetic series is 249 and the sum of 49 terms of the series is 233. Find the 50th term of the series.

18 Prove that $T_n = S_n - S_{n-1}$ for any arithmetic sequence.

19 **a** Find the sum of all integers from 1 to 100 that are multiples of 6.

b Find the sum of all integers from 1 to 100 that are not multiples of 6.

1.04 Geometric sequences

WS Geometric sequences

WS Geometric progressions 1

In a **geometric sequence**, each term is formed by multiplying the previous term by a constant. The constant is called the **common ratio r**.

EXAMPLE 8

Find the common ratio of the geometric sequence:

a $3, 6, 12, \ldots$ **b** $-2, 10, -50, \ldots$ **c** $\frac{1}{2}, \frac{1}{5}, \frac{2}{25}, \ldots$

Solution

a For this sequence, $6 \div 3 = 2$, $12 \div 6 = 2$

So common ratio $r = 2$.

b For this sequence, $10 \div -2 = -5$, $-50 \div 10 = -5$

So common ratio $r = -5$.

c $\frac{1}{5} \div \frac{1}{2} = \frac{2}{5}$ $\frac{2}{25} \div \frac{1}{5} = \frac{2}{5}$

So common ratio $r = \frac{2}{5}$.

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In any geometric sequence, a term is r times more than the previous term. We can write this as a recurrence relation:

$$T_n = rT_{n-1}$$

or $$\frac{T_n}{T_{n-1}} = r$$

EXAMPLE 9

a **i** Find x if $5, x, 45, \ldots$ is a geometric sequence.

ii Find the sequence.

b Is $\frac{1}{4}\ \frac{1}{6}\ \frac{1}{18}\ \ldots$ a geometric sequence?

Solution

a **i** For a geometric sequence:

$$\frac{T_2}{T} = r \text{ and } \frac{T_3}{T_2} = r$$

So $$\frac{T_2}{T} = \frac{T_3}{T_2}$$

$$\frac{x}{5} = \frac{45}{x}$$

$$x^2 = 225$$

$$x = \pm\sqrt{225}$$

$$= \pm 15$$

Note: x is called the **geometric mean** because $x = \sqrt{5 \times 45}$.

ii If $x = 15$ the sequence is $5, 15, 45, \ldots$ $(r = 3)$

If $x = -15$ the sequence is $5, -15, 45, \ldots$ $(r = -3)$

b $$\frac{T_2}{T} = \frac{1}{6} \div \frac{1}{4} = \frac{2}{3}$$

$$\frac{T_3}{T_2} = \frac{1}{18} \div \frac{1}{6} = \frac{1}{3}$$

$\frac{T_2}{T} \neq \frac{T_3}{T_2}$ so the series is not geometric.

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General term of a geometric sequence

Given a geometric sequence with 1st term $T_1 = a$ and common ratio r:

$$T_1 = a \qquad T_3 = T_2 \times r \qquad T_4 = T_3 \times r$$
$$T_2 = T_1 \times r \qquad = (ar) \times r \qquad = (ar^2) \times r$$
$$= ar \qquad = ar^2 \qquad = ar^3$$

Notice that the power of r is one less than the number of the term. So the power of r for T_n is $n - 1$.

*n*th term of a geometric sequence

$$T_n = ar^{n-1}$$

EXAMPLE 10

a **i** Find the 10th term of the sequence 3, 6, 12, …

ii Find the formula for the nth term of the sequence.

b Find the 10th term of the sequence −5, 10, −20, …

c Which term of the sequence 4, 12, 36, … is equal to 78 732?

d The 3rd term of a geometric sequence is 18 and the 7th term is 1458. Find the first term and the common ratio.

Geometric sequences

Solution

a **i** This is a geometric sequence with $a = 3$, $r = 2$ and $n = 10$.

$$T_n = ar^{n-1}$$
$$T_{10} = 3(2)^{10-1}$$
$$= 3 \times 2^9$$
$$= 1536$$

ii $T_n = ar^{n-1}$

$$= 3(2)^{n-1}$$

b $a = -5, r = -2, n = 10$

$$T_n = ar^{n-1}$$
$$T_{10} = -5(-2)^{10-1}$$
$$= -5 \times (-2)^9$$
$$= 2560$$

c This is a geometric sequence with $a = 4$, $r = 3$ and $T_n = 78\,732$.

$$T_n = ar^{n-1}$$
$$78\,732 = 4(3)^{n-1}$$
$$19\,683 = 3^{n-1}$$
$$\log 19\,683 = \log 3^{n-1}$$
$$= (n-1)\log 3$$
$$\frac{\log 19\,683}{\log 3} = n - 1$$
$$\frac{\log 19\,683}{\log 3} + 1 = n$$
$$10 = n$$

So the 10th term is 78 732.

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d Given $T_3 = 18$

$ar^{3-1} = 18$

$ar^2 = 18$ [1]

Given $T_7 = 1458$

$ar^{7-1} = 1458$

$ar^6 = 1458$ [2]

[2] ÷ [1]:

$$\frac{ar^6}{ar^2} = \frac{1458}{18}$$

$$r^4 = 81$$

$$r = \pm\sqrt[4]{81}$$

$$= \pm 3$$

Substitute $r = 3$ into [1]

$a(3)^2 = 18$

$9a = 18$

$a = 2$

Substitute $r = -3$ into [1]

$a(-3)^2 = 18$

$9a = 18$

$a = 2$

The first term is 2 and the common ratio is ±3.

Here is an example of a geometric sequence involving fractions.

EXAMPLE 11

a Find the 8th term of $\frac{2}{3}$ $\frac{4}{15}$ $\frac{8}{75}$ … in index form.

b Find the first value of n for which the terms of the sequence $\frac{1}{5}$, 1, 5, … exceed 3000.

Solution

a

$$\frac{T_2}{T} = \frac{4}{15} \div \frac{2}{3}$$

$$= \frac{2}{5}$$

$$\frac{T_3}{T_2} = \frac{8}{75} \div \frac{4}{15}$$

$$= \frac{2}{5}$$

So $r = \frac{2}{5}$

$$T_n = ar^{n-1}$$

$$= \frac{2}{3}\left(\frac{2}{5}\right)^{8-1}$$

$$= \frac{2}{3}\left(\frac{2}{5}\right)^{7}$$

$$= \frac{2^8}{3(5^7)}$$

b $a = \frac{1}{5}$, $r = 5$

$$T_n > 3000$$

$$ar^{n-1} > 3000$$

$$\frac{1}{5}(5)^{n-1} > 3000$$

$$5^{n-1} > 15\,000$$

$$\log 5^{n-1} > \log 15\,000$$

$$(n-1)\log 5 > \log 15\,000$$

$$n - 1 > \frac{\log 15\,000}{\log 5}$$

$$n > \frac{\log 15\,000}{\log 5} + 1$$

$$> 6.974...$$

So $n = 7$

The 7th term is the first term to exceed 3000.

Exercise 1.04 Geometric sequences

1 Is each sequence geometric? If so, find the common ratio.

a $5, 20, 60, \ldots$ **b** $-4, 3, -2\frac{1}{4}, \ldots$ **c** $\frac{3}{4}, \frac{3}{14}, \frac{3}{49}, \ldots$

d $7, 5\frac{5}{6}, 3\frac{1}{3}, \ldots$ **e** $-14, 42, -168, \ldots$ **f** $1\frac{1}{3}, \frac{8}{9}, \frac{8}{27}, \ldots$

g $5.7, 1.71, 0.513, \ldots$ **h** $2\frac{1}{4}, -1\frac{7}{20}, \frac{81}{100}, \ldots$ **i** $63, 9, 1\frac{7}{8}, \ldots$

j $-1\frac{7}{8}, 15, -120, \ldots$

2 Find the pronumeral in each geometric sequence.

a $4, 28, x, \ldots$ **b** $-3, 12, y, \ldots$ **c** $2, a, 72, \ldots$

d $y, 2, 6, \ldots$ **e** $x, 8, 32, \ldots$ **f** $5, p, 20, \ldots$

g $7, y, 63, \ldots$ **h** $-3, m, -12, \ldots$ **i** $3, x - 4, 15, \ldots$

j $3, k - 1, 21, \ldots$ **k** $\frac{1}{4}, t, \frac{1}{9}, \ldots$ **l** $\frac{1}{3}, t, \frac{4}{3}, \ldots$

3 Find the formula for the nth term of each sequence.

a $1, 5, 25, \ldots$ **b** $1, 1.02, 1.0404, \ldots$ **c** $1, 9, 81, \ldots$

d $2, 10, 50, \ldots$ **e** $6, 18, 54, \ldots$ **f** $8, 16, 32, \ldots$

g $\frac{1}{4}, 1, 4, \ldots$ **h** $1000, -100, 10, \ldots$ **i** $-3, 9, -27, \ldots$

j $\frac{1}{3}, \frac{2}{15}, \frac{4}{75}, \ldots$

4 Find the 6th term of each sequence.

a $8, 24, 72, \ldots$ **b** $9, 36, 144, \ldots$ **c** $8, -32, 128, \ldots$

d $-1, 5, -25, \ldots$ **e** $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$

5 What is the 9th term of each sequence?

a $1, 2, 4, \ldots$ **b** $4, 12, 36, \ldots$ **c** $1, 1.04, 1.0816, \ldots$

d $-3, 6, -12, \ldots$ **e** $\frac{3}{4}, -\frac{3}{8}, \frac{3}{16}, \ldots$

6 Find the 8th term of each sequence.

a $3, 15, 75, \ldots$ **b** $2.1, 4.2, 8.4, \ldots$ **c** $5, -20, 80, \ldots$

d $-\frac{1}{2}, \frac{3}{10}, -\frac{9}{50}, \ldots$ **e** $1\frac{47}{81}, 2\frac{10}{27}, 3\frac{5}{9}, \ldots$

7 Find the 20th term of each sequence, leaving the answer in index form.

a $3, 6, 12, \ldots$ **b** $1, 7, 49, \ldots$ **c** $1.04, 1.04^2, 1.04^3, \ldots$

d $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ **e** $\frac{3}{4}, \frac{9}{16}, \frac{27}{64}, \ldots$

8 Find the 50th term of $1, 11, 121, \ldots$ in index form.

9 Which term of the sequence $4, 20, 100, \ldots$ is equal to 12 500?

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10 Which term of 6, 36, 216, … is equal to 7776?

11 Is 1200 a term of the sequence 2, 16, 128, …?

12 Which term of 3, 21, 147, … is equal to 352 947?

13 Which term of the sequence 8, –4, 2, … is $\frac{1}{128}$?

14 Which term of 54, 18, 6, … is $\frac{2}{243}$?

15 Find the value of n if the nth term of the sequence $-2, 1\frac{1}{2}, -1\frac{1}{8}, \ldots$ is $-\frac{81}{128}$.

16 The first term of a geometric sequence is 7 and the 6th term is 1701.
Find the common ratio.

17 The 4th term of a geometric sequence is –648 and the 5th term is 3888.

a Find the common ratio.

b Find the 2nd term.

18 The 3rd term of a geometric sequence is $\frac{2}{5}$ and the 5th term is $1\frac{3}{5}$.
Find the first term and common ratio.

19 Find the value of n for the first term of the sequence 5000, 1000, 200, … that is less than 1.

20 Find the first term of the sequence $\frac{2}{7}, \frac{6}{7}, 2\frac{4}{7}, \ldots$ that is greater than 100.

1.05 Geometric series

WS Geometric series

WS Geometric series

WS Geometric series assignment

WS Geometric series problems

The sum of a **geometric series** with n terms is given by the formulas:

Sum of a geometric series with *n* terms

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

This formula can also be written as:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

to be used if r is a fraction, that is, $-1 < r < 1$, also written as $|r| < 1$.

Proof

The sum of a geometric series can be written

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \qquad [1]$$

Multiplying both sides by r:

$$rS_n = r(a + ar + ar^2 + \ldots + ar^{n-1})$$
$$= ar + ar^2 + ar^3 + \ldots + ar^n \qquad [2]$$

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[2] – [1]:

$rS_n - S_n = ar^n - a$

$S_n(r-1) = a(r^n - 1)$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

[1] – [2] gives the formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

We can also use these formulas to find the sum of the first n terms of a geometric sequence (also called the ***n*th partial sum**).

EXAMPLE 12

a Find the sum of the first 10 terms of the series 3 + 12 + 48 + …

b Evaluate $60 + 20 + 6\frac{2}{3} + \ldots + \frac{20}{81}$.

c The sum of n terms of 1 + 4 + 16 + … is 21 845. Find the value of n.

Solution

a This is a geometric series with $a = 3$, $r = 4$, $n = 10$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{3(4^{10} - 1)}{4 - 1}$$

$$= \frac{3(4^{10} - 1)}{3}$$

$$= 4^{10} - 1$$

$$= 1\ 048\ 575$$

b $a = 60$, $r = \frac{1}{3}$, $T_n = \frac{20}{81}$

$$T_n = ar^{n-1}$$

$$ar^{n-1} = \frac{20}{81}$$

$$60\left(\frac{1}{3}\right)^{n-} = \frac{20}{81}$$

$$\left(\frac{1}{3}\right)^{n-} = \frac{1}{243}$$

$$\frac{1}{3^{n-}} = \frac{1}{243}$$

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So $3^{n-1} = 243$

$= 3^5$

$n - 1 = 5$

$n = 6$

Since $|r| < 1$, we use the second formula.

$$S_n = \frac{a(1-r^n)}{r-1}$$

$$S_6 = \frac{60\left(1-\left(\frac{1}{3}\right)^6\right)}{1-\frac{1}{3}}$$

$$= \frac{60\left(1-\frac{1}{729}\right)}{\frac{2}{3}}$$

$$= 60 \times \frac{728}{729} \times \frac{3}{2}$$

$$= 89\frac{71}{81}$$

c $a = 1, r = 4, S_n = 21\ 845$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$21\ 845 = \frac{1(4^n - 1)}{4-1}$$

$$= \frac{4^n - 1}{3}$$

$65\ 535 = 4^n - 1$

$65\ 536 = 4^n$

$\log 65\ 536 = \log 4^n$

$= n \log 4$

$$\frac{\log 65\ 536}{\log 4} = n$$

$8 = n$

So 8 terms give a sum of 21 845.

Exercise 1.05 Geometric series

1 Find the sum of 10 terms of each geometric series.

a $6 + 24 + 96 + \ldots$ **b** $3 + 15 + 75 + \ldots$

2 Find the sum of 8 terms of each series.

a $-1 + 7 - 49 + \ldots$ **b** $8 + 24 + 72 + \ldots$

3 Find the sum of 15 terms of each series.

a $4 + 8 + 16 + \ldots$ **b** $\frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \ldots$(to 1 decimal place)

4 Evaluate:

a $2 + 10 + 50 + \ldots + 6250$ **b** $18 + 9 + 4\frac{1}{2} + \ldots + \frac{9}{64}$

c $3 + 21 + 147 + \ldots + 7203$ **d** $\frac{3}{4} + 2\frac{1}{4} + 6\frac{3}{4} + \ldots + 182\frac{1}{4}$

e $-3 + 6 - 12 + \ldots + 384$

5 For the series 7 + 14 + 28 + … find:

a the 9th term

b the sum of the first 9 terms.

6 Find the sum of 30 terms of the series $1.09 + 1.09^2 + 1.09^3 + \ldots$ correct to 2 decimal places.

7 Find the sum of 25 terms of the series $1 + 1.12 + 1.12^2 + \ldots$ correct to 2 decimal places.

8 Find the value of n if the sum of n terms of the series 11 + 33 + 99 + … is equal to 108 251.

9 How many terms of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ give a sum of $\frac{1023}{1024}$?

10 The common ratio of a geometric series is 4 and the sum of the first 5 terms is 3069. Find the first term.

11 Find the number of terms needed to be added for the sum to exceed 1 000 000 in the series 4 + 16 + 64 + …

12 **a** Find the sum of 10 terms of the series 2 + 4 + 8 + …

b Find the sum of 10 terms of the series 1 + 3 + 5 + …

c Find the sum of the first 10 terms of the series 3 + 7 + 13 + …

PUZZLES

1 A poor girl saved a rich king from drowning one day. The king offered the girl a reward of a sum of money in 30 daily payments. He gave her a choice of payments:

Choice 1: $1 the first day, $2 the second day, $3 the third day and so on.

Choice 2: 1 cent the first day, 2 cents the second day, 4 cents the third day and so on, the payment doubling each day.

How much money would the girl receive for each choice? Which plan would give the girl more money?

2 Can you solve Fibonacci's problem?

A man entered an orchard through 7 guarded gates and gathered a certain number of apples. As he left the orchard he gave the guard at the first gate half the apples he had and 1 apple more. He repeated this process for each of the remaining 6 guards and eventually left the orchard with 1 apple. How many apples did he gather? (He did not give away any half-apples.)

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1.06 Limiting sum of an infinite geometric series

Limiting sum of an infinite geometric series

Geomeic series crossword

In some geometric sequences the sum becomes very large as n increases, for example, the series $2 + 4 + 8 + 16 + 32 + \ldots$ We say these series **diverge** (their sum is infinite).

In other geometric sequences, however, such as $8 + 4 + 2 + 1 + \ldots$, the sum does not increase greatly after a few terms, but approaches some constant value. We say these series **converge** (they have a **limiting sum** that is a specific value, sometimes called the **sum to infinity**).

EXAMPLE 13

a Find the sum of 15 terms of $2 + 6 + 18 + \ldots$

b By evaluating the sum of 10 terms and 20 terms correct to 4 decimal places for the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, estimate its limiting sum.

Solution

a $a = 2, r = 3, n = 15$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{15} = \frac{2(3^{15} - 1)}{3 - 1}$$

$$= \frac{2(3^{15} - 1)}{2}$$

$$= 3^{15} - 1$$

$$= 14\,348\,906$$

b $a = 2, r = \frac{1}{2}$: $S_n = \frac{a(1 - r^n)}{1 - r}$

Sum to 10 terms:

$$S_{10} = \frac{2\left[1 - \left(\frac{1}{2}\right)^{10}\right]}{1 - \frac{1}{2}}$$

$$= \frac{2\left[1 - \frac{1}{2^{10}}\right]}{\frac{1}{2}}$$

$$= 3.9961$$

Sum to 20 terms:

$$S_{20} = \frac{2\left[1 - \left(\frac{1}{2}\right)^{20}\right]}{1 - \frac{1}{2}}$$

$$= \frac{2\left[1 - \frac{1}{2^{20}}\right]}{\frac{1}{2}}$$

$$= 4.0000$$

The limiting sum is 4.

Can you see why the series 2 + 6 + 18 + … does not have a limiting sum and the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ has a limiting sum?

It's because of the common ratio. Only geometric series with common ratios that are fractions, $|r| < 1$, will have a limiting sum.

For $S_n = \frac{a(1-r^n)}{1-r}$:

As $n \to \infty$, $r^n \to 0$ when $-1 < r < 1$

We write $\lim_{n\to\infty} r^n = 0$

Limiting sum of a geometric series

$$S = \frac{a}{1-r} \text{ when } |r| < 1.$$

Proof

$$S_n = \frac{a(1-r^n)}{1-r}$$

For $|r| < 1$, $\lim_{n\to\infty} r^n = 0$

$$S_\infty = \frac{a(1-0)}{1-r}$$

$$= \frac{a}{1-r}$$

EXAMPLE 14

a Find the limiting sum of the series $2 + 1 + \frac{1}{2} + \quad + \frac{1}{8} + \ldots$

b Find the sum to infinity of the series $6 + 2 + \frac{2}{3} + \ldots$

c Does the series $\frac{3}{4} + \frac{15}{16} + 1\frac{11}{64} + \ldots$ have a limiting sum?

Solution

a $a = 2, r = \frac{1}{2}$

Since $|r| < 1$, the series has a limiting sum.

$$S = \frac{a}{1-r}$$

$$= \frac{2}{1-\frac{1}{2}}$$

$$= \frac{2}{\frac{1}{2}}$$

$$= 2 \times \frac{2}{1}$$

$$= 4$$

So the limiting sum is 4.

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b $a = 6$

$2 \div 6 = \frac{1}{3}$ and $\frac{2}{3} \div 2 = \frac{1}{3}$ so $r = \frac{1}{3}$

Since $|r| < 1$, the series has a limiting sum.

$$S = \frac{a}{1-r}$$
$$= \frac{6}{1-\frac{1}{3}}$$
$$= \frac{6}{\frac{2}{3}}$$
$$= 6 \times \frac{3}{2}$$
$$= 9$$

So the limiting sum is 9.

c For $\frac{3}{4} + \frac{15}{16} + 1\frac{11}{64} + \ldots$

$r = \frac{15}{16} \div \frac{3}{4}$ or $1\frac{11}{64} \div \frac{15}{16}$

$= 1\frac{1}{4}$

Since $|r| > 1$, this series does not have a limiting sum.

DID YOU KNOW?

A series involving π and e

Here is an interesting series involving π:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

Gottfried Wilhelm Leibniz (1646–1716) discovered this result. It is interesting that while π is an irrational number, it can be written as the sum of rational numbers.

Here is another interesting series involving *e*:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

Remember: 2! = 2 × 1, 3! = 3 × 2 × 1, 4! = 4 × 3 × 2 × 1 and so on.

Research these and other series.

Exercise 1.06 Limiting sum of an infinite geometric series

1 Which series has a limiting sum? Find the limiting sum where it exists.

a $9 + 3 + 1 + \ldots$ **b** $\frac{1}{4} + \frac{1}{2} + 1 + \ldots$ **c** $16 - 4 + 1 - \ldots$

d $\frac{2}{3} + \frac{7}{9} + \frac{49}{54} + \ldots$ **e** $1 + \frac{2}{3} + \frac{4}{9} + \ldots$ **f** $\frac{5}{8} + \frac{1}{8} + \frac{1}{40} + \ldots$

g $-6 + 36 - 216 + \ldots$ **h** $-2\frac{1}{4} + 1\frac{7}{8} - 1\frac{27}{48} + \ldots$ **i** $\frac{1}{9} + \frac{1}{6} + \frac{1}{4} + \ldots$

j $2 - \frac{4}{5} + \frac{8}{25} - \ldots$

2 Find the limiting sum of each series.

a $40 + 20 + 10 + \ldots$ **b** $320 + 80 + 20 + \ldots$ **c** $100 - 50 + 25 - \ldots$

d $6 + 3 + 1\frac{1}{2} + \ldots$ **e** $\frac{2}{5} + \frac{6}{35} + \frac{18}{245} + \ldots$ **f** $72 - 24 + 8 - \ldots$

g $-12 + 2 - \frac{1}{3} + \ldots$ **h** $\frac{3}{4} - \frac{1}{2} + \frac{1}{3} - \ldots$ **i** $12 + 9 + 6\frac{3}{4} + \ldots$

j $-\frac{2}{3} + \frac{5}{12} - \frac{25}{96} + \ldots$

3 Find the difference between the limiting sum and the sum of 6 terms of each series, correct to 2 significant figures.

a $56 - 28 + 14 - \ldots$ **b** $72 + 24 + 8 + \ldots$ **c** $1 + \frac{1}{5} + \frac{1}{25} + \ldots$

d $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ **e** $1\frac{1}{4} + \frac{15}{16} + \frac{45}{64} + \ldots$

4 A geometric series has limiting sum 6 and common ratio $\frac{1}{3}$. Evaluate the first term of the series.

5 A geometric series has a limiting sum of 5 and first term 3. Find the common ratio.

6 The limiting sum of a geometric series is $9\frac{1}{3}$ and the common ratio is $\frac{2}{5}$. Find the first term of the series.

7 A geometric series has limiting sum 40 and its first term is 5. Find the common ratio of the series.

8 A geometric series has limiting sum $-6\frac{2}{5}$ and first term -8. Find its common ratio.

9 The limiting sum of a geometric series is $-\frac{3}{10}$ and its first term is $-\frac{1}{2}$. Find the common ratio of the series.

10 The second term of a geometric series is 2 and its limiting sum is 9. Find the values of first term a and common ratio r.

11 A geometric series has 3rd term 12 and 4th term -3. Find a, r and the limiting sum.

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12 A geometric series has 2nd term $\frac{2}{3}$ and 4th term $\frac{8}{27}$. Find a, r and its limiting sum.

13 The 3rd term of a geometric series is 54 and the 6th term is $11\frac{83}{125}$.
Evaluate a, r and the limiting sum.

14 The 2nd term of a geometric series is $\frac{4}{15}$ and the 5th term is $\frac{32}{405}$.
Find the values of a and r and its limiting sum.

15 The limiting sum of a geometric series is 5 and the 2nd term is $1\frac{1}{5}$.
Find the first term and the common ratio.

16 The series $x + \frac{x}{4} + \frac{x}{16} + \ldots$ has a limiting sum of $\frac{7}{8}$. Evaluate x.

17 **a** For what values of k does the limiting sum exist for the series $k + k^2 + k^3 + \ldots$?

b Find the limiting sum of the series when $k = -\frac{2}{3}$.

c Evaluate k if the limiting sum of the series is 3.

EXT1 **18** **a** For what values of p will the limiting sum exist for the series $1 - 2p + 4p^2 - \ldots$?

b Find the limiting sum when $p = \frac{1}{5}$.

c Evaluate p if the limiting sum of the series is $\frac{7}{8}$.

19 Show that in any geometric series the difference between the limiting sum and the sum of n terms is $\frac{ar^n}{1-r}$.

EXT1 1.07 Proof by mathematical induction

WS Mathematical induction

WS Starting proofs by mathematical induction

WS Proof by induction

There are many different ways of proving mathematical statements.

You have already used proof by deduction that use known facts to prove statements.

The Mathematics Extension 2 course has an entire topic on mathematical proof.

INVESTIGATION

PROOF

Research different types of proofs and find out where each type is used.

1 Find some examples of deductive proofs.

2 Two well-known proofs by contradiction are proving that $\sqrt{2}$ is irrational, and that the set of primes is infinite. Find some other examples of proof by contradiction.

3 What other types of proofs can you find?

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We use proof by **mathematical induction** for statements that involve integer values of n, for example, sequences and series. It relies on proving a specific example and then moving to a general conclusion.

Proof by mathematical induction

Suppose we need to prove that a statement is true for all positive integers, n.

Step 1: Show the statement is true for $n = 1$.

Step 2: Assume the statement is true for some positive integer value $n = k$.

Step 3: Using the assumption, prove that the statement is also true for the next integer $n = k + 1$.

Conclusion: State why the statement is true for all (positive) integers $n \geq 1$.

This method only works when you follow all the steps. If you can only prove a statement true for several values of n but not for $n = k + 1$, it is possible that it is true for some values of n but not others.

EXAMPLE 15

a Can we use mathematical induction to prove that $\sin(\pi - x) = \sin x$ for all x?

b Marcus proves that a statement is true for $n = 1, 2, 3, 4, 5, 6, 7$ and 8. Has he proved the statement true by mathematical induction?

Solution

a Angles can take on any value, not just integer values, so we can't use mathematical induction.

b No, the statement may not be true for $n \geq 9$.

Here are some examples of how to use mathematical induction.

Proof by mathematical induction

EXAMPLE 16

Prove by mathematical induction that for all integers $n \geq 1$:

a $5 + 9 + 13 + \ldots + (4n + 1) = n(2n + 3)$

b $\dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \ldots + \dfrac{1}{3^n} = \dfrac{3^n - 1}{2(3^n)}$

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Solution

a Step 1: Prove the statement true for $n = 1$ (the smallest value of n).

Substituting $n = 1$:

LHS = 5

$$\begin{aligned}\text{RHS} &= n(2n + 3)\\ &= 1[2(1) + 3]\\ &= 5\\ &= \text{LHS}\end{aligned}$$

So the statement is true for $n = 1$.

Step 2: Assume the statement is true for $n = k$.

We assume $5 + 9 + 13 + \ldots + (4k + 1) = k(2k + 3)$ [*]

Step 3: Prove the statement is true for $n = k + 1$.

We want to prove that $5 + 9 + 13 + \ldots + (4k + 1) + [4(k + 1) + 1] = (k + 1)[2(k + 1) + 3]$

Simplifying:

$$5 + 9 + 13 + \ldots + (4k + 1) + (4k + 4 + 1) = (k + 1)(2k + 2 + 3)$$

$$5 + 9 + 13 + \ldots + (4k + 1) + (4k + 5) = (k + 1)(2k + 5)$$

$$\begin{aligned}\text{LHS} &= 5 + 9 + 13 + \ldots + (4k + 1) + (4k + 5)\\ &= k(2k + 3) + (4k + 5) \text{ from the assumption [*] in Step 2}\\ &= 2k^2 + 3k + 4k + 5\\ &= 2k^2 + 7k + 5\\ &= (k + 1)(2k + 5) \text{ (factorising)}\\ &= \text{RHS}\end{aligned}$$

So the statement is true for $n = k + 1$.

Conclusion:

The statement is true for $n = 1$, so it must be true for $n = 1 + 1 = 2$.

The statement is true for $n = 2$, so it must be true for $n = 3$ and so on.

So the statement is true for all integers $n \geq 1$.

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b Step 1: Prove the statement is true for $n = 1$.

$$\text{LHS} = \frac{1}{3}$$

$$\text{RHS} = \frac{3^n - 1}{2(3^n)}$$

$$= \frac{3\ \ -1}{2(3\ \ )}$$

$$= \frac{1}{3}$$

$$= \text{LHS}$$

So the statement is true for $n = 1$.

Step 2: Assume true for $n = k$.

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^k} = \frac{3^k - 1}{2(3^k)} \qquad [*]$$

Step 3: Prove the statement is true for $n = k + 1$.

We want to prove $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^k} + \frac{1}{3^{k+}} = \frac{3^{k+}\ \ -1}{2(3^{k+}\ \ )}$

$$\text{LHS} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^k} + \frac{1}{3^{k+}}$$

$$= \frac{3^k - 1}{2(3^k)} + \frac{1}{3^{k+}} \text{ (from the assumption [*])}$$

$$= \frac{3(3^k - 1)}{3 \times 2(3^k)} + \frac{2 \times 1}{2 \times 3^{k+}}$$

$$= \frac{3^{k+}\ \ - 3}{2(3^{k+1})} + \frac{2}{2(3^{k+1})}$$

$$= \frac{3^{k+}\ \ - 1}{2(3^{k+}\ \ )}$$

$$= \text{RHS}$$

So the statement is true for $n = k + 1$.

Conclusion:

The statement is true for $n = 1$ so by mathematical induction the statement is true for all integers $n \geq 1$.

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We can use mathematical induction to prove other types of statements involving integers such as divisibility (a number is divisible by another number with no remainder).

EXAMPLE 17

Prove by mathematical induction that $n(n + 1)$ is divisible by 2 for any integer $n \geq 1$.

Solution

Step 1: Prove true for $n = 1$.

$n(n + 1) = 1(1 + 1)$

$\quad = 2$

Since 2 is divisible by 2, the statement is true for $n = 1$.

Step 2: Assume it is true for $n = k$.

That is, $k(k + 1)$ is divisible by 2.

We can write $k(k + 1) = 2p$ where p is an integer.

Expanding:

$k^2 + k = 2p \qquad [*]$

Step 3: Prove the statement is true for $n = k + 1$.

We want to prove $(k + 1)[(k + 1) + 1]$ is divisible by 2.

That is, $(k + 1)[(k + 1) + 1] = 2q$ where q is an integer.

Expanding:

$(k + 1)[(k + 1) + 1] = 2q$

$(k + 1)(k + 2) = 2q$

$k^2 + 2k + k + 2 = 2q$

LHS $= k^2 + k + 2k + 2$

$= 2p + 2k + 2$ (from the assumption [*])

$= 2(p + k + 1)$

$= 2q$ for some integer q (since p, k and 1 are all integers)

So $(k + 1)[(k + 1) + 1]$ is divisible by 2.

Conclusion:

The statement is true for $n = k + 1$.

The statement is true for $n = 1$ so by mathematical induction it is true for all integers $n \geq 1$.

EXT1 Exercise 1.07 Proof by mathematical induction

1 State whether each statement can be proved by induction.

a $n(n + 1)(n + 2)$ is divisible by 3 where n is a positive integer

b $\sin x \cot x = \cos x$

c $3 + 8 + 13 + \ldots + (5n - 2) = \dfrac{n(5n+1)}{2}$

d $A = n^2$ is the area of a square where n is its side.

e Polynomial $P(x)$ with degree n has at most n factors.

2 Prove by mathematical induction if n is an integer:

a $3 + 10 + 17 + \ldots + (7n - 4) = \dfrac{n}{2}(7n - 1)$ for all $n \geq 1$

b $5 + 11 + 19 + \ldots + (8n - 3) = n(4n + 1)$ for all $n \geq 1$

c $5 + 10 + 20 + \ldots + 5 \times 2^n = 5(2^{n+1} - 1)$ for all $n \geq 0$

Prove true for $n = 0$ first.

d $1+\dfrac{1}{2}+\dfrac{1}{4}+\ldots+\dfrac{1}{2^{n-}} = 2\left(1-\dfrac{1}{2^n}\right)$ for all $n \geq 1$

e $2 + 5 + 8 + \ldots + (3n - 1) = \dfrac{3n^2+n}{2}$ for all $n \geq 1$

f $2 + 4 + 8 + \ldots + 2^n = 2(2^n - 1)$ for all $n \geq 1$

g $5 + 10 + 15 + \ldots + 5n = \dfrac{5}{2}n(n + 1)$ for all $n \geq 1$

h $-2 - 4 - 6 - \ldots - 2n = -n(n + 1)$ for all $n \geq 1$

i $9 + 14 + 19 + \ldots + (5n + 4) = \dfrac{5n^2+13n}{2}$ for all $n \geq 1$

j $9 + 27 + 81 + \ldots + 3^n = \dfrac{9(3^{n-1}-1)}{2}$ for all $n \geq 2$

k $-4 + 8 - 16 + \ldots - 4(-2)^{n-1} = \dfrac{4[(-2)^n-1]}{3}$ for all $n \geq 1$

l $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{1}{6}n(n + 1)(2n + 1)$ for all $n \geq 1$

m $1 + 27 + 125 + \ldots + (2n - 1)^3 = n^2(2n^2 - 1)$ for all $n \geq 1$

3 Use induction to prove:

a $7^n - 1$ is divisible by 6 for all positive integers n

b $3^{2n} - 1$ is divisible by 8 for all integers $n \geq 1$

c $5^n - 1$ is divisible by 4 for all integers $n \geq 1$

d $n(n + 2)$ is divisible by 4 for all positive even integers n

e The sum of consecutive odd positive integers is divisible by 4

f $5^n + 3^n$ is even for all positive integers n

g $7^n + 3^n$ is divisible by 10 for all odd positive integers n

4 Mila assumed that a statement involving n (a positive integer) is true for $n = k$ and proved it true for $n = k + 1$. Has she proved the statement true by induction?

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1. TEST YOURSELF

For Questions **1** to **4**, select the correct answer **A**, **B**, **C** or **D**.

Formula sheet: Measurement, Sequences and series

Practice quiz

1 The sum of n terms of a geometric sequence is:

A $S_n = \frac{n}{2}[2a + (n-1)d]$ **B** $S_n = \frac{a}{1-r}$

C $S_n = \frac{a(1-r^n)}{1-r}$ **D** $S_n = \frac{n}{2}(a+l)$

2 The limiting sum of an infinite geometric series exists when:

A $r > 1$ **B** $|r| > 1$ **C** $|r| < 1$ **D** $r < 1$

3 The nth term of the statement that can be proved 12, 9, 6, … is:

A $9 + 3n$ **B** $15 - 3n$ **C** $9 - 3n$ **D** $12n - 15$

4 EXT1 An example of a statement that can be proved by mathematical induction is:

A Polynomial $P(x)$ has remainder $P(n)$ when divided by $x - n$

B $2 + 4 + 8 + 16 + \ldots + 2^n = 2(2^n - 1)$

C $\sin(\pi + n) = -\sin n$

D ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$

5 Find a formula for the nth term of each sequence.

a 9, 13, 17, … **b** 7, 0, –7, … **c** 2, 6, 18, …

d 200, 50, $12\frac{1}{2}$, … **e** –2, 4, –8, …

6 For the series 156 + 145 + 134 + …:

a Find the 15th term.

b Find the sum of 15 terms.

c Find the sum of 14 terms.

d Write a relationship between T_{15}, S_{15} and S_{14}.

e Find the value of n for the first negative term of the series.

7 Find whether each sequence is:

i arithmetic **ii** geometric **iii** neither.

a 97, 93, 89, … **b** $\frac{2}{3}\ \frac{1}{2}\ \frac{3}{8}$

c $\sqrt{5}\ \sqrt{20}, \sqrt{45}$ **d** −1.6, −0.4, 0.6, …

e 3.4, 7.5, 11.6, … **f** 48, 24, 12, …

g $-\frac{1}{5}$, 1, −5, … **h** 105, 100, 95, …

i $1\frac{1}{2}$, $1\frac{1}{4}$, 1, … **j** $\log x, \log x^2, \log x^3, \ldots$

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8 The nth term of the sequence 8, 13, 18, … is 543. Evaluate n.

9 The 11th term of an arithmetic sequence is 97 and the 6th term is 32. Find the first term and common difference.

10 A sequence has nth term given by $T_n = n^3 - 5$. Find:

a the 4th term **b** the sum of 4 terms **c** which term is 5827.

11 A sequence has terms 5, x, 45, … Evaluate x if the sequence is:

a arithmetic **b** geometric.

12 EXT1 Stevie proves a statement true for $n = 1, 2, 3, 4$ and 5. Has she proved it by mathematical induction?

13 If x, $2x + 3$ and $5x$ are the first 3 terms of an arithmetic series, calculate the value of x.

14 Find the 20th term of:

a 3, 10, 17, … **b** 101, 98, 95, … **c** 0.3, 0.6, 0.9, …

15 Find the limiting sum of the series 81 + 27 + 9 + …

16 For each series, find the formula for the sum of n terms.

a $5 + 9 + 13 + \ldots$ **b** $1 + 1.07 + 1.07^2 + \ldots$

17 **a** For what values of x does the geometric series $1 + x + x^2 + \ldots$ have a limiting sum?

b Find the limiting sum when $x = \frac{3}{5}$.

c Evaluate x when the limiting sum is $1\frac{1}{2}$.

18 The first term of an arithmetic series is 4 and the sum of 10 terms is 265. Find the common difference.

19 If $x + 2$, $7x - 2$ and $15x + 6$ are consecutive terms in a geometric sequence, evaluate x.

20 Evaluate $8 + 14 + 20 + \ldots + 122$.

21 **a** Calculate the sum of all the multiples of 7 from 1 to 100.

b Calculate the sum of all numbers from 1 to 100 that are not multiples of 7.

22 The sum of n terms of the series $214 + 206 + 198 + \ldots$ is 2760. Evaluate n.

23 Evaluate n if the nth term of the sequence 4, 12, 36, … is 236 196.

24 EXT1 Prove by mathematical induction:

a $15 + 45 + 135 + \ldots + 5(3^n) = \frac{15(3^n - 1)}{2}$ for all integers $n \geq 1$

b $4 + 7 + 10 + \ldots + (3n + 1) = \frac{n(3n + 5)}{2}$ for all integers $n \geq 1$

c $2n(n - 1)$ is divisible by 4 for all integers $n > 1$

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1. CHALLENGE EXERCISE

1 The nth term of a sequence is given by $T_n = \dfrac{n^2}{n+1}$.

a What is the 9th term of the sequence?

b Which term is equal to $18\frac{1}{20}$?

2 For the series $\dfrac{3\pi}{4}, \pi, \dfrac{5\pi}{4}, \ldots$ find the exact value of:

a the common difference

b the 7th term

c the sum of 6 terms.

3 Evaluate the sum of the first 20 terms of the series:

a $3 + 5 + 9 + 17 + 33 + 65 + \ldots$

b $5 - 2 + 10 - 8 + 15 - 32 + \ldots$

4 Which term of the sequence $\dfrac{7}{9}, \dfrac{14}{45}, \dfrac{28}{225}, \ldots$ is equal to $\dfrac{224}{28\,125}$?

5 Find the sum of all integers between 1 and 200 that are not multiples of 9.

6 Find the values of n for which $S_n > 24.99$ for the series $20 + 4 + \dfrac{4}{5} + \ldots$

7 EXT1 Find the values of n for which the sequence given by $T_n = n^2 - 4n$ is negative.

8 The sum of the first 5 terms of a geometric series is 77 and the sum of the next 5 terms is –2464.

a Find the first term and common ratio of the series.

b Find the 4th term of the series.

9 EXT1 Prove by mathematical induction for all positive integers n:

a $a + ar + ar^2 + \ldots + ar^{n-1} = \dfrac{a(r^n - 1)}{r - 1}$

b $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$

10 **a** Find the limiting sum of the series $1 + \cos^2 x + \cos^4 x + \ldots$ where $\cos^2 x \neq 0, 1$.

b Why does this series have a limiting sum?

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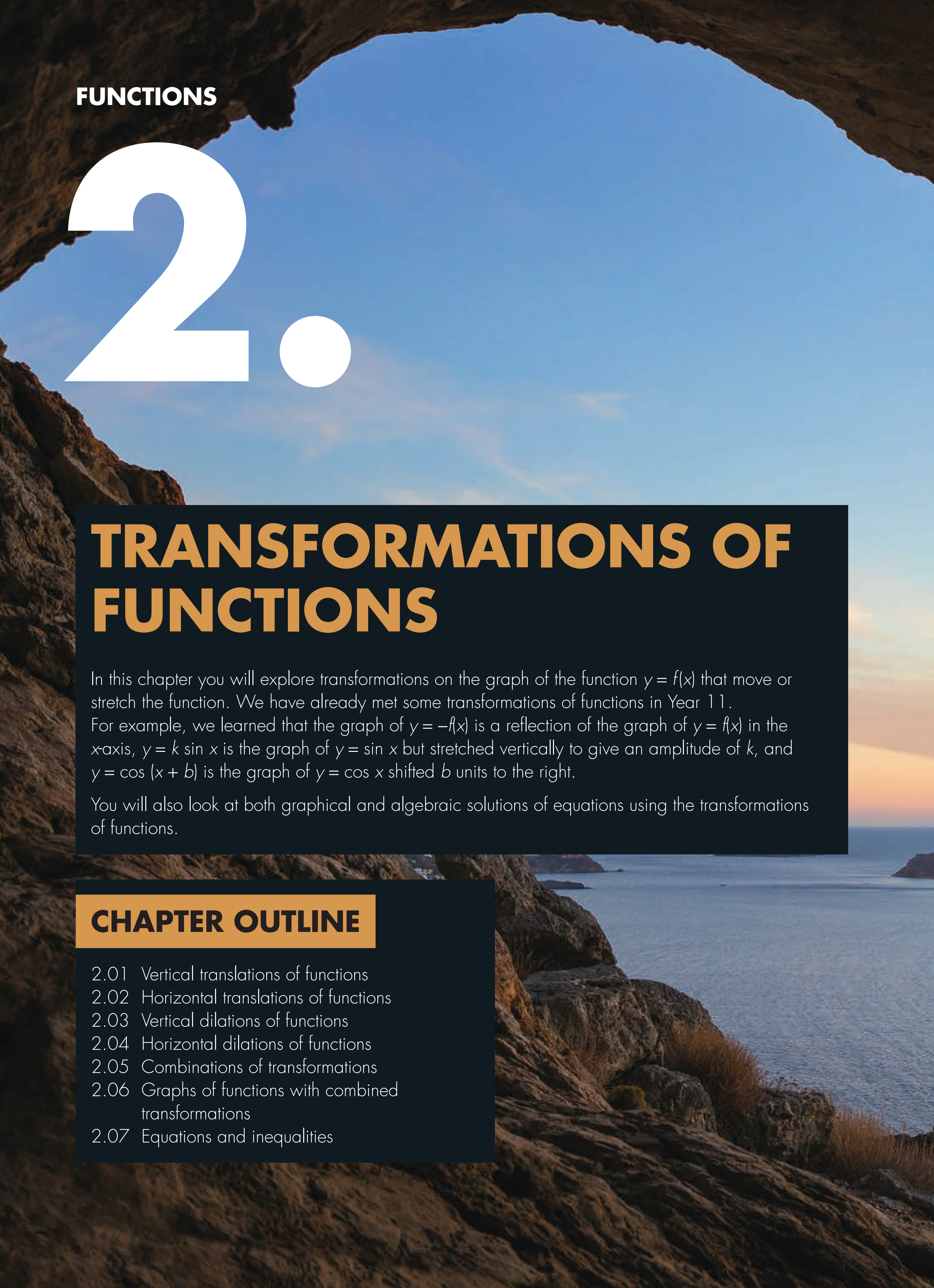


2.

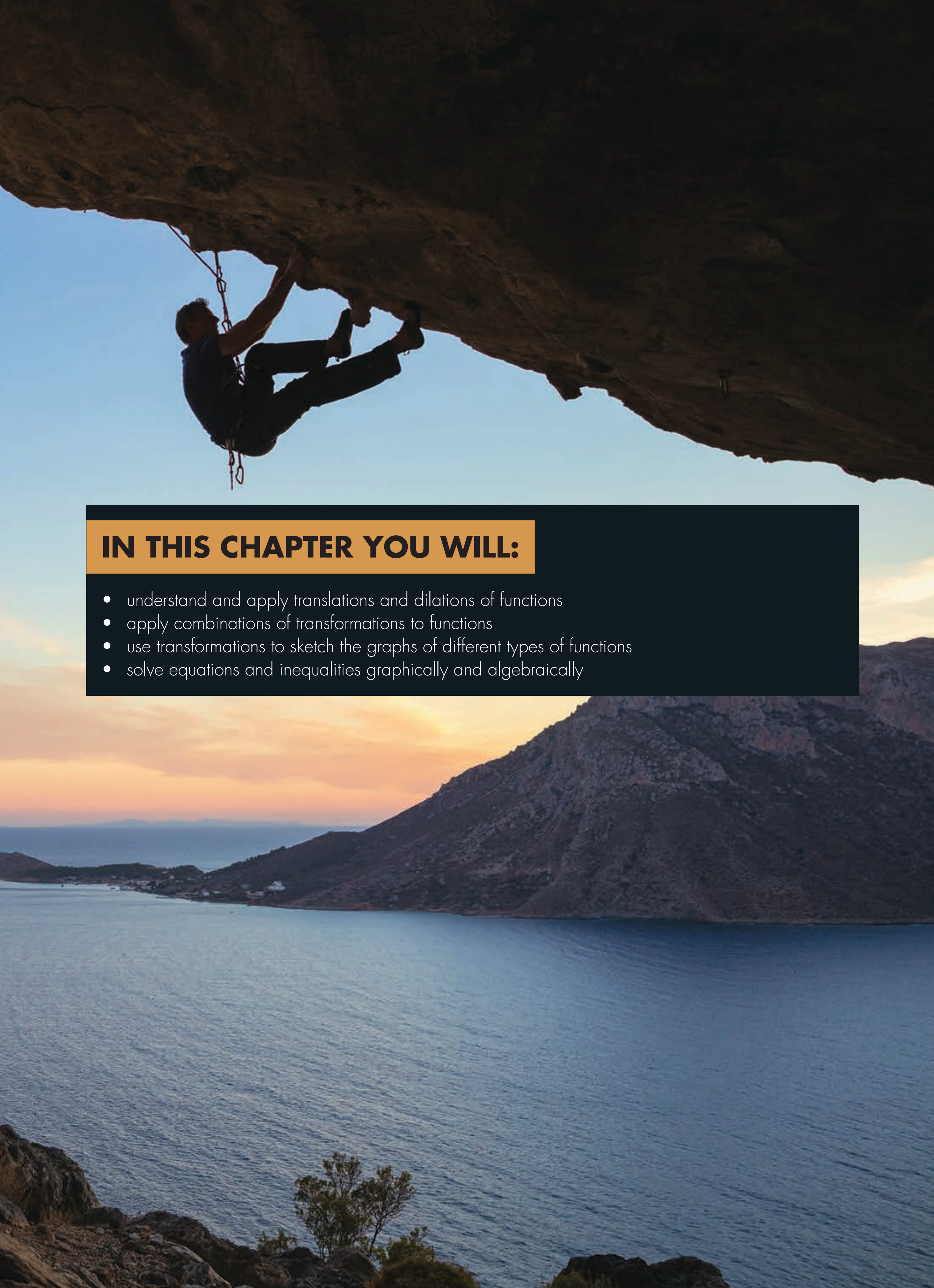
TRANSFORMATIONS OF FUNCTIONS

In this chapter you will explore transformations on the graph of the function $y = f(x)$ that move or stretch the function. We have already met some transformations of functions in Year 11. For example, we learned that the graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis, $y = k \sin x$ is the graph of $y = \sin x$ but stretched vertically to give an amplitude of k, and $y = \cos (x + b)$ is the graph of $y = \cos x$ shifted b units to the right.

You will also look at both graphical and algebraic solutions of equations using the transformations of functions.

CHAPTER OUTLINE

2.01 Vertical translations of functions
2.02 Horizontal translations of functions
2.03 Vertical dilations of functions
2.04 Horizontal dilations of functions
2.05 Combinations of transformations
2.06 Graphs of functions with combined transformations
2.07 Equations and inequalities



IN THIS CHAPTER YOU WILL:

- understand and apply translations and dilations of functions
- apply combinations of transformations to functions
- use transformations to sketch the graphs of different types of functions
- solve equations and inequalities graphically and algebraically

TERMINOLOGY

dilation: The process of stretching or compressing the graph of a function horizontally or vertically.

parameter: a constant in the equation of a function that determines the properties of that function and its graph; for example, the parameters for $y = mx + c$ are m (gradient) and c (y-intercept).

scale factor: The value of k by which the graph of a function is dilated.

transformation: A general name for the process of changing the graph of a function by moving, reflecting or stretching it.

translation: The process of shifting the graph of a function horizontally and/or vertically without changing its size or shape.

2.01 Vertical translations of functions

INVESTIGATION

VERTICAL TRANSLATIONS

Some graphics calculators or graphing software use a dynamic feature to show how a constant c (a **parameter**) changes the graph of a function.

Use dynamic geometry software to explore the effect of c on each graph below. If you don't have dynamic software, substitute different values for c into the equation. Use positive and negative values, integers and fractions.

1 $f(x) = x + c$

2 $f(x) = x^2 + c$

3 $f(x) = x^3 + c$

4 $f(x) = x^4 + c$

5 $f(x) = e^x + c$

6 $f(x) = \ln x + c$

7 $f(x) = \frac{1}{x} + c$

8 $f(x) = |x| + c$

How does the value of c transform the graph? What is the difference between positive and negative values of c?

Notice that c shifts the graph up and down without changing its size or shape. We call this a **vertical translation** (a shift along the y-axis).

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Vertical translation

For the function $y = f(x)$:

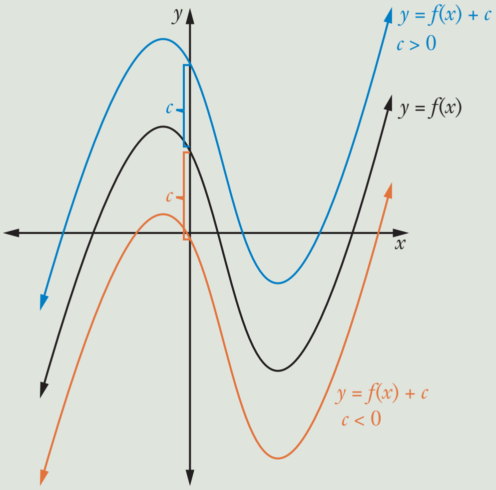
$y = f(x) + c$ translates the graph vertically (along the y-axis).

If $c > 0$, the graph is translated upwards by c units.

If $c < 0$, the graph is translated downwards.

A vertical translation changes the y values of the function.

> In Year 11, we learned that $y = \sin x + c$ is the graph of $y = \sin x$ shifted up c units.



EXAMPLE 1

a Explain how the graph of $y = x^2 + 2$ is related to the graph of $y = x^2$

b If the graph of the function $y = x^2 + 7x + 1$ is translated 4 units down find the equation of the transformed function.

c The point $P(3, -2)$ lies on the function $y = f(x)$ Find the transformed point (the image of P) if the function is translated:

i 6 units down **ii** 8 units up

Solution

a The graph of $y = x^2 + 2$ is a vertical translation 2 units up from the original (parent) function $y = x^2$.

b For a vertical translation 4 units down:

$y = f(x) + c$ where $c = -4$

$y = x^2 + 7x + 1 - 4$

$= x^2 + 7x - 3$

The equation of the transformed function is $y = x^2 + 7x - 3$

c **i** $P(3, -2)$ is translated 6 units down, so subtract 6 from the y value.

The transformed point is $(3, -2 - 6) \equiv (3, -8)$.

ii $P(3, -2)$ is translated 8 units up, so add 8 to the y value

The transformed point is $(3, -2 + 8) \equiv (3, 6)$.

> For points, we use '≡' (identical to) rather than '='.

EXAMPLE 2

a Sketch the graph of $y = x^3 - 3$.

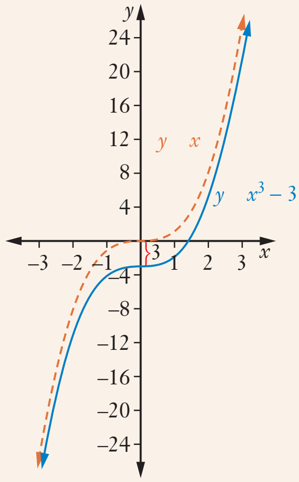
b **i** State the relationship of $y = \frac{1}{x} - 2$ to $y = \frac{1}{x}$.

ii State the domain and range of $y = \frac{1}{x} - 2$

iii Sketch the graph of $y = \frac{1}{x} - 2$.

Solution

a A vertical translation of -3 units shifts the function $y = x^3$ down to the graph of $y = x^3 - 3$.
If you need to find some points on the graph of $y = x^3 - 3$ you could subtract 3 from y values of $y = x^3$.



b **i** $y = \frac{1}{x} - 2$ is a vertical translation 2 units down of $y = \frac{1}{x}$.

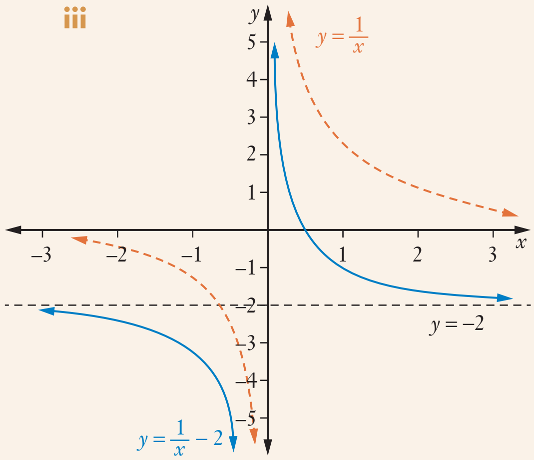
ii Since $x \neq 0$, domain is $(-\infty, 0) \cup (0, \infty)$.

Since $\frac{1}{x} \neq 0$, $\frac{1}{x} - 2 \neq -2$

So range is $(-\infty, -2) \cup (-2, \infty)$.

Since the horizontal asymptote is at $y = -2$, we sketch it as a dotted line.

iii



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Exercise 2.01 Vertical translations of functions

1 Describe how each constant affects the graph of $y = x^2$.

a $y = x^2 + 3$ **b** $y = x^2 - 7$

c $y = x^2 - 1$ **d** $y = x^2 + 5$

2 Describe how each constant affects the graph of $y = x^3$.

a $y = x^3 + 1$ **b** $y = x^3 - 4$ **c** $y = x^3 + 8$

3 Describe how the graph of $y = \frac{1}{x}$ transforms to the graph of $y = \frac{1}{x} + 9$.

4 Find the equation of each translated function.

a $y = x^2$ is translated 3 units downwards

b $f(x) = 2^x$ is translated 8 units upwards

c $y = |x|$ is translated 1 unit upwards

d $y = x^3$ is translated 4 units downwards

e $f(x) = \log x$ is translated 3 units upwards

f $y = \frac{2}{x}$ is translated 7 units downwards

5 Describe the relationship between the graph of $f(x) = x^4$ and:

a $f(x) = x^4 - 1$ **b** $f(x) = x^4 + 6$

6 Find the equation of the transformed function if:

a $y = 2x^3 + 3$ is translated:

i 5 units down **ii** 3 units up

b $y = |x| - 4$ is translated:

i 1 unit up **ii** 2 units down

c $y = e^x + 2$ is translated:

i 1 unit down **ii** 3 units up

d $y = \log_e x - 1$ is translated:

i 11 units up **ii** 7 units down

7 If $P = (1, -3)$ lies on the function $y = f(x)$, find the transformed (image) point of P if the function is translated:

a 2 units up **b** 6 units down **c** m units up

8 Find the original point P on the function $y = f(x)$ if the coordinates of its transformed image are $(-1, 2)$ when the function is translated:

a 1 unit up **b** 3 units down

9 Sketch each set of functions on the same number plane.

a $y = x^2$, $y = x^2 + 2$ and $y = x^2 - 3$

b $y = 3^x$ and $y = 3^x - 4$

c $y = |x|$ and $y = |x| - 3$

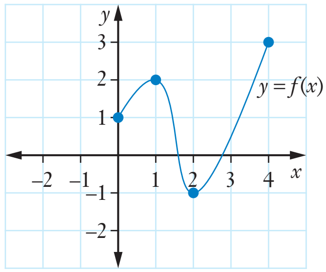
10 **a** Describe the transformation of $y = \frac{1}{x}$ into $y = \frac{1}{x} + 1$.

b Sketch the graph of $y = \frac{1}{x} + 1$.

11 The graph shows $y = f(x)$. Sketch the graph of:

a $y = f(x) - 1$

b $y = f(x) + 2$



12 **a** Show that $\frac{3x+1}{x} = \frac{1}{x} + 3$.

b Hence or otherwise, sketch the graph of $y = \frac{3x+1}{x}$.

2.02 Horizontal translations of functions

WS Translstions f functions

Translations of functions

WS Gaphing translations of functions

INVESTIGATION

HORIZONTAL TRANSLATIONS

Use a graphics calculator or graphing software to explore the affect of parameter b on each graph below. If you don't have dynamic software, substitute different values for b into the equation. Use positive and negative values, integers and fractions for b.

1 $f(x) = (x + b)^2$

2 $f(x) = (x + b)^3$

3 $f(x) = (x + b)^4$

4 $f(x) = e^{x + b)}$

5 $f(x) = \ln(x + b)$

6 $f(x) = \frac{1}{x + b}$

7 $f(x) = |x + b|$

How does the graph change as the value of b changes?

What is the difference between positive and negative values of b?

Notice that the parameter shifts the graph to the left or right without changing its size or shape. We call this a **horizontal translation** (it shifts the function along the x-axis).

For a horizontal translation the shift is in the opposite direction from the sign of b.

To understand why this happens, we change the subject of the equation to x since the translation is a shift along the x-axis. For example:

$$y = (x + 5)^3$$
$$\sqrt[3]{y} = x + 5$$
$$\sqrt[3]{y} - 5 = x$$

This is a shift of 5 units to the left.

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Horizontal translations

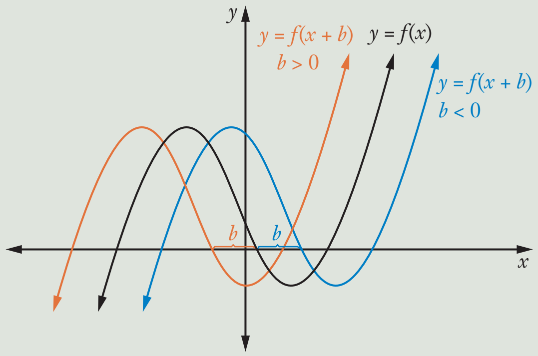
For the function $y = f(x)$:

$y = f(x + b)$ translates the graph horizontally (along the x-axis).

If $b > 0$, the graph is translated to the left by b units.

If $b < 0$, the graph is translated to the right.

A horizontal translation changes the x values of the function.



In Year 11, we learned that $y = \tan(x + b)$ is the graph of $y = \tan x$ shifted left b units.

EXAMPLE 3

a What is the relationship of $f(x) = \log_2(x + 3)$ to $f(x) = \log_2 x$?

b If the graph $y = (x - 4)^3$ is translated 7 units to the right, find the equation of the transformed function.

c The point $P(2, 5)$ lies on the function $y = f(x)$. Find the corresponding (image) point of P given a horizontal translation with $b = 1$.

d The point Q (3, –4) on the graph of $y = f(x - 2)$ is the image of point $P(x, y)$ on $y = f(x)$. Find the coordinates of P.

Solution

a $f(x) = \log_2(x + 3)$ is a horizontal translation 3 units to the left from the parent function $f(x) = \log_2 x$.

b If $y = (x - 4)^3$ is translated 7 units to the right:

$y = f(x + b)$ where $b = -7$

$y = (x - 4 - 7)^3 = (x - 11)^3$

So the equation of the transformed function is $y = (x - 11)^3$

c $y = f(x + b)$ describes a horizontal translation (along the x-axis).

When $b = 1$, x values shift 1 unit to the left.

Image of $P \equiv (2 - 1, 5) \equiv (1, 5)$

d $y = f(x - 2)$ is a horizontal translation 2 units to the right of $y = f(x)$.

So (x, y) becomes $(x + 2, y)$

But $Q(3, -4)$ is the image of $P(x, y)$

So $(x + 2, y) \equiv (3, -4)$

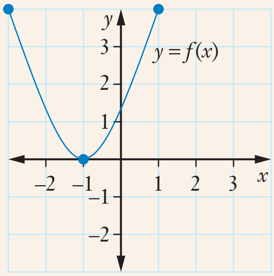
So $x + 2 = 3, y = -4$

$x = 1, y = -4$

So $P \equiv (1, -4)$

EXAMPLE 4

a The graph of $y = f(x)$ shown is transformed into $y = f(x + b)$. Sketch the transformed graph if $b = -3$.



b Sketch the graph of:

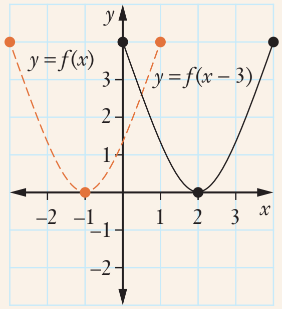
i $y = |x + 3|$

ii $y = \dfrac{1}{x - 2}$

Solution

a The graph $y = f(x + b)$ where $b = -3$ describes a horizontal translation of 3 units to the right.

The transformed graph is 3 units to the right of the original function.



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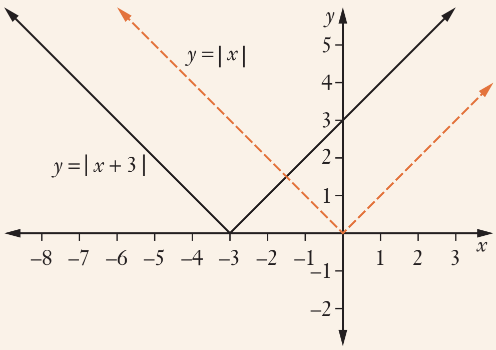
b **i** The function $y = |x + 3|$ is in the form $y = f(x + b)$ where $b = 3$.

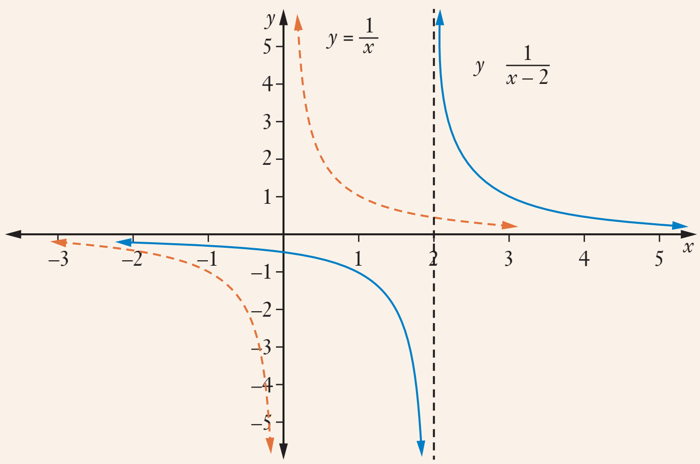
Since $b > 0$, $y = |x|$ is shifted 3 units to the left.



If you need to find some points on the graph of $y = |x + 3|$ you could subtract 3 from x values of $y = |x|$.

ii $y = \dfrac{1}{x - 2}$ is in the form $y = f(x + b)$ where $b = -2$.

Since $b < 0$, $y = \dfrac{1}{x}$ is shifted 2 units to the right.



Exercise 2.02 Horizontal translations of functions

1 Describe how each constant affects the graph of $y = x^2$.

a $y = (x - 4)^2$ **b** $y = (x + 2)^2$

2 Describe how each constant affects the graph of $y = x^3$.

a $y = (x - 5)^3$ **b** $y = (x + 3)^3$

3 Find the equation of each translated graph.

a $y = x^2$ translated 3 units to the left
b $f(x) = 2^x$ translated 8 units to the right
c $y = |x|$ translated 1 unit to the left
d $y = x^3$ translated 4 units to the right
e $f(x) = \log x$ translated 3 units left

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4 Describe how $y = \frac{1}{x}$ transforms to $y = \frac{1}{x-3}$.

5 Describe the relationship between $f(x) = x^4$ and:

a $f(x) = (x+2)^4$ **b** $f(x) = (x-5)^4$

6 Find the equation if:

a $y = -x^2$ is translated

i 4 units to the left **ii** 8 units to the right

b $y = |x|$ is translated

i 3 units to the right **ii** 4 units to the left

c $y = e^{x+2}$ is translated

i 4 units to the left **ii** 7 units to the right

d $y = \log_2(x-3)$ is translated

i 2 units to the right **ii** 3 units to the left

7 If $P = (1, -3)$ lies on the function $y = f(x)$, find the image point of P if the function is transformed to $y = f(x+b)$ where:

a $b = -4$ **b** $b = 9$ **c** $b = t$

8 Find the original point on the function $y = f(x)$ if the coordinates of its image are $(-1, 2)$ when the function is translated:

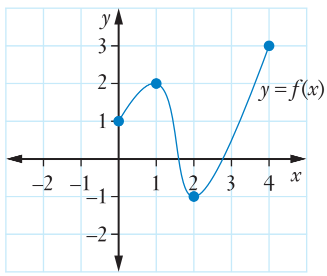
a 4 units to the left **b** 8 units to the right

9 Sketch on the same number plane:

a $y = x^3$ and $y = (x+1)^3$ **b** $f(x) = \ln x$ and $f(x) = \ln(x+2)$

10 The graph shown is $y = f(x)$. Sketch the graph of:

a $y = f(x-1)$ **b** $y = f(x+3)$



11 Find the equation of the transformed function if $f(x) = x^5$ is translated

a 5 units down **b** 3 units to the right

c 2 units up **d** 7 units to the left

12 The point $P(3, -2)$ is the image of a point on $y = f(x)$ after it has been translated 4 units to the left. Find the original point.

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2.03 Vertical dilations of functions

A **dilation** stretches or compresses a function, changing its size and shape.

INVESTIGATION

VERTICAL DILATION

Explore the effect of parameter k on each graph below. If you don't have dynamic software, substitute different values for k into the equation. Use positive and negative values, integers and fractions for k.

1 $f(x) = kx$
2 $f(x) = kx^2$
3 $f(x) = kx^3$
4 $f(x) = kx^4$
5 $f(x) = ke^x$
6 $f(x) = k \ln x$
7 $f(x) = k\left(\frac{1}{x}\right)$
8 $f(x) = k|x|$

How does the graph change as the value of k changes?

What is the difference between positive and negative values of k?

Notice that k stretches the graph up and down along the y-axis and changes its shape. We call this **vertical dilation**. The value of the parameter k controls the amount of stretching (expanding) or shrinking (compressing).

We call k the **scale factor**.

Vertical dilations

For the curve $y = f(x)$:

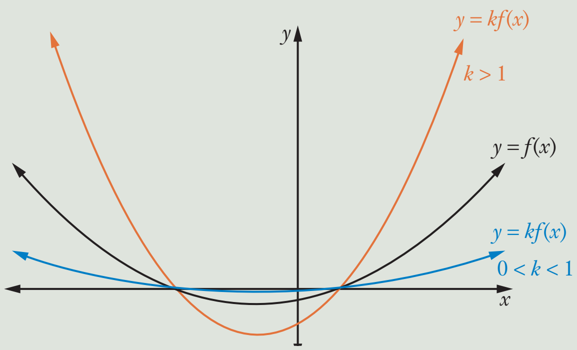
$y = kf(x)$ dilates the curve vertically (along the y-axis) by a scale factor of k.

If $k > 1$, the graph is stretched, or expanded.

If $0 < k < 1$, the graph is shrunk, or compressed.

A vertical dilation changes the y values of the function.

In Year 11, we learned that $y = k \cos x$ is the graph of $y = \cos x$ stretched vertically to give an amplitude of k.



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EXAMPLE 5

a The function $y = x^7$ is dilated vertically by a factor of 3. Find the equation of the transformed function.

b Describe how the function $f(x) = \frac{\log_2 x}{2}$ is related to the function $f(x) = \log_2 x$.

c Find the scale factor of each dilation of a function and state whether the dilation stretches or compresses the graph.

i $y = 7x^2$ **ii** $y = \frac{e^x}{5}$

Solution

a If a function $y = f(x)$ has a vertical dilation with factor k, the equation of its transformed function is $y = kf(x)$.

So if the function $y = x^7$ has a vertical dilation with factor 3, the equation of the transformed function is $y = 3x^7$.

Since $k > 1$, the function is stretched vertically.

b $f(x) = \frac{\log_2 x}{2}$

$= \frac{1}{2}\log_2 x$

So the function is in the form $y = kf(x)$ where $k = \frac{1}{2}$.

Since $0 < k < 1$, the function is compressed vertically.

So $f(x) = \frac{\log_2 x}{2}$ is the result of $f(x) = \log_2 x$ being dilated (compressed) vertically by a scale factor of $\frac{1}{2}$.

c The function $y = kf(x)$ has scale factor k.

i $y = 7x^2$ has scale factor 7 (stretched)

ii $y = \frac{e^x}{5}$

$= \frac{1}{5}e^x$

Scale factor is $\frac{1}{5}$ (compressed).

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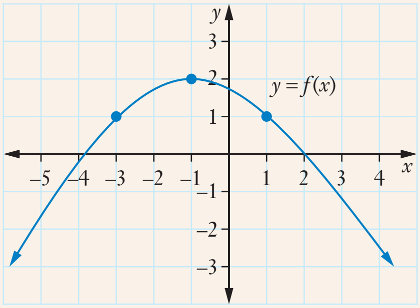
EXAMPLE 6

a The point $N = (-1, 8)$ lies on the function $y = f(x)$. Find the image of N on the function $y = kf(x)$ when:

i $k = 5$ **ii** $k = \frac{1}{2}$

b A function $y = f(x)$ is transformed to $y = kf(x)$. If the image of point A on the transformed function is $(-6, 12)$, find the coordinates of A when $k = 3$.

c The graph shown is $y = f(x)$.
Sketch the graph of $y = 2f(x)$.



d Sketch the graphs of $y = x^2$ and $y = \frac{x^2}{2}$ on the same set of axes.

Solution

a $y = kf(x)$ describes a vertical dilation (along the y-axis).

So the y values of the parent function will change.

i When $k = 5$: y values are multiplied by a factor of 5.

Image of $N \equiv (-1, 8 \times 5) \equiv (-1, 40)$

ii When $k = \frac{1}{2}$: y values will be multiplied by a factor of $\frac{1}{2}$ (or divided by 2).

Image of $N \equiv \left(-1\ \ 8 \times \frac{1}{2}\right) \equiv (-1, 4)$

b When $k = 3$, (x, y) becomes $(x, 3y)$.

$(x, 3y) \equiv (-6, 12)$

$x = -6$

$3y = 12$

$y = 4$

So $A \equiv (-6, 4)$

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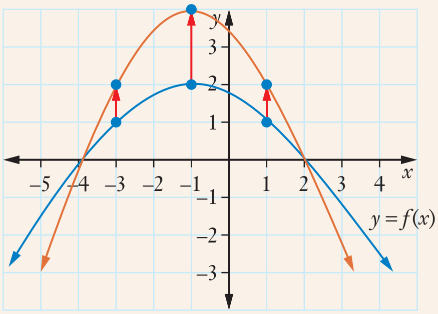
c The graph of $y = 2f(x)$ is a vertical dilation of $y = f(x)$ with factor 2.

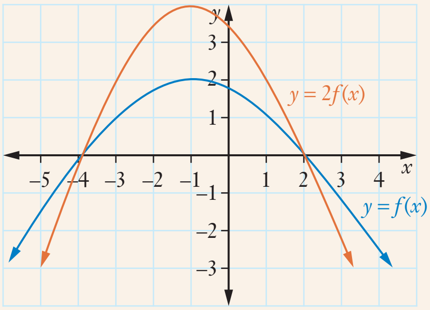
So each y value is doubled and the graph is twice as high as the original graph. For example:

$y = 1$ becomes $y = 2$

$y = 2$ becomes $y = 4$



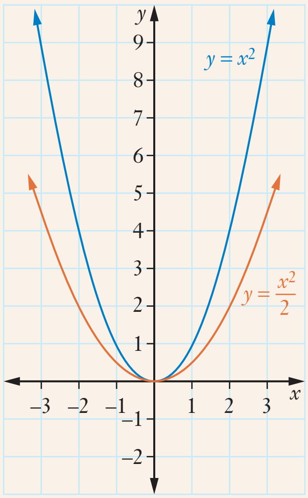
The transformed graph is still a parabola. However it is higher (stretched) and narrower than the original graph.



d $y = \frac{x^2}{2}$ is a vertical dilation of $y = x^2$ with scale factor $\frac{1}{2}$.

This halves the y values.

$(-3, 9)$	becomes	$\left(-3\ \ 4\frac{1}{2}\right)$
$(-2, 4)$	becomes	$(-2, 2)$
$(-1, 1)$	becomes	$\left(-1\ \ \frac{1}{2}\right)$
$(0, 0)$	becomes	$(0, 0)$
$(1, 1)$	becomes	$\left(-1\ \ \frac{1}{2}\right)$
$(2, 4)$	becomes	$(2, 2)$
$(3, 9)$	becomes	$\left(3\ \ 4\frac{1}{2}\right)$



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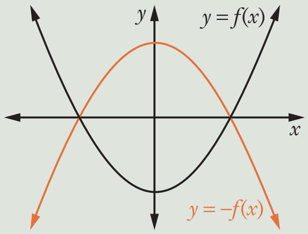
Reflections in the x-axis

You studied reflections in Year 11 in Chapter 7, *Further functions*.

Reflections in the x-axis

$y = -f(x)$ is a reflection of the curve $y = f(x)$ in the x-axis.

This is also a vertical dilation with scale factor $k = -1$.



EXAMPLE 7

a Point $P(2, 4)$ is on the function $y = f(x)$. Find the image of P on the function $y = -f(x)$.

b Sketch the vertical dilation of $f(x) = \dfrac{1}{x}$ with scale factor -1.

Solution

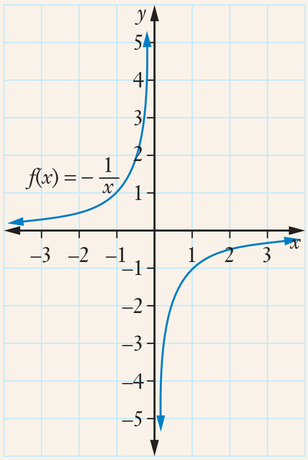
a The function $y = -f(x)$ is a reflection in the x-axis.

The y values are multiplied by -1.

Image of $P \equiv (2, 4 \times [-1]) \equiv (2, -4)$

b A vertical stretch with scale factor -1

is a reflection of $f(x) = \dfrac{1}{x}$ in the x-axis.



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Exercise 2.03 Vertical dilations of functions

1 Describe how the constant affects each transformed graph, given the parent function, and state the scale factor.

a $y = x$

i $y = 6x$ **ii** $y = \frac{x}{2}$ **iii** $y = -x$

b $y = x^2$

i $y = 2x^2$ **ii** $y = \frac{x^2}{6}$ **iii** $y = -x^2$

c $y = x^3$

i $y = 4x^3$ **ii** $y = \frac{x^3}{7}$ **iii** $y = \frac{4x^3}{3}$

d $y = x^4$

i $y = 9x^4$ **ii** $y = \frac{x^4}{3}$ **iii** $y = \frac{3x^4}{8}$

e $y = |x|$

i $y = 5|x|$ **ii** $y = \frac{|x|}{8}$ **iii** $y = -|x|$

f $f(x) = \log x$

i $f(x) = 9 \log x$ **ii** $f(x) = -\log x$ **iii** $f(x) = \frac{2\log x}{5}$

2 Find the equation of each transformed graph and state its domain and range.

a $y = x^2$ dilated vertically with a scale factor of 6

b $y = \ln x$ dilated vertically with a scale factor of $\frac{1}{4}$

c $f(x) = |x|$ reflected in the x-axis

d $f(x) = e^x$ dilated vertically with a scale factor of 4

e $y = \frac{1}{x}$ dilated vertically with a scale factor of 7

3 Find the equation of each transformed function after the vertical dilation given.

a $y = 3^x$ with scale factor 5

b $f(x) = x^2$ with scale factor $\frac{1}{3}$

c $y = x^3$ with scale factor -1

d $y = \frac{1}{x}$ with scale factor $\frac{1}{2}$

e $y = |x|$ with scale factor $\frac{2}{3}$

4 Point $M = (3, 6)$ lies on the graph of $y = f(x)$. Find the coordinates of the image of M when $f(x)$ is:

a dilated vertically with a factor of 4

b reflected in the x-axis

c dilated vertically with a factor of 12

d dilated vertically with a factor of $\frac{5}{6}$

5 The coordinates of the image of $X(x, y)$ are (4, 12) when $y = f(x)$ is vertically dilated. Find the coordinates of X if the scale factor is:

a 3 **b** 2 **c** $\frac{1}{3}$ **d** $\frac{3}{4}$ **e** -1

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6 Sketch each pair of functions on the same set of axes.

a $f(x) = \log_2 x$ and $f(x) = 2\log_2 x$

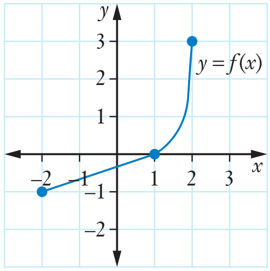
b $y = 3^x$ and $y = 2 \cdot 3^x$

c $y = \dfrac{1}{x}$ and $y = \dfrac{3}{x}$

d $y = |x|$ and $y = 2|x|$

e $y = x^3$ and $y = -x^3$

7 Points on a function $y = f(x)$ are shown on the graph. Sketch the graph of the transformed function showing the image points, given a vertical stretch with factor:



a 3 **b** $\dfrac{1}{2}$ **c** -1

8 Sketch the graph of $y = 2\sqrt{1 - x^2}$.

2.04 Horizontal dilations of functions

Dilaions o functions

Advanced graphs

INVESTIGATION

HORIZONTAL DILATIONS

Use dynamic geometry software to explore the affect of parameter a on each graph below. If you don't have dynamic software, substitute different values for a into the equation. Use positive and negative values, integers and fractions for a.

1 $f(x) = ax$

2 $f(x) = (ax)^2$

3 $f(x) = (ax)^3$

4 $f(x) = (ax)^4$

5 $f(x) = e^{ax}$

6 $f(x) = \ln ax$

7 $f(x) = \dfrac{1}{ax}$

8 $f(x) = |ax|$

How does a transform the graph as the value of a changes?

What is the difference between positive and negative values of a?

Notice that with **horizontal dilations**, the higher the value of a, the more the graph is compressed along the x-axis from left and right. This is inverse variation and the scale factor for horizontal dilations is $\dfrac{1}{a}$.

This is because horizontal dilation affects the x values of the function. To see this, we change the subject of the function to x. For example:

$$y = (3x)^3$$

$$\sqrt[3]{y} = 3x$$

$$\frac{\sqrt[3]{y}}{3} = x$$

or $x = \dfrac{1}{3}\sqrt[3]{y}$

> In Year 11, we learned that $y = \sin ax$ is the graph of $y = \sin x$ compressed horizontally to give a period of $\dfrac{2\pi}{a}$.

This shows a scale factor of $\frac{1}{3}$.

Like horizontal translations, a horizontal stretch works the opposite way to what you would expect, because the equation is in the form $y = f(x)$ rather than $x = f(y)$.

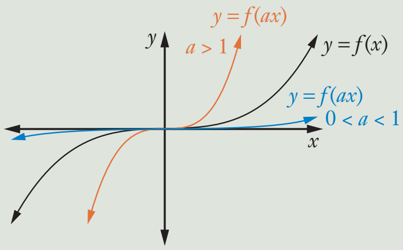
Horizontal dilations

For the curve $y = f(x)$:

$y = f(ax)$ stretches the curve horizontally (along the x-axis) by a scale factor of $\frac{1}{a}$.

If $a > 1$, the graph is compressed.

If $0 < a < 1$, the graph is stretched.



EXAMPLE 8

a Describe how the function $f(x) = x^3$ is related to the function $f(x) = (4x)^3$.

b The function $y = \ln x$ is dilated horizontally by a scale factor of 2. Find the equation of the transformed function.

c Find the scale factor of each function and state whether it stretches or compresses the graph.

i $y = e^{3x}$ **ii** $f(x) = \left|\frac{x}{4}\right|$

Solution

a The function $y = f(ax)$ is a horizontal dilation of $y = f(x)$ with scale factor $\frac{1}{a}$.

So the function $f(x) = (4x)^3$ is a horizontal dilation of $f(x) = x^3$ with scale factor $\frac{1}{4}$.

b If $y = \ln x$ is dilated horizontally by a scale factor of 2:

$$\frac{1}{a} = 2$$

$$a = \frac{1}{2}$$

So $y = \ln\left(\frac{1}{2}x\right)$ or $y = \ln\frac{x}{2}$

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c **i** $y = e^{3x}$ is in the form $y = f(ax)$ where $f(x) = e^x$.

This is a horizontal dilation with $a = 3$.

Scale factor $= \frac{1}{a} = \frac{1}{3}$ (stretched)

ii $f(x) = \left|\frac{x}{4}\right|$ can be written as $f(x) = \left|\frac{1}{4}x\right|$.

The function is in the form $y = f(ax)$ where $f(x) = |x|$.

This is a horizontal dilation with $a = \frac{1}{4}$.

Scale factor $= \frac{1}{a}$

$= \frac{1}{\frac{1}{4}}$

$= 4$ (compressed)

EXAMPLE 9

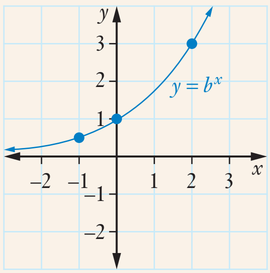
a The points $P(-3, 4)$ and $Q(9, 0)$ lie on the function $y = f(x)$. Find the coordinates of the images of P and Q for the function $y = f(ax)$ when:

i $a = 3$ **ii** $a = \frac{1}{5}$

b When the function $y = f(x)$ is transformed to $y = f(ax)$, the coordinates of the image of $N(x, y)$ are $(16, -5)$. Find the coordinates of N when:

i $a = 4$ **ii** $a = \frac{1}{2}$

c The graph of $y = b^x$ shown is transformed to $y = b^{2x}$. Sketch the graph of the transformed function.



d State the scale factor if the graph $y = |x|$ is transformed to $y = \left|\frac{x}{2}\right|$ and sketch both graphs on the same set of axes.

Solution

a The function $y = f(ax)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{a}$.

i When $a = 3$, scale factor is $\frac{1}{3}$.

All x values are multiplied by $\frac{1}{3}$ (divided by 3).

Image of $P \equiv \left(-3 \times \frac{1}{3}, 4\right) \equiv (-1, 4)$

Image of $Q \equiv \left(9 \times \frac{1}{3}, 0\right) \equiv (3, 0)$

ii When $a = \frac{1}{5}$, scale factor is $\frac{1}{\frac{1}{5}}$ or 5.

All x values are multiplied by 5.

Image of $P \equiv (-3 \times 5, 4) \equiv (-15, 4)$

Image of $Q \equiv (9 \times 5, 0) \equiv (45, 0)$

b We multiply all x values by scale factor $\frac{1}{a}$.

i When $a = 4$, scale factor is $\frac{1}{4}$

So (x, y) becomes $\left(x \times \frac{1}{4}\ \ y\right) = \left(\frac{x}{4}\ \ y\right)$

$\left(\frac{x}{4}\ \ y\right) \equiv (16, -5)$

$\frac{x}{4} = 16$

$x = 64$

$y = -5$

So $N \equiv (64, -5)$

ii When $a = \frac{1}{2}$, scale factor is $\frac{1}{\frac{1}{2}}$ or 2.

So (x, y) becomes $(x \times 2, y) \equiv (2x, y)$

$(2x, y) \equiv (16, -5)$

$2x = 16$

$x = 8$

$y = -5$

So $N \equiv (8, -5)$

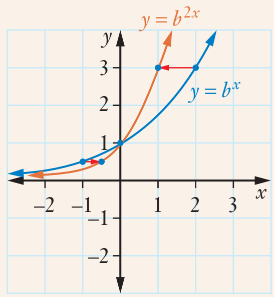
c The graph of $y = b^{2x}$ describes a horizontal dilation of $y = b^x$ with scale factor $\frac{1}{2}$.

So we halve the x values.

$x = -1$ becomes $x = -\frac{1}{2}$

$x = 0$ becomes $x = 0$

$x = 2$ becomes $x = 1$



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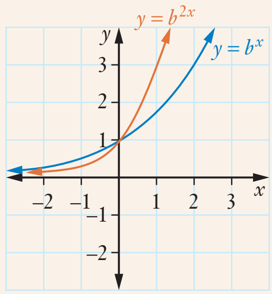
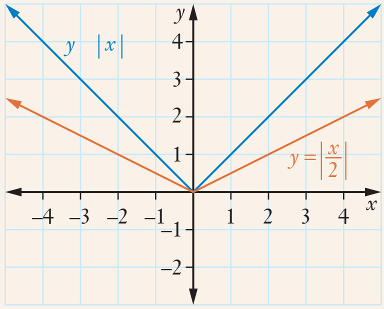
The transformed function is still in the shape of an exponential function, but it has changed shape and size.



d The graph $y = \left|\frac{x}{2}\right|$ is a horizontal dilation of $y = |x|$ with a scale factor $\frac{1}{\frac{1}{2}}$ or 2.

We double the x values.

(−3, 3)	becomes	(−6, 3)
(−2, 2)	becomes	(−4, 2)
(−1, 1)	becomes	(−2, 1)
(0, 0)	becomes	(0, 0)
(1, 1)	becomes	(2, 1)
(2, 2)	becomes	(4, 2)
(3, 3)	becomes	(6, 3)



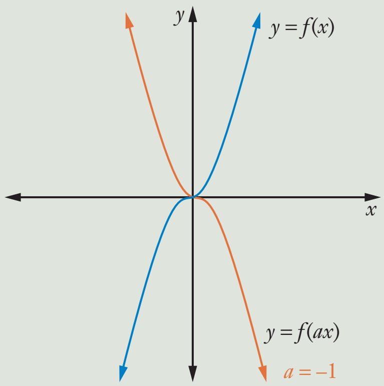
Reflections in the *y*-axis

You studied reflections in the y-axis in Year 11 in Chapter 7, *Further functions*.

Reflections in the *y*-axis

$y = f(-x)$ is a reflection of the curve $y = f(x)$ in the y-axis.

This is a horizontal stretch with scale factor $a = \frac{1}{-1} = -1$.



Notice that for even functions $y = f(x) = f(-x)$.

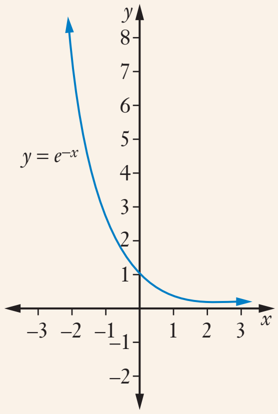
Even functions are already symmetrical about the y-axis so the reflected graph is the same as the original graph.

EXAMPLE 10

Sketch the graph of the horizontal dilation of $y = e^x$ with scale factor -1.

Solution

The horizontal dilation with scale factor -1 is a reflection of $y = e^x$ in the y-axis.



Exercise 2.04 Horizontal dilations of functions

1 Describe the transformation that the constant makes on $f(x) = x^4$ and state the scale factor.

a $f(x) = (8x)^4$ **b** $f(x) = \left(\frac{x}{5}\right)^4$

c $f(x) = \left(\frac{3x}{7}\right)^4$ **d** $f(x) = (-x)^4$

2 Describe whether the constant describes a horizontal or vertical dilation and state the scale factor.

a $y = x^2$

i $y = (2x)^2$ **ii** $y = (5x)^2$ **iii** $y = \left(\frac{x}{3}\right)^2$

b $y = x^3$

i $y = 4x^3$ **ii** $y = \left(\frac{x}{2}\right)^3$ **iii** $y = (-x)^3$

c $y = x^4$

i $y = (7x)^4$ **ii** $y = \frac{x^4}{8}$ **iii** $y = \left(\frac{3x}{4}\right)^4$

d $y = |x|$

i $y = |5x|$ **ii** $y = \left|\frac{x}{2}\right|$ **iii** $y = \left|\frac{3x}{5}\right|$

e $y = 5^x$

i $y = 5^{3x}$ **ii** $y = -5^x$ **iii** $y = 5^{\frac{x}{2}}$

f $f(x) = \log x$

i $f(x) = 8 \log x$ **ii** $f(x) = \log(-x)$ **iii** $f(x) = \log \frac{x}{7}$

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3 Find the equation of each transformed graph and state its domain and range.

a $f(x) = |x|$ is dilated horizontally with a scale factor of $\frac{1}{5}$

b $y = x^2$ is dilated horizontally with a scale factor of 3

c $y = x^3$ is reflected in the y-axis.

d $y = e^x$ is dilated vertically with a scale factor $\frac{1}{9}$

e $y = \log_4 x$ is reflected in the x-axis

4 Point X (–2, 7) lies on $y = f(x)$. Find the coordinates of the image of X on $y = f(ax)$ given:

a $a = 2$ **b** $a = -1$ **c** $a = \frac{1}{3}$

5 The function $y = f(x)$ is transformed into the function $y = f(ax)$. The coordinates of the image point of (x, y) on the original function are (–24, 1) on the transformed function. Find the values of (x, y) if:

a $a = 3$ **b** $a = 2$ **c** $a = \frac{1}{4}$

6 Sketch each pair of functions on the same set of axes:

a $f(x) = \ln x$ and $f(x) = \ln (2x)$ **b** $y = 2^x$ and $y = 2^{\frac{x}{3}}$

c $y = \frac{1}{x}$ and $y = \frac{1}{3x}$ **d** $y = |x|$ and $y = |2x|$

e $f(x) = x^2$ and $f(x) = (3x)^2$ **f** $y = \ln x$ and $y = \ln (-x)$

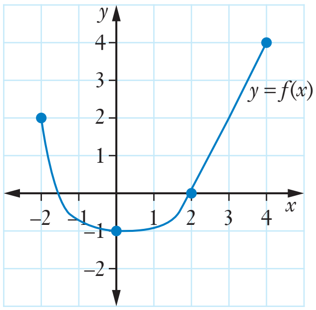
7 Sketch the graphs of $y = e^x$, $y = e^{2x}$ and $y = 2e^x$ on the same set of axes.

8 Explain why a reflection in the y-axis does not change the graph of:

a $y = x^2$ **b** $f(x) = |x|$

9 Sketch the graph of $y = f(ax)$ given the graph of $y = f(x)$ shown, when:

a $a = \frac{1}{2}$ **b** $a = 2$



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WS
Combinaion of transformations

2.05 Combinations of transformations

A function can have any combination of the different types of transformations acting on it.

Transformations

For the curve $y = f(x)$:

$y = f(x) + c$ translates the function vertically:

- up if $c > 0$
- down if $c < 0$

$y = f(x + b)$ translates the function horizontally:

- to the left if $b > 0$
- to the right if $b < 0$

$y = kf(x)$ dilates the function vertically with scale factor k:

- stretches if $k > 1$
- compresses if $0 < k < 1$
- reflects the function in the x-axis if $k = -1$

$y = f(ax)$ dilates the function horizontally with scale factor $\frac{1}{a}$:

- compresses if $a > 1$
- stretches if $0 < a < 1$
- reflects the function in the y-axis if $a = -1$

EXAMPLE 11

a Find the equation of the transformed function if $y = x^4$ is shifted 2 units down and 5 units to the left.

b Find the equation of the transformed function if $y = e^x$ is dilated vertically by a scale factor 3 and translated horizontally 2 units to the right.

Solution

a Starting with $y = x^4$:

A vertical translation 2 units down gives $y = x^4 - 2$.

A horizontal translation 5 units to the left gives $b = 5$.

So the equation becomes $y = (x + 5)^4 - 2$.

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Notice that we could do this the other way around:

A horizontal translation 5 units to the left gives $y = (x + 5)^4$.

A vertical translation 2 units down gives $y = (x + 5)^4 - 2$.

b Starting with $y = e^x$:

A vertical dilation of scale factor 3 gives $y = 3e^x$.

A horizontal translation 2 units to the right gives $b = -2$.

So the equation becomes $y = 3e^{x-2}$.

Notice that we could do this the other way around:

A horizontal translation 2 units to the right gives $y = e^{x-2}$.

A vertical dilation of scale factor 3 gives $y = 3e^{x-2}$.

When the transformations are *both* vertical or *both* horizontal, then the order is important.

EXAMPLE 12

a When the function $y = x^2$ is translated 3 units up (vertically) and vertically dilated by scale factor 4, the equation of the transformed function is $y = 4x^2 + 3$. Find the order in which the transformations were done.

b The equation of the transformed function is $y = (2x + 5)^3$ when the function $y = x^3$ is horizontally dilated by scale factor $\frac{1}{2}$ and translated 5 units (horizontally) to the left. In which order were the transformations done?

c The equation of the transformed function is $y = \ln[3(x - 2)]$ when the function $y = \ln x$ is horizontally dilated by scale factor $\frac{1}{3}$ and translated 2 units (horizontally) to the right. In which order were the transformations done?

Solution

a Starting with $y = x^2$:

A vertical translation 3 units up gives $y = x^2 + 3$.

A vertical dilation by scale factor 4 gives $y = 4(x^2 + 3)$.

This is not the equation of the transformed function.

Try the other way around:

A vertical dilation by scale factor 4 gives $y = 4x^2$.

A vertical translation 3 units up gives $y = 4x^2 + 3$.

So the correct order is the vertical dilation, then the vertical translation.

b Starting with $y = x^3$:

A horizontal dilation of scale factor $\frac{1}{2}$ gives $a = 2$, so the equation is $y = (2x)^3$.

A horizontal translation 5 units to the left gives $b = 5$ so $y = [2(x + 5)]^3$.

This is not the equation of the transformed function.

Try the other way around:

A horizontal translation 5 units to the left gives $y = (x + 5)^3$.

A horizontal dilation of scale factor $\frac{1}{2}$ gives $a = 2$, so the equation is $y = (2x + 5)^3$.

So the correct order is the horizontal translation, then the horizontal dilation.

c Starting with $y = \ln x$:

A horizontal dilation of scale factor $\frac{1}{3}$ gives $y = \ln(3x)$.

A horizontal translation 2 units to the right gives $y = \ln[3(x - 2)]$.

So the correct order is the horizontal dilation, then the horizontal translation.

Doing the horizontal dilation first gives $y = f(a(x + b))$, while doing the horizontal translation first gives $y = f(ax + b)$.

We can state the order we want to perform the transformations.

EXAMPLE 13

Find the equation of the function if $y = x^2$ is first horizontally dilated with scale factor $\frac{1}{2}$, then translated 3 units to the right.

Solution

A horizontal dilation with scale factor $\frac{1}{2}$ gives $a = 2$.

So $y = x^2$ becomes $y = (2x)^2$.

A horizontal translation 3 units to the right gives $b = -3$.

So $y = (2x)^2$ transforms to $y = [2(x - 3)]^2$.

Remember to put brackets around $x - 3$.

We can combine all the transformations into a single expression:

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Equation of a transformed function

$y = kf(a(x + b)) + c$ where a, b, c and k are constants is a transformation of $y = f(x)$:

- a horizontal dilation of scale factor $\frac{1}{a}$
- a horizontal translation of b
- a vertical dilation of k
- a vertical translation of c.

Order of transformations

For $y = kf(a(x + b)) + c$:

1 do horizontal dilation (a), then horizontal translation (b)

2 do vertical dilation (k), then vertical translation (c)

It doesn't matter whether you do horizontal or vertical transformations first.

Notice that the horizontal dilation and translation parameters a and b are inside the brackets (they change x values) and the vertical dilation and translation parameters k and c are outside the brackets (they change the y values).

EXAMPLE 14

a Describe the transformations of $y = e^x$ in the correct order to produce the transformed function $y = \frac{1}{2}e^{x+1} - 3$.

b Describe the transformations of $y = x^2$ in order that give the transformed function $y = 3(2x - 6)^2 + 1$.

c Find the equation of the transformed function if $y = f(x)$ undergoes a vertical dilation with factor 5, a horizontal dilation with factor −1, a translation 4 units to the right and 9 units down.

Solution

a For $y = \frac{1}{2}e^{x+1} - 3$:

Horizontal transformations (a and b): No dilation, $b = 1$ gives a translation 1 unit left.

Vertical transformations (k and c): dilation of scale factor $\frac{1}{2}$ and translation 3 units down. Correct order is:

1 Horizontal translation 1 unit left

2 Vertical dilation of scale factor $\frac{1}{2}$

3 Vertical translation 3 units down

Because vertical transformations can be done first, the order 2–3–1 is also possible.

b For $y = 3(2x - 6)^2 + 1$:

First put the equation in the form $y = kf(a(x + b)) + c$.

$y = 3(2x - 6)^2 + 1$

$\quad = 3[2(x - 3)]^2 + 1$

Horizontal transformations: dilation $a = 2$ and translation $b = -3$.

Vertical transformations: dilation $k = 3$ and translation $c = 1$.

1 Horizontal dilation of scale factor $\frac{1}{2}$

2 Horizontal translation 3 units right

3 Vertical dilation of scale factor 3

4 Vertical translation 1 unit up

The order 3–4–1–2 is also possible.

Alternative method: There is another possible order, if you notice that $y = 3(2x - 6)^2 + 1$ is of the form $y = kf(ax + b) + c$, where the $(ax + b)$ is not factorised, so we can do the horizontal translation first, then horizontal dilation.

The horizontal translation is 6 units right $(b = -6)$ followed by a horizontal dilation of scale factor $\frac{1}{2}$, then 3 and 4 as above.

c We require $y = kf(a(x + b)) + c$.

Horizontal transformations: dilation $a = -1$ and translation $b = -4$.

Vertical transformations: dilation $k = 5$ and translation $c = -9$.

Horizontal transformations: $y = kf(-1(x - 4)) + c$

Add vertical transformations: $y = 5f(-(x - 4)) - 9$

This answer can also be written as $y = 5f(-x + 4) - 9$ or $y = 5f(4 - x) - 9$.

Domain and range

We can find the domain and range of functions without drawing their graphs.

Effect of transformations on domain and range

Horizontal transformations change x values so affect the domain.

Vertical transformations change y values so affect the range.

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EXAMPLE 15

Find the domain and range of:

a $f(x) = -3(x-2)^2 + 5$ **b** $y = 5\sqrt{2x+1}$

Solution

a $y = x^2$ has domain $(-\infty, \infty)$ and range $[0, \infty)$.

Horizontal transformations affect the domain:

No horizontal dilation.

Horizontal translation 2 units right: domain of $x - 2$ is $(-\infty, \infty)$ so domain of $f(x)$ is unchanged.

Vertical transformations affect the range:

Vertical dilation, scale factor -3: Range of y is $[0, \infty)$ so range of $3y$ is 3 times as much, so no change for $[0, \infty)$.

But the $-$ sign in -3 means the y is reflected in the x-axis, so range of $-3y$ is $(-\infty, 0]$.

Vertical translation 5 units up: Range of $-3y$ is $(-\infty, 0]$ so range of $-3y + 5$ is $(-\infty, 5]$.

So $y = -3(x-2)^2 + 5$ has domain $(-\infty, \infty)$ and range $(-\infty, 5]$.

b $y = \sqrt{x}$ has domain $[0, \infty)$ and range $[0, \infty)$.

Horizontal transformations affect the domain:

Domain of $2x + 1$ is $[0, \infty)$ so $2x + 1 \geq 0$.

$2x \geq -1$

$x \geq -\frac{1}{2}$

Vertical transformations affect the range:

Vertical dilation, scale factor 5: Range of $\sqrt{2x+1}$ is $[0, \infty)$, so range of $5\sqrt{2x+1}$ is 5 times as much, so unchanged.

No vertical translation.

So $y = 5\sqrt{2x+1}$ has domain $\left[-\frac{1}{2}\ \infty\right)$ and range $[0, \infty)$.

Exercise 2.05 Combinations of transformations

1 The point (2, –6) lies on the function $y = f(x)$. Find the coordinates of its image if the function is:

a horizontally translated 3 units to the right and vertically translated 5 units down

b translated 4 units up and 3 units to the left

c translated 7 units to the right and 9 units up

d translated 11 units down and 4 units to the left

2 Find the equation of the transformed function where $f(x) = x^5$ is reflected:

a in the x-axis and vertically dilated with scale factor 4

b in the y-axis and horizontally dilated with scale factor 3

3 Find the equation of each transformed function.

a $y = x^3$ is translated 3 units down and 4 units to the left

b $f(x) = |x|$ is translated 9 units up and 1 unit to the right

c $f(x) = x$ is dilated vertically with a scale factor of 3 and translated down 6 units

d $y = e^x$ is reflected in the x-axis and translated up 2 units

e $y = x^3$ is horizontally dilated by a scale factor of $\frac{1}{2}$ and translated down 5 units

f $f(x) = \frac{1}{x}$ is vertically dilated by a factor of 2 and horizontally dilated by a factor of 3

g $f(x) = \sqrt{x}$ is reflected in the y-axis, vertically dilated by a scale factor of 3 and horizontally dilated by a scale factor of $\frac{1}{2}$

h $y = \ln x$ is horizontally dilated by a scale factor of 3 and translated upwards by 2 units

i $f(x) = \log_2 x$ is horizontally dilated by a scale factor of $\frac{1}{4}$ and vertically dilated by a scale factor of 3

j $y = x^2$ is horizontally dilated by a scale factor of 2 and translated down 3 units

4 Describe the transformations to $y = x^3$ in the correct order if the transformed function has equation:

a $y = (x - 1)^3 + 7$

b $y = 4x^3 - 1$

c $y = -5x^3 - 3$

d $y = 2(x + 7)^3$

e $y = 6(2x - 4)^3 + 5$

f $y = 2(3x + 9)^3 - 10$

5 Describe the transformations in their correct order for each of the functions from:

a $y = \log x$ to $y = 2\log(x + 3) - 1$

b $f(x) = x^2$ to $f(x) = -(3x)^2 + 9$

c $y = e^x$ to $y = 2e^{5x} - 3$

d $f(x) = \sqrt{x}$ to $f(x) = 4\sqrt{x - 7} + 1$

e $y = |x|$ to $y = |-2(x + 1)| - 1$

f $y = \frac{1}{x}$ to $y = -\frac{1}{2x} + 8$

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6 The point (8, –12) lies on the function $y = f(x)$. Find the coordinates of the image point when the function is transformed into:

a $y = 3f(x - 1) + 5$ **b** $y = -f(2x) - 7$ **c** $y = 2f(x + 3) - 1$

d $y = 6f(-x) + 5$ **e** $y = -2f(2x - 4) - 3$

7 Given the function $y = f(x)$ find the coordinates of the image of (x, y) if the function is:

a translated 6 units down and 3 units to the right

b reflected in the y-axis and translated 6 units up

c vertically dilated with scale factor 2 and translated 5 units to the left

d horizontally dilated with scale factor 3 and translated 5 units up

e reflected in the x-axis, vertically dilated with scale factor 8, translated 6 units to the left, horizontally dilated with scale factor 5 and translated 1 unit down

8 Find the equation of the transformed function if $y = f(x)$ is:

a translated 2 units down and 1 unit to the left

b translated 5 units to the right and 3 units up

c reflected in the x-axis and translated 4 units to the right

d reflected in the y-axis and translated up 2 units

e reflected in the x-axis and horizontally dilated with a factor of 4

f vertically dilated by a scale factor of 2 and translated 2 units down

9 Find the equation of the transformed function using the correct order of transformations for $y = kf(a(x + b)) + c$.

a $f(x) = \frac{1}{x}$ is reflected in the y-axis, translated up 3 units and dilated vertically by a scale factor of 9

b $y = x^2$ is translated down by 6 units and by 2 units to the left and is horizontally dilated with scale factor $\frac{1}{5}$

c $f(x) = \ln x$ has a vertical dilation with factor 8, a vertical translation of 3 down, a horizontal dilation with factor 2 and a horizontal translation of 5 to the right

d $y = \sqrt{x}$ has a vertical translation of 4 up, a horizontal translation of 4 to the left, a reflection in the y-axis and a vertical dilation with factor 9

e $f(x) = |x|$ is translated up by 7 units, dilated horizontally by a factor of $\frac{1}{6}$ and reflected in the x-axis

f $y = x^3$ is translated 4 units to the left then dilated horizontally with scale factor $\frac{1}{4}$

g $y = 2^x$ is translated up by 5 units, translated 2 units to the right, then is vertically dilated with scale factor 6

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10 Find the domain and range of each function.

a $f(x) = (x + 3)^2 + 5$ **b** $y = 5|-2x| - 2$ **c** $f(x) = \dfrac{1}{2x - 4} + 1$

d $y = 4^{3x} + 2$ **e** $f(x) = 3 \log (3x - 6) - 5$

11 a By completing the square, write the equation for the parabola $y = x^2 + 2x - 7$ in the form $y = (x + a)^2 + b$.

b Describe the transformations on $y = x^2$ that result in the function $y = x^2 + 2x - 7$.

12 Describe the transformations that change $y = x^2$ into the function $y = x^2 - 10x - 3$.

13 The function $y = f(x)$ is transformed to the function $y = kf(a(x + b)) + c$.
Find the coordinates of the image point of (x, y) when:

a $c = 5, b = -3, k = 2$ and $a = \dfrac{1}{2}$

b $c = -2, b = 6, k = -1$ and $a = 3$

14 a Find the equation of the transformed graph if $x^2 + y^2 = 9$ is translated 3 units to the right and 4 units up.

b The circle $x^2 + y^2 = 1$ is transformed into the circle $x^2 - 4x + y^2 + 6y + 12 = 0$. Describe how the circle is transformed.

WS

Gaphing transformed functions

2.06 Graphs of functions with combined transformations

We can find points and sketch the graphs of functions that are changed by a combination of transformations. Translations are the easiest transformations to use since they shift the graph while keeping it the same size and shape.

EXAMPLE 16

Sketch the graph of $y = (x - 2)^2 - 5$.

Solution

$y = (x - 2)^2 - 5$ is transformed from $y = x^2$ by a horizontal translation of 2 units to the right and a vertical translation of 5 units down.

The vertex (turning point) of parabola $y = x^2$ is (0, 0).

So the vertex of $y = (x - 2)^2 - 5$ is $(0 + 2, 0 - 5) \equiv (2, -5)$.

Sketching the graph, we keep the shape of $y = x^2$ and shift it to the new vertex.

We can find the intercepts for a more accurate graph.

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For x-intercepts, $y = 0$:

$$0 = (x - 2)^2 - 5$$
$$5 = (x - 2)^2$$
$$\pm\sqrt{5} = x - 2$$
$$2 \pm \sqrt{5} = x$$

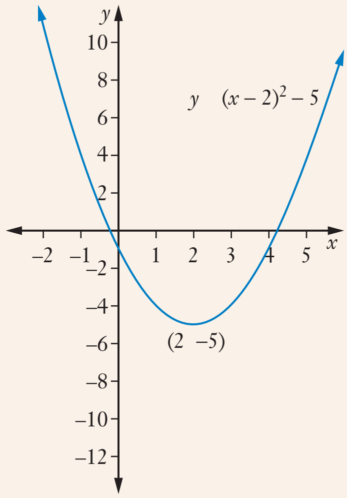
For y-intercepts, $x = 0$:

$$y = (0 - 2)^2 - 5$$
$$= 4 - 5$$
$$= -1$$

So the x-intercepts are approximately 4.2, −0.2.

To find other points on the graph, you can transform points on $y = x^2$ the same way as for the vertex.

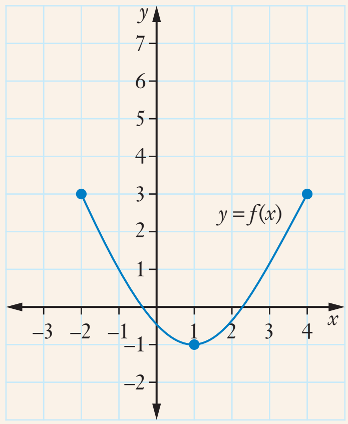
Sketch the graph using a scale on each axis that will show the information. For example, the vertex is at (2, −5) so the y values must go down as far as $y = -5$.



EXAMPLE 17

The graph $y = f(x)$ shown is reflected in the y-axis, dilated vertically with a scale factor of 2 and translated 1 unit up.

Sketch the graph of the transformed function.



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Solution

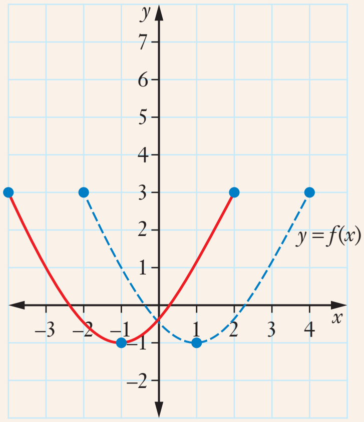
A reflection in the y-axis is a horizontal dilation with scale factor -1.

Multiply each x value by -1.

$x = 1$ becomes $x = -1$

$x = 4$ becomes $x = -4$

$x = -2$ becomes $x = 2$

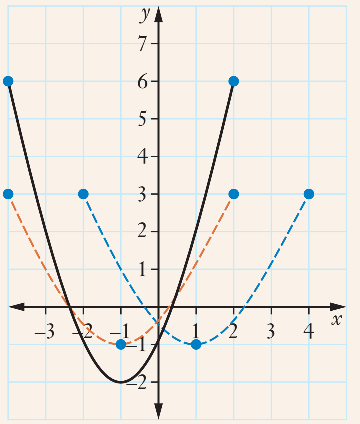


For a vertical dilation with scale factor 2:

Multiply each y value by 2.

$y = -1$ becomes $y = -2$

$y = 3$ becomes $y = 6$

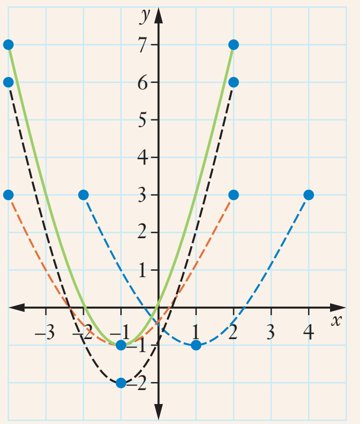


For a vertical translation 1 unit up:

Add 1 to y values.

$y = 6$ becomes $y = 7$

$y = -2$ becomes $y = -1$



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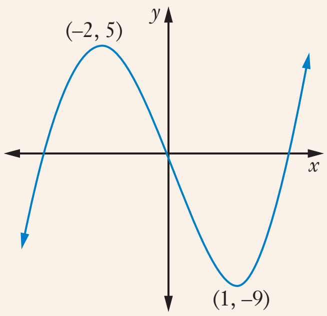
In the previous example, we took one transformation at a time. In the next example, we take transformations together (in the correct order) and plot images of key points on the original (parent) curve.

EXAMPLE 18

a The function $y = f(x)$ is sketched below with stationary (turning) points as shown.



i Describe the transformations if $y = f(x)$ is transformed to $y = 3f(x + 1) - 2$ and how they change the coordinates (x, y) of the parent function.

ii Find the coordinates of the image of each stationary point when the function is transformed.

iii Sketch the graph of $y = 3f(x + 1) - 2$.

b **i** Describe the transformations if $y = |x|$ is transformed to $y = -\left|\frac{x}{2}\right| + 3$ and the image of point (x, y) on the parent function.

ii Sketch the transformed function.

Solution

a **i** Transformations (in order) are:

A horizontal translation 1 unit to the left:

So (x, y) becomes $(x - 1, y)$.

A vertical dilation, scale factor 3:

So $(x - 1, y)$ becomes $(x - 1, 3y)$.

A vertical translation 2 units down:

So $(x - 1, 3y)$ becomes $(x - 1, 3y - 2)$.

ii For $(-2, 5)$:

Image becomes $(-2 - 1, 3 \times 5 - 2) \equiv (-3, 13)$.

For $(1, -9)$:

Image becomes $(1 - 1, 3 \times [-9] - 2) \equiv (0, -29)$.

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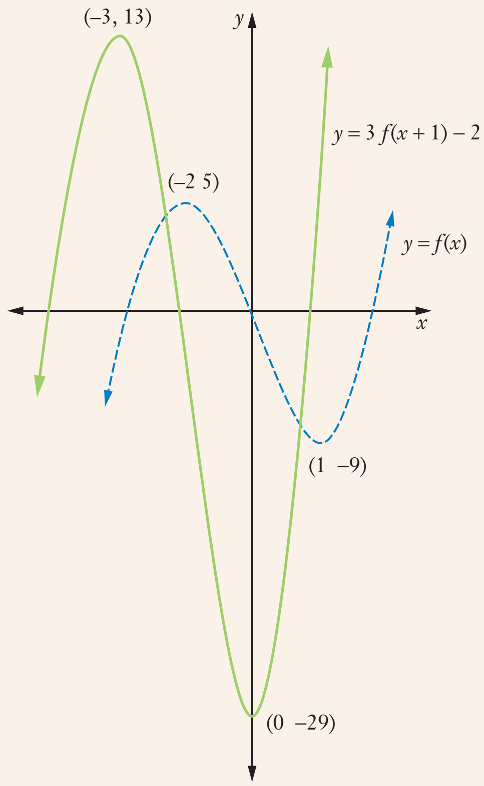
iii $y = f(x)$ passes through (0, 0).

Image becomes $(0 - 1, 3 \times 0 - 2)$
$\equiv (-1, -2)$

Sketch the graph showing this information using a suitable scale on each axis. For example, the y values must go up to 13 and down to −29.



b **i** Transformations (in order) are:

A horizontal dilation, scale factor 2:

So (x, y) becomes $(2x, y)$.

A vertical dilation, scale factor −1 (reflection in the x-axis):

So $(2x, y)$ becomes $(2x, -y)$.

A vertical translation 3 units up:

So $(2x, -y)$ becomes $(2x, -y + 3)$.

ii The intercepts of $y = |x|$ are at (0, 0).

Image of (0, 0) is $(2 \times 0, -0 + 3) \equiv (0, 3)$.

We can find the intercepts on $y = -\left|\frac{x}{2}\right| + 3$

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For x-intercept, $y = 0$:

$$0 = -\left|\frac{x}{2}\right| + 3$$

$$-3 = -\left|\frac{x}{2}\right|$$

$$3 = \left|\frac{x}{2}\right|$$

$$\pm 3 = \frac{x}{2}$$

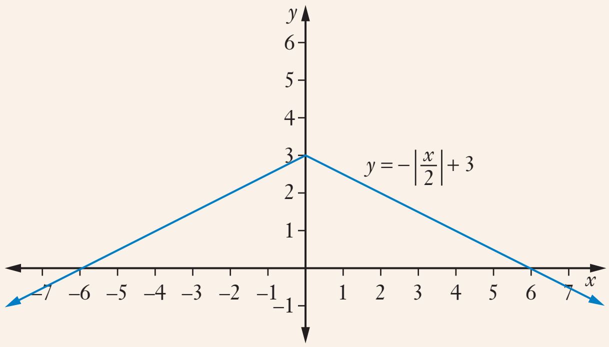
$$\pm 6 = x$$

For y-intercept, $x = 0$:

$$y = -\left|\frac{0}{2}\right| + 3$$

$$= 3$$

Sketching this information using an appropriate scale gives the graph.



Exercise 2.06 Graphs of functions with combined transformations

1 Given $f(x) = x^2$, sketch the graph of:

a $f(x) = x^2 + c$ when

i $c > 0$ **ii** $c < 0$

b $f(x) = (x + b)^2$ when

i $b > 0$ **ii** $b < 0$

c $f(x) = kx^2$ when

i $k > 1$ **ii** $0 < k < 1$ **iii** $k = -1$

d $f(x) = (ax)^2$ when

i $a > 1$ **ii** $0 < a < 1$ **iii** $a = -1$

2 Sketch the graph of the transformed function if the parabola $y = x^2$ is transformed into:

a $y = (x + 2)^2 + 4$ **b** $y = (x - 3)^2 - 1$ **c** $y = (x - 1)^2 + 3$

d $y = -(x + 1)^2 - 2$ **e** $y = 2(x - 1)^2 - 4$

3 Sketch the graph of the transformed function if the cubic function $y = x^3$ is transformed into:

a $y = (x - 1)^3 + 2$ **b** $y = (x - 2)^3 - 3$ **c** $y = -(x + 1)^3 + 4$

d $y = 2(x + 3)^3 - 5$ **e** $y = 3(x - 1)^3 - 2$

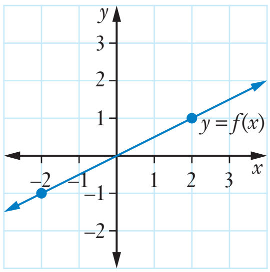
4 A cubic function has stationary points at (6, 1) and (–3, –2).

a Find the images of these points if the function is transformed to $y = -2f(3x) + 1$.

b Sketch the graph of the transformed function.

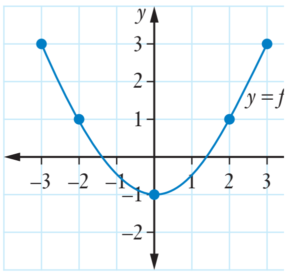
5 Given each function $y = f(x)$, sketch the graph of the transformed function.

a



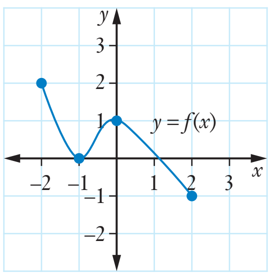
i $y = 3f(x - 1)$ **ii** $y = -f(2x) + 3$

b



i $y = 3f(x + 3) - 2$ **ii** $y = -2f\left(\frac{x}{4}\right) + 3$

c



i $y = 2f(-x) - 1$ **ii** $y = -3f(2x + 4) + 2$

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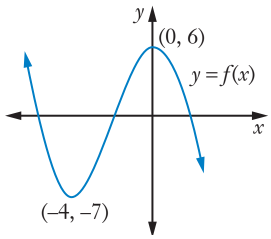
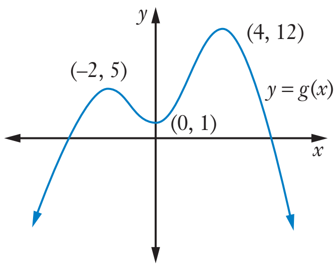
6 For the function $y = f(x)$ with turning points as shown, sketch the transformed function if it is vertically dilated with scale factor 3, translated 4 units down, and horizontally translated 2 units to the left.



7 For the function $y = g(x)$ with turning points as shown, sketch the graph of the transformed function $y = -g[2(x-1)] - 5$.



8 Sketch the graph of:

a $y = -3(x-2)^3 + 1$ **b** $y = 2e^{x+1} - 4$

c $f(x) = 3\sqrt{x-2} - 1$ **d** $y = 2\,|3x| + 4$

e $y = -(3x)^2 + 1$

9 Sketch the graph of:

a $y = 3 - 2\ln x$ **b** $f(x) = -2e^x + 1$

c $y = 1 - (x+1)^3$ **d** $y = \dfrac{2}{x-1} + 3$

e $y = -2(x-3) + 1$

10 a The coordinates of the image of (x, y) when $y = f(x)$ is transformed to $y = 3f(x-2) + 1$ are (–3 2. Find the original point (x, y).

b Sketch the graph of the original function $y = f(x)$ if $y = 3f(x-2) + 1$ is a cubic function with turning points (–3, 2) and (2, –4).

11 The coordinates of the image point of the vertex (x, y) of a parabola are (–24, 18) when $y = f(x)$ is transformed as shown below. Find the coordinates of the original point (x, y) and sketch the graph of the original quadratic function.

a $y = 3f(x-2) - 5$

b $y = -5f[3(x+1)]$

c $y = 2f(2x-6) - 3$

Solving inequalities graphically

2.07 Equations and inequalities

We can use the graphs of transformed functions to solve equations.

EXAMPLE 19

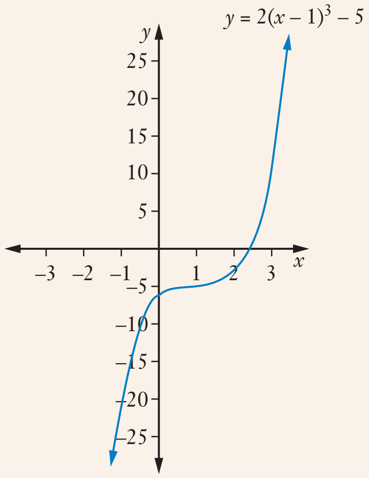
The graph of the cubic function $y = 2(x - 1)^3 - 5$ is shown.

a Solve graphically:

 i $2(x - 1)^3 - 5 = 0$

 ii $2(x - 1)^3 - 5 = 10$

b Solve each of the equations in part **a** algebraically.



Solution

a **i** The solution of $2(x - 1)^3 - 5 = 0$ is where $y = 0$ (x-intercepts).

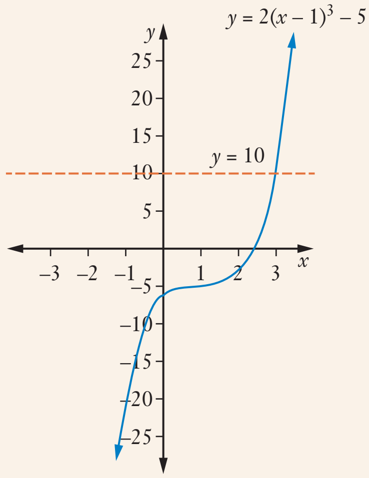
From the graph, the x-intercept is 2.4.

The solution is $x = 2.4$.

ii Draw the line $y = 10$ on the graph.

The solution of $2(x - 1)^3 - 5 = 10$ is where the line intersects the graph.

The solution is $x = 2.9$.



b **i**
$$\begin{aligned} 2(x - 1)^3 - 5 &= 0 \\ 2(x - 1)^3 &= 5 \\ (x - 1)^3 &= 2.5 \\ x - 1 &= \sqrt[3]{2\ 5} \\ x &= \sqrt[3]{2\ 5} + 1 \\ &= 2.36 \end{aligned}$$

ii
$$\begin{aligned} 2(x - 1)^3 - 5 &= 10 \\ 2(x - 1)^3 &= 15 \\ (x - 1)^3 &= 7.5 \\ x - 1 &= \sqrt[3]{7\ 5} \\ x &= \sqrt[3]{7\ 5} + 1 \\ &= 2.96 \end{aligned}$$

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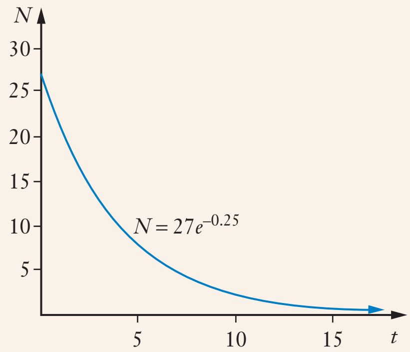
We can use transformed functions to find solutions to practical questions.

EXAMPLE 20

The graph of $N = 27e^{-0.25t}$ shows the number N of cases of measles over t weeks in a country region.



a Use the graph to find the solution to $27e^{-0.25t} = 10$.

b State the meaning of this solution.

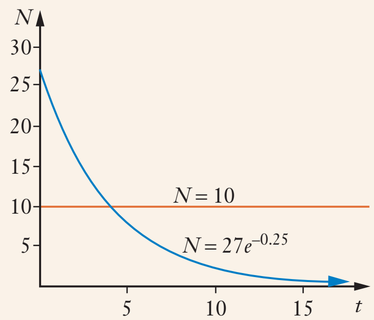
c Solve the equation algebraically.

Solution

a Draw the line $N = 10$ on the graph.

The solution will be where the line intersects the graph.

The solution is $t = 4$.



b This solution means that after 4 weeks there will be 10 cases of measles.

c

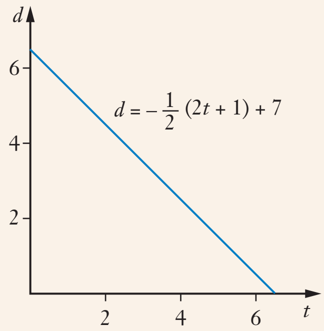
$$\begin{aligned} 27e^{-0.25t} &= 10 \\ e^{-0.25t} &= \frac{10}{27} \\ \ln e^{-0.25t} &= \ln \frac{10}{27} \\ -0.25t &= -0.99325\ \ldots \\ t &= \frac{-099325}{-025} \\ &= 3.97300\ldots \\ &\approx 3.97 \end{aligned}$$

We can solve inequalities graphically.

EXAMPLE 21

a The graph is of the function

$d = -\frac{1}{2}(2t+1)+7$ where d is the distance (in cm) of a marble at t seconds as it rolls towards a barrier.



Solve graphically and explain the solutions:

i $-\frac{1}{2}(2t+1)+7=4$

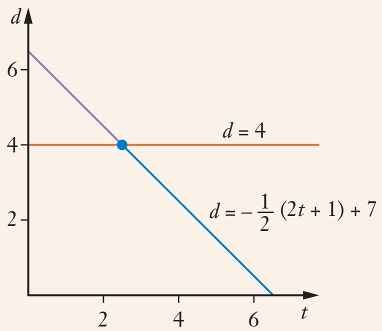
ii $-\frac{1}{2}(2t+1)+7 \geq 4$

b Sketch the graph of $y = 2(x+3)^2 - 5$ and solve graphically:

i $2(x+3)^2 - 5 = 3$ **ii** $2(x+3)^2 - 5 < 3$

Solution

a Draw the line $d = 4$ across the graph.



i From the graph, the solution of $-\frac{1}{2}(2t+1)+7=4$ is $x = 2.5$.

This means that at 2.5 seconds, the marble is 4 cm from the barrier.

ii The solution of $-\frac{1}{2}(2t+1)+7 \geq 4$ is all the t values on and above the line $d = 4$, shown in purple.

For this part of the graph, $t \leq 2.5$.

Because $t \geq 0$ (time is never negative), $0 \leq t \leq 2.5$ is the solution.

This means that for the first 2.5 seconds the marble is 4 cm or more from the barrier.

b The function $y = 2(x+3)^2 - 5$ is a transformation of $y = x^2$.

The vertex of $y = x^2$ is (0, 0).

The image of (0, 0) is $(0 - 3, 0 \times 2 - 5) \equiv (-3, -5)$

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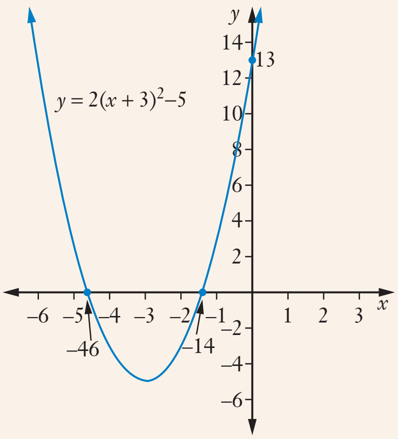
For x-intercept, $y = 0$:

$$0 = 2(x+3)^2 - 5$$
$$5 = 2(x+3)^2$$
$$2.5 = (x+3)^2$$
$$\pm\sqrt{2\ 5} = x + 3$$
$$\pm\sqrt{2\ 5} - 3 = x$$
$$-1.4, -4.6 = x$$

For y-intercept, $x = 0$:

$$y = 2(0+3)^2 - 5$$
$$= 2(9) - 5$$
$$= 13$$

Sketch the graph using a suitable scale on the axes.

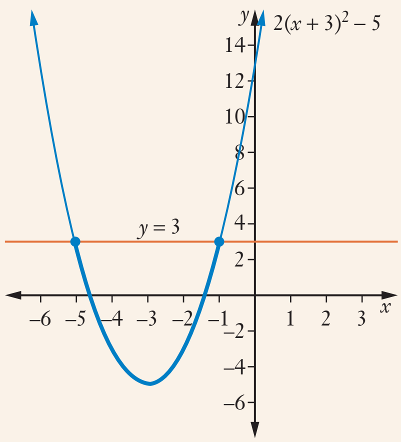


i Draw the line $y = 3$.

From the graph, the solution of $2(x+3)^2 - 5 = 3$ is $x = -5, -1$.

ii The solution of $2(x+3)^2 - 5 < 3$ is all x values below the line $y = 3$.

From the graph, the solution of $2(x+3)^2 - 5 < 3$ is $-5 < x < -1$.



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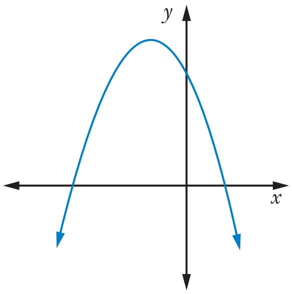
Exercise 2.07 Equations and inequalities

1 For each function $y = f(x)$, state how many solutions there are for the equation $f(x) = 0$.

a



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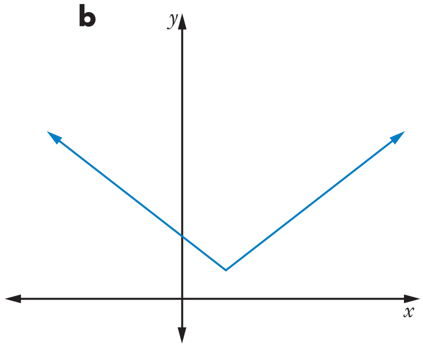

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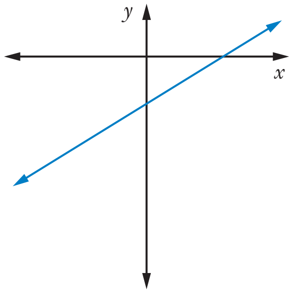

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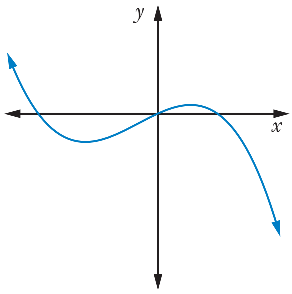

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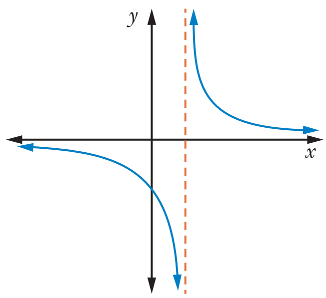

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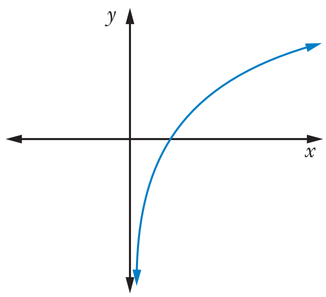

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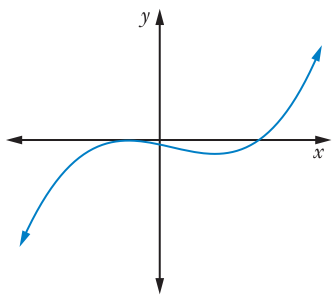

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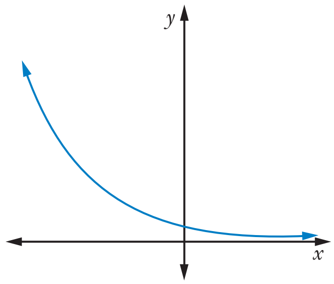

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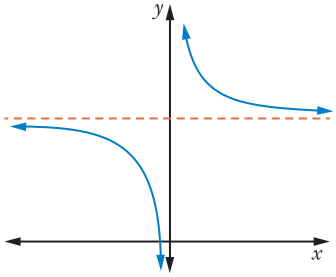
i



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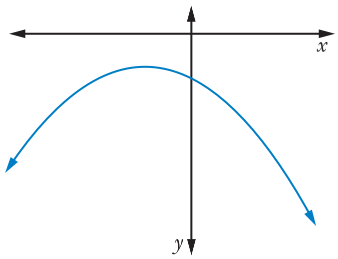


2 The graph of the quadratic function $f(x) = -2(x + 1)^2 + 3$ is shown.

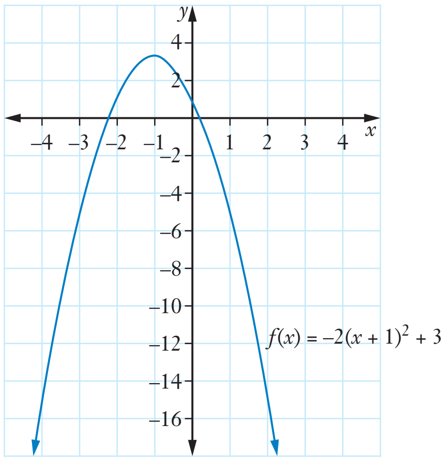
a Solve graphically:

i $-2(x + 1)^2 + 3 = 1$

ii $-2(x + 1)^2 + 3 = -2$

iii $-2(x + 1)^2 + 3 = 0$

b Solve $-2(x + 1)^2 + 3 = 0$ algebraically.



3 The graph of the linear function $f(x) = 3(4x - 5) - 2$ is shown.
Use the graph to solve:

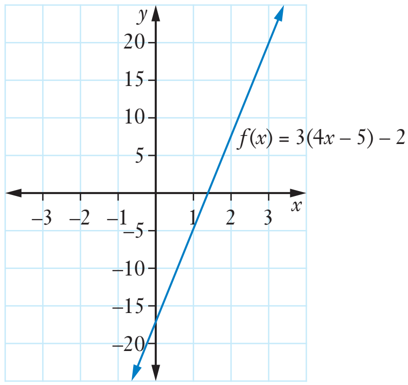
a $3(4x - 5) - 2 = 0$

b $3(4x - 5) - 2 = 5$

c $3(4x - 5) - 2 = -15$

d $3(4x - 5) - 2 > 10$

e $3(4x - 5) - 2 \leq 20$



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4 **a** Sketch the graph of the cubic function $y = -(x + 3)^3 + 1$.

b Solve graphically:

i $-(x + 3)^3 + 1 = 0$

ii $-(x + 3)^3 + 1 = -10$

iii $-(x + 3)^3 + 1 = -20$

c Solve $-(x + 3)^3 + 1 = 0$ algebraically.

5 **a** Sketch the graph of $y = 3|x - 2| + 4$.

b How many solutions does the equation $3|x - 2| + 4 = 1$ have?

c Solve $3|x - 2| + 4 = 10$ graphically and check your solutions algebraically.

6 **a** Sketch the graph of the function $f(x) = \frac{2}{x-3} - 4$, showing asymptotes.

b Solve the equation $\frac{2}{x-3} - 4 = -5$.

c Solve $\frac{2}{x-3} - 4 = -2$.

7 The formula for the area of a garden with side x metres is given by $A = -3(x - 2)^2 + 18$.

a Draw the graph of the area of the garden.

b From the graph, solve the equation $-3(x - 2)^2 + 18 = 10$.

8 A factory has costs according to the formula $C = 2(x + 1)^2 + 3$, where C stands for costs in \$1000s and x is the number of products made.

a Draw the graph of the costs.

b Find the factory overhead (cost when no products are made).

c Solve $2(x + 1)^2 + 3 = 20$ from the graph and explain your answer.

9 Loudness in decibels (dB) is given by $\text{dB} = 10 \log\left(\frac{x}{I}\right)$ where I is a constant.

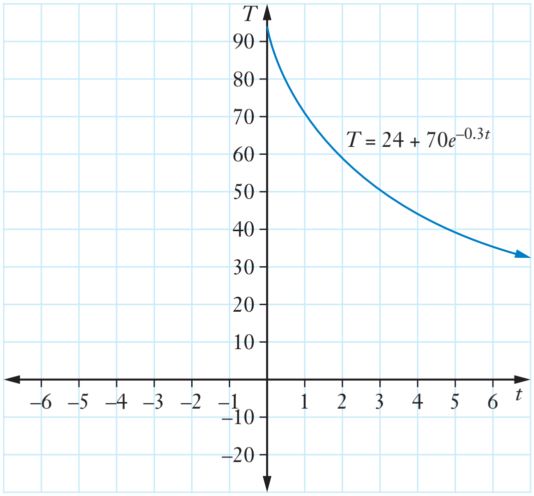
a Sketch the graph of the function given $I = 2$.

b From the graph solve the equation:

i $10 \log\left(\frac{x}{I}\right) = 5$

ii $10 \log\left(\frac{x}{I}\right) = 2$

10 According to Newton's law of cooling, the temperature T of an object as it cools over time t minutes is given by the formula $T = A + Be^{-kt}$. The graph shown is for the formula $T = 24 + 70e^{-0.3t}$ for a metal ball that has been heated and is now cooling down.



a From the graph, solve these equations and explain what the solutions mean.

i $24 + 70e^{-0.3t} = 50$ **ii** $24 + 70e^{-0.3t} = 30$

b Solve these equations algebraically:

i $24 + 70e^{-0.3t} = 80$ **ii** $24 + 70e^{-0.3t} = 26$

c What temperature will the object approach as t becomes large?

Can you give a reason for this?

11 **a** Sketch the graph of $y = (x - 1)^2 - 2$.

b From the graph, solve:

i $(x - 1)^2 - 2 = 2$ **ii** $(x - 1)^2 - 2 \geq 2$ **iii** $(x - 1)^2 - 2 < 2$

12 **a** Sketch the graph of $f(x) = -(2x + 4)^2 + 1$.

b From the graph, solve:

i $-(2x + 4)^2 + 1 = -3$ **ii** $-(2x + 4)^2 + 1 > -3$ **iii** $-(2x + 4)^2 + 1 \leq -3$

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2. TEST YOURSELF

Qz
Practice quiz

For Questions **1** to **3**, choose the correct answer **A**, **B**, **C** or **D**.

1 The function $y = f(x)$ transformed to $y = f(x - 8)$ is:

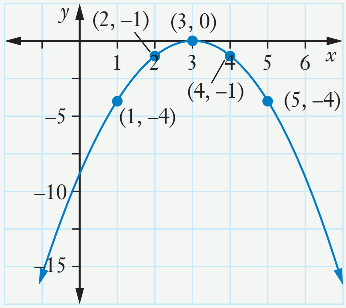
A a vertical translation 8 units up

B a horizontal translation 8 units to the right

C a vertical translation 8 units down

D a horizontal translation 8 units to the left

2 The graph below is a transformation of $y = x^2$. Find its equation.



A $y = (-x + 3)^2$ **B** $y = (-x - 3)^2$

C $y = -(x + 3)^2$ **D** $y = -(x - 3)^2$

3 Find the coordinates of the image of (x, y) when the function $y = f(x)$ is transformed to $y = -2f(x + 1) + 4$.

A $(x + 1, -2y - 4)$ **B** $(x + 1, -2y + 4)$

C $(x - 1, -2y + 4)$ **D** $(-x + 1, 2y + 4)$

4 **a** Draw the graph of $y = e^{x-1} - 2$.

b Use the graph to solve $e^{x-1} - 2 = 8$.

c Solve $e^{x-1} - 2 = 20$ algebraically.

5 The point (24, 36) lies on the graph of $y = f(x)$. Find the coordinates of its image point if the function is transformed to:

a $y = 3f(4x) - 1$ **b** $y = f[3(x + 2)] + 4$ **c** $y = 5f(-x) - 3$

d $y = -2f(x + 7) - 3$ **e** $y = -f(2x - 8) + 5$

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6 Find the equation of each transformed function.

a $y = x^3$ is translated:

i 3 units up **ii** 7 units to the left

b $y = |x|$ is dilated:

i vertically with scale factor 3

ii horizontally with scale factor 2

c $f(x) = \ln x$ is dilated vertically with factor 5 and reflected in the y-axis.

d $f(x) = \frac{1}{x}$ is reflected in the x-axis and translated 4 units to the right.

e $f(x) = 3^x$ is dilated vertically with scale factor 9, dilated horizontally with scale factor $\frac{1}{3}$ and translated 6 units down and 2 units to the right.

7 a State the meaning of the constants a, b, c and k in the function $y = kf(a(x + b)) + c$ and the effect they have on the graph of the function $y = f(x)$.

b Describe the effect on the graph of the function if:

i $k = -1$ **ii** $a = -1$

8 Show that if $y = x^2$ is dilated vertically with scale factor 3, reflected in the x-axis and translated 1 unit up, the transformed function is even.

9 a Draw the graph of $y = 2(x - 3) + 5$.

b From the graph, solve:

i $2(x - 3) + 5 \leq 7$ **ii** $2(x - 3) + 5 > 9$

10 The population of a city over time t years is given by $P = 2e^{0.4(t + 1)}$ where P is population in 10 000s.

a Sketch the graph of the population.

b Use the graph to solve $2e^{0.4(t + 1)} = 5$, and explain the meaning of the solution.

11 Find the equation of the transformed function if $f(x) = x^4$ is horizontally translated 4 units to the left.

12 If (8, 2) lies on the graph of $y = f(x)$, find the coordinates of the image of this point when the function is transformed to $y = -4f[2(x + 1)] - 3$.

13 Solve both graphically and algebraically:

a $2(3x - 6)^2 - 5 = 9$ **b** EXT1 $2(3x - 6)^2 - 5 > 9$ **c** EXT1 $2(3x - 6)^2 - 5 \leq 9$

14 The function $y = f(x)$ is transformed to $y = -7f(x - 3) - 4$.

a Find the coordinates of the image of (x, y).

b If the image point is (–3, 3), find the value of x and y.

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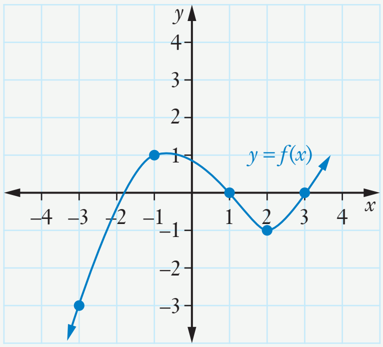
15 From the graph of $y = f(x)$ shown, draw the graph of:

a $y = 2f(x - 1)$

b $y = -f(x) - 2$



16 By drawing the graph of $y = 2(x + 1)^2 - 8$, solve:

a $2(x + 1)^2 - 8 \le 0$

b $2(x + 1)^2 - 8 > 0$

17 Sketch on the same set of axes:

a $y = x^2$ and $y = -4x^2 + 3$

b $y = |x|$ and $y = -|x - 1| + 2$

c $f(x) = e^x$ and $f(x) = \dfrac{e^{x+2}}{2} - 1$

d $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x + 2} + 1$

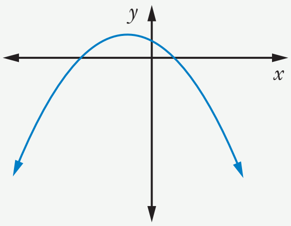
e $y = x^3$ and $y = 2(x - 3)^3 + 1$

f $f(x) = \ln x$ and $f(x) = \ln(-x) + 5$

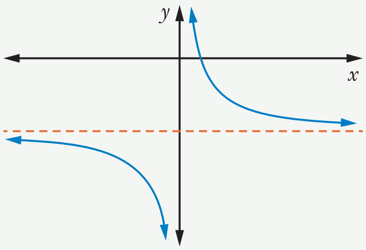
g $y = \sqrt{x}$ and $y = 2\sqrt{x + 4} - 1$

18 Find the number of solutions of $f(x) = 0$ given the graph of each function $y = f(x)$.

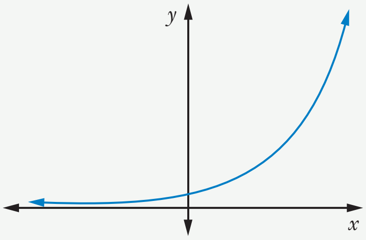
a



b



c



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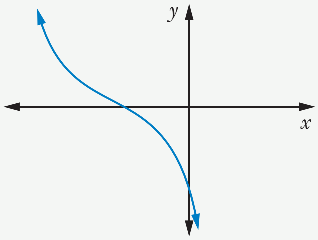

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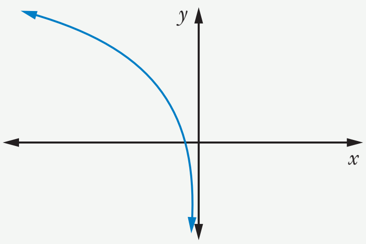

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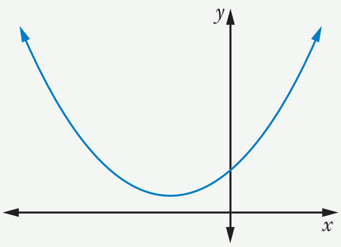

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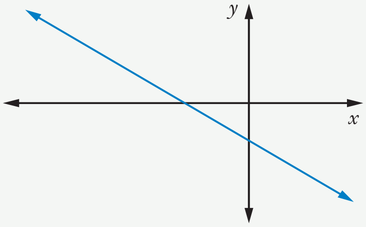

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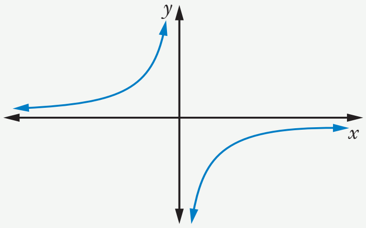

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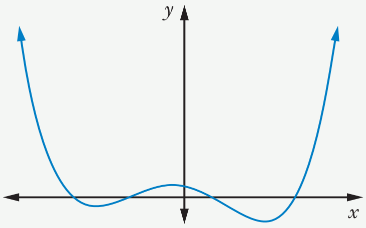

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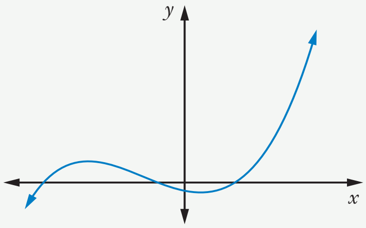

19 **a** Show that $x^2 + y^2 = r^2$ is not a function and describe its graph

b Find 2 functions that together form $x^2 + y^2 = r^2$

c By applying a vertical dilation with scale factor a to both these functions what shape does the combination of these stretched functions make?

20 The point (x, y) lies on the function $y = f(x)$ The image of (x, y) is the point (12, 6) when the function is transformed to $y = -6f(2x + 8)$ Find the coordinates of (x, y).

21 **a** Draw the graph of $y = (x - 2)^2 + 1$.

b From the graph, solve:

i $(x - 2)^2 + 1 = 10$ **ii** $(x - 2)^2 + 1 > 10$ **iii** $(x - 2)^2 + 1 \le 10$

22 Point (x, y) lies on $y = f(x)$ Find the image of (x, y) if the function is transformed to:

a $y = 3f(x + 1) - 5$

b $y = -2f[2(x - 6)] + 4$

c $y = 5f(-x) - 3$

d $y = -3f(-3x + 9) - 1$

23 State whether the function $y = f(x)$ is stretched or compressed if it is dilated:

a vertically with scale factor 7

b horizontally with scale factor $\frac{1}{6}$

c horizontally with scale factor 3

d vertically with scale factor $\frac{1}{4}$

e horizontally with scale factor $\frac{7}{6}$

24 Find the domain and range of:

a $y = 3(x - 7)^2 - 10$ **b** $y = -|x + 1| + 2$ **c** $y = -\frac{2}{x - 3} - 5$

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2. CHALLENGE EXERCISE

1 A ball is thrown into the air from a height of 1 m, reaches its maximum height of 3 m after 1 second and after 2 seconds it is 1 m high.

a The path of the ball follows the shape of a parabola. Find the equation of the height h of the ball over time t seconds.

b After how long does the ball fall to the ground?

c Put the function in the form $h = kf[a(t + b)] + c$ and describe the transformations to change $h = t^2$ into this equation.

2 a If (4, –3) lies on the function $y = f(x)$, find the coordinates of its image point.

i P on $y = 3f(x + 3) + 1$

ii Q on $y = -f(2x) - 3$

iii R on $y = f(2x - 2) + 1$

b Find the equation of the linear function passing through P that is perpendicular to QR.

c If $y = x$ is transformed into this linear function, describe the transformations.

3 a Show that $\dfrac{2x-7}{x-3} = -\dfrac{1}{x-3} + 2$.

b Sketch the graph of $y = \dfrac{2x-7}{x-3}$ and state its domain and range.

c Solve:

i $\dfrac{2x-7}{x-3} \ge 0$ **ii** $\dfrac{2x-7}{x-3} < 2$

4 a If $y = \dfrac{1}{x}$ is dilated horizontally with scale factor 2, explain why the equation of the transformed function is the same as if it was dilated vertically with scale factor 2.

b Is this the same result for the function $y = \dfrac{1}{x^2}$? Why?

5 a What is the equation of the axis of symmetry of the quadratic function $f(x) = ax^2 + bx + c$?

b What types of transformations on this function will change the axis of symmetry?

c Find the equation of the axis of symmetry of the quadratic function:

i $f(x) = 2(x + 1)^2 - 2$

ii $y = -(x - 3)^2 + 7$

iii $y = k(x + b)^2 + c$

v $y = k(ax + b)^2 + c$

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6 The function $y = \sin x$ in the domain $[0, 2\pi]$ is transformed by a reflection in the x-axis, a vertical dilation scale factor 3, a horizontal dilation scale factor 2 and a vertical translation 1 unit down.

a Find the equation of the transformed function.

b State the amplitude, period and centre of the transformed function.

7 EXT1 For the function $f(x) = x(x - 1)(x + 2)$, sketch:

a $y^2 = f(x)$

b $y^2 = 2f(x) - 3$

8 The circle $x^2 + 4x + y^2 - 6y + 12 = 0$ is transformed by a vertical translation 3 units down and a horizontal translation 5 units right. Find the equation of the transformed circle.

9 The function $y = 2^x$ is transformed to $y = 3(2^{-3x-6}) - 5$. Describe the transformations applied to the function.

10 The polynomial $P(x) = x^3 - 3x - 2$ is translated up 2 units and then reflected in the y-axis. Find the equation of the transposed polynomial.

3. VECTORS

In this Mathematics Extension 1 chapter, you will learn what vectors are, how they are related to each other and how to perform basic operations with vectors.

CHAPTER OUTLINE

3.01 EXT1 Vectors
3.02 EXT1 Adding and subtracting vectors
3.03 EXT1 Scalar multiplication
3.04 EXT1 Vectors in the Cartesian plane
3.05 EXT1 Magnitude and direction of a vector
3.06 EXT1 Components of a vector

IN THIS CHAPTER YOU WILL:

- EXT1 understand what vectors and scalars are
- EXT1 sketch vectors using different notations
- EXT1 add and subtract vectors
- EXT1 apply scalar multiplication to vectors
- EXT1 use vectors in the Cartesian plane
- EXT1 find the magnitude and direction of vectors
- EXT1 break vectors up into horizontal and vertical components

TERMINOLOGY

column vector notation: Representation of a two-dimensional vector in the form $\begin{pmatrix} a \\ b \end{pmatrix}$.

component form: Representation of a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in the form $ai + bj$ where i is a unit vector in the x-direction and j is a unit vector in the y-direction.

displacement vector: A vector that represents a displacement between 2 points, shown on the number plane.

equal vectors: Vectors that have the same magnitude and direction.

magnitude of a vector: For the vector $ai + bj$, the length of the vector, given by $\sqrt{a^2 + b^2}$.

parallel vectors: Vectors that have the same direction (**like** parallel vectors) or opposite direction (**unlike** parallel vectors).

position vector: A vector that starts at the origin O.

scalar: A quantity that has magnitude but no direction, denoted by λ or μ.

unit vector: A vector with magnitude 1. The standard unit vectors are i (in the x-direction) and j (in the y-direction)

vector: A quantity with both magnitude and direction which can be represented as a, **a**, or $\overrightarrow{AB}$.

WS

Repeening vectors

EXT1 3.01 Vectors

Scalars and vectors

A **scalar** is a quantity with magnitude (size) but not direction.

A **vector** is a quantity with both magnitude and direction.

Much of the mathematics that you do involves scalars, such as distance, speed, length, weight, time. All real numbers are scalars.

Vectors describe measurements that need direction as well as magnitude, such as displacement, velocity and acceleration.

We write scalars in Greek letters such as λ (lambda) or μ (mu).

We write vectors as a, **a**, or $\overrightarrow{AB}$. When you handwrite vectors, you will usually write a or $\overrightarrow{AB}$.

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We can draw vectors using lines to show magnitude and arrows to show direction.

EXAMPLE 1

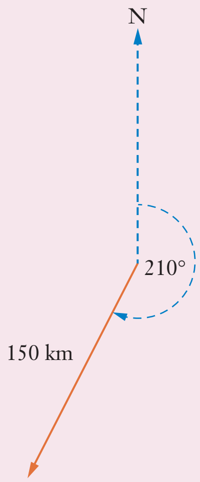
Draw a vector to show:

a a train travelling 150 km on a bearing of 210°

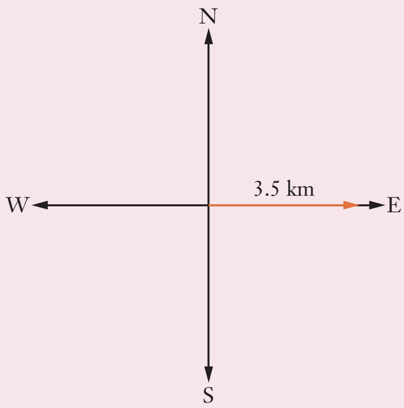
b a person walking 3.5 km due east

Solution

a



b



EXAMPLE 2

A particle starts at the origin and moves 3 cm to the left. It then turns and moves 5 cm to the right, then turns and moves 6 cm to the left.

a Draw vectors showing the movement.

b Where is the particle at the end of its motion?

c What is the total distance the particle travels?

Solution

a

b The particle is at –4.

c Total distance = 3 + 5 + 6 = 14 cm

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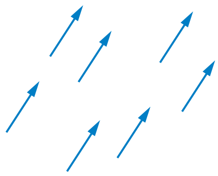
Types of vectors

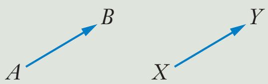
As long as we know its magnitude and direction, we can draw a vector anywhere in a plane. We call this type of vector a **displacement vector**, relative vector or direction vector.



All the vectors above have the same length and direction, so they are **equal vectors**.

Equal vectors

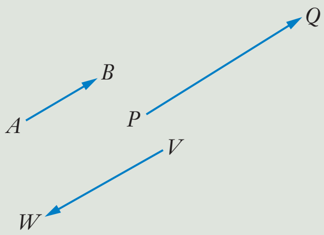
Two vectors are equal if they have the same magnitude and direction.



$\overrightarrow{AB} = \overrightarrow{XY}$

Parallel vectors

Parallel vectors have the same (**like**) or opposite (**unlike**) direction.



$\overrightarrow{AB} \parallel \overrightarrow{PQ}$ (like parallel vectors)

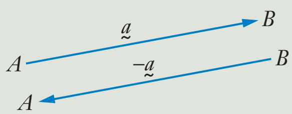
$\overrightarrow{AB} \parallel \overrightarrow{VW}$ (unlike parallel vectors)

If 2 vectors have equal magnitude but opposite directions, then one is the negative of the other.

Negative vectors

Vector $-a$ has the same magnitude as a but the opposite direction.

If $a = \overrightarrow{AB}$ then $-a = -\overrightarrow{AB} = \overrightarrow{BA}$



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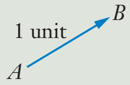
Unit vector

A **unit vector** is a vector with magnitude 1 unit.



EXT1 Exercise 3.01 Vectors

1 State whether each measurement is a vector or a scalar.

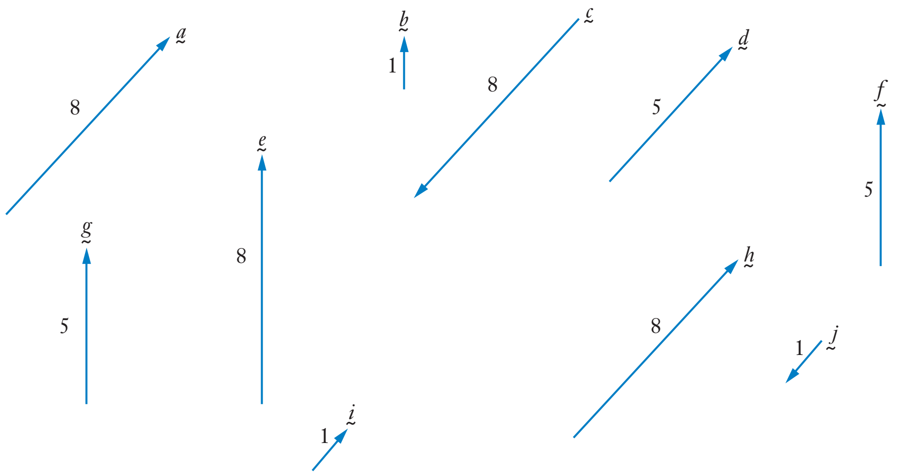
- **a** length
- **b** velocity
- **c** displacement
- **d** force
- **e** time

2 Draw a vector showing each situation.

- **a** A plane flies for 865 km on a bearing of 049°
- **b** A car travels due west for 2 hours at 80 km h^{-1}
- **c** A ball is rolled 3.6 m up a ramp at an angle of inclination of 40°
- **d** A person walks S 20° W for 4.2 km
- **e** A train travels at 145 km h^{-1} for 3 hours on a bearing of 312°

3 Describe the relationship between each pair of vectors.

- **a** $\underset{\sim}{a}$ and $\underset{\sim}{c}$
- **b** $\underset{\sim}{d}$ and $\underset{\sim}{i}$
- **c** $\underset{\sim}{f}$ and $\underset{\sim}{g}$
- **d** $\underset{\sim}{b}$ and $\underset{\sim}{e}$
- **e** $\underset{\sim}{d}$ and $\underset{\sim}{h}$
- **f** $\underset{\sim}{i}$ and $\underset{\sim}{j}$



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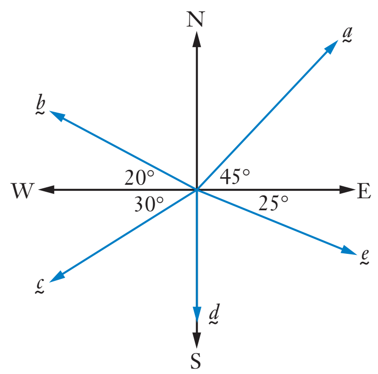
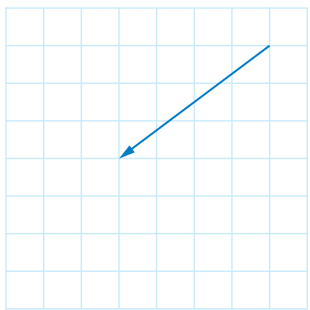
4 Write each vector as:

i a compass bearing

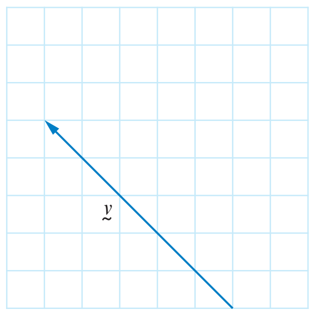
ii a true bearing



5 Copy this diagram and draw a unit vector parallel to the vector below.



6 Copy this diagram and draw $-\underset{\sim}{v}$.



7 Josie leaves home and walks 3.7 km south then turns and walks west for 2.4 km.

a Draw vectors showing Josie's journey.

b Find Josie's distance from home, to 1 decimal place.

c Find Josie's bearing from home, to the nearest degree.

8 A ship sails from Sydney for 297 km on a bearing of 097° then turns and sails on a bearing of 070° for 365 km.

a Draw these bearings as vectors.

b Find the distance of the ship from Sydney to the nearest km.

9 Geoff rides his motorbike 17.2 km on a bearing of 316° then he turns and rides on a bearing of 247° until he is due west of his starting point.

a Draw a diagram showing vectors for Geoff's journey.

b Find how far he is from his starting point.

c Find the bearing of:

i Geoff from the starting point

ii the starting point to Geoff

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10 A ball bearing rolls along a tube for 95 cm then turns into another tube and rolls for 58 cm before it comes to a rest. If the first tube is on a bearing of 125° and the second tube on a bearing of 050°, find:

a the distance the ball bearing is from the start

b the bearing of the ball bearing from the start

11 A particle starts 3 cm to the right of the origin and moves 7 cm to the left. It turns and moves 15 cm to the right and comes to a stop.

a Draw vectors to show the motion of the particle.

b Find where the particle is at the end of its motion.

c Find the total distance it travels.

12 A projectile is tossed straight up in the air from a point 2 m above the ground. It travels upwards 12 m before turning back and returning to the ground. How far does it travel altogether?

13 A particle has initial displacement 5 cm. It moves 4 cm to the right, then 11 cm to the left.

a What is the final displacement of the particle?

b What is the total distance it travels?

14 An object starts at the origin. It moves for 3 s with velocity –4 mm s^{-1}, moves for 5 s with velocity 3 mm s^{-1} and then moves for 2 s with a velocity of –4 mm s^{-1}.

a Draw a diagram showing the motion of the object.

b What is the final displacement of the object?

c What is the total distance it moves?

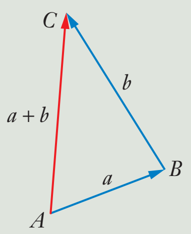
EXT1 3.02 Adding and subtracting vectors

WS Triangle addition of vectors

WS Paallelogam rule for vector addition

Two vectors can be added together by placing them tip to tail. They give a single or **resultant** vector, which is the third side of a triangle.

Triangle law of addition



$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ where AC is the third side of the triangle ABC.

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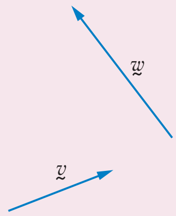
EXAMPLE 3

Given vectors v and w, find:

a $v + w$

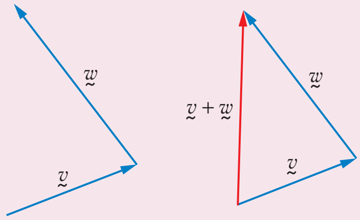
b $w + v$



Solution

a Place the vectors tip to tail as shown.

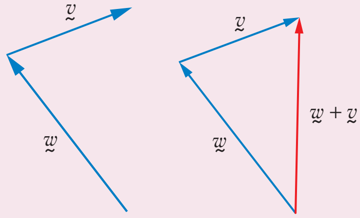
Then we find $v + w$ by drawing a vector from v to w as shown.



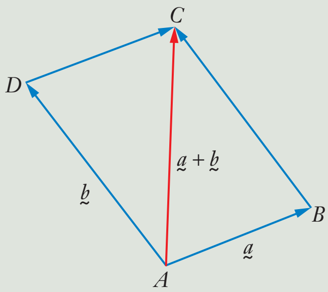
b Place the vectors tip to tail as shown.

Then we find $w + v$ by drawing a vector from w to v as shown.

Notice that $v + w$ and $w + v$ are equal vectors.



Parallelogram law of addition



$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$ where AC is the diagonal between a and b.

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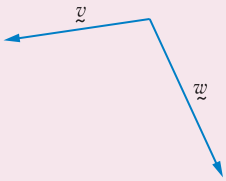
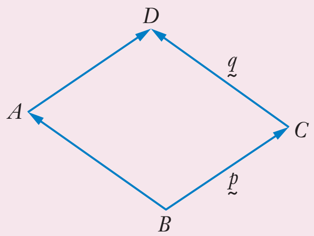
EXAMPLE 4

a Given vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$, find $\underset{\sim}{v} + \underset{\sim}{w}$.

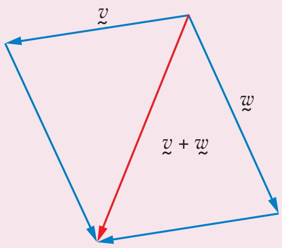


b Write the vector that represents $\underset{\sim}{p} + \underset{\sim}{q}$ in the diagram as shown.



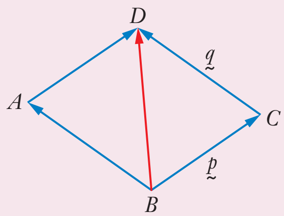
Solution

a Draw a parallelogram as shown.
Resultant vector $\underset{\sim}{v} + \underset{\sim}{w}$ is the diagonal.



b Diagonal BD is the sum of the 2 vectors.

So $\underset{\sim}{p} + \underset{\sim}{q}$ is the vector $\overrightarrow{BD}$.



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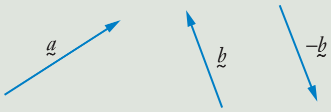
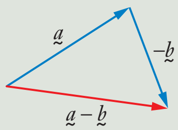
Subtracting vectors using a triangle

Given vectors a and b:



$a - b = a + (-b)$



Zero vector

Adding a vector to its negative gives the zero vector 0.

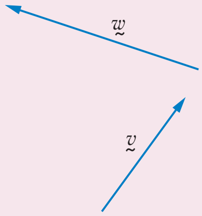
$a + (-a) = 0$

EXAMPLE 5

Given vectors v and w, find:

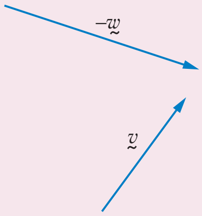
a $v - w$

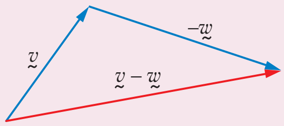
b $w - v$



Solution

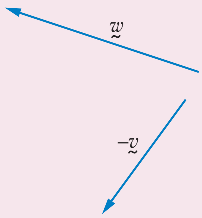
a Add v and $-w$:

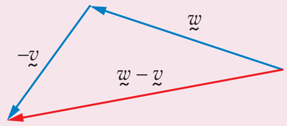




b Add w and $-v$:

Notice that $v - w = -(w - v)$





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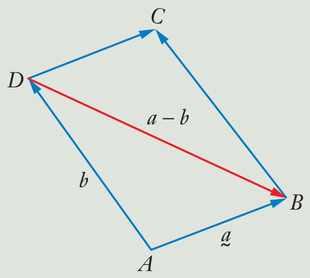
Subtracting vectors using a parallelogram

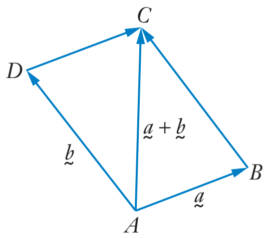


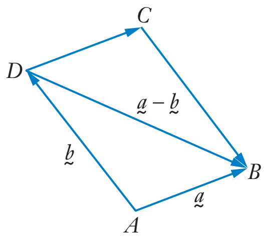
$\overrightarrow{AB} - \overrightarrow{AD} = \overrightarrow{DB}$ where BD is the diagonal between a and b.

We can check this by using addition:

$b + (a - b) = b + a - b = a$

In the parallelogram, one diagonal gives $a + b$ and the other diagonal gives $a - b$.



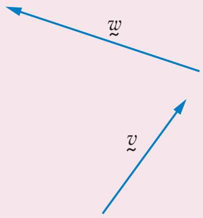


EXAMPLE 6

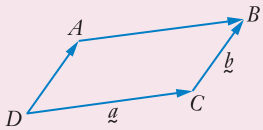
a Given vectors v and w, find:

i $v - w$

ii $w - v$

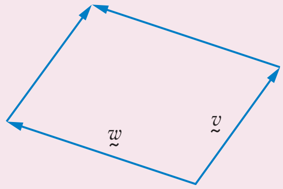


b State the vector that represents $b - a$ in the diagram as shown.

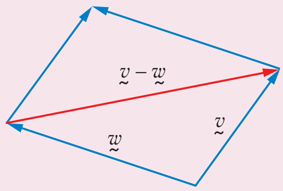


Solution

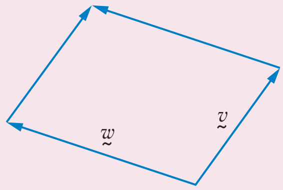
a **i** Draw a parallelogram as shown using v and w.



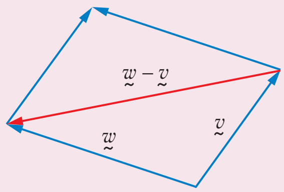
Choose the diagonal that gives $w + (v - w) = v$.



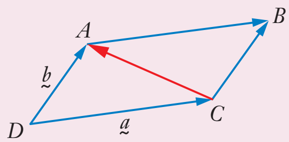
ii Put w and v in a parallelogram as shown.



Choose the diagonal that gives $v + (w - v) = w$.



b Diagonal AC gives $b - a$ since $a + (b - a) = b$.



So $b - a$ is the vector $\overrightarrow{CA}$.

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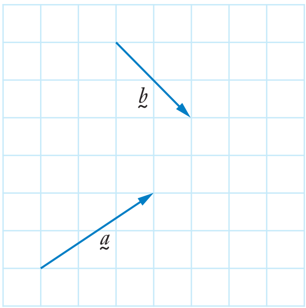
EXT1 Exercise 3.02 Adding and subtracting vectors

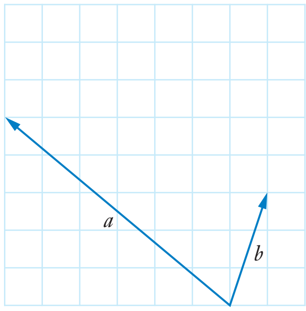
1 Given each set of vectors a and b, draw:

i $a + b$ **ii** $a - b$

a



b



2 Given parallelogram $ABCD$, write the vector that is:

a $\overrightarrow{AB} + \overrightarrow{BC}$

b $\overrightarrow{AD} + \overrightarrow{DB}$

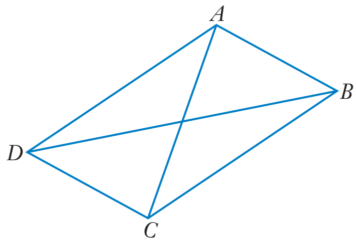
c $\overrightarrow{DC} + \overrightarrow{CA}$

d $\overrightarrow{CD} + \overrightarrow{CB}$

e $\overrightarrow{DA} - \overrightarrow{AB}$

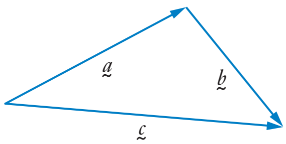
f $\overrightarrow{AB} - \overrightarrow{AD}$

g $\overrightarrow{BC} - \overrightarrow{CD}$

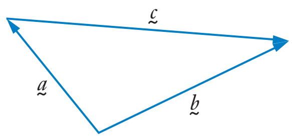


3 Write vector c in terms of a and b for each diagram.

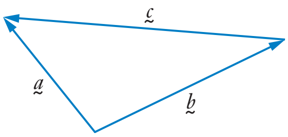
a



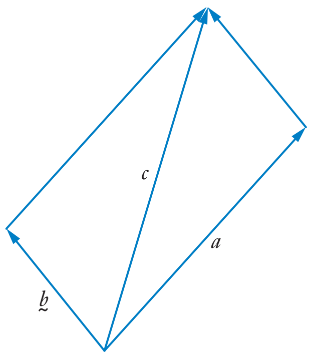
b



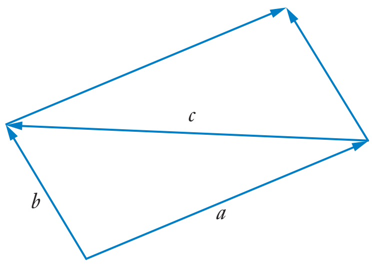
c



d



e

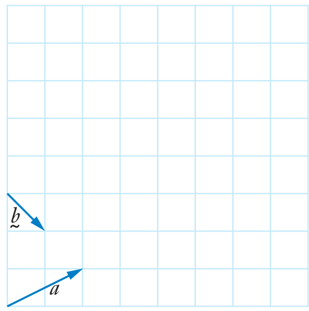


4 Copy the diagram and draw:

a $\underset{\sim}{a} + \underset{\sim}{b}$

b $\underset{\sim}{a} - \underset{\sim}{b}$

c $\underset{\sim}{b} - \underset{\sim}{a}$

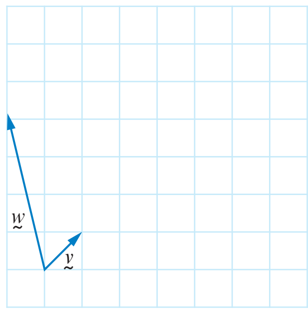


5 Copy the diagram and draw:

a $\underset{\sim}{v} + \underset{\sim}{w}$

b $\underset{\sim}{w} - \underset{\sim}{v}$

c $\underset{\sim}{v} - \underset{\sim}{w}$



6 For the diagram, state the resultant vector of:

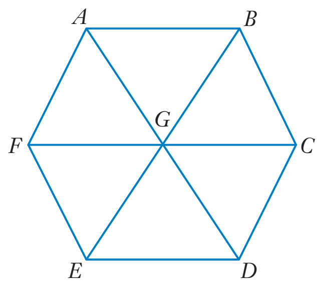
a $\overrightarrow{AB} + \overrightarrow{BG}$

b $\overrightarrow{GE} + \overrightarrow{EF}$

c $\overrightarrow{BG} - \overrightarrow{CG}$

d $\overrightarrow{GC} - \overrightarrow{BC}$

e $\overrightarrow{GE} + \overrightarrow{EF} + \overrightarrow{FA}$



7 For the diagram, state the resultant vector of:

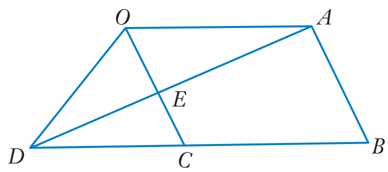
a $\overrightarrow{AD} + \overrightarrow{DO}$

b $\overrightarrow{DE} + \overrightarrow{EC}$

c $\overrightarrow{AE} - \overrightarrow{AO}$

d $\overrightarrow{DB} - \overrightarrow{DA}$

e $\overrightarrow{AE} - \overrightarrow{AC}$



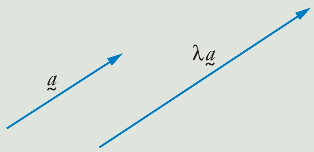
WS Scala multiplication

EXT1 3.03 Scalar multiplication

Scalar multiplication

Multiplying vector $\underset{\sim}{a}$ by a scalar λ gives a vector in the same direction with λ times the magnitude of $\underset{\sim}{a}$.

Two vectors are parallel if one vector is a scalar multiple of the other.



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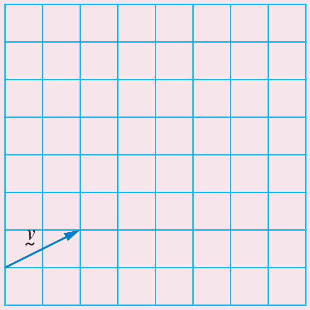
EXAMPLE 7

a Given vector v, draw λv where:

i $\lambda = 3$

ii $\lambda = -2$

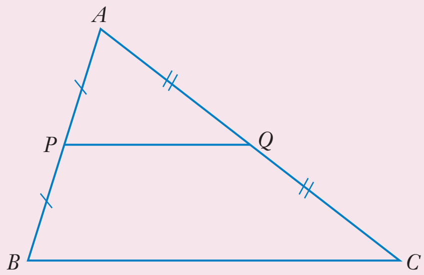
iii $\lambda = \frac{1}{2}$



b In triangle ABC, line PQ is drawn so that it bisects AB and AC. State which vector represents:

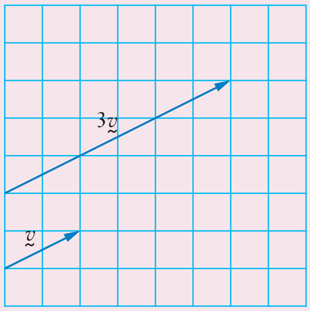
i $2\overrightarrow{AP}$

ii $\frac{1}{2}\overrightarrow{CA}$

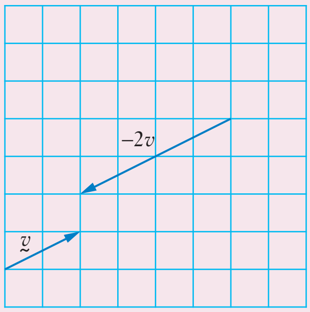


Solution

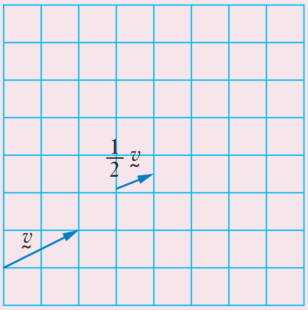
a **i**



ii



iii



b **i** $2\overrightarrow{AP} = \overrightarrow{AB}$

ii $\frac{1}{2}\overrightarrow{CA} = \overrightarrow{CQ} = \overrightarrow{QA}$

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WS

Uni veco

We can reduce a vector to a unit vector by using scalar multiplication.

EXAMPLE 8

Find a scalar λ that will reduce a vector with magnitude 4 to a unit vector.

Solution

Since $|v| = 4$ where $|v|$ is the magnitude of v:

$$\frac{1}{4}|v| = 1$$

$$\text{So } \lambda = \frac{1}{4}$$

Unit vector

The unit vector parallel to vector v is:

$$v \times \frac{1}{|v|} = \frac{v}{|v|} \text{ where } |v| \text{ is the magnitude of } v.$$

WS

5 mm gid paper

EXT1 Exercise 3.03 Scalar multiplication

1 Copy this diagram and draw:

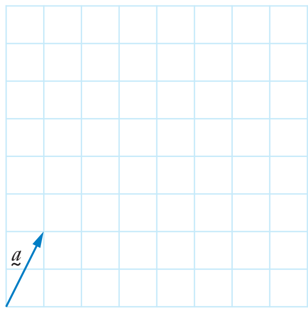
a $3a$

b $2a$

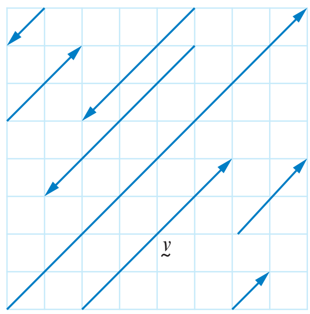
c $\frac{1}{2}a$

d $-a$

e $-4a$



2 Some parallel vectors are shown. Copy the diagram and name each vector in terms of v.



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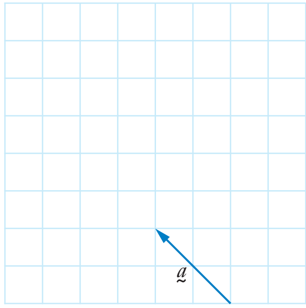
3 Find the scalar that will change each vector below to a unit vector.

a 10 cm **b** 3 m **c** 0.25 units

d $\frac{1}{2}$ unit **e** 7 m

4 Copy this diagram, then draw:



a $2a$

b $-a$

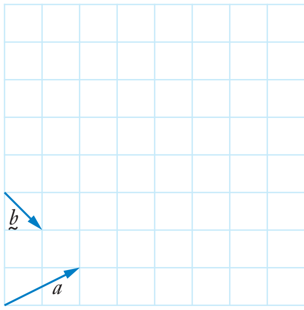
c $4a$

d $\frac{1}{2}a$

e $-3a$

5 Vector v has magnitude 2.5 and direction 120°. Find the scalar μ that gives the unit vector parallel to v.

6 Copy the diagram, then draw:



a $3a$

b $-b$

c $2b$

d $-2a$

e $-\frac{1}{2}a$

7 Vector a has magnitude 4 and direction 023°. Vector b is parallel to a with magnitude 10.

Find the scalar λ, given that:

a $b = \lambda a$ **b** $a = \lambda b$

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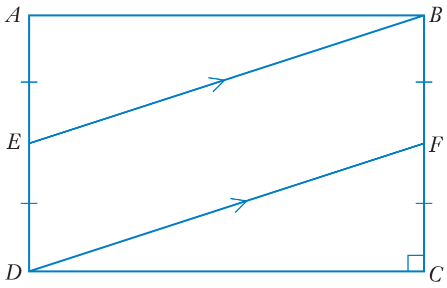
8 In the figure below, state the vector(s) that are:

a $2\overrightarrow{AE}$ **b** $\frac{1}{2}\overrightarrow{CB}$ **c** $-\overrightarrow{DF}$



EXT1 3.04 Vectors in the Cartesian plane

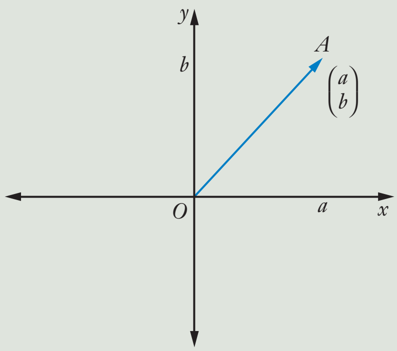
Vetrs inthe Cartesian plane

Position vectors

We can place vectors in the Cartesian (number) plane.

Position vector

A **position vector** $\overrightarrow{OA} = \begin{pmatrix} a \\ b \end{pmatrix}$ is a vector from the origin to the point (a, b).



We could write vector $\overrightarrow{OA}$ as (a, b) but since this could be confused with the point (a, b), we usually write it as a column matrix $\begin{pmatrix} a \\ b \end{pmatrix}$.

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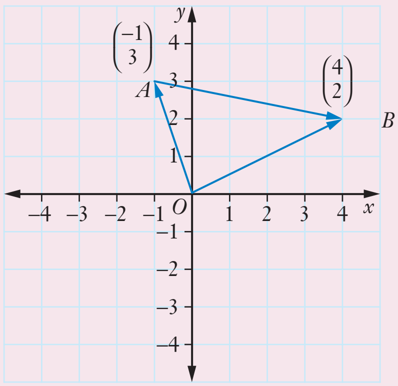
EXAMPLE 9

a Draw $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\overrightarrow{AB}$ on the number plane.

b Write $\overrightarrow{AB}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Solution

a



b $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

So $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

Displacement (relative) vectors

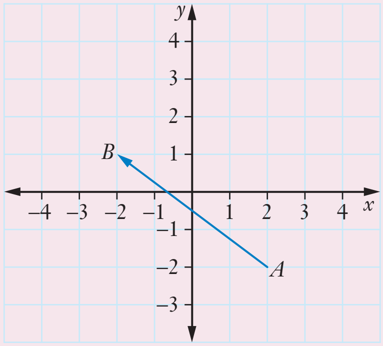
As well as position vectors, we can draw displacement vectors in the Cartesian plane.

EXAMPLE 10

a **i** Draw the position vector $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in the Cartesian plane.

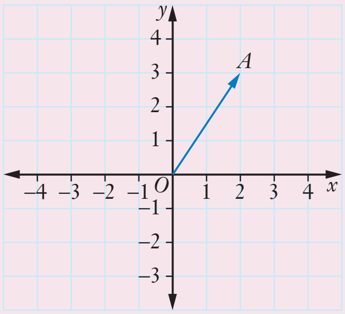
ii Draw 2 displacement vectors equal to vector $\overrightarrow{OA}$.

b The vector $\overrightarrow{AB}$ is shown. Find the position vector that is equal to $\overrightarrow{AB}$.



Solution

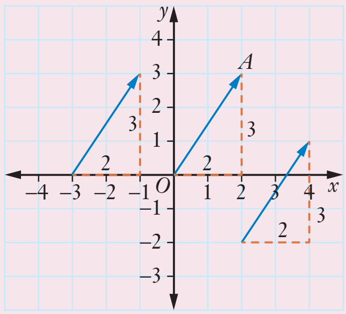
a **i**



ii Vector $\overrightarrow{OA}$ is 2 units to the right and 3 units up from O.

So to draw an equal vector, we translate a point 2 units to the right and 3 units up from any starting point.

Choose 2 starting points (there are many possibilities):



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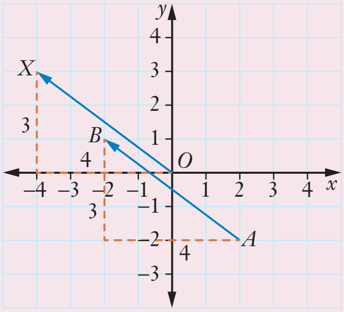
b Vector $\overrightarrow{AB}$ translates a point 4 units to the left of A and up 3 units to B. So for the equal position vector, start from O and shift 4 units to the left and up 3 units.



Point X has coordinates (–4, 3).

So position vector $\overrightarrow{OX} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$.

Sums and differences of vectors in the Cartesian plane

EXAMPLE 11

a Draw $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ on the same number plane, then draw vectors $\overrightarrow{AC} = \overrightarrow{OB}$ and $\overrightarrow{BC} = \overrightarrow{OA}$.

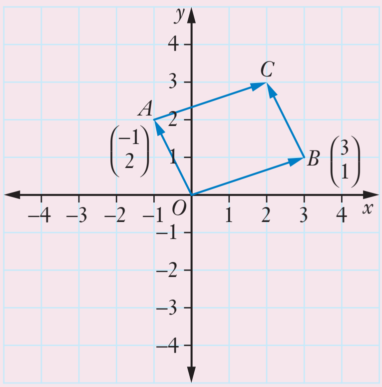
b Find the position vector that is the resultant of $\overrightarrow{OA} + \overrightarrow{OB}$.

c Draw the resultant vector of $\overrightarrow{OB} - \overrightarrow{OA}$.

Write this resultant vector as a position vector.

Solution

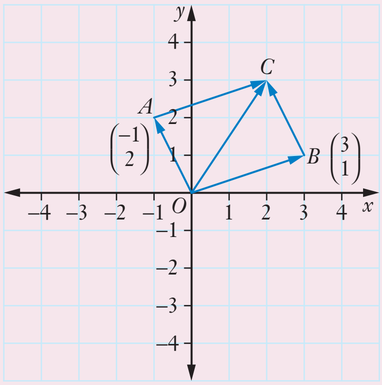
a



A vector equal to $\overrightarrow{OB}$ shifts across 3 units to the right and up 1 unit.

A vector equal to $\overrightarrow{OA}$ shifts across 1 unit to the left and up 2 units.

b



$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$

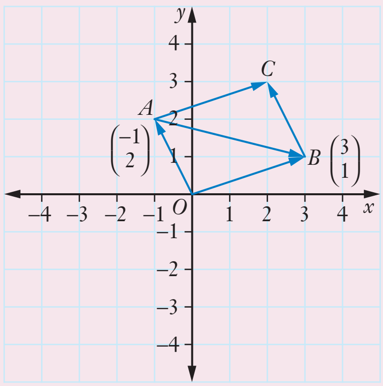
Coordinates of point $C = (2, 3)$

So $\overrightarrow{OC} = \begin{pmatrix}2\\3\end{pmatrix}$

Notice that we could find $\overrightarrow{OC}$ by adding the vectors in **column vector notation**:

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{OB}\\ &= \begin{pmatrix}-1\\2\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix}\\ &= \begin{pmatrix}-1+3\\2+1\end{pmatrix}\\ &= \begin{pmatrix}2\\3\end{pmatrix}\end{aligned}$$

c

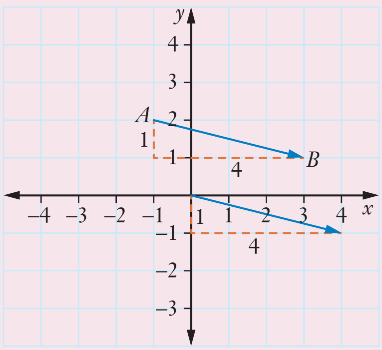


$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

So $\quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

The resultant vector of $\overrightarrow{OB} - \overrightarrow{OA}$ is $\overrightarrow{AB}$ (displacement vector).

For $\overrightarrow{AB}$, point A is translated down 1 unit and 4 units to the right to get B.



So starting at O and moving down 1 unit and 4 units to the right gives the position vector $\begin{pmatrix}4\\-1\end{pmatrix}$.

Notice that we could find this vector by subtracting the vectors in column vector notation:

$$\begin{aligned}\overrightarrow{OB} - \overrightarrow{OA} &= \begin{pmatrix}3\\1\end{pmatrix} - \begin{pmatrix}-1\\2\end{pmatrix}\\ &= \begin{pmatrix}3-(-1)\\1-2\end{pmatrix}\\ &= \begin{pmatrix}4\\-1\end{pmatrix}\end{aligned}$$

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Adding vectors

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix}$$

Subtracting vectors

$$\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ b_1 - b_2 \end{pmatrix}$$

EXAMPLE 12

Given $\overrightarrow{OA} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, find the resultant of $\overrightarrow{OA} + \overrightarrow{OB}$.

Solution

$$\overrightarrow{OA} + \overrightarrow{OB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2+[-1] \\ -3+4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

EXAMPLE 13

Given $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, find the resultant of:

a $\overrightarrow{OA} - \overrightarrow{OB}$

b $\overrightarrow{OB} - \overrightarrow{OA}$

Solution

a $\overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$$= \begin{pmatrix} 1-3 \\ 5-(-2) \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 7 \end{pmatrix}$$

b $\overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

$$= \begin{pmatrix} 3-1 \\ -2-5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -7 \end{pmatrix}$$

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Scalar multiplication

$$\lambda\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix}$$

EXAMPLE 14

Given $\overrightarrow{OA}=\begin{pmatrix} -7 \\ 5 \end{pmatrix}$, find $-4\,\overrightarrow{OA}$.

Solution

$$\overrightarrow{OA}=\begin{pmatrix} -7 \\ 5 \end{pmatrix}$$

$$\begin{aligned} -4\,\overrightarrow{OA} &= -4\begin{pmatrix} -7 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} -4\times -7 \\ -4\times 5 \end{pmatrix} \\ &= \begin{pmatrix} 28 \\ -20 \end{pmatrix} \end{aligned}$$

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EXT1 Exercise 3.04 Vectors in the Cartesian plane

1 Draw each position vector on a number plane.

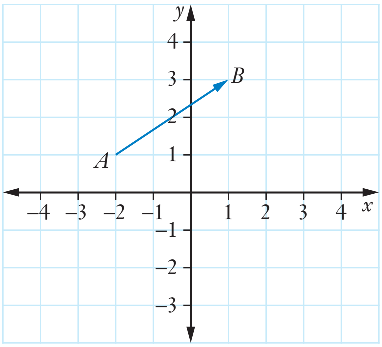
a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **c** $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ **d** $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ **e** $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$

2 Draw a displacement vector equal to each position vector.

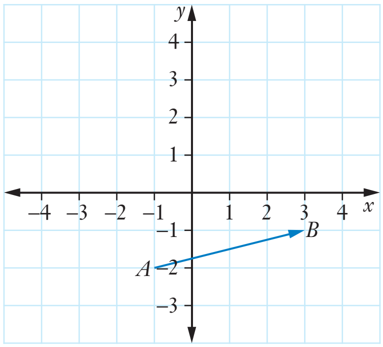
a $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -2 \\ -4 \end{pmatrix}$ **c** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **d** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **e** $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

3 Write the position vector equal to each vector $\overrightarrow{AB}$.

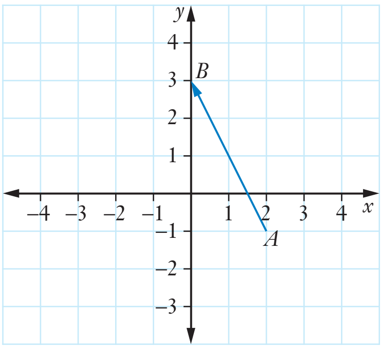
a



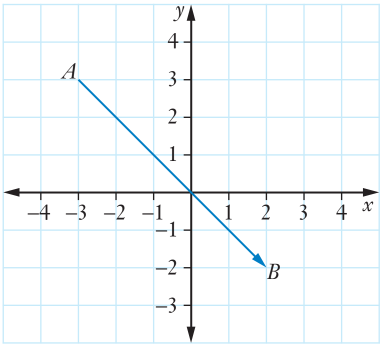
b



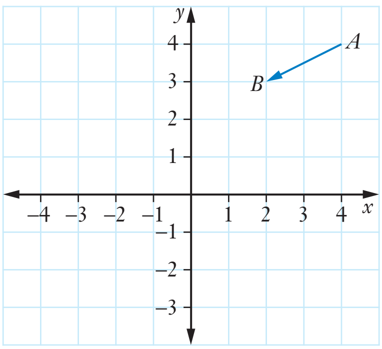
c



d



e



4 Given $\overrightarrow{OA} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, find the resultant vector for:

a $\overrightarrow{OA} + \overrightarrow{OB}$ **b** $\overrightarrow{OB} - \overrightarrow{OA}$ **c** $\overrightarrow{OA} - \overrightarrow{OB}$

5 If $\overrightarrow{OX} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$, find the resultant vector for:

a $\overrightarrow{OX} + \overrightarrow{OY}$ **b** $\overrightarrow{OY} - \overrightarrow{OX}$

6 Find the resultant vector of:

a $\begin{pmatrix} 6 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -8 \\ -5 \end{pmatrix}$ **c** $\begin{pmatrix} 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 9 \\ 2 \end{pmatrix}$

d $\begin{pmatrix} -4 \\ -5 \end{pmatrix} - \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ **e** $\begin{pmatrix} 2 \\ -5 \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \end{pmatrix}$ **f** $5\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

g $-3\begin{pmatrix} 6 \\ -2 \end{pmatrix}$ **h** $\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ **i** $\begin{pmatrix} -1 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

j $4\begin{pmatrix} 2 \\ -7 \end{pmatrix} - 3\begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ -6 \end{pmatrix}$

7 The resultant vector is $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ when adding vectors $\begin{pmatrix} a \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ b \end{pmatrix}$. Find the values of a and b.

8 Given $\overrightarrow{OA} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$, the resultant vector for $\overrightarrow{OB} - \overrightarrow{OA}$ is $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$. Find the values of a and b.

9 The resultant vector of $-3\begin{pmatrix} x \\ y \end{pmatrix}$ is $\begin{pmatrix} -12 \\ 39 \end{pmatrix}$. Evaluate x and y.

10 The resultant vector of $4\begin{pmatrix} p \\ -2 \end{pmatrix} - 3\begin{pmatrix} 2 \\ q \end{pmatrix}$ is $\begin{pmatrix} 14 \\ -8 \end{pmatrix}$. Evaluate p and q.

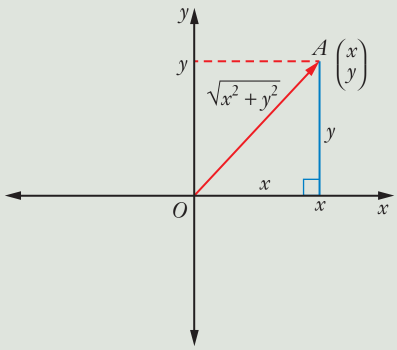
EXT1 3.05 Magnitude and direction of a vector

WS Magniude and direction of a vector

Magnitude of a position vector

Magnitude of the vector $u = \begin{pmatrix} x \\ y \end{pmatrix}$ is:

$$|u| = \sqrt{x^2 + y^2}$$



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Notice that the magnitude of the vector is the distance between O and A.

EXAMPLE 15

Find the magnitude of:

a $v = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

b $w = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ correct to 1 decimal place

Solution

a $|v| = \sqrt{x^2 + y^2}$

$= \sqrt{(-3)^2 + 4^2}$

$= \sqrt{9 + 16}$

$= \sqrt{25}$

$= 5$

So magnitude is 5.

b $|w| = \sqrt{x^2 + y^2}$

$= \sqrt{2^2 + 7^2}$

$= \sqrt{4 + 49}$

$= \sqrt{53}$

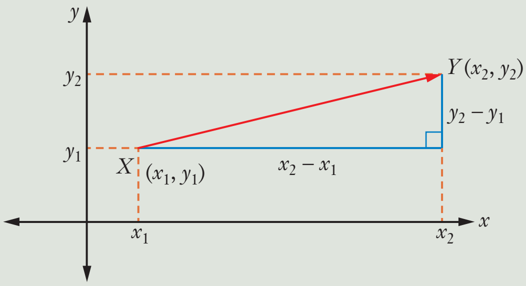
$= 7.3$

So magnitude is 7.3.

Magnitude of a displacement (relative) vector

Magnitude of $\overrightarrow{XY}$ where $\overrightarrow{OX} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ is:

$$\left|\overrightarrow{XY}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Notice that the magnitude of the vector is the distance between X and Y.

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EXAMPLE 16

Find the exact magnitude of $\overrightarrow{AB}$ where:

a $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

b $A = (-6, -5)$ and $B = (-4, 3)$

Solution

a

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(3-1)^2 + (-2-2)^2}$$
$$= \sqrt{4+16}$$
$$= \sqrt{20}$$
$$= 2\sqrt{5}$$

So magnitude is $= 2\sqrt{5}$.

b

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-4-(-6))^2 + (3-(-5))^2}$$
$$= \sqrt{2^2 + 8^2}$$
$$= \sqrt{4+64}$$
$$= \sqrt{68}$$
$$= 2\sqrt{17}$$

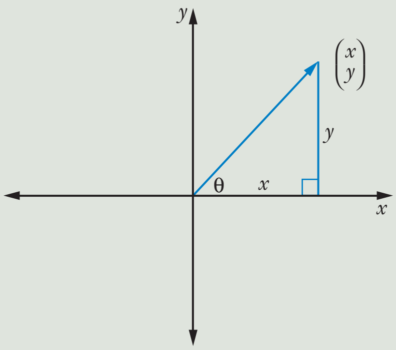
So magnitude is $2\sqrt{17}$.

Direction of a position vector

The direction of vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is θ, the angle of inclination of the vector with the positive x-axis, measured anticlockwise.

$$\tan\theta = \frac{y}{x}$$

$$0° \le \theta \le 360°$$



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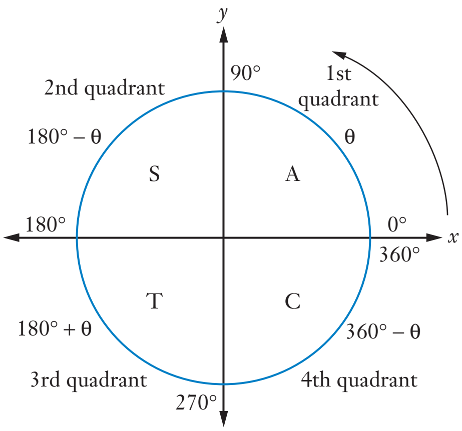
The direction of a vector is measured in the 4 quadrants around the circle as in trigonometry.



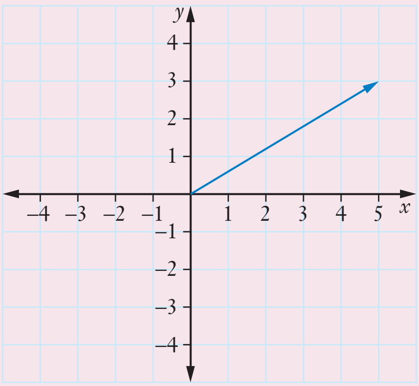
EXAMPLE 17

Find the direction to the nearest degree of:

a $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ **b** $\overrightarrow{OA} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ **c** $\overrightarrow{OA} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$

Solution

a

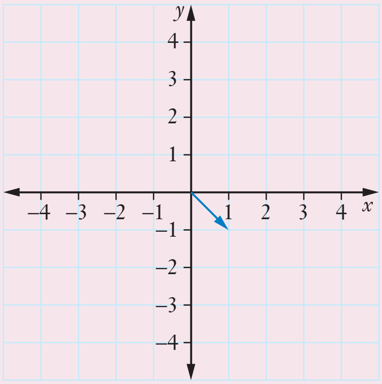


$\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ is in the 1st quadrant.

$$\tan \theta = \frac{y}{x}$$
$$= \frac{3}{5}$$
$$\theta = 30.9637\ldots°$$
$$\approx 31°$$

So the direction is 31°.

b

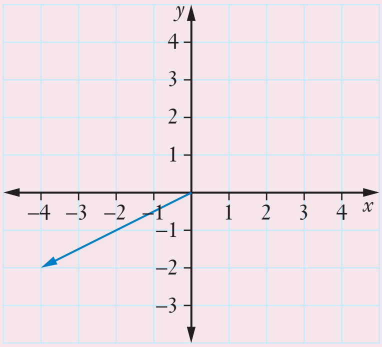


$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is in the 4th quadrant.

$$\tan\theta = \frac{y}{x}$$
$$= \frac{-1}{1}$$
$$= -1$$
$$\theta = 360° - 45° \text{ (4th quadrant)}$$
$$= 315°$$

So direction is 315°.

c



$\begin{pmatrix} -4 \\ -2 \end{pmatrix}$ is in the 3rd quadrant.

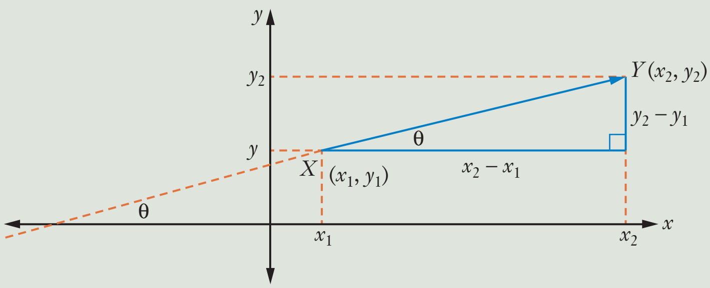
$$\tan\theta = \frac{-2}{-4}$$
$$= \frac{1}{2}$$
$$\theta = 180° + 26.5650...° \text{ (3rd quadrant)}$$
$$= 206.5650...°$$
$$\approx 207°$$

So direction is 207°.

Direction of a displacement vector

The direction of $\overrightarrow{XY}$ where $\overrightarrow{OX} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ is θ, the angle of inclination with the positive x-axis, where:

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_2}$$



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EXAMPLE 18

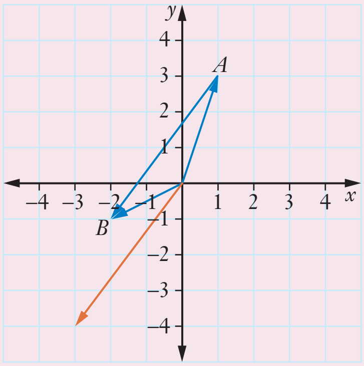
Find the direction of $\overrightarrow{AB}$ to the nearest degree where:

a $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

b $A = (0, -2)$ and $B = (-3, 1)$

Solution

a



Drawing $\overrightarrow{OA}$ and $\overrightarrow{OB}$ shows $\overrightarrow{AB}$.

Drawing $\overrightarrow{AB}$ as a position vector shows that it is in the 3rd quadrant.

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_2}$$

$$= \frac{-1-3}{-2-1}$$

$$= \frac{4}{3}$$

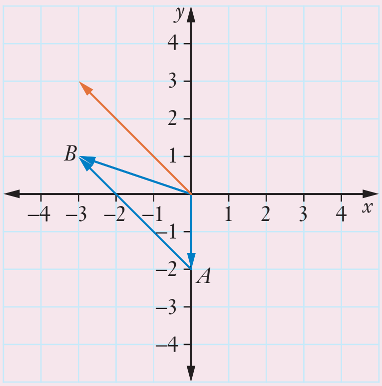
$$\theta \approx 180° + 53° \qquad \text{(3rd quadrant)}$$

$$= 233°$$

So direction of $\overrightarrow{AB}$ is 233°.

Alternatively, since we found the position vector equal to $\overrightarrow{AB}$ to show which quadrant it is in, we could have used $\overrightarrow{AB} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ with the formula $\tan\theta = \frac{y}{x}$.

b



Drawing $A = (0, -2)$ and $B = (-3, 1)$ shows $\overrightarrow{AB}$.

Drawing $\overrightarrow{AB}$ as a position vector shows that it is in the 2nd quadrant.

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_2}$$

$$= \frac{1-(-2)}{-3-0}$$

$$= -1$$

$$\theta = 180° - 45° \qquad \text{(2nd quadrant)}$$

$$= 135°$$

So direction of $\overrightarrow{AB}$ is 135°.

Alternatively, we could use the position vector $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ and the formula $\tan\theta = \frac{y}{x}$.

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EXAMPLE 19

Given $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, find the magnitude and direction of $w = \overrightarrow{OB} - \overrightarrow{OA}$.

Solution

$$w = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3-(-1) \\ 1-2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -1 \end{pmatrix} \qquad \text{(4th quadrant)}$$

$$|w| = \sqrt{x^2 + y^2}$$

$$= \sqrt{4^2 + (-1)^2}$$

$$= \sqrt{16+1}$$

$$= \sqrt{17}$$

$$\approx 4.1$$

$$\tan\theta = \frac{y}{x}$$

$$= -\frac{1}{4}$$

$$\theta \approx 360° - 14° \qquad \text{(4th quadrant)}$$

$$= 346°$$

So magnitude is 4.1 and direction is 346°.

EXT1 Exercise 3.05 Magnitude and direction of a vector

1 Vector a has magnitude 4 cm and direction 108° and vector b has magnitude 7 cm and direction 243°. Find the magnitude and direction of:

a $3a$ **b** $5b$

2 Find the magnitude of each position vector, to 1 decimal place.

a $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} 4 \\ 7 \end{pmatrix}$ **c** $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **d** $\begin{pmatrix} -5 \\ -4 \end{pmatrix}$ **e** $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$

3 Find the exact magnitude of each position vector.

a $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ **c** $\begin{pmatrix} -9 \\ -7 \end{pmatrix}$ **d** $\begin{pmatrix} 2 \\ 9 \end{pmatrix}$ **e** $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

4 Find the direction of each position vector to the nearest degree.

a $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **c** $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$ **d** $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **e** $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$

f $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ **g** $\begin{pmatrix} 5 \\ 9 \end{pmatrix}$ **h** $\begin{pmatrix} -2 \\ -9 \end{pmatrix}$ **i** $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$ **j** $\begin{pmatrix} -2 \\ -8 \end{pmatrix}$

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5 Find the magnitude of $\overrightarrow{AB}$ to 1 decimal place given:

a $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

b $\overrightarrow{OA} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

c $\overrightarrow{OA} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

d $A = (-3, 2)$ and $B = (1, 9)$

e $A = (1, 1)$ and $B = (-2, 4)$

6 Find the direction of $\overrightarrow{AB}$ to the nearest minute if:

a $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

b $\overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

c $\overrightarrow{OA} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

d $A = (-5, 0)$ and $B = (4, 4)$

e $A = (3, -4)$ and $B = (-1, -2)$

7 Find the magnitude and direction of each position vector.

a $\overrightarrow{OX} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$

b $v = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$

c $\overrightarrow{OP} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$

d $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

e $w = \begin{pmatrix} -7 \\ 5 \end{pmatrix}$

8 Find the magnitude and direction of $\overrightarrow{AB}$ given:

a $\overrightarrow{OA} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

b $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$

c $\overrightarrow{OA} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

d $A = (5, -2)$ and $B = (4, 3)$

e $A = (-1, -4)$ and $B = (0, 8)$

9 Given $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, find the magnitude and direction of:

a $\overrightarrow{AB}$

b $\overrightarrow{BA}$

c $\overrightarrow{OA} + \overrightarrow{OB}$

10 Given $\overrightarrow{OX} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$, find the magnitude and direction of:

a $\overrightarrow{XY}$

b $\overrightarrow{OX} + \overrightarrow{OY}$

c $\overrightarrow{OX} - \overrightarrow{OY}$

11 If $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$, find the magnitude and direction of:

a $\overrightarrow{BA}$

b $\overrightarrow{OA} + \overrightarrow{OB}$

c $\overrightarrow{OB} - \overrightarrow{OA}$

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12 Given $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, find the magnitude and direction of:

a $\overrightarrow{OA} + \overrightarrow{OB}$ **b** $\overrightarrow{OB} - \overrightarrow{OA}$ **c** $\overrightarrow{OA} - \overrightarrow{OB}$

13 Given that $\overrightarrow{OA} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$, find the magnitude and direction of:

a $\overrightarrow{OA} + \overrightarrow{OB}$ **b** $\overrightarrow{OA} - \overrightarrow{OB}$

14 The magnitude of vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is 6 and its direction is 150°. Find the exact values of x and y.

15 Vector $\begin{pmatrix} a \\ b \end{pmatrix}$ has magnitude 4 and its direction is 300°. Find the exact values of a and b.

EXT1 3.06 Components of a vector

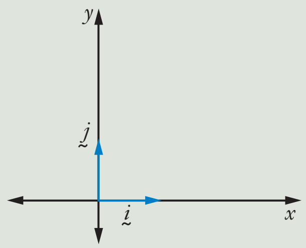
WS Componen of vectors

WS Addiion o vectors using components

We define the unit vector on the x-axis as i and the unit vector on the y-axis as j.

Unit vectors on axes

$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

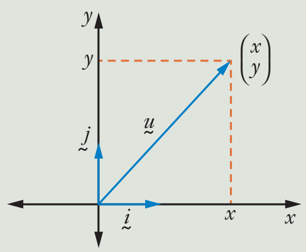


We can write a vector in **component form** by breaking it into its x and y components.

The horizontal and vertical components of any vector can be written in terms of i and j.

Components of a vector

The vector $u = \begin{pmatrix} x \\ y \end{pmatrix}$ can be written as $u = xi + yj$.



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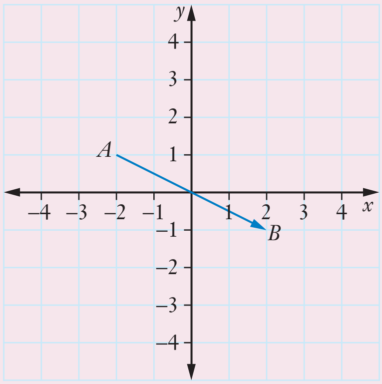
EXAMPLE 20

Write each vector in the form $xi + yj$.

a $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

b $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$

c

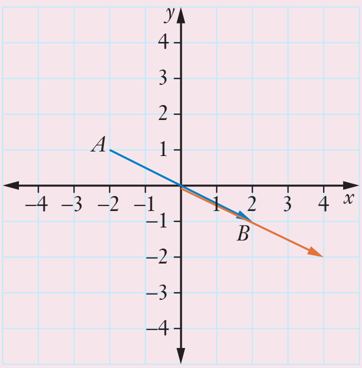


Solution

a $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3i + 2j$

b $\begin{pmatrix} -3 \\ 5 \end{pmatrix} = -3i + 5j$

c



Moving $\overrightarrow{AB}$ 2 units to the right and 1 down gives its position vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

$\overrightarrow{AB} = 4i - 2j$

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Adding and subtracting vectors

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (x_1 + x_2)i + (y_1 + y_2)j$$

$$\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (x_1 - x_2)i + (y_1 - y_2)j$$

EXAMPLE 21

a Given vectors $v = 4i + 7j$ and $w = -i - 3j$, find $v - w$.

b **i** Write $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$ in component form.

ii Find $\overrightarrow{OA} + \overrightarrow{OB}$.

Solution

a
$$\begin{aligned} v - w &= 4i + 7j - (-i - 3j) \\ &= 4i + 7j + i + 3j \\ &= 5i + 10j \end{aligned}$$

b **i**
$$\begin{aligned} \overrightarrow{OA} &= \begin{pmatrix} 3 \\ -1 \end{pmatrix} \\ &= 3i - j \end{aligned}$$

$$\begin{aligned} \overrightarrow{OB} &= \begin{pmatrix} 2 \\ 6 \end{pmatrix} \\ &= 2i + 6j \end{aligned}$$

ii
$$\begin{aligned} \overrightarrow{OA} + \overrightarrow{OB} &= 3i - j + 2i + 6j \\ &= 5i + 5j \end{aligned}$$

We can multiply a vector in its component form by a scalar.

EXAMPLE 22

Given vector $v = -5i + 4j$, find:

a $8v$

b $-3v$

Solution

a
$$\begin{aligned} 8v &= 8(-5i + 4j) \\ &= -40i + 32j \end{aligned}$$

b
$$\begin{aligned} -3v &= -3(-5i + 4j) \\ &= 15i - 12j \end{aligned}$$

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EXAMPLE 23

Find the magnitude and direction of $v = 2i - 3j$.

Magnitude and direction of a vector

Solution

$v = 2i - 3j$ is in the 4th quadrant (draw its graph).

$$|xi + yj| = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}$$

$$= \sqrt{2^2 + (-3)^2} \qquad = \frac{-3}{2}$$

$$= \sqrt{13} \qquad \theta \approx 360° - 56° \quad \text{(4th quadrant)}$$

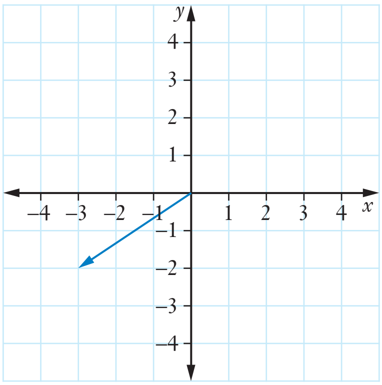
$$\approx 3.6 \qquad = 304°$$

So magnitude is 3.6 and direction is 304°.

EXT1 Exercise 3.06 Components of a vector

1 Write each vector in terms of i and j.

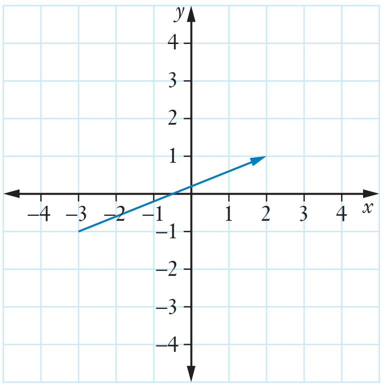
a



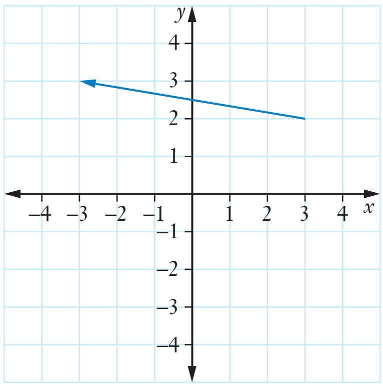
b $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

c $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

d



e



2 Given $v = i - 2j$ and $w = 3i + 5j$, find:

a $v + w$	**b** $v - w$	**c** $w - v$	**d** $4v$
e $-2w$	**f** $v + 3w$	**g** $5v - w$	**h** $6v + 7w$
i $-2v - w$	**j** $-5w - 7v$		

3 Find the magnitude and direction of each vector.

a $v = 4i - 5j$	**b** $v = -3i + 4j$	**c** $v = 9i - 2j$
d $v = -6i - 5j$	**e** $v = -2i + j$	

4 Vector $\overrightarrow{AB}$ has coordinates $A(-2, 5)$ and $B(1, -7)$.

a Write $\overrightarrow{AB}$ in terms of its components.

b Find the magnitude and direction of $\overrightarrow{AB}$.

5 Given $a = 4i - 2j$ and $b = -3i + 7j$, find:

a $3a$	**b** $-b$	**c** $a + b$	**d** $b - a$
e $7a + 2b$	**f** $-5b - 2a$	**g** $9a - 2b$	**h** $-3b - 4a$

6 If $a = 6i + 3j$ and $b = 2i - j$, find the magnitude and direction of:

a $a + b$ **b** $a - b$

7 Find the magnitude and direction of $x + y$ given that $x = 5i + 3j$ and $y = -2i - 5j$.

8 Find the magnitude and direction of $b - a$ if $a = i + 7j$ and $b = 4i - 9j$.

9 Vector $v = ai + 3j$ has magnitude 5. Find the value of a.

10 Vector $v = 6i + bj$ has direction 45°. What is the value of b?

11 Vector $v = ai + bj$ has magnitude 4 and direction 120°. Find the exact values of a and b.

12 Given the vector $v = xi + yj$, prove that $|xi + yj| = \sqrt{x^2 + y^2}$.

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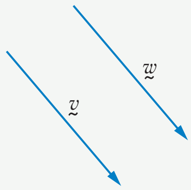
3. TEST YOURSELF

For Questions **1** to **4**, choose the correct answer **A**, **B**, **C** or **D**.

Practice quiz

1 Which 2 statements are true about vectors v and w?



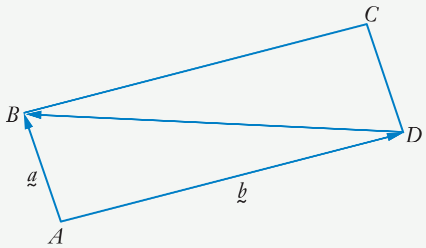
A $v = w$ **B** $v \parallel w$

C $v = -w$ **D** v = scalar of w

2 Which is NOT correct vector notation?

A $\overrightarrow{AB}$ **B** a **C** $\mathbf{a}$ **D** a

3 Vector $\overrightarrow{DB}$ represents:



A $a + b$ **B** $a - b$ **C** $b + a$ **D** $b - a$

4 The direction of the vector $v = (-25, -9)$ is closest to:

A 20° **B** 160° **C** 200° **D** 340°

5 State which measurement is a vector and which is a scalar.

a acceleration **b** distance **c** mass

d displacement **e** velocity

6 Given $\overrightarrow{OA} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -6 \\ -1 \end{pmatrix}$, find the magnitude and direction of:

a $\overrightarrow{OA} + \overrightarrow{OB}$ **b** $\overrightarrow{OB} - \overrightarrow{OA}$

7 Draw each position vector on a number plane.

a $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ **c** $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ **d** $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ **e** $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

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8 The resultant vector of $3\begin{pmatrix} p \\ -4 \end{pmatrix} - \begin{pmatrix} -1 \\ q \end{pmatrix}$ is $\begin{pmatrix} 22 \\ -5 \end{pmatrix}$. Evaluate p and q.

9 A drone flies for 86 m south then turns and flies west for 53 m.

a Draw vectors showing the drone's journey.

b Find the drone's distance and bearing from the starting point.

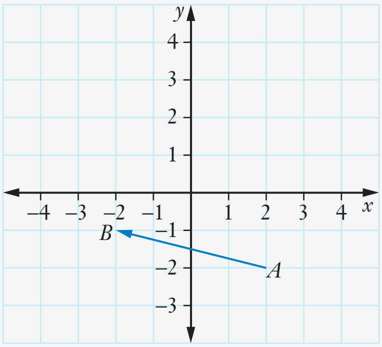
10 Find the scalar that changes a vector with magnitude 12 into a unit vector.

11 Find the magnitude and direction of each vector.

a $v = 4i + j$ **b** $v = i - 3j$

c $v = -2i - 4j$ **d** $v = -4i + 3j$

12 Find the position vector equal to $\overrightarrow{AB}$.



13 If $a = 9i + 5j$ and $b = 4i - 7j$, find the magnitude and direction of:

a $a + b$ **b** $a - b$

14 Find the resultant vector of:

a $\begin{pmatrix} -3 \\ -8 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} 4 \\ -9 \end{pmatrix} - \begin{pmatrix} -5 \\ 2 \end{pmatrix}$ **c** $-5\begin{pmatrix} 3 \\ -5 \end{pmatrix}$

d $\begin{pmatrix} -2 \\ 9 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} -4 \\ -1 \end{pmatrix}$ **e** $3\begin{pmatrix} 2 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

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15 Draw a vector for each situation.

a A plane flies 320 km on a bearing of 288°.

b A car travels due south for 3 hours at 95 km/h.

16 Draw 2 relative (displacement) vectors equal to the position vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

17 A particle starts 2 cm to the left of the origin and moves 3 cm to the right. It then turns and moves 6 cm to the left and then comes to a stop.

a Draw vectors to show the motion of the particle.

b Find where the particle is at the end of its motion.

c Find the total distance it travels.

18 Find the magnitude and direction of $\overrightarrow{OX} = \begin{pmatrix} -5 \\ -12 \end{pmatrix}$.

19 Copy the diagram, then draw:

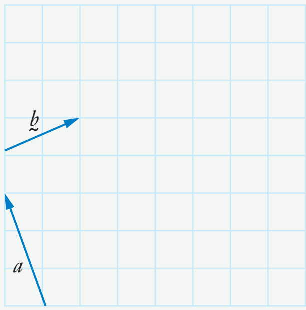
a $2a$

b $-3b$

c $a + b$

d $a - b$

e $b - a$



20 Find the magnitude and direction of $\overrightarrow{AB}$ to 1 decimal place given that $\overrightarrow{OA} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

21 Given $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, find the resultant of:

a $\overrightarrow{OA} + \overrightarrow{OB}$

b $\overrightarrow{OB} - \overrightarrow{OA}$

22 Find the magnitude and direction of $x + y$ given that $x = 2i + j$ and $y = -i - 6j$.

23 Given $v = 5i - 7j$ and $w = -i + 3j$, find:

a $v + w$

b $w - v$

c $-4w$

d $3v + 2w$

e $2v - 6w$

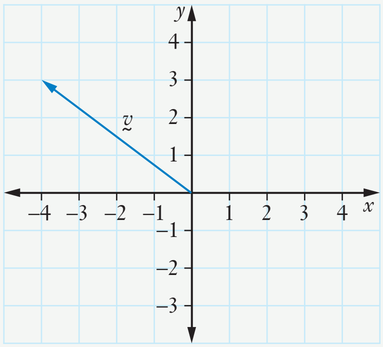
24 Vector $\overrightarrow{AB}$ has coordinates $A(-5, 2)$ and $B(3, -8)$.

a Write $\overrightarrow{AB}$ in terms of the components i and j.

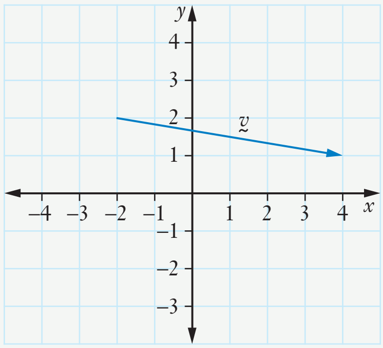
b Find the magnitude and direction of $\overrightarrow{AB}$.

25 Write each vector in terms of i and j.

a



b



c $\underset{\sim}{v} = \begin{pmatrix} -8 \\ 7 \end{pmatrix}$

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3. CHALLENGE EXERCISE

1 Vector $v = ai + bj$ has magnitude 5 and direction 105°.
Find the values of a and b correct to 1 decimal place.

2 Find the magnitude and direction of $\overrightarrow{AB} + \overrightarrow{BC}$ given $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$.

3 The magnitude of vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is $3\sqrt{2}$ and its direction is 225°.

a Evaluate x and y.

b Write the vector in the form $v = xi + yj$.

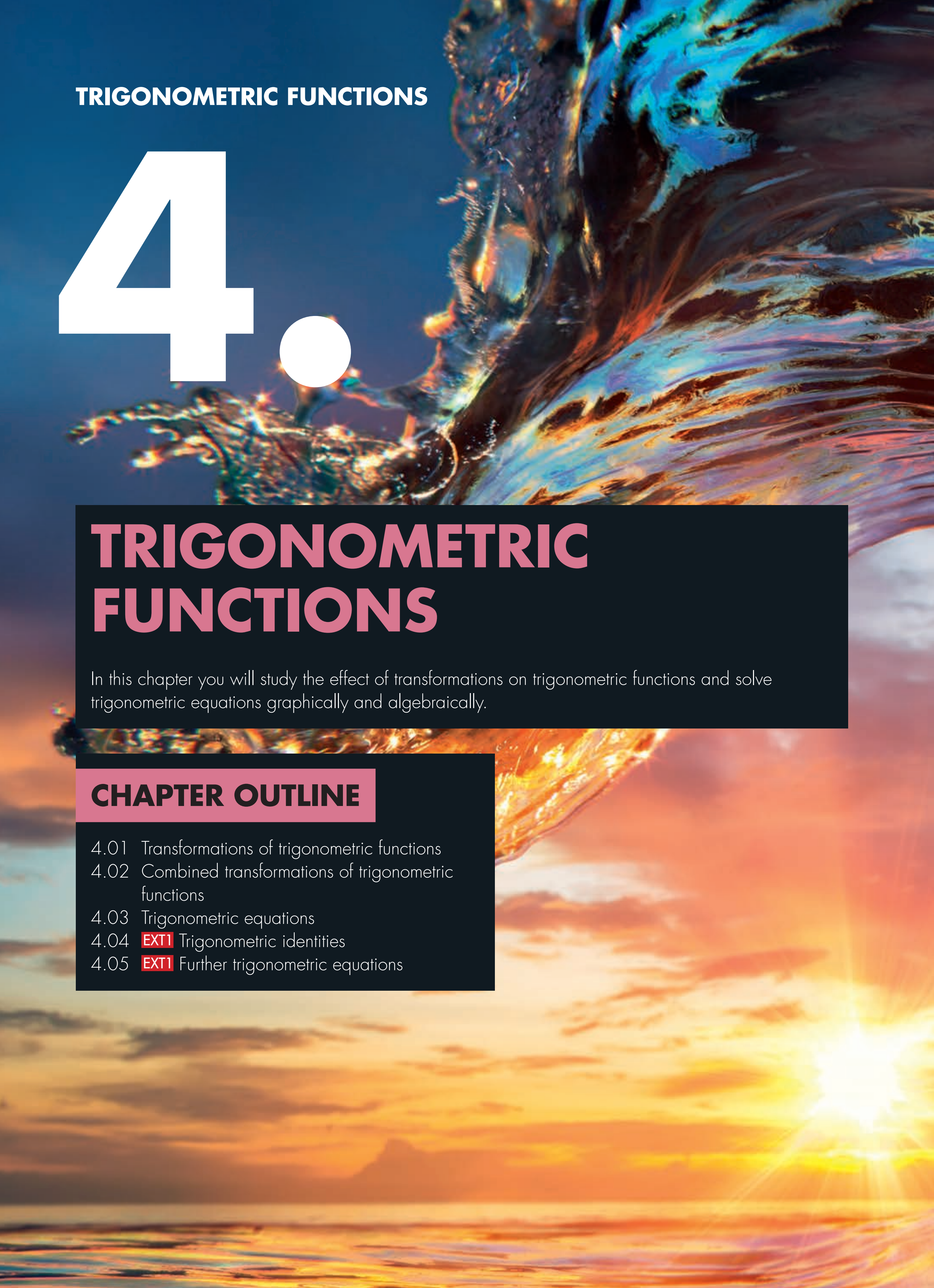
4 A force with magnitude 6 newtons and direction 118° is applied to an object.
Find the horizontal and vertical components of the force, to 1 decimal place.

5 A vector in polar coordinates is given by (r, θ) where r is magnitude and θ is direction.
Write the following vectors in polar form.

a $4i - 3j$

b $\begin{pmatrix} -2 \\ -2\sqrt{3} \end{pmatrix}$

c $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$



4.

TRIGONOMETRIC FUNCTIONS

In this chapter you will study the effect of transformations on trigonometric functions and solve trigonometric equations graphically and algebraically.

CHAPTER OUTLINE

4.01 Transformations of trigonometric functions
4.02 Combined transformations of trigonometric functions
4.03 Trigonometric equations
4.04 EXT1 Trigonometric identities
4.05 EXT1 Further trigonometric equations

IN THIS CHAPTER YOU WILL:

- apply and understand the effect of different transformations on trigonometric functions
- solve trigonometric equations graphically and algebraically
- EXT1 prove and apply trigonometric identities
- EXT1 solve further trigonometric equations graphically and algebraically

TERMINOLOGY

amplitude: The height from the centre of a sine or cosine function to the maximum or minimum values (peaks and troughs of its graph respectively). For $y = k \sin ax$ and $y = k \cos ax$, the amplitude is k.

centre: The mean value of a sine or cosine function that is equidistant from the maximum and minimum values. For $y = k \sin ax + c$ and $y = k \cos ax + c$, the centre is c.

period: The length of one cycle of a periodic function on the x-axis, before the function repeats itself. For $y = k \sin ax$ and $y = k \cos ax$, the period is $\frac{2\pi}{a}$.

phase: A horizontal shift (translation). For $y = k \sin [a(x + b)]$ and $y = k \cos [a(x + b)]$, the phase is b; that is, the graphs of $y = k \sin ax$ and $y = k \cos ax$ respectively are shifted b units to the left.

WS Transforming trigonometric functions

WS Sketching periodic functions - amplitude and period

WS Sketching periodic functions - phase and vertical shift

4.01 Transformations of trigonometric functions

The transformations you studied in Chapter 2, *Transformations of functions*, can be applied to the trigonometric functions.

Vertical dilations

A vertical dilation of $y = f(x)$ is $y = kf(x)$ with scale factor k.

EXAMPLE 1

a Describe the transformation if $y = \cos x$ is transformed to $y = 3 \cos x$.

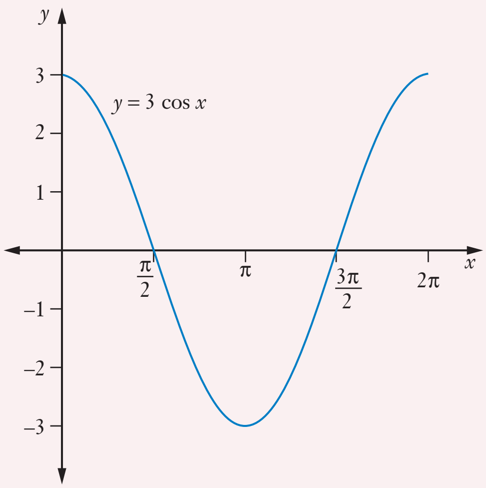
b Sketch the graph of $y = 3 \cos x$ in the domain $[0, 2\pi]$ and state its range.

Solution

a $y = 3 \cos x$ is a vertical dilation of $y = \cos x$ with scale factor 3.

b A vertical dilation multiplies the y values by 3.

$(0, 1)$	becomes	$(0, 3)$
$\left(\frac{\pi}{2} 0\right)$	becomes	$\left(\frac{\pi}{2} 0\right)$
$(\pi, -1)$	becomes	$(\pi, -3)$
$\left(\frac{3\pi}{2}, 0\right)$	becomes	$\left(\frac{3\pi}{2}, 0\right)$
$(2\pi, 1)$	becomes	$(2\pi, 3)$



The range is $[-3, 3]$.

Notice that $y = 3 \cos x$ has amplitude 3.

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Amplitude as a vertical dilation

$y = k \sin x$ or $y = k \cos x$ has **amplitude** k (a vertical dilation with scale factor k).

- If $k > 1$, the function is stretched.
- If $0 < k < 1$, the function is compressed.
- If $k = -1$, the function is reflected in the x-axis.

Note: $y = k \tan x$ does not have an amplitude.

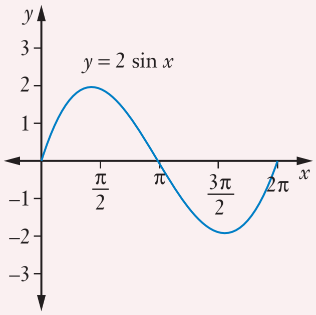
EXAMPLE 2

a Sketch the graph of $f(x) = 2 \sin x$ in the domain $[0, 2\pi]$.

b Find an equation for a cosine function reflected in the x-axis with amplitude 7.

Solution

a This is a vertical dilation of $f(x) = \sin x$ with scale factor 2 (it has amplitude 2).



b $y = \cos x$ is reflected in the x-axis with scale factor $k = -1$.

$y = -\cos x$

Amplitude: $k = 7$

So $y = -7 \cos x$

Horizontal dilations

A horizontal dilation of $y = f(x)$ is $y = f(ax)$ with scale factor $\frac{1}{a}$.

EXAMPLE 3

a Describe the transformation if $f(x) = \sin x$ is transformed to $y = \sin 2x$.

b Draw the graph of $f(x) = \sin 2x$ in the domain $[0, 2\pi]$.

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Solution

a $f(x) = \sin 2x$ is a horizontal dilation of $f(x) = \sin x$ with scale factor $\frac{1}{2}$.

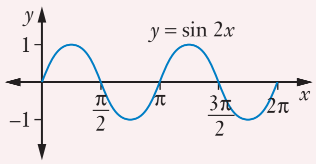
b A horizontal dilation multiplies the x values by $\frac{1}{2}$ (or divides them by 2).

$(0, 0)$	becomes	$(0, 0)$
$\left(\frac{\pi}{2}, 1\right)$	becomes	$\left(\frac{\pi}{4}, 1\right)$
$(\pi, 0)$	becomes	$\left(\frac{\pi}{2}, 0\right)$
$\left(\frac{3\pi}{2}, -1\right)$	becomes	$\left(\frac{3\pi}{4}, -1\right)$
$(2\pi, 0)$	becomes	$(\pi, 0)$

Notice that these image points lie in the domain $[0, \pi]$ and not $[0, 2\pi]$.

The **period** of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$.

To sketch the function in the domain $[0, 2\pi]$ we repeat the sine curve from $x = \pi$ to 2π.



Notice that a horizontal dilation compresses the graph of $y = \sin x$, which changes its period. The function $y = \sin 2x$ has 2 complete sine function cycles in the domain $[0, 2\pi]$.

Period as a horizontal dilation

$y = \sin ax$ has **period** $\frac{2\pi}{a}$.

$y = \cos ax$ has **period** $\frac{2\pi}{a}$.

$y = \tan ax$ has **period** $\frac{\pi}{a}$.

- If $a > 1$, the function is compressed horizontally.
- If $0 < a < 1$, the function is stretched horizontally.
- If $a = -1$, the function is reflected in the y-axis.

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EXAMPLE 4

a Find the period of each function.

i $y = \cos x$ **ii** $f(x) = \sin 5x$ **iii** $y = \tan 2x$

b Sketch each graph in the domain $[0, 2\pi]$.

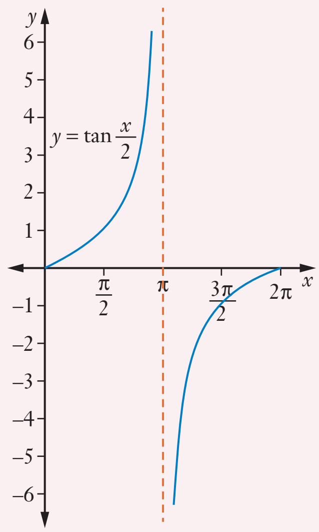
i $y = \tan \frac{x}{2}$ **ii** $y = \sin(-x)$

Solution

a **i** $y = \cos x$ has period 2π.

ii $f(x) = \sin 5x$ has period $\frac{2\pi}{5}$.

iii $y = \tan 2x$ has period $\frac{\pi}{2}$.



b **i** $y = \tan \frac{x}{2}$ is a horizontal dilation of $y = \tan x$.

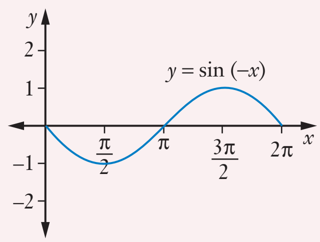
It has period $\frac{\pi}{\frac{1}{2}}$ or 2π.

So there will be one cycle of the tan function in the domain $[0, 2\pi]$.

ii $y = \sin(-x)$ is a reflection of $y = \sin x$ in the y-axis, so $a = -1$.

This will change the x values. Transforming points in the domain $[-2\pi, 0]$ will give image points in the domain $[0, 2\pi]$.

$(0, 0)$	becomes	$(0 \times -1, 0) \equiv (0, 0)$
$\left(-\frac{\pi}{2}, -1\right)$	becomes	$\left(\frac{\pi}{2}, -1\right)$
$(-\pi, 0)$	becomes	$(\pi, 0)$
$\left(-\frac{3\pi}{2}, 1\right)$	becomes	$\left(\frac{3\pi}{2}, 1\right)$
$(-2\pi, 0)$	becomes	$(2\pi, 0)$



Vertical translations

A vertical translation of $y = f(x)$ is $y = f(x) + c$.

EXAMPLE 5

Sketch the graph of $y = \cos x + 2$ in the domain $[-\pi, \pi]$.

Solution

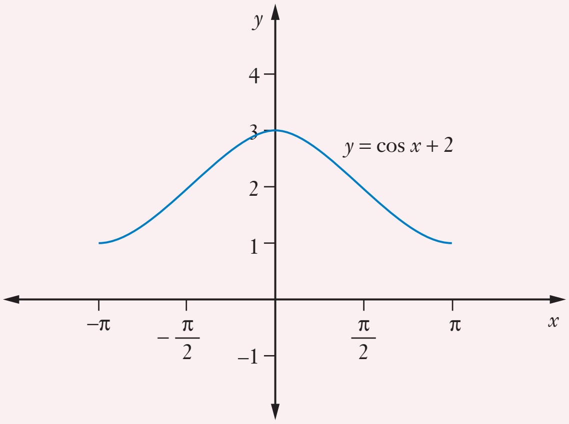
$y = f(x) + c$ is a vertical translation of $y = f(x)$.

So $y = \cos x + 2$ is a vertical translation of $y = \cos x$ up 2 units.

This changes the y values by adding 2 to each.

The domain is $[-\pi, \pi]$.

$(-\pi, -1)$	becomes	$(-\pi, -1 + 2) \equiv (-\pi, 1)$
$\left(-\frac{\pi}{2}, 0\right)$	becomes	$\left(-\frac{\pi}{2}, 2\right)$
$(0, 1)$	becomes	$(0, 3)$
$\left(\frac{\pi}{2}, 0\right)$	becomes	$\left(\frac{\pi}{2}, 2\right)$
$(\pi, -1)$	becomes	$(\pi, 1)$



Notice that the centre of the function is 2.

Centre as a vertical translation

The **centre** of $y = \sin x + c$ and $y = \cos x + c$ is c.

- If $c > 0$, the centre is translated upwards.
- If $c < 0$, the centre is translated downwards.

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EXAMPLE 6

a Find the centre of the function:

i $f(x) = \sin x - 7$ **ii** $y = \cos x + 4$

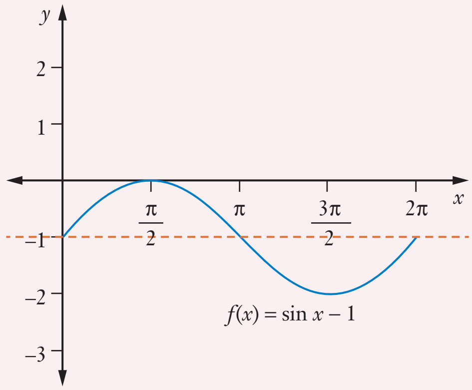
b Sketch the graph of $f(x) = \sin x - 1$.

Solution

a **i** The centre is -7.

ii The centre is 4.

b This is a vertical translation 1 unit down of $f(x) = \sin x$. The centre is -1.



Horizontal translations

A horizontal translation of $y = f(x)$ is given by $y = f(x + b)$.

EXAMPLE 7

Sketch the graph of $y = \sin\left(x - \frac{\pi}{2}\right)$ in the domain $[0, 2\pi]$.

Solution

$y = \sin\left(x - \frac{\pi}{2}\right)$ is a horizontal translation of $y = \sin x$ by $\frac{\pi}{2}$ units to the right.

We change the x values by adding $\frac{\pi}{2}$.

But since we need the *transformed* values of x to be in the domain $[0, 2\pi]$, our *original* values need to be in the domain $\left[0 - \frac{\pi}{2}\ 2\pi - \frac{\pi}{2}\right] \equiv \left[-\frac{\pi}{2}\ \frac{3\pi}{2}\right]$.

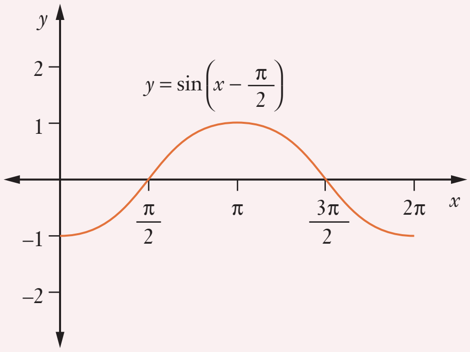
$\left(-\frac{\pi}{2}, -1\right)$ becomes $\left(-\frac{\pi}{2} + \frac{\pi}{2}, -1\right) \equiv (0, -1)$

$(0, 0)$ becomes $\left(0 + \frac{\pi}{2}, 0\right) \equiv \left(\frac{\pi}{2}, 0\right)$

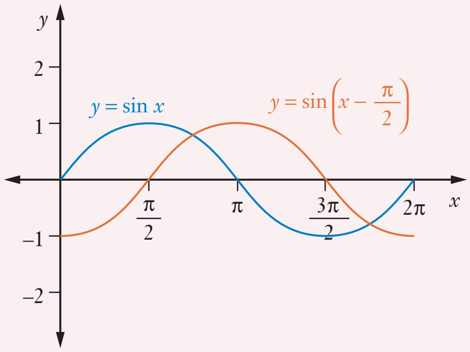
$\left(\frac{\pi}{2}, 1\right)$ becomes $\left(\frac{\pi}{2} + \frac{\pi}{2}, 1\right) \equiv (\pi, 1)$

$(\pi, 0)$ becomes $\left(\pi + \frac{\pi}{2}, 0\right) \equiv \left(\frac{3\pi}{2}, 0\right)$

$\left(\frac{3\pi}{2}, -1\right)$ becomes $\left(\frac{3\pi}{2} + \frac{\pi}{2}, -1\right) \equiv (2\pi, -1)$



Instead of finding points, you could sketch $y = \sin x$ and then shift it $\frac{\pi}{2}$ units to the right.



Phase as a horizontal translation

The **phase** of $y = \sin(x + b)$, $y = \cos(x + b)$ and $y = \tan(x + b)$ is b.

- If $b > 0$, the phase shift is to the left.
- If $b < 0$, the phase shift is to the right.

Phase shift of trigonometric functions

EXAMPLE 8

a Explain the meaning of $\frac{\pi}{4}$ in the equation $y = \tan\left(x + \frac{\pi}{4}\right)$.

b Sketch the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$ in the domain $[0, 2\pi]$.

Solution

a The function $y = \tan\left(x + \frac{\pi}{4}\right)$ has a phase of $\frac{\pi}{4}$ (to the left).

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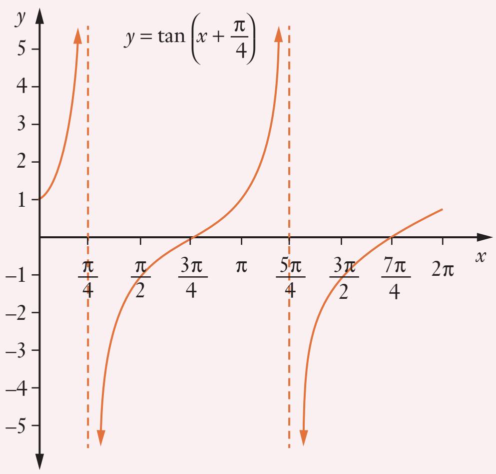
b To find points on the transformed graph, subtract $\frac{\pi}{4}$ from x values.

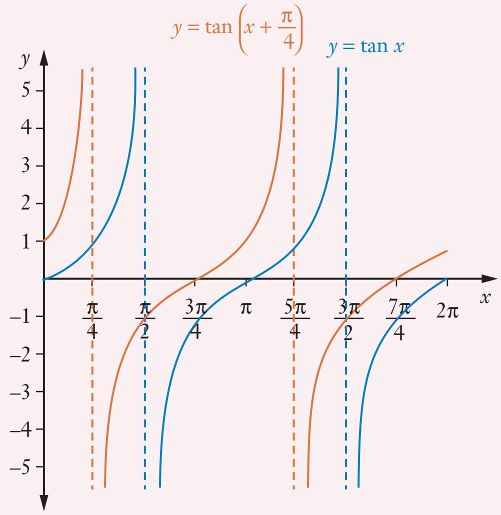
But since we need the *transformed* values of x to be in the domain $[0, 2\pi]$,

our *original* values need to be in the domain $\left[0+\frac{\pi}{4}\ 2\pi+\frac{\pi}{4}\right] \equiv \left[\frac{\pi}{4}\ \frac{9\pi}{4}\right]$.

$\left(\frac{\pi}{4}, 1\right)$	becomes	$(0, 1)$
Undefined at $x=\frac{\pi}{2}$	becomes	Undefined at $x=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$
$\left(\frac{3\pi}{4}, -1\right)$	becomes	$\left(\frac{\pi}{2}, -1\right)$
$(\pi, 0)$	becomes	$\left(\frac{3\pi}{4}, 0\right)$
$\left(\frac{5\pi}{4}, 1\right)$	becomes	$(\pi, 1)$
Undefined at $x=\frac{3\pi}{2}$	becomes	Undefined at $x=\frac{3\pi}{2}-\frac{\pi}{4}=\frac{5\pi}{4}$
$\left(\frac{7\pi}{4}, -1\right)$	becomes	$\left(\frac{3\pi}{2}, -1\right)$
$(2\pi, 0)$	becomes	$\left(\frac{7\pi}{4}, 0\right)$
$\left(\frac{9\pi}{4}, -1\right)$	becomes	$(2\pi, 1)$



Instead of finding points, you could sketch $y = \tan x$ and then shift it $\frac{\pi}{4}$ units to the left.



Exercise 4.01 Transformations of trigonometric functions

1 Describe whether each transformation of a trigonometric function changes its amplitude, period, centre or phase.

a horizontal translation **b** vertical dilation

c horizontal dilation **d** vertical translation

2 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = 5 \sin x$ **b** $f(x) = 2 \tan x$ **c** $y = -\cos x$

d $f(x) = -2 \sin x$ **e** $y = -\tan x$

3 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \sin x + 1$ **b** $y = \tan x - 2$ **c** $f(x) = \cos x - 3$

4 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \cos 4x$ **b** $y = \sin \frac{x}{2}$

c $f(x) = \tan 2x$ **d** $y = \tan \frac{x}{4}$

5 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \cos (x + \pi)$ **b** $y = \tan \left(x - \frac{\pi}{2}\right)$ **c** $y = \sin \left(x - \frac{\pi}{4}\right)$

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6 Find the equation of the transformation of $y = \sin x$ if the transformed function has:

a amplitude 9 **b** a reflection in the x-axis **c** centre –4

d period π **e** a phase shift of π units to the right

7 Find the equation of the transformation of $y = \cos x$ if the transformed function has:

a amplitude 4 **b** a phase of $\frac{\pi}{3}$ units

c centre 8 **d** period $\frac{\pi}{2}$

e a vertical dilation with scale factor 7

8 Find the equation of the transformation of $y = \tan x$ if the transformed function has:

a period 2π **b** a shift of $\frac{\pi}{6}$ units to the right

c a reflection in the y-axis

9 Sketch each graph in the domain $[-\pi, \pi]$.

a $y = 3 \sin x$ **b** $y = \tan(-x)$ **c** $f(x) = \cos 2x$

d $y = \sin(x - \pi)$ **e** $f(x) = -\cos x$

10 **a** EXT1 Find the equation of the transformed function and sketch the graph if the inverse trigonometric function $y = \sin^{-1} x$ is:

i vertically dilated with scale factor 3

ii horizontally dilated with scale factor $\frac{1}{2}$

iii vertically translated up $\frac{\pi}{2}$ units

v horizontally translated 1 unit to the right

b EXT1 Find the equation of the transformed function and sketch the graph if $y = \cos^{-1} x$ is:

i vertically dilated with scale factor $\frac{1}{2}$

ii horizontally dilated with scale factor 3

iii vertically translated down π units

v horizontally translated 2 units to the left

c EXT1 Find the equation of the transformed function and sketch the graph if $y = \tan^{-1} x$ is:

i vertically dilated with scale factor 2

ii horizontally dilated with scale factor 5

iii vertically translated up $\frac{\pi}{4}$ units

v horizontally translated 4 units to the right

4.02 Combined transformations of trigonometric functions

WS Graphing trigonometric functions

WS Sketching periodic functions

We can put all the information about trigonometric functions together.

General equation of trigonometric functions

Function	Amplitude	Period	Phase	Centre
$y = k\sin[a(x+b)] + c$	k	$\dfrac{2\pi}{a}$	b Shift left if $b > 0$ Shift right if $b < 0$	c Shift up if $c > 0$ Shift down if $c < 0$
$y = k\cos[a(x+b)] + c$	k	$\dfrac{2\pi}{a}$		
$y = k\tan[a(x+b)] + c$	No amplitude	$\dfrac{\pi}{a}$		

EXAMPLE 9

a Sketch each function in the domain $[0, 2\pi]$.

i $y = 4\sin\dfrac{x}{2} + 1$ **ii** $y = 3\cos\left(x - \dfrac{\pi}{4}\right)$

b Find the equation of a cosine function that has amplitude 5, period 4π, centre -2 and a phase of 2 units to the left.

Solution

a i $y = 4\sin\dfrac{x}{2} + 1$ has amplitude 4,

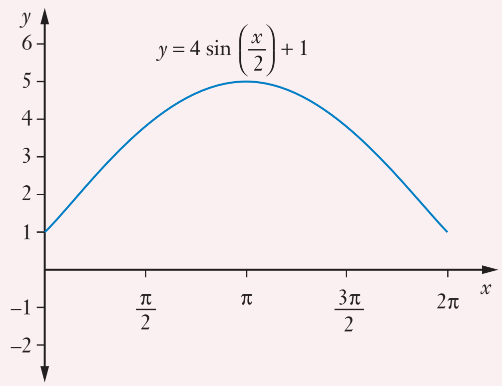
period $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$ and centre 1.

Period 4π means only half the sine function curve will be in the domain $[0, 2\pi]$.

Centre 1 and amplitude 4 means:

Minimum $1 - 4 = -3$

Maximum $1 + 4 = 5$



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ii $y = 3\cos\left(x - \frac{\pi}{4}\right)$ has amplitude 3, period 2π, phase $\frac{\pi}{4}$ to the right and centre 0.

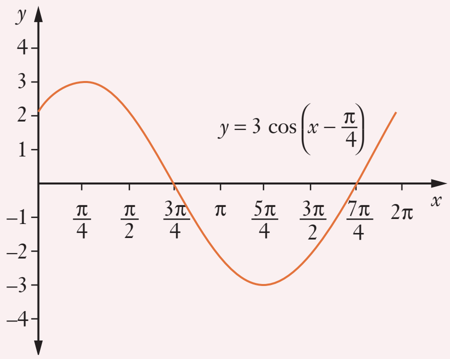
A phase is a horizontal translation, so it changes the x values and the domain.

Add $\frac{\pi}{4}$ to each x value.

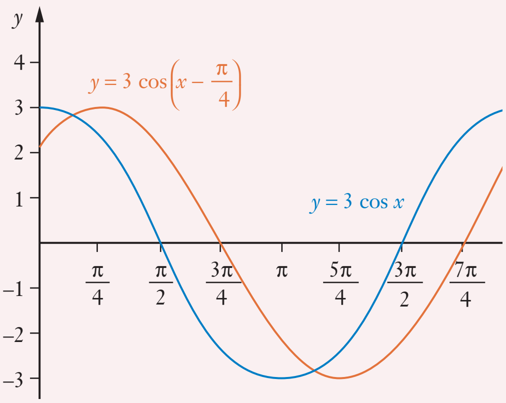
But since we need the *transformed* values of x to be in the domain $[0, 2\pi]$,

our *original* values need to be in the domain $\left[0 - \frac{\pi}{4}\ 2\pi - \frac{\pi}{4}\right] \equiv \left[-\frac{\pi}{4}\ \frac{7\pi}{4}\right]$.

$\left(-\frac{\pi}{4}, 2\ 1\right)$	becomes	$(0, 2.1)$
$(0, 3)$	becomes	$\left(\frac{\pi}{4}, 3\right)$
$\left(\frac{\pi}{4}, 2\ 1\right)$	becomes	$\left(\frac{\pi}{2}, 2\ 1\right)$
$\left(\frac{\pi}{2}, 0\right)$	becomes	$\left(\frac{3\pi}{4}, 0\right)$
$\left(\frac{3\pi}{4}, -2\ 1\right)$	becomes	$(\pi, -2.1)$
$(\pi, -3)$	becomes	$\left(\frac{5\pi}{4}, -3\right)$
$\left(\frac{5\pi}{4}, -2\ 1\right)$	becomes	$\left(\frac{3\pi}{2}, -2\ 1\right)$
$\left(\frac{3\pi}{2}, 0\right)$	becomes	$\left(\frac{7\pi}{4}, 0\right)$
$\left(\frac{7\pi}{4}, 2\ 1\right)$	becomes	$(2\pi, 2.1)$



Alternatively, sketch $y = 3\cos x$ and then shift it $\frac{\pi}{4}$ units to the right.



b $y = k \cos [a(x + b)] + c.$

Amplitude: $k = 5$

Period: $\frac{2\pi}{a} = 4\pi$

$2\pi = 4\pi a$

$\frac{1}{2} = a$

Phase: $b = 2$

Centre: $c = -2$

The equation is $y = 5 \cos \left[\frac{1}{2}(x + 2)\right] - 2$

Exercise 4.02 Combined transformations of trigonometric functions

1 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = 2 \sin x - 3$

b $y = -\tan 2x$

c $f(x) = \cos\left(x + \frac{\pi}{2}\right) + 1$

d $y = \sin\left(-\frac{x}{2}\right) + 2$

e $f(x) = 3 \cos 2x - 2$

2 **a** Find the equation of the transformed function if $y = \sin x$ is vertically dilated with scale factor 5, horizontally dilated with scale factor $\frac{1}{3}$, vertically translated 6 units down and horizontally translated 5 units to the left.

b Describe each transformation as a change in period, amplitude, centre or phase of the function.

3 Find the equation of the transformed function of $y = \cos x$ if it is:

a vertically dilated with scale factor 4, horizontally dilated with scale factor $\frac{1}{6}$, vertically translated 2 units up and horizontally translated $\frac{\pi}{3}$ units to the right

b reflected in the x-axis, reflected in the y-axis, translated 5 units down and π units to the left.

4 Sketch each graph in the domain $[-\pi, \pi]$.

a $y = 3 \sin 2x$

b $y = 2 \tan \frac{x}{2} + 1$

c $f(x) = -2 \cos 3x$

d $y = 5 \sin x - 3$

e $y = \cos(-2x) + 1$

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5 Describe the features of each function in terms of amplitude, period, centre, and phase.

a $y = 3 \tan 4x - 5$

b $y = 8 \cos (x + \pi) - 3$

c $y = 5 \sin [2(x - 3)] + 1$

6 Find the equation of each function.

a a sine function with amplitude 7, period π, phase of 1 unit to the right and centre -3

b a cosine function with amplitude 1, a reflection in the x-axis, period $\frac{2\pi}{5}$ and centre 2

c a tangent function with period 2π, a reflection in the x-axis and a phase of 2 units to the left

d $y = \sin x$ with a vertical dilation scale factor 4, a reflection in the y-axis, a horizontal dilation scale factor 3, a vertical translation 2 units up and a horizontal translation 5 units to the left

7 Describe the features of $y = k \operatorname{cosec} [a(x + b)] + c$.

8 Find the equation of the transformed function if $y = \tan x$ is translated 3 units to the right and then dilated horizontally with scale factor $\frac{1}{4}$.

9 The water depth at a harbour entrance is 5 m at low tide and 25 m at high tide. The time between each low tide is around 12 hours.

a Find the centre of the tidal motion.

b What is the amplitude and period?

c Write an equation for the water depth D metres in terms of time t hours as a cosine function.

10 Find an equation for blood pressure, B, as a sine function of time, t minutes, if the maximum blood pressure is 120 and the minimum is 80, with a heart rate of 60 beats per minute.

11 EXT1 Find the equation of each transformed function and sketch its graph.

a $y = \sin^{-1} x$ is reflected in the x-axis and translated 2 units to the right

b $y = \tan^{-1} x$ is vertically dilated with scale factor 4 and translated 1 unit down

c $y = \cos^{-1} x$ is reflected in the y-axis, horizontally dilated with scale factor 2 and translated $\frac{\pi}{2}$ units up

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WS Solving trigonometric equations graphically

WS Trigonometric equations

4.03 Trigonometric equations

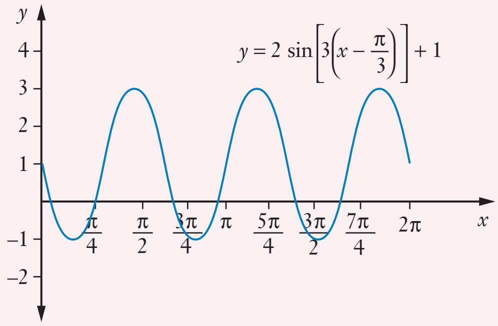
Graphical solutions

We can use the work on transformations to help solve trigonometric equations graphically.

EXAMPLE 10

a The graph of the trigonometric function $y = 2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1$ is shown for $[0, 2\pi]$.

Find the number of solutions to the trigonometric equation $2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1 = 0$ for $[0, 2\pi]$.



b **i** Sketch the graphs of $y = \frac{x}{4} - 1$ and $y = 3\cos x - 2$ for $[0, 2\pi]$.

ii Find the number of solutions to the equation $3\cos x - 2 = \frac{x}{4} - 1$ for $[0, 2\pi]$.

iii Solve the equation graphically.

Solution

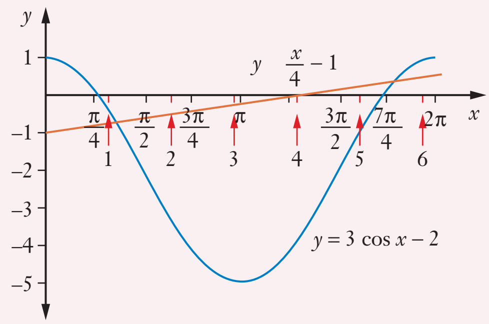
a To solve $2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1 = 0$ graphically, we find the x-intercepts.

The function has 6 x-intercepts in the domain $[0, 2\pi]$.

So the equation has 6 solutions.

b **i** $y = 3\cos x - 2$ has amplitude 3 and centre –2.

$y = \frac{x}{4} - 1$ is a linear function with x-intercept 4 and y-intercept –1.



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ii The solutions to $3\cos x - 2 = \frac{x}{4} - 1$ are shown by where the graphs $y = 3\cos x - 2$ and $y = \frac{x}{4} - 1$ intersect.

The graphs intersect in 2 places in $[0, 2\pi]$ so the equation has 2 solutions.

iii The graphs intersect just after $x = 1$ and $x = \frac{7\pi}{4} \approx 5.5$.

A precise graph drawn on graph paper or using technology would show that the solutions are $x \approx 1.1$ and 5.6.

Algebraic solutions

EXAMPLE 11

Solve for [0°, 360°]:

a $\sin 3x = \frac{1}{2}$ **b** $\cos(2x - 60°) = \frac{1}{\sqrt{2}}$

Solution

a For the domain [0°, 360°]

$0° \le x \le 360°$

$0° \le 3x \le 1080°$

So the new domain is [0°, 1080°] (3 revolutions of the circle)

$\sin 3x = \frac{1}{2}$ sin $x > 0$ in 1st and 2nd quadrants

$3x = 30°, 180° - 30°, 360° + 30°, 360° + (180° - 30°), 720° + 30°, 720° + (180° - 30°)$

$= 30°, 150°, 390°, 510°, 750°, 870°$

$x = 10°, 50°, 130°, 170°, 250°, 290°$

b $0° \le x \le 360°$

$0° \le 2x \le 720°$

$-60° \le 2x - 60° \le 660°$

So the new domain is [–60°, 660°] (2 revolutions of the circle starting at –60°)

$\cos(2x - 60°) = \frac{1}{\sqrt{2}}$ cos $x > 0$ in 1st and 4th quadrants

$2x - 60° = -45°, 45°, 360° - 45°, 360° + 45°$

(720° – 45° is outside the domain)

$= -45°, 45°, 315°, 405°$

$2x = 15°, 105°, 375°, 465°$

$x = 7.5°, 52.5°, 187.5°, 232.5°$

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EXAMPLE 12

Solve each equation for $[0, 2\pi]$.

a $6 \cos 2x - 3 = 0$

b $\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}$

Solution

a For the domain $[0, 2\pi]$

$0 \le x \le 2\pi$

$0 \le 2x \le 4\pi$, so when solving for $2x$ we need to go around the circle twice.

$$6 \cos 2x - 3 = 0$$

$$6 \cos 2x = 3$$

$$\cos 2x = \frac{1}{2}$$

cos $x > 0$ in 1st and 4th quadrants

$$2x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 2\pi + 2\pi - \frac{\pi}{3} \text{ as } 0 \le 2x \le 4\pi$$

$$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

b $0 \le x \le 2\pi$

$$-\frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{7\pi}{4}$$

So the new domain is $\left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$ $\left(\text{1 revolution of the circle starting at } -\frac{\pi}{4}\right)$

$$\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}$$

tan $x > 0$ in 1st and 3rd quadrants

$$x - \frac{\pi}{4} = \frac{\pi}{3}, \pi + \frac{\pi}{3}$$

$$= \frac{\pi}{3}, \frac{4\pi}{3}$$

$$x = \frac{\pi}{3} + \frac{\pi}{4}, \frac{4\pi}{3} + \frac{\pi}{4}$$

$$= \frac{7\pi}{12}, \frac{19\pi}{12}$$

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Exercise 4.03 Trigonometric equations

1 By drawing the graph of $y = 2 \sin 3x$ in the domain $[0, 2\pi]$:

a find the number of solutions of $2 \sin 3x = 1$

b solve $2 \sin 3x = 1$ graphically

2 a Sketch the graphs of $y = -\cos x + 3$ and $y = x - 1$ for $[0, 2\pi]$.

b Solve:

i $-\cos x + 3 = x - 1$ **ii** $-\cos x + 3 = 2$

3 Solve for $[0°, 360°]$:

a $2 \sin 2x = 1$ **b** $\tan 3x = -1$ **c** $\cos (x + 90°) = \dfrac{\sqrt{3}}{2}$

d $\tan (x - 45°) = \sqrt{3}$ **e** $\sin (x + 60°) = 0$

4 Solve for $[0, 2\pi]$:

a $\tan 2x = \sqrt{3}$ **b** $2 \cos 3x + 1 = 0$ **c** $4 \sin^2 \left(x - \dfrac{\pi}{3} \right) = 3$

d $2 \cos^2 2x - 1 = 0$ **e** $\cos (x + \pi) = 1$

5 Solve for $[-\pi, \pi]$:

a $\tan 3x = 1$ **b** $\cos (x + \dfrac{\pi}{4}) = \dfrac{1}{\sqrt{2}}$ **c** $\sin 2x = -1$

d $\cos (x - \dfrac{\pi}{2}) = 0$ **e** $\tan^2 4x = 0$

6 Solve for $[0, 2\pi]$:

a $\cos 2\left(x - \dfrac{\pi}{2} \right) = \dfrac{1}{2}$ **b** $2 \sin\left(3x + \dfrac{3\pi}{2} \right) = 1$

7 The function $T = 15 \cos \dfrac{\pi t}{6} + 20$ models the average monthly temperatures in Nelson Springs, starting in January.

a Find the amplitude, period and centre of the function.

b Solve $15 \cos \dfrac{\pi t}{6} + 20 = 35$ and explain the meaning of the solutions.

8 A set of tidal waves has a maximum height of 20 m and a minimum height of 6 m. The waves break every 10 seconds.

a Find the equation of a sine function that describes the motion of the waves.

b Find the first 4 times that the waves reach their maximum height.

c Find the first time that the waves reach their minimum height.

d When will the height of the waves be in the centre?

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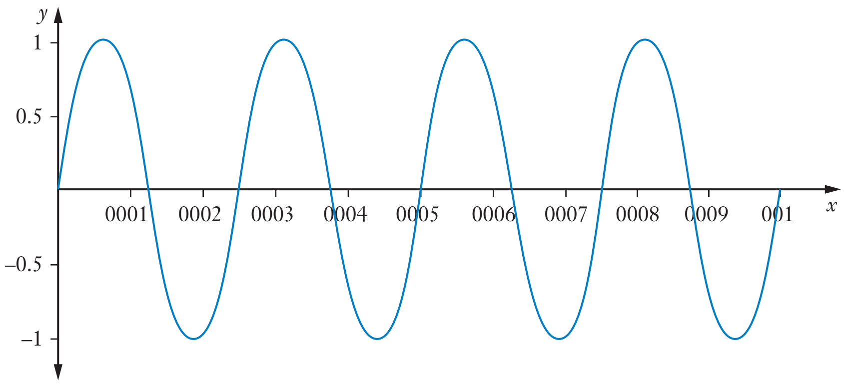
9 Sound waves have the shape of sine functions. The graph below shows the sound wave that occurs when playing the note A above middle C on a piano. Its equation is $y = \sin(880\pi x)$ where x is time in seconds.



a Find the amplitude and period of this sound wave.

b Use the graph to solve for [0, 0.01]:

i $\sin(880\pi x) = 0.5$ **ii** $\sin(880\pi x) = 0$

c Solve algebraically for [0, 0.01] (to 2 significant figures):

i $\sin(880\pi x) = 0.5$ **ii** $\sin(880\pi x) = 0$

d The higher the amplitude, the more volume the sound has (it is louder). (The word 'amplifier' comes from this property.) Find the equation of the note A that is 3 times as loud as the one drawn.

e A note that is higher in pitch has a higher frequency (more cycles) than a lower note. Draw a rough sketch of a middle C note with the same volume as the A note above C.

f The unit of measurement for frequency is hertz (Hz), the number of wave cycles of a sound in 1 second What is the frequency in hertz of note A ?

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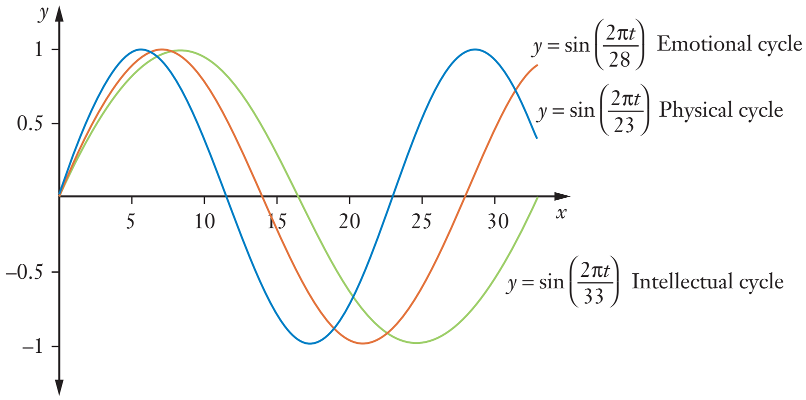
10 Biorhythms is a theory that emotional physical and mental activity in humans can be modelled by 3 sine functions physical $y = \sin \frac{2\pi t}{23}$, emotional $y = \sin \frac{2\pi t}{28}$ and intellectual $y = \sin \frac{2\pi t}{33}$ where t is time in days starting from your date of birt. It was first developed by German doctor Wilhelm Fliess in 1878 but became popular in the 1970s when computers were able to chart the 3 biorhythms Their graphs are sketched belw.



- **a** What is the period of each function? What does this mean?
- **b** When do the physical and intellectual graphs intersect?
- **c** When do the emotional and physical graphs intersect?
- **d** Biorhythms are supposed to be at optimal levels when $y = 1$ (maximum points). Estimate the range of times when all 3 biorhythms are near optimal levels together.

11 A vertical spring is pulled down and then let go. It bounces back up and down again according to the equation $h = 12 \cos t + 15$ where h is the height of the spring in cm and t is time in seconds.

- **a** Describe the significance of the 15 in the equation.
- **b** What are the maximum and minimum heights of the spring?
- **c** What is the height of the spring after π seconds?
- **d** At what times will the spring be at its minimum height?

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WS Trigonometric identities

EXT1 4.04 Trigonometric identities

You learned the following trigonometric identities in the Year 11 course, Chapter 11, *Trigonometric functions*.

WS Trigonometric identities

The sum and difference identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

The double angle identities

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Products to sums and differences

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

The *t*-formulas

If $t = \tan\frac{A}{2}$ then:

$$\sin A = \frac{2t}{1 + t^2} \qquad \cos A = \frac{1 - t^2}{1 + t^2} \qquad \tan A = \frac{2t}{1 - t^2}$$

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EXAMPLE 13

a Write $\sin 2x \sin 3y$ as a sum or difference of ratios.

b Prove that $\sin 3x = 3 \sin x \cos^2 x - \sin^3 x$.

Solution

a $\sin 2x \sin 3y = \frac{1}{2}\left[\cos(2x - 3y) - \cos(2x + 3y)\right]$

b
$$\begin{aligned}\sin 3x &= \sin(2x + x)\\ &= \sin 2x \cos x + \cos 2x \sin x\\ &= (2 \sin x \cos x)\cos x + (\cos^2 x - \sin^2 x)\sin x\\ &= 2\sin x\cos^2 x + \sin x \cos^2 x - \sin^3 x\\ &= 3 \sin x \cos^2 x - \sin^3 x\end{aligned}$$

Auxiliary angle results

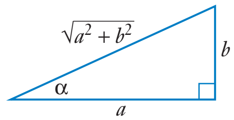
A trigonometric expression involving the sum or difference of sin and cos, such as $3 \sin x - 2 \cos x$, can be simplified to an expression with only one trigonometric ratio, such as $r \sin(x - \alpha)$, where r and α are constants and α is called an **auxiliary angle**.

a sin *x* + *b* cos *x*

$a \sin x + b \cos x = r \sin(x + \alpha)$ where $r = \sqrt{a^2 + b^2}$ and $\tan \alpha = \frac{b}{a}$.

Proof

Let $\tan \alpha = \frac{b}{a}$ and find expressions for $\sin \alpha$ and $\cos \alpha$.



Form a right-angled triangle:

$r = \sqrt{a^2 + b^2}$ (by Pythagoras' theorem), $\therefore \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$, $\cos \alpha = \frac{a}{\sqrt{a^2+b^2}}$

$$\begin{aligned}\text{RHS} &= r \sin(x + \alpha)\\ &= r(\sin x \cos \alpha + \cos x \sin \alpha)\\ &= \sqrt{a^2+b^2}\left(\sin x \times \frac{a}{\sqrt{a^2+b^2}} + \cos x \times \frac{b}{\sqrt{a^2+b^2}}\right)\\ &= a \sin x + b \cos x\\ &= \text{LHS}\end{aligned}$$

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The following auxiliary angle results can be proved in a similar way.

Auxiliary angle results

Given $r = \sqrt{a^2 + b^2}$ and $\tan \alpha = \dfrac{b}{a}$:

- $a \sin x - b \cos x = r \sin (x - \alpha)$
- $a \cos x + b \sin x = r \cos (x - \alpha)$
- $a \cos x - b \sin x = r \cos (x + \alpha)$

EXAMPLE 14

a Write $\sqrt{3} \sin x + \cos x$ in the form $r \sin (x + \alpha)$.

b Write $3 \sin x - 2 \cos x$ in the form $r \sin (x - \alpha)$.

Solution

a $a \sin x + b \cos x = r \sin (x + \alpha)$

For $\sqrt{3} \sin x + \cos x$, $a = \sqrt{3}$, $b = 1$

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{\sqrt{3}^2 + 1^2} \\ &= \sqrt{3 + 1} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \tan \alpha &= \frac{b}{a} \\ &= \frac{1}{\sqrt{3}} \\ \alpha &= 30^\circ \end{aligned}$$

So $\sqrt{3} \sin x + \cos x = 2 \sin (x + 30^\circ)$.

b $a \sin x - b \cos x = r \sin (x - \alpha)$

For $3 \sin x - 2 \cos x$, $a = 3$, $b = 2$

$$\begin{aligned} r &= \sqrt{a^2 + b^2} \\ &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} \tan \alpha &= \frac{b}{a} \\ &= \frac{2}{3} \\ \alpha &\approx 33^\circ 41' \end{aligned}$$

So $3 \sin x - 2 \cos x = \sqrt{13} \sin (x - 33^\circ 41')$

We can also use radians for these identities.

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EXAMPLE 15

a Write $\cos x + \sin x$ in the form $r \cos (x - \alpha)$ where α is in radians.

b Sketch the graph of $y = \cos x + \sin x$ in the domain $[0, 2\pi]$.

Solution

a $a \cos x + b \sin x = r \cos (x - \alpha)$

For $\cos x + \sin x$, $a = 1$, $b = 1$

$$r = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\tan \alpha = \frac{b}{a} = \frac{1}{1} = 1$$

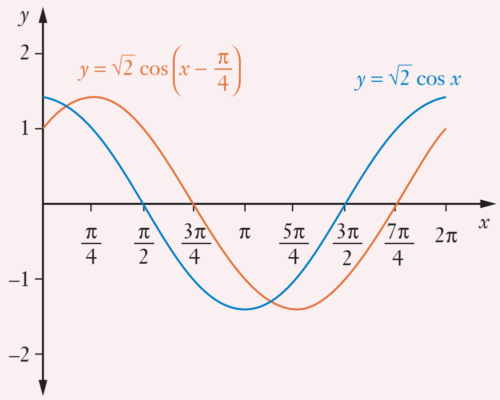
$$\alpha = \frac{\pi}{4}$$

So $\cos x + \sin x = \sqrt{2}\cos (x - \frac{\pi}{4})$

b $y = \cos x + \sin x = \sqrt{2} \cos (x - \frac{\pi}{4})$

This is a transformation of $y = \cos x$ with amplitude $\sqrt{2}$ and phase $\frac{\pi}{4}$ units to the right.

Draw $y = \sqrt{2} \cos x$ and shift the graph $\frac{\pi}{4}$ units to the right or find image points.



TECHNOLOGY

Sketching graphs of trigonometric functions

Sketching graphs of functions in the form $y = a \sin x + b \cos x$ can be difficult because they involve a phase shift and often a non-integer amplitude.

Explore how to sketch graphs of this type using a graphics calculator or graphing software.

If you forget the auxiliary angle formulas, you can refer back to the sum and difference identities instead.

EXAMPLE 16

Write $4 \cos x - 7 \sin x$ in the form $r \cos (x + \alpha)$ where α is in degrees.

Solution

$$r \cos (x + \alpha) = r (\cos x \cos \alpha - \sin x \sin \alpha)$$

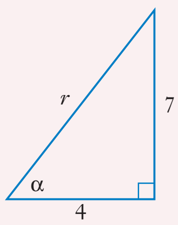
So $4 \cos x - 7 \sin x = r (\cos x \cos \alpha - \sin x \sin \alpha)$

$$= r \cos x \cos \alpha - r \sin x \sin \alpha$$

Comparing LHS with RHS, this means: $4 = r \cos \alpha$ and $7 = r \sin \alpha$

$$\cos \alpha = \frac{4}{r} \quad \text{and} \quad \sin \alpha = \frac{7}{r}$$

Drawing a triangle with this information:



$$r^2 = 4^2 + 7^2$$
$$= 65$$
$$r = \sqrt{65}$$

$$\tan \alpha = \frac{7}{4}$$
$$\alpha \approx 60° 15'$$

So $4 \cos x - 7 \sin x = \sqrt{65} \cos (x + 60° 15')$

EXT1 Exercise 4.04 Trigonometric identities

1 Expand:

a $\cos (a - b)$ **b** $\tan (x + y)$ **c** $\sin (3p + 2q)$

d $\sin (x + y) - \sin (x - y)$ **e** $\cos 2A$

2 Simplify:

a $\cos a \cos b - \sin a \sin b$ **b** $\dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

c $2 \sin x \cos x$ **d** $\sin 5a \cos 3b - \cos 5a \sin 3b$

e $\frac{1}{2}[\cos (A + B) + \cos (A - B)]$

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3 Given $t = \tan \frac{A}{2}$:

a simplify:

i $\frac{1-t^2}{1+t^2}$ **ii** $\frac{6t}{1-t^2}$

b write $\sin 2A$ in terms of t

c prove by writing each expression in terms of t that $\tan A = \frac{\sin A}{\cos A}$

4 Find the exact value of:

a $\sin 75°$ **b** $\tan 15°$

5 By writing $3x$ as $2x + x$, find an expression in terms of $\sin x$ for $\sin 3x$.

6 Prove:

a $\cos 3x = 4\cos^3 x - 3\cos x$ **b** $\tan 2x = \frac{2\tan x}{2 - \sec^2 x}$

7 Write each expression as a sum or difference of angles.

a $\sin 3a \sin 2b$ **b** $\cos 5y \sin 3z$

c $\cos 2p \cos 3q$ **d** $\sin 4x \cos 9y$

8 Write each expression in the form $r \sin (x + \alpha)$ using degrees.

a $2\sin x + \cos x$ **b** $\sin x + \sqrt{3}\cos x$ **c** $\sin x + \cos x$

d $5\sin x + 2\cos x$ **e** $4\sin x + 7\cos x$

9 Write each expression in the form $r \sin (x - \alpha)$ using radians.

a $\sin x - \cos x$ **b** $\sin x - 2\cos x$ **c** $\sin x - \sqrt{3}\cos x$

d $\sqrt{3}\sin x - \cos x$ **e** $5\sin x - 2\cos x$

10 Write each expression in the form $r \cos (x - \alpha)$ using radians.

a $2\cos x + 3\sin x$ **b** $7\cos x + 2\sin x$ **c** $\cos x + \sqrt{3}\sin x$

d $\sqrt{3}\cos x + \sin x$ **e** $3\cos x + 2\sin x$

11 Write each expression in the form $r \cos (x + \alpha)$ using degrees.

a $2\cos x - 3\sin x$ **b** $7\cos x - 2\sin x$ **c** $\cos x - \sqrt{3}\sin x$

d $\sqrt{3}\cos x - \sin x$ **e** $3\cos x - 2\sin x$

12 Write the expression $9\sin x + 7\cos x$ (using degrees) in the form:

a $r \sin (x + \alpha)$ **b** $r \cos (x - \alpha)$

13 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $f(x) = \sin x + \sqrt{3}\cos x$ **b** $y = \cos x - \sin x$ **c** $y = 2\sin x - 3\cos x$

d $f(x) = 4\cos x + 2\sin x$ **e** $f(x) = \sqrt{3}\sin x - \cos x$

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Further trigonometric equations

Trigonometric equations 2

EXT1 4.05 Further trigonometric equations

EXAMPLE 17

Solve $\cos\left(x+\frac{\pi}{3}\right)=\cos\left(x-\frac{\pi}{3}\right)+1$ for $[0, 2\pi]$.

Solution

$$\cos\left(x+\frac{\pi}{3}\right)=\cos\left(x-\frac{\pi}{3}\right)+1$$

$$\cos x\cos\frac{\pi}{3}-\sin x\sin\frac{\pi}{3}=\cos x\cos\frac{\pi}{3}+\sin x\sin\frac{\pi}{3}+1$$

$$-2\sin x\sin\frac{\pi}{3}=1$$

$$\sin x\sin\frac{\pi}{3}=-\frac{1}{2}$$

$$\sin x\times\frac{\sqrt{3}}{2}=-\frac{1}{2}$$

$$\sin x=-\frac{1}{\sqrt{3}}$$

sin $x < 0$ in 3rd and 4th quadrants

$$x\approx\pi+0.61548,\ 2\pi-0.61548$$

$$\approx 3.76,\ 5.67$$

When a trigonometric equation involves products of trigonometric ratios, you can use different methods to solve them.

EXAMPLE 18

Solve $\sin x\cos x=\frac{1}{2}$ for $[0, 2\pi]$.

Solution

Since $\sin 2x = 2\sin x\cos x$

$$\frac{1}{2}\sin 2x=\sin x\cos x$$

Alternatively, you could use the product $\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$.

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Now solve the equation:

$$\sin x \cos x = \frac{1}{2}$$

$$\frac{1}{2}\sin 2x = \frac{1}{2}$$

$$\sin 2x = 1$$

For the domain $0 \le x \le 2\pi$

$0 \le 2x \le 4\pi$, going around the circle twice

$$2x = \frac{\pi}{2}, 2\pi + \frac{\pi}{2}$$

$$= \frac{\pi}{2}, \frac{5\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

We can solve quadratic equations that involve trigonometric ratios.

EXAMPLE 19

a Solve $\cos^2 x + \sin x = -1$ for $[0, 2\pi]$.

b Solve $\sin^2\left[2\left(x + \frac{\pi}{2}\right)\right] - \sin\left[2\left(x + \frac{\pi}{2}\right)\right] = 0$ for $[-\pi, \pi]$.

Solution

a $\cos^2 x + \sin x = -1$

$1 - \sin^2 x + \sin x = -1$ (Substituting $1 - \sin^2 x$ for $\cos^2 x$)

$$0 = \sin^2 x - \sin x - 2$$

$$= (\sin x - 2)(\sin x + 1)$$

$\sin x - 2 = 0$ $\quad$ $\sin x + 1 = 0$

$\sin x = 2$ $\quad$ $\sin x = -1$

No solutions $\quad$ $x = \frac{3\pi}{2}$

So the solution is $x = \frac{3\pi}{2}$.

b For the domain $[-\pi, \pi]$

$$-\pi \le x \le \pi$$

$$-\frac{\pi}{2} \le x + \frac{\pi}{2} \le \frac{3\pi}{2}$$

$$-\pi \le 2\left(x + \frac{\pi}{2}\right) \le 3\pi$$

So the domain is $[-\pi, 3\pi]$

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We can factorise $\sin^2\left[2\left(x+\frac{\pi}{2}\right)\right]-\sin\left[2\left(x+\frac{\pi}{2}\right)\right]=0$ as a quadratic.

Let $u=\sin\left[2\left(x+\frac{\pi}{2}\right)\right]$.

$$u^2-u=0$$
$$u(u-1)=0$$
$$u=0,\qquad u=1$$

Now, replace u with $\sin\left[2\left(x+\frac{\pi}{2}\right)\right]$.

$$\sin\left[2\left(x+\frac{\pi}{2}\right)\right]=0$$
$$2\left(x+\frac{\pi}{2}\right)=-\pi, 0, \pi, 2\pi, 3\pi$$
$$x+\frac{\pi}{2}=-\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$
$$x=-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$$

$$\sin\left[2\left(x+\frac{\pi}{2}\right)\right]=1$$
$$2\left(x+\frac{\pi}{2}\right)=\frac{\pi}{2}, 2\pi+\frac{\pi}{2}$$
$$=\frac{\pi}{2}, \frac{5\pi}{2}$$
$$x+\frac{\pi}{2}=\frac{\pi}{4}, \frac{5\pi}{4}$$
$$x=-\frac{\pi}{4}, \frac{3\pi}{4}$$

So the solutions are $x=-\pi, -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$.

We can use auxiliary angles or t formulas to solve some equations.

EXAMPLE 20

a Find the exact solutions of $\sin x-\cos x=1$ in the domain $[0, 2\pi]$ by using:

i auxiliary angles **ii** t-formulas

b Solve $\sqrt{3}\cos x-\sin x=1$ for $[0, 2\pi]$ by using auxiliary angles.

Auxiliary angle method

Solution

a **i** $\sin x-\cos x$ is in the form $r\sin(x-\alpha)$.

$$a=1, b=1$$
$$r=\sqrt{1^2+1^2}$$
$$=\sqrt{2}$$

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$$\tan\alpha = \frac{1}{1}$$
$$= 1$$
$$\alpha = \frac{\pi}{4}$$

So $\sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

For $[0, 2\pi]$

$$0 \le x \le 2\pi$$
$$-\frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{7\pi}{4}$$

$$\sin x - \cos x = 1$$
$$\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) = 1$$
$$\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

sin $x > 0$ in 1st, 2nd quadrants

$$x - \frac{\pi}{4} = \frac{\pi}{4}, \pi - \frac{\pi}{4}$$
$$= \frac{\pi}{4}, \frac{3\pi}{4}$$
$$x = \frac{\pi}{2}, \pi$$

ii $\sin x - \cos x = 1$

$$\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} = 1 \qquad \text{where } t = \tan\frac{x}{2}$$
$$\frac{2t-(1-t^2)}{1+t^2} = 1$$
$$\frac{2t-1+t^2}{1+t^2} = 1$$
$$2t - 1 + t^2 = 1 + t^2$$
$$2t = 2$$
$$t = 1$$

So $\tan\frac{x}{2} = 1$

For $[0, 2\pi]$

$0 \le x \le 2\pi$

$0 \le \frac{x}{2} \le \pi$, half around the circle only

$$\frac{x}{2} = \frac{\pi}{4}$$
$$x = \frac{\pi}{2}$$

tan $x > 0$ in 1st, 3rd quadrants but domain is $[0, \pi]$

Note: When using t formulas for $\tan\frac{x}{2}$, because $\tan\frac{\pi}{2}$ is undefined, we need to test $x = \pi$ separately.

Substitute $x = \pi$ into $\sin x - \cos x = 1$

LHS $= \sin\pi - \cos\pi$
$= 0 - (-1)$
$= 1$
$=$ RHS, so $x = \pi$ is a solution also.

So solutions are $x = \frac{\pi}{2}, \pi$.

b $\sqrt{3}\cos x - \sin x$ is in the form $r\cos(x + \alpha)$.

$a = \sqrt{3}, b = 1$

$$r = \sqrt{\sqrt{3}^2 + 1^2}$$

$$= \sqrt{4}$$

$$= 2$$

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

So $\sqrt{3}\cos x - \sin x = 2\cos\left(x + \frac{\pi}{6}\right)$.

For $[0, 2\pi]$

$$0 \le x \le 2\pi$$

$$\frac{\pi}{6} \le x + \frac{\pi}{6} \le \frac{13\pi}{6}$$

$$\sqrt{3}\cos x - \sin x = 1$$

$$2\cos\left(x + \frac{\pi}{6}\right) = 1$$

$$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

cos x > 0 in 1st, 4th quadrants

$$x + \frac{\pi}{6} = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$$

$$= \frac{\pi}{3}, \frac{5\pi}{3}$$

$$x = \frac{\pi}{3} - \frac{\pi}{6}, \frac{5\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}, \frac{3\pi}{2}$$

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General solutions

While we often solve trigonometric equations in a particular domain, sometimes we want to find general solutions.

EXAMPLE 21

The displacement (x cm) of a pendulum is given by the equation $x = 5 \sin \frac{2\pi t}{5} + 3$ at t seconds.

a Find the initial displacement.

b Find the maximum and minimum displacement.

c Find the times when the displacement is 5.5 cm (correct to 2 decimal places).

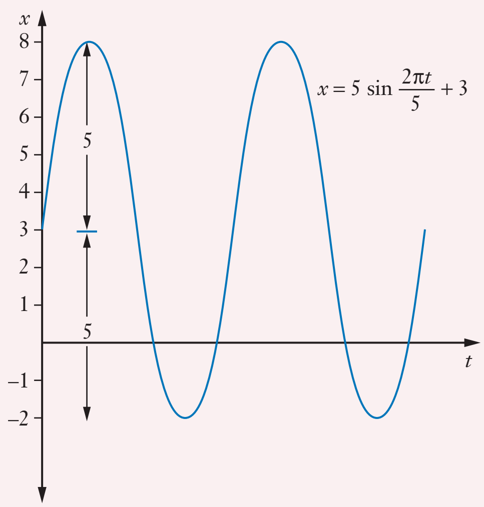
Solution

a Initially, $t = 0$:

$$x = 5 \sin \frac{2\pi(0)}{5} + 3$$
$$= 5 \sin 0 + 3$$
$$= 0 + 3$$
$$= 3$$

So initial displacement is 3 cm.

b



Amplitude is 5 and the centre of motion is 3.

Maximum is $3 + 5 = 8$ cm

Minimum is $3 - 5 = -2$ cm

c When $x = 5.5$

$$5\sin\frac{2\pi t}{5} + 3 = 5.5$$

$$5\sin\frac{2\pi t}{5} = 2.5$$

$$\sin\frac{2\pi t}{5} = \frac{1}{2}$$

$$\frac{2\pi t}{5} = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 2\pi + \pi - \frac{\pi}{6}, 4\pi + \frac{\pi}{6}, 4\pi + \pi - \frac{\pi}{6}, \ldots$$

$$= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$$

$$2\pi t = \frac{5\pi}{6}, \frac{25\pi}{6}, \frac{65\pi}{6}, \frac{85\pi}{6}, \frac{125\pi}{6}, \frac{145\pi}{6}, \ldots$$

$$t = \frac{5}{12}, \frac{25}{12}, \frac{65}{12}, \frac{85}{12}, \frac{125}{12}, \frac{145}{12}, \ldots$$

$= 0.42, 2.08, 5.42, 7.08, 10.42, 12.08, \ldots$ seconds

EXT1 Exercise 4.05 Further trigonometric equations

1 Solve for [0°, 360°]:

- **a** $\cos(x + 58°) + \cos(x - 58°) = 1$
- **b** $\sin(x - 60°) + \sin(x + 60°) = 0.5$
- **c** $\sin(x - 30°) - \sin(x + 30°) = 1$
- **d** $\cos x \sin x = \frac{1}{2}$
- **e** $\sin x \cos x = \frac{1}{2\sqrt{2}}$
- **f** $4\sin^2 x - 1 = 0$
- **g** $2\cos^2 x - \cos x - 1 = 0$
- **h** $\tan^2 x + \tan x = 0$
- **i** $\sin^2 x + \cos x - 1 = 0$
- **j** $6\cos^2 x - 7\cos x + 2 = 0$

2 Solve for [0, 2π], expressing the answer in terms of π where possible, otherwise correct to 2 decimal places.

- **a** $\sin\left(x - \frac{\pi}{4}\right) - \sin\left(x + \frac{\pi}{4}\right) = 0.25$
- **b** $\cos(x - 1.2) + \cos(x + 1.2) = 0.7$
- **c** $\cos\left(x - \frac{\pi}{6}\right) - \cos\left(x + \frac{\pi}{6}\right) = 1$
- **d** $\sin x \cos x = 0.15$
- **e** $\cos x \sin x = 0.23$
- **f** $2\tan^2 x + \tan x - 1 = 0$
- **g** $6\sin^2 x + \sin x - 1 = 0$
- **h** $2\sec^2 x + 3\tan x - 1 = 0$
- **i** $\cos^2 2x + \cos 2x = 0$

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3 Solve for $[0, 2\pi]$:

a $\sin x = \cos x$
b $\cos x = \sqrt{3}\sin x$
c $\sin 2x = \sin x$
d $\tan^2 x - \tan x = 0$
e $2\sin^2 x - \sin x - 1 = 0$
f $2\sin^2 x + 3\cos x - 3 = 0$
g $\sin x \cot x - \sin x = 0$
h $\cos^2 x - 1 = 0$
i $2\sin x \tan x - \tan x + 2\sin x - 1 = 0$
j $3\cos^2 x - 7\cos x + 4 = 0$

4 Solve for $[0°, 360°]$:

a $3\sin\theta + 4\cos\theta = 0$
b $5\cos\theta - 12\sin\theta = -3$
c $\sin\theta - \sqrt{3}\cos\theta = 0$
d $\sin\theta + \cos\theta = -1$
e $4\sin\theta - \cos\theta + 3 = 0$
f $\sin\theta - \cos\theta = 1$
g $\sqrt{2}\cos\theta + \sin\theta = 1$
h $2\sin\theta - \cos\theta = \frac{\sqrt{5}}{2}$
i $3\cos\theta - 5\sin\theta + 2 = 0$
j $\sqrt{2}\cos\theta + \sin\theta + 1 = 0$

5 Solve for $[0, 2\pi]$, expressing the answer in terms of π where possible, otherwise correct to 2 decimal places.

a $7\sin\theta + 3\cos\theta = 0$
b $6\cos\theta - 10\sin\theta = -1$
c $\sin\theta + \sqrt{3}\cos\theta = 1$
d $\sin\theta - \cos\theta + 1 = 0$
e $3\sin\theta + 4\cos\theta + 2 = 0$
f $\cos\theta - 3\sin\theta = 1$
g $\sqrt{2}\sin\theta - \cos\theta = 1$
h $4\sin\theta - 5\cos\theta = \frac{1}{2}$
i $3\sin\theta + 5\cos\theta + 1 = 0$
j $\sin\theta - \cos\theta = \sqrt{\frac{3}{2}}$

6 Solve for $[-180°, 180°]$:

a $\sin(2x - 45°) = \frac{\sqrt{3}}{2}$
b $\sin^2 x = \sin x$

7 Solve for $[0, 2\pi]$:

a $\cos 2x = \cos x$
b $\tan^2 x - 1 = 0$
c $\cos x + \sqrt{3}\sin x = 2$
d $\cos^2 x - \sin x + 1 = 0$
e $\sin 3x + \sin x = 0$

8 **a** Draw the graph of $y = \cos x - \sqrt{3}\sin x$ in the domain $[0, 2\pi]$.
b Find the amplitude, period and phase of the function.
c From the graph, solve $\cos x - \sqrt{3}\sin x = 1$.
d Solve $\cos x - \sqrt{3}\sin x = 1$ algebraically.

9 A spring moves so that its displacement is given by $x = 7\cos\frac{\pi t}{3} + 4$ cm over time t seconds.

a Find the maximum displacement and the times at which this occurs.
b Find the minimum displacement and the times at which this occurs.
c When is the spring at the centre of its motion?
d Solve $7\cos\frac{\pi t}{3} + 4 = 7.5$.

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10 The Pacific Ocean and the Tasman Sea meet at Cape Reinga in New Zealand. The average height of the waves in the Pacific Ocean is measured as $f(t) = 3 \sin t$ on a particular day and the average height of waves in the Tasman Sea is $g(t) = 4 \cos t$ on the same day where height is in metres and t is time in minutes.

a Graph the function $y = f(t) + g(t)$.

b From the graph, find the maximum total height of the waves and the first time this occurs.

11 **a** Sketch the graph of $y = \tan x$.

b Find the general solutions of $\tan x = 0$.

12 Find the general solutions of each equation.

a $\sin x = 1$

b $\cos x = \frac{\sqrt{3}}{2}$

c $\tan x = 1$

d $\cos x = -1$

e $\sin x = 0$

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4. TEST YOURSELF

Formula sheet: Functions, Trigonometry and Vectors

Practice quiz

For Questions **1** to **4**, choose the correct answer **A**, **B**, **C** or **D**.

1 The function $y = 2\cos 3x - 7$ has:

A amplitude 2, period 3 and centre −7

B amplitude 7, period $\frac{1}{3}$ and centre 2

C amplitude 2, period $\frac{2\pi}{3}$ and centre 7

D amplitude 2, period $\frac{2\pi}{3}$ and centre −7

2 The equation of a function with phase π units to the left is:

A $y = \tan(x + \pi)$ **B** $y = \tan(\pi x)$ **C** $y = \tan x + \pi$ **D** $y = \tan(x - \pi)$

3 The solution of $\cos 2x = 1$ in the domain $[0, 2\pi]$ is:

A $x = 0, 2\pi$ **B** $x = 0, \pi, 2\pi$ **C** $x = \frac{\pi}{2}, \frac{3\pi}{2}$ **D** $x = \pi$

4 EXT1 $r\sin x\cos\alpha + r\cos x\sin\alpha$ is equal to:

A $r\sin(x + \alpha)$ **B** $r\cos(x + \alpha)$ **C** $r\sin(x - \alpha)$ **D** $r\cos(x - \alpha)$

5 EXT1 Expand:

a $\sin(a - b)$ **b** $\tan(3a - 5b)$

c $\cos(x - y) + \sin(x + y)$ **d** $\sin 2x$

6 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = 3\cos x$

b $y = \tan\frac{x}{2}$

c $y = \sin x - 2$

d $y = -\sin x$

e $y = 3\cos 2x - 1$

f EXT1 $y = \sqrt{3}\sin x - \cos x$

g EXT1 $y = \cos x + \sin x$

7 The function $h = 3\cos\left(\dfrac{2\pi t}{3}\right) + 10$ shows the water level in a lock in metres over time t hours.

a Find the maximum and minimum levels of water and when they occur.

b Solve $3\cos\left(\dfrac{2\pi t}{3}\right) + 10 = 11$ and explain what the solution means.

8 Simplify:

a $2\cot^2 x + 2$

b $\tan A \operatorname{cosec} A$

c $(\sec A + \tan A)(\sec A - \tan A)$

d $\sin(180° - x)$

9 Solve for $[0, 2\pi]$:

a $4\cos^2 x = 3$

b $2\sin 2x = 1$

c $\cos\left(x - \dfrac{\pi}{2}\right) = -1$

d $\tan^2\left(x + \dfrac{\pi}{6}\right) = 3$

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10 EXT1 Write in the form:

a $r \sin (\theta + \alpha)$

i $2 \sin \theta + 5 \cos \theta$ in degrees **ii** $\sqrt{3} \sin \theta + \cos \theta$ in radians

b $r \sin (\theta - \alpha)$

i $\sin \theta - 2 \cos \theta$ in degrees **ii** $\sin \theta - \cos \theta$ in radians

c $r \cos (\theta + \alpha)$

i $\cos \theta - \sin \theta$ in degrees **ii** $\cos \theta - \sqrt{3} \sin \theta$ in radians

d $r \cos (\theta - \alpha)$

i $2 \cos \theta + 7 \sin \theta$ in degrees **ii** $5 \cos \theta + \sin \theta$ in radians

11 A person's blood pressure has a maximum pressure of 135 and a minimum pressure of 85 and the heartbeat is 70 beats per minute.

a Write an equation showing this blood pressure y as a sine function of time t minutes.

b Draw a graph showing this function.

12 Solve for $[-\pi, \pi]$:

a $2 \cos 2x = 1$ **b** $\tan\left(x - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{3}}$ **c** $\sin\left(x + \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}$

13 EXT1 Find the equation and sketch the graph of the transformed function if $y = \cos^{-1} x$ is translated π units up and horizontally dilated with a scale factor of 2.

14 The high tide mark on a cliff above a river is 80 m and the low tide mark is 50 m. The average time between high tides in the river is approximately 13 hours.

a Write the equation for the height of the tides as a sine function.

b Find the times when the river is halfway between high and low tides.

15 EXT1 Write $\cos 3x \sin 4y$ as a sum or difference of trigonometric ratios.

16 Find the equation of each transformed function of $y = \cos x$ that has:

a period π **b** amplitude 5 **c** a reflection in the x-axis

d a phase of $\frac{\pi}{6}$ units to the right **e** centre 4

17 EXT1 Solve for $[0°, 360°]$:

a $\sin (x + 70°) - \sin (x - 70°) = 0.6$ **b** $\sin x \cos x = \frac{\sqrt{3}}{4}$

c $\tan^2 x = \tan x$ **d** $2 \cos^2 x + \cos x = 1$

e $2 \cos x \tan x = 1$ **f** $2 \sin^2 x + 3 \cos x - 3 = 0$

g $\sin 2x = \sin x$ **h** $\cos x - \sin x = -1$

i $\sqrt{3} \sin x - \cos x = 1$ **j** $4 \sin x + 3 \cos x = 2$

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18 **a** Sketch the graph of $y = 2\sin\frac{x}{2} - 1$ for $[0, 2\pi]$.

b From the graph, solve $2\sin\frac{x}{2} - 1 = 0$.

c Solve $2\sin\frac{x}{2} - 1 = 0$ algebraically.

19 EXT1 Prove that:

a $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

b $\sin A\cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$

20 State the amplitude, period, centre and phase of each function.

a $y = 2\sin 3x - 1$ **b** $y = \cos\left(\frac{x}{2} + \pi\right)$ **c** $y = -3\tan\left(5x - \frac{\pi}{4}\right)$

21 Solve for $[0°, 360°]$:

a $\tan(x + 45°) = 1$ **b** $\sqrt{2}\cos(x - 20°) + 1 = 0$ **c** $3\sin[2(x + 10°)] - 2 = 0$

EXT1 **22** **a** Sketch the graph of the function $f(x) = 2\sin x + 3\cos x$ for $[0, 2\pi]$.

b Solve $2\sin x + 3\cos x = 2$:

i graphically **ii** algebraically

23 Solve for $[-180°, 180°]$:

a $5\tan 2x = -5$ **b** $\cos[3(x - 30°)] + 1 = 0$

24 EXT1 Solve for $[0, 2\pi]$:

a $\sin^3 x = \sin x$ **b** $\cos x = \cos 2x$

c $\sin x\cos x = \frac{1}{2}$ **d** $\cos(x + \pi) + \cos(x - \pi) = \frac{1}{2}$

e $\tan^2 x + \tan x - 2 = 0$ **f** $2\sin x + 4\cos x = 3$

g $\cos x - \sqrt{3}\sin x = 1$ **h** $6\sin x - 8\cos x = 5$

i $\sin\left(x - \frac{\pi}{3}\right) - \sin\left(x + \frac{\pi}{3}\right) - 1 = 0$ **j** $15\sin^2 x - \sin x - 6 = 0$

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4. CHALLENGE EXERCISE

1 a Find the amplitude, period and phase of the function $y = 2\cos\left(2x - \frac{\pi}{2}\right)$.

b Solve $2\cos\left(2x - \frac{\pi}{2}\right) = \sqrt{3}$ for $[0, 2\pi]$.

2 Find the equation of:

a a cosine function with amplitude 8, period 2π and centre 4

b a sine function with amplitude 2, period $\frac{\pi}{4}$, phase $\frac{\pi}{3}$ units to the right and centre 3

c a tangent function with period 2π and phase $\frac{\pi}{2}$ units to the left

3 Sketch the graph of $y = 3\sec 2x$ for $[0, 2\pi]$.

4 Solve $\operatorname{cosec} 2x = \sqrt{2}$ for $[0, 2\pi]$.

5 a Find the equation of the transformation of $y = \cos x$ if it has a vertical translation 5 units down, a horizontal translation $\frac{\pi}{6}$ units to the right, then a vertical stretch, scale factor 4 and a horizontal stretch, scale factor 3.

b Find the amplitude, period, centre and phase of this transformed function.

6 EXT1 Show that $a\sin x + b\cos x = r\sin(\theta + \alpha)$ where $r = \sqrt{a^2 + b^2}$ and $\tan\alpha = \frac{b}{a}$.

7 EXT1 Solve for $[0, 2\pi]$ $\tan\left(x + \frac{\pi}{4}\right) - \tan\left(x - \frac{\pi}{4}\right) = 3$.

8 EXT1 Solve $2\sin^2 2x + \cos 2x - 1 = 0$ for $[0, 2\pi]$.

Practice set 1

In Questions **1** to **8**, select the correct answer **A**, **B**, **C** or **D**.

1 For what values of r does the limiting sum of a geometric series exist?

A $|r| > 1$ **B** $|r| < 1$ **C** $|r| \ge 1$ **D** $|r| \le 1$

2 The transformation of $y = f(x)$ to $y = 3f(2x)$ is:

A Vertical dilation scale factor 3, horizontal dilation scale factor 2

B Horizontal dilation scale factor 3, vertical dilation scale factor 2

C Vertical dilation scale factor 3, horizontal dilation scale factor $\frac{1}{2}$

D Horizontal dilation scale factor 3, vertical dilation scale factor $\frac{1}{2}$

3 Simplify $\dfrac{\sin\theta}{\cos^2\theta\sec\theta}$.

A $\cot\theta$ **B** $\tan^2\theta$ **C** $\tan\theta$ **D** $\cot^2\theta$

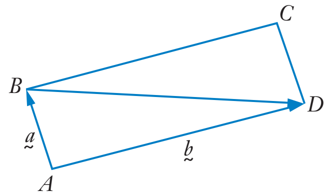
4 EXT1 In the diagram, vector $\overrightarrow{BD}$ represents:

A $\underset{\sim}{a} + \underset{\sim}{b}$

B $\underset{\sim}{a} - \underset{\sim}{b}$

C $\underset{\sim}{b} + \underset{\sim}{a}$

D $\underset{\sim}{b} - \underset{\sim}{a}$



5 The nth term of the sequence 7, 49, 343, … is:

A $7n$ **B** 7^{n-1} **C** $7n - 1$ **D** 7^n

6 $y = \cos(x + \pi) + 3$ has a phase shift of:

A π units to the right **B** 3 units up

C 3 units down **D** π units to the left

7 Find the limiting sum of $\frac{3}{5} + \frac{2}{5} + \frac{4}{15} + \ldots$

A $\frac{1}{5}$ **B** $\frac{9}{10}$ **C** $1\frac{4}{5}$ **D** $\frac{6}{15}$

8 The formula for the sum $1 + 1.03 + 1.03^2 + \ldots + 1.03^{n-1}$ is:

A $S = \dfrac{1.03(1.03^{n-1} - 1)}{1.03 - 1}$ **B** $S = \dfrac{1.03(1.03^n - 1)}{1.03 - 1}$

C $S = \dfrac{1.03^{n-1} - 1}{1.03 - 1}$ **D** $S = \dfrac{1.03^n - 1}{1.03 - 1}$

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9 **a** Sketch the graphs of $y = x^2$ and $y = -(x + 2)^2$ on the same set of axes.
b Describe the transformations that changed $y = x^2$ into the transformed function.

10 Solve each equation for [0°, 360°].

a $\tan 2x + 1 = 0$ **b** $2 \cos 3x = 1$ **c** $2 \sin (x - 90°) = \sqrt{3}$
d $\tan (x - 180°) = \sqrt{3}$ **e** $2 \cos^2 (x + 45°) = 1$

11 Describe the amplitude, period, centre and phase shift of each function.

a $y = 4 \cos 5x$ **b** $y = -2 \sin\left(x - \frac{\pi}{6}\right) + 1$ **c** $y = \tan\left(\frac{x}{4} + 2\right)$

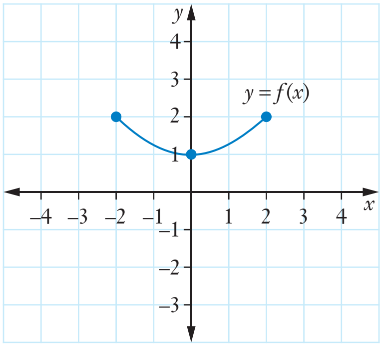
12 Find in index form the 10th term of $\frac{3}{4}, \frac{3}{16}, \frac{3}{64}, \ldots$

13 EXT1 Simplify each expression.

a $\cos a \cos b + \sin a \sin b$ **b** $1 - 2 \sin^2 x$
c $\sin x \cos x$ **d** $\sin (x + y) - \sin (x - y)$
e $\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ **f** $\frac{1 - t^2}{1 + t^2}$ where $t = \tan \frac{A}{2}$

14 Copy the graph $y = f(x)$ and sketch:

a $y = 2f(x)$
b $y = f(x) + 1$
c $y = f(x - 2)$
d $y = f(2x)$



15 EXT1 Given $\overrightarrow{OA} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, find the magnitude and direction (to the nearest minute) of:

a $\overrightarrow{OA} + \overrightarrow{OB}$ **b** $\overrightarrow{OB} - \overrightarrow{OA}$

16 The 2nd term of a geometric sequence is 52 and the 4th term is 13.
Find 2 sequences that satisfy these requirements.

17 Find which term −370 is in the series 17 + 8 − 1 − …

18 EXT1 Prove by mathematical induction that $3 + 6 + 12 + \ldots + 3(2^n) = 3(2^{n+1} - 1)$ for all integers $n \geq 0$.

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19 EXT1 Find the position vector equal to $\overrightarrow{AB}$ where $A = (-4, -5)$ and $B = (2, -8)$.

20 Describe the transformations on $y = x^3$ if the equation of the transformed function is $y = 4(x - 1)^3 - 3$, and state whether any dilations stretch or compress the graph of the function.

21 Solve each equation for $[0°, 360°]$.

a $6 \sin^2 x - 7 \sin x + 2 = 0$

b $2 \sin[2(x - 30)] = 1$

c EXT1 $\sin x - 5 \cos x = 2$

d EXT1 $\cos x - \sin x = 1$

22 A moving sculpture has a ball on the end of a wire that oscillates backwards and forwards between 2 points. The equation of the distance d cm of the ball from the centre of the sculpture at time t seconds is given by $d = 6 \cos (2\pi t) + 10$.

a Find the centre of motion and the maximum distance of the ball from this centre in both directions.

b How long does it take the ball to complete one complete cycle between the 2 points?

23 EXT1 Write $3 \sin \theta + 4 \cos \theta$ in the form $r \sin (\theta + \alpha)$ in degrees.

24 Find the equation of the function if $y = \sqrt{x}$ is dilated vertically with scale factor 4, dilated horizontally with scale factor $\frac{1}{3}$, translated vertically 1 unit down and translated horizontally 7 units to the left.

25 EXT1 If $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$, find the magnitude and direction of:

a $\underset{\sim}{a} + \underset{\sim}{b}$

b $\underset{\sim}{a} - \underset{\sim}{b}$

26 a Find the 50th term of 3, 7, 11, …

b Calculate the sum of 50 terms.

27 The nth term of a series is given by $7n - 3$.

a Find the first 3 terms and the 12th term.

b Evaluate the sum of the first 20 terms.

c Which term is equal to 200?

28 EXT1 Given $\underset{\sim}{v} = 4\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{w} = -3\underset{\sim}{i} + \underset{\sim}{j}$, find:

a $\underset{\sim}{v} + \underset{\sim}{w}$

b $\underset{\sim}{w} - \underset{\sim}{v}$

c $5\underset{\sim}{w}$

d $2\underset{\sim}{v} + 3\underset{\sim}{w}$

e $-3\underset{\sim}{v} - \underset{\sim}{w}$

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29 **a** Sketch the graph of $f(x) = 5\,|x - 2| - 3$.

b From the graph, solve each equation.

i $5\,|x - 2| - 3 = 2$ **ii** $5\,|x - 2| - 3 = 7$ **iii** $5\,|x - 2| - 3 = -3$

c State the domain and range of $f(x)$.

30 Find the exact value of:

a $\cos 120°$ **b** $\sin 300°$ **c** $\tan 225°$

d $\cos(-135°)$ **e** $\tan 690°$

31 EXT1 Prove by mathematical induction that $n(n - 1)$ is divisible by 2 for all integers $n > 1$.

32 EXT1 Vector $\overrightarrow{AB}$ has coordinates $A(-1, 5)$ and $B(6, -4)$.

a Write $\overrightarrow{AB}$ in terms of the components $\underset{\sim}{i}$ and $\underset{\sim}{j}$.

b Find the magnitude and direction of $\overrightarrow{AB}$.

33 The 4th term of an arithmetic sequence is 18 and the 8th term is 62.
Find the formula for the general term of the sequence.

34 Evaluate x if $\sec x = \operatorname{cosec}(2x - 30°)$.

35 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = -7\cos x$ **b** $y = 2\sin x$ **c** $y = \cos x + 1$

d $y = \tan\left(x + \frac{\pi}{2}\right)$ **e** $y = 3\cos 2x$ **f** $y = -4\sin\frac{x}{2} + 3$

36 Prove each identity.

a $\cot x \sec x = \operatorname{cosec} x$ **b** $\sin^2 x \operatorname{cosec}^2 x - \sin^2 x = \cos^2 x$

c EXT1 $\cos 2A = 2\cos^2 A - 1$

37 Find the equation of the transformed function of $y = \sin x$ if the function has:

a amplitude 2

b period 4π

c centre -3

d a reflection in the x-axis and amplitude 5

e a phase shift $\frac{\pi}{2}$ units to the left

f amplitude 5, period 6π, centre 1 and a phase shift of π units to the right

38 EXT1 Find the magnitude and direction of $\underset{\sim}{x} - \underset{\sim}{y}$ given that $\underset{\sim}{x} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{y} = -4\underset{\sim}{i} + 3\underset{\sim}{j}$.

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39 EXT1 Solve each equation for $[0, 2\pi]$.

a $\sin\left(x + \frac{\pi}{6}\right) - \sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$

b $2 \tan x \cos x = \sqrt{3}$

c $4 \sin x + 3 \cos x = 3$

d $\cos x - \sin x = 1$

40 EXT1 The resultant vector of $2\begin{pmatrix} x \\ 3 \end{pmatrix} - 3\begin{pmatrix} -1 \\ y \end{pmatrix}$ is $\begin{pmatrix} -3 \\ -6 \end{pmatrix}$. Evaluate x and y.

41 The geometric series $x + x^2 + x^3 + \ldots$ has a sum to infinity of 5. Find the value of x.

42 EXT1 A particle starts 4 units to the right of the origin. It moves 5 units to the left, then 3 units to the right and stops at that point.

a Draw vectors to show the motion of the particle.

b Find where the particle is at the end of the motion.

c Find the total distance it travels.

43 EXT1 Prove by mathematical induction that $1 + 3 + 9 + \ldots + 3^n = \frac{3^{n+1} - 1}{2}$ for all positive integers n.

44 Solve each equation for $[0, 2\pi]$.

a $\cos x = 0.62$

b $\tan^2 x = 1$

c $2 \sin x - 1 = 0$

d $\cos x = 0$

e $4 \sin^2 x = 3$

f $2 \cos 2x + 1 = 0$

45 Find the first value of n for which the sum of the sequence 20, 4, 0.8, … is greater than 24.85.

46 Evaluate $\frac{2}{25} + \frac{2}{125} + \frac{2}{625} + \ldots$

47 The average temperature T over t months is given by $T = 20 \cos \frac{\pi t}{6} + 18$ where January is $t = 0$.

a Find the amplitude, period and centre of the function. What do these features mean in terms of maximum and minimum temperatures and cycles?

b Find the month with an average temperature of −2°C.

c What is the month with the highest average temperature?

d What is the average temperature in September?

e When is the average temperature 18°C?

48 Solve graphically $(x - 1)^2 - 4 \leq 0$.

49 Show that $f(x) = 3x^2 - 2$ is an even function.

50 EXT1 Find the magnitude and direction of $\underset{\sim}{v} = \begin{pmatrix} -8 \\ -6 \end{pmatrix}$.

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51 **a** Show that log 3, log 9, log 27 … are terms of an arithmetic sequence.

b Find the exact sum of 20 terms of the sequence.

52 EXT1 Write each expression terms of t where $t = \tan \dfrac{A}{2}$.

a $\sin 2A$ **b** $\tan^2 A - 1$

53 **a** Sketch the graph of $y = 3(x-2)^2 - 4$.

b From the graph, solve each inequality.

i $3(x-2)^2 - 4 \geq 8$ **ii** $3(x-2)^2 - 4 < 8$

54 EXT1 Copy the diagram for $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then draw:

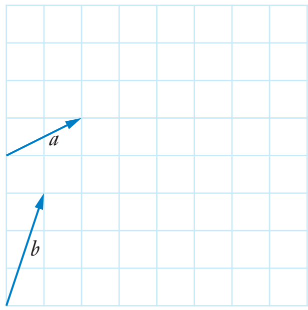
WS 5 mm grid paper

a $-\underset{\sim}{a}$

b $2\underset{\sim}{b}$

c $\underset{\sim}{a} + \underset{\sim}{b}$

d $\underset{\sim}{b} - \underset{\sim}{a}$



EXT1 **55** **a** Write $2 \sin A - 3 \cos A$ in terms of t where $t = \tan \dfrac{A}{2}$.

b Hence or otherwise, solve $2 \sin A - 3 \cos A = 1$ in radians.

56 EXT1 Find the exact magnitude and direction in degrees and minutes of each vector.

a $\underset{\sim}{v} = 2\underset{\sim}{i} + 6\underset{\sim}{j}$ **b** $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j}$ **c** $\underset{\sim}{v} = -4\underset{\sim}{i} - 5\underset{\sim}{j}$

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CALCULUS

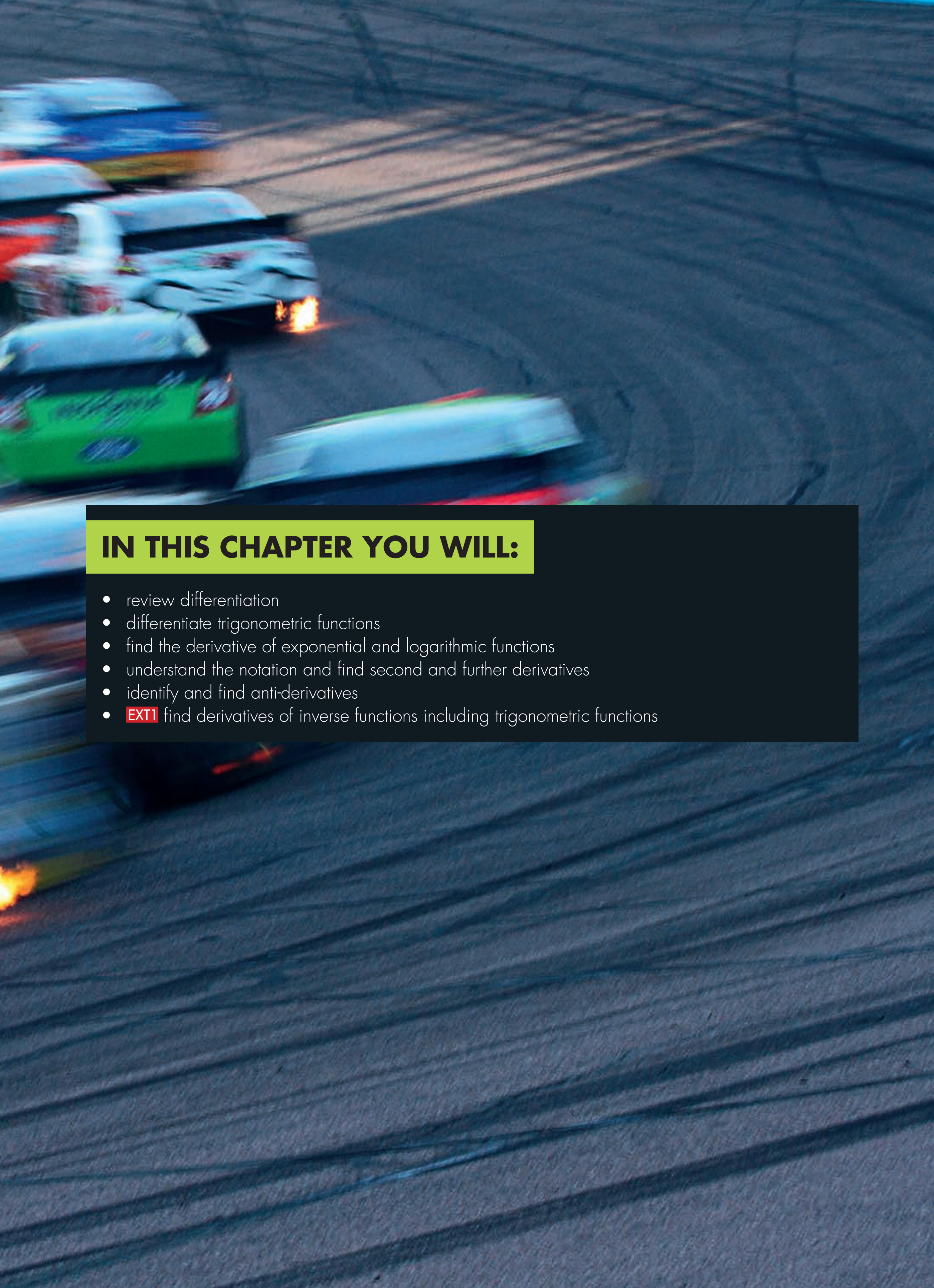
5.

FURTHER DIFFERENTIATION

In this chapter, you will review differentiation and learn how to differentiate trigonometric, exponential and logarithmic functions, and inverse functions, including inverse trigonometric functions. You will also look at higher derivatives and anti-derivatives.

CHAPTER OUTLINE

5.01 Differentiation review
5.02 Derivative of exponential functions
5.03 Derivative of logarithmic functions
5.04 Derivative of trigonometric functions
5.05 Second derivatives
5.06 Anti-derivative graphs
5.07 Anti-derivatives
5.08 Further anti-derivatives
5.09 EXT1 Derivative of inverse functions
5.10 EXT1 Derivative of inverse trigonometric functions



IN THIS CHAPTER YOU WILL:

- review differentiation
- differentiate trigonometric functions
- find the derivative of exponential and logarithmic functions
- understand the notation and find second and further derivatives
- identify and find anti-derivatives
- EXT1 find derivatives of inverse functions including trigonometric functions

TERMINOLOGY

anti-derivative: A function $F(x)$ whose derivative is $f(x)$, that is, $F'(x) = f(x)$. Also called the **primitive** or **integral** function.

anti-differentiation: The process of finding the original function given its derivative.

second derivative: The derivative $f''(x)$ or $\frac{d^2y}{dx^2}$; the derivative of the derivative $f'(x)$ or $\frac{dy}{dx}$.

5.01 Differentiation review

Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-}$$

Product rule

If $y = uv$, then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ or $y' = u'v + v'u$.

Quotient rule

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ or $y' = \frac{u'v - v'u}{v^2}$.

Rates of change

The **average rate of change** between 2 points (x_1, y_1) and (x_2, y_2) is the gradient:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The **instantaneous rate of change** at point (x, y) is the derivative $f'(x)$ or $\frac{dy}{dx}$.

EXAMPLE 1

Water is pumped into a dam according to the formula $Q = 3t^3 + 2t^2 + 270$ where Q is the amount of water in kL and t is time in hours. Find:

a the amount of water in the dam after 6 hours

b the average rate at which the water is pumped into the dam between 3 and 6 hours

c the rate of change after 6 hours

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Solution

a $Q = 3t^3 + 2t^2 + 270$

When $t = 6$

$Q = 3(6)^3 + 2(6)^2 + 270$

$= 990$

So there is 990 kL of water in the dam after 6 hours.

b When $t = 3$

$Q = 3(3)^3 + 2(3)^2 + 270$

$= 369$

Average rate of change $= \dfrac{Q_2 - Q_1}{t_2 - t_1}$

$= \dfrac{990 - 369}{6 - 3}$

$= \dfrac{621}{3}$

$= 207$

So average rate of change is 207 kL h^{-1}.

c $\dfrac{dQ}{dt} = 9t^2 + 4t$

When $t = 6$

$\dfrac{dQ}{dt} = 9(6)^2 + 4(6)$

$= 348$

So the rate of increase after 6 hours is 348 kL h^{-1}.

Exercise 5.01 Differentiation review

1 Differentiate each function.

a $3x^4 - 2x^3 + 7x - 4$ **b** $2x + 5$ **c** $6x^2 - 3x - 2$

2 Find the derivative $f'(x)$ given $f(x) = 4x^5 + 9x^2$.

3 Find $\dfrac{dx}{dt}$ if $x = 2\pi t^3 - 3t^2 + 1$.

4 Find $f'(-2)$ when $f(x) = 8x^3 + 5x - 2$.

5 Differentiate:

a x^{-5} **b** $x^{\frac{2}{3}}$ **c** $\dfrac{1}{x^2}$

d $\sqrt[4]{x}$ **e** $-\dfrac{5}{x^4}$

6 Find the derivative of $y = \sqrt[3]{x}$ at the point where $x = 8$.

7 Differentiate:

a $(3x-1)^7$ **b** $(x^2-x+2)^3$ **c** $\sqrt{7x-2}$

d $\dfrac{1}{3x-2}$ **e** $\sqrt[3]{x^2-3}$

8 Find the derivative of:

a $x^2(x+4)$ **b** $(2x-1)(6x+5)$ **c** $4x(x^2+1)$

d $(4x+3)(x^2-1)^2$ **e** $2x^3\sqrt{x+1}$

9 Differentiate:

a $\dfrac{2x+3}{x-5}$ **b** $\dfrac{x^3}{4x-7}$ **c** $\dfrac{x^2+3}{2x-3}$

d $\dfrac{3x+1}{(2x+9)^2}$ **e** $\dfrac{3x+4}{\sqrt{2x-1}}$

10 Find the gradient of the tangent to the curve:

a $y=x^2-2x+5$ at the point where $x=-2$

b $f(x)=x^3-3$ at the point $(-1,-4)$

11 Find the gradient of the normal to the curve:

a $f(x)=3x^4+x^2-2$ at the point where $x=-1$

b $y=x^2+x-3$ at the point $(-3, 3)$

12 Find the equation of the tangent to the curve:

a $y=2x^2-5x-6$ at the point $(3,-3)$

b $y=5x^3-2x^2-x$ at the point where $x=2$

13 Find the equation of the normal to the curve:

a $f(x)=x^3+2x^2-3x-5$ at the point $(-1,-1)$

b $y=x^2-3x+1$ at the point where $x=3$

14 For the curve $y=x^2-8x+15$, find any values of x for which $\dfrac{dy}{dx}=0$.

15 Find the coordinates of the points at which the curve $y=x^3-2$ has a tangent with gradient 12.

16 Function $f(x)=x^2+x-4$ has a tangent parallel to the line $3x+y-4=0$ at point P. Find the equation of the tangent at P.

17 Find the coordinates of P if the gradient of the tangent to $y=\sqrt{x}$ is $\dfrac{1}{4}$ at point P.

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18 For the curve $y = \dfrac{5x-3}{4x+1}$ at the point where $x = 0$, find the equation of:

a the tangent **b** the normal

19 Find a formula for the rate of change $\dfrac{dQ}{dt}$ given:

a $Q = 3t^2 + 8$ **b** $Q = \dfrac{2}{t-3}$ **c** $Q = \sqrt[3]{2x+3}$

20 The mass M in kg of a snowball as it rolls down a hill over time t seconds is given by $M = t^2 + 3t + 4$.

a Find the average rate at which the mass changes between:

i 2 and 5 seconds **ii** 6 and 8 seconds

b Find the rate at which the mass is changing after:

i 5 seconds **ii** a minute

21 According to Boyle's Law, the pressure of a gas in pascals (Pa) is given by the formula $P = \dfrac{k}{V}$, where k is a constant and V is the volume of the gas in m^3. If $k = 250$ for a certain gas, find the rate of change in the pressure when $V = 10.7$.

22 The height of a ball in metres is given by $h = 4t - 2t^2$ where t is time in seconds.

a Find the height after:

i 1 s **ii** 1.5 s

b How long does it take for the ball to reach the ground?

c Find the velocity of the ball after:

i 0.5 s **ii** 1 s **iii** 2 s

5.02 Derivative of exponential functions

Derivatives of exponential functions

You learned how to differentiate $y = e^x$ in Year 11, in Chapter 10, *Exponential and logarithmic functions*.

Differentiation rules for e^x

$$\frac{d}{dx}e^x = e^x$$

$$\text{If } y = e^{f x)} \text{ then } \frac{dy}{dx} = f'(x)\, e^{f x)}$$

EXAMPLE 2

a If $f(x) = 3e^x$, find the equation of the tangent to the curve at $(2, 3e^2)$.

b Differentiate :

i x^2e^x **ii** e^{8x} **iii** e^{5x-2}

Solution

a
$$f(x) = 3e^x$$
$$f'(x) = 3e^x$$
At $(2, 3e^2)$
$$f'(2) = 3e^2$$
So $m = 3e^2$

Equation:
$$y - y_1 = m(x - x_1)$$
$$y - 3e^2 = 3e^2(x - 2)$$
$$= 3e^2x - 6e^2$$
$$y = 3e^2x - 3e^2$$
(or $3e^2x - y - 3e^2 = 0$)

b **i**
$$y' = u'v + v'u$$
where $u = x^2$ and $v = e^x$
$$u' = 2x \qquad v' = e^x$$
$$y' = 2xe^x + e^xx^2$$
$$= xe^x(2 + x)$$

ii
$$\frac{dy}{dx} = ae^{ax}$$
$$= 8e^{8x}$$

iii
$$\frac{dy}{dx} = f'(x)e^{f\ x)}$$
$$= 5e^{5x-2}$$

We can differentiate other exponential functions.

EXAMPLE 3

Differentiate 2^x.

Solution

$$2 = e^{\ln 2}$$
$$2^x = (e^{\ln 2})^x$$
$$= e^{x \ln 2}$$

$$\frac{dy}{dx} = \ln 2\ e^{x \ln 2}$$ ← ln 2 is a constant
$$= \ln 2 \times 2^x$$
$$= 2^x \ln 2$$

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Derivative of a^x

$$\text{If } y = a^x, \text{ then } \frac{dy}{dx} = a^x \ln a$$

The proof of this has the same steps as in the previous example.

Exercise 5.02 Derivative of exponential functions

1 Differentiate:

a e^{7x} **b** e^{-x} **c** e^{6x-2} **d** $e^{x\ \ +}$

e $e^{x\ \ +5x+7}$ **f** e^{5x} **g** e^{-2x} **h** e^{10x}

i $e^{2x} + x$ **j** $x^2 + 2x + e^{1-x}$ **k** $(x + e^{4x})^5$ **l** xe^{2x}

m $\dfrac{e^{3x}}{x^2}$ **n** x^3e^{5x} **o** $\dfrac{e^{2x+1}}{2x+5}$

2 If $f(x) = e^{3x-2}$ find the exact value of $f'(1)$.

3 Find the derivative of:

a 3^x **b** 10^x **c** 2^{3x-4}

4 Find the gradient of the tangent to the curve $y = e^{5x}$ at the point where $x = 0$.

5 Find the equation of the tangent to the curve $y = e^{2x} - 3x$ at the point $(0, 1)$.

6 For the curve $y = e^{3x}$ at the point where $x = 1$, find the exact gradient of:

a the tangent **b** the normal

7 For the curve $y = e^x$ at the point $(1, e)$, find the equation of:

a the tangent **b** the normal

8 Find the equation of the tangent to the curve $y = 4^{x+1}$ at the point $(0, 4)$.

9 The population of a city is given by $P = 24\ 500e^{0.038t}$ where t is time in years.

a Find the population after:

i 5 years **ii** 10 years

b Find the average rate of change in population between:

i the 1st and 5th years **ii** the 5th and 10th years

c Find the rate of change in population after:

i 5 years **ii** 10 years

10 The displacement of a particle is given by $s = 10e^{2t} - 5t$ cm after t minutes.

a Find the average rate of change in displacement between 1 and 5 minutes.

b Find the rate of change in displacement after:

i 1 minute **ii** 2 minutes **iii** 8 minutes

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11 A radioactive substance has a mass of $M = 20e^{-0.021t}$ in grams over time t years.

- **a** Find the initial mass.
- **b** Find the mass after 50 years.
- **c** Find the average rate of change in mass between 50 and 100 years.
- **d** Find the rate of change in mass after:
  - **i** 50 years **ii** 100 years **iii** 200 years

12 An object moves according to the formula $x = 3e^{2t}$ where x is displacement in cm and t is time in s.

- **a** Find the displacement at 5 s.
- **b** Find the velocity at 5 s.

INVESTIGATION

DERIVATIVE OF A LOGARITHMIC FUNCTION

Draw the derivative (gradient) function of a logarithm function.

What is the shape of the derivative function?

5.03 Derivative of logarithmic functions

WS Derivatives of logarithmic functions

WS Exponential and logarithmic functions

Logarithm rules

If $y = a^x$ then $\log_a y = x$

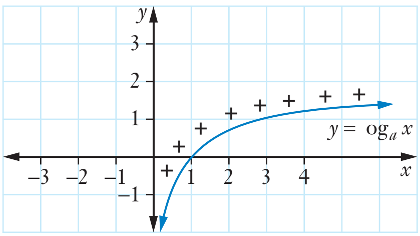
$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

To find the derivative of a logarithmic function, notice that the gradient of the function is always positive but is decreasing.



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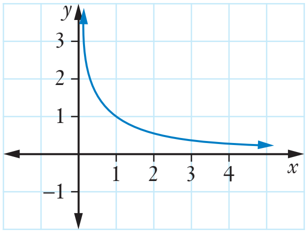
The derivative function of a logarithmic function is a hyperbola.



There is a special rule for $y = \ln x$.

Derivative of $y = \ln x$

If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$ where $x > 0$.

Proof

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Given $y = \ln x = \log_e x$

Then $x = e^y$

$$\frac{dx}{dy} = e^y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{e^y}$$

$$= \frac{1}{x}$$

EXAMPLE 4

a Differentiate $(\ln x + 1)^3$.

b Find the equation of the tangent to the curve $y = \ln x$ at the point $(3, \ln 3)$.

Solution

a $(\ln x + 1)^3$ is a composite function in the form $y = [f(x)]^n$.

$$\frac{dy}{dx} = f'(x)\, nf(x)^{n-1}$$

$$= \frac{1}{x} \times 3(\ln x + 1)^2$$

$$= \frac{3(\ln x + 1)^2}{x}$$

b $\frac{dy}{dx} = \frac{1}{x}$

At $(3, \ln 3)$

$$\frac{dy}{dx} = \frac{1}{3}$$

So $m = \frac{1}{3}$

Equation:

$$y - y_1 = m(x - x_1)$$

$$y - \ln 3 = \frac{1}{3}(x - 3)$$

$$3y - 3\ln 3 = x - 3$$

$$0 = x - 3y - 3 + 3\ln 3$$

Chain rule

If $y = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$ where $f(x) > 0$

Proof

$y = \ln f(x)$ is a composite function.

Let $y = \ln u$ and $u = f(x)$

$$\frac{dy}{du} = \frac{1}{u} \quad \text{and} \quad \frac{du}{dx} = f'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{u} \times f'(x)$$

$$= \frac{1}{f(x)} \times f'(x)$$

$$= \frac{f'(x)}{f(x)}$$

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EXAMPLE 5

a Differentiate:

i $\ln (x^2 - 3x + 1)$ **ii** $\ln \left(\dfrac{x+1}{3x-4}\right)$

b Find the gradient of the normal to the curve $y = \ln (x^3 - 5)$ at the point where $x = 2$.

Solution

a **i**
$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$
$$= \frac{2x-3}{x^2-3x+1}$$

ii It is easier to simplify first using log laws.

$$y = \ln\left(\frac{x+1}{3x-4}\right)$$
$$= \ln (x+1) - \ln (3x-4)$$
$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$
$$= \frac{1}{x+1} - \frac{3}{3x-4}$$
$$= \frac{1(3x-4)}{(x+1)(3x-4)} - \frac{3(x+1)}{(3x-4)(x+1)}$$
$$= \frac{3x-4-3(x+1)}{(x+1)(3x-4)}$$
$$= \frac{3x-4-3x-3}{(x+1)(3x-4)}$$
$$= \frac{-7}{(x+1)(3x-4)}$$

b
$$\frac{dy}{dx} = \frac{3x^2}{x^3-5}$$

When $x = 2$

$$\frac{dy}{dx} = \frac{3(2)^2}{2^3-5}$$
$$= 4$$
$$m_1 = 4$$

The normal is perpendicular to the tangent:

$$m_1 m_2 = -1$$
$$4m_2 = -1$$
$$m_2 = -\frac{1}{4}$$

We can differentiate logarithmic functions with a different base, a.

EXAMPLE 6

Differentiate $y = \log_2 x$.

Solution

$$y = \log_2 x$$
$$= \frac{\ln x}{\ln 2} \text{ using the change of base law}$$
$$= \frac{1}{\ln 2} \ln x$$

$$\frac{dy}{dx} = \frac{1}{\ln 2} \times \frac{1}{x}$$ ← ln 2 is a constant
$$= \frac{1}{x \ln 2}$$

Derivative of $\log_a x$

If $y = \log_a x$, then $\frac{dy}{dx} = \frac{1}{x \ln a}$

The proof of this has the same steps as in the above example.

Exercise 5.03 Derivative of logarithmic functions

1 Differentiate:

a $x + \ln x$	**b** $1 - \ln 3x$	**c** $\ln (3x + 1)$
d $\ln (x^2 - 4)$	**e** $\ln (5x^3 + 3x - 9)$	**f** $\ln (5x + 1) + x^2$
g $3x^2 + 5x - 5 + \ln 4x$	**h** $\ln (8x - 9) + 2$	**i** $\ln (2x + 4)(3x - 1)$
j $\ln \left(\frac{4x+1}{2x-7}\right)$	**k** $(1 + \ln x)^5$	**l** $(\ln x - x)^9$
m $(\ln x)^4$	**n** $(x^2 + \ln x)^6$	**o** $x \ln x$
p $\frac{\ln x}{x}$	**q** $(2x + 1)\ln x$	**r** $x^3 \ln (x + 1)$
s $\ln (\ln x)$	**t** $\frac{\ln x}{x - 2}$	**u** $\frac{e^{2x}}{\ln x}$
v $e^x \ln x$	**w** $5(\ln x)^2$	

2 Find $f'(1)$ if $f(x) = \ln \sqrt{2 - x}$.

3 Find the derivative of $\log_{10} x$.

4 Find the equation of the tangent to the curve $y = \ln x$ at the point $(2, \ln 2)$.

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5 Find the equation of the tangent to the curve $y = \ln(x - 1)$ at the point where $x = 2$.

6 Find the gradient of the normal to the curve $y = \ln(x^4 + x)$ at the point $(1, \ln 2)$.

7 Find the exact equation of the normal to the curve $y = \ln x$ at the point where $x = 5$.

8 Find the equation of the tangent to the curve $y = \ln(5x + 4)$ at the point where $x = 3$.

9 Find the derivative of $\log_3 (2x + 5)$.

10 Find the equation of the normal to the curve $y = \log_2 x$ at the point where $x = 2$.

11 The formula for the time t in years for kangaroo population growth on Kangaroo Island is given by $t = \dfrac{\ln\left(\dfrac{P}{20\,000}\right)}{0021}$.

a What is the initial population?

b Find correct to one decimal place the time it takes for the population to grow to:

i 25 000 **ii** 50 000

c Change the subject of the equation to P.

d Find correct to the nearest whole number the average rate of change in population between 2 and 5 years.

e Find correct to the nearest whole number the rate at which the population is growing after:

i 3 years **ii** 5 years **iii** 10 years

CLASS INVESTIGATION

DERIVATIVE OF TRIGONOMETRIC FUNCTIONS

1 Draw the derivative (gradient) function of sine, cosine and tangent functions.

What is the shape of the derivative function of each graph?

2 By substituting values of x in radians close to 0, find approximations to $\lim_{x \to 0} \dfrac{\sin x}{x}$, $\lim_{x \to 0} \dfrac{\tan x}{x}$ and $\lim_{x \to 0} \dfrac{\cos x}{x}$.

3 Differentiate by first principles to find the derivative of each trigonometric function using the above limits. The sine function will use the EXT1 trigonometric identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$ from Chapter 4, *Trigonometric functions*.

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WS
Derivatives of trigonometric functions

WS
Trigonometric functions and gradient

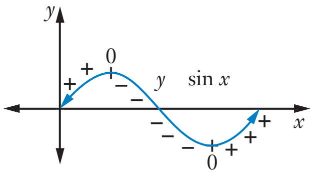
WS
Further trigonometric equations

WS
Differentiating trigonometric functions

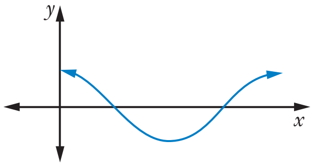
5.04 Derivative of trigonometric functions

Derivative of sin x

We can sketch the derivative (gradient) function of $y = \sin x$.



The sketch of the gradient function is $y = \cos x$.



Derivative of sin x

If $y = \sin x$, then $\dfrac{dy}{dx} = \cos x$

Proof

This proof uses trigonometric results from the investigation on the previous page.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}\\
&= \sin x\lim_{h\to 0}\frac{(\cos h-1)}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x\lim_{h\to 0}\frac{(\cos h-1)}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin x\times 0+\cos x\times 1\\
&= \cos x
\end{aligned}
$$

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EXAMPLE 7

a Differentiate $y = x \sin x$.

b Find the equation of the tangent to the curve $y = \sin x$ at the point $(\pi, 0)$.

Solution

a $y = x \sin x$ is in the form $y = uv$

where $u = x$ and $v = \sin x$

$u' = 1$ and $v' = \cos x$

$y' = u'v + v'u$

$= 1 \times \sin x + \cos x \times x$

$= \sin x + x \cos x$

b $\frac{dy}{dx} = \cos x$

At $(\pi, 0)$

$\frac{dy}{dx} = \cos \pi$

$= -1$

So $m = -1$

Equation:

$y - y_1 = m(x - x_1)$

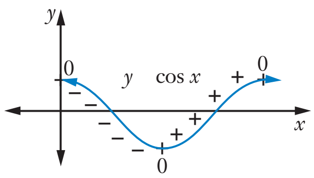
$y - 0 = -1(x - \pi)$

$y = -x + \pi$

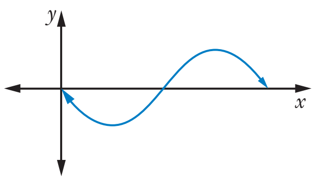
or $x + y - \pi = 0$

Derivative of cos x

We can sketch the derivative (gradient) function of $y = \cos x$.



The sketch of the gradient function below is $y = -\sin x$.



Derivative of cos x

If $y = \cos x$, then $\frac{dy}{dx} = -\sin x$

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You can prove this in a similar way to the derivative of $y = \sin x$. A simpler proof involves changing $\cos x$ into $\sin\left(\frac{\pi}{2} - x\right)$ and using the derivative of $y = \sin x$.

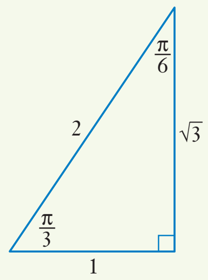
EXAMPLE 8

a Find the derivative of $y = \cos x$ at the point where $x = \frac{\pi}{3}$.

b Find the equation of the tangent to $y = \cos x$ at this point.

Solution

a $\frac{dy}{dx} = -\sin x$



When $x = \frac{\pi}{3}$

$$\frac{dy}{dx} = -\sin\frac{\pi}{3}$$
$$= -\frac{\sqrt{3}}{2}$$

b When $x = \frac{\pi}{3}$

$$y = \cos\frac{\pi}{3}$$
$$= \frac{1}{2}$$

Equation:

$$y - y_1 = m(x - x_1)$$
$$y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$
$$2y - 1 = -\sqrt{3}\left(x - \frac{\pi}{3}\right) \quad \text{(multiplying both sides by 2)}$$
$$= -\sqrt{3}x + \frac{\pi\sqrt{3}}{3}$$
$$6y - 3 = -3\sqrt{3}x + \pi\sqrt{3} \quad \text{(multiplying both sides by 3)}$$
$$3\sqrt{3}x + 6y - 3 - \pi\sqrt{3} = 0$$

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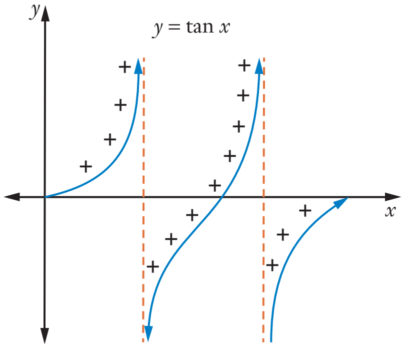
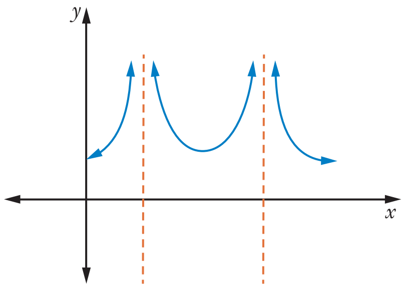
Derivative of tan x

We can sketch the derivative (gradient) function of $y = \tan x$. Notice that the gradient function will have asymptotes in the same place as the original graph, because this is where the tangent is vertical and the gradient is undefined.



The gradient function is $y = \sec^2 x$, where $\sec x = \dfrac{1}{\cos x}$.



Derivative of tan x

$$\text{If } y = \tan x, \text{ then } \frac{dy}{dx} = \sec^2 x$$

You can prove this in a similar way to the derivative of $y = \sin x$. A simpler proof involves changing $\tan x$ into $\dfrac{\sin x}{\cos x}$ and using the quotient rule.

EXAMPLE 9

a Differentiate $y = \frac{\tan x}{3x^2}$.

b Find the gradient of the tangent to the curve $f(x) = \tan x$ at the point where $x = \frac{\pi}{4}$.

Solution

a $y = \frac{\tan x}{3x^2}$ is in the form $y = \frac{u}{v}$.

$u = \tan x \quad \text{and} \quad v = 3x^2$

$u' = \sec^2 x \qquad v' = 6x$

$$\begin{aligned} y' &= \frac{u'v - v'u}{u^2} \\ &= \frac{\sec^2 x \times 3x^2 - 6x \times \tan x}{(3x^2)^2} \\ &= \frac{3x(x \sec^2 x - 2\tan x)}{9x^4} \\ &= \frac{x \sec^2 x - 2\tan x}{3x^3} \end{aligned}$$

b $\frac{dy}{dx} = \sec^2 x$

At $x = \frac{\pi}{4}$

$$\begin{aligned} \frac{dy}{dx} &= \sec^2 \frac{\pi}{4} \\ &= (\sqrt{2})^2 \\ &= 2 \end{aligned}$$

Chain rule

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x) \cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x) \sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x) \sec^2 f(x)$

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Here is the proof for $y = \sin f(x)$. The others are similar.

Proof

$y = \sin f(x)$ is a composite function

where $y = \sin u$ and $u = f(x)$

$$\frac{dy}{du} = \cos u \text{ and } \frac{du}{dx} = f'(x)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \cos u \times f'(x) \\ &= f'(x) \cos u \\ &= f'(x) \cos f(x)\end{aligned}$$

EXAMPLE 10

a Differentiate each function.

i $y = \sin 7x$ **ii** $y = \cos\left(4x^3 + \frac{\pi}{3}\right)$ **iii** $y = \tan(5x - \pi)$

b Find the gradient of the normal to the curve $f(x) = \cos\frac{x}{2}$ at the point where $x = \pi$.

Solution

a **i** $\frac{dy}{dx} = f'(x)\cos f(x)$

$= 7\cos 7x$

ii $\frac{dy}{dx} = -f'(x)\sin f(x)$

$= -12x^2 \sin\left(4x^3 + \frac{\pi}{3}\right)$

iii $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

$= 5\sec^2(5x - \pi)$

b $\frac{dy}{dx} = -f'(x)\sin f(x)$

$= -\frac{1}{2}\sin\frac{x}{2}$

At $x = \pi$:

$\frac{dy}{dx} = -\frac{1}{2}\sin\frac{\pi}{2}$

$= -\frac{1}{2} \times 1$

$= -\frac{1}{2}$

So $m_1 = -\frac{1}{2}$

Normal is perpendicular to the tangent:

$m_1 m_2 = -1$

$-\frac{1}{2} m_2 = -1$

$m_2 = 2$

While trigonometric functions are usually expressed in radians, we can differentiate angles *in degrees* by using the conversion $\pi = 180°$.

EXAMPLE 11

Differentiate $y = \sin x°$.

Solution

$180° = \pi$ radians

$$1° = \frac{\pi}{180}$$

$$x° = \frac{\pi x}{180}$$

So $y = \sin x°$ becomes

$$y = \sin \frac{\pi x}{180}$$

$$\frac{dy}{dx} = f'(x) \cos f(x)$$

$$= \frac{\pi}{180} \cos \frac{\pi x}{180}$$

$$= \frac{\pi}{180} \cos x°$$

We can also differentiate composite functions involving trigonometric functions.

EXAMPLE 12

Differentiate:

a $\tan (e^x)$

b $\ln (\cos x)$

Solution

a $$\frac{dy}{dx} = f'(x) \sec^2 f(x)$$

$$= e^x \sec^2 (e^x)$$

b $$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$= \frac{-\sin x}{\cos x}$$

$$= -\tan x$$

Exercise 5.04 Derivative of trigonometric functions

1 Differentiate:

a $\sin 4x$

b $\cos 3x$

c $\tan 5x$

d $\tan (3x + 1)$

e $\cos (-x)$

f $3 \sin x$

g $4 \cos (5x - 3)$

h $2 \cos (x^3)$

i $7 \tan (x^2 + 5)$

j $\sin 3x + \cos 8x$

k $\tan (\pi + x) + x^2$

l $x \tan x$

m $\sin 2x \tan 3x$

n $\dfrac{\sin x}{2x}$

o $\dfrac{3x + 4}{\sin 5x}$

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p $(2x + \tan 7x)^9$ **q** $\sin^2 x$ **r** $3 \cos^3 5x$

s $e^x - \cos 2x$ **t** $\sin (1 - \ln x)$ **u** $\sin (e^x + x)$

v $\ln (\sin x)$ **w** $e^{3x} \cos 2x$ **x** $\dfrac{e^{2x}}{\tan 7x}$

2 Find the gradient of the tangent to the curve $y = \tan 3x$ at the point where $x = \dfrac{\pi}{9}$.

3 Find the equation of the tangent to the curve $y = \sin (\pi - x)$ at the point $\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$ in exact form.

4 Differentiate $\ln (\cos x)$.

5 Find the exact gradient of the normal to $y = \sin 3x$ at the point where $x = \dfrac{\pi}{18}$.

6 Differentiate $e^{\tan x}$.

7 Find the equation of the normal to the curve $y = 3 \sin 2x$ at the point where $x = \dfrac{\pi}{8}$ in exact form.

8 Show that $\dfrac{d}{dx} [\ln (\tan x)] = \tan x + \cot x$.

9 Differentiate each function.

a $y = \tan x°$ **b** $y = 3 \cos x°$ **c** $y = \dfrac{\sin x°}{5}$

10 Find the derivative of $\cos x \sin^4 x$.

11 The population of salmon in a salmon farm grows and reduces as fish are born and sold.

The population is given by $P = 225 \cos \dfrac{2\pi t}{9} + 750$ where t is time in days.

a What is the centre of the population?

b What is the minimum number of salmon in the farm at any one time?

c What is the maximum population?

d At what times is the population 700?

e At what rate is the population changing after:

i 3 days? **ii** a week? **iii** 10 days? **v** 18 days?

f At what times is the population growing at the rate of 25 fish per day?

12 The tide was measured over time at a beach at Merimbula and given the formula $D = 8 \sin \dfrac{\pi t}{6} + 9$ where D is depth of water in metres and t is time in hours.

a How deep was the water:

i initially? **ii** after 5 hours?

b When was the water 10 m deep?

c At what rate was the depth changing after:

i 3 hours? **ii** 11 hours? **iii** 12 hours?

d At what times was the depth of water decreasing by 3 m h^{-1}?

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WS
The second derivative

WS
First and second derivatives

5.05 Second derivatives

Second derivative

Differentiating $f(x)$ gives $f'(x)$, the first derivative.

Differentiating $f'(x)$ gives $f''(x)$, the **second derivative**.

It is also possible to differentiate further.

Using function notation, differentiating several times gives $f'(x), f''(x), f'''(x)$ and so on.

Using $\frac{dy}{dx}$ notation, differentiating several times gives $\frac{d^2y}{dx^2}$ $\frac{d^3y}{dx^3}$ and so on.

The notation $\frac{d^2y}{dx^2}$ comes from $\frac{d^2}{dx^2}(y)$.

EXAMPLE 13

a Find the first 4 derivatives of $f(x) = x^3 - 4x^2 + 3x - 2$.

b Find the second derivative of $y = (2x + 5)^7$.

c If $f(x) = 4 \cos 3x$, show that $f''(x) = -9 f(x)$

Solution

a
$$f'(x) = 3x^2 - 8x + 3$$
$$f''(x) = 6x - 8$$
$$f'''(x) = 6$$
$$f''''(x) = 0$$

b
$$\begin{aligned}\frac{dy}{dx} &= f'(x) \times nf(x)^{n-1}\\ &= 2 \times 7(2x+5)^6\\ &= 14(2x+5)^6\end{aligned}$$
$$\begin{aligned}\frac{d^2y}{dx^2} &= f'(x) \times nf(x)^{n-1}\\ &= 2 \times 6 \times 14(2x+5)^5\\ &= 168(2x+5)^5\end{aligned}$$

c
$$\begin{aligned}f'(x) &= -f'(x) \times \sin f(x)\\ &= -3 \times 4 \sin 3x\\ &= -12 \sin 3x\end{aligned}$$
$$\begin{aligned}f''(x) &= f'(x) \times \cos f(x)\\ &= 3 \times (-12 \cos 3x)\\ &= -36 \cos 3x\\ &= -9(4 \cos 3x)\\ &= -9f(x) \qquad \text{since } f(x) = 4 \cos 3x\end{aligned}$$

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Exercise 5.05 Second derivatives

1 Find the first 4 derivatives of $x^7 - 2x^5 + x^4 - x - 3$.

2 If $f(x) = x^9 - 5$, find $f''(x)$.

3 Find $f'(x)$ and $f''(x)$ if $f(x) = 2x^5 - x^3 + 1$.

4 Find $f'(1)$ and $f''(-2)$, given $f(t) = 3t^4 - 2t^3 + 5t - 4$.

5 Find the first 3 derivatives of $x^7 - 2x^6 + 4x^4 - 7$.

6 Find the first and second derivatives of $y = 2x^2 - 3x + 3$.

7 If $f(x) = x^4 - x^3 + 2x^2 - 5x - 1$, find $f'(-1)$ and $f''(2)$.

8 Find the first and second derivatives of x^{-4}.

9 If $g(x) = \sqrt{x}$, find $g''(4)$.

10 Given $h = 5t^3 - 2t^2 + t + 5$, find $\frac{d^2h}{dt^2}$ when $t = 1$.

11 Find any values of x for which $\frac{d^2y}{dx^2} = 3$, given $y = 3x^3 - 2x^2 + 5x$.

12 Find all values of x for which $f''(x) > 0$ given that $f(x) = x^3 - x^2 + x + 9$.

13 Find the first and second derivatives of $(4x - 3)^5$.

14 Find $f'(x)$ and $f''(x)$ if $f(x) = \sqrt{2 - x}$.

15 Find the first and second derivatives of $f(x) = \frac{x+5}{3x-1}$.

16 Find $\frac{d^2v}{dt^2}$ if $v = (t + 3)(2t - 1)^2$.

17 Find the value of b in $y = bx^3 - 2x^2 + 5x + 4$ if $\frac{d^2y}{dx^2} = -2$ when $x = \frac{1}{2}$.

18 Find $f''(1)$ if $f(t) = t(2t - 1)^7$.

19 Find the value of b if $f(x) = 5bx^2 - 4x^3$ and $f''(-1) = -3$.

20 If $y = e^{4x} + e^{-4x}$, show that $\frac{d^2y}{dx^2} = 16y$.

21 Prove that $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ given $y = 3e^{2x}$.

22 Show that $\frac{d^2y}{dx^2} = b^2y$ for $y = ae^{bx}$.

23 Find the value of n if $y = e^{3x}$ satisfies the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + ny = 0$.

24 Show that $\frac{d^2y}{dx^2} = -25y$ if $y = 2 \cos 5x$.

25 Given $f(x) = -2 \sin x$, show that $f''(x) = -f(x)$.

26 If $y = 2 \sin 3x - 5 \cos 3x$, show that $\frac{d^2y}{dx^2} = -9y$.

27 Find values of a and b if $\frac{d^2y}{dx^2} = ae^{3x} \cos 4x + be^{3x} \sin 4x$, given $y = e^{3x} \cos 4x$.

28 Find the exact value of $f''(2)$ if $f(x) = x\sqrt{3x-4}$.

29 The displacement of a particle moving in a straight line is given by $x = 2t^3 - 5t^2 + 7t + 8$, where x is in metres and t is in seconds.

a Find the initial displacement.

b Find the displacement after 3 seconds.

c Find the velocity after 3 seconds.

d Find the acceleration after 3 seconds.

30 The height in cm of a pendulum as it swings is given by $h = 8 \cos \pi t + 12$ where t is time in seconds.

a What is the height of the pendulum after 3 s?

b What is the maximum and minimum height of the pendulum?

c What is the velocity of the pendulum after:

i 1 s? **ii** 1.5 s?

d What is the acceleration of the pendulum:

i initially? **ii** after 1 s? **iii** after 1.5 s?

e EXT1 Write the equation for acceleration in terms of h.

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5.06 Anti-derivative graphs

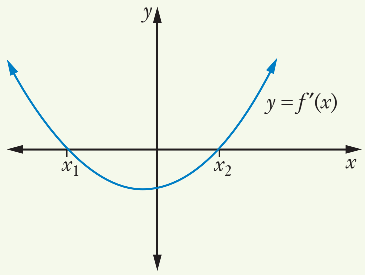
The process of finding the original function $y = f(x)$ given the derivative $y = f'(x)$ is called **anti-differentiation**, and the original function is called the **anti-derivative** function, also called the **primitive** or **integral function**.

WS
Shapes of curves

Anti-derivative graphs

EXAMPLE 14

Sketch the graph of the anti-derivative (primitive function) given the graph of the derivative function below and an initial condition, or starting point, of (0, 2).



Solution

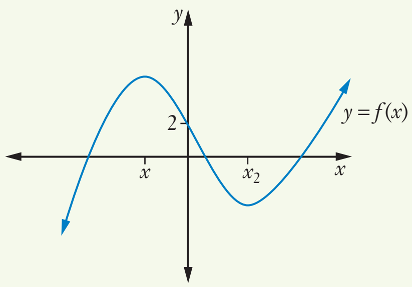
Remember that when you sketch a derivative function, the x-intercepts are where the original function has zero gradient, or stationary (turning) points.

On this graph the stationary points are at $x = x_1$ and $x = x_2$.

Above the x-axis shows where the original function has a positive gradient (it is increasing). On this graph, this is where $x < x_1$ and $x > x_2$.

Below the x-axis shows where the original function has a negative gradient (it is decreasing). On this graph, this is where $x_1 < x < x_2$.

We can sketch this information together with the point (0, 2):



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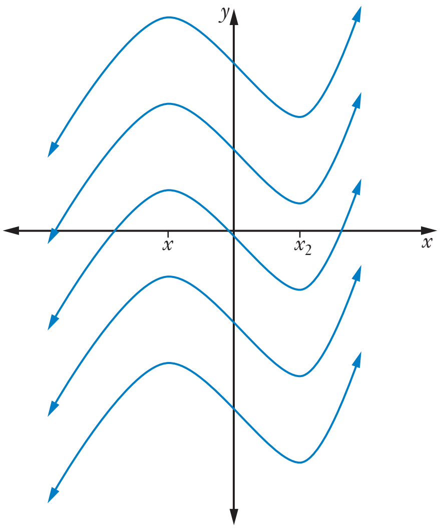
We are not given enough information to sketch a unique graph. There is no way of knowing what the y values of the stationary points are or the stretch or compression of the graph. Also, if we are not given a fixed point on the function, we could sketch many graphs that satisfy the information from the derivative function.

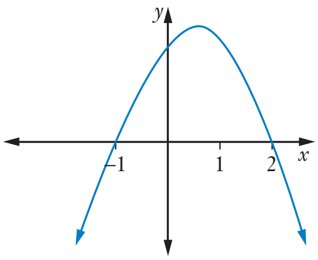


The anti-derivative gives a **family** of curves.

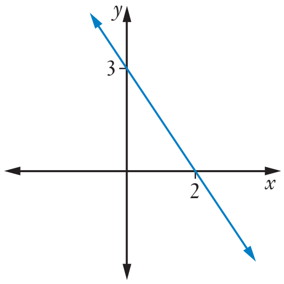
Exercise 5.06 Anti-derivative graphs

1 For each function graphed, sketch the graph of the anti-derivative function given it passes through:

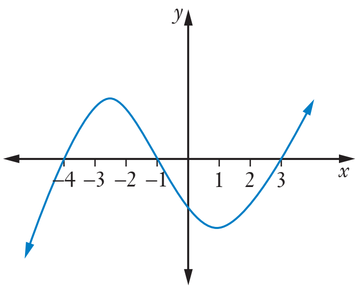
a (0, –1)



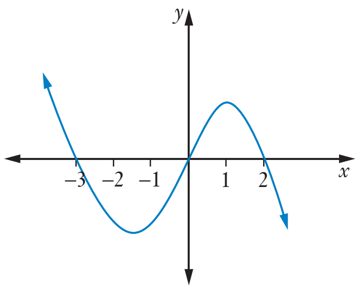
b (1, 2)



c (0, 3)



d (–1, –1)



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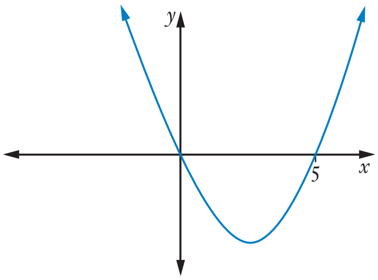
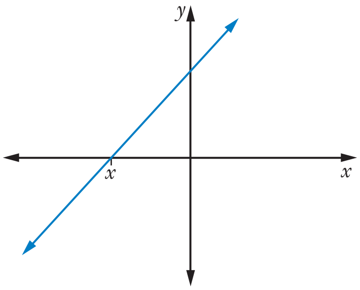
e (0, 1)

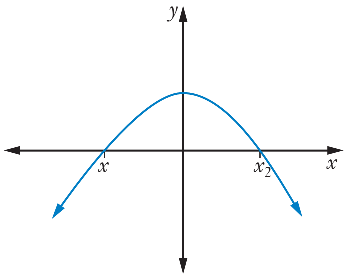


2 Sketch a family of graphs that could represent the anti-derivative function of each graph.

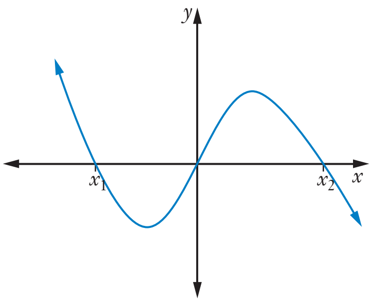
a



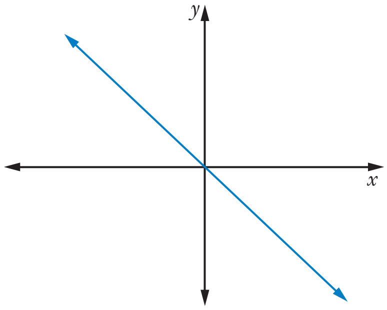
b



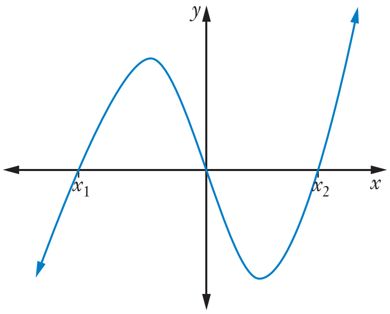
c



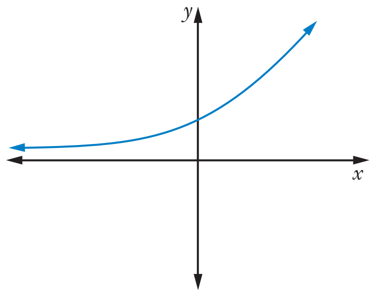
d



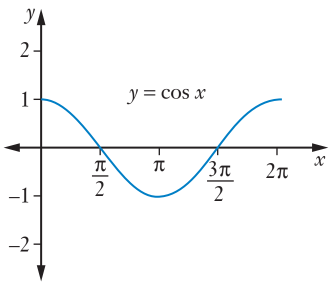
e



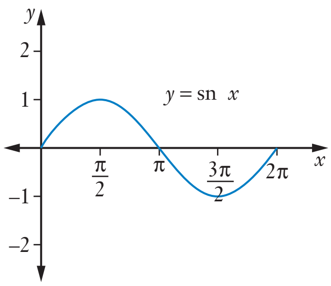
3 The anti-derivative function of the graph below passes through (0, –1). Sketch its graph.



4 Sketch the graph of the anti-derivative function of $y = \cos x$ given that it passes through (0, 0).



5 Sketch a family of anti-derivative functions for the graph below.



INVESTIGATION

ANTI-DERIVATIVE OF $y = x^n$

1 Differentiate:

a x^2 **b** $x^2 + 5$ **c** $x^2 - 7$ **d** $x^2 + 4$ **e** $x^2 - 75$

What would be the anti-derivative of $2x$?

2 Differentiate:

a x^3 **b** $x^3 + 1$ **c** $x^3 + 6$ **d** $x^3 - 2$ **e** $x^3 - 14$

What would be the anti-derivative of $3x^2$?

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3 Differentiate:

a x^4 **b** $x^4 - 3$ **c** $x^4 + 2$ **d** $x^4 + 10$ **e** $x^4 - 1$

What would be the anti-derivative of $4x^3$?

4 Differentiate:

a x^n **b** $x^n + 7$ **c** $x^n + 9$ **d** $x^n - 5$ **e** $x^n - 2$

What would be the anti-derivative of nx^{n-1}?

Can you find a general rule for anti-derivatives that would work for these examples?

5.07 Anti-derivatives

WS Anti-derivatives

WS Antiderivatives

WS Antidifferentiation

Since anti-differentiation is the reverse of differentiation, we can find the equation of an anti-derivative function.

Anti-derivative of x^n

$$\text{If } \frac{dy}{dx} = x^n, \text{ then } y = \frac{1}{n+1}x^{n+1} + C \text{ where } C \text{ is a constant.}$$

Proof

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+} + C\right) = \frac{(n+1)x^n}{n+1}$$
$$= x^n$$

We can apply the same rules to anti-derivatives as we use for derivatives. Here are some of the main ones we use.

Anti-derivative rules

If $\frac{dy}{dx} = k$ then $y = kx$.

If $\frac{dy}{dx} = kx^n$ then $y = \frac{1}{n+1}kx^{n+1} + C$.

If $\frac{dy}{dx} = f(x) + g(x)$ then $y = F(x) + G(x) + C$ where $F(x)$ and $G(x)$ are the anti-derivatives of $f(x)$ and $g(x)$ respectively.

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EXAMPLE 15

Find the anti-derivative of $x^4 - 4x^3 + 9x^2 - 6x + 5$.

Solution

If $f(x) = x^4 - 4x^3 + 9x^2 - 6x + 5$

$$F(x) = \frac{1}{5}x^5 - 4 \times \frac{1}{4}x^4 + 9 \times \frac{1}{3}x^3 - 6 \times \frac{1}{2}x^2 + 5x + C$$

Anti-derivative of 5 is 5*x*.

$$= \frac{x^5}{5} - x^4 + 3x^3 - 3x^2 + 5x + C$$

If we have some information about the anti-derivative function, we can use this to evaluate the constant C.

The anti-derivative function

EXAMPLE 16

a The gradient of a curve is given by $\frac{dy}{dx} = 6x^2 + 8x$. If the curve passes through the point $(1, -3)$, find its equation.

b If $f''(x) = 6x + 2$ and $f'(1) = f(-2) = 0$, find $f(3)$.

Solution

a $\frac{dy}{dx} = 6x^2 + 8x$

So $y = 6 \times \frac{1}{3}x^3 + 8 \times \frac{1}{2}x^2 + C$

$= 2x^3 + 4x^2 + C$

Substitute $(1, -3)$:

$-3 = 2(1)^3 + 4(1)^2 + C$

$= 6 + C$

$-9 = C$

Equation is $y = 2x^3 + 4x^2 - 9$.

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b $f''(x) = 6x + 2$

$f'(x) = 6 \times \frac{1}{2} x^2 + 2 \times \frac{1}{1} x^1 + C$

$= 3x^2 + 2x + C$

Since $f'(1) = 0$:

$0 = 3(1)^2 + 2(1) + C$

$= 5 + C$

$-5 = C$

So $f'(x) = 3x^2 + 2x - 5$

$f(x) = 3 \times \frac{1}{3} x^3 + 2 \times \frac{1}{2} x^2 - 5 \times \frac{1}{1} x^1 + D$

$= x^3 + x^2 - 5x + D$

Since $f(-2) = 0$:

$0 = (-2)^3 + (-2)^2 - 5(-2) + D$

$= -8 + 4 + 10 + D$

$= 6 + D$

$-6 = D$

Equation is $f(x) = x^3 + x^2 - 5x - 6$

$f(3) = 3^3 + 3^2 - 5(3) - 6$

$= 27 + 9 - 15 - 6$

$= 15$

Chain rule

If $\frac{dy}{dx} = (ax + b)^n$, then $y = \frac{1}{a(n+1)}(ax + b)^{n+1} + C$ where C is a constant, $a \neq 0$ and $n \neq -1$.

Proof

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+} + C\right) = \frac{a(n+1)(ax+b)^n}{a(n+1)}$$

$$= (ax + b)^n$$

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EXAMPLE 17

a Find the anti-derivative of $(3x + 7)^8$.

b The gradient of a curve is given by $\frac{dy}{dx} = (2x - 3)^4$. If the curve passes through the point (2, –7), find its equation.

Solution

a $\frac{dy}{dx} = (3x + 7)^8$

$$y = \frac{1}{a(n+1)}(ax + b)^{n+1} + C$$
$$= \frac{1}{3(8+1)}(3x + 7)^{8+1} + C$$
$$= \frac{1}{27}(3x + 7)^9 + C$$
$$= \frac{(3x+7)^9}{27} + C$$

b $\frac{dy}{dx} = (2x - 3)^4$

$$y = \frac{1}{a(n+1)}(ax + b)^{n+1} + C$$
$$= \frac{1}{2(4+1)}(2x - 3)^{4+1} + C$$
$$= \frac{1}{10}(2x - 3)^5 + C$$

Substitute (2, –7):

$$-7 = \frac{1}{10}(2 \times 2 - 3)^5 + C$$
$$= \frac{1}{10}(1)^5 + C$$
$$= \frac{1}{10} + C$$
$$-7\frac{1}{10} = C$$

So the equation is $y = \frac{1}{10}(2x - 3)^5 - 7\frac{1}{10}$

$$= \frac{(2x-3)^5 - 71}{10}$$

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General chain rule

If $\frac{dy}{dx} = f'(x)[f(x)]^n$ then $y = \frac{1}{n+1}[f(x)]^{n+1} + C$ where C is a constant and $n \neq -1$

Proof

$$\frac{d}{dx}\left(\frac{1}{n+1}[f(x)]^{n+} + C\right) = \frac{1}{n+1}f'(x)(n+1)[f(x)]^{n+1-1}$$

$$= f'(x)[f(x)]^n$$

EXAMPLE 18

Find the anti-derivative of:

a $8x^3(2x^4 - 1)^5$

b $x^2(x^3 + 2)^7$

Solution

a Given $f(x) = 2x^4 - 1$

$$f'(x) = 8x^3$$

$$\frac{dy}{dx} = 8x^3(2x^4 - 1)^5$$

$$= f'(x)[f(x)]^n$$

$$y = \frac{1}{n+1}f(x)^{n+1} + C$$

$$= \frac{1}{5+1}(2x^4 - 1)^{5+1} + C$$

$$= \frac{1}{6}(2x^4 - 1)^6 + C$$

$$= \frac{(2x^4 - 1)^6}{6} + C$$

b Given $f(x) = x^3 + 2$

$$f'(x) = 3x^2$$

$$\frac{dy}{dx} = x^2(x^3 + 2)^7$$

$$= \frac{1}{3} \times 3x^2(x^3 + 2)^7$$

$$= \frac{1}{3}f'(x)[f(x)]^n$$

$$y = \frac{1}{3} \times \frac{1}{n+1}f(x)^{n+1} + C$$

$$= \frac{1}{3} \times \frac{1}{7+1}(x^3 + 2)^{7+1} + C$$

$$= \frac{1}{24}(x^3 + 2)^8 + C$$

$$= \frac{(x^3 + 2)^8}{24} + C$$

Exercise 5.07 Anti-derivatives

1 Find the anti-derivative of:

a $2x - 3$ **b** $x^2 + 8x + 1$ **c** $x^5 - 4x^3$

d $(x - 1)^2$ **e** 6 **f** $(3x + 2)^5$

g $8(2x - 7)^4$

2 Find $f(x)$ if:

a $f'(x) = 6x^2 - x$ **b** $f'(x) = x^4 - 3x^2 + 7$ **c** $f'(x) = x - 2$

d $f'(x) = (x + 1)(x - 3)$ **e** $f'(x) = x^{\frac{}{2}}$

3 Express y in terms of x if:

a $\frac{dy}{dx} = 5x^4 - 9$ **b** $\frac{dy}{dx} = x^{-4} - 2x^{-2}$ **c** $\frac{dy}{dx} = \frac{x^3}{5} - x^2$

d $\frac{dy}{dx} = \frac{2}{x^2}$ **e** $\frac{dy}{dx} = x^3 - \frac{2x}{3} + 1$

4 Find the anti-derivative of:

a $\sqrt{x}$ **b** x^{-3} **c** $\frac{1}{x^8}$

d $x^{-\frac{}{2}} + 2x^{-\frac{2}{3}}$ **e** $x^{-7} - 2x^{-2}$

5 Find the anti-derivative of:

a $2x(x^2 + 5)^4$ **b** $3x^2(x^3 - 1)^9$ **c** $8x(2x^2 + 3)^3$

d $15x^4(x^5 + 1)^6$ **e** $x(x^2 - 4)^7$ **f** $x^5(2x^6 - 7)^8$

g $(2x - 1)(x^2 - x + 3)^4$ **h** $(3x^2 + 4x - 7)(x^3 + 2x^2 - 7x)^{10}$

i $(x - 3)(x^2 - 6x - 1)^5$

6 If $\frac{dy}{dx} = x^3 - 3x^2 + 5$ and $y = 4$ when $x = 1$, find an equation for y in terms of x.

7 If $f'(x) = 4x - 7$ and $f(2) = 5$, find an equation for $y = f(x)$.

8 Given $f'(x) = 3x^2 + 4x - 2$ and $f(-3) = 4$, find the value of $f(1)$.

9 Given that the gradient of the tangent to a curve is given by $\frac{dy}{dx} = 2 - 6x$ and the curve passes through $(-2, 3)$, find the equation of the curve.

10 If $\frac{dx}{dt} = (t - 3)^2$ and $x = 7$ when $t = 0$, find x when $t = 4$.

11 Given $\frac{d^2y}{dx^2} = 8$, and $\frac{dy}{dx} = 0$ and $y = 3$ when $x = 1$, find the equation of y in terms of x.

12 If $\frac{d^2y}{dx^2} = 12x + 6$ and $\frac{dy}{dx} = 1$ at the point $(-1, -2)$, find the equation of the curve.

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13 If $f''(x) = 6x - 2$ and $f'(2) = f(2) = 7$, find the equation of the function $y = f(x)$.

14 Given $f''(x) = 5x^4$, $f'(0) = 3$ and $f(-1) = 1$, find $f(2)$.

15 A curve has $\frac{d^2y}{dx^2} = 8x$ and the tangent at $(-2, 5)$ has an angle of inclination of 45° with the x-axis. Find the equation of the curve.

16 The tangent to a curve with $\frac{d^2y}{dx^2} = 2x - 4$ makes an angle of inclination of 135° with the x-axis at the point $(2, -4)$. Find its equation.

17 A function has a tangent parallel to the line $4x - y - 2 = 0$ at the point $(0, -2)$, and $f''(x) = 12x^2 - 6x + 4$. Find the equation of the function.

18 A curve has $\frac{d^2y}{dx^2} = 6$ and the tangent at $(-1, 3)$ is perpendicular to the line $2x + 4y - 3 = 0$. Find the equation of the curve.

19 A function has $f'(1) = 3$ and $f(1) = 5$. Evaluate $f(-2)$ given $f''(x) = 6x + 18$.

20 The velocity of an object is given by $\frac{dx}{dt} = 6t - 5$. If the object has initial displacement of -2, find the equation for the displacement.

21 The acceleration of a particle is given by $\frac{d^2x}{dt^2} = 24t^2 - 12t + 6$ m s^{-2}. Its velocity $\frac{dx}{dt} = 0$ when $t = 1$ and its displacement $x = -3$ when $t = 0$. Find the equation for its displacement.

5.08 Further anti-derivatives

Anti-derivative of exponential functions

$$\text{If } \frac{dy}{dx} = e^x, \text{ then } y = e^x + C$$

Chain rule

$$\text{If } \frac{dy}{dx} = e^{ax+b}, \text{ then } y = \frac{1}{a}e^{ax+b} + C$$

$$\text{If } \frac{dy}{dx} = f'(x)e^{f(x)}, \text{ then } y = e^{f(x)} + C$$

Proof (by differentiation)

$$\frac{d}{dx}\left(\frac{1}{a}e^{ax+b} + C\right) = \frac{1}{a} \times ae^{ax+b}$$
$$= e^{ax+b}$$

$$\frac{d}{dx}[e^{f(x)} + C] = f'(x)e^{f(x)}$$

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EXAMPLE 19

a Find the anti-derivative of $e^{4x} + 1$.

b Find the equation of the function $y = f(x)$ given $f'(x) = 6e^{3x}$ and $f(2) = 2e^6$.

Solution

a $\frac{1}{a}e^{ax+b} + C = \frac{1}{4}e^{4x} + C$

b
$$f'(x) = 6e^{3x}$$
$$f(x) = 6 \times \frac{1}{3}e^{3x} + C$$
$$= 2e^{3x} + C$$

If $f(2) = 2e^6$:
$$2e^6 = 2e^{3\times 2} + C$$
$$= 2e^6 + C$$
$$0 = C$$
So $f(x) = 2e^{3x}$

Anti-derivative of $\frac{1}{x}$

$$\text{If } \frac{dy}{dx} = \frac{1}{x}, \text{ then } y = \ln|x| + C$$

Chain rule

$$\text{If } \frac{dy}{dx} = \frac{f'(x)}{f(x)}, \text{ then } y = \ln|f(x)| + C$$

Proof

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ for $x > 0$, because $\ln x$ is defined only for $x > 0$.

So the anti-derivative of $\frac{1}{x}$ when $x > 0$ is $\ln x$.

Suppose $x < 0$.

Then $\ln(-x)$ is defined because $-x$ is positive.

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$$\frac{d}{dx}[\ln(-x)] = \frac{f'(x)}{f(x)}$$
$$= \frac{-1}{-x}$$
$$= \frac{1}{x}, \qquad x < 0$$

So if $\frac{dy}{dx} = \frac{1}{x}$, then $y = \begin{cases} \ln x + C & \text{if } x > 0 \\ \ln(-x) + C & \text{if } x < 0 \end{cases}$,

or more simply, $\quad y = \ln|x| + C$

EXAMPLE 20

a Find the anti-derivative of $\frac{3}{x}$.

b Find the equation of the function that has $\frac{dy}{dx} = \frac{6x}{x^2 - 5}$ and passes through $(3, 3 \ln 4)$.

Solution

a
$$\frac{dy}{dx} = \frac{3}{x}$$
$$= 3 \times \frac{1}{x}$$
$$y = 3 \ln|x|$$

b
$$\frac{dy}{dx} = \frac{6x}{x^2 - 5}$$
$$= 3 \times \frac{2x}{x^2 - 5}$$
$$= 3 \times \frac{f'(x)}{f(x)} \text{ where } f(x) = x^2 - 5$$
$$y = 3 \ln f|x| + C$$
$$= 3 \ln|x^2 - 5| + C$$

Substitute $(3, 3 \ln 4)$:
$$3 \ln 4 = 3 \ln|3^2 - 5| + C$$
$$= 3 \ln 4 + C$$
$$0 = C$$
So $y = 3 \ln|x^2 - 5|$

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Anti-derivatives of trigonometric functions

If $\frac{dy}{dx} = \cos x$, then $y = \sin x + C$ since $\frac{d}{dx}(\sin x) = \cos x$

If $\frac{dy}{dx} = \sin x$, then $y = -\cos x + C$ since $\frac{d}{dx}(\cos x) = -\sin x$ so $\frac{d}{dx}(-\cos x) = \sin x$

If $\frac{dy}{dx} = \sec^2 x$, then $y = \tan x + C$ since $\frac{d}{dx}(\tan x) = \sec^2 x$

Chain rule

If $\frac{dy}{dx} = \cos(ax + b)$, then $y = \frac{1}{a}\sin(ax + b) + C$

If $\frac{dy}{dx} = \sin(ax + b)$, then $y = -\frac{1}{a}\cos(ax + b) + C$

If $\frac{dy}{dx} = \sec^2(ax + b)$, then $y = \frac{1}{a}\tan(ax + b) + C$

If $\frac{dy}{dx} = f'(x)\cos f(x)$, then $y = \sin f(x) + C$

If $\frac{dy}{dx} = f'(x)\sin f(x)$, then $y = -\cos f(x) + C$

If $\frac{dy}{dx} = f'(x)\sec^2 f(x)$, then $y = \tan f(x) + C$

Proof

$$\frac{d}{dx}\left[\frac{1}{a}\sin(ax+b)+C\right] = \frac{1}{a} \times a\cos(ax + b)$$
$$= \cos(ax + b)$$

The other results can be proved similarly.

EXAMPLE 21

a Find the anti-derivative of $\cos 3x$.

b Find the equation of the curve that passes through $\left(\frac{\pi}{4}, 3\right)$ and has $\frac{dy}{dx} = \sec^2 x$.

Solution

a $y = \frac{1}{a}\sin(ax + b) + C$

$= \frac{1}{3}\sin 3x + C$

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b $y = \tan x + C$

Substitute $\left(\frac{\pi}{4}, 3\right)$:

$3 = \tan\frac{\pi}{4} + C$

$= 1 + C$

$2 = C$

So $y = \tan x + 2$

Exercise 5.08 Further anti-derivatives

1 Find the anti-derivative of:

a $\sin x$ **b** $\sec^2 x$ **c** $\cos x$

d $\sec^2 7x$ **e** $\sin(2x - \pi)$

2 Anti-differentiate:

a e^x **b** e^{6x} **c** $\frac{1}{x}$

d $\frac{3}{3x-1}$ **e** $\frac{x}{x^2+5}$

3 Find the anti-derivative of:

a $e^x + 5$ **b** $\cos x + 4x$ **c** $x + \frac{1}{x}$

d $8x^3 - 3x^2 + 6x - 3 + x^{-1}$ **e** $\sin 5x - \sec^2 9x$

4 Find the equation of a function with $\frac{dy}{dx} = \cos x$ and passing through $\left(\frac{\pi}{2}, -4\right)$.

5 Find the equation of the function that has $f'(x) = \frac{5}{x}$ and $f(1) = 3$.

6 A function has $\frac{dy}{dx} = 4\cos 2x$ and passes through the point $\left(\frac{\pi}{6}, 2\sqrt{3}\right)$.

Find the exact equation of the function.

7 A curve has $f''(x) = 27e^{3x}$ and has $f(2) = f'(2) = e^6$. Find the equation of the curve.

8 The rate of change of a population over time t years is given by $\frac{dP}{dt} = 1350e^{0.054t}$. If the initial population is 35 000, find:

a the equation for population

b the population after 10 years

9 The velocity of a particle is given by $\frac{dx}{dt} = 3e^{3t}$ and the particle has an initial displacement of 5 metres. Find the equation for displacement of the particle.

10 A pendulum has acceleration given by $\frac{d^2x}{dt^2} = -9 \sin 3t$, initial displacement 0 cm and initial velocity 3 cm s^{-1}.

- **a** Find the equation for its velocity.
- **b** Find the displacement after 2 seconds.
- **c** Find the times when the pendulum has displacement 0 cm.

EXT1 5.09 Derivative of inverse functions

EXAMPLE 22

Differentiate the inverse function of $y = x^5 + 2$.

Solution

Inverse function:

$$\begin{aligned} x &= y^5 + 2 \\ x - 2 &= y^5 \\ \sqrt[5]{x-2} &= y \\ (x-2)^{\frac{1}{5}} &= y \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{5}(x-2)^{\frac{1}{5}-1} \\ &= \frac{1}{5}(x-2)^{-\frac{4}{5}} \\ &= \frac{1}{5} \times \frac{1}{(x-2)^{\frac{4}{5}}} \\ &= \frac{1}{5\sqrt[5]{(x-2)^4}} \end{aligned}$$

Sometimes it is hard to differentiate inverse functions directly. We can use this property of differentiation:

$\frac{dy}{dx}$ and $\frac{dx}{dy}$

Given that $y = f(x)$ is a differentiable function:

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

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EXAMPLE 23

Show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ given $y = x^{\frac{1}{3}}$.

Solution

$$y = x^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{1}{3}x^{\frac{1}{3}-1}$$

$$= \frac{1}{3}x^{-\frac{2}{3}}$$

Changing the subject:

$$y = x^{\frac{1}{3}}$$

$$y^3 = x$$

$$\text{or } x = y^3$$

$$\frac{dx}{dy} = 3y^2$$

$$= 3\left(x^{\frac{1}{3}}\right)^2$$

$$= 3x^{\frac{2}{3}}$$

$$\frac{dy}{dx} \times \frac{dx}{dy} = \frac{1}{3}x^{-\frac{2}{3}} \times 3x^{\frac{2}{3}}$$

$$= 1$$

We can use this property to find the derivative of inverse functions.

EXAMPLE 24

a Differentiate the inverse function of $y = x^3 - 1$, leaving your answer in terms of y.

b Find the gradient of the tangent at the point (7, 2) on the inverse function.

Solution

a Inverse function:

$$x = y^3 - 1$$

$$\frac{dx}{dy} = 3y^2$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{3y^2}$$

b At (7, 2):

$$\frac{dy}{dx} = \frac{1}{3y^2}$$

$$= \frac{1}{3 \times 2^2}$$

$$= \frac{1}{12}$$

So the gradient of the tangent at (7, 2) on the inverse function is $\frac{1}{12}$.

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EXAMPLE 25

a Find the derivative of the inverse function $f^{-1}(x)$ of $f(x) = x(x+1)^4$ in terms of y.

b Given that $f^{-1}(-1) = 2$, find the gradient of the tangent to $y = f^{-1}(x)$ at this point.

Solution

a Inverse function is $x = y(y+1)^4$.

It is difficult to change the subject of this equation to y, so we find $\frac{dx}{dy}$.

$\frac{dx}{dy} = u'v + v'u$ where $u = y$ and $v = (y+1)^4$

$$u' = 1 \qquad v' = 4(y+1)^3$$

$$\begin{aligned}\frac{dx}{dy} &= 1 \times (y+1)^4 + 4(y+1)^3 \times y \\ &= (y+1)^4 + 4y(y+1)^3 \\ &= (y+1)^3(y+1+4y) \\ &= (y+1)^3(5y+1)\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}} \\ &= \frac{1}{(y+1)^3(5y+1)}\end{aligned}$$

b Since $f^{-1}(-1) = 2$, the curve passes through $(-1, 2)$.

Substitute $y = 2$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{(2+1)^3(5 \times 2 + 1)} \\ &= \frac{1}{3^3 \times 11} \\ &= \frac{1}{297}\end{aligned}$$

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EXT1 Exercise 5.09 Derivative of inverse functions

1 Show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ given:

a $y = 4x + 3$ **b** $y = x^3$ **c** $y = e^x$

d $y = \ln x$ **e** $y = x^7 - 1$

2 Differentiate $f^{-1}(x)$ given:

a $f(x) = e^x$ **b** $f(x) = \ln x$ **c** $f(x) = \sqrt{x}$

d $f(x) = x^7 - 1$ **e** $f(x) = (x + 2)^3$

3 Find the gradient of the tangent to the inverse function at:

a $(5, 1)$ given $f(x) = x^3 + 4$ **b** $(-1, 1)$ given $f(x) = 2x - 3$

c $(1, 0)$ given $f(x) = e^{3x}$ **d** $(2, 5)$ given $f(x) = \sqrt{x-1}$

e $\left(\frac{1}{9}, 2\right)$ given $f(x) = \frac{1}{x^3 + 1}$

4 **a** Find the derivative of the inverse function f^{-1} given $f(x) = 4x^3$.

b The point $(4, 1)$ lies on f^{-1}. Find the gradient of:

i the tangent **ii** the normal at that point

5 **a** By restricting $f(x)$ to a monotonic increasing domain, find the inverse function of $f(x) = x^2 + 1$.

b Find the derivative of the inverse function f^{-1}.

c Given that $(5, 2)$ lies on f^{-1}, find the gradient of the tangent at this point.

6 Find $\frac{dx}{dy}$ of the inverse function $f^{-1}(x)$ of each function in terms of y.

a $f(x) = x^2 e^x$ **b** $f(x) = 3x \sin 2x$ **c** $y = x(2x - 3)^4$

d $f(x) = \frac{3x - 1}{2x + 5}$ **e** $y = \frac{\ln x}{x + 2}$

7 Find the gradient of the tangent at each point given on the inverse function of:

a $f(x) = (3x + 1)(x - 4)^5$ at $(-10, 3)$

b $y = (x - 3) \cos x$ at $(-2, 0)$

c $f(x) = \frac{x^3}{3x - 4}$ at $(4, 2)$

d $y = x \ln x$ at $(0, 1)$

e $y = \frac{\sin 3x}{x^2}$ at $\left(-\frac{4}{\pi^2}, \frac{\pi}{2}\right)$

WS Derivatives of inverse trigonometric functions

WS Inverse trigonometric functions and gradient

EXT1 5.10 Derivative of inverse trigonometric functions

Derivative of $\sin^{-1}x$

$$\frac{d}{dx}(\sin^{-} x) = \frac{1}{\sqrt{1-x^2}}$$

Proof

Let $y = \sin^{-1} x$

Then $x = \sin y$

$$\frac{dx}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{\cos y}$$

Using $\sin^2 \theta + \cos^2 \theta = 1$:

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \sqrt{1-\sin^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

EXAMPLE 26

Find the equation of the tangent to the curve $y = \sin^{-1} x$ at the point $\left(0 \quad \frac{\pi}{2}\right)$.

Solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

At $\left(0 \quad \frac{\pi}{2}\right)$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-0^2}}$$

$$= 1$$

So $m = 1$

Equation:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\pi}{2} = 1(x - 0)$$

$$= x$$

$$2y - \pi = 2x$$

$$0 = 2x - 2y + \pi$$

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You can use the chain rule to differentiate.

EXAMPLE 27

Differentiate $\sin^{-1}(5x - 1)$.

Solution

$y = \sin^{-1}(5x - 1)$ is a composite function.

Let $y = \sin^{-1} u$ where $u = 5x - 1$.

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \quad \text{and} \quad \frac{du}{dx} = 5$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
&= \frac{1}{\sqrt{1-u^2}} \times 5 \\
&= \frac{5}{\sqrt{1-u^2}} \\
&= \frac{5}{\sqrt{1-(5x-1)^2}} \\
&= \frac{5}{\sqrt{1-(25x^2-10x+1)}} \\
&= \frac{5}{\sqrt{-25x^2+10x}}
\end{aligned}$$

There is a simplified chain rule for differentiating $\sin^{-1}\left(\frac{x}{a}\right)$.

Chain rule

$$\frac{d}{dx}\left[\sin^{-}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{a^2 - x^2}}$$

The proof is similar to the proof of the derivative of $\sin^{-1} x$.

EXAMPLE 28

Differentiate $\sin^{-1}\left(\frac{x}{3}\right)$.

Solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}} \text{ where } a = 3$$
$$= \frac{1}{\sqrt{3^2 - x^2}}$$
$$= \frac{1}{\sqrt{9 - x^2}}$$

Derivative of $\cos^{-1} x$

$$\frac{d}{dx}(\cos^{-} x) = -\frac{1}{\sqrt{1-x^2}}$$

Chain rule

$$\frac{d}{dx}\left[\cos^{-}\left(\frac{x}{a}\right)\right] = -\frac{1}{\sqrt{a^2 - x^2}}$$

The proofs of these results are similar to the proof of $\sin^{-1} x$.

Derivative of inverse cosine function

EXAMPLE 29

Differentiate:

a $\cos^{-1}\left(\frac{x}{7}\right)$

b $\cos^{-1} 2x$

Solution

a
$$\frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - x^2}} \text{ where } a = 7$$
$$= -\frac{1}{\sqrt{7^2 - x^2}}$$
$$= -\frac{1}{\sqrt{49 - x^2}}$$

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b $\cos^{-1} 2x = \cos^{-1}\left(\frac{x}{\frac{1}{2}}\right)$

Method 1: Chain rule

$\cos^{-1} 2x$ is a composite function.

Let $y = \cos^{-1} u$ where $u = 2x$

$\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$ and $\frac{du}{dx} = 2$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= -\frac{1}{\sqrt{1-u^2}} \times 2$$

$$= -\frac{2}{\sqrt{1-(2x)^2}}$$

$$= -\frac{2}{\sqrt{1-4x^2}}$$

Method 2: Formula

$\frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}$ where $a = \frac{1}{2}$

$$= -\frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}}$$

$$= -\frac{1}{\sqrt{\frac{1}{4} - x^2}}$$

$$= -\frac{1}{\sqrt{\frac{1-4x^2}{4}}}$$

$$= -\frac{1}{\frac{\sqrt{1-4x^2}}{2}}$$

$$= -\frac{2}{\sqrt{1-4x^2}}$$

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Derivative of $\tan^{-1} x$

$$\frac{d}{dx}(\tan^{-} x) = \frac{1}{1+x^2}$$

Proof

Let $y = \tan^{-1} x$

Then $x = \tan y$

$$\frac{dx}{dy} = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Using $\tan^2 \theta + 1 = \sec^2 \theta$:

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$

$$= \frac{1}{1+x^2}$$

Chain rule

$$\frac{d}{dx}\left[\tan^{-}\left(\frac{x}{a}\right)\right] = \frac{a}{a^2+x^2}$$

The proof is similar to the proof of the derivative of $\tan^{-1} x$.

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EXAMPLE 30

a Find the gradient of the normal to the curve $y = \tan^{-1} x$ at the point where $x = \frac{1}{\sqrt{3}}$.

b Differentiate $\tan^{-1}\left(\frac{x}{5}\right)$.

Solution

a $\frac{dy}{dx} = \frac{1}{1+x^2}$

When $x = \frac{1}{\sqrt{3}}$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{1}{1+\frac{1}{3}}$$

$$= \frac{1}{\frac{3}{3}+\frac{1}{3}}$$

$$= \frac{1}{\frac{4}{3}}$$

$$= \frac{3}{4}$$

So $m_1 = \frac{3}{4}$

Gradient is perpendicular to the normal.

$m_1 m_2 = -1$

$\frac{3}{4} m_2 = -1$

$m_2 = -1 \times \frac{4}{3}$

$= -\frac{4}{3}$

b $\frac{dy}{dx} = \frac{a}{a^2 + x^2}$ where $a = 5$

$$= \frac{5}{5^2 + x^2}$$

$$= \frac{5}{25 + x^2}$$

EXT1 Exercise 5.10 Derivative of inverse trigonometric functions

1 Differentiate:

a $\cos^{-1} x$ **b** $2 \sin^{-1} x$ **c** $\tan^{-1} x$

d $\cos^{-1} 3x$ **e** $4 \sin^{-1} 2x$ **f** $\sin^{-1} (x^2)$

g $\tan^{-1} (2x - 1)$ **h** $5 \cos^{-1} 8x$ **i** $\cos^{-1} \left(\frac{x}{3}\right)$

j $\tan^{-1} \left(\frac{x}{2}\right)$

2 For each function, find the gradient of:

i the tangent **ii** the normal

a $y = \cos^{-1} x$ at the point $\left(0\ \frac{\pi}{2}\right)$

b $y = \tan^{-1} (2x)$ at the point where $x = \frac{1}{4}$

c $f(x) = (\sin^{-1} x)^3$ at the point where $x = \frac{1}{2}$

d $y = \cos^{-1} \left(\frac{x}{3}\right)$ at the point $\left(0\ \frac{\pi}{2}\right)$

e $y = \tan^{-1} \left(\frac{x}{5}\right)$ at the point where $x = 0$

3 Find the equation of the tangent to the curve $y = \sin^{-1} (2x)$ at the point where $x = 0$.

4 Find the equation of the normal to the curve $y = \tan^{-1} 5x$ at $\left(\frac{1}{5}\ \frac{\pi}{4}\right)$.

5 Find the derivative of:

a $3 \sin^{-1} \left(\frac{x}{6}\right)$ **b** $3 \cos^{-1} \sqrt{x}$ **c** $\cos^{-1} \left(\frac{x}{7}\right)$

d $5 \sin^{-1} (3x + 2)$ **e** $x \cos^{-1} x$ **f** $(\tan^{-1} x + 1)^5$

6 Differentiate:

a $\sin^{-1} (\cos x)$ **b** $\cos^{-1} (\cos x)$ **c** $\sin^{-1} (\ln x)$

d $\tan^{-1} (e^x)$ **e** $\ln (\sin^{-1} x)$ **f** $\frac{1}{\tan^{-} x}$

g $\tan^{-1} (\cos^{-1} x + 1)$ **h** $\tan^{-1} \left(\frac{1}{x}\right)$ **i** $\sin^{-1} \left(\frac{x}{2} + 1\right)$

j $e^{\cos^{-} x}$

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7 Show that $\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0$.

8 Find the second derivative of:

a $\cos^{-1}\left(\frac{x}{3}\right)$ **b** $\ln(\tan^{-1} x)$

9 Find the equation of the tangent to the curve $y = \sin^{-1} x$ at the point where $x = -\frac{1}{2}$.

10 a Find $\frac{d}{dx}\left[\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right]$.

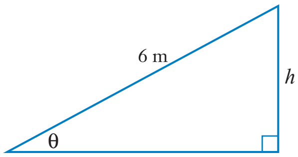
b Draw the graph of $y = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$.

11 Differentiate:

a $\cos^{-1}(e^{2x})$ **b** $\ln(\tan^{-1} x)$ **c** $\tan^{-1}(\ln x)$

d $\sin^{-1}\sqrt{1-x^2}$ **e** $e^{\text{an}^{-}\ x}$

12 A 6 metre long ladder is leaning against a wall at a height of h and angle θ as shown.



a Show that $\theta = \sin^{-1}\left(\frac{h}{6}\right)$.

b The ladder slips down the wall at a constant rate of 0.05 m s^{-1}. Find the rate at which the angle is changing when the height is 2.5 m.

13 A hot air balloon rises into the air at 2 metres per second. Jan is standing 100 m away from the balloon.

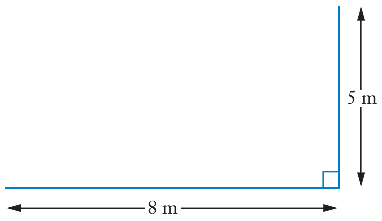
a What is the height of the balloon after t seconds?

b If the angle of elevation from Jan up to the balloon is θ, write an equation for θ in terms of t.

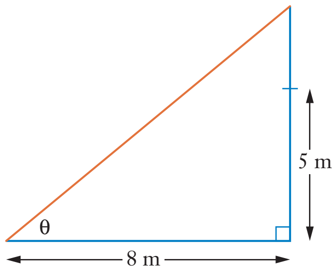
c Find the rate of change in θ (in radians) after:

i 5 seconds **ii** one minute

14 Two walls along a property are 8 m and 5 m long as shown.



A builder extends the 5 m wall as shown at 0.5 metres per minute.



a Write an equation for the angle θ in terms of t.

b Find the rate at which θ is changing after:

i 5 minutes **ii** 20 minutes **iii** an hour

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Summary of differentiation rules

Rule	Chain rule
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-}$
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}[e^{f(x)}] = f'(x)e^{f(x)}$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}[\sin f(x)] = f'(x)\cos f(x)$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}[\cos f(x)] = -f'(x)\sin f(x)$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}[\tan f(x)] = f'(x)\sec^2 f(x)$
EXT1 $\frac{d}{dx}(\sin^{-} x) = \frac{1}{\sqrt{1-x^2}}$	EXT1 $\frac{d}{dx}\left[\sin^{-}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{a^2-x^2}}$
EXT1 $\frac{d}{dx}(\cos^{-} x) = -\frac{1}{\sqrt{1-x^2}}$	EXT1 $\frac{d}{dx}\left[\cos^{-}\left(\frac{x}{a}\right)\right] = -\frac{1}{\sqrt{a^2-x^2}}$
EXT1 $\frac{d}{dx}(\tan^{-} x) = \frac{1}{1+x^2}$	EXT1 $\frac{d}{dx}\left[\tan^{-}\left(\frac{x}{a}\right)\right] = \frac{a}{a^2+x^2}$

Product rule: $\frac{d}{dx}(uv) = u'v + v'u$

Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$

[redacted]

Practice quiz

For Questions **1** to **4**, choose the correct answer **A**, **B**, **C** or **D**.

1 The anti-derivative of $\sin 6x$ is:

A $\frac{1}{6}\cos 6x$ **B** $6\cos 6x$ **C** $-6\cos 6x$ **D** $-\frac{1}{6}\cos 6x$

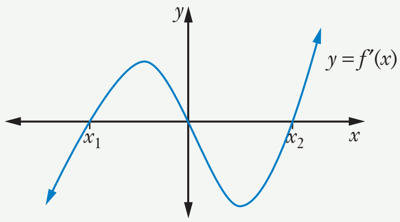
2 The gradient of the tangent to the curve $y = e^{2x} + x$ at $(0, 1)$ is:

A 2 **B** 3 **C** e **D** 1

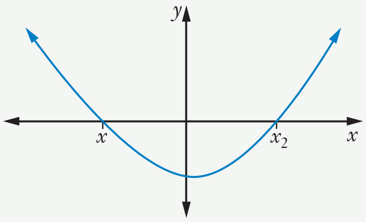
3 If $y = \cos 2x$, then $\frac{d^2y}{dx^2}$ is :

A $-4y$ **B** $-2y$ **C** $4y$ **D** y

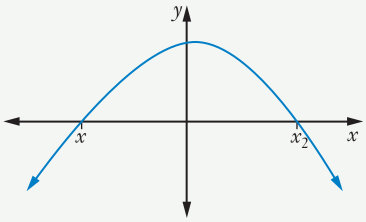
4 The graph of the derivative $y = f'(x)$ is shown. Which of the following graphs could be the graph of $y = f(x)$?



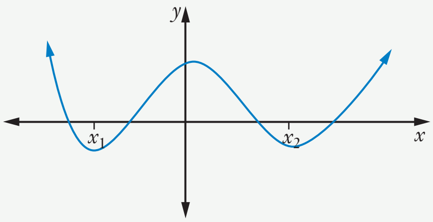
A



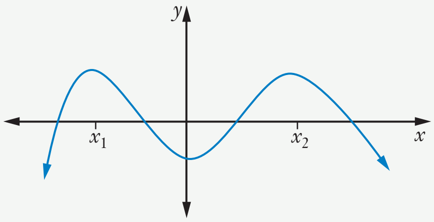
B



C



D



5 Differentiate:

a e^{5x} **b** $2e^{1-x}$ **c** $\ln 4x$ **d** $\ln(4x+5)$

e xe^x **f** $\frac{\ln x}{x}$ **g** $(e^x+1)^{10}$

6 Differentiate:

a $\cos x$ **b** $2\sin x$ **c** $\tan x + 1$ **d** $x \sin x$

e $\frac{\tan x}{x}$ **f** $\cos 3x$ **g** $\tan 5x$

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7 Find the equation of the tangent to the curve $y = 2 + e^{3x}$ at the point where $x = 0$.

8 Find the equation of the tangent to the curve $y = \sin 3x$ at the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

9 If $x = \cos 2t$, show that $\frac{d^2x}{dt^2} = -4x$.

10 EXT1 Differentiate:

a $\sin^{-1} x$ **b** $\tan^{-1} 3x$ **c** $2 \cos^{-1} 5x$

11 Find the exact gradient of the normal to the curve $y = x - e^{-x}$ at the point where $x = 2$.

12 Find the anti-derivative of:

a $10x^4 - 4x^3 + 6x - 3$ **b** e^{5x} **c** $\sec^2 9x$

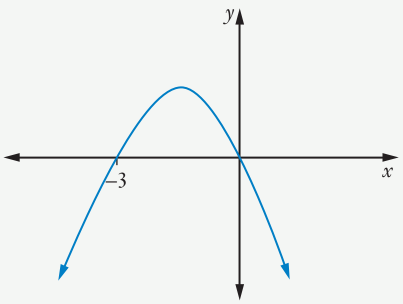
d $\frac{1}{x+5}$ **e** $\cos 2x$ **f** $\sin\left(\frac{x}{4}\right)$

13 Find the gradient of the tangent to the curve $y = 3 \cos 2x$ at the point where $x = \frac{\pi}{6}$.

14 A curve has $\frac{dy}{dx} = 6x^2 + 12x - 5$. If the curve passes through the point $(2, -3)$, find the equation of the curve.

15 EXT1 Find the derivative of $f^{-1}(x)$ if $f(x) = x^5 + 3$.

16 Sketch the graph of the anti-derivative of the following function, given that the anti-derivative passes through $(0, 4)$.



17 Find the equation of the normal to the curve $y = \ln x$ at the point $(2, \ln 2)$.

18 Find the equation of the normal to the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

19 EXT1 Find the derivative of:

a $\sin^{-1} x$ **b** $\cos^{-1}\left(\frac{x}{5}\right)$ **c** $\tan^{-1} x$

d $\sin^{-1} 4x$ **e** $\tan^{-1}\left(\frac{x}{2}\right)$

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20 Differentiate:

a $(5x^2 + 7)^4$ **b** $4x(2x - 3)^7$ **c** $\dfrac{5x-1}{3x+4}$

d $2x^3e^x$ **e** $\dfrac{\tan 3x}{x+1}$

EXT1 **21** **a** Evaluate $\dfrac{d}{dx}(\sin^{-1} x + \cos^{-1} x)$.

b Explain this result.

22 If $f''(x) = 15x + 12$ and $f(2) = f'(2) = 5$, find the equation of $y = f(x)$.

23 If $f(x) = 3x^5 - 2x^4 + x^3 - 2$, find:

a $f(-1)$ **b** $f'(-1)$ **c** $f''(-1)$

EXT1 **24** **a** Find the inverse function $f^{-1}(x)$ given $f(x) = \sqrt{x+1}$.

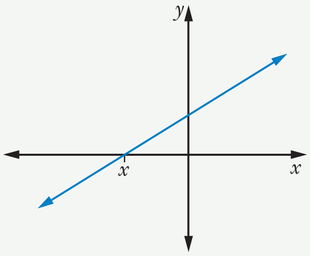
b Find the point P on $f^{-1}(x)$ where $x = 3$.

c Find the equation of the tangent to $f^{-1}(x)$ at P.

25 EXT1 Differentiate $x \tan^{-1} x$.

26 Sketch an example of the graph of an anti-derivative function for each graph.

a



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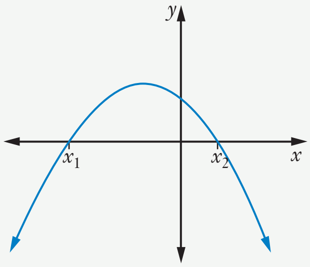
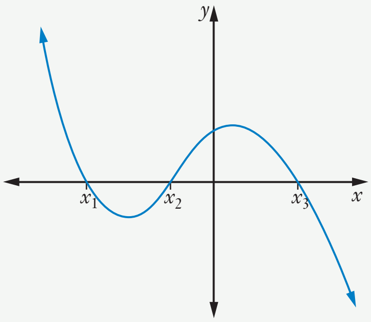


c



27 A function has $f'(3) = 5$ and $f(3) = 2$. If $f''(x) = 12x - 6$, find the equation of the function.

28 EXT1 Find the equation of the tangent to the curve $y = \sin^{-1}\left(\dfrac{x}{3}\right)$ at the point $\left(1\dfrac{1}{2}, \dfrac{\pi}{6}\right)$.

29 Find the anti-derivative of:

a $x^3(3x^4 - 5)^6$ **b** $3x(x^2 + 1)^9$

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5. CHALLENGE EXERCISE

1 Find the exact gradient of the tangent to the curve $y = e^{x + \ln x}$ at the point where $x = 1$.

EXT1 **2** **a** Show that $\tan^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$.

b Find $\frac{d}{dx}\left(\tan^{-1} x + \tan^{-1}\frac{1}{x}\right)$.

c Show that $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$ for all x.

3 Find the first and second derivatives of $\frac{5 - x}{(4x^2 + 1)^3}$.

4 Find the anti-derivative of:

a $2xe^x$ **b** $x^2 \sin(x^3)$

5 Differentiate $e^{x \sin 2x}$.

6 A curve passes through the point (0, −1) and the gradient at any point is given by $(x + 3)(x - 5)$. Find the equation of the curve.

7 EXT1 Differentiate:

a $\sin^{-1}(x^2)$ **b** $\tan^{-1}(e^x)$ **c** $\ln(\sin x + \cos x)$

8 The rate of change of V with respect to t is given by $\frac{dV}{dt} = (2t - 1)^2$.
If $V = 5$ when $t = \frac{1}{2}$, find V when $t = 3$.

9 Find the derivative of $y = \frac{x \log_e x}{e^x}$.

10 EXT1 A car is stopped at point A, 20 km south of an intersection O. Another car leaves the intersection and travels east at 80 km h^{-1}. If this car is at point B:

a Find an equation for angle OAB after t hours.

b Find the rate at which angle OAB is changing after 2 hours (in degrees and minutes per hour, to the nearest minute).

EXT1 **11** **a** Find the inverse function f^{-1} in terms of y given $f(x) = x + e^x$.

b Find the image P on f^{-1} of the point where $f(1) = 1 + e$.

c Find the equation of the tangent to f^{-1} at P.

12 **a** Differentiate $\ln(\tan x)$.

b Find the anti-derivative of $\tan x$.

13 Find the anti-derivative of:

a $x^2 \sin(x^3 - \pi)$ **b** xe^x

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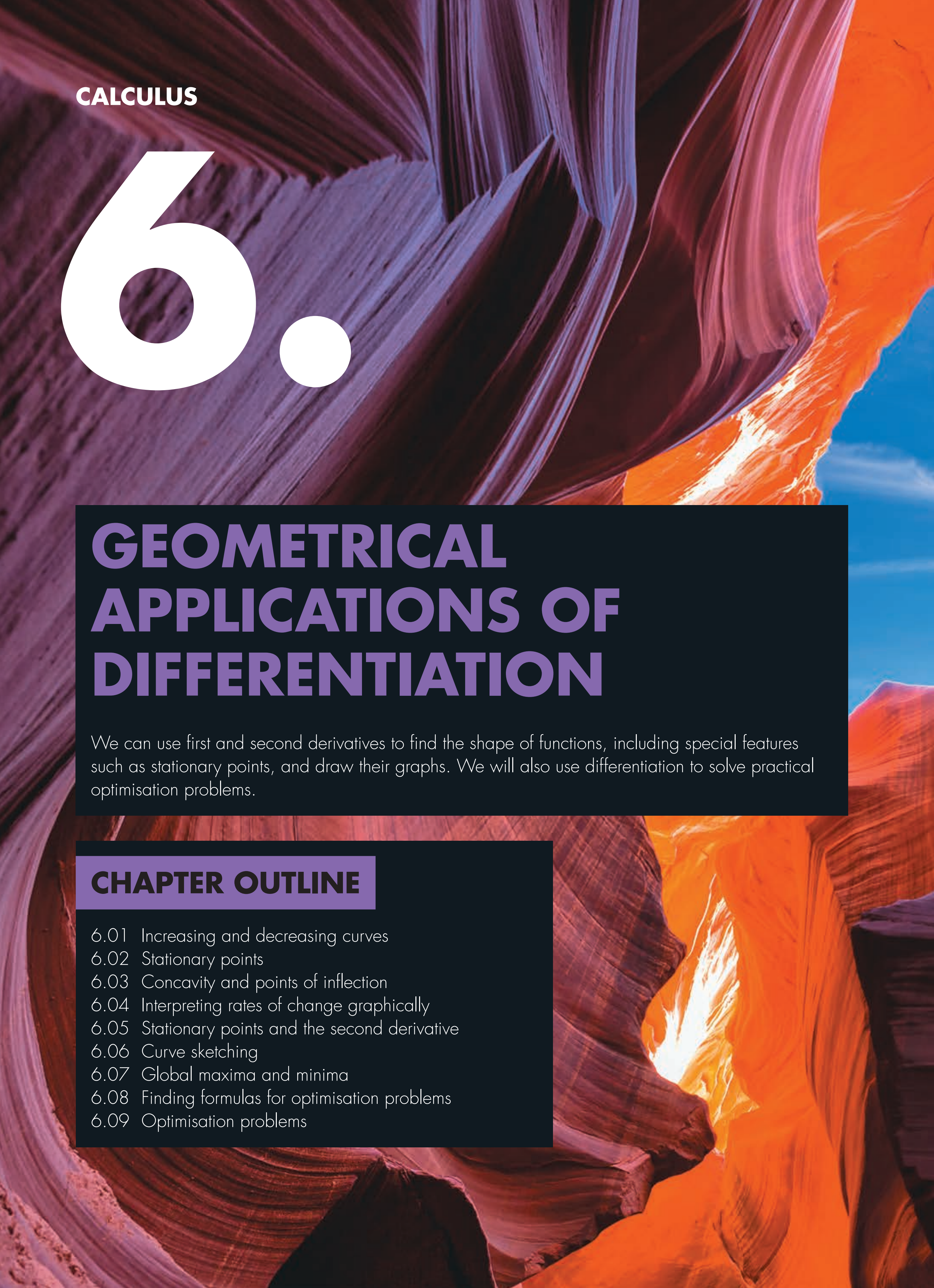


CALCULUS

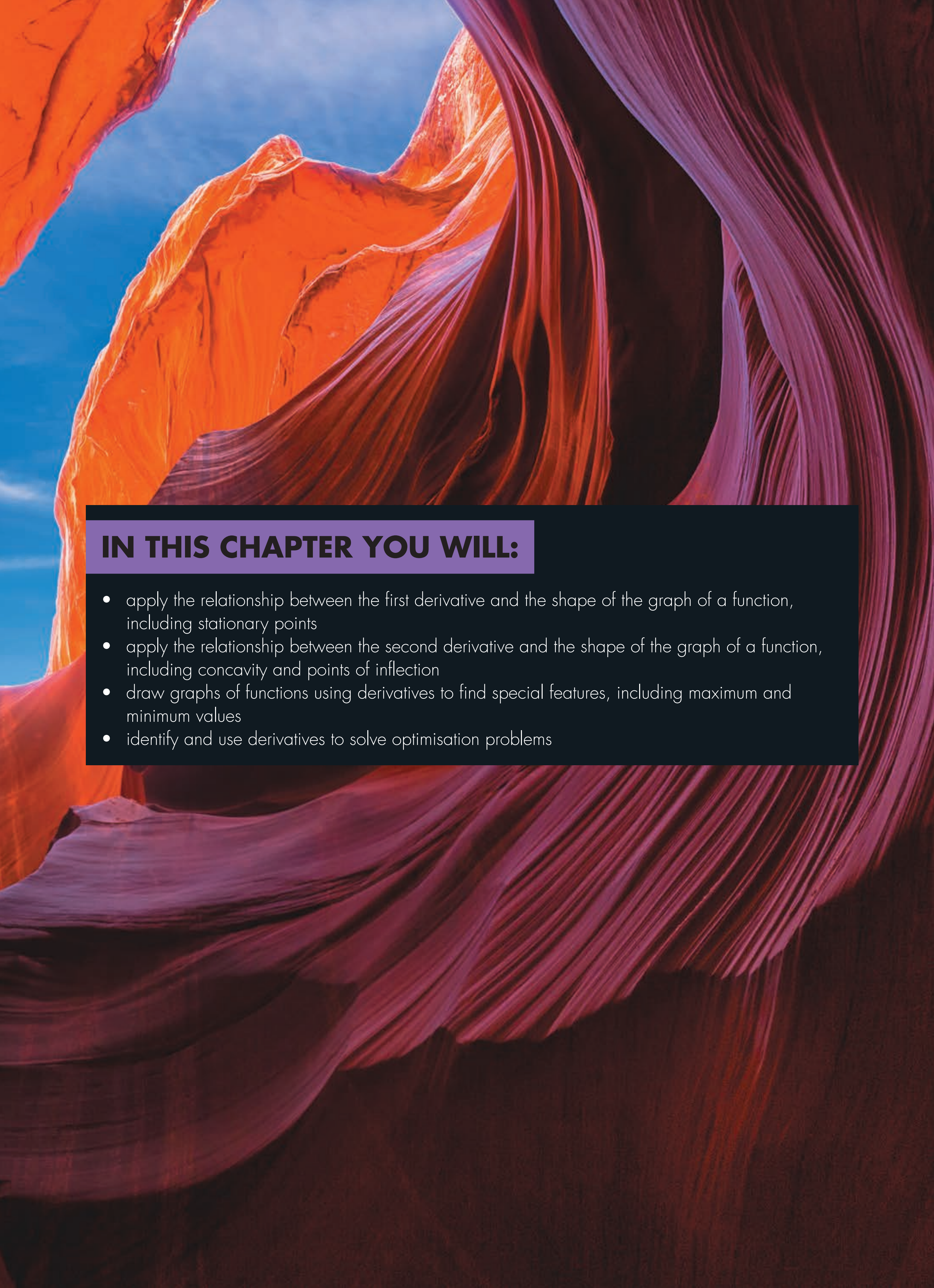
6.

GEOMETRICAL APPLICATIONS OF DIFFERENTIATION

We can use first and second derivatives to find the shape of functions, including special features such as stationary points, and draw their graphs. We will also use differentiation to solve practical optimisation problems.

CHAPTER OUTLINE

6.01 Increasing and decreasing curves
6.02 Stationary points
6.03 Concavity and points of inflection
6.04 Interpreting rates of change graphically
6.05 Stationary points and the second derivative
6.06 Curve sketching
6.07 Global maxima and minima
6.08 Finding formulas for optimisation problems
6.09 Optimisation problems



IN THIS CHAPTER YOU WILL:

- apply the relationship between the first derivative and the shape of the graph of a function, including stationary points
- apply the relationship between the second derivative and the shape of the graph of a function, including concavity and points of inflection
- draw graphs of functions using derivatives to find special features, including maximum and minimum values
- identify and use derivatives to solve optimisation problems

TERMINOLOGY

concavity: The shape of a curve as it bends; it can be concave up or concave down.

global maximum or minimum: The absolute highest or lowest value of a function over a given domain.

horizontal point of inflection: A stationary point where the concavity of the curve changes.

local maximum or minimum: a relatively high or low value of a function shown graphically as a turning point.

maximum point: A stationary point where the curve reaches a peak.

minimum point: A stationary point where the curve reaches a trough.

monotonic increasing or **decreasing:** A function that is always increasing or decreasing.

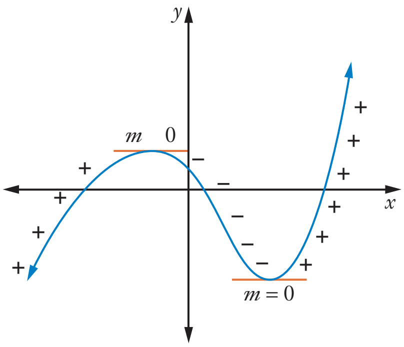
point of inflection: A point at which the curve is neither concave upwards nor downwards, but where the concavity changes.

stationary point: A point on the graph of $y = f(x)$ where the tangent is horizontal and its gradient $f'(x) = 0$. It could be a maximum point, minimum point or a horizontal point of inflection.

6.01 Increasing and decreasing curves

WS The sign o he derivative

You have already seen how the derivative describes the shape of a curve.



Sign of the first derivative

If $f'(x) > 0$, the graph of $y = f(x)$ is increasing.

If $f'(x) < 0$, the graph of $y = f(x)$ is decreasing.

If $f'(x) = 0$, the graph of $y = f(x)$ has a **stationary point**.

Sometimes a curve is **monotonic increasing** or **decreasing** (*always* increasing or decreasing).

Monotonic increasing or decreasing functions

A curve is monotonic increasing if $f'(x) > 0$ for all x.

A curve is monotonic decreasing if $f'(x) < 0$ for all x.

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EXAMPLE 1

a Find all x values for which the curve $f(x) = x^2 - 4x + 1$ is increasing.

b Find any stationary points on the curve $y = x^3 - 48x - 7$.

Solution

a $f'(x) = 2x - 4$

For increasing curve:

$f'(x) > 0$

$2x - 4 > 0$

$2x > 4$

$x > 2$

So the curve is increasing for $x > 2$.

b $y' = 3x^2 - 48$

For stationary points:

$y' = 0$

$3x^2 - 48 = 0$

$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

When $x = 4$:

$y = 4^3 - 48(4) - 7$

$= -135$

When $x = -4$:

$y = (-4)^3 - 48(-4) - 7$

$= 121$

So the stationary points are $(4, -135)$ and $(-4, 121)$.

Exercise 6.01 Increasing and decreasing curves

1 For what x values is the function $f(x) = -2x^2 + 8x - 1$ increasing?

2 Find all values of x for which the curve $y = 2x^2 - x$ is decreasing.

3 Find the domain over which the function $f(x) = 4 - x^2$ is increasing.

4 Find values of x for which the curve $y = x^2 - 3x - 4$ is:

a decreasing **b** increasing **c** stationary

5 Show that the function $f(x) = -2x - 7$ is always (monotonic) decreasing.

6 Prove that $y = x^3$ is monotonic increasing for all $x \neq 0$.

7 Find the stationary point on the curve $f(x) = x^3$.

8 Find all x values for which the curve $y = 2x^3 + 3x^2 - 36x + 9$ is stationary.

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9 Find all stationary points on the curve:

a $y = x^2 - 2x - 3$ **b** $f(x) = 9 - x^2$

c $y = 2x^3 - 9x^2 + 12x - 4$ **d** $y = x^4 - 2x^2 + 1$

10 Find any stationary points on the curve $y = (x - 2)^4$.

11 EXT1 Find all values of x for which the curve $f(x) = x^3 - 3x + 4$ is decreasing.

12 EXT1 Find the domain over which the curve $y = x^3 + 12x^2 + 45x - 30$ is increasing.

13 Find any values of x for which the curve $y = 2x^3 - 21x^2 + 60x - 3$ is:

a stationary **b** EXT1 decreasing **c** EXT1 increasing

14 The function $f(x) = 2x^2 + px + 7$ has a stationary point at $x = 3$. Evaluate p.

15 Evaluate a and b if $y = x^3 - ax^2 + bx - 3$ has stationary points at $x = -1$ and $x = 2$.

16 **a** Find the derivative of $y = x^3 - 3x^2 + 27x - 3$.

b Show that the curve is monotonic increasing for all values of x.

17 Sketch a function with $f'(x) > 0$ for $x < 2$, $f'(2) = 0$ and $f'(x) < 0$ when $x > 2$.

18 Sketch a curve with $\frac{dy}{dx} < 0$ for $x < 4$, $\frac{dy}{dx} = 0$ when $x = 4$ and $\frac{dy}{dx} > 0$ for $x > 4$.

19 Sketch a curve with $\frac{dy}{dx} > 0$ for all $x \neq 1$ and $\frac{dy}{dx} = 0$ when $x = 1$.

20 Sketch a function that has $f'(x) > 0$ in the domain $(-\infty, -2) \cup (5, \infty)$, $f'(x) = 0$ for $x = -2$ and $x = 5$, and $f'(x) < 0$ in the domain $(-2, 5)$.

21 A function has $f(3) = 2$ and $f'(3) < 0$. Show this information on a sketch.

22 The derivative of a function is positive at the point $(-2, -1)$. Show this information on a graph.

23 Find the stationary points on the curve $y = (3x - 1)(x - 2)^4$.

24 Differentiate $y = x\sqrt{x+1}$. Hence find the stationary point on the curve, giving the exact coordinates.

25 The curve $f(x) = ax^4 - 2x^3 + 7x^2 - x + 5$ has a stationary point at $x = 1$. Find the value of a.

26 Show that $f(x) = \sqrt{x}$ has no stationary points.

27 Show that $f(x) = \frac{1}{x^3}$ has no stationary points.

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6.02 Stationary points

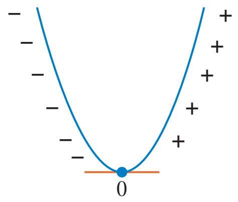
WS
Saionay points

In Year 11, Chapter 8, *Introduction to calculus*, you learned about 3 types of stationary points: minimum point, maximum point and horizontal point of inflection.

Minimum and maximum turning points

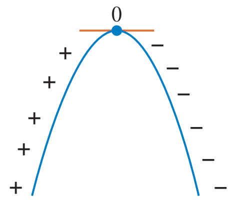
At a local **minimum point**, the curve is decreasing on the LHS and increasing on the RHS.

x	LHS	Minimum	RHS
$f'(x)$	< 0	0	> 0

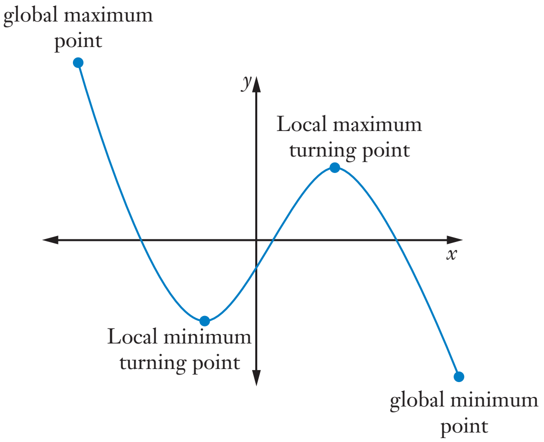


At a local **maximum point**, the curve is increasing on the LHS and decreasing on the RHS.

x	LHS	Minimum	RHS
$f'(x)$	> 0	0	< 0



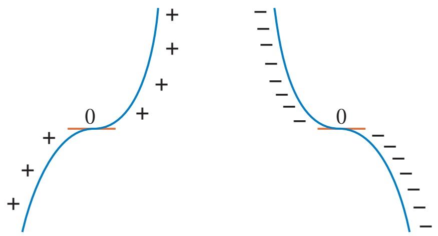
These stationary points are called **local maximum or minimum** points because they are not necessarily the **global maximum or minimum** points on the curve.



Horizontal point of inflection

These curves are increasing or decreasing on **both** sides of the horizontal **point of inflection**.
It is not a turning point since the curve does not turn around at this point.

We will learn more about points of inflection in the next section.



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EXAMPLE 2

Find any stationary points on the curve $f(x) = 2x^3 - 15x^2 + 24x - 7$ and determine their nature.

deterine thir natur' means ind what type of stationary point they are

Solution

$f'(x) = 6x^2 - 30x + 24$

For stationary points:

$$\begin{aligned} f'(x) &= 0 \\ 6x^2 - 30x + 24 &= 0 \\ 6(x^2 - 5x + 4) &= 0 \\ x^2 - 5x + 4 &= 0 \\ (x-1)(x-4) &= 0 \\ x = 1, \quad x &= 4 \end{aligned}$$

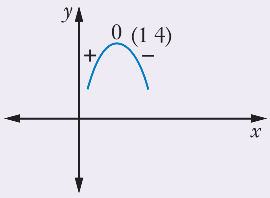
So there are 2 stationary points, where $x = 1$ and $x = 4$.

$$\begin{aligned} f(1) &= 2(1)^3 - 15(1)^2 + 24(1) - 7 \\ &= 4 \end{aligned}$$

So (1, 4) is a stationary point.

To determine its nature, choose a point close to (1, 4) on the LHS and RHS, for example, $x = 0$ and $x = 2$, and test the sign of $f'(x)$.

x	0	1	2
$f'(x)$	24	0	–12
	+		–



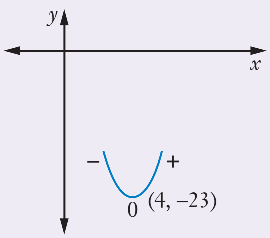
Positive to negative, so (1, 4) is a maximum point.

$$\begin{aligned} f(4) &= 2(4)^3 - 15(4)^2 + 24(4) - 7 \\ &= -23 \end{aligned}$$

So (4, –23) is a stationary point.

To determine its nature, choose, for example, $x = 2$ and $x = 5$.

x	2	4	5
$f'(x)$	–12	0	24
	–		+



Negative to positive, so (4, –23) is a minimum point.

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Exercise 6.02 Stationary points

1 Find the stationary point on the curve $y = x^2 - 1$ and show that it is a minimum point.

2 Find the stationary point on the curve $y = x^4$ and determine its type.

3 The function $f(x) = 7 - 4x - x^2$ has one stationary point. Find its coordinates and show that it is a maximum turning point.

4 Find the turning point on the curve $y = 3x^2 + 6x + 1$ and determine its nature.

5 For the curve $y = (4 - x)^2$, find the turning point and determine its nature.

6 The curve $y = x^3 - 6x^2 + 5$ has 2 turning points. Find them and use the derivative to determine their nature.

7 Find the turning points on the curve $y = x^3 - 3x^2 + 5$ and determine their nature.

8 Find any stationary points on the curve $f(x) = x^4 - 2x^2 - 3$. What type of stationary points are they?

9 The curve $y = x^3 - 3x + 2$ has 2 stationary points. Find their coordinates and determine their type.

10 The curve $y = x^5 + mx^3 - 2x^2 + 5$ has a stationary point at $x = -1$. Find the value of m.

11 For a certain function, $f'(x) = 3 + x$. For what value of x does the function have a stationary point? What type of stationary point is it?

12 A curve has $f'(x) = x(x + 1)$. For what x values does the curve have stationary points? What type are they?

13 **a** Differentiate $P = 2x + \dfrac{50}{x}$ with respect to x.

b Find any stationary points on the curve and determine their nature.

14 For the function $A = \dfrac{h^2 - 2h + 5}{8}$, find any stationary points and determine their nature.

15 Find any stationary points on the function $V = 40r - \pi r^3$ correct to 2 decimal places, and determine their nature.

16 Find any stationary points on the curve $S = 2\pi r + \dfrac{120}{r}$ correct to 2 decimal places, and determine their nature.

17 **a** Differentiate $A = x\sqrt{3600 - x^2}$.

b Find any stationary points on $A = x\sqrt{3600 - x^2}$ (to 1 decimal place) and determine their nature.

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Concaviy

Shape o curves

6.03 Concavity and points of inflection

The first derivative $f'(x)$ is the rate of change of the function $y = f(x)$.

Similarly, the second derivative $f''(x)$ is the rate of change of the first derivative $f'(x)$.

This means the relationship between $f''(x)$ and $f'(x)$ is the same as the relationship between $f'(x)$ and $f(x)$.

Relationship between 1st and 2nd derivatives

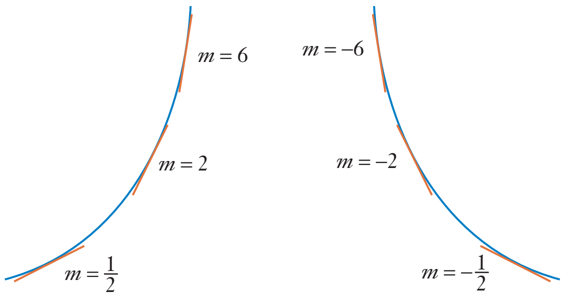
If $f''(x) > 0$ then $f'(x)$ is increasing.

If $f''(x) < 0$ then $f'(x)$ is decreasing.

If $f''(x) = 0$ then $f'(x)$ is stationary.

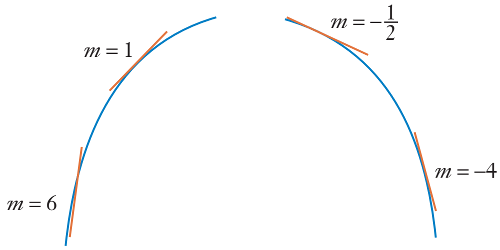
The sign of the second derivative shows the shape of the graph.

If $f''(x) > 0$ then $f'(x)$ is increasing. This means that the gradient of the tangent is increasing.



Notice the upward shape of these curves. The curve lies above the tangents. We say that the curve is **concave upwards**.

If $f''(x) < 0$ then $f'(x)$ is decreasing. This means that the gradient of the tangent is decreasing.



Notice the downward shape of these curves. The curve lies below the tangents. We say that the curve is **concave downwards**.

Sign of 2nd derivative

If $f''(x) > 0$, the curve is concave upwards.

If $f''(x) < 0$, the curve is concave downwards.

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EXAMPLE 3

Find the domain over which the curve $f(x) = 2x^3 - 7x^2 - 5x + 4$ is concave downwards.

Solution

$f'(x) = 6x^2 - 14x - 5$

$f''(x) = 12x - 14$

For concave downwards:

$f''(x) < 0$

$12x - 14 < 0$

$12x < 14$

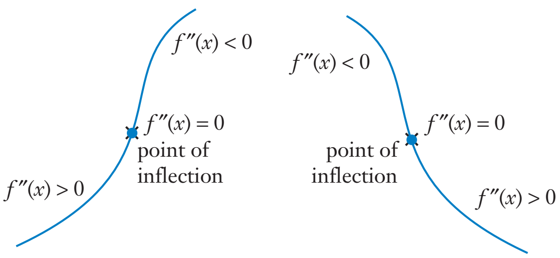
$x < \frac{14}{12}$

$x < 1\frac{1}{6}$

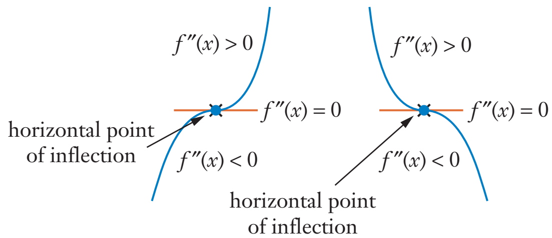
So the domain over which the curve is concave downwards is $(-\infty, 1\frac{1}{6})$

Points of inflection

At the point where $f''(x) = 0$, $f'(x)$ is constant. This means that the gradient of the tangent is neither increasing nor decreasing. This happens when the curve goes from being concave upwards to concave downwards, or concave downwards to concave upwards. We say that the curve is changing **concavity** at a **point of inflection**. The diagrams above show a point of inflection and the change in concavity as the curve changes shape.



The diagrams on the right show a **horizontal point of inflection** that occurs at a stationary point. Notice that the tangent is horizontal at the point of inflection.



Points of inflection

If $f''(x) = 0$, and concavity changes, it is a **point of inflection**.

If $f'(x) = 0$ also, it is a **horizontal point of inflection.**

EXAMPLE 4

a Find the point of inflection on the curve $y = x^3 - 6x^2 + 5x + 9$.

b Does the function $y = x^4$ have a point of inflection?

Solution

a $y' = 3x^2 - 12x + 5$

$y'' = 6x - 12$

For point of inflection, $y'' = 0$ and concavity changes.

$6x - 12 = 0$

$x = 2$

Check that concavity changes by choosing values on the LHS and RHS, for example $x = 1$ and $x = 3$, and testing the sign of the second derivative y''.

x	1	2	3
y''	−6	0	6
	−		+

Since concavity changes (negative to positive), there is a point of inflection at $x = 2$.

When $x = 2$:

$y = 2^3 - 6(2)^2 + 5(2) + 9$

$= 3$

So (2, 3) is a point of inflection.

b $\frac{dy}{dx} = 4x^3$

$\frac{d^2y}{dx^2} = 12x^2$

For point of inflection, $\frac{d^2y}{dx^2} = 0$ and concavity changes.

$12x^2 = 0$

$x^2 = 0$

$x = 0$

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Check that concavity changes by choosing values on the LHS and RHS, for example, $x = \pm 1$, and test the sign of $\frac{d^2y}{dx^2}$.

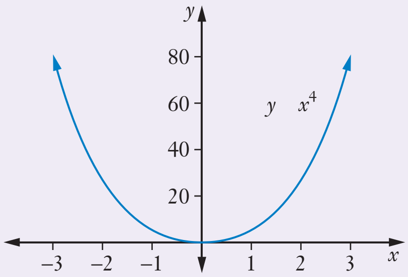
x	-1	0	1
$\frac{d^2y}{dx^2}$	12	0	12
	+		+

Since concavity doesn't change (both sides are positive), (0, 0) is not a point of inflection.

So $y = x^4$ does not have a point of inflection.

We can see this by drawing the graph of $y = x^4$.

This graph has a turning point at (0, 0).



Exercise 6.03 Concavity and points of inflection

1 For what values of x is the curve $y = x^3 + x^2 - 2x - 1$ concave upwards?

2 Find all values of x for which the function $f(x) = (x - 3)^3$ is concave downwards.

3 Prove that the curve $y = 8 - 6x - 4x^2$ is always concave downwards.

4 Show that the curve $y = x^2$ is always concave upwards.

5 Find the domain over which the curve $f(x) = x^3 - 7x^2 + 1$ is concave downwards.

6 Find any points of inflection on the curve $g(x) = x^3 - 3x^2 + 2x + 9$.

7 Find the points of inflection on the curve $y = x^4 - 6x^2 + 12x - 24$.

8 Find the stationary point on the curve $y = x^3 - 2$ and show that it is a point of inflection.

9 EXT1 Find all values of x for which the function $f(x) = x^4 + 2x^3 - 12x^2 + 12x - 1$ is concave downwards.

10 Determine whether there are any points of inflection on the curve:

a $y = x^6$ **b** $y = x^7$ **c** $y = x^5$

d $y = x^9$ **e** $y = x^{12}$

11 Sketch a curve that is always concave up.

12 Sketch a curve where $f''(x) < 0$ for $x > 1$ and $f''(x) > 0$ for $x < 1$.

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13 Find any points of inflection on the curve $y = x^4 - 8x^3 + 24x^2 - 4x - 9$.

14 Show that $f(x) = \frac{2}{x^2}$ is concave upwards for all $x \neq 0$.

15 For the function $f(x) = 3x^5 - 10x^3 + 7$:

- **a** Find any points of inflection.
- **b** Find which of these points are horizontal points of inflection (stationary points).

16 **a** Show that the curve $y = x^4 + 12x^2 - 20x + 3$ has no points of inflection.

- **b** Describe the concavity of the curve.

17 If $y = ax^3 - 12x^2 + 3x - 5$ has a point of inflection at $x = 2$, evaluate a.

18 Evaluate p if $f(x) = x^4 - 6px^2 - 20x + 11$ has a point of inflection at $x = -2$.

19 The curve $y = 2ax^4 + 4bx^3 - 72x^2 + 4x - 3$ has points of inflection at $x = 2$ and $x = -1$. Find the values of a and b.

20 The curve $y = x^6 - 3x^5 + 21x - 8$ has 2 points of inflection.

- **a** Find these points of inflection.
- **b** Show that they are not stationary points.

WS
The shape o a curve

6.04 Interpreting rates of change graphically

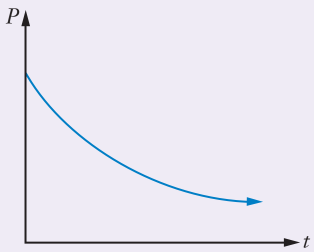
We can find out more about the shape of a graph if we combine the results from the first and second derivatives.

EXAMPLE 5

a For a particular curve, $f(2) = -1$, $f'(2) > 0$ and $f''(2) < 0$. Draw the shape of the curve at this point.

b The curve below shows the population (P) of unemployed people over time t months.

- **i** Describe the sign of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$.
- **ii** How is the population of unemployed people changing over time?
- **iii** Is the rate of change of unemployment increasing or decreasing?



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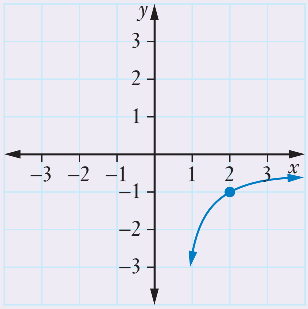
Solution

a $f(2) = -1$ means that the point $(2, -1)$ lies on the curve.

If $f'(2) > 0$, the curve is increasing at this point.

If $f''(2) < 0$, the curve is concave downwards at this point.



b **i** The curve is decreasing, so $\frac{dP}{dt} < 0$.

The curve is concave upwards, so $\frac{d^2P}{dt^2} > 0$.

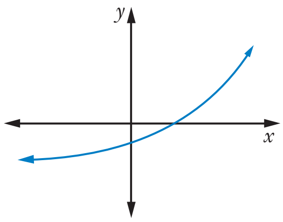
ii Since the curve is decreasing, the number of unemployed people is decreasing.

iii Since the curve is concave upwards, the (negative) gradient is increasing. This means that the rate of change of unemployment is increasing.

Exercise 6.04 Interpreting rates of change graphically

1 For each curve, describe the sign of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

a



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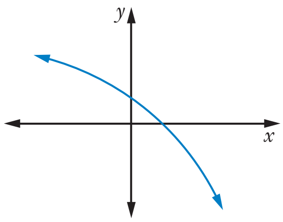

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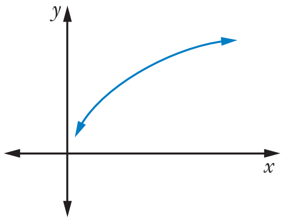

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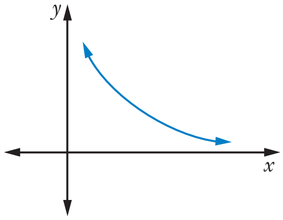
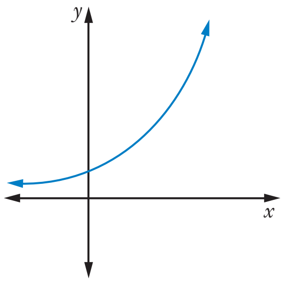
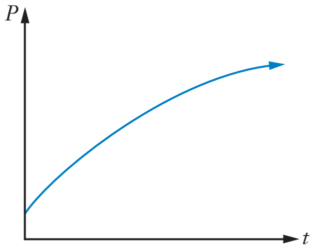


e



2 The curve below shows the population of a colony of sea lions.



a Describe the sign of the first and second derivatives.

b Is the rate of change of the sea lion population increasing?

3 Inflation is increasing, but the rate of increase is slowing. Draw a graph to show this trend.

4 Draw a sketch to show the shape of each curve.

a $f'(x) < 0$ and $f''(x) < 0$

b $f'(x) > 0$ and $f''(x) < 0$

c $f'(x) < 0$ and $f''(x) > 0$

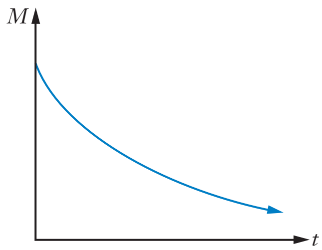
d $f'(x) > 0$ and $f''(x) > 0$

5 The size of classes at a local TAFE college is decreasing and the rate at which this is happening is decreasing. Draw a graph to show this.

6 As an iceblock melts, the rate at which it melts increases. Draw a graph to show this information.

7 The graph shows the decay of a radioactive substance.

Describe the sign of $\dfrac{dM}{dt}$ and $\dfrac{d^2M}{dt^2}$.



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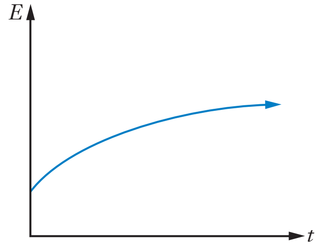
8 The population P of fish in a certain lake was studied over time. At the start of the study the number of fish was 2500.

a During the study, $\frac{dP}{dt} < 0$. What does this say about the number of fish during the study?

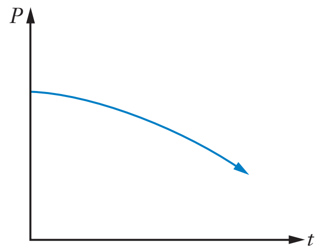
b If at the same time, $\frac{d^2P}{dt^2} > 0$, what can you say about the population rate of change?

c Sketch the graph of the population P against t.

9 The graph shows the level of education of youths in a certain rural area over the past 100 years. Describe how the level of education has changed over this period of time. Include mention of the rate of change.



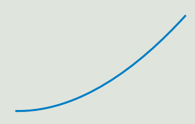
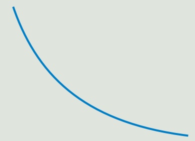
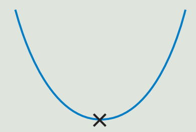
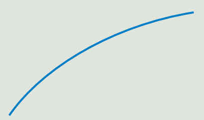
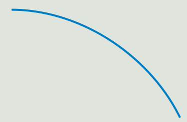
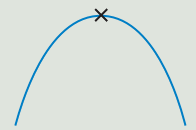
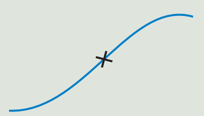
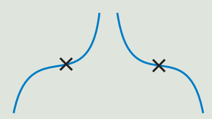
10 The graph shows the number of students in a high school over several years. Describe how the school population is changing over time, including the rate of change.



6.05 Stationary points and the second derivative

Putting the first and second derivatives together gives this summary of the shape of a curve.

Shape of a curve and the derivatives

	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$
$\frac{d^2y}{dx^2} > 0$			
$\frac{d^2y}{dx^2} < 0$			
$\frac{d^2y}{dx^2} = 0$			

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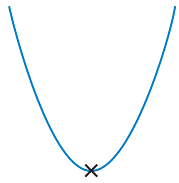
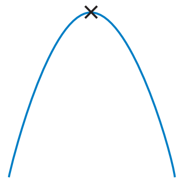
We can use the table to find the requirements for stationary points.

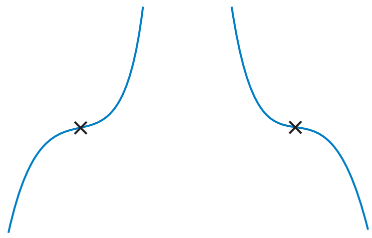
If $f'(x) = 0$ and $f''(x) > 0$, there is a minimum turning point (concave upwards).



If $f'(x) = 0$ and $f''(x) < 0$, there is a maximum turning point (concave downwards).



If $f'(x) = 0$ and $f''(x) = 0$ and concavity changes, then there is a horizontal point of inflection.



Stationary points and the derivatives

Minimum turning point: $f'(x) = 0$ and $f''(x) > 0$

Maximum turning point: $f'(x) = 0$ and $f''(x) < 0$

Horizontal point of inflection: $f'(x) = 0$, $f''(x) = 0$ and concavity changes

Now we can use the second derivative to determine the nature of stationary points.

EXAMPLE 6

a Find the stationary points on the curve $f(x) = 2x^3 - 3x^2 - 12x + 7$ and distinguish between them.

b Find the stationary point on the curve $y = 2x^5 - 3$ and determine its nature.

Solution

a $f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

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For stationary points:

$$f'(x) = 0$$
$$6x^2 - 6x - 12 = 0$$
$$x^2 - x - 2 = 0$$
$$(x + 1)(x - 2) = 0$$
$$x = -1, x = 2$$

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 7$$
$$= 14$$
$$f''(-1) = 12(-1) - 6$$
$$= -18$$
$$< 0 \text{ (concave downwards)}$$

So (–1, 14) is a maximum turning point.

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 7$$
$$= -13$$
$$f''(2) = 12(2) - 6$$
$$= 18$$
$$> 0 \text{ (concave upwards)}$$

So (2, –13) is a minimum turning point.

b $y' = 10x^4$

$y'' = 40x^3$

For stationary points:

$$y' = 0$$
$$10x^4 = 0$$
$$x^4 = 0$$
$$x = 0$$

When $x = 0$:

$$y = 2(0)^5 - 3$$
$$= -3$$
$$y'' = 40(0)^3$$
$$= 0$$

Check that concavity changes by choosing values on the LHS and RHS, for example, $x = \pm 1$.

x	–1	0	1
y''	–40	0	40
	–		+

Since concavity changes, (0, –3) is a horizontal point of inflection.

The table also tells us that the curve changes from concave downwards to concave upwards.

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Exercise 6.05 Stationary points and the second derivative

1 Find the stationary point on the curve $y = x^2 - 2x + 1$ and determine its nature.

2 Find the stationary point on the curve $f(x) = 3x^4 + 1$ and determine what type of point it is.

3 Find the stationary point on the curve $y = 3x^2 - 12x + 7$ and show that it is a minimum turning point.

4 Determine the stationary point on $y = x - x^2$ and show that it is a maximum point.

5 Show that $f(x) = 2x^3 - 5$ has a horizontal point of inflection and find its coordinates.

6 Does the function $f(x) = 2x^5 + 3$ have a stationary point? If it does, determine its nature.

7 Find any stationary points on $f(x) = 2x^3 + 15x^2 + 36x - 50$ and determine their nature.

8 Find the stationary points on the curve $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ and determine whether they are maximum or minimum points.

9 Find any stationary points on the curve $y = (4x^2 - 1)^4$ and determine their nature.

10 **a** Find any stationary points on the curve $y = 2x^3 - 27x^2 + 120x$ and distinguish between them.

b Find any points of inflection on the curve.

11 Find any stationary points on the curve $y = (x - 3)\sqrt{4 - x}$ and determine their nature.

12 Find any stationary points on the curve $f(x) = x^4 + 8x^3 + 16x^2 - 1$ and determine their nature.

13 The curve $y = ax^2 - 4x + 1$ has a stationary point where $x = \frac{1}{2}$.

a Find the value of a.

b Hence, or otherwise, find the stationary point and determine its nature.

14 The curve $y = x^3 - mx^2 + 5x - 7$ has a stationary point where $x = -1$. Find the value of m.

15 The curve $y = ax^3 + bx^2 - x + 5$ has a point of inflection at $(1, -2)$. Find the values of a and b.

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6.06 Curve sketching

We can sketch the graph of a function by using special features such as intercepts, stationary points and points of inflection. Here is a summary of strategies for sketching a curve.

Sketching curves

- Find stationary points $\left(\frac{dy}{dx} = 0\right)$, and determine their nature.
- Find points of inflection $\left(\frac{d^2y}{dx^2} = 0\right)$, and check that concavity changes.
- Find any x-intercepts ($y = 0$), and y-intercepts ($x = 0$).
- Find domain and range.
- Find any asymptotes or other discontinuities.
- Find limiting behaviour of the function.
- Use the symmetry of the function where possible:
  - check if the function is even: $f(-x) = f(x)$
  - check if the function is odd: $f(-x) = -f(x)$

Cuve sketching

Futhr crve sketching

Cuve sketching

Cuve sketching with derivatives

Futhr graphs

Curve sketching

Curve sketching

EXAMPLE 7

a Find any stationary points and points of inflection on the curve $f(x) = x^3 - 3x^2 - 9x + 1$ and hence sketch the curve.

b Sketch the curve of the composite function $y = f(g(x))$ where $f(x) = 2x + 1$ and $g(x) = x^3$, showing any important features.

Solution

a $f'(x) = 3x^2 - 6x - 9$

$f''(x) = 6x - 6$

For stationary points:

$$f'(x) = 0$$

$$3x^2 - 6x - 9 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3, \qquad x = -1$$

$$f(3) = 3^3 - 3(3)^2 - 9(3) + 1$$
$$= -26$$

$$f''(3) = 6(3) - 6$$
$$= 12$$
$$> 0 \quad \text{(concave upwards)}$$

So $(3, -26)$ is a minimum turning point.

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 1$$
$$= 6$$
$$f''(-1) = 6(-1) - 6$$
$$= -12$$
$$< 0 \text{ (concave downwards)}$$

So $(-1, 6)$ is a maximum turning point.

For points of inflection:

$$f''(x) = 0$$
$$6x - 6 = 0$$
$$6x = 6$$
$$x = 1$$

Check concavity changes by choosing values on LHS and RHS e.g. $x = 0$ and $x = 2$.

x	0	1	2
$f''(x)$	−6	0	6

Since concavity changes, $x = 1$ is at a point of inflection.

$$f(1) = 1^3 - 3(1)^2 - 9(1) + 1$$
$$= -10$$

So $(1, -10)$ is a point of inflection.

For x-intercept, $y = 0$:

$$0 = x^3 - 3x^2 - 9x + 1$$

This has no factors so we can't find the x-intercepts.

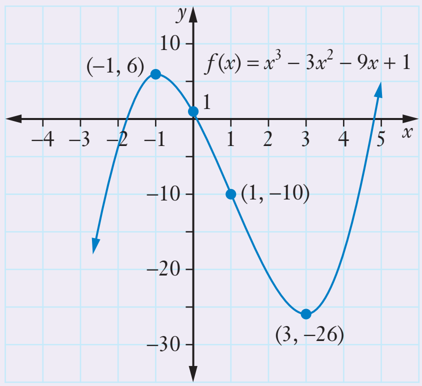
For y-intercept, $x = 0$:

$$f(0) = 0^3 - 3(0)^2 - 9(0) + 1$$
$$= 1$$

$f(x) = x^3 - 3x^2 - 9x + 1$ is a cubic function with no symmetry or discontinuities.

It is not an even or odd function.

Notice that the point of inflection at $(1, -10)$ is not a stationary point. It is the point where the graph naturally changes concavity.



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b $y = f(g(x))$

$= 2x^3 + 1$

$\dfrac{dy}{dx} = 6x^2$

$\dfrac{d^2y}{dx^2} = 12x$

For stationary points:

$\dfrac{dy}{dx} = 0$

$6x^2 = 0$

$x^2 = 0$

$x = 0$

When $x = 0$:

$y = 2(0)^3 + 1$

$= 1$

$\dfrac{d^2y}{dx^2} = 12(0)$

$= 0$

Check concavity either side:

x	-1	0	1
$\dfrac{d^2y}{dx^2}$	-12	0	12

Since concavity changes, (0, 1) is a horizontal point of inflection.

For x-intercepts, $y = 0$

$0 = 2x^3 + 1$

$-1 = 2x^3$

$-0.5 = x^3$

$\sqrt[3]{-0\ 5} = x$

$-0.8 \approx x$

For y-intercept, $x = 0$

$y = 2(0)^3 + 1$

$= 1$

This is (0, 1), the point of inflection.

This is a cubic function. We can make the graph more accurate by finding some extra points.

When $x = -1$:

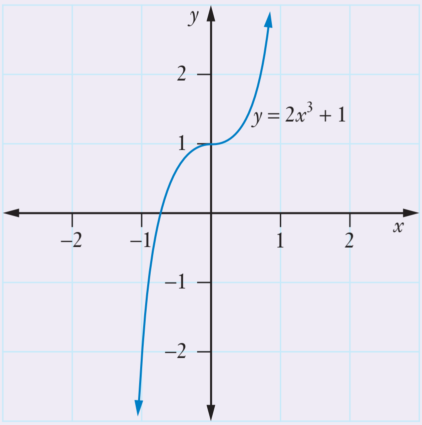
$y = 2(-1)^3 + 1$

$= -1$

When $x = 1$:

$y = 2(1)^3 + 1$

$= 3$



You can use derivatives to help sketch other functions, for example trigonometric, exponential and logarithmic graphs.

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EXAMPLE 8

a Sketch the curve $y = xe^x$, showing any important features.

b EXT1 Given $f(x) = x(x-2)^2$, sketch the graph of $y^2 = f(x)$.

Solution

a $y = xe^x$

$y' = u'v + v'u$ where $u = x$ and $v = e^x$

$u' = 1 \qquad v' = e^x$

$y' = 1 \times e^x + e^x \times x$

$\quad = e^x(1 + x)$

$y'' = u'v + v'u$ where $u = e^x$ and $v = 1 + x$

$u' = e^x \qquad v' = 1$

$y'' = e^x \times (1 + x) + 1 \times e^x$

$\quad = e^x(2 + x)$

For stationary points:

$$y' = 0$$
$$e^x(1 + x) = 0$$
$$1 + x = 0 \qquad (e^x \neq 0)$$
$$x = -1$$

When $x = -1$:

$$y = -1e^{-1}$$
$$= -\frac{1}{e}$$
$$y'' = e^{-1}(2 + -1)$$
$$= \frac{1}{e}$$
$$> 0 \qquad \text{(concave upwards)}$$

So $\left(-1, -\frac{1}{e}\right)$ is a minimum turning point.

For x-intercepts, $y = 0$:

$0 = xe^x$

$x = 0 \qquad (e^x \neq 0)$

For y-intercepts, $x = 0$:

$y = 0e^0$

$\quad = 0$

The general exponential function $y = a^x$ has an asymptote at the x-axis.

Limiting behaviour as $x \to \pm\infty$:

As $x \to \infty$, $xe^x \to \infty$ since x and e^x are both becoming large as x becomes large.

As $x \to -\infty$, $x \to -\infty$ but $e^x \to 0$ when x is negative $\left(\text{since } e^{-x} = \frac{1}{e^x}\right)$.

So $xe^x \to 0^-$ (it approaches zero from the negative side).

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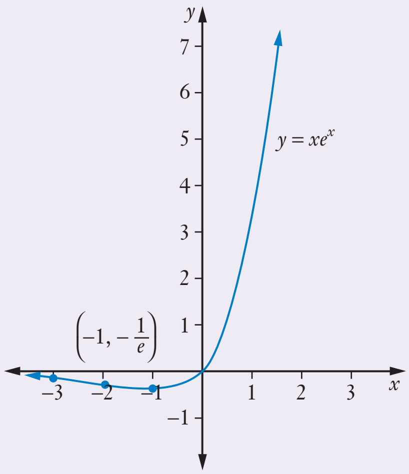
We can sketch this information on a graph.



b EXT1 $y^2 = f(x)$

So $y^2 = x(x-2)^2$

$y = \pm\sqrt{x(x-2)^2}$

For $y = \sqrt{x(x-2)^2}$,

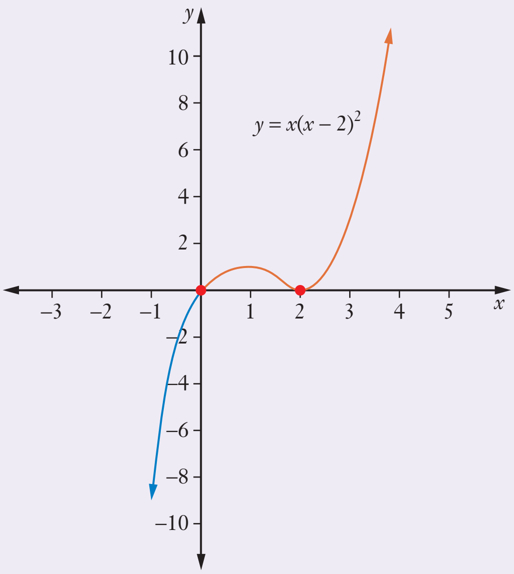
we need $x(x-2)^2 \geq 0$

First, sketch the cubic function $y = x(x-2)^2$ and find the parts of the graph on and above the x-axis.

For x-intercepts, $y = 0$:

$0 = x(x-2)^2$

$x = 0, x = 2$



This cubic function has a positive leading term, x-intercepts at 0 and 2.
$x = 2$ is a double root (multiplicity 2) so it is a turning point, as shown on the graph.

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So $x(x-2)^2 \geq 0$ when $x \geq 0$.

Domain: $[0, \infty)$, Range: $[0, \infty)$ as shown by the brown section of the curve.

$$y = \sqrt{x(x-2)^2}$$
$$= \sqrt{x(x^2 - 4x + 4)}$$
$$= \sqrt{x^3 - 4x^2 + 4x}$$
$$= (x^3 - 4x^2 + 4x)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(3x^2 - 8x + 4)(x^3 - 4x^2 + 4x)^{-\frac{1}{2}}$$
$$= \frac{3x^2 - 8x + 4}{2\sqrt{x^3 - 4x^2 + 4x}}$$
$$= \frac{3x^2 - 8x + 4}{2\sqrt{x(x-2)^2}} \qquad (x \neq 0, 2)$$

For stationary points,

$$\frac{dy}{dx} = 0$$
$$\frac{3x^2 - 8x + 4}{2\sqrt{x^3 - 4x^2 + 4x}} = 0$$
$$3x^2 - 8x + 4 = 0$$
$$(3x-2)(x-2) = 0$$
$$3x - 2 = 0, \quad x - 2 = 0$$
$$x = \frac{2}{3} \qquad x = 2 \text{ (not a stationary point since } x \neq 2)$$

When $x = \frac{2}{3}$:

$$y = \sqrt{\frac{2}{3}\left(\frac{2}{3} - 2\right)}$$
$$\approx 1.089$$

So $\left(\frac{2}{3}, 1\ 089\right)$ is a stationary point.

The second derivative is difficult to find, so we can check the first derivative:

x	0.6	$\frac{2}{3}$	0.7
$\frac{dy}{dx}$	0.13	0	–0.06
	+		–

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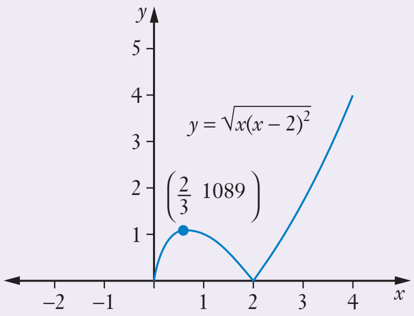
Positive to negative, so $\left(\dfrac{2}{3}, 1\ 089\right)$ is a maximum turning point.

Now, using the brown section of the graph on p. 265, you need to calculate the square root of the y values of that graph to draw the new graph. The x-intercepts at 0 and 2 will be the same because $\sqrt{0} = 0$.

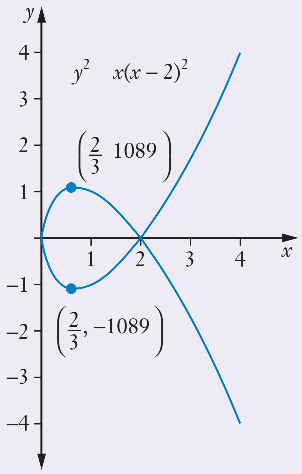
Drawing this information gives the graph below.

A table of values may help.



The graph of $y = -\sqrt{x(x-2)^2}$ is a reflection in the x-axis.

Putting both graphs together gives us the graph of $y^2 = x(x-2)^2$.



Notice that the derivative is undefined at $x = 0$ (where the tangent is vertical) and $x = 2$ (where there are 2 tangents).

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When the second derivative is hard to find, we can use the first derivative to check the type of stationary points.

EXAMPLE 9

Find any stationary points and sketch the function $y = x\sqrt{16 - x^2}$.

Solution

$$y = x\sqrt{16 - x^2}$$

$$y' = u'v + v'u \qquad u = x \qquad v = \sqrt{16 - x^2} = (16 - x^2)^{\frac{1}{2}}$$

$$u' = 1 \qquad v' = -2x \times \frac{1}{2}(16 - x^2)^{-\frac{1}{2}}$$

$$= -\frac{x}{\sqrt{16 - x^2}}$$

$$y' = 1 \times \sqrt{16 - x^2} + \left(-\frac{x}{\sqrt{16 - x^2}}\right) \times x$$

$$= \sqrt{16 - x^2} - \frac{x^2}{\sqrt{16 - x^2}}$$

For stationary points:

$$y' = 0$$

$$\sqrt{16 - x^2} - \frac{x^2}{\sqrt{16 - x^2}} = 0$$

$$16 - x^2 - x^2 = 0 \qquad \text{(multiplying both sides by } \sqrt{16 - x^2}\text{)}$$

$$16 - 2x^2 = 0$$

$$16 = 2x^2$$

$$8 = x^2$$

$$\pm\sqrt{8} = x$$

When $x = \sqrt{8}$:

$$y = \sqrt{8} \times \sqrt{16 - (\sqrt{8})^2}$$

$$= \sqrt{8} \times \sqrt{8}$$

$$= 8$$

So $(\sqrt{8}, 8)$ is a stationary point.

When $x = -\sqrt{8}$:

$$y = -\sqrt{8} \times \sqrt{16 - (-\sqrt{8})^2}$$

$$= -\sqrt{8} \times \sqrt{8}$$

$$= -8$$

So $(-\sqrt{8}, -8)$ is a stationary point.

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Since the second derivative is hard to find, we can check the first derivative on LHS and RHS of $\pm\sqrt{8} \approx 2.8$, to see where the curve is increasing and decreasing.

x	2	2.8	3
y'	+2.3	0	–0.8

Positive to negative, so $(\sqrt{8}, 8)$ is a maximum turning point.

x	–3	–2.8	–2
y'	–0.8	0	+2.3

Negative to positive, so $(-\sqrt{8}, -8)$ is a minimum turning point.

For x-intercepts, $y = 0$:

$$0 = x\sqrt{16 - x^2}$$

$$x = 0, \sqrt{16 - x^2} = 0$$

$$16 - x^2 = 0$$

$$16 = x^2$$

$$\pm 4 = x$$

For y-intercept, $x = 0$:

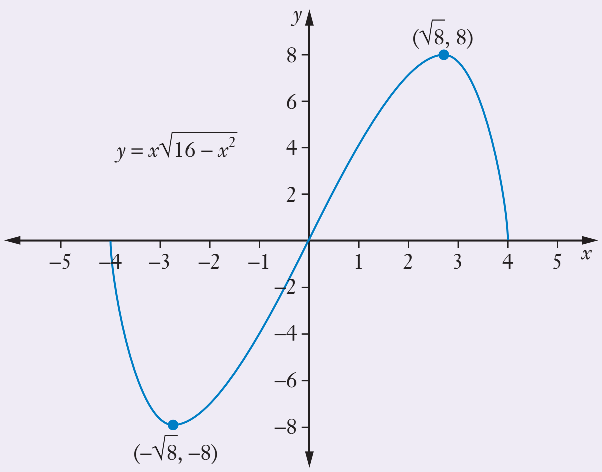
$$y = 0\sqrt{16 - 0^2}$$

$$= 0$$

Domain: $\sqrt{16 - x^2} \geq 0$

This simplifies to $-4 \leq x \leq 4$ or $[-4, 4]$ by solving the inequality or by noticing that the graph of $y = \sqrt{16 - x^2}$ is a semicircle with radius 4.

We can sketch this information on a graph.



Exercise 6.06 Curve sketching

1 Find the stationary point on the curve $f(x) = x^2 - 3x - 4$ and determine its type. Find the x- and y-intercepts of the graph of $f(x)$ and sketch the curve.

2 Sketch the graph of $y = 6 - 2x - x^2$, showing the stationary point.

3 Find the stationary point on the curve of the composite function $y = f(g(x))$ where $f(x) = x^3$ and $g(x) = x - 1$ and determine its nature. Hence sketch the curve.

4 Sketch the graph of $y = x^4 + 3$, showing any stationary points.

5 Find the stationary point on the curve $y = x^5$ and show that it is a point of inflection. Hence sketch the curve.

6 Sketch the graph of $f(x) = x^7$.

7 Find any stationary points on the curve $y = 2x^3 - 9x^2 - 24x + 30$ and sketch its graph.

8 **a** Determine any stationary points on the curve $y = x^3 + 6x^2 - 7$.

b Find any points of inflection on the curve.

c Sketch the curve.

9 Find any stationary points and points of inflection on the curve $y = f(x) + g(x)$ where $f(x) = x^3 - 7x^2 - 1$ and $g(x) = x^2 + 4$ and hence sketch the curve.

10 Find any stationary points and points of inflection on the curve $y = 2 + 9x - 3x^2 - x^3$. Hence sketch the curve.

11 Sketch the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$, showing all stationary points.

12 EXT1 Find the stationary points on the curve $y = f(x)g(x)$ given $f(x) = (x - 4)$ and $g(x) = (x + 2)^2$, and hence sketch the curve.

13 Find all stationary points and points of inflection on the curve $y = (2x + 1)(x - 2)^4$. Sketch the curve.

14 Show that the curve $y = \dfrac{2}{1+x}$ has no stationary points. By considering the domain and range of the function, sketch the curve.

15 Sketch in the domain $[0, 2\pi]$, showing all stationary points:

a $y = \cos 2x$ **b** $y = 5 \sin 4x$

16 Draw the graph of each function, showing stationary points, points of inflection and other features.

a $y = x^2 \ln x$ **b** $y = \dfrac{x}{e^x}$ **c** $y = \dfrac{1}{x^2 - 1}$

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17 EXT1 Sketch the graph of each function, showing features such as stationary points, points of inflection or asymptotes.

a $y = \dfrac{1}{f(x)}$ where $f(x) = x^2 - 16$

b $y = \dfrac{f(x)}{g(x)}$ where $f(x) = x^2$ and $g(x) = x^2 + 4$

c $y = 1 + \dfrac{f(x)}{g(x)}$ given $f(x) = x$ and $g(x) = x^2 - 1$

d $y^2 = f(x)$ if $f(x) = (x - 1)(x - 2)(x + 3)$

6.07 Global maxima and minima

WS Geae and least values

A curve may have local maximum and minimum turning points, but the absolute highest and lowest values of a function over a given domain are called the **global maximum or minimum values** of the function.

EXAMPLE 10

Find the global maximum and minimum values of y for the function $f(x) = x^4 - 2x^2 + 1$ in the domain $[-2, 3]$.

Solution

$f'(x) = 4x^3 - 4x$

$f''(x) = 12x^2 - 4$

For stationary points:

$f'(x) = 0$

$4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

$4x(x + 1)(x - 1) = 0$

$x = 0, \quad x = -1, \quad x = 1$

$f(0) = 0^4 - 2(0)^2 + 1$

$= 1$

$f''(0) = 12(0)^2 - 4$

$= -4$

< 0 (concave downward)

So $(0, 1)$ is a maximum turning point.

$f(-1) = (-1)^4 - 2(-1)^2 + 1$

$= 0$

$f''(-1) = 12(-1)^2 - 4$

$= 8$

> 0 (concave upward)

So $(-1, 0)$ is a minimum turning point.

$f(1) = 1^4 - 2(1)^2 + 1$

$= 0$

$f''(1) = 12(1)^2 - 4$

$= 8$

> 0 (concave upward)

So $(1, 0)$ is a minimum turning point.

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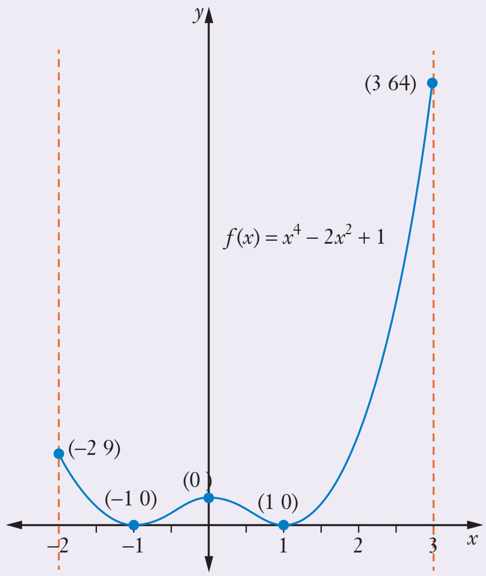
At the endpoints of the domain:

$$f(-2) = (-2)^4 - 2(-2)^2 + 1$$
$$= 9$$
$$f(3) = 3^4 - 2(3)^2 + 1$$
$$= 64$$

Checking, we also notice that $f(x) = x^4 - 2x^2 + 1$ is an even function.

$$f(-x) = (-x)^4 - 2(-x)^2 + 1$$
$$= x^4 - 2x^2 + 1$$
$$= f(x)$$

Drawing this information:



In the domain [−2, 3], the global maximum value is 64 and the global minimum value is 0.

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Exercise 6.07 Global maxima and minima

1 Sketch the graph of $y = x^2 + x - 2$ in the domain $[-2, 2]$ and find the maximum value of y in this domain.

2 Sketch the graph of $f(x) = 9 - x^2$ over the domain $[-4, 2]$. Hence find the maximum and minimum values of the function over this domain.

3 Find the maximum value of $y = x^2 - 4x + 4$ in the domain $[-3, 3]$.

4 Sketch the graph of $f(x) = 2x^3 + 3x^2 - 36x + 5$ for $-3 \le x \le 3$, showing any stationary points. Find the global maximum and minimum values of the function.

5 Find the global maximum for $y = x^5 - 3$ in the domain $[-2, 1]$.

6 Sketch the curve $f(x) = 3x^2 - 16x + 5$ for $0 \le x \le 4$ and find its global maximum and minimum.

7 Find the local and global maximum and minimum of $f(x) = 3x^4 + 4x^3 - 12x^2 - 3$ in the domain $[-2, 2]$.

8 Sketch $y = x^3 + 2$ over the domain $[-3, 3]$ and find its global minimum and maximum.

9 Sketch $y = \sqrt{x+5}$ for $-4 \le x \le 4$ and find its maximum and minimum values.

10 Show that $y = \dfrac{1}{x-2}$ has no stationary points. Find its maximum and minimum values in the domain $[-3, 3]$.

INVESTIGATION

THE LARGEST DISC

One disc 20 cm in diameter and one 10 cm in diameter are cut from a disc of cardboard 30 cm in diameter. Can you find the largest disc that can be cut from the remainder of the cardboard?

6.08 Finding formulas for optimisation problems

Optimisation problems involve finding maximum or minimum values. For example, a salesperson wants to maximise profit; a warehouse manager wants to maximise storage; a driver wants to minimise petrol consumption; a farmer wants to maximise paddock size.

To solve an optimisation problem, we must first find a formula for the quantity that we are trying to maximise or minimise.

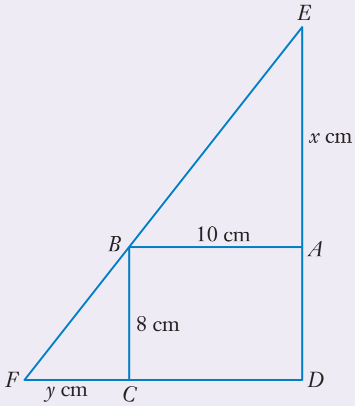
EXAMPLE 11

a A rectangular prism has a base with length twice its width. Its volume is 300 cm^3. Show that the surface area is given by $S = 4x^2 + \frac{900}{x}$.

b $ABCD$ is a rectangle with $AB = 10$ cm and $BC = 8$ cm. Length $AE = x$ cm and $CF = y$ cm.

i Show that $xy = 80$.

ii Show that triangle EDF has area given by $A = 80 + 5x + \frac{320}{x}$.



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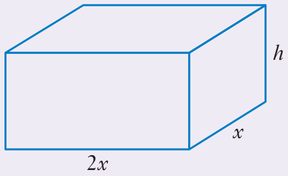
Solution

a Volume:

$$V = lwh$$
$$= 2x \times x \times h$$
$$= 2x^2h$$

$V = 300$:

$$300 = 2x^2h$$
$$\frac{300}{2x^2} = h \qquad [1]$$



Surface area:

$$S = 2(lw + wh + lh)$$
$$= 2(2x^2 + xh + 2xh)$$
$$= 2(2x^2 + 3xh)$$
$$= 4x^2 + 6xh$$

Substitute [1]:

$$S = 4x^2 + 6x \times \frac{300}{2x^2}$$
$$= 4x^2 + \frac{900}{x}$$

b **i** Triangles *AEB* and *CBF* are similar.

So $\frac{10}{y} = \frac{x}{8}$

$$xy = 80 \qquad [1]$$

ii Side $FD = y + 10$ and side $ED = x + 8$

Since $xy = 80$

$$y = \frac{80}{x} \qquad [2]$$

Area:

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(y + 10)(x + 8)$$
$$= \frac{1}{2}(xy + 8y + 10x + 80)$$
$$= \frac{1}{2}\left(80 + 8 \times \frac{80}{x} + 10x + 80\right) \qquad \text{substituting [1] and [2]}$$
$$= \frac{1}{2}\left(160 + \frac{640}{x} + 10x\right)$$
$$= 80 + \frac{320}{x} + 5x$$

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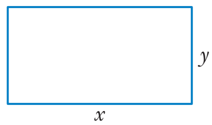
Exercise 6.08 Finding formulas for optimisation problems

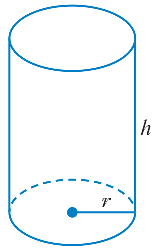
1 The area of a rectangle is to be 50 m^2. Show that its perimeter is given by the equation $P = 2x + \frac{100}{x}$.



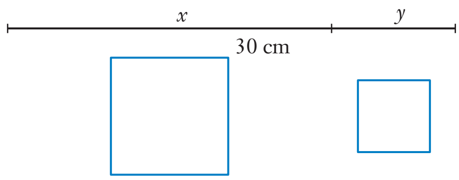
2 A rectangular paddock on a farm is to have a fence with a 120 m perimeter. Show that the area of the paddock is given by $A = 60x - x^2$.

3 The product of 2 numbers is 20. Show that the sum of the numbers is $S = x + \frac{20}{x}$.

4 A closed cylinder is to have a volume of 400 cm^3. Show that its surface area is $S = 2\pi r^2 + \frac{800}{r}$.



5 A 30 cm length of wire is cut into 2 pieces and each piece bent to form a square as shown.



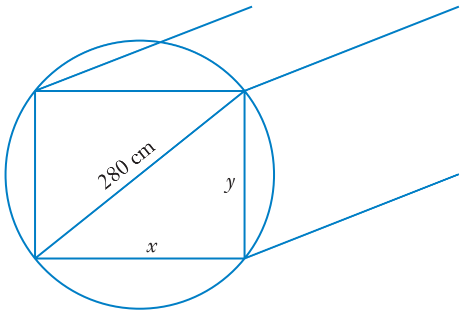
a Show that $y = 30 - x$.

b Show that the total area of the 2 squares is given by $A = \frac{x^2 - 30x + 450}{8}$.

6 A timber post with a rectangular cross-sectional area is to be cut out of a log with a diameter of 280 mm as shown.

a Show that $y = \sqrt{78400 - x^2}$.

b Show that the cross-sectional area is given by $A = x\sqrt{78400 - x^2}$.



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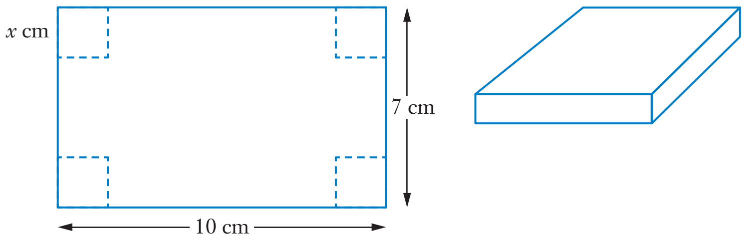
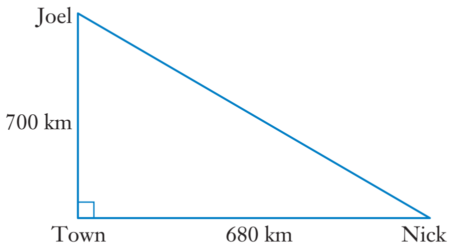
7 A 10 cm by 7 cm rectangular piece of cardboard has equal square corners with side x cm cut out. The sides are folded up to make an open box as shown. Show that the volume of the box is $V = 70x - 34x^2 + 4x^3$.

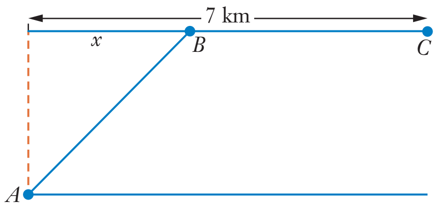


8 A travel agency calculates the expense E per person of organising a holiday in a group of x people as $E = 200 + 400x$. The cost C for each person taking a holiday is $C = 900 - 100x$. Show that the profit to the travel agency on a holiday with a group of x people is given by $P = 700x - 500x^2$.

9 Joel is 700 km north of a town, travelling towards it at an average speed of 75 km h^{-1}. Nick is 680 km east of the town, travelling towards it at 80 km h^{-1}.
Show that, after t hours, the distance between Joel and Nick is given by
$d = \sqrt{952400 - 213800t + 12025t^2}$.



10 Taylor swims from point A to point B across a 500 m wide river, then walks along the river bank to point C. The distance along the river bank is 7 km. If she swims at 5 km h^{-1} and walks at 4 km h^{-1}, show that the time taken to reach point C is given by $t = \dfrac{\sqrt{x^2 + 0.25}}{5} + \dfrac{7 - x}{4}$.



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6.09 Optimisation problems

Opimiaion problems

Futhr optimisation problems

Sating maxima and minima problems

Applicaion of optimisation

Maximum volume

Applicaion of derivatives problems

Applicaion of derivatives assignment

Second derivative assignment

Second derivative problems

You can use derivatives to find the maximum or minimum value of a formula.

Always check that an answer gives a maximum or minimum value.

EXAMPLE 12

The equation for the expense per year, E (in units of \$10 000), of running a certain business is given by $E = x^2 - 6x + 12$, where x is the number (in 100s) of items manufactured.

a Find the expense of running the business if no items are manufactured.

b Find the number of items needed to minimise the expense of the business.

c Find the minimum expense of the business.

Solution

a When $x = 0$:

$E = 0^2 - 6(0) + 12$

$= 12$ (expense is in units of \$10 000)

So the expense of running the business when no items are manufactured is $12 \times \$10\,000 = \$120\,000$ per year.

b For stationary points:

$$\frac{dE}{dx} = 0$$

$$\frac{dE}{dx} = 2x - 6 = 0$$

$$2x = 6$$

$$x = 3 \qquad (x \text{ is in 100s of items})$$

$$\frac{d^2E}{dx^2} = 2$$

$$> 0 \qquad (\text{concave upwards})$$

So $x = 3$ gives a minimum value.

$3 \times 100 = 300$

So 300 items manufactured each year will give the minimum expense.

c When $x = 3$:

$E = 3^2 - 6(3) + 12$

$= 3$

So the minimum expense per year is $3 \times \$10\,000 = \$30\,000$.

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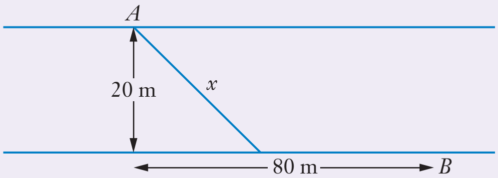
EXAMPLE 13

a The council wants to make a rectangular swimming area at the beach using the seashore on one side and a length of 300 m of shark-proof netting for the other 3 sides. What are the dimensions of the rectangle that encloses the greatest area?

b Kristyn is at point A on one side of a 20 m wide river and needs to get to point B on the other side 80 m along the bank as shown. Kristyn swims to any point on the other bank and then runs along the side of the river to point B.

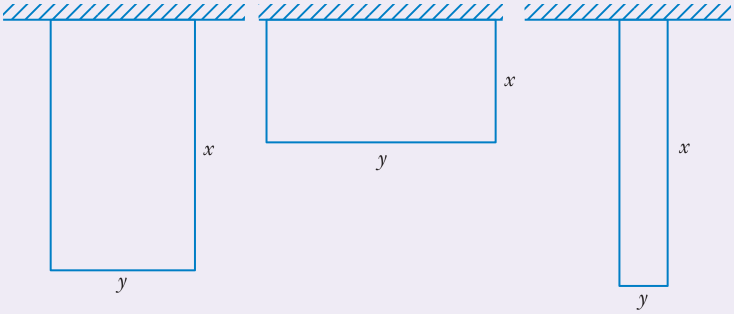


If she can swim at 7 km h^{-1} and run at 11 km h^{-1}, find x, the distance she swims to the nearest metre, to minimise her total travel time.

Optimisation problems

Solution

a Many different rectangles could have a perimeter of 300 m. Let the length of the rectangle be y and the width be x.



Perimeter: $2x + y = 300$ m

$y = 300 - 2x$ [1]

Area:

$A = xy$

$= x(300 - 2x)$ substituting [1]

$= 300\,x - 2x^2$

$$\frac{dA}{dx} = 300 - 4x$$

For stationary points:

$$\frac{dA}{dx} = 0$$

$300 - 4x = 0$

$300 = 4x$

$75 = x$

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$$\frac{d^2A}{dx^2} = -4$$

< 0 Concave downwards

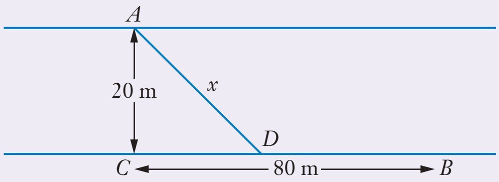
So $x = 75$ gives maximum area.

When $x = 75$: Substituting into [1]

$$y = 300 - 2(75)$$
$$= 150$$

So the dimensions that give the maximum area are 150 m × 75 m.

b First, we need to find a formula for the time Kristyn takes to run the distance $AD + DB$.



$AD = x$ so find DB:

$DB = 80 - CD$

By Pythagoras' theorem, $x^2 = 20^2 + CD^2$

$x^2 - 20^2 = CD^2$

$CD = \sqrt{x^2 - 400}$

$DB = 80 - \sqrt{x^2 - 400}$

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

$$s = \frac{d}{t}$$

$$st = d$$

$$t = \frac{d}{s}$$

Time taken to swim AD:

$$t_1 = \frac{x}{7}$$

Time taken to run DB:

$$t_2 = \frac{80 - \sqrt{x^2 - 400}}{11}$$

Total time:

$$t = t_1 + t_2$$

$$= \frac{x}{7} + \frac{80 - \sqrt{x^2 - 400}}{11}$$

$$= \frac{11x}{77} + \frac{7(80 - \sqrt{x^2 - 400})}{77}$$

$$= \frac{11x + 560 - 7(x^2 - 400)^{\frac{1}{2}}}{77}$$

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$$\frac{dt}{dx} = \frac{11 - 7 \times 2x \times \frac{1}{2}(x^2 - 400)^{-\frac{1}{2}}}{77}$$

$$= \frac{11 - 7x(x^2 - 400)^{-\frac{1}{2}}}{77}$$

For minimum time: $\frac{dt}{dx} = 0$

$$\frac{11 - 7x(x^2 - 400)^{-\frac{1}{2}}}{77} = 0$$

$$11 - 7x(x^2 - 400)^{-\frac{1}{2}} = 0$$

$$\begin{aligned}
11 &= 7x(x^2 - 400)^{-\frac{1}{2}} \\
11 &= \frac{7x}{\sqrt{x^2 - 400}} \\
11\sqrt{x^2 - 400} &= 7x \\
121(x^2 - 400) &= 49x^2 \qquad \text{squaring both sides} \\
121x^2 - 48\,400 &= 49x^2 \\
72x^2 &= 48\,400 \\
x^2 &= 672.222\ldots \\
x &= \sqrt{672222} \\
&\approx 25.9
\end{aligned}$$

To check that t is a minimum:

x	25	25.9	26
$\frac{dt}{dx}$	–0.009	0	0.0006

Since the function is decreasing on LHS and increasing on RHS, t is a minimum at $x = 25.9$.

So Kristyn should swim a distance of 25.9 m to minimise her total travel time.

Exercise 6.09 Optimisation problems

1 The height, in metres, of a ball is given by the equation $h = 16t - 4t^2$, where t is time in seconds. Find when the ball will reach its maximum height, and what the maximum height will be.

2 The cost per hour of a bike ride is given by the formula $C = x^2 - 15x + 70$, where x is the distance travelled in km. Find the distance that gives the minimum cost.

3 The perimeter of a rectangle is 60 m and its length is x m.

a Show that the area of the rectangle is given by the equation $A = 30x - x^2$.

b Hence find the maximum area of the rectangle.

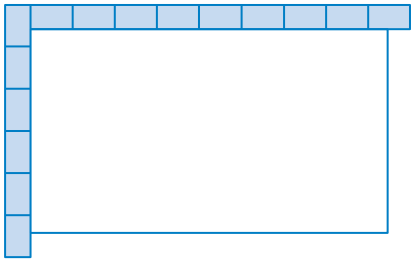
4 A farmer wants to make a rectangular paddock with an area of 4000 m^2. To minimise fencing costs she wants the paddock to have a minimum perimeter.

a Show that the perimeter is given by the equation $P = 2x + \frac{8000}{x}$.

b Find the dimensions of the rectangle that will give the minimum perimeter, correct to 1 decimal place.

c Calculate the cost of fencing the paddock, at $48.75 per metre.

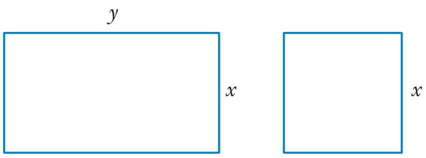
5 Bill wants to put a small rectangular vegetable garden in his backyard using 2 existing walls as part of its border. He has 8 m of garden edging for the border on the other 2 sides. Find the dimensions of the garden bed that will give the greatest area.



6 Find 2 numbers whose sum is 28 and whose product is a maximum.

7 The difference of 2 numbers is 5. Find these numbers if their product is to be minimum.

8 A piece of wire 10 m long is broken into 2 parts, which are bent into the shape of a rectangle and a square as shown. Find the dimensions x and y that make the total area a maximum.



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9 A box is made from an 80 cm by 30 cm rectangle of cardboard by cutting out 4 equal squares of side x cm from each corner. The edges are turned up to make an open box.

a Show that the volume of the box is given by the equation $V = 4x^3 - 220x^2 + 2400x$.

b Find the value of x that gives the box its greatest volume.

c Find the maximum volume of the box.

10 The formula for the surface area of a cylinder is given by $S = 2\pi r(r + h)$ where r is the radius of its base and h is its height.

a Show that if the cylinder holds a volume of 54π m^3, the surface area is given by the equation $S = 2\pi r^2 + \frac{108\pi}{r}$.

b Hence find the radius that gives the minimum surface area.

11 A silo in the shape of a cylinder is required to hold 8600 m^3 of wheat.

a Find an equation for the surface area of the silo in terms of the base radius.

b Find the minimum surface area required to hold this amount of wheat, to the nearest square metre.

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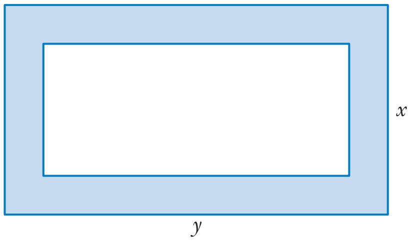
12 A rectangle is cut from a circular disc of radius 6 cm.

a Show that the formula for the area of the rectangle is $A = x\sqrt{144 - x^2}$.

b Find the area of the largest rectangle that can be produced.

13 A poster consists of a photograph bordered by a 5 cm margin. The area of the poster is to be 400 cm^2.

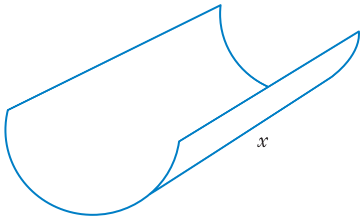


a Show that the area of the photograph is given by the equation $A = 500 - 10x - \dfrac{4000}{x}$.

b Find the maximum area possible for the photograph.

14 A surfboard is in the shape of a rectangle and semicircle, as shown. The perimeter is to be 4 m. Find the maximum area of the surfboard, correct to 2 decimal places.

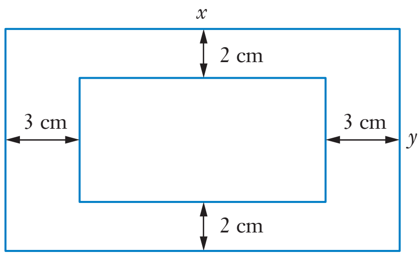
15 A half-pipe is to be made from a rectangular piece of metal of length x m. The perimeter of the rectangle is 30 m.



a Find the dimensions of the rectangle that will give the maximum surface area.

b Find the height from the ground up to the top of the half-pipe with this maximum area, correct to 1 decimal place.

16 The picture frame shown has a border of 2 cm at the top and bottom and 3 cm at the sides. If the total area of the border is to be 100 cm^2, find the maximum area of the frame.



17 A 3 m piece of wire is cut into 2 pieces and bent around to form a square and a circle. Find the size of the 2 lengths, correct to 2 decimal places, that will make the total area of the square and circle a minimum.

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18 Two cars are travelling along roads that intersect at right angles to one another. One starts 200 km away and travels towards the intersection at 80 km h^{-1}, while the other starts at 120 km away and travels towards the intersection at 60 km h^{-1}.

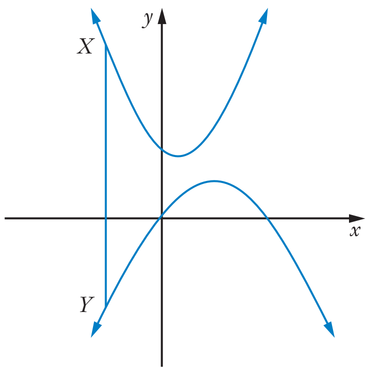
a Show that their distance apart after t hours is given by $d^2 = 10\,000t^2 - 46\,400t + 54\,400$.

b Hence find their minimum distance apart.

19 X is a point on the curve $y = x^2 - 2x + 5$. Point Y lies directly below X and is on the curve $y = 4x - x^2$.

a Show that the distance, d, between X and Y is $d = 2x^2 - 6x + 5$.

b Find the minimum distance between X and Y.



20 A truck travels 1500 km at an hourly cost given by $s^2 + 9000$ cents where s is the average speed of the truck.

a Show that the cost for the trip is given by $C = 1500\left(s + \dfrac{9000}{s}\right)$.

b Find, to the nearest km h^{-1}, the speed that minimises the cost of the trip.

c Find the cost of the trip to the nearest dollar.

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CLASS CHALLENGE

HERON'S PROBLEM

One boundary of a farm is a straight river bank, and on the farm stands a house. Some distance away there is a shed. Each is sited away from the river bank. Each morning the farmer takes a bucket from his house to the river, fills it with water, and carries the water to the shed.

Find the position on the river bank that will allow him to walk the shortest distance from house to river to shed. Further, describe how the farmer could solve the problem on the ground with the aid of a few stakes for sighting.

LEWIS CARROLL'S PROBLEM

After a battle at least 95% of the combatants had lost a tooth, at least 90% had lost an eye, at least 80% had lost an arm, and at least 75% had lost a leg. At least how many had lost all four?

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6. TEST YOURSELF

Practice quiz

Deivaive find-a-word

Second derivative find-a-word

For Questions **1–4** choose the correct answer **A**, **B**, **C** or **D**.

1 A maximum turning point has:

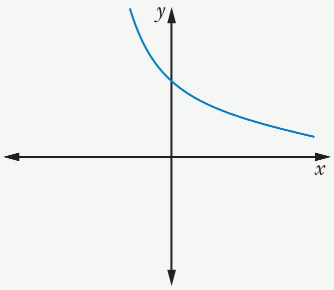
A $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ **B** $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$

C $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$ **D** $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

2 For the graph shown:

A $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$ **B** $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$

C $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$ **D** $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$



3 For a horizontal point of inflection:

A $f''(x) = 0$ **B** $f'(x) = 0$ and $f''(x) = 0$

C $f''(x) = 0$ and concavity changes **D** $f'(x) = 0, f''(x) = 0$ and concavity changes

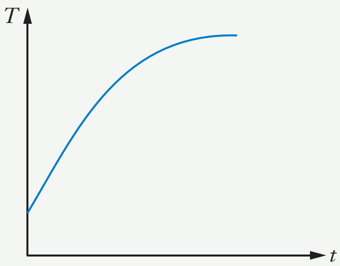
4 The graph below shows temperature T at time t. Which statement describes the shape of the graph?

A The temperature is increasing and the rate of change in temperature is increasing.

B The temperature is decreasing and the rate of change in temperature is increasing.

C The temperature is increasing and the rate of change in temperature is decreasing.

D The temperature is decreasing and the rate of change in temperature is decreasing.



5 Find the stationary points on the curve $y = x^3 + 6x^2 + 9x - 11$ and determine their nature.

6 Find all x values for which the curve $y = 2x^3 - 7x^2 - 3x + 1$ is concave upwards.

7 The height in metres of an object thrown up into the air is given by $h = 20t - 2t^2$, where t is time in seconds. Find the maximum height that the object reaches.

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8 Find the domain over which the curve $y = 5 - 6x - 3x^2$ is decreasing.

9 Find the point of inflection on the curve $y = 2x^3 - 3x^2 + 3x - 2$.

10 A soft drink manufacturer wants to minimise the amount of aluminium in its cans while still holding 375 mL of soft drink. Given that 375 mL has a volume of 375 cm^3:

a show that the surface area of a can is given by $S = 2\pi r^2 + \frac{750}{r}$

b find the radius of the can that gives the minimum surface area.

11 For the function $y = 3x^4 + 8x^3 + 6x^2$:

a find any stationary points

b determine their nature

c sketch the curve for the domain $[-3, 3]$

d find the maximum and minimum values of the function in this domain.

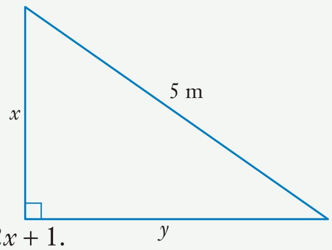
12 A rectangular prism with a square base is to have a surface area of 250 cm^2.

a Show that the volume is given by $V = \frac{125x - x^3}{2}$.

b Find the dimensions that will give the maximum volume.

13 The cost to a business of manufacturing x products a week is given by $C = x^2 - 300x + 9000$. Find the number of products that will give the minimum cost each week.

14 A 5 m length of timber is used to border a triangular garden bed, with the other sides of the garden against the house walls.



a Show that the area of the garden is $A = \frac{1}{2}x\sqrt{25 - x^2}$.

b Find the greatest possible area of the garden bed.

15 Find any points of inflection on the curve $f(x) = x^4 - 6x^3 + 2x + 1$.

16 Find the maximum value of the curve $y = x^3 + 3x^2 - 24x - 1$ in the domain $[-5, 6]$.

17 A function has $f'(2) < 0$ and $f''(2) < 0$. Sketch the shape of the function near $x = 2$.

18 Sketch the graph of the function $f(x) = xe^{2x}$ showing all features.

19 Sketch the graph of the function $y = 2 \cos 4x$ in the domain $[0, \pi]$.

20 EXT1 Given $f(x) = x^2 - 1$ and $g(x) = x + 1$, sketch the graph of:

a $y = f(x) + g(x)$

b $y = f(x)g(x)$

c $y = \frac{g(x)}{f(x)}$

d $y^2 = f(x)g(x)$

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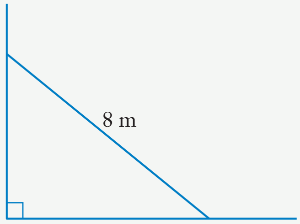
6. CHALLENGE EXERCISE

1 Sketch the curve $y = x(x - 2)^3$ showing any stationary points and points of inflection.

2 EXT1 Find all values of x for which the curve $y = 4x^3 - 21x^2 - 24x + 5$ is increasing.

3 Find the maximum possible area if an 8 m length of fencing is placed across a corner to enclose a triangular space.



4 Find the greatest and least values of $f(x) = 4x^3 - 3x^2 - 18x$ in the domain $[-2, 3]$.

5 Show that the function $f(x) = 2(5x - 3)^3$ has a horizontal point of inflection at (0.6, 0).

6 Two circles have radii r and s such that $r + s = 25$.
Show that the sum of areas of the circles is least when $r = s$.

7 Find the equation of a curve that is always concave upwards with a stationary point at (–1, 2) and y-intercept 3.

EXT1 **8** **a** Given $f(x) = x^2 + 2$, find the stationary point on the reciprocal curve $y = \frac{1}{f(x)}$ and determine its nature.

b Find the domain and range of the curve.

c Find the limit of the curve as x approaches $\pm\infty$ and sketch the curve.

9 **a** Show that $y = x^n$ has a stationary point at (0, 0) where n is a positive integer.

b If n is even, show that (0, 0) is a minimum turning point.

c If n is odd, show that (0, 0) is a point of inflection.

10 EXT1 Sketch the graph of $y = \frac{f(x)}{g(x)}$ showing any stationary points, inflections and other important features, given $f(x) = x^3$ and $g(x) = x^3 + 1$.

11 Find the minimum and maximum values of $y = \frac{x+3}{x^2 - 9}$ in the domain $[-2, 2]$.

12 The cost of running a car at an average speed of V km h^{-1} is given by $c = 100 + \frac{V^2}{75}$ cents per hour. Find the average speed (to the nearest km h^{-1}) at which the cost of a 1000 km trip is a minimum.

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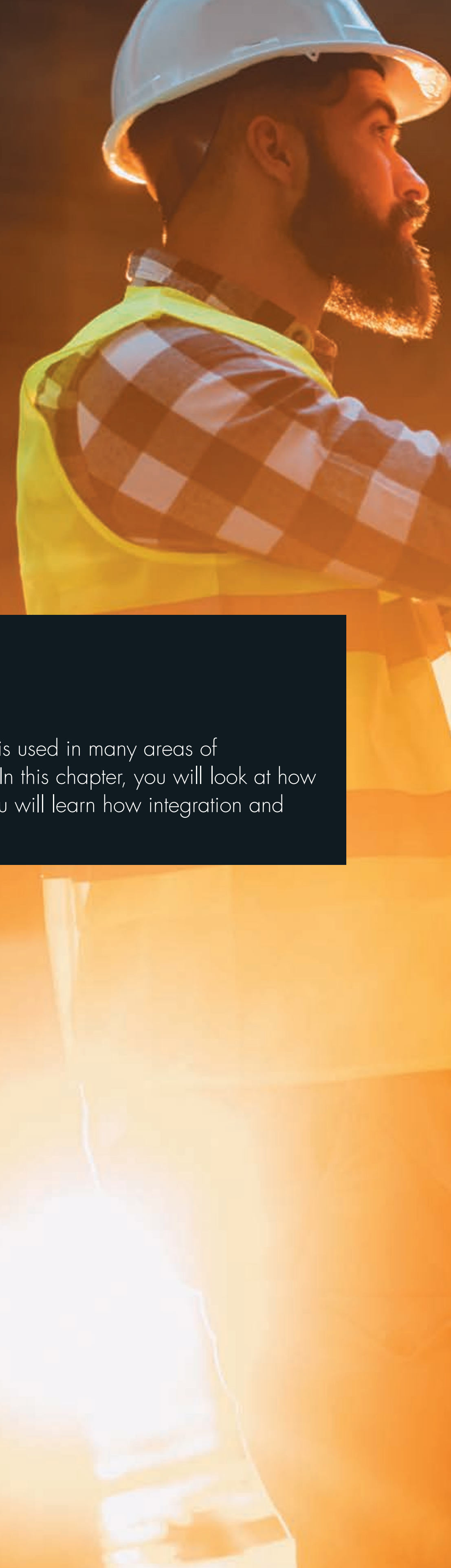
CALCULUS

7.

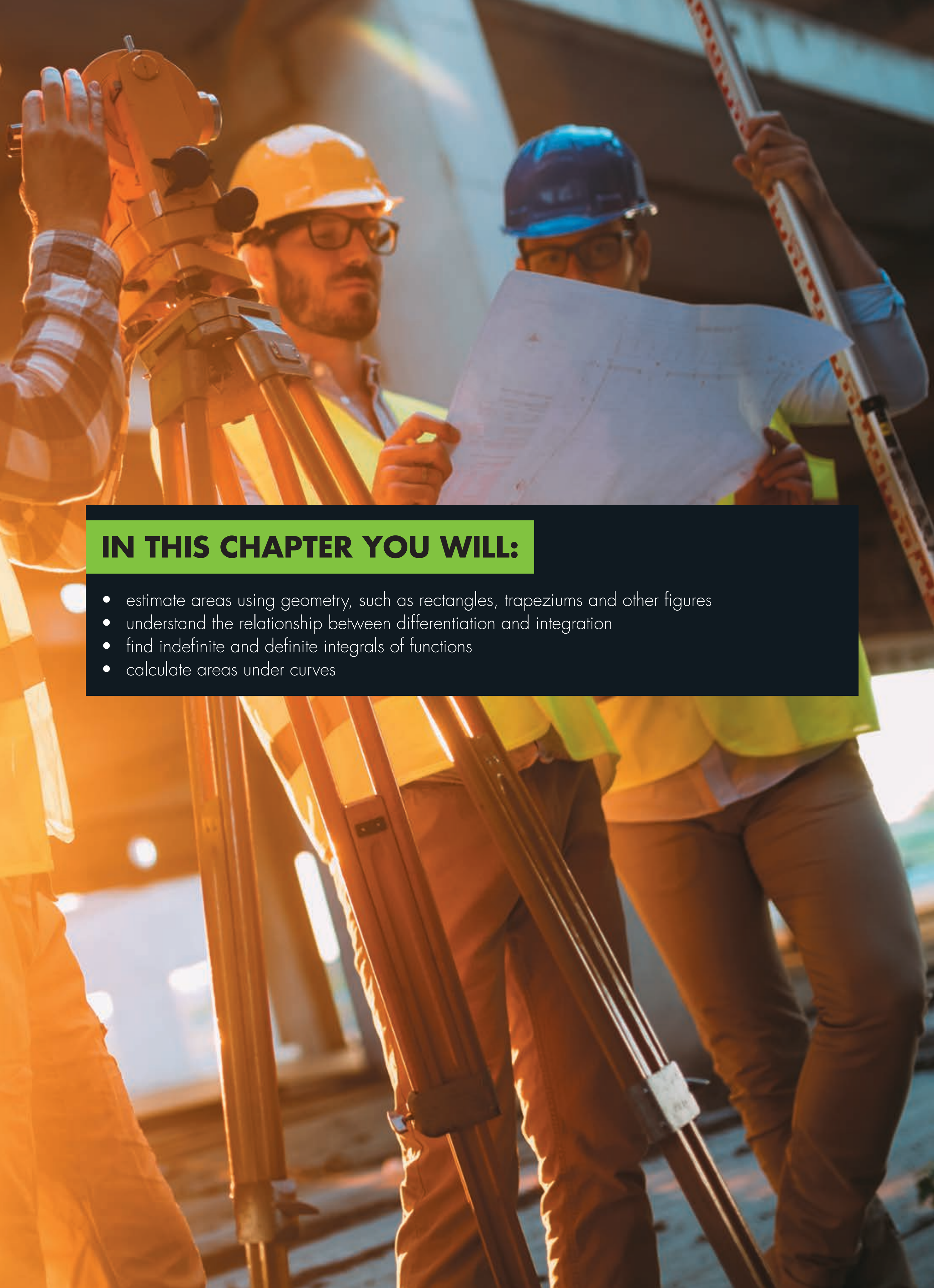


INTEGRATION

Integration is the process of finding an area under a curve. This is used in many areas of knowledge, such as surveying, physics and the social sciences. In this chapter, you will look at how to find both approximate and exact areas under a curve and you will learn how integration and differentiation are related.

CHAPTER OUTLINE

7.01 Approximating areas under a curve
7.02 Trapezoidal rule
7.03 Definite integrals
7.04 Indefinite integrals
7.05 Chain rule
7.06 Integration involving exponential functions
7.07 Integration involving logarithmic functions
7.08 Integration involving trigonometric functions
7.09 Areas enclosed by the *x*-axis
7.10 Areas enclosed by the *y*-axis
7.11 Sums and differences of areas



IN THIS CHAPTER YOU WILL:

- estimate areas using geometry, such as rectangles, trapeziums and other figures
- understand the relationship between differentiation and integration
- find indefinite and definite integrals of functions
- calculate areas under curves

TERMINOLOGY

definite integral: The integral or anti-derivative $y = F(x)$ used to find the area between the curve $y = f(x)$, the x-axis and boundaries $x = a$ and $x = b$, given by $\int_a^b f(x)\,dx = F(b) - F(a)$.

indefinite integral: A general anti-derivative $\int f(x)\,dx$.

integral: An anti-derivative.

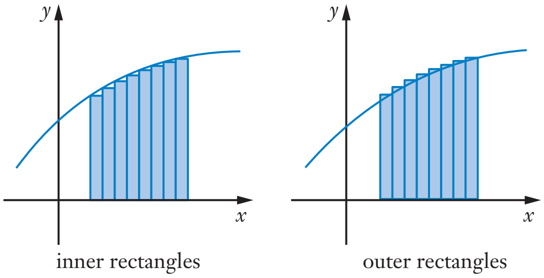
integration: The process of finding an anti-derivative.

trapezoidal rule: A formula for approximating area under a curve by using a trapezium.

WS Areas using rectangles

7.01 Approximating areas under a curve

Mathematicians since the time of Archimedes have used rectangles to approximate irregular areas. In more recent times, we use the number plane to find areas enclosed between a curve and the x-axis. We call this the **area under the curve**.



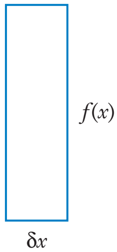
The first diagram has inner or left rectangles that are below the curve, because the top left corners of the rectangles touch the curve.

The second diagram has outer or right rectangles that are above the curve, because the top right corners of the rectangles touch the curve.

The more rectangles we have, the more accurately they approximate the area under the curve.

Integral notation

The diagram at right shows one of the rectangles. The height of each rectangle is $f(x)$ and its width is δx, so its area is $f(x)\,\delta x$. So the sum of all the rectangles is $\Sigma f(x)\,\delta x$ for the different values of x.



We can approximate the area under the curve using a large number of rectangles by making the width of each rectangle very small.

Taking an infinite number of rectangles, $\delta x \to 0$.

$$\text{Area} = \lim_{\delta x \to 0}\left(\sum f(x)\delta x\right)$$
$$= \int f(x)\,dx$$

> δx is 'delta x' and means a small change in x.
> δ is the Greek letter for 'd', for difference.

We use the **integral** symbol $\int$ to stand for the sum of rectangles (the symbol is an S for sum).

We call $\int f(x)\,dx$ an **indefinite integral**.

If we are finding the area under the curve $y = f(x)$ between $x = a$ and $x = b$, we can write $\int_a^b f(x)\,dx$.

We call $\int_a^b f(x)\,dx$ a **definite integral**.

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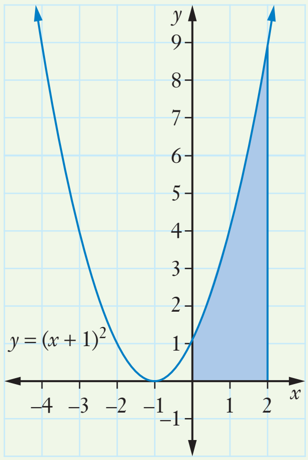
EXAMPLE 1

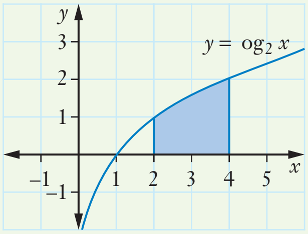
a Find an approximation to the shaded area by using:

i 4 inner rectangles

ii 4 outer rectangles

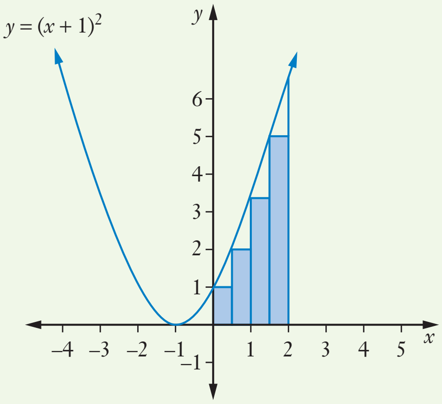


b Find the shaded area below by using a trapezium.



Solution

a **i** Using inner rectangles, the top left corners touch the curve and they lie below the curve.



Each rectangle has height $f(x)$ and width 0.5 units.

Height of 1st rectangle:

$f(0) = (0 + 1)^2 = 1$

Area $= 1 \times 0.5 = 0.5$

Height of 2nd rectangle:

$f(0.5) = (0.5 + 1)^2 = 2.25$

Area $= 2.25 \times 0.5 = 1.125$

Height of 3rd rectangle:

$f(1) = (1 + 1)^2 = 4$

Area $= 4 \times 0.5 = 2$

Height of 4th rectangle:

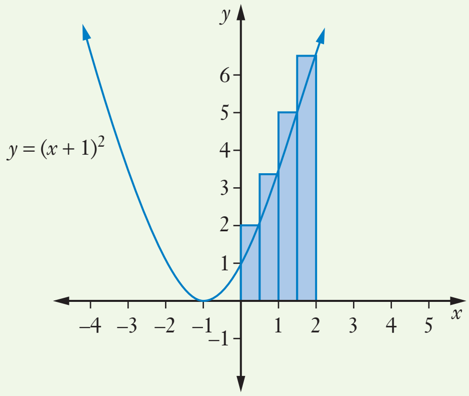
$f(1.5) = (1.5 + 1)^2 = 6.25$

Area $= 6.25 \times 0.5 = 3.125$

Total area $= 0.5 + 1.125 + 2 + 3.125 = 6.75$

So area is 6.75 units^2.

ii Using outer rectangles, the top right corners touch the curve and they go above the curve.



Height of 1st rectangle:

$f(0.5) = (0.5 + 1)^2 = 2.25$

Area $= 2.25 \times 0.5 = 1.125$

Height of 2nd rectangle:

$f(1) = (1 + 1)^2 = 4$

Area $= 4 \times 0.5 = 2$

Height of 3rd rectangle:

$f(1.5) = (1.5 + 1)^2 = 6.25$

Area $= 6.25 \times 0.5 = 3.125$

Height of 4th rectangle:

$f(2) = (2 + 1)^2 = 9$

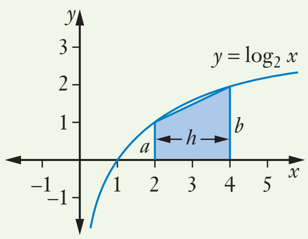
Area $= 9 \times 0.5 = 4.5$

Total area $= 1.125 + 2 + 3.125 + 4.5 = 10.75$

So area is 10.75 units^2

b Area of a trapezium:

$A = \frac{1}{2}h(a + b)$ where h = perpendicular height and a and b are parallel sides.



Here the trapezium has parallel sides at $x = 2$ and $x = 4$.

$$a = f(2) = \log_2 2 = 1$$

$$b = f(4) = \log_2 4 = 2$$

$$h = 4 - 2 = 2$$

$$A = \frac{1}{2}h(a + b) = \frac{1}{2}(2)(1 + 2) = 3$$

So area is 3 units^2.

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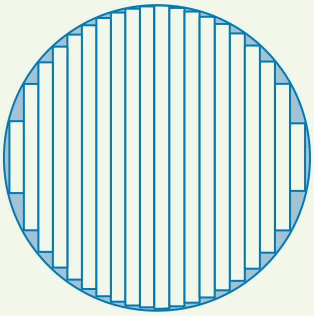
DID YOU KNOW?

Archimedes

Integration has been of interest to mathematicians since very early times. **Archimedes** (287–212 BCE) found the area of enclosed curves by cutting them into very thin layers and finding their sum. He found the formula for the volume of a sphere this way. He also found an estimation of π, correct to 2 decimal places.



Area within curve

Archimedes

TECHNOLOGY

Areas under a curve

We can use a spreadsheet to find approximate areas under a curve using rectangles. Using technology allows us to find sums of large numbers of rectangles without needing to do many calculations. This gives a more accurate approximation to the area under a curve.

For example, we can use a spreadsheet to find the approximate area under the curve $y = (x + 1)^2$ between $x = 0$ and $x = 2$ from Example 1 **a i**.

We find the y values using the formula **=(A2+1)^2** (copy the formula down the column).

The width is **=A3–A2** (copy this value down the column).

The area is **=B2*C2** (copy the formula down the column).

	A	B	C	D
1	‹	y	Width	Area
2	0	1	0.5	0.5
3	0.5	2.25	0.5	1.125
4	1	4	0.5	2
5	1.5	6.25	0.5	3.125
6				
7			Total area	6.75

We can use the spreadsheet to find the area using a much larger number of rectangles, for example, 20 or 40.

	A	B	C	D	E	F	G	H	I
1	x	y	Width	Area		x	y	Width	Area
2	0	1	0.1	0.1		0	1	0.05	0.05
3	0.1	1.21	0.1	0.121		0.05	1.1025	0.05	0.055125
4	0.2	1.44	0.1	0.144		0.1	1.21	0.05	0.0605
5	0.3	1.69	0.1	0.169		0.15	1.3225	0.05	0.066125
6	0.4	1.96	0.1	0.196		0.2	1.44	0.05	0.072
7	0.5	2.25	0.1	0.225		0.25	1.5625	0.05	0.078125
8	0.6	2.56	0.1	0.256		0.3	1.69	0.05	0.0845
9	0.7	2.89	0.1	0.289		0.35	1.8225	0.05	0.091125
10	0.8	3.24	0.1	0.324		0.4	1.96	0.05	0.098
11	0.9	3.61	0.1	0.361		0.45	2.1025	0.05	0.105125
12	1	4	0.1	0.4		0.5	2.25	0.05	0.1125
13	1.1	4.41	0.1	0.441		0.55	2.4025	0.05	0.120125
14	1.2	4.84	0.1	0.484		0.6	2.56	0.05	0.128
15	1.3	5.29	0.1	0.529		0.65	2.7225	0.05	0.136125
16	1.4	5.76	0.1	0.576		0.7	2.89	0.05	0.1445
17	1.5	6.25	0.1	0.625		0.75	3.0625	0.05	0.153125
18	1.6	6.76	0.1	0.676		0.8	3.24	0.05	0.162
19	1.7	7.29	0.1	0.729		0.85	3.4225	0.05	0.171125
20	1.8	7.84	0.1	0.784		0.9	3.61	0.05	0.1805
21	1.9	8.41	0.1	0.841		0.95	3.8025	0.05	0.190125
22	2	9	0.1	0.9		1	4	0.05	0.2
23						1.05	4.2025	0.05	0.210125
24						1.1	4.41	0.05	0.2205
25			Total area	9.17		1.15	4.6225	0.05	0.231125
26						1.2	4.84	0.05	0.242
27						1.25	5.0625	0.05	0.253125
28						1.3	5.29	0.05	0.2645
29						1.35	5.5225	0.05	0.276125
30						1.4	5.76	0.05	0.288
31						1.45	6.0025	0.05	0.300125
32						1.5	6.25	0.05	0.3125
33						1.55	6.5025	0.05	0.325125
34						1.6	6.76	0.05	0.338
35						1.65	7.0225	0.05	0.351125
36						1.7	7.29	0.05	0.3645
37						1.75	7.5625	0.05	0.378125
38						1.8	7.84	0.05	0.392
39						1.85	8.1225	0.05	0.406125
40						1.9	8.41	0.05	0.4205
41						1.95	8.7025	0.05	0.435125
42						2	9	0.05	0.45
43									
44								Total area	8.9175
45									

Use technology with a larger number of rectangles for a more accurate area to this question.

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We can find the area under the same curve by using different methods.

We can use other shapes to find areas under a curve.

EXAMPLE 2

Find an approximation to the area under the curve $y = x^2$ between $x = 0$ and $x = 2$ by using:

a squares

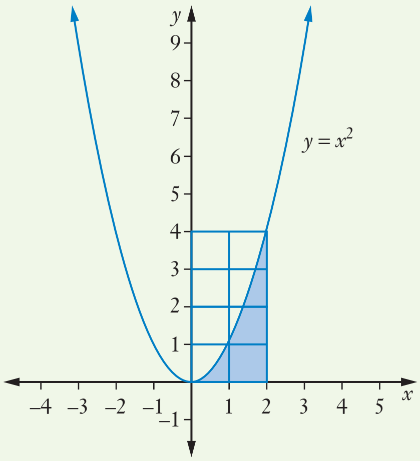
b a triangle

Solution

a On the grid, each square is 1 square unit.
By counting and approximating squares:

$A \approx 3$

So area is 3 units^2.

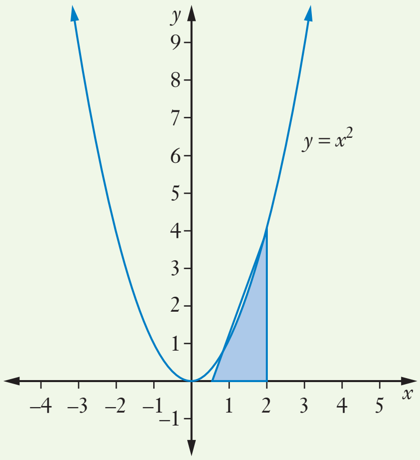


b Using a triangle:

$b = 2 - 0.5 = 1.5$

$$h = f(2)$$
$$= 2^2$$
$$= 4$$
$$A = \frac{1}{2}bh$$
$$= \frac{1}{2} \times 1.5 \times 4$$
$$= 3$$

So area is 3 units^2.



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Exercise 7.01 Approximating areas under a curve

1 Find an approximation to the area under the curve $y = x^2 + 2x$ between $x = 1$ and $x = 2$ by using:

a 2 inner rectangles **b** 2 outer rectangles

2 Find an approximation (to 2 decimal places) to the area under the curve $y = \frac{2}{x+1}$ from $x = 1$ to $x = 3$ using:

a 2 inner rectangles **b** 2 outer rectangles

c 4 inner rectangles **d** 4 outer rectangles

3 Use a trapezium to find an approximate area under the curve:

a $f(x) = x^2$ between $x = 2$ and $x = 3$

b $y = \ln x$ between $x = 4$ and $x = 7$

c $f(x) = x^3 + 1$ between $x = 0$ and $x = 4$

d $f(x) = \sin x$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$ (give answer in exact form)

e $y = 9 - x^2$ between $x = 1$ and $x = 2$

4 Find the approximate area under the curve $f(x) = x^3 + 3$ between $x = 0$ and $x = 4$ by using:

a 2 inner rectangles **b** 2 outer rectangles **c** a trapezium

5 Use a trapezium to find each area under the curve.

a $y = \frac{1}{x}$ between $x = 1$ and $x = 7$

b $y = x^2 + 5$ between $x = 0$ and $x = 1$

c $f(x) = \cos x$ between $x = 0$ and $x = \frac{\pi}{3}$ (in exact form)

d $y = e^x$ between $x = 1$ and $x = 4$ (in exact form)

e $f(x) = x(x - 4)(x - 9)$ between $x = 2$ and $x = 3$

6 a Sketch the graph of $y = 1 - x^2$ and shade the area under the curve (enclosed between the curve and the x-axis).

b Find this approximate area by using a triangle.

7 Find the approximate area under the curve $y = \sqrt{x-1}$ between $x = 2$ and $x = 5$ by using:

a 6 inner rectangles **b** 6 outer rectangles

c a trapezium **d** squares

8 Find the exact area under the curve $y = \sqrt{25 - x^2}$.

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9 **a** Find the exact area under the curve $y = \sqrt{9 - x^2}$.

b Find the approximate area under the curve $y = \sqrt{9 - x^2}$:

i between $x = 1$ and $x = 2$ using a trapezium

ii between $x = 0$ and 3 using 3 outer rectangles

10 Use a triangle to find the approximate area under the curve:

a $y = x^2$ between $x = 0$ and $x = 4$

b $y = \sqrt{x}$ between $x = 0$ and $x = 3$

c $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$

11 Find the approximate area under each curve by using:

i 4 inner rectangles **ii** 4 outer rectangles

a $y = -x^2 + 4x$

b $y = \sin x$ in the domain $[0, \pi]$ (in exact form)

12 Find the approximate area under the curve $y = x^2 + 5$ between $x = 0$ and $x = 5$ (using technology where available) using:

a 10 inner rectangles **b** 10 outer rectangles

7.02 Trapezoidal rule

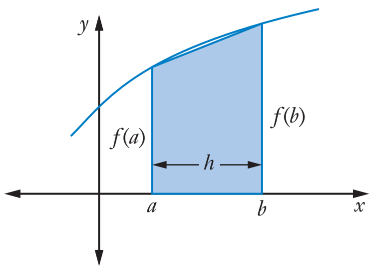
WS
Trapezoidal rule

A trapezium usually gives a much closer approximation to the area under a curve than a rectangle does.

The **trapezoidal rule** is a formula that uses a trapezium to find the area under a curve.

$$A = \frac{1}{2}h[f(a) + f(b)] \text{ where } h = b - a$$

$$= \frac{1}{2}(b - a)[f(a) + f(b)]$$



Trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b - a)[f(a) + f(b)]$$

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EXAMPLE 3

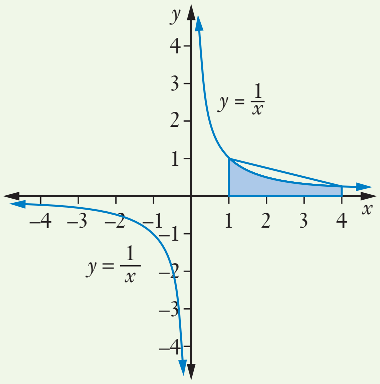
Use the trapezoidal rule to find an approximation for:

a $\int^{4} \frac{1}{x}\, dx$

b $\int_{0} x^3 dx$ using 2 subintervals

Solution

a $\int^{4} \frac{1}{x}\, dx$ is the area under the curve as shaded in the diagram.



$f(x) = \frac{1}{x}$, $a = 1$ and $b = 4$.

$$\int_a^b f(x)\, dx \approx \frac{1}{2}(b - a)[f(a) + f(b)]$$

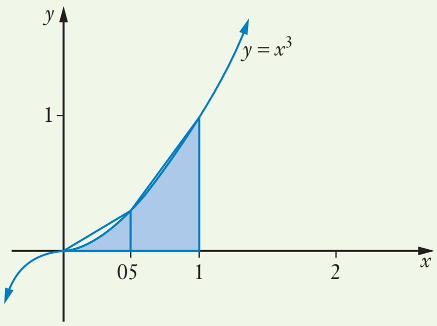
$$\int^4 \frac{1}{x}\, dx \approx \frac{1}{2}(4 - 1)[f(1) + f(4)]$$

$$= \frac{1}{2}(3)\left[\frac{1}{1} + \frac{1}{4}\right]$$

$$= \frac{15}{8}$$

$$= 1\frac{7}{8}$$

b 2 subintervals means 2 trapezia.
We use the trapezoidal formula twice.



$f(x) = x^3$ and $h = 0.5$

$$\int_a^b f(x)\, dx \approx \frac{1}{2}(b - a)[f(a) + f(b)]$$

$$\int_0 x^3\, dx = \int_0^{0.5} x^3\, dx + \int_{0.5} x^3\, dx$$

$$\approx \frac{1}{2}(0.5 - 0)[f(0) + f(0.5)] + \frac{1}{2}(1 - 0.5)[f(0.5) + f(1)]$$

$$= \frac{1}{2}(0.5)[0^3 + 0.5^3] + \frac{1}{2}(0.5)[0.5^3 + 1^3]$$

$$= 0.3125$$

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There is a more general trapezoidal rule when using several subintervals or trapezia.

Trapezoidal rule for *n* subintervals

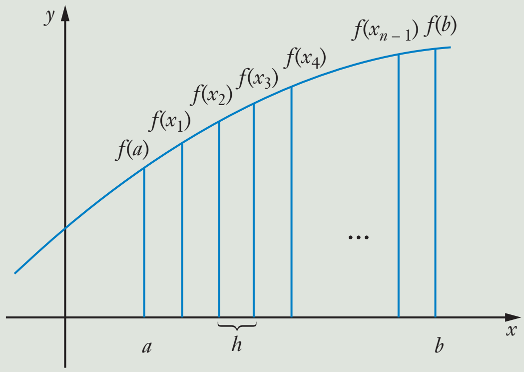
Given n subintervals (trapezia)

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\Big[f(a)+f(b)+2\big\{f(x_1)+ \quad + f(x_{n-1})\big\}\Big]$$

where $a = x_0$ and $b = x_n$, and the values of $x_0, x_1, x_2, \ldots, x_n$ are found by dividing the interval $a \le x \le b$ into n equal subintervals of width $h = \frac{b-a}{n}$.

Since $h = \frac{b-a}{n}$, the formula can also be written as

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\Big[f(a)+f(b)+2\big\{f(x_1)+ \quad + f(x_{n-1})\big\}\Big].$$



Proof

Interval $b - a$ is divided into n trapezia.

So the width of each trapezium is $h = \frac{b-a}{n}$.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}h[f(a)+f(x_1)] + \frac{1}{2}h[f(x_1)+f(x_2)] + \frac{1}{2}h[f(x_2)+f(x_3)] + \ldots + \frac{1}{2}h[f(x_{n-1})+f(b)]$$

$$= \frac{h}{2}[f(a)+f(x_1)+f(x_1)+f(x_2)+f(x_2)+f(x_3)+\ldots+f(x_{n-1})+f(b)]$$

$$= \frac{h}{2}(f(a)+2f(x_1)+2f(x_2)+2f(x_3)+\ldots+2f(x_{n-1})+f(b))$$

$$= \frac{h}{2}\Big[f(a)+f(b)+2\big\{f(x_1)+ \quad + f(x_{n-1})\big\}\Big]$$

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EXAMPLE 4

Trapezoidal rule

a Use the trapezoidal rule with 4 subintervals to find an approximation for $\int_2^3 \frac{2}{x-1}\,dx$ correct to 3 decimal places.

b Use the trapezoidal rule with 7 subintervals to find an approximation for $\int_0^{14} (t^2+3)\,dt$.

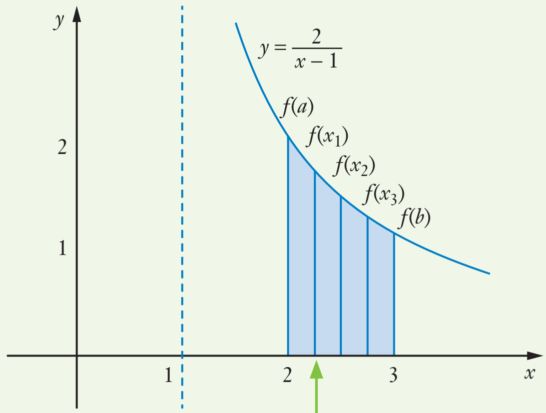
Solution

a From the question and diagram:

$a = 2, b = 3, n = 4.$

The width of each trapezium is

$$h = \frac{b-a}{n}$$
$$= \frac{3-2}{4}$$
$$= \frac{1}{4}$$
$$= 025$$



Note that these x values increase by $h = 0.25$, the width of each trapezium.

Substituting into the general trapezoidal rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+f(b)+2\{f(x_1)+\quad + f(x_{n-1})\}\right]$$

$$\int_2^3 \frac{2}{x-1}\,dx \approx \frac{025}{2}\left[f(2)+f(3)+2\{f(225\,)+f(2\;5)+f(2\,75)\}\right]$$

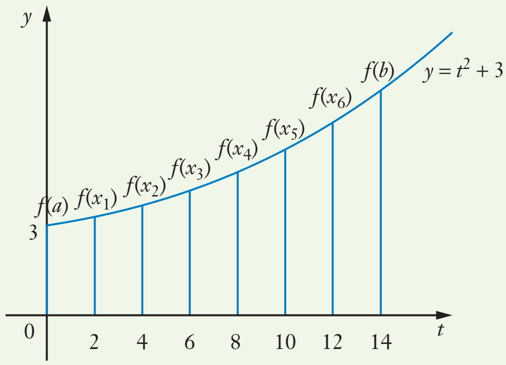
$$= 0125\left[\frac{2}{2-1}+\frac{2}{3-1}+2\left(\frac{2}{225\;-1}+\frac{2}{2\;5-1}+\frac{2}{275\;-1}\right)\right]$$

$$= 0.125[2 + 1 + 2(1.6 + 1.3333\ldots + 1.1428\ldots)]$$
$$= 1.39404\ \ldots$$
$$\approx 1.394$$

b $a = 0, b = 14, n = 7.$

The width of each trapezium is

$$h = \frac{14-0}{7}$$
$$= \frac{14}{7}$$
$$= 2$$



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Substituting into the general trapezoidal rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a)+f(b)+2\left\{f(x_1)+\quad+f(x_{n-1})\right\}\right]$$

$$\int_0^{14}(t^2+3)\,dt \approx \frac{2}{2}\left[f(0)+f(14)+2\left\{f(2)+f(4)+f(6)+f(8)+f(10)+f(12)\right\}\right]$$

$$= [(0^2+3)+(14^2+3)+2\{(2^2+3)+(4^2+3)+(6^2+3)+(8^2+3)$$
$$+(10^2+3)+(12^2+3)\}]$$

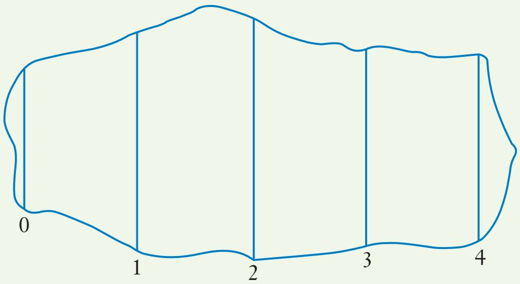
$$= 966$$

We can use the trapezoidal rule to find irregular areas.

EXAMPLE 5

A surveyor needs to find the area of the irregular piece of land shown.

Use the trapezoidal rule to find its approximate area.



Solution

We can use the values in the diagram or put them in the table below.

x	0	1	2	3	4
$f(x)$	3.7	5.9	6.4	5.1	4.9

$a = 0$, $b = 4$, $n = 4$.

From the diagram or table, the height of each trapezium is 1.

$$\text{Area} \approx \frac{h}{2}\left[f(a)+f(b)+2\left\{f(x_1)+\quad+f(x_{n-1})\right\}\right]$$

$$= \frac{1}{2}\left[f(0)+f(4)+2\left\{f(1)+f(2)+f(3)\right\}\right]$$

$$= \frac{1}{2}\left[3\ 7+49+2\left\{5\ 9+6\ 4+5\ 1\right\}\right]$$

$$= 21.7$$

So the area of the land is approximately 21.7 m^2.

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Exercise 7.02 Trapezoidal rule

1 Use the trapezoidal rule to find an approximation for each integral.

a $\int^{2} x^2\,dx$ **b** $\int_0^2 (x^3+1)\,dx$ **c** $\int^{5} \frac{dx}{x}$ **d** $\int^{2} \frac{dx}{x+3}$

2 Find an approximation to $\int^{3} x^3\,dx$ using the trapezoidal rule with:

a 1 subinterval **b** 2 subintervals

3 Use the trapezoidal rule with 2 trapezia to find an approximation to:

a $\int_2^3 \log x\,dx$ **b** $\int_0^2 \frac{dx}{x+4}$

4 Find an approximation to:

a $\int^{4} \log x\,dx$ using 3 trapezia **b** $\int_0^2 (x^2-x)\,dx$ using 4 trapezia

c $\int_0 \sqrt{x}\,dx$ using 5 subintervals **d** $\int^{5} \frac{dx}{x^2}$ using 4 subintervals

e $\int_3^6 \frac{dx}{x-1}$ using 6 trapezia

5 Given the table of values, find the approximate value of each definite integral.

a $\int^{9} f(x)\,dx$

x	1	3	5	7	9
$f(x)$	3.2	5.9	8.4	11.6	20.1

b $\int^{4} f(t)\,dt$

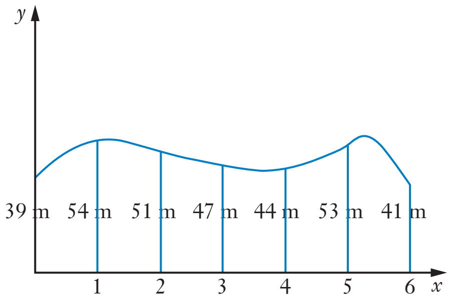
t	1	2	3	4
$f(t)$	8.9	6.5	4.1	2.9

c $\int_2^{14} f(x)\,dx$

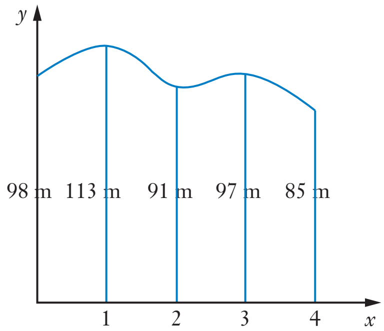
x	2	4	6	8	10	12	14
$f(x)$	25.1	37.8	52.3	89.3	67.8	45.4	39.9

6 Use the trapezoidal rule to find the approximate area of each irregular figure below.

a



b



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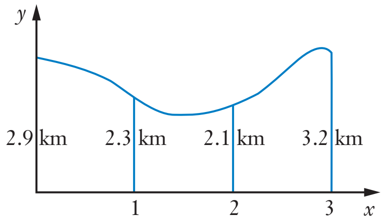
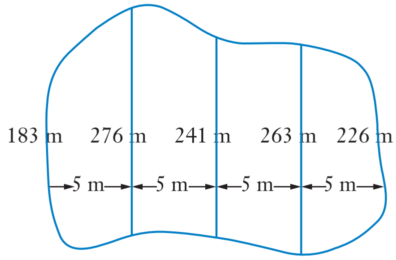
c



d



7.03 Definite integrals

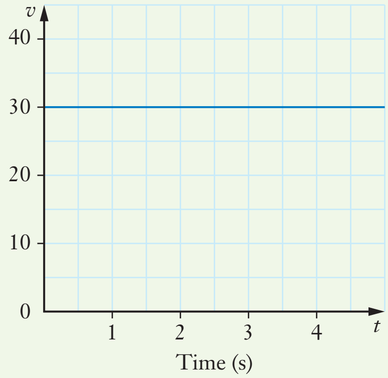
WS Definite integrals

WS Finding definite integrals

We can link the area under a graph to calculus.

EXAMPLE 6

This graph shows the velocity of an object over time as it travels at a constant 30 m s^{-1}.



a Find the distance it travels in:

i 1 s **ii** 2 s **iii** 3 s

b Find the area under the line between:

i $t = 0$ and $t = 1$ **ii** $t = 0$ and $t = 2$ **iii** $t = 0$ and $t = 3$

Solution

a **i** $s = \dfrac{d}{t}$, so $d = st$.

In 1 s, the object travels $30 \times 1 = 30$ m.

ii In 2 s, the object travels $30 \times 2 = 60$ m.

iii In 3 s, the object travels $30 \times 3 = 90$ m.

b **i** The area is $30 \times 1 = 30$ units^2

ii The area is $30 \times 2 = 60$ units^2

iii The area is $30 \times 3 = 90$ units^2

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EXAMPLE 7

This graph shows the speed of an object increasing at a steady rate.

a Find the distance travelled in:

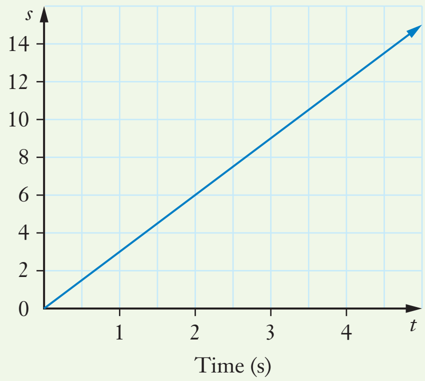
  i 1 s

  ii 4 s

b Find the area under the graph between:

  i $t = 0$ and $t = 1$

  ii $t = 0$ and $t = 4$



Solution

a **i** At $t = 0, s = 0$, and at $t = 1, s = 3$.

$$\text{Average speed} = \frac{0+3}{2} = 1.5 \text{ m s}^{-1}$$

In 1 s, the object travels $1.5 \times 1 = 1.5$ m.

ii At $t = 0, s = 0$, and at $t = 4, s = 12$.

$$\text{Average speed} = \frac{0+12}{2} = 6 \text{ m s}^{-1}$$

In 4 s, the object travels $6 \times 4 = 24$ m.

b **i** $$A = \frac{1}{2}bh = \frac{1}{2} \times 1 \times 3 = 1.5 \text{ units}^2$$

ii $$A = \frac{1}{2}bh = \frac{1}{2} \times 4 \times 12 = 24 \text{ units}^2$$

The graphs in the last 2 examples show velocity (rate of change of displacement) and speed (rate of change of distance) against time. The area under each curve gave the information about the original variable. This is the anti-derivative.

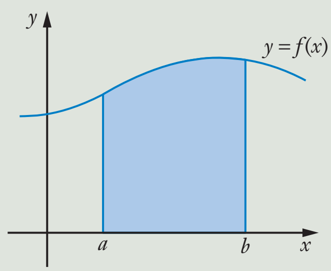
In the same way, the area under any rate of change graph will give the original variable, or the anti-derivative.

Fundamental theorem of calculus

The area enclosed by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is the anti-derivative of function $f(x)$.



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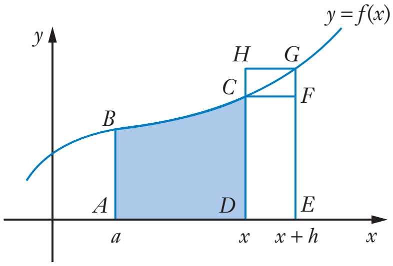
Proof



Consider a continuous curve $y = f(x)$ for all values of $x > a$.

Let area $ABCD$ be $A(x)$

Let area $ABGE$ be $A(x + h)$

Then area $DCGE$ is $A(x + h) - A(x)$

Area $DCFE$ < area $DCGE$ < area $DHGE$

$$f(x) \times h < A(x+h) - A(x) \quad < f(x+h) \times h$$

$$f(x) < \frac{A(x+h) - A(x)}{h} \quad < f(x+h)$$

$$\lim_{h \to 0} f(x) < \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} < \lim_{h \to 0} f(x+h)$$

$$f(x) < A'(x) < f(x)$$

This is the formula for the derivative of $A(x)$ from first principles, from Year 11, Chapter 8, *Introduction to calculus*.

So $A'(x) = f(x)$

$A(x)$ is *an* anti-derivative of $f(x)$.

Let $F(x)$ be the anti-derivative of $f(x)$ with a constant term of 0.

Then $A(x) = F(x) + C$ [1]

Now $A(x)$ is the area under $y = f(x)$ between a and x.

$A(a) = 0$

Substitute in [1]:

$A(a) = F(a) + C$

$0 = F(a) + C$

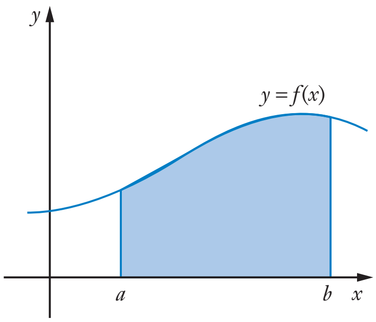
$-F(a) = C$

$A(x) = F(x) - F(a)$

If $x = b$ where $b > a$:

$A(b) = F(b) - F(a)$

So $\int_b^a f(x)\,dx = F(b) - F(a)$



EXAMPLE 8

Evaluate:

a $\int_3^4 (2x+1)\,dx$ **b** $\int_0^5 3x^2\,dx$ **c** $\int_0^2 (-3x^2)\,dx$ **d** $\int_{-} x^3\,dx$

Solution

You learned in Chapter 5, *Further differentiation*, that the anti-derivative of x^n is $\frac{1}{n+1}x^{n+1} + C$. Let $F(x)$ be the anti-derivative where $C = 0$.

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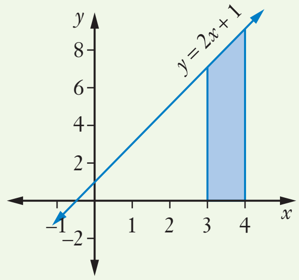
a $\int_3^4 (2x+1)\,dx = \left[x^2 + x\right]_3^4$ — The anti-derivative

$= (4^2 + 4) - (3^2 + 3)$ — $F(b) - F(a)$

$= 20 - 12$

$= 8$

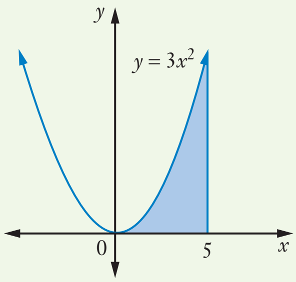
The graph shows the area that this integral calculates. Can you find its area an easier way?



b $\int_0^5 3x^2\,dx = \left[x^3\right]_0^5$

$= 5^3 - 0^3$

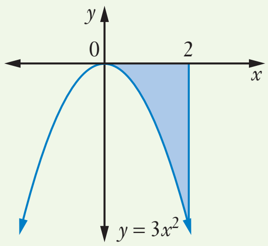
$= 125$



c $\int_0^2 (-3x^2)\,dx = \left[-x^3\right]_0^2$

$= -2^3 - (-0^3)$

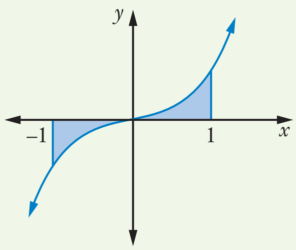
$= -8$



d $\int_{-} x^3\,dx = \left[\frac{x^4}{4}\right]_{-}$

$= \frac{1^4}{4} - \frac{(-1)^4}{4}$

$= 0$



We can also find the definite integral of x^n when n is a fraction or negative.

EXAMPLE 9

Evaluate:

a $\int^2 \frac{2x-3}{x^3}\,dx$

b $\int^8 \sqrt[3]{x}\,dx$

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Solution

a $\int^{2} \frac{2x-3}{x^3}\,dx = \int^{2} \frac{2x}{x^3} - \frac{3}{x^3}\,dx$

$= \int^{2} \frac{2}{x^2} - \frac{3}{x^3}\,dx$

$= \int^{2} (2x^{-2} - 3x^{-3})\,dx$

$= \left[\frac{2x^{-1}}{-1} - \frac{3x^{-2}}{-2}\right]^{2}$

$= \left[-\frac{2}{x} + \frac{3}{2x^2}\right]^{2}$

$= \left[-\frac{2}{2} + \frac{3}{2(2)^2}\right] - \left[-\frac{2}{1} + \frac{3}{2(1)^2}\right]$

$= -1 + \frac{3}{8} + 2 - \frac{3}{2}$

$= -\frac{1}{8}$

b $\int^{8} \sqrt[3]{x}\,dx = \int^{8} x^{\frac{1}{3}}\,dx$

$= \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]^{8}$

$= \left[\frac{3x^{\frac{4}{3}}}{4}\right]^{8}$

$= \left[\frac{3\sqrt[3]{x^4}}{4}\right]^{8}$

$= \frac{3\sqrt[3]{8^4}}{4} - \frac{3\sqrt[3]{1^4}}{4}$

$= \frac{3\times 16}{4} - \frac{3\times 1}{4}$

$= 11\frac{1}{4}$

We can use the definite integral to find original information given a rate of change.

EXAMPLE 10

The velocity of a particle is given by $v = 8t^3 - 3t^2 + 6t + 1$ cm s^{-1}.
Find the change in displacement in the first 3 seconds.

Solution

$v = \frac{dx}{dt} = 8t^3 - 3t^2 + 6t + 1$

$x = \int_0^3 (8t^3 - 3t^2 + 6t + 1)\,dt$

$= \left[8\frac{t^4}{4} - 3\frac{t^3}{3} + 6\frac{t^2}{2} + t\right]_0^3$

$= \left[2t^4 - t^3 + 3t^2 + t\right]_0^3$

$= [2(3)^4 - 3^3 + 3(3)^2 + 3] - [2(0)^4 - 0^3 + 3(0)^2 + 0]$

$= 165$

So the change in displacement in the first 3 seconds is 165 cm.

DID YOU KNOW?

Differentiation vs integration

Many mathematicians in the 17th century were interested in the problem of finding areas under a curve. The Englishman **Isaac Barrow** (1630–77) is said to be the first to discover that differentiation and integration are inverse operations. This discovery is called the **fundamental theorem of calculus**.

Barrow was an outstanding Greek scholar as well as making contributions in the areas of mathematics, theology, astronomy and physics. However, when he was a schoolboy, he was so often in trouble that his father was overheard saying to God in his prayers that if he decided to take one of his children, he could best spare Isaac.

Another English mathematician named Isaac, Sir Isaac Newton (1643–1727), was also a scientist and astronomer, and helped to discover calculus. He was not interested in his school work, but spent most of his time inventing things, such as a water clock and sundial.

Newton left school at 14 to manage the family estate after his stepfather died. However, he spent so much time reading that he was sent back to school. He went on to university and developed the theories in mathematics and science that have made him famous today.

Exercise 7.03 Definite integrals

1 Evaluate each definite integral.

a $\int_0^2 4x\,dx$ **b** $\int^3 (2x+1)\,dx$ **c** $\int_{-}^6 3x^2\,dx$

d $\int^2 (4t-7)\,dt$ **e** $\int_{-} (6y+5)\,dy$ **f** $\int_0^3 6x^2\,dx$

g $\int^2 (x^2+1)\,dx$ **h** $\int_0^2 4x^3\,dx$ **i** $\int_{-}^4 (3x^2-2x)\,dx$

2 Evaluate:

a $\int_{-} x^2\,dx$ **b** $\int_{-2}^3 (x^3+1)\,dx$ **c** $\int_{-2}^2 x^5\,dx$

d $\int^4 \sqrt{x}\,dx$ **e** $\int_0 (x^3-3x^2+4x)\,dx$ **f** $\int^2 (2x-1)^2\,dx$

g $\int_{-} (y^3+y)\,dy$ **h** $\int_3^4 (2-x)^2\,dx$ **i** $\int_{-2}^2 4t^3\,dt$

j $\int_2^4 \frac{x^2}{3}\,dx$ **k** $\int^3 \frac{5x^4}{x}\,dx$ **l** $\int_2^4 \frac{x^4-3x}{x}\,dx$

m $\int^2 \frac{4x^3+x^2+5x}{x}\,dx$ **n** $\int_3^5 \frac{x^3-2x^2+3x}{x}\,dx$ **o** $\int_3^4 \frac{x^2+x+3}{3x^5}\,dx$

3 For each velocity function, find the change in displacement between 2 and 4 seconds.

a $v = 3t^2 + 7$ m s^{-1} **b** $v = 8t - 5$ km h^{-1} **c** $v = 4t^3 + 2t + 3$ cm s^{-1}

d $v = (t+3)^2$ m s^{-1} **e** $v = 5 - 6t + 9t^2$ cm s^{-1}

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4 A high-power hose fills an empty swimming pool at the rate of $r = 25 + 4t^3$ L min^{-1}. Find the volume to the nearest litre after:

a 5 minutes **b** 15 minutes **c** half an hour

INVESTIGATION

AREAS

Look at the results of definite integrals in the examples and exercises. Sketch the graphs where possible and shade in the areas found.

Can you see why the definite integral sometimes gives a negative answer?

Can you see why it will sometimes be zero?

7.04 Indefinite integrals

To find the indefinite integral $\int f(x)\,dx$, we find the anti-derivative of the function.

WS Indefinite integrals

WS Integrating functions

Integral of x^n

$$\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C \text{ where } n \neq -1$$

EXAMPLE 11

Find each indefinite integral.

a $\int 5x^9\, dx$

b $\int \left(\frac{1}{x^3} + \sqrt{x}\right) dx$

Solution

a $\int 5x^9\, dx = 5\int x^9\, dx$

$$= 5 \times \frac{x^{10}}{10} + C$$

$$= \frac{x^{10}}{2} + C$$

b $\int \left(\frac{1}{x^3} + \sqrt{x}\right) dx = \int \left(x^{-3} + x^{\frac{1}{2}}\right) dx$

$$= \frac{x^{-2}}{-2} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= -\frac{1}{2x^2} + \frac{2\sqrt{x^3}}{3} + C$$

DID YOU KNOW?

John Wallis

English clergyman and mathematician **John Wallis** (1616–1703) found that the **area under the curve** $y = 1 + x + x^2 + x^3 + \ldots$ is given by:

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} +$$

He found this result independently of the fundamental theorem of calculus.

We can use the indefinite integral to find original information given a rate of change.

EXAMPLE 12

The rate of air flow into a container is given by $R = 4 + 3t^2$ mm^3 s^{-1}.
If there is initially no air in the container, find the volume of air in it after 12 seconds.

Solution

$$R = \frac{dV}{dt} = 4 + 3t^2$$

$$V = \int (4 + 3t^2)\, dt$$

$$= 4t + t^3 + C$$

When $t = 0$, $V = 0$

$$0 = 4(0) + 0^3 + C$$

$$= C$$

So $V = 4t + t^3$

When $t = 12$:

$$V = 4(12) + 12^3$$

$$= 1776$$

So the container will hold 1776 mm^3 of air after 12 seconds.

Exercise 7.04 Indefinite integrals

1 Find each indefinite integral.

a $\int x^2\, dx$

b $\int 3x^5\, dx$

c $\int 2x^4\, dx$

d $\int (m+1)\, dm$

e $\int (t^2 - 7)\, dt$

f $\int (h^7 + 5)\, dh$

g $\int (y - 3)\, dy$

h $\int (2x + 4)\, dx$

i $\int (b^2 + b)\, db$

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2 Find:

a $\int(x^2+2x+5)\,dx$

b $\int(4x^3-3x^2+8x-1)\,dx$

c $\int(6x^5+x^4+2x^3)\,dx$

d $\int(x^7-3x^6-9)\,dx$

e $\int(2x^3+x^2-x-2)\,dx$

f $\int(x^5+x^3+4)\,dx$

g $\int(4x^2-5x-8)\,dx$

h $\int(3x^4-2x^3+x)\,dx$

i $\int(6x^3+5x^2-4)\,dx$

j $\int(3x^{-4}+x^{-3}+2x^{-2})\,dx$

3 Find each indefinite integral.

a $\int\frac{dx}{x^8}$

b $\int x^{\overline{3}}\,dx$

c $\int\frac{x^6-3x^5+2x^4}{x^3}\,dx$

d $\int(1-2x)^2\,dx$

e $\int(x-2)(x+5)\,dx$

f $\int\frac{3}{x^2}\,dx$

g $\int\frac{dx}{x^3}$

h $\int\frac{4x^3-x^5-3x^2+7}{x^5}\,dx$

i $\int(y^2-y^{-7}+5)\,dy$

j $\int(t^2-4)(t-1)\,dt$

k $\int\sqrt{x}\,dx$

l $\int\frac{2}{t^5}\,dt$

m $\int\sqrt[3]{x}\,dx$

n $\int x\sqrt{x}\,dx$

o $\int\sqrt{x}\left(1+\frac{1}{\sqrt{x}}\right)dx$

4 The rate of change of the angle sum S of a polygon with n sides is a constant 180°. If $S = 360°$ when $n = 4$, find S when $n = 7$.

5 For a certain graph, the rate of change of y values with respect to its x values is given by $R = 3x^2 - 2x + 1$. If the graph passes through the point (–1, 3), find its equation.

6 The rate of change in velocity over time is given by $\frac{dx}{dt} = 4t + t^2 - t^3$.

If the initial velocity is 2 cm s^{-1}, find the displacement after 15 s.

7 The rate of flow of water into a dam is given by $R = 500 + 20t$ L h^{-1}. If there is 15 000 L of water initially in the dam, how much water will there be in the dam after 10 hours?

7.05 Chain rule

You found the anti-derivative of $y = (ax + b)^n$ in Chapter 5, *Further differentiation*. We can write this as an integral.

Chain rule for $(ax + b)^n$

$$\int(ax+b)^n\,dx = \frac{(ax+b)^{n+}}{a(n+1)} + C \quad \text{where } n \neq -1$$

EXAMPLE 13

Find:

a $\int (5x-9)^3\, dx$ **b** $\int_2^3 (3-x)^8\, dx$ **c** $\int^3 \sqrt{4x-3}\, dx$

Solution

a
$$\int (5x-9)^3\, dx = \frac{(5x-9)^4}{5\times 4} + C$$
$$= \frac{(5x-9)^4}{20} + C$$

b
$$\int_2^3 (3-x)^8\, dx = \left[\frac{(3-x)^9}{-1\times 9}\right]_2^3$$
$$= \left[-\frac{(3-x)^9}{9}\right]_2^3$$
$$= -\frac{(3-3)^9}{9} - \left(-\frac{(3-2)^9}{9}\right)$$
$$= 0 + \frac{1}{9}$$
$$= \frac{1}{9}$$

c
$$\int^3 \sqrt{4x-3}\, dx = \int^3 (4x-3)^{\frac{1}{2}}\, dx$$
$$= \left[\frac{(4x-3)^{\frac{3}{2}}}{4\times\frac{3}{2}}\right]^3$$
$$= \left[\frac{\sqrt{(4x-3)^3}}{6}\right]^3$$
$$= \frac{\sqrt{(4\times 3-3)^3}}{6} - \frac{\sqrt{(4\times 1-3)^3}}{6}$$
$$= \frac{\sqrt{9^3}}{6} - \frac{\sqrt{1^3}}{6}$$
$$= \frac{27}{6} - \frac{1}{6}$$
$$= 4\frac{1}{3}$$

In Chapter 5, *Further differentiation*, you also learned about the general chain rule for $f'(x)[f(x)]^n$.

Chain rule for $f'(x)[f(x)]^n$

$$\int f'(x)[f(x)]^n\, dx = \frac{1}{n+1}[f(x)]^{n+} + C \quad \text{where } n \neq -1$$

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EXAMPLE 14

a Find $\int x(x^2+1)^3\,dx$.

b Find the exact value of $\int^2 x^2\sqrt{x^3-1}\,dx$.

Solution

a $f(x) = x^2 + 1$

$f'(x) = 2x$

$$\int x(x^2+1)^3\,dx$$

$$= \int \frac{1}{2}\times 2x(x^2+1)^3\,dx$$

$$= \frac{1}{2}\int 2x(x^2+1)^3\,dx$$

$$= \frac{1}{2}\times\frac{1}{4}(x^2+1)^4 + C$$

$$= \frac{1}{8}(x^2+1)^4 + C$$

applying the chain rule formula

b $f(x) = x^3 - 1$

$f'(x) = 3x^2$

$$\int^2 x^2\sqrt{x^3-1}\,dx$$

$$= \int^2 \frac{1}{3}\times 3x^2(x^3-1)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\int^2 3x^2(x^3-1)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\left[\frac{1}{\frac{3}{2}}(x^3-1)^{\frac{3}{2}}\right]^2$$

$$= \frac{1}{3}\left[\frac{2}{3}\sqrt{(x^3-1)^3}\right]^2$$

$$= \frac{2}{9}\left[\sqrt{(x^3-1)^3}\right]^2$$

$$= \frac{2}{9}\left(\sqrt{(2^3-1)^3} - \sqrt{(1^3-1)^3}\right)$$

$$= \frac{2}{9}\left(\sqrt{7^3} - \sqrt{0^3}\right)$$

$$= \frac{2}{9}(7\sqrt{7})$$

$$= \frac{14}{9}\sqrt{7}$$

Exercise 7.05 Chain rule

1 Find each indefinite integral.

a $\int (3x-4)^2\, dx$ **b** $\int (x+1)^4\, dx$ **c** $\int (5x-1)^9\, dx$

d $\int (3y-2)^7\, dy$ **e** $\int (4+3x)^4\, dx$ **f** $\int (7x+8)^{12}\, dx$

g $\int (1-x)^6\, dx$ **h** $\int \sqrt{2x-5}\, dx$ **i** $2\int (3x+1)^{-4}\, dx$

j $\int 3(x+7)^{-2}\, dx$ **k** $\int \frac{1}{2(4x-5)^3}\, dx$ **l** $\int \sqrt[3]{4x+3}\, dx$

m $\int (2-x)^{\overline{-}\,2}\, dx$ **n** $\int \sqrt{(t+3)^3}\, dt$ **o** $\int \sqrt{(5x+2)^5}\, dx$

2 Evaluate:

a $\int^{2} (2x+1)^4\, dx$ **b** $\int_0 (3y-2)^3\, dy$ **c** $\int^{2} (1-x)^7\, dx$

d $\int_0^2 (3-2x)^5\, dx$ **e** $\int_0 \frac{(3x-1)^2}{6}\, dx$ **f** $\int_4^5 (5-x)^6\, dx$

g $\int_3^6 \sqrt{x-2}\, dx$ **h** $\int_0^2 \frac{5}{(2n+1)^3}\, dn$ **i** $\int^4 \frac{2}{\sqrt{(5x-4)^3}}\, dx$

3 Find each indefinite integral.

a $\int 4x^3(x^4+5)^2\, dx$ **b** $\int 2x(x^2-3)^5\, dx$

c $\int 3x^2(x^3+1)^3\, dx$ **d** $\int (2x+3)(x^2+3x-2)^4\, dx$

e $\int x(3x^2-7)^6\, dx$ **f** $\int x^2(4-5x^3)^2\, dx$

g $\int 4x^5(2x^6-3)^4\, dx$ **h** $\int 3x(5x^2+3)^7\, dx$

i $\int (x+2)(x^2+4x)^5\, dx$ **j** $\int (3x^2-2)(3x^3-6x-2)^3\, dy$

4 Evaluate:

a $\int_0^2 x(2x^2+3)^2\, dx$ **b** $\int_0 x^2(x^3-1)^5\, dx$

c $\int^2 x^4(x^5+2)^3\, dx$ **d** $\int_0 x^3(5-x^4)^7\, dx$

e $\int_2^4 3x(x^2+2)^4\, dx$ **f** $\int_{-} 5x^2(2x^3-7)^3\, dx$

g $\int_{-}^{0} (x-1)(x^2-2x+3)^6\, dx$ **h** $4\int_0 (x^2+2)(x^3+6x-1)^2\, dx$

i $5\int_{-2}^{2} x^2(x^3-1)(x^6-2x^3-1)^4\, dx$

5 A function has $\frac{dy}{dx} = x^2(x^3-2)^4$ and passes through the point (1, 4). Find its equation.

6 The velocity of an object is given by $v = x(x^2-3)^4$ m s^{-1}. If the displacement is 0 after 2 s, find:

a the equation for the displacement

b the displacement after 3 s.

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7.06 Integration involving exponential functions

We can write the anti-derivative of e^x as an integral.

Integral of e^x

$$\int e^x \, dx = e^x + C$$

EXAMPLE 15

Evaluate $\int_0^2 4e^x \, dx$.

Solution

$$\begin{aligned}\int_0^2 4e^x \, dx &= 4\left[e^x\right]_0^2 \\ &= 4(e^2 - e^0) \\ &= 4(e^2 - 1)\end{aligned}$$

Integral of a^x

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

EXAMPLE 16

Find $\int 2^x \, dx$.

Solution

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

$$\int 2^x \, dx = \frac{1}{\ln 2} 2^x + C$$

Chain rule for e^{ax+b}

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$$

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EXAMPLE 17

a Find $\int (e^{2x} - e^{-x})\,dx$.

b Evaluate $\int^{2} 5e^{3x}\,dx$.

Solution

a

$$\begin{aligned}\int (e^{2x} - e^{-x})\,dx &= \frac{1}{2}e^{2x} - \frac{1}{-1}e^{-1} + C\\ &= \frac{1}{2}e^{2x} + e^{-x} + C\end{aligned}$$

b

$$\begin{aligned}\int^{2} 5e^{3x}\,dx &= \left[5 \times \frac{1}{3}e^{3x}\right]^{2}\\ &= \left[\frac{5e^{3x}}{3}\right]^{2}\\ &= \frac{5e^{3\times 2}}{3} - \frac{5e^{3\times 1}}{3}\\ &= \frac{5e^{6}}{3} - \frac{5e^{3}}{3}\\ &= \frac{5e^{3}}{3}(e^{3} - 1)\end{aligned}$$

Exercise 7.06 Integration involving exponential functions

1 Find each indefinite integral.

a $\int e^{4x}\,dx$ **b** $\int e^{-x}\,dx$ **c** $\int e^{5x}\,dx$

d $\int e^{-2x}\,dx$ **e** $\int e^{4x+1}\,dx$ **f** $\int -3e^{5x}\,dx$

g $\int e^{2t}\,dt$ **h** $\int (e^{7x} - 2)\,dx$ **i** $\int (e^{x-3} + x)\,dx$

2 Evaluate in exact form:

a $\int_{0} e^{5x}\,dx$ **b** $\int_{0}^{2} -e^{-x}\,dx$ **c** $\int^{4} 2e^{3x+4}\,dx$

d $\int_{2}^{3} (3x^{2} - e^{2x})\,dx$ **e** $\int_{0}^{2} (e^{2x} + 1)\,dx$ **f** $\int^{2} (e^{x} - x)\,dx$

g $\int_{0}^{3} (e^{2x} - e^{-x})\,dx$

3 Evaluate correct to 2 decimal places:

a $\int^{3} e^{-x}\,dx$ **b** $\int_{0}^{2} 2e^{3y}\,dy$ **c** $\int_{5}^{6} (e^{x+5} + 2x - 3)\,dx$

d $\int_{0} (e^{3t+4} - t)\,dt$ **e** $\int^{2} (e^{4x} + e^{2x})\,dx$

4 Find the indefinite integral of:

a 5^{x} **b** 7^{3x} **c** 3^{2x-1}

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5 **a** Differentiate x^2e^x.

b Hence find $\int x(2+x)e^x \, dx$.

6 A function has $f'(x) = x^2e^{2x}$ and passes through the point (0, 0). Find the equation of the function.

7 A particle moves so that its velocity over time t is given by $v = 2e^t - 1$ m s^{-1}. If displacement $x = 10$ when $y = 0$, find x when $t = 3$.

7.07 Integration involving logarithmic functions

WS Integrals involving exponential and logarithmic functions

WS Inegaion o $\frac{1}{x}$

WS Integration of reciprocals

WS Differentiation and integration involving exponential and logarithmic functions

We can write the anti-derivatives that involve the logarithmic function as an integral.

Integral of $\frac{1}{x}$

$$\int \frac{1}{x} \, dx = \ln|x| + C \quad \text{where } x \neq 0$$

EXAMPLE 18

Evaluate $\int^5 \frac{3}{x} dx$.

Solution

$$\int^5 \frac{3}{x} dx = \left[3\ln|x|\right]^5$$

$= 3 \ln 5 - 3\ln 1$

$= 3 \ln 5 - 3 \times 0$

$= 3 \ln 5$

Integral of $\frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \quad \text{where } f(x) \neq 0$$

EXAMPLE 19

a Find $\int \frac{x^2}{x^3 + 7} dx$.

b Find the exact value of $\int_0^3 \frac{x+1}{x^2 + 2x + 4} dx$.

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Solution

a $f(x) = x^3 + 7,\ f'(x) = 3x^2$

$$\int \frac{x^2}{x^3+7}\,dx = \int \frac{1}{3} \times \frac{3x^2}{x^3+7}\,dx$$
$$= \frac{1}{3}\int \frac{3x^2}{x^3+7}\,dx$$
$$= \frac{1}{3}\ln|x^3 + 7| + C$$

applying the formula

b $f(x) = x^2 + 2x + 4,\ f'(x) = 2x + 2$

$$\int_0^3 \frac{x+1}{x^2+2x+4}\,dx = \frac{1}{2}\int_0^3 \frac{2x+2}{x^2+2x+4}\,dx \qquad \text{as } x + 1 = \frac{1}{2}(2x+2)$$
$$= \frac{1}{2}\Big[\ln + x^2 \quad 2x + 4\Big]_0^3$$
$$= \frac{1}{2}[\ln|3^2 + 2(3) + 4| - \ln|0^2 + 2(0) + 4|]$$
$$= \frac{1}{2}[\ln 19 - \ln 4]$$
$$= \frac{1}{2}\ln\left(\frac{19}{4}\right)$$

Exercise 7.07 Integration involving logarithmic functions

1 Find the integral of each function.

a $\frac{2}{2x+5}$ **b** $\frac{4x}{2x^2+1}$ **c** $\frac{5x^4}{x^5-2}$ **d** $\frac{1}{2x}$ **e** $\frac{2}{x}$

f $\frac{5}{3x}$ **g** $\frac{2x-3}{x^2-3x}$ **h** $\frac{x}{x^2+2}$ **i** $\frac{3x}{x^2+7}$ **j** $\frac{x+1}{x^2+2x-5}$

2 Find:

a $\int \frac{4}{4x-1}\,dx$ **b** $\int \frac{dx}{x+3}$ **c** $\int \frac{x^2}{2x^3-7}\,dx$

d $\int \frac{x^5}{2x^6+5}\,dx$ **e** $\int \frac{x+3}{x^2+6x+2}\,dx$

3 Evaluate correct to one decimal place:

a $\int^3 \frac{2}{2x+5}\,dx$ **b** $\int_2^5 \frac{dx}{x+1}$ **c** $\int^7 \frac{x^2}{x^3+2}\,dx$

d $\int_0^3 \frac{4x+1}{2x^2+x+1}\,dx$ **e** $\int_3^4 \frac{x-1}{x^2-2x}\,dx$

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4 **a** Show that $\frac{3x+3}{x^2-9} = \frac{1}{x+3} + \frac{2}{x-3}$. **b** Hence find $\int \frac{3x+3}{x^2-9}\,dx$.

5 **a** Show that $\frac{x-6}{x-1} = 1 - \frac{5}{x-1}$. **b** Hence find $\int \frac{x-6}{x-1}\,dx$.

6 A function has $f'(x) = \frac{x^2}{3x^3-1}$ and passes through the point (1, 0). Find the equation of the function.

7 A particle has velocity $v = \frac{5t}{t^2+4}$ m s^{-1}. Find its displacement after 5 s if its initial displacement is 4 m.

8 The number of people with measles is increasing at the rate given by $R = \frac{x^2}{3x^3+1}$ people/week. If 3 people had measles initially, find the number with measles after 8 weeks.

7.08 Integration involving trigonometric functions

WS Integrals of trigonometric functions

WS Mixed integration questions

Integrals of trigonometric functions

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

EXAMPLE 20

WS Finding indefinite integrals 1

WS Finding indefinite integrals 2

WS Finding definite integrals

Find the exact value of $\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$.

Solution

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sec^2 x\,dx &= [\tan x]_0^{\frac{\pi}{3}} \\ &= \tan\frac{\pi}{3} - \tan 0 \\ &= \sqrt{3}\end{aligned}$$

Chain rule for trigonometric functions

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

EXAMPLE 21

a Find $\int \sin 3x\,dx$

b Find $\int \cos x^\circ\,dx$.

c Find the exact value of $\int_0^{\frac{\pi}{8}} \sin 2x\,dx$.

Solution

a $\int \sin 3x\,dx = -\frac{1}{3}\cos 3x + C$

c
$$\begin{aligned}\int_0^{\frac{\pi}{8}} \sin 2x\,dx &= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{8}}\\ &= -\frac{1}{2}\cos\left(2\times\frac{\pi}{8}\right) - \left[-\frac{1}{2}\cos(2\times 0)\right]\\ &= -\frac{1}{2}\cos\frac{\pi}{4} + \frac{1}{2}\cos 0\\ &= -\frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{1}{2}\\ &= -\frac{1}{2\sqrt{2}} + \frac{1}{2}\\ &= \frac{2-\sqrt{2}}{4}\end{aligned}$$

b
$$\begin{aligned}\int \cos x^\circ\,dx &= \int \cos\left(\frac{\pi x}{180}\right)dx\\ &= \frac{1}{\frac{\pi}{180}}\sin\left(\frac{\pi x}{180}\right) + C\\ &= \frac{180}{\pi}\sin x^\circ + C\end{aligned}$$

Exercise 7.08 Integration involving trigonometric functions

1 Find the integral of each function.

a $\cos x$	**b** $\sin x$	**c** $\sec^2 x$
d $\frac{\sin x^\circ}{4}$	**e** $\sin 3x$	**f** $-\sin 7x$
g $\sec^2 5x$	**h** $\cos(x+1)$	**i** $\sin(2x-3)$
j $\cos(2x-1)$	**k** $\sin(\pi - x)$	**l** $\cos(x+\pi)$
m $2\sec^2 7x$	**n** $4\sin\left(\frac{x}{2}\right)$	**o** $3\sec^2\left(\frac{x}{3}\right)$

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2 Evaluate each definite integral, giving exact answers where appropriate.

a $\int_0^{\frac{\pi}{2}} \cos x \, dx$ **b** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx$ **c** $\int_{\frac{\pi}{2}}^{\pi} \sin \frac{x}{2} \, dx$

d $\int_0^{\frac{\pi}{2}} \cos 3x \, dx$ **e** $\int_0^{\frac{1}{2}} \sin(\pi x) \, dx$ **f** $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$

g $\int_0^{\frac{\pi}{12}} 3 \cos 2x \, dx$ **h** $\int_0^{\frac{\pi}{10}} -\sin(5x) \, dx$

3 Find:

a $\int (\cos x \cos \frac{\pi}{3} - \sin x \sin \frac{\pi}{3}) \, dx$

b $\int (\sin \pi \cos x - \cos \pi \sin x) \, dx$

4 EXT1 Find the exact value of $\int_0^{\frac{\pi}{6}} \sin x \cos x \, dx$.

5 A curve has $\frac{dy}{dx} = \cos 4x$ and passes through the point $(\pi, \frac{\pi}{4})$.

Find the equation of the curve.

6 A pendulum swings at the rate given by $\frac{dx}{dt} = 12\pi \cos \frac{2\pi t}{3}$ cm s^{-1}.

It starts 2 cm to the right of the origin.

a Find the equation of the displacement of the pendulum.

b Find the exact displacement after:

i 1 s **ii** 5 s

7 The rate at which the depth of water changes in a bay is given by $R = 4\pi \sin \frac{\pi t}{6}$ m h^{-1}.

a Find the equation of the depth of water d over time t hours if the depth is 2 m initially.

b Find the depth after 2 hours.

c Find the highest, lowest and centre of depth of water.

d What is the period of the depth of water?

EXT1 **8** **a** Show $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$.

b Hence find $\int \cos^2 x \, dx$.

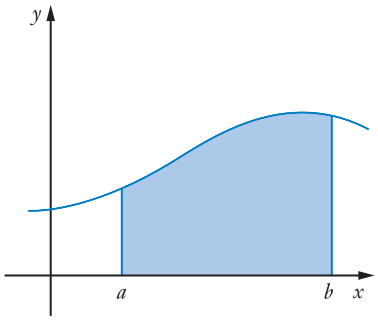
7.09 Areas enclosed by the x-axis

WS
Starting areas

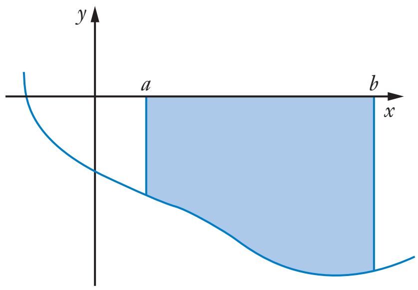
Area under a curve

The definite integral gives the **signed** area under a curve.

Areas above the x-axis give a positive definite integral.



Areas below the x-axis give a negative definite integral.



So, to find areas below the x-axis, we take the absolute value of the definite integral.

Area under a curve

$$\text{Area} = \left|\int_a^b f(x)\,dx\right|$$

It is important to sketch the graph to see where the area is in relation to the x-axis.

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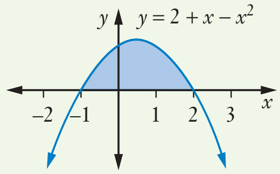
EXAMPLE 22

a Find the area enclosed by the curve $y = 2 + x - x^2$ and the x-axis.

b Find the area bounded by the curve $y = x^2 - 4$ and the x-axis.

Solution

a Sketch the graph of $y = 2 + x - x^2$ and shade the area enclosed between the curve and the x-axis.



The area is above the x-axis, so the definite integral will be positive.

$$\text{Area} = \int_{-}^{2} 2 + x - x^2\, dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-}^{2}$$

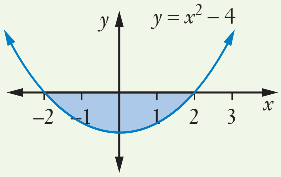
$$= \left(2(2) + \frac{2^2}{2} - \frac{2^3}{3}\right) - \left(2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right)$$

$$= \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= 4\frac{1}{2}$$

So the area is $4\frac{1}{2}$ units^2.

b Sketch the graph of $y = x^2 - 4$.



The definite integral will be negative because the area is below the x-axis.

$$\int_{-2}^{2} (x^2 - 4)\, dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2}$$

$$= \left(\frac{2^3}{3} - 4(2)\right) - \left(\frac{(-2)^3}{3} - 4(-2)\right)$$

$$= \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right)$$

$$= -10\frac{2}{3}$$

$$\text{Area} = \left|-10\frac{2}{3}\right|$$

$$= 10\frac{2}{3} \text{ units}^2$$

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EXAMPLE 23

Area under a curve

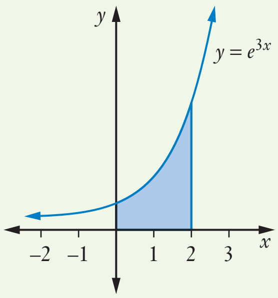
a Find the exact area enclosed between the curve $y = e^{3x}$, the x-axis and the lines $x = 0$ and $x = 2$.

b Find the exact area enclosed between the hyperbola $y = \frac{1}{x}$, the x-axis and the lines $x = 1$ and $x = 2$.

c Find the area enclosed between the curve $y = \cos x$, the x-axis and the lines $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

Solution

a Sketch the graph of $y = e^{3x}$.

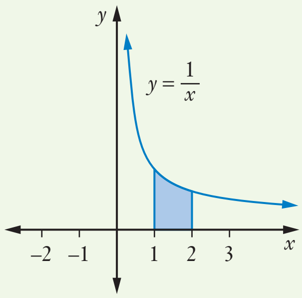


The definite integral will be positive because the area is above the x-axis.

$$\begin{aligned}\text{Area} &= \int_0^2 e^{3x}\,dx \\ &= \left[\frac{1}{3}e^{3x}\right]_0^2 \\ &= \frac{1}{3}e^{3\times 2} - \frac{1}{3}e^{3\times 0} \\ &= \frac{1}{3}e^6 - \frac{1}{3}e^0 \\ &= \frac{1}{3}(e^6 - 1)\end{aligned}$$

So the area is $\frac{1}{3}(e^6 - 1)$ units^2.

b Sketch the graph of $y = \frac{1}{x}$. For $x = 1$ and $x = 2$, we only need to sketch the graph in the 1st quadrant.



The definite integral will be positive because the area is above the x-axis.

$$\begin{aligned}\text{Area} &= \int^2 \frac{dx}{x} \\ &= \left[\ln|x|\right]^2 \\ &= \ln 2 - \ln 1 \\ &= \ln 2 - 0 \\ &= \ln 2\end{aligned}$$

Absolute value not requred as 2 and 1 are postve

So the area is $\ln 2$ units^2.

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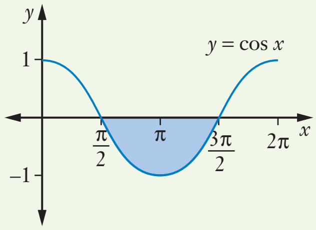
c Sketch the graph of $y = \cos x$.

The definite integral will be negative because the area is below the x-axis.



$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\, dx = [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)$$

$$= -1 - 1$$

$$= -2$$

$$\text{Area} = |-2|$$

$$= 2 \text{ units}^2$$

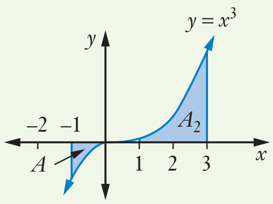
If the area has some parts above the x-axis and some below the x-axis, we need to find these separately.

EXAMPLE 24

Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -1$ and $x = 3$.

Solution

Sketch the graph of $y = x^3$.



There are 2 areas, marked A_1 and A_2 on the diagram.

A_1 is below the x-axis so the integral will be negative.

A_2 is above the x-axis so the integral will be positive.

A_1:

$$\int_{-}^{0} x^3\, dx = \left[\frac{x^4}{4}\right]_{-}^{0}$$

$$= \frac{0^4}{4} - \frac{(-1)^4}{4}$$

$$= -\frac{1}{4}$$

$$\text{So } A_1 = \left|-\frac{1}{4}\right|$$

$$= \frac{1}{4} \text{ units}^2$$

$$A_2 = \int_0^3 x^3\, dx$$

$$= \left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{3^4}{4} - \frac{0^4}{4}$$

$$= \frac{81}{4} \text{ units}^2$$

$$\text{Total area} = A_1 + A_2$$

$$= \frac{1}{4} + \frac{81}{4}$$

$$= 20\frac{1}{2} \text{ units}^2$$

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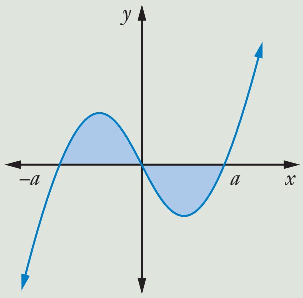
Odd and even functions

Some functions have special properties that we can use to find their areas

Odd functions

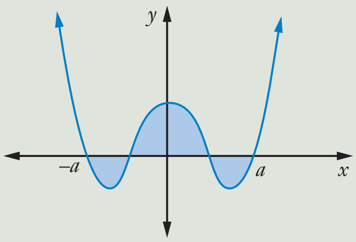
If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\,dx = 0$.

The positive and negative areas cancel each other out.



Even functions

If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.



EXAMPLE 25

Find the area between the curve:

a $y = x^3$, the x-axis and the lines $x = -2$ and $x = 2$

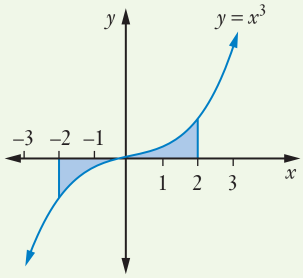
b $y = x^2$, the x-axis and the lines $x = -4$ and $x = 4$

Solution

a Sketch the graph of $y = x^3$ and shade the area bounded by the curve, the x-axis and the boundaries $x = \pm 2$.

$y = x^3$ is an odd function since $f(-x) = -f(x)$.

This means that the shaded areas are symmetrical. We can find the area between $x = 0$ and $x = 2$.



The total area will be twice this area.

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$$\text{Area} = 2\int_0^2 x^3\,dx$$

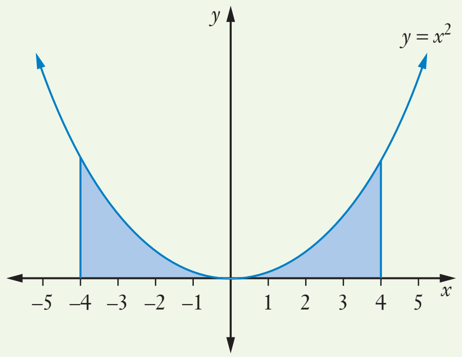
$$= 2\left[\frac{x^4}{4}\right]_0^2$$

$$= 2\left(\frac{2^4}{4} - \frac{0^4}{4}\right)$$

$$= 8$$

So area is 8 units^2.

b Sketch the graph of $y = x^2$ and shade the area enclosed between the curve, the x-axis and the lines $x = \pm 4$.



$y = x^2$ is an even function since $f(-x) = f(x)$.

$$\text{Area} = \int_{-4}^{4} x^2\,dx$$

$$= 2\int_0^4 x^2\,dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^4$$

$$= 2\left(\frac{4^3}{3} - \frac{0^3}{3}\right)$$

$$= 2 \times \frac{64}{3}$$

$$= 42\frac{2}{3}$$

So area is $42\frac{2}{3}$ units^2.

Exercise 7.09 Areas enclosed by the *x*-axis

1 Find the area enclosed between the curve $y = 1 - x^2$ and the x-axis.

2 Find the area bounded by the curve $y = x^2 - 9$ and the x-axis.

3 Find the area enclosed between the curve $y = x^2 + 5x + 4$ and the x-axis.

4 Find the area enclosed between the curve $y = x^2 - 2x - 3$ and the x-axis.

5 Find the area bounded by the curve $y = -x^2 + 9x - 20$ and the x-axis.

6 Find the area enclosed between the curve $y = -2x^2 - 5x + 3$ and the x-axis.

7 Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = 0$ and $x = 2$.

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8 Find the area enclosed between the curve $y = x^4$, the x-axis and the lines $x = -1$ and $x = 1$.

9 Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -2$ and $x = 2$.

10 Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -3$ and $x = 2$.

11 Find the exact area enclosed by the curve $y = 2e^{2x}$, the x-axis and the lines $x = 1$ and $x = 2$.

12 Find the exact area bounded by the curve $y = e^{4x-3}$, the x-axis and the lines $x = 0$ and $x = 1$.

13 Find the area enclosed by the curve $y = x + e^{-x}$, the x-axis and the lines $x = 0$ and $x = 2$, correct to 2 decimal places.

14 Find the area bounded by the curve $y = e^{5x}$, the x-axis and the lines $x = 0$ and $x = 1$, correct to 3 significant figures.

15 Find the area enclosed between the curve $y = \sin x$ and the x-axis in the domain $[0, 2\pi]$.

16 Find the exact area bounded by the curve $y = \cos 3x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{12}$.

17 Find the area enclosed between the curve $y = \sec^2 \frac{x}{4}$, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$, correct to 2 decimal places.

18 Find the area bounded by the curve $y = 3x^2$, the x-axis and the lines $x = -1$ and $x = 1$.

19 Find the area enclosed between the curve $y = x^2 + 1$, the x-axis and the lines $x = -2$ and $x = 2$.

20 Find the area enclosed between the curve $y = x^2$, the x-axis and the lines $x = -3$ and $x = 2$.

21 Find the area enclosed between the curve $y = x^2 + x$, and the x-axis.

22 Find the area enclosed between the curve $y = \frac{1}{x^2}$, the x-axis and the lines $x = 1$ and $x = 3$.

23 Find the area enclosed between the curve $y = \frac{2}{(x-3)^2}$, the x-axis and the lines $x = 0$ and $x = 1$.

24 Find the exact area between the curve $y = \frac{1}{x}$, the x-axis and the lines $x = 2$ and $x = 3$.

25 Find the exact area bounded by the curve $y = \frac{1}{x-1}$, the x-axis and the lines $x = 4$ and $x = 7$.

26 Find the area bounded by the curve $y = \frac{x}{x^2+1}$, the x-axis and the lines $x = 2$ and $x = 4$, correct to 2 decimal places.

27 Find the area bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$.

28 Find the area bounded by the curve $y = \sqrt{x+2}$, the x-axis and the line $x = 7$.

29 Use the trapezoidal rule with 4 subintervals to find the area bounded by the curve $y = \ln x$, the x-axis and the line $x = 5$, correct to 2 decimal places.

30 Find the area bounded by the x-axis, the curve $y = x^3$ and the lines $x = -a$ and $x = a$.

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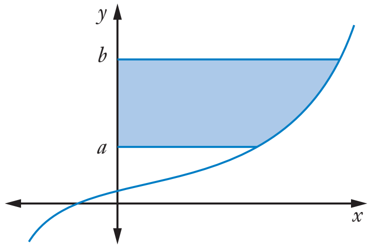
7.10 Areas enclosed by the *y*-axis

Areas of integration

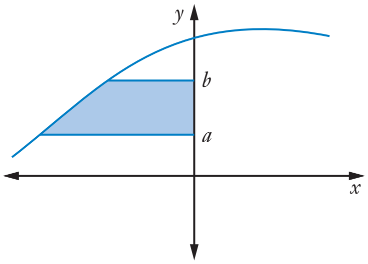


We can find an area bounded by a graph and the y-axis by writing the equation in the form $x = f(y)$.

The definite integral gives the **signed** area.

Areas to the right of the y-axis give a positive definite integral.

Areas to the left of the y-axis give a negative definite integral.



Area bounded by a curve and the *y*-axis

For the curve $x = f(y)$:

$$\text{Area} = \left| \int_a^b f(y)\,dy \right|$$

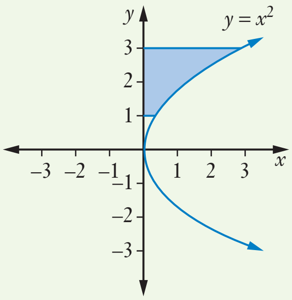
EXAMPLE 26

a Find the area enclosed by the curve $x = y^2$, the y-axis and the lines $y = 1$ and $y = 3$.

b Find the area enclosed by the curve $y = x^2$, the y-axis and the lines $y = 0$ and $y = 4$ in the first quadrant.

Solution

a Sketch the graph of $x = y^2$ and shade the area bounded by the curve, the y-axis and the lines $y = 1$ and $y = 3$.



This is the same shape as the parabola $y = x^2$ with the x and y values swapped.

For example, when $x = 1, y = \pm 1$, when $x = 4, y = \pm 2$.

The area is to the right of the y-axis so the integral will be positive.

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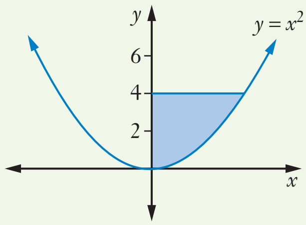
$$\text{Area} = \int_a^b f(y)\,dy$$
$$= \int^{3} y^2\,dy$$
$$= \left[\frac{y^3}{3}\right]^3$$
$$= \frac{3^3}{3} - \frac{1^3}{3}$$
$$= 8\frac{2}{3}$$

So the area is $8\frac{2}{3}$ units^2

b Sketch the graph of $y = x^2$ and shade the area enclosed between the curve, the y-axis and the lines $y = 0$ and $y = 4$.



The area is to the right of the y-axis so the integral will be positive.

Change the subject of the equation to x.

$$y = x^2$$
$$\pm\sqrt{y} = x$$

In the first quadrant:

$$x = \sqrt{y}$$
$$= y^{\frac{1}{2}}$$

$$\text{Area} = \int_a^b f(y)\,dy$$
$$= \int_0^4 y^{\frac{1}{2}}\,dy$$
$$= \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$
$$= \left[\frac{2\sqrt{y^3}}{3}\right]_0^4$$
$$= \frac{2\sqrt{4^3}}{3} - \frac{2\sqrt{0^3}}{3}$$
$$= 5\frac{1}{3}$$

So the area is $5\frac{1}{3}$ units^2.

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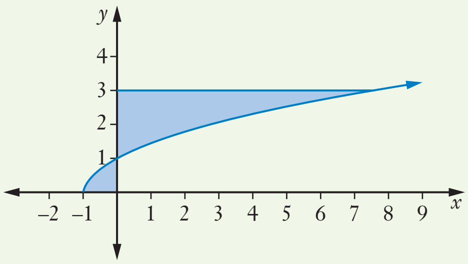
EXAMPLE 27

Find the area enclosed between the curve $y = \sqrt{x+1}$, the y-axis and the lines $y = 0$ and $y = 3$.

Solution

The area between $y = 0$ and $y = 1$ is to the left of the y-axis so the integral will be negative.

The area between $y = 1$ and $y = 3$ is to the right of the y-axis so the integral will be positive.



Changing the subject:

$$y = \sqrt{x+1}$$

$$y^2 = x + 1$$

$$y^2 - 1 = x$$

$$\int_0 (y^2 - 1)\,dy = \left[\frac{y^3}{3} - y\right]_0$$

$$= \left(\frac{1^3}{3} - 1\right) - \left(\frac{0^3}{3} - 0\right)$$

$$= -\frac{2}{3}$$

$$A_1 = \left|-\frac{2}{3}\right|$$

$$= \frac{2}{3}\text{ units}^2$$

$$A_2 = \int^3 (y^2 - 1)\,dy$$

$$= \left[\frac{y^3}{3} - y\right]^3$$

$$= \left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right)$$

$$= 6\frac{2}{3}\text{ units}^2$$

$$\text{Total area} = A_1 + A_2$$

$$= \frac{2}{3} + 6\frac{2}{3}$$

$$= 7\frac{1}{3}\text{ units}^2$$

Exercise 7.10 Areas enclosed by the *y*-axis

1 Find the area bounded by the y-axis, the curve $x = y^2$ and the lines $y = 0$ and $y = 4$.

2 Find the area enclosed between the curve $x = y^3$, the y-axis and the lines $y = 1$ and $y = 3$.

3 Find the area in the first quadrant enclosed between the curve $y = x^2$, the y-axis and the lines $y = 1$ and $y = 4$.

4 Find the area between the lines $y = x - 1$, $y = 0$, $y = 1$ and the y-axis.

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5 Find the area bounded by the line $y = 2x + 1$, the y-axis and the lines $y = 3$ and $y = 4$.

6 Find the area bounded by the curve $y = \sqrt{x}$, the y-axis and the lines $y = 1$ and $y = 2$.

7 Find the area bounded by the curve $x = y^2 - 2y - 3$ and the y-axis.

8 Find the area bounded by the curve $x = -y^2 - 5y - 6$ and the y-axis.

9 Find the area enclosed by the curve $y = \sqrt{3x - 5}$, the y-axis and the lines $y = 2$ and $y = 3$.

10 Find the area in the first quadrant enclosed between the curve $y = \frac{1}{x^2}$, the y-axis and the lines $y = 1$ and $y = 4$.

11 Find the area enclosed between the curve $y = x^3$, the y-axis and the lines $y = 1$ and $y = 8$.

12 Find the area enclosed between the curve $y = x^3 - 2$ and the y-axis between $y = -1$ and $y = 25$.

13 Find the area in the second quadrant enclosed between the lines $y = 4$ and $y = 1 - x$.

14 Find the area enclosed between the y-axis and the curve $x = y(y - 2)$.

15 Find the area in the first quadrant bounded by the curve $y = x^4 + 1$, the y-axis and the lines $y = 1$ and $y = 3$, correct to 2 significant figures.

16 Find the area between the curve $y = \ln x$, the y-axis and the lines $y = 2$ and $y = 4$, correct to 3 significant figures.

7.11 Sums and differences of areas

WS Sums and differences of areas

WS Calculating physical areas

WS Calculating areas between curves

WS Areas between curves

WS Areas between curves 2

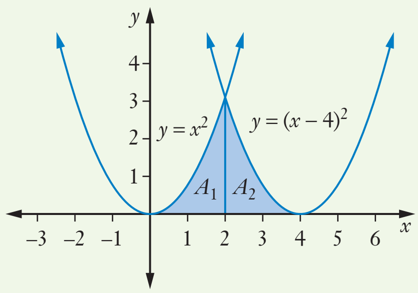
EXAMPLE 28

a Find the area enclosed between the curves $y = x^2$, $y = (x - 4)^2$ and the x-axis.

b Find the area enclosed between the curve $y = x^2$ and the line $y = x + 2$.

Solution

a Sketch the graphs of $y = x^2$ and $y = (x - 4)^2$ (translation of $y = x^2$ by 4 units to the right) and shade the area enclosed between the curves and the x-axis.



Find the x values of their intersection ($x = 2$, from the graph or by solving simultaneous equations).

Shaded area $= A_1 + A_2$

$$= \int_0^2 x^2\,dx + \int_2^4 (x - 4)^2\,dx$$

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Area A_1:

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{2^3}{3} - \frac{0^3}{3}$$

$$= 2\frac{2}{3}$$

Area A_2:

$$\int_2^4 (x-4)^2\,dx = \left[\frac{(x-4)^3}{3}\right]_2^4$$

$$= \frac{(4-4)^3}{3} - \frac{(2-4)^3}{3}$$

$$= 0 + \frac{8}{3}$$

$$= 2\frac{2}{3}$$

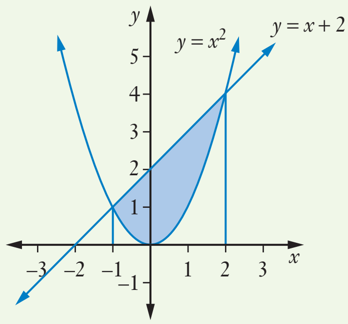
Total area $= A_1 + A_2$

$$= 2\frac{2}{3} + 2\frac{2}{3}$$

$$= 5\frac{1}{3}$$

So area is $5\frac{1}{3}$ units^2.

b Sketch the graphs of $y = x^2$ and $x + 2$ and shade the area enclosed between them.



We can find the x values of the points of intersection of the functions from the graph or by solving simultaneous equations:

$y = x^2$ [1]

$y = x + 2$ [2]

Substituting [1] into [2]:

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2, \qquad x = -1$$

Notice that between $x = -1$ and $x = 2$, the graph of $y = x + 2$ is *above* the graph of $y = x^2$.

So we can find the area by integrating $(x + 2)$ and x^2 between $x = -1$ and $x = 2$ and then finding their difference.

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$$A = \int_{-}^{2} (x+2)\, dx - \int_{-}^{2} x^2\, dx = \int_{-}^{2} (x + 2 - x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-}^{2}$$

$$= \left[\frac{2^2}{2} + 2(2) - \frac{2^3}{3}\right] - \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1^3)}{3}\right]$$

$$= \frac{10}{3} - \left(\frac{-7}{6}\right)$$

$$= \frac{9}{2}$$

$$= 4\frac{1}{2}$$

So area is $4\frac{1}{2}$ units^2.

Exercise 7.11 Sums and differences of areas

1 Find the area bounded by the line $y = 1$ and the curve $y = x^2$.

2 Find the area enclosed between the line $y = 2$ and the curve $y = x^2 + 1$.

3 Find the area enclosed by the curve $y = x^2$ and the line $y = x$.

4 Find the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$.

5 Find the area enclosed between the curve $y = x^2$ and the line $y = x + 6$.

6 Find the area bounded by the curve $y = x^3$ and the line $y = 4x$.

7 Find the area enclosed between the curves $y = (x - 1)^2$ and $y = (x + 1)^2$ and the x-axis.

8 Find the area enclosed between the curve $y = x^2$ and the line $y = -6x + 16$.

9 Find the area enclosed between the curve $y = x^3$, the x-axis and the line $y = -3x + 4$.

10 Find the area enclosed by the curves $y = (x - 2)^2$ and $y = (x - 4)^2$.

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11 Find the area enclosed between the curves $y = x^2$ and $y = x^3$.

12 Find the area enclosed by the curves $y = x^2$ and $x = y^2$.

13 Find the area bounded by the curve $y = x^2 + 2x - 8$ and the line $y = 2x + 1$.

14 Find the area bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$.

15 Find the exact area enclosed between the curve $y = \sqrt{4 - x^2}$ and the line $x - y + 2 = 0$.

16 Find the exact area in the first quadrant between the curve $y = \frac{1}{x}$, the x-axis and the lines $y = x$ and $x = 2$.

17 Find the exact area bounded by the curves $y = \sin x$ and $y = \cos x$ in the domain $[0, 2\pi]$.

18 Find the exact area enclosed between the curve $y = e^{2x}$ and the lines $y = 1$ and $x = 2$.

19 Find the exact area enclosed by the curve $y = \sin x$ and the line $y = \frac{1}{2}$ for $[0, 2\pi]$.

Summary of integration rules

Rule	Chain rule
$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$	$\int f'(x)[f(x)]^n \, dx = \frac{1}{n+1}\left[f(x)^{n+1}\right] + C$
$\int e^x \, dx = e^x + C$	$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$
$\int a^x \, dx = \frac{1}{\ln a} a^x + C$	
$\int \frac{1}{x} \, dx = \ln\lvert x\rvert + C$	$\int \frac{f'(x)}{f(x)} \, dx = \ln\lvert f(x)\rvert + C$
$\int \cos x \, dx = \sin x + C$	$\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$
$\int \sin x \, dx = -\cos x + C$	$\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$
$\int \sec^2 x \, dx = \tan x + C$	$\int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + C$

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[redacted]

WS Derivatives and Integrals

Qz Practice quiz

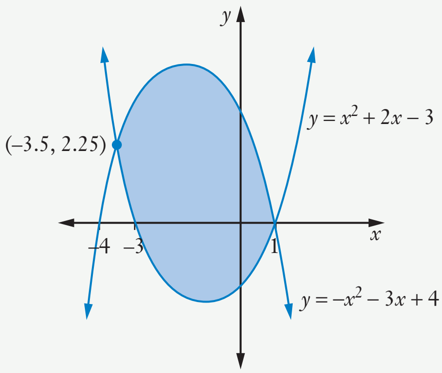
For Questions **1** to **4**, choose the correct answer **A**, **B**, **C** or **D**.

1 Find $\int \sin(6x)\,dx$.

A $\frac{1}{6}\cos(6x) + C$ **B** $6\cos(6x) + C$

C $-6\cos(6x) + C$ **D** $-\frac{1}{6}\cos(6x) + C$

2 Find the shaded area below.



A $\int_{-4}(-x^2 - 3x + 4)\,dx - \int_{-4}(x^2 + 2x - 3)\,dx$

B $\int_{-3\ 5}(-x^2 - 3x + 4)\,dx - \int_{-3\ 5}(x^2 + 2x - 3)\,dx$

C $\int_{-3}(-x^2 - 3x + 4)\,dx + \int_{-3}(x^2 + 2x - 3)\,dx$

D $\int_{-3\ 5}(-x^2 - 3x + 4)\,dx + \int_{-3\ 5}(x^2 + 2x - 3)\,dx$

3 Find $\int 4e^{3x}\,dx$.

A $\frac{4}{3}e^{3x} + C$ **B** $\frac{3}{4}e^{3x} + C$ **C** $12e^{3x} + C$ **D** $\frac{1}{12}e^{3x} + C$

4 Find $\int \frac{x}{x^2 + 3}\,dx$.

A $\frac{2}{(x^2 + 3)^2} + C$ **B** $2\ln|x^2 + 3| + C$

C $\frac{1}{2(x^2 + 3)^2}$ **D** $\frac{1}{2}\ln|x^2 + 3| + C$

5 **a** Use the trapezoidal rule with 2 subintervals to find an approximation to $\int^{2} \frac{dx}{x^2}$.

b Use integration to find the exact value of $\int^{2} \frac{dx}{x^2}$.

6 Find the integral of:

a $3x + 1$ **b** $\frac{5x^2 - x}{x}$ **c** $\sqrt{x}$

d $(2x + 5)^7$ **e** $x^3(3x^4 - 2)^4$

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7 Find $\int 3^x \, dx$.

8 Find the approximate area under the curve $f(x) = x^3$ between $x = 1$ and $x = 3$ by using:

a 4 inner rectangles **b** 4 outer rectangles **c** a trapezium

9 Evaluate:

a $\int_0^2 (x^3 - 1)\, dx$ **b** $\int_{-} x^5 \, dx$ **c** $\int_0 (3x - 1)^4 \, dx$

d $\int_0 x^2 (x^3 - 5)^2 \, dx$ **e** $\int_{-}^2 3x(x^2 + 1)^3 \, dx$

10 Find the area enclosed between the curve $y = \ln x$, the y-axis and the lines $y = 1$ and $y = 3$.

11 Find the area bounded by the curve $y = x^2$, the x-axis and the lines $x = -1$ and $x = 2$.

12 Find $\int \sin x° \, dx$.

13 Find the area enclosed between the curves $y = x^2$ and $y = 2 - x^2$.

14 Find the indefinite integral of:

a e^{4x} **b** $\dfrac{x}{x^2 - 9}$ **c** e^{-x}

d $\dfrac{1}{x + 4}$ **e** $(x - 3)(x^2 - 6x + 1)^8$

15 Evaluate $\int^2 \dfrac{3x^4 - 2x^3 + x^2 - 1}{x^2} dx$.

16 Find the exact area in the first quadrant bounded by $x^2 + y^2 = 9$, the y-axis and the lines $y = 0$ and $y = 3$.

17 Find the area bounded by the curve $y = x^3$, the y-axis and the lines $y = 0$ and $y = 1$.

18 Find the integral of $(7x + 3)^{11}$.

19 Find the area bounded by the curve $y = x^2 - x - 2$, the x-axis and the lines $x = 1$ and $x = 3$.

20 Find the exact area bounded by the curve $y = e^{2x}$, the x-axis and the lines $x = 2$ and $x = 5$.

21 Use the trapezoidal rule with 4 strips to find the area bounded by the curve $y = \ln(x^2 - 1)$, the x-axis and the lines $x = 3$ and $x = 5$.

22 Evaluate $\int_0^4 (3t^2 - 6t + 5)\, dt$.

23 Find the indefinite integral of:

a $\sin 2x$ **b** $3 \cos x$ **c** $\sec^2 5x$ **d** $1 + \sin x$

24 Find the area bounded by the curve $y = x^2 + 2x - 15$ and the x-axis.

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25 The rate at which a metal cools is given by $R = -16e^{-0.4t}$ degrees min^{-1}. If the temperature is initially 215°C, find:

a the equation for the temperature T of the metal

b the temperature, to the nearest degree, of the metal after:
i 5 minutes **ii** half an hour

26 Evaluate:

a $\int_0^{\frac{\pi}{4}} \cos x \, dx$ **b** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx$

27 Find:

a $\int 5(2x-1)^4 \, dx$ **b** $\int \frac{3x^5}{4} \, dx$

28 Find the exact area bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

29 Find:

a $\int x^2(x^3-2)^5 \, dx$ **b** $\int x(5x^2+2)^4 \, dx$

c $\int 5x^3(2x^4-1)^2 \, dx$ **d** $\int (x+2)(x^2+4x-3)^3 \, dx$

30 Find the area bounded by the curve $y = \cos 2x$, the x-axis and the lines $x = 0$ and $x = \pi$.

31 Find (in exact form) the approximate area bounded by the curve $y = \sqrt{x-2}$, the x-axis and the line $x = 4$, using:

a a triangle **b** 2 inner rectangles **c** 2 outer rectangles

32 Find $f(x)$ given $f'(x)$ and a point on the graph of $f(x)$.

a $f'(x) = 3x(2x^2-1)^4$ and passing through $(1, 3)$

b $f'(x) = \sec^2 2x$ and passing through $\left(\frac{\pi}{6} \; \frac{\sqrt{3}}{2}\right)$

c $f'(x) = e^{5x}$ and passing through $(0, \frac{1}{5})$

d $f'(x) = x^3(x^4-15)^3$ and passing through $(2, 0)$

e $f'(x) = \frac{3x^3}{x^4+1}$ and passing through $(0, 2)$

33 The velocity of a particle is given by $v = \frac{t^2}{\sqrt{t^3+9}}$ m s^{-1}.

If the initial displacement is –2 m, find:

a the equation for displacement

b the displacement after 5 s

c when the displacement is 10 m

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7. CHALLENGE EXERCISE

1 **a** Show that $f(x) = x^3 + x$ is an odd function.

b Hence find the value of $\int_{-2}^{2} f(x)\,dx$.

c Find the total area between $y = f(x)$, the x-axis and the lines $x = -2$ and $x = 2$.

2 **a** Show that $\sec x \operatorname{cosec} x = \dfrac{\sec^2 x}{\tan x}$.

b Hence, or otherwise, find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec} x \sec x\,dx$.

3 Find the area enclosed between the curves $y = (x - 1)^2$ and $y = 5 - x^2$.

4 Find the exact value of $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2 2x\,dx$.

5 Evaluate $\int_0 \dfrac{x}{(3x^2 - 4)^2}\,dx$.

6 Use the trapezoidal rule with 4 subintervals to find the area enclosed between the curve $y = \dfrac{3}{x - 2}$, the y-axis and the lines $y = 1$ and $y = 3$.

7 **a** Sketch the curve $y = x(x - 1)(x + 2)$.

b Find the total area enclosed between the curve and the x-axis.

8 Find the area bounded by the parabola $y = x^2$ and the line $y = 4 - x$ correct to 2 decimal places.

9 **a** Find the derivative of $x\sqrt{x + 3}$.

b Hence find $\int \dfrac{x + 2}{\sqrt{x + 3}}\,dx$.

10 **a** Find $\dfrac{d}{dx}(x^2 \ln x)$.

b Hence find the exact value of $\int^{3} 2x(1 + 2\ln x)\,dx$.

11 Find the area enclosed between the curves $y = \sqrt{x}$ and $y = x^3$.

12 **a** Find the sum of 50 terms of the sequence $2^0, 2^{0.2}, 2^{0.4}, 2^{0.6}, \ldots$

b Hence use 50 inner rectangles to find the approximate area under the curve $y = 2^x$ between $x = 0$ and $x = 10$.

c Find this approximate area by using 100 outer rectangles.

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CALCULUS

8.

FURTHER INTEGRATION

In this Mathematics Extension 1 chapter, you will use integration to find volumes of solids of revolution.

You will learn how to integrate by substitution and integrate the inverse trigonometric functions.

CHAPTER OUTLINE

8.01 EXT1 Volumes of solids
8.02 EXT1 Integration by substitution
8.03 EXT1 Definite integrals by substitution
8.04 EXT1 Integration of $\sin^2 x$ and $\cos^2 x$
8.05 EXT1 Integrals involving inverse trigonometric functions

IN THIS CHAPTER YOU WILL:

- EXT1 find volumes of solids by using integration
- EXT1 understand how to integrate using substitution
- EXT1 apply trigonometric identities to integrate $\sin^2 x$ and $\cos^2 x$
- EXT1 integrate functions that result in inverse trigonometric functions

TERMINOLOGY

integrand: A function that is to be integrated.

solid of revolution: A solid formed by rotating the area bounded by the graph of a function about the x-axis or the y-axis.

WS Volumes of integration

WS Volumes

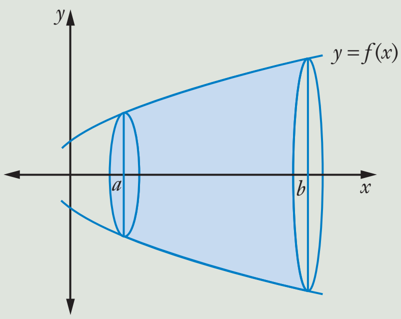
WS Solids of revolution

EXT1 8.01 Volumes of solids

We can find the volume of a solid by rotating an area under a curve about the x-axis or y-axis. We call the figure a **solid of revolution**.

Volumes about the *x*-axis

$$V = \pi\int_a^b y^2\,dx$$



Proof

Take a disc of the solid with width δx.

$V = \pi r^2 h = \pi y^2 \delta x$

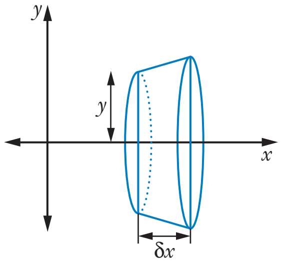
Taking the sum of an infinite number of these discs:

$$V = \lim_{\delta x \to 0}\left[\sum \pi y^2 \delta x\right]$$

$$= \int_a^b \pi y^2\,dx$$

$$= \pi\int_a^b y^2\,dx$$

The **integrand** is y^2 so we integrate y^2 or $[f(x)]^2$.



δx is 'delta x' and means a small change in x.

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EXAMPLE 1

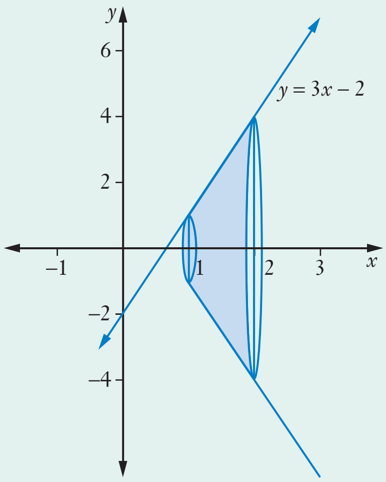
a Find the volume of the solid formed when the line $y = 3x - 2$ is rotated about the x-axis from $x = 1$ to $x = 2$.

b Find the volume of the solid of revolution formed when the curve $x^2 + y^2 = 9$ is rotated about the x-axis between $x = 1$ and $x = 3$.

Volumes of solids of revolution

Solution

a Sketch the line $y = 3x - 2$ and shade the region under the line between $x = 1$ and $x = 2$. The shaded region is rotated about the x-axis as shown.



The integrand is y^2:

$$y = 3x - 2$$

$$y^2 = (3x - 2)^2$$

$$V = \pi \int_a^b y^2\,dx$$

$$= \pi \int^2 (3x - 2)^2\,dx$$

$$= \pi \left[\frac{(3x-2)^3}{3 \times 3}\right]^2$$

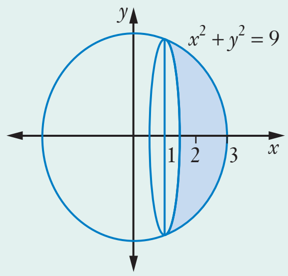
$$= \pi \left[\frac{(3 \times 2 - 2)^3}{9} - \frac{(3 \times 1 - 2)^3}{9}\right]$$

$$= \pi \left[\frac{4^3}{9} - \frac{1^3}{9}\right]$$

$$= 7\pi$$

So the volume is 7π units^3. ← This is the volume in exact form.

b Sketch and rotate the circle about the x-axis between $x = 1$ and $x = 3$. (This solid is called a spherical cap.)



The integrand is y^2:

$x^2 + y^2 = 9$

$y^2 = 9 - x^2$

$$V = \pi \int_a^b y^2 \, dx$$

$$= \pi \int_1^3 (9 - x^2)\, dx$$

$$= \pi \left[9x - \frac{x^3}{3} \right]_1^3$$

$$= \pi \left[\left(9 \times 3 - \frac{3^3}{3} \right) - \left(9 \times 1 - \frac{1^3}{3} \right) \right]$$

$$= \pi \left(27 - 9 - 9 + \frac{1}{3} \right)$$

$$= \pi \left(\frac{28}{3} \right)$$

$$= \frac{28\pi}{3}$$

So the volume is $\frac{28\pi}{3}$ units^3.

EXAMPLE 2

a Find the exact area of the region bounded by the function $y = e^x$, the x-axis and the lines $x = 0$ and $x = 2$.

b Find the exact volume of the solid of revolution formed when the curve $y = e^x$ is rotated about the x-axis from $x = 0$ to $x = 2$.

Solution

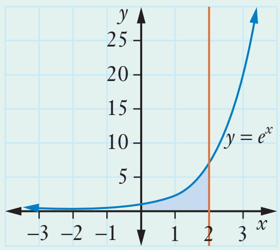
a Area $= \int_0^2 e^x \, dx$

$$= [e^x]_0^2$$

$$= e^2 - e^0$$

$$= e^2 - 1$$

So area is $e^2 - 1$ units^2



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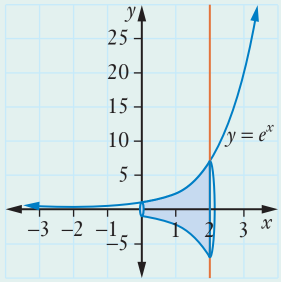
b This region is rotated about the x-axis.



$$y^2 = (e^x)^2$$
$$= e^{2x}$$
$$V = \pi \int_a^b y^2\, dx$$
$$= \pi \int_0^2 e^{2x}\, dx$$
$$= \pi \left[\frac{1}{2} e^{2x} \right]_0^2$$
$$= \pi \left(\frac{1}{2} e^{2 \times 2} - \frac{1}{2} e^{2 \times 0} \right)$$
$$= \pi \left(\frac{1}{2} e^4 - \frac{1}{2} \right)$$
$$= \frac{\pi}{2}(e^4 - 1)$$

So the volume is $\frac{\pi}{2}(e^4 - 1)$ units^3.

INVESTIGATION

SPOT THE ERROR

Find the volume of the solid of revolution formed when the curve $y = x^2 + 1$ is rotated about the x-axis between $x = 0$ and $x = 2$.

Here is Damien's answer:

$$y = x^2 + 1$$
$$y^2 = (x^2 + 1)^2$$
$$V = \pi \int_a^b y^2\, dx$$
$$= \pi \int_0^2 (x^2 + 1)^2\, dx$$
$$= \pi \left[\frac{(x^2+1)^3}{3} \right]_0^2$$
$$= \pi \left[\frac{(2^2+1)^3}{3} - \frac{(0^2+1)^3}{3} \right]$$
$$= \pi \left[\frac{5^3}{3} - \frac{1^3}{3} \right]$$
$$= \pi \left(\frac{124}{3} \right)$$
$$= \frac{124\pi}{3}$$

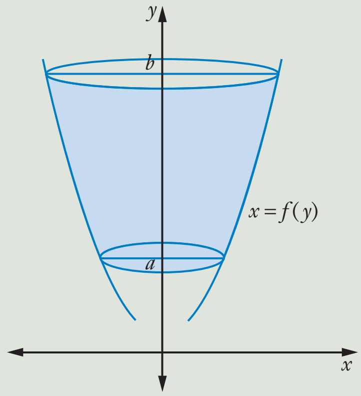
So the volume is $\frac{124\pi}{3}$ units^3.

What was Damien's mistake?

The formula for rotations about the y-axis is similar to the formula for the x-axis. The proof is the same, with the x- and y-values swapped.

Volumes about the *y*-axis

$$V = \pi \int_a^b x^2 \, dy$$



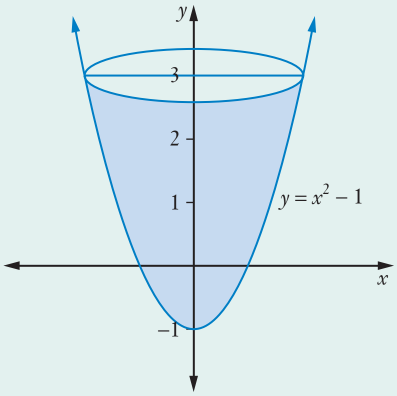
EXAMPLE 3

a Find the volume of the solid formed when $y = x^2 - 1$ is rotated about the y-axis from $y = -1$ to $y = 3$.

b Find the exact volume of the solid of revolution formed if the function $y = \ln x$ is rotated about the y-axis from $y = 1$ to $y = 4$.

Solution

a Sketch the parabola $y = x^2 - 1$ and shade the region left of the curve between $y = -1$ and $y = 3$. Rotate this region about the y-axis as shown. The solid is called a paraboloid.



The integrand is x^2:

$$y = x^2 - 1$$

$$y + 1 = x^2$$

$$V = \pi\int_a^b x^2 \, dy$$

$$= \pi \int_{-}^{3} (y+1)\, dy$$

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$$= \pi\left[\frac{y^2}{2} + y\right]_{-}^{3}$$

$$= \pi\left[\left(\frac{3^2}{2} + 3\right) - \left(\frac{(-1)^2}{2} + (-1)\right)\right]$$

$$= \pi\left(\frac{9}{2} + 3 - \frac{1}{2} + 1\right)$$

$$= 8\pi$$

So the volume is 8π units^3.

b The integrand is x^2:

$$y = \ln x$$

$$x = e^y$$

$$x^2 = (e^y)^2$$

$$= e^{2y}$$

$$V = \pi\int_a^b x^2\,dy$$

$$= \pi\int^4 e^{2y}\,dy$$

$$= \pi\left[\frac{1}{2}e^{2y}\right]^4$$

$$= \pi\left(\frac{1}{2}e^{2\times 4} - \frac{1}{2}e^{2\times 1}\right)$$

$$= \pi\left(\frac{1}{2}e^8 - \frac{1}{2}e^2\right)$$

$$= \frac{\pi e^2}{2}(e^6 - 1)$$

So the volume is $\frac{\pi e^2}{2}(e^6 - 1)$ units^3.

TECHNOLOGY

Volumes of solids

Use graphing software such as GeoGebra or Desmos to sketch and calculate the volumes of solids for different functions.

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EXT1 Exercise 8.01 Volumes of solids

1 Find the volume of the solid of revolution formed when the curve $y = x^2$ from $x = 0$ to $x = 3$ is rotated about the x-axis.

2 Find the volume of the solid formed when the line $y = x + 1$ is rotated about the x-axis between $x = 2$ and $x = 7$.

3 Find the volume of the solid of revolution that is formed when the curve $y = x^2 + 2$ from $x = 0$ to $x = 2$ is rotated about the x-axis.

4 Determine the volume of the solid formed when the curve $y = x^3$ from $x = 0$ to $x = 1$ is rotated about the x-axis.

5 Find the volume, correct to 2 decimal places, of the solid formed when the curve $y = \sec \pi x$ from $x = 0$ to $x = 0.15$ is rotated about the x-axis.

6 Find the exact volume of the solid of revolution formed when the curve $y = e^x$ from $x = 0$ to $x = 3$ is rotated about the x-axis.

7 Find the volume of the solid of revolution formed by rotating the line $y = 4x - 1$ about the x-axis from $x = 2$ to $x = 4$.

8 Find the volume of the hemisphere formed when the curve $x^2 + y^2 = 1$ is rotated about the x-axis between $x = 0$ and $x = 1$.

9 Find the volume of the solid formed when the curve $y = e^{-x} + 1$ is rotated about the x-axis from $x = 1$ to $x = 2$, correct to 1 decimal place.

10 Find, in exact form, the volume of the solid of revolution formed by rotating the curve $y = \sqrt{\sin 2x}$ about the x-axis from $x = 0$ to $x = \frac{\pi}{4}$.

11 Find the volume of the spherical cap formed when the curve $x^2 + y^2 = 4$ is rotated about the y-axis from $y = 1$ to $y = 2$.

12 Find the volume of the paraboloid formed when $y = x^2$ is rotated about the y-axis from $y = 0$ to $y = 3$.

13 Find the exact volume of the solid formed when the curve $y = \frac{1}{\sqrt{x}}$ is rotated about the x-axis from $x = 1$ to $x = 3$.

14 Find the volume of the solid formed when $y = x^2 - 2$ is rotated about the y-axis from $y = 1$ to $y = 4$.

15 The line $y = x + 2$ is rotated about the y-axis from $y = -2$ to $y = 2$. Find the volume of the solid formed.

16 The curve $y = \sqrt{e^x + 1}$ is rotated about the x-axis from $x = 0$ to $x = 1$. Find the exact volume of the solid formed.

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17 Find the exact volume of the solid formed when the curve $y = \frac{2}{\sqrt{2x-1}}$ is rotated about the x-axis from $x = 1$ to $x = 5$.

18 Find the volume of the solid formed when the curve $y = \sqrt{x}$ is rotated about the y-axis between $y = 1$ and $y = 4$.

19 Find the exact volume of the solid formed when the curve $y = \ln x$ is rotated about the y-axis from $y = 1$ to $y = 3$.

20 Find the volume of the solid formed when the line $x + 3y - 1 = 0$ is rotated about the y-axis from $y = 1$ to $y = 2$.

21 Find the volume of the solid formed when the line $x + 3y - 1 = 0$ is rotated about the x-axis from $x = 0$ to $x = 8$.

22 The curve $y = x^3$ is rotated about the y-axis from $y = 0$ to $y = 1$. Find the volume of the solid formed.

23 a Show that the volume of the solid formed by rotating the curve $y = \sqrt{\cos x}$ about the x-axis between $x = 0$ and $x = \frac{\pi}{2}$ is π units^3.

b Use the trapezoidal rule with 4 subintervals to find an approximation to the volume of the solid formed by rotating the curve $y = \sqrt{\cos x}$ about the x-axis from $x = 0$ to $x = \frac{\pi}{2}$.

24 a Find the area of the region bounded by the functions $y = x^2$, $y = (x - 2)^2$ and the x-axis.

b Find the volume of the solid of revolution formed if the area enclosed between $y = x^2$, $y = (x - 2)^2$ and the x-axis is rotated about the x-axis.

25 a Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x + 2$.

b The area bounded by the curve $y = x^2$ and the line $y = x + 2$ is rotated about the x-axis. Find the exact volume of the solid formed.

26 Show that the volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$ by rotating the semicircle $y = \sqrt{r^2 - x^2}$ about the x-axis.

27 a Find an expression for $\sin^2 x$ in terms of $\cos 2x$.

b Hence find the exact volume of the solid if the curve $y = \sin x$ is rotated about the x-axis from $x = 0$ to $x = \frac{\pi}{4}$.

28 a Find the exact area under the curve $y = \frac{1}{\sqrt{1-x^2}}$ between $x = 0$ and $x = 0.5$.

b Find the exact volume of the solid formed when the curve $y = \frac{1}{\sqrt{1-x^2}}$ is rotated about the y-axis from $y = 1$ to $y = 2$.

WS Integration by substitution

WS Integration by substitution

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EXT1 8.02 Integration by substitution

We can use the method of integration by substitution to find integrals that don't fit into any of the common rules. To give you an example of how it works, the questions in this first example can also be done using the chain rule.

EXAMPLE 4

a Find $\int (5x-1)^9\,dx$ using the substitution $u = 5x - 1$.

b Find $\int 12x^2\sqrt{4x^3+1}\,dx$ using the substitution $u = 4x^3 + 1$.

Solution

a Let $u = 5x - 1$ [1]

Then $\frac{du}{dx} = 5$

$du = 5dx$ [2]

$$\int (5x-1)^9\,dx = \frac{1}{5}\int (5x-1)^9 \times 5\,dx$$

$$= \frac{1}{5}\int u^9\,du$$ substituting u and du from [1] and [2]

$$= \frac{1}{5} \times \frac{u^{10}}{10} + C$$

$$= \frac{u^{10}}{50} + C$$

$$= \frac{(5x-1)^{10}}{50} + C$$ replacing u with $5x - 1$

b Let $u = 4x^3 + 1$ [1]

Then $\frac{du}{dx} = 12x^2$

$du = 12x^2\,dx$ [2]

$$\int 12x^2\sqrt{4x^3+1}\,dx = \int \sqrt{4x^3+1} \times 12x^2\,dx$$

$$= \int u^{\frac{1}{2}}\,du$$ substituting u and du from [1] and [2]

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$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2\sqrt{u^3}}{3} + C$$

$$= \frac{2\sqrt{(4x^3+1)^3}}{3} + C \qquad \text{replacing } u \text{ with } 4x^3 + 1$$

The next example cannot be done using any of the common rules.

EXAMPLE 5

Find $\int x\sqrt{x+1}\,dx$ using the substitution $u = x + 1$.

Solution

Let $u = x + 1$ [1]

$u - 1 = x$ [2]

$$\frac{du}{dx} = 1$$

$du = dx$ [3]

$$\int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du \qquad \text{substituting [1], [2] and [3]}$$

$$= \int (u-1) \times u^{\frac{1}{2}}\,du$$

$$= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,du$$

$$= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2\sqrt{u^5}}{5} - \frac{2\sqrt{u^3}}{3} + C$$

$$= \frac{2\sqrt{(x+1)^5}}{5} - \frac{2\sqrt{(x+1)^3}}{3} + C \qquad \text{replacing } u \text{ with } x + 1$$

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EXT1 Exercise 8.02 Integration by substitution

1 Find each integral using the substitution given.

a $\int (3x-4)^7 \, dx$, $u = 3x - 4$

b $\int \sqrt{x+3} \, dx$, $u = x + 3$

c $\int x(x^2-9)^5 \, dx$, $u = x^2 - 9$

d $\int (x+2)(x^2+4x+1)^4 \, dx$, $u = x^2 + 4x + 1$

e $\int \frac{2x+3}{(x^2+3x-1)^3} \, dx$, $u = x^2 + 3x - 1$

f $\int \frac{3x}{\sqrt{2x^2-1}} \, dx$, $u = 2x^2 - 1$

g $\int \frac{x^2}{(x^3+1)^4} \, dx$, $u = x^3 + 1$

2 Find $\int x^2 e^{x\ +1} \, dx$ using the substitution $u = x^3 + 1$.

3 Find $\int e^{\text{an}\, x} \sec^2 x \, dx$ using the substitution $u = \tan x$.

4 By using the substitution $u = x^2 - 4$, find $\int \frac{x}{\sqrt{x^2-4}} \, dx$.

5 By using the substitution $u = 2x + 7$, find $\int x\sqrt{2x+7} \, dx$.

6 Use the substitution $u = \sqrt{x+2}$ to find $\int \frac{x-2}{\sqrt{x+2}} \, dx$.

7 Use the substitution $u = 5 - x$ to find $\int x\sqrt{5-x} \, dx$.

8 Use the substitution $u = 2x - 5$ to find $\int \frac{x}{\sqrt{2x-5}} \, dx$.

9 Use the substitution $u = x + 4$ to find $\int 5x\sqrt{x+4} \, dx$.

10 If the gradient of a curve is given by $\frac{dy}{dx} = \frac{x}{(x^2-1)^2}$ and the curve passes through $(2, -1)$, find the equation of the curve by using the substitution $u = x^2 - 1$.

11 If the gradient of a curve is given by $\frac{dy}{dx} = x(3x^2 - 1)^6$ and the point $(0, 1)$ lies on the curve, find its equation by using the substitution $u = 3x^2 - 1$.

12 A function has $f'(x) = x^2(x^3 - 2)^4$ and passes through the point $\left(1 \quad \frac{14}{15}\right)$. Find its equation.

13 A curve has $\frac{dy}{dx} = \frac{x}{\sqrt{x+1}}$ and y-intercept 2. Using the substitution $u = x + 1$:

a find its domain

b evaluate y when $x = 3$

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14 A function has $f'(x) = 3 \sin x \cos^4 x$ and passes through $(\pi, 0)$. Find its equation by using the substitution $u = \cos x$.

15 A function has $\dfrac{dy}{dx} = e^x(e^x + 1)^5$ and passes through (0, 0) Find its equation by using the substitution $u = e^x + 1$.

16 **a** By using the substitution $u = ax + b$, show that $\int (ax+b)^n\, dx = \dfrac{(ax+b)^{n+}}{a(n+1)} + C$.

b By using the substitution $u = f(x)$, show that $\int f'(x)[f(x)]^n\, dx = \dfrac{1}{n+1}[f(x)]^{n+1} + C$.

EXT1 8.03 Definite integrals by substitution

You can use integration by substitution to evaluate definite integrals.

WS Definite integrals by substitution

EXAMPLE 6

a Find $\int_0^2 x^2\sqrt{x^3+1}\, dx$ using the substitution $u = x^3 + 1$.

b Find the exact value of $\int_0^{\frac{\pi}{6}} \cos^4 x \sin x\, dx$ using the substitution $u = \cos x$.

Integration by substitution

Solution

a Let $u = x^3 + 1$ [1]

Then $\dfrac{du}{dx} = 3x^2$

$du = 3x^2\, dx$ [2]

When evaluating definite integrals by substitution, the values of the integral also need to be changed to u:

When $x = 0$: $u = 0^3 + 1 = 1$

When $x = 2$: $u = 2^3 + 1 = 9$ [3]

$$\int_0^2 x^2\sqrt{x^3+1}\, dx = \frac{1}{3}\int_0^2 \sqrt{x^3+1} \times 3x^2\, dx$$

$$= \frac{1}{3}\int^9 \sqrt{u}\, du \qquad \text{substituting [1], [2] and [3]}$$

$$= \frac{1}{3}\int^9 u^{\frac{1}{2}}\, du$$

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$$= \frac{1}{3}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^9$$

$$= \frac{1}{3}\left[\frac{2\sqrt{u^3}}{3}\right]_1^9$$

$$= \frac{1}{3}\left(\frac{2\sqrt{9^3}}{3} - \frac{2\sqrt{1^3}}{3}\right)$$

$$= \frac{1}{3}\left(18 - \frac{2}{3}\right)$$

$$= 5\frac{7}{9}$$

b Let $u = \cos x$ [1]

Then $\frac{du}{dx} = -\sin x$

$du = -\sin x \, dx$ [2]

When $x = 0$: $u = \cos 0 = 1$

When $x = \frac{\pi}{6}$: $u = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ [3]

$$\int_0^{\frac{\pi}{6}} \cos^4 x \sin x \, dx = -\int_0^{\frac{\pi}{6}} \cos^4 x \times (-\sin x)\, dx$$

$$= -\int_1^{\frac{\sqrt{3}}{2}} u^4 \, du$$ substituting [1], [2] and [3]

$$= -\left[\frac{u^5}{5}\right]_1^{\frac{\sqrt{3}}{2}}$$

$$= -\left(\frac{\left(\frac{\sqrt{3}}{2}\right)^5}{5} - \frac{1^5}{5}\right)$$

$$= -\left(\frac{\frac{\sqrt{3}^4 \times \sqrt{3}}{32}}{5} - \frac{1}{5}\right)$$

$$= -\left(\frac{9\sqrt{3}}{160} - \frac{32}{160}\right)$$

$$= \frac{32 - 9\sqrt{3}}{160}$$

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EXT1 Exercise 8.03 Definite integrals by substitution

1 Use the given substitution to find the exact value of each integral.

a $\int_0 x(3x^2-1)^5\,dx$, $u = 3x^2 - 1$

b $\int^2 \frac{x}{\sqrt{x^2+3}}\,dx$, $u = x^2 + 3$

c $\int_0^2 \frac{2x+3}{(x^2+3x-5)^3}\,dx$, $u = x^2 + 3x - 5$

d $\int_{-3}^{-2} (3+x)^7\,dx$, $u = 3 + x$

e $\int_0 \frac{x^3}{(1+x^4)^5}\,dx$, $u = 1 + x^4$

f $\int_0^4 x\sqrt{x^2+2}\,dx$, $u = x^2 + 2$

g $\int_0 x^2(2x^3-3)^3\,dx$, $u = 2x^3 - 3$

2 Use the given substitution to evaluate each integral, correct to 2 decimal places where necessary.

a $\int_0^3 x\sqrt{(x^2+1)^3}\,dx$, $u = x^2 + 1$

b $\int_{-3} \frac{dx}{\sqrt{x+3}}$, $x = u + 3$

c $\int_0 \frac{x+1}{3(x^2+2x-1)^2}\,dx$, $u = x^2 + 2x - 1$

d $\int_{-3}^{-2} x(3+x)^7\,dx$, $u = x + 3$

e $\int_0 \frac{x}{\sqrt{(1+x)^3}}\,dx$, $u = 1 + x$

3 Use the substitution $u = x^2$ to find the exact value of $\int_0^2 xe^{x}\,dx$.

4 Use the substitution $u = \sin x$ to evaluate $\int_0^{\frac{\pi}{6}} \cos x \sin^2 x\,dx$.

5 Use the substitution $u = x^2 + 1$ to find the area enclosed between the curve $y = x\sqrt{x^2+1}$, the x-axis and the lines $x = 0$ and $x = 3$, to 1 decimal place.

6 Use the substitution $u = x^4 - 2$ to find the area bounded by the curve $y = x^3(x^4 - 2)^4$, the x-axis and the lines $x = 0$ and $x = 1$.

7 Use the substitution $u = x^2 - 1$ to find the area enclosed between the curve $f(x) = 3x(x^2 - 1)^5$, the x-axis and the lines $x = 1$ and $x = 3$.

8 Use the substitution $u = x^3 - 3$ to find the volume of the solid of revolution formed when the curve $y = x(x^3 - 3)^2$ is rotated about the x-axis from $x = 0$ to $x = 1$.

9 Use the substitution $u = 1 + x^3$ to find the volume of the solid formed when the curve $y = x\sqrt{(1+x^3)^3}$ is rotated about the x-axis from $x = -1$ to $x = 1$.

10 Evaluate $\int_0^3 \frac{t}{\sqrt{t+1}}\,dt$ by using the substitution $u = t + 1$.

11 Find the area enclosed between the curve $y = x(x^2 - 2)^4$ and the x-axis by using the substitution $u = x^2 - 2$.

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12 Use the substitution $u = x^2 - 4$ to find the area bounded by the curve $y = x(x^2 - 4)^3$ and the x-axis.

13 Use the substitution $u = x^3 + 2$ to find the volume of the solid formed when the curve $y = x(x^3 + 2)^4$ is rotated about the x-axis from $x = -1$ to $x = 1$.

14 Use the substitution $u = x^5$ to find the volume of the solid of revolution formed when the curve $y = x^2 \sec(x^5)$ is rotated about the x-axis between $x = 0$ and $x = \frac{\pi}{9}$. Answer correct to 3 significant figures.

15 **a** By using the trapezoidal rule with 2 subintervals, find the area bounded by the function $f(x) = \frac{x}{x^3 + 1}$, the x-axis and the lines $x = 2$ and $x = 3$ correct to 2 decimal places.

b Use the substitution $u = x^3 + 1$ to find the exact volume of the solid formed when this area is rotated about the x-axis.

EXT1 8.04 Integration of $\sin^2 x$ and $\cos^2 x$

WS Integrals of $\sin^2 x$ and $\cos^2 x$

WS Integration of $\cos^2 x$ and $\sin^2 x$

We can find the integral of $\sin^2 x$ and $\cos^2 x$ by using the trigonometric result $\cos 2x = \cos^2 x - \sin^2 x$.

Integral of $\cos^2 x$

$$\int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Proof

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= \cos^2 x - (1 - \cos^2 x) \\ &= 2\cos^2 x - 1\end{aligned}$$

$$\cos 2x + 1 = 2\cos^2 x$$

$$\frac{1}{2}(\cos 2x + 1) = \cos^2 x$$

$$\begin{aligned}\int \cos^2 x \, dx &= \int \frac{1}{2}(\cos 2x + 1)\, dx \\ &= \frac{1}{2}\int (\cos 2x + 1)\, dx \\ &= \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right) + C \\ &= \frac{1}{2}x + \frac{1}{4}\sin 2x + C\end{aligned}$$

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EXAMPLE 7

Find the exact value of $\int_0^{\frac{\pi}{8}} \cos^2 x\, dx$.

Solution

$$\begin{aligned}
\int_0^{\frac{\pi}{8}} \cos^2 x\, dx &= \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{8}} \\
&= \left(\frac{1}{2}\times\frac{\pi}{8} + \frac{1}{4}\sin\left(2\times\frac{\pi}{8}\right)\right) - \left(\frac{1}{2}\times 0 + \frac{1}{4}\sin(2\times 0)\right) \\
&= \left(\frac{\pi}{16} + \frac{1}{4}\sin\frac{\pi}{4}\right) - \left(0 + \frac{1}{4}\sin 0\right) \\
&= \frac{\pi}{16} + \frac{1}{4}\times\frac{1}{\sqrt{2}} \\
&= \frac{\pi}{16} + \frac{1}{4\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\pi}{16} + \frac{\sqrt{2}}{8} \\
&= \frac{\pi + 2\sqrt{2}}{16}
\end{aligned}$$

Integral of $\sin^2 x$

$$\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Proof

$$\begin{aligned}
\cos 2x &= \cos^2 x - \sin^2 x \\
&= (1 - \sin^2 x) - \sin^2 x \\
&= 1 - 2\sin^2 x
\end{aligned}$$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\begin{aligned}
\int \sin^2 x\, dx &= \int \frac{1}{2}(1 - \cos 2x)\, dx \\
&= \frac{1}{2}\int (1 - \cos 2x)\, dx \\
&= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C \\
&= \frac{1}{2}x - \frac{1}{4}\sin 2x + C
\end{aligned}$$

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EXAMPLE 8

Find the exact area under the curve $y = \sin^2 x + 1$ between $x = 0$ and $x = \frac{\pi}{2}$.

Solution

$y = \sin^2 x + 1$ is above the x-axis for all x since $\sin^2 x \geq 0$. So the definite integral will be positive.

$$\int_0^{\frac{\pi}{2}} (\sin^2 x + 1)\, dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x + x\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{3}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{3}{2} \times \frac{\pi}{2} - \frac{1}{4}\sin\left(2 \times \frac{\pi}{2}\right)\right] - \left[\frac{3}{2} \times 0 - \frac{1}{4}\sin(2 \times 0)\right]$$

$$= \left(\frac{3\pi}{4} - \frac{1}{4}\sin \pi\right) - \left(0 - \frac{1}{4}\sin 0\right)$$

$$= \frac{3\pi}{4}$$

The area is $\frac{3\pi}{4}$ units^2.

$\cos^2 nx$ and $\sin^2 nx$ identities

$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

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Proof

$$\cos 2nx = \cos^2 nx - \sin^2 nx$$
$$= \cos^2 nx - (1 - \cos^2 nx)$$
$$= \cos^2 nx - 1 + \cos^2 nx$$
$$= 2\cos^2 nx - 1$$
$$\cos 2nx + 1 = 2\cos^2 nx$$
$$\frac{1}{2}(\cos 2nx + 1) = \cos^2 nx$$

Similarly, we can prove the identity for $\sin^2 nx$.

EXAMPLE 9

Find $\int \sin^2 9x\, dx$.

Solution

$$\sin^2 9x = \frac{1}{2}(1 - \cos 18x)$$
$$\int \sin^2 9x\, dx = \int \frac{1}{2}(1 - \cos 18x)\, dx$$
$$= \frac{1}{2}\left(x - \frac{1}{18}\sin 18x\right) + C$$

EXT1 Exercise 8.04 Integration of $\sin^2 x$ and $\cos^2 x$

1 Find the integral of:

a $\cos^2 x - 1$ **b** $\sin^2 x + x^2$ **c** $3\cos^2 x - 5$
d $2\sin^2 x - \cos x$ **e** $7\cos^2 x + \sec^2 7x$ **f** $\cos^2 x + \cos x + 2$
g $\sin 2x - \sin^2 x$ **h** $\sec^2 x + \sin^2 x$ **i** $1 + \cos 3x - \sin^2 x$

2 Evaluate, giving exact values:

a $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$ **b** $\int_0^{\pi} (\cos^2 x + 1)\, dx$ **c** $\int_{\frac{\pi}{2}}^{\pi} (\sin^2 x + x)\, dx$
d $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (2\cos^2 x + 1)\, dx$ **e** $\int_0^{\frac{\pi}{6}} \sin^2 x\, dx$

3 Find the exact volume of the solid of revolution formed if the curve $y = \sin x$ is rotated about the x-axis between $x = 0$ and $x = \frac{\pi}{4}$.

4 Find the exact area bounded by the curve $y = \cos^2 x$, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

5 Find the exact volume of the solid formed if the curve $y = \cos x + 1$ is rotated about the x-axis from $x = 0$ to $x = \frac{\pi}{2}$.

6 Find the primitive function of:

a $\sin^2 3x$ **b** $\cos^2 7x$ **c** $\cos^2 4x$ **d** $\sin^2 10x$

e $\cos^2\left(\frac{x}{2}\right)$ **f** $\sin^2 x°$ **g** $\cos^2 ax$ **h** $3\sin^2\left(\frac{2x}{5}\right)$

7 a Write down an expression for $\sin 3x$ in terms of $\sin x$.

b Hence find $\int_0^{\frac{\pi}{6}} \sin^3 x\, dx$.

8 a Show that $\sin 7x \cos 3x = \frac{1}{2}[\sin(7x + 3x) + \sin(7x - 3x)]$.

b Hence evaluate $\int_0^{\frac{\pi}{2}} \sin 7x \cos 3x\, dx$.

9 Find:

a $\int \sin x \cos x\, dx$ **b** $\int (\cos x + \sin^2 x)\, dx$

c $\int \sin^2\theta \cos^2\theta\, d\theta$ **d** $\int (\cos x + \sin x)^2\, dx$

10 a Find the area enclosed between the curves $y = \sin x$ and $y = \cos x$ and the lines $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

b This area is rotated about the x-axis. Find the volume of the solid of revolution formed.

WS Integrals and inverse trigonometric functions

WS Inverse trigonometric functions

EXT1 8.05 Integrals involving inverse trigonometric functions

Integrals involving inverse trigonometric functions

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$\int -\frac{dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

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EXAMPLE 10

a Evaluate $\int_0 \frac{dx}{\sqrt{1-x^2}}$.

b Find the volume, correct to 2 decimal places, of the solid of revolution formed if the curve $y = \frac{1}{\sqrt{1+x^2}}$ is rotated about the x-axis from $x = 0$ to $x = 3$.

Solution

a
$$\int_0 \frac{dx}{\sqrt{1-x^2}} = [\sin^{-} x]_0$$
$$= \sin^{-1}(1) - \sin^{-1}(0)$$
$$= \frac{\pi}{2} - 0$$
$$= \frac{\pi}{2}$$

b For volumes about the x-axis, the integrand is y^2:

$$y = \frac{1}{\sqrt{1+x^2}}, \qquad y^2 = \frac{1}{1+x^2}$$

$$V = \pi\int_a^b y^2\, dx$$
$$= \pi\int_0^3 \frac{1}{1+x^2}\, dx$$
$$= \pi[\tan^{-} x]_0^3$$
$$= \pi(\tan^{-1}(3) - \tan^{-1}(0)]$$
$$= 3.92$$

So volume is 3.92 units^3.

We looked at the derivatives of $\sin^{-1}\left(\frac{x}{a}\right)$, $\cos^{-1}\left(\frac{x}{a}\right)$ and $\tan^{-1}\left(\frac{x}{a}\right)$ in Chapter 5, *Further differentiation*.

$$\frac{d}{dx}\left[\sin^{-}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\left[\cos^{-}\left(\frac{x}{a}\right)\right] = -\frac{1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\left[\tan^{-}\left(\frac{x}{a}\right)\right] = \frac{a}{a^2 + x^2}$$

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We integrate by finding the anti-derivatives.

Chain rule for inverse trigonometric functions

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int -\frac{dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

EXAMPLE 11

a Evaluate $\int_0^{\bar{\ }2} \frac{dx}{\sqrt{9 - x^2}}$.

b Find $\int \frac{dx}{4 + 9x^2}$.

Solution

a

$$\int_0^{\bar{\ }2} \frac{dx}{\sqrt{9 - x^2}} = \int_0^{\bar{\ }2} \frac{dx}{\sqrt{3^2 - x^2}}$$

$$= \left[\sin^{-}\left(\frac{x}{3}\right)\right]_0^{\bar{\ }2}$$

$$= \sin^{-1}\left(\frac{1\frac{1}{2}}{3}\right) - \sin^{-1}\left(\frac{0}{3}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1} 0$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

b

$$\int \frac{dx}{4 + 9x^2} = \int \frac{dx}{9\left(\frac{4}{9} + \frac{9x^2}{9}\right)}$$

$$= \frac{1}{9}\int \frac{dx}{\frac{4}{9} + x^2}$$

$$= \frac{1}{9}\int \frac{dx}{\left(\frac{2}{3}\right)^2 + x^2}$$

$$= \frac{1}{9}\left[\frac{1}{\left(\frac{2}{3}\right)}\tan^{-}\frac{x}{\left(\frac{2}{3}\right)}\right] + C$$

$$= \frac{1}{9}\left[\frac{3}{2}\tan^{-}\left(\frac{3x}{2}\right)\right] + C$$

$$= \frac{1}{6}\tan^{-}\left(\frac{3x}{2}\right) + C$$

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EXT1 Exercise 8.05 Integrals involving inverse trigonometric functions

1 Find the integral of:

a $\frac{1}{\sqrt{1-x^2}}$ **b** $\frac{-2}{\sqrt{1-x^2}}$ **c** $\frac{1}{1+x^2}$

d $\frac{1}{9+x^2}$ **e** $\frac{1}{\sqrt{4-x^2}}$ **f** $\frac{5}{4+x^2}$

g $\frac{3}{2\sqrt{1-x^2}}$ **h** $\frac{1}{5\sqrt{16-x^2}}$ **i** $\frac{1}{x^2+3}$

2 Find:

a $\int \frac{dx}{\sqrt{1-4x^2}}$ **b** $\int \frac{dt}{\sqrt{9-t^2}}$ **c** $\int \frac{dx}{1+9x^2}$

d $\int \frac{dx}{\sqrt{4-25x^2}}$ **e** $\int \frac{3}{9+16x^2}dx$ **f** $\int \frac{2}{\sqrt{25-4t^2}}dt$

g $\int \frac{dx}{1+5x^2}$ **h** $\int \frac{dx}{4+3x^2}$ **i** $\int \frac{-2}{5\sqrt{25-9x^2}}dx$

3 Find the exact value of:

a $\int_0 \frac{dx}{1+x^2}$ **b** $\int_{-} \frac{dx}{\sqrt{4-x^2}}$ **c** $\int_{-3}^{0} \frac{dt}{t^2+9}$

d $\int_0^{4\frac{}{2}} \frac{dx}{\sqrt{81-x^2}}$ **e** $\int^{\sqrt{3}} \frac{2}{\sqrt{4-x^2}}dx$ **f** $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{3}{2\sqrt{1-x^2}}dx$

g $\int_{-\frac{}{3}}^{\overline{3}} \frac{dx}{\sqrt{4-9x^2}}$ **h** $\int_0^{\overline{7\sqrt{2}}} \frac{3}{\sqrt{1-49x^2}}dx$ **i** $\int_0^{\sqrt{15}} \frac{dx}{5+x^2}$

4 Find the area bounded by the curve $y = \frac{1}{1+x^2}$, the x-axis and the lines $x = 0$ and $x = 2$, to 1 decimal place.

5 **a** Find the area enclosed between the curve $y = \frac{1}{\sqrt{1-x^2}}$, the x-axis and the lines $x = 0$ and $x = \frac{1}{2}$.

b The corresponding area enclosed between the curve and the y-axis is rotated about the y-axis. Find the exact volume of the solid formed.

6 **a** Differentiate $\ln(\sin^{-1} x)$.

b Hence find $\int_{\frac{}{4}}^{\overline{3}} \frac{dx}{\sqrt{1-x^2}\sin^{-1} x}$ correct to 1 decimal place.

7 Use the trapezoidal rule with 4 subintervals to find an approximation to $\int_0^{0.4} \sin^{-} x \, dx$, correct to 1 decimal place.

8 Find the area enclosed between the curve $y = \cos^{-1} x$, the y-axis and the lines $y = 0$ and $y = \frac{\pi}{4}$.

9 Find $\int \frac{x^2}{\sqrt{1-x^6}} \, dx$ by using the substitution $u = x^3$.

10 Find the equation of the curve with $\frac{dy}{dx} = \frac{1}{\sqrt{9-x^2}}$ and passing through $\left(3 \;\; \frac{\pi}{7}\right)$.

11 **a** Show that $\frac{2x^2+5}{(1+x^2)(4+x^2)} = \frac{1}{1+x^2} + \frac{1}{4+x^2}$.

b Hence evaluate $\int^{2} \frac{2x^2+5}{(1+x^2)(4+x^2)} \, dx$ correct to 2 decimal places.

12 For the curve $y = \frac{1}{\sqrt{1+x^2}}$, find the volume of the solid formed by rotating the curve about the x-axis from $x = \frac{1}{\sqrt{3}}$ to $x = 1$.

13 **a** Differentiate $x \cos^{-1} x - \sqrt{1-x^2}$.

b Find the area bounded by the curve $y = \cos^{-1} x$, the x-axis and the lines $x = 0$ and $x = \frac{1}{2}$.

14 Use $\frac{d}{dx}(x \sin^{-1} x + \sqrt{1-x^2})$ to find the area enclosed between the curve $y = \sin^{-1} x$, the x-axis and the lines $x = 0$ and $x = 1$.

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8. TEST YOURSELF

For Questions **1** to **4**, choose the correct answer **A**, **B**, **C** or **D**.

WS Derivatives and integrals

Qz Practice quiz

1 Which formula gives the volume of the solid formed when $y = x^2 + 3$ is rotated about the y-axis between $y = 3$ and $y = 4$?

A $V = \pi \int_3^4 (x^2 + 3)\,dx$ **B** $V = \pi \int_3^4 (x^2 + 3)^2\,dx$

C $V = \pi \int_3^4 (y - 3)\,dy$ **D** $V = \pi \int_3^4 (y - 3)^2\,dx$

2 Which expression is equivalent to $\int \frac{x}{\sqrt{x-1}}\,dx$ when using the substitution $u = x - 1$?

A $\int \frac{u+1}{\sqrt{u}}\,du$ **B** $\int \frac{u}{\sqrt{u}}\,du$

C $\int \frac{x}{\sqrt{u}}\,du$ **D** $\int \frac{1}{\sqrt{u}}\,du$

3 Find $\int \frac{5}{9 + x^2}\,dx$.

A $5 \tan^{-1} 3x + C$ **B** $\frac{5}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$

C $5 \tan^{-1}\left(\frac{x}{3}\right) + C$ **D** $\frac{5}{3} \tan^{-1} 3x + C$

4 Find $\int \sin^2 x\,dx$.

A $\frac{1}{2}x + \frac{1}{4}\sin 2x + C$ **B** $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$

C $\frac{1}{2}x + \frac{1}{4}\cos 2x + C$ **D** $\frac{1}{2}x - \frac{1}{4}\cos 2x + C$

5 The line $y = 2x - 3$ is rotated about the x-axis from $x = 0$ to $x = 3$. Find the volume of the solid of revolution.

6 Find the volume of the solid formed if the curve $y = x^2 + 1$ is rotated about:

a the x-axis from $x = 0$ to $x = 2$

b the y-axis from $y = 1$ to $y = 2$

7 Find the volume of the solid formed if the area bounded by $y = e^{3x}$, the x-axis and the lines $x = 1$ and $x = 2$ is rotated about the x-axis.

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8 **a** Find the area of the region bounded by the function $y = \frac{1}{\sqrt{1 - x^2}}$, the x-axis and the lines $x = 0$ and $x = \frac{1}{2}$.

b Find the volume of the solid formed if this area is rotated about the x-axis. Use the trapezoidal rule and give your answer correct to 2 decimal places.

9 **a** Change the subject of $y = (x + 3)^2$ to x.

b Find the area in the first quadrant bounded by the curve $y = (x + 3)^2$, the y-axis and the lines $y = 9$ and $y = 16$.

c Find the volume of the solid formed if this area is rotated about the y-axis.

10 Evaluate $\int_0 x^3 (x^4 - 3)^4 \, dx$ by using the substitution $u = x^4 - 3$.

11 **a** Find the area bounded by the curve $y = e^x$, the x-axis and the lines $x = 1$ and $x = 2$.

b This area is rotated about the x-axis. Find the volume of the solid of revolution formed.

12 Evaluate:

a $\int_0^{\frac{\pi}{3}} 2\sin^2 x \, dx$ **b** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2 x \, dx$

13 Find the volume of the solid formed if the curve $y = x^3$ is rotated about the y-axis from $y = 0$ to $y = 1$.

14 Find $\int 3x\sqrt{x + 1}$ by using the substitution $u = x + 1$.

15 Find the volume of the solid formed if the curve $y = \sec x$ is rotated about the x-axis from $x = 0$ to $x = \frac{\pi}{6}$.

16 **a** Find the exact area bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

b This area is rotated about the x-axis. Find the volume of the solid formed.

17 The curve $y = 2^x$ is rotated about the x-axis between $x = 1$ and $x = 2$. Use the trapezoidal rule with 2 subintervals to find an approximation of the volume of the solid formed, correct to 3 significant figures.

18 Find the exact volume of the solid formed by rotating the curve $y = \frac{1}{x}$ about the x-axis between $x = 1$ and $x = 3$.

19 **a** Find the exact area enclosed between the curve $y = x\sqrt{x^2 - 1}$, the x-axis and the lines $x = 1$ and $x = 2$.

b Find the volume of the solid formed if this area is rotated about the x-axis.

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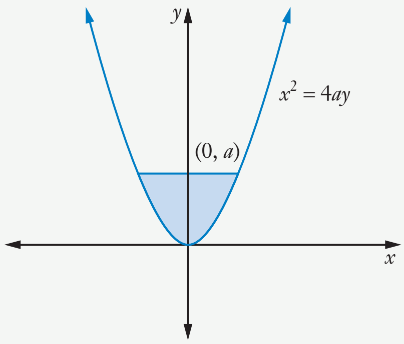
20 For the shaded region shown, find:

a the area

b the volume when this area is rotated about the y-axis.



21 Find $\int xe^{x} \ dx$, using the substitution $u = x^2$.

22 Use the substitution $u = 3x^2 + 1$ to find $\int xe^{3x \ +1} dx$.

23 Use the substitution $u = x^3 - 2$ to evaluate $\int_0^2 x^2 e^{x \ -2} dx$.

24 Evaluate $\int \sin x \cos^2 x \ dx$ using the substitution $u = \cos x$.

25 **a** Find the exact area bounded by the curve $y = \cos x$, the x-axis and the lines $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

b This area is rotated about the x-axis. Find the exact volume of the solid of revolution formed.

8. CHALLENGE EXERCISE

1 Use the trapezoidal rule with 2 trapezia to find the volume of the solid formed when the curve $y = e^x$ is rotated about the y-axis from $y = 3$ to $y = 5$, correct to 2 significant figures.

2 Use the substitution $u = \cos x$ to find $\int \sin^3 x \, dx$.

3 By using the substitution $u = x^2$, find $\int \frac{x}{1+x^4} dx$.

4 Find $\int \frac{x}{\sqrt{1-9x^4}} dx$ using the substitution $u = 3x^2$.

5 Find the volume of the solid formed if the curve $y = \sec^{-1} x$ is rotated about:

a the x-axis from $x = 1$ to $x = 1.5$, using the trapezoidal rule with 5 subintervals (answer correct to 2 decimal places).

b the y-axis from $y = 0$ to $y = \frac{\pi}{4}$

6 **a** Find the area of the region enclosed between the curves $y = x^3$ and $y = x^2$.

b Find the volume of the solid formed if this area is rotated about the x-axis.

7 Find the volume of the solid formed when the curve $y = (x + 5)^2$ is rotated about the y-axis from $y = 1$ to $y = 4$.

8 Use the substitution $u = x^3 + 3x - 2$ to evaluate $\int_0 (x^2 + 1)\sqrt[3]{(x^3 + 3x - 2)^5} \, dx$.

9 **a** Prove that $\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$.

b Find the exact value of $\int_0^{\frac{\pi}{24}} \sin^2 4x \, dx$.

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Practice set 2

In Questions **1** to **7**, select the correct answer **A**, **B**, **C** or **D**.

1 The area of a rectangle with sides x and y is 45. Its perimeter P is given by:

A $P = x + 45x^2$

B $P = x + \dfrac{45}{x}$

C $P = 2x + \dfrac{90}{x}$

D $P = 2x + \dfrac{45}{x}$

2 EXT1 Find $\int \dfrac{dx}{\sqrt{9 - x^2}}$.

A $\dfrac{1}{9}\sin^{-1}\left(\dfrac{x}{9}\right) + C$

B $\sin^{-1}\left(\dfrac{x}{9}\right) + C$

C $\dfrac{1}{3}\sin^{-1}\left(\dfrac{x}{3}\right) + C$

D $\sin^{-1}\left(\dfrac{x}{3}\right) + C$

3 The area enclosed between the curve $y = x^3 - 1$, the y-axis and the lines $y = 1$ and $y = 2$ is given by:

A $\int^2 (x^3 - 1)\,dy$

B $\int^2 (y + 1)\,dy$

C $\int^2 (\sqrt[3]{y} + 1)\,dy$

D $\int^2 (\sqrt[3]{y + 1})\,dy$

4 Find $\int 4x^2(5x^3 + 4)^7\,dx$.

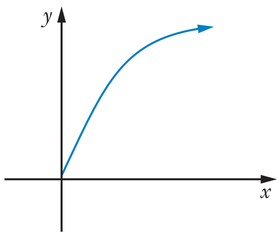
A $\dfrac{4(5x^3 + 4)^8}{15} + C$

B $\dfrac{(5x^3 + 4)^8}{30} + C$

C $\dfrac{(5x^3 + 4)^8}{2} + C$

D $\dfrac{(5x^3 + 4)^8}{120} + C$

5 For the curve shown, which inequalities are correct?



A $\dfrac{dy}{dx} > 0, \dfrac{d^2y}{dx^2} > 0$

B $\dfrac{dy}{dx} > 0, \dfrac{d^2y}{dx^2} < 0$

C $\dfrac{dy}{dx} < 0, \dfrac{d^2y}{dx^2} > 0$

D $\dfrac{dy}{dx} < 0, \dfrac{d^2y}{dx^2} < 0$

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6 A cone with base radius r and height h has a volume of 300 cm^3. Its slant height, l, is given by:

A $l=\sqrt{\dfrac{\pi h^3+900}{\pi h}}$ **B** $l=\sqrt{\dfrac{h^2+900}{\pi h}}$

C $l=\sqrt{\dfrac{h^2+810000}{\pi h}}$ **D** $l=\sqrt{\dfrac{h^3+900}{\pi h}}$

7 The rate at which a waterfall is flowing over a cliff is given by $R = 4t + 3t^2$ m^3 s^{-1}. Find the amount of water flowing after a minute if the amount of water is 10 970 m^3 after 20 seconds.

A 223 220 m^3 **B** 8800 m^3

C 225 370 m^3 **D** 226 250 m^3

8 EXT1 Find the volume of the solid formed when the curve $y=\dfrac{1}{\sqrt{9+x^2}}$ from $x = 0$ to $x=\sqrt{3}$ is rotated about the x-axis.

9 EXT1 Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$.

10 EXT1 Evaluate $\displaystyle\int_2^5 \frac{dx}{x^2+3}$ correct to 2 decimal places.

11 EXT1 Find the exact volume of the solid of revolution formed if $y = x^2 + 2$ is rotated about:

a the x-axis from $x = 0$ to $x = 2$ **b** the y-axis from $y = 2$ to $y = 3$

12 Find all values of x for which the curve $y = (2x - 1)^2$ is decreasing.

13 Find $\int (3x^2-2x+1)\,dx$.

14 Find the maximum value of the curve $y = x^2 + 3x - 4$ in the domain $[-1, 4]$.

15 For the graph of $y = 8 \sin 3x + 5$, find:

a the amplitude **b** the period **c** the centre

16 EXT1 Find $\displaystyle\int \frac{dx}{1+4x^2}$.

17 EXT1 Find $\displaystyle\int_0^1 \sin^{-1} x\, dx$ by using the trapezoidal rule with 4 subintervals.

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EXT1 **18** **a** Sketch the graph of $y = \sin^{-1} 2x$.

b Find the area bounded by the curve $y = \sin^{-1} 2x$, the y-axis, and the lines $y = 0$ and $y = \frac{\pi}{2}$.

19 The area of a rectangle is 4 m^2. Find its minimum perimeter.

20 If $y = \sin 7x$, show that $\frac{d^2y}{dx^2} = -49y$.

21 Find the anti-derivative of $3x^8 + 4x$.

22 Sketch the curve $y = x^3 - 3x^2 - 9x + 2$, showing all stationary points and points of inflection.

23 Find the area enclosed between the curve $y = x^2 - 1$ and the x-axis.

24 EXT1 Given vectors $\overrightarrow{OA} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$, find:

a $-\overrightarrow{OA}$

b $\overrightarrow{OA} + \overrightarrow{OB}$

c $\overrightarrow{OB} - \overrightarrow{OA}$

d $5\overrightarrow{OA} - 6\overrightarrow{OB}$

e the magnitude and direction of $\overrightarrow{OA} - \overrightarrow{OB}$.

25 Find $\int \frac{3x}{2x^2 - 5}\, dx$.

26 If $f(x) = x^3 - 2x^2 + 5x - 9$, find $f'(3)$ and $f''(-2)$.

27 EXT1 Differentiate $\ln(\cos^{-1} x)$.

28 EXT1 Find the derivative of $\tan^{-1}(e^{3x} + 1)$.

29 Evaluate $\int^3 (6x^2 + 4x)\, dx$.

30 Find the domain over which the curve $y = 3x^3 + 7x^2 - 3x - 1$ is concave upwards.

31 Evaluate $\int^2 x\sqrt{3x^2 - 3}\, dx$.

32 Find $\int \sec^2 x(\tan x + 1)^3\, dx$.

33 EXT1 Find $\int \frac{4}{4 + 9x^2}\, dx$.

34 EXT1 Differentiate each function.

a $\sin^{-1} x + \ln x$

b $(\tan^{-1} x)^4$

c $2\cos^{-1} 3x$

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35 **a** If $f(x) = 2x^4 - x^3 - 7x + 9$, find $f(1)$, $f'(1)$ and $f''(1)$.

b What is the geometrical significance of these results? Illustrate by a sketch of $y = f(x)$ at $x = 1$.

36 A piece of wire of length 4 m is cut into 2 parts. One part is bent to form a rectangle with sides x and $3x$, and the other part is bent to form a square with sides y.

a Prove that the total area of the rectangle and square is given by $A = 7x^2 - 4x + 1$.

b Find the dimensions of the rectangle and square when the area has the least value.

37 Given the function $f(x) = x^2$, find the equation of the transformed function if $y = f(x)$ is translated 5 units up, 4 units to the left, stretched horizontally by a factor of 2 and stretched vertically by a factor of 3.

38 The gradient function of a curve is given by $f'(x) = 4x - 3$. If $f(2) = -3$, find $f(-1)$.

39 EXT1 Find $\int (\sin^{-1} x - \sin x)\, dx$ by first finding $\frac{d}{dx}(x \sin^{-1} x + \sqrt{1-x^2})$.

40 EXT1 Find the exact volume of the solid of revolution formed when the curve $y = \sec x$ is rotated about the x-axis from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{3}$.

41 Evaluate $\int_0^3 (2x+1)\, dx$.

42 The following table gives values for $f(x) = \frac{1}{x^2}$.

x	1	2	3	4	5
$f(x)$	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$

Use the table together with the trapezoidal rule to evaluate $\int^5 \frac{dx}{x^2}$ correct to 3 decimal places.

43 **a** Find the stationary point on the curve $y = (x - 2)^3$ and determine its nature.

b Hence sketch the curve.

44 EXT1 Evaluate $\int_0^{\pi} (\cos^2 x + \sec^2 x)\, dx$.

45 EXT1 Evaluate $\int_0^{\frac{1}{4}} \frac{3}{1+x^2}\, dx$ correct to 2 decimal places.

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46 Two families travelling on holidays drive along roads that intersect at right angles. One family is initially 230 km from the intersection and drives towards the intersection at an average of 65 km h^{-1}. The other family is initially 125 km from the intersection and travels towards it at an average of 80 km h^{-1}.

- **a** Show that their distance apart after t hours is given by $d^2 = 10\,625t^2 - 49\,900t + 68\,525$.
- **b** Hence find how long it will take them to reach their minimum distance apart.
- **c** Find their minimum distance apart.

47 For the sequence 100, –50, 25, … find:

a the 10th term **b** the sum of the first 10 terms **c** the limiting sum

EXT1 **48** **a** Find the area in the first quadrant enclosed between the curve $y = x^2 - 1$ and the y-axis between $y = 1$ and $y = 2$, correct to 3 significant figures.

b This area is rotated about the y-axis. Find the exact volume of the solid formed.

49 EXT1 Differentiate $\dfrac{x}{\cos^{-1} x}$.

50 A rectangle is cut from a circular disc of radius 15 cm. Find the area of the largest rectangle that can be produced.

51 Find $\int (3x+5)^7\, dx$.

52 Evaluate $\int^{3} \dfrac{dx}{x}$ correct to 3 decimal places.

53 EXT1 Find the equation of the tangent to the curve $y = \tan^{-1} x$ at the point where $x = \dfrac{1}{\sqrt{3}}$.

54 Find the stationary points on the curve $f(x) = x^4 - 2x^2 + 3$ and distinguish between them.

55 Evaluate $\int^{2} \sqrt{5x-1}\, dx$ as a fraction.

56 Find the area enclosed between the curve $y = (x - 1)^2$ and the line $y = 4$.

57 If a function has a stationary point at (–1, 2) and $f''(x) = 2x - 4$, find $f(2)$.

58 Find the area enclosed between the curves $y = x^2$ and $y = -x^2 + 2x + 4$.

59 Differentiate $x^3 + e^{2x}$.

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60 Water is flowing out of a pool at the rate given by $R = -20$ litres per minute. If the volume of water in the pool is initially 8000 L, find:

a the volume after 5 minutes

b how long it will take to empty the pool.

61 Find the exact value of $\int_0^3 3xe^{x^2+1}\,dx$.

62 The velocity of a particle is given by $v = 12t^2 + 4t + 80$ m s^{-1}. If the particle is initially 3 m to the right of the origin, find its displacement after 5 s.

63 The graph of $y = f(x)$ has a stationary point at (3, 2). If $f''(x) = 6x - 8$, find the equation of $f(x)$.

64 Find the derivative of $\ln(4x+3)^3$.

65 EXT1 Integrate $3x^2\sqrt{x^3-5}$ by using the substitution $u = x^3 - 5$.

66 EXT1 Find the integral of $2x\,e^{x^2-3}$ by using the substitution $u = x^2 - 3$.

67 Find $\int \frac{2x+1}{3x^2+3x-2}\,dx$.

68 Differentiate:

a $\frac{x}{e^{2x}}$

b $\log_3 x$.

69 Find the equation of the tangent to $y = e^{x+1}$ at the point where $x = -1$.

70 Find the stationary point on the curve $y = xe^{2x}$ and determine its nature.

71 EXT1 Use the substitution $u = x^2 - 3$ to find the exact value of $\int_2^3 4x\sqrt{x^2-3}\,dx$.

72 Find the equation of $y = f(x)$ passing through $(\pi, 1)$ and with $f'(x) = -6\sin 3x$.

73 Find $\int_0^{\frac{\pi}{2}} \sin 2x\,dx$.

74 Differentiate:

a $\ln(\sin x)$

b $\tan(e^{5x}+1)$

75 Find an approximation to $\int_0^{\frac{\pi}{4}} \tan x\,dx$ correct to 3 decimal places by using a triangle.

76 Find the area under the curve $y = 4 - x^2$ by using:

a 4 inner rectangles

b 4 outer rectangles

77 EXT1 Find the volume of the solid formed when the curve $y = \sqrt{\cos x}$ is rotated about the x-axis from $x = 0$ to $x = \frac{\pi}{3}$.

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78 Differentiate each function.

a $e^x \sin x$ **b** $\tan^3 x$ **c** $2\cos\left(3x - \frac{\pi}{2}\right)$

79 Find the equation of the tangent to the curve $y = \tan 3x$ at the point where $x = \frac{\pi}{4}$.

80 Differentiate:

a $\sin^3 (e^x)$ **b** $\tan (\ln x + 1)$

81 Find the exact area bounded by the curve $y = \ln (x + 4)$, the y-axis and the lines $y = 0$ and $y = 1$.

82 Find the anti-derivative of each function.

a e^{3x} **b** $\sec^2 \pi x$ **c** $\frac{1}{2x}$

d $\cos\left(\frac{x}{5}\right)$ **e** $\sin 8x$

83 EXT1 Find the volume of the solid of revolution formed when the curve $y = \sin x$ from $x = 0$ to $x = \pi$ is rotated about the x-axis.

84 Find $\int \frac{3x^2 - 2x + 5}{x^2}\,dx$.

85 Find $\int (e^{5x} - \sin \pi x)\,dx$.

86 Find the exact area enclosed between the curve $y = e^x$, the x-axis, the y-axis and the line $x = 2$.

87 Evaluate $\int_{\frac{\pi}{3}}^{\pi} \cos\left(\frac{x}{2} + \pi\right) dx$.

88 EXT1 A function has $f'(x) = x^2 \sec^2 (x^3)$ and $f(0) = 0$.
Use the substitution $u = x^3$ to evaluate $f(1)$.

EXT1 **89** **a** Use the substitution $u = x^2 - \frac{\pi}{2}$ to find $\int x \sin\left(x^2 - \frac{\pi}{2}\right) dx$.

b If a curve has $\frac{dy}{dx} = x \sin\left(x^2 - \frac{\pi}{2}\right)$ and the curve passes through $(0, 0)$, find the smallest value of x in the domain $[0, 2\pi]$ for which $y = \frac{1}{2}$.

STATISTICAL ANALYSIS

9. STATISTICS

In this chapter, you will study different ways of describing, displaying and summarising statistical data. You will look at measures of central tendency and spread, and use these to interpret and compare data.

CHAPTER OUTLINE

9.01 Types of data
9.02 Displaying numerical and categorical data
9.03 Measures of central tendency
9.04 Quartiles, deciles and percentiles
9.05 Range and interquartile range
9.06 Variance and standard deviation
9.07 Shape and modality of data sets
9.08 Analysing data sets

IN THIS CHAPTER YOU WILL:

- identify different types of data
- display data in tables and graphs
- calculate measures of central tendency: the mean, median and mode
- calculate measures of spread: the range, quantiles, interquartile range, variance and standard deviation
- identify outliers
- recognise different modalities and shapes of data sets
- identify bias in data
- compare 2 sets of data

TERMINOLOGY

bar chart: Graph with vertical or horizontal columns, also called a **column graph**.
bimodal: A graph with 2 peaks.
box plot: Graphical display of five-number summary, also called a **box-and-whisker plot**.
categorical data: Data that are named by categories.
continuous data: Numerical data that can take any value that lies within an interval.
decile: One of the values that divide a data set into 10 equal parts.
discrete data: Numerical data that can only take specific distinct values.
dot plot: A column graph of dots.
five-number summary: The lowest and highest values, median, and lower and upper quartiles of a data set.
frequency polygon: Frequency line graph.
histogram: Bar chart of frequencies with no gap between columns.
interquartile range: Measure of spread, the difference between the upper and lower quartiles.
mean: Average score, calculated by dividing the sum of scores by the total number of scores.
median: The middle score when all scores are placed in order.
modality: The number of peaks in a set of data.
mode: The score with the highest frequency.
multimodal: Having many peaks in a set of data.
nominal data: Categorical data that is listed by name with no order.
numerical data: Data whose values are numbers.
ogive: Cumulative frequency polygon.
ordinal data: Categorical data that can be ordered.
outlier: A score that is clearly apart from other scores – it may be much higher or lower than the other scores.
Pareto chart: A chart containing both a bar chart and a line graph where individual values are represented in descending order by the bars and the cumulative total is represented by the line graph.
percentile: One of the values that divide a data set into 100 equal parts.
pie chart: Circular graph showing categories as sectors.
quantile: One of the values that divide a data set into equal parts.
quartile: One of the values that divide a data set into 4 equal parts.
range: Difference between the highest and lowest scores.
skewness: The shape or asymmetry of a graph to one side.
standard deviation: Measure of the spread of data values from the mean. The square root of variance.
stem-and-leaf plot: Graphical display of tens (stem) and units (leaves).
symmetrical distribution: A distribution where the left and right sides are mirror images of each other.
two-way table: A table that combines the effects of 2 separate variables (usually categorical).
variance: Measure of spread, the square of standard deviation.

9.01 Types of data

WS Statistical data match-up

There are many different types of data, for example, the type of public transport people use to go to work, the heights of basketball players or the marks students gain in an exam.

There are 2 main types of data:

- **Categorical data** uses categories described by words or symbols
- **Numerical data** uses numbers or quantities.

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These types can be divided further:

Categorical and numerical data

Categorical data:

- **Nominal data**, which cannot be put in order
- **Ordinal data**, which can be ordered

Numerical data:

- **Discrete data**, which can be counted as separate values
- **Continuous data**, which is measured along a smooth scale

For example:

- Public transport – bus, train, tram, ferry – is **categorical nominal** data since it cannot be put into an order.
- Ratings – strongly disagree, disagree, agree, strongly agree – are **categorical ordinal** since they can be put in order.
- Shoe sizes – $6, 6\frac{1}{2}, 7, 7\frac{1}{2}, \ldots$ – are **numerical discrete** since they can be counted.
- Heights of basketball players – 181 cm, 173.64 cm, 192.1 cm … – are **numerical continuous** since they are along a smooth scale.

EXAMPLE 1

Describe each type of data.

a The breeds of dogs

b Exam marks

c The volume of water in a dam

d Audience size for TV programs

e Makes of cars

f Months of the year

Solution

a Categorical nominal

b Numerical discrete

c Numerical continuous

d Numerical discrete

e Categorical nominal

f Categorical ordinal

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Exercise 9.01 Types of data

1 State whether each type of data is categorical (C) or numerical (N).

a Length of a fence
b Number of koalas in captivity
c Shoe size
d Colour
e Area of land
f Scores on a test
g Number of lollies in a packet
h Gender
i Speed
j Type of swimming strokes
k Attendance at a football match
l Meals on a menu
m Width of a building
n Age
o Weight
p Ranking of quality of a movie
q Surface area of a balloon as it is blown up
r Shirt sizes
s Type of sports offered at a school
t Length of a swimming race

2 State whether each type of data is categorical nominal (N), categorical ordinal (O), numerical discrete (D), numerical continuous (C).

a Survey of radio stations: Excellent, very good, good, poor, very poor
b Weight of truck loads
c Make and model of motorbikes
d Eye colour
e Volume of water in rivers
f Scores on a maths exam
g Number of jellybeans in a packet
h Nationality
i Acceleration
j Olympic sports
k Concert attendance
l Choice of desserts on a menu
m Types of trees in a park

3 Give 3 examples of:

a categorical data
b numerical data
c numerical discrete data
d categorical ordinal data
e numerical continuous data
f categorical nominal data

INVESTIGATION

DATA COLLECTION

Certain organisations are specially set up to collect and analyse data. The Australian Bureau of Statistics (ABS) collects all sorts of data, including the organisation of a regular census.

The census attempts to collect details of every person living in Australia on a particular day. Questions asked include where a person lives, occupation, salary, number of children, religion and marital status. Governments and other organisations use this data to plan future policies in areas such as education, transport, housing. For example, if the number of children in a certain region is increasing, then extra schools could be planned in that area.

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There is evidence that a census was done back in ancient Roman times. Investigate the methods that the Romans or some other ancient civilisation used for collecting data and writing reports.

What information do you think a census should collect? Is there information that you think that is not useful or invades privacy and therefore should not be collected?

Go to the ABS website and find out more about what this organisation does. Other worldwide organisations such as the World Health Organization (WHO) and the United Nations also collect data. Research these and other organisations that collect data, such as universities and the CSIRO.

9.02 Displaying numerical and categorical data

Data can be displayed in many different ways using tables and graphs.

WS Graphing data

WS Two-way tables and Pareto charts

WS Every picture tells a story

WS Cumulative frequency graphs

Numerical data

Frequency tables, **histograms**, **frequency polygons** and other graphs can be used to display numerical data.

EXAMPLE 2

a For the following Year 12 English essay marks (out of 10):

8, 4, 5, 4, 8, 6, 7, 8, 9, 5, 6, 7, 7, 5, 4, 6, 7, 9, 3, 5, 5

- **i** draw a frequency distribution table
- **ii** draw a histogram for this data
- **iii** draw a frequency polygon on the same set of axes as the histogram
- **v** how many scores are less than 5?
- **v** what percentage of scores are over 6?

b The assessment scores for a Year 12 mathematics class are below.

75, 53, 58, 71, 68, 51, 60, 87, 62, 62, 89, 65, 69, 47, 70, 72, 75, 68, 76, 83, 62, 88, 94, 53, 85

- **i** Draw a frequency table that shows the results of the class test using groups of 40–49, 50–59 and so on.
- **ii** Add a column for class centre and cumulative frequency.
- **iii** Draw a cumulative frequency histogram and a cumulative frequency polygon (ogive).
- **v** What percentage of students scored less than 60?

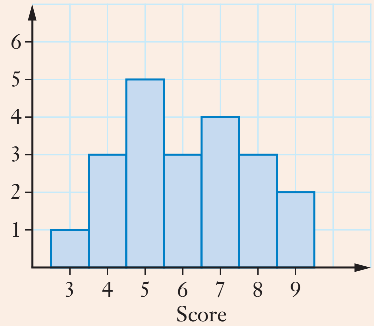
Solution

a **i** The scores range from 3 to 9. We arrange them in a table as shown.

Score	Tally	Frequency
3	𝍷	1
4	𝍷𝍷𝍷	3
5	𝍸	5
6	𝍷𝍷𝍷	3
7	𝍷𝍷𝍷𝍷	4
8	𝍷𝍷𝍷	3
9	𝍷𝍷	2

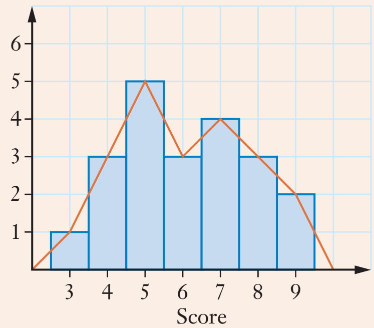
ii The histogram is a bar chart or column graph where the centre of the column is lined up with the score and the columns join together.

Year 12 English essay marks



iii The frequency polygon is a line graph as shown. It starts and ends on the horizontal axis.

Year 12 English essay marks



v Reading from either the table or the graph, scores of 3 and 4 are less than 5. There is one score of 3 and 3 scores of 4.

So there are $1 + 3 = 4$ scores less than 5.

v There are $4 + 3 + 2 = 9$ scores over 6 out of a total of 21 scores.

$$\frac{9}{21} \times 100\% \approx 429\%$$

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b **i**

Scores	Tally	Frequency
40–49	|	1
50–59	||||	4
60–69	卌 |||	8
70–79	卌 |	6
80–89	卌	5
90–99	|	1

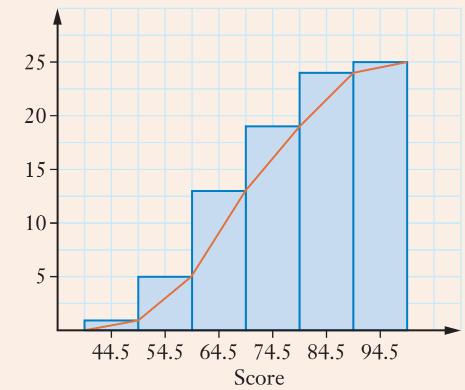
ii The class centre is the average of the highest and lowest possible score in each group. For example, $\frac{40+49}{2} = 44\ 5$.

Add each score to the previous total for cumulative frequencies.

Scores	Class centre	Frequency	Cumulative frequency
40–49	44.5	1	1
50–59	54.5	4	5
60–69	64.5	8	13
70–79	74.5	6	19
80–89	84.5	5	24
90–99	94.5	1	25

iii Use the class centres for the scores on the graph. The cumulative frequency polygon or **ogive** starts at the bottom left of the first column and ends at the top right corner of the last column.

Class test results



v 5 students scored less than 60.

$\frac{5}{25} \times 100\% = 20\%$.

You can also draw **stem-and-leaf plots** to show discrete data.

EXAMPLE 3

The heartbeat rates in beats per minute of a sample of hospital patients were taken:

75, 53, 58, 71, 68, 51, 60, 87, 62, 62, 89, 65, 69, 47, 70, 72, 75, 68, 76, 83, 62, 88, 94, 53, 85

Draw a stem-and-leaf plot to display these scores.

Solution

On the left of a vertical line, put in the 10s for the scores (the stem). On the right, place the unit for each score in order (the leaf).

For example, for a score of 68 show 6 | 8.

Stem	Leaf
4	7
5	1 3 3 8
6	0 2 2 2 5 8 8 9
7	0 1 2 5 5 6
8	3 5 7 8 9
9	4

Note: The stem-and-leaf plot keeps the actual scores whereas grouping them into a frequency distribution table loses this individual information.

Categorical data

We can display categorical data in different tables and graphs, including **two-way tables**, **bar charts**, **pie charts** and **Pareto charts**.

EXAMPLE 4

In a survey of Year 12 students, it was found that 47 students had a dog but not a cat, 19 had both a dog and a cat, 32 had a cat but not a dog, and 54 had neither a dog nor a cat.

a Draw a two-way table showing this data.

b Find the percentage of students who have:

- **i** both a dog and cat
- **ii** a cat but not a dog
- **iii** neither a cat nor a dog

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Solution

a A two-way table separates out the students with dogs from those with cats.

	Has a dog	Does not have a dog
Has a cat	19	32
Does not have a cat	47	54

Note that this table could be the other way around, with the cats at the top and the dogs down the side.

b There are 19 + 32 + 47 + 54 = 152 students altogether.

i 19 students out of 152 have both a dog and a cat: $\frac{19}{152} \times 100\% \approx 12.5\%$

ii 32 students out of 152 have a cat but not a dog: $\frac{32}{152} \times 100\% \approx 21.1\%$

iii 54 students out of 152 have neither a dog nor a cat: $\frac{54}{152} \times 100\% \approx 35.5\%$

EXAMPLE 5

The table shows the eye colour of students.

Represent this data in:

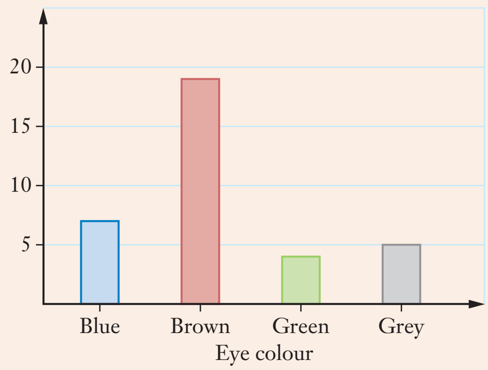
a a bar chart

b a pie chart

Colour	Frequency
Blue	7
Brown	19
Green	4
Grey	5

Solution

a Unlike a histogram, in a bar chart the columns do not need to join up.



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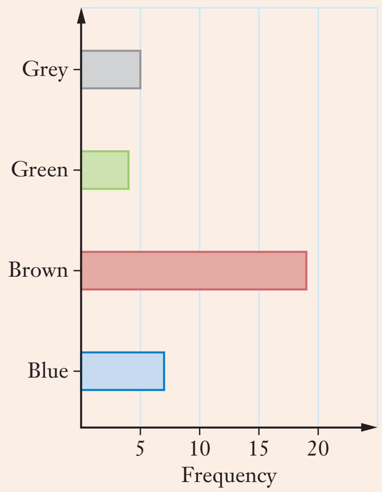
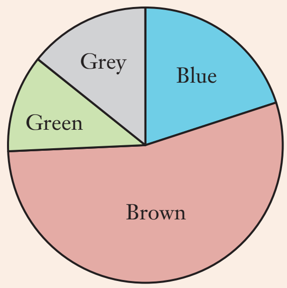
You can also draw the data as a horizontal bar chart like this.



b A pie chart is a circle divided into portions (sectors). Since the angle inside a circle is 360°, each frequency is a proportion of 360°.

There were 35 students surveyed.

Colour	Frequency	Angle
Blue	7	$\frac{7}{35} \times 360° \approx 72°$
Brown	19	$\frac{19}{35} \times 360° \approx 195°$
Green	4	$\frac{4}{35} \times 360° \approx 41°$
Grey	5	$\frac{5}{35} \times 360° \approx 51°$



Note: The number of degrees calculated adds to only 359° because the answers are not exact. This will not greatly affect the pie chart.

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A **Pareto chart** is useful for displaying categorical data from the most to the least important.

Pareto chart template

EXAMPLE 6

The table shows a survey group's preferences for types of TV shows.

a Arrange the table in descending order of frequency and add a percentage frequency column and a cumulative percentage frequency column.

b Draw a Pareto chart to show this data.

Type	Frequency
News	68
Drama	78
Comedy	73
Reality	107
Sport	174
	500

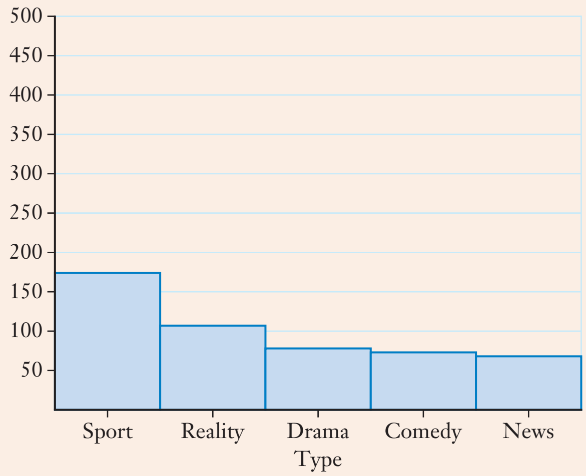
Solution

a

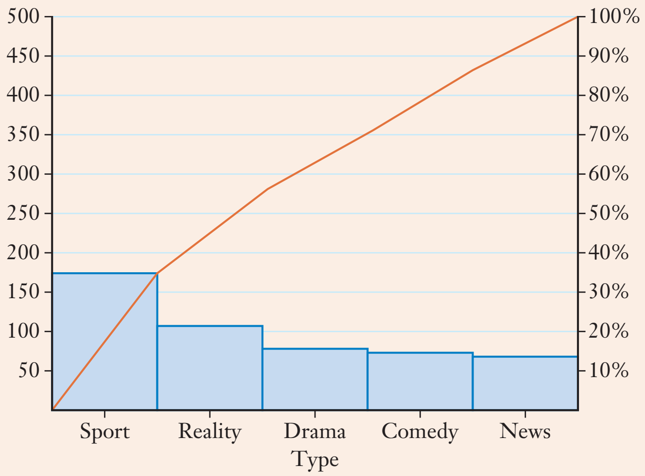
Type	Frequency	Percentage frequency	Cumulative percentage frequency
Sport	174	34.8%	34.8%
Reality	107	21.4%	56.2%
Drama	78	15.6%	71.8%
Comedy	73	14.6%	86.4%
News	68	13.6%	100%
Total	500		

For example, percentage frequency for Sport $= \frac{174}{500} \times 100\% = 348\%$

b Step 1: Draw a bar chart of the frequencies using the left axis.



Step 2: Draw a line graph of cumulative percentages using the right axis.



Statistical graphs

INVESTIGATION

GRAPHS AND SPREADSHEETS

You can draw different types of graphs, including Pareto charts, using a spreadsheet. Enter the data into the spreadsheet, highlight the table and select the chart you want to use. If you are not sure of how to do this, search for online tutorials.

DID YOU KNOW?

Vilfredo Pareto

The Pareto chart is named after **Vilfredo Pareto** (1848–1923), an economist, sociologist, engineer and philosopher. The chart can be used as a tool for quality control.

Research the Pareto chart, the Pareto principle and the 80/20 rule. Find examples of its uses.

Exercise 9.02 Displaying numerical and categorical data

1 For each set of scores on the next page:

- **i** draw a frequency distribution table
- **ii** draw a histogram and frequency polygon
- **iii** find the highest and lowest scores (groups for parts **d** and **e**)
- **v** find the most frequent score (group for parts **d** and **e**).

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a Results of a class quiz:

8, 6, 5, 7, 6, 8, 3, 2, 6, 5, 8, 4, 7, 3, 8, 7, 5, 6, 5, 8, 6, 4, 9, 6, 5

b The number of people ordering pizzas each night:

15, 12, 17, 18, 18 ,15, 16, 13, 15, 17, 18, 12, 17, 14, 16, 15, 17, 18, 19, 15, 15, 12

c The number of people attending a gym:

110, 112, 114, 109, 112, 113, 108, 110, 113, 112, 113, 110, 109, 110, 110, 112, 114, 114, 112, 114, 113

d The results of an assessment task:

45, 79, 65, 48, 69, 50, 62, 74, 38, 69, 88, 96, 90, 58, 52, 68, 63, 61, 79, 74, 50, 65, 77, 91, 56, 77, 63, 81, 90, 59, 67, 50, 61

(Use groups of 30–39, 40–49 and so on.)

e The heights of students (in cm) in a Year 12 class:

159, 173, 182, 166, 172, 179, 181, 163, 178, 169, 183, 158, 162, 167, 174, 175, 180, 174, 176, 159, 161, 171, 174, 179, 180, 159, 157

(Use groups of 155–159, 160–164, 165–169 and so on.)

2 For each data set:

i add a cumulative frequency column and class centre where necessary

ii sketch a cumulative frequency histogram and ogive (cumulative frequency polygon).

a Number of cars in a school car park

Number of cars	Frequency
10	4
11	8
12	11
13	9
14	5

b Results of a science experiment

Score	Frequency
1	7
2	1
3	3
4	0
5	2
6	5

c Number of sales made in a shoe shop

Sales	Class centre	Frequency
0–4		6
5–9		2
10–14		3
15–19		5
20–24		8
25–29		9
30–34		5

d Results of an assessment task

Scores	Class centre	Frequency
0–19		3
20–39		2
40–59		7
60–79		6
80–99		1

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3 The table shows the number of daily rescues at a beach over a period of time.

Rescues	Frequency
4	1
5	3
6	6
7	4
8	5
9	0
10	2

a Draw a histogram showing this data.

b How many times were more than 6 rescues made?

c What was the most common number of daily rescues during the survey?

4 The volume of traffic on a stretch of highway was measured and the results are shown in the table.

Volume/min	Frequency
1–10	7
11–20	10
21–30	8
31–40	5
41–50	4

a Draw a histogram to show this data.

b What was the percentage volume of traffic between 21 and 40 minutes?

5 **a** Draw a two-way table for the following data:

- 27 people play soccer but not tennis
- 35 people play tennis but not soccer
- 28 play neither sport
- 12 play both sports

b What percentage of people play:

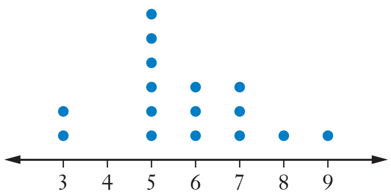
i both sports? **ii** neither sport?

c What percentage of people who play at least one of these sports play only soccer but not tennis?

d What fraction of people who play soccer play tennis as well?

e What percentage of tennis players do not also play soccer?

6 Lauren surveyed her friends and had them rank a film from 1 to 10. The **dot plot** shows the results of her survey.



a How many friends did Lauren survey?

b What was the most common ranking?

c Draw the results in a histogram.

d What percentage of rankings were above 4?

e What fraction of Lauren's friends ranked the film below 4?

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7 The stem-and-leaf plot shows the weights (in kg) of a group of people surveyed at a local gym.

Stem	Leaf
5	4 6 8 9
6	1 3 7 8
7	0 3 4 5 5 9
8	1 2 2 4 7 8 9
9	3 5
10	2 6 7

a Arrange these weights in a frequency distribution table using groups of 50–59, 60–69 and so on, and include class centre and cumulative frequency columns.

b Draw an ogive of this data.

c How many people weighed:

i 80 kg or more? **ii** less than 60 kg?

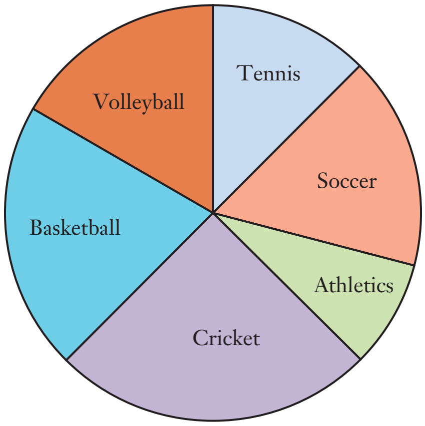
d What percentage of people surveyed weighed from 70 kg to 89 kg?

e What fraction of people weighed between 50 kg and 80 kg?

8 The pie chart shows the number of students taking different school sports.

a What percentage of students play cricket?

b There are 720 students at the school who play these sports. By measuring the angles in the pie chart, complete a table showing the frequencies for the different sports.



9 The table shows the results of a survey of university students asking what degree they were doing.

a What percentage of students were studying law?

b What percentage of students were studying medicine or music?

c Draw a pie chart showing this information.

Degree	Frequency
Medicine	104
Arts	87
Music	58
Science	93
Economics	79
Law	101

10 The two-way table shows the results of a survey into the protective effect of vaccination on a new virus.

	Vaccinated	Not vaccinated
Infected	11	76
Not infected	159	58

a How many people in the survey were infected with the virus?

b What percentage of people surveyed were vaccinated?

c What percentage of vaccinated people had the virus?

d How many people with the virus were not vaccinated?

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11 The two-way table shows the number of people taking part in a trial of a new medication to prevent asthma.

	Taking medication	Control group
Asthmatic	104	105
Not asthmatic	112	109

a How many people took part in the trial?

b What percentage of people were asthmatic?

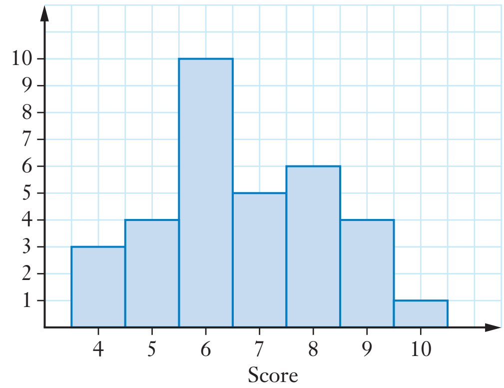
c What percentage of asthmatic people were in the control group?

d How many people who were not asthmatic took part in the trial?

e What fraction of the non-asthmatic people took medication?

12 The frequency histogram shows the scores on a maths quiz.

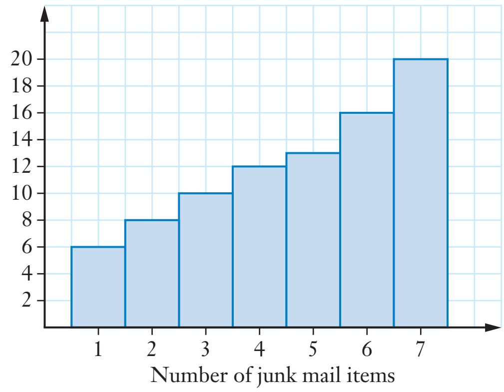
Draw a cumulative frequency histogram and polygon.



13 The cumulative frequency histogram shows data collected in a survey on the number of junk mail items that people received daily in their inbox.

a Draw a frequency distribution table to show the number of junk mail items people receive daily.

b Construct a frequency histogram to show this data.



14 Explain how a stem-and-leaf plot and grouped frequency distribution table can be used for the same data. What are the advantages and disadvantages of each?

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15 Draw a Pareto chart for each set of data.

a A survey into why people like a movie:

Reason	Votes
Acting	33
Storyline	29
Music	12
Characters	26

b Customer complaints about an Internet provider:

Complaint	Frequency
Internet speed	34
Cost	61
Data allowance	59
Technical difficulties	46

c Votes for best café in a suburb:

Café	Votes
Coffee Haus	32
Coffee Bean	48
Café Focus	21
Jumping Bean	63
Caffeine Café	36

9.03 Measures of central tendency

WS Measures of central tendency

Statistical measures

Statistics from a frequency table

WS Mean, median and mode

When we analyse data, we try to find a 'typical', 'normal' or 'average' score. For example we might want to know the average crowd size at football matches through the season. You would usually expect to find this score somewhere in the centre of the data. There are 3 **measures of central tendency**: the mean, the mode and the median.

The mean

$$\text{Mean} = \frac{\text{Sum of scores}}{\text{Total number of scores}}$$

$$\bar{x} = \frac{\Sigma x}{n}$$

The **mean** has symbol $\bar{x}$, n is the number of scores and Σx is the sum of scores. Note: Σ is the Greek letter 'sigma' and is used in mathematics to stand for a sum.

The symbol $\bar{x}$ usually represents the mean of a **sample**. For the mean of a **population**, the correct symbol is μ, the Greek letter mu.

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EXAMPLE 7

There are 5 children in a family, aged 13, 19, 11, 17 and 10. Find the mean of their ages.

Solution

$$\bar{x} = \frac{\Sigma x}{n}$$

$$= \frac{13+19+11+17+10}{5}$$

There are 5 chldren so $n = 5$.

$$= \frac{70}{5}$$

$$= 14$$

So the mean age of the children is 14.

The mean can also be calculated using a calculator's statistics mode.

	Casio scientific	Sharp scientific
Place your calculator in statistical mode.	[key] [key] 1 [key]	[key] [key] [=]
Clear the statistical memory.	[key] 1 [key], [key]-A	[key] [key]
Enter data.	[key] 1 Data to get table. 13 [=] 19 [=] etc. to enter in column. [key] to leave table.	13 [key] 19 [key] etc.
Calculate mean.	[key] 1 [key] $\bar{x}$ [=]	[key] $\bar{x}$
Check the number of scores.	[key] 1 [key] n [=]	[key] n
Change back to normal mode.	[key] [key]	[key] 0

For the mean of larger data sets, it is easier to sort the data into a frequency distribution table to add up the scores.

The mean of data in a frequency table

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$

where Σfx is the sum of each score × its frequency and Σf is the sum of frequencies

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EXAMPLE 8

Find the mean number of hours that members of a class practise piano each week. The number of hours for each student is:

1, 4, 6, 1, 3, 6, 2, 1, 1, 3, 2, 5, 6, 6, 1, 2, 6, 2, 5, 6, 6, 2, 3, 6, 2

Solution

First draw up a frequency distribution table for the hours of practice. Include an fx column for multiplying each score by its frequency.

The table gives us a quick way of finding the sum of scores. For example, we know from the table that there are 6 lots of 2, so we can use $2 \times 6 = 12$. The sum of the fx column gives us the sum of all scores.

Hours (x)	Frequency (f)	Score × frequency (fx)
1	5	5
2	6	12
3	3	9
4	1	4
5	2	10
6	8	48
	$\Sigma f = 25$	$\Sigma fx = 88$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$

$$= \frac{88}{25}$$

$$= 3.52$$

So the mean time students practise each week is 3.52 hours.

On a calculator:

Operation	Casio scientific	Sharp scientific
Clear the statistical memory first (see previous example, previous page).		
Enter data.	[I] [ ] scroll down to STAT, Frequency? ON [I] 1 Data to get table. 1 [=] 2 [=] etc. to enter in x column. 5 [=] 6 [=] etc. to enter in FREQ column. [ ] to leave table.	1 [ ] [ ] 5 [ ] 2 [ ] [ ] 6 [ ] etc.
Calculate mean.	[I] 1 [ ] $\bar{x}$ [=]	[ ] $\bar{x}$

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Although grouped data is not completely accurate because we don't know exactly what scores are in each group, we can still calculate an estimate of the mean.

EXAMPLE 9

From the table, find the mean commuting time that a sample of people take to travel to work.

Minutes	Frequency
0–8	3
9–17	5
18–26	7
27–35	8
36–44	2

Solution

Add **class centre** and fx columns to the table.

Use the class centres as the scores when calculating fx.

Minutes	Class centre (x)	Frequency (f)	Score × frequency (fx)
0–8	4	3	12
9–17	13	5	65
18–26	22	7	154
27–35	31	8	248
36–44	40	2	80
		$\Sigma f = 25$	$\Sigma fx = 559$

$$\begin{aligned}\bar{x} &= \frac{\Sigma fx}{\Sigma f} \\ &= \frac{559}{25} \\ &\approx 22.36\end{aligned}$$

The mean time taken to travel to work is 22.36 minutes.

The mode

The **mode** is the most frequent score.

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There is no mode if all the scores are different, or there could be several scores with the same frequency.

EXAMPLE 10

a Find the mode of these scores:

7, 4, 3, 5, 7, 1, 2

b Find the mode for these shoes sold at a shoe store.

Shoe size	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8	$8\frac{1}{2}$	9	$9\frac{1}{2}$
Frequency	8	9	15	28	53	61	58	29	12	10

Solution

a There are two 7s and only one of the other scores, so the mode is 7.

b The shoe size with the highest frequency is $7\frac{1}{2}$ (there were 61 of them).

So the mode is $7\frac{1}{2}$.

With grouped data, instead of finding the mode we find the **modal class**.

EXAMPLE 11

Find the modal class in this data set showing the ages of people at a caravan park.

Scores	Frequency
10–19	2
20–29	0
30–39	1
40–49	5
50–59	7
60–69	3

Solution

The group or class with the highest frequency is 50–59.

While we do not know the individual score with the highest frequency, we say the modal class is 50–59.

The mode is useful when looking at trends such as the most popular types of clothing. It can also be used for categorical data.

The median

The **median** is the middle score when all scores are in order.

If there are 2 middle scores, the median is the average of those scores.

EXAMPLE 12

Find the median age of a group of people in a band:

18, 15, 20, 18, 17, 16, 11, 13

Solution

Put the ages in order.

11, 13, 15, 16, 17, 18, 18, 20

There are 2 middle ages, 16 and 17, so we find their average.

$$\text{Median} = \frac{16+17}{2} = 16.5$$

So the median age of the band members is 16.5.

The median can also be calculated using a calculator's statistics mode.

Operation	Casio scientific
Clear the statistical memory.	
Enter data.	[I] 1 Data to get table. 18 [=] 15 [=] etc. to enter in column. [ ] to leave table.
Calculate the median.	[I] 1 MinMax med [=]
Change back to normal mode.	[ ] [ ]

You can find the median of data in a frequency table. If there is a large number of scores, you can find the position of the middle score using a cumulative frequency column.

EXAMPLE 13

Find the median of this data set.

Score	Frequency
5	3
6	2
7	4
8	7
9	6
10	3

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Solution

Add a column for cumulative frequencies.

Score	Frequency	Cumulative frequency
5	3	3
6	2	5
7	4	9
8	7	16
9	6	22
10	3	25

There are 25 scores so the position of the middle score is $\frac{25+1}{2} = 13$th.

The 10th to 16th scores are 8, so the 13th score is 8.

You can check this by writing the scores in order.

The median is 8.

The position of the median

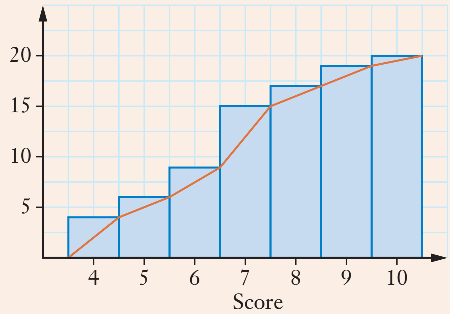
The median of n scores is the $\frac{n+1}{2}$ th score.

If n is even, then the median is the average of the 2 middle scores on both sides of the $\frac{n+1}{2}$ th position.

Another way to find the median is from an ogive (cumulative frequency polygon). We simply use the halfway point on the cumulative frequency axis of the graph.

EXAMPLE 14

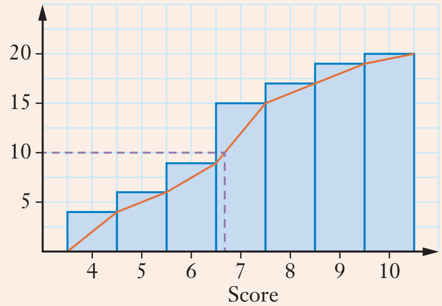
Find the median from the cumulative frequency polygon below.



Solution

There are 20 scores in the data set, so the halfway point is at the 10th score as shown on the cumulative frequency axis. The dotted line meets the ogive inside the 7 column.

The median is 7.



Outliers

Sometimes a set of data contains a score that is unusual, compared with the other scores. This unusual or extreme value is called an **outlier**.

EXAMPLE 15

The prices of houses sold in the town of Greenfield in a particular week are:

\$355 000, \$420 000, \$320 000, \$285 000, \$390 000, \$1 200 000, \$415 000, \$320 000, \$435 000, \$380 000

a Is there an outlier? Why do you think an outlier may be in this data?

b Find the mean house price with and without the outlier.

c Find the median with and without the outlier.

d Find the mode with and without the outlier.

Solution

a The outlier is \$1 200 000 as this is much higher than the other prices. It may be that there is one special house in the area that is much larger than the others, or a certain street with huge houses in it that is unusual for the area.

b Using a calculator:

With the outlier, the mean house price is \$452 000. ← Notice the big difference.

Without the outlier, the mean house price is \$368 888.89.

c With the outlier, the median house price is \$385 000. ← Notice the small difference.

Without the outlier, the median house price is \$380 000.

d The mode is \$320 000 in both cases since this is the most frequent price with or without the outlier.

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Note: When real estate agents talk about house prices, they usually use the median price since this is not as affected by outliers as the mean.

INVESTIGATION

OUTLIERS

Which measures of central tendency do outliers tend to affect most?

Find other examples of data that contain outliers and find the mean, mode and median. How do they change if the outlier is removed? Should outliers be looked at closely and discarded or is there a place for them?

Exercise 9.03 Measures of central tendency

1 For each data set find:

i the mean **ii** the mode **iii** the median

a Number of people auditioning for parts in a play: 5, 5, 7, 6, 5, 6, 1

b Number of minutes for an ambulance to respond to a call: 1, 4, 6, 8, 7, 4, 6, 4, 5

c Ages of students on a basketball team: 15, 18, 14, 19, 18, 17, 11

d Scores on a class quiz: 4, 6, 5, 4, 7, 8

e Prices of petrol (in dollars): 1.43, 1.66, 1.55, 1.49, 1.27, 1.81, 1.49, 1.38

2 Find the mode of each data set:

a

Hair colour	Frequency
Brown	28
Blond	21
Red	8
Black	12
Grey	17

b

Type of cat	Frequency
Siamese	18
Burmese	12
Russian blue	9
Tabby	32
Ginger	26
Persian	19

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3 For each data set find:

i the mean **ii** the median **iii** the mode

a Judges scores on a dance contest

Score	Frequency
3	3
4	4
5	2
6	7
7	6
8	2
9	3

b Number of matches in each match box surveyed

Score	Frequency
50	1
51	6
52	5
53	3
54	4
55	2

c Results in a History assignment

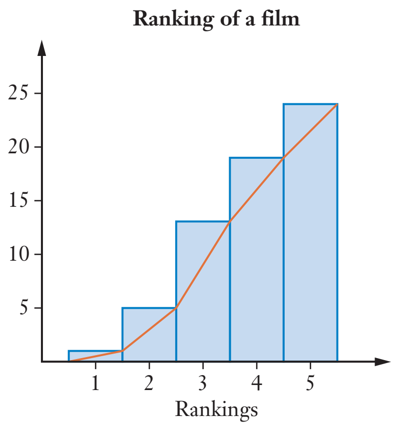
Score	Frequency
14	4
15	2
16	1
17	4
18	3
19	5
20	6

d Attendances at hockey matches

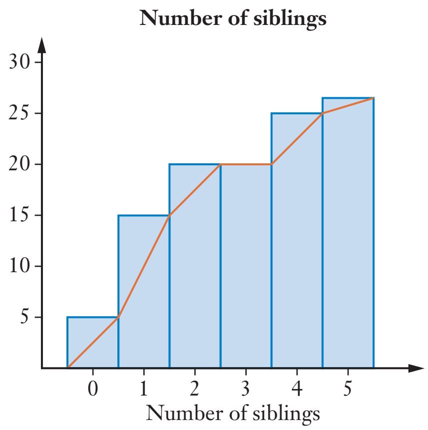
Attendance	Frequency
100	3
101	0
102	2
103	1
104	6
105	5

4 Find the median from each ogive:

a



b



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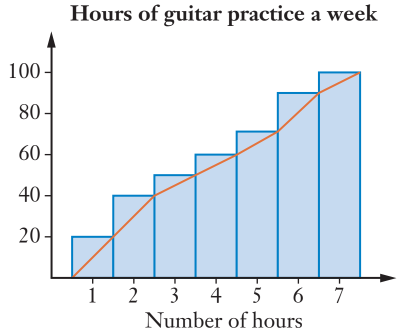
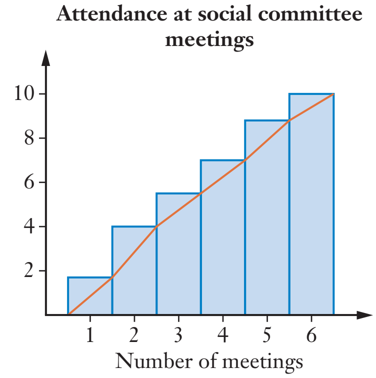
c



d



5 For each data set find:

i the mean **ii** the modal class

a Games of chess played each week by members of a chess club:

Score	Frequency
2–4	5
5–7	4
8–10	7
11–13	4
14–16	3
17–19	2

b Hours per week that gymnasts train:

Score	Frequency
0–4	3
5–9	2
10–14	6
15–19	8
20–24	9
25–29	5

c Results in a Legal Studies exam:

Score	Frequency
10–24	4
25–39	0
40–54	1
55–69	5
70–84	9
85–99	8

d Time it takes for computers to boot up:

Time (s)	Frequency
20–24	12
25–29	8
30–34	9
35–39	7
40–44	8
45–49	11
50–54	12
55–59	6

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6 For each data set:

- **i** add a cumulative frequency column
- **ii** draw a cumulative frequency polygon
- **iii** find the median from the graph (estimate for parts **c** and **d**)

a Number of athletes representing their school over a 30-year period:

Athletes	Frequency
1	5
2	6
3	4
4	8
5	5
6	2

b Number of lollies in a bag:

Number	Frequency
45	3
46	5
47	1
48	7
49	3
50	1

c Hours a week worked by employees in a cafe:

Hours	Frequency
1–5	7
6–10	5
11–15	3
16–20	6
21–25	7
26–30	2

d Time to complete a race:

Time (min)	Frequency
2.5–2.8	3
2.9–3.2	2
3.3–3.6	0
3.7–4.0	6
4.1–4.4	1
4.5–4.8	4
4.9–5.2	4

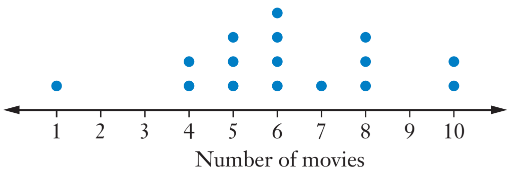
7 For each data set:

- **i** draw a frequency distribution table including cumulative frequency
- **ii** find the mean
- **iii** find the mode or modal class
- **v** draw an ogive
- **v** find the median from the ogive

a Home runs scored over a baseball season

4, 6, 5, 8, 8, 6, 5, 3, 4, 9, 6, 3, 5, 6, 5, 4, 7, 5, 8, 5, 6, 2, 3

b Number of movies seen in a year:



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c Ages of people living in a block of units (use classes of 20–29, 30–39 and so on):

Stem	Leaf
2	1 3 5 5 8 9
3	0 2 4 5 6 6 7 8 8 9
4	2 3 3 6 8
5	0 1 1 1 4 5 5 7 8 8 9
6	3 3 4 5 6 7 7
7	1 5 6
8	1 2 2 4 5 6

8 For each set of data, find:

i the outlier

ii the mean, mode and median

iii the mean, mode and median without the outlier

a Weights (in kg) of people in a lift:

69, 75, 58, 77, 32, 68, 60, 64, 59

b Number of questions attempted in an exam:

Questions	Frequency
1	1
2	0
3	0
4	1
5	3
6	6
7	5
8	3
9	4

c Ages of people at a family party:

Stem	Leaf
2	3 5 6
3	2 7
4	1 3
5	0 0 3 4
6	3 5
7	
8	
9	7

d Rating of a venue for a dance party:

9 For the set of times (in minutes) students are recorded as late for school shown below, find the outlier. Which measures of central tendency (mean, median, mode) does it change?

5, 3, 6, 4, 7, 1, 6, 8, 7, 9, 6, 5, 8, 6, 7, 4, 5, 7, 4

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10 The stem-and-leaf plot below shows the results of a class test:

Stem	Leaf
1	7
2	
3	8 9
4	4 5 6
5	1 3 4 6 7 8
6	4 5 5 7 7 9 9 9 9
7	0 2 3 3 4 5 8
8	3 4 5 7
9	0 1 1 3

a State which score is an outlier and what this means.

b Draw a frequency table including a cumulative frequency column, using groups 10–19, 20–29, 30–39 and so on.

c Use the table to estimate the mean and find the modal class:

i with the outlier included **ii** without the outlier included.

d Draw an ogive excluding the outlier and find the median.

INVESTIGATION

LIMITATIONS OF CENTRAL TENDENCY

Find the mean, mode and median of each set of data. What do you notice?

Set 1: 5, 6, 7, 7, 8, 9

Set 2: 1, 2, 7, 7, 12, 13

How do the 2 sets of data differ? Can we find out by using the measures of central tendency? How else could we describe how they are different from each other?

9.04 Quartiles, deciles and percentiles

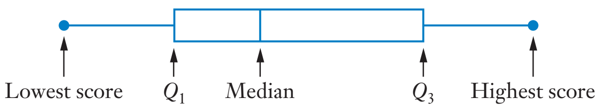
WS Quartiles, deciles and percentiles

Box-and-whisker plots

The measures of central tendency give us good information about data sets, but they don't describe the spread of data. As we have seen, the median divides data sets so that half the values lie below the median and half lie above it. A measure that divides a data set into parts of equal size is called a **quantile**. The median gives only a very rough description of the data set, but with more divisions we can describe the data's spread in more detail.

Quartiles

A **quartile** divides a data set into quarters.



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The 1st quartile (Q_1) is called the **lower quartile** and the 3rd quartile (Q_3) is called the **upper quartile**. The 2nd quartile (Q_2) is the median.

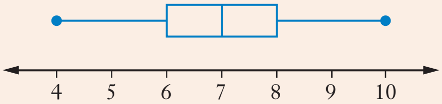
A **box plot** (also called a **box-and-whisker plot**) gives a way of showing a five-number summary: the quartiles and highest and lowest scores.

EXAMPLE 16

a Find Q_1, Q_2 and Q_3 for the number of runs in a softball game:

4, 3, 7, 8, 7, 9, 5, 6, 8, 3, 9

b Find Q_1, Q_3, the median, the highest and lowest score for this data.



c Find the quartiles of the data in this frequency table.

Score	Frequency
1	1
5	3
6	5
7	8
8	11
9	13
10	9

Solution

a Put the 11 scores in order.

3, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9

Find Q_2 (the median) in the usual way: the 6th score is 7.

3, 3, 4, 5, 6, (7), 7, 8, 8, 9, 9

Q_1 is the middle of the scores below the median: the 3rd score, 4.

If a quarile falls between 2 scores we take ther average

3, 3, (4), 5, 6, (7), 7, 8, (8), 9, 9

Q_3 is the middle of the scores above the median: the 9th score, 8.

So $Q_1 = 4$, $Q_2 = 7$, $Q_3 = 8$

On a calculator:

Operation	Casio Scientific
Clear the statistical memory.	
Enter data.	[ ] 1 Data to get table. 4 [=] 3 [=] etc to enter in colum. [ ] to leave table.
Calculate Q_1.	[ ] 1 MinMax Q_1 [=]
Calculate Q_2 (median).	[ ] 1 MinMax med [=]
Calculate Q_3.	[ ] 1 MinMax Q_3 [=]

b From the box plot:

Lowest score = 4, $Q_1 = 6$, Median = 7, $Q_3 = 8$, Highest score = 10.

c Add a cumulative frequency column to the table.

Score	Frequency	Cumulative frequency
1	1	1
5	3	4
6	5	9
7	8	17
8	11	28
9	13	41
10	9	50

There are 50 scores so the median is the $\frac{50+1}{2}$ or 25.5th score (average of 25th and 26th scores).

So $Q_2 = 8$, reading from the cumulative frequency column.

Q_1 is the $\frac{25+1}{2}$ or 13th score (middle of the 1st 25 scores).

So $Q_1 = 7$, reading from the cumulative frequency column.

Q_3 is the $\frac{50+25+1}{2}$ or 38th score (middle of the last 25 scores, halfway between the 26th to 50th scores).

So $Q_3 = 9$, reading from the cumulative frequency column.

So $Q_1 = 7$, $Q_2 = 8$, $Q_3 = 9$.

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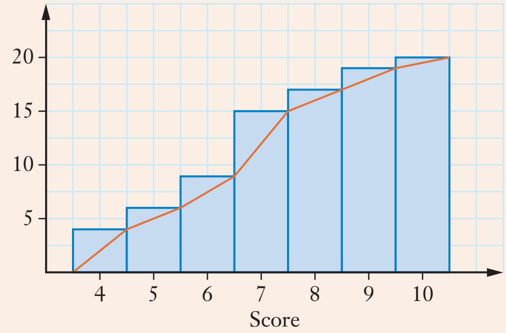
EXAMPLE 17

Find the median and upper and lower quartiles from the ogive.



Solution

There are 20 scores.

The median is halfway:

$$\frac{1}{2} \times 20 = 10$$

So the median is 7, reading across from 10 on the cumulative frequency axis.

To find the 1st (lower) quartile:

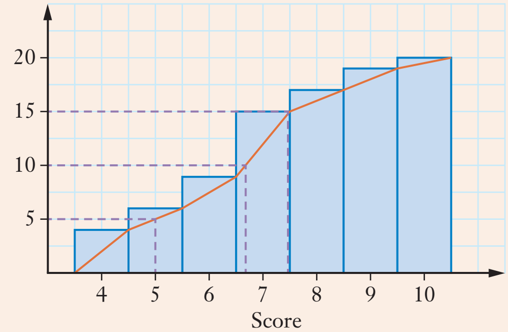
$$\frac{1}{4} \times 20 = 5$$

So reading across from 5, $Q_1 = 5$.

To find the 3rd (upper) quartile:

$$\frac{3}{4} \times 20 = 15$$

So reading across from 15, $Q_3 = 7.5$.



Deciles and percentiles

For a more detailed description of the spread we can divide the data set into smaller parts. **Deciles** divide the data set into 10 parts and **percentiles** divide data sets into 100 parts.

Deciles and percentiles

EXAMPLE 18

The ogive shows the number of hours that a rock group rehearses each week over 25 weeks. The scores have been sorted into groups.

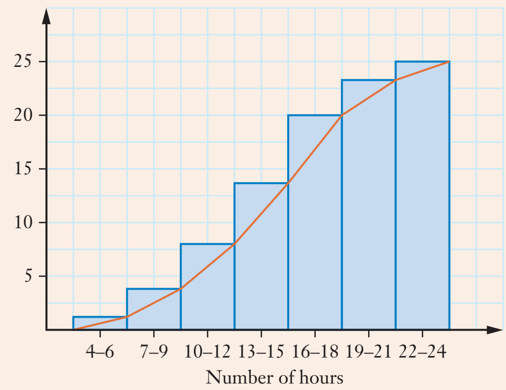
Use the ogive to estimate:

a the 35th percentile

b the 60th percentile

c the 7th decile

We can only **estimate** because the scores have been grouped into classes.



Solution

Redraw the ogive using **class centres** for number of hours, and use the cumulative frequency axis to find answers.

a The 35th percentile is the score that separates the lower 35% of the scores.

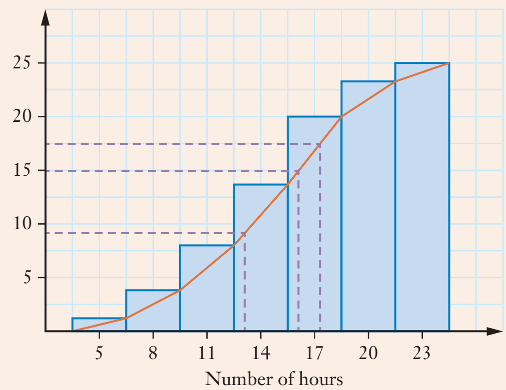
35% of 25 scores = 8.75th score.

The 35th percentile ≈ 13.

b The 60th percentile separates the lower 60% of the scores.

60% of 25 scores = 15th score.

The 60th percentile ≈ 16.



c The 7th decile separates the lower $\frac{7}{10}$ of the scores.

$\frac{7}{10} \times 25$ scores = 17.5th score.

The 7th decile ≈ 17.

You can use a graphics calculator or software to draw graphs and find quartiles, deciles and percentiles more accurately.

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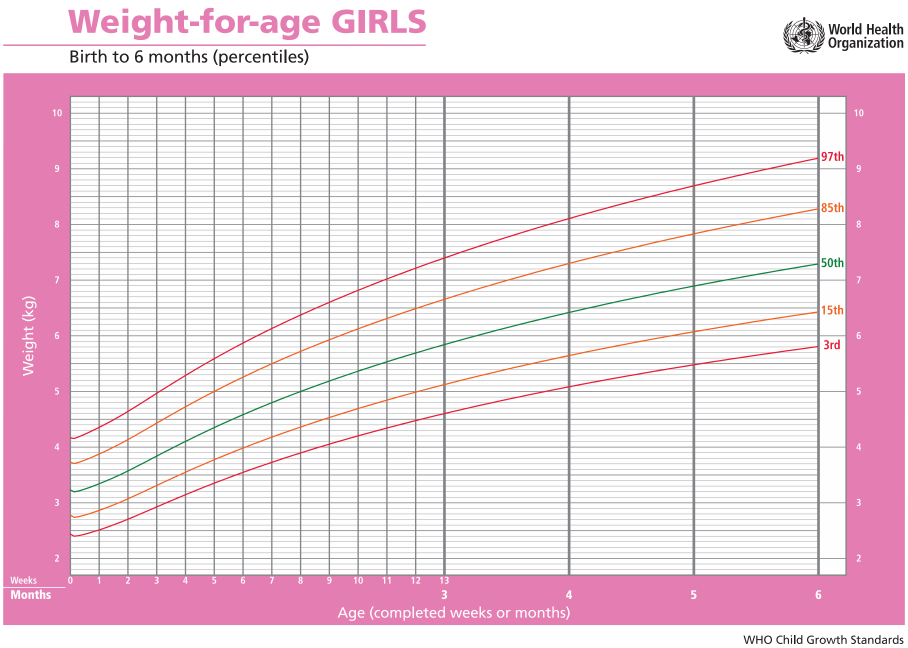
INVESTIGATION

RESEARCHING QUANTILES

Research the words *quartile*, *decile* and *percentile*. When were they first used? Where are they used now? There are other measures such as tercile and quintile. What are they?

Percentiles are used in many applications including graphs of infants' and children's growth rates.



http://www.who.int/childgrowth/standards/cht_wfa_girls_p_0_6.pdf?ua=1

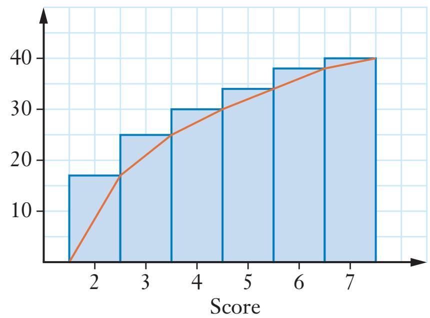
Research and compare different types of growth charts used in Australia and those of the World Health Organization (WHO).

Exercise 9.04 Quartiles, deciles and percentiles

1 For each set of data, find:

i the 1st quartile **ii** the 2nd quartile **iii** the 3rd quartile

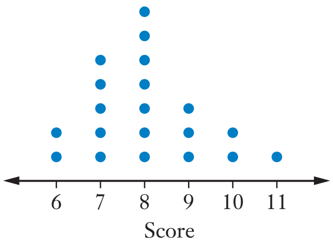
a



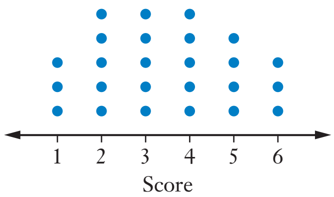
b

Score	Frequency
10	3
11	5
12	4
13	6
14	0
15	2

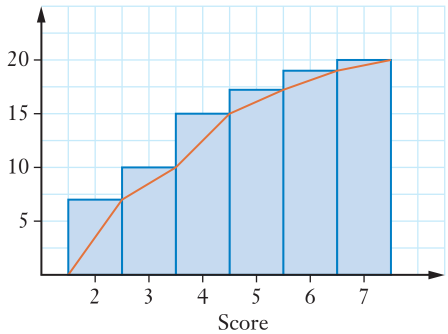
c



d



2 Find the 1st and 3rd quartiles for the following data.



3 For the following data find:

a the 23rd, 55th and 91st percentiles

b the 2nd and 8th decile.

Score	Frequency
23	13
24	19
25	23
26	21
27	9
28	15

4 For the dot plot:

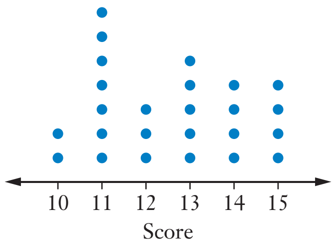
a draw a cumulative frequency polygon

b find:

i the 1st quartile **ii** the 3rd quartile

iii the 35th percentile **v** the 7th decile

v the 1st decile



5 John measured the weights of children in a particular year at school and organised his findings in a table. Estimate the weight that is:

a the median

b the 1st quartile

c the 3rd quartile

d the 60th percentile

Weight (kg)	Frequency
30–34	1
35–39	9
40–44	8
45–49	5
50–54	2

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6 The number of different dress sizes in Huang's Sportwear shop was counted in a stocktake, and the results set out in a table.

Size	Frequency
8	12
10	23
12	20
14	21
16	13
18	11

a How many dresses were counted?

b What percentage of dresses in the store were size 12?

c Find the median dress size.

d Find the 3rd quartile.

e What percentile is size 14?

7 Antonietta surveyed a number of people to find out how many pets they have. Her results are shown.

Number of pets	Frequency
0	7
1	11
2	3
3	2
4	1

a What percentage of the people surveyed had 2 pets?

b Draw a cumulative frequency polygon for this data.

c From the graph, find:

i the median

ii the 1st quartile

iii the 3rd quartile

8 Abdul measured the reaction times of a group of drivers and placed his results in a table.

Time (s)	Frequency
0.65–0.69	2
0.70–0.74	14
0.75–0.79	19
0.80–0.84	8
0.85–0.89	7

a What was the mean reaction time?

b What percentage of people surveyed reacted within 0.75 and 0.79 seconds?

c Draw an ogive to show this data.

d Use the graph to estimate:

i the 30th percentile

ii the median reaction time

iii reaction times between the 1st and 3rd quartiles

9 For each data set, find:

i the median **ii** the lower quartile **iii** the upper quartile

a The number of dogs at a pound over several days:

36, 79, 38, 29, 45, 83, 85, 47, 51, 72, 64

b The number of flying hours that Alexis had during a helicopter flying course:

3, 4, 9, 8, 14, 17, 15, 11, 12

c The distance (in km) travelled by a taxi during several shifts:

128.3, 143.2, 103.7, 99.5, 137.5, 203.4, 154.6, 115.3, 192.3, 125.4

d The number of people attending a choir rehearsal over several weeks:

15, 14, 12, 16, 15, 19, 17, 18

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Range and interquartile range

Interquartile range

Interquartile range

Statistical match-up

Cumulative frequency graphs

9.05 Range and interquartile range

The **range** and **interquartile range** measure the spread of data.

Range and interquartile range

$$\text{Range} = \text{highest score} - \text{lowest score}$$

$$\text{Interquartile range} = Q_3 - Q_1$$

EXAMPLE 19

Find the range and interquartile range of these scores: 8, 13, 5, 15, 20, 21, 17, 16, 9

Solution

$$\begin{aligned}\text{Range} &= \text{highest score} - \text{lowest score}\\ &= 21 - 5\\ &= 16\end{aligned}$$

Put the 9 scores in order to find the quartiles.

5, 8, 9, 13, 15, 16, 17, 20, 21

$Q_2 = 15$

5, 8, | 9, 13, (15), 16, 17, | 20, 21

$Q_1 = \dfrac{8+9}{2} = 8.5$ $\qquad Q_3 = \dfrac{17+20}{2} = 18.5$

$$\begin{aligned}\text{Interquartile range} &= Q_3 - Q_1\\ &= 18.5 - 8.5\\ &= 10\end{aligned}$$

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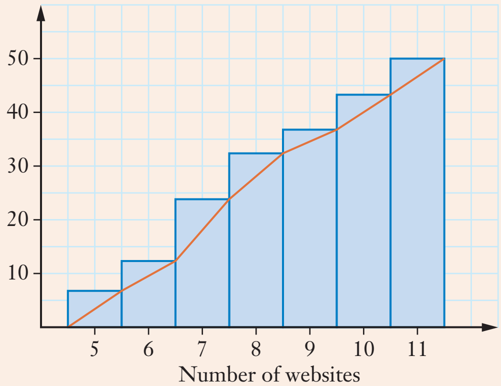
EXAMPLE 20

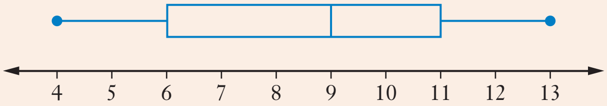
a This set of data shows the results of a survey into the number of travel websites people visit regularly. Find:

i the range **ii** the interquartile range



b From this box plot, find:

i the median **ii** the range **iii** the interquartile range



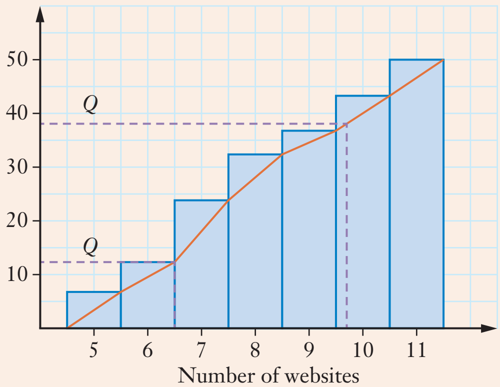
Solution

a **i** Range = 11 – 5 = 6 highest – lowest

ii There are 50 scores.

$\frac{1}{4} \times 50 = 12.5$ so Q_1 will be the 12.5th score.

$\frac{3}{4} \times 50 = 37.5$ so Q_3 will be the 37.5th score.



$Q_1 = 6.5$ and $Q_3 = 10$

Interquartile range $= Q_3 - Q_1$

$= 10 - 6.5$

$= 3.5$

b **i** Median = 9 **ii** Range = 13 – 4 = 9 **iii** Interquartile range = 11 – 6 = 5

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Outliers

Outliers

Outliers are extreme scores. They affect the range because an outlier will be the highest or lowest score. However, outliers do not affect the interquartile range because the interquartile range does not depend on the highest or lowest scores.

Some outliers are more obvious than others. There is a formal definition of outlier that allows us to test if it's an outlier rather than just deciding by inspection.

Outlier

A score is an outlier if it is more than 1.5 times the interquartile range (IQR) below Q_1 or above Q_3.

An outlier is below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$.

EXAMPLE 21

a For the scores 5, 2, 9, 10, 6, 7, 6, 5, 10, 9, 5, 7, 8, 7, 6, determine if 2 is an outlier.

b For this table of data, find a score that looks like an outlier and use the definition to determine if it is an outlier.

Score	Frequency
1	1
2	0
3	0
4	0
5	3
6	5
7	8
8	11
9	13
10	9

Solution

a $Q_1 = 5$ and $Q_3 = 9$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= 9 - 5\\ &= 4\end{aligned}$$

$$\begin{aligned}1.5 \times \text{IQR} &= 1.5 \times 4\\ &= 6\end{aligned}$$

$$\begin{aligned}Q_1 - 1.5 \times \text{IQR} &= 5 - 6\\ &= -1\end{aligned}$$

$$\begin{aligned}Q_1 + 1.5 \times \text{IQR} &= 9 + 6\\ &= 15\end{aligned}$$

Any outlier would have to be less than –1 or greater than 15. So 2 is not an outlier.

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b A score of 1 looks like an outlier.

From Example 16c,

$Q_1 = 7$ and $Q_3 = 9$

$\text{IQR} = Q_3 - Q_1$
$= 9 - 7$
$= 2$

$1.5 \times \text{IQR} = 1.5 \times 2$
$= 3$

$Q_1 - 1.5 \times \text{IQR} = 7 - 3$
$= 4$

$Q_3 + 1.5 \times \text{IQR} = 9 + 3$
$= 12$

Any outlier would have to be less than 4 or greater than 12. So 1 is an outlier.

Exercise 9.05 Range and interquartile range

1 Find the range of each data set.

a 7, 4, 9, 8, 11, 4, 3, 19, 7, 16

b 56, 89, 43, 99, 45, 28, 37, 78

c 103, 108, 99, 112, 126, 87, 101, 123

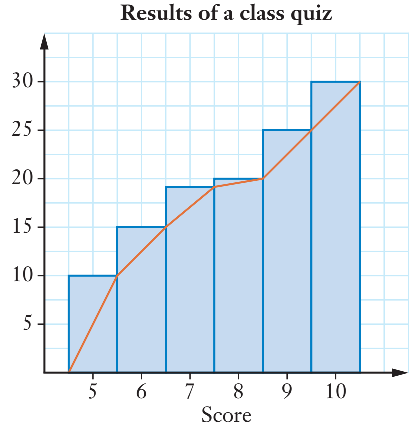
d

Score	Frequency
8	5
9	3
10	7
11	0
12	8
13	7

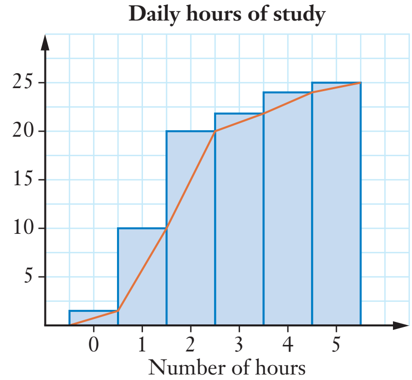
2 For each set of data, find:

i the median **ii** the range **iii** the interquartile range

a



b



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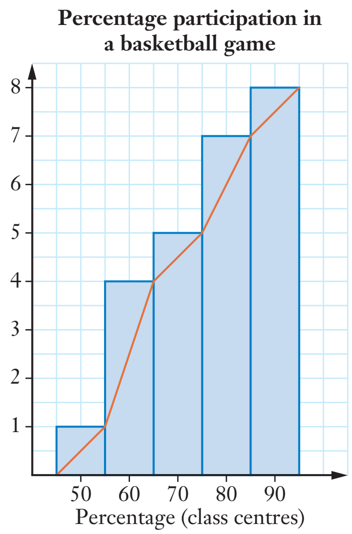
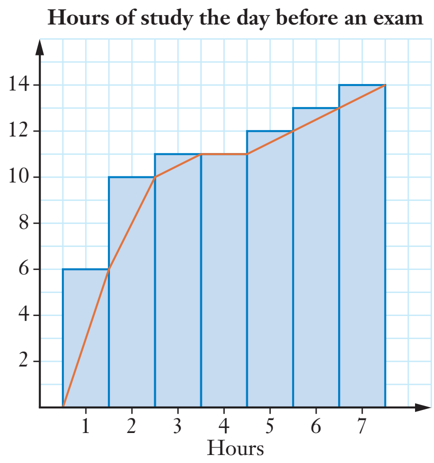
c



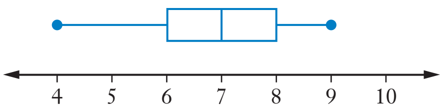
d



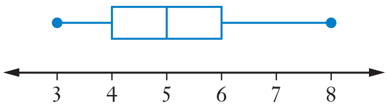
3 For each set of data, find:

i the median **ii** the range **iii** the interquartile range

a



b



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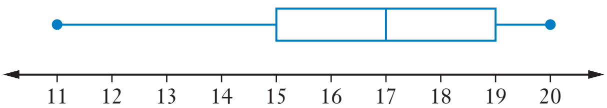
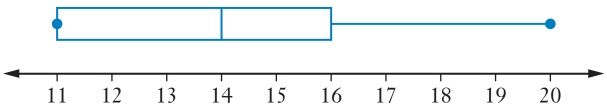


d



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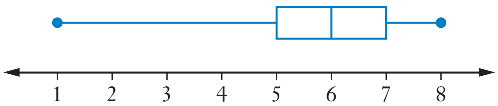

4 For this set of data, find:

a the mean
b the mode
c the median
d the range
e the interquartile range

Rain (mm)	Frequency
5	4
6	7
7	8
8	3
9	3

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5 Find a potential outlier in each set of data and use the definition to see if it really is an outlier.

a 100, 76. 93, 54, 32, 66, 53, 97, 51, 80

b 11, 5, 7, 19, 5, 3, 7, 5, 6, 10, 11. 2, 5, 7, 4, 6, 1

c

Score	Frequency
1	1
2	0
3	0
4	0
5	5
6	4

9.06 Variance and standard deviation

WS Variance and standard deviation

WS Statistical calculations

Statistical measures puzzle

Variance is another measure of spread. It measures how far the scores in a data set are from the mean of the data. You studied variance and standard deviation when studying discrete probability distributions in Year 11, Chapter 12 *Discrete probability distributions*.

The formula for variance, σ^2, is:

$$\sigma^2 = \frac{\Sigma(x - \bar{x})^2}{n}$$

However, you do not have to use it as the calculator's statistical mode can calculate it more easily. The following example will show you what the above formula means, but you don't have to learn it.

EXAMPLE 22

The data below shows the times (in minutes) taken for a fire engine to reach the site of a fire. Find the variance for this data.

4, 7, 3, 9, 4, 5, 1, 3, 5, 9

Solution

First we need to find the mean.

$$\bar{x} = \frac{\Sigma x}{n} = \frac{50}{10} = 5$$

Now we find the difference between each score and the mean. Then we square each difference because we only want positive values. This is shown in the table next page.

x	$x - \bar{x}$	$(x - \bar{x})^2$
1	$1 - 5 = -4$	16
3	$3 - 5 = -2$	4
3	$3 - 5 = -2$	4
4	$4 - 5 = -1$	1
4	$4 - 5 = -1$	1
5	$5 - 5 = 0$	0
5	$5 - 5 = 0$	0
7	$7 - 5 = 2$	4
9	$9 - 5 = 4$	16
9	$9 - 5 = 4$	16
		$\Sigma(x - \bar{x})^2 = 62$

Variance is the mean of these squared differences.

$$\sigma^2 = \frac{\Sigma(x - \bar{x})^2}{n}$$
$$= \frac{62}{10}$$
$$= 6 2$$

Standard deviation is another measure of spread, and it is simply the square root of variance. For the data in the previous example, the standard deviation is $\sqrt{6.2} \approx 2.49$.

We use s for standard deviation of a sample and σ (the lowercase Greek sigma) for the standard deviation of a **population**. In this course we will use s most of the time.

The formula for standard deviation, σ, is:

$$\sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$$

This example shows how to calculate standard deviation and variance using the calculator's statistical mode.

EXAMPLE 23

The table shows the number of hours of karate practice that students at a karate club do each week.
Find correct to one decimal place:

a the standard deviation

b the variance

Practice times (h)	Frequency
2	2
3	4
4	3
5	4
6	1
7	2
8	4

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Solution

a

Operation	Casio scientific	Sharp scientific
Clear the statistical memory.	1 Edit, Del-A	
Enter data.	1 Data to get table. 2 3 etc. to enter in x column. 2 4 etc. to enter in column. to leave table.	2 2 3 4 etc.
Calculate standard deviation.	1 sx	sx

$s = 2.0774 \ldots \approx 2.1$

b Variance $= s^2$

$= 2.0774 \ldots^2$

$= 4.3157 \ldots$

≈ 4.3

Exercise 9.06 Variance and standard deviation

1 For each set of data find:

i the mean **ii** the standard deviation

a Number of minutes kept on hold on the telephone: 7, 4, 9, 8, 4, 6, 2, 4, 5

b Travel time (in minutes) to get into the city: 23, 45, 67, 54, 69, 38, 59, 70, 59, 41

c Height of children (in cm): 101, 112, 131, 122, 130, 143, 152, 107, 112

d Number of repetitions on gym equipment: 8, 6, 9, 5, 5, 7, 6, 4, 8, 9, 6, 3, 6

e Age of performers in a play: 18, 19, 17, 16, 20, 18, 15, 19, 14, 20

2 For each data set, find:

i the standard deviation **ii** the variance

a Weights (in kg): 51, 67, 64, 53, 60, 48, 58, 49, 61, 71, 67, 58

b Class quiz results: 4, 6, 5, 3, 7, 9, 8, 10, 4, 6, 7, 6, 5, 8, 6, 7, 9, 10, 5, 4, 8

c Time spent waiting in a queue (in mins): 11, 14, 15, 25, 31, 54, 36, 39, 31, 41, 44, 50

d Weight of crates (in kg): 87, 88, 56, 91, 68, 73, 55

e Response time (in mins) for helicopter rescue: 1, 5, 7, 3, 8, 6, 5, 5, 4, 8, 9, 3

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3 For each data set find:

i the mean **ii** the standard deviation **iii** the variance

a Number of books read:

Books	Frequency
5	3
6	5
7	6
8	2
9	1
10	3
11	5
12	4

b Piano practice time per week:

Practice (h)	Frequency
1	3
2	0
3	2
4	5
5	7
6	3
7	2

c Weight of luggage:

Weight (kg)	Frequency
31	3
32	0
33	2
34	5
35	7
36	3

d Results of a half-yearly exam:

Score	Frequency
10–19	1
20–29	4
30–39	8
40–49	12
50–59	15
60–69	11
70–79	7

4 In a taste test, the people surveyed had to rank a new biscuit on a scale from 1 to 5. The table shows the results of the survey.

Rank	Frequency
1	5
2	11
3	18
4	21
5	9

a What is the range?

b Find the interquartile range.

c Find the standard deviation.

5 A Year 12 Art class received these results for their major work.

Class	Frequency
20–29	1
30–39	0
40–49	2
50–59	4
60–69	7
70–79	9
80–89	8
90–99	7

a Calculate:

i the mean **ii** the standard deviation

b Remove the outlier and calculate:

i the mean **ii** the standard deviation

6 The data shows the ages of students in an MBA course:

20, 25, 31, 34, 17, 27, 29, 53, 20, 31, 19, 23, 30, 29, 18, 25

a Show that there is an outlier.

b Calculate the standard deviation:

i with the outlier **ii** without the outlier

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7 Jane recorded the number of crocodile sightings each week in a region of the Northern Territory over several weeks.

$$5, 8, 9, 4, 11, 7, 9, 15, 17, 10, 8, 5, 9, 12$$

a What was the mean number of crocodiles sighted per week?

b Find the standard deviation.

c Calculate the variance.

9.07 Shape and modality of data sets

WS Shape of a distribution

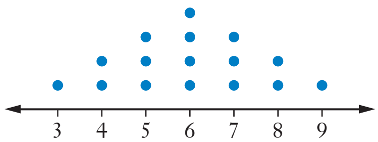
The shape of a frequency distribution

WS Shapes of distributions

The measures of central tendency and spread are called **summary statistics**, and they help us make decisions about data. Other features of data can help with these decisions, and one of these features is the **shape** of the data. The shape often gives us an idea of the centre and spread even before we measure them, while the **modality** describes the number of peaks in the distribution of data.

A data set where the mean, mode and median are equal has a **symmetrical distribution**.

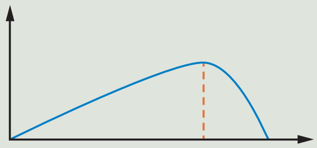
Notice how symmetrical this dot plot is.



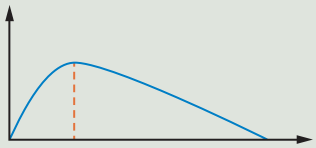
Other graphs that are not so symmetrical can be described by their **skewness**.

The shape of a statistical distribution

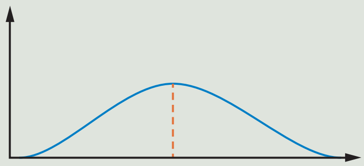
This distribution is **negatively skewed** as most of the area is to the left (or negative direction) of the centre. We can say that the 'tail' points to the low scores in the negative direction.



This distribution is **positively skewed** as most of the area is to the right (or positive direction) of the centre. We can say that the 'tail' points to the high scores in the positive direction.



This distribution is **symmetrical**.



All these graphs are also called **unimodal** since they only have one peak.

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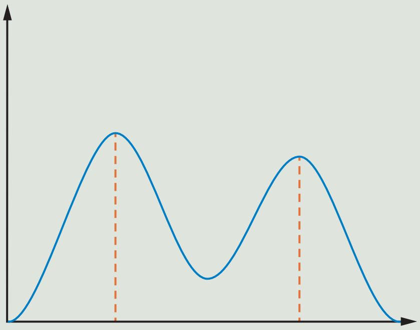
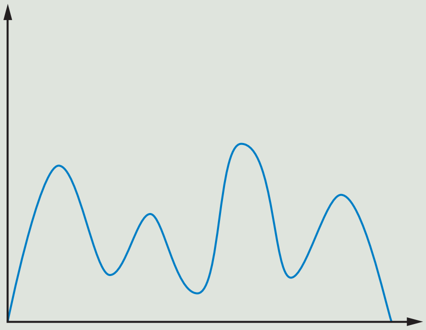
The modality of a statistical distribution

This graph is described as **bimodal** because it has 2 peaks.



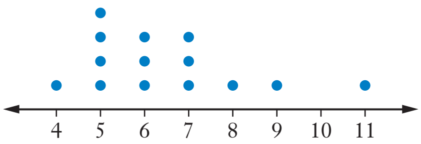
This graph is described as **multimodal** because it has many peaks.



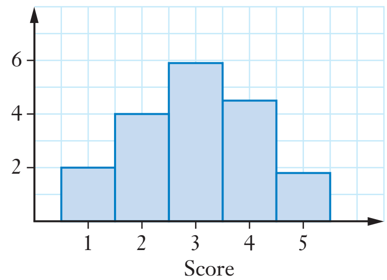
Exercise 9.07 Shape and modality of data sets

1 Describe the shape and modality of each graph.

a



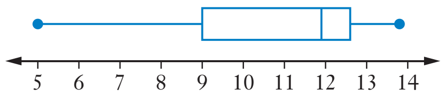
b



c

Stem	Leaf
1	3 4 4 5 5 7 9
2	0 1 1 7 8 8 9 9
3	2 2 5 6 6 8 9 9
4	1 1 3 5
5	0 0 3
6	3 4 5 8
7	8 9

d



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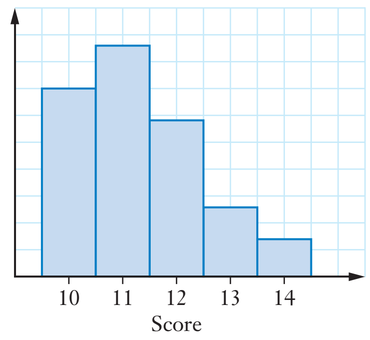
e



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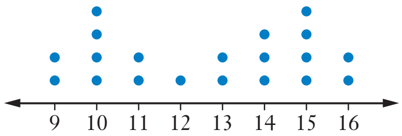

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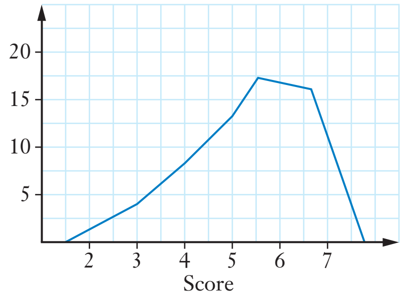

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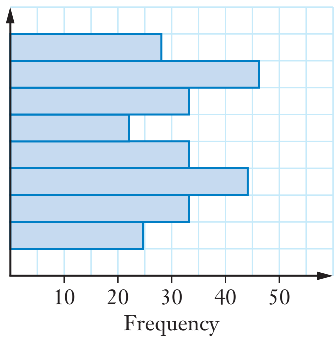
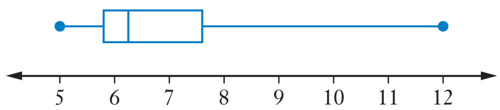


i



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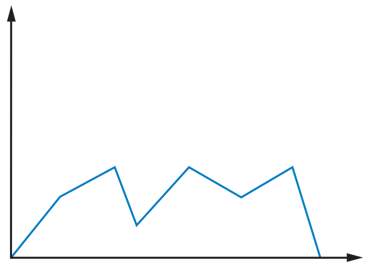

2 **a** Draw a dot plot for this data.

5, 9, 4, 8, 9, 10, 7, 5, 3, 9, 7, 8, 6, 12, 8, 9

b Describe the shape of the dot plot.

3 Describe the shape and modality of each data set.

a

Score	Frequency
1	7
2	9
3	5
4	3
5	1
6	1

b

Score	Frequency
12	3
13	7
14	11
15	14
16	9
17	2

c

Score	Frequency
10–14	3
15–19	6
20–24	7
25–29	4
30–34	7
35–39	5

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d

Score	Frequency
6	1
7	2
8	5
9	8
10	7
11	4

e

Score	Frequency
50– 59	2
60– 69	4
70– 79	6
80– 89	3
90– 99	1

4 Draw graphs with the following shapes.

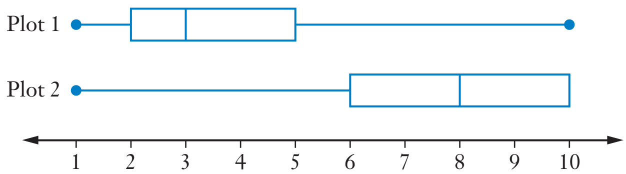
a bimodal **b** skewed negatively

c symmetrical **d** skewed positively

5 Describe the shape and modality of each set of data in the back-to-back stem-and-leaf plot.

Set 1		Set 2
4	1	3 3 4
7 7 6 3	2	1 2 5 6 8 9 9
8 7 5 5 2 0	3	0 2 2 5 6
8 6 6	4	3 6 7
9 8 8 7 1 1	5	0 1
5 4 0	6	3
1 1	7	

6 **a** Describe the shape of the distributions summarised by the parallel box plots.



b What is the difference between their medians?

c Find the difference in their interquartile ranges.

7 The heights of a number of students were measured and the results are below.

159, 175, 181, 153, 177, 168, 175, 163, 155, 184, 167, 179, 157, 149, 160, 171, 180, 160, 162, 169, 163, 179, 145, 187, 161, 148, 182, 151, 150, 178

a Draw a frequency distribution table for the heights, using groups of 145–149, 150–154, 155–159 and so on.

b Describe the type of distribution for this data.

8 Draw a box plot that describes a symmetrical distribution.

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9 This table shows the results of an assessment task.

a Find:

- **i** the mean
- **ii** the median
- **iii** the mode

b Describe the shape of the distribution.

Score	Frequency
4	3
5	5
6	6
7	9
8	6
9	5
10	3

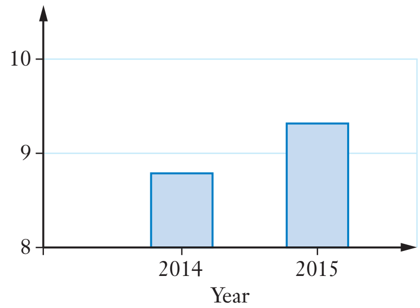
10 Choose a random sample of about 50 people and collect data on the number of siblings (brothers and sisters) each one has. Graph the data and describe the shape and modality of the graph.

INVESTIGATION

MISLEADING GRAPHS

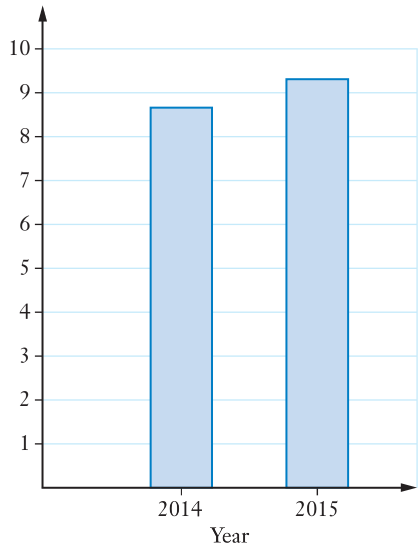
Sydney's median house price increased from $886 408 in 2014 to $929 842 in 2015.

The column graph shows this information. Looking at the graph, you would think that this was a huge price rise because the second column is almost twice as tall as the first column.



Now look at this graph. What is the difference between the 2 graphs? Which one do you think shows the information better? Is one of the graphs misleading? Why?

Search online for other misleading graphs. Collect them into a portfolio and share with the class. Write an account of why each one is misleading and how you could change it to give a better reading of the information.



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Statistics can be misleading in different ways. In the investigation this was caused by the scale on the graph. Sometimes the measures of central tendency or spread can be misleading as well.

9.08 Analysing data sets

Double box plots

Comparing city temperatures

Comparing word lengths

Comparing sports scores

EXAMPLE 24

The table shows the heights of students in a Year 12 class.

a Find the mean and standard deviation of the heights.

b Are the mean and standard deviation misleading for this data? Why?

Height (cm)	Class centre	Frequency
150–154	152	3
155–159	157	18
160–164	162	27
165–169	167	31
170–174	172	12
175–179	177	15
180–184	182	25
185–189	187	11
190–194	192	3

Solution

a Using a calculator:

$\bar{x} = 170.6$

$s = 10.2$

b Looking at the table, the data looks to be bimodal. This might be because the survey is for both males and females.

If this is the case, the mean of 170.6 is misleading because it doesn't tell us about differences in male and female heights. The spread may be less than 10.2 if we split the data into male and female data.

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Comparing two or more sets of data

Sometimes we need to compare different data sets to see how similar or different they are.

EXAMPLE 25

a Two surveys were made into the number of people attending an outdoor cinema: one in 2019 and one in 2020.

For the 2019 survey, the mean was 112 and the standard deviation was 6.7.

For the 2020 survey, the mean was 95 and the standard deviation was 11.9.

Describe how these results differ.

b This back-to-back stem-and-leaf plot shows the results of tests of the life of 2 brands of batteries batteries (measured in hours).

Buzz		Eternity
9 9 1	4	5
5 2 1 1 1 0	5	1 3 4 9
7 7 4 0	6	1 2 3
4 2 0	7	0 1 2 8
4 0	8	1 2 5 5 7
	9	0

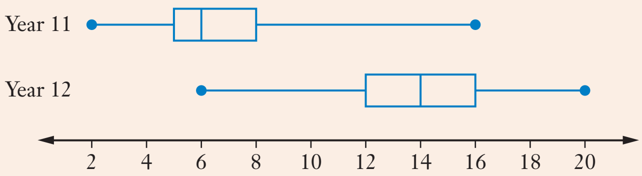
i Describe the shape of the distribution for each brand.

ii Find the mean result for each brand.

iii Find the standard deviation for each brand.

v Compare the results for the 2 brands of batteries.

c The parallel box plots below show the results of 2 surveys into the number of hours 2 groups of students study each week.



i What is the median number of hours studied for each Year group?

ii Calculate the range for each group.

iii What is the interquartile range for each group?

v What is the highest number of hours studied in each group surveyed?

v What is the main difference between the 2 groups?

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Solution

a The mean was lower in the second survey, so it looks as if, on average, fewer people were going to the movies in 2020 than in 2019.

The standard deviation was higher in the second survey, so there was a greater variation in the number of people going to the movies.

b **i** Buzz is slightly positively skewed and Eternity is approximately bimodal.

ii Using a calculator:

The mean for Buzz is 60.4.

The mean for Eternity is 69.4.

iii The standard deviation for Buzz is 12.3 and the standard deviation for Eternity is 14.0.

v The mean was higher on Eternity so these batteries had longer lives overall.

The standard deviation was slightly higher on Eternity, showing slightly more variability in the life of these batteries. That is, the battery lives were more spread out than for Buzz, but there wasn't a big difference between the 2 brands.

c **i** Year 11: median is 6.

Year 12: median is 14.

ii Year 11: range = 16 – 2 = 14

Year 12: range = 20 – 6 = 14

iii Year 11: interquartile range = 8 – 5 = 3

Year 12: interquartile range = 16 – 12 = 4

v Year 11: highest hours = 16

Year 12: highest hours = 20

v Year 12 students generally study for more hours.

CLASS DISCUSSION

ANALYSING DATA SETS

Why do you think the surveys in the example give different results? Are they taken from the same population? How could you tell? What other information could help you decide?

Find other examples online, in newspapers or in magazines that compare 2 or more sets of data. Is the information taken from the same or different populations? Can you tell?

Put these examples in a portfolio and present a report to the class.

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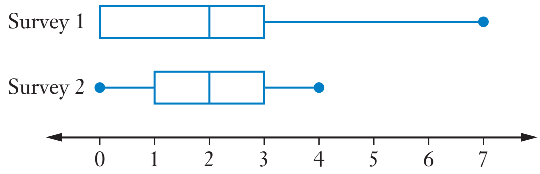
Exercise 9.08 Analysing data sets

1 The parallel box plots show the results of 2 surveys into the number of children in families.



a What was the largest number of children in a family in:

i survey 1? **ii** survey 2?

b What is the median number of children in the families in:

i survey 1? **ii** survey 2?

c Find the interquartile range for:

i survey 1 **ii** survey 2

d Do you think the surveys were taken from the same population or from different populations?

2 The heights (in centimetres) of people were taken from 2 samples.

Sample 1: 158, 169, 177, 147, 160, 153, 167, 164, 171, 159, 162, 166, 172, 148, 157, 176, 155, 170, 149, 150

Sample 2: 181, 166, 179, 181, 164, 173, 168, 165, 176, 181, 157, 162, 178, 182, 157, 172, 180, 173, 159, 151

a Plot this information in a back-to-back stem-and-leaf plot.

b Find the median of each sample.

c Find the range of each sample.

d Do you think the samples were taken from the same population?

3 Surveys were taken at 3 different banks to measure the length of time customers waited before being served. The results are shown in the table.

Time (min)	Bank 1	Bank 2	Bank 3
0–2	29	59	2
3–5	38	26	8
6–8	15	12	11
9–11	9	3	21
12–14	5	0	28
15–17	3	0	20
18–20	1	0	10

a Find **i** the mean and **ii** the median waiting times for customers at each bank.

b Find the standard deviation for each bank.

c Do you think there is a significant difference in waiting times at the banks?

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4 Mrs Spell's piano students earned the following marks in their piano examinations:

Class 1: 91, 86, 74, 92, 85, 89, 63, 71, 80, 91, 85, 72, 54, 78

Class 2: 97, 87, 69, 91, 88, 89, 93, 94, 71, 79, 84, 85, 88

a Sketch parallel box plots showing this information.

b For each class, find:

i the median | **ii** the interquartile range

iii the mean | **v** the range

5 Two speed cameras at different locations recorded speeds (in km h^{-1}) of vehicles travelling over the speed limit:

Camera 1: 85, 66, 75, 69, 72, 83, 80, 69, 74, 77, 73, 74, 90, 84, 65, 73, 69, 89, 76, 103

Camera 2: 122, 142, 120, 118, 116, 135, 140, 123, 135, 124, 120, 119, 138, 131, 122, 119, 125, 130, 130, 113

a Draw a back-to-back stem-and-leaf plot to show this data.

b Find the mean speeds recorded by each camera.

c What do you think was the speed limit at the site where each camera was placed?

6 Jon sat for the HSC in one year and scored 56, 48, 61, 53, 41 and 35 for his maths assessments. He resat his HSC the next year and his maths assessment scores were 73, 58, 67, 74, 59 and 68.

a By how much did his mean scores increase the second year?

b What was the difference in the median scores?

c Calculate the difference in the range of scores for each year.

d By how much does the standard deviation differ over the 2 years?

7 These 2 sets of scores have the same median.

Test 1: 1, 2, 4, 6, 7, 9, 10 **Test 2:** 4, 5, 5, 6, 6, 7, 8

a What is the mean of:

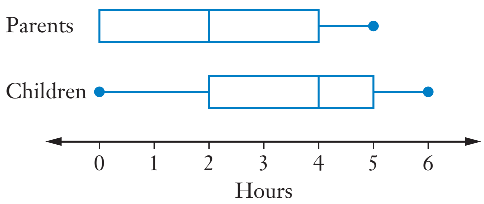
i Test 1? | **ii** Test 2?

b Calculate the standard deviation of:

i Test 1 | **ii** Test 2

c Describe how the 2 sets of scores differ.

8 The parallel box plots show the results of 2 surveys into the number of hours people spend watching TV each day.



a What is the median for each group?

b For each group, find:

i the interquartile range | **ii** the range

c What is the highest number of hours of TV watched?

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9 For the back-to-back stem-and-leaf plot, find:

a the median of each test

b the mean of each test

c the standard deviation of each test

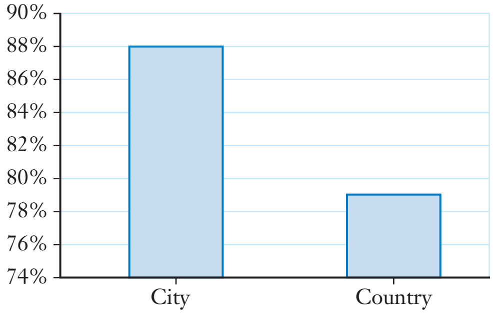
d the difference in range between the 2 tests.

Science		English
	4	3
2 1 1 0	5	8 9
8 6 5 3 3 1	6	0 1 2 2 5
9 9 6 3 2	7	4 6
1 0 0	8	1 2 3
4 2	9	7

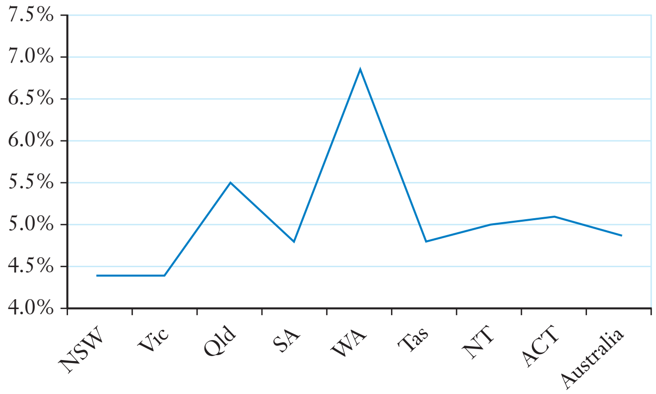
10 The graph compares access to the Internet for city and country households in 2015.

a Describe how this graph is misleading.

b Redraw the graph so it is not misleading.



11 The graph shows the average annual growth in incomes from 2006 to 2011.



Redraw the graph so it is not misleading.

INVESTIGATION

COMPARING SURVEY RESULTS

Conduct a survey among your friends. Make up your own topic, such as what sports they play, subjects they study or their heights. Alternatively, you could carry out an experiment such as counting numbers of people travelling in cars or measuring the time taken for the same journey to school on different days.

To check your results, take another sample and do the same survey or experiment. Are the new results the same? Can you explain why?

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9. TEST YOURSELF

Practice quiz

Statistics review

Statistics crossword

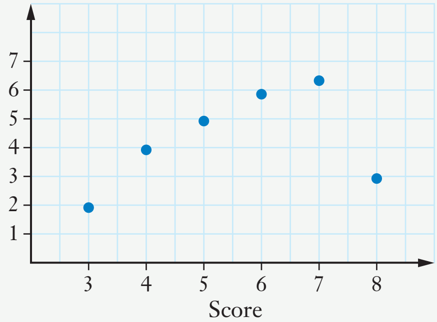
For Questions **1** to **6**, select the correct answer **A**, **B**, **C** or **D**.

1 What type of data is a person's body temperature?

A categorical ordinal
B numerical continuous
C numerical discrete
D categorical nominal

2 This frequency distribution is:

A symmetrical
B positively skewed
C negatively skewed
D multimodal



3 What type of data is a flower's colour?

A categorical ordinal
B numerical continuous
C numerical discrete
D categorical nominal

4 The median of scores 7, 3, 2, 6, 4, 3, 8, 9 is:

A 6
B 4
C 3
D 5

5 Which measure is most affected by an outlier?

A median
B interquartile range
C mode and interquartile range
D mean and range

6 Test 1 has a mean of 4 and standard deviation of 1.2. Test 2 has a mean of 4 and a standard deviation of 2.5. Which statement below is true when comparing the centres and spreads for both tests?

A The centres are the same and test 2 has a larger spread than test 1.
B The spreads are the same and test 2 has a higher centre than test 1.
C The spreads are the same and test 2 has a lower centre than test 1.
D The centres are the same and test 2 has a smaller spread than test 1.

7 Find the mode, median and range of this data set:

8, 6, 8, 4, 5, 6, 8, 5, 7, 4, 7, 8, 6, 8, 9

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8 The table shows the results of a survey to find the distance people must travel to work.

a Find the modal class.

b Find the mean.

c What is the median?

d Find the standard deviation and variance.

Distance (km)	Frequency
0–4	6
5–9	8
10–14	4
15–19	7
20–24	3

9 Find the mean and standard deviation of this data set:

16, 34, 29, 80, 65, 77, 91, 58, 67, 40

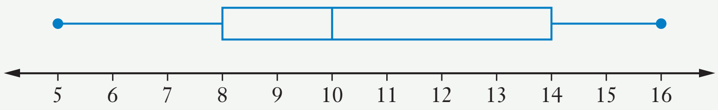
10 From this box plot, find:

a the median

b the range

c the 3rd quartile

d the interquartile range.



11 The table shows the results of a survey into the number of years that people keep a car before selling it.

a Draw a cumulative frequency polygon to show this data.

b Estimate the median age of the cars.

c Estimate:

i the 20th percentile **ii** the 3rd quartile

iii the 91st percentile **v** the 3rd decile **v** the 9th decile

Years	Frequency
0–2	15
3–5	37
6–8	13
9–11	18
12–14	17

12 The table records the number of times people visited a doctor in the past 6 months.

a Find the mean number of visits.

b Find the median number of visits.

c What is the range?

d Find the interquartile range.

Visits	Frequency
0	9
1	8
2	5
3	1
4	3
5	2

13 'Most families have 2 children.' Is this statement about a mean, median or mode?

14 Nikola conducted a survey to find the number of people in each car that travelled across the Anzac Bridge one morning. These were her results:

a Draw a cumulative frequency polygon to illustrate the results.

b From the polygon, find:

i the median number of people in a car

ii the interquartile range

Occupants	Frequency
1	43
2	32
3	12
4	8
5	5

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15 The table shows the results of Mr Cheung's history class.

Score	Frequency
6	7
7	6
8	3
9	6
10	2

a Find the mean score.

b Find the standard deviation.

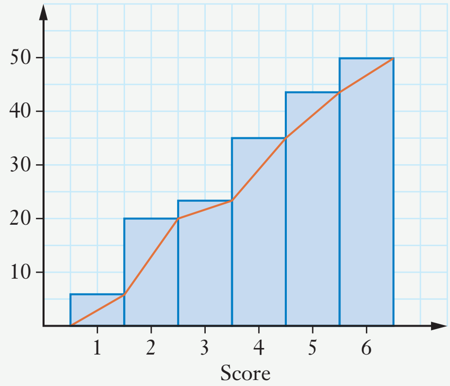
c What is the mode?

16 Find the mean, mode and median of this data set:
45, 49, 49, 48, 43, 45, 41, 40, 49, 48, 44, 40, 42

17 From the graph, find:

a the interquartile range

b the median



18 A class test gave the following scores:
15, 19, 12, 2, 19, 16, 13, 18, 11, 15, 17, 11, 18, 14, 14, 16, 18, 14, 12

a Find:

i the range **ii** the mean

iii the mode **v** the median

b Show that one score is an outlier. Which is it?

c Find, without this outlier:

i the range **ii** the mean

iii the mode **v** the median

d Does the outlier have much effect on all these measures?

19 The two-way table shows the results of a survey into the number of pets microchipped at a veterinary surgery.

	Male	Female	Total
Cats	369	473	842
Dogs	578	664	1242
Total	947	1137	2084

a What percentage of microchipped animals were:

i female? **ii** female cats? **iii** male dogs? **v** dogs?

b Draw this information in a Pareto chart.

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20 The table shows the reasons employees gave for leaving their jobs in a large organisation. Draw a Pareto chart of this data.

Work too difficult	185
Boring work	139
Not paid enough	104
Not getting on with co-workers	56
Unsuitable hours	116

21 A back-to-back stem-and-leaf plot shows the number of mushrooms found in 2 regions of a forest in New Zealand.

North		West
	2	2 5 6
5 4 2	3	4 8 9
8 7 6 3 3	4	0 0 1 3 7
6 5 4 4 2 2	5	1 1 2 3 4 4 7 8

a Calculate the median number of mushrooms recorded in each region.

b Find the mean and standard deviation of North region.

c Find the mean and standard deviation of West region.

d Compare and contrast the 2 regions.

22 Below are the results of 2 English assessments.

Term 1: 8, 7, 9, 8, 6, 5, 8, 7, 7, 5, 9, 9

Term 2: 5, 7, 8, 4, 6, 6, 5, 5, 5, 4, 7

a Draw a box plot for each set of data.

b What is the median of each assessment?

c Find the interquartile range of each set.

d Find the mean and standard deviation for each assessment.

e Compare and contrast the 2 assessments.

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9. CHALLENGE EXERCISE

1 The table shows the heights of students in Year 12.

Height (cm)	Frequency
150–154	7
155–159	5
160–164	15
165–169	9
170–174	8
175–179	15
180–184	6
185–189	2

a Describe the modality of the distribution. Can you explain this?

b Find the mean height and the variance.

2 The mean of a data set is 4.5. The table containing the data has a frequency missing. What is it?

Score	Frequency
2	2
3	5
4	
5	3
6	2
7	4

3 a Sketch a box plot for the following scores.
7, 9, 5, 6, 8, 11, 25, 14, 16

b What does this plot show? What is its shape? Can you make a more symmetrical box plot by taking out one of the scores?

4 For the scores 7, 11, 15, 19, 23

a Find:

i the mean **ii** the standard deviation

b Copy and complete this table:

Score x	$x - \overline{x}$	$(x - \overline{x})^2$
7		
11		
15		
19		
23		
	$\Sigma(x - \overline{x}) =$	$\Sigma(x - \overline{x})^2 =$

c Find the standard deviation by using the formula $\sigma = \sqrt{\dfrac{\Sigma(x - \overline{x})^2}{n}}$.

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5 The mean of 7 scores is 25. If an extra score of 28 is also included, what will be the new mean?

6 Compare and contrast each pair of data sets.

a Set A has a mean of 54 and a standard deviation of 5.6

Set B has a mean of 76 and a standard deviation of 2.1

b Set A has a mean of 11.6 and a standard deviation of 2.7

Set B has a mean of 21.3 and a standard deviation of 9.2

7 A Year 12 class received the following scores out of 10 on their maths quiz.

a Find the mean and standard deviation of the scores.

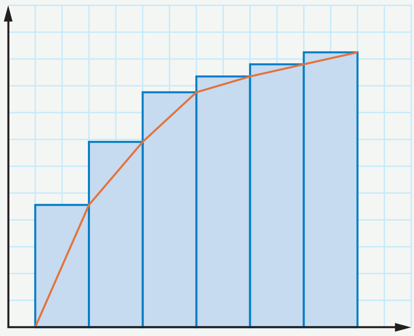
b Draw a box plot for this set of scores.

c Describe the shape of the distribution.

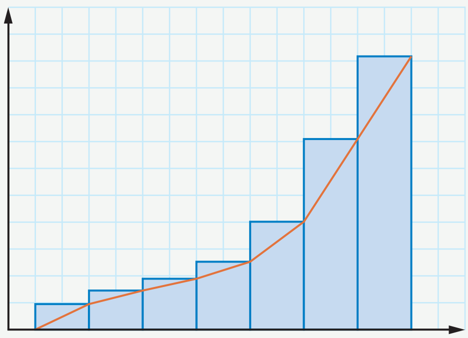
Score	Frequency
4	7
5	11
6	15
7	9
8	4
9	2
10	1

8 Describe the shape of each distribution, given the ogive.

a



b



10. FURTHER VECTORS

In this Mathematics Extension 1 chapter you will look at vectors and their applications to geometry proofs, motion and projectiles.

CHAPTER OUTLINE

10.01 EXT1 Scalar (dot) product
10.02 EXT1 Parallel and perpendicular vectors
10.03 EXT1 Projection of vectors
10.04 EXT1 Proofs using vectors
10.05 EXT1 Motion and vectors
10.06 EXT1 Projectile motion

IN THIS CHAPTER YOU WILL:

- use scalar products
- examine properties of parallel and perpendicular vectors
- project one vector onto another
- apply vector proofs to geometry
- apply vectors to problems involving motion
- understand and solve problems involving projectile motion

TERMINOLOGY

dot product: See **scalar product**.
projectile: An object that is dropped, thrown or thrust and moves under the influence of gravity.
projectile motion: The motion of an object (projectile) along a curved path under the influence of gravity.
projection of a vector: A vector b projected down to a vector a so that the projection is in the same direction as a.
scalar product: The product of 2 vectors as a scalar or value (not a vector), also known as the **dot product**.

Scala product

Componen form of the dot product

Popetis f the dot product

EXT1 10.01 Scalar (dot) product

In Chapter 3, *Vectors*, you learned how to add and subtract vectors and multiply them by a scalar.

Now we will look at multiplying vectors to calculate the scalar product.

Scalar (dot) product

The **scalar (or dot) product** multiplies 2 vectors. It applies the directional growth of one vector to another. This is easy when the vectors have the same direction.

EXAMPLE 1

Find the scalar product $u \quad v$, given $u = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

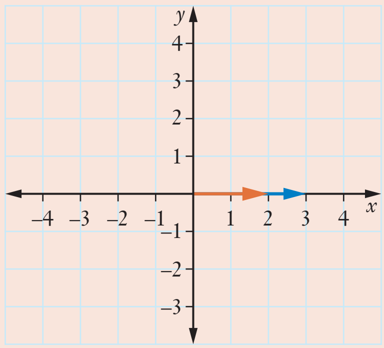
Solution

The 2 vectors are in the same direction.

If we look at the x component, applying the product gives 2×3 or 6.

If we look at the y component, applying the product gives 0×0 or 0.

So $u \quad v = 6$.



Notice that the result of a dot product is a scalar. This is why it is also called a **scalar product**.

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We define the scalar product as the sum of the products of the x and y components.

Scalar (dot) product

Given $u = \begin{pmatrix} x \\ y \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, the **scalar product** is defined as:

$$u \quad v = x_1 x_2 + y_1 y_2$$

EXAMPLE 2

Find the dot product $u \quad v$, given:

a $u = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$

b $u = 3i - 4j$ and $v = i + 2j$

c $u = (6 \cos 30°, -6 \sin 30°)$ and $v = (-2 \cos 45°, -2 \sin 45°)$

Solution

a
$$\begin{aligned} u \quad v &= x_1 x_2 + y_1 y_2 \\ &= 7 \times (-1) + 3 \times 6 \\ &= -7 + 18 \\ &= 11 \end{aligned}$$

b
$$\begin{aligned} u \quad v &= x_1 x_2 + y_1 y_2 \\ &= 3 \times 1 + (-4) \times 2 \\ &= 3 + (-8) \\ &= -5 \end{aligned}$$

c
$$\begin{aligned} u &= (6 \cos 30°, -6 \sin 30°) \\ &= \left(6 \times \frac{\sqrt{3}}{2}, -6 \times \frac{1}{2}\right) \\ &= (3\sqrt{3}, -3) \end{aligned}$$

$$\begin{aligned} v &= (-2 \cos 45°, -2 \sin 45°) \\ &= \left(-2 \times \frac{1}{\sqrt{2}}, -2 \times \frac{1}{\sqrt{2}}\right) \\ &= \left(-\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \quad -\frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right) \\ &= \left(-\frac{2\sqrt{2}}{2} \quad -\frac{2\sqrt{2}}{2}\right) \\ &= \left(-\sqrt{2} \quad -\sqrt{2}\right) \end{aligned}$$

$$\begin{aligned} u \quad v &= x_1 x_2 + y_1 y_2 \\ &= 3\sqrt{3} \times (-\sqrt{2}) + (-3) \times (-\sqrt{2}) \\ &= -3\sqrt{6} + 3\sqrt{2} \end{aligned}$$

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Scalar product of a vector with itself

$$v \cdot v = |v|^2$$

Proof

Given $v = \begin{pmatrix} x \\ y \end{pmatrix}$

$$v^2 = v \quad v$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{pmatrix} x \\ y \end{pmatrix}$$

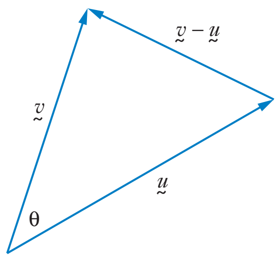
$$= x^2 + y^2$$

$$= |v|^2$$

Scalar product in terms of the angle between the vectors

$$u \cdot v = |u||v| \cos\theta$$

Proof



Using the cosine rule:

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$|v - u|^2 = |u|^2 + |v|^2 - 2|u||v| \cos\theta$$

Since $v \quad v = |v|^2$

$$\text{LHS} = |v - u|^2$$

$$= (v \quad u) \; (v - u)$$

$$= (v - u)^2$$

$$= v^2 \quad 2u \;\; v + u^2$$

$$= |v|^2 - 2u \;\; v + |u|^2$$

So $|v|^2 - 2u \;\; v + |u|^2 = |u|^2 + |v|^2 - 2|u||v| \cos\theta$

$$-2u \;\; v = -2|u||v| \cos\theta$$

$$u \;\; v = |u||v| \cos\theta$$

Now we have 2 ways to find the scalar product.

Scalar product

$$u \;\; v = x_1 x_2 + y_1 y_2 = |u||v| \cos\theta$$

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EXAMPLE 3

a If u has magnitude 7, v has magnitude 5 and the angle between the vectors is 74°, find $u \ \ v$ correct to 1 decimal place.

b Given $u = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$, find:

i $u \ \ v$ **ii** the angle between the vectors to the nearest degree.

Solution

a
$$\begin{aligned} u \ \ v &= |u||v| \cos \theta \\ &= 7 \times 5 \times \cos 74° \\ &= 35 \cos 74° \\ &\approx 9.6 \end{aligned}$$

b **i**
$$\begin{aligned} u \ \ v &= x_1 x_2 + y_1 y_2 \\ &= 4 \times 12 + 3 \times (-5) \\ &= 33 \end{aligned}$$

ii
$$\begin{aligned} |u| &= \sqrt{4^2 + 3^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\begin{aligned} |v| &= \sqrt{12^2 + (-5)^2} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

$$\begin{aligned} u \ \ v &= |u||v| \cos \theta \\ 33 &= 5 \times 13 \times \cos \theta \\ &= 65 \cos \theta \\ \frac{33}{65} &= \cos \theta \\ \theta &\approx 59° \end{aligned}$$

The angle between the vectors is 59°.

EXT1 Exercise 10.01 Scalar (dot) product

1 Find the scalar product $u \ \ v$ to 1 decimal place, given:

a $u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, v = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

b $u = \begin{pmatrix} 4 \\ -1 \end{pmatrix}, v = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$

c $u = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, v = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$

d $u = 4i - j, v = i - 5j$

e $u = -3i + 4j, v = 2i + 3j$

f $u = i - 9j, v = -3i - 2j$

2 Find the exact dot product of:

a $u = (-2\cos 60°, -2\sin 60°)$ and $v = (8\cos 45°, -8\sin 45°)$

b $u = (-10\cos 30°, 10\sin 30°)$ and $v = (-4\cos 45°, -4\sin 45°)$

c $u = (2\cos 45°, -2\sin 45°)$ and $v = (-6\cos 30°, 6\sin 30°)$

3 Find the angle between the vectors, correct to the nearest degree.

a $u = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $v = \begin{pmatrix} -5 \\ 12 \end{pmatrix}$

b $u = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $v = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

c $u = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

d $u = 2i + 5j$ and $v = 3i + j$

e $u = -4i - j$ and $v = i - 5j$

f $u = 3i - 2j$ and $v = -4i - j$

4 The dot product of $\begin{pmatrix} x \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is 14. Find the value of x.

5 The angle between vectors u and v is 60° and their scalar product is 10. If $v = 3i - 4j$, find the magnitude of u.

6 Given $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$:

a Find the angle in degrees and minutes, to the nearest minute, between:

i $\overrightarrow{OA}$ and $\overrightarrow{OB}$

ii $\overrightarrow{OA}$ and $\overrightarrow{OA} - \overrightarrow{OB}$

iii $\overrightarrow{OB}$ and $\overrightarrow{OA} - \overrightarrow{OB}$

b What geometrical relationship does this illustrate?

7 **a** Find the scalar product of $u = 3i - 6j$ and $v = -2i - j$.

b Find the angle between the vectors.

8 The dot product $u \quad v = 24$ where $u = \begin{pmatrix} x \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$. Find the value of x.

9 The angle between vectors $u = i - 3j$ and $v = ai + 5j$ is 120°.

Evaluate a correct to 1 decimal place.

10 Find vector b (to 1 decimal place) given $a = 3i - j$, $a \quad b = -6$ and the angle between a and b is 30°.

11 Show that $v \cdot v = |v|^2$ where:

a $v = 3i - 4j$

b $v = ai + bj$

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EXT1 10.02 Parallel and perpendicular vectors

Paallel and perpendicular vectors

Perpendicular (orthogonal) vectors

The vectors u and v are perpendicular if $u \;\; v = 0$.

Proof

The angle between perpendicular (orthogonal) vectors is 90°.

$$
\begin{aligned}
u \cdot v &= |u||v|\cos\theta \\
&= |u||v|\cos 90° \\
&= |u||v| \times 0 \\
&= 0
\end{aligned}
$$

EXAMPLE 4

Show that vectors $u = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ are perpendicular.

Solution

$$
\begin{aligned}
u \;\; v &= x_1x_2 + y_1y_2 \\
&= 5 \times 3 + (-3) \times 5 \\
&= 15 - 15 \\
&= 0
\end{aligned}
$$

So the vectors are perpendicular.

Parallel vectors

The vectors u and v are parallel if:

$u \;\; v = |u||v|$ for vectors in like directions or

$u \;\; v = -|u||v|$ for vectors in unlike directions

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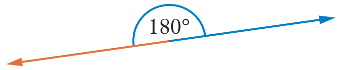
Proof

The angle between parallel vectors in like directions is 0°.

$$u \quad v = |u||v| \cos 0°$$
$$= |u||v| \times 1$$
$$= |u||v|$$

The angle between parallel vectors in unlike directions is 180°.



$$u \quad v = |u||v| \cos 180°$$
$$= |u||v| \times -1$$
$$= -|u||v|$$

EXAMPLE 5

Show that vectors u and v are parallel, given:

a $u = 4i + 3j$ and $v = 6i + 4.5j$

b $u = (6 \cos 30°, -6 \sin 30°)$ and $v = (-4 \cos 30°, 4 \sin 30°)$

Solution

a

$$u \quad v = x_1x_2 + y_1y_2$$
$$= 4 \times 6 + 3 \times 4.5$$
$$= 37.5$$
$$|u| = \sqrt{x^2 + y^2}$$
$$= \sqrt{4^2 + 3^2}$$
$$= 5$$
$$|v| = \sqrt{6^2 + (4\ 5)^2}$$
$$= \sqrt{36 + 2025}$$
$$= 7.5$$
$$|u||v| = 5 \times 7.5$$
$$= 37.5$$
$$= u \quad v$$

So the vectors are parallel (in like directions).

b

$$u = (6 \cos 30°, -6 \sin 30°)$$
$$= \left(6 \times \frac{\sqrt{3}}{2}, -6 \times \frac{1}{2}\right)$$
$$= (3\sqrt{3}, -3)$$
$$v = (-4 \cos 30°, 4 \sin 30°)$$
$$= \left(-4 \times \frac{\sqrt{3}}{2}, 4 \times \frac{1}{2}\right)$$
$$= (-2\sqrt{3}, 2)$$
$$u \cdot v = 3\sqrt{3} \times (-2\sqrt{3}) + (-3) \times 2$$
$$= -18 - 6$$
$$= -24$$
$$|u| = 6 \text{ and } |v| = 4$$
$$|u||v| = 24$$
$$= -u \quad v$$

So the vectors are parallel (in unlike directions)

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EXT1 Exercise 10.02 Parallel and perpendicular vectors

1 Show that each pair of vectors are perpendicular.

a $u = 5i - 2j$ and $v = 4i + 10j$

b $u = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

c $u = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$ and $v = \begin{pmatrix} -12 \\ -4 \end{pmatrix}$

2 Show that each pair of vectors are parallel.

a $u = (2 \cos 60°, -2 \sin 60°)$ and $v = (-10 \cos 60°, 10 \sin 60°)$

b $u = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

c $u = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$ and $v = \begin{pmatrix} 5 \\ -20 \end{pmatrix}$

d $u = 2i + 3j$ and $v = 7i + 10.5j$

e $u = -i + 6j$ and $v = \frac{1}{2}i - 3j$

f $u = -5i + j$ and $v = 20i - 4j$

3 Evaluate x if the vectors $\begin{pmatrix} x \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$ are:

a perpendicular

b parallel

4 State whether each pair of vectors are parallel, perpendicular or neither.

a $u = 4i - 6j$ and $v = -10i + 15j$

b $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$

c $u = -6i + 3j$ and $v = 4i + 8j$

d $u = (-3 \cos 30°, 3 \sin 30°)$ and $v = (-5 \cos 60°, -5 \sin 60°)$

e $u = (5 \cos 45°, 5 \sin 45°)$ and $v = (-8 \cos 45°, -8 \sin 45°)$

f $u = (-4 \cos 30°, -4 \sin 30°)$ and $v = (7 \cos 60°, -7 \sin 60°)$

5 The vectors $u = 5i - 3j$ and $v = 3i + bj$ are perpendicular. Evaluate b.

6 Find the value of n if vectors $\overrightarrow{OA} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -5 \\ n \end{pmatrix}$ are:

a parallel in a like direction

b perpendicular

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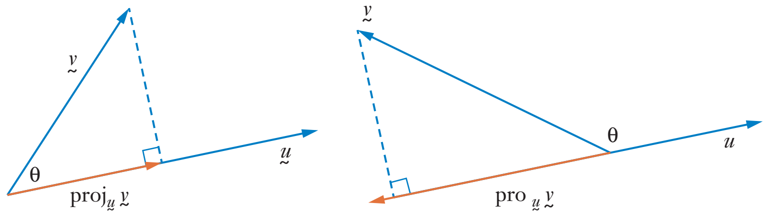
EXT1 10.03 Projection of vectors

A vector can be projected onto another vector as shown.



In both cases, vector v is projected onto vector u by taking the end of vector v on a perpendicular line down to vector u.

Projection of vectors

$\text{proj}_{u} v = \frac{u \quad v}{|u^2|} u$ (**projection of a vector** v onto u)

Proof

From the right-angled triangle above:

$$\cos\theta = \frac{|\text{pro}_{u} v|}{|v|} \qquad [1]$$

Scalar product:

$$u \cdot v = |u||v| \cos\theta$$

$$\frac{u \quad v}{|u||v|} = \cos\theta \qquad [2]$$

From (1) and (2):

$$\frac{|\text{pro}_{u} v|}{|v|} = \frac{u \quad v}{|u||v|}$$

$$\text{pro}_{u} v| = \frac{u \quad v}{|u|} \qquad [3]$$

The unit vector parallel to vector u is $\frac{u}{|u|}$.

But $\text{proj}_{u} v$ is parallel to u so it has the same direction.

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So the unit vector parallel to vector $\text{proj}_{\underset{\sim}{u}} v$ is also $\frac{u}{|u|}$.

$$\text{proj}_{\underset{\sim}{u}} v = \left|\text{pro}_{\underset{\sim}{u}} v\right| \times \frac{u}{|u|}$$

$$= \frac{u\ \ v}{|u|} \times \frac{u}{|u|} \qquad \text{(substituting [3])}$$

$$= \frac{u\ \ v}{|u|^2} u$$

Notice that when there is an obtuse angle between the vectors, $u\ \ v < 0$ so the projection will be parallel but in the opposite direction to u.

EXAMPLE 6

Given $u = 5i - 12j$ and $v = 4i + 3j$, find:

a $\text{proj}_{\underset{\sim}{u}} v$

b $\text{proj}_{\underset{\sim}{u}} v$

Solution

a

$$u\ \ v = x_1 x_2 + y_1 y_2$$
$$= 5 \times 4 + (-12) \times 3$$
$$= 20 - 36$$
$$= -16$$

$$|u| = \sqrt{5^2 + (-12)^2}$$
$$= \sqrt{25 + 144}$$
$$= \sqrt{169}$$
$$= 13$$

$$\text{proj}_{\underset{\sim}{u}} v = \frac{u\ \ v}{|u|^2} u$$
$$= \frac{-16}{13^2}(5i - 12j)$$
$$= \frac{-16}{169}(5i - 12j)$$
$$= -\frac{80}{169} i + \frac{192}{169} j$$

b

$$|v| = \sqrt{4^2 + 3^2}$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25}$$
$$= 5$$

$$\text{proj}_{\underset{\sim}{u}} v = \frac{u\ \ v}{|v|^2} v$$
$$= \frac{-16}{5^2}(4i + 3j)$$
$$= \frac{-16}{25}(4i + 3j)$$
$$= -\frac{64}{25} i - \frac{48}{25} j$$

EXT1 Exercise 10.03 Projection of vectors

1 Given $\underset{\sim}{a} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$, find:

a $\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{a}$

b $\text{proj}_{\underset{\sim}{a}}\, \underset{\sim}{b}$

2 Find $\text{proj}_{\underset{\sim}{u}}\, \underset{\sim}{v}$, given:

a $\underset{\sim}{u} = -5\underset{\sim}{i} + 15\underset{\sim}{j}$ and $\underset{\sim}{v} = -3\underset{\sim}{i} + 7\underset{\sim}{j}$

b $\underset{\sim}{u} = 3\underset{\sim}{i} - 15\underset{\sim}{j}$ and $\underset{\sim}{v} = -5\underset{\sim}{i} - 14\underset{\sim}{j}$

c $\underset{\sim}{u} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

d $\underset{\sim}{u} = -3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{v} = 4\underset{\sim}{i} - 19\underset{\sim}{j}$

e $\underset{\sim}{u} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

3 Given $\underset{\sim}{u} = (-4 \cos 30°, 4 \sin 30°)$ and $\underset{\sim}{v} = (6 \cos 45°, -6 \sin 45°)$, find $\text{proj}_{\underset{\sim}{u}}\, \underset{\sim}{v}$ in exact form.

4 Given $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, find:

a $\text{proj}_{\underset{\sim}{v}}\, \underset{\sim}{u}$

b $\text{proj}_{\underset{\sim}{u}}\, \underset{\sim}{v}$

c $\left|\text{pro}_{\underset{\sim}{v}}\underset{\sim}{u}\right|$

d the direction of $\text{proj}_{\underset{\sim}{v}}\, \underset{\sim}{u}$

5 Show $\left|\text{pro}_{\underset{\sim}{u}}\underset{\sim}{v}\right| = \dfrac{\underset{\sim}{u}\ \ \underset{\sim}{v}}{|\underset{\sim}{u}|}$.

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Geomeic proofs using vectors

EXT1 10.04 Proofs using vectors

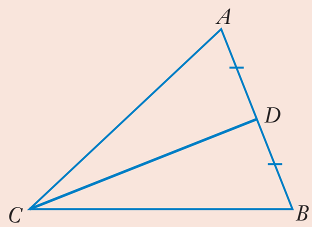
We can use vectors to prove geometrical properties.

EXAMPLE 7

Given $\overrightarrow{CA} = \underset{\sim}{a}$, $\overrightarrow{CB} = \underset{\sim}{b}$ and D is the midpoint of AB, show that:

a $\overrightarrow{BD} = \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b})$

b $\overrightarrow{CD} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$



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Solution

a $\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA}$

$= -\overrightarrow{CB} + \overrightarrow{CA}$

$= -b + a$

$= a - b$

$\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BA}$

$= \frac{1}{2}(a - b)$

b $\overrightarrow{CD} = \overrightarrow{CB} + \overrightarrow{BD}$

$= b + \frac{1}{2}(a - b)$

$= b + \frac{1}{2}a - \frac{1}{2}b$

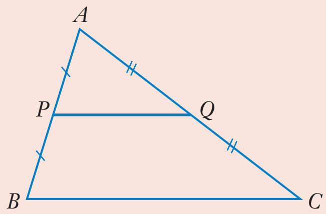
$= \frac{1}{2}a + \frac{1}{2}b$

$= \frac{1}{2}(a + b)$

EXAMPLE 8

a In $\triangle ABC$, PQ bisects AB and AC.
Prove that PQ is parallel to BC and half its length.

b Prove that the diagonals of a parallelogram bisect each other.



Solution

a Let $\overrightarrow{BP} = v$ and $\overrightarrow{CQ} = w$

Since P and Q bisect sides AB and AC respectively:

$\overrightarrow{PA} = v$ and $\overrightarrow{QA} = w$

$\overrightarrow{BA} = 2v$ and $\overrightarrow{CA} = 2w$

$\overrightarrow{PA} = \overrightarrow{PQ} + \overrightarrow{QA}$

$v = \overrightarrow{PQ} + w$

$\overrightarrow{PQ} = v - w$

$\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA}$

$2v = \overrightarrow{BC} + 2w$

$\overrightarrow{BC} = 2v - 2w$

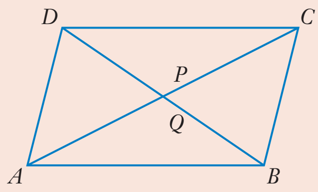
$= 2(v - w)$

$= 2\overrightarrow{PQ}$

So $\frac{1}{2}\overrightarrow{BC} = \overrightarrow{PQ}$ (parallel vectors)

So PQ is half the length and parallel to BC.

b Let the midpoint of $AC = P$ and the midpoint of $BD = Q$.



To prove diagonals bisect each other, we need to prove that P and Q are the same point.

We can do this by proving $\overrightarrow{AP} = \overrightarrow{AQ}$

Since P is the midpoint of AC:

$$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC}$$

Since Q is the midpoint of BD:

$$\overrightarrow{BQ} = \frac{1}{2}\overrightarrow{BD}$$

$$\begin{aligned}
\overrightarrow{AQ} &= \overrightarrow{AB} + \overrightarrow{BQ} \\
&= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD} \\
&= \left(\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AB}\right) + \frac{1}{2}\overrightarrow{BD} \\
&= \frac{1}{2}\overrightarrow{DC} + \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD} \\
&= \frac{1}{2}(\overrightarrow{DC} + \overrightarrow{AB} + \overrightarrow{BD}) \\
&= \frac{1}{2}(\overrightarrow{DC} + \overrightarrow{AD}) \\
&= \frac{1}{2}\overrightarrow{AC}
\end{aligned}$$

So $\overrightarrow{AP} = \overrightarrow{AQ}$ and the diagonals bisect each other.

Some proofs are easier if the vectors are in Cartesian form.

EXAMPLE 9

Given $\overrightarrow{OA} = \begin{pmatrix} a \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ b \end{pmatrix}$ and $AO = OC$, prove:

a triangle ABC is isosceles with $AB = CB$

b $\angle ACB = \angle CAB$

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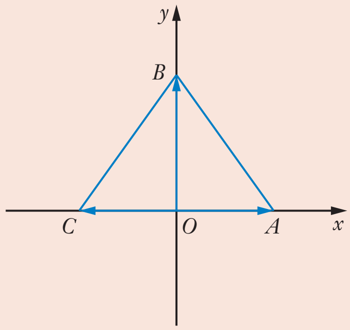
Solution

Sketch triangle ABC as shown.



a Given $\overrightarrow{OA} = \begin{pmatrix} a \\ 0 \end{pmatrix}$, since $AO = OC$ then $\overrightarrow{OC} = \begin{pmatrix} -a \\ 0 \end{pmatrix}$

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{pmatrix} 0 \\ b \end{pmatrix} - \begin{pmatrix} a \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -a \\ b \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(-a)^2 + b^2}$$

$$= \sqrt{a^2 + b^2}$$

$$\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$$

$$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$$

$$= \begin{pmatrix} 0 \\ b \end{pmatrix} - \begin{pmatrix} -a \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\left|\overrightarrow{CB}\right| = \sqrt{a^2 + b^2}$$

$$= \left|\overrightarrow{AB}\right|$$

So ABC is isosceles with $AB = CB$

b For $\angle ACB$:

$$\overrightarrow{CB} \cdot \overrightarrow{CO} = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix}$$

$$= a \times a + b \times 0$$

$$= a^2$$

$$\left|\overrightarrow{CB}\right| = \sqrt{a^2 + b^2}$$

$$\left|\overrightarrow{CO}\right| = a$$

$$u \cdot v = |u||v| \cos \theta$$

$$a^2 = a\sqrt{a^2 + b^2} \cos \angle ACB$$

$$a = \sqrt{a^2 + b^2} \cos \angle ACB$$

$$\frac{a}{\sqrt{a^2 + b^2}} = \cos \angle ACB$$

For $\angle CAB$:

$$\overrightarrow{AB} \cdot \overrightarrow{AO} = \begin{pmatrix} -a \\ b \end{pmatrix} \begin{pmatrix} -a \\ 0 \end{pmatrix}$$

$$= -a \times -a + b \times 0$$

$$= a^2$$

$$\left|\overrightarrow{AB}\right| = \sqrt{a^2 + b^2}$$

$$\left|\overrightarrow{AO}\right| = a$$

$$u \cdot v = |u||v| \cos \theta$$

$$a^2 = a\sqrt{a^2 + b^2} \cos \angle CAB$$

$$\frac{a}{\sqrt{a^2 + b^2}} = \cos \angle ACB$$

So $\angle ACB = \angle CAB$

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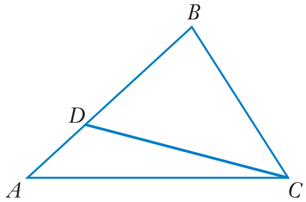
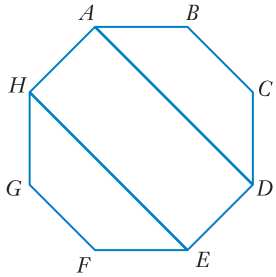
EXT1 Exercise 10.04 Proofs using vectors

1 Prove that $b - a = -(a - b)$.

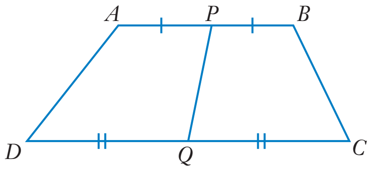
2 In $\triangle ABC$ shown, D is a point on AB where $\overrightarrow{AD} : \overrightarrow{DB} = 1 : 3$.
Given $\overrightarrow{AD} = a$, $\overrightarrow{AC} = b$ and $\overrightarrow{CB} = c$, show that $b + c = 4a$.



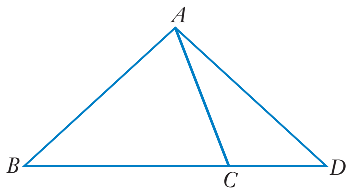
3 In the regular octagon $ABCDEFGH$, show that $\overrightarrow{AD} = \overrightarrow{HE}$.



4 In trapezium $ABCD$, $AB \parallel DC$ and P and Q are the midpoints of AB and DC as shown.
Prove that $AD + BC = 2PQ$.



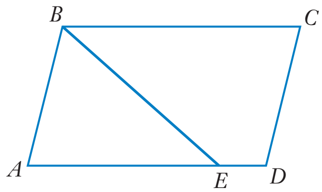
5 In the diagram, $2\overrightarrow{BC} = 3\overrightarrow{CD}$. Given that $\overrightarrow{BA} = a$ and $\overrightarrow{BC} = b$, show that $\overrightarrow{AD} = \frac{5}{3}b - a$.



6 In the diagram, $ABCD$ is a parallelogram and $\overrightarrow{AE} : \overrightarrow{ED} = 9 : 2$, $\overrightarrow{BC} = a$ and $\overrightarrow{AB} = b$.
Prove that:

a $\overrightarrow{ED} = \frac{2}{11}a$

b $\overrightarrow{BE} = \frac{9}{11}a - b$



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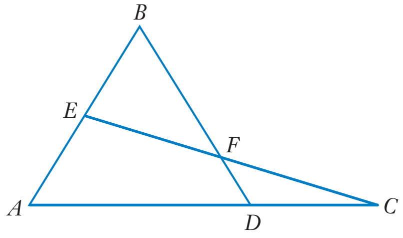
7 In the diagram, $\overrightarrow{BF} : \overrightarrow{FD} = 5 : 3$, $\overrightarrow{AB} = a$ and $\overrightarrow{AD} = b$. E is the midpoint of AB.
Prove that:

a $\overrightarrow{BF} = \frac{5}{8}(b - a)$

b $\overrightarrow{AF} = \frac{5}{8}b + \frac{3}{8}a$

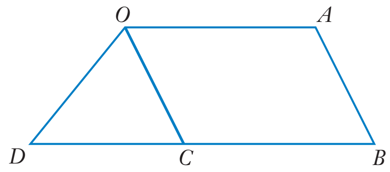
c $\overrightarrow{EF} = \frac{5}{8}b - \frac{1}{8}a$



8 In the diagram, $OABC$ is a parallelogram with $\overrightarrow{OA} = a$, $\overrightarrow{OB} = b$ and $3\overrightarrow{DC} = \overrightarrow{CB}$.
Prove that:

a $\overrightarrow{BD} = -\frac{4}{3}a$

b $\overrightarrow{DA} = \frac{7}{3}a - b$



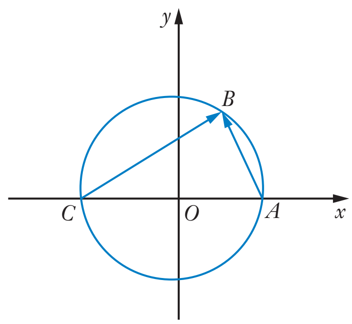
9 Given any quadrilateral, prove that the midpoints of each side form a parallelogram.

10 Use vectors to prove that:

a the diagonals of a rectangle are equal in length

b the diagonals of a rhombus are perpendicular

11 Prove that $\overrightarrow{CB}$ and $\overrightarrow{AB}$ in the diagram are perpendicular.



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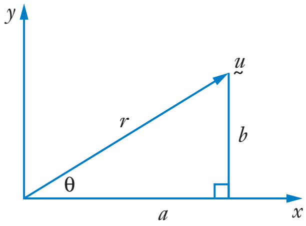
EXT1 10.05 Motion and vectors

A vector can be expressed in terms of its magnitude, r, and direction, θ.

Let $u = ai + bj = \begin{pmatrix} a & b \end{pmatrix}$



$\cos\theta = \dfrac{a}{r}$ $\qquad\qquad$ $\sin\theta = \dfrac{b}{r}$

$a = r\cos\theta$ $\qquad\qquad$ $b = r\sin\theta$

So $u = (r\cos\theta)i + (r\sin\theta)j = (r\cos\theta, r\sin\theta)$

Vector with magnitude *r* and direction θ

If u has magnitude r and direction θ, then:

$$u = (r\cos\theta,\ r\sin\theta)$$

EXAMPLE 10

a A body experiences an acceleration of -10 m s^{-2} in a direction of 135°. Find the vector for the acceleration of the body.

b Given displacement vectors $u = 3i - 4j$ and $v = -2i - j$, find the magnitude and direction of the resultant vector $u + v$.

Solution

a $r = -10 \text{ m s}^{-2}$, $\theta = 135°$.

Let a be the acceleration vector.

$$\begin{aligned} a &= (r\cos\theta, r\sin\theta) \\ &= (-10\cos 135°, -10\sin 135°) \\ &= \left(-10\left(-\frac{1}{\sqrt{2}}\right), -10\left(\frac{1}{\sqrt{2}}\right)\right) \\ &= \left(\frac{10\sqrt{2}}{2} \quad -\frac{10\sqrt{2}}{2}\right) \\ &= (5\sqrt{2}, -5\sqrt{2}) \end{aligned}$$

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b $u + v = 3i - 4j + -2i - j$

$= i - 5j$ (4th quadrant)

Magnitude:

$$|u + v| = \sqrt{x^2 + y^2}$$
$$= \sqrt{1^2 + (-5)^2}$$
$$= \sqrt{1 + 25}$$
$$= \sqrt{26}$$
$$\approx 5.1$$

Direction:

$$\tan \theta = \frac{y}{x}$$
$$= -\frac{5}{1}$$

For the 4th quadrant: $\theta \approx 360° - 79°$

$= 281°$

Velocity as a vector

We can find resultant velocities by adding or subtracting vectors.

EXAMPLE 11

a A yacht is travelling at 80 km h^{-1}. Find its speed under the influence of:

i a head wind of 20 km h^{-1}

ii a tail wind of 20 km h^{-1}

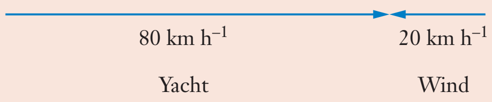
b A plane's controls are set for a speed of 250 km h^{-1} on a bearing of 040°, but there is a westerly wind of 55 km h^{-1} (wind from the west). Find the actual speed and bearing of the plane, taking the wind into account.

Solution

a **i** The head wind will go against the yacht and slow it down.

Speed = 80 – 20

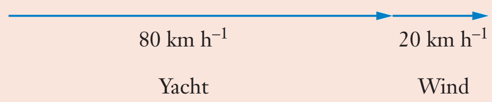
= 60 km h^{-1}



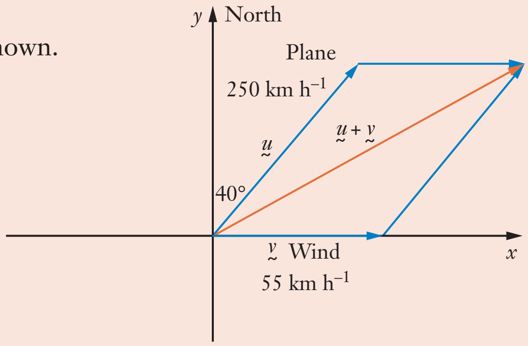
ii The tail wind will go behind the yacht and speed it up.

Speed = 80 + 20

= 100 km h^{-1}



b The direction of the plane will shift towards the direction of the wind as shown. The resultant vector is $\underset{\sim}{u} + \underset{\sim}{v}$.



The direction of $\underset{\sim}{u}$ is $90° - 40° = 50°$.

$\underset{\sim}{u} = (250 \cos 50°, 250 \sin 50°)$

$\underset{\sim}{v} = (55 \cos 0°, 55 \sin 0°)$

Note that the *bearing* (from North) is 040°, but the *direction* (from the *x*-axis) is 50°.

$$\underset{\sim}{u} + \underset{\sim}{v} = (250 \cos 50° + 55 \cos 0°, 250 \sin 50° + 250 \sin 0°)$$

$$= (250 \cos 50° + 55, 250 \sin 50°)$$

$$\text{Magnitude} = \sqrt{x^2 + y^2}$$

$$= \sqrt{(250 \cos 50° + 55)^2 + (250 \sin 50°)^2}$$

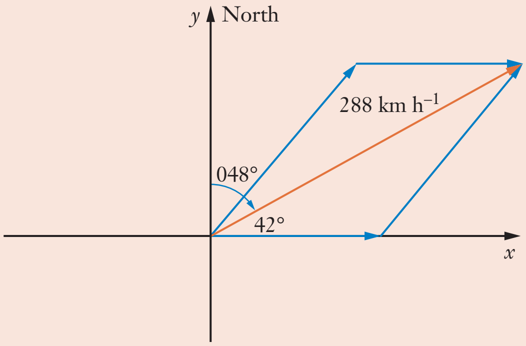
$$\approx 288$$

$$\text{Direction: } \tan \theta = \frac{y}{x} = \frac{250 \sin 50°}{250 \cos 50° + 55}$$

$$\theta \approx 42°$$

The bearing is $90° - 42°$ or 048°.

The plane is travelling at 288 km h^{-1} on a bearing of 048°.



Note that the *direction* is 42°, but the *bearing* is 048°. Make sure you use the correct angle in your answer.

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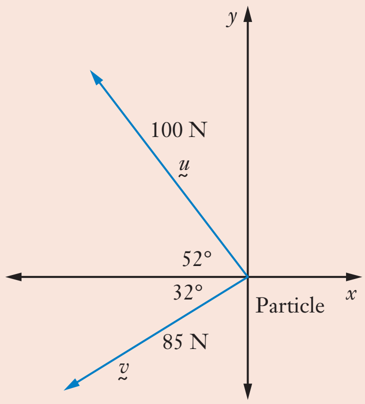
Force as a vector

EXAMPLE 12

Two forces are acting on a particle as shown.

Find the magnitude and direction of the resultant force on the particle.



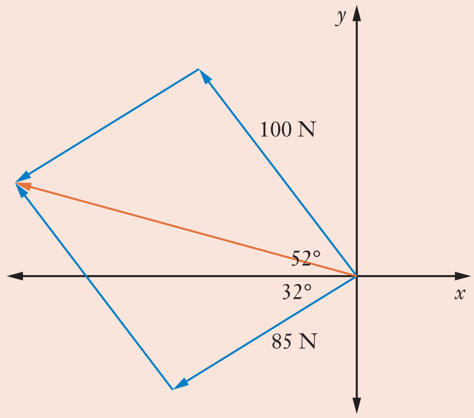
Solution

$\underset{\sim}{u}$ has magnitude 100 N (newtons) and direction $180° - 52° = 128°$ (2nd quadrant).

$\underset{\sim}{u} = (100 \cos 128°, 100 \sin 128°)$

$\underset{\sim}{v}$ has magnitude 85 N and direction $180° + 32° = 212°$ (3rd quadrant)

$\underset{\sim}{v} = (85 \cos 212°, 85 \sin 212°)$



The resultant vector is $\underset{\sim}{u} + \underset{\sim}{v}$.

$\underset{\sim}{u} + \underset{\sim}{v} = (100 \cos 128° + 85 \cos 212°, 100 \sin 128° + 85 \sin 212°)$

$$\begin{aligned}\text{Magnitude} &= \sqrt{x^2 + y^2} \\ &= \sqrt{(100 \cos 128° + 85 \cos 212°)^2 + (100 \sin 128° + 85 \sin 212°)^2} \\ &\approx 137.8\end{aligned}$$

Direction:

$$\begin{aligned}\tan \theta &= \frac{y}{x} \\ &= \frac{100 \sin 128° + 85 \sin 212°}{100 \cos 128° + 85 \cos 212°}\end{aligned}$$

$$\begin{aligned}\theta &\approx 180° - 14° \qquad \text{(2nd quadrant)} \\ &= 166°\end{aligned}$$

So the resultant force has magnitude 137.8 N and direction 166°.

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EXT1 Exercise 10.05 Motion and vectors

1 A displacement vector has magnitude 3.4 km and direction 125°. A second displacement vector has magnitude 2.8 km and direction 28°. Find the magnitude and direction of the resultant vector by adding these vectors.

2 A velocity vector has magnitude 890 m s^{-1} and direction 232°. Another velocity vector has magnitude 549 m s^{-1} and direction 138°. If these 2 velocities are added, find the magnitude and direction of the resultant vector.

3 An airship's controls are set to fly west at 45 km h^{-1}. Find its velocity given:

a a tail wind of 30 km h^{-1} **b** a head wind of 23 km h^{-1}

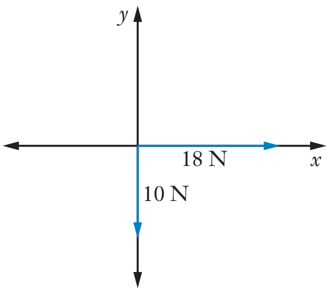
c a northerly wind (wind from the north) of 15 km h^{-1}

4 Juan aims to swim straight across a river at 5 km h^{-1} but a current of 8 km h^{-1} down the river is affecting his progress.

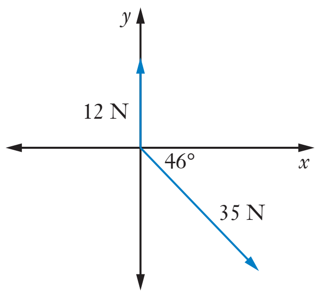
What is Juan's speed under the influence of the current?

5 A particle has 2 forces on it as shown. Find the magnitude and direction of the resultant force in each case.

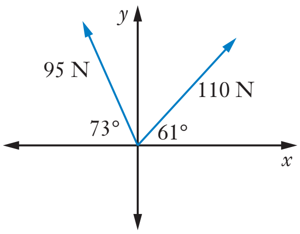
a



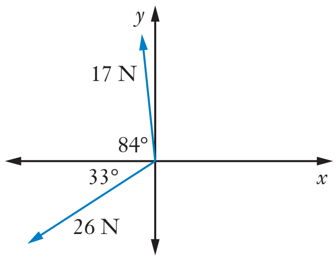
b



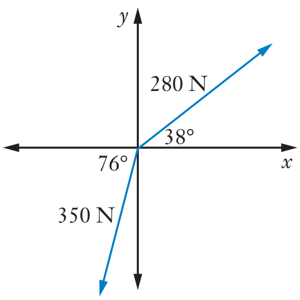
c



d



e



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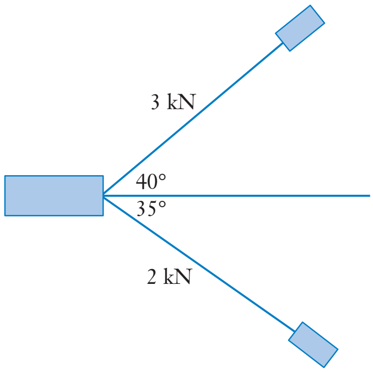
6 Two tugboats are pulling a cruise ship as shown, with the force measured in kilonewtons (kN). Find the magnitude and direction of the resultant force.



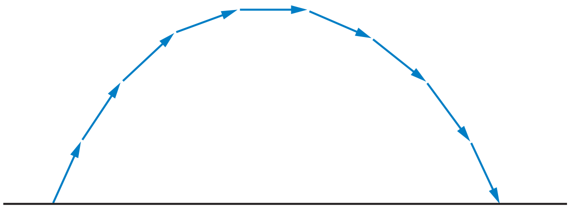
7 The controls of an ultralight plane are set to fly west at 75 km h^{-1}. Due to a cross wind, its resultant velocity is 95 km h^{-1} in a 135° direction. What is the speed and direction of the cross wind?

EXT1 10.06 Projectile motion

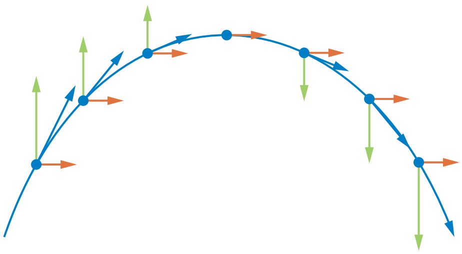
WS Pojecile motion

A **projectile** is any object that is projected up (thrown or fired) and falls down under the acceleration due to gravity (neglecting air resistance). **Projectile motion** describes the path of the projectile, usually as a point moving on the Cartesian plane.

The displacement, velocity and acceleration of a projectile can be represented by vectors. When a projectile is thrown or fired upwards, gravity will cause it to fall back down. Its path is a **parabola**.



The velocity vectors have horizontal and vertical (x and y) components.



Since gravity (a vertical downward force) is the only force on these vectors, it affects the vertical component. The horizontal component remains constant throughout the motion. This is shown in the diagram where the red vectors represent the horizontal components of velocity and the green vectors represent the vertical components of velocity.

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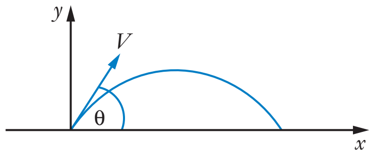
Initial velocity



Suppose a particle is projected with initial speed V at an angle of θ to the horizontal.

We look at the x and y components of the vector for the initial velocity.

$x = V\cos\theta$ (initial velocity in horizontal direction)

$y = V\sin\theta$ (initial velocity in vertical direction)

Initial velocity vector of a projectile

$$\underset{\sim}{V} = V\cos\theta\, \underset{\sim}{i} + V\sin\theta\, \underset{\sim}{j}$$

where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are the unit vectors in the horizontal and vertical directions respectively.

EXAMPLE 13

Find the initial velocity vector of an object that is projected with:

a initial speed 15 m s^{-1} and angle 0°

b initial speed 18 m s^{-1} and angle 60°

Solution

a $V = 15, \theta = 0°$:

$$\begin{aligned}\underset{\sim}{V} &= V\cos\theta\,\underset{\sim}{i} + V\sin\theta\,\underset{\sim}{j}\\ &= 15\cos 0°\,\underset{\sim}{i} + 15\sin 0°\,\underset{\sim}{j}\\ &= 15\underset{\sim}{i} + 0\underset{\sim}{j}\\ &= 15\underset{\sim}{i}\end{aligned}$$

b $V = 18, \theta = 60°$:

$$\begin{aligned}\underset{\sim}{V} &= V\cos\theta\,\underset{\sim}{i} + V\sin\theta\,\underset{\sim}{j}\\ &= 18\cos 60°\,\underset{\sim}{i} + 18\sin 60°\,\underset{\sim}{j}\\ &= 18\times\frac{1}{2}\underset{\sim}{i} + 18\times\frac{\sqrt{3}}{2}\underset{\sim}{j}\\ &= 9\underset{\sim}{i} + 9\sqrt{3}\,\underset{\sim}{j}\end{aligned}$$

Acceleration

The only acceleration that a projectile undergoes is the acceleration due to gravity, which is vertical in a downward direction ($-g$). There is no horizontal acceleration.

$\ddot{x} = 0 \quad \ddot{y} = -g$

So the acceleration vector, $\underset{\sim}{a}$, is:

$$\begin{aligned}\underset{\sim}{a} &= 0\underset{\sim}{i} - g\underset{\sim}{j}\\ &= -g\underset{\sim}{j}\end{aligned}$$

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Acceleration vector of a projectile

$$a = -gj$$

Velocity

We can find the velocity, v, of a projectile anywhere during its motion.

Given initial speed V and direction θ:

$$a = -gj$$

$$v = \int a\,dt$$

$$= \int -g\,j\,dt$$

$$= -gt\,j + c$$

When $t = 0$, $v = V\cos\theta\, i + V\sin\theta\, j$

Substituting:

$$V\cos\theta\, i + V\sin\theta\, j = (-g \times 0)\,j + c$$

$$= c$$

So $v = -gtj + V\cos\theta\, i + V\sin\theta\, j$

$$= V\cos\theta\, i + (-gt + V\sin\theta)\,j$$

Velocity vector of a projectile

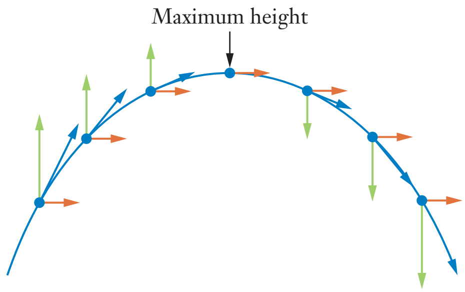
$$v = V\cos\theta\, i + (-gt + V\sin\theta)\,j$$

The **speed** of the projectile is the **magnitude** of its velocity vector at any given time, t.

Note from the equation of the velocity vector that the horizontal (or x-) component of velocity is constant, independent of t.

The maximum height of the projectile occurs when the vertical (or y-) component of velocity is 0 (maximum turning point).

The projectile reaches maximum height at the top of the parabola.



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EXAMPLE 14

A ball is thrown into the air with initial speed 18 m s^{-1} and direction 35°. Acceleration due to gravity is given as 10 m s^{-2}.

a Find the time at which the ball reaches maximum height.

b Find the speed of the ball after 1.2 seconds.

Solution

a
$$\begin{aligned} a &= -gj \\ &= -10j \\ v &= \int a\,dt \\ &= \int -10j\,dt \\ &= -10tj + c \end{aligned}$$

When $t = 0$:

$$\begin{aligned} V &= V\cos\theta\, i + V\sin\theta\, j \\ &= 18\cos 35°\, i + 18\sin 35°\, j \end{aligned}$$

Substituting:

$$\begin{aligned} 18\cos 35°\, i + 18\sin 35°\, j &= (-10\times 0)j + c \\ &= c \end{aligned}$$

$$\begin{aligned} \text{So } v &= -10j + 18\cos 35°\, i + 18\sin 35°\, j \\ &= 18\cos 35°\, i + (-10t + 18\sin 35°)j \end{aligned}$$

At maximum height, the vertical component of $v = 0$

$-10t + 18\sin 35° = 0$

$$\begin{aligned} 18\sin 35° &= 10t \\ \frac{18\sin 35°}{10} &= t \\ 1.0324\ldots &= t \end{aligned}$$

So the maximum height occurs at time 1.03 s.

b $v = 18\cos 35°\, i + (-10t + 18\sin 35°)j$

For speed, we can find the magnitude of the velocity.

When $t = 1.2$:

$$\begin{aligned} |v| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(18\cos 35°)^2 + (-10\times 1\ 2 + 18\sin 35°)^2} \\ &= \sqrt{2202149} \\ &= 14.8396\ldots \end{aligned}$$

So the speed after 1.2 seconds is 14.8 m s^{-1}.

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Displacement

We can find the displacement vector, s, of a projectile anywhere during its motion.

$v = V\cos\theta\, i + (-gt + V\sin\theta)\, j$

$$s = \int v\, dt$$

$$= \int [V\cos\theta\, i + (-gt + V\sin\theta)\, j]\, dt$$

$$= Vt\cos\theta\, i + \left(-\frac{gt^2}{2} + [V\sin\theta]t\right) j + D$$

$$= Vt\cos\theta\, i + \left(-\frac{gt^2}{2} + Vt\sin\theta\right) j + D$$

Suppose the projectile starts at the origin (0, 0):

When $t = 0$, $s = 0i + 0j$

Substituting:

$$0i + 0j = V(0)\cos\theta\, i + \left[-\frac{g(0^2)}{2} + V(0)\sin\theta\right] j + D$$

$$0 = 0 = D$$

So $s = Vt\cos\theta\, i + \left(-\frac{gt^2}{2} + Vt\sin\theta\right) j$

Displacement vector of a projectile

$$s = Vt\cos\theta\, i + \left(-\frac{gt^2}{2} + Vt\sin\theta\right) j$$

Components of projectile motion

	Horizontal (x)	Vertical (y)
Acceleration	$0i$	$-gj$
Velocity	$V\cos\theta\, i$	$(-gt + V\sin\theta)\, j$
Displacement	$Vt\cos\theta\, i$	$\left(-\frac{gt^2}{2} + Vt\sin\theta\, j\right)$

As seen in the following examples, when solving problems involving projectile motion, you cannot just quote the formulas for velocity and displacement above, but must prove them from the acceleration formula.

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EXAMPLE 15

Nirmalan stands on the edge of a cliff 35 m high and throws a stone into the water with a velocity of 16 m s^{-1} at an angle of 30° to the horizontal.

a Find the vector for the stone's displacement in the form $s = ai + bj$ given $g = 10$ m s^{-2}.

b Calculate:

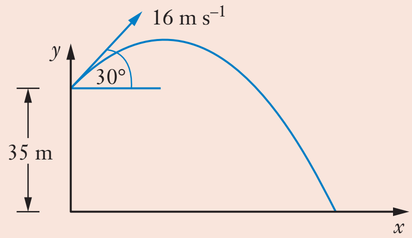
i the time the stone will take to land in the water

ii its distance from the foot of the cliff at that time.

Solution

a Acceleration:

$$a = -gj$$
$$= -10j$$



Velocity:

$$v = \int a\,dt$$
$$= \int -10j\,dt$$
$$= -10tj + C$$

When $t = 0$:

$$v = V\cos\theta\, i + V\sin\theta\, j$$
$$= 16\cos 30°\, i + 16\sin 30°\, j$$
$$= 16 \times \frac{\sqrt{3}}{2}\, i + 16 \times \frac{1}{2}\, j$$
$$= 8\sqrt{3}i + 8j$$

Substituting:

$$8\sqrt{3}i + 8j = -10(0)j + C$$
$$= C$$

So $v = -10tj + 8\sqrt{3}i + 8j$

$$= 8\sqrt{3}i + (-10t + 8)j$$

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Displacement:

$s = \int v\,dt$

$= \int [8\sqrt{3}\,i + (-10t + 8)\,j]\,dt$

$= 8\sqrt{3}ti + (-5t^2 + 8t)\,j + C$

Nirmalan is standing on top of the cliff 35 m high when he throws the stone.

When $t = 0$, $s = 0i + 35\,j$

$= 35\,j$

Substituting:

$35\,j = 8\sqrt{3}(0)i + \left[-5(0)^2 + 8(0)\right]j + D$

$= D$

So $s = 8\sqrt{3}ti + (-5t^2 + 8t)\,j + 35\,j$

$= 8\sqrt{3}ti + (-5t^2 + 8t + 35)\,j$

b **i** When the stone lands, the y component of displacement is 0.

$-5t^2 + 8t + 35 = 0$

Using the quadratic formula:

$$t = \frac{-8 \pm \sqrt{8^2 - 4 \times (-5) \times 35}}{2 \times (-5)}$$

$\approx -1.96, 3.56$

But time cannot be negative.

So it will take 3.56 s for the stone to land in the water.

ii The distance out from the cliff is the x component of displacement.

When $t = 3.56$:

$x = 8\sqrt{3}\,t$

$= 8\sqrt{3}(3.56)$

≈ 49.38

So the stone will land 49.38 m out from the foot of the cliff.

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Speed of a projectile

The speed of a projectile at time t is the magnitude of the velocity vector v at that time.

Maximum height of a projectile

The maximum height occurs when the y component of the velocity vector v is 0, and is the y component of the displacement vector s at that time.

Time of flight

The end of the flight is when the projectile lands on the ground, so the time of flight is when the y component of the displacement vector s is 0.

Range of flight

The horizontal range is how far the projectile reaches when it lands, and is the x component of the displacement vector s at that time.

EXAMPLE 16

Elysse hits a softball at a speed of 30 m s^{-1} and the ball just clears a 1.5 m fence 80 m away.

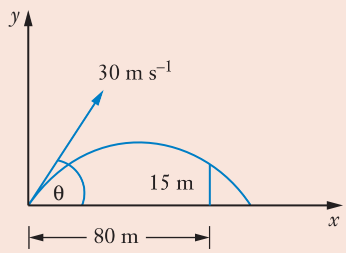
a Find the angle at which the ball was hit (assume no air resistance and use $g = 9.8$ m s^{-2}).

b Find the time of flight of the ball.

c Find how far the ball lands on the other side of the fence.

Solution

a Acceleration:



$$a = -gj$$
$$= -9.8j$$

Velocity:

$$v = \int a\,dt$$
$$= \int -9\ 8j\,dt$$
$$= -9.8tj + c$$

When $t = 0$:

$$V = V\cos\theta\, i + V\sin\theta\, j$$
$$= 30\cos\theta\, i + 30\sin\theta\, j$$

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Substituting:

$30\cos\theta\, i + 30\sin\theta\, j = -9.8(0)j + C$

$= C$

So $v = -9.8tj + 30\cos\theta\, i + 30\sin\theta\, j$

$= 30\cos\theta\, i + (-9.8t + 30\sin\theta)j$

Displacement:

$$s = \int v\, dt$$

$$= \int [30\cos\theta\, i + (-9\ 8t + 30\sin\theta)j]\,dt$$

$$= 30t\cos\theta\, i + (-4.9t^2 + 30t\sin\theta)j + D$$

When $t = 0$, $s = 0i + 0j$

Substituting:

$$0i + 0j = 30(0)\cos\theta\, i + [-4.9(0)^2 + 30(0)\sin\theta]j + D$$

$$0 = D$$

So $s = 30t\cos\theta\, i + (-49t^2 + 30t\sin\theta)j$

The ball just clears the fence, so when $x = 80$, $y = 1.5$:

Equating x and y components:

$80 = 30t\cos\theta$ [1]

$1.5 = -4.9t^2 + 30t\sin\theta$ [2]

These are simultaneous equations involving t and θ.

From [1]:

$$t = \frac{80}{30\cos\theta}$$

$$= \frac{8}{3\cos\theta}$$

Substitute into [2]:

$$1.5 = -4.9\left(\frac{8}{3\cos\theta}\right)^2 + 30\left(\frac{8}{3\cos\theta}\right)\sin\theta$$

$$= -4.9\left(\frac{64}{9\cos^2\theta}\right) + \left(\frac{80}{\cos\theta}\right)\sin\theta$$

$$= -34.8\sec^2\theta + 80\tan\theta$$

$$= -34.8(1 + \tan^2\theta) + 80\tan\theta$$

$$= -34.8 - 34.8\tan^2\theta + 80\tan\theta$$

$$34.8\tan^2\theta - 80\tan\theta + 36.3 = 0$$

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Using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

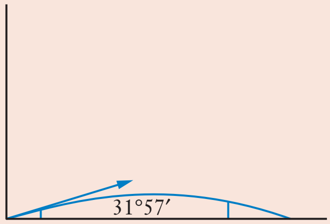
$$\tan\theta = \frac{80 \pm \sqrt{(-80)^2 - 4 \times 348 \times 36\ 3}}{2 \times 34\ 8}$$

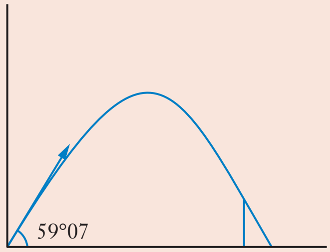
$$= 1.67, 0.62$$

$\theta = \tan^{-1}(1.67)$, $\quad\theta = \tan^{-1}(0.62)$

$= 59°\ 07'$ $\quad= 31°\ 57'$

So the softball is hit at an angle of 59° 07′ or 31° 57′ to just clear the fence.





b The flight ends when the ball reaches the ground and the y component of displacement is 0.

When $\theta = 31°\ 57'$:

$$-4.9t^2 + 30t\sin 31°\ 57' = 0$$
$$-4.9t^2 + 15.9t = 0$$
$$t(-4.9t + 15.9) = 0$$
$$t = 0, \quad -4.9t + 15.9 = 0$$
$$15.9 = 4.9t$$
$$3.24 \approx t$$

When $\theta = 59°\ 07'$:

$$-4.9t^2 + 30t\sin 59°\ 07' = 0$$
$$-4.9t^2 + 25.7t = 0$$
$$t(-4.9t + 25.7) = 0$$
$$t = 0, -4.9t + 25.7 = 0$$
$$25.7 = 4.9t$$
$$5.25 \approx t$$

So the time of flight is 3.24 s or 5.25 s.

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c Range of flight is the x component of displacement ($30t \cos \theta$).

When $\theta = 31° 57'$, time of flight is 3.24 s.

$$\text{Range} = 30(3.24) \cos 31°57'$$
$$\approx 82.5$$

So the ball lands 82.5 m away.

The fence is 80 m away, so the ball lands $82.5 - 80 = 2.5$ m past the fence.

When $\theta = 59° 07'$, time of flight is 5.25 s.

$$\text{Range} = 30(5.25) \cos 59° 07'$$
$$\approx 80.9$$

So the ball lands 80.9 m away.

The fence is 80 m away, so the ball lands $80.9 - 80 = 0.9$ m past the fence.

DID YOU KNOW?

Big Bertha

Big Bertha, a cannon used in World War I, could shoot cannonballs a distance of 120 km.

Research Big Bertha and other cannons to see how far they can shoot cannonballs. Does the size of the cannon or the cannonballs make a difference?

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INVESTIGATION

LAUNCHING ROCKETS

Research and write a report on the factors affecting the launching of a rocket or satellite.

Is the work on projectiles that you are studying here relevant to the launching of rockets or satellites?

What other things need to be considered for the launching of rockets or satellites?

Research the history of space travel.

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EXT1 Exercise 10.06 Projectile motion

1 A particle is projected at an angle of 45° and a speed of 15 m s^{-1}. Neglecting air resistance and taking g as 10 m s^{-2}:

- **a** derive the equation of the displacement vector for the particle
- **b** find the time taken for the particle to reach the ground

2 A bullet is fired at an angle of 60° and with a speed of 120 m s^{-1}. Assuming the acceleration due to gravity is 10 m s^{-2} and neglecting air resistance, find:

- **a** the exact time taken for the bullet to reach its maximum height.
- **b** the bullet's maximum height.

3 A ball is thrown from a window that is 16 m above the ground. If the angle of projection is 60°, initial speed is 5 m s^{-1} and $g = 10$ m s^{-2}, find:

- **a** the time taken for the ball to land (to 1 decimal place)
- **b** how far the ball will land from the base of the building (to 1 decimal place)

4 Pham throws a ball in the air at a velocity of 8.7 m s^{-1} at an angle of 55°. Neglecting air resistance and using $g = 9.8$ m s^{-2}, find:

- **a** the maximum height reached (to 2 decimal places)
- **b** how far away from Pham it will land (to 1 decimal place)

5 Michele throws a frisbee from the window of a building 15.3 m up. If the frisbee has an initial speed of 8.8 m s^{-1} and is thrown at an angle of 73°, find the time taken for it to reach the ground (using $g = 10$ m s^{-2} and neglecting air resistance).

6 A missile is launched at an initial trajectory of 68° and a speed of 1200 m s^{-1}. Neglecting air resistance and the curvature of Earth and taking the acceleration due to gravity as $g = 9.8$ m s^{-2}, calculate:

- **a** the time taken for its flight (to the nearest minute)
- **b** how far away it will strike (to the nearest km)

7
- **a** A particle is projected upwards at an angle to the horizontal of α with speed u. Derive the vectors for acceleration, velocity and displacement for the flight of the particle, taking g as the acceleration due to gravity and neglecting air resistance.
- **b** If $u = 20$ m s^{-1}, $\alpha = 60°$ and $g = 10$ m s^{-2}, find the maximum height reached by the particle.

8 A particle is projected at a speed of 16 m s^{-1} at an angle of elevation of θ. Neglecting air resistance and using $g = 10$ m s^{-2}, find the vector for the displacement of the particle.

9 Find the displacement vector of a particle with initial speed v m s^{-1} and angle of projection β if the particle is projected from a point h above ground level. Use g for the acceleration due to gravity and neglect air resistance.

10 Diana fires an arrow at a speed of 24 m s^{-1}, aiming at a target 1 m high and 35 m away. The air resistance is negligible for angles of projection less than 45°. Find an angle less than 45° through which Diana should fire the arrow to hit the target. Use $g = 10$ m s^{-2}.

11 A horizontal drainpipe 6 m above sea level empties storm water into the sea. If the water comes out horizontally and reaches the sea 2 m out from the pipe, find the initial speed of the water, correct to 1 decimal place. Let g be 10 m s^{-2} and neglect air resistance.

12 A particle is projected at an angle of 60° with speed v. If it reaches the ground after 5.1 s, find the value of v correct to 2 decimal places. Use $g = 9.8$ m s^{-2} and neglect air resistance.

13 Nathan aims a gun at a target on the ground 150 m away. If the initial speed is 125 m s^{-1}, find the angles at which he could fire the gun to hit the target. Use $g = 10$ m s^{-2}.

14 Jamelle stands at the window of a building 6.2 m above ground level. She throws her keys straight out of the window (horizontally) and hopes that her friend Rochelle, who is standing 10.4 m out from the base of the building, will catch them. Ignoring air resistance and using $g = 10$ m s^{-2} for the acceleration due to gravity, find the speed at which Jamelle needs to throw her keys, correct to 1 decimal place.

15 A rocket is fired straight up in the air at a fireworks display. When it reaches a height of 28 m, it explodes and the empty shell is projected at an angle of 60° to the horizontal with a speed of 30 m s^{-1}.

a How long will it take the shell to fall back to the ground after the rocket explodes? Neglect air resistance and take $g = 10$ m s^{-2}. Give your answer to the nearest second.

b How far will the shell land from its launching site (to 1 decimal place)?

16 Cooper kicks a football at 12 m s^{-1} and just clears the goalpost 4 m high and 9 m away. Find the angle of projection through which the football is kicked. Use $g = 10$ m s^{-2} and neglect air resistance.

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17 A waterfall flows at 2 m s^{-1} over a vertical 5 m cliff. How far out from the cliff does it fall? Use $g = 10$ m s^{-2}.

18 An object is projected with horizontal velocity 8 m s^{-1} and vertical velocity 5 m s^{-1}. Find the range of its flight. Use $g = 10$ m s^{-2}.

19 A stone is projected in the air at an angle of $\frac{\pi}{6}$ and a speed of 15 m s^{-1}. How far from a fence 20 m away does it land? Use $g = 9.8$ m s^{-2}.

20 A particle is projected at an initial speed of 10 m s^{-1}. If the horizontal component of the velocity is 6 m s^{-1}, find:

a the vertical component of the velocity

b the angle of projection

c the maximum height of the particle (use $g = 10$ m s^{-2})

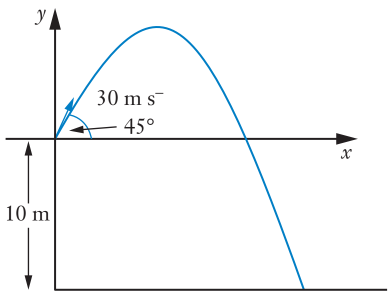
d The speed of the particle after:

i 2 s **ii** 3.1 s

21 A ball is thrown at an angle of 45° with a speed of 30 m s^{-1}, and lands in a 10 m deep ditch. Taking acceleration due to gravity as 9.8 m s^{-2}, find:

a how long it will take to land

b how far out it will land



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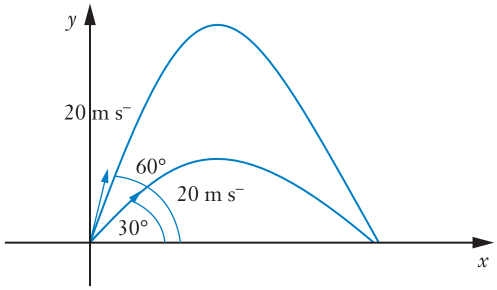
22 A stone is thrown with a speed of 20 m s^{-1} and an angle of 60°. A second stone is thrown at the same time and place, with the same speed but at an angle of 30°. Use $g = 10$ m s^{-2}.

a Which stone finishes its flight first and by how long?

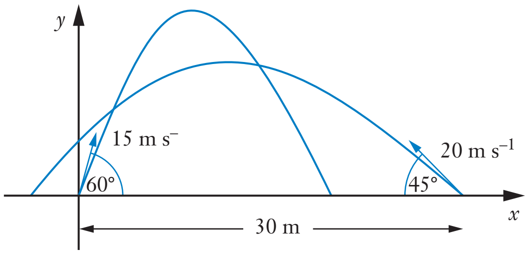
b Show that the 2 stones land at the same place.



23 A ball is thrown from 2 m above the ground with a speed of 22 m s^{-1} and hits a 8 m high target 35 m from where the ball was thrown. Find 2 possible angles at which the ball was thrown. Use $g = 9.8$ m s^{-2}.

24 A projectile is launched at a speed of 45 m s^{-1} and an angle of 45°. A second projectile is launched from the same place 2 seconds later with a velocity of 50 m s^{-1} and an angle of 45° How much later will it take the second projectile to land? Use $g = 98$ m s $^{-2}$ and ignore friction.

25 A ball is thrown up with a speed of 15 m s^{-1} at an angle of 60°. At the same time, another ball is thrown backwards towards the first one, at an angle of 45° and a speed of 20 m s^{-1}. The 2 balls are thrown from points 30 m apart. How far apart will they land? Use $g = 10$ m s^{-2}.



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10. TEST YOURSELF

For Questions **1** to **4**, choose the correct answer **A**, **B**, **C** or **D**.

Fomula sheet: Functions, Trigonometry and Vectors

Practice quiz

1 The angle between vectors u and v is 30° and their scalar product is 20. If $v = 4i - 3j$, find the magnitude of u.

A 8 **B** 2 **C** $2\sqrt{3}$ **D** $\frac{8}{\sqrt{3}}$

2 Two vectors u and v are perpendicular (orthogonal) if:

A $u \quad v = -|u||v|$ **B** $u \quad v = 0$

C $u \quad v = |u||v|\cos\theta$ **D** $u \quad v = |u||v|$

3 Given $u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, find $\text{proj}_{\underset{\sim}{v}} u$.

A $\frac{-1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ **B** $\frac{-1}{25}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

C $\frac{-1}{13}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ **D** $\frac{-1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

4 Which statement about vectors $u = -5i + 2j$ and $v = 0.5i - 0.2j$ is true?

A They are parallel in a like direction.
B They are perpendicular.
C They are neither parallel nor perpendicular.
D They are parallel in an unlike direction.

5 In triangle ABC, $\overrightarrow{AB} = x$ and $\overrightarrow{BC} = 5x$. Prove that $\overrightarrow{AC} = 6x$.

6 Prove that:

a $v \cdot u = u \quad v$

b $v \cdot (u + w) = v \cdot u + v \cdot w$

7 The dot product $\underline{u} \quad v = 24$, where $u = \begin{pmatrix} 4 \\ b \end{pmatrix}$ and $v = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$. Find the value of b.

8 Find the angle between the vectors in each pair, correct to the nearest degree.

a $u = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} -4 \\ -9 \end{pmatrix}$

b $u = 4i + 3j$ and $v = 7i - j$

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9 Use vectors to prove Pythagoras' theorem.

10 Find vector b (to 1 decimal place) given $a = 5i + j$, $a \quad b = 12$ and the angle between a and b is 58°.

11 Carlos hits a baseball at a speed of 60 m s^{-1}, striking the ball from 0.9 m off the ground. Through what angles should he hit the ball so that it just clears a 3 m high wall that is 18 m away? Ignore air resistance and take g as 9.8 m s^{-2}.

12 Given vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$, $\overrightarrow{AC}$ is drawn so that C is the midpoint of BD.

Prove that $\overrightarrow{AC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AD})$.

13 Given $a = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, find:

a $\text{proj}_{\underset{\sim}{b}}\, a$

b $\text{proj}_{\underset{\sim}{a}}\, b$

14 **a** Find the scalar product of $u = 7i - 4j$ and $v = -3i - 2j$.

b Find the angle between the vectors.

15 Show that the following pairs of vectors are parallel.

a $u = (-9 \cos 30°, 9 \sin 30°)$ and $v = (-11 \cos 30°, 11 \sin 30°)$

b $u = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

16 Find the scalar product $u \quad v$ given:

a $u = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$ and $v = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$

b $u = \begin{pmatrix} 10 \\ -3 \end{pmatrix}$ and $v = \begin{pmatrix} -5 \\ 8 \end{pmatrix}$

c $u = 5i - 2j$ and $v = i - 9j$

17 A ball is thrown into the air with an initial speed of 22 m s^{-1} and just clears a 2 m fence 15 m away. Find the angle of projection through which the ball is thrown. Use $g = 9.8$ m s^{-2} and ignore friction.

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18 Find the exact dot product of $u = (-6\cos 60°, 6\sin 60°)$ and $v = (-2\cos 45°, -2\sin 45°)$.

19 The vectors $u = 2i - 5j$ and $v = 4i + yj$ are perpendicular. Evaluate y.

20 The vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$ form 3 vertices of a parallelogram. Find a vector that forms the 4th vertex of the parallelogram.

21 A balloon is drifting south at 2.3 m s^{-1}. Find its velocity if it encounters:

a a tail wind of 3.9 m s^{-1}

b a head wind of 1.5 m s^{-1}

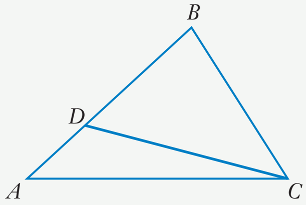
c an easterly wind (wind from the east) of 3.5 m s^{-1}

22 The angle between two vectors $u = 3i - 2j$ and $v = xi + 4j$ is 150°. Evaluate x correct to 2 decimal places.

23 In triangle ABC shown, D is a point on AB where $\overrightarrow{AD} : \overrightarrow{DB} = 2 : 5$.

Given $\overrightarrow{AD} = a$,

$\overrightarrow{AC} = b$ and $\overrightarrow{CB} = c$, show that $c = \dfrac{7a - 2b}{2}$.



24 Given $u = (-8\cos 60°, 8\sin 60°)$ and $v = (2\cos 45°, -2\sin 45°)$, find $\text{proj}_u v$ in exact form.

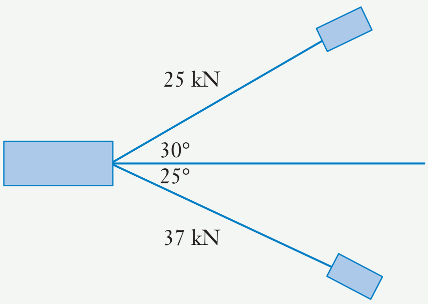
25 Evaluate x if the vectors $\begin{pmatrix} x \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 8 \end{pmatrix}$ are:

a perpendicular

b parallel

26 Mika is rowing at 1.5 m s^{-1} straight across a 560 m wide river. A current of 5.5 m s^{-1} down the river is affecting Mika's progress. What is Mika's speed under the influence of the current?

27 Two tugboats are pulling a cruise ship as shown. Find the magnitude (to 1 decimal place) and the direction (to the nearest degree) of the resultant force.



28 The dot product of $\begin{pmatrix} x \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ is -38. Find the value of x.

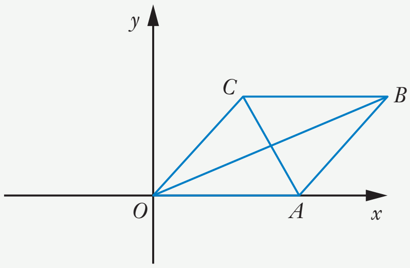
29 Find the value of n if vectors $\overrightarrow{OA} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ n \end{pmatrix}$ are:

a perpendicular

b parallel

30 Find $\text{proj}_u v$ given $u = -9i + 5j$ and $\boldsymbol{v} = -\boldsymbol{i} + 3\boldsymbol{j}$.

31 *OABC* is a parallelogram as shown. If *AC* is perpendicular to *OB*, prove that *OABC* is a rhombus.



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10. CHALLENGE EXERCISE

1 A hot air balloon is drifting east at 53 km h^{-1}. Due to a cross wind, its resultant velocity is 42 km h^{-1} in a 25° direction. What is the speed and direction of the cross wind?

2 If a particle is projected at an angle of 45° with an initial speed of v, show that the range of the particle is 4 times its maximum height (neglecting air resistance).

3 At a school athletics carnival, Carla and Kerry compete in the long jump. Carla jumps at an angle of 30° and with an initial speed of 7.6 m s^{-1}. Kerry jumps at an angle of 35° and with a speed of 6.7 m s^{-1}. Who jumps further, and by how far (to 1 decimal place)? Neglect air resistance and use $g = 9.8$ m s^{-2}.

4 Show that the sum of vectors in regular hexagon $ABCDEF$ is 0, going in a clockwise direction.

5 Vector $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ has magnitude 12 and direction 240°. Find the exact values of a and b.

6 Prove that the diagonals in a rhombus bisect the angles of the rhombus.

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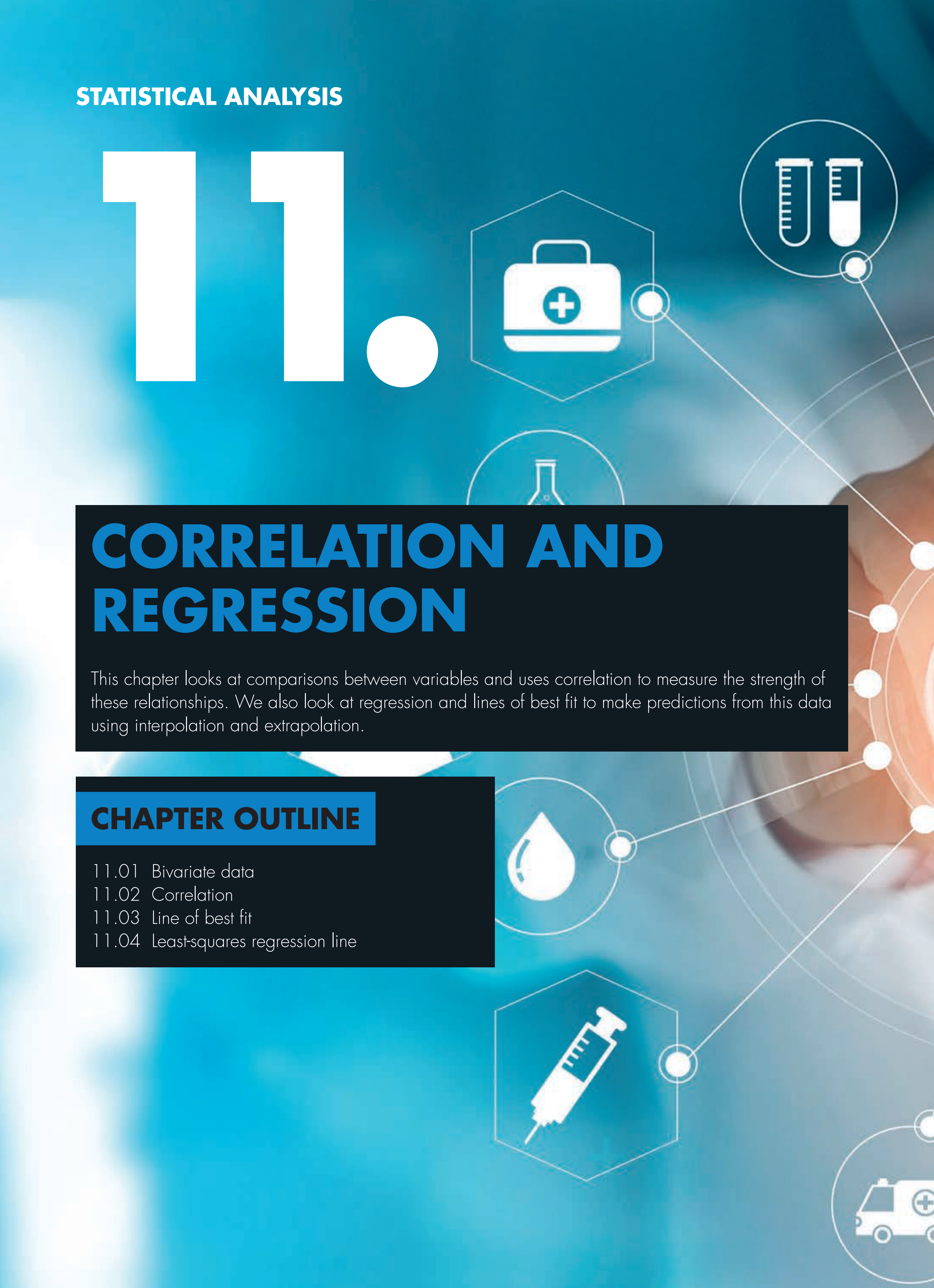

STATISTICAL ANALYSIS

11.

CORRELATION AND REGRESSION

This chapter looks at comparisons between variables and uses correlation to measure the strength of these relationships. We also look at regression and lines of best fit to make predictions from this data using interpolation and extrapolation.

CHAPTER OUTLINE

11.01 Bivariate data
11.02 Correlation
11.03 Line of best fit
11.04 Least-squares regression line

IN THIS CHAPTER YOU WILL:

- interpret scatterplots of bivariate data
- look for correlation in bivariate data and calculate Pearson's correlation coefficient
- apply lines of best fit, including the least-squares regression line
- interpolate and extrapolate from data

TERMINOLOGY

bivariate data: Data relating to 2 variables that have been measured from the same data set.

extrapolation: Making predictions from a model using values outside the range of the original data set.

interpolation: Making predictions from a model using values lying within the range of the original data set.

least-squares regression line: A line of best fit where the squares of the distances from each point in the scatterplot to the line are minimised.

line of best fit: A line drawn through a scatterplot that best models the relationship between 2 variables in bivariate data.

Pearson's correlation coefficient: A calculated value, r, that measures how closely 2 variables are related in a linear relationship. Its value is always between −1 and 1.

scatterplot: A graph showing the value of 2 variables in a bivariate data set.

11.01 Bivariate data

WS Scaeplo

WS A page o scatterplots

WS Body measurements

In Chapter 9, *Statistics*, we looked at data with one variable.

Bivariate data measures 2 variables on the same data set to see if they correlate with (are related to) each other. For example, we might want to see if a person's level of education correlates with the amount of money the person earns.

We draw **scatterplots** to graph bivariate data.

EXAMPLE 1

This table shows the bivariate data for the number of days each year the temperature in a city was over 30°C and the number of people admitted to the local hospital for heat stroke.

Year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Number of days over 30°C	14	17	15	23	21	26	31	33	29	38
Number of patients	64	75	68	72	71	76	83	77	83	92

Draw a scatterplot for this data.

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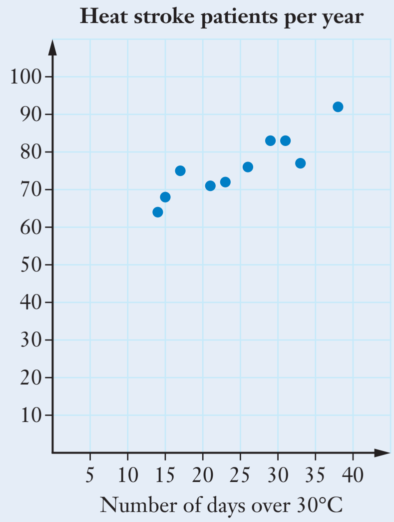
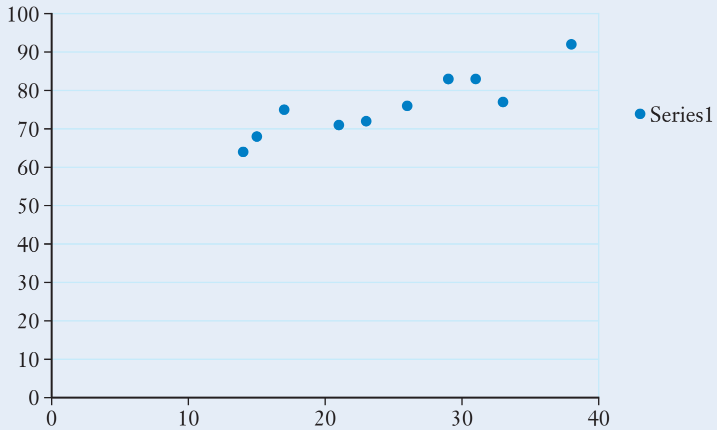
Solution

We draw the graph with number of days over 30°C on the horizontal axis and number of patients on the vertical axis.



You could also put the data from the table into a spreadsheet and choose the scatterplot chart.



We look for patterns in the scatterplot to see if the 2 variables are related. For instance, in the example above there seems to be a linear pattern (it is roughly a straight-line graph).

Not all relationships are linear. We can have scatterplots that have curves or non-linear shapes, or no shape at all.

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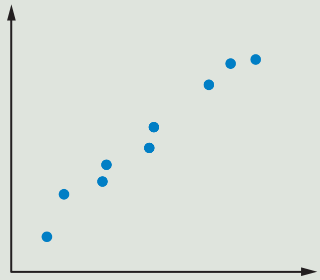
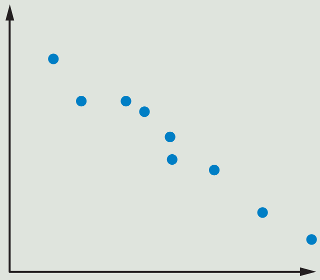
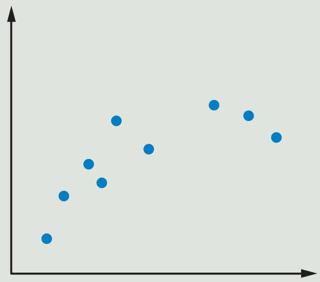
Shape of scatterplots



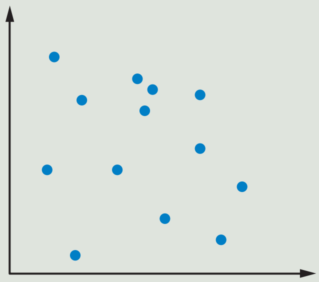
Linear (positive)



Linear (negative)



Non-linear

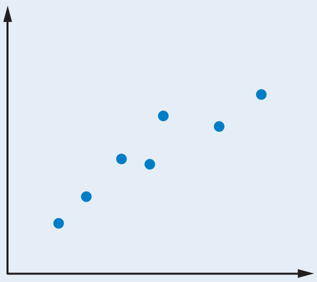


No pattern

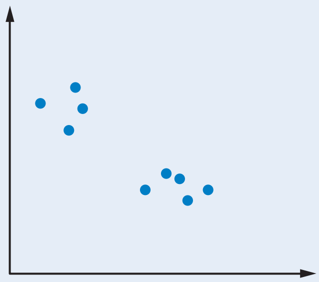
EXAMPLE 2

Describe the shape of each scatterplot.

a



b



Solution

a This data is linear (positive).

b This data is linear (negative).

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Exercise 11.01 Bivariate data

1 **Biometric data** is bivariate data drawn from body measurements such as arm length or height. This table shows the heights and weights of 10 people surveyed.

Height (m)	161.8	175.3	159.5	182.3	166.4	167.9	186.4	164.7	154.8	171.2
Weight (kg)	59.4	73.7	55.3	74.8	63.5	68.2	73.5	62.1	49.9	83.6

a Draw a scatterplot to show this data.

b Describe the shape of the scatterplot.

c Describe the relationship between height and weight (if possible).

2 A group of 8 people were surveyed at random about how many people in their family lived at home. They were also asked about the number of bedrooms in their home.

Number of people in family	4	3	6	7	2	1	5	8
Number of bedrooms	3	2	5	4	2	3	4	6

a Draw a scatterplot showing the data.

b Describe the shape of the scatterplot.

c Describe the relationship (if any) between the number of people in the family and the number of bedrooms in the home.

3 A survey of 10 people measured their IQ with the amount they earned each week.

IQ	114	127	95	130	123	141	136	83	109	96
Amount earned per week ($)	689	945	510	874	751	769	553	350	1250	884

a Draw a scatterplot for this data.

b Describe the shape of the scatterplot.

c Describe any relationship between IQ and amount earned.

4 The table shows the amount of sleep students have and their exam results.

Hours of sleep	6.5	9	7.5	6	7	10	11.5	4	8	8.5
Exam results (%)	87	76	43	87	69	55	60	78	94	72

a Draw a scatterplot of this data.

b Describe the shape of the scatterplot.

c What is the relationship between the amount of sleep and exam results?

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5 The table shows the amount of study time and exam results for some students.

Hours of study/week	15	26	12	17	5	10	2	8	20	3
Exam results (%)	78	86	70	80	63	67	43	77	92	58

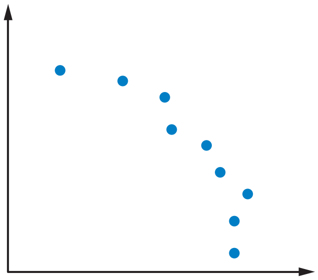
a Draw a scatterplot of the data.

b Describe the form of the scatterplot.

c What is the relationship between the amount of study and the exam results?

6 Describe the pattern of each set of bivariate data.

a



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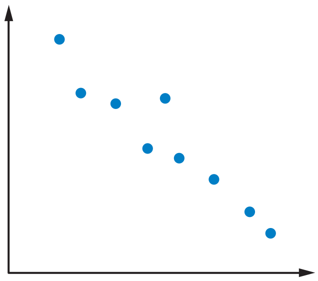

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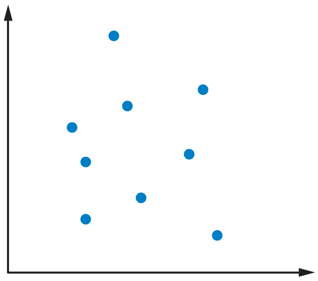

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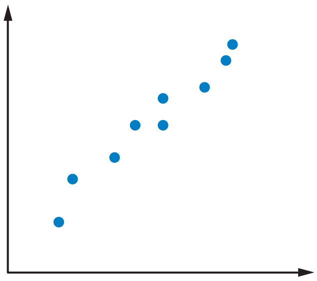

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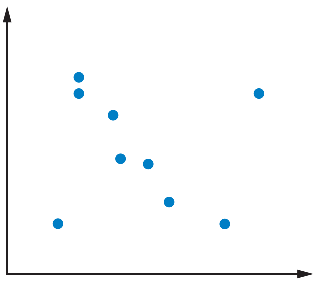

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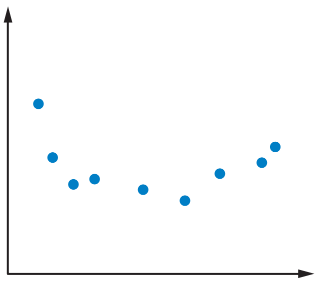

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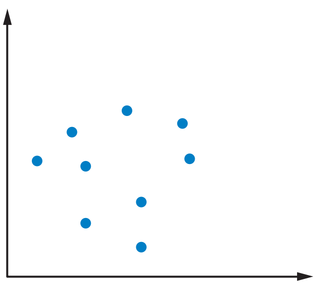

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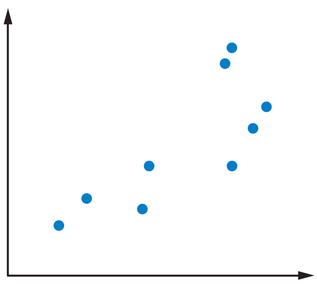

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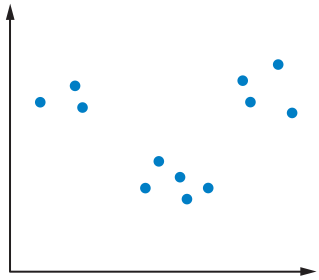
i



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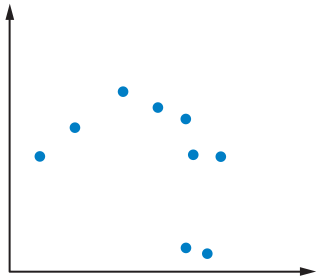


7 **a** Draw a scatterplot for this set of bivariate data.

x	8	3	7	10	2	9	4	5	9	1
y	2	7	1	2	8	4	5	7	3	7

b Describe the shape of the data.

8 State whether each scatterplot has:

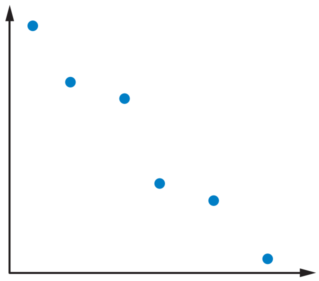
A a positive linear relationship

B a negative linear relationship

C a non-linear relationship

D little or no relationship

a



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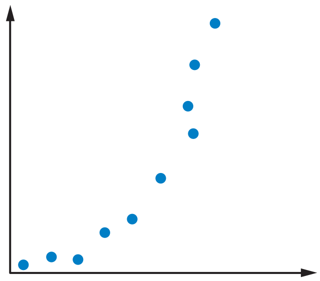

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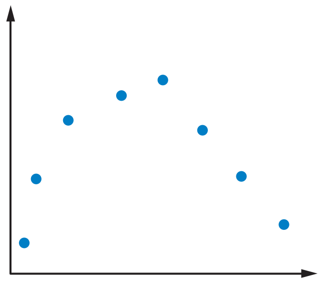

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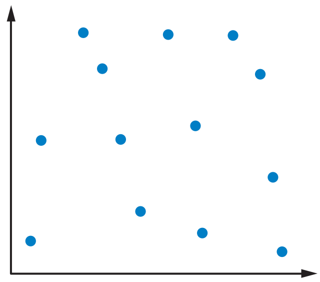

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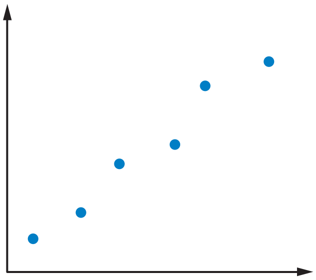
e



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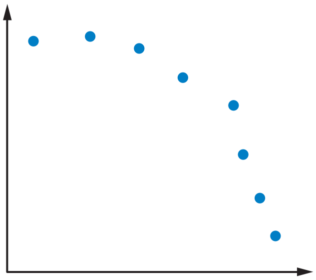

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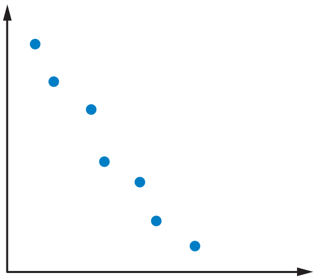

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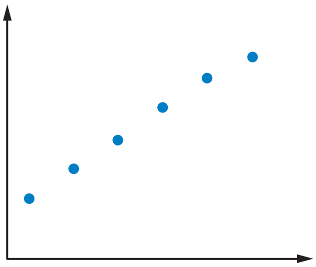

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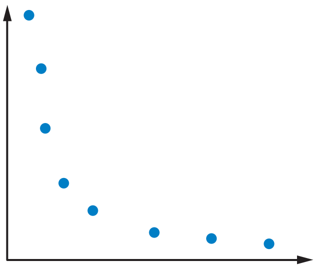

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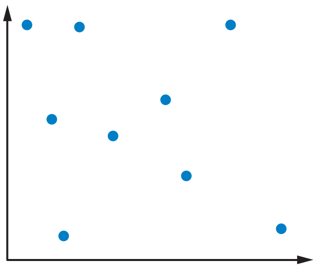


11.02 Correlation

WS

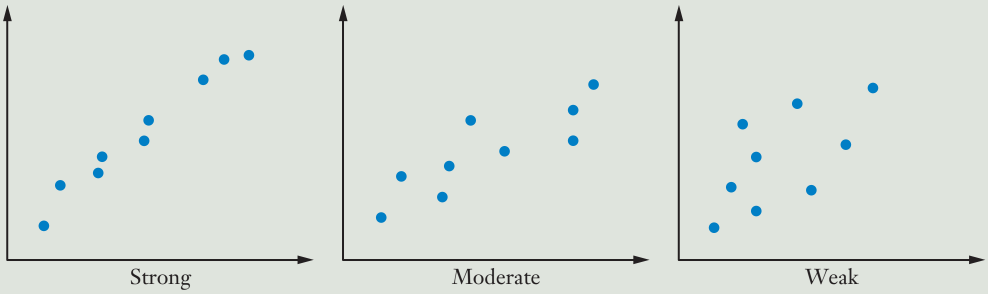
Coelaion

Coelaion matching game

Coelaion

Correlation measures how well 2 variables are related if there seems to be a linear relationship between them. The relationship could be strong, moderate or weak.

Linear scatterplots



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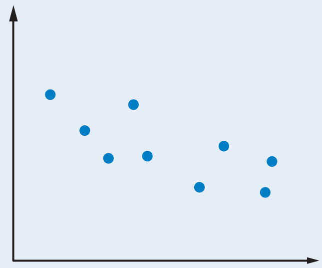
EXAMPLE 3

Describe the strength of the linear pattern in each scatterplot as strong, moderate or weak.

a



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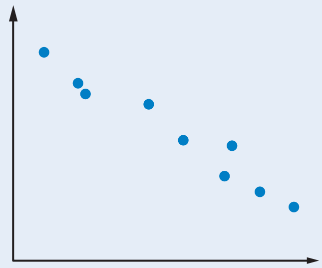
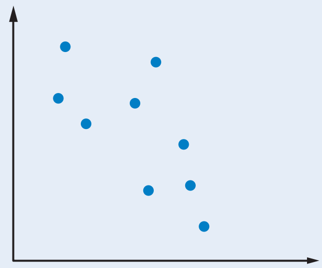
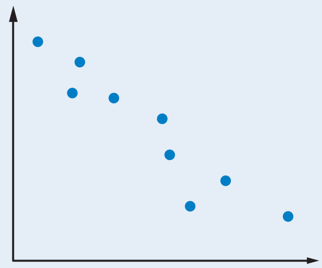


c



d



Solution

a The linear pattern is moderate.

b The linear pattern is strong.

c The linear pattern is weak.

d The linear pattern is strong.

Pearson's correlation coefficient

The terms strong, moderate or weak are very general and not very accurate. We use a measurement (r) called the **Pearsn's correlation coefficient** to determine how closely related variables are in a linear relationship.

The formula for r is complex, but you can use a calculator or spreadsheet to calculate it.

The correlation coefficient always lies between –1 and 1.

Correlation coefficient

- $-1 \leq r \leq 1$ for all correlation coefficients
- $0 < r \leq 1$ for a scatterplot with positive direction, where 1 is perfect positive correlation
- $-1 \leq r < 0$ for a scatterplot with negative direction, where –1 is perfect negative correlation
- $r = 0$ means no correlation

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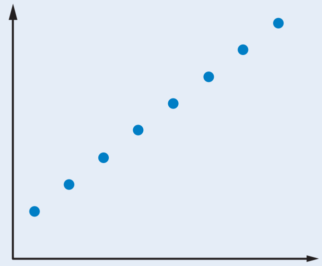
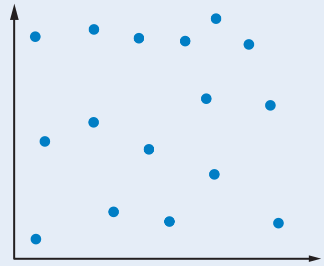
EXAMPLE 4

Match each scatterplot with its correct correlation coefficient.

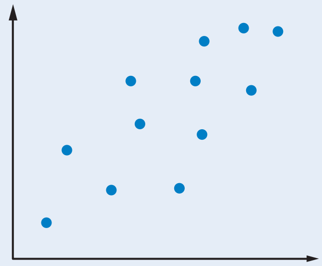
a



b



c



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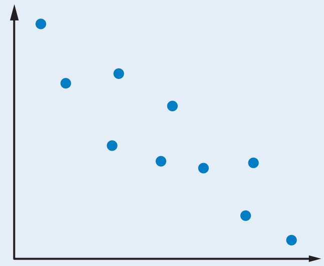
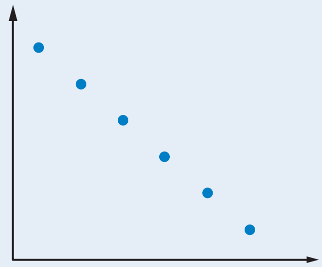


e



A 0.5 **B** 1 **C** -1 **D** -0.4 **E** 0

Solution

a B **b** E **c** A **d** D **e** C

DID YOU KNOW?

Karl Pearson

Karl Pearson (1857–1936), an Englishman, developed the formula for the correlation coefficient r. The coefficient's full name is **Pearson's product moment correlation coefficient**. Karl Pearson was a mathematician and statistician, but he also studied history, law and German literature.

Research Karl Pearson to find out more about his life and studies.

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EXAMPLE 5

A group of students was surveyed for the number of hours that they studied and their result in a maths exam. Find the correlation coefficient for this bivariate data.

Number of hours studied	6	5	12	8	15	9	14
Exam results (%)	71	46	74	67	76	77	83

Solution

Operation	Casio Scientific	Sharp Scientific
Place your calculator in statistical mode.	MODE 2 STAT2: A +BX	MODE 1 STAT 1 LINE
Clear the statistical memory.	SHIFT 1 3 : Edit 2 : Del-A	2ndF DEL
Enter data.	SHIFT 1 2 Data 6 = 5 = etc for 1st column 71 = 46 = etc for 2nd column AC	6 2ndF STO 71 M+ 5 2ndF STO 46 M+ etc.
Calculate r.	SHIFT 1 5 : Reg 3 : r =	ALPHA r =
Change back to normal mode.	MODE 1 : COMP	MODE 0

$r = 0.738$ (correct to 3 decimal places)

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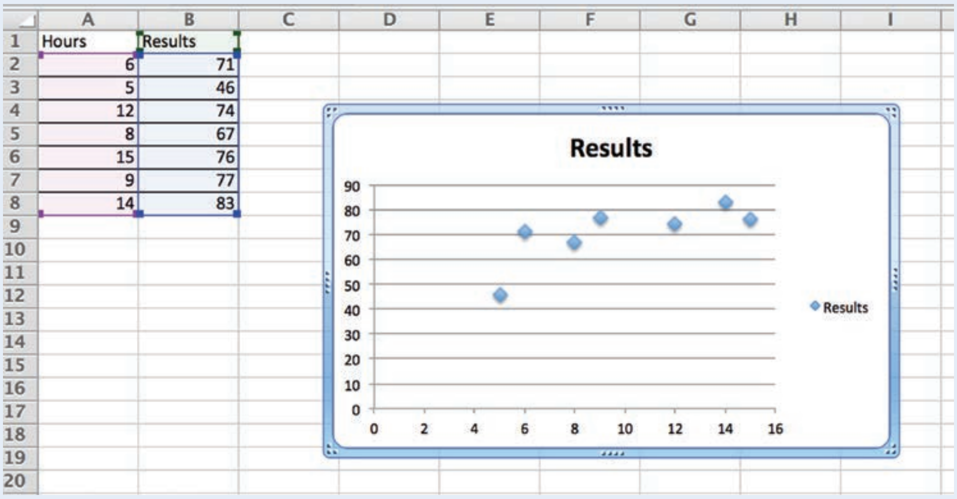
EXAMPLE 6

a Enter the data from Example 5 above into a spreadsheet and draw a scatterplot.

b Use the spreadsheet to find the Pearson's correlation coefficient for the data.

Solution

a Put the values from the table into a spreadsheet and select scatterplot from the charts.



b In an empty cell, type =**PEARSON(A2:A8,B2:B8)**

$r = 0.738$ (to 3 decimal places)

Causality

Two variables can have a high correlation without one *causing* the other. For example, does a person's height cause them to weigh more? Possibly, since most tall people would have heavier, longer bones. However, there are other causes for higher weight that aren't related to height.

EXAMPLE 7

For each set of bivariate data, find which ones have a causal relationship.

a Number of people in a family and the number of TVs

b Speed of a boat and time taken to travel across a lake

Solution

a Not causal: the number of people in a family doesn't determine how many TVs there are.

b Causal: The speed of a boat will determine the time taken to travel across a lake (higher speed means less time).

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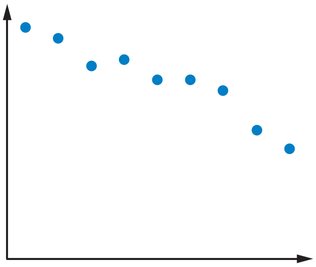
Exercise 11.02 Correlation

1 Match each graph with the correct correlation coefficient.

a



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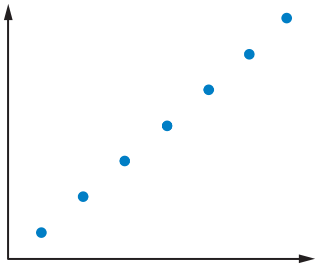

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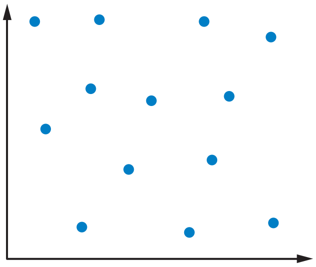

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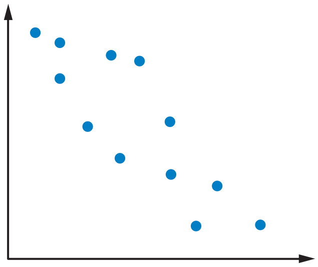

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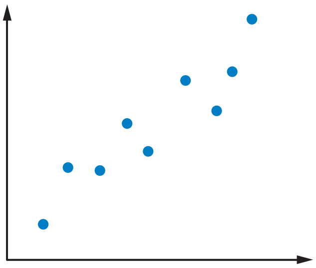

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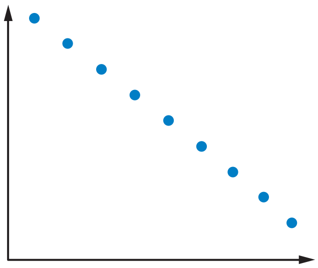

A −0.5 **B** 1 **C** 0.8 **D** −1

E −0.9 **F** 0 **G** −0.7

2 For each table of values:

i draw a scatterplot

ii find the correlation coefficient

a

Height (m)	1.72	1.85	1.61	1.74	1.59	1.79
Weight (kg)	91.3	85.2	58.3	61.9	74.5	102.6

b

Height (cm)	167	180	174	171	154	190
Speed (m s^{-1})	3.7	2.1	3.4	2.2	2.8	4.1

c

Study time (h)	13	21	8	11	18	17
Results (%)	45	89	81	67	74	53

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d

Temperature (°C)	15	18	21	24	26	30	35
Attendance at beach	28	19	54	88	190	245	108

e

Forest cleared (ha)	41	58	87	99	132	168
Number of birds	1200	854	530	201	157	92

f

Number of cars in car park	1100	1450	1809	2004	2234	2569	2871	2906
Pollution (ppm)	1.21	1.54	1.78	2.34	2.99	3.35	4.76	5.97

g

Age	15	19	27	34	49	57
Annual income ($)	2 851	12 600	27 890	38 740	41 834	29 450

h

Exercise (h/week)	14	8	2	10	6	4	32
Weight (kg)	51.8	87.2	74.8	68.4	62.1	63.9	58.9

i

Height (m)	1.59	1.77	1.64	1.78	1.89	1.42
Shoe size	5	7	6	10	9.5	4

j

Exam results (%)	68	92	38	51	77	84
Hours of sleep	7	6	8	6.5	9	7.5

3 For the bivariate data in question 2, which do you think have causality?

4 Find the correlation coefficient of each set of data, correct to 2 decimal places.

a

x	y
3	7
5	9
4	3
11	7
15	12
8	4
9	1

b

x	y
5	67
6	49
3	81
9	23
11	55
8	91
4	61

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c

x	y
8	11
4	8
7	11
2	4
9	12
14	16
23	23

d

x	y
5	21
3	28
6	19
5	17
9	21
4	26
11	15
15	18
9	12

5 Determine whether each pair of variables are likely to have a causal relationship.

a Population of a city and pollution

b Head circumference and weight

c Hours training and fitness

d Weight and health

e Size of house and number of pets

f Size of house and selling price

INVESTIGATION

CAUSALITY

Discuss whether each pair of variables have a high correlation and, if they do, whether one variable causes the other.

1 A person's height and shoe size

2 A person's smoking and lung cancer

3 Amount of study and success in an exam

4 Mathematical and musical ability

5 The number of people at a party and the amount of food and drink consumed

6 The amount of time sunbaking and the incidence of skin cancer

7 The amount of time practising basketball and the number of baskets scored in a game

8 Results in English and Maths exams

9 The length of a person's leg and their walking speed

10 The temperature and the number of people swimming at the beach

Discuss other relationships between variables. Can you find other examples of highly correlated variables where one causes the other? Can you find examples of highly correlated variables where there is no causality?

Discuss causality in each situation described below.

1 The time a sales representative has been with a company and the number of sales gives a correlation coefficient of –0.6.

2 Height of basketball players and number of baskets scored have a correlation coefficient of 0.87.

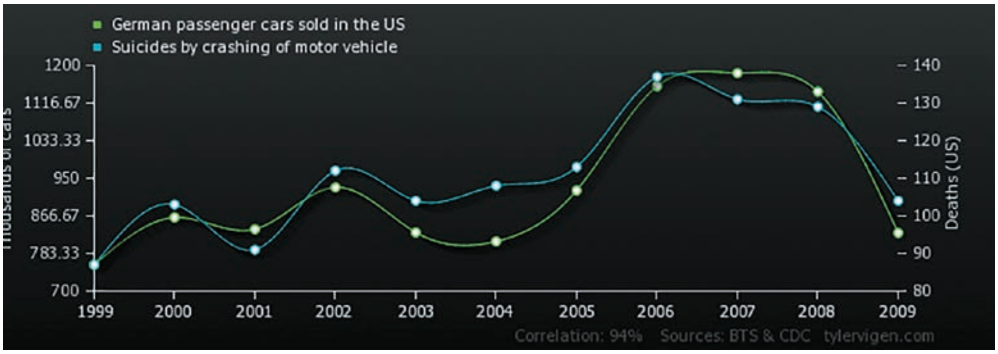
3 Height and self-esteem have a correlation coefficient of 0.32.

4 Temperature and growth of grass have a correlation coefficient of –0.75.

5 Number of hours study and results in the HSC have a correlation coefficient of 0.85.

Collect data from the Internet, newspapers or magazines, or do your own experiments to compare two sets of data. Draw a scatterplot and find the correlation coefficient, then, if there is a high correlation investigate causality. Is the correlation positive or negative? Is it linear?

Here is an example of a high correlation of totally unrelated variables: German cars sold vs suicides by car crashes in the US each year.



http://tylervigen.com/page?page=2

There are many other examples of correlated data that have no causality.

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11.03 Line of best fit

Statistical data is rarely perfect, but we can often see trends in a scatterplot. If there seems to be a linear correlation, we can draw a **regression line** and find its equation. We can then use this line to make predictions.

The easiest regression line to find is the **line of best fit**.

Using a ruler, we draw the line that represents as many points as possible. We try to draw a line where about half the points are above the line and half are below it, so that the distance between the line and the points is kept to a minimum.

WS Lines o bes fit

WS Lines o i

WS Heigh v shoe size

EXAMPLE 8

A ball is rolled down a ramp and its velocity is measured over time.

t (s)	1	2	3	4	5	6
v (m s^{-1})	0.7	1.5	3.6	6.1	7.8	9.9

a Draw a scatterplot of the data and draw a line of best fit.

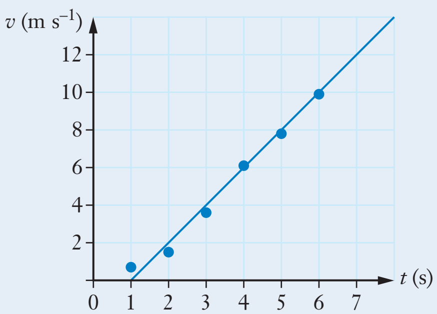
b Use the line of best fit to find the velocity after 3.5 seconds.

c Find the equation of this line.

d Use the equation to find the velocity after 10 seconds.

Solution

a



b After 3.5 seconds, the velocity is 5 m s^{-1}.

c Choose 2 points on the line, say, (2, 2) and (4, 6).

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{4-2} = \frac{4}{2} = 2$$

Equation: $y - y_1 = m(x - x_1)$

$v - 2 = 2(t - 2)$ using $m = 2$ and (2, 2)

$= 2t - 4$

$v = 2t - 2$

d When $t = 10$:

$v = 2(10) - 2$

$= 18$

So after 10 seconds, the velocity is 18 m s^{-1}.

Interpolation and extrapolation

Interpolation is using a model to make predictions about values lying **within** the range of the original data set.

Extrapolation is using a model to make predictions about values **outside** the range of the original data set.

In the above example, finding v when $t = 3.5$ is interpolation while finding v when $t = 10$ is extrapolation.

DID YOU KNOW?

Regression

The word **regression** comes from the Latin *regressio*, meaning 'a return'.

Sir Francis Galton (1822–1911) was the first to use this name. He created the concepts of correlation and regression, and used statistics to develop questionnaires and surveys on human differences.

Research correlation, regression and Sir Francis Galton.

CLASS DISCUSSION

EXTRAPOLATION

Extrapolation is not always accurate. For example, a formula may work well at first, then the conditions may change in a way that means the formula is no longer a good model.

Think of examples of bivariate data from the previous examples. Can you always extrapolate answers from a line of best fit? Why? What are the risks?

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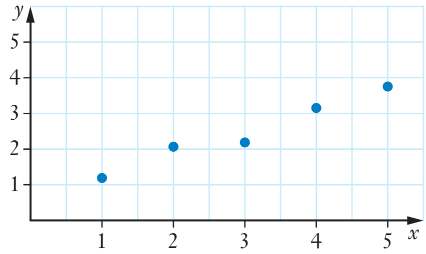
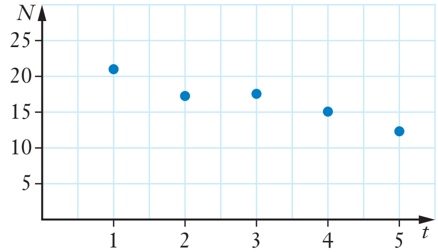
Exercise 11.03 Line of best fit

1 Copy each scatterplot, draw a line of best fit and find its equation.

a



b



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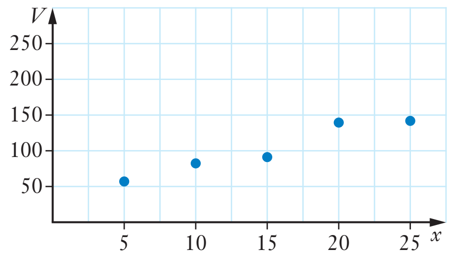

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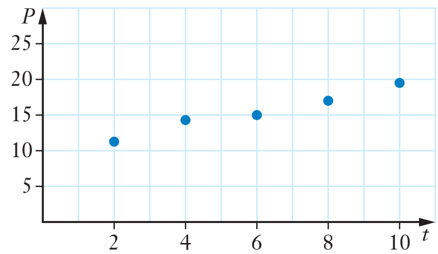

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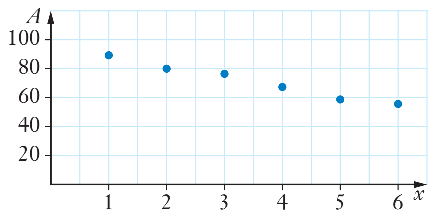

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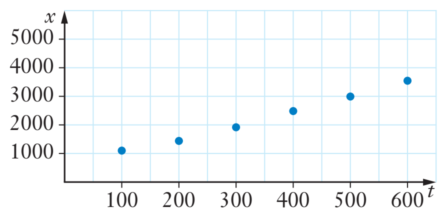

2 The following table shows the results of an experiment testing the temperature of a liquid as it cools down.

t **(min)**	5	10	15	20	25	30
T **(°C)**	87	78	69	56	53	41

a Plot this data on a number plane and draw a line of best fit.

b Find the equation of the line.

c Use the equation to find the temperature after:

i 17 minutes **ii** 35 minutes

d If room temperature is 23°C, is the line of best fit a good model for the cooling of the liquid? Why?

3 The population of birds in a particular area was sampled over several years.

t (years)	1	2	3	4	5	6
P	1030	983	968	954	915	899

a Plot this data on a number plane and draw a line of best fit.
b Find the equation of the line.
c Use the equation to find the population of birds after 7 years.
d At this rate of decline, after how many years would you expect there to be no more birds in this area?

4 The effect of a dose of medicine on a child's temperature over time was measured from a sample of children, with the following results.

t (min)	2	4	6	8	10
T (°C)	39.5	39.1	38.9	38.2	37.7

a Plot this data on a number plane and draw a line of best fit.
b Find the equation of the line.
c Use the equation to find the child's temperature after 15 minutes.
d Is this equation reliable as a measure of temperature after a longer time, say, 1 hour?

5 This table shows the results of a survey into the number of people who attend a new restaurant over a number of weeks.

t (weeks)	1	2	3	4	5	6
No. of people	76	114	163	187	228	274

a Draw a scatterplot and sketch a line of best fit.
b Find the equation of this line.
c Use the equation to find the number of people you would expect to attend the restaurant after 10 weeks.
d Is this equation a good model for the number of people attending the restaurant?

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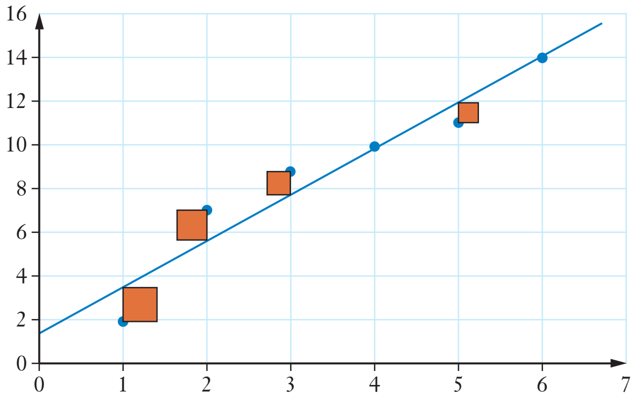
11.04 Least-squares regression line

The line of best fit relies on our eyes and ruler for its accuracy.

There are several different models of regression lines that try to give a more accurate result. The most popular model is called the **least-squares regression line**. It uses a line of best fit for which the squares of the distances from each point in the scatterplot to the line are minimised (see the diagram below). Squaring the distances takes away any negative values (a similar technique to finding standard deviation).



The equation of the least-squares regression line is given by $y = mx + c$ where $m = r\frac{s_y}{s_x}$ and $c = \bar{y} - m\bar{x}$, where r = correlation coefficient, $\bar{x}$ and $\bar{y}$ are the sample means and s_x and s_y are the sample standard deviations.

However, you don't need to use these formulas because the regression line can be found using a scientific calculator, graphics calculator, online calculator or spreadsheet.

EXAMPLE 9

The table shows the results of a survey into the number of cigarettes people smoke during the year and the number of days they are absent from work.

Cigarettes/year	2000	3000	4000	5000	6000	7000	8000	9000	10 000
Absences	23	27	54	49	63	81	107	128	147

Find the least-squares regression line by using:

a a calculator

b a spreadsheet

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Solution

a

Operation	Casio scientific	Sharp scientific
Enter data	SHIFT 1 2 Data 2000 = 3000 = etc for 1st column 23 = 27 = etc for 2nd column AC	2000 2ndF STO 23 M+ 3000 2ndF STO 27 M+ etc.
Calculate *a*.	SHIFT 1 5 Reg 1 A =	ALPHA a =
Calculate *b*.	SHIFT 1 5 Reg 2 B =	ALPHA b =

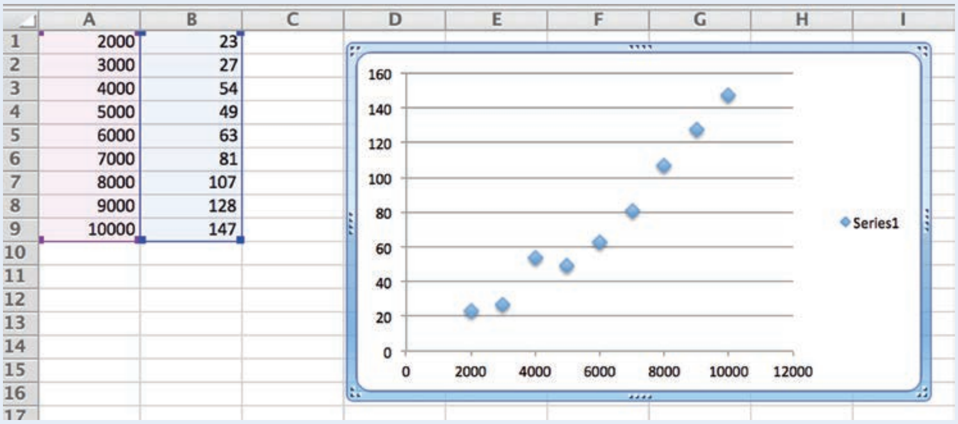
$a = -18.26$

$b = 0.0156$

On a scientific calculator, the equation is in the form $y = a + bx$ or $y = bx + a$

So $y = 0.0156x - 18.26$

b Enter the 2 columns in a spreadsheet and draw a scatterplot.



In Chart Layout select Trendline, then Linear trendline. You can display the equation of the line by going to Trendline again, selecting Trendline options and selecting Display equation on chart.

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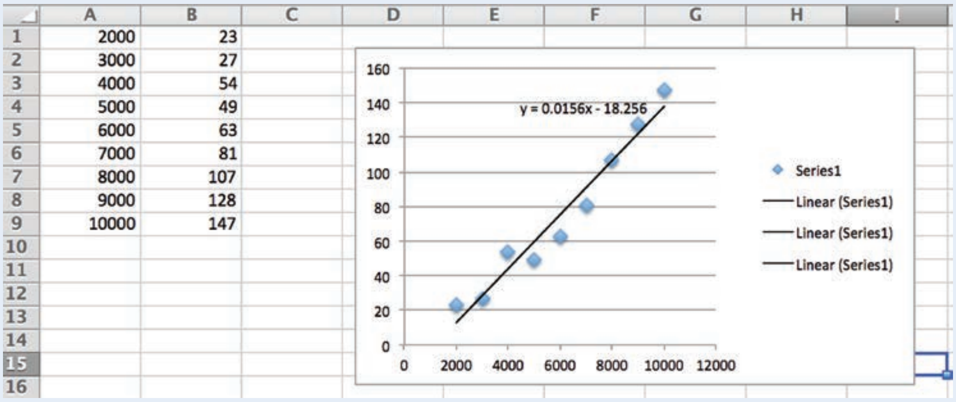

The equation of the least-squares regression line is $y = 0.0156x - 18.256$.

Exercise 11.04 Least-squares regression line

1 For each data set, find:

i the correlation coefficient

ii the gradient of the least-squares regression line

a

x	1	2	3	4	5	6	7
y	3	4	7	10	11	15	17

b

x	2	4	6	8	10	12	14	16
y	8	11	19	29	34	41	45	67

2 For each data set:

i find the equation of the least-squares regression line

ii sketch the scatterplot and regression line on the same axes

a Age of machine and breakdown rates

Age (years)	1	2	3	4	5
Breakdowns	0	2	5	9	15

b Length of time in office and popularity of a political party

Time (years)	1	2	3	4	5	6
Popularity (%)	52.3	43.8	43.7	42.1	37.9	37.6

c Length of drought and yield of crops

Time (years)	1	2	3	4	5	6
Yield (t)	107.3	101.8	100.2	87.6	63.5	47.1

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d Engine size of cars and number of accidents

Size (L)	1.3	1.8	2.0	2.1	2.4	3.8
Accidents	459	447	513	519	506	625

e Age and number wearing glasses

Age	10	20	30	40	50	60	70	80
Glasses	34	28	41	56	87	105	156	209

3 A block of ice was taken out of a freezer and left to thaw. The results are in the table below.

Time t (min)	5	10	15	20	25	30	35	40
Mass m (kg)	23.7	18.8	11.3	8.7	6.2	5.5	2.3	1.5

a Find the correlation coefficient. Is there a high correlation? Is it positive or negative? What does this mean?

b Find the equation of the least-squares regression line.

c Use the equation to estimate the mass of the ice after:

i 18 minutes **ii** an hour

d Discuss why extrapolation may not be useful in this situation.

4 This table shows the weight of gemstones sold at an auction and their selling price.

Weight (carat)	0.05	0.8	1.5	1.7	2.5	2.8	3.1	4.0
Price ($)	144	672	1245	1478	2100	2500	2881	3215

a Draw a scatterplot for this data.

b Find the correlation coefficient. Is there a high correlation between the weight of a gemstone and its cost? Is it positive or negative?

c Find the equation of the least-squares regression line.

d How much would you expect to pay for a 2 carat gemstone?

e How much would you expect a gemstone to weigh if it cost $10 000?

5 The table shows the results of a survey into ages and earnings of a group of people.

Age	15	32	19	28	43	67
Earnings/week ($)	689	1205	840	1154	1587	986

a Draw a scatterplot for this data.

b Find the correlation coefficient.

c Find the equation of the least-squares regression line.

d From this equation find the earnings of a 50-year-old.

e Is this equation a good model to extrapolate?

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11. TEST YOURSELF

For Questions **1** to **5**, select the correct answer **A**, **B**, **C** or **D**.

Practice quiz

Fiing he data find-a-word

1 Which variables are not correlated?

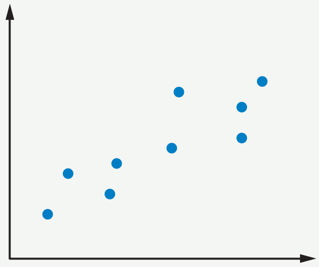
A Hand size and height

B Hours of study and exam results

C Height and hours of employment

D Speed of car and fuel economy

2 Describe the correlation in the scatterplot shown.



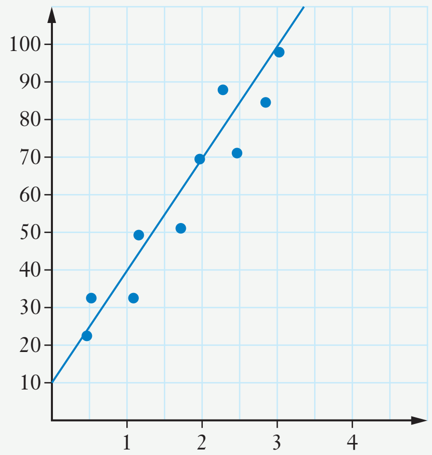
A Weak negative correlation

B Moderate positive correlation

C Moderate negative correlation

D Strong positive correlation

3 Find the equation of the line of best fit.



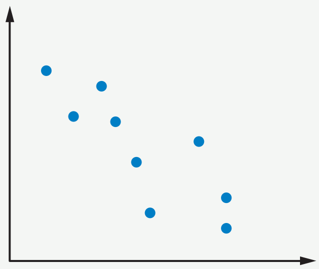
A $y = 60x + 10$

B $y = 10x + 30$

C $y = 30x - 10$

D $y = 30x + 10$

4 Estimate the correlation coefficient of this bivariate data.



A –0.5

B 0.5

C –1

D 1

5 Using the equation of a line of best fit to predict the value of a variable within the domain of the data set is called:

A extrapolation **B** causality **C** interpolation **D** correlation

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6 Make a scatterplot of this table of bivariate data, then:

x	1	2	3	4	5	6	7	8	9
y	17	21	24	29	36	43	44	52	58

a draw a line of best fit and find its equation

b draw a least-squares regression line and find its equation.

7 For each scatterplot, state whether it has:

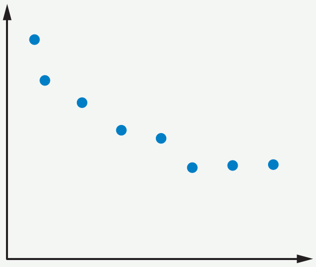
A a positive linear correlation

B a negative linear correlation

C a non-linear correlation

D little or no correlation

a



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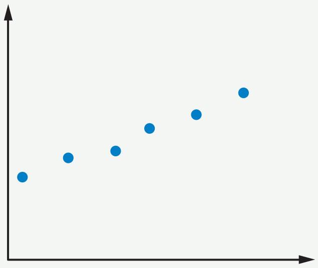

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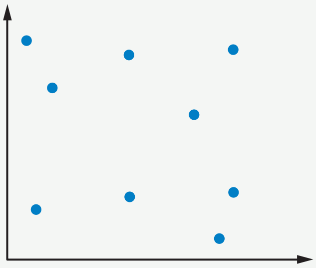

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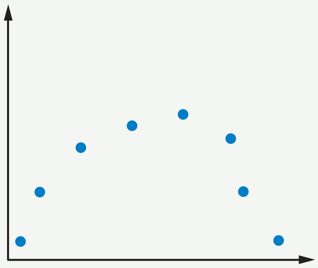

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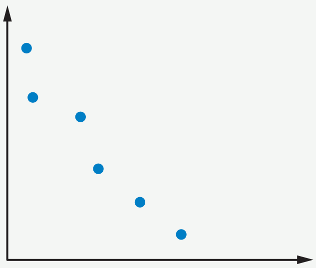

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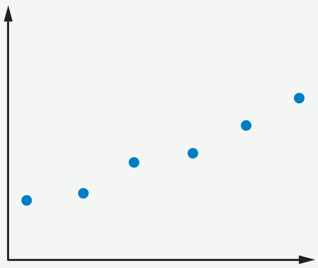

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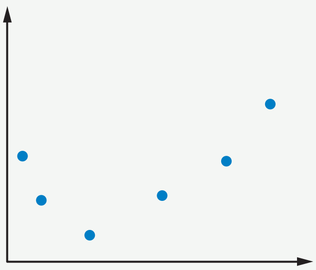

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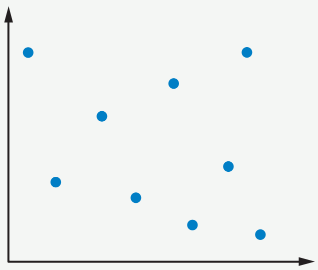

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8 A group of students was surveyed to find whether there was a correlation between a student's music and maths assessment marks. The results are in the table below.

Music	79	58	91	93	65	43	39	64	82	51
Maths	62	63	82	79	73	57	29	52	76	40

a Find the correlation coefficient.

b Find the equation of the least-squares regression line.

c Using the equation, find the maths assessment mark for a student who scores 60 in music.

d Find the music assessment mark for a student who scores 70 in maths.

9 Determine whether each pair of variables are likely to have a causal relationship.

a Height and amount of food eaten

b Number of years playing sport and weight

c Height and arm length

d Number of chickens and number of eggs

e Size of bookshelves and number of books

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11. CHALLENGE EXERCISE

1 The equation of the least-squares regression line is given by $y = mx + c$ where $m = r\dfrac{s_y}{s_x}$ and $c = \overline{y} - m\overline{x}$, where r = correlation coefficient, $\overline{x}$ and $\overline{y}$ are the sample means and s_x and s_y are the sample standard deviations. Evaluate r and $\overline{y}$ given the equation of the least-squares regression line is $y = 2x + 4$, $\overline{x} = 1.2$, $s_x = 1.8$ and $s_y = 4.5$.

2 Sketch a scatterplot that shows a linear correlation of:

a –1 **b** approximately 0.8 **c** 0

d approximately –0.2 **e** 1

3 The table below shows the results of an experiment into the volume of water evaporating from a body of water at different temperatures.

T **(°C)**	10	15	20	25	30	35	40	45
V **(L)**	0.5	1.3	2.9	5.8	10.3	15.7	39.8	76.1

a Draw a scatterplot to show this data.

b Why would a least-squares regression line not give a good approximation for this data?

c Use technology or otherwise to find an equation that might approximately model this data.

4 The table shows heights and shoe sizes of several males.

Shoe size	4	5	6	7	8	9	10	11
Height (m)	1.54	1.65	1.68	1.73	1.59	1.82	1.89	1.95

a Draw a scatterplot to show this data.

b Find the correlation coefficient.

c Find the equation of the least-squares regression line.

d The male in the sample with shoe size 8 is found to be an outlier for this data. For the sample without this outlier, find:

i the correlation coefficient

ii the equation of the least-squares regression line

e Is this equation a good model for shoe sizes and height?

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Practice set 3

In Questions **1** to **9**, select the correct answer **A**, **B**, **C** or **D**.

1 Find the median of 3, 10, 1, 4, 9, 6.

A 5 **B** 1 **C** 4 **D** 6

2 Which pair of variables are not correlated?

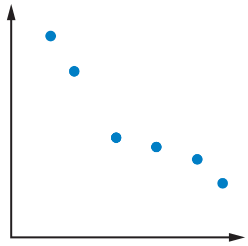
A Size of house and size of family
B Height and foot size
C Number of babies born and number of nappies used
D Distance travelled and average speed over 2 hours

3 EXT1 The angle between vectors u and v is 60° and their scalar product is 24. If $v = 4i + 3j$, find the magnitude of u.

A 2.4 **B** $5\sqrt{3}$ **C** $16\sqrt{3}$ **D** 9.6

4 Describe the correlation in the scatterplot.

A Strong positive linear correlation
B Moderate positive linear correlation
C Moderate negative linear correlation
D Strong negative linear correlation

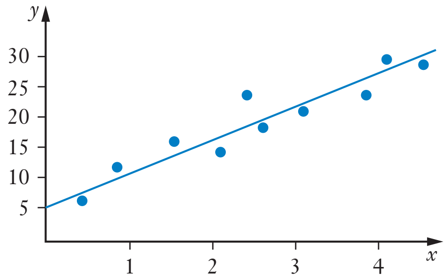


5 EXT1 Two vectors u and v are parallel if (there is more than one answer):

A $u \quad v = -|u||v|$ **B** $u \quad v = 0$
C $u \quad v = |u||v| \cos\theta$ **D** $u \quad v = |u||v|$

6 The equation of this line of best fit is closest to:

A $y = 10x + 5$
B $y = 5x - 5$
C $y = 10x - 5$
D $y = 5x + 5$



7 EXT1 Given $u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$, find $\text{proj}_{v} u$.

A $\begin{pmatrix} -9\ 6 \\ -7\ 2 \end{pmatrix}$ **B** $\begin{pmatrix} -2\ 8 \\ -1\ 8 \end{pmatrix}$ **C** $\begin{pmatrix} 096 \\ -072 \end{pmatrix}$ **D** $\begin{pmatrix} 023 \\ -031 \end{pmatrix}$

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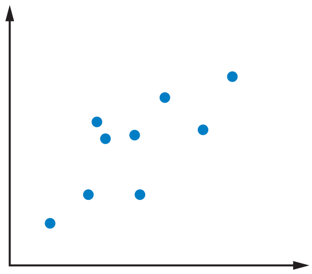
8 The correlation coefficient of the bivariate data shown on this scatterplot is closest to:



A –0.5

B 0.5

C –1

D 1

9 EXT1 The vectors $u = -4i + j$ and $v = 3i - 6j$ are:

A parallel in a like direction

B perpendicular

C neither parallel nor perpendicular

D parallel in an unlike direction

10 These scores are the results of a maths quiz:

7, 9, 5, 8, 9, 5, 6, 8, 7, 9, 5, 5, 7, 6, 5

a Complete a frequency distribution table for these scores.

b Draw a frequency polygon and histogram for this data.

c Draw a cumulative frequency histogram and polygon.

d Find the median.

e Find the interquartile range.

f Draw a box plot to show the five-number summary for this data.

11 For this table of bivariate data:

x	1	2	3	4	5	6	7	8
y	18	23	29	38	41	46	52	60

a draw a line of best fit and find its equation

b draw a least-squares regression line and find its equation

12 Shoppers were asked what they most liked about the shopping centre. Draw a Pareto chart for the survey results.

Variety of shops	34
Amenities	11
Child-friendly	28
Parking	27

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13 EXT1 A quadrilateral $OABC$ is drawn so that $\overrightarrow{OA} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

a Show that $OABC$ is a kite.

b Prove that the diagonals of this kite are perpendicular.

14 Find the mean, standard deviation and variance of the scores 3, 5, 9, 8, 6, 5, 8, 7.

EXT1 **15** **a** Find the dot product of $u = 4i - j$ and $v = -2i - 7j$.

b Find the angle between the vectors.

16 The table shows students' scores on a maths test.

a Find the mean and standard deviation.

b Show that the score of 3 is an outlier.

c Find the mean and standard deviation excluding the outlier.

Score	Frequency
3	1
4	0
5	0
6	2
7	4
8	6
9	8
10	3

17 EXT1 Find the acute angle between each pair of vectors, to the nearest degree.

a $u = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} -2 \\ -7 \end{pmatrix}$

b $u = 5i - 2j$ and $v = 6i - 3j$

18 EXT1 Find the anti-derivative of $(4 - x^2)^{-\frac{\ }{2}}$.

19 Find the mean, mode, median and range of the scores 8, 9, 4, 7, 6, 5, 6.

20 EXT1 Find the exact dot product of $u = (-8 \cos 30°, -8 \sin 30°)$ and $v = (-4 \cos 45°, 4 \sin 45°)$.

21 EXT1 A projectile moves with an initial velocity of 12 m s^{-1} and is projected at an angle of 30°. (Take g as 10 m s^{-2}.) Find:

a the time taken for its flight

b how far it will travel horizontally

c the maximum height that it will reach

22 EXT1 The vectors $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ form 3 vertices of a parallelogram. Find a vector that forms the 4th vertex of the parallelogram.

23 The table below shows the results of a survey into the ratings of a TV show.

Rating	Frequency
1	1
2	7
3	15
4	19
5	10

a How many people were surveyed?

b Find correct to 2 decimal places the mean and standard deviation.

c Find the median.

d Draw a box plot for these results.

24 EXT1 Evaluate x if the vectors $\begin{pmatrix} x \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ are:

a perpendicular **b** parallel

25 Find the correlation coefficient for this set of data, correct to 2 decimal places.

x	3	7	4	8	12	2
y	15	11	9	8	7	18

26 EXT1 Find vector b (to 1 decimal place) given $a = 4i - j$, $a\ \ b = 18$ and the angle between a and b is 47°.

27 EXT1 Prove that $u\ \ v = 0$ given u and v are perpendicular.

28 EXT1 Show that each pair of vectors is parallel.

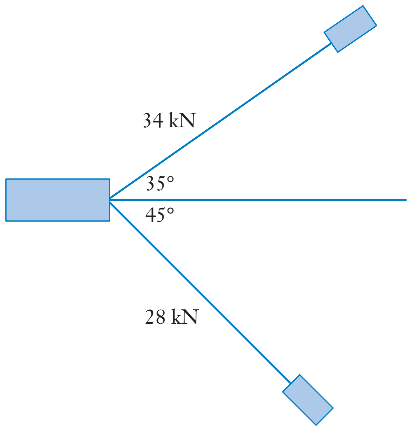
a $u = (-8 \cos 60°, 8 \sin 60°)$ and $v = (-5 \cos 60°, 5 \sin 60°)$

b $u = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$ and $v = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

29 EXT1 Given $a = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$ and $b = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, find:

a $\text{proj}_{b}\, a$ **b** $\text{proj}_{a}\, b$

30 EXT1 Two cars are pulling a truck out of a bog as shown. Find the magnitude and direction of the resultant force to 1 decimal place and the nearest degree.



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31 EXT1 The vectors $u = 3i - 2j$ and $v = 4i + bj$ are perpendicular. Find the value of b.

32 Draw an example of statistical data that is:

a positively skewed

b negatively skewed

c symmetrical

d bimodal

e multimodal

33 Describe each data set as categorical nominal, categorical ordinal, quantitative discrete or quantitative continuous.

a Types of trees planned for a park

b Length of road between towns

c Survey ratings of Poor, Good, Very good, Excellent

d Dress sizes

e Test scores

34 EXT1 A drone is flying with its controls set to north at 28 km h^{-1}. Find its velocity given:

a a tail wind of 14 km h^{-1}

b a head wind of 19 km h^{-1}

c a westerly wind of 7 km h^{-1}

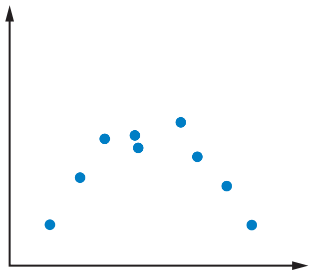
35 EXT1 Find the value of y if vectors $\overrightarrow{OA} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 2 \\ y \end{pmatrix}$ are parallel.

36 EXT1 Vonne swims across a river at 1.2 m s^{-1}. A current of 3.8 m s^{-1} down the river is affecting Vonne's progress. What is his velocity under the influence of the current?

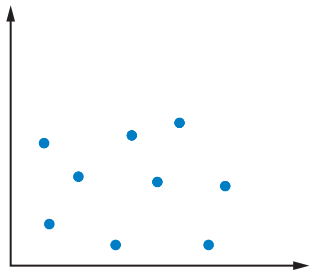
37 EXT1 A triangle OAB is drawn with $\overrightarrow{OA} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$. Prove that the angle sum of triangle OAB is 180°.

38 Match each scatterplot to its description.

a



b



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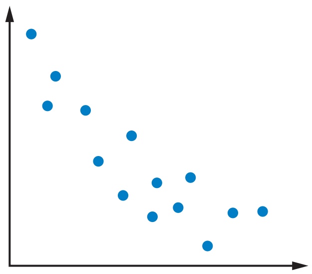

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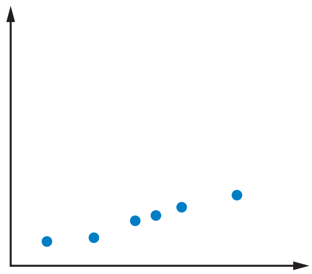

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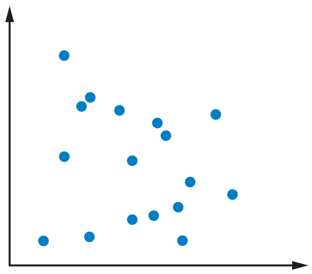

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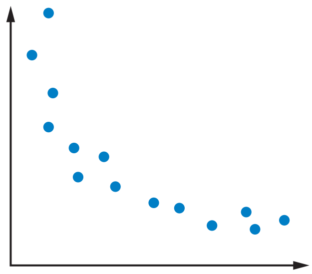

A positive linear correlation

B negative linear correlation

C non-linear correlation

D no correlation

39 EXT1 Find the scalar product $u \ v$ given $u = \begin{pmatrix} -8 \\ 9 \end{pmatrix}$ and $v = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

40 EXT1 Find $\int (1 - \sin^2 x)\, dx$.

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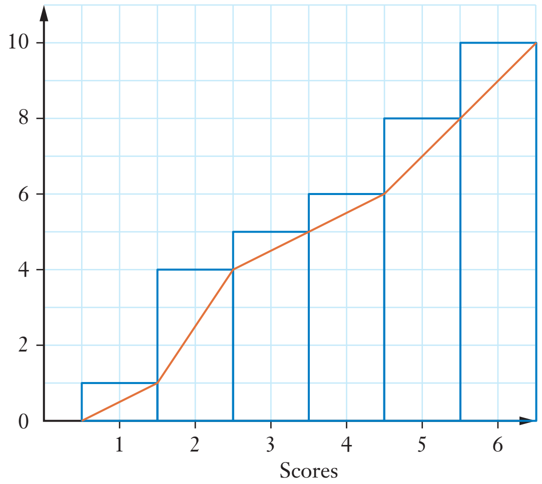
41 For this cumulative frequency polygon, find:

a the median

b the first quartile

c the 60th percentile

d the 4th decile



42 Find all stationary points and points of inflection on the graph of the function $f(x) = 2x^3 - 6x^2 - 48x + 17$.

43 Solve each equation for $[0, 2\pi]$.

a $2 \sin x = 1$

b $\tan^2 x = 1$

c $2 \cos 2x + 1 = 0$

d EXT1 $\sin x + \sqrt{3} \cos x = 1$

44 For the sequence 3, 7, 11, … find:

a the 100th term

b the sum of the first 100 terms

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12. INVESTMENTS, ANNUITIES AND LOANS

Series and sequences have many applications. Financial mathematics is an important part of everyday living as we put money in the bank, pay off credit cards, take out superannuation, buy houses and cars, and many other things. In this chapter you will study the finances of investments, annuities and loans and see how they relate to series.

CHAPTER OUTLINE

12.01 Arithmetic growth and decay
12.02 Geometric growth and decay
12.03 Compound interest
12.04 Compound interest formula
12.05 Annuities
12.06 Annuities and geometric series
12.07 Reducing balance loans
12.08 Loans and geometric series

IN THIS CHAPTER YOU WILL:

- identify arithmetic and geometric growth and decay
- solve practical problems of growth and decay
- solve problems involving compound interest investments using repeated calculations, tables and formulas
- solve problems involving annuities using repeated calculations, tables and geometric series
- solve problems involving reducing balance loans using repeated calculations, tables and geometric series

TERMINOLOGY

annuity: An investment for a fixed period of time where payments are made or received regularly.

compound interest: The interest earned on both the principal and previous interest payments of an investment.

future value: The total value at the close of an investment including all payments and interest earned.

present value: A single payment (called the principal) that will produce a future value over a given time.

reducing balance loan: A loan that is repaid by making regular payments with interest calculated on the amount still owing (the reducing balance of the loan) after each payment.

superannuation: A fixed portion of income that is invested regularly to provide a lump sum or pension when a person retires from the paid workforce; an example of an annuity.

12.01 Arithmetic growth and decay

WS Aihmeic growth and decay

We can use arithmetic sequences and series to describe **growth** (increase) and **decay** (decrease) in practical problems. This is sometimes called **discrete linear growth and decay**.

EXAMPLE 1

A stack of cans on a display at a supermarket has 5 cans on the top row. The next row down has 2 more cans and the next one has 2 more cans and so on.

a Calculate the number of cans in the 11th row down.

b If there are 320 cans in the display altogether, how many rows are there?

Solution

a The first row has 5 cans, the 2nd row has 7 cans, the 3rd row 9 cans and so on. This forms an arithmetic sequence with $a = 5$ and $d = 2$.

For the 11th row, we want $n = 11$:

$$
\begin{aligned}
T_n &= a + (n - 1)d \\
T_{11} &= 5 + (11 - 1) \times 2 \\
&= 5 + 10 \times 2 \\
&= 25
\end{aligned}
$$

So there are 25 cans in the 11th row.

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b If there are 320 cans altogether, this is the sum of cans in all rows.

$$S_n = 320$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$320 = \frac{n}{2}[2 \times 5 + (n-1) \times 2]$$

$$= \frac{n}{2}(10 + 2n - 2)$$

$$= \frac{n}{2}(2n + 8)$$

$$= n^2 + 4n$$

$$0 = n^2 + 4n - 320$$

$$= (n - 16)(n + 20)$$

$$n - 16 = 0, \quad n + 20 = 0$$

$$n = 16, \quad n = -20$$

Since n must be a positive integer, then $n = 16$.

There are 16 rows of cans.

We could substitute $n = 16$ into S_n to check this.

Exercise 12.01 Arithmetic growth and decay

1 A market gardener plants daffodil bulbs in rows, starting with a row of 45 bulbs.
Each successive row has 5 more bulbs than the row before.

- **a** Calculate the number of bulbs in the 34th row.
- **b** Which row would be the first to have more than 100 bulbs in it?
- **c** The market gardener plants 10 545 bulbs altogether. How many rows are there?

2 A stack of logs has 1 on the top, then 3 on the next row down, and each successive row has 2 more logs than the one on top of it.

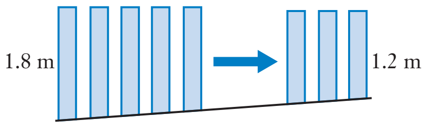
- **a** How many logs are in the 20th row?
- **b** Which row has 57 logs?
- **c** If there are 1024 logs altogether, how many rows are in the stack?

3 A set of books is stacked in layers, where each layer contains 3 books fewer than the layer below. There are 6 books in the top layer, 9 in the next layer, 12 in the next and so on. There are n layers altogether.

a Write down the number of books in the bottom layer.

b Show that there are $\frac{3}{2}n(n + 3)$ books in the stack altogether.

4 A timber fence is to be built on sloping land, with the shortest piece of timber 1.2 m and the longest 1.8 m. There are 61 pieces of timber in the fence.



a What is the difference in height between each piece of timber?

b Assuming no wastage, what length of timber is needed for the fence altogether?

5 A sculpture consists of a set of poles set in a row, with the tallest pole 2.4 m high, the next pole 2.1, the next one 1.8 and so on, down to the last pole which is 0.6 m high.

a How many poles are in the sculpture?

b The poles are made of timber. What length of timber is there altogether in the poles?

6 Johanna has \$2000 in a term deposit that earns simple interest of 2.5% p.a. How much money does she have, including interest, after:

a 1 year? **b** 2 years? **c** 3 years?
d 10 years? **e** 30 years?

7 Each house in a row of terraced houses is to have a new fence. The houses are on a hill so the first fence will be 1 m high, the second will be 1.05 m high, the third 1.1 m high and so on.

a How high will the fence need to be for the 6th house?

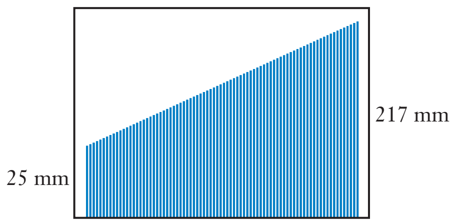
b If the height of the last fence is 1.35 m, how many houses are there?

8 At a courier company, there are different price categories for different weights of parcels. The 1st category is parcels in the range 0–0.5 kg, then 0.5–1 kg, then 1–1.5 kg and so on.

a What is the 10th weight category?

b Which category is 8.5–9 kg?

9 A logo is made with vertical lines equally spaced as shown. The shortest line is 25 mm, the longest is 217 mm, and the sum of the lengths of all the lines is 5929 mm.



a How many lines are in the logo?

b Find the difference in length between adjacent lines.

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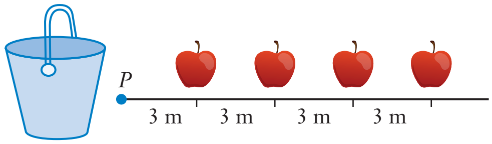
10 In a game, a child starts at point P and runs and picks up an apple 3 m away. She then runs back to P and puts the apple in a bucket. The child then runs to get the next apple 6 m away, and runs back to P to place it in the bucket. This continues until she has all the apples in the bucket.



a How far does the child run from P to pick up the kth apple?

b How far does the child run to fetch all k apples, including return trips to P?

c The child runs 270 m to fetch all the apples and return them to the bucket. How many apples are there?

12.02 Geometric growth and decay

WS Geomeic growth and decay

WS Pacical applications of series

We can use geometric sequences and series to describe growth and decay in practical problems. This is called **geometric growth and decay**. It is also called exponential growth and decay because $T_n = ar^{n-1}$ is an exponential function. You studied exponential growth and decay involving e^x in the Year 11 course.

EXAMPLE 2

A layer of tinting for a car window lets in 95% of light.

a What percentage of light is let in by:

i 2 layers of tinting? **ii** 3 layers of tinting? **iii** 10 layers of tinting?

b How many layers will let in 40% of light?

Solution

a **i** 1 layer lets in 95% of light.

So 2 layers lets in 95% × 95% of light.

$$95\% \times 95\% = 0.95 \times 0.95$$
$$= 0.9025$$
$$= 90.25\%$$

So 2 layers lets in 90.25% of light.

ii 1 layer lets in 95% or 0.95 of light.

2 layers lets in $0.95 \times 0.95 = 0.95^2$ of light.

3 layers lets in $0.95^2 \times 0.95 = 0.95^3$ of light.

$$0.95^3 \approx 0.857$$
$$= 85.7\%$$

So 3 layers lets in 85.7% of light.

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iii The number of layers forms the geometric sequence $0.95, 0.95^2, 0.95^3, \ldots$ with $a = 0.95, r = 0.95$.

For 10 layers, $n = 10$.

$$\begin{aligned} T_n &= ar^{n-1} \\ T_{10} &= 0.95(0.95)^{10-1} \\ &= 0.95(0.95)^9 \\ &= 0.95^{10} \\ &\approx 0.5987 \\ &= 59.87\% \end{aligned}$$

So 10 layers lets in 59.87% of light.

b We want to find n when the nth term is 40% or 0.4.

$$\begin{aligned} T_n &= ar^{n-1} \\ 0.4 &= 0.95(0.95)^{n-1} \\ &= 0.95^n \\ \log 0.4 &= \log 0.95^n \\ &= n \log 0.95 \\ \frac{\log 04}{\log 095} &= n \\ 17.9 &\approx n \end{aligned}$$

So around 18 layers of tinting will let in 40% of light.

EXAMPLE 3

A car bought for \$35 000 depreciates (loses value) by 12% p.a.

a Find its value after:

i 1 year **ii** 2 years **iii** 3 years

b Write the value of the car as a sequence.

c Find what the car is worth after 10 years.

d When will the value of the car drop below \$15 000? Answer to the nearest year.

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Solution

a **i** After 1 year the car is worth \$35 000 – 12% of \$35 000.

$$
\begin{aligned}
\$35\,000 - 12\% \text{ of } \$35\,000 &= \$35\,000(1 - 12\%) \\
&= \$35\,000(1 - 0.12) \\
&= \$35\,000(0.88) \\
&= \$30\,800
\end{aligned}
$$

So the car is worth \$30 800 after 1 year.

ii After 2 years the car is worth \$30 800 – 12% of \$30 800.

$$
\begin{aligned}
\$30\,800 - 12\% \text{ of } \$30\,800 &= \$30\,800(1 - 12\%) \\
&= \$30\,800(0.88) \\
&= \$27\,104
\end{aligned}
$$

So the car is worth \$27 104 after 2 years.

iii After 3 years the car is worth \$27 104 – 12% of \$27 104.

$$
\begin{aligned}
\$27\,104 - 12\% \text{ of } \$27\,104 &= \$27\,104(0.88) \\
&= \$23\,851.52
\end{aligned}
$$

So the car is worth \$23 851.52 after 3 years.

b 1st year = \$30 800, 2nd year = \$27 104, 3rd year = \$23 851.52, …

So 30 800, 27 104, 23 851.52, … is a geometric sequence with $a = 30\,800$ and $r = 0.88$.

c When $n = 10$:

$$
\begin{aligned}
T_n &= ar^{n-1} \\
T_{10} &= 30\,800\,(0.88)^{10-1} \\
&= 30\,800\,(0.88)^9 \\
&= 9747.53
\end{aligned}
$$

So the car is worth \$9747.53 after 10 years.

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d We want $T_n < 15\ 000$.

$$30\ 800(0.88)^{n-1} < 15\ 000$$
$$0.88^{n-1} < 0.487$$
$$\log 0.88^{n-1} < \log 0.487$$
$$(n-1)\log 0.88 < \log 0.487$$
$$(n-1) > \frac{\log 0.487}{\log 0.88} \quad \text{(The inequality reverses because } \log x < 0 \text{ for } 0 < x < 1\text{)}$$
$$n > \frac{\log 0.487}{\log 0.88} + 1$$
$$> 6.63$$

We can substitute $n = 7$ into T_n to check this.

So it will take approximately 7 years for the car to be worth less than \$15 000.

We can also use the limiting sum to model some types of problems.

EXAMPLE 4

a Write $0.\dot{5}$ as a fraction.

b A ball is dropped from a height of 1 metre and bounces up to $\frac{1}{3}$ of its height. It continues bouncing, rising $\frac{1}{3}$ of its height on each bounce.

i Draw a diagram showing the motion.

ii What is the total distance through which the ball travels?

Solution

a $0.\dot{5} = 0.5555555\ldots$

$$= \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} +$$

This is a geometric series with $a = \frac{5}{10} = \frac{1}{2}$ and $r = \frac{1}{10}$.

$$S = \frac{a}{1-r}$$
$$= \frac{\frac{1}{2}}{1-\frac{1}{10}}$$
$$= \frac{\frac{1}{2}}{\frac{9}{10}}$$
$$= \frac{5}{9}$$

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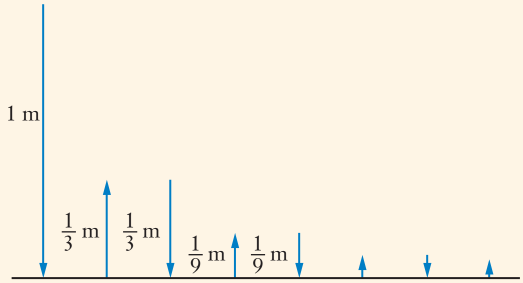
b **i**



ii Notice that there is a series for the ball coming downwards and another series upwards. There is more than one way of calculating the total distance. Here is one way of solving it.

$$\text{Total distance} = 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{9} + \frac{1}{9} + \frac{1}{27} + \frac{1}{27} + \ldots$$
$$= 1 + 2\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \quad\right)$$

$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{3}$

$$S = \frac{a}{1-r}$$
$$= \frac{\frac{1}{3}}{1-\frac{1}{3}}$$
$$= \frac{1}{2}$$

$$\text{Total distance} = 1 + 2\left(\frac{1}{2}\right)$$
$$= 2$$

So the ball travels 2 metres altogether.

INVESTIGATION

LIMITING SUM APPLICATIONS

1 In the above example, in theory will the ball ever stop?

2 Kim owes $1000 on her credit card. If she pays back 10% of the amount owing each month, she will never finish paying it off. Is this true or false?

Exercise 12.02 Geometric growth and decay

1 Water evaporates from a pond at an average rate of 7% each week.

a What percentage of water is left in the pond after:

i 1 week? **ii** 2 weeks? **iii** 3 weeks?

b What percentage is left after 15 weeks?

c If there was no rain, approximately how long (to the nearest week) would it take for the pond to only have 25% of its water left?

2 The price of shares in a particular company is falling by an average of 2% each day.

a What percentage of their initial value do they have after 2 days?

b Approximately how many days will it take for the shares to halve in value?

c After how many days will the shares be worth 10% of their initial value?

3 A painting appreciates (increases its value) by 16% p.a. It is currently worth \$20 000.

a How much will it be worth in:

i 1 year? **ii** 2 years? **iii** 3 years?

b How much will it be worth in 11 years?

c How long will it take for it to be worth \$50 000?

4 A southern brown bandicoot population in Western Australia is decreasing by 5% each year.

a What percentage of the population is left after 5 years?

b After how many years will the population be only 50% of its current level?

c How many years will it take for the population to decrease by 80%?

5 Write each recurring decimal as a fraction.

a $0\dot{4}$ **b** $0\dot{7}$ **c** $1.\dot{2}$ **d** $0.\dot{2}\dot{5}$ **e** $2.\dot{8}\dot{1}$

f $02\dot{3}$ **g** $14\dot{7}$ **h** $101\dot{5}$ **i** $0.1\dot{3}\dot{2}$ **j** $2.\dot{3}6\dot{1}$

6 A frog jumps 0.5 metres. It then jumps 0.1 m and on each subsequent jump it travels 0.2 of the previous distance. Find the total distance through which the frog jumps.

7 A tree grows by $\frac{4}{5}$ of each previous year's growth.

If it was initially 3 m high, find the ultimate height of the tree.

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8 An 8 cm seedling grows by 4.8 cm in the first week, and then keeps growing by 0.6 of its previous week's growth. How tall will it grow?

9 An object rolls 0.5 m in the first second. Then each second after, it rolls by $\frac{5}{6}$ of its previous roll. Find how far it will roll altogether.

10 A 100 m cliff erodes by $\frac{2}{7}$ of its height each year.

a What will the height of the cliff be after 10 years?

b After how many years will the cliff be less than 50 m high?

11 A lamb grows by $\frac{2}{5}$ of its previous growth each month. If a lamb is 45 cm tall:

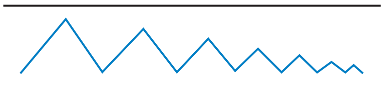
a how tall will it be after 6 months?

b what will its final height be?

12 A weight on an elastic string drops down 60 cm and then bounces back to $\frac{2}{3}$ of its initial height. It keeps bouncing, each time rising back to $\frac{2}{3}$ of its previous height. What is the total distance through which the weight travels?

13 Mary bounces a ball, dropping it from 1.5 m on its first bounce. It then rises up to $\frac{2}{5}$ of its height on each bounce. Find the distance through which the ball travels.

14 A roadside wall has a zigzag pattern on it as shown. The two longest lines are each 2 m long, then the next two lines are $1\frac{3}{4}$ m long and the lines in each subsequent pair are $\frac{7}{8}$ of the length of the previous pair. Find the total length of the lines.



15 Frankie receives a text message that she is asked to send to 8 friends. Frankie forwards this text on to 8 friends and each of them sends it on to 8 friends, and so on.

a Describe the number of people receiving the text as a sequence (including Frankie's text).

b How many people would receive the message in the 9th round of texts?

c How many people would have received the text altogether if it is sent 9 times?

12.03 Compound interest

An **investment** is money that is put into the bank or used to pay for something that will increase in value, or **appreciate**, in the future. An investment can include real estate, art, jewellery or antiques.

The amount invested is called the **present value** or **principal**. The amount the investment is worth after a period of time is called the **future value**.

Future value of investments (FV)

Money in the bank earns interest. Some investments earn simple interest, but most earn **compound interest**. For example, a term deposit or investment account gives the option of adding the interest back into the account (compound interest) or taking the interest as cash or another investment (simple interest).

EXAMPLE 5

Mahmoud invests \$7000 into a term deposit account for 3 years, where it earns 5% p.a.

a How much interest does he earn over the 3 years if the interest is paid into the term deposit account at the end of each year?

b What is the future value of the investment?

Solution

a Interest is $5\% = 0.05$

The interest is added to the principal each time it is paid.

$$\begin{aligned}\text{Amount after 1 year} &= \$7000 + 0.05 \times \$7000 \\ &= \$7000(1 + 0.05) \\ &= \$7000(1.05) \\ &= \$7350\end{aligned}$$

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$$\begin{aligned}\text{Amount after 2 years} &= \$7350 + 0.05 \times \$7350 \\ &= \$7350(1.05) \\ &= \$7717.50\end{aligned}$$

$$\begin{aligned}\text{Amount after 3 years} &= \$7717.50 + 0.05 \times \$7717.50 \\ &= \$7717.50(1.05) \\ &\approx \$8103.38\end{aligned}$$

$$\begin{aligned}\text{Amount of interest} &= \$8103.38 - \$7000 \\ &= \$1103.38\end{aligned}$$

b The future value is \$8103.38.

We also use compound interest to calculate the future value of other investments.

EXAMPLE 6

Rachel and Wade buy a house in Sydney for \$1 250 000. House prices in that area go up by an average of 11.5% p.a. What is their house worth after 2 years?

Solution

Interest is $11.5\% = 0.115$

$$\begin{aligned}\text{Amount after 1 year} &= \$1\,250\,000 + 0.115 \times \$1\,250\,000 \\ &= \$1\,250\,000(1 + 0.115) \\ &= \$1\,250\,000(1.115) \\ &= \$1\,393\,750\end{aligned}$$

$$\begin{aligned}\text{Amount after 2 years} &= \$1\,393\,750(1.115) \\ &= \$1\,554\,031.25\end{aligned}$$

So value after 2 years is \$1 554 031.25

Compound interest tables simplify calculations. The values in the table are called **future value interest factors** as they give the future values of an investment of \$1 at a certain interest rate and time.

Future value interest factors on $1							
Periods	1%	2%	5%	8%	10%	15%	20%
1	1.0100	1.0200	1.0500	1.0800	1.1000	1.1500	1.2000
2	1.0201	1.0404	1.1025	1.1664	1.2100	1.3225	1.4400
3	1.0303	1.0612	1.1576	1.2597	1.3310	1.5209	1.7280
4	1.0406	1.0824	1.2155	1.3605	1.4641	1.7490	2.0736
5	1.0510	1.1041	1.2763	1.4693	1.6105	2.0114	2.4883
6	1.0615	1.1262	1.3401	1.5869	1.7716	2.3131	2.9860
7	1.0721	1.1487	1.4071	1.7138	1.9487	2.6600	3.5832
8	1.0829	1.1717	1.4775	1.8509	2.1436	3.0590	4.2998
9	1.0937	1.1951	1.5513	1.9990	2.3579	3.5179	5.1598
10	1.1046	1.2190	1.6289	2.1589	2.5937	4.0456	6.1917
11	1.1157	1.2434	1.7103	2.3316	2.8531	4.6524	7.4301
12	1.1268	1.2682	1.7959	2.5182	3.1384	5.3503	8.9161
13	1.1381	1.2936	1.8856	2.7196	3.4523	6.1528	10.6993
14	1.1495	1.3195	1.9799	2.9372	3.7975	7.0757	12.8392
15	1.1610	1.3459	2.0789	3.1722	4.1772	8.1371	15.4070
16	1.1726	1.3728	2.1829	3.4259	4.5950	9.3576	18.4884
17	1.1843	1.4002	2.2920	3.7000	5.0545	10.7613	22.1861
18	1.1961	1.4282	2.4066	3.9960	5.5599	12.3755	26.6233

To see how the table works, we can use the example of Mahmoud's term deposit.

EXAMPLE 7

Mahmoud invests $7000 into a term deposit account for 3 years where it earns 5% p.a. Use the table to find the future value of the investment.

Solution

From the table:

For 3 years, $n = 3$ and interest is 5%.

Finding the column for 3 years at 5% gives 1.1576.

1.1576 is the future value on $1.

So future value on $7000 = $7000 × 1.1576
= $8103.20

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Notice that the table gives a slightly different answer from Example 6. This is because the future value interest factors are rounded to 4 decimal places. So using a table is quicker but not as accurate!

We can use the table for investments of less than a year. The value of n stands for time periods, not years.

EXAMPLE 8

Stephanie invests $2000 into a term deposit account for 5 months where it earns 12% p.a., paid monthly. Use the table to find the future value of the investment.

Solution

Interest is 12% p.a.

p.a. or per annum means each year

So interest per month = 12% ÷ 12 = 1%

The value across from $n = 5$ months in the 1% column is 1.0510

Future value on \$2000 = \$2000 × 1.0510

= \$2102

Present value of investments (PV)

Sometimes you want to know how much you would need to invest now to end up with a certain amount in the future. For example, you may be saving up for a holiday or a deposit for a house. This value you need to invest now to achieve a future value is called the **present value**.

EXAMPLE 9

Geordie wants to invest enough money now so that he will have $5000 in 4 years' time to buy a car. Use the table of future value interest factors to calculate how much present value he would need to invest if the interest rate is 5% p.a.

Solution

a We use $n = 4$ and 5%. We know FV = 5000 and we want to find the present value.

Let PV = x

From the table, the value across from $n = 4$ in the 5% column is 1.2155.

This is the future value on $1.

So FV = $x \times 1.2155$ or $1.2155x$

But FV = 5000

So $1.2155x = 5000$

$$x = \frac{5000}{12155}$$
$$= 4113.53$$

So Geordie needs to invest a present value of $4113.53 to have $5000 in 4 years' time.

Exercise 12.03 Compound interest

1 Calculate the future value if $6500 is invested for:

a 2 years at 3% p.a.
b 3 years at 2.5% p.a.
c 4 years at 4.1% p.a.
d 3 years at 1.8% p.a.
e 2 years at 5.3% p.a.

2 Calculate the future value of each investment.

a $2500 for 3 years at 4.5% p.a.
b $10 000 for 4 years at 6.2% p.a.
c $3400 for 5 years at 3.5% p.a.
d $5000 for 3 years at 6% p.a.
e $80 000 for 2 years at 4.5% p.a.

3 Christian and Kate buy a house for $750 000. What is the house worth after 3 years if its value increases by 6% p.a.?

4 Aparna bought a diamond ring for $3000. How much was it worth 3 years later if it appreciated by 5.8% p.a.?

5 A painting bought for $15 000 appreciates by 9% p.a. What is its future value after 4 years?

6 The present value of a necklace is $950. What is its future value after 4 years if it appreciates at 3% p.a.?

7 Hien deposits $4500 into a tour fund where it earns interest of 2.9% p.a. What will be the future value of the tour fund after 3 years?

8 Use the table of future value interest factors on page 536 to calculate the future value of each investment.

a $800 for 7 years at 5% p.a.
b $2000 for 10 years at 1% p.a.
c $5000 for 6 years at 20% p.a.
d $60 000 for 5 years at 10% p.a.
e $100 000 for 8 years at 15% p.a.
f $673.25 for 6 years at 5% p.a.
g $1249.53 for 4 years at 1% p.a.

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h $3000 for 3 months at 12% p.a., paid monthly

i $1000 for 6 months at 12% p.a. paid monthly

j $3500 for 10 months at 12% p.a. paid monthly

9 Use the table of future value interest factors on page 536 to calculate the present value if the future value is $10 000 after:

a 7 years at 2% p.a.

b 5 years at 15% p.a.

c 10 years at 8% p.a.

d 3 years at 1% p.a.

e 4 years at 5% p.a.

12.04 Compound interest formula

WS Compound interest

The calculations on compound interest follow a pattern called a **recurrence relation**.

EXAMPLE 10

Patrick invests $2000 at the beginning of the year at 6% p.a. Find a formula for the amount in the bank at the end of n years.

Solution

$6\% = 0.06$

Amount after 1 year:

$A_1 = \$2000 + 0.06 \text{ of } \$2000$

$= \$2000(1 + 0.06)$

$= \$2000(1.06)$

Amount after 2 years:

$A_2 = A_1 + 006\,A_1$

$= [\$2000(1.06)] + 0.06 \times [\$2000(1.06)]$

$= [\$2000(1.06)](1 + 0.06)$

$= [\$2000(1.06)](1.06)$

$= \$2000(1.06)^2$

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Amount after 3 years:

$A_3 = A_2 + 006\ A_2$

$= [\$2000(1.06)^2] + 0.06 \times [\$2000(1.06)^2]$

$= [\$2000(1.06)^2](1 + 0.06)$

$= [\$2000(1.06)^2](1.06)$

$= \$2000(1.06)^3$

The recurrence relation is $A_{n+1} = A_n + 006\ A_n$

$A_1, A_2, A_3, \ldots$ is a geometric sequence with $a = 2000(1.06)$ and $r = 1.06$.

$T_n = ar^{n-1}$

$= 2000(1.06)(1.06)^{n-1}$

$= 2000(1.06)^n$

So the amount after n years is $\$2000(1.06)^n$.

Compound interest

$A = P(1 + r)^n$

where P = principal (present value)

r = interest rate per period, as a decimal

n = number of periods

A = future value

EXAMPLE 11

Find the amount that will be in the bank after 6 years if \$2000 is invested at 12% p.a. with interest paid:

a yearly **b** quarterly **c** monthly

Solution

$P = 2000$

a $r = 12\% = 0.12, n = 6$

$A = P(1 + r)^n$

$= 2000(1 + 0.12)^6$

$= 2000(1.12)^6$

$= 3947.65$

So the amount is \$3947.65.

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b For quarterly interest, the annual interest rate is divided by 4.

$r = 0.12 \div 4 = 0.03$

Interest is paid 4 times a year.

$n = 6 \times 4 = 24$

$$\begin{aligned} A &= P(1 + r)^n \\ &= 2000(1 + 0.03)^{24} \\ &= 2000(1.03)^{24} \\ &= 4065.59 \end{aligned}$$

So the amount is $4065.59.

c For monthly interest, the annual interest rate is divided by 12.

$r = 0.12 \div 12 = 0.01$

Interest is paid 12 times a year.

$n = 6 \times 12 = 72$

$$\begin{aligned} A &= P(1 + r)^n \\ &= 2000(1 + 0.01)^{72} \\ &= 2000(1.01)^{72} \\ &= 4094.20 \end{aligned}$$

So the amount is $4094.20.

We can find the present value using the compound interest formula.

EXAMPLE 12

Geoff wants to invest enough money to pay for a $10 000 holiday in 7 years' time. If interest is 2.5% p.a., what present value does Geoff need to invest now?

Solution

$A = 10\,000$, $r = 2.5\%$ or 0.025, $n = 7$

We want to find the present value P.

$$\begin{aligned} A &= P(1 + r)^n \\ 10\,000 &= P\,(1 + 0.025)^7 \\ &= P\,(1.025)^7 \end{aligned}$$

$$\frac{10\,000}{1025^{7}} = P$$

$$P \approx 8412.65$$

The present value to invest is $8412.65.

We can use the compound interest formula to find the interest rate or time period by rearranging the formula.

EXAMPLE 13

a Silvana invested \$1800 at 6% p.a. interest and it grew to \$2722.66.
For how many years was the money invested if interest was paid twice a year?

b Find the interest rate if a \$1500 investment is worth \$1738.91 after 5 years.

Solution

a $P = 1800$ and $A = 2722.66$.

Interest is paid twice a year:

$r = 0.06 \div 2 = 0.03$

$$A = P(1 + r)^n$$

$$2722.66 = 1800(1 + 0.03)^n$$

$$= 1800(1.03)^n$$

$$\frac{272266}{1800} = 1.03^n$$

$$1.51259 = 1.03^n$$

$$\log(1.51259) = \log(1.03)^n$$

$$= n \log(1.03)$$

$$\frac{\log(151259)}{\log(103)} = n$$

$$14 \approx n$$

Since interest is paid in twice a year, the number of years will be $14 \div 2 = 7$.

So the money was invested for 7 years.

b $P = 1500, A = 1738.91, n = 5$

$$A = P(1 + r)^n$$

$$1738.91 = 1500(1 + r)^5$$

$$\frac{173891}{1500} = (1 + r)^5$$

$$1.15927 = (1 + r)^5$$

$$\sqrt[5]{115927} = 1 + r$$

$$1.03 = 1 + r$$

$$0.03 = r$$

$$r = 3\%$$

So the interest rate is 3% p.a.

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Exercise 12.04 Compound interest formula

1 Find the amount of money in the bank after 10 years if:

a \$500 is invested at 4% p.a. **b** \$7500 is invested at 7% p.a.

c \$8000 is invested at 8% p.a. **d** \$5000 is invested at 6.5% p.a.

e \$2500 is invested at 7.8% p.a.

2 Sam banks \$1500 where it earns interest at the rate of 6% p.a. Find the amount after 5 years if interest is paid:

a annually **b** twice a year **c** quarterly

3 Chantelle banks \$3000 in an account that earns 5% p.a. Find the amount in the bank after 10 years if interest is paid:

a quarterly **b** monthly

4 Reza put \$350 in the bank where it earns interest of 8% p.a. Find the amount there will be in the account after 2 years if interest is paid:

a annually **b** monthly

5 How much money will there be in an investment account after 3 years if the present value is \$850 and interest of 4.5% p.a. is paid:

a twice a year? **b** quarterly?

6 Find the amount of money there will be in a bank after 8 years if \$1000 earns interest of 7% p.a. with interest paid:

a twice a year **b** quarterly **c** monthly

7 Tanya left \$2500 in a credit union account for 4 years, with interest of 5.5% p.a. paid yearly.

a How much money did she have in the account at the end of that time?

b What would be the difference in the future value if interest was paid quarterly?

8 **a** Find the amount of money there will be after 15 years if Hannah banks \$6000 and it earns 9% p.a. interest, paid quarterly.

b How much more money will Hannah have than if interest was paid annually?

9 How much money will be in a bank account after 5 years if \$500 earns 6.5% p.a. with interest paid monthly?

10 Find the amount of interest earned over 4 years if \$1400 earns 6% p.a. paid quarterly.

11 How much money will be in a credit union account after 8 years if \$8000 earns 7.5% p.a. interest paid monthly?

12 Elva wins a lottery and invests \$500 000 in an account that earns 8% p.a. with interest paid monthly. How much will be in the account after 12 years?

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13 Calculate the principal invested for 4 years at 5% p.a. to achieve a future value of:

a \$5000 **b** \$675 **c** \$12 000 **d** \$289.50 **e** \$12 800

14 What present value is required to accumulate to \$5400 in 3 years with interest of 5.8% p.a. paid quarterly?

15 How many years ago was an investment made if \$5000 was invested at 6% p.a. paid monthly and it is now worth \$6352.45?

16 Find the number of years that \$10 000 was invested at 8% p.a. with interest paid twice a year if there is now \$18 729.81 in the bank.

17 Jude invested \$4500 five years ago at x% p.a. Evaluate x if the amount in his bank account is now:

a \$6311.48 **b** \$5743.27 **c** \$6611.98 **d** \$6165.39 **e** \$6766.46

18 Hamish is given the choice of a bank account in which interest is paid annually or quarterly. If he deposits \$1200, find the difference in the amount of interest paid over 3 years if interest is 7% p.a.

19 Kate has \$4000 in a bank account that pays 5% p.a. with interest paid annually, and Rachel has \$4000 in a different account paying 4% quarterly. Which person will receive more interest over 5 years, and by how much?

20 A bank offers investment account A at 8% p.a. with interest paid twice a year and account B with interest paid at 6% p.a. at monthly intervals. If Georgia invests \$5000 over 6 years, which account pays more interest? How much more does it pay?

21 A hairdresser earns \$36 400 for the first year of work. His salary increases each year by 2%.

a What is his salary in his:
i 5th year of work? **ii** 8th year of work?

b When will his salary reach \$60 000?

22 Yuron earns \$120 000 in his 1st year, then his salary goes up by 3.5% each year after that.

a How much does Yuron earn in his:
i 3rd year? **ii** 12th year? **iii** 20th year?

b What is the first year in which Yuron earns over \$300 000?

23 Masae invests \$5000 in a bank account.

a How many years at 2% p.a. interest will it take for her investment to grow to:
i \$5410? **ii** \$7000?

b At what interest rate would the investment grow to \$6000 after:
i 6 years? **ii** 10 years?

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24 **a** Use the table of future value interest factors on page 536 to find the interest factor for an investment on $1 over 9 years at 8% p.a.

b Prove this interest factor is correct by using the compound interest formula.

25 Show that the future value interest factor of 1.5209 is true for an investment over 3 years at 15% p.a.

12.05 Annuities

A better way to build up money faster is to make regular contributions to an investment. This is called an **annuity**, a name that comes from the same Latin word 'annus' as annual, meaning yearly. However contributions to an annuity could be made more frequently than this. For example, payments into **superannuation** can be made every week or fortnight when an employee is paid.

WS Invemen and annuities

WS Annuiy able problems

WS Annuiy: spreadsheet

Future value of an annuity

EXAMPLE 14

Stevie's grandparents put $100 into a bank account for her on her first birthday. They deposit $100 into the account on each birthday until Stevie is 18 and give her the total amount for her 18th birthday.

How much is in the account at the end of 3 years if interest is 2% p.a.?

Solution

Amount at the end of the 1st year:

Since $100 is deposited at the end of the 1st year on Stevie's 1st birthday, it earns no interest in that year.

$A_1 = \$100$

Amount at the end of the 2nd year:

$A_2 = \$100 + 0.02 \times \$100$

$= \$100(1 + 0.02)$

$= \$100(1.02)$

$= \$102$

But *another* $100 is deposited at the end of the 2nd year on Stevie's 2nd birthday.

So amount $= \$102 + \$100$

$= \$202$

Amount at the end of the 3rd year:

$A_3 = \$202(1.02)$

$= \$206.04$

But *another* $100 is deposited at the end of the 3rd year on Stevie's 3rd birthday.

So amount $= \$206.04 + \$100$

$= \$306.04$

So after 3 years the annuity is worth $306.04.

This is the $300 put in by Stevie's grandparents plus interest of $6.04.

WS

Fuue value table for annuities

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It is assumed that annuity payments are made at the *end* of each period, unless stated otherwise. If they are made at the *beginning* of each period, then the calculations would be different.

You can use the table on the previous page to calculate annuities. You can also download a copy from NelsonNet.

EXAMPLE 15

Stevie's grandparents put $100 into a bank account for her on each birthday, with the final deposit on her 18th birthday. They give Stevie the total amount of the money for her 18th birthday.

a Use the table of future value of annuities factors to calculate how much is in the account after 3 years if interest is 2% p.a.

b How much will Stevie receive on her 18th birthday?

Solution

a From the table:

The value across from $n = 3$ years in the 2% column is 3.0604

This is the future value on $1.

Future value on \$100 = \$100 × 3.0604

= $306.04

b The value across from $n = 18$ years in the 2% column is 21.4123.

Future value on \$100 = \$100 × 21.4123

= $2141.23

So Stevie will receive $2141.23 on her 18th birthday.

We can use the table of future values of an annuity to calculate how much to contribute regularly to achieve a particular future value.

EXAMPLE 16

a Christopher wants to save a certain amount at the end of each year for 5 years until he has $20 000 to buy a car. If the interest rate is 3% p.a., find the amount of each annual contribution Christopher needs to make.

b Alexis wants to save up a $50 000 deposit for a home over 7 years. She wants to make contributions at the end of each quarter. Interest is 8% p.a., paid quarterly. What size contribution would she make?

Solution

a From the future value for annuities table on the previous page, the value across from $n = 5$ in the 3% column is 5.3091.

If we call the contribution x:

Future value = $x \times 5.3091$

So $5.3091x = 20\ 000$

$$x = \frac{20000}{53091}$$

$$= 3767.12$$

So each contribution is \$3767.12

b The contribution is quarterly, or 4 times a year.

Interest rate = 8% ÷ 4 = 2%

Alexis makes 4 contributions each year for 7 years.

Number of periods = 7 × 4 = 28

From the table, the value across from $n = 28$ in the 2% column is 37.0512.

If we call the contribution x:

Future value = $x \times 37.0512$

So $37.0512x = 50\ 000$

$$x = \frac{50000}{370512}$$

$$= 1349.48$$

So each contribution is \$1349.48.

Annuities with regular withdrawals

Another type of annuity is a sum of money earning compound interest that has regular withdrawals or payouts coming out of it.

EXAMPLE 17

Yasmin retires with a lump sum superannuation payment of \$145 000. She puts the money into a financial management company that guarantees 12% p.a. on her annuity, with interest paid monthly. Yasmin withdraws \$1800 at the end of each month as a pension.

Find what Yasmin's annuity is worth after 3 months.

Solution

Monthly interest = 12% ÷ 12 = 1%

$$\text{Amount at the end of 1st month} = \$145\ 000(1 + 0.01)^1$$

$$= \$145\ 000(1.01)$$

$$= \$146\ 450$$

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But Yasmin withdraws $1800

So amount = $146 450 – $1800

= $144 650

Amount at the end of 2nd month = $\$144\ 650(1.01)^1$

= $146 096.50

But Yasmin withdraws $1800

So amount = $146 096.50 – $1800

= $144 296.50

Amount at the end of 3rd month = $\$144\ 296.50(1.01)^1$

= $145 739.47

But Yasmin withdraws $1800

So amount = $144 739.47 – $1800

= $143 939.47

So after 3 months Yasmin's annuity is worth $143 939.47.

Notice that Yasmin's annuity is gradually decreasing. If she took a little less money out each month, she could keep the value of her annuity at around $145 000 or increase its value a little. Try doing the above example with different values to see if Yasmin could draw a pension while keeping her lump sum the same.

TECHNOLOGY

Annuities and spreadsheets

1 The formula for the future value of an annuity is: $FV = a\left[\frac{(1+r)^n - 1}{r}\right]$ where a = regular contribution, r = interest rate and n = number of periods.

You can use this formula to draw up a spreadsheet for future values of an annuity.

Does this formula look familiar? It comes from the sum of a geometric series.

We can use this formula to write a table of future values in a spreadsheet.

Using rows 1 and 2 for headings, we can put 1 in A3 and the formula =**1**+**A3** in A4. Drag this formula down the column for the periods 1, 2, 3, …

We will use $a = 1$ and $r = 0.05$ (5% interest)

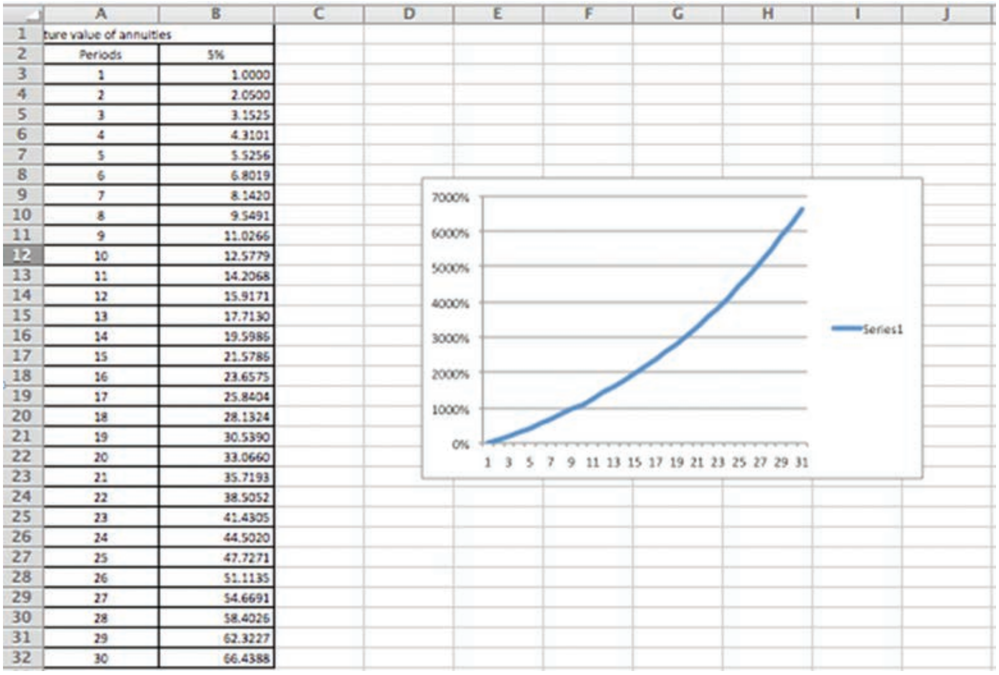
In B3 put the formula =**((1+0.05)^A3–1)/0.05** and drag it down the column.

This gives a set of values for the future value of an annuity at 5% p.a.

Now highlight the column of future values and select the line graph from Charts.

Can you find future values from the graph?

Change the formula to a different interest rate. For example, use 0.08 instead of 0.05 in the formula and drag it down the column. How does this change the graph? Try other interest rate changes and look at how the graph changes.



	A	B
1	ture value of annuities	
2	Periods	5%
3	1	1.0000
4	2	2.0500
5	3	3.1525
6	4	4.3101
7	5	5.5256
8	6	6.8019
9	7	8.1420
10	8	9.5491
11	9	11.0266
12	10	12.5779
13	11	14.2068
14	12	15.9171
15	13	17.7130
16	14	19.5986
17	15	21.5786
18	16	23.6575
19	17	25.8404
20	18	28.1324
21	19	30.5390
22	20	33.0660
23	21	35.7193
24	22	38.5052
25	23	41.4305
26	24	44.5020
27	25	47.7271
28	26	51.1135
29	27	54.6691
30	28	58.4026
31	29	62.3227
32	30	66.4388

2 Use the formula $PV = a\left[\dfrac{(1+r)^n - 1}{r(1+r)^n}\right]$ for the present value of an annuity to draw up a spreadsheet and graph for present value interest factors using similar steps.

For example, for 5% interest, in B3 use the formula **=((1.05)^A3–1)/(0.05*(1.05)^A3)** for the value of 1 in A3, then drag the formula down the column.

How does the graph change if you change the interest rate?

Exercise 12.05 Annuities

1 Calculate the future value of an annuity with a yearly contribution (at the end of each year) of:

- **a** \$5000 for 2 years at 2.5% p.a.
- **b** \$1200 for 3 years at 4% p.a.
- **c** \$875 for 3 years at 3.6% p.a.
- **d** \$10 000 for 2 years at 4.1% p.a.
- **e** \$2000 for 3 years at 3.2% p.a.

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Use the future value table for annuities on page 546 to answer Questions 2 to 7.

2 Find the future value of an annuity with annual contributions of:

a \$6300 for 7 years at 4% p.a.
b \$980 for 5 years at 6% p.a.
c \$7500 for 10 years at 5% p.a.
d \$495.75 for 4 years at 3% p.a.
e \$20 500 for 12 years at 2% p.a.
f \$647.12 for 6 years at 1% p.a.
g \$800 for 15 years at 7% p.a.
h \$598 for 14 years at 9% p.a.
i \$15 000 for 11 years at 13% p.a.
j \$160 000 for 8 years at 4% p.a.

3 Find the future value of an annuity with contributions of:

a \$400 a month for 2 years at 12% p.a. paid monthly
b \$940 a quarter for 5 years at 8% p.a. paid quarterly
c \$2500 twice a year for 8 years at 14% p.a. paid every 6 months
d \$550 three times a year for 5 years at 6% p.a. paid every 4 months
e \$587 a month for 18 months at 12% p.a. paid monthly

4 At the end of each year, Alicia puts \$3500 into a superannuation fund where it earns 9% p.a. How much will she have in superannuation after 30 years?

5 The future value of an annuity is \$35 000 after 12 years. If interest is 6% p.a., find the amount of each yearly contribution.

6 Find the amount of each annual contribution needed to give a future value of:

a \$8450 after 5 years at 7% p.a.
b \$25 000 after 8 years at 3% p.a.
c \$10 000 after 7 years at 4% p.a.
d \$3200 after 5 years at 2% p.a.
e \$1 000 000 after 20 years at 5% p.a.

7 Emlynn wants to put aside a regular amount of money each month for 2 years at 12% p.a., paid monthly, so she will have \$8000 to pay for a film-making course. How much will she need to contribute?

8 Ilona wins \$50 000 in a lottery and invests it in a holiday fund annuity where she withdraws \$5000 at the end of each year. The annuity pays interest of 4% p.a.

a What is the value of her annuity after:
i 1 year? **ii** 2 years? **iii** 3 years?

b What will the annuity be worth after 3 years if Ilona decides to withdraw \$4000 each year instead?

c What will the annuity be worth after 3 years if the interest is 2.7% and Ilona withdraws \$4000 each year?

9 Dave puts his \$125 000 superannuation payout into an annuity and takes out a pension of \$500 a month. The annuity pays interest of 12% p.a., paid monthly.

What is the value of the annuity after:

a 1 month? **b** 2 months? **c** 3 months?

e 3 months if interest is 6% p.a.?

d 3 months if Dave decides to take a pension of \$1000 each month?

10 Graph A shows an annuity of \$15 000 earning 1% interest per month, with a regular withdrawal of \$500 per month. Graph B shows the same \$15 000 annuity paying interest of 2% per month with a regular withdrawal of \$500 each month.

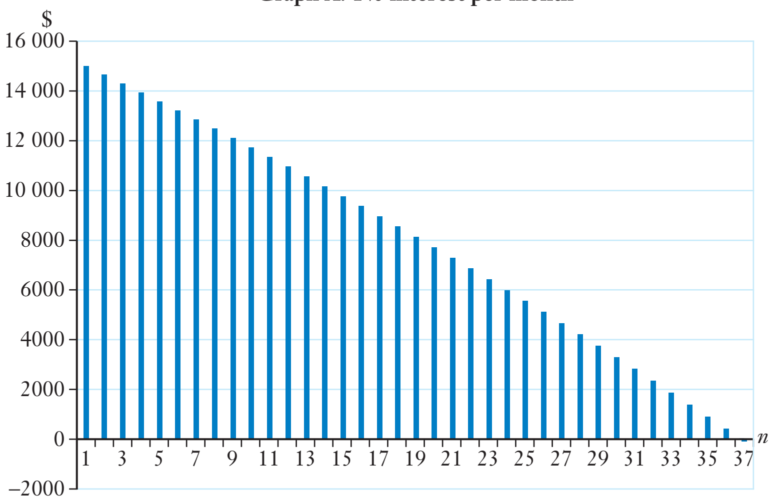
For each graph, determine:

a after how many months the annuity will be worth \$8000

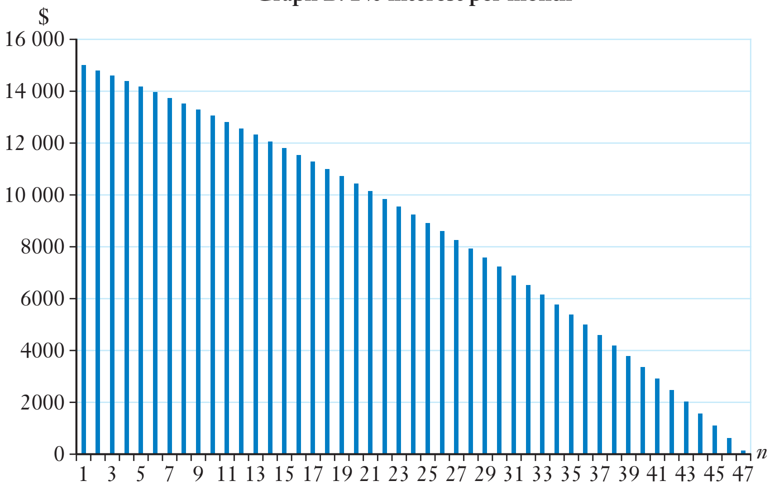
b what the annuity will be worth after a year

c how long it will take for the annuity to run out.

Graph A: 1% interest per month



Graph B: 2% interest per month



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Use the future value table on p. 546 to answer Questions 11 and 12.

11 Ryuji pays $2000 into a superannuation fund at the end of each year at 6% p.a. interest. How many payments would Ryuji make for his superannuation to be greater than:

a $50 000? **b** $80 000?

12 Find the interest rate if $1500 is invested at the end of each year grows to:

a $11 284.95 after 6 years **b** $17 195.85 after 10 years

12.06 Annuities and geometric series

WS Annuiies and geometric series

Annuities

Annuities

EXAMPLE 18

A sum of $1500 is invested at the end of each year in a superannuation fund.
If interest is paid at 6% p.a., how much money will be available at the end of 25 years?

Solution

It is easier to keep track of each annual contribution separately.

Use $A = P(1 + r)^n$ with $P = 1500$ and $r = 0.06$.

The 1st contribution goes in at the end of the 1st year, so it only earns interest for 24 years.

$$A_1 = 1500(1 + 0.06)^{24}$$
$$= 1500(1.06)^{24}$$

The 2nd contribution goes in at the end of the 2nd year, so it earns interest for 23 years.

$$A_2 = 1500(1.06)^{23}$$

Similarly, the 3rd contribution earns interest for 22 years.

$$A_3 = 1500(1.06)^{22}$$

This pattern continues until the final contribution.

The 25th contribution goes in at the end of the 25th year, so it earns interest for 0 years.

$$A_{25} = 1500(1.06)^0$$

The future value is the total of all these contributions together with their interest.

$$\text{FV} = A_1 + A_2 + A_3 + \ldots + A_{25}$$
$$= 1500(1.06)^{24} + 1500(1.06)^{23} + 1500(1.06)^{22} + \ldots + 1500(1.06)^0$$
$$= 1500(1.06)^0 + 1500(1.06)^1 + 1500(1.06)^2 + \ldots + 1500(1.06)^{24}$$
$$= 1500(1.06^0 + 1.06^1 + 1.06^2 + \ldots + 1.06^{24}) \quad \text{(factorising)}$$

$1.06^0 + 1.06^1 + 1.06^2 + \ldots + 1.06^{24}$ is a geometric series with $a = 1.06^0 = 1$, $r = 1.06$ and $n = 25$.

$$S_n = \frac{a(r^n - 1)}{(r-1)}$$

$$S_{25} = \frac{1(106^{25} - 1)}{(106 - 1)}$$

$$\approx 54.86$$

So FV ≈ 1500(54.86)

= 82 296.77

So the total amount of superannuation after 25 years is $82 296.77.

In the previous example, the contributions were made at the *end* of each year. If they were made at the *beginning* of each year, they would all earn an extra year of interest. The 1st contribution would be invested for 25 years, the 2nd for 24 years, and so on until the last contribution for 1 year.

EXAMPLE 19

An amount of $50 is put into an investment account at the end of each month. If interest is paid at 12% p.a. paid monthly, how much is in the account at the end of 10 years?

Solution

We use the compound interest formula where $P = 50$.

$r = 0.12 \div 12 = 0.01$, $n = 10 \times 12 = 120$

The 1st contribution goes in at the end of the 1st month, so it only earns interest for 119 months.

$$A_1 = 50(1 + 0.01)^{119}$$
$$= 50(1.01)^{119}$$

The 2nd contribution goes in at the end of the 2nd month, so it earns interest for 118 months.

$$A_2 = 50(1.01)^{118}$$

The 3rd contribution earns interest for 117 months.

$$A_3 = 50(1.01)^{117}$$

This pattern continues until the final contribution.

The 120th contribution earns interest for 0 months.

$$A_{120} = 50(1.01)^0$$

$$FV = A_1 + A_2 + A_3 + \ldots + A_{120}$$
$$= 50(1.01)^{119} + 50(1.01)^{118} + 50(1.01)^{117} + \ldots + 50(1.01)^0$$
$$= 50(1.01)^0 + 50(1.01)^1 + 50(1.01)^2 + \ldots + 50(1.01)^{119}$$
$$= 50(1.01^0 + 1.01^1 + 1.01^2 + \ldots + 1.01^{119}) \quad \text{(factorising)}$$

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$1.01^0 + 1.01^1 + 1.01^2 + \ldots + 1.01^{119}$ is a geometric series with $a = 1.01^0$ or 1, $r = 1.01$ and $n = 120$.

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$

$$S_{120} = \frac{1(101^{120} - 1)}{(101 - 1)}$$

$$\approx 230.04$$

So FV ≈ 50(230.04)

= 11 501.93

So the total amount after 10 years is $11 501.93.

Exercise 12.06 Annuities and geometric series

1 A sum of $1500 is invested at the end of each year for 15 years at 8% p.a. Find the amount of superannuation available at the end of the 15 years.

2 Liam wants to save up $15 000 for a car in 5 years' time. He invests $2000 at the end of each year in an account that pays 7.5% p.a. interest. How much more will Liam have to pay at the end of 5 years to make up the $15 000?

3 A school invests $5000 at the end of each year at 6% p.a. to go towards a new library. How much will the school have after 10 years?

4 Jacqueline puts aside $500 at the end of each year for 5 years. If the money is invested at 6.5% p.a., how much will she have at the end of the 5 years?

5 Miguel's mother invests $200 for him each birthday up to and including his 18th birthday. The money earns 6% p.a. How much money will Miguel have on his 18th birthday?

6 Xuan is saving up for a holiday. She invests $800 at the end of each year at 7.5% p.a. How much will she have for her holiday after 5 years' time?

7 A couple saves $3000 at the *beginning* of each year towards a deposit on a house. If the interest rate is 5% p.a., how much will the couple have saved after 6 years?

8 Lucia saves up $2000 each year and at the end of the year she invests it at 6% p.a.

a She does this for 10 years. What is her investment worth?

b Lucia continues investing $2000 a year for 5 more years. What is the future value of her investment?

9 Jodie starts work in 2019 and puts \$1000 in a superannuation fund at the end of the year. She keeps putting in this same amount at the end of every year until she retires at the end of 2036. If interest is paid at 10% p.a., calculate how much Jodie will have when she retires.

10 Bol invests \$1000 at the *beginning* of each year. The interest rate is 8% p.a.

a How much will her investment be worth after 6 years?

b How much more would Bol's investment be worth after 6 years if she had invested \$1200 each year?

11 Asam cannot decide whether to invest \$1000 at the end of each year for 15 years or \$500 for 30 years in a superannuation fund. If the interest rate is 5% p.a., which would be the better investment for Asam?

12 Pooja is saving up to go overseas in 8 years' time. She invests \$1000 at the end of each year at 7% p.a. and estimates that the trip will cost her around \$10 000. Will she have enough? If so, how much over will it be? If she doesn't have enough, how much will she need to add to this money to make it up to the \$10 000?

13 Mila puts aside \$20 at the *beginning* of each month for 3 years. How much will she have then if the investment earns 8.2% p.a., paid monthly?

14 **a** Find the future value on an investment of \$1 at the end of each year for 19 years at 7% p.a. using the table of future values of an annuity on page 546.

b Prove that this table value is correct.

12.07 Reducing balance loans

Reducing balance loans

WS Reducing balance loan: spreadsheet

Reducible loans

WS oan repayment tables

People take out loans for many reasons – to buy items such as a car, boat or furniture, to consolidate debts, to buy a home, and for home renovations. A home loan is called a **mortgage**.

An investment or annuity increases in value over time while a loan decreases as the loan is paid off. This is called a **reducing balance loan**. The amount of time taken to pay off the loan is called the term of the loan. A reducing balance loan is similar to an annuity with regular withdrawals.

EXAMPLE 20

Trang borrows \$8000 over 3 years to buy furniture. Interest on the loan is 1.25% per month and monthly repayments are \$277.32. Find the amount owing after:

a 1 month **b** 2 months **c** 3 months

Solution

Use $A = P(1 + r)^n$ with $P = 8000$ and $r = 1.25\% = 0.0125$.
Let A_n be the amount owing after n months.

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a 1st month: Amount owing is \$8000 plus interest less the repayment.

$$A_1 = 8000(1 + 0.0125)^1 - 277.32$$
$$= 8000(1.0125) - 277.32$$
$$= 7822.68$$

Amount owing = \$7822.68

b 2nd month: Amount owing is \$7822.68 plus interest less the repayment.

$$A_2 = 7822.68(1.0125)^1 - 277.32$$
$$= 7643.14$$

Amount owing = \$7643.14

c 3rd month: Amount owing is \$7643.14 plus interest less the repayment.

$$A_3 = 7643.14(1.0125)^1 - 277.32$$
$$= 7461.36$$

Amount owing = \$7461.36

If you know the term of the loan and the amount of the regular contributions, you can calculate the amount of interest owing.

You can use a loan repayments table to calculate the amount you need to contribute to pay off a loan. Here is a table that gives the monthly loan repayments on a \$1000 loan.

	Term (years)					
Interest rate (% p.a.)	**5**	**10**	**15**	**20**	**25**	**30**
2	\$17.53	\$9.20	\$6.44	\$5.06	\$4.24	\$3.70
2.5	\$17.75	\$9.43	\$6.67	\$5.30	\$4.49	\$3.95
3	\$17.97	\$9.66	\$6.91	\$5.55	\$4.74	\$4.22
3.5	\$18.19	\$9.89	\$7.15	\$5.80	\$5.01	\$4.49
4	\$18.42	\$10.12	\$7.40	\$6.06	\$5.28	\$4.77
4.5	\$18.64	\$10.36	\$7.65	\$6.33	\$5.56	\$5.07
5	\$18.87	\$10.61	\$7.91	\$6.60	\$5.85	\$5.37
5.5	\$19.10	\$10.85	\$8.17	\$6.88	\$6.14	\$5.68
6	\$19.33	\$11.10	\$8.44	\$7.16	\$6.44	\$6.00
6.5	\$19.57	\$11.35	\$8.71	\$7.46	\$6.75	\$6.32
7	\$19.80	\$11.61	\$8.99	\$7.75	\$7.07	\$6.65
7.5	\$20.04	\$11.87	\$9.27	\$8.06	\$7.39	\$6.99
8	\$20.28	\$12.13	\$9.56	\$8.36	\$7.72	\$7.34

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EXAMPLE 21

a Piri wants to borrow \$350 000 over 30 years to buy a unit, but she is not sure she can afford to pay the monthly repayments. If interest is 4.5% per month, calculate:

- **i** the amount of each monthly repayment
- **ii** the total amount Piri would pay

b Hamish borrows \$25 000 over 5 years to buy a car. Interest is 2% per month. Find:

- **i** the amount of each monthly repayment
- **ii** the total amount Hamish pays
- **iii** the flat rate of interest on the loan

Solution

a **i** From the table, 30 years at 4.5% p.a. gives \$5.07.

This is on a loan of \$1000, so for \$350 000 we multiply the value by 350.

$\$5.07 \times 350 = \$1774.50$

So Piri would pay \$1774.50 each month.

ii 30 years $= 30 \times 12 = 360$ months

$$\begin{aligned}\text{Total amount repaid} &= \$1774.50 \times 360 \\ &= \$638\,820\end{aligned}$$

b **i** From the table, 5 years at 2% p.a. gives \$17.53.

This is on a loan of \$1000, so for \$25 000 we multiply the value by 25.

$\$17.53 \times 25 = \$438.25$

So Hamish pays \$438.25 each month.

ii 5 years $= 5 \times 12 = 60$ months

$$\begin{aligned}\text{Total amount repaid} &= \$438.25 \times 60 \\ &= \$26\,295\end{aligned}$$

iii
$$\begin{aligned}\text{Interest} &= \$26\,295 - \$25\,000 \\ &= \$1295\end{aligned}$$

$$\frac{1295}{25\,000} \times 100\% = 5.18\%$$

So the flat rate of interest is 5.18%.

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You can use the table to do other calculations.

EXAMPLE 22

a The monthly repayments on a loan of \$70 000 at 5% p.a. are \$553.70. Find the term of the loan.

b A \$150 000 loan with a term of 20 years has monthly instalments of \$1119. Find the interest rate.

Solution

a Let the value in the table be x.

The table is for loans of \$1000.

$\$70\,000 \div \$1000 = 70$

$70 \times x = \$553.70$

$$x = \frac{\$55370}{70} \approx \$7.91$$

Looking at the table in the 5% row, \$7.91 is in the 15 year column.

So the term of the loan is 15 years.

b Let the value in the table be x.

$\$150\,000 \div \$1000 = 150$

$150 \times x = \$1119$

$$x = \frac{\$1119}{150} \approx \$7.46$$

Looking at the table in the 20 year column, \$7.46 is in the 6.5% row.

So the interest rate is 6.5% p.a.

INVESTIGATION

FINANCIAL CALCULATORS

Most bank and other financial websites have calculators rather than tables for loan repayments, values of investments and annuities. Search the websites of banks or general websites that have these and try using these calculators.

Exercise 12.07 Reducing balance loans

1 Calculate the amount owing after 3 months on a loan of:

- **a** \$20 000 at 0.9% per month with repayments of \$432.87 per month
- **b** \$3500 at 1.3% per month with repayments of \$151.57 per month
- **c** \$100 000 at 2.2% per month with repayments of \$2203.22 per month
- **d** \$2000 at 2% per month with repayments of \$105.74 per month
- **e** \$45 800 at 12% p.a. with repayments of \$504.30 per month

2 For each loan below, find:

i the total amount repaid

ii total amount of interest paid

iii the flat interest rate of interest p.a.

a \$5000 over 3 years with a monthly payment of \$166.07

b \$15 900 over 5 years with a monthly payment of \$403.76

c \$80 000 over 12 years with a monthly payment of \$1109.62

d \$235 000 over 25 years with a monthly payment of \$907.09

e \$1348 over 2 years with a monthly payment of \$71.27

Use the table of loan repayments on page 557 to answer the rest of the questions.

3 Find the amount of the monthly repayment on a loan of:

a \$8 000 over 5 years at 6% p.a.

b \$15 000 over 5 years at 8% p.a.

c \$72 000 over 10 years at 7.5% p.a.

d \$430 000 over 20 years at 4% p.a.

e \$312 000 over 15 years at 5.5% p.a.

f \$137 000 over 25 years at 3.5% p.a.

g \$49 000 over 10 years at 7% p.a.

h \$765 000 over 30 years at 2.5% p.a.

i \$925 000 over 25 years at 2% p.a.

j \$1 000 000 over 30 years at 5.5% p.a.

4 Markus takes out a mortgage of \$680 500 over 20 years at 3.5% interest.

a Find his monthly repayment.

b Find the total amount he will pay.

c How much interest does he pay?

d Calculate the flat rate of interest over the whole loan.

5 Find the term of each loan given the monthly payments of:

a \$81.12 for a \$4 000 loan at 8% p.a.

b \$777 for a \$75 000 loan at 4.5% p.a.

c \$937.95 for a \$169 000 loan at 3% p.a.

d \$1560.25 for a \$395 000 loan at 2.5% p.a.

e \$232.20 for a \$20 000 loan at 7% p.a.

f \$3131.25 for a \$625 000 loan at 3.5% p.a.

g \$1302 for a \$120 000 loan at 5.5% p.a.

h \$1809.64 for a \$281 000 loan at 2% p.a.

i \$72.15 for a \$6 500 loan at 6% p.a.

j \$474.15 for a \$81 750 loan at 3.5% p.a.

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6 Find the interest rate of each loan if the monthly instalment is:

- **a** \$57.99 for a \$3000 loan for 5 years
- **b** \$619.65 for an \$81 000 loan for 15 years
- **c** \$2307.36 for a \$456 000 loan for 20 years
- **d** \$2571.56 for a \$212 000 loan for 10 years
- **e** \$6515.36 for a \$947 000 loan for 20 years
- **f** \$178.20 for a \$9000 loan for 5 years
- **g** \$2709.70 for a \$686 000 loan for 30 years
- **h** \$1422 for a \$300 000 loan for 25 years
- **i** \$6814.73 for an \$845 500 loan for 20 years
- **j** \$3127.24 for a \$422 600 loan for 15 years

12.08 Loans and geometric series

WS Loans and geometric series

WS Applicaion of series

We can apply the formulas for compound interest and geometric series to work out the amount of the regular repayments of a reducing balance loan.

EXAMPLE 23

Find the amount of each monthly repayment on a loan of \$20 000 at 12% p.a. over 4 years.

Solution

Let M stand for the monthly repayment.

Number of payments is $4 \times 12 = 48$.

Monthly interest is $12\% \div 12 = 1\% = 0.01$

Each month, we add interest and subtract the repayment.

Amount owing after 1 month:

$$A_1 = 20\,000(1 + 0.01)^1 - M$$
$$= 20\,000(1.01)^1 - M$$

Amount owing after 2 months:

$$A_2 = A_1(1.01)^1 - M$$
$$= [20\,000(1.01)^1 - M](1.01)^1 - M$$
$$= 20\,000(1.01)^2 - M(1.01)^1 - M \quad \text{(expanding brackets)}$$
$$= 20\,000(1.01)^2 - M(1.01^1 + 1) \quad \text{(factorising)}$$

Amount owing after 3 months:

$$\begin{aligned}A_3 &= A_2(1.01)^1 - M\\ &= [20\,000(1.01)^2 - M(1.01^1 + 1)](1.01)^1 - M\\ &= 20\,000(1.01)^3 - M(1.01^1 + 1)(1.01)^1 - M \qquad \text{(expanding brackets)}\\ &= 20\,000(1.01)^3 - M(1.01^2 + 1.01^1) - M\\ &= 20\,000(1.01)^3 - M(1.01^2 + 1.01^1 + 1) \qquad \text{(factorising)}\end{aligned}$$

Continuing this pattern, after 48 months the amount owing is:

$$A_{48} = 20\,000(1.01)^{48} - M(1.01^{47} + 1.01^{46} + 1.01^{45} + \ldots + 1.01^2 + 1.01^1 + 1)$$

But the loan is paid out after 48 months.

So $A_{48} = 0$

$$0 = 20\,000(1.01)^{48} - M(1.01^{47} + 1.01^{46} + 1.01^{45} + \ldots + 1.01^2 + 1.01^1 + 1)$$

$$M(1.01^{47} + 1.01^{46} + 1.01^{45} + \ldots + 1.01^2 + 1.01^1 + 1) = 20\,000(1.01)^{48}$$

$$\begin{aligned}M &= \frac{20000(1\,0\,1)^{48}}{101^{47} + 101^{46} + 101^{45} + \quad + 101^{2} + 101^{1} + 1}\\ &= \frac{20000(1\,0\,1)^{48}}{1 + 1\,0\,1^1 + 1\,0\,1^2 + 1\,0\,1^3 + \quad + 101^{46} + 101^{47}}\end{aligned}$$

The denominator is a geometric series with $a = 1$, $r = 1.01$ and $n = 48$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\begin{aligned}S_{48} &= \frac{1(101^{48} - 1)}{101 - 1}\\ &= \frac{101^{48} - 1}{001}\\ &\approx 61.223\end{aligned}$$

$$\begin{aligned}M &\approx \frac{20000(1\,0\,1)^{48}}{61223}\\ &= 526.68\end{aligned}$$

So the monthly repayment is \$526.68.

We can use this method to find loan repayments for more complex questions.

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EXAMPLE 24

A store charges 9% p.a. for loans, and repayments do not have to be made until the 4th month. Ivan buys \$8000 worth of furniture and pays it off over 3 years.

a How much does Ivan owe after 3 months?

b What are his monthly repayments?

c How much does Ivan pay altogether?

Solution

a Number of payments $= 3 \times 12 - 3 = 33$ (3 months of no repayments)

Monthly interest rate $= 0.09 \div 12 = 0.0075$

Let M stand for the monthly repayment.

The first repayment is made in the 4th month.

After 3 months, the amount owing is

$$A = P(1 + r)^n$$

$$A_3 = 8000(1 + 0.0075)^3$$

$$= 8000(1.0075)^3$$

$$= 8181.35$$

So the amount owing after 3 months is \$8181.35.

b Amount owing after 4 months:

$$A_4 = A_3(1.0075)^1 - M$$

$$= [8000(1.0075)^3](1.0075)^1 - M$$

$$= 8000(1.0075)^4 - M$$

Amount owing after 5 months:

$$A_5 = A_4(1.0075)^1 - M$$

$$= [8000(1.0075)^4 - M](1.0075)^1 - M$$

$$= 8000(1.0075)^5 - M(1.0075)^1 - M$$

$$= 8000(1.0075)^5 - M(1.0075 + 1) \qquad \text{(factorising)}$$

Continuing this pattern, after 36 months the amount owing will be:

$$A_{36} = 8000(1.0075)^{36} - M(1.0075^{32} + 1.0075^{31} + 1.0075^{30} + \ldots + 1.0075^1 + 1)$$

But the loan is paid out after 36 months.

So $A_{36} = 0$

$$0 = 8000(1.00075)^{36} - M(1.0075^{32} + 1.0075^{31} + \ldots + 1.0075^1 + 1)$$

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$$M(1.0075^{32} + 1.0075^{31} + \ldots + 1.0075^{1} + 1) = 8000(1.0075)^{36}$$

$$M = \frac{8000(1\ 0075)^{36}}{10075^{32} + 10075^{31} + \quad + 10075 \quad + 1}$$

$$= \frac{8000(1\ 0075)^{36}}{1 + 1\ 0075^{1} + 10075^{2} + \quad + 10075^{32}}$$

The denominator is a geometric series with $a = 1$, $r = 1.0075$ and $n = 33$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{33} = \frac{1(10075^{33} - 1)}{10075 - 1}$$

$$= \frac{10075^{33} - 1}{00075}$$

$$\approx 37.2849$$

$$M = \frac{8000(1\ 0075)^{36}}{372849}$$

$$\approx 280.79$$

So the monthly repayment is \$280.79.

c $\$280.79 \times 33 = \$9266.07$

So Ivan pays \$9266.07 altogether.

Exercise 12.08 Loans and geometric series

1 An amount of \$3000 is borrowed at 22% p.a. and paid off over 5 years with yearly repayments. How much is each repayment?

2 The sum of \$20 000 is borrowed at 18% p.a. interest calculated monthly over 8 years. How much are the monthly repayments?

3 David borrows \$5000 from the bank and pays back the loan in monthly instalments over 4 years. If the loan incurs interest of 15% p.a. calculated monthly, find the amount of each instalment.

4 Tri and Mai mortgage their house for \$150 000.

- **a** Find the amount of the monthly repayments they will have to make if the mortgage is over 25 years with interest at 6% p.a. compounded monthly.
- **b** If they want to pay their mortgage out after 15 years, what monthly repayments would they need to make?

5 A loan of \$6000 is paid back in equal annual instalments over 3 years. If the interest is 12.5% p.a., find the amount of each annual instalment.

6 Santi buys a car for \$38 000, paying a 10% deposit and taking out a loan for the balance. If the loan is over 5 years with interest of 1.5% monthly, find:

- **a** the amount of each monthly loan repayment
- **b** the total amount that Santi paid for the car.

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7 A \$2000 loan is offered at 18% p.a. with interest charged monthly, over 3 years.

a If no repayment need be paid for the first 2 months, find the amount of each repayment.

b How much will be paid back altogether?

8 Breanna thinks she can afford a mortgage payment of \$800 each month. How much can she borrow, to the nearest \$100, over 25 years at 11.5% p.a.?

9 Get Rich Bank offers a mortgage at $7\frac{1}{2}$% p.a. over 10 years and Capital Bank offers a mortgage at $5\frac{1}{2}$% p.a. over 25 years, both with interest calculated monthly.

a Find the amount of the monthly repayments for each bank on a loan of \$80 000.

b Find the difference in the total amount paid on each mortgage.

10 Majed buys a \$35 000 car. He puts down a 5% deposit and pays the balance back in monthly instalments over 4 years at 12% p.a.

a Find the amount of the monthly payments.

b Find the total amount that Majed pays for the car.

11 Amy borrowed money over 7 years at 15.5% p.a. and she pays \$1200 a month. How much did she borrow?

12 NSW Bank offers loans at 9% p.a. with no repayments for the first 3 months, while Sydney Bank offers loans at 7% p.a. Compare these loans on an amount of \$5000 over 3 years and state which bank offers the better loan and why.

13 Danny buys a home cinema system for \$10 000. He pays a \$1500 deposit and borrows the balance at 18% p.a. over 4 years.

a Find the amount of each monthly repayment.

b How much did Danny pay altogether?

14 A store offers furniture on hire purchase at 20% p.a. over 5 years, with no repayments for 6 months. Ali buys furniture worth \$12 000.

a How much does Ali owe after 6 months?

b What are the monthly repayments?

c How much does Ali pay for the furniture altogether?

15 A loan of \$6000 over 5 years at 15% p.a. interest, charged monthly, is paid back in 5 annual instalments.

a What is the amount of each instalment?

b How much is paid back altogether?

16 a Using the table of loan repayments on page 557, find the amount of the monthly payments on a \$1000 loan over 10 years at 4.5% p.a.

b Show that this table value is correct.

[redacted]

WS Fomula shet: Measurement, Sequences and series

Qz Practice quiz

In Questions **1** to **3**, select the correct answer **A**, **B**, **C** or **D**.

1 An amount of \$2500 invested at 6% p.a. for 5 years with interest paid twice a year has a future value of:

A $\$2500(1.06)^5$ **B** $\$2500(1.03)^5$

C $\$2500(1.03)^{10}$ **D** $\$2500(1.06)^{10}$

2 An investment has a present value of \$68 000 and a future value of \$79 500. Find the flat interest rate on the investment.

A 11.5% **B** 16.9%

C 85.5% **D** 14.5%

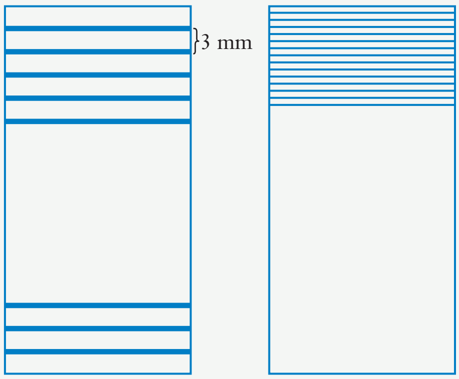
3 A tree is planted when it is 1.2 m tall. Every year its growth is $\frac{3}{8}$ of its previous year's height. Find how tall the tree will grow.

A 1.92 m **B** 2 m **C** 2.5 m **D** 3.2 m

4 A loan of \$22 000 at 12% p. a. is paid off in monthly payments of \$226.29. Find the amount owing after 3 months.

5 Zac puts \$1500 into a savings account that earns 3.7% p.a. How much will Zac have in the account after 3 years?

6 A bamboo blind has 30 slats. It is attached to the window at the top and when the blind is down, the gap between each slat and the next, and between the top slat and the top of the window, is 3 mm. When the blind is up, the slats have no gaps between them.



a Show that when the blind is up, the bottom slat rises 90 mm.

b How far does the next slat rise?

c Explain briefly why the distances the slats rise when the blind is up form an arithmetic sequence.

d Find the distance the 17th slat from the bottom rises.

e What is the sum of the distances that all slats rise?

7 Cristina borrows \$62 500 over 5 years with monthly repayments of \$2022. Find:

a the total amount Cristina pays

b the amount of interest she pays

c the flat interest rate on the total loan

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8 Use the table of future values of an investment on page 536 to find the future value of:

a \$595 over 4 years at 5% p.a.
b \$5 000 over 9 years at 8% p.a.
c \$1651.20 over 6 years at 10% p.a.
d \$13 500 over 11 years at 2% p.a.
e \$9 485 over 18 years at 1% p.a.

9 Murat earned \$20 000 in one year. At the beginning of the 2nd year he received a salary increase of \$450. He now receives the same increase each year.

a What will his salary be after 10 years?
b How much will Murat earn altogether over the 10 years?

10 Ana puts her superannuation payout of \$186 900 into an account that earns 3% p.a. paid monthly. She withdraws \$2500 at the end of each month as a pension.
Find the amount in the account after:

a 1 month
b 2 months
c 3 months

11 Gerri wants to contribute a certain amount of money at the end of each year into a superannuation fund so that she will have \$200 000 at the end of 25 years. If the fund averages 13% p.a., find the amount of the money Gerri would contribute each year.

12 A supermarket stacks boxes with 20 boxes in the bottom stack, 18 boxes in the next stack, 16 in the next and so on.

a How many stacks are there?
b How many boxes are there?

13 Convert each recurring decimal to a fraction.

a $0\dot{4}$
b $0.7\dot{2}$
c $1.\dot{5}\dot{7}$

14 Find the amount invested in a bank account at 9.5% p.a. if the balance in the account is \$5860.91 after 6 years.

15 Haylee borrows \$50 000 for farm machinery at 18% p.a. over 5 years and makes equal yearly repayments on the loan at the end of each year.

a How much does she owe at the end of the first year, just before she makes the first repayment?
b How much is each yearly repayment?

16 **a** If \$2000 is invested at 4.5% p.a., how much will it be worth after 4 years?
b If interest is paid quarterly, how much would the investment be worth after 4 years?

17 Pedro borrows \$200 000 to buy a house. If the interest is 6% p.a. compounded monthly and the loan is over 20 years:

a how much is each monthly repayment?
b how much does Pedro pay altogether?

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18 **a** Find the annual contribution needed for an annuity to have a future value of \$12 000 after 4 years at 5% p.a. if the contribution is made at the *beginning* of the year.

b Find the single investment that would need to be invested at the same interest rate now to have this future value.

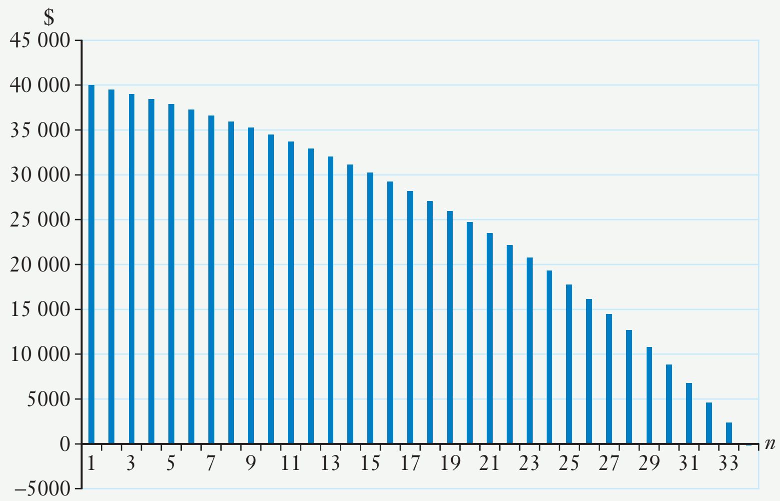
19 Every week during a typing course, Jamal improves his typing speed by 3 words per minute until he reaches 60 words per minute by the end of the course.

a If he can type 18 words per minute in the first week of the course, how many words per minute can he type by week 8?

b How many weeks does the course run for?

20 A ball drops from a height of 1.2 metres then bounces back to $\frac{3}{5}$ of this height. On the next bounce, it bounces up to $\frac{3}{5}$ of this height and so on. Through what distance will the ball travel?

21 The table shows an annuity of \$40 000 earning 5% p.a. with withdrawals of \$2500 at the end of each year.



a What is the annuity worth after:

i 10 years? **ii** 20 years?

b When is the annuity worth:

i \$30 000? **ii** \$10 000?

c How long will it take for the annuity to run out?

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12. CHALLENGE EXERCISE

1 Jane puts \$300 into an account at the beginning of each year to pay for her daughter's education in 5 years' time. If 6% p.a. interest is paid quarterly, how much money will Jane have at the end of the 5 years?

2 A factory sells shoes at \$60 a pair. For 10 pairs of shoes there is a discount, whereby each pair costs \$58. For 20 pairs, the cost is \$56 a pair and so on. Find:

a the price of each pair of shoes on an order of 100 pairs of shoes

b the total price on an order of 60 pairs of shoes

3 Find the amount of money in a bank account if \$5000 earns 8.5% p.a. for 4 years, then 6.5% p.a. for 3 years, with interest paid monthly for all 7 years.

4 A metal is heated to 500°C. A minute later it cools to 425°C, then a minute later it cools down to 361.25°C. If the metal continues to cool in the same way, find:

a its temperature after:
i 10 minutes **ii** 15 minutes

b how long it will take to cool down to:
i 200° **ii** 100°

5 Lukas puts \$1000 into a superannuation account at the beginning of each year where it earns 6% p.a. He retires and collects the superannuation at the end of 25 years.

a How much will the first \$1000 be worth at the end of 25 years, in index form?

b When Lukas deposits the second \$1000 at the end of the 2nd year, how much will it be worth after 25 years?

c How much will the third \$1000 be worth after 25 years?

d How much will the final \$1000 be worth that Lukas deposits at the beginning of the 25th year?

e Show that the total amount in the account after 25 years is $1000 \times \frac{106\,(1\,0\,6^{25} - 1)}{006}$.

f Find the amount that Lukas will have at the end of the 25 years.

6 Kim borrows \$10 000 over 3 years at a rate of 1% interest compounded each month. If she pays off the loan in three equal annual instalments, find:

a the amount Kim owes after one month

b the amount she owes after the first year, just before she pays the first instalment

c the amount of each instalment

d the total amount of interest Kim pays.

7 A superannuation fund paid 6% p.a. for the first 10 years and then 10% p.a. after that. If Thanh put \$5000 into this fund at the beginning of each year, how much would she have at the end of 25 years?

CALCULUS

13.

DIFFERENTIAL EQUATIONS

In this Mathematics Extension 1 chapter you will solve first-order differential equations, recognise and sketch direction fields and look at models that use differential equations.

CHAPTER OUTLINE

13.01 EXT1 Solving $\frac{dy}{dx} = f(x)$

13.02 EXT1 Solving $\frac{dy}{dx} = g(y)$

13.03 EXT1 Solving $\frac{dy}{dx} = f(x)g(y)$

13.04 EXT1 Applications of differential equations

IN THIS CHAPTER YOU WILL:

- EXT1 sketch and interpret direction fields
- EXT1 solve first-order differential equations using a variety of methods, including separation of variables
- EXT1 examine models using differential equations, including exponential growth, Newton's law of cooling and the logistic equation

EXT1 TERMINOLOGY

differential equation: An equation involving a derivative that can be solved to find a function.

direction field: A graph showing the tangents to the family of solutions of a first-order differential equation. Also called a **slope field**.

first-order differential equation: A differential equation involving the first derivative.

logistic equation: The differential equation $\frac{dP}{dt} = kP(N - P)$.

Simple differential equations

EXT1 13.01 Solving $\frac{dy}{dx} = f(x)$

A **differential equation** is any equation involving a derivative, such as $\frac{dy}{dx} = 4x^3 - 7$.

You already know how to solve these by finding the integral or anti-derivative.

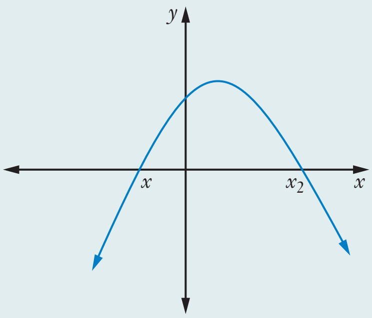
A **first-order differential equation** is an equation involving the first derivative.

Remember that the solution of a differential equation involves a constant C because there is a family of curves that could all give this derivative.

EXAMPLE 1

a Solve $\frac{dy}{dx} = 6x^2 - 4x + 3$ given that when $x = 1, y = 5$.

b Sketch the original function given the graph of its derivative function below.



Solution

a

$$\frac{dy}{dx} = 6x^2 - 4x + 3$$

$$y = \frac{6x^3}{3} - \frac{4x^2}{2} + 3x + C$$

$$= 2x^3 - 2x^2 + 3x + C$$

When $x = 1, y = 5$:

$$5 = 2(1)^3 - 2(1)^2 + 3(1) + C$$

$$= 3 + C$$

$$2 = C$$

So $y = 2x^3 - 2x^2 + 3x + 2$

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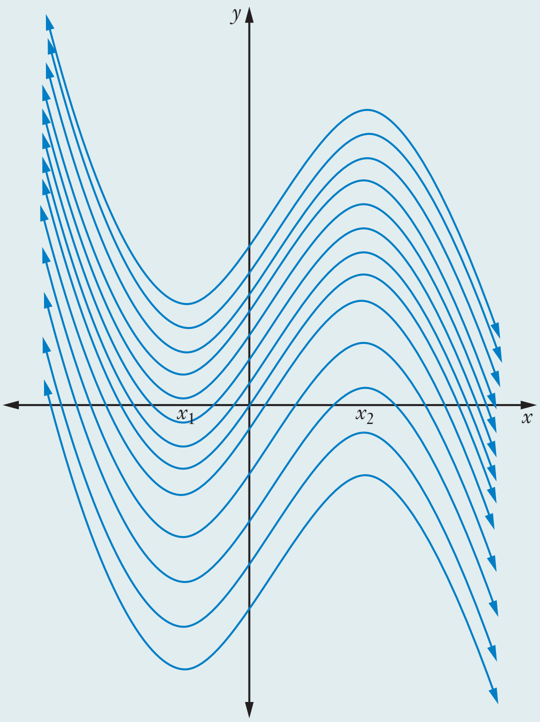
b The graph on the previous page shows the gradient of the original function.

When $x = x_1, x_2$, the gradient is 0. This means the original function has a stationary point at these points.

At x_1: LHS < 0 (decreasing curve) and RHS > 0 (increasing curve) so x_1 is a minimum turning point.

At x_2: LHS > 0 (increasing curve) and RHS < 0 (decreasing curve) so x_2 is a maximum turning point.

Sketching this information gives a family of graphs.



We can find families of functions for specific differential equations by using **direction fields**. These are also called slope fields.

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EXAMPLE 2

a Draw the direction field for $\frac{dy}{dx} = 3x$.

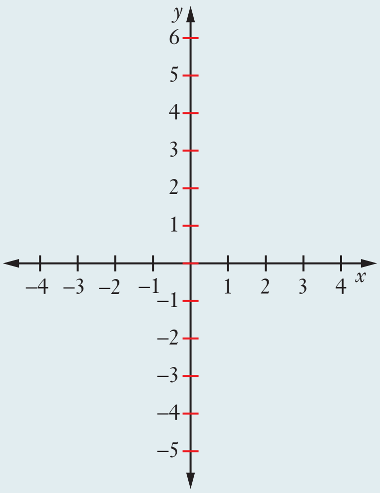
b Find the graph of the function with $\frac{dy}{dx} = 3x$ given that it passes through (0, 1).

c Use calculus to find the equation of this function.

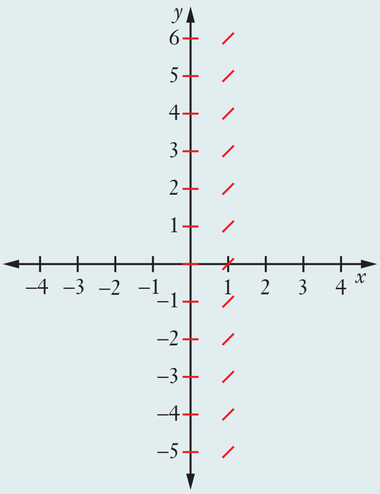
Solution

a When $x = 0$, $\frac{dy}{dx} = 3(0) = 0$ (gradient of the function is 0)

We don't know what the y value of the function is at $x = 0$ so we have many possibilities.



When $x = 1$, $\frac{dy}{dx} = 3(1) = 3$ (gradient of the function is 3)



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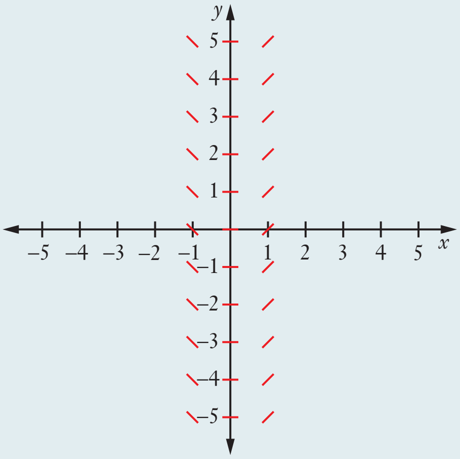
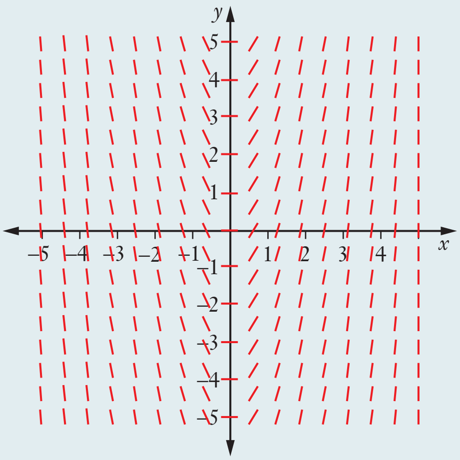
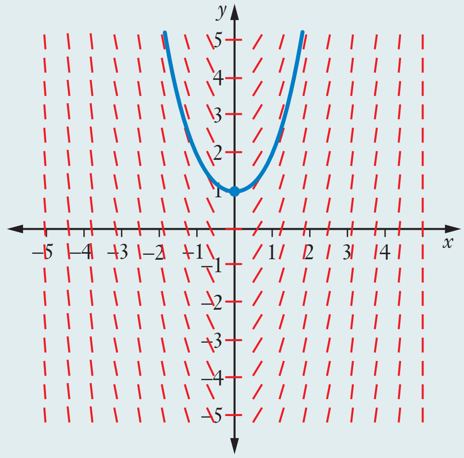
When $x = -1$, $\frac{dy}{dx} = 3(-1) = -3$ (gradient of the function is -3)



We can find the gradient for other values of x and plot them the same way. This gives a direction field. You can use computer applications such as GeoGebra or Desmos to draw this for you. (They both use the term 'slope field'.)



b Plotting the point (0, 1) we follow the shape of the gradients as shown to get an approximate shape of the graph.



c $\frac{dy}{dx} = 3x$

$y = \frac{3x^2}{2} + C$

When $x = 0$, $y = 1$

$1 = \frac{3(0)^2}{2} + C$

$= C$

So the equation is $y = \frac{3x^2}{2} + 1$.

We can find the direction field for many different types of functions.

EXAMPLE 3

a Draw the direction field for $\frac{dy}{dx} = \frac{1}{x^2}$.

b Find the solution to the differential equation $\frac{dy}{dx} = \frac{1}{x^2}$.

c Use an initial point in the direction field to draw the graph of one solution.

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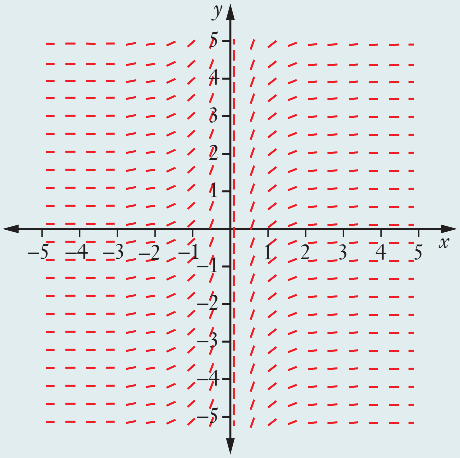
Solution

a When $x = 0$, $\frac{dy}{dx}$ is undefined (vertical gradient)

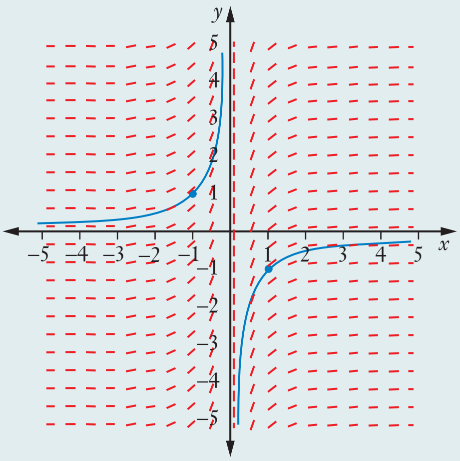
When $x = \pm 1$, $\frac{dy}{dx} = 1$

When $x = \pm 2$, $\frac{dy}{dx} = \frac{1}{4}$, and so on.



b
$$\frac{dy}{dx} = \frac{1}{x^2}$$
$$= x^{-2}$$
$$y = \frac{x^{-}}{-1} + C$$
$$= -\frac{1}{x} + C$$

c The direction field has 2 separate parts in the shape of a hyperbola. Choose 2 points as shown to trace the shape of a hyperbola. (There are many different possible answers.)



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EXT1 Exercise 13.01 Solving $\frac{dy}{dx} = f(x)$

1 Solve each first-order differential equation.

a $\frac{dy}{dx} = 6x + 1$

b $\frac{dy}{dx} = 3x^2 - 4x + 5$

c $f'(x) = \frac{3}{x}$

d $\frac{dy}{dx} = e^{4x}$

e $f'(x) = 3 \cos 2x$

f $f'(x) = \frac{1}{\sqrt{1-x^2}}$

g $\frac{dy}{dx} = \frac{1}{4+x^2}$

2 Solve each differential equation.

a $\frac{dy}{dx} = 6x^2 - 2x - 3$ given $y = 4$ when $x = 0$

b $\frac{dy}{dx} = 2e^{3x}$ given $y = 4$ when $x = 0$

c $f'(x) = \sin 3x$ given $f(\pi) = 2$

d $f'(x) = -\frac{1}{\sqrt{9-x^2}}$ given $f(0) = \frac{\pi}{2}$

e $\frac{dy}{dx} = \frac{2}{3x}$ given $y = 2$ when $x = 1$

3 Draw a direction (slope) field for each differential equation.

a $\frac{dy}{dx} = 2x$

b $\frac{dy}{dx} = x^2$

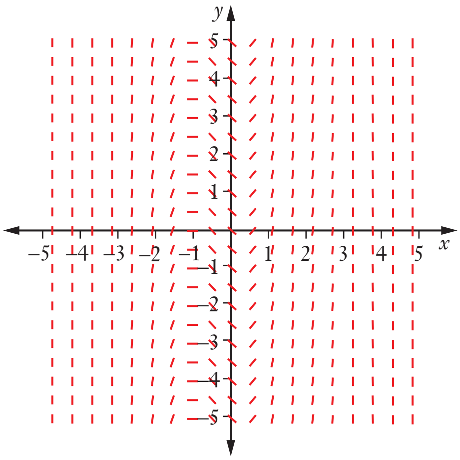
c $f'(x) = 4x^3$

d $f'(x) = -3$

e $\frac{dy}{dx} = \sqrt{x+2}$

4 Copy each direction field and sketch an example of a function in each field.

a



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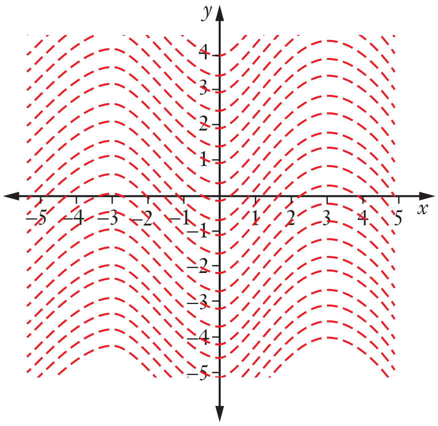
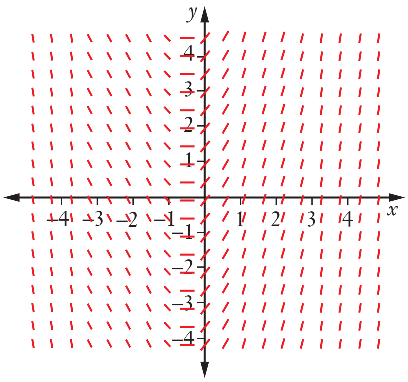
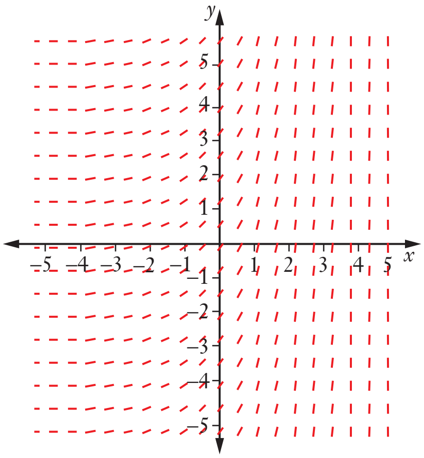
b



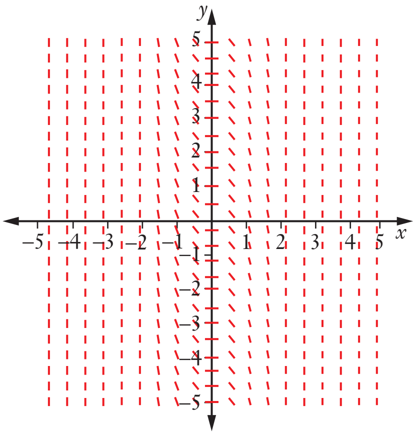
c



d



e



5 **a** Draw a direction field for $\frac{dy}{dx} = \frac{1}{x}$.

b Solve $\frac{dy}{dx} = \frac{1}{x}$ given $y = 2$ when $x = 1$.

c Sketch the graph of this solution on the direction field.

6 Given the differential equation $f'(x) = 2x - 1$:

a draw a direction field

b solve the equation given $f(0) = -3$

7 For $\frac{dy}{dx} = e^x + 1$:

a draw a direction field

b solve $\frac{dy}{dx} = e^x + 1$ given that $y = 1$ when $x = 0$

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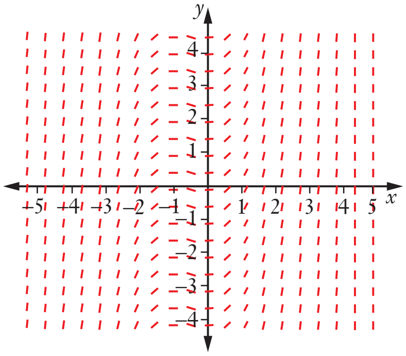
8 Copy each direction field and sketch 3 examples of functions in each field.

a



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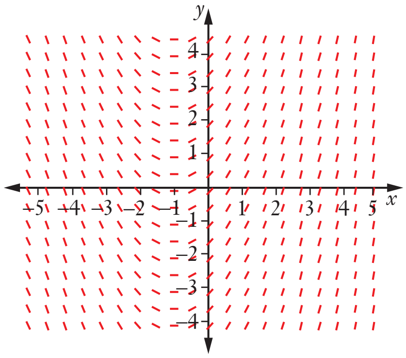
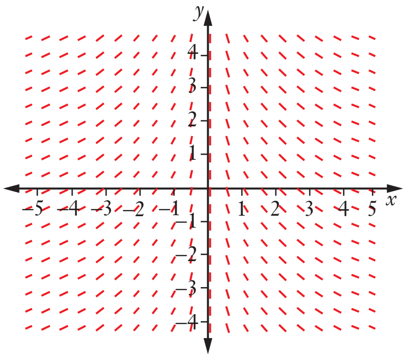
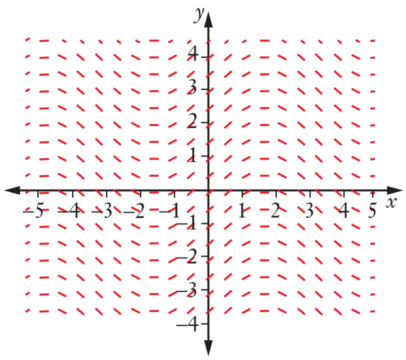


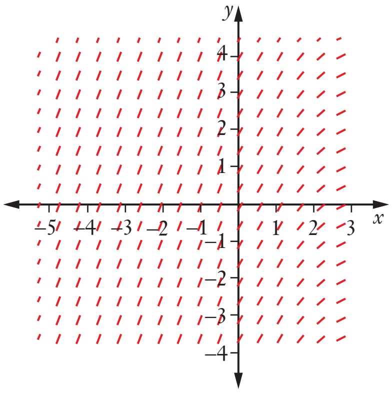
c



d



e



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EXT1

13.02 Solving $\frac{dy}{dx} = g(y)$

WS Differential equations

We can find the direction field for functions that are solutions of differential equations in the form $\frac{dy}{dx} = g(y)$.

EXAMPLE 4

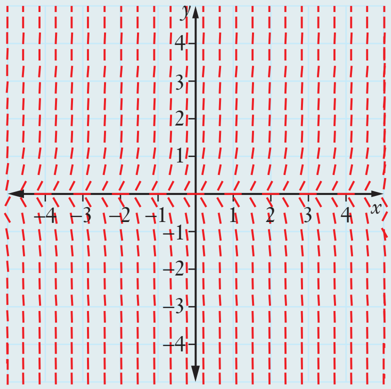
Find the direction field for the differential equation $\frac{dy}{dx} = 8y$.

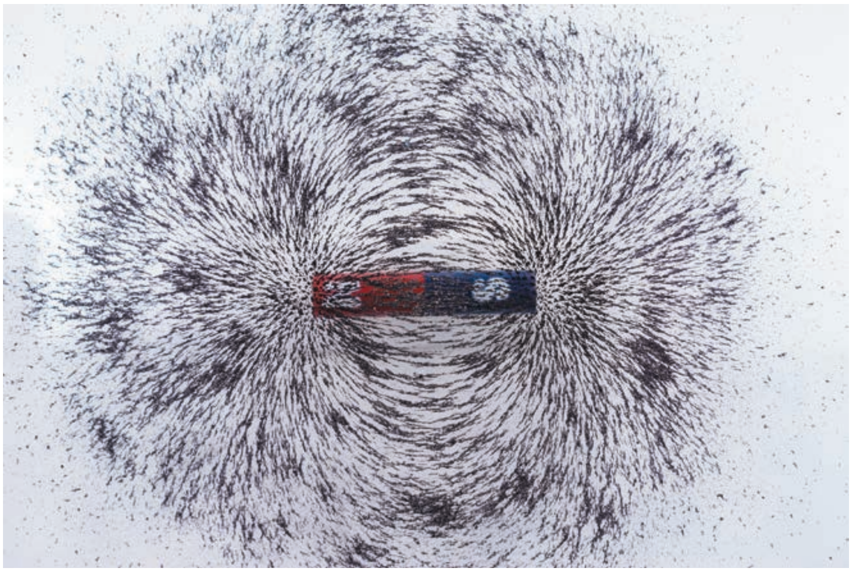
Solution

When $y = 0$, $\frac{dy}{dx} = 8(0) = 0$

When $y = 1$, $\frac{dy}{dx} = 8(1) = 8$

When $y = -1$, $\frac{dy}{dx} = 8(-1) = -8$ and so on.





To solve differential equations in the form $\frac{dy}{dx} = g(y)$ we need to use the following property of derivatives.

$\frac{dy}{dx}$ and $\frac{dx}{dy}$

$\frac{dy}{dx} \times \frac{dx}{dy} = 1$ given that $y = f(x)$ is a differentiable function.

or $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

EXAMPLE 5

Solve each first-order differential equation, given $y \neq 0$.

a $\frac{dy}{dx} = \frac{1}{2y}$

b $\frac{dy}{dx} = 8y$

Solution

a
$$\frac{dy}{dx} = \frac{1}{2y}$$
$$\frac{dx}{dy} = 2y$$
$$x = \int 2y\, dy$$
$$= y^2 + C$$

We change the subject of the equation to y.

$$x = y^2 + C$$
$$x - C = y^2$$
$$\pm\sqrt{x - C} = y$$

b
$$\frac{dy}{dx} = 8y \text{ where } y \neq 0$$
$$\frac{dx}{dy} = \frac{1}{8y}$$
$$x = \int \frac{1}{8y}\, dy$$
$$= \frac{1}{8}\int \frac{1}{y}\, dy$$
$$= \frac{1}{8}\ln|y| + C$$

We change the subject of the equation to y.

$$8x = \ln|y| + 8C$$
$$8x - 8C = \ln|y|$$
$$e^{8x - 8C} = y$$

See note after example.

$$e^{8x}e^{-8C} = y$$
$$Ae^{8x} = y, \text{ where } A = e^{-8C}$$

Because C is a constant, we can write $A = e^{-8C}$ as another constant.

Note: In the solution to Example 5b, we could write $|y| = e^{8x - 8C}$ and then $y = \pm e^{8x - 8C}$, but if we follow through we will get $y = Ae^{8x}$ anyway, with the value of A being positive or negative depending on further information.

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We can solve each differential equation given an initial condition.

EXAMPLE 6

a Solve $\frac{dy}{dx} = 5y^2$ given $\left(0 \ \frac{1}{5}\right)$ lies on the function and $y \neq 0$.

b Solve $f'(x) = \frac{1}{2y-4}$ given $f(4) = 1$ and $y \neq 2$.

Solution

a

$$\frac{dy}{dx} = 5y^2$$

$$\frac{dx}{dy} = \frac{1}{5y^2}$$

$$= \frac{1}{5}y^{-2}$$

$$x = \frac{1}{5}\int y^{-2}\,dy$$

$$= \frac{1}{5} \times \frac{y^{-}}{-1} + C$$

$$= -\frac{1}{5y} + C$$

When $x = 0, y = \frac{1}{5}$

$$0 = -\frac{1}{5\left(\frac{1}{5}\right)} + C$$

$$= -1 + C$$

$$1 = C$$

$$x = -\frac{1}{5y} + 1$$

$$x + \frac{1}{5y} = 1$$

$$\frac{1}{5y} = 1 - x$$

$$\frac{1}{1-x} = 5y$$

$$\frac{1}{5(1-x)} = y$$

b

$$\frac{dy}{dx} = \frac{1}{2y-4}$$

$$\frac{dx}{dy} = 2y - 4$$

$$x = y^2 - 4y + C$$

$f(4) = 1$

$$4 = (1)^2 - 4(1) + C$$

$$= -3 + C$$

$$7 = C$$

So $\quad x = y^2 - 4y + 7$

$$= y^2 - 4y + 4 + 3$$

$$= (y-2)^2 + 3$$

$$x - 3 = (y-2)^2$$

$$\pm\sqrt{x-3} = y - 2$$

$$\pm\sqrt{x-3} + 2 = y$$

Since $f(4) = 1$, we must choose the negative square root.

So the function is $f(x) = -\sqrt{x-3} + 2$.

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EXT1 Exercise 13.02 Solving $\frac{dy}{dx} = g(y)$

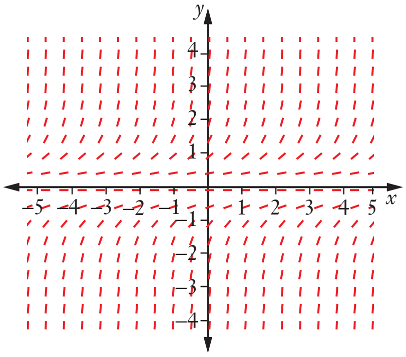
1 Sketch the direction field for each differential equation.

a $f'(x) = 4y + 1$ **b** $f'(x) = y^2$ **c** $\frac{dy}{dx} = e^y$

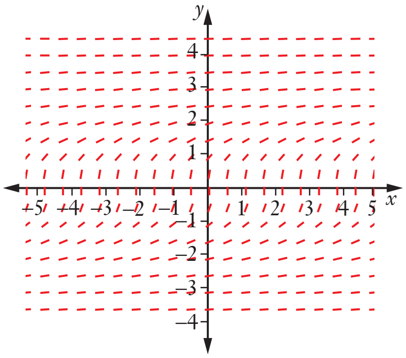
d $\frac{dy}{dx} = \frac{1}{y-1}$ **e** $f'(x) = \frac{1}{y^2}$

2 Copy each direction field and sketch 3 functions in each field.

a



b



3 Solve each first-order differential equation ($y \neq 0$).

a $\frac{dy}{dx} = \frac{2}{y^2}$ **b** $\frac{dy}{dx} = \sqrt{y}$ **c** $\frac{dy}{dx} = \frac{1}{e^y}$

d $\frac{dy}{dx} = y^{-4}$ **e** $f'(x) = y^2$

4 Solve each differential equation.

a $\frac{dy}{dx} = \sqrt{1 - y^2}$ **b** $f'(x) = x^2 + 9$

5 Find solutions to the differential equation $\frac{dy}{dx} = \frac{1}{8y^3}$.

6 Find the function $y = f(x)$ given $\frac{dy}{dx} = \frac{1}{2y+6}$, the function passes through the point (1, 0) and the function is:

a increasing **b** decreasing

7 Find 2 possible solutions to the differential equation $\frac{dy}{dx} = \frac{1}{4(y-3)}$ given that (–2, 0) lies on each function.

8 Solve $\frac{dy}{dx} = \frac{1}{y^2(y^3-1)^4}$ using the substitution $u = y^3 - 1$.

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EXT1 13.03 Solving $\dfrac{dy}{dx} = f(x)g(y)$

Solving $\frac{dy}{dx} = f(x)g(y)$

Differential equations and exponentials

We can find the direction field for functions that solve differential equations in the form $\dfrac{dy}{dx} = f(x)g(y)$.

EXAMPLE 7

Find the direction field for the differential equation $\dfrac{dy}{dx} = 3x^2y$.

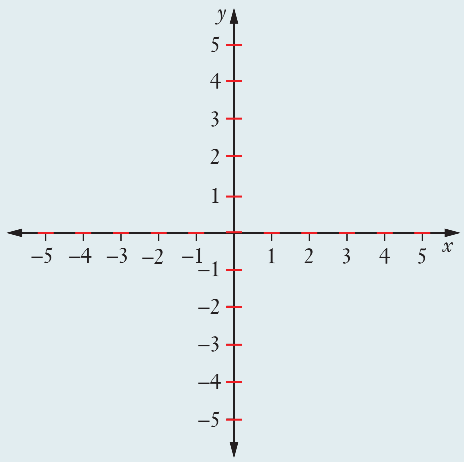
Solution

When $y = 0$, $\dfrac{dy}{dx} = 3x^2(0) = 0$

So all points $(x, 0)$ on the function will have gradient 0.

When $x = 0$, $\dfrac{dy}{dx} = 3(0)^2 = 0$

So all points $(0, y)$ on the function will have gradient 0.



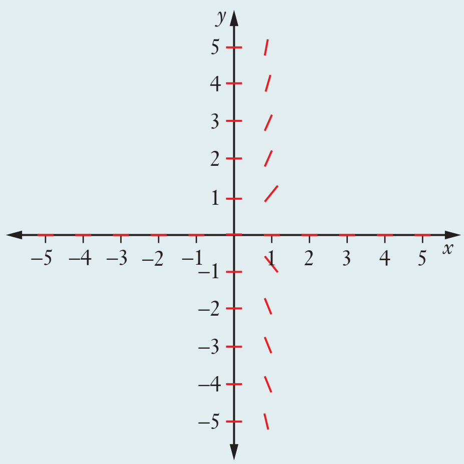
We can check along other vertical lines by varying y values for fixed values of x.

When $x = 1$ and $y = 1$, $\dfrac{dy}{dx} = 3(1)^2(1) = 3$

When $x = 1$ and $y = 2$, $\dfrac{dy}{dx} = 3(1)^2(2) = 6$

When $x = 1$ and $y = -1$, $\dfrac{dy}{dx} = 3(1)^2(-1) = -3$

and so on.

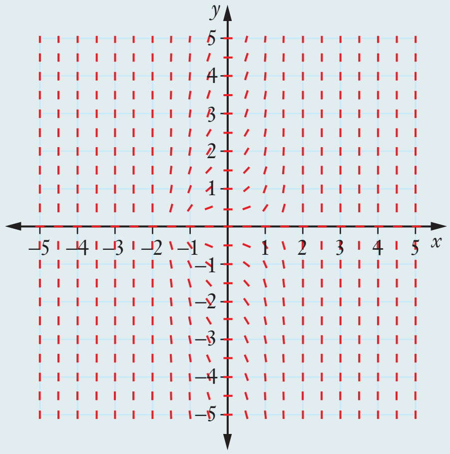


Repeat along other vertical lines.

When $x = 2$ and $y = 1$, $\frac{dy}{dx} = 3(2)^2(1) = 12$ and so on.

When $x = -1$ and $y = 1$, $\frac{dy}{dx} = 3(-1)^2(1) = 3$ and so on.

Drawing the direction field gives:



It is easier to use graphing software to sketch these direction (slope) fields.

We can solve differential equations in the form $\frac{dy}{dx} = f(x)g(y)$ by separating the variables.

This means that we place the x and dx on one side and y and dy on the other.

Separation of variables

To solve $\frac{dy}{dx} = f(x)g(y)$:

separate into $\frac{1}{g(y)}\,dy = f(x)\,dx$ and integrate both sides.

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Not every differential equation is separable.

EXAMPLE 8

Which of the following differential equations are separable?

a $\dfrac{dy}{dx} = \dfrac{2x}{5y^3}$

b $\dfrac{dy}{dx} + \sin x = xy$

c $x\dfrac{dy}{dx} - 2y = 0$

Solution

a $\dfrac{dy}{dx} = \dfrac{2x}{5y^3}$

$5y^3\dfrac{dy}{dx} = 2x$

$5y^3\,dy = 2x\,dx$

So the equation is separable.

b $\dfrac{dy}{dx} + \sin x = xy$

$\dfrac{dy}{dx} = xy - \sin x$

This doesn't allow us to separate out the variables so this is not a separable equation.

c $x\dfrac{dy}{dx} - 2y = 0$

$x\dfrac{dy}{dx} = 2y$

$x\,dy = 2y\,dx$

$\dfrac{1}{2y}\,dy = \dfrac{1}{x}\,dx$

So the equation is separable.

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We can solve separable differential equations.

Differential equations

EXAMPLE 9

Solve the first-order differential equation $\frac{dy}{dx} = \frac{2x}{y^2}$.

Solution

$$\frac{dy}{dx} = \frac{2x}{y^2}$$

$$y^2 \frac{dy}{dx} = 2x$$

$$y^2\, dy = 2x\, dx$$

Note: We could also skip this step, not split $\frac{dy}{dx}$, and integrate both sides with respect to x using the chain rule.

Integrate both sides:

$$\int y^2\, dy = \int 2x\, dx$$

$$\frac{y^3}{3} + c_1 = x^2 + c_2$$

$$\frac{y^3}{3} = x^2 + k$$

We change the subject of the equation to y.

$$y^3 = 3x^2 + C$$

$$y = \sqrt[3]{3x^2 + C}$$

We can also solve each differential equation given an initial point on the function.

EXAMPLE 10

Solve $\frac{dy}{dx} = 3x^2 y$ given the point (0, 1) lies on the function and $y \neq 0$.

Solution

$$\frac{dy}{dx} = 3x^2 y$$

$$dy = 3x^2 y\, dx$$

$$\frac{dy}{y} = 3x^2\, dx$$

Integrate both sides:

$$\int \frac{dy}{y} = \int 3x^2\, dx$$

$$\log_e |y| + c_1 = x^3 + c_2$$

$$\log_e |y| = x^3 + C$$

Substitute (0, 1):

$$\log_e |1| = (0)^3 + C$$

$$0 = C$$

$$\log_e |y| = x^3$$

Change the subject to y:

$$y = e^{x^3}$$

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EXAMPLE 11

a Find the general solutions of $f'(x) = \frac{x^2}{y}$.

b Find the solution given that $f(3) = -2$.

Solution

a $f'(x) = \frac{x^2}{y}$

$\frac{dy}{dx} = \frac{x^2}{y}$

$y\,dy = x^2\,dx$

Integrating both sides:

$\int y\,dy = \int x^2\,dx$

$\frac{y^2}{2} + c_1 = \frac{x^3}{3} + c_2$

$\frac{y^2}{2} = \frac{x^3}{3} + k$

$y^2 = \frac{2x^3}{3} + C$

$y = f(x) = \pm\sqrt{\frac{2x^3}{3} + C}$

b $f(3) = -2$

This satisfies the solution $f(x) = -\sqrt{\frac{2x^3}{3} + C}$

Substituting $x = 3, f(x) = -2$:

$-2 = -\sqrt{\frac{2(3)^3}{3} + C}$

$= -\sqrt{18 + C}$

Squaring both sides:

$(-2)^2 = (-\sqrt{18 + C})^2$

$4 = 18 + C$

$-14 = C$

So the solution is $f(x) = -\sqrt{\frac{2x^3}{3} - 14}$.

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EXT1 Exercise 13.03 Solving $\frac{dy}{dx} = f(x)g(y)$

1 Sketch the direction field for each differential equation.

a $\frac{dy}{dx} = xy^2$ **b** $\frac{dy}{dx} = \frac{x^2}{y}$ **c** $\frac{dy}{dx} = \frac{y}{\sqrt{x+5}}$

d $\frac{dy}{dx} = \frac{y^2-1}{x}$ **e** $f'(x) = x(y+2)^3$

2 Which of these first-order differential equations are separable?

a $\frac{dy}{dx} = \frac{xy}{x+1}$ **b** $\frac{dy}{dx} = x + y^2$ **c** $\frac{dy}{dx} = \frac{y+1}{\sqrt{x}}$

d $y\frac{dy}{dx} = xy - \cos y$ **e** $f'(x) = y^2(x-5)^9$

3 Solve each differential equation.

a $\frac{dy}{dx} = \frac{x}{y}$ **b** $\frac{dy}{dx} = \frac{1}{2xy}$ **c** $\frac{dy}{dx} = \frac{x}{6y^2}$

d $\frac{dy}{dx} = xy$ **e** $\frac{dy}{dx} = 6x^2y^2$ **f** $\frac{dy}{dx} = \frac{x-3}{3y^2}$

g $\frac{dy}{dx} = \frac{4x^2}{y}$ **h** $f'(x) = 12x^3y^2$

i $f'(x) = 6y(x+3)^2$ **j** $\frac{dy}{dx} = \frac{(2x-1)^5}{y^2}$

4 Solve each differential equation.

a $\frac{dy}{dx} = 3x^2y^2$ given (3, 1) is part of the solution

b $f'(x) = \frac{x+1}{\sqrt{y}}$ given $f(0) = 4$

c $\frac{dy}{dx} = \frac{x}{y(x^2-2)^2}$ given that (1, 3) lies on the curve

d $\frac{dy}{dx} = y \sin x$ given $\left(\frac{\pi}{2}, 1\right)$ lies on the curve

e $f'(x) = 6x\sqrt{4-y^2}$ given $f(1) = \pi$

5 Solve $\frac{dy}{dx} = \frac{x+1}{y}$ given the initial point:

a $(-2, 4)$ **b** $(5, -1)$

6 a Differentiate $\log_e(\sin x)$.

b Solve $\sin x \frac{dy}{dx} = \frac{\cos x}{y}$ given $y = -1$ when $x = \frac{\pi}{2}$.

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7 Given $f(1) = 5$, solve $f'(x) = \dfrac{x^2(x^3+1)^3}{y}$.

8 **a** Use the substitution $u = \ln x$ to find $\int \dfrac{\ln x}{x}\,dx$.

b Solve $\dfrac{dy}{dx} = \dfrac{\ln x}{xy}$ given (1, 2) is a point that lies on the curve.

EXT1 13.04 Applications of differential equations

We can use first-order differential equations in many applications, some of which you have seen already.

EXAMPLE 12

The velocity of a particle is given by $\dfrac{dx}{dt} = 4x - 1$ m s^{-1} where x is the displacement. Initially the displacement is 3 m.

a Find the equation for displacement in exact form.

b Find the exact displacement after 5 seconds.

c Find the time when displacement is 10 m, correct to 2 decimal places.

Solution

a

$$\frac{dx}{dt} = 4x - 1$$

So $\dfrac{dt}{dx} = \dfrac{1}{4x-1}$

$$t = \int \frac{1}{4x-1}\,dx$$

$$= \frac{1}{4}\int \frac{4}{4x-1}\,dx$$

$$= \frac{1}{4}\ln|4x-1| + C$$

When $t = 0$, $x = 3$:

$$0 = \frac{1}{4}\ln|4\times 3 - 1| + C$$

$$= \frac{1}{4}\ln 11 + C$$

$$-\frac{1}{4}\ln 11 = C$$

Changing the subject of the equation to x:

$$t = \frac{1}{4}\ln|4x-1| - \frac{1}{4}\ln 11$$

$$= \frac{1}{4}[\ln|4x-1| - \ln 11]$$

$$= \frac{1}{4}\ln\left|\frac{4x-1}{11}\right|$$

$$4t = \ln\left|\frac{4x-1}{11}\right|$$

$$e^{4t} = \frac{4x-1}{11}$$

$$11e^{4t} = 4x - 1$$

$$11e^{4t} + 1 = 4x$$

$$\frac{11e^{4t}+1}{4} = x$$

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b When $t = 5$:

$$x = \frac{11e^{4\times 5} + 1}{4}$$

$$= \frac{11e^{20} + 1}{4}$$

So displacement is $\frac{11e^{20} + 1}{4}$ m after 5 s.

c When $x = 10$:

$$10 = \frac{11e^{4t} + 1}{4}$$

$$40 = 11e^{4t} + 1$$

$$39 = 11e^{4t}$$

$$\frac{39}{11} = e^{4t}$$

$$\ln\left(\frac{39}{11}\right) = 4t$$

$$\frac{\ln\left(\frac{39}{11}\right)}{4} = t$$

$$0.32 = t$$

So displacement is 10 m after 0.32 s.

Exponential growth and decay

For exponential growth and decay, the rate of change of a quantity is proportional to the quantity itself.

$$\frac{dQ}{dt} = kQ$$

For exponential growth $k > 0$.

For exponential decay $k < 0$.

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EXAMPLE 13

The population P of a country town is growing over t years according to the equation

$$\frac{dP}{dt} = 0.024P.$$

a Solve this equation given that the initial population is 35 000.

b Find:

i the population after 10 years

ii the rate at which the population is growing after 10 years

Solution

a

$$\frac{dP}{dt} = 0.024P$$

So $\dfrac{dt}{dP} = \dfrac{1}{0024\ P}$

$$t = \int \frac{1}{0024\ P}\,dP$$

$$= \frac{1}{0024}\int \frac{1}{P}\,dP$$

$$= \frac{1}{0024}\log_e P + C$$

Absolute value symbols not required as $P > 0$.

Changing the subject of the equation to P:

$$t - C = \frac{1}{0024}\log_e P$$

$$0.024(t - C) = \log_e P$$

$$0.024t + c = \log_e P$$

$$e^{0.024t + c} = P$$

$$e^{0.024t} \times e^{c} = P$$

$$Ae^{0.024t} = P$$

When $t = 0$, $P = 35\ 000$:

$$35\ 000 = Ae^{0.024(0)}$$

$$= A$$

So $P = 35\ 000e^{0.024t}$.

b i When $t = 10$:

$$P = 35\ 000e^{0.024(10)}$$

$$= 44\ 493.7$$

So after 10 years the population is 44 494.

ii $\dfrac{dP}{dt} = 0.024P$

After 10 years $P = 44\ 494$

So $\dfrac{dP}{dt} = 0.024 \times 44\ 494$

$$= 1067.8$$

So after 10 years the population is growing at the rate of 1068 people per year.

We could also have differentiated $P = 35\ 000e^{0.024t}$ to find this answer.

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Newton's law of cooling

Objects lose heat according to the law:

$$\frac{dT}{dt} = k(T - T_1)$$

Since the object is cooling, $k < 0$.

EXAMPLE 14

An object is heated to 320°C then left to cool. The temperature in the room is 26°C.

a If the temperature of the object is down to 290°C after 10 minutes, find its temperature after 30 minutes.

b Show that the final temperature of the object is 26°C.

Solution

a $\frac{dT}{dt} = k(T - T_1)$ where $T_1 = 26$

So $\frac{dT}{dt} = k(T - 26)$

$$\frac{dt}{dT} = \frac{1}{k(T-26)}$$

$$t = \int \frac{1}{k(T-26)} dT$$

$$= \frac{1}{k}\int \frac{1}{T-26} dT$$

$$= \frac{1}{k} \ln|T - 26| + C$$

Changing the subject of the equation to T:

$$t - C = \frac{1}{k} \ln|T - 26|$$

$$k(t - C) = \ln|T - 26|$$

$$e^{k(t - C)} = T - 26$$

$$e^{kt} \times e^{kC} = T - 26$$

$$Ae^{kt} = T - 26$$

$$26 + Ae^{kt} = T$$

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When $t = 0$, $T = 320$:

$$320 = 26 + Ae^{k(0)}$$

$$= 26 + A$$

$$294 = A$$

So $T = 26 + 294e^{kt}$

When $t = 10$, $T = 290$:

$$290 = 26 + 294e^{k(10)}$$

$$264 = 294e^{10k}$$

$$\frac{264}{294} = e^{10k}$$

$$\ln\left(\frac{264}{294}\right) = 10k$$

$$\frac{\ln\left(\frac{264}{294}\right)}{10} = k$$

$$-0.0108 = k$$

Note that $k < 0$.

So $T = 26 + 294e^{-0.0108t}$

When $t = 30$:

$$T = 26 + 294e^{-0.0108(30)}$$

$$= 238.9$$

So the temperature is 238.9°C after 30 minutes.

b $T = 26 + 294e^{-0.0108t}$

As $t \to \infty$, $e^{-0.0108t} \to 0$.

So $T \to 26$.

The temperature will eventually reach 26°C.

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There is a more general differential equation called the **logistic equation** that is used for modelling and predicting results in biology, economics and other areas.

INVESTIGATION

LOGISTIC EQUATION

Research the logistic equation as a model for population growth. Search for information on Malthus and Verhulst, who both worked with models of population growth.

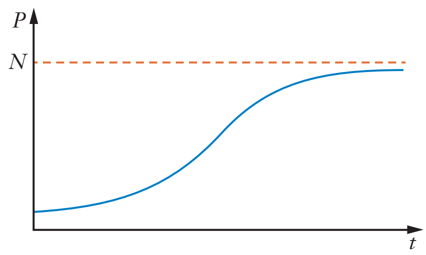
In what areas of biology and chemistry is the logistic equation used to predict growth or decay? Is it used in other areas of study?

Logistic equation

$\frac{dP}{dt} = kP(N - P)$ where N is the limiting condition or carrying capacity.

Sometimes the logistic equation is written in a different form, such as $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$, where K is the **limiting condition**.

Unlike an exponential curve that increases forever, this equation is a more realistic model for population growth, since lack of food and other resources can restrict population to a maximum level.



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EXAMPLE 15

Josh sets up a new salmon farm with 250 fish. The differential equation for P, the population of fish, is given by $\frac{dP}{dt} = 0.0024P(1000 - P)$.

a Show that $\frac{1}{00024\ P(1000-P)} = \frac{5}{12}\left(\frac{1}{P} + \frac{1}{1000-P}\right)$.

b Use the result in part **a** to show that the salmon population is given by the equation $P = \frac{1000}{1+3e^{-2.4t}}$.

c Find the population after:

i 2 years

ii 5 years

d What is the maximum possible population of fish for Josh's salmon farm?

Solution

a $$\text{RHS} = \frac{5}{12}\left(\frac{1}{P} + \frac{1}{1000-P}\right)$$

$$= \frac{5}{12}\left(\frac{1[1000-P]}{P[1000-P]} + \frac{1P}{P[1000-P]}\right)$$

$$= \frac{5}{12}\left(\frac{1000-P+P}{P[1000-P]}\right)$$

$$= \frac{5}{12}\left(\frac{1000}{P[1000-P]}\right)$$

$$= \frac{5000}{12P(1000-P)}$$

$$= \frac{1}{00024\ P(1000-P)} \quad \text{since } \frac{12}{5000} = 00024$$

$= \text{LHS}$

b $\dfrac{dP}{dt} = 0.0024P(1000 - P)$

So $\dfrac{dt}{dP} = \dfrac{1}{00024\ P(1000-P)}$

$$t = \int \frac{1}{00024\ P(1000-P)}\, dP$$

$$= \int \frac{5}{12}\left(\frac{1}{P} + \frac{1}{1000-P}\right) dP \qquad \text{from part } \mathbf{a}$$

$$= \frac{5}{12}\int\left(\frac{1}{P} - \frac{-1}{1000-P}\right) dP$$

$$= \frac{5}{12}[\ln|P| - \ln|1000 - P|] + C$$

$$= \frac{5}{12}\ln\left|\frac{P}{1000-P}\right| + C$$

Changing the subject of the equation to P:

$$t - C = \frac{5}{12}\ln\left|\frac{P}{1000-P}\right|$$

$$\frac{12}{5}(t-C) = \ln\left|\frac{P}{1000-P}\right|$$

$$2.4(t - C) = \ln\left|\frac{P}{1000-P}\right|$$

$$e^{2.4\,(t-C)} = \frac{P}{1000-P}$$

$$e^{2.4t - 2.4C} = \frac{P}{1000-P}$$

$$e^{2.4t} \times e^{-2.4C} = \frac{P}{1000-P}$$

$$Ae^{2.4t} = \frac{P}{1000-P} \qquad \text{where } A = e^{-2.4C}$$

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To find A, note that when $t = 0$, $P = 250$.

$$Ae^{2.4\times 0} = \frac{250}{1000-250}$$

$$A = \frac{1}{3}$$

So $$\frac{1}{3}e^{2.4t} = \frac{P}{1000-P}$$

$$e^{2.4t} = \frac{3P}{1000-P}$$

$$(1000-P)e^{2.4t} = 3P$$

$$1000 - P = \frac{3P}{e^{2.4t}}$$

$$= 3Pe^{-2.4t}$$

$$1000 = P + 3Pe^{-2.4t}$$

$$= P(1 + 3e^{-2.4t})$$

$$\frac{1000}{1+3e^{-2.4t}} = P$$

c **i** When $t = 2$:

$$P = \frac{1000}{1+3e^{-2.4\times 2}}$$

$$= 975.9056 \ldots$$

So after 2 years there are 976 fish.

ii When $t = 5$:

$$P = \frac{1000}{1+3e^{-2.4\times 5}}$$

$$= 999.98$$

So after 5 years there are 1000 fish.

d $$P = \frac{1000}{1+3e^{-2.4t}}$$

As $t \to \infty$, $3e^{-2.4t} \to 0$, so $P \to 1000$.

So the maximum number of fish is 1000.

Note: $\frac{dP}{dt} = 0.0024P(1000 - P)$ is in the form $\frac{dP}{dt} = kP(N - P)$ where $N = 1000$ is the carrying capacity.

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EXT1 Exercise 13.04 Applications of differential equations

1 Solve each differential equation.

a $\frac{dQ}{dt} = 0.2Q$, given that when $t = 0$, $Q = 12$

b $\frac{dN}{dt} = 0.05N$ if $N = 500$ when $t = 0$

c $\frac{dT}{dt} = -1.03T$, given that $T = 450$ initially

d $\frac{dT}{dt} = -0.2(T - 28)$, given that when $t = 0$, $T = 115$

e $\frac{dQ}{dt} = -0.65(Q - 150)$, given that when $t = 0$, $Q = 280$

2 The number of cattle N on a property is growing over t years according to the equation $\frac{dN}{dt} = 0.18N$.

a Solve this equation, given that the initial number of cattle is 600.

b Find the number of cattle after:

i 5 years **ii** 10 years

c Find how long it will take for the number of cattle to reach 2000.

d Find the rate at which the number of cattle is growing after:

i 5 years **ii** 10 years

3 An element of mass M is decaying over t years according to the formula

$$\frac{dM}{dt} = -0.045M.$$

The initial mass is 100 g.

a Solve the differential equation to find the equation for the mass of the element.

b Find the mass after 20 years.

c What is the rate at which the mass is decaying after 20 years?

d Find the half-life of the element (the time it takes to halve its mass).

4 The temperature of an object T over time t minutes is given by $\frac{dT}{dt} = -0.15(T - 20)$. The initial temperature is 150°C.

a Solve the differential equation.

b Find the temperature after:

i 6 minutes **ii** half an hour

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5 The population of fish in a pond over t months is given by $\frac{dP}{dt} = 0.22(P - 60)$ and the initial population is 80.

a Solve the differential equation.

b If the maximum number of fish that can live in the pond is 300, when will the pond reach capacity?

6 A metal is heated to 250°C and then left to cool, which it does according to the formula $\frac{dT}{dt} = -0.04(T - 27)$ where T is the temperature and t is time in minutes.

a Solve the differential equation.

b Find the temperature after:

i 10 minutes ii 20 minutes iii 1 hour

c What is the room temperature?

7 a Show that $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$.

b Solve $\frac{dx}{dt} = x(x + 1)$, giving your answer as an equation for t in terms of x.

8 a Show that $\frac{3}{(x-1)(x+2)} = \frac{1}{x-1} - \frac{1}{x+2}$.

b Solve $\frac{dQ}{dt} = \frac{1}{3}(Q - 1)(Q + 2)$, leaving your solution as t in terms of Q.

9 Solve $\frac{dN}{dt} = N(N - 1)$:

a as a function of t in terms of N

b as a function of N in terms of t, given that when $t = 0$, $N = 10$

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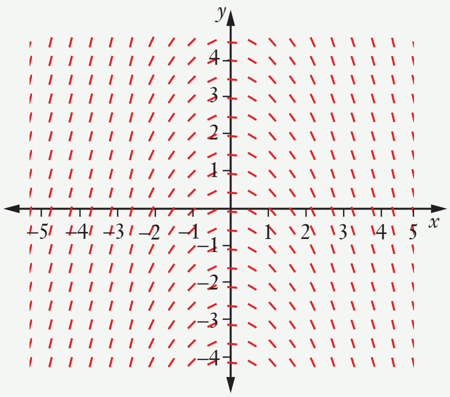
13. TEST YOURSELF

Qz Practice quiz

For Questions **1** to **4**, select the correct answer **A**, **B**, **C** or **D**.

1 Which differential equation could have the direction field below?



A $\frac{dy}{dx} = x^2$

B $\frac{dy}{dx} = -x$

C $\frac{dy}{dx} = -x^2$

D $\frac{dy}{dx} = x$

2 Solve $\frac{dy}{dx} = 3y$, given $y = 20$ when $x = 0$.

A $y = 60e^{0.33x}$ **B** $y = 60e^{3x}$

C $y = 20e^{3x}$ **D** $y = 20e^{0.33x}$

3 What does the equation $\frac{dT}{dt} = 0.06T(2000 - T)$ describe?

A Newtons law of cooling **B** A logistic equation

C Exponential growth **D** Exponential decay

4 Solve $\frac{dy}{dx} = \frac{4x}{3y^2}$ if (0, 0) is a point on the solution.

A $y = 8x^3$ **B** $y = 2\sqrt[3]{x^2}$

C $y = 2x^3$ **D** $y = \sqrt[3]{2x^2}$

5 Solve each first-order differential equation

a $\frac{dy}{dx} = 4x - 3$, given $y = 5$ when $x = 1$

b $\frac{dy}{dx} = \frac{1}{y}$ if $(0, -1)$ lies on the function $y = f(x)$

c $\frac{dQ}{dt} = -0.1Q$, given that initially $Q = 450$

d $\frac{dT}{dt} = -0.05(T - 20)$ if T is initially 300

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6 Draw a direction field for each differential equation.

a $\frac{dy}{dx} = x^2$ **b** $\frac{dy}{dx} = 2x - 1$ **c** $\frac{dy}{dx} = 4y + 3$

d $\frac{dy}{dx} = xy^2$ **e** $\frac{dy}{dx} = \frac{3y}{(x+1)^2}$

7 Solve each differential equation

a $\frac{dy}{dx} = 2y$, given $y = 6$ when $x = 0$

b $\frac{dy}{dx} = \frac{6e^{2x}}{y^2}$ if $y = 3$ when $x = 0$

c $f'(x) = \sqrt{1 - y^2}$, given $f(0) = 0$

d $f'(x) = \sec 2y$, given $f(0) = \pi$

e $\frac{dy}{dx} = \frac{1}{xy}$, given when $x = 1, y = -3$

8 A metal element is heated to 500°C and then left to cool, which it does according to the equation $\frac{dT}{dt} = -0.15(T - 18)$.

a Solve this differential equation.

b Find the temperature after:

i 5 minutes **ii** 15 minutes

c What will the final temperature be?

9 a Show that $\frac{3x-1}{(x-2)(x+3)} = \frac{1}{x-2} + \frac{2}{x+3}$.

b Solve $\frac{dN}{dt} = \frac{(N-2)(N+3)}{3N-1}$, giving the solution in terms of N.

10 Solve each differential equation.

a $\frac{dy}{dx} = \operatorname{cosec} y$, given $y = \frac{\pi}{2}$ when $x = -\frac{\pi}{3}$

b $\frac{dy}{dx} = 6x^2(4 + y^2)$, given (0, 0) lies on the function.

11 The number N of hectares (ha) of forest in a region of Queensland each year is changing as forests are cleared according to the formula $\frac{dN}{dt} = -017(N - 10\,000)$.

a Solve the differential equation given that there are initially 505 000 hectares of forest.

b Find the amount of forest after:

i 10 years **ii** 25 years

c There is a national park that cannot be cleared. How large is it?

12 Given the differential equation $\frac{dy}{dx} = \frac{3x^3}{y^2}$:

a draw a direction field

b solve the equation, given that $y = 4$ when $x = 0$

c draw this solution on the direction field

13 A population Q of bandicoots decreases over time t years according to the formula $\frac{dQ}{dt} = -0.013(Q - 300)$. The initial population was 3200 bandicoots.

a Solve this equation.

b Find the number of bandicoots after:

i 1 year **ii** 3 years

c Find the rate at which the bandicoot population is decreasing after:

i 1 year **ii** 3 years

14 Solve:

a $\frac{dy}{dx} = 3x^2y$, given (0, 2) lies on the function

b $\frac{dy}{dx} = 2x + 2xy^2$ if $y = 1$ when $x = \frac{\sqrt{\pi}}{2}$

c $f'(x) = \frac{6e^{3x}}{5y^4}$, given $f(0) = 0$

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13. CHALLENGE EXERCISE

1 Solve $\frac{dx}{dt} = \frac{x(x^2+3)}{x^2+1}$ as an equation with t in terms of x, given $x = 1$ when $t = 0$.

2 Solve $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{x}$, given $y = 0$ when $x = 1$.

3 A radioactive substance has a half-life of 80 years. Its mass M grams over time t years is given by the formula $\frac{dM}{dt} = -kM$ and its initial mass is M_0.

a Solve the differential equation.

b Find the percentage of the initial mass left after:

i 10 years **ii** 50 years **iii** 100 years

c At what rate is the substance decaying (as a percentage of the initial mass) in grams/year after:

i 10 years? **ii** 50 years? **iii** 100 years?

4 Draw a direction field for the differential equation $\frac{dy}{dx} = \frac{\ln x}{y}$.

5 Solve $\frac{dy}{dx} = 6e^y \sin 3x$, given that when $x = 0$, $y = 0$.

6 The height of a tree after t years is given by $\frac{dQ}{dt} = 0.03Q(45 - Q)$. The tree is 0.3 m high when it is planted.

a Evaluate a and b if $\frac{1}{Q(45-Q)} = \frac{a}{Q} + \frac{b}{45-Q}$.

b Solve the differential equation to find the formula for the height of the tree.

c Find its height after:

i 2 years **ii** 8 years

d What is its ultimate height?

7 In Woodville, 25 people have the flu. The spread of flu through the town is given by $\frac{dN}{dt} = 0.00068N(15\,000 - N)$.

a Solve the differential equation to find the number of people with flu over t weeks.

b Find the number of people with flu after 1 week.

c What is the maximum number of people who will have the flu in Woodville?

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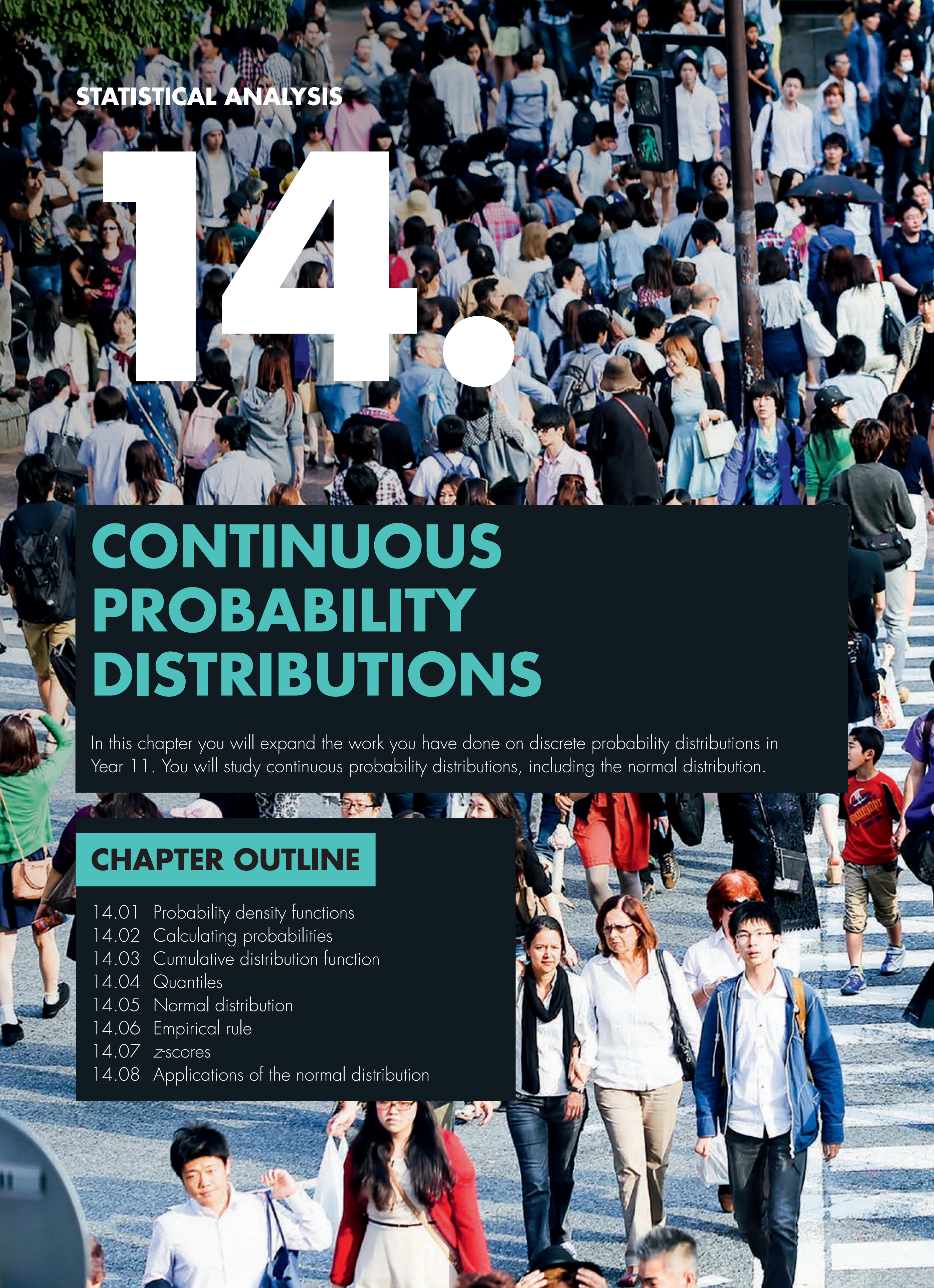

STATISTICAL ANALYSIS

14.

CONTINUOUS PROBABILITY DISTRIBUTIONS

In this chapter you will expand the work you have done on discrete probability distributions in Year 11. You will study continuous probability distributions, including the normal distribution.

CHAPTER OUTLINE

14.01 Probability density functions
14.02 Calculating probabilities
14.03 Cumulative distribution function
14.04 Quantiles
14.05 Normal distribution
14.06 Empirical rule
14.07 *z*-scores
14.08 Applications of the normal distribution

IN THIS CHAPTER YOU WILL:

- recognise continuous random variables
- understand the properties of a probability density function (PDF)
- find cumulative distribution functions (CDF)
- find probabilities of continuous data
- calculate measures of central tendency and spread for continuous probability distributions
- recognise the normal distribution and identify its properties
- calculate probabilities and quantiles for normal distributions
- understand the standard normal distribution and *z*-scores
- apply the normal distribution to solving practical problems

TERMINOLOGY

continuous random variable: A random variable that can have any value along a continuum, for example, the height of a basketball player.

cumulative distribution function: A function $F(x)$ for the probability $P(X \le x)$.

empirical rule: The percentage probabilities (68%, 95%, 99.7%) that normally-distributed scores will lie within 1, 2, and 3 standard deviations, respectively, from the mean.

normal distribution: A continuous probability distribution in which the mean, mode and median are at the centre of a symmetrical bell-shaped graph.

probability density function: A function of a continuous random variable whose integral gives the probability $P(X \le x)$.

random variable: A variable whose values are based on a chance experiment; for example, the number of road accidents in an hour

uniform probability distribution: A probability distribution in which every outcome has the same probability

***z*-score:** Measures how many standard deviations above or below the mean a score is.

14.01 Probability density functions

Probability density functions

Probability density function

EXAMPLE 1

This table gives the results of a survey of different times that runners take to complete a race.

Time (min)	Frequency
0–<4	6
4–<8	8
8–<12	11
12–<16	4
16–<20	2
20–<24	1

a Add a column of relative frequencies.

b Sketch a frequency histogram for the relative frequencies.

c Estimate each probability:

i $P(X < 12)$ **ii** $P(X \ge 16)$ **iii** $P(4 \le X < 8)$

Solution

a

Time (min)	Frequency	Relative frequency
0–<4	6	$\frac{6}{32} = \frac{3}{16}$
4–<8	8	$\frac{8}{32} = \frac{1}{4}$
8–<12	11	$\frac{11}{32}$
12–<16	4	$\frac{4}{32} = \frac{1}{8}$
16–<20	2	$\frac{2}{32} = \frac{1}{16}$
20–<24	1	$\frac{1}{32}$

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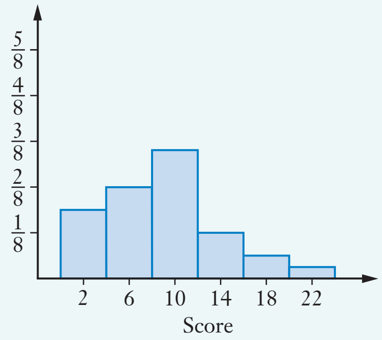
b



c **i** $P(X < 12) = \frac{6}{32} + \frac{8}{32} + \frac{11}{32}$

$= \frac{25}{32}$

ii $P(X \geq 16) = \frac{2}{32} + \frac{1}{32}$

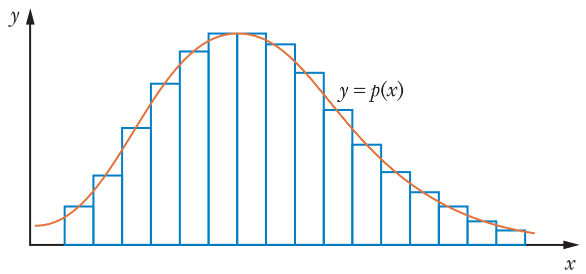
$= \frac{3}{32}$

iii $P(4 \leq X < 8) = \frac{1}{4}$

The times in the above example are values of a **continuous random variable**, but sorted into groups. While we can estimate probabilities using relative frequency, we use other methods when dealing with continuous data.

Continuous probability distributions

With continuous data we can't really draw a histogram as we did in Example 1 or there would be an 'infinite' number of columns with 'zero' widths. Instead, the probability distribution is a continuous curve.



A **continuous probability distribution** is represented by a function $P(X = x)$ or $p(x)$ called a **probability density function** (PDF) where X is the random variable. As with discrete probability distributions, the sum of all probabilities must be 1.

With a continuous probability distribution, we cannot calculate the probability for a single outcome, so $P(X = x) = 0$. Instead, we can only calculate the probability for a range of values such as $P(4 \leq X < 8)$.

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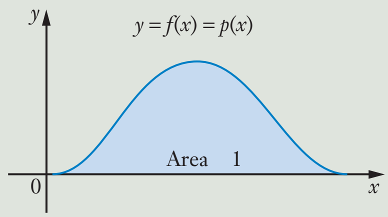
Area under a probability density function

The area under a probability density function is 1.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

where $f(x) \geq 0$ (since $0 \leq p(x) \leq 1$)



EXAMPLE 2

a A function is given by $f(x) = \begin{cases} \dfrac{3x^2}{26} & \text{for } 1 \leq x \leq 3 \\ 0 & \text{for all other } x \end{cases}$.

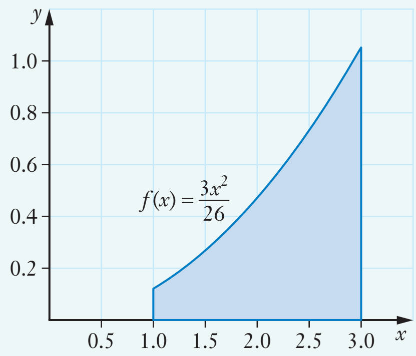
Show that it is a continuous probability distribution.

b A function is given by $f(x) = ax^2$ defined for the domain [0, 5].
Find the value of a for which this is a probability density function.

Solution

a For a continuous probability distribution, the area under the curve must be 1.

Drawing the graph, notice that the area will be 0 for all x values outside $1 \leq x \leq 3$.



$$\begin{aligned}
\text{Area} &= \int_1^3 \frac{3x^2}{26}\,dx \\
&= \frac{1}{26}\int_1^3 3x^2\,dx \\
&= \frac{1}{26}\Big[x^3\Big]_1^3 \\
&= \frac{1}{26}[3^3 - 1^3] \\
&= \frac{1}{26}[26] \\
&= 1
\end{aligned}$$

So $f(x)$ is a continuous probability distribution.

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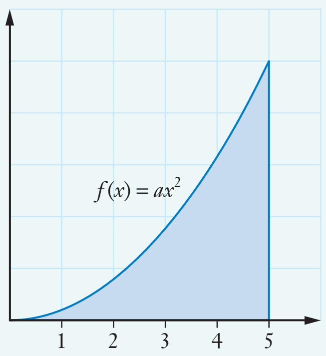
b Drawing the graph gives a parabola in the domain [0, 5].



A PDF has area 1.

So $\int_0^5 ax^2\,dx = 1$

$$\begin{aligned}\int_0^5 ax^2\,dx &= a\int_0^5 x^2\,dx \\ &= a\left[\frac{x^3}{3}\right]_0^5 \\ &= a\left[\frac{5^3}{3} - \frac{0^3}{3}\right] \\ &= a\left[\frac{125}{3}\right] \\ &= \frac{125a}{3}\end{aligned}$$

So $\frac{125a}{3} = 1$

$a = \frac{3}{125}$

Exercise 14.01 Probability density functions

1 State whether each random variable is discrete or continuous.

- **a** The size of T-shirts worn by people
- **b** The speed of cars as they pass a certain point
- **c** The volume of water in a dam
- **d** The number of seats on an aeroplane
- **e** The weight of babies born in August

2 Which of the following functions describe continuous probability distributions?

- **a** $f(x) = 0.2$ in the domain [1, 6]
- **b** $f(x) = \frac{x}{12}$ in the domain [0, 6]
- **c** $f(x) = \begin{cases} \frac{x^3}{324} & \text{for } 0 \le x \le 3 \\ 0 & \text{for all other } x \end{cases}$
- **d** $f(x) = \frac{x^2}{21}$ in the interval $1 \le x \le 4$
- **e** $f(x) = \begin{cases} \frac{x}{8} & \text{for } 1 \le x \le 8 \\ 0 & \text{for all other } x \end{cases}$

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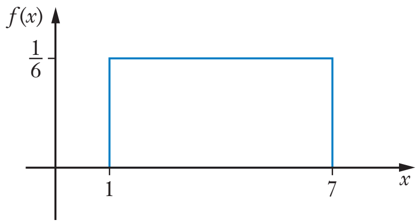
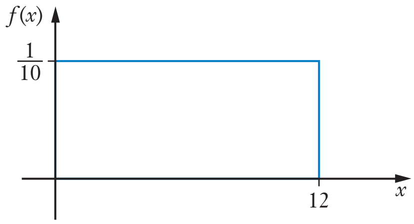
3 Which of the following graphs are of probability density functions?

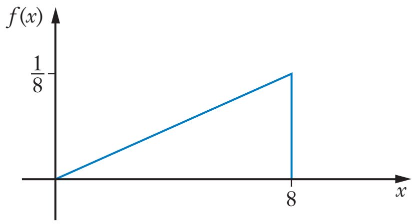
a



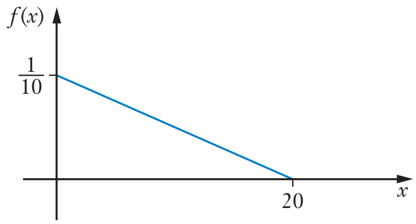
b



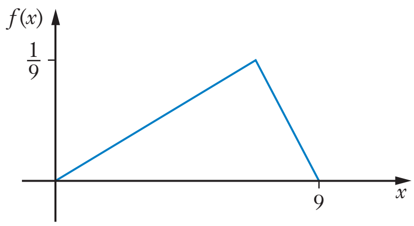
c



d



e



4 A probability density function has the equation $f(x) = \frac{x^4}{3355}$ over the domain $[2, b]$. Evaluate b.

5 Given the continuous probability distribution $f(x) = \begin{cases} kx^3 & \text{for } 0 \le x \le 5 \\ 0 & \text{for all other } x \end{cases}$

find the value of k.

6 A probability density function is given by $f(x) = ae^x$ over a certain domain. Find the exact value of a if the domain is:

a $[1, 3]$ **b** $[1, 7]$ **c** $[0, 4]$

7 A function is given by $f(x) = \frac{x^2}{72}$.

Over what domain starting at $x = 0$ is this a probability density function?

8 A PDF is given by $f(x) = \frac{2x^5}{87\,381}$ over the interval $1 \le x \le b$. Find the value of b.

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14.02 Calculating probabilities

Since $P(X = x) = 0$ for continuous probability distributions, we can only find the probability of a **range** of values $P(a \le X \le b)$.

WS Probability density functions

WS Uniform and triangular probability density functions

Also, since $P(X = a) = 0$ and $P(X = b) = 0$, it makes no difference whether we use $\le$ or $<$, $\ge$ or $>$.

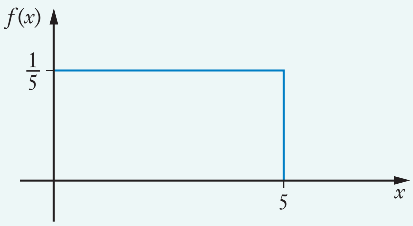
$P(a < X < b) = P(a \le X \le b)$

EXAMPLE 3

For the probability density function, find:

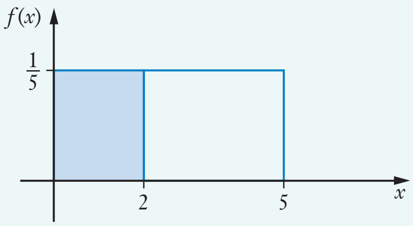
a $P(X \le 2)$

b $P(1 < X < 4)$



Solution

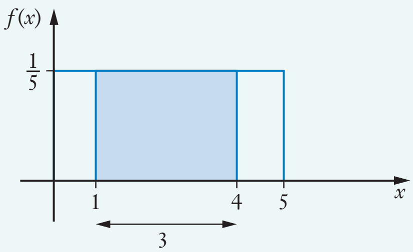
a $P(X \le 2)$ is the shaded area between $x = 0$ and $x = 2$.



$$P(X \le 2) = 2 \times \frac{1}{5} = \frac{2}{5}$$

b $P(1 < X < 4)$ is the shaded area between $x = 1$ and $x = 4$.

Notice that $P(1 < X < 4) = P(1 \le X \le 4)$ since $P(X = 1) = P(X = 4) = 0$



$$P(1 < X < 4) = 3 \times \frac{1}{5} = \frac{3}{5}$$

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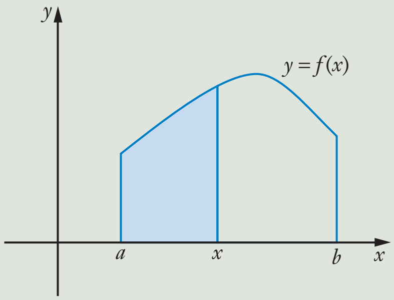
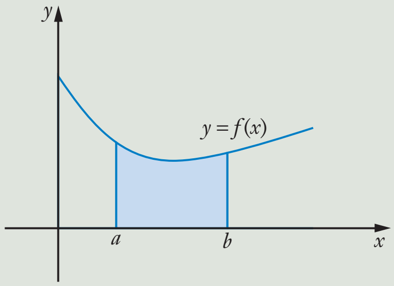
Probabilities in probability density functions

$P(X \le x) = \int_a^x f(x)\,dx$ where $y = f(x)$ is a PDF defined in the domain $[a, b]$.



$P(a \le X \le b) = \int_a^b f(x)\,dx$ where $y = f(x)$ is a PDF and a and b are in the defined domain.



EXAMPLE 4

A function is given by $f(x) = \frac{3x^2}{117}$ defined in the domain [2, 5]. Find:

a $P(X \le 4)$

b $P(3 \le X \le 4)$

Solution

a $P(X \le 4) = \int_2^4 \frac{3x^2}{117}\,dx$ (since domain is [2, 5])

$$= \frac{1}{117}\int_2^4 3x^2\,dx$$

$$= \frac{1}{117}\left[x^3\right]_2^4$$

$$= \frac{1}{117}(4^3 - 2^3)$$

$$= \frac{56}{117}$$

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b $P(3 \le X \le 4) = \int_3^4 \frac{3x^2}{117}\,dx$

$= \frac{1}{117}\int_3^4 3x^2\,dx$

$= \frac{1}{117}\left[x^3\right]_3^4$

$= \frac{1}{117}(4^3 - 3^3)$

$= \frac{37}{117}$

Uniform distributions

In Year 11, Chapter 12, *Discrete probability distributions*, you learned that with a **uniform probability distribution**, every outcome has the same probability.

EXAMPLE 5

A continuous probability function $y = f(x)$ is uniform in the domain [5, 15].

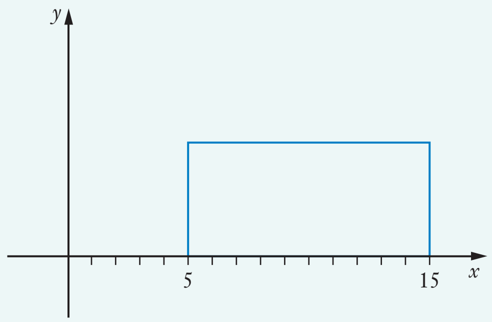
a Sketch the probability density function.

b Find:

i $P(X \ge 8)$ **ii** $P(7 \le X \le 10)$ **iii** $P(8 < X < 11)$

Solution

a A uniform distribution has all equal probabilities so will have the same height. This gives a rectangle.



PDF has area 1:

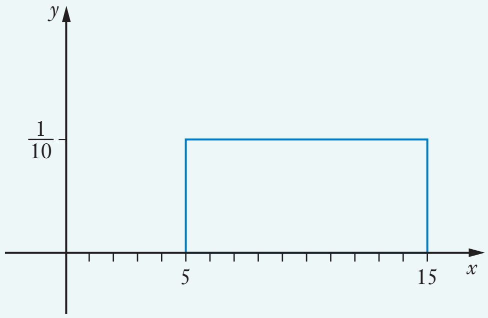
$A = bh$

$= (15 - 5) \times h$

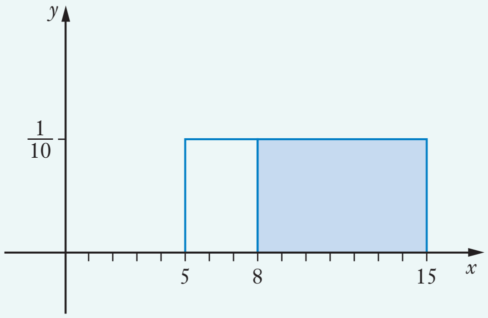
$= 10h$

So $10h = 1$

$h = \frac{1}{10}$



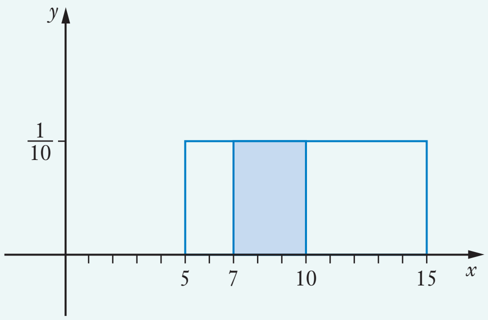
b **i**



$$P(X \geq 8) = (15 - 8) \times \frac{1}{10}$$

$$= \frac{7}{10}$$

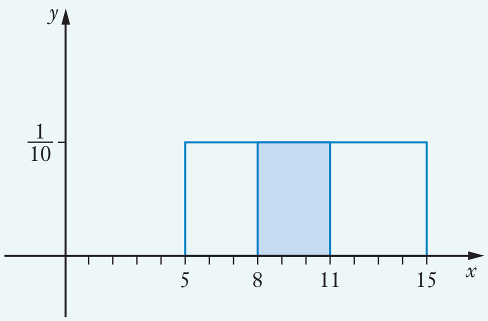
ii



$$P(7 \leq X \leq 10) = (10 - 7) \times \frac{1}{10}$$

$$= \frac{3}{10}$$

iii



$$P(8 < X < 11) = (11 - 8) \times \frac{1}{10}$$

$$= \frac{3}{10}$$

Notice in the example that intervals with the same width have the same probability.

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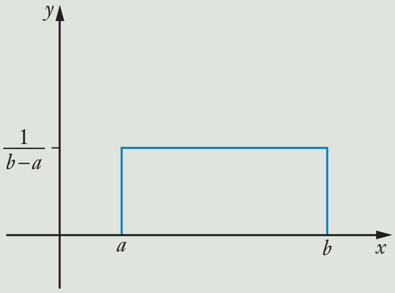
Uniform continuous probability distributions

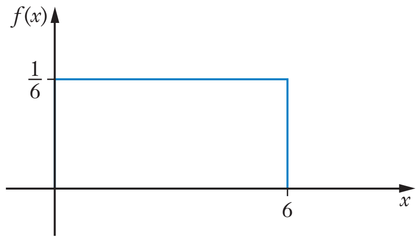
$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for all other } x \text{ values} \end{cases}$$



Equal intervals along the x-axis will have the same probability.

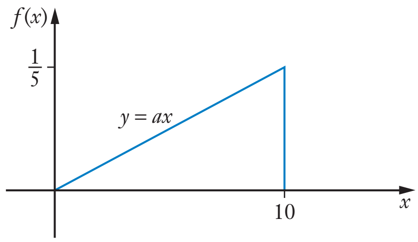
Exercise 14.02 Calculating probabilities

1 For the continuous probability distribution graphed, find:



- **a** $P(X \le 3)$
- **b** $P(1 \le X \le 2)$
- **c** $P(1 \le X \le 4)$
- **d** $P(X < 4)$
- **e** $P(X \ge 4)$

2 A probability density function is shown.



- **a** Find the equation of the linear function $y = ax$.
- **b** Find:
  - **i** $P(X < 9)$
  - **ii** $P(X \le 3)$
  - **iii** $P(4 \le X \le 7)$
  - **v** $P(2 < X < 6)$
  - **v** $P(X > 5)$

3 The continuous probability distribution is defined by $f(x) = ax^2$ in the domain [0, 5].

- **a** Evaluate a.
- **b** Find:

i $P(X \le 3)$	**ii** $P(1 < X < 4)$	**iii** $P(X > 2)$
v $P(X < 1)$	**v** $P(3 \le X < 4)$	

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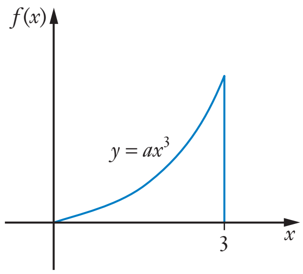
4 The continuous random variable X has the PDF shown.



a Evaluate a.

b Find:

i $P(1 \le X \le 3)$ **ii** $P(X < 2)$

iii $P(1 \le X \le 2)$ **v** $P(X \le 1)$

5 A continuous probability function is given by $f(x) = ke^x$, defined on the domain [1, 6].

a Find the exact value of k.

b Find each exact probability:

i $P(2 \le X \le 5)$ **ii** $P(X < 4)$ **iii** $P(X \ge 3)$

6 a Show that $y = \sin x$ is a probability density function in the domain $\left[0, \frac{\pi}{2}\right]$.

b Find each exact probability:

i $P\left(X \le \frac{\pi}{3}\right)$ **ii** $P\left(0 < X < \frac{\pi}{4}\right)$ **iii** $P\left(X > \frac{\pi}{6}\right)$

7 a Show that the uniform distribution $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for all other } x \text{ values} \end{cases}$

is a probability density function.

b If $a = 3$ and $b = 7$, find:

i $P(X \le 6)$ **ii** $P(X \ge 5)$ **iii** $P(5 \le X \le 6)$

WS Cumulative distribution function

14.03 Cumulative distribution function

In the previous section you integrated the PDF each time to find $P(X \le x)$, a cumulative probability. The **cumulative distribution function** is a general formula for finding $P(X \le x)$ directly.

Cumulative distribution function (CDF)

The cumulative distribution function is given by $F(x) = \int_a^x f(x)\,dx$ where $y = f(x)$ is a PDF defined in the domain $[a, b]$.

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EXAMPLE 6

A continuous probability function is given by $f(x) = \frac{4x^3}{255}$ defined in the domain [1, 4].

a Find the cumulative distribution function.

b Use the CDF to find:

i $P(X \leq 3)$ **ii** $P(X < 1.6)$

Solution

a $F(x) = \int_a^x f(x)\,dx$ where $f(x) = \frac{4x^3}{255}$ is a PDF defined in the domain [1, 4].

$$= \int^x \frac{4x^3}{255}\,dx$$

$$= \frac{1}{255}\int^x 4x^3\,dx$$

$$= \frac{1}{255}\left[x^4\right]^x$$

$$= \frac{1}{255}(x^4 - 1^4)$$

$$= \frac{1}{255}(x^4 - 1)$$

$$= \frac{x^4 - 1}{255}$$

b Using $F(x) = \frac{x^4 - 1}{255}$ to find $P(X \leq x)$:

i For $P(X \leq 3)$:

$$F(3) = \frac{3^4 - 1}{255}$$

$$= \frac{81 - 1}{255}$$

$$= \frac{80}{255}$$

$$= \frac{16}{51}$$

So $P(X \leq 3) = \frac{16}{51}$

ii For $P(X < 1.6)$:

$$F(1.6) = \frac{1\,6^4 - 1}{255}$$

$$= 0.0218$$

So $P(X < 1.6) = 0.0218$

We can use the cumulative distribution function to find probabilities such as $P(X \geq a)$ or $P(a \leq X \leq b)$.

EXAMPLE 7

A continuous probability function is given by $f(x) = \dfrac{3x^2}{335}$ defined in the domain [2, 7].

a Find the cumulative distribution function.

b Use the CDF to find:

i $P(X \geq 4)$ **ii** $P(3.5 \leq X \leq 6.2)$

Solution

a $F(x) = \int_a^x f(x)\,dx$ where $f(x) = \dfrac{3x^2}{335}$ is a PDF defined in the domain [2, 7].

$$= \int_2^x \frac{3x^2}{335}\,dx$$

$$= \frac{1}{335}\int_2^x 3x^2\,dx$$

$$= \frac{1}{335}\left[x^3\right]_2^x$$

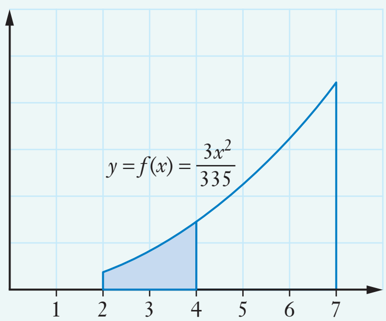
$$= \frac{1}{335}(x^3 - 2^3)$$

$$= \frac{1}{335}(x^3 - 8)$$

$$= \frac{x^3 - 8}{335}$$

b We use $F(x) = \dfrac{x^3 - 8}{335}$ to find $P(X \leq x)$.

i



To find $P(X \geq 4)$, first find $P(X \leq 4)$:

The shaded part of the PDF is $P(X \leq 4)$.

$$P(X \leq 4) = \frac{4^3 - 8}{335}$$

$$= \frac{56}{335}$$

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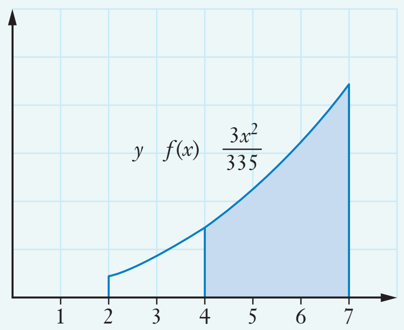


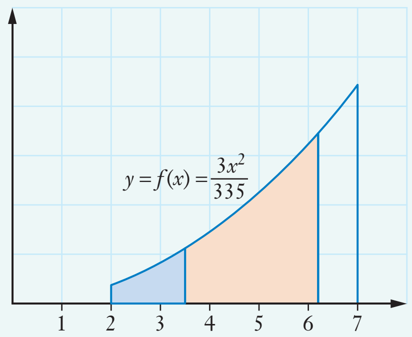
Area under the curve is 1:

$$P(X \geq 4) = 1 - P(X \leq 4)$$

$$= 1 - \frac{56}{335}$$

$$= \frac{279}{335}$$

ii



$P(X \leq 3.5)$ is the area to the left of $x = 3.5$

$P(X \leq 6.2)$ is the area to the left of $x = 6.2$

So $P(3.5 \leq X \leq 6.2) = P(X \leq 6.2) - P(X \leq 3.5)$

$$= \frac{6\ 2^3 - 8}{335} - \frac{3\ 5^3 - 8}{335}$$

$$= 0.688 - 0.104$$

$$= 0.583$$

Mode of a continuous probability distribution

We sometimes want to know what the highest probability is. This is the **mode**.

Mode

The mode is the maximum point of the probability density function.

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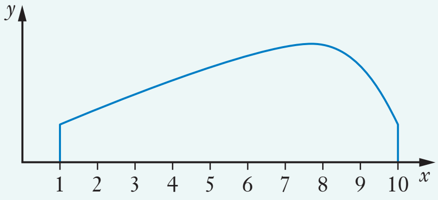
EXAMPLE 8

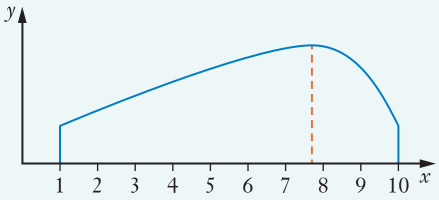
a Find the mode of the continuous probability distribution shown below.



b A continuous probability distribution is defined on the interval $1 \leq x \leq 5$ and has equation $f(x) = \frac{3x(6-x)}{92}$. Find the mode.

Solution

a



The highest point of the PDF is at $x = 7.7$

So the mode is 7.7.

b $f(x) = \frac{3x(6-x)}{92}$

$= \frac{3}{92}(6x - x^2)$

The function is a parabola with $a < 0$ so will have a maximum turning point.
We use calculus to see if this point lies within the defined domain [1, 5].
(We could also use $x = -\frac{b}{2a}$ for the axis of symmetry of a parabola).

$f'(x) = \frac{3}{92}(6 - 2x)$

For stationary points:

$f'(x) = 0$

$\frac{3}{92}(6 - 2x) = 0$

$6 - 2x = 0$

$6 = 2x$

$3 = x$

$x = 3$ lies in the domain [1, 5].

$f''(x) = \frac{3}{92}(-2)$

$= -\frac{3}{46}$

< 0

Concave down so a maximum turning point.

So the mode is 3.

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Exercise 14.03 Cumulative distribution function

1 Find the cumulative distribution function for each continuous probability distribution.

a $f(x) = \frac{x^2}{9}$ defined in the domain [0, 3]

b $f(x) = \frac{4x^3}{1296}$ defined in the domain [0, 6]

c $f(x) = \frac{e^x}{e^4 - 1}$ in the interval $0 \le x \le 4$

d $f(x) = \frac{4(x-2)^3}{625}$ in the domain [2, 7]

e $f(x) = \frac{3x(8-x)}{135}$ in the domain [2, 5]

2 a Find the cumulative distribution function for $f(x) = \begin{cases} \frac{5x^4}{7776} & \text{for } 1 \le x \le 6 \\ 0 & \text{for all other values} \end{cases}$.

b Find:

i $P(X \le 3)$ **ii** $P(X \le 2)$ **iii** $P(X < 5)$

v $P(X > 4)$ **v** $P(2 \le X \le 4)$

3 A continuous probability distribution is given by $f(x) = \frac{4x^3}{2320}$ in the domain [3, 7].

a Find the cumulative distribution function.

b Find:

i $P(X \le 4)$ **ii** $P(X \le 6)$ **iii** $P(X \ge 5)$

v $P(X > 4)$ **v** $P(4 \le X < 6)$

4 A continuous probability distribution is defined by $f(x) = \frac{2e^{2x}}{e^{10} - 1}$ in the domain [0, 5].

a Find the cumulative distribution function.

b Calculate each probability correct to 2 significant figures.

i $P(X \le 2)$ **ii** $P(X \le 4)$ **iii** $P(X > 3)$

v $P(X \ge 2.8)$ **v** $P(2 \le X \le 4)$

5 a Evaluate a if $f(x) = ax^3$ is a continuous probability distribution defined in the domain [0, 9].

b Find the cumulative distribution function.

c Find:

i $P(X \le 5)$ **ii** $P(X \le 4)$ **iii** $P(X > 8)$

v $P(X \ge 3)$ **v** $P(2 \le X \le 6)$

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6 **a** Find the exact value of a if $f(x) = \frac{a}{x}$ is a continuous probability distribution defined in the domain [1, 6].

b Find the cumulative distribution function.

c Find to 2 decimal places:

i $P(X \le 3)$ **ii** $P(X \le 2)$ **iii** $P(X > 5)$

v $P(X \ge 4)$ **v** $P(2 \le X \le 5)$

7 **a** Show that $y = \cos x$ is a probability density function in the domain $\left[\frac{3\pi}{2}, 2\pi\right]$.

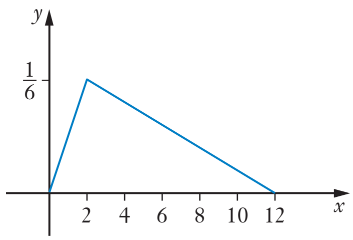
b Find the cumulative distribution function.

c Find each probability in exact form:

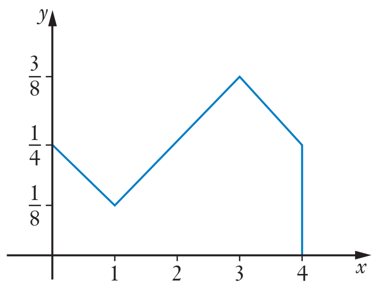
i $P\left(X \le \frac{5\pi}{3}\right)$ **ii** $P\left(X \ge \frac{7\pi}{4}\right)$ **iii** $P\left(\frac{5\pi}{3} \le X \le \frac{11\pi}{6}\right)$

8 Find the mode of each continuous probability distribution.

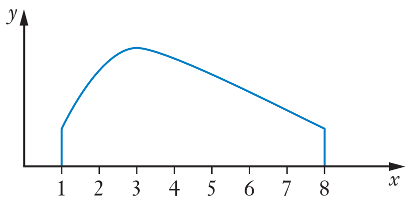
a



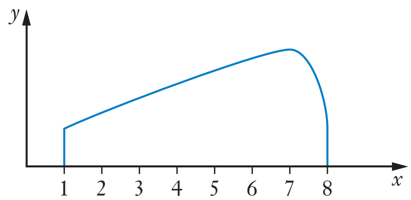
b



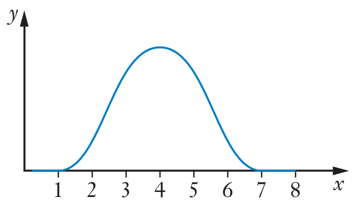
c



d



e



f $f(x) = -\frac{3(x^2 - 8x - 9)}{434}$ defined in the domain [0, 7]

g $f(x) = \frac{4e^{4x}}{e^8(e^{16} - 1)}$ defined in the interval $2 \le x \le 6$

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h $f(x) = -\frac{3(x^2 - 16x + 15)}{1100}$ defined in the domain [1, 11]

i $f(x) = \frac{2(2x^3 - 33x^2 + 168x + 3)}{2105}$ defined in the interval $0 \le x \le 5$

j $f(x) = \frac{3x^2}{342}$ defined in the interval $1 \le x \le 7$

9 **a** Find the mode of the function $f(x) = -\frac{3}{22}(x^2 - 6x + 5)$ defined on the domain [2, 4].

b Find the cumulative distribution function.

c Find $P(X \le a)$ where a is the mode.

10 The times that athletes took to finish a race varied between 3 and 7 minutes and are represented by the continuous probability function $f(x) = \frac{1}{116}(x^3 - 9x^2 + 24x + 1)$ defined in the domain [3, 7].

a Find the cumulative distribution function.

b Find the probability that an athlete will finish this race:

i in less than 5 minutes

ii in 4 minutes or more

iii in between 4 and 5 minutes

c What is the most likely time in which an athlete would finish the race?

14.04 Quantiles

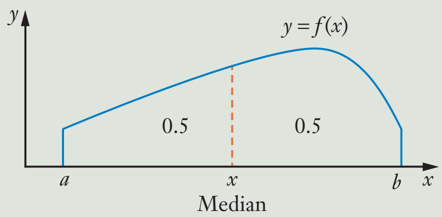
WS Quantiles

Median

For a continuous probability distribution, the **median** is the value of x that splits the distribution into halves. Because the PDF has an area of 1, the area on each side of the median is $\frac{1}{2}$.

Median

The median lies at the point x where $\int_a^x f(x)\,dx = 0.5$ given $y = f(x)$ is a PDF defined in the domain $[a, b]$.



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The median of a continuous probability distribution

EXAMPLE 9

Find the median of the continuous probability distribution defined as $f(x) = \frac{x^2}{21}$ in the domain [1, 4].

Solution

For $f(x) = \frac{x^2}{21}$ defined in the domain [1, 4], first find the cumulative distribution function (CDF):

$$\int^x \frac{x^2}{21}\,dx = \frac{1}{21}\int^x x^2\,dx$$

$$= \frac{1}{21}\left[\frac{x^3}{3}\right]^x$$

$$= \frac{1}{21}\left(\frac{x^3}{3} - \frac{1^3}{3}\right)$$

$$= \frac{1}{21}\left(\frac{x^3 - 1}{3}\right)$$

$$= \frac{x^3 - 1}{63}$$

For the median:

$$\int_a^x f(x)\,dx = 0\ 5$$

$$\frac{x^3 - 1}{63} = 0.5$$

$$x^3 - 1 = 31.5$$

$$x^3 = 32.5$$

$$x = \sqrt[3]{32\ 5}$$

$$\approx 3.2$$

So the median is 3.2.

This means $P(X < 3.2) = 0.5$.

Quartiles, deciles and percentiles

You learned about quartiles, deciles and percentiles in Chapter 9, *Statistics*. They are values that separate a proportion of a set of data. For example, Q_1 > bottom 25% of scores, Q_3 > bottom 75% of scores, 2nd decile > bottom 20% of scores and 67th percentile > bottom 67% of scores.

EXAMPLE 10

A continuous probability distribution is defined as $f(x) = \frac{x^4}{11\,605}$ in the domain [4, 9]. Find, correct to 2 decimal places:

a the 1st quartile

b the 38th percentile

c the 7th decile

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Solution

First, find the CDF.

$$\int_4^x \frac{x^4}{11\,605}\,dx = \frac{1}{11605}\int_4^x x^4\,dx$$

$$= \frac{1}{11\,605}\left[\frac{x^5}{5}\right]_4^x$$

$$= \frac{1}{11\,605}\left(\frac{x^5}{5} - \frac{4^5}{5}\right)$$

$$= \frac{1}{11\,605}\left(\frac{x^5 - 1024}{5}\right)$$

$$= \frac{x^5 - 1024}{58\,025}$$

a 1st quartile: 25%

$$\int_a^x f(x)\,dx = 0.25$$

$$\frac{a^5 - 1024}{58\,025} = 0.25$$

$$a^5 - 1024 = 14\,506.25$$

$$a^5 = 15\,530.25$$

$$a \approx 6.89$$

So the 1st quartile is 6.89.

This means $P(X < 6.89) = 0.25$.

b 38th percentile: 38%

$$\int_a^x f(x)\,dx = 0.38$$

$$\frac{a^5 - 1024}{58\,025} = 0.38$$

$$a^5 - 1024 = 22\,049.5$$

$$a^5 = 23\,073.5$$

$$a \approx 7.46$$

So the 38th percentile is 7.46.

This means $P(X < 7.46) = 0.38$.

c 7th decile: 70%

$$\int_a^x f(x)\,dx = 0.7$$

$$\frac{a^5 - 1024}{58\,025} = 0.7$$

$$a^5 - 1024 = 40\,617.5$$

$$a^5 = 41\,641.5$$

$$a \approx 8.39$$

So the 7th decile is 8.39.

This means $P(X < 8.39) = 0.7$.

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Exercise 14.04 Quantiles

1 Find the median of each continuous random variable correct to 2 decimal places.

a $f(x) = \frac{3x^2}{511}$ defined on the interval $1 \le x \le 8$

b $f(x) = \frac{4x^3}{2401}$ defined in the domain $[0, 7]$

c $f(x) = \frac{5x^4}{16\,807}$ in the interval $0 \le x \le 7$

d $f(x) = \frac{3(x-3)^2}{16}$ in the domain $[1, 5]$

e $f(x) = \frac{(3x+1)^2}{244}$ in the interval $0 \le x \le 4$

f $f(x) = \frac{4x^3}{6560}$ defined in the domain $[1, 9]$

g $f(x) = \frac{3x^2}{1034}$ defined in the domain $[3, 11]$

h $f(x) = \frac{6x^5}{15\,625}$ in the interval $0 \le x \le 5$

i $f(x) = \frac{(2x-1)^4}{16\,105}$ in the domain $[1, 6]$

j $f(x) = \frac{x(x^2-3)^3}{3570}$ defined in the domain $[2, 4]$

2 For each continuous probability distribution, find:

i the 1st quartile **ii** the 2nd decile **iii** the 77th percentile

a $f(x) = \frac{3x^2}{973}$ defined in the domain $[3, 10]$

b $f(x) = \frac{x^3}{324}$ defined in the interval $0 \le x \le 6$

c $f(x) = \frac{5x^4}{3124}$ defined in the interval $1 \le x \le 5$

3 For the continuous probability distribution $f(x) = \frac{3x^2}{512}$ defined in the domain $[0, 8]$, find:

a the median

b the 35th percentile

4 For the continuous probability distribution $f(x) = \frac{x^2}{168}$ defined on the interval $2 \le x \le 8$, find:

a the median **b** the 1st quartile **c** the 3rd quartile

d the 67th percentile **e** the 14th percentile **f** the 8th decile

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5 For the continuous probability distribution defined as $f(x) = \frac{x^2}{576}$ on the interval $0 \le x \le 12$, find:

a the 20th percentile

b the median

c the 3rd quartile

6 For the continuous probability distribution $f(x) = \frac{x^3}{1020}$ defined in the interval $2 \le x \le 8$, find:

a the cumulative probability function

b $P(X \le 5)$

c $P(X > 4)$

d $P(3 \le X \le 7)$

e the median

f the 3rd quartile

g the 9th decile

h the 23rd percentile

14.05 Normal distribution

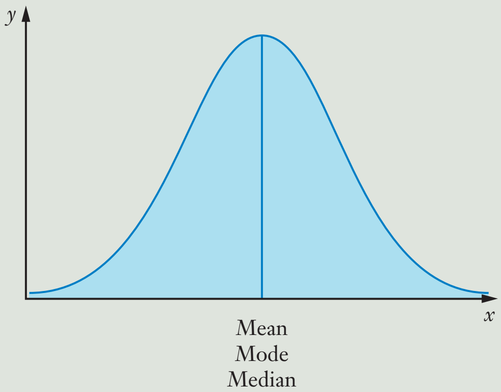
WS The normal distribution

The **normal distribution** is a special continuous probability distribution. Its probability density function is often called a **bell curve** because of its shape.

Normal distribution

The normal distribution is a symmetrical bell-shaped function.

The mean, mode and median are equal, at the centre of the probability density function.



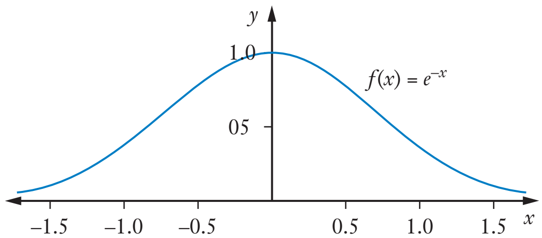
There are many examples of data that are normally distributed, such as IQ, birth weights, ages, reaction times and exam results in a school.

We use the population mean μ and standard deviation σ for the normal distribution.

TECHNOLOGY

The normal distribution

A good estimate for the **shape** of the normal distribution is $f(x) = e^{-x}$.



Use your graphing techniques and technology to sketch this function.

Given a population that is normally distributed with mean μ = 10 and standard deviation σ = 2, use technology to sketch the graph of

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)}{2\sigma}}$$

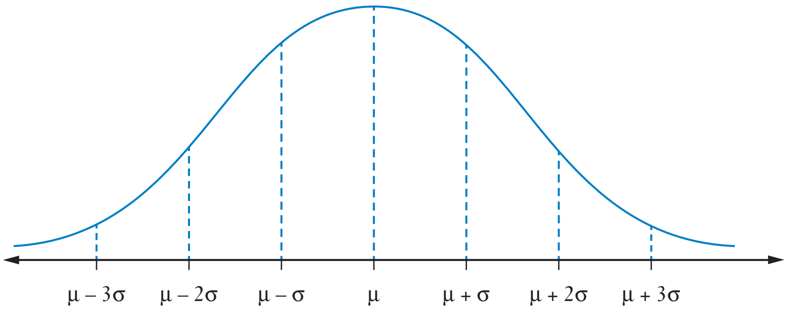
This is the actual equation of the normal distribution.

What do you notice about this graph? Use technology and the trapezoidal rule with many subintervals to approximate the area under the graph. What do you find?

Research the normal distribution and its features.

Graphing a normal distribution

In a normal distribution, most of the data lies within 3 standard deviations of the mean, with the mean in the centre.



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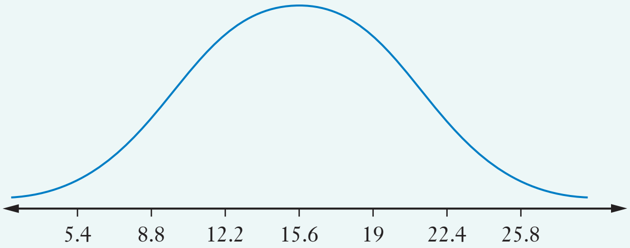
EXAMPLE 11

a A set of data is normally distributed with mean 8.3 and standard deviation 1.2. Sketch the probability distribution function.

b A normal distribution has the probability density function below. Find its mean and standard deviation.



Solution

a

$\mu + \sigma = 8.3 + 1.2$
$= 9.5$

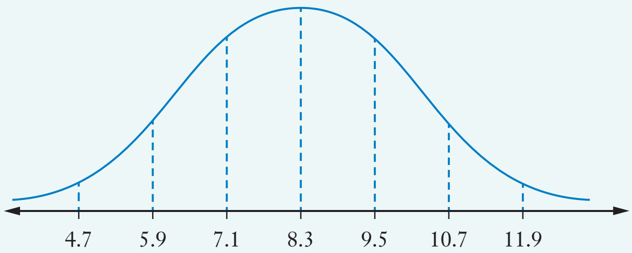
$\mu + 2\sigma = 8.3 + 2(1.2)$
$= 10.7$

$\mu + 3\sigma = 8.3 + 3(1.2)$
$= 11.9$

$\mu - \sigma = 8.3 - 1.2$
$= 7.1$

$\mu - 2\sigma = 8.3 - 2(1.2)$
$= 5.9$

$\mu - 3\sigma = 8.3 - 3(1.2)$
$= 4.7$



b

$\mu = 15.6$ (centre of the distribution)

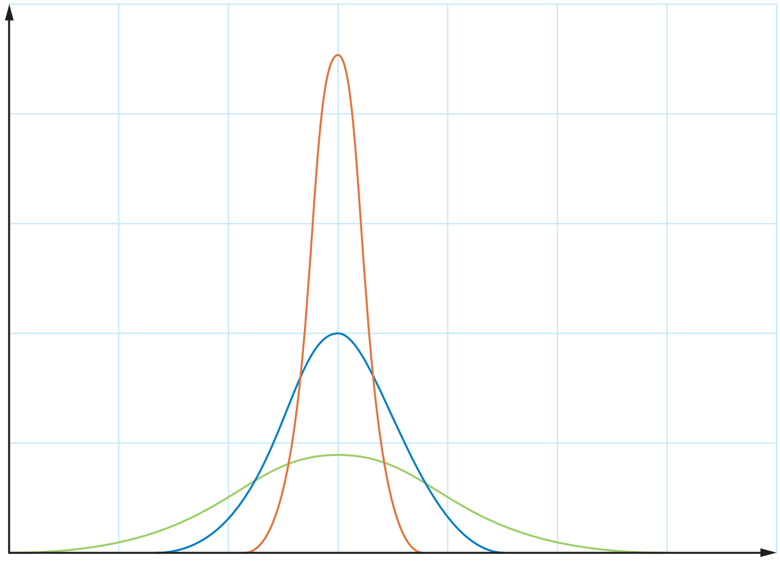
$\mu + \sigma = 19$

$15.6 + \sigma = 19$

$\sigma = 19 - 15.6$

$= 3.4$ (check that all the differences between all values on the graph are 3.4)

The normal distribution can have different shapes depending on the size of the standard deviation. In the diagram, the green curve shows the normal distribution with the highest standard deviation.



Standard normal distribution

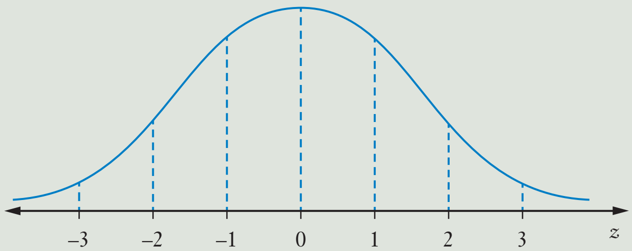
It is difficult to find probabilities in the normal distribution by integration because the equation of the cumulative distribution function is complicated. To get around this, tables of probabilities have been developed from the **standard normal distribution**. This is a normal curve that has been transformed so that the mean is 0 and the standard deviation is 1. The values of a standard normal distribution are called z rather than x, also known as **z-scores** or **standardised scores**.

Standard normal distribution

$\mu = 0$

$\sigma = 1$

Area under the curve is 1.



WS Normal distribution table

Probability tables for the standard normal distribution

Probability tables for the standard normal distribution begin next page. A copy can also be downloaded from *NelsonNet*. Values represent the area to the left of (or less than) the z-score. Row labels show the z-score to one decimal place. Column labels show the second decimal place.

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z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
–3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
–3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
–3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
–3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
–3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
–2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
–2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
–2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
–2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
–2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
–2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
–2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
–2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
–2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
–2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
–1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
–1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
–1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
–1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
–1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
–1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
–1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
–1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
–1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
–1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
–0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
–0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

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EXAMPLE 12

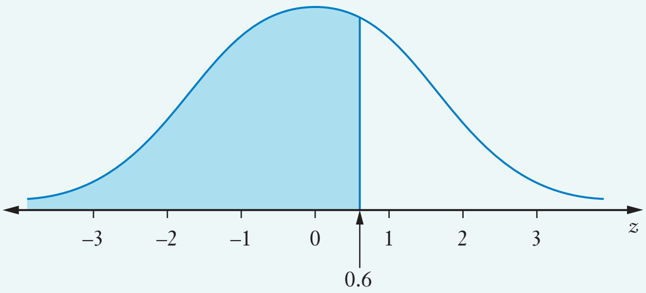
For a standard normal distribution, use the table to find:

a $P(z \leq 0.6)$ **b** $P(z \leq -1.83)$ **c** $P(z < 2.34)$

d $P(z \geq -2.7)$ **e** $P(-0.3 \leq z \leq 1.4)$

Solution

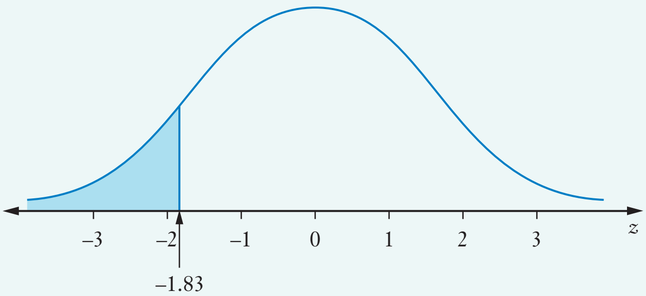
a The table gives the area under the PDF for the standard normal distribution.



Find 0.6 in the left column of the table.

$P(z \leq 0.6) = 0.7257$

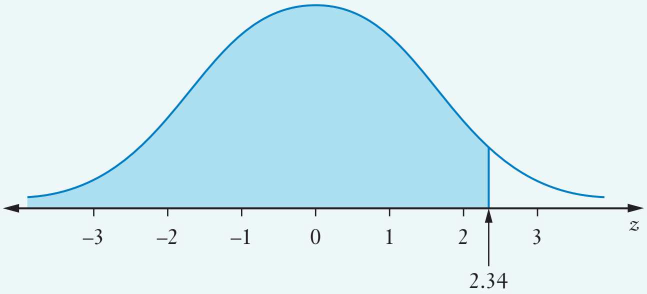
b



For –1.83, find –1.8 in the left column of the table and the entry under 0.03 in this row.

$P(z \leq -1.83) = 0.0336$

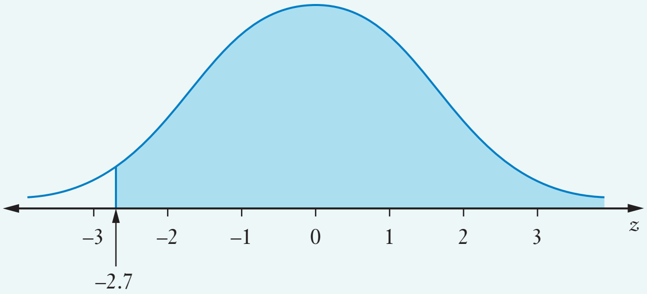
c



For 2.34, find 2.3 in the left column of the table and the entry under 0.04 in this row.

$P(z < 2.34) = 0.9904$

d



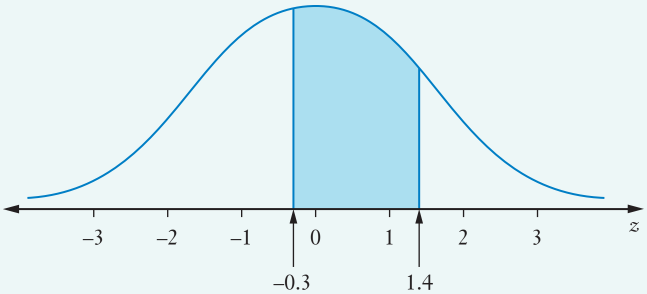
Find –2.7 in the left column of the table.

$P(z \le -2.7) = 0.0035$

$P(z \ge -2.7) = 1 - 0.0035$

$= 0.9965$

e



$P(-0.3 \le z \le 1.4) = P(z \le 1.4) - P(z \le -0.3)$

$= 0.9192 - 0.3821$

$= 0.5371$

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Quartiles, deciles and percentiles

EXAMPLE 13

For a standard normal distribution, use the table on p.633 to find:

a the median **b** the lower quartile **c** the upper quartile

d the 84th percentile **e** the 6th decile

Solution

a The median separates the bottom half of the scores.

We want to find the score with the cumulative probability of 0.5 in the table.

Median = 0 (this is the same value as the mean)

b The lower quartile separates the bottom 0.25 of the data.

In the table there are 2 values close to 0.25:

$P(z \le -0.67) = 0.2514$ and $P(z \le -0.68) = 0.2483$

0.2514 is closer to 0.25 so the lower quartile is approximately −0.67.

(Note: $0.2514 - 0.25 = 0.0014$, and $0.25 - 0.2483 = 0.0017$.)

c The upper quartile separates the bottom 0.75 of the data.

In the table there are 2 values close to 0.75:

$P(z \le 0.67) = 0.7486$ and $P(z \le 0.68) = 0.7517$

0.7486 is closer to 0.75 so the upper quartile is approximately 0.67.

Notice that the values for the lower and upper quartiles are ±0.67 because the normal distribution is symmetrical.

d The 84th percentile separates the bottom 0.84 of the data.

In the table there are 2 values close to 0.84:

$P(z \le 0.99) = 0.8389$ and $P(z \le 1) = 0.8413$

0.8389 is closer to 0.84 so the 84th percentile is approximately 0.99.

e The 6th decile separates the bottom 0.6 of the data.

In the table there are 2 values close to 0.6:

$P(z \le 0.25) = 0.5987$ and $P(z \le 0.26) = 0.6026$

0.5897 is closer to 0.6 so the 6th decile is approximately 0.25.

EXAMPLE 14

Shade the area of the normal distribution where the values are:

a above the top 20% of data

b above the top 10% of data

Solution

a $P(z > a) = 20\%$, so $P(z \leq a) = 80\%$ (0.8 or the 8th decile)

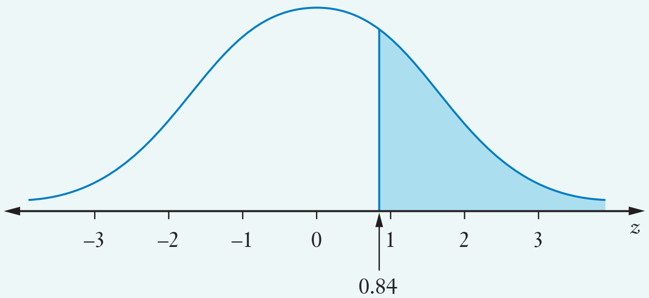
From the table:

$P(z \leq 0.84) = 0.7995$ and $P(z \leq 0.85) = 0.8023$

0.7995 is closer to 0.8 so the 8th decile is approximately 0.84

This means all values to the left of 0.84 lie below 80%.

So the top 20% lies to the right of 0.84.



b $P(z > a) = 10\%$, so $P(z \leq a) = 90\%$ (0.9 or the 9th decile)

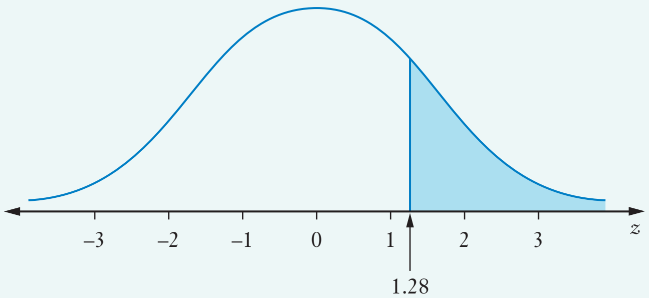
From the table:

$P(z \leq 1.28) = 0.8997$ and $P(z \leq 1.29) = 0.9015$

0.8997 is closer to 0.9 so the 9th decile is approximately 1.28.

This means all values to the left of 1.28 lie below 90%.

So the top 10% lies to the right of 1.28.



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Exercise 14.05 Normal distribution

1 Draw a probability density function for the normal distribution with:

a mean 9 and standard deviation 2

b mean 8.6 and standard deviation 0.3

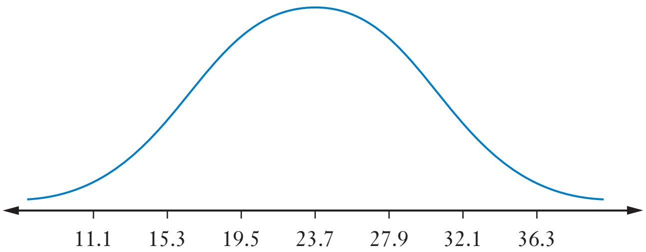
c mean 11.5 and standard deviation 1.4

d mean 27 and standard deviation 2.5

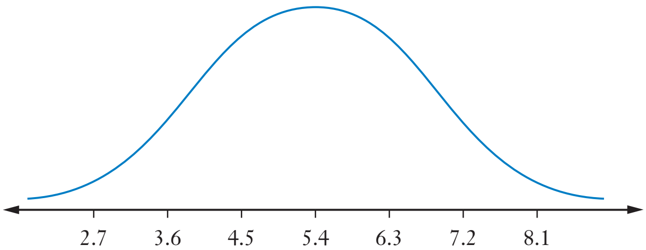
e mean 115.2 and standard deviation 3.2

2 What is the mean and standard deviation of each normal distribution?

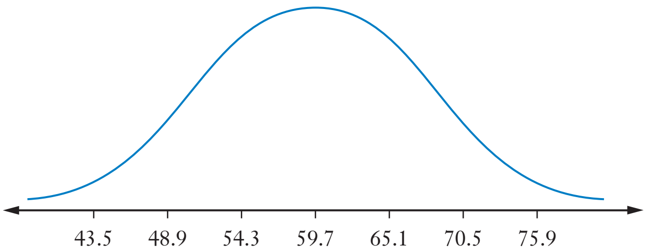
a



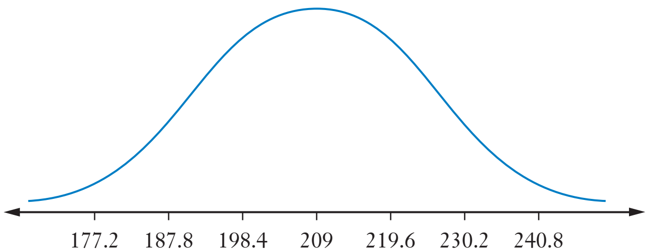
b



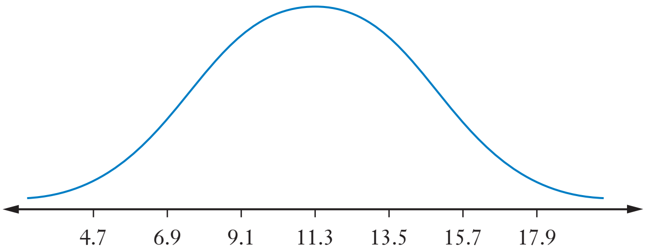
c



d



e



3 Draw the probability density function for a standard normal distribution.

4 Use the probability table for a standard normal distribution on pages 633–4 to find:

a $P(z \leq 0)$ **b** $P(z \leq 1)$

c $P(z \leq 2)$ **d** $P(z \leq 3)$

e $P(z \leq -1)$ **f** $P(z \leq -2)$

g $P(z \leq -3)$ **h** $P(z \leq 1.5)$

i $P(z < -2.67)$ **j** $P(z \leq 3.09)$

5 Use the probability table to find:

a $P(z \geq -0.46)$ **b** $P(z > 2.11)$

c $P(z \geq -2.01)$ **d** $P(-2.4 \leq z \leq -1.76)$

e $P(-2.2 \leq z \leq 2.2)$ **f** $P(1.21 < z < 1.89)$

g $P(-1.45 \leq z \leq 3.1)$ **h** $P(-1 \leq z \leq 1)$

i $P(-2 \leq z \leq 2)$ **j** $P(-3 \leq z \leq 3)$

6 Use the probability table to find:

a the 8th decile **b** the 3rd quartile

c the 29th percentile **d** the 2nd decile

e the 89th percentile **f** the 12th percentile

g the 3rd decile **h** the 1st quartile

i the 63rd percentile

7 Sketch a normal curve and draw on it the area where values lie in the:

a bottom 30% **b** bottom 15%

c top 67% **d** top 24%

e top 12%

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14.06 Empirical rule

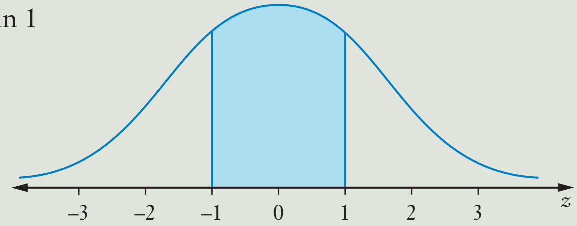
The empirical rule

In Question 5 of the previous exercise you calculated $P(-1 \le z \le 1)$, $P(-2 \le z \le 2)$ and $P(-3 \le z \le 3)$, the probability that values in a normal distribution will fall within 1, 2 or 3 standard deviations of the mean respectively. These probabilities are part of the **empirical rule**.

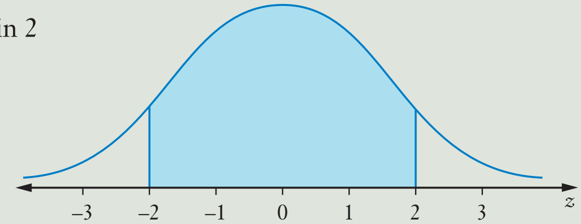
The normal curve

Empirical rule

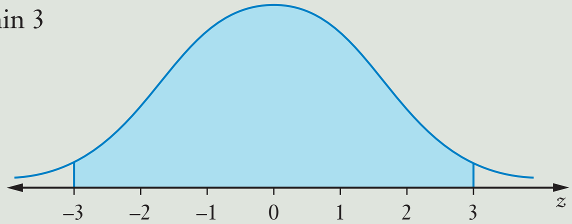
Approximately 68% of scores lie within 1 standard deviation of the mean, with z-scores between −1 and 1.



Approximately 95% of scores lie within 2 standard deviations of the mean, with z-scores between −2 and 2.



Approximately 997% of scores lie within 3 standard deviations of the mean, with z-scores between −3 and 3.



EXAMPLE 15

a A data set is normally distributed with mean 20 and standard deviation 3.

What percentage of scores lie between:

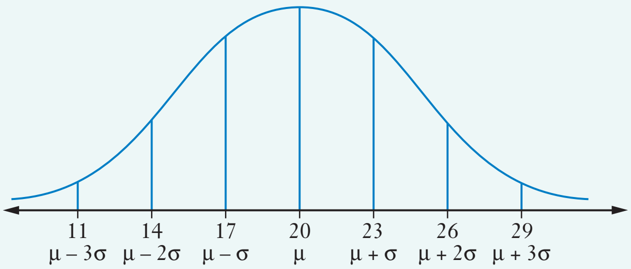
i 17 and 23? **ii** 14 and 26? **iii** 11 and 29?

b A normal distribution has $\mu = 65.2$ and $\sigma = 1.3$. What percentage of scores lie between:

i 61.3 and 65.2? **ii** 65.2 and 66.5? **iii** 62.6 and 69.1?

Solution

a We can draw the PDF for the normal curve.



i Scores between 17 and 23 are within 1 standard deviation of the mean.

So about 68% of scores lie between 17 and 23.

ii Scores between 14 and 26 are within 2 standard deviations of the mean.

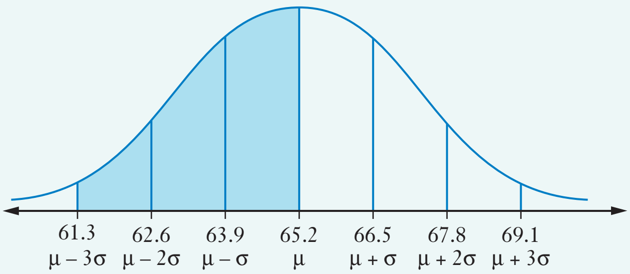
So about 95% of scores lie between 14 and 26.

iii Scores between 11 and 29 are within 3 standard deviations of the mean.

So about 99.7% of scores lie between 11 and 29.

b We can draw the PDF for the normal curve.

i



Scores between 61.3 and 69.1 are within 3 standard deviations of the mean (99.7% of data).

Scores between 61.3 and 65.2 are half this area.

$99.7\% \div 2 = 49.85\%$

So 49.85% of scores lie between 61.3 and 65.2.

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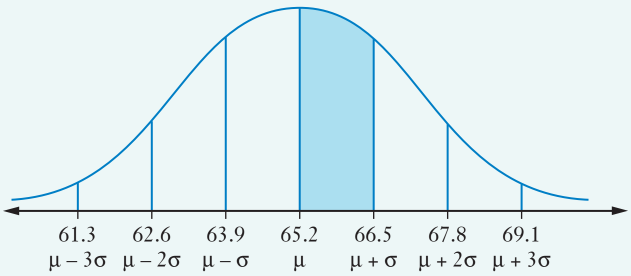
ii



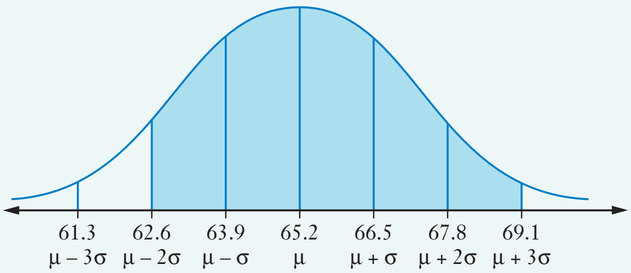
Scores between 63.9 and 66.5 are within 1 standard deviation of the mean (68% of data).

Scores between 65.2 and 66.5 are half this area.

$68\% \div 2 = 34\%$

So 34% of scores lie between 65.2 and 66.5.

iii



Scores between 62.6 and 65.2 are half the area within 2 standard deviations of the mean $\left(\frac{1}{2} \times 95\%\right)$.

Scores between 65.2 and 69.1 are half the area within 3 standard deviations of the mean $\left(\frac{1}{2} \times 997\%\right)$.

$$\begin{aligned}\text{Total area} &= \frac{1}{2} \times 95\% + \frac{1}{2} \times 99.7\% \\ &= 47.5\% + 49.85\% \\ &= 97.35\%\end{aligned}$$

So 97.35% of scores lie between 62.6 and 69.1.

Exercise 14.06 Empirical rule

1 What is the approximate percentage of data in a normal distribution that lies within:

a 1 standard deviation of the mean?

b 2 standard deviations of the mean?

c 3 standard deviations of the mean?

2 A set of data has a mean of 15 and a standard deviation of 1.5. If the data set is normally distributed, find the percentage of data that lies between:

a 13.5 and 16.5 **b** 12 and 18 **c** 10.5 and 19.5

3 A set of data is normally distributed with a mean of 8.4 and a standard deviation of 0.9. Find the percentage of data that lies between:

a 7.5 and 9.3 **b** 6.6 and 10.2 **c** 5.7 and 11.1

4 A set of data is normally distributed with mean 18 and standard deviation 2. Find the percentage of data that lies between:

a 16 and 20 **b** 14 and 22 **c** 12 and 24

d 16 and 18 **e** 18 and 24 **f** 12 and 22

5 A normal distribution has a mean of 65 and standard deviation 4.

a Sketch the graph of its PDF.

b Find the percentage of data that lies between:

i 57 and 73 **ii** 61 and 65 **iii** 65 and 77

v 57 and 69 **v** 61 and 73

6 A normal distribution has a mean of 9.7 and standard deviation 2.1. Find the percentage of data that lies between:

a 7.6 and 11.8 **b** 9.7 and 11.8 **c** 9.7 and 13.9

d 5.5 and 9.7 **e** 3.4 and 11.8

WS The standard normal curve

WS z-scores

WS Areas under the normal curve

z-scores

14.07 z-scores

We can transform any normal distribution into a standard normal distribution by using z-scores.

z-scores

To convert a raw score, x, into a z-score, use the formula:

$$z = \frac{x - \mu}{\sigma}$$

where μ is the mean and σ is the standard deviation of the distribution.

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EXAMPLE 16

A data set is normally distributed with mean 15 and standard deviation 2.

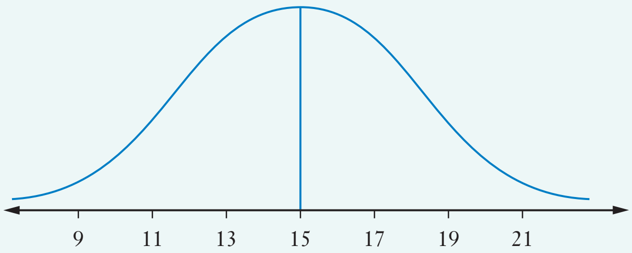
a Draw the probability density function for this distribution.

b Use z-scores to convert each x value to standardised scores:

i $x = 19$ **ii** $x = 15$ **iii** $x = 9$

Solution

a Drawing the PDF gives the curve below.



b $\mu = 15$ and $\sigma = 2$

i For $x = 19$:

$$z = \frac{x-\mu}{\sigma} = \frac{19-15}{2} = 2$$

This is because 19 is 2 standard deviations above the mean.

ii For $x = 15$:

$$z = \frac{x-\mu}{\sigma} = \frac{15-15}{2} = 0$$

This is because 15 is the mean.

iii For $x = 9$:

$$z = \frac{x-\mu}{\sigma} = \frac{9-15}{2} = -3$$

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EXAMPLE 17

A data set is normally distributed with mean 23.8 and standard deviation 1.2.

a Find the z-score for each raw score and describe where it is on the standard normal distribution.

i 25.7 **ii** 20.6 **iii** 28.3

b Which of the scores from part **a** are very unlikely?

c Find the value of a raw score whose z-score is −1.82, correct to 2 decimal places.

Solution

a **i** For $x = 25.7$:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{257 - 23\ 8}{1\ 2}$$

$$\approx 1.58$$

A score of 25.7 lies 1.58 standard deviations above (to the right of) the mean.

ii For $x = 20.6$:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{206 - 23\ 8}{1\ 2}$$

$$\approx -2.67$$

A score of 20.6 lies 2.67 standard deviations below (to the left of) the mean.

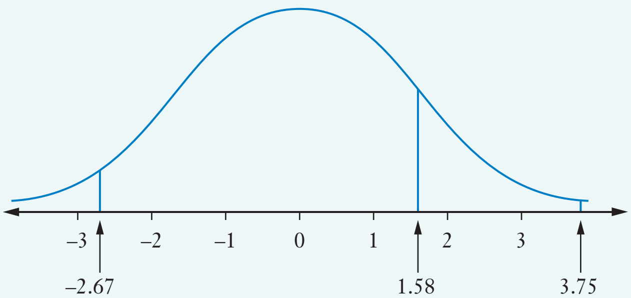
iii For $x = 28.3$:

$$z = \frac{283 - 23\ 8}{1\ 2}$$

$$\approx 3.75$$

A score of 28.3 lies 3.75 standard deviations above (to the right of) the mean.

b We can draw each score on the standard normal distribution.



99.7% of scores lie within 3 standard deviations of the mean.
So a z-score of 3.75 is very unlikely.

This means that a score of 28.3 is very unlikely.

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c

$$z = \frac{x - \mu}{\sigma}$$

$$-1.82 = \frac{x - 23\ 8}{1\ 2}$$

$$-2.184 = x - 23.8$$

$$21.616 = x$$

So the raw score is 21.62.

Exercise 14.07 *z*-scores

1 A data set is normally distributed with mean 18 and standard deviation 1.3.

a Find the *z*-score for each raw score.

i 18	**ii** 19.3	**iii** 20.6	**v** 21.9
v 16.7	**vi** 15.4	**vii** 14.1	

b Which raw score has a *z*-score of:

i 1.5? **ii** −2.1?

2 The length of fish caught in a fishing competition had a mean of 53.1 cm and standard deviation 8.7.

a If the lengths of fish almost certainly lie within 3 standard deviations of the mean, between which lengths would the fish almost certainly lie?

b Find the *z*-score for each length.

i 53.1 cm	**ii** 61.8 cm	**iii** 44.4 cm	**v** 70.5 cm
v 35.7 cm	**vi** 79.2 cm	**vii** 27 cm	**viii** 65 cm

3 A sample of overnight temperatures at Thredbo in June showed a mean temperature of 6.8°C and a standard deviation of 1.1.

a Almost all temperatures lie within 3 standard deviations of the mean. Within what range do almost all temperatures lie?

b Find the *z*-score for each temperature.

i 6.8°C	**ii** 7.9°C	**iii** 9°C	**v** 10.1°C
v 5.7°C	**vi** 4.6°C	**vii** 3.5°C	**viii** 6°C

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4 A survey showed that the mean volume of juice in an orange is 66.4 mL with a standard deviation of 5.8.

a The volume of juice very probably lies within 2 standard deviations of the mean. Between which 2 volumes do they lie?

b Find the z-scores for each volume.

i 66.4 mL	**ii** 72.2 mL	**iii** 78 mL	**v** 83.8 mL
v 60.6 mL	**vi** 54.8 mL	**vii** 49 mL	**viii** 90 mL

c Find the volume that has a z-score of:

i 1.2	**ii** 2.9	**iii** –0.6	**v** –2.3

5 **a** Find the z-score for each raw score below if the mean is 68 and the standard deviation is 4.5.

i 80	**ii** 53.2	**iii** 78.6	**v** 62.1
v 90	**vi** 59.7	**vii** 82.7	**viii** 56.4

b From your answers to part **a**, which scores are most unlikely?

c Which scores in part **a** lie within 2 standard deviations of the mean?

d Which scores in part **a** lie within 3 standard deviations of the mean?

6 The mean diameter of a batch of circular discs is 14.2 mm with standard deviation 1.4.

a What is the z-score for a disc with a diameter of :

i 16 mm?	**ii** 12 mm?

b Find the diameter of a disc with z-score:

i –2.1	**ii** 1.3	**iii** 3.2
v –0.76	**v** 1.95	

7 A set of data that is normally distributed has a mean of 23 and standard deviation 2. Which raw score has a z-score of 2.5?

8 A data set that is normally distributed has a standard deviation of 4.5. A score of 39 has a z-score of 2.7. What is the mean?

9 Find the standard deviation of a normally distributed data set if the mean is 89 and a raw score of 59 has z-score of –0.6.

10 A set of data that is normally distributed has a mean of 53.4 and standard deviation of 5.6. Find the raw score that has a z-score of:

a 0	**b** –2	**c** 1
d 2.8	**e** –1.7	

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11 The standard deviation of a normal distribution is 3.3 and the z-score of 45 is –1. Calculate the mean.

12 The mean of a normally distributed data set is 16 and standard deviation is 1.9.

a Find the scores between which:

i 95% of data lies **ii** 68% of data lies **iii** 99.7% of data lies

b Calculate the z-score of:

i 20 **ii** 13.5

c Find the raw score that has a z-score of:

i –3 **ii** 1.1

13 A normal distribution has a mean of 104.7 and standard deviation 5.1.

a Find the scores that lie within 1 standard deviation of the mean.

b Calculate the z-score of:

i 80 **ii** 103

c Which score has a z-score of:

i 2? **ii** –1.3?

14.08 Applications of the normal distribution

The normal distribution is often used in quality control and predicting outcomes.

WS Applying the normal distribution

WS Normal distribution worded problems 1

WS Normal distribution worded problems 2

WS Continuous random variables assignment

WS Continuous random variables problems

EXAMPLE 18

a A company produces 1 kg packets of sugar. A quality control check found that the weight of the packets was normally distributed with a mean weight of 0.995 kg and standard deviation 0.03 kg. The company policy is to reject any packet with a weight outside 2 standard deviations from the mean.

i What is the smallest weight allowed by the company?

ii What is the largest weight allowed?

iii What percentage of packets will be rejected?

iv What percentage of large packets will be rejected?

b The mean shelf life of a spice is 13.4 weeks and the standard deviation is 1.8 weeks.

i Would a shelf life of 20 weeks be unusual? Why?

ii Find the z-score for a shelf life of 15.5 weeks.

iii What percentage of shelf lives would be expected to be between 13.4 and 15.5 weeks?

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Solution

a $\mu = 0.995$

i Smallest weight allowed:

$$\mu - 2\sigma = 0.995 - 2 \times 0.03$$
$$= 0.935$$

ii Largest weight allowed:

$$\mu + 2\sigma = 0.995 + 2 \times 0.03$$
$$= 1.055$$

iii About 95% of weights lie within 2 standard deviations of the mean, so 5% will lie outside this area.

The company rejects 5% of the packets of sugar.

iv Since the normal distribution is symmetrical, the 5% is made up of 2.5% of larger and 2.5% of smaller packets.

So the company rejects 2.5% of larger packets.

b **i** $\mu = 13.4$

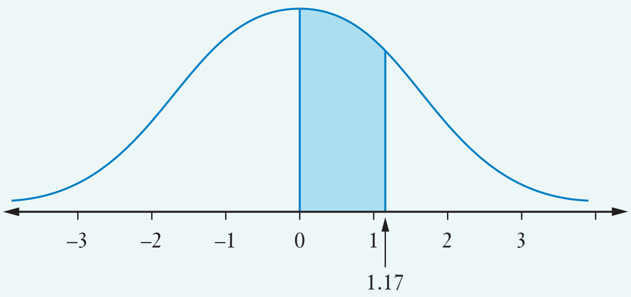
$$\mu - 3\sigma = 13.4 - 3 \times 1.8$$
$$= 8$$
$$\mu + 3\sigma = 13.4 + 3 \times 1.8$$
$$= 18.8$$

So the shelf life of spices almost certainly lies between 8 and 18.8 weeks.

A shelf life of 20 weeks is outside this range, so it would be unusual.

ii
$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{155\ - 13\ 4}{1\ 8}$$
$$= 1.17$$

iii



$\mu = 13.4$ so its z-score $= 0$

$$P(13.4 \le X \le 15.5) = P(0 \le z \le 1.17)$$
$$= P(X \le 1.17) - P(X \le 0)$$
$$= 0.8790 - 0.5000 \qquad \text{(using the table on pages 633–634)}$$
$$= 0.379$$

So 37.9% of shelf lives would be expected to be between 13.4 and 15.5 weeks.

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Using z-scores also allows us to compare 2 data sets.

EXAMPLE 19

In Year 7 at a school the mean weight of students was 59.4 kg and the standard deviation was 3.8 kg. In Year 8, the mean was 63.5 kg and the standard deviation was 1.7 kg.

John in Year 7, and Deng in Year 8, both weighed 68 kg. Which student was heavier in relation to his Year?

Solution

For John:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{68 - 59\ 4}{3\ 8}$$

$$\approx 2.263$$

For Deng:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{68 - 63\ 5}{1\ 7}$$

$$\approx 2.647$$

A z-score of 2.647 is higher than a z-score of 2.263.

So Deng weighed more in relation to his Year group than John.

Exercise 14.08 Applications of the normal distribution

1 A machine at the mint produces coins with a mean diameter of 24 mm and a standard deviation of 0.2 mm.

- **a** What percentage of coins will have a diameter between:
  - **i** 23.4 mm and 24.6 mm?
  - **ii** 24 mm and 24.4 mm?
  - **iii** 23.8 mm and 24.2 mm?
  - **v** 23.4 mm and 24 mm?
  - **v** 23.4 mm and 24.2 mm?
- **b** A coin selected at random has a diameter of 24.8 mm.
  - **i** What is its z-score?
  - **ii** Is this diameter unusual? What could you say about this?
- **c**
  - **i** Convert a diameter of 23.7 mm to a z-score.
  - **ii** Find the probability of producing a coin with a diameter of less than 23.7 mm.
  - **iii** Find the probability of producing a coin with a diameter between 23.7 mm and 23.9 mm.

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2 The maximum temperature in April is normally distributed with a mean of 25.3°C and a standard deviation of 3.4°C.

a What percentage of the time in April would you expect the temperature to be between:

i 21.9° and 28.7°? **ii** 18.5° and 32.1°? **iii** 15.1° and 35.5°?

v 25.3° and 28.7°? **v** 21.9° and 32.1°?

b The temperature drops to 14°.

i What is its z-score? **ii** Is this temperature unusual? Why?

c The temperature rises to 38°.

i What is its z-score? **ii** Is this temperature unusual? Why?

3 A company surveyed the amount of time it took to assemble its product. The times were normally distributed with mean 8 minutes and standard deviation 1.7 minutes.

a What is the z-score for a time of 15 minutes? Would this be an acceptable amount of time to assemble the product? Why?

b Between what times would 95% of times lie?

c What percentage of times would lie between 4.6 and 13.1 minutes?

d **i** Find the z-score for a time of 7 minutes 30 seconds.

ii What percentage of times would lie between 7 minutes 30 seconds and 8 minutes?

e Find the probability of times lying between:

i 6 and 8 minutes **ii** 6 and 9.7 minutes **iii** 5 and 10 minutes

f What percentage of times would lie between:

i 3 and 12 minutes? **ii** 7 and 11 minutes? **iii** 3.5 and 11.9 minutes?

4 A certain brand of perfume is sold in 20 mL bottles. The volume of perfume in the bottles was tested in a quality control check. The mean was found to be 19.9 mL with a standard deviation of 0.4 mL.

a What percentage of bottles have a volume between:

i 19.1 mL and 20.7 mL?

ii 19.9 mL and 21.1 mL?

iii 20 mL and 21 mL?

b Between which two volumes do 99.7% of bottles lie?

c Comment on a bottle that has a volume of 23 mL.

d Find the probability that a bottle of perfume will have a volume between 18.9 and 19.3 mL.

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5 A farmer is only allowed to deliver standard size apples to the markets. The mean diameter must be 7.5 cm with a standard deviation of 0.3 cm. Only apples within 2 standard deviations of the mean are allowed.

a What percentage of apples are allowed?

b What is the largest diameter allowed?

c What is the smallest diameter allowed?

6 A certain brand of car battery has a mean life of 3.1 years with a standard deviation of 0.3 years.

a What is the minimum life you could reasonably expect from a battery of this type?

b What percentage of batteries would have a life between:

i 2.8 and 3.4 years? **ii** 2.5 and 3.5 years?

c Would a life of over 4 years be unusual? Why?

7 A Jack Russell terrier has a mean height of 28 cm and a standard deviation of 0.833 cm.

a What range of heights (to 1 decimal place) would you expect for this breed of dog?

b What is the z-score of a height of 27.2 cm?

c What percentage of dogs would be between 27.2 cm and 30 cm tall?

d Would a dog 24 cm tall be typical of this breed? Why?

8 A factory produces 5 kg bags of bread mix. In a quality check, the mean weight of a bag of bread mix was found to be 4.95 kg, with a standard deviation of 0.15 kg. The factory rejects bags that weigh less than 4.65 kg or more than 5.25 kg.

a What percentage of bags does the factory reject?

b What percentage of bags does the factory reject because their weight is too small?

c The manager of the factory decided that too many bags are being rejected and that in future only those outside the normal range of 3 standard deviations would be rejected.

i What percentage will be rejected?

ii What weights will be rejected?

9 A company manufactures steel rods with a mean diameter of 10.6 cm and a standard deviation of 0.5 cm. The manufacturer rejects rods that are outside acceptable limits.

a If the company rejects 0.3% of rods, find:

i the percentage of rods it accepts

ii the largest diameter it will accept

iii the smallest diameter it will accept

b In one batch of rods, the diameters are 10.9 cm, 9.6 cm, 8.3 cm, 11.4 cm and 12.6 cm. Which ones will be rejected?

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10 A manufacturer of canned fruit guarantees that the minimum weight in each can is 250 g. A random check showed that the mean weight was 252.5 g with a standard deviation of 0.4 g. Comment on this guarantee. Is it realistic? Why?

11 A butcher shop advertises that it will give a free leg of lamb to any customer who can prove that a packet of mince with a mean weight of 1 kg weighs less than 980 g. If the mean weight is 1 kg with a standard deviation of 10 g, what percentage of customers should expect to receive a free leg of lamb?

12 The width of a type of door almost certainly lies within the range 814 mm to 826 mm (3 standard deviations of the mean).

a What is the standard deviation?

b What is the mean width of the door?

c Within what widths do 95% of these doors lie?

13 The mean time for a ferry to travel from one port to another is 3.1 hours. About 47.5% of the time the ferry takes between 3.1 hours and 3.5 hours.

a What is the standard deviation?

b What is the minimum time you would expect the ferry to take?

14 Xavier takes 78 minutes to drive to Epping and 65 minutes to drive to the city. The mean time to Epping is 75.3 minutes with a standard deviation of 2.6, while the mean time to the city is 62.7 minutes with a standard deviation of 1.7.

a Calculate the z-scores for Xavier's trips to Epping and the city.

b Which was the longer trip in comparison with the mean?

15 Kieran scored 78 in an exam where the mean was 69.5 and the standard deviation was 8.5. Cameron scored 71 in an exam where the mean was 61.2 and the standard deviation was 4.8.

a Find Kieran's z-score.

b Calculate Cameron's z-score.

c Which student scored higher in comparison with the other students in each exam?

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14. TEST YOURSELF

For Questions **1** to **3**, choose the correct answer **A**, **B**, **C** or **D**.

Formula sheet: Statistics and probability

Practice quiz

1 Which function does not describe a continuous probability distribution?

A $f(x) = \dfrac{x^2}{21}$ for the interval $1 \le x \le 4$ **B** $f(x) = \dfrac{e^x}{e^3 - 1}$ in the domain $[0, 3]$

C $f(x) = \dfrac{x^4}{625}$ in the domain $[1, 5]$ **D** $f(x) = \dfrac{4x^3}{625}$ in the interval $0 \le x \le 5$

2 The percentage of z-scores between −2 and 2 is:

A 95% **B** 68% **C** 47.5% **D** 49.85%

3 Which random variable is not continuous?

A The temperature of different freezers

B The mass of rocks found at the site of a volcano

C The length of the arms of people

D Shoe sizes of people

4 For the uniform continuous probability distribution shown, find:

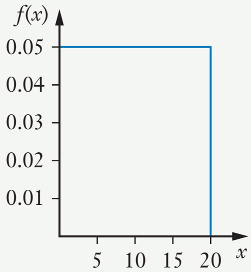
a $P(X \le 15)$ **b** $P(X \le 8)$

c $P(7 \le X \le 18)$ **d** $P(4 < X < 13)$

e $P(X \ge 6)$ **f** the median

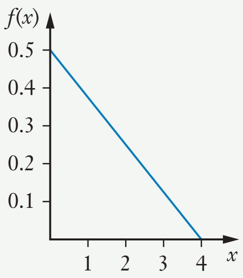
g the 18th percentile **h** the 89th percentile

i the 6th decile **j** the 3rd quartile

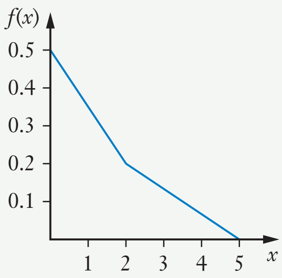


5 State whether each graph represents a probability density function.

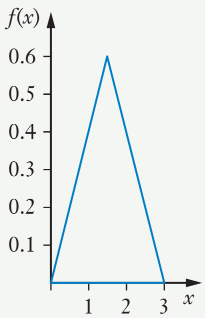
a



b



c



6 Find the median of the continuous random variable $f(x) = \frac{3x^2}{124}$ defined on the interval $1 \le x \le 5$. Answer correct to 2 decimal places.

7 **a** Find the cumulative distribution function for $f(x) = \begin{cases} \frac{3x^2}{511} & \text{for } 1 \le x \le 8 \\ 0 & \text{for all other } x \end{cases}$

b Find:

i $P(X \le 3)$ **ii** $P(X \le 5)$ **iii** $P(X > 6)$

v $P(X \ge 4)$ **v** $P(2 \le X \le 7)$

8 For each continuous probability distribution, find:

i the mode **ii** the median

a $f(x) = \frac{2x}{15}$ defined in the domain $[1, 4]$

b $f(x) = \frac{x^2}{243}$ defined in the interval $0 \le x \le 9$

9 The birth weights of babies born at St John's Hospital were measured and found to be normally distributed with mean 3.2 kg and standard deviation 0.31.

a Find the range of weights in which 99.7% of the weights of these babies would lie.

b Find the z-score (to 2 decimal places) for a weight of:

i 3.9 kg **ii** 3.5 kg

c Use the standard normal probability table on pages 633–634 to find the probability that a baby born at the hospital would have a birth weight between 3.5 kg and 3.9 kg.

10 A function is given by $f(x) = \frac{3x^5}{2048}$. Over what domain starting at $x = 0$ is this a probability density function?

11 Draw a probability density function for the normal distribution with:

a mean 15 and standard deviation 0.5 **b** mean 3.4 and standard deviation 0.2

12 A factory produces 2 kg bags of nails. In a quality check, the mean weight of a bag of nails was found to be 1.95 kg, with a standard deviation of 0.08 kg. The factory rejects bags that weigh less than 1.71 kg or more than 2.19 kg.

a What percentage of bags does the factory reject?

b After a complaint, the manager decided to reject bags that weigh less than 1.87 kg or more than 2.11 kg. What percentage will be rejected?

13 For the PDF $f(x) = \frac{3x^2}{316}$ defined on the interval $3 \le x \le 7$, find:

a the median **b** the 3rd quartile **c** the 4th decile

d the 63rd percentile **e** the 28th percentile

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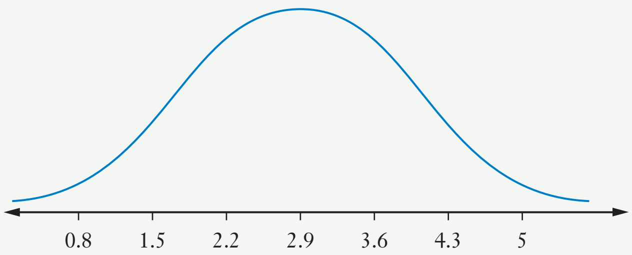
14 Find the mean and standard deviation of this normal distribution.



15 Find the cumulative distribution function for each continuous probability distribution.

a $f(x) = \dfrac{x^4}{625}$ defined on the interval $0 \le x \le 5$

b $f(x) = \dfrac{x^6}{117\,649}$ defined on [0, 7]

c $f(x) = \dfrac{e^x}{e^6 - 1}$ in the interval $0 \le x \le 6$

d $f(x) = \dfrac{x}{40}$ for [1, 9]

16 Use the probability table for a standard normal distribution on pages 633– 634 to find:

a $P(z \le 0.54)$ **b** $P(z \le 1.32)$ **c** $P(z \le -3)$

d $P(z \le -0.71)$ **e** $P(z \ge -1)$ **f** $P(z \ge 2.5)$

g $P(z \ge -1.08)$ **h** $P(-2.3 \le z \le -1.09)$ **i** $P(1.1 \le z \le 3.11)$

17 A set of data is normally distributed with mean 12.5 and standard deviation 1.5. Find the percentage of data that lies between:

a 9.5 and 15.5 **b** 12.5 and 14 **c** 11 and 17

d 10 and 15 **e** 12 and 13

18 A probability density function is given by $f(x) = ae^x$ over a certain domain. Find the exact value of a if the domain is:

a [0, 5] **b** [1, 4]

19 Shade on the standard normal distribution the area where values lie:

a in the bottom 20%

b in the bottom 32%

c in the top 15%

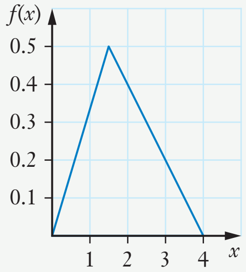
d in the top 30%

e between the bottom 10% and the top 40%

20 Find the mode of each continuous probability distribution.

a $f(x) = \frac{x}{30}$ defined in the domain [2, 8]

b



c $f(x) = \frac{4}{189}(x^3 - 9x^2 + 24x)$ defined in the domain [0, 3]

21 Klare took 9.3 s to finish a race where the mean was 8.3 s and the standard deviation was 1.2. Simon took 8.9 s to run in the next heat where the mean was 8.1 s and the standard deviation was 0.8.

a Find Klare's z-score.

b Calculate Simon's z-score.

c Which person had the better time in comparison with the other runners in their heat?

22 Show that $f(x) = \frac{x^3}{600}$ defined on the interval $1 \le x \le 7$ is a probability density function.

23 Circular tables are made with a mean radius of 1.1 m with standard deviation 0.02 m.

a Find the z-score for a table with a radius of:

i 1.15 m **ii** 1.07 m

b Find the radius of a table with z-score:

i 1 **ii** −2 **iii** 3.1 **v** −0.63 **v** 1.27

24 A normal distribution has a standard deviation of 1.6. A score of 87.9 has a z-score of −1.3. What is the mean?

25 The standard deviation of a normal distribution is 1.9 and the z-score of 52.4 is 1.7. Calculate the mean.

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14. CHALLENGE EXERCISE

1 A probability density function is given by $f(x) = \dfrac{5x^4}{3124}$ over the interval $a \le x \le b$.

Find the value of a and b if the median of the PDF is 4.353 031.

2 A normal distribution has mean μ and standard deviation σ.
Evaluate μ and σ given that 95% of scores lie between 12.4 and 14.

3 Find the mode of the continuous probability distribution $f(x) = \dfrac{4}{249}(x + 2)(x - 4)^2$ defined in the domain [0, 3].

4 Find the cumulative distribution function for the continuous probability distribution $f(x) = \dfrac{3x(x^2 + 1)^2}{62\,000}$ defined on the domain [3, 7].

5 For a normal distribution, $P(X \le 23.8) = 91.92\%$ and $P(X \le 17.15) = 30.85\%$.
Find the mean and standard deviation.

6 A good model for a normal distribution is the function $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)}{2\sigma}}$.

Use technology to show that this is a probability density function given $\mu = 6$ and $\sigma = 1$.

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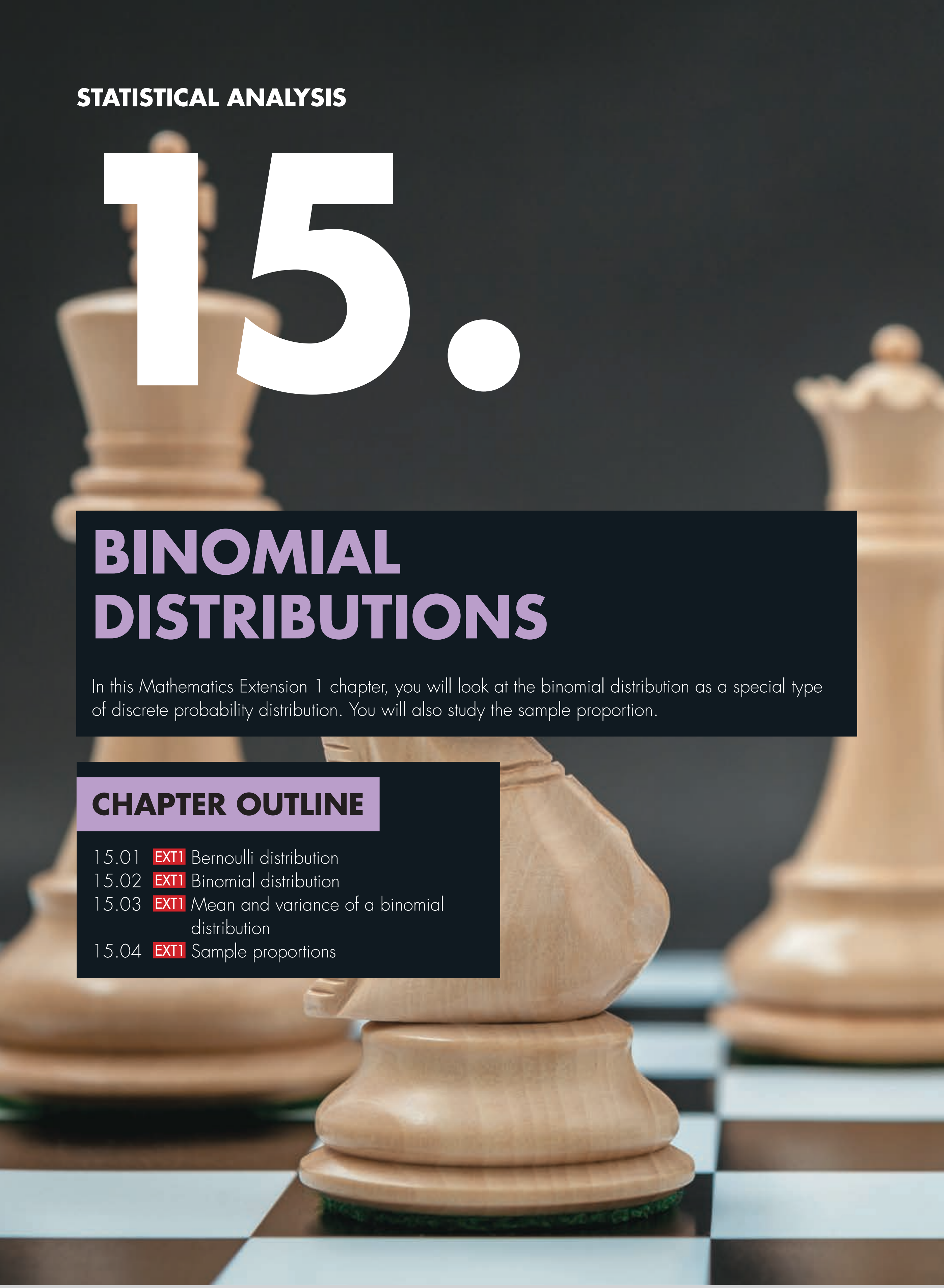

STATISTICAL ANALYSIS

15.

BINOMIAL DISTRIBUTIONS

In this Mathematics Extension 1 chapter, you will look at the binomial distribution as a special type of discrete probability distribution. You will also study the sample proportion.

CHAPTER OUTLINE

15.01 EXT1 Bernoulli distribution
15.02 EXT1 Binomial distribution
15.03 EXT1 Mean and variance of a binomial distribution
15.04 EXT1 Sample proportions

IN THIS CHAPTER YOU WILL:

- EXT1 understand and use a Bernoulli random variable
- EXT1 understand and apply the mean and variance of the Bernoulli distribution
- EXT1 understand the binomial distribution and its relation to the Bernoulli distribution
- EXT1 calculate probabilities and solve problems using the binomial distribution
- EXT1 find the mean and variance of binomial random variables
- EXT1 understand the concept of a sample proportion and its behaviours
- EXT1 explore the behaviour of the sample proportion for large samples

TERMINOLOGY

Bernoulli distribution: A discrete probability distribution for a Bernoulli random variable.
Bernoulli random variable: A random variable that has 2 possible values: 1 for 'success' and 0 for 'failure'.
Bernoulli trial: An experiment with 2 possible outcomes: 'success' and 'failure'.
binomial distribution: A discrete probability distribution of the numbers of successes in a sequence of independent Bernoulli trials.
binomial random variable: A random variable that is the number of successes in n independent Bernoulli trials.
sample proportion: The fraction of successes in n Bernoulli trials.

WS Bernoulli distributions

WS The Bernoulli distribution

EXT1 15.01 Bernoulli distribution

A **Bernoulli random variable** is a discrete random variable used when there are just 2 outcomes: 'success' and 'failure'.

Bernoulli random variable

A Bernoulli random variable is $X = \{0, 1\}$ where 0 = failure and 1 = success

EXAMPLE 1

Which of the following are Bernoulli random variables?

a Turning up a tail when tossing a coin
b The number rolled on a die
c Hitting a bullseye when throwing a dart
d The number of goals scored by a hockey team in a match
e Choosing a red ball from a bag that contains red, white and blue balls

Solution

a This can be desribed by a Bernoulli random variable because there are only 2 outcomes:

$X = \{0, 1\}$ where 1 = tails and 0 = heads.

b This cannot be described by a Bernoulli random variable because there are 6 outcomes:

$X = \{1, 2, 3, 4, 5, 6\}$

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c This can be described by a Bernoulli random variable because there are only 2 outcomes:

$X = \{0, 1\}$ where 1 = hit and 0 = miss.

d This cannot be described by a Bernoulli random variable because there are more than 2 outcomes:

$X = \{0, 1, 2, 3, \ldots\}$

e This can be described by a Bernoulli random variable because there are only 2 outcomes:

$X = \{0, 1\}$ where 1 = red and 0 = not red.

DID YOU KNOW?

Jacob Bernoulli

Bernoulli random variables are named after a Swiss mathematician and astronomer, **Jacob Bernoulli** (1655–1705).

Bernoulli became a minister of religion according to the wishes of his father, but then spent several years studying the discoveries of the mathematicians and scientists of his time. He started teaching mechanics at Basel University and was later appointed as a professor in mathematics.

Jacob was one of many prominent mathematicians in his family.

Who were the other Bernoullis?

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EXAMPLE 2

a Construct a discrete probability distribution table and bar graph for rolling 6 on a die.

b Is this a uniform probability distribution?

Solution

a $P(6) = \frac{1}{6}$

Probability of not rolling a 6:

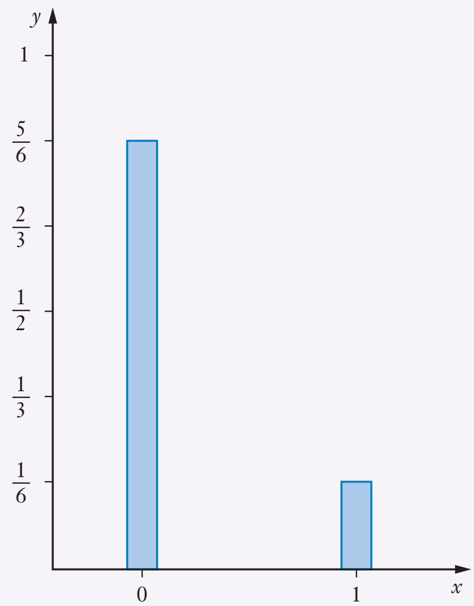
$$P(\overline{6}) = 1 - \frac{1}{6}$$

$$= \frac{5}{6}$$

Probability distribution: 1 = success and 0 = failure

X	0	1
$P(X = x)$	$\frac{5}{6}$	$\frac{1}{6}$

This could also be drawn as a histogram.



b This is not a uniform probability distribution because the probabilities of success ($x = 1$) and failure ($x = 0$) are not the same.

A **Bernoulli distribution** is a general probability distribution for a Bernoulli random variable, using p as the probability of success.

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The probability of failure will be $q = 1 - p$.

Bernoulli distribution

$$P(X = x) = \begin{cases} p & \text{for } x = 1 \\ 1 - p & \text{for } x = 0 \end{cases}$$

We call p a **parameter**. In statistics, a parameter is a constant that describes a feature of a distribution.

EXAMPLE 3

a Write a Bernoulli probability distribution for each situation.

i Passing a test where probability of a pass is 56%

ii Flipping heads on a coin

iii Selecting a red ball at random from a bag containing 7 red and 8 blue balls

b Which of the distributions in part **a** are uniform probability distributions?

Solution

a **i** $P(1) = 56\%$ where $1 = \text{pass}$

$P(0) = 100\% - 56\%$ where $0 = \text{fail}$

$= 44\%$

$$P(X = x) = \begin{cases} 56\% & \text{for } x = 1 \\ 44\% & \text{for } x = 0 \end{cases}$$

ii $P(1) = \frac{1}{2}$ where $1 = \text{heads}$

$P(0) = 1 - \frac{1}{2}$ where $0 = \text{tails}$

$= \frac{1}{2}$

$$P(X = x) = \begin{cases} \frac{1}{2} & \text{for } x = 1 \\ \frac{1}{2} & \text{for } x = 0 \end{cases}$$

iii $P(1) = \frac{7}{15}$ where $1 = \text{red}$

$P(0) = 1 - \frac{7}{15}$ where $0 = \text{blue}$

$= \frac{8}{15}$

$$P(X = x) = \begin{cases} \frac{7}{15} & \text{for } x = 1 \\ \frac{8}{15} & \text{for } x = 0 \end{cases}$$

b Part **ii**, tossing a coin, is a uniform distribution because the probabilities are equal.

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We learned about the mean and variance of a discrete probability distribution in Year 11, Chapter 12, *Discrete probability distributions*.

Mean

The mean (or expected value) of a Bernoulli distribution is $E(X) = p$.

Proof

$$E(X) = \Sigma xp(x)$$
$$= 0 \times (1 - p) + 1 \times p$$
$$= p$$

EXAMPLE 4

Find the mean of each Bernoulli distribution.

a

X	0	1
$P(X = x)$	$\frac{7}{11}$	$\frac{4}{11}$

b $$(p\ x) = \begin{cases} \frac{9}{10} & \text{for } x = 1 \\ \frac{1}{10} & \text{for } x = 0 \end{cases}$$

Solution

a $P(X = 1) = \frac{4}{11}$

So $p = \frac{4}{11}$

Mean $E(X) = p$

$= \frac{4}{11}$

b $p(1) = \frac{9}{10}$

So $p = \frac{9}{10}$

Mean $E(X) = p$

$= \frac{9}{10}$

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Variance

The variance of a Bernoulli distribution is $\text{Var}(X) = p(1-p)$

Proof

$$\begin{aligned}\text{Var}(X) &= \Sigma[x - E(X)]^2 p(x)\\ &= (0-p)^2(1-p) + (1-p)^2 p\\ &= p^2(1-p) + (1-2p+p^2)p\\ &= p^2 - p^3 + p - 2p^2 + p^3\\ &= p - p^2\\ &= p(1-p)\end{aligned}$$

EXAMPLE 5

Find the variance of each Bernoulli distribution.

a

X	0	1
$P(X = x)$	0.4	0.6

b $p(x) = \begin{cases} \frac{3}{5} & \text{for } x = 1 \\ \frac{2}{5} & \text{for } x = 0 \end{cases}$

Solution

a $P(1) = 0.6$

So $p = 0.6$

$$\begin{aligned}\text{Var}(X) &= p(1-p)\\ &= 0.6(1-0.6)\\ &= 0.6 \times 0.4\\ &= 0.24\end{aligned}$$

Note that because $q = 1 - p$, $\text{Var}(X) = pq$.

b $p(1) = \frac{3}{5}$

So $p = \frac{3}{5}$

$$\begin{aligned}\text{Var}(X) &= p(1-p)\\ &= \frac{3}{5}\left(1-\frac{3}{5}\right)\\ &= \frac{3}{5} \times \frac{2}{5}\\ &= \frac{6}{25}\end{aligned}$$

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EXT1 Exercise 15.01 Bernoulli distribution

1 State whether each statement describes a Bernoulli random variable.

a Selecting at random an odd numbered card from a set of numbered cards
b Choosing at random a red playing card from a deck of cards
c The number of tails when tossing 2 coins
d The colour of a traffic light
e A missile hitting a target
f Randomly selecting a boy from a school class
g A film reviewer's star rating of a film
h Selecting a faulty phone in a random quality control check
i The age of a person selected at random in a shopping centre
j Guessing the correct answer to a true/false question

2 Write each Bernoulli distribution as a piecewise function.

a Passing a test if the probability of a pass is $\frac{7}{9}$
b An arrow hitting a target with a probability of 84%
c Getting a head when tossing a coin
d Winning a raffle in which there are 150 tickets sold
e Selecting at random a person taking part in a survey who went on an overseas holiday, if 98 out of 426 people surveyed went on an overseas holiday

3 Draw up a Bernoulli distribution table for each event.

a Selecting a girl at random from a group of 12 boys and 14 girls
b Selecting an ace at random from a deck of cards
c Getting a total of 5 when rolling 2 dice
d Selecting a white card at random from a set of 10 white and 15 red cards
e Selecting a number divisible by 3 when randomly selecting a card from 30 cards, each numbered with a different number from 1 to 30

4 **a** Draw a bar chart for each Bernoulli distribution.
i Getting a tail when tossing a coin
ii Getting a total of 10 when rolling 2 dice
b Is each distribution uniform?

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5 For each Bernoulli distribution, find:

i the mean **ii** the variance

a $P(X = x) = \begin{cases} 0\ 7 & \text{for } x = 1 \\ 0\ 3 & \text{for } x = 0 \end{cases}$

b $p(X = x) = \begin{cases} \dfrac{14}{15} & \text{for } x = 1 \\ \dfrac{1}{15} & \text{for } x = 0 \end{cases}$

c $p(x) = \dfrac{x+1}{3}$ for $x = 0, 1$

d

X	0	1
$P(X = x)$	48%	52%

e

X	0	1
$P(X = x)$	$\dfrac{99}{100}$	$\dfrac{1}{100}$

6 A biased coin has a probability of $\dfrac{3}{4}$ of coming up heads.

a Write a Bernoulli distribution for tossing a head.

b For the distribution, find:

i the mean **ii** the variance

7 On average, 8% of mail is junk mail. Find the mean and variance of a Bernoulli distribution for receiving junk mail.

8 The probability that a basketball player will shoot a basket is $\dfrac{9}{13}$. For the Bernoulli distribution for shooting a basket, find:

a the mean **b** the variance **c** the standard deviation

9 **a** Find the Bernoulli distribution for getting a double when rolling 2 dice.

b For this distribution, find:

i the mean

ii the variance

iii the exact standard deviation

10 The mean of a Bernoulli distribution is $\dfrac{9}{10}$.

a Find the variance.

b Write the Bernoulli distribution as a table.

11 The variance of a Bernoulli distribution is $\dfrac{2}{9}$. Find 2 possible values for the mean.

12 Find two possible Bernoulli distributions with $\text{Var}(X) = 0.2464$.

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WS Binomial distributions

WS Binomial probability experiments

WS Using the binomial probability distribution

EXT1 15.02 Binomial distribution

The **binomial distribution** is an extension of the Bernoulli distribution that comes from running multiple **Bernoulli trials** (experiments in which there are only 2 outcomes – 'success' and 'failure').

Properties of a binomial distribution

- There is a fixed number of trials
- Each trial only has 2 outcomes, success and failure, with probability p and $q = 1 - p$ respectively
- Each trial is independent
- The probabilities are the same for each trial

The **binomial random variable** is the number of successes from n Bernoulli trials.

EXAMPLE 6

State whether each random variable describes a binomial distribution.

a The number of tails when tossing a coin 50 times

b The number of red cards selected when 8 cards are selected at random from a bag of coloured cards

i with replacement **ii** without replacement

Solution

a There is a fixed number of independent trials.

Each has 2 outcomes (tails or heads).

The probability is the same for each trial.

So it is a binomial distribution.

b **i** There is a fixed number of independent trials.

Each has 2 outcomes (red or not red).

The probability is the same for each trial.

So it is a binomial distribution.

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ii There is a fixed number of independent trials.

Each has 2 outcomes.

However the probability is not the same for each trial because each one is conditional on the previous trial.

So it is not a binomial distribution.

We can find the probability of all successes or all failures in multiple events in a binomial distribution.

EXAMPLE 7

Find the probability of each event, leaving the answer in index form.

a Getting 10 heads when tossing a coin 10 times

b Getting 8 red cards when selecting 8 cards with replacement at random from a bag of 10 red and 11 blue cards

Solution

a When tossing a coin 10 times we use the product rule of probability:

$$P(10\ H) = P(H) \times P(H) \times P(H) \times \ldots \quad \text{(10 times)}$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \ \ldots \quad \text{(10 times)}$$

$$= \frac{1}{2^{10}}$$

b $P(R) = \dfrac{10}{21}$ for each trial with replacement.

Using the product rule of probability:

$$P(8\ R) = P(R) \times P(R) \times P(R) \times \ldots \quad \text{(8 times)}$$

$$= \frac{10}{21} \times \frac{10}{21} \times \frac{10}{21} \ldots \quad \text{(8 times)}$$

$$= \frac{10^8}{21^8}$$

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It is more difficult to find the number of ways of getting, say, 3 heads and 7 tails when tossing a coin 10 times.

EXAMPLE 8

a When tossing a coin twice, find the number of ways of tossing:

i 2 heads **ii** 1 head **iii** 0 heads

b When tossing a coin 3 times, find the number of ways of tossing:

i 3 heads **ii** 2 heads **iii** 1 head **v** 0 heads

Solution

a Drawing a probability tree shows all possible outcomes.

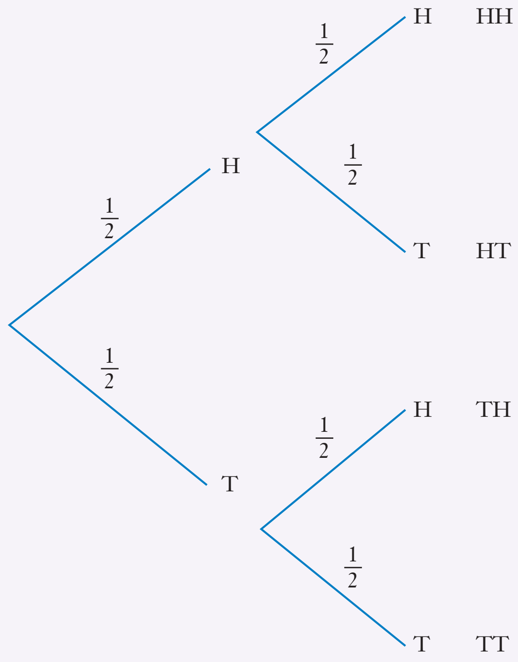
From the probability tree:

i There is 1 way of getting 2 heads (HH)

ii There are 2 ways of getting 1 head (HT and TH)

iii There is 1 way of getting 0 heads (TT)

Notice that there are 4 possible outcomes altogether.



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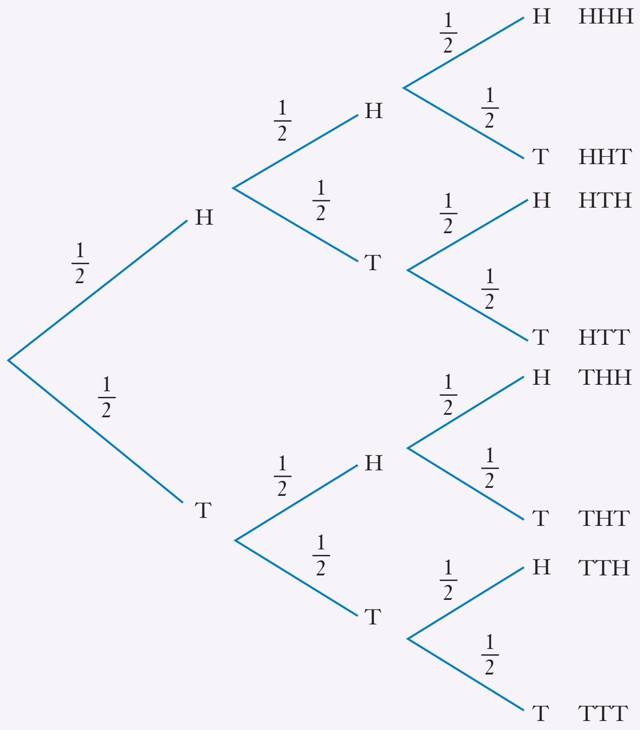
b Drawing a probability tree:



From the probability tree:

i There is 1 way of getting 3 heads (HHH)

ii There are 3 ways of getting 2 heads (HHT, HTH, THH)

iii There are 3 ways of getting 1 head (HTT, THT, TTH)

v There is 1 way of getting 0 heads (TTT)

Notice that there are 8 possible outcomes altogether.

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INVESTIGATION

PROBABILITY AND PASCAL'S TRIANGLE

Copy and complete this table with the results from Example 8 by counting the number of ways of getting each number of heads when tossing a given number of coins.

Number of coins	Number of combinations					
1	1 head 1	0 heads 1				
2	2 heads 1	1 head 2	0 heads 1			
3	3 heads 1	2 heads 3	1 head 3	0 heads 1		
4	4 heads	3 heads	2 heads	1 head	0 heads	
5	5 heads	4 heads	3 heads	2 heads	1 head	0 heads

Can you see a link between these results and Pascal's triangle?

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

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In Year 11, Chapter 3, *Permutations and combinations*, you found the link between Pascal's triangle and coefficients in the expansion of binomial products.

$$
\begin{array}{ccccccccccc}
 & & & & & {}^0C_0 & & & & & \\
 & & & & {}^1C_0 & & {}^1C_1 & & & & \\
 & & & {}^2C_0 & & {}^2C_1 & & {}^2C_2 & & & \\
 & & {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 & & \\
 & {}^4C_0 & & {}^4C_1 & & {}^4C_2 & & {}^4C_3 & & {}^4C_4 & \\
{}^5C_0 & & {}^5C_1 & & {}^5C_2 & & {}^5C_3 & & {}^5C_4 & & {}^5C_5
\end{array}
$$

So the number of outcomes in probability is given by coefficients ${}^nC_r$.

Probabilities in a binomial distribution

If p is the probability of success, then the probability of r successes in n independent trials is given by:

$$
\begin{aligned}
P(X = r) &= {}^nC_r\, p^r(1 - p)^{n-r} \\
&= \binom{n}{r} p^r(1 - p)^{n-r}
\end{aligned}
$$

X is the random variable of the binomial distribution. This is sometimes written as '$X \sim \text{Bin}(n, p)$'.

Proof

Let p be the probability of a success.

Then $1 - p$ is the probability of failure.

There are ${}^nC_r$ ways of getting r successes from n outcomes.

$$P(X = r) = {}^nC_r \times p \times p \times p \times p \times p \ldots \times (1 - p) \times (1 - p) \times (1 - p) \times (1 - p) \ldots$$

r times $\qquad\qquad$ $n - r$ times

$$= {}^nC_r\, p^r(1 - p)^{n-r}$$

The binomial distribution

EXAMPLE 9

a A hydrangea plant has a probability of $\frac{1}{3}$ of producing white flowers. If 4 plants are grown, find the probability that only 1 plant will produce white flowers.

b Maria tosses 8 coins. Find the probability that she tosses 5 heads.

c If I roll 5 standard dice in a game of Yahtzee, find the probability that I roll 3 sixes.

Solution

a $P(W) = \frac{1}{3}$ so $(P\ \overline{W}) = \frac{2}{3}$

$P(r \text{ successes}) = {}^nC_r\, p^r q^{n-r}$

For 1 success in 4 trials:

$P(1 \text{ success}) = {}^4C_1\, p^1 q^{4-1}$

$$P(X = 1) = {}^4C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{4-}$$

$$= {}^4C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3$$

$$= \frac{32}{81}$$

b $P(H) = \frac{1}{2},\ P(\overline{H}) = \frac{1}{2}$

For 5 success in 8 trials:

$P(5 \text{ successes}) = {}^8C_5\, p^5 q^{8-5}$

$$P(X = 5) = {}^8C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{8-5}$$

$$= {}^8C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3$$

$$= \frac{7}{32}$$

c $P(6) = \frac{1}{6},\ P(\overline{6}) = \frac{5}{6}$

For 3 successes in 5 trials:

$P(3 \text{ successes}) = {}^5C_3\, p^3 q^{5-3}$

$$P(X = 3) = {}^5C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{5-3}$$

$$= {}^5C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

$$= \frac{125}{3888}$$

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EXAMPLE 10

A certain type of machine in a car assembly plant has an average probability of 0.1 of breaking down. If the assembly plant has 8 of these machines, what is the probability, correct to 3 decimal places, that at least 6 will be in good working order at any one time?

Solution

$P(\text{broken}) = 0.1$

So $P(\text{working}) = 1 - 0.1$

$= 0.9$

$P(X \geq 6) = P(6) + P(7) + P(8)$

$= {}^8C_6(0.9)^6(0.1)^{8-6} + {}^8C_7(0.9)^7(0.1)^{8-7} + {}^8C_8(0.9)^8(0.1)^{8-8}$

$= {}^8C_6(0.9)^6(0.1)^2 + {}^8C_7(0.9)^7(0.1) + {}^8C_8(0.9)^8(0.1)^0$

$= 0.962$

DID YOU KNOW?

The binomial theorem

The **binomial theorem** has been studied by mathematicians from very early times. For example, **Euclid** discovered the special expansion for $(x + y)^n$ around 300 BCE. Euclid is mainly famous for his work on geometry, but he also studied arithmetic and number theory. He founded the first school of mathematics, in Alexandria in Egypt.

Omar Khayyam working around 1100 C, discovered the expansions of ($x + y)^4$, $(x + y)^5$ and $(x + y)^6$. He is mainly remembered as a poet (he wrote the *Rubaiyat*) but he was also an astronomer and mathematician and wrote a book about algebra.

Blaise Pascal (1623–1662) is associated with the binomial theorem because of his famous 'Pascal's triangle', which is related to the coefficients of binomials. However, he approached the binomial theorem from the standpoint of probability rather than algebra.

Sir Isaac Newton (1642–1727) discovered the general rules for coefficients of binomial expansions, which **Leonhard Euler** (1707–1783) later proved.

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EXT1 Exercise 15.02 Binomial distribution

1 State whether each random variable describes a binomial distribution.

- **a** The number of heads when tossing a coin 20 times
- **b** The number of hits when throwing a ball at a target 45 times
- **c** The number of white marbles when 10 marbles are randomly selected from a bag of coloured marbles (without replacement)
- **d** The number of defective batteries in a sample of 100 batteries
- **e** The total of 2 dice when they are rolled 50 times

2 A coin is tossed 10 times. Find the probability of tossing:

- **a** 6 tails
- **b** 3 tails
- **c** 8 heads
- **d** 1 head
- **e** at least 9 tails

3 A coin is tossed 7 times. Find the probability of tossing:

- **a** 1 tail
- **b** 6 heads
- **c** at least 5 heads

4 Kim throws 8 dice. Find the probability that she throws:

- **a** 2 sixes
- **b** 5 sixes
- **c** 6 sixes
- **d** at least 7 sixes
- **e** fewer than 2 sixes

5 A die is thrown 4 times. Find the probability of throwing

- **a** 2 threes
- **b** 3 ones
- **c** 1 four
- **d** fewer than 3 ones
- **e** at least 3 fives

6 The probability of an arrow hitting a target is 0.7. If 8 arrows are fired, find the probability of hitting the target:

- **a** 5 times
- **b** twice
- **c** 3 times
- **d** 7 times
- **e** at least 6 times

7 The probability that dogs of a certain breed will have black spots is $\frac{2}{5}$.
If a dog of this breed has 5 pups, find the probability that:

- **a** 3 will have black spots
- **b** none will have black spots
- **c** 4 will have black spots
- **d** at least 3 will have black spots
- **e** fewer than 3 will have black spots

8 The probability of a certain brand of light globe being faulty is 1.3%.
In a batch of 8 such light globes, find the probability, as a percentage correct to 1 decimal place, that:

- **a** none will be faulty
- **b** 6 will work
- **c** at least 7 will work

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9 A survey showed that about 70% of people ate Crunchy Muesli for breakfast. If another survey is carried out with 20 people, find the probability that:

a 12 people eat Crunchy Muesli **b** 13 people eat Crunchy Muesli

c 9 people eat Crunchy Muesli

10 The probability on average of a traffic light being red is $\frac{3}{7}$. If there are 8 traffic lights on my way to work, find the probability that:

a 7 will be red **b** 6 will be red **c** more than 5 will be red

d 2 will be red **e** they will be all green

11 If the probability of a manufacturer's machine breaking down is $\frac{3}{11}$, find the probability that at any particular time 7 machines out of a total of 20 machines will be broken down. Leave your answer in index form.

12 A photocopier has a probability of $\frac{2}{5}$ of breaking down at any one time.

a A business owns 2 of these machines. Find the probability that at any one time:

i 1 machine is broken down

ii at least 1 machine is broken down.

b Another business owns 10 of these machines. Find the probability, correct to 2 decimal places, that at any one time:

i 4 of these machines are broken down

ii fewer than 3 machines are broken down

13 **a** A treadmill in a gym has a probability of $\frac{4}{7}$ of being used out of peak times.

If a gym has 9 treadmills, find the probability (to 2 decimal places) that 6 are being used out of peak times.

b A stepper has a probability of $\frac{4}{9}$ of being used out of peak times. If a gym has 6 steppers, find the probability that at least 5 are being used out of peak times.

c Find the probability that 3 treadmills and 2 steppers are being used out of peak times. Answer correct to 2 decimal places.

14 A casino has 20 poker machines and 8 roulette wheels. The probability of a poker machine being faulty is $\frac{1}{6}$ and the probability of a roulette wheel being faulty is $\frac{3}{11}$. Find the probability that at a certain time 3 poker machines and 2 roulette wheels are faulty. Leave your answer in index form.

15 A school owns 12 data projectors and 8 smart boards. The probability of a data projector being used in a certain period is $\frac{3}{4}$ and the probability of a smart board being used in the same period is $\frac{5}{6}$. Find (in index form) the probability that 3 data projectors and 5 smart boards are being used at this time.

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16 A quiz has 20 multiple choice questions, each with a choice of 4 answers. If Jay guesses an answer for each question at random, find the probability that he will score 75% on this quiz, correct to 2 significant figures.

17 A library has 10 different books on ancient history. If 70 people borrow one of these books at random over a period of months, what is the most likely number of people to borrow the book *Catullus and Co*?

18 Silvana rolls a die 50 times.

a What is the most likely number of 6s that she rolls?

b Find the probability of Silvana rolling this number of 6s. Leave your answer in index form.

19 Faisal takes the 4 aces from a deck of cards and randomly selects an ace out of these 4 cards over 100 experiments.

a What is the most likely number of times that he selects the ace of hearts?

b Find the probability that he selects the ace of hearts this number of times (to 3 significant figures).

20 One bag contains 5 blue and 3 yellow balls and another bag contains 7 blue and 2 yellow balls.

a A ball is drawn out at random from each bag. Find the probability of selecting:

i 2 blue balls

ii one yellow and one blue ball

iii at least one yellow ball.

b A bag is chosen at random and a ball drawn out. What is the probability that the ball is blue?

c A bag is chosen at random and a ball drawn out. The result is recorded, then the ball is placed back in the bag. If this is done 20 times, find the probability of drawing out 12 blue balls, correct to 3 decimal places.

21 One bag contains 3 black and 4 white balls and another bag contains 5 black and 3 white balls.

a A bag is chosen at random and a ball drawn out. What is the probability that the ball is white?

b A bag is chosen at random and a ball drawn out. The result is recorded, then the ball is placed back in the bag. If this is done 15 times, find the probability of drawing out 9 white balls (to 3 decimal places).

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EXT1

15.03 Mean and variance of a binomial distribution

WS Binomial probability: mean and standard deviation

Binomial distribution

WS Applications of binomial probability

EXAMPLE 11

Draw a histogram for each binomial distribution and describe its shape.

a Probability of success $p = 0.2$ and number of Bernoulli trials $n = 8$

b $p = 0.5$ and $n = 8$

c $p = 0.7$ and $n = 8$

Solution

$$P(X = r) = {}^nC_r\, p^r q^{n-r}$$
$$= {}^nC_r\, p^r (1 - p)^{n-r}$$

a $p = 0.2$

$$P(X = 0) = {}^8C_0\,(0.2)^0(1 - 0.2)^{8-0}$$
$$= {}^8C_0\,(0.2)^0(0.8)^8$$
$$= 0.168$$

Similarly:

$P(1) = {}^8C_1\,(0.2)^1(0.8)^7 = 0.336$

$P(2) = {}^8C_2\,(0.2)^2(0.8)^6 = 0.294$

$P(3) = {}^8C_3\,(0.2)^3(0.8)^5 = 0.147$

$P(4) = {}^8C_4\,(0.2)^4(0.8)^4 = 0.046$

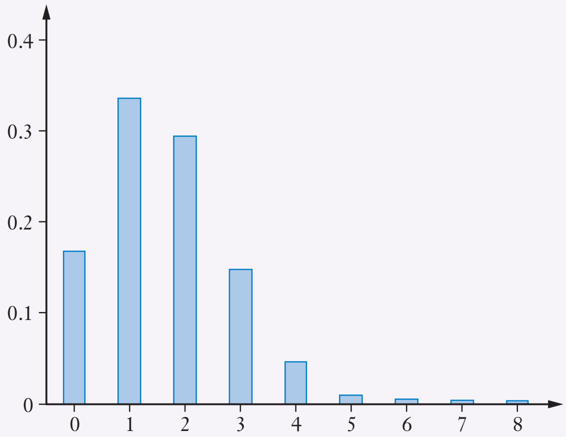
$P(5) = {}^8C_5\,(0.2)^5(0.8)^3 = 0.009$

$P(6) = {}^8C_6\,(0.2)^6(0.8)^2 = 0.001$

$P(7) = {}^8C_7\,(0.2)^7(0.8)^1 = 0.00008$

$P(8) = {}^8C_8\,(0.2)^8(0.8)^0 = 0.000003$

Drawing the graph gives:



The graph is positively skewed.

b $p = 0.5, 1 - p = 0.5$

$P(0) = {}^8C_0\,(0.5)^0(0.5)^8 = 0.0039$

$P(1) = {}^8C_1\,(0.5)^1(0.5)^7 = 0.031\ 25$

$P(2) = {}^8C_2\,(0.5)^2(0.5)^6 = 0.109$

$P(3) = {}^8C_3\,(0.5)^3(0.5)^5 = 0.218\ 75$

$P(4) = {}^8C_4\,(0.5)^4(0.5)^4 = 0.273$

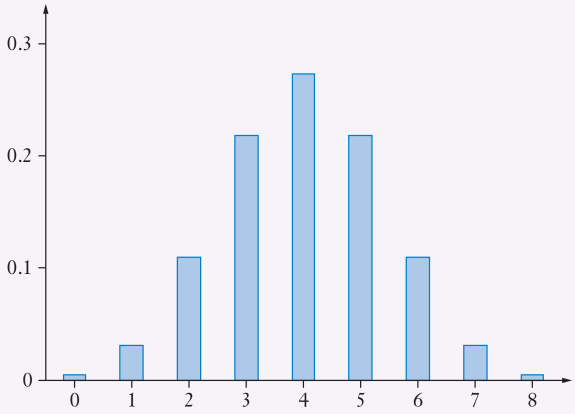
$P(5) = {}^8C_5\,(0.5)^5(0.5)^3 = 0.218\ 75$

$P(6) = {}^8C_6\,(0.5)^6(0.5)^2 = 0.109$

$P(7) = {}^8C_7\,(0.5)^7(0.5)^1 = 0.031\ 25$

$P(8) = {}^8C_8\,(0.5)^8(0.5)^0 = 0.0039$

Drawing the graph gives:



The graph has a symmetrical shape.

c $p = 0.7, 1 - p = 0.3$

$P(0) = {}^8C_0\,(0.7)^0(0.3)^8 = 0.000066$

$P(1) = {}^8C_1\,(0.7)^1(0.3)^7 = 0.0012$

$P(2) = {}^8C_2\,(0.7)^2(0.3)^6 = 0.01$

$P(3) = {}^8C_3\,(0.7)^3(0.3)^5 = 0.047$

$P(4) = {}^8C_4\,(0.7)^4(0.3)^4 = 0.136$

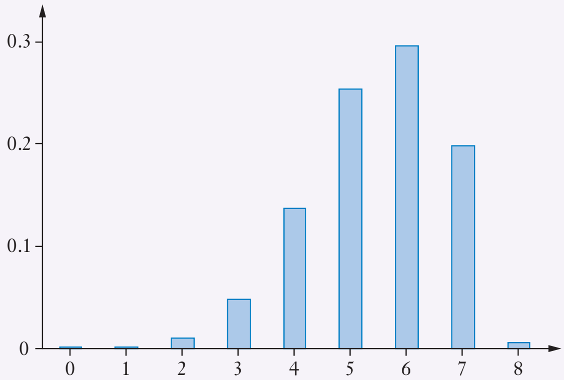
$P(5) = {}^8C_5\,(0.7)^5(0.3)^3 = 0.254$

$P(6) = {}^8C_6\,(0.7)^6(0.3)^2 = 0.296$

$P(7) = {}^8C_7\,(0.7)^7(0.3)^1 = 0.198$

$P(8) = {}^8C_8\,(0.7)^8(0.3)^0 = 0.058$

Drawing the graph gives:



The graph is negatively skewed.

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Shape of a binomial distribution

When $p < 0.5$ the distribution is positively skewed

When $p = 0.5$ the distribution is normal (symmetrical)

When $p > 0.5$ the distribution is negatively skewed

The mean of a Bernoulli distribution is p.

The binomial distribution is n Bernoulli trials.

So the mean of a binomial distribution is np.

Mean

The mean (or expected value) of a binomial distribution is $E(X) = np$.

Proof

$E(X) = E(B_1) + E(B_2) + E(B_3) + \ldots + E(B_n)$ where $B_1, B_2, B_3 \ldots B_n$ are n Bernoulli trials

$= p + p + p + \ldots + p$

$= np$

Variance

The variance of a binomial distribution is $\mathrm{Var}(X) = np(1 - p)$.

Proof

$\mathrm{Var}(X) = \mathrm{Var}(B_1) + \mathrm{Var}(B_2) + \mathrm{Var}(B_3) + \ldots + \mathrm{Var}(B_n)$ where $B_1, B_2, B_3 \ldots B_n$ are n Bernoulli trials

$= p(1 - p) + p(1 - p) + p(1 - p) + \ldots + p(1 - p)$

$= np(1 - p)$

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EXAMPLE 12

Find the mean and variance of each binomial distribution.

a A coin is tossed 100 times and the number of tails recorded.

b A pair of dice is rolled 120 times and the sum of 8 recorded as a 'success'.

Solution

a $P(\text{tail}) = \frac{1}{2}$

So $p = \frac{1}{2}$

$n = 100$

Mean:

$$E(X) = np$$
$$= 100 \times \frac{1}{2}$$
$$= 50$$

Variance:

$$\text{Var}(X) = np(1 - p)$$
$$= 100 \times \frac{1}{2}\left(1 - \frac{1}{2}\right)$$
$$= 50 \times \frac{1}{2}$$
$$= 25$$

b

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(\text{sum of } 8) = \frac{5}{36}$

So $p = \frac{5}{36}$

$n = 60$

Mean:

$$E(X) = np$$
$$= 120 \times \frac{5}{36}$$
$$= 16\frac{2}{3}$$

Variance:

$$\text{Var}(X) = np(1 - p)$$
$$= 120 \times \frac{5}{36}\left(1 - \frac{5}{36}\right)$$
$$= 16\frac{2}{3} \times \frac{31}{36}$$
$$= 14\frac{19}{54}$$

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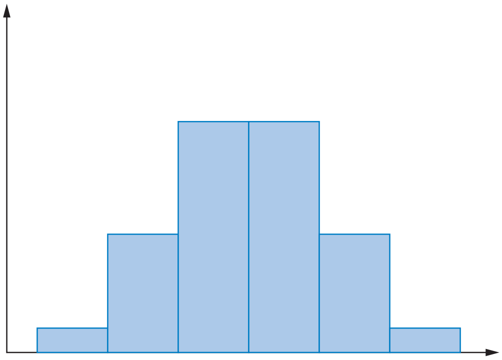
EXT1 Exercise 15.03 Mean and variance of a binomial distribution

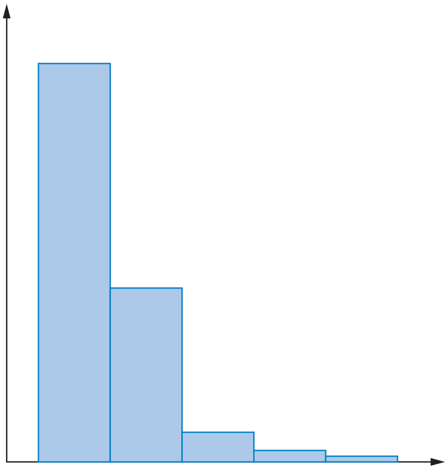
1 Match each graph with the probability of success p.

A $p = 0.2$ **B** $p = 0.9$ **C** $p = 0.5$

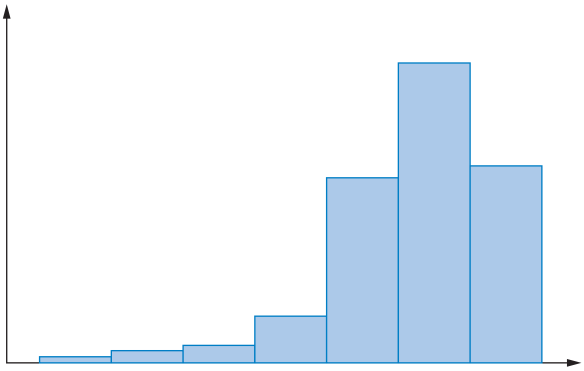
a



b



c



2 Draw a histogram for each binomial distribution and describe its shape.

a $p = 0.1, n = 6$ **b** $p = 0.6, n = 5$ **c** $p = 0.8, n = 6$

3 For each binomial distribution, find:

i the mean **ii** the variance

a Getting heads when tossing a coin 150 times

b Hitting a target when shooting an arrow 60 times, given the probability of $\frac{2}{3}$ that an arrow hits the target

c Rolling 6 when a die is rolled 30 times

d Getting albino pups from a litter of 8 pups given the probability of an albino is 0.12

e Finding defective phones in a batch of 1000 phones if the probability of getting a defective phone is 2.5%

f A movie being rated as a 5 by 100 people when the probability of people rating the movie as a 5 is $\frac{5}{7}$

g A car part being defective in 55 cars when the probability of this part being defective is 2%

h Rolling a double 6 in 200 trials

i Getting a hairless Chinese crested puppy from a litter of 7 puppies when the probability of a hairless puppy is $\frac{5}{9}$

j Selecting a white marble at random in 50 trials from a box containing 23 white and 17 black marbles

4 a When 2 dice are rolled, find the probability that the difference of the numbers on the dice is 2.

b Two dice are rolled 50 times. For the binomial distribution of rolling a difference of 2, find:

i the mean

ii the variance

5 The probability of a new car passing a quality control check is 93%. If a quality control check is made on 250 cars, then for the number of cars that pass, find:

a the mean

b the variance

c the standard deviation

6 The probability of selecting a black jelly bean from a packet is 0.16.

a Find the mean number of black jelly beans selected from 20 packets.

b Find the variance of black jelly beans selected from 50 packets.

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7 The mean of a binomial distribution is 20.5. If there are 50 trials, find the probability of success as a fraction.

8 The mean of a binomial distribution is 12.6 and the probability of success is $\frac{1}{5}$. Find the number of trials.

9 The mean of a binomial distribution is 12.92 and the probability of failure is 0.32. Find the number of trials.

10 Find the binomial distribution with 50 trials that has mean 30.

11 A binomial distribution has mean 68 and variance 13.6. Write the distribution as a table.

12 In a game, a person wins if a die comes up with the number 4 when rolled.

- **a** If X is the discrete random variable for getting a 4 when rolling a die 6 times, find all the possible values of X.
- **b** Draw up a probability distribution table to show all the probabilities (as decimals) of getting a 4 when rolling the die 6 times.
- **c** Draw a bar graph to show this information.
- **d** Describe the shape of the graph.
- **e** Find the probability that a 4 will come up at least 3 times if the die is rolled 6 times.

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WS
Sample proportions

WS
Sample proportion calculations

EXT1 15.04 Sample proportions

A **population proportion** p is the number of people or objects in the population that have a certain characteristic as a fraction of the total population. For example, if 2745 seagulls in a population of 9232 living on the Central Coast have red eyes, then the population proportion is given by $p = \frac{2741}{9232}$.

When the population is large, we usually take a large sample from the population and assume that the **sample proportion** p is a good **estimate** of the population proportion.

Sample proportions

$$p = \frac{x}{n}$$

where x = value of a binomial random variable X with probability of success p and sample size n

The value of $p = \frac{x}{n}$ is different for each sample randomly selected from the population. These sample proportions form a discrete probability distribution called the **sampling distribution of proportions**.

EXAMPLE 13

A company produces batteries and does annual checks to see what proportion of them are defective. In the last 3 years the number of defective batteries was as follows:

Year 1: 31 defective out of a sample of 1500

Year 2: 19 defective out of a sample of 1100

Year 3: 45 defective out of a sample of 2000

a Find the sample proportion for each of the years as a percentage (to 1 decimal place)

b Use these results to state an average percentage of defective batteries for this company.

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Solution

$p = \dfrac{x}{n}$

a Year 1:

$p = \dfrac{31}{1500}$
$= 0.021$
$= 2.1\%$

Year 2:

$p = \dfrac{19}{1100}$
$= 0.017$
$= 1.7\%$

Year 3:

$p = \dfrac{45}{2000}$
$= 0.0225$
$= 2.3\%$

b All the results are around 2%. (Finding the average from the rounded values gives 2.03%.)

So the company has around 2% defective batteries.

Sample size and the normal distribution

We can use the binomial distribution to explore the sampling distribution of proportions p using different values of p and n. Since the calculations become more difficult with larger sample sizes, we can use technology to help draw graphs of sample distributions. Here is an example using a spreadsheet.

EXAMPLE 14

Use a spreadsheet to sketch a bar chart of the sample distribution of p given the population proportion $p = 0.2$ and sample size $n = 8$.

Solution

We can enter the numbers 0 to 8 in column A by putting 0 into A2 and using the formula =**A2+1** in A3 and dragging it down the column.

We can find ${}^8C_r$ by using the formula =**COMBIN()** in column B. The values of r are 0, 1, 2, … 8, so we can enter the first formula in C2 as =**COMBIN(8,A2)** and then drag it down for the other combinations.

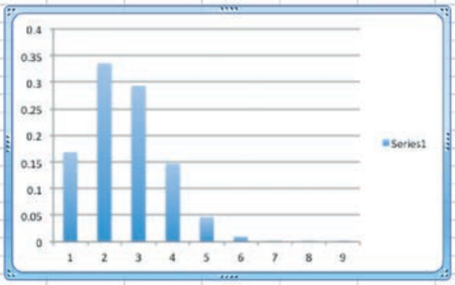
We can use column C to find values of $0.2^0, 0.2^1, 0.2^2, \ldots 0.2^8$ by using the formula =**0.2^A2** and dragging it down the column.

We can use column D to find values of $(1-0.2)^8, (1-0.2)^7, (1-0.2)^6, \ldots (1-0.2)^0$ by using the formula =**(1-0.2)^(8-A2)** and dragging it down the column.

Finally we are ready to put all this information together.

In column E, we put the formula =**B2*C2*D2** and drag it down the column to find all the probabilities $P(0)$, $P(1)$, $P(2)$, … $P(8)$. Highlight column E and choose the bar chart to draw the graph of the distribution.

	A	B	C	D	E
1	n	nCr	0.2^r	(1–0.2)^(8–r	nCr(0.2)^r(1-0.2)^(8–r)
2	0	1	1	0.16777216	0.16777216
3	1	8	0.2	0.2097152	0.33554432
4	2	28	0.04	0.262144	0.29360128
5	3	56	0.008	0.32768	0.14680064
6	4	70	0.0016	0.4096	0.0458752
7	5	56	0.00032	0.512	0.00917504
8	6	28	0.000064	0.64	0.00114688
9	7	8	0.0000128	0.8	8.192E-05
10	8	1	0.00000256	1	0.00000256



Technology can help us find the results for much larger sample sizes. As the probability values in column E become very small, they may be expressed in scientific notation.

EXAMPLE 15

Draw a bar chart of the sample distribution of p, given:

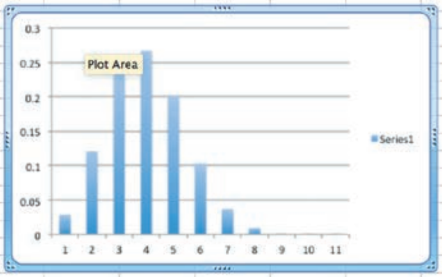
a $p = 0.3$ and $n = 10$ **b** $p = 0.3$ and $n = 25$ **c** $p = 0.3$ and $n = 50$

Solution

We can use the spreadsheet set up from the previous example by changing the numbers and extending the rows.

a Use $p = 0.3$ and $n = 10$.

	A	B	C	D	E
1	n	nCr	0.3^r	(1–0.3)^(10–	nCr(0.2)^r(1-0.2)^(8–r)
2	0	1	1	0.02824752	0.028247525
3	1	10	0.3	0.04035361	0.121060821
4	2	45	0.09	0.05764801	0.233474441
5	3	120	0.027	0.0823543	0.266827932
6	4	210	0.0081	0.117649	0.200120949
7	5	252	0.00243	0.16807	0.102919345
8	6	210	0.000729	0.2401	0.036756909
9	7	120	0.0002187	0.343	0.009001692
10	8	45	0.00006561	0.49	0.001446701
11	9	10	1.9683E-05	0.7	0.000137781
12	10	1	5.9049E-06	1	5.9049E-06



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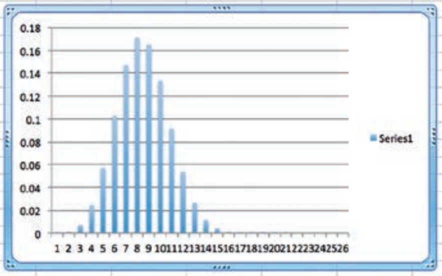
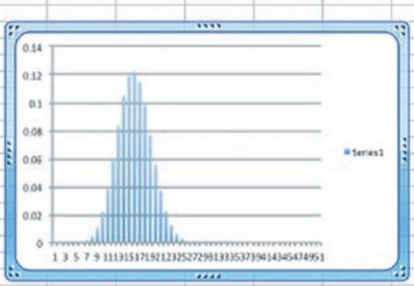
b Use $p = 0.3$ and $n = 25$.

	A	B	C	D	E
1	n	nCr	0.3^r	(1−0.3)^(10−	nCr(0.2)^r(1-0.2)^(8−r)
2	0	1	1	0.00013411	0.000134107
3	1	25	0.3	0.00019158	0.001436859
4	2	300	0.09	0.00027369	0.007389562
5	3	2300	0.027	0.00039098	0.024279989
6	4	12650	0.0081	0.00055855	0.057231402
7	5	53130	0.00243	0.00079792	0.103016524
8	6	177100	0.000729	0.00113989	0.147166462
9	7	480700	0.0002187	0.00162841	0.17119364
10	8	1081575	0.00006561	0.00232631	0.165079581
11	9	2042975	1.9683E-05	0.00332329	0.133635851
12	10	3268760	5.9049E-06	0.00474756	0.091636012
13	11	4457400	1.7715E-06	0.00678223	0.053553514
14	12	5200300	5.3144E-07	0.0096889	0.026776757
15	13	5200300	1.5943E-07	0.01384129	0.011475753
16	14	4457400	4.783E-08	0.01977327	0.004215583
17	15	3268760	1.4349E-08	0.02824752	0.001324897
18	16	2042975	4.3047E-09	0.04035361	0.000354883
19	17	1081575	1.2914E-09	0.05764801	8.05197E-05
20	18	480700	3.8742E-10	0.0823543	1.53371E-05
21	19	177100	1.1623E-10	0.117649	2.42165E-06
22	20	53130	3.4868E-11	0.16807	3.11354E-07
23	21	12650	1.046E-11	0.2401	3.17709E-08
24	22	2300	3.1381E-12	0.343	2.47565E-09
25	23	300	9.4143E-13	0.49	1.3839E-10
26	24	25	2.8243E-13	0.7	4.94252E-12
27	25	1	8.4729E-14	1	8.47289E-14
28					
29					



c Use $p = 0.3$ and $n = 50$.

	A	B	C	D	E
1					
2	0	1	1	1.79847E-08	1.79847E-08
3	1	50	0.3	2.56924E-08	3.85385E-07
4	2	1225	0.09	3.67034E-08	4.04655E-06
5	3	19600	0.027	5.24334E-08	2.77477E-05
6	4	230300	0.0081	7.49048E-08	0.00013973
7	5	2118760	0.00243	1.07007E-07	0.000550934
8	6	15890700	0.000729	1.52867E-07	0.00177086
9	7	99884400	0.0002187	2.18381E-07	0.004770481
10	8	536878650	6.561E-05	3.11973E-07	0.010989144
11	9	2505433700	1.968E-05	4.45676E-07	0.021978287
12	10	10272278170	5.905E-06	6.36681E-07	0.038618991
13	11	37353738800	1.771E-06	9.09544E-07	0.06018544
14	12	1.214E+11	5.314E-07	1.29935E-06	0.08382972
15	13	3.54861E+11	1.594E-07	1.85621E-06	0.105017451
16	14	9.37846E+11	4.783E-08	2.65173E-06	0.118948338
17	15	2.25083E+12	1.435E-08	3.78819E-06	0.122346862
18	16	4.92369E+12	4.305E-09	5.4117E-06	0.114700183
19	17	9.84738E+12	1.291E-09	7.73099E-06	0.098314443
20	18	1.80535E+13	3.874E-10	1.10443E-05	0.077247062
21	19	3.04059E+13	1.162E-10	1.57775E-05	0.055757278
22	20	4.71292E+13	3.487E-11	2.25393E-05	0.037038763
23	21	6.73274E+13	1.046E-11	3.21991E-05	0.022676794
24	22	8.87498E+13	3.138E-12	4.59987E-05	0.012810916
25	23	1.08043E+14	9.414E-13	6.57124E-05	0.006683956
26	24	1.21549E+14	2.824E-13	9.38748E-05	0.003222622
27	25	1.26411E+14	8.473E-14	0.000134107	0.001436369
28	26	1.21549E+14	2.542E-14	0.000191581	0.00059191
29	27	1.08043E+14	7.626E-15	0.000273687	0.00022549
30	28	8.87498E+13	2.288E-15	0.000390982	7.93815E-05
31	29	6.73274E+13	6.863E-16	0.000558546	2.58088E-05
32	30	4.71292E+13	2.059E-16	0.000797923	7.74263E-06
33	31	3.04059E+13	6.177E-17	0.00113989	2.14082E-06
34	32	1.80535E+13	1.853E-17	0.001628414	5.44762E-07
35	33	9.84738E+12	5.559E-18	0.002326305	1.27347E-07
36	34	4.92369E+12	1.668E-18	0.003323293	2.72886E-08
37	35	2.25083E+12	5.003E-19	0.004747562	5.34635E-09
38	36	9.37846E+11	1.501E-19	0.006782231	9.54705E-10
39	37	3.54861E+11	4.503E-20	0.009688901	1.54817E-10
40	38	1.214E+11	1.351E-20	0.013841287	2.26987E-11
41	39	37353738800	4.053E-21	0.019773267	2.99324E-12
42	40	10272278170	1.216E-21	0.028247525	3.52775E-13
43	41	2505433700	3.647E-22	0.040353607	3.68754E-14
44	42	536878650	1.094E-22	0.05764801	3.38652E-15
45	43	99884400	3.283E-23	0.0823543	2.70021E-16
46	44	15890700	9.848E-24	0.117649	1.84105E-17
47	45	2118760	2.954E-24	0.16807	1.05203E-18
48	46	230300	8.863E-25	0.2401	4.90076E-20
49	47	19600	2.659E-25	0.343	1.78751E-21
50	48	1225	7.977E-26	0.49	4.78798E-23
51	49	50	2.393E-26	0.7	8.37548E-25
52	50	1	7.179E-27	1	7.17898E-27
53					
54					



As samples become larger, notice that the sampling distributions become closer to a normal distribution (this is called the **central limit theorem**).

Central limit theorem

For large sample sizes, the sampling distribution of p is approximately normal.

This means we can use the normal distribution for discrete data with a large sample size as well as for continuous data.

Mean and variance of sample proportions distribution

Mean

$$E(p) = p$$

Proof

The mean of a binomial random variable X is given by $E(X) = np$.

$$\begin{aligned}E(p) &= E\left(\frac{X}{n}\right)\\ &= \frac{1}{n}E(X)\\ &= \frac{1}{n}\times np\\ &= p\end{aligned}$$

Variance

$$\text{Var}(p) = \frac{p(1-p)}{n}$$

Proof

The variance of a binomial random variable X is given by $\text{Var}(X) = np(1-p)$.

$$\begin{aligned}\text{Var}(p) &= \text{Var}\left(\frac{X}{n}\right)\\ &= \frac{1}{n^2}\text{Var}(X) && \text{(since variance is the square of standard deviation)}\\ &= \frac{1}{n^2}\times np(1-p)\\ &= \frac{p(1-p)}{n}\end{aligned}$$

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EXAMPLE 16

Find the mean, variance and standard deviation of the sampling distribution with:

a $p = 0.4$ and $n = 50$

b $p = 0.9$ and $n = 125$

Solution

a $E(X)$:

$\mu = p$

$= 0.4$

$\text{Var}(X)$:

$\sigma^2 = \frac{p(1-p)}{n}$

$= \frac{0\ 4(1-04)}{50}$

$= 0.0048$

Standard deviation:

$\sigma = \sqrt{00048}$

$= 0.0693$

b $E(X)$:

$\mu = p$

$= 0.9$

$\text{Var}(X)$:

$\sigma^2 = \frac{p(1-p)}{n}$

$= \frac{0\ 9(1-09)}{125}$

$= 0.00072$

Standard deviation:

$\sigma = \sqrt{000072}$

$= 0.0268$

EXAMPLE 17

The incidence of asthma in a particular population is 21%. A sample of 89 people was taken from this population.

a **i** How many people in this sample would you expect to have asthma?

ii Find the sample proportion p as a fraction.

b Assuming that sample proportion is approximately normally distributed, what is the probability that the percentage of people with asthma is this sample is:

i less than 10%?

ii more than 30%?

Solution

a **i** 21% of 89 = 18.69

So you would expect that 19 people would have asthma in this sample.

ii $p = \frac{19}{89}$

b $\mu = p$

$= 21\%$

$= 0.21$

$$\sigma^2 = \frac{p(1-p)}{n}$$

$$= \frac{021\,(1-021\,)}{89}$$

$= 0.001864$

$$\sigma = \sqrt{0001864}$$

$= 0.04317$

The distribution is approximately normal. To find probabilities we use z-scores.

i For 10% or 0.1:

$$z = \frac{x-\mu}{\sigma}$$

$$= \frac{0\,1-0\,2\,1}{004317}$$

$= -2.55$

Using the probability table for the standard normal distribution on page 633:

$P(z < -2.55) = 0.0054$

This means $P(X < 10\%) = 0.0054$.

So the probability that fewer than 10% of this sample of people has asthma is 0.0054.

ii For 30% or 0.3:

$$z = \frac{x-\mu}{\sigma}$$

$$= \frac{0\,3-0\,2\,1}{004317}$$

$= 2.08$

$P(z < 2.08) = 0.9812$

$P(z > 2.08) = 1 - 0.9812$

$= 0.0188$

So the probability that more than 30% of this sample of people has asthma is 0.0188.

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EXT1 Exercise 15.04 Sample proportions

1 Find the sample proportion p for each situation described.

a Out of a sample of 2150 people, 178 were found to be left-handed.

b Just 7 cars had defective handbrakes out of a sample of 800.

c In a sample of 520 SIM cards, 25 were found to be faulty.

d There were 58 tagged pigeons in a sample of 295.

e A total of 93 people preferred sparkling water in a survey of 627 people.

2 Use technology to draw a bar chart of each distribution of the sample proportion, given:

a $p = 0.4, n = 10$ **b** $p = 0.4, n = 20$ **c** $p = 0.4, n = 40$

d $p = 0.5, n = 25$ **e** $p = 0.5, n = 50$ **f** $p = 0.75, n = 10$

g $p = 0.75, n = 30$ **h** $p = 0.18, n = 15$ **i** $p = 0.18, n = 60$

3 Find the mean and variance of each distribution of the sample proportion, given:

a $p = 0.63, n = 30$ **b** $p = 0.21, n = 55$ **c** $p = 0.7, n = 48$

d $p = 0.42, n = 76$ **e** $p = 0.55, n = 105$

4 A company does regular quality checks on their products and finds that on average 4.7% of their products are faulty. The company does a new quality check on 120 products.

a How many products would you expect to be faulty in this sample?

b Find the mean and standard deviation of the distribution of sample proportions of faulty products.

c If this distribution is approximately normal, find the z-score (to 2 decimal places) for:

i 6% faulty **ii** 5% faulty **iii** 2.3% faulty

v 4% faulty **v** 10% faulty

d Find the probability that the percentage of faulty products in a sample of products is:

i less than 6% **ii** at least 5%

iii at least 10% **v** between 2.3% and 4%

5 In externally marked exam papers, an average of 7.5% of students miss doing the questions on the back page. A random sample of 100 students' exam papers were checked for this student error.

a How many students in the sample would be expected to make this error?

b If the sample proportion is approximately normally distributed, find its mean and standard deviation.

c Find the z-score for each percentage of students making this error:

i 4% **ii** 5% **iii** 8% **v** 10%

d Find the probability that the percentage of students making this error is:

i less than 5% **ii** less than 10% **iii** more than 8%

v more than 4% **v** between 4% and 10%

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6 Car parts are regularly sampled during production to find defective parts. On average, 4% of car parts are defective. A random quality check uses a sample of 180 parts.

a How many car parts would be defective in this sample?

b If sample proportion follows a normal distribution, find its mean and standard deviation.

c Find the z-score for each percentage of a sample of car parts being defective.

i 3% **ii** 2.5% **iii** 5% **v** 4.2%

d Find the probability that the percentage of car parts that are defective in this sample is:

i less than 3% **ii** more than 2.5%

iii more than 5% **v** between 4.2% and 5%

7 Around 14% of the chickens on a chicken farm have brown feathers. A survey is taken of 145 chickens. Find the probability that in this survey the percentage of chickens with brown feathers is:

a less than 8%

b more than 10%

c between 10% and 20%

8 In Havana, Cuba, around 85% of the cars were built before 1960. A sample of 112 cars was taken. Find the probability that in this sample the percentage of cars built before 1960 is:

a less than 80%

b at least 90%

c at least 75%

d between 75% and 80%

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15. TEST YOURSELF

For Questions **1** to **5**, choose the correct answer **A**, **B**, **C** or **D**.

WS
Formula sheet: Statistics and probability

Qz
Practice quiz

1 A distribution of sample proportions with $p = 0.5$ and $n = 12$ is:

A positively skewed **B** normal
C bimodal **D** negatively skewed

2 Which Bernoulli distribution is uniform?

A Getting a 2 when rolling a die
B Getting a red card when selecting a card from a deck of playing cards
C Selecting a red lolly from a packet of mixed lollies of different colours
D Getting a multiple of 5 when choosing a card from 100 cards with a number from 1 to 100 on each one

3 Which 2 random variables below describe a binomial distribution?

A The number of girls in each of 6 Year 10 English classes
B The number of misses when shooting an arrow at a target 60 times
C The number of defective calculators when testing 40 calculators
D The number of green lollies when 12 lollies are selected from a bag of lollies without replacement

4 Which statement does not describe a Bernoulli random variable?

A Rolling the number 6 on a die
B Choosing an ace from a deck of cards
C The number of heads when tossing 3 coins
D Selecting a sweet orange from a sample of oranges

5 The probability of selecting a yellow jellybean from a packet is 0.35.

a Find the expected number of yellow jellybeans selected if 30 packets are sampled.
b Find the variance of yellow jellybeans selected from 45 packets.
c Find the probability of selecting 3 yellow jellybeans from 18 packets.

6 A survey showed that about 63% of people preferred fried eggs to scrambled eggs. If another survey is carried out with 15 people, find the probability that the number of people who preferred fried eggs is:

a 6 **b** 10 **c** 3
d at least 14 **e** less than 2

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7 A survey of 250 people found that 18 people travelled overseas over the previous year.

a Find the sample proportion p as a percentage.

b Find (to 3 decimal places) the mean and standard deviation of the distribution of sample proportions if it is approximately normal.

c Find (to 2 decimal places) the z-score for each percentage of people in the survey who travelled overseas:

i 9% **ii** 5% **iii** 8.5% **v** 6.9% **v** 10%

d Find the probability that the percentage of this sample of people who travelled overseas is:

i less than 5% **ii** 9% or more

iii between 6.9% and 8.5% **v** between 5% and 10%

8 Write the Bernoulli distribution for a missile hitting a target with 86% probability as a:

a piece-wise function **b** table **c** bar chart

9 Find the sample proportion p as a fraction for each situation described.

a Out of a sample of 345 students, 27 played the violin.

b There were 12 faulty airbags on 87 cars tested.

10 The probability that a certain brand of backpack will have faulty straps is 2.8%. In a batch of 19 backpacks, find the probability, as a percentage correct to 1 decimal place, that the number of backpacks with faulty straps is:

a 0 **b** 1 **c** 2 **d** less than 3

11 In a survey of 55 country towns, it was found that 11% of them had a café called the Paragon.

a If the sample proportion is approximately normally distributed, find the mean and standard deviation of the distriution.

b Find the z-score for each percentage of this sample of towns having a Paragon café:

i 10% **ii** 15% **iii** 5% **v** 20%

c Find the percentage probability that the proportion of towns having a Paragon café was:

i less than 5% **ii** more than 15% **iii** between 10% and 20%

12 On average, 2.5% of smart phones are defective. Find the mean and variance of a Bernoulli distribution for defects in smart phones.

13 Draw a histogram for each distribution of the sample proportion and describe its shape, given:

a $p = 0.2, n = 5$ **b** $p = 0.5, n = 5$ **c** $p = 0.8, n = 5$

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14 For each binomial distribution, find:

i the mean **ii** the variance

a Getting tails when tossing a coin 100 times

b Hitting a target when throwing a ball 85 times given the probability of $\frac{5}{9}$ that a ball hits the target

c Rolling a die 25 times and getting a 5

d Getting white flowers from 20 plants given that the probability of getting a white flower is 0.26

e Finding defective car brakes in a batch of 700 brakes if the probability of getting defective brakes is 3.8%

15 For each Bernoulli distribution find:

i the mean **ii** the variance

a $P(X = x) = \begin{cases} 0\;4 & \text{for } x = 1 \\ 0\;6 & \text{for } x = 0 \end{cases}$

b $p(x) = \begin{cases} \frac{5}{7} & \text{for } x = 1 \\ \frac{2}{7} & \text{for } x = 0 \end{cases}$

c $p(x) = \frac{x+1}{3}$ for $x = 0, 1$

d

X	0	1
$P(X = x)$	16%	84%

e

X	0	1
$P(X = x)$	0.29	0.71

16 Find the mean and variance of each distribution of the sample proportion, given:

a $p = 0.3, n = 40$ **b** $p = 0.59, n = 72$ **c** $p = 0.5, n = 66$

d $p = 0.87, n = 100$ **e** $p = 0.42, n = 25$

17 **a** Find the Bernoulli distribution for rolling a total of 5 on 2 dice.

b Find the mean, variance and exact standard deviation of this distribution.

18 A coin is tossed 8 times. Find the probability (as a fraction) of tossing:

a 3 tails **b** 7 tails **c** 6 heads

d 2 heads **e** more than 6 tails

19 On a Saturday morning around 28% of students play sport. A random sample of 140 students is taken one Saturday.

a Find the mean and standard deviation if the sample proportion distribution if it is approximately normal.

b Find the probability that in this sample of students the percentage playing sport is:

i less than 20% **ii** more than 30% **iii** between 25% and 35%

20 The probability of a new calculator passing a quality control check is 93%. If a quality control check is made on 170 calculators, find the mean, variance and standard deviation of the sample proportion distribution of calculators passing the check.

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15. CHALLENGE EXERCISE

1 Find 2 possible Bernoulli distributions with $\text{Var}(X) = 0.1131$.

2 A certain type of exercise bike has a probability of $\frac{2}{11}$ of breaking down at any one time.

a A gym has 7 of these bikes. Find the probability that:

i 2 bikes will be broken down

ii more than 5 bikes will be broken down

b The gym buys 3 more of these exercise bikes. Find the probability, correct to 2 decimal places, that fewer than 2 bikes will be broken down.

3 A quality control check found that out of 150 cricket bats, 5 had a defect.

a Find the sample proportion p as a percentage.

b Find correct to 3 significant figures the mean and standard deviation of the sample proportion distribution if it is approximately normal.

c Find the z-score (to 2 decimal places) for each number of cricket bats in this sample that were defective:

i 6 **ii** 4 **iii** 8 **v** 7

d Find the probability that the number of defective cricket bats in a sample was:

i less than 4 **ii** 4 or more

iii between 7 and 8 **v** between 4 and 5

4 A hotel has 8 standard rooms and 3 suites. The probability that a standard room is left vacant is 9.3% and the probability that a suite is left vacant is 12.9%. On any random night, find the probability that 3 standard rooms and 2 suites are vacant.

5 Use technology to sketch the sample proportion distribution for the each sample size, n, where $p = 0.2$ to show graphically that the distribution approaches the normal distribution as n becomes larger.

a $n = 5$ **b** $n = 15$ **c** $n = 50$

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Practice set 4

In Questions **1** to **10**, select the correct answer **A**, **B**, **C** or **D**.

1 An amount of $6000 is invested at 3.5% p.a. with interest paid quarterly. Find the balance after 10 years.

A $8501.45 **B** $8463.59 **C** $6546.16 **D** $6899.96

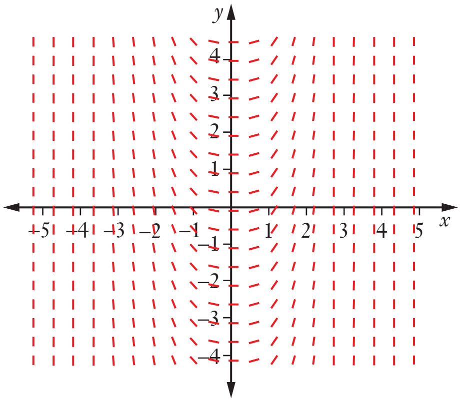
2 EXT1 Which differential equation could have the direction field below?

A $\frac{dy}{dx} = x^2$

B $\frac{dy}{dx} = -x^2$

C $\frac{dy}{dx} = x^3$

D $\frac{dy}{dx} = -x^3$



3 Which function does **not** describe a continuous probability distribution?

A $f(x) = \frac{3x^2}{124}$ for the interval $1 \le x \le 5$

B $f(x) = \frac{8x(x^2+1)^3}{1000}$ in the domain [0, 3]

C $f(x) = 4x^3$ in the domain [0, 1]

D $f(x) = 2 \cos 2x$ in the interval $0 \le x \le \frac{\pi}{4}$

4 EXT1 Solve $\frac{dy}{dx} = \frac{3x^2}{2y}$ if (–1, –4) is a point on the solution.

A $y = -\sqrt{x^3} + 17$ **B** $y = -\sqrt{x^3 + 17}$

C $y = \sqrt{x^3} + 17$ **D** $y = \sqrt{x^3 + 17}$

5 The percentage of scores between z-scores of –3 and 3 is:

A 95% **B** 68% **C** 99.7% **D** 49.85%

6 EXT1 Which Bernoulli distribution is uniform?

A Selecting a 6 when rolling a die

B Selecting a card with a number less than 9 from 20 cards numbered 1 to 20

C Selecting a blue marble from a bag containing 8 blue and 8 white marbles

D Selecting an ace from a deck of playing cards

7 EXT1 The logistic equation is:

A $\frac{dP}{dt} = kP(N - P)$ **B** $\frac{dP}{dt} = -kP$

C $\frac{dP}{dt} = kP$ **D** $\frac{dP}{dt} = k(P - N)$

8 The percentage of data in a normal distribution that lies within 2 standard deviations of the mean is:

A 68% **B** 95% **C** 99.7% **D** 34%

9 The median of a continuous probability distribution $f(x)$ defined in the domain $[a, b]$ is:

A x where $\int_a^b f(x)\,dx = x$ **B** x at the maximum value of $f(x)$

C x where $\int_a^x f(x)\,dx = 0.5$ **D** x where $\int_a^b 0\ 5\,dx = F(x)$

10 EXT1 Solve the differential equation $\frac{dy}{dx} = -002\,y$ given $y = 15$ when $x = 0$.

A $y = -\,0.02e^{-15x}$ **B** $y = 0.02e^{-15x}$

C $y = -\,15e^{0.02x}$ **D** $y = 15e^{-0.02x}$

11 EXT1 The probability of finding a pearl in an oyster is 0.002.

a Find the mean number of pearls in 1500 oysters.

b Find the variance of the number of pearls found in 1500 oysters.

c Find the probability of selecting 2 pearls from 25 oysters, to 3 significant figures.

12 Find each integral.

a $\int e^{3x}\,dx$ **b** $\int (4x - 3)\,dx$ **c** $\int \sec^2 4x\,dx$

d $\int \frac{dx}{x - 3}$ **e** EXT1 $\int \frac{2}{\sqrt{4 - x^2}}\,dx$

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13 EXT1 The probability that a calculator battery is faulty is 1.5%. In a batch of 6 batteries, find the probability, as a percentage correct to 1 decimal place, that:

a none will be faulty **b** 1 will be faulty

c fewer than 2 will be faulty

14 EXT1 A substance is heated to 275°C and then left to cool over t minutes according to the equation $\frac{dT}{dt} = -0.065(T - 26)$.

a Solve this differential equation.

b Find the temperature after:

i 10 minutes **ii** half an hour

c Find the rate at which the substance is cooling after:

i 10 minutes **ii** half an hour

d What will the final temperature be?

15 Use the table of future values of an annuity on page 546 to answer each question.

a Mahmoud wants to deposit $1000 at the end of each year for 20 years so he has a nest egg when he retires from work. If interest is 7% p.a., find how much Mahmoud will have.

b Georgina wants to save a certain amount at the end of each quarter for 3 years so that she will have $10 000 for an overseas trip. If interest is 8% p.a. paid quarterly, how much will Georgina need to save each quarter?

16 Alice promises her son a sum of money on his 18th birthday, made up of $10 for his 1st year of life, $15 for his 2nd year, $20 for his 3rd year and so on, up to 18 years. How much will her son receive?

17 EXT1 Solve each differential equation.

a $\frac{dy}{dx} = 5y$ if (0, 6) lies on the function $y = f(x)$

b $\frac{dT}{dt} = -0.034(T - 32)$ if T is initially 435

c $\frac{dN}{dt} = 00064\ N(5000 - N)$ if when $t = 0$, $N = 1000$

$\left(\text{f rst find } a \text{ and } b \text{ if } \frac{1}{N(5000-N)} = \frac{a}{N} + \frac{b}{5000-N}\right)$

18 A plant grows so that it increases its height each month by 0.2 of its previous month's height. If it grows to 3 m, find its height in the first month.

19 EXT1 Draw a direction field for each differential equation.

a $\frac{dy}{dx} = 2 - x$ **b** $\frac{dy}{dx} = x^2 + 1$ **c** $\frac{dy}{dx} = \frac{2y}{x^2}$

20 **a** Find the cumulative distribution function for $f(x) = \frac{x^2}{114}$ defined for [1, 7].

b Find as a fraction:

i $P(X \leq 5)$ **ii** $P(X \leq 2)$ **iii** $P(X > 3)$

iv $P(X \geq 4)$ **v** $P(3 \leq X \leq 6)$

c Find correct to 2 decimal places:

i the median **ii** the 93rd percentile

21 EXT1 A park has 100 daffodil bulbs. The increase in the number of bulbs N over time t years is given by the logistic equation $\frac{dN}{dt} = 0.00012N(3000 - N)$.

a Show that $\frac{1}{N(3000-N)} = \frac{1}{3000}\left(\frac{1}{N} + \frac{1}{3000-N}\right)$.

b Solve the logistic equation.

c Find the number of daffodil bulbs after:

i 3 years **ii** 6 years

d Find the rate at which the bulbs are increasing after:

i 3 years **ii** 6 years

e What is the maximum possible number of bulbs according to this formula?

22 A farmer places 20 bales of hay in a row in the shed. He then stacks 17 on top of these, then 14 in the next row up and so on, continuing with this pattern.

a How many bales of hay are in the top row?

b How many rows are there?

c How many bales of hay are stacked in the shed?

23 EXT1 Given $u = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ and $v = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$, find:

a $u + v$ **b** $u - v$

c the magnitude and direction of u **d** $u \; v$

e the angle between u and v

24 Ryanna has \$120 000 in a superannuation trust fund. She withdraws \$1600 each month as a pension.

a If the trust fund earns 6% p.a. paid monthly, find the amount left in the fund after:

i 1 month **ii** 2 months **iii** 3 months

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b **i** Show that the amount left after n months is given by:

$120\,000(1.005)^n - 1600(1 + 1.005 + 1.005^2 + \ldots + 1.005^{n-1})$

ii How much will be in the trust fund after 20 months?

25 EXT1 Solve each differential equation.

a $\frac{dy}{dx} = x \sec y$ given $y = \frac{\pi}{2}$ when $x = 2$

b $\frac{dy}{dx} = 3x^2\sqrt{9 - y^2}$ given (0, 0) lies on the function

26 I put \$2000 in the bank, where it earns interest at the rate of 12% p.a., paid quarterly. How much will there be in my account after 3 years?

27 Find the median of the continuous random variable $f(x) = \frac{4x^3}{609}$ defined on the interval $2 \le x \le 5$.

28 Express 0 1 7 as a fraction.

29 Find any stationary points and points of inflection on the graph of the function $y = x^3 - 6x^2 - 15x + 1$.

30 I borrow \$5000 at 18% interest p.a. compounded monthly and make equal monthly payments over 3 years, at the end of which the loan is fully paid out.
Find the amount of each monthly payment.

31 EXT1 Draw the direction field for the differential equation $\frac{dy}{dx} = xy$ and sketch the graph of the solution that passes through (0, 2).

32 The volume of blood in adult humans is normally distributed with mean 4.7 L and standard deviation 0.4.

a What would be the range of blood volumes for 95% of adults?

b What percentage of blood volumes would lie between 4.3 L and 5.1 L?

c Use the table of standard normal probabilities on pages 633–634 to find the z-score for a volume of:

i 6 L **ii** 5.2 L **iii** 3.9 L **v** 4.1 L

d Find the probability for the blood volume X:

i $P(X \le 6)$ **ii** $P(X \le 3.9)$ **iii** $P(X \ge 4.1)$

iv $P(X \ge 5.2)$ **v** $P(3.9 \le X \le 5.2)$

e Find the percentage of blood volumes that lie between:

i 4 L and 5 L **ii** 4.2 L and 5.2 L **iii** 3.6 L and 5.8 L

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33 Use the repayments table for reducing balance loans on \$1000 on page 557 to find the monthly repayments for a loan of:

a \$25 000 at 4% p.a. over 5 years
b \$100 000 at 2.5% p.a. over 25 years
c \$128 500 at 6% p.a. over 15 years
d \$2400 at 7.5% p.a. over 10 years

34 Find the cumulative distribution function for each continuous probability distribution.

a $f(x) = \frac{3(x+2)^2}{335}$ defined in the domain [0, 5]

b $f(x) = \frac{x^3}{156}$ defined in the domain [1, 5]

c $f(x) = 2 \cos x$ in the interval $0 \le x \le \frac{\pi}{6}$

35 Find the mode of the continuous probability distribution $f(x) = \frac{4x - x^2}{9}$ defined in the domain [1, 4].

36 EXT1 Solve each differential equation.

a $\frac{dM}{dt} = -024\,M$ given $M = 65$ when $t = 0$

b $\frac{dy}{dx} = \frac{\sin 2x}{12y^2}$ if $y = 1$ when $x = 0$

c $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{x}$ given (1, 0) lies on the solution

37 EXT1 Find the sample proportion p as a fraction for each situation.

a Out of a sample of 125 students, 68 played football.
b In a rating of a movie, of 150 people surveyed, 19.3% rated the movie as a 5.

38 Use the probability tables for a standard normal distribution on pages 633–4 to find:

a $P(z \le 1.35)$
b $P(z \ge -0.88)$
c $P(z \le -1)$
d $P(z \ge 2.04)$
e $P(-3.12 \le z \le 2.81)$

39 EXT1 For the Bernoulli distribution

$$P(X = x) = \begin{cases} \frac{4}{9} & \text{for } x = 1 \\ \frac{5}{9} & \text{for } x = 0 \end{cases}$$

find:

a the mean
b the variance

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40 Li scored 72% in her first maths exam in which the class mean was 69% with standard deviation 0.8. She scored 65% in her second exam with class mean 55% and standard deviation 1.2. In which exam did Li do better in relation to her class?

41 EXT1 A survey of 75 students found that 21 of them preferred to study in a library.

a Find the sample proportion p as a percentage.

b The sample proportion distribution is approximately normal. Find the mean and standard deviation of the distribution.

c Find (to 2 decimal places) the z-score for the percentage of students in a sample who preferred to study in the library to be:

i 30% **ii** 15% **iii** 25%

v 19.8% **v** 29.5%

d Find the probability that the percentage of students in a sample who preferred to study in the library is:

i less than 20% **ii** 30% or more

iii between 15% and 25% **v** between 19.8% and 29.5%

42 EXT1 Find the mean and variance of each distribution of the sample proportion, given:

a $p = 0.45, n = 80$ **b** $p = 0.2, n = 125$ **c** $p = 0.5, n = 70$

43 I borrow $10 000 over 5 years at 1.85% monthly interest. How much do I need to pay each month?

44 EXT1 Two dice are rolled 7 times and the total recorded. Find correct to 3 decimal places the probability of rolling a total of 8:

a 3 times **b** twice

45 A function is given by $f(x) = \frac{x^4}{1555}$. Over what domain starting at $x = 1$ is this a probability density function?

46 A data set is normally distributed with mean 12.5 and standard deviation 1.3. Find the raw score for each z-score.

a 0.4 **b** –1.5 **c** 2.96 **d** –3

47 EXT1 The probability that a certain poker machine will give a jackpot is 0.5%.

a Write this as a Bernoulli distribution table.

b Find the mean and variance.

48 Find the mode of the continuous probability distribution $f(x) = \frac{3x^2}{512}$ defined in the domain [0, 8].

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49 EXT1 Given the differential equation $\frac{dy}{dx} = 9x^2y$:

a draw a direction field

b solve the equation, given that $y = 4$ when $x = 1$

c draw this solution on the direction field.

50 EXT1 The probability of rolling a 6 on a loaded die is $\frac{3}{5}$. If this die is rolled 10 times, find correct to 3 decimal places the probability of rolling a 6:

a 4 times **b** 6 times **c** at least 9 times

51 A normal distribution has a standard deviation of 2.5. A score of 19 has a z-score of 0.2. What is the mean?

52 A factory produces 350 mL cans of soft drink. In a quality check, the mean volume of soft drink in cans was found to be 349.8 mL, with a standard deviation of 0.2 mL. The factory rejects cans that are outside 2 standard deviations of the mean.

a Find the percentage and volumes of cans that the factory rejects.

b The factory manager decides they are rejecting too many cans and that only cans whose volume is outside 3 standard deviations of the mean will be rejected. What percentage and volumes of cans will the factory reject?

53 EXT1 Draw a histogram for each sample proportion and describe its shape.

a $p = 0.8, n = 6$ **b** $p = 0.1, n = 4$

54 A sum of $2500 is put into a bank account where it earns 2.4% p.a. Find the amount in the bank after 4 years if interest is paid:

a annually **b** quarterly **c** monthly

55 EXT1 Find two possible Bernoulli distributions with $\text{Var}(X) = 0.2244$.

56 EXT1 A quality control check found that out of 60 spare parts, 3 were defective.

a Find the sample proportion p as a percentage.

b Find the mean and standard deviation of the sample proportion distribution if it is approximately normal.

c Find correct to 2 decimal places the probability that the number of defective spare parts in a sample is:

i less than 4 **ii** 5 or more **iii** between 2 and 8

57 EXT1 For each binomial distribution, find, as a fraction where necessary:

i the mean **ii** the variance

a rolling a 1 when rolling a die 30 times

b getting a head when tossing a coin 20 times

c hitting a target in 35 trials given the probability of hitting the target is $\frac{7}{10}$

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ANSWERS

Answers are based on full calculator values and only rounded at the end, even when different parts of a question require rounding. This gives more accurate answers. Answers based on reading graphs may not be accurate.

Chapter 1

Exercise 1.01

1 **a** 14, 17, 20 **b** 23, 28, 33
c 44, 55, 66 **d** 85, 80, 75
e 1, –1, –3 **f** –15, –24, –33
g $2, 2\frac{1}{2}, 3$ **h** 3.1, 3.7, 4.3
i 32, –64, 128 **j** $\frac{27}{320}, \frac{81}{1280}, \frac{243}{5120}$

2 **a** 1456 **b** 63 **c** 78
d 126 **e** 91 **f** 441

3 $\frac{1}{16}, \frac{1}{32}, \frac{1}{64}$

4 38, 51, 66, 83

5 21, 34, 55, 89, 144

6

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

7 28th term **8** 54th term
9 30th term **10** 15th term
11 Yes **12** No
13 Yes **14** $n = 13$
15 $n = 30, 31, 32, \ldots$ **16** –2
17 103 **18** 785
19 **a** $d = 8$ **b** 87
20 $d = 9$ **21** $a = 12, d = 7$
22 173 **23** $a = 5$
24 280 **25** 1133
26 **a** $T_2 - T_1 = T_3 - T_2 = d = \log_5 x$
b $80 \log_5 x$ or $\log_5 x^{80}$ **c** 8.6
27 **a** $T_2 - T_1 = T_3 - T_2 = d = \sqrt{3}$
b $50\sqrt{3}$
28 26 **29** $122b$ **30** 38th term

Exercise 1.02

1 **a** $y = 13$ **b** $x = -4$ **c** $x = 72$
d $b = 11$ **e** $x = 7$ **f** $x = 42$
g $k = 2$ **h** $x = 1$ **i** $t = -2$
j $t = 3$

2 **a** 46 **b** 78 **c** 94
d –6 **e** 67

3 **a** 590 **b** –850 **c** 414
d 1610 **e** –397

4 **a** –110 **b** 12.4 **c** –8.3
d 37 **e** $15\frac{4}{5}$

5 $T_n = 2n + 1$

6 **a** $T_n = 8n + 1$ **b** $T_n = 2n + 98$
c $T_n = 3n + 3$ **d** $T_n = 6n + 74$
e $T_n = 4n - 25$ **f** $T_n = 20 - 5n$
g $T_n = \frac{n+6}{8}$ **h** $T_n = -2n - 28$
i $T_n = 1.2n + 2$ **j** $T_n = \frac{3n-1}{4}$

Exercise 1.03

1 **a** 375 **b** 555 **c** 480
2 **a** 2640 **b** 4365 **c** 240
3 **a** 2050 **b** –2575
4 **a** –4850 **b** 4225
5 **a** 28 875 **b** 3276 **c** –1419
d 6426 **e** 6604 **f** 598
g –2700 **h** 11 704 **i** –290
j 1284
6 21 **7** 8 **8** 11

9 8 and 13 terms

10 $a = 14, d = 4$ **11** $a = -3, d = 5$

12 2025 **13** 3420 **14** 1010

15 **a** $(2x + 4) - (x + 1) = (3x + 7) - (2x + 4) = x + 3$
b $25(51x + 149)$

16 1290 **17** 16

18 $S_n = S_{n-1} + T_n$
So $S_n - S_{n-1} = T_n$

19 **a** 816 **b** 4234

Exercise 1.04

1 **a** No **b** Yes, $r = -\frac{3}{4}$ **c** Yes, $r = \frac{2}{7}$
d No **e** No **f** No
g Yes, $r = 0.3$ **h** Yes, $r = -\frac{3}{5}$ **i** No
j Yes, $r = -8$

2 **a** $x = 196$ **b** $y = -48$
c $a = \pm 12$ **d** $y = \frac{2}{3}$
e $x = 2$ **f** $p = \pm 10$
g $y = \pm 21$ **h** $m = \pm 6$
i $x = 4 \pm 3\sqrt{5}$ **j** $k = 1 \pm 3\sqrt{7}$
k $t = \pm\frac{1}{6}$ **l** $t = \pm\frac{2}{3}$

3 **a** $T_n = 5^{n-1}$ **b** $T_n = 1.02^{n-1}$
c $T_n = 9^{n-1}$ **d** $T_n = 2 \times 5^{n-1}$
e $T_n = 6 \times 3^{n-1}$ **f** $T_n = 8 \times 2^{n-1} = 2^{n+2}$
g $T_n = \frac{1}{4} \times 4^{n-1} = 4^{n-2}$ **h** $T_n = 1000(-0.1)^{n-1}$
i $T_n = -3(-3)^{n-1} = (-3)^n$ **j** $T_n = \frac{1}{3}\left(\frac{2}{5}\right)^{n-}$

4 **a** 1944 **b** 9216 **c** −8192
d 3125 **e** $\frac{64}{729}$

5 **a** 256 **b** 26 244 **c** 1.369
d −768 **e** $\frac{3}{1024}$

6 **a** 234 375 **b** 268.8 **c** −81 920
d $\frac{2187}{156\,250}$ **e** 27

7 **a** 3×2^{19} **b** 7^{19} **c** 1.04^{20}
d $\frac{1}{4}\left(\frac{1}{2}\right)^{19} = \frac{1}{2^{21}}$ **e** $\left(\frac{3}{4}\right)^{20}$

8 11^{49} **9** 6th **10** 5th

11 No **12** 7th **13** 11th

14 9th **15** $n = 5$ **16** $r = 3$

17 **a** $r = -6$ **b** −18

18 $a = \frac{1}{10}, r = \pm 2$

19 $n = 7$ **20** $208\frac{2}{7}$

Exercise 1.05

1 **a** 2 097 150 **b** 7 324 218

2 **a** 720 600 **b** 26 240

3 **a** 131 068 **b** 0.5

4 **a** 7812 **b** $35\frac{55}{64}$ **c** 8403
d 273 **e** 255

5 **a** 1792 **b** 3577

6 148.58 **7** 133.33 **8** $n = 9$

9 10 terms **10** $a = 9$ **11** 10 terms

12 **a** 2046 **b** 100 **c** 2146

Puzzles

1 Choice 1 gives $465.00. Choice 2 gives $10 737 418.23 so choice 2 is better.

2 382 apples

Exercise 1.06

1 **a** Yes, $13\frac{1}{2}$ **b** No
c Yes, $12\frac{4}{5}$ **d** No
e Yes, 3 **f** Yes, $\frac{25}{32}$
g No **h** Yes, $-1\frac{5}{22}$
i No **j** Yes, $1\frac{3}{7}$

2 **a** 80 **b** $426\frac{2}{3}$ **c** $66\frac{1}{3}$ **d** 12
e $\frac{7}{10}$ **f** 54 **g** $-10\frac{2}{7}$ **h** $\frac{9}{20}$
i 48 **j** $-\frac{16}{39}$

3 **a** 0.58 **b** 0.15 **c** 0.000080
d 0.016 **e** 0.89

4 $a = 4$ **5** $r = \frac{2}{5}$ **6** $a = 5\frac{3}{5}$

7 $r = \frac{7}{8}$ **8** $r = -\frac{1}{4}$ **9** $r = -\frac{2}{3}$

10 $a = 3, r = \frac{2}{3}$ or $a = 6, r = \frac{1}{3}$

11 $a = 192, r = -\frac{1}{4}, S = 153\frac{3}{5}$

12 $a = 1, r = \frac{2}{3}, S = 3$ or $a = -1, r = -\frac{2}{3}, S = -\frac{3}{5}$

13 $a = 150, r = \frac{3}{5}, S = 375$

14 $a = \frac{2}{5}, r = \frac{2}{3}, S = 1\frac{1}{5}$

15 $a = 3, r = \frac{2}{5}$ or $a = 2, r = \frac{3}{5}$

16 $x = \frac{21}{32}$

17 **a** $|k| < 1$ **b** $-\frac{2}{5}$ **c** $k = \frac{3}{4}$

18 **a** $-\frac{1}{2} < p < \frac{1}{2}$ **b** $\frac{5}{7}$ **c** $p = \frac{1}{14}$

19 See worked solutions.

Exercise 1.07

See worked solutions for complete proofs.

1 **a, c**

2 **a** Step 3: Prove true for $n = k + 1$.

That is, prove $3 + 10 + 17 + \ldots + (7k - 4) + (7k + 3) = \frac{k+1}{2}(7k + 6)$

$\text{LHS} = 3 + 10 + 17 + \ldots + (7k - 4) + (7k + 3)$

$= \frac{k}{2}(7k - 1) + (7k + 3)$

b Step 3: Prove true for $n = k + 1$.

Prove $5 + 11 + 19 + \ldots + (8k - 3) + (8k + 5) = (k + 1)(4k + 5)$

$\text{LHS} = 5 + 11 + 19 + \ldots + (8k - 3) + (8k + 5)$

$= k(4k + 1) + (8k + 5)$

c Step 3: Prove true for $n = k + 1$.

Prove $5 + 10 + 20 + \ldots + 5{\cdot}2^k + 5{\cdot}2^{k+1} = 5(2^{k+2} - 1)$

$\text{LHS} = 5 + 10 + 20 + \ldots + 5{\cdot}2^k + 5{\cdot}2^{k+1}$

$= 5(2^{k+1} - 1) + 5{\cdot}2^{k+1}$

d Step 3: Prove true for $n = k + 1$.

$1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^k} = 2\left(1 - \frac{1}{2^{k+}}\right)$

$\text{LHS} = 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{k+1-1}}$

$= 2\left(1 - \frac{1}{2^k}\right) + \frac{1}{2^k}$

e Step 3: Prove true for $n = k + 1$.

Prove $2 + 5 + 8 + \ldots + (3k - 1) + (3k + 2) = \frac{3k^2 + 7k + 4}{2}$

$\text{LHS} = 2 + 5 + 8 + \ldots + (3k - 1) + (3k + 2)$

$= \frac{3k^2 + k}{2} + (3k + 2)$

f Step 3: Prove true for $n = k + 1$.

Prove $2 + 4 + 8 + \ldots + 2^k + 2^{k+1} = 2(2^{k+1} - 1)$

$\text{LHS} = 2 + 4 + 8 + \ldots + 2^k + 2^{k+1}$

$= 2(2^k - 1) + 2^{k+1}$

g Step 3: Prove true for $n = k + 1$.

Prove $5 + 10 + 15 + \ldots + 5k + 5(k+ 1) = \frac{5}{2}(k + 1)(k + 2)$

$\text{LHS} = 5 + 10 + 15 + \ldots + 5k + 5(k+ 1)$

$= \frac{5}{2}k(k + 1) + 5(k+ 1)$

h Step 3: Prove true for $n = k + 1$.

Prove $-2 - 4 - 6 - \ldots - 2k - 2(k + 1) = -(k + 1)(k + 2)$

$\text{LHS} = -2 - 4 - 6 - \ldots - 2k - 2(k + 1)$

$= -k(k + 1) - 2(k + 1)$

i Step 3: Prove true for $n = k + 1$.

Prove $9 + 14 + 19 + \ldots + (5k + 4) + (5k + 5 + 4) = \frac{5k^2 + 23k + 18}{2}$

$\text{LHS} = 9 + 14 + 19 + \ldots + (5k + 4) + (5k + 9)$

$= \frac{5k^2 + 13k}{2} + (5k + 9)$

j Step 3: Prove true for $n = k + 1$.

Prove $9 + 27 + 81 + \ldots + 3^k + 3^{k+1} = \frac{9(3^k - 1)}{2}$

$\text{LHS} = 9 + 27 + 81 + \ldots + 3^k + 3^{k+1}$

$= \frac{9(3^{k-} - 1)}{2} + 3^{k+1}$

k Step 3: Prove true for $n = k + 1$.

Prove $-4 + 8 - 16 + \ldots - 4(-2)^{k-1} - 4(-2)^{k}$

$= \frac{4((-2)^{k+} \quad -1)}{3}$

LHS $= -4 + 8 - 16 + \ldots - 4(-2)^{k-1} - 4(-2)^{k}$

$= \frac{4((-2)^{k} - 1)}{3} - 4(-2)^{k}$

l Step 3: Prove true for $n = k + 1$.

Prove $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$

$= \frac{1}{6}(k+1)(k+2)(2k+3)$

LHS $= 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$

$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$

m Step 3: Prove true for $n = k + 1$.

Prove $1^3 + 3^3 + 5^3 + \ldots + (2k-1)^3 + (2k+1)^3$

$= (k+1)^2(2k^2 + 4k + 1)$

LHS $= 2k^4 + 8k^3 + 11k^2 + 6k + 1$

RHS $= 2k^4 + 8k^3 + 11k^2 + 6k + 1$

3 See worked solutions.

4 No, because she didn't prove the statement true for $n = 1$ (or other smallest value).

Test yourself 1

1 C **2** C **3** B **4** B

5 **a** $T_n = 4n + 5$ **b** $T_n = 14 - 7n$

c $T_n = 2 \times 3^{n-1}$ **d** $T_n = 200\left(\frac{1}{4}\right)^{n-}$

e $T_n = (-2)^n$

6 **a** 2 **b** 1185 **c** 1183

d $T_{15} = S_{15} - S_{14}$, $S_{15} = S_{14} + T_{15}$

e $n = 16$

7 **a** i **b** ii **c** i **d** iii

e i **f** ii **g** ii **h** i

i i **j** i

8 $n = 108$

9 $a = -33$, $d = 13$

10 **a** 59 **b** 80 **c** 18th

11 **a** $x = 25$ **b** $x = \pm 15$

12 No, because it could be false for $n > 5$.

13 $x = 3$

14 **a** 136 **b** 44 **c** 6

15 $121\frac{}{2}$

16 **a** $S_n = n(2n+3)$ **b** $S_n = \frac{107^{\,n} - 1}{007}$

17 **a** $|x| < 1$ **b** $2\frac{}{2}$ **c** $x = \frac{1}{3}$

18 $d = 5$

19 $x = -\frac{2}{17}, 2$

20 1300

21 **a** 735 **b** 4315

22 $n = 20$

23 $n = 11$

24 See worked solutions for complete proofs.

a Step 3: Prove true for $n = k + 1$.

Prove $15 + 45 + 135 + \ldots + 5(3)^k + 5(3)^{k+1}$

$= \frac{15(3^{k+} \quad -1)}{2}$

LHS $= 15 + 45 + 135 + \ldots + 5(3)^k + 5(3)^{k+1}$

$= \frac{15(3^k - 1)}{2} + 5(3)^{k+1}$

b Step 3: Prove true for $n = k + 1$.

Prove $4 + 7 + 10 + \ldots + (3k+1) + (3k+4)$

$= \frac{(k+1)(3k+8)}{2}$

LHS $= 4 + 7 + 10 + \ldots + (3k+1) + (3k+4)$

$= \frac{k(3k+5)}{2} + (3k+4)$

c Step 3: Prove true for $n = k + 1$.

Prove $2(k+1)((k+1) - 1)$ is divisible by 4.

$2(k+1)((k+1) - 1) = 4p + 4k$ (substituting $4p$ for $2k^2 - 2k$)

Challenge exercise 1

1 **a** 8.1 **b** 19th

2 **a** $\frac{\pi}{4}$ **b** $\frac{9\pi}{4}$ **c** $\frac{33\pi}{4}$

3 **a** 2 097 170 **b** −698 775

4 6th **5** 17 823

6 $n \geq 5$ **7** $n = 1, 2, 3$

8 **a** $a = 7, r = -2$ **b** −56

9 $a = 5, r = 4$

a Step 3: Prove true for $n = k + 1$.

Prove $a + ar + ar^2 + \ldots + ar^{k-1} + ar^k$

$= \dfrac{a(r^{k+} \ -1)}{r-1}$

LHS $= a + ar + ar^2 + \ldots + ar^{k-1} + ar^k$

$= \dfrac{a(r^k - 1)}{r-1} + ar^k$

b Step 3: Prove true for $n = k + 1$.

Prove $1 + x + x^2 + \ldots + x^{k-1} + x^k = \dfrac{1 - x^{k+}}{1-x}$

LHS $= 1 + x + x^2 + \ldots + x^{k-1} + x^k$

$= \dfrac{1 - x^k}{1-x} + x^k$

10 **a** $\operatorname{cosec}^2 x$

b $r = \cos^2 x$

$-1 < \cos x < 1$ where $\cos^2 x \neq 0, 1$

So $0 < \cos^2 x < 1$

Since $|r| < 1$, the series has a limiting sum.

Chapter 2

Exercise 2.01

1 **a** Vertical translation 3 units up

b Vertical translation 7 units down

c Vertical translation 1 unit down

d Vertical translation 5 units up

2 **a** Vertical translation 1 unit up

b Vertical translation 4 units down

c Vertical translation 8 units up

3 Vertical translation 9 units up

4 **a** $y = x^2 - 3$ **b** $f(x) = 2^x + 8$

c $y = |x| + 1$ **d** $y = x^3 - 4$

e $f(x) = \log x + 3$ **f** $y = \dfrac{2}{x} - 7$

5 **a** Vertical translation 1 unit down

b Vertical translation 6 units up

6 **a** **i** $y = 2x^3 - 2$ **ii** $y = 2x^3 + 6$

b **i** $y = |x| - 3$ **ii** $y = |x| - 6$

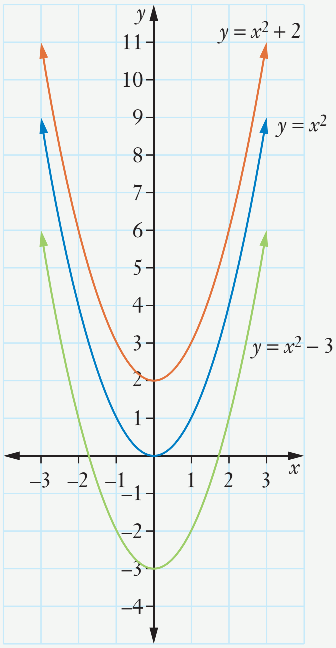
c **i** $y = e^x + 1$ **ii** $y = e^x + 5$

d **i** $f(x) = \log_e x + 10$ **ii** $f(x) = \log_e x - 8$

7 **a** $(1, -1)$ **b** $(1, -9)$ **c** $(1, -3 + m)$

8 **a** $(-1, 1)$ **b** $(-1, 5)$

9 **a**



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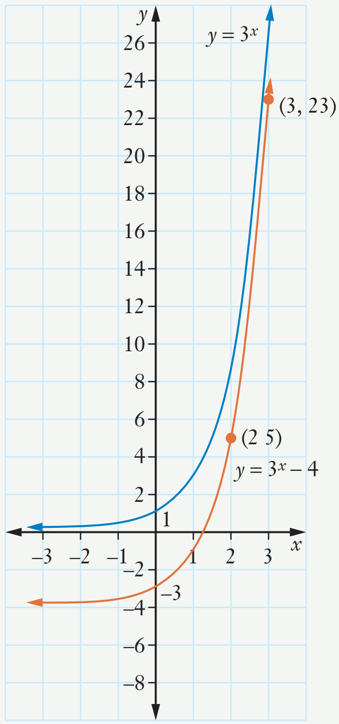

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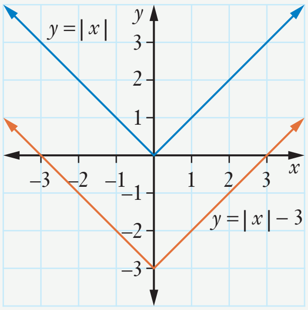
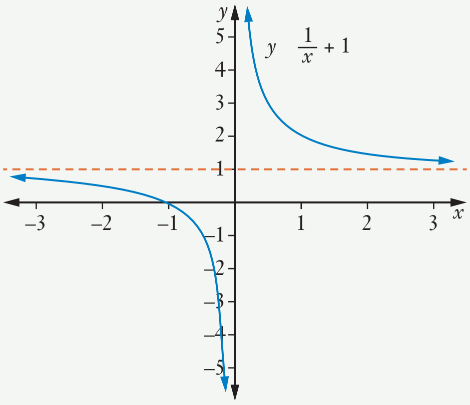
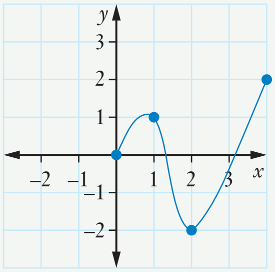


10 a Vertical translation 1 unit up

b



11 a



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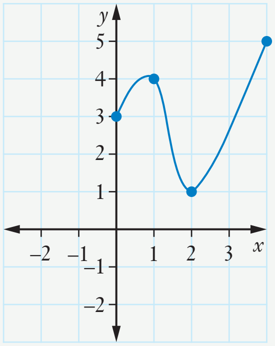
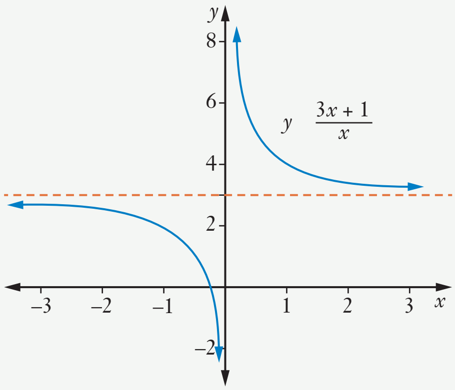


12 b



Exercise 2.02

1 a Horizontal translation 4 units to the right

b Horizontal translation 2 units to the left

2 a Horizontal translation 5 units to the right

b Horizontal translation 3 units to the left

3 a $y = (x + 3)^2$ **b** $f(x) = 2^{x-8}$

c $y = |x + 1|$ **d** $y = (x - 4)^3$

e $f(x) = \log (x + 3)$

4 Horizontal translation 3 units to the right

5 a Horizontal translation 2 units to the left

b Horizontal translation 5 units to the right

6 a i $y = -(x + 4)^2$ **ii** $y = -(x - 8)^2$

b i $y = |x - 3|$ **ii** $y = |x + 4|$

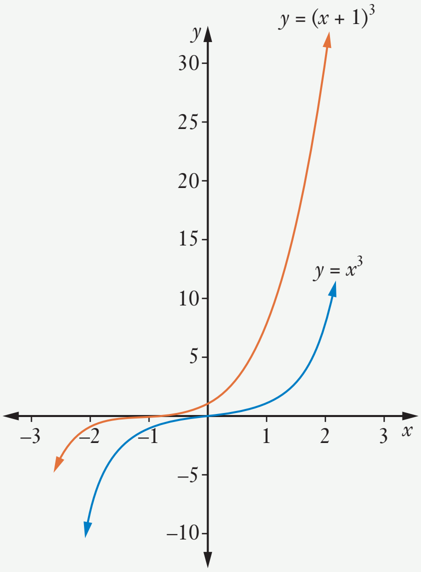
c i $y = e^{x+6}$ **ii** $y = e^{x-5}$

d i $f(x) = \log_2 (x - 5)$ **ii** $f(x) = \log_2 x$

7 a (5, –3) **b** (–8, –3) **c** (1 + *t*, –3)

8 a (3, 2) **b** (–9, 2)

9 **a**



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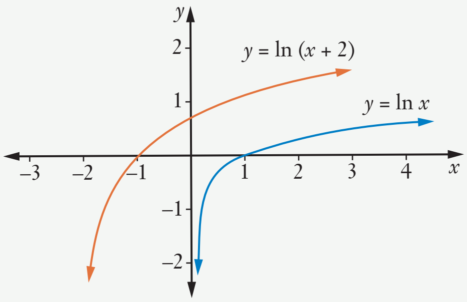
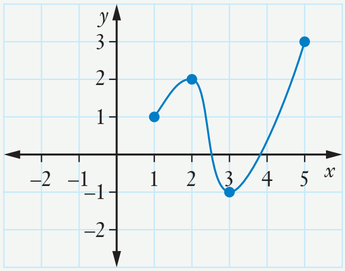


10 **a**



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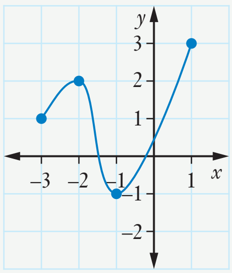


11 **a** $f(x) = x^5 - 5$ **b** $f(x) = (x - 3)^5$
c $f(x) = x^5 + 2$ **d** $f(x) = (x + 7)^5$

12 (7, −2)

Exercise 2.03

1 **a** **i** Vertical dilation scale factor 6 (stretched)
ii Vertical dilation scale factor $\frac{1}{2}$ (compressed)
iii Vertical dilation scale factor −1 (reflection in x-axis)
b **i** Vertical dilation scale factor 2 (stretched)
ii Vertical dilation scale factor $\frac{1}{6}$ (compressed)
iii Vertical dilation scale factor −1 (reflection in x-axis)
c **i** Vertical dilation scale factor 4 (stretched)
ii Vertical dilation scale factor $\frac{1}{7}$ (compressed)
iii Vertical dilation scale factor $\frac{4}{3}$ (stretched)
d **i** Vertical dilation scale factor 9 (stretched)
ii Vertical dilation scale factor $\frac{1}{3}$ (compressed)
iii Vertical dilation scale factor $\frac{3}{8}$ (compressed)
e **i** Vertical dilation scale factor 5 (stretched)
ii Vertical dilation scale factor $\frac{1}{8}$ (compressed)
iii Vertical dilation scale factor −1 (reflection in x-axis)
f **i** Vertical dilation scale factor 9 (stretched)
ii Vertical dilation scale factor −1 (reflection in x-axis)
iii Vertical dilation scale factor $\frac{2}{5}$ (compressed)

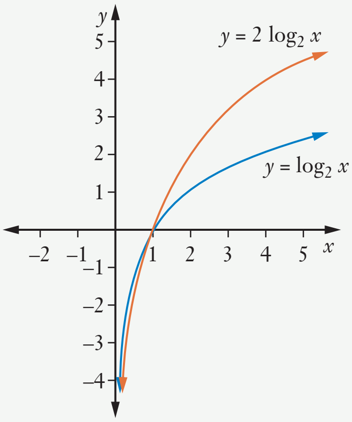
2 **a** $y = 6x^2$; domain $(-\infty, \infty)$; range $[0, \infty)$
b $y = \frac{\ln x}{4}$; domain $(0, \infty)$; range $(-\infty, \infty)$
c $f(x) = -|x|$; domain $(-\infty, \infty)$; range $(-\infty, 0]$
d $f(x) = 4e^x$; domain $(-\infty, \infty)$; range $(0, \infty)$
e $y = \frac{7}{x}$; domain $(-\infty, 0) \cup (0, \infty)$; range $(-\infty, 0) \cup (0, \infty)$

3 **a** $y = 5 \cdot 3^x$ **b** $f(x) = \frac{x^2}{3}$ **c** $y = -x^3$
d $y = \frac{1}{2x}$ **e** $y = \frac{2|x|}{3}$

4 **a** (3, 24) **b** (3, –6) **c** (3, 72) **d** (3, 5)

5 **a** (4, 4) **b** (4, 6) **c** (4, 36) **d** (4, 16)
e (4, –12)

6 **a**



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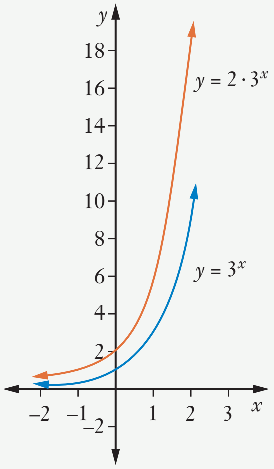

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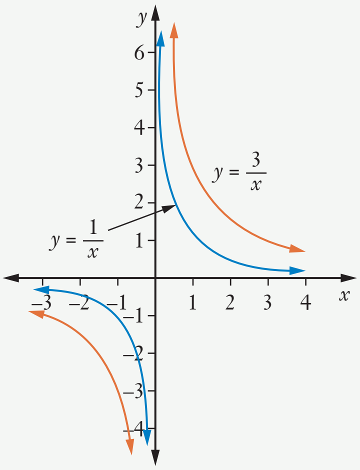
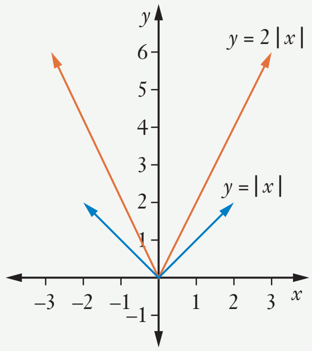
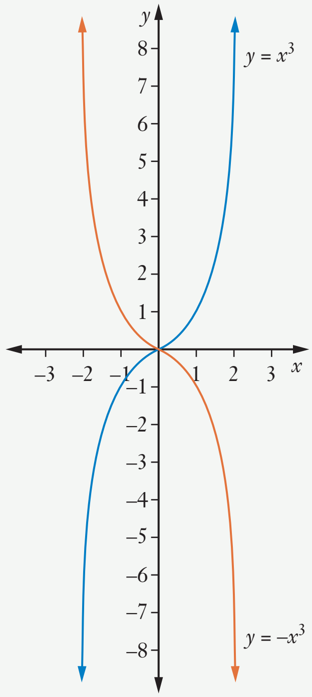


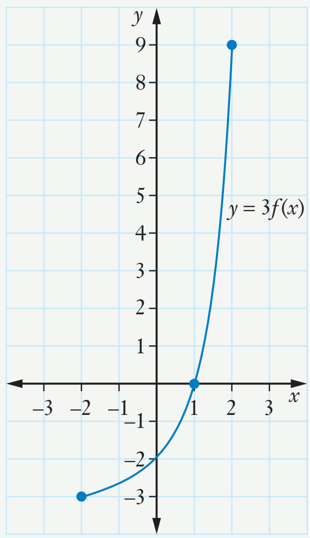
d



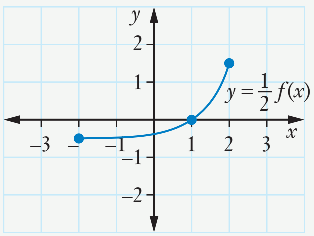
e



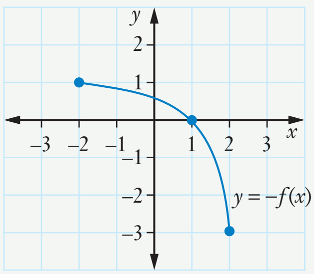
7 **a**



b



c



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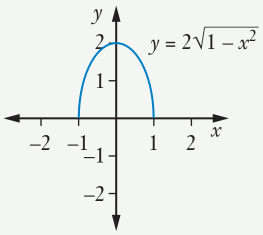


Exercise 2.04

1 **a** Horizontal dilation scale factor $\frac{1}{8}$ (compressed)

b Horizontal dilation scale factor 5 (stretched)

c Horizontal dilation scale factor $\frac{7}{3}$ (stretched)

d Horizontal dilation scale factor –1 (reflection in y-axis). Note that this is the same graph as $f(x) = x^4$.

2 **a** **i** Horizontal dilation scale factor $\frac{1}{2}$ (compressed)

ii Horizontal dilation scale factor $\frac{1}{5}$ (compressed)

iii Horizontal dilation scale factor 3 (stretched)

b **i** Vertical dilation scale factor 4 (stretched)

ii Horizontal dilation scale factor 2 (stretched)

iii Horizontal dilation scale factor –1 (reflection in y-axis)

c **i** Horizontal dilation scale factor $\frac{1}{7}$ (compressed)

ii Vertical dilation scale factor $\frac{1}{8}$ (compressed)

iii Horizontal dilation scale factor $\frac{4}{3}$ (stretched)

d **i** Horizontal dilation scale factor $\frac{1}{5}$ (compressed) (or vertical dilation scale factor 5, stretched)

ii Horizontal dilation scale factor 2 (stretched) (or vertical dilation scale factor $\frac{1}{2}$, compressed)

iii Horizontal dilation scale factor $\frac{5}{3}$ (stretched) (or vertical dilation scale factor $\frac{3}{5}$, compressed)

e **i** Horizontal dilation scale factor $\frac{1}{3}$ (compressed)

ii Vertical dilation scale factor –1 (reflection in x-axis)

iii Horizontal dilation scale factor 2 (stretched)

f **i** Vertical dilation scale factor 8 (stretched)

ii Horizontal dilation scale factor –1 (reflection in y-axis)

iii Horizontal dilation scale factor 7 (stretched)

3 **a** $f(x) = |5x|$; domain $(-\infty, \infty)$; range $[0, \infty)$

b $y = \left(\frac{x}{3}\right)^2$; domain $(-\infty, \infty)$; range $[0, \infty)$

c $y = (-x)^3$; domain $(-\infty, \infty)$; range $(-\infty, \infty)$

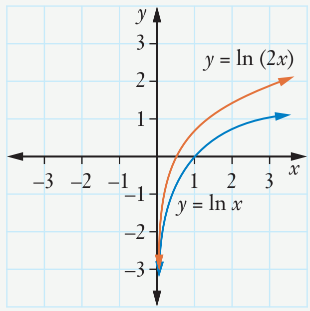
d $y = \frac{e^x}{9}$; domain $(-\infty, \infty)$; range $(0, \infty)$

e $y = -\log_4 x$; domain $(0, \infty)$; range $(-\infty, \infty)$

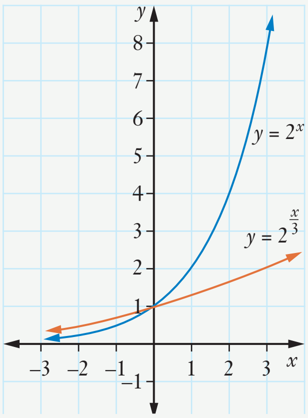
4 **a** (–1, 7) **b** (2, 7) **c** (–6, 7)

5 **a** (–72, 1) **b** (–48, 1) **c** (–6, 1)

6 **a**



b



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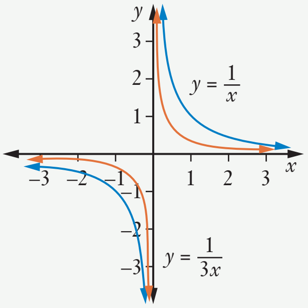

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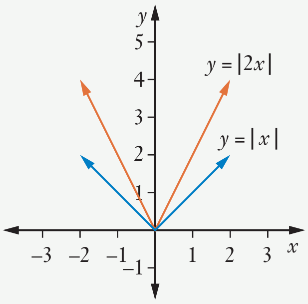

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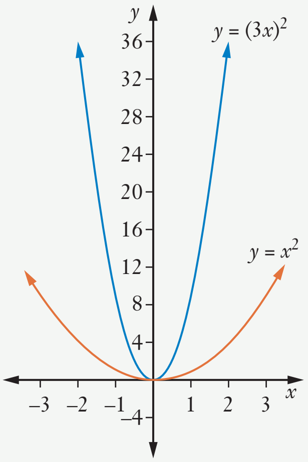

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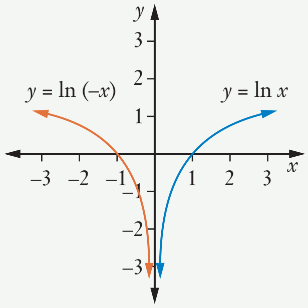

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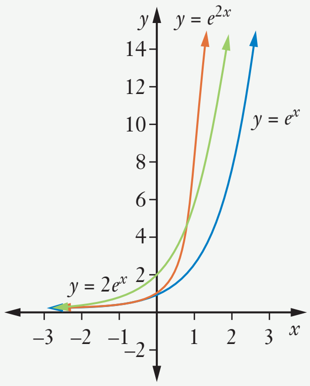
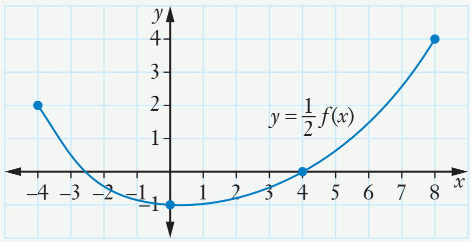


8 A reflection in y-axis transforms $y = f(x)$ into $y = f(-x)$.

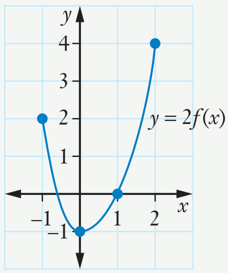
a Since $y = x^2$ is an even function, $f(x) = f(-x)$, so a reflection in the y-axis doesn't change the function.

b Since $y = |x|$ is an even function, $f(x) = f(-x)$ so a reflection in the y-axis doesn't change the function.

9 **a**



b



Exercise 2.05

1 **a** $(5, -11)$ **b** $(-1, -2)$
c $(9, 3)$ **d** $(-2, -17)$

2 **a** $f(x) = -4x^5$ **b** $f(x) = -\frac{1}{243}x^5$

3 **a** $y = (x + 4)^3 - 3$ **b** $f(x) = |x - 1| + 9$
c $f(x) = 3x - 6$ **d** $y = -e^x + 2$
e $y = (2x)^3 - 5$ **f** $f(x) = \frac{6}{x}$
g $f(x) = 3\sqrt{-2x}$ **h** $y = \ln\frac{x}{3} + 2$
i $f(x) = 3\log_2 4x$ **j** $y = \left(\frac{x}{2}\right)^2 - 3$

4 **a** Horizontal translation 1 unit to the right, vertical translation 7 units up
b Vertical dilation scale factor 4, vertical translation 1 unit down
c Vertical dilation scale factor 5, reflection in x-axis, vertical translation 3 units down
d Horizontal translation 7 units to the left, vertical dilation scale factor 2
e Rewrite as $y = 6[2(x - 2)]^3 + 5$. Horizontal dilation scale factor $\frac{1}{2}$, horizontal translation 2 units to the right, vertical dilation scale factor 6, vertical translation 5 units up
f Rewrite as $y = 2[3(x + 3)]^3 - 10$. Horizontal dilation scale factor $\frac{1}{3}$, horizontal translation 3 units to the left, vertical dilation scale factor 2, vertical translation 10 units down

5 **a** Horizontal translation 3 units to the left, vertical dilation scale factor 2, vertical translation 1 unit down
b Horizontal dilation scale factor $\frac{1}{3}$, reflection in x-axis, vertical translation 9 units up
c Horizontal dilation scale factor $\frac{1}{5}$, vertical dilation scale factor 2, vertical translation 3 units down
d Horizontal translation 7 units to the right, vertical dilation scale factor 4, vertical translation 1 unit up
e Reflection in y-axis, horizontal dilation scale factor $\frac{1}{2}$, horizontal translation 1 unit to the left, vertical translation 1 unit down
f Horizontal dilation scale factor $\frac{1}{2}$, reflection in x-axis, vertical translation 8 units up

6 **a** $(9, -31)$ **b** $(4, 5)$
c $(5, -25)$ **d** $(-8, -67)$
e Change to $y = -2f[2(x - 2)] - 3$: $(6, 21)$

7 **a** $(x + 3, y - 6)$ **b** $(-x, y + 6)$
c $(x - 5, 2y)$ **d** $(3x, y + 5)$
e $(5x - 30, -8y - 1)$

8 **a** $y = f(x + 1) - 2$ **b** $y = f(x - 5) + 3$
c $y = -f(x - 4)$ **d** $y = f(-x) + 2$
e $y = -f\left(\frac{x}{4}\right)$ **f** $y = 2f(x) - 2$

9 **a** $f(x) = -\frac{9}{x} + 3$ **b** $y = 5(x + 2)^2 - 6$
c $f(x) = 8\ln\left[\frac{1}{2}(x - 5)\right] - 3$
d $y = 9\sqrt{-(x + 4)} + 4$ **e** $f(x) = -|6x| + 7$
f $y = [4(x + 4)]^3 = 64(x + 4)^3$
g $y = 6(2^{x-2} + 5)$

10 **a** Domain $(-\infty, \infty)$, range $[5, \infty)$
b Domain $(-\infty, \infty)$, range $[-2, \infty)$
c Domain $(-\infty, 2) \cup (2, \infty)$, range $(-\infty, 1) \cup (1, \infty)$
d Domain $(-\infty, \infty)$, range $(2, \infty)$
e Domain $(2, \infty)$, range $(-\infty, \infty)$

11 **a** $y = (x + 1)^2 - 8$
b Horizontal translation 1 unit to the left, vertical translation 8 units down

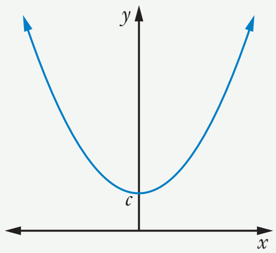
12 Horizontal translation 5 units to the right, vertical translation 28 units down

13 **a** $(2x + 3, 2y + 5)$ **b** $\left(\frac{x}{3} - 6 \quad -y - 2\right)$

14 **a** Circle $(x - 3)^2 + (y - 4)^2 = 9$ or $x^2 - 6x + y^2 - 8x + 16 = 0$
b Translated 2 units to the right, 3 units down

Exercise 2.06

1 **a** **i**



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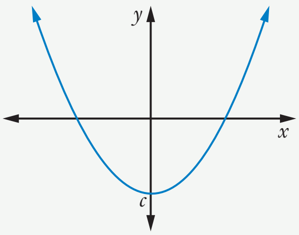
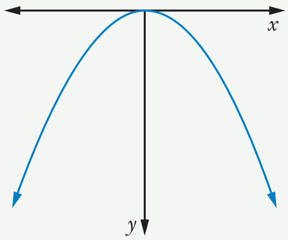
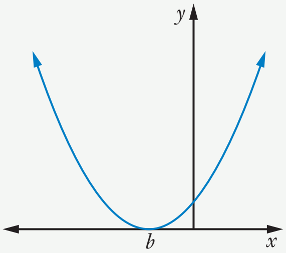


iii



b i



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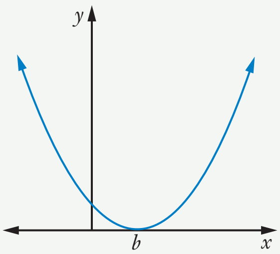
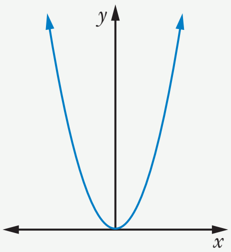


c i



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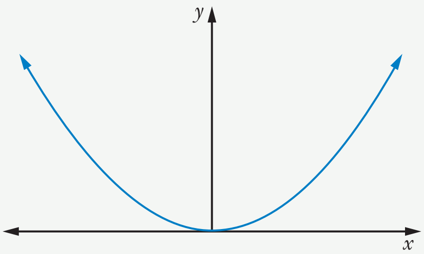
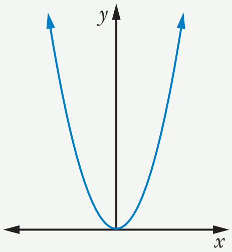
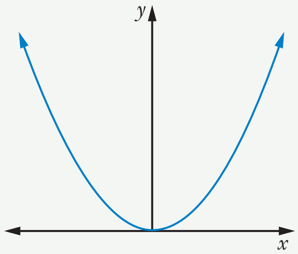


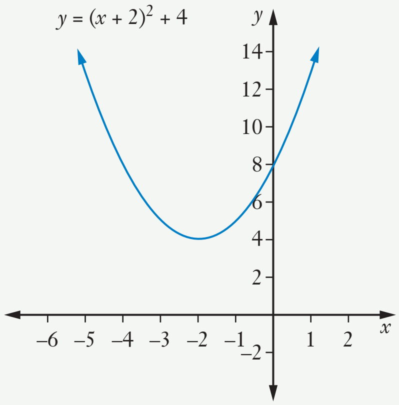
d i, iii



ii



2 a



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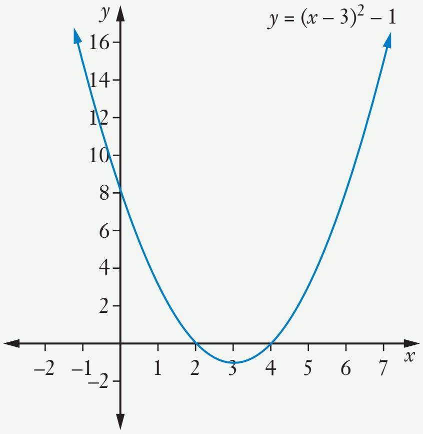

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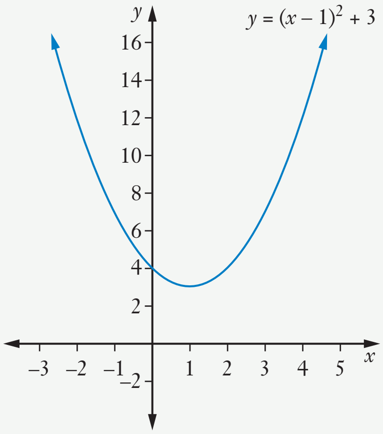
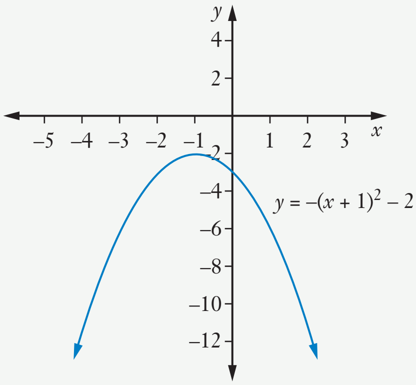
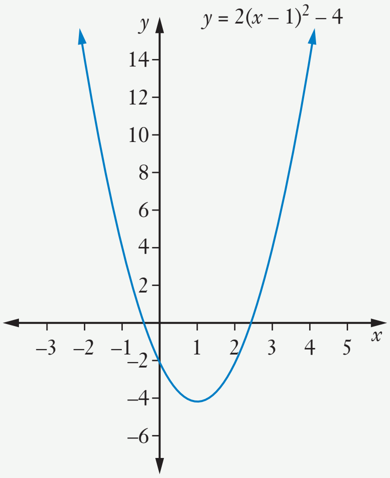


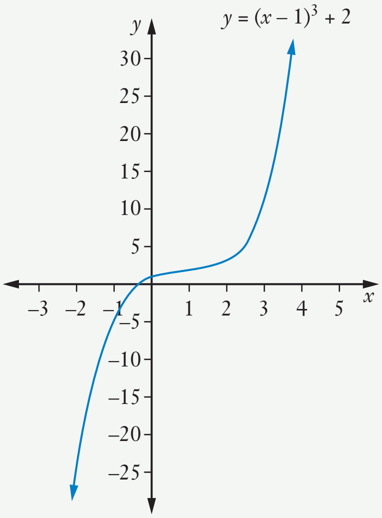
d



e



3 a



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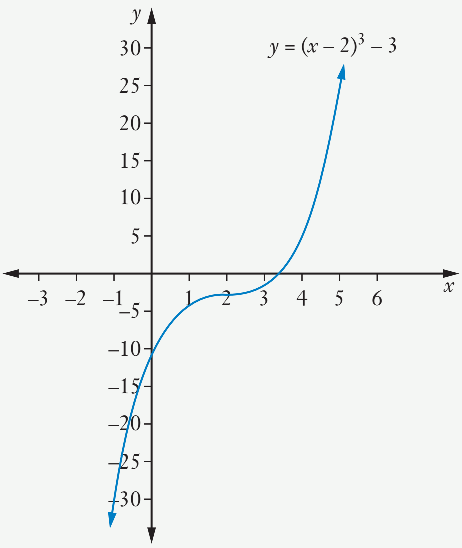
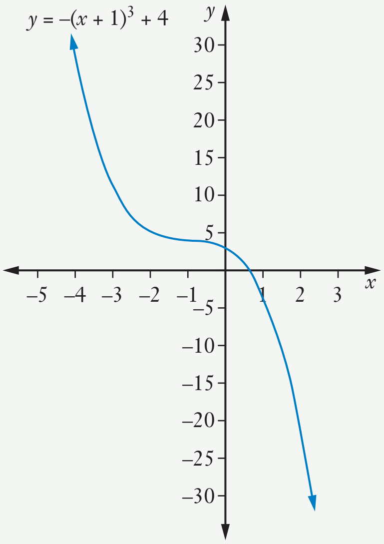
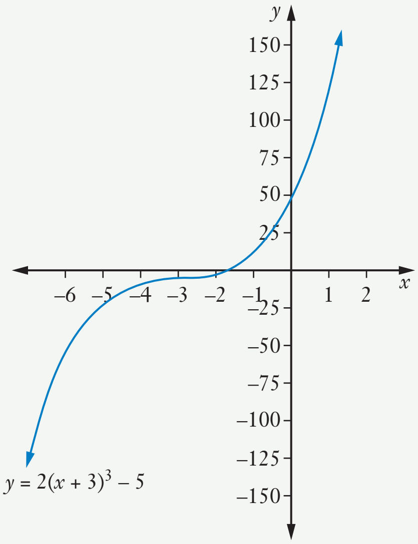


c



d



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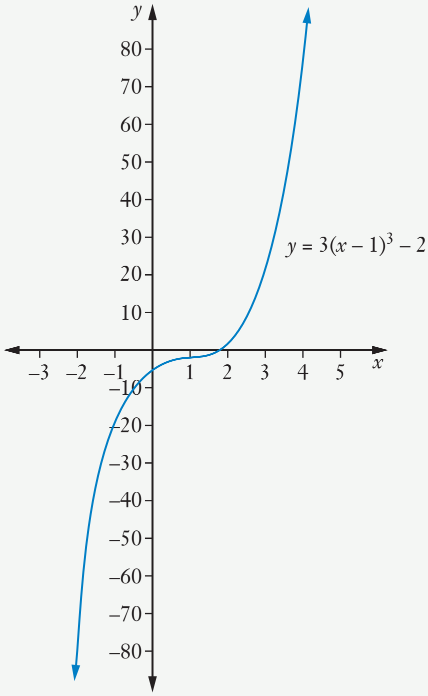
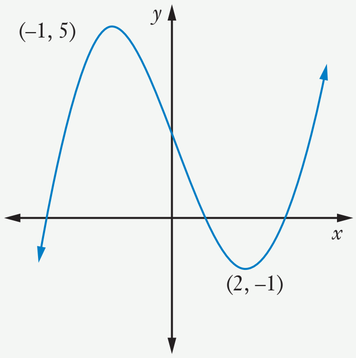
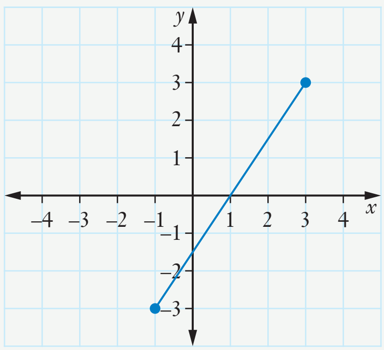


4 a (2, –1) and (–1, 5)

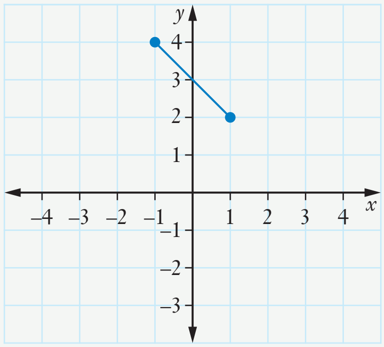
b



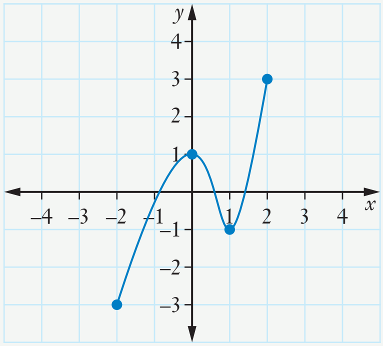
5 a i



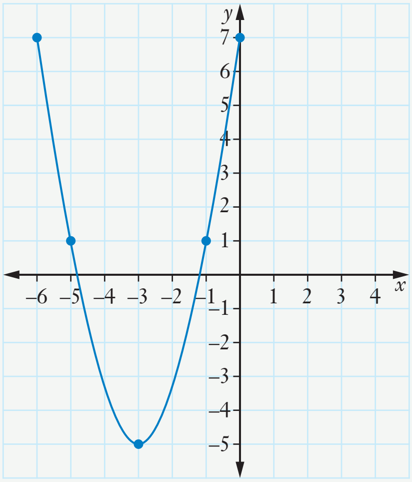
ii



c i



b i



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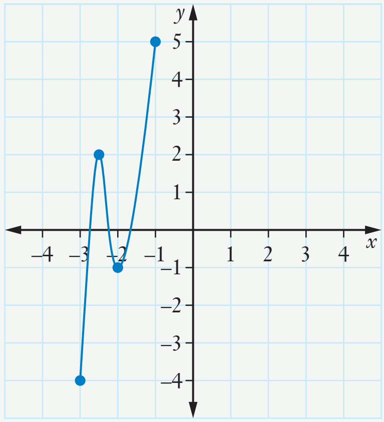

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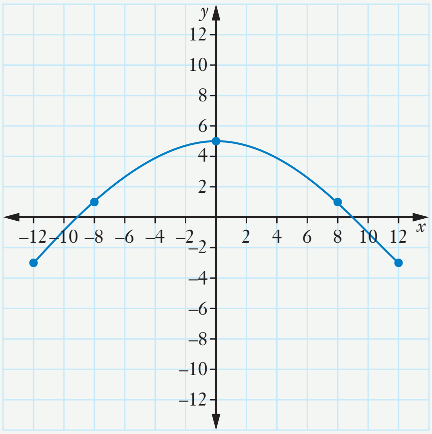

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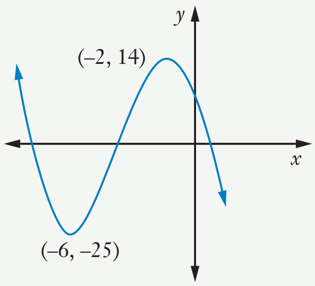

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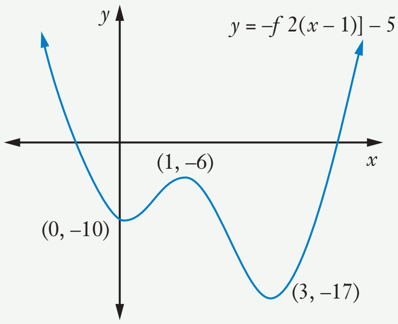
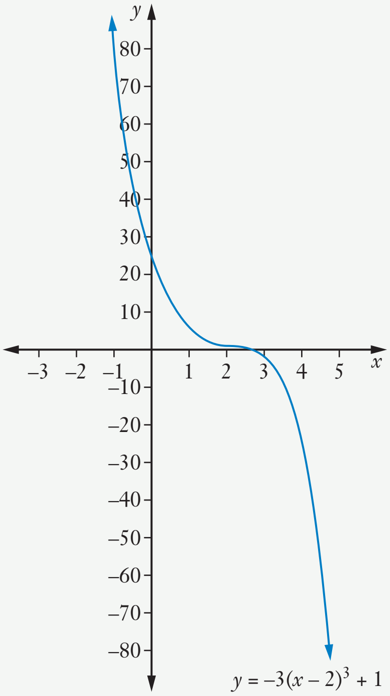
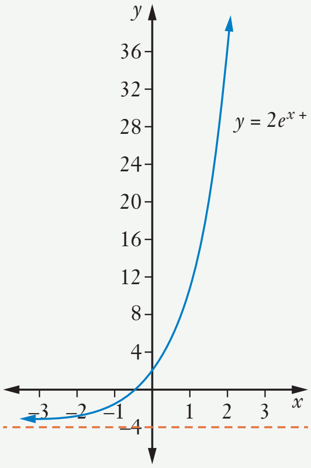


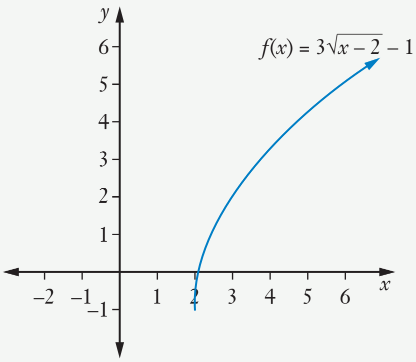
8 a



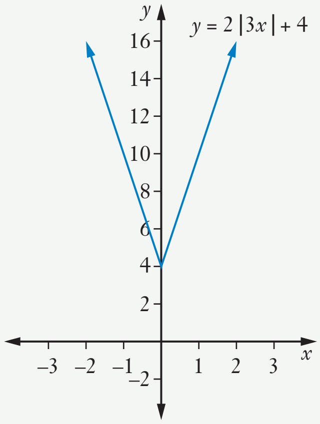
b



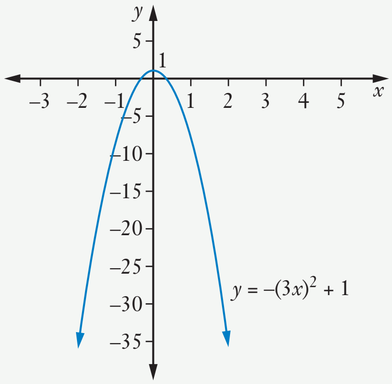
c



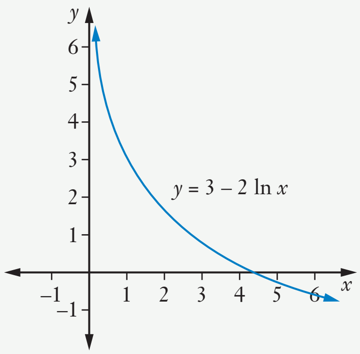
d



e



9 a



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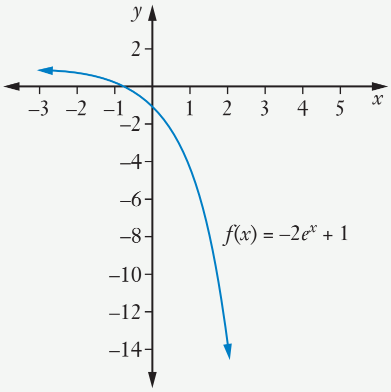

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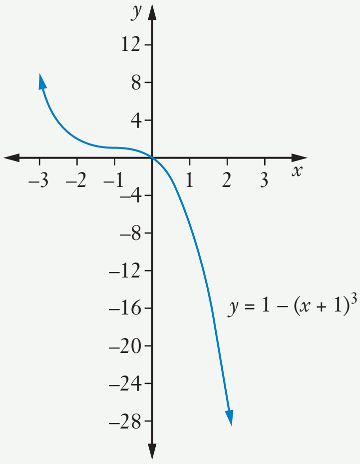
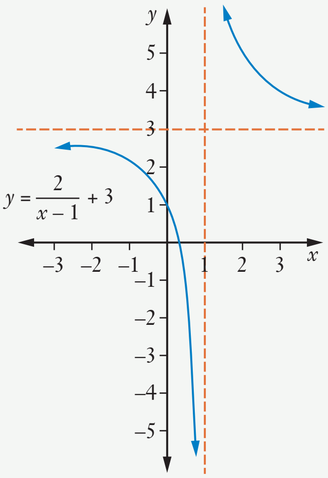
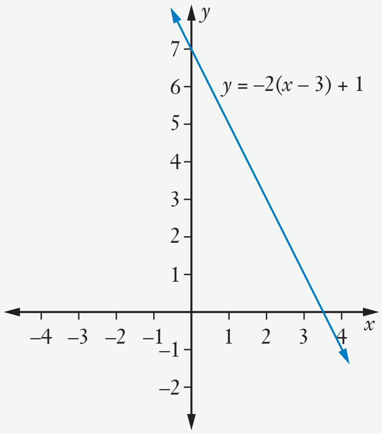


d

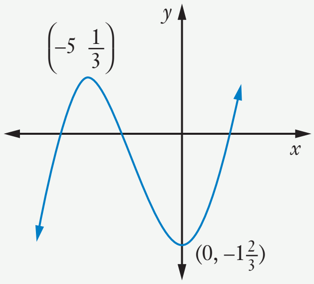


e

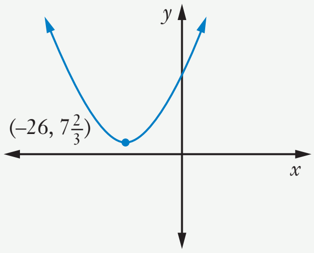


10 **a** $\left(-5, \frac{1}{3}\right)$

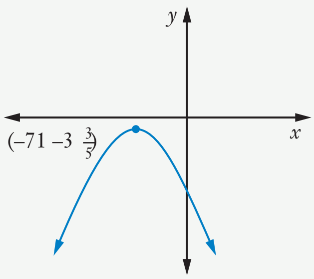
b



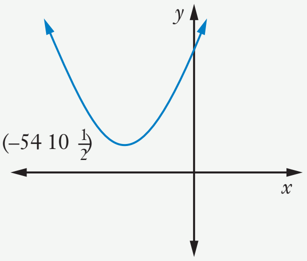
11 **a** $\left(-26, 7\frac{2}{3}\right)$



b (−71, −3.6)



c $(-54, 10\frac{}{2})$



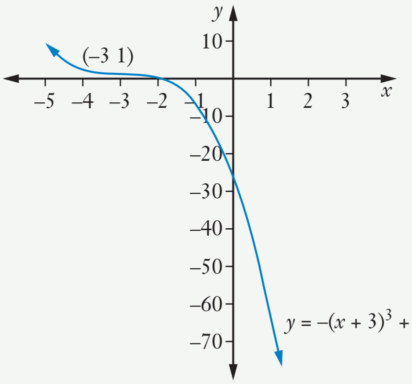
Exercise 2.07

1 **a** 2 **b** 0 **c** 1 **d** 3
e 0 **f** 1 **g** 2 **h** 0
i 1 **j** 0

2 **a** **i** $x = -2, 0$ **ii** $x = 0.6, -2.6$
iii $x = -2.2, 0.2$ **b** $x = 0.2, -2.2$

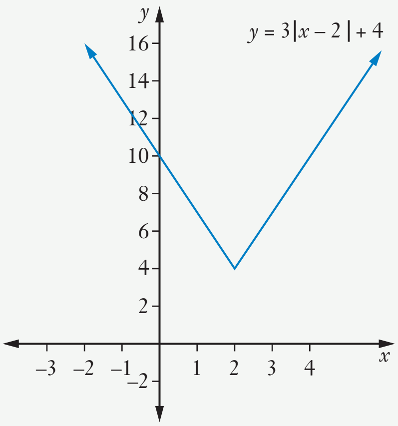
3 **a** $x = 1.4$ **b** $x = 1.9$ **c** $x = 0.2$
d $x > 2.2$ **e** $x \le 3.1$

4 **a**



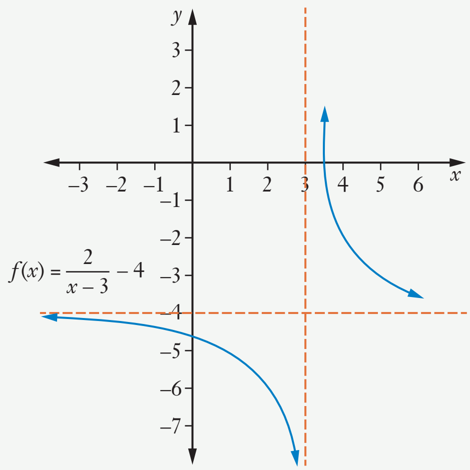
b **i** $x = -2$ **ii** $x = -0.8$ **iii** $x = -0.3$
c $x = -2$

5 **a**



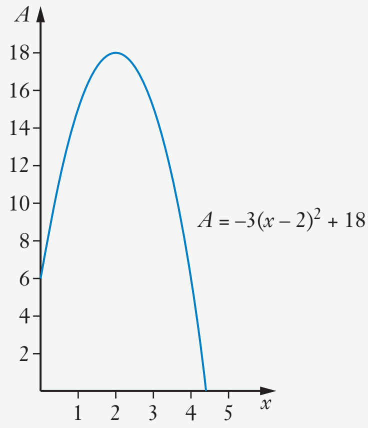
b None **c** $x = 0, 4$

6 **a**



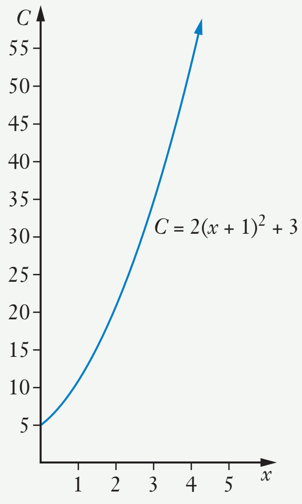
b $x = 1$ **c** $x = 4$

7 **a**



b $x = 0.4, 3.6$

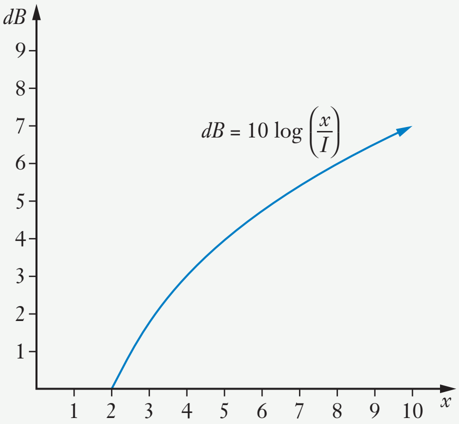
8 **a**



b \$5000

c The costs are \$20 000 when 2 products are made.

9 a



b i $x = 6$ **ii** $x = 3$

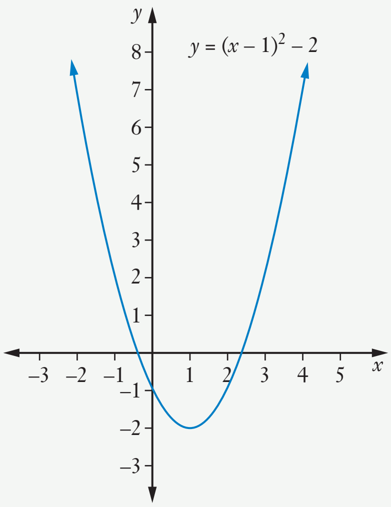
10 a i $t = 3.5$. After 3.5 minutes the temperature is 50°C.

ii $t = 8$. After 8 minutes the temperature is 30°C.

b i $t = 0.74$ **ii** $t = 11.85$

c 24°C (room temperature)

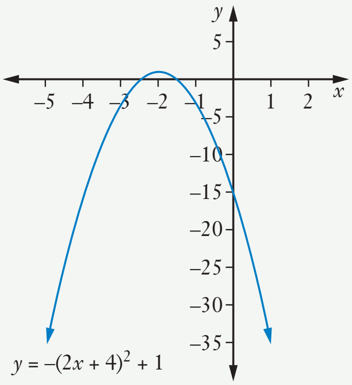
11 a



b i $x = 3, -1$ **ii** $x \leq -1, x \geq 3$

iii $-1 < x < 3$

12 a



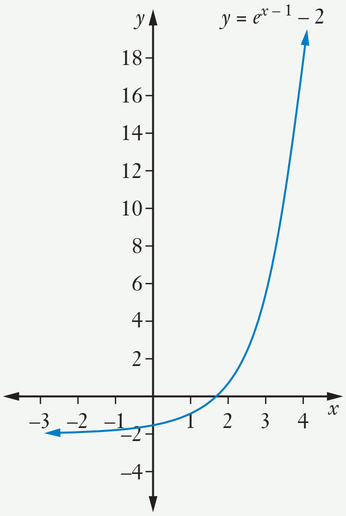
b i $x = -3, -1$ **ii** $-3 < x < -1$

iii $x \leq -3, x \geq -1$

Test yourself 2

1 B **2** D **3** C

4 a



b $x = 3.1$ **c** $x = 4.09$

5 a (6, 107) **b** (6, 40)

c (–24, 177) **d** (17, –75)

e (16, –31)

6 a i $y = x^3 + 3$ **ii** $y = (x + 7)^3$

b i $y = 3|x|$ **ii** $y = \left|\frac{x}{2}\right|$

c $f(x) = 5 \ln(-x)$ **d** $f(x) = -\frac{1}{x - 4}$

e $f(x) = 9 \cdot 3^{3x-2} - 6$

7 **a** c is a vertical translation c units up if $c > 0$ or down if $c < 0$.

b is a horizontal translation b units to the right if $b < 0$ and to the left if $b > 0$.

k is a vertical dilation with scale factor k, stretched if $k > 1$ and compressed if $0 < k < 1$.

a is a horizontal dilation with scale factor $\frac{1}{a}$, compressed if $a > 1$ and stretched if $0 < a < 1$.

b **i** Reflection in the x-axis

ii Reflection in the y-axis

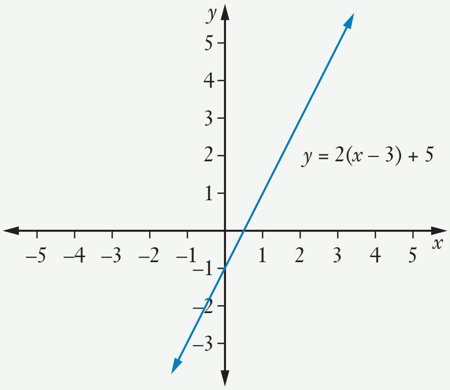
8 $f(x) = -3x^2 + 1$

$f(-x) = -3(-x)^2 + 1$

$= -3x^2 + 1$

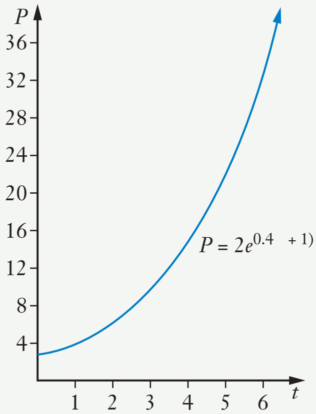
$= f(x)$ so even

9 **a**



b **i** $x \leq 4$ **ii** $x > 5$

10 **a**



b $t = 1.3$. It takes 1.3 years for the population to reach 50 000.

11 $f(x) = (x + 4)^4$

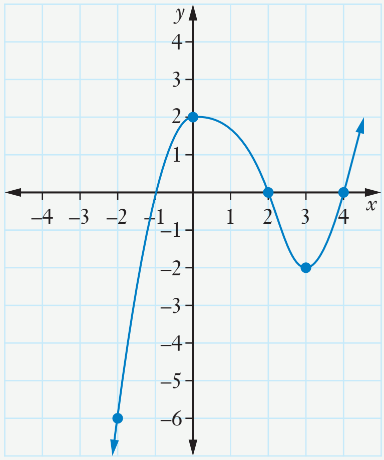
12 $(3, -11)$

13 **a** $x = 2.88, 1.12$ **b** $x < 1.12, x > 2.88$

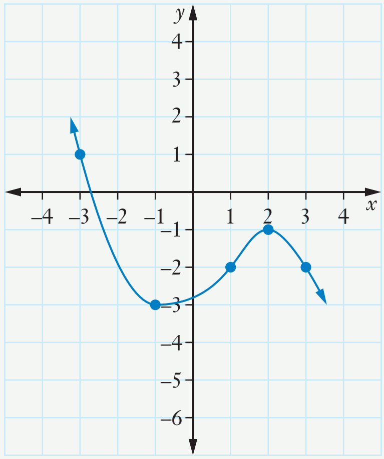
c $1.12 \leq x \leq 2.88$

14 **a** $(x + 3, -7y - 4)$ **b** $x = -6, y = -1$

15 **a**

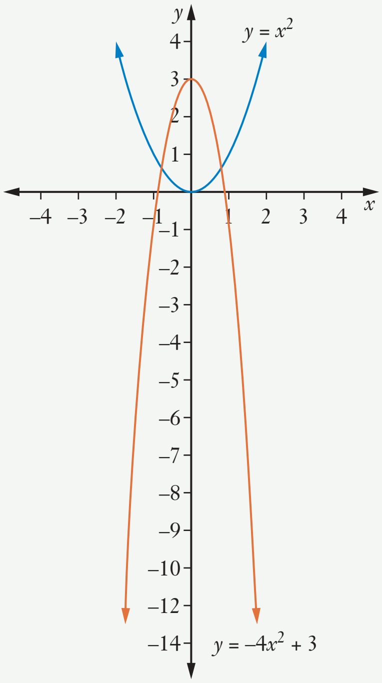


b



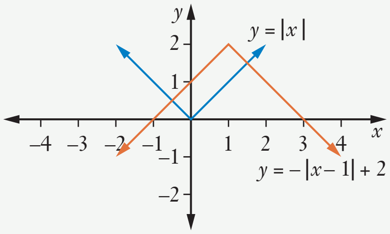
16 **a** $-3 \leq x \leq 1$ **b** $x < -3, x > 1$

17 a



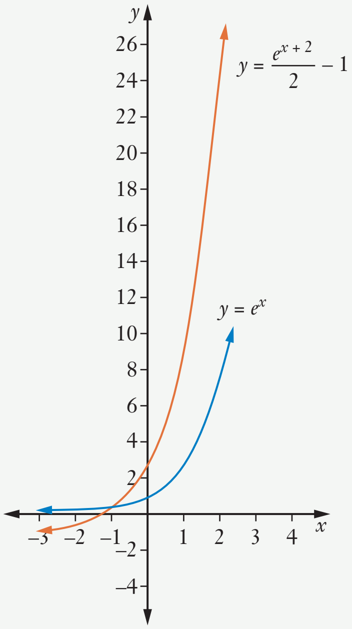
$y = x^2$
$y = -4x^2 + 3$
y
x
4
3
2
1
–1
–2
–3
–4
–5
–6
–7
–8
–9
–10
–12
–14
–4 –3 –2 –1 1 2 3 4

b



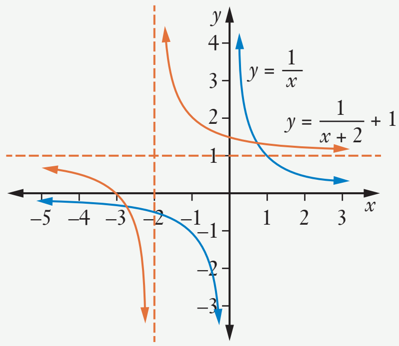
$y = |x|$
$y = -|x - 1| + 2$
y
x
2
1
–1
–2
–4 –3 –2 –1 1 2 3 4

c



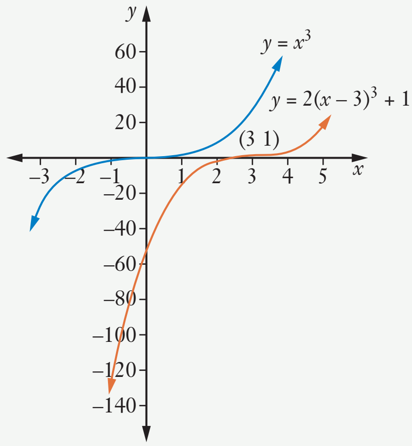
$y = \frac{e^{x+2}}{2} - 1$
$y = e^x$
y
x
26
24
22
20
18
16
14
12
10
8
6
4
2
–2
–4
–3 –2 –1 1 2 3 4

d



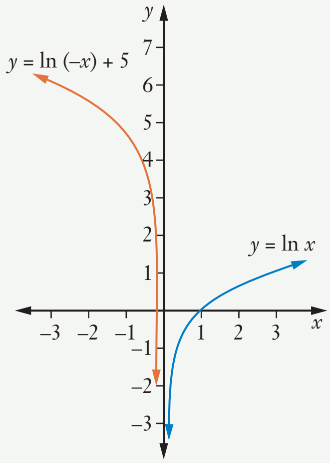
$y = \frac{1}{x}$
$y = \frac{1}{x+2} + 1$
y
x
4
3
2
1
–1
–2
–3
–5 –4 –3 –2 –1 1 2 3

e



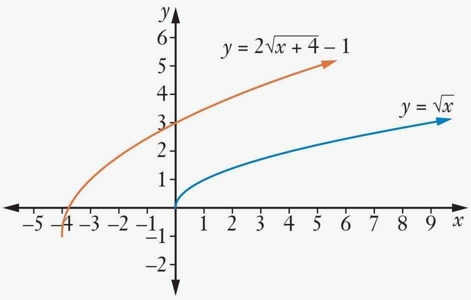
$y = x^3$
$y = 2(x - 3)^3 + 1$
(3 1)
y
x
60
40
20
–20
–40
–60
–80
–100
–120
–140
–3 –2 –1 1 2 3 4 5

f



$y = \ln(-x) + 5$
$y = \ln x$
y
x
7
6
5
4
3
2
1
–1
–2
–3
–3 –2 –1 1 2 3

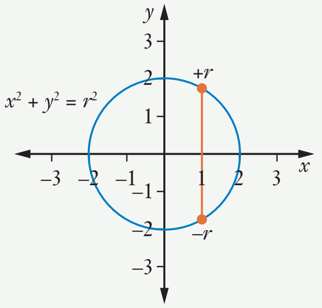
g



$y = 2\sqrt{x + 4} - 1$
$y = \sqrt{x}$
y
x
6
5
4
3
2
1
–1
–2
–5 –4 –3 –2 –1 1 2 3 4 5 6 7 8 9

18 **a** 2 **b** 1 **c** 0 **d** 1
e 1 **f** 0 **g** 1 **h** 0
i 4 **j** 3

19 **a** The circle is not a function since a vertical line cuts the graph in more than one point.

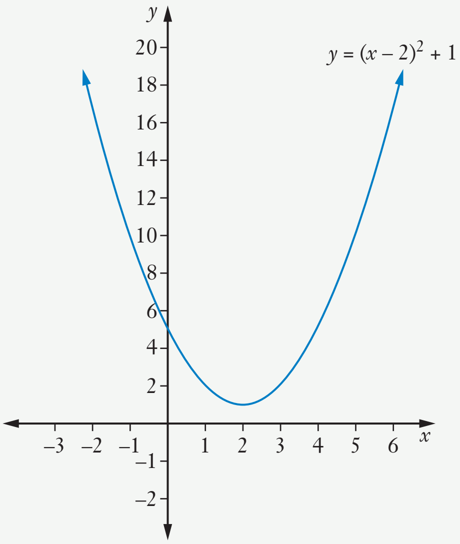


b $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$

c An ellipse (oval)

20 $(32, -1)$

21 **a**



b **i** $x = -1, 5$ **ii** $x < -1, x > 5$
iii $-1 \le x \le 5$

22 **a** $(x - 1, 3y - 5)$ **b** $\left(\frac{1}{2}[x+6], -2y+4\right)$

c $(-x\ 5y - 3)$ **d** $\left(-\frac{1}{3}[x+3], -3y-1\right)$

23 **a** Stretched **b** Compressed
c Stretched **d** Compressed
e Stretched

24 **a** Domain $(-\infty, \infty)$; range $[-10, \infty)$
b Domain $(-\infty, \infty)$; range $(-\infty, 2]$
c Domain $(-\infty, 3) \cup (3, \infty)$; range $(-\infty, -5) \cup (-5, \infty)$

Challenge exercise 2

1 **a** $h = -2t^2 + 4t + 1$ **b** 2.2 seconds

c $h = -2(t - 1)^2 + 3$. Horizontal translation 1 unit to the right, reflection in the x-axis, vertical dilation scale factor 2, vertical translation 3 units up.

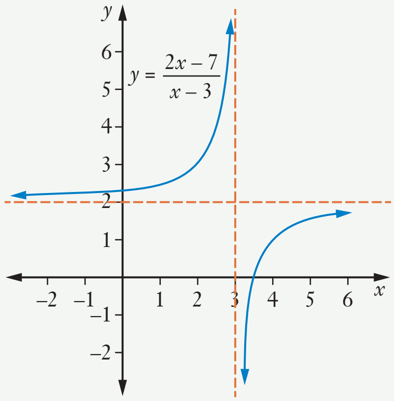
2 **a** **i** $(1, -8)$ **ii** $(2, 0)$ **iii** $(3, -2)$

b $x - 2y - 17 = 0$

c Horizontal dilation with scale factor 2 and horizontal translation 17 units to the right,

OR: vertical dilation with scale factor $\frac{1}{2}$ and vertical translation $\frac{17}{2}$ units down

3 **b**



c **i** $x < 3, x \ge 3.5$ **ii** $x > 3$

4 **a** A horizontal dilation with scale factor $\frac{1}{a} = 2$:

$a = \frac{1}{2}$

$y = \frac{1}{(ax)} = \frac{1}{\frac{1}{2}x} = \frac{2}{x}$

A vertical dilation with scale factor $k = 2$:

$y = \left(\frac{1}{x}\right) = 2\left(\frac{1}{x}\right) = \frac{2}{x}$

So these transformations have the same effect on $y = \frac{1}{x}$.

b No. Horizontal dilation gives $y = \frac{4}{x}$, vertical dilation gives $y = \frac{2}{x}$.

5 **a** $x = -\frac{b}{2a}$ **b** Horizontal translation

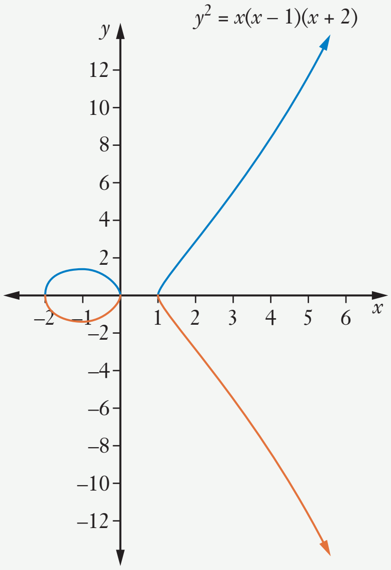
c **i** $x = -1$ **ii** $x = 3$

iii $x = -b$ **v** $x = -\frac{b}{a}$

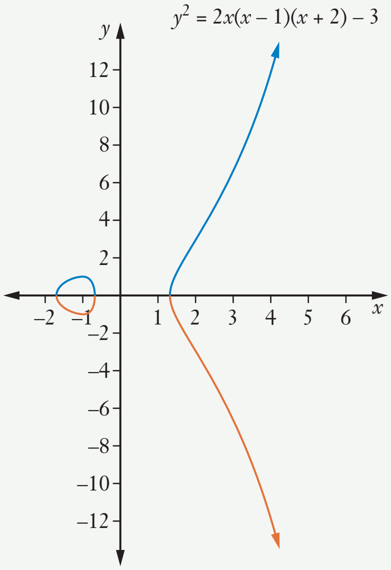
6 **a** $y = -3\sin\frac{x}{2} - 1$

b Amplitude 3, period 4π, centre -1

7 **a**



b



8 $x^2 - 6x + y^2 + 8 = 0$

9 Reflection in y-axis, horizontal dilation scale factor $\frac{1}{3}$, horizontal translation 2 units to the left, vertical dilation scale factor 3, vertical translation 5 units down

10 $y = -x^3 + 3x$

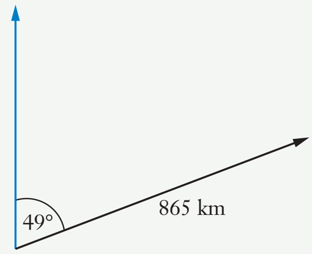
Chapter 3

Exercise 3.01

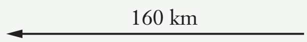
1 **a** Scalar **b** Vector **c** Vector

d Vector **e** Scalar

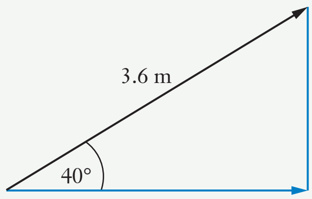
2 **a**



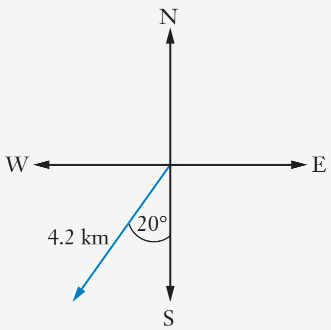
b



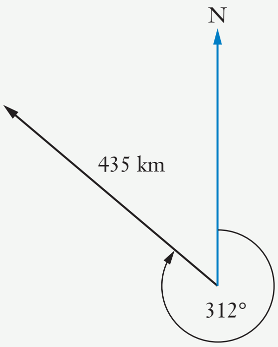
c



d



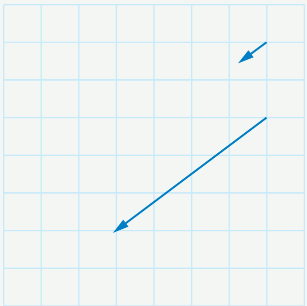
e



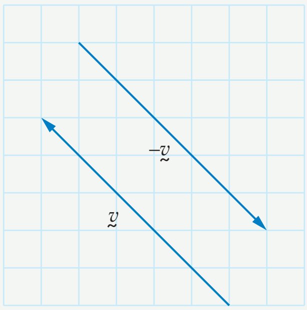
3 **a** Negative vectors **b** Parallel vectors
c Equal vectors **d** Parallel vectors
e Parallel vectors **f** Negative vectors

4 **a** **i** N 45° E **ii** 045°
b **i** N 70° W **ii** 290°
c **i** S 60° W **ii** 240°
d **i** S **ii** 180°
e **i** S 65° E **ii** 115°

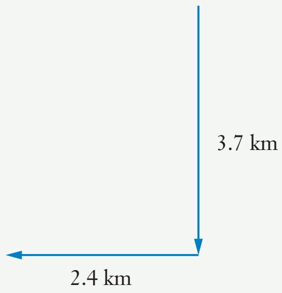
5 Solution is not unique. The vector will have the same direction but different magnitude.



6

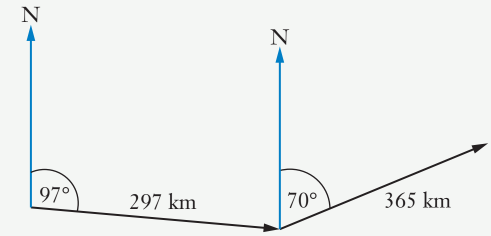


7 **a**



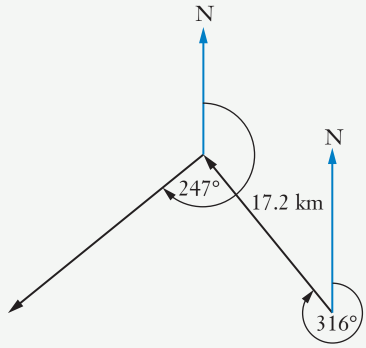
b 4.4 km **c** 213°

8 **a**



b 644 km

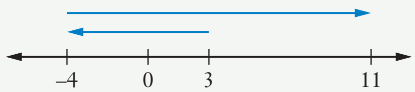
9 **a**



b 41.1 km
c **i** 270° **ii** 090°

10 **a** 123.5 cm **b** 098°

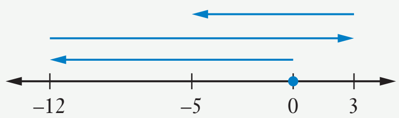
11 **a**



b 11 cm **c** 22 cm

12 26 m **13** **a** –2 cm **b** 15 cm

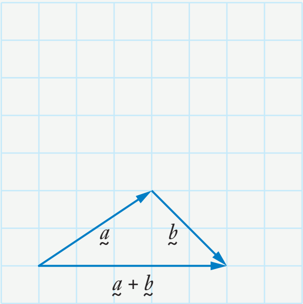
14 **a**



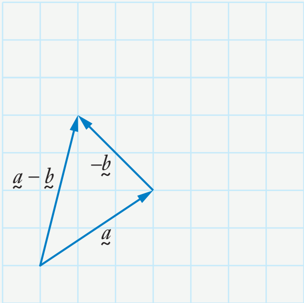
b –5 mm **c** 35 mm

Exercise 3.02

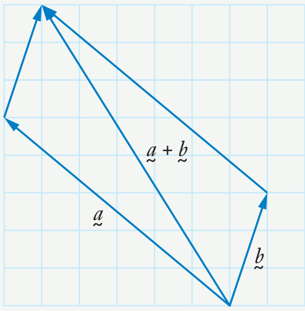
1 a i



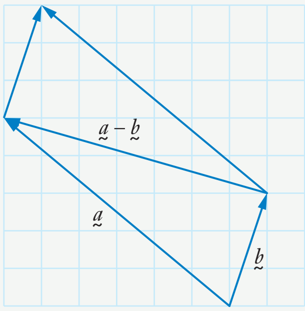
ii



b i



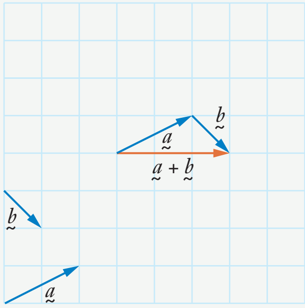
ii



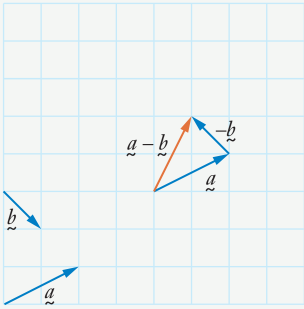
2 a $\overrightarrow{AC}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{DA}$ **d** $\overrightarrow{CA}$
e $\overrightarrow{CA}$ **f** $\overrightarrow{DB}$ **g** $\overrightarrow{AC}$

3 a $c = a + b$ **b** $c = b - a$ **c** $c = a - b$
d $c = a + b$ **e** $c = b - a$

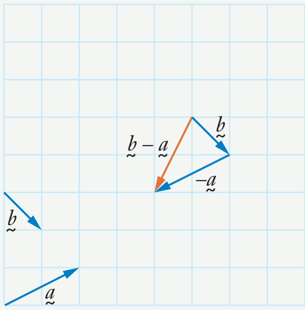
4 a



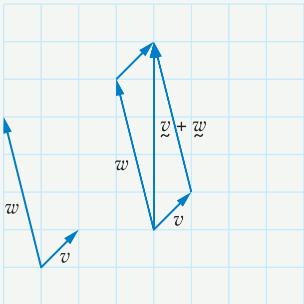
b



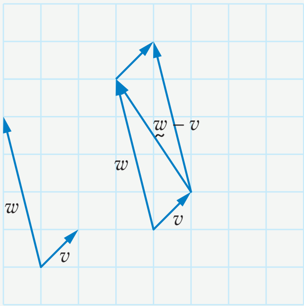
c



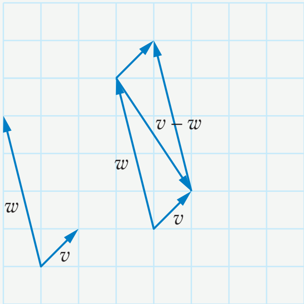
5 a



b



c



6 **a** $\overrightarrow{AG}$ **b** $\overrightarrow{GF}$ **c** $\overrightarrow{BC}$

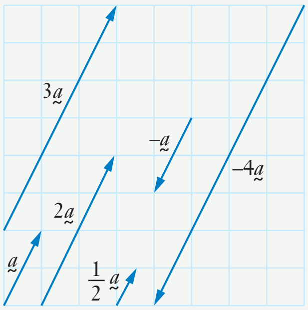
d $\overrightarrow{GB}$ **e** $\overrightarrow{GA}$

7 **a** $\overrightarrow{AO}$ **b** $\overrightarrow{DC}$ **c** $\overrightarrow{OE}$

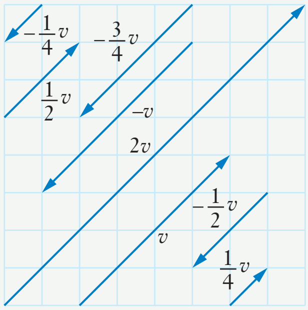
d $\overrightarrow{AB}$ **e** $\overrightarrow{CE}$

Exercise 3.03

1



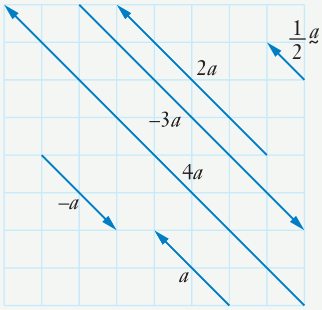
2



3 **a** $\frac{1}{10}$ **b** $\frac{1}{3}$ **c** 4

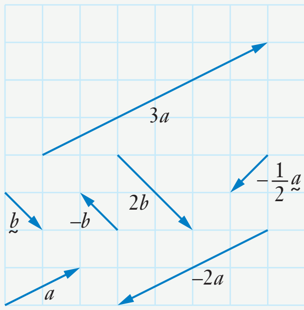
d 2 **e** $\frac{1}{7}$

4



5 $\mu = 0.4$

6



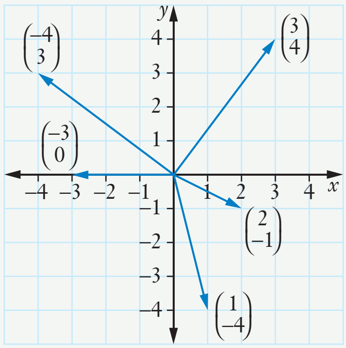
7 **a** $\lambda = 2.5$ **b** $\lambda = 0.4$

8 **a** $\overrightarrow{AD}, \overrightarrow{BC}$ **b** $\overrightarrow{CF}, \overrightarrow{FB}, \overrightarrow{DE}, \overrightarrow{EA}$

c $\overrightarrow{FD}, \overrightarrow{BE}$

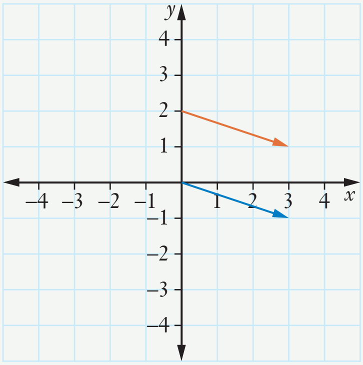
Exercise 3.04

1

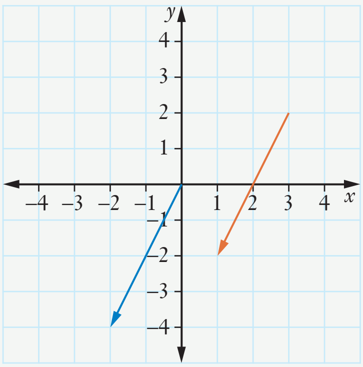


2 Answers are not unique.

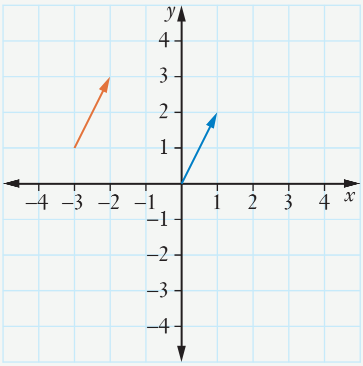
a



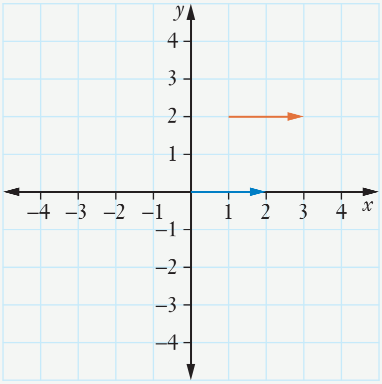
b



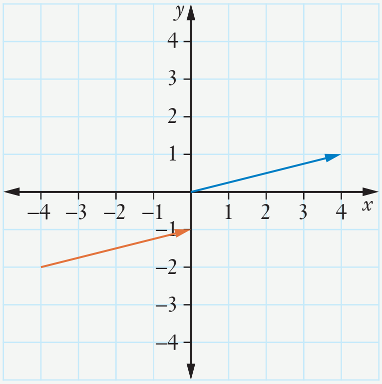
c



d



e



3 **a** $\begin{pmatrix}3\\2\end{pmatrix}$ **b** $\begin{pmatrix}4\\1\end{pmatrix}$ **c** $\begin{pmatrix}-2\\4\end{pmatrix}$

d $\begin{pmatrix}5\\-5\end{pmatrix}$ **e** $\begin{pmatrix}-2\\-1\end{pmatrix}$

4 **a** $\begin{pmatrix}-1\\5\end{pmatrix}$ **b** $\begin{pmatrix}5\\-3\end{pmatrix}$ **c** $\begin{pmatrix}-5\\3\end{pmatrix}$

5 **a** $\begin{pmatrix}-2\\8\end{pmatrix}$ **b** $\begin{pmatrix}-6\\-2\end{pmatrix}$

6 **a** $\begin{pmatrix}4\\3\end{pmatrix}$ **b** $\begin{pmatrix}-6\\-6\end{pmatrix}$ **c** $\begin{pmatrix}-9\\-4\end{pmatrix}$

d $\begin{pmatrix}-8\\-3\end{pmatrix}$ **e** $\begin{pmatrix}3\\-11\end{pmatrix}$ **f** $\begin{pmatrix}15\\20\end{pmatrix}$

g $\begin{pmatrix}-18\\6\end{pmatrix}$ **h** $\begin{pmatrix}5\\-7\end{pmatrix}$ **i** $\begin{pmatrix}2\\-9\end{pmatrix}$

j $\begin{pmatrix}16\\-31\end{pmatrix}$

7 $a = 3, b = 2$

8 $a = -8, b = -6$

9 $x = 4, y = -13$

10 $p = 5, q = 0$

Exercise 3.05

1 **a** 12 cm, 108° **b** 35 cm, 243°

2 **a** 3.2 **b** 8.1 **c** 2.2 **d** 6.4 **e** 7.2

3 **a** $\sqrt{13}$ **b** $\sqrt{74}$ **c** $\sqrt{130}$ **d** $\sqrt{85}$ **e** $\sqrt{17}$

4 **a** 76° **b** 27° **c** 248° **d** 108° **e** 342° **f** 127° **g** 61° **h** 257° **i** 352° **j** 256°

5 **a** 3.6 **b** 11.4 **c** 5.8 **d** 8.1 **e** 4.2

6 **a** 26° 34′ **b** 11° 19′ **c** 108° 26′ **d** 23° 58′ **e** 153° 26′

7 **a** 10.3, 61° **b** 3.2, 252° **c** 4.1, 194° **d** 3.6, 326° **e** 8.6, 144°

8 **a** 7.6, 23° **b** 1.4, 225° **c** 4.5, 63° **d** 5.1, 101° **e** 12.0, 85°

9 **a** 4.1, 194° **b** 4.1, 14° **c** 6.7, 27°

10 **a** 10.8, 338° **b** 6.3, 342° **c** 10.8, 158°

11 **a** 7.1, 352° **b** 3.2, 252° **c** 7.1, 172°

12 **a** 6.1, 99° **b** 6.4, 219° **c** 6.4, 39°

13 **a** 7.1, 315° **b** 7.1, 172°

14 $x = -3\sqrt{3}, y = 3$

15 $a = 2, b = -2\sqrt{3}$

Exercise 3.06

1 **a** $-3i - 2j$ **b** $4i - 3j$ **c** $i + 5j$ **d** $5i + 2j$ **e** $-6i + j$

2 **a** $4i + 3j$ **b** $-2i - 7j$ **c** $2i + 7j$ **d** $4i - 8j$ **e** $-6i - 10j$ **f** $10i + 13j$ **g** $2i - 15j$ **h** $27i + 23j$ **i** $-5i - j$ **j** $-22i - 11j$

3 **a** 6.4, 309° **b** 5, 127° **c** 9.2, 347° **d** 7.8, 220° **e** 2.2, 153°

4 **a** $3i - 12j$ **b** 12.4, 284°

5 **a** $12i - 6j$ **b** $3i - 7j$ **c** $i + 5j$ **d** $-7i + 9j$ **e** $22i$ **f** $7i - 31j$ **g** $42i - 32j$ **h** $-7i - 13j$

6 **a** 8.2, 14° **b** 5.7, 45°

7 3.6, 326°

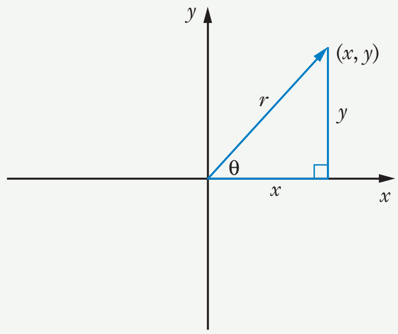
8 16.3, 281°

9 $a = \pm 4$

10 $b = 6$

11 $a = -2, b = 2\sqrt{3}$

12



$r^2 = x^2 + y^2$ (Pythagoras)

$r = \sqrt{x^2 + y^2}$

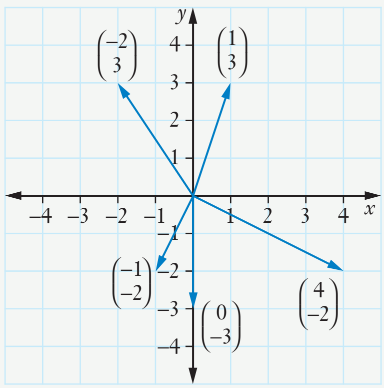
Test yourself 3

1 A, B **2** D **3** B **4** C

5 **a** Vector **b** Scalar **c** Scalar **d** Vector **e** Vector

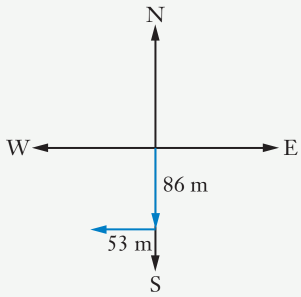
6 **a** 3.2, 252° **b** 11.0, 175°

7



8 $p = 7, q = -7$

9 **a**



b 101 m, 212°

10 $\frac{1}{12}$

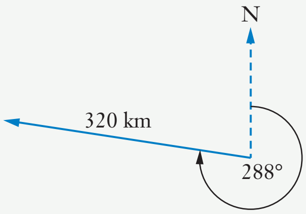
11 **a** 4.1, 14° **b** 3.2, 288°
c 4.5, 243° **d** 5, 143°

12 $\begin{pmatrix}-4\\1\end{pmatrix}$

13 **a** 13.2, 351° **b** 13, 67°

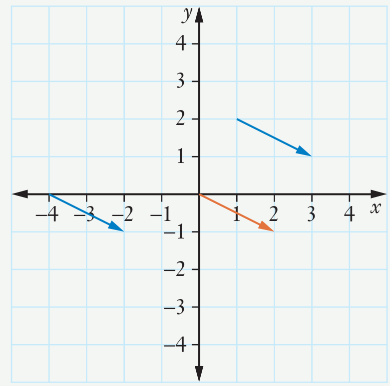
14 **a** $\begin{pmatrix}-4\\-5\end{pmatrix}$ **b** $\begin{pmatrix}9\\-11\end{pmatrix}$ **c** $\begin{pmatrix}-15\\25\end{pmatrix}$
d $\begin{pmatrix}-9\\9\end{pmatrix}$ **e** $\begin{pmatrix}-2\\20\end{pmatrix}$

15 **a**

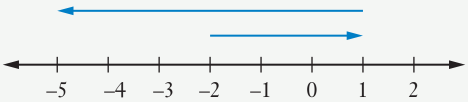


b

16



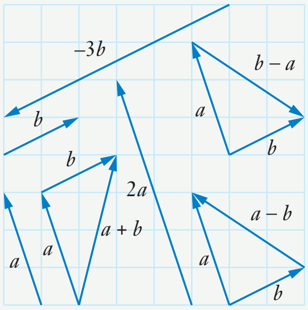
17 **a**



b −5 cm (5 cm to the left of 0)
c 9 cm

18 13, 247°

19



20 5.4, 338°

21 **a** $\begin{pmatrix}-2\\11\end{pmatrix}$ **b** $\begin{pmatrix}-4\\-1\end{pmatrix}$

22 5.1, 281°

23 **a** $4i - 4j$ **b** $-6i + 10j$ **c** $4i - 12j$
d $13i - 15j$ **e** $16i - 32j$

24 **a** $\overrightarrow{AB} = 8i - 10j$ **b** 12.8, 309°

25 **a** $v = -4i + 3j$ **b** $v = 6i - j$
c $v = -8i + 7j$

Challenge exercise 3

1 $a = -1.3$, $b = 4.8$

2 6.4, 231°

3 **a** $x = y = -3$

b $v = -3i - 3j$

4 Horizontal –2.8 N, vertical 5.3 N

5 **a** (5, 323°) **b** (4, 240°)

c (6.3, 108°)

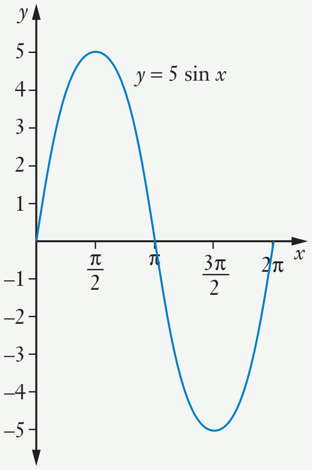
Chapter 4

Exercise 4.01

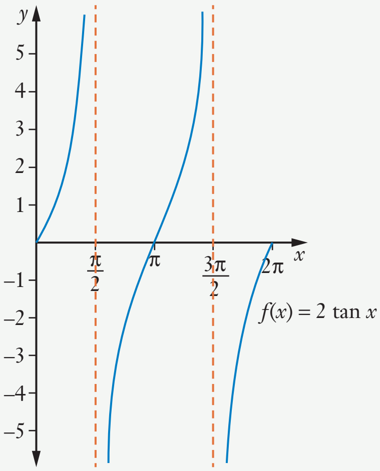
1 **a** Phase shift **b** Amplitude

c Period **d** Centre

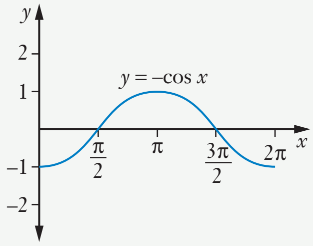
2 **a**



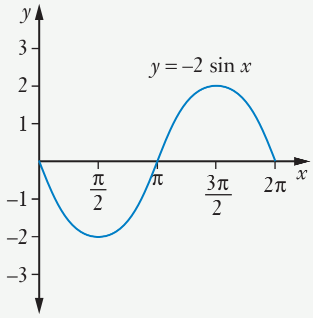
b



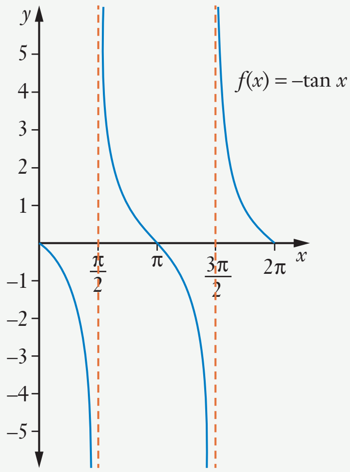
c



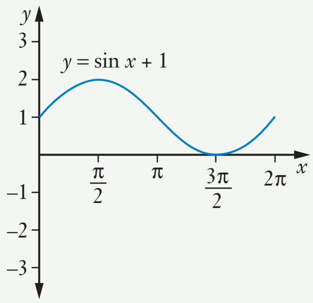
d



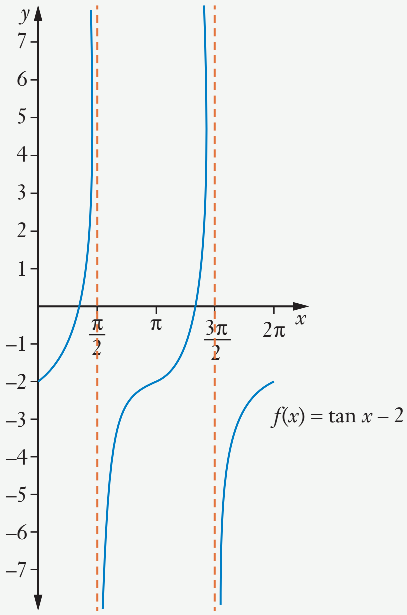
e



3 **a**

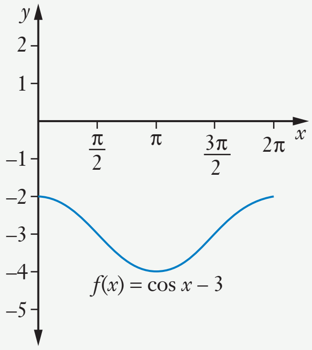


b



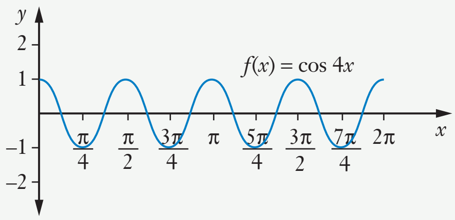
$f(x) = \tan x - 2$

c



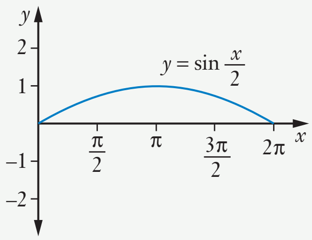
$f(x) = \cos x - 3$

4 a



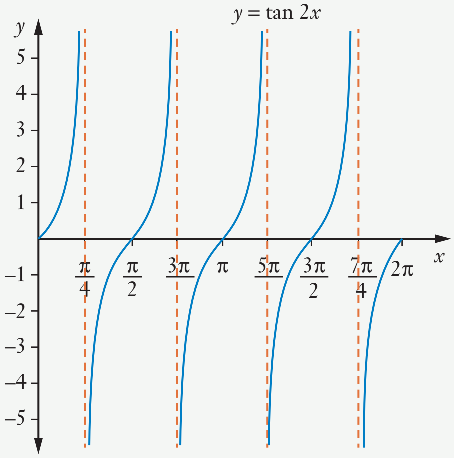
$f(x) = \cos 4x$

b



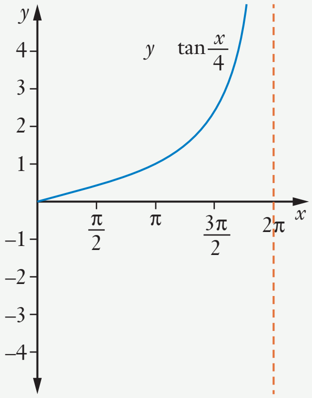
$y = \sin \frac{x}{2}$

c



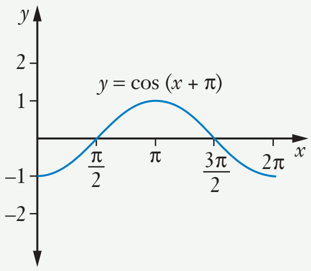
$y = \tan 2x$

d



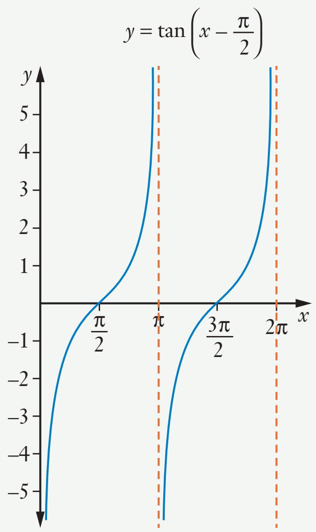
$y \quad \tan \frac{x}{4}$

5 a

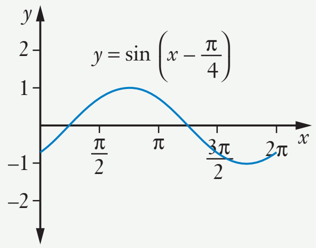


$y = \cos (x + \pi)$

b



c



6 **a** $y = 9 \sin x$ **b** $y = -\sin x$

c $y = \sin x - 4$ **d** $y = \sin 2x$

e $y = \sin(x - \pi)$

7 **a** $y = 4 \cos x$ **b** $y = \cos\left(x + \frac{\pi}{3}\right)$

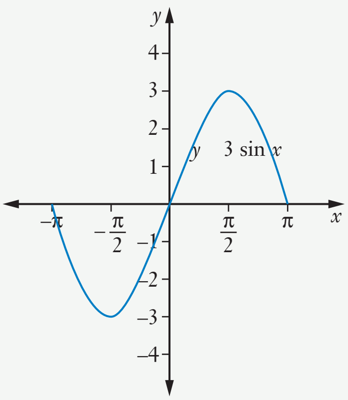
c $y = \cos x + 8$ **d** $y = \cos 4x$

e $y = 7 \cos x$

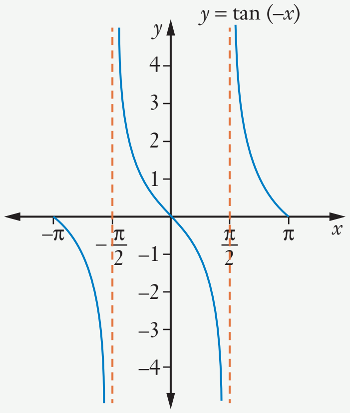
8 **a** $y = \tan \frac{x}{2}$ **b** $y = \tan\left(x - \frac{\pi}{6}\right)$

c $y = \tan(-x)$

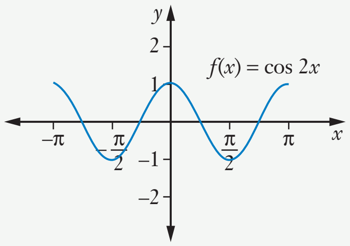
9 **a**



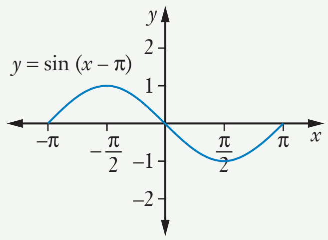
b



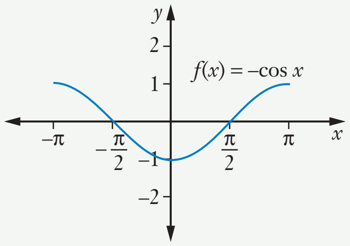
c



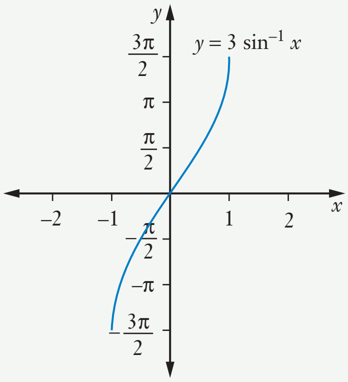
d



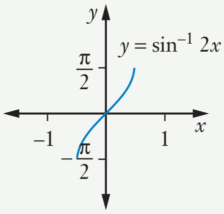
e



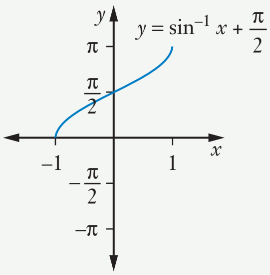
10 **a** **i** $y = 3\sin^{-1} x$



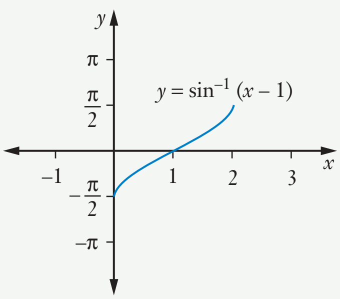
ii $y = \sin^{-1} 2x$



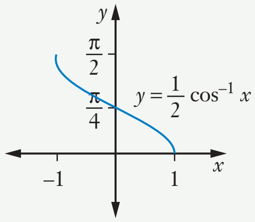
iii $y = \sin^{-1} x + \frac{\pi}{2}$



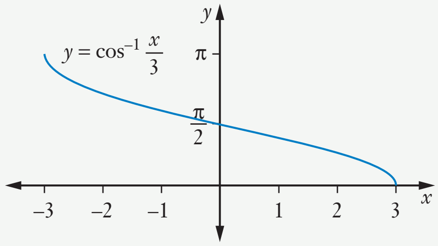
v $y = \sin^{-1}(x - 1)$



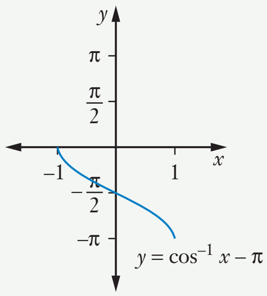
b **i** $y = \frac{1}{2}\cos^{-1} x$



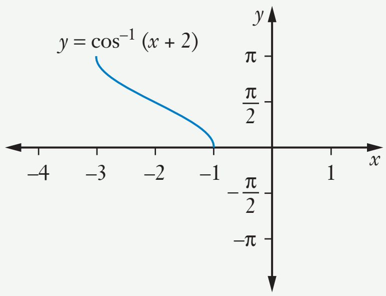
ii $y = \cos^{-1}\frac{x}{3}$



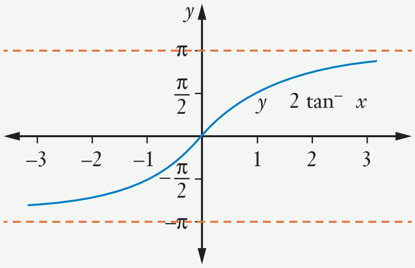
iii $y = \cos^{-1} x - \pi$



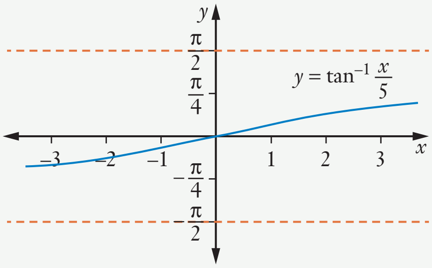
v $y = \cos^{-1}(x + 2)$



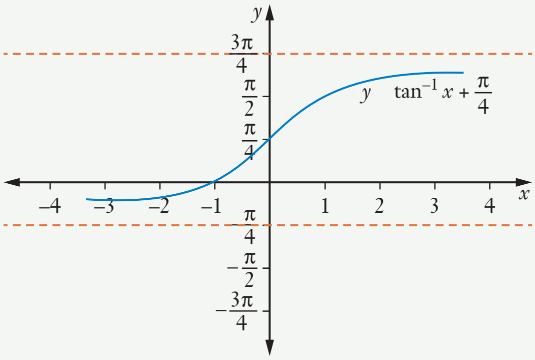
c **i** $y = 2\tan^{-1} x$



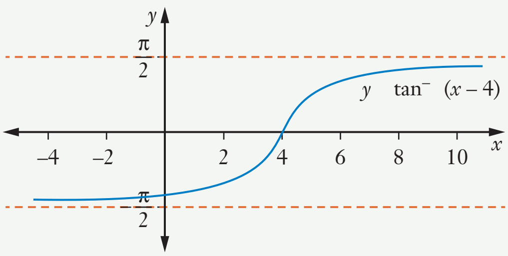
ii $y = \tan^{-1}\frac{x}{5}$



iii $y = \tan^{-1} x + \frac{\pi}{4}$

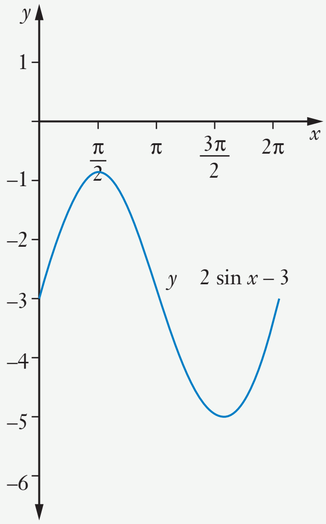


v $y = \tan^{-1}(x - 4)$

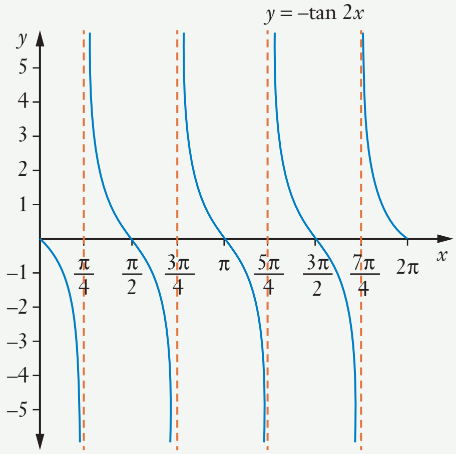


Exercise 4.02

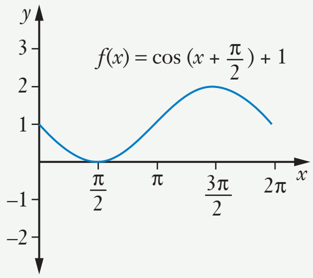
1 **a**



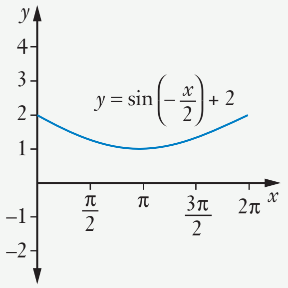
b



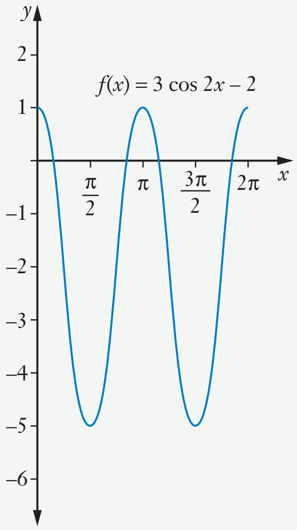
c



d



e



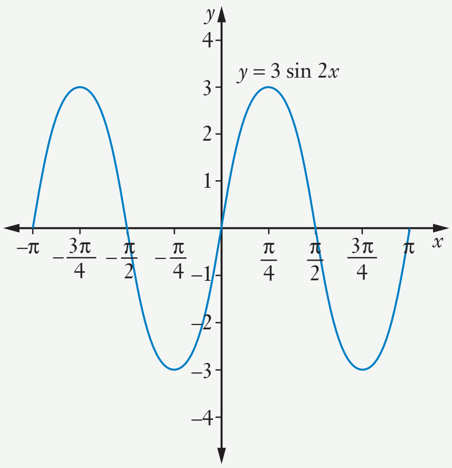
2 **a** $y = 5\sin[3(x + 5)] - 6$

b Amplitude 5, period $\frac{2\pi}{3}$, centre -6, phase shift 5 units to the left

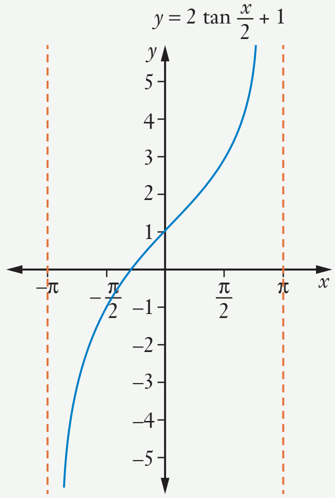
3 **a** $y = 4\cos\left[6\left(x - \frac{\pi}{3}\right)\right] + 2$

b $y = -\cos(x + \pi) - 5$

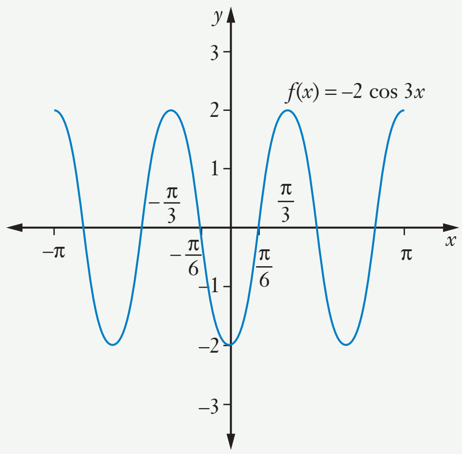
4 **a**



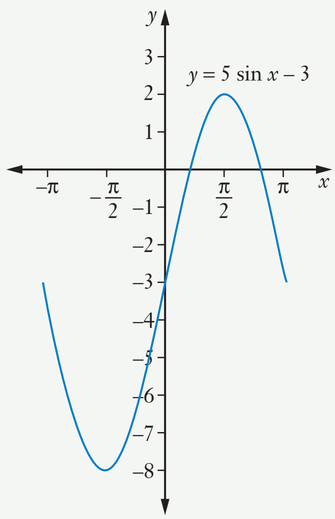
b



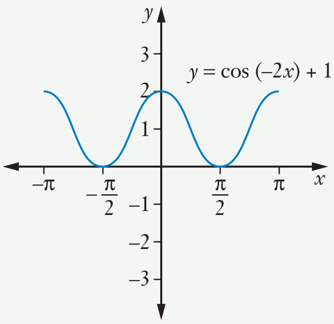
c



d



e



5 **a** No amplitude, period $\frac{\pi}{4}$, centre -5

b Amplitude 8, period 2π, phase shift π units to the left, centre -3

c Amplitude 5, period π, phase shift 3 units to the right, centre 1

6 **a** $y = 7\sin[2(x-1)] - 3$

b $y = -\cos 5x + 2$

c $y = -\tan\left[\frac{1}{2}(x+2)\right]$

d $y = 4\sin\left[-\frac{1}{3}(x+5)\right] + 2$

7 No amplitude, period $\frac{2\pi}{a}$, phase shift b units to the right when $b < 0$, to the left when $b > 0$, centre c

8 $y = \tan(4x - 3)$

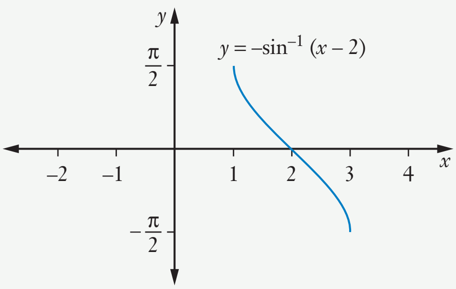
9 **a** 15 m

b Amplitude 10, period 12

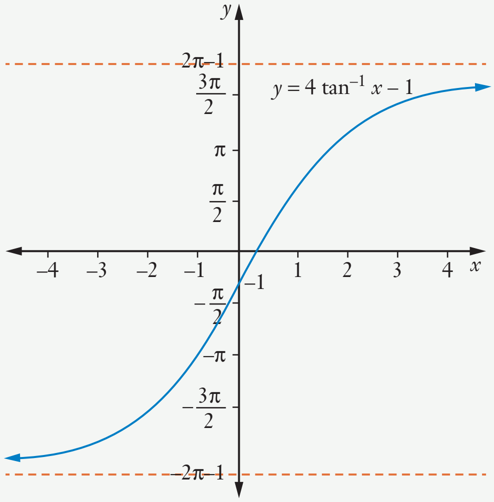
c $D = 10\cos\frac{\pi t}{6} + 15$

10 $B = 20\sin\frac{\pi t}{30} + 100$

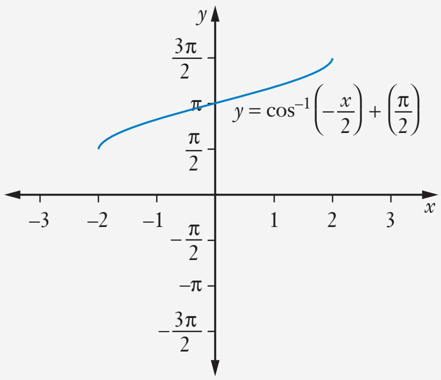
11 **a** $y = -\sin^{-1}(x-2)$



b $y = 4\tan^{-1}x - 1$

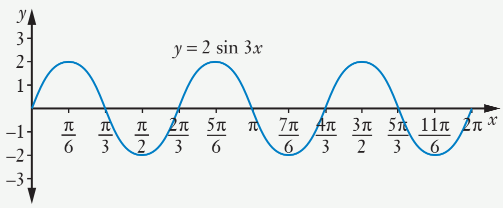


c $y = \cos^{-1}\left(-\frac{x}{2}\right) + \frac{\pi}{2}$



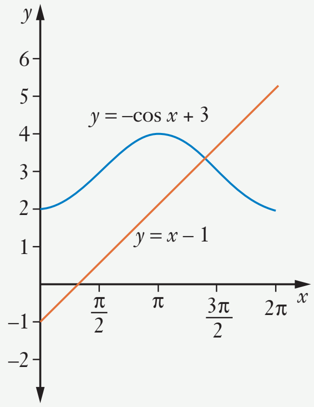
Exercise 4.03

1



a 6 **b** $x = 0.2, 0.9, 2.3, 3, 4.3, 5.1$

2 a



b i $x = 4.4$ **ii** $x = 0, 2\pi$

3 a $x = 15°, 75°, 195°, 255°$

b $x = 45°, 105°, 165°, 225°, 285°, 345°$

c $x = 240°, 300°$ **d** $x = 105°, 285°$

e $x = 120°, 300°$

4 a $x = \frac{\pi}{6} \quad \frac{2\pi}{3} \quad \frac{7\pi}{6} \quad \frac{5\pi}{3}$

b $x = \frac{2\pi}{9} \quad \frac{4\pi}{9} \quad \frac{8\pi}{9} \quad \frac{10\pi}{9} \quad \frac{14\pi}{9} \quad \frac{16\pi}{9}$

c $x = 0, \frac{2\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

d $x = \frac{\pi}{8} \quad \frac{3\pi}{8} \quad \frac{5\pi}{8} \quad \frac{7\pi}{8} \quad \frac{9\pi}{8} \quad \frac{11\pi}{8} \quad \frac{13\pi}{8} \quad \frac{15\pi}{8}$

e $x = \pi$

5 a $x = -\frac{11\pi}{12} \quad -\frac{7\pi}{12} \quad -\frac{\pi}{4} \quad \frac{\pi}{12} \quad \frac{5\pi}{12} \quad \frac{3\pi}{4}$

b $x = -\frac{\pi}{2}, 0$ **c** $x = -\frac{\pi}{4} \quad \frac{3\pi}{4}$

d $x = -\pi, 0, \pi$

e $x = -\pi, -\frac{3\pi}{4} \quad -\frac{\pi}{2} \quad -\frac{\pi}{4}, 0, \frac{\pi}{4} \quad \frac{\pi}{2} \quad \frac{3\pi}{4}, \pi$

6 a $x = \frac{\pi}{3} \quad \frac{2\pi}{3} \quad \frac{4\pi}{3} \quad \frac{5\pi}{3}$

b $x = \frac{2\pi}{9} \quad \frac{4\pi}{9} \quad \frac{8\pi}{9} \quad \frac{10\pi}{9} \quad \frac{14\pi}{9} \quad \frac{16\pi}{9} \quad \frac{19\pi}{9}$

7 a Amplitude 15, period 12, centre 20

b $t = 0, 12, 24, \ldots$ months. Average temperature of 35° occurs in January of each year.

8 a $y = 7 \sin \frac{\pi t}{5} + 13$

b $t = 2.5, 12.5, 22.5, 32.5$ seconds

c 7.5 seconds

d 0, 5, 10, 15, 20, 25, … seconds

9 a Amplitude 1, period $\frac{1}{440}$

b i $x = 0.0002, 0.0009, 0.0025, 0.0032, 0.0045, 0.0055, 0.007, 0.0075, 0.0093$

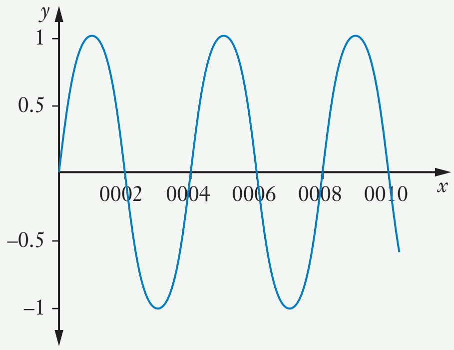
ii $x = 0, 0.001, 0.0021, 0.00035, 0.0045, 0.0056, 0.0068, 0.008, 0.009$

c i $x = 0.00019, 0.00095, 0.0025, 0.0032, 0.0047, 0.0055, 0.0070, 0.0078, 0.0093$

ii $x = 0, 0.0011, 0.0023, 0.0034, 0.0045, 0.0057, 0.0068, 0.0080, 0.0091$

d $y = 3 \sin (880\pi x)$

e



f 440 Hz

10 a Physical 23 days, emotional 28 days, intellectual 33 days

b 7 days, 21 days **c** 6 days, 19 days, 32 days

d around 7 days

11 a 15 is the centre of motion (equilibrium of spring)

b 27 cm maximum, 3 cm minimum

c 3 cm **d** $\pi, 3\pi, 5\pi, \ldots$ seconds

Exercise 4.04

1 a $\cos a \cos b + \sin a \sin b$ **b** $\frac{\tan x + \tan y}{1 - \tan x \tan y}$

c $\sin 3p \cos 2q + \cos 3p \sin 2q$

d $2 \cos x \sin y$ **e** $\cos^2 A - \sin^2 A$

2 a $\cos (a + b)$ **b** $\tan (\alpha - \beta)$ **c** $\sin 2x$

d $\sin (5a - 3b)$ **e** $\cos A \cos B$

3 a i $\cos A$ **ii** $3 \tan A$

b $\frac{4t(1-t^2)}{(1+t^2)^2}$

c See worked solutions.

4 a $\frac{1+\sqrt{3}}{2\sqrt{2}}$ **b** $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

5 $3 \sin x - 4 \sin^3 x$

6 See worked solutions.

7 **a** $\frac{1}{2}[\cos (3a - 2b) - \cos (3a + 2b)]$

b $\frac{1}{2}[\sin (5y + 3z) - \sin (5y - 3z)]$

c $\frac{1}{2}[\cos (2p + 3q) + \cos (2p - 3q)]$

d $\frac{1}{2}[\sin (4x + 9y) + \sin (4x - 9y)]$

8 **a** $\sqrt{5} \sin (x + 26° 34')$ **b** $2 \sin (x + 60°)$

c $\sqrt{2} \sin (x + 45°)$ **d** $\sqrt{29} \sin (x + 21° 48')$

e $\sqrt{65} \sin (x + 60° 15')$

9 **a** $\sqrt{2} \sin \left(x - \frac{\pi}{4}\right)$ **b** $\sqrt{5} \sin (x - 1.11)$

c $2 \sin \left(x - \frac{\pi}{3}\right)$ **d** $2 \sin \left(x - \frac{\pi}{6}\right)$

e $\sqrt{29} \sin (x - 0.38)$

10 **a** $\sqrt{13} \cos (x - 0.98)$ **b** $\sqrt{53} \cos (x - 0.28)$

c $2 \cos \left(x - \frac{\pi}{3}\right)$ **d** $2 \cos \left(x - \frac{\pi}{6}\right)$

e $\sqrt{13} \cos (x - 0.59)$

11 **a** $\sqrt{13} \cos (x + 56° 19')$

b $\sqrt{53} \cos (x + 15° 57')$

c $2 \cos (x + 60°)$

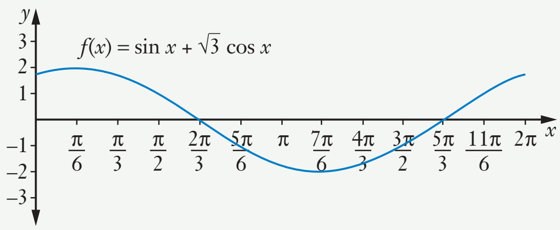
d $2 \cos (x + 30°)$

e $\sqrt{13} \cos (x + 33° 41')$

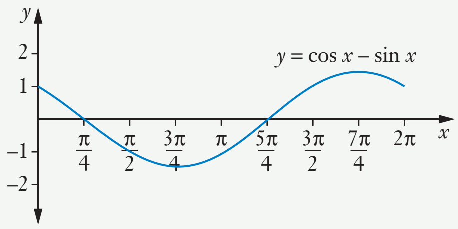
12 **a** $\sqrt{130} \sin (x + 37° 52')$

b $\sqrt{130} \cos (x + 52° 7')$

13 **a**



b



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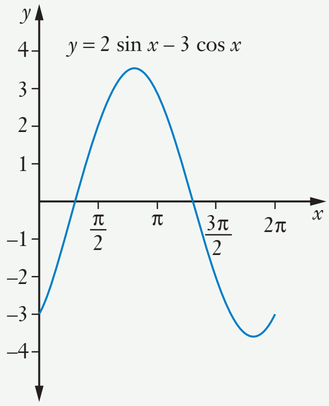
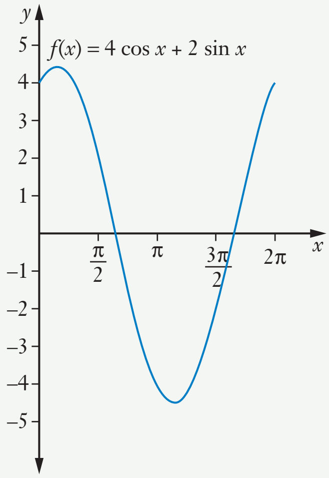
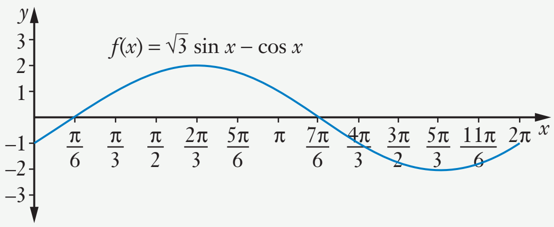


d



e



Exercise 4.05

1 **a** $x = 19° \, 21', 340° \, 39'$ **b** $x = 30°, 150°$

c $x = 180°$ **d** $x = 45°, 225°$

e $x = 22° \, 30', 67° \, 30', 202° \, 30', 247° \, 30'$

f $x = 30°, 150°, 210°, 330°$

g $x = 0°, 120°, 240°, 360°$

h $x = 0°, 135°, 180° \, 315°, 360°$

i $x = 0°, 90°, 270°, 360°$

j $x = 48° \, 11', 60°, 300°, 311° \, 49'$

2 **a** $x = 1.75, 4.53$ **b** $x = 0.26, 6.02$

c $x = \frac{\pi}{2}$

d $x = 0.15, 1.42, 3.29, 4.56$

e $x = 0.24, 1.33, 3.38, 4.47$

f $x = 0.46, 3.61, \frac{3\pi}{4} \; \frac{7\pi}{4}$

g $x = 0.34, 2.80, \frac{7\pi}{6} \; \frac{11\pi}{6}$

h $x = 2.68, 5.82, \frac{3\pi}{4} \; \frac{7\pi}{4}$

i $x = \frac{\pi}{4} \; \frac{\pi}{2} \; \frac{3\pi}{4} \; \frac{5\pi}{4} \; \frac{3\pi}{2} \; \frac{7\pi}{4}$

3 **a** $x = \frac{\pi}{4} \; \frac{5\pi}{4}$ **b** $x = \frac{\pi}{6} \; \frac{7\pi}{6}$

c $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

d $x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

e $x = \frac{\pi}{2} \; \frac{7\pi}{6} \; \frac{11\pi}{6}$

f $x = 0, \frac{\pi}{3} \; \frac{5\pi}{3}, 2\pi$

g $x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

h $x = 0, \pi, 2\pi$

i $x = \frac{\pi}{6} \; \frac{3\pi}{4} \; \frac{5\pi}{6} \; \frac{7\pi}{4}$

j $x = 0, 2\pi$

4 **a** $\theta = 126° \, 52', 306° \, 52'$

b $\theta = 35° \, 58', 189° \, 17'$

c $\theta = 60°, 240°$

d $\theta = 180°, 270°$

e $\theta = 240° \, 43', 327° \, 21'$

f $\theta = 90°, 180°$

g $\theta = 90°, 340° \, 32'$

h $\theta = 56° \, 34', 176° \, 34'$

i $\theta = 51° \, 1', 190° \, 54'$

j $\theta = 160° \, 32', 270°$

5 **a** $\theta = 2.74, 5.88$ **b** $\theta = 0.63, 3.60$

c $\theta = \frac{\pi}{2} \; \frac{11\pi}{6}$ **d** $\theta = 0, \frac{3\pi}{2}, 2\pi$

e $\theta = 2.63, 4.94$ **f** $\theta = 0, 3.78, 2\pi$

g $\theta = 1.23, \pi$ **h** $\theta = 0.97, 3.96$

i $\theta = 2.28, 5.08$ **j** $\theta = \frac{7\pi}{12} \; \frac{11\pi}{12}$

6 **a** $x = -127° \, 30', -97° \, 30', 52° \, 30', 82° \, 30'$

b $x = -180°, 0°, 90°, 180°$

7 **a** $x = 0, \frac{2\pi}{3} \; \frac{4\pi}{3}, 2\pi$

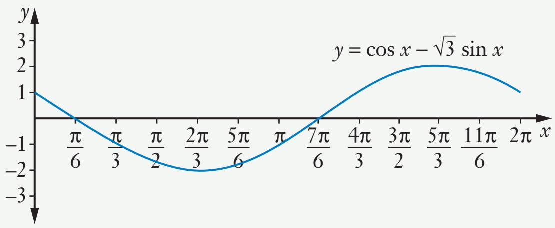
b $x = \frac{\pi}{4} \; \frac{3\pi}{4} \; \frac{5\pi}{4} \; \frac{7\pi}{4}$

c $x = \frac{\pi}{3}$

d $x = \frac{\pi}{2}$

e $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

8 **a**



b Amplitude 2, period 2π, phase shift $\frac{\pi}{3}$ units to the left

c $x = 0, 4, 2\pi$ **d** $x = 0, \frac{4\pi}{3}, 2\pi$

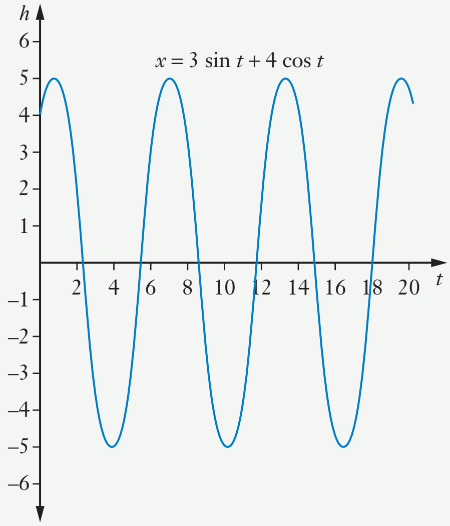
9 **a** 11 cm; $t = 0, 6, 12, 18, \ldots$ s

b −3 cm; $t = 3, 9, 15, 21, \ldots$ s

c $t = 1.5, 4.5, 7.5, 10.5, \ldots$ s

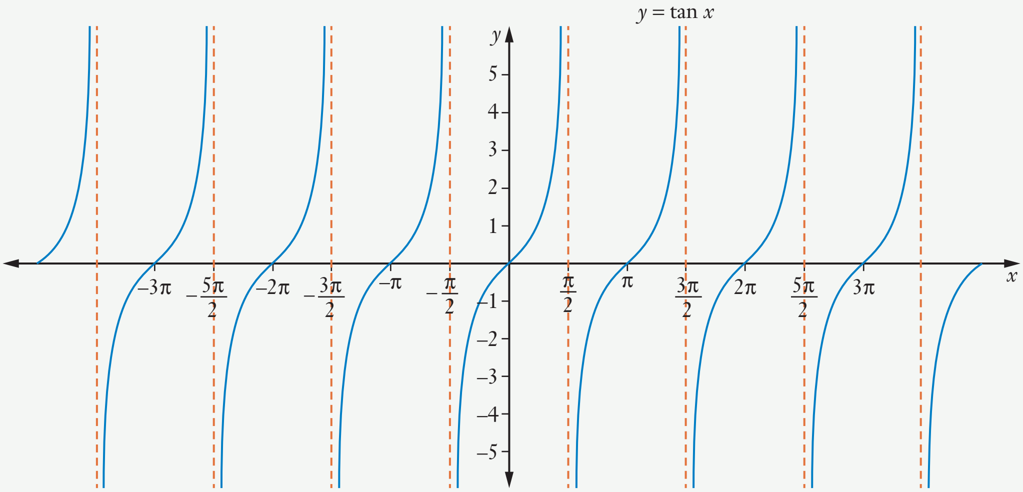
d $t = 1, 5, 7, 11, 13, 17, \ldots$s

10 a



b Maximum 5 metres; $t = 0.6$ min

11 a



b $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

12 a $x = \ldots -\frac{11\pi}{2}, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \ldots$

b $x = \pm\frac{\pi}{6}, \pm\frac{11\pi}{6}, \pm\frac{13\pi}{6}, \pm\frac{23\pi}{6}, \ldots$

c $x = \ldots -\frac{15\pi}{4}, -\frac{11\pi}{4}, -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$

d $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$

e $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Test yourself 4

1 D **2** A **3** B **4** A

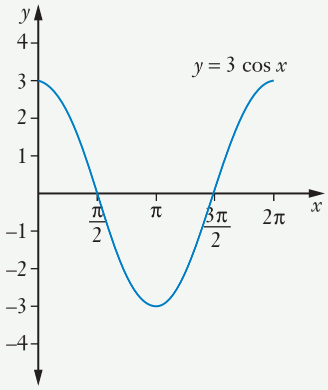
5 a $\sin a \cos b - \cos a \sin b$

b $\dfrac{\tan 3a - \tan 5b}{1 + \tan 3a \tan 5b}$

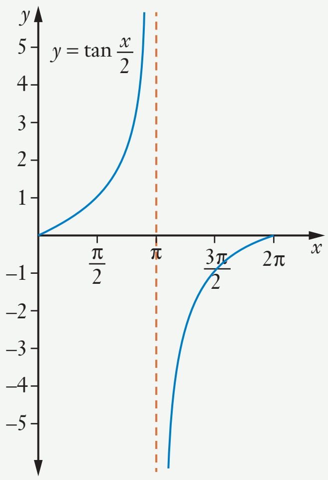
c $\cos x \cos y + \sin x \sin y + \sin x \cos y + \cos x \sin y$

d $2 \sin x \cos x$

6 a



b



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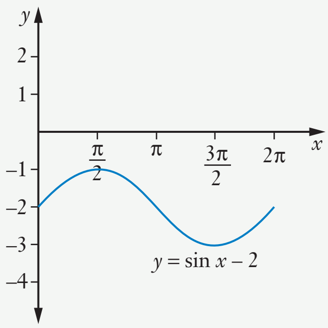
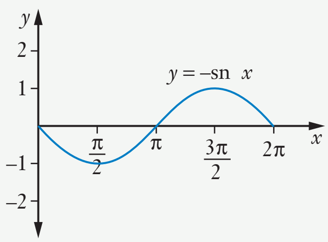
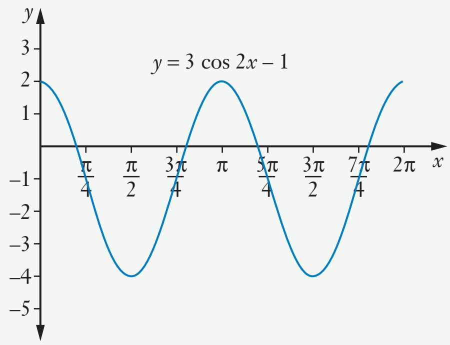


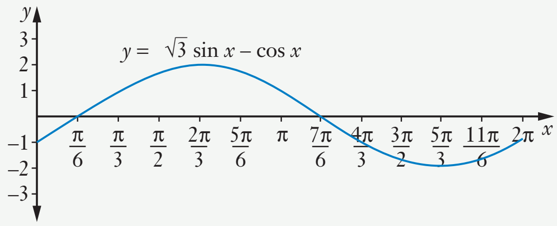
d



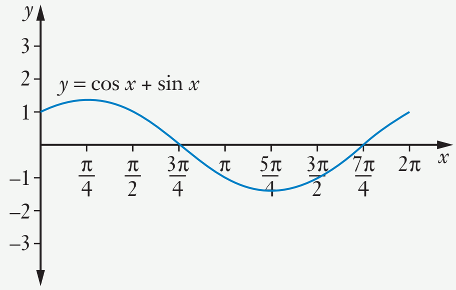
e



f



g



7 a Maximum 13 m; $t = 0, 3, 6, 9, \ldots$ h; minimum 7 m; $t = 1.5, 4.5, 7.5 \ldots$h

b $t = 0.6, 2.4, 3.6, 5.4, 6.6, 8.4, \ldots$h; times when water level in the lock is 11 m

8 a $2\operatorname{cosec}^2 x$ **b** $\sec A$

c 1 **d** $\sin x$

9 a $x = \frac{\pi}{6}\ \frac{5\pi}{6}\ \frac{7\pi}{6}\ \frac{11\pi}{6}$

b $x = \frac{\pi}{12}\ \frac{5\pi}{12}\ \frac{13\pi}{12}\ \frac{17\pi}{12}$

c $x = \frac{3\pi}{2}$

d $x = \frac{\pi}{6}\ \frac{\pi}{2}\ \frac{7\pi}{6}\ \frac{3\pi}{2}$

e $x = 1.15, 1.99, 4.29, 5.13$

10 a i $\sqrt{29}\sin(\theta + 68° 12')$

ii $2\sin\left(\theta + \frac{\pi}{6}\right)$

b **i** $\sqrt{5}\sin(\theta - 63°\,26')$

ii $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)$

c **i** $\sqrt{2}\cos(\theta + 45°)$

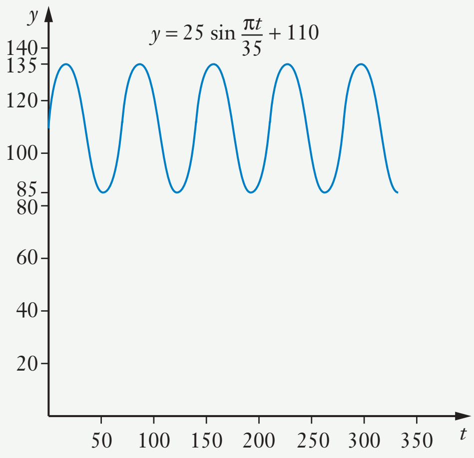
ii $2\cos\left(\theta + \frac{\pi}{3}\right)$

d **i** $\sqrt{53}\cos(\theta - 74°\,3')$

ii $\sqrt{26}\cos(\theta - 0.2)$

11 **a** $y = 25\sin\frac{\pi t}{35} + 110$

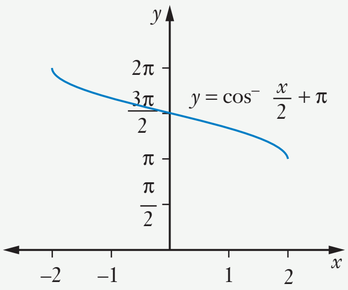
b



12 **a** $x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$ **b** $x = \frac{\pi}{12}, -\frac{11\pi}{12}$

c $x = \pm\frac{\pi}{4}$

13 $y = \cos^{-1}\frac{x}{2} + \pi$



14 **a** $h = 15\sin\frac{2\pi t}{13} + 65$

b 0, 6.5, 13, 19.5, … hours

15 $\frac{1}{2}[\sin(3x + 4y) - \sin(3x - 4y)]$

16 **a** $y = \cos 2x$ **b** $y = 5\cos x$

c $y = -\cos x$ **d** $y = \cos\left(x - \frac{\pi}{6}\right)$

e $y = \cos x + 4$

17 **a** $x = 71°\,23', 288°\,37'$

b $x = 30°, 60°, 210°, 240°$

c $x = 0°, 45°, 180°, 225°, 360°$

d $x = 60°, 180°, 300°$

e $x = 30°, 150°$

f $x = 0°, 60°, 300°, 360°$

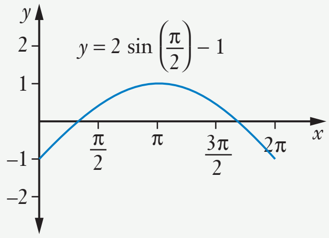
g $x = 0°, 60°, 180°, 300°, 360°$

h $x = 90°, 180°$

i $x = 60°, 180°$

j $x = 119°\,33', 346°\,43'$

18 **a**



b $x = 1, 5.2$ **c** $x = \frac{\pi}{3}, \frac{5\pi}{3}$

19 See worked solutions.

20 **a** Amplitude 2, period $\frac{2\pi}{3}$, centre -1

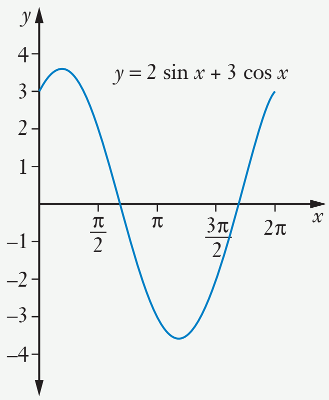
b Amplitude 1, period 4π, centre 0, phase shift 2π units to the left

c No amplitude, reflection in x-axis, period $\frac{\pi}{5}$, centre 0, phase shift $\frac{\pi}{20}$ to the right

21 **a** $x = 0°, 180°, 360°$ **b** $x = 155°, 245°$

c $x = 10°\,54', 59°\,6', 190°\,54', 239°\,6'$

22 **a**



b **i** $x = 1.6, 6$ **ii** $x = 1.57, 5.89$

23 **a** $x = -112.5°, -22.5°, 67.5°, 157.5°$

b $x = 90°, -30°, -150°$

24 **a** $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ **b** $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

c $x = \frac{\pi}{4}, \frac{5\pi}{4}$ **d** $x = 1.8, 4.5$

e $x = \frac{\pi}{4}, \frac{5\pi}{4}, 2.03, 5.18$ **f** $x = 1.3, 5.91$

g $x = 0, \frac{4\pi}{3}, 2\pi$ **h** $x = 1.45, 3.55$

i $x = 2.2, 4.1$

j $x = 0.73, 2.41, 3.79. 5.64$

Challenge exercise 4

1 **a** Amplitude 2, period π, phase shift $\frac{\pi}{4}$ units to the right

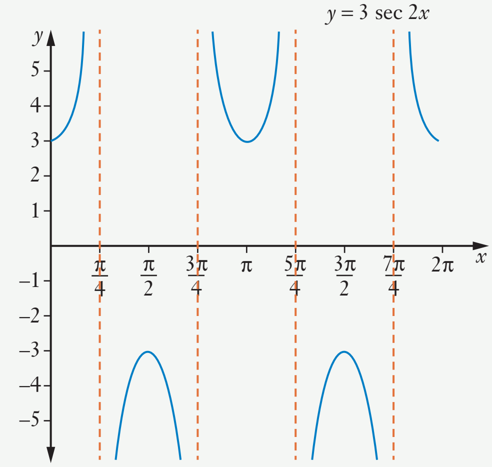
b $x = \frac{\pi}{6} \quad \frac{\pi}{3} \quad \frac{7\pi}{6} \quad \frac{4\pi}{3}$

2 **a** $y = 8 \cos x + 4$

b $y = 2 \sin\left[8\left(x - \frac{\pi}{3}\right)\right] + 3$

c $y = \tan\left[\frac{1}{2}\left(x + \frac{\pi}{2}\right)\right]$

3



4 $x = \frac{\pi}{8} \quad \frac{3\pi}{8} \quad \frac{9\pi}{8} \quad \frac{11\pi}{8}$

5 **a** $y = 4 \cos\left(\frac{1}{3}\left[x - \frac{\pi}{6}\right]\right) - 20$

b Amplitude 4, period 6π, centre –20, phase shift $\frac{\pi}{6}$ to the right

6 See worked solutions.

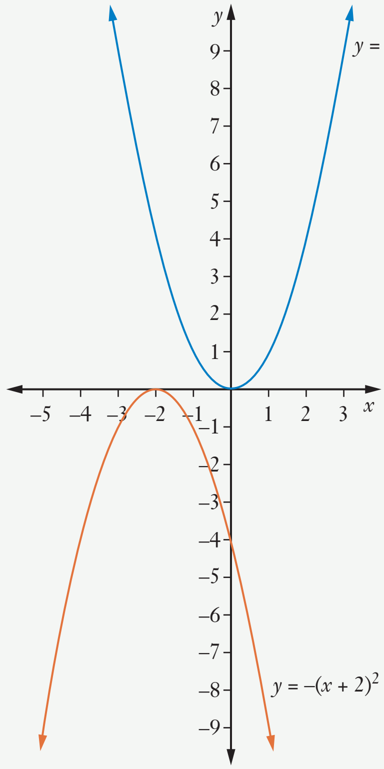
7 $x = 0.42, 2.72, 3.56, 5.86$

8 $x = 0, \frac{\pi}{3} \quad \frac{2\pi}{3}, \pi, \frac{4\pi}{3} \quad \frac{5\pi}{3}, 2\pi$

Practice set 1

1 B **2** C **3** C **4** D
5 D **6** D **7** C **8** D

9 **a**



b Reflection in x-axis, horizontal translation 2 units to the left

10 **a** $x = 67° 30', 157° 30', 247° 30', 337° 30'$

b $x = 20°, 100°, 140° ,220°, 260°, 340°$

c $x = 150°, 210°$

d $x = 60°, 240°$

e $x = 0°, 90°, 180°, 270°, 360°$

11 **a** Amplitude 4, period $\frac{2\pi}{5}$, centre 0

b Amplitude 2, reflection in x-axis, centre 1, phase shift $\frac{\pi}{6}$ units to the right

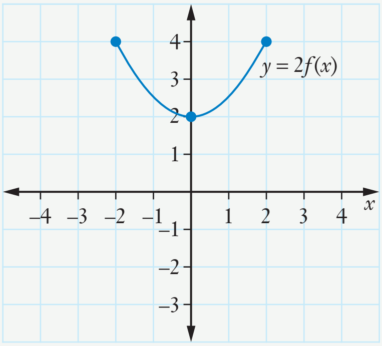
c Period 4π, phase shift 8 units to the left

12 $\frac{3}{4^{10}}$

13 **a** $\cos(a-b)$ **b** $\cos 2x$ **c** $\frac{1}{2}\sin 2x$

d $2\cos x\sin y$ **e** $\tan(\alpha+\beta)$ **f** $\cos A$

14 **a**



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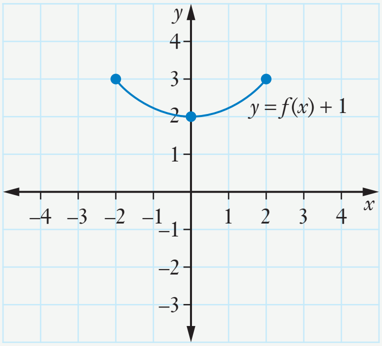

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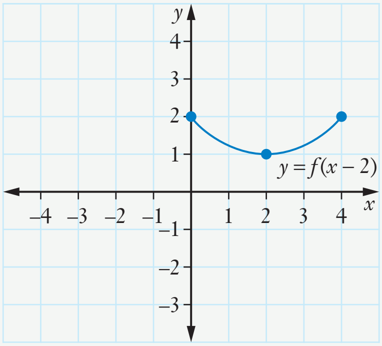
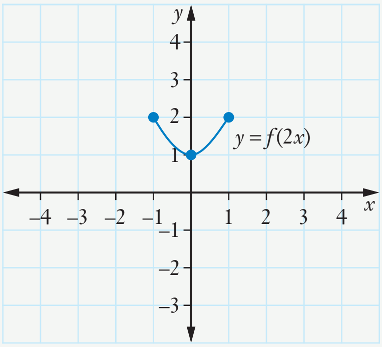


d



15 **a** 13, 337° 23′ **b** 5.4, 111° 48′

16 ±104, 52, ±26, 13, …

17 44th term

18 See worked solutions for complete proof.
Step 3: Prove true for $n = k + 1$

Prove $3 + 6 + 12 + \ldots + 3(2^k) + 3(2^{k+1}) = 3(2^{k+2} - 1)$

$$\begin{aligned}\text{LHS} &= 3 + 6 + 12 + \ldots + 3(2^k) + 3(2^{k+1})\\ &= 3(2^{k+1} - 1) + 3(2^{k+1})\\ &= 6(2^{k+1}) - 3\end{aligned}$$

19 $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$

20 Vertical dilation, scale factor 4 (stretch), horizontal translation 1 unit to the right, vertical translation 3 units down.

21 **a** $x = 30°, 41° 49′, 138° 11′, 150°$

b $x = 45°, 105°, 225°, 285°$

c $x = 101° 47′, 235° 36′$

d $x = 0°, 270°, 360°$

22 **a** Centre 10; max distance 16 cm, 4 cm

b 1 second

23 $5\sin(\theta + 53° 8′)$

24 $y = 4\sqrt{3(x+7)} - 1$

25 **a** 5.8, 329° **b** $\sqrt{2}$, 135°

26 **a** 199 **b** 5050

27 **a** $T_1 = 4, T_2 = 11, T_3 = 18, T_{12} = 81$

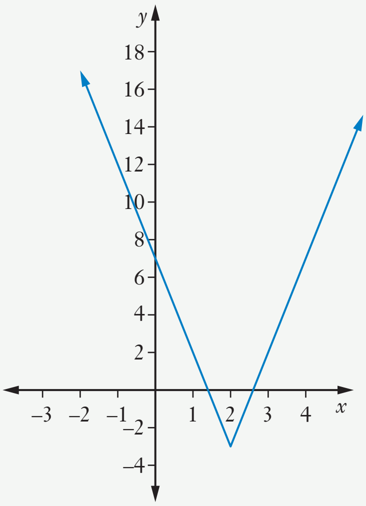
b 1410 **c** 29th term

28 **a** $i - j$ **b** $-7i + 3j$

c $-15i + 5j$ **d** $-i - j$

e $-9i + 5j$

29 **a**



b **i** $x = 1, 3$ **ii** $x = 0, 4$ **iii** $x = 2$

c Domain $(-\infty, \infty)$, range $[-3, \infty)$

30 **a** $-\frac{1}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** 1

d $-\frac{1}{\sqrt{2}}$ **e** $-\frac{1}{\sqrt{3}}$

31 Step 3: Prove true for $n = k + 1$.

$(k + 1)((k + 1) - 1)$ is divisible by 2

i.e $(k + 1)((k + 1) - 1) = 2q$ where q is an integer

$$k(k + 1) = 2q$$

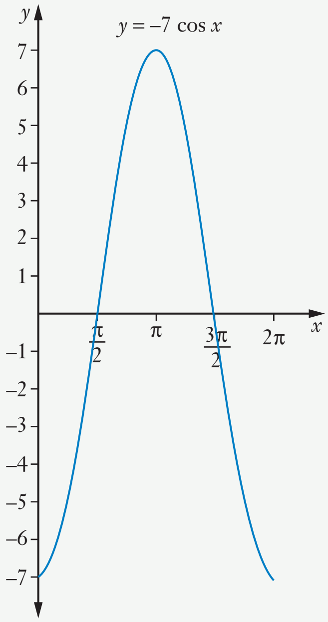
$$k^2 + k = 2q$$

32 **a** $\overrightarrow{AB} = 7i - 9j$ **b** 11.4, 308°

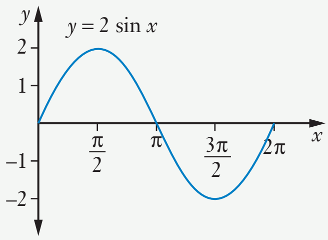
33 $T_n = 11n - 26$

34 $x = 40°$

35 **a**



b



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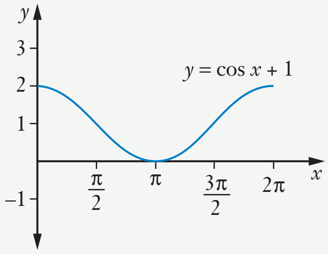
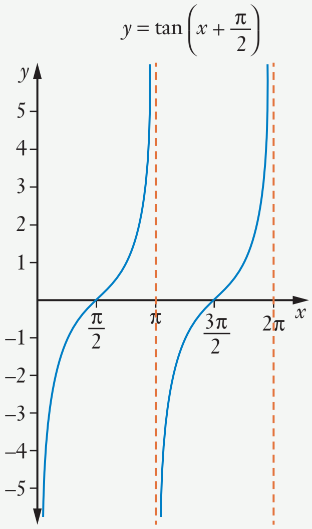


d



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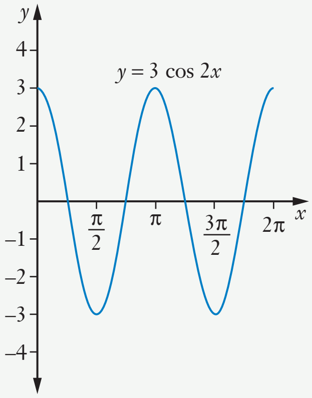
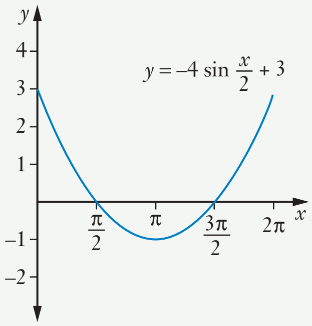


f



36 Proofs (see worked solutions)

37 **a** $y = 2\sin x$ **b** $y = \sin\dfrac{x}{2}$

c $y = \sin x - 3$ **d** $y = -5\sin x$

e $y = \sin\left(x + \dfrac{\pi}{2}\right)$

f $y = 5\sin\left[\dfrac{1}{3}(x - \pi)\right] + 1$

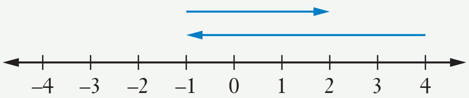
38 7.1, 352°

39 **a** $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ **b** $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

c $x = 0, 1.85, 2\pi$ **d** $x = 0, \dfrac{3\pi}{2}, 2\pi$

40 $x = -3, y = 4$ **41** $x = \dfrac{5}{6}$

42 **a**



b 2 units to the right of O

c 8 units

43 Step 3: Prove true for $n = k + 1$.

i.e. $1 + 3 + 3^2 + \ldots + 3^k + 3^{k+1} = \dfrac{3^{k+2} - 1}{2}$

$\text{LHS} = 1 + 3 + 3^2 + \ldots + 3^k + 3^{k+1}$

$= \dfrac{3^{k+} - 1}{2} + 3^{k+1}$

$= \dfrac{3^{k+1} - 1}{2} + \dfrac{2(3^{k+1})}{2}$

44 **a** $x = 0.9, 5.38$ **b** $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

c $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ **d** $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

e $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

f $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

45 $n = 4$

46 $\dfrac{1}{10}$

47 **a** Centre 18, amplitude 20, so maximum temperature is 38°C, minimum –2°C. Period is 12, so it is an annual cycle.

b July **c** January

d 8°C **e** April and October

48 $-1 \le x \le 3$

49 $f(-x) = 3(-x)^2 - 2$

$= 3x^2 - 2$

$= f(x)$ so even

50 10, 217°

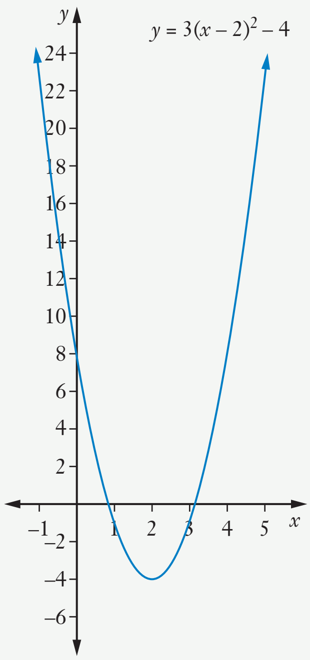
51 **a** $\log 3, \log 3^2, \log 3^3, \ldots = \log 3, 2\log 3, 3\log 3, \ldots$

Arithmetic series, since $2\log 3 - \log 3 = 3\log 3 - 2\log 3 = \log 3$

b $210\log 3$

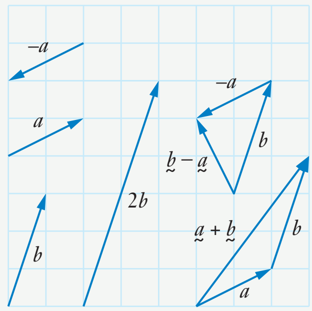
52 **a** $\dfrac{4t(1-t^2)}{(1+t^2)^2}$ **b** $\dfrac{6t^2 - 1 - t^4}{(1-t^2)^2}$

53 **a**



b **i** $x \le 0, x \ge 4$ **ii** $0 < x < 4$

54



55 **a** $\dfrac{4t - 3 + 3t^2}{1 + t^2}$ **b** $x = 1.26, 3.84$

56 **a** $2\sqrt{10}, 71°\ 34'$ **b** $\sqrt{2}, 315°$

c $\sqrt{41}, 231°\ 20'$

Chapter 5

Exercise 5.01

1 **a** $12x^3 - 6x^2 + 7$ **b** 2 **c** $12x - 3$

2 $20x^4 + 18x$ **3** $6\pi t^2 - 6t$ **4** $f'(-2) = 101$

5 **a** $-5x^{-6}$ **b** $\frac{2}{3}x^{-\ 3}$ **c** $-\frac{2}{x^3}$ **d** $\frac{1}{4\sqrt[4]{x^3}}$ **e** $\frac{20}{x^5}$

6 $\frac{1}{12}$

7 **a** $21(3x-1)^6$ **b** $3(2x-1)(x^2-x+2)^2$ **c** $\frac{7}{2\sqrt{7x-2}}$ **d** $-\frac{3}{(3x-2)^2}$ **e** $\frac{2x}{3\sqrt[3]{(x^2-3)^2}}$

8 **a** $3x^2 + 8x$ **b** $24x + 4$ **c** $12x^2 + 4$ **d** $4(x^2-1)(5x^2+3x-1)$ **e** $\frac{x^2(7x+6)}{\sqrt{x+1}}$

9 **a** $-\frac{13}{(x-5)^2}$ **b** $\frac{x^2(8x-21)}{(4x-7)^2}$ **c** $\frac{2(x^2-3x-3)}{(2x-3)^2}$ **d** $\frac{-6x+23}{(2x+9)^3}$ **e** $\frac{3x-7}{\sqrt{(2x-1)^3}}$

10 **a** -6 **b** 3

11 **a** $\frac{1}{14}$ **b** $\frac{1}{5}$

12 **a** $7x - y - 24 = 0$ **b** $51x - y - 72 = 0$

13 **a** $x - 4y - 3 = 0$ **b** $x + 3y - 6 = 0$

14 $x = 4$ **15** $(2, 6), (-2, -10)$

16 $3x + y + 8 = 0$ **17** $(4, 2)$

18 **a** $17x - y - 3 = 0$ **b** $x + 17y + 51 = 0$

19 **a** $6t$ **b** $-\frac{2}{(t-3)^2}$ **c** $\frac{2}{3\sqrt[3]{(2x+3)^2}}$

20 **a** **i** 10 kg s^{-1} **ii** 17 kg s^{-1} **b** **i** 13 kg s^{-1} **ii** 123 kg s^{-1}

21 -2.18 Pa/m^3

22 **a** **i** 2 m **ii** 1.5 m **b** 2 s **c** **i** 2 m s^{-1} **ii** 0 m s^{-1} **iii** -4 m s^{-1}

Exercise 5.02

1 **a** $7e^{7x}$ **b** $-e^{-x}$ **c** $6e^{6x-2}$ **d** $2xe^{x\ +}$ **e** $(3x^2+5)e^{x\ +5x+7}$ **f** $5e^{5x}$ **g** $-2e^{-2x}$ **h** $10e^{10x}$ **i** $2e^{2x} + 1$ **j** $2x + 2 - e^{1-x}$ **k** $5(1+4e^{4x})(x+e^{4x})^4$ **l** $e^{2x}(2x+1)$ **m** $\frac{e^{3x}(3x-2)}{x^3}$ **n** $x^2e^{5x}(5x+3)$ **o** $\frac{4e^{2x+1}(x+2)}{(2x+5)^2}$

2 $3e$

3 **a** $3^x \ln 3$ **b** $10^x \ln 10$ **c** $3(2^{3x-4}) \ln 2$

4 5 **5** $x + y - 1 = 0$

6 **a** $3e^3$ **b** $-\frac{1}{3e^3}$

7 **a** $y = 2ex - e$ **b** $x + 2ey - 2e^2 - 1 = 0$

8 $x \ln 4 - y + 4 = 0$

9 **a** **i** 29 627 **ii** 35 826 **b** **i** 1044 people/year **ii** 1240 people/year **c** **i** 1126 people/year **ii** 1361 people/year

10 **a** $55\ 042\text{ cm min}^{-1}$ **b** **i** 142.8 cm min^{-1} **ii** 1087 cm min^{-1} **iii** $177\ 722\ 205.4\text{ cm min}^{-1}$

11 **a** 20 g **b** 7 g **c** -0.091 g/year **d** **i** -0.147 g/year **ii** -0.051 g/year **iii** -0.0063 g/year

12 **a** 66 079.4 cm **b** $132\ 158.8\text{ cm s}^{-1}$

Exercise 5.03

1 **a** $1 + \frac{1}{x}$ **b** $-\frac{1}{x}$ **c** $\frac{3}{3x+1}$ **d** $\frac{2x}{x^2-4}$ **e** $\frac{15x^2+3}{5x^3+3x-9}$ **f** $\frac{10x^2+2x+5}{5x+1}$ **g** $6x + 5 + \frac{1}{x}$ **h** $\frac{8}{8x-9}$ **i** $\frac{6x+5}{(x+2)(3x-1)}$ **j** $\frac{-30}{(4x+1)(2x-7)}$ **k** $\frac{5}{x}(1+\ln x)^4$ **l** $9\left(\frac{1}{x}-1\right)(\ln x - x)^8$ **m** $\frac{4}{x}(\ln x)^3$

n $6\left(2x+\frac{1}{x}\right)(x^2+\ln x)^5$

o $1+\ln x$ **p** $\frac{1-\ln x}{x^2}$

q $\frac{2x\ln x+2x+1}{x}$

r $3x^2\ln(x+1)+\frac{x^3}{x+1}$ **s** $\frac{1}{x\ln x}$

t $\frac{x-2-x\ln x}{x(x-2)^2}$ **u** $\frac{e^{2x}(2x\ln x-1)}{x(n x)^2}$

v $e^x\left(\frac{1}{x}+\ln x\right)$ **w** $\frac{10\ln x}{x}$

2 $f'(1)=-\frac{1}{2}$ **3** $\frac{1}{x\ln 10}$

4 $x-2y-2+2\ln 2=0$ **5** $x-y-2=0$

6 $-\frac{2}{5}$ **7** $5x+y-\ln 5-25=0$

8 $5x-19y+19\ln 19-15=0$

9 $\frac{2}{(2x+5)\ln 3}$

10 $(2\ln 2)x+y-1-4\ln 2=0$

11 **a** 20 000

b **i** 10.6 years **ii** 43.6 years

c $P=20\,000\,e^{0.021t}$ **d** 452 kangaroos/year

e **i** 447 kangaroos/year

ii 466 kangaroos/year

iii 518 kangaroos/year

Exercise 5.04

1 **a** $4\cos 4x$ **b** $-3\sin 3x$

c $5\sec^2 5x$ **d** $3\sec^2(3x+1)$

e $\sin(-x)$ **f** $3\cos x$

g $-20\sin(5x-3)$ **h** $-6x^2\sin(x^3)$

i $14x\sec^2(x^2+5)$ **j** $3\cos 3x-8\sin 8x$

k $\sec^2(\pi+x)+2x$ **l** $x\sec^2 x+\tan x$

m $3\sin 2x\sec^2 3x+2\tan 3x\cos 2x$

n $\frac{x\cos x-\sin x}{2x^2}$

o $\frac{3\sin 5x-5(3x+4)\cos 5x}{\sin^2 5x}$

p $9(2+7\sec^2 7x)(2x+\tan 7x)^8$

q $2\sin x\cos x=\sin 2x$ **r** $-45\sin 5x\cos^2 5x$

s $e^x+2\sin 2x$ **t** $-\frac{1}{x}\cos(1-\ln x)$

u $(e^x+1)\cos(e^x+x)$ **v** $\frac{\cos x}{\sin x}=\cot x$

w $e^{3x}(3\cos 2x-2\sin 2x)$

x $\frac{e^{2x}(2\tan 7x-7\sec^2 7x)}{\tan^2 7x}$

2 12 **3** $6\sqrt{3}x-12y+6-\pi\sqrt{3}=0$

4 $-\frac{\sin x}{\cos x}=-\tan x$ **5** $-\frac{2}{3\sqrt{3}}=-\frac{2\sqrt{3}}{9}$

6 $\sec^2 x\, e^{\tan x}$

7 $8x+24\sqrt{2}y-72-\pi=0$

8 Proof (see worked solutions).

9 **a** $\frac{\pi}{180}\sec^2 x°$ **b** $-\frac{\pi}{60}\sin x°$ **c** $\frac{\pi}{900}\cos x°$

10 $\sin^3 x(4\cos^2 x-\sin^2 x)$

11 **a** 750 **b** 525 **c** 975

d 2.6, 6.4, 11.6, 15.4 … days

e **i** –136 fish/day **ii** 155 fish/day

iii –101 fish/day **v** 0 fish/day

f 0.23, 4.27, 9.23, 13.27, … days

12 **a** **i** 9 m **ii** 13 m

b 0.2, 5.8, 12.2, 17.8, … h

c **i** 0 m h^{-1} **ii** 3.6 m h^{-1} **iii** 4.2 m h^{-1}

d 1.5, 10.5, 13.5, 22.5, … h

Exercise 5.05

1 $7x^6-10x^4+4x^3-1$; $42x^5-40x^3+12x^2$; $210x^4-120x^2+24x$; $840x^3-240x+24$

2 $72x^7$

3 $f'(x)=10x^4-3x^2, f''(x)=40x^3-6x$

4 $f'(1)=11, f''(-2)=168$

5 $7x^6-12x^5+16x^3$; $42x^5-60x^4+48x^2$; $210x^4-240x^3+96x$

6 $\frac{dy}{dx}=4x-3, \frac{d^2y}{dx^2}=4$

7 $f'(-1)=-16, f''(2)=40$ **8** $-4x^{-5}$; $20x^{-6}$

9 $-\frac{1}{32}$ **10** 26

11 $x=\frac{7}{18}$ **12** $x>\frac{1}{3}$

13 $20(4x-3)^4$ $320(4x-3)^3$

14 $f'(x) = -\dfrac{1}{2\sqrt{2-x}}$; $f''(x) = -\dfrac{1}{4\sqrt{(2-x)^3}}$

15 $f'(x) = -\dfrac{16}{(3x-1)^2}$; $f''(x) = \dfrac{96}{(3x-1)^3}$

16 $\dfrac{d^2v}{dt^2} = 24t + 16$

17 $b = \dfrac{2}{3}$

18 196

19 $b = -2.7$

20 $\dfrac{dy}{dx} = 4e^{4x} - 4e^{-4x}$

$\dfrac{d^2y}{dx^2} = 16e^{4x} + 16e^{-4x} = 16y$

21, 22 Proofs (see worked solutions).

23 $n = -15$

24 $y = 2\cos 5x$; $\dfrac{dy}{dx} = -10\sin 5x$

$\dfrac{d^2y}{dx^2} = -50\cos 5x = -25y$

25 $f(x) = -2\sin x$; $f'(x) = -2\cos x$

$f''(x) = 2\sin x = -f(x)$

26 $y = 2\sin 3x - 5\cos 3x$; $\dfrac{dy}{dx} = 6\cos 3x + 15\sin 3x$

$\dfrac{d^2y}{dx^2} = -18\sin 3x + 45\cos 3x = -9y$

27 $a = -7, b = -24$

28 $f''(2) = \dfrac{3}{4\sqrt{2}} = \dfrac{3\sqrt{2}}{8}$

29 **a** 8 m **b** 38 m **c** 31 m s^{-1}

d 26 m s^{-2}

30 **a** 4 cm

b Maximum 20 cm, minimum 4 cm

c **i** 0 cm s^{-1} **ii** 25.1 cm s^{-1}

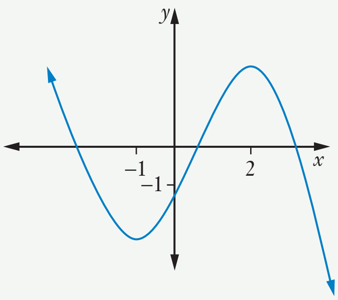
d **i** $-8\pi^2$ or -79 cm s^{-2} **ii** $8\pi^2$ or 79 cm s^{-2}

iii 0 cm s^{-2}

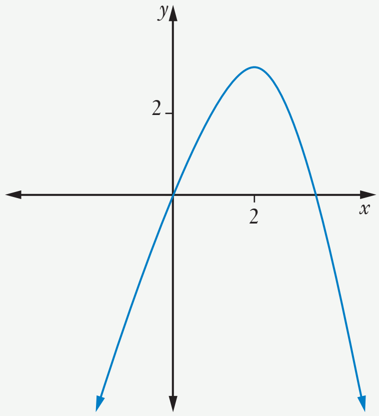
e $\dfrac{d^2h}{dt^2} = -\pi^2(h - 12)$

Exercise 5.06

1 **a**



b



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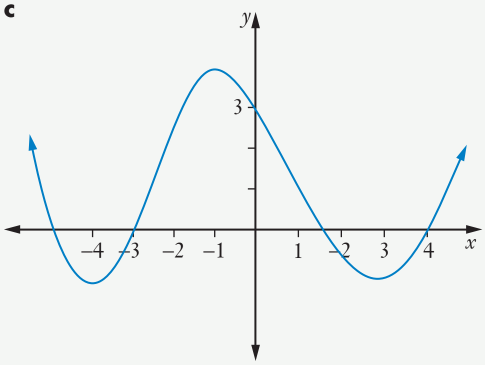

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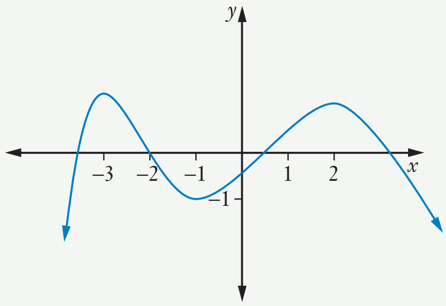

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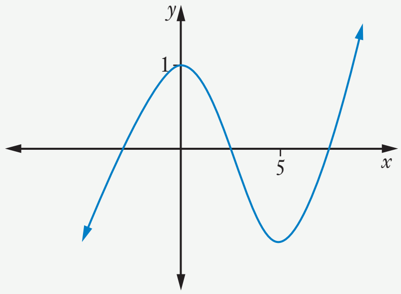
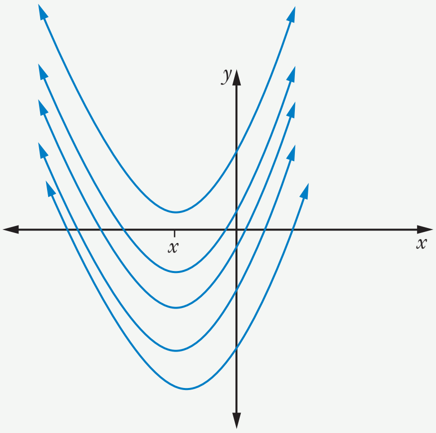


2 a



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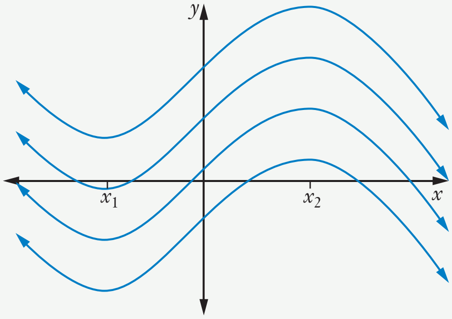

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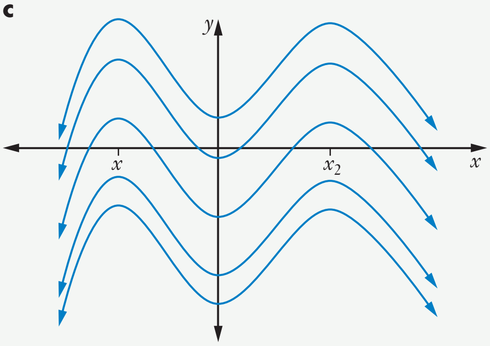

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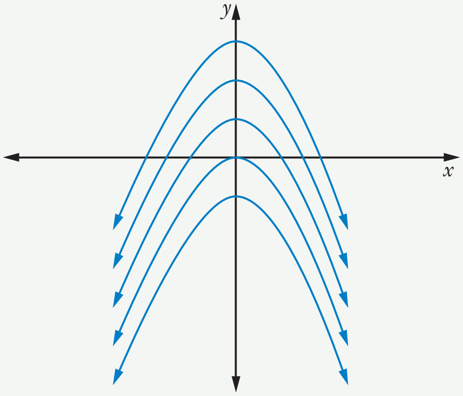

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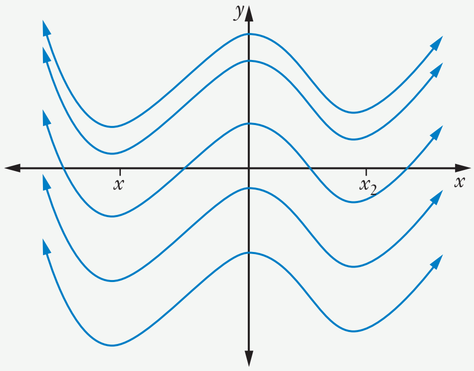

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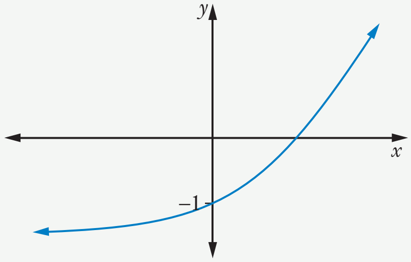

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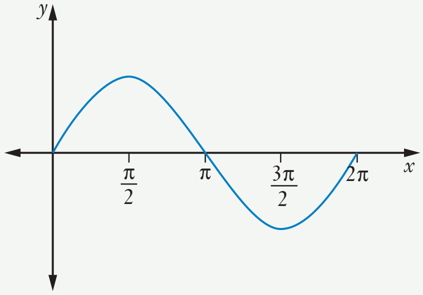

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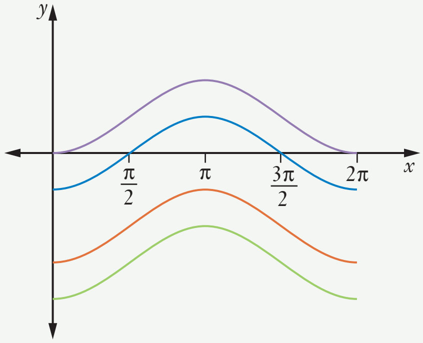


Exercise 5.07

1 a $x^2 - 3x + C$

b $\frac{x^3}{3} + 4x^2 + x + C$

c $\frac{x^6}{6} - x^4 + C$

d $\frac{(x-1)^3}{3} + C$

e $6x + C$

f $\dfrac{(3x+2)^6}{18} + C$

g $\dfrac{4(2x-7)^5}{5} + C$

2 **a** $f(x) = 2x^3 - \dfrac{x^2}{2} + C$

b $f(x) = \dfrac{x^5}{5} - x^3 + 7x + C$

c $f(x) = \dfrac{x^2}{2} - 2x + C$

d $f(x) = \dfrac{x^3}{3} - x^2 - 3x + C$

e $f(x) = \dfrac{2x^{\frac{3}{2}}}{3} + C$

3 **a** $y = x^5 - 9x + C$

b $y = -\dfrac{x^{-3}}{3} + 2x^{-1} + C$

c $y = \dfrac{x^4}{20} - \dfrac{x^3}{3} + C$

d $y = -\dfrac{2}{x} + C$

e $y = \dfrac{x^4}{4} - \dfrac{x^2}{3} + x + C$

4 **a** $\dfrac{2\sqrt{x^3}}{3} + C$

b $-\dfrac{x^{-2}}{2} + C$

c $-\dfrac{1}{7x^7} + C$

d $2x^{\frac{}{2}} + 6x^{\frac{}{3}} + C$

e $-\dfrac{x^{-6}}{6} + 2x^{-1} + C$

5 **a** $\dfrac{(x^2+5)^5}{5} + C$

b $\dfrac{(x^3-1)^{10}}{10} + C$

c $\dfrac{(2x^2+3)^4}{2} + C$

d $\dfrac{3(x^5+1)^7}{7} + C$

e $\dfrac{(x^2-4)^8}{16} + C$

f $\dfrac{(2x^6-7)^9}{108} + C$

g $\dfrac{(x^2-x+3)^5}{5} + C$

h $\dfrac{(x^3+2x^2-7x)^{11}}{11} + C$

i $\dfrac{(x^2-6x-1)^6}{12} + C$

6 $y = \dfrac{x^4}{4} - x^3 + 5x - \dfrac{1}{4}$

7 $f(x) = 2x^2 - 7x + 11$

8 $f(1) = 8$

9 $y = 2x - 3x^2 + 19$

10 $x = 16\frac{}{3}$

11 $y = 4x^2 - 8x + 7$

12 $y = 2x^3 + 3x^2 + x - 2$

13 $f(x) = x^3 - x^2 - x + 5$

14 $f(2) = 20.5$

15 $y = \dfrac{4x^3}{3} - 15x - 14\frac{}{3}$

16 $y = \dfrac{x^3}{3} - 2x^2 + 3x - 4\frac{2}{3}$

17 $f(x) = x^4 - x^3 + 2x^2 + 4x - 2$

18 $y = 3x^2 + 8x + 8$

19 $f(-2) = 77$

20 $x = 3t^2 - 5t - 2$

21 $x = 2t^4 - 2t^3 + 3t^2 - 8t - 3$

Exercise 5.08

1 **a** $-\cos x + C$ **b** $\tan x + C$ **c** $\sin x + C$

d $\dfrac{1}{7}\tan 7x + C$ **e** $-\dfrac{1}{2}\cos(2x - \pi) + C$

2 **a** $e^x + C$ **b** $\dfrac{1}{6}e^{6x} + C$ **c** $\ln|x| + C$

d $\ln|3x-1| + C$ **e** $\dfrac{1}{2}\ln|x^2+5| + C$

3 **a** $e^x + 5x + C$ **b** $\sin x + 2x^2 + C$

c $\dfrac{x^2}{2} + \ln|x| + C$

d $2x^4 - x^3 + 3x^2 - 3x + \ln|x| + C$

e $-\dfrac{1}{5}\cos 5x - \dfrac{1}{9}\tan 9x + C$

4 $y = \sin x - 5$

5 $f(x) = 5\ln|x| + 3$

6 $y = 2\sin 2x + \sqrt{3}$

7 $f(x) = 3e^{3x} - 8e^6x + 14e^6$

8 **a** $P = 25\,000e^{0.054t} + 10\,000$

b 52 900

9 $x = e^{3t} + 4$

10 **a** $\dfrac{dx}{dt} = 3\cos 3t$

b −0.3 cm **c** $0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}, \ldots$ s

Exercise 5.09

1 **a** $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 4 \times \dfrac{1}{4} = 1$

b $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 3x^2 \times \dfrac{1}{3x^2} = 1$

c $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = e^x \times \dfrac{1}{e^x} = 1$

d $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = \dfrac{1}{x} \times x = 1$

e $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 7x^6 \times \dfrac{1}{7x^6} = 1$

2 **a** $\frac{1}{x}$ **b** e^x **c** $2x$

d $\frac{1}{7\sqrt[7]{(x+1)^6}}$ **e** $\frac{1}{3\sqrt[3]{x^2}}$

3 **a** $\frac{1}{3}$ **b** $\frac{1}{2}$ **c** $\frac{1}{3}$

d 4 **e** $-\frac{27}{4}$

4 **a** $\frac{1}{12}\sqrt[3]{\left(\frac{4}{x}\right)^2}$

b **i** $\frac{1}{12}$ **ii** -12

5 **a** $f^{-1}(x) = \sqrt{x-1}$ **b** $\frac{1}{2\sqrt{x-1}}$

c $\frac{1}{4}$

6 **a** $ye^y(2 + y)$ **b** $3(\sin 2y + 2y \cos 2y)$

c $(10y - 3)(2y - 3)^3$

d $\frac{17}{(2y+5)^2}$ **e** $\frac{y+2-y\ln y}{y(y+2)^2}$

7 **a** 47 **b** 1 **c** 0

d 1 **e** $\frac{16}{\pi^3}$

Exercise 5.10

1 **a** $-\frac{1}{\sqrt{1-x^2}}$ **b** $\frac{2}{\sqrt{1-x^2}}$ **c** $\frac{1}{1+x^2}$

d $-\frac{3}{\sqrt{1-9x^2}}$ **e** $\frac{8}{\sqrt{1-4x^2}}$ **f** $\frac{2x}{\sqrt{1-x^4}}$

g $\frac{2}{1+(2x-1)^2} = \frac{1}{2x^2-2x+1}$

h $-\frac{40}{\sqrt{1-64x^2}}$ **i** $-\frac{1}{\sqrt{9-x^2}}$ **j** $\frac{2}{4+x^2}$

2 **a** **i** -1 **ii** 1

b **i** $1\frac{3}{5}$ **ii** $-\frac{5}{8}$

c **i** $\frac{\pi^2}{6\sqrt{3}}$ **ii** $-\frac{6\sqrt{3}}{\pi^2}$

d **i** $-\frac{1}{3}$ **ii** 3

e **i** $\frac{1}{5}$ **ii** -5

3 $y = 2x$

4 $40x + 100y - 25\pi - 8 = 0$

5 **a** $\frac{3}{\sqrt{36-x^2}}$ **b** $-\frac{3}{2\sqrt{x(1-x^2)}}$

c $-\frac{1}{\sqrt{49-x^2}}$ **d** $\frac{15}{\sqrt{-9x^2-12x-3}}$

e $-\frac{x}{\sqrt{1-x^2}} + \cos^{-1} x$ **f** $\frac{5}{1+x^2}(\tan^{-1} x + 1)^4$

6 **a** -1 **b** 1

c $\frac{1}{x\sqrt{1-(\ln x)^2}}$ **d** $\frac{e^x}{1+e^{2x}}$

e $\frac{1}{\sin^{-1} x \cdot \sqrt{1-x^2}}$ **f** $-\frac{1}{(1+x^2)(\tan^{-1} x)^2}$

g $-\frac{1}{\sqrt{1-x^2}\,[1+(\cos^{-1} x+1)^2]}$ **h** $-\frac{1}{1+x^2}$

i $\frac{1}{\sqrt{-x^2-4x}}$ **j** $-\frac{e^{\cos^{-} x}}{\sqrt{1-x^2}}$

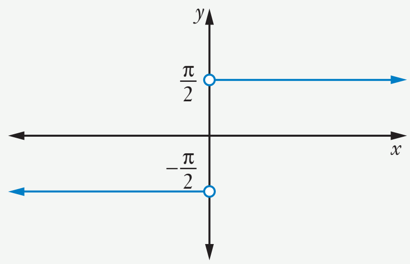
7 $\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$

8 **a** $-\frac{x}{\sqrt{(9-x^2)^3}}$ **b** $\frac{-(1+2x\tan^{-} x)}{(1+x^2)(\tan^{-1} x)^2}$

9 $12x - 6\sqrt{3}y - \sqrt{3}\pi + 6 = 0$

10 **a** 0

b



11 **a** $-\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$ **b** $\frac{1}{(1+x^2)\tan^{-1} x}$

c $\frac{1}{x[1+(\text{ n}\,x)^2]}$ **d** $-\frac{1}{\sqrt{1-x^2}}$

e $\frac{e^{\text{an}^{-} x}}{1+x^2}$

12 **a** $\sin\theta = \frac{h}{6}, \therefore \theta = \sin^{-1}\frac{h}{6}$

b –0.009 radians per second

13 **a** $2t$ m **b** $\theta = \tan^{-1}\left(\frac{t}{50}\right)$

c **i** 0.02 radians s^{-1} **ii** 0.0082 radians s^{-1}

14 **a** $\theta = \tan^{-1}\left(\frac{5+05t}{8}\right)$

b **i** 0033 radians min^{-1} **ii** 0014 radians min^{-1}

iii 0.0031 radians min^{-1}

Test yourself 5

1 D **2** B **3** A **4** C

5 **a** $5e^{5x}$ **b** $-2e^{1-x}$ **c** $\frac{1}{x}$

d $\frac{4}{4x+5}$ **e** $e^x(x+1)$ **f** $\frac{1-\ln x}{x^2}$

g $10e^x(e^x+1)^9$

6 **a** $-\sin x$ **b** $2\cos x$ **c** $\sec^2 x$

d $x\cos x + \sin x$ **e** $\frac{x\sec^2 x - \tan x}{x^2}$

f $-3\sin 3x$ **g** $5\sec^2 5x$

7 $3x - y + 3 = 0$

8 $12x + 4\sqrt{2}y - 4 - 3\pi = 0$

9 $\frac{d^2x}{dt^2} = -4\cos 2t$

10 **a** $\frac{1}{\sqrt{1-x^2}}$ **b** $\frac{3}{1+9x^2}$

c $-\frac{10}{\sqrt{1-25x^2}}$

11 $-\frac{e^2}{e^2+1}$

12 **a** $2x^5 - x^4 + 3x^2 - 3x + C$

b $\frac{1}{5}e^{5x} + C$

c $\frac{1}{9}\tan 9x + C$

d $\ln|x+5| + C$

e $\frac{1}{2}\sin 2x + C$

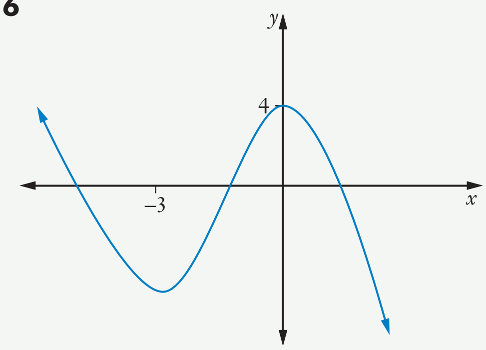
f $-4\cos\frac{x}{4} + C$

13 $-3\sqrt{3}$

14 $y = 2x^3 + 6x^2 - 5x - 33$

15 $\frac{1}{5\sqrt[5]{(x-3)^4}}$

16



17 $2x + y - \ln 2 - 4 = 0$

18 $4x + 8y - 8 - \pi = 0$

19 **a** $\frac{1}{\sqrt{1-x^2}}$ **b** $-\frac{1}{\sqrt{25-x^2}}$ **c** $\frac{1}{1+x^2}$

d $\frac{4}{\sqrt{1-16x^2}}$ **e** $\frac{2}{4+x^2}$

20 **a** $40x(5x^2+7)^3$

b $4(16x-3)(2x-3)^6$

c $\frac{23}{(3x+4)^2}$ **d** $2x^2e^x(x+3)$

e $\frac{3(x+1)\sec^2 3x - \tan 3x}{(x+1)^2}$

21 **a** 0

b $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}; \frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$

22 $f(x) = \frac{5x^3}{2} + 6x^2 - 49x + 59$

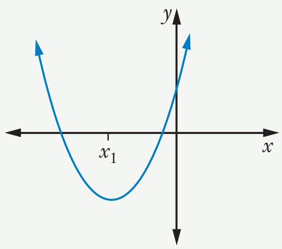
23 **a** –8 **b** 26 **c** –90

24 **a** $f^{-1}(x) = x^2 - 1$ **b** $P = (3, 8)$

c $6x - y - 10 = 0$

25 $\frac{x}{1+x^2} + \tan^{-1}x$

26 **a**



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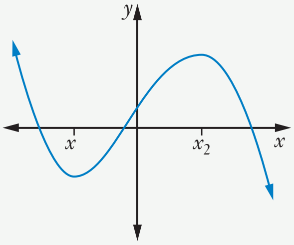

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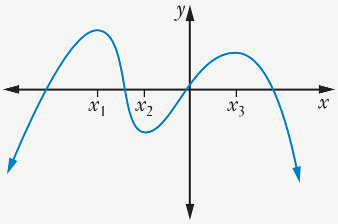


27 $f(x) = 2x^3 - 3x^2 - 31x + 68$

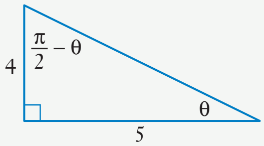
28 $4x - 6\sqrt{3}y + \sqrt{3}\pi - 6 = 0$

29 **a** $\frac{(3x^4 - 5)^7}{84} + C$ **b** $\frac{3(x^2+1)^{10}}{20} + C$

Challenge exercise 5

1 $2e$

2 **a**



b 0

c $\theta + \frac{\pi}{2} - \theta$

3 $\frac{20x^2 - 120x - 1}{(4x^2+1)^4}; \frac{-24(20x^3 - 140x^2 - 3x + 5)}{(4x^2+1)^5}$

4 **a** $e^x + C$ **b** $-\frac{1}{3}\cos(x^3) + C$

5 $(2x\cos 2x + \sin 2x)e^{x\sin 2x}$

6 $y = \frac{x^3}{3} - x^2 - 15x - 1$

7 **a** $\frac{2x}{\sqrt{1-x^4}}$ **b** $\frac{e^x}{1+e^{2x}}$

c $\frac{\cos x - \sin x}{\sin x + \cos x}$

8 $25\frac{5}{6}$

9 $\frac{1 + \ln x - x\ln x}{e^x}$

10 **a** $\theta = \tan^{-1}(4t)$ **b** 3° 32′

11 **a** $x = y + e^y$ **b** $P = (1 + e, 1)$

c $x - (1 + e)y = 0$

12 **a** $\frac{\sec^2 x}{\tan x}$ or $\frac{1}{\sin x \cos x}$ **b** $-\ln|\cos x| + C$

13 **a** $-\frac{\cos(x^3 - \pi)}{3} + C$ **b** $\frac{e^x}{2} + C$

Chapter 6

Exercise 6.01

1 $x < 2$ **2** $x < \frac{1}{4}$

3 $(-\infty, 0)$

4 **a** $x < 1.5$ **b** $x > 1.5$ **c** $x = 1.5$

5 $f'(x) = -2 < 0$ for all x

6 $y' = 3x^2 > 0$ for all $x \neq 0$

7 (0, 0)

8 $x = -3, 2$

9 **a** (1, –4) **b** (0, 9)

c (1, 1) and (2, 0) **d** (0, 1), (1, 0) and (–1, 0)

10 (2, 0)

11 $-1 < x < 1$

12 $(-\infty, -5) \cup (-3, \infty)$

13 **a** $x = 2, 5$ **b** $2 < x < 5$ **c** $x < 2, x > 5$

14 $p = -12$

15 $a = 1\frac{1}{2}, b = -6$

16 **a** $\frac{dy}{dx} = 3x^2 - 6x + 27$

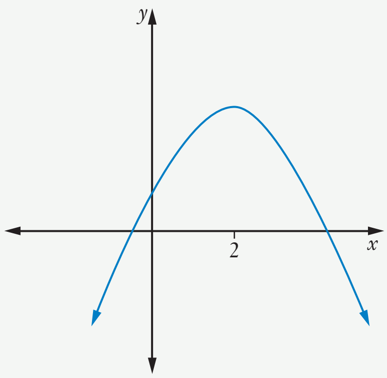
b The quadratic function has $a > 0$

$b^2 - 4ac = -288 < 0$

So $3x^2 - 6x + 27 > 0$ for all x.

The function is monotonic increasing for all x.

17



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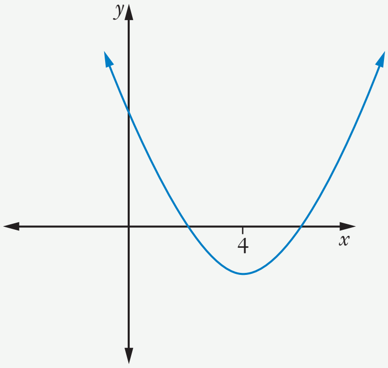

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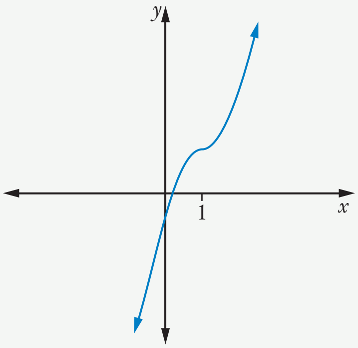

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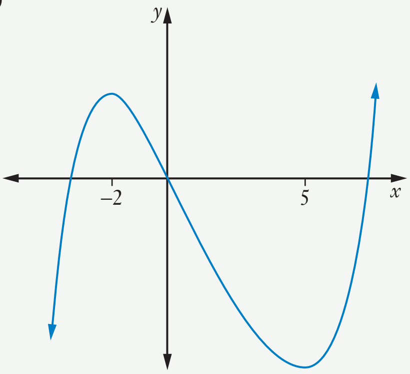

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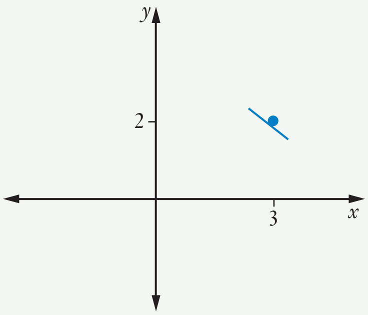

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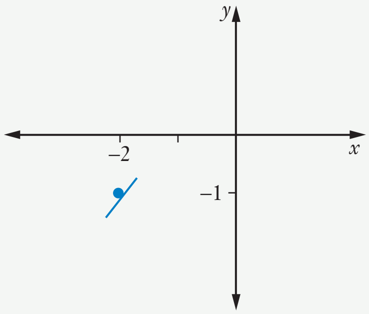


23 (2, 0) and $\left(\frac{2}{3}\ 3\frac{13}{81}\right)$

24 $\frac{3x+2}{2\sqrt{x+1}}$; $\left(-\frac{2}{3}\ -\frac{2\sqrt{3}}{9}\right)$ **25** $a = -1.75$

26 $y' = \frac{1}{2\sqrt{x}} \neq 0$ **27** $y' = -\frac{3}{x^4} \neq 0$

Exercise 6.02

1 (0, –1); show $y' < 0$ on LHS, $y' > 0$ on RHS

2 (0, 0), minimum

3 (–2, 11); show $f'(x) > 0$ on LHS and $f'(x) < 0$ on RHS

4 (–1, –2), minimum **5** (4, 0) minimum

6 (0, 5) maximum, (4, –27) minimum

7 (0, 5) maximum, (2, 1) minimum

8 (0, –3) maximum, (1, –4) minimum, (–1, –4) minimum

9 (1, 0) minimum, (–1, 4) maximum

10 $m = -3$ **11** $x = -3$ minimum

12 $x = 0$ minimum, $x = -1$ maximum

13 **a** $\frac{dP}{dx} = 2 - \frac{50}{x^2}$

b (–5, –20) maximum, (5, 20) minimum

14 $\left(1\ \frac{1}{2}\right)$ minimum

15 (2.06, 54.94) maximum, (–2.06, –54.94) minimum

16 (4.37, 54.92) minimum, (–4.37, –54.92) maximum

17 **a** $\frac{3600 - 2x^2}{\sqrt{3600 - x^2}}$

b (42.4, 1800) maximum, (– 42.4, –1800) minimum

Exercise 6.03

1 $x > -\frac{1}{3}$

2 $x < 3$

3 $y'' = -8 < 0$

4 $y'' = 2 > 0$

5 $\left(-\infty\, 2\ \frac{}{3}\right)$

6 (1, 9)

7 (1, –17) and (–1, –41)

8 (0, –2): $y'' < 0$ on LHS, $y'' > 0$ on RHS

9 $-2 < x < 1$

10 **a** No

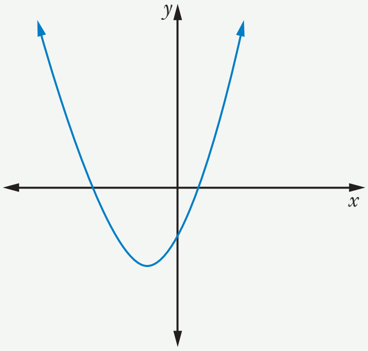
b Yes – point of inflection at (0, 0)

c Yes –point of inflection at (0, 0)

d Yes – point of inflection at (0, 0)

e No

11



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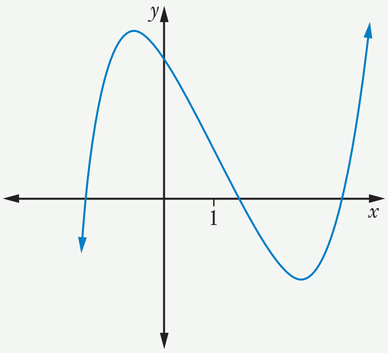


13 None: (2, 31) is not a point of inflection since concavity does not change.

14 Show that $\frac{12}{x^4} > 0$ for all $x \neq 0$.

15 **a** (0, 7), (1, 0) and (–1, 14) **b** (0, 7)

16 **a** $12x^2 + 24 \neq 0$ and there are no points of inflection.

b The curve is always concave upwards.

17 $a = 2$ 18 $p = 4$ 19 $a = 3, b = -3$

20 **a** (0, –8), (2, 2)

b $\frac{dy}{dx} = 6x^5 - 15x^4 + 21$

At (0, –8):

$\frac{dy}{dx} \neq 0$

At (2, 2):

$\frac{dy}{dx} \neq 0$

Exercise 6.04

1 **a** $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} > 0$ **b** $\frac{dy}{dx} < 0, \frac{d^2y}{dx^2} < 0$

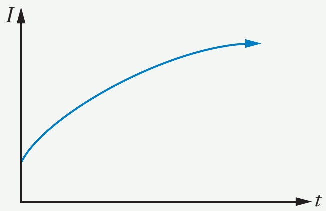
c $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} < 0$ **d** $\frac{dy}{dx} < 0, \frac{d^2y}{dx^2} > 0$

e $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} > 0$

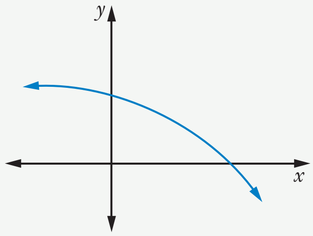
2 **a** $\frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$

b No, the rate is decreasing.

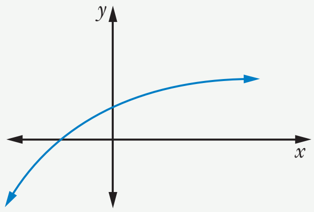
3



4 **a**



b



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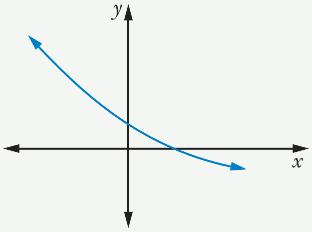

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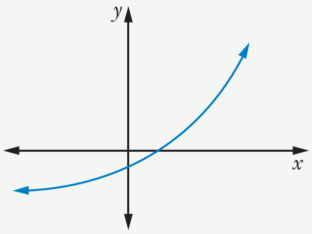

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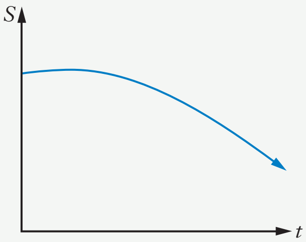

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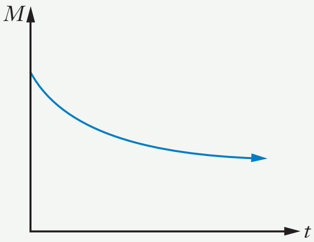
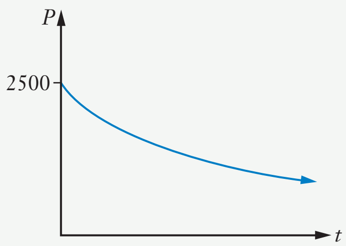


7 $\dfrac{dM}{dt} < 0, \dfrac{d^2M}{dt^2} > 0$

8 **a** The number of fish is decreasing.

b The population rate is increasing.

c



9 The level of education is increasing, but the rate of increase is slowing down.

10 The population is decreasing, and the rate of change in population is decreasing.

Exercise 6.05

1 (1, 0), minimum

2 (0, 1), minimum (flat)

3 $(2, -5)$; $y'' = 6 > 0$ so minimum

4 $(0.5, 0.25)$; $y'' = -2 < 0$ so maximum

5 $(0, -5)$; $f''(x) < 0$ on LHS, $f''(x) > 0$ on RHS

6 Yes – point of inflection at (0, 3)

7 $(-2, -78)$ minimum, $(-3, -77)$ maximum

8 (0, 1) maximum, $(-1, -4)$ minimum, $(2, -31)$ minimum

9 (0, 1) maximum, (0.5, 0) minimum, $(-0.5, 0)$ minimum

10 **a** (4, 176) maximum, (5, 175) minimum

b (4.5, 175.5)

11 (3.67, 0.38), maximum

12 $(0, -1)$ minimum, $(-2, 15)$ maximum, $(-4, -1)$ minimum

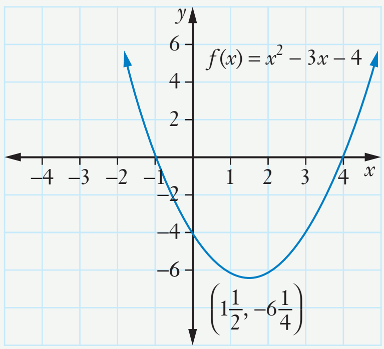
13 **a** $a = 4$ **b** $\left(\frac{1}{2}\ 0\right)$, minimum

14 $m = -4$

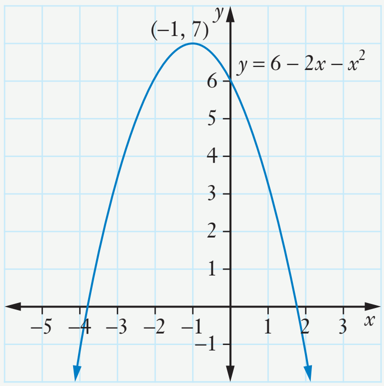
15 $a = 3, b = -9$

Exercise 6.06

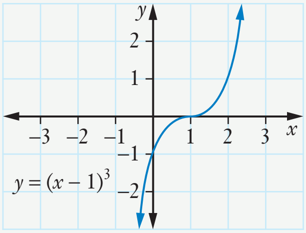
1



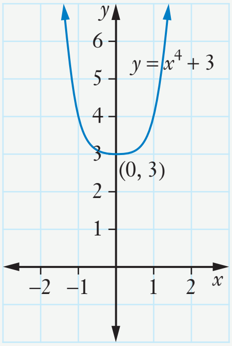
2



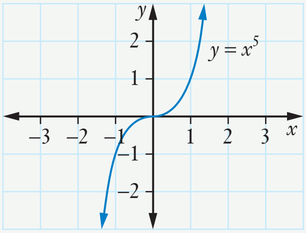
3 (1, 0) point of inflection



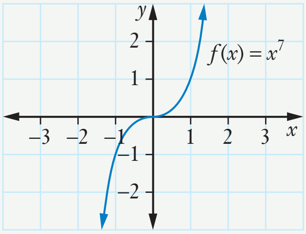
4



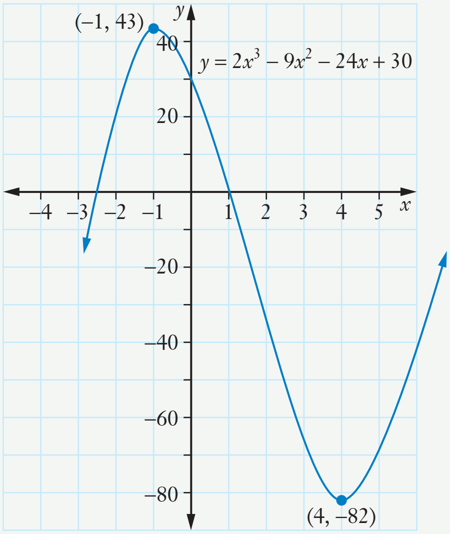
5 (0, 0)



6



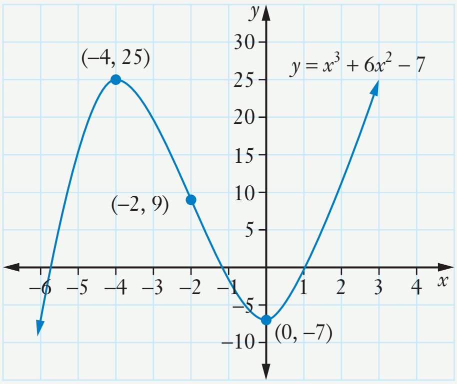
7 (–1, 43) maximum, (4, –82) minimum



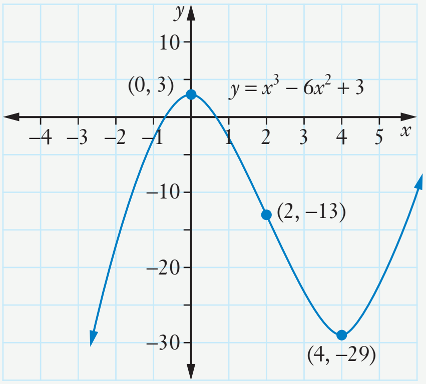
8 **a** (0, –7) minimum, (–4, 25) maximum

b (–2, 9)

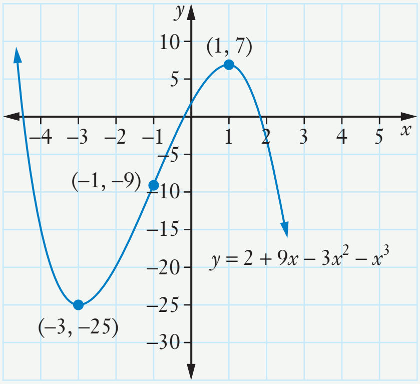
c



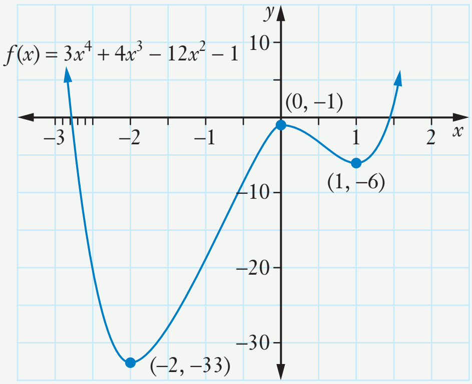
9 (0, 3) maximum, (2, –13) point of inflection, (4, –29) minimum



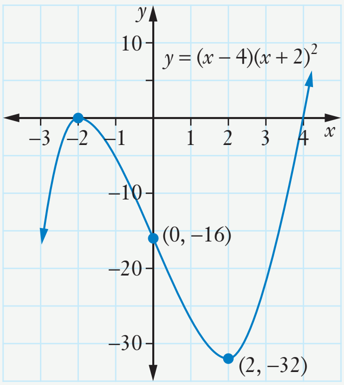
10 (–3, –25) minimum, (–1, –9) point of inflection, (1, 7) maximum



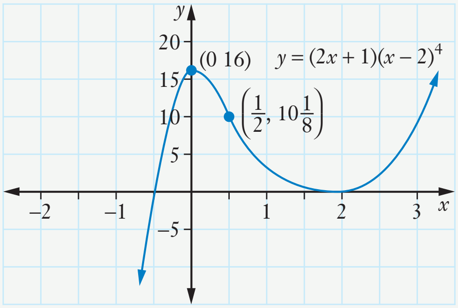
11 (–2, –33) minimum, (0, –1) maximum, (1, –6) minimum



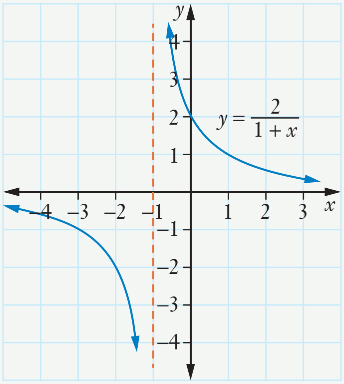
12 (–2, 0) maximum, (0, –16) point of inflection, (2, –32) minimum



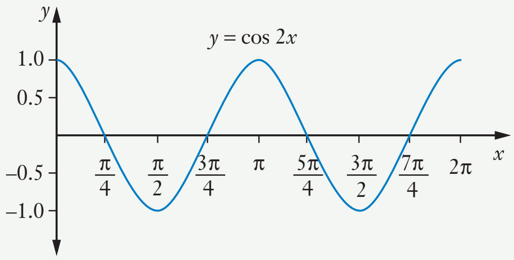
13 (0, 16) maximum, $\left(\frac{1}{2}, 10\frac{1}{8}\right)$ point of inflection, (2, 0) minimum



14 $\frac{dy}{dx} = -\frac{2}{(1+x)^2} \neq 0$ for any x



15 a



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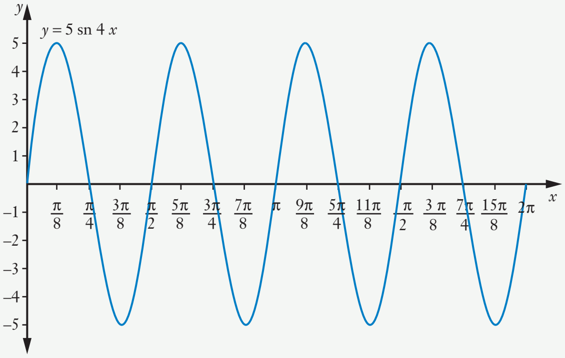
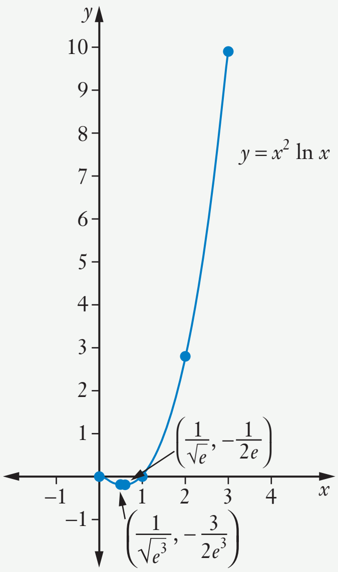
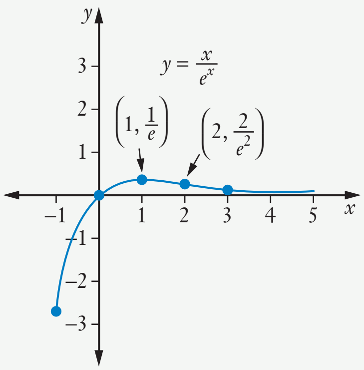


16 a



b



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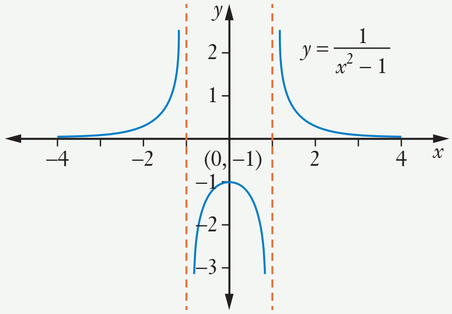
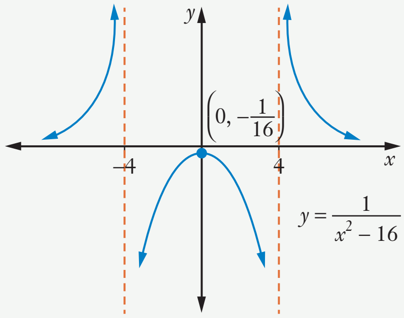


17 a



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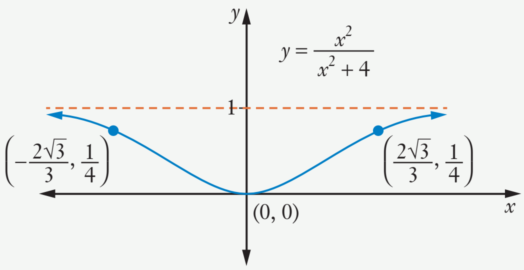

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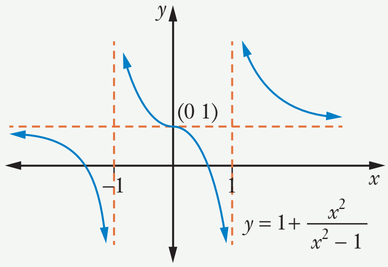
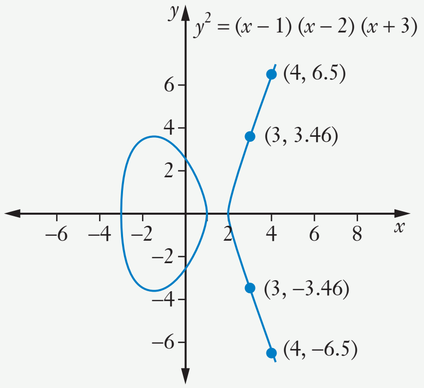
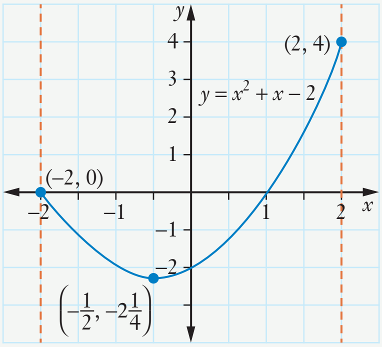


d

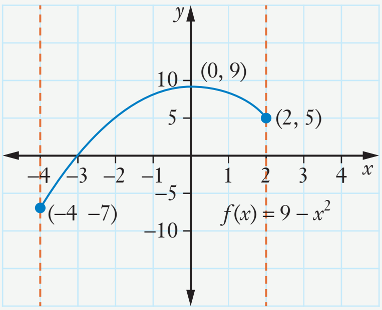


Exercise 6.07

1 Maximum value is 4.

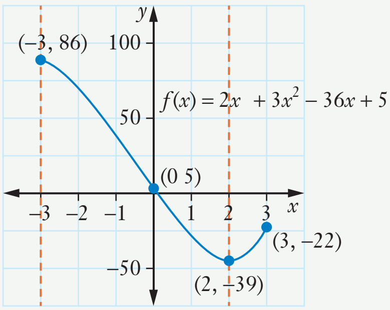


2 Maximum value is 9, minimum value is –7.



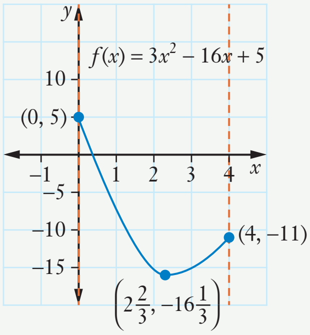
3 Maximum value is 25.

4 Maximum value is 86, minimum value is –39.



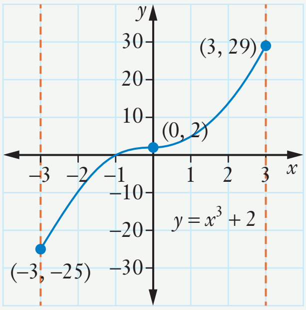
5 Maximum value is –2.

6 Maximum value is 5, minimum value is $-16\frac{1}{3}$.

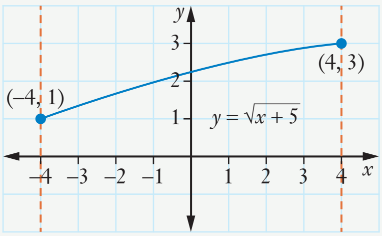


7 Global maximum 29, local maximum –3, global minimum –35, local minimum –35, –8

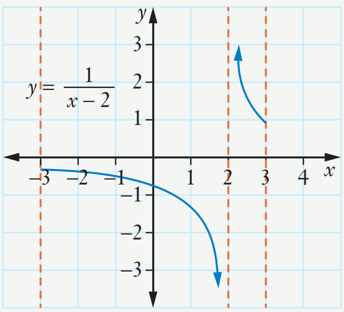
8 Minimum –25, maximum 29



9 Maximum 3, minimum 1



10 Maximum ∞, minimum –∞



Investigation

The disc has radius $\frac{30}{7}$ cm. (This result uses Stewart's theorem – research this.)

Exercise 6.08

See worked solutions for full proofs.

1 $\frac{50}{x} = y$

$P = 2x + 2y$

2 $y = 60 - x$

$A = xy$

3 $\frac{20}{x} = y$

$S = x + y$

4 $\frac{400}{\pi r^2} = h$

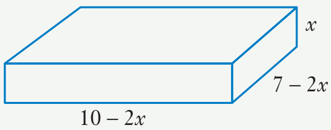
$S = 2\pi r^2 + 2\pi rh$

5 **a** $x + y = 30$

b $A = \left(\frac{1}{4}x\right)^2 + \left(\frac{1}{4}y\right)^2$

6 **a** $x^2 + y^2 = 280^2 = 78\ 400$ **b** $A = xy$

7

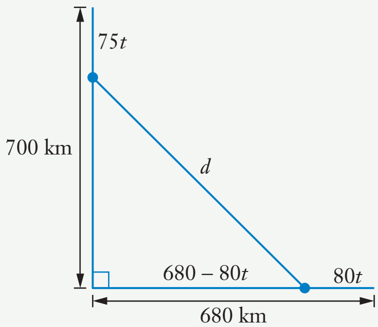


$V = x(10 - 2x)(7 - 2x)$

8 Profit per person = Cost – Expenses

$= (900 - 100x) - (200 + 400x)$

9



$d^2 = (700 - 75t)^2 + (680 - 80t)^2$

10 Distance *AB*: $d = \sqrt{x^2 + 0\ 5^2}$

$t = \frac{\sqrt{x^2 + 025}}{5}$

Distance *BC*: $d = 7 - x$

$t = \frac{7 - x}{4}$

Exercise 6.09

See worked solutions for full proofs.

1 2 s, 16 m **2** 7.5 km

3 **a** $y = 30 - x$

b Maximum area is 225 m^2

4 **a** $\frac{4000}{x} = y$

$P = 2x + 2y$

b 63.2 m by 63.2 m **c** $12 322.88

5 4 m by 4 m **6** 14 and 14

7 –2.5 and 2.5

8 $x = 1.25$ m, $y = 1.25$ m

9 **a** $V = x(30 - 2x)(80 - 2x)$

b $x = 6\frac{2}{3}$ cm **c** 7407.4 cm^3

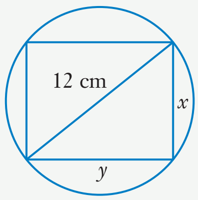
10 **a** $\frac{54}{r^2} = h$

$S = 2\pi r(r + h)$

b Radius is 3 m.

11 **a** $S = 2\pi r^2 + \frac{17\ 200}{r}$ **b** 2324 m^2

12 **a**



$x^2 + y^2 = 12^2$

$A = xy = x\sqrt{144 - x^2}$

b 72 cm^2

13 **a** $\frac{400}{x} = y$

$A = (x - 10)(y - 10)$

b 100 cm^2

14 1.12 m^2

15 **a** 7.5 m by 7.5 m **b** 2.4 m

16 160.17 cm^2

17 1.68 m, 1.32 m

18 **a** $d^2 = (200 - 80t)^2 + (120 - 60t)^2$

b Minimum distance 24 km

19 **a** $d = (x^2 - 2x + 5) - (4x - x^2)$

$= 2x^2 - 6x + 5$

b 0.5

20 a $s = \frac{d}{t}$

So $t = \frac{d}{s}$

$= \frac{1500}{s}$

Cost of trip taking t hours:

$C = (s^2 + 9000)t$

$= (s^2 + 9000)\,\frac{1500}{s}$

$= 1500\left(s + \frac{9000}{s}\right)$

b 95 km h^{-1} **c** $2846

Test yourself 6

1 A **2** C **3** D **4** C

5 (–3, –11) maximum, (–1, –15) minimum

6 $x > 1\frac{}{6}$

7 50 m

8 $x > -1$

9 $(\frac{1}{2}, -1)$

10 a $\frac{375}{\pi r^2} = h$

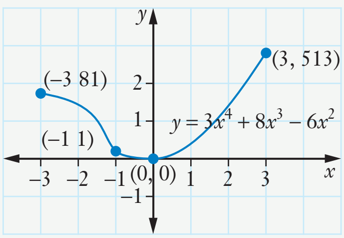
$S = 2\pi r^2 + 2\pi rh$

b 3.9 cm

11 a (0, 0) and (–1, 1)

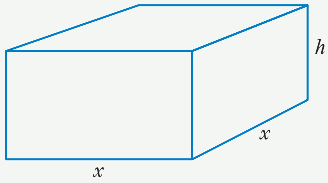
b (0, 0) minimum, (–1, 1) horizontal point of inflection

c



d Maximum value 513, minimum value 0

12 a



$\frac{125 - x^2}{2x} = h$

$V = x^2h$

b 6.45 cm by 6.45 cm by 6.45 cm

13 150 products

14 a $y = \sqrt{25 - x^2}$

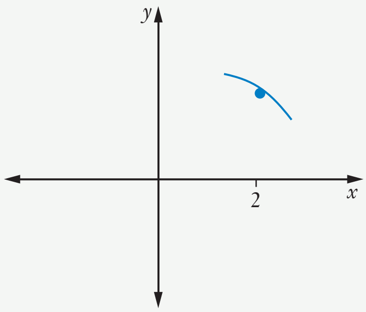
$A = \frac{1}{2}xy$

b 6.25 m^2

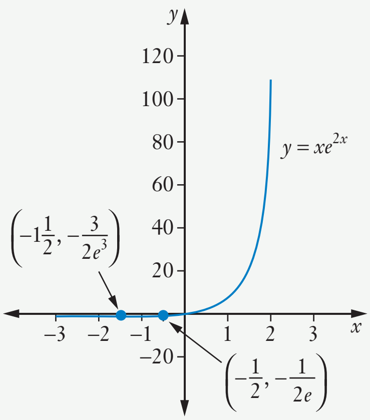
15 (0, 1) and (3, –74)

16 179

17



18



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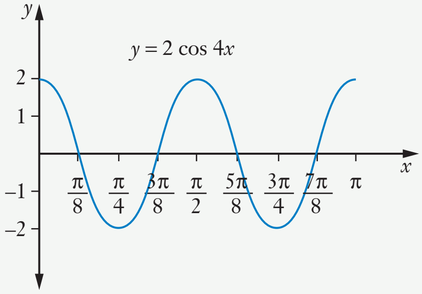
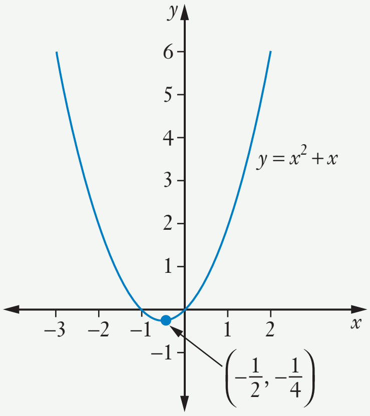
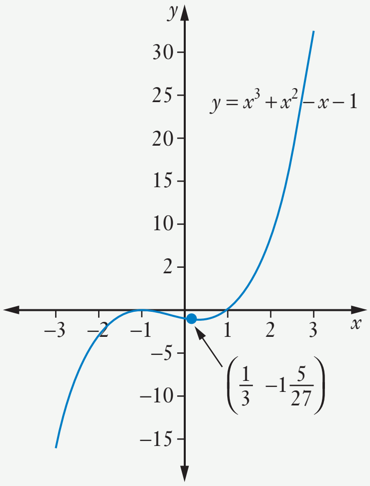


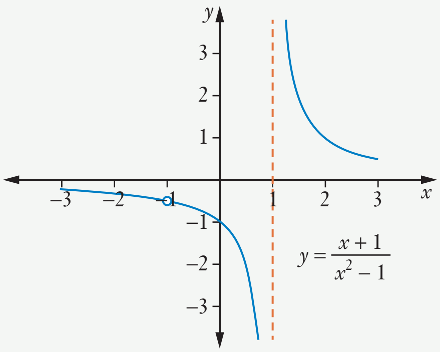
20 a



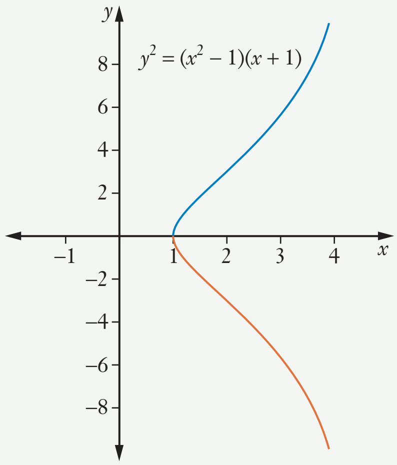
b



c

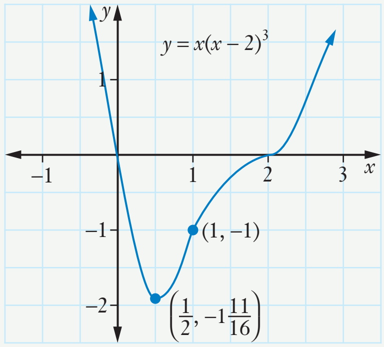


d



Challenge exercise 6

1



2 $x < -\frac{1}{2}, x > 4$

3 16 m^2

4 27; –20.25

5 $f'(0.6) = f'(0.6) = 0$ and concavity changes

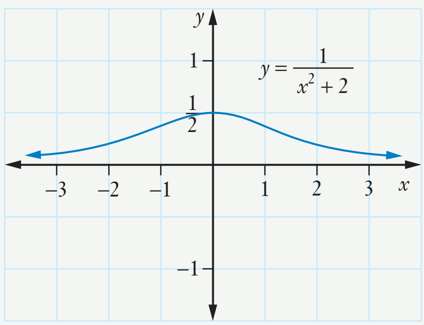
6 Proof. See Worked Solutions. $r = s = 12.5$

7 $y = x^2 + 2x + 3$

8 a $\left(0\ \frac{1}{2}\right)$, maximum

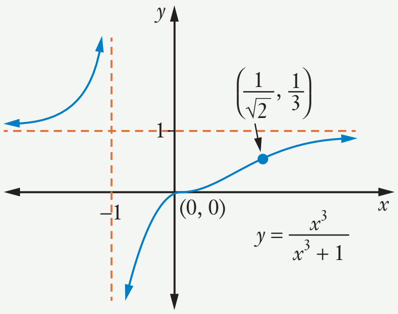
b Domain: $(-\infty, \infty)$, range: $\left(0\ \frac{1}{2}\right]$

c $0^+, 0^+$



9 **a** $y' = 0$ at $(0, 0)$
b $y'' > 0$ on LHS and RHS
c $y'' < 0$ on LHS, $y'' > 0$ on RHS

10



11 Minimum -1, maximum $-\frac{1}{5}$

12 87 km h^{-1}

Chapter 7

Exercise 7.01

1 **a** 4.125 units^2 **b** 6.625 units^2

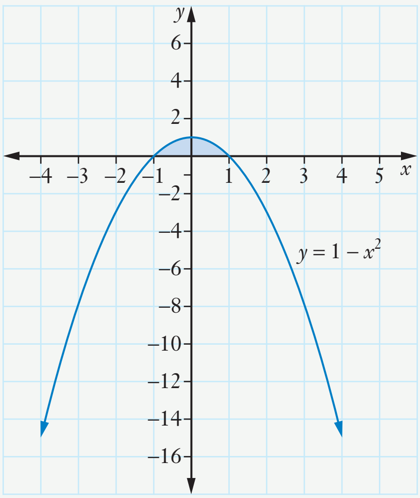
2 **a** 1.17 units^2 **b** 1.67 units^2 **c** 1.27 units^2
d 1.52 units^2

3 **a** 6.5 units^2 **b** 5 units^2 **c** 132 units^2
d $\frac{\pi\left(1+\sqrt{2}\right)}{8\sqrt{2}} = \frac{\pi}{16}\left(2+\sqrt{2}\right)\text{ units}^2$ **e** 6.5 units^2

4 **a** 28 units^2 **b** 156 units^2 **c** 140 units^2

5 **a** $3\frac{3}{7}\text{ units}^2$ **b** 5.5 units^2 **c** $\frac{\pi}{4}\text{ units}^2$
d $\frac{3}{2}(e + e^4)$ **e** 23 units^2

6 **a**



b 1 unit^2

7 **a** 4.41 units^2 **b** 4.91 units^2
c 4.5 units^2 **d** 4.5 units^2

8 $\frac{25\pi}{2}\text{ units}^2$

9 **a** $\frac{9\pi}{2}\text{ units}^2$
b **i** 2.5 units^2 **ii** 8.1 units^2

10 **a** 22.4 units^2 **b** 3.3 units^2 **c** 1 unit^2

11 **a** **i** 6 units^2 **ii** 14 units^2
b **i** $\frac{\pi\sqrt{2}}{4}\text{ units}^2$ **ii** $\frac{\pi\left(\sqrt{2}+2\right)}{4}\text{ units}^2$

12 **a** 60.625 units^2 **b** 73.125 units^2

Exercise 7.02

1 **a** 2.5 **b** 10
c 2.4 **d** 0.225

2 **a** 28 **b** 22

3 **a** 0.39 **b** 0.41

4 **a** 1.08 **b** 0.75 **c** 0.65
d 0.94 **e** 0.92

5 **a** 75.1 **b** 16.5 **c** 650.2

6 **a** 28.9 m^2 **b** 39.25 m^2 **c** 7.45 km^2
d 492.25 m^2

Exercise 7.03

1 **a** 8 **b** 10 **c** 217
d -1 **e** 10 **f** 54
g $3\frac{1}{3}$ **h** 16 **i** 50

2 **a** $\frac{2}{3}$ **b** $21\frac{1}{4}$ **c** 0
d $4\frac{2}{3}$ **e** $1\frac{1}{4}$ **f** $4\frac{1}{3}$
g 0 **h** $2\frac{1}{3}$ **i** 0
j $6\frac{2}{9}$ **k** 100 **l** 54
m $15\frac{5}{6}$ **n** $22\frac{2}{3}$ **o** 0.0126

3 **a** 70 m **b** 38 km **c** 258 cm

d $72\frac{2}{3}$ m **e** 142 cm

4 **a** 750 L **b** 51 000 L **c** 810 750

Exercise 7.04

1 **a** $\frac{x^3}{3}+C$ **b** $\frac{x^6}{2}+C$ **c** $\frac{2x^5}{5}+C$

d $\frac{m^2}{2}+m+C$ **e** $\frac{t^3}{3}-7t+C$ **f** $\frac{h^8}{8}+5h+C$

g $\frac{y^2}{2}-3y+C$ **h** x^2+4x+C $\frac{b^3}{3}+\frac{b^2}{2}+C$

2 **a** $\frac{x^3}{3}+x^2+5x+C$

b $x^4-x^3+4x^2-x+C$

c $x^6+\frac{x^5}{5}+\frac{x^4}{2}+C$

d $\frac{x^8}{8}-\frac{3x^7}{7}-9x+C$

e $\frac{x^4}{2}+\frac{x^3}{3}-\frac{x^2}{2}-2x+C$

f $\frac{x^6}{6}+\frac{x^4}{4}+4x+C$

g $\frac{4x^3}{3}-\frac{5x^2}{2}-8x+C$

h $\frac{3x^5}{5}-\frac{x^4}{2}+\frac{x^2}{2}+C$

i $\frac{3x^4}{2}+\frac{5x^3}{3}-4x+C$

j $-x^{-3}-\frac{x^{-2}}{2}-2x^{-1}+C$

3 **a** $-\frac{1}{7x^7}+C$ **b** $\frac{3x^{\frac{4}{3}}}{4}+C$

c $\frac{x^4}{4}-x^3+x^2+C$ **d** $x-2x^2+\frac{4x^3}{3}+C$

e $\frac{x^3}{3}+\frac{3x^2}{2}-10x+C$ **f** $-\frac{3}{x}+C$

g $-\frac{1}{2x^2}+C$

h $-\frac{4}{x}-x+\frac{3}{2x^2}-\frac{7}{4x^4}+C$

i $\frac{y^3}{3}+\frac{y^{-6}}{6}+5y+C$

j $\frac{t^4}{4}-\frac{t^3}{3}-2t^2+4t+C$

k $\frac{2\sqrt{x^3}}{3}+C$ **l** $-\frac{1}{2t^4}+C$ **m** $\frac{3\sqrt[3]{x^4}}{4}+C$

n $\frac{2\sqrt{x^5}}{5}+C$ **o** $\frac{2\sqrt{x^3}}{3}+x+C$

4 900° **5** $y=x^3-x^2+x+6$

6 $-11\ 079.25$ cm s^{-1} **7** 21 000 L

Exercise 7.05

1 **a** $\frac{(3x-4)^3}{9}+C$ **b** $\frac{(x+1)^5}{5}+C$

c $\frac{(5x-1)^{10}}{50}+C$ **d** $\frac{(3y-2)^8}{24}+C$

e $\frac{(4+3x)^5}{15}+C$ **f** $\frac{(7x+8)^{13}}{91}+C$

g $-\frac{(1-x)^7}{7}+C$ **h** $\frac{\sqrt{(2x-5)^3}}{3}+C$

i $-\frac{2(3x+1)^{-3}}{9}+C$ **j** $-3(x+7)^{-1}+C$

k $-\frac{1}{16(4x-5)^2}+C$ **l** $\frac{3\sqrt[3]{(4x+3)^4}}{16}+C$

m $-2(2-x)^{\overline{2}}+C$ **n** $\frac{2\sqrt{(t+3)^5}}{5}+C$

o $\frac{2\sqrt{(5x+2)^7}}{35}+C$

2 **a** 288.2 **b** $-1\frac{1}{4}$ **c** $-\frac{1}{8}$

d $60\frac{2}{3}$ **e** $\frac{1}{6}$ **f** $\frac{1}{7}$

g $4\frac{2}{3}$ **h** $1\frac{1}{5}$ **i** $\frac{3}{5}$

3 **a** $\frac{1}{3}(x^4+5)^3+C$ **b** $\frac{1}{6}(x^2-3)^6+C$

c $\frac{1}{4}(x^3+1)^4+C$ **d** $\frac{1}{5}(x^2+3x-2)^5+C$

e $\frac{1}{42}(3x^2-7)^7+C$ **f** $-\frac{1}{45}(4-5x^3)^3+C$

g $\frac{1}{15}(2x^6-3)^5+C$ **h** $\frac{3}{80}(5x^2+3)^8+C$

i $\frac{1}{12}(x^2+4x)^6+C$ **j** $\frac{1}{12}(3x^3-6x-2)^4+C$

4 **a** $108\frac{2}{3}$ **b** $-\frac{1}{18}$

c 66 812.75 **d** $10\ 159\frac{1}{32}$

e 564 537.6 **f** $-1236\frac{2}{3}$

g $-19\ 839\frac{3}{14}$ **h** $96\frac{4}{9}$

i −474 618 565.3

5 $y=\frac{1}{15}((x^3-2)^5+61)$

6 **a** $x=\frac{1}{10}(x^2-3)^5-1]$ **b** 777.5 m

Exercise 7.06

1 **a** $\frac{1}{4}e^{4x}+C$ **b** $-e^{-x}+C$ **c** $\frac{1}{5}e^{5x}+C$

d $-\frac{1}{2}e^{-2x}+C$ **e** $\frac{1}{4}e^{4x+1}+C$ **f** $-\frac{3}{5}e^{5x}+C$

g $\frac{1}{2}e^{2t}+C$ **h** $\frac{1}{7}e^{7x}-2x+C$

$e^{x-3}+\frac{x^2}{2}+C$

2 **a** $\frac{1}{5}(e^5-1)$ **b** $e^{-2}-1=\frac{1}{e^2}-1$

c $\frac{2e^7}{3}(e^9-1)$ **d** $19-\frac{1}{2}e^4(e^2-1)$

e $\frac{1}{2}e^4+1\frac{1}{2}$ **f** $e^2-e-1\frac{1}{2}$

g $\frac{1}{2}e^6+e^{-3}-1\frac{1}{2}$

3 **a** 0.32 **b** 268.29

c 37 855.68 **d** 346.85

e 755.19

4 **a** $\frac{1}{\ln 5}5^x+C$ **b** $\frac{1}{3\ln 7}7^{3x}+C$

c $\frac{1}{2\ln 3}3^{2x-1}+C$

5 **a** $x(2+x)e^x$ **b** x^2e^x+C

6 $f(x)=\frac{1}{6}(e^{2x}-1)$ **7** $2e^3+5$ m

Exercise 7.07

1 **a** $\ln|2x+5|+C$ **b** $\ln|2x^2+1|+C$

c $\ln|x^5-2|+C$

d $\frac{1}{2}\ln|x|+C$ or $\frac{1}{2}\ln|2x|+C$

e $2\ln|x|+C$ **f** $\frac{5}{3}\ln|x|+C$

g $\ln|x^2-3x|+C$ **h** $\frac{1}{2}\ln|x^2+2|+C$

i $\frac{3}{2}\ln|x^2+7|+C$ **j** $\frac{1}{2}\ln|x^2+2x-5|+C$

2 **a** $\ln|4x-1|+C$ **b** $\ln|x+3|+C$

c $\frac{1}{6}\ln|2x^3-7|+C$ **d** $\frac{1}{12}\ln|2x^6+5|+C$

e $\frac{1}{2}\ln|x^2+6x+2|+C$

3 **a** 0.5 **b** 0.7 **c** 1.6

d 3.1 **e** 0.5

4 **a** RHS = LHS

b $\ln|x+3|+2\ln|x-3|+C$

5 **a** RHS = LHS

b $x-5\ln|x-1|+C$

6 $f(x)=\frac{1}{9}\ln\left|\frac{3x^3-1}{2}\right|$ **7** 8.95 m

8 4

Exercise 7.08

1 **a** $\sin x+C$ **b** $-\cos x+C$

c $\tan x+C$ **d** $-\frac{45}{\pi}\cos x°+C$

e $-\frac{1}{3}\cos 3x+C$ **f** $\frac{1}{7}\cos 7x+C$

g $\frac{1}{5}\tan 5x+C$ **h** $\sin(x+1)+C$

i $-\frac{1}{2}\cos(2x-3)+C$ **j** $\frac{1}{2}\sin(2x-1)+C$

k $\cos(\pi - x) + C = -\cos x + C$

l $\sin(x + \pi) + C = -\sin x + C$

m $\frac{2}{7}\tan 7x + C$ **n** $-8\cos\frac{x}{2} + C$

o $9\tan\frac{x}{3} + C$

2 **a** 1 **b** $\sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

c $\frac{2}{\sqrt{2}} = \sqrt{2}$ **d** $-\frac{1}{3}$

e $\frac{1}{\pi}$ **f** $\frac{1}{2}$

g $\frac{3}{4}$ **h** $-\frac{1}{5}$

3 **a** $\sin\left(x + \frac{\pi}{3}\right) + C$ or $\frac{\sin x + \sqrt{3}\cos x}{2} + C$

b $\cos(\pi - x) + C = -\cos x + C$

4 $\frac{1}{8}$ **5** $y = \frac{1}{4}\sin 4x + \frac{\pi}{4}$

6 **a** $x = 18\sin\frac{2\pi t}{3} + 2$ cm

b **i** $9\sqrt{3} + 2$ cm **ii** $-9\sqrt{3} + 2$ cm

7 **a** $d = -24\cos\frac{\pi t}{6} + 26$ **b** 14 m

c 50 m; 2 m; 26 m **d** 12 h

8 **a** $\cos 2x = \cos^2 x - \sin^2 x$

b $\frac{1}{4}\sin 2x + \frac{1}{2}x + C$

Exercise 7.09

1 $1\frac{1}{3}$ units^2 **2** 36 units^2 **3** 4.5 units^2

4 $10\frac{2}{3}$ units^2 **5** $\frac{1}{6}$ units^2 **6** 14.3 units^2

7 4 units^2 **8** 0.4 units^2 **9** 8 units^2

10 24.25 units^2 **11** $e^2(e^2 - 1)$ units^2

12 $\frac{1}{4}(e - e^{-3})$ units^2 **13** 2.86 units^2

14 29.5 units^2 **15** 4 units^2

16 $\frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$ units^2 **17** 0.86 units^2

18 2 units^2 **19** $9\frac{1}{3}$ units^2

20 $11\frac{2}{3}$ units^2 **21** $\frac{1}{6}$ units^2

22 $\frac{2}{3}$ units^2 **23** $\frac{1}{3}$ units^2

24 $\ln 3 - \ln 2 = \ln 1.5$ units^2

25 $\ln 2$ units^2

26 0.61 units^2 **27** $5\frac{1}{3}$ units^2

28 18 units^2 **29** 3.98 units^2

30 $\frac{a^4}{2}$ units^2

Exercise 7.10

1 $21\frac{1}{3}$ units^2 **2** 20 units^2

3 $4\frac{2}{3}$ units^2 **4** 1.5 units^2

5 $1\frac{1}{4}$ units^2 **6** $2\frac{1}{3}$ units^2

7 $10\frac{2}{3}$ units^2 **8** $\frac{1}{6}$ units^2

9 $3\frac{7}{9}$ units^2 **10** 2 units^2

11 $11\frac{1}{4}$ units^2 **12** 60 units^2

13 4.5 units^2 **14** $1\frac{1}{3}$ units^2

15 1.9 units^2 **16** 47.2 units^2

Exercise 7.11

1 $1\frac{1}{3}$ units^2 **2** $1\frac{1}{3}$ units^2

3 $\frac{1}{6}$ units^2 **4** $10\frac{2}{3}$ units^2

5 $20\frac{5}{6}$ units^2 **6** 8 units^2

7 $\frac{2}{3}$ units^2 **8** $166\frac{2}{3}$ units^2

9 $\frac{5}{12}$ units^2 **10** $\frac{2}{3}$ units^2

11 $\frac{1}{12}$ units^2 **12** $\frac{1}{3}$ units^2

13 36 units^2 **14** $2\frac{2}{3}$ units^2

15 $(\pi - 2)$ units^2 **16** $\frac{1}{2} + \ln 2$ units^2

17 $2\sqrt{2}$ units^2 **18** $\frac{1}{2}(e^4 - 5)$ units^2

19 $\sqrt{3} - \frac{\pi}{3} = \frac{3\sqrt{3} - \pi}{3}$ units^2

Test yourself 7

1 D **2** B **3** A **4** D

5 **a** 0.535 **b** 0.5

6 **a** $\frac{3x^2}{2}+x+C$ **b** $\frac{5x^2}{2}-x+C$

c $\frac{2\sqrt{x^3}}{3}+C$ **d** $\frac{(2x+5)^8}{16}+C$

e $\frac{(3x^4-2)^5}{60}+C$

7 $\frac{3^x}{\ln 3}+C$

8 **a** 14 units^2 **b** 27 units^2 **c** 28 units^2

9 **a** 2 **b** 0 **c** $2\frac{1}{5}$

d $6\frac{7}{9}$ **e** $228\frac{3}{8}$

10 $e(e^2-1)$ units^2 **11** 3 units^2

12 $-\frac{180}{\pi}\cos x° + C$ **13** $2\frac{2}{3}$ units^2

14 **a** $\frac{1}{4}e^{4x}+C$ **b** $\frac{1}{2}\ln|x^2-9|+C$

c $-e^{-x}+C$ **d** $\ln|x+4|+C$

e $\frac{(x^2-6x+1)^9}{18}+C$

15 $4\frac{1}{2}$ **16** $\frac{9\pi}{4}$ units^2

17 $\frac{3}{4}$ units^2 **18** $\frac{(7x+3)^{12}}{84}+C$

19 3 units^2 **20** $\frac{1}{2}e^4(e^6-1)$ units^2

21 5.36 units^2 **22** 36

23 **a** $-\frac{1}{2}\cos 2x+C$ **b** $3\sin x+C$

c $\frac{1}{5}\tan 5x+C$ **d** $x-\cos x+C$

24 $85\frac{1}{3}$ units^2

25 **a** $T=40e^{-0.4t}+175$

b **i** 180° **ii** 175°

26 **a** $\frac{1}{\sqrt{2}}$ **b** $\sqrt{3}-\frac{1}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$

27 **a** $\frac{(2x-1)^5}{2}+C$ **b** $\frac{x^6}{8}+C$

28 $\frac{1}{\sqrt{2}}$ units^2

29 **a** $\frac{1}{18}(x^3-2)^6+C$ **b** $\frac{1}{50}(5x^2+2)^5+C$

c $\frac{5}{24}(2x^4-1)^3+C$ **d** $\frac{1}{8}(x^2+4x-3)^4+C$

30 2 units^2

31 **a** $\sqrt{2}$ units^2 **b** 1 unit^2

c $1+\sqrt{2}$ units^2

32 **a** $f(x)=\frac{1}{20}[3(2x^2-1)^5+57]$

b $f(x)=\frac{1}{2}\tan 2x$

c $f(x)=\frac{1}{5}e^{5x}$

d $f(x)=\frac{1}{16}[(x^4-15)^4-1]$

e $f(x)=\frac{3}{4}\ln(x^4+1)+2$

33 **a** $x=\frac{2\sqrt{t^3+9}}{3}-4$ **b** 3.7 m

c 7.6 s

Challenge exercise 7

1 **a** Show $f(-x)=-f(x)$

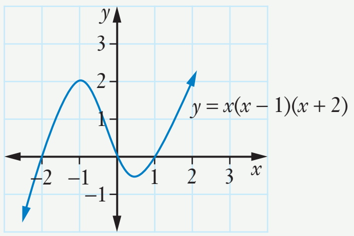
b 0 **c** 12 units^2

2 **a** RHS = LHS **b** $\ln\sqrt{3}=\frac{\ln 3}{2}$

3 9 units^2 **4** $\frac{3-\sqrt{3}}{6}$

5 $\frac{1}{8}$ **6** 7.3 units^2

7 **a**



b $3\frac{}{12}$ units^2

8 11.68 units^2

9 **a** $\frac{3(x+2)}{2\sqrt{x+3}}$ **b** $\frac{2x\sqrt{x+3}}{3}+C$

10 **a** $x(1+2\ln x)$ **b** $18\ln 3$

11 $\frac{5}{12}$ units^2

12 **a** 6879.7 **b** 1375.93 **c** 1527.62

13 **a** 6879.7 **b** 1375.9 units^2

c 1527.6 units^2

Chapter 8

Investigation

Damien's error was in integrating $(x^2 + 1)^2$. The answer he gave was wrong. He should have expanded the expression first, then integrated.

The correct final answer is $\frac{206\pi}{15}$ units^3.

Exercise 8.01

1 $\frac{243\pi}{5}$ units^3 **2** $\frac{485\pi}{3}$ units^3

3 $\frac{376\pi}{15}$ units^3 **4** $\frac{\pi}{7}$ units^3

5 0.51 units^3 **6** $\frac{\pi}{2}(e^6 - 1)$ units^3

7 $\frac{758\pi}{3}$ units^3 **8** $\frac{2\pi}{3}$ units^3

9 4.8 units^3 **10** $\frac{\pi}{2}$ units^3

11 $\frac{5\pi}{3}$ units^3 **12** $\frac{9\pi}{2}$ units^3

13 $\pi \ln 3$ units^3 **14** $\frac{27\pi}{2}$ units^3

15 $\frac{64\pi}{3}$ units^3 **16** πe units^3

17 $2\pi \ln 9$ units^3 **18** $\frac{1023\pi}{5}$ units^3

19 $\frac{\pi}{2}e^2(e^4 - 1)$ units^3 **20** 13π units^3

21 $\frac{344\pi}{27}$ units^3 **22** $\frac{3\pi}{5}$ units^3

23 **a** Proof: see worked solutions.

b 3.1 units^3

24 **a** $\frac{2}{3}$ units^2 **b** $\frac{2\pi}{5}$ units^3

25 **a** 4.5 units^2 **b** $\frac{72\pi}{5}$ units^3

26 Proof: see worked solutions.

27 **a** $\frac{1}{2}(1 - \cos 2x)$ **b** $\frac{\pi}{8}(\pi - 2)$ units^3

28 **a** $\frac{\pi}{6}$ units^2 **b** $\frac{\pi}{2}$ units^3

Exercise 8.02

1 **a** $\frac{(3x-4)^8}{24} + C$ **b** $\frac{2\sqrt{(x+3)^3}}{3} + C$

c $\frac{(x^2-9)^6}{12} + C$ **d** $\frac{(x^2+4x+1)^5}{10} + C$

e $-\frac{1}{2(x^2+3x-1)^2} + C$ **f** $\frac{3\sqrt{2x^2-1}}{2} + C$

g $-\frac{1}{9(x^3+1)^3} + C$

2 $\frac{1}{3}e^{x^3} + C$ **3** $e^{\tan x} + C$

4 $\sqrt{x^2-4} + C$

5 $\frac{\sqrt{(2x+7)^5}}{10} - \frac{7\sqrt{(2x+7)^3}}{6} + C$

6 $\frac{2\sqrt{(x+2)^3}}{3} - 8\sqrt{x+2} + C$

7 $-\frac{10\sqrt{(5-x)^3}}{3} + \frac{2\sqrt{(5-x)^5}}{5} + C$

8 $\frac{\sqrt{(2x-5)^3}}{6} + \frac{5\sqrt{2x-5}}{2} + C$

9 $2\sqrt{(x+4)^5} - \frac{40\sqrt{(x+4)^3}}{3} + C$

10 $y = -\frac{1}{2(x^2-1)} - \frac{5}{6}$

11 $y = \frac{(3x^2-1)^7 + 43}{42}$

12 $f(x) = \frac{(x^3-2)^5}{15} + 1$

13 **a** Domain: $(-1, \infty)$ **b** $y = 4\frac{2}{3}$

14 $f(x) = -\frac{3}{5}(\cos^5 x + 1)$

15 $y = \frac{1}{6}[(e^x + 1)^6 - 64]$

16 Proof (see worked solutions).

Exercise 8.03

1 **a** $1\frac{3}{4}$ **b** $\sqrt{7}-2$ **c** 0
d $\frac{1}{8}$ **e** $\frac{15}{256}$ **f** $\frac{52\sqrt{2}}{3}$
g $-3\frac{1}{3}$

2 **a** 63.05 **b** 4 **c** –0.25
d $-\frac{19}{72}$ **e** 0.24

3 $\frac{1}{2}(e^4-1)$ **4** $\frac{1}{24}$ **5** 10.2 units^2

6 $1\frac{11}{20}$ units^2 **7** 65 536 units^2

8 $\frac{211\pi}{15}$ units^3 **9** $\frac{4\pi}{3}$ units^3 **10** $2\frac{2}{3}$

11 $6\frac{2}{5}$ units^2 **12** 64 units^2

13 $\frac{19\,682\pi}{27}$ units^3 **14** 0.003 26 units^3

15 **a** 0.16 units^2 **b** $\frac{19\pi}{756}$ units^3

Exercise 8.04

1 **a** $\frac{1}{4}\sin 2x-\frac{1}{2}x+C$
b $\frac{1}{2}x-\frac{1}{4}\sin 2x+\frac{x^3}{3}+C$
c $\frac{3}{4}\sin 2x-\frac{7x}{2}+C$
d $x-\frac{1}{2}\sin 2x-\sin x+C$
e $\frac{7x}{2}+\frac{7}{4}\sin 2x+\frac{1}{7}\tan 7x+C$
f $\frac{1}{4}\sin 2x+\sin x+\frac{5x}{2}+C$
g $-\frac{1}{2}\cos 2x+\frac{1}{4}\sin 2x-\frac{1}{2}x+C$
h $\tan x+\frac{1}{2}x-\frac{1}{4}\sin 2x+C$
i $\frac{1}{3}\sin 3x+\frac{1}{2}x+\frac{1}{4}\sin 2x+C$

2 **a** $\frac{\pi}{4}$ **b** $\frac{3\pi}{2}$ **c** $\frac{\pi(2+3\pi)}{8}$
d $\frac{\pi}{3}$ **e** $\frac{\pi}{12}-\frac{\sqrt{3}}{8}=\frac{2\pi-3\sqrt{3}}{24}$

3 $\frac{\pi}{8}(\pi-2)$ units^3 **4** $\frac{\pi-2}{8}$ units^2

5 $\frac{\pi}{4}(3\pi+8)$ units^3

6 **a** $\frac{1}{2}\left(x-\frac{1}{6}\sin 6x\right)+C$
b $\frac{1}{2}\left(x+\frac{1}{14}\sin 14x\right)+C$
c $\frac{1}{2}\left(x+\frac{1}{8}\sin 8x\right)+C$
d $\frac{1}{2}\left(x-\frac{1}{20}\sin 20x\right)+C$
e $\frac{1}{2}(x+\sin x)+C$
f $\frac{1}{2}\left[x-\frac{90}{\pi}\sin(2x)^\circ\right]+C$
g $\frac{1}{2}(x+\frac{1}{2a}\sin 2ax)+C$
h $\frac{3}{2}(x-\frac{5}{4}\sin\frac{4x}{5})+C$

7 **a** $\sin 3x=3\sin x-4\sin^3 x$
b $\frac{1}{4}\left(\frac{8}{3}-\frac{3\sqrt{3}}{2}\right)=\frac{1}{24}(16-9\sqrt{3})$

8 **a** $\sin(7x+3x)+\sin(7x-3x)=2\sin 7x\cos 3x$
b $\frac{1}{10}$

9 **a** $-\frac{1}{4}\cos 2x+C$
b $\sin x+\frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)+C$
c $\frac{1}{8}\left(\theta-\frac{1}{4}\sin 4\theta\right)+C$ **d** $x-\frac{1}{2}\cos 2x+C$

10 **a** $\frac{1}{2}\left(2\sqrt{2}-\sqrt{3}-1\right)$ units^2
b $\frac{\pi}{4}\left(\frac{\pi}{3}-1\right)$ units^3

Exercise 8.05

1 **a** $\sin^{-1}x+C$
b $2\cos^{-1}x+C$ or $-2\sin^{-1}x+C$
c $\tan^{-1}x+C$ **d** $\frac{1}{3}\tan^{-1}\frac{x}{3}+C$
e $\sin^{-1}\frac{x}{2}+C$ **f** $\frac{5}{2}\tan^{-1}\frac{x}{2}+C$
g $\frac{3}{2}\sin^{-1}x+C$ **h** $\frac{1}{5}\sin^{-1}\frac{x}{4}+C$
i $\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C$

2 **a** $\frac{1}{2}\sin^{-1}2x + C$ **b** $\sin^{-1}\frac{t}{3} + C$

c $\frac{1}{3}\tan^{-1}3x + C$ **d** $\frac{1}{5}\sin^{-1}\left(\frac{5x}{2}\right) + C$

e $\frac{1}{4}\tan^{-1}\left(\frac{4x}{3}\right) + C$ **f** $\sin^{-1}\left(\frac{2t}{5}\right) + C$

g $\frac{\sqrt{5}}{5}\tan^{-1}(\sqrt{5}x) + C$

h $\frac{\sqrt{3}}{6}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C$

i $\frac{2}{15}\cos^{-1}\left(\frac{3x}{5}\right) + C$

3 **a** $\frac{\pi}{4}$ **b** $\frac{\pi}{3}$ **c** $\frac{\pi}{12}$

d $\frac{\pi}{6}$ **e** $\frac{\pi}{3}$ **f** π

g $\frac{\pi}{9}$ **h** $\frac{3\pi}{28}$ **i** $\frac{\pi}{3\sqrt{5}}$

4 1.1 units^2

5 **a** $\frac{\pi}{6}$ units^2 **b** $\frac{\pi}{6}(7\sqrt{3} - 12)$ units^3

6 **a** $\frac{1}{\sqrt{1-x^2}\sin^{-1}x}$ **b** 0.3

7 0.9 **8** $\frac{1}{\sqrt{2}}$ units^2

9 $\frac{1}{3}\sin^{-1}x^3 + C$ **10** $y = \sin^{-1}\left(\frac{x}{3}\right) - \frac{5\pi}{14}$

11 **a** RHS = LHS **b** 0.48

12 $\frac{\pi^2}{12}$ units^3

13 **a** $\cos^{-1}x$ **b** $\left(\frac{\pi}{6}+1-\frac{\sqrt{3}}{2}\right)$ units^2

14 $(\frac{\pi}{2} - 1)$ units^2

Test yourself 8

1 C **2** A **3** B

4 B **5** 9π units^3

6 **a** $\frac{206\pi}{15}$ units^3 **b** $\frac{\pi}{2}$ units^3

7 $\frac{\pi}{6}e^6(e^6 - 1)$ units^3

8 **a** $\frac{\pi}{6}$ units^2 **b** 1.83 units^3

9 **a** $x = \pm\sqrt{y} - 3$ **b** $3\frac{2}{3}$ units^2 **c** $\frac{5\pi}{2}$ units^3

10 10.55

11 **a** $e(e - 1)$ units^2 **b** $\frac{\pi}{2}e^2(e^2 - 1)$ units^3

12 **a** $\frac{4\pi - 3\sqrt{3}}{12}$ **b** $\frac{\pi - 6 + 3\sqrt{3}}{24}$

13 $\frac{3\pi}{5}$ units^3 **14** $\frac{6\sqrt{(x+1)^5}}{5} - 2\sqrt{(x+1)^3} + C$

15 $\frac{\pi}{\sqrt{3}}$ units^3

16 **a** $\frac{\sqrt{3}-\sqrt{2}}{2}$ units^2

b $\frac{\pi(\pi - 6 + 3\sqrt{3})}{24}$ units^3

17 9π units^3 **18** $\frac{2\pi}{3}$ units^3

19 **a** $\sqrt{3}$ units^2 **b** $\frac{58\pi}{15}$ units^3

20 **a** $\frac{8a^2}{3}$ units^2 **b** $2\pi a^3$ units^3

21 $\frac{1}{2}e^x + C$ **22** $\frac{1}{6}e^{3x+1} + C$

23 $\frac{1}{3}(e^6 - e^{-2})$ **24** $-\frac{1}{3}\cos^3 x + C$

25 **a** $\frac{1}{\sqrt{2}} - \frac{1}{2} = \frac{\sqrt{2}-1}{2}$ units^2

b $\frac{\pi}{24}(\pi + 6 - 3\sqrt{3})$ units^3

Challenge exercise 8

1 12 units^3 **2** $\frac{1}{3}\cos^3 x - \cos x + C$

3 $\frac{1}{2}\tan^{-1}(x^2) + C$ **4** $\frac{1}{6}\sin^{-1}(3x^2) + C$

5 **a** 0.62 units^3 **b** π units^3

6 **a** $\frac{1}{12}$ units^2 **b** $\frac{2\pi}{35}$ units^3

7 $\frac{215\pi}{6}$ units^3 **8** 0

9 **a** Proof uses $\cos 2nx = \cos^2 nx - \sin^2 nx$: see worked solutions

b $\frac{2\pi - 3\sqrt{3}}{96}$

Practice set 2

1 C **2** D **3** D

4 B **5** B **6** A

7 C **8** $\frac{\pi^2}{18}$ units^3 **9** $\frac{\pi}{6}$

10 0.22

11 **a** $\frac{376\pi}{15}$ units^3 **b** $\frac{\pi}{2}$ units^3

12 $x < \frac{1}{2}$

13 $x^3 - x^2 + x + C$

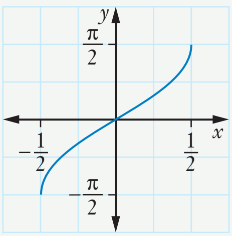
14 24

15 **a** 8 **b** $\frac{2\pi}{3}$ **c** 5

16 $\frac{1}{2}\tan^{-1} 2x + C$

17 0.6

18 **a**



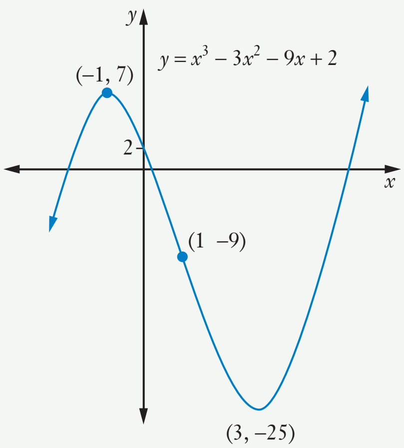
b $\frac{1}{2}$ units^2

19 8 m

20 $\frac{d^2y}{dx^2} = 7(-7\sin 7x)$

21 $\frac{x^9}{3} + 2x^2 + C$

22



23 $1\frac{1}{3}$ units^2

24 **a** $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ **c** $\begin{pmatrix} 1 \\ -7 \end{pmatrix}$

d $\begin{pmatrix} -4 \\ 39 \end{pmatrix}$ **e** $\sqrt{50}$, 98° 8′

25 $\frac{3}{4}\ln|2x^2 - 5| + C$

26 $f'(3) = 20; f''(-2) = -16$

27 $-\frac{1}{\sqrt{1-x^2}\cos^{-1} x}$

28 $\frac{3e^{3x}}{1+\left(e^{3x}+1\right)^2} = \frac{3e^{3x}}{e^{6x}+2e^{3x}+2}$

29 68 **30** $-\frac{7}{9}$ ∞ **31** 3

32 $\frac{1}{4}(\tan x + 1)^4 + C$

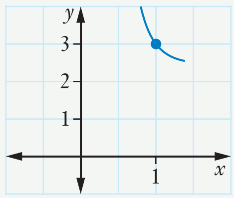
33 $\frac{2}{3}\tan^{-1}\left(\frac{3x}{2}\right) + C$

34 **a** $\frac{1}{\sqrt{1-x^2}} + \frac{1}{x}$ **b** $\frac{4}{1+x^2}(\tan^{-1} x)^3$

c $-\frac{6}{\sqrt{1-9x^2}}$

35 **a** $f(1) = 3, f'(1) = -2, f''(1) = 18$

b Curve is decreasing and concave upwards at (1, 3).



36 **a** $P = 8x + 4y = 4$

$4y = 4 - 8x$

$y = 1 - 2x$

$A = 3x^2 + y^2$

b Rectangle $\frac{2}{7}$ m × $\frac{6}{7}$ m, square with sides $\frac{3}{7}$ m

37 $y = 3\left[\frac{1}{2}(x+4)\right]^2 + 5$

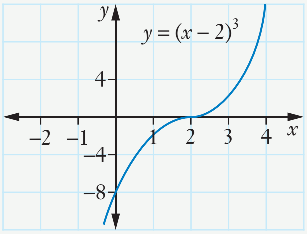
38 $f(-1) = 0$

39 $x \sin^{-1} x + \sqrt{1 - x^2} + \cos x + C$

40 $\pi(\sqrt{3} - 1)$ **41** 12 **42** 0.944

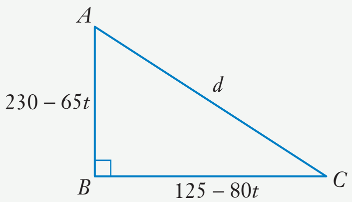
43 **a** (2, 0); point of inflection

b



44 $\frac{\pi}{2}$ **45** 0.73

46 **a**



$d^2 = (230 - 65t)^2 + (125 - 80t)^2$

b 2.3 h **c** 99.8 km

47 **a** $-\frac{25}{128}$ **b** $66\frac{77}{128}$ **c** $66\frac{2}{3}$

48 **a** 1.58 units^2 **b** $\frac{5\pi}{2}$ units^3

49 $\frac{\sqrt{1-x^2}\cos^{-1} x + x}{\sqrt{1-x^2}(\cos^{-1} x)^2}$

50 450 cm^2 **51** $\frac{(3x+5)^8}{24} + C$

52 1.099

53 $9x - 12y + 2\pi - 3\sqrt{3} = 0$

54 (0, 3) maximum, (1, 2) minimum, (–1, 2) minimum

55 $2\frac{8}{15}$ **56** $10\frac{2}{3}$ units^2

57 $f(2) = -16$ **58** 9 units^2 **59** $3x^2 + 2e^{2x}$

60 **a** 7900 L **b** 400 min (6 h 40 min)

61 $\frac{3e}{2}(e^9 - 1)$ **62** 953 m

63 $f(x) = x^3 - 4x^2 - 3x + 20$

64 $\frac{12}{4x+3}$ **65** $\frac{2\sqrt{(x^3-5)^3}}{3} + C$

66 $e^{x-3} + C$

67 $\frac{1}{3} \ln (3x^2 + 3x - 2) + C$

68 **a** $\frac{1 - 2x}{e^{2x}}$ **b** $\frac{1}{x \ln 3}$

69 $x - y + 2 = 0$

70 $\left(-\frac{1}{2}, -\frac{1}{2e}\right)$; minimum

71 $\frac{4(6\sqrt{6} - 1)}{3}$ **72** $y = 2 \cos 3x + 3$

73 1

74 **a** $\cot x$ **b** $5e^{5x} \sec^2 (e^5 + 1)$

75 0.393

76 **a** 6 units^2 **b** 14 units^2

77 $\frac{\sqrt{3}\pi}{2}$ units^3

78 **a** $e^x(\sin x + \cos x)$ **b** $3 \tan^2 x \sec^2 x$

c $-6 \sin (3x - \frac{\pi}{2})$

79 $12x - 2y - 2 - 3\pi = 0$

80 **a** $3e^x \sin^2 (e^x) \cos (e^x)$ **b** $\frac{\sec^2(\ln x + 1)}{x}$

81 $(5 - e)$ units^2

82 **a** $\frac{1}{3} e^{3x} + C$ **b** $\frac{1}{\pi} \tan \pi x + C$

c $\frac{1}{2} \ln x + C$ **d** $5 \sin \left(\frac{x}{5}\right) + C$

e $-\frac{1}{8} \cos 8x + C$

83 $\frac{\pi^2}{2}$ units^3 **84** $3x - 2 \ln x - \frac{5}{x} + C$

85 $\frac{1}{5} e^{5x} + \frac{1}{\pi} \cos \pi x + C$ **86** $(e^2 - 1)$ units^2

87 –1 **88** $f(1) = 0.519$

89 **a** $-\frac{1}{2}\cos\left(x^2 - \frac{\pi}{2}\right) + C$ **b** $x = \sqrt{\frac{3\pi}{2}}$

Chapter 9

Note: Answers obtained from reading graphs are approximate.

Exercise 9.01

1 **a** N **b** N **c** N

d C **e** N **f** N

g	N	**h**	C	**i**	N
j	C	**k**	N	**l**	C
m	N	**n**	N	**o**	N
p	N	**q**	N	**r**	N
s	C	**t**	N		

2 **a** O **b** C **c** N
d N **e** C **f** D
g D **h** N **i** C
j N **k** D **l** N
m N

3 **a** e.g. types of sport, hair colour
b e.g. heights, test scores, prices
c e.g. rankings, test scores, clothing sizes
d e.g. karate belt colour, size of take-away coffee cups, Olympic medals
e e.g. race times, length, rainfall
f e.g. types of trees, housing, cars

Exercise 9.02

1 **a** **i**

Score	Frequency
2	1
3	2
4	2
5	5
6	6
7	3
8	5
9	1

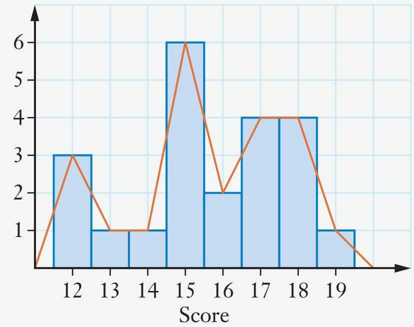
ii

iii Highest 9, lowest 2
v 6

b **i**

Pizzas	Frequency
12	3
13	1
14	1
15	6
16	2
17	4
18	4
19	1

ii

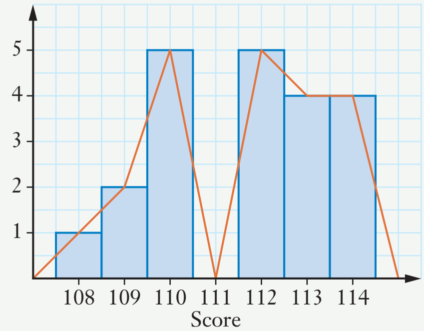


iii Highest 19, lowest 12 **v** 15

c **i**

Gym attendance	Frequency
108	1
109	2
110	5
111	0
112	5
113	4
114	4

ii

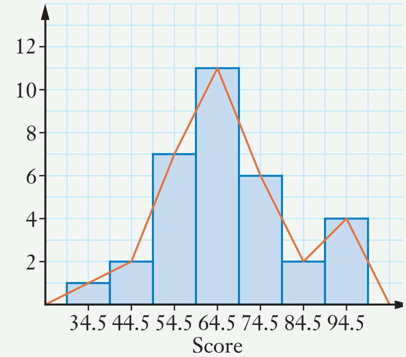


iii Highest 114, lowest 108
v 110 and 112

d **i**

Results	Class centre	Frequency
30–39	34.5	1
40–49	44.5	2
50–59	54.5	7
60–69	64.5	11
70–79	74.5	6
80–89	84.5	2
90–99	94.5	4

ii

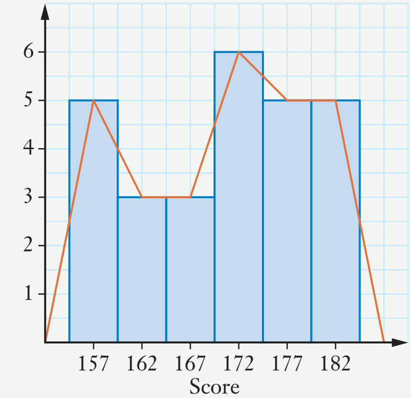


iii Highest 94.5, lowest 34.5 **v** 64.5

e **i**

Height (cm)	Class centre	Frequency
155–159	157	5
160–164	162	3
165–169	167	3
170–174	172	6
175–179	177	5
180–184	182	5

ii

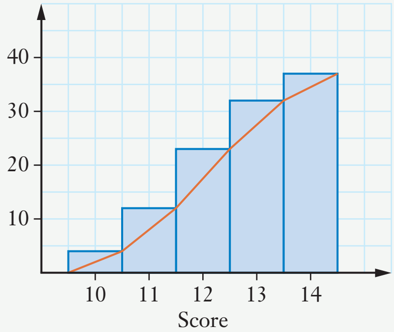


iii Highest 182, lowest 157 **v** 172

2 **a** **i**

Number of cars	Frequency	Cumulative frequency
10	4	4
11	8	12
12	11	23
13	9	32
14	5	37

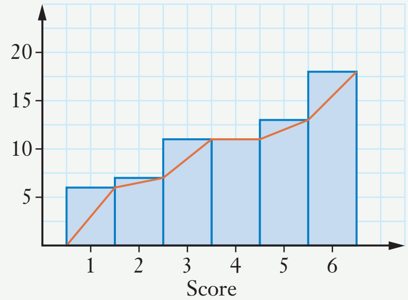
ii



b **i**

Score	Frequency	Cumulative frequency
1	7	7
2	1	8
3	3	11
4	0	11
5	2	13
6	5	18

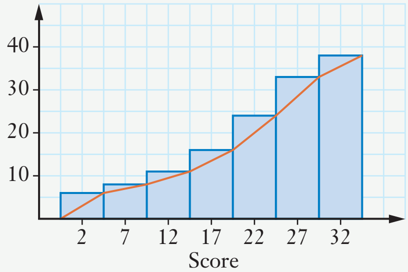
ii



c **i**

Sales	Class centre	Frequency	Cumulative frequency
0–4	2	6	6
5–9	7	2	8
10–14	12	3	11
15–19	17	5	16
20–24	22	8	24
25–29	27	9	33
30–34	32	5	38

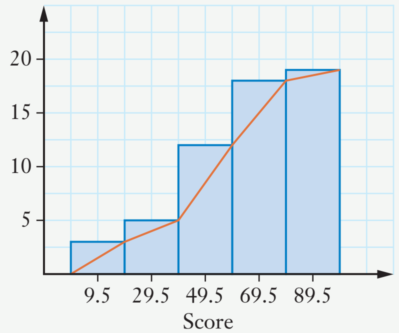
ii



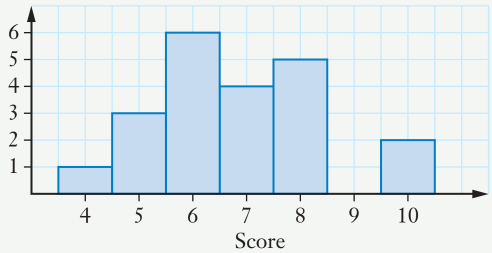
d **i**

Scores	Class centre	Frequency	Cumulative frequency
0–19	9.5	3	3
20–39	29.5	2	5
40–59	49.5	7	12
60–79	69.5	6	18
80–99	89.5	1	19

ii



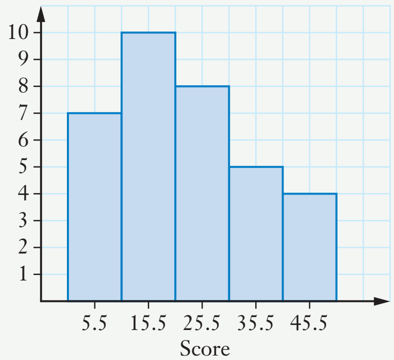
3 **a**



b 11 times

c 6 rescues

4 **a**



b 38.2%

5 **a**

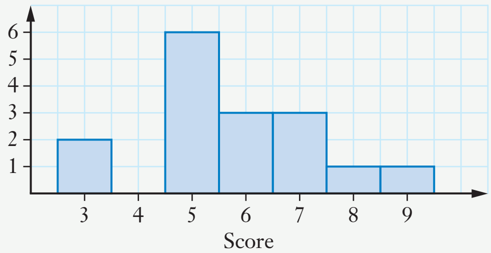
	Play soccer	Do not play soccer
Play tennis	12	35
Do not play tennis	27	28

b **i** 11.8% **ii** 27.5%

c 36.5% **d** $\frac{4}{13}$ **e** 74.5%

6 **a** 16 **b** 5

c

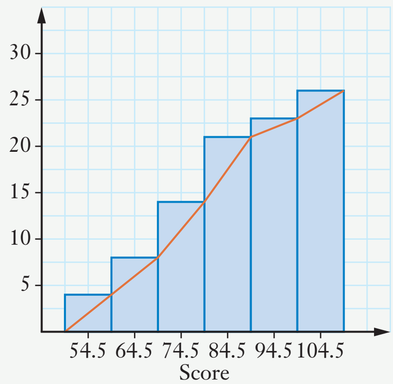


d 87.5% **e** $\frac{1}{8}$

7 **a**

Weight (kg)	Class centre	Frequency	Cumulative frequency
50–59	54.5	4	4
60–69	64.5	4	8
70–79	74.5	6	14
80–89	84.5	7	21
90–99	94.5	2	23
100–109	104.5	3	26

b



c **i** 12 **ii** 4

d 50% **e** $\frac{7}{13}$

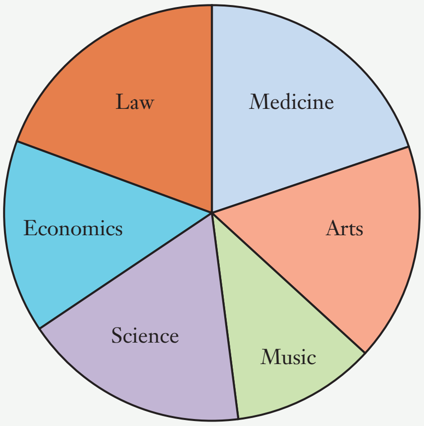
8 a 25%

b

Sport	Frequency
Tennis	90
Soccer	120
Athletics	60
Cricket	180
Basketball	150
Volleyball	120

9 a 19.3% **b** 31%

c



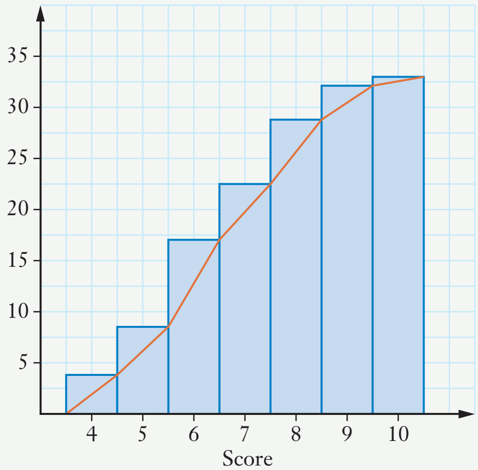
10 a 87 **b** 55.9%

c 6.5% **d** 76

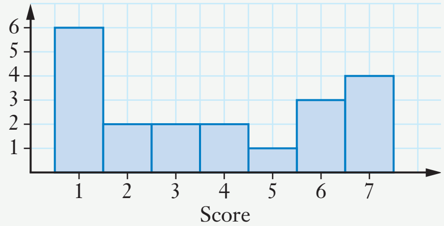
11 a 430 **b** 48.6% **c** 50.2%

d 221 **e** $\frac{112}{221}$

12



13 a

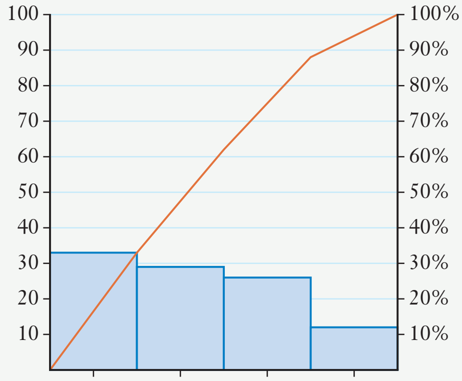


b

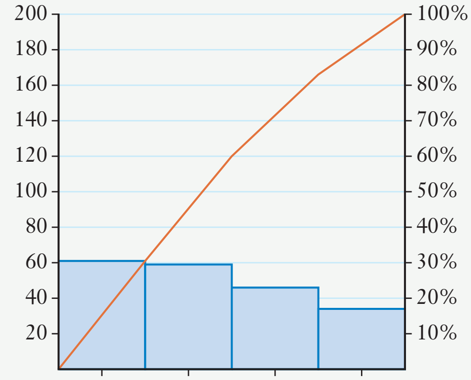
Junk mail items	Frequency
1	6
2	2
3	2
4	2
5	1
6	3
7	4

14 Stem-and-leaf plots list individual scores so retain all details. Not easy to draw, and can be long. A grouped frequency distribution table groups scores so individual data is lost. Easy to draw and compact.

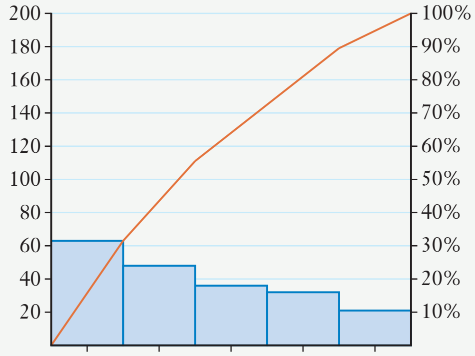
15 a



b



c



Exercise 9.03

1 **a** **i** 5 **ii** 5 **iii** 5

b **i** 5 **ii** 4 **iii** 5

c **i** 16 **ii** 18 **iii** 17

d **i** 5.7 **ii** 4 **iii** 5.5

e **i** 1.51 **ii** 1.49 **iii** 1.49

2 **a** Brown **b** Tabby

3 **a** **i** 6 **ii** 6 **iii** 6

b **i** 52.4 **ii** 52 **iii** 51

c **i** 17.56 **ii** 18 **iii** 20

d **i** 103.3 **ii** 104 **iii** 104

4 **a** 3 **b** 1.5

c 3.5 **d** 3

5 **a** **i** 9.24 **ii** 8–10

b **i** 17 **ii** 20–24

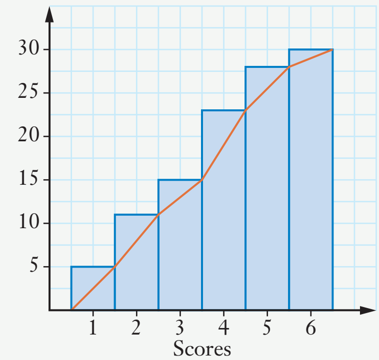
c **i** 68.7 **ii** 70–84

d **i** 39 **ii** 20–24, 50–54 (bimodal)

6 **a** **i**

Athletes	Frequency	Cumulative frequency
1	5	5
2	6	11
3	4	15
4	8	23
5	5	28
6	2	30

ii

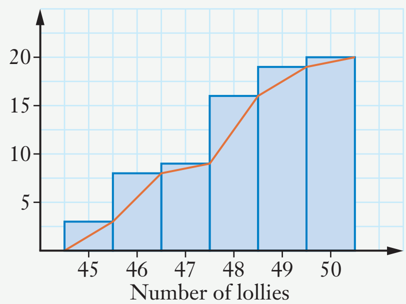


iii 3.5

b **i**

Lollies	Frequency	Cumulative frequency
45	3	3
46	5	8
47	1	9
48	7	16
49	3	19
50	1	20

ii

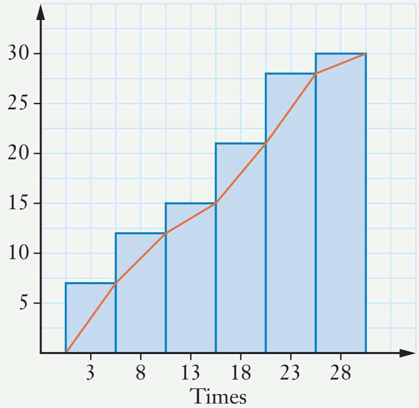


iii 48

c **i**

Time (h)	Class centre	Frequency	Cumulative frequency
1–5	3	7	7
6–10	8	5	12
11–15	13	3	15
16–20	18	6	21
21–25	23	7	28
26–30	28	2	30

ii

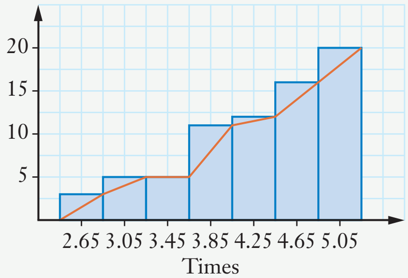


iii 15.5

d **i**

Time (min)	Class centre	Frequency	Cumulative frequency
2.5–2.8	2.65	3	3
2.9–3.2	3.05	2	5
3.3–3.6	3.45	0	5
3.7–4.0	3.85	6	11
4.1–4.4	4.25	1	12
4.5–4.8	4.65	4	16
4.9–5.2	5.05	4	20

ii



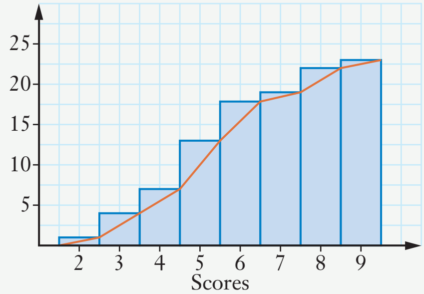
iii 4.0

7 **a** **i**

Score	Frequency	Cumulative frequency
2	1	1
3	3	4
4	3	7
5	6	13
6	5	18
7	1	19
8	3	22
9	1	23

ii 5.35 **iii** 5

v



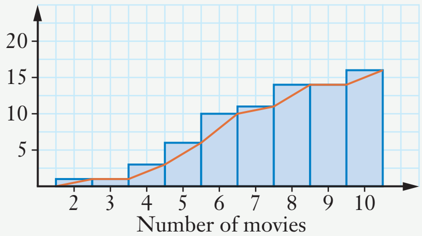
v 5.5

b **i**

Number of movies	Frequency	Cumulative frequency
2	1	1
3	0	1
4	2	3
5	3	6
6	4	10
7	1	11
8	3	14
9	0	14
10	2	16

ii 6.25 **iii** 6

v



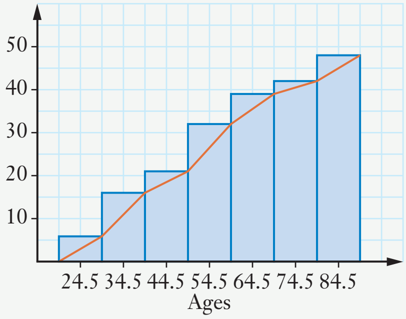
v 6.5

c **i**

Ages	Class interval	Frequency	Cumulative frequency
20–29	24.5	6	6
30–39	34.5	10	16
40–49	44.5	5	21
50–59	54.5	11	32
60–69	64.5	7	39
70–79	74.5	3	42
80–89	84.5	6	48

ii 54.5 **iii** 50–59

v



v 54.5

8 **a** **i** 32 **ii** Mean = 62.4, no mode, median = 64

iii Mean = 66.25, no mode, median = 66

b **i** 1 **ii** Mean = 6.6, mode = 6, median = 7

iii Mean = 6.8, mode = 6, median = 7

c **i** 97 **ii** Mean = 47.1, mode = 50, median = 46.5

iii Mean = 43.2, mode = 50, median = 43

d **i** 3 **ii** Mean = 6.1, mode = 5, median = 6

iii Mean = 6.3, mode = 5, median = 6

9 Outlier 1. It changes the mean.

10 **a** 17. One student had a very low score on the test.

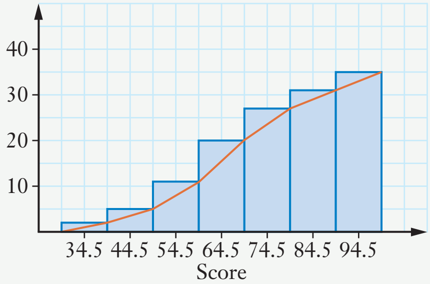
b

Class	Class centre	Frequency	Cumulative frequency
10–19	14.5	1	1
20–29	24.5	0	1
30–39	34.5	2	3
40–49	44.5	3	6
50–59	54.5	6	12
60–69	64.5	9	21
70–79	74.5	7	28
80–89	84.5	4	32
90–99	94.5	4	36

c **i** Mean = 65.6, modal class = 60–69

ii Mean = 67.1, modal class = 60–69

d



Median = 66

Exercise 9.04

1 **a** **i** 2 **ii** 3 **iii** 4.5

b **i** 11 **ii** 12 **iii** 13

c **i** 7 **ii** 8 **iii** 9

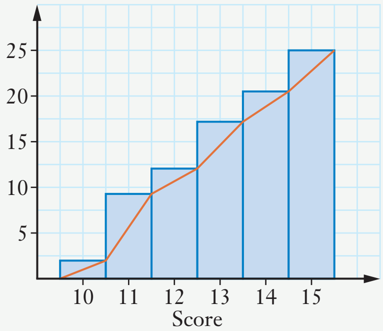
d **i** 2 **ii** 3 **iii** 5

2 $Q_1 = 2$, $Q_3 = 4.5$

3 **a** 23rd percentile = 24, 55th percentile = 25.5, 91st percentile = 28

b 2nd decile = 24, 8th decile = 27

4 **a**



b **i** 11 **ii** 14 **iii** 12

v 13.5 **v** 10.5

5 **a** 42 **b** 37 **c** 38

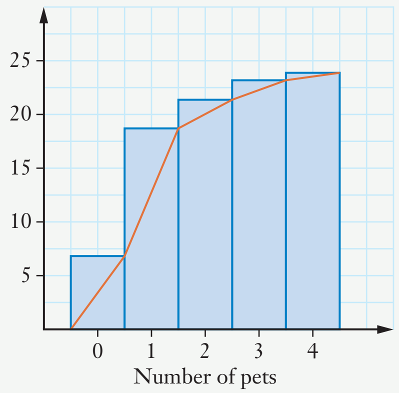
d 43 **e** 37–47

6 **a** 100 **b** 20% **c** 12

d 14 **e** 69th

7 **a** 12.5%

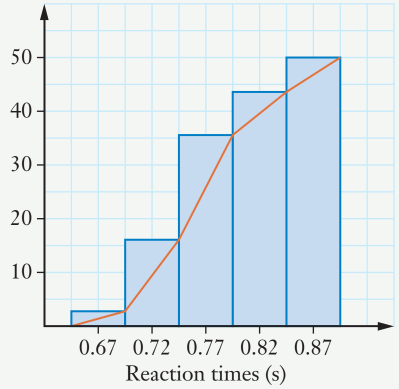
b



c **i** 1 **ii** 0 **iii** 1.5

8 **a** 0.774 **b** 38%

c



d **i** 0.74 **ii** 0.77 **iii** 0.72–0.82

9 **a** **i** 51 **ii** 38 **iii** 79

b **i** 11 **ii** 6 **iii** 14.5

c **i** 132.9 **ii** 115.3 **iii** 154.6

d **i** 15.5 **ii** 14.5 **iii** 17.5

Exercise 9.05

1 **a** 16 **b** 71

c 39 **d** 5

2 **a** **i** 6.5 **ii** 5 **iii** 4

b **i** 2 **ii** 5 **iii** 1

c **i** 40 **ii** 100 **iii** 55

d **i** 2 **ii** 6 **iii** 1

3 **a** **i** 7 **ii** 5 **iii** 2

b **i** 5 **ii** 5 **iii** 2

c **i** 17 **ii** 9 **iii** 4

d **i** 14 **ii** 19 **iii** 5

e **i** 6 **ii** 7 **iii** 2

4 **a** 6.76 **b** 7 **c** 7

d 4 **e** 1.5

5 **a** No outlier **b** 19 **c** 1

Exercise 9.06

1 **a** **i** 5.4 **ii** 2.1

b **i** 52.5 **ii** 14.6

c **i** 123.3 **ii** 16.1

d **i** 6.3 **ii** 1.8

e **i** 17.6 **ii** 1.96

2 **a** **i** 7.2 **ii** 52.1

b **i** 1.96 **ii** 3.8

c **i** 13.5 **ii** 183.2

d **i** 14 **ii** 196.6

e **i** 2.3 **ii** 5.2

3 **a** **i** 8.4 **ii** 2.4 **iii** 5.8

b **i** 4.4 **ii** 1.7 **iii** 2.9

c **i** 34.1 **ii** 1.5 **iii** 2.39

d **i** 51.2 **ii** 14.9 **iii** 222

4 **a** 4 **b** 1.5 **c** 1.14

5 **a** **i** 73.4 **ii** 16.3

b Yes, 20–29 **i** 74.8 **ii** 14.4

6 **a** Yes, 53.

$Q_3 + 1.5 \times \text{IQR} = 30.5 + 15.75 = 46.25$

$Q_1 - 1.5\ \text{IQR} = 20 - 15.75 = 4.25$

So 53 is outside 4.25–46.25.

b **i** 8.4 **ii** 5.3

7 **a** 9.2 **b** 3.5 **c** 12.6

Exercise 9.07

1 **a** Positively skewed, unimodal

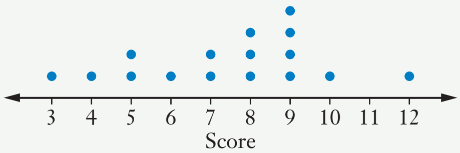
b Symmetrical, unimodal

c Positively skewed, multimodal

d Negatively skewed, unimodal

e Positively skewed, unimodal
f Bimodal
g Negatively skewed, unimodal
h Bimodal
i Positively skewed, unimodal
j Multimodal

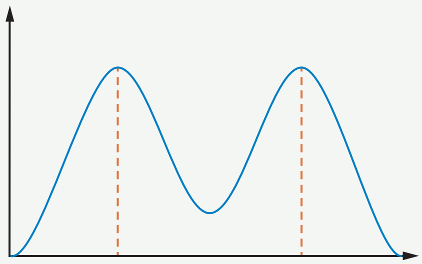
2 a



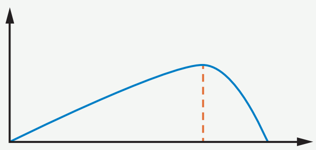
b Positively skewed

3 a Positively skewed, unimodal
b Symmetrical, unimodal
c Bimodal
d Negatively skewed, unimodal
e Symmetrical, unimodal

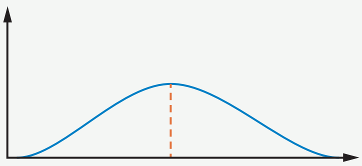
4 a



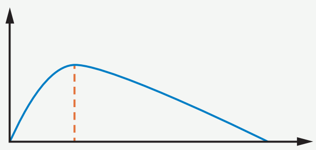
b



c



d



5 Set 1: bimodal, Set 2: positively skewed, unimodal

6 a Plot 1: positively skewed, Plot 2: negatively skewed
b 5 **c** 1

7 a

Score	Class centre	Frequency
145–149	147	3
150–154	152	3
155–159	157	3
160–164	162	6
165–169	167	3
170–174	172	1
175–179	177	6
180–184	182	4
185–189	187	1

b Bimodal

8

9 a i 7 **ii** 7 **iii** 7
b Symmetrical

10 Class discussion

Exercise 9.08

1 a i 7 **ii** 4
b i 2 **ii** 2
c i 3 **ii** 2
d Class discussion

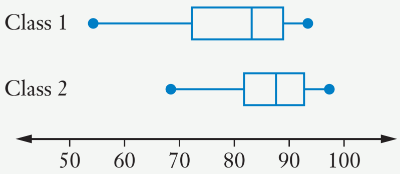
2 a

Sample 1		Sample 2
9 8 7	14	
9 8 7 5 3 0	15	1 7 7 9
9 7 6 4 2 0	16	2 4 5 6 8
7 6 2 1 0	17	2 3 3 6 8 9
	18	0 1 1 1 2

b Sample 1: 161, Sample 2: 172.5
c Sample 1: 30, Sample 2: 31
d Class discussion

3 a i Bank 1: 5.08, Bank 2: 2.77, Bank 3: 11.95
ii Bank 1: 4, Bank 2: 1, Bank 3: 13
b Bank 1: 4.1, Bank 2: 2.4, Bank 3: 4.4
c Class discussion

4 a



b **i** Class 1: 82.5, Class 2: 88

ii Class 1: 17, Class 2: 10.5

iii Class 1: 79.4, Class 2: 85.8

v Class 1: 38, Class 2: 28

5 **a**

Camera 1		Camera 2
9 9 9 6 5	6	
7 6 5 4 4 3 3 2	7	
9 5 4 3 0	8	
0	9	
3	10	
	11	3 6 8 9 9
	12	0 0 2 2 3 4 5
	13	0 0 1 5 5 8
	14	0 2

b Camera 1 773 kmh $^{-1}$ Camera 2 1261 kmh $^{-1}$

c Camera 1 60 kmh $^{-1}$ Camera 2 110 kmh $^{-1}$

6 **a** 17.5 **b** 17 **c** 10

d 2.67

7 **a** **i** 5.57 **ii** 5.86

b **i** 3.16 **ii** 1.25

c The average test score is similar in both tests, but test 1 has a wider spread of scores.

8 **a** Parents: 2, Children: 4

b **i** Parents: 4, Children: 3

ii Parents: 5, Children: 6 **c** 6

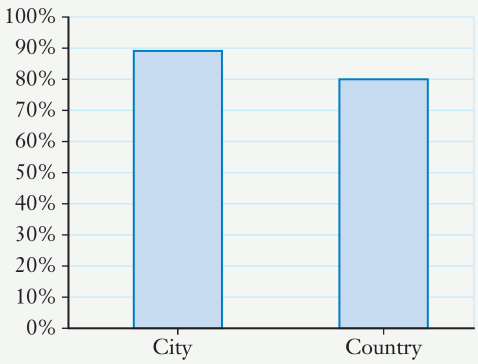
9 **a** Science: 70, English: 63.5

b Science: 69.8, English: 68.8

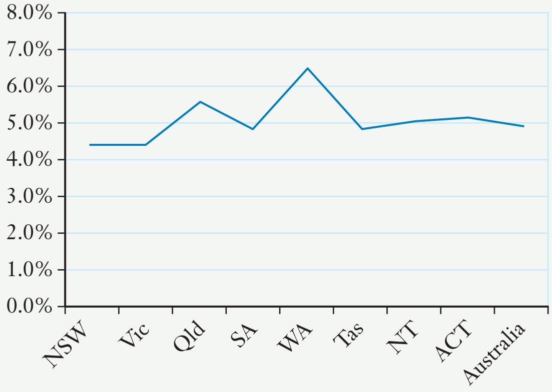
c Science: 12.8, English: 13.4 **d** 10

10 **a** The graph makes it look as if city access is about 3 times as high as country access.

b



11



Test yourself 9

1 B **2** C **3** D

4 D **5** D **6** A

7 Mode 8, median 7, range 5

8 **a** 5–9 **b** 10.75 **c** 9.5

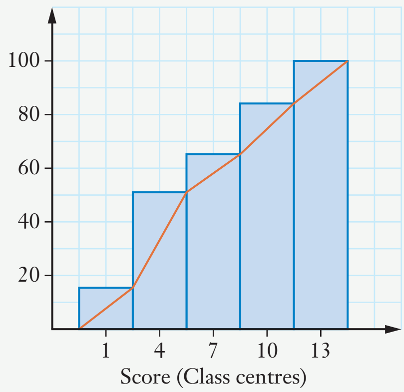
d **i** 6.6 **ii** 43.97

9 Mean 55.7, standard deviation 23.5

10 **a** 10 **b** 11

c 14 **d** 6

11 **a**



b 5.5

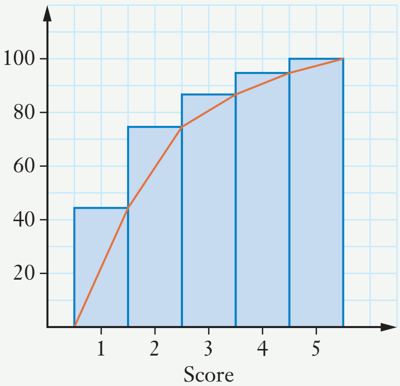
c **i** 3 **ii** 10 **iii** 13

v 4 **v** 12

12 **a** 1.5 **b** 1 **c** 5 **d** 2

13 Mode

14 a



b i 2 **ii** 1.5

15 a 7.6 **b** 1.35 **c** 6

16 Mean 44.8, mode 49, median 45

17 a 3 **b** 3.5

18 a i 17 **ii** 14.42 **iii** 14, 18

v 15

b 2

c i 8 **ii** 15.1 **iii** 14, 18

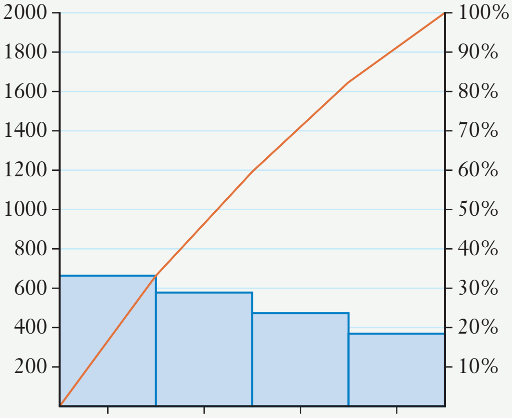
v 15

d Affects range and mean.

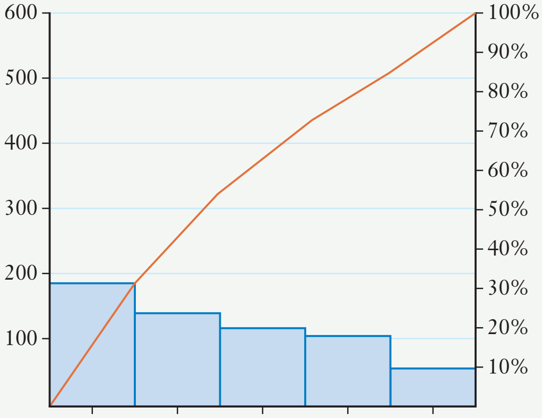
19 a i 54.6% **ii** 22.7% **iii** 27.7%

v 59.6%

b



20



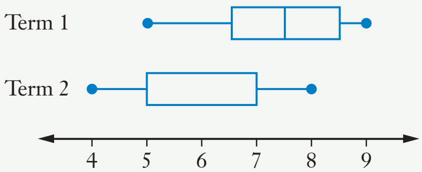
21 a North = 47.5, West = 43

b Mean 46.5, standard deviation 7.8

c Mean 43.4, standard deviation 10.7

d Slightly more mushrooms found in the North region on average, spread higher in West region so results more variable.

22 a



b Term 1: 7.5; Term 2: 5

c Term 1: 2; Term 2: 2

d Term 1: mean 7.3, standard deviation 1.4; Term 2: mean 5.6, standard deviation 1.2

e Students did better on average in Term 1. The 2 assessments had a similar spread.

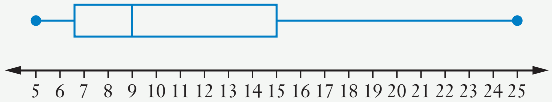
Challenge exercise 9

1 a Bimodal. Female heights may have their own mode and male heights have their own (higher) mode.

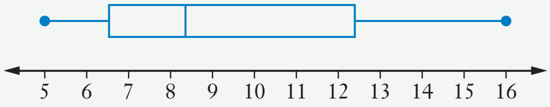
b Mean 168.3, variance 92.2

2 4

3 a



b Positively skewed. There is an outlier at 25.



Without the outlier it is still slightly positively skewed, but it is more symmetrical.

4 **a** **i** 15 **ii** 5.66

b

Score x	$x-\bar{x}$	$(x-\bar{x})^2$
7	–8	64
11	–4	16
15	0	0
19	4	16
23	8	64
	$\Sigma(x-\bar{x})=0$	$\Sigma(x-\bar{x})^2=160$

c 5.66

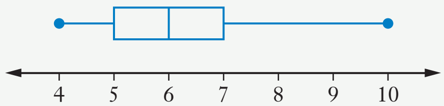
5 25.375

6 **a** Set B has a higher average or centre, and more consistent results (less spread out than set A).

b Set B has a higher average and is less consistent (more spread out).

7 **a** Mean 6.04, standard deviation 1.41

b



c Positively skewed

8 **a** Positively skewed

b Negatively skewed

Chapter 10

Exercise 10.01

1 **a** 7 **b** –15 **c** 0 **d** 9

e 6 **f** 15

2 **a** $4\sqrt{6}-4\sqrt{2}$ **b** $10\sqrt{6}-10\sqrt{2}$

c $-3\sqrt{6}-3\sqrt{2}$

3 **a** 14° **b** 135° **c** 90° **d** 50°

e 87° **f** 132°

4 $x=-1$ **5** 4

6 **a** **i** 60° 15′ **ii** 49° 24′ **iii** 109° 39′

b Exterior angle of a triangle = angle sum of interior opposite angles

7 **a** 0 **b** 90°

8 $x=-12$

9 $a=4.4$ or -24.4

10 $b=-1.5i+1.5j$ or $b=-2.1i-0.3j$

11 **a** $|v|^2=25$

$v\cdot v=3\times 3+-4\times -4=25$

b $|v|^2=a^2+b^2$

$v\cdot v=a\times a+b\times b=a^2+b^2$

Exercise 10.02

1 Show that $u\cdot v=0$ (perp)

2 Show that $u\cdot v=-|u||v|$ (parallel, unlike direction)

3 **a** $x=2.5$ **b** $x=-1.6$

4 **a** Parallel **b** Neither

c Perpendicular **d** Perpendicular

e Parallel **f** Perpendicular

5 $b=5$

6 **a** $n=12.5$ **b** $n=-2$

Exercise 10.03

1 **a** $\begin{pmatrix}4\\2\end{pmatrix}$ **b** $\frac{12}{13}\begin{pmatrix}7\\-4\end{pmatrix}=\begin{pmatrix}646\\-369\end{pmatrix}$

2 **a** $-2.4i+7.2j$ **b** $2.5i-12.5j$

c $\begin{pmatrix}2\\4\end{pmatrix}$ **d** $-6i-4j$

e $\begin{pmatrix}2\ 4\\-1\ 8\end{pmatrix}$

3 $\begin{pmatrix}\frac{3\sqrt{18}+3\sqrt{6}}{4} & \frac{-3\sqrt{6}-3\sqrt{2}}{4}\end{pmatrix}$

4 **a** $\begin{pmatrix}024\\-032\end{pmatrix}$ **b** $\begin{pmatrix}0\ 8\\0\ 4\end{pmatrix}$ **c** 0.4 **d** 307°

5 $\text{pro}_u\, v = \frac{u \cdot v}{|u|^2} u$ where $\frac{u \cdot v}{|u|^2}$ is a scalar

$|\text{pro}_u\, v| = \frac{u \cdot v}{|u|^2}|u| = \frac{u \cdot v}{|u|}$

Exercise 10.04

Proofs (see worked solutions).

Exercise 10.05

1 5.0 km, 84° **2** 1012.6 m s^{-1}, 199°

3 **a** 75 km h^{-1} west **b** 22 km h^{-1} west

c 47.4 km h^{-1}, 252°

4 9.4 km h^{-1}

5 **a** 20.6 N, 331° **b** 27.7 N, 331°

c 188.8 N, 82° **d** 23.7 N, 173°

e 215.5 N, 309°

6 4 kN, 11°

7 67.6 km h^{-1}, 83°

Exercise 10.06

1 **a** $s = \frac{15\sqrt{2}}{2} t\, i + \left(-5t^2 + \frac{15\sqrt{2}}{2} t\right) j$

b $\frac{3\sqrt{2}}{2}$ s

2 **a** $6\sqrt{3}$ s **b** 540 m

3 **a** 2.3 s **b** 5.7 m

4 **a** 2.59 m **b** 7.3 m

5 2.8 s

6 **a** 227 s or 3 min 47 s **b** 102 km

7 **a** $a = -g\, j$

$v = u \cos\alpha\, i + (-gt + u\sin\alpha)\, j$

$s = ut\cos\alpha\, i + \left(-\frac{gt^2}{2} + ut\sin\alpha\right) j$

b 15 m

8 $s = 16t\cos\theta i + (-5t^2 + 16t\sin\theta)\, j$

9 $s = vt\cos\beta i + \left(-\frac{gt^2}{2} + vt\sin\beta + h\right) j$

10 21° **11** 1.8 m s^{-1}

12 28.86 m s^{-1} **13** 3°, 87°

14 9.3 m s^{-1}

15 **a** 6 s **b** 91.7 m

16 63° or 51°

17 2 m **18** 8 m

19 0.1 m or 10 cm short of the fence

20 **a** 8 m s^{-1} **b** 53° **c** 3.2 m

d **i** 13.4 m s^{-1} **ii** 23.8 m s^{-1}

21 **a** 4.8 s **b** 100.9 m

22 **a** Second stone is first by 1.5 s.

b $20\sqrt{3}$ m

23 63°, 39° **24** 2.7 s

25 29.5 m

Test yourself 10

1 D **2** B **3** B **4** D

5 See worked solutions.

6 Let $u = \begin{pmatrix} x \\ y \end{pmatrix}$, $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and $w = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$

7 $b = 22$

8 **a** 87° **b** 45°

9 Proof (see worked solutions).

10 $b = 3i - 3j$ or $b = 1.6i + 4.2j$

11 89° or 8°

12 Proof (see worked solutions).

13 **a** $\begin{pmatrix} -048 \\ -121 \end{pmatrix}$ **b** $\begin{pmatrix} -112 \\ 084 \end{pmatrix}$

14 **a** −13 **b** 116° 34′

15 **a** $u \cdot v = 99$

$|u||v| = 99$

Since $u \cdot v = |u||v|$, parallel in like direction.

b $u \cdot v = -15$

$|u||v| = 15$

Since $u \cdot v = -|u||v|$, parallel in unlike direction.

16 **a** 43 **b** −74 **c** 23

17 81°, 17° **18** $3\sqrt{2} - 3\sqrt{6}$

19 $y = 1.6$ **20** $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ 13 \end{pmatrix}$

21 **a** 6.2 m s^{-1} south **b** 0.8 m s^{-1} south

c 4.2 m s^{-1}, 237°

22 $x = -1.98, -62.02$

23 Proof (see worked solutions).

24 $\begin{pmatrix} \dfrac{\sqrt{2}+\sqrt{6}}{4} \\ -\dfrac{(\sqrt{6}+3\sqrt{2})}{4} \end{pmatrix}$

25 **a** $x = 4$ **b** $x = -2\frac{}{4}$

26 $5.7\ \text{m s}^{-1}$

27 5.5 kN, 357°

28 $x = 13$

29 **a** $n = 12$ **b** $n = -\frac{3}{4}$

30 $-\frac{108}{53}i + \frac{60}{53}j$

31 Proofs (see worked solutions).

Challenge exercise 10

1 $23.2\ \text{km h}^{-1}$, 130°

2 Show that range is $\frac{v^2}{g}$ and maximum height is $\frac{v^2}{4g}$.

3 Carla jumps further by 0.8 m.

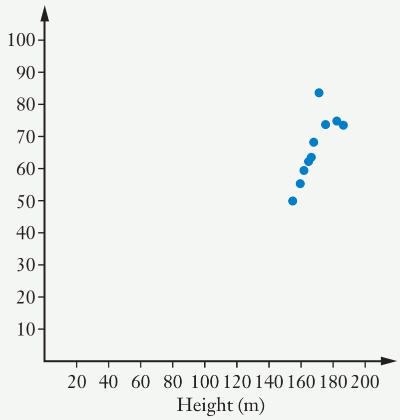
4, 6 Proofs (see worked solutions).

5 $a = -6, b = -6\sqrt{3}$

Chapter 11

Exercise 11.01

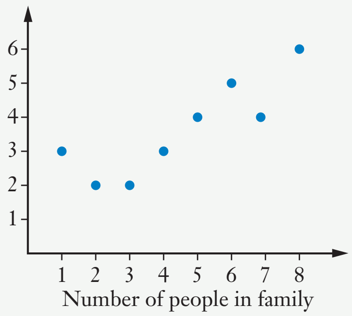
1 **a**



b The shape seems to be linear in a positive direction.

c As height increases, weight increases.

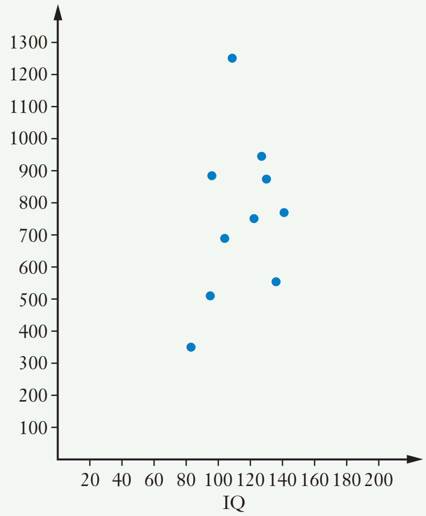
2 **a**



b The scatterplot is non-linear, or linear (positive) with outliers.

c With some exceptions, the more family members, the more bedrooms in the home.

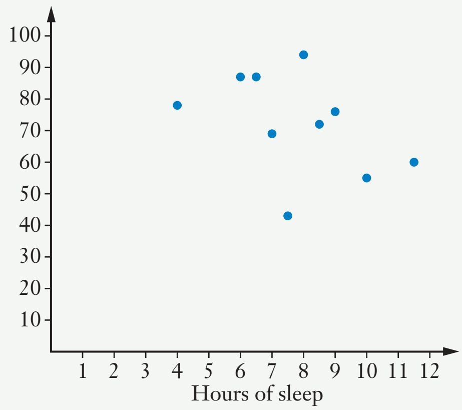
3 **a**



b The scatterplot has no pattern, or is linear (positive) with outliers.

c With some exceptions, the higher the IQ, the more the person earns.

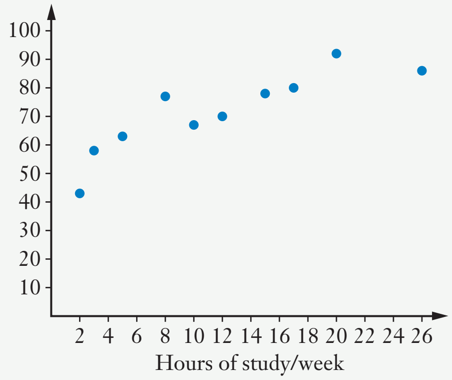
4 **a**



b The scatterplot has no pattern.

c There is little or no correlation found between amount of sleep and exam results.

5 a



b The shape is linear in a positive direction.

c As the number of hours of study increases, exam results increase.

6 a Non-linear

b Linear in negative direction

c No pattern

d Linear in positive direction

e Linear in negative direction with outliers

f Non-linear

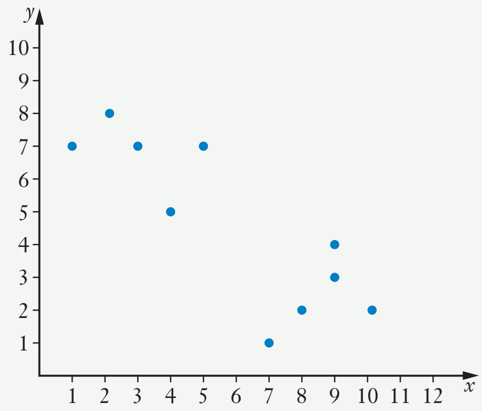
g No pattern

h Linear in positive direction

i Non-linear

j Non-linear

7 a



b Linear in a negative directon

8 a B **b** C **c** C **d** D

e A **f** C **g** B **h** A

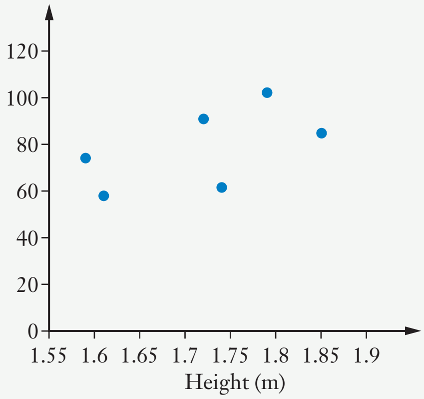
i C **j** D

Exercise 11.02

1 a G **b** B **c** F **d** A

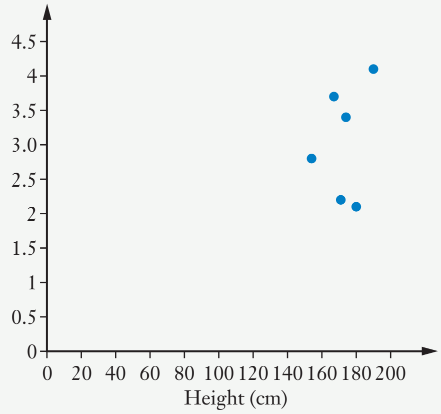
e C **f** D

2 a i



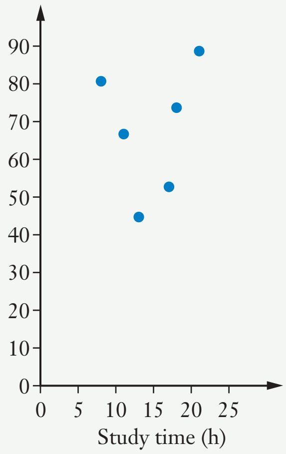
ii 0.571

b i



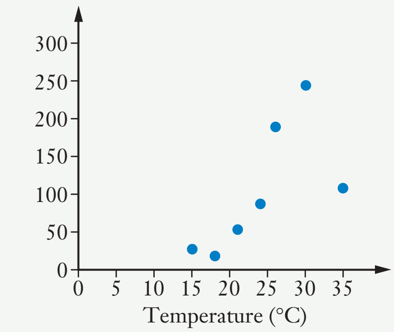
ii 0.284

c i



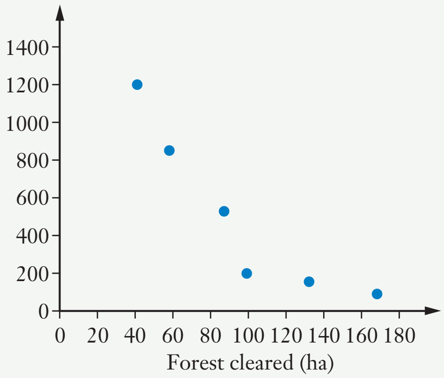
ii 0.181

d **i**



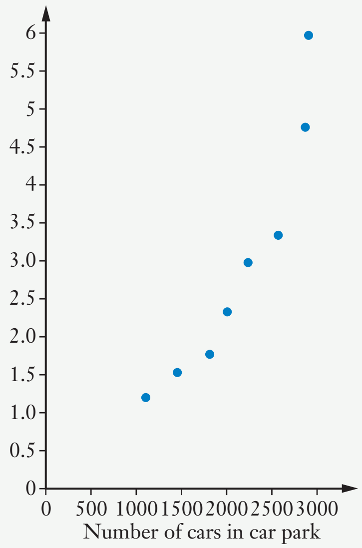
ii 0.685

e **i**



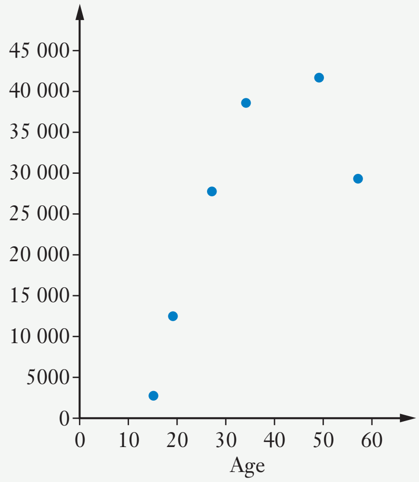
ii −0.908

f **i**



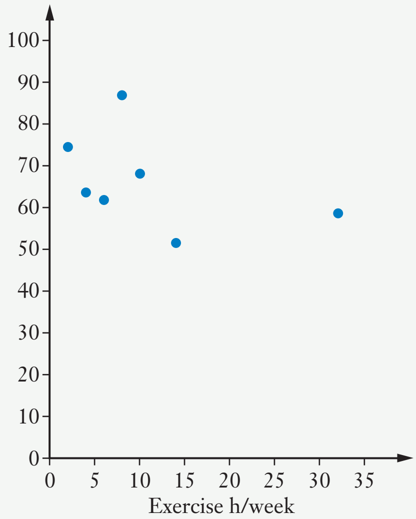
ii 0.930

g **i**



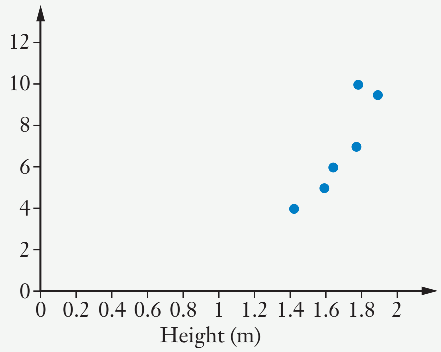
ii 0.749

h **i**



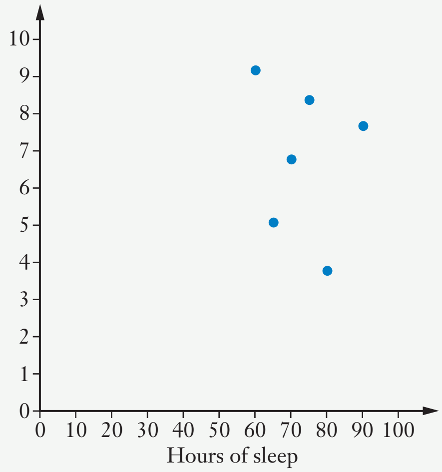
ii −0.430

i **i**



ii 0.905

j **i**



ii −0.182

3 d, e, f, i (Class discussion)

4 **a** 0.40 **b** −0.39 **c** 0.99 **d** −0.60

5 a, c and f (Class discussion)

Exercise 11.03

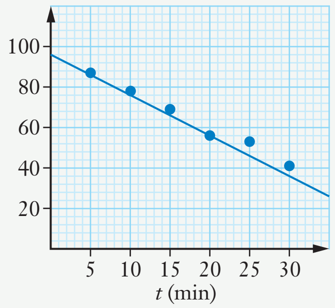
All answers are approximate.

1 **a** $y = 0.6x + 0.7$ **b** $N = -2t + 23$

c $V = 5.6x + 32$ **d** $P = 1.4t + 6.3$

e $A = -8x + 98$ **f** $x = 5.2t + 480$

2 **a**

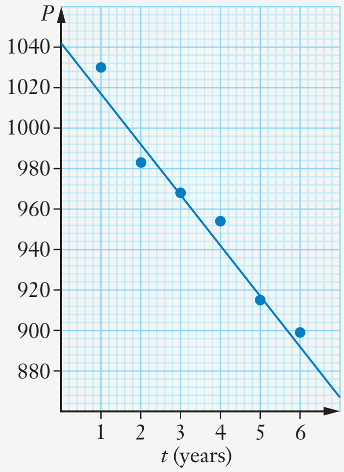


b $T = -2t + 95$

c **i** 61°C **ii** 25°C

d Good model when interpolating, but not for extrapolating because it can only cool to room temperature (23°C) but not below this.

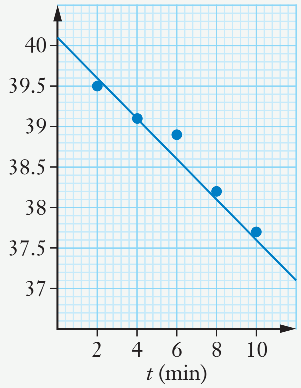
3 **a**



b $P = -25t + 1045$ **c** 870

d 42 years

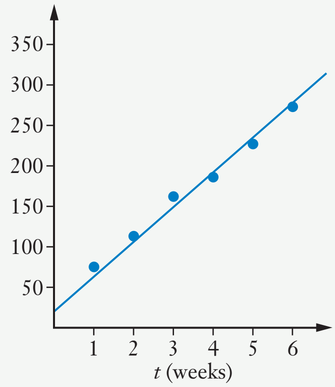
4 **a**



b $T = -0.24\,t + 40.1$ **c** 36.5°

d No (class discussion)

5 **a**



b $N = 37.5t + 42.5$ **c** 418

d No, because the number of people cannot keep increasing forever.

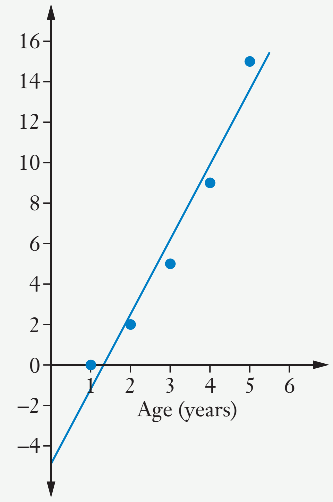
Exercise 11.04

1 **a** **i** $r = 0.9923$ **ii** $m = 2.43$

b **i** $r = 0.976$ **ii** $m = 3.89$

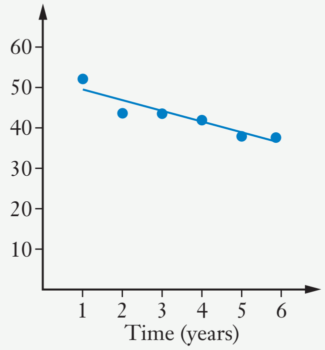
2 **a** **i** $y = 3.7x - 4.9$

ii



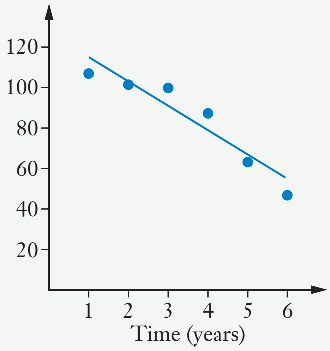
b **i** $y = -2.65x + 52.18$

ii



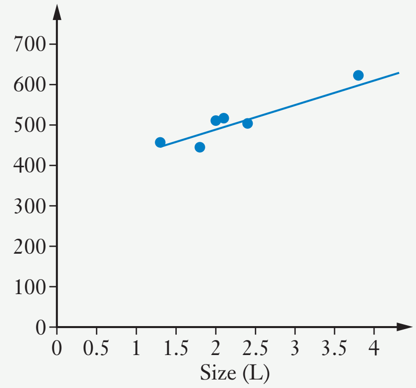
c **i** $y = -1224\,x + 127.43$

ii



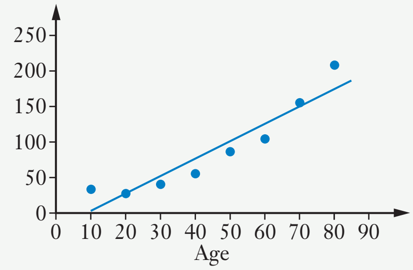
d **i** $y = 69.88x + 355.43$

ii



e **i** $y = 2.49x - 22.36$

ii



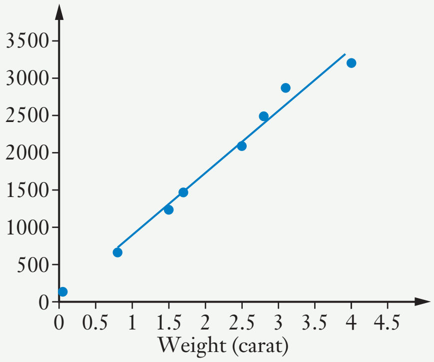
3 **a** $r = -0.954$, which shows a high negative correlation. This means that the regression line has a negative gradient and that the mass decreases as time increases.

b $m = -0.614t + 23.56$

c **i** 125 kg **ii** −13.3 kg

d The answer to **c ii** shows that extrapolation is not useful in this situation. The ice has melted to 0 volume before an hour has passed and the equation is no longer relevant.

4 **a**



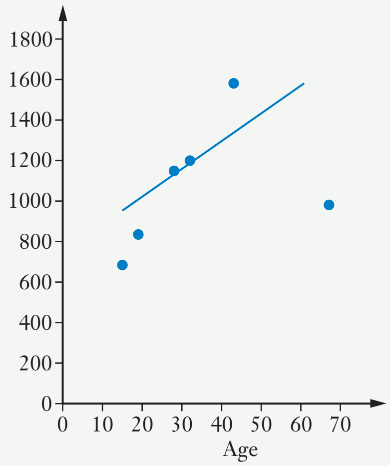
b 0.993. This is a high positive correlation.

c $y = 831.7x + 69.2$

d $1732.59

e 11.94 carats

5 **a**



b 0.395

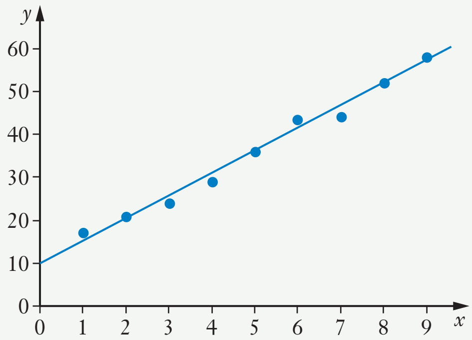
c $y = 6.6x + 853.5$ **d** $1181.92

e No. For example. using the equation to find the age of a person earning $1800 would give a 145-year-old!

Test yourself 11

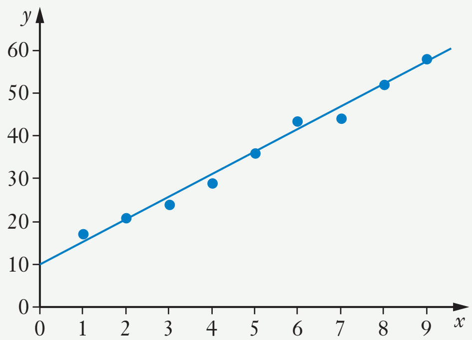
1 C **2** B **3** D **4** A **5** C

6 **a**



$y = 5x + 10$

b



$y = 5.1833x + 10.083$

7 **a** C **b** A **c** D **d** C

e B **f** A **g** C **h** D

8 **a** 0.84 **b** $y = 0.758x + 10.88$

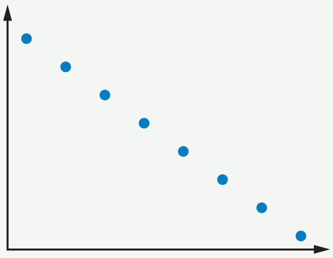
c 56 **d** 78

9 **c, d** are causal (Class discussion)

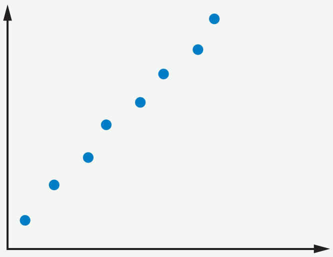
Challenge exercise 11

1 $r = 0.8$, $\bar{y} = 6.4$

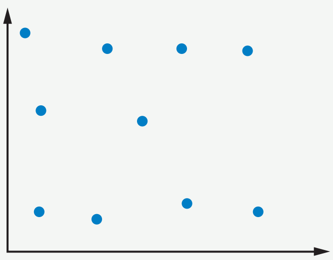
2 **a**



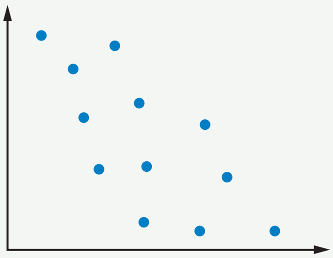
b



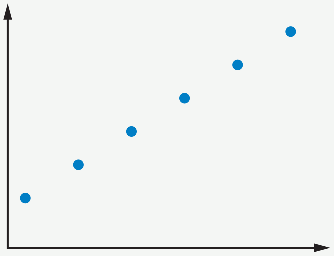
c



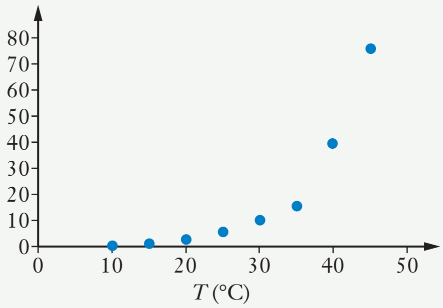
d



e

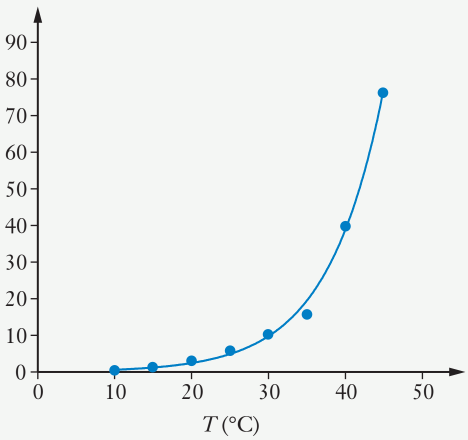


3 **a**



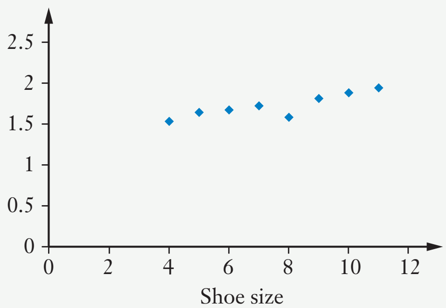
b The correlation is non-linear, so a least squares regression line would not be a good approximation

c



$V = 0.1563e^{0.1379T}$

4 **a**



b 0.876 **c** $y = 0.052x + 1.34$

d **i** 0.992 **ii** $y = 0054\,x + 1.35$

e No. Shoe sizes cannot keep increasing

Practice set 3

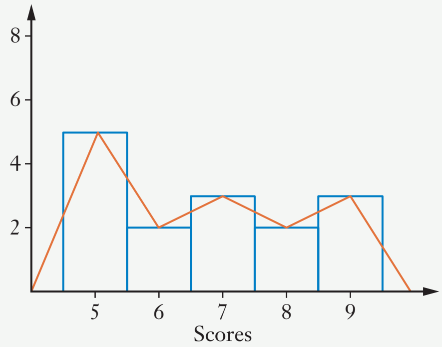
1 A **2** A **3** D **4** D **5** A, D

6 D **7** C **8** B **9** C

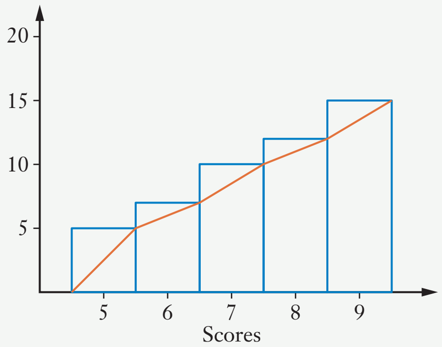
10 **a**

Score	Frequency
5	5
6	2
7	3
8	2
9	3

b



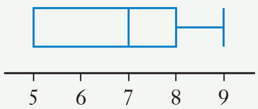
c



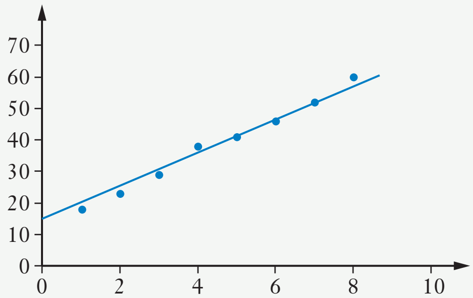
d 7

e 3

f

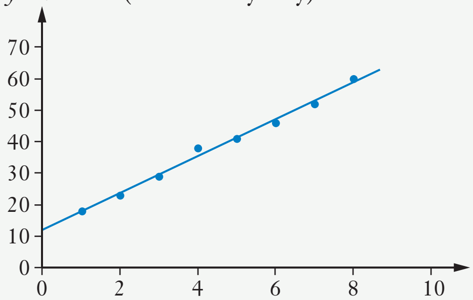


11 a



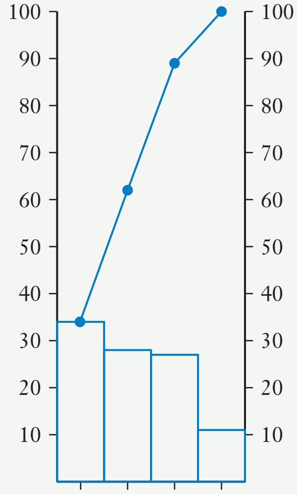
$y = 7x + 15$ (answers may vary)

b



$y = 5.869x + 11.964$

12



13 a $|\overrightarrow{OA}| = 5$, $|\overrightarrow{AB}| = \sqrt{50}$, $|\overrightarrow{BC}| = \sqrt{50}$ and $|\overrightarrow{CO}| = 5$

Since $|\overrightarrow{OA}| = |\overrightarrow{CO}|$ and $|\overrightarrow{AB}| = |\overrightarrow{BC}|$,

$OABC$ is a kite.

b $\overrightarrow{AC} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$\overrightarrow{AC} \cdot \overrightarrow{OB} = 0$

So $\overrightarrow{AC}$ and $\overrightarrow{OB}$ are perpendicular.

14 Mean 6.375, standard deviation 1.87, variance 3.48

15 a -1 **b** $91°55'$

16 a Mean 8.04, standard deviation 1.54

b 3 is not between 4 and 12, so 3 is an outlier.

c Mean 8.26, standard deviation 1.15

17 a 69° **b** 5°

18 $\sin^{-1}\left(\frac{x}{2}\right) + C$

19 Mean 6.43, mode 6, median 6, range 5

20 $8\sqrt{6} - 8\sqrt{2}$

21 a 1.2 s **b** $\frac{36\sqrt{3}}{5} \approx 12.5$ m **c** 1.8 m

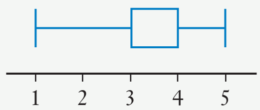
22 $\begin{pmatrix} -4 \\ -9 \end{pmatrix}, \begin{pmatrix} 8 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 11 \end{pmatrix}$

23 a 52

b Mean 3.58, standard deviation 1.01

c 4

d



24 a $x = 12$ **b** $x = -1\frac{1}{3}$

25 -0.82

26 $b = 5.4i + 3.5j$ or $b = 3.1i - 5.6j$

27 $u \cdot v = |u||v| \cos 90° = 0$

28 a $|u||v| = u \cdot v = 40$

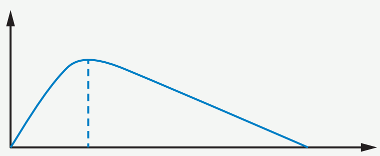
b $|u||v| = 20$, $u \cdot v = -20$

29 a $\begin{pmatrix} 528 \\ 396 \end{pmatrix}$ **b** $\begin{pmatrix} 234 \\ -098 \end{pmatrix}$

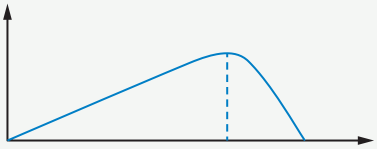
30 4.8 kN, 360° (0°, or straight ahead)

31 $b = 6$

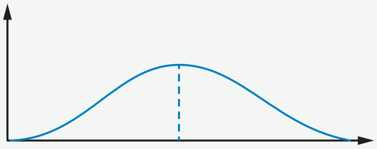
32 a



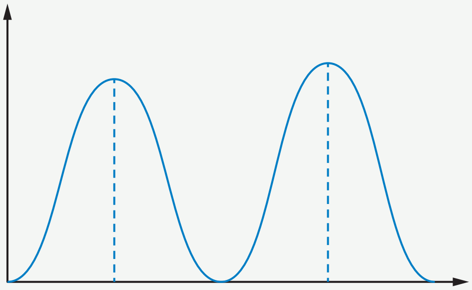
b



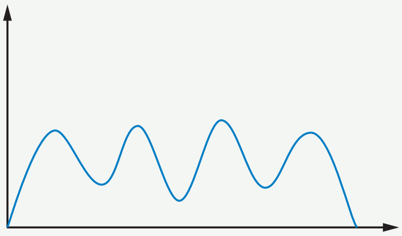
c



d



e



33 **a** categorical nominal
b quantitative continuous
c categorical ordinal
d quantitative discrete
e quantitative discrete

34 **a** 42 km h^{-1} N **b** 9 km h^{-1} N
c 28.9 km h^{-1} N 14° W

35 $y = -3$

36 4 m s^{-1}

37 Solution includes:
$\angle AOB = 69° \, 14'$, $\angle OAB = 93° \, 57'$,
$\angle OBA = 24° \, 43'$
Angle sum $= 69° \, 14' + 86° \, 3' + 24° \, 43'$
$= 180°$

38 **a** C **b** D **c** B **d** A
e D **f** C

39 –59

40 $\frac{1}{2}x + \frac{1}{4}\sin 2x + C$

41 **a** 3.5 **b** 2.1 **c** 4.5 **d** 2.5

42 (4, –143) min, (–2, 73) max, (1, –35) point of inflection

43 **a** $x = \frac{\pi}{6}, \frac{5\pi}{6}$
b $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
d $x = \frac{\pi}{2}, \frac{11\pi}{6}$

44 **a** 399 **b** 20 100

Chapter 12

Exercise 12.01

1 **a** 210 **b** 13th **c** 57
2 **a** 39 **b** 29th **c** 32
3 **a** $3n + 3$
b $S_n = \frac{1}{2}n[2 \times 6 + (n - 1) \times 3]$
4 **a** 0.01 m **b** 91.5 m
5 **a** 7 **b** 10.5 m
6 **a** \$2050 **b** \$2100 **c** \$2150
d \$2500 **e** \$3500
7 **a** 1.25 m **b** 8
8 **a** 4.5–5 kg **b** 18th
9 **a** 49 **b** 4 mm
10 **a** $3k$ m **b** $3k(k + 1)$ m **c** 9

Exercise 12.02

1 **a** **i** 93% **ii** 86.49% **iii** 80.44%
b 33.67% **c** 19 weeks
2 **a** 96.04% **b** 34 days **c** 114 days
3 **a** **i** \$23 200 **ii** \$26 912
iii \$31 217.92
b \$102 345.29 **c** 6.2 years
4 **a** 77.4% **b** 13.5 years **c** 31.4 years

5 **a** $\frac{4}{9}$ **b** $\frac{7}{9}$ **c** $1\frac{2}{9}$ **d** $\frac{25}{99}$

e $2\frac{9}{11}$ **f** $\frac{7}{30}$ **g** $1\frac{43}{90}$ **h** $1\frac{7}{450}$

i $\frac{131}{990}$ **j** $2\frac{361}{999}$

6 0.625 m **7** 15 m

8 20 cm **9** 3 m

10 **a** 4.84 m **b** After 3 years

11 **a** 74.7 cm **b** 75 cm

12 300 cm **13** 3.5 m

14 32 m

15 **a** 1, 8, 64, 512, …

b 16 777 216 people

c 19 173 961 people

Exercise 12.03

1 **a** $6895.85 **b** $6999.79 **c** $7633.37

d $6857.36 **e** $7207.26

2 **a** $2852.92 **b** $12 720.32 **c** $4038.13

d $5955.08 **e** $87 362

3 $893 262 **4** $3552.86

5 $21 173.72 **6** $1069.23

7 $4902.96

8 **a** $1125.68 **b** $2209.20 **c** $14 930

d $96 630 **e** $305 900 **f** $902.22

g $1300.26 **h** $3090.90 **i** $1061.50

j $3866.10

9 **a** $8705.49 **b** $4971.66 **c** $4631.99

d $9705.91 **e** $8227.07

Exercise 12.04

1 **a** $740.12 **b** $14 753.64

c $17 271.40 **d** $9385.69

e $5298.19

2 **a** $2007.34 **b** $2015.87 **c** $2020.28

3 **a** $4930.86 **b** $4941.03

4 **a** $408.24 **b** $410.51

5 **a** $971.40 **b** $972.12

6 **a** $1733.99 **b** $1742.21 **c** $1747.83

7 **a** $3097.06 **b** $13.47

8 **a** $22 800.81 **b** $945.92

9 $691.41 **10** $1776.58

11 $14 549.76 **12** $1 301 694.62

13 **a** $4113.51 **b** $555.32 **c** $9872.43

d $238.17 **e** $10 530.59

14 $4543.28 **15** 4 years

16 8 years

17 **a** $x = 7$ **b** $x = 5$ **c** $x = 8$

d $x = 6.5$ **e** $x = 8.5$

18 $7.68 **19** Kate $224.37

20 Account A, $844.94

21 **a** **i** $39 400.53 **ii** $41 812. 16

b 27th year

22 **a** **i** $128 547 **ii** $175 196.37

iii $230 700.16

b 28th year

23 **a** **i** 4 years **ii** 17 years

b **i** 3.1% **ii** 1.8%

24 **a** 1.9990

b $A = 1.08^9$

$= 1.9990$ to 4 decimal places

25 FV interest factor is 1.5209.

$A = 1.15^3$

$= 1.5209$ to 4 decimal places

Exercise 12.05

1 **a** $10 125 **b** $3745.92 **c** $2720.63

d $20 410 **e** $6194.05

2 **a** $49 759.29 **b** $5524.36

c $94 334.25 **d** $2074.02

e $274 948.05 **f** $3981.08

g $20 103.20 **h** $15 559.48

i $327 214.50 **j** $1 474 272

3 **a** $10 789.40 **b** $22 839.56

c $69 720.25 **d** $9511.37

e $11 513.83

4 $477 076.25

5 $2074.70

6 **a** $1469.39 **b** $2811.42 **c** $1266.10

d $614.91 **e** $30 242.55

7 $296.59

8 **a** **i** $47 000 **ii** $43 880 **iii** $40 635.20
b $43 756.80 **c** $41 833.42
9 **a** $125 750 **b** $126 507.50
c $127 272.58 **e** $125 376.88
d $125 757.53
10 **Graph A: a** 18 months **b** $10 400
c 36 months (3 years)
Graph B: a 27 months (2 years 3 months)
b $12 200
c 46 months (3 years 10 months)
11 **a** 18 **b** 21
12 **a** 9% **b** 3%

Exercise 12.06

1 $40 728.17 2 $3383.22
3 $65 903.97 4 $2846.82
5 $6181.13 6 $4646.71
7 $21 426.03
8 **a** $26 361.59 **b** $46 551.94
9 $45 599.17
10 **a** $7922.80 **b** $1584.56
11 $500 for 30 years ($11 640.86 better off)
12 Yes, $259.80 over 13 $818.72
14 **a** $37.38
b Proof (see worked solutions).

Exercise 12.07

1 **a** $19 234.54 **b** $3177.64
c $99 990.13 **d** $1798.81
e $45 659.71
2 **a** **i** $5978.52 **ii** $978.52
iii 6.5%
b **i** $24 225.60 **ii** $8325.60
iii 10.5%
c **i** $159 785.28 **ii** $79 785.28
iii 8.3%
d **i** $272 127 **ii** $37 127
iii 0.63%
e **i** $1710.48 **ii** $362.48
iii 13.4%
3 **a** $154.64 **b** $304.20 **c** $854.64
d $2605.80 **e** $2549.04 **f** $686.37
g $568.89 **h** $3021.75 **i** $3922
j $5680
4 **a** $3946.90 **b** $947 256
c $266 756 **d** 1.96%
5 **a** 5 years **b** 10 years **c** 20 years
d 30 years **e** 10 years **f** 25 years
g 10 years **h** 15 years **i** 10 years
j 20 years
6 **a** 6% **b** 4.5% **c** 2%
d 8% **e** 5.5% **f** 7%
g 2.5% **h** 3% **i** 7.5%
j 4%

Exercise 12.08

1 $1047.62 2 $394.46 3 $139.15
4 **a** $966.45 **b** $1265.79
5 $2519.59
6 **a** $868.46 **b** $55 907.60
7 **a** $77.81 **b** $2645.42
8 $78 700
9 **a** Get Rich $949.61, Capital Bank $491.27
b $33 427.80 more through Capital Bank
10 **a** $875.60 **b** $43 778.80
11 $61 292.20
12 NSW Bank totals $5791.25, Sydney Bank totals $5557.88 so Sydney Bank is better.
13 **a** $249.69 **b** $13 485.12
14 **a** $13 251.13 **b** $374.07
c $20 199.78
15 **a** $1835.68 **b** $9178.41
16 **a** $10.36
b Proof (see worked solutions).

Test yourself 12

1 C 2 B 3 A
4 $21 980.94 5 $1672.74
6 **a** Each slat rises 3 mm so the bottom one rises up 30×3 mm or 90 mm.
b 87 mm
c 90, 87, 84, … is an arithmetic sequence with $a = 90, d = -3$.
d 42 mm **e** 1395 mm
7 **a** $121 320 **b** $58 820 **c** 18.8%

8 **a** \$723.22 **b** \$9995 **c** \$2925.27
d \$16 785.90 **e** \$11 345.01

9 **a** \$24 050 **b** \$220 250

10 **a** \$184 867.25 **b** \$182 829.42
c \$180 786.49

11 \$1285.19

12 **a** 10 stacks **b** 110 boxes

13 **a** $\frac{4}{9}$ **b** $\frac{13}{18}$ **c** $1\frac{19}{33}$

14 \$3400.01

15 **a** \$59 000 **b** \$15 988.89

16 **a** \$2385.04 **b** \$2392.03

17 **a** \$1432.86 **b** \$343 886.91

18 **a** \$2651.56 **b** \$9872.43

19 **a** 39 words/min **b** 15 weeks

20 4.8 m

21 **a** **i** \$33 000 **ii** \$23 000
b **i** 14 years **ii** 28 years
c 32 years

Challenge exercise 12

1 \$1799.79

2 **a** \$40 **b** \$2880

3 \$8522.53

4 **a** **i** 115.8° **ii** 51.4°
b **i** 6.6 min **ii** 10.9 min

5 **a** $\$1000(1.06)^{25}$ **b** $\$1000(1.06)^{24}$
c $\$1000(1.06)^{23}$ **d** \$1000(1.06)
e $S_{25} = \frac{1000 \times 106 \times (106^{25} - 1)}{106 - 1}$
f \$58 156.38

6 **a** \$10 100 **b** \$11 268.25
c \$4212.41 **d** \$2637.22

7 \$466 563.74

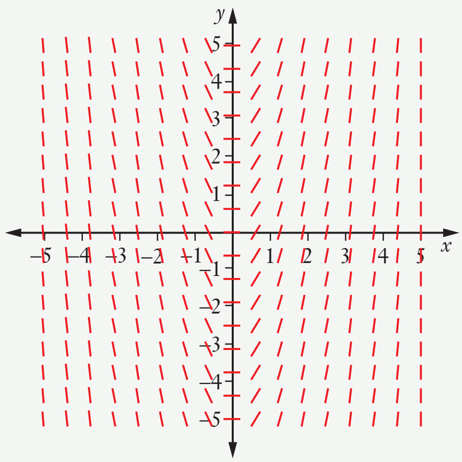
Chapter 13

Exercise 13.01

1 **a** $y = 3x^2 + x + C$
b $y = x^3 - 2x^2 + 5x + C$
c $f(x) = 3 \ln |x| + C$
d $y = \frac{1}{4}e^{4x} + C$
e $f(x) = \frac{3}{2}\sin 2x + C$
f $f(x) = \sin^{-1} x + C$
g $y = \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

2 **a** $y = 2x^3 - x^2 - 3x + 4$
b $y = \frac{2e^{3x}}{3} + 3\frac{1}{3} = \frac{2(e^{3x} + 5)}{3}$
c $f(x) = -\frac{1}{3}\cos 3x + 1\frac{2}{3}$
d $f(x) = \cos^{-1}\frac{x}{3}$
e $y = \frac{2}{3}\ln |x| + 2$

3 **a**



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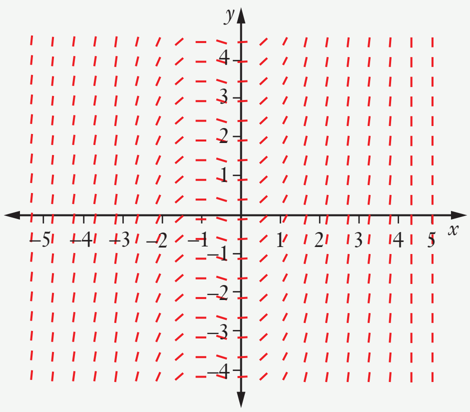

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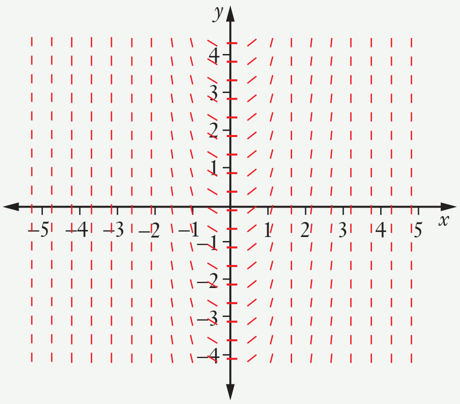

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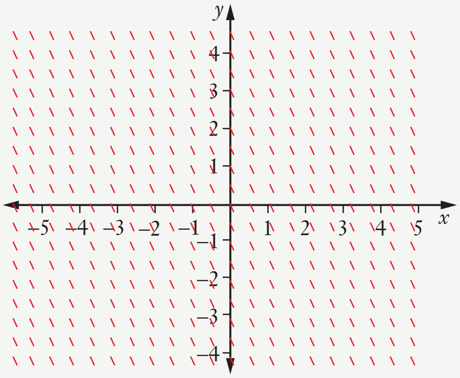

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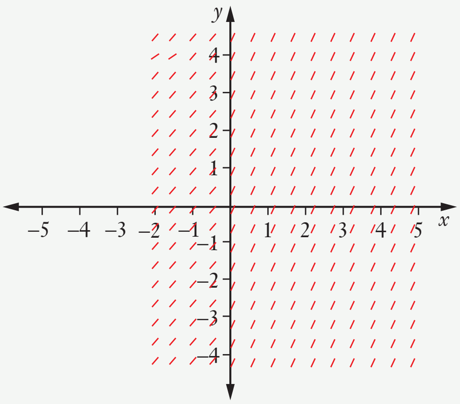
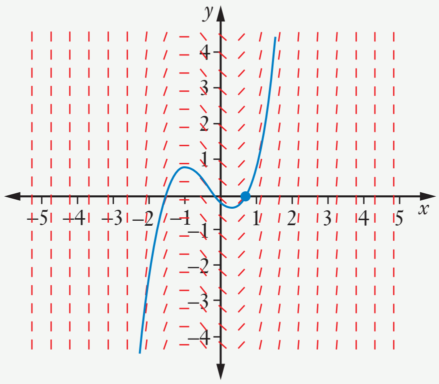


4 a



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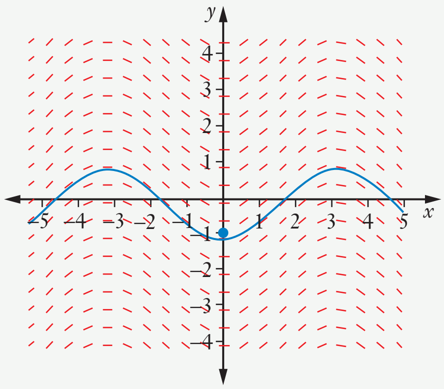

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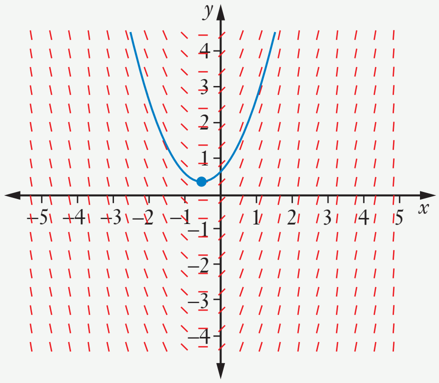

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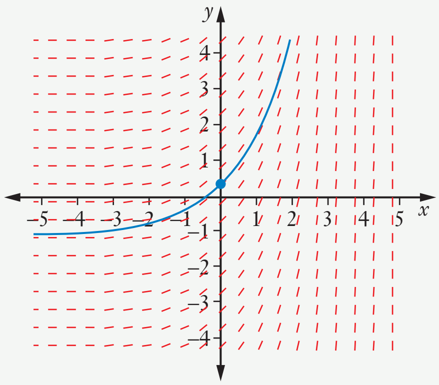

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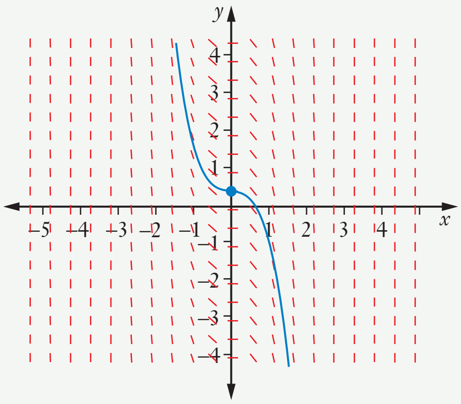
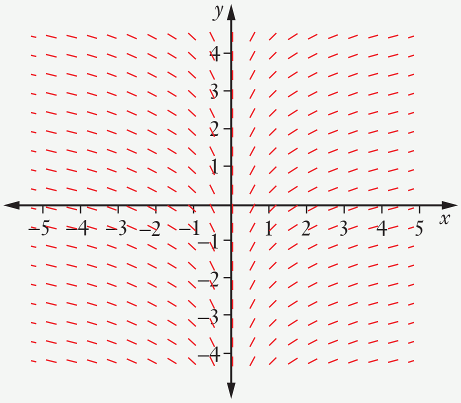
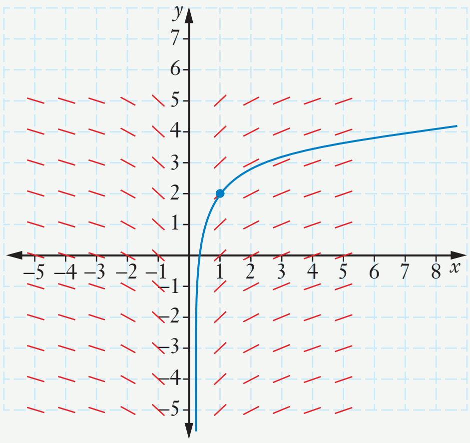


5 a

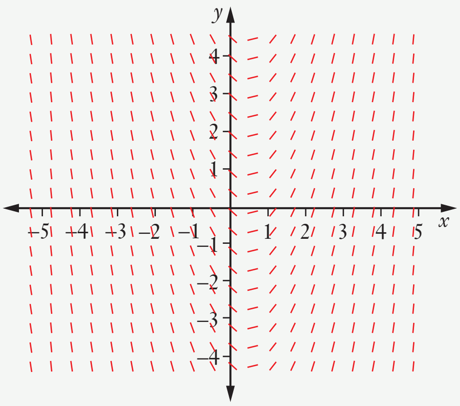


b $y = \ln|x| + 2$

c

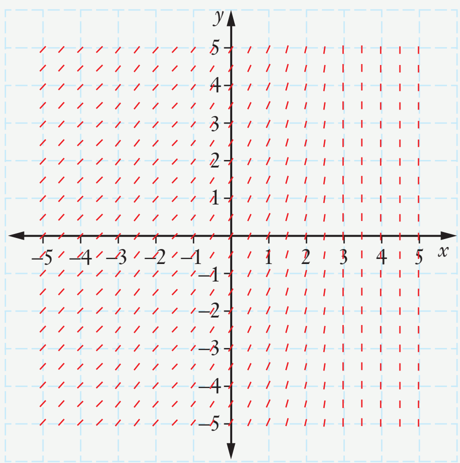


6 a



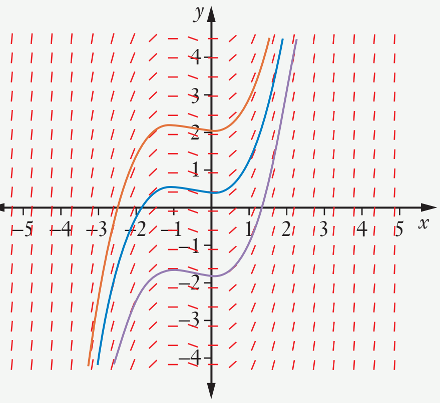
b $f(x) = x^2 - x - 3$

7 a

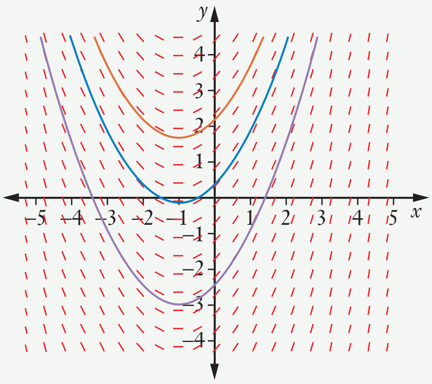


b $y = e^x + x$

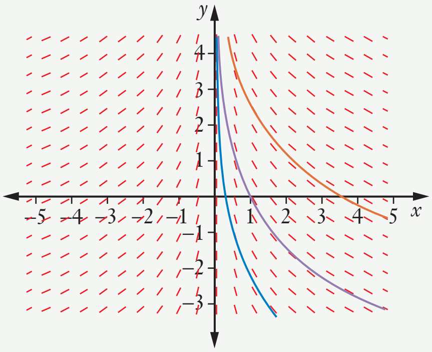
8 a



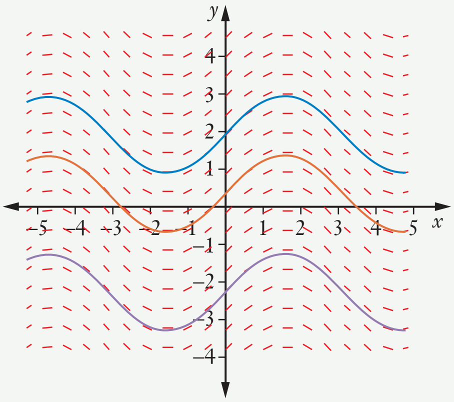
b



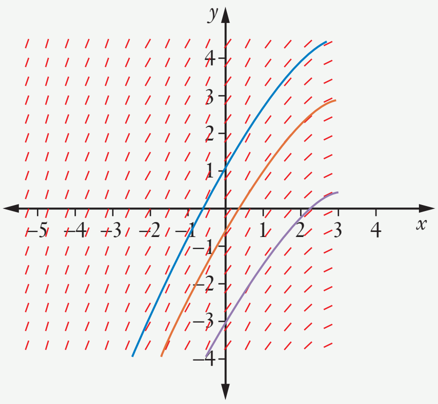
c



d

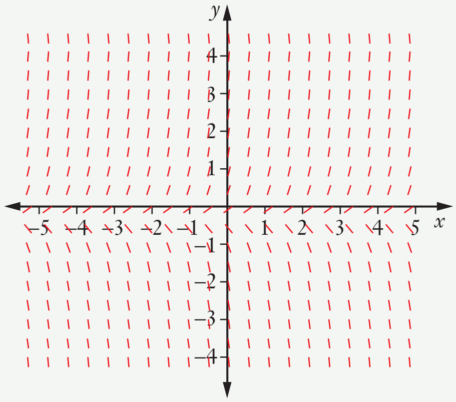


e

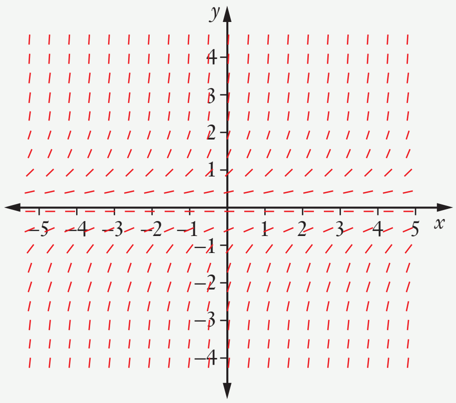


Exercise 13.02

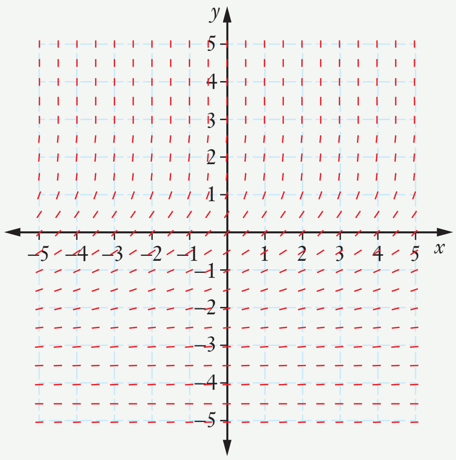
1 a



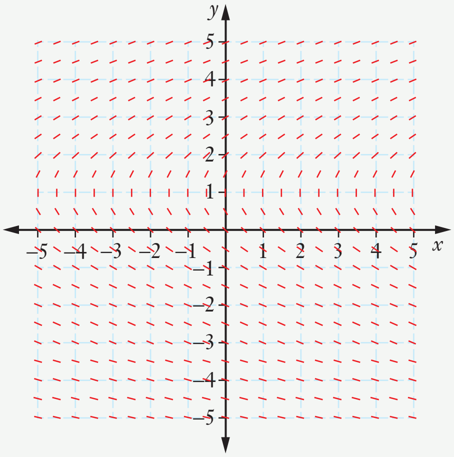
b



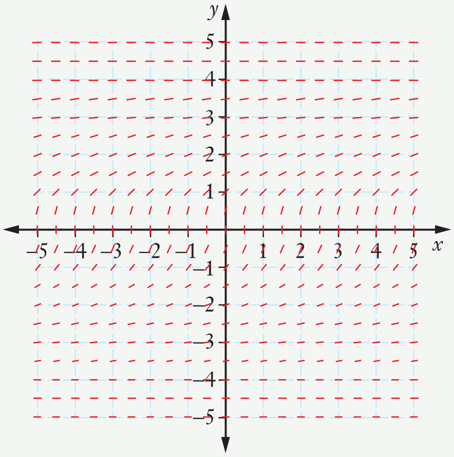
c



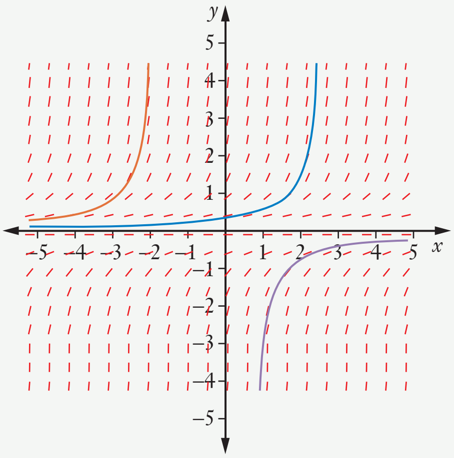
d



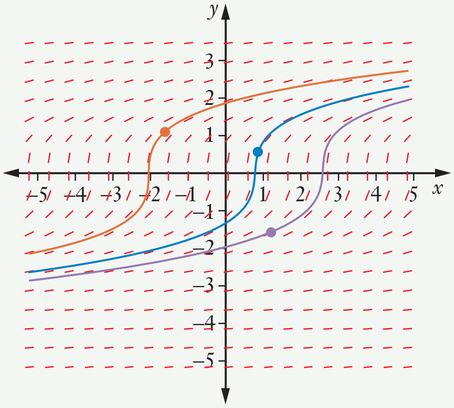
e



2 a



b



3 a $y=\sqrt[3]{6x+C}$

b $y=\left(\dfrac{x}{2}+C\right)^2$

c $y=\ln|x+C|$

d $y=\sqrt[5]{5x+C}$

e $y=-\dfrac{1}{x+C}$

4 a $y=\sin(x+C)$

b $y=3\tan(3x+C)$

5 $y=\sqrt[4]{\dfrac{x+C}{2}}$ or $y=-\sqrt[4]{\dfrac{x+C}{2}}$

6 a $y=\sqrt{x+8}-3$

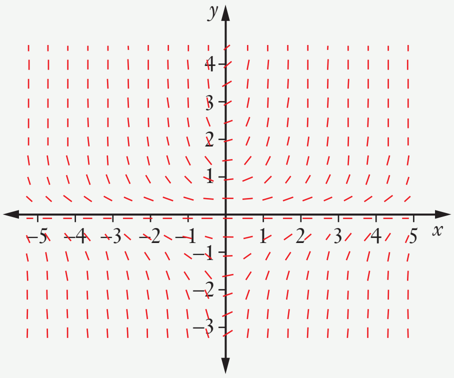
b $y=-\sqrt{x+8}-3$

7 $y=\sqrt{\dfrac{x+20}{2}}+3, y=-\sqrt{\dfrac{x+20}{2}}+3$

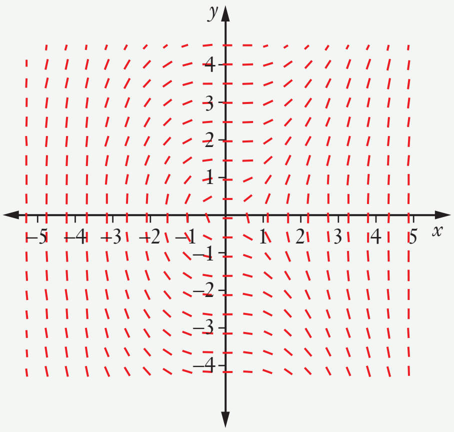
8 $y=\sqrt[3]{\sqrt[5]{15x+C}+1}$

Exercise 13.03

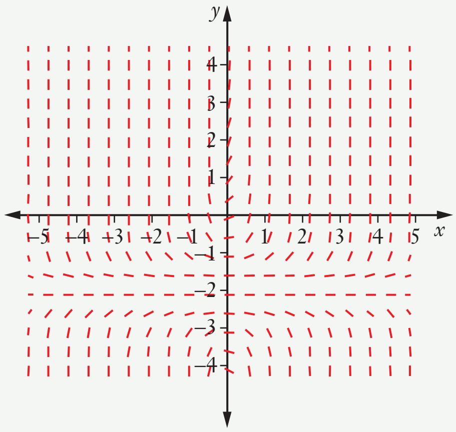
1 a



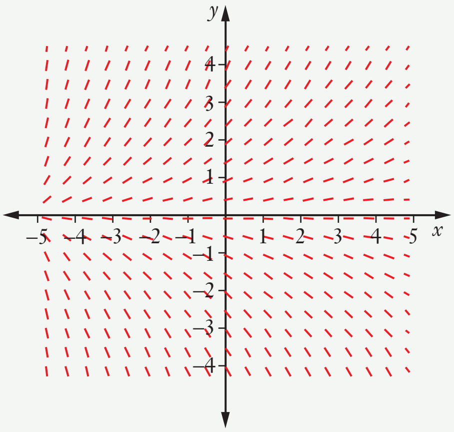
b



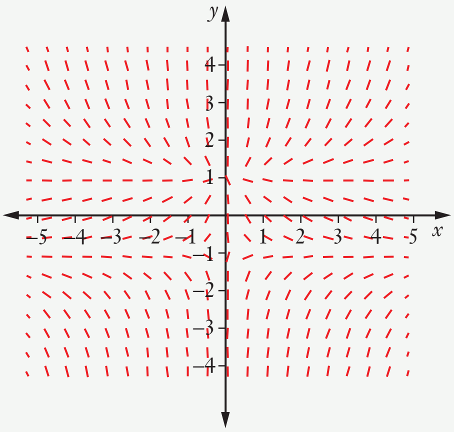
e



c



d



2 a, c, e

3 **a** $y=\pm\sqrt{x^2+C}$ **b** $y=\pm\sqrt{\ln|x|+C}$

c $y=\sqrt[3]{\frac{x^2}{4}+C}$ **d** $y=ke^{\frac{x}{2}}$

e $y=-\frac{1}{2x^3+C}$ **f** $y=\sqrt[3]{\frac{x^2}{2}-3x+C}$

g $y=\pm\sqrt{\frac{8x^3}{3}+C}$ **h** $f(x)=-\frac{1}{3x^4+C}$

i $f(x)=ke^{2(x+3}$ **j** $y=\sqrt[3]{\frac{(2x-1)^6}{4}+C}$

4 **a** $y=\frac{1}{28-x^3}$

b $f(x)=\sqrt[3]{\left(\frac{3x^2+6x+32}{4}\right)^2}$

c $y=\sqrt{-\frac{1}{x^2-2}+8}$

d $y=e^{-\cos x}$ **e** $f(x)=2\sin(3x^2-2)$

5 **a** $y=\sqrt{x^2+2x+16}$ **b** $y=-\sqrt{x^2+2x-34}$

6 **a** $\frac{dy}{dx}=\frac{\cos x}{\sin x}=\cot x$ **b** $y=-\sqrt{2\ln|\sin x|+1}$

7 $y=\sqrt{\frac{(x^3+1)^4+134}{6}}$

8 **a** $\frac{(\text{n}|x|)^2}{2}+C$ **b** $y=\sqrt{(\text{n}|x|)^2+4}$

Exercise 13.04

1 **a** $Q = 12e^{0.2t}$ **b** $N = 500e^{0.05t}$

c $T = 450e^{-1.03}$ **d** $T = 28 + 87e^{-0.2t}$

e $Q = 150 + 130e^{-0.65t}$

2 **a** $N = 600e^{0.18t}$

b **i** 1476 **ii** 3630

c 6.7 years

d **i** 266 cattle/year **ii** 653 cattle/year

3 **a** $M = 100e^{-0.045t}$ **b** 40.7 g

c −1.8 g/year **d** 15.4 years

4 **a** $T = 20 + 130e^{-0.15t}$

b **i** 72.9°C **ii** 21.4°C

5 **a** $P = 60 + 20e^{0.22t}$ **b** 11.3 months

6 **a** $T = 27 + 223e^{-0.04t}$

b **i** 176.5°C **ii** 127.2°C

iii 47.2°C

c 27°C

7 **a** RHS = LHS

b $t = \ln\left|\frac{x}{x+1}\right| + C$

8 **a** $\text{RHS} = \frac{3}{(x-1)(x+2)}$

b $t = \ln\left|\frac{Q-1}{Q+2}\right| + C$

9 **a** $t = \ln\left|\frac{N-1}{N}\right| + C$ **b** $N = \frac{10}{10-9e^{t}}$

Test yourself 13

1 B **2** C **3** B **4** D

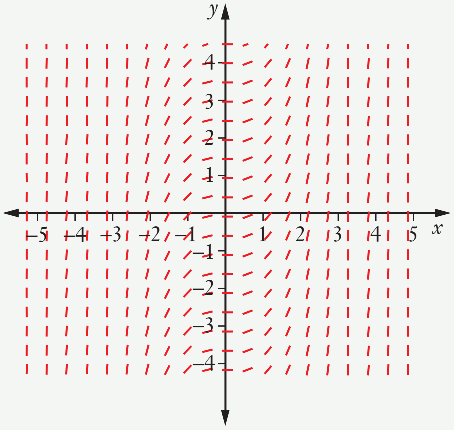
5 **a** $y = 2x^2 - 3x + 6$

b $y = -\sqrt{2x+1}$

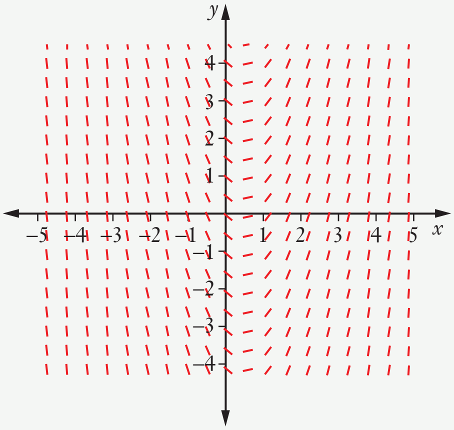
c $Q = 450e^{-0.1t}$

d $T = 20 + 280e^{-0.05t}$

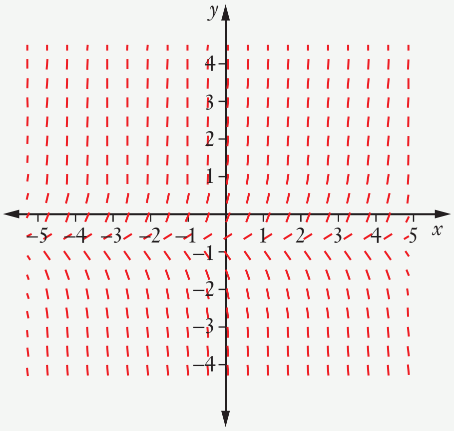
6 **a**



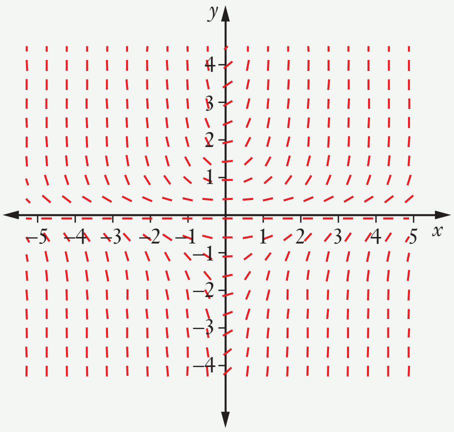
b



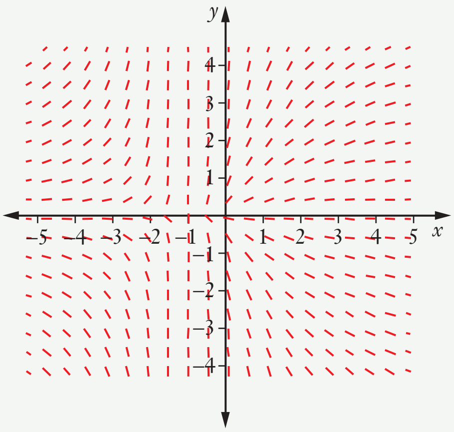
c



d



e



7 **a** $y = 6e^{2x}$ **b** $y = \sqrt[3]{9(e^{2x} + 2)}$

c $y = \sin x$ **d** $y = \frac{1}{2}\sin^{-1} 2x$

e $y = -\sqrt{2\ln|x| + 9}$

8 **a** $T = 18 + 482e^{-0.15t}$

b **i** 245.7°C **ii** 68.8°C

c 18°C

9 **a** RHS = LHS

b $t = \ln [|N - 2||N + 3|^2] + C$

10 **a** $y = \cos^{-1}\left[-\left(x + \frac{\pi}{3}\right)\right]$

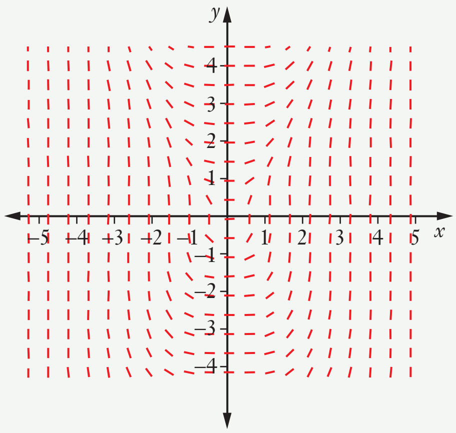
b $y = 2\tan(4x^3)$

11 **a** $N = 10\,000 + 495\,000e^{-0.17t}$

b **i** 100 428.3 ha **ii** 17 060.8 ha

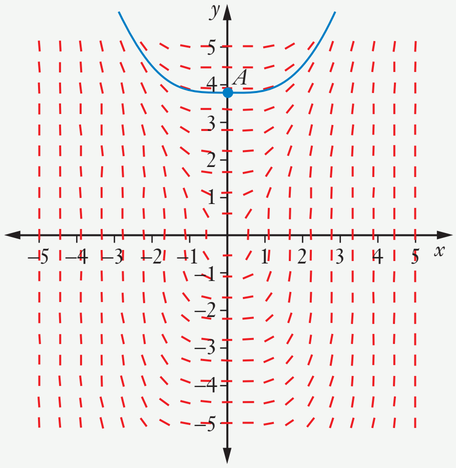
c 10 000 ha

12 **a**



b $y = \sqrt[3]{\frac{9x^4 + 256}{4}}$

c



13 **a** $Q = 300 + 2900e^{-0.013t}$

b **i** 3163 **ii** 3089

c **i** −37 bandicoots/year

ii −36 bandicoots/year

14 **a** $y = 2e^x$ **b** $y = \tan(x^2)$

c $f(x) = \sqrt[5]{2e^{3x} - 2}$

Challenge exercise 13

1 $t = \frac{1}{3}\ln\left(\frac{|x^3 + 3x|}{4}\right)$

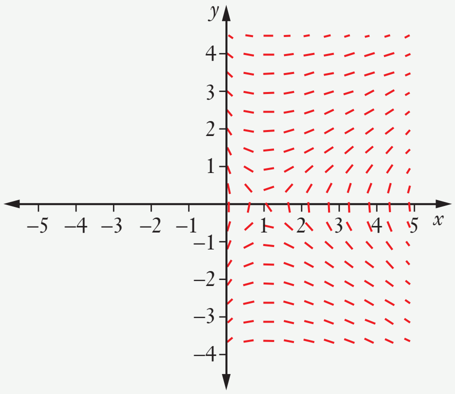
2 $y = \sin(\ln|x|)$

3 **a** $M = M_0 e^{-0.00866t}$

b **i** 91.7% **ii** 64.9% **iii** 42.1%

c **i** 0.79%
ii 0.56%
iii 0.36%

4



5 $y = -\ln|-2\cos x + 3|$

6 **a** $a = b = \frac{1}{45}$ **b** $Q = \frac{45}{1 + 149e^{-135\,t}}$

c **i** 4.1 m **ii** 44.9 m

d 45 m

7 **a** $N = \frac{15\,000}{1 + 599e^{-10\,2t}}$ **b** 14 673

c 15 000

Chapter 14

Exercise 14.01

1 **a** Discrete **b** Continuous

c Continuous **d** Discrete

e Continuous

2 a, d **3** a, d

4 $b = 7$ **5** $k = \frac{4}{625}$

6 **a** $a = \frac{1}{e(e^2 - 1)}$

b $a = \frac{1}{e(e^6 - 1)}$ **c** $a = \frac{1}{e^4 - 1}$

7 [0, 6] **8** $b = 8$

Exercise 14.02

1 **a** $\frac{1}{2}$ **b** $\frac{1}{6}$ **c** $\frac{1}{2}$

d $\frac{2}{3}$ **e** $\frac{1}{3}$

2 **a** $y = \frac{x}{50}$

b **i** $\frac{81}{100}$ **ii** $\frac{9}{100}$ **iii** $\frac{33}{100}$

v $\frac{8}{25}$ **v** $\frac{3}{4}$

3 **a** $a = \frac{3}{125}$

b **i** $\frac{27}{125}$ **ii** $\frac{63}{125}$ **iii** $\frac{117}{125}$

v $\frac{1}{125}$ **v** $\frac{37}{125}$

4 **a** $a = \frac{4}{81}$

b **i** $\frac{80}{81}$ **ii** $\frac{16}{81}$ **iii** $\frac{5}{27}$

v $\frac{1}{81}$

5 **a** $k = \frac{1}{e(e^5 - 1)}$

b **i** $\frac{e(e^3 - 1)}{e^5 - 1}$ **ii** $\frac{e^3 - 1}{e^5 - 1}$ **iii** $\frac{e^2(e^3 - 1)}{e^5 - 1}$

6 **a** $\int_0^{\frac{\pi}{2}} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}}$

$= -\cos\frac{\pi}{2} - (-\cos 0)$

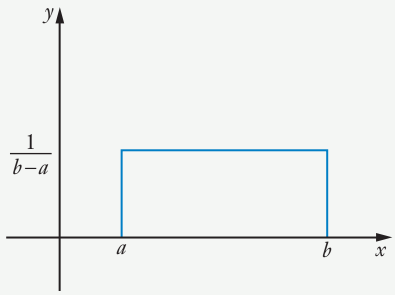
$= -0 + 1$

$= 1$

So $y = \sin x$ is a PDF in the domain $\left[0\ \frac{\pi}{2}\right]$

b **i** $\frac{1}{2}$ **ii** $1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2} - 1}{\sqrt{2}}$ **iii** $\frac{\sqrt{3}}{2}$

7 **a**



$A = bh$

$= (b - a) \times \frac{1}{b - a}$

$= 1$ so PDF

b **i** $\frac{3}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{4}$

Exercise 14.03

1 **a** $F(x)=\frac{x^3}{27}$ **b** $F(x)=\frac{x^4}{1296}$

c $F(x)=\frac{e^x-1}{e^4-1}$ **d** $F(x)=\frac{(x-2)^4}{625}$

e $F(x)=\frac{12x^2-x^3-40}{135}$

2 **a** $F(x)=\frac{x^5-1}{7776}$

b **i** $\frac{121}{3338}$ **ii** $\frac{31}{7776}$ **iii** $\frac{781}{1944}$

v $\frac{2251}{2592}$ **v** $\frac{31}{243}$

3 **a** $F(x)=\frac{x^4-81}{2320}$

b **i** $\frac{35}{464}$ **ii** $\frac{243}{464}$ **iii** $\frac{111}{145}$

v $\frac{429}{464}$ **v** $\frac{13}{29}$

4 **a** $F(x)=\frac{e^{2x}-1}{e^{10}-1}$

b **i** 0.0024 **ii** 0.14 **iii** 0.98

v 0.99 **v** 0.13

5 **a** $a=\frac{4}{6561}$ **b** $F(x)=\frac{x^4}{6561}$

c **i** $\frac{625}{6561}$ **ii** $\frac{256}{6561}$ **iii** $\frac{2465}{6561}$

v $\frac{80}{81}$ **v** $\frac{1280}{6561}$

6 **a** $a=\frac{1}{\ln 6}$ **b** $F(x)=\frac{\ln x}{\ln 6}$

c **i** 0.61 **ii** 0.39 **iii** 0.10

v 0.23 **v** 0.51

7 **a** Show that $\int_{\frac{3\pi}{2}}^{2\pi}\cos x\,dx=1$

b $F(x)=\sin x+1$

c **i** $\frac{2-\sqrt{3}}{2}$ **ii** $\frac{1}{\sqrt{2}}$

iii $\frac{\sqrt{3}-1}{2}$

8 **a** 2 **b** 3 **c** 3 **d** 7

e 4 **f** 4 **g** 6 **h** 8

i 4 **j** 7

9 **a** 3

b $F(x)=-\frac{1}{22}(x^3-9x^2+15x-2)$

c $\frac{1}{2}$

10 **a** $F(x)=\frac{1}{464}(x^4-12x^3+48x^2+4x-201)$

b **i** $\frac{9}{29}$ **ii** $\frac{393}{464}$ **iii** $\frac{73}{464}$

c 7 minutes

Exercise 14.04

1 **a** 6.35 **b** 5.89 **c** 6.09 **d** 3

e 3.11 **f** 7.57 **g** 8.79 **h** 4.45

i 5.29 **j** 3.73

2 **a** **i** 6.47 **ii** 6.05 **iii** 9.19

b **i** 4.24 **ii** 4.01 **iii** 5.62

c **i** 3.79 **ii** 3.62 **iii** 4.75

3 **a** 6.35 **b** 5.64

4 **a** 6.38 **b** 5.12 **c** 7.28 **d** 7.02

e 4.28 **f** 7.44

5 **a** 7.02 **b** 9.52 **c** 10.90

6 **a** $F(x)=\frac{x^4-16}{4080}$ **b** $\frac{203}{1360}$

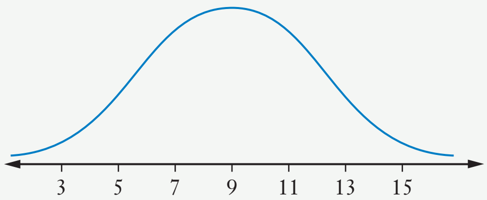
c $\frac{16}{17}$ **d** $\frac{29}{51}$

e 6.73 **f** 7.45

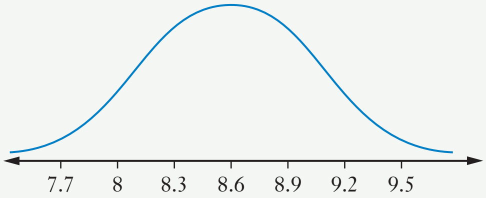
g 7.79 **h** 5.56

Exercise 14.05

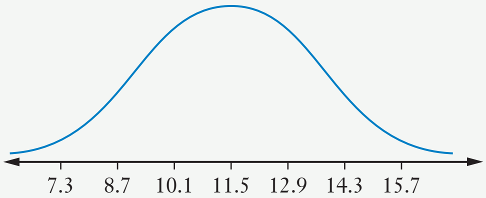
1 **a**



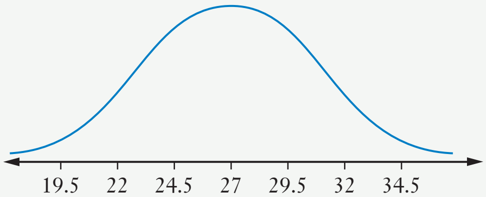
b



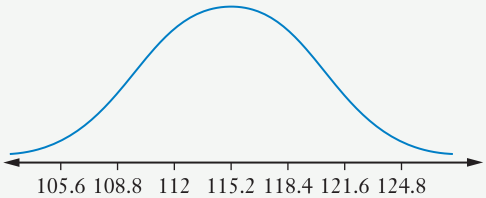
c



d

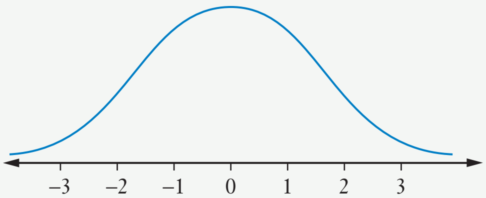


e



2 **a** $\mu = 23.7, \sigma = 4.2$ **b** $\mu = 5.4, \sigma = 0.9$
c $\mu = 59.7, \sigma = 5.4$ **d** $\mu = 209, \sigma = 10.6$
e $\mu = 11.3, \sigma = 2.2$

3

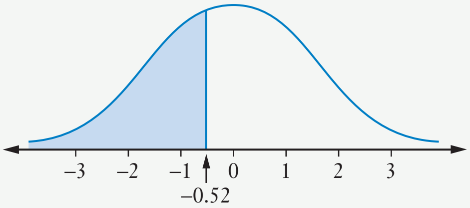


4 **a** 0.5 **b** 0.8413 **c** 0.9772
d 0.9987 **e** 0.1587 **f** 0.0228
g 0.0013 **h** 0.9332 **i** 0.0038
j 0.999

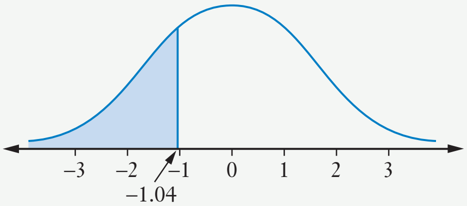
5 **a** 0.6772 **b** 0.0174 **c** 0.9778
d 0.031 **e** 0.9722 **f** 0.0837
g 0.9255 **h** 0.6826 **i** 0.9544
j 0.9974

6 **a** 0.84 **b** 0.67 **c** −0.53
d −0.84 **e** 1.23
f −1.17 or −1.18
g −0.52 **h** −0.67 **i** 0.33

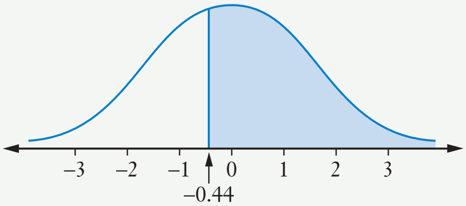
7 **a**



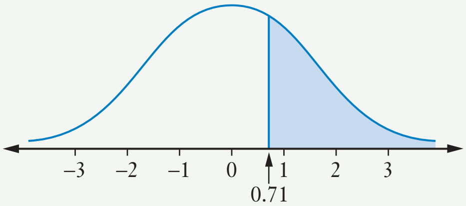
b



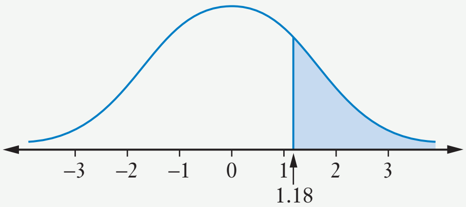
c



d



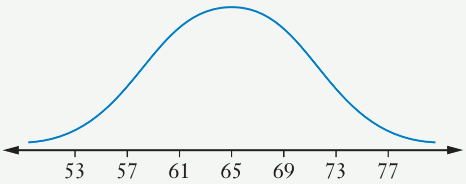
e



Exercise 14.06

1 **a** 68% **b** 95% **c** 99.7%
2 **a** 68% **b** 95% **c** 99.7%
3 **a** 68% **b** 95% **c** 99.7%
4 **a** 68% **b** 95% **c** 99.7%
d 34% **e** 49.85% **f** 97.35%

5 a



b **i** 95% **ii** 34% **iii** 49.85% **v** 81.5% **v** 81.5%

6 a 68% **b** 34% **c** 47.5% **d** 47.5% **e** 83.85%

Exercise 14.07

1 a **i** 0 **ii** 1 **iii** 2 **v** 3 **v** −1 **vi** −2 **vii** −3
b **i** 19.95 **ii** 15.27

2 a 27 cm to 79.2 cm
b **i** 0 **ii** 1 **iii** −1 **v** 2 **v** −2 **vi** 3 **vii** −3 **viii** 1.4

3 a 3.5° to 10.1°
b **i** 0 **ii** 1 **iii** 2 **v** 3 **v** −1 **vi** −2 **vii** −3 **viii** −0.7

4 a 54.8 mL to 78 mL
b **i** 0 **ii** 1 **iii** 2 **v** 3 **v** −1 **vi** −2 **vii** −3 **viii** 4.1
c **i** 73.36 mL **ii** 83.22 mL **iii** 62.92 mL **v** 53.06 mL

5 a **i** 2.7 **ii** −3.3 **iii** 2.4 **v** −1.3 **v** 4.9 **vi** −1.8 **vii** 3.3 **viii** −2.6
b 53.2, 90 and 82.7 **c** 62.1 and 59.7
d 80, 78.6, 62.1, 59.7 and 56.4

6 a **i** 1.29 **ii** −1.57
b **i** 11.26 mm **ii** 16.02 mm **iii** 18.68 mm **v** 13.136 mm **v** 16.93 mm

7 28 **8** 26.85 **9** 50

10 a 53.4 **b** 42.2 **c** 59 **d** 69.08 **e** 43.88 **11** 48.3

12 a **i** 122 to 198 **ii** 14.1 to 17.9 **iii** 10.3 to 21.7
b **i** 2.1 **ii** −1.3
c **i** 10.3 **ii** 18.09

13 a 99.6 to 109.8
b **i** −4.8 **ii** −0.3
c **i** 114.9 **ii** 98.07

Exercise 14.08

1 a **i** 99.7% **ii** 47.5% **iii** 68% **v** 49.85% **v** 83.85%
b **i** 4 **ii** Yes: outside normal range
c **i** −1.5 **ii** 0.0668 **iii** 0.2417

2 a **i** 68% **ii** 95% **iii** 99.7% **v** 34% **v** 81.5%
b **i** −3.3 **ii** Yes: outside normal range
c **i** 3.7 **ii** Yes: outside normal range

3 a 4.1 – no: outside normal range
b 4.6 min to 11.4 min
c 97.35%
d **i** −0.29 **ii** 11.41%
e **i** 0.381 **ii** 0.721 **iii** 0.8418
f **i** 98.9% **ii** 68.32% **iii** 98.5%

4 a **i** 95% **ii** 49.85% **iii** 39.83%
b 18.7 mL to 21.1 mL
c Unusual – outside range
d 0.0606

5 a 95% **b** 8.1 cm **c** 6.9 cm

6 a 2.2 years
b **i** 68% **ii** 88.54%
c Yes – outside normal range

7 a 25.5 cm to 30.5 cm
b −0.96 **c** 82.33%
d No – outside normal range

8 a 5% **b** 2.5% **c** **i** 0.3%
ii greater than 5.4 kg, less than 4.5 kg

9 a **i** 99.7% **ii** 12.1 cm **iii** 9.1 cm
b 8.3 cm, 12.6 cm

10 Yes: Minimum weight within normal limits is 251.3 g.

11 2.5%

12 a 2 mm **b** 820 mm
c 816 mm to 824 mm

13 a 0.2 h **b** 2.5 h

14 **a** Epping 1.04, city 1.35 **b** City

15 **a** 1 **b** 2.04 **c** Cameron

Test yourself 14

1 C **2** A **3** D

4 **a** 0.75 **b** 0.4 **c** 0.55
d 0.45 **e** 0.7 **f** 10
g 3.6 **h** 17.8 **i** 12
j 15

5 **a** Yes **b** Yes **c** No

6 3.98

7 **a** $F(x) = \frac{x^3 - 1}{511}$

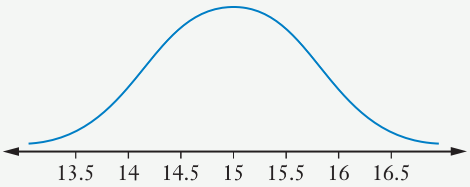
b **i** $\frac{26}{511}$ **ii** $\frac{124}{511}$ **iii** $\frac{296}{511}$
v $\frac{64}{73}$ **v** $\frac{335}{511}$

8 **a** **i** 4 **ii** 2.9
b **i** 9 **ii** 7.1

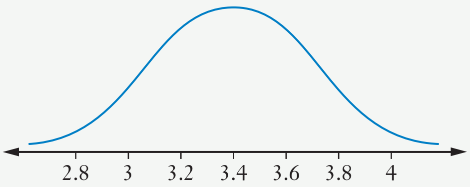
9 **a** 2.27 kg to 4.13 kg
b **i** 2.26 **ii** 0.97
c 0.1541

10 [0, 4]

11 **a**



b



12 **a** 0.3% **b** 18.5%

13 **a** 5.7 **b** 6.4 **c** 5.35
d 6.09 **e** 4.87

14 Mean 2.9, standard deviation 0.7

15 **a** $F(x) = \frac{x^5}{3125}$ **b** $F(x) = \frac{x^7}{823543}$

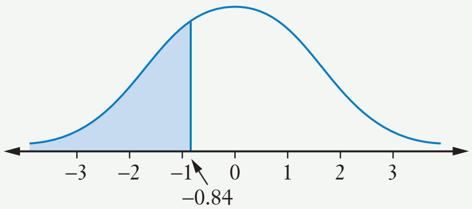
c $F(x) = \frac{e^x - 1}{e^6 - 1}$ **d** $F(x) = \frac{x^2 - 1}{80}$

16 **a** 0.7054 **b** 0.9066 **c** 0.0013
d 0.2389 **e** 0.8413 **f** 0.0062
g 0.8599 **h** 0.1272 **i** 0.1348

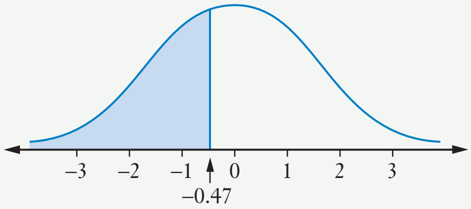
17 **a** 95% **b** 34% **c** 83.85%
d 90.5% **e** 25.86%

18 **a** $a = \frac{1}{e^5 - 1}$ **b** $a = \frac{1}{e(e^3 - 1)}$

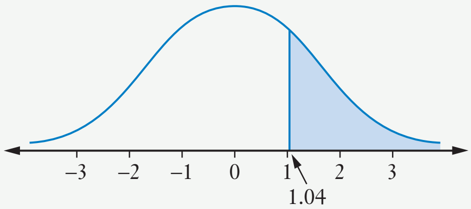
19 **a**



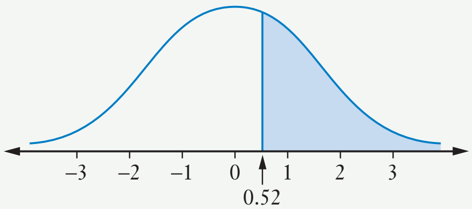
b



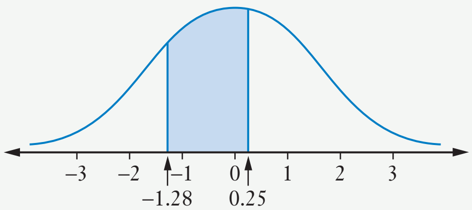
c



d



e



20 **a** 8 **b** 1.5 **c** 2

21 **a** 0.83 **b** 1
c Klare

22 $F(x) = \frac{1}{600}\left[\frac{x^4}{4}\right]^7 = 1$ so PDF

23 **a** **i** 2.5 **ii** −1.5

b **i** 1.12 m **ii** 1.06 m **iii** 1.16 m

v 1.09 m **v** 1.13 m

24 89.98

25 49.17

Challenge exercise 14

1 $a = 1, b = 5$

2 $\mu = 13.2, \sigma = 0.4$

3 0

4 $F(x) = \frac{(x^2+1)^3 - 1000}{124000}$

5 Mean 18.9, standard deviation 3.5

6 Show area is 1.

Chapter 15

Exercise 15.01

1 **a** Yes **b** Yes **c** No **d** No

e Yes **f** Yes **g** No **h** Yes

i No **j** Yes

2 **a** $P(X=x)=\begin{cases} \frac{7}{9} & \text{for } x=1 \\ \frac{2}{9} & \text{for } x=0 \end{cases}$

b $P(X=x)=\begin{cases} 84\% & \text{for } x=1 \\ 16\% & \text{for } x=0 \end{cases}$

c $P(X=x)=\begin{cases} \frac{1}{2} & \text{for } x=1 \\ \frac{1}{2} & \text{for } x=0 \end{cases}$

d $P(X=x)=\begin{cases} \frac{1}{150} & \text{for } x=1 \\ \frac{149}{150} & \text{for } x=0 \end{cases}$

e $P(X=x)=\begin{cases} \frac{49}{213} & \text{for } x=1 \\ \frac{164}{213} & \text{for } x=0 \end{cases}$

3 **a**

X	0	1
$P(X=x)$	$\frac{6}{13}$	$\frac{7}{13}$

b

X	0	1
$P(X=x)$	$\frac{12}{13}$	$\frac{1}{13}$

c

X	0	1
$P(X=x)$	$\frac{8}{9}$	$\frac{1}{9}$

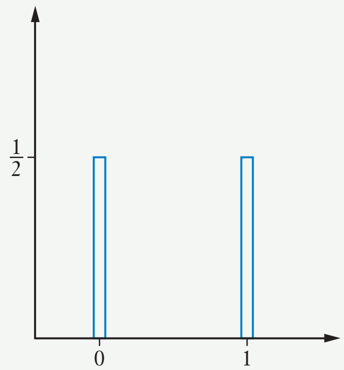
d

X	0	1
$P(X=x)$	$\frac{3}{5}$	$\frac{2}{5}$

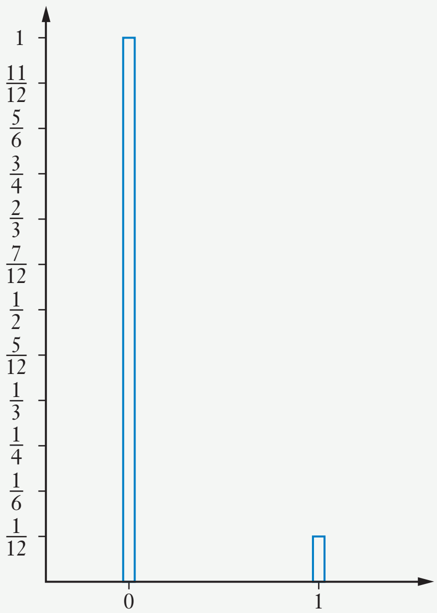
e

X	0	1
$P(X=x)$	$\frac{2}{3}$	$\frac{1}{3}$

4 **a** **i**



ii



b i is uniform.

5 **a** **i** 0.7 **ii** 0.21

b **i** $\frac{14}{15}$ **ii** $\frac{14}{225}$

c **i** $\frac{2}{3}$ **ii** $\frac{2}{9}$

d **i** 52% **ii** 24.96%

e **i** $\frac{1}{100}$ **ii** $\frac{99}{10\,000}$

6 **a** $P(X=x)=\begin{cases}\frac{3}{4} & \text{for } x=1\\ \frac{1}{4} & \text{for } x=0\end{cases}$

b **i** $\frac{3}{4}$ **ii** $\frac{3}{16}$

7 Mean 8%, variance 7.36%

8 **a** $\frac{9}{13}$ **b** $\frac{36}{169}$ **c** $\frac{6}{13}$

9 **a** $P(X=x)=\begin{cases}\frac{1}{6} & \text{for } x=1\\ \frac{5}{6} & \text{for } x=0\end{cases}$

b $E(X)=\frac{1}{6}$, $\text{Var}(X)=\frac{5}{36}$, $\text{sd}=\frac{\sqrt{5}}{6}$

10 **a** $\frac{9}{100}$

b

X	0	1
$P(X=x)$	$\frac{1}{10}$	$\frac{9}{10}$

11 $\frac{1}{3}$ $\frac{2}{3}$

12 $P(X=x)=\begin{cases}056 & \text{for } x=1\\ 044 & \text{for } x=0\end{cases}$ or

$P(X=x)=\begin{cases}044 & \text{for } x=1\\ 056 & \text{for } x=0\end{cases}$

Exercise 15.02

1 **a** Yes **b** Yes **c** No
d Yes **e** No

2 **a** $\frac{105}{512}$ **b** $\frac{15}{128}$ **c** $\frac{45}{1024}$
d $\frac{5}{512}$ **e** $\frac{11}{1024}$

3 **a** $\frac{7}{128}$ **b** $\frac{7}{128}$ **c** $\frac{29}{128}$

4 **a** $\frac{109\,375}{419\,904}$ **b** $\frac{875}{209\,952}$ **c** $\frac{175}{419\,904}$
d $\frac{41}{1\,679\,616}$ **e** $\frac{1\,015\,625}{1\,679\,616}$

5 **a** $\frac{25}{216}$ **b** $\frac{5}{324}$ **c** $\frac{125}{324}$
d $\frac{425}{432}$ **e** $\frac{7}{432}$

6 **a** 0.254 **b** 0.01 **c** 0.0467
d 0.198 **e** 0.552

7 **a** $\frac{144}{625}$ **b** $\frac{243}{3125}$ **c** $\frac{48}{625}$
d $\frac{992}{3125}$ **e** $\frac{2133}{3125}$

8 **a** 90.1% **b** 0.4% **c** 99.6%

9 **a** 11.4% **b** 16.4% **c** 1.2%

10 **a** $\frac{69\,984}{5\,764\,801}$ **b** $\frac{46\,656}{823\,543}$ **c** $\frac{57\,591}{823\,543}$
d $\frac{147\,456}{823\,543}$ **e** $\frac{65\,536}{5\,764\,801}$

11 $^{20}C_7\left(\frac{3}{11}\right)^7\left(\frac{8}{11}\right)^{13}$

12 **a** **i** $\frac{12}{25}$ **ii** $\frac{16}{25}$

b **i** 0.25 **ii** 0.17

13 **a** 0.23 **b** 0.07 **c** 0.03

14 $^{20}C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{17} \times {^8C_2}\left(\frac{3}{11}\right)^2\left(\frac{8}{11}\right)^6$

15 $^{12}C_3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^9 \times {^8C_5}\left(\frac{5}{6}\right)^5\left(\frac{1}{6}\right)^3$

16 0.000 003 4

17 7 people

18 **a** 8 **b** $^{50}C_8\left(\frac{1}{6}\right)^8\left(\frac{5}{6}\right)^{42}$

19 **a** 25 **b** 0.0918

20 **a** **i** $\frac{35}{72}$ **ii** $\frac{31}{72}$ **iii** $\frac{37}{72}$

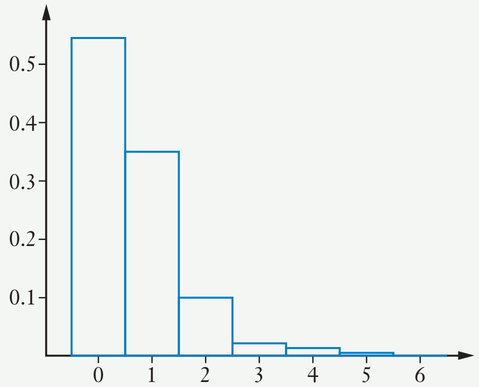
b $\frac{101}{144}$ **c** 0.113

21 **a** $\frac{53}{112}$ **b** 0.127

Exercise 15.03

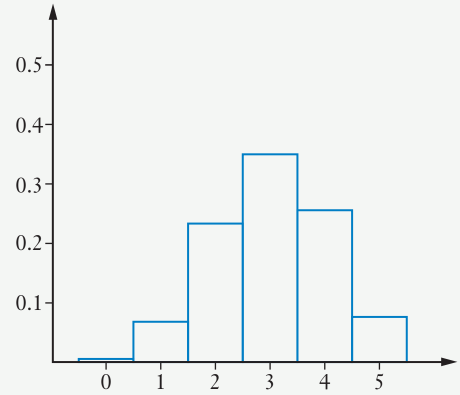
1 **a** C **b** A **c** B

2 **a**



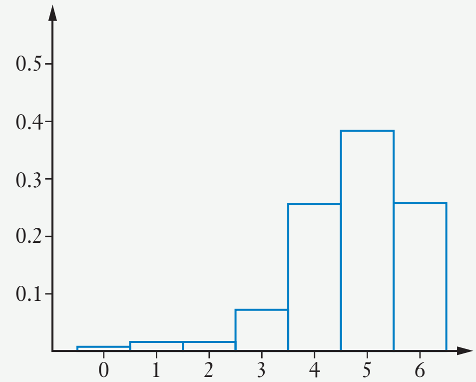
Positively skewed

b



Symmetrical.

c



Negatively skewed

3 **a** **i** 75 **ii** 37.5

b **i** 40 **ii** $13\frac{1}{3}$

c **i** 5 **ii** $4\frac{1}{6}$

d **i** 0.96 **ii** 0.8448

e **i** 25 **ii** 24.375

f **i** $71\frac{3}{7}$ **ii** $20\frac{20}{49}$

g **i** 1.1 **ii** 1.078

h **i** $5\frac{5}{9}$ **ii** $5\frac{65}{162}$

i **i** $3\frac{8}{9}$ **ii** $1\frac{59}{81}$

j **i** $28\frac{3}{4}$ **ii** $12\frac{7}{32}$

4 **a** $\frac{2}{9}$

b **i** $11\frac{1}{9}$ **ii** $8\frac{52}{81}$

5 **a** 232.5 **b** 16.275 **c** 4.034

6 **a** 3.2 **b** 6.72

7 $\frac{41}{100}$ **8** 63 **9** 19

10 $P(X=x)=\begin{cases}\frac{3}{5} & \text{for } x=1\\ \frac{2}{5} & \text{for } x=0\end{cases}$

11

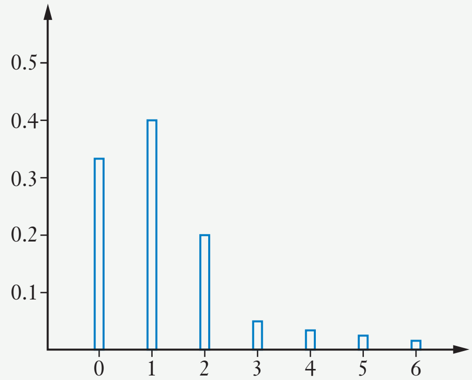
X	0	1
P(X = x)	$\frac{1}{5}$	$\frac{4}{5}$

12 **a** $X = \{0, 1, 2, 3, 4, 5, 6\}$

b

X	0	1	2	3	4	5	6
P(X = x)	0.33	0.4	0.2	0.05	0.008	0.0006	0.00002

c



d Positively skewed

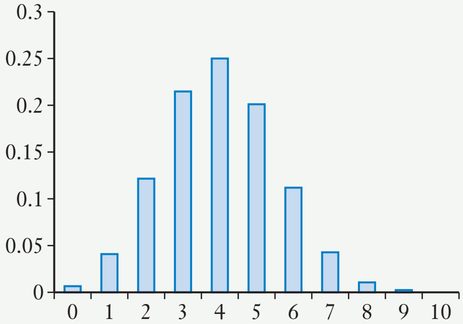
e 0.05862

Exercise 15.04

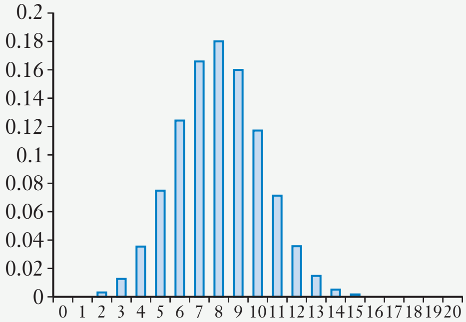
1 **a** $\frac{89}{1075}$ **b** $\frac{7}{800}$ **c** $\frac{5}{104}$

d $\frac{58}{295}$ **e** $\frac{31}{209}$

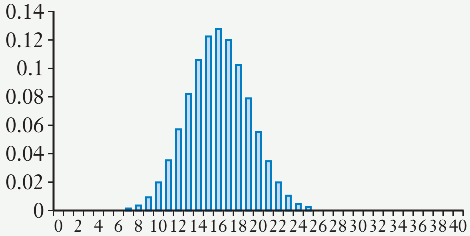
2 **a**



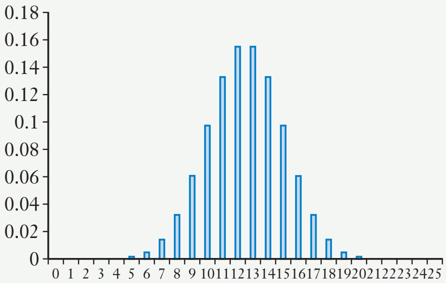
b



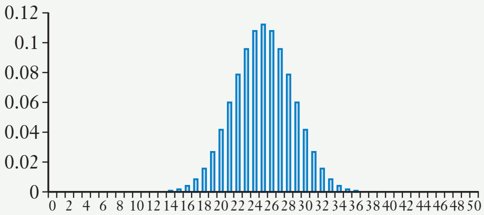
c



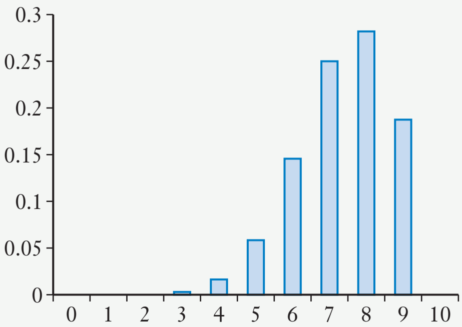
d



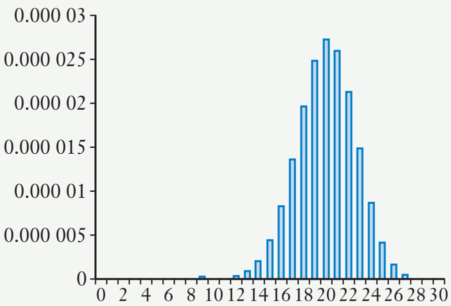
e



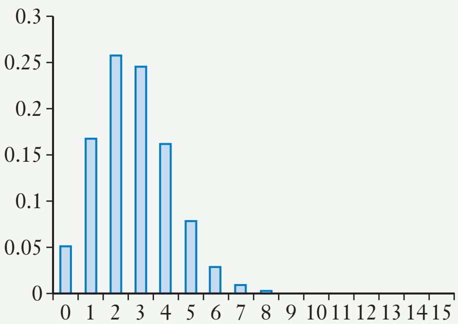
f



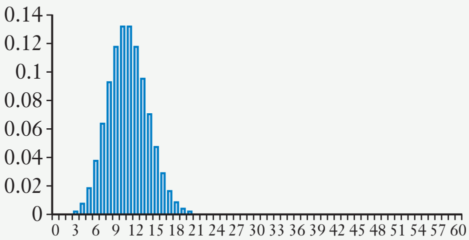
g



h



i



3 **a** $\mu = 0.63, \sigma^2 = 0.00777$
b $\mu = 0.21, \sigma^2 = 0.003$
c $\mu = 0.7, \sigma^2 = 0.0044$
d $\mu = 0.42, \sigma^2 = 0.0032$
e $\mu = 0.55, \sigma^2 = 0.0024$

4 **a** 6 **b** $\mu = 0.047, \sigma = 0.019$
c **i** 0.67 **ii** 0.16 **iii** −1.24
v −0.36 **v** 2.74
d **i** 0.7486 **ii** 0.4364 **iii** 0.0031
v 0.2519

5 **a** 8 **b** $\mu = 0.075, \sigma = 0.026$
c **i** −1.33 **ii** −0.95 **iii** 0.19
v 0.95
d **i** 0.1711 **ii** 0.8289 **iii** 0.4247
v 0.9082 **v** 0.7371

6 **a** 7 **b** $\mu = 0.04, \sigma = 0.0146$
c **i** −0.68 **ii** −1.03
iii 0.68 **v** 0.14
d **i** 0.2483 **ii** 0.8485
iii 0.2483 **v** 0.196

7 **a** 0.0188 **b** 0.9177 **c** 0.8989

8 **a** 0.0694 **b** 0.0694 **c** 0.9985
d 0.0679

Test yourself 15

1 B **2** B **3** B, C **4** C

5 **a** 10.5 **b** 10.24 **c** 0.055

6 **a** 0.041 **b** 0.205 **c** 0.000 75
d 0.0096 **e** 0.000 008 8

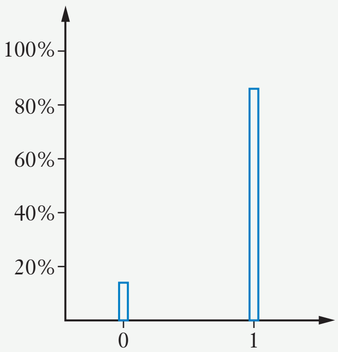
7 **a** 7.2% **b** $\mu = 0.072, \sigma = 0.016$
c **i** 1.10 **ii** −1.35 **iii** 0.80
v −0.18 **v** 1.71
d **i** 0.0885 **ii** 0.1357
iii 0.3595 **v** 0.8679

8 **a** $f(x) = \begin{cases} 86\% & \text{for } x = 1 \\ 14\% & \text{for } x = 0 \end{cases}$

b

x	0	1
$f(x)$	14%	86%

c



9 **a** $\frac{9}{115}$ **b** $\frac{4}{29}$

10 **a** 58.3% **b** 31.9% **c** 8.3% **d** 98.5%

11 **a** $\mu = 0.11, \sigma = 0.042$

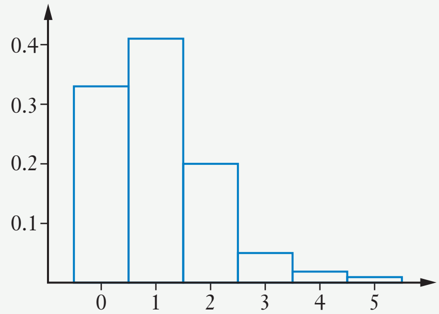
b **i** -0.24 **ii** 0.95 **iii** -1.42

v 2.13

c **i** 7.78% **ii** 17.11% **iii** 57.82%

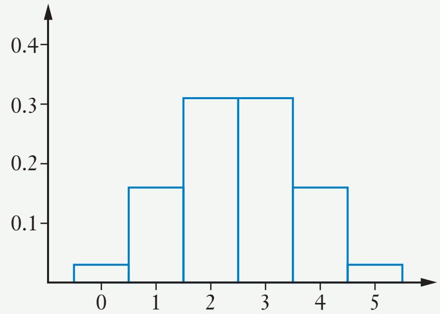
12 Mean 0.025, variance 0.024375

13 **a**



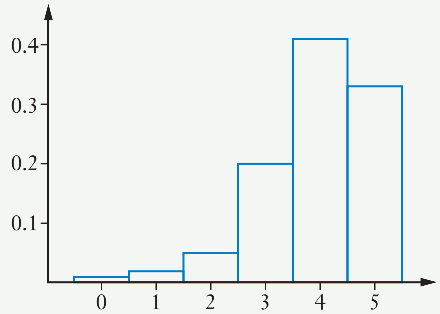
Positively skewed

b



Symmetrical

c



Negatively skewed

14 **a** **i** 50 **ii** 25

b **i** 47.2 **ii** 20.99

c **i** 4.17 **ii** 3.47

d **i** 5.2 **ii** 3.848

e **i** 26.6 **ii** 25.6

15 **a** **i** 0.4 **ii** 0.24

b **i** $\frac{5}{7}$ **ii** $\frac{10}{49}$

c **i** $\frac{2}{3}$ **ii** $\frac{2}{9}$

d **i** 0.84 **ii** 0.1344

e **i** 0.71 **ii** 0.2059

16 **a** $\mu = 0.3, \sigma^2 = 0.0053$

b $\mu = 0.59, \sigma^2 = 0.0034$

c $\mu = 0.5, \sigma^2 = 0.0038$

d $\mu = 0.87, \sigma^2 = 0.0011$

e $\mu = 0.42, \sigma^2 = 0.0097$

17 **a** $f(x) = \begin{cases} \frac{1}{9} & \text{for } x = 1 \\ \frac{8}{9} & \text{for } x = 0 \end{cases}$

b Mean $\frac{1}{9}$, var $\frac{8}{81}$, sd $\frac{2\sqrt{2}}{9}$

18 **a** $\frac{7}{32}$ **b** $\frac{1}{32}$ **c** $\frac{7}{64}$

d $\frac{7}{64}$ **e** $\frac{9}{256}$

19 **a** $\mu = 0.28, \sigma^2 = 0.038$

b **i** 0.0174 **ii** 0.2981 **iii** 0.7523

20 Mean 158.1, var 11.067, sd 3.327

Challenge exercise 15

1 $p = 0.87$ or $p = 0.13$

2 **a** **i** $\frac{4\,960\,116}{19\,487\,171}$ **ii** $\frac{4160}{19\,487\,171}$

b 0.43

3 **a** 3.33% **b** Mean 0.033, SD 0.0146

c **i** 0.46 **ii** -0.45 **iii** 1.37

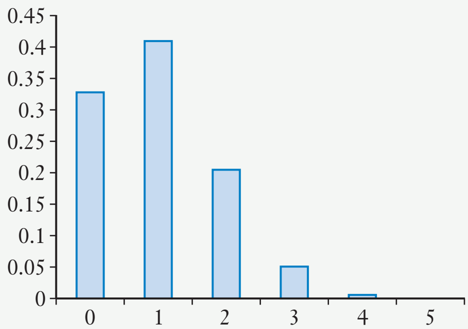
v 0.92

d **i** 0.3264 **ii** 0.6736 **iii** 0.0935

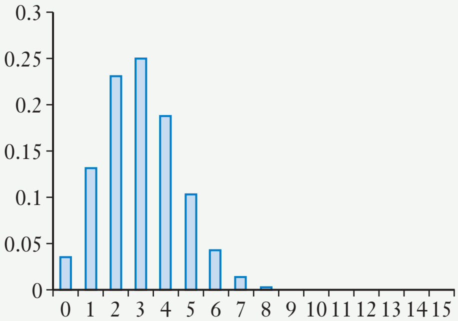
v 0.1736

4 0.0012

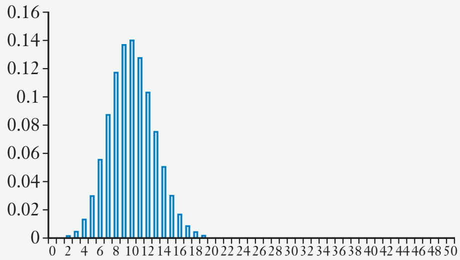
5 **a**



b



c



Practice set 4

1 A **2** C **3** B **4** B **5** C

6 C **7** A **8** B **9** C **10** D

11 **a** 3 **b** 2.994 **c** 0.001 15

12 **a** $\frac{1}{3}e^{3x} + C$ **b** $2x^2 - 3x + C$

c $\frac{1}{4}\tan 4x + C$ **d** $\ln|x - 3| + C$

e $2\sin^{-1}\frac{x}{2} + C$

13 **a** 91.3% **b** 8.3% **c** 99.7%

14 **a** $T = 26 + 249e^{-0.065t}$

b **i** 156°C **ii** 61.4°C

c **i** –84°/mn **ii** –2.3°/min

d 26°C

15 **a** \$40 995.50 **b** \$745.60

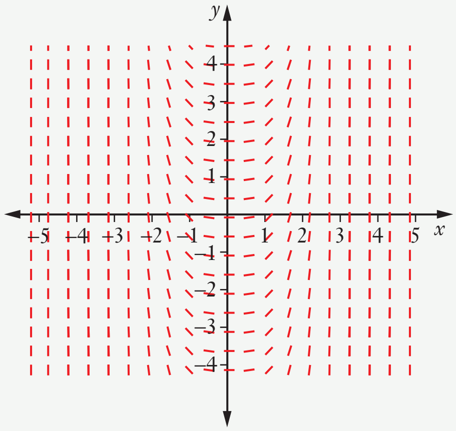
16 \$945

17 **a** $y = 6e^{5x}$ **b** $T = 32 + 403e^{-0.034t}$

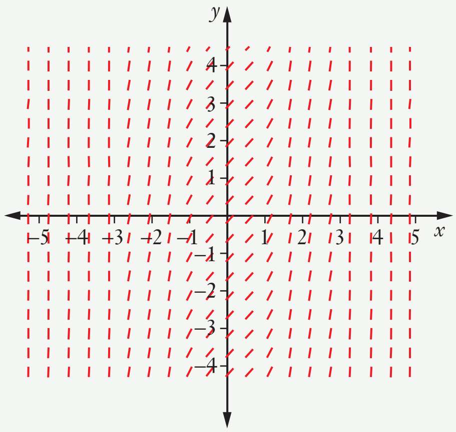
c $N = \dfrac{5000}{1 + 4e^{-32t}}$

18 2.4 m

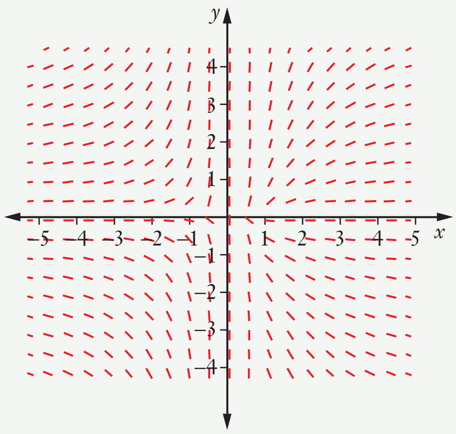
19 **a**



b



c



20 **a** $F(x)=\dfrac{x^3-1}{342}$

b **i** $\dfrac{62}{171}$ **ii** $\dfrac{7}{342}$ **iii** $\dfrac{158}{171}$

v $\dfrac{31}{38}$ **v** $\dfrac{21}{38}$

c **i** 5.56 **ii** 6.83

21 **a** Show RHS = LHS

b $N=\dfrac{3000}{1+29e^{-036\,t}}$

c **i** 277 **ii** 691

d **i** 90 bulbs/year **ii** 191 bulbs/year

e 3000

22 **a** 2 **b** 7 **c** 77

23 **a** $\begin{pmatrix}2\\-16\end{pmatrix}$ **b** $\begin{pmatrix}-8\\8\end{pmatrix}$ **c** 5, 233° 8′

d 33 **e** 59° 29′

24 **a** **i** \$119 000 **ii** \$117 995

iii \$116 984.98

b **i** See worked solutions.

ii \$99 020.88

25 **a** $y=\sin^{-1}\left(\dfrac{x^2-2}{2}\right)$ **b** $y=3\sin(x^3)$

26 \$2851.52

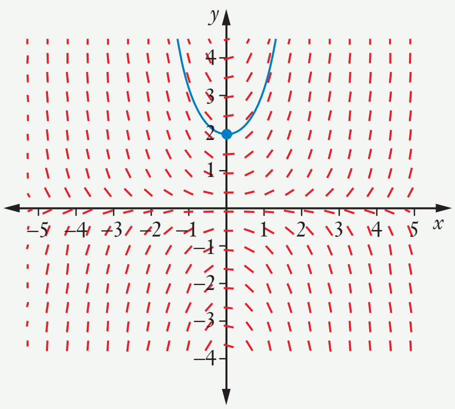
27 4.23

28 $\dfrac{8}{45}$

29 (–1, 9) maximum, (5, –99) minimum, (2, –45) point of inflection

30 \$180.76

31



32 **a** 3.9 L to 5.5 L **b** 68%

c **i** 3.25 **ii** 1.25 **iii** –2 **v** –1.5

d **i** 0.9994 **ii** 0.0228 **iii** 0.9332

v 0.1056 **v** 0.8716

e **i** 73.33% **ii** 78.88% **iii** 99.4%

33 **a** \$460.50 **b** \$449

c \$1084.54 **d** \$28.49

34 **a** $F(x)=\dfrac{(x+2)^3-8}{335}$ **b** $F(x)=\dfrac{x^4-1}{624}$

c $F(x)=2\sin x$

35 2

36 **a** $M=65e^{-0.24t}$

b $y=\dfrac{\sqrt[3]{9-\cos 2x}}{2}$

c $y=\sin(\ln|x|)$

37 **a** $\dfrac{68}{125}$ **b** $\dfrac{29}{150}$

38 **a** 0.9115 **b** 0.8106

c 0.1587 **d** 0.0207

e 0.9966

39 **a** $\dfrac{4}{9}$ **b** $\dfrac{20}{81}$

40 2nd exam

41 **a** 28% **b** $\mu=0.28,\ \sigma=0.0518$

c **i** 0.39 **ii** –2.51 **iii** –0.58 **v** –1.58

v 0.29

d **i** 0.0618 **ii** 0.3483

iii 0.2750 **v** 0.5570

42 **a** $\mu=0.45,\ \sigma^2=0.0031$

b $\mu=0.2,\ \sigma^2=0.00128$

c $\mu=0.5,\ \sigma^2=0.0036$

43 \$277.33

44 **a** 0.052 **b** 0.192

45 [1, 6]

46 **a** 13.02 **b** 10.55 **c** 16.348

d 8.6

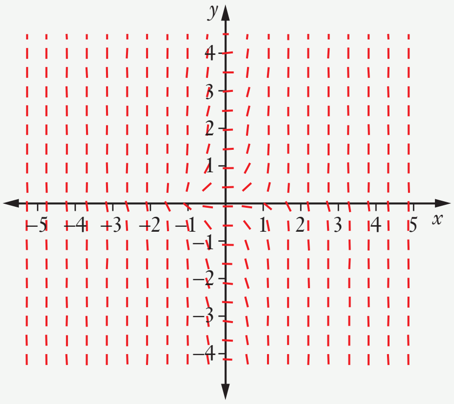
47 **a**

X	0	1
$P(X=x)$	99.5%	0.5%

b $\mu=0.005,\ \sigma^2=0.004\,975$

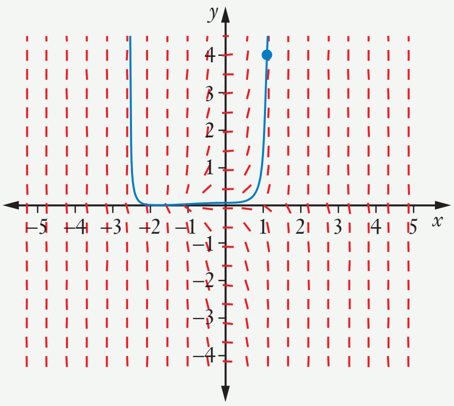
48 8

49 **a**



b $y = 4e^{3x} - 3$

c



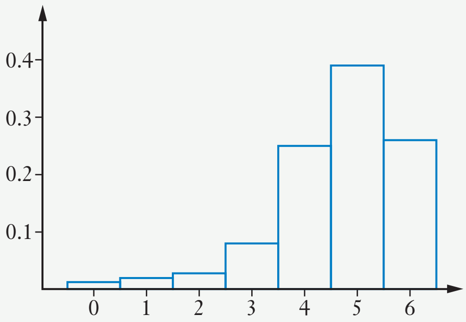
50 **a** 0.111 **b** 0.251 **c** 0.046

51 18.5

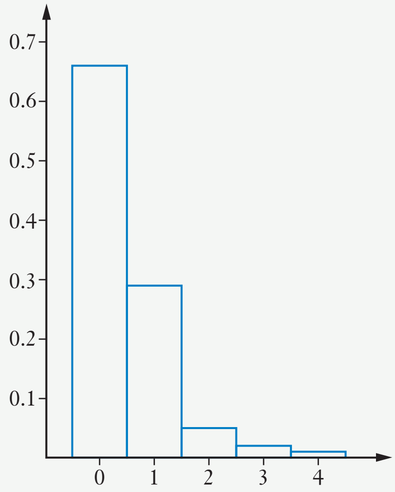
52 **a** 5%, reject cans below 349.4 and above 350.2 mL

b 0.3%, reject cans below 349.2 and above 350.4 mL

53 **a**



b



54 **a** \$2748.78 **b** \$2751.11 **c** \$2751.63

55 $f(x) = \begin{cases} 034 & \text{for } x = 1 \\ 066 & \text{for } x = 0 \end{cases}$ or $f(x) = \begin{cases} 066 & \text{for } x = 1 \\ 034 & \text{for } x = 0 \end{cases}$

56 **a** 5% **b** $\mu = 0.05, \sigma = 0.028$

c **i** 0.73 **ii** 0.12 **iii** 0.72

57 **a** **i** 5 **ii** $4\frac{1}{6}$

b **i** 10 **ii** 5

c **i** $24\frac{1}{2}$ **ii** $7\frac{7}{20}$

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